NEW IDENTITIES OBTAINED FROM GEGENBAUER SERIES EXPANSION

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Abstract. Using the expansion in a Fourier-Gegenbauer series, we prove several identities that extend and generalize known results. In particular, it is proved among other results, that

$$\sum_{n=0}^{\infty} \frac{1}{4^n} \binom{2n}{n} \left( \frac{z-2n}{z-1/2} \right)^3 \left( \frac{z}{n} \right)^3 = \frac{\tan(\pi z)}{\pi}$$

for all complex numbers $z$ such that $\Re(z) > -\frac{1}{2}$ and $z \notin \frac{1}{2} + \mathbb{Z}$.

1. Introduction and Notation

For a complex number $a$ and a nonnegative integer $n$ we define the rising factorial $(a)_n$ as follows.

$$(a)_n \overset{\text{def}}{=} \prod_{0 \leq j < n} (a + j) = \frac{\Gamma(a + n)}{\Gamma(a)} \quad (1.1)$$

where $\Gamma$ is the well-known Eulerian Gamma function (where in the last equality we assume that $a$ is not 0 or a negative integer). We will also introduce a notation for $(a)_n/n!$ namely

$$\langle a \rangle_n \overset{\text{def}}{=} \frac{(a)_n}{n!} = (-1)^n \left( \frac{-a}{n} \right) = \left( \frac{a + n - 1}{n} \right). \quad (1.2)$$

The Gegenbauer (or ultraspherical) polynomials $(C^{(\lambda)}_n)_{n \geq 0}$ can be defined by the generating function

$$\frac{1}{(1 - 2xz + z^2)^{\lambda}} = \sum_{n=0}^{\infty} C^{(\lambda)}_n(x) z^n, \quad -1/2 < \lambda \neq 0, \ |z| < 1. \quad (1.3)$$

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They are orthogonal polynomials on \([-1, 1]\) with respect to the weight function \(\omega_{\lambda}(x) = (1 - x^2)^{\lambda-1/2}\). Furthermore,

\[
\|C_n^{(\lambda)}\|_2^2 = \int_{-1}^{1} (C_n^{(\lambda)}(x))^2 \omega_{\lambda}(x) \, dx,
\]

\[
= \frac{\sqrt{\pi} \Gamma(\lambda + 1/2)}{\Gamma(\lambda)} \cdot \frac{1}{n + \lambda} \left\langle \frac{2\lambda}{n} \right\rangle.
\] (1.4)

Let \(f\) be a measurable function on \((-1, 1)\) such that the integral \(\int_{-1}^{1} \omega_{\lambda}(x) |f(x)| \, dx\) is convergent, (that is \(f\) is integrable with respect to the measure \(\omega_{\lambda}(x) dx\)), we may define the \(\lambda\)-Gegenbauer coefficients \(a_n(f)\) by the formula

\[
a_n(f) = \frac{1}{\|C_n^{(\lambda)}\|_2^2} \int_{-1}^{1} f(x)C_n^{(\lambda)}(x)\omega_{\lambda}(x) \, dx, \quad n = 0, 1, 2, \ldots \) (1.5)

Then we may consider the formal expansion of \(f\) in a \(\lambda\)-Gegenbauer series

\[
S(f) = \sum_{n=0}^{\infty} a_n(f)C_n^{(\lambda)},
\] (1.6)

and the partial sums

\[
S_n(f) = \sum_{k=0}^{n} a_k(f)C_k^{(\lambda)}. \] (1.7)

In order to settle the question of convergence of these series to the function \(f\) we will use two results. First, the “equiconvergence theorem” [9, 9.1.2] that we recall the statement specialized to the case of \(\lambda\)-Gegenbauer polynomials for the convenience of the reader.

**Theorem 1** (Szegö). Let \(f\) be Lebesgue-measurable in \([-1, 1]\), and let the integrals

\[
\int_{-1}^{1} (1 - x^2)^{(\lambda-1/2)} |f(x)| \, dx, \quad \int_{-1}^{1} (1 - x^2)^{(\lambda-1/2)} |f(x)| \, dx
\] (1.8)

exist. If \(S_n(f)\) denotes the \(n\)th partial sum of the expansion of \(f\) in \(\lambda\)-Gegenbauer series, and \(\tilde{S}_n(f)\) is the \(n\)th partial sum of the Fourier (cosine) series of \(\theta \mapsto \tilde{f}(\theta) \overset{\text{def}}{=} |\sin \theta|^\lambda f(\cos \theta)\) then for \(x \in (-1, 1)\)

\[
\lim_{n \to \infty} \left( S_n(f)(x) - (1 - x^2)^{-\lambda/2} \tilde{S}_n(f)(\arccos(x)) \right) = 0
\] (1.9)

Moreover, the convergence is uniform in \([-1 + \epsilon, 1 - \epsilon]\), where \(\epsilon\) is a fixed positive number, \(\epsilon < 1\).

Then, the above theorem is combined with the next theorem [9, Chapter II, (8.14)], (that we also recall the statement), to conclude the convergence of the \(\lambda\)-Gegenbauer series expansion.
Theorem 2 (Zygmund). Suppose that $f$ is integrable, $2\pi$-periodic, and of bounded variation in an interval $I$. Then the Fourier series of $f$ converges to $\frac{1}{2}[f(x^+) + f(x^-)]$ at every point $x$ interior to $I$. If, in addition, $f$ is continuous in $I$, the convergence is uniform in any interval interior to $I$.

When $\lambda = 1/2$, $C_n^{(1/2)}(x) = P_n(x)$ the Legendre polynomial of degree $n$. Legendre polynomials play an important role in our development because most of the formulas take an aesthetically beautiful appearance in this case.

Chebyshev polynomials $(T_n)_{n \geq 0}$ and $(U_n)_{n \geq 0}$ of the first and second kind, respectively, are also special types of Gegenbauer polynomials, but we prefer to define them by the functional identities:

$$T_n(\cos \theta) = \cos \theta, \quad U_n(\cos \theta) \sin \theta = \sin((n + 1)\theta), \quad (\theta \in \mathbb{R}). \quad (1.10)$$

For detailed information on these polynomials we refer the reader to Ismail [4, Chapter 4], Milovanović et al. [6, Chapter 1.2], [7, Chapter 18] and the references cited therein.

In Theorem 3 and Theorem 4, which are the cornerstones of our investigation, it is proved that, for $|x| < 1$, one has

$$\frac{U_m(x)}{(1 - x^2)^{\lambda - 1}} = (m + 1)\sqrt{\pi} \Gamma(\lambda) \sum_{n=0}^{\infty} \frac{\lambda + m + 2n}{m + n + 1} \frac{\lambda}{2n} \frac{\lambda}{m + 2n} C_{m+2n}(x)$$

and

$$\frac{T_m(x)}{(1 - x^2)^{\lambda}} = \frac{\sqrt{\pi} \Gamma(\lambda)}{\Gamma(\lambda + 1/2)} \sum_{n=0}^{\infty} (\lambda + m + 2n) \frac{\lambda}{m + 2n} \frac{\lambda}{2n} C_{m+2n}(x)$$

where $\lambda \in (-1/2, 3)$ in (1.11) and $\lambda \in (-1/2, 1)$ in (1.12). The particular case corresponding to $m = 0$ and $\lambda = 1/2$ can be found in the literature. Levrie’s paper [5] used this expansion to find formulas for $1/\pi$ and $1/\pi^2$, this was considered again by Carantini and D’Aurizio in [2], and more recently by Chen [3]. Let us emphasize on the statement of convergence. Note that

$$\lim_{n \to \infty} (\lambda + m + 2n) \frac{\lambda}{2n} \frac{\lambda}{m + 2n} = \frac{\Gamma(2\lambda)}{(\Gamma(\lambda))^2}.$$

So, it is not straightforward to conclude about the convergence in (1.12). In addition, the $L^2$ theory does not apply in the whole domain of $\lambda$. In particular, when $\lambda = 1/2$ the function $x \mapsto (1 - x^2)^{-1/2}$ is not square-integrable on $(-1, 1)$.

Using (1.11) and (1.12), several series expansions of known functions are obtained using Parseval’s identity. In particular, one appealing
formula is obtained, (Corollary 12),
\[
\sum_{n=0}^{\infty} \frac{1}{4^n} \binom{2n}{n} \frac{z - 2n}{(z^{1/2})^n} = \frac{\tan(\pi z)}{\pi} \tag{1.13}
\]
which is valid for \( \Re(z) > -1/2 \) and \( z \notin 1/2 + \mathbb{Z} \).

The paper is organized as follows. In section 2, two functions related to Chebyshev polynomials are expanded in a \( \lambda \)-Gegenbauer series, and the convergence of the corresponding series is studied. In section 3, when the considered functions are square-integrable with respect to weighted measure \( \omega_{\lambda} \, dx \) Parseval’s theorem is applied. In section 4 many results and applications are demonstrated. We were not exhaustive in our search here because we wanted to keep the length of the paper reasonable.

2. Two expansions in terms of Gegenbauer polynomials

Let \( \lambda \in (-1/2, 3) \) with \( \lambda \neq 0 \), and let \( m \) be a positive integer, we define the function \( f_{\lambda,m} \) on \((-1, 1)\) by
\[
f_{\lambda,m}(x) = \frac{U_m(x)}{(1 - x^2)^{\lambda - 1}}. \tag{2.1}
\]
where \( U_m \) is the Chebyshev polynomial of the second kind. The function \( f_{\lambda,m} \) belongs to \( L^1((-1, 1), \omega_{\lambda} \, dx) \). Our objective is to expand \( f_{\lambda,m} \) in a \( \lambda \)-Gegenbauer series.

**Theorem 3.** Let \( m \) be a nonnegative integer and \( \lambda \in (-1/2, 3) \) with \( \lambda \neq 0 \). Then for all \( x \in (-1, 1) \) the following holds
\[
\frac{U_m(x)}{(1 - x^2)^{\lambda - 1}} = \frac{(m + 1)\sqrt{\pi} \Gamma(\lambda)}{2\Gamma(\lambda + 1/2)} \sum_{n=0}^{\infty} \frac{\lambda + m + 2n}{m + n + 1} \binom{\lambda - 1}{n} \binom{\lambda}{m+n} C_{m+2n}^{(\lambda)}(x)
\]
Moreover, the convergence is uniform with respect to \( x \) on every compact contained in \((-1, 1)\).

**Proof.** The formal expansion of \( f_{\lambda,m} \) in a \( \lambda \)-Gegenbauer series is given by
\[
S(f_{\lambda,m})(x) = \sum_{n=0}^{\infty} a_n(f_{\lambda,m}) C_n^{(\lambda)}(x) \tag{2.2}
\]
with
\[
\|C_n^{(\lambda)}\|_2^2 a_n(f_{\lambda,m}) = \int_{-1}^{1} f_{\lambda,m}(x) C_n^{(\lambda)}(x) \omega_{\lambda}(x) \, dx
\]
\[
= \int_{-1}^{1} U_m(x) C_n^{(\lambda)}(x) \sqrt{1 - x^2} \, dx \tag{2.3}
\]
Now because
\[
U_m(-x)C_n^{(\lambda)}(-x) = (-1)^{n+m} U_m(x) C_n^{(\lambda)}(x), \tag{2.4}
\]
we see that $a_n(f_{\lambda,m}) = 0$ if $n + m$ is odd. In addition, because the $(U_m(x))_{m \geq 0}$ are orthogonal in $L^2((-1,1), \sqrt{1-x^2} \, dx)$ we conclude that $a_n(f_{\lambda,m}) = 0$ if $m > n$. So, we only need to consider the case where $n = m + 2q$ for some nonnegative integer $q$.

Now, using the following formula [7, 18.5.10] which gives the Fourier series expansion of $\theta \mapsto C_n^{(\lambda)}(\cos \theta)$:

$$C_n^{(\lambda)}(\cos \theta) = \sum_{k=0}^{n} \left\langle \frac{\lambda}{k}, \frac{\lambda}{n-k} \right\rangle \cos((2k-n)\theta). \quad (2.5)$$

we get from [2,3]:

$$\|C_{m+2q}^{(\lambda)}\|_2^2 a_{m+2q}(f_{\lambda,m}) = \int_0^{\pi} C_{m+2q}^{(\lambda)}(\cos \theta) \sin((m+1)\theta) \sin \theta \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi} C_{m+2q}^{(\lambda)}(\cos \theta) \cos(m\theta) \, d\theta$$

$$- \frac{1}{2} \int_0^{\pi} C_{m+2q}^{(\lambda)}(\cos \theta) \cos((m+2)\theta) \, d\theta$$

$$= \frac{\pi}{2} \left\langle \frac{\lambda}{q}, \frac{\lambda}{m+q} \right\rangle - \frac{\pi}{2} \left\langle \frac{\lambda}{q+1}, \frac{\lambda}{m+q+1} \right\rangle$$

$$= \frac{\pi}{2} \left\langle \frac{\lambda}{q}, \frac{\lambda}{m+q} \right\rangle \left(1 - \frac{q}{q+\lambda+1} \cdot \frac{q+\lambda+m}{q+m+1} \right)$$

Finally

$$\|C_{m+2q}^{(\lambda)}\|_2^2 a_{m+2q}(f_{\lambda,m}) = \frac{\pi}{2} \frac{m+1}{q+m+1} \left\langle \frac{\lambda}{q}, \frac{\lambda}{m+q} \right\rangle.$$

It follows that

$$a_{m+2q}(f_{\lambda,m}) = \frac{(m+1)\sqrt{\pi} \Gamma(\lambda)}{2\Gamma(\lambda+1/2)} \frac{m+\lambda+2q}{m+1+q} \left\langle \frac{\lambda-1}{q}, \frac{\lambda}{m+q} \right\rangle \quad (2.6)$$

and

$$S(f_{\lambda,m})(x) = \frac{(m+1)\sqrt{\pi} \Gamma(\lambda)}{2\Gamma(\lambda+1/2)} \sum_{n=0}^{\infty} \frac{m+\lambda+2n}{m+1+n} \left\langle \frac{\lambda-1}{n}, \frac{\lambda}{m+n} \right\rangle C_{m+2n}^{(\lambda)}(x) \quad (2.7)$$

Using the facts that

$$\int_{-1}^{1} (1-x^2)^{\lambda-1/2} \, f_{\lambda,m}(x) \, dx = \int_{-1}^{1} |U_m(x)| \sqrt{1-x^2} \, dx < +\infty,$$

$$\int_{-1}^{1} (1-x^2)^{(\lambda-1)/2} \, f_{\lambda,m}(x) \, dx = \int_{-1}^{1} \frac{|U_m(x)|}{(1-x^2)^{(\lambda-1)/2}} < +\infty,$$

(here we use $\lambda < 3$), and that, (according to Theorem [2], the Fourier series of the function $\theta \mapsto U_m(\cos \theta) |\sin \theta|^{2-\lambda}$ is uniformly convergent on every compact interval contained in $(0, \pi)$, we conclude by Theorem...
that the series \( S(f_{\lambda,m})(x) \) is convergent with sum \( f_{\lambda,m}(x) \) for all \( x \in (-1,1) \), the convergence is uniform with respect to \( x \) on every compact interval contained in \((-1,1)\).

Next, let \( \lambda \in (-1/2,1) \) with \( \lambda \neq 0 \), and let \( m \) be a positive integer, we define the function \( h_{\lambda,m} \) on \((-1,1)\) by

\[
h_{\lambda,m}(x) = \frac{T_m(x)}{(1-x^2)^{\lambda}}, \tag{2.8}
\]

where \( T_m \) is the Chebyshev polynomial of the first kind. The function \( h_{\lambda,m} \) belongs to \( L^1((-1,1),\omega_{\lambda} \, dx) \). In the next theorem we expand \( h_{\lambda,m} \) in a \( \lambda \)-Gegenbauer series.

**Theorem 4.** Let \( m \) be a nonnegative integer and \( \lambda \in (-1/2,1) \) with \( \lambda \neq 0 \). Then for all \( x \in (-1,1) \) the following holds

\[
\frac{T_m(x)}{(1-x^2)^{\lambda}} = \frac{\sqrt{\pi} \Gamma(\lambda)}{\Gamma(\lambda+1/2)} \sum_{n=0}^{\infty} (\lambda + m + 2n) C_{\lambda}^{m+2n}(x)
\]

Moreover, the convergence is uniform with respect to \( x \) on every compact contained in \((-1,1)\).

**Proof.** The formal expansion of \( h_{\lambda,m} \) in a \( \lambda \)-Gegenbauer series is given by

\[
S(h_{\lambda,m})(x) = \sum_{n=0}^{\infty} a_n(h_{\lambda,m}) C_n^{(\lambda)}(x) \tag{2.9}
\]

with

\[
\|C_n^{(\lambda)}\|_2^2 a_n(h_{\lambda,m}) = \int_{-1}^{1} h_{\lambda,m}(x) C_n^{(\lambda)}(x) \omega_{\lambda}(x) \, dx
\]

\[
= \int_{-1}^{1} \frac{T_m(x)C_n^{(\lambda)}(x)}{\sqrt{1-x^2}} \, dx,
\]

\[
= \int_0^{\pi} C_n^{(\lambda)}(\cos \theta) \cos(m\theta) \, d\theta. \tag{2.10}
\]

Now because

\[
T_m(-x)C_n^{(\lambda)}(-x) = (-1)^{n+m}T_m(x)C_n^{(\lambda)}(x), \tag{2.11}
\]

we see that \( a_n(h_{\lambda,m}) = 0 \) if \( n + m \) is odd. So let us assume that \( n \) and \( m \) have the same parity. Moreover, because the \( (T_m(x))_{m \geq 0} \) are orthogonal polynomials in \( L^2((-1,1),dx/\sqrt{1-x^2}) \) we conclude that \( a_n(h_{\lambda,m}) = 0 \) if \( m > n \). Therefore, we only need to consider the case where \( n = m + 2q \) for some nonnegative integer \( q \). Now, using again
we find
\[ \|C_{m+2q}^{(\lambda)}\|_2^2 a_{m+2q}(h_{\lambda,m}) = \int_0^\pi C_{m+2q}^{(\lambda)}(\cos \theta) \cos(m\theta) d\theta = \pi \langle \frac{\lambda}{q} \rangle \langle \frac{\lambda}{m+q} \rangle . \] (2.12)

It follows that
\[ a_{m+2q}(h_{\lambda,m}) = \sqrt{\pi} \Gamma(\lambda) \frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1/2)}^2 \frac{(\lambda+m+2q)}{2^{2\lambda}} C_{m+2q}^{(\lambda)}(x). \] (2.13)

Consequently
\[ S(h_{\lambda,m}) = \sqrt{\pi} \Gamma(\lambda) \sum_{q=0}^\infty (\lambda+m+2q) \frac{\langle \frac{\lambda}{q} \rangle \langle \frac{\lambda}{m+q} \rangle}{2^{2\lambda}} C_{m+2q}^{(\lambda)}(x). \] (2.14)

Now, using the facts that
\[ \int_{-1}^1 (1-x^2)^{\lambda-1/2} |h_{\lambda,m}(x)| dx = \int_{-1}^1 \frac{|T_m(x)|}{\sqrt{1-x^2}} dx < +\infty, \]
\[ \int_{-1}^1 (1-x^2)^{\lambda-1/2} |h_{\lambda,m}(x)| dx = \int_{-1}^1 \frac{|T_m(x)|}{(1-x^2)^{\lambda+1/2}} < +\infty, \]
and that, (according to Theorem 2), the Fourier series of the function \( \theta \mapsto \cos(m\theta)|\sin \theta|^{-\lambda} \) is uniformly convergent on every compact interval contained in \((0, \pi)\), we conclude by Theorem 1 that the series \( S(h_{\lambda,m})(x) \) is convergent with sum \( h_{\lambda,m}(x) \) for all \( x \in (-1,1) \) and that the convergence is uniform with respect to \( x \) on every compact interval contained in \((-1,1)\). \( \square \)

**Remark 1.** The formula in Theorem 4 can be written, using the duplication formula for the Gamma function, in the following equivalent form.
\[ \frac{T_m(x)}{(1-x^2)^\lambda} = \sum_{n=0}^\infty (\lambda+m+2n) \binom{m+2n}{n} \frac{\Gamma(\lambda+n)\Gamma(\lambda+m+n)}{2^{1-2\lambda}\Gamma(2\lambda+m+2n)} C_{m+2n}(x). \]

### 3. Parseval’s Theorem in Action

Note that \( f_{\lambda,m} \in L^2((-1,1), \omega_{\lambda}dx) \) for \( \lambda \in (-1/2, 5/2) \) and similarly \( h_{\lambda,m} \in L^2((-1,1), \omega_{\lambda}dx) \) for \( \lambda \in (-1/2, 1/2) \). Thus we may apply Parseval’s theorem. But before proceeding we will need to evaluate two integrals. This is what we do in the next Lemma.
Lemma 5.  
(a) For $\Re(\mu) < 1/2$ and all nonnegative integers $m$ we have

$$J_m(\mu) \overset{\text{def}}{=} \int_0^\pi \frac{\cos(2m\theta)}{\sin^{2\mu} \theta} d\theta = \frac{\sqrt{\pi} \Gamma(1/2 - \mu)}{\Gamma(1 - \mu)} \cdot \frac{(\mu)_m}{(1 - \mu)_m}. \quad (3.1)$$

(b) For $\Re(\mu) < 1/2$ and all nonnegative integers $m$ we have

$$K_m(\mu) \overset{\text{def}}{=} \int_0^\pi \frac{\sin((2m + 1)\theta)}{\sin^{2\mu+1} \theta} d\theta = \frac{\sqrt{\pi} \Gamma(1/2 - \mu)}{\Gamma(1 - \mu)} \cdot \frac{(1 + \mu)_m}{(1 - \mu)_m}. \quad (3.2)$$

Proof. (a) Note that

$$J_m(\mu) - J_{m+1}(\mu) = \int_0^\pi \frac{\cos(2m\theta) - \cos((2m + 2)\theta)}{\sin^{2\mu} \theta} d\theta$$

$$= 2 \int_0^\pi \sin((2m + 1)\theta) \sin^{-1-2\mu} \theta d\theta$$

$$= \left[ \frac{-2 \cos((2m + 1)\theta)}{(2m + 1) \sin^{2\mu-1} \theta} \right]_0^\pi$$

$$+ \frac{2 - 4\mu}{2m + 1} \int_0^\pi \frac{\cos \theta \cos((2m + 1)\theta)}{\sin^{2\mu} \theta} d\theta$$

$$= \frac{1 - 2\mu}{2m + 1} \left( J_m(\mu) + J_{m+1}(\mu) \right).$$

It follows that $(m + \mu)J_m(\mu) = (m + 1 - \mu)J_{m+1}(\mu)$. Thus

$$J_{m+1}(\mu) = \frac{m + \mu}{m + 1 - \mu} J_m(\mu),$$

and consequently

$$J_m(\mu) = \frac{(\mu)_m}{(1 - \mu)_m} J_0(\mu), \quad (3.3)$$

Finally the change of variables $\sin^2 \theta = t$ shows

$$J_0(\mu) = 2 \int_0^{\pi/2} \frac{d\theta}{(\sin^2 \theta)^\mu} = \int_0^1 t^{-\mu-1/2}(1 - t)^{-1/2} dt$$

$$= \frac{\Gamma(1/2) \Gamma(1/2 - \mu)}{\Gamma(1 - \mu)} = \frac{\sqrt{\pi} \Gamma(1/2 - \mu)}{\Gamma(1 - \mu)} \quad (3.4)$$

This proves (a).
(b) This is similar to part (a). Indeed,

\[
K_m(\mu) - K_{m-1}(\mu) = \left( \int_0^\pi \frac{\sin((2m+1)\theta) - \sin((2m-1)\theta)}{\sin^{2\mu+1} \theta} \, d\theta \right)
\]

\[
= 2 \left( \int_0^\pi \cos(2m\theta) \sin^{-2\mu} \theta \, d\theta \right)
\]

\[
= \left[ \frac{\sin(2m\theta)}{m} \sin^{-2\mu} \theta \right]_0^\pi + \frac{\mu}{m} \left( \int_0^\pi \frac{2 \sin(2m\theta) \cos \theta}{\sin^{2\mu+1} \theta} \, d\theta \right)
\]

\[
= \frac{\mu}{m} \left( K_m(\mu) + K_{m-1}(\mu) \right)
\]

It follows that \((m-\mu)K_m(\mu) = (m+\mu)K_{m-1}(\mu)\). Now because \(K_0(\mu) = J_0(\mu)\) we conclude that

\[
K_m(\mu) = \frac{(1 + \mu)_m}{(1 - \mu)_m} J_0(\mu),
\]

and (b) follows using (3.4). \(\square\)

**Theorem 6.**

(a) For all \(\lambda \in (-1/2, 1/2)\) and all nonnegative integers \(m\) the following holds

\[
\sum_{n=0}^{\infty} (\lambda + m + 2n) \frac{\lambda^2}{n!} \frac{\lambda^2}{(n+m)!} = \frac{\tan(\pi \lambda)}{2\pi} \left( 1 + \frac{(\lambda)_m}{(1 - \lambda)_m} \right). \tag{3.5}
\]

(b) For all \(\lambda \in (-1/2, 5/2)\) and all nonnegative integers \(m\) the following holds

\[
\sum_{n=0}^{\infty} \frac{\lambda + m + 2n}{(m + n + 1)^2} \left( \frac{\lambda - n}{n!} \right)^2 \left( \frac{\lambda}{m+n} \right)^2 = \frac{(1 - 2\lambda) \tan(\pi \lambda)}{(m+1)^2 \pi (1 - \lambda)} \left( 1 + \frac{(\lambda - 1)_{m+1}}{(2 - \lambda)_{m+1}} \right) \tag{3.6}
\]

where the right side is defined by continuity when \(\lambda \in \{1/2, 3/2\}\).

(c) For all \(\lambda \in (-1/2, 3/2)\) and all nonnegative integers \(m, q\) the following holds

\[
\sum_{n=m}^{\infty} \frac{\lambda + q + 2n}{q + m + 1 + n} \frac{\lambda - n}{n!} \frac{\lambda}{q+n} \frac{\lambda}{q+m+n} \frac{\lambda}{2n} = \frac{(1 - 2\lambda) \tan(\pi \lambda)}{2\pi (q + 2m + 1)(1 - \lambda)} \left( \frac{(\lambda)_{m+q}}{(2 - \lambda)_{m+q}} + \frac{(\lambda)_m}{(2 - \lambda)_m} \right) \tag{3.7}
\]

where the right side is defined by continuity when \(\lambda = 1/2\).
For all \( \lambda \in (-1/2, 3/2) \) and all nonnegative integers \( m, q \) the following holds

\[
\sum_{n=m+1}^{\infty} \frac{\lambda + q + 2n}{q + n + 1} \left\langle \frac{\lambda}{n}, \frac{\lambda-1}{n-1}, \frac{\lambda}{q+n}, \frac{\lambda}{q+m+1+n} \right\rangle = \frac{(1 - 2\lambda)\tan(\pi\lambda)}{2\pi(q + 1)(1 - \lambda)} \left( \frac{(\lambda)_{m+q+1}}{(2 - \lambda)_{m+q+1}} - \frac{(\lambda)_m}{(2 - \lambda)_m} \right)
\]

(3.8)

where the right side is defined by continuity when \( \lambda = 1/2 \).

Proof. (a) We know that for \( \lambda \in (-1/2, 1/2) \) the function \( h_{\lambda,m} \) defined in (2.8) belongs to \( L^2((-1, 1), \omega_{\lambda}dx) \) and has the expansion given in Theorem 4 in terms of \( \lambda \)-Gegenbauer polynomials. But

\[
\|h_{\lambda,m}\|_2^2 = \int_{-1}^{1} h_{\lambda,m}^2(x) \omega_{\lambda}(x) \, dx,
\]

\[
= \int_{-1}^{1} (T_m(x))^2 \frac{dx}{(1 - x^2)^{\lambda} \sqrt{1 - x^2}},
\]

\[
= \frac{1}{2} \int_0^{\pi} \frac{1 + \cos(2m\theta)}{\sin^{2\lambda} \theta} \, d\theta = \frac{J_0(\lambda) + J_m(\lambda)}{2},
\]

(3.9)

where \( J_m \) was defined and calculated in Lemma 5. It follows that,

\[
\|h_{\lambda,m}\|_2^2 = \frac{\sqrt{\pi} \Gamma(1/2 - \lambda)}{\Gamma(1 - \lambda)} \left( 1 + \frac{(\lambda)_{m+q+1}}{(1 - \lambda)_{m+q+1}} \right).
\]

(3.10)

On the other hand according to Parseval’s theorem, Theorem 4 and (1.4), we have

\[
\|h_{\lambda,m}\|_2^2 = \frac{\sqrt{\pi} \Gamma(1/2 - \lambda)}{\Gamma(1 - \lambda)} \left( 1 + \frac{(\lambda)_{m}}{(1 - \lambda)_{m}} \right).
\]

(3.11)

So, from (3.10) and (3.11) we conclude that

\[
\sum_{n=0}^{\infty} (\lambda + m + 2n) \left\langle \frac{\lambda}{n}, \frac{\lambda}{m+n} \right\rangle^2 \|C_{m+2n}\|_2^2 = \frac{\tan(\pi\lambda)}{2\pi} \left( 1 + \frac{(\lambda)_{m}}{(1 - \lambda)_{m}} \right),
\]

(3.12)

where the expression with the Gamma function was simplified using the reflection formula: \( \Gamma(z)\Gamma(1 - z) = \pi \csc(\pi z) \), see [11, 5.5.3].

(b) Similarly, for \( \lambda \in (-1/2, 1/2) \), the function \( f_{\lambda,m} \) defined by (2.1) belongs to \( L^2((-1, 1), \omega_{\lambda}dx) \) and has the expansion given in Theorem
in terms of $\lambda$-Gegenbauer polynomials. Moreover,

$$
\|f_{\lambda,m}\|_2^2 = \int_{-1}^{1} f_{\lambda,m}^2(x) \omega_\lambda(x) \, dx = \int_{-1}^{1} (1 - x^2)^{3/2 - \lambda} (U_m(x))^2 \, dx,
$$

$$
= \frac{1}{2} \int_{0}^{\pi} \frac{1 - \cos((2m + 2)\theta)}{\sin^{2\lambda - 2} \theta} \, d\theta = \frac{J_0(\lambda - 1) - J_{m+1}(\lambda - 1)}{2},
$$

(3.13)

So, according to Lemma 5 we have for $\lambda \in (0, 1)$:

$$
\|f_{\lambda,m}\|_2^2 = \frac{1}{2} \sqrt{\frac{\pi \Gamma(3/2 - \lambda)}{\Gamma(2 - \lambda)}} \left( 1 - \frac{(\lambda - 1)_{m+1}}{(2 - \lambda)_{m+1}} \right).
$$

(3.14)

Using Parseval’s theorem, Theorem 3 and (1.4) we obtain

$$
\|f_{\lambda,m}\|_2^2 = \frac{(m + 1)^2 \sqrt{\pi \Gamma(3/2 - \lambda)}}{4\Gamma(\lambda + 1/2)} \sum_{n=0}^{\infty} \frac{\lambda + m + 2n}{(m + n + 1)^2} \left( \frac{\lambda - 1}{n} \right)^2 \left( \frac{\lambda}{m+n} \right)^2.
$$

(3.15)

Rearranging and using the reflection formula for the Gamma function the formula in (b) is proved.

(c) Assuming $\lambda \in (-1/2, 1/2)$ we know that both $f_{\lambda,p}$ and $h_{\lambda,q}$ belong to $L^2((-1, 1), \omega_\lambda dx)$. Moreover, assuming that $p$ and $q$ are of the same parity, we have

$$
\langle f_{\lambda,p}, h_{\lambda,q} \rangle = \int_{-1}^{1} f_{\lambda,p}(x)h_{\lambda,q}(x) \omega_\lambda(x) \, dx
$$

$$
= \int_{-1}^{1} (1 - x^2)^{1/2 - \lambda} U_p(x) T_q(x) \, dx
$$

$$
= \int_{0}^{\pi} (\sin \theta)^{1 - 2\lambda} U_p(\cos \theta) T_q(\cos \theta) \sin \theta \, d\theta
$$

$$
= \int_{0}^{\pi} \frac{\sin((p + 1)\theta) \cos(q\theta)}{\sin^{2\lambda - 1} \theta} \, d\theta
$$

$$
= \frac{1}{2} \int_{0}^{\pi} \frac{\sin((p + q + 1)\theta) + \sin((p + 1 - q)\theta)}{\sin^{2\lambda - 1} \theta} \, d\theta
$$

$$
= \begin{cases} 
\frac{1}{2} (K_{(p+q)/2}(\lambda - 1) + K_{(p-q)/2}(\lambda - 1)) & \text{if } p \geq q, \\
\frac{1}{2} (K_{(p+q)/2}(\lambda - 1) - K_{(q-p)/2}(\lambda - 1)) & \text{if } p < q.
\end{cases}
$$

(3.15)

Thus, for nonnegative integers $q$ and $m$ we have

$$
\langle f_{\lambda,q+2m}, h_{\lambda,q} \rangle = K_{q+m}(\lambda - 1) + K_m(\lambda - 1)
$$

$$
= \frac{\sqrt{\pi} \Gamma(3/2 - \lambda)}{2\Gamma(2 - \lambda)} \left( \frac{(\lambda)_{m+q}}{(2 - \lambda)_{m+q}} + \frac{(\lambda)_m}{(2 - \lambda)_m} \right).
$$

(3.16)
But according to Theorem 3 and Theorem 4 we have

\[ f_{\lambda,q+2m}(x) = \frac{(q + 2m + 1)\sqrt{\pi} \Gamma(\lambda)}{2\Gamma(\lambda + 1/2)} \sum_{n=m}^{\infty} \frac{\lambda + q + 2n}{q + m + n + 1} \frac{\langle \lambda - 1 \rangle_{n-m} \langle \lambda \rangle_{n} \langle \lambda \rangle_{q+n} \langle \lambda \rangle_{q+m+n}}{\langle 2\lambda \rangle_{q+2n}} C_{q+2n}(\lambda) \]

\[ h_{\lambda,q}(x) = \frac{\sqrt{\pi} \Gamma(\lambda)}{\Gamma(\lambda + 1/2)} \sum_{n=0}^{\infty} (\lambda + q + 2n) \frac{\langle \lambda \rangle_{n} \langle \lambda \rangle_{q+n} \langle \lambda \rangle_{q+m+n}}{\langle 2\lambda \rangle_{q+2n}} C_{q+2n}(\lambda) \]

Thus \( f_{\lambda,q+2m}, h_{\lambda,q} \) is given by

\[ \frac{(q + 2m + 1)\pi \sqrt{\pi} \Gamma(\lambda)}{2\Gamma(\lambda + 1/2)} \sum_{n=m}^{\infty} \frac{\lambda + q + 2n}{q + m + 1 + n} \frac{\langle \lambda - 1 \rangle_{n-m} \langle \lambda \rangle_{n} \langle \lambda \rangle_{q+n} \langle \lambda \rangle_{q+m+n}}{\langle 2\lambda \rangle_{q+2n}} \]

It follows that

\[ \sum_{n=m}^{\infty} \frac{\lambda + q + 2n}{q + m + 1 + n} \frac{\langle \lambda - 1 \rangle_{n-m} \langle \lambda \rangle_{n} \langle \lambda \rangle_{q+n} \langle \lambda \rangle_{q+m+n}}{\langle 2\lambda \rangle_{q+2n}} \]

\[ = \frac{(1 - 2\lambda) \tan(\pi\lambda)}{2\pi(q + 2m + 1)(1 - \lambda)} \left( \frac{(\lambda)_{m+q}}{(2 - \lambda)_{m+q}} + \frac{(\lambda)_m}{(2 - \lambda)_m} \right) \]

which is (c) for \( \lambda \in (-1/2, 1/2) \). For a given \( q \) and \( m \), let us consider the complex functions

\[ A_n(z) = \frac{z + q + 2n}{q + m + 1 + n} \frac{\langle z - 1 \rangle_{n-m} \langle z \rangle_{n} \langle z \rangle_{q+n} \langle z \rangle_{q+m+n}}{\langle 2\lambda \rangle_{q+2n}} \]

\[ B(z) = \frac{(1 - 2z) \tan(\pi z)}{2\pi(q + 2m + 1)(1 - z)} \left( \frac{(z)_{m+q}}{(2 - z)_{m+q}} + \frac{(z)_m}{(2 - z)_m} \right) \]

Clearly, \( B \) is analytic in the domain \( \Omega = \{ z \in \mathbb{C} : 0 < \Re(z) < 3/2 \} \) with a removable singularity at \( z = 1/2 \). On the other hand, using the fact that \( \langle k^{1-z} \rangle_{k>0} \) converges uniformly on every compact subset of \( \mathbb{C} \) to \( 1/\Gamma(z) \), (this is Gauss’ version of the definition of \( \Gamma(z) \) as a product [7 5.8.1]), we see that \( \langle n^{1-2z} A_n(z) \rangle_{n\geq0} \) converges uniformly on every compact subset of \( \Omega \), (to \( 2^{z-2z}(z - 1)\Gamma(2z)/(\Gamma(z))^2 \)). This implies that the series \( z \mapsto \sum_{n=m}^{\infty} A_n(z) \) is convergent to some analytic function in \( \Omega \). Now, because \( B(\lambda) = \sum_{n=m}^{\infty} A_n(\lambda) \) for \( \lambda \in (0, 1/2) \) we conclude the validity of this equality for \( \lambda \in (0, 3/2) \) by analytic continuation. This proves (e).
(d) Similarly to (c) it is enough to prove the result for \( \lambda \in (-1/2, 1/2) \). According to (3.15), for nonnegative integers \( p \) and \( m \) we have

\[
\langle f_{\lambda,p}, h_{\lambda,p+2m+2} \rangle = \frac{K_{p+m+1}(\lambda - 1) - K_m(\lambda - 1)}{2}
= \frac{\sqrt{\pi} \Gamma(3/2 - \lambda)}{2\Gamma(2 - \lambda)} \left( \frac{(\lambda)_{m+p+1}}{(2 - \lambda)_{m+p+1}} - \frac{(\lambda)_m}{(2 - \lambda)_m} \right),
\]

but according to Theorem 3 and Theorem 4 we have

\[
f_{\lambda,p}(x) = (p + 1)\sqrt{\pi} \Gamma(\lambda) \frac{\lambda + p + 2n}{2\Gamma(\lambda + 1/2)} \sum_{n=0}^{\infty} \frac{\lambda + p + 2n}{p + n + 1} \left( \frac{\lambda - 1}{n} \right) \left( \frac{\lambda}{p+n} \right) C_{p+2n}^{(\lambda)}(x)
\]

\[
h_{\lambda,p+2m+2}(x) = \frac{\sqrt{\pi} \Gamma(\lambda)}{\Gamma(\lambda + 1/2)} \sum_{n=m+1}^{\infty} (\lambda + p + 2n) \left( \frac{\lambda}{n-m-1} \right) \left( \frac{\lambda}{p+m+1+n} \right) C_{p+2n}^{(\lambda)}(x)
\]

hence \( \langle f_{\lambda,p}, h_{\lambda,p+2m+2} \rangle \), is given by

\[
(p + 1)\sqrt{\pi} \Gamma(\lambda) \frac{\lambda + p + 2n}{2\Gamma(\lambda + 1/2)} \sum_{n=m+1}^{\infty} \frac{\lambda + p + 2n}{p + n + 1} \left( \frac{\lambda - 1}{n-m-1} \right) \left( \frac{\lambda}{n} \right) \left( \frac{\lambda}{p+n} \right) C_{p+2n}^{(\lambda)}(x).
\]

It follows that

\[
\sum_{n=m+1}^{\infty} \frac{\lambda + p + 2n}{p + n + 1} \left( \frac{\lambda - 1}{n-m-1} \right) \left( \frac{\lambda}{n} \right) \left( \frac{\lambda}{p+n} \right) C_{p+2n}^{(\lambda)}(x)
= \frac{(1 - 2\lambda)\tan(\pi\lambda)}{2\pi(p + 1)(1 - \lambda)} \left( \frac{(\lambda)_{m+p+1}}{(2 - \lambda)_{m+p+1}} - \frac{(\lambda)_m}{(2 - \lambda)_m} \right),
\]

which is the required formula.

\[\square\]

4. Applications

Since our series expansions of theorems 3 and 4 are valid in the interval \((-1, 1)\), and because we have an explicit evaluation for the considered polynomials at \( x = 0 \), namely, we have \( C_k^{(\lambda)}(0) = 0 \) if \( k \) is odd and \( C_k^{(\lambda)}(0) = (-1)^{k/2} \left( \frac{\lambda}{k/2} \right) \) if \( k \) is even, see [7] Table 18.6.1. The next results follow immediately.
Theorem 7. Let \( m \) be a nonnegative integer, and \( \lambda \in (-1/2, 3) \) with \( \lambda \neq 0 \). Then
\[
\frac{2\Gamma(\lambda + 1/2)}{(2m + 1)\sqrt{\pi} \Gamma(\lambda)} = \sum_{n=m}^{\infty} (-1)^{n-m} \frac{\lambda + 2n}{n + m + 1} \frac{\binom{\lambda-1}{n-m} \binom{\lambda}{n} \binom{\lambda}{m+n}}{\binom{2\lambda}{2n}} \tag{4.1}
\]
\[
= \sum_{n=m}^{\infty} \frac{(-1)^{n-m}}{4^n} \frac{\lambda + 2n}{n + m + 1} \binom{2n}{n} \frac{\binom{\lambda-1}{n-m} \binom{\lambda}{m+n}}{\binom{\lambda+1/2}{n}} \tag{4.2}
\]
\[
= \sum_{n=m}^{\infty} \frac{(-1)^{n-m}}{4^n n!} \frac{\lambda + 2n}{n + m + 1} \frac{2n}{n-m} \frac{\lambda_{n-m}(\lambda)_{m+n}}{(\lambda + 1/2)_n} \tag{4.3}
\]

Theorem 8. Let \( m \) be a nonnegative integer and \( \lambda \in (-1/2, 1) \) with \( \lambda \neq 0 \). Then
\[
\frac{\Gamma(\lambda + 1/2)}{\sqrt{\pi} \Gamma(\lambda)} = \sum_{n=m}^{\infty} (-1)^{n-m} (\lambda + 2n) \frac{\binom{\lambda}{n-m} \binom{\lambda}{n} \binom{\lambda}{m+n}}{\binom{2\lambda}{2n}} \tag{4.4}
\]
\[
= \sum_{n=m}^{\infty} \frac{(-1)^{n-m}}{4^n} (\lambda + 2n) \frac{2n}{n} \frac{\binom{\lambda}{n-m} \binom{\lambda}{m+n}}{\binom{\lambda+1/2}{n}} \tag{4.5}
\]
\[
= \sum_{n=m}^{\infty} \frac{(-1)^{n-m}}{4^n n!} (\lambda + 2n) \frac{2n}{n-m} \frac{(\lambda)_{n-m}(\lambda)_{m+n}}{(\lambda + 1/2)_n} \tag{4.6}
\]

The particular case of \( \lambda = 1/2 \) is interesting. In this case we obtain from Theorem 7 and Theorem 8 the following two identities:
\[
\frac{4}{\pi} = \sum_{n=m}^{\infty} (-1)^{n-m} \frac{(1 + 4n)(2m + 1)}{(n + m + 1)(1 + 2m - 2n) (64)^n} \frac{2n - 2m}{n-m} \frac{2n}{n} \frac{2m + 2n}{m+n}, \tag{4.7}
\]
\[
\frac{2}{\pi} = \sum_{n=m}^{\infty} \frac{(-1)^{n-m}(1 + 4n)}{(64)^n} \frac{2n - 2m}{n-m} \frac{2n}{n} \frac{2m + 2n}{m+n}. \tag{4.8}
\]

In the case \( m = 0 \) the equalities (4.7) and (4.8) are known. (see [5], [2] and [3].)

Because \( C_n^{(1/2)} = P_n \) is the Legendre polynomial of degree \( n \), setting \( \lambda = 1/2 \) in both Theorem 3 and Theorem 4 yields the next results.

Theorem 9. Let \( m \) be a nonnegative integer. Then for all \( x \in [-1, 1] \) the following holds
\[
\sqrt{1-x^2} U_n(x) = \frac{(m + 1)\pi}{4^{m+1}} \sum_{n=0}^{\infty} \frac{1 + 2m + 4n}{(1 - 2n)(n + m + 1)16^n} \frac{2n}{n} \frac{2m + 2n}{n+m} \frac{P_{m+2n}(x)}{P_{m+2n}(x)}
\]

Moreover, the convergence is uniform.
Note that, because \( \sup_{[-1,1]} |P_{m+2n}(x)| = 1 \), the series in theorem 9 is normally convergent on the interval \([-1,1]\), and equality holds on \([-1,1]\).

Similarly,

**Theorem 10.** Let \( m \) be a nonnegative integer. Then for all \( x \in (-1,1) \) the following holds

\[
\frac{T_m(x)}{\sqrt{1-x^2}} = \frac{\pi}{2^{2m+1}} \sum_{n=0}^{\infty} \frac{1+2m+4n}{16^n} \left( \frac{2n}{n+m} \right) P_{m+2n}(x)
\]

Moreover, the convergence is uniform on every compact contained in \((-1,1)\).

As a corollary from the above Theorems 9 and 10 we have the following result which corresponds to the case \( m = 0 \).

**Corollary 11.** For all \( x \in (-1,1) \) the following holds

\[
\sqrt{1-x^2} = \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{1+4n}{(1-2n)(n+1)16^n} \left( \frac{2n}{n} \right) P_{2n}(x)
\]

\[
\frac{1}{\sqrt{1-x^2}} = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{1+4n}{16^n} \left( \frac{2n}{n} \right) P_{2n}(x)
\]

Moreover, the convergence is uniform on every compact contained in \((-1,1)\).

These formulas in Corollary 11 appeared several times in the literature. With exception of the statement about convergence, they can be found in [5], [2] and more recently in [3].

The next result follows from Theorem 6 when some interesting values for \( \lambda \) and \( m \) are selected.

**Corollary 12.** For all complex numbers \( z \) with \( \Re(z) > -\frac{1}{2} \) and \( z \notin \left\{ \frac{1}{2} + k : k \in \mathbb{Z} \right\} \) the following holds

\[
\sum_{n=0}^{\infty} \frac{1}{4^n} \left( \frac{2n}{n} \right) \frac{z - 2n}{(z - 1/2)^n} \left( \frac{z}{n} \right)^3 = \frac{\tan(\pi z)}{\pi}. \quad (4.9)
\]

In particular, for all nonnegative integers \( m \) we have

\[
\sum_{n=0}^{\infty} \frac{1}{4^n+1} \left( \frac{2n}{n} \right) \frac{4m + 1 - 8n}{(m - 1/4)^n} \left( \frac{m + 1/4}{n} \right)^3 = \frac{1}{\pi}. \quad (4.10)
\]

**Proof.** Indeed, both sides of the given equality are analytic in the domain

\[
\Omega = \left\{ z \in \mathbb{C} : \Re(z) > \frac{1}{2}, z - \frac{1}{2} \notin \mathbb{Z} \right\},
\]

and according to [3,5] with \( m = 0 \) both sides are equal when \( z = -\lambda \in (-1/2, 1/2) \) so they are equal in \( \Omega \). \( \square \)
Remark 2. In a similar way as in Corollary [12], analytic extensions to a suitable complex domain of all the results in Theorem [6], Theorem [7], and Theorem [8] can be obtained. We leave this task to the interested reader.

Using (3.6) with $\lambda = 1/2$, we obtain the next corollary.

**Corollary 13.** For all nonnegative integers $m$ we have

$$
\sum_{n=0}^{\infty} \frac{1 + 2m + 4n}{(m+n+1)^2(2n-1)^2} \frac{1}{(256)^n} \binom{2n}{n}^2 \left( \frac{2m+2n}{m+n} \right)^2 = \frac{2^{4m+5}}{\pi^2(2m+1)(2m+3)}.
$$

In particular, when $m = 0$ we get

$$
\sum_{n=0}^{\infty} \frac{1 + 4n}{(n+1)^2(2n-1)^2} \frac{1}{(256)^n} \binom{2n}{n}^4 = \frac{32}{3\pi^2}.
$$

(4.11)

This is given in [5], [2] and [3].

**Corollary 14.** For all positive integers $m$ we have

$$
\sum_{n=0}^{\infty} \frac{1 + 2m + 4n}{(m+2n)(m+1+2n)} \frac{1}{(256)^n} \binom{2n}{n}^2 \left( \frac{2n+2m}{n+m} \right)^2 = \frac{2^{4m+1}}{\pi^2} \cdot \frac{2H_{2m} - H_m}{m^2}
$$

where $H_k = \sum_{j=1}^{k} 1/j$ is the $k$th harmonic number.

**Proof.** According to (3.6) with $\lambda = 3/2$ and $m$ replaced by $m - 1$ we have

$$
\sum_{n=0}^{\infty} \frac{1 + 2m + 4n}{(m+2n)(m+1+2n)} \frac{1}{(256)^n} \binom{2n}{n}^2 \left( \frac{2n+2m}{n+m} \right)^2 = \frac{16^m}{\pi m^2} \lim_{\epsilon \to 0} \Theta(\epsilon)
$$

with

$$
\Theta(\epsilon) = \left( \frac{(1/2 + \epsilon)m}{(1/2 - \epsilon)m} - 1 \right) \cot(\pi\epsilon)
$$

But

$$
\left( \frac{1}{2} + \epsilon \right)_m = \left( \frac{1}{2} \right)_m \prod_{j=1}^{m} \left( 1 + \frac{2}{2j-1} \epsilon \right)
$$

$$
= \left( \frac{1}{2} \right)_m \left( 1 + \left( \sum_{j=1}^{m} \frac{2}{2j-1} \epsilon \right) + O(\epsilon^2) \right)
$$

$$
= \left( \frac{1}{2} \right)_m \left( 1 + (2H_{2m} - H_m)\epsilon + O(\epsilon^2) \right).
$$

Thus $\Theta(\epsilon) = 2(2H_{2m} - H_m)/\pi + O(\epsilon)$ and the required conclusion follows.

In particular, with $m = 1$ we obtain the following companion formula to (4.11):

$$
\sum_{n=0}^{\infty} \frac{(3 + 4n)(1 + 2n)}{(n+1)^3} \frac{1}{(256)^n} \binom{2n}{n}^4 = \frac{32}{\pi^2}.
$$

(4.12)
In a similar spirit. Using (3.7) with $\lambda = 1/2$ we obtain the next corollary

**Corollary 15.** For all nonnegative integers $m$ and $q$ we have

$$
\sum_{n=m}^\infty \frac{1 + 2q + 4n}{(s + 1 + n)(2m + 1 - 2n)} \frac{1}{(256)^n} \binom{2n - 2m}{n - m} \binom{2n}{n} \binom{2q + 2n}{q + n} \binom{2s + 2n}{s + n} = \frac{2^{4q+3}}{\pi^2(2s + 1)(2m + 1)}
$$

where $s = q + m$.

In particular, with $m = 0$, we get for arbitrary nonnegative integer $q$:

$$
\sum_{n=0}^\infty \frac{(1 + 2q + 4n)(2q + 1)}{(q + 1 + n)(1 - 2n)} \frac{1}{(256)^n} \binom{2n}{n} \binom{2q + 2n}{q + n}^2 = \frac{2^{4q+3}}{\pi^2},
$$

and with $q = 0$, we get for arbitrary nonnegative integer $m$:

$$
\sum_{n=m}^\infty \frac{(1 + 4n)(2m + 1)^2}{(m + 1 + n)(2m + 1 - 2n)} \frac{1}{(256)^n} \binom{2n - 2m}{n - m} \binom{2n}{n} \binom{2m + 2n}{m + n} = \frac{8}{\pi^2},
$$

and finally, with $m = q = 0$ we get

$$
\sum_{n=0}^\infty \frac{1 + 4n}{(1 + n)(1 - 2n)} \frac{1}{(256)^n} \binom{2n}{n}^4 = \frac{8}{\pi^2}.
$$

5. **Concluding Remarks**

Many of our results give exact evaluation of some generalized hypergeometric functions at 1 or $-1$. For example, using the standard notation for hypergeometric functions ([7, 16.1]) equation (4.9) is equivalent to

$$
_{5}F_{4}\left(\frac{1}{2}, -z, -z, -z, 1 - \frac{z}{2}; 1, 1, -\frac{z}{2}, \frac{1}{2} - z\right) = \frac{\tan(\pi z)}{\pi z}.
$$

To the author’s knowledge this is new. In the same spirit, Theorem 8 shows that

$$
_{4}F_{3}\left(z, \frac{1}{2} + m, 1 + m + \frac{z}{2}, 2m + z; 1 + 2m, \frac{1}{2} + m + z, m + \frac{z}{2}; -1\right) = \frac{4^m m! \Gamma\left(\frac{1}{2} + m + z\right)}{\sqrt{\pi} \Gamma(1 + 2m + z)}
$$

for every nonnegative integer $m$.

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