Factorization in categories of systems of linear partial differential equations

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26 December 2007

1 Introduction

Factorization of systems of differential equations was first studied for the case of a single linear ordinary differential equation (LODE) with linear ordinary differential operator (LODO) of the form

\[ L = f_0(x)D^n + f_1(x)D^{n-1} + \ldots + f_n(x), \quad D = d/dx, \] (1)

where the coefficients \( f_s(x) \) belong to some differential field \( K \). Factorization is a useful tool for computing a closed form solution of the corresponding linear ordinary differential equation \( Ly = 0 \) as well as determining its Galois group (see for example [46, 47, 54]). For simplicity and without loss of generality we suppose that operators (1) are reduced (i.e. \( f_0(x) \equiv 1 \)) unless we explicitly state the reverse. The most popular case of the differential field \( K = \mathbb{Q}(x) \) of rational functions with rational or algebraic number coefficients is a nontrivial example which is well investigated and will be considered hereafter when we discuss any constructive results.

In this paper we give a review of the current state of the theory of factorization of ordinary and partial differential operators and even more generally,
of systems of linear differential equations of arbitrary type (determined as well as overdetermined). We start with elementary algebraic theory of factorization of linear ordinary differential operators \[1\] developed in the period 1880–1930. After exposing these classical results we sketch more sophisticated algorithmic approaches developed in the last 20 years. The revival of this theory in the last two decades is motivated by the development of powerful computer algebra systems and implementation of nontrivial algebraic and differential algorithms such as factorization of polynomials and indefinite integration of elementary functions.

The main part of this paper will be devoted to modern generalizations of the factorization theory to the most general case of systems of linear partial differential equations and their relation with explicit solvability of nonlinear partial differential equations based on some constructions from the ring theory, theory of partially ordered sets (lattices) and that of abelian categories. Many of the results of this paper may be exposed within the framework of the Picard-Vessiot theory. But we follow a much simpler algebraic approach in order to facilitate the aforementioned generalizations.

The proper theoretical background for the simplest case—factorization of linear ordinary differential operators with rational coefficients—was known already in the end of the XIX century. Paradoxically, mathematicians of that epoch had developed even a nontrivial (theoretical) algorithm of factorization of such operators \[4\]. A review of this theory can be found in \[41\]. In contrast to the well-known property of uniqueness of factorization of usual commutative polynomials into irreducible factors, a simple example \[D^2 = D \cdot D = (D + 1/(x - c)) \cdot (D - 1/(x - c))\] shows that some LODO may have essentially different factorizations with factors depending on some arbitrary parameters. Fortunately according to the results by E. Landau \[28\] and A. Loewy \[30, 31\] exposed below all possible factorizations of a given operator \[L\] over a fixed differential field have the same number of factors in different expansions \[L = L_1 \cdots L_k = \overline{T}_1 \cdots \overline{T}_r\] into irreducible factors and the factors \[L_s, \overline{T}_p\] are pairwise “similar”. (Hereafter we always suppose the order of factors to be greater than 0: \[\text{ord}(L_i) > 0, \text{ord}(\overline{T}_j) > 0\].) We outline the main ideas of this classical theory in Section \[2\]. For simplicity we discuss here only the case of differential operators, a generalization for the case of a general Ore ring (including difference and \(q\)-difference operators, see \[6, 8\]) is straightforward.

Subsequent Sections are devoted to different aspects of the theory of factorization of linear partial differential operators.
2 Factorization of LODO

The basics of the algebraic theory of factorization of LODO was essentially given already in [30, 31], [33]–[36]. Algebraically the main results are just an easy consequence of the fact that the ring $K[D]$ of LODO with coefficients in a given differential field $K$ is Euclidean: for any LODO $L$, $M$ there exist unique LODO $Q$, $R$, $Q_1$, $R_1$ such that

$$L = Q \cdot M + R, \quad L = M \cdot Q_1 + R_1, \quad \text{ord}(R) < \text{ord}(M), \quad \text{ord}(R_1) < \text{ord}(M).$$

For any two LODO $L$ and $M$ using the right or left Euclidean algorithm one can determine their right greatest common divisor $\text{rGCD}(L, M) = G$, i.e. $L = L_1 \cdot G$, $M = M_1 \cdot G$ (the order of $G$ is maximal) and their right least common multiple $\text{rLCM}(L, M) = K$, i.e. $K = \overline{M} \cdot L = \overline{L} \cdot M$ (the order of $K$ is minimal) as well as their left analogues $\text{lGCD}$ and $\text{lLCM}$. All left and right ideals of this ring are principal and all two-sided ideals are trivial. Operator equations

$$X \cdot L + Y \cdot M = B, \quad L \cdot Z + M \cdot T = C \quad (2)$$

with unknown operators $X$, $Y$, $Z$, $T$ are solvable iff $\text{rGCD}(L, M)$ divides $B$ on the right and $\text{lGCD}(L, M)$ divides $C$ on the left. We say that an operator $L$ is (right) transformed into $L_1$ by an operator (not necessary reduced) $B$, and write $L \xrightarrow{B} L_1$, if $\text{rGCD}(L, B) = 1$ and $K = \text{rLCM}(L, B) = L_1 \cdot B = B_1 \cdot L$.

In this case any solution of $Ly = 0$ is mapped by $B$ into a solution $By$ of $L_1y = 0$. Using (2) one may find with rational algebraic operations an operator $B_1$ such that $L_1 \xrightarrow{B_1} L$, $B_1 \cdot B = 1 (\text{mod} L)$. Operators $L$, $L_1$ will be also called similar or of the same kind (in the given differential field $K$). So for similar operators the problem of solution of the corresponding LODE $Ly = 0$, $L_1y = 0$ are equivalent. One can define also the notion of left-hand transformation of $L$ by $B$ into $L_1$: $K = \text{lLCM}(L, B) = B \cdot L_1 = L \cdot B_1$. Obviously left- and right-hand transformations are connected via the adjoint operation. Also one may prove ([35]) that two operators are left-hand similar iff they are right-hand similar. A (reduced) LODO is called prime or irreducible (in the given differential field $K$) if it has no nontrivial factors aside from itself and 1. Every LODO similar to a prime LODO is also prime.

Two (prime for simplicity) LODO $P$ and $Q$ are called interchangeable in the product $P \cdot Q$ and this product will be called interchangeable as well if $P \cdot Q = Q_1 \cdot P_1$, $Q_1 \neq P$, $P_1 \neq Q$. In this case $P$ is similar to $P_1$, $Q$ is similar to $Q_1$ and $P_1 \xrightarrow{Q} P$.

**Theorem 1 (Landau [28] and Loewy [30])** Any two different decompositions of a given LODO $L$ into products of prime LODO $L = P_1 \cdots P_k =$
\( \prod_{1}^{p} \) have the same number of factors \((k = p)\) and the factors are similar in pairs (in some transposed order). One decomposition may be obtained from the other through a finite sequence of interchanges of contiguous factors (in the pairs \( P_{i} \cdot P_{i+1} \)).

All definitions here are constructive over the differential field of rational functions \( K = \mathbb{Q}(x) \): either using the Euclidean algorithm or finding rational solutions of LODO one can determine for example if two given LODO are similar or find all possible (parametric) factorizations of a given LODO with rational functional coefficients \([1, 6, 7, 50]\).

Landau-Loewy theorem also has a useful for the following ring-theoretic interpretation. Namely, every \( L \in K[D] \) generates the corresponding left ideal \( |L| \); \( L_{1} \) divides \( L \) on the right iff \( |L| \subset |L_{1}| \). If we have a factorization \( L = L_{1} \cdots L_{k} \) then we have a chain of ascending left principal ideals \( |L| \subset |L_{2} \cdots L_{k}| \subset |L_{3} \cdots L_{k}| \subset \ldots \subset |L_{k}| \subset |1| = K[D] \). If the factors \( L_{k} \) are irreducible, the chain is maximal, i.e. it is not possible to insert some intermediate ideals between its two adjacent elements. The Landau-Loewy theorem is nothing but the **Jordan-Hölder – Dedekind chain condition**:

**Theorem 2** Any two finite maximal ascending chains of left principal ideals in the ring \( K[D] \) of LODO have equal length.

Similarity of irreducible factors can be also interpreted in this approach. Even more general lattice-theoretic interpretation turned out to be fruitful for a generalization of this simple algebraic theory for the case of factorizations of partial differential operators \([51]\). Namely, let us consider the set of (left) ideals in \( K[D] \) as a partially ordered by inclusion set \( \mathcal{M} \) (called a poset). This poset has the following two fundamental properties:

**Property I** for any two elements \( A, B \in \mathcal{M} \) (left ideals!) one can find a unique \( C = \text{sup}(A, B) \), i.e. such \( C \) that \( C \geq A, C \geq B, \) and \( C \) is “minimal possible””. Analogously there exist a unique \( D = \text{inf}(A, B) \), \( D \leq A, D \leq B, \) \( D \) is “maximal possible”.

Such posets are called **lattices** \([22]\). \( \text{sup}(A, B) \) and \( \text{inf}(A, B) \) correspond to the GCD and the LCM in \( K[D] \).

For simplicity (and following the established tradition) \( \text{sup}(A, B) \) will be hereafter denoted as \( A + B \) and \( \text{inf}(A, B) \) as \( A \cdot B \);

**Property II** For any three \( A, B, C \in \mathcal{M} \) the following **modular identity** holds:

\[
(A \cdot C + B) \cdot C = A \cdot C + B \cdot C
\]

Such lattices are called **modular lattices** or **Dedekind structures**.
As one can prove, modularity implies the Jordan-Hölder-Dedekind chain condition: any two finite maximal chains \( L > L_1 > \cdots > L_k > 0 \) and \( L > M_1 > \cdots > M_r > 0 \) for a given \( L \in \mathcal{M} \) have equal lengths: \( k = r \) (the same for ascending chains). For the interpretation of the notions of similarity, direct sums, Kurosh & Ore theorems on direct sums cf. [51].

But even more fruitful for generalizations is the followings *categorical interpretation* of similarity of LODO and the Jordan-Hölder-Dedekind chain condition. Namely, let us consider the following **abelian category** \( \mathcal{LODO} \) of LODO.

Objects of \( \mathcal{LODO} \) are *reduced* operators \( L = D^n + a_1(x)D^{n-1} + \cdots + a_n(x) \), \( a_i \in \mathbb{K} \). One may ideally think of (finite-dimensional!) the solution spaces \( \text{Sol}(L) \) of such operators in some sufficiently large Picard-Vessiot extension of the coefficient field as another way of representation of an object in \( \mathcal{LODO} \). This helps to understand the meaning of some definitions below, but one shall remember that these solution spaces are *not constructive* unlike the objects-operators.

Morphisms \( \text{Hom}(L, L_1) \) in this category are constructively defined as a *not necessary reduced* LODO \( B \) such that \( L_1 \cdot B = C \cdot L \) for some other LODO \( C \). Non-constructively this \( B \) may be seen as a mapping of solutions of \( L \) into solutions of \( L_1 \). Note that all operators here have coefficients in some *fixed* differential field \( \mathbb{K} \). Two operators \( B_1, B_2 \) generate the same morphism iff \( B_1 = B_2(\text{mod } L) \). Also we should remark that this definition is *not* equivalent to the definition of a transformation of operators \( L \xrightarrow{B} L_1 \) introduced earlier, because for morphisms:

1) \( B \) and \( L \) may have common solutions, i.e. a nontrivial \( r\text{GCD}(B, L) \). This means that the mapping of the solution space \( \text{Sol}(L) \) by \( B \) may have a kernel \( \text{Sol}(r\text{GCD}(B, L)) \). The morphism is not injective in this case.

2) The image of the solution space \( \text{Sol}(L) \) may be smaller than \( \text{Sol}(L_1) \). The morphism is not surjective in this case.

Algebraically this means that \( L_1 \cdot B = C \cdot L \neq \text{rLCM}(B, P) \).

Similarity of operators \( L \) and \( L_1 \) now simply means isomorphism of the objects \( L \) and \( L_1 \) in this category.

The following fact is a direct corollary of our representation of this category as a subcategory of the category of finite-dimensional vector spaces and linear mappings preserving direct sums, products etc.:

**Theorem 3** *The category \( \mathcal{LODO} \) is abelian.*

Among the many useful results for abelian categories (cf. for example [16, 19]) we need the following
Theorem 4 Any abelian category with finite ascending chains satisfies the Jordan-Hölder property.

This will serve us as a maximal theoretical framework for an algebraic interpretation and generalization of the Landau-Loewy theorem for the case of systems of linear partial differential equations below. Again there exist notions of direct sums, Kurosh-Ore theorems on direct sums and the powerful technique of modern homological algebra for abelian categories [16] [19]. We will see below that this rather high level of abstraction allows a very natural generalization of the definition of factorization for arbitrary systems of linear partial differential equations (LPDE).

3 Factorization of LPDO

In contrast to the case of ordinary operators, two main results are seemingly lost for LPDO: Landau-Loewy theorem and the possibility to use some known solution for factorization of operators. In the case of a LODO $L$ obviously if one has its solution $L \phi = 0$ then one can split off a first-order right factor: $L = M \cdot \left( D - \frac{\phi'}{\phi} \right)$. For a LPDO, even if one knows the complete set of solutions, the operator may not be factorizable, as the following classical examples shows:

**Example 1.** The equation $Lu = \left( D_x D_y - \frac{2}{(x+y)^2} \right) u = 0$ with $D_x = \partial/\partial x$, $D_y = \partial/\partial y$ has the following complete solution:

$$Lu = 0 \iff u = -\frac{2(F(x) + G(y))}{x+y} + F'(x) + G'(y),$$

where $F(x)$ and $G(y)$ are two arbitrary functions of one variable each. On the other hand, as an easy calculation shows, the operator $L$ can not be represented as a product of two first-order operators (over any differential extension of the given coefficient field $K = \mathbb{Q}(x,y)$).

**Example 2.** The equation $Lu = \left( D_x D_y - \frac{6}{(x+y)^2} \right) u = 0$ again has the following complete solution:

$$u = \frac{12(F(x) + G(y))}{(x+y)^2} - \frac{6(F'(x) + G'(y))}{x+y} + F''(x) + G''(y), \quad (3)$$

but the operator is again “naively irreducible”. More generally, the equation

$$Lu = u_{xy} - \frac{c}{(x+y)^2} u = 0, \quad c = \text{const} \quad (4)$$
has a complete solution in an “explicit” form similar to (3) iff \( c = n(n + 1) \) for \( n \in \mathbb{N} \). In this case it has the complete solution in the form

\[
    u = c_0 F + c_1 F' + \ldots + c_n F^{(n)} + d_0 G + d_1 G' + \ldots + d_{n+1} G^{(n+1)}
\]

(5)

with some definite \( c_i(x, y), d_i(x, y) \) and two arbitrary functions \( F(x), G(y) \).

Only for \( c = 0 \) the corresponding operator is “naively reducible”: \( L = D_x \cdot D_y \).

The solution technology used here is very old and can be found in [9, 20] under the name of Laplace transformations or Laplace cascade method: after a series of transformations of (4) one gets a naively factorizable LPDE!

Another unpleasant example is also ascribed in [5] to E. Landau: if

\[
    P = D_x + xD_y, \quad Q = D_x + 1, \quad R = D_x^2 + xD_xD_y + D_x + (2 + x)D_y,
\]

(6)

then \( L = Q \circ Q \circ P = R \circ Q \). On the other hand the operator \( R \) is absolutely irreducible, i.e. one can not factor it into product of first-order operators with coefficients in any extension of \( \mathbb{Q}(x, y) \). So there seems to be no hope for an analogue of Landau-Loewy theorem for decomposition of LPDO into product of lower-order LPDO.

We see that a “naive” definition of a factorization of a LPDO as its representation as a product (composition) of lower-order LPDOs lacks some fundamental properties established in the previous Section for factorization of LODO.

Recently [51] an attempt to give a “good” definition of generalized factorization was undertaken. In the next subsection we only briefly sketch the ideas of this approach and describe its nontrivial relation to explicit integrability of nonlinear partial differential equations.

### 3.1 General theory of factorization of an arbitrary single LPDO, ring-theoretic approach

Our goal in [51] was to define a notion of factorization with “good” properties:

- Every LPDO \( L \) shall have only finite chains of ascending generalized factors. In particular \( D_x \) should be irreducible.

- Jordan-Hölder property: all possible generalized factorizations of a given operator \( L \) have the same number of “factors” in different expansions into irreducible factors and the “factors” should be pairwise “similar” in such expansions.
Existence of large classes of solutions should be related to factorization.

Classical theory of integration of LPDO using the Laplace cascade method should be an integral part of this generalized definition.

An obvious extension of the definition may be suggested if one will use ascending chains of arbitrary (not necessary principal) left ideals starting from the left ideal generated by the given operator:

\[ |L\rangle \subset I_1 \subset I_2 \subset \ldots \subset I_k \subset |1\rangle. \tag{7} \]

Unfortunately one can easily see that even for the operator \( D_x \) we have such chains, and they have unlimited length: \( |D_x\rangle \subset |D_x, D_y^m\rangle \subset |D_x, D_y^{m-1}\rangle \subset \ldots |D_x, D_y \rangle \subset |1\rangle \)! So we shall take some special class of ideals, more general, than the principal ideals, but much less rich than arbitrary left ideals. In [51] we gave a definition of such a suitable subclass of left ideals called divisor ideals.

For such special left ideals of the ring of LPDO:

- chains (7) will be finite and different maximal chains for a given \( L \) they have the same length: if \( |L\rangle \subset I_1 \subset I_2 \subset \ldots \subset I_k \subset |1\rangle, |L\rangle \subset J_1 \subset J_2 \subset \ldots \subset J_m \subset |1\rangle \), then \( k = m \) and one can prove a natural lattice-theoretic “similarity” of “factors” in both chain.

- Irreducible LODO will be still irreducible as LPDO.

- For \( \dim = 2, \text{ord} = 2 \) (that is for operators with two independent variables of order two) a LODO is factorizable in this generalized sense (i.e. having a nontrivial chain (7)) iff it is integrable with the Laplace cascade method. We describe this cascade method below in subsection 3.3.

- Algebraically, the problem is reduced from the ring \( \mathbb{Q}(x,y)[D_x, D_y] \) to factorization in rings of formal LODO with noncommutative coefficients \( \mathbb{Q}(x,y, D_x)[D_y] \) and/or \( \mathbb{Q}(x,y, D_y)[D_x] \) (Ore quotients); in these rings all left and right ideals are again principal ideals.

The details, rather involved, may be found in [51]. This approach nevertheless suffers from the following problems:

- The definition of divisor ideals given in [51] is very technical, not intuitive.

- No algorithms for such generalized factorization is known.
A generalization of this ring-theoretic approach to systems of LPDE was proposed recently by M. Singer. Another ring-theoretic approach was considered in [25].

In the next subsection we propose a different, much more intuitive definition of generalized factorization.

### 3.2 General theory of factorization of arbitrary systems of LPDE, approach of abelian categories

Abelian category \( \mathcal{SLPDE} \) of arbitrary systems of LPDE is defined by its objects which are simply systems

\[
S: \begin{cases}
L_{11}u_1 + \ldots + L_{1s}u_s = 0, \\
\ldots \\
L_{p1}u_1 + \ldots + L_{ps}u_s = 0,
\end{cases} \quad L_{ij} \in Q(x_1, \ldots, x_n)[D_{x_1}, \ldots, D_{x_n}], \quad u_k = u_k(x_1, \ldots, x_n). \tag{8}
\]

Morphism \( P : S \rightarrow Q \) of two systems is defined as a matrix of differential operators

\[
P : \begin{cases}
v_1 = P_{11}u_1 + \ldots + P_{1s}u_s, \\
\ldots \\
v_m = P_{m1}u_1 + \ldots + P_{ms}u_s,
\end{cases} \quad P_{ij} \in Q(x_1, \ldots, x_n)[D_{x_1}, \ldots, D_{x_n}]
\]

\( \text{with the condition that any solution set } \{ u_1, \ldots, u_s \} \text{ of the source system (8) is mapped into a subspace of the solution space } \{ v_1, \ldots, u_m \} \text{ of the target system} \)

\[
Q: \begin{cases}
M_{11}u_1 + \ldots + M_{1m}v_m = 0, \\
\ldots \\
M_{q1}v_1 + \ldots + M_{qm}v_m = 0,
\end{cases} \quad M_{ij} \in Q(x_1, \ldots, x_n)[D_{x_1}, \ldots, D_{x_n}], \quad v_k = v_k(x_1, \ldots, x_n). \tag{10}
\]

The standard differential Groebner technique (originally developed in the beginning of the XX century as the so called Janet-Riquier theory \[26, 39, 40, 42\]) makes this definition constructive: (9) is a morphism mapping (8) to (10) iff for any \( i \) the equation \( \sum_{j,k} M_{ij}P_{jk}u_k = 0 \) is reducible to zero modulo the equations of the system (8).

Again, it is easy to see that this category is abelian: for this it is enough to check that \( \mathcal{SLPDE} \) is embeddable into the category of (infinite-dimensional) vector spaces and linear morphisms and this embedding preserves direct sums, products etc.

It seems natural to refer to Theorem 4 to transfer the many properties of factorization proved for the category \( \mathcal{LODO} \) in Section 2 to the case of the category \( \mathcal{SLPDE} \). Unfortunately this is not so simple: the ascending chains of monomorphisms are infinite in general: the same example
The solution to this problem is given by the standard construction of a Serre-Grothendieck factorcategory. We refer to [13, 16, 19] and especially to [17] for a detailed explanation of this important and general construction. One of the important steps of this construction is the construction of inverses of morphisms with “relatively small” kernels; the objects are not formally changed in contrast to the ring-theoretic construction of factorrings and factormodules. In our case we proceed as follows: for a given (say, determined) system of LPDE of the form (8) (with \( s = p \)) and take the subcategory \( S_{n-2} \) of (overdetermined) systems with solution space parameterized by functions of at most \( n-2 \) variables. Then the Serre-Grothendieck factorcategory \( S/S_{n-2} \) has finite ascending chains. Another remarkable feature of this factorcategory is, as we mentioned above, the possibility to consider morphism which had kernels defined by systems from \( S_{n-2} \) as invertible morphisms. This may lead to a more general theory of Bäcklund-type transformations (at least for the case of linear systems), for example of transformations of Moutard type ([9, 20]).

Now we can transfer all theoretical results proved in Section 2 to the case of the factorcategory \( S/S_{n-2} \) and provide a theoretical foundation for the factorization theory of arbitrary linear systems of LPDE.

The obvious drawback still lies in the absence of algorithms for such a generalized factorization. We give an overview of currently known numerous partially algorithmic results in the next Sections 3.3–3.5.

### 3.3 \( dim = 2, ord = 2 \): Laplace transformations and Darboux integrability of nonlinear PDEs

Here we expose the basics of the classical theory [9, 15, 20], which is applicable to hyperbolic linear partial differential equations of order two with two independent variables. For simplicity only the case of an equation with straight characteristics will be discussed here:

\[
Lu = u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0. \tag{11}
\]

The more general case can be found in [20, 2, 52]. If one of the Laplace invariants of (11) \( h = a_x + ab - c \), \( k = b_y + ab - c \) vanishes, one can “naively” factorize the operator in the l.h.s. of (11): \( k \equiv 0 \Rightarrow L = (D_y + a)(D_x + b); \) \( h \equiv 0 \Rightarrow L = (D_x + b)(D_y + a). \)

If \( h \neq 0, k \neq 0 \) then (11) is not factorizable in the “naive” sense. In this case one can perform one of the two Laplace transformations (not to be mixed with Laplace transforms!) which are invertible differential substitutions
(isomorphisms in the category of $\mathcal{SLPDE}$):

\[ u = \frac{1}{h} (D_x + b) u_{(1)} \]

or

\[ u = \frac{1}{k} (D_y + a) u_{(-1)}. \]

In fact each of the above substitutions is the inverse of the other up to a functional factor. Each of these substitutions produces a new operator of the same form (11) but with different coefficients and Laplace invariants. The idea of the Laplace cascade method consists in application of these substitutions a few times, obtaining the (infinite in general) chain

\[ \ldots \leftarrow L_{(-2)} \leftarrow L_{(-1)} \leftarrow L \rightarrow L_{(1)} \rightarrow L_{(2)} \rightarrow \ldots \] (12)

In some cases (namely these cases are considered as integrable in this approach) this gives us on some step an operator $L_{(i)}$ with vanishing $h_{(i)}$ or $k_{(i)}$. Then this chain can not be continued further in the respective direction and one can find an explicit formula for the complete solution of the transformed equation; performing the inverse differential substitutions we obtain the complete solution of the original equation (with quadratures).

One of the main results of [51] are the following Theorems:

**Theorem 5** If \( L = D_x \cdot D_y - a(x, y)D_x - b(x, y)D_y - c(x, y) \) has a nontrivial generalized right divisor ideal (so is factorizable in the sense described in Section 3) iff the chain (12) of Laplace transformations is finite at least in one direction.

**Theorem 6** If \( L = D_x \cdot D_y - a(x, y)D_x - b(x, y)D_y - c(x, y) \) is a lLCM of two generalized right divisor ideals iff the chain (12) of Laplace transformations is finite in both directions.

This shows the meaning of the generalized definition of [51] and provides a partial algorithm for generalized factorization for equations of the form (11).

Although practically efficient for simple cases, this method has the obvious decidability problem: given an operator $L$, how many steps in the chain (12) should be tried? Currently no stopping criterion is known. As the example (11) shows, the number of steps in the chain (equal to $n$ for (11) in the integrable case $c = n(n + 1)$) depends on some subtle arithmetic properties of the coefficients.

There exists a remarkable link of the theory of Laplace transformations to the theory of integrable nonlinear partial differential equations. This
topic was very popular in the XIX century and led to the development of integration methods of Lagrange, Monge, Boole and Ampere. G. Darboux [10] generalized the method of Monge (known as the method of intermediate integrals) to obtain the most powerful method for exact integration of partial differential equations known in the last century.

Recently in a series of papers [2, 48, 57] the Darboux method was cast into a more precise and efficient (although not completely algorithmic) form. For the case of a single second-order nonlinear PDE of the form

\[ u_{xy} = F(x, y, u, u_x, u_y) \]  

the idea consists in linearization: using the substitution \( u(x, y) \rightarrow u(x, y) + \epsilon v(x, y) \) and cancelling terms with \( \epsilon^n, \ n > 1 \), we obtain a LPDE

\[ v_{xy} = Av_x + Bv_y + Cv \]  

with coefficients depending on \( x, y, u, u_x, u_y \). Equations of the type (14) are in fact feasible to the Laplace cascade method, certainly one needs to take into consideration the original equation (13) while performing all the computations of the Laplace invariants and Laplace transformations: (13) allows us to express all the mixed derivatives of \( u \) via \( x, y, u \) and the non-mixed \( u_{x\cdots x}, u_{y\cdots y} \). The following statement can be found in [20], recently it was rediscovered in [2, 48]:

**Theorem 7** A second order, scalar, hyperbolic partial differential equation (13) is Darboux integrable if and only if the Laplace sequence (12) for (14) is finite in both directions.

In [2, 48] this method was also generalized for the case of a general second-order nonlinear PDE

\[ F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \]  

**3.4 dim = 2, ord ≥ 3: Generalized Laplace transformations**

In [52] we have proposed a generalization of the Laplace cascade method for arbitrary strictly hyperbolic equations with two independent variables of the form

\[ \hat{L}u = \sum_{i+j \leq n} p_{i,j}(x, y) \hat{D}_x^i \hat{D}_y^j u = 0, \]  

\[ 12 \]
as well as for $n \times n$ first-order linear systems

$$
(v_i)_x = \sum_{k=1}^{n} a_{ik}(x, y)(v_k)_y + \sum_{k=1}^{n} b_{ik}(x, y)v_k
$$

with strictly hyperbolic matrix $(a_{ik})$.

Here we demonstrate this new method on an example of the constant-coefficient system

$$
\begin{align*}
D_x u_1 &= u_1 + 2u_2 + u_3, \\
D_y u_2 &= -6u_1 + u_2 + 2u_3, \\
(D_x + D_y)u_3 &= 12u_1 + 6u_2 + u_3.
\end{align*}
$$

It has the following complete explicit solution:

$$
\begin{align*}
u_1 &= 2e^yG(x) + e^x(3F(y) + F'(y)) + \exp \frac{x+y}{2}H(x-y), \\
u_2 &= e^yG'(x) + 2e^xF'(y) - 2u_1, \\
u_3 &= D_x u_1 + 3u_1 - 2(e^yG'(x) + 2e^xF'(y)),
\end{align*}
$$

where $F(y)$, $G(x)$ and $H(x-y)$ are three arbitrary functions of one variable each.

The solution technology (cf. [52]) for the details) is again a differential substitution; in the case of the system (17) the transformation is given by:

$$
\begin{align*}
\overline{u}_1 &= u_1, \\
\overline{u}_2 &= u_2 + 2u_1, \\
\overline{u}_3 &= ((D_x + D_y)u_1 - u_1 - 2u_2 - 4u_1).
\end{align*}
$$

The transformed system has a triangular matrix and is easily integrable:

$$
\begin{align*}
D_x \overline{u}_3 &= \overline{u}_3, \\
D_y \overline{u}_2 &= 2\overline{u}_3 + \overline{u}_2, \\
(D_x + D_y)u_1 &= \overline{u}_3 + 2\overline{u}_2 + u_1.
\end{align*}
$$

Again no stopping criterion for the sequences of generalized Laplace transformations is known in the general case. For constant coefficient systems an alternative technology was proposed by F.Schwarz (private communication, 2005): transform the system (17) into a Janet (Gröbner) normal form with term order: LEX, $u_3 > u_2 > u_1$, $x > y$:

$$
\begin{align*}
u_{1,xx} - u_{1,xx} + u_{1,xy} - 3u_{1,xy} + 2u_{1,x} - u_{1,yy} + 2u_{1,y} - u_1 &= 0, \\
u_{2,y} + 3u_{2} - 2u_{1,x} + 8u_{1} &= 0, \\
u_{2,x} - u_2 - \frac{1}{2}u_{1,xx} - \frac{1}{2}u_{1,xy} + 3u_{1,x} + \frac{1}{2}u_{1,y} - \frac{5}{2}u_1 &= 0, \\
u_3 + 2u_{2} - u_{1,x} + u_1 &= 0.
\end{align*}
$$
The first equation factors:

\[ D_x^2 D_y - D_y^2 + D_x D_y^2 - 3D_x D_y + 2D_x - D_y^2 + 2D_y - 1 = (D_x + D_y - 1)(D_y - 1)(D_x - 1). \]

So one can find \( u_1 \) easily and then the other two functions \( u_2 \) and \( u_3 \) are obtained from the remaining equations of the Janet base producing essentially the same solution \([15]\).

**Conjecture:** For constant-coefficient systems this Gröbner basis technology is equivalent to the generalized Laplace technology.

3.5 \( dim \geq 3, \ ord = 2: \) Dini transformations

In \([11]\) another simple generalization of Laplace transformations formally applicable to some second-order operators in the space of arbitrary dimension was proposed. Namely, suppose that an operator \( \hat{L} \) has its principal symbol

\[ Sym = \sum_{i_1+i_2=2} a_{i_1i_2}(\vec{x}) D_{x_{i_1}} D_{x_{i_2}} \]

which factors (as a formal polynomial in formal commutative variables \( D_{x_i} \)) into product of two first-order factors: \( Sym = \hat{X}_1 \hat{X}_2 \) (\( \hat{X}_j = \sum_i b_{ij}(\vec{x}) D_{x_i} \) are first-order operators) and moreover the complete operator \( \hat{L} \) may be written at least in one of the characteristic forms:

\[ L = (\hat{X}_1 \hat{X}_2 + \alpha_1 \hat{X}_1 + \alpha_2 \hat{X}_2 + \alpha_3) \]

\[ L = (\hat{X}_2 \hat{X}_1 + \alpha_1 \hat{X}_1 + \alpha_2 \hat{X}_2 + \alpha_3), \]

where \( \alpha_i = \alpha_i(x, y) \). Since the operators \( \hat{X}_i \) do not necessarily commute we have to take into consideration in \([19]\) and everywhere below the commutation law

\[ [\hat{X}_1, \hat{X}_2] = \hat{X}_1 \hat{X}_2 - \hat{X}_2 \hat{X}_1 = P(x, y)\hat{X}_1 + Q(x, y)\hat{X}_2. \]

This is very restrictive since the two tangent vectors corresponding to the first-order operators \( \hat{X}_i \) no longer span the complete tangent space at a generic point \( (\vec{x}_0) \). \([20]\) is also possible only in the case when these two vectors give an integrable two-dimensional distribution of the tangent subplanes in the sense of Frobenius, i.e. when one can make a change of the independent variables \( (\vec{x}) \) such that \( \hat{X}_i \) become parallel to the coordinate plane \( (x_1, x_2) \); thus in fact we have an operator \( \hat{L} \) with only \( D_{x_1}, D_{x_2} \) in it and we have got no really significant generalization of the Laplace method. If one has only

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but (20) does not hold one can not perform more that one step in the Laplace chain (12) and there is no possibility to get an operator with a zero Laplace invariant (so naively factorizable and solvable).

Below we demonstrate on an example, following an approach proposed by U. Dini in another paper [12], that one can find a better analogue of Laplace transformations for the case when the dimension of the underlying space of independent variables is greater than two. Another particular special transformation was also proposed in [3], [56]; it is applicable to systems whose order coincides with the number of independent variables. The results of [3], [56] lie beyond the scope of this paper.

Let us take the following equation:

\[ Lu = (D_x D_y + x D_x D_z - D_z)u = 0. \] (21)

It has three independent derivatives \( D_z, D_y, D_z \), so the Laplace method is not applicable. On the other hand its principal symbol splits into product of two first-order factors: \( \xi_1 \xi_2 + x \xi_1 \xi_3 = \xi_1 (\xi_2 + x \xi_3) \). This is no longer a typical case for hyperbolic operators in dimension 3; we will use this special feature introducing two characteristic operators \( \hat{X}_1 = D_x, \hat{X}_2 = D_y + x D_z \). We have again a nontrivial commutator \([\hat{X}_1, \hat{X}_2] = D_z = \hat{X}_3 \). The three operators \( \hat{X}_i \) span the complete tangent space in every point \((x, y, z)\). Using them one can represent the original second-order operator in one of two partially factorized forms:

\[ L = \hat{X}_2 \hat{X}_1 - \hat{X}_3 = \hat{X}_1 \hat{X}_2 - 2 \hat{X}_3. \]

Let us use the first one and transform the equation into a system of two first-order equations:

\[ Lu = 0 \iff \begin{cases} \hat{X}_1 u = v, \\ \hat{X}_3 u = \hat{X}_2 v. \end{cases} \] (22)

Cross-differentiating the left hand sides of (22) and using the obvious identity \([\hat{X}_1, \hat{X}_3] = [D_x, D_z] = 0\) we get \( \hat{X}_1 \hat{X}_2 v = D_x (D_y + x D_z) v = \hat{X}_3 v = D_z v \) or \( 0 = D_x (D_y + x D_z) v - D_z v = (D_x D_y + x D_x D_z) v = (D_y + x D_z) D_x v = \hat{X}_2 \hat{X}_1 v. \)

This is precisely the procedure proposed by Dini in [12]. Since it results now in another second-order equation which is “naively” factorizable we easily find its complete solution:

\[ v = \int \phi(x, xy - z) \, dx + \psi(y, z) \]

where \( \phi \) and \( \psi \) are two arbitrary functions of two variables each; they give the general solutions of the equations \( \hat{X}_2 \phi = 0, \hat{X}_1 \psi = 0 \).
Now we can find $u$:

$$u = \int \left( v \, dx + (D_y + xD_z)v \, dz \right) + \theta(y),$$

where an extra free function $\theta$ of one variable appears as a result of integration in (22).

So we have seen that such Dini transformations (22) in some cases may produce a complete solution in explicit form for a non-trivial three-dimensional equation (21). This explicit solution can be used to solve initial value problems for (21).

Dini did not give any general statement on the range of applicability of his trick. In [53] we have proved the following

**Theorem 8** Let $L = \sum_{i+j+k\leq2} a_{ijk}(x, y, z) D_x^i D_y^j D_z^k$ have factorizable principal symbol: $\sum_{i+j+k=2} a_{ijk}(x, y, z) D_x^i D_y^j D_z^k = \hat{S}_1 \hat{S}_2 \pmod{\text{lower-order terms}}$ with generic (non-commuting) first-order LPDO $\hat{S}_1$, $\hat{S}_2$. Then there exist two Dini transformations $L^{(1)}$, $L^{(-1)}$ of $L$.

**Proof.** One can represent $L$ in two possible ways:

$$L = \hat{S}_1 \hat{S}_2 + \hat{T} + a(x, y, z) = \hat{S}_2 \hat{S}_1 + \hat{U} + a(x, y, z) \quad (23)$$

with some first-order operators $\hat{T}$, $\hat{U}$. We will consider the first one obtaining a transformation of $L$ into an operator $L^{(1)}$ of similar form.

In the generic case the operators $\hat{S}_1$, $\hat{S}_2$, $\hat{T}$ span the complete 3-dimensional tangent space in a generic point $(x, y, z)$. Precisely this requirement will be assumed to hold hereafter; operators $L$ with this property will be called generic.

Let us fix the coefficients in the expansions of the following commutators:

$$[\hat{S}_2, \hat{T}] = K(x, y, z) \hat{S}_1 + M(x, y, z) \hat{S}_2 + N(x, y, z) \hat{T}. \quad (24)$$

$$[\hat{S}_1, \hat{S}_2] = P(x, y, z) \hat{S}_1 + Q(x, y, z) \hat{S}_2 + R(x, y, z) \hat{T}. \quad (25)$$

First we try to represent the operator in a partially factorized form: $L = (\hat{S}_1 + \alpha)(\hat{S}_2 + \beta) + \hat{V} + b(x, y, z)$ with some indefinite $\alpha = \alpha(x, y, z)$, $\beta = \beta(x, y, z)$ and $\hat{V} = \hat{T} - \beta \hat{S}_1 - \alpha \hat{S}_2$, $b = a - \alpha \beta - \hat{S}_1(\beta)$.

Then introducing $v = (\hat{S}_2 + \beta)u$ we get the following first-order system:

$$Lu = 0 \iff \begin{cases} (\hat{S}_2 + \beta)u = v, \\ (\hat{V} + b)u = -(\hat{S}_1 + \alpha)v. \end{cases} \quad (26)$$
Next we try to eliminate $u$ by cross-differentiating the left hand sides, which gives

$$
[(\hat{V} + b), (\hat{S}_2 + \beta)]u = (\hat{S}_2 + \beta)(\hat{S}_1 + \alpha)v + (\hat{V} + b)v.
$$

(27)

If one wants $u$ to disappear from this new equation one should find out when $[(\hat{V} + b), (\hat{S}_2 + \beta)]u$ can be transformed into an expression involving only $v$, i.e. when this commutator is a linear combination of just two expressions $(\hat{S}_2 + \beta)$ and $(\hat{V} + b)$:

$$
[(\hat{V} + b), (\hat{S}_2 + \beta)] = \mu(x, y, z)(\hat{S}_2 + \beta) + \nu(x, y, z)(\hat{V} + b).
$$

(28)

This is possible to achieve choosing the free functions $\alpha(x, y, z), \beta(x, y, z)$ appropriately. In fact, expanding the left and right hand sides in (28) in the local basis of the initial fixed operators $\hat{S}_1, \hat{S}_2, \hat{T}$ and the zeroth-order operator 1 and collecting the coefficients of this expansion, one gets the following system for the unknown functions $\alpha, \beta, \mu, \nu$:

$$\begin{align*}
K + \beta P - \hat{S}_2(\beta) &= \nu \beta, \\
M - \hat{S}_2(\alpha) + \beta Q &= \nu \alpha - \mu, \\
N + \beta R &= -\nu, \\
\beta \hat{S}_1(\beta) - \hat{T}(\beta) + \hat{S}_2(a) - \beta \hat{S}_2(\alpha) - \hat{S}_2(\hat{S}_1(\beta)) &= -\nu(a - \alpha \beta - \hat{S}_1(\beta)) - \mu \beta.
\end{align*}$$

After elimination of $\nu$ from its first and third equations we get a first-order non-linear partial differential equation for $\beta$:

$$\hat{S}_2(\beta) = \beta^2 R + (N + P) \beta + K.
$$

(29)

This Riccati-like equation may be transformed into a second-order linear PDE via the standard substitution $\beta = \hat{S}_2(\gamma)/\gamma$. Taking any non-zero solution $\beta$ of this equation and substituting $\mu = \nu \alpha + \hat{S}_2(\alpha) - \beta Q - M$ (taken from the second equation of the system) into the fourth equation of the system we obtain a first-order linear partial differential equation for $\alpha$ with the first-order term $\beta \hat{S}_2(\alpha)$. Any solution of this equation will give the necessary value of $\alpha$. Now we can substitute $[(\hat{V} + b), (\hat{S}_2 + \beta)]u = \mu(\hat{S}_2 + \beta)u + \nu(\hat{V} + b)u = \mu v - \nu(\hat{S}_1 + \alpha)v$ into the left hand side of (27) obtaining the transformed equation $L_{(1)}v = 0$.

If we would start the same procedure using the second partial factorization in (23) we would find the other transformed equation $L_{(-1)}w = 0$. □

4 Other results and conjectures

The theory of integration of linear and nonlinear partial differential equations was among the most popular topics in the XIX century. Enormous amount
of papers were devoted for example to transformations of equations to an integrable form. In particular the papers [29, 37, 38] were devoted to a more general Laplace type transformations. Some of these results were obtained in the framework of the classical differential geometry; cf. [14] for a modern exposition of those results.

In addition to the problems studied above one should mention a class of overdetermined systems of linear partial differential equations with finite-dimensional solution space studied in [58, 32]. There an algorithm for factorization of such systems was proposed.

Another popular in the past decade topic was the theory of “naive” factorization, i.e. representation of a given LPDO as a product of lower-order LPDO: in [24] an algorithm for such factorization was proposed for the case of operators with symbol representable as a product of two coprime polynomials. This result was developed further in [45].

From the theory of Laplace and Dini transformations the following conjectures seem to be natural:

- **If a LPDO is factorizable in the generalized sense, then its principal symbol is factorizable as a multivariate commutative polynomial.**
- **If a LPDO of order $n$ has a complete solution in a quadrature-free form (5) then its symbol splits into $n$ linear factors.**

5 Acknowledgment

The author enjoys the occasion to thank the organizers of the LMS summer lecture course and the complete mini-program “Algebraic Theory of Differential Equations” at the International Centre for Mathematical Sciences, Edinburgh for their efforts which guaranteed the success of the mini-program as well as for partial financial support which made presentation of the results given above possible.

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