BOUNDING THE INVARIANT SPECTRUM WHEN THE SCALAR CURVATURE IS NON-NEGATIVE.

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ABSTRACT. On compact Riemannian manifolds with a large isometry group we investigate the invariant spectrum of the ordinary Laplacian. For either a toric Kähler metric, or a rotationally-symmetric metric on the sphere, we produce upper bounds for all eigenvalues of the invariant spectrum assuming non-negative scalar curvature.

1. INTRODUCTION

Motivation for finding universal bounds on the spectrum of metrics on the sphere comes from a famous result due to Hersch [12]: any metric $g$ on the sphere $S^2$, normalised to have volume $4\pi$, satisfies

$$\lambda_1(g) \leq 2.$$ 

In fact, Hersch also proved that $\lambda_1(g) = 2$ if and only if the metric $g$ is isometric to the canonical round metric.

If $G \leq \text{Iso}(g)$ is a subgroup of the isometry group of a metric $g$, the $G$-invariant spectrum is defined by restricting the Laplacian to $G$-invariant functions. The $k^{th}$ $G$-invariant eigenvalue of the ordinary Laplacian is denoted $\lambda_k^G(g)$. In this setting, the analogue of Hersch’s Theorem for metrics invariant under the standard action of $S^1$ on $S^2$ was studied by Abreu and Freitas. They proved that no counterpart for Hersch’s Theorem holds for the $S^1$-invariant spectrum. More specifically, for any $c \in (0, \infty)$, they were able to construct metrics $g$ with volume $4\pi$ and first invariant eigenvalue $\lambda_1^{S^1}(g) = c$.

However, under additional assumptions on the geometry defined by $g$, they also proved the following theorem which is our primary inspiration in this paper. A related set of results was also proved independently by Engman [9] [10]. To state the result, denote by $\xi_k$ the $\frac{1}{2}(k+1)^{th}$ positive zero of the Bessel function $J_0$ if $k$ is odd, and the $\frac{1}{2}k^{th}$ positive zero of $J_0'$ if $k$ is even.

**Theorem** (Abreu–Freitas, [3]). Let $g$ be a metric on $S^2$ which is invariant under the standard action of $S^1$, with non-negative Gaussian curvature and volume $4\pi$. Then

$$\lambda_k^{S^1}(g) < \frac{1}{2} \xi_k^2.$$ 

In the present note we are concerned with two natural generalisations of this theorem to higher dimensions. In both of them the analogue of non-negative Gaussian curvature is to assume the metrics have non-negative scalar curvature. The two generalisations correspond to two different ways of viewing $S^1$-invariant metrics on $S^2$. The first is to realize $S^2$ as a complex manifold (with complex dimension one), $g$ as a Kähler metric, and to view $S^1$ as a real, one-dimensional torus $T$. In higher dimensions, a well-developed theory of Kähler metrics which are invariant under the Hamiltonian action of a real $n$-torus $T^n$ exists, namely toric Kähler manifolds. In this setting we prove the following result.
Theorem A. Let \((M, \omega)\) be a compact toric Kähler manifold with non-negative scalar curvature. Then there exist constants \(0 < C_k([\omega])\), depending only upon the cohomology class \([\omega] \in H^2(M; \mathbb{R})\), such that

\[
\lambda^T_k \leq C_k.
\]

The \(k = 1\) bound in Theorem A was established by the authors with a slightly different proof in [11]; here a refined method yields all higher eigenvalue bounds.

The second viewpoint we take is to identify \(S^1 \cong SO(2)\) and \(SO(2) \leq SO(3)\) via the embedding

\[
g \in SO(2) \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \in SO(3).
\]

The action is inherited from the natural action of \(SO(3)\) on \(\mathbb{R}^3\) by regarding the sphere as a hypersurface \(S^2 \subset \mathbb{R}^3\). In higher dimensions this leads to the action of the group \(SO(n)\) on the \(n\)-dimensional sphere \(S^n \subset \mathbb{R}^{n+1}\). This example is part of another well-established theory, that of cohomogeneity one metrics. We prove the following result.

Theorem B. There exist constants \(D_k > 0\) such that for any unit-volume, \(SO(n)\)-invariant metric \(g\) on \(S^n\) with non-negative scalar curvature, we have the bounds

\[
\lambda^SO(n)_k(g) \leq D_k.
\]

We should say immediately that Theorem B is not really new and is essentially contained in the work of Colbois, Dryden and El Soufi (c.f. Theorem 1.7 in [6]). However, in our proof one can see very directly how the assumption on the non-negativity of the scalar curvature is used.

2. Background

2.1. Previous results in the literature. In all the results we mention here, \(M\) is assumed to be compact and without boundary. Korevaar [14], building on work by Yang and Yau [16], gave a far-reaching generalisation of the Hersch result to higher eigenvalues proving that, for any oriented Riemannian surface \((S, g)\) of genus \(\gamma\), there exists a universal constant \(C > 0\) such that

\[
\lambda_k(g) \leq \frac{C(\gamma + 1)k}{\text{Vol}(S, g)},
\]

where \(\lambda_k(g)\) is the \(k\)th eigenvalue in the spectrum of the Laplacian.

In the case that the manifold has dimension 3 or higher, Colbois and Dodziuk [5] proved that there is a unit volume metric \(g\) with arbitrarily large \(\lambda_1(g)\); hence there is no hope of a general analogue of the Hersch and Korevaar results for arbitrary manifolds. However, in the case that the manifold \(M\) is a compact, projective, complex manifold, Bourguignon, Li and Yau [4] demonstrated an upper bound for \(\lambda_1(g)\) depending upon some of the data associated with embedding \(M\) into complex projective space \(\mathbb{CP}^N\). This result has been recently extended to higher eigenvalues by Kokarev [13].

The study of the invariant spectrum for the (effective and non-transitive) action for a general compact Lie group \(G\) was taken up by Colbois, Dryden and El Soufi in [6]. They proved that, if the dimension of \(G\) is at least one, then

\[
\sup \left\{ \lambda^G_k(g)\text{Vol}(g)^{2/n} \right\} = \infty,
\]

where the supremum is taken over all \(G\)-invariant metrics conformal to a fixed reference metric \(g_0\).
Finally, in the case of $n$-complex-dimensional toric Kähler manifolds, we mention that Legendre and Sena-Dias [15] have generalised the results of Abreu and Freitas to show that the first $T^n$-invariant eigenvalue, $\lambda_{T^n}^1(g)$, can take any value in $(0, \infty)$ when the metric $g$ is allowed to vary over the set of toric Kähler metrics in a fixed cohomology class. The Colbois–Dryden–El Soufi and Legendre–Sena-Dias results make no assumption about the curvature of the metrics and so do not contradict Theorems A and B.

2.2. Toric Kähler manifolds. In order to prove Theorem A we recall some facts about toric Kähler metrics. We cannot hope to give a comprehensive introduction to this subject but refer readers to the works of Abreu [2] and Donaldson [8] which both contain detailed discussions of the theory. In a nutshell, associated to any Hamiltonian action of an $n$-torus $T^n$ on an $n$-complex-dimensional Kähler manifold $(M, \omega, J)$ is a convex polytope $P \subset \mathbb{R}^n$. The polytope $P$, often referred to in the literature as the moment polytope, depends only upon the cohomology class of the Kähler metric $[\omega] \in H^2(M; \mathbb{R})$. There is also a dense open set $M^0 \subset M$ such that $M^0 \cong P^o \times (0, 2\pi)^n$, where $P^o$ is the interior of $P$. In the natural coordinates, the metric $g(\cdot, \cdot) := \omega(J\cdot, \cdot)$ is given by

$$g = u_{i\bar{j}}dx_i dx_j + u^{ij}d\theta_i d\theta_j,$$

where $u_{i\bar{j}}$ is the Euclidean Hessian of a convex function $u : P^o \to \mathbb{R}$ and $u^{ij}$ is the inverse of this matrix. The function $u$ is referred to as the symplectic potential of the metric $g$. Clearly, in these coordinates, the volume form of the metric is the standard Euclidean form

$$dV = dx_1 \wedge dx_2 \wedge ... \wedge dx_n \wedge d\theta_1 \wedge d\theta_2 \wedge ... \wedge d\theta_n.$$

Crucial to our proof is the formula for the scalar curvature in the $(x_i, \theta_j)$ coordinates which was first given by Abreu [1] (often called the Abreu equation in the literature):

$$\text{Scal}(g) = - \sum_{i,j} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j}.$$

This expression yields the following useful integration-by-parts formula due to Donaldson [7].

Lemma 1 (Donaldson’s Integration-by-parts formula, Lemma 3.3.5 in [7]). Let $u : P^o \to \mathbb{R}$ be a symplectic potential of a toric Kähler metric $g$ and let $F \in C^\infty(P)$. Then

$$\int_P u^{ij} F_{ij} d\mu = \int_{\partial P} 2F d\sigma - \int_P \text{Scal}(g) F d\mu,$$

where $\mu$ is the usual Lesbegue measure on $\mathbb{R}^n$ and $\sigma$ is the integral Lesbegue measure on the boundary $\partial P$.

Here the subscripts on $F$ denote partial differentiation. Equation (2.1) differs slightly from the expression in Lemma 3.3.5 in [7] since we adopt a different convention to describe the singular part of the symplectic potential. We refer the reader to [11] for further discussion of our conventions and the precise definition of the integral Lesbegue measure $\sigma$.

3. Toric Kähler eigenvalue bounds

We now give the proof of Theorem A. Central to the proof is the fact that $\sigma$ does not depend upon the metric $g$. Rather it depends only on the cohomology class of the associated Kähler form $\omega$.

Proof. Fix the moment polytope $P \subset \mathbb{R}^n$ and pick a coordinate $x \in [x_{\text{min}}, x_{\text{max}}]$. Throughout the proof we shall abuse notation by denoting by $f$ functions in this single variable $f : [x_{\text{min}}, x_{\text{max}}] \to \mathbb{R}$ and the extension to the whole polytope $P$ given by $(f \circ \pi)$ where $\pi : P \to [x_{\text{min}}, x_{\text{max}}]$ is the projection to this coordinate.
Let $V_k \subset W^{1,2}$ be the subspace given by

$$V_k = \text{span}\{1, x, x^2, \ldots, x^k\}.$$ 

Hence the min-max characterisation of $\lambda_{kn}^T$ yields

$$\lambda_{kn}^T \leq \sup_{\varphi \in V_k, \varphi \neq 0} \left\{ \frac{\int_P \|\nabla \varphi\|^2 d\mu}{\int_P \varphi^2 d\mu} \right\}.$$ 

We denote by $\varrho : [x_{\min}, x_{\max}] \to \mathbb{R}$ the unique function satisfying

$$\varrho_{xx} = (\varphi_x)^2 \quad \text{and} \quad \varrho(x^*) = \varrho_x(x^*) = 0,$$

where $x^* = \frac{x_{\min} + x_{\max}}{2}$. As $\varrho$ is manifestly convex, the point $x^*$ is a global minimum and so $\varrho$ is non-negative on $[x_{\min}, x_{\max}]$.

The Raleigh quotient thus furnishes us with a map $R : V_k \to \mathbb{R}$

$$R(\varphi) = \frac{\int_P \|\nabla \varphi\|^2 d\mu}{\int_P \varphi^2 d\mu}.$$ 

The numerator can be evaluated using Equation (2.1)

$$\int_P \|\nabla \varphi\|^2 d\mu = \int_P u^{xx} (\varphi_x)^2 d\mu = \int_P u^{xx} \varrho_{xx} d\mu = \int_{\partial P} 2\varrho d\sigma - \int_P \text{Scal}(g) \varrho d\mu.$$ 

As the final term in the previous equation is non-negative, we obtain the bound

$$\int_P \|\nabla \varphi\|^2 d\mu \leq \int_{\partial P} 2\varrho d\sigma.$$ 

Hence

$$R(\varphi) \leq \frac{\int_{\partial P} 2\varrho d\sigma}{\int_P \varphi^2 d\mu},$$

where the bound is independent of the metric. The function $B : V_k \setminus \{0\} \to \mathbb{R}$ given by

$$B(\varphi) = \frac{\int_{\partial P} 2\varrho d\sigma}{\int_P \varphi^2 d\mu},$$

is continuous. We demonstrate this by considering the numerator and denominator separately. We endow the space $V_k$ with the standard Euclidean topology. The map $F_1 : V_k \to V_{2(k-1)}$ given by $F_1(\varphi) = (\varphi_x)^2$ is continuous (this could be seen by writing it out in coordinates). The map $F_2 : V_{2(k-1)} \to V_{2k}$ given by $F_2(f) = \varrho$, where $\varrho$ solves the equation

$$\varrho_{xx} = f \quad \text{and} \quad \varrho(x^*) = \varrho_x(x^*) = 0,$$

is linear and so continuous (note the domain is finite dimensional). The map $F_3 : V_{2k} \to \mathbb{R}$ given by

$$F_3(f) = \int_{\partial P} 2f d\sigma,$$

is also linear and so continuous. The numerator of the function $B$ is given by the composition $F_3 \circ F_2 \circ F_1(\varphi)$ and thus is continuous. A very similar argument yields the continuity of the denominator as a map from $V_k \setminus \{0\}$ to $\mathbb{R}$. Thus we obtain a continuous function $B : V_k \setminus \{0\} \to \mathbb{R}$.

If we scale $\varphi \to \alpha \varphi$ for $\alpha \in \mathbb{R}$, then $\varrho \to \alpha^2 \varrho$. Hence $B$ is determined by the values it takes on the functions $\varphi \in S^k \subset V_k$ (where we take the obvious coordinates on $V_k$). Restricting $B$...
to the sphere yields a continuous function, so the constant $C_k$ can be determined by taking the maximum of $B$ as $\varphi$ varies over the sphere $S^k$.

\begin{center}\begin{frame}4. COHOMOGENEITY ONE METRICS ON THE SPHERE\end{frame}\end{center}

Let $g$ be a unit-volume metric on $S^n$ given by

$$g = dt^2 + \rho^2(t)d\mathbb{S}^2_{n-1},$$

where $d\mathbb{S}^2_{n-1}$ is the unit-volume round metric on the sphere $S^{n-1}$ and $\rho : [0, b] \to \mathbb{R}$ is such that

$$\rho(0) = \rho(b) = 0, \quad \dot{\rho}(0) = -\dot{\rho}(b) = 1 \quad \text{and} \quad \rho > 0 \quad \text{on} \quad (0, b),$$

where dots over a function denotes derivatives with respect to $t$. In these coordinates the scalar curvature is given by

$$\text{Scal}(t) = -2(n-1)\ddot{\rho} + (n-1)(n-2)\frac{1-\dot{\rho}^2}{\rho^2}. \quad (4.1)$$

This immediately yields the following result.

**Lemma 2.** Let $\rho$ be as above and suppose the metric $g$ has non-negative scalar curvature. Then

$$|\dot{\rho}| \leq 1.$$  

\textit{Proof.} Suppose the bound does not hold, then the boundary conditions imply that $\dot{\rho}$ has a global maximum or minimum, $t^*$ say, where $t^* \in (0, b)$. From Equation (4.1), at such a point the scalar curvature is given by

$$\text{Scal}(t^*) = (n-1)(n-2)\frac{1-\dot{\rho}^2(t^*)}{\rho^2(t^*)}.$$  

If $|\dot{\rho}(t^*)| > 1$ there is now an immediate contradiction as the scalar curvature would be strictly negative at this point. \hfill \Box

We will now use a new coordinate $s$ defined by a map $s(t) : [0, b] \to [0, 1]$ given by

$$s(t) := \int_0^t \rho^{n-1}(\tau)d\tau.$$  

As the integrand is positive, $s(t)$ is invertible and we denote its inverse as $t(s)$. In these $s$–coordinates the metric becomes

$$g = \frac{ds^2}{\Phi^{2n-2}} + \Phi^2d\mathbb{S}^2_{n-1}, \quad (4.2)$$

where $\Phi(s) := \rho(t(s))$.

Straightforward calculation, with the convention that derivatives with respect to $s$ are denoted with a prime, yields

$$\dot{\rho}(t(s)) = \Phi'(s)(\Phi(s))^{n-1} = \frac{1}{n}(\Phi^n)' \quad \text{and so}$$

$$\Phi^n(0) = \Phi^n(1) = 0, \quad (\Phi^n)'(0) = -(\Phi^n)'(1) = n.$$  

Combining this with Lemma 2 gives
Lemma 3. Let $\Phi : [0,1] \to \mathbb{R}$ be as above. Then

$$\Phi(s) \leq \Phi_{\text{max}}(s),$$

where

$$\Phi_{\text{max}}(s) = \begin{cases} 
 n^{\sqrt{ns}} & \text{if } 0 \leq s \leq \frac{1}{2}, \\
 n^{\sqrt{n} - n(s - \frac{1}{2})} & \text{if } \frac{1}{2} \leq s \leq 1.
\end{cases}$$

We can now prove Theorem 3. It essentially follows from the fact that $\Phi_{\text{max}}$ is independent of metric.

Proof. Working in the ‘s’ coordinate, we consider the subspace $V_k \subset W^{1,2}$ given by

$$V_k := \text{span}\{1, s, s^2, \ldots, s^k\}.$$ 

We take $\varphi \in S^k \subset V_k$ (where we use the obvious coordinates on $V_k$ to define the sphere) and consider

$$R(\varphi) := \frac{\int_{S^n} |\nabla \varphi|^2 dV_g}{\int_{S^n} \varphi^2 dV_g}.$$

Using the specific form of the metric $g$ in Equation (4.2) we obtain

$$R(\varphi) = \frac{\int_0^1 \Phi^{2n-2}(\varphi')^2 ds}{\int_0^1 \varphi^2 ds} \leq \frac{\int_0^1 \Phi_{\text{max}}^{2n-2}(\varphi')^2 ds}{\int_0^1 \varphi^2 ds},$$

where the last inequality uses Lemma 3. Thus the Raleigh quotient is clearly bounded above by a quantity that does not depend upon the metric $g$. The result follows from the min-max characterisation of the $k^{th}$ eigenvalue. \(\square\)

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