On a conjecture of Bennewitz, and the behaviour of the Titchmarsh-Weyl matrix near a pole.

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Abstract

For any real limit-\( n \) \( 2n \)th-order selfadjoint linear differential expression on \([0, \infty)\), Titchmarsh-Weyl matrices \( M(\lambda) \) can be defined. Two matrices of particular interest are the matrices \( M_D(\lambda) \) and \( M_N(\lambda) \) associated respectively with Dirichlet and Neumann boundary conditions at \( x = 0 \). These satisfy \( M_D(\lambda) = -M_N(\lambda)^{-1} \). It is known that when these matrices have poles (which can only lie on the real axis) the existence of valid HELP inequalities depends on their behaviour in the neighbourhood of these poles. We prove a conjecture of Bennewitz and use it, together with a new algorithm for computing the Laurent expansion of a Titchmarsh-Weyl matrix in the neighbourhood of a pole, to investigate the existence of HELP inequalities for a number of differential equations which have so far proved awkward to analyse.

1 Introduction

In a recent paper \cite{1} a numerical algorithm was reported for the computation of the Titchmarsh-Weyl \( M \) matrices associated with the fourth order differential equation

\[
\mathcal{M}[y] = ((py'')' - (sy'))' + qy = \lambda y.
\]

The motivation for that work was an investigation of the HELP type integral inequality

\[
\left( \int_0^\infty (p |y''|^2 + s |y'|^2 + q |y|^2) dx \right)^2 \leq K \int_0^\infty |y|^2 dx \int_0^\infty |\mathcal{M}[y]|^2 dx.
\]

In this paper we turn our attention to higher order operators and inequalities. We consider operators of the form

\[
\mathcal{M}[y] := \sum_{j=0}^n (-1)^j \frac{d^j}{dx^j} \left( p_j(x) \frac{d^j y}{dx^j} \right);
\]

as we do not wish to be concerned with quasidifferential expressions, we assume that the coefficients \( p_j \) are all smooth. We assume that \( p_n > 0 \) on \((0, \infty)\), and we assume that \( \mathcal{M} \) is regular at \( x = 0 \). \( \mathcal{M} \) will possess selfadjoint realisations in \( L^2[0, \infty) \): we assume that \( x = \infty \) is of limit-point (minimal
deficiency index) type, so that these realisations depend only upon choice of boundary conditions at 
\( x = 0 \). The corresponding HELP inequality is

\[
\left( \int_0^\infty \left\{ \sum_{j=0}^n p_j |y^{(j)}|^2 \right\} \, dx \right)^2 \leq K \left( \int_0^\infty |y|^2 \, dx \right) \left( \int_0^\infty |\mathcal{M}[y]|^2 \, dx \right). \tag{4}
\]

Before we explain the connection between Titchmarsh-Weyl \( M \)-matrices and inequalities (2) and (4), we give a very brief overview of \( M \)-matrices, starting with the scalar case (1 \( \times \) 1 matrices).

Consider a second order Sturm-Liouville equation, say

\[
-(py')' + qy = \lambda wy, \tag{5}
\]

on an interval \([0, \infty)\), with \( x = 0 \) a regular point and \( x = \infty \) a singular point of limit-point type. Suppose that \( y_D \) denotes the solution of (5) subject to the Dirichlet conditions

\[
y_D(0) = 0, \quad py'_D(0) = 1,
\]

while \( y_N \) denotes the solution subject to the Neumann conditions

\[
y_N(0) = -1, \quad py'_N(0) = 0.
\]

Then the Dirichlet \( m \)-function \( m_D(\lambda) \) is, for \( \Im(\lambda) \neq 0 \), the unique function such that

\[
y_N(\cdot) + m_D(\lambda)y_D(\cdot)
\]

is a solution of (5) square integrable over \([0, \infty)\) with respect to \( w \). The Neumann \( m \)-function \( m_N(\lambda) \) is defined by the property that

\[
y_D(\cdot) - m_N(\lambda)y_D(\cdot)
\]

is a square integrable over \([0, \infty)\). The functions \( m_D \) and \( m_N \) are analytic functions of \( \lambda \) on both the upper and lower half planes; moreover \( m_D(\lambda)m_N(\lambda) = -1 \) wherever both \( m_D \) and \( m_N \) are defined.

These ideas were generalized by Hinton and Shaw \cite{9} to certain Hamiltonian systems (which include (reformulations of) higher order Sturm-Liouville systems such as (3) – see Section 3.2 below for details of the fourth order case). Consider a system

\[
\begin{pmatrix}
-v' \\
u'
\end{pmatrix} = S(x; \lambda) \begin{pmatrix} u \\ v \end{pmatrix} \tag{6}
\]

in which \( S \) is a \( 2n \times 2n \) symmetric matrix given in terms of real matrices \( A \) and \( B \) by

\[
S(x; \lambda) = A(x) + \lambda B(x),
\]

where \( B \) is positive semi-definite and, in a certain sense, positive definite on solutions of the differential equation. The dependent variables \( v \) and \( u \) are \( n \)-vector functions of \( x \) and \( \lambda \). Let

\[
\begin{pmatrix}
U_D \\
V_D
\end{pmatrix}
\]

be the \( 2n \times n \) matrix solution of this ODE, partitioned into \( n \times n \) blocks, subject to the initial conditions

\[
U_D(0) = 0, \quad V_D(0) = I,
\]

and let

\[
\begin{pmatrix}
U_N \\
V_N
\end{pmatrix}
\]
be the $2n \times n$ matrix solution subject to initial conditions

$$U_N(0) = -I, \quad V_N(0) = 0.$$  

Then the Dirichlet and Neumann $M$-matrices are defined respectively by the requirements that

$$U_1(\cdot) := U_N(\cdot) + M_D(\lambda)U_D(\cdot), \quad U_2(\cdot) := U_D(\cdot) - M_N(\lambda)U_N(\cdot)$$

be square integrable with respect to $B$ over $[0, \infty)$, for $\Im(\lambda) \neq 0$:

$$\int_0^\infty U_1^*(x)B(x)U_1(x)dx < +\infty, \quad \int_0^\infty U_2^*(x)B(x)U_2(x)dx < +\infty.$$  

The limit-point hypothesis at infinity ensures that $M_N$ and $M_D$ are uniquely determined by these conditions.

We shall also make extensive use of the identity

$$M_D(\lambda)M_N(\lambda) = -I$$

which holds wherever $M_D$ and $M_N$ are defined. For further details see Hinton and Shaw.

Returning to the HELP type inequality (2), we remark that this has been investigated by Russell, while the more general form (4) has been studied by Dias.

In all these investigations the existence of a valid inequality, that is a finite number $K$ in (2) or (4), was shown to depend upon the behaviour of the Titchmarsh-Weyl matrix $M_N$ associated with (1) or (3). The existence of the inequality and value of the best constant is determined by the behaviour of the function

$$\Im(\lambda^2M_N(\lambda)) \quad \text{(the imaginary part of } \lambda^2M_N(\lambda)\text{)}$$

for strictly complex values of the spectral parameter $\lambda$ that lie in the first and third quadrants of the complex plane. Indeed the existence, but not necessarily the value, of the best constant is determined by (8) for values such that $|\lambda| \to 0$. As it is difficult to find examples of $M$ matrices for (1) or (3) which are known in closed form, it is of some importance to be able to investigate this problem numerically.

It is further known that when $0$ lies both in the resolvent set of the realisation of $M$ generated by Neumann boundary conditions ($v(0) = 0$ in the Hamiltonian formulation, or $-(py''')'(0) + sy'(0) = 0$, $py''(0) = 0$ in the fourth order case) and also in the resolvent set of the realisation generated by Dirichlet boundary conditions ($u(0) = 0$, or $y(0) = 0$, $y'(0) = 0$ in the fourth order case) then the inequality (1) fails; a necessary (though not generally sufficient) condition for an inequality is that $0$ be a point of the spectrum of at least one of these two operators. However, when the Titchmarsh-Weyl matrices are meromorphic, a little more can be said on the validity of the inequality. In this case Dias has shown that the poles of $M_N$ occur at the eigenvalues of the realisation of $M$ subject to Neumann conditions $v(0) = 0$ and the poles of $M_D$ occur at the eigenvalues of $M$ subject to Dirichlet conditions $u(0) = 0$.

If $\mu$ is a pole of an $M$-matrix then $\mu$ is simple and the $M$-matrix has an expansion

$$M(\lambda) = \frac{\sigma_{-1}}{\lambda - \mu} + \sigma_0 + \ldots$$

where $\sigma_{-1}$ is real and is called the residue matrix of $M$ at $\mu$. We shall need the concepts of Neumann and Dirichlet translates. In (8) we say that $\mu$ is a Neumann translate if it is an eigenvalue of $M$ subject to Neumann conditions $v(0) = 0$. In a similar way we say that $\mu$ is a Dirichlet translate if it is an eigenvalue of (10), but this time with initial conditions $u(0) = 0$. 

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The residue matrix of $M_N$ associated with a Neumann translate we denote by $\sigma_N$, and the residue matrix of $M_D$ subject to a Dirichlet translate we denote by $\sigma_D$.

Suppose that $\mu_N, \mu_D$ are Neumann and Dirichlet translates respectively. It is shown in [5] that a valid inequality will be found if the differential expression (3) is replaced by either

$$M_N = M - \mu_N$$

or

$$M_D = M - \mu_D$$

provided either of the associated residue matrices $\sigma_N, \sigma_D$ is of full rank. The result for higher order HELP type inequalities is somewhat weaker than that for the second order problem since it is shown in [6] for the second order classical HELP inequality that when the $m$ function is meromorphic then a valid inequality exists if and only if 0 is an eigenvalue of either the Neumann or the Dirichlet problem associated with that expression. In an attempt to strengthen this result for higher order operators Bennewitz (private communication, 1995) has proposed the conjecture that provided

$$\text{rank}(\sigma_N) + \text{rank}(\sigma_D) = n$$

(half the order of the differential expression) then a valid inequality will be found. In this paper we shall prove this conjecture for the general even order HELP inequality.

As we remarked above the existence of a valid inequality is determined by (8) as $|\lambda| \to 0$ and when $M_N$ is meromorphic it must also have a pole at 0. Thus in order to investigate numerically the existence of a valid inequality we must compute the associated residue matrices $\sigma_N, \sigma_D$. In [8] it was noted that this was a difficult numerical problem and in section [9] we report on some new algorithms to solve it.

In section [10] we apply our work to some equations to determine whether or not they are likely to possess associated HELP inequalities.

2 The Bennewitz Conjecture

In order to simplify the algebra we shall assume that at least one of the matrices $M_D(\lambda), M_N(\lambda)$ has a pole at the origin $\lambda = 0$. The Bennewitz Conjecture is as follows.

Conjecture 2.1 (Bennewitz) Suppose that $M_N$ (and hence $M_D$) is meromorphic and that

$$\text{rank}(\text{Res}(M_D, 0)) + \text{rank}(\text{Res}(M_N, 0)) = n.$$

Then there is a valid HELP inequality associated with the differential expression.

2.1 Proof of the Bennewitz Conjecture

In order to prove this result we shall require the following lemmas.

Lemma 2.2 The Titchmarsh-Weyl matrices $M_D$ and $M_N$ are symmetric matrices and are also Nevanlinna functions, in the sense that the matrices $\Im(M_D(\lambda))$ and $\Im(M_N(\lambda))$ are positive definite for $\Re(\lambda) > 0$.

For a proof of this result see Hinton and Shaw [1].

Lemma 2.3 A necessary and sufficient condition for the existence of a HELP inequality is that there exist numbers $\theta_+ \in (0, \pi/2), \theta_- \in (\pi, 3\pi/2), \rho_+ > 0$ and $\rho_- > 0$ such that

$$\Im(-\lambda^2 M_N(\lambda)) > 0 \quad \forall |\lambda| \in (0, \rho_+), \text{ arg}(\lambda) \in [\theta_+, \pi/2)$$

and

$$\Im(\lambda^2 M_N(\lambda)) > 0 \quad \forall |\lambda| \in (0, \rho_-), \text{ arg}(\lambda) \in [\theta_-, 3\pi/2).$$
This result is proved in Dias’ thesis [3], and in a different form for the case $n = 2$ in Russell [11].

Notes

1. In Lemma 2.3 and throughout the rest of this paper, we follow the usual convention for matrices that relations of the form ‘$>$ 0’ and ‘$<$ 0’ indicate positive definiteness and negative definiteness respectively.

2. We could state this lemma in an equivalent form in which there would be just one number $\rho > 0$ equivalent to $\min(\rho_+, \rho_-)$. However for the proof that follows this form is marginally more convenient.

Lemma 2.4 Suppose that $M_D$ or $M_N$ has a simple pole at $\lambda = 0$ with a Laurent expansion

$$\frac{1}{\lambda} M_{-1} + M_0 + \lambda M_1 + \lambda^2 M_2 + \cdots .$$

Then all the coefficients occurring in this expansion are real symmetric matrices.

Proof The symmetry of the coefficients follows from the symmetry of $M_N$ and $M_D$, see Hinton and Shaw [9]. For the rest of the proof we concentrate on $M_N$: the proof for $M_D$ is similar.

From (7) with $x = 0$ it is clear that

$$M_N(\lambda) = U_2(0; \lambda),$$

where \( \begin{pmatrix} U_2 \\ V_2 \end{pmatrix} \) is a ‘square integrable’ solution of the Hamiltonian system (more precisely, is a solution for which $U_2$ is square integrable in the sense of Hinton and Shaw). Also, it can be shown from (5) that

$$V_2(x; \lambda) = V_D(x; \lambda) - M_N(\lambda)V_N(x; \lambda),$$

whence setting $x = 0$ gives

$$V_2(0; \lambda) = I.$$ 

Thus

$$M_N(\lambda) = U_2V_2^{-1}(0; \lambda).$$

(12)

Now it is easy to see that if \( \begin{pmatrix} U_2(x; \mu) \\ V_2(x; \mu) \end{pmatrix} \) is a square integrable solution for $\lambda = \mu$ then \( \begin{pmatrix} U_2(x; \mu) \\ V_2(x; \mu) \end{pmatrix} \) is a square integrable solution for $\lambda = \mu$. Since we are concerned with problems of limit-point type, the square integrable solution for any $\Im(\lambda) \neq 0$ is unique up to postmultiplication by an invertible constant matrix. Any such matrix cancels out upon taking the combination $U_2V_2^{-1}$, and hence

$$\overline{U_2V_2^{-1}(0; \mu)} = U_2V_2^{-1}(0; \overline{\mu}).$$

Thus from (12),

$$M_N(\overline{\lambda}) = \overline{M_N(\lambda)}.$$ 

This implies that the coefficients in the Laurent expansion of $M_N$ about $\lambda = 0$ are real matrices. □

Proof of the Bennewitz Conjecture Under the hypotheses of the conjecture, we can expand $M_D$ and $M_N$ in Laurent series about the point $\lambda = 0$,

$$M_D(\lambda) = \lambda^{-1} M_{-1} + M_0 + \lambda M_1 + O(\lambda^2),$$

$$M_N(\lambda) = \lambda^{-1} \tilde{M}_{-1} + \tilde{M}_0 + \lambda \tilde{M}_1 + O(\lambda^2),$$

(13)

(14)
the expansions being valid in a neighbourhood of \( \lambda = 0 \). By Lemma 2.4, the coefficients in these expansions are real symmetric matrices. We see from the hypothesis of the conjecture that

\[
\text{rank}(M_{-1}) + \text{rank}(\hat{M}_{-1}) = n,
\]

so let \( \text{rank}(M_{-1}) = r, \text{rank}(\hat{M}_{-1}) = n - r \). We also know that \( M_D M_N = M_N M_D = -I \) for all \( \Im(\lambda) > 0 \) (see Hinton and Shaw [6]). Multiplying the Laurent expansions, we obtain the following conditions.

\[
\begin{align*}
M_{-1} \hat{M}_{-1} &= \hat{M}_{-1} M_{-1} = 0, \\
M_{-1} \hat{M}_0 + M_0 \hat{M}_{-1} &= \hat{M}_{-1} M_0 + \hat{M}_0 M_{-1} = 0, \\
M_0 \hat{M}_0 + M_{-1} \hat{M}_1 + M_1 \hat{M}_{-1} &= \hat{M}_0 M_0 + \hat{M}_{-1} M_1 + \hat{M}_1 M_{-1} = -I.
\end{align*}
\]

Thus the columns of \( M_{-1} \) are orthogonal to the columns of \( \hat{M}_{-1} \). We can therefore choose an orthonormal basis of \( \mathbb{R}^n \), say \( \{v_1, \ldots, v_r\} \), such that

\[
\text{Span}\{v_1, \ldots, v_r\} = \text{Span of the columns of } M_{-1},
\]

\[
\text{Span}\{v_{r+1}, \ldots, v_n\} = \text{Span of the columns of } \hat{M}_{-1}.
\]

Let \( V \) be the \( n \times r \) matrix with columns \( v_1, \ldots, v_r \), and let \( \bar{V} \) be the \( n \times (n - r) \) matrix with columns \( v_{r+1}, \ldots, v_n \). We make the following observations.

- From (15),

\[
M_{-1} \bar{V} = 0; \quad \bar{V}^T M_{-1} = 0; \quad \hat{M}_{-1} V = 0; \quad V^T \hat{M}_{-1} = 0.
\]

- Any vector \( v \in \mathbb{R}^n \) can be written in the form

\[
v = V \alpha + \bar{V} \hat{\alpha},
\]

where \( \alpha \in \mathbb{R}^r \) and \( \hat{\alpha} \in \mathbb{R}^{n-r} \).

Using (19) we have

\[
v^T M_N v = \alpha^T V^T M_N V \alpha + \hat{\alpha}^T \bar{V}^T M_N \bar{V} \hat{\alpha} + 2 \alpha^T \bar{V}^T M_N \bar{V} \hat{\alpha},
\]

where we have used the symmetry of \( M_N \) to simplify the last term. We now use the Laurent expansion (14) together with the conditions (18) to simplify this expression. We observe that

\[
V^T M_N V = V^T \left( \lambda^{-1} \hat{M}_{-1} + \hat{M}_0 + \lambda \hat{M}_1 + O(\lambda^2) \right) V
\]

\[
= V^T \hat{M}_0 V + \lambda V^T \hat{M}_1 V + O(\lambda^2).
\]

Now combining (17) and (18) we obtain \( M_{-1} \hat{M}_0 M_{-1} = 0 \), which implies that \( V^T \hat{M}_0 V = 0 \). Thus

\[
V^T M_N V = \lambda V^T \hat{M}_1 V + O(\lambda^2).
\]

At this stage it is not clear that the matrix \( V^T \hat{M}_1 V \) is of full rank, so the \( O(\lambda^2) \) terms might not be negligible. We shall show shortly that in fact \( V^T \hat{M}_1 V \) is of full rank. The next term we need to examine is \( \bar{V}^T M_N \bar{V} \). This is much easier to deal with; the Laurent expansion (14) immediately gives

\[
\bar{V}^T M_N \bar{V} = \frac{1}{\lambda} \bar{V}^T \hat{M}_{-1} \bar{V} + O(1),
\]
and the leading order term here is of full rank since the span of the columns of \( \hat{V} \) is the same as the span of the columns of \( \hat{M}_{-1} \), and \( \hat{M}_{-1} \) is symmetric. Finally, we treat the term \( V^T M_N \hat{V} \). Using (14) and (18) we obtain
\[
V^T M_N \hat{V} = V^T \hat{M}_0 \hat{V} + \lambda V^T \hat{M}_1 \hat{V} + O(\lambda^2).
\] (23)

Once more, since we know little about the ranks of the coefficients in this expansion, it is not clear that the \( O(\lambda^2) \) terms are negligible, a point which will have to be borne in mind later on.

Substituting (21), (22) and (23) back into (20) we obtain
\[
v^T M_N v = \alpha^T \left( \lambda V^T \hat{M}_1 V + O(\lambda^2) \right) \alpha + \hat{\alpha}^T \left( \frac{1}{\lambda} V^T \hat{M}_{-1} \hat{V} + O(1) \right) \hat{\alpha} + 2\alpha^T \left( V^T \hat{M}_0 \hat{V} + \lambda V^T \hat{M}_1 \hat{V} + O(\lambda^2) \right) \hat{\alpha}.
\] (24)

\( \theta \)From the Nevanlinna property of \( M_N \) we know that we must have \( v^T \Im(M_N) v > 0 \) for all non-zero \( v \in \mathbb{R}^n \). Choosing \( \hat{\alpha} = 0 \) we see that this implies, in particular, that
\[
\alpha^T \left( \lambda V^T \hat{M}_1 V + O(\lambda^2) \right) \alpha > 0
\] (25)
for all non-zero \( \alpha \in \mathbb{R}^n \). Now suppose that \( V^T \hat{M}_1 V \) is not of full rank. Then we can choose a non-zero \( \alpha \) such that
\[
\alpha^T (V^T \hat{M}_1 V) \alpha = 0.
\]
Suppose now that with this choice of \( \alpha \) we have an expansion of the form
\[
v^T \Im(M_N) v = \alpha^T (V^T \hat{M}_1 V) \alpha \Im(\lambda^p) + O(\Im(\lambda^{p+1})).
\]
where \( p > 1 \) and the coefficient \( \alpha^T (V^T \hat{M}_1 V) \alpha \) is non-zero. Because \( p > 1 \), we know that \( \Im(\lambda^p) \) is not of one sign on the upper half plane. Thus \( v^T \Im(M_N) v \) cannot be of one sign in the upper half plane, contradicting the Nevanlinna property of \( M_N \). We have thus established that the matrix \( V^T \hat{M}_1 V \) is of full rank; also, from (23), we have therefore established that it is positive definite:
\[
V^T \hat{M}_1 V > 0.
\] (26)
Next we choose \( \alpha = 0, \hat{\alpha} \neq 0 \) in (24): since \( \Im(\frac{1}{\lambda}) < 0 \) when \( \Im(\lambda) > 0 \), we see that \( \hat{V}^T \hat{M}_{-1} \hat{V} \) is negative definite, i.e.
\[
\hat{V}^T \hat{M}_{-1} \hat{V} < 0.
\] (27)
The two results (26) and (27) – together with (24) – imply the Bennewitz conjecture, as we shall show in the remainder of the proof.

\( \theta \)From Lemma 2.3, we first need to show that there exists \( \theta_+ \in (0, \pi/2) \) and \( \rho_+ > 0 \) such that for \( |\lambda| \in (0, \rho_+) \) and \( \arg(\lambda) \in [\theta_+, \pi/2] \), the matrix \( \Im(-\lambda^2 M_N(\lambda)) \) is positive definite.

With \( v = V\alpha + \hat{V} \hat{\alpha} \) as before, (24) gives
\[
v^T \Im(-\lambda^2 M_N) v = \alpha^T \Im(-\lambda^3 V^T \hat{M}_1 V + O(\lambda^4)) \alpha + \hat{\alpha}^T \Im(-\lambda \hat{V}^T \hat{M}_{-1} \hat{V} + O(\lambda^2)) \hat{\alpha} + 2\alpha^T \Im(-\lambda^2 V^T \hat{M}_0 \hat{V} - \lambda^3 V^T \hat{M}_1 \hat{V} + O(\lambda^4)) \hat{\alpha}.
\] (28)

Let \( \lambda = \rho e^{i\theta} \). Fix a number \( \theta_1 \in (\pi/3, \pi/2) \). Then \( -\sin(3\theta) > 0 \) for \( \theta \in [\theta_1, \pi/2] \). Thus (28) implies that there is a constant \( \omega_1 > 0 \) and a number \( \rho_1 > 0 \) such that for all \( \rho \in (0, \rho_1) \) and \( \theta \in [\theta_1, \pi/2] \) the first term in (28) satisfies
\[
\alpha^T \Im(-\lambda^3 V^T \hat{M}_1 V + O(\lambda^4)) \alpha > \omega_1 \rho^3 \|\alpha\|^2.
\] (29)
Similarly, fix \( \theta_2 \in (0, \pi/2) \). Then \( \sin \theta > 0 \) for \( \theta \in [\theta_2, \pi - \theta_2] \). Thus \([27]\) implies that there is a constant \( \omega_2 > 0 \) and a number \( \rho_2 > 0 \) such that for all \( \rho \in (0, \rho_2) \) and \( \theta \in [\theta_2, \pi - \theta_2] \) the second term in \([28]\) satisfies
\[
\hat{\alpha}^T \Im \left( -\lambda \hat{V}^T \hat{M}_{-1} \hat{V} + O(\lambda^2) \right) \hat{\alpha} > \omega_2 \rho \| \hat{\alpha} \|^2.
\]  (30)
We now deal with the last term in \([28]\). Clearly there exist positive constants \( C_1 \) and \( C_2 \) such that for all \( \rho = |\lambda| \in (0, r) \) and \( \theta = \arg(\lambda) \in (0, \pi) \),
\[
2 \alpha^T \Im \left( -\lambda^2 V^T \hat{M}_0 \hat{V} - \lambda^3 V^T \hat{M}_1 \hat{V} + O(\lambda^4) \right) \hat{\alpha} \leq (C_1 \rho^2 |\sin(2\theta)| + C_2 \rho^3) \| \alpha \| \| \hat{\alpha} \|.
\]
We bound the second part using Young’s inequality:
\[
C_2 \rho^3 \| \alpha \| \| \hat{\alpha} \| \leq \frac{1}{2} C_2 \rho \left\{ \rho^3 \| \alpha \|^2 + \rho \| \hat{\alpha} \|^2 \right\}.
\]
We also bound the first part using Young’s inequality:
\[
C_1 \rho^2 |\sin(2\theta)| \| \alpha \| \| \hat{\alpha} \| \leq \frac{1}{2} C_1 |\sin(2\theta)| \left\{ \rho^3 \| \alpha \|^2 + \rho \| \hat{\alpha} \|^2 \right\}.
\]
Combining these inequalities we obtain
\[
2 \alpha^T \Im \left( -\lambda^2 V^T \hat{M}_0 \hat{V} - \lambda^3 V^T \hat{M}_1 \hat{V} + O(\lambda^4) \right) \hat{\alpha} \leq \| \alpha \|^2 \left\{ \frac{C_1}{2} \rho^3 |\sin(2\theta)| + C_2 \rho^4 \right\} + \| \hat{\alpha} \|^2 \left\{ \frac{C_1}{2} \rho |\sin(2\theta)| + C_2 \rho^3 \right\}.
\]  (31)
We now combine \([29]\), \([30]\) and \([31]\). Let \( \theta^* = \max(\theta_1, \theta_2) \in (0, \pi/2) \) and let \( \rho^* = \min(\rho_1, \rho_2, r) \). Then for all \( \rho = |\lambda| \in (0, \rho^*) \) and for all \( \theta = \arg(\lambda) \in [\theta^*, \pi/2] \) we have, from \([28]\),
\[
v^T \Im(-\lambda^2 M_N) v > \left( \omega_1 - \frac{C_1}{2} |\sin(2\theta)| - \frac{C_2}{2} \rho \right) \rho^3 \| \alpha \|^2 + \left( \omega_2 - \frac{C_1}{2} |\sin(2\theta)| - \frac{C_2}{2} \rho \right) \rho \| \hat{\alpha} \|^2.
\]  (32)
Choosing \( \theta_+ \in [\theta^*, \pi/2] \) sufficiently close to \( \pi/2 \) (to make \( |\sin(2\theta)| \) small) and choosing \( \rho_+ \in (0, \rho^* \) sufficiently small, we can ensure that
\[
v^T \Im(-\lambda^2 M_N) v > \frac{\omega_1}{2} \rho^3 \| \alpha \|^2 + \frac{\omega_2}{2} \rho \| \hat{\alpha} \|^2
\]  (33)
for all \( \rho = |\lambda| \in (0, \rho_+) \) and \( \theta = \arg(\lambda) \in [\theta_+, \pi/2] \). This implies that \( \Im(-\lambda^2 M_N(\lambda)) \) is positive definite for all such \( \lambda \), which deals with the first condition in Lemma \([23]\).

To verify the second condition in Lemma \([23]\), we must show that there exists \( \theta_- \in (\pi, 3\pi/2) \) and \( \rho_- > 0 \) such that \( \Im(-\lambda^2 M_N(\lambda)) \) is negative definite for \( |\lambda| \in (0, \rho_-) \) and \( \arg(\lambda) \in [\theta_-, 3\pi/2] \). Looking back at the proof above, it is clear that there are only two changes to the reasoning. First, we need to replace \([29]\) by a result of the form
\[
\alpha^T \Im \left( -\lambda^3 V^T \hat{M}_1 V + O(\lambda^4) \right) \alpha < -\omega_3 \rho \| \alpha \|^2, \quad (\omega_3 > 0 \ \text{constant}),
\]  (34)
while the second term in \([28]\) gives
\[
\hat{\alpha}^T \Im \left( -\lambda \hat{V}^T \hat{M}_{-1} \hat{V} + O(\lambda^2) \right) \hat{\alpha} < -\omega_4 \rho \| \hat{\alpha} \|^2, \quad (\omega_4 > 0 \ \text{constant}),
\]  (35)
to hold for all $|\lambda| \in (0, \rho_4)$ and $\arg(\lambda) \in [\pi + \theta_4, 2\pi - \theta_4]$ for some $\rho_4 > 0$ and $\theta_4 \in (0, \pi/2)$. This we can do because we can choose $\theta_4 \in [0, \pi/2)$ such that $\sin(\theta) < 0$ for all $\theta \in [\pi + \theta_4, 2\pi - \theta_4]$. Choosing $\theta_\pi \geq \pi + \max(\theta_4, \theta_4)$ sufficiently close to $3\pi/2$ (to make $|\sin(2\theta)|$ sufficiently small, as reasoned for (33)) and choosing $\rho_\pi \in (0, \min(\rho_3, \rho_4)]$ sufficiently small, one obtains an inequality of the form

$$v^T \Im(-\lambda^2 M_N)v < -\frac{\omega_3^2}{2} \rho^2 ||\alpha||^2 - \frac{\omega_3^1}{2} \rho ||\tilde{\alpha}||^2$$

for all $\rho = |\lambda| \in (0, \rho_\pi)$ and $\theta = \arg(\lambda) \in [\theta_\pi, 3\pi/2]$. This implies that $\Im(-\lambda^2 M_N(\lambda))$ is negative definite for all such $\lambda$. Both the conditions in Lemma 2.3 have now been verified, and our proof is complete. \(\square\)

### 2.2 A note on the converse of the Bennewitz Conjecture

It seems appropriate to indicate here why we have been unable to prove the converse of the Bennewitz Conjecture: namely, that if

$$\text{rank}(\text{Res}(M_D, 0)) + \text{rank}(\text{Res}(M_N, 0)) = n - d < n$$

then there is no HELP inequality associated with the differential operator. The key to proving such a converse would be Lemma 2.3, which is an if-and-only-if result. Following the notation and proof of the previous section, suppose that $V$ is an $n \times r$ matrix whose columns are $r$ orthonormal vectors spanning the column space of the matrix $M_{-1}$ and let $\hat{V}$ be an $n \times (n - r)$ matrix whose first $n - r - d$ columns are orthonormal vectors spanning the column space of $M_{-1}$ and whose last $d$ column vectors are chosen so that the columns of $\hat{V}$ are orthonormal. Then one can show that the columns of $V$ and of $\hat{V}$ form an orthonormal basis of $\mathbb{R}^n$, as before; equations (13) and (24) can be shown still to hold. Looking for a failure in the first condition of Lemma 2.3, we seek a vector $v$ such that for all $\theta_\pi \in (0, \pi/2)$ and $\rho_\pi > 0$, the inequality

$$v^T \Im(-\lambda^2 M_N(\lambda))v > 0$$

fails for some $\arg(\lambda) \in [\theta_\pi, \pi/2)$ and $|\lambda| \in (0, \rho_\pi)$. It seems reasonable to look for such a vector $v$ in that part of $\mathbb{R}^n$ which is not spanned by the columns of $M_{-1}$ and $M_{-1}$: to this end we must have $\alpha = 0$ in (39), and we must also choose $\tilde{\alpha}$ such that $M_{-1}\hat{V}\tilde{\alpha} = 0$. With these two conditions, (24) becomes

$$v^T \Im(M_N)v = \tilde{\alpha}^T \Im(\hat{V}^T M_0 \hat{V} + \lambda \hat{V}^T \hat{M}_1 \hat{V} + \ldots)\tilde{\alpha},$$

while similarly

$$v^T \Im(-\lambda^2 M_N)v = \tilde{\alpha}^T \Im(-\lambda^2 \hat{V}^T M_0 \hat{V} - \lambda^3 \hat{V}^T \hat{M}_1 \hat{V} + \ldots)\tilde{\alpha}.$$  

Recall that all quantities in these equations, other than $\lambda$, are real. Let $\lambda = r e^{i\theta}$. Then $\Im(-\lambda^2) = -r^2 \sin(2\theta)$, which is negative for $\theta = \pi/2 - \epsilon$, for small $\epsilon$. Thus combining (37) and (38), a necessary condition for a HELP inequality to hold is that

$$\tilde{\alpha}^T \hat{V}^T M_0 \hat{V} \tilde{\alpha} = 0,$$

whence (38a) and (38) become

$$v^T \Im(M_N)v = \tilde{\alpha}^T \Im(\hat{V}^T M_1 \hat{V} + \ldots)\tilde{\alpha},$$

$$v^T \Im(-\lambda^2 M_N)v = \tilde{\alpha}^T \Im(-\lambda^3 \hat{V}^T \hat{M}_1 \hat{V} + \ldots)\tilde{\alpha}.$$  

The Nevanlinna condition $\Im(M_N) > 0$ for $\Im(\lambda) > 0$ implies that the leading term on the right hand side of (11) is strictly positive; the leading term on the right hand side of (12) is then strictly positive for $\arg(\lambda) \in (\pi/3, \pi/2]$, which certainly does not preclude the existence of a HELP inequality: indeed, if it were true for all $v$ and not just those outside the column span of $M_{-1}$ and $M_{-1}$, it would say
that a HELP inequality definitely held. This suggests that we ought to try to prove that (40) must fail, but we have so far been unable to do this.

(Note finally that no new information is obtained by looking for a failure in the second condition of Lemma 2.3).

3 Computing $M_N(\lambda)$ and $M_D(\lambda)$ near a pole by a change of variables

We now turn our attention to the problem of computing the residue matrices of $M_N$ and $M_D$ near a pole. We shall assume once more that the pole is at $\lambda = 0$. Also, since the numerics are the same for $M_D$ as they are for $M_N$, we shall consider a more unified problem: that of computing the residue matrix of an arbitrary Titchmarsh-Weyl matrix $M(\lambda)$ having a pole at $\lambda = 0$.

We describe the solution of the problem in four steps. In the first of these, we define a new matrix $\Psi$ and show that it is well-behaved near the pole of $M$.

3.1 The transformation to ‘safe’ variables – the matrix $\Psi$

Suppose that the Titchmarsh-Weyl matrix $M(\lambda)$ has a pole at $\lambda = 0$. We know that such a pole must be simple \[9\], but we also know that any attempt to compute $M$ directly, by the methods we described in our previous work \[1\], is likely to yield inaccurate results when $|\lambda|$ is small. In an attempt to circumvent this difficulty we shall define a new variable $\Psi$ by

$$\Psi := (\alpha I + M^{-1})^{-1}, \quad (43)$$

where $\alpha$ is a complex constant to be chosen for convenience.

The reason for removing the singularity in this way, rather than by using $\lambda M(\lambda)$ as a new variable, lies in the need to approximate whichever variable is chosen by solving an initial value problem. For $\Psi$ the resulting Riccati-type ODE \[43\] is not singular as $\lambda \to 0$. For $\lambda M(\lambda)$, on the other hand, the corresponding Riccati equation has a $\lambda^{-1}$ singularity occurring in the quadratic term on the right hand side.

In order to show that $\alpha$ may be chosen so that $\Psi$ has a removable singularity at $\lambda = 0$ we need to consider two different cases separately. The first is the case where $M^{-1}$ is well-behaved at $\lambda = 0$; the second is the case where $M^{-1}$ also has a pole at $\lambda = 0$. The second of these two cases seemed, initially, the more pathological, since it includes the case in which the Sturm-Liouville problem with Neumann boundary conditions shares an eigenvalue with the same problem with Dirichlet boundary conditions: however Lemma 3.1 below gives a whole class of problems for which this always happens.

**Lemma 3.1** Suppose that $M^{-1}$ has a removable singularity at $\lambda = 0$. Then there exists a choice of $\alpha$ such that $\Psi$ has a removable singularity at $\lambda = 0$.

**Proof** Let $\bar{M}$ denote $\lim_{\lambda \to 0} M^{-1}(\lambda)$; this exists by hypothesis. Clearly $\Psi$ will have a removable singularity at $\lambda = 0$ if and only if $-\alpha$ is not an eigenvalue of $\bar{M}$. Thus the result is proved. $\square$

**Lemma 3.2** Suppose that $M^{-1}(\lambda)$ and $M(\lambda)$ both have poles at $\lambda = 0$. Then $\Psi$ has a removable singularity at $\lambda = 0$ for any $\alpha$ with $\Im(\alpha) \neq 0$, and for all but finitely many real $\alpha$. Also, in the case $n = 2$, the matrix $\Psi$ has a removable singularity at $\lambda = 0$ for any non-zero $\alpha$. 

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Proof We examine the $2 \times 2$ case first. For any $2 \times 2$ matrix

$$B = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}$$

let $B^A$ denote the matrix of minors of $B$, so that

$$B^A = \begin{pmatrix} b_{2,2} & -b_{1,2} \\ -b_{2,1} & b_{1,1} \end{pmatrix}$$

and $BB^A = B^A B = \det B$. With this notation it is clear that

$$\Psi = \frac{\alpha I + (M^{-1})^A}{\det(\alpha I + M^{-1})}.$$  \hspace{1cm} (44)

Expanding the determinant, we get

$$\Psi = \frac{\alpha I + (M^{-1})^A}{\alpha^2 + \alpha \text{trace}(M^{-1}) + \det(M^{-1})}.$$  \hspace{1cm} (44)

We also know that

$$M^{-1} = \frac{M^A}{\det M}.$$  

Since $M$ has a simple pole, so does $M^A$ (n.b. this step does not generalise to the case of $n \times n$ matrices). Also, $M^{-1}$ has a simple pole by hypothesis. Returning to (44), we can see that $\alpha I + (M^{-1})^A$ has a simple pole. Thus to get a removable singularity for $\Psi$, for any non-zero $\alpha$, it suffices to show that trace($M^{-1}$) has a pole. Let

$$M^{-1}(\lambda) = \frac{1}{\lambda} M_{-1} + M_0 + \lambda M_1 + O(\lambda^2);$$

the only way that trace($M^{-1}$) can fail to have a pole is if trace($M_{-1}$) = 0. If this happens then $M_{-1}$ must be of the form

$$M_{-1} = \begin{pmatrix} \beta & \gamma \\ \gamma & -\beta \end{pmatrix};$$

recalling that $M_{-1}$ is a real non-zero matrix this implies that $M_{-1}$ is of full rank, and hence $M$ must have a zero rather than a pole when $\lambda = 0$. This contradiction proves that trace($M^{-1}$) has a pole, and hence that $\Psi$ has a removable singularity.

We now turn to the case $n > 2$. We know that $M$ and $M^{-1}$ are analytic functions of $\lambda$ with singularities at $\lambda = 0$. Let $\mu_1(\lambda), \ldots, \mu_n(\lambda)$ be the eigenvalues of $\lambda M^{-1}(\lambda)$. As $\lambda M^{-1}(\lambda)$ is analytic, and symmetric in the sense of Kato [10, p. 120], we know from the remark at the bottom of page 121 in [10] that $M^{-1}$ has an analytic Schur decomposition of the form

$$M^{-1}(\lambda)R(\lambda) = R(\lambda)\frac{1}{\lambda}D(\lambda)$$

on a punctured neighbourhood of $\lambda = 0$. Here

$$D = \text{diag}(\mu_1, \ldots, \mu_n),$$

where $\mu_1, \ldots, \mu_n$ are analytic at $\lambda = 0$, while the matrix $R(\lambda)$ is analytic at $\lambda = 0$ and is real orthogonal ($R^{-1} = R^T$) for all sufficiently small real $\lambda$. This orthogonality of $R$ for real $\lambda$ means that $R^{-1}$ is also analytic at $\lambda = 0$. To see this, observe that the only type of singularity which $R^{-1}$ could have would be a pole. A pole would cause $R^{-1}(\lambda)$ to blow up as $\lambda$ approached zero through real values, contradicting the regularity of $R$ by the orthogonality $R^{-1} = R^T$ for real $\lambda$.  

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The Schur decomposition of $\Psi$ is clearly

$$\Psi(\lambda) = R(\lambda)(\lambda^{-1}D(\lambda) + \alpha I)^{-1}R^{-1}(\lambda),$$

and the eigenvalues of $\Psi$ are clearly

$$(\alpha + \frac{1}{\lambda}\mu_1)^{-1}, \ldots, (\alpha + \frac{1}{\lambda}\mu_n)^{-1}.$$ 

These are all analytic functions of $\lambda$. If $j$ is such that $\mu_j \neq 0$ at $\lambda = 0$ then the corresponding eigenvalue of $\Psi$ clearly has a zero at $\lambda = 0$. If $j$ is such that $\mu_j$ has a zero of order at least 2 at $\lambda = 0$, then the corresponding eigenvalue of $\Psi$ has a removable singularity at $\lambda = 0$ provided $\alpha \neq 0$.

If $j$ is such that $\mu_j$ has a simple zero at $\lambda = 0$ then the corresponding eigenvalue of $\Psi$ will have a removable singularity at $\lambda = 0$ for all but one value of $\alpha$. In particular, since $\mu_j$ is real-valued for real $\lambda$, the corresponding eigenvalue of $\Psi$ has a removable singularity for $\Im(\alpha) \neq 0$. Whenever the eigenvalues of $\Psi$ are analytic, so is $\Psi$ itself, from the Schur decomposition (45) in which $R(\lambda)$ and $R^{-1}$ are analytic at $\lambda = 0$. This completes the proof.

**Remark** The proof for $n > 2$ can be extended to show that under the hypothesis of the Bennewitz conjecture – namely, that the ranks of the residue matrices of $M$ and $M^{-1}$ sum to $n$ – the matrix $\Psi$ has a removable singularity for any non-zero $\alpha$. In other words, the result of the case $n = 2$ is recovered in this special case.\[\square\]

### 3.2 The initial value problem for $\Psi$

In the rest of this section we shall consider the case $n = 2$: that of the fourth order Sturm-Liouville problem. We start by recalling the method proposed for the computation of the matrix $M$ in [1]. An interval $[0, X]$ is chosen, with $X$ suitably large; the fourth order Sturm-Liouville equation is cast in the form

$$Jz' = Sz,$$

where $J$ is the symplectic matrix

$$J = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$S$ is the symmetric matrix

$$S = \begin{pmatrix} \lambda w - q & 0 & 0 & 0 \\ 0 & -s & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/p \end{pmatrix},$$

and $z$ is the vector of quasi-derivatives

$$z = \begin{pmatrix} y \\ y' \\ -(py'')' + sy' \\ py'' \end{pmatrix}.$$ 

Then we consider the matrix initial value problem consisting of the differential equation

$$JZ' = SZ$$
and the initial condition

\[ Z(X) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

Let \( Z(x) \) denote the solution of this problem, where \( 0 \leq x \leq X \). We partition \( Z \) as

\[ Z(x) = \begin{pmatrix} U(x) \\ V(x) \end{pmatrix}, \]

and form the corresponding initial value problem for the matrix \( UV^{-1} \). We solve this initial value problem, starting from \( x = X \), to find \( UV^{-1}(0) \). Our approximation to \( M \) is then given by

\[ M \approx (UV^{-1}(0))^{-1}. \]  

(The formula would be exact if we had \( X = +\infty \).) If we replace \( M \) in (43) by the expression on the right hand side of (47) then we get

\[ \Psi \approx (\alpha I + UV^{-1}(0))^{-1}. \]

Clearly, then, the process of approximating \( \Psi \) can be reduced to that of deriving an initial value problem for the matrix

\[ \Gamma(x) := (\alpha I + UV^{-1}(x))^{-1}. \]

The initial condition is obvious: \( \Gamma(X) = \alpha^{-1}I \). The differential equation is also quite straightforward to derive. If the matrix \( S \) in (46) is partitioned as

\[ S = \begin{pmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2} \end{pmatrix}, \]

then it turns out that

\[ \Gamma' = -\{ \Gamma S_{2,1}(I - \alpha \Gamma) + (I - \alpha \Gamma)S_{1,2} \Gamma + (I - \alpha \Gamma)S_{1,1}(I - \alpha \Gamma) + \Gamma S_{2,2} \Gamma \}. \]

We solve this equation using the NAG routine D02QGF, which allows reverse communication for evaluation of the right hand side of the differential equation: this helps to keep the programme structure simple when the right hand side is complicated. D02QGF is a variable-order, variable-step Adams code and is therefore able to cope with mild stiffness. In practice, we noted in [1] that stiffness is not usually a problem unless \( X \) has been chosen much larger than necessary.

### 3.3 Determining the Taylor expansion of \( \Psi \)

Determining an approximate Taylor expansion of an analytic function from numerical values of the function is not easy. The number of coefficients in the expansion which can be computed reliably depends on the accuracy with which the function values can be computed, on the rate of decay of the Taylor coefficients as one proceeds up the series and, ultimately, on the precision of the machine arithmetic.

Our problem is slightly compounded by the fact that we have a function \( \Psi \) with a removable singularity at the point around which we wish to expand it (\( \lambda = 0 \)). We cannot compute \( \Psi(0) \); indeed we cannot compute \( \Psi(\lambda) \) for any \( \lambda \) with zero imaginary part. Our approach has been to compute \( \Psi \) at a sequence of points

\[ \lambda = \frac{\mu}{2j}, \quad j = 0, 1, 2, \ldots, \]
where \( \mu \) is a fixed complex number with \( \Im(\mu) \neq 0 \), and solve a Vandermonde system (using the algorithm of Björck and Pereyra \( [2] \)) to obtain approximations to the Taylor coefficients. We shall now consider how the different sources of error and the ill-conditioning of the Vandermonde system will affect the approximations to the Taylor coefficients which we obtain. For simplicity we shall set aside our matrix-valued function \( \Psi \) and consider a complex-valued function \( f \) given by a Taylor expansion

\[
f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m + a_{m+1} z^{m+1} + \cdots
\]  

(49)

Clearly the following system of equations holds:

\[
\begin{align*}
\alpha_0 + \alpha_1 \mu + \alpha_2 \mu^2 + \cdots + \alpha_m \mu^m &= g(\mu) \\
\alpha_0 + \alpha_2 \mu^2 + \cdots + \alpha_m \mu^m &= g(\mu/2) \\
\alpha_0 + \alpha_4 \mu^4 + \cdots + \alpha_m \mu^m &= g(\mu/4) \\
&\vdots \\
\alpha_0 + \alpha_{2^j} \mu^{2^j} + \cdots + \alpha_m \mu^m &= g(\mu/2^j)
\end{align*}
\]  

(50)

where the function \( g \) is given by

\[
g(z) = f(z) - a_{m+1} z^{m+1} - a_{m+2} z^{m+2} - \cdots.
\]

A Vandermonde matrix is an \((m + 1) \times (m + 1)\) matrix of the form

\[
V = \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
\alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_m \\
\alpha_0^2 & \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_m^2 \\
\alpha_0^4 & \alpha_1^4 & \alpha_2^4 & \cdots & \alpha_m^4 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_0^{2^j} & \alpha_1^{2^j} & \alpha_2^{2^j} & \cdots & \alpha_m^{2^j}
\end{pmatrix},
\]

where \( \alpha_0, \ldots, \alpha_m \) are distinct complex numbers. If we let \( a = (a_0, a_1 \mu, a_2 \mu^2, \ldots, a_m \mu^m)^T \) and \( g = (g(\mu/2), g(\mu/2^2), \ldots, g(\mu/2^j))^T \) then we can cast our system as a dual Vandermonde problem of the form

\[
V^T a = g,
\]

where \( \alpha_0 = 1/2^m \), \( \alpha_1 = 1/2^{m-1} \), \ldots, \( \alpha_m = 1 \). Since the \( \alpha_j \) are positive real numbers arranged in ascending order, the error analysis of Higham \( [3] \) is now applicable to the solution of this system by the Björck-Pereyra algorithm. In particular this error analysis shows, when we know \( g \) ‘exactly’, that the Björck-Pereyra algorithm introduces essentially no more error into \( a \) than is already implied by the storage of \( g \) in machine arithmetic. Since we do not have any of the special sign properties on the vector \( g \) which would make for a better error estimate, this suggests that the contribution of the machine precision \( \epsilon \) to the error (measured in the norm \( \| \cdot \|_1 \) will be a term of the order

\[
m\epsilon \|V^{-1}\|_\infty \leq C m \epsilon 2^m,
\]  

(51)

where \( C \) is independent of \( m \) \( [3] \). With a machine precision of \( 10^{-16} \) this suggests that, on grounds of roundoff alone, it will not be possible to obtain reasonable accuracy in the vector \( a \) for \( m \) much greater than \( 6 \). This was borne out in the experiments which we conducted. Of course we could use \( \| \cdot \|_\infty \) to measure the error, instead of \( \| \cdot \|_1 \); however this would make no difference, as the ratio \( \|x_1\|_1/\|x\|_\infty \) is never greater than \( m \) for any non-zero \( m \)-vector \( x \), and we have already seen that \( m \) must be quite small.

We now turn to the contribution to the error arising from \( g \): we do not know \( g \) exactly because we do not know \( f \) exactly. We must approximate \( g \) by \( f \). This entails an error

\[
g(z) - f(z) = -a_{m+1} z^{m+1} - a_{m+2} z^{m+2} - \cdots,
\]  

(52)
Similarly, Next we tackle the term \( \sum_{p=1}^{\infty} a_{m+p} \text{Interpolant of } z^{m+p} \), the second equality coming from (52). Now let coefficients for the function \( f \) for the moment, we observe that \( a_0, \ldots, a_m \) from (50), the coefficients \( a_0, \ldots, a_m \) are the interpolation coefficients for the function \( g \) at the points \( \mu, \mu/2, \ldots, \mu 2^{-m} \). As interpolation is a linear process,

\[
\text{Interpolant of } g = \text{Interpolant of } f + \text{Interpolant of } (g - f)
\]

\[
= \text{Interpolant of } f - \sum_{p=1}^{\infty} a_{m+p} \text{Interpolant of } z^{m+p}, \quad (53)
\]

denote the \( m \)th degree polynomial interpolant to the function \( z \mapsto z^{m+p} \) at the points 1, 2\(^{-1}\), \ldots, 2\(^{-m}\). By (53) the change in the computed value of \( a_\nu \mu^\nu \) caused by approximating \( g \) by \( f \) is given by

\[
\sum_{p=1}^{\infty} a_{m+p} c^{(p)}_\nu.
\]

The coefficients \( c^{(p)}_\nu \) can be computed explicitly using the Björck-Pereyra algorithm. The solution is given by

\[
c^{(p)}_\nu = \sum_{j=0}^{m-\nu} (-1)^{m-\nu-j} \beta_{m-\nu-j}(1, 2^{-1}, \ldots, 2^{-m+1}) \mu^{m+p} \frac{1}{2^j} \prod_{r=0}^{j-1} (1 - 2^{m+p-r}) \prod_{r=1}^{j} (1 - 2^r), \quad (54)
\]

where \( \beta_k(y_0, \ldots, y_{m-1}) \) denotes the sum of all products of \( k \) distinct elements of the set \( \{y_0, \ldots, y_{m-1}\} \). We use this formula to get a bound on \( c^{(p)}_\nu \). First, we consider \( \beta_k(1, 2^{-1}, \ldots, 2^{-m+1}) \). This is a sum of \( \binom{m}{k} \) terms, of which the greatest is \( 1.2^{-1} \ldots 2^{-k+1} = 2^{-k(k-1)/2} \). Thus

\[
\beta_{m-\nu-j}(1, 2^{-1}, \ldots, 2^{-m+1}) \leq \binom{m}{m-\nu-j} 2^{-m-\nu-j}(m-\nu-j)/2. \quad (55)
\]

Next we tackle the term

\[
\prod_{r=0}^{j-1} (1 - 2^{m+p-r}) = \prod_{r=0}^{j-1} 2^{m+p-r} \prod_{r=1}^{j} (1 - 2^{m-p+r})
\]

\[
\leq \prod_{r=0}^{j-1} 2^{m+p-r}
\]

\[
= 2^{(m+p-(j-1)/2)j} \quad (56)
\]

Similarly,

\[
\prod_{r=1}^{j} (1 - 2^r) = \prod_{r=1}^{j} 2^r (1 - 2^{-r})
\]

\[
= 2^{j(j+1)/2} \prod_{r=1}^{j} (1 - 2^{-r})
\]

\[
\geq 2^{j(j+1)/2} \prod_{r=1}^{j} \exp(-2.2^{-r})
\]

\[
\geq 2^{j(j+1)/2} \exp(-2). \quad (57)
\]
Substituting (55), (56) and (57) back into (54) we obtain the estimate

\[ |c_p^{(p)}| \leq \sum_{j=0}^{m-\nu} \binom{m}{m-\nu-j} 2^{-\nu m} \mu^{m+p} \left( \frac{1}{2^j} \right)^{m+p-j} \exp(2) \frac{2^{(m+p-j-1)/2}j}{2j+1/2} \]

which simplifies to give

\[ |c_p^{(p)}| \leq \sum_{j=0}^{m-\nu} \binom{m}{m-\nu-j} |\mu|^{m+p} \exp(2) 2^{-(m-\nu)^2/2+(m-\nu)(2j+1)/2-j(j+1)/2}. \]  

By the change of dummy summation variable \( j = m - \nu - k \), this gives

\[ |c_p^{(p)}| \leq \sum_{k=0}^{m-\nu} \binom{m}{k} |\mu|^{m+p} \exp(2) 2^{-k(k-1)/2}. \]

For our purposes, it suffices to make a very blunt estimate at this stage: throw away the powers of 2, and extend the summation up to \( k = m \). Since

\[ \sum_{k=0}^{m} \binom{m}{k} = 2^m, \]

we get the bound

\[ |c_p^{(p)}| \leq |\mu|^p |2\mu|^m \exp(2) \leq \exp(2) |2\mu|^{m+p}. \]

Thus, neglecting the integration error, the change in the computed value of \( a_\nu \mu^\nu \) caused by replacing \( g \) by \( f \) will be bounded by

\[ \exp(2) \sum_{p=1}^{\infty} |a_{m+p}| |2\mu|^{m+p}. \]

Provided \( 2\mu \) is strictly within the radius of convergence of the power series, this will tend to zero as \( m \) tends to infinity; moreover, if \( a_{m+1} \neq 0 \), the leading order term for small \( \mu \) will be

\[ |a_{m+1}| \exp(2) |2\mu|^{m+1}. \]

In order to assess the likely order of magnitude for a suitable value of \( \mu \) it seems reasonable to ask that this error term be of the same order of magnitude as the error arising from roundoff. The constant \( C \) in (51) is independent of \( m \) so we can neglect it; the coefficient \( a_{m+1} \) we obviously do not know, but if we assume that it is \( O(1) \) then we obtain

\[ m.2^m \epsilon \approx |2\mu|^{m+1}. \]

We have already seen that with \( \epsilon = 10^{-16} \), the choice of \( m = 6 \) is likely to be the biggest possible. In some sense, this modest value of \( m \) justifies the assumption that \( a_{m+1} \) is an \( O(1) \) quantity. It also gives \( m.2^m \epsilon \approx 4 \times 10^{-5} \), which suggests \( \mu \approx 0.1 \). With \( m = 5 \), on the other hand, we have \( m.2^m \epsilon \approx 1.7 \times 10^{-8} \), which suggests \( \mu \approx 0.025 \).

Based on these observations we devised the following algorithm for computing the first \( k+1 \) Taylor coefficients of the matrix \( \Psi \), where \( k < 5 \).

1. Make the tolerance \( TOL \) for the computation of \( \Psi \) as small as possible within the constraints of reasonable run-times. This depends on the machine at one’s disposal.

2. Start with \( |\mu| \approx 0.025 \) (say) and \( n = k + 1 \).
3. Compute approximations to $a_0, \ldots, a_m$: from these extract approximations to $a_0, \ldots, a_k$, the coefficients of interest.

4. If $m < 7$, increase the value of $m$ and compute new coefficient approximations.

5. While the approximations seem to be improving and $m < 7$, keep increasing $m$ and computing new approximations.

6. When the sequence of coefficient approximations appears to start to diverge stop increasing $m$.
   Discard the latest (starting-to-diverge) approximations.

7. Now regard $m$ as fixed and start to double $\mu$. Follow the same process as above, doubling $\mu$ while this seems to improve the values of $a_0, \ldots, a_k$. Stop either when the user’s target accuracy is achieved, or when the approximations seem to start to diverge, or when a doubling of $\mu$ would give $|\mu| > 0.5$. Return a warning flag ($IFAIL = 2$) if the target accuracy has not been reached.

The error due to integration is never explicitly controlled in this process, though step 6 should ensure that $m$ is never taken large enough to magnify the integration error to an unacceptable level.

Typically, for computing the Taylor coefficients of our matrix $\Psi$, we might start with $\mu = 0.025i$, and compute $\Psi$ using an initial value solver with $TOL = 10^{-11}$. If $\Psi$ has a Taylor expansion

$$\Psi(\lambda) = \Psi_0 + \lambda \Psi_1 + \lambda^2 \Psi_2 + \lambda^3 \Psi_3 + \lambda^4 \Psi_4 + \cdots,$$

we usually find that only the coefficients $\Psi_0, \ldots, \Psi_3$ can be computed with an accuracy of $10^{-4}$ or better; the accuracy of $\Psi_3$ might be $10^{-4}$, of $\Psi_2$ about $10^{-6}$, of $\Psi_1$ about $10^{-8}$, while $\Psi_0$ might have an accuracy of $10^{-10}$, achieved with $m = 6$ or $m = 7$ and with a value of $\mu$ of about $0.2i$. The precise details depend on the problem in question. The deterioration in the accuracy of the coefficients as one proceeds up the series need not be a problem if one intends to use them simply to compute values of $\Psi(\lambda)$ by a truncated Taylor expansion for small values of $|\lambda|$.

### 3.4 Recovering the residue of $M$ from $\Psi$

We suppose that the first few terms of the Taylor expansion of $M$ have been determined:

$$\Psi(\lambda) = \Psi_0 + \lambda \Psi_1 + \lambda^2 \Psi_2 + \lambda^3 \Psi_3 + \cdots \quad (62)$$

We want to determine the first few coefficients in the Laurent expansion of $M$:

$$M(\lambda) = \frac{1}{\lambda} M_{-1} + M_0 + \lambda M_1 + \lambda^2 M_2 + \cdots \quad (63)$$

As we shall see, from the first $n$ terms in the Taylor expansion of $\Psi$ we can determine at most the first $n - 1$ terms in the Laurent expansion of $M$. Thus, although we have eliminated the problems associated with trying to compute $M$ near the pole, we have paid a price in terms of having to compute more Taylor coefficients than we get repaid in Laurent coefficients.

Equation (63) may be rearranged to yield

$$M^{-1} = \Psi^{-1} - \alpha I = \frac{\Psi^{-1}}{\det(\Psi)} - \alpha I$$

$$= \frac{\Psi A - \alpha \det(\Psi) I}{\det(\Psi)}.$$
Inverting both sides,

\[ M = \frac{\det(\Psi)(\Psi^A - \alpha \det(\Psi)I)^{-1}}{\det(\Psi - \alpha \det(\Psi)I)} \]

Now we expand the determinant in the denominator to get

\[ \det(\Psi - \alpha \det(\Psi)I) = \det(\Psi) \left\{ 1 - \alpha \text{trace}(\Psi) + \alpha^2 \det(\Psi) \right\}. \]

Thus we obtain the formula

\[ M = \frac{\Psi - \alpha \det(\Psi)I}{1 - \alpha \text{trace}(\Psi) + \alpha^2 \det(\Psi)}. \]  

(64)

We shall obtain the Laurent series for \( M \) by Taylor expansion of the numerator and denominator in (64). From (62) we obtain the expansions

\[ \text{trace}(\Psi) = \text{trace}(\Psi_0) + \lambda \text{trace}(\Psi_1) + \lambda^2 \text{trace}(\Psi_2) + \lambda^3 \text{trace}(\Psi_3) + \cdots, \]  

(65)

\[ \det(\Psi) = \det \Psi_0 + \lambda \det \Psi_1 + \lambda^2 (\det \Psi_1 + \text{trace}(\Psi_0^A \Psi_2)) + \lambda^3 (\text{trace}(\Psi_0^A \Psi_2) + \text{trace}(\Psi_0^A \Psi_3)) + \lambda^4 (\text{trace}(\Psi_0^A \Psi_4) + \text{trace}(\Psi_1^A \Psi_3) + \det(\Psi_2)) + \cdots. \]  

(66)

We know that the denominator in (64) must have a zero at \( \lambda = 0 \) because \( M \) has a pole at \( \lambda = 0 \) by hypothesis: thus

\[ 1 - \alpha \text{trace}(\Psi) + \alpha^2 \det(\Psi) = \lambda a_1 + \lambda^2 a_2 + \lambda^3 a_3 + \lambda^4 a_4 + \cdots, \]  

(67)

where the coefficients \( a_1, a_2, a_3 \) and \( a_4 \) are evident by comparing (64) and (67). We shall also write

\[ \Psi - \alpha \det(\Psi)I = A_0 + \lambda A_1 + \lambda^2 A_2 + \lambda^3 A_3 + \lambda^4 A_4 + \cdots, \]  

(68)

where

\[
\begin{align*}
A_0 &= \Psi_0 - \alpha \det(\Psi_0)I, \\
A_1 &= \Psi_1 - \alpha \text{trace}(\Psi_0^A \Psi_1)I, \\
A_2 &= \Psi_2 - \alpha (\det(\Psi_1) + \text{trace}(\Psi_0^A \Psi_2))I, \\
A_3 &= \Psi_3 - \alpha (\text{trace}(\Psi_0^A \Psi_3) + \text{trace}(\Psi_1^A \Psi_2))I, \\
A_4 &= \Psi_4 - \alpha (\text{trace}(\Psi_0^A \Psi_4) + \text{trace}(\Psi_1^A \Psi_3) + \det(\Psi_2))I.
\end{align*}
\]
If \( a_1 \neq 0 \), then we may combine (64), (67) and (68) to give us the Laurent expansion

\[
M(\lambda) = \lambda^{-1} \frac{A_0}{a_1} + \left( \frac{A_1}{a_1} + \frac{a_2}{a_1^2} A_0 \right) + \lambda \left( \frac{A_2}{a_1} - \frac{a_2}{a_1^2} A_1 + \left( \frac{a_2^2}{a_1^2} - \frac{a_3}{a_1^3} \right) A_0 \right) + \lambda^2 \left( \frac{A_3}{a_1} - \frac{a_3}{a_1^2} A_2 + \left( \frac{a_3^2}{a_1^2} - \frac{a_4}{a_1^3} \right) A_1 + \left( \frac{2a_1a_3 - a_2^2a_2}{a_1^4} - \frac{a_4}{a_1^3} \right) A_0 \right) + \cdots
\]

This happens when \( \Psi(0) = \frac{1}{a_1} I \), which happens when the residue matrix \( \text{Res}(\lambda, 0) \) has full rank, giving \( M^{-1} \) a zero at \( \lambda = 0 \). The code checks that the value of \( a_1 \), and uses (69) if \(|a_1| > TOL\), \((70)\) if \(|a_1| < TOL\), where \( TOL \) is the tolerance used by D02QGF in the computation of \( \Psi \). The cases \( a_1 = a_2 = 0 \) cannot arise with \( 2 \times 2 \) matrices: for \( a_2 = 0 \) necessarily implies \( A_1 = 0 \). Since \( \Psi(0) = \frac{1}{a_1} I \), we see that \( \det(\Psi(0)) \neq 0 \); this implies that \( M^{-1} = (\Psi^4 - \det(\Psi)I) / \det(\Psi) \) has a double zero at \( \lambda = 0 \), implying that \( M \) has a double pole (or worse). This is not possible, as both \( M \) and \( M^{-1} \) have, at worst, simple poles.

\section{Numerical Experiments}

Our primary objective in these experiments was to compute the residues of Titchmarsh-Weyl matrices for a number of fourth order Sturm-Liouville equations and to use these, together with the Bennewitz Conjecture, to decide whether HELP inequalities hold for these equations.

Before listing our example problems, we mention the following useful result. To avoid complicated conditions on quasiderivatives we state the result for smooth coefficients in the differential operator.

\begin{lemma}
Suppose that \( \mathcal{L} \) is a fourth order differential operator on the domain of functions \( f \) in \( L^2[0, \infty) \) which are four times continuously differentiable and are such that \( \mathcal{L}f \in L^2[0, \infty) \). Suppose that \( \mathcal{L} \) has the form \( \mathcal{L} = \ell^2 \) where \( \ell \) is a second order operator

\[
\ell(y)(x) = -y''(x) + q(x)y(x)
\]

and \( q \) is twice continuously differentiable with \( q(0) = 0 \), \( q'(0) = 0 \). Suppose also that \( \mathcal{L} \) is strong limit-point at infinity. Then any eigenfunction \( y \) of \( \mathcal{L} \) subject to Neumann boundary conditions \( y''(0) = 0 = y''(0) \) which is not in the null-space of \( \ell \) generates an eigenfunction \( z = \ell y \) of \( \mathcal{L} \) subject to Dirichlet conditions \( z(0) = 0 = z'(0) \).

\textbf{Proof} Suppose \( y \) is as described, so that \( \mathcal{L} y = \lambda y \) for some real \( \lambda \), and let \( z = \ell y \). Because \( y \) is not in the null-space of \( \ell \), \( z \) is non-trivial. Also \( z(0) = -y''(0) + q(0)y(0) = 0 \) because \( y''(0) = 0 \) and \( q(0) = 0 \), and \( z'(0) = -y''(0) + q(0)y'(0) + q'(0)y(0) = 0 \) since \( y''(0) = 0 \) and \( q(0) = q'(0) = 0 \). Thus \( z \) satisfies the Dirichlet boundary conditions. Clearly \( \mathcal{L} z = \ell^2 y = \ell \mathcal{L} y = \ell y = \lambda y \).
\[ \lambda z, \text{ so } z \text{ satisfies the differential equation } Lz = \lambda z: \text{ as the coefficient } q \text{ is twice continuously differentiable, this makes it easy to see that } z \text{ is four times continuously differentiable. Finally, } z \text{ is square integrable. This follows because} \]

\[ \langle z, z \rangle = \langle \ell y, \ell y \rangle = \langle \ell^2 y, y \rangle = \lambda \langle y, y \rangle, \]

the penultimate equality using the fact that \( \ell^2 = L \) is strong limit-point at infinity together with the fact that \( \ell y(0) = (\ell y)'(0) = 0. \) □

**Equation 1** The differential equation

\[ y^{(iv)} - (s(x)y')' + q(x)y = \lambda y \]

on the interval \([0, \infty), \) with coefficients

\[ s(x) = \frac{8x^2(x^4 - 3x^2 - 5)}{(x^2 + 1)^2}, \]

\[ q(x) = \frac{4[4x^{12} - 24x^{10} - 7x^8 + 96x^6 + 46x^4 - 60x^2 - 15]}{(x^2 + 1)^4}. \]

This equation is strong limit-point at infinity. Imposing a Dirichlet boundary condition \( y(0) = y'(0) = 0 \) reveals that this problem was carefully crafted so that \( \lambda = 0 \) would be an eigenvalue of multiplicity 2; the reader may check that

\[ y(x) = (x^2 + x^4)e^{-x^2}, \quad y(x) = (x^3 + x^5)e^{-x^2} \]

are the corresponding eigenfunctions. We arranged this because we suspected that it would result in a problem for which we would have

\[ \text{rank}(M_D) = 2, \quad \text{rank}(M_N) = 0, \quad (71) \]

although in fact we do not know of any result which would guarantee this. The numerical results in Table 1 suggest that (71) is indeed true (the determinant of \( M_D \) vanishes nowhere). If it is, then the hypotheses of the Bennewitz conjecture are satisfied and we have an equation for which a HELP inequality holds. We should mention that there seems to be a dearth of fourth order examples with multiple eigenvalues in the literature: indeed, we could not find any.

Notice that we carried out the computations for two different values of \( \alpha, \) to provide an additional check on our results; we also quote what the code thinks is the imaginary part of the residue matrix. This ought to be zero, so it provides an indication of the error. We also quote the error indicator returned by the code: this is reassuringly of the same order of magnitude as the imaginary part of the computed residue matrix.

**Equation 2** The differential equation

\[ y^{(iv)} - 2(x^2 y')' + (x^4 - 2)y = \lambda y \]

on the interval \([0, \infty). \) This equation is strong limit-point at infinity; its differential operator is the formal square of the second order operator \( \ell y = -y'' + x^2 y. \) By Lemma 4.1 we know that all but at most one of the Neumann eigenvalues will be Dirichlet eigenvalues: in fact if Dirichlet boundary conditions \( y(0) = 0 = y'(0) \) are imposed, then the eigenvalues are \( \lambda_k = 16(k + 1)^2, \) while if Neumann boundary conditions \( y''(0) = 0 = y''(0) \) are imposed, then the eigenvalues are \( \lambda_k = 16k^2. \) Thus at each of the points \( \lambda = 16(k + 1)^2, k = 0, 1, \ldots, \) both the Titchmarsh-Weyl matrices \( M_N \) and \( M_D \) will have poles. Given that \( M_N = -M_D^{-1} \) it is clear that this
Equation 3

The differential equation (2) associated with this differential equation, provided the operator is defined

\[ M \] defined by

\[ M\psi = y^{(iv)} - 2(x^2 y')' + (x^4 - 2)y - 16(k + 1)^2 y, \]

where \( k \) is some non-negative integer. This is a result which Diaz conjectured in his thesis but was unable to prove.

Consulting the numerical results in Table 2, we see that our code obtains approximations to the residue matrices which are as near to rank 1 as one could expect: they are matrices whose elements are not small but whose determinants are \( O(10^{-13}) \).

**Equation 3** The differential equation

\[ y^{(iv)} - 2(e^x y')' + (e^{2x} - e^x) y = \lambda y \quad (72) \]

on the interval \([0, \infty)\). The differential operator here is the formal square of the second order operator \( \ell y = -y'' + \exp(x)y \). With Dirichlet conditions \( y(0) = 0 = y'(0) \) the eigenvalues

\[
\text{Determinant of residue matrix: } 2303.17 + i0.0008
\]

\[
\text{Value of } |a_1|: 6.3 \times 10^{-10}.
\]

\[
\text{Integration tolerance for } \Psi: 10^{-9}
\]

\[
\text{Determinant of residue matrix: } 2303.18 - i0.02
\]

\[
\text{Value of } |a_1|: -1.7 \times 10^{-12}.
\]

\[
\text{Integration tolerance for } \Psi: 10^{-9}
\]

| Using \( \alpha = 1 \), truncating \([0, \infty)\) to \([0, 10]\): |
|-------------------------------------------------|
| \[ \text{Res}(M, \lambda = 0) = \left( \begin{array}{cc} -384.83 & -167.96 \\ -167.96 & -79.29 \end{array} \right) + i \left( \begin{array}{cc} 4.2 \times 10^{-2} & -1.7 \times 10^{-2} \\ -1.7 \times 10^{-2} & 6.6 \times 10^{-3} \end{array} \right) \] |
| Code error estimate (sup norm): \( 3.4 \times 10^{-2} \) |
| Determinant of residue matrix: \( 2303.17 + i0.0008 \) |
| Value of \( |a_1| \): \( 6.3 \times 10^{-10} \) |
| Integration tolerance for \( \Psi \): \( 10^{-9} \) |

| Using \( \alpha = 1 + i \), truncating \([0, \infty)\) to \([0, 10]\): |
|-------------------------------------------------|
| \[ \text{Res}(M, \lambda = 0) = \left( \begin{array}{cc} -384.79 & -167.94 \\ -167.94 & -79.28 \end{array} \right) + i \left( \begin{array}{cc} 2.5 \times 10^{-3} & 1.0 \times 10^{-3} \\ 1.0 \times 10^{-3} & 4.1 \times 10^{-4} \end{array} \right) \] |
| Code error estimate (sup norm): \( 2.1 \times 10^{-3} \) |
| Determinant of residue matrix: \( 2303.18 - i0.02 \) |
| Value of \( |a_1| \): \( -1.7 \times 10^{-12} \) |
| Integration tolerance for \( \Psi \): \( 10^{-9} \) |

### Table 1: Results for Equation 1

| Using \( \alpha = 1 \), truncating \([0, \infty)\) to \([0, 20]\): |
|-------------------------------------------------|
| \[ \text{Res}(M, \lambda = 16) = \left( \begin{array}{cc} -82.62549 & -40.74366 \\ -40.74366 & -20.09121 \end{array} \right) + i \left( \begin{array}{cc} -7.3 \times 10^{-6} & -3.6 \times 10^{-6} \\ -3.6 \times 10^{-6} & -1.8 \times 10^{-6} \end{array} \right) \] |
| Code error estimate (sup norm): \( 3.5 \times 10^{-6} \) |
| Determinant of residue matrix: \( -4.5 \times 10^{-14} - i2.3 \times 10^{-18} \) |
| Value of \( |a_1| \): \( 9.7 \times 10^{-9} \) |
| Integration tolerance for \( \Psi \): \( 10^{-9} \) |

| Using \( \alpha = 1 + i \), truncating \([0, \infty)\) to \([0, 20]\): |
|-------------------------------------------------|
| \[ \text{Res}(M, \lambda = 16) = \left( \begin{array}{cc} -82.62548 & -40.74366 \\ -40.74366 & -20.09121 \end{array} \right) + i \left( \begin{array}{cc} -1.8 \times 10^{-5} & -8.7 \times 10^{-6} \\ -8.7 \times 10^{-6} & -4.3 \times 10^{-6} \end{array} \right) \] |
| Code error estimate (sup norm): \( 4.7 \times 10^{-6} \) |
| Determinant of residue matrix: \( 2.3 \times 10^{-15} - i5.7 \times 10^{-18} \) |
| Value of \( |a_1| \): \( -9.7 \times 10^{-3} \) |
| Integration tolerance for \( \Psi \): \( 10^{-9} \) |

### Table 2: Results for Equation 2
are not known in closed form. However it is a relatively straightforward matter to compute approximations using the code SLEUTH \cite{7}. For example, computing at different tolerances and using different truncations of $[0, \infty)$, the following approximations seem to be correct to all decimal places quoted:

\begin{align*}
\lambda_0 & = 35.560604, \quad \lambda_1 = 128.113477, \quad \lambda_2 = 297.84692
\end{align*}

For Neumann boundary conditions $y'''(0) = 0 = y''(0)$ the corresponding approximate eigenvalues obtained were

\begin{align*}
\lambda_0 & = 6.199245, \quad \lambda_1 = 43.002631, \quad \lambda_2 = 136.295990.
\end{align*}

¿From this numerical evidence there is no overlap between the first few eigenvalues of the Dirichlet and Neumann spectra. To investigate whether or not there are likely to be HELP inequalities associated with this equation, we must compute the residues of the Titchmarsh-Weyl matrices at these eigenvalues using our code. The results are shown in Table 3. These residue matrices appear (to within the error we expected at the given tolerance) to be of rank 1. This suggests that there is no HELP inequality associated with (72).

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