Integrated Wishart bridge processes and generalised Hartman-Watson law

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Abstract

This article is concerned with the joint law of an integrated Wishart bridge process and the trace of an integrated inverse Wishart bridge process over the interval $[0, t]$. Its Laplace transform is obtained by studying the Wishart bridge processes and the absolute continuity property of Wishart laws.

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1 Introduction

Suppose $X$ is a solution to the stochastic differential equation on the cone $\tilde{S}_n^+$ of $n \times n$ symmetric positive semi-definite matrices

$$dX_t = \sqrt{X_t} dW_t a + a^\top dW_t^\top \sqrt{X_t} + (bX_t + X_t b + \alpha a^\top a) \, dt, \quad t \geq 0,$$

(1.1)

where $X_0 = x \in \tilde{S}_n^+$, $W$ is an $n \times n$ matrix-valued Brownian motion, $a$ in the space $GL(n)$ of $n \times n$ invertible matrices, $b$ in the cone $\tilde{S}^-$ of negative semi-definite matrices such that $ab = ba$ and $\alpha \in \{1, 2, \ldots, n-1\} \cup (n-1, \infty)$.

The process $X$ satisfying (1.1), first introduced in Bru (1991), is called a Wishart process of dimension $n$, index $\alpha$ and parameters $a, b$ with initial value $x$ and is denoted $WIS(n, \alpha, a, b, x)$. It was shown in Cuchiero et al. (2011) that the stochastic differential equation (1.1) has a unique weak solution for $\alpha \geq n - 1$ as well as for $\alpha \in \{1, 2, \ldots, n - 1\}$ with the additional condition of rank$(x) \leq \alpha$. For $\alpha \geq n + 1$, Mayerhofer et al. (2011) showed that the solution to (1.1) exists as a strong solution and is unique for $t \geq 0$. Moreover, it was shown in Mayerhofer et al. (2011) that if the initial value $x$ belongs to the space $S_n^+$ of $n \times n$ positive definite matrices, the solution to (1.1) also belongs to $S_n^+$.

Given $\alpha \geq n + 1$ and $x \in \tilde{S}_n^+$, then for $t \geq 0$, the determinant of $X$ satisfies the stochastic differential equation

$$\ln(\det(X_t)) = \int_0^t (\alpha - n - 1) \, Tr(a^\top a X_s^{-1}) + 2 \, Tr(b) \, ds$$

$$+ \int_0^t 2 \, Tr(\sqrt{X_s^{-1}} dW_s a), \quad 0 \leq t < \tau,$$

(1.2)

where $\tau = \inf\{t \geq 0 : \det(X_t) = 0\}$. It was shown in Mayerhofer et al. (2011) Theorem 3.4 that for $\alpha \geq n + 1$ and $x \in S_n^+$, $\tau = \infty$ almost surely.

Given $t \geq 0$, the Laplace transform of $X_t$ can be computed directly from solving the matrix Riccati ordinary differential equation (see Ahdida and Alfonsi (2013) Proposition
4) and is given by

\[
\mathbb{E} e^{-Tr(uX_t)} = \frac{\exp \left\{ Tr \left[ u \left( I - 2\sigma_t u \right)^{-1} e^{bt}Xe^{bt} \right] \right\}}{\det (I - 2\sigma_t u)^{\alpha/2}}, \quad u \in \tilde{S}_n^+,
\]

where \( \sigma_t = \int_0^t e^{bs}a^Ta e^{bT}ds \). Therefore, by comparing the above expression to the Laplace transform of the non-central Wishart random variable computed in Letac and Massam (2008), we deduce that \( X_t \) follows the non-central Wishart distribution with \( \alpha \) degrees of freedom, covariance matrix \( \sigma_t \) and non-centrality matrix \( e^{bt}xe^{bt}\sigma_t^{-1} \), denoted \( \mathcal{W}_n(\alpha, \sigma_t, e^{bt}xe^{bt}\sigma_t^{-1}) \).

We denote the space of \( n \times n \) matrix-valued continuous function defined on \( [0, t] \) by \( \mathcal{C}([0, t], \mathbb{R}^{n \times n}) \), the law of a Wishart process \( X \) on \( \mathcal{C}([0, \infty), \mathbb{R}^{n \times n}) \) and its respective semi-group by \( ^nQ_{x}^{\alpha,a,b} \), or simply \( Q_{x}^{\alpha,a,b} \) when there is no ambiguities about the dimension \( n \). Moreover, we assume \( \Omega = \mathcal{C}([0, \infty), \mathbb{R}^{n \times n}) \), the set of \( \mathbb{R}^{n \times n} \)-valued continuous functions defined on \( [0, \infty) \), and denote \( X \) the coordinate process \( X_t(\omega) = \omega_t \).

For \( \alpha \geq n + 1 \), the Wishart law \( Q_{x}^{\alpha,a,b} \) is absolutely continuous with respect to the parameters \( \alpha \) and \( b \), their respective Cameron-Martin-Girsanov formulae are given as follows:

**Lemma 1.1** (Absolute continuity of Wishart laws). Let \( \alpha \geq n + 1 \), \( t \geq 0 \) and \( Q_{x}^{\alpha,a,b} \) be the law of \( \text{WIS}(n, \alpha, a, b, x) \) on \( \mathcal{C}([0, \infty), \mathbb{R}^{n \times n}) \).

(i) For \( u \in \tilde{S}_n^\alpha \) such that \( ua = au \),

\[
dQ_{x}^{\alpha,a,b+u} = \exp \left\{ Tr \left[ \frac{1}{2}(a^Ta)^{\alpha-1}u (X_t - X_0) \right. \right. \\
- \frac{1}{2}\alpha ut - \int_0^t (a^Ta)^{\alpha-1} (u^2 + bu) X_s ds \left. \right\} dQ_{x}^{\alpha,a,b}.
\] (1.3)
For \( x \in S_n^+ \) and \( \nu \in [(n + 1 - \alpha)/2, \infty) \),

\[
dQ^{\alpha+2\nu,a,b}_x = \left( \frac{\det X_t}{\det x} \right)^{\nu/2} \exp \left\{ -Tr \left[ \nu bt \\ + (\alpha - n - 1 + \nu) \frac{\nu}{2} \int_0^t (a^\top a) X_s^{-1} ds \right] \right\} dQ^{\alpha,a,b}_x. \tag{1.4}
\]

### Main result

This article is concerned with the joint conditional Laplace transform of the pair for

\[
\left( \int_0^t X_s ds, \int_0^t Tr (a^{-1} a X_s^{-1}) ds \right), \tag{1.5}
\]

for a \( WIS(n, \alpha, a, b, x) \) process \( X \) and \( \alpha \geq n + 1 \), given \( X_t \) for a fixed \( t \geq 0 \).

Let us first state the main result of this article,

**Theorem 1.1.** Let \( X \) be a \( WIS(n, \alpha, a, b, x) \) process and \( \alpha \geq n + 1 \), then

\[
\mathbb{E} \left( \exp \left\{ -Tr \left[ u^2 \int_0^t X_s ds \right] - \frac{\lambda^2}{2} Tr \left[ \int_0^t (a^\top a) X_s^{-1} ds \right] \right\} \mid X_t = y \right) = q_t^{\alpha+2\nu,\alpha,\beta+\delta_u}(x, y) \left( \frac{\det y}{\det x} \right)^{-\nu\alpha/2} \exp \left\{ Tr \left[ \nu \lambda bt + \left( \frac{1}{2} (a^\top a)^{-3/2} u^2 \right)^{1/2} (y - x) - \alpha t \right] \right\}, \tag{1.6}
\]

where

\[
u_{\lambda} = \sqrt{\lambda^2 + (\alpha - n - 1)^2 - \alpha + n + 1},
\]

and \( q_t^{\alpha,a,b}(x, y) \) denotes the density of a \( WIS(n, \alpha, a, b, x) \) semi-group.
Formula (1.6) is an extension of that given in Proposition 2.4 of Donati-Martin et al. (2004), where \( a \) is assumed to be the identity matrix \( \text{id} \) and \( b \) is 0. The proof for Theorem 1.1 relies on, as in that of Donati-Martin et al. (2004), the absolute continuity of Wishart law with respect to the dimension parameter \( \alpha \) and the drift parameter \( b \) as well as the law of a Wishart bridge process over \([0, t]\), which will be defined in the next section.

## 2 Wishart bridge processes

A bridge of a Wishart process can be thought of as a Wishart process with its two end points “pinned down” over a fixed time interval. We define the law of a Wishart bridge process as a regular conditional probability measure, analogous to that of a squared Bessel bridge process as defined in Revuz and Yor (1999) Chapter XI.

We denote the space of \( n \times n \) matrix-valued continuous function defined on \( A \subseteq [0, \infty) \) by \( C(A, \mathbb{R}^{n \times n}) \), the law of \( X \) on \( C([0, \infty), \mathbb{R}^{n \times n}) \) and its respective semi-group by \( nQ_{x}^{\alpha,a,b} \), or simply \( Q_{x}^{\alpha,a,b} \) when there is no ambiguities about the dimension \( n \). Throughout this article, we assume \( \Omega = C([0, \infty), \mathbb{R}^{n \times n}) \) and denote \( X \) the coordinate process \( X_t(\omega) = \omega_t \).

For every \( t \geq 0 \), let us consider the space \( \mathbb{W}_t = C([0, t], \mathbb{R}^{n \times n}) \) endowed with the topology generated by the uniform metric \( \rho \) and the Borel \( \sigma \)-algebra \( B(\mathbb{W}_t) \) generated by this topology. Therefore the metric space \( (\mathbb{W}_t, \rho) \) is complete and separable (see Billingsley (1968)). Consequently, there exists a unique regular conditional distribution of \( nQ_{x}^{\alpha,a,b}(\cdot | X_t) \), namely a family of probability measures \( nQ_{x,y,t}^{\alpha,a,b} \) on \( \mathbb{W}_t \) such that for every \( B \in B \),

\[
nQ_{x}^{\alpha,a,b}(B) = \int nQ_{x,y,t}^{\alpha,a,b}(B) \mu_t(dy),
\]

where \( \mu_t \) is the density of \( X_t \) under \( nQ_{x}^{\alpha,a,b} \).

Therefore we can define a Wishart bridge process by specifying its law as follow:

**Definition 2.1.** A continuous process of which law is \( nQ_{x,y,t}^{\alpha,a,b} \) is called an \( n \)-dimensional Wishart Bridge process (with parameters \( \alpha, a, b \)) from \( x \) to \( y \) over \([0, t]\) and is denoted by \( Wis_{t}^{n,a,a,b}(x, y) \).
As for the law of a Wishart process, we simply write $Q_{x,y,t}^{a,a,b}$ for the law of a Wishart bridge process when there is no ambiguities about the dimension. Loosely speaking, the law of a Wishart bridge can be understood in a sense that for every $B \in \mathcal{B}(\mathbb{W}_t)$,

$$Q_{x,y,t}^{a,a,b}(B) = Q_{x}^{a,a,b}(B|X_t = y),$$

where $X$ is the coordinate process.

From the definition of a regular conditional probability (see, for example Ikeda and Watanabe (1989)), we observe that for every $B \in \mathcal{B}(\mathbb{W}_t)$, the map $y \mapsto Q_{x,y,t}^{a,a,b}(B)$ is measurable and for every measurable function $f$ on $\mathbb{W}_t \times \mathbb{R}^{n \times n}$,

$$\int f(\omega, \omega_t)Q_{x}^{a,a,b}(d\omega) = \int \int f(\omega, y)Q_{x,y,t}^{a,a,b}(d\omega) \mu_t(dy). \tag{2.1}$$

Throughout this article, we follow the notation in Revuz and Yor (1999) Chapter III, denoting a semi-group $P_t$ acting on an element $f$ in $C_0(\mathbb{R}^{n \times n}, \mathbb{R})$ by $P_tf$, that is

$$P_tf = \int f(y)P_t(x, dy),$$

where $C_0(\mathbb{R}^{n \times n}, \mathbb{R})$ denotes the set of real-valued continuous functions on $\mathbb{R}^{n \times n}$ vanishing at infinity. And the function $p_t(x, y)$ such that

$$\int f(y)P_t(x, dy) = \int f(y)p_t(x, y)dy,$$

for every Borel measurable function $f$ is called the density of the semi-group $P_t$.

We also make use of the square bracket $P_t[f]$ instead of $P_t(f)$ to avoid confusion with probability measures.

### 2.1 Integrated Wishart bridge processes

Suppose $X$ is a Wishart process with law $Q_{x}^{a,a,b}$, we call the process $Y$ defined by

$$Y_t = \int_0^t X_s ds, \quad t \geq 0,$$
an integrated Wishart process. An explicit formula for the conditional Laplace transform of $Y_t$ given $X_t$ at a fixed $t \geq 0$ was derived in Donati-Martin et al. (2004) for $\alpha \geq n + 1$, $a = \text{id}$ and $b = 0$ using the absolute continuity property of Wishart laws. Similarly, the aforementioned formula can be extended to a more general class of Wishart processes by using the absolute continuity property of Wishart laws.

**Theorem 2.1.** Let $\alpha \geq n + 1$, $a \in GL(n)$ and $b \in \tilde{S}_n^-$ be commutative. Then for $t \geq 0$,

$$Q_{x,y}^b \left[ \exp \left\{ -Tr \left[ (u^2 + bu) \int_0^t (a^\top a)^{-1} X_s ds \right] \right\} \right] = \left\{ \frac{q_{t+u}(x,y)}{q_{t}(x,y)} \exp \left\{ Tr \left[ -\frac{1}{2} u \left( (a^\top a)^{-1} (y - x) - \alpha t \right) \right] \right\} , \ u \in \mathcal{D}, \right.$$  \hspace{1cm} (2.2)

where

$$\mathcal{D} = \left\{ u \in S_n : u + b \in \tilde{S}_n^-, au = ua \right\},$$

and $X$ is the coordinate process, $Q_{x,y}^b$ and $q_t^b$ denotes the $WIS_{t}^{n,\alpha,a,b}(x,y)$ law and the density of a $WIS(n, \alpha, a, b, x)$ semigroup respectively.

**Proof.** For every measurable Borel measurable function $f$, it follows from (2.1) and the Cameron-Martin-Girsanov formula (1.3) that

$$\int Q_{x,y}^b \left[ \exp \left\{ -\int_0^t (a^\top a)^{-1} (u^2 + bu) X_s ds \right\} \right] f(y)q_t^b(x,y)dy = Q_{x}^{b+u} \left[ f(X_t) \right] = \int Q_{x,y}^{b+u} \left[ \exp \left\{ Tr \left[ -\frac{1}{2} u \left( (a^\top a)^{-1} (X_t - x) - \alpha t \right) \right] \right\} f(y)q_{t+u}^b(x,y)dy$$

$$= \int \exp \left\{ Tr \left[ -\frac{1}{2} u \left( (a^\top a)^{-1} (y - x) - \alpha t \right) \right] \right\} f(y)q_{t+u}^b(x,y)dy.$$

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Therefore, we have

\[
Q^b_{x,y} \left[ \exp \left\{ -\text{Tr} \left( \int_0^t \left( u^2 + bu \right) (a^\top a)^{-1} X_s \right) ds \right\} \right] q^b_t(x, y) \\
= \exp \left\{ \text{Tr} \left[ -\frac{1}{2} u \left( (a^\top a)^{-1} (y - x) - \alpha t \right) \right] \right\} q^b_t(x, y),
\]

almost surely.

Replacing \( u^2 + bu \) in Theorem 2.1 with \( u^2 \) and solve

\[
u^2 = (a^\top a)^{-1} (\delta_u^2 + b\delta_u),
\]

for \( \delta_u \), we obtain the followings,

**Corollary 2.1.** Let \( \alpha \geq n + 1, a \in GL(n) \) and \( b \in \tilde{S}_n^- \) be commutative. Then for \( t \geq 0 \),

\[
Q^b_{x,y} \left[ \exp \left\{ -\text{Tr} \left( u^2 \int_0^t X_s ds \right) \right\} \right] \\
= \frac{q_{t+u}^b(x, y)}{q_t^b(x, y)} \exp \left\{ \text{Tr} \left[ \frac{1}{2} (a^\top a)^{-3/2} (u^2 + bu)^{1/2} u (y - x - \alpha t) \right] \right\}, \quad u \in \mathcal{D}, \quad (2.3)
\]

where

\[
\delta_u = \frac{1}{2} \left( -b + \sqrt{b^2 - 4a^\top au^2} \right),
\]

\[
\mathcal{D} = \left\{ u \in S_n : \delta_u + b \in \tilde{S}_n^-, au = ua \right\},
\]

and \( X \) is the coordinate process, \( Q^b_{x,y} \) and \( q_t^b \) denotes the \( \text{WIS}^{n,\alpha,a,b}_t(x, y) \) law and the density of a \( \text{WIS}(n, \alpha, a, b, x) \) semigroup respectively.

In the case of \( \text{WIS}(n, \alpha, I_n, 0, x) \), as considered in Donati-Martin et al. (2004), Corollary 2.1 allows us to find an explicit expression for the Laplace transform of an integrated Wishart bridge process. This extends formula (2.8) of Donati-Martin et al. (2004), where the Laplace transform of the trace of an integrated Wishart bridge process was considered.
We summarise this result in the corollary below, which can also be considered as the matrix extension of formula (2.m) of Pitman and Yor (1982).

**Corollary 2.2.** Let $\alpha \geq n + 1$. For every $u \in \tilde{S}_n^-$,

$$Q_{x,y} \left[ \exp \left\{ -T r \left( u^2 \int_0^t X_s ds \right) \right\} \right] = \frac{q_t^u(x, y)}{q_t^0(x, y)} \exp \left( T r \left[ -\frac{1}{2} (\alpha t + x - y) \right] \right),$$

where $Q_{x,y}$ and $q_t^0(x, y)$ denote the $W I S_t^{n,0,I,0}(x, y)$ law and the density of a $W I S(n, \alpha, I, 0, x)$ semigroup respectively.

### 2.2 Generalised Hartman-Watson law

The generalised Hartman-Watson law of a Wishart process $X$ for $a = \text{id}$ and $b = 0$, namely the conditional distribution of

$$T r \left( \int_0^t X_s^{-1} ds \right),$$

given $X_t$, was studied in Donati-Martin et al. (2004) through its Laplace transform. By using the Wishart bridge processes and absolute continuity property of Wishart laws, the Laplace transform of the generalised Hartman-Watson law given in Donati-Martin et al. (2004) can also be obtained for $a \neq \text{id}$.

As in Theorem 2.1, by the definition of Wishart bridge processes and the Cameron-Martin-Girsanov formula (1.4), we have the followings,

**Theorem 2.2.** Let $\alpha \geq n + 1$, $\nu \in \left( (n + 1 - \alpha) / 2, \infty \right)$ and $t \geq 0$, then

$$Q_t^\alpha_{x,y} \left[ \exp \left\{ - (\alpha - n - 1 + \nu) \frac{1}{2} T r \left[ (a^T a) \int_0^t X_s^{-1} ds \right] \right\} \right] = \frac{q_t^{\alpha + 2\nu}(x, y)}{q_t^{\alpha}(x, y)} \left( \frac{\det y}{\det x} \right)^{-\nu/2} \exp \{ \nu T r(b) t \},$$

where $Q_t^\alpha_{x,y}$ and $q_t^{\alpha}(x, y)$ denote the $W I S_t^{n,\alpha,a,b}(x, y)$ law and the density of a $W I S(n, \alpha, a, b, x)$ semigroup respectively.
Proof. As in the proof of Theorem 2.1 by applying (1.4) and (2.1).

We can therefore compute the Laplace transform of the generalised Hartman-Watson law, which extends Proposition 2.4 of Donati-Martin et al. (2004) to a wider class of Wishart processes.

Corollary 2.3. Let $\alpha \geq n + 1$ and $t \geq 0$, then for every $u \in \mathbb{R}$,

$$Q^\alpha_{x,y} \left[ \exp \left\{ -\frac{u^2}{2} \text{Tr} \left[ (a^\top a) \int_0^t X_s^{-1} ds \right] \right\} \right] = \frac{q^{\alpha + 2\nu_u}_t(x, y)}{q^\alpha_t(x, y)} \left( \frac{\det y}{\det x} \right)^{-\frac{\nu_u}{2}} \exp \{ \nu_u \text{Tr}(b)t \},$$

where

$$\nu_u = \sqrt{u^2 + (\alpha - n - 1)^2} - \alpha + n + 1,$$

$Q^\alpha_{x,y}$ and $q^\alpha_t$ denote the $WIS_{n,\alpha,a,b}^t(x, y)$ law and the density of a $WIS(n, \alpha, a, b, x)$ semigroup respectively.

2.3 Proof of Theorem 1.1

Combining the arguments made in the proofs of Theorem 2.1 and Theorem 2.2, we have the following,

Theorem 2.3. Let $a \in GL(n)$, $b \in \hat{S}_n^-$ be commutative, $\alpha \geq n + 1$. Then for every $u \in \hat{S}_n$ such that $u + b \in \hat{S}_n^-$, $ua = au$ and $\lambda \in \mathbb{R}$,

$$Q^{\alpha,a,b}_{x,y,t} \left[ \exp \left\{ -\text{Tr} \left[ (u^2 + bu) \int_0^t (a^\top a)^{-1} X_s ds \right] - \frac{\lambda^2}{2} \text{Tr} \left[ \int_0^t (a^\top a) X_s^{-1} ds \right] \right\} \right]$$

$$= \frac{q^{\alpha + 2\nu_u,a,b+u}_t(x, y)}{q^{\alpha,a,b}_t(x, y)} \left( \frac{\det y}{\det x} \right)^{-\frac{\nu_u}{2}} \exp \left\{ \text{Tr} \left[ \nu_u bt - \frac{1}{2} u \left( (a^\top a)^{-1} (y - x) - \alpha t \right) \right] \right\},$$

where

$$\nu_u = \sqrt{\lambda^2 + (\alpha - n - 1)^2} - \alpha + n + 1,$$
$Q_{x,y}^b$ and $q_t^b$ denote the $WIS_t^{n,a,a,b}(x,y)$ law and the density of a $WIS(n,\alpha,a,b,x)$ semi-group respectively.

Therefore, Theorem 2.3 can be reformulated to give the joint Laplace transform of the pair

$$
\left(\int_0^t X_s ds, \int_0^t \text{Tr} \left(a^{-1} a X_s^{-1}\right) ds\right),
$$

under the Wishart bridge law.

**Corollary 2.4.** Let $a \in GL(n)$, $b \in \tilde{S}_n^-$ be commutative, $\alpha \geq n + 1$. Then,

$$
Q_{x,y,t}^{\alpha,a,b} \left[ \exp \left\{ -\text{Tr} \left[ u^2 \int_0^t X_s ds \right] - \frac{\lambda^2}{2} \text{Tr} \left[ \int_0^t (a^\top a) X_s^{-1} ds \right] \right\} \right] = 
q_{t}^{\alpha+2\nu, a+b+\delta_u}(x, y) \left( \frac{\det y}{\det x} \right)^{-\nu_{\lambda}/2} \exp \left\{ \text{Tr} \left[ \nu_{\lambda} b t + \left( \frac{1}{2} (a^\top a)^{-3/2} (u^2 + bu)^{1/2} (y - x) - at \right) \right] \right\},
$$

where

$$
\begin{align*}
    u &\in \mathcal{D}, \quad \lambda \in \mathbb{R}, \\
    \delta_u &= \frac{1}{2} \left( -b + \sqrt{b^2 - 4a^\top a u^2} \right), \\
    \mathcal{D} &= \left\{ u \in S_n : \delta_u + b \in \tilde{S}_n^-, au = ua \right\}, \\
    \nu_{\lambda} &= \sqrt{\lambda^2 + (\alpha - n - 1)^2} - \alpha + n + 1,
\end{align*}
$$

$Q_{x,y}^b$ and $q_t^b$ denote the $WIS_t^{n,a,a,b}(x,y)$ law and the density of a $WIS(n,\alpha,a,b,x)$ semi-group respectively.

Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and a Wishart process $X$ defined on it. Then Theorem 1.1 follows from Corollary 2.4 by identifying $\mathbb{E}(\cdot | X_t = y)$ to $\mathbb{E}(\cdot | \sigma(X_t))(\omega_y)$ where $\omega_y \in \{ \omega \in \Omega : X_t(\omega) = y \}$. 

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