On accumulation points of volumes of log surfaces

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Abstract. Let $C \subset [0, 1]$ be a set satisfying the descending chain condition. We show that every accumulation point of volumes of log canonical surfaces $(X, B)$ with coefficients in $C$ can be realized as the volume of a log canonical surface with big and nef $K_X + B$ and with coefficients in $\overline{C} \cup \{1\}$. As a corollary, if $C \subset \mathbb{Q}$, then all accumulation points of volumes are rational numbers. This proves a conjecture of Blache. For the set of standard coefficients $C_2 = \{1 - 1/n \mid n \in \mathbb{N}\} \cup \{1\}$ we prove that the minimal accumulation point is between $1/(7^2 \cdot 42^2)$ and $1/42^2$.

Keywords: log canonical surfaces, volume, accumulation points.

§1. Introduction

Let $(X, B)$ be a projective log surface with log canonical singularities. The volume $\text{vol}(K_X + B)$ measures the asymptotic growth of the space of global sections of pluri-log-canonical divisors:

$$h^0(X, [m(K_X + B)]) = \frac{\text{vol}(K_X + B)}{2} m^2 + o(m^2).$$

One says that $K_X + B$ is big if $\text{vol}(K_X + B) > 0$. If this is the case, let $\pi : (X, B) \to (X_{\text{can}}, B_{\text{can}})$ be the contraction to the log canonical model. Then $\pi^*(K_{X_{\text{can}}} + B_{\text{can}})$ is the positive part of $K_X + B$ in the Zariski decomposition and

$$\text{vol}(K_X + B) = \text{vol}(K_{X_{\text{can}}} + B_{\text{can}}) = (K_{X_{\text{can}}} + B_{\text{can}})^2.$$ 

The log canonical divisor $K_{X_{\text{can}}} + B_{\text{can}}$ is ample, but not Cartier in general. Hence $\text{vol}(K_X + B)$, being positive, is not necessarily an integer.

Assume that $C \subset [0, 1]$ and let $S(C)$ be the set of log canonical projective surfaces $(X, B)$ such that the coefficients of $B$ belong to $C$. Using the minimal model program for log surfaces, one has the following equalities of sets of volumes:

\[ K^2(C) := \{ \text{vol}(K_X + B) \mid (X, B) \in S(C), \ K_X + B \text{ is big} \} \]
\[ = \{ (K_X + B)^2 \mid (X, B) \in S(C), \ K_X + B \text{ is big and nef} \} \]
\[ = \{ (K_X + B)^2 \mid (X, B) \in S(C), \ K_X + B \text{ is ample} \}. \]

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The three most commonly used sets of coefficients are \( C_0 = \{0\} \), \( C_1 = \{0, 1\} \) and \( C_2 = \{1 - 1/n \mid n \in \mathbb{N}\} \cup \{1\} \). The last set is called the set of standard coefficients. It appears naturally when one studies groups of automorphisms of smooth varieties and in the adjunction formula. An easy observation is that these three sets satisfy the descending chain condition (DCC).

Another useful DCC set arises in the study of elliptic 3-folds. This is the set of hyperstandard coefficients

\[
C_3 = \left\{ 1 - \frac{r}{n} \mid n \in \mathbb{N}, \ r = 0, 1, \ldots, \frac{11}{12}, 1 \right\}.
\]

A fundamental result of the first author states that if \( C \) is a DCC set, then the set \( \mathbb{K}^2(C) \) is also DCC ([1], Theorem 8.2). The DCC is an assertion about the accumulation points of \( \mathbb{K}^2(C) \). Blache [2] and Kollár ([3], Theorem 39) constructed several examples of accumulation points of volumes of log surfaces without boundary, and Urzúa and Yáñez [4] recently used Kollár’s construction to show that all positive integers are actually accumulation points of \( \mathbb{K}^2(C_0) \). Yet the overall picture of the accumulation points is unknown even for \( C_0 \).

We recall the definition of a nonklt centre of \((X, B)\) in §2. We call a nonklt centre admissible if it is neither a simple elliptic singularity \( P \in X \) nor a smooth curve \( B_0 \subset B \) disjoint from \( \text{Sing}(X) \) and from the rest of \( B \). The first main result of this paper is the following theorem.

**Theorem 1.1.** Suppose that \( C \subset [0, 1] \) satisfies the descending chain condition. Then \( v_\infty \in \mathbb{R}_{>0} \) is an accumulation point of \( \mathbb{K}^2(C) \) if and only if there is a log canonical surface \((X, B) \in S(\overline{C} \cup \{1\})\) with the following properties.

(i) \( K_X + B \) is ample and \( v_\infty = (K_X + B)^2 \).

(ii) One of the following conditions holds:

(a) The set of codiscrepancies of divisors over \( X \) with respect to \((X, B)\) contains an accumulation point of \( C \);

(b) \((X, B)\) has an accessible nonklt centre.

(iii) If \( 1 \) is not in the closure \( \overline{C} \) of \( C \), then each irreducible component of \( |B| \) has geometric genus at most 1.

**Remark 1.2.** The condition (ii)(a) is equivalent to the existence of a log surface \((X, B) \in S(\overline{C} \cup 1)\) with big and nef \( K_X + B \) such that a coefficient of \( B \) is in \( \text{Acc}(C) \).

In §3 we prove the ‘if part’ of Theorem 1.1 by constructing a sequence of log canonical surfaces with volumes converging to \( v_\infty \). The harder part of the theorem is the ‘only if part’. The proof is carried out in §4. There are several interesting consequences of Theorem 1.1.

**Corollary 1.3.** Let \( C \subset [0, 1] \) be a DCC set. Then

\[
\text{Acc}(\mathbb{K}^2(C)) \subset \mathbb{K}^2(\overline{C} \cup \{1\}),
\]

where \( \text{Acc} (\cdot) \) denotes the set of accumulation points. In particular, if \( \overline{C} \subset \mathbb{Q} \), then \( \text{Acc}(\mathbb{K}^2(C)) \subset \mathbb{Q} \).
Remark 1.4. Theorem 1.1 gives an easy way to find many accumulation points of $\mathbb{K}^2(C)$. For example, it follows easily that $\mathbb{N} \subset \text{Acc}(\mathbb{K}^2(C_0))$; see Example 3.6.

Corollary 1.3 and Example 3.6 prove half of a conjecture of Blache ([2], Conjecture 3(a)). On the other hand, the second half ([2], Conjecture 3(b)) saying that $\min \text{Acc}(\mathbb{K}^2(C_0)) = 1$, is false. For example, since the surface in [5], Ch. 5, of volume $1/462$ has a non-empty accessible nonklt centre as in Theorem 1.1, (ii) (b), it follows that $1/462$ is an accumulation point of $\mathbb{K}^2(C_0)$.

The fact that $\mathbb{N} \subset \text{Acc}(\mathbb{K}^2(C_0))$ and the existence of many accumulation points smaller than 1 was also established in recent work of Urzúa and Yáñez [4].

Corollary 1.5. Let $C \subset [0,1]$ be a DCC set such that $\text{Acc}(C) = \{1\}$, for example, $C = C_2$. Then $\text{Acc}(\mathbb{K}^2(C)) = \mathbb{K}^2_{\text{nonklt}}(C)$, where

$\mathbb{K}^2_{\text{nonklt}}(C) = \{(K_X + B)^2 | (X, B) \in S(C), K_X + B \text{ is ample, nonklt}(X, B) \neq \emptyset\}.

If $C$ is finite, for example, $C = C_0$ or $C_1$, then $\text{Acc}(\mathbb{K}^2(C)) \subset \mathbb{K}^2_{\text{nonklt}}(C)$.

Our second main result is an explicit lower bound for $\text{Acc}(\mathbb{K}^2(C_2))$ for the standard coefficient set

$C_2 = \left\{1 - \frac{1}{n}, \ n \in \mathbb{N}\right\} \cup \{1\}$.

The following theorem will be proved in § 5.

Theorem 1.6.

$$\min \text{Acc} \mathbb{K}^2(C_2) = \min \mathbb{K}^2_{\text{nonklt}}(C_2) \geq \frac{1}{86436} = \frac{1}{7^2 \cdot 42^2}.$$ 

On the other hand, applying Theorem 1.1 to Example 5.3.1 in [6], we find that $1/42^2$ is an accumulation point of $\mathbb{K}^2(C_2)$. Here are our bounds for the minimal accumulation points of the three most commonly used sets.

Theorem 1.7. The following assertions hold.

(i) When $C = C_0, \ C_1$ and $C_2$, one has $\min \text{Acc}(\mathbb{K}^2(C)) = \min \mathbb{K}^2_{\text{nonklt}}(C)$.

(ii) $\frac{1}{86436} = \frac{1}{7^2 \cdot 42^2} \leq \min \mathbb{K}^2_{\text{nonklt}}(C_2) \leq \frac{1}{1764} = \frac{1}{42^2}$.

(iii) When $C = C_0$ or $C_1$, one has

$\frac{1}{86436} = \frac{1}{7^2 \cdot 42^2} \leq \min \mathbb{K}^2_{\text{nonklt}}(C) \leq \frac{1}{462} = \frac{1}{11 \cdot 42}$.

Remark 1.8. An effective lower bound for $\mathbb{K}^2(C)$ for any DCC set $C$ is given in [7], 4.8, but it is too small to be realistic. For the sets $C_0, C_1, C_2$ it works out to be about $10^{-3 \cdot 10^{10}}$; see [5], § 10. Our bounds for $\text{Acc}(\mathbb{K}^2(C_2))$ are considerably ‘higher’.

In § 6 we discuss iterated accumulation points of $\mathbb{K}^2(C)$, that is, accumulation points of accumulation points and so on. In particular, we prove that their ‘accumulation complexity’ is unbounded even for the simplest set $C_0 = \{0\}$. Giancarlo Urzúa has informed us that this also follows from the construction in [4].

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§ 2. Preliminaries

In §§ 2, 3 and 4 we work over an algebraically closed field of arbitrary characteristic. In §§ 5 and 6, characteristic 0 is assumed since we use the Kawamata–Viehweg vanishing theorem. We recall the standard definitions of the minimal model program (see, for example, [8]). We will need them for surfaces only. Given an $\mathbb{R}$-divisor $B = \sum_j b_j B_j$ on a normal surface $X$ and a real number $a$, we write

$$B^a = \sum_{b_j = a} b_j B_j, \quad B^{> a} = \sum_{b_j > a} b_j B_j, \quad B^{< a} = \sum_{b_j < a} b_j B_j.$$  

A log surface consists of a normal projective surface $X$ and an effective $\mathbb{R}$-divisor $B$ such that $K_X + B$ is $\mathbb{R}$-Cartier. Let $f : Y \to X$ be a morphism from a normal surface $Y$. Then we often write $B_Y$ for the $\mathbb{R}$-divisor on $Y$ given by the identity $K_Y + B_Y = f^*(K_X + B)$.

A log resolution is a resolution of singularities $f : Y \to X$ such that $\text{Exc}(f) \cup f_*^{-1}B$ has a simple normal crossing support, where $\text{Exc}(f)$ is the exceptional locus of $f$ and $f_*^{-1}B$ is the strict transform of $B$ on $Y$. We can write

$$K_Y = f^*(K_X + B) + \sum_i a_i E_i.$$  

One usually writes $a_i = a(E_i, X, B)$ and calls it the discrepancy of $E_i$ with respect to $(X, B)$. A pair $(X, B)$ is said to be log canonical (resp. Kawamata log terminal) if $a_i \geq -1$ (resp. $a_i > -1$) for all $i$. The numbers $b(E_i, X, B) := -a(E_i, X, B)$ are called codiscrepancies. We use the abbreviations lc and klt for ‘log canonical’ and ‘Kawamata log terminal’ respectively. Thus, $(X, B)$ is lc (resp. klt) if the coefficients in $B_Y$ are $\leq 1$ (resp. $< 1$).

A nonklt centre of a log surface $(X, B)$ is the image of a divisor $E$ on $Y$ whose coefficient in $B_Y$ is $\geq 1$, that is, $= 1$ in the log canonical case. In other words, $E$ is either a component of $B = \sum j b_j B_j$ with $b_j \geq 1$, or the image of an exceptional divisor with $a(E, X, B) \leq -1$. This definition is independent of the log resolution. The nonklt locus of $(X, B)$, denoted by nonklt$(X, B)$, is the union of all nonklt centres.

A log surface $(X, B)$ is said to be dlt (divisorially log terminal) if it is log canonical and there is a finite subset $S \subset X$ such that $(X \setminus S, \text{supp } B \setminus S)$ is a smooth normal-crossing pair and $S$ contains no nonklt centres.

**Definition 2.1.** Let $(X, B = \sum b_j B_j)$ be a log surface, $b_i \geq 0$. We do not assume that it is lc, and we allow some coefficients $b_i$ to be $> 1$, but the divisor $K_X + B$ must be $\mathbb{R}$-Cartier. A dlt blowup of $(X, B)$ is a partial resolution of singularities $f : Y \to X$ such that

(i) $Y$ is $\mathbb{Q}$-factorial;

(ii) the pair $(Y, \sum_j \min(1, b_j) f_*^{-1}B_j + \sum_i E_i)$ is dlt, where $E_i$ are the exceptional divisors of $f$;

(iii) one has $B_Y \geq f_*^{-1}B + \sum E_i$, that is, all discrepancies $a(E_i, X, B)$ are $\leq 1$. If $(X, B)$ is lc, then $B_Y = f_*^{-1}B + \sum E_i$.

It is a result of Hacon that dlt blowups exist in any dimension; see, for example, [9], Theorem 10.4. For surfaces, this is an elementary fact: a dlt blowup can
be obtained by taking a log resolution followed by contracting back the exceptional curves with discrepancy \( a(E, X, B) > -1 \). The configuration of such curves is log terminal, whence rational, and the contraction exists by \([10]\).

**Definition 2.2.** We will say that \( f : Y \to X \) is an *effective resolution* of a log surface \((X, B)\) if \( Y \) is smooth and \( B_Y \geq 0 \), that is, the discrepancies \( a_i \) are \( \leq 0 \). For example, the minimal resolution of surface singularities is effective.

**Definition 2.3.** Let \((X, B)\) be a log canonical surface, \( Z \) a nonklt centre (so a point or a curve), \( f : Y \to X \) a log resolution, and \( E \) an irreducible component of \( B_Y^{-1} \) on \( Y \) such that \( Z = f(E) \). We say that \( Z \) is an *inaccessible* nonklt centre if \( E \) is disjoint from the rest of \( B_Y^{-0} \). Otherwise we call \( Z \) an *accessible* nonklt centre. It is easy to see that this definition is independent of the choices of the log resolution \( f \) and the component \( E \).

**Remark 2.4.** One easily sees that a log canonical surface \((X, B)\) has non-empty accessible nonklt locus if and only if \( \text{supp } B_Y^{>0} \) is singular at some point of \( B_Y^{-1} \).

It follows from the classification of log canonical singularities (see, for example, \([11]\)) that a nonklt centre is inaccessible if and only if it is either a point \( p \in X \) which is a simple elliptic singularity with \( p \notin B \), or a smooth curve \( B_j \) which occurs in \( B \) with coefficient \( b_j = 1 \), lies in the smooth part of \( X \) and is disjoint from the rest of \( B^{>0} \).

**Lemma 2.5.** Let \( Z \) be an accessible nonklt centre of a log canonical surface \((X, B)\). Then there is an effective resolution of singularities \( f : Y \to X \) such that \( Z = f(B_1), B_Y = \sum_{i=1}^n b_i B_i, n \geq 2, b_1 = 1, b_2 > 0, \) and \( B_1 \cup B_2 \neq \emptyset \).

**Proof.** \( Y \) can be taken as to be the minimal resolution of a dlt blowup of \((X, B)\). \( \square \)

A set \( C \subset \mathbb{R} \) is called a DCC set (resp. an ACC set) if it satisfies the descending (resp. ascending) chain condition: every strictly decreasing (resp. increasing) subsequence of \( C \) terminates.

**Definition 2.6.** Let \( C \subset [0, 1] \). The *derivative set* is defined as follows:

\[
C' = \left\{ 1 - \frac{1 - \sum n_j b_j}{m} \bigg| b_j \in C, m, n_j \in \mathbb{N}, 1 - \sum n_j b_j \geq 0 \right\} \cup \{1\}. \tag{2.1}
\]

It is easy to see that if \( C \) is a DCC set, then so is \( C' \). Note that \( C'_0 = C'_1 = C'_2 = C_2 \).

A standard way in which the derivative set appears is the adjunction formula \([12]\). Let \((Y, E + \Delta) \in S(C)\) be a log canonical surface with a reduced curve \( E \). In particular, \( K_Y + E + \Delta \) is an \( \mathbb{R} \)-Cartier divisor and the curve \( E \) is at worst nodal. Then the restriction of \( K_Y + E + \Delta \) to \( E \) is equal to \( K_E + \text{Diff}_E (\Delta) \), where \( \text{Diff}_E (\Delta) = \sum b'_k Q_k \) is called the *different* and is an effective divisor with coefficients \( b'_k \in C' \). An explicit formula for the coefficients \( b'_k \) is as follows.

**Lemma 2.7.** Let \((Y, E + \sum b_j \Delta_j)\) be a log canonical surface as above. Then, for each \( k \),

(i) either \( b'_k = 1 \);

(ii) or \( Q_k \in Y \) is smooth and \( m = 1, n_j = (E \cdot \Delta_j)_{Q_k} \) in the equation (2.1);
(iii) or \( Q_k \in Y \) is a singularity and the exceptional locus of its minimal resolution
\[ g: Z \to Y \]
\( Z \to Y \) is a chain of curves \( F_1, \ldots, F_r \) with \( F_i^2 = -p_i \leq 2 \), and \( EF_1 = 1 \), \( EF_i = 0 \) when \( i > 1 \), and then

(a) \( m = \det(p_1, \ldots, p_r) \) is the determinant of the matrix \( M \) corresponding to the
resolution, with \( p_i \) on the main diagonal and \((-1)\) on a diagonal next to the main one, and the number \( m \) is equal to the index of \( Q_k \in Y \), and

(b) \( n_j = \sum_{i=1}^r (f_i^{-1} \Delta_j \cdot F_i) \det(p_{i+1}, \ldots, p_r) \), with \( \det = 1 \) when \( i = r \).

Proof. Assume that \( Q_k \in Y \) is singular and \( b'_k < 1 \). By the classification of log canonical surface singularities with coefficients in \( \{1\} \) (see [11]), the resolution graph of \( Q_k \in Y \) is a chain. The log discrepancies \((1 + f_i)\) of \( K_Y + E + \Delta \) along \( F_i \) are solutions of an \( r \times r \)-system of linear equations with matrix \( M \). The formula for \( M^{-1} \) in terms of cofactors gives the formula stated. The number \( \det(p_{i+1}, \ldots, p_r) \) is the minor of \( M \) obtained by removing the first column and the \( i \)th row. \( \square \)

In particular, we have the following well-known fact.

Corollary 2.8. For the set \( C_2 \), if \( b'_k = 1 - 1/n < 1 \), then \( n = mn' \), where \( m = \det(p_1, \ldots, p_r) \) is the index of \( Q_k \in Y \), and

(i) either \( \text{supp} g^{-1}_* \Delta \) intersects only the last curve \( F_r \), once, and the corresponding coefficient \( g^{-1}_* \Delta \) is equal to \( 1 - 1/n' \);

(ii) or \( \text{supp} g^{-1}_* \Delta \) is disjoint from \( \bigcup F_i \), and \( m = n, n' = 1 \).

In particular, \( n(K_Y + E + \Delta) \) is integral and Cartier in a neighbourhood of \( Q_k \).

We now introduce an auxiliary ACC set.

Lemma 2.9. Let \( C \subset [0, 1] \) be a DCC set and \( m \in \mathbb{N} \). Then the set
\[
T_m(C) = \left\{ \frac{1-b}{\{mb\}} \mid b \in C, \{mb\} \neq 0 \right\} \cup \{1\}
\]
is an ACC set and thus attains the maximum \( t_m(C) \in \mathbb{R}_{\geq 1} \). For every \( b \in C \) with \( \{mb\} \neq 0 \), one has \( b + t_m\{mb\} \geq 1 \). Here \( \{x\} \) denotes the fractional part of \( x \).

Lemma 2.10. When \( C = C_0, C_1 \) or \( C_2 \), one has \( t_m = 1 \) for all \( m \).

Proofs of these lemmas are straightforward and we omit them.

§ 3. Construction of accumulation points

In this section we prove one direction of Theorem 1.1: if there is a log canonical surface \((X, B) \in S(\overline{C} \cup \{1\})\) with ample \( K_X + B \) satisfying the conditions (ii) and (iii), then \( (K_X + B)^2 \) is an accumulation point of \( \mathbb{K}^2(C) \).

3.1. Accumulation points of volumes due to accumulating coefficients.

Theorem 3.1. Let \( C \subset [0, 1] \) be a DCC set. Let \((X, B) \in S(\overline{C})\) be such that \( K_X + B \) is big and nef and at least one coefficient of \( B \) is an accumulation point of \( C \). Then \( (K_X + B)^2 \) is an accumulation point of \( \mathbb{K}^2(C) \).

Proof. Write \( B = \sum_{j \in J} b_j B_j \). Let \( J_\infty \) be the set of \( j \) such that \( b_j \) is an accumulation point of \( C \). Then \( J_\infty \) is non-empty by assumption. Since bigness is an open condition on the space \( N^1(X)_{\mathbb{R}} \) of numerical classes of \( \mathbb{R} \)-divisors on \( X \), there is
a strictly increasing sequence \( b_j^{(s)} \in C \) converging to \( b_j \) for every \( j \in J_\infty \) such that \( K_X + B^{(s)} \) is big, where

\[
B^{(s)} := \sum_{j \in J_\infty} b_j^{(s)} B_j + \sum_{j \in J - J_\infty} b_j B_j.
\]

Since \((X, B)\) is log canonical and \( 0 \leq B^{(s)} < B \) for every \( s \), all the log surfaces \((X, B^{(s)})\) are log canonical. By construction, \((X, B^{(s)}) \in S(C)\). Since \( K_X + B \) is nef, one has \( \text{vol}(K_X + B^{(s)}) < \text{vol}(K_X + B) \). Finally, since volume is a continuous function on \( N^1(X)_\mathbb{R} \), we have

\[
\lim_{s \to \infty} \text{vol}(K_X + B^{(s)}) = \text{vol}(K_X + B) = (K_X + B)^2. \quad \Box
\]

### 3.2. Accumulation points of volumes due to nonklt loci.

**Definition 3.2.** We denote by \( C_B \) the set of coefficients occurring in \( B \).

**Theorem 3.3.** Let \((X, B)\) be a log canonical surface with ample \( K_X + B \) and non-empty accessible nonklt locus. Then, for every \( \epsilon > 0 \), there is a log canonical surface \((X', B') \in S(C_B)\) with ample \( K_{X'} + B' \) such that

(i) \( X' \) is birational to \( X \) and \( B' \) is the strict transform of \( B \);

(ii) \( \text{vol}(K_X + B) - \epsilon < \text{vol}(K_{X'} + B') < \text{vol}(K_X + B) \).

**Proof.** Consider a resolution as in Lemma 2.5 and \( K_Y + B_Y = f^*(K_X + B) \). Blowing up the points of intersection of \( B_1 \) and \( B_2 \) several times if necessary, we can assume that \( f \) is effective and \( B_1, B_2 \) meet transversally at a single point \( p \in Y \).

In our notation, \( b_1 = 1 \) and \( b_2 > 0 \). We blow up \( p \in Y \) and then blow up its pre-image on the strict transforms of \( B_1 \) on the blown-up surfaces. Let \( h: Y^{(s)} \to Y \) be the resulting morphism after \( s \) blowups. The inverse image of \( B_1 + B_2 \) on \( Y^{(s)} \) has the following dual graph:

\[
\begin{align*}
\circ & \quad \circ & \quad \circ & \quad \circ \\
B_1^{(s)} & \quad E_s & \quad E_{s-1} & \quad E_2 & \quad E_1 & \quad B_2^{(s)} \\
(1) & \quad (2) & \quad (2) & \quad (2)
\end{align*}
\]

where the \( B_j^{(s)} \) are the strict transforms of the \( B_j \) \((j = 1, 2)\), \( E_i \) \((1 \leq i \leq n)\) are the exceptional curves, and the numbers above the vertices are the negatives of the self-intersection numbers of the corresponding curves.

Let \( K_{Y^{(s)}} + B_{Y^{(s)}} = h^*(K_Y + B_Y) \). If we write

\[
B_{Y^{(s)}} = h_*^{-1}B_Y + \sum_{1 \leq i \leq n} e_i E_i
\]

as a sum of distinct components, then \( e_1 = e_2 = \cdots = e_n = b_2 > 0 \). Now let

\[
B'_{Y^{(s)}} = B_{Y^{(s)}} - b_2 \sum_{1 \leq i \leq s} i E_i = h_*^{-1}B_Y + b_2 \sum_{1 \leq i \leq s} \frac{s - i}{s} E_i.
\]

Since \( K_{Y^{(s)}} + B_{Y^{(s)}} \) is big and nef and \( B'_{Y^{(s)}} < B_{Y^{(s)}} \), one has

\[
\text{vol}(K_{Y^{(s)}} + B'_{Y^{(s)})} < \text{vol}(K_{Y^{(s)}} + B_{Y^{(s)})}.
\]
On the other hand, one computes that
\[(K_{Y^{(s)}} + B'_{Y^{(s)}})^2 = (K_{Y^{(s)}} + B_{Y^{(s)}})^2 - \frac{b_2^2}{s} = (K_X + B)^2 - \frac{b_2^2}{s}.\]
Take
\[s > \max\left\{\frac{b_2^2}{\epsilon}, \frac{b_2^2}{(K_X + B)^2}\right\}.
\]
Then
\[\max\{0, (K_X + B)^2 - \epsilon\} < (K_{Y^{(s)}} + B'_{Y^{(s)}})^2 < (K_X + B)^2.\]
 Since \(h_s(K_{Y^{(s)}} + B'_{Y^{(s)}}) = 0\) for \(1 \leq i \leq s - 1\). Thus all the exceptional curves \(F \not= E_s\) of \(Y^{(s)} \to X\) satisfy \((K_{Y^{(s)}} + B'_{Y^{(s)}})F \leq 0\). Clearly, they are all contracted by the morphism \(g: Y^{(s)} \to X^{(s)}\). Since the coefficient of \(E_s\) in \(B'_{Y^{(s)}}\) is zero, one has \((X^{(s)}, B_{X^{(s)}}) \in S(C_B)\). Moreover,
\[(K_{X^{(s)}} + B^{(s)})^2 \geq (K_{Y^{(s)}} + B'_{Y^{(s)})}^2 = (K_X + B)^2 - \frac{b_2^2}{s} > (K_X + B)^2 - \epsilon.\]
We take \((X', B') = (X^{(s)}, B^{(s)})\).

**Corollary 3.4.** Let \((X, B)\) be a log canonical surface with ample \(K_X + B\) and non-empty accessible nonklt locus. For every \(\epsilon > 0\) there is a log canonical surface \((X', B') \in S(C_B)\) with ample \(K_{X'} + B'\) such that
(i) \(X'\) is birational to \(X\) and \(B'\) is the strict transform of \(B\);
(ii) \((X', B')\) has no accessible nonklt locus;
(iii) \(\text{vol}(K_X + B) - \epsilon < \text{vol}(K_{X'} + B') < \text{vol}(K_X + B)\).

**Proof.** By Theorem 3.3 there is a log surface \((X^{(1)}, B^{(1)}) \in S(C_B)\) with ample \(K_{X^{(1)}} + B^{(1)}\) such that \((X^{(1)})\) is birational to \(X\), \(B^{(1)}\) is the strict transform of \(B\), and
\[\text{vol}(K_X + B) - \frac{\epsilon}{2} < \text{vol}(K_{X^{(1)}} + B^{(1)}) < \text{vol}(K_X + B).\]
If \((X^{(1)}, B^{(1)})\) has a non-empty accessible nonklt locus, then we can apply Theorem 3.3 again to \((X^{(1)}, B^{(1)})\) to get \((X^{(2)}, B^{(2)})\). By induction, if we have constructed \((X^{(s)}, B^{(s)})\) in \(S(C_B)\) and if \((X^{(s)}, B^{(s)})\) has a non-empty accessible nonklt locus, then we can apply Theorem 3.3 to \((X^{(s)}, B^{(s)})\) to obtain a log surface \((X^{(s+1)}, B^{(s+1)}) \in S(C_B)\) with ample \(K_{X^{(s+1)}} + B^{(s+1)}\) such that
\[\text{vol}(K_{X^{(s)}} + B^{(s)}) - \frac{\epsilon}{2^{s+1}} < \text{vol}(K_{X^{(s+1)}} + B^{(s+1)}) < \text{vol}(K_{X^{(s)}} + B^{(s)}).\]
Since \(\mathbb{K}^2(C_B)\) is a DCC set, this process terminates after, say, \(N - 1\) steps. Then \((X^{(N)}, B^{(N)})\) has ample \(K_{X^{(N)}} + B^{(N)}\) and empty accessible nonklt locus. It is birational to \(X\) and \(B^{(N)}\) is the strict transform of \(B\). Moreover,
\[(K_X + B)^2 - \epsilon < (K_X + B)^2 - \sum_{s=1}^{N} \frac{1}{2^s} \epsilon < (K_{X^{(N)}} + B^{(N)})^2 < (K_X + B)^2.\]
We now take \((X', B') = (X^{(N)}, B^{(N)})\). \(\square\)
Corollary 3.5. Let \((X, B)\) be a log canonical surface with ample \(K_X + B\) and non-empty accessible nonklt locus. Then there is an infinite sequence of log canonical surfaces \((X^{(s)}, B^{(s)}) \in S(C_B)\) with ample \(K_{X^{(s)}} + B^{(s)}\) such that

(i) \(X^{(s)}\) is birational to \(X\) and \(B^{(s)}\) is the strict transform of \(B\);
(ii) \((X^{(s)}, B^{(s)})\) has no accessible nonklt locus;
(iii) the volumes \(\text{vol}(K_{X^{(s)}} + B^{(s)})\) are strictly increasing and

\[
\text{vol}(K_X + B) = \lim_{s \to \infty} \text{vol}(K_{X^{(s)}} + B^{(s)}).
\]

3.3. Part ‘if’ of Theorem 1.1.

Proof of Theorem 1.1: part ‘if’. We first assume that \(1 \in \overline{C}, \) so \(\overline{C} \cup \{1\} = \overline{C}. \) If \((X, B)\) satisfies (ii)(a), then there is a birational morphism \(f : Y \to X\) extracting a single curve, with codiscrepancy contained in \(\text{Acc}(C)\). As usual, we write \(K_B + B_Y = f^*(K_X + B)\). Note that \(B_Y\) is effective and the log surface \((Y, B_Y)\) satisfies the hypotheses of Theorem 3.1. Hence we have \((K_X + B)^2 = (K_Y + B_Y)^2 \in \text{Acc}(\mathbb{K}^2(C))\).

If \((X, B)\) satisfies (ii)(b), then, by Corollary 3.5,

\[
\text{vol}(K_X + B) \in \text{Acc}(\mathbb{K}^2(C_B)).
\]

In this case we can assume that \((X, B)\) does not satisfy (ii)(a). Hence \(C_B \subset C\) and, therefore, \(\text{vol}(K_X + B) \in \text{Acc}(\mathbb{K}^2(C))\).

We now assume that \(1 \notin \overline{C}\). Then each irreducible component of \([B]\) has geometric genus at most 1 by hypothesis (iii). By Corollary 3.5 we get a sequence \((X^{(s)}, B^{(s)}) \in S(C \cup \{1\})\) with volumes converging to \((K_X + B)^2\) and with an additional property: each component of \([B^{(s)}]\) has geometric genus \(\leq 1\). Since there is no accessible nonklt locus, each component of \([B^{(s)}]\) must be a smooth curve lying in the smooth locus of \(X^{(s)}\) and disjoint from the rest of \(B^{(s)}\). By adjunction, it must have genus \(\geq 2\). Thus, \([B^{(s)}] = 0\) and \((X^{(s)}, B^{(s)}) \in S(C)\). \(\square\)

Example 3.6. First consider the log canonical surface \((\mathbb{P}^1 \times \mathbb{P}^1, B)\), where \(B\) is a reduced divisor consisting of three horizontal lines \(L_i\) and \(n \geq 3\) vertical lines \(V_j\). Then \(K_{\mathbb{P}^1 \times \mathbb{P}^1} + B\) is ample and \((K_{\mathbb{P}^1 \times \mathbb{P}^1} + B)^2 = 2(n - 2)\). This exhausts all positive even integers as \(n\) varies. By Theorem 1.1, every number \(2(n - 2)\) is an accumulation point of \(\mathbb{K}^2(C_0)\).

Now let \(F_1\) be the Hirzebruch surface obtained by blowing up one point of \(\mathbb{P}^2\) and let

\[
B = \Gamma_0 + \Gamma_1 + \Gamma_2 + \sum_{1 \leq j \leq n} F_j \quad (n \geq 2),
\]

where \(\Gamma_0\) is the section with \(\Gamma_0^2 = -1\), \(\Gamma_1\) and \(\Gamma_2\) are two distinguished sections with \(\Gamma_1^2 = \Gamma_2^2 = 1\), and the \(F_j\) are distinct fibres not passing through \(\Gamma_1 \cap \Gamma_2\). Then the pair \((F_1, B)\) is log canonical with ample \(K_{F_1} + B\) and \((K_{F_1} + B)^2 = 2n - 3\). This exhausts all positive odd integers as \(n\) varies. By Theorem 1.1, every number \(2n - 3\) with \(n \geq 2\) is an accumulation point of \(\mathbb{K}^2(C_0)\).

As a result of this construction, we obtain that \(\mathbb{N} \subset \text{Acc}(\mathbb{K}^2(C_0))\). This proves one part of Conjecture 3(a) in [2].
§ 4. The limit surface

Lemma 4.1. Let $Z$ be a projective normal surface and let $B = \sum_{j} b_j B_j$ be an effective $\mathbb{R}$-Cartier divisor on $Z$ with $b_0 < 1$. Then there are finitely many effective resolutions $g_t: Z_t \to Z$, $1 \leq t \leq r$, such that if $g: Y \to Z$ is an effective resolution, then there is a $t$ such that the birational map $g_t^{-1} \circ g: Y \to Z_t$ is a morphism and it is an isomorphism over a Zariski-open neighbourhood of the strict transform $g_t^{-1}B_0$ of $B_0$.

Proof. Since any resolution $f: Y \to Z$ dominates the minimal resolution of $Z$, we can replace $Z$ by its minimal resolution and replace the log canonical divisor $K_Z + B$ by its strict transform. In other words, we can assume that $Z$ is smooth.

We will take $g_t: Z_t \to Z$ to be the effective resolutions (see Definition 2.2) that blow up only points of $B_0$ and their strict transforms. The lemma is equivalent to saying that there are only finitely many such effective resolutions.

Suppose that $f: Y \to Z$ is such a resolution. Since $b_0 < 1$, $f$ cannot blow up a smooth point of $B_0$ and their strict transforms. Hence there are only finitely many choices for first-level blowups at the points of $B_0$.

Over every point $p \in B_0$ which is blown up, there are finitely many blowups that make the support of the total transform of $B$ a divisor with simple normal crossings over $p$. Let $p_1, \ldots, p_k$ be the inverse images of $p$ lying on the strict transform of $B_0$. Then the codiscrepancy of the exceptional divisor of the $k$th blowup over $p_i$ and at the strict transform of $B_0$ takes the form $a_i - k(1 - b_0)$, where $a_i$ is the coefficient of the (unique) boundary component in the pullback of $K_Z + B$ that intersects the strict transform of $B_0$ at $p_i$. Under the condition that $b_0 < 1$ and $a_i - k(1 - b_0) \geq 0$, there are only finitely many further blowups over $p_i$. □

Let $C \subset [0, 1]$. We define the following subset of $\mathbb{K}^2(C)$:

$$\mathbb{K}^2_{\text{kl}}(C) := \{(K_X + B)^2 \mid (X, B) \in S(C) \text{ is klt, } K_X + B \text{ is ample}\}.$$ 

Lemma 4.2. Suppose that $C \subset [0, 1]$ is a DCC set and $v_\infty \in \text{Acc}(\mathbb{K}^2(C))$. Then there are a smooth projective surface $(Z, D)$ with a reduced divisor $D$ and a sequence of smooth projective surfaces $Y^{(s)}$ with boundary divisors $B_{Y^{(s)}}$ such that

(i) there is a birational morphism $g_s: Y^{(s)} \to Z$ which blows up only nodes of $D$ and of its total transforms;

(ii) $(Y^{(s)}, B_{Y^{(s)}})$ is log canonical, $K_{Y^{(s)}} + B_{Y^{(s)}}$ is big and nef, $g_{ss}B_{Y^{(s)}} \leq D$;

(iii) $\text{vol}(K_{Y^{(s)}} + B_{Y^{(s)}})$ form a strictly increasing sequence converging to $v_\infty$;

(iv) the log canonical model $(X^{(s)}, B^{(s)})$ of $(Y^{(s)}, B_{Y^{(s)}})$ lies in $S(C)$;

(v) there are no $(-1)$-curves $E$ on $Y^{(s)}$ such that $(K_{Y^{(s)}} + B_{Y^{(s)}})E = 0$, that is, $Y^{(s)} \to X^{(s)}$ is the minimal resolution;

(vi) if $v_\infty \in \mathbb{K}^2_{\text{kl}}(C)$, then, in addition, $(X^{(s)}, B^{(s)})$ is klt and

$$g_t^*(K_Z + g_{ss}B_{Y^{(s)}}) \leq K_{Y^{(t)}} + B_{Y^{(t)}}$$

for every $s < t$.

Proof. Since $\mathbb{K}^2(C)$ is a DCC set ([1], Theorem 8.2), $v_\infty$ is approached by a strictly increasing sequence $(K_{X^{(s)}} + B^{(s)})^2$ in $\mathbb{K}^2(C)$, where $(X^{(s)}, B^{(s)})$ are surfaces in $S(C)$ whose log canonical divisors $K_{X^{(s)}} + B^{(s)}$ are ample.
By Theorem 7.6 in [1] (see also Theorem 4.7 in [7]) there is a diagram

\[ \begin{array}{ccc}
  Y(s) & \longrightarrow & Z(s) \\
  \downarrow & & \downarrow \\
  X(s) & & 
\end{array} \]

such that

(a) \( f_s : Y(s) \to X(s) \) is the minimal resolution of \( (X(s), B(s)) \);
(b) \( g_s : Y(s) \to Z(s) \) is birational and, defining \( K_Y(s) + B_Y(s) = f_s^*(K_X(s) + B(s)) \),
\[ D(s) = g_s(\text{supp} B_Y(s) \cup \text{Exc}(f_s)), \]
the log surfaces \( (Z(s), D(s)) \) form a bounded class.

Since the log surfaces \( (Z(s), D(s)) \) form a bounded class, it suffices to assume that \( (Z(s), D(s)) \) are fixed, to be denoted by \( (Z, D) \) (see [7], Remark 5.7). By Lemmas 5.5 and 5.6 in [7], we can assume that \( Z \) is smooth and \( g_s : Y(s) \to Z \) blows up only nodes of \( D \) and of its total transforms.

Thus we have found \( (Z, D) \) and \( (Y(s), B_Y(s)) \) satisfying (i)–(v). The last very important property (vi) is Theorem 8.5 in [1]; see also Theorem 5.8 in [7]. □

We now are ready to complete the proof of Theorem 1.1.

Proof of the ‘only if’ part of Theorem 1.1. Suppose that \( v_\infty \in \text{Acc}(\mathbb{R}^2(C)) \). Let \( (Z, D) \), \( (Y(s), B_Y(s)) \) and \( (X(s), B(s)) \) be as in Lemma 4.2.

klt case. We first assume that \( v_\infty \in \text{Acc}(\mathbb{R}^2_{\text{Klt}}(C)) \). Then, by Lemma 4.2, \( (X(s), B(s)) \) have klt singularities for all \( s \) and, for every \( s < t \),
\[ g_t^*(K_Z + B_Z(s)) \leq K_Y(t) + B_Y(t), \tag{4.1} \]
where the divisor \( B_Z(s) \) is equal to \( g_s*(B_Y(s)) \). In particular, it follows that \( B_Z(s) \leq B_Z(t) \) for any \( s < t \).

Let \( B_Z = \lim_{s \to \infty} B_Z(s) \). Since the divisors \( K_Z + B_Z(s) \) are nef, so is their limit \( K_Z + B_Z \). Moreover, the bigness of \( K_Z + B_Z(s) \) implies that of \( K_Z + B_Z(s) \leq K_Z + B \). By (4.1),
\[ (K_Y(s) + B_Y(s))^2 \leq (K_Z + B_Z(s))^2 \leq (K_Y(t) + B_Y(t))^2 \]
and we see that
\[ (K_Z + B_Z)^2 = \lim_{s \to \infty} (K_Z + B_Z(s))^2 = \lim_{s \to \infty} (K_X(s) + B(s))^2 = v_\infty. \]

We need to show that \( (Z, B_Z) \) is log canonical. Clearly, the coefficients of the divisor \( B_Z \) belong to \([0,1] \). Let \( E \) be a divisor over \( Z \). By (4.1) we have, for every \( s < t \),
\[ b(E, Z, B_Z(s)) \leq b(E, Y(t), B_Y(t)) \leq 1, \]
where \( b \) denotes the codiscrepancies of the log surfaces. It follows that
\[ b(E, Z, B_Z) = \lim_{s \to \infty} b(E, Z, B_Z(s)) \leq b(E, Y(t), B_Y(t)) \leq 1. \]
Hence \((Z, B_Z)\) is log canonical.

However, it is possible that the set \(C_{B_Z}\) of coefficients is not contained in \(\overline{C} \cup \{1\}\) as required. We need to contract all components of \(B_Z\) whose coefficients are not in \(\overline{C} \cup \{1\}\). To do this, we consider the contraction \((Z, B_Z) \to (Z_{\text{can}}, B_{\text{can}}) =: (X, B)\) of the surface \((Z, B_Z)\) onto its log canonical model. We will show that \((X, B) \in S(\overline{C} \cup \{1\})\) and \((X, B)\) satisfies the conditions (i), (ii), (iii) of Theorem 1.1.

**Claim.** \((X, B) \in S(\overline{C} \cup \{1\})\).

**Proof.** Being the log canonical model of \((Z, B_Z)\), the surface \((X, B)\) is log canonical. We need to show that \(C_B \subset \overline{C} \cup \{1\}\) or, equivalently, every component \(B_j\) of \(B_Z\) with coefficient \(b_j \notin \overline{C} \cup \{1\}\) is contracted by the morphism \(Z \to X\).

Since \(b_j < 1\), up to taking a subsequence there is a resolution \(h: Z' \to Z\) such that for each \(s\), the birational map \(g'_s = h^{-1} \circ g_s: Y^{(s)} \to Z'\) is a morphism, and it is an isomorphism over a Zariski-open neighbourhood of \(h^{-1}B_j\) by Lemma 4.1.

Taking a subsequence, we can assume that, for every \(s\), \(b(B_j, K_{Y^{(s)}} + B_{Y^{(s)}}) \notin C\). It follows that \(B_j\) is contracted by the morphism \(Y^{(s)} \to X^{(s)}\) and, therefore, \((K_{Y^{(s)}} + B_{Y^{(s)})}B_j, Y^{(s)}) = 0\), where \(B_j, Y^{(s)}\) is the strict transform of \(B_j\) on \(Y^{(s)}\).

We compute \((K_Z + B_Z)B_j\) on \(Z'\). Note that for every fixed curve \(E\) on \(Z'\), the codiscrepancies \(b(E, Y^{(s)}, B_{Y^{(s)}})\) converge to \(b(E, Z, B_Z)\). Only the codiscrepancies of the finitely many curves in \(h^{-1}(D)\) that intersect \(B_j\) are relevant for the computation of \((K_Z + B_Z)B_j\). Thus we have

\[
(K_Z + B_Z)B_j = h^*(K_Z + B_Z)h_*^{-1}B_j = \lim_{s \to \infty} g'_s(K_{Y^{(s)}} + B_{Y^{(s)}})h_*^{-1}B_j = \lim_{s \to \infty} (K_{Y^{(s)}} + B_{Y^{(s)}})B_j, Y^{(s)} = 0. \quad \square
\]

**Claim.** \((X, B)\) satisfies condition (ii) of Theorem 1.1.

**Proof.** Suppose that \((X, B)\) does not satisfy (ii) (b). We need to show that \((X, B)\) satisfies (ii) (a).

Since \((X, B)\) does not satisfy (ii) (b), \(\text{supp} B_Z\) is smooth along nonklt \((Z, B_Z)\), so that the maps \(g_s: Y^{(s)} \to Z\) blow up only klt points of \((Z, B_Z)\) by Lemma 4.2 (i). By Lemma 4.1 there are only finitely many possible blowups \(f: Y \to Z\) such that the pullback \(f^*(K_Z + B_Z) = K_Y + B_Y\) is still a log canonical divisor with an effective boundary \(B_Y\). Since \(B_{Y^{(s)}}\) is effective and \(g_{s*}(B_{Y^{(s)})} \leq B_Z\), the morphisms \(Y^{(s)} \to Z\) must be among these finitely many blowups. Thus, up to taking a subsequence we can assume that \(Y^{(s)} = Y\) and the sets \(\text{supp} B_{Y^{(s)}}\) are the same for all \(s\).

Since \((K_{Y^{(s)}} + B_{Y^{(s)})}^2\) is strictly increasing, up to taking a subsequence, the boundary divisors \(B_{Y^{(s)}}\) form a strictly increasing sequence. There must be a component of \(\text{supp} B_{Y^{(s)}}\), say \(B_0\), that is not contracted by \(f_s: Y^{(s)} \to X^{(s)}\) for any \(s\), and the sequence \(b(B_0, Y^{(s)}, B_{Y^{(s)}})\) is strictly increasing and converges to \(b(B_0, Z, B_Z), \) the coefficient of \(B_0\) in \(B_Z\). Since \(B_0\) is not contracted by \(f_s: Y^{(s)} \to X^{(s)}\) and \(C_{B(s)} \subset C\), the coefficient of \(B_0\) in \(B_{Y^{(s)}}\) belongs to \(C\). This means that \(b(B_0, Z, B_Z)\) is an accumulation point of \(C\), and condition (ii) (a) of Theorem 1.1 is verified. \(\square\)

**Claim.** If \(1 \notin \overline{C}\), then \((X, B)\) satisfies condition (iii) of Theorem 1.1.
Proof. Suppose that \([B] \neq 0\) and that \(B_0\) is a component of \([B]\). Let \(B_{0,Y(s)}\) be the strict transform of \(B_0\) on \(Y^{(s)}\). Then
\[
\lim_{s \to \infty} b(B_{0,Y(s)}, Y^{(s)}, B_{Y(s)}) = 1.
\]
Since \(1 \notin \overline{C}\), the number \(b(B_{0,Y(s)}, Y^{(s)}, B_{Y(s)})\) does not lie in \(C\) for sufficiently large \(s\). On the other hand, the log canonical model \((X^{(s)}, B_s)\) belongs to \(\mathcal{S}(C)\). This can happen only when \(B_{0,Y(s)}\) is contracted by \(f_s: Y^{(s)} \to X^{(s)}\) to an lc singularity and, therefore, the geometric genus of \(B_0\) (and \(B_{0,Y(s)}\)) is at most 1. □

We conclude that the theorem is proved in the case when \(v_\infty \in \text{Acc}(\mathbb{K}_\text{klt}^2(C))\).

**nonklt case.** Suppose that \(v_\infty \in \text{Acc}(\mathbb{K}_\text{klt}^2(C)) \setminus \text{Acc}(\mathbb{K}_\text{klt}^2(C))\). Then there are at most finitely many \(s\) such that \((X^{(s)}, B_s)\) is klt. Choosing an infinite subsequence, we can assume that \(\text{nonklt}(X^{(s)}, B^{(s)}) \neq \emptyset\) for all \(s\).

For each \(s\), let \(\mu_s: \overline{X}^{(s)} \to X^{(s)}\) be a \(\mathbb{Q}\)-factorial dlt blowup extracting only divisors with discrepancy \(-1\). Let \(K_{\overline{X}^{(s)}} + B_{\overline{X}^{(s)}} = \mu_s^*(K_{X^{(s)}} + B^{(s)})\). We write \(B_{\overline{X}^{(s)}} = B^{(1)}_{\overline{X}^{(s)}} + B^{(2)}_{\overline{X}^{(s)}}\), where \(B^{(1)}_{\overline{X}^{(s)}}\) is the fractional part of \(B_{\overline{X}^{(s)}}\) and \(B^{(2)}_{\overline{X}^{(s)}}\) is the (non-zero) reduced part.

We can choose a strictly decreasing sequence \(\epsilon_s \in [0, 1]\) with \(\lim_{s \to \infty} \epsilon_s = 0\) such that
1) \(1 - \epsilon_s \notin C\);
2) \(K_{\overline{X}^{(s)}} + B^{(1)}_{\overline{X}^{(s)}}\) is lc and big, where \(B^{(1)}_{\overline{X}^{(s)}} = B^{(1)}_{\overline{X}^{(s)}} + (1 - \epsilon_s)B^{(2)}_{\overline{X}^{(s)}}\);
3) \((K_{\overline{X}^{(s)}} + B^{(1)}_{\overline{X}^{(s)}})^2\) is strictly increasing and \(\lim_{s \to \infty} (K_{\overline{X}^{(s)}} + B^{(1)}_{\overline{X}^{(s)}})^2 = v_\infty\).

Let \(\mathcal{D} = C \cup \{1 - \epsilon_s\}_s\). Then \(\mathcal{D}\) is a DCC set and \((\overline{X}^{(s)}, B^{(1)}_{\overline{X}^{(s)}}) \in \mathcal{S}(\mathcal{D})\) are klt. As in the klt case, we can then construct a log canonical surface \((Z, B_Z)\) with big and nef \(K_Z + B_Z\), whose log canonical model lies in \(\mathcal{S}(\mathcal{D} \cup \{1\})\).

We need to show that the log canonical model \((X, B)\) of \((Z, B_Z)\) is actually in \(\mathcal{S}(C \cup \{1\})\). By construction, each coefficient \(b\) of the divisor \(B\) either belongs to \(\text{Acc}(\mathcal{D}) \cup \{1\}\) or occurs in the divisors \(B^{(1)}_{\overline{X}^{(s)}}\) infinitely many times. Since \(\text{Acc}(\mathcal{D}) = \text{Acc}(C) \cup \{1\}\) and each of the coefficients \(1 - \epsilon_s\) occurs only once, we have \(b \in \overline{C} \cup \{1\}\).

An argument similar to those used in the klt case show that \((X, B)\) satisfies conditions (ii) and (iii) of Theorem 1.1. □

§ 5. Lower bounds of accumulation points for \(\mathbb{K}_2^2(C_2)\)

Let \(C \subset [0, 1]\) be a DCC subset. By [1], the set \(\mathbb{K}_2^2(C)\) (and hence also \(\text{Acc}(\mathbb{K}_2^2(C))\)) is DCC. The paper [7] gives an explicitly computable lower bound for \(\mathbb{K}_2^2(C)\), but we have already mentioned that it is much too small to be useful. However, we have the following quite efficient bound.

**Definition 5.1.**
\[
v_1(C) = \min\{(K_X + B)^2 \mid (X, B) \in \mathcal{S}(C \cup \{1\}), K_X + B \text{ is ample and } [B] \neq 0\}.
\]
Theorem 5.2 (Kollár [6], (6.2.1), (5.3.1)). One has

$$v_1(C_2) = \frac{1}{1764} = \frac{1}{42^2}. $$

By Corollary 1.5,

$$\min \operatorname{Acc}(\mathbb{K}^2(C_2)) = \min \mathbb{K}^2_{\operatorname{nonklt}}(C_2),$$

where for the latter we only consider log surfaces \((X, B)\) with \(\operatorname{nonklt}(X, B) \neq \emptyset\). If \([B] \neq 0\), then we are done by Theorem 5.2. Below, we deal with the remaining case, when \([B] = \emptyset\) but \((X, B)\) has an isolated nonklt centre, a point \(p \in X\). The number \(t_m(C)\) in the following theorem was defined in §2.

Theorem 5.3. Let \((X, B = \sum b_i B_i)\) be a log canonical surface with coefficients in a DCC set \(\mathcal{C}\). Suppose that \(K_X + B\) is ample and that \(H^0(X, [m(K_X + B)]) \neq 0\). Then

$$\operatorname{vol}(K_X + B) \geq \frac{v_1(C)}{(1 + mt_m(C))^2}. $$

Proof. Let \(C \sim [m(K_X + B)]\) be an effective curve. Hence we have \(m(K_X + B) \sim_{\mathbb{R}} C + \{mB\}\). We define

$$D = C + \max(B, \{mB\}).$$

If \(B = \sum b_i B_i\), then the coefficient of \(B_j\) in \(\max(B, \{mB\})\) is equal to \(b_j\) when \(\{mb\} = 0\), and to 1 when \(\{mb\} \neq 0\). Thus, \(K_X + D \geq K_X + B\) and the integral part of \(D\) satisfies \([D] \geq m(K_X + B)\). Hence there is a big divisor \(E\) with \(\operatorname{Supp} E = [D]\). We also note that \(K_X + D\) is \(\mathbb{R}\)-Cartier because \(B\) does not pass through the non-\(\mathbb{Q}\)-factorial singularities of \(X\) since \((X, B)\) is lc.

We want to give an upper bound for the divisor \(K_X + D\) in terms of \(K_X + B\). We seek a number \(t\) such that

$$(1 + mt)(K_X + B) \geq K_X + D$$

$$\iff B + t(C + \{mB\}) \geq C + \max(B, \{mB\}).$$

This is true if \(t \geq 1\) and one has \(b + t\{mb\} \geq 1\) for every \(b \in \mathcal{C}\) with \(\{mb\} \neq 0\). By Lemma 2.9, this holds if we set \(t = t_m(C)\). This gives us

$$(1 + mt_m)^2 \operatorname{vol}(K_X + B) \geq \operatorname{vol}(K_X + D). \quad (5.1)$$

The log surface \((X, D)\) may be not log canonical. Let \(f : Y \to X\) be a dlt blowup as in Definition 2.1. Put \(K_Y + D_Y = f^*(K_X + D)\). If \(D_Y = \sum d_j D_j + \sum e_i E_i\), then \(e_i \geq 1\). We define

$$D'_Y = \sum \min(1, d_j) D_j + \sum \min(1, e_i) E_i.$$

Then

$$f^*(K_X + D) \geq K_Y + D'_Y \geq f^*(K_X + B),$$

the latter being true because \(K_X + D \geq K_X + B\) and \((X, B)\) is log canonical.

It follows that the divisor \(K_Y + D'_Y\) is big. Its reduced part \(|D'_Y| = f^{-1}([D])\) supports a big divisor. Thus, on the log canonical model \((Z, D_Z)\) \((Y, D'_Y)\), the
reduced part $|D_Z|$ is not equal to 0. We have $\text{vol}(K_Y + D_Y) = \text{vol}(K_Z + D_Z)$ and, by Theorem 5.2, $\text{vol}(K_Z + D_Z) \geq v_1(C)$. Altogether,

$$\text{vol}(K_X + D) = \text{vol}(f^*(K_X + D)) \geq \text{vol}(K_Y + D_Y) \geq v_1(C).$$

Together with the equation (5.1), this proves the statement. □

Next, we find a section of a multiple of $K_X + B$ provided that nonklt$(X, B)$ contains an isolated point.

**Proposition 5.4.** Suppose that $(X, B)$ is in $\mathcal{S}(\mathbb{C})$ and $K_X + B$ is ample. Let $p$ be an isolated nonklt centre of $(X, B)$. Let $m \geq 2$ be an integer such that $m(K_X + B)$ is integral and Cartier near the point $p$. Then

$$h^0(X, [(m(K_X + B))] \geq 1.$$  

**Proof.** Let $f : Y \to X$ be a log resolution. We write $K_Y + B_Y = f^*(K_X + B)$ and $E := B_Y^{-1}$. By assumption, the divisor $E$ is reduced, non-empty and has a connected component contained in $f^{-1}(p)$. Since $(m - 1)(K_Y + B_Y)$ is big and nef, we have

$$H^1(Y, K_Y + [(m - 1)(K_Y + B_Y)]) = 0$$

by the Kawamata–Viehweg vanishing theorem. The standard exact sequence $0 \to \mathcal{O}_Y(-E) \to \mathcal{O}_Y \to \mathcal{O}_E \to 0$ together with the above vanishing gives

$$H^0(Y, K_Y + [(m - 1)(K_Y + B_Y)] + E) \to H^0(E, (K_Y + [(m - 1)(K_Y + B_Y)] + E)|_E).$$

Let $E_0$ be a connected component of $E$ contained in $f^{-1}(p)$. The coefficients of the divisors $B_j$ in $B_Y$ that intersect $E_0$ satisfy $0 \leq b_j < 1$. They occur in the different. Since $m(K_X + B)$ is integral and Cartier at $p$, it follows that in a neighbourhood of $E_0$ one has

$$K_Y + [(m - 1)(K_Y + B_Y)] + E = m(K_Y + B_Y)$$

and is trivial. Therefore, the above two groups $H^0$ are non-zero. Thus, the divisor

$$f_*(K_Y + [(m - 1)(K_Y + B_Y)] + E) = K_X + [(m - 1)(K_X + B)] + B^{-1}$$

is effective. It is now easy to see that for a number of the form $b = 1 - 1/n$, $n \in \mathbb{N}$, and for any $m \in \mathbb{N}$ one has $[(m - 1)b] = [mb]$. Thus, the last divisor is equal to

$$K_X + [(m - 1)(K_X + B)] + B^{-1} = m(K_X + B).$$

□

**Remark 5.5.** Suppose that $(X, B) \in \mathcal{S}(\mathbb{C})$ has an ample $K_X + B$ and $k$ distinct isolated nonklt centres $p_i, 1 \leq i \leq k$. Let $m \geq 2$ be an integer such that $m(K_X + B)$ is integral and Cartier near each point $p_i$. Then the argument used in Proposition 5.4 gives that $h^0(X, [(m(K_X + B)]) \geq k.$

**Proposition 5.6.** For every $DCC$ set $C \subset [0, 1] \cap \mathbb{Q}$ there is an $m(C) \in \mathbb{N}$ such that for any log surface $(X, B)$ in $\mathcal{S}(C)$ with ample $K_X + B$ and any point $p \in X$ which is an isolated nonklt centre of $(X, B)$, the divisor $m(C)(K_X + B)$ is integral and Cartier near $p$. 
Proof. Let $f : Y \to X$ be a dlt blowup as in Definition (2.1). Let $B_Y = E + \Delta$ be the decomposition into integral and fractional parts. We have $E \neq 0$ by assumption. Write $E = \sum E_i$. By Shokurov’s connectedness theorem ([12], Lemma 5.7), $E$ is connected. As in §2, the adjunction to $E$ gives

$$0 \equiv (K_Y + E + \Delta)|_E = K_E + \text{Diff}_E(\Delta), \quad \text{Diff}_E(\Delta) = \sum b'_k Q_k, \ b'_k \in \mathcal{C}' .$$

It follows that

(i) either $p_a(E) = 1$, $E$ is a smooth elliptic curve or a cycle of $\mathbb{P}^1$’s, and $p \notin B$. Then $K_X + B$ is Cartier near $p$;

(ii) or $E$ is a $\mathbb{P}^1$ or a chain of $\mathbb{P}^1$’s and $\Delta$ intersects only the end curves.

Restriction to an end curve gives an identity $\sum b'_k = 1$ or $2$, with $b'_k < 1$, $b'_k \in \mathcal{C}'$. Since $\mathcal{C}'$ is a DCC set, this identity has only finitely many solutions. For a fixed $b'_k < 1$ we have

$$b'_k = 1 - \frac{1 - \sum n_j b_j}{n}$$

and there are only finitely many solutions for $n, n_j, b_j$. By Lemma 2.7, $n$ is the index of the singularity $Q_k \in Y$.

We conclude that there are only finitely many possibilities for $b_j$ and the Cartier indices of $K_Y + B_Y$ at the singular points over $p$. Since all the $b_j$ belong to $\mathbb{Q}$ by assumption, there is a fixed number $m$ (depending only on $\mathcal{C}$) such that $m(K_Y + B_Y)$ is integral and Cartier. If $p$ is a point as in (i) above, then near $p$ the divisor $K_X + B = K_X$ is already Cartier. Otherwise $X$ is klt at $p$. By the base-point-free theorem for lc surfaces, it follows that $m(K_X + B)$ is Cartier. □

**Lemma 5.7.** Let $(X, B)$ be a log surface in $\mathcal{S}(C_2)$ and let $p \in X$ be a point that is an isolated nonklt centre of $(X, B)$. Then the divisor $m(K_X + B)$ is integral and Cartier near $p$ when $m = 1, 2, 3, 4$ or $6$.

Proof. As in the proposition just proved, for a dlt blowup $Y \to X$ on the exceptional divisor $E = \sum E_i$ there are several singularities with indices $m_j$ and an identity of the form $\sum (1 - 1/n_j) = 1$ or $2$. It is well known that the only solutions $(n_j)$ of this identity are $(2, 2), (3, 3, 3), (2, 4, 4), (2, 3, 6), (2, 2, 2, 2)$. By Corollary 2.8, the indices $m_j$ divide $n_j$: $n_j = m_j n'_j$, and the corresponding coefficients of $B$ are $b_j = 1 - 1/n'_j$. If $m$ is the G.C.D. of these numbers (that is, $m = 1, 2, 3, 4$ or $6$), then $m(K_Y + B_Y)$ is integral and Cartier, and so is $m(K_X + B)$ by the base-point-free theorem. □

Putting this together, we obtain the following theorem.

**Theorem 5.8.** For every log surface $(X, B) \in \mathcal{S}(C_2)$ with an ample divisor $K_X + B$ and nonklt$(X, B) \neq \emptyset$, one has

$$(K_X + B)^2 \geq \frac{1}{72} \cdot 42^2 .$$

Proof. If $[B] \neq 0$, then $(K_X + B)^2 \geq 1/42^2$ by Theorem 5.2. Otherwise there is an isolated nonklt centre. Then we apply Theorem 5.3 and take into account that $t_m = 1$ by Lemma 2.10 and $m \leq 6$ by Lemma 5.7. □

We conclude by completing the proof of Theorem 1.7.
Theorem 6.2. For any set \( \mathcal{C} \subset [0,1] \) and any \( k \in \mathbb{N} \) one has \( \text{Acc}^k(\mathbb{K}^2(\mathcal{C})) \neq \emptyset \).

Proof. Consider \( (X, B) = (\mathbb{P}^2, \sum_{j=1}^{n} L_j) \) with \( n \geq 4 \), where the \( L_j \) are \( n \) distinct lines in general position. Then \((X, B)\) is a log canonical surface with ample \( K_X + B \) and \((K_X + B)^2 = (n - 3)^2\). Let \( p_j \in L_j \cap L_n \) for \( 1 \leq j \leq n - 1 \) be the \( n - 1 \) nodes of \( B \) on \( L_n \). We apply the construction in Theorem 3.3 to all these points at once.

Let \( h: Z = Y^{(s_1, \ldots, s_{n-1})} \to \mathbb{P}^2 \) be obtained by blowing up \( s_j \) times at the point \( p_j \) and its pre-images on the strict transforms of \( L_j \), \( 1 \leq j \leq n - 1 \). As in Theorem 3.3,
we define a divisor $B_Z$ by putting $K_Z + B_Z = h^*(K_X + B)$ and a divisor $B'_Z$ by putting
\[ B'_Z = B_Z - \sum_{j=1}^{n-1} \sum_{i=1}^{s_j} \frac{i}{s_j} E_{j,i} = h_*^{-1}B + \sum_{j=1}^{n-1} \sum_{i=1}^{s_j} \frac{s_j - i}{s_j} E_{j,i}. \]

By construction, $K_Z + B'_Z$ intersects the curves $E_{j,i}$ non-negatively, and one has $(K_Z + B'_Z)h_*^{-1}L_j = n - 4$ for $1 \leq j \leq n - 1$ and $n - 3 - \sum_{j=1}^{n-1} 1/s_j$ for $j = L_n$. Thus, for sufficiently large $s_j$ (for example, $s_j \geq n - 1$), the divisor $K_Z + B'_Z$ is nef and of volume
\[ (K_Z + B'_Z)^2 = (n - 3)^2 - \sum_{j=1}^{n-1} \frac{1}{s_j}, \]
and on the log canonical model $(Z, B'_Z)$ the curve $h_*^{-1}L_n$ is not contracted and $(Z, B'_Z)$ has a non-empty accessible nonklt locus.

Each of the above positive numbers is an accumulation point of $\mathbb{K}^2(C_0)$ by Theorem 1.1. Fixing $s_1, \ldots, s_{n-2}$ and letting $s_{n-1} \to \infty$, we show that
\[ (n - 3)^2 - \sum_{j=1}^{n-2} \frac{1}{s_j} \in \text{Acc}^2(\mathbb{K}^2(C_0)). \]
Continuing in this way, we eventually see that $(n - 3)^2 \in \text{Acc}^n(\mathbb{K}^2(C_0))$. (And indeed it is easy to see that one can do much better than this when $n \geq 5$.) Now for every $n \geq \max(k, 4)$ one has
\[ \text{Acc}^k(\mathbb{K}^2(C)) \cap \text{Acc}^n(\mathbb{K}^2(C_0)) \neq \emptyset. \]

**Definition 6.3.** We say that a set $C \subset \mathbb{R}$ is of finite accumulation complexity if $\text{Acc}^k(C) = \emptyset$ for $k \gg 0$. (For example, the sets $C_0, C_1, C_2, C_3$ have this property.)

**Remark 6.4.** We expect that for every set $C \subset [0, 1]$ of finite accumulation complexity and for every $d > 0$, the set $\mathbb{K}^2(C) \cap [0, d]$ is of finite accumulation complexity. In particular, the function $f(k) = \min \text{Acc}^k(C)$ tends to $\infty$ as $k \to \infty$.

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