2-graded polynomial identities for the Jordan algebra of the symmetric matrices of order two

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Abstract

The Jordan algebra of the symmetric matrices of order two over a field $K$ has two natural gradings by $\mathbb{Z}_2$, the cyclic group of order 2. We describe the graded polynomial identities for these two gradings when the base field is infinite and of characteristic different from 2. We exhibit bases for these identities in each of the two cases. In one of the cases we perform a series of computations in order to reduce the problem to dealing with associators while in the other case one employs methods and results from Invariant theory. Moreover we extend the latter grading to a $\mathbb{Z}_2$-grading on $B_n$, the Jordan algebra of a symmetric bilinear form in a vector space of dimension $n$ ($n = 1, 2, \ldots, \infty$). We call this grading the scalar one since its even part consists only of the scalars. As a by-product we obtain finite bases of the $\mathbb{Z}_2$-graded identities for $B_n$. In fact the last result describes the weak Jordan polynomial identities for the pair $(B_n, V_n)$.

Keywords: Graded identities, Jordan identities, Finite basis of identities, Weak identities, Weak Jordan identities

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Introduction

Polynomial identities in simple associative algebras over a field have always been of significant interest to ring theorists. Most of the research in this area has been done under the assumption of the base field being infinite, and even stronger, of characteristic 0. In spite of the extensive research in this area little is known about the concrete form of the identities satisfied by such algebras. In fact (assuming the base field $K$ infinite) the identities of the matrix algebras $M_n(K)$ are known only for $n = 2$ and char $K \neq 2$ (see for example [10], [3], [13]). One may study other types of identities. Thus the trace identities of the matrix algebra $M_n(K)$, char $K = 0$, were described independently by Procesi [15], and by Razmyslov (see for example [16]). Here we recall that of great importance for Ring theory have also been the methods developed in the course of studying the trace identities of $M_n(K)$. Also weak identities were introduced and used successfully by Razmyslov in describing the identities of associative and Lie algebras, see for an account and various applications [16].

Later on in the 80-ies, a powerful theory developed by Kemer provided a description of the ideals of identities (also called T-ideals) in the free associative algebra over a field of characteristic 0. One finds details concerning Kemer’s theory in [10]. One of the principal tools in that theory was the usage of $\mathbb{Z}_2$-graded algebras and their graded identities. Clearly this provided a strong impulse to the study of graded identities in associative algebras. The interested reader can look at [1] and at [14] and their references for some of the important results concerning gradings and graded identities.

The latter paper dealt with graded identities in the Lie algebra $sl_2(K)$, and it is one of the few about the topic. It is somewhat surprising that graded identities in Jordan algebras have not been studied in detail yet except for the paper [18] where the author described the $\mathbb{Z}_2$-graded identities of the Jordan superalgebra of a bilinear form.

In this paper we study the $\mathbb{Z}_2$-graded identities for the Jordan algebra $B_2$ of the symmetric matrices of order two over an infinite field $K$ of characteristic different from 2. Up to a graded isomorphism there are two nontrivial $\mathbb{Z}_2$-gradings on $B_2$. We exhibit finite bases (that is generators) of the corresponding ideals of graded identities. In one of the cases our methods are more general than needed and we are able to describe a basis of the weak Jordan identities of the pairs $(B_n, V_n)$ and $(B, V)$ where $B_n$ and $B$ stand for the Jordan algebra of a nondegenerate symmetric bilinear form on the vector spaces $V_n$ and $V$, respectively, dim $V_n = n$ and dim $V = \infty$. It is well known that the Jordan algebra of the symmetric $2 \times 2$ matrices is a Jordan algebra of a nondegenerate symmetric bilinear form on a vector space of dimension 2 therefore there is nothing wrong with our notation. Recall that the Jordan algebras $B_n$ and $B$ are simple and special and their associative enveloping algebras are the Clifford algebras $C_n$ and $C$, respectively. The weak (associative) identities of the pairs $(C_n, V_n)$ and $(C, V)$ were described in [12] over an infinite field of characteristic $\neq 2$, see also [3] for the case char $K = 0$. 

2
1 Preliminaries

Throughout $K$ stands for an infinite field of characteristic different from 2; all vector spaces and algebras (not necessarily associative) are considered over $K$. If $A$ is an associative algebra then $\overline{A}$ stands for the vector space of $A$ equipped with the Jordan product $a \circ b = (ab + ba)/2$. It is immediate to see that $\overline{A}$ is a Jordan algebra; the Jordan algebras of this type and their subalgebras are called special, otherwise they are exceptional. Let $V$ be a vector space with a nondegenerate symmetric bilinear form $\langle u, v \rangle$, and let $B = K \oplus V$. One defines a multiplication $\circ$ on $B$ as follows. If $\alpha, \beta \in K$ and $u, v \in V$ then $(\alpha + u) \circ (\beta + v) = (\alpha \beta + \langle u, v \rangle) + (\alpha v + \beta u)$. It is not difficult to check that $B$ is a Jordan algebra. We denote it by $B$ whenever $\dim V = \infty$, and by $B_n$ when $\dim V = n$. In fact the above is an abuse of notation since the algebras $B_n$ and $B$ depend on the form $\langle u, v \rangle$. Clearly equivalent symmetric bilinear forms define isomorphic Jordan algebras, and vice versa. Let us observe that if the field $K$ is algebraically closed then up to an isomorphism there is only one algebra $B_n$. Over an arbitrary field one has to interpret $B_n$, respectively $B$, as a class of Jordan algebras which are not necessarily isomorphic but are classified by the inequivalent nondegenerate symmetric bilinear forms on the corresponding vector space. In order to keep the notation consistent we shall denote the vector space in the latter case by $V_n$, that is $\dim V_n = n$. If $A$ is any algebra one defines the associator of $a, b, c \in A$ as $(a, b, c) = (ab)c - a(bc)$ where $ab$ is the product in $A$. Let $J_2$ be the vector space of the symmetric $2 \times 2$ matrices, it is a subalgebra of the Jordan algebra $M_2(K)^+$.

Since for every two traceless matrices $a, b \in M_2$ one has that $a \circ b$ is a scalar multiple of the unit matrix (and the trace is nondegenerate) one gets $J_2 \in B_2$. In view of the previous remark we consider $B_2$ as the class of the Jordan algebras of a nondegenerate symmetric bilinear form on a vector space $V_2$, $\dim V_2 = 2$.

We fix the basis $I = e_{11} + e_{22}, a = e_{11} - e_{22}, b = e_{12} + e_{21}$ of $J_2$. Here $e_{ij}$ are the usual matrix units. One gets immediately $a^2 = b^2 = I$, $a \circ b = 0$.

If $G$ is a group and $A$ is an algebra then $A$ is $G$-graded if $A = \bigoplus_{g \in G} A_g$, a direct sum of vector subspaces such that $A_g A_h \subseteq A_{gh}$ for all $g, h \in G$. The elements of $A_g$ are homogeneous of degree $g$. If $a \in A_g$, then we shall denote its homogeneous degree by $|a|$. We shall consider only gradings by the additive cyclic group $\mathbb{Z}_2$. So a graded algebra in this paper means $A = A_0 \oplus A_1$ where $A_i A_j \subseteq A_{i+j}$, $i, j = 0, 1$, and the latter sum taken modulo 2. Sometimes, when needed, we shall use upper indices to denote the graded components of $A$. That is when necessary we shall write $A = A^{(0)} \oplus A^{(1)}$ instead of $A = A_0 \oplus A_1$.

Let $X$ be an infinite countable set, $X = \{x_1, x_2, \ldots\}$, and denote by $K(X)$ and by $J(X)$ the free (unitary) associative and the free Jordan algebra freely generated by $X$ over $K$. A polynomial $f = f(x_1, \ldots, x_n) \in K(X)$ is a polynomial identity (a PI or an identity) for the associative algebra $A$ if $f(a_1, \ldots, a_n) = 0$ for every $a_i \in A$. The set $T(A)$ of all identities for $A$ is an ideal in $K(X)$ that is closed under endomorphisms; such ideals are called T-ideals. It is easy to show that every T-ideal is the ideal of identities of certain algebra. In the same manner one defines Jordan identities and T-ideals in $J(X)$. An identity $f$ is a
consequence of the identity \( g \) (or \( f \) follows from \( g \) as an identity) if \( f \) lies in the T-ideal generated by \( g \). Similarly \( f \) and \( g \) are equivalent as identities if each of them follows from the other. A set of identities is called a \textit{basis} of the T-ideal \( I \) if this set generates \( I \) as a T-ideal.

As a general rule finding a basis of the identities of a given algebra may be an extremely difficult (if not hopeless) problem. Here we point out that in the associative case, a celebrated result of Kemer states that every nontrivial T-ideal has a finite basis in characteristic 0 (see \cite{10}). This had been for more than 30 years the famous Specht problem. We also remark that except for the matrix algebras \( M_n(K) \), \( n \leq 2 \), and \( \text{char } K \neq 2 \), for the Grassmann algebra \( E \) and its tensor square \( E \otimes E \) (the latter only when \( \text{char } K = 0 \)) no other "interesting" T-ideal has an explicit finite basis known.

The polynomial identities of the Jordan algebras \( B_n \), and \( B \) were described by Vasilovsky in \cite{17} under some minor restrictions on the characteristic of the base field. Namely Vasilovsky found bases of the corresponding T-ideals. (Recall that Iltyakov in \cite{8} dealt with the case of \( B_n \) in characteristic 0.) The structure of the relatively free algebras \( J(X)/T(B_n) \) and \( J(X)/T(B) \) was given in \cite{3}, \cite{11}.

Let \( A \) be an algebra and \( V \) a subspace such that \( V \) generates \( A \) as an algebra. A polynomial \( f(x_1, \ldots, x_n) \) is a \textit{weak identity} for the pair \( (A, V) \) if \( f(v_1, \ldots, v_n) = 0 \) for all \( v_i \in V \). Depending on the choice of \( A \) and \( V \) one defines rules for consequences of a weak identity. Since we shall deal with a particular situation we do not treat the most general case here but instead we refer the reader to \cite{12} and \cite{6} for further information about weak identities in a general setting.

Let \( A = J \) be a Jordan algebra and \( V \) a subspace that generates \( J \) as an algebra. In this case one speaks of weak Jordan identities. Note that here the weak identities are polynomials in the free Jordan algebra. We denote by \( T(J, V) \) the set of the weak identities for the pair \( (J, V) \) and call it the \textit{weak T-ideal} of that pair. If \( f(x_1, \ldots, x_n) \in J(X) \) then the weak T-ideal \( (f)_w^w \) defined by \( f \) is the ideal of \( J(X) \) generated by all polynomials \( f(g_1, \ldots, g_n), g_i \in J(X) \).

This determines the rule for taking consequences of a given polynomial, or set of polynomials, as weak identities.

We shall need some facts about gradings and graded identities. Let \( X = Y \cup Z \) be a disjoint union of the infinite sets \( Y = \{y_1, y_2, \ldots \} \) and \( Z = \{z_1, z_2, \ldots \} \). We define a \( \mathbb{Z}_2 \)-grading on the free Jordan algebra \( J(X) \) as follows. If \( m \) is a monomial then it is of \( \mathbb{Z}_2 \)-degree 0 (that is an even element) if its total degree in the variables \( Z \) is even; otherwise it is of \( \mathbb{Z}_2 \)-degree 1 that is, an odd element.

Put \( J(X)_i \) the span of all monomials of \( \mathbb{Z}_2 \)-degree \( i \), \( i = 0, 1 \), then \( J(X) = J(X)_0 \oplus J(X)_1 \) is a grading on \( J(X) \). Let \( J \) be a graded Jordan algebra, \( J = J_0 \oplus J_1 \), a polynomial \( f(y_1, \ldots, y_m, z_1, \ldots, z_n) \) is a graded identity for \( J \) if \( f \) vanishes whenever one substitutes the variables \( y_i \) by any elements of \( J_0 \) and the \( z_i \) by any elements of \( J_1 \). Clearly the set \( T_2(J) \) of all graded identities of \( J \) is an ideal that is closed under endomorphisms of \( J \) that respect its grading.

We consider unitary algebras only. It is easy to show that the unit element 1 lies in the even component of the graded algebra. Suppose \( J \in B_n \) is graded,
where $i = \sqrt{-1} \in K$. Thus $K \subseteq J_0$. Let the vector space $J_0$ have a basis consisting of $1$ and $v_1, \ldots, v_k \in V_n$. If $\alpha + v \in J_1, \alpha \in K, v \in V_n$ then $v_i \circ (\alpha + v) \in J_1$ but on the other hand $v_i \circ (\alpha + v) = \alpha v_i + \langle v_i, v \rangle \in J_0$. Hence $\alpha = 0$ and $\langle v_i, v \rangle = 0$. Therefore $J_i$ is a subspace of $V_n$ and moreover $J_1$ is orthogonal to $\text{sp}(v_1, \ldots, v_k)$, the span of $v_1, \ldots, v_k$. The same argument as above transfers verbatim to $B$. In this way we prove the following proposition.

**Proposition 1** Every $\mathbb{Z}_2$-grading on the Jordan algebras $B_n$ and $B$ is defined by a splitting of the vector spaces $V_n$, respectively $V$, as a direct sum of two orthogonal subspaces.

In the paper [2] the authors described all gradings on the Jordan algebras $B_n$. So our Proposition 1 is a particular case of the results in [2]. We give a proof of it here since in our particular case the proof is rather simple and straightforward without relying on the technique developed in [2].

**Corollary 2** Up to a graded isomorphism there are two nontrivial $\mathbb{Z}_2$-gradings on the Jordan algebra $J = J_2$. These are given by $J = J_0 \oplus J_1$ where either $J_0 = \text{sp}(I, a), J_1 = \text{sp}(b)$, or $J_0 = \text{sp}(I), J_1 = \text{sp}(a, b)$.

In the next two sections we shall handle the graded identities in each of the two possibilities for the grading on $J_2$. The reader will observe that the latter grading has as an even part the scalars only. That is why we call the first of the gradings described in the Corollary the nonscalar and the second the scalar grading, respectively.

**Remark 3** The algebra $J_2$ admits nontrivial gradings by groups other than $\mathbb{Z}_2$. The existence of such gradings may depend on the ground field $K$. Let us consider for example $G = \mathbb{Z}_3$, the cyclic group of order 3. Suppose $J_2 = (J_2)_0 \oplus (J_2)_1 \oplus (J_2)_2$ is a nontrivial $G$-grading on $J_2$. One concludes, according to [2, Theorem 1] that $(J_2)_0 = K$ and, as $G$ has no elements of order two, $\dim((J_2)_1) = \dim((J_2)_2) = 1$. Furthermore $(J_2)_1 = Kw_1$ and $(J_2)_2 = Kw_2$ where $w_1$ and $w_2$ are traceless symmetric matrices such that $w_1^2 = w_2^2 = 0$ and $w_1 \circ w_2 = 1$. In other words the span of $w_1$ and $w_2$ is a hyperbolic plane with respect to the bilinear form. Matrices with these properties exist if and only if $i = \sqrt{-1} \in K$. If this is the case one may choose for example

$$w_1 = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \\
w_2 = \frac{1}{2} \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}.$$  

We observe that one may reach the above conclusion by computing directly, without relying on the general result of [2]. It is easy to show that $\dim(J_2)_0 = 1$ and that $(J_2)_1$ and $(J_2)_2$ consist of traceless matrices. Hence $(J_2)_r \circ (J_2)_s \subseteq (J_2)_0 \cap (J_2)_s = 0$ where $\{r, s\} = \{1, 2\}$. Then $w_1$ and $w_2$ are traceless singular square zero matrices. The remaining computations are easy and immediate.
2 The nonscalar grading

In this section we fix the grading \( B_2 = J = J_0 \oplus J_1 \) on the Jordan algebra of the symmetric \( 2 \times 2 \) matrices given by \( J_0 = \text{sp}(I, a), J_1 = \text{sp}(b) \). First we list some elementary facts about the graded identities for this grading. Let \( T = T_2(J) \) be the ideal of the graded identities for \( J \). In this section we shall not use the symbol \( \circ \) for the product in a Jordan algebra, we shall use instead the common symbol \( \cdot \) for the product and, as a rule, we shall simply omit it. Recall that \(|u|\) stands for the \( \mathbb{Z}_2 \)-degree of an element \( a \) while \( \deg a \) is the usual degree of a homogeneous element in the free algebra.

Denote by \( I \) the ideal of graded identities generated by the polynomials

\[
\begin{align*}
    x_1(x_2x_3) & - x_2(x_1x_3) \quad \text{if} \ |x_1| = |x_2| & (1) \\
    (y_1y_2, z_1, z_2) & - (y_1(y_2, z_1, z_2) + y_2(y_1, z_1, z_2) - 2z_1(y_2, y_1, y_2)) & (2) \\
    (y_1y_2, y_3, z_1) & - (y_1(y_2, y_3, z_1) + y_2(y_1, y_3, z_1)) & (3) \\
    (z_1z_2, x_1, x_2) & & (4) \\
    (y_1, y_2, z_1, x, y_3) & - (y_1, y_3, z_1, x, y_2) & (5)
\end{align*}
\]

Here and in what follows the long associators without inner parentheses will be left normed. That is \( (x_1, x_2, x_3, x_4, x_5) \) stands for \( ((x_1, x_2, x_3), x_4, x_5) \), and so on. Recall that if one considers the vector space of a Jordan algebra \( J \) equipped with the trilinear composition \((a, b, c), a, b, c \in J \), then \( J \) becomes a Lie triple system. Passing from Jordan algebras to Lie triple systems one may prove, see for example \([9, \text{pp. 343, 344}]\), that every (long) associator is a linear combination of left normed ones. We call these left normed associators proper ones. Also the letter \( y \), with or without an index, stands for an even variable; \( z \) with or without index for an odd variable, and \( x \) for any variable (even or odd).

**Lemma 4** The graded identities from (1) to (5) hold for the Jordan algebra \( J \). In other words \( I \subseteq T \).

**Proof.** The proof consists of a straightforward (and easy) computation, so we omit it. \( \Box \)

Since we consider algebras over infinite fields then every graded identity is equivalent to a finite collection of multihomogeneous ones (namely its multihomogeneous components). Therefore we shall consider only multihomogeneous identities. Our aim in this section is to prove that in fact \( I = T \).

Denote \( L = J(X)/I \) where \( J(X) \) is the free graded Jordan algebra. We shall work in \( L \); we shall keep the same notation for the images of the variables \( y \) and \( z \) in \( L \). The graded identity \([4]\) implies that the elements \( z_iz_j \) lie in the associative centre of \( L \). Therefore the subalgebra of \( L \) generated by all \( z_iz_j \) is associative.

Since the ideal \( I \) is homogeneous in the grading it follows that the algebra \( L \) is graded, the grading on it being induced by that on \( J(X) \), so \( L = L_0 \oplus L_1 \).

**Proposition 5** The subalgebra \( L_0 \) of \( L \) is associative. Also the subalgebra of \( L \) generated by \( L_1 \) is associative.
Lemma 6

The following polynomials lie in I.

(a) \((x_1, x_2, x_3)\), \(|x_1| = |x_3|\);

(b) \((y_1 z_1, y_2, y_3) - y_1 (z_1, y_2, y_3)\);
Then we obtain

Now we induct once again on

This finishes the case

Moreover according to (7) we do the same in the second

In order to prove the former is a graded identity in

It is clear that (c) follows from the graded identity (5).

Proof. Since \((x_1 x_2 x_3) = (x_1 x_2) x_3 - x_1(x_2 x_3) = (x_3 x_2) x_1 - x_1(x_2 x_3) = 0\) in \(L\), by (1) we obtain (a). Similarly by (1) we have \(((y_1 z_1) y_2) y_3 = ((y_2 z_1) y_3)y_1\)

Thus, by (6) which proves (b). It is clear that (c) follows from the graded identity (5). in (d) it suffices to consider first the case \(\sigma = 1\), the identity permutation, and then apply the graded identity (5). Suppose first \(k = 1\). It follows from (1) that \((y_1 y_2 z_1) z_2 = ((y_1 y_2) z_2) z_1\) and that

Therefore we obtain

\[(y_1, y_2, z_1) z_2 = (y_1, y_2, z_2) z_1 \quad \text{(6)}\]

It follows from the latter identity, together with (b) and (a), that

\[(y_1, y_2, z_1)(z_2 y_3) = (y_1, y_2, z_2 y_3) z_1 = (y_3(y_1, y_2, z_2)) z_1 = (y_1, y_2, z_2)(y_3 z_1) \quad \text{(7)}\]

Now write \((z_1, y_1, y_2, z_2, y_3) = ((z_1, y_1, y_2) z_2) y_3 - (z_1, y_1, y_2)(z_2 y_3)\). Then according to the graded identity (1) we transpose \(z_1\) and \(z_2\) in the first summand on the right-hand side. Moreover according to (7) we do the same in the second summand, and this finishes the case \(k = 1\).

Let \(k > 1\), and suppose that for every integer \(\leq k - 1\) the polynomial of (d) lies in \(I\). In order to settle this case it suffices to check that the following equalities hold in \(L\).

\[(z_1, y_1, \ldots, y_2k) z_2 = (z_2, y_1, \ldots, y_2k) z_1;\]

\[(z_1, y_1, \ldots, y_{2k+1} z_2 = (z_2, y_1, \ldots, y_{2k+1} z_1).\]

In order to prove the former is a graded identity in \(L\) we observe that by (6)

\[(z_1, y_1, \ldots, y_{2k} z_2 = (z_2, y_{2k-1}, y_{2k}) (z_1, y_1, \ldots, y_{2k-2}).\]

Now we induct once again on \(k\), supposing that

\[(z_1, y_1, \ldots, y_{2k-2} z_2 = (z_2, y_1, \ldots, y_{2k-2}) z_1.\]

Then we obtain

\[(z_2, y_{2k-1}, y_{2k}) (z_1, y_1, \ldots, y_{2k-2} = z_1(z_2, y_{2k-1}, y_{2k}, y_1, \ldots, y_{2k-2}).\]
Now we apply (c) and get the required equality. The latter equality is verified in a similar way. By (7), and by (b) and (a) (and once again by induction) we have

\[(z_1, y_1, \ldots, y_{2k})(y_{2k+1}z_2) = (z_2, y_{2k-1}, y_{2k})(y_{2k+1}(z_1, y_1, \ldots, y_{2k-2}))
\]

\[= (z_2, y_{2k-1}, y_{2k})(z_1, y_1, \ldots, y_{2k-2})
\]

\[= (z_1y_{2k+1})(z_2, y_{2k-1}, y_{2k}, y_1, \ldots, y_{2k-2}).
\]

Now apply (d) to get the required equality.

Thus both equalities hold in \(L\); the identity (d) follows easily from them. \(\Diamond\)

We shall need some more graded identities in \(L\).

Lemma 7 The following polynomials are graded identities for \(L\).

(i) \((y_1, z_2, (y_2z_1)) - (y_2(y_1, z_1, z_2) + z_1(y_1, y_2, z_2))\);

(ii) \(z_1(z_2, z_3, y_1);\)

(iii) \((z_1z_2)(z_3, x, y_1) - (z_1, z_2, y_1, x, z_3);\)

(iv) \((y_1, z_1, z_2)(y_2, z_3, z_4) - z_1(y_1, z_2, z_3, y_2, z_4);\)

(v) \((y_1, y_2, z_1)(y_3, y_4, z_2) - z_1(z_2, y_1, y_2, y_3, y_4).\)

In other words the above polynomials lie in \(I\).

Proof. In order to prove (i) observe that \((y_1, z_2, (y_2z_1)) = (y_1z_2)(y_2z_1) - y_1(z_2(y_2z_1)),\) and that

\[y_1(z_2(y_2z_1)) = y_1(y_2(z_1z_2) + (y_2, z_1, z_2)) = (y_1y_2)(z_1z_2) + y_1(y_2, z_1, z_2).
\]

Also we have the following equality in \(L\)

\[(y_1z_2)(y_2z_1) = z_1(y_2(y_1z_2)) = z_1((y_1y_2)z_2 - (y_2, y_1, z_2))
\]

\[= (y_1y_2)(z_1z_2) + (y_1y_2, z_1, z_2) - z_1(y_1, y_2, z_2).
\]

Subtracting the last two identities and applying the graded identity (2) we obtain (i).

Similarly (ii) follows from (1) and (4) since

\[z_1((z_2z_3)y_1) = (z_2z_3)(z_1y_1) = ((z_2z_3)z_1)y_1 = ((z_1z_2)z_3)y_1
\]

and moreover \(z_1(z_2(z_3y_1)) = (z_1z_2)(z_3y_1) = ((z_1z_2)z_3)y_1.\)

The graded identity (iii) holds since \(z_1z_2\) is in the associative centre of \(L\) and furthermore, by (4) one has \((z_1z_2y_1), x, z_3) = 0 in \(L.\)

Now we deduce (iv). It follows from the graded identities (4) and (1) that

\[((y_1z_2)(y_2, z_3, z_4) = (y_2, z_3((y_1z_1)z_2), z_4) = (y_2, (z_1z_2)(y_1z_3), z_4)
\]

\[= (z_1z_2)(y_2, (y_1z_3), z_4).
\]
On the other hand \(((y_1 z_1) z_2)(y_2, z_3, z_4) = (z_1 z_2)(y_1 (y_2, z_3, z_4) + z_3(y_2, y_1, z_4))\) by (i). Since \(z_1 z_2\) is in the (associative) centre we have
\[
((y_1 z_1) z_2)(y_2, z_3, z_4) = ((z_1 z_2) y_1)(y_2, z_3, z_4) + ((z_1 z_2) z_3)(y_2, y_1, z_4).
\]
Hence \((y_1, z_1, z_2)(y_2, z_3, z_4) = ((z_1 z_2) z_3)(y_2, y_1, z_4)\). By (1) the elements \(z_1 z_2\) and \((y_1, z_2) z_3\) are in the centre. Thus \(z_1 (y_1, z_2, z_3, y_2, z_4) = (z_1 (z_2 z_3))(y_1, y_2, z_4)\). Now \(z_1 (z_2 z_3) = (z_1 z_2) z_3\) hence \((y_1, z_1, z_2)(y_2, z_3, z_4) = z_1 (y_1, z_2, z_3, y_2, z_4)\).

It remains to prove (v). In the proof of Lemma 6 (d) we showed that \(((y_1 y_2) z_1) z_2 = ((y_1 y_2) z_2) z_1\). Therefore, by (1)
\[
(y_1, y_2, z_1)(y_3, y_4, z_2) = (y_1, y_2, (y_3, y_4, z_2)) z_1 = -(y_3, y_4, z_2, y_1, y_2) z_1
\]
\[
= (z_2, y_3, y_4, y_1, y_2) z_1.
\]
Now we apply the graded identity from Lemma 6 (c) and order the variables \(y\) in the last associator.

The following proposition shows that the choice of the elements of \(\Omega_0\) can be really arbitrary.

**Proposition 8** Let \(u_1\) and \(u_2\) be two nonzero associators in \(L\) of the same multidegree. Then \(u_1 = \pm u_2\).

**Proof.** Let \(u \in \Omega\). We start with the following easy remark. Let us substitute every even variable of \(u\) by the matrix \(a\), and every odd variable of \(u\) by the matrix \(b\). Then the evaluation of \(u\) on the algebra \(J\) will be \(\pm a\) whenever \(|u| = 0\), and \(\pm b\) whenever \(|u| = 1\). This statement is checked by an obvious induction on \(\deg u\). If \(\deg u = 3\) then we can have \((a, a, b) - (b, a, a) = b\) and \((b, b, a) = -(a, b, b) = a\). If \(u = (u_1, u_2, u_3)\) then the \(u_i\) are associators of lower degree than \(u\), and we apply the induction.

As we observed earlier, every associator \(u \in \Omega\) is a linear combination of proper ones. We shall prove the proposition for proper (that is left normed) associators. First we shall show that every such associator \(u\) can be written as \(u = (z_1, \ldots, z_{2m}) u_t\) where \(t = 0, 1\), and \(u_0 = (z_{2m+1}, y_1, \ldots, y_{2k})\), while \(u_1 = (z_{2m+1}, y_1, \ldots, y_{2k}, z_{2m+2}, y_{2k+1})\). Here \(m \geq 0\).

We shall induct on the total degree \(n\) of \(u\) in the variables \(z\), and furthermore on \(\ell\) where the total degree \(\deg u = 2\ell + 1\). If \(n = 0\) there are no such nonzero associators. Suppose \(n = 1\). If \(u = (x_1, x_2, x_3, \ldots)\) then exactly one of \(x_1\) and \(x_3\) is an odd variable, hence we may assume (up to a sign) it is \(x_1\). Then apply Lemma 6 (c), and the result holds for every \(\ell\).

Suppose \(n = 2\). If \(\ell = 1\) then \(\deg u = 3\), and \(u = (z_1, z_2, y_1) = (z_2, z_1, y_1)\). Take \(\ell \geq 2\). One cannot have \(u = (z_1, z_2, y_1, \ldots)\) since the dots would stand for even variables (at least two), and \(u = 0\). Thus \(u = (z_1, y_1, \ldots, y_p, z_2, y_{p+1}, \ldots)\) with \(p \geq 1\). (The indices of the variables may be permuted but we use this simpler notation.) Also the integer \(p\) is even since otherwise \(u = 0\). Moreover the rightmost dots stand for even variables. Since the associator \((z_1, y_1, \ldots, y_p, z_2)\) is even (in the grading) and we have \(n = 2\) odd variables the rightmost dots are
For the variables $y$ either $1$ or $-1$ therefore $u$ proper ones, and then apply the results proved above for the proper associators. We write each of them as a linear combination of multihomogeneous degree. We write each of them as a linear combination of

It is clear from the last argument that we can permute the variables $z$ to some permutation of the even and, separately, of the odd variables. Then in order to get $u$ to simplify the notation we write $A$ fore we have to deal with the case when $A_1$ has exactly two odd variables. Once again by induction ($n = 2$) we can assume $A_1 = (z_1, y_1, \ldots, y_{2k-2}, z_2, y_{2k-1})$ up to some permutation of the even and, separately, of the odd variables. Then $|A_1| = 0$ and therefore $x_2$ must be some $z$, and since $n = 3$, $x_1$ is some $y$. In order to simplify the notation we write $u = (A_1, y, z)$ and $A_1 = (A_2, z', y')$ where $A_2$ is an associator, $|A_2| = 1$ and $A_2$ contains exactly one odd variable. Therefore $u = (A_2, z', y', z)$. Now we apply the identity (iii) from Lemma 7 and then use the fact that the algebra generated by $L_1$ is associative (and commutative) in order to get

\[
u = \pm (A_2z')(y', z, y) = \pm (A_2z')(z, y', y) = \pm ((z'z) A_2, y', y) = \pm (z'z)(A_2, y', y).
\]

It is clear from the last argument that we can permute the variables $z$ at will; for the variables $y$ it follows from Lemma 6.

Now let $u$ and $w$ be two associators (not necessarily proper ones), of the same multihomogeneous degree. We write each of them as a linear combination of proper ones, and then apply the results proved above for the proper associators. Therefore $u$ and $w$ differ only by a scalar multiple. But this multiple must be either $1$ or $-1$ due to the remark made at the beginning of the proof.

Another consequence of Lemma 7 and the proof of Proposition 8 is the following.

**Corollary 9** Let $u \in A$. If we substitute any variable $x$ of $u$ by an associator $w$ such that $|x| = |w|$ then we get a linear combination of elements of $A$.

**Proof.** The algebra $L_0$ is associative and commutative, and the same holds for the subalgebra of $L_1$ generated by $L_1$. If, in some substitution, there appears an element of the type $z_i z_j$ it can be ”eaten” by the associator (or by the element $z_{j_1} \ldots z_{j_t}$ in the case of elements of type (i)) in the definition of $A$. We finish the proof of the corollary by means of a case-by-case analysis, substituting in each of the elements of $A$, associators for some variable. These cases are straightforward; one needs also the graded identities from Lemma 4. For example, if we substitute the variable $y$ for $(y_2, z_3, z_4)$ in $(y_1, z_1, z_2)y$, by Lemma 7 we conclude that we get $z_1(y_1, z_2, z_3, y_2, z_4)$.

Recall that we denote by $S$ the span of the set $A$ (defined just before Lemma 6). The following lemmas assure that certain elements belong to $S$.  


Lemma 10 The polynomial \( N = ((y_1 \ldots y_k), y, z) \) lies in the span of the elements of type (ii) of the set \( S \).

Proof. Let \( S'' \) be the set of the elements of type (ii) of the same multidegree as \( N \). We shall show that \( N \) lies in the span \( V \) of \( S'' \). We induct on \( k \). If \( k = 1 \) we have nothing to prove. Then write, using the graded identity \( \text{[3]} \)
\[
N = (y_1 \ldots y_{k-1})(y_k, y, z) + y_k((y_1 \ldots y_{k-1}), y, z).
\]
The element \((y_1 \ldots y_{k-1})(y_k, y, z)\) is clearly of the type (ii). In order to prove that \( y_k((y_1 \ldots y_{k-1}), y, z) \in V \) we apply the inductive assumption to the element \((y_1 \ldots y_{k-1}), y, z\). Hence it suffices to prove that all elements of the type
\[
y((y_1 \ldots y_p)(z, y_{p+1}, \ldots, y_q)), \quad p < k, \quad q - p \equiv 0 \pmod{2}
\]
are linear combinations of elements of type (ii). But the latter element equals
\[
(y((y_1 \ldots y_p)(z, y_{p+1}, \ldots, y_q)) - (y, (y_1 \ldots y_p), (z, y_{p+1}, \ldots, y_q))).
\]
The first summand is of type (ii). The second summand equals, up to a sign, \((y_1 \ldots y_p, y, (z, y_{p+1}, \ldots, y_q)).\) First consider the element \(((y_1 \ldots y_p), y, z).\) Applying to it the graded identity \( \text{[3]} \) several times we obtain a linear combination of elements of the type (ii) from the set \( A \). Now according to the previous Corollary \( \text{[3]} \) if we substitute an associator for a variable in an element of \( A \), we get once again elements of \( A \) as long as the \( \mathbb{Z}_2 \)-degree is preserved. This finishes the proof. \( \diamond \)

Lemma 11 The polynomial \( N = ((y_1 \ldots y_k), z_1, z_2) \) lies in \( S \).

Proof. As in the previous lemma we induct on \( k \). We shall show that \( N \) is in the span \( V \) of the elements of types (iii) and (iv). The base of the induction \( k = 1 \) is obvious. It follows from the graded identity \( \text{[4]} \) that
\[
N = (y_1 \ldots y_{k-1})(y_k, z_1, z_2) + y_k((y_1 \ldots y_{k-1}), z_1, z_2) - 2z_1(z_2, (y_1 \ldots y_{k-1}), y_k).
\]
The first summand from the right is an element of \( S \) (of type (iv)). By the induction we can assume that \(((y_1 \ldots y_{k-1}), z_1, z_2) \in V \) is a linear combination of elements of types (iii) and (iv). Moreover the elements of types (iii) and (iv) are products of even elements; therefore they lie in the associative algebra \( L_0 \). Thus multiplying these by \( y_k \) yields once again elements of the same types. It remains to prove that the last summand lies in \( V \). Applying Lemma \( \text{[10]} \) to \((z_2, (y_1 \ldots y_{k-1}), y_k)\) we write it as a linear combination of elements of type (ii). Therefore it suffices to prove that elements of the form
\[
M = z_1((y_1 \ldots y_n)(z_2, y_{n+1}, \ldots, y_m)), \quad n < k, \quad m - n \text{ even}, \quad \text{are in } V.\]
But we have
\[
M = (y_1 \ldots y_n)((z_1(z_2, y_{n+1}, \ldots, y_m)) - (z_1, (z_2, y_{n+1}, \ldots, y_m), (y_1, \ldots, y_n)).
\]
Here the first summand on the right is of type (iv). The second is also in \( V \) due to the inductive assumption combined with Corollary \( \text{[4]} \). \( \diamond \)
Lemma 12 If $s \in S$ then $sz \in S$.

Proof. First we notice that the elements of $A$ of the types (ii), (iii), (iv) can be obtained by the elements of type (i) after the substitution of a variable $x$ by an associator $u$ such that $|u| = |x|$. Thus the lemma will follow from Corollary 9 if we prove that $((y_1 \ldots y_k)(z_1 \ldots z_p))z \in S$. If the number $p$ is even then $z_1 \ldots z_p$ lies in the associative centre of $L$ and hence it will "eat" the variable $z$ and we get an element of type (i). So suppose $p$ is odd. Then the product $z_2 \ldots z_p$ is central and it suffices to show that the element $R = ((y_1 \ldots y_k)z_1)z \in S$. One sees easily that

$$R = ((y_1 \ldots y_k), z_1, z) + (y_1 \ldots y_k)(z_1z).$$

But the first summand on the right lies in $S$ due to Lemma 11 while the second is already of type (i). $\diamond$

Lemma 13 The element $N = ((y_1 \ldots y_k), (y_{k+1} \ldots y_n), z) \in S$.

Proof. We induct on $n - k$. If $n - k = 1$ this is Lemma 10. Suppose $n - k > 1$. The lemma will follow from the following claim. If we substitute in $(y_1 \ldots y_r)(z, y_{r+1}, \ldots, y_s)$ a variable $y$ by a product of $n - k$ variables $y$ the resulting expression lies in $S$. The claim clearly holds (for products of any length) if we substitute some of the variables $y_1$ to $y_r$. If we substitute some of $y_{r+1}, \ldots, y_s$, then we first apply Lemma 10 and then the induction. $\diamond$

Lemma 14 The element $R = (y_1 \ldots y_k)((y_{k+1} \ldots y_n)z)$ lies in $S$.

Proof. The proof follows from Lemma 13 by using an argument similar to that of the proof of Lemma 12. $\diamond$

Proposition 15 The set $A$ spans the relatively free graded algebra $L$.

Proof. First we claim that the elements $R = ((y_1 \ldots y_{p+1})z_1)((y_{p+1} \ldots y_q)z_2) \in S$. Indeed since $(z_1, y, z_2)$ is a graded identity we have that $R$ can be written as $R = z_1((y_1 \ldots y_{p+1})(y_{p+1} \ldots y_q)z_2)$. Therefore our claim follows from Lemma 12 and from Lemma 14. Moreover the product of an even number of elements of $L_1$ lies in the associative centre of $L$. Hence the fact that $R \in S$, together with Corollary 9 imply that the product of two elements from $A$ lies in $S = sp(A)$. Thus $S$ is a subalgebra of $L$. Since $X \subseteq S$ by definition, and $X$ generates $L$ as an algebra we obtain $S = L$ as required. $\diamond$

Let $u_1, u_2 \in A$. We shall call $u_1$ and $u_2$ similar if

$$u_1 = (y_{i_1}^{n_1} \ldots y_{i_k}^{n_k})a_1, \quad u_2 = (y_{i_1}^{n_1} \ldots y_{i_k}^{n_k})a_2.$$  

Here the $a_i, i = 1, 2$, are of the form $(z_{j_1} \ldots z_{j_p})w_i, p \geq 0,$ and $w_i$ are associators. Note that we do not require that $a_1 = a_2$. In other words $u_1$ and $u_2$ are similar if the even variables that appear in them outside the associators, are the same (counting the multihomogeneous degrees).

We have gathered all necessary information for the main result in this section.
**Theorem 16** Let $K$ be an infinite field, char $K \neq 2$. Then the ideal $T$ of the graded identities for the Jordan algebra $J$ of the symmetric $2 \times 2$ matrices is generated (as a graded $T$-ideal) by the identities (1) to (3). In other words $T = I$.

**Proof.** We split the proof into three steps.

**Claim 1.** Let $u_1, \ldots, u_n$ be elements of $A$ having the same multidegree. Suppose no two of them are similar and that $\sum \alpha_i u_i \in T$ is a graded identity for $J$ where $\alpha_i \in K$ are scalars. Then all $\alpha_i = 0$.

Let $u_i = c_i a_i$ where $a_i$ are as in the definition of similarity, and the $c_i$ are products of even variables. Then $c_i \neq c_j$ whenever $i \neq j$. Since $L_0$ is associative (and commutative) we can assume that the variables $y$ in each $c_i$ are written in ascending order.

Let $\sum \alpha_i u_i = f(y_1, \ldots, y_p, z_1, \ldots, z_q)$. Suppose further $f \neq 0$. Define

$$g(y_1, \ldots, y_p, z_1, \ldots, z_q) = f(y_1 + 1, \ldots, y_p, z_1, \ldots, z_q).$$

The polynomial $g$ is a graded identity for the Jordan algebra $J$. We draw the reader’s attention that $g$ is not multihomogeneous. Since the base field is infinite then all its multihomogeneous components are also graded identities for $J$. One of its homogeneous components is exactly $f$. Take the homogeneous component $h$ of $g$ that is nonzero and of the lowest degree in $y_1$. (That is we take for $h$ the nonzero polynomial obtained from $f$ after substituting the largest possible number of variables $y_1$ by 1.) The polynomial $h$ is obtained from $f$ by means of the following procedure. First we take the sum of all $\alpha_i c_i u_i$ where the degree of $y_1$ in $c_i$ is the largest possible, and discard the remaining summands.

Then we substitute in these summands, all entries of $y_1$ in $c_i$ by 1 (and keep the entries of $y_1$ in the associators.). This gives exactly $h$ since whenever 1 is substituted in an associator, the associator vanishes. Now the polynomial $h$ does not contain $y_1$ outside associators. By repeating the above argument to $h(y_1, y_2 + 1, y_3, \ldots, y_p, z_1, \ldots, z_q)$ we shall get a nonzero polynomial that does not contain $y_2$ outside associators, and so on. Finally we shall get a non-zero polynomial $f_1$ that does not contain any variable $y_i$ outside its associators. Clearly $f_1 \in T$ since $f \in T$. But $f_1$ is obtained by $f$ by removing some of the summands and discarding the $c_i$ parts of the remaining summands. Now as the $c_1, \ldots, c_n$ are pairwise distinct we get that there is only one $a_i$ in $f_1$. That is $f_1 = \alpha_i a_i$ for some $i$. On the other hand if $\alpha_i a_i \in T$ this means it must be a graded identity for $J$. But this is possible only if $\alpha_i = 0$ in which case $f_1 = 0$, and $\alpha_i c_i a_i$ does not participate in $f$. Then we repeat the above procedure to $f$ (having discarded $\alpha_i c_i a_i$) and we continue by induction.

**Claim 2.** The set $A$ is linearly independent modulo the graded ideal $T$.

It follows from Claim 1 that it suffices to consider only the elements of $A$ where all variables $y$ appear in the associators only. Thus we have to show that the elements $z_{j_1} \ldots z_{j_k}, u^1, z_{j_1} u^1, u^0$, are independent. Note that that the $u^i$ are associators, not variables. Then these elements are of pairwise different multidegrees, and cannot be linearly dependent. Our claim follows.
Claim 3. The inclusion $T \subseteq I$ holds.

Let $f \in T$ be a multihomogeneous polynomial. Since $I \subseteq T$, by Proposition 15 it follows that $f \equiv \sum \alpha_i u_i \pmod{I}$. Here $\alpha_i \in K$ and $u_i \in A$. But Claims 1 and 2 yield that all $\alpha_i = 0$, and $f \in I$. The claim is proved.

In order to finish the proof of the theorem it suffices to recall that $T \subseteq I$ and $I \subseteq T$. \hfill $\diamond$

\section{The scalar grading}

In the scalar grading on $J$, as we commented before, the component $J_0$ is the one dimensional span of the unit matrix $I$, and hence we identify $J_0 = K$. It turns out that the scalar grading is somewhat "easier" to resolve even in a more general situation. Here we shall describe the graded identities of the Jordan algebras $B$ and $B_n$ of a nondegenerate symmetric bilinear form on the vector spaces $V$ and $V_n$, respectively. Here dim $V = \infty$, dim $V_n = n$. We have $B^{(0)} = K$, and $B^{(1)} = V$, respectively $B_n^{(0)} = K$, $B_n^{(1)} = V_n$. (We shall use upper indices for the grading in order not to confuse them with the corresponding algebras $B_0$ and $B_1$.) We keep the notation $X$ for the variables in the free Jordan algebra, $X = Y \cup Z$ where $Y$ are the even and $Z$ are the odd variables. We have the following graded identity in $B$.

$$(y, x_1, x_2) = 0$$ (8)

Its validity in $B$ is immediate: the even elements are scalars. Note that it follows from (8) that $(z_1 z_2, x_1, x_2) = 0$.

We denote in this section by $I$ the ideal of graded identities defined by the polynomial (8), and by $L = J(X)/I$ the corresponding relatively free algebra.

Let $f(y_1, \ldots, y_p, z_1, \ldots, z_q)$ be a multihomogeneous polynomial. Then modulo the graded identity in (8) we write it as

$$f(y_1, \ldots, y_p, z_1, \ldots, z_q) = y_1^{n_1} \cdots y_p^{n_p} g(z_1, \ldots, z_q)$$

where $g$ is some polynomial on the variables $z$ only. Therefore $f$ is a graded identity for $B$ if and only if $g$ is. But $g$ is a Jordan polynomial in the variables $z$ only. Therefore $f$ is a graded identity for $B$ if and only if $g$ is a weak Jordan identity for the pair $(B, V)$. In this way we have to describe these weak Jordan identities.

Define $M$ to be the subalgebra of $L = J(X)/I$ generated by the variables $Z$.

Lemma 17 The algebra $M = M^{(0)} \oplus M^{(1)}$ is $\mathbb{Z}_2$-graded. The subalgebra $M^{(0)}$ is spanned by all products $(z_{i_1} z_{j_1}) \cdots (z_{i_k} z_{j_k})$ while the vector space $M^{(1)}$ is spanned by all $z_{i_0}(z_{i_1} z_{j_1}) \cdots (z_{i_k} z_{j_k})$.

Proof. It is clear that the above decomposition of $M$ is a grading. The other two statement of the lemma are equally trivial. \hfill $\diamond$
We denote by $C$, respectively $C_n$, the Clifford algebra of the vector space $V$, respectively $V_n$. As we already mentioned, $C$ and $C_n$ are the associative envelopes of the special Jordan algebras $B$ and $B_n$, respectively. The weak associative identities for the pairs $(C, V)$ and $(C_n, V_n)$ were described in [5, 6] over a field of characteristic 0, and in [12] over an infinite field of characteristic different from 2. The paper [12] made use of the invariants of the orthogonal group as described by De Concini and Procesi in [2]. Note that the description of these invariants in [7] is characteristic-free.

Such a description is given in terms of double tableaux. We recall briefly the main notions and results we shall need here. Let $t_{ij}$, $i = 1, 2, \ldots, j = 1, 2, \ldots, n$ be commuting variables, and consider the (formal) vectors $t_i = (t_{i1}, \ldots, t_{in})$. Let $U$ be the free $K[t_{ij}]$-module freely generated by $t_i$, $i = 1, 2, \ldots$ A nondegenerate symmetric bilinear form on $U$ is defined as $t_i \circ t_j = t_{i1}t_{j1} + \cdots + t_{in}t_{jn}$. The description of the polynomial algebra $R = K[(t_i \circ t_j)]$ was given in [7, Section 5]. We consider the double tableau

$$T = \begin{pmatrix}
p_{11} & p_{12} & \cdots & p_{1m_1} \\
p_{21} & p_{22} & \cdots & p_{2m_2} \\
\cdots & \cdots & \cdots & \cdots \\
p_k1 & p_{k2} & \cdots & p_{km_k}
\end{pmatrix} \begin{pmatrix}
q_{11} & q_{12} & \cdots & q_{1m_1} \\
q_{21} & q_{22} & \cdots & q_{2m_2} \\
\cdots & \cdots & \cdots & \cdots \\
q_{k1} & q_{k2} & \cdots & q_{km_k}
\end{pmatrix}$$

(9)

In it we have $m_1 \geq m_2 \geq \ldots \geq m_k \geq 0$, and $p_{ij}$ and $q_{ij}$ are integers.

Suppose $T = (p_{11} p_{12} \ldots p_m \mid q_{11} q_{12} \ldots q_m)$ is a double row tableau filled with positive integers. We associate to it the polynomial $\varphi(T) \in R$:

$$\varphi(T) = \sum (-1)^\sigma (t_{p_1} \circ t_{q_{\sigma(1)}})(t_{p_2} \circ t_{q_{\sigma(2)}}) \cdots (t_{p_m} \circ t_{q_{\sigma(m)}}).$$

Here $\sigma$ runs over the symmetric group $S_m$ and $(-1)^\sigma$ is the sign of $\sigma$. Clearly $\varphi(T) = \det((t_{pi} \circ t_{qj}))$ where $1 \leq i, j \leq m$. If $p_i = p_j$ or $q_i = q_j$ for some $i \neq j$ then $\varphi(T) = 0$.

In general, if $T^{(1)}$, $T^{(2)}$, $\ldots$, $T^{(k)}$ are the rows of the double tableau $T$ filled with positive integers then we associate to $T$ the polynomial $\varphi(T) = \varphi(T^{(1)})\varphi(T^{(2)}) \cdots \varphi(T^{(k)})$.

The double tableau $T$ of the form (9) is **doubly standard** if the inequalities $p_{i1} < p_{i2} < \ldots < p_{im}$, $q_{i1} < q_{i2} < \ldots < q_{im}$, $p_{ij} \leq q_{ij}$, $q_{ij} \leq p_{i+1,j}$ hold for all $i$ and $j$. In other words if we form the ordinary tableau by inserting each row of $q_{ij}$ just below its counterpart $p_{ij}$, the resulting tableau will be standard (that is its entries increase strictly along the rows, and increase with possible repetitions along the columns). One of the main results of [7] is the following.

**Theorem 18 ([7, Theorem 5.1])** The polynomials $\{\varphi(T)\}$ where $T$ runs over all doubly standard tableaux (9) of positive integers, such that $m_1 \leq n$, form a basis of the vector space $R$ over $K$.

We shall call the double tableaux of the type (9) simply tableaux if all their entries are positive integers. If only $p_{11} = 0$ and all remaining entries of $T$ are
positive integers we call it 0-tableau. If $T$ is a 0-tableau consisting of a single row we associate to it the polynomial 
$$\tilde{\varphi}(T) = \sum (-1)^{\sigma} t_{q_{\sigma(1)}} (t_{p_2} \circ t_{q_{\sigma(2)}}) \ldots (t_{p_m} \circ t_{q_{\sigma(m)}}).$$
If $T^{(1)}$, $T^{(2)}$, \ldots, $T^{(k)}$ are the rows of the 0-tableau $T$ then we associate to $T$ the polynomial $\tilde{\varphi}(T) = \tilde{\varphi}(T^{(1)}) \tilde{\varphi}(T^{(2)}) \ldots \tilde{\varphi}(T^{(k)})$. Pay attention to the fact that the rows $T^{(2)}$, \ldots, $T^{(k)}$ are tableaux but not 0-tableaux.

Clearly all the above holds if we substitute $R$ by $M$ (that is $t_i$ by $z_i$). Formally speaking one has to let $n \to \infty$ in order to justify the statements for the infinite dimensional case but this is evidently true.

**Proposition 19** The vector space $M^{(0)}$ has a basis consisting of all polynomials associated to doubly standard tableaux. Also $M^{(1)}$ has a basis consisting of all polynomials associated to doubly standard 0-tableaux.

**Proof.** The assertion for $M^{(0)}$ follows immediately from the above cited Theorem 18 of De Concini and Procesi, [7]. The one for $M^{(1)}$ also follows from [7] in the following way. Our symmetric bilinear form is nondegenerate. Let $T$ be some 0-tableau and consider $\bar{\varphi}(T)$. Take a new variable $z_0$, then $z_0 \circ \bar{\varphi}(T)$ is represented by a double tableau, and we apply the argument above. Then $z_0 \circ \bar{\varphi}(T)$ will be a linear combination of standard tableaux (the straighening algorithm from [4]). But in a standard tableau the leftmost entry of the first row must correspond to $z_0$, and we are done. ♦

Recall that when dealing with weak identities we consider them in the free Jordan algebra $J(X)$.

**Theorem 20** 1. The weak Jordan identities for the pair $(B, V)$ are consequences of the polynomial $(x_1 x_2, x_3, x_4)$.

2. The weak Jordan identities for the pair $(B_n, V_n)$ follow from $(x_1 x_2, x_3, x_4)$ and $f_n = \sum (-1)^{\sigma} x_{\sigma(1)} (x_{n+2} x_{\sigma(2)}) \ldots (x_{2n+1} x_{\sigma(n+1)})$.

**Proof.** The first assertion of the theorem is a straightforward application of Theorem 18. The same for the second assertion. (Note that the polynomial $f_n$ "kills" all tableaux whose first row is of length $\geq n + 1$.) ♦

**Corollary 21** 1. The ideal of the graded identities for the Jordan algebra $B$ (with the scalar grading) coincides with the ideal I generated by the polynomial from $\delta$.

2. The ideal of the graded identities for the Jordan algebra $B_n$ with the scalar grading is generated by $\delta$ and by the identity 
$$g_n = \sum (-1)^{\sigma} z_{\sigma(1)} (z_{n+2} z_{\sigma(2)}) \ldots (z_{2n+1} z_{\sigma(n+1)}), \quad \sigma \in S_n+1.$$

3. The graded identities for the Jordan algebra of the symmetric $2 \times 2$ matrices (with the scalar grading) follow from $\delta$ and $\sum (-1)^{\sigma} z_{\sigma(1)} (z_4 z_{\sigma(2)}) (z_5 z_{\sigma(3)})$ where $\sigma$ runs over $S_3$. 

17
Proof. Statement (3) is a particular case of (2). Also (1) and (2) are immediate due to Theorem 20 and to the remarks preceding Lemma 17.

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