Minimal Coupling in Koopman-von Neumann Theory

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Classical mechanics (CM), like quantum mechanics (QM), can have an operatorial formulation. This was pioneered by Koopman and von Neumann (KvN) in the 30’s. They basically formalized, via the introduction of a classical Hilbert space, earlier work of Liouville who had shown that the classical time evolution can take place via an operator, nowadays known as the Liouville operator. In this paper we study how to perform the coupling of a point particle to a gauge field in the KvN version of CM. So we basically implement at the classical operatorial level the analog of the minimal coupling of QM. We show that, differently than in QM, not only the momenta but also other variables have to be coupled to the gauge field. We also analyze in details how the gauge invariance manifests itself in the Hilbert space of KvN and indicate the differences with QM. As an application of the KvN method we study the Landau problem proving that there are many more degeneracies at the classical operatorial level than at the quantum one. As a second example we go through the Aharonov-Bohm phenomenon showing that, at the quantum level, this phenomenon manifests its effects on the spectrum of the quantum Hamiltonian while at the classical level there is no effect whatsoever on the spectrum of the Liouville operator.

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1 Introduction

It is well-known that in classical statistical mechanics the evolution of probability densities \( \rho(q, p) \) is given by the Liouville equation

\[
\partial_t \rho(q, p) = -\hat{L}\rho(q, p) \tag{1.1}
\]

where \( \hat{L} = \partial_p H \partial_q - \partial_q H \partial_p \) and \( H(q, p) \) is the Hamiltonian of the system. The \( \rho(q, p) \) must only be integrable functions (i.e. \( \rho \in L^1 \)-space) because of their meaning as probability densities \((i.e. \int \rho \, dq \, dp < \infty)\). As they are only \( L^1 \)-functions they do not make up an Hilbert space. In 1931 KvN \cite{KvN} postulated the same evolution equation for complex distributions \( \psi(q, p) \) making up an \( L^2 \) Hilbert space:

\[
\partial_t \psi(q, p) = -\hat{L}\psi(q, p) \tag{1.2}
\]

If we postulate eq. (1.2) for \( \psi(q, p) \), then it is easy to prove that functions \( \rho \) of the form:

\[
\rho = |\psi|^2 \tag{1.3}
\]

evolves with the same equation as \( \psi \). This is so because the operator \( \hat{L} \) contains only first order derivatives. This is not what happens in QM where the evolution of the \( \psi(q) \) is via the Schrödinger operator \( \hat{H} \) while the one of the associated \( \rho = |\psi|^2 \) is via a totally different operator. The reason is that the Schrödinger operator \( \hat{H} \), differently than the Liouville operator \( \hat{L} \), contains second order derivatives. By postulating the relations (1.3) and (1.2) for the \( \psi \), KvN managed to build an operatorial formulation for classical mechanics (CM) equipped with an Hilbert space structure and producing the same results as the Liouville formulation. Of course there are phases in a complex \( \psi \) and one could wonder which is their role in CM. This problem is addressed in ref.\cite{ref3}. We will briefly review the KvN formalism in section 2 of this paper.

The question we want to address in this paper is how to couple a gauge field to the point particle degrees of freedom once we work in the operatorial formulation of CM. We know that in QM there is a simple rule known as "minimal coupling" (MC) \cite{ref4}:

\[
\vec{p} \rightarrow \vec{p} - \frac{e}{c} \vec{A} \tag{1.4}
\]

where \( \vec{A} \) is the gauge field. This rule says that it is enough to replace \( \vec{p} \) with \( \vec{p} - \frac{e}{c} \vec{A} \) in the Hamiltonian \( H(q, p) \rightarrow H(q, p - \frac{e}{c} \vec{A}) \) in order to get the interaction of the particle with the gauge field and then represent \( \vec{p} \) operatorially as \( \frac{\hbar}{i} \frac{\partial}{\partial x} \). Since the operatorial formulation of CM has an Hilbert space structure like QM and it has operators like \( \hat{L} \) analog of the Schrödinger \( \hat{H} \) operator of QM, we would like to find out the "minimal coupling" rules which would transform the \( \hat{L} \) without a gauge field interaction into the one \( \hat{L}_A \) with interaction.
Let us start by just using the rule (1.4) inside the $H$ which appears in $\hat{L}$:

$$\hat{L} = (\partial_q H) \partial_q - (\partial_q H) \partial_p$$

(1.5)

without modifying the derivative $\frac{\partial}{\partial p}$. We can work out the simple case of a particle moving under a constant magnetic field directed along $z$. Let us choose the gauge field as:

$$\begin{cases}
A_x = 0 \\
A_y = Bx \\
A_z = 0
\end{cases}$$

(1.6)

Using the MC (1.4):

$$p_y \longrightarrow p_y - \frac{eB}{c} x$$

(1.7)

the Hamiltonian becomes

$$H_A = \frac{p_x^2}{2m} + \frac{1}{2m} \left(p_y - \frac{eB}{c} x\right)^2 + \frac{p_z^2}{2m}$$

(1.8)

and $\hat{L}$ would turn into

$$\hat{L}_A = \frac{p_x}{m} \frac{\partial}{\partial x} + \frac{1}{m} \left( p_y - \frac{eB}{c} x \right) \left( \frac{\partial}{\partial y} + \frac{eB}{c} \frac{\partial}{\partial p_x} \right) + \frac{p_z}{m} \frac{\partial}{\partial z}$$

(1.9)

If we compare this Liouville operator with the one containing no interaction with the magnetic field which is

$$\hat{L} = \frac{p_x}{m} \frac{\partial}{\partial x} + \frac{p_y}{m} \frac{\partial}{\partial y} + \frac{p_z}{m} \frac{\partial}{\partial z}$$

(1.10)

we see that the tricks to pass from (1.10) to (1.9) are the substitutions:

$$\begin{cases}
p_y \rightarrow p_y - \frac{eB}{c} x \\
\frac{\partial}{\partial y} \rightarrow \frac{\partial}{\partial y} + \frac{eB}{c} \frac{\partial}{\partial p_x}
\end{cases}$$

(1.11)

These are the MC rules for the Liouville operator in the case of a constant magnetic field.

For the Schrödinger operator the MC rules would have been the quantum operatorial version of just the first relation of (1.11)

$$\frac{\hbar}{i} \frac{\partial}{\partial y} \rightarrow \frac{\hbar}{i} \frac{\partial}{\partial y} - \frac{eB}{c} x$$

(1.12)

which would turn into the second one if we had represented $x = -\frac{\hbar}{i} \frac{\partial}{\partial p_x}$. Somehow in QM the two rules (1.11) would be just one and the same differently than in CM where we cannot identify $p_y$ with $\frac{\partial}{\partial y}$. 

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In section 3 of this paper we will generalize the MC rule (1.11) to the case of an arbitrary magnetic field. Those rules, even if derived from the same simple principle as above, will involve various complicated combinations of the derivative operators. We shall show anyhow that those complicated combinations could be put in a very simple and illuminating form using the concept of superfield which naturally appears in a functional approach to the KvN theory. This functional approach will be briefly reviewed in section 2.

Let us now go back to our derivation of the minimal coupling rules for \( \hat{L} \). If in the original \( H \) there were a potential \( V(y) \) then the \( \hat{L} \) of eq. (1.5) would have contained a derivative with respect to \( p_y \). Now in (1.7) we changed \( p_y \) to get the MC and so one would be led to conclude that also \( \frac{\partial}{\partial p_y} \) has to be changed in \( \hat{L} \) and not just \( \frac{\partial}{\partial y} \) as it appears in (1.11). Actually this is not the case. We will see in section 3 that the derivatives with respect to the momenta never have to be changed in the MC correspondence.

The reader, realizing that we have an Hilbert space, may wonder if the gauge invariance manifests itself on the states via a phase like it does in QM. This is so only in a particular representation. The issue of gauge invariance will be thoroughly examined in section 3 of this paper.

In section 4 we will apply the KvN theory to the Landau problem. We will compare the results to the quantum case and show that there are many more degeneracies at the classical level than at the quantum one. Finally in section 5 we will study the Aharonov-Bohm phenomenon both at the CM level via the KvN formalism and at the QM level via the Schrödinger equation. We will show that at the quantum level the spectrum of the Schrödinger Hamiltonian is changed by the presence of the gauge potential while the spectrum of the classical Liouville operator is left unchanged. This spectrum drives the motion of the system at the classical level as it is explained in ref. [2]. Further calculational details omitted in the various sections of this paper are confined to few appendices which conclude the paper.

2 Functional Approach to the KvN Theory

As we indicated in the previous section, KvN postulated for the Hilbert space states \( \psi \) of CM the same evolution (1.2) as for the probability densities \( \rho \) (1.1). So the propagation kernel for \( \psi \) will be the same as the propagation kernel for \( \rho \). This last one has an immediate physical meaning being the transition probability \( P(\varphi^a t|\varphi^a_{0t0}) \) of finding the particle in the phase space point \( \varphi^a = (q, p) \) at time \( t \) if it was in configuration \( \varphi^a_0 \) at time \( t_0 \). In CM this \( P(\varphi^a t|\varphi^a_{0t0}) \) is nothing else than a Dirac delta

\[
P(\varphi^a t|\varphi^a_{0t0}) = \delta[\varphi^a - \phi^a_{cm}(t; \varphi_{0t0})]
\]  

(2.1)
where $\phi_d^a$ is the classical solution of the Hamilton equations of motion $\dot{\phi}^a = \omega^{ab} \partial H / \partial \phi^b$ ($\omega^{ab}$ symplectic matrix) with initial condition $(\varphi_0, t_0)$. We know that in general, if we have a probability $P(f|i)$ to go from configuration $(i)$ to configuration $(f)$, the following decomposition holds:

$$P(f|i) = \sum_{k_i} P(f|k_{N-1})P(k_{N-1}|k_{N-2}) \cdots P(k_1|1)$$  \hspace{2cm} (2.2)

In our case this becomes

$$P(\varphi^a t|\varphi_0^a t_0) = \lim_{N \to \infty} \prod_{j=1}^N d^{2n} \varphi_i \delta^{(2n)}[\varphi_j^a - \phi_d^a(t_j|\varphi_{j-1} t_{j-1})]$$  \hspace{2cm} (2.3)

where we have sliced the interval of time $t - t_0$ in $N$ intervals labelled by $t_j$. In the continuum limit we could formally write the relation above as

$$P(\varphi^a t|\varphi_0^a t_0) = \int \mathcal{D} \varphi \tilde{\delta}[\varphi^a(t) - \phi_d^a(t)]$$  \hspace{2cm} (2.4)

where $\mathcal{D} \varphi$ is a functional integration and $\tilde{\delta}[\ ]$ a functional Dirac delta. As the $\phi_d^a(t)$ are the solutions of the Hamilton equation, we could rewrite the $\tilde{\delta}[\ ]$ in (2.4) as

$$\tilde{\delta}[\varphi^a(t) - \phi_d^a(t)] = \tilde{\delta}[\dot{\varphi}^a - \omega^{ab} \partial_b H] \text{det} \delta^a_b \partial_t - \partial_b (\omega^{ad} \partial_d H)]$$  \hspace{2cm} (2.5)

Next we could Fourier transform the Dirac delta on the RHS of (2.5) introducing $2n$ extra variables $\lambda$, and we could exponentiate the determinant on the RHS of (2.5) using $4n$ anticommuting variables $c^a, \bar{c}^a$. The final result is the following

$$P(\varphi^a t|\varphi_0^a t_0) = \int \mathcal{D}' \varphi \mathcal{D} \lambda \mathcal{D} c \mathcal{D} \bar{c} \exp \left[ i \int_{t_0}^t \mathrm{d} t \mathcal{L} \right]$$  \hspace{2cm} (2.6)

where

$$\mathcal{L} = \lambda_a \dot{\varphi}^a + i \bar{c}^a \dot{c}^a - \lambda_a \omega^{ab} \partial_b H - i \bar{c}^a \omega^{ad} (\partial_d \partial_b H) c^b$$  \hspace{2cm} (2.7)

and with $\mathcal{D}''$ we indicate that the integration is over paths with fixed end points in $\varphi$. All this is described in many more details in ref. [4]. Associated to this $\mathcal{L}$ there is an Hamiltonian which is

$$\mathcal{H} = \lambda_a \omega^{ab} \partial_b H + i \bar{c}^a \omega^{ad} (\partial_d \partial_b H) c^b$$  \hspace{2cm} (2.8)

We notice that, instead of just the original $2n$ phase space coordinates $\varphi^a$, we now have $8n$ variables $(\varphi^a, \lambda, c\bar{c}, \bar{c})$ whose geometrical meaning has been studied in details in ref. [4]. We will indicate with $\mathcal{M}$ the original phase space coordinatized by $\varphi^a$ and with $\bar{\mathcal{M}}$ the space coordinatized by $(\varphi^a, \lambda, c\bar{c}, \bar{c})$. What is nice is that all these $8n$ variables can be put together in a single object known as superfield in the literature on supersymmetry. In order to construct it we first enlarge the standard time $t$ to two
Grassmannian partners $\theta, \bar{\theta}$ which make, together with the time $t$, what is known as superspace and then we build the following object

$$\Phi^a(t, \theta, \bar{\theta}) = \varphi^a(t) + \theta c^a(t) + \bar{\theta} \omega^{ab} \bar{c}_b(t) + i \bar{\theta} \theta \omega^{ab} \lambda_b(t)$$  \hspace{1cm} (2.9)$$

It is easy to prove that

$$i \int d\theta d\bar{\theta} H[\Phi] = \mathcal{H}$$  \hspace{1cm} (2.10)$$

where $H$ is the usual Hamiltonian of classical mechanics in $\mathcal{M}$ but where we have replaced the standard bosonic variables $\varphi^a$ with the superfield variables $\Phi^a$. By expanding $H(\Phi)$ in $\theta$ and $\bar{\theta}$ it is straightforward to prove the following formula

$$H(\Phi^a) = H(\varphi^a) + \theta N_H - \bar{\theta} \bar{N}_H + i \bar{\theta} \theta \mathcal{H}$$  \hspace{1cm} (2.11)$$

where the precise form of $N_H, \bar{N}_H$ is not necessary in this paper and can be found in ref. [4]. The same steps we did above for the Hamiltonian can be done for any function $O(\varphi)$ of the phase space $\mathcal{M}$, i.e. replace $\varphi$ with $\Phi$ and expand $O(\Phi)$ in $\theta, \bar{\theta}$

$$O(\Phi^a) = O(\varphi^a) + \theta N_O - \bar{\theta} \bar{N}_O + i \bar{\theta} \theta O$$  \hspace{1cm} (2.12)$$

where $O = \lambda^a \omega^{ab} \partial_b O + i \bar{c}_a \omega^{ad} (\partial_d \partial_b O) c^b$. The index ”$a$” in $\varphi^a$ indicates either the first $n$ configurational variables $q$ or the second $n$ momentum variables $p$, so in the case of $n = 1$ the substitution in $H$ on the LHS of (2.10) is:

$$q \rightarrow \Phi^q = q + \theta c^q + \bar{\theta} \bar{c}_p + i \bar{\theta} \theta \lambda_p$$  \hspace{1cm} (2.13)$$

$$p \rightarrow \Phi^p = p + \theta c^p - \bar{\theta} \bar{c}_q - i \bar{\theta} \theta \lambda_q$$  \hspace{1cm} (2.14)$$

We have put the index $q$ (or $p$) on $c, \bar{c}, \lambda$ just to indicate that we refer to the first $n$ (or the second $n$) of the $c, \bar{c}, \lambda$ variables. A relation analog to (2.10) holds also at the Lagrangian level

$$i \int d\theta d\bar{\theta} L[\Phi] = L + (s.t.)$$  \hspace{1cm} (2.15)$$

where $L$ and the $H$ of formulas (2.7) and (2.8) will be reduced to:

$$L_B = \lambda^a \dot{\varphi}^a - \lambda_\mu \omega^{ab} \partial_b H$$  \hspace{1cm} (2.17)$$

$$H_B = \lambda_\mu \omega^{ab} \partial_b H$$  \hspace{1cm} (2.18)$$

The subscript $B$ is for Bosonic but we will neglect it from now on.
and the superfields (2.13) and (2.14) to:

\[ \Phi^q = q + i \bar{\theta} \theta p \]  \hspace{1cm} (2.19)
\[ \Phi^p = p - i \bar{\theta} \theta q \]  \hspace{1cm} (2.20)

KvN postulated for the wave functions \( \psi \) the same evolution as for the \( \rho \), so the kernel of propagation for the \( \psi \): \( K(\varphi^a t | \varphi^a_0 t_0) \), also known as transition amplitude, will have the same path integral expression as the transition probability \( P(\varphi^a t | \varphi^a_0 t_0) \). This last has the expression (2.6) where actually the integration over the \( c^a, \bar{c}_a \) can be dropped because we are not propagating these variables. Their overall integration would give one as proved in ref. [4]. So the equality of \( P(\varphi^a t | \varphi^a_0 t_0) \) and \( K(\varphi^a t | \varphi^a_0 t_0) \) would give:

\[ P(\varphi^a t | \varphi^a_0 t_0) = K(\varphi^a t | \varphi^a_0 t_0) = \int D\varphi D\lambda \exp \left[ i \int dt L_B \right] \]  \hspace{1cm} (2.21)

The two path integrals for \( P(\ | \ ) \) and for \( K(\ | \ ) \) are the same if we want to propagate probability densities \( \rho \) or wave functions \( \psi \) both living in the \( \varphi \)-space only. If instead we want to propagate the analogous objects living in the \( (\varphi, c) \)-space, then the evolution of \( \rho(\varphi, c) \) would be via the \( \mathcal{L} \) of eq. (2.7) but the evolution of the \( \psi(\varphi, c) \) would be via a different Lagrangian studied in ref. [6].

In a manner similar to what is done in field theory we could also introduce the generating functional which would have the form

\[ Z_{CM}[j] = \int D\varphi D\lambda \exp \left[ i \int dt \left( L_B + j \varphi \partial \varphi + j \lambda \partial \lambda \right) \right] \]  \hspace{1cm} (2.22)

where \( j_\varphi, j_\lambda \) are currents associated to the two variables \( \varphi, \lambda \). We shall now derive the operatorial formalism associated to this path integral and compare it with the KvN operatorial version of CM given by eq. (1.2). Having a path integral we can introduce the concept of commutator as Feynman did in the quantum case: given two functions \( O_1(\varphi, \lambda) \) and \( O_2(\varphi, \lambda) \), let us evaluate the following quantity under our path integral:

\[ \langle [O_1, O_2] \rangle \equiv \lim_{\epsilon \to 0} \langle (O_1(t + \epsilon)O_2(t) - O_2(t + \epsilon)O_1(t)) \rangle \]  \hspace{1cm} (2.23)

which is defined as the expectation value of the commutator. In our case what we get is

\[ \langle [\varphi^a, \varphi^b] \rangle = 0 \]  \hspace{1cm} (2.24)
\[ \langle [\varphi^a, \lambda^b] \rangle = i \delta^a_b \]  \hspace{1cm} (2.25)

The first commutators confirm that we are doing CM and not QM. In fact they tell us that the \( q \) commute not only among themselves but also with the \( p \). The second

\[ ^4 \text{The reader may be puzzled that the same kernel propagates both } \psi \text{ and } |\psi|^2. \text{ This is not a problem and does not lead to any contradiction as proved in appendix A of ref. [2].} \]
commutators instead tell us that the $\lambda_a$ are something like the momenta conjugate to $\varphi^a$. In order to satisfy the (2.25) they can be realized as
\[
\hat{\varphi}^a = \varphi^a, \quad \hat{\lambda}_a = -i \frac{\partial}{\partial \varphi^a}
\] (2.26)

Via this operatorial realization of $\lambda_a$ also $H$ of eq. (2.18) can be turned into an operator
\[
H_B \rightarrow \hat{H} = -i \omega^{ab} \partial_b H \partial_a
\] (2.27)
and it is easy to see that $\hat{H} = -i \hat{L}$ where $\hat{L}$ is the Liouville operator of eq. (1.5). So this confirms that the operatorial formalism generated by our path integral is nothing else than the KvN one.

If we specify things to $n = 1$, the relations (2.25) are just
\[
[q, \lambda_q] = i
\] (2.28)
\[
[p, \lambda_p] = i
\] (2.29)
and the representation we used for $\lambda_q$ in (2.20) is not the only possible one. In fact in (2.26) we realized $\varphi^a$ as a multiplicative operator and $\lambda_a$ as a derivative one but, using the fact that (2.28) and (2.29) are two distinct relations, we could have realized $q$ and $\lambda_p$ as multiplicative operators and $\lambda_q$ and $p$ as derivative ones:
\[
\begin{align*}
q &\rightarrow \hat{q} \\
\lambda_p &\rightarrow \hat{\lambda}_p \\
\lambda_q &\rightarrow -i \frac{\partial}{\partial q} \\
p &\rightarrow i \frac{\partial}{\partial \lambda_p}
\end{align*}
\] (2.30)

In the representation (2.26) we would diagonalize $\hat{\varphi}^a$
\[
\hat{\varphi}^a |q, p\rangle = \varphi^a |q, p\rangle
\] (2.31)

and the kernel (2.21) could be formally represented as
\[
K(\varphi^a t | \varphi^a_0 t_0) = \langle qpt | q_0 p_0 t_0 \rangle
\] (2.32)

In the mixed representation (2.30) we would diagonalize instead these other operators:
\[
\hat{\varphi}^a |q, \lambda_p\rangle = q |q, \lambda_p\rangle
\]
\[
\hat{\lambda}_p |q, \lambda_p\rangle = \lambda_p |q, \lambda_p\rangle
\] (2.33)

In this representation we have to do a "partial Fourier transform" in order to go from the states $|qp\rangle$ to the $|q \lambda_p\rangle$ ones. We say "partial Fourier transform" because somehow
we are just "replacing" $p$ in the state $|qp\rangle$ with $\lambda_p$ which is somehow the momentum conjugate to $p$ as can be easily seen from the Lagrangian $L_p$ of eq. (2.17). Using the transformation formula:

$$\langle q'p'|q,\lambda_p \rangle = \frac{1}{\sqrt{2\pi}} \delta(q - q') \exp[ip' \lambda_p]$$  \hspace{1cm} (2.34)

we can transform $\psi$ from the $|q,p\rangle$ basis to the $|q,\lambda_p\rangle$ one:

$$\tilde{\psi}(q,\lambda_p) \equiv \langle q,\lambda_p | \psi \rangle = \int dq'dp' \langle q\lambda_p | q'p' \rangle \cdot \langle q'p' | \psi \rangle = \frac{1}{\sqrt{2\pi}} \int \psi(q,p) e^{-i\lambda_pp} dp$$  \hspace{1cm} (2.35)

In the same manner we can transform the kernel (2.32) and get:

$$\tilde{K}(q\lambda_p t | q_0 \lambda_{p_0} t_0) \equiv \langle q\lambda_p t | q_0 \lambda_{p_0} t_0 \rangle = \int dq'dp_0 dq' dp' \langle q\lambda_p t | q'p' t \rangle \langle q'p' t | q_0 p_0 t_0 \rangle \langle q_0 p_0 t_0 | q_0 \lambda_{p_0} t_0 \rangle = \int dq'dp_0 dq' dp' \frac{1}{\sqrt{2\pi}} e^{-i\lambda_pp'} \delta(q - q') K(q'p' t | q_0 p_0 t_0) \frac{1}{\sqrt{2\pi}} e^{i\lambda_{p_0} p_0} \delta(q_0 - q_0) = \frac{1}{2\pi} \int dp_0 dp' c \exp[-i\lambda_pp'] \exp[i\lambda_{p_0} p_0] K(qp' t | q_0 p_0 t_0)$$  \hspace{1cm} (2.36)

This mixed representation is the most useful one in analyzing the issue of how to go from this formulation of CM to QM \cite{7}. It is also the representation where gauge invariance manifests itself via the multiplication by a phase like in QM. We will examine all this in details in the next section.

### 3 Minimal Coupling in $\mathcal{H}$ and Gauge Invariance

In the previous section we have introduced the concept of superfield, formulas (2.9)-(2.13)-(2.14)-(2.19)-(2.20). Using them it is then easy to put in a compact form the minimal coupling rules (1.11) used in the case of a constant magnetic field. First, via the representation (2.26), the relation (1.11) can be turned into the following one

$$\begin{cases} p_y \rightarrow p_y + \frac{eB}{c} x \\ \lambda_y \rightarrow \lambda_y + \frac{eB}{c} \lambda_{px} \end{cases}$$  \hspace{1cm} (3.1)

Next let us notice what we did in (2.9)-(2.10) in order to go from the Hamiltonian ($H$) in $\mathcal{M}$ to the new Hamiltonian ($\tilde{H}$) in $\mathcal{M}$: we just replaced $\varphi^a$ with the superfield $\Phi^a$.  

\hspace{1cm} \footnote{This is the "analog" of the transformation formula $\langle q|p \rangle = \frac{1}{\sqrt{2\pi \hbar}} \exp[ipq/\hbar]$ between momenta and position eigenstates but here the role of position and momenta is taken respectively by $p$ and $\lambda_p$.}
Let us do the same for the minimal coupling in $\mathcal{M}$ space for the case of a constant magnetic field given by eq. (1.7). It means the following:

\[
p_y \rightarrow p_y - \frac{e}{c} B x
\]

\[
\Phi^p_y \rightarrow \Phi^p_y - \frac{e}{c} B \Phi^x
\]

(3.2)

(3.3)

Expanding (3.3) in $\theta$, $\bar{\theta}$ and using (2.19) and (2.20) we get

\[
p_y - i \bar{\theta} \theta \lambda_y \rightarrow p_y - i \bar{\theta} \theta \lambda_y - \frac{eB}{c} (x + i \bar{\theta} \theta \lambda_p_z)
\]

(3.4)

Comparing the terms with equal number of $\theta$ and $\bar{\theta}$ (3.4) becomes

\[
\begin{cases}
p_y \rightarrow p_y - \frac{eB}{c} x \\
\lambda_y \rightarrow \lambda_y + \frac{eB}{c} \lambda_p_z
\end{cases}
\]

(3.5)

which are exactly the substitution rules (3.1) for the minimal coupling for $\mathcal{H}$. So we can say that the superfield formalism provides a compact way, eq. (3.3), to write the complicated minimal coupling (3.1).

Let us now check if this compact way of expressing things via superfields is an accident of the case of a constant magnetic field or if it holds in general. The Hamiltonian $H$ of a free particle in a generic magnetic field is obtained via the minimal coupling (1.4) and is

\[
H = \frac{1}{2m} \left\{ \left( p_x - \frac{e}{c} A_x \right)^2 + \left( p_y - \frac{e}{c} A_y \right)^2 + \left( p_z - \frac{e}{c} A_z \right)^2 \right\}
\]

(3.6)

The associated $\mathcal{H}_\beta$ of (2.18) is then:

\[
\mathcal{H}_\beta = \frac{\lambda_x}{m} \left( p_x - \frac{e}{c} A_x \right) + \frac{\lambda_y}{m} \left( p_y - \frac{e}{c} A_y \right) + \frac{\lambda_z}{m} \left( p_z - \frac{e}{c} A_z \right) - \lambda_{p_x} \frac{\partial H}{\partial x} - \lambda_{p_y} \frac{\partial H}{\partial y} - \lambda_{p_z} \frac{\partial H}{\partial z} =
\]

\[
= \frac{1}{m} \left( \lambda_x + \frac{e}{c} \frac{\partial A_x}{\partial x} \lambda_{p_x} + \frac{e}{c} \frac{\partial A_y}{\partial y} \lambda_{p_y} + \frac{e}{c} \frac{\partial A_z}{\partial z} \lambda_{p_z} \right) \left( p_x - \frac{e}{c} A_x \right) + \frac{1}{m} \left( \lambda_y + \frac{e}{c} \frac{\partial A_x}{\partial x} \lambda_{p_x} + \frac{e}{c} \frac{\partial A_y}{\partial y} \lambda_{p_y} + \frac{e}{c} \frac{\partial A_z}{\partial z} \lambda_{p_z} \right) \left( p_y - \frac{e}{c} A_y \right) + \frac{1}{m} \left( \lambda_z + \frac{e}{c} \frac{\partial A_x}{\partial x} \lambda_{p_x} + \frac{e}{c} \frac{\partial A_y}{\partial y} \lambda_{p_y} + \frac{e}{c} \frac{\partial A_z}{\partial z} \lambda_{p_z} \right) \left( p_z - \frac{e}{c} A_z \right)
\]

(3.7)

So we see that this last expression can be obtained from the $\mathcal{H}$ of the free particle

\[
\mathcal{H} = \frac{1}{m} \lambda_x p_x + \frac{1}{m} \lambda_y p_y + \frac{1}{m} \lambda_z p_z
\]

(3.8)
via the simple substitution

\[
\begin{cases}
  p_i \rightarrow p_i - \frac{e}{c}A_{q_i} \\
  \lambda_{q_i} \rightarrow \lambda_{q_i} + \frac{e}{c} \sum_j \left( \lambda_{p_j} \frac{\partial A_{q_i}}{\partial q_j} \right)
\end{cases}
\] (3.9)

This is the minimal coupling for $\mathcal{H}$ in a generic magnetic field generalizing the one in a constant magnetic field given by eq. (3.5). We have indicated with $A_{q_i}$ the spatial components of the gauge field. We want now to see if (3.9) can be derived from the superfield generalization of the standard MC, i.e.:

\[
\begin{aligned}
  p_i &\rightarrow p_i - \frac{e}{c}A_{q_i}(q) \\
  \Phi^{p_i} &\rightarrow \Phi^{p_i} - \frac{e}{c}A_{q_i}(\Phi^q)
\end{aligned}
\] (3.10)

Let us first notice that, if we neglect the $c, \bar{c}$ variables, we have that the following relation holds:

\[
i\int d\theta d\bar{\theta}A_{q_i}[\Phi^q] = -\sum_j \lambda_{p_j} \frac{\partial A_{q_i}}{\partial q_j} \equiv A_{q_i}
\] (3.12)

This can be checked easily using eq. (2.12). If we now expand (3.11) in $\theta, \bar{\theta}$ and we neglect the $c, \bar{c}$, what we get is:

\[
p_i - i\bar{\theta}\lambda_{q_i} \rightarrow p_i - i\bar{\theta}\lambda_{q_i} - \frac{e}{c}(A_{q_i} + i\bar{\theta}A_{q_i})
\] (3.13)

Comparing the terms above with the same number of $\theta, \bar{\theta}$ we obtain

\[
\begin{cases}
  p_i \rightarrow p_i - \frac{e}{c}A_{q_i} \\
  \lambda_{q_i} \rightarrow \lambda_{q_i} + \frac{e}{c} \sum_j \left( \lambda_{p_j} \frac{\partial A_{q_i}}{\partial q_j} \right)
\end{cases}
\] (3.14)

which are exactly the relations (3.9). So this proves that (3.11) is the most compact way to write the minimal coupling for $\mathcal{H}$.

The reader may ask which is the physical reason why $\lambda_{q_i}$ should be changed, like we do in eq. (3.9), when we turn on the magnetic field. To answer this question we have to analyze the issue of the gauge invariance of the system. Let us remember that the Lagrangian associated to the $H$ of eq. (3.6) was

\[
L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{e}{c}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z)
\] (3.15)
where

\[
\begin{align*}
\dot{x} &= \frac{1}{m}(p_x - e A_x) \\
\dot{y} &= \frac{1}{m}(p_y - e A_y) \\
\dot{z} &= \frac{1}{m}(p_z - e A_z)
\end{align*}
\] (3.16)

The velocities which appear above are measurable quantities and so they must be gauge-invariant. Since \(A_q\) transform under a gauge transformation as \(A_q + \partial_q \alpha(q)\) with \(\alpha(q)\) an arbitrary function, the \(p_i\) must transform as

\[p_i \rightarrow p_i + \frac{e}{c} \partial_q \alpha(q)\] (3.17)

The Hamiltonian \(H\), being a combination of gauge-invariant quantities like \(p_i - \frac{e}{c} A_q i\), is gauge invariant while the \(L\) of eq. (3.15) changes by a total derivative. As the Hamiltonian \(H\) is basically the energy of the system it must be gauge invariant. Let us now ask ourselves how \(\lambda\) should change under a gauge transformation. If we adopt the compact notation (3.11) we used for the minimal coupling of \(H\), it seems natural that the superfield analog of the gauge transformation (3.17) should be

\[\Phi^p i \rightarrow \Phi^p i + \frac{e}{c}[\partial_q \alpha](\Phi^q)\] (3.18)

and

\[A_q(\Phi^q) \rightarrow A_q(\Phi^q) + [\partial_q \alpha](\Phi^q)\] (3.19)

where by \([\partial_q \alpha](\Phi^q)\) we mean that we insert \(\Phi^q\) in place of \(q\) in the function that we obtain by making the derivative of \(\alpha(q)\) with respect to \(q_i\). Expanding (3.18) in \(\theta, \bar{\theta}\) we get (3.17) as first component and the following one as last component

\[\lambda_q \rightarrow \lambda_q + i \int d\theta d\bar{\theta} \frac{e}{c}[\partial_q \alpha](\Phi^q) = \lambda_q - \frac{e}{c} \lambda_p j \partial_q \alpha(q)\] (3.20)

Similarly expanding (3.20) in \(\theta, \bar{\theta}\) we get the usual transformation \(A_q \rightarrow A_q + \partial_q \alpha\) as first component and the following one as last component:

\[A'_{q_i} = A_{q_i} + \partial_q \bar{\alpha}(q, \lambda_p)\] (3.21)

where

\[\bar{\alpha}(q, \lambda_p) = - \sum_j \lambda_{p_j} \frac{\partial \alpha}{\partial q_j}\] (3.22)

It is then easy to see that the combination \(\lambda_q - \frac{ie}{c} \int d\theta d\bar{\theta} A_q(\Phi^q) = \lambda_q - \frac{e}{c} A_q\), which is the \(\lambda_q\) components of eq. (3.11), is gauge invariant if we gauge transform \(\lambda_q\) as in (3.20) and \(A_q(\Phi^q)\) as in (3.19). So the RHS of (3.9) are gauge invariant...
quantities and, as a consequence, also the $H_B$ of eq. (3.4) is gauge invariant because it is built out of the combinations (3.9). Of course all this is very formal and stems from the extension of the standard gauge transformations to the superfields, (3.18)-(3.19). While $H$, being the energy, must be gauge invariant there is apparently no reason why $H$ should be gauge invariant. Similarly, while we know that $p_i$ should change under a gauge transformation like in (3.17) in order to make gauge invariant the velocities (3.16) which are observables, there is apparently no physical reason why $\lambda_{qi}$ should change under a gauge transformation as in (3.20). Actually there is a physical reason and it is the following. As the velocities (3.16) are gauge invariant then their evolution has to be gauge invariant too. The evolution can occur via $H$ and the standard Poisson brackets

$$\{\varphi^a, \varphi^b\}_{pb} = \omega^{ab}$$

or via $H$ and some extended Poisson brackets $\{\ , \ \}_{epb}$ which were introduced in ref. 4 (for a brief review see appendix A). In this extended formalism the Hamiltonian $H$ (3.7) can be written, using the notation (3.12), in the following compact way:

$$H = \frac{1}{m} \sum_i \left( \lambda_{qi} - \frac{e}{c} A_{qi} \right) \left( p_i - \frac{e}{c} A_{qi} \right)$$

The gauge invariant velocities

$$v_{qi} = \frac{1}{m} \left( p_i - \frac{e}{c} A_{qi} \right)$$

evolve via the extended Poisson brackets in the following manner:

$$\dot{v}_{qi} = \{v_{qi}, H\}_{epb}$$

For example the result for $v_x$ is

$$\dot{v}_x = \frac{e}{mc} \left( B_z v_y - B_y v_z \right)$$

If we now use a different gauge and just change the gauge field $A_{qi}$ and $p_i$, but not $\lambda_{qi}$, we would get as new $H$:

$$H' = \frac{1}{m} \sum_i \left( \lambda_{qi} - \frac{e}{c} A'_{qi} \right) \left( p'_i - \frac{e}{c} A'_{qi} \right)$$

where with $A'_{qi}$, $p'_i$ and $A'_{qi}$ we indicate the quantities gauge transformed according to (3.18)-(3.19).

The evolution of the velocity $v_x$, via the gauge transformed $H'$, would turned out to be:

$$\dot{v}_x = \{v_x, H'\}_{epb} = \frac{e}{mc} \left( B_z v_y - B_y v_z \right) + \frac{e}{mc} \left[ (\partial_x^2 \alpha) v_x + (\partial_y \partial_x \alpha) v_y + (\partial_z \partial_x \alpha) v_z \right]$$

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So we notice that the evolution is not anymore gauge invariant because it depends on the gauge parameters \( \alpha \) which appear on the RHS of (3.29). This is absurd because the velocities are gauge invariant quantities and so their evolution must maintain their gauge invariance. This lack of gauge invariance is the price we would have paid by not allowing \( \lambda_q \) to change under a gauge transformation or, equivalently, by not allowing \( \lambda \) to enter \( \mathcal{H} \) via the MC combination of eq. (3.9). Note that the \( \lambda_p \), differently than the \( \lambda_q \), do not have to be changed at all in order to maintain the gauge invariance of the system. This answers one question we raised in the introduction and concludes the analysis of the issue of the gauge invariance.

Up to now we have regarded \( \mathcal{H} \) as a function endowed with its own extended Poisson brackets. We want now to proceed to analyze the same issue of gauge invariance when we turn \( \mathcal{H} \) into an operator \( \hat{\mathcal{H}} \) like we did in formula (2.27). Let us first briefly review what happens in QM, following ref. [3]. In classical mechanics the gauge transformations leave \( q \) invariant but change \( p_i \) as follows

\[
p'_i = p_i + \frac{e}{c} \partial_{q_i} \alpha(q) \quad (3.30)
\]

and, as a consequence, the Poisson brackets are left invariant under these transformations.

\[
\{q_i, p_j\} = \{q_i, p'_j\} = \delta_{ij} \quad (3.31)
\]

Then the quantization rules for the transformed variables are

\[
\{q_i, p'_j\} = \delta_{ij} \quad \rightarrow \quad \{\hat{q}_i, \hat{p}'_j\} = i\hbar \delta_{ij} \quad (3.32)
\]

This implies that \( \hat{p}'_j \) can be realized operatorially like the original \( \hat{p}_j \), i.e. \( \hat{p}'_j = -i\hbar \frac{\partial}{\partial q_i} \).

The quantum Hamiltonian is then

\[
\hat{H} = \left(-i\hbar \frac{\partial}{\partial q_i} - \frac{e}{c} \hat{A}_q \right)^2 / 2m \quad (3.33)
\]

As the \( \hat{p}'_i \) have been realized as before, if we do a gauge transformation we get that only \( \hat{A}_i \) change in \( \hat{H} \) and the new Hamiltonian is:

\[
\hat{H}' = \left[-i\hbar \frac{\partial}{\partial q_i} - \frac{e}{c} (\hat{A}_q + \partial_{q_i} \hat{\alpha}) \right] / 2m \quad (3.34)
\]

It is easy to check that one can pass from \( \hat{H} \) to \( \hat{H}' \) via a unitary transformation

\[
\hat{H}' = U \hat{H} U^{-1} \quad (3.35)
\]

where \( U = \exp \left( i \frac{e}{\hbar} \alpha(q) \right) \). So, differently than for the function \( H \) of CM, the Hamiltonian is not gauge invariant, but what is important is that the expectation values are gauge invariant. In fact, if the \( H \) transforms as (3.33), the states change as

\[
|\psi'\rangle = U |\psi\rangle \quad (3.36)
\]
In the $|q\rangle$ representation this becomes
\[ \psi'(q) = \exp \left( i \frac{e}{\hbar} \alpha(q) \right) \psi(q) \]
(3.37)
which is the usual transformation by a phase under gauge transformations. We can notice that the expectation values of $\langle \psi'|\hat{p}_i'|\psi'\rangle$ and $\langle \psi|\hat{p}_i|\psi\rangle$ are related exactly as the classical momenta in (3.30):
\[ \langle \psi'|\hat{p}_i'|\psi'\rangle = \langle \psi|\hat{p}_i|\psi\rangle + \frac{e}{c} \partial_{q_i} \alpha(q) \]
(3.38)
Let us now turn to the KvN operatorial theory and check how the gauge transformations are implemented. At the operatorial level we have to construct everything so that the expectation values of $\langle \psi'|\hat{\lambda}_q' - e\hat{A}_q'|\psi'\rangle$ and $\langle \psi|\hat{\lambda}_q - e\hat{A}_q|\psi\rangle$ (3.39) would be gauge invariant. Let us start by noticing that the commutation relations $[\varphi^a, \lambda_b] = i\delta^a_b$ are the operatorial counterpart of the extended Poisson brackets $\{\varphi^a, \lambda_b\}_{\text{epb}} = \delta^a_b$ and the gauge transformed coordinates $\varphi'^a, \lambda'_b$ under (3.17) and (3.20) have the same epb $\{\varphi'^a, \lambda'_b\}_{\text{epb}} = \delta^a_b$ as the original variables. So we expect that also the associated commutators among the gauge transformed operators would be the same as the original one:
\[ [\varphi'^a, \lambda'_b] = i\delta^a_b \]
(3.40)
This means that we can represent the $\varphi'^a$ and $\lambda'_b$ in the same manner as the $\varphi^a$ and $\lambda_b$. As a consequence the gauge transformed version of the quantities in (3.39) is:
\[ \langle \psi'|\hat{\lambda}_q' - e\hat{A}_q'|\psi'\rangle = \langle \psi|\hat{\lambda}_q - e\hat{A}_q|\psi\rangle \]
(3.41)
Note that, via the introduction of the following operator
\[ \bar{U} = \exp \left\{ - \frac{e}{c} \hat{\lambda}_{pq} [\partial_{q_p}, \tilde{\alpha}]\tilde{\lambda}(\tilde{q}) \right\} \]
(3.42)
we can write the following transformations:
\[ \hat{p}_i - \frac{e}{c} \hat{A}_q - \frac{e}{c} [\partial_{q_i}, \alpha](\tilde{q}) = \bar{U} \left[ \hat{p}_i - \frac{e}{c} \hat{A}_q \right] \bar{U}^{-1} \]
\[ \hat{\lambda}_q - \frac{e}{c} \hat{A}_q - \frac{e}{c} [\partial_{q_i}, \tilde{\alpha}](\tilde{q}, \tilde{\lambda}_p) = \bar{U} \left[ \hat{\lambda}_q - \frac{e}{c} \hat{A}_q \right] \bar{U}^{-1} \]
(3.43)
This implies that (3.41) will be gauge invariant provided we transform the states as follows
\[ |\psi'\rangle = \bar{U} |\psi\rangle = \exp \left\{ - \frac{e}{c} \hat{\lambda}_{pq} [\partial_{q_p}, \tilde{\alpha}]\tilde{\lambda}(\tilde{q}) \right\} |\psi\rangle \]
(3.44)
Let us now represent this transformation law on the two basis given by (2.31) and (2.33). In the basis (2.33) we have from (3.44)

\[ \psi'(q, \lambda_p) \equiv \langle q, \lambda_p | \psi' \rangle = \langle q, \lambda_p | \exp\left\{ -\frac{ie}{c} \hat{\lambda}_p \partial_q \alpha(q) \right\} | \psi \rangle = \exp\left( -\frac{ie}{c} \alpha(q) \right) \langle q, \lambda_p | \psi \rangle = \exp\left( \frac{ie}{c} \tilde{\alpha} \right) \psi(q, \lambda_p) \] (3.45)

So in this basis the gauge transformation is just the multiplication by a local phase factor \( \tilde{\alpha} \) in the space \((q, \lambda_p)\) where \( \tilde{\alpha} \) has been defined in (3.22).

Now let us represent (3.44) in the \((q, p)\) basis (2.31) and let us make use of the transformation formula (2.34). What we get is:

\[ \psi'(q, p) = \langle qp | \psi' \rangle = \int dq' dp' \langle qp | q' \lambda'_p \rangle \langle q' \lambda'_p | \tilde{U} | \psi \rangle = \int \frac{d\lambda'_p}{\sqrt{2\pi}} \exp[ip\lambda'_p] \exp\left[ i\frac{e}{c} \tilde{\alpha}(q, \lambda'_p) \right] \langle q\lambda'_p | \psi \rangle \] (3.46)

Inserting a further completeness we obtain

\[ \psi'(q, p) = \int d\lambda'_p dp' \delta\left( p - p' - \frac{e}{c} \partial \alpha \right) \psi(q, p') = \psi\left( q, p - \frac{e}{c} \partial \alpha \right) \] (3.47)

So in the \((q, p)\) representation of our Hilbert space the gauge transformations are not implemented by the multiplication by a local phase, like in the \((q, \lambda_p)\) representation, but by just a shift in the argument \( p \) of the wave function. It is easy to show, as we will do in the appendix B, that the Liouville eq. (1.2) is invariant in form under the gauge transformations. In appendix C we will show that the phase \( \exp\left( \frac{ie}{c} \tilde{\alpha} \right) \) of (3.45) is exactly the one that can ”pass through”\(^6\) the Hamiltonian \( \hat{H} \) of KvN if we change the gauge field as we do in (3.19). In that appendix we shall also explore which gauge fields have to be inserted in \( \hat{H} \) to allow for a general \(^7\) phase \( \exp[i\alpha(q, \lambda_p)] \) to ”pass through” \( \hat{H} \).

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\(^6\) By ”pass through” we mean a procedure explained in appendix C.

\(^7\) By ”general” we mean one not of the form \( \exp\left( \frac{ie}{c} \tilde{\alpha} \right) \).
4 Landau Problem

In this section we will make a first application of the minimal coupling scheme that we have developed previously for the KvN formalism. This first application is the Landau problem. We will first review it in quantum mechanics and then turn to the classical operatorial version of KvN. The Landau problem is concerned with the dynamics of a particle under a constant magnetic field directed along \( z \). We make the following choice for the gauge potential:

\[
A_x = 0, \quad A_y = Bx, \quad A_z = 0
\] (4.1)

The Schrödinger Hamiltonian is then

\[
\hat{H} = \frac{1}{2m} \left[ \hat{p}_y^2 + \left( \hat{p}_y - \frac{eB}{c} \hat{x} \right)^2 + \hat{p}_z^2 \right]
\] (4.2)

As \( \hat{p}_y, \hat{p}_z \) commute with \( \hat{H} \) we can diagonalize all these three operators simultaneously. The eigenfunctions will then be labelled by the eigenvalues of \( \hat{H} \), i.e. \( E \), and by those of \( \hat{p}_y, \hat{p}_z \) which are \( p^0_y \) and \( p^0_z \). Their form will be

\[
\psi_{E,p^0_y,p^0_z}(x,y,z) = \frac{1}{2\pi\hbar} \exp \left[ \frac{i}{\hbar} \left( p^0_y y + p^0_z z \right) \right] \psi(x)
\] (4.3)

The stationary eigenvalue problem

\[
\hat{H} \psi_{E,p^0_y,p^0_z} = E \psi_{E,p^0_y,p^0_z}
\] (4.4)

leads to the following differential equation for \( \psi(x) \)

\[
- \frac{\hbar^2}{2m} \psi''(x) + \frac{1}{2m} \left( p^0_y - \frac{eB}{c} \right)^2 \psi(x) = \left( E - \frac{p^0_y^2}{2m} \right) \psi(x)
\] (4.5)

Indicating with \( E_t \) the quantity

\[
E_t \equiv E - \frac{p^0_z^2}{2m}
\] (4.6)

and making a change of variables from \( x \) to \( x' \), with \( x' \equiv p^0_y - \frac{eB}{c} \), eq. (4.5) is turned into the following one

\[
- \frac{\hbar^2}{2m} \psi''(x') + \frac{1}{2m} \left[ \frac{c}{eB} \right]^2 x'^2 \psi(x') = \left[ \frac{c}{eB} \right]^2 E_t \psi(x')
\] (4.7)

We can immediately notice that this is like an harmonic oscillator eigenvalue problem with the frequency replaced by

\[
\omega \equiv \frac{c}{eBm}
\] (4.8)
and with the energy replaced by \( \left[ \frac{c}{eB} \right]^2 E_t \). This quantity is discretized like in the harmonic oscillator problem:

\[
\left[ \frac{c}{eB} \right]^2 E_t = \hbar \omega \left( n + \frac{1}{2} \right) \tag{4.9}
\]

Combining (4.6) and (4.8) we get from (4.9)

\[
E_{n,p_0^y} = \frac{eB \hbar}{mc} \left( n + \frac{1}{2} \right) + \frac{p_0^z}{2m} \tag{4.10}
\]

So the eigenfunctions (4.3) can be labelled by the quantum numbers \((n, p_0^y, p_0^z)\): i.e. \(\psi_{n,p_0^y,p_0^z}\). Note that these wave functions are degenerate because all those with different values of \(p_0^y\) have the same \(E_{n,p_0^y}\). So there is an infinite degeneracy.

Let us now analyze the same problem at the classical level using the operatorial formalism of KvN. Using (3.24) and the gauge choice (4.1), the \(\mathcal{H}\) is

\[
\mathcal{H} = \frac{1}{m} \lambda_x p_x + \frac{1}{m} \left( \lambda_y - \frac{e}{c} A_y \right) \left( p_y - \frac{e}{c} A_y \right) + \frac{1}{m} \lambda_z p_z \tag{4.11}
\]

We will now turn \(\mathcal{H}\) into an operator \(\hat{\mathcal{H}}\) using the "mixed" representation (2.30) and what we get is:

\[
\hat{\mathcal{H}} = \frac{1}{m} \partial_x \partial_{\lambda p_x} + \frac{1}{m} \left( -i \frac{\partial}{\partial y} + \frac{eB}{c} \lambda_p \right) \left( i \frac{\partial}{\partial \lambda p_y} - \frac{eB}{c} \lambda_p \right) + \frac{1}{m} \partial_z \partial_{\lambda p_z} \tag{4.12}
\]

Let us now diagonalize this operator. The reason to do that is because in the KvN theory the equation to solve is (1.2) which, because of (2.27), can be written as

\[
i \partial_t \psi = \hat{\mathcal{H}} \psi \tag{4.13}
\]

So, like for the Schrödinger equation, one should first diagonalize \(\hat{\mathcal{H}}\)

\[
\hat{\mathcal{H}} \psi_E = \bar{E} \psi_E \tag{4.14}
\]

and then write a generic wave function as

\[
\psi(t) = \sum_E C_E e^{-i \bar{E} t} \psi_E \tag{4.15}
\]

where the \(C_E\) are derived from the expansion of the initial \(\psi\) on the \(\psi_E\). We want to underline that the \(\bar{E}\) that appear in eqs. (1.14)-(1.13) have nothing to do with the physical energy of the system. They are simply the possible eigenvalues of the evolution operator \(\hat{\mathcal{H}}\) and, using them and the associated eigenfunctions, we can deduce the evolution of the \(\psi\) like it is done in formula (1.13). For more details about this and on the manner to reconstruct the standard deterministic motion of CM see ref. [2].
Let us now turn to (4.12) and diagonalize it like in (4.14). Note that the operators: 
\[-i \frac{\partial}{\partial y}, i \frac{\partial}{\partial \lambda_y}, -i \frac{\partial}{\partial z}, i \frac{\partial}{\partial \lambda_z}\]
commute with \(\hat{H}\) and so we can diagonalize simultaneously these five operators. The generic eigenfunction (in the mixed representation \(q, \lambda_p\)) has the form
\[
\psi(q, \lambda_p) = \frac{1}{(2\pi)^2} \exp[i\lambda_0^0 y - i\lambda_y p_y^0] \exp[i\lambda_z^0 z - i\lambda_z p_z^0] \psi(x, \lambda_p)
\] (4.16)
where \(\lambda_0^0, p_y^0, \lambda_z^0, p_z^0\) are eigenvalues of the operators 
\[-i \frac{\partial}{\partial y}, i \frac{\partial}{\partial \lambda_y}, -i \frac{\partial}{\partial z}, i \frac{\partial}{\partial \lambda_z}\] respectively. We see the similarity with the quantum case except for the fact that the dimension of the space is double. Inserting (4.16) in (4.14) we get the equation
\[
\left[ \frac{1}{m} \frac{\partial^2}{\partial x^2} + \frac{1}{m} \left( \lambda_y^0 + \frac{eB}{c} \lambda_y^0 \right) \left( p_y^0 - \frac{eB}{c} x \right) + \frac{1}{m} \lambda_z^0 p_z^0 \right] \psi(x, \lambda_p) = \tilde{E} \psi(x, \lambda_p)
\] (4.17)
Via the new quantity
\[
\tilde{E}_+ \equiv \tilde{E} - \frac{1}{m} \lambda_z^0 p_z^0
\] (4.18)
we can rewrite eq. (4.17) as
\[
\left[ \frac{1}{m} \frac{\partial^2}{\partial x^2} + \frac{1}{m} \left( \lambda_y^0 + \frac{eB}{c} \lambda_y^0 \right) \left( p_y^0 - \frac{eB}{c} x \right) \right] \psi(x, \lambda_p) = \tilde{E}_+ \psi(x, \lambda_p)
\] (4.19)
Doing now the following change of variables
\[
x' \equiv x - \frac{c}{eB} p_y^0 \\
\lambda_{p'_x} \equiv \lambda_{p_x} + \frac{c}{eB} \lambda_{y}^0
\] (4.20)
we can rewrite eq. (4.19) as
\[
\left[ \frac{1}{m} \frac{\partial^2}{\partial x'^2} - \frac{1}{m} \left( \frac{eB}{c} \right)^2 \lambda_{p'_x} x' \right] \psi(x', \lambda_{p'_x}) = \tilde{E}_+ \psi(x', \lambda_{p'_x})
\] (4.21)
The dimensions of the various quantities are such that we can write the above equation as
\[
\left[ \frac{1}{m} \frac{\partial^2}{\partial x'^2} - m\omega^2 \lambda_{p'_x} x' \right] \psi(x', \lambda_{p'_x}) = \tilde{E}_+ \psi(x', \lambda_{p'_x})
\] (4.22)
where \(\omega \equiv \frac{eB}{mc}\) has the dimension of an angular velocity and it is related to the well-known Larmor frequency of rotation of a particle in a constant magnetic field. Eq. (4.22) is the KvN eigenvalue equation for an harmonic oscillator which is studied in
details in appendix D which we advice the reader to go through before going on. The spectrum of the $\hat{H}$ for the harmonic oscillator (see appendix D) is given by

$$E_{\pm N}^{osc} = N\omega, \quad N = \cdots, -2, -1, 0, 1, 2, \cdots$$  \hspace{1cm} (4.23)

So, using (4.18), the final spectrum of eq. (4.17) is

$$\tilde{E} = N\left(\frac{eB}{mc}\right) + \frac{1}{m}\lambda_0^0 p_z^0$$  \hspace{1cm} (4.24)

and the wave functions are any linear combination of the following eigenfunctions (see appendix D):

$$\psi_{N,n,\lambda_y^0 p_y^0,\lambda_z^0 p_z^0} = \frac{1}{2\pi} \exp[i\lambda_y^0 y - i\lambda_y^0 p_y^0] \exp[i\lambda_z^0 z - i\lambda_z^0 p_z^0] \cdot \psi_n^{osc}(Z_+)\psi_n^{osc}(Z_-)$$  \hspace{1cm} (4.25)

where $\psi_n^{osc}$ are the eigenfunctions of the quantum 1-dim harmonic oscillator\footnote{Note that we have a discretization phenomenon even at the classical level for the eigenvalues of the Liouvillian. This is related to the requirement of single valuedness of the KvN states as explained in appendix D.} with $\hbar$ replaced by an arbitrary quantity $\Delta$ which has the dimension of an action and with $Z_+, Z_-$ defined as:

$$Z_+ = \frac{x' + \Delta \lambda_{p_z}}{\sqrt{2}}, \quad Z_- = \frac{x' - \Delta \lambda_{p_z}}{\sqrt{2}}$$  \hspace{1cm} (4.26)

We see from the form of the wave functions in (4.25) that the degeneracy is much more than in the quantum case. Not only the eigenfunctions with different values of $p_y^0$ have the same $\tilde{E}$, but the same happens for those eigenfunctions with different values\footnote{$n = -N, -N + 1, -N + 2, \cdots$ if $N$ is negative and $n = 0, 1, 2, \cdots$ if $N$ is positive.} of $\lambda_y^0$, $n$ and of $\lambda_z^0, p_z^0$. So it is a much more wider degeneracy than in the quantum case. The reader may wonder why this happens. We feel that this may be due to the fact that the ”wave functions” in the KvN formalism have a number of variables $(q, \lambda_p)$ that is double than in QM.

The reader may be puzzled by the presence in the eigenfunctions (4.25) of the arbitrary quantity $\Delta$ which did not appear in the original $\hat{H}$. This is not a problem. In fact once we are given an initial wave function $\tilde{\psi}(q, \lambda_p)$ not depending on $\Delta$, we will expand it on the basis (4.25) as:

$$\tilde{\psi}(q, \lambda_p) = \sum_{N,n,\cdots} \left(C_{N,n,\lambda_y^0 p_y^0,\lambda_z^0 p_z^0} \psi_{N,n,\lambda_y^0 p_y^0,\lambda_z^0 p_z^0}\right)$$  \hspace{1cm} (4.27)

where $C_{N,n,\cdots}$ are coefficients which will depend on $\Delta$ and this dependence will compensate the one contained in the $\psi_{N,n,\cdots}$. The evolution in $t$ will not reintroduce the dependence on $\Delta$ because the eigenvalues of $\hat{H}$ do not depend on $\Delta$ as shown in appendix D. Solving a classical system via its KvN states is like working in the Schrödinger picture. We could also work out the analog of the Heisenberg picture and this is done for the Landau problem in appendix E.
5 Aharonov-Bohm Phenomenon

The second application of the MC that we will study is the well-known Aharonov-Bohm (AB) effect, [8]. This is a phenomenon which proves that the QM wave functions are changed by the presence of a gauge potential even in regions where the magnetic field associated to this gauge potential is zero. The change in the wave functions can be detected by an interference experiment. The classical motion instead feels only the magnetic field and not the gauge potential. In this section we will study this phenomenon not by looking at wave functions but at the spectrum of respectively the Schrödinger operator \( \hat{H} \) and the classical KvN Liouville operator \( \hat{H} \). The geometrical set up that we will use for the AB effect is illustrated in Figure 1 and it has been suggested in the book of Sakurai [3]. Basically we have two infinitely long cylinders one inside the other. We will study the Schrödinger operator \( \hat{H} \) in the region in between the two cylinders. We will show, as Sakurai [3] indicated, that the spectrum of \( \hat{H} \) changes once we turn on the magnetic field inside the smaller cylinder. In the region in between the two cylinders the magnetic field is zero because the smaller cylinder shields completely the magnetic field. So we are exactly in an AB configuration: zero magnetic field and non-zero gauge potential. Using the same geometrical configuration, we will study the spectrum of the KvN-Liouville operator \( \hat{H} \) and we will prove that it does not change once we turn on the magnetic field differently than what happens in QM. We feel that this, in the framework of the operatorial formulation of CM, is the best mathematical proof that there is no AB effect in CM.

Let us now study the Schrödinger operator in the geometrical set up of Figure 1 and let us do it at first without magnetic field. The Schrödinger operator in cylindrical coordinates

\[
\begin{align*}
0 & = \rho \cos \varphi \\
y & = \rho \sin \varphi \\
z & = z
\end{align*}
\]

is for a free particle:

\[
\hat{H} = -\frac{\hbar^2}{2\mu} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} \right)
\]

and the eigenvalues equation is

\[
-\frac{\hbar^2}{2\mu} \left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} \right) \psi(\rho, \varphi, z) = E \psi(\rho, \varphi, z)
\]  

As the operators \( \frac{\partial}{\partial \varphi} \) and \( \frac{\partial}{\partial z} \) commute with \( \hat{H} \) we can diagonalize these three operators simultaneously and have

\[
\psi(\rho, \varphi, z) = \frac{1}{2\pi} \exp \left[ \frac{ip_0 z}{\hbar} \right] \exp[im\varphi] R(\rho)
\]

\[
\text{for } p_0 > 0, \quad \text{and } P(\rho) = \frac{1}{\rho} \exp \left[ \frac{ip_0 z}{\hbar} \right] \exp[im\varphi] R(\rho)
\]

\[
\text{for } p_0 < 0
\]
where $p_0^2$ is a fixed value and $m$ is an integer. Inserting (5.4) in (5.3) we get the following equation for $R(\rho)$

$$R''(\rho) + \frac{R'(\rho)}{\rho} + \left(\bar{s} - \frac{m^2}{\rho^2}\right)R(\rho) = 0$$

(5.5)

where

$$\bar{s} \equiv \frac{2\mu E}{\hbar^2} - \frac{p_0^2}{\hbar^2}$$

(5.6)

If $\bar{s} = 1$ then eq. (5.5) would be the well-known Bessel equation [9]. We can get it by using the new variables $r \equiv \sqrt{\bar{s}} \rho$. In "$r$" eq. (5.5) is

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} + \left(1 - \frac{m^2}{r^2}\right)R = 0$$

(5.7)

As we want the wave function to be confined between the two cylinders, the boundary conditions should be

$$R(\sqrt{sa}) = R(\sqrt{sb}) = 0$$

(5.8)

where $a$ and $b$ are the radii of respectively the smaller and the larger cylinder, see Figure 1. The general solution of eq. (5.7), with $m$ integer, is given by the linear combination of the Bessel functions of the first and second kind which are [9]:

$$J_m(r) = \left(\frac{r}{2}\right)^m \sum_{n=0}^{\infty} \frac{(-1)^n (r/2)^{2n}}{n! \Gamma(n + m + 1)}$$

$$Y_m(r) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}[J_{m+\epsilon}(r) - (-1)^m J_{-m-\epsilon}(r)]$$

(5.9)

The general solution will then be

$$R(\sqrt{\bar{s}} \rho) = AJ_m(\sqrt{\bar{s}} \rho) + BY_m(\sqrt{\bar{s}} \rho)$$

(5.10)

Imposing the boundary conditions (5.8) we will get the spectrum of the system. In order to simplify things we will consider the limiting case in which the radius of the internal cylinder $a$ goes to zero. In this case the boundary conditions (5.8) become

$$R(0) = 0, \quad R(\sqrt{\bar{s}} b) = 0$$

(5.11)

It is well known [9] that the Bessel functions of the second kind $Y$ are singular in the origin, so we restrict ourselves to solutions (5.10) of the form:

$$R(\sqrt{\bar{s}} \rho) = AJ_m(\sqrt{\bar{s}} \rho)$$

(5.12)

With this choice the first of the boundary conditions (5.11) is automatically satisfied because [9] $J_m(0) = 0$ for $m \geq 1$. We have to satisfy only the second one of the conditions (5.11) which implies:

$$J_m(\sqrt{\bar{s}} b) = 0$$

(5.13)
This relation tells us that we have to look for the zeros of the above Bessel functions. Let us call them \( \alpha_{k,m} \) where \( m \) indicates to which Bessel function we refer to and \( k \) labels the various zeros of the \( m \)-Bessel function in increasing order, so \( k = 1, 2, 3, \ldots \). The solutions of eq. (5.13) can then be formally written as

\[
\sqrt{s}b = \alpha_{k,m}
\]  

(5.14)

Replacing \( \bar{s} \) in the equation above with its expression (5.6), we get that

\[
E_{k,m} = \hbar^2 \frac{\alpha_{k,m}^2}{2\mu b^2} + \frac{p_z^2}{2\mu}
\]

(5.15)

These are the energy levels. If we choose \( m = 1 \) and the second zero \( (k = 2) \) which is \( \alpha_{2,1} \approx 3.83 \), we get

\[
E_{2,1} = \hbar^2 \frac{7.33}{\mu b^2} + \frac{p_z^2}{2\mu}
\]

(5.16)

see Figure 2.

Let us now turn on the magnetic field \([8]\) which we want to be zero everywhere except for the \( B_z \) component on the line \( x^2 + y^2 = 0 \) and with a fixed flux \( \Phi_B \). A choice of

the gauge potential \([8]\) is

\[
\begin{align*}
A_x &= \frac{-y\Phi_B}{2\pi(x^2 + y^2)} \\
A_y &= \frac{x\Phi_B}{2\pi(x^2 + y^2)} \\
A_z &= 0
\end{align*}
\]

(5.17)

Turning to cylindrical coordinates we have that (5.17) is equivalent to

\[
A_\rho = 0, \quad A_\phi = \frac{\Phi_B}{2\pi \rho}, \quad A_z = 0
\]

(5.18)

So with the above choice of gauge the minimal coupling affects only the \(-i\hbar \frac{\partial}{\partial \varphi}\):

\[
-i\hbar \frac{\partial}{\partial \varphi} \rightarrow -i\hbar \frac{\partial}{\partial \varphi} - \frac{e \Phi_B}{c 2\pi}
\]

(5.19)

The associated Schrödinger operator (5.2) is then:

\[
\hat{H} \rightarrow \hat{H}_B = -\frac{\hbar^2}{2\mu} \left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \left( \frac{\partial}{\partial \varphi} - \frac{ie}{ch} \Phi_B \right)^2 + \frac{\partial^2}{\partial z^2} \right]
\]

(5.20)

and the eigenvalue equation is

\[
-\frac{\hbar^2}{2\mu} \left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} - 2ie \Phi_B \frac{1}{\rho^2} \frac{\partial}{\partial \varphi} - \frac{e^2}{c^2 \hbar^2} \frac{\Phi_B^2}{\rho^2} + \frac{\partial^2}{\partial z^2} \right] \psi = E\psi(\rho, \varphi, z)
\]

(5.21)
Like for (5.4) we can choose solutions of the form
\[
\psi(\rho, \varphi, z) = \frac{1}{2\pi} \exp \left[ i \frac{p_0 z}{\hbar} \right] \exp[i m \varphi] R(\rho)
\] (5.22)
which, inserted in (5.21), give the following differential equation for \(R(\rho)\)
\[
- \frac{\hbar^2}{2\mu} \left[ \frac{R''}{R} + \frac{1}{\rho} \frac{R'}{R} - \frac{1}{\rho^2} \left( m - \frac{e\Phi_B}{\hbar} \right)^2 - \frac{p_0^2}{\hbar^2} \right] = E
\] (5.23)
Indicating with \(\alpha \equiv \frac{e\Phi_B}{\hbar c}\) and with \(\bar{s} = \frac{2\mu E}{\hbar^2} - \frac{p_0^2}{\hbar^2}\), eq. (5.23) can be written as
\[
R''(\rho) + \frac{R'(\rho)}{\rho} + \left[ \bar{s} - \left( \frac{m - \alpha}{\rho} \right)^2 \right] R(\rho) = 0
\] (5.24)
If we compare the previous equation with (5.5) we notice that having turned on the magnetic field has only shifted \(m \rightarrow m_B \equiv m - \alpha\). Here \(m_B\) will not be anymore an integer but a real number. Doing the same change of variables as before, \(r \equiv \sqrt{\bar{s}} \rho\), we can transform (5.24) into
\[
\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} + \left( 1 - \frac{m_B^2}{r^2} \right) R = 0
\] (5.25)
For this equation with \(m_B\) real there are two linearly independent solutions which are two Bessel functions of the first kind with opposite indeces:
\[
J_{m_B}(r) = \left( \frac{r}{2} \right)^{m_B} \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{r}{2})^{2n}}{n! \Gamma(n + m_B + 1)}
\]
\[
J_{-m_B}(r) = \left( \frac{r}{2} \right)^{-m_B} \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{r}{2})^{2n}}{n! \Gamma(-n + m_B + 1)}
\] (5.26)
One immediately notices that \(J_{m_B}(0) = 0\) while \(J_{-m_B}(0)\) diverges. As before we must have as boundary conditions \(J_{m_B}(0) = 0\) and so the general solution of (5.25) is
\[
R(r) = AJ_{m_B}(r)
\] (5.27)
The other boundary condition gives the following relation
\[
J_{m_B}(\sqrt{\bar{s}}b) = 0
\] (5.28)
from which, as before, we can derive the energy levels
\[
E_{k,m_B} = \hbar \frac{\alpha_{k,m_B}^2}{2\mu b^2} + \frac{\bar{s}^2}{2\mu}
\] (5.29)
If $\Phi_B$ is such to give, for example, $\alpha = 0.1$, then we will have to consider the Bessel functions $J_{m-0.1}$. The second zero of $J_{0.9}$, analog to the second one of $J_1$ that we considered before, is $\alpha_{2,0.9} = 3.70$, see Figure 2. Inserting this value in (5.29) we get

$$E_{2,0.9} = \hbar^2 6.84 \mu b^2 + \frac{p_z^2}{2\mu} < E_{2,1}$$

(5.30)

The last inequality indicates that $E_{2,0.9}$ is smaller than the corresponding level $E_{2,1}$ of the case without magnetic field calculated in (5.16). So this is a clear indication, like it was suggested in ref. [3], that the presence of a gauge potential modifies the spectrum of the Schrödinger operator even if the wave function is restricted to an area with zero magnetic field.

Let us now perform the same analysis in the classical case using the KvN operator $\hat{H}$. We have to be careful here because, we have to go to cylindrical coordinates both for the $q$ and the $p$ variables and for their derivatives $\frac{\partial}{\partial q}, \frac{\partial}{\partial p}$ which enter $\hat{H}$. In the case without magnetic field the Lagrangian is

$$L = \frac{1}{2} \mu \dot{\rho}^2 + \frac{1}{2} \mu \rho^2 \dot{\varphi}^2 + \frac{1}{2} \mu \dot{z}^2$$

(5.31)

and so the momenta conjugate to $\rho, \varphi, z$ are:

$$\begin{align*}
    p_\rho &= \mu \dot{\rho} \\
    p_\varphi &= \mu \rho^2 \dot{\varphi} \\
    p_z &= \mu \dot{z}
\end{align*}$$

(5.32)

The relations between $p_x, p_y, p_z$ and $p_\rho, p_\varphi, p_z$ can be easily worked:

$$\begin{align*}
    p_x &= \mu \dot{x} = \mu \dot{\rho} \cos \varphi - \mu \rho \dot{\varphi} \sin \varphi = p_\rho \cos \varphi - \frac{p_\varphi}{\rho} \sin \varphi \\
    p_y &= \mu \dot{y} = \mu \dot{\rho} \sin \varphi + \mu \rho \dot{\varphi} \cos \varphi = p_\rho \sin \varphi + \frac{p_\varphi}{\rho} \cos \varphi \\
    p_z &= p_z
\end{align*}$$

(5.33)

As the $z$ and $p_z$ are the same in the two coordinate systems we can summarize the basic transformations in the following set

$$\begin{align*}
    x &= \rho \cos \varphi \\
    y &= \rho \sin \varphi \\
    p_x &= p_\rho \cos \varphi - \frac{p_\varphi}{\rho} \sin \varphi \\
    p_y &= p_\rho \sin \varphi + \frac{p_\varphi}{\rho} \cos \varphi
\end{align*}$$

(5.34)

We should note that the transformations of the momenta $p_x, p_y$ are not just functions of the new momenta $p_\rho, p_\varphi$ but also of the cylindrical coordinates $\varphi, \rho$. Let us remember
that \( \hat{H} \) in cartesian coordinates contains the derivatives \( \frac{\partial}{\partial q}, \frac{\partial}{\partial p} \) and so we should check how they are related to the derivatives in cylindrical coordinates. Using (5.34) it is a long but easy calculation to show that:

\[
\begin{pmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial p_x} \\
\frac{\partial}{\partial p_y}
\end{pmatrix}
= \begin{pmatrix}
\cos\varphi & -\sin\varphi/\rho & -p_x\sin\varphi/\rho^2 & p_x/\rho \cos\varphi + p_y\sin\varphi \\
\sin\varphi & \cos\varphi/\rho & p_x\cos\varphi/\rho^2 & p_x/\rho \sin\varphi - p_y\cos\varphi \\
0 & 0 & \cos\varphi & -\rho \sin\varphi \\
0 & 0 & \sin\varphi & \rho \cos\varphi
\end{pmatrix}
\begin{pmatrix}
\frac{\partial}{\partial \rho} \\
\frac{\partial}{\partial \varphi} \\
\frac{\partial}{\partial p} \rho \\
\frac{\partial}{\partial p \varphi}
\end{pmatrix}
\]

(5.35)

Equipped with these transformations we can then easily transform \( \hat{H} \) from cartesian coordinates to cylindrical ones. In the case of a free particle we get

\[
\hat{H} = -\frac{i}{\mu} \frac{p_x}{\partial x} - \frac{i}{\mu} \frac{p_y}{\partial y} - \frac{i}{\mu} \frac{p_z}{\partial z} = -\frac{i}{\mu} \frac{p_\rho}{\partial \rho} - \frac{i}{\mu} \frac{p_{\varphi}}{\rho \partial \varphi} - \frac{i}{\mu} \frac{p_z}{\rho^3 \partial p_\rho} 
\]

(5.36)

where in the second step we have used (5.34) and (5.35). Next we will turn on the magnetic field whose gauge potential is in (5.17). The minimal coupling for \( \hat{H} \) is given in (3.24) and we need it to build the \( \mathbf{A}_q \). For our potential these \( \mathbf{A}_q \) are

\[
\begin{align*}
\mathbf{A}_x &= -\lambda_{p_x} \frac{\Phi_B}{\pi} \frac{xy}{(x^2 + y^2)^2} + \lambda_{p_y} \frac{\Phi_B}{2\pi} \frac{x^2 - y^2}{(x^2 + y^2)^2} \\
\mathbf{A}_y &= \lambda_{p_y} \frac{\Phi_B}{2\pi} \frac{x^2 - y^2}{(x^2 + y^2)^2} + \lambda_{p_x} \frac{\Phi_B}{\pi} \frac{xy}{(x^2 + y^2)^2} \\
\mathbf{A}_z &= 0
\end{align*}
\]

(5.37)

In the expression above we have now to turn the \( \lambda_{p_x}, \lambda_{p_y} \) into operators, like in (2.26), and next we have to change everything into cylindrical coordinates using (5.34) and (5.35). The result is

\[
\begin{align*}
\hat{\mathbf{A}}_x &= i \frac{\Phi_B}{2\pi} \frac{1}{\rho^2} \left[ \sin\varphi \frac{\partial}{\partial p_\rho} - \rho \cos\varphi \frac{\partial}{\partial p_\varphi} \right] \\
\hat{\mathbf{A}}_y &= -i \frac{\Phi_B}{2\pi} \frac{1}{\rho^2} \left[ \cos\varphi \frac{\partial}{\partial p_\rho} + \rho \sin\varphi \frac{\partial}{\partial p_\varphi} \right] \\
\hat{\mathbf{A}}_z &= 0
\end{align*}
\]

(5.38)

Inserting (5.37)-(5.38) into (3.24) and turning the \( \lambda_q \) into operators, after a long but trivial calculation, we get

\[
\hat{\mathbf{H}}_A \equiv -\frac{i}{\mu} \frac{p_\rho}{\partial \rho} - \frac{i}{\mu \rho^2} \left( p_\varphi - e \frac{\Phi_B}{2\pi c} \right) \frac{\partial}{\partial \varphi} - \frac{i}{\mu} \frac{p_z}{\partial z} - \frac{i}{\mu \rho^3} \left( p_\varphi - e \frac{\Phi_B}{2\pi c} \right)^2 \frac{\partial}{\partial p_\rho}
\]

(5.39)
Notice that we could obtain this $\hat{H}_A$ from the $\hat{H}$ of eq. (5.36) by just doing the replacement

$$p_\varphi \longrightarrow p_\varphi - \frac{e\Phi_B}{2\pi c} \quad (5.40)$$

The eigenvalue equation in the free case

$$\hat{H} \psi(\rho, p_\rho, \varphi, p_\varphi, z, p_z) = \tilde{E} \psi(\rho, p_\rho, \varphi, p_\varphi, z, p_z) \quad (5.41)$$

can be solved by noticing that $\hat{H}$ commutes with the four operators $-i \frac{\partial}{\partial \varphi}, -i \frac{\partial}{\partial z}, p_\varphi, p_z$.

So these five operators can be diagonalized simultaneously and the solution will have the form

$$\psi = \frac{1}{2\pi} \tilde{R}(\rho, p_\rho) \delta(p_\varphi - p^0_\varphi) \delta(p_z - p^0_z) \exp(i\varphi) \exp(i\lambda^0_z z) \quad (5.42)$$

where $n$ is an integer and $p^0_\varphi, p^0_z, \lambda^0_z$ are the eigenvalues of $\hat{p}_\varphi, \hat{p}_z, -i \frac{\partial}{\partial z}$.

Inserting (5.42) in (5.41) we get the following equation for $\tilde{R}(\rho, p_\rho)$

$$\left(-i \frac{p_{\varphi}}{\mu} \frac{\partial}{\partial \rho} + \frac{p^0_{\varphi} n}{\mu \rho^2} - i \frac{p^0_{\varphi} p^2_{\varphi}}{\mu \rho^2} + \frac{\lambda^0_{\varphi} p^0_\varphi}{\mu} - \tilde{E} \right) \tilde{R}(\rho, p_\rho) = 0 \quad (5.43)$$

We showed before that we could turn $\hat{H}$ into the $\hat{H}_A$ of (5.39) by just doing the substitution (5.40) $p_\varphi \rightarrow p_\varphi - \frac{e\Phi_B}{2\pi c}$. It is then clear that we can turn also the solutions (5.42) into the solutions of the eigenvalue equation

$$\hat{H}_A \psi_A = \tilde{E}_A \psi_A \quad (5.44)$$

by just doing the substitution (5.40) into (5.42). The result is:

$$\psi_A = \frac{1}{2\pi} \tilde{R}_A(\rho, p_\rho) \delta(p_\varphi - p^0_\varphi - \frac{e\Phi_B}{2\pi c}) \delta(p_z - p^0_z) \exp(i\varphi) \exp(i\lambda^0_z z) \quad (5.45)$$

Once we insert this into eq. (5.44) we will get for $\tilde{R}_A(\rho, p_\rho)$ the following equation:

$$\left(-i \frac{p_{\varphi}}{\mu} \frac{\partial}{\partial \rho} + \frac{p^0_{\varphi} n}{\mu \rho^2} - i \frac{p^0_{\varphi} p^2_{\varphi}}{\mu \rho^2} + \frac{\lambda^0_{\varphi} p^0_\varphi}{\mu} - \tilde{E}_A \right) \tilde{R}_A(\rho, p_\rho) = 0 \quad (5.46)$$

So eq. (5.46) is the same as the free one (5.43) and as a consequence the spectrum $\tilde{E}_A$ is the same as the $\tilde{E}$ of (5.43). This is the proof that the spectrum of the Liouville operator is not changed by the presence of the gauge potential. Of course the two eigenfunctions which have the same eigenvalues $\tilde{E} = \tilde{E}_A$ are different because they are labelled by different eigenvalues of the operator $\hat{p}_\varphi$. In fact the eigenfunction $\psi$ of eq. (5.42) has eigenvalue $p^0_\varphi$ for the operator $\hat{p}_\varphi$ while the eigenfunction $\psi_A$ of eq. (5.45) has eigenvalue $p^0_\varphi + \frac{e\Phi_B}{2\pi c}$. So the two eigenfunctions are related by a shift in one of their
"classical" numbers $p_\varphi^0$. The difference with the quantum case is that the corresponding eqs. (5.5) and (5.24) cannot be turned one into the other like we did in the KvN case. This is so because in (5.5) $m$ is an integer and not a continuous real eigenvalue like $p_\varphi^0$ is in the KvN case. Also in classical mechanics we had an integer eigenvalue, $n$ for $-i\frac{\partial}{\partial \varphi}$, but we managed to down-load on $p_\varphi^0$, and not on $n$, the difference between the free and the interacting case.

The reader may object that, even if the classical spectrum is the same in the two cases, the eigenfunctions are different and then the evolution may lead to different results. Actually it is not so because, as we see from (4.15), we have to integrate over all the possible eigenvalues which label the eigenfunctions. In our case the different eigenfunctions (5.42)-(5.45) have only the "classical" number $p_\varphi^0$ shifted. Since $p_\varphi^0$ can assume every real number, when in (4.15) we integrate over all the $p_\varphi^0$ a shift in them has no effect on the final result.

We feel that this proof that the spectrum of the classical KvN operator is unchanged by the presence of the gauge potential, while the spectrum of the Schrödinger operator is changed, is the most convincing proof of the AB phenomenon.

6 Conclusions

In this paper we have studied which is the minimal coupling procedure for the KvN operatorial approach to CM. We have shown that the MC involves not only the momenta but also their derivatives. We managed to encapsulate these two MC into a single one using the concept of superfield. We have then applied this technique to the Landau problem and to the Aharonov-Bohm phenomenon. In the first case (Landau problem) we showed that in the KvN formalism there is a sort of discretization phenomenon in the eigenvalues of the evolution operator. Moreover we proved that there are many more degeneracies in the classical than in the quantum case. For the second problem (the Aharonov-Bohm one) we showed that at the quantum level there is a change in the spectrum of the Schrödinger operator once the gauge potential is present while there is no change in the spectrum of the classical KvN operator. We feel this is the most convincing proof of the AB effect. The paper contains also a complete analysis of the issue of gauge invariance in the Hilbert space of KvN.

Having now all the tools to write down, in the KvN formalism, the interaction between a particle and a gauge field, what we should do next is to see how the gauge fields interact among themselves in the KvN operatorial approach. This has already been started and work is in progress on it.

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Appendices

A Appendix

In this appendix we will briefly review the extended canonical formalism associated to the Hamiltonian $\mathcal{H}$ of eq. (2.3). From the Lagrangian (2.7) one could derive the equations of motion for the $8n$-variables ($\varphi^a, \lambda_a, c^a, \bar{c}_a$) by the simple variational principle. These equations are:

$$\dot{\varphi}^a - \omega^{ab} \partial_b H = 0 \quad (A.1)$$

$$[\delta_b^a \partial_t - \omega^{ac} \partial_c \partial_b H]c^b = 0 \quad (A.2)$$

$$\delta_b^a \partial_t \bar{c}_a + \bar{c}_a \omega^{ac} \partial_c \partial_b H = 0 \quad (A.3)$$

$$[\delta_b^a \partial_t + \omega^{ac} \partial_c \partial_b H]\lambda_a = -i\bar{c}_a \omega^{ac} \partial_d \partial_b H c^d \quad (A.4)$$

We could ask ourselves if these same equations could be derived from the Hamiltonian $\mathcal{H}$. The answer is yes. If we introduce the following extended Poisson brackets structure in the extended space ($\varphi^a, \lambda_a, c^a, \bar{c}_a$)

$$\{\varphi^a, \lambda_b\}_{epb} = \delta^a_b \quad (A.5)$$

$$\{\bar{c}_b, c^a\}_{epb} = -i\delta^a_b \quad (A.6)$$

(while all the other brackets are zero) we get that the equations of motion (A.1)-(A.4) can be derived as

$$\frac{dO}{dt} = \{O, \mathcal{H}\}_{epb} \quad (A.7)$$

where $O$ is any of the variables ($\varphi^a, \lambda_a, c^a, \bar{c}_a$) or any function of them. More details can be found in ref. [4].

B Appendix

In this appendix we will prove that the Liouville eq. (1.2) is invariant under the gauge transformations. Let us write (1.2) in the abstract form:

$$i \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle \quad (B.1)$$

Let us now do a gauge transformation by a parameter $\alpha$. The new ket will be

$$|\psi'(t)\rangle = \tilde{U} |\psi(t)\rangle \quad (B.2)$$

where $\tilde{U}$ is the expression in formula (3.42). We will prove that this state satisfies the following equation

$$i \frac{d}{dt} |\psi'(t)\rangle = \hat{H}' |\psi'(t)\rangle \quad (B.3)$$
where $\hat{\mathcal{H}}'$ is the operator obtained from $\hat{\mathcal{H}}$ by doing a gauge transformation

$$
\hat{\mathcal{H}}' = \frac{1}{m} \left( \lambda_{q_i} + \frac{e}{c} \lambda_{p_k} \partial_{q_k} A_{q_i}' \right) \left( p_i - e \lambda_{p_i} \partial_{q_i} \Phi' \right)
$$

where the $A_{q_i}'$ is the gauge transformed vector potential and $\Phi'$ is the gauge transformed scalar potential. (B.4) is the $H$ in the gauge ($A'$, $\Phi'$) and it is derived from $H = \frac{(p_i - A_{q_i}')^2}{2m} + e\Phi'$. Let us now evaluate the LHS of (B.3)

$$
i \frac{d}{dt} \psi'(t) = i \left( \frac{d}{dt} \tilde{U}(t) \psi(t) \right) = i \left[ \frac{d}{dt} \tilde{U}(t) \right] \psi(t) + i \tilde{U}(t) \frac{d}{dt} \psi(t) = -e \frac{\partial \tilde{\alpha}}{c} \tilde{U} \psi(t)
$$

$$
+ \tilde{U}(t) \hat{\mathcal{H}} \psi(t) = -e \frac{\partial \tilde{\alpha}}{c} \tilde{U} \psi(t) + \tilde{U}(t) \hat{\mathcal{H}} \tilde{U}^{-1} \psi(t) = \left\{ -e \frac{\partial \tilde{\alpha}}{c} \tilde{U} + \hat{\mathcal{H}} \right\} \psi(t)
$$

(B.5)

where

$$\hat{\mathcal{H}} \equiv \tilde{U} \hat{\mathcal{H}} \tilde{U}^{-1}
$$

In the first steps above we have used the expression (3.42) for $\tilde{U}$ and formula (3.22). The explicit expression for $\hat{\mathcal{H}}$ is

$$
\hat{\mathcal{H}} = \frac{1}{m} \left( \lambda_{q_i}' + \frac{e}{c} \lambda_{p_k} \partial_{q_k} A_{q_i}(q') \right) \left( p_i' - \frac{e}{c} A_{q_i}(q') \right) - e \lambda_{p_i} \partial_{q_i} \Phi(q')
$$

(B.7)

where

$$
\begin{align*}
\lambda_{q_i}' &= \tilde{U} \lambda_{q_i} \tilde{U}^{-1} = \lambda_{q_i} + \frac{e}{c} \lambda_{p_k} \partial_{q_k} \alpha(q) \\
p_i' &= \tilde{U} p_i \tilde{U}^{-1} = p_i - \frac{e}{c} \partial_{q_i} \alpha(q) \\
q_i' &= \tilde{U} q_i \tilde{U}^{-1} = q_i \\
\lambda_{q_i}' &= \tilde{U} \lambda_{q_i} \tilde{U}^{-1} = \lambda_{q_i}
\end{align*}
$$

(B.8)

Remembering that

$$
\begin{align*}
A_{q_i}' &= A_{q_i} + \partial_{q_i} \alpha \\
\Phi' &= \Phi - \frac{1}{c} \partial \alpha
\end{align*}
$$

(B.9)

we can rewrite the $\hat{\mathcal{H}}$ of (B.4) as

$$
\hat{\mathcal{H}} = \frac{1}{m} \left( \lambda_{q_i} + \frac{e}{c} \lambda_{p_k} \partial_{q_k} \alpha(q) + \frac{e}{c} \lambda_{p_k} \partial_{q_k} A_{q_i}(q) \right) \left( p_i - \frac{e}{c} \partial_{q_i} \alpha(q) - \frac{e}{c} A_{q_i}(q) \right)
$$

$$
- e \lambda_{p_i} \partial_{q_i} \Phi = \frac{1}{m} \left( \lambda_{q_i} + \frac{e}{c} \lambda_{p_k} \partial_{q_k} A_{q_i}' \right) \left( p_i - \frac{e}{c} A_{q_i}' \right) - e \lambda_{p_i} \partial_{q_i} \left[ \Phi'(q) + \frac{1}{c} \partial \alpha \right] =
$$

$$
= \hat{\mathcal{H}}' + \frac{e}{c} \frac{\partial}{\partial t} \tilde{\alpha}
$$

(B.10)

Using this result (B.3) becomes

$$
i \frac{d}{dt} \psi'(t) = \hat{\mathcal{H}}' \psi'(t)
$$

(B.11)

which is what we wanted to prove.
Appendix

We know that one of the effects of the introduction of the MC is that local phases multiplying the states can be absorbed by a gauge transformation of the gauge field. What we mean is the following: if

$$\psi'(q) = \exp \left[ i \frac{e}{\hbar} \alpha(q) \right] \psi(q)$$

(C.1)

and

$$\hat{H} = \frac{1}{2m} \left( -i \hbar \frac{\partial}{\partial q} - \frac{e}{c} A(q) \right) \left( -i \hbar \frac{\partial}{\partial q} - \frac{e}{c} A(q) \right)$$

(C.2)

then

$$\hat{H}' \psi' = \exp \left[ i \frac{e}{\hbar} \alpha(q) \right] \hat{H} \psi$$

(C.3)

where $\hat{H}'$ is obtained from $\hat{H}$ by replacing $\hat{A}$ with its gauge transformed $A' = A + \frac{\partial \alpha}{\partial q}$. Now if we do the minimal coupling at the level of $\hat{H}$, like in (3.24), the phase that can "pass through", like in (C.3), is $\exp \left[ i \frac{e}{\hbar} \tilde{\alpha} \right]$ with $\tilde{\alpha}$ given in (3.22):

$$\hat{H}' \psi' = \exp \left[ i \frac{e}{\hbar} \tilde{\alpha} \right] \hat{H} \psi$$

(C.4)

The proof goes as follows. Let us use the mixed representation (2.30) for the $\hat{H}$ associated to (3.24):

$$\hat{H} = \frac{1}{m} \left( -i \frac{\partial}{\partial q} + \frac{e}{c} \lambda \frac{\partial A}{\partial q} \right) \left( i \frac{\partial}{\partial \lambda} - \frac{e}{c} A(q) \right)$$

(C.5)

and let us then transform $\hat{H}$ into a $\hat{H}'$ where $A \rightarrow A' + \frac{\partial \alpha}{\partial q}$. In the same mixed representation (2.30) the wave function will be of the form $\psi(q, \lambda_p)$ and we can also construct the following new state:

$$\psi'(q, \lambda_p) = \exp \left[ -i \frac{e}{c} \lambda_p \partial_q \alpha(q) \right] \psi(q, \lambda_p) = \exp \left[ i \frac{e}{c} \tilde{\alpha} \right] \psi(q, \lambda_p)$$

(C.6)

Equipped with these tools it is then a long but easy calculation to prove (C.4). To conclude we can say that the local phase transformations of the form (C.3) on the KvN states $\psi(q, \lambda_p)$ are the classical counterpart of the local phase transformations (C.1) on the quantum Hilbert states $\psi(q)$.

At this point a question which arises naturally is the following: if instead of the very particular phase transformation (C.6) on the KvN states we perform a general local phase transformation of the form

$$\psi'(q, \lambda_p) = \exp [i \alpha(q, \lambda_p)] \psi(q, \lambda_p)$$

(C.7)
which gauge fields do we have to introduce in the \( \hat{H} \) in order to absorb the phase like in (C.4)? The answer is the following: let us start from the \( \hat{H} \) of the free particle

\[
\hat{H} = \frac{1}{m} \frac{\partial^2}{\partial q \partial \lambda_p} \tag{C.8}
\]

and do a general phase transformation

\[
\psi'(q, \lambda_p) = \exp[i\alpha(q, \lambda_p)]\psi(q, \lambda_p) \tag{C.9}
\]

If we now perform in (C.8) the following MC

\[
-i \frac{\partial}{\partial q} \rightarrow -i \frac{\partial}{\partial q} + A_q, \quad i \frac{\partial}{\partial \lambda_p} \rightarrow i \frac{\partial}{\partial \lambda_p} - A_{\lambda_p} \tag{C.10}
\]

where \( A_q \) and \( A_{\lambda_p} \) are two gauge fields which transform as follows

\[
A'_q = A_q - \frac{\partial\alpha(q, \lambda_p)}{\partial q}, \quad A'_{\lambda_p} = A_{\lambda_p} - \frac{\partial\alpha(q, \lambda_p)}{\partial \lambda_p} \tag{C.11}
\]

then the new \( \hat{H}_A \):

\[
\hat{H}_A \equiv \frac{1}{m} \left(-i \frac{\partial}{\partial q} + A_q\right) \left(i \frac{\partial}{\partial \lambda_p} - A_{\lambda_p}\right) \tag{C.12}
\]

would satisfy the following relation

\[
\hat{H}'_A \psi' = \exp[i\alpha(q, \lambda_p)]\hat{H}_A \psi \tag{C.13}
\]

where \( \hat{H}'_A \) is the gauge transformed of \( \hat{H}_A \) via eq. (C.11).

We should notice that (C.9) is a more general gauge transformation than the one in (C.6). To implement (C.9) we need two gauge fields \( A_q, A_{\lambda_p} \) while for the (C.6) we could build everything from one field \( A(q) \), see (C.3). The transformation (C.6) is a particular case of (C.9) and the same is true for the gauge fields which enter (C.3). In fact comparing (C.3) with (C.12) we see that the two general gauge fields of (C.12) should be of the following particular form in order to reproduce (C.3):

\[
A_q = \frac{e}{c} \lambda_p \frac{\partial A}{\partial q}, \quad A_{\lambda_p} = \frac{e}{c} A(q) \tag{C.14}
\]

It is interesting to notice that the interaction introduced by the two general gauge fields \( A_q, A_{\lambda_p} \) appearing in (C.12) is something new and not related to any magnetic field in \( q \)-space unless the fields have the particular form (C.14). We hope to come back in the future to the study of the forces generated by the general gauge potential of (C.12).
In this appendix we want to build and diagonalize the $\hat{H}$ associated to an harmonic oscillator in one dimension. The Hamiltonian is

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2$$  \hspace{1cm} (D.1)

So the $\mathcal{H}$ (2.18) is:

$$\mathcal{H} = \lambda \frac{p}{m} - m\omega^2 \lambda q$$  \hspace{1cm} (D.2)

which, in its operatorial version, using the representation (2.26), is:

$$\hat{\mathcal{H}} = -i \left( \frac{p}{m} \frac{\partial}{\partial q} - m\omega^2 q \frac{\partial}{\partial p} \right)$$  \hspace{1cm} (D.3)

This operator is very similar to the components of the angular momentum in standard quantum mechanics, so we can diagonalize it with similar techniques. First of all we can turn the standard phase space coordinates $(q, p)$ into the following new ones $(r, \theta)$ defined as:

$$\sqrt{m\omega}q = r\cos\theta, \quad \frac{p}{\sqrt{m}} = r\sin\theta$$  \hspace{1cm} (D.4)

The Hamiltonian becomes:

$$\hat{\mathcal{H}} = -i \frac{r}{\sqrt{m}} \sin\theta \left( \frac{\partial r}{\partial q} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial q} \frac{\partial}{\partial \theta} \right) + i \sqrt{m\omega} r \cos\theta \left( \frac{\partial r}{\partial p} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial p} \frac{\partial}{\partial \theta} \right) = i\omega \frac{\partial}{\partial \theta}$$  \hspace{1cm} (D.5)

The eigenfunctions of the previous operator are:

$$\psi(r, \theta) = F(r)e^{-iN\theta}$$  \hspace{1cm} (D.6)

and the associated eigenvalues are $\tilde{E} = N\omega$. Imposing the single valuedness of the wave functions we get that

$$\psi(r, \theta + 2\pi) = \psi(r, \theta) \Rightarrow e^{-2i\pi N} = 1$$  \hspace{1cm} (D.7)

which implies: $N \in \{0, \pm 1, \pm 2, \ldots\}$. So the discretization of the eigenvalues of the Liouvillian is a direct consequence of the requirement of single valuedness of the KvN wave functions, requirement that was already present in Koopman’s original paper [1].

The proof of the discretization of the spectrum of the Liouville operator for an harmonic oscillator can be worked out also in the mixed representation (2.30) and (2.33). In this representation the $\mathcal{H}$ of eq. (2.18) becomes:

$$\hat{\mathcal{H}} = \frac{1}{m} \frac{\partial}{\partial q} \frac{\partial}{\partial \lambda_p} - m\omega^2 \lambda q$$  \hspace{1cm} (D.8)
Let us introduce the following new variables

\[ Z_+ \equiv \frac{q + \Delta \lambda_p}{\sqrt{2}}, \quad Z_- \equiv \frac{q - \Delta \lambda_p}{\sqrt{2}} \]  

(D.9)

where \( \Delta \) is a constant which has the dimension of an action. In terms of these new variables the \( \hat{H} \) of (D.8) can be written as

\[ \hat{H} = \frac{1}{2m} \left[ m \omega^2 Z_+^2 - \Delta \omega \right] - \frac{1}{\Delta} \left[ m \omega^2 Z_-^2 - \Delta \omega \right] = \frac{1}{\Delta} \left[ H_{osc}(Z_+, \frac{\partial}{\partial Z_+}) - H_{osc}(Z_-, \frac{\partial}{\partial Z_-}) \right] \]  

(D.10)

As indicated in the second step above, we notice that \( \hat{H} \) is the difference of two quantum harmonic oscillators respectively in \( Z_- \) and \( Z_+ \), where the role of \( \bar{h} \) is taken by our constant \( \Delta \). The eigenstates of \( \hat{H} \):

\[ \hat{H} \psi(Z_+, Z_-) = \tilde{E} \psi(Z_+, Z_-) \]  

(D.11)

can be easily obtained. They are

\[ \psi_n(Z) = \left( \sqrt{\pi} 2^n n! \sigma_0 \right)^{-1/2} H_n \left( \frac{Z}{\sigma_0} \right) \exp \left( -\frac{Z^2}{2 \sigma_0^2} \right), \quad n = 0, +1, +2, \cdots \]  

(D.12)

where \( H_n \) are the Hermite polynomials and \( \sigma_0 = \sqrt{\frac{\Delta}{m \omega}} \). The eigenvalues are:

\[ \tilde{E}_{n,m} = \frac{1}{\Delta} \left[ (m + \frac{1}{2}) \Delta \omega - \left( n + \frac{1}{2} \right) \Delta \omega \right] = (m - n) \omega = N \omega \]  

(D.13)

where \( N \) can take every positive or negative integer value: \( N = 0, \pm 1, \pm 2, \cdots \). This confirms the discretization phenomenon we found before. Let us notice that the quantity \( \Delta \) disappears from the spectrum, so it is just an artifact of the \((q, \lambda_p)\) representation and it is needed in (D.9) only for dimensional reasons. There was no need of it in the first derivation, eqs. (D.3)-(D.7), of the discretization phenomenon. Second let us notice that, due to the difference of the two oscillators quantum numbers \( m \) and \( n \) above, the zero-point "energy" \( \tilde{E} \) is zero differently than in the quantum case. Note also that there is an \( \infty \)-order degeneracy in the sense that associated to the eigenstate \( \tilde{E} = N \omega \) there is the set of eigenfunctions: \( \psi = \psi_n(Z_+) \psi_{n+N}(Z_-) \), where \( n \) can be any integer if \( N \geq 0 \) while \( n > -N \) if \( N < 0 \).

This doubling of oscillators in the classical case is basically due to the fact that the classical KvN wave functions depend on a number of variables \((q, \lambda_p)\) which is double with respect to the quantum case.

\[ ^{11} \text{We have put quotation marks around the word "energy" because, as we explained in section 4, the } \tilde{E} \text{ is not the energy but one of the eigenvalues of the evolution operator.} \]


E Appendix

In section 4 we have analyzed the properties of the Landau problem in the Schrödinger picture of the KvN formalism. In this appendix we want to study the same problem in the corresponding Heisenberg picture\[12\]. In particular we want to find out which are the constants of motion, i.e. the operators that commute with the generator of the time evolution $\mathcal{H}$. These operators will give us some indications concerning the trajectory of the classical particle in a constant magnetic field.

Let us remember the form of the Liouvillian in the Landau problem:

$$\mathcal{H} = \frac{1}{m} \lambda_x p_x + \frac{1}{m} \lambda_y \left( \frac{e}{c} B \lambda_p^x \right) \left( p_y - \frac{eB}{c} x \right) + \frac{1}{m} \lambda_z p_z$$  \hspace{1cm} (E.1)

Defining $v_y = \frac{1}{m} \left( p_y - \frac{e}{c} A_y \right)$ and noticing that $[x, \mathcal{H}] = ip_x/m$, we get the following relation:

$$[v_y, \mathcal{H}] = \frac{1}{m} \left[ p_y - \frac{e}{c} A_y, \mathcal{H} \right] = -\frac{e}{m^2c} [Bx, \lambda_x] p_x = -\frac{ieB}{m^2c} p_x$$  \hspace{1cm} (E.2)

If we introduce the Larmor frequency: $\omega = \frac{eB}{mc}$ we can then easily prove that $x_0 \equiv x + v_y/\omega$ is a constant of motion. In fact, using (E.2) we get:

$$[x_0, \mathcal{H}] = \left[ x + \frac{v_y}{\omega}, \mathcal{H} \right] = \frac{i}{m} p_x + \frac{mc}{eB} \cdot \left( -\frac{ieB}{m^2c} p_x \right) = 0$$  \hspace{1cm} (E.3)

In the same way the commutators of $y$ and $v_x$ with the Liouvillian are:

$$[y, \mathcal{H}] = \frac{i}{m} \left( p_y - \frac{eB}{c} x \right), \quad [v_x, \mathcal{H}] = \frac{iB}{mc} \left( p_y - \frac{eB}{c} x \right)$$  \hspace{1cm} (E.4)

and so we obtain that also $y_0 \equiv y - v_x/\omega$ commutes with $\mathcal{H}$:

$$[y_0, \mathcal{H}] = [y, \mathcal{H}] - \frac{mc}{eB} [v_x, \mathcal{H}] = 0$$  \hspace{1cm} (E.5)

Now classically a particle in a constant magnetic field directed along $z$ describes an helicoidal orbit whose projection on the $x,y$-plane is a circumference with a radius equal to the Larmor one $\varrho_{Lar}$:

$$\varrho_{Lar}^2 \equiv \frac{1}{\omega^2} (v_x^2 + v_y^2)$$  \hspace{1cm} (E.6)

Using eqs. (E.2)-(E.4) it is possible to prove that also the Larmor radius is a constant of the motion:

$$[\varrho_{Lar}^2, \mathcal{H}] = \frac{1}{\omega^2} [v_x^2, \mathcal{H}] + \frac{1}{\omega^2} [v_y^2, \mathcal{H}] = 0$$  \hspace{1cm} (E.7)

\[12\] All the objects appearing in this appendix are operators. Therefore we will not use explicitly the hat-symbol (" ∧ ") to indicate them.
The Larmor radius can be written also in terms of the $x, x_0$ and $y, y_0$ operators in the following way:

$$
\ell_{\text{Lar}}^2 = \frac{1}{\omega^2} (v_x^2 + v_y^2) = \frac{1}{\omega^2} \left[ \omega (y - y_0) \right]^2 + \frac{1}{\omega^2} \left[ \omega (x_0 - x) \right]^2 = (x - x_0)^2 + (y - y_0)^2 \quad (E.8)
$$

Therefore $(x_0, y_0)$ is the center of a circumference which is the projection of the orbit of the particle onto the plane $(x, y)$ and $\ell_{\text{Lar}}$ is the corresponding radius. Note that in the KvN operatorial formalism the operators $x_0$ and $y_0$ are suitable combinations of the $\varphi$ operators and they commute among themselves. This implies that they can be determined with arbitrary precision. In quantum mechanics, instead, one can prove that the following relation holds:

$$
[x_0, y_0] = -\frac{i\hbar c}{eB} \quad (E.9)
$$

and therefore, differently than in classical mechanics, there is an uncertainty relation involving the coordinates of the center of the circumference.

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Figure Caption

Figure 1: Aharonov-Bohm geometrical set up.

Figure 2: Zeros of Bessel functions: $m=1$ (continuous line), $m=0.9$ (dashed line).
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