On a question of H.A. Schwarz

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To Manfred Denker on the occasion of his 75th birthday

1 Introduction

The question to which the title of this paper refers is not usually associated with the name of Hermann Amandus Schwarz and this needs some explanation. To explain the problem let us consider a tetrahedron with vertices \(1, 2, 3, 4\); let \(d_{ij}\) (with \(i < j\)) be the length of the side \(ij\). Let \(V\) denote the volume of the tetrahedron. Let \(CM_0(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34})\) denote the determinant of the matrix

\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & x_{12}^2 & x_{13}^2 & x_{14}^2 \\
1 & x_{12}^2 & 0 & x_{23}^2 & x_{24}^2 \\
1 & x_{13}^2 & x_{23}^2 & 0 & x_{34}^2 \\
1 & x_{14}^2 & x_{24}^2 & x_{34}^2 & 0
\end{pmatrix}
\]

This turns out to be a sum of 10 monomials the coefficients of which are \(\pm 2\); it is therefore convenient to write

\[
CM(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}) = CM_0(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34})/2.
\]

The Cayley-Menger formula, which is actually due to Lagrange (1773), states that

\[
(12V)^2 = CM(d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34}).
\]

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The work described here is part of a project to document the life and work of Kurt Heegner whose papers have been deposited in the Handschriftenabteilung of the SUB Göttingen. This was a collaboration with Hans Opolka and Norbert Schappacher; H. Opolka undertook the task of producing transcriptions of Heegner’s most important mathematical manuscripts and gave the first analysis of them. In this paper we shall not touch on the biography of Heegner here.

Two last words of thanks - first to Tim Browning (Bristol) who made some stimulating and helpful remarks. Finally to the team behind PARI/gp which I have used extensively in the work described here.
The question referred to in the title is whether the variety given by

\[ y^2 = CM(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}) \]

is rational over \( \mathbb{Q} \), that is, whether we can find six elements \( \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6 \) of

\[ \mathbb{Q}(x_{12}, x_{13}, x_{23}, x_{24}, x_{34}) \]

so that this field is \( \mathbb{Q}(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) \). Note that \( CM(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}) \) is irreducible, [6].

The question as to the rationality of a variety is a subtle one. Lüroth’s theorem shows that in the case of one variable any subfield of \( \mathbb{Q}(x) \) of finite index is itself of the form \( \mathbb{Q}(\xi) \). In higher dimensions the analogue of Lüroth’s theorem does not hold and the question, even over \( \mathbb{C} \), is delicate and has received considerable attention in recent times; for a discussion of such problems for classes of varieties not all that different from the one with which we shall be concerned see [11]. As it happens our case is rather special and the central problem is the question as to the set of algebraic number-fields for which the variety is rational.

Before going further it is instructive to look at the case of triangles. The analogue of the Cayley-Menger determinant is the Heron function,

\[ H(x_{12}, x_{23}, x_{13}) = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & x_{12} & x_{13} \\ 1 & x_{12} & 0 & x_{23} \\ 1 & x_{13} & x_{23} & 0 \end{vmatrix} \]

which is equal to \( 2x_{12}^2x_{23}^2 + 2x_{23}^2x_{13}^2 + 2x_{13}^2x_{12}^2 - x_{12}^4 - x_{23}^4 - x_{13}^4 \), or, if one prefers, to \((x_{12} + x_{23} + x_{13})(x_{12} + x_{23} - x_{13})(x_{12} - x_{23} + x_{13})(-x_{12} + x_{23} + x_{13})\). If we have a triangle \( 1, 2, 3 \) with side-lengths \( d_{12}, d_{23}, d_{13} \) and area \( A \) then the formula usually ascribed to Heron (but probably older) asserts \((4A)^2 = H(d_{12}, d_{23}, d_{13})\).

Note that \( H \) is reducible but neither \( CM \) nor its higher dimensional analogues are – see [6]. In this case the corresponding variety is rational; the problem has a long history and attracted even the attention of Gauss who wrote about it in a letter to H. Schumacher dated 21st October, 1847 and gives a parametric solution. This is essentially the same as one given by Schottky in [22] who discusses it at some length. One should note it is far from true that any solution of either diophantine problem corresponds to to a geometric solution. In the case of triangles with rational area (Heron triangles) we obtain the geometric solutions by restricting the triplet
\((d_{12}, d_{23}, d_{13})\) so that all components are positive and the three triangle inequalities are satisfied, that is, six inequalities in all.

With these ideas we can now discuss Schwarz’ role in the history of this problem. He moved to Berlin as Weierstrass’ successor in 1892, just before he turned 50. Two years earlier he published his Collected Works and published only one paper during his tenure as professor in Berlin. He seems to have decided on playing the “Grand Old Man”. He had married, in 1868, Kummer’s daughter Marie and apparently felt himself obliged to uphold the Kummer tradition. He was very active in the Berlin Mathematische Gesellschaft and sought out promising young students and encouraged them. Kurt Heegner was one of the last, probably the last, and wrote warmly of Schwarz’ efforts on his behalf. Kummer had taken up the problem of finding all quadrilaterals with rational sides and rational diagonals. This was a problem going back to the 7th century Indian mathematician Brahmagupta and Kummer [17] introduced a novel technique, employing, as we now say, elliptic curves, to study it. One can, apart from the problem of understanding which solutions are geometric, consider this as the question of finding all the rational solutions of \(CM(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}) = 0\). He was quite possibly aware of the “Cayley-Menger” formula in later years – his work on Kummer surfaces has its origins in the theory of the Fresnel wave surface and Hamilton’s discovery of a singularity of this surface and his prediction, spectacular in 1833 when it was made, of the phenomenon of conical refraction. Much of Cayley’s work of the 1840s deals with tetrahedroids which embody the algebraic approach to the wave surface. Kummer put forward, in the Berlin Mathematische Gesellschaft, the problem finding all rational tetrahedra, that is, again leaving aside the problem of determining which solutions are geometric, of finding the rational (or integral) solutions of \(y^2 = CM(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34})\). This became a favourite problem of Schwarz and he encouraged several young mathematicians to work on it, including Kurt Heegner. One of the people who took up the gauntlet was Otto Schulz whose extremely informative doctoral thesis [23] contains in particular an historical summary which is the source for the information given above. The problem of the rationality of the variety \(y^2 = CM(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34})\) is attributed by Schulz to Schwarz and one can presume that Schwarz was motivated by the Heron example. Schottky, who was later also in Berlin, discussed, in 1916 the Heron example in connection with the approach we shall describe here [22] p.150ff.]. Another Berlin student, a contemporary of Schulz, was Fritz Neiß (Neiss), whose much shorter thesis, deals with the same problem and offers some interesting insights. Schulz give examples of rational subvarieties of the variety above. Indeed one class of examples goes back to an extension of a theorem of Brahmagupta by Kummer to the effect that the subvariety of \(CM(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}) = 0\) defined by the quadrilateral being inscribed in a circle is rational. This can be extended to tetrahedra by an
observation of Friedrich Ankum which we shall describe presently.

Schulz actually proves

**Theorem** (O. Schulz) *The variety*

\[ y^2 = CM(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}) \]

is rational over \( \mathbb{Q}(\sqrt{-1}) \).

He was clearly dissatisfied with this result (as it involved \( \mathbb{Q}(\sqrt{-1}) \) rather than \( \mathbb{Q} \)) and relegated it to an appendix, [23, p.268ff]. His argument is fairly simple and we shall come to the essential point later. The purpose of this paper is to give a more sophisticated proof based on ideas of Heegner which are preserved in the Handschriftenabteilung of the SUB Göttingen, Codex Ms. K. Heegner 1:32, 1:33 (“Rationale Vierecke und Tetraeder und ihre Beziehung zu den elliptischen und hyperelliptischen Funktionen”). It illuminates the problem further and is based on a theorem of F. Schottky on Weddle surfaces, [21, 22].

The results of the Berlin group were not published in any mathematical journals; perhaps had the First World War not broken out at this time this would have changed. The theses of Schulz and Neiss were circulated in printed versions, as was required at the time, but they escaped the notice of L.E. Dickson and his team preparing the “History of the Theory of Numbers”, [7] and remained unknown outside a very small circle of readers. Heegner’s never completed his investigations to his own satisfaction and poverty and failing health in his later years meant that what he did achieve remained unpublished – although he did drop a hint at the end of his final paper [14]. His ideas remain extremely interesting but now have to be embedded in the developments in diophantine analysis of recent decades.

One could see the present paper as “Tales of forgotten genius”, to borrow a phrase from Ian Stewart, [2]. There have been many papers over the years dealing with problems of rational quadrilaterals and tetrahedra but few have found the direction that was explored in Berlin. Comparatively recently, in connection with Heron triangles, a related method was found by Robin Hartshorne and Ronald van Luijk, at that time a student of Hendrik Lenstra - see [13, 18] which brought the ideas from algebraic geometry into the study of this apparently fairly elementary problem.

## 2 Ankum’s observation

One important discovery of the group around Schwarz is due to Friedrich Ankum: he noted that \( CM(d_{12}, d_{13}, d_{14}+t, d_{23}, d_{24}+t, d_{34}+t) \) is a quadratic function of \( t \). Ankum never completed a thesis and his observation has

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1Interestingly this paper is included in the bibliography of [4] but is not cited in the body of the text as Prof. Flynn has kindly checked. Presumably the relevant passage was not included in the final text of the book.
been reported by Schulz, Neiss and Heegner. Heegner recognized that the equation in \((x, y, z)\) \(CM(d_{12}, d_{13}, x, d_{23}, y, z) = 0\) is a special type of Kummer surface, a tetrahedroid, and that Ankum’s observation is an expression of the fact that the point at infinity \([0 : 1 : 1 : 1]\) (with the usual convention of introducing the additional projective coordinate as the zeroth) is a node of the tetrahedroid. In fact tetrahedroids were the first class of Kummer surfaces to be investigated, by Cayley, from 1846 onwards. They are the algebraic version of the Fresnel wave surface when questions of the real locus are left out of the discussion. For all of this see [15, Chapters IX,X].

The theory of tetrahedroids is not well covered in the contemporary literature. There is a brief but useful account by A.-S. Elsenhans and J. Jahnel in [10]. There is a more extensive account by J.W.S. Cassels and E.V. Flynn in [4], from a rather different point of view. Elsenhans and Jahnel are interested in the arithmetic of Kummer surfaces and tetrahedroids provide interesting examples. Cassels and Flynn’s goal is to study the arithmetic of curves of genus 2. Kummer surfaces are closely associated with them and in particular with their Jacobians but are more amenable to study. Tetrahedroids are associated with curves with reducible Jacobians which happens quite often\(^2\). Both of these accounts stress the analysis of a selected node and this proves not to be so convenient for our purposes. Rather it turns out that the, more global, combinatorial approach taken by Hudson, [15], is appropriate. We shall also need Cayley’s theory of the symmetroid and the associated Weddle surface, briefly described in [4] but we need the account given in [15] or in the final chapter of [16]. Finally the two papers of W. Edge, [8, 9] provide an interesting perspective. The geometrical literature does not pay much attention to fields of definition and our account is a matter of making all of the details of the classical account concrete with the help of computer algebra, in the spirit of [4].

Let

\[
T_{a,b,c}(X_0, X_1, X_2, X_3) = \frac{1}{2} \begin{vmatrix}
0 & 1 & 1 & 1 & X_0^2 \\
1 & 0 & a^2 & b^2 & X_1^2 \\
1 & a^2 & 0 & c^2 & X_2^2 \\
1 & b^2 & c^2 & 0 & X_3^2 \\
X_0^2 & X_1^2 & X_2^2 & X_3^2 & 0
\end{vmatrix},
\]

i.e. \(X_0^2 CM(a, b, X_1/X_0, c, X_2/X_0, X_3/X_0)\). Then \(T_{a,b,c}(X_0, X_1, X_2, X_3) = 0\) is the projective equation of a tetrahedroid and which we denote by \(T_{a,b,c}\); this is the family considered by Heegner.

\(^2\)There are two useful databases of curves of genus 2, one due to M. Stoll at \texttt{http://www.mathe2.uni-bayreuth.de/stoll}\ and one due to A.R. Booker, J. Sijssing, A.V. Sutherland, J. Voight, R. v. Bommel and D. Yasaki at the data base of L-Functions and modular forms,(see [2]), at \texttt{http://www.lmfdb.org}\. A perusal of these tables is most instructive.
We shall work over a field $k$ of characteristic 0 (usually $\mathbb{Q}$); we shall associate with the “triangle” 123 the element $\delta_{a,b,c} = H(a, b, c)k^{\times 2}$ of $k^{\times}/k^{\times 2}$.

Neiss observed the following identity which he deduced from Sylvester’s identity for determinants

$$H(d_{12}, d_{13}, d_{23})H(d_{12}, d_{24}, d_{34}) = D_{12}^2 + (d_{12}CM(d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34}))^2$$

where $D_{12} = D_{12}(d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34})$ is

$$\begin{vmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & d_{12} & d_{14} \\
1 & d_{12} & 0 & d_{24} \\
1 & d_{13} & d_{12} & d_{34}
\end{vmatrix}.$$ 

From this one can make two deductions. If $CM(d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34}) = 0$, which would be the case if we were dealing with a plane quadrilateral, then the square-classes $\delta_{a,b,c}$ for all three element subsets of $\{1, 2, 3, 4\}$ are equal. If $CM(d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34}) \neq 0$ then square classes $\delta_{a,b,c}$ of two triangles are equal if $-1$ is a square in $k$ and differ multiplicatively by a norm from $k(\sqrt{-1})^x$ otherwise. We assume that none of the corresponding $H$ is zero.

Finally Schulz noted that $D_{12}(d_{12}, d_{13}, d_{14} + s, d_{23}, d_{24} + s, d_{34} + s)$ is of the form $As + B$ where

$$A = -2((d_{12}^2 - d_{13}^2 + d_{23}^2)d_{14} + (d_{12}^2 + d_{13}^2 - d_{23}^2)d_{24} - 2d_{12}^2d_{34}),$$

a refined version of Ankum’s observation from which Schulz the formula which is the basis of his proof. We shall use use another version of this formula derived from the theory of Kummer varieties.

### 3 Some classical mathematics

In this section we shall recall some classical ideas, mainly due to Cayley, which we shall use. Let $\mathcal{P}$ be a set of six points in $\mathbb{P}^3$, no four of which lie in a plane. Then the space of quadratic forms which vanish at these points is four dimensional; let $S_1, S_2, S_3, S_4$ be a basis. If we take a point $\xi \in \mathbb{P}^3$ outside $\mathcal{P}$ then the subspace of quadratic forms vanishing at $\xi$ is one-dimensional. However, by Bézout’s theorem, $[12] \S 1.7$, two quadrics in $\mathbb{P}^3$ meet in eight points (cf. Cramer’s paradox). Thus there is a further point $\xi'$ associated with $\xi$. This means that the map $S : \mathbb{P}^3 \setminus \mathcal{P} \to \mathbb{P}^3; x \mapsto [S_1(x) : S_2(x) : S_3(x) : S_4(x)]$ is of degree two and, being quadratic, is Galois and associated with a (non-linear) involution of $\mathbb{P}^3$. The details can be found in $[24]$.

We recall that a trope of a Kummer surface is the intersection of the Kummer surface with a plane tangential to it along a conic. There are
sixteen tropes and they correspond to the nodes of the dual Kummer surface - [4, Ch.4].

To this framework one can associate three varieties, \( K_P, K_P^* \) and \( W_P \). They are defined as follows. \( K_P \) is the vanishing set of the degree four polynomial \( \det(z_1 S_1 + z_2 S_2 + z_3 S_3 + z_4 S_4) \) where the \( S_i \) are considered as matrices. This is called the \textit{symmetroid}. It is a Kummer surface and fairly easy to determine in terms of \( P \). Next let \( DS(x) \) be the formal derivative of \( S \) and let \( JS(x) \) its determinant, i.e. the Jacobian. Then, by definition, the vanishing set of \( JS(x) \) is the Weddle surface, \( W_P \). There is a 1–1 correspondence between \( K_P \) and \( W_P \), see [4, (5.1.5)]: it is cubic and arises from the minors of the determinant. Finally let \( K_P^* \) be the image of \( W_P \) under \( S \). It is again a Kummer surface, the dual of \( K_P \) and the map \( S \) is a partial desingularization of \( K_P^* \); fifteen of the nodes are blown up to lines in \( W_P \). This comes about as follows; for each pair of points \( \{P, P'\} \) in \( P \) the image under \( S \) of the line joining \( P \) and \( P' \) is a single point, and is then a node of \( K_P^* \). This yields the fifteen nodes; the final one is not in the image of \( S \). For any partition of \( P \) into two disjoint subsets \( \Pi, \Pi' \) each of three elements the planes defined by \( \Pi \) and \( \Pi' \) meet in a line and the image of this line is a trope in \( K_P^* \). We obtain ten of the sixteen tropes in this way. The combinatorics of these nodes and tropes describe the relations between them in a very convenient fashion. Tetrahedroids are characterised amongst Kummer surfaces by the fact that the nodes split into four sets of four each of which is contained in a plane. The four planes define the tetrahedron giving the tetrahedroid its name.

The locus of the fixed points of the involution described above are contained in \( W_P \) and, as it is irreducible is identical with it. An argument involving Galois theory and Bézout’s theorem shows that the ideal generated \( JS(x)^2 \) is generated is in the image of \( S \) and that it is generated by a polynomial of degree four. That is, there is a polynomial \( F \) of degree four so that \( JS(x)^2 = u.F(S(x)) \) where \( u \) is a scalar - see [15, p.170],[21, 22]. This result seems to be due to Schottky and we shall refer to it as “Schottky’s theorem”. Whereas the nodes of \( K_P^* \) are determined by the constructions above the polynomial \( F \) is not so easily determined in general. In our case it is quite easy as the dual \( T_{a,b,c}^* \) of \( T_{a,b,c} \) turns out to be the zero set of

\[
T_{a,b,c}^*(X_0, X_1, X_2, X_3) = \frac{1}{2} \begin{vmatrix}
0 & c^2 & b^2 & a^2 & X_0^2 \\
c^2 & 0 & 1 & 1 & X_1^2 \\
b^2 & 1 & 0 & 1 & X_2^2 \\
a^2 & 1 & 1 & 0 & X_3^2 \\
X_0^2 & X_1^2 & X_2^2 & X_3^2 & 0
\end{vmatrix}
\]

We note that

\[
T_{a,b,c}^*(abcX_0, cX_1, bX_2, aX_3) = (abc)^2 T_{a,b,c}^*(X_0, X_1, X_2, X_3),
\]
that

\[ T_{\lambda a, \lambda b, \lambda c}(X_0, X_1, X_2, X_3) = \lambda^2 T_{a, b, c}(\lambda X_0, X_1, X_2, X_3) \]

and that

\[ T^*_{\lambda a, \lambda b, \lambda c}(X_0, X_1, X_2, X_3) = \lambda^2 T^*_{a, b, c}(\lambda^{-1} X_0, X_1, X_2, X_3). \]

The classical theory ([4] p.40 and passim,[15] p.171) shows that \( K^*_P \) is the dual of \( K_P \). They are isomorphic over the complex numbers but not necessarily otherwise, [4] Ch. 4, esp. Theorem 4.5.1. In the case of tetrahedroids they are isomorphic over the field of definition. The formulæ above show the relationships.

We asserted above that \( K_P \) can be determined from \( P \). To do this we can arrange the coordinate system so that four of the points of \( P \) are \([1 : 0 : 0 : 0], [0 : 1 : 0 : 0], [0 : 0 : 1 : 0] \) and \([0 : 0 : 0 : 1] \). There remain two additional points which we write as \([p_1 : p_2 : p_3 : p_4]\) and \([q_1 : q_2 : q_3 : q_4]\).

In fact, by assumption, none of the \( p_j \) are zero and so we could also assume that \([p_1 : p_2 : p_3 : p_4] = [1 : 1 : 1 : 1]; \) this makes computations easier but sometimes obscures the structure of formulæ. In the general case we can (assuming \( p_3 q_2 \neq p_2 q_3 \)) take, as a basis the set of quadratic forms

\[
\begin{align*}
(p_4 q_3 - p_3 q_4)x_1 x_2 + (p_2 q_4 - p_4 q_2)x_1 x_3 + (p_3 q_2 - p_2 q_3)x_1 x_4 \\
p_3 q_3(p_2 q_1 - p_1 q_2)x_1 x_2 + p_2 q_2(p_1 q_3 - p_3 q_1)x_1 x_3 + p_1 q_1(p_3 q_2 - p_2 q_3)x_2 x_3 \\
(p_2 q_4 q_3 - p_1 p_3 q_2 q_4)x_1 x_2 + p_2 q_2(p_1 q_4 - p_4 q_1)x_1 x_3 + p_1 q_1(p_3 q_2 - p_3 q_3)x_2 x_4 \\
p_3 q_3(p_4 q_1 - p_1 q_4)x_1 x_2 + (p_1 p_2 q_3 q_4 - p_3 p_4 q_1 q_2)x_1 x_3 + q_1(q_2 - q_3)x_3 x_4
\end{align*}
\]

denoted by \( S_1, S_2, S_3 \) and \( S_4 \) respectively. One can evaluate \( J S(x) \) from this but a simple observation, due to Caspary and Hutchinson,[15] p.170], that it is, up to a multiple,

\[
W_{p,q}(x) = \begin{vmatrix}
   x_1^2 & p_1 x_1 & q_1 x_1 & p_1 q_1 \\
   x_2^2 & p_2 x_2 & q_2 x_2 & p_2 q_2 \\
   x_3^2 & p_3 x_3 & q_3 x_3 & p_3 q_3 \\
   x_4^2 & p_4 x_4 & q_4 x_4 & p_4 q_4
\end{vmatrix}
\]

The determination of \( K_P \) is as follows. We let \( V \) be a symmetric matrix with diagonal elements 0 and we write \( \Delta(v_{12}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34}) \) for \( \det(V) \). Then \( K_P \) is the intersection in \( \mathbb{P}^5 \) of \( \Delta(v_{12}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34}) = 0 \) with the two hyperplanes \( \sum_{i<j} p_j v_{i,j} = 0 \) and \( \sum_{i<j} q_j v_{i,j} = 0 \). This is associated with the so-called “irrational form” ([15] §19) of the equation of a Kummer surface. Also from this one can give the map from \( K_P \to W_P \) explicitly; for all this see [15] pp.171,172]. The maps are cubic, as noted above.

One can relatively easily derive the “irrational form” from information about the nodes and tropes of the Kummer surface - [15] pp.34–36]. The “irrational form” involves three products of pairs of linear forms defining
tropes. Write these as \( L_1(x)L_1'(x) \), \( L_2(x)L_2'(x) \) and \( L_3(x)L_3'(x) \); the equation takes the form

\[
\sqrt{L_1(x)L_1'(x)} + \sqrt{L_2(x)L_2'(x)} + \sqrt{L_3(x)L_3'(x)} = 0.
\]

In language more acceptable now this means

\[
(L_1(x)L_1'(x) + L_2(x)L_2'(x) - L_3(x)L_3'(x))^2 - 4L_1(x)L_1'(x)L_2(x)L_2'(x) = 0.
\]

We shall take

\[
L_1 = -(a - b + c)(cx_0 + x_2 + x_3), \quad L_1' = (-a + b + c)(-cx_0 + x_2 + x_3)
\]

\[
L_2 = (a + b + c)(cx_0 + x_2 - x_3), \quad L_2' = (a + b - c)(-cx_0 + x_2 - x_3)
\]

\[
L_3 = 2(-cx_1 - bx_2 + ax_3), \quad L_3' = 2(cx_1 - bx_2 + ax_3)
\]

One verifies that \( L_1 + L_1' + L_2 + L_2' + L_3 + L_3' = 0 \) and with \( q_1 = (a + b - c)(a - b + c) \), \( q_2 = -(a + b + c)(-a + b + c) \), \( q_3 = -(a + b - c)(a + b - c) \) and \( q_4 = (a + b + c)(a - b + c) \) then one has

\[
L_1q_2q_3 + L_1'q_1q_4 + L_2q_1q_3 + L_2'q_2q_4 + L_3q_1q_2 + L_3'q_3q_4 = 0.
\]

One finds now

\[
16c^2T_{a,b,c} = -((L_1L_1' + L_2L_2' - L_3L_3')^2 - 4L_1L_1'L_2L_2')
\]

The identity Schulz used and his version of Ankum’s observation is a variant of this.

It follows from [15 p.171] that the set \( P \) corresponding to \( T_{a,b,c} \) can be taken to be the four basis points, which we write as \( P_1, P_2, P_3, P_4, P_5, P_6 \), with \( P_5 = [1 : 1 : 1 : 1] \) and \( P_6 = [q_1 : q_2 : q_3 : q_4] \). One can now calculate the nodes of the Kummer surface which is the image under \( S \) of the Weddle surface. The four planes of the tetrahedron are simple to find and with respect to the natural set of coordinates associated with them we can identify the image as \( T_{2a,2b,2c} \). All of the coordinates are rational in \( a, b, c \). The condition that the Kummer surface be a tetrahedroid is that \( q_1q_2 = q_3q_4 \), see [21 p.366]. Note that \( q_1q_2 = -H(a,b,c) \). The calculations at this point are routine and the results are not so elegant that they need be exhibited here. It may be the case that these are contained in the formulæ in [11 §11] – see [4 p.40] and [8 p.953]. I have not attempted to verify this.

4 Completion of the proof of Schulz’ theorem

The result of the previous section shows that if \( P \) and \( S = (S_1, S_2, S_3, S_4) \) are as above then there is an element \( \gamma \in \text{GL}(4, \mathbb{Q}(a,b,c)) \) so that for some element \( u \in \mathbb{Q}(a,b,c)^\times \) one has

\[
W_{1,q}(x)^2 = kT_{a,b,c}(\gamma(S(x))),
\]

9
where $1 = (1, 1, 1, 1)$ and $q = (q_1, q_2, q_3, q_4)$. We can now specialize $x$ to be a point so that $S(x)$ lies in the hyperplane defined by $L_1$ but does not lie in the Weddle surface. To see this we can argue as follows. The trope defined by $L_1$ intersects the image of $S$ along a conic. The description of $S$ given above shows that this is the image of a line which is the intersection of two planes each containing three nodes. As $L_1 \circ S$ is a quadratic form we conclude that it vanishes along a rational line. In particular the variety $L_1 \circ S = 0$ is rational over $Q$ for any admissible choice of $a, b, c$ and so the rational points are Zariski dense in the quadric. This means that the rational points of $L_1 \circ S = 0$ are not contained in the Weddle surface and it follows that there is a point $x$ not in the Weddle surface so that $L_1(S(x)) = 0$. Evaluating (*) at $x$ we conclude that $T_{a,b,c}(\gamma(S(x)))$ lies in $-Q(a, b, c)^{x^2}$ and therefore $k$ lies in $-Q(a, b, c)^{x^2}$. This completes the proof of Schulz’ theorem as all the expressions are rational in $a, b, c$.

We note that the approach given here shows that there is an explicit quadratic function $S^*$, depending rationally on $a, b, c$, a rational function $\kappa(a, b, c)$ and an explicit quartic function namely $W_{1,q}$ also depending rationally on $a, b, c$ so that

$$W_{1,q}(x) = -\kappa(a, b, c)^2T_{a,b,c}(S^*(x))$$

We have not given an explicit expression for $\kappa(a, b, c)$ but it would be possible to do so by comparing coefficients.

The tetradedroids have nodes and tropes which are rational (see [10]) but there are other constructs which are not rational. Heegner gave another representation of the varieties $CM(d) = 0$ and $V^2 = CM(d)$ involving $Q(\sqrt{-H(d_{12}, d_{13}, d_{23}))}$ or one of the further. In the case of rational quadrilaterals this is independent of the choice of “face”; in the case of rational tetrahedra it is not as the factors corresponding to two different faces can differ multiplicatively by a norm from $Q(\sqrt{-1})$. Heegner stressed the significance of classifying the solutions by, in the first case, the field and in the second case, by the associated quaternion algebra. There are quite a number of questions which arise in this context. One is whether there is a combinatorial structure such as the theory of the Markoff tree in the case the the Markoff equation and its generalizations, [3 Ch. 2], [19 pp.106–110].

finally we note that the proof of Schulz’ theorem given here is a geometrical version of the original proof. It does not rule out that either the variety considered in the theorem, or the more refined varieties considered by Heegner are rational over $Q$. The proof given here, and Schulz’ own one show that $V^2 = -CM(d)$ is rational over $Q$. In the case of Heron triangles both $Y^2 = H(a, b, c)$ and $Y^2 = -H(a, b, c)$ are rational. To see this let $U = (-a + b + c)/(a + b + c)$, $V = (a - b + c)/(a + b + c)$ and $Z = Y/(a + b + c)^2$. Then the two equations become $Z^2 = UV(2 - U - V)$ and $Z^2 = -UV(2 - U - V)$. For fixed $V$ these are quadratic in $U$ and
have rational points, $U = 0$ and $U = 2 - V$. They are then rational and
with varying $V$ we get the required parametrization. The question as to
the rationality of $V^2 = CM(d)$ itself over $\mathbb{Q}$, or, for that matter, over $\mathbb{R}$
is still very much open. One notes that the real loci of $T_{a,b,c}(x) = 0$ seem,
on the basis of graphical experiments, always to be connected. If it were
otherwise it would represent an obstruction to $y^2 = T_{a,b,c}(x)$ being rational.
At the present the evidence, such as it is, is against a positive resolution of
Schwarz’ problem over $\mathbb{Q}$, or even $\mathbb{R}$.

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