Geometric definition, rapid prototyping, and cutting force analysis of cylindrical milling tools with arbitrary helix angle variations

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Abstract
In this work, a method is developed for geometric definition and analysis of cylindrical milling tools having various free-form variations of helix angle. The method is based on replicating a position vector on each cutting edge to generate a point set for the whole tool envelope using piece-wise rotation and magnification matrices which are varied according to mathematical laws describing the intended variable shape of the tool. The computed point sets are applied in additive manufacturing of samples of such tools having non-conventional shape features by transforming the point sets to stereolithography formats that are sliced to guide the 3D-printing processes. This manufacturing route that simplifies the realization of arbitrary helix profiles on milling tools is a major contribution of this work since such tools are gaining popularity for their passive damping of vibrations and reduction of cutting forces but are notoriously difficult to manufacture, limiting their exploitation. Analyses are shown about the effects of the considered variable profiles on milling cutting force. These suggest that cutting forces can be greatly suppressed by the proposed free-form helix angle variations. For example, relative to a conventional fixed helix tool of same mean helix angle, the innovative tools recorded 40.33%–84.42% and 60.53%–67.81% force reductions at axial depths of cut of 1 and 5 mm. This demonstrates that innovative variable tool profiles which can be realized through the simplified rapid prototyping technique are promising for advanced sustainable manufacturing of parts with preferred surface conditions.

Keywords
Milling tool, smooth variable helix angle, non-smooth variable helix angle, cutting forces and vibrations, adaptation of helix profiles

Introduction
Milling process performance indicators can be controlled by controlling, amongst other factors, tool shape and toolpath geometry. Milling tools are often produced with multiple helix-shaped flutes. The helix shape helps to smoothen and elongate the duration and geometric length of contact between a cutting edge and a workpiece thus minimizing impacts. Also, helix shapes induce oblique angles needed for easier chip evacuation. Furthermore, proper helix angle alteration on a tool can be used to mitigate chatter and reduce regenerative cutting forces. Helix angle alteration can be effected across flutes or on flutes, therefore, it is necessary for clarity to define the descriptive expressions for different type of alterations that will be discussed in this work. A uniform helix tool refers to a tool having the same helix angle specification on all the flutes. Therefore, uniform helix tools can be classified as either uniform fixed helix tools for which the helix angle does not vary on the flutes or uniform variable helix tools for which the helix angle varies on the flutes. The basic characteristics of uniform helix tools is that the shape of the transverse cross-section does not change with axial location. It is possible for the variable helix angle to change in a smooth (continuous) or
non-smooth (discontinuous) manner. Non-uniform helix tools, therefore, clearly suggest tools for which helix angle specification differs for at least one of the flutes, and the so-described tools have transverse cross-sections that change with the change of axial location. The words uniform and non-uniform connote similar meaning when referring to other geometric parameters like pitch angle. Where applicable, the described terminology is strictly used in the following brief literature review in order the put the related works in the same context.

It is observed in Altuntas et al.\(^6\) that non-uniform pitches can suppress regenerative chatter of milling around a desired speed range. A method for finding the optimal sequence of non-uniform pitch angles based on the criteria of regenerative chatter suppression for a given chatter frequency, spindle speed, and number of cutting edges is established in Budak.\(^7\) In Sims et al.\(^8\) it is reported that productivity improvements can result from non-uniform pitches and non-uniform fixed helix angles, and it is further reported that new milling cyclic-fold bifurcations can result from the non-uniformities. A study of the effects on cutting forces and regenerative chatter stability of free-form serration of cutting edges of end mills modeled with cubic splines finds that for the studied systems, edge force components cause force amplitudes of serrated cutters to differ from those of the regular cutters and that chatter suppression is expected from serration only when the feed rate is smaller than the double of the serration amplitude.\(^9\) Increasing the feed rate in the range where chip thickness is higher than the amplitude of the serration waves is found to destabilize the regenerative response of a milling process causing the dynamics to approach that of regular end mills.\(^10\) Non-uniform fixed helix angles are demonstrated in Yusoff and Sims\(^11\) to be capable of a fivefold improvement in chatter suppression in an optimal case. By presenting the smooth variable helix angle as a function of lag angle, harmonic variation of helix angle of milling tools is investigated in Dombovari and Stepan\(^5\) revealing high potential for chatter suppression and high material removal rates at the low spindle-speed domain. In Comak and Budak\(^12\) a design method is presented for selecting the best combination of non-uniform fixed helix angles and non-uniform pitches for maximizing chatter-free material removal rate. The reducing effects of serration on cutting forces and milling chatter instability are optimized in Tehranizadeh et al.\(^13\) using a genetic algorithm. The study reveals that milling tools with optimum serration can reduce cutting forces by up to 30\% in comparison to standard serrated tools. It has been reported in Guo et al.\(^14\) for a milling process that beyond helix angle of 30°, the effect of helix shape on chatter stability becomes minor.

Other works in literature have tried to ascertain how uniform fixed helix angles of cutting tools affect cutting forces. Smaller helix angles of end mills were experimentally found to develop greater cutting forces because of reduced period of contact between the tool and the workpiece.\(^15\) It is observed in Hosokawa et al.\(^16\) that the tangential and normal force components of milling process decrease significantly with the increase of helix angle. Other experimental investigations\(^17,18\) observed that the decrease of tangential and normal force components with the increase of helix angle is accompanied by increase of the axial force component. Worthy to note are the seemingly contradicting analytical and experimental results in Ozturk et al.\(^5\) which indicate that tangential, normal, and axial force coefficients are simultaneously increased with increase of helix angle. A finite element simulation presented in Li et al.\(^19\) showed that the decrease of resultant cutting force with the increase of helix angle reaches a minimum beyond which further increase of helix angle causes a rise of resultant cutting force. A similar trend is derived analytically in Wan et al.\(^20\) by considering the mechanistic model of cutting force in the feed-normal direction only. A generalized formulation of chip thickness is used in the modeling of cutting forces for a range of tools with irregular cutting edge geometries like serration, non-uniform pitches, and non-uniform fixed helix angles.\(^21\) In Okafor and Sultan,\(^22\) the geometries of bull-nosed wavy end-mills with harmonic variable helix angles were approximated with cubic spline interpolation in order to estimate the cutting forces developed during cutting of inconel 718 under emulsion cooling. The authors recorded peak cutting forces prediction errors that ranged from −0.09\% to 22.7\%. Based on the model in Okafor and Sultan,\(^22\) the effects of the parameters of the harmonic variable helix angles on cutting forces were simulated in Sultan and Okafor\(^23\) and show that increase in both wavelength and helix angle cause decrease of resultant cutting forces. By interpolating cutting edge data points using B-spline parametric curves, the impacts of geometries of a range of milling tools from end mill to serrated tapered ball end mill on chip load and cutting forces were revealed in Hosseini et al.\(^24\)

The mentioned works focused on the geometric modeling of the cutting edges of tools with non-uniform pitches and/or non-uniform fixed helix angles. Also covered in the literature are special cases of smooth geometric variation due to serration which induces smooth variation of radial position of cutting edge elements and smooth variation (of harmonic type) of helix angles. The works are focused on the effects of the above-named geometric non-uniformities and variations on cutting forces and regenerative chatter vibration. In other words, none of the reviewed works have considered tool geometry as an analytical basis for the manufacturing of the tools they analyzed. It is known that the manufacturing of varying-geometry tools is limited by the extreme difficulty of realizing unconventional shape features on the cutting edges, limiting their exploitation. The presented study is focused on the geometric definition of cylindrical milling tools with harmonic and various novel non-smooth variable and
free-form smooth variable helix angles with the aim of establishing and demonstrating a simplified framework for rapid prototyping or numerically controlled manufacturing of milling tools with such arbitrary shape features. Harmonic cutting edges are reported to be extremely difficult to manufacture through the currently available methods. The more sophisticated free-form cutting edges proposed here are expected to be even more difficult to manufacture using conventional approaches. Therefore, the proposed method is a major contribution for allowing exact realization of any form of variable helix angle on cutting edges. The impact of the shape variations on the cutting forces is examined, too, to provide a technical motivation for the advantages of the novel tool shapes.

Generic formulations applicable to both uniform and non-uniform variable helix angles will be developed for tool geometry and cutting forces. However, uniform cases will be used in this paper as examples for clear representation of results.

Section 2 provides a detailed geometric description of milling tools having free-form variable helix angles, and describes the appropriate graphical representation of such tools and the process of transforming the geometric models to CAD models for additive manufacturing of the tools. The important geometric and kinematics parameters are then used to analyze the cutting forces developed by the tools in Section 3. Variable helix angle examples are introduced and the cutting forces associated with the examples are computed and discussed in Section 4. The important outcomes of the work are summarized in the conclusion in Section 5.

**Geometry of cylindrical milling tools with variable helix angles**

This section presents a description of the geometry of cutting edges of arbitrary configuration in the three dimensional space using vector geometry. It also discusses the issue of proper graphical representation of the spatially varying geometric parameters of such tools which are obscured in the three dimensional view of the tools. Finally, it explains how the variable helix angles can be integrated into a rapid prototyping technique for the tools based on layered additive manufacturing.

**Cutting edge geometry**

The shape of a milling tool is a collection of many material points in space whose locations govern the geometry of the tool. These points are necessary for full geometric definition which in turn is necessary for the manufacturing and dynamical analysis of milling tools. Therefore, expressions representing the material points should be reproduced from independently specified basic geometric parameters and their dependent or derived geometric parameters. The basic geometric parameters of cylindrical milling tools are the tool envelope diameter $D$, the tool tip pitch angles and variable helix angles, and the derived geometric parameters like lag angles and off-tip pitch angles. This subsection describes how these basic geometric parameters can be used to specify the tool tip cutting edge positions to generate overall spatial distribution of the cutting edges. Milling tools are often equipped with multiple helix-shaped flutes. The $N$ flutes are sequentially indexed with the numbers $j = 1, 2, 3, \ldots, N$ but for symbolic brevity, geometric and dynamical quantities will be subscripted with $j$ only in the exceptional cases when the expressions for quantities are dependent on $j$, when expressions are dependent on the quantities for adjacent flutes or when the quantities contributed from different edges are summed to describe the overall quantities for the tool. Consider that a milling tool has a body fixed frame $\{X_{1}, X_{2}, X_{3}\}$ with the origin located at the tool center at the tip of the tool, see Figure 1(a).

The figure shows a representative cutting edge between the tool tip and a section at the axial coordinate $z$ along the $X_{3}$ axis. At $z = 0$ (the transverse plane containing the tool tip), the position vector of the cutting edge material element as seen in Figure 1(a) is

$$r(0) = \frac{D}{2} \begin{bmatrix} \sin \theta(0) \\ \cos \theta(0) \\ 0 \end{bmatrix}.$$  \hspace{1cm} (1)$$

The position vector of the cutting edge element at $z$ becomes

$$r(z) = R(z) + S(z) r(0),$$  \hspace{1cm} (2)$$

where the rotation matrix

$$S(z) = \begin{bmatrix} \cos \alpha(z) & \sin \alpha(z) & 0 \\ -\sin \alpha(z) & \cos \alpha(z) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$  \hspace{1cm} (3)$$

is due to the angular lag $\alpha(z)$ of the cutting edge element at the axial coordinate $z$ relative to the cutting edge element at $z = 0$ for the cutting edge depicted in Figure 1(a) to have a general variable helix angle, and $R(z) = [0 0 z]^T$ is the position vector of the center of the tool at $z$. Since the needed position vector $r(z)$ is dependent on $z$ and the angle $\alpha(z)$, the remaining part of this subsection is dedicated to the description of the procedures for computing the lag angle. Over a differential axial depth element of $dz$ at the axial coordinate $z$, the helix angle $\beta(z)$ can be considered as fixed as shown in Figure 1(b) such that the corresponding differential angular lag is $d\alpha = \frac{\pi}{2} \tan \beta(z) dz$. Therefore, noting that points higher up a cutting edge lag behind those lower down the edge in terms of angular coordinates at a rate $\frac{\pi}{2} \tan \beta(z)$, the angular lag at height $z$ relative to the cutting edge element at the tip becomes
The angular lag can be computed for helix angle functions of both analytical and numerical forms using composite numerical integration schemes. The generalized composite numerical integration schemes formulated and applied for computing helix-induced functions in Ozoegwu and Eberhard can be used.

With \( \alpha(z) \) computed, all the elevated material points of the cutting edge \( r(z) \) can be computed to completely specify the location and configuration of the cutting edge, and the analyses are repeated for all the cutting edges.

Traditionally, milling tools are manufactured with fixed helix angles meaning that the helix angles and pitch of such tools are generally given as \( \beta(z) = \beta \) and \( \psi(z) \neq \psi_0 \). Using equation (4), then the lag angle varies linearly with axial coordinate as \( \alpha(z) = \frac{2}{D} \tan \beta z \) for such traditional tools. The geometry, kinematics, and dynamics of such tools are relatively easy to analyze. As hinted in the introduction, it has been found in many investigations that variation of helix angles of milling tools can help to reduce cutting force which is known as the major militating factor against milling productivity and efficiency. As a result, there has been rising interest in such tools. While fixed and harmonic helix angles have mostly been considered, other variable helix angles are now being considered for further force reduction.

**Tool envelope generation**

The material points of cutting edges formulated in the foregoing can be adjusted using rotation and magnification operations to generate both the rake and flank surfaces which together form the envelope of a tool. The internal material points can also be generated in like manner as the envelope. To generate a tool envelope, consider that a cylindrical annulus is defined between radii \( R_0 = D/2 \) and \( R_A < R_0 \) to enclose the flutes of the tool. The radial range of the annulus can be discretized into radial segments \( \Delta R = (R_0 - R_A)/\Lambda \) to generate an arithmetic series of annuli at \( R_i = R_0 - i \Delta R \) for \( i = 0, 1, 2, \ldots, \Lambda \). Every material point \( r(z) \) on the cutting edge can then be transformed (a magnification) to a material point on the rake face as

\[
 r_{\text{rake}}(z) = R_i r(z),
\]

where

\[
 R_i = \frac{2}{D} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} R^{-1}_i \end{bmatrix}.
\]

The magnification \( R_i \) is formulated in a way that does not affect the component of \( r(z) \) in the \( X_3 \) axis. To estimate the coordinates on the flank face of the \( j \)-th flute, the \((j - 1)\)-th pitch \( \psi_{j-1,z} = \psi_{j-1}(z) \) is broken into angular steps \( \Delta \psi_{j-1,z} = \psi_{j-1,z}/\Lambda \) to generate the series \( \psi_{j-1,z,i} = i \Delta \psi_{j-1,z} \) for \( i = 1, 2, \ldots, \Lambda - 1 \) where

\[
 \psi_{j,z}(z) = \psi_{j-1,0} - \alpha_j(z) + \alpha_{j-1}(z).
\]

The lag angles \( \alpha_j(z) \) are computed using equation (4) for smooth variable helix angles and computed using equation (29) for non-smooth variable helix angles. The flank face coordinates can then be generated as

\[
 \alpha(z) = \frac{2}{D} \int_0^z \tan \beta(\zeta) d\zeta.
\]
\begin{equation}
R_i^{\text{flank}}(z) = S_i \psi_i^{R}(z),
\end{equation}

where \( i \) varies from 1 to \( \Lambda - 1 \) to avoid the duplication of the points corresponding to \( i = 0 \) and \( \Lambda \). The rotation matrix \( S_i(z) \) is

\begin{equation}
S_i(z) = \begin{bmatrix}
\cos \psi_{z,i} & -\sin \psi_{z,i} & 0 \\
\sin \psi_{z,i} & \cos \psi_{z,i} & 0 \\
0 & 0 & 1
\end{bmatrix},
\end{equation}

where \( \psi_{z,i} \) is a shortened representation of \( \psi_{i-1,z,i} \). The flank face coordinates can also be generated from other distributions of rake face coordinates. For example, the Lag angle \( \psi(z) \) (a computed function needed for evaluation of the material points of the flank surface of flutes). The latter two geometric functions should be clearly visible and measurable on a proper two dimensional representation of a tool, and such a representation is proposed now. The pitch angle \( \theta(z) \) of all the flutes can be seen with equal detail on every transverse section of the tool. Relative to the transverse section at the tool tip, the lag angle \( \alpha(z) \) of all the flutes can also be seen on any elevated transverse section. Therefore, the horizontal axis of a representative two dimensional chart should simultaneously represent the pitch angle \( \theta(z) \) and lag angle \( \alpha(z) \). These functions should be charted on the same axes for all the flutes of a tool. Since pitch angles are measured between adjacent flutes and lag angles are measured relative to the tool tip, the horizontal axis should be given in terms of cumulative angular pitch (CAP) defined as

\begin{equation}
\Psi_j(z) = \sum_{j=1}^{j} \psi_j(z),
\end{equation}

and the lag angle should be given as shifted angular lag (SAL) (shifted by the cumulative pitch at the tip \( \Psi_j(0) = \sum_{j=1}^{j-1} \psi_j(0) \)) defined as

\begin{equation}
\alpha_j^{\text{shifted}}(z) = \psi_j(z) + \Psi_j(0).
\end{equation}

The geometry charts based on equations (13) and (14) will place the plots for the \( (j+1) \)-th flute to the right of the plots for the \( j \)-th flute. That is, plots for \( j = 1, 2, \ldots, N \) are placed from left to right on the same plane. An illustrative two-dimensional chart for a two-flute tool is shown in Figure 2(a).

The angular pitch at the axial coordinate \( z \) is read from the graph as \( \theta_j(z) = \Psi_j(z) \) for \( j = 1 \) and \( \theta_j(z) = \Psi_j(z) - \sum_{j=1}^{j-1} \psi_j(z) \) otherwise. Likewise, angular lag at the axial coordinate \( z \) is read from the graph as \( \alpha_j(z) = \alpha_j(z) + \Psi_j(0) \). For example, in Figure 2(a), the angular pitches at the tool tip are read as \( \psi_{1,0} = A_0 B_0 \) and \( \psi_{2,0} = B_0 C_0 \). At the axial coordinate \( z \), the angular pitches are read as

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure2.png}
\caption{Two-dimensional representations of arbitrary tools: (a) typical two-flute tool and (b) typical four-flute tool.}
\end{figure}

Graphical representation and geometric constraints of variable helix angle tools

Cylindrical milling tools with variable helix angles can be represented using the computed point set for the tool envelope. The problem is that a three dimensional representation will obscure some of the flutes. On the other hand, two dimensional graphs can be used to chart the important geometric parameters of all the flutes with equal detail. The most important variable geometric parameters for a flute are the helix angle \( \beta(z) \) (an independently specified function), the lag angle \( \alpha(z) \) (a computed function needed for evaluation of the material points of the cutting edge \( r(z) \)) and the pitch angle \( \psi(z) \) (a computed function needed for evaluation of the material points of the flank surface of flutes).
\( \psi_1(z) = A_1B_1 \) and \( \psi_2(z) = B_1C_1 \) while the angular lags are read as \( \alpha_1(z) = B_1B_2 \) and \( \alpha_2(z) = C_1C_2 \). A very important geometric constraint on non-uniform tools is that the pitch angle must be greater than zero within the flute depth \( w_h \) since flutes must not intersect one another. The constraint does not exist for uniform tools since the lag angle is the same for all the cutting edges such that no two cutting edges approach each other or recede from each other. This condition simply means that \( \phi(w_h) > 0 \) which can be seen from equation (7) to mean

\[
\alpha_j(w_h) - \alpha_{j-1}(w_h) < \psi_{j-1,0}.
\]  

This inequality can only be solved analytically for \( w_h \) when \( \alpha_j(z) \) are obtainable in closed forms which is the case for fixed helix angle tools. Noting that \( \alpha_j(z) = \frac{z}{B}\tan\beta_jz \) for this special case of non-uniform fixed tools, the condition becomes \( w_h < \min(\frac{z}{B}\tan\beta_jz) \) for all \( \beta_j > \beta_{j-1} \) where the function \( \text{min}(\cdot) \) returns the minimum value of the argument. The angular lag functions \( \alpha_j(z) \) for the more general non-uniform variable helix angle can only be computed numerically meaning that the geometric constraint \( \psi(w_h) > 0 \) can only be read graphically as illustrated for an arbitrary four-flute tool in Figure 2(b). The flute depth cannot be higher than the axial coordinate corresponding to the point indicated with an arrow due to geometric restriction between the first and the second flutes. This demonstrates another important use of the graphical representation.

The geometric constraints described so far relates to the pitch angle due to the limit imposed by the flute depth. The pitch angle itself is dependent on the lag angle which in turn is dependent on the helix angle. The lag angle expressed in equation (4) can only have a meaningful numerical value when \( \cos\beta(z) \neq 0 \). This constitutes an important geometric constraint which reflects the nature of the dependence of lag angle on the helix angle.

**Additive manufacturing of the tools**

Based on the geometric description, a point set representing a tool envelope can be generated. Since the positions of the points are exactly known, the coordinates of a sample point set can be written in the format of a stereolithography (stl) file which can then be sliced in a 3D printing preprocessor for rapid prototyping of the tool represented by the point set. To generate a point set for a tool, the following steps are repeated for all the flutes:

1. Specify the basic geometric parameters of the tool which are the tool diameter \( D \), the number of flutes \( N \) and the flute depth \( w_h \).
2. Specify the pitch angle at the tool tip \( \phi_0 \) and the helix angle function \( \beta(z) \).

3. Compute the position \( r(0) \) of the cutting edge of the flute at the tool tip using the specified angular position at the tool tip \( \theta(0) \).
4. Compute the lag angle \( \alpha(z) \) at axial depth \( z \) for all the depth steps.
5. Compute the rotation matrix \( S(z) \) and then compute all the material points \( r(z) \) for the cutting edge of the flute.
6. Generate the coordinate data for the rake face \( r_{rake}^{flank}(z) \).
7. Compute the pitch angle \( \psi(z) = \psi_{j-1}(z) \), then compute the transformation matrices \( S_h(z) \) and generate the coordinate positions for the flank side \( r_{flank}^{flank}(z) \).
8. The coordinate data of the rake and flank surfaces are combined in a structured manner to form the point set for the tool envelope.

If the number of layers of points is \( M_{ps} \), the number of points per layer per flute is \( 2\pi \) and the number of flutes is \( N \) then the number of points in the point set is \( 2\pi\lambda(M_{ps} + 1) \) and the number of stl loops (also the number of triangular meshes) is \( 2(2\pi\lambda - 1)M_{ps} \). The generation of the loops is automated using Matlab, the generated stl file for the envelope is converted to binary solid format using Meshmixer, and then the tool can be printed, for an example see Figure 3.

**Cutting force analysis**

The cutting force is evaluated to provide a basis for quantified assessment of the benefits of the novel variable helix angle functions. A more beneficial variable helix angle reduces the magnitude of cutting forces and hence reduces the amplitude of cutting force-induced periodic vibrations in relative terms. Reduction of cutting power and energy which are directly proportional to cutting force is a major criterion of sustainable machining.

The popular so-called mechanistic methods to compute cutting forces are the linear and power force
laws.\textsuperscript{26} The laws are empirically established as 
\[ F = K_c h w + K_n w \] 
and 
\[ F = K h^2 w \] 
for cutting force components where \( K_c \), \( K_n \), and \( K \) are force coefficients. They are used as constitutive equations in the formulation of cutting force models for geometrically and kinematically advanced machining processes like the milling process which has time-varying chip thickness and time-varying cutting force orientation. A review of the development and early applications of the mechanistic models can be found in Ehmann et al.\textsuperscript{27} These basic force laws were applied in modeling of cutting forces in the works cited in the introductory section. The linear force law was used in Engin and Altintas\textsuperscript{26,29} to develop generalized mathematical models for cutting forces on solid end mills. The linear force law was also utilized in a unification of cutting force modeling for turning, boring, drilling, and milling operations.\textsuperscript{30} In what follows here, a unified formulation which represents both the linear and power force models is developed for cylindrically advanced machining processes like the milling tools with arbitrary helix angle variations.

Every differential element of the cutting edge is subject to differential force components from which the cutting force can be integrated, see Figure 4(a). The cutting edge element is subjected to the differential normal \( dF_n(t,z) \), tangential \( dF_t(t,z) \), and helix-aligned \( dF_h(t,z) \) forces. Both \( dF_n(t,z) \) and \( dF_t(t,z) \) act normal to the edge element but while the latter is tangential the former points toward the tool axis. The force \( dF_h(t,z) \) acts along the cutting edge element. The differential forces are shown in Figure 4(a) at a general axial height \( z \). The response of the tool is normally considered in the feed direction \((X_1)\), feed-normal direction \((X_2)\) and, sometimes, axial direction \((X_3)\), therefore, the differential force vector 
\[ d\mathbf{F}(t,z) = [dF_n(t,z) \ dF_t(t,z) \ dF_h(t,z)]^T, \]
when referred to the \( \{X_1, X_2, X_3\} \) system, becomes
\[ d\mathbf{F}(t,z) = A(t,z)d\mathbf{F}(t,z), \]
where \( A(t,z) \) is the rotation matrix given as
\[ A(t,z) = S_1^T(z)S_2^T(z)S_3^T(t,z) \] (17)
and
\[ S_1(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta(z) & \sin\beta(z) \\ 0 & -\sin\beta(z) & \cos\beta(z) \end{bmatrix}, \] (18)
\[ S_2(z) = \begin{bmatrix} \cos\beta(z) & 0 & \sin\beta(z) \\ 0 & 1 & 0 \\ -\sin\beta(z) & 0 & \cos\beta(z) \end{bmatrix}, \] (19)
\[ S_3(z,t) = \begin{bmatrix} \cos\theta(t,z) & -\sin\theta(t,z) & 0 \\ \sin\theta(t,z) & \cos\theta(t,z) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]

The angle \( \theta(t,z) \), illustrated in planar view for the second flute of a three-flute tool at height \( z \) in Figure 4(b), is the instantaneous angular position of the rotating cutting edge element at the axial height \( z \). The instantaneous angular position \( \theta(t,z) \) is the sum of the geometric (or static) part \( \theta(0) - \alpha(z) \) where \( \theta(0) \) and \( \alpha(z) \) are as given in the preceding section and the rotational (kinematic) part \( \pi\Omega t/30 \) where \( \Omega \) is the spindle speed in rpm and \( t \) is the time instant. Therefore,
\[ \theta(t,z) = \frac{\pi\Omega}{30} t + \theta(0) - \alpha(z). \] (20)

The differential cutting force vector \( d\mathbf{F}(t,z) \) is a function of chip thickness \( h(t,z) \)\textsuperscript{26} and is expressed in a general form here as
\[ d\mathbf{F}(t,z) = H(t,z)K(h(t,z))dz. \] (21)
For the power force model, \( \mathbf{K}(h(t,z)) \) is the column of force coefficients given as
\[
\mathbf{K}(h(t,z)) = \mathbf{K} = \{K_1, K_n, K_b\}^T \quad \text{and} \quad H(t,z) = g(t,z)h(t,z)^T.
\] (22)

For the linear model,
\[
\mathbf{K}(h(t,z)) = h(t,z)\mathbf{K}_c + \mathbf{K}_e
\] (23)

where \( \mathbf{K}_c \) and \( \mathbf{K}_e \) are the columns of chip-forming and edge force coefficients given as
\[
\mathbf{K}_c = \{K_{c_1}, K_{c_n}, K_{c_b}\}^T, \quad \mathbf{K}_e = \{K_{e_1}, K_{e_n}, K_{e_b}\}^T \quad \text{and} \quad H(t,z) = g(t,z).
\] (24)

According to Ozoegwu and Eberhard,\(^{31}\) the chip thickness is given as \( h(t,z) = \tau(z)\sin\theta(t,z) \) where the function \( g(t,z) \) switches on and off the cutting action of the cutting edge according to whether the edge is in cutting action or not. It is given as
\[
g(t,z) = 0.5(1 + \text{sign}(\sin(\theta(t,z) - \arctan(\phi)))
\]
\[
- \sin(\theta - \arctan(\phi))),
\] (25)

where \( \phi = (\sin \theta_s - \sin \theta_0)/(\cos \theta_s - \cos \theta_0) \). The start angle \( \theta_s \) and end angle \( \theta_e \) of cutting action are \( \theta_0 = 0 \) and \( \theta_e = \arccos(1 - 2p) \) for up-milling and \( \theta_s = \arccos(2p - 1) \) and \( \theta_e = \pi \) for down-milling. The radial immersion is given as \( \rho = B/D \) where \( B \) is the radial depth of cut, see Figure 4(b). The expressions for \( g(t,z), \theta_s, \) and \( \theta_e \) for the configuration that combines up- and down-milling can be seen in Ozoegwu et al.\(^{32}\) The time interval \( \tau(z) = \frac{\pi}{\beta_0 z} \phi(z) \) is the time interval between a cutting edge and the preceding cutting edge at the axial height \( z \). The integrated cutting force vector summed from all the cutting edges becomes
\[
\mathbf{F}(t) = \sum_{j=1}^{N} \int_0^w H_j(t,z)\mathbf{A}_j(t,z)\mathbf{K}(h_j(t,z))dz,
\] (26)

where \( w \) is the axial depth of cut. The mean cutting force can be computed as
\[
\overline{F} = \frac{1}{t_f} \int_{t_0}^{t_f} |\mathbf{F}(t)|dt,
\] (27)

where \( t_f \) is the time interval of the milling process. It is not unusual to imply from many published works the simplification that \( df_i(t,z) \) aligns with the \( X_3 \) axis meaning that \( \mathbf{A}(t,z) = S_i^f(t,z) \).

The criterion for identifying a better variable helix angle function is a better reduction of the magnitudes of cutting forces. In addition, reduction of the mean cutting force \( \overline{F} \) is viewed as a measure of reduced machining energy. This criterion is used in what follows to compare different variable helix angles designs.

### Examples

Different variable helix angle functions \( \beta(z) \) are compared in this section. The comparison is benchmarked against milling tools having uniform tool tip pitch angles \( \phi_0 \) and fixed helix angles meaning that \( \phi(z) = \phi_0 \) and \( \beta(z) = \beta \neq \pi/2 \).

### Variable helix angles

In Okafor and Sultan,\(^{22}\) harmonic helix angles were interpolated using cubic splines while in Dombovari and Stepan,\(^{5}\) helix angles were derived from explicitly specified harmonic lag angles. Since the helix angle is a primarily specified geometric parameter of milling tools, the harmonic helix angle is expressed explicitly here in the form
\[
\beta(z) = C(1 + \lambda \sin \left( \frac{2\pi n}{w_h} z + \phi \right))
\]
\[
= C + a_h \sin \left( \frac{2\pi}{L_h \cos C} z + \phi \right),
\] (28)

where \( n \) is the number of wave cycles needed in the depth range of the helix profile \( w_h \), \( \phi \) is phase angle of the variation, \( a_h = CA \) is the amplitude of the variation, and \( L_h = w_h/(n \cos C) \) is the wavelength. Since the helix angle is usually positive, the parameter \( \lambda \) is here selected from the range \( |\lambda| \approx 1 \). The parameter is used to control the amplitude of harmonic variation about the flute mean helix angle \( C \) assuming that \( n \) is an integer. It must be noted that when \( \lambda = 0 \), harmonic helix angle becomes fixed helix angle. For the harmonic tools, the constraint \( \cos(\beta(z)) \neq 0 \) implies that \( C \neq \frac{x}{2}(1 + \lambda \sin \left( 2\pi \frac{\phi}{L_h} z + \phi \right))^{-1} < \frac{\pi}{2} \). This means that \( C < \frac{x}{2}(1 + \lambda \sin \left( 2\pi \frac{\phi}{L_h} z + \phi \right))^{-1} \) when the parametric effect of \( C \) is investigated.

Tools with non-smooth variable helix angles can be modeled in a piece-wise manner by tagging the cutting edge segments between the depths \( z_{i-1} = (i - 1)\Delta z \) and \( z_i = i\Delta z \) with numbers \( S_i \) for \( i = 1, 2, 3, \ldots, P \) and then representing the lag angle variation for the depth \( z \) varying within the depth interval tagged \( S_i \) as
\[
\alpha(z) = \begin{cases} 
\frac{\pi}{2} \tan \beta_0 z & \text{for } S_i = 1, \\
\frac{\pi}{2} \tan \beta_{S_{i-1}}(z - z_{S_{i-1}}) + \frac{\pi}{2} \sum_{j=1}^{S_i-1} \tan \beta_{j-1}(z_j - z_{j-1}) & \text{otherwise.}
\end{cases}
\] (29)
Table 1. Parameters for geometric plots and cutting force simulation.

| Parameter                     | Symbol | Value          | Unit  |
|-------------------------------|--------|----------------|-------|
| Spindle speed                 | \(\Omega\) | 5000           | rpm   |
| Tool diameter                 | \(D\)  | 20             | mm    |
| Tool tip pitch                | \(q_0\) | (180 180) (two-tooth) | deg   |
|                               |        | (90 90 90) (four-tooth) |     |
| Tool mean helix angle         | \(\beta\) | 25             | deg   |
| Flute depth                   | \(w_f\) | 5 (for dynamical simulation) | mm |
|                               |        | 50 (for geometric plots) |     |
| Feed speed                    | \(v\)  | 0.0025         | m/s   |
| Up-milling radial immersion   | \(\rho\) | 0.5            |       |
| Linear force law coefficient  | \(K_c\) | 10^6 [2959.815 1086.252 661.602]^T | Nm⁻² |
|                               | \(K_e\) | 10^3 [10.926 4.010 2.442]^T | Nm⁻¹ |

The smoothed free-forms rely on numerically specified non-smooth values of helix angles at discrete axial depth locations which are then smoothed with polynomial and various spline interpolations to ensure smooth variable helix angles. The simplest interpolated variable helix angle is the linear form \(\beta(z) = \frac{\beta(w_0) - \beta(0)}{w_0 - 0} z + \beta(0)\) where \(\beta(0)\) is the helix angle at \(z = 0\) and \(\beta(w_0)\) is the helix angle at \(z = w_0\). The linear variation is the first order member of the set of polynomial variations that has countable infinite members ranging from zero-th to infinite order variations. The zero-th order polynomial variation corresponds to the case of fixed helix angle characterized with the linear lag angle variation \(\alpha(z) = \frac{1}{2} \tan \beta\). A \(p\)-th order polynomial variation of the helix angle of a cutting edge requires a knowledge of not less than \(p + 1\) known interpolation values, \(\beta_i = \beta(i \Delta z)\) for \(i = 0, 1, 2, \ldots, (P - 1)\), of the helix angle at equally spaced depth mesh points \(0, \Delta z, 2\Delta z, \ldots, P \Delta z\) where \(P \geq p + 1\), \(P \Delta z = w_0\), and \(\Delta z = w_0/P\). The general polynomial helix angle model can then be written as

\[
\beta(z) = a^T(z) T^{-1} S v, \tag{30}
\]

where \(a(z) = \{1 \ z \ \cdots \ z^p\}^T\) is the vector of polynomial basis and the matrices \(T \in \mathbb{R}^{(p + 1) \times (p + 1)}\), \(S \in \mathbb{R}^{(p + 1) \times (p + 1)}\), and the vector of numerically specified discrete values of helix angles \(v \in \mathbb{R}^{p + 1}\) are given as

\[
T_{mn} = \begin{cases} 
  p + 1, & \text{for } m = 1 \text{ and } n = 1, \\
  (\Delta z)^m + n^2 \sum_{j = -p + 1}^{1} j^m + n^2, & \text{otherwise},
\end{cases} 
\tag{31}
\]

\[
S_{mn} = \begin{cases} 
  1, & \text{for } m = 1 \text{ and } n = p, \\
  (n-p)^{m-1}(\Delta z)^{m-1}, & \text{otherwise},
\end{cases} 
\tag{32}
\]

\[v_m = \beta_i + m \Delta z. \tag{33}\]

While polynomial interpolations are based on the nodal boundaries of \(p\) adjoining interpolation intervals, spline interpolations are realized by imposing boundary conditions on the interpolated interval. Typically, such boundary conditions specify that the interpolated function and its derivatives are continuous at the nodes. While equation (29) is proposed specifically for computation of the lag angles of non-smooth variable helix flutes, the lag angles of the smoothed versions should be computed by applying numerical integration rules in equation (4).

**Numerical results and discussions**

In what follows, the milling dynamics of the example variable helix angles relative to the benchmark of a conventional fixed helix angle are compared. This is judged in terms of the extent of suppression of cutting forces induced by the considered variable helix angles relative to the benchmark. The harmonic helix tools have the shape parameters \(C, n, \phi, \text{ and } \lambda\). The numerical parameters adopted for simulations are summarized in Table 1. To ensure typical order of magnitudes, the values of the elements of \(K_c\) and \(K_e\) given in Table 1 are transformations of the force coefficients in Altuntas et al., see Appendix A for more detail of the transformation. Results for two-flute tools are generated as shown in Figure 5 for 1 mm axial depth of cut. The result of the effects of variation of the harmonic helix parameter \(C\) on \(F\) when \(\lambda = 0.5\), \(n = 5\), and \(\phi = 0^\circ\) is shown in Figure 5(a). It is seen that initial rise of \(C\) induces a fall in \(F\) (this is analogous to the established results in Sultan and Okafor for harmonic tools) followed by rise of \(F\) on continued rise of \(C\) (this is analogous to the established results in Li et al. and Wan et al. for fixed helix tools). A novel result that is obvious from the figure is that the trend can reoccur for harmonic helix tools on continued increase of \(C\) within the limit imposed by the earlier discussed constraint on \(C\). The minimum mean cutting force of 16.75 N occurred at \(C = 52.84^\circ\). Rise of number of wave cycles \(n\) when \(\lambda = 0.5\), \(C = 25^\circ\), and \(\phi = 0^\circ\) is seen in Figure 5(b) to cause an initial net fall in \(F\) followed by relatively mild and unsteady undulations with continued rise of \(n\). The minimum mean cutting force of
17.83 N occurred at \( n = 4 \): Rise of phase angle \( \phi \) when \( n = 5 \), \( C = 25^\circ \), and \( \lambda = 0.5 \) is seen in Figure 5(c) to cause an S-shaped continuous rise in \( F \) after an initial minor fall. The minimum mean cutting force of 17.87 N occurred at \( f = 3 \): 60\(/C176\). The parameter \( l \) did not affect the cutting force, therefore, the variation amplitude \( ah \) has the same effect as \( C \). The wavelength \( Lh \) has opposite effect as \( n \). The fall in force with rise of \( Lh \) reported over just three data points in Sultan and Okafor23 may correspond to a typical segment of rise in force with increase of \( n \). To explain the fall of \( F \) with initial rise of \( n \), the variation of \( n \) from 0 to 0.25 during which there will be effective rise of helix angle is considered. By virtue of the results in Wan et al.20 and Sultan and Okafor23 which correlate falling force with rise of helix angle, initial rise of \( n \) is, therefore, expected to cause a fall in \( F \) as seen in Figure 5(b).

To check for the possible benefits of free-form helix angles, a comprehensive comparison involving five four-fluted tools with variable helix angle types that span all the considered cases is carried out. Since the helix angle has a strong effect on cutting forces, the benchmark tool (tool 1) and all the five compared tools are defined such as to have equal mean helix angle \( \beta = 25^\circ \). All the tools have the same basic geometric parameters and are subjected to the same cutting process summarized in Table 1. The variable helix angle types of the compared tools are summarized in Table 2.

The variable helix angle parameters of tool 2 are \( C = 25^\circ \), \( \lambda = 0.5 \), and \( \phi = 0^\circ \). The optimum number of wave cycles for tool 2 is found to be \( n = 4 \) at both axial depths of cut of 1 and 5 mm using graphical plots shown in Figure 6 generated under uniform variation of \( n \). Tool 3 is non-smooth with optimal assignment of sampled helix angles along the cutting edges. The helix angles are sampled to have a mean of \( \beta = 25^\circ \) and the optimal sequence of the values along the cutting edges which minimizes the mean cutting forces are found through a random search algorithm based on the Matlab command \texttt{randperm}. The command is used to generate many permutations of the sampled helix angles and a looped computation of \( F \) is carried out to identify the optimal permutation. For axial depths of cut of \( w = 1 \) and 5 mm, results of samples of such looped computation are shown in Figure 7 for 500 permutations and the optimal permutations are shown in Table 3. It can be noted from Figure 7 that simple permutation of non-smooth helix angles along the cutting edges can greatly affect cutting forces. Though it is possible to identify the global optimal permutation by looping through all the possible permutations (3628800 in number) which can be generated using the Matlab command \texttt{perms}, the figure shows that random search which utilizes much smaller number of permutations can reveal several near-optimal permutations. Tool 4, tool 5, and tool 6 are the cubic spline, piecewise cubic Hermite, and cubic polynomial smoothing of tool 3. The envelope point sets and their graphical representations are shown in Figure 8 for all the tools. The percentages of mean cutting force reduction of the tools are summarized in Table 2 for the two axial depths of cut. In the table, positive values indicate force reduction.
since the percentage mean cutting force reduction is calculated as \( \Delta \overline{F}_i = 100 \frac{\overline{F}_i - \overline{F}_0}{\overline{F}_0} \) where \( \overline{F}_0 \) is the mean force for the benchmark tool \( 1 \) and \( \overline{F}_i \) is the mean force for tool \( i \) for \( i = 2, 3, \ldots, 6 \). The computed percentage mean cutting force reductions in Table 2 and the time evolution of the cutting force components in Figures 9

| \( z \) (mm) | 0  | 0.5 | 1  | 1.5 | 2  | 2.5 | 3  | 3.5 | 4  | 4.5 |
|--------------|----|-----|----|-----|----|-----|----|-----|----|-----|
| \( \beta(z) \) at \( w = 1 \) | 25 | 34  | 30 | 10  | 13 | 40  | 35 | 16  | 20 | 27  |
| \( \beta(z) \) at \( w = 5 \) | 20 | 25  | 10 | 40  | 30 | 27  | 35 | 13  | 16 | 34  |
and 10 show that the proposed non-smooth variable helix angle (tool 3) and the smoothed versions (cubic spline (tool 4), Hermite (tool 5), and polynomial (tool 6)) are much more effective than the harmonic variable helix angle (tool 2) for reduction of cutting force. At 1 mm axial depth of cut, the non-smooth tool 3 attained a very high force reduction of 84.42%. Smoothing weakened the force reduction capacity of the tool at this axial depth of cut. The non-smooth tool 3 attained a force reduction of 65.50% at 5 mm axial depth of cut, and at this axial depth of cut, cubic Hermite smoothing enhanced the force reduction capacity of the tool. One reason for this significant reduction of cutting force even when the mean helix angle is fixed could be that the differential forces are directed differently at different segments of a cutting edge causing the components to cancel each other to some extent in the feed, feed-normal, and axial directions. This is the most likely explanation as it is not rational to expect reductions due to the higher helix segments which are counteracted by the lower helix segments to be the explanation. Reduced force means reduced form error arising from deflections, reduced cutting energy, and reduced tool wear (prolonged tool life), therefore, non-smooth and free-form variable helix angles raise the prospects of machining of accurate parts with reduced energy demands over prolonged tool life.

**Conclusions**

Milling tools with shape variations are increasingly being deployed in production because of their favorable dynamics but their potential and possible options are not fully exploited because definition of variable helix shapes on flutes is difficult to perform systematically. Motivated by this limitation, an analytical method that simplifies the realization of generalized variable helix angles on flutes through additive manufacturing is formulated and demonstrated. The geometric features of a tool were captured to generate a point set which represents the tool envelope to be exported to 3D printers for additive manufacturing. A simplified two dimensional graphical representation for the three dimensional tools is established for variable geometries and the associated geometric constraints. The dynamics of the tools is analyzed to justify their benefits in terms of cutting force reduction. The innovative tools with non-smooth variable helix angles and smooth free-form helix angles showed significant promise for cutting force reduction. Relative to a conventional fixed helix tool of same mean helix angle, the innovative tools recorded 40.33%–84.42% force reductions at axial depth of cut of 1 mm and 60.53%–67.81% force reductions at axial depth of cut of 5 mm. Therefore, geometric design of tools can be exploited in suppressing cutting forces, and the other benefits associated with...
cutting force reduction like form error reduction, vibration reduction, machining energy reduction, and tool life elongation.

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Appendix

A: Transformation between linear and power force coefficient vectors

Given equation (21), equivalence of the linear and power cutting force models expressed in scalar form reads

\[ hK_{ac} + K_{tc} = h^2K_i \quad \text{for } i = t, n, \text{and } a. \]  

(A1)

If the parameters of the power force law are known and the aim is to determine the equivalent parameters of the linear force law, then six equations are needed. A way to get a set of six equations for this purpose is to specify two chip thickness values \( h_1 \) and \( h_2 \). Equation (A1) can then be expressed in matrix form from which results

\[
K_i = Q^{-1}F_p \quad \text{where} \quad K_i = \begin{bmatrix} K_c & K_e \end{bmatrix},
\]

\[
Q = \begin{bmatrix} h_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & h_1 & 1 & 0 \\ 0 & 0 & 0 & h_1 & 1 \\ h_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & h_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & h_2 \end{bmatrix} \quad \text{and} \quad F_p = \begin{bmatrix} h_1^2K_i \\ h_1^2K_n \\ h_1^2K_a \\ h_2^2K_i \\ h_2^2K_n \\ h_2^2K_a \end{bmatrix}.
\]

(A2)

A similar transformation of linear force parameters to power force parameters can be derived. The parameters of the force law used in Altintas et al.\textsuperscript{6} are

Figure A1. Power force to linear force transformation example.

\( K_i = 697 \text{ MPa} \) and \( K_n = 255.799 \text{ MPa} \). Since edge force coefficients are not involved, the power force law is assumed. The other force parameters \( K_c \) and \( \gamma \), and the chip thicknesses \( h_1 \) and \( h_2 \) are assumed on practical order of magnitude bases. A helix-aligned force coefficient of \( K_{ac} = 155.799 \text{ MPa} \) and an exponent of \( \gamma = 0.85 \) are assumed for the power law. The values \( h_1 = 0.015 \) and \( h_2 = 0.03 \text{ mm} \) per tooth are assumed for chip thickness. Inserting these values in equation (A2) gave the \( K_c \) and \( K_e \) given in Table 1. For an illustration of the equivalence, a power force data is plotted against the transformed linear force data in Appendix Figure A1 for the four-tooth miller cutting at 5 mm depth of cut. Based on the figure, the transformation is acceptable since it is more about maintaining order of magnitude.

B: Notation

\( a_0 \) \quad \text{wave amplitude of harmonic tool}

\( B \) \quad \text{radial depth of cut}

\( D \) \quad \text{tool diameter}

\( F(t, z) \) \quad \text{cutting force vector}

\( F_c(t, z) \) \quad \text{tangential cutting force component}

\( F_n(t, z) \) \quad \text{radial (thrust) cutting force component}

\( F_a(t, z) \) \quad \text{helix-aligned cutting force component}

\( F(t, z) \) \quad \text{a rotational transformation of } F(t, z)

\( T \) \quad \text{mean cutting force}

\( h(t, z) \) \quad \text{chip thickness}

\( K \) \quad the column of non-linear cutting force coefficients \( \{ K_i, K_n, K_a \}^T \)

\( K_i, K_n, K_a \) \quad \text{tangential, radial, and helix-aligned cutting force coefficients}

\( K_c, K_{nc}, K_{ac} \) \quad \text{the column of linear chip-forming force coefficients} \( \{ K_{nc}, K_{nc}, K_{ac}\}^T \)

\( K_{nc}, K_{nc}, K_{ac} \) \quad \text{tangential, radial, and helix-aligned chip-forming force coefficients}

\( K_e \) \quad the column of linear edge force coefficients \( \{ K_{nc}, K_{nc}, K_{ac}\}^T \)
$K_{te}$, $K_{ae}$, $K_{ae}$
tangential, radial, and helix-aligned edge force coefficients

$L_h$
wavelength of harmonic tool

$M_{ps}$
number of sampling points on a cutting edge

$n$
number of wave cycles of harmonic tool

$N$
number of flutes

$P$
number of fixed helix segments of non-smooth flute

$r(z)$
positions on flank face

$r_{flank}(z)$
positions on flank face

$r_{rake}(z)$
positions on rake face

$r_{internal}(z)$
positions at the internal material

$r_{i,q}$
points of milling tool

$R_i$
radial coordinates of rake face

$R_0$
outer flute radius

$R_L$
inner flute radius

$v$
feed speed

$w_h$
flute depth

$w$
axial depth of cut

$z$
axial coordinate

$\alpha(z)$
lag angle

$\alpha_{shifted}(z)$
shifted angular lag

$\beta(z)$
helix angle

$\bar{\beta}$
tool mean helix angle

$\Lambda$
number of sampling points on rake or flank face

$\Omega$
spindle speed

$\phi$
phase angle of harmonic tool

$\psi(z)$
pitch angle

$\psi_0$
tool tip pitch angle

$\Psi_j(z)$
cumulative angular pitch

$\rho$
radial immersion

$y_0$
tool tip pitch

$\tau(z)$
rotational time lag between two adjacent flutes

$\theta(t, z)$
instantaneous angular position of a cutting edge element

$\theta_0, \theta_1$
end and start angular positions of cutting action