Remarks on nonlinear Schrödinger equations with harmonic potential

Rémi Carles

Mathématiques Appliquées de Bordeaux et UMR 5466 CNRS
351 cours de la Libération
33 405 Talence cedex, France
carles@math.u-bordeaux.fr

Abstract

Bose-Einstein condensation is usually modeled by nonlinear Schrödinger equations with harmonic potential. We study the Cauchy problem for these equations. We show that the local problem can be treated as in the case with no potential. For the global problem, we establish an evolution law, which is the analogue of the pseudo-conformal conservation law for the nonlinear Schrödinger equation. With this evolution law, we give wave collapse criteria, as well as an upper bound for the blow up time. Taking the physical scales into account, we finally give a lower bound for the blow up time.

1 Introduction

This paper is devoted to existence and blow up results for the nonlinear Schrödinger equation with isotropic harmonic potential,

\[
\begin{aligned}
&i\hbar \partial_t u^\hbar + \frac{\hbar^2}{2} \Delta u^\hbar = \frac{\omega^2}{2} x^2 u^\hbar + \lambda |u^\hbar|^{2\sigma} u^\hbar, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n, \\
u^\hbar_{|t=0} = u^\hbar_0,
\end{aligned}
\]

(1.1)

where \( \lambda \in \mathbb{R} \), and \( \omega, \sigma > 0 \). Similar equations are considered for Bose-Einstein condensation (see for instance [6], [13], [14]), with \( \sigma = 1 \); the real \( \lambda \) may be positive or negative, depending on the considered chemical element, and is proportional to \( \hbar^2 \). With the operators introduced in [3] and [4] (see Eq. (1.3)), we prove existence results which are analogous to the well-known results for the nonlinear Schrödinger equation with no potential (see for instance [1]). These operators simplify the proof of some results of [3], [13] and [14], as well as the general approach for (1.1). In addition, we state two evolution laws (Lemma 3.1), which can be considered as the analogue of the pseudo-conformal evolution law of the free nonlinear Schrödinger...
field, and allow us to prove blow up results. Precisely, if we assume that \( \lambda \) is negative (attractive nonlinearity) and \( \sigma \geq \frac{2}{n} \), then under the condition
\[
\frac{1}{2} \| \hbar \nabla u_0^h \|_{L^2}^2 + \frac{\lambda}{\sigma + 1} \| u_0^h \|_{L^{2\sigma+2}}^{2\sigma+2} < 0,
\]
the wave collapses at time \( t^h_\ast \leq \frac{\pi}{2\omega} \) (Prop. 3.2). Notice that this condition is exactly the same as the well-known condition for the nonlinear Schrödinger equation with no potential \( (\omega = 0, \text{ see e.g. } [5], [12]) \). In particular, blow up occurs for focusing cubic nonlinearities \( (\lambda < 0 \text{ and } \sigma = 1) \) in space dimensions two and three, but not in space dimension one. Next, we prove that if \( \lambda \) is negative and proportional to \( \hbar^2 \), \( \sigma = 1 \) (the physical case), and \( n = 2 \text{ or } 3 \), then the wave collapse time can be bounded from below by \( \frac{\pi}{2\omega} - \Lambda \hbar^{\alpha} \), for some constant \( \Lambda \) and positive number \( \alpha \) (Cor. 4.2). When \( n = 1 \), we consider the case of a quintic nonlinearity \( (\sigma = 2) \), which should be the right model for Bose-Einstein Condensation in low dimension (see [10]). Notice that all these results are proved for fixed \( \hbar \), with constants independent of \( \hbar \in [0,1] \).

The following quantities are formally independent of time,
\[
N^h = \| u^h(t) \|_{L^2}^2,
\]
\[
E^h = \frac{1}{2} \| \hbar \nabla_x u^h(t) \|_{L^2}^2 + \frac{\omega^2}{2} \| xu^h(t) \|_{L^2}^2 + \frac{\lambda}{\sigma + 1} \| u^h(t) \|_{L^{2\sigma+2}}^{2\sigma+2}.
\]
(1.2)

If \( N^h \) and \( E^h \) are defined at time \( t = 0 \), we prove that the solution \( u^h \) is defined locally in time, with the conservation of \( N^h \) and \( E^h \), provided that \( \sigma < \frac{2}{n - 2} \) when \( n \geq 3 \). If \( \lambda \geq 0 \), then the solution \( u^h \) is defined globally in time. If \( \lambda < 0 \), several cases occur.

- If \( \sigma < 2/n \), then the solution is defined globally in time.
- If \( \sigma \geq 2/n \), then the solution is defined globally in time if \( u_0^h \) is sufficiently small.
- If \( \sigma \geq 2/n \) and \( E^h < \frac{\omega^2}{2} \| xu_0^h \|_{L^2}^2 \), then the solution collapses at time \( t^h_\ast \leq \frac{\pi}{2\omega} \).

The operators on which our analysis relies are
\[
J^h_j(t) = \frac{\omega}{\hbar} x_j \sin(\omega t) - i \cos(\omega t) \partial_j ; \quad H^h_j(t) = \omega x_j \cos(\omega t) + i \hbar \sin(\omega t) \partial_j.
\]
(1.3)

We denote \( J^h(t) \) (resp. \( H^h(t) \)) the operator-valued vector with components \( J^h_j(t) \) (resp. \( H^h_j(t) \)).

Lemma 1.1 \( J^h \) and \( H^h \) satisfy the following properties.

- The commutation relation,
\[
[ J^h(t), i\hbar \partial_t + \frac{\hbar^2}{2} \Delta - \frac{\omega^2}{2} x^2 ] = [ H^h(t), i\hbar \partial_t + \frac{\hbar^2}{2} \Delta - \frac{\omega^2}{2} x^2 ] = 0.
\]
(1.4)
• Denote 
\[ M^{\hbar}(t) = e^{-i\omega^2 \frac{t^2}{2} \tan(\omega t)} \text{, and } Q^{\hbar}(t) = e^{i\omega^2 \frac{t^2}{2} \cot(\omega t)} \text{, then} \]

\[ J^{\hbar}(t) = -i \cos(\omega t)M^{\hbar}(t)\nabla_x M^{\hbar}(-t), \]

\[ H^{\hbar}(t) = i\hbar \sin(\omega t)Q^{\hbar}(t)\nabla_x Q^{\hbar}(-t). \] (1.5)

• The modified Sobolev inequalities. For \( n \geq 2 \), and \( 2 \leq r < \frac{2n}{n-2} \), define \( \delta(r) \) by

\[ \delta(r) \equiv n \left( \frac{1}{2} - \frac{1}{r} \right). \] (1.6)

Then for any \( 2 \leq r < \frac{2n}{n-2} \) (\( 2 \leq r \leq \infty \) if \( n = 1 \)), there exists \( C_r \) such that,

\[ \| v(t) \|_{L^r} \leq C_r \| v(t) \|_{L^2}^{1-(\frac{1}{r})} \left( \| J^{\hbar}(t)v(t) \|_{L^2} + \| H^{\hbar}(t)v(t) \|_{L^2} \right)^{\delta(r)}. \] (1.7)

• For any function \( F \in C^1(\mathbb{C}, \mathbb{C}) \) of the form \( F(z) = zG(|z|^2) \), we have,

\[ H^{\hbar}(t)F(v) = \partial_z F(v)H^{\hbar}(t)v - \partial_{\bar{z}} F(v)\overline{H^{\hbar}(t)v}, \ \forall t \notin \frac{\pi}{\omega} \mathbb{Z}, \]

\[ J^{\hbar}(t)F(v) = \partial_z F(v)J^{\hbar}(t)v - \partial_{\bar{z}} F(v)\overline{J^{\hbar}(t)v}, \ \forall t \notin \frac{\pi}{2\omega} + \frac{\pi}{\omega} \mathbb{Z}. \] (1.8)

Remark. Property (1.8) is a direct consequence of (1.5). Property (1.7) is a consequence of the usual Sobolev inequalities and (1.5).

Notations. We work with initial data which belong to the space

\[ \Sigma := \{ u \in L^2(\mathbb{R}^n) ; xu, \nabla u \in L^2(\mathbb{R}^n) \}. \]

Notice that \( \Sigma = D(\sqrt{-\Delta + |x|^2}) \), we work in the same space as in [11].

The notation \( r' \) stands for the Hölder conjugate exponent of \( r \).

The paper is organized as follows. In Sect. 2, we study the local Cauchy problem for (1.1), and we give sufficient conditions for the solution of (1.1) to be defined globally in time. In Sect. 3, we give a sufficient condition under which the solution blows up in finite time, and provide an upper bound for the breaking time. In Sect. 4, we give a lower bound for the breaking time, that shows that the upper bound underscored in Sect. 3 is the physical breaking time in the semi-classical limit.

2 Existence results

The solution of (1.1) with \( \lambda = 0 \) is given by Mehler’s formula (see e.g. [7]),

\[ u^{\hbar}(t, x) = \left( \frac{\omega}{2i\pi \hbar \sin(\omega t)} \right)^{n/2} \int_{\mathbb{R}^n} e^{i\frac{\omega}{\sin(\omega t)} \left( \frac{x^2}{2} - \frac{\omega^2}{2} \cos(\omega t) - x \cdot y \right)} u^{\hbar}_0(y) dy =: U^{\hbar}(t)u^{\hbar}_0(x). \]
This formula defines a group $U^h(t)$, unitary on $L^2$, for which Strichartz estimates are available, that is, mixed time-space estimates, which are exactly the same as for $U_0^h(t) = e^{i \frac{t}{h} \Delta}$. Recall the main properties from which such estimates stem (see [5], or [9] for a more general argument).

- The group $U^h(t)$ is unitary on $L^2$, $\|U^h(t)\|_{L^2 \to L^2} = 1$.

- For $0 < t \leq \frac{\pi}{2\tau}$, the group is dispersive, with $\|U^h(t)\|_{L^1 \to L^{\infty}} \leq C|ht|^{-n/2}$.

We postpone the precise statement of Strichartz estimates to Sect. 4. Duhamel’s formula associated to (1.1) reads

\[ u^h(t, x) = U^h(t)u^h_0(x) - i\lambda h^{-1} \int_0^t U^h(t-s) (|u^h|^{2\sigma} u^h)(s, x) ds. \]

Replacing $U^h(t)$ by $U^h_0(t)$ yields Duhamel’s formula associated to

\[ \begin{cases} i\hbar \partial_t u + \frac{\hbar^2}{2} \Delta u = \lambda |u^h|^{2\sigma} u^h, \\ u^h_{t=0} = u^h_0. \end{cases} \tag{2.1} \]

The local Cauchy problem for this equation is now well-known in many cases (see for instance [8] for a review). In particular, the local well-posedness in $\Sigma$ is established thanks to the operators $\hbar \nabla_x$ and $x/\hbar + it \nabla_x$ (Galilean operator). This result is proved thanks to Strichartz inequalities, and to the following properties.

- The above two operators commute with $i\hbar \partial_t + \frac{\hbar^2}{2} \Delta$.

- They act on the nonlinearity $|u^h|^{2\sigma} u^h$ like derivatives.

- Gagliardo-Nirenberg inequalities.

From Lemma [4], the operators $H^h$ and $J^h$ meet all these requirements. Mimicking the classical proofs for (2.1) easily yields,

**Proposition 2.1** Let $u^h_0 \in \Sigma$. If $n \geq 3$, assume moreover $\sigma < \frac{2}{(n-2)}$. Then there exists $T^h > 0$ such that (1.1) has a unique solution $u^h \in C([0, T^h], \Sigma)$. Moreover $N^h$ and $E^h$ defined by (1.2) are constant for $t \in [0, T^h]$.

If $\lambda > 0$, the conservations of mass and energy provide a priori estimates on the $\Sigma$-norm of $u^h(t)$, and prove global existence in $\Sigma$.

If $\lambda < 0$ and $\sigma < 2/n$, then the energy $E$ controls the $\Sigma$-norm of $u^h(t)$. Indeed, from Gagliardo-Nirenberg inequalities [4,7],

\[ \|u^h(t)\|_{L^{2\sigma+2}} \leq C \|u^h(t)\|_{L^2}^{1-\delta} \left( \|hJ^h(t)u^h\|_{L^2} + \|H^h(t)u^h\|_{L^2} \right)^{\delta(2\sigma+2)}. \]

Notice that the following identity holds point-wise,

\[ |\omega x u^h(t, x)|^2 + |h \nabla_x u^h(t, x)|^2 = |hJ^h(t)u^h(t, x)|^2 + |H^h(t)u^h(t, x)|^2, \]
and one can rewrite the energy as

$$E^\hbar = \frac{1}{2}\|hJ^\hbar(t)u^\hbar\|_{L^2}^2 + \frac{1}{2}\|H^\hbar(t)u^\hbar\|_{L^2}^2 + \frac{\lambda}{\sigma + 1}\|u^\hbar(t)\|_{L^{2\sigma+2}}^{2\sigma+2}. \quad (2.2)$$

Therefore, using the conservation of mass $N^\hbar$ yields

$$\|hJ^\hbar(t)u^\hbar\|_{L^2}^2 + \|H^\hbar(t)u^\hbar\|_{L^2}^2 \leq 2E^\hbar + C(\|hJ^\hbar(t)u^\hbar\|_{L^2} + \|H^\hbar(t)u^\hbar\|_{L^2})^{n\sigma},$$

and if $\sigma < 2/n$, then the quantity $\|hJ^\hbar(t)u^\hbar\|_{L^2}^2 + \|H^\hbar(t)u^\hbar\|_{L^2}^2$ remains bounded for all times (for any fixed $\hbar$).

Similarly, global existence can be proved for small data.

**Proposition 2.2** Let $u_0^\hbar \in \Sigma$, and if $n \geq 3$, assume $\sigma < 2/(n - 2)$. Then $u^\hbar$ is defined globally in time and belongs to $C([0, +\infty], \Sigma)$ in the following cases.

- $\lambda \geq 0$ (defocusing nonlinearity).
- $\lambda < 0$ (focusing nonlinearity) and $\sigma < 2/n$.
- $\lambda < 0$, $\sigma \geq 2/n$ and $\|u_0^\hbar\|_\Sigma$ sufficiently small.

**Remark.** In particular, in space dimension one, the solution $u^\hbar$ is always globally defined for cubic nonlinearities ($\sigma = 1$).

### 3 Wave collapse

Split the energy $E^\hbar$ into $E_1^\hbar + E_2^\hbar$, with

$$E_1^\hbar(t) = \frac{1}{2}\|hJ^\hbar(t)u^\hbar\|_{L^2}^2 + \frac{\lambda}{\sigma + 1}\cos^2(\omega t)\|u^\hbar(t)\|^{2\sigma+2}_{L^{2\sigma+2}},$$

$$E_2^\hbar(t) = \frac{1}{2}\|H^\hbar(t)u^\hbar\|_{L^2}^2 + \frac{\lambda}{\sigma + 1}\sin^2(\omega t)\|u^\hbar(t)\|^{2\sigma+2}_{L^{2\sigma+2}}.$$

**Lemma 3.1** The quantities $E_1^\hbar$ and $E_2^\hbar$ satisfy the following evolution laws,

$$\frac{dE_1^\hbar}{dt} = \frac{\omega\lambda}{2\sigma + 2}(n\sigma - 2)\sin(2\omega t)\|u^\hbar(t)\|^{2\sigma+2}_{L^{2\sigma+2}},$$

$$\frac{dE_2^\hbar}{dt} = \frac{\omega\lambda}{2\sigma + 2}(2 - n\sigma)\sin(2\omega t)\|u^\hbar(t)\|^{2\sigma+2}_{L^{2\sigma+2}}.$$

**Remark.** This lemma can be regarded as the analogue of the pseudo-conformal conservation law, discovered by Ginibre and Velo ([8]) for the case with no potential ($\omega = 0$).
Sketch of the proof. Expanding $|\hbar J^h(t)u^h(t, x)|^2$ yields,

$$
|\hbar J^h(t)u^h(t, x)|^2 = \omega^2 x^2 \sin^2(\omega t)|u^h(t, x)|^2 + \hbar^2 \cos^2(\omega t)|\partial_j u^h(t, x)|^2 \\
+ \hbar \omega x_j \text{Im}(\overline{\partial_j u}).
$$

When differentiating the above relation with respect to time and integrating with respect to the space variable, one is led to computing the following quantities,

$$
\partial_t \int |x_j u^h(t, x)|^2 dx = 2\hbar \text{Im} \int x_j \overline{\partial_j u}^h,
$$

$$
\partial_t \int |\partial_j u^h(t, x)|^2 dx = -2\frac{\omega^2}{\hbar} \text{Im} \int x_j \overline{\varphi_j u}^h - 2\frac{\lambda}{\hbar} \text{Im} \int \partial_j^2 \overline{\varphi u}^h|u^h|^{2\sigma} u^h,
$$

$$
\partial_t \text{Im} \int (x_j \overline{\varphi_j u}) = \frac{\hbar}{2} \int |\nabla x u^h|^2 + \frac{\omega^2}{2\hbar} \int x^2 |u^h|^2 + \frac{\lambda}{\hbar} \int |u^h|^{2\sigma+2} - \hbar \text{Re} \int x_j \partial_j u^h \Delta u^h + \frac{\omega^2}{\hbar} \text{Re} \int x_j \partial_j u^h x^2 u^h
$$

$$
+ 2\frac{\lambda}{\hbar} \text{Re} \int x_j \partial_j u^h |u^h|^{2\sigma} u^h.
$$

It follows,

$$
\frac{d}{dt} \int |\hbar J^h(t)u^h(t, x)|^2 dx = \frac{\omega \sigma \lambda}{\sigma + 1} \sin(2\omega t) \int |u|^{2\sigma+2}
$$

$$
- 2\lambda \hbar \cos^2(\omega t) \text{Im} \int \partial_j^2 \overline{\varphi u}^h|u^h|^{2\sigma} u.
$$

Notice that it is sensible that the right hand side is zero when $\lambda = 0$; from the commutation relation (3.1), the $L^2$-norm of $J^h(t)u^h$ is conserved when $\lambda = 0$, since $J^h(t)u^h$ then solves a linear Schrödinger equation.

Finally, the first part of Lemma 3.1 follows from the identity,

$$
\frac{d}{dt} \|u^h(t)\|_{L^{2\sigma+2}}^2 = -\hbar(\sigma + 1) \text{Im} \int |u|^{2\sigma} \Delta u.
$$

The second part of Lemma 3.1 follows from the relation $E^h_1 + E^h_2 = E^h = \text{cst}$. □

As an application of this lemma, we can prove wave collapse when $E^h_1(0) < 0$.

**Proposition 3.2** Let $u^h_0 \in \Sigma$, and if $n \geq 3$, assume $\sigma < 2/(n - 2)$. Assume that the nonlinearity is attractive ($\lambda < 0$) and $\sigma \geq 2/n$. Assume that

$$
\frac{1}{2} \|\hbar \nabla u^h_0\|_{L^2}^2 + \frac{\lambda}{\sigma + 1} \|u^h_0\|_{L^{2\sigma+2}}^{2\sigma+2} < 0.
$$

Then $u^h$ blows up at time $t^*_h \leq \pi/2\omega$,

$$
\exists t^*_h \leq \frac{\pi}{2\omega}, \quad \lim_{t \to t^*_h} \|\nabla x u^h(t)\|_{L^2} = \infty, \quad \text{and} \quad \lim_{t \to t^*_h} \|u^h(t)\|_{L^\infty} = \infty.
$$

6
Proof. From our assumptions, if $u^h \in C([0, T]; \Sigma)$ with $T \leq \pi/2\omega$,

$$E^h_1(0) = E^h - \frac{1}{2} \|\omega xu^h_0\|_{L^2}^2 < 0,$$
and for all $t \in [0, T]$, $\frac{dE^h_1}{dt} \leq 0$. (3.2)

On the other hand, $E^h_1$ can be written as,

$$E^h_1(t) = -\frac{1}{2} \cos(2\omega t) \|\omega xu^h(t, x)\|_{L^2}^2 + E^h \cos^2(\omega t)
+ \frac{\omega^2}{2} \sin(2\omega t) \Im \int (\overline{u^h} \cdot \nabla_x u^h).
$$

In particular, Cauchy-Schwarz inequality yields,

$$E^h_1(t) \geq -\frac{1}{2} \cos(2\omega t) \|\omega xu^h(t, x)\|_{L^2}^2 + E^h \cos^2(\omega t)
- \frac{1}{2} \sin(2\omega t) \|\nabla_x u^h(t)\|_{L^2} \|\nabla_x u^h(t)\|_{L^2}.
$$

So long as $\nabla_x u^h$ remains bounded in $L^2$, so does $xu^h$. This follows from the conservations of mass and energy, along with Gagliardo-Nirenberg inequality.

Assume $u^h \in C([0, \pi/2\omega]; \Sigma)$. Then letting $t$ go to $\pi/2\omega$ yields

$$E_1(\frac{\pi}{2\omega}) \geq \frac{1}{2} \left\|\omega xu^h(\frac{\pi}{2\omega}, x)\right\|_{L^2}^2,
$$
which is impossible from (3.2). Thus, there exists $t^*_h \leq \pi/2\omega$ such that

$$\lim_{t \to t^*_h} \|\nabla_x u^h(t)\|_{L^2} = \infty.
$$

From the conservation of energy,

$$\lim_{t \to t^*_h} \|u^h(t)\|_{L^{2\sigma+2}}^{2\sigma+2} = \infty,
$$
and the last part of the proposition stems from the conservation of mass. □

Remark. Notice that the blow up condition also reads

$$E^h < \frac{\omega^2}{2} \|xu^h_0\|_{L^2}^2.
$$

In term of energy, this means that the blow up occurs for higher values of the Hamiltonian than in the case with no potential, where the condition reads $E^h < 0$. This sufficient blow up condition varies continuously with $\omega \geq 0$.

**Corollary 3.3** Assume $\sigma \geq 2/n$, $\lambda < 0$. Let $v^0_0 \in \Sigma$. For $k \in \mathbb{R}$, define $u^h_0 = kv^0_0$. Then for $|k|$ sufficiently large, $u^h(t, x)$ collapses at time $t^*_h \leq \pi/2\omega$, as in Prop. 3.3.

Proof. For $|k|$ large, $E^h_1(0)$ becomes negative, and one can use the results of Prop. 3.3. □
4 Lower bound for the breaking time

In this section, we specify the dependence of the coupling constant $\lambda$ upon physical constants, and assume $\lambda = a\hbar^2$. We also assume that the nonlinearity is cubic, $\sigma = 1$. Physically, $a$ is the $s$-wave scattering length. It is negative in the case of Bose-Einstein condensation for $^7$Li system (\[2\], \[1\]). We prove that if the space dimension $n$ is two or three, then the nonlinear term $a\hbar^2|u\hbar|^2u\hbar$ in (1.1) is negligible in the semi-classical limit $\hbar \to 0$, up to some time depending on $\hbar$. This will give us a lower bound for the breaking time $t^\hbar_*$ when $\hbar \to 0$, and prove that

$$t^\hbar_* \to \frac{\pi}{2\omega},$$

As previously noticed, no blow up occurs for $\sigma = 1$ and $n = 1$, that is why we restrict our attention to $n = 2$ or 3. In the one-dimensional case, it has been proved in \[10\] that the right model for Bose-Einstein consists in replacing the cubic nonlinearity $|u\hbar|^2u\hbar$ by the quintic nonlinearity $|u\hbar|^4u\hbar$. This case is critical for global existence issues (see Prop. 2.2, Prop. 3.2), and is treated at the end of this section.

Define the function $v^\hbar$ as the solution of the linear Cauchy problem,

$$\begin{cases}
i\hbar\partial_t v^\hbar + \frac{\hbar^2}{2} \Delta v^\hbar = \frac{\omega^2}{2} x^2 v^\hbar, \\v^\hbar|_{t=0} = v^\hbar_0.
\end{cases}$$

(4.1)

4.1 The case $n = 2$ or 3

When $n = 2$ or 3, recall that we consider now the initial value problem for $u^\hbar$,

$$\begin{cases}
i\hbar\partial_t u^\hbar + \frac{\hbar^2}{2} \Delta u^\hbar = \frac{\omega^2}{2} x^2 u^\hbar + a\hbar^2 |u^\hbar|^2 u^\hbar, \\
u^\hbar|_{t=0} = u^\hbar_0,
\end{cases}$$

(4.2)

where $a$ is fixed. Our first result is independent of the sign of $a$.

**Proposition 4.1** Assume $n = 2$ or 3. Let $u^\hbar_0 \in \Sigma$ be such that $\|u^\hbar_0\|_{L^2}, \|\nabla_x u^\hbar_0\|_{L^2}$ and $\|xu^\hbar_0\|_{L^2}$ are bounded, uniformly with $\hbar \in [0,1]$. Then there exist $C, \Lambda, \alpha > 0$ and a finite real $q$ such that

$$\sup_{0 \leq t \leq \pi/2\omega - \Lambda \hbar^a} \|A^\hbar(t)(u^\hbar - v^\hbar)(t)\|_{L^2} \leq Ch^{1/2},$$

where $A^\hbar(t)$ can be either of the operators $Id$, $J^\hbar(t)$ or $H^\hbar(t)$.

**Remark.** Notice that the assumption $\|\nabla_x u^\hbar_0\|_{L^2}$ be bounded uniformly with $\hbar$ means that $u^\hbar_0$ has no $\hbar$-dependent oscillation.
From Lemma 1.1, (1.4), $J^h v^h$ and $H^h v^h$ solve a linear Schrödinger equation with harmonic potential, and in particular their $L^2$-norms are conserved with time,

$$\|J^h(t)v^h\|_{L^2} = \|\nabla_x u_0^h\|_{L^2}, \quad \|H^h(t)v^h\|_{L^2} = \|\omega_x u_0^h\|_{L^2}.$$ 

We can deduce the following,

**Corollary 4.2** Let $n = 2$ or $3$, and $u_0^h \in \Sigma$ be such that $\|u_0^h\|_{L^2}$, $\|\nabla_x u_0^h\|_{L^2}$ and $\|\omega_x u_0^h\|_{L^2}$ are bounded, uniformly with $h \in [0, 1]$. Assume $a < 0$ and

$$\|\nabla u_0^h\|_{L^2}^2 + a \|u_0^h\|_{L^4}^4 < 0.$$

Then there exists $\Lambda, \alpha > 0$ such that

$$\forall h \in [0, 1], \quad t_h^* \geq \frac{\pi}{2\omega} - \Lambda h^\alpha.$$

To prove Prop. 4.1, we first state precisely the Strichartz estimates we will use. Recall the classical definition (see e.g. [5]),

**Definition 1** A pair $(q, r)$ is admissible if $2 \leq r < \infty$ (resp. $2 \leq r \leq \infty$ if $n = 1$, $2 \leq r < \infty$ if $n = 2$) and

$$\frac{2}{q} = \delta(r) \equiv n \left( \frac{1}{2} - \frac{1}{r} \right).$$

Strichartz estimates provide mixed type estimates (that is, in spaces of the form $L^q_t(L^r_x)$, with $(q, r)$ admissible) of quantities involving the unitary group $U_0(t) = e^{i\frac{\omega}{2} \Delta}$.

A simple scaling argument yields similar estimates when $U_0$ is replaced with $e^{i\frac{\omega}{2} \Delta}$, with precise dependence upon the parameter $h$. As noticed in Sect. 2, the same Strichartz estimates hold when $e^{i\frac{\omega}{2} \Delta}$ is replaced by $U^h(t)$ (provided that only finite time intervals are involved).

**Proposition 4.3** Let $I$ be a interval contained in $[0, \pi/2\omega]$. For any admissible pair $(q, r)$, there exists $C_r$ such that for any $f \in L^2$,

$$\|U^h(t)f\|_{L^q(I; L^r)} \leq C_r h^{-1/q}\|f\|_{L^2}.$$

For any admissible pairs $(q_1, r_1)$ and $(q_2, r_2)$, there exists $C_{r_1, r_2}$ such that for $F = F(t, x)$,

$$\left\| \int_{I \cap (s \leq t)} U^h(t - s) F(s) ds \right\|_{L^{q_1}(I; L^{r_1})} \leq C_{r_1, r_2} h^{-1/q_1 - 1/q_2} \|F\|_{L^q(I; L^r)}.$$(4.3)

The above constants are independent of $I \subset [0, \pi/2\omega]$ and $h \in [0, 1]$. 

9
We now state two technical lemmas on which the proof of Prop. 4.1 relies.

**Lemma 4.4** If \( n = 2 \) or 3, there exists \( q, r, s, k \) satisfying

\[
\begin{align*}
\frac{1}{r'} &= 1 + 2s, \\
\frac{1}{q'} &= 1 + 2k, \\
\frac{1}{q} &= 1 + 2r,
\end{align*}
\]

and the additional conditions:

- The pair \((q, r)\) is admissible,
- \( 0 < \frac{1}{k} < \delta(s) < 1 \).

**Remark.** Notice that in particular, \( q \) is finite.

**Proof of Lemma 4.4.** With \( \delta(s) = 1 \), the first part of (4.4) becomes

\[
\delta(r) = \frac{n}{2} - 1,
\]

and this expression is less than 1 for \( n = 2 \) or 3. Still with \( \delta(s) = 1 \), the second part of (4.4) yields

\[
\frac{2}{k} = 2 - \frac{n}{2},
\]

which lies in \( ]0, 2[ \) for \( n = 2 \) or 3. By continuity, these conditions are still satisfied for \( \delta(s) \) close to 1 and \( \delta(s) < 1 \). \( \square \)

**Lemma 4.5** Assume \( n = 2 \) or 3, and let \( a^h \) solve

\[
\begin{align*}
\frac{i\hbar \partial_t a^h}{2} + \frac{\hbar^2}{2} \Delta a^h &= \frac{\omega^2}{2} x^2 a^h + \hbar^2 F^h(a^h) + \hbar^2 S^h, \\
a^h_{t=0} &= 0.
\end{align*}
\]

Assume that there exists \( C_0 > 0 \) such that for any \( t < \pi/2 \omega \),

\[
\left\| F^h(a^h)(t) \right\|_{L^2} \leq \frac{C_0}{\left( \frac{\pi}{2\omega} - t \right)^{2\delta(r)}} \left\| a^h(t) \right\|_{L^2}.
\]

Then there exist \( C, \Lambda > 0 \) independent of \( h \in [0, 1] \) such that

\[
\sup_{0 \leq t \leq \frac{\pi}{2\omega} - \Lambda h^\alpha} \left\| a^h(t) \right\|_{L^2} \leq C h^{1-1/2} \left\| S^h \right\|_{L^2'(0, \pi/2\omega - \Lambda h^\alpha; L^2')} ,
\]

where \( \alpha = \frac{1}{2\delta(r) - 1} \).
Proof of Lemma 4.3. From (4.3) with \( q_1 = q_2 = q \), for any \( t < \pi/2\omega \),

\[
\|a^h\|_{L^q(0,t;L^q)} \leq C t^{1/2} \|S^h\|_{L^q(0,t;L^q)} + C t^{1/2} \|F^h(a^h)\|_{L^q(0,t;L^q)}.
\] (4.5)

From our assumptions,

\[
\|F^h(a^h)\|_{L^q(0,t;L^q)} \leq \left\| \frac{C_0}{(\frac{\pi}{\omega} - s)^{2\delta(\omega)}} \|a^h(s)\|_{L^\infty} \right\|_{L^q(0,t)}.
\]

Apply Hölder’s inequality in time with (4.4),

\[
\|F^h(a^h)\|_{L^q(0,t;L^q)} \leq C \left( \int_0^t \frac{ds}{(\frac{\pi}{\omega} - s)^{\delta(\omega)}} \right)^{2/\delta} \|a^h\|_{L^q(0,t;L^q)} \leq C \frac{1}{(\frac{\pi}{\omega} - t)^{\delta(\omega)}} \|a^h\|_{L^q(0,t;L^q)}.
\]

Plugging this estimate into (4.5) yields, for \( t \leq \Delta t \),

\[
\|a^h\|_{L^q(0,t;L^q)} \leq C t^{1/2} \|S^h\|_{L^q(0,t;L^q)} + C t^{1/2} (\Lambda \alpha)^{2/\delta(\omega)} \|\alpha^h\|_{L^q(0,t;L^q)}.
\]

From (4.4), the power of \( h \) in the last term is canceled for \( \alpha = \frac{1}{2\delta(\omega)} - 1 \). If in addition \( \Lambda \) is sufficiently large, the last term of the above estimate can be absorbed by the left hand side (up to doubling the constant \( C \) for instance),

\[
\|a^h\|_{L^q(0,t;L^q)} \leq C t^{1/2} \|S^h\|_{L^q(0,t;L^q)}.
\]

The last three estimates also imply,

\[
\|F^h(a^h)\|_{L^q(0,t;L^q)} \leq C \|S^h\|_{L^q(0,t;L^q)}. \tag{4.6}
\]

The lemma then follows from Prop. 4.3, (4.3), with this time \( q_1 = \infty \) and \( q_2 = q \), along with (4.6).

Proof of Proposition 4.4. Denote \( w^h = u^h - v^h \) the remainder we want to assess. It solves the initial value problem,

\[
\begin{align*}
    i\hbar \partial_t w^h + \frac{\hbar^2}{2} \Delta w^h &= \frac{\omega^2}{2} x^2 w^h + a h^2 |u^h|^2 u^h, \\
    w^h_{|t=0} &= 0.
\end{align*}
\] (4.7)

We first want to apply Lemma 4.5 with \( a^h = w^h \). Since \( u^h = v^h + w^h \), we can take \( F^h(w^h) = a |u^h|^2 w^h \), \( S^h = a |u^h|^2 v^h \).
The point is now to control the $L^s$-norm of $u^h$. Notice that we can easily control the $L^2$-norm of $v^h$. Indeed, as we already emphasized, for any time $t$,

$$\|v^h(t)\|_{L^2} = \|u^h_0\|_{L^2}, \quad \|J^h(t)v^h\|_{L^2} = \|\nabla u^h_0\|_{L^2}.$$ 

From Lemma 1.1, (1.5), and Gagliardo-Nirenberg inequality, we also have,

$$\|v^h(t)\|_{L^2} \leq C \left| \frac{\cos(\omega t)}{\frac{\pi}{2\omega} - t} \right|^\delta,$$

Therefore, the assumptions of Prop. 4.1 imply that there exists $C_0 > 0$ independent of $h$ such that for any $t < \pi/2\omega$,

$$\|v^h(t)\|_{L^2} \leq \frac{C_0}{(\frac{\pi}{2\omega} - t)^\delta}.$$ 

Now $w^h_{t=0} = 0$ and we know from Prop. 2.1 that there exists $T^h$ such that the $\Sigma$-norm of $w^h$ is continuous on $[0, T^h]$. In particular, there exists $t^h > 0$ such that the following inequality,

$$\|w^h(t)\|_{L^2} \leq \frac{C_0}{(\frac{\pi}{2\omega} - t)^\delta}, \quad (4.8)$$ 

holds for $t \in [0, t^h]$. So long as (4.8) holds, we have obviously

$$\|u^h(t)\|_{L^2} \leq \frac{2C_0}{(\frac{\pi}{2\omega} - t)^\delta}.$$ 

This estimate allows us to apply Lemma 4.5, which yields, along with (4.4), and provided that $t \leq \pi/2\omega - \Lambda h$,

$$\|w^h\|_{L^\infty(0,t;L^2)} \leq CH^{-1/2}\|u^h\|^2_{L^2(0,t;L^2')},$$

$$\leq C H^{-1/2}\|u^h\|^2_{L^2(0,t;L^2)}\|v^h\|_{L^2(0,t;L^2)} \leq C\Lambda^{-2/3}h^{1/2}.$$ 

Now apply the operator $J^h$ to (1.1). From Lemma 1.1, $J^h u^h$ solves the same equation as $w^h$, with $|u^h|^2 u^h$ replaced by $J^h(|u^h|^2 u^h)$. From (1.8),

$$|J^h(t)(|u^h|^2 u^h)(t,x)| \leq 4|u^h(t,x)|^2|J^h(t)u^h(t,x)|.$$ 

Writing $J^h u^h = J^h v^h + J^h w^h$ and proceeding as above yields, so long as (4.8) holds,

$$\|J^h u^h\|_{L^\infty(0,t;L^2)} \leq C\Lambda^{-2/3}h^{1/2}. \quad (4.10)$$
Combining (4.9) and (4.10), along with Gagliardo-Nirenberg inequality, yields, so long as (4.8) holds,

\[ \|u^\h(t)\|_{L^\infty} \leq C \frac{1}{(\frac{\pi}{2\omega} - t)^{\delta_Q}} \Lambda^{-2/\omega} h^{1/q}. \] (4.11)

Possibly enlarging the value of \( \Lambda \), (4.11) shows that (4.8) remains valid up to time \( \pi/2\omega - \Lambda h^\alpha \). This proves Prop. 4.1 when \( A^\h(t) = Id \) or \( J^\h(t) \), from (4.9) and (4.10). The case \( A^\h(t) = H^\h(t) \) is then an easy by-product. \( \square \)

4.2 The case \( n = 1 \)

We finally prove the analogue of the above results in space dimension one. When \( n = 1 \), one can do without Strichartz estimates, and simply use the Sobolev embedding \( H^1 \subset L^\infty \),

\[ \|f\|_{L^\infty} \leq C \|f\|_{L^2}^{1/2} \|\partial_x f\|_{L^2}^{1/2}. \]

The wave \( u^\h \) now solves

\[ \begin{cases} 
  i\h \partial_t u^\h + \frac{\h^2}{2} \partial_x^2 u^\h = \frac{\omega^2}{2} x^2 u^\h + \h^2 |u^\h|^4 u^\h, \\
  u^\h_{|t=0} = u_0^\h. 
\end{cases} \] (4.12)

We start with the analogue of Lemma 4.5.

**Lemma 4.6** Assume \( n = 1 \), and let \( a^\h \) solve

\[ \begin{cases} 
  i\h \partial_t a^\h + \frac{\h^2}{2} \partial_x^2 a^\h = \frac{\omega^2}{2} x^2 a^\h + \h^2 F^\h(a^\h) + \h^2 S^\h, \\
  a^\h_{|t=0} = 0. 
\end{cases} \] (4.13)

Assume that there exists \( C_0 > 0 \) such that for any \( t < \pi/2\omega \),

\[ \|F^\h(a^\h)(t)\|_{L^2} \leq \frac{C_0}{(\frac{\pi}{2\omega} - t)^{\delta_Q}} \|a^\h(t)\|_{L^2}. \]

Then there exists \( C > 0 \) independent of \( \h \in [0, 1] \) such that for any \( \Lambda \geq 1 \),

\[ \sup_{0 \leq t \leq \pi/2\omega - \Lambda h} \|a^\h(t)\|_{L^2} \leq C \h \int_0^{\pi/2\omega - \Lambda h} \|S^\h(t)\|_{L^2} dt. \]

**Proof.** Multiply (4.13) by \( \overline{a^\h} \), integrate with respect to \( x \), and take the imaginary part of the result. This yields, from Cauchy-Schwarz inequality,

\[ \frac{d}{dt} \|a^\h(t)\|_{L^2} \leq 2\h \|F^\h(a^\h)(t)\|_{L^2} + 2\h \|S^\h(t)\|_{L^2} \]

\[ \leq \frac{2C_0\h}{(\frac{\pi}{2\omega} - t)^{\delta_Q}} \|a^\h(t)\|_{L^2} + 2\h \|S^\h(t)\|_{L^2}. \]
The lemma then follows from the Gronwall lemma. □

We can now prove the analogue of Prop. 4.1.

**Proposition 4.7** Assume \( n = 1 \). Let \( u^0_\hbar \in \Sigma \) be such that \( \| u^0_\hbar \|_{L^2}, \| \partial_x u^0_\hbar \|_{L^2} \) and \( \| xu^0_\hbar \|_{L^2} \) are bounded, uniformly with \( \hbar \in [0, 1] \). Then there exist \( C, \Lambda > 0 \) such that

\[
\sup_{0 \leq t \leq \pi/2\omega - \Lambda \hbar} \| A^\hbar(t)(u^\hbar - v^\hbar)(t) \|_{L^2} \leq C,
\]

where \( A^\hbar(t) \) can be either of the operators \( \text{Id}, J^\hbar(t) \) or \( H^\hbar(t) \).

**Proof.** The proof follows the proof of Prop. 4.1 very closely, if we take \( q = \infty, (s, k) = (\infty, 4) \). Denote \( w^\hbar = u^\hbar - v^\hbar \) the remainder we want to assess. It solves the initial value problem,

\[
\begin{aligned}
  i\hbar \partial_t w^\hbar + \frac{h^2}{2} \partial_x^2 w^\hbar &= \omega^2 x^2 w^\hbar + a\hbar^2 |u^\hbar|^4 u^\hbar, \\
  w^\hbar|_{t=0} &= 0.
\end{aligned}
\]

We first want to apply the above lemma with \( a^\hbar = w^\hbar \). Since \( u^\hbar = v^\hbar + w^\hbar \), we can take

\[
F^\hbar(w^\hbar) = a |u^\hbar|^4 w^\hbar, \quad S^\hbar = a |u^\hbar|^4 v^\hbar.
\]

The point is now to control the \( L^\infty \)-norm of \( u^\hbar \). Notice that we can easily control the \( L^\infty \) norm of \( v^\hbar \). Indeed, as we already emphasized, for any time \( t \),

\[
\| v^\hbar(t) \|_{L^2} = \| u^0_\hbar \|_{L^2}, \quad \| J^\hbar(t) v^\hbar \|_{L^2} = \| \partial_x u^0_\hbar \|_{L^2}.
\]

From Lemma 1.1, (1.5), and Gagliardo-Nirenberg inequality, we also have,

\[
\| v^\hbar(t) \|_{L^\infty} \leq \frac{C}{|\cos(\omega t)|^{1/2}} \| v^\hbar(t) \|_{L^2}^{1/2} \| J^\hbar(t) v^\hbar \|_{L^2}^{1/2} \leq \frac{C}{(\pi/2\omega - t)^{1/2}} \| v^\hbar(t) \|_{L^2}^{1/2} \| J^\hbar(t) v^\hbar \|_{L^2}^{1/2}.
\]

Therefore, the assumptions of Prop. 4.7 imply that there exists \( C_0 > 0 \) independent of \( \hbar \) such that for any \( t < \pi/2\omega \),

\[
\| v^\hbar(t) \|_{L^\infty} \leq \frac{C_0}{(\pi/2\omega - t)^{1/2}}.
\]

So long as

\[
\| w^\hbar(t) \|_{L^\infty} \leq \frac{C_0}{(\pi/2\omega - t)^{1/2}}, \tag{4.14}
\]

14
holds, we have obviously

\[ \| u^h(t) \|_{L^\infty} \leq \frac{2C_0}{(\frac{\pi}{2\omega} - t)^{1/2}}. \]

This estimate allows us to apply the above lemma, which yields, provided that \( t \leq \frac{\pi}{2\omega} - \Lambda h \),

\[ \| w^h \|_{L^\infty(0,t;L^2)} \leq C \| u^h \|_{L^4(0,t;L^\infty)} \] (4.15)

\[ \leq CA^{-1}. \]

Similarly, applying the operator \( J^h \) to (4.7) yields, so long as (4.8) holds,

\[ \| J^h w^h \|_{L^\infty(0,t;L^2)} \leq CA^{-1}. \] (4.16)

Combining (4.15) and (4.16), along with Gagliardo-Nirenberg inequality, yields, so long as (4.14) holds,

\[ \| w^h(t) \|_{L^\infty} \leq C \frac{1}{\left(\frac{\pi}{2\omega} - t\right)^{1/2}} \Lambda^{-1}. \] (4.17)

Taking \( \Lambda \) large enough, (4.17) shows that (4.14) remains valid up to time \( \frac{\pi}{2\omega} - \Lambda h \). This proves Prop. 4.7 when \( A^h(t) = \text{Id} \) or \( J^h(t) \), from (4.15) and (4.16). The case \( A^h(t) = H^h(t) \) is then an easy by-product. \( \square \)

**Corollary 4.8** Let \( n = 1 \), and \( u_0^h \in \Sigma \) be such that \( \| u_0^h \|_{L^2}, \| \partial_x u_0^h \|_{L^2} \) and \( \| x u_0^h \|_{L^2} \) are bounded, uniformly with \( h \in [0,1] \). Assume \( a < 0 \) and

\[ \frac{1}{2} \| \partial_x u_0^h \|_{L^2}^2 + \frac{a}{3} \| u_0^h \|_{L^6}^6 < 0. \]

Then there exists \( \Lambda > 0 \) such that

\[ \forall h \in [0,1], \quad \sigma^h \geq \frac{\pi}{2\omega} - \Lambda h. \]

Acknowledgment. The results in this paper were improved thanks to remarks made by T. Colin.

**References**

[1] C. C. Bradley, C. A. Sackett, and R. G. Hulet, *Bose-Einstein condensation of Lithium: Observation of limited condensate number*, Phys. Rev. Lett. 78 (1997), 985–989.
[2] C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, *Evidence of Bose-Einstein condensation in an atomic gas with attractive interactions*, Phys. Rev. Lett. **75** (1995), 1687–1690.

[3] R. Carles, *Équation de Schrödinger semi-classique avec potentiel harmonique et perturbation non-linéaire*, Séminaire X-EDP, 2001–2002 (Palaiseau), École Polytech., Exp. No. III, 12p. (French).

[4] ____, *Semi-classical Schrödinger equations with harmonic potential and nonlinear perturbation*, preprint, 2001, available at www.math.u-bordeaux.fr/~carles/English/publi.html.

[5] T. Cazenave, *An introduction to nonlinear Schrödinger equations*, Text. Met. Mat., vol. 26, Univ. Fed. Rio de Jan., 1993.

[6] C. Cohen-Tannoudji, *Cours du collège de france*, 1998–99, available at www.lkb.ens.fr/~laloe/PHYS/cours/college-de-france/.

[7] R.P. Feynman and A.R. Hibbs, *Quantum mechanics and path integrals (International Series in Pure and Applied Physics)*, Maidenhead, Berksh.: McGraw-Hill Publishing Company, Ltd., 365 p., 1965.

[8] J. Ginibre and G. Velo, *On a class of nonlinear Schrödinger equations. ii scattering theory, general case*, J. Funct. Anal. **32** (1979), 33–71.

[9] M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), no. 5, 955–980.

[10] E. B. Kolomeisky, T. J. Newman, J. P. Straley, and X. Qi, *Low-dimensional Bose liquids: Beyond the Gross-Pitaevskii approximation*, Phys. Rev. Lett. **85** (2000), no. 6, 1146–1149.

[11] Yong-Geun Oh, *Cauchy problem and Ehrenfest’s law of nonlinear Schrödinger equations with potentials*, J. Diff. Eq. **81** (1989), no. 2, 255–274.

[12] C. Sulem and P.-L. Sulem, *The nonlinear Schrödinger equation, self-focusing and wave collapse*, Springer-Verlag, New York, 1999.

[13] Takeya Tsurumi and Miki Wadati, *Stability of the D-dimensional nonlinear Schrödinger equation under confined potential*, J. Phys. Soc. Japan **68** (1999), no. 5, 1531–1536.

[14] Jian Zhang, *Stability of attractive Bose-Einstein condensates*, J. Statist. Phys. **101** (2000), no. 3-4, 731–746.