Hypoelliptic Dunkl equations in the space of distributions on $\mathbb{R}^d$

Khalifa TRIMÈCHE
Faculty of Sciences of Tunis, Department of Mathematics,
CAMPUS, 1060 Tunis, Tunisia
email : khlifa.trimeche@fst.rnu.tn

Abstract
In this paper we define and study the Dunkl convolution product and the Dunkl transform on spaces of distributions on $\mathbb{R}^d$. By using the main results obtained, we study the hypoelliptic Dunkl convolution equations in the space of distributions.

Keywords : Dunkl intertwining operator; Dual Dunkl intertwining operator; Dunkl translation operator; Dunkl convolution product; Dunkl transform; Hypoelliptic distributions.

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1 Introduction

We consider the differential-difference operators $T_j, j = 1, 2, \cdots, d$, on $\mathbb{R}^d$ introduced by C.F.Dunkl in [5]. These operators are very important in pure mathematics and in Physics. They provide a useful tool in the study of special functions with root systems [6,9,4]. Moreover the commutative algebra generated by these operators has been used in the study of certain exactly solvable models of quantum mechanics, namely the Calogero-Sutherland-Moser models, which deal with systems of identical particles in a one dimensional space (see [10,15,16]).

C.F.Dunkl has proved in [7] that there exists a unique isomorphism $V_k$ from the space of homogeneous polynomials $\mathcal{P}_n$ on $\mathbb{R}^d$ of degree $n$ onto itself satisfying the transmutation relations

$$T_j V_k = V_k \frac{\partial}{\partial x_j}, \quad j = 1, 2, \cdots, d, \quad (1.1)$$

and

$$V_k(1) = 1. \quad (1.2)$$
This operator is called Dunkl intertwining operator. It has been extended to an isomorphism from $\mathcal{E}(\mathbb{R}^d)$ (the space of $C^\infty$-functions on $\mathbb{R}^d$) onto itself satisfying the relations (1.1) and (1.2) (see[23]).

The operator $V_k$ possesses the integral representation

$$\forall x \in \mathbb{R}^d, \quad V_k(f)(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y), \quad f \in \mathcal{E}(\mathbb{R}^d), \quad (1.3)$$

where $\mu_x$ is a probability measure on $\mathbb{R}^d$ with support in the closed ball $B(0, \|x\|)$ of center 0 and radius $\|x\|$ (see [19, 23]).

We have shown in [23] that for each $x \in \mathbb{R}^d$, there exists a unique distribution $\eta_x$ in $\mathcal{E}'(\mathbb{R}^d)$ (the space of distributions on $\mathbb{R}^d$ of compact support) with support in $B(0, \|x\|)$ such that

$$V_k^{-1}(f)(x) = \langle \eta_x, f \rangle, \quad f \in \mathcal{E}(\mathbb{R}^d).$$

We have studied also in [23] the transposed operator $tV_k$ of the operator $V_k$. It has the integral representation

$$\forall y \in \mathbb{R}^d, \quad tV_k(f)(y) = \int_{\mathbb{R}^d} f(x) d\nu_y(x). \quad (1.4)$$

where $\nu_y$ is a positive measure on $\mathbb{R}^d$ with support in the set $\{x \in \mathbb{R}^d/ \|x\| \geq \|y\|\}$ and $f$ in $\mathcal{D}(\mathbb{R}^d)$ (the space of $C^\infty$-functions on $\mathbb{R}^d$ with compact support). This operator is called Dual Dunkl intertwining operator.

We have proved in [23] that the operator $tV_k$ is an isomorphism from $\mathcal{D}(\mathbb{R}^d)$ onto itself, satisfying the transmutation relations

$$\forall y \in \mathbb{R}^d, \quad tV_k(T_j f)(y) = \frac{\partial}{\partial y_j} tV_k(f)(y), \quad j = 1, 2, \ldots, d. \quad (1.5)$$

Using the operator $V_k$, C.F.Dunkl has defined in [7] the Dunkl kernel $K$ by

$$\forall x \in \mathbb{R}^d, \forall z \in \mathbb{C}^d, \quad K(x, -iz) = V_k(e^{-i\langle \cdot, z \rangle})(x). \quad (1.6)$$

Using this kernel C.F.Dunkl has introduced in [7] a Fourier transform $F_D$ called Dunkl transform.

By using the operators $V_k$ and $tV_k$ we have defined in [25] the Dunkl translation operators and we have determined their properties. With the aid of these operators we define and study in this work the Dunkl convolution product on spaces of distributions. We present also the properties of the Dunkl transform of distributions. The results obtained have permitted to characterize the hypoelliptic Dunkl convolution equations in the space of distributions in terms of their Dunkl transform. This characterization was first given by L.Ehrenpreis [8] and next by L.Hörmander [11] in the case of the classical Fourier transform on $\mathbb{R}^d$. In [1][2] the authors have studied this characterization for the Hankel, Jacobi and Chébli-Trimèche transforms. We remark that their proof of the existence of a parametrix is complicated. In this paper we give a very simple proof of this result for the Dunkl transform on $\mathbb{R}^d$, which can also be applied to the cases of the preceding transforms.
2 The eigenfunction of the Dunkl operators

In this section we collect some notations and results on Dunkl operators and the Dunkl kernel (see [6, 7, 12, 13, 14]).

2.1 Reflection Groups, Root Systems and Multiplicity Functions

We consider $\mathbb{R}^d$ with the euclidean scalar product $\langle \cdot, \cdot \rangle$ and $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$. On $\mathbb{C}^d$, $\| \cdot \|$ denotes also the standard Hermitian norm, while $\langle z, w \rangle = \sum_{j=1}^d z_j w_j$.

For $\alpha \in \mathbb{R}^d \setminus \{0\}$, let $\sigma_\alpha$ be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to $\alpha$, i.e.

$$\sigma_\alpha(x) = x - \left( \frac{2\langle \alpha, x \rangle}{\| \alpha \|^2} \right) \alpha.$$  \hfill (2.1)

A finite set $R \subset \mathbb{R}^d \setminus \{0\}$ is called a root system if $R \cap \mathbb{R} \alpha = \{ \pm \alpha \}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. For a given root system $R$ the reflections $\alpha_\alpha, \alpha \in R$, generate a finite group $W \subset O(d)$, the reflection group associated with $R$. All reflections in $W$ correspond to suitable pairs of roots. For a given $\beta \in \mathbb{R}^d \setminus \cup_{\alpha \in R} H_\alpha$, we fix the positive subsystem $R_+ = \{ \alpha \in R; \langle \alpha, \beta \rangle > 0 \}$, then for each $\alpha \in R$ either $\alpha \in R_+$ or $-\alpha \in R_+$.

A function $k : R \to \mathbb{C}$ on a root system $R$ is called a multiplicity function if it is invariant under the action of the associated reflection group $W$. If one regards $k$ as a function on the corresponding reflections, this means that $k$ is constant on the conjugacy classes of reflections in $W$. For abbreviation, we introduce the index

$$\gamma = \gamma(R) = \sum_{\alpha \in R_+} k(\alpha).$$  \hfill (2.2)

Moreover, let $\omega_k$ denotes the weight function

$$\omega_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)}.$$  \hfill (2.3)

which is $W$-invariant and homogeneous of degree $2\gamma$.

For $d = 1$ and $W = \mathbb{Z}_2$, the multiplicity function $k$ is a single parameter denoted also $k$ and

$$\forall x \in \mathbb{R}, \quad \omega_k(x) = |x|^{2k}.$$  \hfill (2.4)

We introduce the Mehta-type constant

$$c_k = \left( \int_{\mathbb{R}^d} e^{-\|x\|^2} \omega_k(x) dx \right)^{-1}.$$  \hfill (2.5)

which is known for all Coxeter groups $W$ (see [5, 9]).
2.2 Dunkl Operators and Dunkl kernel

The Dunkl operators $T_j, j = 1, \cdots, d,$ on $\mathbb{R}^d,$ associated with the finite reflection group $W$ and the multiplicity function $k,$ are given for a function $f$ of class $C^1$ on $\mathbb{R}^d$ by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{x \in \mathbb{R}_+} k(\alpha) \alpha_j \langle f(x) - f(\sigma_\alpha(x)), \alpha \rangle. \quad (2.6)$$

In the case $k = 0,$ the $T_j, j = 1, 2, \cdots, d,$ reduce to the corresponding partial derivatives. In this paper, we will assume throughout that $k \geq 0$ and $\gamma > 0.$

For $f$ of class $C^1$ on $\mathbb{R}^d$ with compact support and $g$ of class $C^1$ on $\mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} T_j f(x) g(x) \omega_k(x) dx = -\int_{\mathbb{R}^d} f(x) T_j g(x) \omega_k(x) dx, \quad j = 1, 2, \cdots, d. \quad (2.7)$$

For $y \in \mathbb{R}^d,$ the system

$$\begin{cases}
T_j u(x, y) = y_j u(x, y), \quad j = 1, 2, \cdots, d, \\
u(0, y) = 1,
\end{cases} \quad (2.8)$$

admits a unique analytic solution on $\mathbb{R}^d,$ denoted by $K(x, y)$ and called Dunkl kernel.

This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d.$

**Example 2.1.**

If $d = 1$ and $W = \mathbb{Z}_2,$ the Dunkl kernel is given by

$$K(z, t) = j_{-1/2}(izt) + \frac{zt}{2\gamma + 1} j_{1/2}(izt), \quad z, t \in \mathbb{C}, \quad (2.9)$$

where for $\alpha' \geq -1/2, j_\alpha$ is the normalized Bessel function defined by

$$j_\alpha(u) = 2^\alpha \Gamma(\alpha + 1) \frac{J_\alpha(u)}{u^\alpha} = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n (u/2)^{2n}}{n! \Gamma(n + \alpha + 1)}, \quad u \in \mathbb{C}, \quad (2.10)$$

with $J_\alpha$ the Bessel function of first kind and index $\alpha$ (see [7]).

The Dunkl kernel possesses the following properties.

(i) For $z, t \in \mathbb{C}^d,$ we have $K(z, t) = K(t, z), K(z, 0) = 1,$ and $K(\lambda z, t) = K(z, \lambda t)$ for all $\lambda \in \mathbb{C}.$

(ii) For all $\nu \in \mathbb{Z}_+^d, x \in \mathbb{R}^d,$ and $z \in \mathbb{C}^d$ we have

$$|D_z^\nu K(x, z)| \leq \|x\|^{|
u|} \exp \left[ \max_{w \in W} \langle wx, Rez \rangle \right]. \quad (2.11)$$

In particular

$$|D_z^\nu K(x, z)| \leq \|x\|^{|
u|} \exp[\|x\|\|Rez\|], \quad (2.12)$$

$$|K(x, z)| \leq \exp[\|x\|\|Rez\|], \quad (2.13)$$
and for all \( x, y \in \mathbb{R}^d \):
\[
|K(ix, y)| \leq 1,
\]
(2.14)
with
\[
D_\nu \chi = \frac{\partial^{|
u|}}{\partial z_1^{\nu_1} \cdots \partial z_d^{\nu_d}} \text{ and } |
u| = \nu_1 + \cdots + \nu_d.
\]

(iii) For all \( x, y \in \mathbb{R}^d \) and \( w \in W \) we have
\[
K(-ix, y) = \overline{K(ix, y)} \text{ and } K(wx, wy) = K(x, y).
\]
(2.15)

(iv) The function \( K(x, z) \) admits for all \( x \in \mathbb{R}^d \) and \( z \in \mathbb{C}^d \) the following Laplace type integral representation
\[
K(x, z) = \int_{\mathbb{R}^d} e^{(y; z)} d\mu_x(y),
\]
(2.16)
where \( \mu_x \) is a probability measure on \( \mathbb{R}^d \) with support in the closed ball \( B(0, \|x\|) \) of center 0 and radius \( \|x\| \) and we have
\[
\text{supp } \mu_x \cap \{ y \in \mathbb{R}^d /\|y\| = \|x\| \} \neq \emptyset.
\]
(2.17)
More precisely the measure \( \mu_x \) satisfies
- \( \text{supp } \mu_x \) is contained in \( \text{co}\{wx, w \in W\} \) the convex hull of the orbit of \( x \) under \( W \).
- \( \text{supp } \mu_x \cap \{wx, w \in W\} \neq \emptyset \).

(see [19]).

**Remark 2.1**
When \( d = 1 \) and \( W = \mathbb{Z}_2 \), the relation (2.16) is of the form
\[
K(x, z) = \frac{\Gamma(\gamma + 1/2)}{\sqrt{\pi \Gamma(\gamma)}} |x|^{-2\gamma} \int_{-|x|}^{|x|} (|x| - y)^{-1}(|x| + y)^{\gamma} e^{y z} dy.
\]
(2.18)
Then in this case the measure \( \mu_x \) is given for all \( x \in \mathbb{R} \setminus \{0\} \) by:
\[
d\mu_x(y) = \mathcal{K}(x, y) dy,
\]
with
\[
\mathcal{K}(x, y) = \frac{\Gamma(\gamma + 1/2)}{\sqrt{\pi \Gamma(\gamma)}} |x|^{-2\gamma} (|x| - y)^{-1}(|x| + y)^{\gamma} 1_{[-|x|, |x|]}(y),
\]
(2.19)
where \( 1_{[-|x|, |x|]} \) is the characteristic function of the interval \( [-|x|, |x|] \).

We remark that by change of variables, the relation (2.18) takes the following form
\[
\forall x \in \mathbb{R}^d, \forall z \in \mathbb{C}^d, \quad K(x, z) = \frac{\Gamma(\gamma + 1/2)}{\sqrt{\pi \Gamma(\gamma)}} \int_{-1}^1 e^{t x z} (1 - t^2)^{\gamma-1} (1 + t) dt.
\]
(2.20)
3 The Dunkl intertwining operator and its dual

**Notation** We denote by
- $C(\mathbb{R}^d)$ (resp. $C_c(\mathbb{R}^d)$) the space of continuous functions on $\mathbb{R}^d$ (resp. with compact support).
- $C^p(\mathbb{R}^d)$ (resp. $C^p_c(\mathbb{R}^d)$) the space of functions of class $C^p$ on $\mathbb{R}^d$ (resp. with compact support).

We provide the preceding spaces with the classical topology.
- $E(\mathbb{R}^d)$ the space of $C^\infty$-functions on $\mathbb{R}^d$ equipped with the topology of uniform convergence on all compact for the functions and their derivatives.
- $D(\mathbb{R}^d)$ the space of $C^\infty$-functions on $\mathbb{R}^d$ with compact support. We have

$$D(\mathbb{R}^d) = \bigcup_{a \geq 0} D_a(\mathbb{R}^d),$$

where $D_a(\mathbb{R}^d)$ is the space of $C^\infty$-functions on $\mathbb{R}^d$, with support in the closed ball $B(o,a)$ of center $o$ and radius $a$.

The topology on $D_a(\mathbb{R}^d)$ is defined by the seminorms

$$p_n(\psi) = \sup_{|\mu| \leq n} |D^\mu \psi(x)|, \ n \in \mathbb{N},$$

where

$$D^\mu = \frac{\partial^{|\mu|}}{\partial x_1^{\mu_1} \cdots \partial x_d^{\mu_d}}, \ \mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}^d.$$  

These seminorms are equivalent to the seminorms

$$q_m(\psi) = \sup_{|\mu| \leq m} |T^\mu \psi(x)|, \ m \in \mathbb{N},$$

where

$$T^\mu = T_1^{\mu_1} \circ T_2^{\mu_2} \circ \cdots \circ T_d^{\mu_d}.$$  

The space $D(\mathbb{R}^d)$ equipped with the inductive limit topology is a Fréchet space.

- $S(\mathbb{R}^d)$ the space of $C^\infty$-functions on $\mathbb{R}^d$ which are rapidly decreasing as their derivatives. The topology on this space is defined by the seminorms

$$P_{r,s}(\psi) = \sup_{|\mu| \leq r} (1 + ||x||^2)^s|D^\mu \psi(x)|, \ r, s \in \mathbb{N}.$$  

These seminorms are equivalent to the seminorms

$$Q_{k,\ell}(\psi) = \sup_{|\mu| \leq k} (1 + ||x||^2)^\ell|T^\mu \psi(x)|, \ k, \ell \in \mathbb{N}.$$  

Equipped with this topology $S(\mathbb{R}^d)$ is a Fréchet space.

We consider also the following spaces.
The Dunkl intertwining operator $V_k$ is defined on $C(\mathbb{R}^d)$ by
\[ \forall x \in \mathbb{R}^d, V_k(f)(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y), \] (3.1)
where $\mu_x$ is the measure given by the relation (2.16) (see [23]).

We have
\[ \forall x \in \mathbb{R}^d, \forall z \in \mathbb{C}^d, K(x, z) = V_k(e^{<.,z>})(x). \] (3.2)

The operator $\mathcal{V}_k$ satisfying for $f$ in $C_c(\mathbb{R}^d)$ and $g$ in $C(\mathbb{R}^d)$, the relation
\[ \int_{\mathbb{R}^d} \mathcal{V}_k(f)(y) g(y) dy = \int_{\mathbb{R}^d} V_k(g)(x) f(x) \omega_k(x) dx. \] (3.3)
is given by
\[ \forall y \in \mathbb{R}^d, \mathcal{V}_k(f)(y) = \int_{\mathbb{R}^d} f(x) d\nu_y(x), \] (3.4)
where $\nu_y$ is a positive measure on $\mathbb{R}^d$ whose support satisfies
\[ \text{supp } \nu_y \subset \{ x \in \mathbb{R}^d/ \| x \| \geq \| y \| \} \quad \text{and} \quad \text{supp } \nu_y \cap \{ x \in \mathbb{R}^d/ \| x \| = \| y \| \} \neq \emptyset. \] (3.5)

This operator is called the dual Dunkl intertwining operator (see [23]).

The following theorems give some properties of the operators $V_k$ and $\mathcal{V}_k$ (see [23]).

**Theorem 3.1**

(i) The operator $V_k$ is a topological isomorphism from $\mathcal{E}(\mathbb{R}^d)$ onto itself satisfying the transmutation relations
\[ \forall x \in \mathbb{R}^d, \quad T_j V_k(f)(x) = V_k \left( \frac{\partial}{\partial y_j} f \right)(x), \quad j = 1, 2, \cdots, d, \quad f \in \mathcal{E}(\mathbb{R}^d). \] (3.6)

(ii) For each $x \in \mathbb{R}^d$, there exists a unique distribution $\eta_x$ in $\mathcal{E}'(\mathbb{R}^d)$ with support in the closed ball $B(0, \| x \|)$ such that for all $f$ in $\mathcal{E}(\mathbb{R}^d)$ we have
\[ V_k^{-1}(f)(x) = \langle \eta_x, f \rangle. \] (3.7)
Moreover
\[ \text{supp } \eta_x \cap \{ y \in \mathbb{R}^d/ \| y \| = \| x \| \} \neq \emptyset. \] (3.8)

**Theorem 3.2**
(i) The operator $^t V_k$ is a topological isomorphism form $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}(\mathbb{R}^d)$) onto itself, satisfying the transmutation relations

$$\forall y \in \mathbb{R}^d, \quad ^t V(T_j f)(y) = \frac{\partial}{\partial y_j} ^t V(f)(y), \quad j = 1, 2, \ldots, d, f \in \mathcal{D}(\mathbb{R}^d). \quad (3.9)$$

(ii) For all $f$ in $\mathcal{D}(\mathbb{R}^d)$ we have

$$\text{supp } f \subset B(o,a) \iff \text{supp } ^t V_k(f) \subset B(o,a). \quad (3.10)$$

where $B(o,a)$ is the closed ball of center $o$ and radius $a > 0$.

(iii) For each $y \in \mathbb{R}^d$, there exists a unique distribution $Z_y$ in $\mathcal{S}'(\mathbb{R}^d)$ with support in the set $\{ x \in \mathbb{R}^d / \|x\| \geq \|y\| \}$ such that for all $f$ in $\mathcal{D}(\mathbb{R}^d)$ we have

$$^t V_k^{-1}(f)(y) = \langle Z_y, f \rangle. \quad (3.11)$$

Moreover

$$\text{supp } Z_y \cap \{ x \in \mathbb{R} / \|x\| = \|y\| \} \neq \emptyset. \quad (3.12)$$

**Example 3.1**

When $d = 1$ and $W = \mathbb{Z}_2$, the Dunkl intertwining operator $V_k$ is defined by (3.1) with for all $x \in \mathbb{R}\{0\}$, $d\mu_x(y) = K(x,y)dy$, where $K$ given by the relation (2.19).

The dual Dunkl intertwining operator $^t V_k$ is defined by (3.4) with $d\nu_y(x) = K(x,y)\omega_k(x)dx$, where $K$ and $\omega_k$ given respectively by the relations (2.19) and (2.3).

**Example 3.2**

The Dunkl intertwining operator $V_k$ of index $\gamma = \sum_{i=1}^d \alpha_i$, $\alpha_i > 0$, associated with the reflection group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ on $\mathbb{R}^d$, is given for all $f$ in $\mathcal{E}(\mathbb{R}^d)$ and for all $x \in \mathbb{R}^d$ by

$$V_k(f)(x) = \prod_{i=1}^d \left( \frac{\Gamma(\alpha_i + 1/2)}{\sqrt{\pi} \Gamma(\alpha_i)} \right) \int_{[-1,1]^d} f(t_1x_1, t_2x_2, \cdots, t_dx_d) \times \prod_{i=1}^d (1 - t_i^2)^{\alpha_i - 1}(1 + t_i)dt_1 \cdots dt_d, \quad (3.13)$$

(see [27]).

**Definition 3.1.** The dual Dunkl intertwining operator on $\mathcal{E}'(\mathbb{R}^d)$ denoted also by $^t V_k$ is defined by

$$\langle ^t V_k(S), \varphi \rangle = \langle S, V_k(\varphi) \rangle, \quad \varphi \in \mathcal{E}(\mathbb{R}^d). \quad (3.14)$$

The operator $^t V_k$ possesses the following properties (See [25] p.26-27).

i) It is a topological isomorphism from $\mathcal{E}'(\mathbb{R}^d)$ onto itself. Its inverse is given by

$$\langle ^t V_k^{-1}(S), \varphi \rangle = \langle S, V_k^{-1}(\varphi) \rangle, \quad \varphi \in \mathcal{E}(\mathbb{R}^d). \quad (3.15)$$
ii) Let $T_{f\omega_k}$ be the distribution of $E'(\mathbb{R}^d)$ given by the function $f\omega_k$, with $f \in D(\mathbb{R}^d)$. Then we have

$$tV_k(T_{f\omega_k}) = T_{tV_k(f)}.$$  

(3.16)

iii) Let $T_g$ be the distribution of $E'(\mathbb{R}^d)$ given by the function $g$ in $D(\mathbb{R}^d)$. Then we have

$$tV_k^{-1}(T_g) = T_{tV_k^{-1}(g)\omega_k}.$$  

(3.17)

4 Dunkl transform

In this section we define the Dunkl transform and we give the main results satisfied by this transform (see [7, 13, 14]).

**Notations**

We denote by

- $L^p_k(\mathbb{R}^d), p \in [1, +\infty]$, the space of measurable functions on $\mathbb{R}^d$ such that

$$\|f\|_{k,p} = \left( \int_{\mathbb{R}^d} |f(x)|^p \omega_k(x) dx \right)^{1/p} < +\infty, \text{ if } 1 \leq p < +\infty,$$

$$\|f\|_{k,\infty} = \text{ess sup}_{x \in \mathbb{R}^d} |f(x)| < +\infty.$$

- $H(\mathbb{C}^d)$ the space of entire functions on $\mathbb{C}^d$ which are rapidly decreasing and of exponential type. We have

$$H(\mathbb{C}^d) = \bigcup_{a \geq 0} H_a(\mathbb{C}^d)$$

where $H_a(\mathbb{C}^d)$ is the space of entire functions $\Psi$ on $\mathbb{C}^d$ satisfying

$$\forall \, m \in \mathbb{N}, \sup\limits_{z \in \mathbb{C}^d} (1 + ||z||^2)^m |\Psi(z)| e^{-a||Imz||} < +\infty.$$

The Dunkl transform of a function $f$ in $D(\mathbb{R}^d)$ is given by

$$\forall \, y \in \mathbb{R}^d, \quad F_D(f)(y) = \int_{\mathbb{R}^d} f(x)K(x,-iy)\omega_k(x)dx.$$  

(4.1)

This transform has the following properties.

i) For $f$ in $L^1_k(\mathbb{R}^d)$ the function $F_D(f)$ belongs to $C(\mathbb{R}^d)$, tends to zero as $t$ goes to infinity, and we have $||F_D(f)||_{k,\infty} \leq ||f||_{k,1}$.

(ii) Let $f$ be in $D(\mathbb{R}^d)$. If $\hat{f}(x) = f(-x)$ and $f_w(x) = f(wx)$ for $x \in \mathbb{R}^d$, $w \in W$, then for all $y \in \mathbb{R}^d$ we have

$$F_D(f)(y) = \overline{F_D(f)(y)} \quad \text{and} \quad F_D(f_w)(y) = F_D(f)(wy).$$  

(4.2)
iii) There is a one-to-one correspondence between the space of all radial functions \( f \) in \( L^1_k(\mathbb{R}^d) \) and the space of integrable functions \( F \) on \([0, +\infty[\) with respect to the measure \( \frac{r^{\gamma + d - 1}dr}{\Gamma(\gamma + \frac{d}{2})^{2\gamma + \frac{d}{2}}} \), via
\[
f(x) = F(||x||) = F(r), \text{ with } r = ||x||.
\]
Moreover, the Dunkl transform \( \mathcal{F}_D(f) \) of \( f \) is related to the Fourier-Bessel transform \( \mathcal{F}_B^{\gamma + \frac{d}{2} - 1}(F) \) of \( F \) by
\[
\forall y \in \mathbb{R}^d, \mathcal{F}_D(f)(y) = \frac{2^{\gamma + \frac{d}{2}}}{c_k} \mathcal{F}_B^{\gamma + \frac{d}{2} - 1}(F)(||y||). \tag{4.3}
\]
The transform \( \mathcal{F}_B^{\gamma + \frac{d}{2} - 1} \) is given by
\[
\forall \lambda \geq 0, \mathcal{F}_B^{\gamma + \frac{d}{2} - 1}(\lambda) = \int_0^\infty g(r)j_{\gamma + \frac{d}{2} - 1}(\lambda r)\frac{r^{2\gamma + d - 1}}{\Gamma(\gamma + \frac{d}{2})^{2\gamma + \frac{d}{2}}} dr, \tag{4.4}
\]
with \( j_{\gamma + \frac{d}{2} - 1}(\lambda r) \) the normalized Bessel function. (See [18] p.585-589, and [24]).

iv) For all \( f \) in \( \mathcal{S}(\mathbb{R}^d) \) we have
\[
\mathcal{F}_D(f) = \mathcal{F} \circ V_k(f), \tag{4.5}
\]
where \( \mathcal{F} \) is the classical Fourier transform on \( \mathbb{R}^d \) given by
\[
\forall y \in \mathbb{R}^d, \mathcal{F}(f)(y) = \int_{\mathbb{R}^d} f(x)e^{-i(x,y)}dx, \text{ if } f \in D(\mathbb{R}^d), \tag{4.6}
\]
The following theorems are proved in [13, 14].

**Theorem 4.1.** The transform \( \mathcal{F}_D \) is a topological isomorphism

i) from \( D(\mathbb{R}^d) \) onto \( H(\mathbb{C}^d) \),

ii) from \( \mathcal{S}(\mathbb{R}^d) \) onto itself.

The inverse transform is given by
\[
\forall x \in \mathbb{R}^d, \mathcal{F}_D^{-1}(h)(x) = \frac{c_k^2}{2^{2\gamma + d}} \int_{\mathbb{R}^d} h(y)K(x, iy)\omega_k(y)dy. \tag{4.7}
\]

**Remark 4.1**

Another proof of Theorem 4.1 is given in [25].

**Theorem 4.2.** Let \( f \) be in \( L^1_k(\mathbb{R}^d) \) such that the function \( \mathcal{F}_D(f) \) belongs to \( L^1_k(\mathbb{R}^d) \). Then we have the following inversion formula for the transform \( \mathcal{F}_D \):
\[
f(x) = \frac{c_k^2}{2^{2\gamma + d}} \int_{\mathbb{R}^d} \mathcal{F}_D(f)(y)K(x, iy)\omega_k(y)dy, \text{ a.e.} \tag{4.8}
\]

**Theorem 4.3.**
i) Plancherel formula for $\mathcal{F}_D$.
For all $f$ in $\mathcal{D}(\mathbb{R}^d)$ we have
\[
\int_{\mathbb{R}^d} |f(x)|^2 \omega_x(x) dx = \frac{c_k^2}{2^{2\gamma+d}} \int_{\mathbb{R}^d} |\mathcal{F}_D(f)(y)|^2 \omega_k(y) dy. \tag{4.9}
\]

ii) Plancherel Theorem for $\mathcal{F}_D$.
The renormalized Dunkl transform $f \to 2^{-\gamma-d/2}c_k \mathcal{F}_D(f)$ can be uniquely extended to an isometric isomorphism on $L^2_k(\mathbb{R}^d)$.

5 Dunkl convolution product and Dunkl transform of distributions

5.1 Dunkl translation operators and Dunkl convolution product of functions
The definitions and properties of Dunkl translation operators and Dunkl convolution product of functions presented in this subsection are given in the seventh section of [25] p. 33 - 37.

The Dunkl translation operators $\tau_x, x \in \mathbb{R}^d,$ are defined on $\mathcal{E}(\mathbb{R}^d)$ by
\[
\forall \ y \in \mathbb{R}^d, \tau_x f(y) = (V_k)_x(V_k)_y[V_k^{-1}(f)(x+y)]. \tag{5.1}
\]
For $f$ in $\mathcal{D}(\mathbb{R}^d)$ the function $\tau_x f$ can be expressed by using the dual Dunkl intertwining operator as follows
\[
\forall \ y \in \mathbb{R}^d, \tau_x f(y) = (V_k)_x(tV_k^{-1})_y[tV_k(f)(x+y)]. \tag{5.2}
\]
Using the relations (5.1) and (5.2) we deduce that the Dunkl translation operators can also be written in the following forms
\[
\forall \ (x, y) \in \mathbb{R}^d \times \mathbb{R}^d, \tau_x f(y) = (V_k)_{x \otimes V_k}(\Psi)(x, y), \tag{5.3}
\]
with
\[
\Psi(x, y) = V_k^{-1}(f)(x+y), \ f \in \mathcal{E}(\mathbb{R}^d),
\]
and
\[
\forall \ y \in \mathbb{R}^d, \tau_x f(y) = tV_k^{-1}[\tilde{\mu}_x \ast tV_k(f)](y), \tag{5.4}
\]
where $\mu_x$ is the measure given by (2.16), $\tilde{\mu}_x$ the measure defined by
\[
\int_{\mathbb{R}^d} g(y)d\tilde{\mu}_x(y) = \int_{\mathbb{R}^d} g(-y)d\mu_x(y), \ g \in C(\mathbb{R}^d), \tag{5.5}
\]
and $\tilde{\mu}_x \ast g$ the function given by
\[
\forall \ y \in \mathbb{R}^d, \tilde{\mu}_x \ast g(y) = \int_{\mathbb{R}^d} g(y-t)d\tilde{\mu}_x(t). \tag{5.6}
\]
The operators $\tau_x$, $x \in \mathbb{R}^d$, satisfy the properties
i) For all $x \in \mathbb{R}^d$, the operators $\tau_x$, is continuous from $\mathcal{E}(\mathbb{R}^d)$ into itself.
ii) The function $x \mapsto \tau_x$, is of class $C^\infty$ on $\mathbb{R}^d$.
iii) For all $x, y \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$ we have the product formula

$$\tau_{x}K(y,z) = K(x,z)K(y,z).$$  \hfill (5.7)

iv) For all $f$ in $\mathcal{E}(\mathbb{R}^d)$, we have

$$\tau_{x}f(0) = f(x), \quad \tau_{x}f(y) = \tau_{y}f(x).$$ \hfill (5.8)

and

$$T_{j}(\tau_{x}f) = \tau_{x}(T_{j}f), \quad j = 1, \ldots, d. \hfill (5.9)$$

$$\tau_{x}(T_{j}f) = \tau_{x}(T_{j}f), \quad j = 1, \ldots, d. \hfill (5.10)$$

v) For $f$ in $\mathcal{D}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, the function $y \mapsto \tau_{x}f(y)$ belongs to $\mathcal{D}(\mathbb{R}^d)$ and we have

$$\forall y \in \mathbb{R}^d, \quad F_D(\tau_x f)(y) = K(ix,y)F_D(f)(y). \hfill (5.11)$$

vi) For all $f$ in $C(\mathbb{R})$ we have

$$\forall x \in \mathbb{R}, \quad \tau_x f(x) = \frac{1}{2} \int_{-\infty}^{\infty} f(\sqrt{x^2+y^2-2xyl})(1+\frac{x-y}{\sqrt{x^2+y^2-2xyl}})\Phi_k(t)dt$$

$$+ \frac{1}{2} \int_{-\infty}^{\infty} f(-\sqrt{x^2+y^2-2xyl})(1-\frac{x-y}{\sqrt{x^2+y^2-2xyl}})\Phi_k(t)dt, \hfill (5.13)$$

where

$$\Phi_k(t) = \frac{\Gamma(k+\frac{d}{2})}{\sqrt{\pi \Gamma(k)}}(1+t)(1-t^2)^{k-1}. \hfill (5.14)$$

$2^{nd}$ cas.: For all $f$ in $C(\mathbb{R}^d)$ radial we have

$$\forall x \in \mathbb{R}^d, \quad \tau_y f(x) = V_k[f_0(\sqrt{||x||^2+||y||^2+2\langle x, y \rangle}))(y), \hfill (5.15)$$

with $f_0$ the function on $[0, +\infty[$ given by

$$f(x) = f_0(||x||). \hfill (5.16)$$

The Dunkl convolution product of $f$ and $g$ in $\mathcal{D}(\mathbb{R}^d)$ is the functions $f \ast_D g$ defined by

$$\forall x \in \mathbb{R}^d, \quad f \ast_D g(x) = \int_{\mathbb{R}^d} \tau_x f(-y)g(y)d\omega_k(y). \hfill (5.17)$$
This convolution is commutative and associative and admits the following properties

i) For \( f, g \in \mathcal{D}(\mathbb{R}^d) \) (resp. \( S(\mathbb{R}^d) \)) the function \( f \ast_D g \) belongs to \( \mathcal{D}(\mathbb{R}^d) \) (resp. \( S(\mathbb{R}^d) \)) and we have
\[
\forall y \in \mathbb{R}^d, \quad \mathcal{F}_D(f \ast_D g)(y) = \mathcal{F}_D(f)(y)\mathcal{F}_D(g)(y).
\] (5.18)

ii) For \( f, g \in \mathcal{D}(\mathbb{R}^d) \) (resp. \( S(\mathbb{R}^d) \)) we have
\[
^tV_k(f \ast_D g) = ^tV_k(f) \ast^tV_k(g)
\] (5.19)
where \( \ast \) is the classical convolution product of functions on \( \mathbb{R}^d \).

\[ \text{5.2 Dunkl Convolution product of distributions} \]

**Définition 5.1.** The Dunkl Convolution product of a distribution \( S \) in \( \mathcal{D}'(\mathbb{R}^d) \) and a function \( \varphi \) in \( \mathcal{D}(\mathbb{R}^d) \) is the function \( S \ast_D \varphi \) defined by
\[
\forall x \in \mathbb{R}^d, \quad S \ast_D \varphi(x) = \langle S_y, \tau_{-y}\varphi(x) \rangle.
\] (5.20)

**Remark 5.1**

If \( S = T_f \omega_k \) is the distribution in \( \mathcal{D}'(\mathbb{R}^d) \) given by the function \( f\omega_k \) with \( f \) in \( C(\mathbb{R}^d) \), we have
\[
S \ast_D \varphi = f \ast_D \varphi.
\] (5.21)

**Theorem 5.1.** The function \( S \ast_D \varphi \) is of class \( C^\infty \) on \( \mathbb{R}^d \) and we have
\[
T^\mu(S \ast_D \varphi) = S \ast_D (T^\mu\varphi) = (T^\mu S) \ast_D \varphi,
\] (5.22)
where
\[
T^\mu = T_1^{\mu_1} o T_2^{\mu_2} o \cdots o T_d^{\mu_d}, \quad \text{with} \quad (\mu_1, \mu_2, \ldots, \mu_d) \in \mathbb{N}^d,
\]
and \( T_j, \; j = 1, 2, \ldots, d \), the Dunkl operator defined on \( \mathcal{D}'(\mathbb{R}^d) \) by
\[
\langle T_j S, \varphi \rangle = -\langle S, T_j \varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^d).
\] (5.23)

**Proof**

i) We shall prove first that the function \( S \ast_D \varphi \) is continuous on \( \mathbb{R}^d \). Let \( x^0 \in \mathbb{R}^d \) and \( B(x^0, r) \) the closed ball of center \( x^0 \) and radius \( r > 0 \). We consider \( \varphi \) in \( \mathcal{D}(\mathbb{R}^d) \) such that \( \text{supp} \varphi \subset B(o, a) \), \( a > 0 \). From the relation (5.4) and the fact that the support of \( \mu \) is contained in the closed ball of center \( o \) and radius \( ||x|| \), the relation (3.10) implies that
\[
\forall x \in B(x^0, r), \quad \text{supp} \tau_{-y}\varphi(x) \subset B(o, a + r + ||x^0||).
\]
We put \( K = B(o, a + r + ||x^0||) \) and \( \Phi(x, y) = \tau_{-y}\varphi(x) \).
As the distribution \( S \) belongs to \( \mathcal{D}'(\mathbb{R}^d) \) then there exist a seminorm \( p_n \) and a positive constant \( C \) such that for all \( \theta \) in \( \mathcal{D}(\mathbb{R}^d) \) with \( \text{supp} \theta \subset K \), we have
\[
|\langle S, \theta \rangle| < Cp_n(\theta).
\] (5.24)
On the other hand as the function $\Phi$ is of class $C^\infty$ on $\mathbb{R}^d \times \mathbb{R}^d$, then from (5.3) for all $\alpha, \beta \in \mathbb{N}^d$ the function $D^\alpha \Phi(x, y)$ is continuous on $\mathbb{R}^d \times K$. Thus:

\[ \forall \varepsilon > 0, \exists B(x^0, r_0), r_0 < r, \forall x \in B(x^0, r_0) \implies p_n(\Phi(x, y) - \Phi(x^0, y)) < \frac{\varepsilon}{C} \]

This relation and (5.24) imply

\[ \forall x \in B(x^0, r_0), \quad |S * D \varphi(x) - S * D \varphi(x^0)| \leq C p_n(\Phi(x, y) - \Phi(x^0, y)) < \varepsilon. \]

The function $S * D \varphi$ is then continuous at $x^0$, and thus it is continuous on $\mathbb{R}^d$.

ii) We shall prove now that the function $S * D \varphi$ admits a partial derivative $\frac{\partial}{\partial x_j} S * D \varphi$ at $x^0 \in \mathbb{R}^d$ and we have

\[ \frac{\partial}{\partial x_j} S * D \varphi(x^0) = \langle S_y, \frac{\partial}{\partial x_j} \tau \varphi(x) \rangle. \quad (5.25) \]

Let $h \in \mathbb{R}^d \setminus \{0\}$, by applying the Taylor formula we obtain

\[
\begin{align*}
\Phi(x^0_1, \ldots, x^0_j + h, \ldots, x^0_d, y) &= \Phi(x^0, y) + h \frac{\partial}{\partial x_j} \Phi(x^0, y) \\
 &\quad + h^2 \int_0^1 (1 - t) \frac{\partial^2}{\partial x_j^2} \Phi(x^0_1, \ldots, x^0_j + th, \ldots, x^0_d, y) dt.
\end{align*}
\]

Thus

\[
\frac{S * D \varphi(x) - S * D \varphi(x^0)}{h} = \langle S_y, \frac{\partial}{\partial x_j} \Phi(x^0, y) \rangle = h \langle S_y, R(x^0, y, h) \rangle,
\]

where

\[
R(x^0, y, h) = \int_0^1 (1 - t) \frac{\partial^2}{\partial x_j^2} \Phi(x^0_1, \ldots, x^0_j + th, \ldots, x^0_d, y) dt. \quad (5.26)
\]

It suffices to show that $\langle S_y, R(x^0, y, h) \rangle$ remains bounded when $h$ tends to zero. We put

\[ M = \sup_{(x, y) \in B(x^0, r) \times K} |D^n_y \frac{\partial^2}{\partial x_j^2} \Phi(x, y)|. \]

From (5.26) we deduce that

\[ \forall y \in K, \quad |D^n_y R(x^0, y, h)| \leq M. \]

From (5.24) we deduce that

\[ |\langle S_y, R(x^0, y, h) \rangle| \leq CM. \]

Thus the function $S * D \varphi$ admits the partial derivative $\frac{\partial}{\partial x_j} S * D \varphi(x^0)$ at $x^0 \in \mathbb{R}^d$ and we have (5.25).

These result are true on $\mathbb{R}^d$ and in particular we have

\[ \forall x \in \mathbb{R}^d, \quad \frac{\partial}{\partial x_j} S * D \varphi(x) = \langle S_y, \frac{\partial}{\partial x_j} \tau \varphi(x) \rangle. \quad (5.27) \]
By applying the i) to this partial derivative we deduce that it is continuous on $\mathbb{R}^d$.
Similar proofs as for i) and ii) show that the function $S * D \varphi$ admits continuous partial derivatives of all order with respect to all variables. Then the function $S * D \varphi$ is of class $C^\infty$ on $\mathbb{R}^d$. On the other hand using the definition of Dunkl operator $T_j$ and the relations (5.27), (5.9) we obtain
\[
\forall x \in \mathbb{R}^d, \quad T_j(S * D \varphi)(x) = \langle S_y, T_j \tau_y \varphi(x) \rangle = \langle S_y, \tau_y(T_j \varphi)(x) \rangle = S * D (T_j \varphi)(x). \quad (5.28)
\]
By iteration we get
\[
\forall x \in \mathbb{R}^d, \quad T_\mu(S * D \varphi)(x) = S * D (T_\mu \varphi)(x).
\]
On the other hand from (5.28) and (5.10) we have
\[
\forall x \in \mathbb{R}^d, \quad T_j(S * D \varphi)(x) = \langle S_y, \tau_y(T_j \varphi)(x) \rangle = \langle T_j S_y, \tau_y \varphi(x) \rangle = (T_j S) * D \varphi(x).
\]
By applying this relation to the other Dunkl operators and their composite we obtain
\[
\forall x \in \mathbb{R}^d, \quad T_\mu(S * D \varphi)(x) = (T_\mu S) * D \varphi(x).
\]
This completes the proof of the theorem.

**Remark 5.2.**
We have
\[
\forall x \in \mathbb{R}^d, \quad \delta * D \varphi(x) = \langle \delta, \tau_y \varphi(x) \rangle = \varphi(x).
\]

### 5.3 Tensoriel product of distributions (see [21][3])

**Theorem 5.2.** Let $S, \mathcal{U}$ be two distributions in $\mathcal{D}'(\mathbb{R}^d)$. Then

i) There exists a unique distribution in $\mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d)$ such that for all $\varphi, \psi$ in $\mathcal{D}(\mathbb{R}^d)$ we have
\[
\langle W, \varphi \otimes \psi \rangle = \langle S, \varphi \rangle \langle \mathcal{U}, \psi \rangle. \quad (5.29)
\]
The distribution $W$ is the tensoriel product of the distributions $S$ and $\mathcal{U}$ and it is denoted by $S \otimes \mathcal{U}$.

ii) For all $\phi$ in $\mathcal{D}(\mathbb{R}^d \times \mathbb{R}^d)$ we have
\[
\langle S \otimes \mathcal{U}, \phi(x, y) \rangle = \langle S_x, \langle \mathcal{U}_y, \phi(x, y) \rangle \rangle = \langle \mathcal{U}_y, \langle S_x, \phi(x, y) \rangle \rangle. \quad (5.30)
\]
The tensoriel product of distributions satisfies the following properties.

i) Let $S, \mathcal{U}, \mathcal{V}$ be in $\mathcal{D}'(\mathbb{R}^d)$. There exists a unique distribution $S \otimes \mathcal{U} \otimes \mathcal{V}$ in $\mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ such that for all $\varphi, \psi, \theta$ in $\mathcal{D}(\mathbb{R}^d)$ we have
\[
\langle S \otimes \mathcal{U} \otimes \mathcal{V}, \varphi \otimes \psi \otimes \theta \rangle = \langle S, \varphi \rangle \langle \mathcal{U}, \psi \rangle \langle \mathcal{V}, \theta \rangle. \quad (5.31)
\]
We deduce that
\[(S \otimes U) \otimes V = S \otimes (U \otimes V).\] (5.32)

ii) For all \(S, U\) in \(\mathcal{D}'(\mathbb{R}^d)\) we have
\[D^\alpha (S \otimes U) = (D^\alpha S) \otimes U.\] (5.33)

iii) For all \(S, U\) in \(\mathcal{D}'(\mathbb{R}^d)\) we have
\[\text{supp}(S \otimes U) = (\text{supp} S) \times (\text{supp} U).\] (5.34)

5.4 Dunkl convolution product of distributions

To define the Dunkl convolution product of the distributions \(S\) and \(U\) in \(\mathcal{D}'(\mathbb{R}^d)\) we must consider the expression
\[\langle S_x \otimes U_y, \tau_x \varphi(y) \rangle, \varphi \in \mathcal{D}(\mathbb{R}^d).\] (5.35)

But the function \((x, y) \rightarrow \tau_x \varphi(y)\) defined on \(\mathcal{D}(\mathbb{R}^d \times \mathbb{R}^d)\) is not with compact support. Then the expression (5.35) is not well defined in the general case. It will have a sense if the set
\[\text{supp}(\tau_x \varphi(y)) \cap \text{supp}(S \otimes U)\]
is compact for all \(\varphi \in \mathcal{D}(\mathbb{R}^d)\). We say in this case that the supports of \(S\) and \(U\) satisfy the “supports condition”.

**Definition 5.2.** Let \(S\) and \(U\) be two distributions in \(\mathcal{D}'(\mathbb{R}^d)\) such that their supports satisfy the ” supports condition “. The Dunkl convolution product of \(S\) and \(U\) is the distribution \(S *_D U\) in \(\mathcal{D}'(\mathbb{R}^d)\) defined by
\[\langle S *_D U, \varphi \rangle = \langle S_x \otimes U_y, \tau_x \varphi(y) \rangle.\] (5.36)

**Remark 5.3.**
Using (5.30) the relation (5.36) can also be written in the form
\[\langle S *_D U, \varphi \rangle = \langle S_x, (U_y, \tau_x \varphi(y)) \rangle = \langle U_y, (S_x, \tau_x \varphi(y)) \rangle.\] (5.37)

**Proposition 5.1.** Let \(S\) be in \(\mathcal{E}'(\mathbb{R}^d)\) and \(U\) in \(\mathcal{D}'(\mathbb{R}^d)\). Then the supports of these distributions satisfies the ” supports condition “.

The Dunkl convolution product of distributions is commutative and associative and satisfies the following properties.

i) Let \(S\) and \(U\) be in \(\mathcal{D}'(\mathbb{R}^d)\) such that their support satisfy the ” supports condition “. Then for all \(\mu \in \mathbb{N}^d\) we have
\[T^\mu (S *_D U) = (T^\mu S) *_D U = S *_D (T^\mu U).\] (5.38)

ii) Let \(S\) be in \(\mathcal{D}'(\mathbb{R}^d)\) and \(f \in \mathcal{D}(\mathbb{R}^d)\). Then we have
\[S *_D T_{f_\omega_k} = T_{(S *_D f)_\omega_k}.\] (5.39)
where \( T_{f \omega_k} \) is the distribution in \( \mathcal{D}'(\mathbb{R}^d) \) given by the function \( f \omega_k \).

**Remark 5.4**

Let \( S \) and \( U \) be two distributions in \( \mathcal{E}'(\mathbb{R}^d) \), with \( \text{supp} \, S \subset B(o,a), \, a > 0 \), and \( \text{supp} \, U \subset B(o,b), \, b > 0 \). Then the distribution \( S \ast_D U \) belongs to \( \mathcal{E}'(\mathbb{R}^d) \) and we have \( \text{supp} \, S \ast_D U \subset B(o,a+b) \).

**Proposition 5.2.** Let \( S \) and \( U \) be two distributions in \( \mathcal{E}'(\mathbb{R}^d) \). Then we have

\[
i^t V_k(S \ast_D U) = i^t V_k(S) \ast i^t V_k(U).
\]

where \( \ast \) is the classical convolution product of distributions on \( \mathbb{R}^d \).

**Proof**

From (3.14) for \( \varphi \) in \( \mathcal{E}(\mathbb{R}^d) \) we have

\[
\langle i^t V_k(S \ast_D U), \varphi \rangle = \langle S \ast_D U, V_k(\varphi) \rangle.
\]

By using (5.37) and (5.1) we obtain

\[
\langle i^t V_k(S \ast_D U), \varphi \rangle = \langle S_x, \{ U_y, \tau_x(V_k(\varphi))(y) \} \rangle = \langle S_x, \{ U_y, (V_k)_x(\varphi(x+y)) \} \rangle.
\]

By applying (3.14) we get

\[
\langle i^t V_k(S \ast_D U), \varphi \rangle = \langle i^t V_k(S), \{ i^t V_k(U), \varphi(x+y) \} \rangle = \langle i^t V_k(S), i^t V_k(U), \varphi \rangle.
\]

Thus

\[
i^t V_k(S \ast_D U) = i^t V_k(S) \ast i^t V_k(U).
\]

**5.5 Dunkl Convolution product of tempered distributions**

The results of this subsection are proved in [26].

**Définition 5.3** Let \( S \) be in \( \mathcal{S}'(\mathbb{R}^d) \) and \( \varphi \) in \( \mathcal{S}(\mathbb{R}^d) \). The Dunkl Convolution product of \( S \) and \( \varphi \) is the function \( S \ast_D \varphi \) defined by

\[
\forall \, x \in \mathbb{R}^d, \, S \ast_D \varphi(x) = \langle S_y, \tau_{-y} \varphi(x) \rangle.
\]

**Proposition 5.3.** For \( S \) in \( \mathcal{S}'(\mathbb{R}^d) \) and \( \varphi \) in \( \mathcal{S}(\mathbb{R}^d) \) the function \( S \ast_D \varphi \) belongs to \( \mathcal{E}(\mathbb{R}^d) \) and we have

\[
T^\mu(S \ast_D \varphi) = S \ast_D (T^\mu \varphi) = (T^\mu S) \ast_D \varphi,
\]

where

\[
T^\mu = T_{\mu_1} \circ T_{\mu_2} \circ \cdots \circ T_{\mu_d}, \text{ with } \mu = (\mu_1, \mu_2, \cdots, \mu_d) \in \mathbb{N}^d.
\]

**5.6 Dunkl transform of distributions**

**Definition 5.4**
i) The Dunkl transform of a distribution $S$ in $\mathcal{S}'(\mathbb{R}^d)$ is defined by
\[
\langle \mathcal{F}_D(S), \psi \rangle = \langle S, \mathcal{F}_D(\psi) \rangle, \psi \in \mathcal{S}(\mathbb{R}^d).
\] (5.42)

ii) We define the Dunkl transform of a distribution $S$ in $\mathcal{E}'(\mathbb{R}^d)$ by
\[
\forall \ y \in \mathbb{R}^d, \mathcal{F}_D(S)(y) = \langle S, K(-iy, x) \rangle.
\] (5.43)

Remarks 5.5
i) When the distribution $S$ in $\mathcal{E}'(\mathbb{R}^d)$ is given by the function $g\omega_k$ with $g \in \mathcal{D}(\mathbb{R}^d)$, and denoted by $Tg\omega_k$, the relation (5.43) coincides with (4.1).

ii) From (3.14) and (2.16) the relation (5.43) can also be written in the form
\[
\forall \ y \in \mathbb{R}^d, \mathcal{F}_D(S)(y) = \mathcal{F} \circ tV_k(S)(y)
\] (5.44)
where $\mathcal{F}$ is the classical Fourier transform of distributions in $\mathcal{E}'(\mathbb{R}^d)$ given by
\[
\forall \ y \in \mathbb{R}^d, \mathcal{F}(U)(y) = \langle U, e^{-i\langle \cdot, y \rangle} \rangle.
\] (5.45)

Notation.
We denote by $\mathcal{H}(\mathbb{C}^d)$ the space of entire functions on $\mathbb{C}^d$ which are rapidly increasing and of exponential type. We have
\[
\mathcal{H}(\mathbb{C}^d) = \bigcup_{a \geq 0} \mathcal{H}_a(\mathbb{C}^d)
\]
where $\mathcal{H}_a(\mathbb{C}^d)$ is the space of entire functions $\Psi$ on $\mathbb{C}^d$ satisfying
\[
\exists N \in \mathbb{N}, \sup_{z \in \mathbb{C}^d} (1 + ||z||^2)^{-N} |\Psi(z)|e^{-a||Imz||} < +\infty.
\]
We topology this space with the classical topology.

The following theorem is given in [25] p.27.

Theorem 5.3. The transform $\mathcal{F}_D$ is a topological isomorphism from
i) $\mathcal{S}'(\mathbb{R}^d)$ onto itself.
ii) $\mathcal{E}'(\mathbb{R}^d)$ onto $\mathcal{H}(\mathbb{C}^d)$.

Theorem 5.4. Let $S$ be in $\mathcal{S}'(\mathbb{R}^d)$ and $\varphi$ in $\mathcal{S}(\mathbb{R}^d)$. Then the distribution on $\mathbb{R}^d$ given by $(S \ast_D \varphi)\omega_k$ belongs to $\mathcal{S}'(\mathbb{R}^d)$ and we have
\[
\mathcal{F}_D(T(S \ast_D \varphi)\omega_k) = \mathcal{F}_D(\varphi)\mathcal{F}_D(S).
\] (5.46)

We consider the radial positive function $\varphi$ in $\mathcal{D}(\mathbb{R}^d)$, with support in the closed ball of center 0 and radius 1, satisfying
\[
\int_{\mathbb{R}^d} \varphi(x)\omega_k(x)dx = 1,
\]
and \( \phi \) the function on \([0, +\infty[\) given by
\[
\varphi(x) = \phi(\|x\|) = \phi(r), \text{ with } r = \|x\|.
\]
For \( \varepsilon \in [0, 1] \), we denote by \( \varphi_\varepsilon \) the function on \( \mathbb{R}^d \) defined by
\[
\forall x \in \mathbb{R}^d, \quad \varphi_\varepsilon(x) = \frac{1}{\varepsilon^{d+\gamma}} \phi\left(\frac{\|x\|}{\varepsilon}\right).
\] (5.47)

**Theorem 5.5.** Let \( S \) be in \( \mathcal{S}'(\mathbb{R}^d) \). We have
\[
\lim_{\varepsilon \to 0} S *_D \varphi_\varepsilon = S,
\] (5.48)
the limit is in \( \mathcal{S}'(\mathbb{R}^d) \).

**Definition 5.5.** Let \( S \) be in \( \mathcal{S}'(\mathbb{R}^d) \) and \( V \) in \( \mathcal{E}'(\mathbb{R}^d) \). The Dunkl convolution product of \( S \) and \( V \) is the distribution \( S *_D V \) on \( \mathbb{R}^d \) defined by
\[
\langle S *_D V, \psi \rangle = \langle S_x, (V_y, \tau_x\psi(y)) \rangle, \quad \psi \in \mathcal{D}(\mathbb{R}^d).
\] (5.49)

**Remark 5.6.**

The relation (5.49) can also be written in the form
\[
\langle S *_D V, \psi \rangle = \langle S, \tilde{V} *_D \psi \rangle, \quad \psi \in \mathcal{D}(\mathbb{R}^d),
\] (5.50)
with \( \tilde{V} \) the distribution in \( \mathcal{E}'(\mathbb{R}^d) \) defined by
\[
\langle \tilde{V}, f \rangle = \langle V, \hat{f} \rangle, \quad f \in \mathcal{E}(\mathbb{R}^d),
\]
where \( \hat{f} \) given by
\[
\forall x \in \mathbb{R}^d, \quad \hat{f}(x) = f(-x).
\]

**Theorem 5.6.** Let \( S \) be in \( \mathcal{S}'(\mathbb{R}^d) \) and \( V \) in \( \mathcal{E}'(\mathbb{R}^d) \). Then the distribution \( S *_D V \) belongs to \( \mathcal{S}'(\mathbb{R}^d) \) and we have
\[
\mathcal{F}_D(S *_D V) = \mathcal{F}_D(V) \cdot \mathcal{F}_D(S).
\] (5.51)

**Proof**

We deduce the result from (5.50), (5.39) and Theorem 5.4.

**Definition 5.6.** We define the dual Dunkl intertwining operator \( ^tV_k \) on \( \mathcal{S}'(\mathbb{R}^d) \) by
\[
^tV_k(S) = \mathcal{F}^{-1} \circ \mathcal{F}_D(S).
\] (5.52)

**Theorem 5.7.**

i) The operator \( ^tV_k \) is a topological isomorphism from \( \mathcal{S}'(\mathbb{R}^d) \) onto itself.

ii) Let \( S \) be in \( \mathcal{S}'(\mathbb{R}^d) \) and \( U \) in \( \mathcal{E}'(\mathbb{R}^d) \). Then we have
\[
^tV_k(S *_D U) = ^tV_k(S) * ^tV_k(U).
\] (5.53)

where \( * \) is the classical convolution product of tempered distributions on \( \mathbb{R}^d \).
Proof
i) We deduce the result from Theorem 5.3 and the properties of the classical Fourier transform of tempered distributions on $\mathbb{R}^d$.

ii) From Theorem 5.6 we have
$$\mathcal{F}_D(S \ast_D U) = \mathcal{F}_D(S) \mathcal{F}_D(U).$$

Using (5.52) we obtain
$$\mathcal{F} \circ \gamma_k(S \ast_D U) = \mathcal{F} \circ \gamma_k(S) \mathcal{F} \circ \gamma_k(U).$$

Thus
$$\mathcal{F}(\gamma_k(S \ast_D U)) = \mathcal{F}(\gamma_k(S) \ast \gamma_k(U)).$$

We deduce (5.53) from this relation and the injectivity of the transform $\mathcal{F}$ on $\mathcal{S}'(\mathbb{R}^d)$.

Remark 5.7.
When the distribution $S$ is in $\mathcal{E}'(\mathbb{R}^d)$ another proof of the relation (5.54) has been given in Proposition 5.2.

6 Hypoelliptic Dunkl convolution equations in the space of distributions

Let $S$ be in $\mathcal{E}'(\mathbb{R}^d)$. In this section we study convolution equations of the form
$$S \ast_D U = V,$$
where $U$ and $V$ are distributions in $\mathcal{D}'(\mathbb{R}^d)$.

We say that the equation (6.1) is hypoelliptic if all solution $U$ is given by a function $f \omega_k$ with $f$ in $\mathcal{E}(\mathbb{R}^d)$ whenever $V$ is given by a function $g \omega_k$ with $g$ in $\mathcal{E}(\mathbb{R}^d)$.

When (6.1) is hypoelliptic we say also that the distribution $S$ is hypoelliptic.

The main result of this section is the characterization of hypoelliptic Dunkl convolution equations in terms of their Dunkl transform.

We say that the distribution $S$ in $\mathcal{E}'(\mathbb{R}^d)$ satisfies the $H$-property if
i) There exists $A,M > 0$ such that $|\mathcal{F}_D(S)(x)| \geq ||x||^{-A}$ for all $||x|| \geq M$.

ii) $\lim_{||z|| \to \infty, z \in \mathbb{Z}} \frac{|\text{Im}z|}{||z||} = \infty$, where $\mathbb{Z} = \{ z \in \mathbb{C}^d, \mathcal{F}_D(S)(z) = 0 \}$, with $||z||^2 = \sum_{j=1}^{d} (\text{Re}z_j)^2 + (\text{Im}z_j)^2$.

Proposition 6.1. Let $S$ be in $\mathcal{E}'(\mathbb{R}^d)$. If $S$ is hypoelliptic then $S$ satisfies the i) of the $H$-property.

To prove this proposition we need the following Lemma.

Lemma 6.1. Let $\phi$ be a positive function in $D(\mathbb{R}^d)$ such that $\phi(0) = 1$ and which is even for $d = 1$ and radial for $d \geq 2$. Then there exist positive constants $C$ and $X$ such that for $||x|| \geq X$ we have
$$\mathcal{F}_D(K(\cdot, \cdot)\phi)(x) \geq \frac{C}{||x||^{2\gamma + d}}.$$ (6.2)
Proof

i) We suppose that \( d \geq 2 \).

We have

\[
\forall x \in \mathbb{R}^d, \quad F_D(K(\cdot,x)\phi)(x) = \int_{\mathbb{R}^d} |K(\cdot,t)|^2 \phi(t) \omega_k(t) dt.
\]

As \( \phi \) is radial then there exists a function \( \varphi \) on \([0, +\infty[ \) such that

\[
\phi(t) = \varphi(||t||) = \varphi(r), \quad \text{with } r = ||t||.
\]

By using polar coordinates we obtain

\[
\forall x \in \mathbb{R}^d, \quad F_D(K(\cdot,x)\phi)(x) = \int_0^{\infty} \left( \int_{S^{d-1}} |K(ix, r\sigma)|^2 \omega_k(\sigma) d\sigma \right) \varphi(r) r^{2\gamma + d-1} dr.
\]

As the function \( \varphi \) is positive, then for all \( x \in \mathbb{R}^d \setminus \{0\} \) we have

\[
F_D(K(\cdot,x)\phi)(x) \geq \frac{1}{||x||^{2\gamma + d}} \int_0^{1} \left( \int_{S^{d-1}} |K(i\beta, u\sigma)|^2 \omega_k(\sigma) d\sigma \right) \varphi(\frac{u}{||x||}) u^{2\gamma + d-1} du.
\]

We denote by \( I_\beta(||x||) \) the integrals of the second member. From the properties of the function \( \varphi \) we deduce that there exists \( X > 0 \) such that for all \( u \in [0,1] \) and \( ||x|| \geq X \), we have \( \varphi(\frac{u}{||x||}) \geq \frac{1}{2} \). Then

\[
I_\beta(||x||) \geq \frac{1}{2} \int_0^{1} \left( \int_{S^{d-1}} |K(i\beta, u\sigma)|^2 \omega_k(\sigma) d\sigma \right) u^{2\gamma + d-1} du.
\]

As the second member is continuous on \( S^{d-1} \) with respect to the variable \( \beta \), then for \( ||x|| \geq X \), we have

\[
I_\beta(||x||) \geq \frac{1}{2} \min_{\beta \in S^{d-1}} \int_0^{1} \left( \int_{S^{d-1}} |K(i\beta, u\sigma)|^2 \omega_k(\sigma) d\sigma \right) u^{2\gamma + d-1} du,
\]

and there exists \( \beta_0 \in S^{d-1} \) such that for \( ||x|| \geq X \), we have

\[
I_\beta(||x||) \geq \frac{1}{2} \int_0^{1} \left( \int_{S^{d-1}} |K(i\beta_0, u\sigma)|^2 \omega_k(\sigma) d\sigma \right) u^{2\gamma + d-1} du. \quad (6.4)
\]

As the function

\[
u \to \int_{S^{d-1}} |K(i\beta_0, u\sigma)|^2 \omega_k(\sigma) d\sigma
\]

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is continuous on $[0,1]$, then if
\[
\int_0^1 \int_{S^{d-1}} |K(i \beta_0, u \sigma)|^2 \omega_k(\sigma) d\sigma u^{2\gamma + d-1} du = 0,
\]
we deduce that
\[
\forall u \in [0,1], \int_{S^{d-1}} |K(i \beta_0, u \sigma)|^2 \omega_k(\sigma) d\sigma = 0.
\]
By taking $u = 0$, we obtain
\[
\int_{S^{d-1}} \omega_k(\sigma) d\sigma = 0,
\]
which contradicts
\[
\int_{S^{d-1}} \omega_k(\sigma) d\sigma = d_k = \frac{2}{c_k \Gamma(\gamma + \frac{d}{2})}.
\]
Then
\[
\int_0^1 \left( \int_{S^{d-1}} |K(i \beta_0, u \sigma)|^2 \omega_k(\sigma) d\sigma u^{2\gamma + d-1} du \right) \neq 0. \quad (6.5)
\]
We denote the first member of this relation by $2C$. By using the relations (6.3),(6.4) and (6.5), we deduce that for $||x|| \geq X$, we have
\[
\mathcal{F}_D(K(ix,.)\phi)(x) \geq \frac{C}{||x||^{2\gamma + d}}.
\]
ii) We suppose that $d = 1$.

The same proof as for i) gives the relation (6.2).

**Proof of Proposition 6.1.**

We assume that the i) of the $H$-property does not hold. Then we can find a sequence $(x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^d$ such that $||x_n|| \geq 2^n$ and
\[
\forall n \in \mathbb{N}, |\mathcal{F}_D(S)(x_n)| < ||x_n||^{-n}. \quad (6.6)
\]
We consider the sequence $\{U_p\}_{p \in \mathbb{N}}$ of distributions in $\mathcal{D}'(\mathbb{R}^d)$ given by
\[
U_p = \sum_{n=0}^p T_K(-ix_n,) \omega_k,
\]
where $T_K(-ix_n,) \omega_k$ the distribution in $\mathcal{D}'(\mathbb{R}^d)$ given by the function $K(-ix_n,.)\omega_k$. Let $\varphi$ be in $\mathcal{D}(\mathbb{R}^d)$. For all $p,q \in \mathbb{N}$ with $p > q$, we have
\[
\langle U_p, \varphi \rangle - \langle U_q, \varphi \rangle = \sum_{n=q}^p \langle T_K(-ix_n,) \omega_k, \varphi \rangle.
\]
\[ p \sum_{n=q}^{p} F_D(\varphi)(x_n). \]  (6.7)

But from Theorem 4.1 the function \( F_D(\varphi) \) is rapidly decreasing. Then there exists a positive constant \( C \) such that

\[ \forall y \in \mathbb{R}^d, \ |F_D(\varphi)(y)| \leq \frac{C}{1 + ||y||}. \]

Thus

\[ \forall n \in \mathbb{N}, \ |F_D(\varphi)(x_n)| \leq \frac{C}{||x_n||} \leq \frac{C}{2^n}. \]  (6.8)

By applying this relation to (6.7) we obtain

\[ |\langle U_p, \varphi \rangle - \langle U_q, \varphi \rangle| \leq C \sum_{n=q}^{p} \frac{1}{2^n} \to 0, \text{ as } q \to +\infty. \]

Then

\[ \langle U_p, \varphi \rangle \to L(\varphi), \text{ as } p \to +\infty. \]

We deduce that \( L \) is a distribution \( U \) in \( \mathcal{D}'(\mathbb{R}^d) \) and \( U_p \) converges to \( U \) in \( \mathcal{D}'(\mathbb{R}^d) \) as \( p \) tends to infinity. Thus

\[ U = \sum_{n=0}^{\infty} T_K(-ix_n,)\omega_k, \]  (6.9)

and for all \( \varphi \) in \( \mathcal{D}(\mathbb{R}^d) \) we have

\[ \langle U, \varphi \rangle = \sum_{n=0}^{\infty} F_D(\varphi)(x_n). \]  (6.10)

We shall prove now that the distribution \( S \ast_D U \) of \( \mathcal{D}'(\mathbb{R}^d) \) is given by a function \( f \omega_k \) with \( f \) in \( \mathcal{E}(\mathbb{R}^d) \).

From (5.37),(6.6) and (5.11), for all \( \varphi \) in \( \mathcal{D}(\mathbb{R}^d) \) we have

\[ \langle S \ast_D U, \varphi \rangle = \langle S_y, \langle U_t, \tau_y \varphi(t) \rangle \rangle \]

\[ = \langle S_y, \sum_{n=0}^{\infty} K(iy, x_n) F_D(\varphi)(x_n) \rangle. \]

By applying Theorem 4.1 and Definition 5.4 ii) we obtain

\[ \langle S \ast_D U, \varphi \rangle = \sum_{n=0}^{\infty} F_D(\varphi)(x_n) \langle S_y, K(iy, x_n) \rangle \]

\[ = \sum_{n=0}^{\infty} F_D(\varphi)(x_n) F_D(S)(-x_n). \]  (6.11)
This relation can also be written in the form

\[ \langle S \ast D U, \varphi \rangle = \sum_{n=0}^{\infty} F_D(S)(-x_n) \int_{\mathbb{R}^d} K(-it, x_n) \varphi(t) \omega_k(t) dt. \]

By using (2.14) and the fact that \( \varphi \) belongs to \( \mathcal{D}(\mathbb{R}^d) \) and \( F_D(S) \) satisfies the relation (6.6), we can interchange the series and the integral and we obtain

\[ \langle S \ast D U, \varphi \rangle = \int_{\mathbb{R}^d} \left[ \sum_{n=0}^{\infty} F_D(S)(-x_n)K(-it, x_n) \right] \varphi(t) \omega_k(t) dt. \]

Thus the distribution \( S \ast D U \) is given by the function \( f_{\omega_k} \), with

\[ f(t) = \sum_{n=0}^{\infty} F_D(S)(-x_n)K(-it, x_n). \]

Let \( t_0 \in \mathbb{R}^d \) and \( B(t_0, r) \) the open ball of center \( t_0 \) and radius \( r > 0 \). By using (6.6) and (2.12) we deduce that for all \( \nu \in \mathbb{N}^d \) there exists a positive constant \( C \) such that

\[ \sup_{t \in B(t_0, r)} |D^\nu (F_D(S)(-x_n)K(-it, x_n))| \leq C ||x_n||^{-n+|\nu|}. \]

As the series \( \sum_{n=0}^{\infty} ||x_n||^{-n+|\nu|} \) converges, we deduce that the function \( f \) admits continuous partial derivatives of all order on \( B(t_0, r) \) and then on \( \mathbb{R}^d \). Thus \( f \) belongs to \( \mathcal{E}(\mathbb{R}^d) \).

In the following we want to show that the distribution \( U \) does not given by a function \( g_{\omega_k} \) with \( g \) in \( \mathcal{E}(\mathbb{R}^d) \). If not we take a positive function \( \phi \) in \( \mathcal{D}(\mathbb{R}^d) \) such that \( \phi(0) = 1 \) which is even for \( d = 1 \) and radial for \( d \geq 2 \) and \( F_\Delta^{-1}(\phi) \) is non negative. We consider \( \Delta_k \) the Dunkl Laplacian given by

\[ \Delta_k = \sum_{j=1}^{d} T_j^2, \]

and \( p \in \mathbb{N} \) such that \( p > \frac{1}{2}(\gamma + \frac{d}{2}) \). Using (6.10) and (2.8) we obtain for all \( y \in \mathbb{R}^d \):

\[ \langle K(iy, \cdot) \Delta_k^{2p} U, \phi \rangle = \langle \Delta_k^{2p} U, K(iy, \cdot) \phi \rangle \]

\[ = \langle \Delta_k^{2p} T_{p\omega_k}, K(iy, \cdot) \phi \rangle \]

\[ = \langle T_{(\Delta_k^{2p} g)\omega_k}, K(iy, \cdot) \phi \rangle \]

\[ = \int_{\mathbb{R}^d} K(iy, t) \phi(t) \Delta_k^{2p} g(t) \omega_k(t) dt. \]

By taking \( y = x_j \), we have

\[ \langle K(iy, \cdot) \Delta_k^{2p} U, \phi \rangle = \int_{\mathbb{R}^d} K(ix_j, t) \phi(t) \Delta_k^{2p} g(t) \omega_k(t) dt. \]
As the function $\phi \triangle^2 p g$ belongs to $L^1_k(\mathbb{R}^d)$, then from the properties of the Dunkl transform we have
\[
\lim_{j \to +\infty} \langle K(iy,.) \triangle^2 p U, \phi \rangle = 0. \tag{6.13}
\]
On the other hand from (6.12) we have
\[
\langle K(iy,.) \triangle^2 p U, \phi \rangle = \langle U, \triangle^2 p (K(iy,.)\phi) \rangle
= \sum_{n=0}^{\infty} \mathcal{F}_D(\triangle^2 p (K(iy,.)\phi))(x_n).
\]
Thus
\[
\langle K(iy,.) \triangle^2 p U, \phi \rangle = \sum_{n=0}^{\infty} ||x_n||^4 p \mathcal{F}_D(K(iy,.)\phi)(x_n). \tag{6.14}
\]
On the other hand for all $y \in \mathbb{R}^d$ we have
\[
\forall z \in \mathbb{R}^d, \mathcal{F}_D(K(iy,.)\phi)(z) = \int_{\mathbb{R}^d} K(iy,t)K(-it,z)\phi(t)\omega_k(t)dt. \tag{6.15}
\]
Thus from Theorem 4.1 and the relation (5.11) we obtain
\[
\forall z \in \mathbb{R}^d, \mathcal{F}_D(K(iy,.)\phi)(z) = \frac{2^{2\gamma+d}}{c_k^2} F^{-1}_D(\mathcal{F}_D^{-1}(\phi))(z).
\]
As the function $F^{-1}_D(\phi)$ is positive and even for $d = 1$ and radial for $d \geq 2$, then for $d \geq 2$ we have
\[
\forall t \in \mathbb{R}^d, F^{-1}_D(\phi)(t) = F(||t||),
\]
where $F$ is a positive function on $[0, +\infty[$.

Using (5.15),(5.13) and example 3.1 we have
\[
\forall z \in \mathbb{R}^d, \mathcal{F}_D(K(iy,.)\phi)(z) = \frac{2^{2\gamma+d}}{c_k^2} V_k[F(\sqrt{||y||^2 + ||z||^2 - 2(y,z)})](z) \text{ if } d \geq 2.
\]
\[
\forall z \in \mathbb{R}, \mathcal{F}_D(K(iy,.)\phi)(z) = C V_k[(F^{-1}_D(\phi))(\sqrt{y^2 + z^2 - 2yz})](z) \text{ if } d = 1,
\]
with
\[
C = 2^{2k+1}(\Gamma(k + \frac{1}{2}))^2.
\]
Thus the function $\mathcal{F}_D(K(iy,.))\phi(z)$ is positive.

On the other hand by taking $y = x_j$ we deduce from (6.14) the following relation
\[
\sum_{n=0}^{\infty} ||x_n||^4 p \mathcal{F}_D(K(ix_j,.)\phi)(x_n) \geq ||x_j||^4 p \mathcal{F}_D(K(ix_j,.\phi)(x_j). \tag{6.16}
\]
But from the relation (6.15) we have
\[
\mathcal{F}_D(K(ix_j,.\phi)(x_j) = \int_{\mathbb{R}^d} |K(ix_j, t^2 \phi(t)\omega_k(t)dt.
\]
By applying Lemma 6.1 there exist positive constants $C$ and $X$ such that for $||x_j|| \geq X$ we have

$$F_D(K(ix_j,.)\phi)(x_j) \geq \frac{C}{||x_j||^{2\gamma+d}}.$$  

From this inequality and (6.14),(6.16) we obtain for $||x_j|| \geq X$:

$$(K(ix_j,.)\Delta^2_k U, \phi) \geq C|x_j|^{4p-2\gamma-d}.$$  

Thus

$$\langle K(ix_j,.)\Delta^2_k U, \phi \rangle \rightarrow +\infty, \text{ as } j \rightarrow +\infty.$$  

This contradicts (6.14). Hence the distribution $U$ is not given by a function $g_\omega_k$ with $g$ in $E(\mathbb{R}^d)$.

**Proposition 6.2.** Let $S$ be in $E'(\mathbb{R}^d)$. If $S$ is hypoelliptic then $S$ satisfies the ii) of the $H$-property.

**Proof**

Suppose that the ii) of the $H$-property does not hold. Then there exists a sequence $(z_n)_{n\in\mathbb{N}} \subset \mathbb{C}^d$ and a positive constant $M$ such that for all $n \in \mathbb{N}$,

$$F_D(S)(z_n) = 0 \text{ and } |\text{Im}z_n| \leq M\log|z_n|.$$  

Let $\phi$ be in $D(\mathbb{R}^d)$. According to Theorem 4.1 i) there exists $a \in \mathbb{N}$ such that for very $p \in \mathbb{N}$ we can find $C_p > 0$ for which

$$\forall z \in \mathbb{C}^d, |F_D(\phi)(z)| \leq C_p e^{a||z||-p\log(1+||z||)}.$$  

If we take $p \in \mathbb{N}$ such that $p > Ma + 2$, we get

$$||z_n||^2|F_D(\phi)(z_n)| \leq C_p.$$  

(6.17)

Let $(a_n)_{n\in\mathbb{N}}$ be a complex sequence such that the series $\sum_{n=0}^{\infty} |a_n|$ is convergent. We consider the sequence $\{\mathcal{V}_q\}_{q\in\mathbb{N}}$ of distributions in $D'(\mathbb{R}^d)$ given by

$$\mathcal{V}_q = \sum_{n=0}^{q} a_n T||z_n||^2K(iz_n,.)(\omega_k).$$  

For all $q, r \in \mathbb{N}$ with $q > r$, we have

$$\langle \mathcal{V}_q, \phi \rangle - \langle \mathcal{V}_r, \phi \rangle = \langle \sum_{n=r}^{q} a_n T||z_n||^2K(iz_n,.)(\omega_k), \phi \rangle = \sum_{n=r}^{q} a_n ||z_n||^2F_D(\phi)(-z_n).$$  

Thus using (6.17) we obtain

$$|\langle \mathcal{V}_q, \phi \rangle - \langle \mathcal{V}_r, \phi \rangle| \leq C_p \sum_{n=r}^{q} |a_n| \rightarrow 0, \text{ as } r \rightarrow +\infty.$$  

(6.18)

Then

$$\langle \mathcal{V}_q, \phi \rangle \rightarrow L(\phi), \text{ as } q \rightarrow +\infty.$$  

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We deduce that $L$ is a distribution $V$ in $\mathcal{D}'(\mathbb{R}^d)$ and $V_q$ converges to $V$ in $\mathcal{D}'(\mathbb{R}^d)$ as $q$ tends to infinity. Thus
\[
V = \sum_{n=0}^{\infty} a_n T|z_n|^2 K(iz_n, \cdot)\omega_k, \quad (6.19)
\]
and from (6.18) we deduce that
\[
|\langle V, \phi \rangle| \leq C_p \sum_{n=0}^{\infty} |a_n|. \quad (6.20)
\]

On the other hand by making a proof similar to those which has given the relation (6.11) we obtain
\[
\langle S \ast_D V, \phi \rangle = \langle \sum_{n=0}^{\infty} a_n |z_n|^2 \mathcal{F}_D(S)(z_n)\mathcal{F}_D(\phi)(-z_n) = 0. \n\]
Thus
\[
S \ast_D V = 0
\]
As $S$ is hypoelliptic, we deduce that the distribution $V$ is given by a function $f_{\omega_k}$ with $f$ in $\mathcal{E}(\mathbb{R}^d)$. Then we have
\[
V = Tf_{\omega_k}. \quad (6.21)
\]
From (2.8) we have
\[
\forall n \in \mathbb{N}, \quad K(iz_n, 0) = 1.
\]
Thus for all closed ball of center $o$ and radius $r > 0$ we have
\[
\forall n \in \mathbb{N}, \quad \sup_{y \in B(o, r)} |K(iz_n, y)| \geq 1. \quad (6.22)
\]
On the other hand using (6.20) and (6.21) we obtain
\[
\sup_{y \in B(o, r)} |f(y)| \leq C_p \sum_{n=0}^{\infty} |a_n|.
\]
Thus
\[
\forall n \in \mathbb{N}, \quad \sup_{y \in B(o, r)} ||z_n||^2 |K(iz_n, y)| \leq C_p. \quad (6.23)
\]
From this relation and (6.22) we deduce that
\[
\forall n \in \mathbb{N}, \quad ||z_n|| \leq C_p. \quad (6.24)
\]
which is a contradiction with our choice of the sequence $(z_n)_{n \in \mathbb{N}}$. This completes the proof.

**Proposition 6.3.** Let $S$ be in $\mathcal{E}'(\mathbb{R}^d)$. If $S$ satisfies the $H$-property, then there
exists a parametrix for $S$, that is, there exist $V$ in $\mathcal{E}'(\mathbb{R}^d)$ and $\psi$ in $\mathcal{D}(\mathbb{R}^d)$ such that $\delta = S * D V + T_{\psi_{\omega_k}}$, where $\delta$ represents the Dirac functional.

**Proof**

Using (5.44) the $H$-property can also be written in the form

i) There exist $A, M > 0$ such that $|\mathcal{F}(\{V_k(S)\})(x)| \geq ||x||^{-A}$ for all $||x|| \geq M$.

ii) $\lim_{||z|| \to \infty, z \in \mathbb{Z}} \frac{\log ||z||}{||z||} = 0,

where $\{z \in \mathbb{C}, \mathcal{F}(\{V_k(S)\})(z) = 0\}$

We see that the $H$-property is true for the distribution $\{V_k(S)\}$ of $\mathcal{E}'(\mathbb{R}^d)$ in the case of the classical Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$. Then from [11] there exists a parametrix for $\{V_k(S)\}$, that is, there exist $V_0$ in $\mathcal{E}'(\mathbb{R}^d)$ and $\psi_0$ in $\mathcal{D}(\mathbb{R}^d)$ such that

$$\delta = \{V_k(S) * V_0 + T_{\psi_0}. \tag{6.25}$$

As the operator $\{V_k\}$ is a topological isomorphism from $\mathcal{E}'(\mathbb{R}^d)$ onto itself, we deduce from (6.25) and (3.17) that

$$\delta = \{V_k(S) * \{V_k\}(\{V_k^{-1}(V_0)\}) + \{V_k\}(\{V_k^{-1}(T_{\psi_0})\}). \tag{6.26}$$

Thus

$$\delta = \{V_k(S) * \{V_k\}(V) + \{V_k\}(T_{\psi_{\omega_k}}). \tag{6.26}$$

with

$$\{V_k^{-1}(V_0)\} = V, \text{ and } \{V_k^{-1}(\psi_0)\} = \psi.$$

The distribution $V$ and the function $\psi$ belong respectively to $\mathcal{E}'(\mathbb{R}^d)$ and $\mathcal{D}(\mathbb{R}^d)$. On the other hand from Proposition 5.2 we have

$$\{V_k(S * D V) = \{V_k(S) * \{V_k\}(V).$$

Thus the relation (6.26) can also be written in the form

$$\{V_k^{-1}(\delta) = S * D V + T_{\psi_{\omega_k}}.$$

But

$$\{V_k^{-1}(\delta) = \delta.$$ 

Thus

$$\delta = S * D V + T_{\psi_{\omega_k}}.$$

**Theorem 6.1.** We assume that the distribution $S$ in $\mathcal{E}'(\mathbb{R}^d)$ is such that $Z = \{z \in \mathbb{C}, \mathcal{F}_D(S)(z) = 0\}$ is infinite. The following assertions are equivalent.

i) $S$ is hypoelliptic.

ii) $S$ satisfies the $H$-properties.

iii) There exists a parametrix for $S$, that is, there exist $V$ in $\mathcal{E}'(\mathbb{R}^d)$ and $\psi$ in $\mathcal{D}(\mathbb{R}^d)$ such that $\delta = S * D V + T_{\psi_{\omega_k}}$.

**Proof**

From Propositions 6.1 and 6.2 it suffices to prove that iii) $\implies$ ii). Assume that the distribution $\mathcal{U}$ is in $\mathcal{D}'(\mathbb{R}^d)$ and that $S * D \mathcal{U}$ is given by a function $f_{\omega_k}$, with $f$ in $\mathcal{E}(\mathbb{R}^d)$.

From iii) we have

$$\delta = S * D V + T_{\psi_{\omega_k}}.$$
with $V$ in $\mathcal{E}'(\mathbb{R}^d)$ and $\psi$ in $\mathcal{D}(\mathbb{R}^d)$.

Thus
\[ U = U *_{D} \delta = U *_{D} (S *_{D} V + T_{\psi \omega_k}). \]

Using the commutativity and the associativity of the Dunkl convolution product of distributions in $\mathcal{E}'(\mathbb{R}^d)$, we obtain
\[ U = V *_{D} (S *_{D} U + T_{\psi \omega_k}) = V *_{D} T_{f \omega_k} + U *_{D} T_{\psi \omega_k}. \]

By applying (5.39) we obtain
\[ U = V *_{D} T_{f \omega_k} + U *_{D} T_{\psi \omega_k}. \]

But from Theorem 5.1 the function $V *_{D} f + U *_{D} \psi$ belongs to $\mathcal{E}(\mathbb{R}^d)$. Thus $S$ is hypoelliptic.

**Example 6.1.**

We suppose that $d \geq 2$ and we consider the equation
\[ \triangle_k U = V, \]
$U$ and $V$ are distributions in $\mathcal{D}'(\mathbb{R}^d)$. We say that the Dunkl Laplacian $\triangle_k$ is hypoelliptic if all solution $U$ is given by a function $f_{\omega_k}$ with $f$ in $\mathcal{E}(\mathbb{R}^d)$ whenever $V$ is given by a function $g_{\omega_k}$ with $g$ in $\mathcal{E}(\mathbb{R}^d)$. As we have
\[ \triangle_k U = (\triangle_k \delta) *_{D} U, \]
where $\delta$ is the Dirac distribution on $\mathbb{R}^d$.

Then the hypoellipticity of $\triangle_k$ is equivalent to the hypoellipticity of the distribution $\triangle_k \delta$ in $\mathcal{E}'(\mathbb{R}^d)$ given by
\[ \langle \triangle_k \delta, \varphi \rangle = \langle \delta, \triangle_k \varphi \rangle = \triangle_k \varphi(o), \varphi \in \mathcal{E}(\mathbb{R}^d). \]

The relation (2.8) implies that
\[ \forall z \in \mathbb{C}^d, F_D(\triangle_k \delta)(z) = \sum_{j=1}^{d} z_j^2. \]  \hspace{1cm} (6.27)

i) From (6.27) we deduce that
\[ \forall x \in \mathbb{R}^d, F_D(\triangle_k \delta)(x) = ||x||^2. \]

Thus for $||x|| \geq 1$ we have
\[ |F_D(\triangle_k \delta)(x)| \geq ||x||^{-1}. \] \hspace{1cm} (6.28)

ii) The relation (6.27) implies also that
\[ \mathcal{Z} = \{ z \in \mathbb{C}^d, F_D(\triangle_k \delta)(z) = 0 \} = \{ (x, y) \in \mathbb{R}^d \times \mathbb{R}^d; ||x|| = ||y|| and \langle x, y \rangle = 0 \}. \]
Thus
\[
\lim_{\|z\| \to \infty, z \in \mathbb{Z}} \frac{\|\text{Im} z\|}{\log \|z\|} = \lim_{\|y\| \to \infty} \frac{\|y\|}{\log (2^{\frac{3}{2}}\|y\|)} = +\infty. \quad (6.29)
\]
The relations (6.28),(6.29) show that the distribution $\Delta_k \delta$ satisfies the $H$-property. Thus Theorem 6.1 implies that the distribution $\Delta_k \delta$ is hypoelliptic. The Dunkl Laplacian is then hypoelliptic. This result was first proved in [17] by another method.

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