Spin models constructed from Hadamard matrices

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Abstract A spin model (for link invariants) is a square matrix $W$ which satisfies certain axioms. For a spin model $W$, it is known that $W^T W^{-1}$ is a permutation matrix, and its order is called the index of $W$. Jaeger and Nomura found spin models of index 2, by modifying the construction of symmetric spin models from Hadamard matrices.

The aim of this paper is to give a construction of spin models of an arbitrary even index from any Hadamard matrix. In particular, we show that our spin models of indices a power of 2 are new.

Keywords Spin model · Association scheme · Hadamard matrix

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1 Introduction

The notion of spin model was introduced by Jones [8] to construct invariants of knots and links. The original definition due to Jones requires that a spin model be a symmetric matrix, but later by Kawagoe, Munemasa, and Watatani [9], a general definition allowing non-symmetric matrices is given. In this paper, we consider spin models which are not necessarily symmetric.

Let $X$ be a non-empty finite set. We denote by $Mat_X(\mathbb{C}^*)$ the set of square matrices with non-zero complex entries whose rows and columns are indexed by $X$. For $W \in Mat_X(\mathbb{C}^*)$ and $x, y \in X$, the $(x, y)$-entry of $W$ is denoted by $W(x, y)$. A spin model
$W \in \text{Mat}_X(\mathbb{C}^*)$ is defined to be a matrix which satisfies two conditions (type II and type III; see Sect. 2).

One of the examples of spin models is a Potts model, defined as follows. Let $X$ be a finite set with $r$ elements, and let $I, J \in \text{Mat}_X(\mathbb{C}^*)$ be the identity matrix and the all 1’s matrix, respectively. Let $u$ be a complex number satisfying

$$\left( u^2 + u^{-2} \right)^2 = r \quad \text{if} \quad r \geq 2, \quad \quad u^4 = 1 \quad \text{if} \quad r = 1. \quad (1)$$

Then a Potts model $A_u$ is defined as

$$A_u = u^3 I - u^{-1} (J - I).$$

As examples of spin models, we know only Potts models [6, 8], spin models on finite abelian groups [3, 4], Jaeger’s Higman-Sims model [6], Hadamard models [7, 11], and tensor products of these. Apart from spin models on finite abelian groups, non-symmetric Hadamard models are essentially the only known family of non-symmetric spin models.

If $W$ is a spin model, then by [7, Proposition 2], $R = W^T W^{-1}$ is a permutation matrix. The order of $R$ as a permutation is called the index of the spin model $W$.

A Hadamard matrix of order $r$ is a square matrix $H$ of size $r$ with entries $\pm 1$ satisfying $HH^T = I$. In [7], Jaeger and Nomura constructed non-symmetric Hadamard models, which are spin models of index 2:

$$W = \left( \begin{array}{cc} \left( \begin{array}{ccc} 1 & 1 \\ 1 & 1 \end{array} \right) \otimes A_u & \left( \begin{array}{ccc} 1 & -1 \\ -1 & 1 \end{array} \right) \otimes \xi H \\ \left( \begin{array}{ccc} -1 & 1 \\ 1 & -1 \end{array} \right) \otimes \xi H^T & \left( \begin{array}{ccc} 1 & 1 \\ 1 & 1 \end{array} \right) \otimes A_u \end{array} \right), \quad (2)$$

where $\xi$ is a primitive 8-th root of unity, $A_u \in \text{Mat}_X(\mathbb{C}^*)$ is a Potts model, and $H \in \text{Mat}_X(\mathbb{C}^*)$ is a Hadamard matrix.

Note that non-symmetric Hadamard models are a modification of the earlier Hadamard models ([7], see also [7, Sect. 5]), defined by

$$W' = \left( \begin{array}{cc} \left( \begin{array}{ccc} 1 & 1 \\ 1 & 1 \end{array} \right) \otimes A_u & \left( \begin{array}{ccc} 1 & -1 \\ -1 & 1 \end{array} \right) \otimes \omega H \\ \left( \begin{array}{ccc} 1 & -1 \\ -1 & 1 \end{array} \right) \otimes \omega H^T & \left( \begin{array}{ccc} 1 & 1 \\ 1 & 1 \end{array} \right) \otimes A_u \end{array} \right), \quad (3)$$

where $\omega$ is a 4-th root of unity.

To construct spin models of index $m > 2$, it seems natural to consider an $m \times m$ block matrix $W = (W_{i,j})_{i,j \in \mathbb{Z}_m}$ such that each block $W_{ij}$ is the tensor product of two matrices like those in (2) and (3):

$$W_{ij} = S_{ij} \otimes T_{ij} \quad (i, j \in \mathbb{Z}_m). \quad (4)$$

Such matrices appeared in [5, Proposition 6.2], with the matrices $S_{ij} \in \text{Mat}_{\mathbb{Z}_m}(\mathbb{C}^*)$ given by

$$S_{ij}(\ell, \ell') = \eta^{(\ell - \ell')(i - j)} \quad (\ell, \ell' \in \mathbb{Z}_m), \quad (5)$$
where η is a primitive m-th root of unity.

In this paper, we construct an infinite class of spin models of even index containing non-symmetric Hadamard models. Also, we construct an infinite class of symmetric spin models containing Hadamard models. Our main result is as follows:

**Theorem 1.1** Let r be a positive integer, and let m be an even positive integer. Define $Y = \{1, \ldots, r\}$, $X_i = \{(i, \ell, x) \mid \ell \in \mathbb{Z}_m, x \in Y\}$ for $i \in \mathbb{Z}_m$, and $X = X_0 \cup \cdots \cup X_{m-1}$. Let $A_u, H \in \text{Mat}_Y(\mathbb{C}^*)$ be a Potts model and a Hadamard matrix, respectively. Define $V_{ij}$ for $i, j \in \mathbb{Z}_m$ by

$$V_{ij} = \begin{cases} A_u & \text{if } i - j \text{ is even}, \\ H & \text{if } (i, j) \equiv (0, 1) \pmod{2}, \\ H^T & \text{if } (i, j) \equiv (1, 0) \pmod{2}. \end{cases}$$

(6)

Then the following statements hold:

(i) Let $a$ be a primitive $2m^2$-th root of unity. Let $W \in \text{Mat}_X(\mathbb{C}^*)$ be the matrix whose $(\alpha, \beta)$ entry is given by $a^{2m(\ell - \ell') + \epsilon(i, j)} V_{ij}(x, y)$ for $\alpha = (i, \ell, x), \beta = (j, \ell', y) \in X$, where $\epsilon(i, j) = (i - j)^2 + m(i - j)$. Then $W$ is a spin model of index $m$.

(ii) Let $\eta$ be a primitive m-th root of unity, and let $b$ be an $m^2$-th root of unity. Let $W' \in \text{Mat}_X(\mathbb{C}^*)$ be the matrix whose $(\alpha, \beta)$ entry is given by $\eta^{(\ell - \ell')(i - j)} b^{\delta(i, j)} V_{ij}$ for $\alpha = (i, \ell, x), \beta = (j, \ell', y) \in X$, where $\delta(i, j) = (i - j)^2$. Then $W'$ is a symmetric spin model.

Note that, in order for $a^{\epsilon(i, j)}$ and $b^{\delta(i, j)}$ to be well-defined, we need to identify $\mathbb{Z}_m$ with the subset $\{0, 1, \ldots, m - 1\}$ of integers.

**Remark 1.2** In Theorem 1.1 (i), if we define $S_{ij}$ by (5) with $\eta = a^{2m}$, and $T_{ij}$ by $T_{ij} = a^{\epsilon(i, j)} V_{ij}$, then the $(X_i, X_j)$-block of the matrix $W$ is given by (4). Similarly, in Theorem 1.1(ii), (4) holds with $T_{ij} = b^{\delta(i, j)} V_{ij}$.

The spin models $W, W'$ given in Theorem 1.1 are determined by a Hadamard matrix $H$ of order $r$, a complex number $u$ satisfying (1), and a primitive $2m^2$-th root of unity $a$ or an $m^2$-th root of unity $b$, respectively. Throughout this paper, we denote by $W_{H,u,a}, W'_{H,u,b}$ the spin models given by Theorem 1.1(i), (ii), respectively.

Observe that, for any spin models $W_i (i = 1, 2)$ of indices $m_i$, their tensor product $W_1 \otimes W_2$ is also a spin model of index $\text{LCM}(m_1, m_2)$. In Sect. 5, we show that the non-symmetric spin model $W_{H,u,a}$ whose index is a power of 2 is new in the following sense:

**Theorem 1.3** Let $H$ be a Hadamard matrix of order $r$. Let $W_{H,u,a}$ be a spin model given in Theorem 1.1(i), whose index $m$ is a power of 2. If $r > 4$, then $W_{H,u,a}$ cannot be decomposed into a tensor product of known spin models.

We note that the list of known spin models is given in Sect. 5. Jaeger and Nomura [7, p. 278] expected that new non-symmetric spin models of index a power of 2 should be found, and our results confirm this expectation.


2 Type II and type III conditions on block matrices of tensor products

First we define a spin model. A type II matrix on a finite set $X$ is a matrix $W \in \text{Mat}_X(\mathbb{C}^*)$ which satisfies the type II condition:

$$\sum_{x \in X} \frac{W(\alpha, x)}{W(\beta, x)} = n \delta_{\alpha, \beta} \quad (\text{for all } \alpha, \beta \in X). \quad (7)$$

Let $W^{-} \in \text{Mat}_X(\mathbb{C}^*)$ be defined by $W^{-}(x, y) = W(y, x) - 1$. Then the type II condition is written as $WW^{-} = nI$. Hence, if $W$ is a type II matrix, then $W$ is non-singular with $W^{-1} = n^{-1}W^{-}$.

A type II matrix $W \in \text{Mat}_X(\mathbb{C}^*)$ is called a spin model if $W$ satisfies the type III condition:

$$\sum_{x \in X} \frac{W(\alpha, x)W(\beta, x)}{W(\gamma, x)} = D \frac{W(\alpha, \beta)}{W(\alpha, \gamma)W(\gamma, \beta)} \quad (\text{for all } \alpha, \beta, \gamma \in X) \quad (8)$$

for some nonzero real number $D$ with $D^2 = n$, which is independent of the choice of $\alpha, \beta, \gamma \in X$.

Let $m$ be a positive integer. In this section, assuming that $W$ is an $m \times m$ block matrix with blocks of the form (4), we will establish conditions on $T_{ij}$ under which $W$ satisfies the type II and type III conditions. Some parts of these conditions are already given in [5, Proposition 5.1, Proposition 6.2].

Let $\eta$ be a primitive $m$-th root of unity, and let $S_{ij}$ be the matrix of size $m$ defined by (5) for $i, j \in \mathbb{Z}_m$. Let $r$ be a positive integer, and define $Y = \{1, \ldots, r\}$, $X_i = \{(i, \ell, x) | \ell \in \mathbb{Z}_m, x \in Y\}$ for $i \in \mathbb{Z}_m$, and $X = X_0 \cup \cdots \cup X_{m-1}$. Let $T_{ij} \in \text{Mat}_Y(\mathbb{C}^*)$ be a matrix for $i, j \in \mathbb{Z}_m$, and let $W_{ij}$ be the matrix defined by (4). Let $W \in \text{Mat}_X(\mathbb{C}^*)$ be the matrix whose $(X_i, X_j)$-block is $W_{ij}$ for $i, j \in \mathbb{Z}_m$. Then

$$W((i, \ell, x), (j, \ell', y)) = S_{ij}(\ell, \ell')T_{ij}(x, y). \quad (9)$$

**Lemma 2.1** ([5, Proposition 5.1]) The matrix $W$ is a type II matrix if and only if $T_{ij}$ is a type II matrix for all $i, j \in \mathbb{Z}_m$.

**Lemma 2.2** The matrix $W$ satisfies the type III condition (8) if and only if the following equality holds for all $i_1, i_2, i_3 \in \mathbb{Z}_m$ and $x_1, x_2, x_3 \in Y$:

$$\sum_{x \in Y} \frac{T_{i_1,i_0}(x_1, x)T_{i_2,i_0}(x_2, x)}{T_{i_3,i_0}(x_3, x)} = \frac{D}{m} \cdot \frac{T_{i_1,i_2}(x_1, x_2)}{T_{i_1,i_3}(x_1, x_3)T_{i_3,i_2}(x_3, x_2)}, \quad (10)$$

where $i_0 = i_1 + i_2 - i_3 \mod m$.

**Proof** The type III condition (8) for $\alpha = (i_1, \ell_1, x_1)$, $\beta = (i_2, \ell_2, x_2)$, $\gamma = (i_3, \ell_3, x_3)$ is equivalent to

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\[
\sum_{i,\ell \in \mathbb{Z}_m} \eta(\ell_1 - \ell)(i_1 - i) \eta(\ell_2 - \ell)(i_2 - i) \sum_{x \in \mathcal{Y}} T_{i_1,i}(x_1, x) T_{i_2,i}(x_2, x) T_{i_3,i}(x_3, x) = D \eta(\ell_1 - \ell_2)(i_1 - i_2) \cdot \\
T_{i_1,i}(x_1, x) T_{i_2,i}(x_2, x) T_{i_3,i}(x_3, x),
\]

By a direct computation, we obtain

\[
\eta(\ell_1 - \ell)(i_1 - i) \eta(\ell_2 - \ell)(i_2 - i) \eta(\ell_3 - \ell)(i_3 - i) \eta(\ell_1 - \ell_2)(i_1 - i_2) \cdot \\
T_{i_1,i}(x_1, x) T_{i_2,i}(x_2, x) T_{i_3,i}(x_3, x).
\]

So (8) is equivalent to

\[
\sum_{i \in \mathbb{Z}_m} \left( \sum_{\ell \in \mathbb{Z}_m} \eta(\ell_1 + \ell_2 - \ell_3 - \ell)(i_1 + i_2 - i_3 - i) \right) \sum_{x \in \mathcal{Y}} T_{i_1,i}(x_1, x) T_{i_2,i}(x_2, x) T_{i_3,i}(x_3, x) = D \\
T_{i_1,i}(x_1, x) T_{i_2,i}(x_2, x) T_{i_3,i}(x_3, x).
\]

Since \( \eta \) is a primitive \( m \)-th root of unity and \( i_0 = i_1 + i_2 - i_3 \mod m \), we have

\[
\sum_{\ell \in \mathbb{Z}_m} \eta(\ell_1 + \ell_2 - \ell_3 - \ell)(i_1 + i_2 - i_3 - i) = m \delta_{i,i_0}.
\]

Thus (11) is equivalent to (10). \( \square \)

We remark that in [5, Proposition 6.2] only the necessity of (10) for the type III condition is proved.

Let \( z_m \) be the permutation matrix of order \( m \):

\[
\begin{pmatrix}
1 & \cdots & 1 \\
\end{pmatrix}
\]

We define the permutation matrix \( R \) of size \( n = m^2 r \) by \( R = I_m \otimes z_m \otimes I_r \), where \( I_m \) and \( I_r \) are the identity matrices of size \( m \) and \( r \), respectively. The order of \( R \) is \( m \).

**Lemma 2.3** The matrix \( W \) satisfies \( W^T W^{-1} = R \) if and only if \( T_{ij} = \eta^{i-j} T_{ji}^T \) holds for all \( i, j \in \mathbb{Z}_m \).

**Proof** For \( \alpha = (i, \ell, x) \) and \( \beta = (j, \ell', x) \in X \),

\[
W^T (\alpha, \beta) = W(\beta, \alpha) = \eta^{(\ell' - \ell)(j-i)} T_{j,i}(y, x),
\]
\[(RW)(\alpha, \beta) = ((I_m \otimes z_m \otimes I_r)W)((i, \ell, x), (j, \ell', y)) \]
\[= W((i, \ell - 1, x), (j, \ell', y)) \]
\[= \eta^{(\ell - 1 - \ell')((i-j)}T_{ij}(x, y) \]
\[= \eta^{(\ell' - \ell)(j-i)}\eta^{-(i-j)}T_{ij}(x, y). \]

Therefore \( R = W^T W^{-1} \) if and only if \( T_{ji}(y, x) = \eta^{-(i-j)}T_{ij}(x, y) \) holds for all \( i, j \in \mathbb{Z}_m \) and \( x, y \in Y \).

\[ \square \]

### 3 Proof of Theorem 1.1

From Remark 1.2, the results in Sect. 2 can be used for the matrices \( W \) and \( W' \) given in Theorem 1.1, if we define \( T_{ij} \) according to Remark 1.2.

For a mapping \( g \) from \( \mathbb{Z}_2 \) to \( \mathbb{Z} \), we denote by \( \lambda_g \) the mapping from \( \mathbb{Z}_4 \) to \( \mathbb{Z} \) defined by

\[
\lambda_g(i_1, i_2, i_3, i_4) = g(i_1, i_4) + g(i_2, i_4) - g(i_3, i_4) + g(i_1, i_3) + g(i_3, i_2) - g(i_1, i_2).
\]

(12)

Recall that we regard \( \mathbb{Z}_m \) as the subset \( \{0, 1, \ldots, m - 1\} \) of \( \mathbb{Z} \), and \( \delta, \epsilon : \mathbb{Z}_2 \to \mathbb{Z} \) are defined by \( \delta(i, j) = (i - j)^2 \), \( \epsilon(i, j) = \delta(i, j) + m(i - j) \), respectively.

**Lemma 3.1** For all \( i_1, i_2, i_3, i_4 \in \mathbb{Z} \), we have

\[
\lambda_\delta(i_1, i_2, i_3, i_4) = (i_1 + i_2 - i_3 - i_4)^2,
\]

\[
\lambda_\epsilon(i_1, i_2, i_3, i_4) = (i_1 + i_2 - i_3 - i_4)(i_1 + i_2 - i_3 - i_4 + m).
\]

In particular, if \( i_0 = i_1 + i_2 - i_3 \) (mod \( m \)), then

\[
\lambda_\delta(i_1, i_2, i_3, i_0) \equiv 0 \pmod{m^2},
\]

\[
\lambda_\epsilon(i_1, i_2, i_3, i_0) \equiv 0 \pmod{2m^2}.
\]

**Proof** Straightforward. \( \square \)

In [7, §5.1], the following is used to construct non-symmetric or symmetric Hadamard models:

**Lemma 3.2** [7, §5.1] Let \( A_u, H \in \text{Mat}_Y(\mathbb{C}^*) \) be a Potts model and a Hadamard matrix, respectively. Then the following holds for all \( x_1, x_2, x_3 \in Y \):

\[
\sum_{y \in Y} \frac{A_u(x_1, y)A_u(x_2, y)}{A_u(x_3, y)} = D_u \frac{A_u(x_1, x_2)}{A_u(x_1, x_3)A_u(x_3, x_2)},
\]

(13)

\[
\sum_{y \in Y} A_u(x_1, y)H(y, x_2)H(y, x_3) = D_u \frac{H(x_1, x_2)H(x_1, x_3)}{A_u(x_2, x_3)},
\]

(14)
\[
\sum_{y \in Y} A_u(x_1, y) H(x_2, y) H(x_3, y) = D_u \frac{H(x_2, x_1) H(x_3, x_1)}{A_u(x_2, x_3)}, \tag{15}
\]

\[
\sum_{y \in Y} \frac{H(y, x_1) H(y, x_2)}{A_u(x_2, y)} = D_u A_u(x_1, x_2) H(x_3, x_1) H(x_3, x_2), \tag{16}
\]

\[
\sum_{y \in Y} \frac{H(x_1, y) H(x_2, y)}{A_u(x_2, y)} = D_u A_u(x_1, x_2) H(x_1, x_3) H(x_2, x_3), \tag{17}
\]

where

\[
D_u = \begin{cases} \ -u^2 - u^{-2} & \text{if } |Y| \geq 2, \\ \ u^2 & \text{if } |Y| = 1. \end{cases}
\]

We now prove Theorem 1.1. Since \( A_u \) and \( H \) are type II matrices, so are the matrices \( T_{ij} = a^{c(i,j)} V_{ij} \) or \( b^{c(i,j)} V_{ij} \). Thus, Lemma 2.1 implies that \( W_{H,u,a} \) and \( W'_{H,u,b} \) are type II matrices.

We claim

\[
\sum_{y \in Y} \frac{V_{i_1,i_0}(x_1, y) V_{i_2,i_0}(x_2, y)}{V_{i_3,i_0}(x_3, y)} = D_u \frac{V_{i_1,i_2}(x_1, x_2)}{V_{i_1,i_3}(x_1, x_3) V_{i_3,i_2}(x_3, x_2)} \tag{18}
\]

for all \( i_1, i_2, i_3 \in \mathbb{Z}_m \) and \( x_1, x_2, x_3 \in Y \), where \( i_0 = i_1 + i_2 - i_3 \mod m \). Indeed, let \( i_1, i_2, i_3 \in \mathbb{Z}_m \). Then

\[
(18) \iff \begin{cases} (13) & \text{if } (i_1, i_2, i_3) \equiv (0, 0, 0), (1, 1, 1) \pmod 2, \\ (14) & \text{if } (i_1, i_2, i_3) \equiv (0, 1, 1) \pmod 2, \\ (15) & \text{if } (i_1, i_2, i_3) \equiv (1, 0, 0) \pmod 2, \\ (16) & \text{if } (i_1, i_2, i_3) \equiv (1, 1, 0) \pmod 2, \\ (17) & \text{if } (i_1, i_2, i_3) \equiv (0, 0, 1) \pmod 2. \end{cases}
\]

Moreover, when \( (i_1, i_2, i_3) \equiv (1, 0, 1), (0, 1, 0) \pmod 2 \), (18) is equivalent to (14), (15), respectively, with \( x_1 \) and \( x_2 \) switched. Therefore, (18) holds in all cases by Lemma 3.2.

First, we show that \( W_{H,u,a} \) and \( W'_{H,u,b} \) satisfy the condition (10). From Lemma 3.1 we have

\[
a^{c(i_1,i_2,i_3,i_0)} = 1 \quad \text{and} \quad b^{c(i_1,i_2,i_3,i_0)} = 1.
\]

In view of (12), these imply

\[
c^{g(i_1,i_0)+g(i_2,i_0)-g(i_3,i_0)} = c^{g(i_1,i_2)-g(i_1,i_3)-g(i_3,i_2)}, \tag{19}
\]

where \((c, g) = (a, \epsilon), (b, \delta)\). Combining (18) and (19), we obtain

\[
\sum_{y \in Y} c^{g(i_1,i_0)} V_{i_1,i_0}(x_1, y) c^{g(i_2,i_0)} V_{i_2,i_0}(x_2, y) = D_u \frac{c^{g(i_1,i_2)} V_{i_1,i_2}(x_1, x_2)}{c^{g(i_1,i_3)} V_{i_1,i_3}(x_1, x_3) c^{g(i_3,i_2)} V_{i_3,i_2}(x_3, x_2)}
\]

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for all $i_1, i_2, i_3 \in \mathbb{Z}_m$ and $x_1, x_2, x_3 \in Y$. Thus (10) holds by setting $D = mD_u$. It follows from Lemma 2.2 that $W_{H,u,a}$ and $W'_{H,u,b}$ satisfy the type III condition (8), and hence they are spin models. Since $\delta(i,j) = \delta(j,i)$, $W'_{H,u,b}$ is symmetric.

Finally, we show that $W_{H,u,a}$ has index $m$. Since $\eta = \epsilon(i,j) - \epsilon(j,i)$, we have $\eta^2 - \eta = \epsilon(i,j) - \epsilon(j,i) = a^{2m(i-j)} = a^{2m(j-i)}$. So, $T_{ij} = \eta^i \eta^{-j} T_{ji}$ holds for all $i, j \in \mathbb{Z}_m$. From Lemma 2.3, $W_{H,u,a}$ has index $m$. This completes the proof of Theorem 1.1.

4 Properties of spin models in Theorem 1.1

For a positive integer $r$, we let $u$ be a complex number satisfying (1).

Lemma 4.1 If $r \leq 4$, then $u$ is a root of unity. Otherwise, $|u| \neq 1$. If $r \geq 4$ or $r = 1$, then $u^4 > 0$.

Proof If $u$ is a root of unity and $r > 1$, then $r = (u^2 + u^{-2})^2 - |u|^4 + 2 + |u|^{-4} = 4$. It is easy to see that $u$ is indeed a root of unity if $r \leq 4$. If $r \geq 4$ or $r = 1$, then we have $u^4 > 0$ from (1).

For a matrix $W \in \text{Mat}_X(\mathbb{C}^*)$, we define

$$E(W) = \left\{ \frac{|W(x,y)|}{|W(x,x)|} \bigg| x, y \in X \right\} \subset \mathbb{R}_{>0}.$$ 

Then

$$E(W_1 \otimes W_2) = E(W_1)E(W_2)$$ (20)

holds for any matrices $W_1, W_2$ with nonzero entries.

For the remainder of this section, let $W_{H,u,a}, W'_{H,u,b}$ be the spin models given in Theorem 1.1(i) and (ii), respectively. This means that $m$ is an even positive integer, $a$ is a primitive $2m^2$-th root of unity, $b$ is an $m^2$-th root of unity, and $H$ is a Hadamard matrix of order $r$.

Lemma 4.2 We have

$$E(W_{H,u,a}) = E(W'_{H,u,b}) = \left\{ 1, |u|^{-4}, |u|^{-3} \right\} \quad \text{if } r > 4,$$

$$\left\{ 1 \right\} \quad \text{otherwise}.$$

Proof Immediate from Theorem 1.1 and Lemma 4.1.

Lemma 4.3

(i) Suppose $r \geq 4$ or $r = 1$. Then the entries of $W_{H,u,a}, W'_{H,u,b}$ which have absolute value 1 are $2m^2$-th roots of unity, $m^2$-th roots of unity, respectively. Moreover, $W_{H,u,a}$ contains a primitive $2m^2$-th root of unity as one of its entries.

(ii) Suppose $r = 2$, and put $v = \text{LCM}(2m^2, 16), v' = \text{LCM}(m^2, 16)$. Then the entries of $W_{H,u,a}, W'_{H,u,b}$ are $v$-th roots of unity, $v'$-th roots of unity, respectively. Moreover, $W_{H,u,a}$ contains a primitive $v$-th root of unity as one of its entries.
**Proof** Firstly, suppose $r > 4$. From Lemma 4.1, the entries of $W_{H,u,a}$, $W'_{H,u,b}$ with absolute value 1 are

$$
\pm a^{2m(\ell-\ell')(i-j)+\epsilon(i,j)} (i - j : \text{odd}),
\pm \eta^{(\ell-\ell')(i-j)} b^{\delta(i,j)} (i - j : \text{odd}),
$$

(21)

which are $2m^2$-th roots of unity, $m^2$-th roots of unity, respectively. Putting $i = 1$, $j = \ell = \ell' = 0$ in (21), we obtain $a^{1+m}$ which is a primitive $2m^2$-th root of unity.

Next, suppose $r \leq 4$. Then the entries of $W_{H,u,a}$, $W'_{H,u,b}$ are given by

$$
u a^{2m(\ell-\ell')(i-j)+\epsilon(i,j)} (v \in \{u^3, -u^{-1}, \pm 1\}),
$$

(22)

$$
u \eta^{(\ell-\ell')(i-j)} b^{\delta(i,j)} (v \in \{u^3, -u^{-1}, \pm 1\}),
$$

(23)

respectively, all of which are roots of unity.

If $r = 4$ or 1, then from (1), $u^4 = 1$. From (22), (23), the entries of $W_{H,u,a}$, $W'_{H,u,b}$ are $2m^2$-th roots of unity, $m^2$-th roots of unity, respectively. Putting $i = 1$, $j = \ell = \ell' = 0$ in (22), we obtain $a^{1+m}$ which is a primitive $2m^2$-th root of unity.

Finally, suppose $r = 2$. Since $u$ is a primitive 16-root of unity by (1), the expressions in (22), (23) are $\nu$-th roots of unity, an $\nu'$-th roots of unity, respectively. Putting $v = u^3$, $i = 1$, $j = \ell = \ell' = 0$ in (22), we obtain $u^3 a^{1+m}$ which is a primitive $\nu$-th root of unity.

For $S \in \text{Mat}_X(\mathbb{C}^*)$, we denote by $\mu(S)$ the least common multiple of the orders of the entries of $S$ which have a finite order. If none of the entries of $S$ has a finite order, then we define $\mu(S) = \infty$. For a nonzero complex number $\zeta$, we denote by the same symbol $\mu(\zeta)$ the order of $\zeta$ if $\zeta$ has a finite order.

**Lemma 4.4** Suppose $m \equiv 0 \pmod{4}$. Then for $W = W_{H,u,a}$ or $W = W'_{H,u,b}$, we have $\mu(W) | 2m^2$.

**Proof** Immediate from Lemma 4.3. □

In Table 1, we summarize the properties of $W = W_{H,u,a}$, $W'_{H,u,b}$ obtained from Lemmas 4.1, 4.2, and 4.4.

For $W \in \text{Mat}_X(\mathbb{C}^*)$ and for a permutation $\sigma$ of $X$, we define $W^\sigma$ by $W^\sigma(\alpha, \beta) = W(\sigma(\alpha), \sigma(\beta))$ for $\alpha, \beta \in X$. Observe that if $W$ is a spin model, then $W^\sigma$ is also a spin model. If $W$ is a spin model, then from (7), (8), $-W$ and $\pm \sqrt{-1} W$ are also spin models. Two spin models $W_1$, $W_2$ are said to be **equivalent** if $c W_1^{\sigma} = W_2$ for some permutation $\sigma$ of $X$ and a complex number $c$ with $c^4 = 1$.

Two Hadamard matrices are said to be **equivalent** if one can be obtained from the other by negating rows and columns, or and permuting rows and columns.

**Lemma 4.5** Let $H_1$, $H_2 \in \text{Mat}_Y(\mathbb{C}^*)$ be equivalent Hadamard matrices. Then $W_{H_1,u,a}$ is equivalent to $W_{H_2,u,a}$, and $W'_{H_1,u,b}$ is equivalent to $W'_{H_2,u,b}$.
Table 1 Summary of properties

| W          | Index | Size | $r$    | $\mu(W)$ | $E(W)$ |
|------------|-------|------|--------|-----------|--------|
| $W_{H,u,a}$ | $m$   | $m^2 r$ | $r = 1$ | $2m^2$ | $[1]$ |
|            |       |       | $r = 2$ | $\mu(W) \mid \text{LCM}(2m^2, 16)$ | $[1]$ |
|            |       |       | $r = 4$ | $2m^2$ | $[1]$ |
|            |       |       | $r > 4$ | $2m^2$ | $\{1, |u|^{-4}, |u|^{-3}\}$ |
| $W'_{H,u,b}$ | $1$   | $m^2 r$ | $r = 1$ | $\mu(W)m^2$ | $[1]$ |
|            |       |       | $r = 2$ | $\mu(W) \mid \text{LCM}(m^2, 16)$ | $[1]$ |
|            |       |       | $r = 4$ | $\mu(W)m^2$ | $[1]$ |
|            |       |       | $r > 4$ | $\mu(W)m^2$ | $\{1, |u|^{-4}, |u|^{-3}\}$ |

Proof Let $(W_1, W_2, c, g) = (W_{H_1,u,a}, W_{H_2,u,a}, a, e)$ or $(W_{H_1,u,b}, W'_{H_2,u,b}, b, \delta)$.

If $H_2$ is obtained by a permutation of columns of $H_1$, then there exists a permutation $\pi$ of $Y$ such that $H_2(x, \pi(y)) = H_1(x, y)$ for all $x, y \in Y$. We define a permutation $\sigma$ of $X$ by

$$\sigma((i, \ell, x)) = \begin{cases} (i, \ell, \pi(x)) & \text{if } i \text{ is odd}, \\ (i, \ell, x) & \text{otherwise}. \end{cases}$$

Then for $\alpha = (i, \ell, x), \beta = (j, \ell', y) \in X$,

$$W_2^\sigma(\alpha, \beta) = W_2(\sigma(\alpha), \sigma(\beta))$$

$$= \begin{cases} c_{ij}(i, j)A_u(x, y) & \text{if } i \equiv j \equiv 0 \pmod{2}, \\ c_{ij}(i, j)H_1(x, \pi(y)) & \text{if } i \equiv j + 1 \equiv 0 \pmod{2}, \\ c_{ij}(i, j)H_1^T(\pi(x), y) & \text{if } i + 1 \equiv j \equiv 0 \pmod{2}, \\ c_{ij}(i, j)A_u(\pi(x), \pi(y)) & \text{if } i \equiv j + 1 \equiv 0 \pmod{2} \end{cases}$$

$$= \begin{cases} c_{ij}(i, j)A_u(x, y) & \text{if } i \equiv j \equiv 0 \pmod{2}, \\ c_{ij}(i, j)H_1(x, y) & \text{if } i \equiv j + 1 \equiv 0 \pmod{2}, \\ c_{ij}(i, j)H_1^T(x, y) & \text{if } i + 1 \equiv j \equiv 0 \pmod{2}, \\ c_{ij}(i, j)A_u(x, y) & \text{if } i \equiv j + 1 \equiv 0 \pmod{2} \end{cases}$$

$$= W_1(\alpha, \beta).$$

If $H_2$ is obtained by a permutation of rows of $H_1$, then there exists a permutation $\pi'$ of $Y$ such that $H_2(\pi'(x), y) = H_1(x, y)$ for all $x, y \in Y$. We define a permutation $\sigma'$ of $X$ by

$$\sigma'((i, \ell, x)) = \begin{cases} (i, \ell, \pi'(x)) & \text{if } i \text{ is even}, \\ (i, \ell, x) & \text{otherwise}. \end{cases}$$

Similar calculation shows $W_2^\sigma(\alpha, \beta) = W_1(\alpha, \beta)$.

If $H_2$ is obtained by negating a column $y_1$ of $H_1$, then $H_2(x, y_1) = -H_1(x, y_1)$, $H_2(x, y) = H_1(x, y)$ for all $x \in Y$ and $y \in Y - \{y_1\}$. We define a permutation $\rho$ of $X$
by

\[ \rho((i, \ell, x)) = \begin{cases} (i, \ell + \delta_{x,y_1} \frac{m}{2}, x) & \text{if } i \text{ is odd}, \\ (i, \ell, x) & \text{otherwise}. \end{cases} \]

Note that \( S_{ij}(\ell, \ell') = (-1)^{i-j} S_{ij}(\ell + \frac{m}{2}, \ell') = (-1)^{i-j} S_{ij}(\ell, \ell' + \frac{m}{2}) \). Thus for \( \alpha = (i, \ell, x), \beta = (j, \ell', y) \in X \),

\[ W_2^\rho(\alpha, \beta) = W_2(\rho(\alpha), \rho(\beta)) \]

\[ = \begin{cases} c^{g(i,j)} S_{ij}(\ell, \ell') A_u(x, y) & \text{if } i \equiv j \equiv 0 \pmod{2}, \\ c^{g(i,j)} S_{ij}(\ell, \ell' + \delta_{y,y_1} \frac{m}{2}) H_2(x, y) & \text{if } i \equiv j + 1 \equiv 0 \pmod{2}, \\ c^{g(i,j)} S_{ij}(\ell + \delta_{x,y_1} \frac{m}{2}, \ell') H_2^T(x, y) & \text{if } i + 1 \equiv j \equiv 0 \pmod{2}, \\ c^{g(i,j)} S_{ij}(\ell + \delta_{x,y_1} \frac{m}{2}, \ell' + \delta_{y,y_1} \frac{m}{2}) A_u(x, y) & \text{if } i \equiv j + 1 \equiv 1 \pmod{2} \end{cases} \]

\[ = \begin{cases} c^{g(i,j)} S_{ij}(\ell, \ell') A_u(x, y) & \text{if } i \equiv j \equiv 0 \pmod{2}, \\ (-1)^{\delta_{y,y_1} c^{g(i,j)} S_{ij}(\ell, \ell' + \delta_{y,y_1} \frac{m}{2}) H_2(x, y) & \text{if } i \equiv j + 1 \equiv 0 \pmod{2}, \\ (-1)^{\delta_{y,y_1} c^{g(i,j)} S_{ij}(\ell + \delta_{x,y_1} \frac{m}{2}, \ell') H_2^T(x, y) & \text{if } i + 1 \equiv j \equiv 0 \pmod{2}, \\ c^{g(i,j)} S_{ij}(\ell + \delta_{x,y_1} \frac{m}{2}, \ell' + \delta_{y,y_1} \frac{m}{2}) A_u(x, y) & \text{if } i \equiv j + 1 \equiv 1 \pmod{2} \end{cases} \]

\[ = W_1(\alpha, \beta). \]

If \( H_2 \) is obtained by negating a row \( x_1 \) of \( H_1 \), then \( H_2(x_1, y) = -H_1(x_1, y), H_2(x, y) = H_1(x, y) \) for all \( x \in Y - \{x_1\} \) and \( y \in Y \). We define a permutation \( \rho' \) of \( X \) by

\[ \rho'(i, \ell, x) = \begin{cases} (i, \ell + \delta_{x,x_1} \frac{m}{2}, x) & \text{if } i \text{ is even}, \\ (i, \ell, x) & \text{otherwise}. \end{cases} \]

Similar calculation shows \( W_2^{\rho'}(\alpha, \beta) = W_1(\alpha, \beta) \).

\[ \square \]

5 Decomposability

Lemma 5.1 Let \( S_1, S_2 \) be finite subsets of positive real numbers. Suppose \( 1 \in S_1 \cap S_2 \) and \( |S_1 S_2| = 3 \). Then

\[ (|S_1|, |S_2|) \in \{(2, 2), (1, 3), (3, 1)\}. \]

If \( |S_1| = |S_2| = 2 \), then \( S_1 S_2 = \{1, a, a^2\} \) or \( \{1, a, a^{-1}\} \) for some positive real number \( a \neq 1 \).

Proof By way of contradiction, we prove that if \( |S_1| \geq 3 \) and \( |S_2| \geq 2 \) then \( |S_1 S_2| > 3 \). Since \( S_1 \cup S_2 \subset S_1 S_2 \), we obtain \( S_2 \subset S_1 = S_1 S_2 \). Let \( S_1 = \{1, \lambda, \mu\} \)
(\lambda, \mu \neq 1, \lambda \neq \mu). Then we may put \( S_2 = \{1, \lambda\} \) without loss of generality. Then we have \( \lambda^2 \in S_1 S_2 = S_1 \), so \( \mu = \lambda^2 \) and \( S_1 S_2 = \{1, \lambda, \lambda^2, \lambda^3\} \). This implies \( |S_1 S_2| = 4 \), a contradiction.

Suppose \( |S_1| = |S_2| = 2 \). Then \( S_1 = \{1, a\} \), \( S_2 = \{1, b\} \) for some \( a, b \neq 1 \). Then \( |S_1 S_2| = 3 \) implies \( a = b \) or \( a = b^{-1} \).

**Lemma 5.2** Let \( A \in \text{Mat}_{Z_2}(\mathbb{C}^*) \) be a matrix all of whose entries are roots of unity. Let \( B \in \text{Mat}_{Z_2}(\mathbb{C}^*) \) be a matrix which satisfies \( \mu(B) < \infty \). Then \( \mu(A \otimes B) \) is a divisor of \( \text{LCM}(\mu(A), \mu(B)) \).

**Proof** Let \( Z'_2 = \{(x_2, y_2) \in Z_2 \times Z_2 \mid o(B(x_2, y_2)) < \infty\} \). Then

\[
\mu(A \otimes B) = \text{LCM}(o(A(x_1, y_1)B(x_2, y_2)) \mid x_1, y_1 \in Z_1, (x_2, y_2) \in Z'_2),
\]

which is a divisor of \( \text{LCM}(\mu(A), \mu(B)) \). \( \square \)

Some examples of spin models are listed in Sect. 1, i.e., Potts model, non-symmetric Hadamard models, and Hadamard models. We remark that non-symmetric Hadamard models and Hadamard models are special cases of spin models given in Theorem 1.1(i), (ii), respectively. In addition to these examples, the following spin models are known.

**Spin models on finite abelian groups** Bannai-Bannai-Jaeger [3] gives solutions to modular invariance equation for finite abelian groups, and every solution gives a spin model. Let \( U \) be a finite abelian group, and \( e = \exp(U) \) denote the exponent of \( U \). Let \( \{\chi_a \mid a \in U\} \) be the set of characters of \( U \) with indices chosen so that \( \chi_a(b) = \chi_b(a) \) for all \( a, b \in U \). Let \( U = U_1 \oplus \cdots \oplus U_h \) be a decomposition of \( U \) into a direct sum of cyclic groups \( U_1, U_2, \ldots, U_h \). For each \( i \in \{1, 2, \ldots, h\} \) let \( a_i \) be a generator and \( n_i \) be the order of the cyclic group \( U_i \). For each \( x \in U \), we define the matrix \( A_x \in \text{Mat}_U(\mathbb{C}) \) by

\[
A_x(\alpha, \beta) = \delta_{\alpha, \beta - a} \quad (\alpha, \beta \in U).
\]

For any \( x = \sum_{i=1}^h x_i a_i \) (\( 0 \leq x_i < n_i \)), let

\[
t_x = t_0 \prod_{i=1}^h \eta_i^{x_i} \chi_{a_i}(a_i)^{x_j(x_j-1)} \prod_{1 \leq \ell < k \leq h} \chi_{a_\ell}(a_k)^{x_\ell x_k}, \quad (24)
\]

where \( \eta_i = \chi_{a_i}(a_i)^{-\frac{n_i(n_i-1)}{2}} \) and

\[
t_0^2 = D^{-1} \prod_{x \in U} \sum_{j=1}^h \eta_j^{-x_j} \chi_{a_j}(a_j)^{x_j(x_j-1)} \prod_{1 \leq \ell < k \leq h} \chi_{a_\ell}(a_k)^{-x_\ell x_k}, \quad (25)
\]

where \( D^2 = |U| \). Let \( \theta_x = t_x / t_0 \) for any \( x \in U \). Then, for any \( x \in U \), \( \theta_x \) is a root of unity and \( \theta_x^{2|U|} = 1 \). Especially, we get

\[
\theta_x^{2|U|} = 1. \quad (26)
\]
The matrix
\[ W = \sum_{x \in U} t_x A_x \]  
(27)
is a spin model.

**Jaeger’s Higman-Sims model** In [6], Jaeger constructed a spin model \( W_J \) on the Higman-Sims graph of size 100. We denote by \( A \) the adjacency matrix of the Higman-Sims graph. We put \( W_J = -\tau^5 I - \tau A + \tau^{-1} (J - A - I) \), where \( \tau \) satisfies \( \tau^2 + \tau^{-2} = 3 \). Then \( W_J \) is a symmetric spin model.

Now every known spin model belongs to one of the following five families:

(a) \( A_u \): Potts model of size \( r \geq 2 \). If \( r = 2 \), then \( \mu(A_u) = 16 \). If \( r = 4 \), then \( \mu(A_u) = 2 \) or 4. If \( r = 2, 4 \), then \( E(A_u) = \{1\} \). If \( r > 4 \), then \( E(A_u) = \{1, |u|^{-4}\} \), and hence \( |E(A_u)| = 2 \).

(b) \( W_U \): spin model on a finite abelian group \( U \). We have various kinds of indices and \( E(W_U) = \{1\} \).

(c) \( W_J \): Jaeger’s Higman-Sims model of size 100. We have \( E(W_J) = \{1, \tau^{-4}, \tau^{-6}\} \) with \( \tau^2 + \tau^{-2} = 3 \), and hence \( |E(W_J)| = 3 \).

(d) \( W_{H,u,a} \): spin models given in Theorem 1.1(i).

(e) \( W_{H,u,b} \): spin models given in Theorem 1.1(ii).

By way of contradiction, we now give a proof of Theorem 1.3. Let \( H \) be a Hadamard matrix of order \( r > 4 \). Let \( s \) be a positive integer and \( a \) a primitive \( 2^{s+1} \)-th root of unity. For the remainder of this section, we denote by \( W \) the spin model \( W_{H,u,a} \) given in Theorem 1.1(i) of index \( 2^s \). By Lemma 4.2 we obtain
\[ E(W) = \{1, |u|^{-4}, |u|^{-3}\}. \]  
(28)
We assume that
\[ W = W_1 \otimes W_2 \otimes \cdots \otimes W_v, \]  
(29)
where each of \( W_1, W_2, \ldots, W_v \) is a known spin model listed in (a)–(e) and their sizes are not equal to 1. Since \( |E(W)| = 3 \) from (28), using Lemma 5.1 we may assume without loss of generality
\[ (|E(W_1)|, |E(W_2)|, \ldots, |E(W_v)|) = (1, \ldots, 1, 2, 2) \text{ or } (1, \ldots, 1, 3). \]
A known spin model \( W' \) with \( |E(W')| = 1 \) belongs to the family (b) or to the families (a), (d) and (e) with \( r \leq 4 \). Therefore, (29) can be reduced to the following cases:

\[ W = W_1 \otimes W_2 \otimes W_3 \quad \text{with } E(W_1) = \{1\}, \quad |E(W_2)| = |E(W_3)| = 2, \]  
(30)
\[ W = W_1 \otimes W_2 \quad \text{with } E(W_1) = \{1\}, \quad |E(W_2)| = 3, \]  
(31)
where in (30), (31), \( W_1 \) is a tensor product of spin models on finite abelian groups and spin models in the families (a), (d) and (e) with \( r \leq 4 \). Note that \( W_1 \) could possibly be of size 1 in (30).
First, we treat the case (30). Then Lemma 5.1 implies \( E(W_2 \otimes W_3) = \{1, \beta, \beta^2\} \), or \( \{1, \beta, \beta^{-1}\} \) for some \( \beta \). On the other hand, \( E(W_2 \otimes W_3) = E(W_1)E(W_2 \otimes W_3) = E(W) = \{1, |u|^{-4}, |u|^{-3}\} \) by (28). This is a contradiction.

Next, we treat the case (31). We have \( E(W_2) = E(W_1)E(W_2) = E(W) = \{1, |u|^{-4}, |u|^{-3}\} \) from (28). Since \( \{1, |u|^{-4}, |u|^{-3}\} \neq \{1, \tau^{-4}, \tau^{-6}\} \), \( W_2 \) cannot be the spin model (c). Therefore, \( W_2 \) belongs to the family (d) or (e). This means \( W_2 = W_{H', u', a'} \) or \( W_2 = W'_{H', u', b'} \), where \( H' \) is a Hadamard matrix of order \( r' = (u'^2 + u' - 2)2 \). Since \( |E(W_2)| = 3 \), Lemma 4.2 implies \( r' > 4 \) and \( E(W_2) = \{1, |u'|^{-4}, |u'|^{-3}\} \). Then we have \( |u'| = |u| \), as \( E(W) = E(W_2) \). Now the second part of Lemma 4.1 implies \( u^4 > 0 \) and \( u^4 > 0 \), hence

\[
u^4 = u^4,
\]

and further \( r = r' \) by (1). Therefore the size of \( W_2 \) is \( 2^{2s'}r \) for some integer \( s' \) with \( 0 < s' < s \), and the size of \( W_1 \) is \( 2^{2(s-s')} \). In particular, we obtain \( s > 1 \).

Since the tensor product of spin models on finite abelian groups is also a spin model on a finite abelian group, we may suppose that

\[
W_1 = W_{11} \otimes W_{12} \otimes W_{13},
\]

where \( W_{11} \) is a spin model on a finite abelian group \( U \), \( W_{12} \) is a tensor product of spin models in the family (a) with \( r \leq 4 \), and \( W_{13} \) is a tensor product of spin models in the families (d) and (e) with \( r \leq 4 \).

We put \( |U| = 2^{n_1} \). Since the size \( 2^{n_1} \) of \( W_{11} \) cannot exceed that of \( W_1 \), we have \( n_1 \leq 2(s-s') \). Then the size of \( W_{12} \otimes W_{13} \) is \( 2^{2(s-s')-n_1} \). The diagonal entry of \( W_{11} \) is a complex number \( t_0 \) given by (25). The diagonal entries of \( W_{12}, W_{13} \) are 16-th roots of unity. We denote by \( \kappa_2, \kappa_3 \) the diagonal entries of \( W_{12}, W_{13} \), respectively. Comparing the diagonal entries of (33), we have \( u^3 = t_0\kappa_2\kappa_3u^3 \), thus

\[
W = (t_0^{-1}W_{11}) \otimes (\kappa_2^{-1}W_{12}) \otimes (\kappa_3^{-1}W_{13}) \otimes (u^3 u^{-3} W_2).
\]

From (26), we have

\[
\mu(t_0^{-1}W_{11}) | 2^{n_1+1}.
\]

From (a), we have

\[
\mu(\kappa_2^{-1}W_{12}) | 2^4.
\]

From (a) and Lemma 4.4, we have

\[
\mu(\kappa_3^{-1}W_{13}) | 2^{2(s-s')-n_1+1}.
\]

Since \( W_2 \) is a spin model belonging to the family (d) or (e), Lemma 4.3 and (32) imply

\[
\mu(u^3 u^{-3} W_2) | 2^{2s'+1}.
\]

From (34)–(38) and Lemma 5.2, we have

\[
\mu(W) | \text{LCM}(2^{n_1+1}, 2^4, 2^{2(s-s')-n_1+1}, 2^{2s'+1}).
\]
Since \( n_1 < 2s \), we have \( \max(n_1 + 1, 4, 2(s - s') - n_1 + 1, 2s' + 1) \leq 2s \). This implies \( \mu(W) \mid 2^{2s} \), which contradicts Lemma 4.3(i).

### 6 Spin models in Theorem 1.1 with \( r \leq 4 \)

In this section, we treat the case of \( r \leq 4 \) in Theorem 1.3. We show that if \( r = 1, 4 \) in Theorem 1.1, then \( WH,u,a \) is not new.

If \( r = 4 \) in Theorem 1.1(i), then \( WH,u,a \) is a tensor product of a Hadamard matrix of order 4 and \( W(1),u,a \). Indeed, up to equivalence, there is a unique Hadamard matrix of order \( r = 4 \). By Lemma 4.5, we may assume without loss of generality

\[
H = \begin{pmatrix}
1 & -1 & -1 & -1 \\
-1 & 1 & -1 & -1 \\
-1 & -1 & 1 & -1 \\
-1 & -1 & -1 & 1
\end{pmatrix}.
\]

Then \( Au = u^3H \) with \( (u^2 + u^{-2})^2 = 4 \). Therefore we have \( WH,u,a = H \otimes W(1),u,a \).

Similarly, a spin model \( W'H,u,b \) in Theorem 1.1(ii) can be decomposed as \( H \otimes W'(1),u,b \).

#### Lemma 6.1

Let \( m \equiv 0 \pmod{4} \). Let \( W(1),u,a \) be a spin model given in Theorem 1.1 of index \( m \), where \( u^4 = 1 \) and \( a \) is a primitive \( 2m^2 \)-th root of unity. Then \( W(1),u,a \) is equivalent to \( W(1),1,au^3 \).

**Proof** First we assume that \( u = -1 \). Then \( a^{\epsilon(i,j)}(-1)^{i-j-1} = -(a)^{\epsilon(i,j)} \) holds for all \( i, j \in \mathbb{Z}_{m^2} \). From this, we have \( W(1),-1,a = -W(1),1,-a \). Therefore \( W(1),-1,a \) is equivalent to \( W(1),1,-a \).

Next we assume that \( u^2 = -1 \). Since \( m \equiv 0 \pmod{4} \), we have

\[
u(wu^3)^{\epsilon(i,j)} = \begin{cases} 
  a^{\epsilon(i,j)}u & \text{if } i - j \text{ is even,} \\
  a^{\epsilon(i,j)} & \text{if } i - j \text{ is odd.}
\end{cases}
\]

From this, we have \( uW(1,1,au^3) = W(1,u,a) \). Therefore \( W(1),u,a \) is equivalent to \( W(1),1,au^3 \). \( \square \)

#### Lemma 6.2

Let \( m \) be even, and \( \xi \) be a primitive \( 2m^2 \)-th root of unity. Then we have

\[
\sum_{x=0}^{m^2-1} \xi^{-x(x-m)} = m. \quad (39)
\]

**Proof** If \( (39) \) holds for \( \xi = \exp(2\pi \sqrt{-1}/(2m^2)) \), then by considering the action of the Galois group, we see that \( (39) \) holds for any primitive \( 2m^2 \)-th root of unity \( \xi \). Therefore we may assume \( \xi = \exp(2\pi \sqrt{-1}/(2m^2)) \) without loss of generality.
$m$ is even, we may write $m = 2k$. Then

$$\sum_{x=0}^{m^2-1} \xi^{-x(x-m)} = \sum_{x=0}^{m^2-1} \xi^{-(x-k)^2-k^2}$$

$$= \xi^k \sum_{x=0}^{m^2-1} \xi^{-(x-k)^2}$$

$$= \frac{\xi^k}{2} \sum_{x=0}^{m^2-1} (\xi^{-(x-k)^2} + \xi^{-(x-k+m^2)^2})$$

$$= \frac{\exp(\pi \sqrt{-1}/4)}{2} \sum_{x=0}^{2m^2-1} \xi^{-(x-k)^2}$$

$$= 1 + \sqrt{-1} \frac{2m^2-1}{2\sqrt{2}} \xi^{-x^2}.$$ 

Now the result follows from [10, Theorem 99].

Of particular interest among spin models on finite abelian groups are spin models on finite cyclic groups. The spin model defined below is a special case of spin models on finite cyclic groups constructed by [1]. Let $m$ be even, and $a$ be a primitive $2m^2$-th root of unity. We restrict (24) and (25) to $\mathbb{Z}_{m^2}$, that is, $h = 1$. In (24) and (25), we put $\eta_1 = a^{-m+1}$, $\chi_{a_1}(a_1) = a^2$. Then (24) and (25) become

$$t_x = t_0 a^{x(x-m)} \quad (x \in \mathbb{Z}_{m^2}),$$

$$t_0^2 = m^{-1} \sum_{x=0}^{m^2-1} a^{-x(x-m)} = 1,$$

respectively, where we used Lemma 6.2 in (41). Thus we may take $t_0 = 1$. Then the matrix $W$ given in (27) has entries

$$W(\alpha, \beta) = a^{(\beta-\alpha)(\beta-\alpha-m)} \quad (\alpha, \beta \in \mathbb{Z}_{m^2}).$$

We note that this spin model $W$ on $\mathbb{Z}_{m^2}$ was constructed originally in [2, Theorem 2].

**Proposition 6.3** Let $m \equiv 0$ (mod 4). Let $W(1, u, a)$ be a spin model given in Theorem 1.1 (i) of index $m$, where $u^4 = 1$ and $a$ is a primitive $2m^2$-th root of unity. Then $W(1, u, a)$ is equivalent to $W$ defined in (42).

**Proof** From Lemma 6.1 it is sufficient to prove that $W(1, 1, au^3)$ is equivalent to $W$. By assumption, $m = 4k$ for some positive integer $k$. Since $a^{8k^2}$ is a primitive 4-th root of
unity, there exists \( t \in \mathbb{Z}_4 \) such that \( u^3 = a^{8k^2t} \). We define a bijection \( \psi : \mathbb{Z}_m^2 \rightarrow \mathbb{Z}_{m^2} \) by

\[
\psi(i, \ell) = (4k^2t + 1)i + 4k\ell
\]

for \((i, \ell) \in \mathbb{Z}_m^2\). Then for all \( i, j, \ell, \ell' \in \mathbb{Z}_m \),

\[
(\psi(j, \ell') - \psi(i, \ell))(\psi(j, \ell') - \psi(i, \ell) - m) = ((4k^2t + 1)(j - i) + 4k(\ell' - \ell))(4k(\ell - \ell' + (i - j)^2 + 4k(i - j)) + 32k^2(-kt(j - i)(l' - l) + kt(j - i)(kt(j - i) + 1)/2 + (l' - l)(l' - l - 1)/2) \equiv (8k^2t + 1)(8k(\ell - \ell')(i - j) + (i - j)^2 + 4k(i - j)) \pmod{32k^2}).
\]

Thus

\[
W(\psi(i, \ell), \psi(j, \ell')) = a^{(\psi(j, \ell') - \psi(i, \ell))(\psi(j, \ell') - \psi(i, \ell) - m)} = a^{(8k^2t+1)(8k(\ell-\ell')(i-j)+(i-j)^2+4k(i-j))} = (au^3)^{2m(\ell-\ell')(i-j)+(i-j)^2+m(i-j)} = W_{(1),1,au^3}((i, \ell, 1, (j, \ell', 1)), \text{ and we conclude that } W \text{ is equivalent to } W_{(1),1,au^3}.
\]

To conclude the paper, we note that the decomposability and identification with known spin models are yet to be determined for the following cases.

1. \( W_{H,u,a} : r = 1, m \equiv 2 \pmod{4} \),
2. \( W'_{H,u,b} : r = 1 \),
3. \( W_{H,u,a} \text{ and } W'_{H,u,b} : r = 2 \),
4. \( W_{H,u,a} \text{ and } W'_{H,u,b} : r > 4 \) and \( m \) is not a power of 2.

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