Conditional Expectation as Quantile Derivative

Dirk Tasche*

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Abstract

For a linear combination $\sum u_j X_j$ of random variables, we are interested in the partial
derivatives of its $\alpha$-quantile $Q_\alpha(u)$ regarded as a function of the weight vector $u = (u_j)$. It
turns out that under suitable conditions on the joint distribution of $(X_j)$ the derivatives exist
and coincide with the conditional expectations of the $X_i$ given that $\sum u_j X_j$ takes the value
$Q_\alpha(u)$. Moreover, using this result, we deduce formulas for the derivatives with respect to
the $u_i$ for the so-called expected shortfall $E\left[ \frac{1}{\delta} \sum u_j X_j - Q_\alpha(u) \right]^\delta \left[ \sum u_j X_j \leq Q_\alpha(u) \right]$, with
$\delta \geq 1$ fixed. Finally, we study in some more detail the coherence properties of the expected
shortfall in case $\delta = 1$.

Key words: quantile; value-at-risk; quantile derivative; conditional expectation; expected
shortfall; conditional value-at-risk; coherent risk measure.

1 Introduction

The last decade has seen a growing interest in quantiles of probability distributions by practi-
tioners mainly in the financial industry. Since quantiles have a simple interpretation in terms of
over- or undershoot probabilities they have found entrance in current risk management practice
in form of the value-at-risk concept (cf. [10]).

In particular, there is need for computing derivatives of quantiles of weighted sums of ran-
dom variables with respect to the weights. [2] represents an early example for the use of these
derivatives. More recently, in [15] was shown for general risk measures that their derivatives with
respect to the asset weights are the key to the solution of the “capital allocation” problem (see
also [4] for the case of “coherent” risk measures). The problem to allocate the total risk to risk
sources in connection with the need to differentiate risk measures appears also in other scientific
disciplines. For an example in statistics see [16].

In case of normally distributed random vectors the formulae for the derivatives are obvious
(cf. [7]). In [11], a result (Theorem 1) was provided for general distributions and even non-

*Zentrum Mathematik (SCA), TU München, 80290 München, Germany; email: tasche@ma.tum.de
linear combinations of random variables. A similar formula can be found in [1]. Unfortunately, the results in [11] and [6] lack of an intuitive explanation. Nevertheless, readily interpretable formulae are available for the case of linear combinations of random variables (cf. [9] or [8]). The primary intention with the present paper is to give a sufficient condition as general as possible on the underlying distribution for the formulae in [9] and [8] to remain valid. In addition, we will transfer the result on quantiles onto the so-called expected shortfall (also called conditional value-at-risk). Finally, because of the theoretical importance of the expected shortfall we will discuss its role as coherent risk measure in the sense of [1].

This paper is organized as follows: In section 2 we recall some properties of conditional densities and introduce the technical assumptions needed for the main result on differentiation of quantiles. In section 3, this result is presented in an easy to digest (eq. (6)) and a rigorous version (Theorem 3.3). We then apply the result to the expected shortfall in section 4. The last section is devoted to a more detailed study of the latter with respect to its coherence properties.

2 Some facts on conditional densities

We are going to present sufficient conditions for quantiles of a sum $\sum u_i X_i$ to be differentiable with respect to the weights $u_i$. These conditions will heavily rely on the existence of a conditional density of one component of the $X_i$ given the others. So we will start our study by summarizing some facts on conditional densities.

First we recall the notion of conditional density in the context of a random vector $(X_1, \ldots, X_d)$. We write $B(\mathbb{R}^d)$ for the $\sigma$-algebra of Borel sets on $\mathbb{R}^d$. By the indicator function $I(A, \omega) = I(A, \omega)$ of a set $A$ we mean the function defined by

$$I(A, \omega) \overset{\text{def}}{=} \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A. \end{cases}$$

**Definition 2.1** Let $d \geq 2$ and let $(X_1, \ldots, X_d)$ be an $\mathbb{R}^d$-valued random vector. A measurable function $\phi : \mathbb{R}^d \to [0, \infty), (t, x_2, \ldots, x_d) \mapsto \phi(t, x_2, \ldots, x_d)$ is called conditional density of $X_1$ given $(X_2, \ldots, X_d)$ if for all $A \in B(\mathbb{R})$, $B \in B(\mathbb{R}^{d-1})$

$$P[X_1 \in A, (X_2, \ldots, X_d) \in B] = E \left[ I((X_2, \ldots, X_d) \in B) \int_A \phi(t, X_2, \ldots, X_d)dt \right].$$

An equivalent formulation for Definition 2.1 is

$$\int_A \phi(t, X_2, \ldots, X_d)dt = P\left[X_1 \in A \mid X_2, \ldots, X_d \right],$$

i.e. $\phi(\cdot, X_2, \ldots, X_d)$ is a density of the conditional distribution of $X_1$ given $(X_2, \ldots, X_d)$. Recall the well-known fact that the existence of a joint density of $(X_1, \ldots, X_d)$ is sufficient but not
necessary for the existence of a conditional density of $X_1$ given $(X_2, \ldots, X_d)$. On the other hand, the existence of such a conditional density implies that the unconditional distribution of $X_1$ has a density $f$ that is given by $f(t) = E[\phi(t, X_2, \ldots, X_d)]$. Moreover, a situation can occur where the distribution of $(X_2, \ldots, X_d)$ is purely discrete and a conditional density of $X_1$ given $(X_2, \ldots, X_d)$ exists.

For our purpose, the following three easy conclusions from the existence of a conditional density are important.

**Lemma 2.2** Let $d \geq 2$ and let $(X_1, \ldots, X_d)$ be an $\mathbb{R}^d$-valued random vector with a conditional density $\phi$ of $X_1$ given $(X_2, \ldots, X_d)$. Then for any weight vector $(u_1, \ldots, u_d) \in \mathbb{R} \setminus \{0\} \times \mathbb{R}^{d-1}$ we have

(i) the function $t \mapsto |u_1|^{-1} E \left[ \phi \left( u_1^{-1} \left( t - \sum_{j=2}^{d} u_j X_j \right), X_2, \ldots, X_d \right) \right]$ is a density of $\sum_{i=1}^{d} u_i X_i$,

(ii) for $i = 2, \ldots, d$, almost surely for $t \in \mathbb{R}$

$$E \left[ X_i \left| \sum_{j=1}^{d} u_j X_j = t \right. \right] = \frac{E \left[ X_i \phi \left( u_1^{-1} \left( t - \sum_{j=2}^{d} u_j X_j \right), X_2, \ldots, X_d \right) \right]}{E \left[ \phi \left( u_1^{-1} \left( t - \sum_{j=2}^{d} u_j X_j \right), X_2, \ldots, X_d \right) \right]}$$

(iii) almost surely for $t \in \mathbb{R}$

$$E \left[ X_1 \left| \sum_{j=1}^{d} u_j X_j = t \right. \right] = \frac{E \left[ \frac{t - \sum_{j=2}^{d} u_j X_j}{u_1} \phi \left( u_1^{-1} \left( t - \sum_{j=2}^{d} u_j X_j \right), X_2, \ldots, X_d \right) \right]}{E \left[ \phi \left( u_1^{-1} \left( t - \sum_{j=2}^{d} u_j X_j \right), X_2, \ldots, X_d \right) \right]}.$$

To say Lemma 2.2 with words: if there is a conditional density of $X_1$ given the other components, then subject to the condition $u_1 \neq 0$ the distribution of $\sum_{i=1}^{d} u_i X_i$ is absolutely continuous with density specified in Lemma 2.2 (i), and the conditional expectations of the $X_i$ given the sum $\sum_{j=1}^{d} u_j X_j$ can be calculated via the formulae in Lemma 2.2 (ii) and (iii).

In the subsequent section, it will turn out that the quantiles of the sum $\sum_{j=1}^{d} u_j X_j$ are differentiable with respect to the weights $u_i$ if the quantities mentioned in Lemma 2.2 are smooth in a certain sense. This observation motivates the following definition.

**Assumption 2.3** Let $d \geq 2$ and let $(X_1, \ldots, X_d)$ be an $\mathbb{R}^d$-valued random vector with a conditional density $\phi$ of $X_1$ given $(X_2, \ldots, X_d)$. We say that $\phi$ satisfies Assumption 2.3 in an open set $U \subset \mathbb{R} \setminus \{0\} \times \mathbb{R}^{d-1}$ if the following three conditions hold:

(i) For fixed $x_2, \ldots, x_d$ the function $t \mapsto \phi(t, x_2, \ldots, x_d)$ is continuous in $t$.

(ii) The mapping

$$(t, u) \mapsto E \left[ \phi \left( u_1^{-1} \left( t - \sum_{j=2}^{d} u_j X_j \right), X_2, \ldots, X_d \right) \right], \quad \mathbb{R} \times U \to [0, \infty)$$
is finite-valued and continuous.

(iii) For each \( i = 2, \ldots, d \) the mapping

\[
(t, u) \mapsto E \left[ X_i \phi \left( u_1^{-1} \left( t - \sum_{j=2}^{d} u_j X_j \right), X_2, \ldots, X_d \right) \right], \quad \mathbb{R} \times U \rightarrow \mathbb{R}
\]

is finite-valued and continuous.

Note that (i) from Assumption 2.3 in general does imply neither (ii) nor (iii). Furthermore, (ii) and (iii) may be valid even if the components of the random vector \((X_1, \ldots, X_d)\) do not have finite expectations.

Remark 2.4 Here is a list of some situations in which Assumption 2.3 is satisfied:

1) \((X_1, \ldots, X_d)\) is normally distributed and its covariance matrix has full rank.

2) \((X_1, \ldots, X_d)\) and \(\phi\) satisfy (i) and for each \((s, v) \in \mathbb{R} \times U\) there is some neighbourhood \(V\) such that the random fields

\[
\left( \phi \left( u_1^{-1} (t - \sum_{j=2}^{d} u_j X_j), X_2, \ldots, X_d \right) \right)_{(t,u) \in V}
\]

and for \( i = 2, \ldots, d \)

\[
\left( X_i \phi \left( u_1^{-1} (t - \sum_{j=2}^{d} u_j X_j), X_2, \ldots, X_d \right) \right)_{(t,u) \in V}
\]

are uniformly integrable.

3) \(E[|X_i|] < \infty, \ i = 2, \ldots, d\), and \(\phi\) is bounded and satisfies (i).

4) \(E[|X_i|] < \infty, \ i = 2, \ldots, d\). \(X_1\) and \((X_2, \ldots, X_d)\) are independent. \(X_1\) has a continuous density.

5) There is a finite set \(M \subset \mathbb{R}^{d-1}\) such that \(P[(X_2, \ldots, X_d) \in M] = 1\), and (i) is satisfied.

Note that Remark 2.4 3) is a special case of 2) and that 4) and 5) resp. are special cases of 3). Perhaps, 4) is the case most interesting for applications. It corresponds to the situation where a sample of \((X_1, \ldots, X_d)\) and a weight vector \(u\) are given and the density of \(\sum_{j=1}^{d} u_j X_j\) is estimated by kernel estimation.

3 Quantile Derivatives

If \(X\) is a real valued random variable and \(\alpha\) is any number between 0 and 1, the \(\alpha\)-quantile of \(X\) is the 100\(\alpha\)-% threshold of \(X\), i.e. the lowest bound to be exceeded by \(X\) only with probability 100\((1 - \alpha)\%\). We will make use of the following formal definition.
Definition 3.1 Let $X$ be a real valued random variable and let $\alpha \in (0, 1)$. Then the $\alpha$-quantile $Q_\alpha(X)$ of $X$ is defined by

$$Q_\alpha(X) \overset{\text{def}}{=} \inf\{x \in \mathbb{R} | P[X \leq x] \geq \alpha\}.$$ 

In general, the case $P[X \leq Q_\alpha(X)] > \alpha$ is possible, but in this paper solely $P[X \leq Q_\alpha(X)] = \alpha$ will occur. The reason is that our method for proving differentiability of the quantiles will be based on the implicit function theorem.

Let us briefly outline the reasoning. We want to study the mapping

$$Q_\alpha(u) \overset{\text{def}}{=} Q_\alpha\left(\sum_{j=1}^{d} u_j X_j\right),$$

regarded as a function of the weight vector $u$. Assume for the moment that we already know that $Q_\alpha(u)$ is differentiable with respect to the components of $u$. If there is a conditional density of $X_1$ given $(X_2, \ldots, X_d)$, then by Lemma 2.2 (i) the distribution of $\sum_{j=1}^{d} u_j X_j$ is continuous, and we obtain by (2) for all $u$ with $u_1 > 0$

$$\alpha = P[\sum_{j=1}^{d} u_j X_j \leq Q_\alpha(u)] = E\left[u_1^{-1}(Q_\alpha(u) - \sum_{j=2}^{d} u_j X_j) \int_{-\infty}^{\phi(t, X_2, \ldots, X_d)} dt \right].$$

(4)

Ignoring the question whether or not differentiation under the expectation is permitted, by differentiating with respect to $u_i$, $i = 2, \ldots, d$, we obtain from (4)

$$0 = u_1^{-1} E\left[\frac{\partial Q_\alpha(u)}{\partial u_i} - X_i \phi\left(u_1^{-1}(Q_\alpha(u) - \sum_{j=2}^{d} u_j X_j), X_2, \ldots, X_d\right)\right].$$

(5)

Solving (5) for $\frac{\partial Q_\alpha(u)}{\partial u_i}$ and applying formally Lemma 2.2 (ii) now yields

$$\frac{\partial Q_\alpha(u)}{\partial u_i} = E\left[X_i \bigg| \sum_{j=1}^{d} u_j X_j = Q_\alpha(u)\right], \quad i = 2, \ldots, d.$$ 

(6)

An analogous computation could be done in the cases $u_1 < 0$ and $i = 1$ and would yield (6) also for $u_1 < 0$ or $i = 1$. Equation (6) has been presented in [3] without examination of the question whether $Q_\alpha$ is differentiable and in [8] for the case of $(X_1, \ldots, X_d)$ with a joint density.

In order to make this approach mathematically rigorous by invoking the implicit function theorem, we have to verify some smoothness conditions for the expression $P\left[\sum_{j=1}^{d} u_j X_j \leq z\right]$ considered as a function of $u$ and $z$.

Lemma 3.2 Let $(X_1, \ldots, X_d)$ be an $\mathbb{R}^d$-valued random vector. Assume that there is a conditional density $\phi$ of $X_1$ given $(X_2, \ldots, X_d)$ satisfying Assumption 2.3 in some open set $U \subset \mathbb{R} \setminus \{0\} \times \mathbb{R}^{d-1}$. Define the random field $(Z(u))_{u \in U}$ by

$$Z(u) \overset{\text{def}}{=} \sum_{i=1}^{d} u_i X_i, \quad u \in U.$$ 

(7)
Then the function \( F : \mathbb{R} \times U \to [0, \infty) \), defined by
\[
F(z, u) \overset{\text{def}}{=} \mathbb{P}[Z(u) \leq z],
\]
is partially differentiable in \( z \) and \( u_i, i = 1, \ldots, d \), with jointly continuous derivatives
\[
\frac{\partial F(z, u)}{\partial z} = |u_1|^{-1} \mathbb{E} \left[ \phi \left( u_1^{-1} \left( z - \sum_{j=2}^{d} u_j X_j \right), X_2, \ldots, X_d \right) \right], \quad (8)
\]
\[
\frac{\partial F(z, u)}{\partial u_1} = - \frac{\text{sign}(u_1)}{u_1^2} \mathbb{E} \left[ \left( z - \sum_{j=2}^{d} u_j X_j \right) \phi \left( u_1^{-1} \left( z - \sum_{j=2}^{d} u_j X_j \right), X_2, \ldots, X_d \right) \right], \quad (9)
\]
and
\[
\frac{\partial F(z, u)}{\partial u_i} = - |u_1|^{-1} \mathbb{E} \left[ X_i \phi \left( u_1^{-1} \left( z - \sum_{j=2}^{d} u_j X_j \right), X_2, \ldots, X_d \right) \right], \quad i = 2, \ldots, d. \quad (10)
\]

**Proof.** The joint continuity of the expressions for the partial derivatives follows from Assumption 2.3 (ii) and (iii). Equation (8) is obvious since by Lemma 2.2 (i) the right-hand side of (8) as function of \( z \) is a continuous density of \( Z(u) \).

By the representations
\[
F(z, u) = \mathbb{E} \left[ \int_{-\infty}^{u_1^{-1} \left( z - \sum_{j=2}^{d} u_j X_j \right)} \phi(t, X_2, \ldots, X_d) \, dt \right]
\]
in case \( u_1 > 0 \) and
\[
F(z, u) = \mathbb{E} \left[ \int_{u_1^{-1} \left( z - \sum_{j=2}^{d} u_j X_j \right)}^{\infty} \phi(t, X_2, \ldots, X_d) \, dt \right]
\]
in case \( u_1 < 0 \) respectively, the application of Theorem A.(9.1) from [5] on differentiation under the integral yields the desired formulae (9) and (10). \( \square \)

With Lemma 3.2 we are in a position suitable to give a rigorous formulation to (8). Keep in mind that by Lemma 2.2 equation (8) on the one hand and (11) and (12) on the other hand have essentially the same meaning.

**Theorem 3.3** Let \( \alpha \in (0, 1) \) be fixed, and let \((X_1, \ldots, X_d)\) be an \( \mathbb{R}^d \)-valued random vector with a conditional density \( \phi \) of \( X_1 \) given \((X_2, \ldots, X_d)\) that satisfies Assumption 2.3 in some open set \( U \subset \mathbb{R} \setminus \{0\} \times \mathbb{R}^{d-1} \). Define the random field \((Z(u))_{u \in U}\) by (7) and the function \( Q_\alpha : U \to \mathbb{R} \) by (3).
If the density \( t \mapsto |u_1|^{-1} E \left[ \phi \left( u_1^{-1} \left( t - \sum_{j=2}^{d} u_j X_j \right), X_2, \ldots, X_d \right) \right] \) of \( Z(u) \) is positive at \( t = Q_\alpha(u) \), then \( Q_\alpha \) is partially differentiable in some neighbourhood of \( u \) with continuous derivatives

\[
\frac{\partial Q_\alpha}{\partial u_1}(u) = u_1^{-1} \left( Q_\alpha(u) - \frac{E \left[ \phi \left( u_1^{-1} (Q_\alpha(u) - \sum_{j=2}^{d} u_j X_j), X_2, \ldots, X_d \right) \right]}{E \left[ \phi \left( u_1^{-1} (Q_\alpha(u) - \sum_{j=2}^{d} u_j X_j), X_2, \ldots, X_d \right) \right]} \right)
\]

and

\[
\frac{\partial Q_\alpha}{\partial u_i}(u) = \frac{E \left[ X_i \phi \left( u_1^{-1} (Q_\alpha(u) - \sum_{j=2}^{d} u_j X_j), X_2, \ldots, X_d \right) \right]}{E \left[ \phi \left( u_1^{-1} (Q_\alpha(u) - \sum_{j=2}^{d} u_j X_j), X_2, \ldots, X_d \right) \right]}, \ i = 2, \ldots, d. \tag{12}
\]

**Proof.** As Assumption 2.3 implies that the distributions of \( (Z(u))_{u \in U} \) are continuous, we have for all \( u \in U \)

\[
P \left[ Z(u) \leq Q_\alpha(u) \right] = \alpha. \tag{13}
\]

From (13), by Lemma 3.2 and the implicit function theorem we obtain the assertion. \( \square \)

## 4 Shortfall Derivatives

As a quantile at a fixed level gives only local information about the underlying distribution, a promising way to escape from this shortcoming is to consider the so-called expected shortfall over or under the quantile. These quantities, to be defined in the following theorem, can be interpreted as moments of the difference between the underlying random sum and a quantile in a worst case situation specified by the confidence level of the quantile.

**Theorem 4.1** Let \( \delta \geq 1 \) be fixed. Let \((X_1, \ldots, X_d), \phi, U, \) and \( \alpha \) be as in Theorem 3.3, and assume additionally

\[
E \left[ \left| X_i \right|^\delta \right] < \infty, \ i = 1, \ldots, d.
\]

Let \( Z(u) \) and \( Q_\alpha(u) \) be as in Theorem 3.3 and let

\[
S_{\alpha,\delta}(u) \overset{\text{def}}{=} E \left[ \left| Z(u) - Q_\alpha(u) \right|^\delta \left| Z(u) \leq Q_\alpha(u) \right. \right] \quad \text{and} \quad S^*_\alpha(u) \overset{\text{def}}{=} E \left[ \left| Z(u) - Q_\alpha(u) \right|^\delta \left| Z(u) \geq Q_\alpha(u) \right. \right], \ u \in U.
\]

If the density \( t \mapsto |u_1|^{-1} E \left[ \phi \left( u_1^{-1} \left( t - \sum_{j=2}^{d} u_j X_j \right), X_2, \ldots, X_d \right) \right] \) of \( Z(u) \) is positive at \( t = Q_\alpha(u) \), then for each \( i = 1, \ldots, d \) the partial derivatives of \( S_{\alpha,\delta} \) and \( S^*_\alpha \) with respect to \( u_i \) exist
and are continuous in some neighbourhood of $u$. They can be computed by

$$\frac{\partial S_{\alpha,\delta}(u)}{\partial u_i} = \delta E \left[ (\frac{\partial Q_\alpha(u)}{\partial u_i} - X_i) Z(u) - Q_\alpha(u) \right]^{\delta-1} \left| Z(u) \leq Q_\alpha(u) \right|$$

$$\frac{\partial S_{\alpha,\delta}^*(u)}{\partial u_i} = \delta E \left[ (X_i - \frac{\partial Q_\alpha(u)}{\partial u_i}) Z(u) - Q_\alpha(u) \right]^{\delta-1} \left| Z(u) \geq Q_\alpha(u) \right|, \quad u \in U,$$

where the formulas for $\frac{\partial Q_\alpha(u)}{\partial u_i}$ are given in Theorem 3.3.

**Proof.** For any event $A$ define the indicator function $I(A, \omega) = I(A)$ by (11). Fix $\delta \geq 1$ and let $G(z, u) \overset{\text{def}}{=} E \left[ |Z(u) - z|^{\delta} I(Z(u) \leq z) \right]$. Then the proof for e.g. (14) can be based on the representation

$$G(z, u) = \delta \int_{-\infty}^{0} P[Z(u) \leq z + t] (-t)^{\delta-1} dt,$$

by using Lemma 3.2 and Theorem A.(9.1) from [5]. We omit the details. \qed

Casually, one might be more interested in conditional moments of the underlying random variable itself than in moments of the difference between the random variable and a quantile. The following corollary to Theorem 4.1 covers this case. Note that in contrast to our notation some people call solely the quantities $T_{\alpha,1}(u)$ and $T_{\alpha,1}^*(u)$, defined in (15) and (16) respectively, expected shortfall.

**Corollary 4.2** Let an integer $n \geq 1$ be fixed. Let $(X_1, \ldots, X_d)$, $\phi$, $U$, and $\alpha$ be as in Theorem 3.3 and assume additionally

$$E[|X_i|^n] < \infty, \quad i = 1, \ldots, d.$$

Define $Z(u)$ and $Q_\alpha(u)$ as in Theorem 3.3 and let

$$T_{\alpha,n}(u) \overset{\text{def}}{=} E \left[ Z(u)^n \right| Z(u) \leq Q_\alpha(u) \right]$$

$$T_{\alpha,n}^*(u) \overset{\text{def}}{=} E \left[ Z(u)^n \right| Z(u) \geq Q_\alpha(u) \right], \quad u \in U.$$

If the density $t \mapsto |u_1|^{-1} E \left[ \phi \left( u_1^{-1} \left( t - \sum_{j=2}^{d} u_j X_j \right) , X_2, \ldots, X_d \right) \right]$ of $Z(u)$ is positive at $t = Q_\alpha(u)$, then for each $i = 1, \ldots, d$ the partial derivatives of $T_{\alpha,n}$ and $T_{\alpha,n}^*$ with respect to $u_i$ exist and are continuous in some neighbourhood of $u$. They can be computed by

$$\frac{\partial T_{\alpha,n}(u)}{\partial u_i} = n E \left[ X_i Z(u)^{n-1} \right| Z(u) \leq Q_\alpha(u) \right]$$

$$\frac{\partial T_{\alpha,n}^*(u)}{\partial u_i} = n E \left[ X_i Z(u)^{n-1} \right| Z(u) \geq Q_\alpha(u) \right], \quad u \in U.$$
Proof. Follows through the representations

\[
E \left[ Z(u)^n \mid Z(u) \leq Q_\alpha(u) \right] = \sum_{k=0}^{n} \binom{n}{k} E \left[ (Z(u) - Q_\alpha(u))^k \mid Z(u) \leq Q_\alpha(u) \right] Q_\alpha(u)^{n-k}
\]

and

\[
E \left[ Z(u)^n \mid Z(u) \geq Q_\alpha(u) \right] = \sum_{k=0}^{n} \binom{n}{k} E \left[ (Z(u) - Q_\alpha(u))^k \mid Z(u) \geq Q_\alpha(u) \right] Q_\alpha(u)^{n-k}
\]

from Theorem 4.1.

Equation (17) in case \( n = 1 \) was derived in [14] for random variables \( X_1, \ldots, X_d \) with a joint density.

5 Expected shortfall as risk measure

Since quantiles (or the value-at-risk) as risk measures have some severe deficiencies (cf. [1] or [13]), in the scientific literature other risk measures are preferred. One among those risk measures is the Conditional Value-at-Risk (CVaR, expected shortfall with \( n = 1 \) in Corollary 4.2) because of its close relationship to the “coherent” risk measures introduced in [1].

Indeed, there seems to be some confusion in the literature whether in general CVaR is a coherent risk measure or not. For instance, in [1] and [3] is argued that CVaR is not a coherent risk measure, whereas [12] says it is. The discrepancy between these statements is easy to explain since the definitions of CVaR the different authors used are not identical. We will not examine here which definition is more useful, but will show that even in the situation of the CVaR defined to be an elementary expectation – as in [1] and [3] –, it enjoys to a great extent the coherence properties. We start with an elementary but useful lemma.

Lemma 5.1 Let \( Y \) be a random variable in \( L_1(\Omega, \mathcal{F}, P) \) and \( y \in \mathbb{R} \) such that \( P \left[ Y \leq y \right] \) is positive. Then for all \( F \in \mathcal{F} \) with \( P \left[ F \right] \geq P \left[ Y \leq y \right] \) we have

\[
E \left[ Y \mid F \right] \geq E \left[ Y \mid Y \leq y \right]. \tag{18}
\]

Proof. (18) is trivial in case \( P \left[ F \cap \{ Y \leq y \} \right] = 0 \). Hence assume \( P \left[ F \cap \{ Y \leq y \} \right] \) to be positive. Define the indicator function \( I(A) \) of the event \( A \in \mathcal{F} \) as in (1). We then obtain

\[
E \left[ Y \mid Y \leq y \right] = y + \frac{E \left[ (Y - y) I(\{ Y \leq y \} \cap F) \right] + E \left[ (Y - y) I(\{ Y \leq y \} \cap \Omega \setminus F) \right]}{P \left[ Y \leq y \right]}
\]

\[
\leq y + E \left[ Y - y \mid \{ Y \leq y \} \cap F \right] P \left[ Y \leq y \right] P \left[ F \right] \leq y + E \left[ Y - y \mid \{ Y \leq y \} \cap F \right] P \left[ Y \leq y \mid F \right]
\]

\[
\leq y + \frac{E \left[ (Y - y) I(\{ Y \leq y \} \cap F) \right] + E \left[ (Y - y) I(\{ Y > y \} \cap F) \right]}{P \left[ F \right]}
\]

\[
= E \left[ Y \mid F \right]. \tag*{\Box}
\]
Properties (i) to (iv) in the following proposition are just the constituting properties of coherent risk measures (cf. [3]). From this point of view, Proposition 5.2 says that CVaR restricted to sets of continuous random variables is in fact a coherent risk measure.

**Proposition 5.2** Let \( \alpha \in (0, 1) \) be fixed. Assume that \( \mathcal{M} \) is a convex cone (i.e. \( X, Y \in \mathcal{M}, h > 0 \Rightarrow hX + Y \in \mathcal{M} \)) in \( L_1(\Omega, \mathcal{F}, P) \). Assume further that for each \( X \in \mathcal{M} \) we have \( P[X \leq Q_\alpha(X)] = \alpha \) and \( X + a \in \mathcal{M} \) for all \( a \in \mathbb{R} \).

For \( X \in \mathcal{M} \) define 
\[
\rho(X) \overset{\text{def}}{=} -E[X \mid X \leq Q_\alpha(X)].
\]

Then we have

(i) \( \rho \) is monotonous, i.e. \( X, Y \in \mathcal{M}, X \leq Y \text{ a.s.} \Rightarrow \rho(X) \geq \rho(Y). \)

(ii) \( \rho \) is subadditive, i.e. \( X, Y \in \mathcal{M} \Rightarrow \rho(X + Y) \leq \rho(X) + \rho(Y). \)

(iii) \( \rho \) is positively homogeneous, i.e. \( X \in \mathcal{M}, h > 0 \Rightarrow \rho(hX) = h \rho(X). \)

(iv) \( \rho \) is translation invariant, i.e. \( X \in \mathcal{M}, a \in \mathbb{R} \Rightarrow \rho(X + a) = \rho(X) - a. \)

**Proof.** (iii), (iv) are trivial. Concerning (i), by Lemma 5.1 for \( X \leq Y \) we have
\[
\rho(X) = -E[X \mid X \leq Q_\alpha(X)] \geq -E[X \mid Y \leq Q_\alpha(Y)] \geq -E[Y \mid Y \geq Q_\alpha(Y)] = \rho(Y).
\]

Similarly, concerning (ii):
\[
\rho(X + Y) = -E[X \mid X + Y \leq Q_\alpha(X + Y)] - E[Y \mid X + Y \leq Q_\alpha(X + Y)] \\
\leq -E[X \mid X \geq Q_\alpha(X)] - E[Y \mid Y \geq Q_\alpha(Y)] \\
= \rho(X) + \rho(Y).
\]

This completes the proof. \( \square \)

Note that the context specified by Lemma 2.2 (and by Assumption 2.3) fits into the assumptions for Proposition 5.2. We state this fact formally in the subsequent example.

**Example 5.3** Suppose that \( (X_1, \ldots, X_d) \) is an \( \mathbb{R}^d \)-valued random vector with conditional density \( \phi \) of \( X_1 \) given \( (X_2, \ldots, X_d) \). Assume that \( E[|X_i|] < \infty, i = 1, \ldots, d \). Let \( U \subset (0, \infty) \times \mathbb{R}^{d-1} \) be an open convex cone and define
\[
\mathcal{M} \overset{\text{def}}{=} \{a + \sum_{i=1}^d u_i X_i \mid a \in \mathbb{R}, u \in U\}.
\]

Then \( \mathcal{M} \) satisfies the conditions of Proposition 5.2.
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