Cognitive swarming in complex environments with attractor dynamics and oscillatory computing

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Abstract Neurobiological theories of spatial cognition developed with respect to recording data from relatively small and/or simplistic environments compared to animals’ natural habitats. It has been unclear how to extend theoretical models to large or complex spaces. Complementarily, in autonomous systems technology, applications have been growing for distributed control methods that scale to large numbers of low-footprint mobile platforms. Animals and many-robot groups must solve common problems of navigating complex and uncertain environments. Here, we introduce the ‘NeuroSwarms’ control framework to investigate whether adaptive, autonomous swarm control of minimal artificial agents can be achieved by direct analogy to neural circuits of rodent spatial cognition. NeuroSwarms analogizes agents to neurons and swarming groups to recurrent networks. We implemented neuron-like agent interactions in which mutually visible agents operate as if they were reciprocally-connected place cells in an attractor network. We attributed a phase state to agents to enable patterns of oscillatory synchronization similar to hippocampal models of theta-rhythmic (5–12 Hz) sequence generation. We demonstrate that multi-agent swarming and reward-approach dynamics can be expressed as a mobile form of Hebbian learning and that NeuroSwarms supports a single-entity paradigm that directly informs theoretical models of animal cognition. We present emergent behaviors including phase-organized rings and trajectory sequences that interact with environmental cues and geometry in large, fragmented mazes. Thus, NeuroSwarms is a model artificial spatial system that integrates autonomous control and theoretical neuroscience to potentially uncover common principles to advance both domains.

Keywords swarming · multi-robot groups · place cells · oscillations · spatial navigation · emergence

1 Introduction

Spatial cognition in rodents has been extensively studied in non-naturalistic environments such as linear or circular tracks, radial arm mazes, and T-mazes, or small open-field arenas such as squares or cylinders of approximately 1–2 m² area. Such experimental conditions have allowed individual place fields of hippocampal pyramidal neurons (O’Keefe and Dostrovsky, 1971) and the activity of other spatial cells (Knierim, 2006; Moser et al., 2008; Savelli et al., 2008; Poulter et al., 2018; Wang et al., 2018) to be exquisitely controlled and analyzed, leading to a detailed neural coding account of distributed representations that subserve spatial learning, memory, and planning in mammals (O’Keefe and Nadel, 1978; Moser and Paulsen, 2001; Knierim and Hamilton, 2011; Monaco and Abbott, 2011; Pfeiffer and Foster, 2013; Hartley et al., 2014; Burgess, 2014; Schiller et al., 2015; Foster, 2017), potentially extending to general cognitive computations in humans (Bellmund et al., 2018; Kunz et al., 2019). However, the multiplicity of Poisson-distributed hippocampal place fields exposed in larger environments (Fenton et al., 2008; Rich et al., 2014) and species differences in mapping 3-dimensional contexts (Yartsev and Ulanovsky, 2013; Casali et al.,...
suggest large and/or complex environments as the next frontier in understanding spatial navigation.

Computational models of rodent spatial networks have typically emulated the restricted environments of experimental studies (for computational efficiency, ease of analysis, and compatibility with published data). Despite these limitations, recent theoretical results have demonstrated the importance of sensory and cortical feedback in stabilizing and shaping hippocampal and entorhinal cortical spatial representations (Monaco et al., 2011; Poll et al., 2016; Rennó-Costa and Tort, 2017; Ocko et al., 2018); this relationship has been supported by experimental approaches to the animals’ own active sensing behaviors such as lateral head scanning (Monaco et al., 2014; Yadav and Doreswanny, 2017) and closed-loop control of orienting distal cues (Jayakumar et al., 2019). Additionally, extending prior theoretical frameworks such as the attractor map formalism (Zhang, 1996; Tsodyks, 1999; Samsonovich and McNaughton, 1997; Knierim and Zhang, 2012) to large spatial contexts has revealed substantial increases in the computational and mnemonic capacities of these network models (Hedrick and Zhang, 2016). Thus, a theoretical approach to naturalistic and dynamical spatial coding in large or complex environments may require closed-loop systems that integrate sensory information with internal spatial maps in continuously adapting loops. Complementary to the animal studies, investigating the behaviors and performance of completely specified but artificial spatial systems including virtual agents and/or mobile robotics platforms may help to elucidate the computational principles of spatial cognition in naturalistic contexts (Hasselmo, 2018; Tomov et al., 2018; Savelli and Knierim, 2019; Gauzzer et al., 2019).

Hippocampal phenomena that have been theorized to support biological spatial cognition include (1) self-stabilizing activity patterns in attractor map networks and (2) temporal-phase organization relative to global oscillations. On the basis of these phenomena, we introduce a brain-inspired dynamical controller for self-organized swarms of autonomous agents. Our key realization was that each virtual or robotic agent can be represented as a spatial neuron (e.g., a place cell) whose place preference follows the location of the agent in its local environment. The analogy of a multi-agent group (or swarm) to a space-coding neural network follows immediately. If we further suppose that inter-agent distances map to ‘synaptic weights’ and, consequently, that relative agent movements map to changes in those weights, then spatial configurations of the swarm constitute an attractor map network (Zhang, 1996; Tsodyks, 1999; Samsonovich and McNaughton, 1997) and the swarm’s internal motion dynamics constitute learning based on synaptic modification (Hebb, 1949; Oja, 1982).

Additionally, spatial activity in the rodent hippocampus is strongly modulated by continuous theta oscillations (5–12 Hz) during locomotion (Vanderwolf, 1969; Buzsáki, 2005). The resulting ‘phase precession,’ a monotonic advance in timing from late to early within each theta cycle, may enhance the precision and temporal organization of spatial codes (O’Keefe and Recce, 1993; Jensen and Lisman, 2000; Foster and Wilson, 2007; Drieu et al., 2018) in ways that support decision-making and/or deliberative planning during subsequent sleep or quiescent states (Buzsáki, 1989; Johnson and Redish, 2007; Buzsáki and Moser, 2013; Wikenheiser and Redish, 2015; Papale et al., 2016; Muessig et al., 2019). Recently, in contrast to phase precession, we discovered a novel class of spatial phase-coding neurons in open field environments termed ‘phaser cells’ that were located predominantly in the rat lateral septum and characterized by a strong coupling of theta-phase timing to firing rates (Monaco et al., 2019a). This coupling supports an intrinsic neuronal mechanism of phase-coding that may theoretically transform spatial information to synchronize downstream targets using the global theta oscillation (Monaco et al., 2019a). Accordingly, to examine the effects of temporal phase organization, our dynamical swarm controller considers each agent to have an internal phase variable (analogous to the theta-phase of a place cell or phaser cell) that may interact via oscillatory coupling with its neighbors’ phases. Such oscillator-based swarming has been previously generalized as the ‘swarmalators’ framework (O’Keefe et al., 2017). Thus, together with attractor dynamics, these phenomena may provide brain-like solutions to problems of decentralized self-organization and distributed communication in autonomous swarming.

We refer to our conception of this swarm controller as NeuroSwarms (Monaco et al., 2019b; Fig. 1A). In this paper, we derive an operational NeuroSwarms implementation (Section 2), present emergent swarming behaviors in simulations of a fragmented and heterogeneous environment (Section 3.1), demonstrate NeuroSwarms as a dual system which can be expressed through single-entity simulations that help inform biological theory (Section 3.2), evaluate adaptations of reward-approach behaviors in a large hairpin maze (Section 3.3), and discuss implications for autonomous systems design and biological spatial cognition in large, complex environments (Section 4).
Fig. 1 Conceptual schematic and theoretical neuroscientific inspiration for the NeuroSwarms controller. A. An artificial spatial system of mobile virtual or robotic agents communicate over sparse recurrent channels (bottom) just as spatial neurons in biological neural circuits produce reverberating activity patterns that reflect energy minima in the dynamical state-space of the system (e.g., fixed-point attractors; top; adapted from Knierim and Zhang, 2012). B. Example simulation of the spatial self-organization of an activity bump on an attractor map. In an attractor map network, the environment is represented by a continuum of locations with overlapping place fields, leading to network connectivity that produces self-reinforcing spatial activity patterns. Adapted from Zhang (1996). C. Schematic of a minimal model of temporal-phase coding in which an excitatory external input (green) is rhythmically modulated by a continuous inhibitory oscillation (blue) such as the hippocampal theta rhythm. Adapted from Monaco et al. (2019a) as permitted by the CC-BY 4.0 International License (creativecommons.org/licenses/by/4.0/).

2 Methods

2.1 Hippocampal model mechanisms

*Self-stabilizing attractor maps.* Hippocampal place cells fire within a contiguous region of the animal’s local environment, or ‘place field’ (O’Keefe and Dostrovsky, 1971). Place fields are thought to collectively form cognitive maps (O’Keefe and Nadel, 1978) that are stabilized (at least in part) via attractor dynamics, such as fixed points or continuous manifolds of the network energy surface, that drive activity toward low-dimensional spatial or task representations (Fig. 1B; Knierim and Zhang, 2012). Attractor map models have shown that recurrent connectivity between place cells with nonlinear integration of inputs is nearly sufficient to achieve stable spatial attractors (Zhang, 1996; Samsonovich and McNaughton, 1997; Tsodyks, 1999). For instance, a rate-based network following

\[
\frac{dr_i}{dt} = -r_i + g \left( \sum_j J_{ij} r_j + I_i \right), \tag{1}
\]

where \(r_i\) is the rate of unit \(i\), \(I_i\) is the unit’s total input, and \(g\) is a sigmoidal nonlinearity, only further requires that the recurrent weights \(J_{ij}\) encode the degree of place-field overlap between units (i.e., the strength of learned spatial associations). Such an encoding follows from a kernel function of field-center distances, e.g.,

\[
J_{ij} := F(x_i - x_j) = A \exp\left(\frac{-|x_i - x_j|^2}{\sigma^2}\right) - B, \tag{2}
\]

where \(x_k\) is the field-center position of unit \(k\), \(\sigma\) is the Gaussian scale constant, and \(A\) and \(B\) determine the strength of local excitation vs. long-range inhibition, respectively. While this formulation violates Dale’s law, it illustrates the typical parsimony of attractor map models (Tsodyks, 1999). A network constructed from equa-
tions (1) and (2) supports self-organization of its activity into a singular, contiguous ‘bump’ that emerges as the network relaxes (Fig. 1B; Zhang, 1996). The activity bump can then respond to input changes due to, e.g., movement through the environment or internal processing. These conditions are encapsulated within NeuroSwarms by analogizing (1) inter-agent visibility (e.g., for a line-of-sight communication channel) to sparse, recurrent network connectivity, and (2) a local kernel of inter-agent distances to spatially-weighted synapses.

Oscillatory organization. Hippocampal phase precession relative to the theta rhythm is clearest on linear tracks (O’Keefe and Recce, 1993), on which place fields typically arrange in an unambiguous sequential order that may enable learning of temporally compressed ‘theta sequences’ (Feng et al., 2015). While the phenomenon is less clear in open, 2-dimensional environments, phase precession has been observed in particular traversals of the firing fields of place cells as well as grid cells in entorhinal cortex (Climer et al., 2013; Jeewajee et al., 2014). However, our analysis of phaser cells in the lateral septum (Monaco et al., 2019a) revealed a more direct phase code for 2-dimensional position that was consistent with a minimal model of temporal phase-coding (Fig. 1C). To study how the phaser cell code may contribute to sequence formation in open environments, the NeuroSwarms controller adds an internal phase variable to each agent and couples the modulation frequency of that phase to each agent’s total input (see equation (12) below).

Internalized place fields for neural control. There are two reasons why neural swarming control must decouple each agent’s physical location from its internal self-localization. First, the multiplicity of agents is a qualitative difference with brain circuits; every hippocampal neuron in a biological network corresponds to the same single agent (e.g., a rat) and has particular inputs from internal processing or sensory inputs (Wang et al., 2016) that contribute to the appearance and location of the cell’s place field. Given the analogy of agents to neurons (Fig. 1A), an individual rat has many place fields but an individual swarm agent should have only one. Further, the location of an agent’s place field cannot be identical to the agent’s physical location, which depends on the momentary vicissitudes of a entity operating in the external world (or a simulation thereof). Second, experimental studies have compellingly demonstrated that spatial path planning in hippocampal networks may rely on activating sequences of place cells representing remote locations (Gupta et al., 2010; Pfeiffer and Foster, 2013; Ólafsdóttir et al., 2018; Momennejad et al., 2018), indicating that internal representations should be separable from an animal’s (or agent’s) current physical location. Thus, NeuroSwarms assigns a distinct cue-based place preference to each agent.

2.2 Mobile Hebbian learning with a global oscillation

Hebbian learning in neural network models typically increments or decrements a synaptic weight according to a learning rate and a measure of the activity correlation between the pre-synaptic (input) and the post-synaptic (output) neurons (Hebb, 1949; Levy and Steward, 1979; Oja, 1982; Eichenbaum, 2018). For the NeuroSwarms controller, the conceptual similarity of the synaptic strength relation in a neural network and the physical distance relation in a multi-agent group allows us to construct a neural activation and learning model for the motion of artificial mobile agents.

Following the Gaussian attractor-map kernel from equation (2), we explicitly relate a recurrent synaptic weight matrix $W \in \mathbb{R}^{N_s \times N_r}$, prior to learning-based updates, to swarm state via

$$W_{ij} = V_{ij} \exp\left(-D_{ij}^2 / \sigma^2\right),$$

(3)

for inter-agent visibility $V \in \{0,1\}^{N_s \times N_r}$, inter-agent distances $D$, and spatial constant $\sigma$. To provide for environmental interactions, we consider a minimal reward-approach mechanism for a set of reward coordinates that serve as attractive locations. Thus, we likewise incorporate a feedforward matrix $W^{r} \in \mathbb{R}^{N_s \times N_r}$ for reward learning,

$$W_{ik} = V_{ik}^r \exp\left(-D_{ik}^r / \kappa\right),$$

(4)

for agent-reward visibility $V^{r} \in \{0,1\}^{N_s \times N_r}$, agent-reward distances $D^{r}$, and spatial constant $\kappa$. The reward weights are based on an exponential kernel to allow for long-range approach behaviors. (We emphasize that the NeuroSwarms framework encompasses the general equivalence between synaptic weights and agent distances, but the particular implementation that we present here is one of many possible designs.)

To define neuron-like inputs, we consider that each agent’s internal place-field location derives from the conjunction of sensory cue inputs related to a preferred location. We define time-continuous sensory cue inputs $c \in \mathbb{R}^{N_s \times N_c}$ following

$$\tau_{c} \dot{c}_{ik} = V_{ik}^c V_{ik}^{c*} - c_{i},$$

(5)

for cue $k$, agent-cue visibility $V_{ik} \in \{0,1\}^{N_s \times N_c}$, fixed agent-cue preferences $V_{ik}^{c*} \in \{0,1\}^{N_s \times N_c}$, and integration time-constant $\tau_{c}$. Thus, the product in equation (5) yields the integer number of preferred cues visible from
each agent’s position. This means that place-field size is not independently controlled but determined by the relative cue richness of the environment: more cues will generally increase agent heterogeneity and spatial selectivity. Similarly, we compute reward inputs \( r \in \mathbb{R}^{N_s \times N_r} \) following
\[
\tau_r \dot{r}_{ik} = V_{ik}' - r_i ,
\]
for reward \( k \), and integration time-constant \( \tau_r \). Unlike sensory cues, all agents respond equally to all (visible) rewards. Lastly, we define recurrent swarming inputs \( q \in \mathbb{R}^{N_s \times N_r} \) following
\[
\tau_q \dot{q}_{ij} = V_{ij} \cos(\theta_j - \theta_i) - q_{ij} ,
\]
for ‘post-synaptic’ agent \( i \), ‘pre-synaptic’ agent \( j \), integration time-constant \( \tau_q \), and internal oscillatory phase state \( \theta \). The cosine term in equation (7) confers phase-modulation of the input in which excitation/inhibition depends on whether units \( i \) and \( j \) are in/out of sync.

Having defined the input signals, we consider ‘net currents’ as gain-modulated, visibility-normalized quantities for sensory cue inputs,
\[
I_{c_i} = \frac{g_c}{\sum_k V_{ik}} \sum_k c_{ik} ,
\]
reward inputs,
\[
I_{r_i} = \frac{g_r}{\sum_k V_{ik}} \sum_k W_{ik} r_{ik} ,
\]
and recurrent swarming inputs,
\[
I_{q_i} = \frac{g_q}{\sum_j V_{ij}} \sum_j W_{ij} q_{ij} ,
\]
where the parameter gains \( g_c, g_r \), and \( g_q \) sum to 1. Because the net inputs are bounded in equations (8)–(10), we simply apply linear rectification rather than a saturating nonlinearity (cf. equation (1)) to their sum to calculate post-synaptic activation
\[
p = [I_c + I_r + I_q]_+ ,
\]
which is the remaining component needed to compute Hebbian (or any two-factor) learning. In terms of swarming, however, the agents are phase-coupled via equation (7). Thus, in the same way that a spiking neuron can be reduced to a phase description of its orbit on the phase plane, we consider that \( p \) drives the agents’ internal phase state, e.g.,
\[
\dot{\theta} = \omega_0 + \omega_f p ,
\]
where \( \omega_f \) sets the maximum increase in input-modulated angular frequency above the baseline frequency \( \omega_0 \). The net effect of this mechanism is that agents have place-cell-like spatial tuning with phaser-cell-like phase coding and synchronization.

The core of the NeuroSwarms controller comprises the learning-based updates to \( W \) and \( W' \). A na"ive Hebbian rule, such as \( dW_{ij} = \eta p_i q_j \), would cause weights to grow unbounded, leading to ictogenesis in recurrent networks or spatial singularities in swarms. Instead, after updating agent activations via equation (11), we compute updated weights \( W' \) as
\[
W'_{ij} = W_{ij} + \Delta t \eta V_{ij} p_i (q_{ij} - p_i W_{ij}) ,
\]
with simulation time-step \( \Delta t \) and learning rate \( \eta \), which effectuates a pre-synaptic normalization according to Oja’s rule (Oja, 1982). Similarly, the updated feedforward weights \( W'' \) are computed for reward \( k \) as
\[
W''_{ik} = W_{ik} + \Delta t \eta_i V_{ik} p_i (r_{ik} - p_i W_{ik}) .
\]
The normalization effected by equations (13) and (14) is due to a subtractive term, quadratic in the post-synaptic activation \( p \), that depresses the growth of overly active synapses. In place-cell network models, feedback inhibition typically serves to spread out place fields to more efficiently map an environment (Savelli and Knierim, 2010; Monaco and Abbott, 2011), but the lack of explicit inhibition in NeuroSwarms allows synaptic depression to provide a similar repulsive role due to the distance–weight equivalence of equation (3).

2.3 Neural swarm control: closing the loop

To integrate with swarming, the controller attempts to drive the agents’ kinematic states to the equivalent desired inter-agent distances, in effect replacing the typical attraction and repulsion fields of conventional swarming models (e.g., Gazi and Passino, 2011). The updated weights \( W' \) and \( W'' \) can be converted directly to desired distances by inverting the Gaussian swarming kernel in equation (3),
\[
D'_{ij} = \sqrt{-2\sigma^2 \log W'_{ij}} ,
\]
and the exponential reward kernel in equation (4),
\[
D''_{ij} = -\kappa \log W''_{ij} .
\]
To compute the resultant swarm motion, the desired positional shift of agent \( i \) is averaged across its visible neighbors, i.e.,
\[
f_i = \frac{1}{2N_s} \sum_{j=1}^{N_s} V_{ij} (D'_{ij} - D''_{ij}) \frac{x_j - x_i}{|x_j - x_i|} ,
\]
and the resultant reward-related motion is similarly computed as the average
\[
f_i^r = \frac{1}{\sum_k V_{ik}^s} \sum_k V_{ik}^s (D_{ik}^r - D_{ik}^s) \frac{x_k^r - x_i}{|x_k^r - x_i|}. \tag{18}
\]
The net positional shift is calculated as a linear combination of the swarm- and reward-related shifts,
\[
\Delta x = \alpha f + (1 - \alpha) f^r,
\]
where \(\alpha = 0.5\) for all simulations presented. The remaining processing of \(\Delta x\) in our NeuroSwarms implementation serves to embed the foregoing dynamics within ‘physical’ simulations of irregular or complex 2-dimensional environments. First, our example environments (Fig. 2) of \(\sim 500\)-point height (for arbitrary points units) were processed for wall proximity and normal vectors for all interior locations. Thus, a ‘barrier aware’ positional shift \(\Delta x^b\) is calculated as
\[
\Delta x^b_i = (1 - \beta_s) \Delta x_i + \beta_s |\Delta x_i| n_i, \tag{20}
\]
for an exponential kernel \(\beta_s = \exp(-d/\lambda)\) of distance \(d\) to the nearest wall with a constant of \(\lambda = 20\) points, and the normal vectors \(n_i\) of the nearest wall. These shifts update the internal place-field locations \(x_s \leftarrow x_s + \Delta x^b\) of each swarm agent. Second, ‘physical’ agent locations are updated based on the instantaneous velocity needed for each agent to approach their internal field locations, \(\mathbf{v}_s = (x_s - x)/\Delta t\), which is processed through a momentum filter,
\[
\mathbf{v}_\mu = \mu \mathbf{v} + (1 - \mu) \mathbf{v}_s, \tag{21}
\]
with the actual velocity (prior to updating) \(\mathbf{v}\) and coefficient \(\mu\); a speed-limiting nonlinearity based on a kinetic-energy maximum \(E_{\text{max}}\),
\[
\mathbf{v}_{\text{max}} = \sqrt{2E_{\text{max}}/m}, \tag{22}
\]
\[
\mathbf{v}_{ki} = \mathbf{v}_{\text{max}}, \tanh \left( \frac{|\mathbf{v}_{\mu}|}{\mathbf{v}_{\text{max}}} \right) \frac{\mathbf{v}_{\mu}}{|\mathbf{v}_{\mu}|}, \tag{23}
\]
where \(m\) is agent mass; and barrier avoidance as in equation (20),
\[
\mathbf{v}_i = (1 - \beta) \mathbf{v}_{ki} + \beta |\mathbf{v}_{ki}| n_i, \tag{24}
\]
for proximity \(\beta\) and normal vectors \(n\). Finally, agent locations are updated by \(x \leftarrow x + \mathbf{v}\Delta t\).

2.3.1 Single-entity simulations

In keeping with the neuroscientific motivation for NeuroSwarms (Fig. 1), our implementation allows for singleton simulations analogous to conventional models of neural networks in an animal such as a rat navigating a maze. With only minor adjustments, NeuroSwarms can operate with a single agent (i.e., \(N = 1\)) that owns a collection of ‘virtual’ or ‘mental’ swarming particles (e.g., \(N_s = 300\)) that guide the agent’s spatial behavior. In this sense, the virtual swarm represents a highly dynamic spatial field that provides the agent with various options for constructing a path through the environment. The dynamics of the virtual swarm are as described above up to equation (20). An array \(V^i \in \{0,1\}^{N_s}\) indicates which particles’ positions are visible to the agent and serves to additionally mask the learning updates in equations (13) and (14). To produce motion, single-agent velocity is instead calculated using a cubic-activation-weighted average of the swarm,
\[
\mathbf{v}_s = \frac{1}{\Delta t \sum_j V_j^s p_j^3} \sum_{i=1}^{N_s} V_i^s p_i^3 (x_s - x), \tag{25}
\]
prior to processing the ‘physical’ embedding of the agent’s motion in equations (21)–(24). Thus, the agent constructs a path toward the most highly activated and visible swarm particles.

2.4 NeuroSwarms simulations

Simulated environments (Fig. 2) contain fixed-position rewards and cues depicted as gold stars and purple shapes, respectively. Environments are otherwise defined by a set of linear barrier segments (e.g., walls) that form a closed shape defining an interior space that becomes the set of allowable agent locations. Simulations are initialized by setting all velocities, input signals, and activations to zero, randomly choosing internal phase states, and randomly assigning agent positions to allowable locations within a set of ‘spawn discs’ defined in the environment. Random number generator seeds are reused for simulations meant to compare parameter values, unless otherwise specified. Environments are specified as vector image files in Tiny SVG format with XML text nodes defining reward, cue, and spawn disc locations. Unless noted, parameters were set to the default values displayed in Table 1. The python source code will be made available upon reasonable request.
Cognitive swarms with attractors and oscillators

Fig. 2 Example post-initialization ($t = 0.1$ s) swarm states for NeuroSwarms simulations. (Left) A single-agent simulation in the ‘multireward’ arena, which contains 3 rewards (gold stars; northwest, southwest, southeast), 7 cues (purple shapes), and 3 spawn discs. White enclosed areas constitute the set of allowable locations for swarm agents; black regions constitute barriers and disallowed locations. Initial particle positions are sampled from the spawn discs and initial phases are random. Green circle in southwest; the single agent; dots: 300 virtual swarm particles with internal phase state indicated by color. (Right) A multi-agent simulation in the ‘hairpin’ maze, which contains 5 connected hallways, 3 rewards, 7 cues, and 4 spawn discs. Circles: 300 swarm agents with internal phase state indicated by color; reward (gold star) size is for visual differentiation only and has no effect in the model.

Table 1 Parameters, default values, and descriptions (with units) for the NeuroSwarms controller implementation.

| Symbol | Value | Description                                |
|--------|-------|--------------------------------------------|
| $\Delta t$ | 0.01 s | integration time-step of simulation       |
| duration | 180.0 s | total simulation time                      |
| $N$    | 300   | no. of ‘physical’ agents (multi-agent)     |
| $N_s$  | 300   | no. of internal fields (multi-agent) or virtual particles (single-entity) |
| $D_{\text{max}}$ | 1.0 | max. inter-agent visibility range          |
| $E_{\text{max}}$ | 3e3 kg\cdot$\text{points}^2$/$\text{s}^2$ | max. kinetic energy                        |
| $\mu$  | 0.9   | momentum coefficient of agent motion       |
| $m_{\text{multi}}$ | 0.3 kg | mean agent mass (multi-agent)              |
| $m_{\text{single}}$ | 3.0 kg | agent mass (single-entity)                 |
| $\sigma$ | 1.0 | spatial scale of swarm interaction         |
| $r$    | 1.0   | spatial scale of reward interaction        |
| $\eta_r$ | 1.0 | learning rate for reward connections       |
| $\eta_e$ | 1.0 | learning rate for swarm connections        |
| $\omega_0$ | 0.0 cycles/s | baseline oscillatory frequency             |
| $\omega_I$ | 1.0 cycles/s | max. increase in oscillatory frequency due to neural activation |
| $g_c$  | 0.4   | gain of sensory cue inputs                 |
| $g_r$  | 0.2   | gain of reward inputs                      |
| $g_s$  | 0.4   | gain of swarming inputs                    |
| $\tau_c$ | 0.5 s | time-constant of sensory cue inputs        |
| $\tau_r$ | 0.5 s | time-constant of reward inputs             |
| $\tau_s$ | 0.1 s | time-constant of swarming inputs           |
| $d_{\text{rad}}$ | 0.0 points | points, reward contact radius             |

† These parameter values are multiplicatively scaled to the notional environment size, defined in points as the radius of a disc with the same area as the set of allowable locations in the environment’s interior.

3 Results

3.1 Emergent swarming behaviors

We designed the multireward arena (Fig. 2, left) to characterize emergent swarming and reward approach behaviors, and the hairpin maze (Fig. 2, right) to assess behavioral adaptation in large, fragmented environments. We observed several emergent dynamical behaviors in simulations of both multi-agent swarming and single-entity locomotion (Section 2, Methods). The most notable and persistent behaviors included the emergence of phase-sorted spatial formations such as line segments, rings, or concentric loops (Fig. 3). These behaviors were analogous in form to (1) the ‘phase wave’ states observed in certain swarmalator regimes (O’Keeffe et al., 2017), and (2) the hippocampal phenomena of theta sequences and theta-rhythmic phase assemblies (Foster and Wilson, 2007; Drieu et al., 2018). Further, by inspection of simulation movies, we observed two dynamical features. First, agent subgroups forming line segments and rings continuously phase-synchronized in a shared oscillation that was independent from the absolute movement or rotation of the formation in space. Second, line or ring formations would often break apart and re-form new configurations that typically involved other agents or formations that were able to phase-synchronize with elements of the subgroup. These alternating disintegrative and aggregative dynamics may be consistent with analyses of
persistent homologies in place-cell networks with transient connectivity (Babichev and Dabaghian, 2017).

These spatiotemporal dynamics are evident across frame captures of multi-entity (Fig. 3A) and single-entity (Fig. 3B) simulations. While phase-ordered groups could appear far from rewards (Fig. 3A, last two frames, smaller red circles), swarm agents typically approached a reward location and formed a rotating ring centered on the reward position (Fig. 3A, southeast corner, last three frames). Such reward rings appeared in single-entity simulations, but the virtual swarm particles (Section 2.3.1) additionally exhibited particularly extended line segments that often traced out phase-ordered trajectory sequences; e.g., the agent followed an extended sequence to the reward located in the southeast corner (Fig. 3B, last two frames). Further, we observed the size of reward rings decreased over time, reflecting a relaxation of phase and momentum given the centrally organizing reward location.

When the reward kernel’s spatial scale \( \kappa \) (equation (4); Table 1) was increased, streams of virtual swarm particles formed around distal rewards as the particles’ motion was modulated by agent visibility interacting with the geometry of the environment. As shown in the first frame of Fig. 3C, a step-like pattern formed near the northwest reward location while a wavy pattern formed near the southeast reward location. Both virtual swarm formations presented path choices to the single agent located in the large central compartment of the arena. As expected (Section 2.3.1), virtual swarm particles that were not visible to the agent remained fixed in place due to masking of the weight updates in equations (13) and (14). In addition to single rings, double and even triple concentric loops of nested, non-overlapping, phase-sorted rings were observed in some simulations. An example of a double loop forming is shown in the southeast corner at \( t = 16.74 \) s (Fig. 3C). Strikingly, we did not predict, expect, or adapt the NeuroSwarms controller design to observe these emergent behaviors; we simply implemented abstractions for place-cell spatial tuning, phaser-cell oscillatory synchronization, and a distance-weight equivalence for Hebbian learning with notions of visibility and environmental geometry that provided spatial barriers to communication (Section 2). These behaviors would be unexpected as well from conventional swarming algorithms (Gazi and Passino, 2011).

3.2 Reward-based behavior in a compartmented arena

To assess the spatial performance of NeuroSwarms, we examined the ability of single-entity behavior to find all three rewards in the multimarket arena. We focused on the parameter constants governing the spatial scale of swarm (\( \sigma \)) and reward (\( \kappa \)) interactions (equations (3) and (4); Table 1) and found (\( \sigma, \kappa \)) values for which the agent approached multiple rewards regardless of its initial spawn location. Due to the random initialization of location, we selected 40 simulations for analysis in which the agent was spawned in the southwest corner (as in Fig. 2, left). The agent successfully captured one, two, or all three rewards in 11, 28, and 1 simulation(s) at elapsed times ranging from 4–108, 20–179, and \( \sim 160 \) s, respectively. Frame captures of reward approaches are shown in Fig. 4A for the simulation in which all three rewards were found. The ability of the agent to approach multiple fixed rewards over time was an unexpected and emergent behavior: based on our NeuroSwarms implementation, we had predicted that the rewards would serve as stable attractors in the absence of additional mechanisms such as adaptation or reward learning. However, in accordance with those expectations, we observed simulations which failed to explore much of the environment after approaching a single reward location. For the same parameters but a different random seed than shown in Fig. 4A, a failed exploration occurred (Fig. 4B) when the virtual particles split into two fixed-point, out-of-phase attractors that essentially ‘trapped’ the agent.

To counteract these unsuccessful equilibria, we implemented a ‘reward capture’ mechanism in the NeuroSwarms controller based on a minimum contact radius, \( d_{\text{rad}} \). This feature causes rewards to cease being attractive locations to the virtual swarm particles upon contact by the agent, thus releasing the agent from reward-related attractors before further exploration is prevented. Indeed, having capturable rewards with \( d_{\text{rad}} = 12 \) points enabled a simulation that was otherwise identical to the failed case (Fig. 4B) to successfully navigate the arena to capture all three rewards (Fig. 4C). Thus, a notion of reward adaptation or reward consumption may be crucial to achieving continuous exploration.

For the 40 single-entity simulations with fixed rewards, the bottom panel of Fig. 5A reveals strong attractors at the southeast and northwest corners of the arena associated with reward locations. To demonstrate the effect of the contact radius on exploration when rewards were capturable, the trajectories resulting from contact radii of 1, 4, 10, and 15 points are shown in the top row of Fig. 5A; these values produced 1, 3, 8, and 30 (out of 40) simulated trajectories, respectively, that successfully contacted all three rewards (Fig. 5A, red traces). In a few simulations, the single-entity agent spawned in the southwest corner, found the southeast
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Fig. 3 Temporal evolution of swarming and single-entity approaches to rewards. A. Three agent-clusters were initially populated in the multireward arena. The internal place-field location of each agent is indicated by a small black dot (e.g., $t = 1.26$ s, inset, top right). Phase sorting is indicated by sequentially ordered colors of the circle markers representing agent positions. A reward-centered phase ring was created ($t = 9.71$ s) with a decreasing diameter over time (southeast corner, $t = 22.49$ s and $t = 24.32$ s). NeuroSwarms parameters: $\sigma = 1.5$, $g_c = 0.2$, $g_r = 0.3$, $g_s = 0.5$; Table 1. B. A single-entity agent (green circle) was guided by $N_v = 300$ virtual particles (phase-colored dots). Swarm particles formed phase sequences leading the agent from the southwest corner to the reward location in the southeast corner of the arena by $t = 20.3$ s. NeuroSwarms parameters: $\sigma = 4$, $\kappa = 1.5$, $g_c = 0.2$, $g_r = 0.3$, $g_s = 0.5$; Table 1. C. Step-like patterns of particles appeared near rewards that were occluded from the perspective of the single agent by corners in the environmental geometry. While the agent became ‘indecisive’ around $t = 10.24$ s, as it was pulled simultaneously in both directions, the agent ultimately found its way to the southeast reward by $t = 16.74$ s. NeuroSwarms parameters: $\sigma = 4$, $\kappa = 8$, $g_c = 0.2$, $g_r = 0.3$, $g_s = 0.5$; Table 1.

reward first, and then later returned to the southwest corner in order to collect all three rewards; such a wandering trajectory suggests that the model might qualify as an ergodic system under these conditions, but that hypothesis would be appropriately addressed by future analytical studies. These results demonstrate that the sensitivity of reward capture modulates exploratory variability by mitigating the effect of reward-related attractors. Histograms of the time-to-capture profile across agent spawn sites and reward locations reflect the structure of the environment as well as the different possible sequences of reward contact (Fig. 5B). Thus, the contact radius for capturable rewards exerted substantial control over the likelihood of the single-entity agent finding all rewards in the environment.

3.3 Behavioral adaptation in large hairpin mazes

A key challenge facing current state-of-the-art swarm controllers is the inability to rapidly adapt to dynamic changes in complex environments. The hairpin maze is well suited to study such adaption because swarm agents spawned from certain hallways do not have line-of-sight visibility to rewards that may be located in adjacent hallways. A form of behavioral adaption can be assessed based on whether agents spawned into reward-free hallways can nonetheless navigate to rewards in other parts of the maze.

We examined multi-agent swarming dynamics in the hairpin maze under several conditions: pure swarming (equation (13); Fig. 6A); swarming with sensory cue inputs (equation (8); Fig. 6B); and swarming with sensory cue inputs and reward approach (equation (14); Fig. 6C). The sample frames shown in Fig. 6 demonstrate the emergence of phase-ordered structures in each of these conditions with the clear distinction that tightly configured reward rings became prevalent when reward learning was activated (Fig. 6C). In that condition, with the same NeuroSwarms features as studied above in the multireward arena, it was also clear that agents in the second and third hallways had difficulty leaving to find another hallway with a reward. We expected this was due to (1) the parity of swarming and reward spatial constants ($\sigma, \kappa$), which perhaps overemphasized swarming at the cost of reward-following in highly-partitioned environments, and (2) the need for more sensitive reward-capture. Thus, we simulated a condition with fixed and capturable rewards using $d_{rad} = 10$ points but also increased the spatial constants...
Single-entity reward-approach behavior with fixed or capturable rewards. The agent was initialized to the spawn disc in the southwest corner of the multireward arena. A rare example in which the single agent (green circle) captured all three rewards when rewards were fixed (i.e., they remained attractive despite previous contact with the agent): southwest reward at \( \sim 8.9 \) s, southeast reward at \( \sim 33 \) s, and northwest reward at \( \sim 160 \) s. Movie frames show the initial contacts with each reward (gold stars). NeuroSwarms parameters: \( \sigma = 4, \kappa = 1.5, g_c = 0.2, g_r = 0.3, g_s = 0.5 \); Table 1. B. With the same parameters as (A) but initialized with a different random seed, this final frame of a simulation shows the converged state after the agent was attracted to the southwest corner and remained there for the duration. The red ellipse highlights that the agent was ‘stuck’ between two fixed-point attractors that formed through mutual phase-desynchronization. C. With the identical parameters and random seed as (B), rewards were made to be ‘capturable’ at a minimum contact radius of \( d_{rad} = 12 \) points. Thus, rewards ceased to be attractive locations once the agent made initial contact. The agent captured the southwest reward at \( \sim 5 \) s, the southeast reward at \( \sim 27 \) s, and the northwest reward at \( \sim 60 \) s. Transparent/white stars indicate captured rewards. NeuroSwarms parameters: \( \sigma = 4, \kappa = 1.5, g_c = 0.2, g_r = 0.3, g_s = 0.5, d_{rad} = 12 \); Table 1.

Multi-agent trajectories for this enhanced reward-exploration regime are shown in Fig. 6D: with fixed rewards (top panel), the reward attractors dominate the dynamics and agents generally stayed within their initial hallways; with capturable rewards (bottom panel), there was substantially more path variability between agents, spatial coverage increased (cf. the spiral patterns characteristic of agents’ exits from reward locations after contact), and many more agents were able to traverse from one hallway to the next.

To assess the converged state of multi-agent dynamics in the hairpin maze, we simulated \( N = 300 \) agents for 300 s using the same parameters and fixed rewards as the top panel of Fig. 6D. The temporal progression of swarm state across the simulation frames presented in Fig. 7 shows distinct stages exhibited by the four initial clusters of the swarm. The two clusters that spawned in reward-free hallways eventually found their way around the barriers to adjacent hallways after milling in various line segment or ring formations for nearly a minute (Fig. 7). All of the clusters successfully converged onto the three reward locations in the maze, but the two that traversed hallways left some agents behind. The progression of those swarm clusters from initial positions to ring/arc formations to linear trajectory sequences to fixed-point reward attractors illustrates a high degree of spontaneous adaptation to the circumstances in the hairpin maze. These dynamics were self-organized and emergent, providing behaviors that resulted in nearly complete convergence to reward locations. Thus, NeuroSwarms demonstrated autonomous spatial navigation to unknown, occluded, and remote rewards in a large and complex environment.
4 Discussion

We introduced the NeuroSwarms controller as a model for studying neural control paradigms of artificial swarming agents. We demonstrated that NeuroSwarms also acts as a two-way bridge between artificial systems and theoretical models of animal cognition. This reciprocity arises due to a single-entity paradigm in which NeuroSwarms controls a single agent in which an internal, virtual ‘cognitive swarm’ guides the agent’s spatial behavior. Both modes of operation, multi-agent and single-entity, share the same underlying neural mechanisms (with differences described in Section 2.3.1). This duality enables developments in artificial systems to also inform advances in neurobiological theories of spatial cognition. Additionally, this duality will aid discovery of neural dynamics in large, uncertain, and/or complex environments based on closed-system approaches to distributed spatial coding. We presented behaviors responding to environmental complexities such as multiple reward sites (that optionally interact with the system by being ‘consumed’ by agents), heterogeneous agent-based preferences for neutral-valued spatial cues, and geometric constraints that occlude agents’ visibility of cues, rewards, and other agents.

Swarms governed by NeuroSwarms self-organize into emergent, transitory configurations in position and phase that directly recall spatial attractor dynamics (Zhang, 1996; Tsodyks, 1999; Samsonovich and McNaughton, 1997; Hedrick and Zhang, 2016; Knierim and Zhang, 2012) and sequential oscillatory phenomena (O’Keefe and Recce, 1993; Foster and Wilson, 2007; Drieu et al., 2018; Monaco et al., 2019a) that have been theorized to operate within hippocampal circuits. We explicitly designed NeuroSwarms to combine features of attractor maps and oscillatory computing using robust transformations (such as the spatial kernels of distance converted to synaptic strengths in...
Dynamics of a multi-agent swarm in a large hairpin maze. Movie frame captures are shown for simulations with $N = 300$ agents in a rectangular environment ($885 \times 519$ points including borders) partitioned into 5 hallways in a hairpin pattern. Three hallways contain rewards which are substantially occluded from the other maze sections. Emergent formations are circled in red.

A. Frames from a pure swarming simulation, without reward or sensory cue influence. NeuroSwarms parameters: $d_{\text{max}} = 1.5$, $\eta = 1$, $\eta_r = 0$, $g_c = 0$, $g_r = 0$, $g_s = 1$; Table 1. B. Frames from a capturable-rewards simulation with 1:1 swarm/cue input gains but no reward influence. NeuroSwarms parameters: $d_{\text{max}} = 1.5$, $\eta = 1$, $\eta_r = 0$, $g_c = 0.5$, $g_r = 0$, $g_s = 0.5$; Table 1. C. Frames from a capturable-rewards simulation with equalized swarm, reward, and cue input gains. NeuroSwarms parameters: $d_{\text{max}} = 1.5$, $\eta_s = 1$, $\eta_r = 1$, $g_c = g_r = g_s = 1/3$; Table 1. D. Multi-agent trajectories are shown from two 80 s simulations: fixed rewards (top) and capturable rewards with $d_{\text{rad}} = 10$ points (bottom). Compare with multireward arena simulations in Fig. 5A. NeuroSwarms parameters: $d_{\text{max}} = 1.5$, $\sigma = 2$, $\kappa = 6.6$, $g_c = 0.1$, $g_r = 0.1$, $g_s = 0.8$; Table 1.

To leverage inertial, energetic, and cost benefits of small-scale robots, critical future applications of autonomous technologies may depend on coordinating large numbers of agents with minimal onboard sensing and communication resources. However, a critical problem for autonomous multi-robot groups is that state-of-the-art control schemes break down as robotic agents are scaled down (decreasing agent resources) and the numerical size of swarms is scaled up (increasing communication and coordination requirements) (Murray, 2007; Hamann et al., 2016; Yang et al., 2018; Chung et al., 2018). NeuroSwarms addresses the hypothesis that a similar distributed scaling problem may have been solved by the evolved neural architecture of mammalian brains. Compared to signal comprehension, signal production errors may be particularly deleterious to large-scale, distributed and decentralized computations (Salahshour et al., 2019). Thus, onboard suites for future cognitive swarming platforms based on NeuroSwarms principles should emphasize reliable transmission of low-bandwidth data packets (e.g., spikes or continuous phase signals). Low fidelity inputs are more easily compensated by distributed processing;
i.e., sensor designs should emphasize energy and cost to maximize deployment duration and swarm size.

In summary, we made the explicit analogy from agents and swarms to neurons and neural circuits. This analogy permitted the tools of theoretical neuroscience to be leveraged in developing a model artificial spatial system. The NeuroSwarms controller required two features to support cognitive swarming: (1) an internal phase state, and (2) decoupling of physical location from internal self-localization. The phase state naturally encapsulated neural activation (cf. equation (12)) and could be used to drive spike generation, if desired, in future models. Phase-based organization additionally leveraged the expressive complexity of mobile oscillators revealed by the swarmalator formalism (O’Keeffe et al., 2017; Monaco et al., 2019b). The separation of position vs. self-localization allowed swarm motion dynamics to be interpreted as Hebbian learning in an oscillatory place-coding neural network (Section 2.2). Thus, theorized hippocampal phenomena such as attractor map formation and oscillatory sequence generation provide a framework for advances in decentralized swarm control and, reciprocally, the theoretical neuroscience of spatial navigation in complex, changing environments.

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Conflict of interest

The authors have no conflicts of interest to declare.

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