Three-body Physics in Strongly Correlated Spinor Condensates

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Spinor condensates have proven to be a rich area for probing many-body phenomena richer than that of an ultracold gas consisting of atoms restricted to a single spin state. In the strongly correlated regime, the physics controlling the possible novel phases of the condensate remains largely unexplored, and few-body aspects can play a central role in the properties and dynamics of the system through manifestations of Efimov physics. The present study solves the three-body problem for bosonic spinors using the hyperspherical adiabatic representation and characterizes the multiple families of Efimov states in spinor systems as well as their signatures in the scattering observables relevant for spinor condensates. These solutions exhibit a rich array of possible phenomena originating in universal few-body physics, which can strongly affect the spin dynamics and three-body mean-field contributions for spinor condensates. The collisional aspects of atom-dimer spinor condensates are also analyzed and effects are predicted that derive from Efimov physics.

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In recent years, the development of optical traps has stimulated the realization of spinor condensates [1]. The coupling of the degenerate spin degrees of freedom leads to novel quantum phenomena such as the formation of spin domains, spin textures, spin mixing dynamics, and counterintuitive quantum phases. These have been intensively investigated both experimentally [2–13] and theoretically [14–23]. Such phenomena have been shown to be sensitive to the (typically weak) interatomic interactions, characterized by multiple scattering lengths associated with the various atomic hyperfine spins states. Of particular interest is the fact that strongly correlated spinor condensates can enable explorations of spinor physics in exotic dynamical regimes. Although the scattering lengths for most alkali species are modest or even small, one key exception is 85Rb [24]. There have been several proposals to create strongly correlated spinor condensates [25–31] where the scattering lengths substantially exceed the range of interatomic interactions, i.e. the van der Waals length \( r_{vdW} \). This enables the spin states to effectively interact even at large distances. In this scenario one should also consider few-body correlations, notably effects associated with the existence of Efimov states [32–34].

In single-spin condensates, when the interactions are enhanced by the presence of a Feshbach resonance [35], an infinity of Efimov states emerges that strongly affects scattering observables at ultracold energies [32–34]. More recently, advances have been made in our understanding of universal Efimov physics in an ultracold quantum gas. Despite the complex nature of the interatomic interactions, recent experimental [36–43] and theoretical [44–48] studies have shown surprisingly that the usual three-body parameter is universal, depending only on \( r_{vdW} \), which now permits even more quantitative predictions of interesting few-body phenomena to be made.

The present exploration of Efimov physics in a spinor condensate system shows that the additional spin degrees of freedom can fundamentally modify the Efimov trimer’s energy spectrum and that scattering processes can strongly affect the condensate spin dynamics. In the context of nuclear physics, where isospin symmetry plays an important role, the work of Bulgac and Efimov [49] demonstrated a much richer structure for Efimov physics when the spin degree of freedom is considered. In this case, multiple families of Efimov states can coexist, depending on the particular spin states and different scattering lengths in the problem [50]. This is in striking contrast to the standard Efimov scenario where only a single spin state is available. For the multilevel bosonic systems examined here, with the topologically distinct case of a spinor condensate, the atomic hyperfine states provide the internal atomic structure. Several interesting effects are predicted for the three-body scattering observables controlling the dynamical evolution of spinor condensates. Similarly, our results point to the possibility of exploring spinor physics in an atom-dimer mixture. These results emerge from a calculation of the collisional properties of this system, including a characterization of the signatures of Efimov physics.

The study of few-body physics in spinor condensates requires proper inclusion of the multichannel nature of interatomic interactions, originating from the underlying atomic hyperfine structure. Our study begins from the multichannel generalization of the zero-range Fermi pseudopotential [51–53] for s-wave interactions (a.u.), namely:

\[
\hat{\psi}(r) = \frac{4\pi A}{m} \delta^3(r) \frac{\partial}{\partial r},
\]

where \( \delta^3(r) \) is the usual three-dimensional Dirac-\( \delta \) function and \( A \) the scattering length matrix written in the two-body spin basis denoted by \( \{ | \sigma \rangle \} \). Within this framework, the three-body problem is solved in the adiabatic hyperspherical representation, using the Green’s function method developed in Refs. [54, 55]. In this representation, the hyperradius \( R \) determines the overall size of the system, and the internal motion is described by a set of
five hyperangles, collectively denoted by Ω. Briefly, the adiabatic fixed-R eigenvalue equation reads:

$$\left[ \hat{\Lambda}^2(\Omega) + \frac{1}{2\mu R^2} + \hat{V}(R, \Omega) + \hat{E}_\Sigma \right] \Phi(R; \Omega) = U(R) \Phi(R; \Omega),$$

(2)

where $\mu = m/\sqrt{3}$ is the three-body reduced mass, $\hat{\Lambda}$ is the grand angular momentum operator \[^{54}\], $\hat{V}$ is the sum of pairwise interactions, and $\hat{E}_\Sigma$ is the sum of the atomic energy levels, which is diagonal in the three-body spin basis $\{|\Sigma\rangle\}$. The channel functions $\Phi(R; \Omega)$ and the three-body potentials $U(R)$ describe the physical properties of the system and are obtained by solving Eq. (2) for fixed values of $R$. Application of the zero-range potential model reduces the problem to solving a transcendental equation whose roots, $s_{2b}(R)$, determine $U_s(R)$ through

$$U_s(R) = \frac{s_{2b}(R)^2 - 1/4}{2\mu R^2},$$

(3)

(See outline of our formulation in Ref. \[^{55}\].) For three-identical bosons, for instance, solving Eq. (2) in the limit $R/a \to 0$ yields a single imaginary root, independent of $R$, with numerical value $s_0 \approx 1.0062i$. Insertion of $s_0$ into Eq. (3) produces the attractive $1/R^2$ potential that supports an infinity of three body bound states characteristic of the Efimov effect. In the present study, the threshold energy levels, $\hat{E}_\Sigma$ in Eq. (2), are degenerate and are set equal to zero. In spinor condensates at vanishingly small magnetic fields, the atomic levels are $(2f + 1)$-fold degenerate ($f$ is the atomic hyperfine angular momentum and $m_f = -f, ..., f$, its azimuthal component). In fact, this degeneracy leads to fundamentally different three-body physics than is obtained for the usual Efimov case with atoms in a single spin state.

The interatomic interaction for spinor condensates is spin-dependent, and we assume the scattering length operator in Eq. (1) can be represented as

$$\hat{A} = \sum_{F_{2b},M_{F_{2b}}} a_{F_{2b},M_{F_{2b}}} |F_{2b},M_{F_{2b}}\rangle \langle F_{2b},M_{F_{2b}}|,$$

(4)

where $F_{2b}$ and $M_{F_{2b}}$ are the two-body total spin and its projection. Due to bosonic symmetry only the symmetric spin states ($F_{2b} = \text{even}$) are allowed to interact with rotationally-invariant scattering lengths $\{a_0, a_2, ..., a_{2f}\}$. These scattering lengths set important length scales in the problem, and their relative magnitudes and signs determine many-body properties such as the miscibility of the different spin components. Moreover, the scattering lengths also determine the nature of the three-body interactions and many of the scattering properties of the system, potentially impacting the spin dynamics of condensates.

Figure shows the three-body potentials for $f=1$ atoms for the allowed values of the total three-body hyperfine spin, $|F_{2b} - f| \leq F_{3b} \leq F_{2b} + f$. These are, of course, independent of $M_{F_{3b}} = F_{2b} + m_f$. The $F_{3b} = 0$ states are spatially antisymmetric and thus noninteracting in the potential model of Eq. (1). The results in Fig. 1 were obtained by solving Eq. (2) in the spin basis $\{|\Sigma\rangle\}$ and with $a_0 = 10^4 r_{vdW}$ and $a_2 = 10^5 r_{vdW}$. Figure (a) emphasizes the three-body physics for $R \leq \{a_0, a_2\}$ (shaded region) where two attractive potentials exist for $F_{3b} = 1$ (red solid line) and $3$ (blue dash-dotted line). Both potentials are associated with $s_0 \approx 1.0062i$ and allow for the coexistence of two families of Efimov states, (represented in Fig. 1 by the horizontal solid and dashed-dotted lines) —a feature absent in systems of single state atoms. For $R > a_0$, the $F_{3b} = 1$ potential turns into an atom-dimer channel describing collisions between a $|F_{2b} = 0, M_{F_{2b}} = 0\rangle$ dimer, with energy $-1/m_0^2$, and a $|m_f = 0\rangle$ atom. For $a_0 < R < a_2$, the shaded region in Fig. (b) only the $F_{3b} = 3$ family of Efimov states persists. For $R > a_2$, this $F_{3b} = 3$ potential, and two other $F_{3b} = 1$ and $F_{3b} = 2$ potentials, converge to the dimer energy $-1/m^2$ and describe atom-dimer collisions in states $|F_{2b} = 2, M_{F_{2b}} = -1, 0, 1\rangle + |m_f = 1, 0, -1\rangle$. This offers an
TABLE I: Values of $s_\nu$ relevant for $f=1$ and 2 spinor condensates covering all possible regions of $R$ for the different ranges of the relevant scattering lengths. For $f=1$ we list the lowest few values of $s_\nu$ for each $F_{3\nu}$ while for $f=2$ we only list the values of $s_\nu$ and their multiplicity (superscript), instead of the specific value of $F_{3\nu}$ where they occur.

\[
\begin{array}{lcccc}
(f = 1) & F_{3\nu} = 1 & F_{3\nu} = 2 & F_{3\nu} = 3 \\
R < |a_{(0,2)}| & 1.0062 \times 10^{-1}, 2.1662 & 2.1662 & 1.0062, 4.4653 \\
|a_0| < R < |a_2| & 0.7429 & 2.1662 & 1.0062, 4.4653 \\
|a_2| < R < |a_0| & 0.4097 & 4 & 2 \\
R \gg |a_{(0,2)}| & 2 & 4 & 2 \\
\end{array}
\]

\[
\begin{array}{lcccc}
(f = 2) & F_{3\nu} = 0, 1, \ldots, 6 \\
\end{array}
\]

\[
\begin{array}{lcccc}
R < |a_{(2,4)}| & 1.0062(1), 2.1662(2) \\
|a_0| < R < |a_{(2,4)}| & 1.0062(1), 0.4905(1) \\
|a_2| < R < |a_{(0,4)}| & 1.0062(1), 0.7473(1), 0.6608(1) \\
|a_4| < R < |a_{(2,2)}| & 1.0062(1), 0.5528(1), 0.3788(1), 0.5219(1) \\
|a_{(0,2)}| < R < |a_2| & 1.0062(1), 0.6608(1) \\
|a_{(2,4)}| < R < |a_0| & 1.0062(1), 0.5528(1), 0.5219(1) \\
|a_{(2,4)}| < R < |a_{(0,4)}| & 0.6861(1) \\
R \gg |a_{(2,4)}| & 2(5), 4(2) \\
\end{array}
\]

Table I summarizes the values of $s_\nu$ relevant for $f=1$ and 2 spinor condensates, covering all possible regions of $R$ and for different magnitudes of the relevant scattering lengths. For $f=1$ the values of $s_\nu$ are listed according to the value of $F_{3\nu}$ while for $f=2$ they are not assigned in detail (see the complete assignment in Ref. \[55\]). Notably, for $f=1$ ensembles, the imaginary values of $s_\nu$ agree exactly with the ones for single level atoms, except with the important distinction that such roots can be degenerate in the spinor case. It is well known that the existence of overlapping series of states can lead to formation of ultra long-lived states \[23, 44\]. In our present case, the $F_{3\nu} = 1$ and 3 Efimov states can interact for finite (but small) magnetic fields and such controllability can not only produce long-lived states but also, due to their weakly bound character, affect the spin dynamics of the condensate. The occurrence of such effects, however, will depend on the three-body short-range physics \[46\]. Further analysis of this parameter space is beyond the scope of the present study. This feature opens up exciting possibilities for the study of Efimov trimers in spinor condensates in a well-controlled manner. More interestingly, $f=2$ ensembles exhibit several values of $s_\nu$ that differ from those for single level atoms. The physics controlling the appearance of these new roots is due to the fact that the atoms in the two-body states $|F_{2\alpha}M_{F_{2\alpha}}\rangle$ are not in a pure quantum state. Instead, they are in a mixture of states, with the amount of mixing controlled by the angular momentum algebra.

The above results illustrate the rich structure of Efimov states in spinor systems. Such richness also appears in the three-body scattering observables. Here, we use a WKB model \[52\] to determine the scattering length and energy dependence of collision rates for all relevant scattering processes for $f=1$ spinor condensates. The results for $F_{3\nu} = 1, 2$ and 3, are summarized in Table II. Notice that scattering observables can display log-periodic interference and resonant effects due to the multiple families of Efimov states and collision pathways available in spinor ensembles. Interference and resonance effects are parameterized according to, respectively,

\[
M_{s_\nu}^{(\alpha)}(a) = \alpha e^{-2\eta} \left[ \sin^2 \left( s_\nu \ln \frac{2}{r_0} \right) + \sinh^2 \eta \right],
\]

\[
P_{s_\nu}^{(\alpha)}(a) = \beta \frac{\sinh 2\eta}{\sin 2\eta} \left( s_\nu \ln \frac{2}{r_0} \right) + \sinh^2 \eta,
\]

where $r_0 = r_{vdW} e^{-\phi/s_\nu}$ is the three-body parameter, incorporating the short-range physics through the phase $\phi \approx 40$, and $\eta$ is the three-body inelasticity parameter \[23, 44\] which encapsulates the probability for decay into deeply bound molecular states. In the above equations, $\alpha$ and $\beta$ are universal constants that can be evaluated for each $F_{3\nu}$. In Table II $K_3^{(1)}$ and $K_3^{(2)}$ denote the collision rate for three-body recombination into weakly bound $F_{2\nu} = 0$ and 2 dimers, respectively. Such dimers can still remain trapped and further dissociate into free atoms via collision with other atoms with dissociation rates $D_3^{(0)} \propto K_3^{(0)}(k^4a_0)$ and $D_3^{(2)} \propto K_3^{(2)}(k^4a_2)$, where $k^2 = 2\mu E$ with $E > 0$ being the three-body collision energy. This interplay between dimer formation and dissociation can provide an interesting dynamical regime in spinor condensates which is absent when the scattering lengths are small. Three-body recombination into deeply bound molecular states, $K_3^{(d)}$, can only lead to losses and can display resonant enhancements due to formation of Efimov states or can be suppressed due to repulsive three-body interactions. Note that the total rate is obtained by multiplying $K_3$ in Table II by the appropriate factors that account for the various degeneracies in the problem.

While inelastic collisions determine the stability of condensates, three-body elastic processes determine how different spins interact. Moreover, they can also display resonant effects due to Efimov states, which can strongly affect the spin dynamics. Within the mean-field description of spinor condensates \[11\], such effects can be incorporated through the three-body scattering length operator

\[
\hat{A}_{3\nu} = \sum_{F_{3\nu}M_{F_{3\nu}}} a_{F_{3\nu}}^{(3\nu)} |F_{3\nu}M_{F_{3\nu}}\rangle \langle F_{3\nu}M_{F_{3\nu}}| (F_{2\nu})_3 |F_{2\nu})_3 .
\]

This is a natural extension of the three-body scattering length (units of length$^4$) as defined in \[32, 53, 60\].
scattering length dependence of $a_{3b}$ is listed in Table I for $f=1$ atoms, showing the different ways it is influenced by Efimov physics. (Note that for $F_{2b}=2$, $a_{3b}$ is not defined since for this case a higher centrifugal barrier suppresses collisions at ultracold energies. One interesting case emerges, for instance, when $a_{0}<0$ or $a_{2}<0$ where $a_{3b}$ can display resonant effects if $a_{0}$ or $a_{2}$ are tuned near a Efimov resonance—parametrized in Table II by the tangent-like, log-periodic function

$$T_{n}^{a}(a) = \alpha + \beta \frac{2 \sin \left( |s_{0}| \ln \frac{\alpha}{s_{0}} \right) \cos \left( |s_{0}| \ln \frac{\alpha}{s_{0}} \right)}{\sin^{2} \left( |s_{0}| \ln \frac{\alpha}{s_{0}} \right) + \sinh^{2} \eta},$$

(8)

where $\alpha$ and $\beta$ are, again, universal constants. [Note that oscillations in $a_{3b}$, parametrized as $O_{n}^{a}(a)$ in Ref. 53, are also allowed.] In this case $|a_{3b}| \gg |\{a_{0}, a_{2}\}$ and three-body correlations can dominate mean-field interactions, allowing for both attractive ($a_{3b}<0$) and repulsive ($a_{3b}>0$) three-body interactions.

From the mean-field perspective, in order to understand three-body contributions for spinor condensates, it is convenient to write $A_{3b}$ in a way that makes explicit the importance of spin-exchange interactions. For $f=1$ atoms we can rewrite Eq. (7) as

$$A_{3b} = \alpha_{3b} + \alpha_{3b}^{ex} \sum_{i<j} \vec{f}_{i} \cdot \vec{f}_{j},$$

(9)

where $\vec{f}_{i}$ ($i=1, 2$ and $3$) is the atomic hyperfine angular momentum for the atom $i$, and the three-body direct and exchange interactions given, respectively, by

$$\alpha_{3b} = (2a_{3b}^{(3)} + 3a_{3b}^{(1)})/5$$

$$\alpha_{3b}^{ex} = (a_{3b}^{(3)} - a_{3b}^{(1)})/5.$$

This form for $A_{3b}$ is in close analogy to the two-body spinor case, in which $A_{2b}=\alpha_{2b} + \alpha_{2b}^{ex} \vec{f}_{1} \cdot \vec{f}_{2}$, where $\alpha_{2b}=(a_{0}+3a_{2})/3$ and $\alpha_{2b}^{ex}=(a_{2}-a_{0})/3$. Mean-field contributions, however, are introduced through the corresponding two- and three-body coupling constants $g_{2b}=4(\pi/m)a_{2b}$, $g_{2b}^{ex}=4(\pi/m)a_{2b}^{ex}$, $g_{3b}=3(12\pi/m)a_{3b}$, and $g_{3b}^{ex}=3(12\pi/m)a_{3b}^{ex}$, respectively.

Observe that the $a_{3b}^{1/2}$ (or $a_{3b}$) dependence of $g_{2b}$ and $g_{3b}^{ex}$ can quickly make the three-body direct and spin-exchange mean-field energies, $g_{3b}n^{2}$ and $g_{3b}^{ex}n^{2}$, comparable to their two-body counterparts, $g_{2b}n$ and $g_{2b}^{ex}n$. In fact, resonant effects in $a_{3b}$ due to Efimov states can strongly affect both ferromagnetic ($g_{3b}^{ex}<0$) and antiferromagnetic ($g_{3b}^{ex}>0$) phases in spinor condensates. Whether $g_{3b}^{ex}<0$, $g_{3b}^{ex}>0$, or both can be stabilized by a repulsive three-body interaction ($g_{3b}>0$ and $g_{3b}^{ex}>0$) to form localized, self-bound, quantum droplets of spinor characters in a similar spirit of Fig. 1(b). The study of mean-field three-body contributions and their possible effects in spinor condensates, however, will be a subject of future investigations.

Finally, our present study has also shown the possibility of creating atom-dimer spinor condensates where $F_{2b} \neq 0$ dimers can exchange $M_{2b}$ by colliding with other atoms in $|m_{f}\rangle$ states. For instance, for $f=1$ atoms, Fig. 1(b) shows $F_{2b} = 2$ dimers can collide with atoms in states $|-1\rangle$, $|0\rangle$ and $|1\rangle$ and their collisional properties are listed in Table SI of Ref. 53. Similar to atomic spinor condensates, in the mean-field approximation the relevant parameters for this atom-dimer mixture is the elastic atom-dimer scattering length matrix $A_{ad}$ whose elements $a_{ad}$ are listed in Table SI. As one can see, Efimov resonances can also strongly affect the mean-field energy. Moreover, if both $a_{0}>0$ and $a_{2}>0$ are large, both $F_{2b}=0$ and 2 dimers can remain trapped, leading to an interesting regime where reactive scattering can affect the dynamics of the system.
In summary, we have explored universal aspects of Efimov physics in spinor systems and found a rich variety of scattering phenomena that strongly affect the spin dynamics in strongly correlated spinor condensates. The multiple, co-existing, families of Efimov states characteristic of spinor systems can lead to non-trivial spin dynamics, dominated by three-body correlations, as well as allowing for the existence of ultralong lived Efimov states. We also study few-body aspects of atom-dimer spinor condensates and show that it can offer novel regimes for studying spin-like-physics.

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of the Clebsch-Gordan coefficients as
\[ |F_{2b,M_{F_{2b}}}⟩ = \sum_{m_{f_1},m_{f_2}} (f_{1}m_{f_1},f_{2}m_{f_2}|F_{2b},M_{F_{2b}}⟩|m_{f_1},m_{f_2}). \] (S1)

For \( f = 1 \) atoms these spin functions are, for \( F_{2b} = 0, 1 \) and 2,
\[
|00⟩ = \frac{1}{\sqrt{3}} |−1,1⟩ + \frac{1}{\sqrt{3}} |0,0⟩ + \frac{1}{\sqrt{3}} |−1,−1⟩,
\] (S2)
\[
|10⟩ = \frac{1}{\sqrt{2}} |−1,1⟩ + \frac{1}{\sqrt{2}} |1,−1⟩,
\] (S3)
\[
|1±1⟩ = \frac{1}{\sqrt{2}} |0,±1⟩ ± \frac{1}{\sqrt{2}} |±1,0⟩,
\] (S4)
\[
|20⟩ = \frac{1}{\sqrt{2}} |−1,−1⟩ + \frac{1}{\sqrt{2}} |1,1⟩,
\] (S5)
\[
|2±1⟩ = \frac{1}{\sqrt{2}} |0,±1⟩ ± \frac{1}{\sqrt{2}} |±1,0⟩,
\] (S6)
\[
|2±2⟩ = |±2±2⟩.
\] (S7)

As one can see, \( F_{2b} = 1 \) states are antisymmetric and must be excluded from our s-wave interaction model.

Similarly, three-body spin functions of total angular momentum \( |F_{3b} − f_3| ≤ F_{3b} ≤ F_{3b} + f_3 \) and projection \( M_{F_{3b}} = M_{F_{2b}} + m_{f_3} \) can be expressed in terms of the corresponding two spin states Eq. (S1) as
\[
|F_{3b,M_{F_{3b}}}⟩ = \sum_{M_{F_{2b}},m_{f_3}} (f_{1}f_{2}f_{3}M_{f_{3}}|F_{3b},M_{F_{3b}}⟩|F_{2b},M_{F_{2b}}⟩|m_{f_3}).
\] (S8)

We will, however, analyse the three-body spin functions for \( f = 1 \) atoms for each value of \( F_{3b} \) separately since symmetry considerations that allow us to disregard some states are less evident than in the two-body case. Here we will consider only states with \( M_{F_{3b}} = 0 \) but the same considerations also applies for \( M_{F_{3b}} ≠ 0 \).

For \( F_{3b} = 0 \) one can, by inspection, determine that the corresponding spin function
\[
|00(1)⟩ = \frac{1}{\sqrt{3}} |−1,1⟩ − \frac{1}{\sqrt{3}} |0,0⟩ + \frac{1}{\sqrt{3}} |−1,−1⟩,
\] (S9)
is antisymmetric under permutations of any two spins. Similar to the two-body case, a three-body antisymmetric spin state requires an antisymmetric spacial wave function in order to form a symmetric total wave function. Since in our model only s-wave interactions are allowed, antisymmetric three-body states are noninteracting, and we neglect them from our analysis and calculations.

For \( F_{3b} = 1 \), the analysis is more complicated. Now, there exist three spin states with \( F_{3b} = 1 \), each one corresponding to the allowed values for \( F_{2b} \). They are given by,
\[
|10(0)⟩ = |00⟩|0⟩,
\] (S10)
\[
|10(1)⟩ = −\frac{|−1,1⟩}{\sqrt{2}} + \frac{|1,−1⟩}{\sqrt{2}},
\] (S11)
\[
|10(2)⟩ = −\frac{|−2⟩}{\sqrt{10/3}} + \frac{|2⟩}{\sqrt{5/3}},
\] (S12)

**SUPPLEMENTARY MATERIAL**

I. TWO- AND THREE-BODY SPIN FUNCTIONS

In this section we will give explicit expressions for both two- and three-body spin states for \( f = 1 \) atoms as well as discuss some symmetry properties relevant for our present study. As usual, the two-body spin functions of total angular momentum \( |f_1 − f_2| ≤ F_{2b} ≤ f_1 + f_2 \) and projection \( M_{F_{2b}} = m_{f_1} + m_{f_2} \) can be expressed in terms
Besides the fact that permutations of spins 1 and 2 are symmetric ($F_{3b}=$even) or antisymmetric ($F_{3b}=$odd), no clear symmetry property can be derived by inspection. The three spin functions for $F_{3b} = 1$ can be symmetrized to form a pair of mixed symmetry states (one symmetric and other antisymmetric with respect to permutations of spins 1 and 2) and a fully symmetric spin state \[|10(0, 2)\] given by
\[
|10(0, 2)\rangle = \frac{2}{3} |10(2)\rangle + \sqrt{\frac{2}{3}} |10(0)\rangle . \tag{S13}
\]

Although in our calculations it is crucial include all these states, we determined that collision processes involving three atoms in mixed symmetry states are suppressed at low energies due to stronger centrifugal barriers in the three-body potentials. Therefore, in this case, the $F_{3b} = 1$ totally symmetric state is dominant. For atom-dimer collisions, however, both symmetric and mixed symmetry states should be considered. For $F_{3b} = 2$ states, the corresponding spin functions
\[
|20(1)\rangle = \frac{|11\rangle - |10\rangle}{\sqrt{6}} + \frac{|00\rangle}{\sqrt{3}}, \tag{S14}
\]
\[
|20(2)\rangle = \frac{|11\rangle - |00\rangle}{\sqrt{6}} + \frac{|21\rangle - |10\rangle}{\sqrt{3}}, \tag{S15}
\]
form a pair of mixed symmetry states. Similar to $F_{3b} = 1$, we determined that for $F_{3b} = 2$, collision processes involving three atoms in such states are also suppressed at low energies due to stronger centrifugal barriers in the three-body potentials. Consequently, they are neglected for the analysis of collisions between three atoms but are included in atom-dimer collisions. For $F_{3b} = 3$, there exists only one spin function
\[
|30(2)\rangle = \frac{|21\rangle - |10\rangle}{\sqrt{3}} + \frac{|00\rangle}{\sqrt{2}}, \tag{S16}
\]
that forms a totally symmetric spin state and, therefore, it is included in our analysis.

II. THREE-BODY GREEN’S FUNCTION APPROACH FOR SPINOR SYSTEMS

In the present study we use the Green’s functions formulation developed in Refs. [2, 3] to solve the three-body spinor problem in the adiabatic hyperspherical representation. Here, we briefly outline the main steps of this formalism for spinor systems. We solve Eq. (2) of the main text [setting $E_3^\Sigma = 0$] by expressing the corresponding Lippmann-Schwinger equation [2] for each component of the channel function $\Phi$ as

$$
\Phi_{\Sigma}(R; \Omega) = -2\mu R^2 \sum_{\Sigma', k} \int d\Omega' G_{\Sigma\Sigma'}(\Omega, \Omega') \times \delta_{\Sigma, k}'(R, \Omega') \Phi_{\Sigma'}(R; \Omega'), \tag{S17}
$$

where

$$
\delta_{\Sigma, k}'(R, \Omega) = \langle \Sigma | v(r_{ij}) | \Sigma' \rangle, \tag{S18}
$$
is the interaction term, given in our present study by Eq. (1) of the main text. The superscript $(k)$ means that particle $k$ is a spectator while particles $i$ and $j$ interact. The Green’s functions are set to satisfy $[\hat{A} - (s^2 - 4)]G_{\Sigma\Sigma'}(\Omega, \Omega') = \delta(\Omega, \Omega')$, where $2\mu R^2 U(R) = s^2 - 1/4$.

Our choice for the three-body spin basis is the one formed by product states $\{\Sigma\} = \{m_{f_1} m_{f_2} m_{f_3}\}$. This choice simplifies the formulation and, as we will see, the solutions for a given value of $F_{3b}$ and $M_{F_{3b}}$ can be obtained in the last step of our formulation.

As shown in Ref. [2] (see also Ref. [4]), the solutions of Eq. (S17) can be obtained by determining (for fixed values of $R$) the values of $s$ in which the determinant of the matrix

$$
Q = \left[ \begin{array}{c} 3\frac{\pi}{2R} & M^{(1)} & M^{(2)} & P_- & M^{(3)} & P_+ \end{array} \right] - 1, \tag{S19}
$$

vanishes. Here, the matrices $(P_+ \Sigma\Sigma') = \langle \Sigma | P_{12} | \Sigma' \rangle$ and $(P_- \Sigma\Sigma') = \langle \Sigma | P_{13} | \Sigma' \rangle$ represent, respectively, cyclic and anti-cyclic permutations of the three-body spin basis, and

$$
M_{\Sigma\Sigma'}^{(i)} = \left\{ \begin{array}{ll} A_{\Sigma\Sigma'}^{(i), s} \cot(s \pi/2), & i = 1, \\
-4 \text{sin}(x \pi/6), & i = 2, 3, \end{array} \right. \tag{S20}
$$
is the two-body scattering length matrix written in the three-body spin basis,

$$
A_{\Sigma\Sigma'}^{(i)} = \langle \Sigma | \hat{A}_{\Sigma}^{(i)} | \Sigma' \rangle. \tag{S21}
$$

Up to this point, our formulation does not account for any symmetry property of the system. In order to select the desired spin symmetry, we simply “project” the matrix $Q$ above to the particular subspace we are interested. Therefore, the solutions with proper symmetry can be obtained by solving (for fixed values of $R$)

$$
\det [S(Q) S^T] = 0, \tag{S22}
$$

where

$$
(S)_{\Sigma\Sigma'} = \langle \Sigma | \left( \sum_{F_{3b}} |F_{3b} M_{F_{3b}}(F_{2b})\rangle\langle F_{3b} M_{F_{3b}}(F_{2b})| \right) |\Sigma' \rangle. \tag{S23}
$$

For $f = 1$ atoms, for instance, the $F_{3b} = 1$ solutions are determined by solving the transcendental equation,

$$
3\frac{\pi}{2R} (a_0 + a_2)s \cot(s \pi/6) + 3\frac{\pi}{2R} (a_0 a_2)s^2 \cot(s \pi/6)^2 - 2\frac{\pi}{2} a_0 (a_0 + a_2) \sin(s \pi/6) + 2 \frac{\pi}{2} a_0 a_2 \sin(s \pi/6) \sin(s \pi/6) R^2 + 1 = 0. \tag{S24}
$$

(Imaginary roots can be obtained by mapping $s \to is$.) As we can see, the above transcendental equation depends on both two-body scattering lengths, $a_0$ and $a_2$. I
On the other hand, the $F_{3b} = 2$ and $3$ solutions are obtained, respectively, through,

$$
\frac{3^{1/4}a_2s \cot\left(\frac{\pi}{4} s\right)}{2^{1/2}R} + \frac{2^{1/2}a_2s \sin\left(\frac{\pi}{4} s\right)}{3^{1/4}s} = 1, \quad \text{(S25)}
$$

and

$$
\frac{3^{1/4}a_2s \cot\left(\frac{\pi}{2} s\right)}{2^{1/2}R} + \frac{2^{1/2}a_2s \sin\left(\frac{\pi}{2} s\right)}{3^{1/4}s} = 1, \quad \text{(S26)}
$$

which depend only on $a_2$. In Table SI we list the solutions of Eqs. (S24)–(S26) for the regions in $R$ in which the $s$ is constant. The values of $s$ listed represent the imaginary roots and/or the lowest real root. In Table SI we also list the corresponding values of $s$ relevant for $2 = 2$ spinor condensates. Note that $F_{3b} = 0$ and $1$ the possible values for $s$ depend only on $a_2$, while for $F_{3b} = 2$ they depend on $a_0$, $a_2$ and $a_4$, $F_{3b} = 3$ and $4$ on $a_2$ and $a_4$, and $F_{3b} = 5$ and $6$ only on $a_4$.

### III. THREE-BODY SCATTERING LENGTH OPERATOR

In this section we will derive the form of the three-body scattering operator in which spin-exchange processes are more evident and in which it is adequate for studies of three-body effects in the mean-field approach. We follow closely the derivation of the corresponding two-body scattering length matrix established in Ref. [5]. We start by expressing $A_{3b}$ [Eq. (7) of the main text] in terms of the projector operators for each value of $F_{3b}$, $P_{F_{3b}}$, as

$$
\hat{A}_{3b} = \sum_{F_{3b}} a^{(F_{3b})}_{3b} |F_{3b}, M_{F_{3b}}(F_{2b})\rangle \langle F_{3b}, M_{F_{3b}}(F_{2b})|,
$$

$$
= \sum_{F_{3b}} a^{(F_{3b})}_{3b} P_{F_{3b}}, \quad \text{(S27)}
$$

where

$$
P_{F_{3b}} = \sum_{M_{F_{3b}} F_{2b}} |F_{3b}, M_{F_{3b}}(F_{2b})\rangle \langle F_{3b}, M_{F_{3b}}(F_{2b})|. \quad \text{(S28)}
$$

Here, since the three-body states form a complete set of orthonormal states, the projection operators satisfy the following

$$
\sum_{F_{3b}} P_{F_{3b}} = 1. \quad \text{(S29)}
$$

Spin-exchange terms are determined by expressing them in terms of the projection operators. Using Eq. (S29) and the usual angular momentum relations, we have

$$
\sum_{i<j} \tilde{f}_i \cdot \tilde{f}_j = \frac{1}{2} \left( F_{3b}^2 - \sum_k f_k^2 \right)
$$

$$
= \frac{1}{2} \left( F_{3b}^2 - \sum_k f_k^2 \right) \sum \mathcal{P}_{F_{3b}},
$$

$$
= \frac{1}{2} \left[ F_{3b}(F_{3b} + 1) - 3f(f + 1) \right] \mathcal{P}_{F_{3b}}, \quad \text{(S30)}
$$

where $\tilde{f}_i$ ($i=1, 2$ and $3$) is the atomic hyperfine angular momentum for the atom $i$. For $f = 1$ atoms, the above equation gives

$$
\sum_{i<j} \tilde{f}_i \cdot \tilde{f}_j = 3\mathcal{P}_{3} - 2\mathcal{P}_{1} - 3\mathcal{P}_{0}, \quad \text{(S31)}
$$

and using $\mathcal{P}_{3} = 1 - \mathcal{P}_{0} - \mathcal{P}_{1} - \mathcal{P}_{2}$ from Eq. (S29) we have

$$
\mathcal{P}_{1} = \frac{3 - \sum_{i<j} \tilde{f}_i \cdot \tilde{f}_j - 3\mathcal{P}_{2} - 6\mathcal{P}_{0}}{5}. \quad \text{(S32)}
$$

Now, using Eqs. (S29) and (S32), we can express $A_{3b}$ in Eq. (S27) as

$$
\hat{A}_{3b} = a_{3b} + \alpha_{3b} \sum_{i<j} \tilde{f}_i \cdot \tilde{f}_j, \quad \text{(S33)}
$$

where the direct and spin-exchange terms are given by

$$
a_{3b} = \frac{3a_{1b}^{(1)} + 2a_{3b}^{(3)}}{5} \quad \text{and} \quad \alpha_{3b} = \frac{a_{3b}^{(3)} - a_{3b}^{(1)}}{5}. \quad \text{(S34)}
$$

Note that, since $F_{3b} = 0$ and $2$ interactions are suppressed (see discussion in Section I), we neglected in Eq. (S33) the projection on the corresponding subspaces by setting $\mathcal{P}_{0} = 0$ and $\mathcal{P}_{2} = 0$. Also we note that, similar to the two-body case, the three-body spin-exchange term only describes processes in which $M_{3b}$ is conserved, i.e., $|m_{f_1}m_{f_2}m_{f_3}| = 000 \leftrightarrow |\pm 10 \mp 1$ or $|\pm 100 \leftrightarrow |\pm 1 \pm 1 \mp 1$, as well as cyclic permutations of $|m_{f_1}m_{f_2}m_{f_3}|$.

### IV. ANALYTICAL FORMULAS FOR ATOM-DIMER COLLISIONS

In deriving the scattering length dependence of atom-dimer collisional processes, the influence of Efimov physics in inelastic processes generates interference and resonant effects which are parametrized, respectively, by

$$
M_{s_0}^\eta(a) = \alpha e^{-2\eta \left[ \sin^2 \left( \frac{|s_0|}{\phi_{s_0}} \right) + \sinh^2 \eta \right]}, \quad \text{(S35)}
$$

$$
P_{s_0}^\eta(a) = \beta \frac{\sin 2\eta}{\sin \left( \frac{|s_0|}{\phi_{s_0}} \right) + \sin^2 \eta}, \quad \text{(S36)}
$$

where $\alpha$ and $\beta$ are universal constants and $r_o = \phi_{vdW} e^{-|s_0|}$ is the three-body parameter, incorporating the three-body short-range physics through
the phase $\phi$ \[8\]. $\eta$ is the three-body inelasticity parameter \[8\], \[8\] which encapsulates the probability for decay into deeply bound molecular states. Similarly, Efimov physics also manifest in atom-dimer elastic processes through resonant and interference effects parametrized by

$$T^0_\eta(a) = \alpha + \beta \frac{2 \sin \left( |s_0| \ln \frac{a}{a_0} \right) \cos \left( |s_0| \ln \frac{a}{a_0} \right)}{\sin^2 \left( |s_0| \ln \frac{a}{a_0} \right) + \sinh^2 \eta}. \quad (S37)$$

$$O^0_\eta(a) = \alpha + \beta e^{-2\eta} \left( \sin^2 \left( |s_0| \ln \frac{a}{a_0} \right) + \sinh^2 \eta \right). \quad (S38)$$

In Table SII we summarize the atom-dimer decay rates and elastic parameters for $f = 1$ atoms.

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[8] Y. Wang, J. P. D’Incao, and B. D. Esry, Advances in Atomic, Molecular, and Optical Physics 62, 1 (2013).
TABLE SII: Scattering length dependence for \( F_{3b} = 1, 2 \) and 3 collision processes relevant for \( f=1 \) atom-dimer spinor condensates. Here, \( K^{(0/4)}_{\text{ad}} \) and \( K^{(2/4)}_{\text{ad}} \) are the decay rate of weakly bound \( F_{2b} = 0 \) and 2 dimers into deeply bound states. Whenever \( a_0 > 0 \) and \( a_2 > 0 \), \( K^{(0/2)}_{\text{ad}} \) gives the decay rate for \( F_{2b} = 0, F_{2b} = 2 \) dimers into \( F_{2b} = 2 \) (\( F_{2b} = 0 \)) dimers due to collisions of atoms in states \( |m_f\rangle \). \( a^{(F_{2b})}_{\text{ad}} \) is the atom-dimer scattering length. Here, \( k_{\text{ad}} = 2\mu_{\text{ad}}E_{\text{col}}, M(a), P(a), T(a) \) and \( O(a) \) are given in Eqs. \( \text{S35-S38} \), \( s_1 \approx 0.7429 \) (|\( a_2 | \gg |a_0 | \rangle \) and \( s_1 \approx 0.4097 \) (|\( a_0 | \gg |a_2 | \rangle \) for \( F_{3b} = 1 \) and \( s_1 \approx 2.1662 \) for \( F_{3b} = 2 \).

| \( F_{3b} = 1 \) | \( a_2 \gg a_0 \) | \( a_2 \gg |a_0| \) | \( |a_2| \gg a_0 \) |
|---|---|---|---|
| \( K^{(0/4)}_{\text{ad}}/a_0 \) | \( P^{(0)}_{\text{ad}}(a_0) \) | \( \frac{a_2}{a_0} \) | \( P^{(0)}_{\text{ad}}(a_0) \) |
| \( K^{(2/4)}_{\text{ad}}/a_2 \) | \( \frac{a_2}{a_0} \) | \( \frac{a_2}{a_0} \) | \( - \) |
| \( K^{(2/0)}_{\text{ad}}/a_2 \) | \( M^{(2)}_{\text{ad}}(a_0) \left( \frac{a_2}{a_0} \right)^{2s_1} \) | \( - \) | \( - \) |
| \( K^{(0/2)}_{\text{ad}}/a_2 \) | \( K^{(2/0)}_{\text{ad}} \left( \frac{a_2}{a_0} \right)^{2s_1} \) | \( k_{\text{ad}} \) | \( - \) |
| \( a^{(0)}_{\text{ad}}/a_0 \) | \( T^{(0)}_{\text{ad}}(a_0) \) | \( - \) | \( T^{(0)}_{\text{ad}}(a_0) \) |
| \( a^{(2)}_{\text{ad}}/a_2 \) | \( 1 + O^{(2)}_{\text{ad}}(a_0) \left( \frac{a_2}{a_0} \right)^{4s_1} \) | \( 1 + T^{(2)}_{\text{ad}}(a_0) \left( \frac{a_2}{a_0} \right)^{4s_1} \) | \( - \) |

| \( F_{3b} = 2 \) | \( a_2 \gg r_0 \) | \( F_{3b} = 3 \) | \( a_2 \gg r_0 \) |
|---|---|---|---|
| \( K^{(2/2)}_{\text{ad}}/a_2 \) | \( \frac{r_{\text{vdW}}}{a_2} \) | \( K^{(2/2)}_{\text{ad}}/a_2 \) | \( P^{(0)}_{\text{ad}}(a_2) \) |
| \( a^{(2)}_{\text{ad}}/a_2 \) | \( 1 + \left( \frac{r_{\text{vdW}}}{a_2} \right)^{4s_1} \) | \( a^{(2)}_{\text{ad}}/a_2 \) | \( T^{(2)}_{\text{ad}}(a_2) \) |