ENTROPY-BASED GOODNESS OF FIT TEST FOR A COMPOSITE HYPOTHESIS

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Abstract. In this paper, we consider the entropy-based goodness of fit test (Vasicek’s test) for a composite hypothesis. The test measures the discrepancy between the nonparametric entropy estimate and the parametric entropy estimate obtained from an assumed parametric family of distributions. It is shown that the proposed test is asymptotically normal under regularity conditions, but is affected by parameter estimates. As a remedy, a bootstrap version of Vasicek’s test is proposed. Simulation results are provided for illustration.

1. Introduction

For decades, the goodness of fit (gof) test for statistical models has been a core issue in statistical analysis. The gof test has a long history and various methodologies have been developed by many researchers. See, for instance, D’Agostino and Stephens [3]. The entropy based gof test, the entropy based gof test has been very popular among practitioners in diverse fields. In particular, the entropy test of Vasicek [11] has been studied extensively in the literature. His approach involves a nonparametric estimate (m-spacing estimate) of Shannon’s entropy. Thus far, a number of articles exist on the distributional properties of Vasicek’s test: see, for instance, Kashimov [5], van Es [10], Beirant et al. [2], Song [8], and the references therein. Among them, Song [8] rigorously verifies that Vasicek’s estimator is consistent and asymptotically normal under certain regularity conditions. This result is easily applicable to simple vs. simple gof tests. However, attention has not yet been paid to composite hypothesis tests. In the literature, it is well known that gof tests are often affected by parameter estimation, and their limiting distributions rely on the choice of parameter estimators. This phenomenon is prominent in the empirical process of the gof tests, as seen in Durbin [4], and often leads practitioners to a burdensome situation. This difficulty may be overcome by using the transformation method proposed by Khmaladze [6], Bai [1] and Lee [7], which, however, is not
easy to implement owing to a time consuming computational process. In this
study, we focus on the entropy test to measure the discrepancy between the
nonparametric entropy estimate (Vasicek’s estimate) and the parametric en-
tropy estimate obtained from the assumed parametric family of distributions.
Although simple and natural, to our knowledge, no literature has explicitly
considered this test. It may be because the proposed test is severely affected
by parameter estimation, and thus, is not as useful in actual implementation.
Conventionally, gof methods depending on asymptotic theories do not perform
well for small samples, and particularly, in the implementation of Vasicek’s test,
the choice of spacing parameter \( m \) escalates this difficulty. As a remedy, it is
natural to adopt the parametric bootstrap approach and construct a bootstrap
version of the tests. Thus, we propose a bootstrap version of Vasicek’s test
for a composite hypothesis and investigate its finite sample behavior through
a simulation study. The organization of this paper is as follows. In Sec-
tion 2, we introduce Vasicek’s test and show that a certain asymptotic exp-
ansion form holds for Vasicek’s test with plugged-in estimators and leads to a result having
asymptotic normality. Further, we introduce a bootstrap version of Vasicek’s
test and demonstrate that its usage is justifiable. In Section 3, we conduct a
simulation study to evaluate the proposed bootstrap test and compare its per-
formance with other existing tests. Concluding remarks are provided in Section
4.

2. Main result

Given an i.i.d. random sample \( X_1, \ldots, X_n \) with common distribution \( F \),
Vasicek (1992) proposed as an estimate of \( H(F) = -\int \log f(x)f(x)dx \) the
following:

\[
V_{mn} = \frac{1}{n} \sum_{i=1}^{n} \log \frac{1}{2m}(X_{(i+m)} - X_{(i-m)}),
\]

where \( f = F' \), \( X_{(i)} \) denotes the ordered r.v.s., and \( X_{(i)} = X_{(1)} \) for \( i < 1 \)
and \( X_{(i)} = X_{(n)} \) for \( i > n \). Later, Song (2000) showed that if the following
conditions are fulfilled

(R1) \( E(\log f(X_1))^2 < \infty \);
(R2) \( \sup_{\phi(F) < x < \psi(F)} F(x)(1-F(x)) \frac{f'(x)}{f(x)} < \infty \);
(R3) \( m = m_n \) satisfies \( \log n/m = o(1) \) and \( m(\log n)^{2/3}/n^{1/3} = o(1) \) as \( n \to \infty \),

where \( \phi(F) = \sup \{ x : F(x) = 0 \} \) and \( \psi(F) = \inf \{ x : F(x) = 1 \} \), then

\[
n^{1/2}(V_{mn} - H(F) + \log 2m + \gamma - R_{2m-1}) \overset{d}{\to} N(0, \sigma^2(F)),
\]

where \( R_n = \sum_{i=1}^{n} 1/i, \gamma = \lim_{n \to \infty} (R_n - \log n), \) and \( \sigma^2(F) = \text{Var}(\log f(X_1)) \).
The result in (2.2) is applicable to a goodness of fit test for a composite hypothesis such as

\[ H_0 : X_i \sim F = F_{\theta_0} \quad \text{vs.} \quad H_1 : X_i \sim F \notin \{F_{\theta_0}\}, \]

where \( \{F_{\theta_0}\} \) is a parameter family indexed with \( \theta \in \Theta, \) a subset of \( \mathbb{R}^d, \) \( d \geq 1, \) and \( \theta_0 \) is an interior point of \( \Theta. \) The result in (2.2) indicates that under (R1) and (R2), with \( F \) replaced by \( F_{\theta_0}, \) and (R3),

\[
n^{1/2}(V_{mn} - H(F_{\theta_0}) + \log 2m + \gamma - R_{2m-1}) \xrightarrow{\text{d}} N(0, \sigma^2(F_{\theta_0})),
\]

where \( f_\theta \) is continuous in \( \theta, \) \( H(F_{\theta}) = - \int \log(f_\theta(x))f_\theta(x)dx \) and \( \sigma^2(F_{\theta}) = \text{Var}_\theta \log(f_\theta(X_1)) \) for all \( \theta. \) Here, \( \text{Var}_\theta \) and \( E_\theta \) denote the variance and expectation under \( F_{\theta}, \) respectively.

The argument in (2.3) suggests a test based on the difference between \( H_n \) and \( H(\hat{\theta}_n), \) where \( \hat{\theta}_n \) is a consistent estimator of \( \theta_0, \) since \( H(\theta_0) \) is unknown. As usual, we assume \( n^{1/2}(\hat{\theta}_n - \theta_0) = O_P(1) \) under \( H_0. \) In view of the proof of Theorem 1 of Song [8], it can be seen that

\[
V_{mn} = H_n - \log 2m - \gamma + R_{2m-1} + o_P(n^{-1/2}),
\]

where \( H_n = -\frac{1}{n} \sum_{i=1}^n \log f_{\theta_0}(X_i). \) Hence,

\[
V_{mn} - H(F_{\theta_0}) + \log 2m + \gamma - R_{2m-1} = H_n - H(F_{\theta_0}) + o_P(n^{-1/2}).
\]

Then, we have the result addressed below.

**Theorem 1.** Suppose that

(R1) \( E_{\theta_0}(\log f(X_1))^2 < \infty. \)

(R2) \( \sup_{0 < x < \psi(f_{\theta_0})} f_{\theta_0}(x)(1 - F_{\theta_0}(x)) \left| \frac{f_{\theta_0}(x)}{F_{\theta_0}(x)} \right| < \infty. \)

Further, assume (R3),

(R4) \( E_{\theta_0} \frac{\partial}{\partial \theta} \log f_{\theta_0}(X_1) = 0 \) and \( u(\theta) = \frac{\partial}{\partial \theta} H(\theta) = - \int \log f_\theta(x) \frac{\partial}{\partial \theta} f_\theta(x)dx \) is continuous in \( \theta. \)

(R5) \( \hat{\theta}_n - \theta_0 = n^{-1} \sum_{i=1}^n l_{\theta_0}(X_i) + o_P(n^{-1/2}), \) where \( l_\theta \) is a \( d \times 1 \) vector function with \( E_{\theta_0}(l_{\theta_0}(X_1) = 0 \) and \( \text{Var}_{\theta_0} \left| l_{\theta_0}(X_1) \right| < \infty, \) where \( \| \cdot \| \) is a Euclidean norm.

Then, under \( H_0, \)

\[
T_n := n^{1/2}(V_{mn} - H(F_{\theta_0}) + \log 2m + \gamma - R_{2m-1}) \xrightarrow{\text{d}} N(0, \tau^2)
\]

with \( \tau^2 = \text{Var}_{\theta_0}(\log f_{\theta_0}(X_1) + H(F_{\theta_0}) + l_{\theta_0}(X_1)^T u(\theta_0)). \)

**Proof.** By the mean value theorem, \( H(F_{\theta_0}) - H(F_{\theta_0}) = (\hat{\theta}_n - \theta_0)^T u(\theta_0) + \rho_n, \) where \( \| \rho_n \| \leq \eta_n \sup_{0 < \theta - \theta_0 \leq \eta_n} \| u(\theta) - u(\theta_0) \| \) with \( \eta_n = \| \hat{\theta}_n - \theta_0 \| \) owing to (R4) and (R5). Combining this and (2.5), the theorem is validated. \( \square \)
Theorem 1 indicates that the parameter estimation affects the null limiting distribution, and further, there is a serious difficulty in estimating $\tau^2$ when the explicit form of $l_\theta$ is unknown.

Remark. One may consider another test based on $H_n = \frac{1}{n} \sum_{i=1}^n \log f_{\hat{\theta}_n}(X_i)$. Put $\Delta_n = n^{1/2}(\hat{H}_n - H_n)$ and suppose that the following holds:

(R6) $\frac{\partial f_\theta}{\partial \theta}$ is continuous in $\theta$ and for some $\epsilon > 0$,

$$\frac{1}{n} \sum_{i=1}^d \sup_{|\theta_i - \theta_{0,i}| \leq \epsilon} \left| \frac{\partial f_\theta(x)}{\partial \theta} \right| \leq g(x), \quad E_{\theta_0} g(X_1) < \infty,$$

where $\theta_i$ denotes the $i$-th entry of $\theta$. By the mean value theorem, we can express

$$\Delta_n = \sqrt{n}(\theta_0 - \hat{\theta}_n)^T \frac{1}{n} \sum_{i=1}^d \frac{\partial f_{\theta_i'}}{\partial f_{\theta_n}},$$

where $\theta_i'$ is between $\theta_0$ and $\hat{\theta}_n$. Then, using (R6), we can easily see that for all $i = 1, \ldots, d$,

$$\frac{1}{n} \sum_{i=1}^d \frac{\partial f_{\theta_i'}}{\partial f_{\theta_n}} - \frac{1}{n} \sum_{i=1}^d \frac{\partial f_{\theta_0}}{\partial f_{\theta_n}} = o_P(1).$$

Together with (R4), this entails (2.3), and thus, $\Delta_n = o_p(1)$. Then, in view of (2.4), (2.5), and Theorem 1, we have that under (R1)'', (R2)'', and (R3)–(R6),

$$T_n' := n^{1/2}(\hat{H}_n - H(F_{\hat{\theta}_n})) \overset{d}{\to} N(0, \tau^2).$$

Meanwhile, Song’s approach can be also extended to a rowwise independent double array of random variables, say, $X_{n1}, \ldots, X_{nn}$. Suppose that $X_{ni}, i = 1, \ldots, n$ follows from $F_{\theta_n}$ where $\{\theta_n\}$ is a sequence in $\Theta$ that converges to an interior point $\theta_0 \in \Theta$ as $n$ tends to $\infty$. In this case, we can consider the estimator

$$DV_{mn} = \frac{1}{n} \sum_{i=1}^n \log \frac{n}{2m}(X_{n,(i+m)} - X_{n,(i-m)}),$$

where $X_{n,(i)}$ are analogously defined as $X_{(i)}$.

In what follows, we assume

(R1)'' For some $\epsilon > 0$,

$$\int_{||\theta - \theta_0|| \leq \epsilon} (1 + (\log f_\theta(x))^2) f_\theta(x) dx < \infty,$$

(R2)'' $\phi(F_0) = \phi$ and $\psi(F_\theta) = \psi$ for all $\theta$. Further, for some $\epsilon > 0$,

$$\sup_{\phi < x < \psi} \sup_{||\theta - \theta_0|| \leq \epsilon} F_\theta(x)(1 - F_\theta(x)) \frac{|f_\theta'(x)|}{f_\theta^2(x)} < \infty.$$
Then, if we put

\[ DH_n = -\frac{1}{n} \sum_{i=1}^{n} \log f_{\theta_n}(X_{ni}), \]

following essentially the same lines as in the proof of Theorem 1 of Song [8], one can check that provided (R3) holds,

\[ DV_{mn} + \log 2m + \gamma - R_{2m-1} = DH_n + o_P(n^{-1/2}). \]

Then, if the following condition is satisfied:

\[ (R4) \quad E_{\theta_0} \int f_{\theta_0}(x) \, dx = 0 \quad \text{for all } \theta \quad \text{and } \, \alpha(\theta) \quad \text{in } (R4) \quad \text{is continuous in } \theta, \]

and if the estimator \( \hat{\theta}_n \) of \( \theta_n \) based on \( X_{ni}, i = 1, \ldots, n \) satisfies:

\[ (R5) \quad \hat{\theta}_n - \theta_n = n^{-1} \sum_{i=1}^{n} l_{\theta_n}(X_{ni}) + o_P(n^{-1/2}), \]

where \( l_{\theta_n} \) is continuous in \( \theta \), \( E_{\theta_0} \alpha(x) = 0 \) for all \( \theta \), and \( \int \sup_{||\theta - \theta_0||_2 \leq \epsilon} ||\alpha(x)||_2 f_{\theta}(x) \, dx < \infty \) for some \( \epsilon > 0 \),

using the dominated convergence theorem and Lindeberg’s central limit theorem, we can have

\[ \sqrt{n}(DH_n - H(F_{\hat{\theta}_n})) \overset{d}{\to} N(0, \tau^2), \]

and subsequently, we have the following.

**Theorem 2.** Under (R1)”, (R2)”,” (R3), (R4)” and (R5)”

\[ (2.6) \quad \sqrt{n}(DV_{mn} - H(F_{\hat{\theta}_n}) + \log 2m + \gamma - R_{2m-1}) \overset{d}{\to} N(0, \tau^2). \]

The argument in (2.6) suggests that a bootstrap test can be designed for the composite hypothesis test in Theorem 1. Here, we use the parametric bootstrap method as in Stute et al. [9]. Given sample \( X_1, \ldots, X_n \), we estimate \( \theta_0 \) by \( \hat{\theta}_n \) and generate the bootstrap sample from \( F_{\hat{\theta}_n} \), say, \( X_1^*, \ldots, X_n^* \), and put

\[ V_{mn}^* = \frac{1}{n} \sum_{i=1}^{n} \log \frac{n}{2m}(X_{i+m}^* - X_{i-m}^*). \]

Then, if \( \hat{\theta}_n \to \theta_0 \) a.s., \( \hat{\theta}_n \) is the estimator based on the bootstrap sample, and

\[ (R5)” \quad \hat{\theta}_n - \theta_n = n^{-1} \sum_{i=1}^{n} l_{\theta_n}(X_{ni}^*) + o_P(n^{-1/2}), \]

we can conclude that under (R1)”,” (R2)”,” (R3) and (R4)”

\[ (2.7) \quad T_{mn}^* := \sqrt{n}(DV_{mn}^* - H(F_{\hat{\theta}_n}^*) + \log 2m + \gamma - R_{2m-1}) \overset{d}{\to} N(0, \tau^2) \quad \text{a.s.} \]

By obtaining \( |T_{mn}^*| \) for the bootstrapped sample \( B \) times, say, \( |T_{mn}^b|, b = 1, \ldots, B \), we can calculate sample quantiles, say \( c = c(n, \alpha) \), given any significance level \( \alpha \).

Then, we reject \( H_0 \) if \( |T_{mn}| \geq c \). This bootstrap method provides a more stable test, unaffected by the choice of spacing parameter \( m \), especially in handling
small samples, as seen in the simulation study below, where we focus on the finite sample behavior of $T^*_n$ and investigate its empirical sizes and powers.

3. Simulation

In this simulation study, we evaluate the bootstrap Vasicek’s test $T^*_n$ (T) and compare its performance with the Kolmogorov-Smirnov (KS), Cramer-von Mises (CV), and Anderson-Darling (AD) tests. To be fair, we also employ the bootstrap versions of KS, CV, and AD tests.

For this, we consider

Group 1: Laplace(0, 1), Normal(0, 1), and Student’s t(3) distributions

and

Group 2: Gamma, Inverse-Gaussian (IG), and Weibull distributions with skewness equal to 1.414. The shape parameter of the Gamma, IG, and Weibull are 2, 4.5, and 1.259, respectively, and the scale parameter of the distribution is equal to 1 in all cases.

The figures in Tables 1-6 (Tables 1-3 for Group 1 and Tables 4-6 for Group 2) exhibit the proportion of the number of rejections of the null hypothesis out of 500 repetitions with $B = 500$. Here, we use $(n = 20, m = 4, 5, 6, 7)$, $(n = 50, m = 6, 7, 8, 9)$, $(n = 100, m = 8, 9, 10, 11)$, nominal level 0.05, and repetition number 1,000. In all the cases, the sizes turn out to be close to the nominal level regardless of the choice of $n, m$ and the power tends to increase as the sample size increases. In particular, it is shown that none of the tests outperform the others perfectly: our test significantly outperforms other tests in the cases of Student’s t vs. Normal and Weibull vs. Inverse-Gaussian. As seen in the tables, the choice of $m$ can affect the performance of the test in power. Thus, it may be an important issue to choose an optimal $m$ that produces the best powers, but it is difficult to set up a rule theoretically to obtain such $m$. Our past experience suggests that one may choose $m = c_1 + c_2 n^{1/3}$ for some suitable $c_1, c_2 > 0$, but this cannot be directly applied to all situations. In practice, for a given gof test, one could obtain an optimal $m$ empirically through a simulation. Overall, our findings show that the bootstrap Vasicek’s test performs adequately and is compatible with other existing tests.

4. Concluding remarks

To perform a gof test for a composite hypothesis, we suggested to use of a bootstrap Vasicek’s test. A simulation study indicates that the bootstrap test performs adequately in terms of size and power. The comparison study with other tests such as the KS, CV, and AD tests indicates that none of these tests outperform the others completely. Vasicek’s test appears to outperform the others in some situations and is proven to be a useful tool to perform a gof test. Manifestly, it would be of great interest to extend our method to dependent data sets, especially the residuals from time series models such as
autoregressive and GARCH models. Thus, we leave this as a task of our future study.

Table 1. Laplace null model: sizes and powers

|   | distribution | $n$  | $m$  | T    | KS   | CV   | AD   |
|---|--------------|------|------|------|------|------|------|
| Size | Laplace(0,1) | 20   | 4    | 0.064| 0.052| 0.052| 0.048|
|     |              | 20   | 5    | 0.050| 0.052| 0.056| 0.054|
|     |              | 20   | 6    | 0.058| 0.050| 0.052| 0.050|
|     |              | 20   | 7    | 0.064| 0.056| 0.058| 0.054|
|     |              | 50   | 6    | 0.048| 0.048| 0.072| 0.062|
|     |              | 50   | 7    | 0.046| 0.048| 0.046| 0.044|
|     |              | 50   | 8    | 0.036| 0.056| 0.068| 0.064|
|     |              | 50   | 9    | 0.058| 0.036| 0.038| 0.042|
|     |              | 100  | 8    | 0.046| 0.058| 0.054| 0.046|
|     |              | 100  | 9    | 0.056| 0.056| 0.038| 0.046|
|     |              | 100  | 10   | 0.050| 0.060| 0.040| 0.048|
|     |              | 100  | 11   | 0.042| 0.052| 0.050| 0.052|
| Power | Normal(0,1) | 20   | 4    | 0.195| 0.076| 0.084| 0.078|
|     |              | 20   | 5    | 0.222| 0.096| 0.076| 0.068|
|     |              | 20   | 6    | 0.252| 0.096| 0.094| 0.082|
|     |              | 20   | 7    | 0.240| 0.088| 0.094| 0.082|
|     |              | 50   | 6    | 0.442| 0.180| 0.170| 0.164|
|     |              | 50   | 7    | 0.482| 0.168| 0.152| 0.138|
|     |              | 50   | 8    | 0.492| 0.180| 0.158| 0.142|
|     |              | 50   | 9    | 0.508| 0.212| 0.192| 0.152|
|     |              | 100  | 8    | 0.690| 0.364| 0.404| 0.342|
|     |              | 100  | 9    | 0.692| 0.386| 0.420| 0.370|
|     |              | 100  | 10   | 0.696| 0.420| 0.420| 0.378|
|     |              | 100  | 11   | 0.718| 0.406| 0.396| 0.348|
| Power | $t(3)$       | 20   | 4    | 0.070| 0.058| 0.058| 0.072|
|     |              | 20   | 5    | 0.080| 0.064| 0.072| 0.092|
|     |              | 20   | 6    | 0.048| 0.060| 0.058| 0.066|
|     |              | 20   | 7    | 0.072| 0.078| 0.062| 0.066|
|     |              | 50   | 6    | 0.078| 0.068| 0.064| 0.076|
|     |              | 50   | 7    | 0.060| 0.074| 0.058| 0.068|
|     |              | 50   | 8    | 0.082| 0.098| 0.100| 0.011|
|     |              | 50   | 9    | 0.056| 0.086| 0.076| 0.009|
|     |              | 100  | 8    | 0.046| 0.104| 0.092| 0.110|
|     |              | 100  | 9    | 0.064| 0.098| 0.088| 0.100|
|     |              | 100  | 10   | 0.054| 0.078| 0.070| 0.074|
|     |              | 100  | 11   | 0.088| 0.078| 0.074| 0.088|
Table 2. Normal null model: sizes and powers

| distribution | $n$ | $m$ | $T$   | KS   | CV   | AD   |
|--------------|-----|-----|-------|------|------|------|
| Size         |     |     |       |      |      |      |
| Normal(0,1)  | 20  | 4   | 0.044 | 0.062| 0.060| 0.058|
|              | 20  | 5   | 0.054 | 0.064| 0.050| 0.044|
|              | 20  | 6   | 0.068 | 0.048| 0.060| 0.060|
|              | 20  | 7   | 0.050 | 0.070| 0.052| 0.056|
|              | 50  | 6   | 0.050 | 0.068| 0.058| 0.064|
|              | 50  | 7   | 0.052 | 0.050| 0.046| 0.044|
|              | 50  | 8   | 0.058 | 0.056| 0.044| 0.044|
|              | 50  | 9   | 0.056 | 0.042| 0.054| 0.062|
|              | 100 | 8   | 0.032 | 0.066| 0.058| 0.050|
|              | 100 | 9   | 0.052 | 0.058| 0.054| 0.056|
|              | 100 | 10  | 0.066 | 0.036| 0.046| 0.042|
|              | 100 | 11  | 0.034 | 0.046| 0.050| 0.040|
| Power        |     |     |       |      |      |      |
| $t(3)$       | 20  | 4   | 0.136 | 0.226| 0.290| 0.320|
|              | 20  | 5   | 0.104 | 0.252| 0.302| 0.324|
|              | 20  | 6   | 0.116 | 0.304| 0.326| 0.366|
|              | 20  | 7   | 0.092 | 0.246| 0.302| 0.332|
|              | 50  | 6   | 0.276 | 0.470| 0.550| 0.570|
|              | 50  | 7   | 0.248 | 0.524| 0.612| 0.626|
|              | 50  | 8   | 0.158 | 0.444| 0.514| 0.548|
|              | 50  | 9   | 0.120 | 0.474| 0.586| 0.622|
|              | 100 | 8   | 0.528 | 0.704| 0.786| 0.824|
|              | 100 | 9   | 0.514 | 0.748| 0.844| 0.864|
|              | 100 | 10  | 0.404 | 0.722| 0.808| 0.844|
|              | 100 | 11  | 0.348 | 0.712| 0.832| 0.850|
| Power        |     |     |       |      |      |      |
| Laplace(0,1) | 20  | 4   | 0.062 | 0.232| 0.268| 0.274|
|              | 20  | 5   | 0.048 | 0.230| 0.248| 0.274|
|              | 20  | 6   | 0.048 | 0.244| 0.296| 0.292|
|              | 20  | 7   | 0.034 | 0.208| 0.230| 0.236|
|              | 50  | 6   | 0.210 | 0.458| 0.580| 0.576|
|              | 50  | 7   | 0.128 | 0.456| 0.558| 0.556|
|              | 50  | 8   | 0.110 | 0.434| 0.536| 0.548|
|              | 50  | 9   | 0.068 | 0.400| 0.522| 0.528|
|              | 100 | 8   | 0.428 | 0.690| 0.812| 0.806|
|              | 100 | 9   | 0.380 | 0.716| 0.824| 0.840|
|              | 100 | 10  | 0.318 | 0.742| 0.836| 0.836|
|              | 100 | 11  | 0.258 | 0.704| 0.814| 0.816|
### Table 3. Student’s $t$ null model: sizes and powers

|    | distribution | $n$ | $m$ | $T$ | KS  | CV  | AD  |
|----|--------------|-----|-----|-----|-----|-----|-----|
| Size | $t(3)$       |     |     |     |     |     |     |
|     | 20           | 4   | 0.066 | 0.060 | 0.050 | 0.056 |
|     | 20           | 5   | 0.060 | 0.072 | 0.054 | 0.050 |
|     | 20           | 6   | 0.056 | 0.058 | 0.052 | 0.056 |
|     | 20           | 7   | 0.054 | 0.048 | 0.048 | 0.053 |
|     | 50           | 6   | 0.048 | 0.048 | 0.040 | 0.040 |
|     | 50           | 7   | 0.042 | 0.028 | 0.042 | 0.046 |
|     | 50           | 8   | 0.048 | 0.036 | 0.042 | 0.042 |
|     | 50           | 9   | 0.044 | 0.052 | 0.046 | 0.054 |
|     | 100          | 8   | 0.028 | 0.060 | 0.056 | 0.050 |
|     | 100          | 9   | 0.052 | 0.044 | 0.042 | 0.040 |
|     | 100          | 10  | 0.058 | 0.058 | 0.052 | 0.050 |
|     | 100          | 11  | 0.074 | 0.044 | 0.050 | 0.054 |
| Power | Laplace(0,1) | 20  | 4   | 0.068 | 0.044 | 0.038 | 0.036 |
|     | 20           | 5   | 0.086 | 0.044 | 0.046 | 0.042 |
|     | 20           | 6   | 0.066 | 0.052 | 0.054 | 0.056 |
|     | 20           | 7   | 0.072 | 0.058 | 0.060 | 0.056 |
|     | 50           | 6   | 0.080 | 0.058 | 0.048 | 0.046 |
|     | 50           | 7   | 0.072 | 0.052 | 0.056 | 0.058 |
|     | 50           | 8   | 0.082 | 0.058 | 0.044 | 0.028 |
|     | 50           | 9   | 0.074 | 0.042 | 0.052 | 0.042 |
|     | 100          | 8   | 0.104 | 0.056 | 0.048 | 0.044 |
|     | 100          | 9   | 0.116 | 0.078 | 0.060 | 0.052 |
|     | 100          | 10  | 0.070 | 0.064 | 0.048 | 0.052 |
|     | 100          | 11  | 0.068 | 0.058 | 0.038 | 0.040 |
| Power | Normal(0,1)  | 20  | 4   | 0.312 | 0.060 | 0.054 | 0.048 |
|     | 20           | 5   | 0.358 | 0.060 | 0.052 | 0.040 |
|     | 20           | 6   | 0.356 | 0.052 | 0.042 | 0.030 |
|     | 20           | 7   | 0.328 | 0.044 | 0.030 | 0.026 |
|     | 50           | 6   | 0.804 | 0.052 | 0.048 | 0.050 |
|     | 50           | 7   | 0.854 | 0.044 | 0.048 | 0.046 |
|     | 50           | 8   | 0.816 | 0.048 | 0.050 | 0.050 |
|     | 50           | 9   | 0.810 | 0.030 | 0.042 | 0.013 |
|     | 100          | 8   | 0.990 | 0.044 | 0.060 | 0.130 |
|     | 100          | 9   | 0.996 | 0.062 | 0.096 | 0.158 |
|     | 100          | 10  | 0.992 | 0.064 | 0.084 | 0.144 |
|     | 100          | 11  | 0.984 | 0.068 | 0.088 | 0.136 |
Table 4. Gamma null model: sizes and powers

| distribution      | n  | m  | T   | KS  | CV  | AD  |
|-------------------|----|----|-----|-----|-----|-----|
| **Size** Gamma \( k=2 \) | 20 | 4  | 0.060 | 0.062 | 0.058 | 0.050 |
|                   | 20 | 5  | 0.048 | 0.054 | 0.052 | 0.046 |
|                   | 20 | 6  | 0.044 | 0.056 | 0.048 | 0.048 |
|                   | 20 | 7  | 0.046 | 0.072 | 0.078 | 0.074 |
|                   | 50 | 6  | 0.040 | 0.046 | 0.050 | 0.056 |
|                   | 50 | 7  | 0.034 | 0.034 | 0.038 | 0.038 |
|                   | 50 | 8  | 0.044 | 0.048 | 0.046 | 0.050 |
|                   | 50 | 9  | 0.046 | 0.036 | 0.038 | 0.034 |
|                   | 100| 8  | 0.056 | 0.050 | 0.05  | 0.056 |
|                   | 100| 9  | 0.058 | 0.058 | 0.052 | 0.054 |
|                   | 100| 10 | 0.048 | 0.048 | 0.056 | 0.054 |
|                   | 100| 11 | 0.056 | 0.056 | 0.054 | 0.060 |
| **Power** Inverse Gaussian \( \lambda=4.5 \) | 20 | 4  | 0.068 | 0.078 | 0.080 | 0.084 |
|                   | 20 | 5  | 0.072 | 0.078 | 0.096 | 0.098 |
|                   | 20 | 6  | 0.044 | 0.110 | 0.100 | 0.100 |
|                   | 20 | 7  | 0.034 | 0.082 | 0.080 | 0.082 |
|                   | 50 | 6  | 0.110 | 0.124 | 0.132 | 0.152 |
|                   | 50 | 7  | 0.084 | 0.114 | 0.152 | 0.168 |
|                   | 50 | 8  | 0.098 | 0.128 | 0.152 | 0.174 |
|                   | 50 | 9  | 0.084 | 0.148 | 0.158 | 0.166 |
|                   | 100| 8  | 0.164 | 0.198 | 0.240 | 0.278 |
|                   | 100| 9  | 0.150 | 0.224 | 0.268 | 0.298 |
|                   | 100| 10 | 0.126 | 0.248 | 0.272 | 0.308 |
|                   | 100| 11 | 0.128 | 0.208 | 0.248 | 0.264 |
| **Power** Weibull \( k=1.259 \) | 20 | 4  | 0.078 | 0.054 | 0.064 | 0.064 |
|                   | 20 | 5  | 0.066 | 0.070 | 0.072 | 0.070 |
|                   | 20 | 6  | 0.034 | 0.066 | 0.050 | 0.048 |
|                   | 20 | 7  | 0.054 | 0.062 | 0.082 | 0.084 |
|                   | 50 | 6  | 0.056 | 0.078 | 0.076 | 0.072 |
|                   | 50 | 7  | 0.072 | 0.064 | 0.062 | 0.066 |
|                   | 50 | 8  | 0.060 | 0.062 | 0.056 | 0.062 |
|                   | 50 | 9  | 0.070 | 0.058 | 0.066 | 0.068 |
|                   | 100| 8  | 0.074 | 0.080 | 0.090 | 0.088 |
|                   | 100| 9  | 0.066 | 0.070 | 0.072 | 0.070 |
|                   | 100| 10 | 0.088 | 0.096 | 0.094 | 0.084 |
|                   | 100| 11 | 0.086 | 0.074 | 0.092 | 0.088 |
Table 5. Inverse Gaussian null model: sizes and powers

| distribution | n   | m   | T   | KS  | CV  | AD  |
|--------------|-----|-----|-----|-----|-----|-----|
| **Size**     |     |     |     |     |     |     |
| Inverse Gaussian $\lambda=4.5$ |     |     |     |     |     |     |
| 20 | 4  | 0.086 | 0.060 | 0.074 | 0.076 |
| 20 | 5  | 0.040 | 0.052 | 0.048 | 0.054 |
| 20 | 6  | 0.042 | 0.050 | 0.050 | 0.044 |
| 20 | 7  | 0.062 | 0.040 | 0.044 | 0.040 |
| 50 | 6  | 0.062 | 0.064 | 0.056 | 0.048 |
| 50 | 7  | 0.050 | 0.066 | 0.074 | 0.068 |
| 50 | 8  | 0.056 | 0.048 | 0.052 | 0.052 |
| 50 | 9  | 0.056 | 0.038 | 0.044 | 0.042 |
| 100 | 8  | 0.054 | 0.052 | 0.066 | 0.054 |
| 100 | 9  | 0.052 | 0.062 | 0.048 | 0.050 |
| 100 | 10 | 0.048 | 0.052 | 0.048 | 0.058 |
| 100 | 11 | 0.058 | 0.052 | 0.052 | 0.046 |
| **Power**    |     |     |     |     |     |     |
| Gamma $k=2$  |     |     |     |     |     |     |
| 20 | 4  | 0.162 | 0.266 | 0.324 | 0.322 |
| 20 | 5  | 0.150 | 0.270 | 0.314 | 0.324 |
| 20 | 6  | 0.106 | 0.306 | 0.324 | 0.328 |
| 20 | 7  | 0.076 | 0.280 | 0.336 | 0.348 |
| 50 | 6  | 0.416 | 0.562 | 0.644 | 0.654 |
| 50 | 7  | 0.346 | 0.562 | 0.632 | 0.638 |
| 50 | 8  | 0.306 | 0.518 | 0.600 | 0.618 |
| 50 | 9  | 0.296 | 0.586 | 0.660 | 0.678 |
| 100 | 8 | 0.680 | 0.814 | 0.868 | 0.874 |
| 100 | 9 | 0.628 | 0.824 | 0.884 | 0.898 |
| 100 | 10 | 0.608 | 0.840 | 0.880 | 0.888 |
| 100 | 11 | 0.574 | 0.832 | 0.864 | 0.870 |
| Weibull $k=1.259$ | | | | | | |
| 20 | 4  | 0.342 | 0.476 | 0.516 | 0.524 |
| 20 | 5  | 0.286 | 0.468 | 0.514 | 0.532 |
| 20 | 6  | 0.256 | 0.504 | 0.540 | 0.560 |
| 20 | 7  | 0.220 | 0.522 | 0.550 | 0.552 |
| 50 | 6  | 0.730 | 0.834 | 0.880 | 0.890 |
| 50 | 7  | 0.700 | 0.826 | 0.872 | 0.876 |
| 50 | 8  | 0.632 | 0.818 | 0.854 | 0.870 |
| 50 | 9  | 0.616 | 0.816 | 0.872 | 0.870 |
| 100 | 8 | 0.978 | 0.994 | 0.998 | 0.998 |
| 100 | 9 | 0.944 | 0.984 | 0.994 | 0.994 |
| 100 | 10 | 0.938 | 0.984 | 0.998 | 0.998 |
| 100 | 11 | 0.924 | 0.968 | 0.994 | 0.994 |
Table 6. Weibull null model: sizes and powers

| Size     | Weibull $k=1.259$ | $n$ | $m$ | T   | KS    | CV    | AD    |
|----------|-------------------|-----|-----|-----|-------|-------|-------|
|          |                   | 20  | 4   | 0.058 | 0.044 | 0.032 | 0.042 |
|          |                   | 20  | 5   | 0.034 | 0.060 | 0.052 | 0.046 |
|          |                   | 20  | 6   | 0.046 | 0.050 | 0.062 | 0.066 |
|          |                   | 20  | 7   | 0.036 | 0.054 | 0.050 | 0.056 |
|          |                   | 50  | 6   | 0.040 | 0.044 | 0.040 | 0.058 |
|          |                   | 50  | 7   | 0.058 | 0.048 | 0.052 | 0.062 |
|          |                   | 50  | 8   | 0.044 | 0.054 | 0.044 | 0.056 |
|          |                   | 50  | 9   | 0.042 | 0.050 | 0.052 | 0.052 |
|          |                   | 100 | 8   | 0.060 | 0.050 | 0.046 | 0.050 |
|          |                   | 100 | 9   | 0.054 | 0.074 | 0.058 | 0.062 |
|          |                   | 100 | 10  | 0.050 | 0.078 | 0.074 | 0.080 |
|          |                   | 100 | 11  | 0.072 | 0.048 | 0.044 | 0.052 |
| Power    | Inverse Gaussian $\lambda=4.5$ | 20  | 4   | 0.228 | 0.158 | 0.190 | 0.190 |
|          |                   | 20  | 5   | 0.248 | 0.168 | 0.210 | 0.214 |
|          |                   | 20  | 6   | 0.248 | 0.144 | 0.192 | 0.198 |
|          |                   | 20  | 7   | 0.222 | 0.174 | 0.212 | 0.208 |
|          |                   | 50  | 6   | 0.668 | 0.378 | 0.524 | 0.582 |
|          |                   | 50  | 7   | 0.660 | 0.412 | 0.518 | 0.564 |
|          |                   | 50  | 8   | 0.638 | 0.330 | 0.486 | 0.560 |
|          |                   | 50  | 9   | 0.602 | 0.372 | 0.470 | 0.534 |
|          |                   | 100 | 8   | 0.942 | 0.688 | 0.840 | 0.890 |
|          |                   | 100 | 9   | 0.938 | 0.664 | 0.834 | 0.904 |
|          |                   | 100 | 10  | 0.920 | 0.692 | 0.828 | 0.898 |
|          |                   | 100 | 11  | 0.940 | 0.704 | 0.852 | 0.914 |
| Power    | Gamma $k=2$       | 20  | 4   | 0.068 | 0.070 | 0.068 | 0.066 |
|          |                   | 20  | 5   | 0.060 | 0.064 | 0.058 | 0.070 |
|          |                   | 20  | 6   | 0.056 | 0.058 | 0.076 | 0.084 |
|          |                   | 20  | 7   | 0.044 | 0.080 | 0.074 | 0.072 |
|          |                   | 50  | 6   | 0.092 | 0.074 | 0.092 | 0.084 |
|          |                   | 50  | 7   | 0.086 | 0.082 | 0.086 | 0.078 |
|          |                   | 50  | 8   | 0.090 | 0.082 | 0.086 | 0.094 |
|          |                   | 50  | 9   | 0.084 | 0.082 | 0.088 | 0.092 |
|          |                   | 100 | 8   | 0.140 | 0.094 | 0.124 | 0.132 |
|          |                   | 100 | 9   | 0.118 | 0.108 | 0.136 | 0.130 |
|          |                   | 100 | 10  | 0.114 | 0.084 | 0.108 | 0.128 |
|          |                   | 100 | 11  | 0.150 | 0.110 | 0.146 | 0.156 |

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