SPHERICAL INDICATRICES WITH THE MODIFIED ORTHOGONAL FRAME

MOHAMD SALEEM LONE, MURAT KEMAL KARACAN, YILMAZ TUNCER, AND HASAN ES

ABSTRACT. In this paper, we study spherical images of the modified orthogonal vector fields and Darboux vector of a regular curve which lies on the unit sphere in Euclidean 3-space.

1. Introduction

Many interesting properties of a space curve $\alpha$ in $E^3$ can be investigated by means of the concept of spherical indicatrix of the tangent, principal normal, binormal and Darboux vector to $\alpha$ [7]. Most commonly researchers use the Frenet frame of a curve to characterize the properties of curves. In addition to Frenet frame various frames have been designed. The study of curves with respect to these frames are qualified problems [3]. In [1,2,5-7], the authors have characterized the spherical indicatrices of a curve in different frames.

In this paper, we obtain spherical representations of a curve with respect to the modified orthogonal frame in Euclidean 3-space.

2. Preliminaries

Let $\alpha(s)$ be a $C^3$ space curve in Euclidean 3-space $E^3$, parametrized by arc length $s$. We also assume that its curvature $\kappa(s) \neq 0$ anywhere. Then an orthonormal frame $\{t, n, b\}$ exists satisfying the Frenet-Serret equations

$$
\begin{bmatrix}
t'(s) \\
n'(s) \\
b'(s)
\end{bmatrix} =
\begin{bmatrix}
0 & \kappa & 0 \\
-\kappa & 0 & \tau \\
0 & -\tau & 0
\end{bmatrix}
\begin{bmatrix}
t(s) \\
n(s) \\
b(s)
\end{bmatrix},
$$

where $t$ is the unit tangent, $n$ is the unit principal normal, $b$ is the unit binormal, and $\tau(s)$ is the torsion. For a given $C^1$ function $\kappa(s)$ and a continuous function $\tau(s)$, there exists a $C^3$ curve $\varphi$ which has an orthonormal frame $\{t, n, b\}$ satisfying the Frenet-Serret frame (2.1). Moreover, any other curve $\tilde{\varphi}$ satisfying the same conditions, differs from $\varphi$ only by a rigid motion.

Now let $\alpha(s)$ be a general analytic curve which can be reparametrized by its arc length. Assuming that the curvature function has discrete zero points or $\kappa(s)$ is not identically zero, we have an orthogonal frame $\{T, N, B\}$ defined as follows:

$$
T = \frac{d\varphi}{ds}, \quad N = \frac{dT}{ds}, \quad B = T \times N,
$$

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where $T \times N$ is the vector product of $T$ and $N$. The relations between \{T, N, B\} and previous Frenet frame vectors at non-zero points of $\kappa$ are
\begin{equation}
T = t, N = \kappa n, B = \kappa b.
\end{equation}
Thus, we see that $N(s_0) = B(s_0) = 0$ when $\kappa(s_0) = 0$ and squares of the length of $N$ and $B$ vary analytically in $s$. From Eq. (2.2), it is easy to calculate
\begin{equation}
\begin{bmatrix}
T'(s) \\
N'(s) \\
B'(s)
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 0 \\
-\kappa^2 & \frac{\kappa'}{\kappa} & \tau \\
0 & -\tau & \frac{\kappa'}{\kappa}
\end{bmatrix}
\begin{bmatrix}
T(s) \\
N(s) \\
B(s)
\end{bmatrix}
\end{equation}
and
\begin{equation}
\tau = \tau(s) = \frac{\det(\alpha', \alpha'', \alpha''')}{\kappa^2}
\end{equation}
is the torsion of $\alpha$. From Frenet-Serret equations, we know that any point, where $\kappa^2 = 0$ is a removable singularity of $\tau$. Let $\langle, \rangle$ be the standard inner product of $E^3$, then \{T, N, B\} satisfies:
\begin{equation}
\langle T, T \rangle = 1, \langle N, N \rangle = \langle B, B \rangle = \kappa^2, \langle T, N \rangle = \langle T, B \rangle = \langle N, B \rangle = 0.
\end{equation}
The orthogonal frame defined in Eq. (2.3) satisfying Eq. (2.4) is called as modified orthogonal frame [8].

3. Darboux vector with modified orthogonal frame

Let $\alpha$ be a unit speed curve and let \{T, N, B\} be the modified orthogonal frame at point $\alpha(s)$ along curve $\alpha$ in $E^3$ and also derivative vectors of the orthogonal frame \{T, N, B\} be given as
\begin{equation}
\begin{aligned}
T' &= N, \\
N' &= -\kappa^2 T + \frac{\kappa'}{\kappa} N + \tau B, \\
B' &= -\tau N + \frac{\kappa'}{\kappa} B.
\end{aligned}
\end{equation}
Moreover, let $w$ be
\begin{equation}
w = aT + bN + cB.
\end{equation}
Then we can find components of vector $w$ such that it satisfies equalities:
\begin{equation}
\begin{aligned}
T' &= w \times T, \\
N' &= w \times N, \\
B' &= w \times B.
\end{aligned}
\end{equation}
Also one can do the computations below:
\begin{equation}
\begin{aligned}
w \times T &= b(N \times T) + c(B \times T) = cN - bB, \\
w \times N &= a(T \times N) + c(B \times N) = -c\kappa^2 T + aB, \\
w \times B &= a(T \times B) + b(N \times B) = b\kappa^2 T + aN.
\end{aligned}
\end{equation}
From Eq.(3.4) we can compute components of the vector $w$ as follows
\begin{equation}
a = \tau, \quad b = 0, \quad c = 1, \kappa = \text{const}.
\end{equation}
Thus we write vector $w$ and $\|w\|$ as
\begin{equation}
w(s) = \tau(s) T(s) + B(s), \quad \kappa = \text{const}.
\end{equation}
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\[ \|w\| = \sqrt{\kappa^2 + \tau^2}, \quad \kappa = \text{const.} \]

\( w \) in Eq.(3.6) is called Darboux vector of the curve \( \alpha \). Now we easily can write the followings equalities

\[
\begin{align*}
T \times T' &= B \\
N \times N' &= \frac{\kappa'}{\kappa} N + \tau B = \tau \kappa^2 T + \kappa^2 B = \kappa^2 (\tau T + B) \\
B \times B' &= B \times (-\tau N + \frac{\kappa'}{\kappa} B) = \tau \kappa^2 T.
\end{align*}
\]

Moreover we can write \( N \times N' \) as

\[
N \times N' = \kappa^2 w.
\]

From Eq.(3.8), we infer the result that \( w \) and \( N \times N' \) are linear dependent. If \( \kappa \) equals one then Frenet and the orthogonal frames coincide for every \( s \in I \). Let \( \phi \) be angle between \( B \) and darboux vector \( w \). Then we have the following equalities

\[
\begin{align*}
\sin \phi &= \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \quad \text{or} \quad \tau = \|w\| \sin \phi \\
\cos \phi &= \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}} \quad \text{or} \quad \kappa = \|w\| \cos \phi.
\end{align*}
\]

Let \( M \) be a hypersurface in \( E^n \) and the \( S \) be the shape operator of \( M \) with \( D \) acting as covariant derivative. Then, we have the famous Gauss equation:

\[
\overline{D}XY = DXY + \langle S(X), Y \rangle N
\]

for each \( X, Y \in \chi(M) \).

4. ARCLENGTH OF SPHERICAL REPRESENTATIONS OF THE CURVE

Definition 4.1. Let \( \alpha \) be a unit speed regular curve in Euclidean 3-space with modified orthogonal frame \( T, N \) and \( B \). The unit tangent vectors along the curve \( \alpha \) generate a curve \( (T) \) on the sphere of radius one about the origin. The curve \( (T) \) is called the spherical indicatrix of \( T \) or more commonly, \( (T) \) is called tangent indicatrix of the curve \( \alpha \). If \( \alpha = \alpha(s) \) is a natural representation of the curve \( \alpha \), then \( (T) = T(s) \) will be a representation of \( (T) \). Similarly one considers the principal normal indicatrix \( (N) = N(s) \) and binormal indicatrix \( (B) = B(s) \) [8].

Theorem 4.2. Let \( \alpha : I \subset R \rightarrow E^3 \) be a curve and \( T(s), N(s), B(s) \) be its tangent, normal and the binormal, then the following hold:

\[
\begin{align*}
(1) \quad &D_{T_T}T_T = -T + \frac{\tau}{\kappa^2} B. \\
(2) \quad &D_{T_N}T_N = \frac{\phi'}{\kappa^2 \|w\|} (\kappa \sin \phi T + \cos \phi B) - \frac{1}{\kappa^2} N. \\
(3) \quad &D_{T_B}T_B = \frac{1}{\tau} T - \frac{1}{\kappa^2} B.
\end{align*}
\]
Proof. Let $T = T(s)$ be the tangent vector field of the curve $\alpha : I \subset R \to E^3$. The spherical curve $\alpha_T = T$ on $S^2$ is called 1st spherical representation of the tangents of $\alpha$. Let $s$ be the arclength parameter of $\alpha$. If we denote the arclength of the curve $\alpha_T$ by $s_T$, then we may write

$$\alpha_T(s) = T(s).$$

By differentiating both sides of Eq. (4.1) with respect to $s$ and using norm, we get

$$T_T = \frac{N ds_T}{ds},$$

$$\frac{ds_T}{ds} = \kappa$$

and

$$T_T = \frac{N}{\kappa}.$$ 

If we differentiate with respect to $s$ again, we obtain

$$\frac{d}{ds} (T_T) = \frac{d}{ds} \left( \frac{1}{\kappa} N \right)$$

$$\frac{dT_T}{ds_T} = \left\{ \left( \frac{1}{\kappa} \right)' N + \frac{1}{\kappa} \left( -\kappa^2 T + \frac{\kappa'}{\kappa} N + \tau B \right) \right\} \frac{ds}{ds_T}$$

$$\frac{dT_T}{ds_T} = \left[ \left( \frac{1}{\kappa} \right)' N - \kappa T + \frac{\kappa'}{\kappa^2} N + \frac{\tau}{\kappa} B \right] \frac{1}{\kappa}$$

(4.2) 

$$D_{T_T} T_T = -T + \frac{\tau}{\kappa^2} B.$$ 

which is part (1) of the theorem. The spherical curve $\alpha_N = N(s)$ on $S^2$ is called 2nd spherical representation or the spherical representation of $N$ of the curve $\alpha$. Let $s \in I$ be the arclength of the curve $\alpha$. If we denote the arclength of $N$ by $s_N$ and take $\kappa = \text{constant}$, we can write

$$\alpha_N(s_N) = N(s).$$

Differentiating both sides of Eq. (4.3) with respect to $s$, we get

$$\frac{d\alpha_N}{ds_N} = \left( -\kappa^2 T + \tau B \right) \frac{ds}{ds_N}.$$ 

(4.4) 

Taking norm of the Eq. (4.4), we obtain

$$\frac{ds_N}{ds} = \kappa \sqrt{\kappa^2 + \tau^2} \quad \text{or} \quad \frac{ds_N}{ds} = \kappa \|w\|.$$ 

Hence we get $T_N$ as

$$T_N = -\frac{\kappa}{\|w\|} T + \frac{1}{\kappa} \frac{\tau}{\|w\|} B$$

or

$$T_N = -\cos \phi T + \frac{1}{\kappa} \sin \phi B.$$ 

(4.5)
If we differentiate Eq.(4.5) with respect to s and by Eq.(3.9), we have
\[
\frac{dT_N}{ds} = \left[ \phi' \sin \phi T - \cos \phi N + \frac{\phi'}{\kappa} \cos \phi B + \frac{\sin \phi}{\kappa} (\tau N) \right] \frac{ds}{ds_N} \\
= \left[ \phi' (\kappa \sin \phi T + \cos \phi B) - (\kappa \cos \phi + \tau \sin \phi) N \right] \frac{1}{\kappa^2 \|w\|} \\
= \left[ \phi' (\kappa \sin \phi T + \cos \phi B) - \|w\| N \right] \frac{1}{\kappa^2 \|w\|}
\]
or
\[
 DT_N T_N = \frac{\phi'}{\kappa^2 \|w\|} (\kappa \sin \phi T + \cos \phi B) - \frac{1}{\kappa^2} N,
\]
which is (2) of the theorem.

The spherical curve \( B = B(s) \) on \( S^2 \) is called 3rd spherical representation or the spherical representation of \( B \) of the curve \( \alpha \). Let \( s \in I \) be the arclength of the curve \( \alpha \). If we denote the arclength of \( B \) by \( s_B \) and take \( \kappa = \text{constant} \), we can write
\[
\alpha_B(s_B) = B(s).
\]
Differentiating Eq.(4.8) with respect to \( s \), we get
\[
T_B \frac{ds}{ds_B} = -\tau N.
\]
Taking the norm of the Eq.(4.9), we obtain
\[
\frac{ds_B}{ds} = \kappa \tau.
\]
So we can write
\[
T_B = -N \frac{1}{\kappa}.
\]
Again differentiating(4.10) with respect to \( s \), we get
\[
\frac{dT_B}{ds} = \frac{d}{ds} \left( \frac{N}{\kappa} \right) \frac{ds}{ds_B} = -\frac{1}{\kappa} N' \frac{1}{\kappa \tau} = -\frac{1}{\kappa^2 \tau} N' = -\frac{1}{\kappa^2 \tau} (-\kappa^2 T + \tau B)
\]
or
\[
 DT_B T_B = \frac{1}{\tau} T - \frac{1}{\kappa^2} B,
\]
which is (3) of the theorem.

5. The pole curves

**Theorem 5.1.** Let \( \alpha : I \subset R \rightarrow E^3 \) be a curve and \( C \) be the vector field defined as \( C = \frac{w}{\|w\|} \), then we have
\[
 DT_C T_C = -\sin \phi T - \frac{\cos \phi}{r} B + \frac{\|w\|}{\phi'} N.
\]

**Proof.** Let \( s_C \) be the arclength parameter of \( (C) \) and the unit vector at direction of Darboux vector \( w \) according to orthogonal frame \( E = \{T, N, B\} \) be \( T_C \). Moreover
the geodesic curvature of \((C)\) according to \(E^3\) be \(k_C = \|D_{T_C} T_C\|\). Then we can write equation of \((C)\) as
\[
\alpha(s_C) = C = \frac{w}{\|w\|} = \frac{\tau}{\|w\|} T + \frac{1}{\|w\|} B.
\]
From Eq.(3.9), we get
\[
C = (\sin \phi) T + \left(\frac{\cos \phi}{\kappa}\right) B.
\]
Differentiating Eq.(4.13) with respect to \(s_c\) and by Eq.(2.3), we get
\[
\frac{dC}{ds} = \phi' \kappa \left\{ (\kappa \cos \phi T - \sin \phi B) + (\kappa \sin \phi - \tau \cos \phi) N \right\} \frac{ds}{ds_C},
\]
Using Eq.(3.9), we can write (4.14) as
\[
\frac{dC}{ds_C} = \frac{\phi'}{\kappa} (\kappa \cos \phi T - \sin \phi B) \frac{ds}{ds_C}.
\]
Since \(T_C = \frac{dC}{ds_C}\), by taking the norm of Eq.(4.15), we get
\[
\frac{ds_C}{ds} = \frac{\phi'}{\kappa} \sqrt{\kappa^2 \cos^2 \phi + \sin^2 \phi \kappa^2} = \phi'
\]
and
\[
T_C = \cos \phi T - \sin \phi \frac{\kappa}{\kappa} B.
\]
Differentiating Eq.(4.16) with respect to \(s\) and using Eq.(2.3) and Eq.(3.9), we calculate \(D_{T_C} T_C\) as
\[
D_{T_C} T_C = \left\{ -\phi' \sin \phi T + \cos \phi N \right\} \frac{ds}{ds_C} + \left\{ -\phi' \sin \phi T + \cos \phi N - \left(\frac{\sin \phi}{\kappa}\right)' B - \left(\frac{\sin \phi}{\kappa}\right)' B' \right\} \frac{ds}{ds_C}
\]
\[
= \left\{ -\phi' \sin \phi T + \cos \phi N - \left(\frac{\phi' \cos \phi}{\kappa} - \frac{\kappa' \sin \phi}{\kappa^2} \right) B - \sin \phi \left\{ \frac{\kappa}{\kappa} - \frac{\kappa'}{\kappa} \right\} \frac{1}{\phi'} \right\} \frac{ds}{ds_C}
\]
\[
D_{T_C} T_C = \left\{ -\sin \phi T + \cos \phi N - \frac{\phi'}{\kappa} B + \frac{1}{\phi'} \left( \frac{\sin \phi}{\kappa} \frac{\kappa}{\|w\|} + \frac{\tau \sin \phi}{\kappa} \frac{\kappa}{\|w\|} \right) N \right\} \frac{1}{\phi'}
\]
\[
= \left\{ -\sin \phi T + \cos \phi N - \frac{\phi'}{\kappa} B + \frac{1}{\phi'} \left( \frac{\sin \phi}{\kappa} \frac{\kappa}{\|w\|} + \frac{\tau \sin \phi}{\kappa} \frac{\kappa}{\|w\|} \right) N \right\} \frac{1}{\phi'}
\]
\[
= -\sin \phi T - \cos \phi \frac{B}{\kappa} + \frac{1}{\phi'} \left( \frac{\kappa}{\|w\|} + \frac{\tau}{\|w\|} \right) \sin \phi \frac{\kappa}{N}
\]
\[
D_{T_C} T_C = -\sin \phi T - \cos \phi \frac{B}{\kappa} + \frac{1}{\phi'} \left( \frac{\kappa^2}{\|w\|} + \frac{\tau^2}{\|w\|} \right) \sin \phi \frac{\kappa}{N}
\]
\[
= -\sin \phi T - \cos \phi \frac{B}{\kappa} + \frac{1}{\phi'} \left( \frac{\kappa^2}{\|w\|} + \frac{\tau^2}{\|w\|} \right) \frac{N}{\kappa}
\]
or

\begin{equation}
D_{T_C}T_C = - \sin \phi T - \frac{\cos \phi}{\kappa} B + \frac{\|w\|}{\phi'} N. 
\end{equation}

6. Geodesic curvatures of the curves \((T), (N), (B)\) and \((C)\) according to \(S^2\)

**Theorem 6.1.** Let \(\gamma_T, \gamma_N, \gamma_B\) and \(\gamma_C\) be geodesic curvatures of \((T), (N), (B)\) and \((C)\) respectively, then the following holds:

\begin{enumerate}
\item \[\gamma_T = \sqrt{\left(\frac{\phi'}{\kappa \|w\|}\right)^2 + \left(\frac{\kappa^2 - 1}{\kappa}\right)^2}.\]
\item \[\gamma_N = \sqrt{\frac{\|w\|^2}{\kappa^2 T^2}} + \kappa^2.\]
\item \[\gamma_C = \frac{\|w\|}{\phi'}.\]
\end{enumerate}

**Proof.** Let \(\gamma_T, \gamma_N, \gamma_B\) and \(\gamma_C\) be geodesic curvatures of \((T), (N), (B)\) and \((C)\) respectively. Using Eqs.(4.2), (4.6), (4.11) and (4.17), we get

\[
\gamma_T = \|D_{T_T}T_T + \langle S(T_T), T_T \rangle T\| = \|D_{T_T}T_T + 1.T\| = \|\frac{T}{\kappa^2} - \frac{T}{\kappa^2} B + T\| = \frac{T}{\kappa} = \tan \phi,
\]

\[
\gamma_N = \|D_{T_N}N + \langle S(T_N), T_N \rangle N\|
\]

\[
= \|\frac{\phi'}{\kappa^2 \|w\|} (\kappa \sin \phi T + \cos \phi B) - \frac{1}{\kappa^2} N + 1.N\|
\]

\[
= \|\frac{\phi'}{\kappa^2 \|w\|} (\kappa \sin \phi T + \cos \phi B) + \left(\frac{\kappa^2 - 1}{\kappa^2}\right) N\|
\]

\[
= \sqrt{\left(\frac{\phi'}{\kappa^2 \|w\|} (\kappa \sin \phi T + \cos \phi B)\right)^2 + \left(\frac{\kappa^2 - 1}{\kappa^2}\right)^2} \kappa^2
\]

\[
= \sqrt{\left(\frac{\phi'}{\kappa \|w\|}\right)^2 \frac{1}{\kappa^4} \kappa^2 + \left(\frac{\kappa^2 - 1}{\kappa}\right)^2}
\]

\[
= \sqrt{\left(\frac{\phi'}{\kappa \|w\|}\right)^2 + \left(\frac{\kappa^2 - 1}{\kappa}\right)^2}
\]

\[
\gamma_B = \|D_{T_B}B + 1.B\| = \left\|\frac{1}{\tau^2} - \frac{1}{\kappa^2} B + B\right\| = \sqrt{\frac{1}{\tau^2} + \frac{1}{\kappa^2}} + \kappa^2
\]

\[
= \sqrt{\frac{\kappa^2 + \tau^2}{\kappa^2 \tau^2}} + \kappa^2
\]

\[
= \sqrt{\frac{\|w\|^2}{\kappa^2 \tau^2}} + \kappa^2
\]
and
\[
\gamma_C = \left\| D_{T_{C}} T_{C} + \langle S(T_{C}), T_{C} \rangle C \right\|
\]
\[
= \left\| D_{T_{C}} T_{C} + 1.C \right\|
\]
\[
= \left\| \left( -\sin \phi T - \frac{\cos \phi B}{\kappa} + \frac{\|w\|}{\phi'} N \right) + \left( \sin \phi T + \frac{\cos \phi}{\kappa} B \right) \right\|
\]
\[
= \left\| \frac{||w||}{\phi'} N \right\| = \frac{||w||}{\phi'}.
\]

\[\Box\]

**Theorem 6.2.** Let frenet vector fields of a curve $\alpha : I \rightarrow E^3$ be $T$, $N$, $B$ and also let Darboux vector field of curve $\alpha$ be $w = \tau T + B$. Let $\alpha_w$, $\alpha_T$, $\alpha_N$ and $\alpha_B$ be, respectively, spherical indicatrices of vector fields $w$, $T$, $N$ and $B$ of a curve $\alpha$ in Euclidean 3–space $E^3$. Then

(i) $\alpha_T$ is a spherical involute for $\alpha_w$,

(ii) If curve $\alpha$ has the constant curvature, then $\alpha_B$ is a spherical involute for $\alpha_w$.

(iii) If curve $\alpha$ is a helix and $\alpha_N$ has the constant curvature, then $\alpha_N$ is a spherical involute for $\alpha_w$.

**Proof.** Tangents of $(C)$ constant pole and motion pole curves are common. we already know for spherical indicatrices of curve $\alpha : I \rightarrow E^3$ the following equalities

(5.2) \[
\left\{\begin{array}{l}
\alpha_T = T, \alpha_B = B, \alpha_w = \frac{\tau}{||w||} T + \frac{1}{||w||} B (\kappa = \text{const.}) .
\end{array}\right.
\]

Similary, we also know that

(5.3) \[
\left\{\begin{array}{l}
\alpha_T' = T_T = N(s) \frac{ds}{ds_T}, \\
\alpha_N' = T_N = (-\kappa^2 T + \tau B) \frac{ds}{ds_N}, \\
\alpha_B' = T_B = -\tau T \frac{ds}{ds_B}, \\
\alpha_C' = T_C = \phi' \left( \cos \phi T - \sin \phi \frac{B}{\kappa} \right) \frac{ds}{ds_C}.
\end{array}\right.
\]

(i) From inner product of unit tangent vector fields of $T_T$ and $T_C$, we get

(5.4) \[
\langle T_T , T_C \rangle = \frac{ds}{ds_T} \phi' \left( N, \cos \phi T - \sin \phi \frac{B}{\kappa} \right) = 0.
\]

So we conclude that $\alpha_T$ is a spherical involute of $\alpha_w$.

(ii) From inner product of unit tangent vector fields of $T_B$ and $T_C$, we get

\[
\langle T_B , T_C \rangle = \left\langle -\tau N \frac{ds}{ds_B}, \phi' \left( \cos \phi T - \sin \phi \frac{B}{\kappa} \right) \frac{ds}{ds_C} \right\rangle
\]
\[
= -\tau \phi' \frac{ds}{ds_B} \frac{ds}{ds_C} \left\langle N, \cos \phi T - \sin \phi \frac{B}{\kappa} \right\rangle = 0.
\]

So $\alpha_B$ is a spherical involute of $\alpha_w$. 
(iii) From inner product of unit tangent vector fields of $T_N$ and $T_C$, we get
\[
\langle T_N, T_C \rangle = \left( -\kappa^2 T + \tau B \right) \frac{ds}{ds_N} \phi' \left( \cos \phi \frac{T - \sin \phi}{\kappa} \frac{B}{\kappa} \right) \frac{ds}{ds_C} \\
= \phi' \frac{ds}{ds_N} \frac{ds}{ds_C} \left( -\kappa^2 T + \tau B, \cos \phi \frac{T - \sin \phi}{\kappa} \frac{B}{\kappa} \right) \\
= -\phi' \frac{ds}{ds_N} \frac{ds}{ds_C} \kappa \left( \cos \phi \frac{T - \sin \phi}{\kappa} \frac{B}{\kappa} \right) \\
= -\kappa \phi' \frac{ds}{ds_N} \frac{ds}{ds_C} \left[ \tau \cos \phi \kappa + \sin \phi \right] \\
\langle T_N, T_C \rangle = \phi' \kappa \|w\| \frac{ds}{ds_N} \frac{ds}{ds_C}
\]
Since $\alpha$ is a helix, $\frac{\vec{z}}{\kappa} = - \cot \phi$ is constant. We obtain $\phi' = 0$. Thus the proof is finished.

\[\square\]

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