Universality Constraints on Three-Body Recombination for Cold Atoms: from $^4$He to $^{133}$Cs

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(Dated: October 4, 2018)

Abstract

For atoms with large scattering length, the dependence of the 3-body recombination rate on the collision energy is determined by the scattering length and the Efimov 3-body parameters and can be expressed in terms of universal functions of a single scaling variable. We use published results on the 3-body recombination rate for $^4$He atoms to constrain the universal functions. We then use those universal functions to calculate the 3-body recombination rate for other atoms with large scattering length at nonzero temperature. The constraints from the $^4$He results are strong if the scattering length is near the minimum of the 3-body recombination rate at threshold. We apply our results to $^{133}$Cs atoms with a large positive scattering length, and compare them with experimental results from the Innsbruck group.

PACS numbers: 21.45.+v, 34.50.-s, 03.75.Nt

Keywords: Few-body systems, three-body recombination, scattering of atoms and molecules.
INTRODUCTION

Vitaly Efimov discovered in 1970 that 3-body systems of nonrelativistic particles with short-range interactions have remarkable universal properties if the S-wave scattering length $a$ is large compared to the range. If $a = \pm \infty$, there are infinitely many 3-body bound states (Efimov states or Efimov trimers) with an accumulation point at the scattering threshold and a geometric spectrum [1]:

$$E_{T}^{(n)} = (e^{-2\pi/s_0})^{n-n_\ast}h^2\kappa_*^2/m,$$

where $\kappa_\ast$ is the binding wavenumber of the branch of Efimov states labeled by $n_\ast$. This geometric spectrum is a signature of a discrete scaling symmetry with discrete scaling factor $e^{\pi/s_0}$ [2]. In the case of identical bosons, $s_0 \approx 1.00624$ and the discrete scaling factor is $e^{\pi/s_0} \approx 22.7$. Efimov showed that the discrete scaling symmetry is also relevant for finite $a$ [3]. There is an infinite sequence of negative values of $a$ differing by $e^{\pi/s_0}$ and approaching $-\infty$ for which there is an Efimov trimer at the 3-atom scattering threshold. There is also an infinite sequence of positive values of $a$ differing by $e^{\pi/s_0}$ and approaching $+\infty$ for which there is an Efimov trimer at the atom-dimer scattering threshold. We will refer to the universal phenomena characterized by a discrete scaling symmetry that can occur in 3-body systems with a large scattering length as Efimov physics. Another example of Efimov physics that was discovered more recently is an infinite sequence of positive values of $a$ differing by $e^{\pi/s_0}$ and approaching $+\infty$ for which the 3-body recombination rate into the shallow dimer vanishes at the threshold [4, 5, 6]:

$$a = (e^{\pi/s_0})^{n_{a_{0}}}, \text{ where } a_{0} \approx 0.32 \kappa_{\ast}^{-1}.$$

In some systems, such as $^4$He atoms, the universal aspects of Efimov physics are determined by two parameters: the scattering length $a$ and the Efimov parameter $\kappa_\ast$. However, the situation is more complicated for the alkali atoms that are used in most cold atom experiments as they form diatomic molecules with many deep-lying bound states (deep dimers). Efimov physics is modified by the existence of the deep dimers, because Efimov trimers can decay into an atom and a deep dimer. The deep dimers also provide additional 3-body recombination channels. If there are deep dimers, the universal aspects of Efimov physics are determined by three parameters: $a$, $\kappa_\ast$, and a parameter $\eta_\ast$ that determines the widths of Efimov trimers [7].

The first experimental evidence for Efimov physics has recently been presented by the Innsbruck group [8]. They carried out experiments with ultracold $^{133}$Cs atoms in the lowest hyperfine state, using a magnetic field to control their scattering length. They observed a resonant enhancement in the 3-body recombination rate at $a \approx -850 \ a_0$ that can be attributed to an Efimov trimer near the 3-atom threshold. At the temperature 10 nK, the loss rate as a function of $a$ can be fit rather well by the universal formula for zero temperature derived in Ref. [7] with a width parameter $\eta_\ast = 0.06(1)$. The Innsbruck group also observed a local minimum in the 3-body recombination rate near $a \approx 210 \ a_0$ that might be attributable to Efimov physics. One complication is that $a$ is not large compared to the van der Waals length scale for Cs atoms: $(mC_6/\hbar^2)^{1/4} \approx 200 \ a_0$. Universal predictions might not be quantitatively accurate for such a small value of $a$. Another complication is that the measurements were carried out at 200 nK. At this temperature, there could be large corrections to the universal predictions for zero temperature.

The 3-body recombination rate can be calculated at nonzero temperature by carrying out a thermal average of the 3-body recombination rate as a function of the collision energy. The universal predictions for nonzero collision energy have not yet been calculated. There have
been several previous efforts to calculate the 3-body recombination rate for atoms with large scattering length at nonzero temperature. D’Incao, Suno, and Esry calculated the 3-body recombination rate at nonzero temperature for a simple model potential whose range and depth were tuned to give a single S-wave bound state and a large scattering length [9]. They considered the qualitative effect of temperature on both the resonant enhancement in the case \( a < 0 \) and on the local minimum in the case \( a > 0 \). Jonsell [10] and Yamashita, Frederico, and Tomio [11] considered the effects of temperature on the resonant enhancement in the case \( a < 0 \). Jonsell used the adiabatic hyperspherical approximation to calculate the 3-body elastic scattering rate as a function of the collision energy [10]. Yamashita et al. calculated the position and width of the Efimov resonance using a numerically exact method [11]. The two groups used their results to estimate the 3-body recombination rate at nonzero temperature. They were able to describe qualitatively the temperature dependence of the data from the Innsbruck experiment. Neither of these calculations is a definitive universal prediction for the 3-body recombination rate at nonzero temperature.

In this paper, we use published results on the 3-body recombination rate for \(^4\text{He}\) atoms to constrain the universal functions that determine its dependence on the collision energy. Those universal functions are then used to calculate the 3-body recombination rate at nonzero temperature for other atoms with large positive scattering length. The constraints from the \(^4\text{He}\) results are strong if the scattering length is near the minimum of the 3-body recombination rate at threshold. We apply our results to \(^{133}\text{Cs}\) atoms with a large positive scattering length, and compare them with the experimental results from the Innsbruck group.

THREE-BODY RECOMBINATION

Three-body recombination is a 3-atom collision process in which two of the atoms bind to form a diatomic molecule (a dimer). The 3-body recombination rate \( R \) is a function of the momenta \( \mathbf{p}_1, \mathbf{p}_2, \) and \( \mathbf{p}_3 \) of the three incoming atoms. Galilean invariance implies that \( R \) does not depend on the total momentum \( \mathbf{P}_{\text{tot}} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 \). It can therefore be expressed as a function of a pair of Jacobi momenta \( \mathbf{p}_{12} \) and \( \mathbf{p}_{3,12} \). It is convenient to parameterize the Jacobi momenta in terms of the collision energy \( E \), four angles describing the orientations of the Jacobi momenta, and a hyperangle \( \alpha_3 \) determined by the ratio of the magnitudes of the Jacobi momenta. We denote the 3-body recombination rate averaged over the angles and the hyperangle by \( K_3(E) \):

\[
K_3(E) = \langle R(\mathbf{p}_{12}, \mathbf{p}_{3,12}) \rangle_{\hat{p}_{12}, \hat{p}_{3,12}, \alpha_3}.
\]  

If the scattering length \( a \) is positive and large compared to the range of the interaction, one of the dimers that can be produced by the recombination process is the shallow dimer with binding energy

\[
E_D = \frac{\hbar^2}{(ma^2)}.
\]  

If there are deep dimers, they can be produced by the recombination process for either sign of \( a \). The recombination rate can be decomposed into the contribution from the shallow dimer and the sum of the contributions from all the deep dimers:

\[
K_3(E) = K_{\text{shallow}}(E) + K_{\text{deep}}(E).
\]
The 3-body recombination rate into the shallow dimer can be further decomposed into contributions from the channels in which the total orbital angular momentum of the three atoms has definite quantum number $J$:

$$K_{\text{shallow}}(E) = \sum_{J=0}^{\infty} K^{(J)}(E).$$

(5)

The scaling behavior near the scattering threshold for each of the angular momentum contributions is known [12]: $K^{(J)}(E) \sim E^{2J}$, where $\lambda_J = 0, 3, 2,$ and $3$ for $J = 0, 1, 2,$ and $3$. At the scattering threshold $E = 0$, only the $J = 0$ term is nonzero.

In a gas of atoms with number density $n_A$, the rate of decrease in the number density due to 3-body recombination defines an experimentally measurable loss rate constant $L_3$:

$$\frac{d}{dt} n_A = -L_3 n_A^3.$$  

(6)

The event rate constant $\alpha$ for 3-body recombination is defined so that the number of recombination events per time and per volume is $\alpha n_A^3$. If the number of atoms lost from the system per recombination event is $n_{\text{lost}}$, the rate constant is $L_3 = n_{\text{lost}} \alpha$. The binding energy of the dimer is released through the kinetic energies of the recoiling atom and dimer. If their kinetic energies are large enough that the atom and dimer both escape from the system and if they don’t undergo further collisions before escaping, then $n_{\text{lost}} = 3$. On the other hand, if they both remain in the system and if we regard the dimer as a distinct chemical species, then $n_{\text{lost}} = 2$.

In an ensemble of identical bosons with number density $n_A$ and 3-particle momentum distribution $f(p_1, p_2, p_3)$, the event rate per volume and per time is

$$\alpha n_A^3 = \frac{1}{6} \int R(p_{12}, p_{3,12}) f(p_1, p_2, p_3) \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_3}{(2\pi)^3}.$$ 

(7)

The factor of $1/6$ compensates for overcounting the states of the three identical particles by integrating over the entire range of the three momenta. The 3-particle momentum distribution is normalized so that the number of 3-particle states per volume-cubed is $n_A^3/3!$:

$$\frac{1}{6} \int f(p_1, p_2, p_3) \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_3}{(2\pi)^3} = \frac{n_A^3}{6}.$$ 

(8)

Dividing Eq. (7) by Eq. (8), we get an expression for $\alpha$ as a ratio of two integrals.

If the system is in thermal equilibrium at temperature $T$, the 3-particle momentum distribution $f(p_1, p_2, p_3)$ is the product of three Bose-Einstein distributions. If $T$ is large compared to the critical temperature for Bose-Einstein condensation, $k_B T_c \gg 3.3 \hbar^2 n_A^{2/3}/m$, the 3-particle momentum distribution can be approximated by a Boltzmann distribution proportional to $\exp(-E_{\text{tot}}/(k_B T))$, where $E_{\text{tot}}$ is the total energy of the three atoms. The total energy is the sum of the center-of-mass energy and the collision energy $E$: $E_{\text{tot}} = P_{\text{tot}}^2/(6m)+E$. The Boltzmann factor is therefore the product of a Boltzmann factor for the center-of-mass energy and a Boltzmann factor for the collision energy. Since the recombination rate does not depend on the total momentum $P_{\text{tot}}$, the event rate constant $\alpha(T)$ reduces to the Boltzmann average over the collision energy $E$:

$$\alpha(T) = \frac{1}{6} \int_0^\infty dE E^2 \frac{dE}{e^{-E/(k_B T)}} K_3(E).$$ 

(9)
The weight factor $E^2$ comes from using hyperspherical variables for the Jacobi momenta. The integral in the denominator can be evaluated analytically to give $2(k_B T)^3$.

**UNIVERSAL BEHAVIOR**

In this section and in the subsequent two sections, we consider atoms for which the only diatomic molecule is the shallow dimer and there are no deep dimers. The total 3-body recombination rate $K_3(E)$ is therefore equal to $K_{\text{shallow}}(E)$. An example of an atom with a large positive scattering length and no deep dimers is $^4\text{He}$. In this section, we summarize the universal information that is known on the recombination rate $K_3(E)$ for this case.

At the scattering threshold $E = 0$, only the $J = 0$ term in the decomposition of $K_{\text{shallow}}(E)$ into orbital angular momentum channels in Eq. (5) is nonzero. An analytic expression for this term has recently been derived [13, 14]:

$$K^{(0)}(E = 0) = \frac{768\pi^2(4\pi - 3\sqrt{3})}{\sin^2(\pi s_0) + \cos^2[\ln(a/a_{*0})]} \frac{\hbar a^4}{m}. \quad (10)$$

The parameter $a_{*0}$ differs from $\kappa_*^{-1}$ by a multiplicative constant that is known only to a couple digits of accuracy: $a_{*0} \approx 0.32 \kappa_*^{-1}$. The coefficient of $\hbar a^4/m$ in Eq. (10) is a log-periodic function of $a/a_{*0}$ that oscillates between zero and a maximum value $C_{\text{max}}$ as a function of $\ln(a)$. The analytic expression for the maximum value is

$$C_{\text{max}} = \frac{768\pi^2(4\pi - 3\sqrt{3})}{\sin^2(\pi s_0)}. \quad (11)$$

Its numerical value is $C_{\text{max}} \approx 402.7$. The expression in Eq. (10) has zeroes when $a$ is $(e^{\pi/s_0})^n a_{*0}$, where $n$ is an integer. Since $\sin^2(\pi s_0) \approx 139$ is so large, $K^{(0)}(0)$ can be approximated with an error of less than 1% of $C_{\text{max}} \hbar a^4/m$ by

$$K^{(0)}(E = 0) \approx C_{\text{max}} \sin^2[\ln(a/a_{*0})] \hbar a^4/m. \quad (12)$$

This approximate functional form of the rate constant was first deduced in Refs. [4, 5]. The coefficient $C_{\text{max}}$ and the relation between $a_{*0}$ and $\kappa_*$ was calculated numerically in Refs. [6, 13]. The maxima of the coefficient of $\hbar a^4/m$ in $K^{(0)}(0)$ in Eq. (12) occur when $a$ is $(e^{\pi/s_0})^n 4.76 a_{*0}$. The maxima of $K^{(0)}(0)$ in Eq. (12) occur when $a$ is $(e^{\pi/s_0})^n 14.3 a_{*0}$.

If the collision energy $E$ is small compared to the natural energy scale $\hbar^2/(m\ell^2)$ set by the range $\ell$, the 3-body recombination rate $K_{\text{shallow}}(E)$ is a universal function of the collision energy $E$, the scattering length $a$, and the Efimov parameter $a_{*0}$. The calculation of the rate can be reduced to the calculation of universal functions of a scaling variable $x$ defined by

$$x = (ma^2E/\hbar^2)^{1/2}. \quad (13)$$

Only the $J = 0$ term in $K_{\text{shallow}}(E)$ depends on $a_{*0}$, and that dependence is log-periodic with discrete scaling factor $e^{\pi/s_0} \approx 22.7$. The constraints of universality are therefore particularly simple for $J \geq 1$. These terms are determined by a single universal function for each $J$:

$$K^{(J)}(E) = f_J(x) \hbar a^4/m. \quad (14)$$

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1 This constant differs from the constant $C_{\text{max}}$ in Ref. [13] by a factor of 6.
The constraints of universality are more intricate for $J = 0$. They can be deduced from Efimov’s radial law, which follows from unitarity and from the fact that there is only one attractive adiabatic hyperspherical potential in the scale-invariant region of the hyperradius $R$: $\ell \ll R \ll a$. Efimov’s radial law implies that the $J = 0$ term in the 3-body recombination rate must have the form 

$$K^{(0)}(E) = \frac{k}{x^4} \left| s_{32}(x) + \frac{s_{31}(x)s_{12}(x)e^{2is_0 \ln(a/a_0)}}{1 - s_{11}(x)e^{2is_0 \ln(a/a_0)}} \right|^2 \frac{\hbar a^4}{m},$$

(15)

where $k$ is a constant and the functions $s_{ij}(x)$ are entries of a symmetric $3 \times 3$ unitary matrix. The values of $s_{11}(x)$ and $s_{12}(x)$ at the threshold are known analytically up to phases [14]:

$$s_{11}(0) = e^{-2\pi s_0} e^{2i\delta_1},$$

(16a)

$$s_{12}(0) = \sqrt{1 - e^{-4\pi s_0}} e^{2i\delta_2}.$$  

(16b)

The entries $s_{32}(x)$ and $s_{31}(x)$ go to zero as $x^2$ as the threshold is approached. Their ratio at the threshold is also known analytically [14]:

$$s_{31}(x)/s_{32}(x) \to \sqrt{\coth(\pi s_0)} e^{2i(\delta_1 - \delta_2)} \quad \text{as} \quad x \to 0.$$  \hspace{1cm} (17)

This limiting behavior together with the analytic values in Eqs. (16) are sufficient to deduce the analytic result in Eq. (10) up to a multiplicative normalization constant provided that $\delta_1 = \pm \pi/2$.

**ANALYSIS OF $^4$He ATOMS**

$^4$He atoms are a classic example of atoms with a large positive scattering length. There are several modern potentials that are believed to describe the interactions of $^4$He atoms with low energy accurately. The only one for which the 3-body recombination rate has been calculated is the HFD-B3-FCI1 potential [16]. The scattering length for the HFD-B3-FCI1 potential is $a = 172 a_0$. This is much larger than either the effective range $r_s = 14 a_0$ or the van der Waals length scale $\ell_{vdW} = 10.2 a_0$. The large scattering length $a$ implies that $^4$He atoms have universal properties that are determined by $a$. These properties can be calculated as expansions in powers of the range divided by $a$. We refer to the leading order in this expansion as the scaling limit or the zero-range limit. A thorough analysis of the universal properties of $^4$He atoms in the scaling limit has been presented by Braaten and Hammer [15]. Various 3-body observables for $^4$He atoms have been calculated by Platter and Phillips to next-to-next-to-leading order in the expansion in powers of the range divided by $a$ [17]. The agreement with numerical results from exact solutions of the 3-body Schrödinger equation is impressive [18, 19].

The $^4$He potential supports a diatomic molecule with a single energy level. The binding energy of this shallow dimer in the HFD-B3-FCI1 potential is $E_D = 1.600$ mK [20]. Experience has shown that the universal predictions for 3-body observables are significantly more accurate if the dimer binding energy is used as the 2-body input instead of $a$ [15]. Using the universal expression for the dimer binding energy in Eq. (3), we obtain the scattering length

$$a_{^4\text{He}} = 164.5 a_0.$$  \hspace{1cm} (18)

\footnote{A convenient conversion constant for $^4$He atoms $\hbar^2/m = 43.2788$ K $a_0^2$.}
The 3-body recombination rate $K_3(E)$ for $^4$He atoms interacting through the HFD-B3-FCI1 potential \[16\] has been calculated as a function of the collision energy from the threshold to 10 mK \[12\]. The results were separated into the contributions from $J = 0, 1, 2$, and 3. The individual contributions and their sum are shown in Fig. 1. We can determine the value of $a_{*0}$ for $^4$He atoms interacting through the HFD-B3-FCI1 potential from the value of the 3-body recombination loss rate at threshold: $K_3(0) = 7.10 \times 10^{-28}$ cm$^6$/s. Inserting $a_{\text{He}} = 164.5 \, a_0$ in Eq. (10) and solving for $a_{*0}$, we obtain

$$a_{*0}^{\text{He}} = 143.1 \, a_0.$$  \hspace{1em} (19)

The result is approximately equal to $1.150 \, a_{\text{He}}$. The near equality between $a_{*0}^{\text{He}}$ and $a_{\text{He}}$ reflects the fact that $K_3(0)$ for $^4$He is much smaller than the maximum value $C_{\text{max}} \hbar a_{\text{He}}^4/m = 3.67 \times 10^{-26}$ cm$^6$/s allowed by universality, which implies that $a/a_{*0}$ is close to a zero of the sine function in the numerator of Eq. (12). The value of $a_{*0}$ for $^4$He can also be determined from the binding energy of the excited $^4$He trimer, which is $E_3^{(1)} = 2.62$ mK for the HFD-B3-FCI1 potential \[20\]. The resulting value, $a_{*0} \approx 146\, a_0$, is consistent with the value in Eq. (19). Its accuracy is limited by the accuracy to which the numerical value of the product $a_{*0} \kappa_0 \approx 0.32$ is known.

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**FIG. 1:** The 3-body recombination rate $K_3(E)$ for $^4$He atoms (in units of cm$^6$/s) as a function of the collision energy $E$ (in units of mK) from Ref. \[12\]. The four lowest curves at $E = 10$ mK starting from the bottom are the contributions to $K_3(E)$ from $J = 0, 1, 3$ and 2 (thin solid, dashed, dotted and dot-dashed curve, respectively). The solid curve at the top is their sum.

We can use the $^4$He results calculated in Ref. \[12\] to determine the universal functions in the contributions to the 3-body recombination rate for $J = 1, 2$, and 3. By fitting the results of Ref. \[12\] for $J = 1, 2$, and 3, the universal functions $f_J(x)$ defined by Eq. (14) can be determined in the range $0 < x < 2.5$. The behavior of these universal functions near the
threshold are

\begin{align}
    f_1(x) &\approx 3.43 x^6, \\
    f_2(x) &\approx 261 x^4, \\
    f_3(x) &\approx 90.3 x^6.
\end{align}  

(20a)  

(20b)  

(20c)

The results of Ref. \cite{12} for $J = 0$ do not contain enough information to determine all the independent universal functions in the expression in Eq. (15) that comes from Efimov's radial law. However, if we make a few plausible simplifying assumptions, the universal functions for $J = 0$ can be strongly constrained. According to Eq. (16a), the absolute value of the function $s_{11}(x)$ is approximately 0.002 at the threshold $x = 0$. Our first simplifying assumption is that this function remains small compared to 1 and can be neglected for $x < 2.5$. The expression for the 3-body recombination rate in Eq. (15) can then be reduced to

\[
K^{(0)}(E) \approx C_{\text{max}} \left| \begin{array}{c}
    \sin[s_0 \ln(a/a_{s0})](1 + h_1(x)) + \cos[s_0 \ln(a/a_{s0})]h_2(x) \\
    + i \left( \sin[s_0 \ln(a/a_{s0})]h_3(x) + \cos[s_0 \ln(a/a_{s0})]h_4(x) \right)
\end{array} \right|^2 \frac{2\hbar a^4}{m},
\]

(21)

where the functions $h_i(x)$ are real-valued and vanish at $x = 0$. In the case of $^4$He, the coefficients in Eq. (21) are

\begin{align}
    \sin[s_0 \ln(a/\bar{a}_{\text{He}})] &= 0.140, \\
    \cos[s_0 \ln(a/\bar{a}_{\text{He}})] &= 0.990.
\end{align}  

(22a)  

(22b)

A dramatic feature of the $J = 0$ contribution to $K_3(E)$ for $^4$He in Fig. 1 is the deep local minimum at the collision energy $E_{\text{min}} = 2.1$ mK. The rate at the minimum is $K^{(0)}(E_{\text{min}}) = 2.3 \times 10^{-39}$ cm$^6$/s. The rate decreases by more than an order of magnitude as $E$ goes from 0.6 mK to 2.1 mK and then it increases by almost an order of magnitude as $E$ goes from 2.1 mK to 10 mK. This dramatic behavior can be reproduced by slowly varying universal functions if the real part of the amplitude inside the absolute value sign in Eq. (21) changes sign at $E_{\text{min}}$ and if the imaginary part of the amplitude is small compared to the real part except near $E_{\text{min}}$. Our second simplifying assumption is therefore that the imaginary part of that amplitude can be neglected for $x < 2.5$. Thus the expression for the rate that follows from our two simplifying assumptions is

\[
K^{(0)}(E) = C_{\text{max}} \left| \sin[s_0 \ln(a/a_{s0})](1 + h_1(x)) + \cos[s_0 \ln(a/a_{s0})]h_2(x) \right|^2 \frac{2\hbar a^4}{m}. 
\]

(23)

We expect this to be a good approximation except near those values of $a$ and $E$ at which it vanishes, which in the case of $^4$He is at $E = 2.1$ mK.

The approximation for $K^{(0)}(E)$ in Eq. (23) still involves two unknown functions $h_1(x)$ and $h_2(x)$. The $J = 0$ results for $^4$He in Ref. \cite{12} can be used to determine only a specific linear combination of those two functions:

\[
h_{\text{He}}(x) = 0.140 h_1(x) + 0.990 h_2(x).
\]

(24)

To determine this linear combination, we fit the $J = 0$ results of Ref. \cite{12} using the expression

\[
K_{\text{He}}^{(0)}(E) = C_{\text{max}} [0.140 + h_{\text{He}}(x)]^2 \frac{2\hbar a_{\text{He}}^4}{m} + K_{\text{He}}^{(0)}(E_{\text{min}}) (2E/(E_{\text{min}} + E))^2. 
\]

(25)
The last term in Eq. (25) is a simple model for the term in \( K(0)(E) \) that comes from the square of the imaginary part of the amplitude in Eq. (21). It has the correct value at the threshold, where it must vanish, and at \( E_{\text{min}} = 2.1 \) mK. This model affects the fit for \( h_{\text{He}}(x) \) only in the region very close to \( E_{\text{min}} \). The resulting function \( h_{\text{He}}(x) \) vanishes at \( x = 0 \), decreases monotonically to \(-0.135 \) at \( x = 1.15 \), and then decreases further to \(-0.206 \) at \( x = 2.5 \).

OTHER SYSTEMS WITHOUT DEEP DIMERS

If the 3-body recombination rate had been calculated for another system of identical bosons with a large positive scattering length, such as \(^4\text{He} \) atoms with a potential other than HFD-B3-FCI1, we would be able to use those results to determine a different linear combination of \( h_1(x) \) and \( h_2(x) \) than in Eq. (24). We would then know two independent functions \( h_1(x) \) and \( h_2(x) \), and we would be able to calculate \( K(0)(E) \) for any system of identical bosons with large positive scattering length. Although we cannot determine \( h_1(x) \) and \( h_2(x) \) from the results for \(^4\text{He} \), we can put constraints on them if assume that \( h_{\text{He}}(x) \) is not unnaturally small because of a near cancellation between the two terms in Eq. (24). The constraint in Eq. (24) on \( h_1(x) \) and \( h_2(x) \) can be implemented by expressing them in the form

\[
\begin{align*}
h_1(x) &= \frac{t(x)}{0.140} h_{\text{He}}(x), \quad (26a) \\
h_2(x) &= \frac{1 - t(x)}{0.990} h_{\text{He}}(x), \quad (26b)
\end{align*}
\]

where \( t(x) \) is an unknown function of \( x \). If \( t(x) \) is in the range \( 0 < t(x) < 1 \), the two terms in Eq. (24) contribute to \( h_{\text{He}}(x) \) with the same sign. If \( t(x) \) is in the range

\[
-2 < t(x) < +3,
\]

the two terms in Eq. (25) can have opposite signs, but the absolute value of their sum cannot be smaller than that of the larger of the two terms by more than a factor of 3. This range is consistent with our assumption that \( h_{\text{He}}(x) \) is not unnaturally small. Given this assumption, we can generate a range of predictions for the 3-body recombination rate for other systems with large scattering lengths by varying \( t(x) \) in the range given by Eq. (27).

The 3-body recombination rate \( K_3(E) \) is shown in Fig. 2 as a function of the collision energy \( E \) for three values of the ratio \( a/a_{40} \) of the scattering length to the Efimov parameter: \( a/a_{40} = 1 \), for which \( K_{\text{shallow}}(0) = 0 \), \( a/a_{40} = 1.150 \), which corresponds to \(^4\text{He} \), and \( a/a_{40} \approx 4.76 \), for which \( K_{\text{shallow}}(0) = C_{\text{max}} ha^4/m \). The rate \( K_3(E) \) is the sum of the \( J = 0 \) term in Eq. (23) and the \( J = 1, 2, \) and 3 terms in Eq. (13). The rate for each value \( a/a_{40} \) is shown as an error band that corresponds to varying the parameter \( t \) defined by Eqs. (26) over the range in Eq. (27). Since the error band comes only from the \( J = 0 \) term, it always remains above the contribution from the \( J = 2 \) term. In the \(^4\text{He} \) case, \( a/a_{40} = 1.150 \), the band reduces to a curve given by the heavy solid line. For \( a/a_{40} = 1 \), the rate is dominated by the \( J = 2 \) term and the error band is always fairly narrow. It extends above the \( J = 2 \) curve by a factor that is about 1.5 for \( E < 0.25 E_D \) and increases to about 3.6 at \( E = 6.25E_D \). For \( a/a_{40} = 4.76 \), the error band is narrow for \( E < 10^{-2} E_D \), but its width increases quickly with \( E \). For \( E > 0.25 E_D \), the error band extends above the \( J = 2 \) curve by about two orders of magnitude.
FIG. 2: The 3-body recombination rate $K_3(E)$ for atoms with no deep dimers and a large scattering length. The rate $K_3(E)$ (in units of $\hbar a^4/m$) is shown as a function of the collision energy $E$ (in units of $\hbar^2/(ma^2)$) for various values of the ratio $a/a_0$: 1 (solid lines), 1.150 (heavy solid line), and 4.76 (dashed lines). The band of predictions for each value of $a/a_0$ corresponds to the range $-2 < t < +3$ for the variable $t$ in Eq. (26).

Given a range of predictions for $K_3(E)$, we can obtain a range of predictions for the 3-body recombination event rate constant $\alpha(T)$ given by Eq. (9). Since the results in Ref. [12] only allow us to determine the function $h_{He}(x)$ in the region $x < 2.5$, the integral in the numerator of Eq. (9) can only be evaluated up to the energy $E_{\text{max}} = 6.25E_D$. As long as $T < 0.74E_D$, more than 99% of the weight of the normalizing integral in the denominator of Eq. (9) is in the region $x < 2.5$. In Fig. 3 $\alpha(T)$ is shown as a function of $a$ for fixed $a_0$ and several values of $T$. The range of predictions is shown by plotting 6 curves corresponding to integer values of the variable $t$ defined in Eq. (26) ranging from $-2$ to $+3$. The widths of the error bands reduce to zero at the $^4\text{He}$ value $a = 1.150a_0$. The error bands remain narrow in a region that includes the value $a = a_0$ at which $K_{\text{shallow}}(E = 0)$ vanishes. We therefore obtain almost unique predictions for $\alpha(T)$ near its local minimum as a function of $a$. The depth of the minimum decreases as the temperature $T$ increases.

EFFECTS OF DEEP DIMERS

In this section and in the subsequent section, we consider atoms that have deep dimers. In the scaling (or zero-range) limit, the cumulative effect of all the deep dimers on Efimov physics can be taken into account rigorously through one additional parameter $\eta_*$ [7]. The effects of Efimov physics can be dramatic only if $\eta_*$ is much less than 1. If the universal expression for a scattering amplitude for the case of no deep dimers is known as an analytic
FIG. 3: The 3-body recombination event rate constant \( \alpha(T) \) for atoms with no deep dimers and a large scattering length. The rate constant \( \alpha(T) \) (in units of \( \hbar a_s^4/m \)) is shown as a function of \( a \) for fixed \( a_s0 \) and several values of the temperature \( T \): \( T = 0 \) (solid line), \( 0.007 T_s0 \) (dashed line), \( 0.07 T_s0 \) (dotted line), where \( k_B T_s0 = \hbar^2/(ma_s0^2) \). The 6 curves for each temperature \( T \) correspond to integer values of \( t \) from \( -2 \) to \( +3 \).

function of \( a_s0 \), the corresponding result for a system with deep dimers can be obtained without any additional calculation simply by making the substitution

\[
\ln a_s0 \longrightarrow \ln a_s0 - i \eta_s/s_0.
\]

Making this substitution in the factor \( \sin[s_0 \ln(a/a_s0)] \) in Eq. (12), the resulting expression for the recombination rate at threshold into the shallow dimer is

\[
K_{\text{shallow}}(E = 0) \approx C_{\text{max}} \left( \sin^2[s_0 \ln(a/a_s0)] + \sinh^2 \eta_s \right) \hbar a^4/m.
\]

This expression shows that one effect of the deep dimers is to eliminate the zeroes of \( K_{\text{shallow}}(0) \). Making this substitution in Eq. (23), the resulting expression for the \( J = 0 \) contribution to \( K_{\text{shallow}}(E) \) is

\[
K^{(0)}(E) = C_{\text{max}} \left[ \cosh^2 \eta_s \left( \sin[s_0 \ln(a/a_s0)](1 + h_1(x)) + \cos[s_0 \ln(a/a_s0)]h_2(x) \right)^2 \\
+ \sinh^2 \eta_s \left( \cos[s_0 \ln(a/a_s0)](1 + h_1(x)) - \sin[s_0 \ln(a/a_s0)]h_2(x) \right)^2 \right] \hbar a^4/m.
\]

If there are deep dimers, there is an additional contribution \( K_{\text{deep}}(E) \) from 3-body recombination rate into deep dimers. A generalization of Efimov’s radial law implies that \( K_{\text{deep}}(E) \) must have the form

\[
K_{\text{deep}}(E) = \frac{k|s_{21}(x)|^2(1 - e^{-4\eta_s})}{x^4 |1 - s_{11}(x)e^{2i[s_0 \ln(a/a_s0)-2\eta_s]}|^2} \hbar a^4/m.
\]
where \( k \) is the same constant and \( s_{11}(x) \) and \( s_{31}(x) \) are the same entries of a symmetric \( 3 \times 3 \) unitary matrix that appear in Eq. (15). We can obtain an analytic expression for the value at the threshold \( E = 0 \) by using the value of \( s_{11}(0) \) given in Eq. (16a) and the limiting behavior of \( s_{31}(x) \) as \( x \to 0 \):

\[
K_{\text{deep}}(E = 0) = \frac{C_{\text{max}} \sinh(2\pi s_0) \sinh(2\eta_*)}{4 \left( \sinh^2(\pi s_0 + \eta_*) + \cos^2[s_0 \ln(a/a_0)] \right)} \frac{\hbar a^4}{m}. \tag{32}
\]

Since \( \sinh^2(\pi s_0 + \eta_*) > 139 \) is so large, \( K_{\text{deep}}(0) \) can be approximated with an error of less than 1% by omitting the \( \cos^2 \) term in the denominator of Eq. (32). To within the same accuracy, we can replace \( \sinh(2\pi s_0) \) and \( \sinh(\pi s_0 + \eta_*) \) by exponentials to get the simpler expression

\[
K_{\text{deep}}(E = 0) \approx \frac{(1 - e^{-4\eta_*})C_{\text{max}}}{4} \frac{\hbar a^4}{m}. \tag{33}
\]

One of the simplifying assumption that we used to obtain the expression for \( K^{(0)}(E) \) in Eq. (23) was that \( |s_{11}(x)| \ll 1 \) for \( x < 2.5 \). We can also use this assumption to simplify the expression for \( K_{\text{deep}}(E) \) in Eq. (31). The only dependence of \( K_{\text{deep}}(E) \) on the collision energy \( E \) then comes from the factor \( |s_{13}(x)|^2 \) in the numerator. The \( ^4\text{He} \) results do not give significant constraints on this factor, because in the expression for \( K^{(0)}(E) \) in Eq. (15), \( s_{13}(x) \) appears in the combination \( s_{12}(x)s_{13}(x) \). We will therefore make the simplifying assumption that \( |s_{13}(x)|^2 \) is a sufficiently slowly varying function of \( x \) for \( x < 2.5 \) that we can approximate \( K_{\text{deep}}(E) \) by its value at \( E = 0 \), which is given in Eq. (33).

The dependence of the 3-body recombination event rate constant \( \alpha(T) \) on the scattering length \( a \) is illustrated in Figs. 4 and 5. The rate \( K_3(E) \) in the numerator of Eq. (9) is the sum of \( K_{\text{shallow}}(E) \), for which the \( J = 0 \) term is given in Eq. (30) and the \( J = 1, 2, \) and 3 terms are given in Eq. (14), and \( K_{\text{deep}}(E) \), which is approximated by Eq. (33). For each value of \( T \) and \( \eta_* \), the range of predictions is shown by plotting 6 curves corresponding to integer values of the variable \( t \) defined in Eq. (26) ranging from -2 to +3. The widths of the error bands reduce to zero at the \( ^4\text{He} \) value \( a = 1.150a_0 \). In Fig. 4, \( \eta_* \) is fixed at the value 0.01 and \( \alpha(T) \) is plotted as a function of \( a \) for various values of \( T \). A convenient unit for the temperature is \( T_{s0} = h^2/(k_Bma_0^2) \). As \( T \) increases from 0 to 0.007 \( T_{s0} \), the prediction for \( a \) near \( a_0 \) is essentially unchanged. As \( T \) increases further, the local minimum decreases in depth and it tends to disappear around \( T = 0.1T_{s0} \). In Fig. 5, \( T \) is fixed at the value 0.007 \( T_{s0} \) and \( \alpha(T) \) is plotted as a function of \( a \) for various values of \( \eta_* \). As \( \eta_* \) increases, the depth of the local minimum decreases until it disappears around \( \eta_* = 0.06 \).

Our approximation for the 3-body recombination rate \( K_3(E) \) depends on the fitted functions \( h_1(x) \) and \( h_2(x) \) in Eq. (31) and \( f_J(x) \) in Eq. (14). If the only energies that contribute significantly to the Boltzmann average in Eq. (9) are ones that satisfy \( E \ll E_D \), we can use a much simpler approximation. We can approximate \( K_3(E) \) by the sum of the \( J = 0 \) term of \( K_{\text{shallow}}(E) \) at \( E = 0 \), which is given in Eq. (29), the leading term in the low-energy expansion of the \( J = 2 \) term, which is given by Eqs. (14) and (20b), and \( K_{\text{deep}}(0) \), which is given in Eq. (33). Inserting these expression into Eq. (9), we get a simple expression for the rate constant for 3-body recombination:

\[
\alpha_{\text{total}}(T) \approx 67.1 \left( \sin^2[s_0 \ln(a/a_0)] + \frac{1}{2} \cosh(2\eta_*)(1 - e^{-2\eta_*}) + 7.8 \left( \frac{k_BT}{E_D} \right)^2 \right) \frac{\hbar a^4}{m}. \tag{34}
\]
FIG. 4: The 3-body recombination event rate constant $\alpha(T)$ for atoms with deep dimers and a large scattering length. The rate constant $\alpha(T)$ (in units of $\hbar a_0^4/m$) is shown as a function of $a$ for $\eta_s = 0.01$ and several values of the temperature $T$: $T = 0$ (solid line), $0.007 T_{*0}$ (dashed line), $0.07 T_{*0}$ (dotted line), where $k_B T_{*0} = \hbar^2/(ma_0^2)$. The 6 curves for each temperature $T$ correspond to integer values of $t$ from $-2$ to $+3$.

APPLICATION TO $^{133}$Cs ATOMS

The Innsbruck group has carried out beautiful measurements of the 3-body recombination rate for ultracold $^{133}$Cs atoms in the $|f = 3, m_f = +3\rangle$ hyperfine state [8]. By varying the magnetic field from 0 to 150 G, they were able to change the scattering length from $-2500 a_0$ through 0 to $+1600 a_0$. In this range of magnetic field, the $|f = 3, m_f = +3\rangle$ state is the lowest hyperfine state, so 2-body losses are energetically forbidden. Thus, the dominant loss mechanism is 3-body recombination. The van der Waals length scale for Cs atoms is $(mC_6/\hbar^2)^{1/4} \approx 200 a_0$. The range of scattering lengths studied by the Innsbruck group includes a region of large negative $a$ and a region of large positive $a$ separated by a region of small $|a|$. In the two regions of large scattering length, few-body physics should be universal. An interesting open question is whether there is any relation between the Efimov parameters $\kappa_s$ and $\eta_s$ that characterize the two universal regions.

In the region of negative $a$, the Innsbruck group measured the loss rate constant $L_3$ as a function of $a$ at three different temperatures: $T = 10$ nK, 200 nK, and 250 nK. They observed a dramatic enhancement of the loss rate for $a$ near $-850 a_0$. At $T = 10$ nK, the loss rate as a function of $a$ can be fit rather well by the universal formula for $T = 0$ in Ref. [7] with parameters $a'_* = -850(20) a_0$ and $\eta_* = 0.06(1)$. Thus, the large enhancement in the loss rate can be explained by the resonant enhancement from an Efimov trimer near the 3-atom threshold. More recently, the Innsbruck group has measured the position of

---

3 A convenient conversion constant for $^{133}$Cs atoms is $\hbar^2/m = 1.30339 \text{ K} a_0^2$. 

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FIG. 5: The 3-body recombination event rate constant $\alpha(T)$ for atoms with deep dimers and a large scattering length. The rate constant $\alpha(T)$ (in units of $\hbar a_0^4/m$) is shown as a function of $a$ for temperature $0.007 T_{*0}$, where $k_B T_{*0} = h^2/(ma_0^2)$, and several values of $\eta_*$: 0 (solid line), 0.01 (dashed line), and 0.06 (dotted line). The 6 curves for each value of $\eta_*$ correspond to integer values of $t$ from $-2$ to $+3$.

the maximum loss rate as a function of the temperature [21]. Its behavior as a function of temperature can be explained at least qualitatively by the dependence of the binding energy and width of the Efimov resonance on the scattering length [10, 11].

In the region of positive $a$, the Innsbruck group measured the loss rate constant $L_3$ at $T = 200$ nK for values of $a$ ranging up to $1228 a_0$. Their results in the region $0 < a < 600 a_0$ are shown as solid triangles in Fig. 6. The vertical axis is the recombination length $\rho_3$ defined by

$$\rho_3 = \left( \frac{2m}{\sqrt{3\hbar}} L_3 \right)^{1/4}.$$  \hspace{1cm} (35)

They observed a local minimum in the loss rate for $a$ near $200 a_0$. This value is near the van der Waals length scale $(mC_6/h^2)^{1/4} \approx 200 a_0$, so there may be large corrections to the universal predictions. The universal prediction for the loss rate at $T = 0$ is $\alpha = K_3(0)/6$, where $K_3(0)$ is the sum of Eqs. (22) and (33). By fitting the data for $a > 500 a_0$ to this expression, they obtained $a_+ = 1060(70) a_0$ for the scattering length $e^{\pi/(2s_0)} a_{*0}$ at which the coefficient of $a^4$ achieves its maximum value. Universality would then imply that the minimum should be at $a_{*0} = 223(15) a_0$. The fit was insensitive to the value of $\eta_*$ and yielded only the upper bound $\eta_* < 0.2$. The Innsbruck group also determined the location of the minimum directly by measuring the fraction of atoms that were lost after a fixed time. The result was $a_{\min} = 210(10) a_0$, which is consistent with the value obtained by fitting the data for $a > 500 a_0$.

We now consider whether our results obtained by scaling results from $^4$He atoms can be applied to this system of $^{133}$Cs atoms. In the experiments with positive scattering length,
the typical peak number density was $5 \times 10^{13}$ cm$^{-3}$. The corresponding critical temperature for Bose-Einstein condensation is $T_c \approx 160$ nK. Thus, the temperature $T = 200$ nK was not far above $T_c$. We ignore this complication and calculate thermal averages using the Maxwell-Boltzmann distribution as in Eq. (9) instead of the Bose-Einstein distribution. When $a = a_{\text{min}}$, the universal prediction for the binding energy $E_D$ of the shallow dimer in Eq. (3) gives $3 \times 10^4$ nK, which is about two orders of magnitude higher than the temperature. The binding energy $E_D$ decreases to about $3 \times 10^3$ nK at $a = 600 a_0$. Thus the temperature $T = 200$ nK is safely in the region $T < 0.7 E_D$ in which the thermal average in Eq. (9) can be calculated accurately from scaling the results for $^4$He atoms.

In Fig. (6), we compare the universal predictions for the recombination length $\rho_3$ defined in Eq. (35) with the Innsbruck data. We assume $n_{\text{lost}} = 3$, so that $L_3 = 3 \alpha$, and we set $a_{*0} = 210 a_0$. We plot $\rho_3$ for $T = 200$ nK as a function of $a$ for several values of $\eta_*$. For each value of $\eta_*$, a band of predictions is obtained by plotting 6 curves corresponding to integer values of the variable $t$ defined in Eq. (26) ranging from $-2$ to $+3$. The band has zero width at $a = 242 a_0$, and the band remains narrow for $a$ near $a_{\text{min}} = 210 a_0$. The predictions fall well below the data in the region $a < 300 a_0$, especially if $\eta_*$ is small enough that there is still a local minimum. This suggests that $a_{\text{min}}$ is too small for the universal predictions to be quantitatively accurate. For $a > 500 a_0$, the scattering length is large enough that we expect the universal predictions to be valid. The results of Ref. [8] for $a > 400 a_0$ lie within the error bands of our universal predictions. Unfortunately, the widths of the error bands are too large for $a > 500 a_0$ to allow us to determine the Efimov parameters $a_{*0}$ and $\eta_*$. 

![Graph](image_url)

FIG. 6: The 3-body recombination length $\rho_3$ for $^{133}$Cs atoms as a function of $a$ for $T = 200$ nK. The data points are from Ref. [8]. The curves are the universal prediction for several values of $\eta_*$: 0 (solid lines), 0.01 (dashed lines), and 0.06 (dotted lines). The 6 curves for each value of $\eta_*$ correspond to integer values of $t$ from $-2$ to $+3$. 

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SUMMARY

In this work, we have used previously published results on the 3-body recombination rate of $^4$He atoms as a function of the collision energy to constrain the universal functions that govern 3-body recombination processes for atoms with large scattering length. Since the scattering length for $^4$He atoms differs from the Efimov parameter $a_{\ast 0}$ by only about 15%, the constraints are very strong near a minimum in the 3-body recombination rate as a function of scattering length. We were unable to fit the $^{133}$Cs data from the Innsbruck group near the local minimum of the recombination rate by adjusting the Efimov parameters $a_{\ast 0}$ and $\eta_{\ast}$. This is not a surprise, because the local minimum occurs at a value of $a$ that is not large compared to the van der Waals length scale. The universal predictions should be accurate at larger values of the scattering length for which the recombination rate was measured by the Innsbruck group. Unfortunately the constraints from $^4$He atoms have large error bands at those larger values of $a$. Our universal predictions obtained by scaling results for $^4$He atoms can be applied to other systems of identical bosons for which the local minimum of the recombination rate occurs at a large value of the scattering length. For such a system, our results can be used to give a reliable determination of the Efimov parameters.

In recent years, an effective field theory has been developed for nonrelativistic few-body systems with large scattering length [22, 23]. It allows the model-independent calculation of few-body observables using a systematic expansion in $\ell/a$. The leading term in this expansion is the “universal” term. This effective field theory can be applied to a large variety of physical systems ranging from atomic physics to nuclear physics. It has been used to calculate the atom-dimer scattering phase shift as a function of the collision energy up to the dimer-breakup threshold [15]. The results were used to give universal predictions for the resonant dimer relaxation rate at nonzero temperature [24]. The predictions were compared with measurements by the Innsbruck group, which revealed a resonant enhancement of the inelastic loss rate from a system of ultracold $^{133}$Cs atoms and dimers [25]. This effective field theory can also be used to calculate the universal functions that determine the 3-body recombination rate as a function of the collision energy. Such a calculation would allow the Efimov parameters $\kappa_{\ast}$ and $\eta_{\ast}$ for $^{133}$Cs atoms to be determined by fitting the Innsbruck data for $a > 500 a_0$. It could also be applied to any other system of identical bosons with large positive scattering length. Once the Efimov parameters have been determined accurately, it will be possible to make quantitative tests of the correlations between different aspects of Efimov physics that are predicted by universality.

We thank H.-W. Hammer for valuable discussions. This research was supported in part by the Department of Energy under grants DE-FG02-05ER15715 (EB) and DE-FG02-93ER40756 (LP), by an Ohio University postdoctoral fellowship and by the Korea Research Foundation under grant KRF-2006-612-C00003 (DK).

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