Higher-Page Bott-Chern and Aeppli Cohomologies and Applications

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Abstract. For every positive integer $r$, we introduce two new cohomologies, that we call $E_r$-Bott-Chern and $E_r$-Aeppli, on compact complex manifolds. When $r = 1$, they coincide with the usual Bott-Chern and Aeppli cohomologies, but they are coarser, respectively finer, than these when $r \geq 2$. They provide analogues in the Bott-Chern-Aeppli context of the $E_r$-cohomologies featuring in the Frölicher spectral sequence of the manifold. We apply these new cohomologies in several ways to characterise the notion of page-$(r-1)$-$\partial \bar{\partial}$-manifolds that we introduced very recently. We also prove analogues of the Serre duality for these higher-page Bott-Chern and Aeppli cohomologies and for the spaces featuring in the Frölicher spectral sequence. We obtain a further group of applications of our cohomologies to the study of Hermitian-symplectic and strongly Gauduchon metrics for which we show that they provide the natural cohomological framework.

1 Introduction

Let $X$ be an $n$-dimensional compact complex manifold.

(I) The Frölicher spectral sequence (FSS) of $X$ is a well-known collection of complexes canonically associated with the complex structure of $X$. Its 0-th page

$$\cdots \xrightarrow{\bar{\partial}} C^\infty_{p,q}(X, \mathbb{C}) \xrightarrow{\bar{\partial}} C^\infty_{p,q+1}(X, \mathbb{C}) \xrightarrow{\bar{\partial}} \cdots$$

is given by the Dolbeault complexes of $X$, consisting of the spaces of $\mathbb{C}$-valued pure-type $C^\infty$ differential forms on $X$ together with differentials $d_0$ defined by the $\bar{\partial}$-operator. As in any spectral sequence, each of the next pages can be computed as the cohomology of the previous page. In particular, the 1-st page

$$\cdots \xrightarrow{d_1} E_1^{p,q}(X) \xrightarrow{d_1} E_1^{p+1,q}(X) \xrightarrow{d_1} \cdots$$

features the Dolbeault cohomology groups $H_{\bar{\partial}}^{p,q}(X, \mathbb{C}) = E_1^{p,q}(X)$ of $X$ as vector spaces, while the differentials $d_1$ are induced by $\partial$ as $d_1(\{\alpha\}) = \{\partial\alpha\}$. In general, for every $r \in \mathbb{N}$, the differentials $d_r$ on the $r$-th page

$$\cdots \xrightarrow{d_r} E_r^{p,q}(X) \xrightarrow{d_r} E_r^{p+r,q-r+1}(X) \xrightarrow{d_r} \cdots$$

are of type $(r, -r + 1)$.

Thus, the higher pages (namely, those corresponding to $r \geq 2$) of the FSS are refinements of the Dolbeault cohomology of $X$. The main interest of this spectral sequence is that it provides a link between the complex structure of $X$ and its differential structure. Indeed, the FSS converges to the De Rham cohomology of $X$ in the sense that there are (non-canonical) isomorphisms:

$$H^k_{\text{DR}}(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} E_r^{p,q}(X), \quad k \in \{0, \ldots, 2n\}.$$
As with any spectral sequence, there exists \( r \in \mathbb{N}^* \) such that all the differentials \( d_s \) vanish identically for all \( s \geq r \). This is equivalent to saying that the spectral sequence becomes stationary from the \( r \)-th page in the sense that
\[
E_r^{p,q}(X) = E_{r+1}^{p,q}(X) = \ldots, \quad p, q \in \{0, \ldots, n\}.
\]
This common vector space is denoted by \( E_\infty^{p,q}(X) \). We say in this case that the spectral sequence degenerates at the \( r \)-th page (or at \( E_r \)) and we denote this property by \( E_r(X) = E_\infty(X) \). Obviously, the smaller \( r \), the stronger the degeneration property.

(II) On the other hand, the classical Bott-Chern cohomology ([BC65]) provides a different kind of refinement of the Dolbeault cohomology, while the equally classical Aeppli cohomology ([Aep62]) is a coarser replacement thereof. They have a wide range of applications, especially in non-Kähler geometry, but also in Arakelov geometry and in Index Theory. (See e.g. [Bis13] and the references therein.) As is well known, they are respectively defined by
\[
H^{\bullet, \bullet}_{BC}(X, \mathbb{C}) = \frac{\ker \partial \cap \ker \bar{\partial}}{\text{Im} (\partial \bar{\partial})} \quad \text{and} \quad H^{\bullet, \bullet}_{A}(X, \mathbb{C}) = \frac{\ker(\partial \bar{\partial})}{\text{Im} \partial + \text{Im} \bar{\partial}}
\]
and the identity induces natural maps
\[
H^{p+q}_{DR}(X, \mathbb{C}) \quad \vdash  \quad H^{p,q}_{BC}(X) \quad \vdash  \quad H^{p,q}_{A}(X) \quad \vdash  \quad E_1^{p,q}(X).
\]

Moreover, a well-known Serre-type duality between the Bott-Chern cohomology of any bidegree and the Aeppli cohomology of the complementary bidegree (see e.g. [Sch07]), established by the well-defined, canonical, non-degenerate pairings:
\[
H^{p,q}_{BC}(X) \times H^{n-p,n-q}_{A}(X) \longrightarrow \mathbb{C}, \quad \{\alpha\}_{BC}, \{\beta\}_A \longmapsto \int_X \alpha \wedge \beta,
\]
for all \( p, q = 0, \ldots, n \), can be viewed as a generalisation to the transcendental context of the classical duality in algebraic geometry between curves and divisors.

Thus, in many respects, the classical Bott-Chern and Aeppli cohomologies are related to the first page of the FSS, namely to the Dolbeault cohomology of \( X \), to which they are canonically isomorphic on the important class of \( \partial\bar{\partial} \)-manifolds. These manifolds, introduced in [DGMS75], behave cohomologically as compact Kähler manifolds, hence, in particular, they support a classical Hodge decomposition, namely canonical isomorphisms:
\[
H^k_{DR}(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} E_1^{p,q}(X), \quad k \in \{0, \ldots, 2n\},
\]
induced by the identity map thanks to the hypothesis that every Dolbeault cohomology class $c \in E^{\bullet, \bullet}_1(X)$ can be represented by a $d$-closed pure-type form. But such an $X$ need not carry any Kähler metric and, in fact, the class of $\partial\bar{\partial}$-manifolds strictly contains the class of compact Kähler (and even that of class $\mathcal{C}$) manifolds.

(III) In [PSU20], we introduced, for every integer $r \in \mathbb{N}^*$, the class of page-$(r-1)$-$\partial\bar{\partial}$-manifolds by requiring the analogue of the Hodge decomposition to hold when the Dolbeault cohomology spaces $E^{\bullet, \bullet}_1(X)$ are replaced by the spaces $E^{\bullet, \bullet}_r(X)$ featuring on the $r$-th page of the FSS. In particular, page-0-$\partial\bar{\partial}$-manifolds coincide with the classical $\partial\bar{\partial}$-manifolds, but for every $r \geq 1$, the class of page-$r$-$\partial\bar{\partial}$-manifolds is new. Moreover, the page-$r$-$\partial\bar{\partial}$-property becomes weaker and weaker as $r$ increases in the sense that, for every $r \in \mathbb{N}$, the following implication holds:

$$X \text{ is a page-}r\text{-}\partial\bar{\partial}\text{-manifold} \implies X \text{ is a page-}(r+1)\text{-}\partial\bar{\partial}\text{-manifold}.$$  

Thus, the theory of page-$r$-$\partial\bar{\partial}$-manifolds enables the extension of Hodge Theory to the higher pages of the FSS and, in particular, a further extension of the class of compact Kähler manifolds. Quite a few examples of classes of page-$r$-$\partial\bar{\partial}$-manifolds that are not $\partial\bar{\partial}$-manifolds in the classical sense were exhibited in [PSU20].

1.1 The new cohomologies

The main goal of this work is to continue the construction of the higher-page Hodge Theory begun in [PSU20] by introducing higher-page analogues of the Bott-Chern and Aeppli cohomologies that provide a natural link between their classical counterparts and the Frölicher spectral sequence and, simultaneously, parallel the higher-page Frölicher cohomologies $E^{\bullet, \bullet}_r(X)$ when $r \geq 2$.

The new terminology that we introduce in this paper can be divided into two main groups.

(a) A $(p, q)$-form $\alpha$ on $X$ is said to be $E_r$-exact if $\alpha$ represents the zero $E_r$-cohomology class on the $r$-th page of the Frölicher spectral sequence of $X$. Also, $\alpha$ is said to be $\overline{E}_r$-exact if $\bar{\alpha}$ is $E_r$-exact. When $r = 1$, these notions coincide with $\partial\bar{\partial}$-exactness, respectively $\partial$-exactness.

The notion of $E_r\overline{E}_r$-exactness is introduced in (iii) of Definition 3.1 as a weakening, inspired by the characterisations of $E_1$-exactness and $\overline{E}_r$-exactness given in Proposition 2.3, of the standard notion of $\partial\bar{\partial}$-exactness. When $r = 1$, the notion of $E_1\overline{E}_1$-exactness is defined as $\partial\bar{\partial}$-exactness.

The notion of $E_r\overline{E}_r$-closedness is introduced in (i) of Definition 3.1 as a strengthening of the standard notion of $\partial\bar{\partial}$-closedness. When $r = 1$, the notion of $E_1\overline{E}_1$-closedness is defined as $\partial\bar{\partial}$-closedness.

All the exactness notions become weaker and weaker as $r$ increases, in contrast to the closedness conditions that become stronger and stronger.

(b) The $E_r$-Bott-Chern, respectively the $E_r$-Aeppli, cohomology groups of bidegree $(p, q)$ of $X$ are introduced in Definition 3.4 by quotienting out

- the smooth $d$-closed $(p, q)$-forms by the $E_r\overline{E}_r$-exact ones;
- respectively, the smooth $E_r\overline{E}_r$-closed $(p, q)$-forms by those lying in $\text{Im} \partial + \text{Im} \bar{\partial}$.

When $r = 1$, these cohomology groups coincide with the standard Bott-Chern, respectively Aeppli, cohomology groups.
Recall that $\partial \overline{\partial}$-manifolds are usually defined in terms of the equivalence of various types of exactness properties for $d$-closed pure-type forms. One of our main results in this paper, which also provides some of the main applications of the two kinds of cohomologies that we introduce, is the following statement, which extends this to page-$r$-$\partial \overline{\partial}$-manifolds (see Theorem 4.1).

**Theorem 1.1.** For a compact complex manifold $X$, denote by $E^{p,q}_{r,BC}(X)$ and $E^{p,q}_{r,A}(X)$ the higher-page Bott-Chern and Aeppli cohomology groups introduced in Definition 3.4. The following properties are equivalent:

1. $X$ is a page-$(r-1)$-$\partial \overline{\partial}$-manifold.

2. For any $d$-closed $(p,q)$-form $\alpha$, the following properties are equivalent:
   
   $\alpha$ is $d$-exact $\iff$ $\alpha$ is $E_r$-exact $\iff$ $\alpha$ is $\overline{E}_r$-exact $\iff$ $\alpha$ is $E_r \overline{E}_r$-exact.

3. For all $p,q \in \{0, \ldots, n\}$, the natural map $E^{p,q}_{r,BC}(X) \to E^{p,q}_{r,A}(X)$ is an isomorphism.

4. For all $p,q \in \{0, \ldots n\}$, the map $E^{p,q}_{r,BC}(X) \to E^{p,q}_{r,A}(X)$ is injective.

5. One has $\dim E^{k}_{r,BC}(X) = \dim E^{k}_{r,A}(X)$ for all $k \in \{0, \ldots, 2n\}$.

### 1.2 First set of further applications: duality results

Using the Hodge theory based on pseudo-differential Laplacians introduced in [Pop16] and [Pop19], we prove in §2, respectively §3, the analogues of the Serre duality for the $E_r$-cohomology, respectively the $E_r$-Bott-Chern and $E_r$-Aeppli cohomologies, for all $r \geq 2$. The case $r = 1$ is standard. These results can be summed up as follows (cf. Theorem 2.1, Corollaries 2.4 and 2.9, Theorem 3.11).

**Theorem 1.2.** Let $X$ be a compact complex manifold with $\dim C X = n$. Fix an arbitrary $r \in \mathbb{N}^\star$.

For every $p,q \in \{0, \ldots, n\}$, the canonical bilinear pairings

$$E^{p,q}_r(X) \times E^{n-p,n-q}_r(X) \to \mathbb{C}, \quad (\{\alpha\}_{E_r}, \{\beta\}_{E_r}) \mapsto \int_X \alpha \wedge \beta,$$

and

$$E^{p,q}_{r,BC}(X) \times E^{n-p,n-q}_{r,A}(X) \to \mathbb{C}, \quad (\{\alpha\}_{E_{r,BC}}, \{\beta\}_{E_{r,A}}) \mapsto \int_X \alpha \wedge \beta,$$

are well defined and non-degenerate.

This means that every space $E^{n-\cdot}_r(X)$ can be viewed as the dual of $E^{\cdot}_r(X)$ and every space $E^{\cdot,\cdot}_{r,BC}(X)$ can be viewed as the dual of $E^{\cdot,\cdot}_{r,A}(X)$.

### 1.3 Second set of further applications: special metrics

A final group of applications of the new cohomologies that we introduce in this paper features in section 6 where it is shown that the higher-page Aeppli cohomologies provide the natural framework for the study of Hermitian-symplectic (H-S) and strongly Gauduchon (sG) metrics and manifolds. (See Proposition 6.2.)
In particular, we introduce the strongly Gauduchon (sG) cone \( SG_X \subset E^{n-1,n-1}_{2,A}(X, \mathbb{R}) \) (in a way different to the one it was defined in [Pop15] and [PU18]), the Hermitian-symplectic (H-S) cone \( HS_X \subset E^{1,1}_{3,A}(X, \mathbb{R}) \) and the SKT cone \( SKT_X \subset E^{1,1}_{1,A}(X, \mathbb{R}) = H^{1,1}_A(X, \mathbb{R}) \) of \( X \) (see Definition 6.3) that are then shown to be open convex cones in their respective cohomology vector spaces (see Lemma 6.4).

We go on to use these cones to obtain:

- a new numerical characterisation of sGG manifolds (introduced in [Pop15] and [PU18]) besides those obtained in [PU18] (one of which was \( b_1 = 2h_0^0 \), while the inequality \( b_1 \leq 2h_0^0 \) holds on every manifold – see Theorem 1.6 in [PU18]);

- a numerical characterisation of compact complex SKT manifolds on which every SKT metric is Hermitian-symplectic.

Specifically, we prove the following fact. (See Corollary 6.6 for further details.) As usual, \( e \) stands each time for the dimension of the corresponding higher-page Aeppli cohomology space of \( X \) denoted by \( E \).

**Proposition 1.3.** Let \( X \) be a compact complex manifold with \( \dim \mathbb{C}X = n \).

(i) The inequality \( e_{2,A}^{n-1,n-1} \leq e_{1,A}^{n-1,n-1} \) holds. Moreover, \( X \) is an sGG manifold if and only if \( e_{2,A}^{n-1,n-1} = e_{1,A}^{n-1,n-1} \).

(ii) The inequality \( e_{3,A}^{1,1} \leq e_{1,A}^{1,1} \) holds. Moreover, every SKT metric on \( X \) is Hermitian-symplectic (H-S) if and only if \( e_{3,A}^{1,1} = e_{1,A}^{1,1} \).

Recall that Streets and Tian asked in [ST10, Question 1.7] whether every H-S manifold of dimension \( n \geq 3 \) is Kähler. If the answer to this question is affirmative, then (ii) of Proposition 1.3 implies that every SKT manifold \( X \) for which \( e_{3,A}^{1,1} = e_{1,A}^{1,1} \) is Kähler. Actually, we always have \( e_{3,A}^{1,1} \leq e_{1,A}^{1,1} \) on any \( X \).

## 2 Serre-type duality for the Frölicher spectral sequence

Let \( X \) be a compact complex manifold with \( \dim \mathbb{C}X = n \). For every \( r \in \mathbb{N} \), we let \( E_r^{p,q}(X) \) stand for the space of bidegree \( (p, q) \) featuring on the \( r^{th} \) page of the Frölicher spectral sequence of \( X \).

As is well known, the first page of this spectral sequence is given by the Dolbeault cohomology of \( X \), namely \( E_1^{p,q}(X) = H^{p,q}_\partial(X) \) for all \( p, q \). Moreover, the classical Serre duality asserts that every space \( H^{p,q}_\partial(X) \) is the dual of \( H^{n-p,n-q}_\partial(X) \) via the canonical non-degenerate bilinear pairing

\[
H^{p,q}_\partial(X) \times H^{n-p,n-q}_\partial(X) \longrightarrow \mathbb{C}, \quad ([\alpha], [\beta]) \mapsto \int_X \alpha \wedge \beta.
\]

In this section, we will extend this duality to all the pages of the Frölicher spectral sequence. For the sake of perspicuity, we will first treat the case \( r = 2 \) and then the more technically involved case \( r \geq 3 \).
2.1 Serre-type duality for the second page of the Fröhlicher spectral sequence

The main ingredient in the proof of the next statement is the Hodge theory for the $E_2$-cohomology introduced in [Pop16] via the construction of a pseudo-differential Laplace-type operator $\tilde{\Delta}$.

**Theorem 2.1.** For every $p, q \in \{0, \ldots, n\}$, the canonical bilinear pairing

$$E_2^{p,q}(X) \times E_2^{n-p,n-q}(X) \longrightarrow \mathbb{C}, \quad (\{\alpha\}_{E_2}, \{\beta\}_{E_2}) \mapsto \int_X \alpha \wedge \beta,$$

is well defined (i.e. independent of the choices of representatives of the cohomology classes involved) and non-degenerate.

**Proof.**

- To prove well-definedness, let $\{\alpha\}_{E_2} \in E_2^{p,q}(X)$ and $\{\beta\}_{E_2} \in E_2^{n-p,n-q}(X)$ be arbitrary classes in which we choose arbitrary representatives $\alpha, \beta$. Thus, $\partial \alpha = 0$, $\partial \alpha \in \text{Im} \bar{\partial}$ (since $[\alpha]_{\bar{\partial}} \in \ker d_1$) and $\beta$ has the analogous properties. In particular, $\partial \beta = \bar{\partial} v$ for some $(n-p+1, n-q-1)$-form $v$. Any other representative of the class $\{\alpha\}_{E_2}$ is of the shape $\alpha + \partial \eta + \bar{\partial} \zeta$ for some $(p-1, q)$-form $\eta \in \ker \bar{\partial}$ and some $(p, q-1)$-form $\zeta$. (Indeed, $[\partial \eta]_{\bar{\partial}} = d_1([\eta]_{\bar{\partial}})$.) We have

$$\int_X (\alpha + \partial \eta + \bar{\partial} \zeta) \wedge \beta = \int_X \alpha \wedge \beta + (-1)^{p+q} \int_X \eta \wedge \bar{\partial} \beta + (-1)^{p+q} \int_X \zeta \wedge \bar{\partial} \beta \quad \text{(by Stokes)}$$

$$= \int_X \alpha \wedge \beta + (-1)^{p+q} \int_X \eta \wedge \bar{\partial} v \quad \text{(since } \bar{\partial} \beta = \bar{\partial} v \text{ and } \bar{\partial} \beta = 0)$$

$$= \int_X \alpha \wedge \beta + \int_X \bar{\partial} \eta \wedge v = \int_X \alpha \wedge \beta \quad \text{(by Stokes and } \bar{\partial} \eta = 0).$$

Similarly, the integral $\int_X \alpha \wedge \beta$ does not change if $\beta$ is replaced by $\beta + \partial a + \bar{\partial} b$ with $a \in \ker \bar{\partial}$.

- To prove non-degeneracy for the pairing $(1)$, we fix an arbitrary Hermitian metric $\omega$ on $X$ and use the pseudo-differential Laplacian associated with $\omega$ introduced in [Pop16, §1.1]:

$$\tilde{\Delta} := \partial p'' \partial^* + \partial^* p'' \partial + \bar{\partial} \partial^* + \partial \bar{\partial} : C^\infty_{p,q}(X) \longrightarrow C^\infty_{p,q}(X), \quad p, q = 0, \ldots, n,$

where $p'' : C^\infty_{p,q}(X) \longrightarrow \mathcal{H}^{p,q}_{\Delta''}(X) := \ker \Delta''$ is the orthogonal projection w.r.t. the $L^2$ inner product defined by $\omega$ onto the $\Delta''$-harmonic space in the standard 3-space decomposition

$$C^\infty_{p,q}(X) = \mathcal{H}^{p,q}_{\Delta''}(X) \oplus \text{Im} \bar{\partial} \oplus \text{Im} \bar{\partial}^*.$$

Recall that $\Delta'' = \bar{\partial} \partial^* + \partial \bar{\partial} : C^\infty_{p,q}(X) \longrightarrow C^\infty_{p,q}(X)$ is the usual $\bar{\partial}$-Laplacian induced by $\omega$ and the above decomposition is $L^2$-orthogonal. It was proved in [Pop16, Theorem 1.1.] that for every $p, q \in \{0, \ldots, n\}$, the linear map

$$\mathcal{H}^{p,q}_{\Delta''}(X) := \ker(\tilde{\Delta} : C^\infty_{p,q}(X) \longrightarrow C^\infty_{p,q}(X)) \hookrightarrow E_2^{p,q}(X), \quad \alpha \mapsto \{\alpha\}_{E_2},$$

is an isomorphism. This is a Hodge isomorphism showing that every double class $\{\alpha\}_{E_2} \in E_2^{p,q}(X)$ contains a unique $\tilde{\Delta}$-harmonic representative.
**Claim 2.2.** For every $\alpha \in C_{p,q}^\infty(X)$, the equivalence holds: $\tilde{\Delta} \alpha = 0 \iff \Delta(\star \tilde{\alpha}) = 0$, where $\star = \star_\omega$ is the Hodge-star operator associated with $\omega$.

Suppose for a moment that this claim has been proved. To prove non-degeneracy for the pairing (1), let $\{\alpha\}_{E_2} \in E_2^{p,q}(X)$ be an arbitrary non-zero class whose unique $\tilde{\Delta}$-harmonic representative is denoted by $\alpha$. So, $\alpha \neq 0$ and $\star \tilde{\alpha} \in H^{n-p,n-q}_\Delta(X) \setminus \{0\}$. In particular, $\star \tilde{\alpha}$ represents an element in $E_2^{n-p,n-q}(X)$ and the pair $(\{\alpha\}_{E_2}, \{\star \tilde{\alpha}\}_{E_2})$ maps under (1) to $\int_X \alpha \wedge \star \tilde{\alpha} = \int_X |\alpha|^2_\omega dV_\omega = ||\alpha||^2_{L^2_\omega} \neq 0$. Since $p,q$ and $\alpha$ were arbitrary, we conclude that the pairing (1) is non-degenerate.

- **Proof of Claim 2.2.** Since $\tilde{\Delta}$ is a sum of non-negative operators of the shape $\Lambda^* A$, we have

$$\ker \tilde{\Delta} = \ker(p^\prime \partial) \cap \ker(p^\prime \partial^*) \cap \ker \partial \cap \ker \partial^*.$$ 

Thus, the orthogonal 3-space decomposition following above yields the following equivalence:

$$\alpha \in \ker \tilde{\Delta} \iff (i) \partial \alpha \in \Im \partial \oplus \Im \partial^* \quad (\text{ii}) \partial^* \alpha \in \Im \partial \oplus \Im \partial^* \quad \text{and} \quad (\text{iii}) \alpha \in \ker \partial \cap \ker \partial^*.$$ 

Let $\alpha \in \ker \tilde{\Delta}$. Since $\star : \Lambda^{p,q}T^*X \rightarrow \Lambda^{n-q,n-p}T^*X$ is an isomorphism, the well-known identities $\star \star = (-1)^{p+q} \partial^* \partial$ and $\partial^* = - \partial \star \star$ yield:

$$\partial^* \alpha = 0 \iff \partial \tilde{\alpha} = 0 \iff \partial^* (\star \tilde{\alpha}) = 0 \quad \text{and} \quad \partial^* \alpha = 0 \iff \partial^* \tilde{\alpha} = 0 \iff \partial^* (\star \tilde{\alpha}) = 0.$$ 

Thus, $\alpha$ satisfies condition (iii) if and only if $\star \tilde{\alpha}$ satisfies condition (iii).

Meanwhile, $\alpha$ satisfies condition (ii) if and only if there exist forms $\xi, \eta$ such that $\partial^* \alpha = \partial \xi + \partial^* \eta$. The last identity is equivalent to

$$\partial^* \tilde{\alpha} = \partial \xi + \partial^* \eta \iff -(\star \star) \partial (\star \tilde{\alpha}) = \pm \partial \star \star (\star \xi) \pm \star (- \partial \star \star \eta) \iff \partial (\star \tilde{\alpha}) = \pm \partial^* (\star \xi) \pm \partial (\star \eta).$$ 

Thus, $\alpha$ satisfies condition (ii) if and only if $\star \tilde{\alpha}$ satisfies condition (i).

Similarly, $\alpha$ satisfies condition (i) if and only if there exist forms $u, v$ such that $\partial \alpha = \partial u + \partial^* v$. The last identity is equivalent to

$$\partial \tilde{\alpha} = \partial \bar{u} + \partial^* \bar{v} \iff -\star \partial \star (\star \tilde{\alpha}) = -\star \partial (\star \tilde{\alpha}) = -\partial^* (\star \tilde{\alpha}) = \partial^* (\star \bar{u}) + \partial (\star \bar{v}).$$ 

Thus, $\alpha$ satisfies condition (i) if and only if $\star \tilde{\alpha}$ satisfies condition (ii).

This completes the proof of Claim 2.2 and implicitly that of Theorem 2.1. $\square$

### 2.2 Serre-type duality for the pages $r \geq 3$ of the Frölicher spectral sequence

In this section, we construct elliptic pseudo-differential operators $\tilde{\Delta}^{(\omega)}_{(r)}$ associated with any given Hermitian metric $\omega$ on $X$ whose kernels are isomorphic to the spaces $E_r^{p,q}(X)$ in every bidegree $(p,q)$. This extends to arbitrary $r \in \mathbb{N}^*$ the construction performed in [Pop16] for $r = 2$. We then apply this construction to prove the existence of a (non-degenerate) duality between every space $E_r^{p,q}(X)$ and the space $E_{r-p,n-q}^{n-p,n-q}(X)$ that extends to every page in the Frölicher spectral sequence the classical Serre duality (corresponding to $r = 1$).

Let $X$ be an arbitrary compact complex $n$-dimensional manifold. Fix $r \in \mathbb{N}$ and a bidegree $(p,q)$ with $p,q \in \{0, \ldots, n\}$. A smooth $\mathbb{C}$-valued $(p,q)$-form $\alpha$ on $X$ will be said to be $E_r$-closed if it represents an $E_r$-cohomology class, denoted by $\{\alpha\}_{E_r} \in E_r^{p,q}(X)$, on the $r$-th page of the Frölicher
spectral sequence of $X$. Meanwhile, $\alpha$ will be said to be $E_r$-exact if it represents the zero $E_r$-cohomology class, i.e. if $\{\alpha\}_{E_r} = 0 \in E^p,q_r(X)$. The $\mathbb{C}$-vector space of $C^\infty$ $E_r$-closed (resp. $E_r$-exact) $(p, q)$-forms will be denoted by $\mathcal{Z}^{p,q}_r(X)$ (resp. $\mathcal{E}^{p,q}_r(X)$). Of course, $\mathcal{E}^{p,q}_r(X) \subset \mathcal{Z}^{p,q}_r(X)$ and $E^{p,q}_r(X) = \mathcal{Z}^{p,q}_r(X)/\mathcal{E}^{p,q}_r(X)$.

The following statement was proved in [CFGU97]. It renders explicit the $E_r$-closedness and $E_r$-exactness conditions. In particular, it gives a more concrete description, equivalent to the more formal standard one, of the spaces $E^{p,q}_r(X)$ and the differentials $d_r$ featuring in the Frölicher spectral sequence. It was also used in [Pop19].

**Proposition 2.3.** (i) Fix $r \geq 2$. A form $\alpha \in C^\infty_{p,q}(X)$ is $E_r$-closed if and only if there exist forms $u_l \in C^\infty_{p+l,q-l}(X)$ with $l \in \{1, \ldots, r-1\}$ satisfying the following tower of $r$ equations:

\[
\begin{align*}
\bar{\partial}\alpha &= 0 \\
\partial\alpha &= \bar{\partial}u_1 \\
\bar{\partial}u_1 &= \bar{\partial}u_2 \\
& \vdots \\
\bar{\partial}u_{r-2} &= \bar{\partial}u_{r-1}.
\end{align*}
\]

We say in this case that $\bar{\partial}\alpha = 0$ and $\partial\alpha$ runs at least $(r-1)$ times.

(ii) Fix $r \geq 2$. The map $d_r : E^{p,q}_r(X) \to E^{p+r,q-r+1}_r(X)$ acts as $d_r(\{\alpha\}_{E_r}) = \{\partial u_{r-1}\}_{E_r}$ for every $E_r$-class $\{\alpha\}_{E_r}$, any representative $\alpha$ thereof and any choice of forms $u_l$ satisfying the above tower of $E_r$-closedness equations for $\alpha$.

(iii) Fix $r \geq 2$. A form $\alpha \in C^\infty_{p,q}(X)$ is $E_r$-exact if and only if there exist forms $\zeta \in C^\infty_{p-1,q}(X)$ and $\xi \in C^\infty_{p,q-1}(X)$ such that

\[
\alpha = \partial\zeta + \bar{\partial}\xi,
\]

with $\xi$ arbitrary and $\zeta$ satisfying the following tower of $(r-1)$ equations:

\[
\begin{align*}
\bar{\partial}\zeta &= \partial v_{r-3} \\
\bar{\partial}v_{r-3} &= \partial v_{r-4} \\
& \vdots \\
\bar{\partial}v_1 &= \partial v_0 \\
\bar{\partial}v_0 &= 0,
\end{align*}
\]

for some forms $v_0, \ldots, v_{r-3}$. (When $r = 2$, $\zeta_2 = \zeta_0$ must be $\bar{\partial}$-closed.)

We say in this case that $\bar{\partial}\zeta$ reaches 0 in at most $(r-1)$ steps.

(iv) The following inclusions hold in every bidegree $(p, q)$:

\[
\cdots \subset \mathcal{E}^{p,q}_r(X) \subset \mathcal{E}^{p,q}_{r+1}(X) \subset \cdots \subset \mathcal{Z}^{p,q}_{r+1}(X) \subset \mathcal{Z}^{p,q}_r(X) \subset \cdots,
\]

with $\{0\} = \mathcal{E}^{0,q}_0(X) \subset \mathcal{E}^{0,q}_1(X) = (\text{Im } \bar{\partial})^{0,q}$ and $\mathcal{Z}^{p,q}_1(X) = (\text{ker } \bar{\partial})^{p,q} \subset \mathcal{Z}^{p,q}_0(X) = C^\infty_{p,q}(X)$.

**Proof.** See [CFGU97].

The immediate consequence that we notice is the well-definedness of the pairing that parallels on any page of the Frölicher spectral sequence the classical Serre duality.
**Corollary 2.4.** Let $X$ be a compact complex manifold with $\dim \mathbb{C}X = n$. For every $r \in \mathbb{N}^*$ and every $p, q \in \{0, \ldots, n\}$, the canonical bilinear pairing

$$E^p_q(X) \times E^{n-p, n-q}_r(X) \longrightarrow \mathbb{C}, \quad (\{\alpha\}_{E_r}, \{\beta\}_{E_r}) \mapsto \int_X \alpha \wedge \beta,$$

is well defined (i.e. independent of the choices of representatives of the $E_r$-classes involved).

**Proof.** By symmetry, it suffices to prove that $\int_X \alpha \wedge \beta = 0$ whenever $\alpha \in C^\infty_{p,q}(X)$ is $E_r$-exact and $\beta \in C^\infty_{n-p,n-q}(X)$ is $E_r$-closed. By Proposition 2.3, these conditions are equivalent to

$$\bar{\partial} \beta = 0, \quad \partial \beta = \bar{\partial} u_1, \quad \partial u_1 = \bar{\partial} u_2, \ldots, \partial u_{r-2} = \bar{\partial} u_{r-1},$$

for some forms $u_j$ and to $\alpha = \partial \zeta + \bar{\partial} \xi$ for some form $\zeta$ satisfying

$$\bar{\partial} \zeta = \partial v_{r-3}, \quad \bar{\partial} v_{r-3} = \partial v_{r-4}, \ldots, \partial v_1 = \bar{\partial} v_0, \bar{\partial} v_0 = 0$$

for some forms $v_k$. We get

$$\int_X \alpha \wedge \beta = \int_X \partial \zeta \wedge \beta + \int_X \bar{\partial} \xi \wedge \beta.$$

Every integral on the r.h.s. above is seen to vanish by repeated integration by parts. Specifically,

$$\int_X \partial \xi \wedge \beta = \pm \int_X \xi \wedge \bar{\partial} \beta = 0 \quad \text{since} \quad \bar{\partial} \beta = 0,$$

while for every $l \in \{1, \ldots, r-2\}$ we have

$$\int_X \partial \zeta \wedge \beta = \pm \int_X \zeta \wedge \partial \beta = \pm \int_X \zeta \wedge \bar{\partial} u_1 = \pm \int_X \bar{\partial} \zeta \wedge u_1 = \pm \int_X \partial v_{r-3} \wedge u_1$$

$$= \pm \int_X v_{r-3} \wedge \partial u_1 = \pm \int_X v_{r-3} \wedge \bar{\partial} u_2 = \pm \int_X \bar{\partial} v_{r-3} \wedge u_2 = \pm \int_X \partial v_{r-4} \wedge u_2$$

$$\vdots$$

$$= \pm \int_X v_0 \wedge \partial u_{r-2} = \pm \int_X v_0 \wedge \bar{\partial} u_{r-1} = \pm \int_X \bar{\partial} v_0 \wedge u_{r-1} = 0,$$

since $\bar{\partial} v_0 = 0$. \qed

We will now prove that the above pairing is also non-degenerate, thus defining a Serre-type duality on every page of the Frölicher spectral sequence. Much of the following preliminary discussion appeared in [Pop19, §2.2 and Appendix], so we will only recall the bare bones.

Let us fix an arbitrary Hermitian metric $\omega$ on $X$. For every bidegree $(p, q)$, $\omega$-harmonic spaces (also called $E_r$-harmonic spaces):

$$\cdots \subset H^p_{r+1} \subset H^p_r \subset \cdots \subset H^p_1 \subset C^\infty_{p,q}(X)$$

were inductively constructed in [Pop17, §3.2, especially Definition 3.3. and Corollary 3.4.] such that every subspace $H^p_{r,q} = H^p_{r,q}(X, \omega)$ is isomorphic to the corresponding space $E^p_{r,q}(X)$ on the $r^{th}$ page of the Frölicher spectral sequence.

Moreover, these spaces fit into the inductive construction described in the next
Proposition 2.5. Let \((X, \omega)\) be a compact Hermitian manifold with \(\dim \mathbb{C} X = n\).

(i) For every bidegree \((p, q)\), the space \(C^\infty_{p, q}(X)\) splits successively into mutually \(L^2_\omega\)-orthogonal subspaces as follows:

\[
C^\infty_{p, q}(X) = \text{Im} \, d_0 \oplus \bigoplus_{r \in \mathbb{N}^*} \text{Im} \, d_1^{(\omega)} \oplus \bigoplus_{r \in \mathbb{N}^*} \text{Im} \, d_1^{(\omega)*}
\]

where, for \(r \in \mathbb{N}^*\), the operators \(d_1^{(\omega)}\) are defined as

\[
d_1^{(\omega)} = d_1^{(\omega) p, q} = p_r \partial D_{r-1} p_r : \mathcal{H}_r^{p, q} \longrightarrow \mathcal{H}_{r+r}^{p, q-r+1}
\]

using the \(L^2_\omega\)-orthogonal projections \(p_r = p_r^{p, q} : C^\infty_{p, q}(X) \longrightarrow \mathcal{H}_r^{p, q}\) onto the \(\omega\)-harmonic spaces \(\mathcal{H}_r^{p, q}\) and where we inductively define

\[
D_{r-1} := ((\Delta^{(1)}(1))^{-1} \bar{\partial}^* \partial) \ldots ((\Delta^{(r-1)})^{-1} \bar{\partial}^* \partial) \quad \text{and} \quad D_0 = \text{Id.}
\]

(So, \(p_1 = p''\).) See (iii) below for the inductive definition of the pseudo-differential Laplacians \(\tilde{\Delta}^{(r)}\).

Thus, the triples \((p_r, d_1^{(\omega)}, \mathcal{H}_r^{p, q})\) are defined by induction on \(r \in \mathbb{N}^*\): once the triple \((p_{r-1}, d_1^{(\omega)}, \mathcal{H}_{r-1}^{p, q})\) has been constructed for all the bidegrees \((p, q)\), it induces \(p_r\), which induces \(d_1^{(\omega)}\), which induces \(\mathcal{H}_r^{p, q}\) defined as the \(L^2_\omega\)-orthogonal complement of \(\text{Im} \, d_1^{(\omega)}\) in \(\ker \, d_1^{(\omega)}\).

The operators \(d_1^{(\omega)}\) can also be considered to be defined on the whole spaces of smooth forms:

\[
d_1^{(\omega)} = p_r \partial D_{r-1} p_r : C^\infty_{p, q}(X) \longrightarrow C^\infty_{p+r, q-r+1}(X).
\]

(ii) The above definition of \(d_1^{(\omega)}\) follows from the requirement that the following diagram be commutative:

\[
\begin{array}{ccc}
E_{r}^{p, q}(X) & \xrightarrow{d_1} & E_{r}^{p+r, q-r+1}(X) \\
\downarrow \approx & & \downarrow \approx \\
\mathcal{H}_{r}^{p, q} & \xrightarrow{d_r = p_r \partial D_{r-1} p_r} & \mathcal{H}_{r}^{p+r, q-r+1},
\end{array}
\]
where the maps $d_r : E^{p,q}_r(X) \to E^{p+1,q-r+1}_r(X)$ are the differentials on the $r$th page of the Frölicher spectral sequence. Thus, the maps $d_r^{(\omega)}$ are the metric realisations, at the level of the harmonic spaces, of the canonical maps $d_r$.

(iii) For every $r \in \mathbb{N}^*$, the adjoint of $d_r^{(\omega)}$ is

$$(d_r^{(\omega)})^* = p_r D_{r-1}^* \partial^* p_r : \mathcal{H}^{p+r,q-r+1}_r \to \mathcal{H}^{p,q}_r.$$ 

It induces the “Laplacian”

$$\tilde{\Delta}^{(\omega)}_{(r+1)} = d_r^{(\omega)} (d_r^{(\omega)})^* + (d_r^{(\omega)})^* d_r^{(\omega)} : \mathcal{H}^{p,q}_r \to \mathcal{H}^{p,q}_r$$

given by the explicit formula

$$\tilde{\Delta}^{(\omega)}_{(r+1)} = p_r [(\partial D_{r-1} p_r) (\partial D_{r-1} p_r)^* + (p_r \partial D_{r-1})^* (p_r \partial D_{r-1}) + \tilde{\Delta}^{(r)}] p_r,$$

which is the restriction and co-restriction to $\mathcal{H}^{p,q}_r$ of the pseudo-differential Laplacian

$$\tilde{\Delta}^{(r+1)} : (\partial D_{r-1} p_r) (\partial D_{r-1} p_r)^* + (p_r \partial D_{r-1})^* (p_r \partial D_{r-1}) + \tilde{\Delta}^{(r)} : C^\infty_p(X) \to C^\infty_p(X).$$

(iv) For every $r \in \mathbb{N}^*$ and every bidegree $(p, q)$, the following orthogonal 3-space decomposition holds:

$$\mathcal{H}^{p,q}_r = \text{Im} d_r^{(\omega)} \oplus \mathcal{H}^{p,q}_{r+1} \oplus \text{Im} (d_r^{(\omega)})^*,$$

where $\text{ker} d_r^{(\omega)} = \text{Im} d_r^{(\omega)} \oplus \mathcal{H}^{p,q}_{r+1}$. In particular, this confirms that $\mathcal{H}^{p,q}_{r+1}$ is the orthogonal complement for the $L^2$-inner product of $\text{Im} d_r^{(\omega)}$ in $\text{ker} d_r^{(\omega)}$. Moreover,

$$\mathcal{H}^{p,q}_{r+1} = \text{ker} \tilde{\Delta}^{(\omega)}_{(r+1)} = \text{ker} d_r^{(\omega)} \cap \text{ker} (d_r^{(\omega)})^* \simeq E^{p,q}_{r+1}(X),$$

for every $r \in \mathbb{N}$ and all $p, q \in \{0, \ldots, n\}$.

Proof. The verification of the details of these statements was done in [Pop19, §2.2 and Appendix]. □

We saw in (i) of Proposition 2.3 how the $E_r$-closedness property of a differential form is characterised in explicit terms. We will now define by analogy the property of $E_r^*$-closedness when a Hermitian metric has been fixed.

**Definition 2.6.** Let $(X, \omega)$ be an $n$-dimensional compact complex Hermitian manifold. Fix $r \geq 1$ and a bidegree $(p, q)$. A form $\alpha \in C^\infty_{p,q}(X)$ is said to be $E^*_r$-closed with respect to the metric $\omega$ if and only if there exist forms $v_l \in C^\infty_{p-l,q+l}(X)$ with $l \in \{1, \ldots, r-1\}$ satisfying the following tower of $r$ equations:

$$\begin{align*}
\bar{\partial}^* \alpha & = 0 \\
\partial^* \alpha & = \bar{\partial}^* v_1 \\
\partial^* v_1 & = \bar{\partial}^* v_2 \\
& \vdots \\
\partial^* v_{r-2} & = \bar{\partial}^* v_{r-1}.
\end{align*}$$
We say in this case that $\bar{\partial}^* \alpha = 0$ and $\partial^* \alpha$ runs at least $(r - 1)$ times.

We can now use the $E_r$-closedness and $E_{r+1}^*$-closedness properties to characterise the $H_r$-harmonicity property defined above.

**Proposition 2.7.** Let $(X, \omega)$ be an $n$-dimensional compact complex Hermitian manifold. Fix $r \geq 1$ and a bidegree $(p, q)$. For any form $\alpha \in C_{p,q}^\infty(X)$, the following equivalence holds:

$$\alpha \in H_r^{p,q} \iff \alpha \text{ is } E_r\text{-closed and } E_{r+1}^*\text{-closed.}$$

**Proof.** We know from Proposition 2.5 that $\alpha \in H_r^{p,q}$ if and only if $\alpha \in H_r^{p,q}$ and $\alpha \in \ker d_r^\omega \cap \ker (d_r^\omega)^*$. Now, for $\alpha \in H_r^{p,q}$, the definition of $d_r^\omega$ shows that $\alpha \in \ker d_r^\omega$ if and only if $\alpha \in \ker (d_r^\omega)^*$. Since $\alpha \in \ker (d_r^\omega)^*$, the definition of $(d_r^\omega)^*$ shows that $\alpha \in \ker (d_r^\omega)^*$ if and only if $\alpha \in \ker (d_r^\omega)^*$ and this last fact is equivalent to $\alpha$ being $E_{r+1}$-closed. Similarly, for $\alpha \in H_r^{p,q}$, the definition of $(d_r^\omega)$ shows that $\alpha \in \ker (d_r^\omega)$ if and only if $\alpha \in \ker (d_r^\omega)$ and this last fact is equivalent to $\alpha$ being $E_r^*$-closed. $\square$

**Corollary 2.8.** In the setting of Proposition 2.7, the following equivalence holds:

$$\alpha \text{ is } E_r\text{-closed } \iff \star \alpha \text{ is } E_r^*\text{-closed.}$$

**Proof.** We know from (i) of Proposition 2.3 that $\alpha$ is $E_r$-closed if and only if there exist forms $u_l \in C_{p+l,q-l}^\infty(X)$ for $l = 1, \ldots, r - 1$ such that

$$(- \star \partial \star) \star \bar{\alpha} = 0, \quad (- \star \bar{\partial} \star) \star \bar{\alpha} = (- \star \partial \star) \star \bar{u}_1, \ldots, (- \star \bar{\partial} \star) \star \bar{u}_{r-2} = (- \star \partial \star) \star \bar{u}_{r-1}.$$  

Indeed, we have transformed the $E_r$-closedness condition of (i) in Proposition 2.3 by conjugating and applying the Hodge star operator several times. Since $- \star \partial \star = \bar{\partial}^*$ and $- \star \bar{\partial} \star = \partial^*$, the above conditions are equivalent to $\star \bar{\alpha}$ being $E_{r+1}^*$-closed (with the forms $\star \bar{u}_l$ playing the part of the forms $v_l$). $\square$

An immediate (and new to our knowledge) consequence of this discussion is the analogue on every page $E_r$ of the Frölicher spectral sequence of the classical **Serre duality**. The well-definedness was proved in Corollary 2.4. The case $r = 1$ is the Serre duality, while the case $r = 2$ was proved in Theorem 2.1.

**Corollary 2.9.** Let $X$ be a compact complex manifold with $\dim \mathbb{C} X = n$. For every $r \in \mathbb{N}^*$ and every $p, q \in \{0, \ldots, n\}$, the canonical bilinear pairing

$$E_r^{p,q}(X) \times E_r^{-p,n-q}(X) \rightarrow \mathbb{C}, \quad \{\alpha\}_{E_r}, \{\beta\}_{E_r} \mapsto \int_X \alpha \wedge \beta,$$

is non-degenerate.

**Proof.** Let $\{\alpha\}_{E_r} \in E_r^{p,q}(X) \setminus \{0\}$. If we fix an arbitrary Hermitian metric $\omega$ on $X$, we know from Proposition 2.5 that the associated harmonic space $H_r^{p,q}$ is isomorphic to $E_r^{p,q}(X)$ and that the class $\{\alpha\}_{E_r}$ contains a (unique) representative $\alpha$ lying in $H_r^{p,q}$. By Proposition 2.7, this is equivalent to
α being both $E_r$-closed and $E_r^*$-closed, while by Corollary 2.8, this is further equivalent to $\ast \alpha$ being both $E_r^*$-closed and $E_r$-closed, hence to $\ast \alpha$ lying in $H_r^{p-n, n-q}$.

In particular, $\ast \alpha$ represents a non-zero class $\{\ast \alpha\}_{E_r} \in E_r^{n-p, n-q}(X)$. We have

\[
(\{\alpha\}_{E_r}, \{\ast \alpha\}_{E_r}) \mapsto \int_X \alpha \wedge \ast \alpha = ||\alpha||^2 > 0,
\]

where $|| \cdot ||$ stands for the $L^2_\omega$-norm. This shows that for every non-zero class $\{\alpha\}_{E_r} \in E_r^{p, q}(X)$, the map $(\{\alpha\}_{E_r}, \cdot) : E_r^{n-p, n-q}(X) \rightarrow \mathbb{C}$ does not vanish identically, proving the non-degeneracy of the pairing. \qed

**Remark 2.10.** The numerical version of Cor. 2.9 was proved in [Pop17] and the present version and its method of proof were already announced and used in various works by the first and third named authors. See, e.g. [BP18, §3.4]. Subsequently, Cor 2.9 was also proved via a different method in [Ste20], reducing it to the classical Serre duality for the first page of the Frölicher spectral sequence. The method there also gives an alternative way to approach the other duality results in this article. Quite recently, A. Milivojevic [Mil20] has also found a third proof, by yet another method. The harmonic methods used here and in the next section give some finer information besides the duality, which is interesting in its own right and is useful in applications (see e.g. Lemma 6.4).

## 3 Higher-page Bott-Chern and Aeppli cohomologies: definition, Hodge theory and duality

Let $X$ be an $n$-dimensional compact complex manifold. Fix an arbitrary positive integer $r$ and a bidegree $(p, q)$. In §2.2, we defined the notions of $E_r$-closedness and $E_r$-exactness for forms $\alpha \in C^{\infty}_{p, q}(X)$ as higher-page analogues of $\bar{\partial}$-closedness (that can now be called $E_1$-closedness) and $\bar{\partial}$-exactness (that can now be called $E_1$-exactness). We then gave these notions explicit descriptions in Proposition 2.3.

In the same vein, we say that $\alpha$ is $E_r$-*closed* if $\bar{\alpha}$ is $E_r$-closed and we say that $\alpha$ is $E_r$-*exact* if $\bar{\alpha}$ is $E_r$-exact. In particular, characterisations of $E_r$-closedness and $E_r$-exactness are obtained by permuting $\partial$ and $\bar{\partial}$ in the characterisations of $E_r$-closedness and $E_r$-exactness of Proposition 2.3.

Moreover, we can take our cue from Proposition 2.3 to define higher-page analogues of $\partial \bar{\partial}$-closedness and $\partial \bar{\partial}$-exactness in the following way.

**Definition 3.1.** Suppose that $r \geq 2$.

(i) We say that a form $\alpha \in C^{\infty}_{p, q}(X)$ is $E_r \bar{E}_r$-**closed** if there exist smooth forms $\eta_1, \ldots, \eta_{r-1}$ and $\rho_1, \ldots, \rho_{r-1}$ such that the following two towers of $r-1$ equations are satisfied:

\[
\partial \alpha = \bar{\partial} \eta_1, \quad \bar{\partial} \alpha = \partial \rho_1, \\
\bar{\partial} \eta_1 = \partial \eta_2, \quad \partial \rho_1 = \partial \rho_2, \\
\vdots \\
\bar{\partial} \eta_{r-2} = \partial \eta_{r-1}, \quad \partial \rho_{r-2} = \partial \rho_{r-1}.
\]
Lemma 3.2. Fix an arbitrary $\alpha$ in the two towers of $(r - 1)$ equations under (i) by saying that $\partial \alpha$, resp. $\bar{\partial} \alpha$, runs at least $(r - 1)$ times.

(iii) We say that a form $\alpha \in C_{p,q}^\infty(X)$ is $E_\alpha E_r$-exact if there exist smooth forms $\zeta, \xi, \eta$ such that

$$\alpha = \partial \zeta + \bar{\partial} \xi + \partial \eta$$

and such that $\zeta$ and $\eta$ further satisfy the following conditions. There exist smooth forms $v_{r-3}, \ldots, v_0$ and $u_{r-3}, \ldots, u_0$ such that the following two towers of $r - 1$ equations are satisfied:

$$\begin{align*}
\bar{\partial} \zeta &= \partial v_{r-3} \\
\bar{\partial} v_{r-3} &= \bar{\partial} v_{r-4} \\
\vdots \\
\bar{\partial} v_0 &= 0, \\
\partial \eta &= \bar{\partial} u_{r-3} \\
\partial u_{r-3} &= \partial u_{r-4} \\
\vdots \\
\partial u_0 &= 0.
\end{align*}$$

(iv) We refer to the properties of $\zeta$, resp. $\eta$, in the two towers of $(r - 1)$ equations under (iii) by saying that $\bar{\partial} \zeta$, resp. $\partial \eta$, reaches 0 in at most $(r - 1)$ steps.

When $r - 1 = 1$, the properties of $\bar{\partial} \zeta$, resp. $\partial \eta$, reaching 0 in $(r - 1)$ steps translate to $\bar{\partial} \zeta = 0$, resp. $\partial \eta = 0$.

To unify the definitions, we will also say that a form $\alpha \in C_{p,q}^\infty(X)$ is $E_1 E_1$-closed (resp. $E_1 E_1$-exact) if $\alpha$ is $\bar{\partial} \partial$-closed (resp. $\partial \bar{\partial}$-exact).

As with $E_r$ and $\bar{E}_r$, it follows at once from Definition 3.1 that the $E_r \bar{E}_r$-closedness condition becomes stronger and stronger as $r$ increases, while the $E_r \bar{E}_r$-exactness condition becomes weaker and weaker as $r$ increases. In other words, the following inclusions of vector spaces hold:

$$\{\bar{\partial} \partial \text{-exact forms}\} \subset \cdots \subset \{E_r \bar{E}_r\text{-exact forms}\} \subset \{E_{r+1} \bar{E}_{r+1}\text{-exact forms}\} \subset \cdots$$

The following statement collects a few other immediate relations among these notions.

Lemma 3.2. Fix an arbitrary $r \in \mathbb{N}^*$.

(i) A pure-type form $\alpha$ is simultaneously $E_r$-closed and $\bar{E}_r$-closed if and only if $\alpha$ is simultaneously $\partial$-closed and $\bar{\partial}$-closed.

(ii) If $\alpha$ is $E_r \bar{E}_r$-exact, then each of the classes $\{\alpha\} E_r$ and $\{\alpha\} \bar{E}_r$ contains a $\bar{\partial} \partial$-exact form and $\alpha$ is both $E_r$-exact and $\bar{E}_r$-exact.

(iii) Fix any bidegree $(p, q)$ and let $\alpha \in C_{p,q}^\infty(X)$. If $\alpha$ is $E_r \bar{E}_r$-exact for some $r \in \mathbb{N}^*$, then $\alpha$ is $d$-exact.

Proof. (i) is obvious. To see (ii), let $\alpha = \partial \zeta + \bar{\partial} \xi + \partial \eta$ be $E_r \bar{E}_r$-exact, with $\zeta$ and $\eta$ satisfying the conditions under (ii) of Definition 3.1. Then

$$\{\alpha\} E_r = \{\alpha - \partial \zeta - \bar{\partial} \eta\} E_r = \{\partial \bar{\partial} \zeta\} E_r$$

and

$$\{\alpha\} \bar{E}_r = \{\alpha - \partial \zeta - \bar{\partial} \eta\} \bar{E}_r = \{\partial \bar{\partial} \zeta\} \bar{E}_r,$$

while $\alpha = \partial \zeta + \bar{\partial}(-\partial \xi + \eta)$ is $E_r$-exact and $\alpha = \partial(\zeta + \bar{\partial} \xi) + \partial \eta$ is $\bar{E}_r$-exact.
To prove (iii), let $\alpha = \partial \zeta + \bar{\partial} \xi + \bar{\partial} \eta$, where $\zeta$ and $\eta$ satisfy the conditions in the two towers under (ii) of Definition 3.1. Going down the first tower, we get

$$\partial \zeta = d \zeta - \bar{\partial} \zeta = d \zeta - \partial v_{-3} = d(\zeta - v_{-3}) + \partial v_{-4} = \cdots = d(\zeta - v_{-3} + \cdots + (-1)^r v_0).$$

In particular, $\partial \zeta$ is $d$-exact.

Similarly, going down the second tower, we get

$$\bar{\partial} \eta = d(\eta - u_{-3} + \cdots + (-1)^r u_0).$$

In particular, $\bar{\partial} \eta$ is $d$-exact.

Since $\partial \bar{\partial} \xi$ is also $d$-exact, we infer that $\alpha$ is $d$-exact. Explicitly, we have

$$\alpha = d[(\zeta + \eta) + \bar{\partial} \xi - w_{-3} + \cdots + (-1)^r w_0],$$

where $w_j := u_j + v_j$ for all $j$. □

The main takeaway from Lemma 3.2 is that $E, \overline{E}$-exactness implies $E_r$-exactness, $\overline{E}_r$-exactness and $d$-exactness. Let us now pause briefly to notice a property involving the spaces $C^p,q_{E^r}$ of $E^r$-exact $(p, q)$-forms, resp. $C^p,q_{\overline{E}^r}$ of $\overline{E}_r$-exact $(p, q)$-forms.

**Lemma 3.3.** Fix an arbitrary $r \in \mathbb{N}^*$. For any bidegree $(p, q)$, the following identity of vector subspaces of $C^\infty_{p,q}(X)$ holds:

$$C^p,q_{\overline{E}^r} + C^p,q_{E^r} = \text{Im} \partial + \text{Im} \bar{\partial}.$$

**Proof.** For any bidegree $(p, q)$, consider the vector spaces (see (iv) of Definition 3.1 for the terminology):

$$E^p,q_{\partial,r} := \{ \alpha \in C^\infty_{p,q}(X) \mid \partial \alpha \text{ reaches 0 in at most } r \text{ steps} \},$$

$$E^p,q_{\bar{\partial},r} := \{ \beta \in C^\infty_{p,q}(X) \mid \bar{\partial} \beta \text{ reaches 0 in at most } r \text{ steps} \}.$$

From the definitions, we get: $C^p,q_{r} = \partial(E^p,q_{\partial,r-1}) + \text{Im} \bar{\partial}$ and $C^p,q_{\overline{E}^r} = \text{Im} \partial + \bar{\partial}(E^p,q_{\partial,r-1})$. This trivially implies the contention. □

We now come to the main definitions of this subsection.

**Definition 3.4.** Let $X$ be an $n$-dimensional compact complex manifold. Fix $r \in \mathbb{N}^*$ and a bidegree $(p, q)$.

(i) The $E^r$-Bott-Chern cohomology group of bidegree $(p, q)$ of $X$ is defined as the following quotient complex vector space:

$$E^p,q_{r,BC}(X) := \frac{\{ \alpha \in C^\infty_{p,q}(X) \mid d \alpha = 0 \}}{\{ \alpha \in C^\infty_{p,q}(X) \mid \alpha \text{ is } E_r \overline{E}_r \text{-exact} \}}.$$

(ii) The $E^r$-Aeppli cohomology group of bidegree $(p, q)$ of $X$ is defined as the following quotient complex vector space:

$$E^p,q_{r,A}(X) := \frac{\{ \alpha \in C^\infty_{p,q}(X) \mid \alpha \text{ is } E_r \overline{E}_r \text{-closed} \}}{\{ \alpha \in C^\infty_{p,q}(X) \mid \alpha \in \text{Im} \partial + \text{Im} \bar{\partial} \}}.$$
When \( r = 1 \), the above groups coincide with the standard Bott-Chern, respectively Aeppli, cohomology groups (see [BC65] and [Aep62]). Note that, by (i) of Lemma 3.2, the representatives of \( E_r \)-Bott-Chern classes can be alternatively described as the forms that are simultaneously \( E_r \)-closed and \( \bar{E}_r \)-closed, while by Lemma 3.3, the \( E_r \)-Aeppli-exact forms can be alternatively described as those forms lying in \( \mathcal{O}^{p,q} + \bar{\mathcal{O}}^{p,q} \).

Also note that the inclusions of vector spaces spelt out just before Lemma 3.2 and their analogues for the \( E_r \)- and \( \bar{E}_r \)-cohomologies imply the following inequalities of dimensions:

\[
\cdots \leq \dim E^{p,q}_{r,BC}(X) \leq \dim E^{p,q}_{r-1,BC}(X) \leq \cdots \leq \dim E^{p,q}_{1,BC}(X) = \dim H^{p,q}_{BC}(X)
\]

and their analogues for the \( E_r \)-Aeppli cohomology spaces.

The first step towards extending to the higher pages of the Frölicher spectral sequence the standard Serre-type duality between the classical Bott-Chern and Aeppli cohomology groups of complementary bidegrees is the following

**Proposition 3.5.** Let \( X \) be a compact complex manifold with \( \dim_C X = n \). For every \( r \in \mathbb{N}^* \) and all \( p, q \in \{0, \ldots, n\} \), the following bilinear pairing is well defined:

\[
E^{p,q}_{r,BC}(X) \times E^{n-p,n-q}_{r,A}(X) \longrightarrow \mathbb{C}, \quad (\{\alpha\}_{E_{r,BC}}, \{\beta\}_{E_{r,A}}) \mapsto \int_X \alpha \wedge \beta,
\]

in the sense that it is independent of the choice of representative of either of the classes \( \{\alpha\}_{E_{r,BC}} \) and \( \{\beta\}_{E_{r,A}} \).

**Proof.** The proof consists in a series of integrations by parts (mathematical ping-pong).

- To prove independence of the choice of representative of the \( E_r \)-Bott-Chern class, let us modify a representative \( \alpha \) to some representative \( \alpha + \partial \zeta + \bar{\partial} \xi + \bar{\partial} \eta \) of the same \( E_r \)-Bott-Chern class. This means that \( \partial \zeta + \bar{\partial} \xi + \bar{\partial} \eta \) is \( E_r \bar{E}_r \)-exact, so \( \zeta \) and \( \eta \) satisfy the towers of \( r-1 \) equations under (ii) of Definition 3.1. We have

\[
\int_X (\alpha + \partial \zeta + \bar{\partial} \xi + \bar{\partial} \eta) \wedge \beta = \int_X \alpha \wedge \beta \pm \int_X \zeta \wedge \partial \beta \pm \int_X \xi \wedge \bar{\partial} \beta \pm \int_X \eta \wedge \bar{\partial} \beta.
\]

Since \( \beta \) is \( E_r \bar{E}_r \)-closed, it is also \( \partial \bar{\partial} \)-closed (see (i) of Lemma 3.2), so the last but one integral above vanishes.

Using the \( r-1 \) equations in the first tower under (i) of Definition 3.1 (with \( \beta \) in place of \( \alpha \)) and the first tower under (ii) of the same definition, we get:

\[
\int_X \zeta \wedge \partial \beta = \int_X \zeta \wedge \bar{\partial} \eta_1 = \pm \int_X \bar{\partial} \xi \wedge \eta_1 = \pm \int_X \partial v_{r-3} \wedge \eta_1 = \pm \int_X v_{r-3} \wedge \partial \eta_1 = \pm \int_X v_{r-3} \wedge \bar{\partial} \eta_2 = \pm \int_X \bar{\partial} v_{r-3} \wedge \eta_2 = \pm \int_X \partial v_{r-4} \wedge \eta_2 = \pm \int_X v_{r-4} \wedge \partial \eta_2
\]

\[
= \pm \int_X v_0 \wedge \bar{\partial} \eta_{r-1} = \pm \int_X \bar{\partial} v_0 \wedge \eta_{r-1} = 0,
\]

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where the last identity follows from $\partial v_0 = 0$.

Playing the analogous mathematical ping-pong while using the second tower under both (i) and (ii) of Definition 3.1, we get:

$$
\int_X \eta \wedge \bar{\partial} \beta = \int_X \eta \wedge \partial \rho_1 = \pm \int_X \partial \eta \wedge \rho_1 = \pm \int_X u_{r-3} \wedge \rho_1 = \pm \int_X u_{r-3} \wedge \bar{\partial} \rho_1
$$

$$
\vdots
$$

$$
= \pm \int_X u_0 \wedge \partial \rho_{r-1} = \pm \int_X \partial u_0 \wedge \rho_{r-1} = 0,
$$

where the last identity follows from $\partial u_0 = 0$.

We conclude that $\int_X (\alpha + \partial \zeta + \bar{\partial} \xi + \bar{\partial} \eta) \wedge \beta = \int_X \alpha \wedge \beta$.

To prove independence of the choice of representative of the $E_r$-Aeppli class, let us modify a representative $\beta$ to some representative $\beta + \partial \zeta + \bar{\partial} \xi$ of the same $E_r$-Aeppli class. So, $\zeta$ and $\xi$ are arbitrary forms. We get:

$$
\int_X \alpha \wedge (\beta + \partial \zeta + \bar{\partial} \xi) = \int_X \alpha \wedge \beta \pm \int_X \partial \alpha \wedge \zeta \pm \int_X \bar{\partial} \alpha \wedge \xi = 0,
$$

where the last identity follows from $\partial \alpha = 0$ and $\bar{\partial} \alpha = 0$. $\square$

We now take up the issue of the non-degeneracy of the above bilinear pairing. For the sake of expediency, we start by defining the dual notion to the $E_r \bar{E}_r$-closedness of Definition 3.1 after we have fixed a metric.

**Definition 3.6.** Let $(X, \omega)$ be a compact complex Hermitian manifold. Fix an integer $r \geq 2$ and a bidegree $(p, q)$.

We say that a form $\alpha \in C^\infty_{p, q}(X)$ is $E_r \bar{E}_r$-closed with respect to the Hermitian metric $\omega$ if there exist smooth forms $a_1, \ldots, a_{r-1}$ and $b_1, \ldots, b_{r-1}$ such that the following two towers of $r-1$ equations are satisfied:

$$
\partial^* \alpha = \bar{\partial}^* a_1
$$

$$
\bar{\partial}^* a_1 = \partial^* a_2
$$

$$
\vdots
$$

$$
\partial^* a_{r-2} = \bar{\partial}^* a_{r-1},
$$

$$
\bar{\partial}^* b_{r-2} = \partial^* b_{r-1}.
$$

That this notion is indeed dual to the $E_r \bar{E}_r$-closedness via the Hodge star operator $\star = \star_\omega$ associated with the metric $\omega$ is the content of the following analogue of Corollary 2.8.

**Lemma 3.7.** In the setting of Definition 3.6, the following equivalence holds for every form $\alpha \in C^\infty_{p, q}(X)$:

$$
\alpha \text{ is } E_r \bar{E}_r\text{-closed } \iff \star \alpha \text{ is } E_r^* \bar{E}_r^*\text{-closed}.
$$

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Proof. Thanks to conjugations, to the fact that \( \ast \ast = \pm \text{Id} \) (with the sign depending on the parity of the total degree of the forms involved) and to \( \ast \) being an isomorphism, the two towers of \( r - 1 \) equations that express the \( E_r \overline{E}_r \)-closedness of \( \alpha \) (cf. (i) of Definition 3.1) translate to

\[
\begin{align*}
(- \ast \overline{\partial} \ast)(\ast \overline{\alpha}) &= (- \ast \overline{\partial} \ast)(\ast \overline{\eta}_1), \\
(- \ast \overline{\partial} \ast)(\ast \overline{\eta}_1) &= (- \ast \overline{\partial} \ast)(\ast \overline{\eta}_2), \\
&\vdots \\
(- \ast \overline{\partial} \ast)(\ast \overline{\eta}_{r-2}) &= (- \ast \overline{\partial} \ast)(\ast \overline{\eta}_{r-1}),
\end{align*}
\]

Now, put \( a_j := \ast \overline{\eta}_j \) and \( b_j := \ast \overline{\rho}_j \) for all \( j \in \{1, \ldots, r - 1\} \). Since \( - \ast \overline{\partial} \ast = \overline{\partial} \ast \) and \( - \ast \overline{\partial} \ast = \overline{\partial} \ast \), these two towers amount to \( \ast \overline{\alpha} \) being \( E_r \overline{E}_r \)-closed. (See Definition 3.6).

We now come to two crucial lemmas from which Hodge isomorphisms for the \( E_r \)-Bott-Chern and the \( E_r \)-Aeppli cohomologies will follow. Based on the terminology introduced in (ii) of Definition 3.1, we define the vector spaces:

\[
\begin{align*}
\mathcal{F}^{p, q}_{\overline{\partial}, r} &:= \{ \alpha \in C_{p, q}^\infty(X) \mid \partial \alpha \text{ runs at least } r \text{ times} \}, \\
\mathcal{F}^{p, q}_{\overline{\partial}, r} &:= \{ \beta \in C_{p, q}^\infty(X) \mid \overline{\partial} \beta \text{ runs at least } r \text{ times} \}
\end{align*}
\]

and their analogues \( \mathcal{F}^{p, q}_{\overline{\partial}^*, r} \) and \( \mathcal{F}^{p, q}_{\overline{\partial}^*, r} \) when \( \partial \) is replaced by \( \overline{\partial}^* \) and \( \overline{\partial} \) is replaced by \( \overline{\partial}^* \). Note that the space of \( E_r^* \overline{E}_r^* \)-closed \((p, q)\)-forms defined in Definition 3.6 is precisely the intersection \( \mathcal{F}^{p, q}_{\overline{\partial}^*, r-1} \cap \mathcal{F}^{p, q}_{\overline{\partial}, r-1} \).

**Lemma 3.8.** Let \((X, \omega)\) be a compact complex Hermitian manifold. Fix an integer \( r \geq 2 \), a bidegree \((p, q)\) and a form \( \alpha \in C_{p, q}^\infty(X) \).

The following two statements are equivalent.

(i) \( \alpha \) is \( E_r^* \overline{E}_r^* \)-closed (w.r.t. \( \omega \));

(ii) \( \alpha \) is \( L^2_\omega \)-orthogonal to the space of smooth \( E_r \overline{E}_r \)-exact \((p, q)\)-forms.

**Proof.** “(i) \( \implies \) (ii)” Suppose that \( \alpha \) is \( E_r^* \overline{E}_r^* \)-closed. This means that \( \alpha \) satisfies the two towers of \((r - 1)\) equations in Definition 3.6. Let \( \beta = \overline{\partial} \zeta + \overline{\partial} \partial \xi + \overline{\partial} \eta \) be an arbitrary \( E_r \overline{E}_r \)-exact \((p, q)\)-form. So, \( \zeta \) and \( \eta \) satisfy the respective towers of \((r - 1)\) equations under (ii) of Definition 3.1. For the \( L^2_\omega \)-inner product of \( \alpha \) and \( \beta \), we get:

\[
\langle \langle \alpha, \beta \rangle \rangle = \langle \langle \overline{\partial}^* \alpha, \zeta \rangle \rangle + \langle \langle \overline{\partial}^* \partial \alpha, \xi \rangle \rangle + \langle \langle \overline{\partial} \alpha, \eta \rangle \rangle.
\]

Since \( \overline{\partial}^* \partial \alpha = \overline{\partial}^* \overline{\partial}^* \alpha_1 = 0 \), the middle term on the r.h.s. of (4) vanishes.

For the first term on the r.h.s. of (4), we use the towers of equations satisfied by \( \alpha \) and \( \zeta \) to get:

\[
\begin{align*}
\langle \langle \partial^* \alpha, \zeta \rangle \rangle &= \langle \langle \overline{\partial} \alpha_1, \zeta \rangle \rangle = \langle \langle \alpha_1, \zeta \rangle \rangle = \langle \langle \alpha_1, \partial \nu_{r-3} \rangle \rangle = \langle \langle \partial^* \alpha_1, \nu_{r-3} \rangle \rangle = \langle \langle \partial^* \alpha_2, \nu_{r-3} \rangle \rangle \\
&= \langle \langle \alpha_2, \partial \nu_{r-3} \rangle \rangle = \langle \langle \alpha_2, \partial v_{r-4} \rangle \rangle \\
&\vdots \\\n&= \langle \langle \alpha_{r-1}, \partial v_0 \rangle \rangle = 0,
\end{align*}
\]
where the last identity followed from the property \( \partial v_0 = 0 \).

For the last term on the r.h.s. of (4), we use the towers of equations satisfied by \( \alpha \) and \( \eta \) to get:

\[
\langle \langle \partial^* \alpha, \eta \rangle \rangle = \langle \langle \partial^* b_1, \eta \rangle \rangle = \langle \langle b_1, \partial \eta \rangle \rangle = \langle \langle b_1, \partial u_{r-3} \rangle \rangle = \langle \langle \partial^* b_1, u_{r-3} \rangle \rangle = \langle \langle \partial^* b_2, u_{r-3} \rangle \rangle
\]

\[
= \langle \langle b_2, \partial u_{r-3} \rangle \rangle = \langle \langle b_2, \partial u_{r-4} \rangle \rangle
\]

\[
\vdots
\]

\[
= \langle \langle b_{r-1}, \partial u_0 \rangle \rangle = 0,
\]

where the last identity followed from the property \( \partial u_0 = 0 \).

“(ii) \implies (i)” Suppose that \( \alpha \) is orthogonal to all the smooth \( E_r \overline{E}_r \)-exact \((p, q)\)-forms \( \beta \). These forms are of the shape \( \beta = \partial \xi + \partial \bar{\xi} + \partial \eta \), where \( \xi \) is subject to no condition, while \( \zeta \in E^{p-1,q}_{\bar{\partial},r-1} \) and \( \eta \in E^{p,q-1}_{\bar{\partial},r-1} \). (See notation introduced in the proof of Lemma 3.3).

The orthogonality condition is equivalent to the following three identities:

\[
(a) \langle \langle \partial^* \partial^* \alpha, \xi \rangle \rangle = 0, \quad (b) \langle \langle \partial^* \alpha, \zeta \rangle \rangle = 0, \quad (c) \langle \langle \partial^* \alpha, \eta \rangle \rangle = 0
\]

holding for all the forms \( \zeta, \xi, \eta \) satisfying the above conditions.

Since \( \xi \) is subject to no condition, (a) amounts to \( \partial^* \partial^* \alpha = 0 \). This means that \( \partial^* \alpha \in \ker \partial^* \) and \( \partial^* \partial^* \alpha \). Condition (b) requires \( \partial^* \alpha \perp E^{p-1,q}_{\bar{\partial},r-1} \), while (c) requires \( \partial^* \alpha \perp E^{p,q-1}_{\bar{\partial},r-1} \).

- Unravelling condition (b). The forms \( \zeta \in E^{p-1,q}_{\bar{\partial},r-1} \) are characterised by the existence of forms \( v_{r-3}, \ldots, v_0 \) satisfying the first tower of \((r-1)\) equations in (iii) of Definition 3.1. That tower imposes the condition \( v_{r-j} \in E^{p-1,q}_{\bar{\partial},r-j+1} \cap F_{\bar{\partial},j-2} \) for every \( j \in \{3, \ldots, j\} \). (We have dropped the superscripts to lighten the notation.)

Now, every form \( \zeta \in \ker \Delta'' \) satisfies the condition \( \partial^* \partial \zeta = 0 \), hence \( \zeta \in E^{p,q}_{\bar{\partial},1} \subset E^{p,q}_{\bar{\partial},r-1} \). From condition (b), we get \( \partial^* \alpha \perp \ker \Delta'' \). Since \( \ker \partial^* \) (to which \( \partial^* \alpha \) belongs by condition (a)) is the orthogonal direct sum between \( \ker \Delta'' \) and \( \im \partial^* \), we get \( \partial^* \alpha \in \im \partial^* \), so

\[
\partial^* \alpha = \partial^* a_1
\]

for some form \( a_1 \). Condition (b) becomes:

\[
0 = \langle \langle \partial^* \alpha, \zeta \rangle \rangle = \langle \langle a_1, \partial \zeta \rangle \rangle = \langle \langle a_1, \partial v_{r-3} \rangle \rangle = \langle \langle \partial^* a_1, v_{r-3} \rangle \rangle \quad \text{for all} \quad v_{r-3} \in E^{p,q}_{\bar{\partial},r-2} \cap F_{\bar{\partial},1}.
\]

In other words, \( \partial^* a_1 \perp (E^{p,q}_{\bar{\partial},r-2} \cap F_{\bar{\partial},1}) \).

We will now use the 3-space decomposition (13) of \( C_{p,q}^\infty(X) \) for the case \( r = 2 \). (See Proposition 7.2 in Appendix one.) It is immediate to check the inclusion \( E^{p,q}_{\bar{\partial},r-2} \cap F_{\bar{\partial},1} \supset H_2 \oplus (\im \partial^* + \im \partial (E^{p,q}_{\bar{\partial},1})) \). Therefore, condition (b) implies that \( \partial^* a_1 \perp (H_2 \oplus (\im \partial^* + \im \partial (E^{p,q}_{\bar{\partial},1}))) \).

Since the orthogonal complement of \( H_2 \oplus (\im \partial + \im \partial (E^{p,q}_{\bar{\partial},1})) \) is \( \partial^* (E^{p,q}_{\bar{\partial},1}) + \im \partial^* \) by the 3-space decomposition (13) for \( r = 2 \), we infer that \( \partial^* a_1 \in \partial^* (E^{p,q}_{\bar{\partial},1}) + \im \partial^* \). Therefore, there exist forms \( b_1 \in \ker \partial^* \) and \( a_2 \) such that

\[
\partial^* a_1 = \partial^* b_1 + \partial^* a_2.
\]

Since \( \partial^* b_1 = 0 \), equations (5) and (6) yield:

\[
\partial^* \alpha = \partial^* (a_1 - b_1)
\]

\[
\partial^* (a_1 - b_1) = \partial^* a_2.
\]
Thus, condition (b) becomes:

\[ 0 = \langle \partial^* \alpha, \zeta \rangle = \langle \bar{\partial}^* (a_1 - b_1), \zeta \rangle = \langle (a_1 - b_1, \partial v_{r-3}) \rangle = \langle \partial^* (a_1 - b_1), v_{r-3} \rangle \]
\[ = \langle \bar{\partial}^* a_2, v_{r-3} \rangle = \langle (a_2, \partial v_{r-4}) \rangle = \langle \partial^* a_2, v_{r-4} \rangle \quad \text{for all} \quad v_{r-4} \in E_{\partial, r-3} \cap F_{\partial, 2}. \]

In other words, \( \partial^* a_2 \perp (E_{\partial, r-3} \cap F_{\partial, 2}) \).

Now, it is immediate to check the inclusion \( E_{\partial, r-3} \cap F_{\partial, 2} \supset H_3 \oplus (\text{Im} \bar{\partial} + \partial (E_{\bar{\partial}, 2})) \). Since the orthogonal complement of \( H_3 \oplus (\text{Im} \bar{\partial} + \partial (E_{\bar{\partial}, 2})) \) is \( \partial^* (E_{\bar{\partial}, r-2}) + \text{Im} \partial^* \) by the 3-space decomposition (13) for \( r = 3 \), we infer that \( \partial^* a_2 \in \partial^* (E_{\bar{\partial}, r-1}) + \text{Im} \partial^* \). Therefore, there exist forms \( b_2 \in E_{\bar{\partial}, r-2} \) and \( a_3 \) such that

\[ \partial^* a_2 = \partial^* b_2 + \partial^* a_3. \tag{8} \]

Since the condition \( b_2 \in E_{\bar{\partial}, r-2} \) translates to the equations

\[ \partial^* b_2 = \partial^* c_1 \quad \text{and} \quad \partial^* c_1 = 0, \tag{9} \]

for some form \( c_1 \), equations (7) and (8) yield:

\[
\begin{align*}
\partial^* \alpha &= \bar{\partial}^* (a_1 - b_1 - c_1) \\
\partial^* (a_1 - b_1 - c_1) &= \bar{\partial}^* (a_2 - b_2) \\
\partial^* (a_2 - b_2) &= \bar{\partial}^* a_3.
\end{align*}
\]

Continuing in this way, we inductively get the following tower of \( (r - 1) \) equations:

\[
\begin{align*}
\partial^* \alpha &= \bar{\partial}^* (a_1 - b_1 - c_1) - c_1^{(3)} - \cdots - c_r^{(r-2)} \\
\partial^* (a_1 - b_1 - c_1 - c_1^{(3)} - \cdots - c_r^{(r-2)}) &= \bar{\partial}^* (a_2 - b_2 - c_2^{(3)} - \cdots - c_r^{(r-2)}) \\
& \quad \vdots \\
\partial^* (a_{r-2} - b_{r-2}) &= \bar{\partial}^* a_{r-1},
\end{align*}
\]

where \( b_j \in E_{\bar{\partial}, j} \) for all \( j \in \{1, \ldots, r - 2\} \), so \( b_j \) satisfies the following tower of \( j \) equations:

\[
\begin{align*}
\bar{\partial}^* b_j &= \partial^* c_1^{(j)} \\
\bar{\partial}^* c_1^{(j)} &= \partial^* c_2^{(j)} \\
& \quad \vdots \\
\bar{\partial}^* c_2^{(j)} &= \partial^* c_1^{(j)} \\
\bar{\partial}^* c_1^{(j)} &= 0,
\end{align*}
\]

for some forms \( c_i^{(j)} \).

Consequently, conditions (a) and (b) to which \( \alpha \) is subject imply that \( \alpha \in F_{\partial^*, r-1} \) (cf. tower (10)), which is the first of the two conditions required for \( \alpha \) to be \( E^r_{\partial^* E} \)-closed under Definition 3.6.

- **Unravelling condition (c).** Proceeding in a similar fashion, with \( \partial^* \) and \( \bar{\partial}^* \) permuted, we infer that conditions (a) and (c) to which \( \alpha \) is subject imply that \( \alpha \in F_{\bar{\partial}^*, r-1} \), which is the second of the two conditions required for \( \alpha \) to be \( E^r_{\bar{\partial}^* E} \)-closed under Definition 3.6.

- We conclude that \( \alpha \) is indeed \( E^r_{\partial^* E} \)-closed. \( \square \)

The immediate consequence of Lemma 3.8 is the following **Hodge isomorphism** for the \( E_r \)-Bott-Chern cohomology.
Corollary and Definition 3.9. Let \((X, \omega)\) be a compact complex Hermitian manifold. For every bidegree \((p, q)\) and every \(r \in \mathbb{N}^*\), every \(E_r\)-Bott-Chern cohomology class \(\{\alpha\}_{E_r, BC} \in E^{p,q}_{r, BC}(X)\) can be represented by a unique form \(\alpha \in C^\infty_{p,q}(X)\) satisfying the following three conditions:

\[\alpha\] is \(\partial\)-closed, \(\bar{\partial}\)-closed and \(E^r\)-closed.

Any such form \(\alpha\) is called \(E_r\)-Bott-Chern harmonic with respect to the metric \(\omega\).

There is a vector-space isomorphism depending on the metric \(\omega\):

\[E^{p,q}_{r, BC}(X) \simeq \mathcal{H}_r^{p,q}(X),\]

where \(\mathcal{H}_r^{p,q}(X) \subset C^\infty_{p,q}(X)\) is the space of \(E_r\)-Bott-Chern harmonic \((p, q)\)-forms associated with \(\omega\).

Of course, the above isomorphism maps every class \(\{\alpha\}_{E_r, BC} \in E^{p,q}_{r, BC}(X)\) to its unique \(E_r\)-Bott-Chern harmonic representative.

The analogous statement for the \(E_r\)-Aeppli cohomology follows at once from standard material. Indeed, it is classical that the \(L^2_\omega\)-orthogonal complement of \(\text{Im} \partial\) (resp. \(\text{Im} \bar{\partial}\)) in \(C^\infty_{p,q}(X)\) is \(\ker \partial^*\) (resp. \(\ker \bar{\partial}^*\)). The immediate consequence of this is the following Hodge isomorphism for the \(E_r\)-Aeppli cohomology.

Corollary and Definition 3.10. Let \((X, \omega)\) be a compact complex Hermitian manifold. For every bidegree \((p, q)\), every \(E_r\)-Aeppli cohomology class \(\{\alpha\}_{E_r, A} \in E^{p,q}_{r, A}(X)\) can be represented by a unique form \(\alpha \in C^\infty_{p,q}(X)\) satisfying the following three conditions:

\[\alpha\] is \(E_r\)-\(E^r\)-closed, \(\partial^*\)-closed and \(\bar{\partial}^*\)-closed.

Any such form \(\alpha\) is called \(E_r\)-Aeppli harmonic with respect to the metric \(\omega\).

There is a vector-space isomorphism depending on the metric \(\omega\):

\[E^{p,q}_{r, A}(X) \simeq \mathcal{H}_r^{p,q}(X),\]

where \(\mathcal{H}_r^{p,q}(X) \subset C^\infty_{p,q}(X)\) is the space of \(E_r\)-Aeppli harmonic \((p, q)\)-forms associated with \(\omega\).

Of course, the above isomorphism maps every class \(\{\alpha\}_{E_r, A} \in E^{p,q}_{r, A}(X)\) to its unique \(E_r\)-Aeppli harmonic representative.

We can now conclude from the above results that there is a Serre-type canonical duality between the \(E_r\)-Bott-Chern cohomology and the \(E_r\)-Aeppli cohomology of complementary bidegrees.

Theorem 3.11. Let \(X\) be a compact complex manifold with \(\dim_{\mathbb{C}} X = n\). For all \(p, q \in \{0, \ldots, n\}\), the following bilinear pairing is well defined and non-degenerate:

\[E^{p,q}_{r, BC}(X) \times E^{n-p,n-q}_{r, A}(X) \longrightarrow \mathbb{C}, \quad \left(\{\alpha\}_{E_r, BC}, \{\beta\}_{E_r, A}\right) \mapsto \int_X \alpha \wedge \beta.\]
Proof. The well-definedness was proved in Proposition 3.5. The non-degeneracy is proved in the usual way on the back of the above preliminary results, as follows.

Let \( \{ \alpha \}_{E_r, BC} \in E^p,q_{r, BC}(X) \) be an arbitrary non-zero class. Fix an arbitrary Hermitian metric \( \omega \) on \( X \) and let \( \alpha \) be the unique \( E_r \)-Bott-Chern harmonic representative (w.r.t. \( \omega \)) of the class \( \{ \alpha \}_{E_r, BC} \) (whose existence and uniqueness are guaranteed by Corollary and Definition 3.9). In particular, \( \alpha \neq 0 \).

Based on the characterisations of the \( E_r \)-Bott-Chern and \( E_r \)-Aeppli harmonics given in Corollaries and Definitions 3.9 and 3.10, Lemma 3.7 and the standard equivalences \( (\alpha \in \ker \partial \iff \ast \alpha \in \ker \partial^*) \) and \( (\alpha \in \ker \bar{\partial} \iff \ast \bar{\alpha} \in \ker \bar{\partial}^*) \) ensure that \( \ast \alpha \) is \( E_r \)-Aeppli harmonic. In particular, \( \ast \alpha \) represents an \( E_r \)-Aeppli class \( \{ \ast \alpha \}_{E_r, A} \in E^{n-p, n-q}_{r, A}(X) \). Moreover, pairing \( \{ \alpha \}_{E_r, BC} \) with \( \{ \ast \bar{\alpha} \}_{E_r, A} \) yields \( \int_X \alpha \wedge \ast \bar{\alpha} = ||\alpha||^2 \neq 0 \), where \( || \cdot || \) stands for the \( L^2 \)-norm.

Similarly, starting off with a non-zero class \( \{ \beta \}_{E_r, A} \in E^{p, n-p-n-q}_{r, A}(X) \) and selecting its unique \( E_r \)-Aeppli harmonic representative \( \beta \), we get that \( \beta \neq 0 \), \( \ast \beta \) is \( E_r \)-Bott-Chern harmonic (hence it represents a class in \( E_{r, BC}^p,q(X) \)) and the classes \( \{ \ast \beta \}_{E_r, BC} \) and \( \{ \beta \}_{E_r, A} \) pair to \( \pm ||\beta||^2 \neq 0 \). \( \Box \)

4 Characterisations of page-\( r \)-\( \partial \bar{\partial} \)-manifolds

In this section, we apply the higher-page Bott-Chern and Aeppli cohomologies to give various characterisations of the page-\( r \)-\( \partial \bar{\partial} \)-manifolds introduced in [PSU20]. We will write \( E_{r, BC}(X) := \bigoplus_{p,q} E_{r, BC}^p,q(X) \), \( E_{r, A}^k(X) := \bigoplus_{p+q=k} E_{r, BC}^p,q(X) \), \( e_{r, BC}^k(X) := \dim E_{r, BC}^k(X) \) and similarly for \( E_{r, A} \).

Theorem 4.1. For a compact complex manifold \( X \), the following properties are equivalent:

(A) \( X \) is a page-(\( r - 1 \))-\( \partial \bar{\partial} \)-manifold.

(B) The map \( E_{r, BC}(X) \to E_{r, A}(X) \) induced by the identity is an isomorphism.

(C) One has \( e_{r, BC}^k(X) = e_{r, A}^k(X) \) for all \( k \).

(D) The map \( E_{r, BC}(X) \to E_{r, A}(X) \) induced by the identity is injective.

(E) For any d-closed \( (p, q) \)-form \( \alpha \), the following properties are equivalent:

\[ \alpha \text{ is } d\text{-exact} \iff \alpha \text{ is } E_r\text{-exact} \iff \alpha \text{ is } \bar{E}_r\text{-exact} \iff \alpha \text{ is } E_r \bar{E}_r\text{-exact.} \]

Let us first sketch how one might approach this theorem in an elementary way. Fix \( r \in \mathbb{N}^* \) and a bidegree \( (p, q) \). As in section 2.2, let \( Z_{r, q}^p \) and \( C_{r, q}^p \) stand for the space of \( E_r \)-closed, resp. \( E_r \)-exact, smooth \( (p, q) \)-forms on \( X \). Let \( D_{r, q}^p \) stand for the space of \( E_r \bar{E}_r \)-exact smooth \( (p, q) \)-forms on \( X \).

Lemma 4.2. (i) The following inclusions of vector subspaces of \( C_{p+1, q}^\infty(X) \) hold:

\[
\text{Im} (\partial \bar{\partial})^{p, q} \subset \partial (Z_{r, q}^p) \subset \bigcap_{p+q \leq r} D_{r, q}^{p+1, q} \subset C_{r, q}^{p+1, q} \cap \ker d
\]

(ii) Every \( E_r \)-class \( \{ \alpha \}_{E_r} \in E_r^{p, q}(X) \) can be represented by a d-closed form if and only if \( \partial (Z_{r, q}^p) \subset \text{Im} (\partial \bar{\partial}) \). In other words, this happens if and only if the first inclusion in (11) is an equality.
Proof. (i) To prove the first inclusion, it suffices to show that every \( \partial \)-exact \((p, q)\)-form is \( E_r \)-closed. Let \( \alpha = \bar{\partial} \beta \) be a \((p, q)\)-form. Then, \( \partial \alpha = 0 \) and \( \partial \alpha = \bar{\partial}(-\partial \beta) \). Putting \( u_1 := -\partial \beta \), we have \( \partial u_1 = 0 \), so we can choose \( u_2 = 0, \ldots, u_{r-1} = 0 \) to satisfy the tower of equations under (i) of Proposition 2.3. This shows that \( \alpha \) is \( E_r \)-closed.

To prove the second inclusion, let \( \alpha \in Z^{p, q}_r \). By (i) of Proposition 2.3, this implies that \( \bar{\partial} \alpha = 0 \), so if we write \( \partial \alpha = \partial \zeta + \bar{\partial} \xi + \partial \eta \) with \( \zeta = \alpha, \xi = 0 \) and \( \eta = 0 \), we satisfy the conditions under (ii) of Definition 3.1 with \( v_j = 0 \) and \( u_j = 0 \) for all \( j \in \{0, \ldots, r - 3\} \). This proves that \( \partial \alpha \) is \( E_r \overline{E}_r \)-exact.

The third inclusion on the first row is a consequence of (iii) and (iv) of Lemma 3.2, while the “vertical” inclusion is a translation of (iv) of the same lemma.

(ii) Let \( \{\alpha\} \in E^{p, q}_r(X) \) be an arbitrary class and let \( \alpha \) be an arbitrary representative of it. Then, \( \{\alpha\} \in E^{p, q}_r(X) \) can be represented by a \( d \)-closed form if and only if there exists an \( E_r \)-exact form \( \rho = \partial a + \bar{\partial} b \), with \( a \) satisfying the conditions \( \partial a = \partial c_{r-3}, \bar{\partial} c_{r-3} = \partial c_{r-4}, \ldots \partial c_0 = 0 \) for some forms \( c_j \), such that \( \partial(\alpha - \rho) = 0 \). This last identity is equivalent to

\[
\bar{\partial} \partial b = \partial \alpha.
\]

Thus, the class \( \{\alpha\} \in E_r \) contains a \( d \)-closed form if and only if the form \( \partial \alpha \), which already lies in \( \partial(Z^{p, q}_r) \), is \( \bar{\partial} \partial \)-exact. This proves the contention. \( \square \)

**Theorem 4.3.** Let \( X \) be a compact complex manifold with \( \dim_c X = n \). Fix an arbitrary integer \( r \geq 2 \). The following properties are equivalent.

(A) \( X \) is a page-(\( r - 1 \))-\( \partial \bar{\partial} \)-manifold.

(F) For all \( p, q \in \{0, \ldots, n\} \), the following identities of vector subspaces of \( C^{\infty}_{p+1, q}(X) \) hold:

(i) \( \text{Im}(\partial \bar{\partial}) = \partial(Z^{p, q}_r) \) and (ii) \( \mathcal{C}^{p, q}_r \cap \ker d = \text{Im} d \).

**Proof.** By (ii) of Lemma 4.2, identity (i) in (F) is equivalent to every \( E_r \)-class of type \((p, q)\) being representable by a \( d \)-closed form. On the other hand, if this is the case, then identity (ii) in (F) is equivalent to the map \( E^{p, q}_r(X) \ni \{\alpha\} \mapsto \{\alpha\} \in H^{p+q}_{\overline{DR}}(X, \mathbb{C}) \) (with \( \alpha \in \ker d \)) being well defined and injective, which means \( X \) is page-(\( r - 1 \))-\( \partial \bar{\partial} \). \( \square \)

Note that identity (ii) in (F) of Theorem 4.3 is a reformulation of the first equivalence in (E) of Theorem 4.1. However, it appears to be nontrivial to obtain the whole statement of Theorem 4.1 in an elementary way, so we will now make use of some algebro-cohomological machinery. As a preparation, let us compute the higher-page Bott-Chern and Aeppli cohomologies for all indecomposable double complexes. Recall that a double complex is a bigraded vector space \( A \) with maps \( \partial_1, \partial_2 \) of bidegree \((1, 0)\) and \((0, 1)\) s.t. \( d := \partial_1 + \partial_2 \) satisfies \( d^2 = 0 \). It is bounded if \( A^{p, q} = 0 \) for all but finitely many \((p, q) \in \mathbb{Z}^2\) and indecomposable if it cannot be written as a nontrivial direct sum \( A = B \oplus C \). There are two kinds of indecomposable double complexes (see [KQ20], [Ste20]).

**Squares**
These are double complexes generated by an element \( a \) of pure bidegree \((p,q)\) s.t. \( \partial_1 \partial_2 a \neq 0 \):

\[
\langle \partial_2 a \rangle \longrightarrow \langle \partial_2 \partial_1 a \rangle \\
\uparrow \uparrow \\
\langle a \rangle \longrightarrow \langle \partial_1 a \rangle.
\]

**Zigzags**

A zigzag is a double complex generated by pure-type elements \( a_1, \ldots, a_l \) some fixed total degree \( k = p + q \) and satisfying \( \partial_1 \partial_2 a_i = 0 \) for all \( i \) and \( \partial_1 a_i = \partial_2 a_{i+1} \neq 0 \) for all \( i \) s.t. \( a_i \neq 0 \neq a_{i+1} \). In particular, \( \partial_2 a_1 \) and \( \partial_1 a_l \) may or may not be zero.

\[
\langle \partial_2 a_1 \rangle \\
\uparrow \\
\langle a_1 \rangle \longrightarrow \langle \partial_1 a_1 \rangle \\
\uparrow \\
\langle a_2 \rangle \longrightarrow \ldots \\
\uparrow \\
\langle a_l \rangle \longrightarrow \langle \partial_1 a_l \rangle.
\]

The length of a zigzag is given by its dimension as a vector space. It has to be \( 2l - 1 \), \( 2l \) or \( 2l + 1 \), depending on whether the outermost differentials vanish or not. A zigzag of length one is called a dot.

In order to compute \( E_{r,BC} \) and \( E_{r,A} \) on those double complexes, first recall that \( E_{r,BC} \) is a quotient of \( H_{BC} \) and \( E_{r,A} \) is a subspace of \( H_A \). In particular, if \( H_{BC} \) or \( H_A \) are zero on some double complex, so are their lower dimensional counterparts. Further note that \( \partial_1 \partial_2 = 0 \) on zigzags and we even have \( \partial_1 = \partial_2 = 0 \) on dots. These observations yield

**Observation 4.4.** For a square \( S \) we have \( E_{r,A}(S) = E_{r,BC}(S) = 0 \), while for a dot \( D = \langle a \rangle \) we have \( E_{r,BC}(D) = E_{r,A}(D) = \langle a \rangle \) for all \( r \geq 1 \).

For higher length zigzags \( Z \), generated by \( a_1, \ldots, a_l \), we get that \( H_A(Z) = \langle a_1, \ldots, a_l \rangle \) keeps the lower antidiagonal, while \( H_{BC}(Z) = \langle \partial_2 a_1, \ldots, \partial_2 a_l, \partial_1 a_l \rangle \) remembers the higher antidiagonal. To describe their higher-page analogues, it suffices to understand the kernel, resp. cokernel, of the projection \( H_{BC}(Z) \to E_{r,BC}(Z) \), resp. the inclusion \( E_{r,A}(Z) \to H_A(Z) \). These are described as follows.

**Lemma 4.5.** Let \( Z \) be a zigzag of length at least two, generated by \( a_1, \ldots, a_l \). For any \( i \not\in \{1, \ldots, l\} \), set \( a_i := 0 \). Then, for any \( r \geq 2 \):

1. **Even length type I:** If \( \partial_2 a_1 = 0 \) and \( \partial_1 a_l \neq 0 \), one has:

   \[
   \ker (H_{BC}(Z) \to E_{r,BC}(Z)) = \langle \partial_1 a_1, \ldots, \partial_1 a_{r-1} \rangle \\
   \coker (E_{r,A}(Z) \to H_A(Z)) = \langle a_{l-r+2}, \ldots, a_l \rangle.
   \]
2. **Even length type II**: If $\partial_2 a_1 \neq 0$ and $\partial_1 a_l = 0$, one has:

$$\ker (H_{BC}(Z) \to E_{r, BC}(Z)) = \langle \partial_1 a_{l-r+2}, \ldots, \partial_1 a_l \rangle$$

$$\operatorname{coker} (E_{r, A}(Z) \to H_A(Z)) = \langle a_1, \ldots, a_{r-1} \rangle.$$ 

3. **Odd length type M**: If $\partial_2 a_1 = 0 = \partial_2 a_l$, one has $E_{r, A}(Z) = H_A(Z)$ and

$$\ker (H_{BC}(Z) \to E_{r, BC}(Z)) = \langle \partial_1 a_1, \ldots, \partial_1 a_{r-1}, \partial 2 a_{l-r+2}, \ldots, \partial_2 a_l \rangle.$$ 

4. **Odd length type L**: If $\partial_2 a_1 \neq 0 \neq \partial_2 a_l$, one has $H_{BC}(Z) = E_{r, BC}(Z)$ and

$$\operatorname{coker} (E_{r, A}(Z) \to H_A(Z)) = \langle a_1, \ldots, a_{r-1}, a_{l-r+2}, \ldots, a_l \rangle.$$ 

Note that for large $r$, some of the written generators could be zero or there could be some overlap in the last two cases.

**Proof.** Let us only do the computation for $E_{r, BC}$. The elements that get modded out here in addition to the $\partial_1 \partial_2$-exact ones are the $E_r$-exact ones and the $\bar{E}_r$-exact ones. By the definition of $E_r$-exactness, this means that, whenever a zigzag has top left corner generated by $a_1$ with $\partial_2 a_1 = 0$, i.e.:

$$\langle a_1 \rangle \longrightarrow \langle \partial_1 a_1 \rangle$$

$$\langle a_2 \rangle \longrightarrow \langle \partial_1 a_2 \rangle$$

$$\ddots$$

the classes of $\partial_1 a_1, \ldots, \partial_1 a_{r-1}$ are zero in $E_{r, BC}(Z)$. Along the same lines, if a zigzag has bottom right corner generated by $a_1$ with $\partial_2 a_1 = 0$, i.e.:

$$\langle a_1 \rangle \longrightarrow \langle \partial_2 a_1 \rangle$$

$$\langle a_2 \rangle \longrightarrow \langle \partial_2 a_2 \rangle$$

$$\ddots$$

the classes of $\partial_2 a_1, \ldots, \partial_2 a_{r-1}$ are zero in $E_{r, BC}(Z)$. This yields the result for $E_{r, BC}$. The calculation for $H_A$ is analogous. 

Let us also record what this yields for the dimensions of the new cohomology groups.

**Corollary 4.6.** Let $Z$ be an indecomposable bounded double complex and let $r \geq 2$. 

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1. If $Z$ is a square, then $e_{r,BC}(Z) = 0 = e_{r,A}(Z)$.

2. If $Z$ is a dot, then $e_{r,BC}(Z) = 1 = e_{r,A}(Z)$.

3. If $Z$ is a zigzag of odd length $2l + 1 \geq 3$ of type $L$, one has $e_{r,BC}(Z) = h_{BC}(Z) = l + 1$ and $e_{r,A}(Z) = \max\{l - 2(r - 1), 0\}$.

4. If $Z$ is a zigzag of odd length $2l + 1 \geq 3$ of type $M$, one has $e_{r,BC}(Z) = \max\{l - 2(r - 1), 0\}$ and $e_{r,A}(Z) = h_{A}(Z) = l + 1$.

5. If $Z$ is a zigzag of even length $2l$, one has $e_{r,BC}(Z) = e_{r,A}(Z) = \max\{l - r + 1, 0\}$.

Recall ([KQ20], [PSU20], [Ste20]) that any bounded double complex $A$ can be written as a direct sum of indecomposable ones, all indecomposable ones are either squares or zigzags and $A$ has the page-$r$-$\partial_1\partial_2$-property if and only if in any decomposition into indecomposables, there are no odd length zigzags other than dots (length one) and no even length zigzags of length greater than $2r$. Thus, we get

**Corollary 4.7.** For any bounded double complex $A$ such that all the numerical quantities involved are finite, there is an inequality:

$$e_{r,A}(A) + e_{r,BC}(A) \geq e_r(A) + \bar{e}_r(A) \geq 2b(A).$$

Equality holds if $A$ satisfies the page-$(r - 1)$-$\partial_1\partial_2$-property

**Proof.** Since all the quantities involved are additive under direct sums, it suffices to show this for indecomposable double complexes $Z$. The middle and right hand side were computed in [Ste20]: $e_r(Z) + e_r(Z)$ equals 0 for a square and for a zigzag of length $2l \leq 2(r - 1)$, while it equals 2 for all other zigzags. Also, $b(Z) = 1$ for odd length zigzags and $b(Z) = 0$ otherwise. In particular, for an arbitrary double complex, the middle quantity is just twice the number of all zigzags which have odd length or even length at least $2r$. By the previous Corollary 4.6, $e_{r,A}(Z) + e_{r,BC}(Z) = 0$ for squares and even length zigzags of length $2l \leq 2(r - 1)$, while it equals 2 for dots and zigzags of length $2r$ and is greater than or equal to 2 for all other zigzags.

**Remark 4.8.** Somewhat unexpectedly, the equality $e_{r,A}(A) + e_{r,BC}(A) = 2b(A)$ does not imply the page-$(r - 1)$-$\partial_1\partial_2$-property for $r \geq 2$, contrary to the case $r = 1$ ([AT13]). For example, both sides are equal to 2 for $r \geq 2$ and $A$ a zigzag of length 3. As one may see, for example from a Hopf surface, this behaviour really occurs in geometric situations. A different generalisation of the case $r = 1$ has been obtained in [PSU20].

**Corollary 4.9.** Let $A$ be a bounded double complex such that $e^k_{r,BC}(A)$ and $e^k_{r,A}(A)$ are finite. The following properties are equivalent:

(A') $A$ has the page-$(r - 1)$-$\partial_1\partial_2$-property.

(B') The map $E_{r,BC}(A) \to E_{r,A}(A)$ is an isomorphism.

(C') One has $e^k_{r,BC}(A) = e^k_{r,A}(A)$ for all $k \in \mathbb{Z}$.
Proof. Note that property \((B')\) implies property \((C')\). Thus, it suffices to show that property \((B')\) is satisfied for squares, dots and even length zigzags of length \(\leq 2(r-1)\) and property \((C')\) is violated for all other zigzags. This is a direct consequence of Lemma 4.5 and Corollary 4.6.

Proof of Theorem 4.1. Only the last two points remain to be proved. By the duality between \(E_{r, BC}(X)\) and \(E_{r, A}(X)\), both spaces have the same dimension, so \((D) \iff (B)\).

Property \((E)\) can be restated as the fact that all the maps originating from \(E_{r, BC}(X)\) (including the one with target in \(E_{r, A}(A)\)) in the following commutative diagram are injective:

\[
\begin{array}{ccc}
E_{r, BC}^{p,q}(X) & \rightarrow & E_{r}^{p,q}(X) \\
\downarrow & & \downarrow \\
E_{r}^{p,q}(X) & \rightarrow & H_{dR}^{p+q}(X, \mathbb{C}) \\
\downarrow & & \downarrow \\
\bar{E}_{r}^{p,q}(X) & \rightarrow & E_{r}^{p,q}(X)
\end{array}
\]

In particular, \((E) \Rightarrow (D)\). Since the composition of two maps being injective implies the right-hand map is injective, we get \((D) \Rightarrow (E)\). \(\square\)

Remark 4.10. As one sees from the above proof, if the equivalent conditions of Theorem 4.1 hold, then the maps \(E_{r, BC}^{p,q}(X) \rightarrow E_{r}^{p,q}(X)\) and \(E_{r}^{p,q}(X) \rightarrow \bar{E}_{r}^{p,q}(X)\) are isomorphisms as well.

Remark 4.11. The equivalence between \((A)\) and \((D)\) is an extension for \(r > 1\) of [AT17, Thm 3.1]. Together with Proposition 3.5, Theorem 4.1 also implies the following extension of [AT17, Thm 5.2].

Corollary 4.12. For a compact complex manifold \(X\) of dimension \(n\), the canonical bilinear pairing in \(E_{r\text{-Bott-Chern}}\) cohomology \(E_{r, BC}^{p,q}(X) \times E_{r, BC}^{n-p,n-q}(X) \rightarrow \mathbb{C}\), given by wedge product and integration, is well defined. Moreover, \(X\) is a page-\((r-1)\)-\(\partial\bar{\partial}\)-manifold if and only if the pairing is non-degenerate.

5 Relations with the \(E_r\)-sG manifolds

Let \(X\) be an \(n\)-dimensional compact complex manifold and let \(\omega\) be a Gauduchon metric on \(X\). This means that \(\omega\) is a smooth, positive definite \((1, 1)\)-form on \(X\) such that \(\partial\bar{\partial}\omega^{n-1} = 0\). Gauduchon metrics were introduced and proved to always exist in [Gau77]. It was noticed in [Pop19] that \(\partial\omega^{n-1}\) is \(E_{r}\)-closed for every \(r \in \mathbb{N}^*\) and the following definition was introduced.

Definition 5.1. ([Pop19]) Let \(r \in \mathbb{N}^*\). A Gauduchon metric \(\omega\) on \(X\) is said to be \(E_r\)-sG if \(\partial\omega^{n-1}\) is \(E_{r}\)-exact.

We say that \(X\) is an \(E_r\)-sG manifold if an \(E_r\)-sG metric \(\omega\) exists on \(X\).

The \(E_1\)-sG property coincides with the strongly Gauduchon (sG) property introduced in [Pop13]. The following implications and their analogues for \(X\) are obvious:

\[\omega \text{ is } E_1\text{-sG} \implies \omega \text{ is } E_2\text{-sG} \implies \omega \text{ is } E_3\text{-sG}\]

and so is the fact that, for biddegree reasons, the \(E_r\)-sG property coincides with the \(E_3\)-sG property for all \(r \geq 4\).

The link between the page-\((r-1)\)-\(\partial\bar{\partial}\) and the \(E_r\)-sG properties is spelt out in the following
Proposition 5.2. Let \( r \in \mathbb{N}^* \) and let \( X \) be a page-\((r-1)\)-\(\partial\bar{\partial}\)-manifold. Then, every Gauduchon metric on \( X \) is \( E_r \)-sG. In particular, \( X \) is an \( E_r \)-sG manifold.

Proof. Let \( \omega \) be a Gauduchon metric on \( X \). Then, \( \partial \omega^{n-1} \) is \( \bar{\partial} \)-closed and \( \partial \)-closed, hence \( d \)-closed. It is also \( \partial \)-exact, or equivalently, \( \bar{\partial} \)-exact, hence also \( \bar{\partial} \)-exact.

Now, thanks to Theorem 4.1, the page-\((r-1)\)-\(\partial\bar{\partial}\)-property of \( X \) implies the equivalence between \( \bar{\partial} \)-exactness and \( \bar{\partial} \)-exactness for \( d \)-closed pure-type forms. Consequently, \( \partial \omega^{n-1} \) must be \( E_r \)-exact, so \( \omega \) is an \( E_r \)-sG metric. \( \square \)

Let \( X_{u,v} \) be a Calabi-Eckmann manifold, i.e. any of the complex manifolds \( C^\infty \)-diffeomorphic to \( S^{2u+1} \times S^{2v+1} \) constructed by Calabi and Eckmann in [CE53]. Recall that the \( X_{0,v} \)'s and the \( X_{u,0} \)'s are Hopf manifolds. By [Pop14], \( X_{u,v} \) does not admit any sG metric. However, we now prove the existence of \( E_2 \)-sG metrics when \( uv > 0 \).

Proposition 5.3. Let \( X_{u,v} \) be a Calabi-Eckmann manifold of complex dimension \( \geq 2 \). Let \( u \leq v \).

(i) If \( u > 0 \), then \( X_{u,v} \) does not admit sG metrics, but it is an \( E_2 \)-sG manifold.

(ii) If \( u = 0 \), \( X_{u,v} \) does not admit \( E_r \)-sG metrics for any \( r \).

Proof. By Borel’s result in [Hir78, Appendix Two by A. Borel], we have

\[
H_{\bar{\partial}}^{\bullet \bullet}(X_{u,v}) \cong \frac{\mathbb{C}[x_{1,1}]}{(x_{u+1,1})} \otimes \langle x_{v+1,1}, x_{0,1} \rangle.
\]

In other words, a model for the Dolbeault cohomology of the Calabi-Eckmann manifold \( X_{u,v} \) is provided by the CDGA (see [NT78])

\[
\langle V \langle x_{0,1}, x_{1,1}, y_{u+1,u}, x_{v+1,v} \rangle, \bar{\partial} \rangle,
\]

with differential

\[
\bar{\partial}x_{0,1} = 0, \quad \bar{\partial}x_{1,1} = 0, \quad \bar{\partial}y_{u+1,u} = x_{u+1,1}^{u+1}, \quad \bar{\partial}x_{v+1,v} = 0.
\]

Thus, if \( u > 0 \), we have a minimal model. (For \( u = 0 \) we have only a cofibrant model in the sense of [NT78].)

Moreover, \( \partial \) acts on generators as follows [NT78]:

\[
\partial x_{0,1} = x_{1,1} \quad \text{ (hence } \partial x_{1,1} = 0), \quad \partial y_{u+1,u} = 0, \quad \partial x_{v+1,v} = 0.
\]

Next we determine the spaces \( E_r^{n,n-1} \), for any \( r \geq 1 \), where \( n = u + v + 1 \).

Let us first focus on the case (i), i.e. \( u > 0 \). We need to consider the Dolbeault cohomology groups \( H_{\bar{\partial}}^{n-1,n-1}(X_{u,v}) \) and \( H_{\bar{\partial}}^{n,n-1}(X_{u,v}) \). They are given by

\[
H_{\bar{\partial}}^{u,v,u+v}(X_{u,v}) = \langle x_{0,1}, x_{1,1}^{u-1}, x_{v+1,v} \rangle, \quad H_{\bar{\partial}}^{u+1,v+1,u+v}(X_{u,v}) = \langle x_{1,1}, x_{v+1,v} \rangle.
\]

Now we consider

\[
H_{\bar{\partial}}^{u,v,u+v}(X_{u,v}) \xrightarrow{\bar{\partial}} H_{\bar{\partial}}^{u+1,v+1,u+v}(X_{u,v}) \rightarrow 0.
\]

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Since $\partial(x_{0,1}, x_{1,1}^{u-1}, x_{v+1,v}) = \partial(x_{0,1}) \cdot x_{1,1}^{u-1} \cdot x_{v+1,v} = x_{1,1}^{u} \cdot x_{v+1,v}$, the first map is surjective. Therefore, $E^{u,n-1}_2(X_{u,v}) = 0$. Thus, any Gauduchon metric on $X_{u,v}$ is an $E_2$-sG metric.

Next we focus on the case (ii), i.e. $u = 0$, so $v \geq 1$. In this case $H^{u+1,v}_\partial(X_{0,v}) = \langle x_{v+1,v} \rangle$. Notice that the Dolbeault cohomology groups $H^{v-r+1,v+r-1}_\partial(X_{0,v})$ are all zero for every $r \geq 1$. Therefore, $E^{r+1,v}_r(X_{0,v}) = \{0\}$ for every $r \geq 1$. Meanwhile, from

$$\{0\} = E^{r-r+1,v+r-1}_r(X_{0,v}) \xrightarrow{d_r} E^{v+1,v}_r(X_{0,v}) \longrightarrow 0,$$

we get $E^{v+1,v}_r(X_{0,v}) = H^{v+1,v}_\partial(X_{0,v})$ for every $r \geq 2$. So, the existence of an $E_r$-sG metric on $X_{0,v}$ would imply the existence of an sG metric, which would contradict [Pop14].

As a by-product of Borel’s description of the Dolbeault cohomology of the Calabi-Eckmann manifolds $X_{u,v}$ used in the above proof, one gets that the Frölicher spectral sequence of $X_{u,v}$ satisfies $E_1 \neq E_2 = E_\infty$ when $u > 0$, whereas it degenerates at $E_1$ when $u = 0$. This latter fact implies that no Hopf manifold $X_{0,v}$ can have a pure De Rham cohomology, hence cannot be a page-$r$-$\partial\bar{\partial}$-manifold for any $r$. Indeed, if the De Rham cohomology were pure, then $X_{0,v}$ would be a $\partial\bar{\partial}$-manifold, a fact that is trivial to contradict.

By the previous result, all the Calabi-Eckmann manifolds that are not Hopf manifolds are $E_2$-sG manifolds. However, the next observation shows that they are not page-$r$-$\partial\bar{\partial}$-manifolds for any $r \in \mathbb{N}$.

**Lemma 5.4.** Let $u, v \geq 0$ and let $X = S^{2u+1} \times S^{2v+1}$ be equipped with any of the Calabi-Eckmann complex structures. Assume that either $u \neq v$ or $u = v = 1$. Then, the De Rham cohomology of $X$ is not pure.

**Proof.** It is a well-known consequence of the result in [Hir78, Appendix Two by A. Borel] recalled above that, for any Calabi-Eckmann manifold, the Frölicher spectral sequence degenerates at $E_2$. Thus, if the De Rham cohomology were pure, $X$ would be page-$r$-$\partial\bar{\partial}$. By [PSU20, Thm 3.1], this would imply $h_{BC}(X) = h_\partial(X)$, where $h_{BC}(X) := \sum_{p,q} h^{p,q}_{BC}(X)$ and $h_\partial(X) := \sum_{p,q} h^{p,q}_{\partial}(X)$ are the total Bott-Chern and Hodge numbers. For $u \neq v$, the Bott-Chern numbers were computed in [Ste20, Cor. H], yielding $h_{BC}(X) = 4u + 5 > 4u + 4 = h_\partial(X)$. A calculation in [TT17] implies that this continues to hold for $u = v = 1$.

**Remark 5.5.** This lemma is very likely to also hold for arbitrary $u = v > 1$. From the proof of [Ste20, Cor H] it follows that to settle this issue, it suffices to determine whether $h^{u+1,v+1}_{BC} = 1$ for the higher-dimensional Calabi-Eckmann manifolds with $u = v$. Since we are only interested in the existence of a counterexample here, we do not pursue this issue.

### 6 Further applications to H-S, SKT and sGG manifolds

We start by recalling the following definitions (see [ST10] for the notion of Hermitian-symplectic (H-S) metric and manifold; [Pop13] for the notion of strongly Gauduchon (sG) metric and manifold; [PU18] for the notion of sGG manifold).
Definition 6.1. Let $X$ be a compact complex manifold with $\dim \mathbb{C} X = n$.

(a) A Hermitian metric $\omega$ on $X$ is said to be

(i) **SKT** if $\partial \bar{\partial} \omega = 0$;

(ii) **Hermitian-symplectic (H-S)** if there exists a form $\rho^{0,2} \in C^{\infty}_{0,2}(X, \mathbb{C})$ such that

$$d(\rho^{0,2} + \omega + \rho^{0,2}) = 0;$$

(iii) **strongly Gauduchon (sG)** if $\partial \omega^{n-1}$ is $\bar{\partial}$-exact.

(b) If $X$ admits an SKT, or an H-S, or an sG metric $\omega$, then $X$ is said to be an **SKT manifold**,
resp. an **H-S manifold**, resp. an **sG manifold**.

(c) If every Gauduchon metric on $X$ is strongly Gauduchon (sG), then $X$ is said to be an **sGG manifold**.

The notion of sG metric is the analogue in bidegree $(n-1, n-1)$ of the notion of H-S metric. Indeed, by [Pop13, Proposition 4.2], a metric $\omega$ is sG if and only if there exists a form $\Omega^{n-2,n} \in C^{\infty}_{n-2,n}(X, \mathbb{C})$ such that

$$d(\Omega^{n-2,n} + \omega^{n-1} + \Omega^{n-2,n}) = 0.$$

Our first observation is that the higher-page Aeppli cohomologies introduced in this paper provide the natural cohomological framework for the study of the above metric notions.

Proposition 6.2. Let $X$ be a compact complex manifold with $\dim \mathbb{C} X = n$ and let $\omega$ be a Hermitian metric on $X$.

(i) The metric $\omega$ is strongly Gauduchon (sG) if and only if $\omega^{n-1}$ is $E_2\bar{E}_2$-closed. In particular, in this case, $\omega^{n-1}$ induces an $E_2$-Aeppli cohomology class $\{\omega^{n-1}\}_{E_2,A} \in E^{n-1,n-1}_{2,A}(X)$.

(ii) The metric $\omega$ is Hermitian-symplectic (H-S) if and only if $\omega$ is $E_3\bar{E}_3$-closed. In particular, in this case, $\omega$ induces an $E_3$-Aeppli cohomology class $\{\omega\}_{E_3,A} \in E^{1,1}_{3,A}(X)$.

When $n = 3$, $\omega$ is Hermitian-symplectic (H-S) if and only if $\omega$ is $E_2\bar{E}_2$-closed.
In particular, in this case, $\omega$ induces an $E_2$-Aeppli cohomology class $\{\omega\}_{E_2,A} \in E^{1,1}_{2,A}(X)$.

Proof. (i) The sG condition on $\omega$ is defined by requiring $\partial \omega^{n-1}$ to be $\bar{\partial}$-exact. By conjugation, this is equivalent to $\bar{\partial} \omega^{n-1}$ being $\partial$-exact. These two conditions express the fact that $\omega$ satisfies the two towers of 1 equation each (hence corresponding to the case $r = 2$) in (i) of Definition 3.1, so they are equivalent to $\omega^{n-1}$ being $E_2\bar{E}_2$-closed.

(ii) The H-S condition on $\omega$ is equivalent to the existence of a form $\rho^{2,0} \in C^{\infty}_{2,0}(X, \mathbb{C})$ such that $\partial \omega = -\bar{\partial} \rho^{2,0}$ and $\partial \rho^{2,0} = 0$. By conjugation, these conditions are equivalent to the existence of a form $\rho^{0,2} \in C^{\infty}_{0,2}(X, \mathbb{C})$ such that $\bar{\partial} \omega = -\partial \rho^{0,2}$ and $\bar{\partial} \rho^{0,2} = 0$. These four conditions express the fact that $\omega$ satisfies the two towers of 2 equations each (hence corresponding to the case $r = 3$) in (i) of Definition 3.1, or equivalently that $\omega$ is $E_3\bar{E}_3$-closed.

When $n = 3$, the condition $\partial \rho^{2,0} = 0$ is automatic since

$$i\partial \rho^{2,0} \wedge \bar{\partial} \rho^{0,2} = \partial \rho^{2,0} \wedge * \bar{\partial} \rho^{0,2} = |\partial \rho^{2,0}|^2 \omega dV_\omega \geq 0 \quad \text{and} \quad \int_X i\partial \rho^{2,0} \wedge \bar{\partial} \rho^{0,2} = \int_X \partial (i\rho^{2,0} \wedge \bar{\partial} \rho^{0,2}) = 0,$$
where \( * = *_\omega \) is the Hodge star operator induced by \( \omega \) and the primitivity (trivial for bidegree reasons) of the \((0,3)\)-form \( \partial \rho^{0,2} \) was used to infer that \( * \bar{\partial} \rho^{0,2} = i \bar{\partial} \rho^{0,2} \). In fact, the last identity follows from the general formula:

\[
* v = (-1)^{k(k+1)/2} i^{p-q} \frac{\omega^{n-p-q} \wedge v}{(n-p-q)!}, \quad \text{where } k := p + q,
\]
satisfied by any primitive form \( v \) of any bidegree \((p, q)\) (see e.g. [Voi02, Proposition 6.29, p. 150]). □

Using the real structure of the spaces \( E^{1,1}_{r, A}(X) \), we will now consider their real versions \( E^{1,1}_{r, A}(X, \mathbb{R}) \) and define cones of higher Aeppli cohomology classes representable by the kinds of metrics discussed above.

**Definition 6.3.** Let \( X \) be a compact complex manifold with \( \dim \mathbb{C} X = n \). We define the:

(i) **strongly Gauduchon (sG) cone** of \( X \) as the set:

\[
SG_X := \left\{ \{ \omega^{n-1} \}_{E_2,A} \in E^{n-1,n-1}_{2,A}(X, \mathbb{R}) \mid \omega \text{ is an sG metric on } X \right\} \subset E^{n-1,n-1}_{2,A}(X, \mathbb{R});
\]

(ii) **Hermitian-symplectic (H-S) cone** of \( X \) as the set:

\[
HS_X := \left\{ \{ \omega \}_{E_3,A} \in E^{1,1}_{3,A}(X, \mathbb{R}) \mid \omega \text{ is an H-S metric on } X \right\} \subset E^{1,1}_{3,A}(X, \mathbb{R});
\]

(iii) **SKT cone** of \( X \) as the set:

\[
SKT_X := \left\{ \{ \omega \}_{E_{1,1}} \in E^{1,1}_{1,1}(X, \mathbb{R}) \mid \omega \text{ is an SKT metric on } X \right\} \subset E^{1,1}_{1,1}(X, \mathbb{R}) = H^{1,1}_A(X, \mathbb{R}).
\]

The strongly Gauduchon cone was defined in a different way in [Pop15] and in [PU18], although it was denoted in the same way. We keep the same notation for the object defined above since it will be seen in Corollary 6.6 to play an identical role in characterising sGG manifolds. Meanwhile, recall that the Gauduchon cone \( G_X \subset E^{n-1,n-1}_{1,A}(X, \mathbb{R}) = H^{n-1,n-1}_A(X, \mathbb{R}) \) of \( X \) was introduced in an analogous way in [Pop15] (i.e. as the set of the Aeppli cohomology classes of all the \( \omega^{n-1} \) induced by Gauduchon metrics \( \omega \)) and shown there to be an open convex cone in \( H^{n-1,n-1}_A(X, \mathbb{R}) \). The same conclusion holds for the new cones introduced above.

**Lemma 6.4.** The sG, H-S and SKT cones of \( X \) are open convex cones in their respective cohomology vector spaces.

**Proof.** Let us spell out the arguments for \( HS_X \). They run analogously for \( SG_X \) and \( SKT_X \).

The fact that \( HS_X \) is a convex cone as a subset of \( E^{1,1}_{3,A}(X, \mathbb{R}) \) is obvious: linear combinations with positive coefficients of H-S metrics are H-S metrics and taking the \( E^{1,1}_{3,A} \)-cohomology class is a linear operation.

To see that \( HS_X \) is open in \( E^{1,1}_{3,A}(X, \mathbb{R}) \), we will use the Hodge theory for the higher Aeppli cohomology developed in §3. Let \( \{ \omega \}_{E_{3,A}} \in HS_X \), where \( \omega \) is an H-S metric on \( X \). Since every
class in $E_{3,A}^{1,1}(X, \mathbb{R})$ is representable by a unique $E_3$-Aeppli harmonic (w.r.t. $\omega$) form, by Corollary and Definition 3.10, the class \( \{ \omega + \alpha_h \}_{E_{3,A}} \) still lies in $\mathcal{H}_X$ for every $E_3$-Aeppli harmonic form $\alpha_h \in C_{\infty}^0(X, \mathbb{R})$ whose $C^0$-norm induced by $\omega$ is small enough. Note that, by Corollary and Definition 3.10, the space $\mathcal{H}_{3,A}^{1,1}(X)$ has a real structure as well. □

We now take advantage of the fact that all the higher-page Aeppli cohomologies are defined using the same space of exact forms, which leads to the inclusions:

\[
\cdots \subset E_{r+1,1}^{1,1}(X) \subset E_{r,1}^{1,1}(X) \subset \cdots \subset E_{1,1}^{1,1}(X),
\]

to deduce the following

Lemma 6.5. The Gauduchon, $sG$, $H$-S and SKT cones of $X$ are related to one another as follows:

\[
\mathcal{SG}_X = \mathcal{G}_X \cap E_{2,A}^{n-1,n-1}(X, \mathbb{R}) \quad \text{and} \quad \mathcal{HS}_X = \mathcal{SKT}_X \cap E_{3,A}^{1,1}(X, \mathbb{R}).
\]

Proof. It follows at once from Proposition 6.2, Definition 6.3 and the above inclusions. □

Corollary 6.6. The following equivalences hold:

\[
\mathcal{SG}_X = \mathcal{G}_X \iff E_{2,A}^{n-1,n-1}(X) = E_{1,A}^{n-1,n-1}(X) \iff e_{2,A}^{n-1,n-1} = e_{1,A}^{n-1,n-1}
\]
\[
\iff X \text{ is an } sGG \text{ manifold}
\]

and

\[
\mathcal{HS}_X = \mathcal{SKT}_X \iff E_{3,A}^{1,1}(X) = E_{1,A}^{1,1}(X) \iff e_{3,A}^{1,1} = e_{1,A}^{1,1}
\]
\[
\iff \text{every SKT metric on } X \text{ is Hermitian-symplectic (H-S),}
\]

where $e$ stands each time for the dimension of the corresponding higher-page Aeppli cohomology space of $X$ denoted by $E$.

Proof. It follows at once from Lemmas 6.4 and Lemmas 6.5. □

7 Appendix

We refer to the appendix of [Pop19] for the details of the inductive construction of a Hodge theory for the pages $E_r$, with $r \geq 3$, of the Frölicher spectral sequence. Here, we will only remind the reader of the conclusion of that construction and will point out a reformulation of it that was used in the present paper.

Let $X$ be an $n$-dimensional compact complex manifold. We fix an arbitrary Hermitian metric $\omega$ on $X$. As recalled in §2.2, for every bidegree $(p, q)$, the $\omega$-harmonic spaces (also called $E_r$-harmonic spaces)

\[
\cdots \subset \mathcal{H}_{r+1}^{p,q} \subset \mathcal{H}_r^{p,q} \subset \cdots \subset \mathcal{H}_1^{p,q} \subset C_{\infty}^p(X)
\]

were constructed by induction on $r \in \mathbb{N}^*$ in [Pop17, §3.2] such that every $\mathcal{H}_r^{p,q}$ is isomorphic to the corresponding space $E_r^{p,q}(X)$ featuring on the $r^{th}$ page of the Frölicher spectral sequence of $X$. 32
Moreover, pseudo-differential “Laplacians” \( \tilde{\Delta}^{(r+1)} : \mathcal{H}^{p,q}_r \rightarrow \mathcal{H}^{p,q}_r \) were inductively constructed in the appendix to [Pop19] such that

\[
\ker \tilde{\Delta}^{(r)} = \mathcal{H}^{p,q}_r, \quad r \in \mathbb{N}^*,
\]

where \( \tilde{\Delta}^{(1)} = \Delta'' = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial} \) is the usual \( \bar{\partial} \)-Laplacian.

The conclusion of the construction in the appendix to [Pop19] was the following statement. It gives a 3-space orthogonal decomposition of each space \( C^{\infty}_{p,q}(X) \), for every fixed \( r \in \mathbb{N}^* \), that parallels the standard decomposition \( C^{\infty}_{p,q}(X) = \ker \Delta'' \oplus \text{Im} \bar{\partial} \oplus \text{Im} \bar{\partial}^* \) for \( r = 1 \).

**Proposition 7.1.** (Corollary 4.6 in [Pop19]) Let \((X, \omega)\) be a compact complex n-dimensional Hermitian manifold. For every \( r \in \mathbb{N}^* \), put \( D_{r-1} := ((\tilde{\Delta}^{(1)})^{-1} \bar{\partial} \bar{\partial}) \cdots ((\tilde{\Delta}^{(r-1)})^{-1} \bar{\partial} \bar{\partial}) \) and \( D_0 = \text{Id} \).

(i) For all \( r \in \mathbb{N}^* \) and all \((p, q)\), the kernel of \( \tilde{\Delta}^{(r+1)} : C^{\infty}_{p,q}(X) \rightarrow C^{\infty}_{p,q}(X) \) is given by

\[
\ker \tilde{\Delta}^{(r+1)} = \left( \ker (p_r \partial D_{r-1}) \cap \ker (\partial D_{r-1} p_r)^* \right) \cap \left( \ker (p_{r-1} \partial D_{r-2}) \cap \ker (\partial D_{r-2} p_{r-1})^* \right) \cap \cdots \cap \left( \ker (p_1 \partial) \cap \ker (\partial p_1)^* \right) \cap \left( \ker \bar{\partial} \cap \ker \bar{\partial}^* \right).
\]

(ii) For all \( r \in \mathbb{N}^* \) and all \((p, q)\), the following orthogonal 3-space decomposition (in which the sums inside the big parentheses need not be orthogonal or even direct) holds:

\[
C^{\infty}_{p,q}(X) = \ker \tilde{\Delta}^{(r+1)} \oplus \left( \text{Im} \bar{\partial} + \text{Im} (\partial p_1) + \text{Im} (\partial D_1 p_2) + \cdots + \text{Im} (\partial D_{r-1} p_r) \right) \oplus \left( \text{Im} \bar{\partial}^* + \text{Im} (p_1 \partial) + \text{Im} (p_2 \partial D_1) + \cdots + \text{Im} (p_r \partial D_{r-1}) \right),
\]

where \( \ker \tilde{\Delta}^{(r+1)} \oplus (\text{Im} \bar{\partial} + \text{Im} (\partial p_1) + \text{Im} (\partial D_1 p_2) + \cdots + \text{Im} (\partial D_{r-1} p_r)) = \ker \bar{\partial} \cap \ker (p_1 \partial) \cap \ker (p_2 \partial D_1) \cap \cdots \cap \ker (p_r \partial D_{r-1}) \) and \( \ker \tilde{\Delta}^{(r+1)} \oplus (\text{Im} \bar{\partial}^* + \text{Im} (p_1 \partial) + \text{Im} (p_2 \partial D_1) + \cdots + \text{Im} (p_r \partial D_{r-1})) = \ker \bar{\partial}^* \cap \ker (p_1 \partial)^* \cap \ker (p_2 \partial D_1)^* \cap \cdots \cap \ker (p_r \partial D_{r-1})^* \).

For each \( r \in \mathbb{N}^* \), \( p_r = p^{p,q}_r \) stands for the \( L^2 \)-orthogonal projection onto \( \mathcal{H}^{p,q}_r \).

We will now cast the 3-space decomposition (12) in the terms used in the present paper. Recall that in the proof of Lemma 3.3, we defined the following vector spaces for every \( r \in \mathbb{N}^* \) and every bidegree \((p, q)\) based on the terminology introduced in (iv) of Definition 3.1:

\[
\mathcal{E}^{p,q}_{\partial, r} := \{ \alpha \in C^{\infty}_{p,q}(X) \mid \partial \alpha \text{ reaches 0 in at most } r \text{ steps} \},
\]

\[
\mathcal{E}^{p,q}_{\bar{\partial}, r} := \{ \beta \in C^{\infty}_{p,q}(X) \mid \bar{\partial} \beta \text{ reaches 0 in at most } r \text{ steps} \}.
\]

When a Hermitian metric \( \omega \) has been fixed on \( X \) and the adjoint operators \( \bar{\partial}^* \) and \( \bar{\partial}^\ast \) with respect to \( \omega \) have been considered, we define the analogous subspaces \( \mathcal{E}^{p,q}_{\partial^*, r} \) and \( \mathcal{E}^{p,q}_{\bar{\partial}^*, r} \) of \( C^{\infty}_{p,q}(X) \) by replacing \( \partial \) with \( \partial^* \) and \( \bar{\partial} \) with \( \bar{\partial}^* \) in the definitions of \( \mathcal{E}^{p,q}_{\partial, r} \) and \( \mathcal{E}^{p,q}_{\bar{\partial}, r} \).

Part (ii) of Proposition 7.1 can be reworded as follows.
Proposition 7.2. Let \((X, \omega)\) be a compact complex \(n\)-dimensional Hermitian manifold. For every \(r \in \mathbb{N}^*\) and for all \(p, q \in \{0, \ldots, n\}\), the following orthogonal 3-space decomposition (in which the sums inside the big parantheses need not be orthogonal or even direct) holds:

\[
C^\infty_{p, q}(X) = \mathcal{H}^{p, q}_r \oplus \left( \text{Im} \bar{\partial} + \partial(\mathcal{E}^{p-1, q}_{\bar{\partial}, r-1}) \right) \oplus \left( \partial^*(\mathcal{E}^{p+1, q}_{\bar{\partial}^*, r-1}) + \text{Im} \bar{\partial}^* \right),
\]

where \(\mathcal{H}^{p, q}_r\) is the \(E_r\)-harmonic space induced by \(\omega\) (see §2.2 and earlier in this appendix) and the next two big parantheses are the spaces of \(E_r\)-exact \((p, q)\)-forms, respectively \(E^\star_r\)-exact \((p, q)\)-forms:

\[
\text{Im} \bar{\partial} + \partial(\mathcal{E}^{p-1, q}_{\bar{\partial}, r-1}) = C^{p, q}_r \quad \text{and} \quad \partial^*(\mathcal{E}^{p+1, q}_{\bar{\partial}^*, r-1}) + \text{Im} \bar{\partial}^* = \mathcal{C}^{p, q}_r.
\]

Moreover, we have

\[
\mathcal{Z}^{p, q}_r = \mathcal{H}^{p, q}_r \oplus \left( \text{Im} \bar{\partial} + \partial(\mathcal{E}^{p-1, q}_{\bar{\partial}, r-1}) \right) = \mathcal{H}^{p, q}_r \oplus C^{p, q}_r,
\]

\[
\star \mathcal{Z}^{p, q}_r = \mathcal{H}^{p, q}_r \oplus \left( \partial^*(\mathcal{E}^{p+1, q}_{\bar{\partial}^*, r-1}) + \text{Im} \bar{\partial}^* \right) = \mathcal{H}^{p, q}_r \oplus \mathcal{C}^{p, q}_r
\]

where \(\mathcal{Z}^{p, q}_r\) and \(\star \mathcal{Z}^{p, q}_r\) are the spaces of smooth \(E_r\)-closed, resp. \(E^\star_r\)-closed, \((p, q)\)-forms.

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