Reliable computation of PID gain space for general second-order time-delay systems

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ABSTRACT
This paper addresses the problem of determining the stability gain space of a PID controller for general second-order time-delay systems. First, a review of existing results and the associated drawbacks is presented. Subsequently, a new algorithm to compute the entire PID stability gain space is developed. The new algorithm is based upon existing results on the relationship between the stability of a quasi-polynomial and its derivatives, an extended version of the Hermit–Biehler theorem, and also the Nyquist criterion. The algorithm entails extraction of an admissible range for the PID parameter $K_p$, and then based on this range, a stability region in the $(K_i - K_d)$ plane is computed. Well-known examples are studied to demonstrate the reliability and accuracy of the results.

1. Introduction
Despite a great deal of development in advanced controller design methods during the past decades, proportional-integral-derivative (PID) controllers are still widely used in industrial process systems. By various accounts, over 95% of the currently working controllers are of the PID type (Brisk, 2004). This widespread application is based on several advantages such as simplicity in control structure, control stability, effective performance, model-free tuning, robust behaviour, and easy maintenance (Astrom & Hagglund, 2006; Panda, 2008). Even where more advanced controllers are employed, the PID controller yet remains in charge of regulating the first control automation level (Visioli, 2006).

Modern digital industrial control systems are required to operate much more efficiently than their previous and older analogue counter parts. As a result, the current trend in tuning PID controllers is to move away from traditional design heuristics such as the Ziegler–Nichols method (Ziegler & Nichols, 1942), and towards powerful optimisation-based synthesis methods (Ho, 2003). In case of the latter, knowing the closed-loop stability region with respect to the $(K_i, K_d, K_p)$ parameters is critical and necessary to be used as constraints of the tuning optimisation problem. In recent years, a number of methods to determine this space have been proposed. For example, Ho, Datta, and Bhattacharyya (1997) developed a linear programming method to determine the range of stabilising PI and PID controller parameters for a given system model. Based on the Nyquist criterion, Söylemez, Munro, and Baki (2003) developed a fast method for calculating stabilising PID controller parameters, and Tan, Kaya, and Atherton (2003) presented a new method for estimation of stabilising PI and PID controllers by plotting the stability boundary locus.

The algorithms cited previously are only applicable to finite-dimensional systems. This is unfortunate since time-delay is an omnipresent feature of all industrial processes. More recently, attention has been shifted to algorithms which deal directly with non-finite dimensional time-delay systems. Initial works, such as Silva, Datta, and Bhattacharyya (2001, 2002) and Martelli (2005), considered first-order time-delay systems. Second-order integrating systems with time-delay were studied in Ou, Tang, Gu, and Zhang (2005) and Ou, Zhang, and Gu (2006). Integral time-delay systems have also been studied in Ou, Zhang, and Gu (2005) and Padula and Visioli (2012). In addition to special system models, some solutions for general time-delay systems have also been proposed which are complex to evaluate (for example, see Almodaresi, Bozorg, & Taghirad, 2016; Bozorg & Termeh, 2011; Hohenbichler, 2009; Ou, Zhang, & Yu, 2009) or have a large computational cost (for example, see Yu, Le, Xian, & Wang, 2013). There are also seminal works in the domain of fractional order PID controllers some of which are Caponetto and Dongola (2013), Badri and Tavazoei (2013), and Hamamci (2008).

Second-order process models are of special interest since they are the lowest order model which can exhibit
principal characteristics of real-life processes. In Wang (2007), a new graphical approach is proposed for the stabilisation of second-order time-delay systems with two real poles. However, according to Martelli (2009), the stability conditions were not explicit and no finite number of required computational steps is specified. Moreover, as demonstrated in this paper, in some instances, the algorithm is not reliable. A second-order time-delay system with an additional finite zero has been considered in Martelli (2009). In this work, a new algorithm is developed to compute the entire stability gain space. While the algorithm is precise, it is complex and difficult to implement. Second-order time-delay systems with a pair of complex poles have been studied in Wang and Zhang (2010) by using a similar approach as presented in Wang (2007). Computation of stabilising PI and PID parameters for second-order time-delay systems with real unstable poles has also been considered in Farkh, Laabidi, and Ksouri (2014).

The contributions of this paper are two-fold. First, we demonstrate some inaccuracies in some of the existing results cited above and highlight what appears to be the source of these inaccuracies. Second, we propose an algorithm for computation of regions for PID stabilising gain space of general (real or complex poles) second-order time-delay systems. The algorithm computes efficiently and is based on an extended version of the Hermite–Biehler theorem and the Nyquist criterion. We also demonstrate that it is consistent with fundamental theorems on stability of quasi-polynomials and therefore returns a reliable solution.

2. Review on some previous results

Consider the standard closed-loop system as shown in Figure 1.

Let the plant model $G(s)$ be a second-order time-delay system defined as,

$$G(s) = \frac{k}{(1 + T_1 s)(1 + T_2 s)} e^{-\theta s}.$$  \hspace{1cm} (1)

where $k > 0$ is the steady-state gain, $\theta > 0$ is the time-delay, and $T_1$ and $T_2$ are the time constants of the plant with arbitrary sign. System (1), as considered in Wang (2007), is a special class of the general second-order time-delay systems and contains two real poles only. An ideal PID-type controller is considered in the following form:

$$C(s) = K_p + \frac{K_i}{s} + K_ds.$$ \hspace{1cm} (2)

From system model (1) and controller (2), the characteristic equation for the closed-loop system in Figure 1 is

$$\Delta(s) = s(1 + T_1 s)(1 + T_2 s) + k(K_i + K_p s + K_ds^2) e^{-\theta s}.$$ \hspace{1cm} (3)

Equations such as (3) are known as quasi-polynomials. One of the earliest studies on the stability of quasi-polynomials was carried out by Pontryagin (Pontryagin, 1955). Based on Pontryagin’s results, a significant extension of the Hermite–Biehler theorem has been developed to study the stability of quasi-polynomials (see Bellman & Cook, 1963).

The algorithm outlined in Wang (2007) first determines an admissible range for $K_p$, and then for a specific value of $K_p$ within this range, an associated stability region for $(K_i - K_d)$ is computed by graphical means. Central to the methodology is the following result which sets out a necessary and sufficient condition for the existence of stabilising PID controllers.

**Lemma 2.1** (Wang, 2007): A necessary and sufficient condition for existence of a stabilising PID controller with stability degree $\sigma > 0$ for the plant described in (1) is

$$\frac{m}{6(n + kK_pe^{\sigma\theta}) + \sigma^2 m} < \frac{1}{\sigma^2},$$ \hspace{1cm} (4)

where

$$m = 3\theta(2 - \theta\sigma)((1 - T_1\sigma)T_2 + (1 - T_2\sigma)T_1)$$

$$+ 6T_1T_2(1 - \theta\sigma) + \theta^2(1 - \theta\sigma)(1 - T_1\sigma)(1 - T_2\sigma)$$

$$n = (1 - T_1\sigma)(1 - T_2\sigma)(1 + \theta\sigma - \theta^2\sigma^2) - 2T_1T_2\sigma^2$$

$$+ \sigma(1 - 2\theta\sigma)((1 - T_1\sigma)T_2 + (1 - T_2\sigma)T_1).$$ \hspace{1cm} (5)

The value of $\sigma$ determines the line $s = -\sigma$ which will constrain all the closed-loop poles to its left.

From Lemma 2.1 for a given plant and stability degree $\sigma > 0$, Inequality (4) becomes a function of the controller parameter $K_p$. It therefore appears that there always exists some $K_p$, which will satisfy (4). Accordingly, there must exist at least one PID controller for any second-order system of the form specified in (1) which will place the closed-loop poles of the system to the left of an arbitrary line on the negative real axis. Note that from Lemma 2.1,
a necessary and sufficient condition for existence of stabilising PID controller is
\[
\frac{6(n + kK_p e^{\sigma \theta}) + \sigma^2 m}{m} = \frac{6(n + kK_p e^{\sigma \theta})}{m} + \sigma^2 > \sigma^2,
\]
(6)
or equivalently,
\[
\frac{n + kK_p e^{\sigma \theta}}{m} > 0.
\]
(7)

It is clear that for any values of \(m\) and \(n\), a \(K_p\) can be found (positive or negative) such that (7) is satisfied. This assertion is correct for delay-free second-order systems, but not generally correct, as shown in Example 2.1.

**Example 2.1:** Consider the following second-order time-delay system:
\[
G(s) = \frac{1}{(1 - 0.6s)(1 - 0.8s)} e^{-3s}.
\]
(8)

Take \(\sigma = 0.1\). From Lemma 2.1,
\[
\frac{6.9458}{6(1.3158 + K_p e^{0.03}) + 0.069458} < \frac{1}{0.01},
\]
(9)

which indicates that for \(K_p > -0.9747\), there exists at least one stabilising PID controller. In addition, Remark 2 in Wang (2007) emphasises that the stability region is nonempty since for \(K_p > -0.9747\) there exists some \(\epsilon > 0\) such that,
\[
\epsilon < \frac{6.9458}{6(1.3158 + K_p e^{0.03}) + 0.069458} < \frac{1}{0.01}.
\]
(10)

To see why this cannot be true, consider that if there exists a PID controller which places the closed-loop poles to the left of \(s = -0.1\), then there must exist at least one PID controller which places the poles to the left of the origin. However, according to Theorem 1 in Wang (2007), there does not exist any stabilising PID controller for this plant. This theorem states that for a stabilising PID controller to exist, (11) must be satisfied in the interval \((0, \pi)\),
\[
\tan(z) = -\frac{2T_1 T_2 + \theta (T_1 + T_2)}{\theta^2 - T_1 T_2 z^2 + \theta (T_1 + T_2) z}.
\]
(11)

However, from Figure 2, it is clear that for the plant specified by (8), Equation (11) does not have any roots in the said interval.

There is also a secondary contradiction which appears to stem from an earlier proposition in Silva et al. (2002) on the relationship between the stability of a quasi-polynomial and its delay-free version. It states ‘a minimal requirement for any control design is that the delay-free closed-loop system be stable’ (also see, Remark 2 and Theorem 1 in Wang, 2009, and Condition 1 in Wang, 2012, and also Farkh et al., 2014; Marquez-Rubio, del Muro-Cuellar, and Ramirez, 2014; Wang & Zhang, 2010). This statement is also quoted elsewhere in the literature (for example, see Silva et al., 2001; Silva, Datta, and Bhattacharyya, 2004) and forms the basis for several recent works in this topic (Farkh et al., 2014; Keel & Bhattacharyya, 2008; Marquez-Rubio et al., 2014; Ou et al., 2005; Silva et al., 2004; Wang, 2007, 2009, 2012; Xu, Datta, & Bhattacharyya, 2003).

Despite extensive referrals, a formal proof of the proposition appears to be missing. In addition, there are counterexamples which show that the statement might not be correct. For example, Kharitonov, Niculescu, Moreno, and Michiels (2005) show that for the proportional-only control, the stability gain space of delayed and delay-free versions of a plant can have an empty intersection.

**Example 2.2** (Wang, 2007): Consider the following second-order time-delay system:
\[
G(s) = \frac{1}{(1 - 0.5s)(1 - s)} e^{-0.2s}.
\]
(12)

From, Equation (15a) in Wang (2007), the delay-free version of the plant is stable if and only if,
\[
\begin{align*}
K_i &> 0, \\
K_d &> 1.5, \\
K_p + 1 &> \frac{0.5K_i}{K_d - 1.5}.
\end{align*}
\]
(13)
Consider the closed-loop system as shown in Figure 1. This result can be simply obtained from Routh Hurwitz table. Now, consider the following two PID controllers:

\[ C_1(s) = -0.95 + \frac{0.01}{s} + 1.4s \]
\[ C_2(s) = -0.98 + \frac{0.01}{s} + 1.6s \]  

Since \( C_1 \) and \( C_2 \) invalidate the first and the third conditions of (13), respectively, they will not stabilise the delay-free plant, and by implication of the statement should not also stabilise the delayed version either. However, both of these controllers will actually stabilise the delayed version of the system. The step responses of the closed-loop system with \( C_1 \) and \( C_2 \) are shown in Figure 3.

3. Main results

In this section, we consider a general form of second-order time-delay systems with real or complex poles. For this general form, we first develop new necessary conditions to compute the stability gain space of the PID controller. We then make direct use of the Hermite–Biheler theorem and the Nyquist criterion to develop a simple and reliable algorithm for computation of the PID stabilising gain space.

3.1 Problem formulation and preliminary results

Consider the closed-loop system as shown in Figure 1 with the general second-order time-delay system \( G(s) \),

\[ G(s) = \frac{k}{s^2 + \alpha s + \beta} e^{-\delta s}. \]  

Without loss of generality, we assume \( k > 0 \) in the remainder of this paper. The controller \( C(s) \) we consider is in the form of (2). From (15) and (2), the characteristic equation of the system in Figure 1 becomes

\[ \Delta(s) = s(s^2 + \alpha s + \beta) + k(K_i + K_p s + K_ds^2) e^{-\delta s}. \]  

Throughout the paper, we consider only the PID parameter \( K_i \) to be positive. Alternatively, the results can be simply extended to the region of \( K_i < 0 \) using methods presented in this work. Now, multiplying both sides of (16) by \( e^{\delta s} \) gives

\[ f(s) = \Delta(s) e^{\delta s} = (s^3 + \alpha s^2 + \beta s) e^{\delta s} + k(K_i + K_p s + K_ds^2). \]  

Since \( e^{\delta s} \) has no finite zeros, the quasi-polynomial (16) is Hurwitz stable if and only if (17) is Hurwitz stable. That is, the unity feedback loop in Figure 1 is stable if and only if all zeros of (17) are in the left half complex plane. The following theorem provides a necessary and sufficient condition for stability of quasi-polynomials such as \( f(s) \) in (17).

**Theorem 3.1** (Bellman & Cook, 1963): Consider the following quasi-polynomial:

\[ f(s) = \sum_{i=0}^{n} \sum_{l=1}^{r} \delta_{il} s^{n-i} e^{\theta_l s}, \]  

where \( \theta_1 < \theta_2 < \cdots < \theta_r, \theta_1 + \theta_1 > 0 \) and \( \delta_{0r} \neq 0 \). The quasi-polynomial (18) is stable if and only if,

1. \( f_r(w) \) and \( f_l(w) \) have only real roots and these roots interlace,
2. \( f_l'(w^*) f_r(w^*) - f_l(w^*) f_r'(w^*) > 0 \) for some \( w^* \in (-\infty, \infty) \).

where \( f_l(w) \), \( f_r(w) \), \( f_l'(w) \), and \( f_r'(w) \) denote the real and imaginary parts of \( f(jw) \) and \( f'(jw) \) (the first derivative of \( f(jw) \) with respect to \( w \)), respectively.

Theorem 3.1 is known as the extended Hermite–Biheler theorem. Other important theorems which have been used in the development of our proposed conditions are the following two theorems.

**Theorem 3.2** (Bellman & Cook, 1963): Consider the quasi-polynomial (18) in Theorem 3.1. Let \( M \) and \( N \) denote the highest powers of \( s \) and \( e^{\theta_l s} \). Also, suppose that \( f_r(w) \) and \( f_l(w) \) are the real and imaginary parts of \( f(jw) \). Let \( \eta \) be a constant such that the coefficients of the highest degree in \( f_r(w) \), and \( f_l(w) \) do not vanish at \( w = \eta \). Then, a necessary and sufficient condition for \( f_r(w) \) or \( f_l(w) \) to have
only real roots is that in the interval $-2l\pi + \eta \leq w \leq 2l\pi + \eta$, $f_i(w)$ or $f_i(w)$ have exactly $4IN + M$ real roots for a sufficiently large $l$.

**Theorem 3.3** (Kharitonov et al., 2005): Consider the quasi-polynomial $f(s)$ in (18). If $f(s)$ is stable, $f(s)$ (derivation of $f(s)$ with respect to $s$) is also a stable quasi-polynomial.

A straightforward conclusion of Theorem 3.3 is that if $f(s)$ is an unstable quasi-polynomial, then $f(s)$ is also unstable.

### 3.2 Necessary conditions for PID stabilisation

In this section, we use Theorem 3.3 to derive tight parameter bounds which are then used as a set of necessary conditions for existence of a PID stabilising controller for the plant considered in (15).

**Lemma 3.1:** Consider the quasi-polynomial $f(s)$ in (17). Then, $f(s)$ is stable only if

$$
\begin{cases}
\alpha > -\frac{9}{\pi} \\
\alpha > -\left(\frac{3}{\beta} + \frac{\theta \beta}{\alpha}\right) \\
\alpha > -\left(\frac{9}{\beta} + \frac{\theta \beta}{2\beta}\right)
\end{cases}
$$

(19)

**Proof:** Computing the third derivation of $f(s)$ in (17) with respect to $s$ yields

$$
\frac{d^3}{ds^3} f(s) = f_a(s) e^{\beta s}.
$$

(20)

where

$$
f_a(s) = \left(\theta^3 s^3 + \theta^2 (\alpha \theta + 9)s^2 + \theta (\beta \theta^2 + 6 \alpha \theta + 18)s + (\beta \theta^2 + 2 \alpha \theta + 2)\right).
$$

(21)

From Theorem 3.3, $f(s)$ is stable only if $\frac{d^3}{ds^3} f(s)$ in (20) is stable, and clearly $\frac{d^3}{ds^3} f(s)$ is stable if and only if $f_a(s)$ in (21) is stable. From the Routh Hurwitz table, $f_a(s)$ is Hurwitz stable if and only if all inequalities of (19) are satisfied. This completes the proof.

The conditions proposed in Lemma 3.1 are consistent with existing result by Lee, Wang, and Xiang (2010). Using their results, it can be easily shown that the third statement of (19) is a necessary and sufficient condition for existence of PID stabilising controllers for the special class of plant (15) in which there are only one stable and one unstable pole. The plant considered in Lee et al. (2010) can contain more than two poles; however, unlike the system considered in this work, it does not permit two unstable (or two stable) poles at the same time and also oscillatory transients. Moreover, the entire stability area was not considered in Lee et al. (2010). Lemma 3.2 is another result which can be obtained using Theorem 3.3. It is used as a necessary condition to compute the stabilising gain space of the PID parameters, in the final result of the paper.

**Lemma 3.2:** Consider the quasi-polynomial $f(s)$ in (17). Let

$$
f_c(s) = \left(\theta^2 s^3 + \theta^2 \alpha + 6 \theta s^2 + (4 \alpha \theta + \theta^2 \beta + 6)s + 2 \alpha + 2 \beta \theta\right),
$$

(22)

and $n$ be the number of unstable poles of $f_c(s)$ in (22). Then, $f(s)$ is stable only if the Nyquist plot of $H(s) = \frac{2ke^{-wa}}{f_c(s)}$ encircles the $(-1, 0)$ point $-n$ times.

**Proof:** From (17),

$$
\frac{d^2}{ds^2} f(s) = f_c(s) e^{\beta s} + 2kK_d,
$$

(23)

where $f_c(s)$ is defined in (22). Evidently, $f_c(s)$ in (23) can be considered as the closed-loop characteristic equation for an equivalent open-loop system $H(s) = \frac{2ke^{-wa}}{f_c(s)}$ when the feedforward controller is $K_d$. Consequently, an admissible range for the PID parameter $K$ can be derived using the Nyquist diagram of $H(s)$. This completes the proof.

**Remark 3.1:** Based on Lemma 3.2, if $f_c(s)$ is Hurwitz stable, then an admissible range for $K_d$ is found as follows:

$$
-GM_1 < K_d < GM_2
$$

(24)

where $GM_1$ and $GM_2$ are the gain margin of $-H(s)$ and $H(s)$, respectively.

### 3.3 An admissible range for $K_p$

Substituting $s = jw$ into (17) yields

$$
f(jw) = (-\alpha \omega^2 + j(\beta \omega - \omega^3)) \ e^{j\omega} + jkK_p \omega - kK_d \omega^2.
$$

(25)

Taking $z = \theta w$, the real and imaginary parts of the characteristic equation (25) can be separated as follows:

$$
f_r(z) = \left(\alpha(z\theta)^3 - \beta(z\theta)^2\right) \sin(z) - \alpha(z\theta)^2 \cos(z) - kK_d(z\theta)^2 + kK_i,
$$

(26)

$$
f_i(z) = \left(\beta(z\theta)^3 - \alpha(z\theta)^2\right) \cos(z) - \alpha(z\theta)^2 \sin(z) + kK_p(z\theta),
$$

(27)
In (27), only the proportional term of the PID controller appears. Accordingly, Theorem 3.2 can be used to obtain a range for $K_p$ under which $f_i(z)$ has only real roots. By rewriting $f_i(z)$, we have

$$f_i(z) = \frac{z}{\theta}(\tilde{f}_i(z) + kK_p),$$

where

$$\tilde{f}_i(z) = \left(\beta - \left(\frac{z}{\theta}\right)^2\right) \cos(z) - \alpha \left(\frac{z}{\theta}\right) \sin(z).$$  \hfill (29)

Consider the following lemmas which are used in deriving an admissible range for the PID parameter $K_p$ in the next section.

**Lemma 3.3:** Consider $f_i(z)$ and $\tilde{f}_i(z)$ as defined by (27) and (29). Then, $f_i(z)$ has only simple real roots if and only if $\tilde{f}_i(z) + kK_p$ has $2l + 1$ zeros in the interval $0 < z \leq 2\pi$ for a sufficiently large value of $l$.

**Proof:** From Theorem 3.2, we have $M = 3$, and $N = 1$. Hence, $f_i(z)$ only has simple real roots if and only if (1) the number of its roots in the interval $-2\pi + \eta \leq z \leq 2\pi + \eta$ for sufficiently large values of $l$ is equal to $4l + 3$, and (2) the term $\cos(z)$ which is the coefficient of the highest power in $f_i(z)$ does not vanish for $z = \eta$.

To satisfy the second condition, let $\eta = 0$. Then, $f_i(z)$ must have $4l + 3$ zeros in $-2\pi \leq z \leq 2\pi$ for sufficiently large values of $l$ to satisfy the first condition. Considering the relationship between $f_i(z)$ and $\tilde{f}_i(z) + kK_p$ in (28), and that $\tilde{f}_i(z) + kK_p$ is an even function of $z$, implies $f_i(z)$ has $4l + 3$ zeros in $-2\pi \leq z \leq 2\pi$ if and only if $\tilde{f}_i(z) + kK_p$ has $2l + 1$ zeros in the interval $0 < z \leq 2\pi$. \hfill \blacksquare

**Remark 3.2:** Since the number of roots is a natural number, the value of $l$ in Lemma 3.3 must be from the set of half natural numbers $L = 0.5, 1, 1.5, 2, 2.5, \ldots$.

**Lemma 3.4:** Let $A = -\alpha \left(\frac{z}{\theta}\right)$, and $B = \beta - \left(\frac{z}{\theta}\right)^2$, where $\alpha$, $\beta$, and $\theta$ are constants. Assume, $\beta \geq \alpha^2/2$, and $F(z) = \sqrt{A^2 + B^2}$. Then, $F(z)$ is a strictly decreasing function if $0 < z < \theta \sqrt{\beta - \alpha^2/2}$, and strictly increasing if $z > \theta \sqrt{\beta - \alpha^2/2}$, with respect to $z$.

**Proof:** Let $F'(z)$ be the first derivative of $F(z)$ with respect to $z$. Therefore,

$$F'(z) = \frac{4z(z^2 - \theta^2(\beta - \alpha^2/2))}{\theta^4F(z)},$$

From (30), if $0 < z < \theta \sqrt{\beta - \alpha^2/2}$, then $F'(z) < 0$, and if $z > \theta \sqrt{\beta - \alpha^2/2}$, then $F'(z) > 0$. Finally, $F'(z) = 0$ if $z = \theta \sqrt{\beta - \alpha^2/2}$. This completes the proof. \hfill \blacksquare

It is known that for every $A$, $B$, and $z$,

$$A \sin(z) + B \cos(z) = \sqrt{A^2 + B^2} \cos\left(z - \tan^{-1}\frac{A}{B}\right).$$

(31)

From (29) and (31), we have

$$\tilde{f}_i(z) = \sqrt{A^2 + B^2} \cos\left(z - \tan^{-1}\frac{A}{B}\right),$$

(32)

where $A = -\alpha \left(\frac{z}{\theta}\right)$, and $B = \beta - \left(\frac{z}{\theta}\right)^2$. From Lemma 3.4, for $\beta \geq \alpha^2/2$, the amplitude of $\tilde{f}_i(z)$ in (32) has a minimum point which is used in computing the admissible range of $K_p$. Specifically, for the roots interlacing property that appears in the next section, all local maximum and minimum points of $\tilde{f}_i(z) + kK_p$ must be positive and negative, respectively. We summarise this condition in Lemma 3.5.

**Lemma 3.5:** Let $\tilde{f}_i(z)$ be as defined in (29). Define $E_1$ and $E_2$ as follows:

• $E_1$: the minimum value of all local maximum points of $\tilde{f}_i(z)$.
• $E_2$: the maximum value of all local minimum points of $\tilde{f}_i(z)$.

Therefore, the extremum points of $\tilde{f}_i(z) + kK_p$ do not have any change of sign if and only if

$$\frac{-1}{k} E_1 < K_p < \frac{-1}{k} E_2$$

(33)

**Proof:** The proof is trivial by considering the definitions of $E_1$ and $E_2$. \hfill \blacksquare

**Remark 3.3:** One of the local maximum points of $\tilde{f}_i(z)$ is at $z = 0$. As a result, from Lemma 3.5, $\tilde{f}_i(z) + kK_p$ for $z = 0$ in the $K_p$ admissible range must have positive sign. Then, from (27), a necessary condition for Lemma 3.5 to be satisfied is $\tilde{f}_i(0) + kK_p > 0$ or, equivalently, $K_p > -\frac{A}{B}$.

It is now possible to propose a new algorithm which can be effectively used to derive an admissible range for the PID parameter $K_p$.

**Algorithm 3.1**

Step 1: Specify the plant parameters $\alpha$, $\beta$, and $\theta$.

Step 2: From Lemma 3.5, determine the values of $E_1$ and $E_2$ and compute the interval (33).

Step 3: Determine some values of $l \in L$ ($L$ is defined in Remark 3.2) and for each one a relevant interval $(K_{p1}, K_{p2})$ such that for all values of $K_p$ in the interval, the condition of Lemma 3.3 is satisfied.
Step 4: The admissible range of \( K_p \) is the intersection region of \((E_1, E_2)\) and the union of the intervals \((K_{p1}, K_{p2})\).

**Remark 3.4:** For the special case \( \beta \leq \alpha^2/2 \), it is possible to propose some direct equations to compute the admissible range of \( K_p \) (for example, see Wang, 2007). However, in the general case, using such results can lead to an inaccurate range, especially when \( \beta \gg \alpha^2/2 \).

**Example 3.1:** Consider the plants,

\[
G_1(s) = \frac{2}{s^2 + 2s + 3} e^{-1.5s}, \quad G_2(s) = \frac{20}{s^2 + 2s + 20} e^{-3s}.
\]

Figure 4 shows the function \( \tilde{f}_i(z) \) along with two limit values of \( E_1 = 3 \) and \( E_2 = -2.984 \) for \( G_1(s) \). Therefore, for Step 2 in Algorithm 3.1, we have

\[
-1.5 < K_p < 1.474.
\]  

For \( G_2(s) \), the function \( \tilde{f}_i(z) \) and its constraints \( E_1 = 9.279 \) and \( E_2 = -9.5 \) are shown in Figure 5. Step 2 in Algorithm 3.1 gives

\[
-0.4639 < K_p < 0.475.
\]  

For the third step, first let \( l = 2 \), \( K_{p1} = -0.4639 \), and \( K_{p2} = -0.1245 \) which are found by considering the points \( p_{11} \) and \( p_{21} \) in Figure 5 as the maximum possible shift of \( \tilde{f}_i(z) \) (downward and upward) for which it keeps its five roots in the interval \( 0 < z \leq 4\pi \). Alternatively, let \( l = 2.5 \), \( K_{p1} = -0.3854 \), and \( K_{p2} = 0.475 \) which are determined by \( p_{12} \) and \( p_{22} \) in Figure 5. By combining the results, we have \( K_p \in (-0.4639, 0.475) \) for Step 3. Finally, from Steps 2 and 3, the admissible range for the PID parameter \( K_p \) is found to be as (36).

### 3.4 Stability region in the \((K_i - K_d)\) plane

In this section, for any given value of \( K_p \) within its admissible range, we derive a region for \((K_i - K_d)\) in which the interlacing property for the roots of \( f_i(z) \) and \( f_j(z) \) in (26) and (27) is satisfied.

**Lemma 3.6:** Consider \( f(jw) \) as defined in (25). Let \( f_i(z) \) and \( f_j(z) \) be the real and imaginary parts of \( f(jw) \) defined in (26) and (27). Then, the roots of \( f_i(z) \) and \( f_j(z) \) for \( K_i > 0 \) interlace, if and only if the following set of (infinite) inequalities holds:

\[
(-1)^h K_d < (-1)^h a(z_h) K_i + (-1)^h b(z_h), \quad h = 1, 2, \ldots
\]  

\[
(-1)^h K_d + (-1)^h b(z_h) < 0, \quad h = 1, 2, \ldots
\]  

\[
(-1)^h K_d + (-1)^h a(z_h) K_i < 0, \quad h = 1, 2, \ldots
\]
where $z_h$, for $h = 1, 2, \ldots$ are the positive roots of $f_i(z)$, and
\[
    a(z) = \frac{\theta^2}{z^2}, \quad b(z) = -\frac{\alpha}{k} \cos(z) + \left(\frac{z}{k\theta} - \frac{\beta \theta}{k z}\right) \sin(z).
\]

**Proof:** From (26) and (27),
\[
    \begin{align*}
        f_i(z) &= A \sin(z - \frac{\theta}{2}) + B \cos(z - \frac{\theta}{2}) + k K_i \frac{z}{K D_{\theta}} \\
        f_r(z) &= A \sin(z) + B \cos(z) + k K_i - k K_d \frac{z}{K D_{\theta}} , \tag{39}
    \end{align*}
\]
where
\[
    A = \frac{z^3}{\theta^3} - \beta \frac{z}{\theta}, \quad B = -\alpha \frac{z^2}{\theta^2}. \tag{40}
\]

From (31), (39) can be rewritten as
\[
    \begin{align*}
        f_i(z) &= \sqrt{A^2 + B^2} \cos(z - \tan^{-1} \frac{A}{B} - \frac{\theta}{2}) + k K_i \frac{z}{K D_{\theta}} \\
        f_r(z) &= \sqrt{A^2 + B^2} \cos(z - \tan^{-1} \frac{A}{B}) + k K_i - k K_d \frac{z}{K D_{\theta}} . \tag{41}
    \end{align*}
\]

Suppose the parameters $K_i$, $K_p$, and $K_d$ are equal to zero. The two functions $f_i(z)$ and $f_r(z)$ will then have a relative phase shift equal to $\pi/2$ but with equal amplitude and period. Therefore, the roots of $f_i(z)$ and $f_r(z)$ interlace. If any of the PID parameters are nonzero, a dynamic (non-periodic) y-intercept is added to each of the functions $f_i(z)$ and $f_r(z)$ and one of the following cases must occur:

Case1: Some roots of $f_i(z)$ do not interlace any roots of $f_r(z)$ (see Figure 6 as an example).
Case2: The roots of $f_i(z)$ and $f_r(z)$ interlace (see Figure 7 as an example).

Case3: Some roots of $f_i(z)$ and $f_r(z)$ interlace pairwise (see Figure 8 as an example).

Since $K_i > 0$, then $f_i(z_0) = k K_i > 0$ (since $z_0 = 0$). Therefore, from (39), $f_r(z)$ can be restated as follows:
\[
    f_r(z) = \frac{k z^2}{\theta^2} [-K_d + a(z) K_i + b(z)] , \tag{42}
\]
where $a(z)$ and $b(z)$ are defined in (38). Since $\frac{k z^2}{\theta^2}$ has a positive value, condition (37) is equivalent to
\[
    (-1)^h f_i(z_h) > 0, \quad h = 1, 2, \ldots \tag{43}
\]
Satisfying (43) implies that between every two consecutive roots of $f_i(z)$, there exists at least one root of $f_r(z)$.
and therefore only Case 2 can occur. This completes the proof.

**Summary:** For any given value of \( K_p \) within its admissible range resulted by Algorithm 3.1, the intersection region of the infinite number of inequalities in (37) determines a set of \((K_i, K_d)\) values under which the interlacing property of Theorem 3.1 for the roots of \( f_i(z) \) in (26) and the roots of \( f_j(z) \) in (27) is satisfied. For each value of \( K_p \), (37) represents a half-plane in the Cartesian coordinate system \((K_i - K_d)\). If the set of inequalities (37) are substituted with the following set of equalities,

\[
(-1)^h K_d = (-1)^h a(z_h) K_i + (-1)^h b(z_h), \quad h = 1, 2, \ldots \tag{44}
\]

for each fixed value of \( K_p \), (44) represents a straight line in the \((K_i - K_d)\) plane. Then, (37) guarantees the interlacing property by defining a set of infinite number of boundaries in the \((K_i - K_d)\) plane. Therefore, Lemma 3.6 provides a theoretical result which is not practically usable. However, in Lemma 3.7, we show that the intersection region in question is actually determined by only a finite set of lines described by (44).

From (38), \( a(z_h) \) is always positive and a decreasing function of \( z_h \). Therefore, we will focus on the variations of \( b(z_h) \) for a given value of \( K_p \).

**Lemma 3.7:** Consider \( f_i(z) \) and \( b(z) \) as defined in (27) and (38). Let \( z_h \) be the \( h \)-th positive root of \( f_i(z) \) and \( m \) be the minimum value of \( h \) such that \( z_m > \sqrt{\beta} \). Then,

- \( 0 > b(z_h) > b(z_{h+2}) \) for odd values of \( h \geq m \).
- \( 0 < b(z_h) < b(z_{h+2}) \) for even values of \( h \geq m \).

**Proof:** The proof is long and hence presented in the Appendix.

We are now able to present the final result of this section which is based on Lemma 3.7 and a finite set of inequalities (44).

**Theorem 3.4:** Consider \( f_i(z) \) and \( f_j(z) \) and the \( y \)-intercept \( b(z) \) as defined in (26), (27), and (38), respectively. Suppose \( K_i^+ \) and \( K_d^+ \) are the lower and upper boundaries for \( K_d \) derived using Lemma 3.2. Let \( z_h \) be the \( h \)-th positive root of \( f_i(z) \) and \( m \) be the minimum value of \( h \) where \( z_m > \sqrt{\beta} \). Let \( e \) be the minimum even value of \( h \) for which we have \( z_e \geq z_m \) and \( b(z_e) \geq K_d^+ \), and \( o \) be the minimum odd value of \( h \) for which \( z_o \geq z_m \). Then, \( f(z) \) in (17) is Hurwitz stable, if and only if the following conditions hold:

1. The value of \( K_p \) lies in the union of the intervals obtained from Algorithm 3.1.
2. The values of \( K_i \) and \( K_d \) lie in the intersection of planes determined by inequalities (37) for \( h = 1, 3, \ldots, o \) and the line \( K_d = K_d^- \) (from the below), the inequalities (37) for \( h = 2, 4, \ldots, (e - 2) \) and the line \( K_d = K_d^+ \) (from the above), and by the vertical line \( K_i = 0 \) from the right-hand side.

**Proof:** The first condition of Theorem 3.4 guarantees that the roots of \( f_i(z) \) in (27) are all simple and real. Accordingly, from Lemma 3.2, outside the interval \((K_i^+, K_d^-)\), there does not exist any stabilising PID controllers for the second-order time-delay system (15). From Lemma 3.7, for even positive roots of \( f_i(z) \) which are greater than \( z_{(e - 2)} \), the \( y \)-intercept \( b(z_h) \) is a strictly increasing function of \( z \). In addition, \( a(z_h) \) in (38) is always positive. Therefore, inequality (37) for \( h = 2, 4, \ldots, (e - 2) \) and the line \( K_d = K_d^- \) compute an upper boundary in the \((K_i - K_d)\) plane which is denoted by \( K_d^+ \). By a similar approach, noting the decreasing value of \( b(z_h) \) for odd values of \( h \) where \( z_h \geq z_o \), and recalling that \( a(z_h) \) is a positive and strictly decreasing function with respect to \( h \), inequality (37) for \( h = 1, 3, \ldots, o \) and the line \( K_d = K_d^- \) give a lower boundary in the \((K_i - K_d)\) plane denoted by \( K_d^- \). Considering the positive values for \( K_i \) establishes a region in the \((K_i - K_d)\) plane with the three boundaries \( K_d^+ \), \( K_d^- \), and \( K_i = 0 \) within which the interlacing property of Theorem 3.1 is satisfied. At the same time, the two conditions of the theorem guarantee the simplicity and realness of the roots of \( f_i(z) \) in (26). Evidently, the first condition of Theorem 3.1 is satisfied if and only if the two conditions of Theorem 3.4 are satisfied. Now, from (26) and (27),

\[
\begin{cases}
  f_i(z = 0) = 0, & f'_i(z = 0) = \frac{1}{\theta}(\beta + K_p) \\
  f_j(z = 0) = kK_i, & f'_j(z = 0) = 0
\end{cases}
\tag{45}
\]

It is already shown that the PID parameter \( K_p \) in its admissible range is always larger than \(-\frac{\beta}{\theta}\) (see Remark 3.3). In addition, since the parameters \( k \) and \( \theta \) in (15) are positive, for \( w = 0 \), we have

\[
f'_j(0) f_j(0) - f'_i(0) f_i(0) = \frac{k}{\theta} K_i (\beta + K_p) > 0. \tag{46}
\]

Consequently, the second item of Theorem 3.1 is also satisfied, and this completes the proof.

We summarise our results in a new algorithm to determine the stabilising gain space of the PID parameters for the second-order time-delay system (15) under the unity feedback structure as shown in Figure 1:

**Algorithm 3.2:**

Step 1: Specify the plant parameters \( \alpha, \beta, \) and \( \theta \).
Step 2: Check the necessary conditions for existence of a stabilising PID controller as defined in Lemma 3.1. If they are satisfied, go to Step 3.
Step 3: Using Algorithm 3.1, find an admissible range for the proportional gain $K_p$. In addition, compute the boundary values $K_d^-$ and $K_d^+$ as the lower and the upper boundaries of PID parameter $K_d$ using Lemma 3.2.

Step 4: Select a fixed value for $K_p$ within its admissible range.

Step 5: Determine $e$ and $o$ as defined in Theorem 3.4.

Step 6: Compute the intersection region of the three boundaries $K_i^{id}$ from below, $K_i^{+d}$ from above, and $K_i = 0$ from the left-hand side in the $(K_i - K_d)$ plane.

Step 7: Go to Step 4.

**Example 3.2:** Consider again the plant $G_1(s)$ in (34) from Example 3.1. This system satisfies the necessary conditions outlined in Lemma 3.1. From Example 3.1, the admissible range for the PID parameter $K_p$ is $(-1.5, 1.4740)$. In addition, from Lemma 3.2, the admissible range for $K_d$ is $(-3.25, 5.7116)$. Taking $K_p = 1$ for illustration purposes, we find $o = 3$ and $e = 8$. The two lines (44) for $h = 1, 3$ and $K_d = K_d^-$ determine $K_i^{-d}$, and the lines (44) for $h = 2, 4, 6$ and $K_d = K_d^+$ determine $K_i^{+d}$. All of the stated lines and the resulting intersection region with the vertical line $K_i = 0$ are plotted in Figure 9. The steps are then repeated for different values of $K_p$. The resulting intersection regions are shown in Figure 10.

**Example 3.3:** Consider the plant (12) in Example 2.2. Transforming the system into the standard form (15) gives

$$G(s) = \frac{2}{s^2 - 3s + 2} e^{-0.2s}$$

(47)

This system satisfies the necessary conditions obtained in Lemma 3.1. Moreover, from Algorithm 3.1, the range for $K_p$ is determined as follows:

$$-1 < K_p < 0.5825,$$

(48)

which, in this instance, is similar to the interval obtained for $K_p$ in Wang (2007). The transfer function $H(s)$ defined in Lemma 3.2 becomes

$$H(s) = \frac{4 e^{-0.2s}}{0.04s^3 + 1.08s^2 + 3.68s - 5.2}$$

(49)

Computing the poles of $H(s)$ shows it has an unstable pole at $s = 1.0662$. The Nyquist plot of $H(s)$ is shown in Figure 11. The interval between $P_1$ and $P_2$ refers to where the Nyquist diagram encircles the point $(-1, 0)$ once anticlockwise. Accordingly, from the Nyquist theorem and
Lemma 3.2, the admissible range for the PID parameter $K_d$ can be determined as follows:

$$P_2 < -\frac{1}{K_d} < P_1, \Rightarrow 1.3001 < K_d < 4.4209. \quad (50)$$

where $P_1 = -0.2262$ and $P_2 = -0.7692$.

For illustration purposes, we select two different values $-0.95$ and $-0.98$ for $K_p$. This leads to the two stability regions in the $(K_i - K_d)$ plane shown in Figure 12. Both controllers $C_1(s)$ and $C_2(s)$ which were given in (14) are within this region. Accordingly, the proposed algorithm is reliable and consistent with the results obtained in Example 2.2.

4. Conclusion

This paper studied the problem of determining the stability gain space of PID controllers for general second-order time-delay systems. Based on an extension of the Hermite–Biehler theorem, and the Nyquist criterion, a new simple algorithm for determining the stability gain space has been proposed. The algorithm not only applies to both stable and unstable systems, but can also be utilised for systems with real or complex poles. To investigate the validity of the proposed algorithm, three different types of second-order time-delay systems have been studied.

Disclosure statement

No potential conflict of interest was reported by the authors.

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### Appendix

Consider the following lemma which is necessary to prove Lemma 3.7.

**Lemma A.1**: Let \( A = -\frac{q}{z} \), and \( B = \frac{\theta_0}{kz} - \frac{z}{k} \), where \( \alpha, \beta, \theta, \) and \( k \) are fixed values. Let \( \theta \) and \( z \) are both positive and \( z \) is a variable. For \( z > \theta \sqrt{|B|} \), both \( F_1(z) = \sqrt{A^2 + B^2} \) and \( F_2(z) = zF_1(z) \) are strictly increasing functions of \( z \).

**Proof**: Let \( F_1'(z) \) be the first derivative of \( F_1(z) \) with respect to \( z \). Therefore,

\[
F_1'(z) = \frac{1}{k^2\theta^2 z^3} \left( z^4 - \theta^4 \beta^2 \right) F_1(z), \tag{A1}
\]

From (A1), if \( z > \theta \sqrt{|B|} \), then \( F_1'(z) > 0 \). In addition,

\[
F_2'(z) = F_1(z) + zF_1'(z). \tag{A2}
\]

Since both \( z \) and \( F_1'(z) \) are positive for \( z > \theta \sqrt{|B|} \), then \( F_2'(z) \) is a positive function when \( z > \theta \sqrt{|B|} \) and the proof is complete. \( \blacksquare \)

#### A.1 Proof of Lemma 3.7

From (28), the positive roots of \( f_i(z) \) are the same as the positive roots of \( \tilde{f}_i(z) + kK_p \). It is also clear that the positive roots of \( \tilde{f}_i(z) + kK_p \) are the same as the positive roots of \( \tilde{f}_i(z) = \frac{\alpha}{z^2}(\tilde{f}_i(z) + kK_p) \). From (29) and (31),

\[
\tilde{f}_i(z) = \sqrt{A^2 + B^2} \cos \left( z - \tan^{-1} \frac{A}{B} \right) + \frac{K_p\theta}{z}, \tag{A3}
\]

and from (31) and (38),

\[
b(z) = \sqrt{A^2 + B^2} \cos \left( z - \tan^{-1} \frac{A}{B} + \frac{\pi}{2} \right)
= -\sqrt{A^2 + B^2} \sin \left( z - \tan^{-1} \frac{A}{B} \right) \tag{A4}
\]

where \( A = -\frac{q}{z} \), and \( B(z) = B = \frac{\theta_0}{kz} - \frac{z}{k} \). Let \( z_h \) be the \( h \)-th positive root of \( \tilde{f}_i(z) \). Suppose \( m \) is the least value of \( h \) for which \( z_m > \theta \sqrt{|B|} \). From Lemma A.1,

\[
\left| \sqrt{A^2 + B^2(z_h)} \right| < \left| \sqrt{A^2 + B^2(z_{h+1})} \right|, \tag{A5}
\]

and,

\[
\left| z_h \sqrt{A^2 + B^2(z_h)} \right| < \left| z_{h+1} \sqrt{A^2 + B^2(z_{h+1})} \right|, \tag{A6}
\]

Therefore,

\[
|F(z_h)| > |F(z_{h+1})|, \quad h = m, m + 1, \ldots. \tag{A7}
\]
where \( F(z) = \frac{K_0}{z\sqrt{A^2 + B^2}} \). From (A3),

\[
\bar{f}_k(z_m) = 0 \iff \cos \left( z_h - \tan^{-1} \frac{A}{B(z_h)} \right) = -F(z_h).
\]

(A8)

Then, from (A7) and (A8), for \( h = m, m + 1, \ldots \),

\[
|\cos \left( z_h - \tan^{-1} \frac{A}{B(z_h)} \right)| > |\cos \left( z_{h+1} - \tan^{-1} \frac{A}{B(z_{h+1})} \right)|.
\]

(A9)

It is known that for any \( h = 1, 2, \ldots \),

\[
\sin^2 \left( z_h - \tan^{-1} \frac{A}{B(z_h)} \right) + \cos^2 \left( z_h - \tan^{-1} \frac{A}{B(z_h)} \right) = 1.
\]

(A10)

Hence, from (A9) and (A10) for \( h = m, m + 1, \ldots \)

\[
|\sin \left( z_h - \tan^{-1} \frac{A}{B(z_h)} \right)| < |\sin \left( z_{h+1} - \tan^{-1} \frac{A}{B(z_{h+1})} \right)|.
\]

(A11)

Finally, from (A4), (A5), and (A11),

\[
|b(z_h)| < |b(z_{h+1})|, \quad h = m, m + 1, \ldots
\]

(A12)

Consider now the sign of \( b(z_h) \) for \( h \geq m \). We will show that if \( z_h > \theta \sqrt{|B|} \), then \( b(z_h) \) will be negative and positive for odd and even values of \( h \), respectively. First, it is clear that,

\[
-1 \leq \cos \left( z_h - \tan^{-1} \frac{A}{B(z_h)} \right) \leq 1, \quad \forall h.
\]

(A13)

Then, from (A8),

\[
-1 \leq F(z_h) \leq 1, \quad \forall h.
\]

(A14)

As a result of (A14), we have \(-1 \leq F(z_m) \leq 1\). Therefore, considering (A7) makes it clear that,

\[
|F(z_h)| < 1, \quad \forall h > m.
\]

(A15)

Now, consider the two functions,

\[
H_1(z) = \cos \left( z - \tan^{-1} \frac{A}{B} \right) + F(z)
\]

\[
H_2(z) = \cos \left( z - \tan^{-1} \frac{A}{B} + \frac{\pi}{2} \right),
\]

(A16)

where \( A \) and \( B \) were defined first in (A3) and (A4). From (A8), the roots of \( \bar{f}_k(z) \) are the same as the roots of \( H_1(z) \). From (A4), \( b(z) \) has the same sign as \( H_2(z) \). Let \( K_p = 0 \), hence, \( F(z) = 0 \). In this case, regarding the phase shift between \( H_1(z) \) and \( H_2(z) \) for odd and even roots of \( H_1(z) \), the sign of \( H_2(z) \) is negative and positive, respectively. If \( K_p \neq 0 \), then \( F(z) \) is added to \( H_1(z) \). In this case, if the magnitude of \( F(z) \) is less than one, then the sign of \( H_2(z) \) for odd and even roots of \( H_1(z) \) will not vary with respect to the previous case. Therefore, from (A15), \( b(z_h) \) retains its negative sign for odd and positive sign for even values of \( h > m \). Now, from (A12), for \( h > m \), the magnitude of \( b(z_h) \) is an increasing function of \( z_h \). Therefore, for odd values of \( h > m \), \( b(z_h) \) is decreasing and for even values of \( h > m \), \( b(z_h) \) is increasing. This completes the proof.