ON THE $\theta$-SPLIT SIDE OF THE LOCAL RELATIVE TRACE FORMULA

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Abstract. The author derives an expression for one side of the local relative trace formula at the level of Lie algebras, by combining methods of Arthur and Harish-Chandra with the structure theory of reductive symmetric spaces.

1. Introduction

The local trace formula of Arthur, derived in [2], is a tool in local harmonic analysis that identifies a sum of distributions involving (weighted) orbital integrals with a sum of distributions involving (weighted) characters. This allows one to use harmonic analysis to better understand the representation theory of a $p$-adic group $H$. Specifically, one obtains information about the representation $R$ of $H \times H$ on the space $C^\infty_c(H)$ of locally constant and compactly supported complex-valued functions on $H$ that is given by

$$(R(h, g)\phi)(x) = \phi(h^{-1}xg).$$

To derive this formula, Arthur takes two test functions $f_1$ and $f_2$ in $C^\infty_c(H)$, and expresses the averaged operator $R(f_1, f_2)$ as an integral operator with kernel

$$K(x, y) = \int_H f_1(h) f_2(x^{-1}hy) \, dh.$$  

After a suitable truncation procedure, he integrates this kernel along the diagonal to obtain a (geometric) expression for a modification of the trace of $R(f_1, f_2)$. He then combines this truncation procedure with the Plancherel formula of Harish-Chandra to obtain a spectral expression for this modification. The local trace formula is the statement that these two expressions are equal, which provides a connection between the geometry of $H$ and its representation theory.

We would like to express Arthur’s development in the context of symmetric spaces. The diagonal embedding of $H$ into $H \times H$ defines a reductive symmetric space $H\backslash H \times H$ and an isomorphism of $C^\infty_c(H\backslash H \times H)$ with $C^\infty_c(H)$ by associating to $\phi$ in $C^\infty_c(H)$ the function

$$(x_1, x_2) \mapsto \phi(x_1^{-1}x_2)$$

on $H\backslash H \times H$. This isomorphism intertwines $R$ with the right regular representation of $H \times H$ on $C^\infty_c(H\backslash H \times H)$, and we can interpret $K(x, y)$ as the kernel of this representation. Optimistically, one hopes that Arthur’s development may be generalized to address the right regular representation of a $p$-adic group $G$ on $C^\infty_c(H\backslash G)$ for a large class of reductive symmetric spaces $H\backslash G$. In this article, we give this generalization at the level of Lie algebras for the geometric side of the formula.

The motivating article for pursuing the local trace formula at the level of Lie algebras is [24], in which Waldspurger proves that

$$J(f_1, f_2) = J(\hat{f}_1, \hat{f}_2)$$
for a bilinear distribution $J(f_1, f_2)$ that equals a sum of certain (orbital) integrals over the conjugacy classes of maximal tori in $H_F$. He derives this identity from Arthur’s local trace formula with the exponential map, and uses it to prove several substantial results from local harmonic analysis, including the representability by a specific function of certain invariant distributions on $\mathfrak{g}$, extending earlier work of Harish-Chandra in [12]. In addition to being a powerful tool in harmonic analysis, this formula also provides a technically simpler analog of Arthur’s local trace formula that nevertheless exhibits much of its structure. For example, one encounters in it weighted orbital integrals on $\mathfrak{g}$ with essentially the same weights. For these reasons, one would very much like to generalize this identity from reductive groups to reductive symmetric spaces.

To discuss this generalization in more detail, we require some notation. We will be working over a $p$-adic field $F$ with ring of integers $\mathcal{O}$, and a chosen uniformizing element $\varpi$. We will assume further that $F$ is of characteristic 0 and of residue characteristic greater than 2. Let $G$ be a reductive, algebraic group defined over $\mathcal{O}$, which we will assume to be split and connected, and let $\theta : G \to G$ be an involution on $G$, which we will also assume to be defined over $\mathcal{O}$. Write $H := G^\theta$ for the fixed points of $\theta$. Let the $F$-points of $G$ and $H$ be written $G_F$ and $H_F$ respectively. The quotient of groups $H_F \backslash G_F$ has the structure of a reductive symmetric space. We will occasionally omit the subscript $F$ when considering $F$-points, following the convention that any quotient written as $H \backslash G$ will be $H_F \backslash G_F$. If $\mathfrak{g}$ denotes the Lie algebra of $G_F$ and $\mathfrak{h}$ the Lie algebra of $H_F$, then the Killing form (or the involution $\theta$) provides an orthogonal complement $\mathfrak{h}^\perp$ to $\mathfrak{h}$ and a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ from the inclusion $\mathfrak{h} \subset \mathfrak{g}$. The subspace $\mathfrak{h}^\perp$ is the tangent space to the symmetric variety $H_F \backslash G_F$ at the identity coset $H_F$.

We begin the development of the relative local trace formula with a modified version of the infinitesimal Plancherel identity:

$$\int_{\mathfrak{h}^\perp} f(g^{-1}Xg) \, dX = \int_{\mathfrak{h}} \hat{f}(g^{-1}Xg) \, dX.$$ 

This already incorporates Arthur’s application of Harish-Chandra’s Plancherel formula. What remains is to truncate both sides of this identity suitably, integrate over the symmetric space $H \backslash G$, and refine the resulting expressions into two equal distributions of possibly weighted orbital integrals and characters. In other words, our task will be to study the identity

$$\int_{H \backslash G} \phi(g) \int_{\mathfrak{h}^\perp} f(g^{-1}Xg) \, dX \, dg = \int_{H \backslash G} \phi(g) \int_{\mathfrak{h}} \hat{f}(g^{-1}Xg) \, dX \, dg$$

for a carefully chosen function $\phi \in C^\infty_c(H \backslash G)$.

In this paper, we attend to the left-hand side, which we call the $\theta$-split side, and give an essentially complete formulation for the corresponding part of the trace formula. This formulation will involve only weighted orbital integrals whose weights are only slightly modified from those of Arthur. The infinitesimal spectral side is less straightforward, and we will discuss its development in subsequent papers, currently in preparation. In [20], the author has given an independent derivation of both sides for the case of $F^\times \backslash SL_2(F)$, a concrete example that clarifies some of the technical difficulties that must be overcome in the general case.

1.1. Outline. In section 2 we review some of the structure theory of reductive symmetric spaces, and derive the formal expansion of the $\theta$-split side of the relative local trace formula. We derive a Weyl integration formula for the vector space $\mathfrak{h}^\perp$ and prove that the $\theta$-split side
where $L$ is a set of Levi subgroups, $T_M$ is a set of representatives of the $M_F$-conjugacy classes of certain tori that are elliptic in $M_F$, and

$$\omega_M(g, \mu) := \sum_{S \in T_G(T)} \frac{|W_G(S)|}{|W_H(S)|} \int_{Z_H(T) \setminus (A_M \cdot Z_H(T))_F} \phi(gstg) \, dt.$$ 

The set $T_G(T)$ is a set of representatives of the $H_F$-conjugacy classes of $\theta$-split tori contained in the $G_F$-conjugacy class of $T$. In specific cases, the set $T_G(T)$ appears to be rather difficult to compute explicitly.

In section 3 we carefully choose a suitable truncation function that correctly generalizes Arthur’s truncation procedure. We fix a particular $F$-split maximal torus $A$ and let $\phi$ equal the characteristic function of the set

$$\{ g : \text{Cartan } (\theta(g)^{-1}g) \in \text{Hull } \{ W_G(A) \cdot \mu \}^* \}$$

for some dominant coweight $\mu$ of $A$. We call this characteristic function $\bar{\omega}(g, \mu)$. When $\mu$ is sufficiently regular in a sense that depends on $g$, the terms defining $\omega_M$ beautifully combine to yield a simple combinatorial formula:

$$\omega_M(g, \mu) = \# \{ \nu \in X_*(A_M)^- \cap \text{Im } \tau : \nu \in \text{Hull } \{ \mu_B - H_B(g) + H_B(\theta(g)) \}^* \}.$$ 

Here $X_*(A_M) \cap \text{Im } \tau$ is the subset of $X_*(A_M) \cong A_M/A_M(O)$ comprising those coweights that have representatives (or lifts) in $A_M$ that lie in the image of the function

$$\tau(x) = \theta(x)^{-1}x$$

on $G$. Aside from the complication introduced by this set, these weight factors are essentially those defined by Arthur in the local trace formula for reductive algebraic groups. We also show that these combinatorial weight factors are equal to the Euler-Poincaré characteristics of certain line bundles on certain toric varieties, when $\mu$ is sufficiently regular. These Euler-Poincaré characteristics are polynomials in the coefficients of $\mu$, by general results from algebraic geometry, and we will call these polynomials $\nu_M(g, \mu)$.

Finally, after proving that the integrals in the formal expansion of the $\theta$-split side are sufficiently well behaved in section 4 we apply the Lebesgue dominated convergence theorem to prove the main result of this paper, which is the next proposition, proven in 5. We briefly explain this result. If we replace the weight factors $\omega_M$ on the $\theta$-split side by the polynomials $\nu_M$, we obtain a distribution that we will call $J_-(f, \nu)$. This distribution does not equal the $\theta$-split side, but rather serves as a uniquely defined polynomial approximation at infinity.

**Proposition 1.1.** For any two dominant coweights $\mu_1$ and $\mu_2$, let $\mu := \mu_1 + d\mu_2$. Then

$$\lim_{d \to \infty} \theta \text{-split side } - J_-(f, \nu) = 0.$$ 

This proposition completes the fundamental development of the $\theta$-split side of the relative local trace formula. Indeed, once one has derived a polynomial approximation $J_+(\hat{f}, \nu)$ for the other side, and proven the analogous limit, one can subtract these two limits to prove that

$$\lim_{d \to \infty} J_-(f, \nu) - J_+(\hat{f}, \nu) = 0.$$
Since \( J_-(f, \nu) \) and \( J_+(\hat{f}, \nu) \) are both polynomials in the coordinates of \( \mu \), this is only possible if they are equal for all values of \( \mu \):

\[
J_-(f, \nu) = J_+(\hat{f}, \nu).
\]

This is the sought relative local trace formula for indeterminate \( \mu \). In this paper, however, we will only concern ourselves with one side of this formula, and so we will set as our goal the derivation of the limit of proposition 1.1.

It is worth emphasizing that even though we let \( \mu \) become very big in the derivation of this identity, we do not pass to the limit. The relative local trace formula, like the formula of Arthur, may be said to depend as a polynomial on \( \mu \). This provides a formula for each coefficient of this polynomial, but only the formula given by the constant coefficient does not reduce to the relative local trace formula for some smaller subgroup of \( G_F \).

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2. Preliminary Structure Theory

To begin, we will review some definitions and theorems concerning reductive symmetric spaces that may be found in [23] for algebraically closed fields, and in various papers by Helminck (eg. [5], [8], [10]) more generally. We address \( p \)-adic fields of characteristic zero and residue characteristic greater than 2, as well as their algebraic closures.

2.1. Tori Adapted to \( \theta \). Much of the structure theory of \( H_F \backslash G_F \) is encoded in the relationship between \( \theta \) and the structure theory of \( G_F \). Because there always exists a torus preserved by the action of \( \theta \) (eg. [5]), we can study the action of \( \theta \) on its roots and weights. One complication, however, is that this action can vary from one torus to the next. The next definition, in part, identifies those tori on which the action of \( \theta \) is particularly simple.

**Definition 2.1.** Suppose that \( T \subset G \) is a torus. Then \( T \) is \( \theta \)-stable if \( \theta(T) \subset T \); \( T \) is \( \theta \)-split if \( \theta(t) = t^{-1} \) for all \( t \in T \); \( T \) is \( \theta \)-fixed if \( \theta(t) = t \) for all \( t \in T \); and \( T \) is \( (\theta, F) \)-split if \( T \) is both \( \theta \)-split and \( F \)-split.

Suppose that \( T \) is a \( \theta \)-stable torus. Let \( X^*(T) \) denote the group of rational characters of \( T \), and let \( X_*(T) \) denote the group of rational cocharacters of \( T \). \( \theta \) then acts by composition on \( X_*(T) \) and by precomposition on \( X^*(T) \). While a \( \theta \)-stable torus \( T \) will not usually be either \( \theta \)-split or \( \theta \)-fixed, it will certainly contain \( \theta \)-fixed and \( \theta \)-split subtori. Set

\[
T^- := \{ t \in T : \theta(t) = t^{-1} \} \quad \text{and} \quad T^+ := \{ t \in T : \theta(t) = t \}.
\]

Geometrically, the product map

\[
\mu : T^+ \times T^- \to T; \quad (t_1, t_2) \mapsto t_1 \cdot t_2
\]

is an isogeny. It provides a decomposition \( T = T^+ \cdot T^- \) where \( T^- \cap T^+ \) is a finite group. In fact, this intersection has the form \((\mathbb{Z}/2\mathbb{Z})^n \) for some \( n \).

We will write \( N_G(T) \) for the normalizer of \( T \), \( Z_G(T) \) for its centralizer, and \( W_G(T) \) for its Weyl group. The possibly restricted abstract root systems of \( t \) and \( T \) in \( G \) will be written \( \Phi(t, G) \) and \( \Phi(T, G) \) respectively. We will also need to generalize the notion of a Cartan subalgebra from Lie algebras to this symmetric space setting.
Definition 2.2. A Cartan subspace of $\mathfrak{h}^\perp$ is a maximal abelian subspace of $\mathfrak{h}^\perp$ that contains only semisimple elements.

It follows from this definition that the Lie algebra of a maximal $\theta$-split torus is a Cartan subspace of $\mathfrak{h}^\perp$, and that every such Cartan subspace arises in this way. We will be interested at first in the $H_F$-conjugacy classes of Cartan subspaces of $\mathfrak{h}^\perp$, or, equivalently, the $H_F$-conjugacy classes of maximal $\theta$-split tori. Over $\bar{F}$, these collapse into a single conjugacy class.

Theorem 2.3. $H_F$ acts transitively on the maximal $\theta$-split tori of $G_F$.

One consequence of this theorem is that all Cartan subspaces of $\mathfrak{h}^\perp$ have the same dimension, which is then a uniquely defined number.

Definition 2.4. The rank of $\mathfrak{h}^\perp$ is the dimension of any Cartan subspace in $\mathfrak{h}^\perp$.

There is also a notion of regularity adapted to these Cartan subspaces, which is not equivalent to regularity in $\mathfrak{g}$.

Definition 2.5. We say that an element $X$ in $\mathfrak{h}^\perp$ is $\theta$-regular if the centralizer $\mathfrak{h}_X^\perp$ has minimal dimension among all centralizers of elements in $\mathfrak{h}^\perp$. Otherwise, we say that $X$ is $\theta$-singular.

For example, if $\theta$ is the trivial involution, then $\mathfrak{h}^\perp$ is trivial, and 0 is $\theta$-regular, but not regular in $\mathfrak{g}$. The expression $\mathfrak{h}_{\theta-\text{reg}}$ will denote the set of elements in $\mathfrak{h}^\perp$ that are $\theta$-regular. Often, we will also consider the centralizers of maximal $\theta$-fixed and $\theta$-split tori, which are described in the following lemma.

Lemma 2.6. $H_F$ acts transitively on the maximal $\theta$-split tori of $G_F$. If $T$ is a maximal torus in $H$, then $Z_G(T)$ is a maximal torus in $G$. If $T$ is a maximal $\theta$-split torus in $G$ then $[Z_G(T), Z_G(T)] < H$.

From the theory of algebraic groups, applied to the subgroup $H_F$, we know that any two maximal $\theta$-fixed tori that are also $F$-split are conjugate by an element of $H_F$. The same, however, is not necessarily true for $(\theta, F)$-split tori. In fact, even $SL_2$ provides a counterexample.

Example 2.7. Take $\mathfrak{g} = \mathfrak{sl}_2$, and $\theta$ to be the involution

$$\theta : \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & -a \end{pmatrix}.$$ 

Then $\mathfrak{h} = \mathfrak{t}$, the subspace of diagonal matrices, and

$$\mathfrak{h}^\perp = \left\{ \begin{pmatrix} b \\ 0 \\ c \end{pmatrix} : b, c \in F \right\}.$$

For every $c \in F^\times$, there is a corresponding Cartan subspace of $\mathfrak{h}^\perp$:

$$\mathfrak{a}_c := \left\{ \begin{pmatrix} 0 \\ ct \\ 0 \\ t \end{pmatrix} : t \in F \right\}.$$

Two such Cartan subspaces, say $\mathfrak{a}_c$ and $\mathfrak{a}_d$, are conjugate by an element of $H_F$ if and only if they are conjugate by a diagonal matrix, which implies that $c = dr^4$ for some $r \in F^\times$. In other words, they are conjugate precisely when $c$ and $d$ both represent the same fourth power class. On the other hand,
is $F$-split precisely when its characteristic polynomial splits over $F$, ie. when $c$ is a square. When $F$ is $p$-adic, there will exist squares in $F$ that are not fourth powers, and so we can construct two distinct $(\theta, F)$-split tori that are not $H_F$-conjugate.

Some reductive symmetric spaces lack this complication and all maximal $(\theta, F)$-split tori are $H_F$-conjugate. For most symmetric spaces, however, two maximal $(\theta, F)$-split tori will only be conjugate over a slightly enlarged set.

**Theorem 2.8.** (Proposition 10.3 in [6]) Let $A_1$ and $A_2$ be two maximal $(\theta, F)$-split tori, and let $A_F$ be a $F$-split maximal torus such that $A_1 \subset A_F \subset G$. Then

$$g^{-1}A_1g = A_2$$

for some $g \in (Z_G(A)H)_F$

where $A$ is the algebraic torus whose $F$-points are $A_F$.

**Corollary 2.9.** All maximal $(\theta, F)$-split tori in $G_F$ have the same rank.

The proof of this theorem is instructive, and we will summarize part of it, as it applies to maximal $\theta$-split tori. Suppose that $S$ and $T$ are maximal $\theta$-split tori that are conjugate over $G_F$. Then there exists a $g_S$ in $G_F$ such that

$$g_S^{-1}tg_S = s \in S$$

for an arbitrary $t$ in $T$.

By applying $\theta$ to this equation and taking inverses,

$$\theta(g_S)^{-1}t\theta(g_S) = s.$$

This then implies that $\theta(g_S)g_S^{-1}$ belongs to the centralizer $Z_G(T)$ of $T$. But the derived group of $Z_G(T)$ is a subgroup of $H$, and so

$$g_S \in (\tilde{T}H)_F$$

where $\tilde{T}$ is the center of $Z_G(T)$. One difficulty in the structure theory theory of reductive symmetric spaces is that we cannot say very much more about $g_S$. Certainly $g_S$ does not necessarily belong to $H_F$, and so one is usually forced to work with this slightly larger set. We summarize some of these observations as a proposition.

**Proposition 2.10.** If $g_S$ conjugates a maximal $\theta$-split torus $S$ to another maximal $\theta$-split torus $T$, then $g_S$ is an element of the set $(\tilde{T}H)_F$. In particular, the groups $N_G(T)_F$ and $N_{\tilde{T}H}(T)_F$ are equal and so elements in the Weyl group $W_G(T)$ have representatives in $(\tilde{T}H)_F$.

Suppose that we are given a maximal $\theta$-split torus $T$. From a representative $g_S$ of a double coset in $N_{\tilde{T}H}(T)_F \setminus (\tilde{T}H)_F/H_F$, we can define a torus $S := g_S^{-1}Tg_S$ and an isomorphism

$$T \rightarrow S : t \mapsto g_S^{-1}tg_S.$$

Notice that this isomorphism commutes with the involution $\theta$ and that there is an isomorphism of restricted root systems:

$$\Phi(T, G) \cong \Phi(S, G).$$

In addition, the next proposition shows that the $H_F$-conjugacy classes of maximal $\theta$-split tori in $G_F$ that are conjugate to $T$ are parametrized by elements $g_S$ that precisely represent the double cosets $N_{\tilde{T}H}(T)_F \setminus (\tilde{T}H)_F/H_F$. We will need this result in section 3.
Proposition 2.11. The \( H_F \)-conjugacy classes of maximal \( \theta \)-split tori that are \( G_F \)-conjugate to \( T \) are parametrized by elements of \( N_{\tilde{T}H}(T)_F/(\tilde{T}H)_F/H_F \):

\[
K_\theta(F, T) := \{ \text{maximal } \theta \text{-split tori that are } G_F \text{-conjugate to } T \}/(H_F - \text{conjugacy})
\]

\[
\leftrightarrow N_{\tilde{T}H}(T)_F/(\tilde{T}H)_F/H_F.
\]

Proof. This bijection is given explicitly by the map

\[
g_S \mapsto S.
\]

We have seen that this map is surjective, so it remains to check that it is also injective. To wit, if \( S = S' \) for representatives \( g_S \) and \( g_{S'} \) of different double cosets, then \( x := g_{S'}g_S^{-1} \in (\tilde{T}H\tilde{T})_F \) normalizes \( T \). But such an \( x \) is equal to \( t_1ht_2 \) for \( t_1, t_2 \in T_F \) and \( h \in H_F \), and therefore \( h \) must normalize \( T_F \) and \( \tilde{T}_F \). This implies that \( x = t_1ht_2h^{-1}h \in (\tilde{T}H)_F \), and that \( g_S \) and \( g_{S'} \) both represent the same double coset of \( N_{\tilde{T}H}(T)_F/(\tilde{T}H)_F/H_F \), proving injectivity. \( \square \)

There is another interpretation of these double cosets that is perhaps computationally simpler. The map \( \tau : g \mapsto \theta(g)g^{-1} \) sends \( N_{\tilde{T}H}(T)_F/(\tilde{T}H)_F/H_F \) to a set of \( \theta \)-twisted \( N_{\tilde{T}H}(T)_F \)-conjugacy classes in \( \tau((\tilde{T}H)_F) \subset T_F \). This perspective provides some intuitive justification that \( K_\theta(F, T) \) is finite. A proof may be found in, eg. \([8]\).

2.2. A Weyl Integration Formula for \( \mathfrak{h}^\perp \). In this section, we will prove an analog of the Weyl integration formula for \( \mathfrak{h}^\perp \), loosely following part of \([17]\). This differs from the development of the formula for Lie algebras mainly in the computation of a Jacobian, which requires a couple algebraic tricks. We present this computation first.

Proposition 2.12. Let \( T \) be a maximal \( \theta \)-split torus of \( G \) with Lie algebra \( \mathfrak{t} \). Let \( U = U(\mathfrak{t}) \) be the set of points in \( \mathfrak{h}^\perp_{\theta-\text{reg}} \cap \mathfrak{t} \).

- Set \( \mathfrak{t}' := \mathfrak{h}^\perp_{\theta-\text{reg}} \cap \mathfrak{t} \). The map

\[
\phi_t : \mathfrak{t}' \times Z_H(T) \setminus H \to U; \quad (A, h) \mapsto h^{-1}Ah
\]

is \( |W_H(T)| \)-to-one, regular, and surjective.

- Its Jacobian is

\[
\prod_{\alpha \in \Phi(\mathfrak{t}, G)} \alpha(A)^{m_\alpha} \]

where \( m_\alpha \) is the number of roots of some maximal Cartan subalgebra containing \( \mathfrak{t} \) whose restriction to \( \mathfrak{t} \) is \( \alpha \).

Proof. First, we need to show that the map \( \phi_t \) is well-defined. Indeed, for any \( z \in Z_H(T) \) and any \( h \in H \),

\[
\phi_t(A, z) = h^{-1}z^{-1}Azh = h^{-1}Ah = \phi_t(A, h)
\]

so this map is defined on the set \( \mathfrak{t}' \times Z_H(T) \setminus H \).

Next, we determine the extent to which injectivity can fail. Suppose that

\[
\phi_t(A_1, h_1) = \phi_t(A_2, h_2)
\]

then
Then $h_1^{-1}A_1 h_1 = h_2^{-1}A_2 h_2$, which means that $h_2 h_1^{-1} A_1 h_1 h_2^{-1} = A_2$. Since $A_1$ and $A_2$ are both elements of $t'$, it must be the case that

$$h_2 h_1^{-1} \in N_H(h_{A_1}) = N_H(T).$$

$N_H(T)$ therefore acts transitively on the preimage of $h_1^{-1} A_1 h_1$ and the stabilizer of any element under this action is $Z_H(T)$. By the orbit-stabilizer theorem, choosing an element in the preimage of $\phi_t(A_1, h_1)$ defines a 1-1 correspondence between this preimage and $W_H(T)$. Since $U$ was chosen so that $\phi_t$ would be surjective, to complete the proof of this proposition, we only need to compute the Jacobian of this map.

We begin with some general observations about the interplay between $\theta$ and the restricted root system of $T$. First, because $\mathfrak{t} \subset \mathfrak{h}^\perp$, we see that $\theta(\alpha) = -\alpha$ for any $\alpha \in \Phi(\mathfrak{t}, G)$. Therefore, $\theta$ interchanges the root subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$.

Fix a set of positive abstract roots, or, equivalently, a minimal parabolic subgroup over $\bar{F}$ containing $Z_G(\mathfrak{t})$, and then choose an orthonormal basis $\{v_1, v_2, \ldots, v_k\}$ of root vectors for the direct sum of the $\mathfrak{g}_\alpha$ with $\alpha > 0$. Take $\mathfrak{g}_i := \oplus_\alpha \mathfrak{g}_\alpha$ so that $\mathfrak{g} = \mathfrak{g}_i \oplus \mathfrak{g}_1$, where $\mathfrak{g}_i$ is the centralizer of $\mathfrak{t}$. Then over $\bar{F}$,

$$\{v_1, v_2, \ldots, v_k\} \cup \{\theta(v_1), \theta(v_2), \ldots, \theta(v_k)\}$$

is a basis for $\mathfrak{g}_1$;

$$\{v_1 - \theta(v_1), v_2 - \theta(v_2), \ldots, v_k - \theta(v_k)\}$$

is a basis for $\mathfrak{h}_1^\perp := \mathfrak{h}^\perp \cap \mathfrak{g}_1$;

$$\{v_1 + \theta(v_1), v_2 + \theta(v_2), \ldots, v_k + \theta(v_k)\}$$

is a basis for $\mathfrak{h}_1 := \mathfrak{h} \cap \mathfrak{g}_1$.

Under the adjoint action, elements of $\mathfrak{t} \subset \mathfrak{h}^\perp$ map elements of $\mathfrak{h}_1$ to elements of $\mathfrak{h}_1^\perp$ and vice versa. This map is injective because $\mathfrak{g}_1$ does not intersect $\mathfrak{g}_i$.

Next, we define an involution $\sigma$ on $\mathfrak{g}_1$ whose $+1$-eigenspace is the vector space generated by the vectors in $\{v_1, \ldots, v_k\}$ and whose $-1$-eigenspace is the vector space generated by $\{\theta(v_1), \ldots, \theta(v_k)\}$. As a consequence of this definition, $\sigma$ interchanges $\mathfrak{h}_1$ and $\mathfrak{h}_1^\perp$. In fact,

$$\sigma \circ \text{ad}(A) \circ \sigma(v_i) = \sigma \circ \text{ad}(A)(v_i) = \sigma(\alpha_i(A)v_i) = \alpha_i(A)v_i = \text{ad}(A)v_i$$

where $\alpha_i$ is the root corresponding to $v_i$. Similarly,

$$\sigma \circ \text{ad}(A) \circ \sigma(\theta(v_i)) = \sigma \circ \text{ad}(A)(-\theta(v_i)) = \sigma(-\alpha_i(A)\theta(v_i)) = \alpha_i(A)\theta(v_i) = \text{ad}(A)\theta(v_i)$$

and so $\sigma$ commutes with $-\text{ad}(A)$.

We would like to make the computation of the Jacobian at the point $(A, h)$ explicit with these bases and this involution. Let $B$ be the Killing form of $\mathfrak{g}$ and notice that the vectors $v_i - \theta(v_i)$ and $v_i + \theta(v_i)$ form an orthogonal basis with respect to it. Because of our assumptions on the characteristic of $F$, the norm of $B(v_i \pm \theta(v_i), v_i \pm \theta(v_i))$ equals one, so this basis is effectively orthonormal. We can therefore use these bases to compute the Jacobian, provided we equip the tangent spaces at elements of $U$ and $t \times Z_H(T) \setminus H$ with volume forms induced by the Killing form. e.g. the exterior product of the elements of a basis of functionals dual to an orthonormal basis.

Because $\phi_t$ is $H_F$-equivariant, we can assume that $h = e$, the identity element of $H_F$. The derivative of $\phi_t$ is

$$d(\phi_t)_{(A,e)} : t \oplus \mathfrak{h}_1 \to \mathfrak{h}_1^\perp; \quad (Y, X) \mapsto [X, A] + Y.$$

This means that the restricted maps

$$-\text{ad}(A) : \mathfrak{h}_1 \to \mathfrak{h}_1^\perp \text{ and } -\text{ad}(A) : \mathfrak{h}_1^\perp \to \mathfrak{h}_1$$
have well-defined Jacobians, and by conjugating by $\sigma$, we see that they coincide. As a consequence,

$$(\text{Jacobian of } \text{ad}(X)|_{h_1})^2 = (\text{Jacobian of } \text{ad}(X)|_{g_1}) = \prod_{\alpha \in \Phi(t,G)} \alpha(X)^{m_{\alpha}}.$$ 

The sought Jacobian is

$$(\text{Jacobian of } \text{ad}(X)|_{h_1}) = \prod_{\alpha \in \Phi(t,G)} \alpha(X)^{m_{\alpha}}$$

which completes this proof. \qed

Before we turn to the derivation of the formal development of the $\theta$-split side of the trace formula, we make a few remarks about the proof of this proposition. First, the Jacobian

$$\prod_{\alpha \in \Phi(t,G)} \alpha(X)^{m_{\alpha}}$$

has been adjusted by a constant, according to our choice of measures, and might not equal the norm of an element of $F$. Nevertheless, it will still be a well-defined real number. Second, the normalizing constant $|W_H(T)|$ depends on the underlying field $F$, even when $T$ is split over $F$. For example, in the case of $SL_2$, equipped with the involution of example 2.7, this constant equals half the number of fourth roots of unity in $F$ when $T$ is a maximal $(\theta, F)$-split torus. This particular dependence on $F$, however, will disappear by the time we state the relative local trace formula in a final form.

We also mention two corollaries to the proof of this proposition. First, if $t \subset h^\perp$ is a Cartan subalgebra, and $A \in t$ is semisimple, then for some $r$,

$$\det(\text{Ad}(A) + \lambda I; g) = \left( \prod_{\alpha \in \Phi(t,G)} \alpha(A)^{m_{\alpha}} \right)^{\lambda^r + \cdots}$$

So $A \in t'$ if and only if this leading coefficient (of $\lambda^r$), which is a polynomial over $F$, does not vanish. Second, we can break up the Zariski-open set of all $\theta$-regular semisimple elements of $h^\perp$ into a disjoint union of sets of the form $U(t)$:

$$\bigsqcup_{t \subset h^\perp} U(t) = \bigsqcup_{t \subset h^\perp} \text{Ad}(H)(t').$$

Here the disjoint sum runs over a set of representatives of the $H_F$-conjugacy classes of Cartan subalgebras $t$ in $h^\perp$.

**Theorem 2.13.** Let $f \in C_c^\infty(g)$. Then

$$\int_{h^\perp} f(X) \, dX = \sum_{T \in T_{G/H}} \frac{1}{|W_H(T)|} \int_{\nu} \left| \prod_{\alpha \in \Phi(t,G)} \alpha(X)^{m_{\alpha}} \right|^\frac{1}{2} \int_{Z_{H(T)} \backslash H} f(h^{-1}Xh) \, dh \, dX$$

where $T_{G/H}$ denotes a set of representatives of the $H_F$-conjugacy classes of maximal $\theta$-split tori in $G_F$. 

Proof. Because the set of semisimple elements that belong to some \( t' \) is dense in \( \mathfrak{h}^\perp \),
\[
\int_{\mathfrak{h}^\perp} f(X) \, dX = \sum_{T \in T_{G/H}} \int_{U(t')} f(X) \, dX.
\]
We can then pull back each summand by the corresponding \( \phi_t \), and this sum becomes
\[
\sum_{T \in T_{G/H}} \frac{1}{|W_H(T)|} \left| \int_{U'} \prod_{\alpha \in \Phi(t,G)} \alpha(X)^{m_\alpha} \right|^2 \int_{Z_H(T) \backslash H} f(h^{-1}Xh) \, dh \, dX
\]
by the previous proposition.

2.3. A Formal Expression for the \( \theta \)-split Side. The next integration formula provides an expression for one side of the trace formula for an unspecified function \( \phi \) in \( C_c^\infty(H \backslash G) \). We will derive and study this formula for unspecified weight factors for the remainder of this section. In section 3 we will choose and study a particular \( \phi \) that generalizes Arthur’s approach to this setting.

**Theorem 2.14.** Let \( f \) be a function in \( C_c^\infty(\mathfrak{g}) \). Then
\[
\int_{H \backslash G} \int_{\mathfrak{h}} \hat{f}(g^{-1}Xg) \, dX \phi(g) \, dg
= \sum_{T \in T_{G/H}} \frac{1}{|W_H(T)|} \left| \int_{U'} \prod_{\alpha \in \Phi(t,L)} \alpha(X)^{m_\alpha} \right|^2 \int_{Z_G(T) \backslash G} f(g^{-1}Xg) \omega_T(g) \, dg \, dX
\]
where
\[
\omega_T(g) := \int_{Z_H(T) \backslash Z_G(T)} \phi(tg) \, dt
\]
for any left \( H \)-invariant function function \( \phi \in C_c^\infty(H \backslash G) \). The function \( \omega_T \) is called a weight factor.

**Proof.** Because the Fourier transform commutes with the coadjoint action of \( G \) on \( C_c^\infty(\mathfrak{g}) \), one has a modified Plancherel identity:
\[
\int_{\mathfrak{h}} \hat{f}(g^{-1}Xg) \, dX = \int_{\mathfrak{h}^\perp} f(g^{-1}Xg) \, dX.
\]
Each side of this identity depends as a function of \( g \) only on the coset \( Hg \), so we can integrate over the symmetric space.

\[
(1) \quad \int_{H \backslash G} \phi(g) \int_{\mathfrak{h}} \hat{f}(g^{-1}Xg) \, dX \, dg = \int_{H \backslash G} \phi(g) \int_{\mathfrak{h}^\perp} f(g^{-1}Xg) \, dX \, dg
\]
for any left \( H \)-invariant \( \phi \in C_c^\infty(H \backslash G) \), which we introduce to guarantee convergence. By the Weyl integration formula for \( \mathfrak{h}^\perp \), (1) equals
\[
\int_{H \backslash G} \sum_{T \in T_{G/H}} \frac{1}{|W_H(T)|} \left| \int_{U'} \prod_{\alpha \in \Phi(t,L)} \alpha(X)^{m_\alpha} \right|^2 \int_{Z_H(T) \backslash H} f((hg)^{-1}X(hg)) \phi(g) \, dh \, dX \, dg.
\]
The integration by stages formula for algebraic groups allows us to change variables from $H \setminus G \times Z_H(T) \setminus H$ to $Z_H(T) \setminus G$ and from $Z_H(T) \setminus G$ to $Z_G(T) \setminus G \times Z_H(T) \setminus Z_G(T)$. By this formula, and a change in the order of integration, we obtain

$$ \sum_{T \in T_{G/H}} \frac{1}{|W_H(T)|} \int_{ \mathcal{V} } \prod_{\alpha \in \Phi(TG)} \alpha(X)^{m_\alpha} \left( \int_{Z_G(T) \setminus G} f(g^{-1}Xg) \left( \int_{Z_H(T) \setminus Z_G(T)} \phi(tg) \, dt \right) \, dg \, dX \right)^{\frac{1}{2}} $$

which completes the proof. □

We call the left hand side of this identity the $\theta$-fixed side of the trace formula. The right hand side is called the $\theta$-split side. The integrals over the $G_F$-orbits of $\theta$-regular semisimple elements in $\mathfrak{h}^{\perp}$ will be called $\theta$-split orbital integrals. We set

$$ D_{Z_G(T)}^G(X) := \prod_{\alpha \in \Phi(TG)} \alpha(X)^{m_\alpha} $$

where $\mathfrak{t}$ is a Cartan subalgebra containing $X$ in $\mathfrak{h}^{\perp}$.

2.4. Preliminary Refinements. Before we introduce a specific weight factor, we will refine this expression for the $\theta$-split side in three ways: first, we simplify the expression for $\omega_T(\cdot, \mu)$; second, we refine the sum over the set $T_{G/H}$; and third, we shift the dependence of $\omega_T$ on tori $T$ to Levi subgroups parametrizing subsets of $T_{G/H}$.

2.4.1. Centralizers of Tori. The weight factors of theorem 2.14 are expressed as integrals over reductive symmetric spaces of the form $Z_H(T) \setminus Z_G(T)$, where $T$ is a maximal $\theta$-split torus. In this subsection, we consider the image of this quotient for maximal $\theta$-fixed and $\theta$-split tori under the map

$$ \tau : H \setminus G \to G; \quad g \mapsto \theta(g)^{-1}g. $$

In both cases, this image will be a subgroup of the $F$-points of an abelian group containing a $\theta$-split torus. For $\theta$-fixed tori, this follows from lemma 2.6.

Lemma 2.15. \textit{[23]} $Z_H(T) \setminus Z_G(T)$ is a quotient of tori, and $\tau(Z_H(T) \setminus Z_G(T))$ is a subset of the torus $Z_G(T)^{-}$.

In the definition of a weight factor in the statement of theorem 2.14 on the other hand, we encounter the symmetric space $Z_H(T) \setminus Z_G(T)$, where $T$ is some maximal $\theta$-split torus. The integrand, however, will depend only on the image of an element under the map $\tau$.

Lemma 2.16. \textit{Remember that we have written $\hat{T}$ for the center of $Z_G(T)$. The image of $Z_H(T) \setminus Z_G(T)$ under the map $\tau$ is an abelian subgroup of the $F$-points of $\hat{T}$.}

Proof. Over the algebraic closure, any $g \in Z_G(T)$ can be written as a product $sd$ where $s$ belongs to the center of $Z_G(T)_F$ and $d$ belongs to its derived group. By lemma 2.6, the derived group $[Z_G(T), Z_G(T)]$ is a subgroup of $H$, and so we can compute $\tau$ at $g$:

$$ \tau(Z_H(T)sd) = \tau(Z_H(T)s) = \theta(s)^{-1}s \in \hat{T}_F. $$

Because

$$ Z_H(T)_F \setminus Z_G(T)_F \hookrightarrow (Z_H(T) \setminus Z_G(T))_F $$

is sent by $\tau$ to $\hat{T}_F$, the result follows. □

Notice that this image need not be the entire set $\hat{T}_F$. For example, if $G$ is itself a $\theta$-split torus, then $\tau$ can only equal squares.
2.4.2. Sums Over $H_F$-conjugacy Classes. We also wish to refine the sum over $T_{G/H}$ in theorem 2.14 in two ways. First, we would like to group elements of $T_{G/H}$ according to $G_F$-conjugacy. Second, we would like to make the contribution from each Levi subgroup of $G$ more apparent. To accomplish these goals, we require some notation.

Fix an $F$-split $\theta$-stable torus $A$ that contains a maximal $(\theta, F)$-split torus $A^-$ and satisfies $A(\mathcal{O}) = A \cap K$. The existence of such a torus follows from the existence of a $\theta$-stable torus in any $\theta$-stable reductive subgroup, including $Z_G(A^-)$ (see [3]). For any Levi subgroup $M_F$, let $A_M$ denote the $F$-split component of the center of $M_F$. Set $\mathcal{L}^-$ equal to the set of Levi subgroups $M_F$ in $G_F$ containing $A$ that are equal to the centralizer of $A^-_M$. The involution $\theta$ therefore preserves elements of $\mathcal{L}^-$. For each $M$ in $\mathcal{L}^-$, we choose a system of representatives for the $M_F$-conjugacy classes of tori that contain some maximal $\theta$-split tori $T$ that is elliptic in $M_F$. This implies that $A^-_M$ is the $(\theta, F)$-split part of $T$. Let $T_M$ denote this system of representatives. We also choose a system of representatives for the $H_F$-conjugacy classes of maximal $\theta$-split tori in $G_F$ that are $G_F$-conjugate to $T$, and let $T_{H/G}(T)$ denote this system.

As with the next three lemmas, all of this notation is modified from notation in [16], and we follow closely the development given there. Our goal is to replace the sum over $T_{H/G}$ with an iterated sum over $\mathcal{L}^-$, $T_M$, and $T_{H/G}(T)$.

**Lemma 2.17.** Suppose that $T \in T_M$ and that $g^{-1}Tg \subset M$ is a $\theta$-split torus for some $g \in G_F$. Then $g \in N_{\bar{T}H}(M)$.

*Proof.* Because $g^{-1}Tg \subset M$ and $g^{-1}Tg$ is a maximal $\theta$-split torus, $g^{-1}Tg$ must contain $A^-_M$. Since $A^-_M$ is the $F$-split part of $T$, by comparing ranks, we see that $A^-_M$ is also the $F$-split part of $g^{-1}Tg$. In other words, $g$ belongs to the normalizer of $A^-_M$, which is a subset of the normalizer of $M$. Further, if $g^{-1}tg = s$ for $t \in T$ and $s \in g^{-1}Tg$, then $\theta(g)^{-1}t\theta(g) = s$ also, and so $\theta(g)g^{-1}$ belongs to the centralizer of $T$. Therefore, $g$ belongs to $(\bar{T}H)_F$, as claimed.

**Lemma 2.18.** Let $T_F \in T_M$. Then the number of $M_F$-conjugacy classes of maximal $\theta$-split tori in $M_F$ that are $G_F$-conjugate to $T_F$ is

$$|W_{\bar{T}H}(A_M)| \cdot \frac{|W_M(T)|}{|W_G(T)|}.$$  

*Proof.* By the preceding lemma, $g^{-1}T_Fg \subset M_F$ implies that $g \in N_{\bar{T}H}(M)_F$. So the number of $M_F$-conjugacy classes of tori in $M_F$ that contain a $\theta$-split torus and are $G_F$-conjugate to a $\theta$-split torus $T$ is equal to

$$|(M \cap \bar{T}H)_F \backslash N_{\bar{T}H}(M)_F \cap N_{\bar{T}H}(T)_F| = [(M \cap \bar{T}H)_F \backslash N_{\bar{T}H}(M)_F : N_{M\cap\bar{T}H}(T)_F \cap N_{\bar{T}H}(T)_F].$$

This last expression is the index of two finite groups, and we can compute the order of each individually:

$$|N_{M\cap\bar{T}H}(T)_F \cap N_{\bar{T}H}(M)_F| = |N_M(T)_F \cap N_G(T)_F| = \frac{|W_G(T)|}{|W_M(T)|}$$

$$|(M \cap \bar{T}H)_F \backslash N_{\bar{T}H}(M)_F| = |W_{\bar{T}H}(A_M)|.$$

Therefore

$$[(M \cap \bar{T}H)_F \backslash N_{\bar{T}H}(M)_F : N_{M\cap\bar{T}H}(T)_F \cap N_{\bar{T}H}(T)_F] = \frac{|W_M(T)|}{|W_{\bar{T}H}(A_M)| \cdot |W_G(T)|}$$

as required. □
We would like to express the $\theta$-split side, in part, as a sum over the set $\mathcal{L}^-$, which may contain distinct Levi subgroups that are $G_F$-conjugate, or even $H_F$-conjugate. With these lemmas, however, one may quickly write down the normalizing constants that such an expression requires.

**Lemma 2.19.** The sum on the $\theta$-split side of the trace formula can be rewritten in the following way:

$$
\sum_{T \in \mathcal{T}_{G/H}} \frac{1}{|W_H(T)|} (\cdot) = \sum_{M \in \mathcal{L}^-} \sum_{T \in \mathcal{T}_M} \frac{1}{|W_{\tilde{T}H}(A_M)| \cdot |W_M(T)|} \sum_{S \in \mathcal{T}_G(T)} \frac{|W_G(S)|}{|W_H(S)|} (\cdot)
$$

where we have written $(\cdot)$ to denote the rest of each term (this can equal any function defined on the set of $H_F$-conjugacy classes of $\theta$-split tori in $G$).

**Proof.** By the preceding lemmas,

$$
\sum_{M \in \mathcal{L}^-} \sum_{T \in \mathcal{T}_M} \frac{|W_G(T)|}{|W_{\tilde{T}H}(A_M)| \cdot |W_M(T)|} (\cdot)
$$

is a sum over representatives of the $G_F$-conjugacy classes that contain maximal $\theta$-split tori in $G_F$. Summing over representatives of the $H_F$-conjugacy classes that are $G_F$-conjugate to a given $\theta$-split torus provides the lemma. \hfill \Box

This last decomposition is better suited to our purposes, in part, because we will be able to absorb the sum over $\mathcal{T}_G(T)$ into the weight factors in a natural way.

2.4.3. **Weight Factors.** Remember that we have defined weight factors as integrals over symmetric spaces associated to certain tori:

$$
\omega_S(g) := \int_{Z_H(S) \backslash Z_G(S)} \phi(tg) \, dt.
$$

Suppose that $S \in \mathcal{T}_G(T)$ for some $T \in \mathcal{T}_G(M)$. Then there exists a $g_S$ in $G_F$ that conjugates $S$ to $T$, and so we may express these weight factors in terms of the torus $T$:

$$
\omega_S(g) = \int_{Z_H(T) \backslash Z_G(T)} \phi(g_S t g_S^{-1} g) \, dt.
$$

Let $A_M$ be the $F$-split component of the center of $M$. By proposition 2.6, $Z_G(T)_F$ contains a cocompact subgroup $(A_M^- \cdot Z_H(T))_F$. We can write the weight factor as

$$
\int_{A_M^- \cdot Z_H(T) \backslash Z_G(T)} \int_{Z_H(T) \backslash A_M^- \cdot Z_H(T)} \phi(g_S t g_S^{-1} g) \, ds \, dt.
$$

Because $A_M^- \cdot Z_H(T) \backslash Z_G(T)$ is compact, the outer integral is not an essential part of the truncation procedure, and we may absorb the integral over $(A_M^- \cdot Z_H(T))_F \backslash Z_G(T)_F$ into the orbital integrals. Changing variables $g \mapsto g_S g$, we then let

$$
\omega_M(g) := \sum_{S \in \mathcal{T}_G(T)} \frac{|W_G(S)|}{|W_H(S)|} \int_{Z_H(T)_F \backslash (A_M^- \cdot Z_H(T))_F} \phi(g_S s g) \, ds.
$$

One motivation for this refinement is that $\tau$ maps the symmetric space $Z_H(S)_F \backslash (A_M^- \cdot Z_H(S))_F$ into the $F$-split torus $A_M^-$. 
2.5. The Trace Formula. We now assemble these ingredients into a skeleton for the \( \theta \)-split side of the trace formula.

**Definition 2.20.** For any family of functions \( \omega_M \in \mathcal{C}_c^\infty(G) \) parametrized by Levi subgroups \( M \) of \( G \) with \((\theta, F)\)-split center, set \( J_-(f, \omega) \) equal to the sum

\[
\sum_{M \in \mathcal{L}^-} \sum_{T \in \mathcal{T}_M} \frac{1}{|W_T^H(A_M)| \cdot |W_M(T)|} \int_{T^*} |D^G(X)|^{1/2} \int_{A^-_M \cdot Z_H(T) \backslash G} f(g^{-1}Xg) \omega_M(g) \, dg \, dX.
\]

This compact notation affords a succinct statement of the \( \theta \)-split side.

**Theorem 2.21.** Suppose one has a split, reductive algebraic group \( \mathbb{G} \) over \( \mathcal{O} \), equipped with an involution \( \theta : \mathbb{G} \to \mathbb{G} \) over \( \mathcal{O} \) and suppose that \( \mathbb{G}(F) \) contains a \((\theta, F)\)-split maximal torus. Let \( H = \mathbb{G}^\theta \). Then for any \( f \in \mathcal{C}_c^\infty(\mathfrak{g}) \),

\[
\int_{H \backslash G} \int_{\mathfrak{h}} \hat{f}(g^{-1}Xg) \phi(g) \, dX \, dg = J_-(f, \omega)
\]

where

\[
\omega_M(g) = \sum_{S \in \mathcal{I}_G(T)} \frac{|W_G(S)|}{|W_H(S)|} \int_{Z_H(T) \backslash (A_M \cdot Z_H(T))} \phi(g_{st}g) \, dt
\]

for some \( \phi \in \mathcal{C}_c^\infty(H \backslash G) \).

We would like to refine this expression for the \( \theta \)-split side further, but to do so, we need to introduce specific weight factors and study them in some depth. This is the goal of the next section of this paper.

3. The Weight Factors

3.1. More Structure Theory. Given an \( F \)-split maximal torus \( A \) such that \( A \cap K \) equals \( A(\mathcal{O}) \), we may identify \( X_*(A) \) with \( A/A \cap K \), and for \( a \in A \), we may write \( \nu_a \) for the image of \( a \) under the map \( A \to A/(A \cap K) \). When \( A_M \) and \( A \) are two \( F \)-split tori of \( G \) such that \( A_M \subset A \), we will treat \( X_*(A_M) \) as a subset of \( X_*(A) \) because there is a natural inclusion \( X_*(A_M) \hookrightarrow X_*(A) \). For example, we may say without confusion that elements of \( a \in A_M \) map to elements \( \nu_a \in X_*(A) \).

Next, we consider parabolic subgroups that are adapted in some way to \( \theta \). We will assume that these groups contain a fixed \( F \)-split \( \theta \)-stable maximal torus \( A \) with \( A \cap K \) equal to \( A(\mathcal{O}) \) and \( A^- \) a maximal \((\theta, F)\)-split torus. That such a torus exists follows from the existence of an \( F \)-split \( \theta \)-stable torus in any split reductive group, including \( Z_G(A^-) \) (eg. \[5\]).

**Definition 3.1.** Let \( M \) be a \( \theta \)-stable Levi subgroup of \( G \) and write

\[ \mathcal{P}(M) := \{ \text{parabolics that are minimal among those containing } M \} \]

We say that \( P \in \mathcal{P}(M) \) is \( \theta \)-split if \( P \) and \( \theta(P) \) are opposite parabolics with respect to \( M \). We write

\[ \mathcal{P}(M)^- := \{ \theta \text{-split parabolics that are minimal among those containing } M \} \]

As with \( \mathcal{P}(M) \), we can describe the elements of \( \mathcal{P}(M)^- \) in terms of Weyl facets.
Proposition 3.2. [6] Let $A_M$ be the $F$-split component of the $\theta$-stable Levi subgroup $M = Z_G(A_M)$. Then parabolics in $\mathcal{P}(M)^-$ correspond bijectively with the Weyl chambers of the restricted root system $\Phi(A_M, G)$. This bijection is given by associating to each Weyl chamber in $X_*(A_M)$ of the restricted root system the unique Weyl facet of $\Phi(A_M, G)$ in $X_*(A_M)$ containing it, and hence a parabolic subgroup in $\mathcal{P}(M)$.

Let $B$ be a Borel subgroup containing $A$ and let $\mu \in X_*(A)$ be regular in the sense that it does not belong to any $X_*(A_M)$ except $X_*(A)$. The Weyl group acts on $X_*(A)$ and the orbit of $\mu$ under this action contains a unique element dominant with respect to $B$. Call this element $\mu_B$. For any parabolic $P$ containing $B$, let $\mu_P$ denote the projection of $\mu_B$ onto the Weyl facet associated to $P$ in $a := X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$. Note that each $\mu_P$ is in fact well-defined (see eg. section 12 of [16]). By varying $B$ over the Borel subgroups containing $A$, we obtain a family of cocharacters $\mu_B$ indexed by $\mathcal{P}(A)$.

Recall the homomorphism

$$H_G : G \to \Lambda_G := \{\text{cocharacters}\}/\{\text{coroots}\}$$

defined to be trivial on $K$ and equal to the projection

$$A \to A/(A \cap K)$$
on $A$. By the Cartan decomposition, these two properties determine $H_G$. For any parabolic $P$ with Levi complement $M$ and unipotent radical $N$, we define

$$H_P(mnk) := H_M(m)$$

where $m \in M$, $n \in N$, and $k \in K$. By the Iwasawa decomposition, this uniquely determines a function

$$H_P : G \to \Lambda_M := \{\text{cocharacters}\}/\{\text{coroots of } M\}.$$ 

Last, for any set $S$ of points in $\Lambda_M$, we define Hull $S$ to be the convex hull of the projections of these points to $a_M := \Lambda_M \otimes_{\mathbb{Z}} \mathbb{R}$ under the map

$$\Lambda_M \to \Lambda_M \otimes_{\mathbb{Z}} \mathbb{R} = a_M.$$ 

We will also write Hull $S$ for the preimage of this convex hull under this map. The meaning of Hull $S$ will therefore depend on whether it is a subset of $\Lambda_M$ or $a_M$.

It will often be convenient to break up Hull $S$ according to the images of its elements in $\Lambda_G$. If each element of $S$ maps to the same element in $\Lambda_G$ under the natural map $\Lambda_M \to \Lambda_G$, then write Hull $S^*$ for the set of elements in Hull $S$ that have the same image in $\Lambda_G$ as every element of $S$. We will use a set of this form to define the truncating function $\phi$.

3.2. A First Weight Factor. Fix an $F$-split $\theta$-stable maximal torus $A$ with $A(\mathcal{O})$ equal to $A \cap K$ and $A^-$ a maximal $(\theta, F)$-split torus, as well as a Borel subgroup $B \supset A$, which provides a choice of simple roots $\Delta$, the notion of a dominant coweight in $X_*(A)$, and the notion of a positive coroot. We will call the set of dominant coweights $X_*(A)_{dom}$.

Thanks to the Cartan decomposition, there is a proper map

$$\text{Cartan} : G \to K \setminus G/K \cong X_*(A)_{dom}$$
to the set of dominant coweights of $A$, as well as the “invariant” map

$$\text{inv} : G \times G \to X_*(A)_{dom} \cong \Lambda_A$$

$$(g, h) \mapsto \text{Cartan}(h^{-1}g).$$
Definition 3.3. Set
\[
\bar{\omega}(g, \mu) := \begin{cases} 
1, & \text{inv}(g, \theta(g)) = \text{Cartan}(\theta(g)^{-1}g) \in \text{Hull}\{W_G(A) \cdot \mu\}^* \\
0, & \text{otherwise}.
\end{cases}
\]

We note that when the algebraic group $G$ is not split over $F$, the Cartan decomposition is slightly more complicated, but nevertheless may be used to define an analogous function, as in [2], that yields a manageable truncation procedure.

Notice that each element in the Weyl orbit of a dominant coweight $\mu$ maps to the same element under the map $X_*(A) \to \Lambda_G$, so that we can indeed break up the convex hull of this orbit in $X_*(A)$ according to this common image. To show that this function satisfies the conditions of theorem 2.14 we need to check that it is locally constant and compactly supported.

Lemma 3.4. The function $g \mapsto \bar{\omega}(g, \mu)$ belongs to $C^\infty_c(H \setminus G)$ for any $\mu \in X_*(A)_{\text{dom}}$.

Proof. The morphism Cartan is proper and locally constant, so it suffices to show that precomposition by the map
\[
\tau : H \setminus G \to G; \quad Hg \mapsto \theta(g)^{-1}g
\]
sends $C^\infty_c(G)$ to $C^\infty_c(H \setminus G)$. But this map is a closed immersion (see [19]), and this is sufficient. \quad \square

Our next goal will be to understand the asymptotic behavior of $\omega_M$ as $\mu$ becomes very large in the direction of the Borel subgroup $B$.

3.3. Orthogonal Sets and Arthur’s Key Geometric Lemma. In this section, we state Arthur’s key geometric lemma. Detailed accounts include section 5 of the original article [2], and the sections leading up to and including section 22 in the expository article [16].

We will need the notion of a $(G, A)$-orthogonal set. Recall that a $(G, A)$-orthogonal set is a set of points $x_B$ in $X_*(A)$, indexed by the Borel subgroups $B$ containing $A$, such that for each pair of adjacent Borel subgroups $B$ and $B'$,
\[
x_B - x_{B'} = r\check{\alpha}
\]
where $r$ is an integer and $\check{\alpha}$ is the unique coroot that is positive for $B$ and negative for $B'$. These sets are called positive if $r$ is positive for each adjacent pair. This notion can also be extended to include a set of points $x_B \in a$ by requiring only that $r$ be a real number, and not necessarily an integer. The properties of these sets are discussed in depth in [2] or [16]. We will also occasionally impose an additional regularity condition on these sets:

Definition 3.5. A $(G, A)$-orthogonal set $(x_B)$ is called special if $x_B$ is $B$-dominant for every Borel subgroup $B \supset A$.

There is a generalization of a $(G, A)$-orthogonal set that is important in the theory of the local trace formula. For a given Levi subgroup $M$, define a $(G, M)$-orthogonal set to be a family of points $x_P$ in $\Lambda_M$, indexed by the parabolic subgroups $P$ that contain $M$, subject to the condition that for each pair of adjacent parabolic subgroups $P$ and $P'$,
\[
x_P - x_{P'} = r\beta_{P,P'}
\]
where $r$ is an integer and $\beta_{P,P'}$ is the the smallest element in the projection of $R_U \cap R_{U'}$ to $\Lambda_M$. Here $R_U$ and $R_{U'}$ denote the roots that occur in the Lie algebras of the unipotent radicals $U$ and $U'$ of the parabolic subgroups $P$ and $P'$. These generalized orthogonal sets are called
positive if \( r \) is positive for each adjacent pair of parabolic subgroups containing \( M \). They are called special if the condition
\[
\langle \alpha, x_P \rangle > 0 \quad \text{for all } \alpha \in R_U
\]
holds for every parabolic subgroup \( P \) that contains \( M \). As with \((G, A)\)-orthogonal sets, this notion can be extended to include sets of points in \( a_M \) by requiring only that \( r \) be a real number.

Arthur’s key geometric lemma concerns coweights that are sufficiently regular in the sense that \( \langle \alpha, \mu \rangle \) is very large for each positive root \( \alpha \). Exactly how large \( \langle \alpha, \mu \rangle \) needs to be will depend on an element \( g \) of \( G \), and the location of the vertex in the Bruhat-Tits building \( \mathcal{B}(G) \) of \( G \) that \( g \) represents (recall that \( G \) maps surjectively onto the set of vertices in \( \mathcal{B}(G) \)). This dependence can be captured by a single function on \( \mathcal{B}(G) \).

More precisely, on \( a \) one can choose a Weyl group invariant Euclidean norm \( \| \cdot \|_E \) which extends uniquely to a \( G \)-invariant metric on \( \mathcal{B}(G) \). We denote this metric by \( d(x_1, x_2) \) where \( x_1 \) and \( x_2 \) are points in \( \mathcal{B}(G) \). Let \( x_0 \) be the basepoint of \( \mathcal{B}(G) \) (whose stabilizer contains \( K \)), and set
\[
d(x) := d(x, x_0) \\
d(g) := d(g \cdot x_0, x_0).
\]

It is the quantity \( d(g) \) that we need to state Arthur’s key geometric lemma.

Remember that the invariant map can be defined on \( \mathcal{B}(G) \). Abusing notation slightly, one could write
\[
\text{inv} : \mathcal{B}(G) \times \mathcal{B}(G) \to X_*(A)_{\text{dom}}.
\]
For motivation, notice also that \( \bar{\omega} \) is essentially defined by the inequality
\[
\text{inv}(g, \theta(g)) \leq \mu
\]
where \( x \leq y \) means that \( y - x \) is a sum of positive coroots. Here is Arthur’s key geometric lemma.

**Proposition 3.6.** Let \( x_1, x_2 \in \mathcal{B}(G) \), and suppose that \( \mu \in X_*(A)_{\text{dom}} \). There is a constant \( c \) such that whenever
\[
\langle \alpha, \mu \rangle \geq c \cdot [1 + d(x_1) + d(x_2)] \quad \text{for all } \alpha \in \Delta
\]
the family of points \( \{\mu_B - H_B(x_2) + H_B(x_1)\} : B \text{ is a Borel subgroup containing } A \) is a \((G, A)\)-orthogonal set, and for any \( a \in A \), the inequality \( \text{inv}(ax_2, x_1) \leq \mu \) is satisfied precisely when
\[
\nu_a \in \text{Hull} \{\mu_B - H_B(x_2) + H_B(x_1)\}^*.
\]

This geometric lemma will facilitate the derivation of an asymptotic description of the preliminary weight factors, in the next section.

### 3.4. Asymptotic Behavior of \( \omega_M \).

When \( \phi(g) = \bar{\omega}(g, \mu) \), we find in the trace formula the following weight factors:
\[
\omega_M(g, \mu) := \sum_{S \in T_G(T)} \frac{|W_G(S)|}{|W_H(S)|} \int_{Z_H(T)_F \setminus (A_M \cdot Z_H(T))_F} \bar{\omega}(g_{SSg}, \mu) \, ds.
\]

The sum over \( T_G(T) \) appears to be difficult to describe in some special cases, but it is fortunately possible to remove it from this equation, and derive a simplified expression for these weight factors in the process.
Proposition 3.7. The weight factors can be written in the following way:

\[ \omega_M(g, \mu) = \int_{H_F \setminus (H \cdot A_M)_F} \tilde{\omega}(sg, \mu) \, ds. \]

Proof. We have already seen in section 2 that the elements \( g_S \) precisely represent the double cosets

\[ N_{\tilde{T}H}(T)_F \setminus (\tilde{T}H)_F / H_F. \]

As a first approximation to these double cosets, we consider the double cosets

\[ Z_{\tilde{T}H}(T)_F \setminus (\tilde{T}H)_F / H_F. \]

The Weyl group \( W_G(T) = W_{\tilde{T}H}(T) \) acts on this set from the left, and the elements \( g_S \) represent the \( W_G(T) \)-orbits under this action. Notice that \( |W_G(T)| = |W_G(S)| \), because \( T \) and \( S \) are \( G_F \)-conjugate. On the other hand, suppose that \( n \in N_{\tilde{T}H}(T) \) sends a coset to itself:

\[ nZ_G(T)_FgS_{H_F} = Z_G(T)_FgS_{H_F}. \]

With a few algebraic manipulations, this becomes

\[ g_S^{-1}ng_SZ_G(S)_FH_F = Z_G(S)_FH_F \]

which implies that \( g_S^{-1}ng_S \) represents an element of \( W_G(S) \) that has a representative in \( H_F \). Since elements of \( W_H(S) \) certainly fix these cosets, the cardinality of the fixator of this double coset equals the cardinality of the image of the injection \( W_H(S) \to W_G(S) \). The orbit-stabilizer theorem then implies that the size of the \( W_G(T) \)-orbit containing the double coset represented by \( g_S \) is \( |W_G(S)| \cdot |W_H(S)|^{-1} \). We can therefore write

\[ \sum_{S \in T_G(T)} \frac{|W_G(S)|}{|W_H(S)|} \int_{Z_H(T)_F \setminus (T \cdot Z_H(T))_F} \tilde{\omega}(g_Ssg, \mu) \, ds = \sum_{g_i \in S} \sum_{g_S \in S} \int_{Z_H(T)_F \setminus (T \cdot Z_H(T))_F} \tilde{\omega}(g_Ssg, \mu) \, ds. \]

where \( S \) now indexes a system of representatives of the more tractable double cosets

\[ Z_{\tilde{T}H}(T)_F \setminus (\tilde{T}H)_F / H_F. \]

The image of \( Z_{\tilde{T}H}(T)_F \) under \( \tau \) is a normal subgroup of the abelian group \( \tilde{T} \). Since \( \tau \) maps \( S \) to a system of representatives of the cosets of this subgroup that belong to the image of \( \tau \), these integrals may be combined:

\[ \int_{H_F \setminus (H \cdot \tilde{T})_F} \tilde{\omega}(sg, \mu) \, ds. \]

Restricting to those elements that map to elements of \( A_M \) under \( \tau \) yields the proposition. \( \square \)

We could also express the weight factor \( \omega_M \) as

\[ \omega_M(g, \mu) = \int_{A_M \cap \text{Im} \tau} \tilde{\omega}(s^{\frac{1}{2}}g, \mu) \, ds \]

where \( s^{\frac{1}{2}} \) is written formally; the integrand depends only on \( s =: \tau(s^{\frac{1}{2}}) \).

Definition 3.8. For each \( M \in \mathcal{L}^- \), define

\[ \omega_M^{\text{asym}}(g, \mu) := \# \{ \nu \in X_*(A_M) \cap \text{Im} \tau : \nu \in \text{Hull} \{ \mu_B - H_B(g) + H_B(\theta(g)) \}^* \} \]

where we have written \( X_*(A_M) \cap \text{Im} \tau \) for the set of coweights in \( A_M / A_M \cap K \) that have a representative in \( A_M \cap \text{Im} \tau \).
The next proposition connects these new functions to the preliminary weight factors in the local trace formula.

**Proposition 3.9.** There is a constant \( c \) such that when \( \langle \alpha, \mu \rangle \geq c \cdot [1 + d(r, g) + d(r, \theta(g))] \) for all simple roots \( \alpha \),

\[
\omega_M(g, \mu) = \omega_M^{\text{asym}}(g, \mu).
\]

**Proof.** We have shown that \( \omega_M \) may be expressed as the integral over \( A_M \cap \Im \tau \) of the characteristic function \( \bar{\omega}(s^{\frac{1}{2}}g, \mu) \). This can be written in a form amenable to Arthur’s key geometric lemma:

\[
\text{meas}_{A_M}\{s \in A_M \cap \Im \tau : \text{inv}(sg, \theta(g)) \in \text{Hull} \{\mu_B\}^*\}.
\]

Arthur’s lemma states that there exists a constant \( c \) such that when \( \langle \mu, \alpha \rangle \geq c \cdot [1 + d(g) + d(\theta(g))] \) for all simple roots \( \alpha \),

\[
\text{inv}(sg, \theta(g)) \in \text{Hull} \{\mu_B\}^* \iff \nu_s \in \text{Hull} \{\mu_B - H_B(g) + H_B(\theta(g))\}^*.
\]

It then follows that

\[
\omega_M(g, \mu) = \#\{\nu \in X_*(A_M) \cap \Im \tau : \nu \in \text{Hull} \{\mu_B - H_B(g) + H_B(\theta(g))\}^*\}
\]

when \( \mu \) is sufficiently regular. \( \square \)

### 3.5. Further Refinements of this Weight Factor

In this section, we relate the asymptotic weight factor \( \omega_M^{\text{asym}} \) to the involution \( \theta \). Specifically, we show that it depends only on those ingredients of the structure theory of \( G \) that are associated to the \((\theta, F)\)-split torus \( A_M^- \). We begin by discussing \((G, A)\)-orthogonal sets in more detail.

**Lemma 3.10.** [16] Let \( \{x_B : B \in \mathcal{P}(A)\} \) be a positive \((G, A)\)-orthogonal set. Then,

\[
\text{Hull}\{x_B : B \in \mathcal{P}(A)\}^* = \{x \in a : x \leq_B x_B, \forall B \in \mathcal{P}(A)\},
\]

where \( a = X_*(A) \otimes \mathbb{R} \).

\( \theta \) defines an involution on \( X_*(A) \) by composition and thence an involution on \( a \). Let \( a^- \) be the \(-1\)-eigenspace of this restricted involution.

**Definition 3.11.** Let \( C \) be a Weyl facet in \( a^- \). We say that \( C \) is \( \theta \)-split if \( \theta(C) = -C \) and \( \theta \)-fixed if \( \theta(C) = C \). For example, \( \theta \)-split Weyl facets correspond to \( \theta \)-split parabolic subgroups, and \( \theta \)-fixed Weyl facets correspond to \( \theta \)-stable parabolic subgroups.

On each \( \theta \)-split Weyl facet \( C \), there is an involution \( -\theta : C \to C \) defined by \( x \mapsto -\theta(x) \). On each \( \theta \)-fixed Weyl facet \( C \), there is the restricted involution \( \theta : C \to C \). If \( C \) is a \( \theta \)-split Weyl facet, then the fixator in \( C \) of \( -\theta \) is \( C \cap a^- \).

Let \( C \) be a \( \theta \)-split Weyl facet. Let \( \{e_i\} \) be the set of generating coweights for the one-dimensional Weyl facets in the boundary of \( C \). \( -\theta \) permutes the \( e_i \). Let \( \tau \) denote the permutation, so that \( -\theta(e_i) = e_{\tau(i)} \). Since \( -\theta \) is an involution, \( \tau \) has order two. Observe that if \( x = a_1 e_1 + \cdots + a_n e_n \), then \( x \in a^- \) if and only if \( a_i = a_{\tau(i)} \) for all \( i \).

**Proposition 3.12.** Let \( x = a_1 e_1 + \cdots + a_n e_n \in C \), \( x_B = b_1 e_1 + \cdots + b_n e_n \), and \( x_B^- = c_1 e_1 + \cdots + c_n e_n \), where \( c_i = \min(b_i, b_{\tau(i)}) \). Then whenever \( x \in a^- \), \( x \leq_B x_B \) is equivalent to \( x \leq_B x_B^- \).
Proof. Indeed, $x \in a^-$ implies that $\theta(x) = -x$ or $a_i = a_{\tau(i)}$ for all $i$. $x \leq_B x_B$ implies that $a_i \leq b_i$ for all $i$, and therefore that $a_i = a_{\tau(i)} \leq b_{\tau(i)}$ also. Consequently, $a_i \leq \min(b_i, b_{\tau(i)})$, and $x \leq_B x_B$. On the other hand $x \leq_B x_B$ implies $x \leq_B x_B$ immediately, because $x_B \leq_B x_B$ and the relation $\leq_B$ is transitive.

In general, when $x = a_1 e_1 + \cdots + a_n e_n \in C$, a $\theta$-split Weyl facet, we shall write $x^- = c_1 e_1 + \cdots + c_n e_n$, where $c_i = \min(a_i, a_{\tau(i)})$. This is well-defined because $x$ uniquely determines each $a_i$ (Weyl chambers are unbounded simplices). Observe that $x^-$ is a coweight whenever $x$ is.

When $\{x_B : B \in \mathcal{P}(A)\}$ is a $(G, A)$-orthogonal set, $B$ is a given Borel, and $P \supset B$ is a parabolic subgroup adjacent to $B$, we set $x_P$ equal to the projection of $x_B$ onto the Weyl facet corresponding to $P$ in $a$. Because $\{x_B : B \in \mathcal{P}(A)\}$ is $(G, A)$-orthogonal, this element is well-defined.

**Corollary 3.13.** If $\{x_B : B \in \mathcal{P}(A)\}$ is a $(G, A)$-orthogonal set in $a$, then $\{x_B^- : B \in \mathcal{P}(A)^-\}$ is a $(G, A^-)$-orthogonal set in $a^-$. 

Proof. Since $x \leq_P y$ in $a^-$ if and only if $x \leq_P y$ in $a$, we will not distinguish these relations. Let $x_B^-$ and $x_P^-$ project to different elements of the Weyl facet $C$ separating the Weyl chambers corresponding to $P$ and $P'$ in the Weyl fan of $a^-$. Then there exists an element $y \in C$ such that $y \leq_B x_B^-$ but $y \nleq_{B'} x_B^-$. By proposition 3.12, then, $y \leq_B x_B$ while $y \nleq_{B'} x_B$, so that $\{x_B : B \in \mathcal{P}(A)\}$ is not a $(G, A)$-orthogonal set, contrary to assumption.

**Corollary 3.14.** Let $\{x_B : B \in \mathcal{P}(A)\}$ be a positive $(G, A)$-orthogonal set for some $\theta$-stable torus $A$. Then

$$Hull\{x_B : B \in \mathcal{P}(A)\}^* \cap a^- = Hull\{x_P^- : P \in \mathcal{P}(A)^-\}^*$$

Proof. Let $x \in C \cap a^-$ for some Weyl facet $C$, and let $x_P$ be the unique element in $\{x_B : B \in \mathcal{P}(A)\} \cap C$. By the preceding proposition and corollary, when $x$ is dominant,

$$x \in Hull\{x_B : B \in \mathcal{P}(A)\}^* \cap a^- \iff x \leq_B x_B$$

$$\iff x \leq_B x_B^-$$

$$\iff x \in Hull\{x_P^- : P \in \mathcal{P}(A)^-\}^*$$

as required.

Let $P$ be a parabolic subgroup with Levi component $M$. The map $H_P : G \to \Lambda_M$ may be composed with the natural map $\Lambda_M \to a_M$, providing a second map $H_P : G \to a_M \subset a$. We will not distinguish between these two maps with additional notation.

**Lemma 3.15.** Let $P$ be a $\theta$-split parabolic subgroup containing $A$. Then $H_P(\theta(g)) - H_P(g)$ is $\theta$-split in $a$.

Proof. Let $P = MN$ be the Levi decomposition of the parabolic subgroup $P$, and set $g = mnk$, where $m \in M$, $n \in N$, and $k \in K$. Because $P$ is $\theta$-split, $\theta(n)$ belongs to the unipotent subgroup of $P$, and so $H_P(\theta(g)) = H_M(\theta(m))$. On the other hand, $H_P(g) = H_M(m)$ by definition. Since $\theta$ commutes with the composition of the maps $A \to X_*(A) \to \Lambda_M \to a_M$ and preserves $K$, the Cartan decomposition then implies that $\theta(H_M(m)) = H_M(\theta(m))$. With these observations in mind, we can apply $\theta$ to $H_P(\theta(g)) - H_P(g)$ and simply compute:

$$\theta(H_P(\theta(g)) - H_P(g)) = \theta(H_P(\theta(g))) - \theta(H_P(g)) = \theta(H_M(\theta(m))) - \theta(H_M(m))$$

$$= H_M(m) - H_M(\theta(m)) = H_P(g) - H_P(\theta(g)).$$
So $H_P(\theta(g)) - H_P(g)$ is $\theta$-split in $a$.

**Proposition 3.16.** We can express the asymptotic weight factors using only $\theta$-split parabolics and tori:

$$\omega^\text{asym} _M(g,\mu) = \# \{ \nu \in X_s(A_M) \cap \text{Im} \tau : \nu \in \text{Hull}((\mu_p)^- - H_P(g) + H_P(\theta(g)) : P \in \mathcal{P}(A)^-)^* \}.$$  

**Proof.** Let $a^-$ be the $-1$-eigenspace of $\theta$. Any coweight $\nu$ in $X_s(A_M)$ maps to $a^-$, so we are only required to show that

$$\text{Hull}((\mu_p^- - H_P(g) + H_P(\theta(g)) : P \in \mathcal{P}(A)^-)^* \cap a^-.$$  

By the second corollary, this follows from the identity,

$$\mu_p^- - H_P(g) + H_P(\theta(g)) = (\mu_p - H_P(g) + H_P(\theta(g)))^-.$$

Set $\mu_p = a_1e_1 + \cdots + a_ne_n$ and $H_P(\theta(g)) - H_P(g) = b_1e_1 + \cdots + b_ne_n$. By lemma 3.15, $H_P(\theta(g)) - H_P(g)$ is $\theta$-split, and so $b_i = b_{\tau(i)}$ for all $i$.

$$(\mu_p - H_P(g) + H_P(\theta(g)))^- = ((a_1 + b_1)e_1 + \cdots + (a_n + b_n)e_n)^- = (\min(a_1 + b_1, a_{\tau(1)} + b_{\tau(1)})e_1 + \cdots + \min(a_n + b_n, a_{\tau(n)} + b_{\tau(n)})e_n) = (\min(a_1 + b_1, a_{\tau(1)} + b_1)e_1 + \cdots + \min(a_n + b_n, a_{\tau(n)} + b_n)e_n) = (\min(a_1, a_{\tau(1)})e_1 + \cdots + \min(a_n, a_{\tau(n)})e_n) + (b_1e_1 + \cdots + b_ne_n) = \mu_p^- - H_P(g) + H_P(\theta(g)).$$

The proposition now follows from the observation that $X_s(A_M^\theta) \cap \text{Im} \tau$ equals $X_s(A_M) \cap \text{Im} \tau$.  

### 3.6. Polynomial Weight Factors

The simplest way to show that the functions $\omega^\text{asym} _M$ describe polynomials for $\mu$ sufficiently regular is perhaps an appeal to the theory of toric varieties. One can extract from [4] the relevant results, which are described in this context in [16]. These express the number of lattice points inside an orthogonal set in terms of the Euler-Poincaré characteristic

$$EP(L) := \sum_i (-1)^i \dim H^i(V, L)$$

of a line bundle $L$ on a (toric) variety $V$. For any automorphism $s$ of $V$, we will also need the function

$$EP(s, L) := \sum_i (-1)^i \text{tr} (s; H^i(V, L))$$

so that $EP(L) = EP(1, L)$ (1 denotes the identity automorphism).

For each Levi subgroup $M$ in $\mathcal{L}^-$, we define a toric variety. Specifically, let $\hat{G}$ and $\hat{M}$ denote the Langlands duals of $G$ and $M$ respectively. The quotient $Z(\hat{M})/Z(\hat{G})$ is a torus that we will call $T_M$. Its character group is $\Lambda_M$, and $\theta$ acts on $\Lambda_M$. Let $\Lambda_M^\theta$ be the set of points in $\Lambda_M$ that $\theta$ fixes, and let $\Lambda_{M^\theta \backslash M}$ be the quotient of $\Lambda_M$ by $\Lambda_M^\theta$. The action of $\theta$ on $\Lambda_M$ provides an action of $\theta$ on $T_M$. The Weyl fan of the adjoint group $\hat{G}/Z(\hat{G})$ induces a Weyl fan in the $\theta$-split part $T_M^\theta$ of $T_M$, whose character group is $\Lambda_{M^\theta \backslash M}$. Let the toric variety defined by this induced fan be called $Y_{M^\theta \backslash M}$. This variety is complete, non-singular, and equipped with an action of $T_M^\theta$.

One reason we are interested in this toric variety is that the group of isomorphism classes of $T_M^\theta$-equivariant line bundles on $Y_M$ is isomorphic to the group of orthogonal sets in $\Lambda_{M^\theta \backslash M}$. More explicitly, the fixed points of $T_M^\theta$ on $Y_{M^\theta \backslash M}$ may be indexed in a natural way by the elements of $\mathcal{P}(M)^\theta$. If we are given a $T_M^\theta$-equivariant line bundle on $Y_M$, we can thereby extract the
character $y_P$ by which $T^-_M$ acts on the line over the fixed point indexed by $P$. The points $y_P$ form an orthogonal set in $\Lambda_{M^\theta\setminus M}$. The connection between orthogonal sets in $\Lambda_{M^\theta\setminus M}$ and their corresponding line bundles extends to the level of weight factors, as in the following proposition.

**Proposition 3.17.** Let $\mathcal{L}$ be the line bundle on $Y_{M^\theta\setminus M}$ corresponding to the orthogonal set

$$P \mapsto x_P$$

in $\Lambda_{M^\theta\setminus M}$. If this orthogonal set is positive, then

$$EP(\mathcal{L}) = |\{\nu \in \Lambda_{M^\theta\setminus M} : \nu \in \text{Hull}\{x_P : P \in \mathcal{P}(M)^-\}^*\}|.$$

More generally, if $L$ is a subgroup of $\Lambda_{M^\theta\setminus M}$ with finite index, then

$$\frac{1}{|Z_L|} \sum_{s \in Z_L} EP(s, \mathcal{L}) = |\{\nu \in L : \nu \in \text{Hull}\{x_P : P \in \mathcal{P}(M)^-\}^*\}|$$

where

$$Z_L := \text{Hom}(\Lambda_{M^\theta\setminus M}/L, \mathbb{C}^\times).$$

Our main application of this proposition is the following proposition, which expresses the asymptotic weight factors in terms of these line bundles.

**Corollary 3.18.** For $\mu$ sufficiently regular, the family of points

$$B \mapsto \mu_B - H_B(g) + H_B(\theta(g))$$

form an orthogonal set in $\Lambda_{M^\theta\setminus M}$. Let $\mathcal{L}$ be the corresponding line bundle on $Y_{M^\theta\setminus M}$ and let $L$ denote a sublattice of finite index in $\Lambda_{M^\theta\setminus M}$. Then,

$$\frac{1}{|Z_L|} \sum_{s \in Z_L} EP(s, \mathcal{L}) = |\{\nu \in L : \nu \in \text{Hull}\{\mu_B - H_B(g) + H_B(\theta(g))\}^*\}|.$$

The group of isomorphism classes of $T^-_M$-equivariant line bundles is a finitely generated abelian group $E$, and is isomorphic to the group of orthogonal sets in $\Lambda_{M^\theta\setminus M}$. There is a homomorphism of $\Lambda_{M^\theta\setminus M}$ into this group that sends each element $x$ of $\Lambda_{M^\theta\setminus M}$ to the constant orthogonal set, all of whose vertices are $x$. The quotient $E/\Lambda_{M^\theta\setminus M}$ is isomorphic to $\text{Pic}(Y_{M^\theta\setminus M})$, which is a free abelian group, and there is a polynomial $F$ of degree $\dim Y_{M^\theta\setminus M}$ on the $\mathbb{Q}$-vector space $E/\Lambda_{M^\theta\setminus M} \otimes \mathbb{Z} \otimes \mathbb{Q}$ such that

$$EP(\mathcal{L}) = F(\mathcal{L})$$

where we have precomposed $F$ with the surjection $E \rightarrow E/\Lambda_{M^\theta\setminus M}$ so that both these expressions are functions on $E$. In particular, we may regard $EP$ as a polynomial function on $E$ (or $E/\Lambda_{M^\theta\setminus M}$).

More generally, for any $s \in T^-_M$, we can apply the localization theorem for equivariant $K$-theory to express $EP(s, \mathcal{L})$ as a sum of contributions from each connected component of the fixed point set $Y^s_{M^\theta\setminus M}$ of $s$ on $Y_{M^\theta\setminus M}$. Each of these connected components is a complete non-singular toric variety for some quotient of $T^-_M$, and so

$$EP(s, \mathcal{L}) = \sum_{P \in \mathcal{P}(M)^-} \langle s, y_P \rangle F_{s,P}(M)$$

for polynomial functions $F_{s,P}$ on $(E/\Lambda_{M^\theta\setminus M}) \otimes \mathbb{Z} \otimes \mathbb{Q}$. For fixed $s$, then, $EP(s, \mathcal{L})$ is a polynomial on $E$, as is any sum of functions with this form. By the corollary, then, we deduce that the weight factors $\omega^\text{asyp}_{M}$ on the $\theta$-split side asymptotically equal polynomials.
Definition 3.19. There is an injective map $X_*(A_M) \to \Lambda_{M^\theta \backslash M}$, so we may write the image of $X_*(A_M)^- \cap \text{Im } \tau$ as a union of lattices $L_1, L_2, \ldots, L_k$ in $\Lambda_{M^\theta \backslash M}$. Define

$$\nu_M(g, \mu) := \sum_{i=1}^k \frac{1}{|Z_{L_i}|} \sum_{s \in \mathcal{Z}_{L_i}} EP(s, \mathcal{L})$$

where $\mathcal{L}$ is the line bundle on $Y_M$ for which

$$\frac{1}{|Z_{L_i}|} \sum_{s \in \mathcal{Z}_{L_i}} EP(s, \mathcal{L}) = \left| \{ \nu \in L_i : \nu \in \text{Hull } \{ \mu_B - H_B(g) + \bar{H}_B(\theta(g)) \}^\ast \} \right|$$

when $\mu$ is sufficiently regular.

The weight factors $\nu_M(g, \mu)$ are polynomials in $\mu$, asymptotically equal to $\omega_M^\text{asym}(g, \mu)$, and thence asymptotically equal to $\omega_M(g, \mu)$. It is the coefficients of these weight factors, especially the constant coefficient, that we would like to use as weight factors in the final relative local trace formula.

4. Analysis

4.1. The Adjoint Quotient. We begin by recalling some facts about the “adjoint quotient” map, over which orbital integrals are especially well-behaved. Set $A_G := \text{Spec } \mathcal{O}_G$, where $\mathcal{O}_g$ is the $F$-algebra of polynomial functions on $g$ and $\mathcal{O}_G$ is its set of $G$-invariants. The inclusion $\mathcal{O}_G \hookrightarrow \mathcal{O}_g$

induces this adjoint quotient map

$$\pi_G : g \to A_G$$

which essentially maps each element of $g$ to its characteristic polynomial. The Jacobian of the restriction of this map to a Cartan subalgebra is $|D^G(X)|^{1/2}$. The fiber $\pi_G^{-1}(x)$ is a union of conjugacy classes for any $x \in A_G$. In particular,

$$\pi_G^{-1}(0) = \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_r =: \mathcal{N}$$

where $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$, and $\mathcal{O}_r$ are the nilpotent $G$-orbits of $g$ ordered by increasing dimension. We define $\mathbb{A}_G$ to be the set of points where this morphism is étale (or where the Jacobian is nonzero). If $M \subset G$, then the inclusion $\mathcal{O}^G \hookrightarrow \mathcal{O}^M_G$ induces another map $\mathbb{A}_M \to \mathbb{A}_G$, and the Jacobian of this map is $|D^G_M(X)|^{1/2}$.

4.2. Shalika Germs. Because we have ordered the nilpotent $G$-orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_r$, and $\mathcal{O}_r$ of $g$ by increasing dimension, the sets $\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_i$ are closed in $\mathcal{O}$ for all $i$. In fact, $\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_i$ is closed in $\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_{i+1}$ and $\mathcal{O}_{i+1}$ is a complementary open subset. These sets form a stratification of the nilpotent elements of $g$. We review some basic properties of these orbits.

$G$ acts on each $\mathcal{O}_i$ by conjugation. Therefore, the exact sequence

$$0 \to \mathcal{D}(\mathcal{O}_1) \to \mathcal{D}(\mathcal{O}_1 \cup \mathcal{O}_2) \to \mathcal{D}(\mathcal{O}_2) \to 0$$

gives rise to an exact sequence

$$0 \to \mathcal{D}(\mathcal{O}_1)^G \to \mathcal{D}(\mathcal{O}_1 \cup \mathcal{O}_2)^G \to \mathcal{D}(\mathcal{O}_2)^G.$$
According to [18], any $G$-invariant distribution on $\mathcal{O}_2$ lifts to a $G$-invariant distribution on $\mathfrak{g}$. The right hand map in this sequence is therefore surjective:

$$0 \to \mathcal{D}(\mathcal{O}_1)^G \to \mathcal{D}(\mathcal{O}_1 \cup \mathcal{O}_2)^G \to \mathcal{D}(\mathcal{O}_2)^G \to 0.$$ 

This implies that $\dim \mathcal{D}(\mathcal{O}_1)^G = \dim \mathcal{D}(\mathcal{O}_2)^G = 1$, and thence $\dim \mathcal{D}(\mathcal{O}_1 \cup \mathcal{O}_2)^G = 2$. In this manner, one proves by induction that

$$\dim \mathcal{D}(\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_i)^G = i.$$ 

Let $\mu_j$ be the integral over the $j^{th}$ nilpotent orbit. Then the distributions in the set $\{\mu_j : j \leq i\}$ span the space $\mathcal{D}(\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_i)^G$. Choose functions $f_i$ such that

$$\mu_j(f_i) = \delta_{ij}.$$ 

The orbital integrals over each $f_i$ can be used to define Shalika germs on $\mathfrak{h}_\theta^\perp_{\theta-\text{reg}}$.

**Definition 4.1.** Let $\tilde{\Gamma}_i$ denote the germ of the function

$$X \mapsto \begin{cases} \int_{Z_G(X) \setminus G} f_i(g^{-1}Xg) \, dg & X \in \mathfrak{h}_\theta^\perp_{\text{reg}} \\ 0 & X \notin \mathfrak{h}_\theta^\perp_{\text{reg}} \end{cases}$$

at the origin. The measures $dg$ are chosen to be $G$-invariant and compatible with any isomorphisms $Z_G(X) \cong Z_G(Y)$ induced by conjugation. We call $\tilde{\Gamma}_i$ the Shalika germ corresponding to the nilpotent orbit $\mathcal{O}_i$.

At semisimple elements, we can write orbital integrals as integrals over compact sets, which are well-behaved. This basic result of Harish-Chandra is not true at nilpotent elements, whose orbits are not necessarily closed. We will nevertheless follow his development, which starts with the following critical lemma.

**Lemma 4.2.** [12] Let $M \subset G$ be the centralizer of a torus in $G$, and let $\omega_\theta \subset \mathfrak{g}$ and $\omega_M \subset \mathfrak{n}_M^\perp$ be two compact sets. Then

$$\{g \in M \setminus G : g^{-1}Xg \in \omega_\theta \text{ for some } X \in \pi_M^{-1}(\omega_M)\}$$

has compact closure in $M \setminus G$.

The next few propositions are essentially classical, but we do not impose the condition that $X$ is regular semisimple in $\mathfrak{g}$. Rather, we assume that $X$ is $\theta$-regular and semisimple in $\mathfrak{h}^\perp$. But the proofs are mostly the same.

**Proposition 4.3.** Let $f \in C_c^\infty(\mathfrak{g})$ and let $T$ be a maximal $\theta$-split torus with Lie algebra $\mathfrak{t}$. Let $u$ be any locally constant left $Z_G(T)$-invariant function on $G$, which we will eventually assume to be a weight factor. Then

$$X \mapsto O_X(f) := \int_{Z_G(T) \setminus G} f(g^{-1}Xg) \, u(g) \, dg$$

is locally constant on $\mathfrak{t}' := \mathfrak{t} \cap \mathfrak{h}_\theta^\perp_{\theta-\text{reg}}$.

**Proof.** Let $\omega_M$ be a compact open neighborhood of $\pi_{Z_G(T)}(X)$, where $X \in \mathfrak{t}'$. In Harish-Chandra’s lemma, take $\omega_\theta = \text{supp } f$ and $M = Z_G(T)$, where $M$ is not necessarily a torus. Then

$$\mathcal{X} := \{g \in Z_G(T) \setminus G : g^{-1}Xg \in \text{supp } f \text{ for some } X \in \pi_M^{-1}(\omega_M)\}$$
has compact closure. We can therefore write the orbital integral as an integral over a compact set:

\[
\int_{Z_G(T) \backslash G} f(g^{-1}Xg) \, u(g) \, dg = \int_X f(g^{-1}Xg) \, u(g) \, dg.
\]

By compactness, the right hand integral is locally constant as a function of \(g\) and \(X\), so the orbital integral is too.

**Corollary 4.4.** For any maximal \(\theta\)-split torus \(T\) with Lie algebra \(\mathfrak{t}\), we can represent \(\tilde{\Gamma}_i\) by a function that restricts to a locally constant function on \(\mathfrak{t}'\).

Near the origin of \(\mathfrak{g}\), orbital integrals may be expressed as sums of nilpotent orbital integrals. This is the Shalika germ expansion.

**Proposition 4.5.** Let \(T\) be a maximal \(\theta\)-split torus with Lie algebra \(\mathfrak{t}\). For each \(f \in C_c^\infty(\mathfrak{g})\), there is a closed and open neighborhood \(U\) of \(0 \in \mathfrak{g}\) such that

\[
O_X(f) = \sum_{i=1}^r \mu_i(f) \cdot \tilde{\Gamma}_i(X)
\]

for all \(X \in U \cap \mathfrak{t}'\).

**Proof.** For any \(f \in C_c^\infty(\mathfrak{g})\), the function

\[
\phi = f - \sum \mu_i(f) \cdot f_i
\]

has vanishing nilpotent orbital integrals, and so every element in \(\mathcal{D}(\mathcal{N})^G\) maps \(\phi\) to 0. Since \(\mathcal{D}(\mathcal{N})^G\) is the dual space of \(C_c^\infty(\mathcal{N})_G\), the space of coinvariants for \(G\), \(\phi\) also maps to 0 under the projection \(C_c^\infty(\mathcal{N}) \to C_c^\infty(\mathcal{N})_G\). Now, as in [3],

\[
\lim_{0 \in V} C_c^\infty(\pi_G^{-1}V)_G \cong C_c^\infty(\mathcal{N})_G
\]

where the colimit runs over the compact open neighborhoods \(V\) of \(0\) in \(A_G\). So if \(\phi\) maps to 0 in \(C_c^\infty(\mathcal{N})_G\), then there must be a \(V\) such that \(\phi\) maps to 0 in \(C_c^\infty(\pi_G^{-1}V)_G\). Every distribution in \(\mathcal{D}(\pi_G^{-1}V)^G\) therefore takes \(\phi\) to 0. If we write \(U = \pi_G^{-1}V\), then \(O_X(\phi) = 0\) for all \(X \in U \cap \mathfrak{t}'\) or

\[
O_X(f) = \sum_{i=1}^r \mu_i(f) \cdot \tilde{\Gamma}_i(X)
\]

as required. \(\square\)

An important property of these germs is that they satisfy a partial homogeneity relation that holds for squares in the field \(F\).

**Proposition 4.6.** Let \(d_i = \dim \mathcal{O}_i\). Then

\[
\tilde{\Gamma}_i(\alpha^2 X) = |\alpha|^{-d_i} \tilde{\Gamma}_i(X)
\]

for all \(\alpha \in F\).

**Proof.** Set \(f_{\alpha^2}(X) = f(\alpha^2 X)\). Because \(\mu_i(f_{\alpha^2}) = |\alpha|^{-d_i} \mu_i(f)\) (see eg. [12]), we can verify this identity directly on a small neighborhood of \(0 \in \mathfrak{g}\). Explicitly,

\[
\Gamma_i(\alpha^2 X) = O_X((f_i)_{\alpha^2}) = \sum_j \mu_j((f_i)_{\alpha^2}) \Gamma_j(X) = \sum_j |\alpha|^{-d_j} \mu_j(f_i) \Gamma_j(X) = |\alpha|^{-d_i} \Gamma_i(X)
\]

as required. \(\square\)
We use this last lemma to define canonical representatives for these Shalika germs. Choose a function \( \Gamma'_i \) whose germ at the origin is \( \hat{\Gamma}_i \), as well as a lattice \( L \) containing 0 on which the homogeneity relation \((2)\) holds for \( \Gamma'_i \). For any \( X \in \mathfrak{h}^+ \), one can choose a square \( c^2 \) in \( F \) such that \( c^2 X \in L \) and define

\[
\Gamma_i(X) := |c|^d \Gamma'_i(c^2 X).
\]

We will usually refer to these canonical representatives as Shalika germs as well.

Orbital integrals also have Shalika germ expansions near arbitrary semisimple elements. We will derive these from the Shalika germ expansions near the origin of the Lie algebra \( \mathfrak{l} \) of reductive subgroups \( L \) in \( G \) that arise as centralizers of semisimple elements of \( \mathfrak{g} \). Write \( \hat{\Gamma}_F^L \) for the Shalika germ on \( \mathfrak{l} \) that corresponds to a given nilpotent \( L \)-orbit \( \mathcal{O} \) in \( \mathfrak{l} \). Let \( \Gamma^L_{\mathcal{O}} \) be the corresponding canonical representative, which is defined on \( \mathfrak{l} \). With this notation, for example, \( \Gamma^L_{\mathcal{O}} = \Gamma_i \). Write \( Z \) for the center of \( G \) and \( \mathfrak{z} \) for its Lie algebra. To derive these generalized expansions, we will first relate \( \Gamma^L_{\mathcal{O}} \) to \( \Gamma^L_{[G,G]} \).

**Lemma 4.7.** The Shalika germs of a reductive group \( G \) can be expressed in terms of the Shalika germs of its derived group:

\[
\Gamma^G_i(X + Z) = \sum_{[G,G]-\text{orbits } \mathcal{O} \subset \mathcal{O}_i} \Gamma^L_{[G,G]}(X).
\]

where \( X \in [\mathfrak{g}, \mathfrak{g}] \) and \( Z \in \mathfrak{z} \).

**Proof.** The normal subgroup \([G,G]_F Z_F \) of \( G_F \) has finite index, so \( D := ([G,G]_F Z_F) \backslash G_F \) is a finite group. Each nilpotent \( G \)-orbit \( \mathcal{O}_i \) can be written as a finite union of nilpotent \([G,G]\)-orbits. In fact, by fixing one of these nilpotent \([G,G]\)-orbits, say \( \mathcal{O} \), we can write \( \mathcal{O}_i = \bigcup_{x \in D} x^{-1} \mathcal{O} x \).

For each \( x \in D \), we can choose some \( f_x \in C^\infty_c([\mathfrak{g}, \mathfrak{g}]) \) such that \( \Gamma^L_{x^{-1} \mathcal{O} x}(X) \) and \( \mathcal{O}_x(f_x) \) have equal germs at the origin. Let \( f'_x \) be the sum of the \( f_x \) as \( x \) varies through \( D \):

\[
f'_x = \sum_{x \in D} f_x.
\]

Let \( f_i \in C^\infty_c(\mathfrak{g}) \) restrict to \( f'_i \in C^\infty_c([\mathfrak{g}, \mathfrak{g}]) \) and suppose that \( f_i(X + Z) = f_i(X) \) for all \( Z \) in some small lattice \( L \) containing 0 in \( \mathfrak{z} \). Then for any \( X + Z \) in a small neighbourhood of the origin, with \( X \in [\mathfrak{g}, \mathfrak{g}] \) and \( Z \in \mathfrak{z} \),

\[
\Gamma^G_i(X + Z) = \mathcal{O}_{X+Z}(f_i) = \mathcal{O}_x(f'_i) = \sum_{x \in D} \mathcal{O}_x(f_x) = \sum_{x \in D} \Gamma^L_{[G,G]_x^{-1} \mathcal{O}_x}(X)
\]

as required. \( \Box \)

**Corollary 4.8.** Shalika germs \( \Gamma^G_i \) are translation invariant by elements of \( \mathfrak{z} \cap \mathfrak{h}^+ \).

This corollary is already enough to define a Shalika germ expansion near central semisimple elements of \( \mathfrak{g} \). At arbitrary semisimple elements \( S \), the Shalika germ expansion near \( S \) in the centralizer \( \mathfrak{g}_S \) provides a Shalika germ expansion near \( S \) in \( \mathfrak{g} \), according to the next proposition.

**Proposition 4.9.** Let \( S \in \mathfrak{h}^+ \) be semisimple, and suppose that \( T \) is a maximal \( \theta \)-split torus in \( Z_G(S) \) with Lie algebra \( \mathfrak{t} \). For any \( f \in C^\infty_c(\mathfrak{g}) \),

\[
\mathcal{O}_X(f) = \sum_i \mu_{S+Y_i}(f) \cdot \Gamma^Z_G(X)
\]

for all \( X \in U \cap \mathcal{U}' \), where \( U \) is an open neighborhood of \( S \) in \( \mathfrak{g}_S \) and \( \{Y_i\} \) is a complete set of representatives for the nilpotent orbits of \( Z_G(S) \) in \( \mathfrak{g}_S \).
Proof. Let \( \phi \in C_c^\infty(\mathfrak{g}) \). The Shalika germ expansion at 0 for \( G = Z_G(S) \) and \( \phi_S(X) = \phi(X + S) \) gives

\[
\int_{Z_G(X) \setminus Z_G(S)} \phi_S(g^{-1}Xg) \, dg = \sum_i \mu_{X}^{Z_G(S)}(\phi_S) \cdot \Gamma_i^{Z_G(S)}(X)
\]

for all \( X \in U' \cap \mathfrak{t}' \), where \( U' \) is an open neighborhood of \( S \) in \( \mathfrak{g}_S \). Because the Shalika germs \( \Gamma_i^{Z_G(S)} \) are translation invariant by elements in the center of \( Z_G(S) \),

\[
\int_{Z_G(S + X) \setminus Z_G(S)} \phi(g^{-1}(X + S)g) \, dg = \sum_i \mu_{X + Y_i}^{Z_G(S)}(\phi) \cdot \Gamma_i^{Z_G(S)}(X)
\]

for all \( X + S \in U' \cap \mathfrak{t}' \). The proposition follows from a trick due to Harish-Chandra that is implicit in the proof of lemma 29 in \cite{12}: we can choose \( \phi \) such that

\[
\int_{Z_G(X) \setminus Z_G(S)} \phi(g^{-1}Xg) \, dg = \int_{Z_G(X) \setminus G} f(g^{-1}Xg) \, dg
\]

for all \( X \) over some compact neighborhood of the origin in \( \mathbb{A}_H \). \qed

The Shalika germ expansions by themselves almost guarantee that orbital integrals near semisimple elements become locally integrable when multiplied by \( |D^H(X)|^2 \), except that certain germs might have degrees of homogeneity that are too large. Fortunately, we can prove that the only terms that could possibly be problematic are identically zero.

**Definition 4.10.** Let \( \mathcal{O}(X) \) denote the \( G \)-orbit of \( X \in \mathfrak{g} \). For any subset \( S \subseteq \mathfrak{g} \), write \( \mathcal{O}(S) := \bigcup_{X \in S} \mathcal{O}(X) \).

**Proposition 4.11.** Let \( T \) be some maximal \( \theta \)-split torus with Lie algebra \( \mathfrak{t} \). Then

\[
\dim \mathcal{O}_i > \dim Z_G(T) \setminus G \implies \Gamma_i(X) = 0 \text{ for all } X \in \mathfrak{t}'.
\]

**Proof.** Suppose that \( \dim \mathcal{O}_i > \dim Z_G(T) \setminus G \). Let \( \{X_j : j \geq 0\} \) be a convergent sequence in \( \mathcal{O}(\mathfrak{t}') \) with limit \( X \). Then for each \( j \geq 0 \),

\[
\dim \mathcal{O}(X_j) = \dim Z_G(T) \setminus G < \dim \mathcal{O}_i.
\]

Since the function \( Y \mapsto \dim \mathcal{O}(Y) = \text{rank ad } Y \) is upper semicontinuous, we see that \( X \notin \mathcal{O}_i \). Thus

\[
\overline{\mathcal{O}(\mathfrak{t}') \cap \mathcal{O}_i} = \emptyset.
\]

We claim that for any \( Y \in \mathcal{O}_i \) and any open set \( U \) containing \( Y \) but disjoint from \( \mathcal{O}_j \) with \( j < i \), there is a function \( f_i \) supported in \( U \) such that \( \Gamma_i(X) \) and \( O_X(f_i) \) have the same germs at the origin.

We prove this by induction. To begin, suppose that \( i = r \). Then \( \mathcal{O}_r \) is relatively open in \( \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_r \), so that \( \mathcal{O}_k \cap U \) is relatively open in \( \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_r \) and does not intersect \( \mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{r-1} \). Take any nonnegative and nonzero real-valued function \( \tilde{f}_r \) that is supported in \( U \). Then \( \mu_j(\tilde{f}_r) = 0 \) whenever \( j < r \) and \( \mu_r(\tilde{f}_r) > 0 \). The function

\[
f_r = \frac{\tilde{f}_r}{\mu_r(\tilde{f}_r)}
\]

has the sought properties.
We now suppose that the claim is true for all \( j > k \). \( \mathcal{O}_k \) is relatively open in \( \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_k \), so that \( \mathcal{O}_r \cap U \) is relatively open in \( \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_k \) and does not intersect \( \mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{k-1} \). Again, we take any nonnegative and nonzero real-valued function \( \tilde{f}_k \) that is supported on \( U \). Set

\[
f_k := \frac{\tilde{f}_k}{\mu_k(\tilde{f}_k)} - \sum_{j > k} \frac{\mu_j(\tilde{f}_k)}{\mu_k(\tilde{f}_k)} f_j
\]

where each \( f_j \) has been chosen with two properties: first, \( f_j \) is supported on an open set that does not intersect \( \mathcal{O}_i \) for any \( i < j \); second, the functions \( \Gamma_j(X) \) and \( O_X(f_j) \) have the same germs at the origin. These functions \( f_j \) exist according to the induction hypothesis. For \( h < k \),

\[
\mu_h(f_k) = \frac{\mu_h(\tilde{f}_k)}{\mu_k(\tilde{f}_k)} - \sum_{j > k} \frac{\mu_j(\tilde{f}_k)}{\mu_k(\tilde{f}_k)} \mu_h(f_j) = \frac{\mu_h(\tilde{f}_k)}{\mu_k(\tilde{f}_k)} = 0.
\]

For \( h > k \),

\[
\mu_h(f_k) = \frac{\mu_h(\tilde{f}_k)}{\mu_k(\tilde{f}_k)} - \sum_{j > k} \frac{\mu_j(\tilde{f}_k)}{\mu_k(\tilde{f}_k)} \mu_h(f_j) = \frac{\mu_h(\tilde{f}_k)}{\mu_k(\tilde{f}_k)} - \frac{\mu_h(\tilde{f}_k)}{\mu_k(\tilde{f}_k)} = 0.
\]

For \( h = k \),

\[
\mu_h(f_k) = \frac{\mu_h(\tilde{f}_k)}{\mu_k(\tilde{f}_k)} - \sum_{j > k} \frac{\mu_j(\tilde{f}_k)}{\mu_k(\tilde{f}_k)} \mu_h(f_j) = \frac{\mu_k(\tilde{f}_k)}{\mu_k(\tilde{f}_k)} = 1.
\]

Therefore, this function has all the sought properties, and the claim follows inductively.

Now take any \( Y \in \mathcal{O}_i \) that does not belong to the closure of \( \mathcal{O}(t') \), i.e. such that there is an open set \( U \) containing \( Y \) that does not intersect \( \mathcal{O}(t') \). We can choose \( \tilde{f}_i \) with support in this set, and define \( \tilde{\Gamma}_i(X) \) to be the germ of the function

\[
X \mapsto \begin{cases} 
\int_{Z_G(X) \setminus G} f_i(g^{-1}Xg) \, dg & \text{if } X \in h_{\text{reg}}^+, \\
0 & \text{if } X \notin h_{\text{reg}}^+.
\end{cases}
\]

This function is identically 0 because we are integrating outside the support of \( f_i \). Thus \( \tilde{\Gamma}_i(X) = 0 \) and \( \Gamma_i(X) = 0 \).

This last result allows us to give the following lemma in full generality.

**Lemma 4.12.** The function \( \mathfrak{t} \rightarrow \mathbb{C} \) defined by extending

\[
X \mapsto |D_{Z_G(T)}^G(X)|^\frac{1}{2} \int_{Z_G(T) \setminus G} f(g^{-1}Xg) \, dg
\]

by 0 from \( \mathfrak{t}_{\text{reg}} \) to \( \mathfrak{t} \) is locally bounded (and therefore locally integrable) on each Cartan subalgebra \( \mathfrak{t} \) of \( \mathfrak{h}^\perp \).

**Proof.** The Shalika germ expansion of this integral at an element \( Y \) of \( \mathfrak{t} \) is

\[
|D_{Z_G(T)}^G(X)|^\frac{1}{2} \int_{Z_G(T) \setminus G} f(g^{-1}Xg) \, dg = \sum_{\text{nilpotent } Z_G(Y)\text{-orbits indexed by } i} \mu_i(f) \cdot |D_{Z_G(T)}^G(X)|^\frac{1}{2} \cdot \Gamma_i^e(Z_G(Y))(X).
\]
It is enough to verify this lemma termwise, and check that each \( |D^{G}_{Z_G(Y)}(X)|^{\frac{1}{2}} \cdot \Gamma^{Z_G(Y)}_{i}(X) \) is locally bounded. Near \( Y \), each term is (up to a constant)

\[
|D^{Z_G(Y)}_{Z_G(T)}(X)|^{\frac{1}{2}} \cdot \Gamma^{Z_G(Y)}_{i}(X).
\]

Because \( Y \) belongs to the center of \( Z_G(Y) \) and this function is translation invariant under central elements, it suffices to check integrability when \( Y = 0 \). To do this, we consider the degrees of homogeneity for each factor:

\[
\text{degree of } |D^{Z_G(Y)}_{Z_G(T)}(X)|^{\frac{1}{2}} = \frac{1}{2} (\dim Z_G(Y) - \dim Z_G(T))
\]

\[
\text{degree of } \Gamma^{Z_G(Y)}_{i}(X) = - \dim \mathcal{O}_i
\]

\[
\geq -\frac{1}{2} (\dim Z_G(Y) - \dim Z_G(T))
\]

or else \( \Gamma^{Z_G(Y)}_{i}(X) = 0 \). This means that

\[
\text{degree of } |D^{Z_H(Y)}_{Z_G(T)}(X)|^{\frac{1}{2}} \cdot \Gamma^{Z_G(Y)}_{i}(X) \geq 0
\]

which implies that this term is locally bounded near \( Y \) if it is locally bounded near elements \( X \) for which \( Z_G(X) \) is a proper subgroup of \( Z_G(Y) \). In this way, we can inductively reduce to the case in which \( Z_G(Y) \) is a torus, which is immediate.

\[\square\]

4.3. **Bounding the Weighted Terms.** The crucial step that allows us to replace the initial weight factors \( \omega \) by their simplifications \( \nu \) is an application of the Lebesgue dominated convergence theorem. This requires several lemmas on the absolute summability of not just orbital integrals, but also of their weighted analogs. These lemmas are perhaps easiest to approach using the language of abstract norms on \( F \)-varieties. We will state some results in this section without proof. A detailed exposition of the missing arguments is [16], covering material due to Harish-Chandra.

An abstract norm on an \( F \)-variety is just a function \( \| \cdot \| \) whose value is always greater than or equal to 1. Among abstract norms, there is a notion of equivalence: one abstract norm is equivalent to another if the first is bounded by a constant times a power of the second and vice versa. We will only be interested in a certain equivalence class of abstract norms.

Namely, if \( U \) is an affine scheme whose space of global sections \( \mathcal{O}_U \) are generated by the functions \( f_1, f_2, \ldots, f_n \), then we can define an abstract norm as follows:

\[
\|x\|_U := \max \left\{ \frac{1}{|f_1(x)|}, \frac{1}{|f_2(x)|}, \ldots, \frac{1}{|f_n(x)|} \right\}
\]

where \( x \in U \). For general schemes \( X \), we choose a collection of affine open subsets of \( X \) that cover \( X \), eg. \( U_1, U_2, \ldots, U_{n-1} \), and \( U_n \). Define the norm

\[
\|x\|_X := \inf \{ \|x\|_{U_i} : x \in U_i \}
\]

for all \( x \in X \). While this abstract norm depends on the choice of the the affine sets \( U_i \) as well as the generating functions \( f_i \) on each, the equivalence class it defines is independent of all these choices, and so we define a norm on a general scheme to be an abstract norm that belongs to this equivalence class. An example of such a norm that we have already encountered is the function \( \| \cdot \|_G := \exp d(\cdot) \) on the algebraic group \( G \). The next proposition is explained in [16] and lists some of the more elementary properties of norms.
Proposition 4.13. Let $X$ and $Y$ be affine schemes of finite type over $F$ and let $\| \cdot \|_X$ and $\| \cdot \|_Y$ be norms on $X(F)$ and $Y(F)$ respectively.

- Let $\phi : Y \to X$ be a morphism and denote by $\phi^* \| \cdot \|_X$ the abstract norm on $Y(F)$ obtained by composing $\| \cdot \|_X$ with $\phi : Y(F) \to X(F)$. Then $\| \cdot \|_Y$ dominates $\phi^* \| \cdot \|_X$. If $\phi$ is finite, then $\| \cdot \|_Y$ is equivalent to $\phi^* \| \cdot \|_X$.
- Suppose $Y$ is a closed subscheme of $X$. Then the restriction of $\| \cdot \|_X$ to $Y(F)$ is equivalent to $\| \cdot \|_Y$.
- If $F$ is locally compact, then a subset $B(F)$ of $X(F)$ has compact closure if and only if it is bounded, i.e. the norm function $\| \cdot \|_X$ is bounded on $B(F)$.
- All three of $\sup \{ \|x\|_X, \|y\|_Y \}, \|x\|_X + \|y\|_Y,$ and $\|x\|_X \cdot \|y\|_Y$ are valid norms on $(X \times Y)(F) = X(F) \times Y(F)$.
- Let $U := X_f$ denote the principal open subset of $X$ determined by a regular function $f$ on $X$, so that $U(F) = \{ x \in X(F) : f(x) \neq 0 \}$. Then $\|u\|_U := \sup \{ \|u\|_X, |f(u)|^{-1} \}$ is a norm on $U(F)$.
- Suppose we are given a finite cover of $X$ by affine open subsets $U_1, U_2, \ldots, U_r$ as well as a norm $\| \cdot \|_i$ on $U_i(F)$ for each $i = 1, 2, \ldots, r$. For $x \in X(F)$, define $\|x\|_S$ to be the infinite of the numbers $\|x\|_i$, where $i$ ranges over the set of indices for which $x \in U_i(F)$. Then $\| \cdot \|$ is a norm on $X(F)$.
- Let $G$ be a group scheme of finite type over $F$, and suppose we are given an action of $G$ on $X$. Let $B$ be a bounded subset of $G(F)$. Then there exist $c, R > 0$ such that $\|bx\|_X \leq c \|x\|_X^R$ for all $b \in B, x \in X(F)$.

If we are given a morphism of schemes $\varphi : X \to Y$ and a norm $\| \cdot \|_X$ on $X$, we can define an abstract norm on $Y$:

$$\|y\|_Y := \inf \{ \|x\|_X : \varphi(x) = y \}.$$

We will call this norm the push-forward of $\| \cdot \|_X$ and sometimes denote it $\varphi_* \| \cdot \|_X$. When this abstract norm is a norm on the image of $\varphi$, it will be a norm for all choices of $\| \cdot \|_X$, and we say that the morphism $\varphi$ has the norm descent property. The behavior of the norm descent property under composition of morphisms is described by the next lemma.

Lemma 4.14. Consider morphisms $f : X \to Y$ and $g : Y \to Z$ of affine schemes of finite type over $F$. Put $h = g \circ f : X \to Z$. Assume that the map $f : X(F) \to Y(F)$ is surjective. Then

- If $f$ and $g$ satisfy the norm descent property, then so does $h$.
- If $h$ satisfies the norm descent property, then so does $g$.

A sufficient condition for a morphism to have the norm descent property is for it to admit sections on its image locally in the Zariski topology. For example, Bruhat theory defines an open immersion associated to each Levi subgroup $M$:

$$\tilde{U} \times M \times U \hookrightarrow G$$

which is given by multiplication. The image of $\tilde{U} \times \{1\} \times U$ projects to an open subset in $M\backslash G$, and defines a section over this open subset. By translating this $(G$-equivariant) morphism by elements of $G$, we obtain sections over the open sets in some covering of $M\backslash G$, which implies that the morphism

$$G \to M\backslash G$$

has the norm descent property. This classical result implies that

$$\|g\|_{M\backslash G} := \inf \{ \|mg\|_G : m \in M \}$$
is a norm on the affine $F$-variety $M \backslash G$.

Other morphisms with the norm descent property include finite morphisms and quotients of the form $G \rightarrow T \backslash G$, where $T$ is any torus of $G$. In other words,

$$\|g\|_{T \backslash G} := \inf \{\|tg\|_G : t \in T\}$$

is a norm on $T \backslash G$. Especially important is the case in which $T$ is the $F$-split torus $A_M$. In the next lemma, we will use these norms to bound the weight factors.

**Lemma 4.15.** For any Levi subgroup $M$,

$$\omega_M(g, \mu) \leq a(1 + \log \|g\|_M + \|\mu\|_E)^r$$

$$\nu_M(g, \mu) \leq a(1 + \log \|g\|_M + \|\mu\|_E)^r$$

for some constants $a$ and $r$, which we can assume to be equal and independent of $M$.

**Proof.** We begin by bounding the preliminary weight factor $\omega_M$:

$$\omega_M(g, \mu) \leq |\{\nu \in X_\circ(A_M) : K\theta(g)^{-1}agK \in \text{Conv} (\mu) \text{ and } H_A(a) = \nu_a = \nu\}|$$

$$\leq |\{\nu \in \Lambda_M : K\theta(g)^{-1}mgK \in \text{Conv} (\mu) \text{ and } H_M(m) = \nu\}|$$

and this last bound is left $M$-invariant. The requirement that $\text{inv} (mg, \theta(g)) \in \text{Conv} (\mu)$ implies, by the triangle inequality in the Bruhat-Tits building, that

$$d(m) \leq d(g) + d(\theta(g)) + \|\mu\|_E$$

and so this is bounded by

$$|\{\nu \in \Lambda_M : d(m) \leq d(g) + d(\theta(g)) + \|\mu\|_E \text{ and } H_M(m) = \nu\}|$$

$$\leq c|\{\nu \in \Lambda_M : \|\bar{\nu}\| \leq d(g) + d(\theta(g)) + \|\mu\|_E\}|$$

where $\bar{\nu}$ is the image of $\nu$ under the natural map $\Lambda_M \rightarrow a_M$ and $c$ is some constant. This is in turn bounded by

$$a(1 + \log \|g\|_G + \|\mu\|_E)^r$$

for constants $a$ and $r$. By the left $M$-invariance of some of our bounds, we can strengthen the term $\log \|g\|_G$ to $\log \|g\|_M\|_G$, giving the sought bound. The explicit analysis of section 3.6 implies $\nu_M$ is similarly bounded, keeping in mind the inequality $\|H_F(g)\|_E \leq \log \|g\|_G$. 

The classical trace formula is a sum of terms bounded by expressions like the one described in the next lemma. We will need to know that these converge to control the asymptotic behavior of the $\theta$-split side.

**Lemma 4.16.** Let $\| \cdot \|_{Z_G(X) \backslash G}$ by a norm on the homogeneous space $Z_G(T) \backslash G$, where $T$ is a maximal $\theta$-split torus with Lie algebra $\mathfrak{t}$. The integral

$$\int_{\mathfrak{t}} |D_{Z_G(T)}^G(X)|^{\frac{1}{2}} \int_{Z_G(T) \backslash G} f(g^{-1}Xg) (\log \|g\|_{Z_G(X) \backslash G})^r \ dg \ dX$$

converges absolutely for any $f \in C^\infty_c (\mathfrak{g})$ and any integer $r > 0$.

**Proof.** According to the elementary theory of real functions, for any positive $\epsilon$, the inequality $(\log x)^r < x^\epsilon$ holds for real numbers $x$ taken to be sufficiently large. It is therefore enough to prove that

$$\int_{\mathfrak{t}} |D_{Z_G(T)}^G(X)|^{\frac{1}{2}} \int_{Z_G(T) \backslash G} f(g^{-1}Xg) \|g\|_{Z_G(T) \backslash G}^\epsilon \ dg \ dX$$
converges for some positive $\epsilon$.

Choose a norm $\| \cdot \|_g$ on $\mathfrak{g}$. Let $U(t)$ denote the $G$-orbit of $t'$ so that, by proposition 4.13, the following is a norm on $U(t)$:

$$\| X \|_{rs} = \max \left( \| X \|_g, \frac{1}{| D_Z^G (T) (X) |} \right).$$

Because $f$ is compactly supported, we can bound $\| X \|_g$ on its support. This allows us to bound $\| g^{-1} X g \|_{rs}$ by some constant multiple of

$$\max \left( 1, \frac{1}{| D_Z^G (T) (g^{-1} X g) |} \right) = \max \left( 1, \frac{1}{| D_Z^G (T) (X) |} \right)$$

whenever the integrand does not vanish. Since proposition 4.13 again implies that $\| g \|_{Z_G (T) \setminus G}$ is dominated by $\| g^{-1} X g \|_{rs}$, it is enough to prove the convergence of

$$\int_{t'} | D_Z^G (T) (X) |^{\frac{1}{2}} \int_{Z_G (X) \setminus G} f(g^{-1} X g) \frac{1}{| D_Z^G (T) (X) |} \, dg \, dX.$$

The integrand is compactly supported, and so it is enough to prove that it is also locally integrable. We have already proven that the normalized orbital integral

$$| D_Z^G (T) (X) |^{\frac{1}{2}} \int_{Z_G (X) \setminus G} f(g^{-1} X g) \, dg$$

is locally bounded. Because $| D_Z^G (T) (X) |^{-\epsilon}$ is locally integrable when $\epsilon$ is sufficiently small, we can find an $\epsilon$ for which the product of these two expressions is also locally integrable, and this completes the proof of the lemma.

\section{5. The $\theta$-split Side of the Relative Local Trace Formula}

With the results of section 4 in hand, we can prove the limit of proposition 1.1, which is the goal of this paper. As discussed in the introduction, the next proposition implies that $J_-(f, \nu)$ is the correct expression for the geometric side of the local relative trace formula for Lie algebras. One obtains more concrete expressions by evaluating this polynomial at specific values of $\mu$.

\textbf{Proposition 5.1.} For any two dominant coweights $\mu_1$, $\mu_2$, let $\mu := \mu_1 + d\mu_2$. Then

$$\lim_{d \to \infty} J_-(f, \omega) - J_-(f, \nu) = 0$$

\textbf{Proof.} To prove that

$$\lim_{d \to \infty} J_-(f, \omega) - J_-(f, \nu) = 0$$

it is sufficient to prove that for an arbitrary torus $T$,

$$\lim_{d \to \infty} \int_{t'} | D_Z^G (T) (X) |^{\frac{1}{2}} \int_{A_{M \setminus G}} f(g^{-1} X g) \omega_M (g, \mu_1 + d\mu_2) \, dg \, dX$$

$$- \int_{t'} | D_Z^G (T) (X) |^{\frac{1}{2}} \int_{A_{M \setminus G}} f(g^{-1} X g) \nu_M (g, \mu_1 + d\mu_2) \, dg \, dX$$

$$= \lim_{d \to \infty} \int_{t'} | D_Z^G (T) (X) |^{\frac{1}{2}} \int_{A_{M \setminus G}} f(g^{-1} X g) (\omega_M (g, \mu_1 + d\mu_2) - \nu_M (g, \mu_1 + d\mu_2)) \, dg \, dX = 0.$$
Now by the lemma 4.15, there exist constants $a$, $b$, and $r$ such that
\[
|\omega_M(g, \mu_1 + d\mu_2) - \nu_M(g, \mu_1 + d\mu_2)| \leq a(b + \log g_{AM\setminus G} + d)^r.
\]

But by Arthur’s key geometric lemma, this difference of weight factors vanishes unless $d$ is less than some constant times $(1 + \log g_{AM\setminus G})$. Since $g_{AM\setminus G}$ is certainly less than $g_{AM\setminus G}$, we obtain a simple bound of this difference that is uniform in $d$:
\[
|\omega_M(g, \mu_1 + d\mu_2) - \nu_M(g, \mu_1 + d\mu_2)| \leq a(b + \log g_{AM\setminus G})^r
\]
for possibly different constants $a$ and $b$. We have proven in section 4 that
\[
\int_t \left| D^G(X) \right|^\frac{1}{2} \int_{AM\setminus G} f(g^{-1}Xg) \, dg \, dX \quad \text{and} \quad \int_t \left| D^G(X) \right|^\frac{1}{2} \int_{AM\setminus G} f(g^{-1}Xg) (\log g_{AM\setminus G})^r \, dg \, dX
\]
are absolutely summable, so we may apply the Lebesgue dominated convergence theorem and evaluate the limit inside the integral. The proof of the proposition then follows from the observation that
\[
\omega_M(g, \mu_1 + d\mu_2) = \nu_M(g, \mu_1 + d\mu_2)
\]
for sufficiently large $d$. \qed

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