RELATIVISTIC FLOWS AFTER SHOCK EMERGENCE

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ABSTRACT

We investigate relativistic flows that occur after a shock wave generated inside a star reaches the surface. First, the effect of sphericity is included through successive approximations by adding correction terms to an already known self-similar ultrarelativistic solution for plane-parallel geometry. The inclusion of sphericity increases the early acceleration compared with the original plane-parallel flow. Second, we obtain semianalytic solutions for a mildly relativistic flow in which the rest-mass energy density is nonnegligible in the equation of state (EOS). To take this into account, we use energy and pressure instead of density and pressure as thermodynamic variables. These solutions assume self-similar evolution except for the initial conditions. Third, we carry out numerical calculations with a special relativistic hydrocode based on Godunov’s method to check the applicability of the sphericity corrections and semianalytic solutions. The EOS used in our calculations includes the rest-mass energy density. Comparisons with the numerical calculations support the validity of the sphericity corrections. The evolution of the pressure and Lorentz factor of each fluid element in the semianalytic solution for mildly relativistic flows matches the numerical results, at least for early phases. We also numerically investigate the final free-expansion phase. For spherically symmetric or plane-parallel ultrarelativistic flows, we could not observe this phase even when the self-similar variables grew to 10^6. However, flows in which the ratio of pressure to density at shock breakout is less than unity did reach free expansion. We derive the final energy distributions for these flows and compare with previous work.

Subject headings: hydrodynamics — relativity — shock waves

1. INTRODUCTION

A great deal of study has been devoted to shock propagation in stellar explosions. If sufficient energy is liberated inside a star, a shock wave will form and travel toward the surface. Once it arrives there, it disappears. This event is referred to as shock breakout or shock emergence. Material from the stellar envelope, accelerated by the shock wave, expands into the surrounding space as a rarefaction wave. The ejected matter continues to accelerate until a sufficient amount of its thermal energy is converted into kinetic energy. Finally, it expands freely.

Gandel’mann & Frank-Kamenetskii (1956) were the first to pose this problem. Sakurai (1960) solved it with a self-similar analysis, assuming a one-dimensional plane-parallel (PP) configuration. The stellar surface marks the boundary; the atmosphere fills one side of it, and on the other side is vacuum. The initial density distribution is assumed to follow a power law \( \rho_0 \propto x^\delta \), where \( x \) is the distance from the surface. In particular, \( \delta = 3 \) represents a radiative stellar envelope and \( \delta = 1.5 \) a convective one. Sakurai’s self-similar solution is, however, severely restricted in terms of applicability to a real star, because sphericity cannot be ignored in describing the flow far from the surface. Kazhdan & Murzina (1992) worked to include sphericity by adding correction terms proportional to the dimensionless parameter \( x_0 = 1 - \rho/R_* \), where \( \rho \) is the distance from the star’s center and \( R_* \) is the stellar radius. Matzner & McKee (1999) conducted a more detailed study. They investigated the effects of the progenitor’s structure and the inclusion of sphericity, deriving semianalytic distributions for such physical quantities as the velocity and density. These studies ignored the effects of special relativity and radiative cooling, and thus the speed of the front in these studies continues to increase to infinity.

Tan et al. (2001) studied transrelativistic flows. They estimated the maximum velocity of the ejecta, taking into account the effects of sphericity and radiative diffusion. They also studied the effects of aspherical structure and gravity.

Ignoring the rest-mass energy, Nakayama & Shigeyama (2005) obtained a self-similar solution in the ultrarelativistic limit. They assumed the same initial configuration as used by Sakurai (1960). In addition, the equation of state (EOS) was taken to be \( \varepsilon = 3p \) instead of \( \varepsilon = 3p + \rho \), where \( p \) is the pressure, \( \rho \) is the density, and \( \varepsilon \) is the internal energy density per unit volume. This is the indispensable assumption to derive a self-similar solution. Pan & Sari (2006) have presented an analytical expression describing the flow after shock emergence.

In this paper, we concentrate on two issues inherent to the above ultrarelativistic self-similar solutions: First, the assumption of spherically symmetric (SS) geometry is a better approximation of a real system than one of planarity. The farther a wave is from the stellar surface, the greater the deviation from the PP case. The end of acceleration cannot be described within the range in which the PP approximation is valid, that is, \( |x_0| \ll 1 \); the pressure, which varies as \( p \propto \rho^{4/3} \propto (1 - x_0)^{-4} \), does not significantly drop in this range. To take sphericity into account, we need to introduce a characteristic length, the stellar radius \( R_* \), which prohibits an analysis with self-similar solutions. Thus, in §2 we conduct successive approximations in terms of a parameter \( \zeta = a/R_* \) (where \( a \) is the distance of each fluid element from the stellar surface at shock emergence), which we use to characterize sphericity. From the set of fundamental equations for ultrarelativistic hydrodynamics, we find that the sphericity is involved to order \( O(\varepsilon^{1/[2(2\sqrt{3} - 3)]+1}) \) after shock emergence and to \( O(\zeta) \) before shock emergence. We take this effect into account only up to \( O(\varepsilon^{1/[2(2\sqrt{3} - 3)]+1}) \).
The second issue is that the acceleration does not terminate in Nakayama & Shigeyama’s (2005) solution. The Lorentz factor $\gamma$ of a fluid element increases indefinitely with time $t$ as measured from the moment of shock emergence: $\gamma \propto t^{0.732}$ for any $\delta$. The energy source for acceleration is the internal thermal energy, which is converted to kinetic energy during expansion. It is reasonable to consider that when the internal energy is sufficiently depleted, the ejecta are no longer accelerated. This implies that these self-similar solutions cannot describe a stage of free expansion. This might be a result of neglecting the rest-mass energy in the $\epsilon = 3\rho$ EOS. The EOS has to be $\epsilon = 3\rho + \rho$ in later phases, when the gas becomes only mildly relativistic. In $\S$ 3, we derive a semianalytic solution including this term in the EOS.

In $\S$ 4, we describe numerical calculations conducted using the Godunov method for relativistic hydrodynamics. The aim is to examine the validity of the sphericity treatment and the semianalytic solutions for mildly relativistic flows. Calculations are carried out from the moment of shock emergence for both PP and SS configurations and various initial energy strengths, setting $\delta = 3$ and using the initial conditions from the analytical solutions at $t = 0$ of Nakayama & Shigeyama (2005). These results are presented in $\S$ 4.2.

2. INCLUSION OF SPHERICITY

2.1. Formulation

In this section, we derive sphericity corrections for the PP flow after shock emergence. First, we present the basic equations for an ultrarelativistic SS flow. Then we transform these equations into nondimensional forms, and finally, we employ successive approximations to obtain zeroth- and first-order equations. The zeroth-order equations reproduce the self-similar solutions obtained by Nakayama & Shigeyama (2005). The first-order ones then provide the amount of correction.

We confine ourselves to dealing with a perfect fluid subject to no external force, including gravity. If a shock wave is strong enough, gravity will have practically no effect on the fluid’s motion. We assume an initial density distribution $\rho_0 = b(R_s - r)^m$, where $b$ is a constant. The basic equations of relativistic hydrodynamics are derived from the conservation of energy–momentum and mass. Here we present the equations for SS flows in a fixed frame $(r, t)$:

$$\frac{\partial}{\partial t} [\gamma^2 (\varepsilon + \beta^2 p)] + \frac{1}{r^2} \frac{\partial}{\partial r} [r^2 \gamma^2 (\varepsilon + p) \beta] = 0,$$

$$\frac{\partial}{\partial t} [\gamma^2 (\varepsilon + p) \beta] + \frac{1}{r^2} \frac{\partial}{\partial r} [r^2 \gamma^2 (\varepsilon + p) \beta^2] + \frac{\partial p}{\partial r} = 0,$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho \beta) = 0,$$

where $\varepsilon$ denotes the internal energy density, $p$ is the pressure, and $\rho$ is the density, all measured in the fluid frame; $\rho^\prime = \gamma \rho$ is the density measured in the fixed frame, where the fluid moves with velocity $c_\beta$, and $\gamma$ denotes the Lorentz factor, defined as $\gamma = 1/(1 - \beta^2)^{1/2}$. We set the speed of light to $c = 1$ in what follows. We assume the ultrarelativistic EOS

$$\varepsilon = 3\rho.$$  

With this EOS, equations (1)–(3) transform into the expressions presented by Blandford & McKee (1976):

$$\frac{d}{dt} (p s^4) = \gamma^2 \frac{\partial p}{\partial t} \frac{d}{dt} \ln (p^s s^4) = -\frac{4}{r^2} \frac{\partial}{\partial r} (p^2 \beta),$$

$$\frac{d}{dt} (p \rho^{-4/3}) = 0,$$

where the convective derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \beta \frac{\partial}{\partial r}$$

has been introduced.

In order to describe the motion after shock emergence, it is convenient to use a Lagrangian coordinate $a$ defined as the position of a fluid element at $t = 0$ as measured from the stellar surface. Then, from the definition of the velocity of the fluid element, we obtain

$$\frac{dr}{dt} = v.$$  

For a PP flow, a single dimensionless variable ($\eta$) is sufficient to describe the self-similar motion of the gas. In contrast, for an SS flow, where we must deal with the wave across the stellar surface, the stellar radius $R_s$ appears as a length scale, and the flow is not self-similar any more. Another nondimensional variable is necessary besides $\eta$. We can write these variables as

$$\eta = \frac{t}{\gamma^2 a}, \quad \xi = \frac{a}{R_s}.$$  

The dependent variables are written using nondimensional functions $F$, $G$, $H$, and $I$ such that

$$p = p_1(a) F(\eta, \xi),$$

$$\gamma^2 = \gamma_1^2(a) G(\eta, \xi),$$

$$\rho = \rho_1(a) H(\eta, \xi), \quad r = (R_s - a) I(\eta, \xi).$$

Here $p_1(a)$, $\gamma_1^2(a)$, and $\rho_1(a)$ are the profiles at $t = 0$. They obey power laws

$$p_1(a) = c_p(A, b) a^{(6-m)/(m+1)},$$

$$\gamma_1^2(a) = c_\gamma(A, b) a^{-m/(m+1)},$$

$$\rho_1(a) = \gamma_1 p_1 = c_\rho(A, b) a^{(6-m)/(m+1)},$$

where

$$m = (2\sqrt{3} - 3)\delta.$$  

and $c_p(A, b)$, $c_\gamma(A, b)$, and $c_\rho(A, b)$ are constants that depend on the quantities $A$ and $b$, which characterize the flow before emergence of the shock. The parameter $A$ specifies the amount of energy in the flow through a relation with the shock Lorentz factor ($\Gamma$). $\Gamma^2 = A(\Gamma - 1)^m$. (For a more detailed description, see Nakayama & Shigeyama 2005.) We transform the independent variables from $(t, a)$ to $(\eta, \xi)$ using the following relations:

$$\frac{d}{dt} = \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial t} = \frac{\eta}{t} \frac{\partial \eta}{\partial t},$$

$$\frac{\partial}{\partial a} = \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial a} + \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial a} = -\frac{1}{m+1} \frac{\eta}{a} \frac{\partial \eta}{\partial \xi} + \frac{\xi}{a} \frac{\partial \xi}{\partial \xi}.$$
Then equations (5)–(6) and (8) can be converted into the following nondimensional expressions:

\[
\frac{\partial}{\partial \eta} (\ln F + 2 \ln G) = \frac{I^2 H}{\sqrt{G}} \left( \frac{\delta - m}{m + 1} - \frac{\eta}{m + 1} \frac{\partial F}{\partial \eta} + \frac{\xi \partial F}{\partial \xi} \right),
\]

\[
\frac{\partial \ln G}{\partial \eta} + \frac{3}{2} \frac{\partial \ln F}{\partial \eta} = I^2 H \sqrt{G} 
\times \left( - \frac{\eta}{m + 1} \frac{\partial \ln G}{\partial \eta} + \xi \frac{\partial \ln G}{\partial \xi} - \frac{m}{m + 1} \right) - 4 \gamma^2 \xi \frac{G}{1 - \xi},
\]

\[
\frac{\partial \ln F}{\partial \eta} = \frac{4}{3} \frac{\partial \ln H}{\partial \eta}, \quad \frac{\partial I}{\partial \eta} = \frac{\gamma^2 \xi}{1 - \xi}.
\]

Expansions of these formulae in \(\gamma^{-2}\) can be truncated at the first contributing term because the motion is assumed to be ultra-relativistic. The last term in equation (18) is of order \(\xi^{1/(m+1)}\).

Therefore, we expand equations (10)–(11) as power series in \(\xi\), starting with the term proportional to \(\xi^{1/(m+1)}\):

\[
F(\eta, \xi) = F_0(\eta)[1 + \xi^{1/(m+1)} \Gamma_1(\eta)],
\]

\[
G(\eta, \xi) = G_0(\eta)[1 + \xi^{1/(m+1)} \Gamma_1(\eta)],
\]

\[
H(\eta, \xi) = H_0(\eta)[1 + \xi^{1/(m+1)} \Gamma_1(\eta)],
\]

\[
I(\eta, \xi) = I_0(\eta)[1 + \xi^{1/(m+1)} \Gamma_1(\eta)].
\]

The zeroth-order equations become

\[
\frac{d l_0}{d \eta} = 0, \quad \frac{d \ln F_0}{d \eta} = \frac{4}{3} \frac{d \ln H_0}{d \eta},
\]

\[
\frac{d \ln F_0}{d \eta} + \frac{3}{2} \frac{d \ln G_0}{d \eta} = \frac{I_0^2 H_0}{\sqrt{G_0}} \left( \frac{\delta - m}{m + 1} - \frac{\eta}{m + 1} \frac{d F_0}{d \eta} \right).
\]

\[
\frac{d \ln G_0}{d \eta} + \frac{3}{2} \frac{d \ln F_0}{d \eta} = - \frac{I_0^2 H_0 \sqrt{G_0}}{m + 1} \left( \eta \frac{d \ln G_0}{d \eta} - m \right).
\]

These correspond to the planar case. The first-order equations are

\[
\frac{d l_1}{d \eta} = c, R^{-m/(m+1)} \frac{d F_1}{d \eta} = \frac{4}{3} \frac{d H_1}{d \eta},
\]

\[
\frac{(m + 1) \sqrt{G_0}}{I_0^2 H_0} \left( \frac{d F_1}{d \eta} + \frac{d G_1}{d \eta} \right) = \left( 2 \eta + H_1 \right) \left( \frac{\delta - m}{m + 1} - \eta \frac{d F_0}{d \eta} \right) - \eta \frac{d F_1}{d \eta} + \frac{G_1}{2}
\]

\[- \eta \frac{d F_1}{d \eta} + F_0 \Gamma_1,
\]

\[
\frac{m + 1}{I_0^2 H_0 \sqrt{G_0}} \left( \frac{d G_1}{d \eta} + 3 \frac{d F_1}{d \eta} + 4c \frac{R^{-m/(m+1)} G_0}{d \eta} \right) = - \eta \frac{d G_1}{d \eta} + G_1 - \left( 2 \eta + H_1 \right) \left( \eta \frac{d \ln G_0}{d \eta} - m \right).
\]

Solutions of equations (24)–(29) are obtained through numerical integration with initial conditions

\[
F_0(0) = G_0(0) = H_0(0) = I_0(0) = 1, \quad \Gamma_1(0) = 0.
\]

As this implies, we adopt the distributions for PP geometry at \(t = 0\) as the initial conditions for the SS geometry, because we can neglect deviations from PP flow at shock emergence. All sphericity corrections prior to shock emergence are \(O(\xi)\), which is of higher order than those after shock emergence, \(O(\xi^{1/(m+1)})\).

2.2. Solutions

The results are shown in Figure 1. The dash-dotted lines represent the flow with the sphericity correction, and the dashed lines are the self-similar solutions. The effect of sphericity is to cause the density \(H\) and pressure \(P\) to drop more steeply than in the PP case.

It is appropriate to confirm the range over which the correction is valid. This is one incentive to conduct numerical calculations. In §4.2.2, we examine the validity of our correction.

3. MILDLY RELATIVISTIC FLOW

3.1. Formulation

In later phases, \(p\) becomes comparable to \(\rho\), and we will no longer be able to ignore the rest-mass energy density in the EOS. We must then use the EOS \(\varepsilon = 3p + \rho\) instead of \(\varepsilon = 3p\). In order to include the term for the rest-mass energy density, we replace the dependent variable \(\rho\) in equations (1)–(3) with the enthalpy per unit volume: \(w = 4p + \rho\). As a result, the basic equations become

\[
\frac{\partial}{\partial t} (w \gamma^2 \beta) + \frac{\partial}{\partial x} (w \gamma^2 \beta^2 + p) = 0,
\]

\[
\frac{\partial}{\partial t} (w \gamma^2 - p) + \frac{\partial}{\partial x} (w \gamma^2 \beta) = 0,
\]

\[
\frac{\partial}{\partial t} [\gamma(w - 4p)] + \frac{\partial}{\partial x} (\beta \gamma(w - 4p)) = 0.
\]

These expressions combine to yield

\[
\frac{w}{2} \frac{d s^2}{d t} + \gamma^2 \frac{d p}{d t} = \frac{d \rho}{d t},
\]

\[
\frac{1}{2} \frac{d \ln \gamma^2}{d t} + \frac{d \ln w}{d t} - \frac{1}{w} \frac{d p}{d t} = - \frac{d \beta}{d t},
\]

\[
\frac{d}{d t} [\ln (w - 4p)] + \frac{d}{w} (p - w) = 0.
\]

To use the Lagrangian coordinate as in the self-similar ultrarelativistic PP solution, another equation, not involving \(\rho\), is needed. This can be obtained by differentiating the definition of the fluid velocity with respect to \(a\) as

\[
\frac{d}{d t} \frac{d x}{d a} = \frac{d v}{d a}.
\]

We seek a solution of the form

\[
p = p_1(a) F(\eta), \quad \gamma^2 = \gamma_1^2(a) G(\eta),
\]

\[
w = 4p_1(a) W(\eta), \quad \frac{\partial \gamma}{\partial a} = J(\eta).
\]
one fluid element with preshock position solid and dotted lines from the numerical calculations were generated by tracing at shock emergence. The sphericity correction was obtained for numerical calculations were carried out with the parameter an ultrarelativistic SS flow from a numerical calculation, respectively. The two involvings sphericity, an ultrarelativistic PP flow from a numerical calculation, and dotted, solid, and dotted lines correspond to the self-similar solutions, the solutions (39)–(40) into equations (35)–(38) and obtain nondimensional equations,

\[
\frac{2W' G'}{F'} F + \left[ 1 + \frac{\eta}{(m+1)G} \right] F = \frac{\delta - m}{m+1}\frac{1}{G}, \\
2 \left( 1 + \frac{\eta}{m+1} \right) F' + \frac{4 W'}{F} = \frac{2m-1}{m+1} F',
\]

\[
4 \frac{W'}{F} = \left( 3 + \frac{F}{W} \right) \frac{F'}{F'},
\]

\[
J' = \frac{1}{2(m+1)G} \left( \frac{G'}{G} + m \right).
\]

The initial conditions are

\[
F(0) = 1, \quad G(0) = 1, \quad J(0) = 1,
\]

\[
W(0) = 1 + \frac{\rho_1(a)}{4\rho_1(a)}.
\]

If \(W(0) = 1\), the solutions agree with the results of Nakayama & Shigeayama (2005). Otherwise, they are not self-similar and \(F, G, W, J\) depend not only on \(\eta\) but also on \(a\) and \(A\) through equation (46). We check the validity of the assumed form of the solution by comparison with results obtained from direct numerical integrations of the relativistic hydrodynamic equations in § 4.

3.2. Solutions

The quantities \(F\) and \(G\), integrated with various values of \(W(0)\), are shown in Figure 2. It is obvious that mildly relativistic flows, unlike an ultrarelativistic flow, which forgets its history, depend on the initial energy distribution.

From the numerical integration, we find that \(F, G, W, J\) undergo power-law evolution in the limit \(\eta \to \infty\). The indices are shown in Table I for \(\eta > 10^5\). The values \(dx/dy\) in \(\eta\) in this table indicate that even though the semianalytic solutions include the rest-mass energy in the EOS, the acceleration does not completely cease. For \(\rho_1/\rho_i \approx 1\), the acceleration almost terminates as \(\eta \to \infty\). In contrast, the acceleration does not terminate at all for \(\rho_1/\rho_i \gtrsim 1\).

4. NUMERICAL CALCULATIONS

4.1. Calculation Procedure

We carried out numerical calculations using the Godunov method for relativistic hydrodynamics in the Lagrangian coordinate system (Marti & Müller 1996). To write down the basic equations for relativistic hydrodynamics in conservative form, we have to introduce some new variables, \(V, s,\) and \(Q\). These are related to the quantities \(\rho\) and \(w\) in the local rest frame of the fluid through

\[
V \equiv \frac{1}{\rho_0\gamma}, \quad s \equiv \frac{w_0\gamma v}{\rho}, \quad Q = \frac{w_0\gamma^2 -p}{\rho_0\gamma} - 1.
\]

For PP geometry, the basic equations are expressed in these variables as

\[
\frac{dV}{dt} = \frac{\partial v}{\partial s}, \quad \frac{ds}{dt} = -\frac{\partial p}{\partial s}, \quad \frac{dQ}{dt} = -\frac{\partial (\rho v)}{\partial s},
\]
where the Lagrangian coordinate is defined as

$$dm = -\rho \gamma_1 da = -\rho \gamma dx.$$  

(49)

For SS geometry,

$$\frac{dV}{dt} = \frac{\partial}{\partial m_r}(4\pi r^2 v), \quad \frac{ds}{dt} = -\frac{\partial}{\partial m_r}(4\pi r^2 \rho),$$

(50)

$$\frac{dQ}{dt} = -\frac{\partial}{\partial m_r}(4\pi r^2 \rho v)$$

(51)

and the Lagrangian coordinate is

$$dm_r = -4\pi(R_\ast - a)^2 \rho_1 \gamma_1 da = 4\pi r^2 \rho_1 \gamma dr.$$  

(52)

These equations are integrated numerically by evaluating the derivatives on the right-hand sides using the Godunov (1959) method. The difference lies in the EOS: $\varepsilon = 3p + \rho$ in the numerical calculations, while $\varepsilon = 3p$ in the self-similar solutions.

Calculations were carried out for both PP and SS geometry from the moment of shock emergence ($t = 0$). We set the initial conditions as for the self-similar solutions at time $t = 0$. The parameters used for the calculations are $\delta = 3, A = 1.0 \times 10^{16}, 1.0 \times 10^{12}, \text{and } 1.0 \times 10^{10}, \text{and } R_\ast = 1.0 \times 10^9 \text{ m. Here } A \text{ has dimensions of some power of time. Thus, the values of } A \text{ are expressed using seconds as the unit. Note that } \gamma_1 \text{ depends on } A \text{ as}

$$\gamma_1 \propto A^{1/2(m+1)} \sim A^{0.2}.$$  

(53)

Hence, $\gamma_1$ varies by a factor of about 10 for the above values of $A$.

We obtain $p, \rho,$ and $\gamma$ directly from the numerical results. The dimensionless quantities $F, G$, and $H$ can be obtained from $p/p_1, \gamma/p_1,$ and $\rho/p_1,$ respectively. On the other hand, when we made calculations starting before shock emergence, the moment of emergence could not be determined exactly. As a result, we could not determine $p_1, \gamma_1,$ or $\rho_1$ to sufficient accuracy. This was caused by the finite thickness of the shock front in the numerical calculations. Therefore, in the following we dedicate ourselves to calculations after shock emergence.

4.2. Results

4.2.1. Connection to the Initial States

It is preferable to describe the results after shock emergence in terms of the original position of each fluid element, $x_0$, rather than the Lagrangian coordinate $a$, to aid in actual application to astrophysical phenomena. For PP geometry, the definition of the Lagrangian coordinate provides an exact relation between $a$ and $x_0$:

$$\int_0^{x_0} \rho_0 \, dt = \int_0^a \rho_1' \, da.$$  

(54)

Substituting $\rho_0 = b \rho_0^4$ and the expression for $\rho_1'$ (eq. [14]) into both sides of this equation, we obtain

$$x_0 = \left[2A(m+1) \right]^{1/[(m+1)/3]} a^{1/(m+1)}

\times \left\{ \frac{2A[1 + (2\sqrt{3} - 3)\delta]}{[3 + 2\sqrt{3}(1 + \delta)]^{(1+\delta)/2}[1+\sqrt{3}(1+\delta)]} \right\}^{\lambda},$$

(55)

where $\lambda \equiv (\delta - m)/[(m+1)(\delta + 1)].$ If $\delta = 3$, then $x_0 = 1.56,0.48,0.18,0.418$. Although the corresponding relation $r_0 = r_0(a)$ for SS is determined by

$$\int_0^a 4\pi (R_\ast - a)^2 \rho_1' \, da = \int_{r_0}^{R_\ast} 4\pi r^2 \rho_0 \, dr_0,$$

(56)

it is difficult to express $r_0$ as a function of $a$ in analytical form unless one truncates higher terms in $a/R_\ast$ and $x_0/R_\ast$. To lowest order, the form is the same as in the PP case,

$$r_0 \approx \left[2A(m+1) \right]^{1/[(m+1)/3]} a^{1/(m+1)}

\times \left\{ \frac{2A[1 + (2\sqrt{3} - 3)\delta]}{[3 + 2\sqrt{3}(1 + \delta)]^{(1+\delta)/2}[1+\sqrt{3}(1+\delta)]} \right\}^{\lambda}.$$  

(57)
Using these relations, after shock emergence a fluid element can be traced back to its original position.

In addition, we can see how the parameters characterizing the explosion, such as the injected energy $E_{\text{inj}}$ and the ejected mass $M_{\text{ej}}$, affect the flow through the parameter $A$. The value of $A$ can be related to $E_{\text{inj}}$ and $M_{\text{ej}}$ by

$$E_{\text{inj}} = \int \gamma^2 \rho \, dx \propto \Gamma^{-m} M_{\text{ej}} \propto A M_{\text{ej}}.$$  \hfill (58)

Thus we obtain

$$A \propto E_{\text{inj}} / M_{\text{ej}}.$$  \hfill (59)

It is desirable to obtain the proportionality constant of equation (59), which connects the shock velocity with $E_{\text{inj}}$ and $M_{\text{ej}}$. Tan et al. (2001) proposed an analytical formula (their eqs. [4] and [9]) to estimate such a coefficient (see also eqs. [17], [18], and [19] of Matzner & McKee 1999). In our case, we can obtain an expression for $A$ as a function of $E_{\text{inj}}$ and $M_{\text{ej}}$ by substituting $\rho = \rho(-t)^d$ into equation (4) of Tan et al. (2001).

4.2.2. Effects of Sphericity

We now discuss the validity of our sphericity correction. Figure 1 shows the trajectories of a fluid element with preshock position $x_0/R_0 = 4.49 \times 10^{-3}$ and $p_1/\rho_1 = 33.1$ at shock emergence (the ratio of the internal to the kinetic energy is about 132). The solutions of equations (20)–(22) including the sphericity correction (dash-dotted lines) match the numerical results for the SS case with $A = 1.0 \times 10^{16}$ (dotted lines) at small $\eta$. On the contrary, the planar flows start to deviate from the very beginning. The degree of accuracy of the sphericity correction is shown in Table 2. Our description for an SS flow is a good approximation in the range $\eta \leq 0.5$, within an accuracy of 5%.

Next we compare the ultrarelativistic PP and SS flows from the numerical calculations (shown in Fig. 1). The density of the SS flow decreases more drastically than that in the PP case (solid line, bottom). This results from the difference in geometry; the volume of a spherical shell expands more than a PP shell does. Then the pressure in the SS flow also decreases more than in the PP case (top), because the flow is adiabatic. The Lorentz factor in the SS flow increases more dramatically than that in the PP version. As the acceleration advances, the ratio of the internal energy to the kinetic energy, which is about $4\rho/\rho$ for $\gamma \rightarrow \infty$ in the observer’s frame, drops with decreasing $\rho$ because $p/\rho$ decreases as $\rho^{1/3}$ in an adiabatic flow. As a consequence, the acceleration of an SS flow is more quickly advanced and terminates on a shorter timescale than in the PP case.

4.2.3. Intermediate Distribution

For $\eta \rightarrow \infty$, the values of $F$, $G$, and $H$ from the self-similar solution have power-law distributions with respect to $\eta$. We refer to these as the intermediate distributions. $F$, $G$, and $H$ from the numerical calculations can also be fitted by power-law distributions. The indices are shown in Table 3 as obtained for $\eta \sim 10–100$. Comparing the self-similar flow and the ultrarelativistic SS flow obtained numerically, the differences in the indices are 1.1%, 5.7%, and 0.63% for $d \ln F/d \ln \eta$, $d \ln G/d \ln \eta$, and $d \ln H/d \ln \eta$, respectively. The difference in $d \ln G/d \ln \eta$ is particularly prominent, which implies that the rest-mass energy density acts to suppress the acceleration. That is to say, $d \ln G/d \ln \eta$ in the numerical calculations is smaller than that for the self-similar solutions. The values of $p_1/\rho_1$ range from 10 to 70 depending on $a$ in this particular numerical calculation.

Figure 3 shows the evolution of several fluid elements in a PP flow obtained from a numerical calculation. Although each line has a different value of $p_1/\rho_1$ in the range from 8.5 to 33, the trajectories are insensitive to it, which indicates that the flow is very close to self-similar. If we allow $p_1/\rho_1 \rightarrow \infty$, the results will approach the self-similar solutions.

The numerical results show that an SS flow behaves differently from a PP flow. This is illustrated in Figure 4. The trajectory of each fluid element of an SS flow depends on the initial position $a$. It is insufficient to use $\eta$ to characterize the flow. The values of the indices vary in the ranges indicated in Table 3.

4.2.4. Mildly Relativistic Flow

The acceleration in the numerical calculations is suppressed as compared with that from the self-similar solutions because of the rest-mass energy in the EOS used for the numerical calculations.
The numerical results for mildly relativistic flows are shown in Figure 5 for a PP configuration and in Figure 6 for an SS configuration. In Figures 7 and 8, we compare the semianalytic solutions (PP) for a mildly relativistic flow with the corresponding results from numerical calculations. The power-law indices for the intermediate distribution are given in Table 4. The semianalytic solutions provide a precise description of a mildly relativistic flow up to $\gamma_0/\gamma \approx 100$.

4.2.5. Final Distribution

We next investigate the final energy distributions for the mildly relativistic flow and the PP flow from numerical calculations.

The energy per unit mass excluding the rest-mass energy is given by

$$\epsilon = \frac{\tau}{\rho \gamma}, \quad \tau \equiv (\rho + 4p)\gamma^2 - p - \rho \gamma.$$ (60)

In the stage of free expansion, the ejecta are accelerated to $\gamma = \gamma_0 \gg 1$. Thus, $\gamma_0 \sim \epsilon$. Using the definition of the Lagrangian coordinate, we calculate the energy spectrum, defined as the total mass $M(>\gamma)$ with energy per unit mass greater than $\gamma_0$, as

$$M(>\gamma) = \int \rho \gamma \gamma_0^2 \eta^{(m+1)/2} \gamma_0^{-2} d\eta.$$ (61)

Furthermore, it is useful to present the energy $E(>\gamma)$ of the ejecta with Lorentz factors greater than $\gamma_0$. From the numerical results for a mildly relativistic planar flow, when $\eta \to \infty$ we obtain the relation between $a$ and the final Lorentz factor $\gamma_0$:

$$\gamma_0 \propto a^{-0.64}.$$ (62)

Using this relation, we find

$$E(>\gamma) = \int \gamma \rho \gamma_0^2 \eta^{(m+1)/2} \gamma_0^{-2} d\eta$$
$$\propto b A^{2-(m+1)/6} [2(m+1)]^{-1.6}.$$ (63)
Tan et al. (2001) derived an empirical formula for the final Lorentz factor \( f \) of each element as a function of important parameters such as \( \rho_0, E_{\text{inj}}, M_{\text{ejc}}, \) and \( x \) that characterize the shock emergence phenomenon. For a PP flow, they obtain

\[
\gamma_f \sim \sqrt{1 + \left( \frac{2.03 + [q(1 + q^2)^{0.12}]^{3/2}}{[\rho_0(R_\ast - x)]^{0.187}} \right)}.
\]

(64)

where \( q \) is defined as

\[
q = 0.736 \left( \frac{E_{\text{inj}}}{\bar{m}} \right)^{1/2} \left[ \frac{\bar{m}M_{\text{ejc}}}{\rho_0(R_\ast - x)} \right]^{0.187}.
\]

(65)

Here \( \bar{m}(r) \) is the fraction of \( M_{\text{ejc}} \) within \( r \) excluding any remnant mass \( M_{\text{rem}} \), that is, \( \bar{m}(r) = [m(r) - M_{\text{rem}}]/M_{\text{ejc}} \). \( E_{\text{inj}} \) is the injected energy in units of the rest-mass energy of \( M_{\text{ejc}} \), that is, \( E_{\text{inj}} = E_{\text{inj}}/M_{\text{ejc}} \). In the limit \( q \to \infty \), this equation tends to reproduce the result of Johnson & McKee (1971) and thus \( M(>\gamma_f) \propto \gamma_f^{-2.1} \), which is also derived from an ultrarelativistic self-similar solution (Nakayama & Shigeyama 2005). On the other hand, Sakurai’s (1960) solutions give \( M(>\gamma_f) \propto \gamma_f^{-3.6} \). These results suggest that the more relativistic the shock is, the more that energy can be transferred to the matter closer to the surface. This fact has already been pointed out and formulated by Tan et al. (2001, their eqs. [35]–[37]; see also Johnson & McKee 1971).

Next we discuss the energy spectrum for spherical flows. From the numerical result \( A = 1 \times 10^{10} \) for the mildly relativistic SS flow,

\[
M(>\gamma_f) \propto \sqrt{\frac{E_{\gamma_f} - E_{\gamma_0}}{\gamma_f}}.
\]

(66)

The energy spectrum for spherical symmetry is thus

\[
M(>\gamma_f) = \int_{\rho_{\gamma_f}}^{\rho_0} 4\pi r^2 dr = \int \rho_1(4\pi(R_\ast - a)^2 da
\]

\[
\sim 4\pi R_\ast^2 \int \rho_{\gamma_1} da \propto hA^{(d+1)/(m+1)} \gamma_f^{-2.7}.
\]

(67)

In terms of the energy,

\[
E(>\gamma) \propto \gamma_f^{-1.7}.
\]

(68)

Although the definition of \( M(>\gamma_f) \) in the PP configuration differs from that of the SS configuration, the difference in geometry is negligible so long as we deal with very thin shells.
Sphericity does not significantly alter the exponents of $E(\gamma f)$ in $E(\gamma f)$ and $M(\gamma f)$ from those of the PP flow, but it does reduce the absolute values of $E(\gamma f)$ and $M(\gamma f)$ compared with the PP flow. Following Tan et al. (2001), we define a deficit factor to characterize the reduction of the final velocity of an SS flow from the corresponding PP flow:

$$f_{\text{sph}} = \frac{(\gamma f^2)}{(\gamma f^2)_{\text{planar}}}. \quad (69)$$

We show the deficit factors of the flows with $A = 1.0 \times 10^{10}$ and $A = 1.0 \times 10^{12}$ in Figure 9, which clearly illustrates that the effect of sphericity is more prominent in the outer layers, which was seen in the flows with $E_{\text{ini}} = 0.3$ calculated by Tan et al. (2001). On the contrary, Tan et al. also showed that nonrelativistic SS flows become identical to the corresponding PP flows in the outer layers. The figure also indicates that the spherical expansion in more intense explosions with larger $A$ suppresses the acceleration to a greater degree. Note, however, that the PP flow with $A = 1.0 \times 10^{12}$ does not reach the free-expansion phase in our calculation. Thus, the deficit factor $f_{\text{sph}}$ for $A = 1.0 \times 10^{12}$ shown in this figure will continue to decrease as more time elapses.

### 5. CONCLUSIONS AND DISCUSSION

We have derived formulae for adding a sphericity correction to ultrarelativistic PP flows and presented semianalytic solutions for mildly relativistic flows after shock emergence. Both are demonstrated to agree well with numerical results, at least in the earliest phases of evolution after shock emergence.

Although the semianalytic solutions include the rest-mass energy in the EOS, this inclusion does not lead to a termination of the acceleration if the ratio of pressure to rest-mass energy density, $p_1/\rho_1$, is large enough. The semianalytic solutions cannot describe the whole evolution of a flow for finite values of $p_1/\rho_1$. This may be attributed to the assumption that a flow can be described with one nondimensional parameter, aside from the initial conditions. Upon reaching the stage of free expansion, a planar flow appears to deviate from self-similarity even if it is ultrarelativistic. Furthermore, we neglected the effect of radiative diffusion, which plays a critical role in suppressing the acceleration.

| Solution | $x_0/R_*$ | $p_1/\rho_1$ | $d \ln F/d \ln \eta$ | $d \ln G/d \ln \eta$ | $d \ln H/d \ln \eta$ |
|----------|------------|--------------|------------------|------------------|------------------|
| PP (numerical) | $1.05 \times 10^{-3}$ | 1.02 | -1.21 | 0.03 | -0.98 |
| | $1.72 \times 10^{-3}$ | 0.727 | -1.25 | 0.02 | -0.98 |
| | $2.88 \times 10^{-3}$ | 0.507 | -1.26 | 0.03 | -0.97 |
| SS (numerical) | $7.96 \times 10^{-4}$ | 1.24 | -3.31 | 0.00 | -2.99 |
| | $1.54 \times 10^{-3}$ | 0.782 | -3.45 | 0.00 | -2.99 |
| | $2.59 \times 10^{-3}$ | 0.546 | -3.52 | 0.00 | -3.00 |
We conducted a numerical calculation for an ultrarelativistic flow setting the parameter $A = 1.0 \times 10^{12}$. The calculation was continued until $\eta$ reached $\sim 10^6$, but it still did not reach the stage of free expansion. We could not confirm by numerical calculations whether an ultrarelativistic flow leads to free expansion or not.

On the other hand, when a flow is not so ultrarelativistic at the time of shock emergence, the acceleration comes to an end in both SS and PP configurations. However, there remains a difference between the two: a spherical flow stops accelerating rather suddenly, whereas a PP flow approaches the end of acceleration asymptotically.

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