A HITCHIN-KOBAYASHI CORRESPONDENCE
FOR COHERENT SYSTEMS
ON RIEMANN SURFACES

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ABSTRACT. A ‘coherent system’ $(E, V)$, consists of a holomorphic bundle plus a linear subspace of its space of holomorphic sections. Based on the usual notion in Geometric Invariant Theory, a notion of slope stability has been defined for such objects (by Le Potier, and also by Raghavendra and Vishwanath). In this paper we show that stability in this sense is equivalent to the existence of solutions to a certain set of gauge theoretic equations. One of the equations is essentially the vortex equation (i.e. the Hermitian-Einstein equation with an additional zeroth order term), and the other is an orthonormality condition on a frame for the subspace $V \subset H^0(E)$. 
§1. Introduction

The Hitchin-Kobayashi correspondence for holomorphic bundles relates the existence of Hermitian-Einstein metrics to the property of (slope) stability. Analogous correspondences are known to hold for a number of so-called augmented holomorphic bundles. In this paper we examine the case of coherent systems over Riemann surfaces.

Introduced in [LeP] and [RV], a coherent system is an ‘augmented bundle’ consisting of a holomorphic bundle together with a linear subspace of its space of holomorphic sections. In order to construct moduli spaces for such objects, a notion of stability is required. As defined by LePotier, and also by Rhagavendra and Vishwanath, the definition involves a real parameter, which we label $\alpha$. Using this notion, moduli spaces of $\alpha$-stable coherent systems are constructed in [KN], [RV], and [LeP]. The constructions given in these references are all based on geometric invariant theory, and relate the above notion of stability to standard notions of stability in GIT. The parameter $\alpha$ corresponds to a choice of linearization for the group action in the GIT setting.

In [BDGW] we introduced a set of geometric equations on coherent systems. These equations, which we called the orthonormal vortex equations, are very similar to the vortex equations defined in [B] on holomorphic pairs and their generalization defined on $k$-pairs in [BeDW]. For a given coherent system $(\mathcal{E}, V)$ the equations determine a metric on $\mathcal{E}$ and also a frame of $k$ linearly independent sections for $V \subset H^0(X, \mathcal{E})$. In addition to the parameter $\alpha$, a second parameter (which we label $\tau$) enters the equations. The two parameters are related by an identity involving numerical invariants of $(\mathcal{E}, V)$, viz. the rank and degree of $\mathcal{E}$ and the dimension of $V$.

By a Hitchin-Kobayashi correspondence for coherent systems, we mean a correspondence between the property $\alpha$-stability, and the existence of solutions to the orthonormal vortex equations. One direction in this correspondence was proven in [BDGW], namely

Theorem A [BDGW, Prop. 2.9]. Suppose that for some suitable choice of parameters $\alpha$ and $\tau$, a coherent system $(\mathcal{E}, V)$ admits a solution to the orthonormal vortex equations. Then $(\mathcal{E}, V)$ is a direct sum of $\alpha$-stable coherent systems. If $r$ is coprime to either $d$ or $k$, and if $\tau$ is generic, then $(\mathcal{E}, V)$ is $\alpha$-stable.

In this paper we prove the converse, namely

Theorem B (3.13). Fix $\alpha > 0$, and let $(\mathcal{E}, V)$ be an $\alpha$-stable coherent system. Then, with a suitably compatible choice for $\tau$, there is a unique smooth solution to the orthonormal vortex equations on $(\mathcal{E}, V)$.

The proof of Theorem A is a relatively minor modification of the proof in [B] of the analogous statement for holomorphic pairs. That, in turn, is based on the original argument of Kobayashi showing that the Hermitian-Einstein condition implies stability. The modifications required are described in [BDGW].

The proof of Theorem B is the main result of this paper and is given in Section 3. Before proceeding to this main result, we need to give some definitions and establish some basic properties of $\alpha$-stability. This is done in Section 2.

§2. Definitions and Basic Properties of Stable Coherent Systems

Definition 2.1. Let $X$ be a Riemann surface and let $E \rightarrow X$ be a fixed smooth...
complex bundle. A coherent system on $E$ is a pair $(\mathcal{E}, V)$ where $\mathcal{E}$ is a holomorphic bundle isomorphic to $E$, and $V$ is a linear subspace of $H^0(X, \mathcal{E})$. We say that $(\mathcal{E}, V)$ is of type $(d, r, k)$ if $\deg(\mathcal{E}) = d$, $\text{rank}(\mathcal{E}) = r$, and $\dim(V) = k$.

The equations we consider are equations for a metric on $E$ and a basis for $V$. We assume that $X$ has a fixed (Kähler) metric. Then, if we denote a Hermitian bundle metric on $E$ by $H$, and let $\{\phi_1, \ldots, \phi_k\}$ be a set of linearly independent holomorphic sections spanning $V$, the orthonormal vortex equations can be written as

$$i\Lambda F_H + \sum_{i=1}^{k} \phi_i \otimes \phi_i^* = \tau I$$

$$< \phi_i, \phi_j > = \alpha I_k$$

In the first equation, $F_H$ is the curvature of the metric connection on $E$, $\Lambda$ denotes the contraction with the Kähler form on $X$, the adjoint in $\phi^*$ is with respect to the metric, $I$ is the identity section of $\text{End}E$, and $\tau$ is a real parameter. The left hand side of the second equation is the $k \times k$ matrix whose i-j entry is the $L^2$ inner product of $\phi_i$ and $\phi_j$ in $\Omega^0(X, E)$. On the right hand side, $I_k$ is the unit $k \times k$ matrix and $\alpha$ is a real parameter. By taking $\int_X \text{Tr}$ of the first equation, we see that $\tau$ and $\alpha$ must be related by

$$d + \alpha k = r\tau \quad (2.2)$$

**Definition 2.2.**

1. Define the subobjects of $(\mathcal{E}, V)$ to be subbundles $\mathcal{E}' \subset \mathcal{E}$ together with subspaces $V' \subset V \cap H^0(X, \mathcal{E}')$.
2. A subpair is proper unless $\mathcal{E}' = 0$ and $V' = 0$, or $\mathcal{E}' = \mathcal{E}$ and $V' = V$.
3. A morphism $u : (\mathcal{E}, V_E) \to (\mathcal{F}, W)$ between two coherent systems consists of a sheaf map $u : \mathcal{E} \to \mathcal{F}$ such that $u(V_E) \subset V_F$, where here $u$ denotes the induced map on the space of global section.

**Definition 2.3.** For a given $\alpha \in \mathbb{R}$, define the $\alpha$-degree of $(\mathcal{E}', V')$ to be

$$\deg_\alpha(\mathcal{E}', V') = \deg(\mathcal{E}') + \alpha \dim(V')$$

The $\alpha$-slope of $(\mathcal{E}', V')$ is then

$$\mu_\alpha(\mathcal{E}', V') = \frac{\deg_\alpha(\mathcal{E}', V')}{\text{rank}(\mathcal{E}')}$$

We say the Coherent Systems $(\mathcal{E}, V)$ is $\alpha$-stable if for all subsystems $(\mathcal{E}', V')$

$$\mu_\alpha(\mathcal{E}', V') < \mu_\alpha(\mathcal{E}, V)$$

If the strong inequality is replaced by a weak one, then we say the coherent system is $\alpha$-semistable.

**Definition 2.4.** A coherent system $(\mathcal{E}, V)$ is decomposable if the bundle decomposes as $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ and $V = V_1 \oplus V_2$, with $V_i \subset H^0(X, \mathcal{E}_i)$ for $i = 1, 2$. That is, a decomposable coherent system splits as $(\mathcal{E}, V) = (\mathcal{E}_1, V_1) \oplus (\mathcal{E}_2, V_2)$.

It follows immediately that
Lemma 2.5. Suppose that a type \((d, r, k)\) coherent system decomposes into coherent systems of type \((d_1, r_1, k_1)\) and \((d_2, r_2, k_2)\), i.e. \((\mathcal{E}, V) = (\mathcal{E}_1, V_1) \oplus (\mathcal{E}_2, V_2)\). Then for any \(\alpha\),
\[
\mu_\alpha(\mathcal{E}, V) = \frac{r_1}{r} \mu_\alpha(\mathcal{E}_1, V_1) + \frac{r_2}{r} \mu_\alpha(\mathcal{E}_2, V_2).
\]

Definition 2.6. A coherent system \((\mathcal{E}, V)\) is called polystable if it is decomposable and decomposes into a sum of \(\alpha\)-stable coherent systems, each with \(\alpha\)-slope \(\mu_\alpha(\mathcal{E}, V)\).

Proposition 2.7. Let \((\mathcal{E}, V)\) be a coherent system of type \((d, r, k)\), and let \((\mathcal{E}', V')\) be a subsystem of type \((d', r', k')\) such that \(\mu_\alpha(\mathcal{E}, V) = \mu_\alpha(\mathcal{E}', V')\). Then either \(r k' = r' k\) and \(\mu(\mathcal{E}') = \mu(\mathcal{E})\),
or
\[
\alpha = \frac{r' d - rd'}{rk' - r' k}.
\]
In particular, if \(r\) is coprime to either \(d\) or \(k\), and \(\alpha\) is not a rational number with denominator of magnitude less than or equal to \(r k\), then all \(\alpha\)-semistable coherent systems are \(\alpha\)-stable.

For convenience, we will say that \(\alpha\) is generic if it is not of the above form, i.e. if it is not a rational number with denominator of magnitude less than or equal to \(r k\). Similarly, values of \(\tau\) which correspond, via (1.5), to generic values of \(\alpha\) will be called generic.

It has been noted by (by Lepotier and by King and Newstead) that under the above definitions, coherent systems do not form a convenient category. In particular, quotients of coherent systems are not necessarily included in the category. (Consider, as an extreme example of this possibility, the quotient of \((\mathcal{E}, V)\) by \((\mathcal{E}, W)\), where \(W \subset V\).) Following King and Newstead, one thus needs to introduce a larger category, which we will denote by \(\mathcal{CS}\). The objects in \(\mathcal{CS}\) are triples \((\mathcal{E}, V, \sigma)\), where \(\mathcal{E}\) is any sheaf, \(V\) is a finite dimensional vector space, and \(\rho : V \otimes \mathcal{O}_X \rightarrow \mathcal{E}\) is a sheaf map (not necessarily injective). A morphism between two such objects, say \((\mathcal{E}, V, \rho)\) and \((\mathcal{F}, W, \sigma)\), consists of a sheaf map \(f : \mathcal{E} \rightarrow \mathcal{F}\) and a linear map \(L : V \rightarrow W\) such that the following diagram commutes
\[
\begin{array}{ccc}
V \otimes \mathcal{O}_X & \xrightarrow{\rho} & \mathcal{E} \\
\downarrow L \otimes 1 & & \downarrow f \\
W \otimes \mathcal{O}_X & \xrightarrow{\sigma} & \mathcal{F}
\end{array}
\]

Then:

1. The coherent systems correspond to the objects \((\mathcal{E}, V, \rho)\) in which \(\mathcal{E}\) is torsion free and \(\rho\) is injective. We will denote the corresponding coherent system by \((\mathcal{E}, V)\), where \(V \subset H^0(X, \mathcal{E})\) is the image \(\rho(V \otimes \mathcal{O}_X)\).
2. The notion of the type of a coherent system extends to objects in \(\mathcal{CS}\).
3. Subobjects of \((\mathcal{E}, V, \rho)\) are objects \((\mathcal{E}', V', \rho')\) with a morphism made up of inclusions \(i_E : \mathcal{E} \rightarrow \mathcal{E}'\) and \(i_V : V \rightarrow V'\),
4. The definition of \(\alpha\)-stability can be extended to objects in \(\mathcal{CS}\), and one can show (cf [KN]) that the \(\alpha\)-semistable objects are in fact coherent systems.
When considering semistable objects, we thus need not distinguish between coherent systems and objects in the larger category $CS$. The larger category will however be needed at some places in the ensuing discussion, especially in the construction of filtrations of coherent systems (cf. Lemma 2.9).

**Lemma 2.8.** Fix $\alpha > 0$, and let $(\mathcal{E}_1, V_1)$ and $(\mathcal{E}_2, V_2)$ be $\alpha$-semistable coherent systems. Let $u : (\mathcal{E}_1, V_1) \rightarrow (\mathcal{E}_2, V_2)$ be a morphism of coherent systems. Suppose that $\mu_\alpha(\mathcal{E}_1, V_1) \geq \mu_\alpha(\mathcal{E}_2, V_2)$ and that in the case of equality, at least one of the coherent systems is $\alpha$-stable.

1. If the induced map $u : V_1 \rightarrow V_2$ is injective, then $u : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ is an isomorphism.
2. If the induced map $u : V_1 \rightarrow V_2$ is the zero map, then $u : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ is the zero map.

**Proof.**

1. Suppose that $u : V_1 \rightarrow H^0(\mathcal{E}_2)$ is injective. Consider the subsytems $(\text{Ker}(u), 0)$ and $(\text{Im}(u), u(V_1))$ of $(\mathcal{E}_1, V_1)$ and $(\mathcal{E}_2, V_2)$ respectively. By the injectivity assumption, we have
   \[0 \rightarrow (\text{Ker}(u), 0) \rightarrow (\mathcal{E}_1, V_1) \rightarrow (\text{Im}(u), u(V_1)) \rightarrow 0.\]
   If $\text{Ker}(u) \neq O$, then the $\alpha$-semistability condition implies
   \[\mu_\alpha(\text{Ker}(u), 0) \leq \mu_\alpha(\mathcal{E}_1, V_1), \tag{2.6}\]
   \[\mu_\alpha(\text{Im}(u), u(V_1)) \leq \mu_\alpha(\mathcal{E}_2, V_2). \tag{2.7}\]
   Moreover, the inequality (2.6) (respectively (2.7)) is strict if $(\mathcal{E}_1, V_1)$, (respectively $(\mathcal{E}_2, V_2)$) is $\alpha$-stable. Also,
   \[r_k \mu_\alpha(\text{Ker}(u), 0) + r_I \mu_\alpha(\text{Im}(u), u(V_1)) = r_1 \mu_\alpha(\mathcal{E}_1, V_1), \tag{2.8}\]
   where $r_k$ and $r_I$ denote the ranks of $\text{Ker}(u)$ and $\text{Im}(u)$ respectively. We thus find that either
   \[\mu_\alpha(\mathcal{E}_1, V_1) > \mu_\alpha(\mathcal{E}_2, V_2) \geq \mu_\alpha(\mathcal{E}_1, V_1),\]
   or
   \[\mu_\alpha(\mathcal{E}_1, V_1) = \mu_\alpha(\mathcal{E}_2, V_2) > \mu_\alpha(\mathcal{E}_1, V_1).\]
   Thus $\text{Ker}(u) = O$ and $u$ is an isomorphism.

2. If $V_1 \subset H^0(\text{Ker}(u))$, then we can consider the subsystems $(\text{Ker}(u), V_1)$ and $(\text{Im}(u), 0)$. Part (2) then follows by similar arguments to the ones above.

**Remark:** It follows from part (1) that, under the assumptions of Lemma 2.8, $u : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ is an isomorphism if and only if the induced map $u : V_1 \rightarrow V_2$ is an isomorphism.

**Lemma 2.9.** *(Harder Narasimhan filtration for unstable coherent systems)* Fix $\alpha > 0$, and let $(\mathcal{E}, \mathcal{V}, \rho)$ be an object in $CS$ which is not $\alpha$-semistable. Then there is a unique filtration by sub-coherent systems
   \[0 = (\mathcal{E}, V_1) \subset (\mathcal{E}, V_2) \subset \cdots \subset (\mathcal{E}, V_n) \subset \cdots \tag{2.9}\]
such that

1. \( E_k = E \),
2. if \((E, V, \rho)\) corresponds to a coherent system (i.e. if \(\rho\) is injective) then \(V_k = V\). Otherwise, \(V_k = \rho(V \otimes \mathcal{O}_X)\)
3. for \(1 \leq i \leq k\), \((E_i/E_{i-1}, V_i/V_{i-1})\) is an \(\alpha\)-semistable sub-system of \((E/E_{i-1}, V/V_{i-1})\),
4. the \(\alpha\)-slopes are ordered such that

\[
\mu_\alpha(E_1, V_1) > \mu_\alpha(E_2/E_1, V_2/V_1) > \ldots > \mu_\alpha(E_k/E_{k-1}, V_k/V_{k-1}) .
\] (2.10)

Proof. As for holomorphic bundles, the result follows from the fact that the slopes of subobjects are bounded above. If \(\mu(E') \leq \mu_{\text{max}}\) for all subbundles, then for any subobject \((E', V', \rho')\) we have

\[
\frac{\text{deg}(E') + \alpha \dim(V')}{\text{rank}(E')} \leq \mu_{\text{max}} + \frac{\alpha k}{r},
\]

where \(k = \dim(V)\) and \(r = \text{rank}(E)\). It follows that there are subobjects of maximal \(\alpha\)-slope, and that amongst these there is one, say \((E_1, V_1, \rho_1)\), of maximal rank. By construction, this maximal subobject is \(\alpha\)-semistable, and thus corresponds to a sub-coherent system, which we denote by \((E_1, V_1)\). Also, by the maximality property of this sub-object, we can assume that \(V_1 = V \cap H^0(E_1)\).

The rest of the filtration is constructed by iterating this construction. \(\square\)

Lemma 2.10. [RV, Cor 1.8] (Seshadri filtration for \(\alpha\)-semistable coherent systems)

Fix \(\alpha\), and let \((E, V)\) be a coherent system which is \(\alpha\)-semistable. Then there is a filtration by sub-coherent systems

\[
0 = (E_0, V_0) \subset (E_1, V_1) \subset \ldots (E_k, V_k) = (E, V)
\] (2.11)

such that

1. \(\mu_\alpha(E_i/E_{i-1}, V_i/V_{i-1}) = \mu_\alpha(E, V)\), and
2. \((E_i/E_{i-1}, V_i/V_{i-1})\) is an \(\alpha\)-stable pair for all \(i\).

Moreover, the isomorphism class of the pair

\[
gr_\alpha(E, V) = \bigoplus(E_i/E_{i-1}, V_i/V_{i-1})
\]

is independent of the filtration.

§3. THE MAIN RESULT

In this section we prove Theorem B. The proof follows the method introduced by Donaldson in [Do], and later used in [H] and in [GP]. Thus we reformulate the vortex equations as minimization criteria for a functional based on a symplectic moment map.

We begin with some background and notation. Let \(E \longrightarrow X\) be the underlying smooth bundle for \(E\), i.e. \(E\) is \(E\) together with a holomorphic structure. Specification of a holomorphic structure is equivalent to a choice of a \(\bar{\partial}\) operator on \(\mathcal{O}^0(E)\).
We denote the set of all such operators by \( C \). A dimension-\( k \) coherent system on \( E \) consists of a pair \((E, V)\) where \( V \) is a \( k \)-dimensional linear subspace of \( H^0(X, E) \). If we use \( k \)-frames in \( H^0(X, E) \) to describe such subspaces, then the configuration space of all such coherent systems can be described as follows. Define:

\[
\mathcal{X}^k = C \times (\Omega^0(E))^k ,
\]

(3.1)

\[
\mathcal{X}_0^k = \{(\partial E, \phi_1, \ldots, \phi_k) \in \mathcal{X}^k : \text{the sections are linearly independent}\} ,
\]

(3.2)

\[
\mathcal{H}^k = \{(\partial E, \phi_1, \ldots, \phi_k) \in \mathcal{X}^k : \partial E(\phi_i) = 0 \text{ for } i = 1, \ldots k\} ,
\]

(3.3)

\[
\mathcal{X}^{CS} = \mathcal{X}_0^k / \text{GL}(k) , \text{ where GL}(k) \text{ acts on } (\Omega^0(E))^k .
\]

(3.4)

The space of dimension-\( k \) coherent systems is then the subvariety of \( \mathcal{X}^{CS} \) given by

\[
\mathcal{H}^{CS} = (\mathcal{X}_0^k \cap \mathcal{H}^k) / \text{GL}(k) .
\]

(3.5)

Suppose now that we fix a Hermitian metric on \( E \). Then we can use this fixed bundle metric to define the unitary gauge group \( G \). Also the spaces \( C \) and \( \Omega^0(E) \) acquire Kähler, and thus symplectic, structures. Denoting the symplectic forms by \( \omega_C \) and \( \omega_0 \) respectively, we can give \( C \times (\Omega^0(E))^k \) the symplectic structure with symplectic form \( \omega = \omega_C + \omega_0 + \cdots + \omega_0 \). The gauge group \( G \) acts by

\[
g(\partial E, \phi_1, \ldots, \phi_k) = (g \circ \partial E \circ g^{-1}, g\phi_1, \ldots, g\phi_k)
\]

and this action preserves \( \omega \). In addition, the action of \( U(k) \) on \( (\Omega^0(E))^k \) is also symplectic, and commutes with the action of \( G \).

**Proposition 3.1.** There are moment maps for the actions of \( G \) and \( U(k) \) on \( \mathcal{X}^k \). These are given, respectively, by

\[
\Psi_G(\partial E, \phi_1, \ldots, \phi_k) = \Lambda F,
\]

\[
\Psi_U(\partial E, \phi_1, \ldots, \phi_k) = -i < \phi_i, \phi_j >
\]

Here \( F \) denotes the curvature of the metric connection determined by \( \partial E \) and the fixed Hermitian metric on \( E \).

Notice that for any non-zero real number \( \alpha \), the union of the \( \text{GL}(k) \)-orbits through the level set \( \Psi^{-1}_U(-i\alpha I_k) \) is exactly \( \mathcal{X}^k_0 \). It follows from this that \( \mathcal{X}^{CS} \) can be described as the symplectic reduction \( \Psi_U^{-1}(-i\alpha I_k) / U(k) \). Similarly, \( \mathcal{H}^{CS} = (\Psi_U^{-1}(-i\alpha I_k) \cap \mathcal{H}^k) / U(k) \).

The actions (on \( \mathcal{X}^k \)) of \( G \) and \( \text{GL}(k) \) commute, and thus the \( G \) action descends to the quotient \( \mathcal{X}^{CS} = \mathcal{X}_0^k / \text{GL}(k) \). The subvariety \( \mathcal{H}^{CS} \) is preserved by this action. Similarly, in the symplectic descriptions, \( G \) acts on \( \Psi_U^{-1}(-i\alpha I_k) / U(k) \) and this action is symplectic with respect to the reduced symplectic structure. Furthermore, there is a moment map for this action given as follows.

\[
\Psi^{CS}_G(\mathcal{E}, V) = \Lambda F - i\Sigma \phi_i \otimes \phi_i
\]

(3.7)

where \( \{\phi_i, \ldots, \phi_k\} \) is any frame for \( V \) such that \( < \phi_i, \phi_j > = \alpha I_k \).

Theorem B can now be reformulated as
Theorem 3.2. If \((E, V)\) is an \(\alpha\)-stable coherent system, then there is a unique smooth solution to
\[
\Psi^{CS}_\Theta(E, V) = -i\tau I
\]
on the \(\mathcal{G}_C\) orbit through \((E, V)\).

Following [Do], we use the moment map (3.7) to define an analog of the functional \(J\) defined in [Do]. As in [Do] we use \(\nu\) to denote the Trace norm on matrices, and set
\[
N(A) = (\int \nu(A)^2)^{1/2}.
\tag{3.8}
\]

Definition 3.3. With \(\tau \operatorname{rank}(E) = \deg(E) + \alpha k\), define \(J: \mathcal{X}^{CS} \to \mathbb{R}\) by
\[
J(E, V) = N(\Psi^{CS}_\Theta(E, V) + i\tau I).
\tag{3.9}
\]

The basic idea of the proof of Theorem 3.2 is to study the restriction of \(J\) to the \(\mathcal{G}_C\)-orbit through \((E, V)\). The main steps involved are

1. to show that a minimizing sequence on this orbit converges, say to \((E_\infty, V_\infty)\),
2. to show that \((E_\infty, V_\infty)\) is on the orbit through \((E, V)\), and
3. to show that \(J = 0\) at this minimizer, i.e. \(J(E_\infty, V_\infty) = 0\).

**Step 1: Convergence of a minimizing sequence.**

Let \(\mathcal{O}(E, V)\) be the \(\mathcal{G}_C\)-orbit through \((E, V)\), and let \(\{(E_n, V_n)\}\) be a minimizing sequence for \(J|_{\mathcal{O}(E, V)}\). Let \((E_n, V_n)\) be represented by \((\overline{\partial}_n, \phi^1_n, \ldots, \phi^k_n)\) in \(\Psi_U^{-1}(-i\alpha I_k)\). Then for each \(n\) we have
\[
J(E_n, V_n) = N(\Lambda F_n - i\Sigma \phi^1_n \otimes \phi^1_n^* + i\tau I),
\tag{3.10a}
\]
\[
< \phi^1_n, \phi^1_n^* > = \alpha I_k,
\tag{3.10b}
\]
where the notation \(F_n\) refers to the curvature of metric connection determined by the fixed Hermitian metric on \(E\) and the holomorphic structure \(\overline{\partial}_n\).

Proposition 3.4. There exist \((\overline{\partial}_\infty, \phi^1_\infty, \ldots, \phi^k_\infty)\) in \(L^2_1(C \times (\Omega^0(E))^k)\) such that after passing to a subsequence, and up to equivalence under \(\mathcal{G} \times U(k)\),

1. \(\overline{\partial}_n \to \overline{\partial}_\infty\) in \(L^2_1\),
2. \(\phi^1_n \to \phi^1_\infty\) in \(L^2_1\).

(Here we may assume that the unitary connection corresponding to \(\overline{\partial}_0\) is used to define the \(L^2_1\) norm.) Furthermore, \(\phi^1_\infty \neq 0\) and \(\overline{\partial}_\infty \phi^1_\infty = 0\) for \(i = 1, \ldots, k\). Also, \(< \phi^1_\infty, \phi^1_\infty > = \alpha I_k\).

**Proof.** The norm used in the definition of \(J\) is equivalent to the usual \(L^2\) norm. We may thus assume that there is a uniform bound on \(||\Lambda F_n - i\Sigma \phi^1_n \otimes \phi^1_n^* + i\tau I||_{L^2_2}\).

But this is equivalent to a uniform bound on
\[
\mathcal{YMH}_\tau((\overline{\partial}_n, \phi^1_n, \ldots, \phi^k_n)) = ||F_n||_{L^2_2} + \Sigma ||D_n \phi^1_n||_{L^2_2} + ||\Sigma \phi^1_n \otimes \phi^1_n^* - \tau I||_{L^2_2},
\]
where \(F_n = D^2_n\), i.e. \(D_n\) is the metric connection determined by the fixed Hermitian metric on \(E\) and the holomorphic structure \(\overline{\partial}_n\). Indeed, since \(\overline{\partial}_n \phi^1_n = 0\) for all \(i\), the two functionals differ only by a constant determined by the topology of the sphere.
bundle $E$ (cf. [B]). Thus there are uniform bounds on each of $||F_n||^2_{L^2}$, $||D_n\phi^n_i||^2_{\frac{1}{2}}$ and $||\Sigma\phi^n_i \otimes \phi^{n*}_i - \tau I||^2_{L^2}$.

Since the base manifold $X$ is a Riemann surface, it follows by a theorem of Uhlenbeck (cf. [U]) that the bound on $||F_n||^2_{L^2}$ is enough to ensure the weak convergence in $L^2$ norm of the $D_n$. Since $\overline{\partial}_n$ is the antiholomorphic part of $D_n$, this in turn leads to the weak convergence of $\overline{\partial}_n$.

Since $||\phi^n_i||^2_{L^2} = \alpha$ for all $n$ and $i$, we may assume that for fixed $i$, $\{\phi^n_i\}$ converges weakly in $L^2$. Furthermore, by using the bound on $||D_n\phi^n_i||^2_{L^2}$ and the convergence of $\overline{\partial}_n$, we obtain a bound on $||D_0\phi^n_i||^2_{L^2}$. The weak convergence of $\{\phi^n_i\}$ may thus be taken to be in $L^2$. We will denote the limit point by $\{\overline{\partial}_\infty, \phi_1^\infty, \ldots, \phi_k^\infty\}$.

By construction, we have $\overline{\partial}_n\phi^n_i = 0$ for each $n$ and $i$. Thus $\overline{\partial}_\infty\phi_i^\infty = (\overline{\partial}_\infty - \overline{\partial}_n)(\phi_i^\infty + (\overline{\partial}_n - \overline{\partial}_0)(\phi_i^\infty - \phi^n_i) + \overline{\partial}_0(\phi^n_i - \phi^n_i)$.

Defining $x_n$, $y_n \in \Omega^{0,1}(\text{End}(E))$ by $\overline{\partial}_\infty - \overline{\partial}_n = x_n$ and $\overline{\partial}_n - \overline{\partial}_0 = y_n$, we get $||\overline{\partial}_\infty\phi_i^\infty||_{L^2} \leq ||x_n||_{L^4}||\phi_i^\infty||_{L^4} + ||y_n||_{L^4}||\phi_i^\infty - \phi^n_i||_{L^4} + ||\phi_i^\infty - \phi_i||_{L^2}$.

But, since $L^2 \subset L^4$ is compact, we can assume $||x_n||_{L^4} \longrightarrow 0$, $||\phi_i^\infty - \phi_i^n||_{L^4} \longrightarrow 0$, and also that $||y_n||_{L^4}$ is bounded. Thus $\overline{\partial}_\infty\phi_i^\infty = 0$.

Examining $<\phi_i, \phi_j>$, we find $| <\phi_i^\infty, \phi_j^\infty> - \alpha I_k | \leq | <\phi_i^\infty, \phi_j^\infty> - <\phi_i^n, \phi_j^n> | + | <\phi_i^n, \phi_j^n> - \alpha I_k | \leq | <\phi_i^\infty - \phi_i^n, \phi_j> - <\phi_i^n, \phi_j^\infty - \phi_j^n> | \leq \sum_{i,j} ||\phi_i^\infty - \phi_i^n||_{L^2}||\phi_j||_{L^2} + \sum_{i,j} ||\phi_j^\infty - \phi_j^n||_{L^2}||\phi_j||_{L^2} \leq k \alpha \sum_{i} ||\phi_i^\infty - \phi_i^n||_{L^2}$

Thus $<\phi_i^\infty, \phi_j^\infty> = \alpha I_k$. In particular, $\phi_i^\infty \neq 0$ for all $i$.

\begin{flushright}
$\square$
\end{flushright}

**Definition 3.5.** Let $E_\infty$ be the holomorphic bundle determined by $\overline{\partial}_\infty$, and let $V_\infty$ be the subspace of $H^0(X, E_\infty)$ spanned by $\{\phi_1^\infty, \ldots, \phi_k^\infty\}$. Notice that because of (c), the dimension of $V_\infty$ is $k$. Thus $\{(\overline{\partial}_\infty, \phi_1^\infty, \ldots, \phi_k^\infty)\}$ defines a coherent system $(E_\infty, V_\infty)$ in $\mathcal{H}^{CS}$.

**Step 2: the minimizer is on the orbit.**

We now show that if $(E, V)$ is $\alpha$-stable, then $(E_\infty, V_\infty)$ is on the same $\mathfrak{G}_C$-orbit as $(E, V)$. We do this by first showing that there is a non-trivial homomorphism $h : (E, V) \longrightarrow (E_\infty, V_\infty)$, and then proving that $h$ must be an isomorphism if $(E, V)$ is $\alpha$-stable.

Since all the $(\overline{\partial}_n, \phi^n_1, \ldots, \phi^n_k)$ in the minimizing sequence lie on the same $\mathfrak{G}_C \times GL(k)$-orbit in $\mathfrak{A}^k$, for each $n$ we can find $g_n \in \mathfrak{G}_C$ and $A^{(n)} \in GL(k)$ such that $\overline{\partial}_n = g_n(\overline{\partial}_0)$ and $A^{(n)}_{ij}\phi^n_j = g_n(\phi^n_i)$. Let $\tilde{g}_n = g_n/||g_n||_{L^2}$, and set $\tilde{A}^{(n)} = A^{(n)}/||A^{(n)}||$. Since $\{\tilde{A}^{(n)}\}$ is a bounded sequence of $k \times k$ matrices, it has a convergent subsequence. Say $\tilde{A}^{(n)} \longrightarrow \tilde{A}^\infty$. 


Lemma 3.6. The sequence \( \{\tilde{g}_n\} \) has a subsequence which converges in \( L^2_1 \), say

\[
\tilde{g}_n \to \tilde{g}_\infty
\]

Furthermore, \( \tilde{g}_\infty \neq 0 \), \( \overline{\partial}_\infty \circ \tilde{g}_\infty = \tilde{g}_\infty \circ \overline{\partial}_0 \), and \( \tilde{g}_\infty(V) \subset V_\infty \). In particular, \( \tilde{g}_\infty \) gives a nontrivial homomorphism from \( (E, V) \) to \( (E_\infty, V_\infty) \).

Proof. The following argument showing the convergence of \( \{\tilde{g}_n\} \) can be found in [H]. Using the operators \( \overline{\partial}_n \) and \( \overline{\partial}_0 \) we define an elliptic operator

\[
\overline{\partial}_{n,0} : \Omega^0(X, E \otimes E^*) \to \Omega^{0,1}(X, E \otimes E^*) .
\]

We can write \( \overline{\partial}_{n,0} = \overline{\partial}_{\infty,0} + \beta_n \), with \( \beta_n \to 0 \) in \( L^2_1 \). From the fact that

\[
\overline{\partial}_{n,0}(\tilde{g}_n) = \overline{\partial}_{n,0}(g_n) = 0 ,
\]

we get

\[
||\overline{\partial}_{\infty,0}(\tilde{g}_n)||_{L^2} \leq ||\beta_n||_{L^4}||\tilde{g}_n||_{L^4} . \tag{3.11}
\]

Thus, elliptic estimates give

\[
||\tilde{g}_n||_{L^2_1} \leq C(||\beta_n||_{L^4}||\tilde{g}_n||_{L^4} + ||\tilde{g}_n||_{L^2}) . \tag{3.12}
\]

But \( ||\tilde{g}_n||_{L^2_1} = 1 \). Also, since since \( L^2_1 \subset L^4_0 \) is compact, we have \( ||\beta_n||_{L^4} \to 0 \) and thus (3.12) gives a uniform bound on \( ||\tilde{g}_n||_{L^2_1} \). After renaming a subsequence, we thus get \( \tilde{g}_n \to \tilde{g}_\infty \) in \( L^2_1 \). Since \( ||\tilde{g}_n||_{L^2} = 1 \), we can conclude that \( \tilde{g}_\infty \neq 0 \). It now follows from (3.11) that \( \overline{\partial}_{\infty,0}(\tilde{g}_\infty) = 0 \), i.e. \( \overline{\partial}_{\infty} \circ \tilde{g}_\infty = \tilde{g}_\infty \circ \overline{\partial}_0 \).

Finally, if we set \( c_n = |A^{(n)}|/||g_n||_{L^2} \), we can write

\[
\tilde{g}_n(\phi_i^0) = c_n A_{ij}^{(n)} \phi_j^n .
\]

Thus for each \( n \), \( \tilde{g}_n(\phi_i^0) \) is in the space spanned by \( \{\phi_j^n\}^{j=1}_{j=1} \). If we define

\[
d_n = \sum_{p=1}^{k} \frac{|<\tilde{g}_n(\phi_i^0), \phi_p^n>|^2}{||\phi_p^n||^2} - ||\tilde{g}_n(\phi_i^0)||^2 ,
\]

so \( d_n \) is the distance from \( \tilde{g}_n(\phi_i^0) \) to the space spanned by \( \{\phi_j^n\}^{j=1}_{j=1} \), then clearly \( d_n = 0 \) for all \( n \). Thus, using the convergence of the \( \phi_i^n \) and the \( \tilde{g}_n \), we get

\[
0 = \lim_{n \to \infty} d_n = \sum_{p=1}^{k} \frac{|<\tilde{g}_\infty(\phi_i^0), \phi_p^{inf,ty}>|^2}{||\phi_p^{inf,ty}||^2} - ||\tilde{g}_\infty(\phi_i^0)||^2 .
\]

That is, \( \tilde{g}_\infty(\phi_i^0) \) is in the space spanned by \( \{\phi_j^{inf,ty}\}^{j=1}_{j=1} \).

\[\square\]

It remains to establish that any such homomorphism is an isomorphism. As in [Do], the proof in the general case will involve induction on the rank of the bundle. We first observe that the result is true for rank one bundles.
Lemma 3.7. If the rank of \( E \) is one, then \( \tilde{g}_\infty \) is a constant multiple of the identity. In particular, \( \tilde{g}_\infty : (\mathcal{E}, V) \rightarrow (\mathcal{E}_\infty, V_\infty) \) is an isomorphism.

Proof. Since \( \mathcal{E} \) and \( \mathcal{E}_\infty \) are line bundles of equal degree, any non trivial homomorphism \( \tilde{g}_\infty : \mathcal{E} \rightarrow \mathcal{E}_\infty \) is a constant multiple of the identity. \( \square \)

Suppose now that \( \text{rank}(E) > 1 \) but that \( \tilde{g}_\infty : (\mathcal{E}, V) \rightarrow (\mathcal{E}_\infty, V_\infty) \) is not an isomorphism. Since \( \tilde{g}_\infty(V) \subset V_\infty \), the canonical factorization of \( \tilde{g}_\infty : \mathcal{E} \rightarrow \mathcal{E}_\infty \) (cf. [Do]) gives a factorization

\[
\begin{array}{cccccc}
0 & \longrightarrow & (\mathcal{K}, V_K) & \longrightarrow & (\mathcal{E}, V) & \longrightarrow & (\mathcal{I}, V/V_k) & \longrightarrow & 0 \\
& & \tilde{g}_\infty & & \downarrow & & \downarrow & & \\
0 & \longleftarrow & (\mathcal{Q}, V_Q) & \longleftarrow & (\mathcal{E}_\infty, V_\infty) & \longleftarrow & (\tilde{I}, \tilde{g}_\infty(V)) & \longleftarrow & 0 \\
\end{array}
\]

(3.13)

In this diagram, \( \text{rank}(\mathcal{I}) = \text{rank}(\tilde{I}) \), \( \text{dim}(V/V_k) = \text{dim}(\tilde{g}_\infty(V)) \), \( \text{deg}(\mathcal{I}) \leq \text{deg}(\tilde{I}) \), while \( \text{rank}(\mathcal{E}) = \text{rank}(\mathcal{E}_\infty) \), \( \text{dim}(V) = \text{dim}(V_\infty) \), and \( \text{deg}(\mathcal{E}) = \text{deg}(\mathcal{E}_\infty) \). Furthermore, since \( (\mathcal{E}, V) \) is \( \alpha \)-stable, we have \( \mu_\alpha(\mathcal{K}, V_K) < \mu_\alpha(\mathcal{E}, V) < \mu_\alpha(\mathcal{I}, V/V_k) \).

Strictly speaking, the quotients \( (\mathcal{I}, V/V_k) \) and \( (\mathcal{Q}, V_Q) \) may not be coherent systems, and the above factorization makes sense only in the category \( \mathcal{C}S \). In fact, \( (\mathcal{I}, V/V_k) \) is a coherent system, i.e. the sheaf map in the corresponding object in \( \mathcal{C}S \) is indeed injective. (This is because \( V_K = V \cap H^0(X, \mathcal{K}) \).)

The only abuse of notation is thus in the term \( (\mathcal{Q}, V_Q) \). We can tolerate this since we will not make use of this term in what follows.

We now need the following Lemmas, which are the analogs of Lemmas 2 and 3 in [Do].

Lemma 3.8. Suppose that \( \text{rank}(\mathcal{E}) > 1 \), and that \( (\mathcal{E}, V) \) has a subsystem \( (\mathcal{E}_1, V_1) \) with \( \mu_\alpha(\mathcal{E}_1, V_1) \geq \mu_\alpha(\mathcal{E}, V) \). Set \( \mathcal{E}_2 = \mathcal{E}/\mathcal{E}_1 \) and \( V_2 = V/V_1 \), and let

\[
\mu_\alpha(\mathcal{E}_2, V_2) = \mu(\mathcal{E}_2) + \alpha \frac{\text{dim}(V_2)}{\text{rank}(\mathcal{E}_2)}.
\]

Let \( \{\phi_1, \ldots, \phi_k\} \) be any basis for \( V \) such that the first \( k_1 \) sections form a basis for \( V_1 \).

Then

\[
J(\tilde{\partial}_E, \phi_1, \ldots, \phi_k) \geq R_1(\mu_\alpha(\mathcal{E}_1, V_1) - \mu_\alpha(\mathcal{E}, V)) + R_2(\mu_\alpha(\mathcal{E}, V) - \mu_\alpha(\mathcal{E}_2, V_2)) \quad (3.14)
\]

Proof. 

Let \( (\mathcal{E}_1, V_1) \) be a subsystem of \( (\mathcal{E}, V) \). Let \( \{\phi_1, \ldots, \phi_k\} \) be an orthonormal frame for \( V \) such that \( V' \) is spanned by the first \( k_1 \) elements, \( \{\phi_1, \ldots, \phi_{k_1}\} \). With respect to the smooth splitting \( E = E_1 \oplus (E/E_1) \), we get a decomposition of each section into its components in \( E_1 \) and \( E_2 = (E/E_1) \), i.e.

\[
\phi_i = \phi'_i + \phi'^*_i.
\]

The expression \( \Sigma \phi_i \otimes \phi^*_i \) thus has a block decomposition as

\[
\Sigma \phi_i \otimes \phi^*_i = \left( \sum_{i=1}^{k_1} \phi_i \otimes \phi^*_i + \sum_{i=k_1+1}^k \phi'_i \otimes (\phi'_i)^* \right) \oplus \left( \sum_{i=k_1+1}^k \phi'^*_i \otimes (\phi'^*_i)^* \right)
\]

(3.15)
Notice that we can write
\[ \int \{ \text{Tr} \left( \sum_{i=1}^{k_1} \phi_i \otimes \phi_i^* + \sum_{i=k_1+1}^{k} \phi_i' \otimes (\phi_i')^* \right) \} = k_1 + \int t^2, \]
where \( t^2 \) is a non-negative real number. It follows that
\[ \int \{ \text{Tr} \left( \sum_{i=k_1+1}^{k} \phi_i^\perp \otimes (\phi_i^\perp)^* \right) \} = (k - k_1) - \int t^2. \]
Recall also the block decomposition of \( i \Lambda F_H \), viz.
\[ i \Lambda F_H = \begin{pmatrix} i \Lambda F_{H_1} + \Pi & * \\ * & i \Lambda F_{H_2} - \Pi \end{pmatrix}, \]
where \( \Pi \) is a positive definite endomorphism coming from the second fundamental form of the inclusion of \( E_1 \) in \( E \).

By exactly the same argument as in Lemma 2 of [Do], we thus get
\[ \nu(i \Lambda F_H + \Sigma \phi \otimes \phi^* - 2 I) \geq |\text{Tr}(i \Lambda F_{H_1} + \sum_{i=1}^{k_1} \phi_i \otimes \phi_i^* - \tau I_1 + \text{Tr} \Pi + t^2)| \]
\[ + |\text{Tr}(i \Lambda F_{H_2} + \sum_{i=K_{k_1}+1}^{k} \phi_i^\perp \otimes (\phi_i^\perp)^* - \tau I_1 - \text{Tr} \Pi)|, \]
But \( \tau = \mu_\alpha(\mathcal{E}, V) \) and by assumption \( \mu_\alpha(\mathcal{E}_1, V_1) \geq \mu_\alpha(\mathcal{E}, V) \geq \mu_\alpha(\mathcal{E}_2, V_2) \). Thus we get
\[ J(\overline{\partial}_E, \phi_1, \ldots, \phi_k) \geq R_1(\mu_\alpha(\mathcal{E}_1, V_1) - \tau) + R_2(\mu_\alpha(\mathcal{E}, V) - \tau). \]

**Lemma 3.9.** Let \((\mathcal{E}, V)\) be an \( \alpha \)-stable coherent system, with \( \text{rank}(\mathcal{E}) > 1 \). Suppose that \((\mathcal{E}, V)\) is given as an extension of coherent systems
\[ 0 \to (\mathcal{E}_1, V_1) \to (\mathcal{E}, V) \to (\mathcal{E}_2, V_2) \to 0 \]
and suppose that Theorem 3.2 is true for coherent systems on bundles of lower rank than \( \mathcal{E} \). Let \( \{\phi_1, \ldots, \phi_k\} \) be any basis for \( V \) such that the first \( k_1 \) sections form a basis for \( V_1 \). Then
\[ J(\overline{\partial}_E, \phi'_1, \ldots, \phi'_k) \leq R_1(\mu_\alpha(\mathcal{E}, V) - \mu_\alpha(\mathcal{E}_1, V_1)) + R_2(\mu_\alpha(\mathcal{E}_2, V_2) - \mu_\alpha(\mathcal{E}, V)), \] (3.15)
for some \((\overline{\partial}_E, \phi'_1, \ldots, \phi'_k)\) on the \( G_\mathbb{C} \times \text{GL}(k, \mathbb{C}) \) orbit through \((\overline{\partial}_E, \phi_1, \ldots, \phi_k)\).

**Proof.** Suppose first that \((\mathcal{E}_1, V_1)\) and \((\mathcal{E}_2, V_2)\) are both \( \alpha \)-stable. Let \( \overline{\partial}_i \) denote the holomorphic structures on \( \mathcal{E}_i \). By our inductive hypothesis, and possibly after a complex gauge transformation of \( \mathcal{E}_1 \), we can pick \( \overline{\partial}_1 \) and \( \phi_1, \ldots, \phi_{k_1} \) such that
\[ i \Lambda F_1 + \sum_{i=1}^{k_1} \phi_i \otimes \phi_i^* = \tau_1 I_1 \] (3.16a)
\[ \phi_1, \ldots, \phi_{k_1} \geq 0 I_1. \] (3.16b)
Similarly, after a complex gauge transformation on $E_2$, we can pick $\overline{\partial}_2$ and a basis $\{\rho_1, \ldots, \rho_{k_2}\}$ for $V_2$ such that

$$i\Lambda F_2 + \Sigma_{i=1}^{k_2} \rho_i \otimes \rho^*_i = \tau_2 I_2 \quad (3.17a)$$
$$< \rho_i, \rho_j >= \alpha I_{k_2} \quad (3.17b)$$

The gauge transformations on $E_1$ and $E_2$ combine to produce a gauge transformation taking $E$ to an isomorphic holomorphic bundle, which we again relabel as $\mathcal{E}$. The $\overline{\partial}$-operator corresponding to the holomorphic structure on $\mathcal{E}$ then has a block decomposition as

$$\overline{\partial}_E = \begin{pmatrix} \overline{\partial}_1 & \beta \\ 0 & \overline{\partial}_2 \end{pmatrix},$$

where $\beta \in \Omega^{0,1}(Hom(E_2, E_1))$ is a representative of the extension class. We obtain a properly normalized basis for $V$ as follows. We fix a real number $0 < \lambda < 1$, and pick smooth sections $\sigma_i \in \Omega^0(X, E_1)$ such that $\tilde{\phi}_i = \sigma_i + \lambda \rho_i$ is a lift of $\lambda \rho_i$ to $\mathcal{E}$, i.e. $\tilde{\phi}_i \in H^0(X, \mathcal{E})$. Furthermore, the $\sigma_i$ can be chosen such that

$$< \phi_i, \sigma_j > = 0 \quad (3.18a)$$
$$< \sigma_i, \sigma_j > = \alpha(1 - \lambda^2)I_{k_1} \quad (3.18b)$$

Then $\{\phi_1, \ldots, \phi_{k_1}, \tilde{\phi}_1, \ldots, \tilde{\phi}_{k_2}\}$ is an orthogonal basis the basis for $V$, with all vectors of length $\sqrt{\alpha}$. Using $\overline{\partial}_E$ and this basis for $V$, we thus get the block decomposition

$$i\Lambda F + \Sigma_{i=1}^{k_1} \tilde{\phi}_i \otimes \tilde{\phi}^*_i$$
$$= \begin{pmatrix} \tau_1 I_1 - i\Lambda \beta \wedge \beta^* + \Sigma_{i=1}^{k_2} \sigma_i \otimes \sigma^*_i & B \\ \tau_2 I_2 - i\Lambda \beta^* \wedge \beta + (\lambda^2 - 1)\Sigma_{i=1}^{k_2} \rho_i \otimes \rho^*_i \end{pmatrix},$$

where

$$B = \overline{\partial}_{1,2}^* \beta + \lambda \Sigma_{i,j} \sigma_i \otimes \tilde{\rho}^*_j.$$

After a gauge transformation of the form $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, the term $B$ changes to

$$B^{(u)} = B + \Delta(u) + u\lambda^2 \Sigma_{i=k_1+1}^{k_2} \rho_i \otimes \rho^*_i,$$

where $\Delta$ is the “$\overline{\partial}_{1,2}$-Laplacian”. In fact, the endomorphism $u \in \Omega^0(Hom(E_2, E_1))$ can be chosen so that $B^{(u)} = 0$. This can be seen as follows: The operator $\Delta + \lambda^2 \Sigma_{i=k_1+1}^{k_2} \rho_i \otimes \rho^*_i$ is elliptic, and its kernel can be identified with those morphisms $u : (E_2, V_2) \to (E_2, V_2)$ such that $u(V_2) = 0$. In view of our assumptions on $(E_2, V_2)$ and $(E_2, V_2)$, it follows by Lemma 2.8 that this kernel is trivial. The equation $B^{(u)} = 0$ can thus be solved for $u$.

Suppose then that $B = 0$. The holomorphic structure on $\mathcal{E}$ is thus determined by $(\overline{\partial}_1, \overline{\partial}_2, \beta)$. Now consider the 1-parameter family $\{ (g_s, A_s) \} \in \text{GL}(k, \mathbb{C})$, where

$$g_s = \begin{pmatrix} 1 & 0 \\ 0 & s^{-1} \end{pmatrix}, A_s = \begin{pmatrix} 1 & 0 \\ 0 & \gamma(s) \end{pmatrix}, \text{and } \gamma(s) = \frac{s}{\sqrt{(1 - \lambda^2) + s^2 - 1}}.$$
This generates a 1-parameter family, say \((E_s, V_s)\), in the \(\mathfrak{G}_C(E) \times GL(k, \mathbb{C})\) orbit through \((E, V)\). The holomorphic structure of \(E_s\) is given by \((\bar{\partial}_1, \bar{\partial}_2, s\beta)\) and a basis for \(V_s\) is given by \(\{\phi_1, \ldots, \phi_{k_1}, \tilde{\phi}_1(s), \ldots, \tilde{\phi}_{k_2}(s)\}\), where

\[
\tilde{\phi}_{k_2}(s) = \gamma(s)\sigma_i + \frac{\lambda}{s}\gamma(s)\rho_i.
\]

Observe that \(\{\phi_1, \ldots, \phi_{k_1}, \tilde{\phi}_1(s), \ldots, \tilde{\phi}_{k_2}(s)\}\) is an orthogonal basis in which all vectors have length \(\sqrt{\alpha}\). Also, in the limit \(s \to 0\), we have \(\gamma(s) \to 0\) and \(\frac{s}{\gamma(s)} \to 1\). We thus get

\[
i\Psi^{CS}(E_s, V_s) - \tau I = \begin{pmatrix} (\tau_1 - \tau)I_1 + \epsilon_1(s) & 0 \\ 0 & (\tau_2 - \tau)I_2 - \epsilon_2(s) \end{pmatrix},
\]

where

\[
\epsilon_1(s) = -s^2i\lambda\beta \wedge \beta^* + \gamma(s)^2\Sigma_{i=1}^{k_2}\sigma_i \otimes \sigma_i^*,
\]

\[
\epsilon_2(s) = s^2i\Lambda\beta^* \wedge \beta + (1 - \frac{\lambda^2}{s^2}\gamma(s)^2)\Sigma_{i=1}^{k_2}\rho_i \otimes \rho_i^*.
\]

Clearly \(\epsilon_1(s) \to 0\), \(\epsilon_2(s) \to 0\) as \(s \to 0\). Also, since \((E, V)\) is \(\alpha\)-stable, we have \(\tau_1 < \tau < \tau_2\). The proof of Lemma 3 in [Do] is now easily adapted to show

\[
J(E_s, V_s) = J_1 - 2s^2|\beta|^2 - k(1 - \lambda^2)\gamma(s)^2\alpha,
\]

where

\[
J_1 = R_1(\mu_\alpha(E, V) - \mu_\alpha(E_1, V_1)) + R_2(\mu_\alpha(E_2, V_2) - \mu_\alpha(E, V)).
\]

Thus for small enough \(s\), \(J(E_s, V_s) < J_1\), as required.

If \((E_1, V_1)\) and \((E_2, V_2)\) are not \(\alpha\)-stable, then by Lemmas 2.9 and 2.10 they have filtrations by \(\alpha\)-stable coherent systems. Exactly as in [Do], the above argument can be modified to take this into account. This completes the proof of Lemma 3.9.

We now apply Lemmas 3.8 and 3.9 to the exact sequences in (3.13). Reasoning as in [Do], we get

\[
J(E, V) < R_K(\mu_\alpha(E, V) - \mu_\alpha(K, V_K)) + R_I(\mu_\alpha(I, V/V_K) - \mu_\alpha(E, V))
\]

\[
\leq R_Q(\mu_\alpha(E_\infty, V_\infty) - \mu_\alpha(Q, V_Q)) + R_I(\mu_\alpha(\tilde{I}, g_\infty(V)) - \mu_\alpha(E_\infty, V_\infty))
\]

\[
\leq J(E_\infty, V_\infty).
\]

This is impossible, since \((E_\infty, V_\infty)\) is minimizing for \(J|_O(E, V)\). We have thus proven

**Proposition 3.10.** If \((E, V)\) is an \(\alpha\)-stable coherent system, then \((E_\infty, V_\infty)\) is on the same \(\mathfrak{G}_C\)-orbit as \((E, V)\).
Finally, we must show that by minimizing $J|_{\mathcal{O}(\mathcal{E},V)}$ we obtain a unique solution to the orthonormal vortex equations. Here it is convenient to use the smooth function $||\Psi^CS_{\mathfrak{g}}(\mathcal{E},V) + i\tau I||^2_{L^2}$, rather than the function $J(\mathcal{E},V) = N(\Psi^CS_{\mathfrak{g}}(\mathcal{E},V) + i\tau I)$. Because of the equivalence of the usual $L^2$ norm and the norm used to define $J$, we have
\[ C_1 ||\Psi^CS_{\mathfrak{g}}(\mathcal{E},V) + i\tau I||^2_{L^2} \leq J(\mathcal{E},V) \leq C_2 ||\Psi^CS_{\mathfrak{g}}(\mathcal{E},V) + i\tau I||^2_{L^2}, \]
for some fixed constants $C_1, C_2$. It will thus suffice to show that $||\Psi^CS_{\mathfrak{g}}(\mathcal{E},V) + i\tau I||^2_{L^2}$ has a unique smooth minimum on the orbit $\mathcal{O}(\mathcal{E},V)$, and that its minimum value is zero.

**Lemma 3.11.** Let $\mathfrak{g}^* = \mathfrak{g}/S^1$, where $S^1$ is identified with the subgroup of $\mathfrak{g}$ consisting of the constant multiples of the identity. Then $\mathfrak{g}^*$ acts symplectically on $\mathcal{H}^{CS}$, and this action extends to a holomorphic action of $(\mathfrak{g}_C)^* = \mathfrak{g}_C/C^*$. The moment map for the action of $\mathfrak{g}^*$ is $\Psi^C\mathfrak{g}(\mathcal{E},V) + i\tau I$.

**Proof.** Everything except the last sentence is immediate. The claim concerning the moment map follows from the fact that $\int_X \text{Tr}(\Psi^C_{\mathfrak{g}}(\mathcal{E},V)) = -i\tau$. Thus $\Psi^C_{\mathfrak{g}}(\mathcal{E},V) + i\tau I$ is the projection of $\Psi^CS_{\mathfrak{g}}(\mathcal{E},V)$ onto the (dual of)the Lie algebra of $\mathfrak{g}^*$, as required. \qed

It thus follows by general properties of moment maps in such situations (cf. [K]), that if $\mathfrak{g}^*$ acts freely (or with finite stabilizer subgroups) on all points of the orbit $\mathcal{O}(\mathcal{E},V)$, then $\Psi^C_{\mathfrak{g}}(\mathcal{E},V) + i\tau I = 0$ at a minimum for $||\Psi^C_{\mathfrak{g}}(\mathcal{E},V) + i\tau I||^2_{L^2}$ on the orbit. Furthermore, there is at most one such minimum on each orbit.

**Lemma 3.12.** If $p = (\mathcal{E},V)$ is an $\alpha$-stable point in $\mathcal{H}^{CS}$, then the isotropy subgroup of $\mathfrak{g}^*$ at $p$ is at most finite.

**Proof.** This follows, in the usual way, from the fact that $(\alpha)$-stable objects are simple. Thus the only automorphisms of $(\mathcal{E},V)$ are the constant multiples of the identity. \qed

It thus follows from the above that if $(\mathcal{E},V)$ is an $\alpha$-stable coherent system with rank$(\mathcal{E}) \geq 1$, then on $\mathcal{O}(\mathcal{E},V)$ there is a unique solution to the equation
\[ \Psi^C_{\mathfrak{g}}(\mathcal{E}',V') + i\tau I = 0. \]

The smoothness of the solution follows from standard elliptic regularity arguments. This concludes the proof of Theorem 3.2, and thus establishes

**Corollary 3.13 (Theorem B).** If $(\mathcal{E},V)$ is an $\alpha$-stable coherent system, then there is a unique smooth metric on $\mathcal{E}$, say $H$, and frame $\{\phi_1, \ldots, \phi_k\}$ for $V$ such that
\[ i\Lambda F^\mathfrak{g}_{\mathcal{E},H} + \sum \phi_i \otimes \phi_i^* = \tau I \]
\[ <\phi_i, \phi_j> = \alpha_{ik}. \]

**Remark.** In the special case where $\mathcal{E}$ is a line bundle, $(\mathcal{E},V)$ is automatically $\alpha$-stable. Furthermore, the orthonormal vortex equations reduce to a set of equations which can be viewed as a Kazdan-Warner type of equation with constraints. The existence theorem which follows from the above result is, as far as we know, new even in this context.
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