On approximation of smoothing probabilities for hidden Markov models

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Abstract

We consider the smoothing probabilities of hidden Markov model (HMM). We show that under fairly general conditions for HMM, the exponential forgetting still holds, and the smoothing probabilities can be well approximated with the ones of double sided HMM. This makes it possible to use ergodic theorems. As an applications we consider the pointwise maximum a posteriori segmentation, and show that the corresponding risks converge

Key words: Hidden Markov models, smoothing, segmentation

1. Introduction

Let $Y = \{Y_t\}_{t=-\infty}^{\infty}$ be a double-sided stationary MC with states $S = \{1, \ldots, K\}$ and irreducible aperiodic transition matrix $(P(i,j))$. Let $X = \{X_t\}_{t=-\infty}^{\infty}$ be the (double-sided) process such that: 1) given $\{Y_t\}$ the random variables $\{X_t\}$ are conditionally independent; 2) the distribution of $X_0$ depends on $\{Y_t\}$ only through $Y_0$. The process $X$ is sometimes called the hidden Markov process (HMP) and the pair $(Y,X)$ is referred to as hidden Markov model (HMM). The name is motivated by the assumption that the process $Y$ (sometimes called as regime) is non-observable. The distributions $P_i := P(X_1 = i | Y_1)$ are called emission distributions. We shall assume that the emission distributions are defined on a measurable space $(X,\mathcal{B})$, where $X$ is usually $\mathbb{R}^d$ and $\mathcal{B}$ is the Borel $\sigma$-algebra. Without loss of generality, we shall assume that the measures $P_i$ have densities $f_i$ with respect to some reference measure $\mu$. Since our study is mainly motivated by statistical learning, we would like to be consistent with the notation used there (Ephraim and Merhav, 2002).

HMM’s are widely used in various fields of applications, including speech recognition (Rabiner, 1989, Jelinek, 2001), bioinformatics (Koski, 2001; Durbin et al., 1998), language processing, image analysis (Li et al., 2000) and many others. For general overview about HMM’s, we refer to textbook (Cappé et al. 2005) and overview paper (Ephraim and Merhav, 2002).

The central objects of the present papers are the smoothing probabilities $P(Y_i = s|X_{-i}, \ldots, X_0)$, where $i, z, n \in \mathbb{Z}$ and $s \in S$. They are important tools for making the inferences about the regime at time $t$. By Levy’s martingale convergence theorem, it immediately follows that as $n \to \infty$,

$$P(Y_t = s|X_{-t}, \ldots, X_0) \to P(Y_t = s|X_{-t}, \ldots, X_{-1}) =: P(Y_t = s|X^\infty_{-t}), \quad \text{a.s.}$$

Let $P(Y_t \in s|X^\infty_{-t})$ denote the $K$-dimensional vector of probabilities from the right side of (1). By martingale convergence theorem, again, as $z \to -\infty$

$$P(Y_t = s|X_{-z-1}, \ldots, X_{-1}, X_0, X_1, \ldots) \to P(Y_t = s|X^\infty_{-z-1}) =: P(Y_t = s|X^\infty_{-z}), \quad \text{a.s.}$$

The double-sided smoothing process $\{P(Y_t \in s|X^\infty_{-t})\}_{t=-\infty}^{\infty}$ is stationary and ergodic, hence for this process the ergodic theorems hold. To be able to use these ergodic theorems for establishing the limit theorems in terms of smoothing...
probabilities \(P(Y_i = s | X_i, \ldots, X_n)\), it is necessary to approximate it with double-sided smoothing process. This approach is, among others, used in (Bickel et al., 1998). In other words, we are interested in bounding the difference \(\|P(Y_i \in \{X_i\}) - P(Y_i \in \{X_{\infty}^n\})\|\), where \(\| \cdot \|\) stands for total variation distance. Our first main result, Corollary 2.1 states that under so-called cluster assumption \(A\), there exists a bounded random variable \(C\) and a constant \(\rho_0 \in (0, 1)\) such that for every \(z_2, z_1, t, n\) such that \(z_2 \leq z_1 \leq 1 \leq t \leq n\),

\[
\|P(Y_i \in \{X_i\}) - P(Y_i \in \{X_{\infty}^n\})\| \leq C_1 \rho_1 t^{-1}. \quad \text{a.s.}
\]

Similar results can be found in the literature for the special case, where transition matrix has all positive entries or the emission densities \(f_i\) are all positive (Gland and Mevel, 2000; Gerencser and Molnar-Saska, 2002; Cappé et al., 2005). Both conditions are restrictive, and the assumption \(A\) relaxes them. We go one more step further, considering the approximation of smoothing probabilities with two-sided limits. Our second main result, Theorem 3.1 states that under \(A\), for every \(z \leq 1 \leq t \leq k \leq n\)

\[
\|P(Y_i \in \{X_i\}) - P(Y_i \in \{X_{\infty}^n\})\| \leq C_1 \rho_1 t^{-1} + C_1^p \rho_1^k \quad \text{a.s.},
\]

where \(\rho_0 \in (0, 1)\) is a fixed constant, \(C_1\) is a finite random variable as in the previous bound and \((C_1^p)\) is a finite ergodic process. Of course, without the ergodic property, the existence of \((C_1^p)\) would be trivial, however, as shown in the proof of Theorem 3.1 the ergodic property makes the bound useful in applications. The condition \(A\) was introduced in (Lember and Koloydenko, 2010, 2008), where under the conditions slightly stronger than \(A\) the existence of infinite Viterbi alignment was shown. The technique used in these papers differs heavily rom the one in the present paper, yet the same assumption appears. This implies that \(A\) is indeed essential for HMM’s.

Our motivation in studying the limit theorems of smoothing processes comes from the segmentation theory. Generally speaking, the segmentation problem consists of estimating the unobserved realization of underlying Markov chain \(Y_1, \ldots, Y_n\) given the \(n\) observations from from HMP \(X_1, \ldots, X_n\): \(x^0 := \{x_1, \ldots, x_n\}\). Formally, we are looking for a mapping \(g : \{X^n\} \to S^n\) called classifier that maps every sequence of observations into a state sequence, see (Koloydenko and Lember, 2010; Kuljus and Lember, 2010) for details. For finding the best \(g\), it is natural to associate to every state sequence \(s^0 \in \{X^n\}\) a measure of goodness \(R(s^0|x^0)\), referred to as the risk of \(s^0\). The solution of the segmentation problem is then the state sequence with minimal risk. In the framework of pattern recognition theory, the risk is specified via loss-function \(L : S^n \times S^n \to [0, \infty)\), where \(L(a^0, b^0)\) measures the loss when the actual state sequence is \(a^0\) and the prognosis is \(b^0\). For any state sequence \(s^0 \in S^n\), the risk is then

\[
R(s^0|x^0) := E[L(Y^n, s^0)|X^n = x^0] = \sum_{a^0 \in S^n} L(a^0, s^0)P(Y^n = a^0|X^n = x^0). \quad (2)
\]

In this paper, we consider the case when the loss function is given as

\[
L(a^0, b^0) = \frac{1}{n} \sum_{i=1}^n l(a_i, b_i), \quad (3)
\]

where \(l(a_i, b_i) \geq 0\) is the loss of classifying the \(i\)-th symbol \(a_i\) as \(b_i\). Typically, for every state \(s\), \(l(s, s) = 0\). Most frequently, \(l(s, s') = I_{|s-s'|} \) and then the risk \(R(s^0|x^0)\) just counts the expected number of misclassified symbols. Given a classifier \(g\), the quantity \(R(g, x^0) := R(g(x^0)|x^0)\) measures the goodness of it when applied to the observations \(x^0\). When \(g\) is optimal in the sense of risk, then \(R(g, x) = \min_{s} R(s^0|x^0) =: R(x^0)\). We are interested in the random variables \(R(g, X^n)\). When \(g\) is maximum likelihood classifier – so called Viterbi alignment – and HMM satisfies \(A\), then (under an additional mild assumption), it can be shown that there exists a constant \(R\), such that \(R(g, X^n) \to R\) a.s. (Caliebe, 2006; Lember and Koloydenko, 2010). In this paper, we show that under \(A\), the similar results holds for optimal alignment: there exists a constant \(R\) such that \(R(X^n) \to R\) a.s.. Those numbers (clearly \(R \geq R\)) depend only on the model and they measure the asymptotic goodness of the segmentation. If \(l(s, s') = I_{|s-s'|}\), then \(R\) and \(R\) are the asymptotic symbol-by-symbol misclassification rates when Viterbi alignment or the best alignment (in given sense) are used in segmentation, respectively.
2. Approximation of the smoothing probabilities

2.1. Preliminaries

Throughout the paper, let \( x^n_u \) where \( u, v \in \mathbb{Z}, u \leq v \) be a realization of \( X_u, \ldots, X_v \). We refer to \( x^n_u \) as the observations. When \( u = 1 \), then it is omitted from the notation, i.e. \( x^1 = x^n_u \). Let \( p(x^n_u) \) stand for the likelihood of the observations \( x^n_u \). For every \( u \leq t \leq v \) and \( s \in S \), we also define the forward and backward variables \( \alpha(x^n_u, s) \) and \( \beta(x^n_{v+1}|s) \) as follows

\[
\alpha(x^n, s) := p(x^n_u|Y = s)P(Y = s), \quad \beta(x^n_{v+1}|s) := \begin{cases} 1, & \text{if } t = v; \\ p(x^n_{v+1}|Y = s), & \text{if } t < v. \end{cases}
\]

Here \( p(x^n_u|Y = s) \) and \( p(x^n_{v+1}|Y = s) \) are conditional densities (see also (1)). The backward variables can be calculated recursively (backward recursion):

\[
\beta(x^n_{u+1}|s) = \sum_{i \in S} P(s, i)f_i(x_{u+1})\beta(x^n_{u+2}|i).
\]

For every \( t \in \mathbb{Z} \), we shall denote by \( \pi_t(x^n_u) \) the \( K \)-dimensional vector of conditional probabilities \( P(Y_t \in \cdot |X^n_u = x^n_u) \). Our first goal is to bound the difference \( \pi_t(x^n_u) - \pi_t(x^n_{u-1}) \), where \( z_2 \leq z_1 \leq 1 \leq t \leq n \). For that, we shall follow the approach in (Cappé et al., 2005). It bases on the observation that given the observations \( x^n_t \), the underlying chain \( Y_1, \ldots, Y_n \) is a conditionally inhomogeneous MC, i.e. for every \( z \leq k < n \) and \( j \in S \)

\[
P(Y_{k+1} = j|Y_k = j, X^n_u = x^n_u) = P(Y_{k+1} = j|Y_k = y_k, X^n_u = x^n_u) =: F_k(y_k, j),
\]

where for every \( i \in S \), \( F(i, j) \) is called the forward smoothing probability (Cappé et al., 2005, Prop. 3.3.2), also (Ephraim and Merhav, 2002 (5.2.1)). It is known (Cappé et al., 2005, 5.21), (Ephraim and Merhav, 2002, (3.30)) that if \( \beta(x^n_{u+1}|l) > 0 \), then

\[
F_k(i, j) = \frac{P(i, j)f_i(x_{k+1})\beta(x^n_{k+1}|j)}{\beta(x^n_{k+1}|i)}.
\]

When \( \beta(x^n_{k+1}|l) = 0 \), we define \( F_k(i, j) = 0 \) \( \forall j \in S \). Note that the matrix \( F_k \) depends on the observations \( x^n_{k+1} \), only. This dependence is sometimes denoted by \( F_k(x^n_{k+1}) \). With the matrices \( F_k \), for every \( t \) such that \( z \leq k \leq n \), it holds (e.g. Cappé et al., 2005 (4.30))

\[
\pi_t(x^n_u) = \pi_t(x^n_{u-1})\left( \prod_{i = z}^{t-1} F_i(x^n_{i+1}) \right),
\]

where \( \cdot \) stands for transposition. For \( n \geq t \geq 1 \geq z_1 \geq z_2 \), thus

\[
(\pi_t(x^n_{z_1}) - \pi_t(x^n_{z_2})) = (\pi_t(x^n_{z_1}) - \pi_t(x^n_{z_2}))\left( \prod_{i = z_1}^{t-1} F_i(x^n_{i+1}) \right).
\]

Let \( \pi_1 \) and \( \pi_2 \) be two probability measures on \( S \). If \( A \) is a transition matrix on \( S \), then \( A^i \pi_1 \), \( (i = 1, 2) \) is a vector that corresponds to a probability measure. We are interested in total variation distance between the measures \( A^i \pi_1 \) and \( A^i \pi_2 \). The approach in this paper uses the fact that the difference between measures can be bounded as follows (Cappé et al., 2005, Cor. 4.3.9)

\[
\|A^i \pi_1 - A^i \pi_2\| = \|A^i(\pi_1 - \pi_2)\| \leq \|\pi_1 - \pi_2\|\delta(A),
\]

where \( \delta(A) \) is Dobrushin coefficient of \( A \) defined as follows

\[
\delta(A) := \frac{1}{2} \sup_{i, j \in S} ||A(i, \cdot) - A(j, \cdot)||.
\]

Here, \( A(i, \cdot) \) is the \( i \)-th row of the matrix. Hence, applying (7) to (6), we get (Cappé et al., 2005, Prop. 4.3.20)

\[
\|\pi_t(x^n_{z_1}) - \pi_t(x^n_{z_2})\| \leq \|\pi_t(x^n_{z_1}) - \pi_t(x^n_{z_2})\|\left( \prod_{i = z_1}^{t-1} F_i(x^n_{i+1}) \right) \leq 2\delta\left( \prod_{i = z_1}^{t-1} F_i(x^n_{i+1}) \right).
\]
Another useful fact is that for two transition matrices $A, B$, it holds (see, e.g. \cite{Cappé2005}) that $\delta(AB) \leq \delta(A)\delta(B)$, hence, the right hand side of (8) can be further bounded above with $2 \prod_{k=1}^{n-1} \delta(F_i[x_{k+1}^n])$. The Dobrushin coefficient of $A$ can be estimated above using so-called Doeblin condition: If there exists $\epsilon > 0$ and a probability measure $\mu = (\lambda_1, \ldots, \lambda_K)$ on $S$ such that
\begin{equation}
A(i, j) \geq \epsilon \lambda_j, \quad \forall i, j \in S,
\end{equation}
then $\delta(A) \leq 1 - \epsilon$ \cite{Cappé2005}. Lemma 4.3.13. This condition holds, for example, if all entries of $A$ are positive. If $F_i$ satisfies Doeblin conditions, then the right hand side converges to zero exponentially with $t$.

2.2. Cluster-assumption

Recall that $f_i$ are the densities of $P(X_t \in \{Y_t = i\}$ with respect to some reference measure $\mu$ on ($X, \mathcal{B}$). For each $i \in S$, let $G_i := \{x \in X : f_i(x) > 0\}$. We can subset $C \subset S$ a cluster if the following conditions are satisfied:
\[ \min_{j \in C} P_j(\cap_{i \in C} G_i) > 0, \quad \text{and} \quad \max_{j \in C} P_j(\cap_{i \in C} G_i) = 0. \]

Hence, a cluster is a maximal subset of states such that $G_C = \cap_{i \in C} G_i$, the intersection of the supports of the corresponding emission distributions, is ‘detectable’. Distinct clusters need not be disjoint and a cluster can consist of a single state. In this latter case such a state is not hidden, since it is exposed by any observation it emits. When $K = 2$, then $S$ is the only cluster possible, since otherwise the underlying Markov chain would cease to be hidden. Let $C$ be a cluster. The existence of $C$ implies the existence of a set $X_C \subset \cap_{i \in C} G_i$ and $\epsilon > 0, K < \infty$ such that $\mu(X_C) > 0$, and $\forall x \in X_C$, the following statements hold: (i) $\epsilon < \min_{i \in C} f_i(x)$; (ii) $\max_{i \in C} f_i(x) < K$; (iii) $\max_{j \in C} f_j(x) = 0$. For proof, see \cite{Lember2010}.

**Assumption A**: There exists a cluster $C \subset S$ such that the sub-stochastic matrix $R = (P(i, j))_{i,j \in C}$ is primitive (i.e. there is a positive integer $r$ such that the $r$th power $R$ is strictly positive).

Clearly assumption A is satisfied, if the matrix $P$ has all positive elements. Since any irreducible aperiodic matrix is primitive, the assumption A is also satisfied, if the the densities $f_i$ satisfy the following condition: For every $x \in X$, $\min_{i \in C} f_i(x) > 0$, i.e. for all $i \in S, G_i = X_C$. Thus A is more general than the strong mixing condition \cite{Cappé2005} Assumption 4.3.21 and \cite{Cappé2005} Assumption 4.3.29. For more general discussion about A, see \cite{Lember2010}.

In the following, we assume A. Let $C$ be the corresponding cluster, and let $X_C$ be the corresponding set.

**Proposition 2.1.** Let $x_{k+1}^n$ be such that $p(x_{k+1}^n) > 0$ and $x_{k+1}^n \in X_C^n$. Then
\begin{equation}
\delta(\prod_{i=1}^{k+1} F_i[x_{i+1}^n]) \leq 1 - \epsilon \min_{i,j} R(i, j) \left( \frac{\epsilon}{K} \right)^r =: \rho < 1. \tag{10}
\end{equation}

**Proof.** Let $A := \prod_{i=0}^{r-1} F_{k+i}$. Using backward recursion, it follows that for every $r \geq 1$
\begin{align*}
A(i, j) & = \frac{\sum_{i_1 \cdots i_r} \cdots \sum_{i_1 \cdots i_r} P(i, i_1)f_{i_1}(x_{k+1}) \cdots f_{i_r}(x_{k+r})P(i_{r-1}, j)f_{j}(x_{k+r})}{\sum_{i_1 \cdots i_r} \cdots \sum_{i_1 \cdots i_r} P(i, i_1)f_{i_1}(x_{k+1}) \cdots f_{i_r}(x_{k+r})P(i_{r-1}, j)f_{j}(x_{k+r})}\beta(x_{k+r+1}^n | j) \\
& \geq \frac{\epsilon}{K} \frac{\sum_{i_1 \cdots i_r} \cdots \sum_{i_1 \cdots i_r} P(i, i_1)f_{i_1}(x_{k+1}) \cdots P(i_{r-1}, j)f_{j}(x_{k+r})\beta(x_{k+r+1}^n | j)}{\sum_{i_1 \cdots i_r} \cdots \sum_{i_1 \cdots i_r} P(i, i_1)f_{i_1}(x_{k+1}) \cdots P(i_{r-1}, j)f_{j}(x_{k+r})\beta(x_{k+r+1}^n | j)}
\end{align*}
Since $x_{k+1}^n \in X_C^n$, then by (iii), (ii) and (i), thus for every $i, j \in S$
\begin{align*}
A(i, j) & = \frac{\sum_{i_1 \cdots i_r} \cdots \sum_{i_1 \cdots i_r} P(i, i_1)f_{i_1}(x_{k+1}) \cdots P(i_{r-1}, j)f_{j}(x_{k+r})\beta(x_{k+r+1}^n | j)}{\sum_{i_1 \cdots i_r} \cdots \sum_{i_1 \cdots i_r} P(i, i_1)f_{i_1}(x_{k+1}) \cdots P(i_{r-1}, j)f_{j}(x_{k+r})\beta(x_{k+r+1}^n | j)}
\geq \frac{\epsilon}{K} \frac{\sum_{i_1 \cdots i_r} \cdots \sum_{i_1 \cdots i_r} P(i, i_1)f_{i_1}(x_{k+1}) \cdots P(i_{r-1}, j)f_{j}(x_{k+r})\beta(x_{k+r+1}^n | j)}{\sum_{i_1 \cdots i_r} \cdots \sum_{i_1 \cdots i_r} P(i, i_1)f_{i_1}(x_{k+1}) \cdots P(i_{r-1}, j)f_{j}(x_{k+r})\beta(x_{k+r+1}^n | j)} \\
& \geq \frac{\epsilon}{K} \frac{\sum_{i_1 \cdots i_r} \cdots \sum_{i_1 \cdots i_r} P(i, i_1)f_{i_1}(x_{k+1}) \cdots P(i_{r-1}, j)f_{j}(x_{k+r})\beta(x_{k+r+1}^n | j)}{\sum_{i_1 \cdots i_r} \cdots \sum_{i_1 \cdots i_r} P(i, i_1)f_{i_1}(x_{k+1}) \cdots P(i_{r-1}, j)f_{j}(x_{k+r})\beta(x_{k+r+1}^n | j)}
\end{align*}
where

$$\epsilon = \frac{\min_{i,j} R(i,j)}{\max_{i,j} R(i,j)}, \quad \lambda_j := \frac{\beta(x^{(i+1)}_j)}{\sum_j \beta(x^{(i+1)}_j)}.$$ 

Since

$$p(x^{(i+1)}_j) = \sum_j \alpha(x^{(i+1)}_j, j) \beta(x^{(i+1)}_j) > 0,$$

there must be a $j \in S$ such that $\beta(x^{(i+1)}_j) > 0$. So $(\lambda_j)_{j \in S}$ is a probability measure and Doeblin condition holds. \(\square\)

**Lemma 2.1.** Let $x^n_t$ be the sequence of observations with positive likelihood, i.e. $p(x^n_t) > 0$. Then, for every $t$ such that $z_2 \leq z_1 \leq t \leq n$,

$$\|\pi_1[x^n_t] - \pi_1[x^n_s]\| \leq 2\rho^{(k)(t)},$$

(11)

where $\rho \in (0, 1)$ is as in (10) and

$$f(t) := \frac{t - 2}{r}, \quad \kappa(x^n_t) := \sum_{j=0}^{j(t)-1} I_{X^n_j} x^n_{(k+1)r+1}.$$

**Proof.** Recall that for two transition matrices $A, B, \delta(AB) \leq \delta(A)\delta(B)$, so

$$\delta\left( \prod_{j=1}^{t-1} F_i \right) = \delta\left( \prod_{j=0}^{t-1} \left( \prod_{i=0}^{(k+1)r} F_i \right) \right) \leq \prod_{j=0}^{t-1} \delta\left( F_i \right) = \prod_{j=0}^{t-1} \delta(A_u),$$

where $A_u := \prod_{i=0}^{(k+1)r} F_i x^n_{(k+1)r+1}$. From Proposition 2.1, with $k = ur + 1$,

$$\delta(A_u) \leq \begin{cases} \rho, & \text{if } x^n_{(k+1)r+1} \in X^n_r; \\ 1, & \text{else}. \end{cases}$$

From (3), it holds

$$\|\pi_1[x^n_t] - \pi_1[x^n_s]\| \leq 2\delta\left( \prod_{j=1}^{t-1} F_i x^n_{(k+1)r+1} \right) \leq 2 \prod_{j=0}^{t-1} \delta(A_u) \leq 2\rho^{(k)(t)}.$$

Let $s_1 \in C$. By irreducibility and cluster assumption, there is a path $s_1, \ldots, s_{r+1}$ such that $s_1 \in C$ and $P(Y_1 = s_1, \ldots, Y_{r+1} = s_{r+1}) > 0$. By (i), for any $s_2, \ldots, s_{r+1} \in C$, it holds $P(X^n_{r+1} \in X^n_s | Y_2 = s_2, \ldots, Y_{r+1} = s_{r+1}) > 0$ implying that $P(X^n_{r+1} \in X^n_s) > 0$. By stationarity of $X$, for every $k \geq 0$, it holds $P(X^n_{k+1} \in X^n_s) = P(X^n_{k+1} \in X^n_s) := p_r > 0$. The process $(X^n_t)_{t \geq 1}$ is ergodic, so

$$\lim_{t \to \infty} \frac{\kappa(X^n_t)}{t} = \lim_{t \to \infty} \frac{1}{r} \frac{\kappa(X^n_t)}{f(t)} = \frac{p_r}{r} > 0, \quad \text{a.s..}$$

(12)

**Corollary 2.1.** Assume A. Then, there exists a non-negative finite random variable $C_1$ as well as constant $\rho_1 \in (0, 1)$ such that for every $z, t, n$ such that $z_2 \leq z_1 \leq 1 \leq t \leq n$,

$$\|P(Y_t \in \cdot | X^n_z) - P(Y_t \in \cdot | X^n_{z_1})\| \leq C_1 \rho_1^{t-1}, \quad \text{a.s..}$$

(13)

**Proof.** The right hand side of (11) does not depend on $n$ (as soon as it is bigger than $t$), hence from Lemma 2.1

$$\sup_{n \geq t} \|P(Y_t \in \cdot | X^n_z) - P(Y_t \in \cdot | X^n_{z_1})\| \leq 2\rho^{(k)}(t), \quad \text{a.s..}$$

Thus, if $t \to \infty$ then by (12), it holds

$$\limsup_{t \to \infty} \frac{1}{t} \log \left( \sup_{n \geq t} \|P(Y_t \in \cdot | X^n_z) - P(Y_t \in \cdot | X^n_{z_1})\| \right) \leq \frac{p_r}{r} \log \rho_1, \quad \text{a.s..}$$

(14)
Let \( \rho \) be such that \( \frac{\rho}{\kappa} \log \rho < \rho < 0 \). Let
\[
T(\omega) := \max \{ t \geq 1 : \frac{\log 2 + \kappa(X') \log \rho}{t} > \rho \}. \tag{15}
\]
From (14), it follows that for almost every \( \omega \), \( T(\omega) < \infty \) and for \( t > T(\omega) \),
\[
\log \left( \sup_{n \geq t} \| P(Y_t \in \cdot | X^n) - P(Y_t \in \cdot | X^n_z) \| \right) \leq t_0
\]
and, hence, for \( t > T(\omega) \) and \( n \geq t \),
\[
\| P(Y_t \in \cdot | X^n) - P(Y_t \in \cdot | X^n_z) \| \leq e^{t_0} = (\rho_1)' t_0,
\]
where \( \rho_1 := e^{\rho} \). The inequality (13) holds with \( C_1 := 2 \rho^{-T+1} \).

The inequality (14) is similar to Theorem 2.2 in [Gland and Mevel, 2000]. The forgetting equation in form (13) is used in [Gerencser and Molnar-Saska, 2002].

**Corollary 2.2.** There exist a constant \( \rho_1 \in (0, 1) \) and an ergodic process \( \{ C_z \}_{z=1}^\infty \) so that for any \( z_2 \leq z_1 \leq z \leq t \leq n \)
\[
\| P(Y_t \in \cdot | X^n) - P(Y_t \in \cdot | X^n_z) \| \leq C_1 \rho_1^{t-z}, \text{ a.s.} \tag{16}
\]

**Proof.** The existence of \( C_z \) follows exactly as in case \( z = 1 \). The ergodicity of \( \{ C_z \} \) follows from the fact that the random variables \( \{ C_z \} \) are stationary coding of the ergodic process \( X \).

**Theorem 2.1.** Assume A. Then there exist a constant \( \rho_0 \in (0, 1) \) and an infinite ergodic process \( \{ C'_z \}_{z=1}^\infty \) so that for every \( z \leq 1 \leq t \leq k \leq n \)
\[
\| P(Y_t \in \cdot | X^n) - P(Y_t \in \cdot | X^n_{z}) \| \leq C_1 \rho_0^{t-1} + C_2 \rho_0^{k-z}, \text{ a.s.}, \tag{17}
\]
where \( C_1 \) is a finite random variable as in Corollary 2.2.

**Proof.** We reverse the time by defining \( Y'_k = Y_{-k}, X'_k = X_{-k} \). Thus, \( P(Y'_t \in \cdot | X'_t) = P(Y_t \in \cdot | X^n_t) \). It is easy to see that when HMM \( (Y, X) \) satisfies A, then so does the reversed HMM \( (Y', X') \). From Corollary 2.2 it follows that there exists \( \rho_2 \in (0, 1) \) and ergodic process \( \{ C'_{-z} \} \) so that for any \( n_2 \leq n_1 \leq k \leq -t \leq -1 \leq -z \)
\[
\| P(Y_t \in \cdot | X^n_{-z}) - P(Y_t \in \cdot | X^n_{-z}) \| = \| P(Y'_t \in \cdot | X^n_{-z}) - P(Y'_t \in \cdot | X^n_{-z}) \| \leq C'_1 \rho_0^{t-1} + C'_2 \rho_0^{k-1} \text{ a.s.},
\]
where \( C'_1 := C'_{-z} \). The right side does not depend on \( n_1, n_2 \) and \( z \). Hence, letting \( z \to -\infty \) and using Levy martingale convergence theorem, for every \( 1 \leq t \leq k \leq n_1 \leq n_2 \)
\[
\| P(Y_t \in \cdot | X^n_{-z}) - P(Y_t \in \cdot | X^n_{-z}) \| \leq C_1 \rho_0^{k-1}, \text{ a.s.}. \tag{18}
\]
Letting now \( n_1 \to \infty \) and using Levy martingale convergence theorem again, for every \( 1 \leq t \leq k \leq n \)
\[
\| P(Y_t \in \cdot | X^n_{-z}) - P(Y_t \in \cdot | X^n_{-z}) \| \leq C_1 \rho_0^{k-1}, \text{ a.s.}. \tag{19}
\]
Applying the same theorem to (13), with \( z_2 \to -\infty \) and \( z = z_1 \), we get that for every \( z \leq 1 \leq t \leq n \),
\[
\| P(Y_t \in \cdot | X^n_{-z}) - P(Y_t \in \cdot | X^n_{-z}) \| \leq C_1 \rho_0^{t-1} \text{ a.s.} \tag{19}
\]
Hence, with, \( \rho_0 = \max(\rho_1, \rho_2) \), from the inequalities (18) and (19), the inequality (17) follows.
3. Convergence of risks

Recall that \( l : S \times S \to [0, \infty) \) is the pointwise loss. Let, for any \( 1 \leq t \leq n \), and \( s \in S \),
\[
R_t(s|x^n) = E[l(Y_t, s_i)|X^n = x^n] = \sum_{a \in S} l(a, s)P(Y_t = a|X^n = x^n).
\]
Thus, \( R_t(x^n) \) is the conditional risk of classifying \( Y_t = s \) given the observations \( x^n \). The risk of the whole state sequence \( x^n \) as defined in (6) with \( L \) as in (3) is easily seen to be
\[
R(s|x^n) = \frac{1}{n} \sum_{i=1}^{n} R_t(s_i|x^n).
\]
Let for every \( t \in \mathbb{Z} \) and \( s \in S \),
\[
R_t(s|X^n_\infty) := E[l(Y_t, s_i)|X^n_\infty] = \sum_{a \in S} l(a, s)P(Y_t = a|X^n_\infty).
\]
For \( t \geq 1 \), thus
\[
|R_t(s|X^n_\infty) - R_t(s|X^n_1)| \leq l(s)||P(Y_t \in \cdot|X^n_1) - P(Y_t \in \cdot|X^n_\infty)||,
\] (20)
where \( l(s) = \max_a l(a, s) \). Finally, recall that \( R(x^n) := \min_s R(s|x^n) \).

**Theorem 3.1.** Suppose A holds. Then there exists a constant \( R \) such that \( R(X^n) \to R \) a.s. and in \( L_1 \).

**Proof.** The process \( X \) is ergodic, so for a constant \( R \),
\[
\frac{1}{n} \sum_{i=1}^{n} \min_s R_t(s|X^n_\infty) \to R, \quad \text{a.s. and in } L_1.
\] (21)
Let \( M < \infty \) be such that \( P(C'_n \leq M) = q > 0 \). Let, for every \( n, k(n) = \max[k \leq n : C'_k \leq M] \). Since the process \( C' \) is ergodic, in the process \( n \to \infty, k(n) \to \infty \), a.s. From (20), it follows, that with \( A := \max_n l(a, s) \),
\[
| \min_s R_t(s|X^n_\infty) - \min_s R_t(s|X^n_1) | \leq A ||P(Y_t \in \cdot|X^n_1) - P(Y_t \in \cdot|X^n_\infty)||.
\]
Hence
\[
|R(X^n)| \leq \frac{1}{n} \sum_{i=1}^{n} \min_s R_t(s|X^n_\infty) \leq A \frac{1}{n} \sum_{t=1}^{n} \sum_{s} ||P(Y_t \in \cdot|X^n_1) - P(Y_t \in \cdot|X^n_\infty)|| \leq A \frac{1}{n} \sum_{t=1}^{k(n)} \sum_{s} ||P(Y_t \in \cdot|X^n_1) - P(Y_t \in \cdot|X^n_\infty)|| + A \frac{2(n-k(n))}{n}.
\] (22)
By inequality (17), for every \( 1 \leq t \leq k(n) \)
\[
\sum_{i=1}^{k(n)} ||P(Y_t \in \cdot|X^n_1) - P(Y_t \in \cdot|X^n_\infty)|| \leq C_1 \sum_{i=1}^{k(n)} \rho_i^{(n-1)} + M \sum_{i=1}^{k(n)} \rho_i^{(n-1)} \leq (C_1 + M) \sum_{n=0}^{\infty} \rho_i < \infty, \quad \text{a.s.}
\]
Let \( \tau_1 := \min[i \geq 0 : C'_i \leq M] \), \( \tau_j := \min[i > \tau_{j-1} : C'_i \leq M] \). Since \( \{C'_i\} \) is ergodic, the random variables \( T_j = \tau_{j+1} - \tau_j, j = 1, 2, \ldots \) are identically distributed. By Kac’s return time theorem, \( ET_j = q^{-1} \). Finally, denote
\[
j(n) = \max[j : \tau_j \leq n].
\]
Thus \( k(n) = j(n) \) and \( n - k(n) \leq T_{j(n)} \). Since \( T_{j(n)} \) is a.s. finite, clearly \( j(n), k(n) \to \infty \) as \( n \) grows. From the finite expectation of \( ET_{j(n)} \), it follows that
\[
\frac{T_{j(n)}}{j(n)} \to 0, \quad \text{a.s.,}
\]
implies that
\[
\frac{n - k(n)}{n} \leq \frac{T_{j(n)}}{j(n)} \to 0, \quad \text{a.s.}
\] (23)
Hence, the right hand side of (23) goes to 0, a.s. and from (21), it now follows that \( R(X^n) \to R \) a.s. Risks are nonnegative, so the convergence in \( L_1 \) follows from Sheffe’s lemma. \( \square \)
Given $l$, the constant $R$ — asymptotic risk — depends on the model, only. It measures the average loss of classifying one symbol using the optimal classifier. For example, if $l$ is symmetric, then the optimal classifier (in the sense of misclassification error) makes in average about $Rn$ classification errors. Clearly this is the lower bound: no other classifier does better. The constant $R$ might be hard to determine theoretically, but Theorem 3.1 guarantees that it can be approximated by simulations.

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