REAL-TIME PROPAGATOR IN THE
FIRST QUANTISED FORMALISM

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ABSTRACT

We argue that a basic modification must be made to the first quantised formalism of string theory if the physics of ‘particle creation’ is to be correctly described. The analogous quantisation of the relativistic particle is performed, and it is shown that the proper time along the world line must go both forwards and backwards (in the usual quantisation it only goes forwards). The matrix propagator of the real time formalism is obtained from the two directions of proper time.

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1. The Issue

One of the most well known problems concerns the ultimate fate of an evaporating black hole. A common viewpoint is that this question can only be settled in a complete theory of quantised gravity plus matter. Strings provide such a theory; further, black hole solutions have been constructed for strings\(^1\). Why then don’t we have a complete understanding of the fate of the black hole?

One problem seems to be that it is not easy to get the string black hole to evaporate. Topological methods can study the black hole to all string loop orders\(^2\) (evaporation should appear from one loop onwards). The fact that the black hole doesn’t appear to evaporate has led to the speculation that the functional measure for matter fields is such for the string case that there is indeed no evaporation. Matrix model techniques should also describe the string theory to all loop orders, but again there is no clear evaporating solution.

In this talk we argue that there is a more basic problem encountered in studying any process of particle (string) creation using the techniques used in studying strings. Strings are studied in the first quantised language. For the simpler but analogous case of a free scalar field theory, this corresponds to writing the propagation amplitude between two space-time points as the sum over all trajectories of a relativistic scalar particle travelling between those two points. Such a computation generates a propagator between the ‘in’ and the ‘out’ vacua:\(^3\)

\[
D(x', x) = \int D[\text{paths}(x \to x')] e^{iS} = \langle \text{out} | T[\phi(x')\phi(x)] | \text{in} \rangle / \langle \text{out} | 0 | \text{in} \rangle = \langle \text{out} | 0 | \text{in} \rangle
\]

Using such a propagator in the string case (and enforcing the vanishing \(\beta\)-function conditions) leads to Einstein equations incorporating backreaction of created particles in the form\(^4\)

\[
G_{\mu\nu} = \frac{\langle \text{out} | T_{\mu\nu} | 0 \rangle_{\text{in}}}{\langle \text{out} | 0 \rangle_{\text{in}}}
\]

But what we need on the RHS of Einstein’s equations is a true expectation value of \(T_{\mu\nu}\) in the physical state, not an ‘in-out’ scattering amplitude. In fact when the ‘in’ and ‘out’ vacua are not the same, perturbation theory needs the \(2x2\) matrix propagator of the real-time formalism. Thus if particle creation and backreaction are to be correctly studied in the first quantised language used for strings then we must be able to find in a natural way the matrix propagator from the ‘sum over paths’ formalism. We will show how this emerges naturally with a careful quantisation of the relativistic scalar particle, and will then comment on the string case.

2. Quantising the Relativistic Particle

The geometric action for a scalar particle is

\[
S = \int_{X_i}^{X_f} mds = \int_{X_i}^{X_f} m(X^\mu \tau X_{\mu\tau})^{1/2}d\tau = \int Ld\tau
\]
where $\tau$ is an arbitrary parametrisation of the world line. The canonical momenta
\[
P_\mu = \frac{\partial L}{\partial X^\mu,\tau} = \frac{m X^\mu,\tau}{(X^\mu,\tau X^\mu,\tau)^{1/2}}
\]
satisfy the constraints
\[
P^\mu P_\mu - m^2 = 0
\]
We choose the range of the parameter $\tau$ as $[0,1]$. Following the approach given in Kaku\textsuperscript{5}, we impose the constraint at each $\tau$ through a $\delta$-function:
\[
\delta(p^2(\tau) - m^2) = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\lambda(\tau) e^{-i\lambda/2(p^2(\tau) - m^2)}
\]
The path integral amplitude to propagate from $X_i$ to $X_f$ becomes
\[
G(X_2, X_1) \equiv N \int \frac{D[X]D[P]D[\lambda]}{\text{Vol}[\text{Diff}]} e^{i\int_0^1 d\tau[p_\mu(\tau)X^\mu,\tau(\tau) - \lambda/2(p^2(\tau) - m^2)]}
\]
where $N$ is a normalisation constant, $P_\mu X^\mu,\tau = m(X^\mu,\tau X^\mu,\tau)^{1/2}$ is the original Lagrangian in Eq. (3) and we have divided by the volume of the symmetry group, which which is related to $\tau$-diffeomorphisms in the manner discussed below. (The $\delta$-function constraint on the momenta and dividing by $\text{Vol}[\text{Diff}]$ remove the two phase space co-ordinates redundant in the description of the particle path.)

The action is invariant under
\[
S: \quad \delta X^\mu(\tau) = \epsilon(\tau)\lambda(\tau)P^\mu(\tau)
\]
\[
\delta P_\mu(\tau) = 0
\]
\[
\delta \lambda(\tau) = (\epsilon(\tau)\lambda(\tau)),\tau
\]
$\lambda$ transforms as an einbein under the diffeomorphism $\tau \rightarrow \tau'(\tau)$. Note that for regular $\epsilon(\tau)$, $\lambda$ either changes sign for no $\tau$ or for all $\tau$. We take $\text{Diff}$ as the group of regular diffeomorphisms connected to the identity; then $\lambda$ does not change sign under the allowed diffeomorphisms. These diffeomorphisms cannot gauge-fix $\lambda(\tau)$ to any preassigned function $\lambda_1(\tau)$. $\lambda(\tau)$ and $\lambda_1(\tau)$ must have the same value of
\[
\int_0^1 \lambda(\tau)d\tau \equiv \Lambda
\]
$\Lambda$ may be interpreted as the length of the world line. The restriction Eq. (9) is usually assumed to mean that the length of the world line is the only remaining parameter after gauge-fixing. But what we find instead is that there is a further complication: there is a discrete infinity of classes, each with one or more continuous parameters. One class comes from configurations $\lambda(\tau)$ which are everywhere positive. This class can be gauge-fixed with the allowed diffeomorphisms to have
\[
\dot{\lambda}(\tau) = 0, \quad \int_0^1 d\tau \lambda(\tau) = \Lambda
\]
with \( 0 < \Lambda < \infty \). Similarly, the set of everywhere negative \( \lambda(\tau) \) can be gauge-fixed as in Eq. (10) but with \( -\infty < \Lambda < 0 \). Thus these classes give for the Fourier transform of Eq. (7)

\[
G_{\Lambda>0}(p) = \frac{1}{4\pi} \int_0^\infty d\lambda(\tau) e^{-i\lambda/2(p^2(\tau) - m^2)} = \frac{i}{p^2 - m^2 + i\epsilon},
\]

\[
G_{\Lambda<0}(p) = \frac{1}{4\pi} \int_{-\infty}^0 d\lambda(\tau) e^{-i\lambda/2(p^2(\tau) - m^2)} = \frac{-i}{p^2 - m^2 - i\epsilon}
\]

respectively. (We have added the \( \epsilon \) term for convergence to each sector; this term was not in the action.) Keeping the first class alone gives the Feynman propagator for particles, while the second gives its complex conjugate.

But in the path integral over \( \lambda \) in Eq. (7) we also have the class of \( \lambda(\tau) \) which are positive for \( 0 < \tau < \tau_1 \), negative for \( \tau_1 < \tau < 1 \). The group of orientation preserving diffeomorphisms can gauge fix this to

\[
\dot{\lambda}(\tau) = 0 \quad \text{for} \quad \tau \neq \tau_1, \quad \int_0^{\tau_1} d\tau \lambda(\tau) = \Lambda_1, \quad \int_{\tau_1}^1 d\tau \lambda(\tau) = \Lambda_2
\]

with \( 0 < \Lambda_1 < \infty, -\infty < \Lambda_2 < 0 \). We would like to identify this sector as the contribution to the amplitude to start with a state of type \( 1 \) and end with a state of type 2 (the off-diagonal element \( D_{12} \) of the matrix propagator). A general sector has a given number of alternations in the sign of \( \lambda \), and in each interval of constant sign we gauge fix \( \lambda \) to a constant \( \Lambda_i \). We can perform the integration over the variables \( \Lambda_i \) appearing in each sector, but we should also specify a ‘transition amplitude’ for each point \( \tau_i \) where \( \lambda \) changes sign. We allow this amplitude to depend on the states on both sides of \( \tau_i \), and also on whether \( \lambda \) changes from positive to negative or vice versa. This amplitude is not supplied by the original action; it is supplementary information needed for determining the propagator.

We will argue with examples below that we should identify \( \lambda > 0 \) with particles of the type 1 field and \( \lambda < 0 \) with particles of the type two field in the language of the real time formalism for perturbation theory. To obtain the matrix propagator element \( D_{ij} \) \((i, j = 1, 2)\) we need to add together all sectors for \( \lambda(\tau) \) beginning as type \( i \) and ending as type \( j \). When the einbein \( \lambda \) changes sign we may say that the proper time along the world line has reversed direction.

3. Examples

First we note another way of arriving at the proper time representation of the propagator. The Feynman propagator for a scalar field in Minkowski space can be written as

\[
D_F(p) = \frac{i}{p^2 - m^2 + i\epsilon} = \int_{\lambda=0}^\infty \tilde{\lambda} e^{i\tilde{\lambda}(p^2 - m^2 + i\epsilon)}
\]

Eq. (13) can be used to express \( G_F(p) \) in a first quantised language, with \( p^2 = -\tilde{\square} \). The Hamiltonian on the world line is \( \tilde{\square} + m^2 \), evolution takes place in a fictitious time for a duration \( \tilde{\lambda} \), and this length \( \tilde{\lambda} \) of the world line is summed over all values from 0 to \( \infty \).
Now consider the scalar field in Minkowski space at temperature $\beta^{-1}$. For the closed time path beginning and ending at $t = -\infty$ the matrix propagator is

$$D(p) = \left\{ \begin{array}{ll} \frac{-i}{p^2 - m^2 + \pi \epsilon} + 2\pi n(p)\delta(p^2 - m^2) & 2\pi[(n(p) + \theta(-p_0))\delta(p^2 - m^2)] \\ 2\pi[n(p) + \theta(p_0)]\delta(p^2 - m^2) & \frac{-i}{p^2 - m^2 + \pi \epsilon} + 2\pi n(p)\delta(p^2 - m^2) \end{array} \right. \tag{14}$$

($n(p)$ is the number density.) We can write this as

$$D(p) = \int_0^\infty d\lambda e^{-i\lambda H - \epsilon\lambda M}, \quad H = \begin{pmatrix} -(p^2 - m^2) & 0 \\ 0 & (p^2 - m^2) \end{pmatrix}$$

$$M = \begin{pmatrix} 1 + 2n(p) & -2\sqrt{n(p)(n(p) + 1)} \\ -2\sqrt{n(p)(n(p) + 1)} & 1 + 2n(p) \end{pmatrix} \tag{15}$$

Comparing with Eq. (13) we observe the following. To obtain the matrix propagator in a many-body situation we need to consider both evolutions $e^{-i\lambda H}$ and $e^{i\lambda H}$ on the world line. A regulating factor is needed to define the first quantised path integral, near the mass shell. But the regulator matrix $M$ need not be diagonal in the two kinds of world line evolution.

We can now relate the world line evolution in Eq. (15) to the general discussion of proper time quantisation in the previous section. The two propagations $\pm(i\lambda(p^2 - m^2))$ correspond to the two signs of $\lambda$ that arose in the gauge fixing of the relativistic particle. We find that to obtain Eq. (14) from the proper time quantisation of the relativistic particle we must take the amplitude for $\lambda$ to go from positive to negative as $\text{sech}(\beta|p_0|/2)e^{2\beta p_0/2}$, and for negative to positive as $\text{sech}(\beta|p_0|/2)e^{-2\beta p_0/2}$. Indeed, extending the computation of Eq. (11) to all ‘sectors’ arising in the gauge fixing, and summing over sectors with the above amplitudes for orientation reversal, we reproduce the matrix propagator Eq. (14).

As another example, consider the free scalar field propagating in 1+1 spacetime with metric

$$ds^2 = C(\eta)[d\eta^2 - dx^2], \quad -\infty < \eta < \infty, \quad 0 < x < 2\pi \tag{16}$$

$$C(\eta) = A + Be(\eta), \quad A > B \geq 0 \tag{17}$$

This describes a Universe with ‘sudden’ expansion, so we expect particle creation. Let the physical state be the ‘in’ vacuum $|0 >_\text{in}$. Then we need the matrix propagator

$$D(z_2, z_1) = \begin{pmatrix} i_n < 0|T[\phi(z_2)\phi(z_1)]|0 >_\text{in} \\
_i < 0|\phi(z_2)\phi(z_1)|0 >_\text{in} \end{pmatrix} \begin{pmatrix} i_n < 0|\phi(z_1)\phi(z_2)|0 >_\text{in} \\
_i < 0|T[\phi(z_2)\phi(z_1)]|0 >_\text{in} \end{pmatrix} \tag{18}$$

Our goal is to see if there exists a choice of regulator matrix $M$ such that evaluating

$$D = \int_0^\infty d\lambda e^{-i\lambda H - \epsilon\lambda M}, \quad H = \begin{pmatrix} (\Box + m^2) & 0 \\ 0 & -(\Box + m^2) \end{pmatrix} \tag{19}$$

gives the matrix propagator Eq (18). Here $\Box$ acts on functions on spacetime $f(\eta, x)$. 
Fourier modes \( \sin(nx) \), \( \cos(nx) \) of the scalar field decouple from each other. Consider for one such mode the basis of solutions of the field equation

\[
f_n^1 = \frac{\cos(nx)}{\sqrt{2\pi}} e^{-i\omega_n^+ \eta}, \quad \eta < 0
\]
\[
f_n^1 = \frac{\cos(nx)}{\sqrt{2\pi}} \left[ \frac{1}{2} \left( 1 + \frac{\omega_n^-}{\omega_n^+} \right) e^{-i\omega_n^+ \eta} + \frac{1}{2} \left( 1 - \frac{\omega_n^-}{\omega_n^+} \right) e^{i\omega_n^- \eta} \right], \quad \eta > 0
\]

(\( \omega_n^\pm \) are the frequencies of the mode for \( \eta > 0, \eta < 0 \) respectively.) In the flat space example above we had space-time translation invariance, so the ‘regulator matrix’ \( M \) was diagonal in the 4-momentum \( p \). In the present example there is no translational invariance in \( \eta \), so the modes \( f_n^1, f_n^2 \) can ‘connect’ across the point on the world line where the einbein \( \lambda \) changes sign, in the proper time path integral. Thus for each \( n \) we need to consider a 4x4 matrix, which acts on a column vector \((f_n^1, f_n^2)^+, (f_n^1, f_n^2)^-\). Here the first pair of functions propagate on the world line as \( e^{i(\bar{\omega} + m^2)} \) while the second pair propagates as \( e^{-i(\bar{\omega} + m^2)} \). The result we get is that the matrix propagator Eq. (18) is obtained for

\[
M = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & \frac{4B_n}{1+4B_n} & \frac{-2}{1+4B_n} \\
\frac{-2}{1+4B_n} & \frac{4B_n}{1+4B_n} & 1 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

(21)

where

\[
B_n = -\frac{\omega_n^+ - \omega_n^-}{2\omega_n^+ + \omega_n^-}
\]

(22)

(\( B_n \) are proportional to the Bogoliubov co-efficients.) If we had not allowed for the reversal of the proper time direction (i.e. \( M \) was the identity matrix) then we would get the ‘in-out’ propagator Eq. (1) instead of Eq. (18).

4. Strings

We show that the real time matrix propagator is ‘natural’ for the first quantised string in the same sense that interactions are natural for a string. For interactions we note that one closed string can split into two closed strings (the pants diagram) with the world sheet smooth at all points. In a similar sense we can get a smooth analogue of the einbein sign change discussed above for the particle. We have a Minkowski signature target space, which requires a Minkowski signature world sheet. The world sheet metric and action are

\[
\begin{align*}
ds^2 &= g_{ab}dx^a dx^b, \\
S &= \int d^2 x \sqrt{-g} g^{\mu\nu} X_\mu X_\nu
\end{align*}
\]

(23)

where \( x^0 = \tau, x^1 = \sigma \) are the world sheet coordinates and \( X^\mu \) are the target space (i.e. spacetime) coordinates. Consider the metric

\[
g_{ab} = \begin{pmatrix}
-tanh \tau & sech \tau \\
sech \tau & tanh \tau
\end{pmatrix}, \quad \sqrt{-g} = 1
\]

(24)
\[\tau \rightarrow -\infty, \quad g_{ab} \rightarrow \text{diag}\{1, -1\}, \quad S \rightarrow \int \partial_\tau X^\mu \partial_\tau x_\mu - \partial_\sigma X^\mu \partial_\sigma X_\mu\]

\[\tau \rightarrow \infty, \quad g_{ab} \rightarrow \text{diag}\{-1, 1\}, \quad S \rightarrow \int (-)\partial_\tau X^\mu \partial_\tau x_\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu\]

(25)

In the particle limit for the string, \(\partial_\tau X^\mu\) gives the \(X^\mu, \tau\) for the particle case and \(\partial_\sigma X^\mu\) gives the mass to the particle. Thus in the limits \(\tau \rightarrow \pm \infty\), \(S\) corresponds to the Hamiltonians \(\mp(p^2 - m^2)\) on the particle world line, and we reproduce the ‘orientation change’ along the world line with the string world sheet metric everywhere regular \((\sqrt{-g} = 1)\). Since we have to sum over all geometries on the world sheet we are forced to consider metrics such as Eq. (24). It is appropriate that in a theory of gravity (i.e. string theory) we are naturally led to the real time propagator because generic solutions of such a theory give curved space with inequivalent ‘in’ and ‘out’ vacua and consequent particle creation.

We note that the field equations for the background spacetime in string theory are derived not from a Lagrangian but from requiring consistent propagation of the string in the background. It is the causal ‘in-in’ propagator that must be ‘consistent’, not the ‘in-out’ vacuum propagator, if we are to get the correct backreaction equations and not an equation like Eq. (2). We have shown how the ‘first quantised’ language can obtain the causal propagator, and moreover that the required ‘orientation reversal’ is natural to strings.

Our central point, viz. that proper time need not go only forwards, may have relevance beyond the first quantization approach to strings. The quantization of gravity is very much like the quantization of the relativistic particle. In the canonical approach, the wavefunction depends on 3-geometries, and a semiclassical 4-geometry emerges only if one makes a WKB ansatz for the dominant mode of the 3-metric. The ‘time’ direction so obtained looks continuous presumably only at scales much larger than one period of the WKB waveform. Thus the semiclassical nature of the 4-geometry would break down when the geometry is examined at sufficiently small scales. This can happen for instance when we try to relate the Kruskal co-ordinates to the Schwarzschild co-ordinates at the horizon of a black hole. One might think that the problem would be avoided in a path-integral formalism using a continuous proper time to describe the 4-geometries, but here one should consider the possibility that the proper time may not run only forwards. Issues like the third quantization of geometries then appear in the same way that we obtained the effect of many particles (second quantization) from the study of just one relativistic trajectory.

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6. References

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