Two non-conjugate embeddings of $S_3 \times \mathbb{Z}_2$ into the Cremona group II

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Abstract. We prove more precisely the following main result of [Isk1]. There is two non-conjugate embeddings of $S_3 \times \mathbb{Z}_2$ into the Cremona group that is given by the linear action of this group on the plane and the action on the two-dimensional torus.

1. Introduction and the main result

1.1. Remind the main result which is inspired by [LPR]. Consider two following actions of the group $G \cong S_3 \times \mathbb{Z}_2$ on the rational surfaces $P$ and $T$ (where $S_3$ is the symmetric group and $\mathbb{Z}_2$ is the cyclic group of order two).

(I) $P$ is the plane $x + y + z = 0$: $S_3$ acts by the permutations on the coordinates and $\mathbb{Z}_2$ is given by $(x, y, z) \rightarrow (-x, -y, -z)$;

(II) $T$ is the two-dimensional torus $xyz = 1$: $S_3$ acts by the permutations on the coordinates and $\mathbb{Z}_2$ is given by $(x, y, z) \rightarrow (x^{1-}, y^{1-}, z^{1-})$.

In the both cases we have embedding of this group into two-dimensional Cremona group $Cr_2(\mathbb{C})$. The question is: are the images of these embeddings of $G$ conjugate in $Cr_2(\mathbb{C})$ or not?

Our answer is the following.

1.1.1. Proposition. The images of these embeddings of $G$ are not conjugate in $Cr_2(\mathbb{C})$. In the other words, there is not exist $G$-equivariant birational map between $G$-surfaces $P$ and $T$.

1.2. Commentary. Together with [LPR] this result means that the simple algebraic group of type $G_2$ is not Cayley group in the following sense. Let $G$ as in [LPR] be the connected algebraic group (we let it only in this commentary). Let $\mathfrak{g}$ be its Lie algebra. Consider $G$ as the algebraic variety $X$ with the action of $G$ on itself by the conjugation $gxg^{-1}$, $g \in G$, $x \in X$, and the Lie algebra $\mathfrak{g}$ as the algebraic variety $Y$ with adjoint action $Ad_G g(y) = gyg^{-1}$, $g \in G$, $y \in Y$. The group $G$ is called Cayley group if there exist $G$-equivariant birational map $\lambda: X \rightarrow Y$, i. e. $\lambda(gxg^{-1}) = Ad_G g(\lambda(x))$. The question about Cayley property is reduced to some property of character lattice $T$ of the maximal torus $T \in G$ with the natural action of Weyl group $W$ on $T$. Many canonical linear algebraic groups are proved to be non-Cayley ones. However, there also exist a lot of exceptions.

The maximal torus of a group of type $G_2$ is two-dimensional, and its Weil group is $W \cong S_3 \times \mathbb{Z}_2$. The question about Cayley property is exactly problem [LPR]. Our result proves that $G_2$ is not Cayley. It is surprising that in [LPR] proved that the group $G_2 \times G_m^2$ (where $G_m$ is the multiplicative group of the field with the trivial action of Weyl group) is a Cayley group. Thus, $G_2$ is stable Cayley in this sense. It would be interesting to know, is the group $G_2 \times G_m$ Cayley or not?

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In our work [Isk1] we give the sketchy proof of proposition [1.1] based on the general method of factorization of $G$-equivariant birational maps between rational $G$-surfaces (see [Ma1], [Isk3]). $G$ here is a finite group. However we give complete and more precise proof of it because of great interest in this fact.

1.3. As in [Isk1] we use the conception of rational $G$-surface for proof. Here $G \subset Cr_2(\mathbb{C})$ is the finite subgroup in the Cremona group (see, for instance, [Ma1], [Isk2]). In our case $G \simeq S_3 \times \mathbb{Z}_2$ and it acts on the compactifications of $Y$ and $X$ of surfaces $P$ and $T$ (which correspond to cases (I) and (II) respectively). Describe these actions more precisely.

The case (I). The surface $Y = \mathbb{P}^2$. Let $(u_0, u_1, u_2)$ be the homogeneous coordinates on $\mathbb{P}^2$ and $x = \frac{u_1}{u_0}$, $y = \frac{u_2}{u_0}$ and $z = -\frac{u_1 + u_2}{u_0}$. Then the action of $G$ on $Y$ is given in the matrix form as follows (we use the homogenous coordinates).

The involution $\sigma_{xy} = (xy) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$;

the 3-cycle $\sigma_{xyz} = (xyz) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}$;

the involution $\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ is the generator of $\mathbb{Z}_2$.

Fixed elements: the point $(1, 0, 0)$.

Invariant elements: the line $L_0 = (u_0 = 0)$. The group $\mathbb{Z}_2$ acts on $L_0$ trivially and $S_3$ acts as the standard irreducible two-dimensional linear representation. A unique $G$-invariant 0-orbit of length 2 is $\{(0, 1, \lambda), (0, \lambda, 1)\}$, where $\lambda = e^{\frac{2\pi i}{3}}$ is the cubic root of 1. There are also two $G$-invariant 0-orbits of length 3, which are given by $\{(0, 0, 1), (0, 1, -1), (0, 1, 0)\}$ and $\{(0, 1, 1), (0, 1, -2), (0, -1, 1)\}$.

There is $G$-equivariant pencil of lines with $G$-fixed point $(1, 0, 0)$ and $G$-invariant section (the line $L_0$). The group $S_3$ acts on the base of this pencil and on the section $L_0$. The group $\mathbb{Z}_2$ acts on the fibers by $t \rightarrow -t$.

The case (II). The surface $X$ is the most natural smooth compactification of two-dimensional torus $T$: $(xyz = 1)$. Consider the threefold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with homogeneous coordinates $(x_1, x_0) \times (y_1, y_0) \times (z_1, z_0)$ and coordinates $(x = \frac{x_1}{x_0}, y = \frac{y_1}{y_0}, z = \frac{z_1}{z_0})$. Then this compactification is given by

$$x_1y_1z_1 = x_0y_0z_0.$$  \hfill (1)

Under the Segre map $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7$ the equation (1) is the $G$-invariant equation of the hyperplane $\mathbb{P}^6 \supset X$. The surface $X$ is given there by the quadratic equations that are restrictions on $\mathbb{P}^6$ of the equations which give the Segre map. Under this embedding $X$ is a smooth projective del Pezzo of degree 6 in $\mathbb{P}^6$.

$G$ acts in (1) as follows. $S_3$ acts by the coordinate permutations induced by the permutations of factors of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (for example $\sigma_{xy}: (x_1, x_0) \longleftrightarrow (y_1, y_0)$). $\mathbb{Z}_2$ acts as follows.

$$(x_1, y_1, z_1) \longleftrightarrow (x_0, y_0, z_0).$$

These actions are linear under the Segre map.
According to the classification of $G$-minimal rational $G$-surfaces (see [Isk2, Mo]), $X \subset \mathbb{P}^6$ is the minimal del Pezzo $G$-surface of degree 6 with $G$-invariant Picard group $\text{Pic}^G(X) = \mathbb{Z}(-K_X)$, where $K_X$ is the canonical class. The configuration of $(-1)$-curves on $X$ is a unique $G$-orbit and have the form of regular 6-angle. Its equation is the infinite hyperplane section

$$x_0y_0z_0 = 0.$$  \hspace{1cm} (2)

These lines are one-dimensional fibers of three birational projections $X \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$, two one-dimensional fibers in each of them.

Further we need other $G$-birational equivalent to $X$ models $X_0, X_1, X_2$ and classification of some 0-dimensional (of small length) and 1-dimensional orbits on them.

1.4. Start from the surface $X$ given by (1). Pic $X$ is generated by $(-1)$-curves and all $(-1)$-curves forms a unique $G$-orbit, which is equivalent to $-K_X$, so $\text{Pic}^G_X = \mathbb{Z}(-K_X)$. We may consider an affine model $T$: $(xyz = 1)$ for studying of $G$-orbits, because the orbit on infinite is known (it is all $(-1)$-curves) and there is not 0-orbit of length less than 6.

1.4.1. Lemma. Let $A \subset T$ be the $G$-equivariant 0-orbit of length $d < 6$. Then $A$ is one of the following.

$d = 1$: a unique fixed point $P = (1, 1, 1)$;

$d = 2$: a unique orbit $\{P_1 = (\lambda, \lambda, \lambda), P_{-1} = (\lambda^{-1}, \lambda^{-1}, \lambda^{-1})\}$, where $\lambda = e^{2\pi i} \frac{\pi}{3}$ is a cubic root of 1 (notice that $\lambda^{-1} = \lambda^2$);

$d = 3$: a unique orbit $\{Q_1 = (1, -1, -1), Q_2 = (-1, 1, -1), Q_3 = (-1, -1, 1)\}$;

$d = 4$: there is no such orbits;

$d = 5$: there is no such orbits, because $5 \nmid 12 = |G|$.

Indeed, one can see this from the equation $xyz = 1$ and the action of $G$ on $T$. In the case $d = 4$ the stabilizer of the point is $\langle \sigma_{xyz} \rangle = \mathbb{Z}_3$ acting by $(x, y, z) \mapsto (y, z, x)$, which means that $x = y = z$.

1.4.2. Now find some of $G$-orbits, which consist of rational curves.

$T = \Gamma_x + \Gamma_y + \Gamma_z \sim -K_X$. This $G$-orbit consists of the curves of genus 0 which contain the fixed point $P = (1, 1, 1)$ and is given by $x = 1, y = 1, z = 1$. In the projective form these equations on $\mathbb{P}^1 \times \mathbb{P}^1$ are as follows.

\begin{align*}
\Gamma_x: & \quad y_1z_1 - y_0z_0 = 0, \\
\Gamma_y: & \quad x_1z_1 - x_0z_0 = 0, \\
\Gamma_z: & \quad x_1y_1 - x_0y_0 = 0.
\end{align*} \hspace{1cm} (3)

Each of these curves $\Gamma$ is smooth one of genus 0 with $-K_X \Gamma = 2, \Gamma^2 = 0$, and each two of them intersect by a unique point $P$. Blow up this point. Then after $G$-equivariant contraction of these curves we have three $G$-conjugate points on the image of exceptional $(-1)$-curve of blow-up on the model $X_2 \subset \mathbb{P}^3$, see below (this is example of $G$-equivariant link $\Phi_{6,1}$, see section 2).

This birational $G$-map $\gamma: X \dashrightarrow X_2$ is a projection from the tangent plane to the fixed point $P$ with the embedding $X \subset \mathbb{P}^6$ described above. Under this projection the point $P$ blows up, and three conics passing through $P$ contract.
\[ D = \Delta_x + \Delta_y + \Delta_z \sim -2K_X. \] This is a triple of smooth curves of genus 0 with equations 
\[ y = z, \quad z = x, \quad \text{and} \quad x = y \] respectively. In the projective form we have 
\[ \Delta_x: x_1y_1^2 - x_0y_0^2 = 0, \]
\[ \Delta_y: y_1z_1^2 - y_0z_0^2 = 0, \]
\[ \Delta_z: z_1x_1^2 - z_0x_0^2 = 0. \] (4)

All these curves \( \Delta \) intersect in three points \( P, P_1 \) and \( P_{-1} \) with \( -K_X \Delta = 4 \) and \( \Delta^2 = 2 \).

1.4.3. Curves \( \Gamma \) and \( \Delta \) are components of the fibers of \( G \)-invariant pencil of rational curves \( \Pi = |-K_X - 2P - P_{-1} - P_1| \) (i.e. the pencil of curves from the linear system \( |-K_X| \) which contain points \( P_1, P_{-1} \) and twice point \( P \)). Moreover, \( \Gamma_x + \Delta_x \sim \Gamma_y + \Delta_y \sim \Gamma_z + \Delta_z \) are the fibers of this pencil. Equations (3) and (4) involve that \( \Gamma \cap \Delta = \{P, Q_1\} \), \( \Gamma y \cap \Delta_y = \{P, Q_2\} \), \( \Gamma_z \cap \Delta_z = \{P, Q_3\} \).

\[ E = E_x + E_y + E_z \sim -K_X. \] This triple of smooth curves of genus 0 is given by \( x = -1, \ y = -1, \ z = -1 \) accordingly. One can see from the equations that \( E_z \supset Q_2, Q_3, E_y \supset Q_1, Q_3, E_x \supset Q_1, Q_2 \), and \( Q_i \) are the only intersection points for the components of \( E \). As above, \( -K_X E = 2, \ E^2 = 0 \), so after blow up of \( Q_1, Q_2 \) and \( Q_3 \) strict transforms \( E_x', E_y', E_z' \) are \( G \)-conjugate triple of \((-2)\)-curves which do not intersect each other. They may be \( G \)-equivariant contracted to three ordinary double points on the cubic surface in \( \mathbb{P}^3 \).

1.5. Consider now \( G \)-birational model \( X_0 \) of \( G \)-surface \( X \), namely, standard projectivisation of torus \( T \subset A^3 \subset \mathbb{P}^3 \)
\[ X_0: xyz = w^3, \] (5)
where \((x, y, z, w)\) are homogenous coordinates on \( \mathbb{P}^3 \). This model differs from \( X \) only on infinity \( w = 0 \). There are not 6 lines as on \( X \) but three ones, which are given by \( x = w = 0, \ y = w = 0 \) and \( z = w = 0 \). There are 3 ordinary double points \((0, 0, 1, 0), (0, 1, 0, 0), \) and \((1, 0, 0, 0)\). \( G \) acts on \( X_0 \) not linear (as on \( X \subset \mathbb{P}^6 \) does). More precisely, \( S_3 \) acts linearly by permutations of \( x, y, z \) and \( Z_2 \) acts by birational involution \((x, y, z, w) \mapsto (x^{-1}, y^{-1}, z^{-1}, w^{-1})\).

However there is another cubic model \( X_1 \subset \mathbb{P}^3 \) such that \( G \) acts linearly on it. There are three ordinary double points as on \( X_0 \). This model is given by \( G \)-equivariant birational projection
\[ p_1: X \subset \mathbb{P}^6 \dashrightarrow \mathbb{P}^3 \supset X_1 \]
from the plane \( \langle Q_1, Q_2, Q_3 \rangle \), which is a linear span of \( Q_1, Q_2, Q_3 \in X \) in \( \mathbb{P}^6 \). As we notice in 1.4 conics \( E_x, E_y, E_z \) contract to the singular points.

Choose a homogenous coordinates \((x, y, z, w)\) in \( \mathbb{P}^3 \) such that the image of \( P \) is a point \((0, 0, 0, 1)\) and \( Z_2 \) acts by involution \((x, y, z, w) \mapsto (-x, -y, -z, w)\). Then \( S_3 \) acts by coordinate permutations as on \( X_0 \). Under the projection \( p_1: X \dashrightarrow X_1 \) three singular points (the images of \( E_x, E_y, \) and \( E_z \)) lie on the infinity \( w = 0 \). The images of conics \( \Gamma_x, \Gamma_y, \) and \( \Gamma_z \) are three lines passing through \( P_0 = p_1(P) \). Thus, \( P_0 \) is the Eckardt point, whose tangent plane is \( G \)-invariant, i.e. it is given by linear equation \( x + y + z = 0 \).

So, the equation of \( X_1 \subset \mathbb{P}^3 \) is the following.
\[ X_1: xyz - aw^2(x + y + z) = 0, \quad a \in \mathbb{C}. \] (6)
We can put \( a = \frac{1}{3} \) for convenience. Then the singular points are \((1, 0, 0, 0), (0, 1, 0, 0),\) 
\((0, 0, 1, 0)\), and the images of \(P_1\) and \(P_{-1}\) are \((1, 1, 1, 1)\) and \((-1, -1, -1, -1)\). Three lines \(x = 0, y = 0, z = 0\) lie in the tangent plane \(x + y + z = 0\).

1.6. **Remark.** As at each cubic surface, there is Geiser’s birational involution concerned with \(P_0\), i.e. the projection from this point to the plane composed with the automorphism of the double covering. Obviously this birational involution is \(G\)-equivariant and in our case it is biregular because \(P_0\) is the Eckardt point. \(G\)-equivariant birational Bertini involution is concerned with two \(G\)-conjugate points \(P_1, P_{-1} \in X\). Its action on Pic \(X_1\) differs from the standard one because \(X_1\) is singular and the points \(P_1, P_{-1}\) are not in the general position. This means that the third intersection point of the line \(\langle P_1, P_{-1} \rangle \subset \mathbb{P}^3\) and \(X_1\) (i.e. \(P_0\)) lies on the \((-1)\)-curves.

1.7. There is also one \(G\)-equivariant birational model of \(X\). Namely, quadric \(X_2 \subset \mathbb{P}^3\) of type

\[
X_2: xy + yz + zx = 3w^2. \tag{7}
\]

It is given from \(X_1\) by \(G\)-equivariant birational transform \((x, y, z, w) \mapsto (x^{-1}, y^{-1}, z^{-1}, w^{-1})\). Indeed, this coordinate change with the multiplication to the common factor transform the equation \([\overline{3}]\) with \(a = \frac{1}{3}\) to the equation \([\overline{1}]\). The action of \(G\) on \((x, y, z, w)\) are still linear. Moreover, \(X_2\) is nothing but the image of \(X \subset \mathbb{P}^6\) under the linear projection

\[
p_2: X \subset \mathbb{P}^6 \longrightarrow \mathbb{P}^3 \supset X_2
\]

from the tangent plane to \(X\) at the \(G\)-fixed point \(P \in X\). The projection \(p_2\) blows up \(P\) to the conic \(C_0: (w = 0) \subset X_2\) and contracts three conics \(\Gamma_x, \Gamma_y, \Gamma_z \subset X\) to the \(G\)-invariant 0-orbit \(A = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)\} \subset C_0\).

The curve \(C_0\) is the image of the point \(P_0 \in X_1\) and the 0-orbit \(A\) is the image of the lines \(x = 0, y = 0, z = 0\) under our map \(X_1 \longrightarrow X_2\).

1.8. **Lemma.**

(i): The variety \(X_2\) is smooth and there is no \(G\)-fixed points on it;

(ii): Conics \(C_0 = (w = 0)\) and \(C_1 = (x + y + z = 0)\) are \(G\)-invariant and on \(C_0\) acts only \(S_3\) by a unique two-dimensional irreducible representation and \(G\) acts effective on \(C_1\);

(iii): \(G\)-invariant 0-orbits of length \(d < 6\) on \(X_2\) are:

\(d = 2\): \(\{P_1 = (1, 1, 1, 1), P_{-1} = (-1, -1, -1)\}\), the images on \(X_2\) of \(P_1\) and \(P_{-1}\) on \(X\) (or \(X_1\)); \(\{R_1 = (1, \lambda, \lambda^{-1}), R_2 = (\lambda, 1, \lambda^{-1})\}\) = \(C_0 \cap C_1\), the basic points of the conic pencil on \(X_2\) generated by \(C_0\) and \(C_1\), \(\lambda = e^{2\pi i}\)

\(d = 3\): \(A = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)\}\),

\(B = \{(-1, 2, 2, 0), (2, -1, 2, 0), (2, 2, -1, 0)\}\), \(A, B \subset C_0\);

\(d = 4\): there is no such orbits;

\(d = 5\): there is no such orbits;

(iv): let

\[
\Pi_0 = (t_0(x + y + z) + t_1w = 0), \quad (t_0, t_1) \in \mathbb{P}^1,
\]

be the pencil of conics generated by \(C_0\) and \(C_1\); then on the base of this pencil \(\mathbb{P}^1\) acts only \(Z_2 = G/S_3: (t_0, t_1) \mapsto (-t_0, t_1)\) with two fixed points \((1, 0)\) and \((0, 1)\); there is two reducible fibers \(F_1\) and \(F_{-1}\) on \(\Pi_0\) with intersection points \(P_1\) and \(P_{-1}\) of the components with coordinates \((1, -3)\) and \((1, 3)\) accordingly; on the irreducible fibers \(G\) acts without fixed points;
**v:** there is another pencil of conics

\[ \Pi_1 = \{ u_0(x - y) + u_1(y - z) + u_2(z - x) = 0; \ u_0 + u_1 + u_2 = 0, \ (u_0, u_1, u_2) \in \mathbb{P}^2 \} \]

passing through \( P_1, P_{-1} \in X_2; \) it is an image of \( \Pi \) from clause [1.4.3] under the projection \( p_2: X \rightarrow X_2; \) on the base of this pencil \( \mathbb{P}^1 = (u_0 + u_1 + u_2 = 0) \) as usual acts \( S_3 \) by an two-dimensional irreducible representation, \( \mathbb{Z}_2 \) acts on each fiber with fixed points \( \Pi_1 \cap C_0 \) and invariant 2-section \( \Pi_1 \cap C_1. \)

There is two reducible fibers \( G_1 \) and \( G_2 \) on \( \Pi_1 \) with intersection points of the components \( R_1 \) and \( R_2 \) with coordinates \( (\lambda^{-1}, 1, \lambda) \) and \( (\lambda^{-1}, 1, \lambda, 1) \) on the base accordingly.

All these statements can be straightforward check using the equations and the action of \( G. \)

2. **Proof of the main result**

2.1. The idea of the proof of proposition [1.1.1] is the same as in the [Isk1]. We should show that there is no \( G \)-equivariant birational map between \( G \)-surfaces \( X \) and \( Y = \mathbb{P}^2 \), defined in section [1.3] According to the general theory (see [Isk3]), each \( G \)-equivariant birational map can be decomposed (by algorithm given by \( G \)-equivariant Sarkisov program) to the sequence of \( G \)-equivariant elementary birational maps-links. The classification of such links is obtained in the algebraic case if \( G \) is finite Galois group, acting on smooth \( k \)-minimal rational surfaces over the perfect field \( k \) (see [Isk3]).

However, according to the general conception of rational \( G \)-surfaces (see [Ma1], [Isk2]), this classification may be applied to the geometrical case, i.e. to the case of action of the finite group \( G \) on smooth projective minimal rational \( G \)-surfaces. Their classification is also known (see [Isk2] and \( G \)-equivariant 2-dimensional Mori theory [Mo]).

Some differences between geometrical and algebraical cases concerned with the description of \( G \)-orbits. For example, in the geometrical case (with non-trivial \( G \)-action) \( G \)-orbits are closed proper subset. In the algebraical case, for instance, for del Pezzo surfaces of great degrees, the existence of one \( k \)-point (orbit of degree 1) involves everywhere density of such \( k \)-points in the Zariski topology.

The theorem of decomposition of any birational \( G \)-map to a sequence of links is known (see [Isk3], 2.5) and all links are determined by their centers, i.e. \( G \)-orbits of dimension 0, all points of which are in the general position on the corresponding \( G \)-surface. So, for proof of the main result we need the following.

a) Choose from the general classification (see [Isk3], 2.6) the links which is concerned with our case. Indicate minimal \( G \)-surfaces on which acts these links (as \( G \)-equivariant birational maps), starting from \( X \).

b) Classify all zero-dimensional \( G \)-orbits of length \( d < \deg X_i = K^2_{X_i} \) on all of such surfaces \( X_i \) and choose those of them, whose points are in the general position (this is necessary condition for the existence of links).

Remind, that the points \( x_1, x_2, \ldots, x_d \in X \) are in the general position, if

1) \( X \) is del Pezzo surface, \( \sigma: X' \rightarrow X \) is the simultaneous blow up of all \( x_1, \ldots, x_d \) and \( X' \) is also del Pezzo surface, or

2) \( \pi: X \rightarrow \mathbb{P}^1 \) is a conic bundle, then none of this points lye on the degenerated fibers and at most one point of \( x_1, \ldots, x_d \) lies on the fiber; this is necessary and
sufficient condition of the existence of link with center in $x_1, \ldots, x_d$, i. e. the birational transform

\[
X \dashrightarrow \X' \to \PP^1
\]

which is blow up of $x_1, \ldots, x_d$ and a contraction of strict transforms of the fibers on which they lie.

2.2. Suppose that there exists $G$-equivariant birational map $\chi: X \to Y$. Then $\chi$ can be decomposed to the composition of links. According to the particular algorithm of decomposition and considering all possibilities for links we will see that there is no link whose image is $\PP^2$. So we will obtain a contradiction with the decomposition theorem. So there is no such $\chi$, which means that two embeddings $G$ to the Cremona group that we define in (I) and (II), section 1.1, are not conjugate in $Cr_2(\C)$.

By construction from the algorithm choose the very ample $G$-invariant linear system $\mathcal{H}' = \mathcal{H}_Y$ on $Y$, for example, $\mathcal{H}' = | - K_Y |$. Let $\mathcal{H} = \mathcal{H}_X = \chi_*^{-1} \mathcal{H}'$ be its strict transform on $X$. By the definition of strict transform the linear system $\mathcal{H}$ has no fixed components and $\dim \mathcal{H} = \dim \mathcal{H}'$ ($\mathcal{H}$ has basic points if $\chi$ is not morphism). $\Pic^G(X) = \ZZ(-K_X)$, so $\mathcal{H} \sim -aK_X$, $a \in \ZZ_{>0}$.

The map $\chi: X \to Y$ is not isomorphism, so, by $G$-equivariant Noether inequality (see [Isk3], 2.4), linear system $\mathcal{H}$ has maximal singularity, which is a zero-dimensional $G$-orbit $x \in X$ of length $d = d(x)$ and multiplicity $\mult_2 \mathcal{H} = r = r(x) > a$ at all points $x_i \in x$. We have $\mathcal{H}^2 = a^2 K_X^2 > r^2 d$ ($\mathcal{H}$ is mobile linear system without fixed components, passes through all points of $G$-orbits $x$ with multiplicity $r$, and determines a birational map) so $d < K_X^2$, in our case $d < K_X^2 = 6$. The length of the orbit is divided by the order of the group. So, the maximal singularities by which we can construct links can be only $G$-orbits of length $d = 1, 2, 3$ or $4$. We find such orbits in lemma [4.4].

2.3. Now, for every such $G$-orbit we should check are its points in the general position or not and if yes, then choose from the classification ([Isk3], 2.6) appropriate link $\Phi: X \dashrightarrow X_1$ (do not confuse with $X_1$ from paragraph 1.5). Algorithm of decomposition is the following. The map $\chi_1: X_1 \dashrightarrow Y$ in the composition $\chi = \chi_1 \circ \Phi$ is given by the linear system $\mathcal{H}_1 = \Phi_*(\mathcal{H}) \sim -a_1 K_{X_1}$ with $a_1 < a$. We have $a \in \ZZ_{>0}$, so the decomposition should be terminated on the isomorphism after the finite number of steps. Start to find links. There is no $G$-orbit of length $4$ on $X$ (see [4.4]), so start from $d = 3$.

The case $d = 3$. There is one such orbit $(Q_1, Q_2, Q_3)$. Show that in this case there is no such “untwisting” link $\Phi$. Indeed, the points $Q_1, Q_2, Q_3$ on $X$ are not in the general position. On the blow up of these three points $\sigma: X' \to X$ strict transforms $E'_x, E'_y, E'_z$ of conics $E_x, E_y, E_z$ (see 1.4.2) are $G$-invariant triple of $(-2)$-curves, so $-K_X$ is not ample $(-K_X \cdot E'_y = 0)$.

(As we see in paragraph 1.5, the projection $p_1: X \dashrightarrow X_1$ from the plane $(Q_1, Q_2, Q_3) \subset \PP^6 \dashrightarrow \PP^3$ blows up these three points and contracts $E_x, E_y, E_z$ to the ordinary double points on $X_1$.)

Notice that in [Isk1] we mistakenly say that there is such link $\Phi_{6,3}$. The mistake is the following. We do not check the condition of generality (though this is not affect to the final result).
2.3.1. Remark. The matter why the link $\Phi_{6,3}$ does not exists is easy. For existence of link with center in the maximal singularity $x \subset X$ it is necessary and sufficient that under the blow up $\sigma: X' \to X$ of the cycle $x$ on the surface $X'$ there is two extremal rays, one of which is the exceptional divisor. Pic$^G(X') = \mathbb{Z} \oplus \mathbb{Z}$, so the Mori cone is generated by these extremal rays and $-K_{X'}$ is ample by Kleiman’s criterion. The link $\Phi$ is nothing but the blow up $\sigma$ and the extremal contraction of this second ray.

In the classical terms the non-generality of position of $Q_1$, $Q_2$ and $Q_3$ means that this $G$-orbit could not be a maximal singularity in the linear system $\mathcal{H}$, because if $r > a$, then $E_x$, $E_y$ and $E_z$ are fixed components of $\mathcal{H}$. Indeed, $\mathcal{H} \subset |{-}aK_X - rQ_1 - rQ_2 - rQ_3|$, the cycles $E_x - Q_2 - Q_3$, $E_y - Q_1 - Q_3$, $E_z - Q_1 - Q_2$ are effective and must have non-negative intersection with $\mathcal{H}$. But $\mathcal{H} \cdot (E_x - Q_2 - Q_3) = (-aK_x - rQ_1 - rQ_2 - rQ_3) \cdot (E_x - Q_2 - Q_3) = 2a - 2r < 0$.

So we have a contradiction.

2.3.2. The case $d = 2$. There is a unique such orbit $x = \{P_1, P_{-1}\}$ (see [14, 1]). The pair $P_1$, $P_{-1}$ is in the general position on $X$. Indeed, each of these points does not lie on the $(-1)$-curves (which lie on the infinity $x_0y_0z_0 = 0$), and there is no curve of genus 0 and degree 2 (with respect to $-K_X$) on which both of these points lie.

There is corresponding link $\Phi = \Phi_{6,2}: X \dashrightarrow X_1$ (see [Isk3], 2.6). Find out what is $X_1$. Let $\sigma: X' \to X$ be the blow up of $x = \{P_1, P_{-1}\}$. Then $-K_{X'}$ is ample and $K_{X'}^2 = 4$. The linear system $|{-}K_{X'}|$ gives $G$-equivariant embedding $X' \subset \mathbb{P}^4$ into the del Pezzo surface of degree 4 (i.e. into an intersection of two quadrics). The image of the exceptional $(-1)$-curves are two skew lines whose linear span is a hyperplane $\mathbb{P}^3$. This image is $G$-invariant and intersects $X' \subset \mathbb{P}^4$ by another $G$-invariant pair of skew lines which is reducible curve of genus 1 together with the pair that we blow up.

Contract this residual pair of lines. We get del Pezzo surface $X_1 \subset \mathbb{P}^6$ of degree 6.

In the general algebraic case the surface $X_1$ may be not isomorphic to $X$, i.e. $\Phi_{6,2}$ is not always a birational involution (but I do not know such examples). However, in our particular case $\Phi_{6,3}$ is a birational involution of Bertini type. Indeed, if $\sigma: X'' \to X' \to X$ is a blow up of $G$-orbit $\{Q_1, Q_2, Q_3\}$ together with $\{P_1, P_{-1}\}$, then $K_{X'}^2 = 1$. But $-K_{X''}$ is not ample (because it is degenerated del Pezzo surface of degree 1 with three $(-2)$-curves, or, after their contraction, three ordinary double points). Nevertheless, the linear system $|{-}2K_{X''}|$ gives contraction of $(-2)$-curves composed with double covering $X'' \to Q^* \subset \mathbb{P}^3$ of the quadric cone branched over a fixed point and a curve on $Q^*$ with three ordinary double points.

The involution of this double covering $X'' \to X$ induces the birational involution $\Phi_{6,2}$. It differs from the standard Bertini involution given by the Picard group, because 5 blowing up points are not in the general position. The point $P$ and ordinary double points are still fixed under this involution. It also induces the involution on the cubic surface $X_1$ that we consider in remark 1.6.

The birational involution $\Phi_{6,2}$ maps the pencil of rational curves $\Pi = |{-}K_X - 2P - P_1 - P_{-1}|$ onto itself and changes the components of its three fibers: $\Gamma_x \leftrightarrow \Delta_x$, $\Gamma_y \leftrightarrow \Delta_y$, $\Gamma_z \leftrightarrow \Delta_z$ (see 1.4.3).

Apparently, we can write the particular equation for the residual $G$-invariant pair of the curves of genus 0 on $X$ (but we do not need that). This 3-linear 3-homogenous equation should be as follows.
The curve $H \sim -K_X$ is $G$-invariant, passes through $P_1$, $P_{-1}$, and have the ordinary double singularities in them. This means that it is reducible.

The link $\Phi_{6,2}: X \rightarrow X$ maps the linear system $H \sim -aK_X$ with the maximal singularity $x = \{P_1, P_{-1}\}$ of multiplicity $r = r(x)$ to the linear system $\mathcal{H}_1 \sim -a_1 K_X$ with the basic cycle $x_1 = x = \{P_1, P_{-1}\}$ with the multiplicity $r_1 < a_1$ (this is not maximal singularity for $\mathcal{H}_1$). The formulas for the coefficients are given in \cite{Isk3}, theorem 2.6, case $K_X^2 = 6$, case d):

$$a_1 = 2a - r(x) < a, \quad r_1 = 3a - 2r(x) < a_1. \quad (8)$$

So, $x = \{P_1, P_{-1}\}$ already is not the maximal singularity for $\mathcal{H}_1$. The map $\chi_1 = \chi \circ \Phi^{-1}: X \rightarrow Y$ defined by $\mathcal{H}_1$ is not isomorphism, so, by lemma–Noether inequality, $\mathcal{H}_1$ must have the maximal singularity. Now it can be only the point $P$ of multiplicity $r_1 = r(P) > a_1$.

2.3.3. The case $d = 1$. The only fixed point $x_1 = P$. It does not lie on the $(-1)$-curves, so it is in a general position. The corresponding link $\Phi_{6,1}: X \rightarrow X_2$, where $X_2 \subset \mathbb{P}^3$ is a smooth quadric with $\text{Pic}^G(X_2) = \frac{1}{2}\mathbb{Z}(-K_X)$ which we consider in section 1.7, is the birational projection $p_2: X \rightarrow X_2 \subset \mathbb{P}^3$ from the tangent plane in $\mathbb{P}^6$ to $X$ at $P$. This link blows up the point $P$ and contracts three conics $\Gamma_x, \Gamma_y, \Gamma_z$ to the $G$-orbit $A$ of length 3 on the conic $C_0 \subset X_2$ which is the image of blowing up $P$. The stabilizer of $P \in X$ is the whole $G$ which acts on the projectivization of tangent plane to $C_0$ as $S_3$ by the standard irreducible 2-dimensional linear representation. We completely considered this situation in the lemma 1.8.

There is $G$-invariant curve $C_0 \sim -\frac{1}{2}K_{X_2}$, which generates $\text{Pic}^G(X_2)$ on $X_2$. This means that, under the notations from 1.8, the formulas for the action of $\Phi_{6,1}$ are the following.

$$-K_X \mapsto -\frac{3}{2}K_{X_2} - 2A \sim 3C_0 - 2A,$$

$$P \mapsto -\frac{1}{2}K_{X_2} - A \sim C_0 - A, \quad (9)$$

$$\mathcal{H}_1 \subset \{ -a_1 K_X - r_1 P \} \mapsto \mathcal{H}_2 \subset \{ -a_2 K_{X_2} - r_2 A \} \quad \text{where}$$

$$a_2 = \frac{3}{2}a_1 - \frac{1}{2}r_1, \quad r_2 = 2a_1 - r_1.$$ 

Under our hypothesis of maximality of the singularity in $P$ we have $r_0 = \text{mult}_P \mathcal{H}_1 > a_1$, so $a_2 < a_1$ and $r_2 < a_2$. This means that the $G$-orbit $A$ is not the maximal singularity for $\mathcal{H}_2$.

2.4. Now we are on the quadric $X_2 \subset \mathbb{P}^3$ with linear system $\mathcal{H}_2 \sim -a_2 K_{X_2}$. Denote this system by $\mathcal{H}$ and put $a = a_2$ for simplicity. The map $\chi_2: X_2 \rightarrow Y = \mathbb{P}^2$ is not isomorphism, so, by Noether inequality, there is a maximal singularity, $G$-orbit $x$ of multiplicity $r = r(x) > a$ and of length $d < K_{X_2}^2 = 8$. $d|12 = |G|$ so the only possibilities for $d$ are $d = 1, 2, 3, 4, 6$. We can exclude the cases $d = 1$ and $d = 4$ by lemma 1.8. So consider the other cases $d = 2, 3, 6$. 

$$H: \sum_{g \in G} g((x_1 - \lambda x_0)(y_1 - \lambda y_0)(z_1 - \lambda^2 z_0) + (x - \lambda x_0)(y_1 - \lambda^2 y_0)(z_1 - \lambda^2 z_0)) = 0.$$
2.4.1. The case $d = 2$. There is two $G$-orbits $\{R_1, R_2\}$ and $\{P_1, P_{-1}\}$ (see 1.8). Start from $x = \{R_1, R_2\} = C_0 \cap C_1$ of multiplicity $r = r(x) > a$. Such $R_1$, $R_2$ do not lie on one line on the quadric $X_2$, because each line intersects $C_0$ by a unique point ($C_0$ is the hyperplane section). So, $R_1$ and $R_2$ are in the general position. There exists link $\Phi_{8,2}$ (see [Isk3], 2.6, the case $K_X^2 = 8$, a), $d = 2$), which is a blow up $X_2 \to X_1$ of the points $\{R_1, R_2\}$. There is $G$-invariant structure of conic bundle $\pi : X_1 \to \mathbb{P}^1$ on $X_1$, which is given by the pencil $\Pi_0$ from 1.8 (iv). Let $\mathcal{H}_1$ be the strict transform on $X_1$ of linear system $\mathcal{H}$. Then $\mathcal{H}_1 \sim -a_1 K_{X_1} + b_1 f_1$, where $f_1 \cong C'_0$ is $G$-invariant fiber. We have

$$K_{X_1}^2 = 6, \quad -K_{X_1} = E + C'_0 + C'_1 \sim E + 2f_1,$$

where $E$ is the exceptional divisor on $X_1 \to X_2$, i.e. $G$-invariant pair of $(-1)$-curves and $C'_0$, $C'_1$ are strict transforms of $C_0$, $C_1$. By [Isk3], 2.6

$$a_1 = 2a - r(x) < a,$$

$$b_1 = 2(r(x) - a).$$

From [Isk3] one can see that $a_1 < a$ and $b_1 > 0$. This means that by the Noether inequality (because $X_1 \not\cong Y$) there is a maximal singularity in $\mathcal{H}_1$. But:

2.4.2. Lemma. There is no zero-dimensional $G$-orbits $x_1$ of length $d_1$ on $X_1$, whose points are in the general position in the sense of 2.1 2), i.e. there is no point in it that lies on the degenerated fiber and there are no two points that lie on the same fiber of morphism $\pi : X_1 \to \mathbb{P}^1$ (this is a condition of existence of “untwisting” link for $\mathcal{H}_1$ on the conic bundle with $b_1 \geq 0$).

Proof. By lemma 1.8 (iv), a cyclic group of order 2 acts on $\mathbb{P}^1$, so for generality the length $d_1$ must be less or equal than 2. There is no $G$-fixed points on $X_1$ (as on $X_2$) The orbit $\{P_1, P_{-1}\}$ of length 2 by lemma 1.8 lies on the degenerated fibers. On the non-degenerated fibers acts $S_3$ without fixed points.

So, the decomposition by the algorithm in this situation is impossible. This means that $\{R_1, R_2\}$ is not maximal, and we need to come back to the quadric $X_2$. Consider the other possibilities.

2.4.3. The case $d = 2$ with maximal singularity $x = \{P_1, P_{-1}\}$, $r(x) > a$. As in 2.4.1, the pair $\{P_1, P_{-1}\}$ is in the general position and there exists link $\Phi_{8,2}$ (see [Isk3], theorem 2.6, the case $K_X^2 = 8$, c), $d = 2$), which is the blow up $X \to X_1$ of $P_1$, $P_{-1}$. There is a structure of conic bundle $\pi : X_1 \to \mathbb{P}^1$ on $X_1$ that is given by the pencil $\Pi_1$ from 1.8 (v).

The difference from the previous case is the following. The action of $G = S_3 \times \mathbb{Z}_2$ on $\pi : X_1 \to \mathbb{P}^1$ is given by two commuting actions. $S_3$ acts on the base $\mathbb{P}^1$ without fixed points and this action can be shifted to $X_1$ as the changing of the fibers. $\mathbb{Z}_2$ acts on $X_1 \to \mathbb{P}^1$ only on the fibers. The fixed points in this case are the intersection of the fibers with the curve $C'_0 \subset X_1$, which is the pre-image of the curve $C_0 \subset X_2$. So, $\mathbb{Z}_2$ acts as the classical De Jonquieres involution. The fixed curve $C'_0$ determines $\pi : X_1 \to \mathbb{P}^1$. Indeed, the factor by this involution is the ruled surface $\mathbb{F}_1 \to \mathbb{P}^1$ branched over $C'_0$. The pair of the $(-1)$-curves $E$ covers the exceptional section $S_1 \subset \mathbb{F}_1$.

Notice that there is no fixed points on the base $\mathbb{P}^1$ (so there is no $G$-invariant fiber as in the previous case). But the class of the fiber $[f_1]$ in $\text{Pic}^G(X)$ is $G$-invariant, so the action of this link is the same as in [Isk3].
By the Noether inequality there must be the maximal singularity in the linear system \( \mathcal{H} \sim -a_1K_{X_1} + b_1f_1 \) which is \( G \)-orbit \( x_1 \subset X_1 \) of multiplicity \( r_1 = r(x_1) > a_1 \), whose points are in the general position in the sense of 2.4.2.

There is no \( G \)-fixed points on \( X_1 \) and the only \( G \)-invariant orbit of length 2, \( \{ R_1, R_2 \} \), lies in the degenerated fibers (see 1.8 (v)).

The both \( G \)-orbits of length 3, \( A \) and \( B \), satisfy the generality condition. Both of them lie on \( C'_0 \) and they change under the involution of double covering \( \pi|_{C'_0} : C'_0 \to \mathbb{P}^1 \) (do not confuse with involution of the action of \( \mathbb{Z}_2 \)).

There are also \( G \)-orbits of length 6. By the generality condition they lie also on \( C'_0 \), one point on the fiber. The other intersection points of \( C'_0 \) with fibers are complementary \( G \)-orbit of length 6.

There is an “untwisting” link concerned with every maximal singularity \( x_1 \subset X_1 \) of length 3 or 6. That is, the elementary transform

\[
\Phi : X_1 \xrightarrow{\mathcal{E}_{x_1}} X'_1.
\]

Remind that the elementary transform blows up the points of orbit \( x_1 \subset X_1 \) and contracts the strict transforms of the fibers on which these points lie. In the general case an elementary transform is not a birational involution. But in our case it is. Indeed, the transform \( \mathcal{E}_{x_1} \) induces the isomorphism on the curve \( C'_0 \), and, as we notice above, \( C'_0 \) completely determines \( \pi : X_1 \to \mathbb{P}^1 \), so \( X'_1 = X_1 \) in (11).

The birational involution \( \Phi = \mathcal{E}_x \) acts by the following formulas from [Isk3], 2.6.

\[
\begin{align*}
-K_{X_1} &\mapsto -K_{X_1} + d_1f_1 - 2x_1, & a'_1 &= a_1, \\
f_1 &\mapsto f_1, & b'_1 &= b_1 + d_1(a_1 - r_1), \\
x_1 &\mapsto d_1f_1 - x_1, & r'_1 &= 2(r_1 - a_1),
\end{align*}
\]

where \( d_1 \) is the length of the orbit \( x_1 \). As the last formula shows, the multiplicity of the maximal singularities decreases. After the finite number of such transforms we can “untwist” all maximal singularities (which means that the multiplicities of the base points in the modified linear system will be not greater than \( a \)). Then, by the Noether inequality, the coefficient at \( b_1 \) at the fiber will be less than 0.

In this situation the link \( \Phi_{8,2}^{-1} : X_1 \to X_2 \) (the inverse to \( \Phi_{8,2} \)) contracts \( G \)-invariant pair of \((-1)\)-curves \( E \) (the sections of the pencil \( \pi : X_1 \to \mathbb{P}^1 \)) to \( G \)-orbit of length two \( x = \{ P_1, P_{-1} \} \). It acts by the following formulas.

\[
\begin{align*}
-K_{X_1} &\mapsto -K_{X_2} - x, & a &= a_1 + \frac{2}{3}b_1, \\
2f_1 &\mapsto -K_{X_2} - 2x, & r(x) &= a_1 + b_1.
\end{align*}
\]

These formulas involves \( a < a_1 \) and \( r(x) < a \), because \( b_1 < 0 \). This means that this link is untwisted and \( x = \{ P_1, P_{-1} \} \) already is not the maximal singularity.

2.5. The last cases are those whose singularities of \( \mathcal{H} \) are \( G \)-orbits of length 3 or 6. There are only two orbits of length 3: \( A \) and \( B \) (see 1.8). First consider
2.5.1. **The case** $d = 6$. Suppose that there is a maximal singularity, $G$-orbit $x$ of length 6, all whose points are in the general position and the multiplicity $r = r(x) > a$, where $\mathcal{H} \sim -aK_{X_2}$.

Then there is an untwisting link $\Phi_{8,6}$, the Geiser’s involution (see [Isk3], 2.6). Indeed, if we blow up $x$, then we get del Pezzo surface of degree 2, on which the classical Geiser’s involution acts biregular. Apply this involution and contract the blowing up divisor. The link $\Phi_{8,6}$ acts by the following formulas.

\[ a_1 = 7a - 6r(x) < a, \]
\[ r(x_1) = 8a - 7r(x) < a_1. \]

2.5.2. **The case** $d = 3$. If the maximal singularity is $x = A$, then the link $\Phi_{8,3}$ with center in this singularity is the inverse map $\Phi_{6,3}^{-1} : X_2 \to X$ and we are in the situation from which we start. If the maximal singularity is $x = B$ (we missed this case in [Isk1]), then the link $\Phi_{8,3} : X_2 \to X'$ acts from $X_2$ to del Pezzo surface of degree 6 $X'$. But $X' \cong X$, because the birational involution on $X$ with the center in $\{P_1, P_{-1}\}$ that we consider in 2.3.2 changes $\{\Gamma_2, \Gamma_y, \Gamma_z\} \leftrightarrow \{\Delta_x, \Delta_y, \Delta_z\}$. Thus, it changes the cycles $A$ and $B$ on $C_0$ that correspond to these directions in the tangent plane to $P \in X$.

So, we consider all possibilities but have not found any $G$-equivariant $G$-map $X \to Y = \mathbb{P}^2$. So there is no such map, which prove the proposition 1.1.1.

2.6. **Commentary.** In the proof we determine that the smooth relative minimal $G$-models on which there exists $G$-equivariant map from $X$ are only the quadric $X_2$ or the conic bundle with two different actions of $G$. It is interesting to find by this method all of $G$-surfaces on which $Y = \mathbb{P}^2$ may be $G$-equivariant mapped.

It is also interesting the following. If we consider only $S_3$ instead of $G$ with the similar actions on $X$ and $Y$, then these surfaces are $S_3$-birational equivalent. Indeed, in this case there are three fixed points $P$, $P_1$ and $P_{-1}$ on $X$. Therefore, there are two fixed points $P_1$, $P_{-1}$ on the quadric $X_2$. The stereographic projection from one of them is the required $G$-equivariant birational isomorphism, so both of these embeddings of $S_3$ it $Cr_2(\mathbb{C})$ are conjugate. From the other hand, the involution $(x, y) \to (x^{-1}, y^{-1})$ of course is conjugate to the linear one.

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