SOME NEW SERIES FOR 1/π
MOTIVATED BY CONGRUENCES

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Abstract. In this paper, we deduce a family of six new series for 1/π; for example,
\[
\sum_{n=0}^{\infty} \frac{41673840n + 4777111}{5780^n} W_n \left( \frac{1444}{1445} \right) = \frac{147758475}{\sqrt{95} \pi}
\]
where \( W_n(x) = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{n}{k} \right)^2 \left( \frac{1444}{1445} \right)^k x^k \). To do so, we manage to transform our series to series of the type
\[
\sum_{n=0}^{\infty} \frac{an + b}{m^n} \sum_{k=0}^{n} \binom{n}{k}^4
\]
studied by Shaun Cooper in 2012. In addition, we pose 17 new series for 1/π motivated by congruences; for example, we conjecture that
\[
\sum_{k=0}^{\infty} \frac{4290k + 367}{3136^k} \binom{2k}{k} T_k(14, 1) T_k(17, 16) = \frac{5390}{\pi},
\]
where \( T_k(b, c) \) is the coefficient of \( x^k \) in the expansion of \( (x^2 + bx + c)^k \).

1. Introduction

Let \( n \in \mathbb{N} = \{0, 1, 2, \ldots\} \). In 1894 J. Franel \cite{8} introduced the usual Franel numbers \( f_n = \sum_{k=0}^{n} \binom{n}{k}^3 \) (\( n \in \mathbb{N} \)) and the Franel numbers \( f_n^{(4)} = \sum_{k=0}^{n} \binom{n}{k}^4 \) (\( n \in \mathbb{N} \)) of order four. By Zeilberger’s algorithm (cf. \cite{9}), the sequence \( (f_n^{(4)})_{n \geq 0} \) satisfies the following recurrence first claimed by Franel:
\[
(n + 2)^3 f_{n+2}^{(4)} = 4(1 + n)(3 + 4n)(5 + 4n)f_n^{(4)} + 2(3 + 2n)(7 + 9n + 3n^2)f_{n+1}^{(4)}.
\]

M. Rogers and A. Straub \cite{11} confirmed the author’s conjectural series for 1/π involving Franel polynomials.

In 2005 Y. Yang used modular forms of level 10 to discover the following curious identity relating Franel numbers of order four to Ramanujan-type series for 1/π:
\[
\sum_{k=0}^{\infty} \frac{4k + 1}{36^k} f_k^{(4)} = \frac{18}{\sqrt{15} \pi}.
\]

Key words and phrases. Ramanujan-type series for 1/π, congruences, binomial coefficients, symbolic computation.

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This has not been published by Yang, but more identities of this kind were
deduced by S. Cooper [4] in 2012 via modular forms. For the classical
Ramanujan-type series for $1/\pi$, one may consult [1, 2, 10] and the nice
survey given by Cooper [5, Chapter 14].

For $n \in \mathbb{N}$ the polynomial

$$W_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \binom{2(n-k)}{k} x^k$$

at $x = -1$ coincides with $(-1)^n f_n^{(4)}$, this can be easily verified since the
sequence $((-1)^n W_n(-1))_{n\geq0}$ satisfies the same recurrence as $(f_n^{(4)})_{n\geq0}$. In
2011 the author [16, (3.1)-(3.10)] proposed ten identities of the form

$$\sum_{k=0}^{\infty} \frac{ak + b}{m^k} W_k \left( \frac{1}{m} \right) = \frac{C}{\pi},$$

where $a, b, m$ are integers with $am \neq 0$, and $C^2$ is rational. They were later
confirmed in [6].

In this paper we establish six new series for $1/\pi$ involving $W_n(x)$.

**Theorem 1.1.** We have the following identities:

1. $$\sum_{k=0}^{\infty} \frac{45k + 8}{40^k} W_k \left( \frac{9}{10} \right) = \frac{215\sqrt{15}}{12\pi},$$

2. $$\sum_{k=0}^{\infty} \frac{1360k + 389}{(-60)^k} W_k \left( \frac{16}{15} \right) = \frac{205\sqrt{15}}{\pi},$$

3. $$\sum_{k=0}^{\infty} \frac{735k + 124}{200^k} W_k \left( \frac{49}{50} \right) = \frac{10125\sqrt{7}}{56\pi},$$

4. $$\sum_{k=0}^{\infty} \frac{376380k + 69727}{(-320)^k} W_k \left( \frac{81}{80} \right) = \frac{260480\sqrt{5}}{3\pi},$$

5. $$\sum_{k=0}^{\infty} \frac{348840k + 47461}{1300^k} W_k \left( \frac{324}{325} \right) = \frac{1314625\sqrt{2}}{12\pi},$$

6. $$\sum_{k=0}^{\infty} \frac{41673840k + 4777111}{5780^k} W_k \left( \frac{1444}{1445} \right) = \frac{147758475\sqrt{95}}{95\pi}.$$
**Conjecture 1.1.** We have the following identities:

\[
\sum_{k=0}^{\infty} \frac{4k + 1}{6^k}W_k \left( -\frac{1}{8} \right) = \frac{\sqrt{72 + 42\sqrt{3}}}{\pi}, \tag{1.7}
\]

\[
\sum_{k=0}^{\infty} \frac{392k + 65}{(-108)^k}W_k \left( -\frac{49}{12} \right) = \frac{387\sqrt{3}}{\pi}, \tag{1.8}
\]

\[
\sum_{k=0}^{\infty} \frac{168k + 23}{112^k}W_k \left( \frac{63}{16} \right) = \frac{1652\sqrt{3}}{9\pi}, \tag{1.9}
\]

\[
\sum_{k=0}^{\infty} \frac{1512k + 257}{(-320)^k}W_k \left( -\frac{405}{64} \right) = \frac{1184\sqrt{35}}{5\pi}, \tag{1.10}
\]

\[
\sum_{k=0}^{\infty} \frac{56k + 9}{324^k}W_k \left( \frac{25}{4} \right) = \frac{1134\sqrt{35}}{125\pi}, \tag{1.11}
\]

\[
\sum_{k=0}^{\infty} \frac{13000k - 1811}{(-1296)^k}W_k \left( -\frac{625}{9} \right) = \frac{49356\sqrt{39}}{5\pi}, \tag{1.12}
\]

\[
\sum_{k=0}^{\infty} \frac{9360k - 1343}{1300^k}W_k \left( \frac{900}{13} \right) = \frac{21515\sqrt{39}}{3\pi}, \tag{1.13}
\]

\[
\sum_{k=0}^{\infty} \frac{56355k + 2443}{(-5776)^k}W_k \left( -\frac{83521}{361} \right) = \frac{4669535\sqrt{2}}{68\pi}, \tag{1.14}
\]

\[
\sum_{k=0}^{\infty} \frac{5928k + 253}{5780^k}W_k \left( \frac{1156}{5} \right) = \frac{28951\sqrt{2}}{4\pi}. \tag{1.15}
\]

**Remark 1.1.** Motivated by congruences, the author actually found (1.1)-(1.15) in 2020.

Van Hamme [20] thought that classical Ramanujan-type series for $1/\pi$ should have their $p$-adic analogues involving the $p$-adic Gamma function. This does not hold in general for generalized Ramanujan-type series, for example, the author [12, Conjecture 1.5] discovered the identity

\[
\sum_{n=0}^{\infty} \frac{6n - 1}{256^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2(n-k)}{n-k} \cdot 12^{n-k} = \frac{8\sqrt{3}}{\pi}
\]

(which was later confirmed in [6]) and conjectured its related $p$-adic congruence

\[
\sum_{n=0}^{p-1} \frac{6n - 1}{256^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2(n-k)}{n-k} \cdot 12^{n-k} \equiv -p \pmod{p^2}
\]

(with $p$ any prime greater than 3) which has nothing to do with the Legendre symbol $(\frac{2}{p})$. 

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For the author’s philosophy to generate series for $1/\pi$ via congruences, one may consult the survey [12] and the recent paper [17, Section 1].

The so-called “holonomic alchemy” (cf. [6]) does not work for proving our Theorem 1.1, for the reason see Lemma 2.1 and Remark 2.1. We will prove Theorem 1.1 in the next section via transforming (1.1)-(1.6) to series of the type

$$\sum_{k=0}^{\infty} \frac{ak + b}{m^k} f_k$$

studied by Cooper [4], and present related conjectural congruences in Section 3.

In Sections 4 and 5, we will pose 10 other new conjectural series for $1/\pi$ motivated by congruences.

2. Proof of Theorem 1.1

Lemma 2.1. For $|z| \leq 1/30$, we have

$$\sum_{k=0}^{\infty} \frac{z^k}{(1 + 4z)^{k+1}} W_k \left( \frac{1}{1 + 4z} \right) = \sum_{n=0}^{\infty} f_n^{(4)} z^n\right)^n$$

and

$$\sum_{k=0}^{\infty} \frac{k z^k}{(1 + 4z)^{k+1}} W_k \left( \frac{1}{1 + 4z} \right) = \sum_{n=0}^{\infty} n(f_n^{(4)} + 4s_n) z^n,$$

where

$$s_n := \sum_{0 \leq j < n} (-1)^{n-1-j} \binom{n-1}{j} \binom{n+j}{j} \binom{2j}{j} \binom{2(n-1-j)}{n-1-j}.$$  (2.3)

Remark 2.1. Note that the identity (2.2) contains a sophisticated term $s_n$ defined by (2.3). It is difficult to see how $s_n$ is related to the Franel numbers of order 4. This is why the “holonomic alchemy” (cf. [6]) is not helpful to our proof of Theorem 1.1.

Proof of Lemma 2.1. Let $N$ be any nonnegative integer. Then

$$\sum_{k=0}^{N} \frac{z^k}{(1 + 4z)^{k+1}} W_k \left( \frac{1}{4z + 1} \right)$$

$$= \sum_{k=0}^{N} z^k \sum_{j=0}^{k} \binom{k+j}{2j} \binom{2j}{j} \binom{2(k-j)}{k-j} (1 + 4z)^{-j-k-1}$$

$$= \sum_{k=0}^{N} z^k \sum_{j=0}^{k} \binom{k+j}{2j} \binom{2j}{j} \binom{2(k-j)}{k-j} \sum_{r=0}^{\infty} \binom{-j-k-1}{r} (4z)^r$$
Similarly, where is the Legendre polynomial of degree and hence

Clearly

\[
\sum_{n=0}^{\infty} z^n \sum_{k=0}^{\min\{n,N\}} k (k+j) \binom{k+j}{j} 2 \left( \frac{2(k-j)}{k-j} \right) \binom{n+j}{n-k} \frac{n-j}{k-j} (-4)^{n-k} = 0
\]

Similarly,

\[
\sum_{k=0}^{N} \frac{k z^k}{(1+4z)^{k+1}} W_k \left( \frac{1}{4z+1} \right)
\]

\[
= \sum_{n=0}^{\infty} z^n \sum_{j=0}^{\min\{n,N\}} (-4)^{n-j} \left( \frac{2j}{j} \right)^2 \left( \frac{2(k-j)}{k-j} \right) k (n-j) \frac{n-j}{k-j} (-4)^{k-j}
\]

Clearly \((2m)^m \leq (1+1)^{2m} = 4^m\) for all \(m \in \mathbb{N}\). Thus

\[
\left| \sum_{k=j}^{\min\{n,N\}} \binom{n-j}{k-j} \frac{(2(k-j))}{(-4)^{k-j}} \right| \leq \sum_{k=j}^{\infty} \binom{n-j}{k-j} = 2^{n-j}
\]

and hence

\[
\left| \sum_{j=0}^{\min\{n,N\}} (-4)^{n-j} \left( \frac{2j}{j} \right)^2 \binom{n+j}{2j} \sum_{k=j}^{\min\{n,N\}} \binom{n-j}{k-j} \frac{(2(k-j))}{(-4)^{k-j}} \right|
\]

\[
\leq \sum_{j=0}^{\min\{n,N\}} 4^n \binom{n+j}{2j} 2^{n-j} \leq 8^n \sum_{j=0}^{n} \binom{n}{j} \binom{n+j}{j} \left( \frac{2-1}{2} \right)^j = 8^n P_n(2),
\]

where

\[
P_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \left( \frac{x-1}{2} \right)^k
\]

is the Legendre polynomial of degree \(n\). Similarly,

\[
\left| \sum_{j=0}^{\min\{n,N\}} (-4)^{n-j} \left( \frac{2j}{j} \right)^2 \binom{n+j}{2j} \sum_{k=j}^{\min\{n,N\}} k (n-j) \frac{n-j}{n-k} \frac{(2(k-j))}{(-4)^{k-j}} \right|
\]

\[
\leq \sum_{j=0}^{\min\{n,N\}} 4^n \binom{n+j}{2j} \binom{2j}{j} \min\{n,N\} 2^{n-j} \leq n8^n P_n(2).
\]
By the Laplace-Heine formula (cf. [19, p. 194]),

$$P_n(2) \sim \frac{(2 + \sqrt{3})^{n+1/2}}{\sqrt{2n\pi \sqrt{3}}}$$ as $n \to +\infty$.

As $8(2 + \sqrt{3}) < 29.86$, we have $n8^n P_n(2) < 30^n$ if $n$ is sufficiently large.

Recall that $|z| < 1/30$.

In view of the above,

$$\lim_{N \to +\infty} \sum_{k=0}^{N} \frac{z^k}{(1 + 4z)^k} W_k \left( \frac{1}{1 + 4z} \right)$$

$$= \lim_{N \to +\infty} \sum_{n=0}^{N} z^n \sum_{j=0}^{n} (-4)^{n-j} \binom{2j}{j} \binom{n+j}{2j} \sum_{k=j}^{n} \binom{n-j}{n-k} \binom{1}{k-j}$$

$$= \sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} (-4)^{n-j} \binom{2j}{j} \binom{n+j}{2j} \binom{n-j}{n-j} \binom{1}{n-j}$$

$$= \sum_{n=0}^{\infty} z^n (-1)^n W_n(-1) = \sum_{n=0}^{\infty} f_n^{(4)} z^n.$$

Similarly,

$$\lim_{N \to +\infty} \sum_{k=0}^{N} \frac{k z^k}{(1 + 4z)^k} W_k \left( \frac{1}{1 + 4z} \right) - \sum_{n=0}^{\infty} n f_n^{(4)} z^n$$

$$= \lim_{N \to +\infty} \sum_{n=0}^{N} z^n \sum_{j=0}^{n} (-4)^{n-j} \binom{2j}{j} \binom{n+j}{2j} \sum_{k=j}^{n} \binom{n-j}{n-k} \binom{1}{k-j}$$

$$= -\sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} (-4)^{n-j} \binom{2j}{j} \binom{n+j}{2j} (n-j) \sum_{k=j}^{n} \binom{n-j}{n-k-1} \binom{1}{k-j}$$

$$= -\sum_{n=0}^{\infty} z^n \sum_{j=0}^{n} (-4)^{n-j} (n-j) \binom{n+j}{j} \binom{2j}{j} \binom{n-j-3/2}{n-j-1}$$

$$= \sum_{n=0}^{\infty} n z^n \sum_{0 \leq j < n} 4^{n-j} \binom{n-1}{j} \binom{n+j}{j} \binom{2j}{j} \binom{-1/2}{n-j-1}$$

$$= \sum_{n=0}^{\infty} n z^n \sum_{0 \leq j < n} (-1)^{n-j-1} \binom{n-1}{j} \binom{n+j}{j} \binom{2j}{j} \binom{2(n-j-1)}{n-j-1}.$$

So we have the desired result. □
Lemma 2.2. For any \( n \in \mathbb{N} \) we have
\[
5n(4n+1)((n+2)s_{n+2} - 16ns_n) = (30n^3 + 54n^2 + 7n - 2)f_{n+1}^{(4)} + 2(60n^3 + 58n^2 + 17n + 2)f_n^{(4)}. \tag{2.4}
\]

Proof. Let \( u_n \) denote the left-hand side or the right-hand side of (2.4). Via Zeilberger’s algorithm, we find that
\[
(1 + n)(3 + n)^3(5 + 4n)u_{n+2} \times (344 + 2572n + 8198n^2 + 13329n^3 + 10875n^4 + 4190n^5 + 600n^6)
= 2(2 + n)(9 + 4n)P(n)u_{n+1} + 4(1 + n)(2 + n)(3 + 4n)(5 + 4n)(9 + 4n)Q(n)u_n
\]
for all \( n = 0, 1, 2, \ldots \), where
\[
P(n) = 62208 + 506208n + 1799416n^2 + 3578972n^3 + 4250502n^4 + 3104119n^5 + 1401609n^6 + 380700n^7 + 56940n^8 + 3600n^9
\]
and
\[
Q(n) = 40108 + 127005n + 164335n^2 + 110729n^3 + 40825n^4 + 7790n^5 + 600n^6.
\]
Note also that \( u_0 = 0, u_1 = 2150 \) and \( u_2 = 103680 \). As both sides of (2.4) give the same integer sequence \( (u_n)_{n \geq 0} \), we have (2.4) as desired. \( \square \)

Now we are able to present an auxiliary theorem.

Theorem 2.3. Let \( a, b \) and \( x \) be complex numbers with \( |x - 1| \geq 7.5 \). Then
\[
\frac{10}{x} (x-1)^2(x-2) \sum_{n=0}^{\infty} \frac{an + b}{(4x)^n} W_n \left(1 - \frac{1}{x}\right) = \sum_{k=0}^{\infty} \frac{2ax(5x - 7)k + a(10x - 13) + 10b(x - 1)(x - 2)}{(4x)^{n+1}} f_k^{(4)} \tag{2.5}
\]

Proof. Note that \( |1/(4x - 4)| \leq 1/30 \). Applying (2.1) with \( z = 1/(4x - 4) \), we get
\[
\sum_{n=0}^{\infty} \frac{1}{(4x)^n} W_n \left(1 - \frac{1}{x}\right) = \frac{x}{x-1} \sum_{k=0}^{\infty} \frac{f_k^{(4)}}{(4x-4)^k}. \tag{2.6}
\]

If we have
\[
\sum_{n=0}^{\infty} \frac{n}{(4x)^n} W_n \left(1 - \frac{1}{x}\right) = \frac{x}{10(x-1)^2(x-2)} \sum_{k=0}^{\infty} \frac{(10x - 14)(kx + 1) + 1}{(4x-4)^k} f_k^{(4)}, \tag{2.7}
\]
then combining (2.6) with (2.7) we immediately get (2.5). The identity (2.7) is equivalent to the following one with \( z = 1/(4x - 4) \):
\[
5(1 - 4z) \sum_{k=0}^{\infty} \frac{kz^k}{(1 + 4z)^{k+1}} W_k \left(\frac{1}{1 + 4z}\right) = \sum_{k=0}^{\infty} \frac{((5 - 8z)(1 + 4z)k + 4z(5 - 6z)) f_k^{(4)} z^k}{(5 - 6z)^k}. \tag{2.8}
\]
Below we prove (2.8) for $|z| \leq 1/30$. For convenience, we write $[z^m]f(z)$ with $m \in \mathbb{N}$ to denote the coefficient of $z^m$ in the power series expansion of $f(z)$.

By Lemma 2.1, for any $n \in \mathbb{N}$ we have

$$[z^{n+1}](1 - 16z^2) \sum_{k=1}^{\infty} \frac{kz^{k-1}}{(1 + 4z)^{k+1}} W_k \left( \frac{1}{1 + 4z} \right)$$

$$= [z^{n+2}](1 - 16z^2) \sum_{m=0}^{\infty} m(f_m^{(4)} + 4s_m)z^m$$

$$= (n + 2)(f_{n+2}^{(4)} + 4s_{n+2}) - 16n(f_n^{(4)} + 4s_n)$$

$$= (n + 2)f_{n+2}^{(4)} - 16nf_n^{(4)} + 4((n + 2)s_{n+2} - 16ns_n).$$

Now let $n \in \mathbb{Z}^+$. By the recurrence of $(f_m^{(4)})_{m \geq 0}$, we have

$$4n(4n + 1)(4n - 1)f_{n-1}^{(4)} = (n + 1)^3 f_{n+1}^{(4)} - 2(2n + 1)(3n^2 + 3n + 1)f_n^{(4)}$$

and hence

$$n(4n + 1)((32n + 52)f_{n+1}^{(4)} + (96n + 56)f_n^{(4)} - 32(4n - 1)f_{n-1}^{(4)})$$

$$= 4n(4n + 1)(8n + 13)f_{n+1}^{(4)} + 8n(4n + 1)(12n + 7)f_n^{(4)} - 8(n + 1)^3 f_{n+1}^{(4)} + 16(2n + 1)(3n^2 + 3n + 1)f_n^{(4)}$$

$$= 4f_{n+1}^{(4)} + 8(60n^3 + 58n^2 + 17n + 2)f_n^{(4)}$$

$$= 20n(4n + 1)((n + 2)s_{n+2} - 16ns_n)$$

with the aid of Lemma 2.2. Combining this with the last paragraph, we get

$$[z^{n+1}]5(16z^2 - 1) \sum_{k=1}^{\infty} \frac{kz^{k-1}}{(1 + 4z)^{k+1}} W_k \left( \frac{1}{1 + 4z} \right)$$

$$= -5(n + 2)f_{n+2}^{(4)} + 80nf_n^{(4)} - 20((n + 2)s_{n+2} - 16ns_n)$$

$$= -5(n + 2)f_{n+2}^{(4)} + 80nf_n^{(4)} - (32n + 52)f_{n+1}^{(4)}$$

$$- (96n + 56)f_n^{(4)} + 32(4n - 1)f_{n-1}^{(4)}$$

$$= [z^{n+1}](32z^2 - 12z - 5) \left( 4 \sum_{k=0}^{\infty} (k + 1)f_k^{(4)}z^k + \sum_{k=1}^{\infty} kf_k^{(4)}z^{k-1} \right)$$

$$- [z^{n+1}](32z^2 + 8z) \sum_{k=0}^{\infty} f_k^{(4)}z^k.$$
In view of (2.2),
\[
5(16z^2 - 1) \sum_{k=1}^{\infty} \frac{kz^{k-1}}{(1 + 4z)^{k+1}} W_k \left( \frac{1}{1 + 4z} \right)
\]
\[
= 5(16z^2 - 1) \sum_{m=1}^{\infty} m(f_m^{(4)} + 4s_m)z^{m-1}
\]
\[
= 5(16z^2 - 1)(6 + 68z + 900z^2 + \ldots) = -30 - 340z - 4020z^2 - \ldots
\]
Combining this with the final result in the last paragraph, we find that
\[
5(16z^2 - 1) \sum_{k=1}^{\infty} \frac{kz^{k-1}}{(1 + 4z)^{k+1}} W_k \left( \frac{1}{1 + 4z} \right)
\]
\[
= (4z + 1)(8z - 5) \left( 4 \sum_{k=0}^{\infty} (k + 1)f_k^{(4)}z^k + \sum_{k=1}^{\infty} kf_k^{(4)}z^{k-1} \right) - 8z(4z + 1) \sum_{k=0}^{\infty} f_k^{(4)}z^k
\]
and hence
\[
5(4z - 1) \sum_{k=1}^{\infty} \frac{kz^{k-1}}{(1 + 4z)^{k+1}} W_k \left( \frac{1}{1 + 4z} \right)
\]
\[
= (8z - 5) \left( 4 \sum_{k=0}^{\infty} (k + 1)f_k^{(4)}z^k + \sum_{k=1}^{\infty} kf_k^{(4)}z^{k-1} \right) - 8z \sum_{k=0}^{\infty} f_k^{(4)}z^k.
\]
This yields the desired (2.8).
The proof of Theorem 2.3 is now complete.  \qed

**Proof of Theorem 1.1.** In light of Theorem 2.3, we have
\[
\sum_{k=0}^{\infty} \frac{45k + 8}{40^k} W_k \left( \frac{9}{10} \right) = \frac{1075}{72} \sum_{k=0}^{\infty} \frac{4k + 1}{36^k} f_k^{(4)},
\]
\[
\sum_{k=0}^{\infty} \frac{1360k + 389}{(-60)^k} W_k \left( \frac{16}{15} \right) = \frac{9225}{32} \sum_{k=0}^{\infty} \frac{4k + 1}{(-64)^k} f_k^{(4)},
\]
\[
\sum_{k=0}^{\infty} \frac{735k + 124}{200^k} W_k \left( \frac{49}{50} \right) = \frac{10125}{784} \sum_{k=0}^{\infty} \frac{60k + 11}{196^k} f_k^{(4)},
\]
\[
\sum_{k=0}^{\infty} \frac{376380k + 69727}{(-320)^k} W_k \left( \frac{81}{80} \right) = \frac{5209600}{243} \sum_{k=0}^{\infty} \frac{17k + 3}{(-324)^k} f_k^{(4)},
\]
\[
\sum_{k=0}^{\infty} \frac{348840k + 47461}{1300^k} W_k \left( \frac{324}{325} \right) = \frac{1314625}{243} \sum_{k=0}^{\infty} \frac{65k + 9}{1296^k} f_k^{(4)},
\]
\[
\sum_{k=0}^{\infty} \frac{41673840k + 4777111}{5780^k} W_k \left( \frac{1444}{1445} \right) = \frac{147758475}{1444} \sum_{k=0}^{\infty} \frac{408k + 47}{5776^k} f_k^{(4)}.
\]
By S. Cooper [4],
\[ \sum_{k=0}^{\infty} \frac{4k+1}{36^k} f_k^{(4)} = \frac{6\sqrt{15}}{5\pi}, \quad \sum_{k=0}^{\infty} \frac{4k+1}{(-64)^k} f_k^{(4)} = \frac{32\sqrt{15}}{45\pi}, \]
\[ \sum_{k=0}^{\infty} \frac{60k+11}{196^k} f_k^{(4)} = \frac{14\sqrt{7}}{\pi}, \quad \sum_{k=0}^{\infty} \frac{17k+3}{(-324)^k} f_k^{(4)} = \frac{81\sqrt{5}}{20\pi}, \]
\[ \sum_{k=0}^{\infty} \frac{65k+9}{1296^k} f_k^{(4)} = \frac{81\sqrt{2}}{4\pi}, \quad \sum_{k=0}^{\infty} \frac{408k+47}{5776^k} f_k^{(4)} = \frac{76\sqrt{95}}{5\pi}. \]

So we have the desired (1.1)-(1.6). This concludes the proof. \(\square\)

3. Congruences related to the identities (1.1)-(1.6)

In [12, Section 3] the author introduced the polynomials
\[ S_n(x) = \sum_{k=0}^{n} \binom{n}{k}^4 x^k \quad (n = 0, 1, 2, \ldots) \quad (3.1) \]
and made conjectures on \( \sum_{k=0}^{p-1} S_k(x) \) modulo \( p^2 \) (with \( p \) an odd prime) for each integer \( x \) among the numbers
\[ 1, \ -2, \ \pm 4, \ -9, \ 12, \ 16, \ -20, \ 36, \ -64, \ 196, \ -324, \ 1296, \ 5776. \]
See also [15, Conjectures 49-51].

Theorem 1.1 and its proof are actually motivated by the following conjecture.

**Conjecture 3.1.** Let \( p \) be an odd prime and let \( x \) be a \( p \)-adic integer with \( x \not\equiv 0 \pmod{p} \). Then
\[ \sum_{k=0}^{p-1} \frac{1}{(4x)^k} W_k \left( 1 - \frac{1}{x} \right) \equiv \sum_{k=0}^{p-1} S_k(4x-4) \pmod{p}. \quad (3.2) \]

When
\[ x \in \left\{ 2, \pm \frac{5}{4}, \pm 4, \ 5, \ 10, \ -15, \ 50, \ -80, \ 325, \ 1445 \right\}, \]
we have the further congruence
\[ \sum_{k=0}^{p-1} \frac{1}{(4x)^k} W_k \left( 1 - \frac{1}{x} \right) \equiv \sum_{k=0}^{p-1} S_k(4x-4) \pmod{p^2}. \quad (3.3) \]

The identity (1.1) is motivated by the following conjecture on related congruences.

**Conjecture 3.2.** (i) For any \( n \in \mathbb{Z}^+ \) we have
\[ \frac{10^{n-1}}{4n} \sum_{k=0}^{n-1} (45k+8)40^{n-1-k} W_k \left( \frac{9}{10} \right) \in \mathbb{Z}^+. \]
(ii) Let \( p \neq 2, 5 \) be a prime. Then
\[
\sum_{k=0}^{p-1} \frac{45k + 8}{40^k} W_k \left( \frac{9}{10} \right) \equiv \frac{p}{16} \left( 129 \left( \frac{-15}{p} \right) - 1 \right) \pmod{p^2}.
\]
When \( \left( \frac{-15}{p} \right) = 1 \), for any \( n \in \mathbb{Z}^+ \) the number
\[
\sum_{k=0}^{pn-1} \frac{45k + 8}{40^k} W_k \left( \frac{9}{10} \right) - p \sum_{k=0}^{n-1} \frac{45k + 8}{40^k} W_k \left( \frac{9}{10} \right)
\]
divided by \((pn)^2\) is a \( p \)-adic integer.

The identity (1.2) is motivated by the following conjecture on related congruences.

**Conjecture 3.3.** (i) For any \( n \in \mathbb{Z}^+ \) we have
\[
\frac{15^{n-1}}{n} \sum_{k=0}^{n-1} (1360k + 389)(-60)^{n-1-k} W_k \left( \frac{16}{15} \right) \in \mathbb{Z}^+.
\]

(ii) Let \( p > 5 \) be a prime. Then
\[
\sum_{k=0}^{p-1} \frac{1360k + 389}{(-60)^k} W_k \left( \frac{16}{15} \right) \equiv \frac{p}{2} \left( 779 \left( \frac{-15}{p} \right) - 1 \right) \pmod{p^2}.
\]
When \( \left( \frac{-15}{p} \right) = 1 \), for any \( n \in \mathbb{Z}^+ \) the number
\[
\sum_{k=0}^{pn-1} \frac{1360k + 389}{(-60)^k} W_k \left( \frac{16}{15} \right) - p \sum_{k=0}^{n-1} \frac{1360k + 389}{(-60)^k} W_k \left( \frac{16}{15} \right)
\]
divided by \((pn)^2\) is a \( p \)-adic integer.

The identity (1.3) is motivated by the following conjecture on related congruences.

**Conjecture 3.4.** (i) For any \( n \in \mathbb{Z}^+ \) we have
\[
\frac{50^{n-1}}{4n} \sum_{k=0}^{n-1} (735k + 124)200^{n-1-k} W_k \left( \frac{49}{50} \right) \in \mathbb{Z}^+.
\]

(ii) Let \( p \neq 2, 5 \) be a prime. Then
\[
\sum_{k=0}^{p-1} \frac{735k + 124}{200^k} W_k \left( \frac{49}{50} \right) \equiv \frac{p}{32} \left( 3969 \left( \frac{-7}{p} \right) - 1 \right) \pmod{p^2}.
\]
When \( \left( \frac{7}{p} \right) = 1 \), for any \( n \in \mathbb{Z}^+ \) the number
\[
\sum_{k=0}^{pn-1} \frac{735k + 124}{200^k} W_k \left( \frac{49}{50} \right) - p \sum_{k=0}^{n-1} \frac{735k + 124}{200^k} W_k \left( \frac{49}{50} \right)
\]
divided by \((pn)^2\) is a \( p \)-adic integer.
The identity (1.4) is motivated by the following conjecture on related congruences.

**Conjecture 3.5.** (i) For any \( n \in \mathbb{Z}^+ \) we have

\[
\frac{80^{n-1}}{n} \sum_{k=0}^{n-1} (376380k + 69727)(-1)^k 320^{n-1-k} W_k \left( \frac{81}{80} \right) \in \mathbb{Z}^+.
\]

(ii) Let \( p \neq 2, 5 \) be a prime. Then

\[
\sum_{k=0}^{p-1} \frac{376380k + 69727}{(-320)^k} W_k \left( \frac{81}{80} \right) \equiv \frac{p}{3} \left( 209198 \left( -\frac{5}{p} \right) - 17 \right) \pmod{p^2}.
\]

When \( \left( \frac{-5}{p} \right) = 1 \), for any \( n \in \mathbb{Z}^+ \) the number

\[
\sum_{k=0}^{pn-1} \frac{376380k + 69727}{(-320)^k} W_k \left( \frac{81}{80} \right) - p \sum_{k=0}^{n-1} \frac{376380k + 69727}{(-320)^k} W_k \left( \frac{81}{80} \right)
\]

divided by \((pn)^2\) is a \( p \)-adic integer.

The identity (1.5) is motivated by the following conjecture on related congruences.

**Conjecture 3.6.** (i) For any \( n \in \mathbb{Z}^+ \) we have

\[
\frac{325^{n-1}}{n} \sum_{k=0}^{n-1} (348840k + 47461)1300^{n-1-k} W_k \left( \frac{324}{325} \right) \in \mathbb{Z}^+,
\]

and this number is odd if and only if \( n \in \{2^a : a \in \mathbb{N} \} \).

(ii) Let \( p \neq 2, 5, 13 \) be a prime. Then

\[
\sum_{k=0}^{p-1} \frac{348840k + 47461}{1300^k} W_k \left( \frac{324}{325} \right) \equiv \frac{p}{3} \left( 142384 \left( -\frac{2}{p} \right) - 1 \right) \pmod{p^2}.
\]

When \( p \equiv 1, 3 \pmod{8} \), for any \( n \in \mathbb{Z}^+ \) the number

\[
\sum_{k=0}^{pn-1} \frac{348840k + 47461}{1300^k} W_k \left( \frac{324}{325} \right) - p \sum_{k=0}^{n-1} \frac{348840k + 47461}{1300^k} W_k \left( \frac{324}{325} \right)
\]

divided by \((pn)^2\) is a \( p \)-adic integer.

The identity (1.6) is motivated by the following conjecture on related congruences.

**Conjecture 3.7.** (i) For any \( n \in \mathbb{Z}^+ \) we have

\[
\frac{1445^{n-1}}{n} \sum_{k=0}^{n-1} (41673840k + 4777111)5780^{n-1-k} W_k \left( \frac{1444}{1445} \right) \in \mathbb{Z}^+,
\]

and this number is odd if and only if \( n \in \{2^a : a \in \mathbb{N} \} \).
(ii) Let \( p \neq 2, 5, 17 \) be a prime. Then
\[
\sum_{k=0}^{p-1} \frac{41673840k + 4777111}{5780^k} W_k \left( \frac{1444}{1445} \right) = p \left( \frac{-95}{p} \right) - 2 \pmod{p^2}.
\]

When \( \left( \frac{-95}{p} \right) = 1 \), for any \( n \in \mathbb{Z}^+ \) the number
\[
\sum_{k=0}^{p-1} \frac{5928k + 253}{5780^k} W_k \left( \frac{1156}{5} \right) - p \sum_{k=0}^{n-1} \frac{5928k + 253}{5780^k} W_k \left( \frac{1156}{5} \right)
\]
divided by \((pn)^2\) is a \( p \)-adic integer.

The conjectural identities (1.7)-(1.15) are motivated by related congruences stated in [18, Conjectures 10.34-10.42].

4. A NEW TYPE SERIES FOR \( 1/\pi \) INVOLVING GENERALIZED CENTRAL TRINOMIAL COEFFICIENTS

For \( b, c \in \mathbb{Z} \) and \( n \in \mathbb{N} \) the generalized trinomial coefficient \( T_n(b, c) \) denotes the coefficient of \( x^n \) in the expansion of \((x^2 + bx + c)^n\).

In 2011, the author [16, 13] posed over 60 conjectural series for \( 1/\pi \) of the following seven types with \( a, b, c, d, m \) integers and \( mbcd(b^2 - 4c) \) nonzero.

- **Type I.** \( \sum_{k=0}^{\infty} \frac{a+dk}{m^k} \binom{2k}{k}^2 T_k(b,c) \).
- **Type II.** \( \sum_{k=0}^{\infty} \frac{a+dk}{m^k} \binom{3k}{k} T_k(b,c) \).
- **Type III.** \( \sum_{k=0}^{\infty} \frac{a+dk}{m^k} \binom{4k}{k} T_k(b,c) \).
- **Type IV.** \( \sum_{k=0}^{\infty} \frac{a+dk}{m^k} \binom{2k}{2k} T_{2k}(b,c) \).
- **Type V.** \( \sum_{k=0}^{\infty} \frac{a+dk}{m^k} \binom{2k}{2k} T_{3k}(b,c) \).
- **Type VI.** \( \sum_{k=0}^{\infty} \frac{a+dk}{m^k} T_k(b,c)^3 \).
- **Type VII.** \( \sum_{k=0}^{\infty} \frac{a+dk}{m^k} T_k(b,c)^2 \).

Though some of these new families of conjectural series for \( 1/\pi \) have been proved (see, e.g., [3]), the three conjectural series for \( 1/\pi \) of type VI and two of type VII remain open.

In a recent published paper [17] the author proposed four conjectural series for \( 1/\pi \) of a new type:

- **Type VIII.** \( \sum_{k=0}^{\infty} \frac{a+dk}{m^k} T_k(b,c) T_k(b_*,c_*)^2 \),
where \( a, b, b_*, c, c_*, d, m \) are integers with \( mbcd_*,cc_*d(b^2 - 4c)(b_*^2 - 4c_*)(b_2^2c_* - b_*^2c) \neq 0 \).

Here we introduce series for \( 1/\pi \) involving generalized central trinomial coefficients of the following novel type:

- **Type IX.** \( \sum_{k=0}^{\infty} \frac{a+dk}{m^k} \binom{2k}{k} T_k(b,c) T_k(b_*,c_*) \),
where \( a, b, b_*, c, c_*, d, m \) are integers with \( mbcd_*,cc_*d(b^2 - 4c)(b_*^2 - 4c_*)(b^2c_* - b_*^2c) \neq 0 \).
Conjecture 4.1. We have the following identities:

\[
\sum_{k=0}^{\infty} \frac{4290k + 367}{3136^k} \binom{2k}{k} T_k(14, 1) T_k(17, 16) = \frac{5390}{\pi} \tag{IX1}
\]

and

\[
\sum_{k=0}^{\infty} \frac{540k + 137}{3136^k} \binom{2k}{k} T_k(2, 81) T_k(14, 81) = 98 \frac{3}{\pi} (10 + 7\sqrt{5}). \tag{IX2}
\]

The conjectural identity (IX1) is motivated by the following conjecture on congruences.

Conjecture 4.2. (i) For any integer \( n > 1 \), we have

\[
n\binom{2n}{n} \mid \sum_{k=0}^{n-1} (4290k + 367)3136^{n-k} \binom{2k}{k} T_k(14, 1) T_k(17, 16).
\]

(ii) Let \( p \) be an odd prime with \( p \neq 7 \). Then

\[
\sum_{k=0}^{p-1} \frac{4290k + 367}{3136^k} \binom{2k}{k} T_k(14, 1) T_k(17, 16) \equiv \frac{p}{2} \left( 1430 \left( -\frac{1}{p} \right) + 39 \left( \frac{3}{p} \right) - 735 \right) \pmod{p^2}.
\]

Moreover, when \( p \equiv 1 \pmod{12} \), for any \( n \in \mathbb{Z}^+ \) the number

\[
\sum_{k=0}^{pn-1} \frac{4290k + 367}{3136^k} \binom{2k}{k} T_k(14, 1) T_k(17, 16)
\]

\[
- p \sum_{k=0}^{n-1} \frac{4290k + 367}{3136^k} \binom{2k}{k} T_k(14, 1) T_k(17, 16)
\]

divided by \((pn)^2 \binom{2n}{n}\) is a \( p \)-adic integer.

(iii) For any prime \( p > 7 \), we have

\[
\left( -\frac{1}{p} \right) \sum_{k=0}^{p-1} \frac{1}{3136^k} \binom{2k}{k} T_k(14, 1) T_k(17, 16)
\]

\[
\equiv \begin{cases} 
4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \text{ & } p = x^2 + 15y^2 (x, y \in \mathbb{Z}), \\
2p - 12x^2 \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \text{ & } p = 3x^2 + 5y^2 (x, y \in \mathbb{Z}), \\
0 \pmod{p^2} & \text{if } \left( -\frac{15}{p} \right) = -1.
\end{cases}
\]

Remark 4.1. Note that the imaginary quadratic field \( \mathbb{Q}(\sqrt{-15}) \) has class number two.

The conjectural identity (IX2) is motivated by the following conjecture on congruences.
Conjecture 4.3. (i) For any integer $n > 1$, we have
\[
2n \binom{2n}{n} \mid \sum_{k=0}^{n-1} (540k + 137)3136^{n-1-k} \binom{2k}{k} T_k(2, 81)T_k(14, 81).
\]

(ii) Let $p$ be an odd prime with $p \neq 7$. Then
\[
\sum_{k=0}^{p-1} \frac{540k + 137}{3136^k} \binom{2k}{k} T_k(2, 81)T_k(14, 81)
\equiv \frac{p}{3} \left( 270 \left( \frac{-1}{p} \right) - 104 \left( \frac{-2}{p} \right) + 245 \left( \frac{-5}{p} \right) \right) \pmod{p^2}.
\]
Moreover, when $p \equiv \pm 1, \pm 9 \pmod{40}$, for any $n \in \mathbb{Z}^+$ the number
\[
\sum_{k=0}^{pn-1} \frac{540k + 137}{3136^k} \binom{2k}{k} T_k(2, 81)T_k(14, 81)
\]
\[
-p \left( \frac{-1}{p} \right) \sum_{k=0}^{n-1} \frac{540k + 137}{3136^k} \binom{2k}{k} T_k(2, 81)T_k(14, 81)
\]
divided by $(pn)^2 \binom{2n}{n}$ is a $p$-adic integer.

(iii) For any prime $p > 7$, we have
\[
\left( \frac{-1}{p} \right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{3136^k} T_k(2, 81)T_k(14, 81)
\]
\[
\begin{cases}
4x^2 - 2p \pmod{p^2} & \text{if } \left( \frac{2}{p} \right) = \left( \frac{p}{2} \right) = 1 & \text{if } p = x^2 + 30y^2, \\
8x^2 - 2p \pmod{p^2} & \text{if } \left( \frac{2}{p} \right) = 1, \left( \frac{p}{2} \right) = \left( \frac{x}{p} \right) = -1 & \text{if } p = 2x^2 + 15y^2, \\
20x^2 - 2p \pmod{p^2} & \text{if } \left( \frac{2}{p} \right) = 1, \left( \frac{p}{2} \right) = \left( \frac{p}{x} \right) = 1 & \text{if } p = 5x^2 + 6y^2, \\
2p - 12x^2 \pmod{p^2} & \text{if } \left( \frac{3}{p} \right) = 1, \left( \frac{p}{3} \right) = \left( \frac{3}{x} \right) = 1 & \text{if } p = 3x^2 + 10y^2, \\
0 \pmod{p^2} & \text{if } \left( \frac{-30}{p} \right) = -1,
\end{cases}
\]
where $x$ and $y$ are integers.

Remark 4.2. Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-30})$ has class number four.

5. Other New Conjectural Series for $1/\pi$

As mentioned in [14, Remark 4.4], an identity of MacMahon implies that the polynomial
\[
F_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n + 2k}{2k} \binom{2k}{k} x^{n-k}
\]
at $x = -4$ coincides with the Franel number $f_n = \sum_{k=0}^{n} \binom{n}{k}^3$. Conjecture 4.4 of Sun [14] lists ten conjectural series for $1/\pi$ involving $F_n(x)$ with $x \neq -4$;
eight of them were later confirmed in [6], but the following two remain open:

\[ \sum_{k=0}^{\infty} \frac{357k + 103}{2160^k} \binom{2k}{k} F_k(-324) = \frac{90}{\pi}, \]  
(5.1)

\[ \sum_{k=0}^{\infty} \frac{k}{3645^k} \binom{2k}{k} F_k(486) = \frac{10}{3\pi}. \]  
(5.2)

Here we pose the following new conjecture.

**Conjecture 5.1.** We have the following identities:

\[ \sum_{k=0}^{\infty} \frac{6k + 1}{(-1728)^k} \binom{2k}{k} F_k(-324) = \frac{24}{25\pi} \sqrt{375 + 120\sqrt{10}}, \]  
(5.3)

\[ \sum_{k=0}^{\infty} \frac{4k + 1}{(-160)^k} \binom{2k}{k} F_k(-20) = \frac{\sqrt{30}}{5\pi} \cdot \frac{5 + \sqrt{145 + 30\sqrt{6}}}{\sqrt{145 + 30\sqrt{6}}}, \]  
(5.4)

\[ \sum_{k=0}^{\infty} \frac{1290k + 289}{27648^k} \binom{2k}{k} F_k(-2160) = \frac{96\sqrt{15}}{\pi}, \]  
(5.5)

\[ \sum_{k=0}^{\infty} \frac{804k + 49}{276480^k} \binom{2k}{k} F_k(12096) = \frac{120\sqrt{15}}{\pi}, \]  
(5.6)

\[ \sum_{k=0}^{\infty} \frac{24k + 5}{135}^k \binom{2k}{k} F_k\left(-\frac{27}{8}\right) = \frac{3}{2\pi} (5\sqrt{6} + 4\sqrt{15}). \]  
(5.7)

**Remark 5.1.** The author actually found (5.3)-(5.7) in 2020. As all of them converge quickly, one can easily check them via Mathematica or Maple.

The identity (5.3) is motivated by [14, Conjecture 4.6]. The reader might wonder how we found the right-hand side of the identity (5.3). We thought that the left-hand side of (5.3) times \(\pi\) is an algebraic number and found the form of this algebraic number via calculating its first 100 digits and using the Maple command `identify`.

The identities (5.4) and (5.5) are motivated by related congruences stated in [18, Conjectures 10.47–10.48].

The identity (5.6) is motivated by the following conjecture on related congruences.

**Conjecture 5.2.** (i) Let \(n > 1\) be an integer. Then

\[ \frac{1}{n \binom{2n}{n}} \sum_{k=0}^{n-1} (804k + 49)276480^{-k} \binom{2k}{k} F_k(12096) \in \mathbb{Z}^+, \]

and this number is odd if and only if \(n \in \{2^a + 1 : a \in \mathbb{N}\}\).

(ii) Let \(p > 5\) be a prime. Then

\[ \sum_{k=0}^{p-1} \frac{804k + 49}{276480^k} \binom{2k}{k} F_k(12096) \equiv p \left( 95 \left( \frac{-15}{p} \right) - 46 \left( \frac{30}{p} \right) \right) \pmod{p^2}. \]
Moreover, if \( p \equiv 1, 3 \pmod{8} \) then for any \( n \in \mathbb{Z}^+ \) the number
\[
\sum_{k=0}^{n-1} \frac{804k + 49}{276480^k} \binom{2k}{k} F_k(12096) - p \left( \frac{-15}{p} \right) \sum_{k=0}^{n-1} \frac{804k + 49}{276480^k} \binom{2k}{k} F_k(12096)
\]
divided by \((pn)^2\binom{2n}{n}\) is a \( p \)-adic integer.

(iii) Let \( p > 5 \) be a prime. Then
\[
\sum_{k=0}^{p-1} \frac{(2k)^k}{276480^k} F_k(12096)
\]
gives
\[
\begin{cases}
4x^2 - 2p \pmod{p^2} & \text{if } \left( \frac{x}{p} \right) = \left( \frac{2}{p} \right) = \left( \frac{F_k}{p} \right) = 1 & \text{if } p = x^2 + 210y^2, \\
8x^2 - 2p \pmod{p^2} & \text{if } \left( \frac{x}{p} \right) = \left( \frac{2}{p} \right) = \left( \frac{F_k}{p} \right) = -1 & \text{if } p = 2x^2 + 105y^2, \\
2p - 12x^2 \pmod{p^2} & \text{if } \left( \frac{x}{p} \right) = \left( \frac{2}{p} \right) = 1 & \text{if } p = 3x^2 + 70y^2, \\
20x^2 - 2p \pmod{p^2} & \text{if } \left( \frac{x}{p} \right) = \left( \frac{2}{p} \right) = \left( \frac{F_k}{p} \right) = -1 & \text{if } p = 5x^2 + 42y^2, \\
2p - 24x^2 \pmod{p^2} & \text{if } \left( \frac{x}{p} \right) = \left( \frac{2}{p} \right) = 1 & \text{if } p = 6x^2 + 35y^2, \\
28x^2 - 2p \pmod{p^2} & \text{if } \left( \frac{x}{p} \right) = \left( \frac{2}{p} \right) = -1 & \text{if } p = 7x^2 + 30y^2, \\
40x^2 - 2p \pmod{p^2} & \text{if } \left( \frac{x}{p} \right) = \left( \frac{2}{p} \right) = -1 & \text{if } p = 10x^2 + 21y^2, \\
56x^2 - 2p \pmod{p^2} & \text{if } \left( \frac{x}{p} \right) = \left( \frac{2}{p} \right) = -1 & \text{if } p = 14x^2 + 15y^2, \\
0 \pmod{p^2} & \text{if } \left( \frac{210}{p} \right) = -1,
\end{cases}
\]
where \( x \) and \( y \) are integers.

Remark 5.2. Note that the imaginary quadratic field \( \mathbb{Q}(\sqrt{-210}) \) has class number eight.

The identity (5.7) is motivated by the following conjecture on related congruences.

Conjecture 5.3. (i) Let \( n \) be any positive integer. Then
\[
\frac{4^{n-1}}{n(2n-1)} \sum_{k=0}^{n-1} (24k + 5)135^{n-1-k}2^k \binom{2k}{k} F_k \left( -\frac{27}{8} \right) \in \mathbb{Z}^+, 
\]
and this number is congruent to 5 modulo 8.

(ii) Let \( p > 5 \) be a prime. Then
\[
\sum_{k=0}^{p-1} \frac{(24k + 5)2^k}{135^k} \binom{2k}{k} F_k \left( -\frac{27}{8} \right) \equiv p \left( 4 \left( -\frac{6}{p} \right) + \left( -\frac{15}{p} \right) \right) \pmod{p^2}.
\]
Moreover, if \( \left( \frac{10}{p} \right) = 1 \) then for any \( n \in \mathbb{Z}^+ \) the number
\[
\sum_{k=0}^{pn-1} \frac{(24k + 5)2^k}{135^k} \binom{2k}{k} F_k \left( -\frac{27}{8} \right) - p \left( -\frac{6}{p} \right) \sum_{k=0}^{n-1} \frac{(24k + 5)2^k}{135^k} \binom{2k}{k} F_k \left( -\frac{27}{8} \right)
\]
divided by \((pn)^2\binom{2n}{n}\) is a \( p \)-adic integer.
(iii) Let $p > 5$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{2k(2k)}{135^k} F_k \left( -\frac{27}{8} \right)$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{2}{p}) = (\frac{x}{5}) = (\frac{y}{5}) = 1 \& \ p = x^2 + 30y^2, \\
8x^2 - 2p \pmod{p^2} & \text{if } (\frac{2}{p}) = 1, \ (\frac{x}{p}) = (\frac{y}{p}) = -1 \& \ p = 2x^2 + 15y^2, \\
2p - 12x^2 \pmod{p^2} & \text{if } (\frac{2}{p}) = 1, \ (\frac{x}{p}) = (\frac{y}{p}) = -1 \& \ p = 3x^2 + 10y^2, \\
20x^2 - 2p \pmod{p^2} & \text{if } (\frac{2}{p}) = 1, \ (\frac{x}{p}) = (\frac{y}{p}) = -1 \& \ p = 5x^2 + 6y^2, \\
0 \pmod{p^2} & \text{if } (\frac{30}{p}) = -1, \end{cases}$$

where $x$ and $y$ are integers.

In 2012 the author (cf. [13, (8)]) conjectured that

$$\sum_{n=0}^{\infty} \frac{28n + 5 \binom{2n}{n}}{576^n} \sum_{k=0}^{n} \frac{5^n(2k)^2(2n-k)^2}{k!} \left( -\frac{25}{16} \right)^k = \frac{9}{\pi}(2 + \sqrt{2}),$$

which remains open up to now. Here we pose a similar conjecture.

**Conjecture 5.4.** We have the following identity:

$$\sum_{n=0}^{\infty} \frac{182n + 31 \binom{2n}{n}}{576^n} \sum_{k=0}^{n} \frac{(2k)^2(2n-k)^2}{k!} \left( -\frac{25}{16} \right)^k = \frac{189}{2\pi}. \quad (5.8)$$

This is motivated by the author’s following conjecture on related congruences.

**Conjecture 5.5.** Let $p > 3$ be a prime. Then

$$\sum_{n=0}^{p-1} \frac{182n + 31 \binom{2n}{n}}{576^n} \sum_{k=0}^{n} \frac{(2k)^2(2n-k)^2}{k!} \left( -\frac{25}{16} \right)^k \equiv \frac{p}{2} \left( 63 \left( -\frac{1}{p} \right) - 1 \right) \pmod{p^2}.$$  

Also,

$$\sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{576^n} \sum_{k=0}^{n} \frac{(2k)^2(2n-k)^2}{k!} \left( -\frac{25}{16} \right)^k \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{x}{p}) = 1 \& \ p = x^2 + 7y^2 \ (x, y \in \mathbb{Z}), \\
0 \pmod{p^2} & \text{if } (\frac{x}{p}) = -1, \ i.e., \ p \equiv 3, 5, 6 \pmod{7}. \end{cases}$$

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SOME NEW SERIES FOR 1/π MOTIVATED BY CONGRUENCES

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