EXTENDING MAPS BY INJECTIVE \(\sigma\)-Z-MAPS
IN HILBERT MANIFOLDS

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Abstract. The aim of the paper is to prove that if \(M\) is a metrizable manifold modelled on a Hilbert space of dimension \(\alpha \geq \aleph_0\) and \(F\) is its \(\sigma\)-Z-set, then for every completely metrizable space \(X\) of weight no greater than \(\alpha\) and its closed subset \(A\), for any map \(f: X \to M\), each open cover \(\mathcal{U}\) of \(M\) and a sequence \((A_n)\) of closed subsets of \(X\) disjoint from \(A\) there is a map \(g: X \to M\) \(\mathcal{U}\)-homotopic to \(f\) such that \(g\big|_A = f\big|_A\). It is shown that if \(f\big(\partial A\big)\) is contained in a locally closed \(\sigma\)-Z-set in \(M\) or \(f\big(X \setminus A\big) \cap f\big(\partial A\big) = \emptyset\), the map \(g\) may be taken so that \(g\big|_{X \setminus A}\) be an embedding. If, in addition, \(X \setminus A\) is a connected manifold modelled on the same Hilbert space as \(M\), then there is a \(\mathcal{U}\)-homotopic to \(f\) map \(h: X \to M\) such that \(h\big|_A = f\big|_A\) and \(h\big|_{X \setminus A}\) is an open embedding.

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In [17], West has proved that any homotopy \((f_t: X \to M)_{t \in I}\) from a separable completely metrizable space \(X\) into a separable metrizable manifold \(M\) modelled on an infinite-dimensional Fréchet space can be approximated by homotopies \((h_t: X \to M)_{t \in I}\) such that \(h_t = f_t\) for \(t = 0, 1\) and each \(h_t\) with \(t \in (0, 1)\) is a closed embedding. He has also made a note that the homotopy \((h_t)_{t \in I}\) may be modified in such a way that it be an embedding of \(X \times (0, 1)\) into \(M\). This claim was applied by Anderson and McCharen in their joint paper [2] on extending homeomorphisms between Z-set of (separable) Fréchet manifolds. Unfortunately, this statement of West is incorrect (see Example 4.7 below). In this paper we try to give sufficient conditions under which a map of a complete metric space \(X\) into a Hilbert manifold \(M\) could be approximated, in the limitation topology and relative a given closed set \(A \subset X\), by mappings \(g\) which are embeddings on \(X \setminus A\). Moreover, we request that \(g\) is a \(Z\)-embedding on any of the countably many closed subsets of \(X \setminus A\), or, when \(X \setminus A\) is a manifold modelled on the same Hilbert space as \(M\), that \(g\big|_{X \setminus A}\) be an open embedding of \(X \setminus A\) into \(M\). Our proofs totally differ from that of West [17] and depend on the argument used by Toruńczyk in [16, Proof of 3.1].
The paper is organized as follows. In Section 1 we establish notation and terminology and cite theorems which shall be applied in the next sections. The second part is devoted to the proof of the main lemma on extending maps which take values in ANR’s. Section 3 deals with extending maps from completely metrizable spaces into infinite-dimensional Hilbert manifolds by injections whose images are $\sigma$-$Z$-sets. In the last part we give conditions under which a map is extendable by an embedding.

1. THEOREMS TO QUOTE

In this paper $I$ denotes the unit interval $[0, 1]$. The letters $X, Y, Z, K,$ etc. stand for metrizable spaces. By a map we mean a continuous function. Whenever $g$ is a function, $\text{im} \ g$ stands for the image of $g$. If, in addition, $g$ takes values in a topological space, by $\text{im} \ g$ we denote the closure of the image in the whole space. If $A$ is a subset of $X$, $\text{int} A$, $\overline{A}$ and $\partial A$ stand for, respectively, the interior, the closure and the boundary of $A$ in the whole space $X$. If $B \subset X$ is a superset of $A$, by $\text{int}_B A$, $\text{cl}_B A$ and $\partial_B A$ we denote the interior, closure and boundary of $A$ relative to $B$. We use $w(X)$ to denote the topological weight of $X$.

Whenever $Y$ is a metrizable space, $\text{cov}(Y)$ and $\text{Metr}(Y)$ denote the collections of all open covers of $Y$ and of all metrics on $Y$ inducing its topology, respectively. For every $g \in \text{Metr}(Y)$, $B_{\rho}(y, r)$ and $\overline{B}_{\rho}(y, r)$ stand for the open and the closed $\rho$-ball (respectively) in $Y$ with center at $y \in Y$ and of radius $r > 0$. Following Toruńczyk [15], for a map $f$ of a metrizable space $X$ into $Y$, $B(f, U)$ with $U \in \text{cov}(Y)$ consists of all maps $g : X \to Y$ which are $U$-close to $f$, that is, $g$ belongs to $B(f, U)$ iff for every $x \in X$ there is $U \in \mathcal{U}$ such that $\{f(x), g(x)\} \subset U$. Similarly, for $g \in \text{Metr}(Y)$ and a map $\alpha : Y \to (0, +\infty)$, $B_{\alpha}(f, \alpha)$ is the set of all maps $g : X \to Y$ such that $g(f(x), g(x)) \leq \alpha(f(x))$ for any $x \in X$. On the space $C(X, Y)$ of all maps of $X$ into $Y$ we always consider the limitation topology in which $\{B(f, U) : U \in \text{cov}(Y)\}$ (respectively $\{B_{\alpha}(f, \alpha) : \alpha \in C(Y, (0, +\infty))\}$) with fixed $g \in \text{Metr}(Y)$ may serve as a base of open (closed) neighbourhoods of a given map $f$. For basic properties of this topology the reader is referred to [15], [5]. The set of all [closed; open] embeddings of $X$ into $Y$ is denoted by $\text{Emb}(X, Y)$ [$\text{Emb}^c(X, Y)$; $\text{Emb}^o(X, Y)$].

By a Hilbert manifold we mean a metrizable space which admits an open cover by sets homeomorphic to some infinite-dimensional Hilbert space. Any metrizable manifold modelled on an infinite-dimensional Fréchet space is a Hilbert manifold (since every Fréchet space is homeomorphic to a Hilbert one — see [15, 16]). Hilbert manifolds are completely metrizable ANR’s. For simplicity, we say that a space is an $\alpha$-manifold, where $\alpha$ is an infinite cardinal, if it is a manifold modelled on the Hilbert space of dimension $\alpha$. 
If \( \mathcal{U} \) is any collection of subsets of \( Y \) (\( \mathcal{U} \) need not cover \( Y \) and the members of \( \mathcal{U} \) need not be open) and \( F: X \times I \to Y \) is a homotopy, then \( F \) is said to be a \( \mathcal{U} \)-homotopy iff for any \( x \in X \) either \( F(\{x\} \times I) \) consists of a single point or there is \( U \in \mathcal{U} \) such that \( F(\{x\} \times I) \subseteq U \).

Two maps \( f, g: X \to Y \) are \( \mathcal{U} \)-homotopic in \( Y \) if there is a \( \mathcal{U} \)-homotopy \( X \times I \to Y \) which connects \( f \) and \( g \). Additionally, if \( f \big|_A = g \big|_A \), \( f \) and \( g \) are said to be \( \mathcal{U} \)-homotopic in \( Y \) relative \( A \) if there is a \( \mathcal{U} \)-homotopy \( F \) connecting \( f \) and \( g \) such that \( F(a,t) = f(a) \) for every \( a \in A \) and \( t \in [0,1] \).

The following is known as Toruńczyk’s Lemma (see [15, Lemma 1.1], and [5] for proof).

1.1. Lemma. Let \( Y \) be completely metrizable, \( F \) a subspace of \( C(X,Y) \) and \( U_1, U_2, \ldots \) open subsets of \( C(X,Y) \). If \( U_n \cap F \) is dense in \( F \) for each \( n \geq 1 \), then maps in \( F \) are approximable by elements of \( F_0 \cap \bigcap_{n=1}^{\infty} U_n \), where \( F_0 \) denotes the closure of \( F \) in the topology of \( \varrho \)-uniform convergence in \( C(X,Y) \) and \( \varrho \in \text{Metr}(Y) \).

In the sequel we shall also apply the next two results of Toruńczyk and the one of Henderson and Schori.

1.2. Lemma ([15, Lemma 1.3]). If \( Y \) is an ANR and \( A \) is a closed subset of \( X \), then the map \( C(X,Y) \ni f \mapsto f \big|_A \in C(A,Y) \) is open.

1.3. Lemma ([15]). If \( X \) is completely metrizable and of weight no greater than \( \alpha \geq \aleph_0 \) and \( M \) is an \( \alpha \)-manifold, then the set \( \text{Emb}^\alpha(X,M) \) is dense in \( C(X,Y) \).

1.4. Theorem ([11]). If \( M \) and \( N \) are two \( \alpha \)-manifolds (where \( \alpha \) is an infinite cardinal) and \( M \) has no more than \( \alpha \) components (i.e. \( w(M) = \alpha \)), then the set \( \text{Emb}^\alpha(M,N) \) is dense in \( C(M,N) \).

The next two lemmas are easy to prove.

1.5. Lemma. Let \( K \) be compact and \( p: Y \times K \to Y \) be the natural projection. Then the map \( C(X,Y \times K) \ni f \mapsto p \circ f \in C(X,Y) \) is open.

1.6. Lemma. If \( F \) is a closed subset of \( Y \), then the limitation topology of \( C(X,F) \) coincides with the one induced by the limitation topology of \( C(X,Y) \), when \( C(X,F) \) is considered as a subset of \( C(X,Y) \).

Recall that a subset \( P \) of \( Y \) is locally closed (in \( Y \)) if every point \( p \) of \( P \) has an open in \( Y \) neighbourhood \( U \) such that \( P \cap U \) is relatively closed in \( U \). Equivalently, \( P \) is locally closed iff \( \overline{P} \setminus P \) is closed in \( Y \).

1.7. Lemma. Let \( A \) be a closed subset of \( X \) and \( P \) a locally closed ANR-set in \( Y \), and let a map \( f: X \to Y \) be such that \( f(X \setminus A) \subseteq P \). Then, for every \( \mathcal{U} \in \text{cov}(P) \) there exists a neighbourhood \( G \) of \( f \big|_{X \setminus A} \) in \( C(X \setminus A, P) \) such that each map \( g: X \to Y \) satisfying \( g \big|_A = f \big|_A \) and \( g \big|_{X \setminus A} \in G \) is \( \mathcal{U} \)-homotopic to \( f \) in \( Y \) relative \( A \).
Proof. It is well known that ANR’s are the so-called locally equiconnected spaces ([8], [9], [12]), that is, there is an open subset Ω of $P \times P$ containing the diagonal and a map $\lambda: \Omega \times [0, 1] \to P$ such that $\lambda(x, y, 0) = x$, $\lambda(x, y, 1) = y$ and $\lambda(z, z, t) = x$ for every $(x, y) \in \Omega$, $z \in P$ and $t \in [0, 1]$. Let $d \in \text{Metr}(Y)$. If $P$ is closed in $Y$, put $\theta \equiv 1$ (as a member of $C(P, (0, \infty))$). Otherwise let $\theta: P \ni z \mapsto \frac{1}{2}\text{dist}_d(z, P \setminus P) \in (0, +\infty)$. Now every $z \in P$ has an open in $P$ neighbourhood $V_z$ such that

\[(1-1) \quad V_z \times V_z \subset \Omega \text{ and } \lambda(V_z \times V_z \times [0, 1]) \subset U \cap B_d(z, \theta(z)) \]

for some $U \in \mathcal{U}$. Put $\mathcal{V} = \{V_z: z \in P\} \in \text{cov}(P)$ and $G = B(f|_{X \setminus A}), \mathcal{V}) \subset C(X \setminus A, P)$ and assume that $g: X \to Y$ is as in the statement of the lemma. Define $F: X \times [0, 1] \to Y$ by $F(x, t) = \lambda(f(x), g(x), t)$ for $x \in X \setminus A$ and $F(x, t) = f(x)$ for $x \in A$. By (1-1), $F$ is well defined and we only need to check that it is continuous at points of $A$. Fix a sequence $(x_n, t_n) \in (X \setminus A) \times [0, 1]$ convergent to $(a, t) \in A \times [0, 1]$. If $f(a) \in P$, then $(f(x_n), g(x_n), t_n) \to (f(a), f(a), t)$ ($n \to \infty$) and the continuity of $\lambda$ gives $\lim_{n \to \infty} F(x_n, t_n) = f(a)$. Now suppose that $f(a) \notin P$. Since $f(x_n) \in P$, we see that $f(a) \in P \setminus P$. Take $z_n \in P$ for which $g(x_n), f(x_n) \in V_{z_n}$. Then, by (1-1), both $F(x_n, t_n)$ and $f(x_n)$ belong to $B_d(z_n, \theta(z_n))$, so $d(F(x_n, t_n), f(x_n)) \leq 2\theta(z_n)$. But $\theta(z_n) \leq \frac{1}{2}d(z_n, f(a)) \leq \frac{1}{2}d(z_n, f(x_n)) + \frac{1}{2}d(f(x_n), f(a)) \leq \frac{1}{2}\theta(z_n) + \frac{1}{2}d(f(x_n), f(a))$ and thus $\theta(z_n) \leq d(f(x_n), f(a))$. Finally we obtain $d(F(x_n, t_n), f(a)) \leq d(F(x_n, t_n), f(x_n)) + d(f(x_n), f(a)) \leq 3d(f(x_n), f(a)) \to 0$.

\[\square\]

2. Main lemma

The proof of the following result is based on the Proof of 3.1 given by Toruńczyk in [16].

2.1. Lemma. Let $P$ be a completely metrizable ANR-set in a metric space $(Y, g)$ and $A, A_1, A_2, A_3, \ldots$ be closed subsets of $X$ such that $A \cap A_n = \emptyset \neq A_n$ for any $n \in \mathbb{N}$. Suppose also that, $G_n$ is a dense $G_\delta$-set in $C(A_n, P)$ ($n \in \mathbb{N}$) and that a map $\lambda: X \setminus A \to (0, \infty)$ satisfies $\inf \lambda(A_n) > 0$ for all $n \in \mathbb{N}$. Then, given a mapping $f: X \to Y$ with $f(X \setminus A) \subset P$, and a neighbourhood $G$ of $f|_{X \setminus A}$ in $C(X \setminus A, P)$, there is a mapping $h: X \to Y$ satisfying the following conditions:

(A1) $h|_A = f|_A$, $h(X \setminus A) \subset P$,

(A2) $h|_{A_n} \in G_n$ for all $n \in \mathbb{N}$,

(A3) $h|_{X \setminus A} \in G$ and $g(h(x), f(x)) < \lambda(x)$ for all $x \in X \setminus A$.

Proof. Take maps $\delta: X \to I$ and $\theta: P \to (0, 1]$ such that

\[(2-1) \quad B_\theta(f|_{X \setminus A}, \theta) \subset G\]
and (2-2) \( \delta \leq \lambda, \quad \delta^{-1}(\{0\}) = A, \quad \inf \delta(A_n) > 0 \) for each \( n \in \mathbb{N} \).

Let \( d \) be the maximum metric on the space \( P \times (0, 1] \), that is,

\[
d((p, s), (q, t)) = \max(\rho(p, q), |s - t|).
\]

There is an open subset \( U \) of \( C(X \setminus A, P \times (0, 1]) \) such that

\[
\inf \lambda_{X \setminus A} = Y_x \quad \text{if} \quad \exists \lambda \in \mathbb{R}.
\]

Example. 2.3. \( \exists \in \mathbb{R} \)

Let \( m = \emptyset \), then for every neighbourhood \( U \) of \( \lambda \) and \( (2-2) \) (note that \( (2-1) \) and \( (2-3) \) and the remainder of (A3) follows from (2-3).)

2.2. Similar to that of Lemma 2.1 (but simpler), we omit it.

Fix \( n \geq 1 \). By Lemma 1.2, the set \( U \setminus A_n := \{ F \setminus A_n : F \in U \} \) is open in \( C(A_n, P \times (0, 1]) \) and the map

\[
\varphi_n : U \ni F \mapsto F \setminus A_n \in U \setminus A_n
\]

is open as well. Let \( m_n = \frac{1}{n} \inf \delta(A_n) > 0 \). By (2-3), \( F(A_n) \subset P \times [m_n, 1] \) for each \( F \in U \). This yields that \( U \setminus A_n \subset C(A_n, P \times [m_n, 1]) \).

By Lemma 1.6, \( U \setminus A_n \) is open in \( C(A_n, P \times [m_n, 1]) \) and \( \varphi_n \), as a map of \( U \) into \( C(A_n, P \times [m_n, 1]) \), is open as well. Now let \( p : P \times (0, 1) \to P \) be the natural projection and put

\[
\psi_n : U \ni v \mapsto p \circ v \in C(A_n, P).
\]

By Lemma 1.5, \( \psi_n \) is open. We conclude from this that the set \( D_n = \varphi_n^{-1}(\psi_n^{-1}(G_n)) \) is a dense \( G_\delta \) subset of \( U \).

Now Lemma 1.1 shows that the set \( D = \bigcap_{n=1}^\infty D_n \) is dense in \( U \). Take \( u \in D \subset C(X \setminus A, P \times (0, 1]) \) and define \( h : X \to Y \) as follows: \( h \setminus A = f \setminus A \) and \( h \setminus X \setminus A = p \circ u \). Thanks to (2-3) and (2-2), \( h \) is continuous. What is more, by construction, \( h \) satisfies (A1) and (A2). Finally, \( h \setminus X \setminus A \in G \) because of (2-1) and (2-3) and the remainder of (A3) follows from (2-3) and (2-2) (note that \( u \in U \)).

In the sequel we shall also need the next result. Since its proof is similar to that of Lemma 2.1 (but simpler), we omit it.

2.2. Lemma. Let \( A \) and \( P \) be subsets of \( X \) and \( Y \), respectively, such that \( A \) is closed in \( X \) and \( P \) is an ANR. Let \( S \) be a dense subset of \( C(X \setminus A, P) \). If \( f \in C(X, Y) \) is such that \( f(X \setminus A) \subset P \) and \( f(\partial A) \cap P = \emptyset \), then for every neighbourhood \( G \) of \( f \setminus X \setminus A \) in \( C(X \setminus A, P) \) there is a map \( h : X \to Y \) such that \( h \setminus A = f \setminus A \) and \( h \setminus X \setminus A \in G \cap S \).

We end the section with the following

2.3. Example. As this example shows, the assumption of Lemma 2.1 that \( \inf \lambda(A_n) > 0 \) for each \( n \) cannot be omitted. Let \( X = [0, +\infty); \ Y = P = I; \ A = \{0\}; \ A_1 = [1, +\infty); \ G_1 = \{ g \in C(A_1, P) : g(x) \neq 0 (x \to \infty) \}; \ f : X \to Y, \ f \equiv 0; \lambda : X \to I, \lambda(t) = 1 \) for \( t \leq 1 \) and
\[ \lambda(t) = \frac{1}{t} \text{ for } t \geq 1; \text{ and } U = \{P\}. \] Note that \( G_1 \) is open and dense in \( C(X, Y) \) and there is no map \( h : X \to Y \) such that \( |h(x) - f(x)| \leq \lambda(x) \) for each \( x \in X \) and \( h|_{A_x} \in G_1 \).

3. Extending maps by injective \( \sigma\)-\( Z \)-maps

We begin with

3.1. Definition. For a subset \( B \) of \( Y \), let \( S_Y(X, B) \) be the collection of all maps \( g : X \to Y \) such that \( \overline{\text{im}} \ g \subset B \) (the closure taken in \( Y \)). Note that if \( B \) is open or \( G_\delta \), so is \( S_Y(X, B) \).

Following Toruńczyk [13, 14, 15], we say that a closed subset \( A \) of \( X \) is a \( Z \)-set in \( X \), if the space \( C(Q, X \setminus A) \), where \( Q \) denotes the Hilbert cube, is dense in \( C(Q, X) \). (If \( X \) is an ANR, this definition is equivalent to the original one by Anderson [1].) Similarly, \( A \) is said to be a strong \( Z \)-set in \( X \) iff for every \( U \in \text{cov}(X) \) there is a map \( u : X \to X \) which is \( U \)-close to the identity map \( \text{id}_X \) on \( X \) and \( A \cap \overline{\text{im}} \ u = \emptyset \) (cf. e.g. [4], [3], [6, 7]). In other words, \( A = \tilde{A} \) is a \( Z \)-set in \( X \) iff \( \tilde{S}_X(Q, X \setminus A) \) is dense in \( C(Q, X) \), and \( A \) is a strong \( Z \)-set in \( X \) iff \( \tilde{S}_X(X, X \setminus A) \) is dense in \( C(X, X) \). Countable unions of [strong] \( Z \)-sets are called [strong] \( \sigma\)-\( Z \)-sets. If \( X \) is complete metrizable and \( B \) is its \( \sigma\)-\( Z \)-set, then the set \( \tilde{S}_X(Q, X \setminus B) \) is a dense \( G_\delta \)-set, and if \( B \) is a strong \( \sigma\)-\( Z \)-set, the same is true with \( Q \) replaced by \( X \) (this follows from Toruńczyk’s Lemma).

Not every \( Z \)-set in an ANR is a strong \( Z \)-set ([4, Key example, p. 56]). However, by a theorem due to Henderson [10], \( Z \)-sets in Hilbert manifolds are strong \( Z \)-sets (for other results in this matter see [3]). This fact will be used by us several times.

We say that a map \( f : X \to Y \) is a \( Z \)-map \( [\sigma\)-\( Z \)-map] iff \( \overline{\text{im}} \ f \) is a \( Z \)-set \( [\sigma\)-\( Z \)-set] in \( Y \). We similarly define \( Z \)-embeddings and \( \sigma\)-\( Z \)-embeddings (cf. [15]) Note that \( Z \)-maps are closed. It is well known ([15]) that if \( X \) is completely metrizable and of weight no greater than \( \alpha \geq \aleph_0 \) and \( M \) is an \( \alpha \)-manifold, then \( Z \)-embeddings of \( X \) into \( M \) are dense in \( C(X, M) \). In the following result we only need to know that \( Z \)-maps are dense.

3.2. Lemma. Every Hilbert manifold \( M \) contains a \( \sigma\)-\( Z \)-set \( F \) such that each closed subset of \( M \) disjoint from \( F \) is a \( Z \)-set in \( M \).

Proof. Take a sequence of \( Z \)-maps \( f_n : M \to M \) which converges uniformly to \( \text{id}_M \) with respect to a fixed metric of \( M \) and put \( F = \bigcup_{n=1}^{\infty} \overline{\text{im}} f_n \). \( \square \)

3.3. Corollary. Let \( M \) be an \( \alpha \)-manifold \( (\alpha \geq \aleph_0) \), \( K \) a \( \sigma\)-\( Z \)-set in \( M \) and let \( X \) be a completely metrizable space of weight no greater than \( \alpha \). Then there is a dense \( G_\delta \)-subset \( G \) of \( C(X, M) \) which consists of \( Z \)-embeddings whose images are disjoint from \( K \).
Proof. Take $F$ as in Lemma 3.2 and put

$$G = \text{Emb}^\gamma(X, M) \cap \mathcal{S}_M(X, M \setminus (F \cup K)).$$

By Henderson's theorem [10], $F \cup K$ is a strong $\sigma$-$Z$-set in $M$ and thus $\mathcal{S}_M(X, M \setminus (F \cup K))$ is a dense $\mathcal{G}_\delta$-set in $\mathcal{C}(X, M)$. Thanks to Lemma 1.3, also $\text{Emb}^\gamma(X, M)$ is a dense $\mathcal{G}_\delta$-set in $\mathcal{C}(X, M)$. Finally, Lemma 1.1 yields that $G$ is dense as well. The remainder of the assertion is clear.

Now we are able to state and prove the main result of this section.

3.4. Theorem. Let $A$ be a closed subset of $X$ such that $X \setminus A$ is completely metrizable and $w(X \setminus A) \leq \alpha$ (where $\alpha \geq \aleph_0$) and let $M \subset Y$ be an $\alpha$-manifold. Let $A_1, A_2, \ldots$ be nonempty closed subsets of $X$ disjoint from $A$ and let $B \subset M$ be a $\sigma$-$Z$-set in $M$. Let a map $\lambda: X \setminus A \to (0, +\infty)$ be such that $\inf \lambda(A_n) > 0$ for each $n \in \mathbb{N}$. If $f: X \to Y$ is such a map that $f(X \setminus A) \subset M$, then for every neighbourhood $G$ of $f|_{X \setminus A}$ in $\mathcal{C}(X \setminus A, M)$ and $g \in \text{Metr}(Y)$ there is a map $h: X \to Y$ satisfying the following conditions:

(Z1) $h|_A = f|_A$, $h(X \setminus A) \subset M \setminus B$,

(Z2) $h|_{A_n}$ is a $Z$-embedding into $M$ for all $n \in \mathbb{N}$,

(Z3) $h|_{X \setminus A} \in G$ and $g(h(x), f(x)) < \lambda(x)$ for all $x \in X \setminus A$.

Proof. Enlarging the sets $A_n$, we may and do assume that

$$X \setminus A = \bigcup_{n=1}^{\infty} A_n.$$

Since $w(X \setminus A) \leq \alpha$ and $X \setminus A$ is completely metrizable, by Corollary 3.3, for each $n \in \mathbb{N}$ there is a dense $\mathcal{G}_\delta$-subset $\mathcal{G}_n$ of $\mathcal{C}(A_n, M)$ consisting of $Z$-embeddings whose images are disjoint from $B$. Now putting $P = M$ and applying Lemma 2.1, we obtain a map $h: X \to Y$ such that the conditions (A1)–(A3) are fulfilled. This yields (Z3) and (Z2). Finally, (Z1) is also satisfied because of (A2) and (3-1).

3.5. Remark. As the above proof shows, the map $h$ appearing in the statement of Theorem 3.4 may be chosen in such a way that it additionally fulfills the following condition:

(Z4) $h|_{X \setminus A}$ is a one-to-one $\sigma$-$Z$-map as a map of $X \setminus A$ into $M$ (this is guaranteed by (3-1) after enlarging the sets $A_n$).

What is more, with use of Lemma 1.7, $h$ may be forced to satisfy also:

(H1) the maps $h|_{X \setminus A}$ and $f|_{X \setminus A}$ are $\mathcal{U}$-homotopic in $M$ and

(H2) if $M$ is locally closed in $Y$, then $h$ and $f$ are $\mathcal{U}$-homotopic in $Y$ where $\mathcal{U}$ is an arbitrarily given relatively open cover of $M$. Both the points (H1) and (H2) may be added also in the statements of Proposition 4.2 and Theorem 4.5. This observation shall be used in the sequel (see Corollaries 3.6, 4.3, 4.4 and 4.6).
As a consequence of Theorem 3.4 we get a generalization of West’s theorem [17] and a strengthened version of Lemma 1.3:

3.6. Corollary. Let $X$ be completely metrizable with $w(X) \leq \alpha$, $M$ an $\alpha$-manifold and $B$ its $\sigma$-$Z$-set.

(a) For every homotopy $F: X \times I \to M$ and each $U \in \text{cov}(M)$ there is a homotopy $H: X \times I \to M$ $U$-close to $F$ such that $F(\cdot, t) = H(\cdot, t)$ for $t = 0, 1$ and $H|_{X \times [\varepsilon, 1-\varepsilon]}$ is a $Z$-embedding with image disjoint from $B$ for any $\varepsilon \in (0, \frac{1}{2})$.

(b) For every map $f: X \to M$ and each $U \in \text{cov}(M)$ there is a $U$-homotopy $F: X \times I \to M$ such that $F(\cdot, 0) = f$ and for any $\varepsilon \in (0, 1)$, $F|_{X \times [\varepsilon, 1]}$ is a $Z$-embedding with image disjoint from $B$. In particular, $f_t = F(\cdot, t)$ (with $t \in (0, 1]$) is a $Z$-embedding of $X$ into $M$.

4. Extending maps by embeddings

In this section we give sufficient conditions under which an arbitrary map $f: X \to M$ is approximable by maps $g: X \to M$ such that $g|_A = f|_A$ and $g|_{X \setminus A}$ is an embedding.

The proof of the following is left as a simple exercise.

4.1. Lemma. If $h: X \to Y$ is such a map that $h|_U$ is an embedding, where $U$ is open in $X$, then $\overline{h(\partial U)} \cap h(U) = \emptyset$.

The property stated in the above lemma forces us to make some restrictions on a map which we want to extend by an embedding.

For need of the next result, note that a subset $F$ of a Hilbert manifold $M$ is a locally closed $\sigma$-$Z$-set in $M$ iff $F$ is a $Z$-set in some open in $M$ neighbourhood of $F$.

4.2. Proposition. Let $A$ be a closed subset of $X$ such that $X \setminus A$ is completely metrizable and of weight no greater than $\alpha$ ($\alpha \geq s_0$). Let $M \subset Y$ be an $\alpha$-manifold and $B$ its $\sigma$-$Z$-set. If $f: X \to Y$ is such a map that there is a closed subset $K$ of $Y$ such that $f(X \setminus A) \subset M \setminus K$ and $M \cap (\overline{f(\partial A)} \setminus K)$ is a $\sigma$-$Z$-set in $M$, then for each neighbourhood $G$ of $f|_{X \setminus A}$ in $C(X \setminus A, M)$ there is a map $h: X \to Y$ satisfying the following conditions:

(E1) $h|_A = f|_A$, $h(X \setminus A) \subset M \setminus (B \cup K)$ and $h|_{X \setminus A} \in G$,

(E2) $h|_{X \setminus A}$ is a $\sigma$-$Z$-embedding of $X \setminus A$ into $M$ whose image is locally closed in $M$,

(E3) if $f(\partial A) \subset M \setminus K$ and $f|_{\partial A}$ is an embedding, then so is $h|_{X \setminus A}$.

Proof. Put $M' = M \setminus K$ and

\begin{equation}
B' = \overline{f(\partial A)} \cap M'.
\end{equation}

\[\overline{B} = \overline{f(\partial A)} \cap M' \subset M \setminus K \setminus \{f(\partial A)\}
\]
Then $M'$ is an $\alpha$-manifold, $B'$ its closed $\sigma$-$Z$-set and thus it is a $Z$-set in $M'$, and $f(X \setminus A) \subset M'$. Take $\mathcal{U} \in \cov(M)$ with $B(f|_{X \setminus A}, \mathcal{U}) \subset G$ and let $\mathcal{V} \in \cov(M')$ be a star refinement of the cover $\{U \cap M': U \in \mathcal{U}\}$ of $M'$. Now by Theorem 3.4, there is a map $h': X \to Y$ such that

\begin{equation}
(4-2) \quad h'|_A = f|_A, \quad h'(X \setminus A) \subset M' \setminus B'
\end{equation}

and $h'|_{X \setminus A}$ and $f|_{X \setminus A}$ are $\mathcal{V}'$-close. Fix metrics $d$ and $\varrho$ on $X$ and $Y$, respectively, such that for all $x, y \in X$:

\begin{equation}
(4-3) \quad \varrho(h'(x), h'(y)) \leq d(x, y).
\end{equation}

Further, we define $\lambda \in C(X, [0, +\infty))$ and closed subsets $A_1, A_2, \ldots$ of $X$ as follows. If $\partial A = \emptyset$, put $\lambda = 1$ and $A_n = X \setminus A$ ($n \geq 1$). Otherwise, let $\lambda(x) = \dist_d(x, \partial A)$ and $A_n = \lambda^{-1}(1/n, +\infty)) \setminus A$. Note that

\begin{equation}
(4-4) \quad A_k \subset \text{int} A_{k+1} \ (k \geq 1), \quad \bigcup_{n=1}^\infty A_n = X \setminus A.
\end{equation}

Again by Theorem 3.4, there exists a map $h: X \to Y$ for which $h|_A = h'|_A$, $h(X \setminus A) \subset M' \setminus (B' \cup B)$, $h|_{X \setminus A}$ is a $Z$-embedding into $M'$ for all $n \in \mathbb{N}$, $\varrho(h(x), h'(x)) < \lambda(x)$ for any $x \in X \setminus A$ and $h|_{X \setminus A}$ is $\mathcal{V}'$-close to $h'|_{X \setminus A}$. We easily get (E1). We infer from the connections $h(X \setminus A) \subset M' \setminus B'$ and $h|_A = h'|_A = f|_A$ that

\begin{equation}
(4-5) \quad h(X \setminus A) \cap \overline{h(\partial A)} = \emptyset.
\end{equation}

Note that if $\partial A = \emptyset$, all the conditions (E1)–(E3) are basically fulfilled. Therefore from now on, we assume that the boundary of $A$ is nonempty.

Let $x_n, x \in X \setminus A$ be such that $h(x_n) \to h(x)$ ($n \to \infty$). We claim that there is $j \geq 1$ such that $x_n \in A_j$ for almost all $n$. Suppose, for the contrary, that there is a subsequence $(y_n)_n$ of $(x_n)_n$ such that $y_n \notin A_n$. This says that $\lambda(y_n) < 1/n$ and thus there is $a_n \in \partial A$ for which

\begin{equation}
(4-6) \quad d(y_n, a_n) \to 0 \ (n \to \infty).
\end{equation}

But then $\varrho(h(y_n), h'(y_n)) \leq \lambda(y_n) \to 0$, so $h'(y_n) \to h(x)$. What is more, $\varrho(h'(y_n), h'(a_n)) \leq d(y_n, a_n) \to 0$ (by (4-3)). This yields $h(a_n) = h'(a_n) \to h(x)$ and therefore $h(x) \in \overline{h(\partial A)}$, which denies (4-5) (since $x \in X \setminus A$). So, there is $j \geq 1$ such that $x_n \in A_j$ for almost all $n$. But then $x_n \to x$, because of the facts that $h|_{A_j}$ is a closed embedding of $A_j$ into $M'$, $h(x) \in M'$ and $h|_{X \setminus A}$ is one-to-one (thanks to (4-4)). We have shown that $h|_{X \setminus A}$ is an embedding. Similarly, under the assumptions of (E3), $h|_{X \setminus A}$ is an embedding. Indeed, let $y_n \in X \setminus A$, $a \in \partial A$ and $h(y_n) \to h(a)$. By (E1), (4-5), the assumptions of (E3) and thanks to the closedness of $h(A_j)$ in $M'$, $y_n \notin A_j$ for almost all $n$. This implies that $\lambda(y_n) \to 0$ and there is a sequence $(a_n)_n$ of elements of $\partial A$ for
which (4-6) is fulfilled. As in the previous part of the proof, we show that \( h'(y_n) \to h(a) \) and thus \( h'(a_n) \to h(a) = h'(a) \). This, combined with (4-2) and the assumptions of (E3), gives \( a_n \to a \) and therefore \( y_n \to a \), by (4-6).

It suffices to prove that \( h(X \setminus A) \) is locally closed in \( M \). Since \( F := h(X \setminus A) \subset M' \) and \( M' \) is open in \( M \), it is enough to show that \( F \) is locally closed in \( M' \). If \( y \in F \), then there is \( n \geq 1 \) and \( x \in \text{int} \, A_n \) for which \( h(x) = y \). Since \( h|_{X \setminus A} \) is an embedding, there is an open in \( M' \) set \( V \subset M' \) such that \( h(\text{int} \, A_n) = V \cap F \). But then \( y \in V \cap F = V \cap h(A_n) \) and the latter set is closed in \( V \).

Under the notation of the statement of Proposition 4.2, the most interesting case of this result appears when \( K = \emptyset \) or \( K = f(\partial A) \):

4.3. **Corollary** (cf. [2, Theorem 3.1]). Let \( X \) be completely metrizable of weight no greater than \( \alpha \) and \( A \) its closed subset. Let \( M \) be an \( \alpha \)-manifold and \( B \) its \( \sigma \)-Z-set. Let \( f \in C(X,M) \) and \( U \in \text{cov}(M) \). If \( f(X \setminus A) \cap \overline{f(\partial A)} = \emptyset \) or \( f(\partial A) \) is a Z-set in \( M \), then there is a map \( h: X \to M \) \( U \)-homotopic to \( f \) such that \( h|_A = f|_A \), \( h|_{X \setminus A} \) is a \( \sigma \)-Z-embedding whose image is locally closed in \( M \) and disjoint from \( B \) and \( h(X \setminus A) = h(X \setminus A) \cup h(\partial A) \). What is more, if \( f|_{\partial A} \) is an embedding and \( f(\partial A) \) is a Z-set in \( M \), then \( h \) is an embedding [a Z-embedding].

Observe that if the image of a map \( f: X \to M \) is contained in a locally closed \( \sigma \)-Z-set \( L \) of \( M \), then \( K = L \setminus L \) is closed, \( f(X) \cap K = \emptyset \) and \( \overline{f(X)} \setminus K \) is a \( \sigma \)-Z-set in \( M \). This notice, Proposition 4.2 and Corollary 4.3 give (cf. [2, Lemma 2.4]):

4.4. **Corollary.** Let \( X \) be completely metrizable, \( w(X) \leq \alpha \), \( M \) an \( \alpha \)-manifold and \( B \) its \( \sigma \)-Z-set. Let \( U \in \text{cov}(M) \).

(A) \( F: X \times I \to M \) is a homotopy such that the closure of \( F(X \times \{0,1\}) \) is a Z-set in \( M \), then there is a homotopy \( H: X \times I \to M \)

\( U \)-close to \( F \) such that \( H(\cdot,t) = F(\cdot,t) \) for \( t = 0,1 \) and \( H|_{X \times \{0,1\}} \) is a \( \sigma \)-Z-embedding whose image is disjoint from \( B \). If, in addition, \( F|_{X \times \{0,1\}} \) is an embedding, so is \( H \).

(B) \( f: X \to M \) is a map whose image is contained in a locally closed \( \sigma \)-Z-set in \( M \), then there is a \( U \)-homotopy \( F: X \times I \to M \) such that \( F(\cdot,0) = f \) and \( F|_{X \times \{0,1\}} \) is a \( \sigma \)-Z-embedding whose image is locally closed and disjoint from \( B \). If, in addition, \( f \) is an embedding, so is \( F \).

Our last goal is to give a sufficient condition under which a map can be extended by an open embedding.

4.5. **Theorem.** Let \( A \) be a closed subset of \( X \) such that \( X \setminus A \) is an \( \alpha \)-manifold which has no more than \( \alpha \) components (i.e. \( w(X \setminus A) = \alpha \)).
Let $M$ be a subset of $Y$ such that $M$ is an $\alpha$-manifold and let $B$ be a $Z$-set in $M$. Let $f : X \to Y$ be a map such that $f(X \setminus A) \subset M$. If $f$ is approximable by maps $g : X \to Y$ such that $g|_A = f|_A$, $g(X \setminus A) \subset M$ and $g(X \setminus A) \cap g(\partial A) = \emptyset$, then for each neighbourhood $G$ of $f|_{X \setminus A}$ in $C(X \setminus A, M)$ there is a map $h : X \to Y$ satisfying the following conditions:

(O1) $h|_A = f|_A$ and $h(X \setminus A) \subset M \setminus B$,

(O2) $h|_{X \setminus A}$ is an open embedding,

(O3) $h|_{X \setminus A} \in G$.

Proof. Thanks to the assumption, we may and do assume that $f(X \setminus A) \cap f(\partial A) = \emptyset$. Put $M' = M \setminus f(\partial A)$ and $B' = B \setminus M'$. Then $M'$ is an $\alpha$-manifold, $B'$ is a $Z$-set in $M'$ and $f(X \setminus A) \subset M'$. Let $S = \text{Emb}^\alpha(X \setminus A, M') \cap \mathcal{S}_{M'}(X \setminus A, M' \setminus B')$. Since $\mathcal{S}_{M'}(X \setminus A, M' \setminus B')$ is open and dense in $C(X \setminus A, M')$, thus $S$ is dense as well (by Theorem 1.4). Now the assertion follows from Lemma 2.2.

4.6. Corollary. Let $M$ and $N$ be two $\alpha$-manifolds such that $w(M) = \alpha$.

(A) If $F : M \times I \to N$ is a homotopy such that the closure of $F(X \times \{0,1\})$ is a $Z$-set in $N$, then for every $U \in \text{cov}(N)$ there is a $U$-close to $F$ homotopy $H : M \times I \to N$ such that $H(\cdot, t) = F(\cdot, t)$ for $t = 0, 1$ and $H|_{M \times (0,1)}$ is an open embedding.

(B) If $f : M \to N$ is such a map that $\overline{\text{im}} f$ is a $Z$-set in $N$, then for every $U \in \text{cov}(N)$ there is a $U$-homotopy $F : M \times I \to N$ such that $F(\cdot, 0) = f$ and $F|_{M \times (0,1)}$ is an open embedding.

We end the paper with

4.7. Example. Let $f_t = \text{id}_M : M \to M$ ($t \in I$), where $M$ is a manifold. By Lemma 4.1, there is no homotopy $(h_t)_{t \in I}$ such that $h_0 = h_1 = \text{id}_M$ and $h|_{M \times (0,1)}$ is an embedding. This shows that the paper of West [17] contains an oversight. However, it was applied by Anderson and McCharen [2] once—in [2, Lemma 2.4]. Fortunately, the assertion of that lemma is true, which follows from our Corollary 4.4.

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