AXIOMATIC $KK$-THEORY FOR REAL C*-ALGEBRAS

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Abstract. We establish axiomatic characterizations of $K$-theory and $KK$-theory for real C*-algebras. In particular, let $F$ be an abelian group-valued functor on separable real C*-algebras. We prove that if $F$ is homotopy invariant, stable, and split exact, then $F$ factors through the category $KK$. Also, if $F$ is homotopy invariant, stable, half exact, continuous, and satisfies an appropriate dimension axiom, then there is a natural isomorphism $K(A) \to F(A)$ for a large class of separable real C*-algebras $A$. Furthermore, we prove that a natural transformation $F(A) \to G(A)$ of homotopy invariant, stable, half-exact functors which is an isomorphism when $A$ is complex is necessarily an isomorphism when $A$ is real.

1. Introduction

In this paper we establish axiomatic characterizations of $K$-theory and $KK$-theory for real C*-algebras, following that established in the complex case by [5]. These results and techniques will be used in a forthcoming paper to prove a classification of real simple purely infinite C*-algebras along the lines of that in the complex case proven by [8] and [7].

Let $F$ be an abelian group-valued functor on separable real C*-algebras. We prove that if $F$ is homotopy invariant, stable, and split exact, then $F$ factors through the category $KK$. Also, if $F$ is homotopy invariant, stable, half exact, continuous, and satisfies the dimension axiom, then there is a natural isomorphism $K(A) \to F(A)$ for a large class of separable real C*-algebras $A$. These results are proven along the same lines as the proofs in [5].

If $F$ is homotopy invariant, stable, and split exact then we use $F$ to construct a $CRT$-module $F^{CRT}(A)$ (see [4], [2], and [3]) for any separable real C*-algebra. This $CRT$-module structure allows us to prove that a natural transformation $F(A) \to G(A)$ of homotopy invariant, stable,
half-exact functors must be an isomorphism if it is an isomorphism in the special case that $A$ is complex.

These main results are all in Section 3. In Section 2 we re-develop the essential preliminaries concerning $KK$-theory in the real case, including the equivalence of the two principal models of $KK$-theory: the Kasparov bimodule construction and the Fredholm picture.

2. The Fredholm Picture of $KK$-Theory

We take the following definition from Section 2.3 of [9] to be the standard definition of $KK$-theory for real C*-algebras. It is essentially the same as that in [6] where it was simultaneously developed for both real and complex C*-algebras.

**Definition 2.1.** Let $A$ be a graded separable real C*-algebra and $B$ be a real C*-algebra with a countable approximate identity.

(i) A Kasparov $(A-B)$-bimodule is a triple $(E, \phi, T)$ where $E$ is a countably generated graded Hilbert $B$-module, $\phi : A \to \mathcal{L}(E)$ is a graded $*$-homomorphism, and $T$ is an element of $\mathcal{L}(E)$ of degree 1 such that

$$(T - T^*)\phi(a), (T^2 - 1)\phi(a), \text{ and } [T, \phi(a)]$$

lie in $\mathcal{K}(E)$ for all $a \in A$.

(ii) Two triples $(E_i, \phi_i, T_i)$ are unitarily equivalent if there is a unitary $U$ in $\mathcal{L}(E_0, E_1)$, of degree zero, intertwining the $\phi_i$ and $T_i$.

(iii) If $(E, \phi, T)$ is a Kasparov $(A-B)$-bimodule and $\beta : B \to B'$ is a homomorphism of C*-algebras, then the pushed-forward Kasparov $(A-B')$-bimodule is defined by

$$\beta_*(E, \phi, T) = (E \hat{\otimes}_\beta B', \phi \hat{\otimes} 1, T \hat{\otimes} 1).$$

(iv) Two Kasparov $(A-B)$-bimodules $(E_i, \phi_i, T_i)$ for $i = 0, 1$ are homotopic if there is a Kasparov bimodule $(A-B \hat{\otimes} C[0, 1])$, say $(E, \phi, T)$, such that $(\varepsilon_i)_*(E, \phi, T)$ and $(E_i, \phi_i, T_i)$ are unitarily equivalent for $i = 0, 1$, where $\varepsilon_i$ denotes the evaluation map.

(v) A triple $(E, \phi, T)$ is degenerate if the elements

$$(T - T^*)\phi(a), (T^2 - 1)\phi(a), \text{ and } [T, \phi(a)]$$

are zero for all $a \in A$. By Proposition 2.3.3 of [9], degenerate bimodules are homotopic to trivial bimodules.

(vi) $KK(A, B)$ is defined to be the set of homotopy equivalence classes of Kasparov $(A-B)$-bimodules.
The following theorem summarizes the principal properties of $KK$-theory for real $C^*$-algebras from Chapter 2 of [9].

**Proposition 2.2.** $KK(A, B)$ is an abelian group for separable $A$ and $\sigma$-unital $B$. As a functor on separable real $C^*$-algebras (contravariant in the first argument and covariant in the second argument), it is homotopy invariant, stable, and has split exact sequences in both arguments. Furthermore, there is a natural associate pairing (the intersection product)

$$\otimes_C: KK(A, C \otimes B) \otimes KK(C \otimes A', B') \to KK(A \otimes A', B \otimes B').$$

We now turn to the Fredholm picture of $KK$-theory, which was developed in [5] with only the situation of complex $C^*$-algebras in mind. However, the approach goes through the same for real $C^*$-algebras, as follows.

**Definition 2.3.** Let $A$ and $B$ be real separable $C^*$-algebras.

(i) A $KK(A, B)$-cycle is a triple $(\phi_+, \phi_-, U)$, where $\phi_+: A \to \mathcal{M}(\mathcal{K} \otimes B)$ are $\ast$-homomorphisms, and $U$ is an element of $\mathcal{M}(\mathcal{K} \otimes B)$ such that

$$U\phi_+(a) - \phi_-(a)U, \quad \phi_+(a)(UU^* - 1), \quad \text{and} \quad \phi_-(a)(UU^* - 1)$$

lie in $\mathcal{K} \otimes B$ for all $a \in A$.

(ii) Two $KK(A, B)$-cycles $(\phi_+^i, \phi_-^i, U^i)$ are homotopic if there is a $KK(A, B \otimes C[0, 1])$-cycle $(\phi_+, \phi_-, U)$ such that $(\varepsilon_i \phi_+, \varepsilon_i \phi_-, \varepsilon_i(U)) = (\phi_+^i, \phi_-^i, U^i)$, where $\varepsilon_i : \mathcal{M}(\mathcal{K} \otimes B \otimes C[0, 1]) \to \mathcal{M}(\mathcal{K} \otimes B)$ is induced by evaluation at $i$.

(iii) A $KK(A, B)$-cycle $(\psi_+, \psi_-, V)$ is degenerate if the elements

$$V\psi_+(a) - \psi_-(a)V, \quad \psi_+(a)(VV^* - 1), \quad \text{and} \quad \psi_-(a)(VV^* - 1)$$

are zero for all $a \in A$.

(iv) The sum $(\phi_+, \phi_-, U) \oplus (\psi_+, \psi_-, V)$ of two $KK(A, B)$-cycles is the $KK(A, B)$-cycle

$$\begin{pmatrix}
\phi_+ & 0 \\
0 & \psi_+
\end{pmatrix}, \begin{pmatrix}
\phi_- & 0 \\
0 & \psi_-
\end{pmatrix}, \begin{pmatrix}
U & 0 \\
0 & V
\end{pmatrix}$$

where the algebra $M_2(\mathcal{M}(\mathcal{K} \otimes B))$ is identified with $\mathcal{M}(\mathcal{K} \otimes B)$ by means of some isomorphism $M_2(\mathcal{K}) \cong \mathcal{K}$, which is unique up to homotopy by Section 1.17 of [6].

(v) Two cycles $(\phi_+^i, \phi_-^i, U^i)$ are said to be equivalent if there exist degenerate cycles $(\psi_+^i, \psi_-^i, V^i)$ such that

$$(\phi_+^0, \phi_-^0, U^0) \oplus (\psi_+^0, \psi_-^0, V^0) \quad \text{and} \quad (\phi_+^1, \phi_-^1, U^1) \oplus (\psi_+^1, \psi_-^1, V^1)$$

are homotopic.
(vi) $\text{KK}(A, B)$ is defined to be the set of equivalence classes of $\text{KK}(A, B)$-cycles.

The following lemma is the real version of Lemma 2.3 of [5].

**Lemma 2.4.** $\text{KK}(A, B)$ is an abelian group, for separable real $C^*$-algebras $A$ and $B$. As a functor it is contravariant in the first argument and covariant in the second argument.

**Proof.** For the first statement, we show that a cycle $(\phi_+, \phi_-, U)$ has inverse $(\phi_-, \phi_+, U^*)$. Indeed, the sum

$$
\left(\begin{array}{cc}
\phi_+ & 0 \\
0 & \phi_-
\end{array}\right), \left(\begin{array}{cc}
\phi_- & 0 \\
0 & \phi_+
\end{array}\right), \left(\begin{array}{cc}
U & 0 \\
0 & U^*
\end{array}\right)
$$

is homotopic to a degenerate cycle via the operator homotopy

$$
W_t = \begin{pmatrix}
\cos(t)U & -\sin(t) \\
\sin(t) & \cos(t)U^*
\end{pmatrix}, \quad t \in \left[0, \frac{\pi}{2}\right].
$$

The functoriality is established as in [5] in Sections 2.4 through 2.7.

The next proposition establishes the isomorphism between the two pictures of $\text{KK}$-theory. First we review some preliminaries regarding graded $C^*$-algebras and Hilbert modules.

If $B$ is a real $C^*$-algebra, then the *standard even grading* on $M_2(B)$ is obtained by setting $M_2(B)^{(0)}$ to be the set of diagonal matrices and $M_2(B)^{(1)}$ the set of matrices with zero diagonal. The *standard even grading* on $\mathcal{K} \otimes B$ is obtained by choosing an isomorphism $\mathcal{K} \otimes B \cong M_2(\mathcal{K} \otimes B)$. This in turn induces a canonical (modulo a unitary automorphism) grading on $\mathcal{M}(\mathcal{K} \otimes B)$.

Let $\mathbb{H}_B$ be the Hilbert $B$-module consisting of all sequences $\{b_n\}_{n=1}^{\infty}$ in $B$ such that $\sum_{n=1}^{\infty} b_n^*b_n$ converges. Giving $B$ the trivial grading (that is $B^{(0)} = B$ and $B^{(1)} = \{0\}$), let $\mathbb{H}_B = \mathbb{H}_B \oplus \mathbb{H}_B$ be the graded Hilbert $B$-module with $\mathbb{H}_B^{(0)} = \mathbb{H}_B \oplus 0$ and $\mathbb{H}_B^{(1)} = 0 \oplus \mathbb{H}_B$. Then the induced grading on $\mathcal{L}(\mathbb{H}_B)$ is identical with the standard even grading of $M_2(\mathcal{M}(\mathcal{K} \otimes B))$. Under the isomorphism $M_2(\mathcal{M}(\mathcal{K} \otimes B)) \cong \mathcal{M}(\mathcal{K} \otimes B)$, this grading coincides with the one described in the previous paragraph.

**Theorem 2.5.** Let $A$ and $B$ be a real separable $C^*$-algebras. Then $\text{KK}(A, B)$ is isomorphic to $\text{KK}(A, B)$.

**Proof.** We give $A$ and $B$ the trivial grading and $\mathcal{M}(\mathcal{K} \otimes B)$ is given the standard even grading described above.
For a $\text{KK}(A, B)$-cycle $x = (\phi_,\phi_-,U)$ we define

$$\alpha(x) = \left( \hat{H}_B, \begin{pmatrix} \phi_+ & 0 \\ 0 & \phi_- \end{pmatrix}, \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix} \right).$$

It is readily verified that $\alpha(x)$ is a Kasparov $(A-B)$ bimodule. Furthermore, for $x = (\phi_+,\phi_-,U)$ and $y = (\psi_+,\psi_-,V)$ it is easy to see that $\alpha(x+y)$ and $\alpha(x)+\alpha(y)$ are unitarily equivalent via a degree 0 unitary.

We must show that $\alpha$ induces a well-defined homomorphism

$$\overline{\alpha} : \text{KK}(A, B) \rightarrow \text{KK}(A, B).$$

Note first that $\alpha$ sends degenerate elements to degenerate elements. Next, suppose $(\phi_+,\phi_-,U)$ is a $\text{KK}(A, B \otimes C[0, 1])$-cycle implementing a homotopy between $x = (\phi^0_+,\phi^0_-,U_0)$ and $y = (\phi^1_+,\phi^1_-,U_1)$. That is, $\varepsilon_i(\phi_+) = \phi^i_+$, $\varepsilon_i(\phi) = \phi^i_-$, and $\varepsilon_i(U) = U_i$; where $\varepsilon_i : \mathcal{M}(\mathcal{K} \otimes B \otimes C[0, 1]) \rightarrow \mathcal{M}(\mathcal{K} \otimes B)$ is the map induced by the evaluation map $\varepsilon_i$ at $t$. Consider the Kasparov $(A-B \otimes C[0, 1])$-bimodule

$$z = \left( \hat{H}_{B \otimes C[0, 1]}, \begin{pmatrix} \phi_+ & 0 \\ 0 & \phi_- \end{pmatrix}, \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix} \right).$$

Since $\hat{H}_{B \otimes C[0, 1]} \hat{\otimes} \varepsilon_i B \cong \hat{H}_B$ it follows that

$$(\varepsilon_0)_*(z) = \left( \hat{H}_B, \begin{pmatrix} \phi^0_+ & 0 \\ 0 & \phi^0_- \end{pmatrix}, \begin{pmatrix} 0 & U^*_0 \\ U_0 & 0 \end{pmatrix} \right)$$

and

$$(\varepsilon_1)_*(z) = \left( \hat{H}_B, \begin{pmatrix} \phi^1_+ & 0 \\ 0 & \phi^1_- \end{pmatrix}, \begin{pmatrix} 0 & U^*_1 \\ U_1 & 0 \end{pmatrix} \right)$$

Therefore $\alpha(x)$ and $\alpha(y)$ are homotopic. Hence $\overline{\alpha}$ is well-defined.

To show that $\overline{\alpha}$ is surjective, let $y = (E, \phi, T)$ be a Kasparov $(A,B)$-bimodule. By Proposition 2.3.5 of [9], we may assume that $E \cong \hat{H}_B$ and that $T = T^\ast$. Thus, with respect to the graded isomorphism $\mathcal{L}(\mathbb{H}_B) \cong M_2(\mathcal{M}(\mathcal{K} \otimes B))$, we can write

$$\phi = \begin{pmatrix} \phi_+ & 0 \\ 0 & \phi_- \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}$$

where $\phi_+$ and $\phi_-$ are homomorphisms from $A$ to $\mathcal{M}(\mathcal{K} \otimes B)$ and $U$ is an element of $\mathcal{M}(\mathcal{K} \otimes B)$. Then $y = \overline{\alpha}(x)$ where $x = (\phi_+,\phi_-,U)$.

Finally, we show that $\overline{\alpha}$ is injective. Suppose that $\overline{\alpha}(x) = \overline{\alpha}(y)$ where $x = (\phi_+,\phi_-,U)$ and $y = (\psi_+,\psi_-,V)$. Then there is a Kasparov $(A, B \otimes C[0, 1])$-bimodule $z = (E, \phi, T)$ such that

$$(\varepsilon_0)_*(z) \cong \left( \hat{H}_B, \begin{pmatrix} \phi_+ & 0 \\ 0 & \phi_- \end{pmatrix}, \begin{pmatrix} 0 & U^*_0 \\ U_0 & 0 \end{pmatrix} \right)$$
and
\[(\varepsilon_1)_*(z) \cong \left( \hat{\mathbb{H}}_B, \begin{pmatrix} \psi_+ & 0 \\ 0 & \psi_- \end{pmatrix}, \begin{pmatrix} 0 & V^* \\ V & 0 \end{pmatrix} \right).\]

As above, we may assume that \( E = \hat{\mathbb{H}}_{B \otimes C[0,1]} \) and that \( T = T^* \). Then \( z \) has the form
\[ z = \left( \hat{\mathbb{H}}_{B \otimes C[0,1]}, \begin{pmatrix} \theta_+ & 0 \\ 0 & \theta_- \end{pmatrix}, \begin{pmatrix} 0 & W \\ W^* & 0 \end{pmatrix} \right) \]
where \( \theta_+ \) and \( \theta_- \) are homomorphisms to \( \mathcal{M}(\mathcal{K} \otimes B \otimes C[0,1]) \) and \( W \) is an element of \( \mathcal{M}(\mathcal{K} \otimes B \otimes C[0,1]) \). Then \((\theta_+, \theta_-, W)\) is a Kasparov \((A, B \otimes C[0,1])\)-bimodule implementing a homotopy between \( x \) and \( y \).

\[\Box\]

3. The Universal Property of \( KK\)-Theory

Let \( F \) be a functor from the category \( C^*\mathbb{R}\text{-Alg} \) of separable real C*-algebras to the category \( \text{Ab} \) of abelian groups. We say that \( F \) is

(i) homotopy invariant if \((\alpha_1)_* = (\alpha_2)_*\) whenever \( \alpha_1 \) and \( \alpha_2 \) are homotopic homomorphisms on the level of real C*-algebras.

(ii) stable if \( e_* : F(A) \to F(K \otimes A) \) is an isomorphism for the inclusion \( e : A \hookrightarrow K \otimes A \) defined via any rank one projection.

(iii) split exact if any split exact sequence of separable C*-algebras
\[ 0 \to A \to B \to C \to 0 \]
induces a split exact sequence
\[ 0 \to F(A) \to F(B) \to F(C) \to 0. \]

(iv) half exact if any short exact sequence of separable C*-algebras
\[ 0 \to A \to B \to C \to 0 \]
induces an exact sequence
\[ F(A) \to F(B) \to F(C). \]

In what follows we will see that if \( F \) is homotopy invariant and half exact, then it is split exact.

The following theorem is the version for real C*-algebras of Theorem 3.7 of [5] and Theorem 22.3.1 of [1].

**Theorem 3.1.** Let \( F \) be a functor from \( C^*\mathbb{R}\text{-Alg} \) to \( \text{Ab} \) that is homotopy invariant, stable, and split exact. Then there is a unique natural pairing \( \alpha : F(A) \otimes KK(A, B) \to F(B) \) such that \( \alpha(x \otimes 1_A) = x \) for all \( x \in F(A) \) and where \( 1_A \in KK(A, A) \) is the class represented by the identity homomorphism.
Furthermore, the pairing respects the intersection product on $KK$-theory in the sense that

$$\alpha(x \otimes y) \otimes z = \alpha(x \otimes (y \otimes_B z)) : F(A) \otimes KK(A, B) \otimes KK(B, C) \to F(C).$$

Proof. Let $\Phi \in KK(A, B)$. Using Theorem 2.3, we represent $\Phi$ with a $KK(A, B)$ cycle and as in Lemma 3.6 of [5], we may assume that this cycle has the form $(\phi_+, \phi_-, 1)$. We use the same construction as in Definitions 3.3 and 3.4 in [5]. In that setting $F$ is assumed to be a functor from separable complex C*-algebras to any abelian category. This construction produces a homomorphism $\Phi_* : F(A) \to F(B)$ and we then define $\alpha(x \otimes \Phi) = \Phi_*(x)$. The proof of Theorems 3.7 and 3.5 of [5] carry over in the real case to show that $\alpha$ is natural, is well-defined, satisfies $\alpha(x \otimes 1_A) = x$, and is unique.

That $\alpha$ respects the Kasparov product follows from the uniqueness statement. \qed

For any real C*-algebra $A$ we define $SA = C_0(\mathbb{R}, A)$ and $S^{-1}A = \{f \in C_0(\mathbb{R}, \mathbb{C} \otimes A) \mid f(-x) = \overline{f(x)}\}$. For any functor $F$ on $C^*\mathbb{R}$-$\text{Alg}$ and any integer $n$, we define $F_n(A) = F(S^n(A))$ where

$$S^n(A) = \begin{cases} S^n(A) & n > 0 \\ A & n = 0 \\ (S^{-1})^{-n} & n < 0. \end{cases}$$

Corollary 3.2. Let $F$ be a functor from $C^*\mathbb{R}$-$\text{Alg}$ to $\text{Ab}$ that is homotopy invariant, stable, and split exact. Then $F_*(A)$ has the structure of a graded module over the ring $K_*(\mathbb{R})$. In particular, $F(S^8A) \cong F(A)$ and $F(S^{-1}SA) \cong F(A)$.

Proof. For all separable $A$ and $\sigma$-unital $B$, the pairing of Proposition 2.2 gives $KK_*(A, B)$ the structure of a module over $KK_*(\mathbb{R}, \mathbb{R})$. Taking $A = B$, we define a graded ring homomorphism $\beta$ from $K_*(\mathbb{R}) \cong KK_*(\mathbb{R}, \mathbb{R})$ to $KK_*(A, A)$ by multiplication by $1_A \in KK(A, A)$.

Then for any $x \in F_m(A)$ and $y \in K_n(\mathbb{R})$ we define $x \cdot y = \alpha(x \otimes \beta(y)) \in F_{n+m}(A)$. The second statement follows from the $KK$-equivalence between $\mathbb{R}$ and $S^8\mathbb{R}$, and that between $\mathbb{R}$ and $S^{-1}S\mathbb{R}$ from Section 1.4 of [2]. \qed

For all integers $n$ and $m$ there is a $KK$-equivalence between $S^nS^mA$ and $S^{n+m}A$, so it follows that the pairing Theorem 3.1 extends to a well-defined graded pairing

$$\alpha : F_*(A) \otimes KK_*(A, B) \to F_*(B).$$
Let $\mathbf{KK}$ be the category whose objects are separable real $\mathrm{C}^*\text{-algebras}$ and the set of morphisms from $A$ to $B$ is $KK(A,B)$. There is a canonical functor $KK$ from $\mathbf{C^*R-Alg}$ to $\mathbf{KK}$ that takes an object $A$ to itself and which takes a $\mathrm{C}^*$-homomorphism $f: A \to B$ to the corresponding element $[f] \in KK(A,B)$.

**Corollary 3.3.** Let $F$ be a functor from $\mathbf{C^*R-Alg}$ to $\mathbf{Ab}$ that is homotopy invariant, stable, and split exact. Then there exists a unique functor $\hat{F}: \mathbf{KK} \to \mathbf{A}$ such that $\hat{F} \circ KK = F$.

**Proof.** The functor $\hat{F}$ takes an object $A$ in $\mathbf{KK}$ to $F(A)$ in $\mathbf{Ab}$ and takes a morphism $y \in KK(A,B)$ to the homomorphism $F(A) \to F(B)$ defined by $y \mapsto \alpha(x \otimes y)$. The composition $\hat{F} \circ KK = F$ clearly holds on the level of objects. On the level of morphisms we must verify the formula $\alpha(x \otimes [f]) = f^*([x])$ for $f: A \to B$ and $x \in F(A)$. This formula follows by the naturality of the pairing $\alpha$, the formula $\alpha(x \otimes 1_A) = x$, and the formula $f^*(1_A) = [f] \in KK(A,B)$ which is verified as in Section 2.8 of [5]. □

**Proposition 3.4.** Let $F$ be a functor from $\mathbf{C^*R-Alg}$ to $\mathbf{Ab}$ that is homotopy invariant and half exact. Then for any short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

there is a natural boundary map $\partial: F(SC) \to F(A)$ that fits into a (half-infinite) long exact sequence

$$\cdots \to F(SB) \xrightarrow{g_*} F(SC) \xrightarrow{\partial} F(A) \xrightarrow{f_*} F(B) \xrightarrow{g_*} F(C) \cdots .$$

**Proof.** Use the mapping cone construction as in Section 21.4 of [1]. □

**Corollary 3.5.** A functor $F$ from $\mathbf{C^*R-Alg}$ to $\mathbf{Ab}$ that is homotopy invariant and half exact is also split exact.

**Proof.** The splitting implies that $g_*$ is surjective. Thus in the sequence of Proposition 3.4, $\partial = 0$ and $f_*$ is injective. □

**Proposition 3.6.** Let $F$ be a functor from $\mathbf{C^*R-Alg}$ to $\mathbf{Ab}$ that is homotopy invariant, stable, and half exact. Then for any short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

there is a natural long exact sequence (with 24 distinct terms)

$$\cdots \to F_{n+1}(C) \xrightarrow{\partial} F_n(A) \xrightarrow{f_*} F_n(B) \xrightarrow{g_*} F_n(C) \xrightarrow{\partial} F_{n-1}(A) \to \cdots .$$

**Proof.** From Corollary 3.5 and Corollary 3.2, $F$ is periodic; so Proposition 3.4 gives the long exact sequence. □
We say that a homotopy invariant, stable, half-exact functor $F$ from $C^*\mathbb{R}$-Alg to the category $\text{Ab}$ of abelian groups

(v) satisfies the dimension axiom if there is an isomorphism $F_*(\mathbb{R}) \cong K_*(\mathbb{R})$ as graded modules over $K_*(\mathbb{R})$

(vi) is continuous if for any direct sequence of real C*-algebras $(A_n, \phi_n)$, the natural homomorphism

$$\lim_{n \to \infty} F_*(A_n) \to F_*(\lim_{n \to \infty} A_n)$$

is an isomorphism.

**Theorem 3.7.** Let $F$ be a functor from $C^*\mathbb{R}$-Alg to $\text{Ab}$ that is homotopy invariant, stable, half exact and satisfies the dimension axiom. Then there is a natural transformation $\beta: K_n(A) \to F_n(A)$. If $F$ is also continuous, then $\beta$ is an isomorphism for all real C*-algebras in the smallest class of separable C*-algebras which contains $\mathbb{R}$ and is closed under KK-equivalence, countable inductive limits, and the two-out-of-three rule for exact sequences.

**Proof.** Let $z$ be a generator of $F(\mathbb{R}) \cong \mathbb{Z}$ and for $x \in K_n(A) \cong KK(\mathbb{R}, S^n A)$ define a $K_*(\mathbb{R})$-module homomorphism $\beta: K_*(A) \to F_*(A)$ by $\beta(x) = \alpha(z \otimes x)$. Taking $A = \mathbb{R}$, Theorem [3.1] yields that $\beta(1_0) = z$ where $1_0$ is the unit of the ring $K_*(\mathbb{R}) = KK_*(\mathbb{R}, \mathbb{R})$. Therefore, $\beta$ is an isomorphism for $A = \mathbb{R}$. Then bootstrapping arguments show that $\beta$ is an isomorphism for all real C*-algebras in the class described. □

From Section 2.1 of [3] we have distinguished elements

$$c \in KK_0(\mathbb{R}, \mathbb{C}), \quad r \in KK_0(\mathbb{C}, \mathbb{R})$$

$$\varepsilon \in KK_0(\mathbb{R}, T), \quad \zeta \in KK_0(T, \mathbb{C})$$

$$\psi_U \in KK_0(\mathbb{C}, \mathbb{C}), \quad \psi_T \in KK_0(T, T)$$

$$\gamma \in KK_{-1}(\mathbb{C}, T), \quad \tau \in KK_1(T, \mathbb{R})$$

For any homotopy invariant, stable, split exact functor $F$ on $C^*\mathbb{R}$-Alg, define the united $F$-theory of a real C*-algebra $A$ to be

$$F^{CRT}(A) = \{ F_*(A), F_*(\mathbb{C} \otimes A), F_*(T \otimes A) \}$$
together with the collection of natural homomorphisms

\[ c_n : F_n(A) \to F_n(C \otimes A) \]
\[ r_n : F_n(C \otimes A) \to F_n(A) \]
\[ \varepsilon_n : F_n(A) \to F_n(T \otimes A) \]
\[ \zeta_n : F_n(T \otimes A) \to F_n(C \otimes A) \]
\[ (\psi_u)_n : F_n(C \otimes A) \to F_n(C \otimes A) \]
\[ (\psi_T)_n : F_n(T \otimes A) \to F_n(T \otimes A) \]
\[ \gamma_n : F_n(C \otimes A) \to F_{n-1}(C \otimes A) \]
\[ \tau_n : F_n(T \otimes A) \to F_{n+1}(A) \]

induced by the elements \( c, r, \varepsilon, \zeta, \psi_u, \psi_T, \gamma, \tau \) via the pairing of Theorem 3.1.

**Proposition 3.8.** Let \( F \) be a homotopy invariant, stable, split exact functor from \( C^*\mathbb{R}\)-\textbf{Alg} to \( \textbf{Ab} \) and let \( A \) be a separable real \( C^* \)-algebra. Then \( F^{\text{CRT}}(A) \) is a CRT-module. Moreover, if in addition \( F \) is half exact, then \( F^{\text{CRT}}(A) \) is acyclic.

**Proof.** To show that \( F^{\text{CRT}}(A) \) is a CRT-module, we must show that the CRT-module relations

\[ rc = 2 \quad \psi_u \beta_u = -\beta_u \psi_u \quad \xi = r \beta_u^2 c \]
\[ cr = 1 + \psi_u \quad \psi_T \beta_T = \beta_T \psi_T \quad \omega = \beta_T \gamma \zeta \]
\[ r = \tau \gamma \quad \varepsilon \beta_o = \beta_T^2 \varepsilon \quad \beta_T \varepsilon \tau = \varepsilon \beta_T + \eta_T \beta_T \]
\[ c = \zeta \varepsilon \quad \zeta \beta_T = \beta_T^2 \zeta \quad \varepsilon r \zeta = 1 + \psi_T \]
\[ (\psi_u)^2 = 1 \quad \gamma \beta_u^2 = \beta_T \gamma \quad \gamma \tau = 1 - \psi_T \]
\[ (\psi_T)^2 = 1 \quad \tau \beta_T^2 = \beta_o \tau \quad \tau = -\tau \psi_T \]
\[ \psi_T \varepsilon = \varepsilon \quad \gamma = \gamma \psi_u \quad \tau \beta_T \varepsilon = 0 \]
\[ \zeta \gamma = 0 \quad \eta_o = \tau \varepsilon \quad \varepsilon \xi = 2 \beta_T \varepsilon \]
\[ \zeta = \psi_u \zeta \quad \eta_T = \gamma \beta_u \zeta \quad \xi \tau = 2 \beta_T \]

hold among the operations \( \{c_n, r_n, \varepsilon_n, \zeta_n, (\psi_u)_n, (\psi_T)_n, \gamma_n, \tau_n\} \) on \( F^{\text{CRT}}(A) \). But in the proof of Proposition 2.4 of [3], it is shown that these relations hold at the level of \( KK \)-elements. Therefore, using the associativity of the pairing of Theorem 3.1 the same relations hold among the operations of \( F^{\text{CRT}}(A) \).
Suppose now that \( F \) is also half-exact. To show that \( K^\text{CRT}(A) \) is acyclic, we must show that the sequences

\[
\cdots \to F_n(A) \xrightarrow{\eta_0} F_{n+1}(A) \xrightarrow{c} F_{n+1}(\mathbb{C} \otimes A) \xrightarrow{r\beta_U^{-1}} F_{n-1}(A) \to \cdots
\]

\[
\cdots \to F_n(A) \xrightarrow{\eta_2} F_{n+2}(A) \xrightarrow{\varepsilon} F_{n+2}(T \otimes A) \xrightarrow{r\beta_T^{-1}} F_{n-1}(A) \to \cdots
\]

\[
\cdots \to F_{n+1}(\mathbb{C} \otimes A) \xrightarrow{\gamma} F_{n}(T \otimes A) \xrightarrow{\xi} F_{n}(\mathbb{C} \otimes A) \xrightarrow{1-\psi_U} F_{n}(\mathbb{C} \otimes A) \to \cdots
\]

are exact. These can be derived from the short exact sequences

\[
0 \to S^{-1}\mathbb{R} \otimes A \to \mathbb{R} \otimes A \to \mathbb{C} \otimes A \to 0
\]

\[
0 \to S^{-2}\mathbb{R} \otimes A \to \mathbb{R} \otimes A \to T \otimes A \to 0
\]

\[
0 \to SC \otimes A \to T \otimes A \to \mathbb{C} \otimes A \to 0
\]

from Sections 1.2 and 1.4 of [2].

Indeed, focusing on the first one for our argument, the short exact sequence

\[
0 \to S^{-1}\mathbb{R} \otimes A \xrightarrow{f} \mathbb{R} \otimes A \xrightarrow{g} \mathbb{C} \otimes A \to 0
\]

gives rise to the long exact sequence

\[
\cdots \to F_n(A) \xrightarrow{\eta_0} F_{n+1}(A) \xrightarrow{c} F_{n+1}(\mathbb{C} \otimes A) \xrightarrow{r\beta_U^{-1}} F_{n-1}(A) \to \cdots
\]

where the homomorphisms are given by the multiplication of the elements \([f] \in KK_2(\mathbb{R}, \mathbb{R}), [g] \in KK(\mathbb{R}, \mathbb{C}), \) and \( \partial \in KK_{-1}(\mathbb{C}, \mathbb{R}) \). In the case of the functor \( KK(B, -) \), it was shown in the proof of Proposition 2.4 of [3] that the resulting sequence has the form

\[
\cdots \to KK_n(B, A) \xrightarrow{\eta_0} KK_{n+1}(B, A) \xrightarrow{c} KK_{n+1}(B, \mathbb{C} \otimes A) \xrightarrow{r\beta_U^{-1}} \cdots
\]

for all separable real C*-algebras \( B \). It then follows easily that the \( KK \)-element equalities \([f] = \eta_0, [g] = c, \partial = r\beta_U^{-1} \) hold. This proves that Sequence 1 is exact. Sequences 2 and 3 are shown to be exact the same way.

\[\square\]

**Theorem 3.9.** Let \( F \) and \( G \) be homotopy invariant, stable, half exact functors from \( C^\text{*-Alg} \) to \( \text{Ab} \) with a natural transformation \( \mu_A : F(A) \to G(A) \). If \( \mu_A \) is an isomorphism for all complex C*-algebras \( A \) in \( C^\text{*-Alg} \), then \( \mu_A \) is an isomorphism for all real C*-algebras in \( C^\text{*-Alg} \).
Proof. Let $A$ be a real separable C*-algebra. The natural transformation $\mu_A$ induces a homomorphism $\mu_A^{CRT} : F^{CRT}(A) \to G^{CRT}(A)$ of acyclic CRT-modules which is, by hypothesis, an isomorphism on the complex part. Then the results in Section 2.3 of [4] imply that $\mu_A^{CRT}$ is an isomorphism. □

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