Simple wave-optical superpositions as prime number sieves

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We encode the sequence of prime numbers into simple superpositions of identical waves, mimicking the archetypal prime number sieve of Eratosthenes. The primes are identified as zeros accompanied by phase singularities in a physically generated wave-field for integer valued momenta. Similarly, primes are encoded in the diffraction pattern from a simple single aperture and in the harmonics of a single vibrating resonator. Further, diffraction physics connections to number theory reveal how to encode all Gaussian primes, twin-primes, and how to construct wave fields with amplitudes equal to the divisor function at integer spatial frequencies. Remarkably, all of these basic diffraction phenomena reveal that the naturally irregular sequence of primes can arise from trivially ordered wave superpositions.

We may construe the distribution of primes as a puzzle in physics. Can the seemingly random yet highly orchestrated prime number distribution correspond to states of a physical system? Tentative affirmations have arisen from research into physics and number theory connections, particularly via the Riemann zeta function. Building upon Euler’s connection between generalized harmonic series and products over primes, Riemann described a Fourier-like analysis to synthesize the prime counting function. Hinging upon this construction is the placement of non-trivial zeros of the associated zeta function, which remains elusive.

Diverse studies have revealed the ‘physics of the Riemann hypothesis’, ranging from classical and quantum billiard balls, quantum scattering and bound states to statistical physics, condensed matter and more. Notably, Berry and Keating related the zeros of the Riemann zeta function to eigenvalues in wave systems to imply validity of the Riemann hypothesis. Berry and Keating related the zeros of the Riemann hypothesis, ranging from classical and quantum mechanics, to imply validity of the Riemann hypothesis.

Statistically speaking, the distribution of prime numbers can be modeled by the Riemann zeta function. The Riemann zeta function is defined over the complex numbers, with a branch cut along the real line from 1 to 0. The Riemann zeta function is a complex function that is defined for all complex numbers except 1. The Riemann zeta function is related to the prime numbers through the Euler product formula, which states that the Riemann zeta function can be expressed as an infinite product over all prime numbers.

The product is over all prime numbers, and is given by:

\[ \zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \]

where \( s \) is a complex variable, and \( \frac{1}{1-p^{-s}} \) is a geometric series over the prime numbers.

The Riemann hypothesis states that all non-trivial zeros of the Riemann zeta function lie on the critical line \( \Re(s) = \frac{1}{2} \). This hypothesis has been verified for the first 10 trillion zeros, but a proof of the hypothesis remains elusive.

The Riemann hypothesis has important implications for the distribution of prime numbers. The hypothesis states that the distribution of prime numbers is closely related to the distribution of zeros of the Riemann zeta function. This relationship is captured by the explicit formula for the prime counting function, which is given by:

\[ \pi(x) = \frac{x}{\log(x)} - \frac{x}{2\log^2(x)} + \frac{x}{3\log^3(x)} - \cdots \]

where \( \pi(x) \) is the number of primes less than or equal to \( x \).

The Riemann hypothesis implies that the error term in the prime number theorem is \( O(x^{1/2 + \epsilon}) \), for any \( \epsilon > 0 \), which is significantly better than the error term in the prime number theorem without the Riemann hypothesis, which is \( O(x^{1/2+\epsilon}) \).

In this letter we are interested in whether simple wave superposition can give rise to the prime number sequence, in the absence of dynamical chaos or dedicated factorization checks. Can analog wave optical prime sieves perform basic primality tests?

Wilson’s theorem, that \( p \) divides \( (p-1)! + 1 \) for prime \( p \), could be employed to design a field with amplitude \( \cos(\pi(x-1)/x + \pi/x) \), the integer floor of which provides an indicator function for primes over integer \( x \) positions. Whilst oscillatory, the construction is nonetheless contrived. A step further could be to consider the product \( \prod_{n=2}^{N} \sin(\pi x/n - \pi) \), which identifies all composite integer (non-prime) \( x \) as a zero, since each sinc factor eliminates multiples of all but one integer of interest. Though based upon single-slit diffraction, it is difficult to envisage an experimental implementation of this series. We shall instead consider much simpler wave superpositions, which could be realized in an experiment – essentially the opposite of the sinc idea, whereby composites are discarded by removing zeros in the wave field.

Suppose a superposed set \( \Psi_N(r) \) of \( N \times N \) identical scalar wave sources is located in the x-y plane, with each source defined by wavefunction \( \psi(r) \) at position \( r_j = (x_j, y_j) \); here the optic axis is along the z direction. Farfield diffraction of \( \Psi_N(r) \) can be written as the Fourier transform of \( \psi(r) \) convolved with a set of Dirac deltas \( \delta(r - r_j) \), i.e.,

\[ \hat{\Psi}_N(q) = F[\psi(r) * \Sigma_j \delta(r - r_j)] \]

where the spatial frequency or momentum is denoted by \( q = (q_x, q_y) \).

For an \( N \times N \) diffraction grating with \( x_j = (j - (N + 1)/2)/N \) for \( j = 1, 2, 3, \ldots, N \), in dimensionless units, and likewise for the \( y \) positions, the far-field wave is proportional to,

\[ \hat{\Psi}_N(q) = \hat{\psi}(q) \frac{\sin(\pi q_x) \sin(\pi q_y)}{\sin(\pi q_x/N) \sin(\pi q_y/N)} \]

(1)

where \( \hat{\psi}(q) = F[\psi(r)] \) is the Fourier transform of \( \psi(r) \). For ideal pinholes, \( \hat{\psi}(q) \) tends to unity and \( \hat{\Psi}_N(q_x, 0) \) or \( \hat{\Psi}_N(0, q_y) \) then has form matching one of the graphs in Fig. 1 for a given \( N \). Re-scaling \( q \) by \( N \) reveals Eq. 1 as a modified Dirichlet kernel from Fourier-series analysis, which arises from a symmetric sum of plane waves. In fact the mathematics employed throughout this Letter
revolves around complex exponential sums with rational frequencies, which are ubiquitous in analytic number theory [15].

FIG. 1. Symmetric sets of waves encode the prime-number sieve of Eratosthenes. Diffraction of identical wave sources arranged in simple $N \times N$ grids creates interference patterns with zeros at all integer momenta $q$, except at multiples of $N$, where the amplitude equals $\pm N$. Hermite-Gauss mode sources (shown in green) can be used to place additional zeros at one or both of the red circles. Superposition then encodes sources (shown in green) can be used to place additional zeros arranged in simple $N$ sieve of Eratosthenes. Diffraction of identical wave sources

$\exp(2\pi ik\hat{z})\psi(r, z)$ of a paraxial scalar wave satisfies the paraxial Helmholtz equation $\{\partial^2_x + \partial^2_y + 2i k \partial_z\} \psi(r, z) = 0$ [16], where $k$ is the wave number. Among plane waves and other forms, exact solutions of this equation are given by the Hermite-Gauss modes, which are also eigenfunctions of the quantum harmonic oscillator. In the $x$-$y$ plane, up to a complex constant, the $(0,0), (1,0)$ and $(0,1)$ order modes can be written as $\psi_{00}(r) = \exp(-r^2/\sigma^2)$, $\psi_{10}(r) = x \psi_{00}(r)$ and $\psi_{01}(r) = y \psi_{00}(r)$, respectively, where $r = |r|$ and $\sigma$ is the beam waist at the plane $z = 0$. A superposition of $(0,2)$ and $(2,0)$ modes gives the quadratic form $\psi_Q(r) = r^2 \psi_{00}(r)$. Any of these waves in \( \lim \sigma \to \infty \) would suffice for this discussion, which correspond to ‘polynomial waves’ [17, 18]. All wave forms and superpositions in this work were chosen for simplicity.

For bounded $f(x)$ in the implicit Fourier convention of Eq. [1] the relation $F[\partial_x f(x)] = 2\pi i q_x F[f(x)]$ holds. Similarly, $F[x f(x)] = i/(2\pi) \delta_{q_x} F[f(x)]$. As such, $F[\psi_{10}(r)]$ contains a zero at the $q_x$ origin, which can be shifted along $q_y$ by $N$ units of momenta after applying a linear phase-ramp $\exp(2\pi i x N)$ to tilt the $N \times N$ wave array. Equivalent remarks hold if the identical wave sources are instead chosen to be $F[\psi_{01}(r)]$. Phase ramps for each $N \times N$ array could be difficult in an experiment but can be avoided by using the $2^{nd}$ order mode sources, since $F[\psi_Q(r)]$ contains a ring of zeros in the far-field, due to $2^{nd}$ order gradients arising from the quadratic term $r^2$. By careful choice of $\sigma$, the ring radius can be set to $N$ units of momenta. Hence the required isolated zeros can be inscribed in the far-field to prevent candidate primes becoming non-zero in the sieve generated by the symmetric superposition of all waves.

The classical Eratosthenes algorithm iteratively eliminates composites by crossing out multiples of primes identified at each step. Our wave-optical approach automatically sieves all integer multiples and is hence less efficient. Nonetheless Eq. [2] represents a close analogy,

$\hat{\Psi}_M(q) = \sum_{N=2}^M \hat{\Psi}_N(q) \frac{\sin(\pi q_x)}{\sin(\pi q_x/N)} \frac{\sin(\pi q_y)}{\sin(\pi q_y/N)}$,  \(2\)

where $\hat{\Psi}_N(q) = (-1)^N \hat{\psi}(q_x - N, q_y)$, for a phase shifted array when each source is of type $\psi_{10}(r)$, or $\hat{\psi}_N(q) = \hat{\psi}_Q(q)$ for the $2^{nd}$ order mode sources of width $\sigma_N$ without a phase ramp. Using these superpositions, zeros in the far-field pattern at integer momenta on the $q_x$-$q_y$ axes indicate primes for integers $|q| \leq M^2$, with non-zero amplitudes specifying composite numbers. Sifting occurs indefinitely beyond $|q| = M^2$ but some composite momenta will then also correspond to zero amplitudes. For a given signal to noise ratio, this sifting is physically limited by energy, giving rise to diminishing wave amplitudes at large $|q|$. Displaying the square root of the amplitude, Fig. 2 shows some examples of Eq. [2] with $\psi_{10}(r)$ and $\sigma =
0.05, $M = 31$, producing Fig. 2(a) and the corresponding trace along $q_y = 0$ in Fig. 2(c). The ring of size $|q| = 7$ in Fig. 2(b) shows an example of $F[\psi_0(r)]$ for $\sigma = 1/(N\pi)$ where $N = 7$ and Fig. 2(d) shows a horizontal trace over discrete $q_x$ computed from Eq. 2 using $\psi_0(r)$ sources with the sum maximum $M = 31$. Note the almost zero values for even integers, non-integers, and $q_x = 25$ in Fig. 2(c), which is masquerading as a possible prime. The amplitude at $q_x = 25$ evaluates to the tiny value of $24/(25\pi 3)$ but is strictly zero for prime $q_x$. These observations reveal important physical limitations for realizing wave-optical prime sieves. Similar remarks hold for the discrete plot in Fig. 2(c), for an un-shifted $\hat{\psi}_Q(q)$ based source spectrum.

![Image](54x375 to 299x558)

**FIG. 2.** The prime sieve of Eratosthenes realized with superposed Hermite-Gauss modes. (a) Square root of the far-field horizontal trace from the origin along $q_y = 0$ in (c). (b) Ring of zeros for a quadratic $2^{nd}$ order mode with dimensionless radius $q = 7$, superpositions of which yield the prime sieve in (d), for integer $q$ along either $q$ axis.

Simpler wave-optical prime sieves are possible if mimicry of Eratosthenes’ algorithm is jettisoned. Given that the trigonometric ratios in Eq. 1 provide the essential divisibility tests for $q_x$ or $q_y$, it is instructive to devise more basic complex exponential sums over ordered rational frequencies. To this end, consider

$$\hat{\psi}_Q(q) \equiv \sum_{N=1}^{M} \sum_{j=1}^{N} e^{2\pi i \alpha j/N} = \sum_{N=1}^{M} (-1)^{(1+N)/N} \frac{\sin(\pi \alpha)}{\sin(\pi \alpha/N)},$$

(3)

where $\alpha$ and $j/N$ could correspond to respective momentum and position or vice versa. Note that the sign alternation for $\alpha$ divisible by $N$ cancels that of the sine term, hence Eq. 3 is a simpler one-dimensional version of Eq. 1 which can be furnished with a given source term $\psi(\alpha)$ through convolution, if relevant. From a diffraction physics perspective, the inner sum in Eq. 3 can be viewed as a phase singularity for integer $\alpha$ not divisible by $N$, since the phasor sum inscribes a circle as a regular polygon in the Argand plane for $\alpha = 1$, winding $\alpha$ times about this polygon. When $N$ divides $\alpha$, the inner sum instead represents a plane wave of amplitude $N$, since all phasors add along a line in the Argand plane. As with all sums here, only the real part of the superposition is of physical significance.

The number-theoretic properties of Eq. 3 are interesting. The inner sum evaluates to $N$ for $\alpha$ divisible by $N$, and is zero for all other integer $\alpha$. $\hat{\psi}_Q(q)$ is therefore the divisor function $\sigma_1(\alpha)$ from number theory, describing the sum of all integers that divide integer $\alpha$. Equation 3 thus represents another type of sieve, for which candidate prime $\alpha$ are identified as the fixed points $\hat{\psi}_Q(q) = \alpha$. An interpretation in terms of Thomae’s ruler function can also be made for ideal pin-holes, as explained next.

Consider $T_M(\alpha') = F^{-1}[\hat{\psi}_Q(q)]$, with the inclusion of wave sources $\psi(\alpha')$ normalized by $M$, where $\alpha'$ is Fourier-conjugate to $\alpha$. The Fourier shift theorem gives $T_M(\alpha') = M^{-1} \sum \delta(\alpha' - j/N)$, where the sums range as in Eq. 3 and $\ast$ denotes convolution. Each $M^{th}$ row of these delta functions, evenly spaced on the unit interval, can be written as $\Pi(\alpha')\Pi_{\alpha}(\alpha')$, where $\Pi(\alpha')$ is the unit step function and $\Pi_{\alpha}(\alpha')$ is a Dirac comb with teeth separated by $1/N$ units on the $\alpha'$ axis. The inverse transform is then $\psi(\alpha') \ast (\Pi(\alpha') \Pi_{\alpha}(\alpha'))$, where the sum runs from $N$ to $M$. Consider now a rational point $\alpha' = n/m$ with co-prime $(n,m)$, such that $M$ is divisible by $m$ and choose ideal pin-holes $\psi(\alpha') \rightarrow \delta(\alpha')$. Observe that $\Pi_{\alpha}(n/m) = \Pi_{\alpha}(1/m)$, since scaling $1/m$ by $n$ simply selects another tooth of the comb or else there are no teeth at either location and both sides are then zero. If $M$ is divisible by $m$, then $\Pi_{\alpha}(n/m)$ term gives unity for $m = N = M$, as the fraction $1/m$ uniquely identifies just one tooth out of all combs in the double sum $F^{-1}[S(\alpha)]$. This is true for all $m$ when $M$ tends to infinity, hence $\lim_{M \rightarrow \infty} T_M(n/m) = 1/m$. The definition of Thomae’s function is that $T(n/m) = 1/m$ for co-prime $(n, m)$ and zero for irrational arguments, which implies that $\lim_{M \rightarrow \infty} T_M(\alpha') = T(\alpha')$, as claimed.

In this context, the distribution of the primes is given by the fixed points in the spectrum $\hat{\psi}_Q(\alpha)$ of Thomae’s function $T(\alpha')$ for integer $\alpha$. Superpositions such as $\hat{\psi}_Q(\alpha)$ can be realized in simplified diffraction experiments. For example, identifying $\alpha'$ with position $x$, the set of wave sources sampled by the 2D Dirac distribution $\sum \delta(x - j/N) \delta(y - N)$ produces a far-field diffraction pattern matching Eq. 3 along the $q_x$ axis for $q_y = 0$. Extra $j = 0$ terms were included in Eq. 3 for aesthetic purposes to plot this distribution as the set of black squares in Fig. 3(a), where the numbers indicate the value of $N$ in the sum (y points down the page), which ranges up to maximum $M = 11$ (cf. the set of natural line angles in the discrete Hough transform). Figure 3(a)
alone reveals the asymptote towards Thomae’s function in the effective superposition, since there are roughly 1/2 as many vertically aligned sources in the middle than the outermost columns, 1/3 as many sources at either one 3\textsuperscript{rd} or two 3\textsuperscript{rd}’s of horizontal distance x, and so on. The Fourier transform of this distribution of ideal pinholes was computed and M + 1 was subtracted from the wave amplitude to account for the additional sources arising from the j = 0 terms. Further division by q\textsubscript{x} created the plot of normalized wave amplitude over discrete momenta q = q\textsubscript{x} (at q\textsubscript{y} = 0) shown in Fig. 3(b), where the red dots are primes. As expected, the unity values arise from fixed points in the normalized divisor function σ\textsubscript{1}(q)/q up until q = 11. For this chosen value of M, primes are also uniquely identified by zeros in the wave field at integer q up until q = M\textsuperscript{2} = 121, since there are no terms in the effective S(q) sum to contribute non-zero amplitude at discrete momenta q. Beyond this q, composite momenta can also give rise to zeros and the sieve no longer faithfully identifies candidate prime q. Small variations of Eq. 3 such as the inclusion of source types other than ideal pinholes, index changes, or reinterpretations of α, can be used to adapt such wave sieves for other number theoretic wave fields of interest. For example, any integer can be represented as a unique product of squared and square-free integers, where a square-free integer contains no squares in its prime factor decomposition. By considering only rows N = 1, 2, 4, 9... in Fig. 3(a), the partial set of source locations then sifts integer momenta by the squares of all primes, resulting in zeros in the far-field diffraction pattern at integer momenta corresponding to all square-free numbers [22].

An acoustic example is a vibrating pipe open at both ends. With α identified as position x along the pipe, the real part of Eq. 3 describes the longitudinal wave displacement as a set of modes \cos(\pi x q_{NM}/(2L_M)) for a pipe of length L_M = LCM(1, 2, 3...M) and particular harmonics q_{NM} = jL_M/N, where LCM is the least common multiple. At time t = 0 for x ≤ M, fixed points of the longitudinal wave displacement identify prime x. For M < x ≤ M\textsuperscript{2}, zero wave displacement indicates prime x. Dynamics can be included with, say, a linear dispersion relationship. The displacement x at a pipe end would be prime or zero at prime instances of time, in units scaled by the dispersion relation.

Other connections between basic diffraction physics and number theory are possible with further variants of Eq. 2 or Eq. 3. Two final number theoretical wave fields are worth discussing - a superposition containing the set of Gaussian primes and another that sieves twin primes. For integers a and b, the Gaussian integers a + ib are complex numbers which can be uniquely factorized by other Gaussian integers known as ‘Gaussian primes’ that have norm a\textsuperscript{2} + b\textsuperscript{2} equal to a prime number [4]. As such, many prime numbers (but not all) can be factored by Gaussian primes. When α in Eq. 2 is interpreted as r\textsuperscript{2} = x\textsuperscript{2} + y\textsuperscript{2}, this superposition represents a sum of paraxial spherical waves on the optic axis (z = 0), which automatically sifts Gaussian primes for integer x and y. The Gaussian sieve continuum in Fig. 4(a) was computed from Eq. 3 up to M = 23, taking the square root of the intensity. The field is most intense at the origin, since all sources lie on the optic axis. The bright squares in Fig. 4(b) are the same data, where each square shows the fixed points of the wave at integer (x, y) positions, corresponding precisely to all Gaussian primes. In experiment, the various j/N phase curvatures could arise from different source locations on the optic axis. Indeed, a simple longitudinal grating can be realized by endowing Eq. 3 with a little more diffraction physics.

FIG. 3. (a) Primes encoded in a single symmetric aperture. (b) Normalized far-field diffraction from pinholes at the square locations in (a) identifies primes as unit amplitudes along the q\textsubscript{x} axis for momentum |q| = q ≤ 11, beyond which zeros uniquely identify primes until q > 11\textsuperscript{2}.

FIG. 4. Gaussian and twin primes (a) The magnitude of Eq. 3 for α = r\textsuperscript{2}. (b) Fixed points in (a) uniquely identify primes of form x\textsuperscript{2} + y\textsuperscript{2}. (c) Twin primes sieved by superposing diffraction patterns separated by 2 units of momenta along the q\textsubscript{x} axis using Eq. 2, plotted as the square root of the intensity. (d) Normalized plot over discrete q along the q\textsubscript{x} axis, where the 1\textsuperscript{st} of each twin prime is uniquely zero, shown in red.

Consider near-field diffraction of a unit-amplitude
plane wave through an ideal pin-hole on the optic axis, with the phase advanced to zero at the observation plane. The diffracted field is proportional to \( \exp[2\pi i r^2/(2\lambda z)]/\lambda z \), after Fresnel propagation by the distance \( z \). The scattered wave from a complementary aperture, such as an opaque infinitesimal particle, is \( 1 - \exp[2\pi i r^2/(2\lambda z)]/\lambda z \), by Babinet’s principle [20]. Consider now a distribution of such sources at non-overlapping inverse distances \( 1/(2\lambda z) = j/N + N \) on the optic axis. By analogy with Fig. 3(a), this diffraction is from a single longitudinal grating. It can be shown that \( \sum 2(j/N + N) \cos[2\pi r^2(j/N + N)] - 1 = N(2N + 1) \) when \( N \) divides \( r^2 \) and zero for all other dimensionless \( r^2 \) integers. For a successive sum over all \( N \), kinematical diffraction of a plane wave from this set of point-particles on the optic axis creates a near-field prime sieve superposition with wave amplitude proportional to \( \sigma_1(r^2) + 2\sigma_2(r^2) \), where \( \sigma_2(\alpha) \) is the \( 2^{nd} \) order divisor function (the sum of squares of all factors of \( \alpha \)). Normalisation of the amplitude by \( r^2(2r^2+1) \) gives unity for Gaussian prime \( r^2 \) and larger values for all other integer \( r^2 \). Explicit scattering cross sections for identical scatterers can be included by convolution of a suitable function in the transverse \((x,y)\) coordinates, or \( r \) for radially symmetric particles.

Any of our sieves can be applied concurrently to remove multiple distributions of integers, using wave superposition. For example, twin primes were sieved by adding Eq. 2 to an identical wave shifted along \( q_x \) by two integer units of momenta. The wave \( \psi_N(q) \) was set to unity for simplicity to yield the square root of intensity shown in Fig. 3(c), with \( M = 62 \) in Eq. 2. The corresponding graph over discrete \( q_x \) in Fig. 3(d) was computed after normalising the wave magnitude by \( q_x^2 + (q_x + 2)^2 \) and subtracting one, such that twin primes appear as unique integer zeros. Generalizations to sieve other prime gaps, tuples etc. are possible. While finite energy constraints are fundamental, technical issues such as non-paraxial diffraction can be overcome as Eq. 3 can be viewed as a simple sum over plane waves.

In conclusion, upon identifying direct connections between elementary number theory and basic diffraction physics, a wide variety of prime number sieves has been demonstrated using wave superposition. Examples were chosen to easily locate prime numbers in frequency, space and time, which placed constraints on the architecture of the source distributions. Given that wave superposition naturally leads to discrete phenomena originating from beats and modes, wave fields are littered with integers in general. It is thus interesting to consider whether the set of prime numbers implicitly resides within more general wave-field superpositions.

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