Review of some classical gravitational superenergy tensors using computational techniques

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Abstract. We use computational algorithms recently developed by ourselves to study completely four index divergence-free quadratic in Riemann tensor polynomials in GR. Some results are new and others reproduce and/or correct known ones. The algorithms are part of a Mathematica package called Tools of Tensor Calculus (TTC).

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1. Introduction

Recently, interest in tensors containing interaction fields, and in particular the Riemann tensor, has increased. The so-called superenergy tensors are good examples, having some particular properties [1]. This interest is not new. The tensors of Bel–Robinson [2], Bel [3] and Sachs [4], among others, are some examples which have been studied in the past and have zero divergence in certain conditions. Collinson [5] has made a much deeper study on all four index quadratic in Riemann and divergence-free tensors, in four dimensions for any spacetime. Nevertheless these kinds of calculations are very hard to do by hand and some errors could arise. Bearing this in mind, we have recently developed some computational algorithms able to handle index tensor polynomial expressions. The paradigms of these polynomials are those built with the Riemann tensor as in

\[ \ldots \frac{1}{2} R_{ij} \, R_{mn} \, R_{kl} \, R_{ij} + 3 R_{im} \, R_{kn} \, R_{nj} \, R_{ij} \ldots \] (1)

The implementation of the algorithms is part of the Mathematica package called TTC (Tools of Tensor Calculus) [6]. These algorithms allow us to perform a complete simplification of this kind of expression. The simplification algorithms do not produce the most beautiful expression but, besides a zero result, produce a canonical one.

Other authors who have worked on algorithmic simplification of index tensor polynomials include Ilyin and Kryukov [7], Portugal [8] and Fulling et al [9]. Parker and Christensen [11] do not appear on this list because, although they have a big package on tensor calculus, they don’t work algorithmically in this topic but with libraries of rules. In a recent article [10] we have solved the part of the simplification problem related to dummy indices and moniterm symmetry properties. Cyclic and, in general, multiterm symmetry properties were not considered. Also, dimensional properties were left aside for further work, due to their complexity. Actually [12, 13], we can handle multiterm and dimensional properties as the...
algorithm has the advantage that it includes the possibility of storing a compacted version of the generated rules in order to increase functionality. This means that, although a first computation over a polynomial can be really slow, others on the same or on other sessions are faster than the first one. This allows us to play easily with results by permuting indices or performing other operations.

There are some problems with trusting programs which do automatic computations, especially when no exact coincidences arise when compared with older papers. All we can say is that our algorithm has been tested, besides in the present paper, on a large set of known Riemann properties (all those included in [11], among others) and calculations related to the Gauss–Bonnet invariant [14]. On the other hand, the fact that the implementation of the algorithm is part of a package, TTC (from now on we will use TTC to represent both the simplification algorithm and/or the package), allows us to test the results on explicit spacetimes. We have tested the calculations on the Vaydia metric, confirming our results.

The aims of this paper are as follows.

• To show the TTC capabilities on a real calculus. This can be of interest to researchers working on algebraic computation and/or working on superenergy tensors, or on any other field where these techniques can be applied.
• To review some known results and correct some others. Specifically we have corrected a minor part of the results of Collinson, and some signs in the original Sachs paper [4].
• To prove some theorems about existence and uniqueness of superenergy tensors.

By superenergy tensors we mean, in this paper, four index quadratic in Riemann tensors having some chosen properties. Following the work of Collinson, we want to find divergence-free tensors in a pseudo-Riemannian manifold (with non-fixed dimensions and for the four-dimensional case) and from these we will analyse the index symmetry properties. We leave aside other properties such as the dominant superenergy property (DSEP) and others explained in [1].

The paper is arranged as follows. In section 2 we briefly explain what TTC does when simplifying index tensor polynomials in general. In section 3 we define exactly which kind of tensors we want to find and which are the steps to reach our main result, that is to find all four index quadratic in Riemann and divergence-free tensors in a pseudo-Riemannian manifold without using dimensional properties. Next we specialize these results for Ricci flat pseudo-Riemannian manifolds and for four dimensions. We analyse the tensors found with respect to index symmetry properties, specifically we give existence and uniqueness theorems for totally symmetric tensors.

2. Computational simplification of index tensor polynomials

As stated above, by simplification we mean finding a canonical, and therefore unique, index tensor polynomial for any given monomial. In this process the following must be taken into account.

• Dummy indices: the output expression must be unique with respect to the freedom of renaming dummy indices.
• Index symmetry properties of each individual tensor appearing in the monomial: since in this paper we work with the Riemann tensor, $R$, the following properties, and those which arise when differentiating and contracting, must be taken into account

$$R_{ijkl} = - R_{jikl} = - R_{ijlk} = R_{klij}$$

$$R_{ijkl} + R_{ijlk} + R_{iklj} = 0$$
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\[ R_{ijkl,m} + R_{ijmn,l} + R_{ijkl;n} = 0 \]
\[ R_{ijkl;m} + R_{ijmn;k} = 0 \]
\[ R_{ijkl}^{m...n} = a R_{ijkl}^{n...m} + \text{(Riemann terms)} \]

where the last expression refers to Ricci commuting relations. At this point it is important to set up the convention for Ricci commuting relations and the definition of the Ricci tensor used by TTC which follows [15].

If \( v \) is a vector field then

\[ v_{;i,j,k} - v_{;i,k,j} = v_{;m} R^{m}_{;ijk} \]
\[ R_{ij} = R^{m}_{;imj} \]

It is important to note the change of sign of the Ricci tensor with respect to that used in Sachs [4], Collinson [5] and Robinson’s [17] original papers.

- Properties due to the finite dimension of the manifold. These kinds of properties arise due to the possibility of antisymmetrization over \( n + 1 \) indices appearing in the monomial, \( n \) being the dimension of the manifold. In the next example \( n = 2 \)

\[ ... + A_{ij} B_{k} + ... \Rightarrow A_{ij} B_{j} = 0. \]

In the following subsection we briefly explain how the algorithms work. For details, refer to [12, 13].

2.1. Dummy index problem

Actually, TTC uses a codification of each monomial, which is an improved version of [10], following the criteria:

(a) Free indices are renamed: IndexF[1], IndexF[2],… all of them in upper position (contravariant).

(b) Dummy indices are renamed: Index[1], Index[2],… all of them in upper position (contravariant).

(c) We find the permutation of free and dummy indices in order to find the most lexicographically ordered monomial expression. Note that at this stage the codification is independent of the order of the free indices. We call this global codification.

Finally we take the inverse of the permutation performed on the free indices over the global codification of the monomial. The final result is called codification of the original monomial.

Let us see an apparently simple example. Consider the monomial

\[ T_{ij} = A_{ikmn} A_{j}^{nkm} \]

with \( A \) being some four index tensor. Its global codification version is reached by taking \( j \) as IndexF[1] and \( i \) as IndexF[2], or equivalently, by permuting the free indices:

\[ A[\text{IndexF}[1], \text{Index}[1], \text{Index}[2], \text{Index}[3], \text{IndexF}[2], \text{Index}[2], \text{Index}[3], \text{Index}[1]] \]

which corresponds to \( T'_{ij} = A_{ikmn} A_{j}^{mnk} \). \( T' \) is not the original monomial, but the one obtained by permuting free indices. In general the global codification version of a monomial is a different monomial. The codification version of \( T \) is reached by taking the inverse permutation over the free indices

\[ A[\text{IndexF}[1], \text{Index}[2], \text{Index}[3], \text{Index}[1], \text{IndexF}[2], \text{Index}[2], \text{Index}[3], \text{Index}[1]] \]

which corresponds to \( T_{ij} = A_{ikmn} A_{j}^{nkm} \).

All rules found through a calculation are stored in a codification version, but all monomials with the same global codification version can use a permutation version of the stored rules, so the task of finding rules is essentially independent of the order of free indices.
2.2. Index tensor properties

First the user must declare each tensor entering the monomial, together with their symmetries. Some properties are automatically applied as Ricci commuting relations. TTC acts on each monomial, applying in all possible ways the individual properties of each tensor over the starting monomial and over each new resulting monomial. When this loop ends (no new monomials are created) TTC solves the resulting system of equations and applies the rules over the original monomial. Subsequent monomials can use the same or a permuted version of these rules if the *global codification* version is the same as one of the monomials in the rules.

2.3. Dimensional properties

As stated in the introduction, dimensional properties arise due to the possibility of antisymmetrizing \( n + 1 \) indices of the starting monomial and equating the result to zero, \( n \) being the manifold dimension. Nevertheless, we must be sure to take all possible sets of \( n + 1 \) indices in the monomial, even if they are dummy or *hidden* (as in the curvature \( R \)). The algorithm takes the monomial and writes it in a metric expanded version. The possible sets of \( n + 1 \) indices are in the metric part of the monomial. Let us see some examples in four dimensions.

(a) In the first example there are no dimensional properties:

\[
R \rightarrow R_{mn pq} g^{mp} g^{nq}.
\]

(b) Only one set of five indices can be effectively antisymmetrized:

\[
R^a b \rightarrow R^a_{m n s} R_{o p q r} g^{mn} g^{o p} g^{q r} g^{o p} g^{s t}.
\]

(c) The last example is the case handled in the present paper. Several sets of five indices can be effectively antisymmetrized. Note that for \( n \geq 6 \) there are no dimensional properties,

\[
R^{a b c d} \rightarrow R^{a b m n} R^{s t v} g^{mn} g^{o r} g^{s v} g^{t e} g^{d e}.
\]

TTC finds all these properties and simplifies them using non-dimensional ones. Then it solves the resulting system of equations and applies the resulting rules where necessary. Although the procedure described is correct, the algorithm has actually been improved in order to optimize the number of sets of indices to be antisymmetrized [13].

3. Collinson-like calculus

Collinson found by hand that in a four-dimensional pseudo-Riemannian manifold there are ten independent divergence-free tensors having four indices and quadratic in Riemann. Due to Ricci commuting relations we understand that a polynomial *quadratic in Riemann* can have terms linear in the second derivative of the Riemann tensor itself. Here we use TTC to perform the computerized version of his calculations, but the plan is close to the Collinson paper.

First we must generate the starting basis

\[
T^{a b c d} = A^{a b c d g h k l m n} R^{g h k} R^{l m n} + B^{a b c d l m n p q} R^{l m n p q},
\]

where \( A^{a b c d g h k l m n} \) and \( B^{a b c d l m n p q} \) are polynomic tensors built with the metric \( g^{a b} \) using the free indices plus the contracted ones in all possible ways, symbolically

\[
A^{a b c d g h k l m n} = \sum \alpha_\sigma \sigma \left( g^{a b} g^{c d} g^{e f} g^{g h} g^{k l} g^{m n} \right)
\]

\[
B^{a b c d l m n p q} = \sum \beta_\sigma \sigma \left( g^{a b} g^{c d} g^{k l} g^{m n} g^{p q} \right)
\]
σ being the permutations over metric indices, and \( \alpha_\sigma \) and \( \beta_\sigma \) being constant parameters.

TTC performs this calculation, simplifying the result using non-dimensional properties and dimensional ones. In the second case we obtain, as in Collinson’s paper, 63 independent monomials. This polynomial is what we call the starting basis \( T \), which has 63 independent coefficients (including the global one). In doing this, TTC finds three independent dimensional properties which include those found by Buchdahl [16].

Secondly, we take the divergence of \( \nabla a \)

\[
T^{abcd, \alpha} \equiv \nabla a
\]

TTC calculates this and simplifies it by using non-dimensional and dimensional properties. After this we must solve the coefficients of the equation

\[
T^{abcd, \alpha} = 0,
\]

and so we find the family of polynomials, \( T \), fulfilling (3). This computation has been done in an \( n \)-dimensional pseudo-Riemannian manifold with no other restriction (not using dimensional properties) and restricted to \( n = 4 \) and \( R_{ij} = 0 \). The results have been analysed under a few relevant index symmetry properties. The following subsections present results from the computations.

### 3.1. Without dimensional properties

**Theorem 1.** In an \( n \)-dimensional pseudo-Riemannian manifold and not using dimensional properties:

(a) There exist 14 independent quadratic in Riemann four index divergence-free tensors.

(b) There are no totally symmetric quadratic in Riemann and divergence-free tensors.

(c) The complete family of quadratic in Riemann and divergence-free tensors \( T^{abcd} \) totally symmetric in \( (bcd) \) is

\[
T^{abcd} = a_S T^{abcd}_S + a_R T^{abcd}_R,
\]

\[
T^{abcd}_S = Q^{abcd};
\]

\[
Q^{abcd} = -\frac{1}{3} g^{ac} R^d_i R^{ib} + 2 R^{ab:dc} - \frac{4}{3} R^{bd:ac} + \frac{4}{3} g^{ac} R^{bd:ij} - 2 g^{ac} R^{i,di} + \frac{4}{3} R^{ib} R^{ac} + \frac{4}{3} g^{ac} R^{i,di} - \frac{2}{3} g^{ac} R^{i,dp} R^{dp} + \frac{2}{3} R^{ab} R^{cd}
\]

\[
T^{abcd}_R = X^{ab} S^{cd};
\]

\[
X^{ab} = K U^{ab} + L V^{ab} - \frac{1}{2} W^{ab}
\]

\[
U^{ab} = G^{ab,s} s - 2 G^{ab,a} + 2 G^{a} R^{pbk} - \frac{1}{3} G_{pq} R^{pq} g^{ab}
\]

\[
V^{ab} = R^{ab} - R_{ba} g^{ab} - R S^{ab}
\]

\[
W^{ab} = G^{mpr} R^{pq} - \frac{1}{4} g^{ab} G_{mpqr} R_{mpqr}
\]

where \( a_S, a_R, K \) and \( L \) are four independent constant parameters.

\( T_S \) is the tensor found by Sachs [4] with an opposite sign on the term \( \frac{4}{3} R^{ib} R^{ac} \) with respect to his original definition. This sign represents a real correction and has nothing to do with the different sign convention in the Ricci definition. \( T_R \) is the family defined by Robinson [17].
Theorem 2. In a Ricci flat \((R_{ij} = 0)\) n-dimensional pseudo-Riemannian manifold and not using dimensional properties:

(a) There exist six independent quadratic in Riemann four index divergence-free tensors.

(b) There is only one tensor \(T^{abcd}\) quadratic in Riemann, divergence free and totally symmetric:

\[
T^{abcd} = M^{(abcd)} + 16 P^{(abcd)} - 4 Q^{(ab)(cd)} - 4 Q^{(a)(bcd)}
\]

\[
M^{abcd} = g^{ab} g^{cd} R_{mnop} R^{mnop}
\]

\[
P^{abcd} = R_{a}^{\phantom{a}bc} R^{mnop}
\]

\[
Q^{abcd} = g^{ab} R_{mno}^{\phantom{mno}c} R^{mnop}
\]

This tensor coincides with the symmetrization of the one defined by Senovilla [1], which he called the generalized Bel–Robinson tensor. Actually we are trying to extend this result to the cosmological vacuum.

3.2. Using dimensional properties

Theorem 3 (Collinson-corrected). In a four-dimensional pseudo-Riemannian manifold:

(a) There exist nine independent quadratic in Riemann four index divergence-free tensors.

Using the notation of Collinson [5], the Collinson family is

\[
T^{abcd} = a_1 T_1^{abcd} + a_2 T_2^{abcd} + a_3 T_3^{abcd} + a_4 T_4^{abcd} + a_5 T_5^{abcd} + a_6 T_6^{abcd} + a_8 T_8^{abcd} + a_9 T_9^{abcd} + a_{10} T_{10}^{abcd}
\]

where \(a_i\) are arbitrary numerical coefficients.

The tensors \(T_1\) to \(T_9\) (without \(T_7\)) can be written in terms of \(T_{10}\)

\[
T_1^{abcd} = T_2^{abcd} = T_3^{abcd} = g^{cd} T_{10}^{ai b i}
\]

\[
T_6^{abcd} = T_5^{abcd} = T_4^{abcd} = g^{cd} T_{10}^{abi} - \frac{1}{2} g^{cd} T_{10}^{ai b i}
\]

\[
T_8^{abcd} = T_{10}^{abcd} = T_{10}^{abcd} - 2 T_{10}^{abcd}
\]

and \(T_{10}\) is defined through \(T_{10}^{abcd} = 6 Q_1^{abcd} + Q_2^{(b)(cd)} + Q_3^{abcd}\) being

\[
Q_1^{abcd} = R_{i j}^{abcd} + R^{aibi} R^{cd i j} = R_{i j}^{abcd} + R^{aibi} R^{cd i j} + g^{ab} R^{dici} R^{bij}
\]

\[
Q_2^{abcd} = 4 R^{abcd} + 6 g^{cd} R^{bici} + 6 g^{cd} R^{bici} + 6 R^{abcd} + 6 R^{abcd} + 6 R^{abcd}
\]

\[
Q_3^{abcd} = 8 g^{ac} R_{ij}^{d k} R^{k b i j} - 8 R_{ij}^{cd} R^{b i a j} + 8 R_{ij}^{d} R^{b i a j} - 8 R_{ij}^{b d} R^{c i a j} + 2 R_{ij}^{d} R^{c i a j} - 2 R_{ij}^{d i} R^{c a j} + 3 g^{ab} R^{d i} R^{b i} + 3 g^{ab} R_{ij}^{d k} R^{k c i}
\]

(b) There are no totally symmetric quadratic in Riemann and divergence-free tensors.

We note that the Collinson tensor \(T_7\) does not appear, which is identically zero, since \(T_{10}\) is in fact symmetric with respect to the last two indices (this is not mentioned in Collinson’s paper). We have checked the symmetry of \(T_{10}\) by hand, corroborating our results.

Theorem 4. In a Ricci flat \((R_{ij} = 0)\) four-dimensional pseudo-Riemannian manifold:

(a) There exists one quadratic in Riemann four index divergence-free tensor.

(b) The unique divergence-free tensor is totally symmetric and is the Bel–Robinson tensor [2].
4. Concluding remarks

Through four theorems, we have reviewed and improved some results on classical gravitational superenergy tensors. The results have been reached using computational algorithms which do automatic calculations. As we have commented in the introduction, some problems of trust could arise. Nevertheless, we think that such problems are not exclusively related to computational techniques. How can we be sure that all calculations of Collinson, or anyone, are correct? Collinson is a good pioneer of large calculations on the topic, but to validate his results we hope that others will obtain the same results using independent calculations. We also hope that any other program [7, 8, 14, 18] will be able to do our calculations with an independent code and/or algorithms.

Assuming that these computational techniques are validated, and limiting our scope to superenergy tensors, we think that they are a very good complement to research based on defining such tensors having some chosen properties. We can analyse the uniqueness of these definitions with respect to the desired properties. But we can also analyse more complex problems, especially those related to the existence of conserved quantities. These studies can be done on a generic spacetime and using Einstein field equations. We hope that in the near future we will be able to handle the case of the cosmological vacuum and scalar field interactions.

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