Wave Packet Transmission of Bloch Electron Manipulated by Magnetic field

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We study the phenomenon of wave packet revivals of Bloch electrons and explore how to control them by a magnetic field for quantum information transfer. It is showed that the single electron system can be modulated into a linear dispersion regime by the “quantized” flux and then an electronic wave packet with the components localized in this regime can be transferred without spreading. This feature can be utilized to perform the high-fidelity transfer of quantum information encoded in the polarization of the spin. Beyond the linear approximation, the re-localization and self-interference occur as the novel phenomena of quantum coherence.

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INTRODUCTION

Most recently, many theoretical investigations about quantum information transfer (QIT) based on quantum spin systems are carried out in order to implement scalable quantum computation \cite{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13}. Here, the quantum spin system usually behaves as a quantum data bus to intergrade many qubits. These investigations mainly aim at transferring quantum state through a solid state data bus with minimal spatial and dynamical control over the on-chip interactions between qubits. In this paper we will pay attention to a fundamental aspect of QIT and generally study the wave packet spreading and revival of Bloch electrons in one-dimensional lattice systems.

For the problems of wave packet evolution, we can cast back for much earlier investigations by Schrödinger and others about the quantum mechanical descriptions of localization of macroscopic objects \cite{14}. They demonstrated that a class of wave packets (now we call them coherent and squeezed states) of harmonic oscillator can keep their shapes during propagation and their centers of mass (CM) follow a classical trajectory. As a semi-classical solution of the Schrödinger equation, a superposition of the much higher excitation states with almost-homogeneous spectrum form an coherent-state type wave packet in the Coulomb potential, which can show the phenomena of non-spreading evolution and self-interference on classical orbits \cite{15}. This prediction has been demonstrated in the experiment involving the laser-induced excitation of atomic Rydberg wave packets \cite{16, 17}.

We can refer such non-spreading wave packet evolution with a complete auto-correlation \cite{14} as a perfect QIT if we could encode the quantum information in the spin polarization of electron. The investigation in this paper is motivated by our recent explorations about the QIT based on the quantum system possessing a commensurate structure of energy spectrum matched with a symmetry (SMS), which ensures a perfect QIT both in one and higher dimensional cases \cite{3, 18}. Actually the almost-homogeneous spectrum for the coherent-state type wave packet just satisfies the condition of SMS. In particular the non-spreading transfer of a zero-momentum wave packet is attractive for the task of quantum information transmission since a static superposition can behaves as a quantum storage. This is very similar to the scheme of the quantum storage of photon based on atomic ensemble where two stored photonic wave packets localized in the same position with different polarizations can function to decode the information of qubit \cite{19, 20}.

This paper will focus on a realistic, but simplest Bloch electron system (see the Fig. 1) in a magnetic field where the on-site Coulomb interactions are ignored. In this sense the spin polarization is always conversed during the time evolution of an arbitrary state and then quantum information encoded in the spin polarization of electron can be well protected. Thus the locality of electron wave packet becomes a crucial element to maximize the fidelity of QIT. We show that, by the “quantized” flux threading the ring lattice, the effective dispersion relation of a Bloch electron can be modulated into a liner dispersion regime that possesses SMS structure, and then an electronic wave packet with the components localized in this linear regime can be transferred without changes of its shape. This feature can be utilized to perform the high-fidelity QIT encoded in the polarization of the spin. The phenomena of wave packet revivals and self-interference can also be demonstrated for the cases beyond the lin-
ear dispersion regime. These novel quantum coherence effects may suggest a feasible protocol to implement the perfect QIT of Bloch electrons manipulated by the external magnetic field.

MODEL OF FLUX-CONTROLLED BLOCH ELECTRON IN A RING AND ITS LINEARIZATION

In this section, we present the Bloch electron model under consideration, a simple tight-binding model in an external magnetic field. Here, the Coulomb interaction is ignored for simply. We restrict our attention to the external magnetic field. Here, the Coulomb interaction under consideration, a simple tight-binding model in an

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The time $t$ is in the unit of $100/J$.

can be obtained through the Fourier transformation

$\sum_{j,\sigma} e^{i\phi N} a_{j,\sigma}^\dagger a_{j+1,\sigma} + h.c.$, (1)

deepens on the magnetic flux $\phi$ through the ring in the unit of flux quantum $\Phi_0 = \hbar/e$. Here, $a_{j,\sigma}^\dagger$ is the creation operator of Bloch electron at $j$th site with spin $\sigma = \uparrow, \downarrow$. The flux $\phi$ does not exert force on the Bloch electrons, but can change the local phase of its wave function due to the Aharonov-Bohm (AB) effect. Note that the interaction between the field and electrons is independent of the intrinsic degree of freedom-spin. This will be crucial to employ such kind of setup to transfer quantum information encoded in the polarization of the spin. Because of the AB effect, the role of the magnetic flux cannot be removed trivially.

Now we consider the evolution of the GWP

$\psi_{\sigma}(k_0, N_A) = \frac{1}{\sqrt{\Omega_1}} \sum_j e^{-\frac{j^2}{2}(N_A)^2} e^{ik_0 j} |j\rangle_\sigma$ (2)

with the velocity $k_0$, where $|j\rangle = a_{j,\sigma}^\dagger |0\rangle$, the half-width of the wave packet

$2\sqrt{\ln 2/\alpha} << N$, (3)

and the normalization factor $\Omega_1 = \sum_j e^{-\alpha^2 (j-N_A)^2}$. The limitation for the width of the GWP ensures the locality the state and avoids the overlap between the head and the tail of the wave packet. We will see that as time evolution, the head and tail do meet in certain situation and the interference phenomenon occurs.

In the following, we will show that the appropriate magnetic flux can ensure the transfer of the wave packet without spreading. The well-known Bloch dispersion relation

$\epsilon_k = -2J \cos(k + \frac{2\pi \phi}{N})$, (4)

comes just in this region, then the effective Hamiltonian becomes

$H_{eff} = v_p$, (8)

is the Bloch momentum operator. Obviously, this displacement effect time can directly result in a non-spreading wave packets transmission in the linear dispersion regime. Indeed. Fig. 3 illustrates how the “quantized” magnetic flux $\phi = \phi_n$, where $n = 0, 1, 2, ..., \pi/2$, can speed up the zero-momentum Gaussian wave packet (GWP) $|\psi(0, N_A)\rangle$ centered at site $N_A$ with the width $1/\alpha$. The details of the will be given in the next section.
Actually, with the linearized Hamiltonian $H_{\text{eff}}$, the time evolution of some states can be described as a spatial translation by the evolution operator

$$U(t) = \exp(-i\mu t) \equiv T(vt),$$

with a displacement $x = vt$. Here the the translational operator is defined by

$$T(x_0) |j\rangle_\sigma = |j + x_0\rangle_\sigma$$

for arbitrary $|j\rangle_\sigma$.

For small $\alpha$, $|\psi(0, N_A)\rangle_\sigma$ is also a GWP of width $\alpha$ around $k = 0$ in $k$-space, and then the wave function at instance $t$ is a translated GWP

$$|\Phi(t)\rangle_\sigma = T(vt) |\psi(0, N_A)\rangle_\sigma = e^{i\pi \alpha |\phi| N} |\psi(0, N_\epsilon(t))\rangle_\sigma,$$

which is centered at $N_\epsilon(t) = N_A + vt$. The overall phase factor $e^{i\pi \alpha |\phi| N}$ has no effect on the final result. It is seen that the wave packet moves with velocity $v$. Since all the wave functions satisfy the periodic boundary condition

$$|\psi(0, N + i)\rangle_\sigma = |\psi(0, i)\rangle_\sigma,$$

we have

$$\begin{align*}
\langle i | \Phi(t) \rangle_\sigma &= \langle i - N | \Phi(t) \rangle_\sigma & \text{for } i > N, \\
\langle i | \Phi(t) \rangle_\sigma &= \langle i + N | \Phi(t) \rangle_\sigma & \text{for } i < 1.
\end{align*}$$

Obviously, the wave packet moves along the ring keeping the initial shape without any spreading as illustrated in Fig. 2 schematically.

**NON-SPREADING WAVES PACKETS**

**EVOLUTION AND SOLID STATE FLYING QUBIT**

To analyze the cyclic motion of Bloch electrons, the numerical simulation is performed for a zero-momentum GWP with $\alpha = 0.1$ in the system of $N = 100$ and $\phi = N/4 = 25$. The simulated time evolution of wave packet is plotted in Fig. 3. For the cases with different values of $\phi$ and $\alpha$, the autocorrelation functions

$$|A(t)| = \sum_\sigma |\langle \sigma | \Phi(t) \psi(0, N_A) \rangle_\sigma|^2,$$

which can be used to describe the properties of the electron propagation, are investigated numerically. The results for $\phi = 20, 25, 33$ and $\alpha = 0.1, 0.3$ are plotted in Fig. 3(a) and 3(b), which show that a zero-momentum GWP with small $\alpha$ can be transferred without spreading when $\phi$ are around each $\phi_n$.

Meanwhile, the flux threading the ring can control the shape and destination of the final wave packet. It is observed from the above analysis that the flux plays an important role for manipulating non-spreading wave packet.

Actually, such phenomenon can also be understood by the following transformation of the basis vector of Hilbert space. The existence of the flux is equivalent to adding a speed to boost the zero-momentum wave packet since the magnetic flux provides an extra phase to the basis in the position-space, i.e.,

$$|j\rangle_\sigma = e^{2\pi i \phi j/N} |\phi_j\rangle_\sigma.$$  \hfill (16)

In other words, a GWP with small $\alpha$ and momentum $k_0 = \pi(2n + 1)/2$, can transfer along the ring without spreading approximately. We will demonstrate this in the last section about open chain system.

In the above studies, the spin state of the Bloch electron is a conservative quantity that can not be influence during the propagation no matter how the spatial shape of the wave function changes. From an abstract point of view, the spatial properties of the carrying particle, i.e. the Bloch electron, seems to be irrelevant since only amplitudes and relative phases are used to encode quantum information. However, when the propagation of the Bloch electron can be exploited to transfer the information of qubit, the non-spreading propagation of the carrier is very crucial for the expected high-fidelity of
quantum state transfer from a location to another.

With the above consideration we can imagine the electronic wave packets with spin polarization as an analogue of photon “flying qubit”, the type II (polarized) photon qubit where the quantum information was encoded in its two polarization states. We define the solid state “flying qubit”, at a single location $A$ in a quantum wire, as the two Bloch electronic wave packets $|1\rangle_A = |1\rangle |N_A\rangle$ and $|0\rangle_A = |0\rangle |N_A\rangle$ be encoding as

$$|1\rangle |N_A\rangle = \frac{1}{\sqrt{\Omega_1}} \sum_j e^{\frac{-\alpha^2}{2}(j-N_A)^2} e^{i\frac{\pi}{2} j} a_j^\dagger |0\rangle,$$

$$|0\rangle |N_A\rangle = \frac{1}{\sqrt{\Omega_1}} \sum_j e^{\frac{-\alpha^2}{2}(j-N_A)^2} e^{i\frac{\pi}{2} j} a_j^\dagger |0\rangle \quad (17)$$

Because of the intrinsic linearity of the Schrödinger equation, it is self-consistent to encode an arbitrary state

$$|\psi\rangle = \cos(\frac{\theta}{2}) |1\rangle_A + \sin(\frac{\theta}{2}) e^{i\varphi} |0\rangle_A \quad (18)$$

as

$$|\psi(\theta, \varphi)\rangle_A = \frac{1}{\sqrt{\Omega_1}} \sum_j e^{-\frac{\alpha^2}{2}(j-N_A)^2} e^{i\frac{\pi}{2} j} a_j^\dagger |0\rangle \quad (19)$$

and then it is transferred to another place $B$ with a very high fidelity due to the feature of non-spreading propagation of Bloch electron.

SELF-INTERFERENCE AND REVIVAL OF SPREADING WAVE PACKET

We now turn our attention to the problem of non-spreading propagation of Bloch electronic wave packets beyond the linear dispersion regime. We consider a zero-momentum GWP in an external field with $\phi$ far from $\phi_n$ (or a GWP with small momentum $k_0$ but $\phi = 0$). Because of the nonlinear dispersion relation, such kind of wave packet spreads while its center is moving. It is clear that, when the head of the wave packet catches up with its tail, quantum interference phenomena set in.

In order to demonstrate this phenomenon, numerical simulation is performed for a GWP with $\alpha = 0.3$ and $k_0 = 0.05\pi$ (or a zero-momentum GWP with the external magnetic flux $\phi = k_0 N/2\pi$) in the 100-site ring system. The time evolution of the GWP obtained by numerical simulation is plotted in Fig. 4(a). The interference fringe appears when the GWP spreads, which demonstrates the self-interference phenomenon. The profile of the fringe can be estimated analytically as follows.

Consider a GWP $|\psi(k_0, N_A)\rangle$ at $t = 0$ in the coordinate space. When $\phi = 0$, the Hamiltonian $H[0]$ can be approximately written as

$$H_{eff} = -J \sum_{k, \sigma} \varepsilon_k a_{k, \sigma}^\dagger a_{k, \sigma} \quad (20)$$

for $\varepsilon_k \sim (2 - k^2)$. On the other hand, $|\psi(k_0, N_A)\rangle_{\sigma}$ is also a GWP around $k_0$ in the $k$-space. Then for GWP with small $k_0$ will evolves into a GWP

$$|\Phi(t)\rangle_{\sigma} = A_2 \sum_j e^{i\varphi(j, t)} e^{-\frac{\alpha^2}{2}(j-N_c)^2} |j\rangle_{\sigma} \quad (21)$$

with the spreading width

$$\alpha' = \alpha/\sqrt{1 + 4\alpha^4J^2 t^2}, \quad (22)$$

centered at $N_c = N_A + 2J k_0 t$, where $A_2$ is normalization factor, and

$$\varphi(j, t) = k_0 j + 2J t - J k_0^2 t + J(j - N_c)^2 \alpha^2 \alpha'^2 \quad (23)$$

is the time-dependent phase, i.e., the momentum of the moving GWP. Such kind of wave packet spreads while its center is still moving. Since all the wave functions satisfy the periodic boundary condition $|N + j\rangle = |j\rangle$, at certain instant $t$, there is an overlap between the head and tail parts of the wave packet and then the quantum interference phenomena occurs. In other words, when the GWP spreads over the circumference of the ring, we need to consider the virtual superposition of the “head” and “tail”.

Since the widely spreading GWP can wind the ring many times, the virtual superposition can be considered as the re-normalized wave function

$$\sigma (j | \Phi_{us}(t)\rangle_{\sigma} = \frac{1}{\sqrt{\Omega_{us}}} \sigma (j \pm lN | \Phi(t)\rangle_{\sigma} \quad (24)$$

for $l = 1, 2, \ldots, 3$.

FIG. 4: (color online) The self-interference (a) and quantum revival (c) phenomena of the GWPs obtained by numerical simulation. (b) Plots of the self-interference fringe for 100-site ring obtained by numerical simulation (solid line) and theoretical analysis (circle) at $t_0 = 8\pi/90 J$. (d) Plots of the autocorrelation functions, $|\langle A(t)\rangle|$, for the zero-momentum GWPs with $\alpha = 0.1$ in 100-site ring (circle) and chain (solid line) systems. The unit of time $t$ is $100/J$ in (a), (c), and 1000/J in (d).
eral cases, where $\Delta E$ in general case, we have energy-level spacing between any two eigen states. Then in the theory of SMS, the revival time is $\tau = \frac{1}{2} \frac{\pi}{\alpha} (N + 1)^2$ (27)

Here, $| \langle j | \Phi(\delta t) \rangle \rangle^2$ and

$c = \exp[-\alpha^2 N(j - N_c + N/2)]$ (28)

only provide the modulation to the fringe. The spatial period

$\Delta = \frac{2\pi}{K} = \left| \frac{\pi}{NJ\alpha^2\Delta\delta\tau} \right|$ (29)

characterizes the interference fringe.

In Fig. 4(b), the interference fringe at $\delta \tau = 90/J$ obtained by numerical simulation and the analytical approximate result are plotted. It shows that the theoretical analysis is in agreement with the result of numerical simulation.

Now we consider the special case of zero-momentum GWPs moving along a lattice without magnetic field. In this case, although the dispersion relation for such kind of GWPs is nonlinear, the quantum revival is still possible since the $k^2$-dispersion also meets the condition of SMS [3, 18]. To demonstrate this numerical simulation for the time evolution of a GWP with $\alpha = 0.1$ and $N_A = (N + 1)/2$ in the ring of 100-sites and chain. The results are plotted in Fig. 4(d), which show that the revival time is well in agreement with the analytical estimation.

**THE WAVE PACKET DYNAMICS IN THE OPEN CHAIN**

In usual, a practical Bloch electron system is of open chain, in which the magnetic filed has no longer effect on shape of the GWP. Corresponding to the single-particle spectrum

$\varepsilon_k = -v \cos k$ (32)

where $v = 2J$, the eigenvectors of $H$ are

$| \psi_{k,\sigma} \rangle = a_{k,\sigma}^\dagger |0\rangle = \sum_{i=1}^{N} \sqrt{\frac{2}{N+1}} \sin(ki) |i\rangle \sigma$, (33)
where
\[ k = \frac{\pi l}{N + 1} \] (34)
for \( l = 1, ..., N \) can be regarded as a pseudo-momentum. Nevertheless, the GWP \( |\psi_{\sigma}(k_0, N_A)| \) located at \( N_A \) with momentum \( k_0 \sim \pi/2 \) at \( t = 0 \) will also evolve into
\[ |\Phi_{k,\sigma}(t)\rangle = |\psi_{k,\sigma}(k_0, N_A + vt)\rangle. \] (35)

Since all the eigenvectors satisfy the open boundary condition, we have
\[ |\psi_{k,\sigma}(k_0, N_A - vt)\rangle = -|\psi_{k,\sigma}(k_0, 2N - 2N_A + vt)\rangle \] (36)
for \( N_A - vt > N \). It indicates that the wave packet reflects at the boundaries with “\( \pi \)-phase shift”. Then the wave packet bounds back and forth along the chain as illustrated in Fig. 5(a) schematically.

Similarly, under the transformation
\[ e^{ik_0j\sigma} |j\rangle_\sigma \rightarrow |j\rangle_\sigma, \] (37)
the propagation of a moving GWP with \( k_0 = 2\pi\phi/N \) is equivalent to that of a zero-momentum GWP in the system with extra phase exp\((2\pi\phi/N)\) on the hopping term. Numerical simulation for the time evolution of a GWP with \( \alpha = 0.1 \) and \( k_0 = \pi/2 \) in a chain of \( N = 100 \) is plotted in Fig. 5(b). The autocorrelation functions \( |A(t)| \) are also calculated for \( \alpha = 0.1, 0.3 \) and \( k_0 = 2\pi\phi/N \), \( \phi = 20, 25, 33 \), which are plotted in Fig. 3(c) and 3(d). It shows that a GWP with small \( \alpha \) and momentum \( k_0 = \pi(2n+1)/2 \), can be transferred along the chain without spreading approximately. From the autocorrelation functions for rings and chains, it is easy to find that the period of the revivals of GWP in a ring is approximately the half of that in a chain. This is in agreement with the analytical results that the period is \( \tau = (N + 1)/J \) for the chain and \( \tau = N/(2J) \) for the ring. Comparing the revival times for GWPs with and the linear and nonlinear dispersion relations, we have the conclusion that the former is suitable for implementing the fast QIT in the solid.

**SUMMARY**

In summary, the quantum transmission of a Bloch electron in the one-dimensional lattice is studied by theoretical analysis and numerical simulation. It is found that a zero-momentum GWP can be transferred without spreading approximately if an optimal magnetic flux is applied. This feature can be employed to perform the high-fidelity QIT encoded in the polarization of the Bloch electrons. Meanwhile, beyond such optimal range of the field, the time evolution of the GWP is also investigated in the non-liner dispersion regime. The novel quantum coherence effects found in this paper, such as the wave packet revivals and self-interference, can motivate a feasible protocol based on the practical systems to implement the perfect QIT of Bloch electrons controlled by the external magnetic field.

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