Homogenization of parabolic equations with a continuum of space and time scales.

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Abstract

This paper addresses the issue of homogenization of linear divergence form parabolic operators in situations where no ergodicity and no scale separation in time or space are available. Namely, we consider divergence form linear parabolic operators in $\Omega \subset \mathbb{R}^n$ with $L^\infty(\Omega \times (0,T))$-coefficients. It appears that the inverse operator maps the unit ball of $L^2(\Omega \times (0,T))$ into a space of functions which at small (time and space) scales are close in $H^1$-norm to a functional space of dimension $n$. It follows that once one has solved these equations at least $n$-times it is possible to homogenize them both in space and in time, reducing the number of operations counts necessary to obtain further solutions. In practice we show that under a Cordes type condition that the first order time derivatives and second order space derivatives of the solution of these operators with respect to harmonic co-ordinates are in $L^2$ (instead of $H^{-1}$ with Euclidean coordinates). If the medium is time independent then it is sufficient to solve $n$ times the associated elliptic equation in order to homogenize the parabolic equation.

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1 Introduction and main results

Let $\Omega$ be a bounded and convex domain of class $C^2$ of $\mathbb{R}^n$. Let $T > 0$. Consider the following parabolic PDE

\[ \begin{cases} 
\partial_t u = \text{div} \left( a(x,t) \nabla u(x,t) \right) + g & \text{in } \Omega \times (0,T) \\
u(x,t) = 0 & \text{for } (x,t) \in (\partial \Omega \times (0,T)) \cup (\Omega \cup \{t = 0\}).
\end{cases} \]  

(1.1)

Write $\Omega_T := \Omega \times (0,T)$. $g$ is a function in $L^2(\Omega_T)$. $(x,t) \to a(x,t)$ is a mapping from $\Omega_T$ into the space of symmetric positive definite matrices with entries in $L^\infty(\Omega_T)$. Assume $a$ to be uniformly elliptic on the closure of $\Omega_T$. This paper addresses the issue of the homogenization of (1.1) in space and time in situations where scale separation and ergodicity at small scales are not available (see [15], [43] and [3] for an introduction to classical homogenization theory). For that purpose, we will introduce in subsection 1.2 theorems establishing under Cordes type conditions the increase of regularity of solutions of (1.1) when derivatives are taken with respect to harmonic coordinates instead of Euclidean coordinates. In subsections 1.2, 1.3 these results will be used to homogenize (1.1) in space and in time. More precisely, assume $a$ to be written on a fine tessellation with $N$ degrees of freedom. If $a$ is time independent, then by solving $n$-times an elliptic boundary value-problem associated to (1.1) (at a cost of $O(N(\ln N)^{n+3})$ operations using the Hierarchical matrix method [11]) it is possible approximate the solutions of (1.1) by solving an homogenized operator with $N^\alpha$ degrees of freedom ($\alpha < 1$, $\alpha = 0.2$ for instance) or with a fixed number $M$ of degrees of freedom (numerical experiments given at the end of this paper have been conducted with $N = 16641$ and $M = 9$), this problem is of practical importance for oil extraction and reservoir modeling in geophysics. If $a$ is also characterized by a continuum of time scales, then the method presented here does not reduce the number of operation counts necessary to solve (1.1) only one time. However if one needs to solve (1.1) $K$ ($K > n$) times (with different right hand sides) then by solving (1.1) $n$ times it is possible to obtain an approximation of the solutions of (1.1) by solving an homogenized (in space and time) parabolic equation written on a coarse tessellation with coarse time steps.

1.1 Compensation phenomenon

Let $F$ be the solution of the following parabolic equation

\[ \begin{cases} 
\partial_t F = \text{div} \left( a(x,t) \nabla F(x,t) \right) & \text{in } \Omega_T \\
F(x,t) = x & \text{for } (x,t) \in (\partial \Omega \times (0,T)) \\
\text{div} \left( a(x,0) \nabla F(x,0) \right) = 0 & \text{in } \Omega.
\end{cases} \]  

(1.2)
By (1.2) we mean that \( F := (F_1, \ldots, F_n) \) is a \( n \)-dimensional vector field such that each of its entries satisfies

\[
\begin{aligned}
\partial_t F_i &= \text{div} (a(x,t) \nabla F_i(x,t)) \quad \text{in} \quad \Omega_T \\
F_i(x,t) &= x_i \quad \text{for} \quad (x,t) \in (\partial \Omega \times (0,T)) \\
\text{div} (a(x,0) \nabla F_i(x,0)) &= 0 \quad \text{in} \quad \Omega.
\end{aligned}
\] (1.3)

Observe that if \( a \) is time independent then \( F \) is the solution of an elliptic boundary value problem.

**Definition 1.1.** Write

\[
\sigma := \iota \nabla F a \nabla F.
\] (1.4)

Write \( \beta_\sigma \) the Cordes parameter associated to \( \sigma \) defined by

\[
\beta_\sigma := \text{esssup}_{(x,t) \in \Omega_T} \left( n - \frac{(\text{Trace}[\sigma])^2}{\text{Trace}[\iota \sigma \sigma]} \right).
\] (1.5)

Observe that since

\[
\beta_\sigma = \text{esssup}_{(x,t) \in \Omega_T} \left( n - \frac{\sum_{i=1}^n \lambda_{i,\sigma}(x,t))^2}{\sum_{i=1}^n \lambda_{i,\sigma}(x,t)} \right).
\] (1.6)

where \( (\lambda_{i,M}) \) denotes the eigenvalues of \( M \), \( \beta_\sigma \) is a measure of the anisotropy of \( \sigma \).

### 1.1.1 Time independent medium.

In this subsection we assume that \( a \) does not depend on time \( t \). Write for \( p \geq 2 \), \( W^{2,p}_D \) (\( D \) for Dirichlet boundary condition) the Banach space \( W^{2,p}_D(\Omega) \cap W^{1,p}_0(\Omega) \). Equip \( W^{2,p}_D(\Omega) \) with the norm

\[
\|v\|^2_{W^{2,p}_D(\Omega)} := \int_{\Omega} \left( \sum_{i,j} (\partial_i \partial_j v)^2 \right)^{\frac{p}{2}}.
\] (1.7)

Equip the space \( L^p(0,T,W^{2,p}_D(\Omega)) \) with the norm

\[
\|v\|^p_{L^p(0,T,W^{2,p}_D(\Omega))} = \int_0^T \int_{\Omega} \left( \sum_{i,j} (\partial_i \partial_j v)^2 \right)^{\frac{p}{2}} \, dx \, dt.
\] (1.8)

**Theorem 1.1.** Assume that \( \partial_t a \equiv 0 \), \( g \in L^2(\Omega_T), \) \( \Omega \) is convex, \( \beta_\sigma < 1 \) and \( (\text{Trace}[\sigma])^{\frac{n}{n-1}} \in L^\infty(\Omega) \) then \( u \circ F^{-1} \in L^2(0,T,W^{2,2}_D(\Omega)) \) and

\[
\|u \circ F^{-1}\|_{L^2(0,T,W^{2,2}_D(\Omega))} \leq \frac{C}{1 - \beta_\sigma^{\frac{1}{2}}} \|g\|_{L^2(\Omega_T)}.
\] (1.9)
Remark 1.1. The constant $C$ can be written
\[ C = \frac{C_n}{(\lambda_{\min}(a))^\frac{1}{4}} \|((\text{Trace}[\sigma]))^{\frac{n}{4}}\|_{L^\infty(\Omega)}. \]

Through this paper, we write
\[ \lambda_{\min}(a) := \inf_{(x,t) \in \Omega_T} \inf_{I \in \mathbb{R}^n, |I|=1} l.a(x,t).l. \quad (1.10) \]

Remark 1.2. According to theorem 1.1 although the second order derivatives of $u$ with respect to Euclidean coordinates are only in $L^2(0,T,H^{-1}(\Omega))$, they are in $L^2(\Omega_T)$ with respect to harmonic coordinates.

Remark 1.3. Observe that if $a$ is time independent then $F$ and $\sigma$ are time independent and $F$ is the solution of the following elliptic problem:
\[ \begin{cases} \text{div} \, a \nabla F = 0 & \text{in} \quad \Omega \\ F(x) = x & \text{on} \quad \partial \Omega. \end{cases} \quad (1.11) \]

In dimension one $F$ is trivially an homeomorphism. In dimension 2 this property follows from topological constraints \cite{2} (even with $a_{i,j} \in L^\infty(\Omega)$), \cite{6} (one can also deduce from \cite{6} that for $n = 2$, if $a$ is smooth then the conditions $\beta_\sigma < 1$ and $(\text{Trace}[\sigma])^{-1} \in L^\infty(\Omega)$ are satisfied). In dimension three and higher $F$ can be non-bijective even if $a$ is smooth, we refer to \cite{6} and \cite{21}, however in dimension 3 the assumption $(\text{Trace}[\sigma])^{\frac{n}{4}} \in L^\infty(\Omega_T)$ implies that $F$ is an homeomorphism. If $n \geq 4$ we need to assume that $F$ is an homeomorphism to prove the theorem.

Remark 1.4. In fact the condition $(\text{Trace}(\sigma))^{-1} \in L^p(\Omega_T)$ for $p < \infty$ depending on $n$ is sufficient to obtain theorem 1.1 and the following compensation theorems. For the sake of clarity this paper has been restricted to $(\text{Trace}(\sigma))^{-1} \in L^\infty(\Omega_T)$.

Remark 1.5. Write
\[ \mu_\sigma := \text{esssup}_{\Omega_T} \frac{\lambda_{\max}(\sigma)}{\lambda_{\min}(\sigma)}. \quad (1.12) \]

It is easy to check that $\mu_\sigma$ is bounded by an increasing function of $(1 - \beta_\sigma)^{-1}$ and in dimension two $\beta_\sigma < 1$ is equivalent to $\mu_\sigma < \infty$.

Remark 1.6. Theorem 1.1 has been called compensation phenomenon because the composition by $F^{-1}$ increases the regularity of $u \in L^2(0,T,H^{1}(\Omega))$. The choice of this name has been motivated by F. Murat and L. Tartar’s work on H-convergence \cite{53} which is also based on a regularization property called compensated compactness or div-curl lemma introduced in the 70’s by Murat and Tartar \cite{52}, \cite{67} (we also refer to \cite{25} for refinements of the div-curl lemma).
The compensation phenomena presented in this subsection can be observed numerically. In figure 1, the value of $a$ is set to be equal to 1 or 100 with probability $1/2$ on each triangle of a fine mesh characterized by 16641 nodes and 32768 triangles. (1.1) has been solved numerically on that mesh with $g = 1$. $u$, $u \circ F^{-1}$, $\partial x u$ and $\partial x(u \circ F^{-1})$ have been plotted at time $t = 1$ in figure 2.
Figure 2: $u$, $u \circ F^{-1}$, $\partial_x u$ and $\partial x(u \circ F^{-1})$ at time $t = 1$ for the time independent site percolating medium.
In situations where \( g \in L^\infty(0, T, L^2(\Omega)) \), \( \partial_t g \in L^2(0, T, H^{-1}(\Omega)) \) or \( g \in L^p(\Omega_T) \) with \( p > 2 \), one can obtain a higher regularity for \( u \circ F^{-1} \). This is the object of the following theorems.

**Theorem 1.2.** Assume that \( \Omega \) is convex, \( g \in L^\infty(0, T, L^2(\Omega)) \), \( \partial_t g \in L^2(0, T, H^{-1}(\Omega)) \), \( \partial_t a \equiv 0 \), \( \beta_\sigma < 1 \) and \( (\text{Trace}[\sigma])^{\frac{2}{\gamma}} \in L^\infty(\Omega_T) \) then for all \( t \in [0, T] \), \( u \circ F^{-1}(. , t) \in W_{D, t}^2(\Omega) \) and

\[
\| u \circ F^{-1}(. , t) \|_{W_{D, t}^2(\Omega)} \leq \frac{C}{1 - \beta_\sigma^\frac{1}{2}} \left( \| g \|_{L^\infty(0, T, L^2(\Omega))} + \| \partial_t g \|_{L^2(0, T, H^{-1}(\Omega))} \right).
\]

(1.13)

**Remark 1.7.** The constant \( C \) can be written

\[
C = \frac{C_{n, \Omega, p}}{\lambda_{\min}(a)} \left( \| (\text{Trace}[\sigma])^{\frac{2}{\gamma}} \|_{L^\infty(\Omega_T)} \right)^{\frac{1}{2}} \left( 1 + \frac{1}{\lambda_{\min}(a)} \right)^{\frac{1}{2}}.
\]

**Theorem 1.3.** Assume that \( \Omega \) is convex, \( g(\cdot, 0) \in L^2(\Omega) \), \( \partial_t g \in L^2(0, T, H^{-1}(\Omega)) \), \( g \in L^p(\Omega_T) \), \( \partial_t a \equiv 0 \), \( \beta_\sigma < 1 \) and \( (\text{Trace}[\sigma])^{\frac{2}{\gamma}} \in L^\infty(\Omega_T) \) then there exists a real number \( p_0 > 2 \) depending only on \( n, \Omega \) and \( \beta_\sigma \) such that for each \( p \) such that \( 2 \leq p < p_0 \) one has

\[
\| u \circ F^{-1} \|_{L^p(0, T, W_{D, t}^{2, p}(\Omega))} \leq \frac{C}{1 - \beta_\sigma^\frac{1}{2}} \left( \| g \|_{L^p(\Omega_T)} \right)
\]

\[
+ \| g(\cdot, 0) \|_{L^2(\Omega)} + \| \partial_t g \|_{L^2(0, T, H^{-1}(\Omega))}.
\]

(1.14)

**Remark 1.8.** The constant \( C \) can written

\[
C = \frac{C_{n, \Omega, p}}{\lambda_{\min}(a)} \left( \| (\text{Trace}[\sigma])^{\frac{2}{\gamma}} \|_{L^\infty(\Omega_T)} \right)^{\frac{1}{2}} \left( 1 + \frac{1}{\lambda_{\min}(a)} \right)^{\frac{1}{2}}.
\]

Write

\[
\| v \|_{C^\gamma(\Omega)} := \sup_{x, y \in \Omega, x \neq y} \frac{|v(x) - v(y)|}{|x - y|^{\gamma}}.
\]

(1.15)

**Theorem 1.4.** Assume that \( n \leq 2, \Omega \) is convex, \( g(\cdot, 0) \in L^2(\Omega) \), \( \partial_t g \in L^2(0, T, H^{-1}(\Omega)) \), \( g \in L^p(\Omega_T) \), \( \partial_t a \equiv 0 \), \( \beta_\sigma < 1 \), \( (\text{Trace}[\sigma])^{-1} \in L^\infty(\Omega_T) \) and \( g \in L^2[0, T; L^{p^*}(\Omega)] \) with \( 2 < p^* \). Then there exists \( p \in (2, p^*) \) and \( \gamma(p) > 0 \) such that

\[
\left( \int_0^T \| \nabla (u \circ F^{-1})(\cdot , t) \|_{C^\gamma(\Omega)}^2 dt \right)^{\frac{1}{2}} \leq \frac{C}{1 - \beta_\sigma^\frac{2}{p}} \left( \| g \|_{L^p(\Omega_T)} \right)
\]

\[
+ \| g(\cdot, 0) \|_{L^2(\Omega)} + \| \partial_t g \|_{L^2(0, T, H^{-1}(\Omega))}.
\]

(1.16)
Remark 1.9. The constant $C$ in (1.16) depends on $n$, $p$, $\Omega$, $\lambda_{\min}(a)$ and $\|\text{Trace}(\sigma)^{-1}\|_{L^\infty(\Omega_T)}$. It is easy to check that if $n = 1$ then the theorem is valid with $\gamma = 1/2$.

In the following theorems $\Omega$ is not assumed to be convex.

**Theorem 1.5.** Assume $n \geq 2$ and $\partial_t a \equiv 0$. Let $p > 2$. There exist a constant $C^* = C^*(n, \partial \Omega) > 0$ a real number $\gamma > 0$ depending only on $n$, $\Omega$ and $p$ such that if $\beta_\sigma < C^*$ then

$$
\left( \int_0^T \| \nabla (u \circ F^{-1})(\cdot, t) \|_{C^\gamma(\Omega)}^2 dt \right)^{\frac{1}{2}} \leq C \left( \| g \|_{L^p(\Omega_T)} + \| g(\cdot, 0) \|_{L^2(\Omega)} + \| \partial_t g \|_{L^2(0,T,H^{-1}(\Omega))} \right)
$$

(1.17)

Remark 1.10. The constant $C$ in (1.17) depends on $n$, $\gamma$, $\Omega$, $C^*$, $\lambda_{\min}(a)$ and $\|\text{Trace}(\sigma)^{-1}\|_{L^\infty(\Omega_T)}$.

It is easy to check that if $a = e(x)S(x,t)$ where $e$ is a time independent symmetric uniformly elliptic matrix with $L^\infty(\Omega)$ entries and $S$ is a regular uniformly positive function then the results given in this sub-section and the homogenization schemes of sub-section 1.2 remain valid with the time independent harmonic coordinates associated to $e$, i.e. solution of $- \text{div} e \nabla F = 0$.

1.1.2 Medium with a continuum of time scales.

In this subsection the entries of $a$ are merely in $L^\infty(\Omega_T)$. We need to introduce the following Cordes type condition.

**Condition 1.1.** We say that condition 1.1 is satisfied if and only if there exists $\delta \in (0, \infty)$ and $\epsilon > 0$ such that

$$
\text{esssup}_{\Omega_T} \frac{\delta^2 \text{Trace}[\sigma]^\prime + 1}{\delta \text{Trace}[\sigma] + 1} \leq \frac{1}{n + \epsilon}.
$$

(1.18)

Write

$$
z_\sigma := \text{esssup}_{\Omega_T} \frac{\text{Trace}[\sigma]^\prime}{\text{Trace}[\sigma]^2}.
$$

(1.19)

Observe that $z_\sigma$ is a measure of anisotropy of $\sigma$, in particular $1 \leq z_\sigma \leq n$ and $z_\sigma = 1$ if $\sigma$ is isotropic. Write

$$
y_\sigma := \| \text{Trace}[\sigma]\|_{L^\infty(\Omega_T)} \| (\text{Trace}[\sigma])^{-1}\|_{L^\infty(\Omega_T)}.
$$

(1.20)

**Proposition 1.1.** If $\| \text{Trace}[\sigma]\|_{L^\infty(\Omega_T)} < \infty$ and $\| (\text{Trace}[\sigma])^{-1}\|_{L^\infty(\Omega_T)} < \infty$ then condition 1.1 is satisfied with

$$
\delta := n \| (\text{Trace}[\sigma])^{-1}\|_{L^\infty(\Omega_T)}
$$

(1.21)

and with $\epsilon := \frac{2n y_\sigma - n}{2ny_\sigma}$ provided that $z_\sigma \leq 1 + \frac{\epsilon}{n}$.
Remark 1.11. Observe that in dimension one \(z_0 = 1\), thus for \(n = 1\) condition (1.1) is satisfied is \(\text{Trace}[\sigma] \in L^\infty(\Omega_T)\) and \((\text{Trace}[\sigma])^{-1} \in L^\infty(\Omega_T)\).

We have the following theorems

**Theorem 1.6.** Assume that \(\Omega\) is convex, and condition (1.4) is satisfied then \(u \circ F^{-1} \in L^2(0, T, W^{2,2}_D(\Omega))\), \(\partial_t (u \circ F^{-1}) \in L^2(\Omega_T)\) and

\[
\|u \circ F^{-1}\|_{L^2(0, T, W^{2,2}_D(\Omega))} + \|\partial_t (u \circ F^{-1})\|_{L^2(\Omega_T)} \leq C\|g\|_{L^2(\Omega_T)}^{1.22}
\]

where \(C\) depends on \(\Omega, n, \delta\) and \(\epsilon\).

**Remark 1.12.** According to theorem 1.6 although the second order space derivatives and first order time derivatives of \(u\) with respect to Euclidean coordinates are only in \(L^2(0, T, H^{-1}(\Omega))\), they are in \(L^2(\Omega_T)\) with respect to harmonic coordinates.

Similarly we obtain the following theorems in situations where \(g \in L^p(\Omega_T)\) with \(p > 2\).

**Theorem 1.7.** Assume that \(\Omega\) is convex, and condition (1.4) is satisfied then there exists a number \(p_0 > 2\) depending on \(n, \Omega, \epsilon\) such that for \(p \in (2, p_0)\), \(u \circ F^{-1} \in L^p(0, T, W^{2,2}_D(\Omega))\), \(\partial_t (u \circ F^{-1}) \in L^p(\Omega_T)\) and

\[
\|u \circ F^{-1}\|_{L^p(0, T, W^{2,2}_D(\Omega))} + \|\partial_t (u \circ F^{-1})\|_{L^p(\Omega_T)} \leq C\|g\|_{L^p(\Omega_T)}^{1.23}
\]

where \(C\) depends on \(\Omega, n, \delta\) and \(\epsilon\).

**Theorem 1.8.** Assume that \(\Omega\) is convex, and condition (1.4) is satisfied then there exists a number \(\alpha_0 > 2\) depending on \(n, \Omega, \epsilon\) such that for \(\alpha \in (0, \alpha_0)\), \(\nabla (u \circ F^{-1}) \in L^2(0, T, C^\alpha(\Omega))\) and

\[
\|\nabla (u \circ F^{-1})(., t)\|_{L^2(0, T, C^\alpha(\Omega))} \leq C\|g\|_{L^p(\Omega_T)}^{1.24}
\]

where \(C\) depends on \(\Omega, \delta, n,\) and \(\epsilon\).

These compensation phenomena can be observed numerically. We consider in dimension \(n = 2\),

\[
a(x, y, t) = \frac{1}{6} \left( \sum_{i=1}^{5} \frac{1.1 + \sin(2\pi x'/\epsilon_i)}{1.1 + \sin(2\pi y'/\epsilon_i)} + \sin(4x'^2y'^2) + 1 \right)^{1.25}
\]

with \(x' = x + \sqrt{2}t, y' = y - \sqrt{2}t, \epsilon_1 = \frac{1}{7}, \epsilon_2 = \frac{1}{17}, \epsilon_3 = \frac{1}{27}, \epsilon_4 = \frac{1}{37}, \) and \(\epsilon_5 = \frac{1}{67}\). This medium has been plotted in figure 3 at time 0 (observe that \(\lambda_{\text{max}}(a)/\lambda_{\text{min}}(a) \approx 100\)).

Figure 3 has been solved numerically on that mesh with \(g \equiv 1\) on the fine mesh characterized by 16641 nodes and 32768 triangles. Figure 4 shows \(\partial_x u\) and \(\partial_x (u \circ F^{-1})\) at time 0.3.

In figure 5 and 6, the value of \(x_0\) is set to \(x_0 := (0.75, -0.25)\) and the curves \(t \to u(x_0, t), u \circ F^{-1}(x_0, t), \nabla u(x_0, t), \nabla u \circ F^{-1}(x_0, t)\) are plotted from \(t = 0\) to \(t = 0.3\).
Figure 3: $a$ at time 0.

Figure 4: $\partial_x u$ and $\partial_x (u \circ F^{-1})$ at time $t = 0.3$ for the Multi-scale Trigonometric time dependent Medium.
(a) $u$ and $u \circ F^{-1}$.

Figure 5: $t \rightarrow u(x_0, t), u \circ F^{-1}(x_0, t)$ from $t = 0$ to $t = 0.3$ with $x_0 := (0.75, -0.25)$

(b) $\partial_x u$ and $\partial_x(u \circ F^{-1})$.

Figure 6: $t \rightarrow \nabla u(x_0, t), \nabla u \circ F^{-1}(x_0, t)$ from $t = 0$ to $t = 0.3$ with $x_0 := (0.75, -0.25)$
1.2 Homogenization in space.

Let $X_h$ be a finite dimensional subspace of $H^1_0(\Omega) \cap W^{1,\infty}(\Omega)^1$ with the following approximation property: there exists a constant $C_X$ such that for all $f \in W^{2,2}_D(\Omega)$

$$\inf_{v \in X_h} \|f - v\|_{H^1_0(\Omega)} \leq C_X h \|f\|_{W^{2,2}_D(\Omega)}.$$

(1.26)

It is known and easy to check that the set of piecewise linear functions on a triangulation of $\Omega$ satisfies condition (1.26) provided that the length of the edges of the triangles are bounded by $h$ ($C_X$ in (1.26) being given by the aspect ratio of the triangles).

For media characterized by a continuum of time scales we will consider twice differentiable elements satisfying the following usual inverse inequalities (see section 1.7 of [29]): for $v \in X_h$,

$$\|v\|_{W^{2,2}_D(\Omega)} \leq C_X h^{-1} \|v\|_{H^1_0(\Omega)}.$$  

(1.27)

and

$$\|v\|_{H^1_0(\Omega)} \leq C_X h^{-1} \|v\|_{L^2(\Omega)}.$$  

(1.28)

In this paper we will use splines to ensure that condition (1.27) is satisfied (observe that it requires the quasi-uniformity of the (coarse) mesh, i.e. a bound on the aspect ratio of the (coarse) triangles).

For $t \in (0,T)$ let us define

$$V_h(t) := \{ \varphi \circ F(x,t) : \varphi \in X_h \}.$$  

(1.29)

Write $L^2(0,T;H^1_0(\Omega))$ the usual Sobolev space associated to the norm

$$\|v\|^2_{L^2(0,T;H^1_0(\Omega))} := \int_0^T \|v(\cdot,t)\|^2_{H^1_0(\Omega)} dt.$$  

(1.30)

Write $Y_T$ the subspace of $L^2(0,T;H^1_0(\Omega))$ such that for each $v \in Y_T$ and $t \in [0,T]$, $x \to v(x,t)$ belongs to $V_h(t)$.

Write $u_h$ the solution in $Y_T$ of the following system of ordinary differential equations:

$$\begin{cases}
(\psi, \partial_t u_h)_{L^2(\Omega)} + a[\psi, u_h] = (\psi, g)_{L^2(\Omega)} & \text{for all } t \in (0,T) \text{ and } \psi \in V_h(t) \\
u_h(x,0) = 0.
\end{cases}$$  

(1.31)

Write

$$a[v, w] := \int_\Omega \nabla v(x,t) a(x,t) \nabla w(x,t) dx.$$  

(1.32)
1.2.1 Time independent domain

We have the following theorem

**Theorem 1.9.** Assume that $\partial_t a \equiv 0$, $\Omega$ is convex, $\beta \sigma < 1$ and $(\text{Trace}[\sigma])^{-1} \in L^\infty(\Omega_T)$ then

$$
\|(u - u_h)(., T)\|_{L^2(\Omega)} + \|u - u_h\|_{L^2(0, T; H^1_0(\Omega))} \leq C h \|g\|_{L^2(\Omega_T)}. \tag{1.33}
$$

**Remark 1.13.** The constant $C$ depends on $C_X$, $n$, $\Omega$, $\lambda_{\min}(a)$ and $\|\text{Trace}[\sigma]^{-1}\|_{L^\infty(\Omega_T)}$. If $n \geq 5$ it also depends on $\|\text{Trace}[\sigma]\|_{L^\infty(\Omega_T)}$ and if $n = 1$ it also depends on $\lambda_{\max}(a)$.

1.2.2 Medium with a continuum of time scales.

**Theorem 1.10.** Assume that $\Omega$ is convex, and condition 1.1 is satisfied then

$$
\|(u - u_h)(T)\|_{L^2(\Omega)} + \|u - u_h\|_{L^2(0, T; H^1_0(\Omega))} \leq C h \|g\|_{L^2(\Omega_T)}. \tag{1.34}
$$

**Remark 1.14.** The constant $C$ depends on $C_X$, $n$, $\Omega$, $\delta$ and $\epsilon$, $\lambda_{\min}(a)$ and $\lambda_{\max}(a)$.

The system of ordinary differential equations is still characterized by a continuum of time scales in situations where the entries of $a$ merely belong to $L^\infty(\Omega_T)$. They need to be discretized (homogenized) in time in order to be solved numerically. This will be the object of the next subsection. Loosely speaking, although is associated to a fine tessellation and fine time steps, it is possible to approximate its operator on a coarse tessellation with coarse time steps.

1.3 Homogenization in space and time.

Let $M \in \mathbb{N}^*$. Let $(t_n = n \frac{T}{M})_{0 \leq n \leq M}$ be a discretization of $[0, T]$. Let $(\varphi_i)$ be a basis of $X_h$. Write $Z_T$ the subspace of $Y_T$ such that $w \in Z_T$ if and only if $w$ can be written

$$
w(x, t) = \sum_i c_i(t) \varphi_i(F(x, t)). \tag{1.35}
$$

and the functions $t \to c_i(t)$ are constants on each intervals $(t_n, t_{n+1}]$. Write $V$ the subspace of $Y_T$ such that its elements $\psi$ can be written

$$
\psi(x, t) = \sum_i d_i \varphi_i(F(x, t)). \tag{1.36}
$$

where the parameters $d_i$ are constants (on $[0, T]$). For $w \in Y_T$, define $w_n \in V$ by

$$
w_n(x, t) := \sum_i c_i(t_n) \varphi_i(F(x, t)). \tag{1.37}
$$
Write \( v \) the solution in \( Z_T \) of the following system of implicit ordinary
differential equations (such that \( v(x, 0) \equiv 0 \): for \( n \in \{0, \ldots, M - 1\} \) and \( \psi \in V \),
\[
(\psi(t_{n+1}), v_{n+1}(t_{n+1}))_{L^2(\Omega)} = (\psi(t_n), v_n(t_n))_{L^2(\Omega)} \\
+ \int_{t_n}^{t_{n+1}} \left((\partial_t \psi(t), v_{n+1}(t))_{L^2(\Omega)} \\
- a[\psi(t), v_{n+1}(t)] + (\psi(t), g(t))_{L^2(\Omega)}\right) dt.
\]
(1.38)

The following theorem shows the stability of the implicit scheme (1.38).

**Theorem 1.11.** Let \( v \in Z_T \) be the solution of (1.38). We have
\[
\|v(T)\|_{L^2(\Omega)} + \|v\|_{L^2(0, T, H^1_0(\Omega))} \leq C\|g\|_{L^2(\Omega_T)},
\]
(1.39)

**Remark 1.15.** The constant \( C \) depends on \( n, \Omega \) and \( \lambda_{\text{min}}(a) \).

The following theorem gives an error bound on the accuracy of time
discretization scheme (1.38) when \( a \) does not depend on time.

**Theorem 1.12.** Let \( v \in Z_T \) be the solution of (1.38) and \( u_h \) be the solution of (1.31). Assume that \( \partial_t a \equiv 0 \). We have
\[
\|(u_h - v)(T)\|_{L^2(\Omega)} + \|u_h - v\|_{L^2(0, T, H^1_0(\Omega))} \leq C|\Delta t| \\
\left(\|\partial_t g\|_{L^2(0, T, H^{-1}(\Omega))} + \|g(., 0)\|_{L^2(\Omega)}\right).
\]
(1.40)

**Remark 1.16.** The constant \( C \) depends on \( n, \Omega \) and \( \lambda_{\text{min}}(a) \).

The following theorem gives an error bound on the accuracy of the time
discretization scheme (1.38) when \( a \) has no bounded time derivatives.

**Theorem 1.13.** Assume that \( \Omega \) is convex, and condition 1.1 is satisfied.
Let \( v \in Z_T \) be the solution of (1.38) and \( u_h \) be the solution of (1.31), we have
\[
\|(u_h - v)(T)\|_{L^2(\Omega)} + \|u_h - v\|_{L^2(0, T, H^1_0(\Omega))} \leq C\frac{|\Delta t|}{h}\|g\|_{L^2(\Omega_T)}
\]
(1.41)

where \( C \) depends on \( \Omega, n, \delta, \epsilon, \lambda_{\text{min}}(a) \) and \( \lambda_{\text{max}}(a) \).

**Remark 1.17.** Observe that the accuracy of the time discretization scheme (1.38) requires that \(|\Delta t| << h\) when \( a \) has no bounded time derivatives.

We refer to section 3 for numerical experiments.
1.4 Literature and further remarks.

For early works on homogenization with random mixing coefficients we refer to [59], [45], [58], [61], [73], [46], [47], [36]. Papanicolaou and Varadhan [60] have considered a two-component Markov process \((x(t), y(t))\) where \(y(t)\) is rapidly varying (and is not assumed to be ergodic in dimension one) and enters in the coefficients of the stochastic process driving \(x(t)\). They have studied the convergence properties of \(x(t)\) as the fluctuations of \(y(t)\) becomes more rapid using the martingale approach to diffusion, developed by Stroock and Varadhan [65], [64], [66], [63].

The numerical homogenization method implemented in this paper is a finite element method. The idea of using oscillating tests functions can be back tracked to the work of Murat and Tartar on homogenization and H-convergence, we refer in particular to [68] and [53]. Those papers also contain convergence proofs for the finite element method in an abstract setting for a sequence of H-converging elliptic operators (recall that the framework of H-convergence is independent from ergodicity or scale separation assumptions and are based on the compactness of any sequence of solutions of 
\[-\text{div} \ a_\epsilon \nabla u_\epsilon = g \]
with uniformly bounded and elliptic conductivities \(a_\epsilon\), we also refer to the initial work of Spagnolo [62] for G-convergence).

The numerical implementation and practical application of oscillating test functions in numerical finite element homogenization have been called multi-scale finite element methods and have been studied by several authors [8], [28], [54], [11], [10], [34], [38], [5]. The work of Hou and Wu [11] has been a large source of inspiration in numerical applications (particularly for reservoir modeling in geophysics, we refer to [71], [48], [1] and [72] for recent developments) since it was leading to a coarse scale operator while keeping the fine scale structures of the solutions. With the method introduced by Hou and Wu, the construction of the base functions is decoupled from element to element leading to a scheme adapted to parallel computers. A proof of the convergence of the method is given in periodic settings when the size of the heterogeneities is smaller than the grid size and an oversampling technique is proposed to remove the so called cell resonance error [42] when the size of the heterogeneities is comparable to the grid size.

Allaire and Brizzi [4] have observed that multiscale finite element method with splines would have a higher accuracy and have introduced the composition rule (we also refer to [8]). In [55], it has been observed that if \(u\) is the solution of the divergence form elliptic equation

\[
\begin{align*}
- \text{div} \ (a(x) \nabla u(x)) &= g & \text{in } \Omega \\
\quad u &= 0 & \text{in } \partial \Omega.
\end{align*}
\] (1.42)
and $F$ are harmonic coordinates defined by

$$\begin{aligned}
\text{div} \ a \nabla F &= 0 \quad \text{in } \Omega \\
F(x) &= x \quad \text{on } \partial \Omega.
\end{aligned} \tag{1.43}$$

then under the Cordes type condition $\beta_{\sigma} < 1$ on $\sigma$ given by (1.4), one has

$$\|u \circ F^{-1}\|_{W^{2,p}(\Omega)} \leq C\|g\|_{L^p(\Omega)}. \tag{1.44}$$

It has been deduced from this compensation phenomenon that numerical homogenization methods based on oscillating finite elements can converge in the presence of a continuum of scales if one uses global harmonic coordinates to obtain the test functions instead of solutions of a local cell problem \[55\].

In dimension three and higher it has been known since the work of Fenchenko and Khruslov \[33\], \[44\] that the homogenization of divergence form elliptic operators $-\text{div} \ a \nabla u = g$ can lead to a non local homogenized operator if

the sequence of matrices $a_\epsilon$ is uniformly elliptic but with entries uniformly bounded only in $L^1(\Omega)$. From a numerical point of view this non-local effects imply that a nonlocal numerical homogenization method cannot be avoided to obtain accuracy. Hence in \[55\], it is shown that the accuracy of local methods depend on the aspect ratio of the triangles of the tessellation with respect to harmonic coordinates (which is not the case if one uses non local finite elements, we refer to \[55\] for further discussions on the apparition of non local effects in numerical homogenization). Recently Briane has shown \[20\] that this non-local effect is absent in dimension two in the H-convergence setting.

The phenomenon is similar here, however observe that if one has solved the initial parabolic equation at least $n$ times and those solutions are (locally) linearly independent it is also possible to use them as new coordinates for numerical homogenization. Observe that in dimension higher than three the harmonic coordinates are not always invertible, an idea to bypass this difficulty could be either to choose the change of coordinates locally and adaptively or to enrich the coordinates by writing down the initial equations as degenerate equations in a space of higher dimension \[69\], these points have not been explored. For divergence form elliptic equations, recall that fast methods based on hierarchical matrices$^2$ are available \[13\], \[9\], \[12\], \[10\], \[11\] for solving (1.42) and (1.43) in $O(N(\ln N)^n+3)$ operations ($N$ being the number of interior nodes of the fine mesh).

The issue of numerical homogenization partial differential equations with heterogeneous coefficients has received a great deal of attention and many methods have been proposed. A few of them are cited below.

$^2$As for the fast multipole method and the hierarchical multipole method designed by L. Greengard and V. Rokhlin \[39\], these methods are based on the singular value decomposition of operators Green’s function.
• Multi-scale finite volume methods [49].
• Heterogeneous Multi-scale Methods [27], [23].
• Wavelet based homogenization [37], [26], [24], [16], [7], [18].
• Residual free bubbles methods [19].
• Discontinuous enrichment methods [32], [31].
• Partition of Unity Methods [35].
• Energy Minimizing Multi-grid Methods [70].

Following the methods of [55], it is possible to implement a finite-volume method based on the compensation theorems given in this paper. The elements given in this paper contain the fine scale structure of $F$, as it has been done in [55], it is possible to approximate the initial parabolic operator by a homogenized parabolic operator associated to the coarse mesh (the test functions in this case would be piecewise linear on the coarse mesh and the approximation error associated to the homogenized operator would depend on the aspect ratio of the triangles of the coarse mesh in the metric induced by $F$).

Finally, in this paper $a$ has been assumed to be bounded and uniformly elliptic. Without these assumptions the diffusion associated to homogenized operator can be anomalously slow [14], [56] or fast (super-diffusive) [57]. If $a$ has an unbounded skew symmetric component, the homogenization of (1.1) can give rise to a degenerate operator [57].

2 Proofs

2.1 Compensation.

2.1.1 Time independent medium.

We will need the following lemmas. Let $A_T$ be the bilinear form on $L^2(0, T; H^1_0(\Omega))$ defined by

$$A_T[v, w] := \int_0^T a[v, w](t) \, dt$$  \hfill (2.1)

where

$$a[v, u](t) := \int_\Omega t \nabla v(x, t) a(x, t) \nabla u(x, t) \, dx.$$  \hfill (2.2)

We write $A_T[u] := A_T[u, u]$.

**Lemma 2.1.** We have

$$\|u(\cdot, T)\|_{L^2(\Omega)}^2 + A_T[u] \leq \frac{C_{n, \Omega}}{\lambda_{\min}(a)} \|g\|_{L^2(\Omega_T)}^2,$$  \hfill (2.3)
Proof. Multiplying (1.1) by $u$ and integrating with respect to time we obtain
that
\[
\frac{1}{2} \left\| u(\cdot, T) \right\|_{L^2(\Omega)}^2 + \mathcal{A}_T[u] = (u, g)_{L^2(\Omega_T)}.
\]  
(2.4)

Using Poincaré and Minkowskii inequalities leads us to (2.3).

Lemma 2.2. Assume $\partial_t a \equiv 0$. We have
\[
\left\| \partial_t u(\cdot, T) \right\|_{L^2(\Omega_T)}^2 + a[u(\cdot, T)] \leq \left\| g \right\|_{L^2(\Omega_T)}^2.
\]  
(2.5)

Proof. Multiplying (1.1) by $\partial_t u$ and integrating by parts we obtain that
\[
\left\| \partial_t u(\cdot, t) \right\|_{L^2(\Omega)}^2 + a[\partial_t u, u] = (\partial_t u, g)_{L^2(\Omega)}.
\]  
(2.6)

Observing that
\[
a[\partial_t u, u] = \frac{1}{2} \partial_t (a[u]) - \frac{1}{2} \int_{\Omega} \nabla u \partial_t a \nabla u.
\]  
(2.7)

we conclude by integration with respect to time and using Minkowski inequality.

Lemma 2.3. Assume $\partial_t a \equiv 0$. We have
\[
\left\| \partial_t u(\cdot, T) \right\|_{L^2(\Omega)}^2 + \mathcal{A}_T[\partial_t u] \leq \frac{C_n \Omega}{\lambda_{\text{min}}(a)} \left\| \partial_t g \right\|_{L^2(0,T,H^{-1}(\Omega))}^2 + \left\| g(\cdot, 0) \right\|_{L^2(\Omega)}^2.
\]  
(2.8)

Proof. We obtain from (1.1) that
\[
\partial_t^2 u = \text{div} \left( a(x,t) \nabla \partial_t u(x,t) \right) + \text{div} \left( \partial_t a(x,t) \nabla u(x,t) \right) + \partial_t g.
\]  
(2.9)

Multiplying (2.9) by $\partial_t u$ and integrating with respect to time we obtain that
\[
\frac{1}{2} \left\| \partial_t u(\cdot, T) \right\|_{L^2(\Omega)}^2 + \mathcal{A}_T[\partial_t u] = \int_0^T (\partial_t u, \partial_t g)_{L^2(\Omega)} \, dt - \int_0^T (\partial_t a)[\partial_t u, u] \, dt
\]  
\[+ \frac{1}{2} \left\| \partial_t u(\cdot, 0) \right\|_{L^2(\Omega)}^2.
\]  
(2.10)

We conclude by the $H^{-1}$-duality inequality and Minkowski inequality.

We now need a variation of Campanato’s result \cite{23} on non-divergence form elliptic operators. Let us write for a symmetric matrix $M$,
\[
\nu_M := \frac{\text{Trace}(M)}{\text{Trace}(tMM)}.
\]  
(2.11)
We consider the following Dirichlet problem:

\[ L_M v = f \]  \hspace{1cm} (2.12)

with \( L_M := \sum_{i,j=1}^{n} M_{ij}(x) \partial_i \partial_j \). The following theorems 2.1 and 2.2 are straightforward adaptations of theorem 1.2.1 of [51]. They are proven in [51] under the assumption that \( M \) is bounded and elliptic. It is easy to check that the conditions \( \beta_M < 1 \) and \( \nu_M \in L^\infty(\Omega) \) are sufficient for the validity of those theorems. We refer to [55] for that adaptation.

**Theorem 2.1.** Assume that \( \beta_M < 1, \nu_M \in L^\infty(\Omega) \) and \( \Omega \) is convex. Then if \( f \in L^2(\Omega) \) the Dirichlet problem (2.12) has a unique solution satisfying

\[ \|v\|_{W^{2,2}_D(\Omega)} \leq \frac{C}{1-\beta_M} \|\nu_M f\|_{L^2(\Omega)}. \]  \hspace{1cm} (2.13)

**Remark 2.1.** \( \beta_M \) is the Cordes parameter associated to \( M \).

**Theorem 2.2.** Assume that \( \beta_M < 1, \nu_M \in L^\infty(\Omega) \) and \( \Omega \) is convex. Then, there exists a real number \( p_0 > 2 \) depending only on \( n, \Omega \) and \( \beta_M \) such that for each \( f \in L^p(\Omega) \), \( 2 \leq p < p_0 \) the Dirichlet problem (2.12) has a unique solution satisfying

\[ \|v\|_{W^{2,p}_D(\Omega)} \leq \frac{C_{n,\Omega,p}}{1-\beta_M} \|\nu_M f\|_{L^p(\Omega)}. \]  \hspace{1cm} (2.14)

Let us now prove the compensation theorems. Choose

\[ M := \frac{\sigma}{|\text{det}(\nabla F)|^{\frac{1}{2}}} \circ F^{-1}. \]  \hspace{1cm} (2.15)

It is easy to check that \( \beta_{\sigma} < 1 \) implies that \( F \) is an homeomorphism from \( \Omega \) onto \( \Omega \), thus (2.15) is well defined. Moreover observe that \( \beta_M = \beta_{\sigma} \) and

\[ \|\nu_M\|_{L^\infty(\Omega_T)} \leq \frac{C_n}{(\lambda_{\min}(a))^{\frac{1}{2}}} \|(\text{Trace}[\sigma])^{\frac{n}{2}-1}\|_{L^\infty(\Omega_T)}. \]  \hspace{1cm} (2.16)

Fix \( t \in [0,T] \). Choose

\[ f := \frac{(\partial_t u - g)}{|\text{det}(\nabla F)|^{\frac{1}{2}}} \circ F^{-1}. \]  \hspace{1cm} (2.17)

Observe that by the change of variable \( y = F(x) \) one obtains that if \( \partial_t a \equiv 0 \) (which implies that \( F \) is time independent), \( \partial_t u \in L^2(\Omega) \) and \( g(.,t) \in L^2(\Omega) \) that \( f \in L^2(\Omega) \) and

\[ \|f\|_{L^2(\Omega)} \leq \|\partial_t u\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}. \]  \hspace{1cm} (2.18)
It follows from theorem 2.2 that there exists a unique \( v \in W^{2,2}_{D}(\Omega) \) satisfying

\[
\|v\|^2_{W^{2,2}_{D}(\Omega)} \leq \frac{C\|\nu M\|^2_{L^{\infty}(\Omega_T)}}{(1 - \beta_{\sigma}^2)^2} \left( \|\partial_t u\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Omega)} \right).
\] (2.19)

and the following equation

\[
\partial_t \hat{u}(y, t) = \sum_{i,j} \left( \sigma(F^{-1}(y, t), t) \right)_{i,j} \partial_t \partial_j y(y, t) + \hat{g}(y, t).
\] (2.20)

We use the notation \( \hat{g} := g \circ F^{-1} \) and \( \hat{u} := u \circ F^{-1} \). Using the change of variable \( y = F(x) \) and using the property \( \text{div} a \nabla F = 0 \) when \( \partial_t a \equiv 0 \) we obtain that (2.20) can be written

\[
\partial_t u = \text{div} \left( a \nabla (v \circ F) \right) + g.
\] (2.21)

If \( \partial_t u \in L^2(\Omega) \) and \( g(., t) \in L^2(\Omega) \) we can use the uniqueness property of the solution of the divergence form elliptic Dirichlet problem

\[
\text{div} \left( a \nabla u \right) = \partial_t u - g.
\] (2.22)

to obtain that \( v \circ F = u \). Thus using lemma 2.3 we have proven theorem 1.2. Moreover assume that \( g \in L^2(\Omega_T) \) and \( \partial_t u \in L^2(\Omega_T) \). It follows that for \( t \in [0, T] \) - \( B, g(., t) \in L^2(\Omega) \) and \( \partial_t u(., t) \in L^2(\Omega) \) where \( B \) is a subset of \( [0, T] \) of 0-Lebesgue measure. It follows from the previous arguments that on \( [0, T] \) - \( B, u \circ F^{-1}(., t) \in W^{2,2}_{D}(\Omega) \) and satisfies

\[
\|u \circ F^{-1}(., t)\|^2_{W^{2,2}_{D}(\Omega)} \leq \frac{C\|\nu M\|^2_{L^{\infty}(\Omega_T)}}{(1 - \beta_{\sigma}^2)^2} \left( \|\partial_t u(., t)\|^2_{L^2(\Omega)} + \|g(., t)\|^2_{L^2(\Omega)} \right).
\] (2.23)

Integrating (2.23) with respect to time we obtain that

\[
\|u \circ F^{-1}\|^2_{L^2(0, T, W^{2,2}_{D}(\Omega))} \leq \frac{C\|\nu M\|^2_{L^{\infty}(\Omega_T)}}{(1 - \beta_{\sigma}^2)^2} \left( \|\partial_t u\|^2_{L^2(\Omega_T)} + \|g\|^2_{L^2(\Omega_T)} \right).
\] (2.24)

Thus using lemma 2.2 we have obtained theorem 1.1.

Let us now prove theorem 1.4. Assume that there exists \( q_0 > 2 \) such that for \( 2 \leq p < q_0, \partial_t u \in L^p(\Omega_T) \) and \( g \in L^p(\Omega_T) \). Let us now apply theorem 2.2 with \( p < \min(p_0, q_0) \), \( M \) given by (2.15) and \( f \) given by (2.17). It follows that for \( t \in [0, T] - B \) (where \( B \) is a subset of \( [0, T] \) of 0-Lebesgue measure), \( g(., t) \in L^p(\Omega) \) and \( \partial_t u(., t) \in L^p(\Omega) \). We deduce from theorem
and the argumentation related to equation (2.22) that on $[0, T) - B$, $u \circ F^{-1}(., t) \in W^{2,p}_D(\Omega)$ and

$$
\|u \circ F^{-1}(., t)\|_{W^{2,p}_D(\Omega)}^p \leq \frac{C_{n,p,\Omega}\|\nu_M\|_{L^\infty(\Omega_T)^p}}{(1 - \beta_{\tilde{\sigma}}^p)}(\|\partial_t u(., t)\|_{L^p(\Omega_T)}^p + \|g(., t)\|_{L^p(\Omega_T)}^p).
$$

(2.25)

Integrating (2.25) with respect to time we obtain that $u \circ F^{-1} \in L^p(0, T, W^{2,p}_D(\Omega))$ and

$$
\|u \circ F^{-1}\|_{L^p(0, T, W^{2,p}_D(\Omega))} \leq C_{n,p,\Omega}\|\nu_M\|_{L^\infty(\Omega_T)^p}(1 - \beta_{\tilde{\sigma}}^p).
$$

(2.26)

It remains to show that under the assumptions of theorem 1.3, $\partial_t u \in L^p(\Omega_T)$.

In order to bound $\|\partial_t u(., t)\|_{L^p(\Omega)}$ we use general Sobolev inequalities (chapter 5.6 of [30]).

- If $n \geq 3$, write $p^* = 2n/(n - 2)$. We have for $2 < p \leq p^*$,

$$
(\int_{\Omega} (\partial_t u)^p \ dx)^{\frac{2}{p}} \leq C_{n,\Omega}(\int_{\Omega} (\partial_t u)^{p^*} \ dx)^{\frac{2}{p^*}}
$$

(2.27)

thus, using Gagliardo-Nirenberg-Sobolev inequality

$$
(\int_{\Omega} (\partial_t u)^p \ dx)^{\frac{2}{p}} \leq C_{n,p,\Omega}\frac{1}{\lambda_{\min}(a)}[\partial_t u].
$$

(2.28)

- If $n = 2$, we write for $(x_1, x_2, x_3) \in \Omega \times (0, 1)$, $v(x_1, x_2, x_3) := \partial_t u(x_1, x_2)$. Using Gagliardo-Nirenberg-Sobolev inequality in dimension three we obtain that for $2 < p \leq 6$

$$
(\int_{\Omega} (\partial_t u)^p \ dx)^{\frac{2}{p}} \leq C_{n,p,\Omega}(\int_{\Omega} (\nabla \partial_t u)^2 \ dx).
$$

(2.29)

Which leads us to (2.28).

- If $n = 1$ then using Morrey’s inequality we obtain that with $\gamma := 1/2$,\n
$$
\|\partial_t u\|_{C^{0,\gamma}(\Omega)}^2 \leq C_{\Omega}\frac{1}{\lambda_{\min}(a)}[\partial_t u].
$$

(2.30)

We conclude the proof of theorem 1.3 by using lemma 2.3.

We deduce theorem 1.4 from Morrey’s inequality and theorem 1.3.
Hölder continuity for $n \geq 3$ or non-convexity of $\Omega$. In this paragraph we will not assume $\Omega$ to be convex. Let $N^{p,\lambda}(\Omega)$ $(1 < p < \infty, 0 < \lambda < n)$ be the weighted Morrey space formed by the functions $v : \Omega \to \mathbb{R}$ such that $\|v\|_{N^{p,\lambda}(\Omega)} < \infty$ with

$$\|v\|_{N^{p,\lambda}(\Omega)} := \sup_{x_0 \in \Omega} \left( \int_{\Omega} |x-x_0|^{-\lambda} |v(x)|^p \right)^{\frac{1}{p}}. \quad (2.31)$$

To obtain the Hölder continuity of $u \circ F^{-1}$ in dimension $n \geq 3$ we use corollary 4.1 of [50]. We give the result of S. Leonardi below in a form adapted to our context. Consider the Dirichlet problem (2.12). We do not assume $\Omega$ to be bounded. We write $W^{2,p,\lambda}(\Omega)$ the functions in $W^2_D(\Omega)$ such that their second order derivatives belong to $N^{p,\lambda}(\Omega)$.

**Theorem 2.3.** There exist a constant $C^* = C^*(n,p,\lambda,\partial \Omega) > 0$ such that if $\beta_M < C^*$ and $f \in N^{p,\lambda}(\Omega)$ then the Dirichlet problem (2.12) has a unique solution in $W^{2,p,\lambda} \cap W^{1,p}_0(\Omega)$. Moreover, if $0 < \lambda < n < p$ then $\nabla v \in C^{\alpha}(\Omega)$ with $\alpha = 1 - \frac{n}{p}$ and

$$\|\nabla v\|_{C^{\alpha}(\Omega)} \leq \frac{C}{\lambda_{\min}(M)} \|f\|_{N^{p,\lambda}(\Omega)} \quad (2.32)$$

where $C = C(n,p,\lambda,\partial \Omega)$.

The proof of theorem 1.5 is an application of theorem 2.3. We just need to observe that from Hölder inequality we have for $0 < \epsilon < 0.5$

$$\|f\|_{N^{p,\epsilon}(\Omega)} \leq C_{n,p,\Omega,\epsilon} \|f\|_{L^p(1+\epsilon)(\Omega)}. \quad (2.33)$$

From this point the proof is similar to the proof of theorem 1.3.

### 2.1.2 Medium with a continuum of time scales.

We will need theorems 1.6.2 and 1.6.3 of [51]. For the sake of completeness we will remind those theorems below in version adapted to our framework. Consider the following parabolic problem:

$$\partial_t v = \sum_{i,j=1}^n M_{ij}(x) \partial_i \partial_j v + f. \quad (2.34)$$

We assume $M$ to be symmetric bounded and elliptic and $v = 0$ at $t = 0$ and on the boundary $\partial \Omega$. Write

$$\eta_M := \sup_{x \in \Omega_t} \frac{\text{Trace}[f^t M M] + 1}{(\text{Trace}[M] + 1)^2}. \quad (2.35)$$
and
\[ \alpha_M := \sup_{x \in \Omega_T} \frac{\text{Trace}[M] + 1}{\text{Trace}[\hat{\sigma} \hat{\sigma}] + 1}. \]  
(2.36)

Write for \( p \geq 2 \)
\[ S_p(\Omega_T) := \left\{ v \in L^p(0,T,W^{2,p}_D(\Omega)) ; \partial_t v \in L^p(\Omega_T) ; v(.,0) \equiv 0 \right\} \]  
(2.37)
and
\[ \|v\|_{S_p(\Omega_T)}^p := \int_{\Omega_T} \left( \sum_{i,j} \left( \partial_i \partial_j v \right)^2 + \left( \partial_t v \right)^2 \right)^{\frac{p}{2}} \, dy \, dt. \]  
(2.38)

**Theorem 2.4.** Assume \( \Omega \) to be convex and that there exists \( \epsilon \in (0,1) \) such that \( \eta_M \leq 1/(n+\epsilon) \), then for each \( f \in L^2(\Omega_T) \) the Cauchy-Dirichlet problem \( (2.34) \) admits a unique solution in \( S_2(\Omega_T) \) which satisfies the bound
\[ \|v\|_{S_2(\Omega_T)} \leq \frac{\alpha_M}{1 - \sqrt{1-\epsilon}} \|f\|_{L^2(\Omega_T)}. \]  
(2.39)

**Theorem 2.5.** Assume \( \Omega \) to be convex and that there exists \( \epsilon \in (0,1) \) such that \( \eta_M \leq 1/(n+\epsilon) \), then there exists a number \( p_0 > 2 \) depending on \( \Omega, n, \epsilon \) such that for each \( f \in L^{p_0}(\Omega_T) \) the Cauchy-Dirichlet problem \( (2.34) \) admits a unique solution in \( S_{p_0}(\Omega_T) \) which satisfies the bound
\[ \|v\|_{S_{p_0}(\Omega_T)} \leq C_{p_0} \frac{\alpha_M}{1 - \sqrt{1-\epsilon}} \|f\|_{L^{p_0}(\Omega_T)}. \]  
(2.40)

**Remark 2.2.** In fact theorem 1.6.3 of [51] is written with \( 1 - C(p)\sqrt{1-\epsilon} \) in the denominator of \( (2.40) \) but it is easy to modify it to obtain \( (2.40) \) by lowering the value of \( p_0 \) changing the value of \( C_p \).

Let \( \delta > 0. \) Let us now apply theorem \( 2.4 \) on \( [0,T/\delta] \) with
\[ M := \delta \sigma \circ F^{-1}(y,\delta t) \]  
(2.41)
and
\[ f := \delta (g \circ F^{-1})(y,\delta t). \]  
(2.42)

Observe that if condition \( 1.1 \) is satisfied then \( F \) is an homeomorphism and \( M \) is well defined, bounded and elliptic. Moreover \( \eta_M < \infty \) and \( \alpha_M < \infty \) since
\[ \text{esssup}_{\Omega_T} \frac{\text{Trace}[M] + 1}{\text{Trace}[\sigma] + 1} = \text{esssup}_{\Omega_T} \frac{\delta^2 \text{Trace}[\sigma] + 1}{(\delta \text{Trace}[\sigma] + 1)^2}. \]  
(2.43)

It follows that the following equation admits a unique solution in \( S_2(\Omega\frac{T}{\delta}) \).
\[ \partial_t w(y, t) = \sum_{i,j} M_{i,j}(y, t) \partial_i \partial_j w(y, t) + k(y, t) \]  
(2.44)
with \( k(y,t) = \delta \hat{g}(y,\delta t) \). And we have
\[
\int_0^T \int_\Omega \left( (\partial_t w)^2 + \sum_{i,j} (\partial_i \partial_j w)^2 \right) dy dt \leq \frac{C}{(1 - \sqrt[4]{1 - \epsilon})^2} \| f \|_{L^2(\Omega_T)}.
\] (2.45)

Using the change of variables \( t \to \delta t \) and writing
\[
w(y, t) := v(y, \delta t).
\] (2.46)
we obtain that \( v \) satisfies the following equation on \( \Omega_T \)
\[
\partial_t v(y, t) = \sum_{i,j} \left( \sigma(F^{-1}(y, t), t) \right)_{i,j} \partial_i \partial_j v(y, t) + \hat{g}(y, t).
\] (2.47)

Using the change of variable \( y = F(x) \) and observing that \( \partial_t F = \text{div} \ a \nabla F \)
we obtain that \( v \circ F \) satisfies
\[
\partial_t (v \circ F) = \text{div} \ (a \nabla (v \circ F)) + g.
\] (2.48)

It follows from the uniqueness of the solution of (2.48) that \( u = v \circ F \). In resume we have obtained theorem 1.6 (we use lemma 2.4 to control the constants). The proof of 1.7 is similar and based on theorem 2.5. The proof of 1.8 follows from 1.7 and Morrey’s inequality.

Let us now prove proposition 1.1. Write \( x = \text{Trace}[\sigma] \) and \( z = n \text{Trace}[\sigma] \) (observe that \( 1 \leq z \leq n \)). It is easy to check that condition 1.4 can be written
\[
-\delta^2 x^2 \left( \frac{\epsilon + n}{n} z - 1 \right) + 2x\delta - (n + \epsilon - 1) \geq 0.
\] (2.49)
Choose \( \delta = n \| (\text{Trace}[\sigma])^{-1} \|_{L^\infty(\Omega_T)} \). Observing that \( \delta x \geq n \) and \( \delta x \leq ny_\sigma \) it is easy to conclude the proof of proposition 1.1. Similarly obtains the following lemma by straightforward computation from equation (2.49).

**Lemma 2.4.** Assume that condition 1.7 is satisfied then \( \mu_\sigma < C(n, \epsilon, \delta) \)
\[
\| (\text{Trace}[\sigma])^{-1} \|_{L^\infty(\Omega_T)} \leq C(n, \epsilon, \delta)
\] (2.50)

and
\[
\| \text{Trace}[\sigma] \|_{L^\infty(\Omega_T)} \leq C(n, \epsilon, \delta).
\] (2.51)

### 2.2 Convergence of the finite element method.

Write \( R_h \) the projection operator mapping \( L^2(0, T; H^1_0(\Omega)) \) onto \( Y_T \) defined by: for all \( v \in Y_T \)
\[
A_T[v, u - R_h u] = 0.
\] (2.52)
Write \( \rho := u - R_h u \) and \( \theta := R_h u - u_h \).
Lemma 2.5.

\[ \frac{1}{2} \|(u - u_h)(T)\|_{L^2(\Omega)}^2 + \mathcal{A}_T[u - u_h] = \int_{\Omega_T} \rho \partial_t (u - u_h) + \mathcal{A}_T \rho, u - u_h \cdot \mathcal{T} \cdot (u - u_h) \cdot \mathcal{T}. \] (2.53)

Proof. Subtracting \( (1.1) \) (integrated against \( \psi \)) and \( (1.31) \) we obtain that

\[ (\psi, \partial_t (u - u_h)) + a[\psi, u - u_h] = 0 \quad \text{for all } \psi \in V_h(t). \] (2.54)

Integrating by parts with respect to time we deduce that

\[ (\psi, (u - u_h)(., t)) + a[\psi, u - u_h] = \int_{\Omega_t} \partial_t \psi (u - u_h). \] (2.55)

Taking \( \psi = \theta \) in \( (2.55) \) we deduce that

\[ \| (u - u_h)(., t) \|_{L^2(\Omega)}^2 + \mathcal{A}_T[u - u_h] = \int_{\Omega_t} \partial_t \theta (u - u_h) + (\rho, (u - u_h)(., t)) \]

\[ + \mathcal{A}_T \rho, u - u_h \cdot \mathcal{T} \cdot (u - u_h) \cdot \mathcal{T}. \] (2.56)

Observing that

\[ \int_0^t (\partial_t \theta, u - u_h) + (\rho, (u - u_h)(., t)) = \frac{1}{2} \|(u - u_h)(., t)\|_{L^2(\Omega)}^2 + \int_0^t (\rho, \partial_t (u - u_h)). \] (2.57)

we deduce the lemma.

\[ \Box \]

2.2.1 Time independent medium.

Lemma 2.6.

\[ \| (u - u_h)(T) \|_{L^2(\Omega)}^2 + \mathcal{A}_T[u - u_h] \leq 2 \left( \| \rho \|_{L^2(\Omega_T)} \| \partial_t u - \partial_t u_h \|_{L^2(\Omega_T)} + \mathcal{A}_T \rho \right). \] (2.58)

Proof. Lemma 2.6 is a straightforward consequence of lemma 2.5 and Cauchy-Schwartz and Minkowski inequalities.

\[ \Box \]

Lemma 2.7. We have

\[ \| u_h(., T) \|_{L^2(\Omega)}^2 + \mathcal{A}_T[u_h] \leq \frac{C_{n, \Omega}}{\lambda_{\min}(a)} \| g \|_{L^2(\Omega_T)}^2. \] (2.59)

Proof. Taking \( \psi = u_h \) in \( (1.31) \) and integrating with respect to time we obtain that

\[ \frac{1}{2} \| u_h(., T) \|_{L^2(\Omega)}^2 + \mathcal{A}_T[u_h] = (u_h, g)_{L^2(\Omega_T)}. \] (2.60)

Using Poincaré and Minkowski inequalities leads us to \( (2.59) \).
Lemma 2.8. Assume $\partial t a \equiv 0$. We have
\[ \|\partial_t u_h\|_{L^2(\Omega_T)}^2 + a[u_h(\cdot, T)] \leq \|g\|^2_{L^2(\Omega_T)}. \tag{2.61} \]

Proof. The proof is similar to lemma 2.2. We need to take $\psi = \partial_t u_h$ in (1.31).

Let $t \in [0, T]$ and $v \in H^1_0(\Omega)$, we will write $R_{h,t}v(\cdot, t)$ the solution of:
\[ \int_{\Omega} t \nabla \psi a(x, t)(\psi, v - R_{h,t}v) \, dx = 0 \text{ for all } \psi \in V_h(t). \tag{2.62} \]

We will need the following lemma,

Lemma 2.9. Assume the mapping $x \rightarrow F(x, t)$ to be invertible, then for $v \in H^1_0(\Omega)$ we have
\begin{itemize}
  \item For $n = 1$,
    \[ (a[v - R_{h,t}v])^{\frac{1}{2}} \leq C_X h \|v \circ F^{-1}(\cdot, t)\|_{W^{2,2}_D} \|a \nabla F\|_{L^\infty(\Omega_T)}^{\frac{1}{2}}. \tag{2.63} \]
  \item For $n \geq 2$,
    \[ (a[v - R_{h,t}v])^{\frac{1}{2}} \leq C_X h \|v \circ F^{-1}(\cdot, t)\|_{W^{2,2}_D} \times C_n \mu_\sigma^{\frac{n-1}{4}} \|(\text{Trace}[\sigma])^{-1}\|_{L^\infty(\Omega_T)}^{\frac{n-2}{4}}. \tag{2.64} \]
\end{itemize}

Remark 2.3. Recall that $\mu_\sigma$ is given by equation (1.12) and it is easy to check that $\mu_\sigma$ is bounded by an increasing function of $(1 - \beta_\sigma)^{-1}$.

Proof. Using the change of coordinates $y = F(x, t)$ we obtain that (we write $\tilde{\psi} := v \circ F^{-1}$)
\[ a[v] = Q[\tilde{\psi}] \tag{2.65} \]
with
\[ Q[w] := \int_{\Omega} t \nabla w(y, t) Q(y, t) \nabla w(y, t) \, dy \tag{2.66} \]
and
\[ Q(y, t) := \frac{\sigma}{\det(\nabla F) \circ F^{-1}}. \tag{2.67} \]

Using the definition of $R_{h,t}v$ we obtain that
\[ Q[\tilde{\psi} - R_{h,t}v] = \inf_{\varphi \in \mathcal{X}_h} Q[\tilde{\psi} - \varphi]. \tag{2.68} \]

Using property (1.26) we obtain that
\[ Q[\tilde{\psi} - R_{h,t}v] \leq \lambda_{\max}(Q) C_X h^2 \|\tilde{\psi}\|^2_{W^{2,2}_D(T)}. \tag{2.69} \]

It is easy to obtain that
\* \( n = 1. \)
\[
\lambda_{\text{max}}(Q) \leq \|a \nabla F\|_{L^\infty(\Omega_T)}. \tag{2.70}
\]

\* \( n \geq 2. \)
\[
\lambda_{\text{max}}(Q) \leq C_n \mu_{\sigma}^{\frac{n-1}{2}} \left\| \text{Trace}[\sigma] \right\|_{L^\infty(\Omega_T)}^{-1} \| \frac{n-1}{2} \|. \tag{2.71}
\]

**Lemma 2.10.** Assume that \( \partial_t a \equiv 0, \Omega \) is convex, \( \beta_{\sigma} < 1 \) and \( (\text{Trace}[\sigma])^{-1} \in L^\infty(\Omega_T) \) then
\[
\mathcal{A}_T[\rho] \leq C h^2 \| g \|^2_{L^2(\Omega_T)}. \tag{2.72}
\]

**Remark 2.4.** The constant \( C \) depends on \( C_X, n, \Omega, \lambda_{\min}(a) \) and \( \left\| (\text{Trace}[\sigma])^{-1} \right\|_{L^\infty(\Omega_T)} \). If \( n \geq 5 \) it also depends on \( \left\| \text{Trace}[\sigma] \right\|_{L^\infty(\Omega_T)} \) and if \( n = 1 \) it also depends on \( \lambda_{\text{max}}(a) \).

**Proof.** The proof is a straightforward application of lemma 2.9 and theorem 1.1. Observe that in dimension one \( a \nabla F = (\int_\Omega a^{-1})^{-1} \).

**Lemma 2.11.** Assume that \( \partial_t a \equiv 0, \Omega \) is convex, \( \beta_{\sigma} < 1 \) and \( (\text{Trace}[\sigma])^{-1} \in L^\infty(\Omega_T) \) then
\[
\| \rho \|_{L^2(\Omega_T)} \leq C h^2 \| g \|_{L^2(\Omega_T)}. \tag{2.73}
\]

**Remark 2.5.** The constant \( C \) depends on \( C_X, n, \Omega, \lambda_{\min}(a) \) and \( \left\| (\text{Trace}[\sigma])^{-1} \right\|_{L^\infty(\Omega_T)} \). If \( n \geq 5 \) it also depends on \( \left\| \text{Trace}[\sigma] \right\|_{L^\infty(\Omega_T)} \) and if \( n = 1 \) it also depends on \( \lambda_{\text{max}}(a) \).

**Proof.** The proof follows from standard duality techniques (see for instance theorem 5.7.6 of [17]). We choose \( v \in L^2(0, T, H^1_0(\Omega)) \) to be the solution of the following linear problem: for all \( w \in L^2(0, T, H^1_0(\Omega)) \)
\[
\mathcal{A}_T[w, v] = (w, \rho)_{L^2(\Omega_T)}. \tag{2.74}
\]
Choosing \( w = \rho \) in equation (2.74) we deduce that
\[
\| \rho \|^2_{L^2(\Omega_T)} = \mathcal{A}_T[\rho, v - \mathcal{R}_{h,t}v]. \tag{2.75}
\]
Using Cauchy Schwartz inequality we deduce that
\[
\| \rho \|^2_{L^2(\Omega_T)} \leq \left( \mathcal{A}_T[\rho] \right)^{\frac{1}{2}} \left( \mathcal{A}_T[v - \mathcal{R}_{h,t}v] \right)^{\frac{1}{2}}. \tag{2.76}
\]
Using theorem 1.1 we obtain that
\[
\| \mathcal{R}_{h,t} \|_{L^2(0,T,W^{1,2}_D(\Omega))} \leq C \| \rho \|_{L^2(\Omega_T)}. \tag{2.77}
\]
Using lemma 2.9 we obtain that
\[
\left( \mathcal{A}_T[v - \mathcal{R}_{h,t}v] \right)^{\frac{1}{2}} \leq Ch \| \rho \|_{L^2(\Omega_T)}. \tag{2.78}
\]
It follows that
\[ \| \rho \|_{L^2(\Omega_T)} \leq Ch(A_T[\rho])^{\frac{1}{2}}. \] (2.79)
We deduce the lemma by applying lemma 2.10 to bound \( A_T[\rho] \).

**Theorem 2.6.** Assume that \( \partial_t a \equiv 0 \), \( \Omega \) is convex, \( \beta_\sigma < 1 \) and \( (\text{Trace}[\sigma])^{-1} \in L^\infty(\Omega_T) \) then
\[ \| (u - u_h)(.,T) \|^2_{L^2(\Omega)} + \| u - u_h \|^2_{L^2(0,T;H^1_0(\Omega))} \leq Ch^2 \|g\|^2_{L^2(\Omega_T)}. \] (2.80)

**Remark 2.6.** The constant \( C \) depends on \( C_X, n, \Omega, \lambda_{\text{min}}(a) \) and \( \| (\text{Trace}[\sigma])^{-1} \|_{L^\infty(\Omega_T)} \). If \( n \geq 5 \) it also depends on \( \| \text{Trace}[\sigma] \|_{L^\infty(\Omega_T)} \) and if \( n = 1 \) it also depends on \( \lambda_{\text{max}}(a) \).

**Proof.** The proof is a straightforward application of lemmas 2.11, 2.10, 2.8, 2.6 and 2.2.

2.2.2 Medium with a continuum of time scales.

In this subsection we will assume that the finite elements are in \( H^2(\Omega) \cap H^1_0(\Omega) \) and satisfy inverse inequality (1.27).

**Lemma 2.12.**
\[ \frac{1}{2} \| (u - u_h)(t) \|_{L^2(\Omega)}^2 + A_t[u - u_h] = \int_{\Omega_t} \frac{\hat{\rho}}{|\det \nabla F| \circ F^{-1}} \left( \hat{g} + \sum_{i,j=1}^n \sigma_{i,j} \circ F^{-1} \partial_i \partial_j \hat{u}_h - \partial_t \hat{u}_h \right). \] (2.81)

**Proof.** Consider equation (2.53). We have
\[ \int_{\Omega_t} \rho \partial_t(u - u_h) = \int_{\Omega_t} \frac{\hat{\rho}}{|\det \nabla F| \circ F^{-1}} \partial_t(\hat{u} - \hat{u}_h) + \int_{\Omega_t} \rho \partial_t F(\nabla F)^{-1} \nabla(u - u_h). \] (2.82)

Using equation (1.2) we obtain that
\[ \int_{\Omega_t} \rho \partial_t F(\nabla F)^{-1} \nabla(u - u_h) = -A_t[\rho, u - u_h] - \sum_{i,j=1}^n \int_{\Omega_t} \hat{\rho} Q_{i,j} \partial_i \partial_j (\hat{u} - \hat{u}_h). \] (2.83)

**Lemma 2.13.**
\[ \| \frac{\partial_t \hat{u}_h}{|\det(\nabla F)|^{\frac{1}{2}} \circ F^{-1}} \|_{L^2(\Omega_T)} \leq 2 \| g \|_{L^2(\Omega_T)} + C \| \hat{u}_h \|_{L^2(0,T;W^{2,2}_D(\Omega))}. \] (2.84)

where the constant \( C \) depends on \( n, \lambda_{\text{max}}(a), \| \text{Trace}[\sigma] \|_{L^\infty(\Omega_T)}, \mu_\sigma \).
Proof. Using the change of variable $y = F(x, t)$ in (1.31) we obtain that for all $\varphi \in X_h$

$$\begin{cases}
(\varphi, \frac{\partial_t \hat{u}_h}{\det(\nabla F) \circ F^{-1}})_{L^2(\Omega)} = \\
\sum_{i,j=1}^{n} \int_{\Omega} (\varphi, Q_{i,j} \partial_i \partial_j \hat{u}_h)_{L^2(\Omega)} + (\varphi, \frac{\hat{g}}{\det(\nabla F) \circ F^{-1}})_{L^2(\Omega)}
\end{cases}$$

(2.85)

Recall that $Q$ is given by (2.67). We choose $\varphi = \partial_t \hat{u}$ and observe that

$$\frac{\sigma}{\det \nabla F} = \frac{\sigma}{\det \sigma |\det a|^{\frac{1}{4}}}.$$  \hspace{1cm} (2.86)

Thus

$$\frac{\sigma}{\det \nabla F} \leq C(\sigma, \lambda_{\max}(a), \Vert \text{Trace}[\sigma] \Vert_{L^\infty(\Omega_T)}, \mu_\sigma).$$  \hspace{1cm} (2.87)

We deduce the lemma by Minkowski inequality. 

Combining lemma 2.12 and lemma 2.13 we obtain the following lemma

**Lemma 2.14.**

$$\frac{1}{2} \Vert(u - u_h)(T)\Vert_{L^2(\Omega)}^2 + A_T[u - u_h] \leq \Vert \rho \Vert_{L^2(\Omega_T)} \left( \Vert g \Vert_{L^2(\Omega_T)} + C \Vert \hat{u}_h \Vert_{L^2(0,T;W^{2,2}_D(\Omega))} \right).$$

(2.88)

where the constant $C$ depends on $n, \lambda_{\max}(a), \Vert \text{Trace}[\sigma] \Vert_{L^\infty(\Omega_T)}, \mu_\sigma$.

**Lemma 2.15.** Assume that $\Omega$ is convex, and condition (1.1) is satisfied then

$$\Vert \rho \Vert_{L^2(\Omega_T)} \leq Ch^2 \Vert g \Vert_{L^2(\Omega_T)}. \hspace{1cm} (2.89)$$

**Remark 2.7.** The constant $C$ depends on $C_X, n, \Omega, \delta$ and $\epsilon, \lambda_{\min}(a)$ and $\lambda_{\max}(a)$.

**Proof.** The proof is similar to the proof of lemma 2.11. As in 2.11 we choose $v \in L^2(0,T; H^1_0(\Omega))$ to be the solution of the following linear problem: for all $w \in L^2(0,T; H^1_0(\Omega))$

$$A_T[w, v] = (w, \rho)_{L^2(\Omega_T)}.$$  \hspace{1cm} (2.90)

Choosing $w = \rho$ in equation (2.90) we deduce that

$$\Vert \rho \Vert_{L^2(\Omega_T)}^2 = A_T[\rho, v - R_{h,t}v].$$  \hspace{1cm} (2.91)

Using Cauchy Schwartz inequality we deduce that

$$\Vert \rho \Vert_{L^2(\Omega_T)}^2 \leq \left( A_T[\rho] \right)^{\frac{1}{2}} \left( A_T[v - R_{h,t}v] \right)^{\frac{1}{2}}.$$  \hspace{1cm} (2.92)
Using theorem 1.6 we obtain that
\[ \| \hat{\varepsilon} \|_{L^2(0,T,W_D^{2,2}(\Omega))} \leq C \| \rho \|_{L^2(\Omega_T)}. \]  
(2.93)

Using lemma 2.9 we obtain that
\[ (A_T[v - \mathcal{R}_{h,t}v])^{\frac{1}{2}} \leq Ch \| \rho \|_{L^2(\Omega_T)}. \]  
(2.94)

It follows that
\[ \| \rho \|_{L^2(\Omega)} \leq Ch \| A_T[\rho] \|^\frac{1}{2}. \]  
(2.95)

We deduce the lemma by applying lemma 2.9 and theorem 1.6 to bound $A_T[\rho]$.

**Lemma 2.16.** Assume that $\Omega$ is convex, and that $\text{Trace}[\sigma] \in L^\infty(\Omega_T)$.

\[ \| \hat{u}_h \|_{L^2(0,T,W_D^{2,2}(\Omega))} \leq \frac{C}{h} \| g \|_{L^2(\Omega_T)}. \]  
(2.96)

**Remark 2.8.** The constant $C$ depends on $C_X$, $n$, $\Omega$, $\lambda_{\text{min}}(a)$ and $\lambda_{\text{max}}(a)$ and $\| \text{Trace}[\sigma]\|_{L^\infty(\Omega_T)}$.

**Proof.** Using the inverse inequality (1.27) of the finite elements we obtain that
\[ \| \hat{u}_h \|_{L^2(0,T,W_D^{2,2}(\Omega))} \leq \frac{C_X}{h} \| \nabla \hat{u}_h \|_{L^2(0,T,W_D^{2,2}(\Omega))}. \]  
(2.97)

Using the change of variables $y = F(x)$ we obtain that
\[ \| \nabla \hat{u}_h \|^2_{L^2(0,T,W_D^{2,2}(\Omega))} \leq C A_T[u_h] \]  
(2.98)

where $C$ depends on $n$, $\lambda_{\text{min}}(a)$ and $\| \text{Trace}[\sigma]\|_{L^\infty(\Omega_T)}$. We deduce the lemma by using lemma 2.7.

**Theorem 2.7.** Assume that $\Omega$ is convex, and condition 1.1 is satisfied then
\[ \frac{1}{2} \| (u - u_h)(T) \|^2_{L^2(\Omega)} + A_T[u - u_h] \leq Ch \| g \|^2_{L^2(\Omega_T)}. \]  
(2.99)

**Remark 2.9.** The constant $C$ depends on $C_X$, $n$, $\Omega$, $\delta$ and $\epsilon$, $\lambda_{\text{min}}(a)$ and $\lambda_{\text{max}}(a)$.

**Proof.** The proof is a straightforward application of lemma 2.14, lemma 2.15 and lemma 2.16.

30
2.2.3 Homogenization and discretization in time.

We use the notation of subsection 1.3 First let us observe that the numerical scheme associated to \( \mathcal{L}_3 \) is stable. Indeed choosing \( \psi = v_{n+1} \) one gets

\[
\left| v_{n+1}(t_{n+1}) \right|^2_{L^2(\Omega)} = \langle v_{n+1}(t_n), v_n(t_n) \rangle_{L^2(\Omega)}
\]

\[
+ \frac{1}{2} \left( \left| v_{n+1}(t_{n+1}) \right|^2_{L^2(\Omega)} - \left| v_{n+1}(t_n) \right|^2_{L^2(\Omega)} \right)
\]

\[
- \int_{t_n}^{t_{n+1}} \left( a[v_{n+1}(t)] + (v_{n+1}(t), g(t))_{L^2(\Omega)} \right) dt.
\]

(2.100)

It follows by Cauchy-Schwartz inequality that

\[
\frac{1}{2} \left| v_{n+1}(t_{n+1}) \right|^2_{L^2(\Omega)} \leq \frac{1}{2} \left| v_n(t_n) \right|^2_{L^2(\Omega)}
\]

\[
- \int_{t_n}^{t_{n+1}} \left( a[v_{n+1}(t)] + (v_{n+1}(t), g(t))_{L^2(\Omega)} \right) dt.
\]

(2.101)

Hence using Poincaré and Minkowski inequalities one obtains that

\[
\left| v_{n+1}(t_{n+1}) \right|^2_{L^2(\Omega)} + \int_{t_n}^{t_{n+1}} a[v_{n+1}(t)] dt \leq \left| v_n(t_n) \right|^2_{L^2(\Omega)}
\]

\[
+ \frac{C_{n,\Omega}}{\lambda_{\text{min}}(a)} \int_{t_n}^{t_{n+1}} |g(t)|^2_{L^2(\Omega)} dt.
\]

(2.102)

which implies theorem 1.11 and the stability of the scheme. Integrating (1.31) with respect to time we obtain that for \( \psi \in V \),

\[
(\psi(t_{n+1}), u_h(t_{n+1}))_{L^2(\Omega)} = (\psi(t_n), u_h(t_n))_{L^2(\Omega)} + \int_{t_n}^{t_{n+1}} \left( (\partial_t \psi(t), u_h(t))_{L^2(\Omega)}
\]

\[
- a[\psi(t), u_h(t)] + (\psi(t), g(t))_{L^2(\Omega)} \right) dt.
\]

(2.103)

Let us write \( (u^i) \) the coordinates of \( u_h \) associated to the basis \( (\varphi_i \circ F) \), i.e.

\[
u_h(x, t) := \sum_i u^i(t) \varphi_i(F(x, t)).
\]

(2.104)

Let us define

\[
\nu_n(x, t) := \sum_i u^i(t_n) \varphi_i(F(x, t)).
\]

(2.105)

Subtracting (2.103) and (1.38) we obtain that for \( \psi \in Z_T \),

\[
(\psi(t_{n+1}), (\nu_n + v_n)(t_{n+1}))_{L^2(\Omega)} = (\psi(t_n), (\nu_n + v_n)(t_n))_{L^2(\Omega)}
\]

\[
+ \int_{t_n}^{t_{n+1}} \left( (\partial_t \psi(t), (u_h - v_{n+1})(t))_{L^2(\Omega)}
\]

\[
- a[\psi(t), (u_h - v_{n+1})(t)] \right) dt.
\]

(2.106)
Choosing \( \psi = u_{n+1} - v_{n+1} \) we deduce using Cauchy-Schwartz inequality that

\[
\frac{1}{2} |(u_{n+1} - v_{n+1})(t_{n+1})|_{L^2(\Omega)}^2 + \int_{t_n}^{t_{n+1}} a[(u_{n+1} - v_{n+1})(t)] \, dt \leq \\
\frac{1}{2} |(u_n - v_n)(t_n)|_{L^2(\Omega)}^2 + \int_{t_n}^{t_{n+1}} \left( (\partial_t(u_{n+1} - v_{n+1}))(t), (u_h - u_{n+1})(t) \right)_{L^2(\Omega)} \\
- a[(u_{n+1} - v_{n+1})(t), (u_h - u_{n+1})(t)] \, dt.
\]  

(2.107)

**Time independent medium.** Observe that if the medium is time independent then (2.107) can be written

\[
\frac{1}{2} |(u_{n+1} - v_{n+1})(t_{n+1})|_{L^2(\Omega)}^2 + \int_{t_n}^{t_{n+1}} a[(u_{n+1} - v_{n+1})(t)] \, dt \leq \\
\frac{1}{2} |(u_n - v_n)(t_n)|_{L^2(\Omega)}^2 - \int_{t_n}^{t_{n+1}} a[(u_{n+1} - v_{n+1})(t), (u_h - u_{n+1})(t)] \, dt
\]

(2.108)

which leads us to

\[
\frac{1}{2} |(u_{n+1} - v_{n+1})(t_{n+1})|_{L^2(\Omega)}^2 + \int_{t_n}^{t_{n+1}} a[(u_{n+1} - v_{n+1})(t)] \, dt \leq \\
\frac{1}{2} |(u_n - v_n)(t_n)|_{L^2(\Omega)}^2 \\
+ \int_{t_n}^{t_{n+1}} \int_{t_n}^{t_{n+1}} 1(t < s) a[(u_{n+1} - v_{n+1})(t), \partial_s u_h(s)] \, ds \, dt.
\]

Write \( \Delta t := t_{n+1} - t_n \). Using Minkowski inequality we obtain that

\[
a[(u_{n+1} - v_{n+1})(t), \partial_s u_h(s)] \leq \frac{1}{2\Delta t} a[(u_{n+1} - v_{n+1})(t)] \\
+ \frac{1}{2}\Delta t a[\partial_s u_h(s)].
\]

(2.110)

It follows from (2.109) that

\[
|(u_{n+1} - v_{n+1})(t_{n+1})|_{L^2(\Omega)}^2 + \int_{t_n}^{t_{n+1}} a[(u_{n+1} - v_{n+1})(t)] \, dt \leq \\
|(u_n - v_n)(t_n)|_{L^2(\Omega)}^2 + |\Delta t|^2 \int_{t_n}^{t_{n+1}} a[\partial_s u_h(s)] \, ds.
\]

(2.111)

Observing that

\[
\int_{t_n}^{t_{n+1}} a[(u_{n+1} - v_{n+1})(t)] \, dt \geq 0.5 \int_{t_n}^{t_{n+1}} a[(u_h - v_{n+1})(t)] \, dt \\
- \int_{t_n}^{t_{n+1}} a[(u_h - u_{n+1})(t)] \, dt
\]

(2.112)
and
\[
\int_{t_n}^{t_{n+1}} a[(u_h - u_{n+1})(t)] dt \leq |\Delta t|^2 \int_{t_n}^{t_{n+1}} a[\partial_s u_h(s)] ds \tag{2.113}
\]
we obtain that
\[
\begin{align*}
&\left| (u_{n+1} - v_{n+1})(t_{n+1}) \right|^2_{L^2(\Omega)} + 0.5 \int_{t_n}^{t_{n+1}} a[(u_h - v_{n+1})(t)] dt \\
&\left| (u_n - v_n)(t_n) \right|^2_{L^2(\Omega)} + \frac{3}{2} |\Delta t|^2 \int_{t_n}^{t_{n+1}} a[\partial_s u_h(s)] ds.
\end{align*}
\]
(2.114)

In conclusion we have obtained the following following lemma

**Lemma 2.17.** Let \( v \in Z_T \) be the solution of \((1.38)\). We have
\[
\| (u_h - v)(T) \|^2_{L^2(\Omega)} + \int_0^T a[(u_h - v)(t)] dt \leq 3|\Delta t|^2 \int_0^T a[\partial_s u_h(s)] ds.
\]
(2.115)

Combining lemma 2.3 with lemma 2.17 we obtain the following theorem:

**Theorem 2.8.** Let \( v \in Z_T \) be the solution of \((1.38)\). We have
\[
\| (u_h - v)(T) \|^2_{L^2(\Omega)} + \int_0^T a[(u_h - v)(t)] dt \leq \left( \frac{4}{\lambda_{\min}(a)} \| \partial_t g \|^2_{L^2(0,T,H^{-1}(\Omega))} + \| g(\cdot,0) \|^2_{L^2(\Omega)} \right)
\]
(2.116)

**Time dependent medium.** Observe that
\[
\partial_t(u_{n+1} - v_{n+1}) = \partial_t F(\nabla F)^{-1}\nabla(u_{n+1} - v_{n+1}).
\]

It follows after writing \( \partial_t F = \text{div} a \nabla F \), integration by parts and using the change of variables \( y = F(x,t) \) in \((2.107)\) that
\[
\begin{align*}
\frac{3}{2} \left| (u_{n+1} - v_{n+1})(t_{n+1}) \right|^2_{L^2(\Omega)} + \int_{t_n}^{t_{n+1}} a[(u_{n+1} - v_{n+1})(t)] dt &\leq \\
\frac{1}{2} \left| (u_n - v_n)(t_n) \right|^2_{L^2(\Omega)} - 2 \int_{t_n}^{t_{n+1}} a[(u_{n+1} - v_{n+1})(t),(u_h - u_{n+1})(t)] dt \\
- \sum_{i,j} \int_{t_n}^{t_{n+1}} (\tilde{u}_h - \tilde{u}_{n+1}) Q_{i,j} \partial_i \partial_j (\tilde{u}_{n+1} - \tilde{v}_{n+1}) dt dy.
\end{align*}
\]
(2.117)
Hence using Minkowski inequality we obtain that

\[
(c_{n+1} - v_{n+1})(t_{n+1})^2_{L^2(\Omega)} + \int_{t_n}^{t_{n+1}} a[(u_{n+1} - v_{n+1})(t)] dt \leq \\
(c_n - v_n)(t_{n})^2_{L^2(\Omega)} + 4 \int_{t_n}^{t_{n+1}} a[(u_{h} - u_{n+1})(t)] dt \\
- 2 \sum_{i,j} \int_{t_n}^{t_{n+1}} \int_{\Omega} (\hat{u}_h - \hat{v}_{n+1})Q_{i,j}\partial_i\partial_j(\hat{u}_{n+1} - \hat{v}_{n+1}) dt dy.
\]  

Using Minkowski inequality we obtain that

\[
\left| \sum_{i,j} \int_{t_n}^{t_{n+1}} \int_{\Omega} (\hat{u}_h - \hat{v}_{n+1})Q_{i,j}\partial_i\partial_j(\hat{u}_{n+1} - \hat{v}_{n+1}) dt dy \right| \leq \\
C_A h^2 \int_{t_n}^{t_{n+1}} \int_{\Omega} |\hat{u}_h - \hat{v}_{n+1}|^2 dt dy \\
+ \frac{\lambda_{\text{max}}(Q)}{C_A} \int_{t_n}^{t_{n+1}} \sum_{i,j} \int_{\Omega} |\partial_i\partial_j(\hat{u}_{n+1} - \hat{v}_{n+1})|^2 dt dy.
\]  

Using the inverse inequality (1.27) and the change of variable \( y = F(x) \) we obtain that

\[
\int_{t_n}^{t_{n+1}} \sum_{i,j} \int_{\Omega} |\partial_i\partial_j(\hat{u}_{n+1} - \hat{v}_{n+1})|^2 dt dy \leq \frac{C_X}{h^2\lambda_{\text{min}}(Q)} \\
\int_{t_n}^{t_{n+1}} a[(u_{n+1} - v_{n+1})(t)] dt.
\]  

In resume, choosing \( C_A = \frac{4C_X\lambda_{\text{max}}(Q)}{h^2\lambda_{\text{min}}(Q)} \) we have obtained that

\[
(c_{n+1} - v_{n+1})(t_{n+1})^2_{L^2(\Omega)} + 0.5 \int_{t_n}^{t_{n+1}} a[(u_{n+1} - v_{n+1})(t)] dt \leq \\
(c_n - v_n)(t_{n})^2_{L^2(\Omega)} + 8 \int_{t_n}^{t_{n+1}} a[(u_{h} - u_{n+1})(t)] dt \\
+ \frac{8C_X\lambda_{\text{max}}(Q)}{h^2\lambda_{\text{min}}(Q)} h^2 \int_{t_n}^{t_{n+1}} \int_{\Omega} |\hat{u}_h - \hat{v}_{n+1}|^2 dt dy.
\]  

And a computation similar to the one leading to (2.11) gives us

\[
(c_{n+1} - v_{n+1})(t_{n+1})^2_{L^2(\Omega)} + \frac{1}{4} \int_{t_n}^{t_{n+1}} a[(u_{h} - v_{n+1})(t)] dt \leq \\
(c_n - v_n)(t_{n})^2_{L^2(\Omega)} + 9 \int_{t_n}^{t_{n+1}} a[(u_{h} - u_{n+1})(t)] dt \\
+ \frac{8C_X\lambda_{\text{max}}(Q)}{h^2\lambda_{\text{min}}(Q)} h^2 \int_{t_n}^{t_{n+1}} \int_{\Omega} |\hat{u}_h - \hat{v}_{n+1}|^2 dt dy.
\]
Moreover using the change of variables $F(x) = y$ and the inverse inequality (1.28) we obtain that

$$\int_{t_n}^{t_{n+1}} a\left((u_h - u_{n+1})(t)\right) dt \leq \frac{C_X \lambda_{\text{max}}(Q)}{h^2} \int_{t_n}^{t_{n+1}} \int_{\Omega} \left| \tilde{u}_h - \tilde{u}_{n+1} \right|^2 dt dy.$$  

(2.123)

Let us also observe that

$$\int_{t_n}^{t_{n+1}} \int_{\Omega} \left| \tilde{u}_h - \tilde{u}_{n+1} \right|^2 dt dy \leq |\Delta t|^2 \int_{t_n}^{t_{n+1}} \int_{\Omega} \left| \partial_t \tilde{u}_h \right|^2 dt dy.$$  

(2.124)

It follows that

$$\left| (u_{n+1} - v_{n+1})(t_{n+1}) \right|^2 _{L^2(\Omega)} + \frac{1}{4} \int_{t_n}^{t_{n+1}} a\left((u_h - v_{n+1})(t)\right) dt \leq$$

$$\left| (u_n - v_n)(t_n) \right|^2 _{L^2(\Omega)} + C_B \frac{|\Delta t|^2}{h^2} \int_{t_n}^{t_{n+1}} \int_{\Omega} \left| \partial_t \tilde{u}_h \right|^2 dt dy.$$  

(2.125)

with

$$C_B = C_n C_X \lambda_{\text{max}}(Q)(1 + \frac{1}{\lambda_{\text{min}}(Q)}).$$  

(2.126)

We deduce that

$$\left\| (u_h - v)(T) \right\|^2 _{L^2(\Omega)} + \frac{1}{4} \int_0^T a\left((u_h - v)(t)\right) dt \leq C_B \frac{|\Delta t|^2}{h^2} \int_0^T \int_{\Omega} \left| \partial_t \tilde{u}_h \right|^2 dt dy.$$  

(2.127)

Using lemma 2.4 to control $C_B$ and combining (2.127) with theorem 1.6 we obtain the following theorem.

**Theorem 2.9.** Assume that $\Omega$ is convex, and condition 1.1 is satisfied. Let $v \in Z_T$ be the solution of (1.38), we have

$$\left\| (u_h - v)(T) \right\|^2 _{L^2(\Omega)} + \frac{1}{4} \int_0^T a\left((u_h - v)(t)\right) dt \leq C \frac{|\Delta t|^2}{h^2} \|g\|^2 _{L^2(\Omega_T)}.$$  

(2.128)

where $C$ depends on $\Omega$, $n$, $\delta$, $\epsilon$, $\lambda_{\text{min}}(a)$ and $\lambda_{\text{max}}(a)$.
3 Numerical Experiments

The purpose of this section is to give several illustrations of the implementation of this method. The domain is the unit square in dimension two. Equation (1.1) is solved on a fine tessellation characterized by 16129 interior nodes (degrees of freedoms).

Three different coarse tessellations are considered, one with 9 degrees of freedoms (noted dof in the tables), one with 49 and the last one with 225.

The parabolic operator associated to equation (1.1) has been homogenized onto these coarse meshes using the method the method presented in this paper. We have chosen splines to span the space $X_h$ introduced in subsection 1.2.

3.1 Time independent examples.

Example 1. Time independent site percolation.

In this example we consider the site percolating medium associated to figure 1. The fine mesh is characterized by 16641 nodes. (1.1) has been homogenized to three different coarse meshes with 9, 49 and 225 interior nodes using the method described here and splines for the space $X_h$. (1.1) has been solved with the fine mesh operator and the coarse mesh operators with $g = 1$ and $g = \sin(2.4x - 1.8y + 2\pi t)$. The fine mesh and coarse mesh errors are given in tables 1, 2, 3, 4. Figure 7 shows $u$ computed on 16641 interior nodes and $u_h$ computed on 9 interior nodes in the case $g = 1$ at time 1.

![Figure 7: $u$ computed on 16641 interior nodes and $u_h$ computed on 9 interior nodes.](image)

(a) $u$.  
(b) $u_h$. 

Figure 7: $u$ computed on 16641 interior nodes and $u_h$ computed on 9 interior nodes.
Table 1: Coarse Mesh Error. Time Independent Site Percolation with $g = 1$.

| dof | $L^1$ | $L^\infty$ | $L^2$ | $H^1$ |
|-----|-------|-------------|-------|-------|
| 9   | 0.0142| 0.0389      | 0.0168| 0.0366|
| 49  | 0.0077| 0.0450      | 0.0101| 0.0482|
| 225 | 0.0035| 0.0228      | 0.0060| 0.0293|

Table 2: Fine Mesh Error. Time Independent Site Percolation with $g = 1$.

| dof | $L^1$ | $L^\infty$ | $L^2$ | $H^1$ |
|-----|-------|-------------|-------|-------|
| 9   | 0.0196| 0.0843      | 0.0251| 0.1193|
| 49  | 0.0136| 0.0698      | 0.0184| 0.1028|
| 225 | 0.0040| 0.0243      | 0.0070| 0.0485|

Table 3: Coarse Mesh Error. Time Independent Percolation Case with $g = \sin(2.4x - 1.8y + 2\pi t)$.

| dof | $L^1$ | $L^\infty$ | $L^2$ | $H^1$ |
|-----|-------|-------------|-------|-------|
| 9   | 0.0236| 0.0569      | 0.0262| 0.0477|
| 49  | 0.0181| 0.0571      | 0.0215| 0.0558|
| 225 | 0.0119| 0.0774      | 0.0167| 0.0939|

Table 4: Fine Mesh Error. Time Independent Percolation with $g = \sin(2.4x - 1.8y + 2\pi t)$.

| dof | $L^1$ | $L^\infty$ | $L^2$ | $H^1$ |
|-----|-------|-------------|-------|-------|
| 9   | 0.0424| 0.1099      | 0.0512| 0.1712|
| 49  | 0.0277| 0.0985      | 0.0348| 0.1451|
| 225 | 0.0174| 0.0886      | 0.0242| 0.1192|
Example 2. Time independent high conductivity channel.

In this example $a$ is random and characterized by a fine and long ranged high conductivity channel. We choose $a(x) = 100$, if $x$ is in the channel, and $a(x) = O(1)$ and random, if $x$ is not in the channel. The media is illustrated in figure 8.

Table 5: Coarse Mesh Error, high conductivity channel.

| dof | $L^1$  | $L^\infty$ | $L^2$  | $H^1$ |
|-----|--------|------------|--------|-------|
| 9   | 0.0159 | 0.0496     | 0.0207 | 0.0477|
| 49  | 0.0067 | 0.0389     | 0.0102 | 0.0345|
| 225 | 0.0035 | 0.0228     | 0.0060 | 0.0293|

Table 6: Fine Mesh Error, high conductivity channel.

| dof | $L^1$  | $L^\infty$ | $L^2$  | $H^1$ |
|-----|--------|------------|--------|-------|
| 9   | 0.0178 | 0.0564     | 0.0257 | 0.0947|
| 49  | 0.0079 | 0.0388     | 0.0129 | 0.0660|
| 225 | 0.0040 | 0.0243     | 0.0070 | 0.0485|

Figure 8: High Conductivity Channel superposed on a random medium.

Tables 5 and 6 give the coarse and fine meshes errors.
3.2 Time dependent examples.

In the following examples we consider media characterized by a continuum of time scales. In the following examples the ODE obtained on the coarse mesh from the homogenization of (1.1) have also been homogenized in time according to the method described in subsection 1.3.

**Example 3.** Time Dependent Multiscale trigonometric.

In this example $a$ is given by equation (1.25). Although the number fine time steps to solve (1.1) is 2663, only 134 coarse time steps have been used to solve the homogenized equation. Hence if one also takes into account homogenization in space, the compression factor is of the order of 35000 for the coarse mesh with 9 interior nodes.

Figure 9 shows the curves of $t \rightarrow a(x_0, t)$ and $t \rightarrow F(x_0, t)$ for a given $x_0 \in \Omega$.

![Figure 9: Multiscale time dependent trigonometric medium.](image)

(a) $t \rightarrow a(x_0, t)$.  
(b) Top view of $t \rightarrow F(x_0, t)$.

The coarse and fine mesh relative $L^1, L^2, L^\infty$, and $H^1$ errors with respect to time have been plotted in figures 10, 11, 12 and 13. The initial increase of the relative error has its origin in the initial value $u \equiv 0$ at time 0.

The coarse and fine meshes errors are given in tables 7 and 8 for $g = 1$ at $t = 0.1$, those errors are given in tables 9 and 10 for $g = \sin(2.4x - 1.8y + 2\pi t)$ at $t = 0.1$.
Table 7: Coarse Mesh Error. Multiscale trigonometric time dependent Medium. 
\( g = 1 \).

| dof | \( L^1 \) | \( L^\infty \) | \( L^2 \) | \( H^1 \) |
|-----|----------|----------|----------|----------|
| 9   | 0.0018   | 0.0045   | 0.0019   | 0.0039   |
| 49  | 0.0012   | 0.0054   | 0.0015   | 0.0060   |

Table 8: Fine Mesh Error. Multiscale trigonometric time dependent Medium. 
\( g = 1 \).

| dof | \( L^1 \) | \( L^\infty \) | \( L^2 \) | \( H^1 \) |
|-----|----------|----------|----------|----------|
| 9   | 0.0031   | 0.0096   | 0.0034   | 0.0242   |
| 49  | 0.0014   | 0.0059   | 0.0016   | 0.0166   |

Table 9: Coarse mesh error. Multiscale trigonometric time dependent Medium. 
\( g = \sin(2.4x - 1.8y + 2\pi t) \).

| dof | \( L^1 \) | \( L^\infty \) | \( L^2 \) | \( H^1 \) |
|-----|----------|----------|----------|----------|
| 9   | 0.0043   | 0.0087   | 0.0044   | 0.0085   |
| 49  | 0.0033   | 0.0079   | 0.0035   | 0.0084   |

Table 10: Fine mesh error. Multiscale trigonometric time dependent medium. 
\( g = \sin(2.4x - 1.8y + 2\pi t) \).

| dof | \( L^1 \) | \( L^\infty \) | \( L^2 \) | \( H^1 \) |
|-----|----------|----------|----------|----------|
| 9   | 0.0082   | 0.0199   | 0.0087   | 0.0379   |
| 49  | 0.0038   | 0.0104   | 0.0040   | 0.0244   |
(a) Coarse mesh $L^1$ error.  (b) Fine mesh $L^1$ error.

Figure 10: $L^1$ error. Multiscale Trigonometric time dependent Medium.

(a) Coarse mesh $L^2$ error.  (b) Fine Mesh $L^2$ error.

Figure 11: $L^2$ error. Multiscale Trigonometric time dependent Medium.
Figure 12: $L^\infty$ error. Multiscale Trigonometric time dependent Medium.

Figure 13: $H^1$ error. Multiscale Trigonometric time dependent Medium.
Example 4. Time Dependent Random Fourier Modes.

In this example \( a(x, y, t) = e^{h(x, y, t)} \) where \( h \) is given by the following equation

\[
h(x, y, t) = \sum_{|k| \leq R} (a_k \sin(2\pi k \cdot x') + b_k \cos(2\pi k \cdot x'))
\]

where \( R = 4, x' = x + \sqrt{2}t, y' = y - \sqrt{2}t, \) \( a_k \) and \( b_k \) are independent identically distributed random variables on \([-0.2, 0.2]\). In this example, one can compute that \( \frac{\lambda_{\text{max}}(a)}{\lambda_{\text{min}}(a)} = 95.7 \).

Figure 14: Random Fourier Mode

Figure 14 is a plot of \( a \) at time 0. Although the number fine time steps to solve (1.1) is 1332 only 67 coarse time steps have been used to solve the homogenized equation. Hence if one also takes into account homogenization in space, the compression factor is of the order of 35000 for the coarse mesh with 9 interior nodes.

The coarse and fine mesh relative \( L^1, L^2, L^\infty, \) and \( H^1 \) errors with respect to time (up to time \( t = 0.1 \)) have been plotted in figures 15, 16, 17 and 18. Those errors are also given up to \( t = 1 \) in figures 19, 20, 21, 22.

The coarse and fine meshes errors are given in tables 11 and 12 for \( g = 1 \) at \( t = 0.1 \).
(a) coarse mesh $L^1$ error.  
(b) fine mesh $L^1$ error.

Figure 15: $L^1$ error. Random Fourier modes up to $t = .1$.

Table 11: Coarse mesh error at $t = 0.1$. Time dependent random Fourier modes.

| dof | $L^1$  | $L^\infty$ | $L^2$  | $H^1$  |
|-----|--------|------------|--------|--------|
| 9   | 0.0025 | 0.0028     | 0.0064 | 0.0052 |
| 49  | 0.0032 | 0.0098     | 0.0034 | 0.0100 |

Table 12: Fine mesh error at $t = 0.1$. Time dependent random Fourier modes.

| dof | $L^1$  | $L^\infty$ | $L^2$  | $H^1$  |
|-----|--------|------------|--------|--------|
| 9   | 0.0063 | 0.0344     | 0.0079 | 0.0481 |
| 49  | 0.0042 | 0.0207     | 0.0049 | 0.0337 |
(a) coarse mesh $L^2$ error.  

(b) Fine Mesh $L^2$ error.

Figure 16: $L^2$ error. Random Fourier modes up to $t = .1$.

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(a) coarse mesh $L_\infty$ error.  

(b) fine mesh $L_\infty$ error.

Figure 17: $L_\infty$ error. Random Fourier modes up to $t = .1$. 

45
(a) coarse mesh \( H^1 \) error. 

(b) fine mesh \( H^1 \) error.

Figure 18: \( H^1 \) error. Random Fourier modes up to \( t = .1 \).

(a) coarse mesh \( L^1 \) error. 

(b) fine mesh \( L^1 \) error.

Figure 19: \( L^1 \) error. Random Fourier modes up to \( t = 1 \).
(a) coarse mesh $L^2$ error.  
(b) Fine Mesh $L^2$ error.

Figure 20: $L^2$ error. Random Fourier modes up to $t = 1$.

(a) coarse mesh $L_\infty$ error.  
(b) fine mesh $L_\infty$ error.

Figure 21: $L_\infty$ error. Random Fourier modes up to $t = 1$. 

47
Figure 22: $H^1$ error. Random Fourier modes up to $t = 1$. 

(a) coarse Mesh $H^1$ error.  
(b) fine Mesh $H^1$ error.
Table 13: Coarse Mesh Error for the time dependent random fractal medium with spline element

| dof | $L^1$ | $L^\infty$ | $L^2$ | $H^1$ |
|-----|-------|------------|-------|-------|
| 9   | 0.0046| 0.0074     | 0.0052| 0.0065|
| 49  | 0.0036| 0.0046     | 0.0036| 0.0059|

Table 14: Fine Mesh Error for the time dependent random fractal medium with spline element

| dof | $L^1$ | $L^\infty$ | $L^2$ | $H^1$ |
|-----|-------|------------|-------|-------|
| 9   | 0.0039| 0.0082     | 0.0043| 0.0222|
| 49  | 0.0033| 0.0054     | 0.0034| 0.0168|

**Example 5. Time Dependent Random Fractal**

In this example, $a$ is given by a product of discontinuous functions oscillating randomly at different space and time scales. Namely $a(x,t) := a_1(x,t)a_2(x,t) \cdots a_n(x,t)$ with $n = 6$ and $a_i(x,t) = c_{i}^{pq}(t)$ for $x \in \left[\frac{q}{2^i}, \frac{q+1}{2^i}\right) \times \left[\frac{p}{2^i}, \frac{p+1}{2^i}\right)$. The coefficients $c_{i}^{pq}(t)$ are chosen at random with uniform law in $[\frac{1}{\gamma}, \gamma]$ with $\gamma = 0.7$ and independently in subdivision in space and in time, thus they are assumed to be constant in each time interval $0.1 \times \left[\frac{k}{4^i}, \frac{k+1}{4^i}\right)$.

In this example we have $\frac{\lambda_{max}(a)}{\lambda_{min}(a)} = 160.3295$. Although the number of fine time steps to solve (1.1) is 3482, only 175 coarse time steps have been used to solve the homogenized equation which corresponds to a reduction of the complexity of the scheme by a factor of 35000 in the case of the coarse tessellation with 9 interior nodes. $a$ and the map $(F_1, F_2)$ are drawn in figure 23. $L^1$, $L^2$, $L^\infty$ and $H^1$ errors are given in figure 24 to 27. Coarse and fine mesh errors are given in table 13 and 14 at time $t = 0.1$. We have chosen $g = 1$ in this numerical experiment, one obtains similar results by choosing $g = \sin(2.4x - 1.8y + 2\pi t)$.
Figure 23: $a$ and $(F_1, F_2)$ at time $t = 0$, $t = 0.1$ for the time dependent random fractal medium.
Figure 24: $L^1$ error for the time dependent random fractal medium at $t = 0.1$.  

Figure 25: $L^2$ error for the time dependent random fractal medium at $t = 0.1$.  

(a) coarse mesh $L^1$ error.  

(b) fine mesh $L^1$ error.  

(a) coarse mesh $L^2$ error.  

(b) Fine Mesh $L^2$ error.
Figure 26: $L^\infty$ error for the time dependent random fractal medium at $t = 0.1$.

(a) coarse mesh $L^\infty$ error.       (b) fine mesh $L^\infty$ error.

Figure 27: $H^1$ error for the time dependent random fractal medium at $t = 0.1$.

(a) coarse Mesh $H^1$ error.       (b) fine Mesh $H^1$ error.
References

[1] Jørn E. Aarnes. On the use of a mixed multiscale finite element method for greater flexibility and increased speed or improved accuracy in reservoir simulation. *Multiscale Model. Simul.*, 2(3):421–439 (electronic), 2004.

[2] Giovanni Alessandrini and Vincenzo Nesi. Univalent $\sigma$-harmonic mappings: connections with quasiconformal mappings. *J. Anal. Math.*, 90:197–215, 2003.

[3] G. Allaire. *Shape Optimization by the Homogenization Method*. Springer, 2001.

[4] G. Allaire and R. Brizzi. A multi-scale finite element method for numerical homogenization. Technical report, CMAPX, 2004.

[5] G. Allaire and R. Brizzi. A multi-scale finite element method for numerical homogenization. *SIAM MMS*, 4:790–812, 2005.

[6] Alano Ancona. Some results and examples about the behavior of harmonic functions and Green’s functions with respect to second order elliptic operators. *Nagoya Math. J.*, 165:123–158, 2002.

[7] Amir Averbuch, Gregory Beylkin, Ronald Coifman, and Moshe Israeli. Multiscale inversion of elliptic operators. In *Signal and image representation in combined spaces*, volume 7 of *Wavelet Anal. Appl.*, pages 341–359. Academic Press, San Diego, CA, 1998.

[8] Ivo Babuška, Gabriel Caloz, and John E. Osborn. Special finite element methods for a class of second order elliptic problems with rough coefficients. *SIAM J. Numer. Anal.*, 31(4):945–981, 1994.

[9] M. Bebendorf and Y. Chen. Efficient solution of nonlinear elliptic problems using hierarchical matrices with broyden updates. preprint 51/2005, *Max-Planck-Institut MiS, Leipzig*, 2005.

[10] Mario Bebendorf. Approximate inverse preconditioning of fe systems for elliptic operators with non-smooth coefficients. *Preprint 7/2004, Max-Planck-Institute MiS, Leipzig*, 2005.

[11] Mario Bebendorf. Efficient inversion of galerkin matrices of general second-order elliptic differential operators with nonsmooth coefficients. *Math. Comp.*, 74:1179–1199, 2005.

[12] Mario Bebendorf. Why approximate lu decompositions of finite element discretizations of elliptic operators can be computed with almost linear complexity. *Preprint 8/2005, Max-Planck-Institute MiS, Leipzig*, 2005.

[13] Mario Bebendorf and Wolfgang Hackbusch. Existence of $\mathcal{H}$-matrix approximants to the inverse FE-matrix of elliptic operators with $L^\infty$-coefficients. *Numer. Math.*, 95(1):1–28, 2003.
Gérard Ben Arous and Houman Owhadi. Multiscale homogenization with bounded ratios and anomalous slow diffusion. *Comm. Pure Appl. Math.*, 56(1):80–113, 2003.

A. Bensoussan, J. L. Lions, and G. Papanicolaou. *Asymptotic analysis for periodic structure*. North Holland, Amsterdam, 1978.

Gregory Beylkin and Nicholas Coult. A multiresolution strategy for reduction of elliptic PDEs and eigenvalue problems. *Appl. Comput. Harmon. Anal.*, 5(2):129–155, 1998.

S. C. Brenner and L. R. Scott. *The Mathematical Theory of Finite Element Methods*. Springer, 2002. second edition.

M. E. Brewster and G. Beylkin. A multiresolution strategy for numerical homogenization. *Appl. Comput. Harmon. Anal.*, 2(4):327–349, 1995.

Franco Brezzi and Donatella Marini. Subgrid phenomena and numerical schemes. In *Frontiers in numerical analysis (Durham, 2002)*, Universitext, pages 1–16. Springer, Berlin, 2003.

Marc Briane, Graeme W. Milton, and Vincenzo Nesi. Change of sign of the corrector’s determinant for homogenization in three-dimensional conductivity. *Arch. Ration. Mech. Anal.*, 173(1):133–150, 2004.

S. Campanato. Un risultato relativo ad equazioni ellittiche del secondo ordine di tipo non variazionale. *Ann. Scuola Norm. Sup. Pisa (3)*, 21:701–707, 1967.

Shanqin Chen, Weinan E, and Chi-Wang Shu. The heterogeneous multiscale method based on the discontinuous Galerkin method for hyperbolic and parabolic problems. *Multiscale Model. Simul.*, 3(4):871–894 (electronic), 2005.

A. Chertock and D. Levy. On wavelet-based numerical homogenization. *Multiscale Modeling and Simulation*, 3:65–88, 2004.

R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes. Compensated compactness and Hardy spaces. *J. Math. Pures Appl. (9)*, 72(3):247–286, 1993.

Mihai Dorobantu and Björn Engquist. Wavelet-based numerical homogenization. *SIAM J. Numer. Anal.*, 35(2):540–559 (electronic), 1998.

W. E, B. Engquist, X. Li, W. Ren, and E. Vanden-Eijnden. The heterogeneous multiscale method: A review. Technical report, preprint. [http://www.math.princeton.edu/multiscale/review.pdf](http://www.math.princeton.edu/multiscale/review.pdf)
[28] Yalchin R. Efendiev, Thomas Y. Hou, and Xiao-Hui Wu. Convergence of a nonconforming multiscale finite element method. SIAM J. Numer. Anal., 37(3):888–910 (electronic), 2000.

[29] A. Ern and J.-L. Guermond. Theory and practice of finite element methods., volume 159 of Applied Mathematical Sciences. Springer, 2004. Two volumes bound as one.

[30] L. Evans. Partial Differential equations, volume 19 of Graduate Studies in Mathematics. American Mathematical Society, 1997.

[31] Charbel Farhat, Isaac Harari, and Leopoldo P. Franca. The discontinuous enrichment method. Comput. Methods Appl. Mech. Engrg., 190(48):6455–6479, 2001.

[32] Charbel Farhat, Isaac Harari, and Ulrich Hetmaniuk. The discontinuous enrichment method for multiscale analysis. Comput. Methods Appl. Mech. Engrg., 192(28-30):3195–3209, 2003. Multiscale computational mechanics for materials and structures (Cachan, 2002).

[33] V. N. Fenchenko and Ê. Ya. Khruslov. Asymptotic behavior of solutions of differential equations with a strongly oscillating coefficient matrix that does not satisfy a uniform boundedness condition. Dokl. Akad. Nauk Ukrain. SSR Ser. A, (4):24–27, 95, 1981.

[34] Jacob Fish and Amir Wagiman. Multiscale finite element method for a locally nonperiodic heterogeneous medium. Comput. Mech., 12(3):164–180, 1993.

[35] Jacob Fish and Yuan Zheng. Multi-scale enrichment based on partition of unity. Int. J. Num. Meth. Engrg., 2005.

[36] Josselin Garnier. Homogenization in a periodic and time-dependent potential. SIAM J. Appl. Math., 57(1):95–111, 1997.

[37] A. C. Gilbert. A comparison of multiresolution and classical one-dimensional homogenization schemes. Appl. Comput. Harmon. Anal., 5(1):1–35, 1998.

[38] Olivier Goubet. Séparation des variables dans le problème de Stokes. Application à son approximation multiéchelles éléments finis. C. R. Acad. Sci. Paris Sér. I Math., 315(12):1315–1318, 1992.

[39] L. Greengard and V. Rokhlin. A fast algorithm for particle simulations. J. Comput. Phys., 73(2):325–348, 1987.

[40] Viet Ha Hoang and Christoph Schwab. High-dimensional finite elements for elliptic problems with multiple scales. Multiscale Model. Simul., 3(1):168–194 (electronic), 2004/05.

[41] Thomas Y. Hou and Xiao-Hui Wu. A multiscale finite element method for elliptic problems in composite materials and porous media. J. Comput. Phys., 134(1):169–189, 1997.
[42] Thomas Y. Hou, Xiao-Hui Wu, and Yu Zhang. Removing the cell resonance error in the multiscale finite element method via a Petrov-Galerkin formulation. *Commun. Math. Sci.*, 2(2):185–205, 2004.

[43] V. V. Jikov, S. M. Kozlov, and O. A. Oleinik. *Homogenization of Differential Operators and Integral Functionals*. Springer-Verlag, 1991.

[44] E. Ya. Khruslov. Homogenized models of composite media. In *Composite media and homogenization theory (Trieste, 1990)*, volume 5 of *Progr. Nonlinear Differential Equations Appl.*, pages 159–182. Birkhäuser Boston, Boston, MA, 1991.

[45] C. Kipnis and S.R.S. Varadhan. Central limit theorem for additive functional of reversible markov processes and application to simple exclusion. *Comm. Math. Phys.*, 104:1–19, 1986.

[46] S.M. Kozlov. The method of averaging and walks in inhomogeneous environments. *Russian Math. Surveys*, 2(40):73–145, 1985.

[47] S.M. Kozlov. Averaging of difference schemes. *Math. USSR. Sbornik*, 2(57):351–369, 1987.

[48] S. H. Lee, P. Jenny, and H. A. Tchelepi. A finite-volume method with hexahedral multiblock grids for modeling flow in porous media. *Comput. Geosci.*, 6(3-4):353–379, 2002. Locally conservative numerical methods for flow in porous media.

[49] S. H. Lee, P. Jenny, and H. A. Tchelepi. Multi-scale finite-volume method for elliptic problems in subsurface flow simulation. *Journal of Computational Physics*, 187:47–67, 2003.

[50] S. Leonardi. Weighted Miranda-Talenti inequality and applications to equations with discontinuous coefficients. *Comment. Math. Univ. Carolin.*, 43(1):43–59, 2002.

[51] A. Maugeri, D. K. Palagachev, and L. G. Softova. *Elliptic and Parabolic Equations with Discontinuous Coefficients*, volume 109 of *Mathematical Research*. Wiley-VCH, 2000.

[52] François Murat. Compacité par compensation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 5(3):489–507, 1978.

[53] François Murat and Luc Tartar. $H$-convergence. In *Topics in the mathematical modelling of composite materials*, volume 31 of *Progr. Nonlinear Differential Equations Appl.*, pages 21–43. Birkhäuser Boston, Boston, MA, 1997.

[54] Assad A. Oberai and Peter M. Pinsky. A multiscale finite element method for the Helmholtz equation. *Comput. Methods Appl. Mech. Engrg.*, 154(3-4):281–297, 1998.

[55] H. Owhadi and L. Zhang. Metric based upscaling. 2005. Accepted for publication in Communications on Pure and Applied Mathematics.
[56] Houman Owhadi. Anomalous slow diffusion from perpetual homogenization. *Ann. Probab.*, 31(4):1935–1969, 2003.

[57] Houman Owhadi. Averaging versus chaos in turbulent transport? *Comm. Math. Phys.*, 247(3):553–599, 2004.

[58] G. Papanicolaou and S.R.S. Varadhan. Boundary value problems with rapidly oscillating random coefficients. In *Colloquia Mathematica Societatis János Bolyai*, volume 27, 1979.

[59] G. Papanicolaou and S.R.S. Varadhan. Diffusions with random coefficients. In N. Ikeda, editor, *Statistics and Probability. Essays in Honour of C.R. Rao*, pages 547–552. North-Holland, Amsterdam, 1982.

[60] G. C. Papanicolaou, D. Stroock, and S. R. S. Varadhan. Martingale approach to some limit theorems. In *Papers from the Duke Turbulence Conference (Duke Univ., Durham, N.C., 1976), Paper No. 6*, pages ii+120 pp. Duke Univ. Math. Ser., Vol. III. Duke Univ., Durham, N.C., 1977.

[61] G. C. Papanicolaou and S. R. S. Varadhan. A limit theorem with strong mixing in Banach space and two applications to stochastic differential equations. *Comm. Pure Appl. Math.*, 26:497–524, 1973.

[62] S. Spagnolo. Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche. *Ann. Scuola Norm. Sup. Pisa (3) 22 (1968), 571-597; errata, ibid. (3), 22:673, 1968.

[63] D. W. Stroock and S. R. S. Varadhan. Martingales. I, II, III. In *Topics in probability theory (Sem., Courant Inst. Math. Sci., New York, 1971–1972)*, pages 113–161. Courant Inst. Math. Sci., New York, 1973.

[64] Daniel W. Stroock and S. R. S. Varadhan. Diffusion processes and martingales. I. In *Martingales (Report Meeting, Oberwolfach, 1970)*, pages 60–66. Lecture Notes in Math., Vol. 190. Springer, Berlin, 1971.

[65] Daniel W. Stroock and S. R. S. Varadhan. Diffusion processes and martingales. II. In *Martingales (Report Meeting, Oberwolfach, 1970)*, pages 67–75. Lecture Notes in Math., Vol. 190. Springer, Berlin, 1971.

[66] Daniel W. Stroock and S. R. Srinivasa Varadhan. *Multidimensional diffusion processes*, volume 233 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1979.

[67] L. Tartar. Compensated compactness and applications to partial differential equations. In *Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV*, volume 39 of *Res. Notes in Math.*, pages 136–212. Pitman, Boston, Mass., 1979.
[68] L. Tartar. Homogénéisation et compacité par compensation. In Séminaire Goulaouic-Schwartz (1978/1979), pages Exp. No. 9, 9. École Polytech., Palaiseau, 1979.

[69] S.R.S. Varadhan. private discussion. 2005.

[70] W. L. Wan, Tony F. Chan, and Barry Smith. An energy-minimizing interpolation for robust multigrid methods. SIAM J. Sci. Comput., 21(4):1632–1649 (electronic), 1999/00.

[71] X. H. Wu, Y. Efendiev, and T. Y. Hou. Analysis of upscaling absolute permeability. Discrete Contin. Dyn. Syst. Ser. B, 2(2):185–204, 2002.

[72] X. H. Wu, Y. Efendiev, and T. Y. Hou. Accurate multiscale finite element methods for two-phase flow simulations. submitted to J. Comput. Phys., 2005.

[73] V. V. Zhikov, S. M. Kozlov, O.A. Oleinik, and Kha T’en Ngoan. Averaging and g-convergence of differential operators. Russian Math. Surveys, 34(5):69–147, 1979.