Teleparallel Darkness.

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First a review of Teleparallel theory is done with special emphasis on the derivation of conservation equations within this theory and in particular of energy-momentum conservation. Given that we are allowed to speak about the existence of negative energy, the question is that in its interaction with matter, we need not have matter conservation: It is only the sum of both which should remain constant. This does not only leads to an accelerated expansion without the need of a cosmological constant, but it may also contribute to explain the origin of dark matter, and poses questions about the origin of inflation at earlier times. The prediction of the proposed model can be qualitatively compared to recent results of Cosmic Microwave Background (CMB) analysis.

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I. CONSERVATION EQUATIONS

We follow most of the conventions and notations of references [1] and [2]. We also adopt the (+,−,−,−) Lorentzian metric convention. Let us also clarify that greek indices are used for the coordinate holonomic quantities and latin indices are used for non-holonomic ones.

One way of viewing the kind of Riemannian spaces in which Teleparallel theories are formulated, is by taking as departure point the hypothesis that physics somehow establishes a canonical, smooth, path-independent isomorphism between the tangent spaces of any two points of the manifold and hence that we may take some orthonormal, but otherwise arbitrary reference basis, which we will call $\vec{u}_a$, and refer all vectors at all points to that basis (of course, we may identify $\vec{u}_o$ with its preimage at any given point). Parallel transport of vectors and tensors can then be introduced as meaning to transport them keeping constant their components with respect to this basis. Then Cartan covariant derivative is just the variation with respect to this reference basis expressed, for example, in terms of the coordinate basis. Mathematically, this has the consequence of accepting only parallelizable manifolds as physically meaningful. This is nothing more than just a topological condition on the sort of Riemannian spaces we deal with. In particular the coordinate vectors $\partial_\mu(x)$, no matter which sets of coordinates $x$ we take, should be expressible in terms of the reference basis:

$$\partial_\mu(x) = h^a_\mu(x) \vec{u}_a$$

(1)

Let us also remember that the relationship between the Levi-Civita covariant derivative $\overset{o}{\nabla}$ due to the symmetric Riemannian connection and the “Cartan” covariant derivative $\nabla$ of the Weitzenböck connection is given by the difference between the Christoffel symbols of both covariant derivatives:

$$\Gamma^\rho_{\mu\nu} = \overset{o}{\Gamma}^\rho_{\mu\nu} + K^\rho_{\mu\nu}$$

(2)

Where $K^\rho_{\mu\nu}$ is the contorsion tensor given by:

$$K^\rho_{\mu\nu} = \frac{1}{2} (g^{\rho\sigma} [T_{\mu\nu\sigma} + T_{\nu\sigma\mu}] - T^\rho_{\mu\nu})$$

(3)

and hereafter (within the sign conventions adopted in this paper) $T^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu} - \Gamma^\rho_{\mu\nu}$ is the torsion tensor of the Weitzenböck connection. Riemannian connection has curvature and no torsion and Weitzenböck connection has torsion and no curvature. Now, as a first point in the discussion we must clarify what can be considered a conservation equation for the energy-momentum tensor within Teleparallel theory. Suppose we had some (symmetric) energy-momentum tensor $S^\rho_\mu$ and let $w^\rho$ be the components of some vector which is Cartan covariant constant. Such vectors do exist because any linear constant combination of vectors of the reference basis $\vec{u}_a$ is Cartan covariant constant. It is only when we express them in terms of the coordinate reference base $\partial_\mu(x)$ that they seem to be position-dependent. $S^\nu_\mu w^\rho$ represents the flow of the component of energy-momentum in the direction of the four-vector $\vec{w}$, so $\overset{o}{\nabla}_\nu (S^\nu_\mu w^\rho) = 0$ expresses the conservation of such $\vec{w}$ component. However we can write this as:

$$0 = \overset{o}{\nabla}_\nu (S^\nu_\mu w^\rho) = \nabla_\nu (S^\nu_\mu w^\rho) - S^\rho_\mu w^\nu K^\nu_{\rho\nu}$$

$$= w^\rho (\nabla_\nu S^\nu_\mu - S^\nu_\mu \rho T^\nu_{\nu\rho})$$

(4)

If we want conservation of energy-momentum in all directions, then it must hold:

$$\nabla_\nu S^\nu_\mu - S^\nu_\mu \rho T^\nu_{\nu\rho} = 0$$

(5)

To the best of my knowledge this condition has never been put in such a explicit covariant form by any another author. This condition is not the same as the condition $\overset{o}{\nabla}_\nu S^\nu_\mu = 0$. If we substitute the Cartan covariant derivative by its classical counterpart, we reach another expression for this conservation law:

$$\overset{o}{\nabla}_\nu S^\nu_\mu - S^\nu_\mu \rho K^\rho_{\nu\mu} = 0$$

(6)

Of course, these equations reduce to the zero divergence condition in Minkowski space, however the important
point is that (for second rank tensors in Teleparallel spaces) they represent the correct generalization of the zero divergence condition of Minkowski space.

There is a second way of obtaining the same result which might be somewhat more transparent and also more easily generalizable to other types of tensors. Suppose we are given some tensor $S^\mu\nu$ and then we express it partially in terms of the holonomic base $\partial_\alpha$ and partially in terms of the arbitrary reference base $\vec{u}_a$. A vector $\vec{V}$ can be expressed in terms of either base:

$$\vec{V} = V^\alpha \partial_\alpha = V^a \vec{u}_a = V^a \vec{u}_a,$$

hence $V^\alpha = V^a h^a_\alpha$. So let us form the quantities:

$$S^{\mu\nu} = S^{\mu\nu} h^a_\nu$$  \hspace{1cm} (7)

As long as we keep fixed the arbitrary reference base, this quantities will only be transformed in their first index in any coordinate transformation. And they will be transformed as vectors because the second index is not transformed. We will just have a set of four vectors instead of a tensor. Hence we can apply to them the divergence theorem, which says ([3], pg. 43):

$$\int_M \text{div} \vec{V} \, dv = \int_{\partial M} \vec{V} \cdot \vec{N} \, d\vec{v}$$  \hspace{1cm} (8)

Where $\vec{N}$ is just the normal, oriented to the exterior, of $\partial M$. So to get a conservation equation we just must impose the condition $\text{div} \vec{V} = 0$. A expression for this condition in coordinates is ([4], pg. 384):

$$\text{div} \left( V^\alpha \frac{\partial}{\partial x^\alpha} \right) = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \left( V^\alpha \sqrt{-g} \right) = 0$$  \hspace{1cm} (9)

In our case we have the vectors $S^{\mu\alpha}$, so the conservation equations can be written as:

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( S^{\mu\alpha} \sqrt{-g} \right) = 0$$  \hspace{1cm} (10)

One equation for each component with respect to the reference base. Now we know that:

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \sqrt{-g} = h^\alpha_\beta \frac{\partial}{\partial x^\mu} \Gamma^\beta_{\beta\mu}$$  \hspace{1cm} (11)

So we can write the conservation equation as:

$$0 = \frac{\partial S^{\mu\alpha}}{\partial x^\mu} + S^{\mu\alpha} \Gamma^\beta_{\beta\mu}$$  \hspace{1cm} (12)

Now let us multiply by $h_{\alpha\gamma}$ (the inverse matrix of $h^{\alpha\alpha}$), taking into account that:

$$S^{\mu\alpha} = S^{\mu\beta} h^a_\alpha \iff S^{\mu\alpha} h_{\alpha\gamma} = S^{\mu\gamma}$$  \hspace{1cm} (13)

So we have:

$$0 = \frac{\partial S^{\mu\beta}}{\partial x^\mu} h^0_\beta h_{\alpha\gamma} + S^{\mu\beta} h_{\alpha\gamma} \frac{\partial h^0_\beta}{\partial x^\mu} + S^{\mu\gamma} \Gamma^\beta_{\beta\mu}$$  \hspace{1cm} (14)

Remembering also that:

$$h_{\alpha\gamma} \frac{\partial h^0_\beta}{\partial x^\mu} = \Gamma^\beta_{\beta\mu}$$  \hspace{1cm} (15)

We reach the conclusion that the conservation equation can be written as:

$$0 = \frac{\partial S^{\mu\gamma}}{\partial x^\mu} + S^{\mu\beta} \Gamma^\gamma_{\beta\mu} + S^{\mu\gamma} \Gamma^\beta_{\beta\mu}$$  \hspace{1cm} (16)

However this is not a manifestly covariant equation, but we can put it a more convenient form taking into account the definition of the covariant derivative, written in the form:

$$\frac{\partial S^{\mu\gamma}}{\partial x^\mu} = \nabla_\mu S^{\mu\gamma} - S^{\mu\beta} \Gamma^\gamma_{\beta\mu} - S^{\beta\gamma} \Gamma^\mu_{\beta\mu}$$  \hspace{1cm} (17)

So substituting we reach the conclusion that the conservation equation can be written in a manifestly covariant form:

$$0 = \nabla_\alpha S^{\alpha\gamma} + S^{\beta\gamma} T^\alpha_{\beta\alpha}$$  \hspace{1cm} (18)

Which is the result we found before so, in some sense, nothing new has been invented: what has been done is just to rewrite for parallelizable spaces the condition imposed by the divergence theorem for having conserved quantities. But now, this also gives a method for writing conservation equations for higher order tensors. Let $S^{\alpha\beta\gamma}$ be a third order tensor. By the same method it is easy to find that the conservation equation can be written as:

$$0 = \nabla_\alpha S^{\alpha\beta\gamma} + S^{\beta\gamma} T^\alpha_{\beta\alpha}$$  \hspace{1cm} (19)

Very similar to previous equation (18): just one more index. The main reason why it is so easy to formulate covariant conservation equations in these spaces is that we can add magnitudes which belong to different points, because we can refer all of them to a common reference (otherwise arbitrary) base $\vec{u}_a$. In a general Riemannian space, it is only possible add magnitudes defined in different points if they are scalars, if they are vectors or higher order tensors, the comparison between them depends on the path used to carry them from one point to the other, so talking about global values makes very little sense, unless the magnitude is a scalar. So it is natural to have conservation equations for scalars (let us say electric charge), but it is difficult to even think about what is meant for a conservation equation of a global magnitude which is not a scalar: it must be clearly stated the coordinate system and how to add values. And certainly energy or momentum are not scalars.

So there is another reason for considering more seriously parallelizable spaces as the sort of abstract mathematical spaces to be used in physical theories: not only they are the only kind of spaces in which spinorial structures can be defined as was shown in [5], but they are also needed to formulate conservation equations of magnitudes which are not scalars.

II. ENERGY-MOMENTUM CONSERVATION.

As it is well known Einstein’s tensor $G^{\mu\nu}$ verifies $\nabla_{\nu} G^{\mu\nu} = 0$, so from Einstein’s equation we do not get a
conservation equation for the energy-momentum tensor of matter alone. For the energy-momentum tensor $T$ of matter, accepting it to be a symmetric tensor, we rather get that

$$\nabla_\nu T^\mu_\nu - T^\mu_\nu \rho^\nu_\nu = 0$$

Comparing it with (5) we see it is not quite exactly the same, a further term is present. Let us remember that we are dealing with a lagrangian density $\Lambda$ for the gravitational field which, in Teleparallel theories, might in general be written as (we take a system of units in which $c = 1$):

$$\Lambda = \kappa_g (a_1 \Lambda_1 + a_2 \Lambda_2 + a_3 \Lambda_3)$$

where:

$$\Lambda_1 = g^\alpha_\nu T^\alpha_\nu \rho_\alpha T^\beta_\beta \mu$$
$$\Lambda_2 = g^\alpha_\nu T^\alpha_\nu \beta \lambda T^\beta_\alpha \sigma$$
$$\Lambda_3 = T^\nu_\nu T^{\rho\mu}_\rho T^{\mu\nu}_\nu$$

The “default” values: $a_1 = 1$, $a_2 = -1/2$, $a_3 = -1/4$, produce a lagrangian density equivalent to General Relativity, meaning that for those values the difference between $\bar{R}$ and $(a_1 \Lambda_1 + a_2 \Lambda_2 + a_3 \Lambda_3)$ is just a total divergence. As a matter of fact, adding $2g^{\alpha\beta} \nabla_\alpha T^\lambda_\beta$ one obtains $\bar{R}$. However, for the moment, we are not going to fix the value of those coefficients, we want to keep open the possibility of choosing other values for them.

One way of obtaining the gravitational energy-momentum tensor is to directly reproduce the reasoning which in classical mechanics leads to the conservation of energy. Taking $h$ as the determinant of $h^{\alpha\nu}$ we then write (we follow the standard work [6], as a matter of fact is just the first step of applying Noether’s method considering the translational invariance of the lagrangian, see also [7], section 2):

$$\frac{\partial \Lambda(h)}{\partial x^\alpha} = \frac{\partial \Lambda(h)}{\partial h^{\alpha\nu}} \frac{\partial h^{\alpha\nu}}{\partial x^\mu} + \frac{\partial \Lambda(h)}{\partial h^{\alpha\nu}_\mu} \frac{\partial h^{\alpha\nu}_\mu}{\partial x^\mu}$$

The field equations can be written as:

$$\frac{1}{h} h^a \sigma \left[ \frac{\partial \Lambda(h)}{\partial h^a \beta} - \frac{\partial \Lambda(h)}{\partial h^a \beta \nu, \gamma} \right] + T^\nu_\nu = 0$$

And taking them into account one immediately is led to:

$$\frac{1}{h} \frac{\partial}{\partial x^\gamma} \left[ h \left( h^{a\nu}_\mu, \frac{\partial \Lambda(h)}{\partial h^{a\nu}_\mu} - \Lambda^\gamma_\mu \right) \right] - T^\nu_\nu T^\sigma_\nu = 0$$

One would like to identify the term within the round brackets in last equation with the energy-momentum tensor, but it is not a tensor, so the idea is to decompose it into tensorial and non-tensorial terms, hence we write previous equation as:

$$\frac{1}{h} \frac{\partial}{\partial x^\gamma} \left[ h (Q^\gamma_\mu + N^\gamma_\mu) \right] - T^\nu_\nu T^\sigma_\nu = 0$$

Where $Q^\gamma_\mu$ is the tensorial part and $N^\gamma_\mu$ is a non-tensorial term. Expressing the partial derivative of the $Q$ tensor as a Cartan covariant derivative one arrives immediately to the following equation:

$$0 = \nabla_\gamma Q^\gamma_\mu - Q^\gamma_\mu T^\nu_\nu +$$

$$+T^\nu_\nu N^\gamma_\mu + Q^\gamma_\mu T^\sigma_\nu + \frac{\partial N^\gamma_\mu}{\partial x^\nu} - T^\nu_\nu T^\sigma_\nu$$

Now the problem is to eliminate the non-tensorial terms of this equation. Of course, it must be possible to do so, because it is not possible to have an equality between entities which transform in different ways (just put the first two terms at one side and the other terms at the other side). The needed calculations for eliminating, or transforming, those non-tensorial terms are a bit cumbersome, but with some work it can be shown that by taking $Q^\gamma_\mu$ as:

$$Q^\gamma_\mu = \kappa_g \left[ a_1 \left( g^\alpha_\nu T^\alpha_\nu T^\beta_\beta \mu - g^\alpha_\nu T^\alpha_\nu T^\gamma_\gamma \beta \mu \right) \right]$$
$$+ a_2 \left[ g^\alpha_\nu T^\alpha_\nu T^\beta_\beta \mu - g^\alpha_\nu T^\alpha_\nu T^\gamma_\gamma \beta \mu \right]$$
$$+ a_3 \left[ g^\alpha_\nu T^\alpha_\nu T^\beta_\beta \mu - g^\alpha_\nu T^\alpha_\nu T^\gamma_\gamma \beta \mu \right]$$

one arrives to the conclusion that

$$T^\nu_\nu N^\gamma_\mu + Q^\gamma_\mu T^\sigma_\nu + \frac{\partial N^\gamma_\mu}{\partial x^\nu} = T^\nu_\nu T^\sigma_\nu$$

The expression for $Q^\gamma_\mu$ might seem strange at first sight, but it happens to be exactly $-j^\gamma_\mu$ (the fully covariant version of the so called gauge current found in [1], see also [8], [9]):

$$\frac{1}{h} h^a \sigma \frac{\partial \Lambda(h)}{\partial h^a \gamma} = j^\gamma_\mu = -Q^\gamma_\mu$$

So substituting (30) into (28), we are led to the result:

$$\nabla_\gamma j^\gamma_\mu - j^\gamma_\mu T^\nu_\nu + T^\nu_\nu T^\sigma_\nu = 0$$

So taking into account both equations (32) and (20) one gets:

$$\nabla_\gamma \left( T^\gamma_\nu + j^\gamma_\nu \right) - (T^\gamma_\nu + j^\gamma_\nu) T^\sigma_\nu = 0$$

Which has exactly the form of equation (5) and so it can be interpreted as just expressing the conservation of total energy-momentum: the energy-momentum of the material fields $T^\gamma_\nu$ plus the energy-momentum of the gravitational field $j^\gamma_\nu$. It also clarifies the meaning of the term $T^\nu_\nu T^\sigma_\nu$. This term specifies the energy-momentum interchange between the gravitational field and the material one. It is the term which prevents energy-momentum of the gravitational field or energy-momentum of the material field from being conserved separately by themselves. The interpretation of $j^\gamma_\nu$ as the correct energy-momentum tensor for the gravitational field can be further underlined if we rewrite the field equations (25) as:

$$\frac{1}{h} h^a \sigma \frac{\partial \Lambda(h)}{\partial h^a \gamma} = j^\gamma_\nu + T^\nu_\nu$$
and compares it with the equations for the electromagnetic field in Minkowski space written as:

\[ \frac{\partial}{\partial x^\gamma} \left( \frac{\partial A_\nu}{\partial A_{\alpha \gamma}} \right) = -J^\nu \]  

(35)

Informally, it is usually accepted that this last equation says that currents are the sources of the electromagnetic field. Then, in the same sense, the previous equation might be interpreted as saying that energy-momentum current density has the uncomfortable characteristic that in general it is not symmetric. But if we accept the default values for the coefficients \( a_1, a_2, a_3 \) then there is an easy way out of the problem: Einstein’s equations are a total of ten equations, however they do not completely determine the metric tensor. Diffeomorphisms comprise the gauge freedom in General Relativity: any two solutions which are related by a diffeomorphism represent the same physical solution. Now, if instead of considering the metric tensor as the final solution of a gravitational problem, we ask a complete solution of such a problem to be given by the specification of the sixteen functions \( h^a_{\alpha} \) which give the coordinate basis vectors in terms of the arbitrary constant reference basis, then we have some further freedom, because once the metric tensor is given, we may have several “square roots” \( h^a_{\alpha} \) which produce the same metric tensor: There are sixteen arbitrary \( h^a_{\alpha} \) functions and only ten independent conditions imposed by the metric tensor. We have lots of “gauge freedom”, so let us use part of that freedom to decree that the antisymmetric part of the energy-momentum tensor \( j^{\sigma \nu} \) should be zero. This “gauge condition” amounts just to a set of six equations, because in a four dimensional space an antisymmetric second rank tensor has only six independent components. So, although it is a crude way of counting degrees of freedom, we have increased by six the number of unknown functions when substituting the metric tensor as solution by the \( h^a_{\alpha} \); but we have also added six additional equations, so we expect not to have changed the “gauge freedom”. The gauge condition can be written as:

\[ j^{[\mu \nu]} = 0 = (g^{\mu \gamma} T^\nu_{\beta \mu} - g^{\mu \nu} T^\gamma_{\beta \mu}) g^{\lambda \delta} T^\alpha_{\lambda \alpha} - \frac{1}{2} (g^{\mu \gamma} T^\nu_{\beta \lambda} - g^{\mu \nu} T^\gamma_{\beta \lambda}) T^{\beta \lambda}_{\mu} \]  

(36)

Of course it is a covariant condition: true in one coordinate system means true in all, so we are not limiting the set of coordinates in which the theory is formulated: we are not imposing conditions on the sort of diffeomorphisms which might be used. As a matter of fact, it has previously been argued that Teleparallel theories may have too much gauge freedom (see [10], [11]) and that they suffer from a problem of non-predictability of torsion. Although a formal proof should be investigated, we clearly expect this gauge condition to fix the origin of such problems. At least, the introduction of this condition invalidates the reasoning supporting such assertions, because clearly these additional six equations have not been taken into account when studying the predictability of torsion. Furthermore: being an algebraic condition on the torsion tensor, not every boundary condition is acceptable, because the boundary condition must also obey the gauge condition. It must be noted that those problems reappear again [12] in more recent works when analyzing dark energy as is done for example in [13] and [14].

Accepting such a gauge condition, the energy-momentum field turns out to be symmetric. And furthermore it is immediate to check that it has zero trace. Needless to say it is a perfectly covariant local definition of energy-momentum for the gravitational field.

There is one further point which merits some comment. Teleparallel theories have some degree of freedom in the way the coefficients are chosen. However it is only for the case in which we obtain the Teleparallel equivalent to General Relativity (TEGR) when the resulting equations happen to be symmetric (we obtain Einstein’s tensor). So it is only in this case in which we have the freedom to impose that the energy-momentum tensor of the gravitational field should be symmetric. So this condition eliminates the rest of possibilities for the coefficients.

Anyway, we leave this section with the final expression for the energy-momentum tensor of the gravitational field in the only case which interests us: for the “default values” of the parameters which make Teleparallel theory almost equivalent General Relativity, in the sense that Einstein’s equations are also obtained:

\[ j^{\lambda \gamma} = \kappa g \left[ g^{\mu \nu} (T_{\alpha \alpha} T^{\gamma \nu \lambda} + T^{\lambda \nu \gamma}) - 2 T_{\alpha \alpha}^{\gamma} T^{\beta \beta \lambda} \right. \\
\left. + T_{\alpha \beta}^{\gamma} T^{\beta \beta \lambda} - \frac{1}{2}(T_{\beta \alpha}^{\gamma} T^{\beta \beta \lambda} + T_{\alpha \beta}^{\gamma} T^{\beta \beta \lambda}) + T_{\gamma \alpha \beta} T^{\gamma \alpha \lambda} + \left( \frac{\Lambda_1}{2} - \frac{1}{2} \Lambda_2 - \frac{1}{4} \Lambda_3 \right) g^{\gamma \lambda} \right] \]  

(37)

### III. THE FLAT UNIVERSE CASE.

Let us write the gravitational energy-momentum tensor in the specially important case of flat (homogeneous isotropic) universe. The cases of positive and negative curvature can be analyzed similarly (they are also parallelizable spaces) although it is more cumbersome, and it is done elsewhere [15]. As a matter of fact, the class of parallelizable spaces is quite big, as it is shown in [16] (taking into account that every noncompact space, on which a spinorial structure might be defined, is parallelizable [9]). We use a “cartesian” coordinate system with coordinates \( x^0 = ct, x^1 = x, x^2 = y, x^3 = z \), and we postulate the following matrix of gravitational potential vectors (the role played by the coordinate vectors is similar to that of vector potentials):

\[ h^a_{\alpha} = \text{diag} \left( 1, a(t), a(t), a(t) \right) \]  

(38)
Using this potentials, the metric is just the very well known diagonal metric of flat space:

\[ g_{\mu\nu} = \text{diag}(1, -a^2(t), -a^2(t), -a^2(t)) \] (39)

The energy-momentum tensor for such a space is:

\[ j_0^0 = -6\kappa_g \left( \frac{\dot{a}(t)}{a(t)} \right)^2 \quad j_1^1 = j_2^2 = j_3^3 = 2\kappa_g \left( \frac{\dot{a}(t)}{a(t)} \right)^2 \] (40)

Where the dot signals ordinary differentiation with respect to ct. The first point which deserves attention is that energy density is negative, so it seems that there is at least a known field whose energy density takes negative values. Being proportional to the square of the Hubble parameter, it can be said that it is purely “kinetical” in this case: it is proportional to the square of the speed at which \(a(t)\) changes, and the minus sign tells us that absorption of (positive) energy will decrease this speed, as energy density becomes less negative. In the other two cases, of positive and negative curvature, energy density also happens to turn out negative.

It might be argued that experimental evidence disagrees with gravitational field having negative energy. After all, there is quite good experimental indirect evidence of the radiation of gravitational energy from binary pulsars \([17][18]\). They lose energy through radiation. Well, this is just a problem of how to interpret experimental evidence. An outgoing wave carrying energy outwards can be interpreted as an outgoing wave of negative energy incoming particles. A situation similar to a waterpool with an outlet just in the center of its floor, which is suddenly opened: there is an outgoing wave of incoming particles. The main difference in this respect is the sign of the energy of the incoming particles. So to speak, negative energy gravitons are just falling into the potential well.

It should be almost intuitively obvious: If there are a couple of pulsars losing energy through gravitational radiation, then they may even collapse and form a black hole, so in this situation the gravitational field increases with time. This is not like the case of two electrical particles attracting each other, they radiate electromagnetic energy as they approach each other, but the field decreases because the dipolar moment decreases when they are nearer. Now, if the gravitational field were a positive energy field, it is not obvious how could the gravitational field increase if radiation consisted in the emission of positive energy gravitons. And if one accepts it to be a negative energy field, then it is very difficult to accept that its radiation is formed by positive energy particles.

In the case of a dust-filled universe \(a(t) = C_0 t^{2/3}\), so the gravitational energy density is proportional to \(-t^{-2}\) which, when multiplied by \(a^3 \propto t^2\) to take into account the increase in volume, just gives constant energy (per comoving volume): dust does not contribute to any variation of energy of the gravitational field, it does not interchange energy with the gravitational field.

In the case of a universe filled with just radiation the solution for \(a(t)\) is of the form \(a(t) = C_0 t^{1/2}\). So the gravitational energy density is also proportional to \(-t^{-2}\) (it is the square of a logarithmic derivative, so no matter the exponent it will be proportional to \(-t^{-2}\)), which when multiplied by \(a^3(t)\), to take into account the increase of volume, gives the result that energy of gravitational field changes as \(-t^{-1/2}\), which is an increase and which is just the rate needed to compensate the rate at which energy of radiation decreases: \(pa^4\) is constant, so \(pa^3\) decreases as \(a^{-1} \propto t^{-1/2}\). The absorption of positive energy from radiation just decreases the rate at which universe expands. In the dust-filled case, the decrease in speed is just to compensate the increase in volume, so that total energy is the same. As energy of light is absorbed by the gravitational field, its “kinetic” energy increases (it decreases its “speed”). A radiation dominated universe expands at a slower rate \((t^{1/2})\) than a dust filled one \((t^{2/3})\), absorption of energy decreases its speed.

Even more clear, for the flat universe Einstein’s equation can be used to calculate the energy-momentum tensor of matter:

\[ T_0^0 = 6\kappa_g \left( \frac{\dot{a}}{a} \right)^2 \] (41)

\[ T_1^1 = T_2^2 = T_3^3 = 2\kappa_g \left( \frac{\dot{a}^2 + 2\ddot{a}a}{a^2} \right) \] (42)

Looking at equation (41) we see that the energy density of the matter fields is just the same (but positive) as the energy density of the gravitational field, so that total energy is zero. This agrees with the idea of a zero initial condition for the universe.

**IV. DARK ENERGY AND DARK MATTER.**

We have already argued that equation (5) expresses the conservation of energy-momentum, so let us write that equation as:

\[ \Box \nu S_{\mu}^{\nu} = \nabla_{\nu} S_{\mu}^{\nu} - S_{\mu}^{\rho} T^{\nu}_{\mu \rho} = 0 \] (43)

We have also seen in equation (20) that Einstein’s equation implies:

\[ \Box \nu T_{\mu}^{\nu} = -T_{\sigma}^{\nu} T^{\sigma}_{\nu \mu} \] (44)

Where \(T\) is the energy-momentum tensor of matter fields. These means that energy-momentum of material fields is not conserved if the right hand side is different from zero. Let us suppose we are dealing with a perfect fluid in an isotropic homogeneous flat universe. The energy-momentum tensor can be written in such case as:

\[ T_{\mu\nu} = \text{diag} \left( \rho, pa^2, pa^2, pa^2 \right) \] (45)

Where \(\rho = \rho(t)\) is the mass-energy density, \(p = p(t)\) the pressure and the \(a^2(t)\) factors come from the metric. The
right hand of equation \((44)\) can be easily computed and one obtains:
\[
\nabla_\mu T_\mu^\nu = \left( -\frac{3}{a}\dot{a}p(t), 0, 0, 0 \right) \quad (46)
\]

The first thing which stands out is that if \(p(t) \neq 0\) then in an expanding universe, mass-energy of the material field (by itself) is not conserved. As a matter of fact the temporal component of this equation is:
\[
\dot{\rho} + 3\rho \frac{\dot{a}}{a} = -\frac{3}{a}\dot{a}p \quad (47)
\]

Which just happens to be another form of writing the “classical conservation equation”:
\[
\nabla_\mu T_\mu^\nu = 0 \iff \dot{\rho} + 3(\rho + p)\frac{\dot{a}}{a} = 0 \quad (48)
\]

We have already seen such a non-conservation behaviour when we considered the radiation-filled universe before: a positive pressure means that the gravitational field drains the positive energy from the “material” field. As a matter of fact we have also seen the case when we considered the radiation-filled universe before: a small “function” \(\lambda\) in the righthand side of equation \((47)\) says that this process produces a negative pressure component which fuels the acceleration of the expansion.

Of course, in such a case we do not know what exactly to put in the righthand side of \((47)\). If they exist, we have for the moment no idea about the characteristics of matter generation processes from gravitational fields. Anyway these matter-emission processes must nowadays have very low probability of occurrence because otherwise they would have already been detected. Let us just put a small “function” \(\lambda\) in the righthand side of equation \((47)\), and we’ll try to guess about it later. Considering all pressure coming from matter-emission processes, so that all matter is just dust, then we may write:
\[
\lambda = -3p(t)\frac{\dot{a}}{a} \iff 3p(t) = -\lambda \frac{a}{\dot{a}} \quad (49)
\]

So although the process may have very low probability of occurrence, if \(\lambda\) were constant, the negative pressure would increase with the Hubble time \((T_H = a/\dot{a})\) or in other words, with the expansion of the universe. It would of course overcome the mass term in the equation which determines the acceleration of the expansion of the universe (one of Friedmann’s equations):
\[
3\frac{\dot{a}}{a} = -\frac{1}{4\kappa_g} \left[ \rho(t) + 3p(t) \right] = -\frac{1}{4\kappa_g} \left[ \rho(t) - \lambda \frac{a}{\dot{a}} \right] \quad (50)
\]

From that moment, positive acceleration sets in. We do not need to have a cosmological constant to explain the acceleration. So dark energy in principle could be explained as the (quantum) process of disintegration of gravity into matter. Somehow a graviton emits a material particle and falls into a lower energy state. This can be taken as a possible solution to the cosmological constant puzzle \([19]\).

In fact we cannot consider this \(\lambda\) to be a constant, it may depend on the strength of the gravitational field. We would need a quantum theory of gravitation to be able to calculate \(\lambda(t)\). However, we may get an idea of the order of magnitude of \(\lambda\) by considering zero the acceleration, taking \([20]\) the Hubble constant \(H_0 = \dot{a}/a\) to be 71 \((\text{km/s})/\text{Mpc}\), and taking the density of the universe to be the critical one \(\approx 9.47 \times 10^{-27}\text{kg/m}^3\). We get \(\lambda \approx 2.18 \times 10^{-44}\text{kg/(m}^3\text{s}^2\)). This should be the order of magnitude of the rate at which matter is created at the expense of the gravitational field nowadays. Of course, it says nothing about what sort of particles are created. But one may have some intuition: background gravitons have been expanding since the big bang, so now they are very long wave particles. Considerig their energy to be \(E_g = -h\nu\) and if they are to be responsible of most matter creation, then maybe they can only be involved in (relatively) low-energy processes. So the most likely massive particles to be generated in this sort of processes nowadays would be very low mass ones, for example, some sort of low-mass sterile neutrino \([21, 22]\) would be an excellent candidate for dark (non-detectable) matter. And if energy is so scarce, it is not strange that neutrinos so created are not relativistic. It is interesting to note that, although with different motivations, growing matter scenarios have already been proposed \([23, 24]\).

There is nothing to prevent the gravitational field from falling even further down in energy levels (as it is the usual objection to negative energies), only that nowadays the rate is extremely slow. It does seem that gravitational systems are in fact unstable. Supposedly, a quantum theory of gravitation should be able to explain this rate.

It must be noticed that we still have another Friedmann equation (with \(k = 0\) as we are considering flat space), or just remembering that total energy density is zero, adding both contributions we have:
\[
\rho = 6\kappa_g \left( \frac{\dot{a}}{a} \right)^2 \iff \rho = \frac{3H^2}{8\pi G} \quad (51)
\]

The material energy density must be equal to the critical density so, not having dark energy \(\Omega_\Lambda\) as a component of density, dark matter must be a more substantial contribution to total density. Dark energy used to serve two purposes: contributing to the total energy density and providing the acceleration, for which we now have of another possible mechanism. Substituting the values of \(\rho\) and \(\rho\) in equation \((50)\), we obtain:
\[
3\frac{\dot{a}}{a} = -\frac{1}{4\kappa_g} \left[ 6\kappa_g \left( \frac{\dot{a}}{a} \right)^2 - \lambda \frac{a}{\dot{a}} \right] \quad (52)
\]
To get any further we must guess the form of \( \lambda \). The equation that defines \( \lambda \) in terms of matter creation is:

\[
\lambda = \dot{\rho} + 3\dot{\rho} = \frac{1}{a^3} \frac{d}{dt}(pa^3)
\]

(53)

Where \( pa^3 \) is the matter inside a comoving constant volume. One is tempted to consider that the number of gravitons should be approximately constant because the dissintegration rate into matter is really very weak. Unfortunately that might not be the case. Even if we dismiss the dissintegration rate into matter (the classical dust-only universe in which there is no interchange of energy between matter and the gravitational field) the universe still expands. This means that the frequency of gravitons should decrease and hence their total energy (or the gravitational energy within a comoving volume) should change if their number is constant. So energy conservation forces us to assume that their number cannot be constant: there must be processes in which a graviton dissintegrates itself into other gravitons of lesser frequency (energy conservation imposes that the frequencies of the outgoing gravitons should add up to the frequency of the incoming graviton).

One would like to think that this increase in the wavelength of the gravitons is what, when averaged over many gravitons, causes the universe expansion. But the mechanism must surely be more complicated because, for example, absorption of energy from photons, which should decrease the wavelength of gravitons and according to this idea help expanding the universe, however slows down its expansion compared the classical dust-only universe.

We have very little base to guess the form of the matter creation function \( \lambda \), but we may suppose that the increase of matter inside a comoving volume is proportional to both the number of gravitons within that volume and to the energy of those gravitons, so we guess that we can approximately express the matter creation rate to be proportional to the total energy of the gravitons involved. Now the energy density of the gravitational field (of the gravitons) is proportional to \((\dot{a}/a)^2\), so the total energy of gravitons within constant comoving volume must be \(a^3(\dot{a}/a)^2\) and hence the \( \lambda \) function becomes:

\[
\lambda = \frac{6k_g}{\tau} \left( \frac{\dot{a}}{a} \right)^2 a^3 = \frac{6k_g}{\tau} \left( \frac{\dot{a}}{a} \right)^2
\]

(54)

Where we have introduced the factor \( 6k_g \) in the expression just for commodity and we have written the proportionality constant as \( \tau \) because this constant happens to be a time. Anyway, going back to (52), it leads to:

\[
2\frac{\ddot{a}}{a} = \frac{1}{\tau} \frac{\dot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2
\]

(55)

We can write this equation as:

\[
2\frac{\ddot{a}}{a} = \frac{1}{\tau H} \left( 1 - \frac{1}{T_H} \right)
\]

(56)

Where \( T_H = a/\dot{a} \) is the Hubble time. If we make the approximation that the Hubble time is at every instant the age of the universe (which of course is not true, but just to get an intuition of what’s going on in the equation), \( \tau \) equals the moment in which the acceleration is zero. Before that moment \( T_H \) is less than \( \tau \) and the acceleration is negative, after it \( T_H \) is bigger and the acceleration is positive. So if the age of the universe is about 14.000 million years and acceleration was zero about 6000 million years ago, we can infer the order of magnitude of \( \tau \approx 8 \times 10^9 \text{years} \). The inverse is a small proportionality constant. So let us try to solve equation (55) more exactly. It has no closed-form general solution (at least none which Maple© can find), but we can try to get a Taylor series in terms of \( t \). However if \( \tau \to \infty \), the solution must tend to be that of the dust-only universe: \( C_d t^{2/3} \), which has no Taylor series around \( t = 0 \). So it is better to try first a solution of the form \( C_d t^{2/3} y(t) \) to get rid of the \( t^{2/3} \) term. Then \( y(t) \) can be consider as a correction to the dust-only case and its value should be nearly one in the vicinity \( t = 0 \), if \( \tau \) is big enough. Making such substitution one is immediately led to:

\[
2\frac{\ddot{y}}{y} + \frac{4}{\tau} \frac{\dot{y}}{y} = \frac{1}{\tau} \left( \frac{2}{3\tau} + \frac{\dot{y}}{y} \right) - \left( \frac{\dot{y}}{y} \right)^2
\]

(57)

And now we try a solution of the form \( y = e^{\alpha t} \). We need not put any constant multiplying this function because it is going to disapper, as only quotients between \( y(t) \) and its derivatives occur in previous equation (which is also the reason for such a trial). Substituting we get:

\[
2\ddot{s} + 3\dot{s}^2 + \frac{4}{\tau} \dot{s} - \frac{1}{3\tau^2} \dot{s} - \frac{2}{3\tau} = 0
\]

(58)

This equation has neither a general closed form solution, but we can get its Taylor expansion by trying:

\[
s(t) = \sum_{n=1}^{\infty} S_n t^n
\]

(59)

We do not need a \( S_0 \) term because it amounts to a multiplicative constant. Substituting and equating to zero the terms in \( 1/t \), constant, and so on, we get:

\[
S_1 = \frac{1}{6\tau}; \quad S_2 = \frac{1}{144\tau^2}; \quad \cdots
\]

(60)

Given that \( \tau \approx 8 \times 10^9 \text{years} \), to get \( S_2 \tau^2 = S_1 t \) we must wait till 24\( \tau \), or till 192\( \times \times 10^9 \text{years} \) after the big bang. We will not usually consider the second term, so for “small” times our approximate solution is:

\[
a(t) = C_d t^{2/3} e^{t/6\tau}
\]

(61)

There are a couple of things which is interesting to find. First of all, matter density is given by:

\[
\rho = 6k_g \left( \frac{\dot{a}}{a} \right)^2 = 6k_g \left( \frac{2}{3t} + \frac{1}{6\tau} + \frac{t}{72\tau^2} + \cdots \right)^2
\]

(62)
So it also begins in a state of infinite density, however the final density is a more delicate subject: this is not a bad spot to remember that we are basing our development in just an educated guess about the properties of matter creation, so any sort of predictions for very long times should be taken with caution.

The time at which acceleration is zero is not exactly $\tau$ in this model, but it is given by the condition:

$$0 = 2\frac{\ddot{a}}{a} - \frac{1}{\tau a} \left( \frac{\dot{a}}{a} \right)^2 \implies \frac{\dot{a}}{a} = \frac{1}{\tau} \implies \frac{2}{3t} + \dot{s} = \frac{1}{\tau}$$

(63)

Keeping just the first term in the expansion of $\dot{s}$, and calling $T_a$ the instant in which the acceleration is zero, this equation leads to:

$$T_a = \frac{4}{5} \tau$$

(64)

If one keeps also the second term in the expansion of $\dot{s}$ then one obtains $T_a = 0.7896\tau$ but, given the rough value we are using for $T_a$, it does not make too much sense to try use such precision. Anyway, it means that, if we take $T_a$ to be $8 \times 10^9$ years, then a more reasonable value for $\tau$ is $10^{10}$ years.

More interesting is with how much matter must the universe begin with. If we multiply density by the cube of the scale factor, we get the mass per unit of comoving volume:

$$\rho a^3 = C_{dd}^3 6k_g e^{t/2\tau} \left( \frac{4}{9} \frac{t}{5\tau} + \left( \frac{1}{96} + \frac{1}{54} \right) \frac{t^2}{\tau^2} + \cdots \right)$$

(65)

In the limit $t \to 0$, the mass per comoving volume is:

$$\mu(0) = \frac{8}{3} C_{dd}^3 k_g$$

(66)

Taking our instant of time to be $t_0 = 14 \times 10^9$ years, and $\tau = 10^{10}$ the current value of the mass per comoving volume is:

$$\mu_0 \approx 10.2 C_{dd}^3 k_g$$

(67)

Or 3.8 times the original mass per comoving volume, so this certainly means that dark matter is more than normal matter nowadays, because all the excess is due to matter creation. Now, the development we have made must surely fail for sufficiently early times. It cannot be accepted that the disintegration rate of gravitons has no more physics behind it than the law we have written. It is no more than an approximation forced by ignorance. In particular, for early enough times one should need a quantum theory of gravitation to predict the correct rate of disintegration (and into what do gravitons disintegrate). So if these figures have any sense, they must just reflect the amount of dark matter generated in the matter-dominated era, but as we move further back into the radiation era things should be described in a quite different unknown way. So, it is not unreasonable to suppose that the original matter density of this model is already a mixture of dark matter and normal matter. So last figure amounts to saying that about $1/3.8 = 26.3\%$ of all matter now present already existed at the beginning of the matter dominated era. If one accepts that nowadays normal matter is just about $4.9\%$ of the critical mass, this implies that at the beginning of matter dominated era, dark matter was $21.4\%$ of the matter now existing, or that it was already $4.37$ times more than normal matter. Recombination time does not coincide with the moment of matter/radiation equality, but one can look at the approximations we are using as taking as instantaneous all the radiation era, so it does not make too much sense to distinguish between those moments. Hence this result can be qualitatively compared with the results of CMB analysis, which tell us the composition of the universe at recombination. According to Planck Collaboration data [25], the barion density is given by $\Omega_b h^2 = 0.022$ (well, with more precision depending on which data sets you use) and the dark matter density is given by $\Omega_\Lambda h^2 = 0.12$, so the relation between them is $\Omega_\Lambda/\Omega_b = 5.45$; one may think that a $4.37$ value is not too bad a result given the crude model we are using.

However, even this “agreement” must be taken with lots of caution, apart from the precision in the data we have used:

- First and most important: results of CMB analysis are model-dependent, so of course it is not really correct to mix results for one model with predictions of another. As a matter of fact, only the precision that present mass (per comoving volume) is $3.8$ times more now than at the beginning of matter-dominated era, can be taken as independent of CMB analysis, because it is based on Hubble constant, and the moment on which acceleration of the universe was zero. Values which can be measured independently of CMB analysis.

- And second: we cannot say exactly at what moment are these predictions valid, because we have approximated all the radiation era as having zero duration.

So it is only the qualitative result of predicting a greater abundance of dark matter over normal matter what can be argued in favor of the model. It must be noted that, according to this model, such composition has changed substantially since recombination, dark matter should be much more nowadays: about $95.1\%$, or $19.4$ times more than normal matter, if one applies the figures from the $\Lambda$CDM model. So, although this description transmits a different qualitative story, there are too many conditions and guesses in these figures. Also we would need a better determination of $t_0/\tau$ because, appearing in an exponent, its effect is quite pronounced.

Finally, this model poses questions about the inflation era, because basically nowadays it seems we are viewing graviton disintegration as a highly suppressed process. If a graviton disintegrates into another graviton
and some further particle(s), the energy difference goes to
the other particle(s). It is then not unreasonable to guess
that the process is more likely if both gravitons have en-
ergies of the same order of magnitude, which means that
they must be of sufficiently high energy to create more
massive particles. In sufficiently early times we should
expect to have much more energetic gravitons, because
the lengths involved are much more smaller, and hence
frequencies should be expected to be higher. So it is ex-
pected that many other processes would not be forbidden
and graviton disintegration into matter, which drives the
acceleration of expansion, should be quite a normal pro-
cess, creating in principle all sorts of particles. So one
would not be surprised to find another exponential ex-
ansion era at sufficiently early times. So to speak the
idea is that, if there is such a \( \tau \), it should be many or-
ders of magnitude smaller early in the radiation era and
also the intuition is that this early exponential expansion
would end when disintegration processes become highly
suppressed as the frequency of gravitons decreases.

V. CONCLUSIONS

If one takes seriously energy-momentum conservation
equations in Teleparallel theory, then it is unavoidable to
reach the conclusion that negative pressure means mat-
ter creation from the gravitational field. Saying it other-
wise, it means that matter can be created at the expense
of an acceleration of the universe expansion. This elimi-
nates the problem of the cosmological constant, there is
no need for it. It also means that there is much more
dark matter than usually thought, as we do not have the
cosmological constant to add up to the universe den-
sity. In the model introduced the amount of dark matter
varies with time, as more dark matter is generated in the
expansion (the model qualitatively predicts the relation
between dark matter and normal matter). Furthermore,
if one accepts that gravitons are unstable, one may have
the intuition that graviton disintegration processes are
nowadays highly suppressed: we can only observe them
indirectly through the acceleration of universe expansion,
however those processes might have been much more fre-
quent in sufficiently early times generating the inflation
era. If such a point of view could be held, one could even
invoke Occam’s razor: what could be the point in postu-
ating some other fields, like an hypothetical inflaton, if
one could be able to do just with gravity?

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