Dynamic Analysis of the Rod-Fastened Rotor Considering the Characteristics of Circumferential Tie Rods

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Abstract: The research on the dynamic performance of the rod-fastened rotor (RFR) has always been a hotspot. However, the structural complexity of RFR has brought significant challenges to the dynamic study of the RFR. The tie rods provide preload for the rotor shaft segment, while the coordinate deformation of the tie rods will occur during the process of vibration. In addition, the tie rods and the rotor shaft segments are structurally connected in parallel. These factors all will influence the dynamic performance of the RFR. In this paper, for a RFR system, the vibration equation of the RFR considering all factors of the tie rods is deduced in detail. The influence of various factors on the dynamic performance of the rotor is investigated. Results show that the preload directly affects the dynamic performance of the RFR system. When the preload is small, the tie rod has a larger influence on the natural frequencies of the rotor. However, when the preload force reaches a certain value, the influence of the tie rod on the natural frequencies of the rotor is almost negligible. The research results provide a theoretical reference for the understanding of and further research on RFR.

Keywords: tie rods; rod-fastened rotors; dynamics; flexural stiffness

1. Introduction

Gas turbines are widely used in transportation, energy, aviation, and national defense and are hailed as “the crown jewels” in the machinery manufacturing industry with the most market application prospects. The dynamic performance of the rotor directly affects the performance and life of the entire gas turbine. Therefore, the rotor design is important. The rod-fastened rotor (RFR) of a heavy-duty gas turbine is a typical disk-rod rotor, which has two structural forms: a central rod-fastened rotor and a circumferential rod-fastened rotor. The central rod-fastened rotor relies on a central tie rod to clamp the rotor together, while the circumferential rod-fastened rotor uses several circumferentially evenly distributed tie rods to clamp the rotor together. The contact interface and tie rods are the most distinctive features of the RFR. Compared with the integral rotor, the RFR has the advantages of low weight, ease of cooling, easy assembly, and flexible selection of disk material. By contrast with the integral rotor, the RFR mainly depends on the contact interface between the disks to transmit force and power. Therefore, the RFR cannot be simplified as an integral rotor to make a further study.

A lot of work on the RFR has been reported, including the contact interface processing, the research of the mechanical model of the RFR, and the dynamic analysis of the RFR system. By experimental measurement, Greenwood and Williamson [1] found that the height of micro-convex bodies on the rough surface is approximate to the Gaussian distribution. They assumed that the machined surface was a smooth surface covered by a certain density of equal-curvature spherical micro-convex bodies obeying Gaussian distribution.
The micro-convex bodies did not affect each other, and the contact of each micro-convex body conforms to the Hertz contact formula. In the plane contact, the relational expression of the number of micro-convex bodies, the average contact area, the load, and the distance between the two planes are derived in the elastic range. It lays the foundation for studying the rough surface contact problem with the statistical parameter model, and the model is called the GW model. McCool J.I. [2] investigated the isotropic and heterogeneous rough surface contact problem, in which the radius of curvature of the micro-convex body is arbitrarily distributed. The results are consistent with the results of the GW model, extending the scope of application of the GW model. In the literature [3], based on the elastic contact theory and statistical parameter description, the calculation method of normal contact stiffness of rough surface considering the influence of surface waviness is given. Zhao Y.W. [4] proposed an elastoplastic contact model of a rough surface, including elastic, elastoplastic, and plastic deformation states. It compensates for the serious deviations caused by the GW model based on elastic deformation when it is subjected to large normal loads. In the literature [5,6], the GW model was modified by transforming the interaction between the micro-convex bodies into elastic deformations of the nominal contact surface, and the contact between the rough surface and the smooth plane was analyzed in the elastic and elastoplastic range. The results showed that with the increase of the normal load, the plastic deformation caused by the micro-convex body and the interaction between them became increasingly obvious, which must be fully considered. Majumdar A. and Bhushan B. [7] proposed the fractal contact model based on the self-affine fractal features of the machined surface. That is the MB model. The fractal dimension and fractal roughness with an independent scale are used as the characterization parameters of the rough surface. In the literature [8], the authors considered that the contact area and the micro-convex had scale correlation, which solved the contradiction between the elastic deformation and the plastic deformation transformation of the MB model so that it conformed to the prediction of classical contact mechanics. In the literature [9,10], the tangential contact properties between the two rough planes and between the rough plane and the sphere are studied, respectively. It is found that the normal load, material properties, surface morphology, tangential load, and friction coefficient all have influences on the tangential contact stiffness. Allara M. [11] studied the contact properties of chamfered cuboids under tangential force in the condition of elastic contact conditions. In the literature [12–14], the authors studied the contact properties of simple geometrical bodies such as spheres and ellipsoids and extended them with the GW model to obtain the displacement expression under the tangential load. It had become a widely used method for studying the tangential contact performance of rough surfaces. Komvopoulos K. [15] investigated the effects of surface morphology, elastic layer thickness, and matrix material on the maximum Von Mises stress and shear stress distribution by sliding contact analysis between the smooth sphere and a flat surface with a rough surface contact layer. In the literature [16–21], the finite element method is used to study the contact problem between rough surfaces. The above reports were the research of contact interface, which is essential to the study of the RFR. What follows are some studies directly on the dynamic characteristics of the RFR. Rao Z.S. [3] analyzed comprehensively the structural characteristics of the combined rotor, and proposed a mechanical model of RFR which is proved by experiment and calculation. Yuan Qi et al. [22] investigated the vibration characteristics of a gas turbine rotor, which considered the contact interface and preload. In their study, an improved 2-D finite element method (FEM) considered the contact interface was presented, which is useful to evaluate the critical speed of the rotor. Zhang Y.C. et al. [23] provided a calculation method of contact stiffness by modal test and FEM analysis. Jam J.E. et al. [24] advanced a FEM dynamic analysis model of the RFR, which is proved to be effective. Lu M.J. et al. [25] analyzed the vibration characteristic of the RFR by the FEM method. He Peng et al. [26] investigated the dynamic characteristics of the RFR based on elastic-plastic contact. Jin Gao et al. [27] studied the influence of the moment and preload on the bending stiffness of contact interfaces. Yang L. et al. [28] advanced a calculation method of vibration characteristics of the
rotor with initial bending, and it has been proved correct. Zhuo M. et al. [29] investigated
the influence of the thermal effects on the tensile forces of rods in the combined rotor of
heavy-duty gas turbine. Xu H. et al. [30] studied the effect of detuning of clamping force of
tie rods on dynamic performance of the combined rotor. Liu Heng [31,32], Qi Yuan [33],
Liang Hu [34] investigated the non-linear dynamic characteristics of a circumferential
RFR. The contact interface and tie rod are the most obvious features in the rod-fastened
rotor. These reports mostly focus on the dynamic model of RFR and the calculation of
contact stiffness of the rough contact interfaces. However, the characteristics of tie rods
are also essential. It will also influence the dynamic performance of the rotor. Thus the
dynamic analysis of RFR considering the characteristics of circumferential tie rods needs to
be studied.

In this study, for a RFR system, the vibration equation of RFR is deduced in detail
using the lumped parameter method. The equation fully considers all factors brought by
the tie rods. The effects of these factors on the dynamics performance of the rotor system
were investigated systematically.

2. Theoretical Analysis
2.1. The Equivalent Flexural Stiffness (EFS) of Contact Interface

The contact model of the shaft segment with the contact interface consists of two
shaft segments and a contact interface (see Figure 1). The contact interface is in a series
relationship with two elastic shaft segments. In the actual contact, the contact interface
consists of rough contact surfaces of two shaft segments interacting with each other and has
no thickness on the macroscopic level. The calculation method for the equivalent flexural
stiffness (EFS) of the contact interface is given below.

![Figure 1. The ith elastic shaft segment with the contact interface.](image)

Based on the statistical model in rough surface contact established by Greenwood
and Williamson [35], the contact course between two rough surfaces is actually the contact
behavior of the micro-convex bodies distributed on them. The relation between the pressure
and the distance of the rough contact surface is given by:

$$P = \frac{4}{3\sigma\sqrt{2\pi}}\eta A_{nom} E'\beta_0^{\frac{1}{2}} \int_{d_0}^{\infty} (z - d_0)^{\frac{1}{2}} e^{-\frac{z^2}{8\sigma^2}} dz$$  \hspace{1cm} (1)

Here, $P$ is the pressure of the contact rough surfaces. $\sigma$ is RMS (root mean square) of
micro-convex body height distribution. $\eta$ is the distribution density of the micro-convex
body. $\beta_0$ is the average micro-convex body radius of curvature. $E'$ is the equivalent Young’s
modulus. $A_{nom}$ is the nominal contact area of the contact surface. $d_0$ is the distance of two
reference contact planes when the preload is applied. $z$ is the parameter of the micro-convex
body height.

The EFS of the contact interface can be given by:

$$G_r = \frac{\partial M}{\partial \theta} = -\frac{2}{\sigma\sqrt{2\pi}}\eta E'\beta_0^{\frac{1}{2}} \int_{A_{nom}} \int_{d_0}^{\infty} (z - d_0 - y\theta)^{\frac{3}{2}} e^{-\frac{z^2}{8\sigma^2}} \cdot y^2 \cdot \eta^2 dz dA_{nom}$$  \hspace{1cm} (2)
Here, $M$ is the bending moment which is applied to the contact interface. $y$ is the radial parameter of the contact surface. $\theta$ is the angle of rotation between two contact planes.

By numerical methods, it can firstly calculate $d_0$ according to Equation (1), then substitute $d_0$ into Equation (2) to obtain EFS $G_r$ at different rotation angles.

### 2.2. The EFS of the Shaft Segment with the Contact Interface

The presence of the contact interface makes the calculation of the flexural stiffness of the shaft segment different from the corresponding integral rotor. Many factors can influence the flexural stiffness of the shaft segment, such as the preload, the structural parameters, and the contact state of the contact interface, and the length of the shaft segment.

When the $i$th flexible shaft segment of the rotor has a contact interface, as shown in Figure 1, $l_1, l_2$ are the lengths of the shaft segments on both sides of the contact interface, respectively. $(EI)_i$ is the flexural stiffness of the shaft segment and $T_0$ is the axial preload. When the contact surfaces have sufficient lateral friction to ensure that the contact interface does not experience lateral slippage, the contact can be further equivalent to a torsional spring-hinge structure in which the torsion stiffness of the torsion spring is equal to the EFS $G_r$ of the contact interface.

Since the flexible shaft segment and the torsion spring are connected in series, EFS of the entire shaft segment is written as:

$$\frac{(EI)_{\text{eq}}}{(EI)_i} = \frac{l_1 + l_2}{l_1 + l_2 + \frac{T_0}{(EI)_i}} = \frac{1}{\frac{l_1 + l_2}{(EI)_i} + \frac{T_0}{(EI)_i}} = \frac{1}{1 + \frac{(EI)_i}{(EI)_i} (EI)_i}$$

(3)

where $L$ is the total shaft length $L = l_1 + l_2$.

From Equation (3), it can be seen that the flexural stiffness of the elastic shaft segment with the contact interface is always smaller than the bending stiffness of the corresponding integral shaft segment. The greater the EFS of the contact interface $G_r$, and the longer shaft segment will have a smaller influence on EFS of the shaft segment. That is to say, in this case, EFS of the shaft segment with the contact interface will be close to the flexural stiffness of the corresponding integral shaft segment.

### 2.3. The Coordinate Deformation of the Tie Rods

As shown in Figure 2, during the process of vibration of RFR, the coordinate deformation of the tie rods will occur, which will produce a changeable additional bending moment on the fastened shaft segment of the rotor. This will affect the dynamic behavior of the rotor.
It can be seen from Figure 3 that the rod-fastened rotor is fastened by eight tie rods along the circumferential direction. The installation radius of the tie rod is \( r \). The length of the shaft segment is taken as \( l \), and the center curvature radius of the shaft segment is \( R(z) \). It is easily obtained as follows:

\[
\frac{1}{R(z)} = \frac{M(z)}{EI}
\]

(4)

Here, \( M(z) \) is the bending moment applied to the rotor.

Figure 3. Circumferential tie rod distribution on the cross-section of the rotor.

For the vibration deformation of the RFR, the tie rods will cause the following effects: the rods are lengthened below the middle interface, so the tension increases; the rods are shortened above the middle interface, so the tension reduces. According to the coordination deformation relationship for the lowest No. 2 tie rod, according to the geometric deformation relationship, can be given by:

\[
\frac{l + \Delta l}{l} = \frac{R + r}{R}
\]

(5)

In Equation (5) \( \Delta l = \Delta l_1 + \Delta l_2 \) is the stretching amount of rod microsegment. Therefore the increasing tension is:

\[
\Delta F = EA_r \frac{\Delta l}{T} = EA_r \frac{r}{R}
\]

(6)

Here, \( A_r \) is the cross-sectional area of tie rod.

The uppermost No. 1 tie rod, according to the geometric deformation relationship, can be given by:

\[
\frac{l - \Delta l}{l} = \frac{R + r}{R}
\]

(7)

Thus the reduced tension is:

\[
\Delta F = EA_r \frac{\Delta l}{T} = EA_r \frac{r}{R}
\]

(8)

The tension change of all rods can be equivalent to an additional moment \( M'_0 \), and can be written as:

\[
M'_0(z) = 4EA_r \frac{r^2}{R} = 4\frac{EA_r}{(EI)_{eq}} \frac{r^2}{R} M(z)
\]

(9)
2.4. The Displacement and Load Transfer Equations of the Tie Rods

The rotor shaft segments and tie rods are in a parallel relationship in the structure, and the displacement and load transfer of the tie rods will directly influence the dynamic performance of the rotor. Therefore, the displacement and load transfer equations of the tie rods are given below in detail.

For the rod-fastened rotor, the tie rods and the rotor are both discretized at the corresponding cross-sections, which is favorable for analyzing the interaction relationship between the tie rods and the rotor. The preload of the rotor is the compressing force \( T_0 \), and the preload of the tie rods is the pulling force \( T_0' = -T_0 \), and the minus sign means that their directions are opposite. The preload is always tangent to the shaft segment.

The force analysis of the \( j \)th shaft segment of the tie rod in the \( x-z \) plane and the \( y-z \) plane is shown in Figure 4. \( x_{rj}, y_{rj} \) are the displacement of rod shaft segment in the \( x \) and \( y \) direction, \( z \) is the axial coordinate, \( l_j \) is the length of rod shaft segment, \( \varphi_{rj}, \psi_{rj} \) are the rotating angle of tie rods in the \( x-z \) plane and \( y-z \) plane, \( M_{rj}, N_{rj} \) are the bending moment of tie rod shaft segment, \( S_{rj}, Q_{rj} \) are the shear force of tie rod shaft segment, and \( \omega \) is the rotational angular velocity. It can be seen that the balanced equation of the force and bending moment of the tie rod shaft segment in the \( x-z \) plane are written as:

\[
S_{rj} - T_0' \varphi_{rj-1} = S_{rj-1} - T_0' \varphi_{rj} \quad (10)
\]

\[
M_{rj} = M_{rj-1} - l_j S_{rj-1} \quad (11)
\]

![Figure 4. The force analysis of the jth tie rod shaft segment.](image)

It is known from the literature [36] that the differential equation of massless shaft segment with pre-tension is given by:

\[
\frac{\partial^2}{\partial z^2} \left( \frac{E l_s}{E I_{rx}} \frac{\partial^2 y}{\partial z^2} \right) - T_0' \frac{\partial^2 y}{\partial z^2} = 0 \quad (12)
\]

It is known that the bending moment of tie rods \( M_r = E l_s \frac{\partial^2 y}{\partial z^2} \). \( E I_{rx} \) is the bending stiffness of the tie rod shaft segment. Thus Equation (12) can be simplified as:

\[
\frac{\partial^2 M_r}{\partial z^2} - \frac{T_0'}{E I_{rx}} M_r = 0 \quad (13)
\]

By integrating Equation (13) and using the corresponding boundary conditions, the rotating angle and displacement in the \( x-z \) plane can be obtained as:

\[
\varphi_{rj} = \varphi_{rj-1} + \frac{l_j}{E l_s} \left(1 + \frac{1}{6} \frac{T_0'}{E I_{rx}} l_j^2 M_{rj-1} - \frac{l_j^2}{2 E I_{rx}} S_{rj-1} \right) \quad (14)
\]

\[
x_{rj} = x_{rj-1} + l_j \varphi_{rj-1} + \frac{1}{2} \frac{l_j^2}{E l_s} M_{rj-1} - \frac{1}{6} \frac{l_j^3}{E I_{rx}} S_{rj-1} \quad (15)
\]
Therefore, the following matrix form is written as:

\[
\begin{pmatrix}
    x_r \\
    \varphi_r \\
    M_r \\
    S_r
\end{pmatrix}
\bigg|_j
= \begin{pmatrix}
    1 & I & \frac{-r^2}{2EI_{rs}} & \frac{-l^2}{6EI_{rs}} \\
    0 & 1 & \frac{l(1+\frac{1}{2}\beta^2)}{EI_{rs}} & \frac{-l^2}{6EI_{rs}} \\
    0 & 0 & 1 & -1 \\
    0 & 0 & \frac{l(1+\frac{1}{2}\beta^2)}{EI_{rs}} & (1 + \frac{l^2}{EI_{rs}})
\end{pmatrix}
\begin{pmatrix}
    x_r \\
    \varphi_r \\
    M_r \\
    S_r
\end{pmatrix}
\bigg|_{j-1}
\quad (16)
\]

In the 3-2 plane, the same form can be obtained.

For the \( j \)-th shaft segment of the tie rod, from Equation (16), the following can be obtained:

\[
\begin{pmatrix}
    x_r \\
    \varphi_r \\
    M_r \\
    S_r
\end{pmatrix}
\bigg|_j
= \begin{pmatrix}
    1 & I & \frac{-r^2}{2EI_{rs}} & \frac{-l^2}{6EI_{rs}} \\
    0 & 1 & \frac{l(1+\frac{1}{2}\beta^2)}{EI_{rs}} & \frac{-l^2}{6EI_{rs}} \\
    0 & 0 & 1 & -1 \\
    0 & 0 & \frac{l(1+\frac{1}{2}\beta^2)}{EI_{rs}} & (1 + \frac{l^2}{EI_{rs}})
\end{pmatrix}
\begin{pmatrix}
    x_r \\
    \varphi_r \\
    M_r \\
    S_r
\end{pmatrix}
\bigg|_{j-1}
\]

\[
\begin{pmatrix}
    x_r \\
    \varphi_r \\
    M_r \\
    S_r
\end{pmatrix}
\bigg| _{j-1}
= B_{j}^{* -1}
\begin{pmatrix}
    x_r \\
    \varphi_r \\
    M_r \\
    S_r
\end{pmatrix}
\bigg|_j
- B_{j}^{* -1}
\begin{pmatrix}
    1 & I & \frac{-r^2}{2EI_{rs}} & \frac{-l^2}{6EI_{rs}} \\
    0 & 1 & \frac{l(1+\frac{1}{2}\beta^2)}{EI_{rs}} & \frac{-l^2}{6EI_{rs}} \\
    0 & 0 & 1 & -1 \\
    0 & 0 & \frac{l(1+\frac{1}{2}\beta^2)}{EI_{rs}} & (1 + \frac{l^2}{EI_{rs}})
\end{pmatrix}
\begin{pmatrix}
    x_r \\
    \varphi_r \\
    M_r \\
    S_r
\end{pmatrix}
\bigg|_{j-1}
\quad (17)
\]

Here,

\[
B_{j}^{* -1} = \begin{pmatrix}
    \frac{6EI_{rs}}{l(1-2\beta^2)} & \frac{-2EI_{rs}}{l(1-2\beta^2)} & \frac{2EI_{rs}}{l(1-2\beta^2)} & \frac{-2EI_{rs}}{l(1-2\beta^2)} \\
    \frac{6EI_{rs}}{l(1+\beta^2)} & \frac{-2EI_{rs}}{l(1+\beta^2)} & \frac{2EI_{rs}}{l(1+\beta^2)} & \frac{-2EI_{rs}}{l(1+\beta^2)} \\
    \frac{6EI_{rs}}{l(1-2\beta^2)} & \frac{-2EI_{rs}}{l(1-2\beta^2)} & \frac{2EI_{rs}}{l(1-2\beta^2)} & \frac{-2EI_{rs}}{l(1-2\beta^2)} \\
    \frac{6EI_{rs}}{l(1+\beta^2)} & \frac{-2EI_{rs}}{l(1+\beta^2)} & \frac{2EI_{rs}}{l(1+\beta^2)} & \frac{-2EI_{rs}}{l(1+\beta^2)}
\end{pmatrix}, \beta' = \frac{1}{6} \frac{T_0'}{T_0}
\]

In fact, \( \beta' \) is the influence coefficient of the pre-tension force \( T_0' \) on the displacement and load transmission of the tie rod shaft segments.

Due to the coordinate deformation of the rod and rotor in the vibration process, it is easy to know the following:

\[
\begin{pmatrix}
    x \\
    \varphi
\end{pmatrix}
\bigg|_j
= \begin{pmatrix}
    x_r \\
    \varphi_r
\end{pmatrix}
\bigg|_j
\quad (19)
\]

That is, the displacement and the rotation angle of the rotor and the rod at the corresponding nodes are equal during the process of vibration.

Therefore, from Equations (18) and (19), the following can be obtained:

\[
\begin{pmatrix}
    M_r \\
    S_r
\end{pmatrix}
\bigg|_{j-1}
= B_{j}^{* -1}
\begin{pmatrix}
    x \\
    \varphi
\end{pmatrix}
\bigg|_j
- B_{j}^{* -1}
\begin{pmatrix}
    1 & I & \frac{-r^2}{2EI_{rs}} & \frac{-l^2}{6EI_{rs}} \\
    0 & 1 & \frac{l(1+\frac{1}{2}\beta^2)}{EI_{rs}} & \frac{-l^2}{6EI_{rs}} \\
    0 & 0 & 1 & -1 \\
    0 & 0 & \frac{l(1+\frac{1}{2}\beta^2)}{EI_{rs}} & (1 + \frac{l^2}{EI_{rs}})
\end{pmatrix}
\begin{pmatrix}
    x \\
    \varphi
\end{pmatrix}
\bigg|_{j-1}
\quad (20)
\]

2.5. The Equation of Motion of Rod-Fastened Rotor (RFR)

The RFR is made of shaft segments clamped by several tie rods that are evenly distributed in the circumferential direction. Therefore, the contact interface and the tie rods are the most distinctive features of the rod-fastened rotor. The presence of the contact interface gives the RFR structural discontinuity so that the flexural stiffness of the RFR is smaller than the flexural stiffness of the corresponding integral rotor. Therefore, the presence of tie rods makes the rotor structure more complex. First, the preload supplied by the tie rods directly affects the contact state of the contact interface, which in turn affects the bending stiffness of the shaft segment. Then, the shaft segments and tie rods are in a parallel relationship in the structure. The preload causes the shaft segment to bear the compressing force, while the tie rods are subjected to the tension force, which will affect the transmission of the displacement and load of the rotor, thereby affecting the flexural stiffness of the rotor. Finally, in the process of rotor vibration, the coordinate deformation of the tie rods will have a certain effect on the flexural stiffness of the shaft segment. This section gives the vibration equation of the RFR considering all influencing factors by the lumped parameter method.

The preload of the rotor shaft segment is the compressing force, and the direction is along the direction of the rotor deflection curve. \( M_{ij}, N_{ij} \) are the bending moment applied
to the rotor by the tie rods in the x-z plane and y-z plane because of the parallel structure relationship between the rotor and rod. \( S_{j}, Q_{j} \) are the shear force applied to the rotor by the tie rods in the x-z plane and y-z plane. \( M_{0j}, N_{0j} \) are the additional bending moments that resulted from the coordinate deformation of rods in the process of vibration. The other parameters of the rotor are similar to the parameters of the tie rods in meaning. From Figure 5, the balanced equation of force and bending moment of the shaft segment of the rotor in the x-z plane can be written as:

\[
S_j + T_0\varphi_{j-1} + S_{rj-1} = S_{j-1} + T_0\varphi_{j} + \sum P_{xj} + S_{rj} \tag{21}
\]

\[
M_j = M_{j-1} - l_jS_{j-1} + T_0l_j\varphi_{j-1} - T_0(x_j - x_{j-1}) - M_{0j-1} + M_{0j} - M_{rj-1} + M_{rj} - M_{kj} \tag{22}
\]

![Figure 5. The force analysis of the jth fastened rotor shaft segment.](image)

In Equation (22), \( M_{0j} = M_{j-1} \cdot \frac{4EAr^2}{EI} \) is the additional bending moment resulted from the coordinate deformation of the tie rods.

Equation (22) can be simplified as:

\[
\mu M_j = \mu M_{j-1} - l_jS_{j-1} - M_{rj-1} + M_{rj} - M_{kj} \tag{23}
\]

In Equation (23), \( \mu = 1 - \frac{4EA\pi r^2}{EI} \). In fact, \( \mu \) is the correction factor considering the coordinate deformation of the tie rods. \( \mu = 1 \) means that the coordinate deformation of tie rods is not considered, while \( \mu \neq 1 \) means that it is taken into consideration.

It is known from the literature [36] that the differential equation of massless shaft segment with pressure is expressed as:

\[
\frac{\partial^2}{\partial x^2} \left( EI_x \frac{\partial^2 y}{\partial z^2} \right) + T_0 \frac{\partial^2 y}{\partial z^2} = 0 \tag{24}
\]

It is known that \( M = EI_x \frac{\partial^2 y}{\partial z^2} \), \( EI_x \) is the bending stiffness of the rotor shaft segment, so Equation (24) can be simplified as:

\[
\frac{\partial^2 M}{\partial z^2} + \frac{T_0}{EI_x} M = 0 \tag{25}
\]

By integrating Equation (25) and using the corresponding boundary conditions, the rotating angle and displacement in the x-z plane can be obtained as:

\[
\varphi_j = \varphi_{j-1} + \frac{l_j}{EI_x} (1 - \frac{1}{6} \frac{T_0}{EI_x} l_j^2) (\mu M_{j-1} - M_{rj-1} + M_{rj}) - \frac{l_j^2}{2EI_x} (S_{j-1} - S_{rj-1} + S_{rj}) \tag{26}
\]

\[
x_j = x_{j-1} + \frac{l_j}{2} \frac{\partial^2}{\partial z^2} (\mu M_{j-1} - M_{rj-1} + M_{rj}) - \frac{l_j^2}{6} EI_x (S_{j-1} - S_{rj-1} + S_{rj}) \tag{27}
\]

From the above, we know that the flexural stiffness of the RFR shaft segment with the contact interface is different from that of the corresponding integral rotor. Therefore,
\[(E_1)_{eq}\] should be used in place of \(E_1\) to consider the influence of the contact interfaces. In the following, the flexural stiffness of the shaft segment with the contact interface is replaced by \((E_1)_{eq}\). Therefore, the following matrix form can be written as:

\[
\begin{pmatrix}
x \\ \varphi \\ M \\ S
\end{pmatrix}_j = 
\begin{pmatrix}
1 & I & 1 & l \\
0 & 1 + \frac{\mu}{(E_1)_{eq}} & 0 & 0 \\
0 & 0 & 1 + \frac{\mu}{(E_1)_{eq}} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\ \varphi \\ M \\ S
\end{pmatrix}_{j-1} + 
\begin{pmatrix}
0 \\ 0 \\ 0 \\ 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
x \\ \varphi \\ M \\ S
\end{pmatrix}_j = 
\begin{pmatrix}
0 \\ 0 \\ 0 \\ 0
\end{pmatrix}
\begin{pmatrix}
x \\ \varphi \\ M \\ S
\end{pmatrix}_j
\]

\[\sum P_x = \left( \begin{array}{c}
M_k \\
N_k
\end{array} \right)_j = \left( \begin{array}{c}
-\theta_y \\
-\theta_x \\
-\beta \\
0
\end{array} \right)_j + \left( \begin{array}{c}
0 \\
0 \\
-\theta_2 \omega \\
0
\end{array} \right)_j
\]

In Equation (30), \(d_{xx}, d_{yy}, d_{xy}, d_{yx}\) are the damping coefficient of bearing, \(k_{xx}, k_{yy}, k_{yx}, k_{xy}\) are the stiffness coefficient of bearing, \(d_{fx}, d_{fy}, d_{fxy}, d_{fyx}\) are the externally applied damping coefficient, \(k_{f1}, k_{f2x}, k_{f2y}, k_{f2y}, k_{f2x}, k_{f2y}\) are the externally applied stiffness coefficient, \(P_{cx}, P_{cy}\) are the external force exerted on the shaft.

From Equation (28), the following can be obtained:

\[
\begin{pmatrix}
M \\ S
\end{pmatrix}_j = (A^{*-1})_{j+1} + B^{*-1}_{j+1} + B^{*-1}_{j+2} \left( \begin{array}{c}
1 \\
0
\end{array} \right)_j \left( \begin{array}{c}
x \\ \varphi
\end{array} \right)_j - (A^{*-2})_{j+1} + B^{*-2}_{j+1} \left( \begin{array}{c}
1 \\
0
\end{array} \right)_{j+1} \left( \begin{array}{c}
x \\ \varphi
\end{array} \right)_j - B^{*-1}_{j+2} \left( \begin{array}{c}
x \\ \varphi
\end{array} \right)_{j+2}
\]

Here, \(A^{*-1}\) is the influence coefficient of the compressing force \(T_{0r}\) on the displacement and load transmission of the fastening rotor shaft segments.
Similarly, for the \((j+1)\)-th shaft segment, this can be written as:

\[
\begin{pmatrix}
M \\
S
\end{pmatrix}
_j = (A^{s-1} + B^{s-1})^{j+1} + B^{s-1} \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ \varphi \end{pmatrix}_j - (A^{s-1} + B^{s-1})^{j+1} \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ \varphi \end{pmatrix}_j - B^{s-1} \begin{pmatrix} x \\ \varphi \end{pmatrix}_j
\]

(33)

From Equation (28), the balanced equation of the force and bending moment of the corresponding mass points is:

\[
\begin{pmatrix}
-M_k \\
\sum P_x
\end{pmatrix}
_j - \begin{pmatrix}
M \\
S
\end{pmatrix}_j + \left( 1 - \frac{\beta_0 \rho_0^2}{(EI_y)_l} \right) \mu \begin{pmatrix}
M \\
S
\end{pmatrix}_j - \begin{pmatrix}
M_x \\
S_x
\end{pmatrix}_j = 0
\]

(34)

\[
\begin{pmatrix}
-N_k \\
\sum P_y
\end{pmatrix}
_j - \begin{pmatrix}
N \\
Q
\end{pmatrix}_j + \left( 1 - \frac{\beta_0 \rho_0^2}{(EI_y)_l} \right) \mu \begin{pmatrix}
N \\
Q
\end{pmatrix}_j - \begin{pmatrix}
N_x \\
Q_x
\end{pmatrix}_j = 0
\]

(35)

From Equations (29)–(35), the equation of motion of the shaft segment of the rotor considering all factors brought by the tie rods is as follows:

\[
M_j \ddot{X}_j + C_j \dot{X}_j + K_{\alpha}X_{j-1} + K_{\beta}X_{j} + K_{\psi}X_{j+1} + K_{\phi}X_{j+2} = 0
\]

(36)

In Equation (36), the specific parameters are listed in the Appendix A.

The free vibration equation of the rotor fully considering all factors brought by the tie rods can be obtained by assembling the equations of motion of all shaft segments:

\[
M\ddot{X} + C\dot{X} + KX = 0
\]

(37)

Here, \(M\) is the total mass matrix, \(C\) is the total damping matrix, and \(K\) is the total stiffness matrix.

### 3. Results and Discussion

In this section, first, a simple rotor shaft segment with an annular contact surface is used as an example to investigate the effect of the parameters of the contact surface on EFS of the contact interface and the effect of the preload on EFS of the shaft segment with the contact interface. Then, for a RFR, a dynamic model considering all factors brought by the tie rods is built, which includes the contact interface, preload, the coordinate deformation of the tie rods, the parallel relationship between the tie rods and rotor, etc. The influence of various factors on the dynamic performance of the rotor is investigated in detail. In addition, the effect of the tie rod parameters on the rotor system is discussed.

As shown in Figure 3, the contact interface in the RFR is a pair of annular contact surfaces with an inner radius of 29 mm and an outer radius of 44 mm. The parameters of the contact surface measured in [35] are used here. On the nominal contact surface, RMS of the micro-convex body height distribution is \(\sigma = 2.01\ \mu m\), the average radius of curvature of the top of the micro-convex body is \(\beta_0 = 95\ \mu m\), and the micro-convex body distribution density is \(\eta = 0.5625 \times 10^8\ L/\text{m}^2\).

Based on the above parameters, the equivalent bending stiffness of the contact interface between the shaft segments at different preloads and rotation angle can be calculated by numerical calculation methods. The result is shown in Figures 6 and 7. When the rotation angle \(\theta\) is \(1 \times 10^{-7}\) rad, and the preload increases from 0 to \(5 \times 10^7\) N, the equivalent flexural stiffness of the contact interface is shown in Figure 6. It can be seen that when the preload is small, EFS of the contact interface increases rapidly. As the further increase of
the preload, EFS of the contact interface will slowly increase to a constant value. When the preload is $8 \times 10^3$ N, $5 \times 10^4$ N, $5 \times 10^5$ N, respectively, changing the rotation angle from $1 \times 10^{-8}$ rad to $5 \times 10^{-4}$ rad, change of EFS of the contact interface is shown in Figure 7. It can be observed that when the rotation angle is small, EFS of the contact interface is almost unchangeable. When the rotation angle is larger than $0.5 \times 10^{-4}$ rad, EFS of the contact interface varies exponentially with the rotation angle. In the dynamic analysis of the rotor, the rotation angle is very small, so it is correct to ignore the influence of the rotation angle on the equivalent bending stiffness of the contact interface. Therefore, the influence of the rotation angle on EFS of the contact interface will be ignored in the subsequent calculations.

![Figure 6](image6.png)

**Figure 6.** EFS of the contact surface changes with preload.

![Figure 7](image7.png)

**Figure 7.** Equivalent flexural stiffness (EFS) of the contact surface changes with the rotation angle.

Figures 8 and 9 show the changes of EFS of the contact interface with RMS of the micro-convex body distribution $\sigma$ and the average micro-convex body radius of curvature of the top $\beta_0$. 

...
Figures 8 and 9 show the changes of EFS of the contact interface with RMS of the micro-convex body distribution $\sigma$ and the average micro-convex body radius of curvature of the top $\beta_0$.

**Figure 8.** EFS of the contact surface changes with the root mean square of the micro-convex body distribution.

It can be seen from Figure 8 that the greater the RMS of the height distribution of the micro-convex body $\sigma$, the smaller the EFS of the contact interface. That is, the rougher the surface, the lower the EFS of the contact interface, so EFS of the contact surface can be increased by reducing the surface roughness. Figure 9 shows that the average micro-convex body radius of curvature of the top $\beta_0$ has almost no effect on EFS of the contact surface.

If the total length of the shaft segment $L = 50$ mm with annular contact interface, according to Equation (12), the changes of EFS of the shaft segment with the preload can be calculated as shown in Figure 10. The blue color curve represents the shaft segment with the contact interface, and the red color curve represents the corresponding integral shaft segment. As can be seen from the figure, when the preload is small, EFS of the shaft
segment with the contact interface increases rapidly with the preload. As the preload increases further, the increasing speed of EFS of the shaft segment with the contact interface gradually becomes slower. When the preload reaches $10^7$ N, EFS of the shaft segment with the contact interface tends to be consistent with the bending stiffness of the corresponding integral shaft segment. A further increase of the preload will no longer change EFS of the shaft segment.

Figure 10. EFS of the shaft segment with contact interface changes with preload.

3.1. The Rotor Model

Figure 11a shows the model of the rotor. It can be seen that the rotor consists of two shaft heads and four intermediate disks, clamped together by eight long tie rods. The rotor contains a total of five pairs of annular contact interfaces and is supported by rolling bearings. By the lumped parameter method [37], the rotor is discretized into 26 concentrated masses and 25 massless flexible shaft segments. Each rod is divided into 7 concentrated masses and 6 flexible shaft segments (see Figure 11b). The length of the rod shaft segment is the same as that of the corresponding shaft segment of the rotor so that the interaction between the tie rod and the rotor can be analyzed easily. Bearing support is located at both ends of the rotor, and the bearing support is simplified as a rigid support.

(a) The rotor model

Figure 11. Cont.
Figure 11. The rotor model.

3.2. The Influence of Various Factors on the Dynamic Performance of RFR

According to the previous analysis, it can be seen that the preload, the contact interface, the coordinate deformation of the tie rods, and the parallel relationship between the tie rods and the rotor will all affect the total stiffness matrix $K$ of the rotor. Therefore, based on the numerical method to solve the free vibration equation of the rotor system, the influence of each influencing factor on the dynamic characteristics of the RFR will be investigated in detail. For ease of description and comparison, five different situations are listed in Table 1.

Table 1. The different situations.

| Situations | Contact Interfaces | The Influence Coefficient of the Preload of the Rotor $\beta$ | The Correction Factor of Coordinate Deformation of Tie Rods $\mu$ | The Tie Rods $EI_r$ | The Influence Coefficient of the Preload of the Tie Rods $\beta'$ |
|------------|--------------------|-------------------------------------------------|-------------------------------------------------|-----------------|-----------------|
| A          | Considering        | $\beta = 0$                                    | $\mu = 1$                                      | $EI_r = 0$      | $\beta' = 0$    |
| B          | Considering        | $\beta \neq 0$                                 | $\mu = 1$                                      | $EI_r = 0$      | $\beta' = 0$    |
| C          | Considering        | $\beta = 0$                                    | $\mu \neq 1$                                  | $EI_r = 0$      | $\beta' = 0$    |
| D1         | Considering        | $\beta = 0$                                    | $\mu = 1$                                      | $EI_r = 0$      | $\beta' \neq 0$ |
| D2         | Considering        | $\beta = 0$                                    | $\mu = 1$                                      | $EI_r = 0$      | $\beta' \neq 0$ |
| E          | Considering        | $\beta \neq 0$                                 | $\mu \neq 1$                                  | $EI_r = 0$      | $\beta' \neq 0$ |

3.2.1. The Effect of Preload

The RFR with circumferential tie rods depends on the tie rods to fasten the disks together. The existence of preload not only affects the contact state of the contact interface but also affects the transmission of the rotor displacement and load. These factors will affect the flexural stiffness of the rotor, which in turn affects the natural frequency of the rotor.

Suppose the RFR is simplified as the corresponding integral rotor, the natural frequencies of the rotor system are shown in Table 2. When only considering the influence of preload on the contact state of the contact interface, it corresponds to situation A. The rate of change of natural frequencies of the RFR is shown in Figure 12. The rate of change of each order natural frequency of the RFR is written as:

$$rF_i = \frac{F_{Ai} - F_0}{F_0} \quad i = 1, 2, 3.$$  \hspace{1cm} (38)

Table 2. The natural frequencies of the corresponding integral rotor.

| Natural frequencies/Hz | First-Order | Second-Order | Third-Order |
|------------------------|-------------|--------------|-------------|
| 123.890                | 374.675     | 732.560      |             |
Table 2. The natural frequencies of the corresponding integral rotor.

|          | First Order | Second Order | Third Order |
|----------|-------------|--------------|-------------|
| Natural frequencies / Hz | 123.890     | 374.675      | 732.560     |

Figure 12. The rate of change of natural frequencies of rod-fastened rotor (RFR) considering the contact interface.

As can be seen from Figure 12, the preload affects the contact stiffness of the contact interface between the disks, which in turn affects the natural frequency of each order of the rotor. The greater the preload, the closer the natural frequency of the RFR to the natural frequency of the corresponding integral rotor. It also can be seen that the preload has a larger influence on the first-order natural frequency of the rotor and a smaller influence on the second-order natural frequency, which is determined by the structure of the rotor and the mode shape of the rotor.

When considering the influence of preload on the contact state of the contact interface and the displacement and load transmission of the shaft segments at the same time, this corresponds to situation B. The rate of change of natural frequencies of the RFR is shown in Figure 13. In order to study the influence of the transmission of the rotor displacement and load caused by preload on the natural frequencies of the RFR, it is necessary to compare situation A and situation B. When comparing two different situations, such as situation A and situation B, the rate of change of each order natural frequency of the RFR is written as

\[
 rF_i = \frac{F_{B_i} - F_{A_i}}{F_{A_i}} \quad i = 1, 2, 3. \quad (39)
\]

Here, \( F_{A_i} \) and \( F_{B_i} \) are the \( i \)-order natural frequency of RFR corresponding to situation A and situation B, respectively. The definition of the rate of change of the natural frequencies is similar in other comparisons.

By comparing situation B with situation A in Figure 12, the amplitude of the rate of change of each order natural frequency increases with the increase of the preload. This means that the preload affects the displacement and load transmission of the shaft segment of the rotor, which slightly reduces the flexural stiffness of the rotor and results in each order natural frequency having a certain decrease. The effect is gradually increasing with the increase of the preload. It also can be seen that the change of displacement and load transmission of the shaft segment caused by preload has a larger influence on the first-order natural frequency of the rotor and a smaller influence on the third-order natural frequency.
3.2.2. The Effect of the Coordinate Deformation of the Tie Rods

The change rate of natural frequencies of the RFR considering the coordinated deformation of the tie rods with the increase of the preload, corresponding to situation C, is shown in Figure 14.

According to Equation (36), it is easy to know that the coordinated deformation of the tie rods enhances the local flexural stiffness of the rod-fastened shaft segment of the rotor. From Figure 14, it can be seen that the coordinated deformation of the tie rods increases the first-order and third-order natural frequencies of the rod-fastened rotor, while decreases the second-order natural frequency of the rod-fastened rotor. The reason is that the rotor system is a complex system composed of multiple degrees of freedom, and the fastened shaft segment is exactly at the node of the second-order mode shape of the rotor. With the
increase of preload, the effect of the coordinated deformation of the tie rods on the natural frequencies of the rod-fastened rotor is gradually reduced. The coordinated deformation of the rods has a greater influence on the first-order and second-order natural frequencies but has little influence on the third-order natural frequency. When the preload is smaller than 1e4N, the effect is relatively larger.

3.2.3. The Effect of the Parallel Relationship between the Tie Rods and the Rotor

The tie rods and the rotor are in a parallel relationship in the structure, and the tie rods are subjected to pre-tension. It is necessary to study the effect of the flexural stiffness of the tie rods on the natural frequency of the rotor. The rate of change of natural frequencies of the rod-fastened rotor considering the effect of the parallel relationship of the tie rods and the rotor with the increase of the preload, corresponding to situation D, is given in Figure 15.

![Figure 15. The rate of change of natural frequencies of the RFR considering the parallel relationship between the tie rods and rotor.](image)

It can be seen from Figure 15 that the parallel relationship of the tie rods and rotor slightly increases the natural frequencies of the rotor system. The parallel relationship between the tie rods and the rotor has a relatively larger influence on the first-order natural frequency of the rod-fastened rotor. With the increase of preload, the effect of the parallel relationship between the tie rods and rotor on each order natural frequency gradually decreases. In addition, the influence coefficient of the preload of the tie rods $\beta^1$ has little influence on each order natural frequency of the rotor system. Only when the preload is larger than $5 \times 10^4$ N does it have a relatively larger influence on the first-order frequency of the rotor system.

3.2.4. Considering the Effects of All Factors Caused by the Tie Rods

The RFR with circumferential tie rods depends on the tie rods to clamp the disks together. The contact interface and tie rods are the most distinctive features of the RFR. The tie rods are under pre-tension, while the discs are under pre-pressure. There is a force interaction between the tie rods and the discs in the process of vibration, which brings significant challenges to the dynamic simulation of the rotor system. To accurately analyze the rotor system, all factors caused by the tie rods are taken into consideration corresponding to situation E.
From Figure 16, it can be seen that considering the various factors brought by the tie rods, the tie rods have a certain influence on the natural frequencies of the rotor. In particular, when the preload of the rotor is relatively low, the influence of the contact interface and the tie rod on the natural frequency of the rotor is relatively larger. In this case, the presence of the contact interface causes the natural frequencies to decrease, and the presence of the tie rod increases the first-order and third-order natural frequencies and decreases the second-order natural frequency. When the preload of the rotor reaches a certain value, the natural frequencies of the rotor approach the natural frequencies of the corresponding integral rotor. The influence of the tie rod on the natural frequency of the rotor is small. Therefore, for the rod-fastened rotors of the gas turbine, the previous simplified calculation of neglecting tie rods is accurate.

Figure 16. The rate of change of natural frequencies of the RFR considering all factors caused by tie rods.

3.2.5. The Effect of the Tie Rod Parameters on the Rotor System

The number of tie rods, the installation radius of the tie rod, and the tie rod radius all have a certain effect on the dynamic characteristics of the RFR. The results are shown in Figures 17–19.

Figure 17. The rate of change of natural frequencies of the RFR with a different number of tie rods.
Analysis, the vibration angle is very small, so it is correct to ignore the effect of the preloaded contact interface. The results are shown in Figures 17–20.

The natural frequencies of the rotor system increase monotonously, while the second-order natural frequency decreases monotonously. The effect of the number of tie rods on the first-order and second-order natural frequencies is larger than the effect on the third-order natural frequency. Moreover, when the preload is relatively small, the effect of the number of tie rods on the natural frequencies is larger.

Figure 18 shows the effect of the installation radius of the tie rods on the natural frequencies of the rotor system. With the increasing number of tie rods, the first-order and third-order natural frequencies increase monotonously, while the second-order natural frequency decreases monotonously. The effect of the number of tie rods on the first-order and second-order natural frequencies is larger than the effect on the third-order natural frequency. Moreover, when the preload is relatively small, the effect of the number of tie rods on the natural frequencies is larger.

Figure 17 plots the effect of the number of tie rods on the natural frequencies of the rotor system. With the increasing number of tie rods, the first-order and third-order natural frequencies increase monotonously, while the second-order natural frequency decreases monotonously. The effect of the number of tie rods on the first-order and second-order natural frequencies is larger than the effect on the third-order natural frequency. Moreover, when the preload is relatively small, the effect of the number of tie rods on the natural frequencies is larger.

The rate of change of natural frequencies of the RFR with different installation radius of tie rods.

The number of tie rods, the installation radius of the tie rod, and the tie rod radius all affect the natural frequencies of the rotor system. The preload affects the contact stiffness of the contact interface of disks, which in turn affects the natural frequency of each order of the rotor system. The greater preload makes the natural frequency of the RFR closer to the natural frequencies of the corresponding integral rotor within a certain range of preload. The preload affects the stiffness of the contact interface. The preload affects the displacement and load transmission of the shaft segment of the rotor, which reduces the flexural stiffness of the rotor and makes each order's natural frequency decrease to a certain extent.

Figure 18 shows the effect of the installation radius of the tie rods on the natural frequencies of the rotor system. With the increase of the installation radius of the tie rods, the effect of the installation radius of the tie rods on the natural frequencies of the rotor system is similar to the effect of the number of tie rods on the natural frequencies of the rotor system. The larger installation radius of the tie rods has a larger influence on the natural frequencies of the rotor system.

The rate of change of natural frequencies of the RFR with different number of tie rods.

According to the results, the following conclusions can be drawn:

1. The number of tie rods, the installation radius of the tie rod, and the tie rod radius all affect the natural frequencies of the rotor system.
2. The preload affects the contact stiffness of the contact interface of disks, which in turn affects the natural frequency of each order of the rotor system.
3. The greater preload makes the natural frequency of the RFR closer to the natural frequencies of the corresponding integral rotor within a certain range of preload.
4. The preload affects the displacement and load transmission of the shaft segment of the rotor, which reduces the flexural stiffness of the rotor and makes each order's natural frequency decrease to a certain extent.

Figure 19 shows the effect of the radius of tie rods on the natural frequencies of the rotor system.
Figure 19 plots the effect of the tie rod radius on the natural frequencies of the rotor system. It can be seen that the increase of the tie rod radius makes the natural frequencies of the rotor system increase very slightly. However, the effect is almost negligible.

4. Conclusions

According to the results, the following conclusions can be drawn:

(1) When the rotation angle of the shaft segment with the contact surface is small, EFS of the contact surface is almost constant. When the rotation angle is large, EFS of the contact surface changes almost exponentially with the rotation angle. In the rotor dynamics analysis, the vibration angle is very small, so it is correct to ignore the influence of the rotation angle on EFS of the contact interface.

(2) The greater the RMS of the height distribution of the micro-convex body $\sigma$, the smaller EFS of the contact interface. That is to say, the rougher the surface, the lower EFS of the contact surface. Therefore, EFS of the contact surface can be increased by reducing the surface roughness. The average radius of the curvature of the top of the micro-convex body $\beta_0$ has almost no effect on the equivalent flexural stiffness of the contact surface.

(3) When the preload is small, EFS of the shaft segment with the contact interface increases rapidly with the increase of preload. When the preload increases further, the increasing speed of EFS of the shaft segment with the contact interface gradually becomes slow. When the preload reaches a certain value, EFS of the shaft segment with the contact interface tends to be consistent with the flexural stiffness of the corresponding integral shaft segment.

(4) The preload affects the contact stiffness of the contact interface of disks, which in turn affects the natural frequency of each order of the rotor system. The greater preload makes the natural frequency of the RFR closer to the natural frequency of the corresponding integral rotor within a certain range of preload. The preload affects the displacement and load transmission of the shaft segment of the rotor, which slightly reduces the flexural stiffness of the rotor and makes each order’s natural frequency decrease to a certain extent.

(5) During the process of vibration of the rotor system, the coordinated deformation of the tie rods enhances the local bending stiffness of the rod-fastened shaft segment of the rotor. The coordinated deformation of the tie rods increases the first-order and third-order natural frequencies of the RFR, while decreases the second-order natural frequency of RFR. When the preload is relatively small, the effect is larger.

(6) With the increasing number of tie rods and the installation radius of the tie rods, the first-order and third-order natural frequencies increase monotonously, while the second-order natural frequency decreases monotonously. The effect of the number of tie rods and the installation radius of the tie rods on the first-order and second-order natural frequencies is larger than the effect on the third-order natural frequency. The effect of the tie rod radius on the natural frequencies of the rotor system is very small and almost negligible.

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Appendix A

$$M_j = \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & \theta_y & 0 \\ 0 & 0 & 0 & \theta_x \end{pmatrix}, \quad C_j = \begin{pmatrix} d_{Fx} + d_{xx} & d_{Fx} + d_{xy} & d_{xx} + d_{xy} & 0 & 0 \\ d_{Fyx} + d_{yx} & d_{Fy} + d_{yy} & 0 & 0 & \theta_z \omega \end{pmatrix}$$

$$K_\alpha = \begin{pmatrix} k_1 & 0 & k_2 & 0 \\ 0 & k_1 & 0 & k_2 \\ k_3 & 0 & k_4 & 0 \\ 0 & k_3 & 0 & k_4 \end{pmatrix}, \quad K_\beta = \sum_{p=1}^{4} K_{\beta p}$$

$$K_{\beta 1} = \begin{pmatrix} k_{Fx} + k_{xx} & k_{Fxy} + k_{xy} & 0 & 0 \\ k_{Fyx} + k_{yx} & d_{Fy} + d_{yy} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_{\beta 2} = \begin{pmatrix} -k_1 & 0 & k_5 & 0 \\ 0 & -k_1 & 0 & k_5 \\ -k_3 & 0 & k_6 & 0 \\ 0 & -k_3 & 0 & k_6 \end{pmatrix}$$

$$K_{\beta 3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k_7 & 0 & k_8 & 0 \\ 0 & k_7 & 0 & k_8 \end{pmatrix}, \quad K_{\beta 4} = \begin{pmatrix} k_9 & 0 & k_{10} & 0 \\ 0 & k_9 & 0 & k_{10} \\ k_{11} & 0 & k_{12} & 0 \\ 0 & k_{11} & 0 & k_{12} \end{pmatrix}$$

$$K_\phi = \sum_{p=1}^{3} K_{\phi p}$$

$$K_{\phi 1} = \begin{pmatrix} -k_9 & 0 & k_{13} & 0 \\ 0 & -k_9 & 0 & k_{13} \\ -k_{11} & 0 & k_{14} & 0 \\ 0 & -k_{11} & 0 & k_{14} \end{pmatrix}, \quad K_{\phi 2} = \begin{pmatrix} k_{15} & 0 & k_{16} & 0 \\ 0 & k_{15} & 0 & k_{16} \\ k_{17} & 0 & k_{18} & 0 \\ 0 & k_{17} & 0 & k_{18} \end{pmatrix}$$

$$K_\phi = \begin{pmatrix} -k_{15} & 0 & k_{19} & 0 \\ 0 & -k_{15} & 0 & k_{19} \\ k_{20} & 0 & k_{21} & 0 \\ 0 & k_{20} & 0 & k_{21} \end{pmatrix}$$

$$k_1 = -\left(\frac{2(1 - \beta)}{l \beta (1 + 2 \beta)}\right) \quad k_2 = -\left(\frac{(2 \beta^2 - \beta + 1) T_0}{\beta (1 + 2 \beta)}\right)$$

$$k_3 = \frac{6(1 - 2 \beta) E l_x}{l^2 (1 + 2 \beta) \mu} + \frac{12 (1 + \beta') E l_{rx}}{l^2 (1 - 2 \beta') \mu} \quad k_4 = \frac{2(1 - 6 \beta) E l_x}{l (1 + 2 \beta) \mu} + \frac{6(1 + 2 \beta') E l_{rx}}{l (1 - 2 \beta') \mu}$$

$$k_5 = \frac{(2 \beta - 1)(\beta + 1) T_0}{\beta (1 + 2 \beta)} \quad k_6 = \frac{4 E l_x}{l (1 + 2 \beta) \mu} + \frac{6 E l_{rx}}{l (1 - 2 \beta') \mu}$$

$$k_7 = \left(\frac{l}{\mu}\right) \left(\frac{12 E l_{rx}(1 + \beta')}{l^3 (1 - 2 \beta')}\right)_{j+1} \quad k_8 = \left(\frac{l}{\mu}\right) \left(\frac{6 E l_{rx}(1 + 2 \beta')}{l^3 (1 - 2 \beta')}\right)_{j+1}$$
\[ k_9 = \frac{12EI_x(1-\beta')}{l^3(1+2\beta)} + \frac{12EI_{rx}(1+\beta')}{l^3(1-2\beta')} \quad k_{10} = \frac{6(1-2\beta)EI_x}{l^2(1+2\beta)} + \frac{6(1+2\beta')EI_{rx}}{l^2(1-2\beta')} \]

\[ k_{11} = \frac{6EI_x}{l^2(1+2\beta)} + \frac{6EI_{rx}}{l^2(1-2\beta')} \mu \quad k_{12} = \frac{4EI_x}{l(1+2\beta)} + \frac{4EI_{rx}}{l(1-2\beta')} \mu \]

\[ k_{13} = \frac{6EI_x}{l^2(1+2\beta)} + \frac{6EI_{rx}}{l^2(1-2\beta')} \mu \quad k_{14} = \frac{2EI_x}{l(1+2\beta)} + \frac{2EI_{rx}}{l(1-2\beta')} \mu \]

\[ k_{15} = -\left(\frac{12EI_{rx}(1+\beta')}{l^3(1-2\beta')}\right)_{j+2} \quad k_{16} = -\left(\frac{6EI_{rx}(1+2\beta')}{l^2(1-2\beta')}\right)_{j+2} \]

\[ k_{17} = \left(\frac{1}{\mu}\right)_{j+1} \left(\frac{6EI_{rx}}{l^2(1-2\beta')}\right)_{j+2} + \left(\frac{1}{\mu}\right)_{j} \left(\frac{12EI_{rx}(1+\beta')}{l^3(1-2\beta')}\right)_{j+1} \]

\[ k_{18} = \left(\frac{1}{\mu}\right)_{j+1} \left(\frac{4EI_{rx}}{l(1-2\beta')}\right)_{j+2} + \left(\frac{1}{\mu}\right)_{j} \left(\frac{6EI_{rx}}{l(1-2\beta')}\right)_{j+1} \]

\[ k_{19} = \left(\frac{-6EI_{rx}}{l^2(1-2\beta')}\right)_{j+2} \quad k_{20} = \left(\frac{1}{\mu}\right)_{j+1} \left(\frac{6EI_{rx}}{l^2(1-2\beta')}\right)_{j+2} \]

\[ k_{21} = \left(\frac{1}{\mu}\right)_{j+1} \left(\frac{-2EI_{rx}}{l^2(1-2\beta')}\right)_{j+2} \]

References

1. Greenwood, J.A.; Williamson, J.B.P. Contact of nominally flat surfaces. Proceedings of the Royal Society of London. *Math. Phys. Sci. Ser. A* 1966, 295, 300–319. [CrossRef]

2. McCool, J.I. The distribution of micro contact area, load, pressure and flash temperature under Greenwood-Williamson model. *ASME J. Tribol.* 1988, 110, 106–111. [CrossRef]

3. Rao, Z.S. A Study of Dynamic Characteristic and Contact Stiffness of the Rod Fastening Composite Special Rotor. Ph.D. Thesis, Harbin Institute of Technology, Harbin, China, 1992.

4. Zhao, Y.; Maietta, D.M.; Chang, L. An asperity microcontact model incorporating the transition from elastic deformation to fully plastic flow. *ASME J. Tribol.* 2000, 122, 86–93. [CrossRef]

5. Ciavarella, M.; Greenwood, J.A.; Paggi, M. Inclusion of “interaction” in the Greenwood and Williamson contact theory. *Wear* 2008, 265, 729–734. [CrossRef]

6. Zhao, Y.; Chang, L. A model of asperity interactions in elastic-plastic contact of rough surfaces. *ASME J. Tribol.* 2001, 123, 857–864. [CrossRef]

7. Majumdar, A.; Bhushan, B. Fractal Model of Elastic-Plastic Contact between Rough Surfaces. *J. Tribol.* 1991, 113, 1–11. [CrossRef]

8. Morag, Y.; Etsion, I. Resolving the contradiction of asperities plastic to elastic mode transition in current contact models of fractal rough surfaces. *Wear* 2007, 262, 624–629. [CrossRef]

9. Hagman, L.A.; Olofsson, U. A model for micro-slip between flat surfaces based on deformation of ellipsoidal elastic asperity-ties-parametric study and experimental investigation. *Tribol. Int.* 1998, 31, 209–217. [CrossRef]

10. Fujimoto, T.; Kagami, J.; Kawaguchi, T.; Hatazawa, T. Micro-displacement characteristics under tangential force. *Wear* 2000, 241, 136–142. [CrossRef]

11. Allarà, M. A model for the characterization of friction contacts in turbine blades. *J. Sound Vib.* 2008, 320, 527–544. [CrossRef]

12. Cohen, O.; Kligerman, Y.; Etsion, I. A model for contact and static friction of nominally flat rough surfaces under full stick con-tact condition. *J. Tribol.* 2008, 130, 031401. [CrossRef]

13. Olofsson, U.; Hagman, L. A model for micro-slip between flat surfaces based on deformation of ellipsoidal elastic bodies. *Tribol. Int.* 1997, 30, 599–603. [CrossRef]

14. Sevostianov, I.; Kachanov, M. Normal and tangential compliances of interface of rough surfaces with contacts of elliptic shape. *Int. J. Solids Struct.* 2008, 45, 2723–2736. [CrossRef]

15. Komvopoulos, K.; Gong, Z.Q. Stress analysis of a layered elastic solid in contact with a rough surface exhibiting fractal behavior. *Int. J. Solids Struct.* 2007, 44, 2109–2129. [CrossRef]

16. Hyun, S.; Pei, L.; Molinari, J.-F.; Robbins, M.O. Finite-element analysis of contact between elastic self-affine surfaces. *Phys. Rev. E* 2004, 70, 026117. [CrossRef]
17. Pei, L.; Hyun, S.; Molinari, J.; Robbins, M. Finite element modeling of elasto-plastic contact between rough surfaces. *J. Mech. Phys. Solids* **2005**, *53*, 2385–2409. [CrossRef]
18. Prasanta, S.; Niloy, G. Finite element contact analysis of fractal surfaces. *J. Phys. D Appl. Phys.* **2007**, *40*, 4245–4252.
19. Sellgren, U.; Björklund, S.; Andersson, S. A finite element-based model of normal contact between rough surfaces. *Wear* **2003**, *254*, 1180–1188. [CrossRef]
20. Brizmer, V.; Kligerman, Y.; Etsion, I. Elastic–plastic spherical contact under combined normal and tangential loading in full stick. *Tribol. Lett.* **2006**, *25*, 61–70. [CrossRef]
21. Sellgren, U.; Andersson, S. The tangential stiffness of conformal interfaces between rough surfaces. *Tribol. Trans.* **2003**, *46*, 22, 3–12.
22. Yuan, Q.; Gao, R.; Feng, Z.; Wang, J. Analysis of dynamic characteristics of gas turbine rotor considering contact effects and pre-tightening force. In Proceedings of the ASME Turbo Expo, Berlin, Germany, 9–13 June 2008.
23. Zhang, Y.; Du, Z.; Shi, L.; Liu, S. Determination of contact stiffness of rod-fastened rotors based on modal test and finite element analysis. *J. Eng. Gas Turbines Power* **2010**, *132*. [CrossRef]
24. Jam, J.E.; Meisami, F.; Nia, N.G. Vibration analysis of tie-rod/tie-bolt rotors using FEM. *Int. J. Eng. Sci. Technol.* **2011**, *3*, 7292–7300.
25. Lu, M.; Geng, H.; Yang, B.; Yu, L. Finite element method for disc-rotor dynamic characteristics analysis of gas turbine rotor considering contact effects and rod preload. In Proceedings of the 2010 IEEE International Conference on Mechatronics and Automation, Xi’an, China, 4–7 August 2010.
26. Peng, H.; Liu, Z.; Wang, G.; Zhang, M. Rotor Dynamic Analysis of Tie-Bolt Fastened Rotor Based on Elastic-Plastic Contact. In Proceedings of the ASME Turbo Expo, Vancouver, BC, Canada, 6–10 June 2011; pp. 365–373. [CrossRef]
27. Gao, J.; Yuan, Q.; Li, P.; Feng, Z.; Zhang, H.; Lv, Z. Effects of bending moments and pre-tightening forces on the flexural stiffness of contact interfaces in rod-fastened rotors. *J. Eng. Gas Turbines Power* **2012**, *134*. [CrossRef]
28. Yang, L.; Xu, H.; Yu, L. Study on the calculation methods of vibration characteristics of the rotor with initial bending. In Proceedings of the 2017 IEEE International Conference on Mechatronics and Automation, Takamatsu, Japan, 6–9 August 2017.
29. Zhuo, M.; Yang, L.H.; Xia, K.; Yu, L. Thermal effects on the tensile forces of rods in rod-fastened rotor of heavy-duty gas turbine. *Proc. Imeche Part C J. Mech. Sci.* **2019**, *233*, 2753–2762. [CrossRef]
30. Xu, H.; Yang, L.; Xu, T.; Wu, Y. Effect of Detuning of Clamping Force of Tie Rods on Dynamic Performance of Rod-Fastened Jeffcott Rotor. *Math. Probl. Eng.* **2021**, *2021*, 1–11. [CrossRef]
31. Liu, H. Nonlinear Dynamic Analysis of a Flexible Rod Fastening Rotor Bearing System. *Chin. J. Mech. Eng.* **2010**, *46*. [CrossRef]
32. Yi, J.; Liu, H.; Liu, Y.; Jing, M. Global nonlinear dynamic characteristics of rod-fastening rotor supported by ball bearings. *Proc. Inst. Mech. Eng. Part. K J. Multi Body Dyn.* **2014**, *229*, 208–222. [CrossRef]
33. Yuan, Q.; Gao, J.; Li, P. Nonlinear dynamics of the rod-fastened Jeffcott rotor. *J. Vib. Acoust.* **2014**, *136*, 021011. [CrossRef]
34. Hu, L.; Liu, Y.; Zhao, L.; Zhou, C. Nonlinear dynamic behaviors of circumferential rod fastening rotor under unbalanced pre-tightening force. *Arch. Appl. Mech.* **2016**, *86*, 1621–1631. [CrossRef]
35. He, P. Analysis of Dynamic Characteristics of Distributed Rod Fastening Rotor. Master’s Thesis, Harbin Institute of Technology, Harbin, China, 2009.
36. Wang, W. Study on Design Method of Contact Interface Strength and Rotor-Bearing System Dynamics for Heavy-Duty Gas Turbine Combination Rotor. Ph.D. Thesis, Xi’an Jiaotong University, Xi’an, China, 2012.
37. Lie, Y.; Heng, L. *Bearing—Rotor System Dynamics*; Xi’an Jiaotong University Press: Xi’an, China, 2001.