The ground-state energies and radii for $^4$He, $^{16}$O, and $^{40}$Ca are calculated with the unitary-model-operator approach (UMOA). In the present study, we employ the similarity renormalization group (SRG) evolved nucleon-nucleon ($NN$) and three-nucleon ($3N$) interactions based on the chiral effective field theory. This is the first UMOA calculation with both $NN$ and $3N$ interactions. The calculated ground-state energies and radii are consistent with the recent $ab$ initio results with the same interaction. We evaluate the expectation values with two- and three-body SRG evolved radius operators, in addition to those with the bare radius operator. With the aid of the higher-body evolution of radius operator, it is seen that the calculated radii tend to be SRG resolution-scale independent. We find that the SRG evolution gives minor modifications for the radius operator.

I. INTRODUCTION

Recent nuclear $ab$ initio studies are encouraged by the development, in particular, of the nuclear interactions from the chiral effective field theory ($\chi$EFT) [1, 2]. In the $\chi$EFT, nuclear interactions are obtained through the perturbation expansion of the chiral Lagrangian which is the effective Lagrangian of the quantum chromodynamics. By taking into account the higher-order expansion terms, the systematic improvement of the nuclear interactions can be expected, for recent example, see Refs. [3–5]. As another advantage of employing the $\chi$EFT, the consideration of higher-order terms in the perturbation series allows us the systematic derivation of the three-nucleon ($3N$) interaction. With the development of the $\chi$EFT interactions, the impacts of the $3N$ force on nuclear structure calculation have been discussed extensively, for example, in light nuclei [6–9], medium-mass nuclei [10–18], and infinite nuclear matter [19–22].

Besides the progress in nuclear forces, the advancement in many-body methods is also necessary. To deal with nuclear many-body problems, one can use the $ab$ initio calculation methods such as no-core shell model (NCSM) [23], quantum Monte Carlo methods [24], nuclear lattice EFT calculations [25], coupled-cluster method [26], self-consistent Green’s function method [27], in-medium similarity renormalization group approach [28], and many-body perturbation theory [29, 30]. Over the past decade, the tremendous advancements were made in nuclear $ab$ initio studies. Nowadays, the capability of the $ab$ initio calculations has reached mass region $A \sim 100$ [18, 29, 31]. In addition to these methods, we can apply the unitary-model-operator approach (UMOA) [32, 33] to solve the nuclear many-body Schrödinger equation. In the UMOA, a unitary transformation of the Hamiltonian is constructed so that the one-particle-one-hole and two-particle-two-hole excitations are limited from the transformed Hamiltonian. So far, we calculated the ground-state energies and radii for some closed shell nuclei with only the nucleon-nucleon ($NN$) interactions. In this work, we include the $3N$ interaction effect to the UMOA calculation for the first time.

Due to the non-perturbative nature of the nuclear force, in most cases, it is not possible to apply directly the nuclear interactions to the many-body calculations. To bridge the gap between nuclear forces and many-body calculations, we evolve the nuclear Hamiltonian with the similarity renormalization group (SRG) flow equation [34]. Through the SRG evolution, we obtain the Hamiltonian whose coupling between low- and high-momentum regions is suppressed. With such nuclear interactions, recent $ab$ initio results significantly underestimate the nuclear radii, see for instance Refs. [16, 17, 28, 35]. Since the nuclear size can affect the single-particle level structure of a nucleus, the reproduction of nuclear radii is one of the fundamental issues to discuss the nuclear structure. As seen in NCSM calculations for few-body systems [36, 37], we should also evolve consistently other operators than the Hamiltonian. In this work, we demonstrate the effect of the SRG evolution on radius operator.

This paper is organized as follows. The Hamiltonian and radius operators employed in this work are introduced in Sec. II. Section III describes briefly the formalism of the UMOA. In Sec. IV, the numerical results for $^4$He, $^{16}$O, and $^{40}$Ca are given. After confirming convergence and consistency with the other $ab$ initio results,
the effects of the SRG evolution on radius operator are discussed. The summary of the present work is given in Sec. V.

II. HAMILTONIAN AND RADIUS OPERATORS

Our starting Hamiltonian is composed of the kinetic, $NN$, and $3N$ terms:

$$H = T + V^{NN} + V^{3N}. \quad (1)$$

Here, $T$ is the kinetic energy operator. The $V^{NN}$ and $V^{3N}$ indicate the $NN$ and $3N$ interactions, respectively. Usually, the bare Hamiltonian $H$ is too "hard" to apply for many-body calculation methods. It causes the slow convergence with respect to the size of model space and calculations demand the huge amount of computational resources. To obtain the converged results from the feasible model-space calculations, the similarity-renormalization group (SRG) evolution [34] is employed in this work. We consider the unitary transformation of the original Hamiltonian:

$$H(\alpha) = U^\dagger_{\text{SRG}}(\alpha) H U_{\text{SRG}}(\alpha). \quad (2)$$

Here, $U_{\text{SRG}}(\alpha)$ is the unitary transformation operator and is evolved by the flow equation:

$$\frac{dU_{\text{SRG}}(\alpha)}{d\alpha} = U_{\text{SRG}}(\alpha) \eta(\alpha). \quad (3)$$

The $\alpha$ is the resolution scale parameter of the flow equation in unit of fm$^4$. The $\eta$ is called the generator of the SRG evolution and is taken as $\eta(\alpha) = [T, H(\alpha)]$. Note that the initial condition for $U_{\text{SRG}}(\alpha)$ is $U_{\text{SRG}}(0) = 1$. Alternative to $\alpha$, it is common to use $\lambda_{\text{SRG}} = \alpha^{-1/4}$ for controlling the flow equation Eq. (3). The Hamiltonian is transformed by Eq. (2) from $\lambda_{\text{SRG}} = \infty$ fm$^{-1}$ to lower values where the interaction is "soft" enough for convergence of the many-body calculation methods. As discussed, for example in Ref. [38], the SRG evolution, however, induces the many-body forces:

$$H(\lambda_{\text{SRG}}) = T + V^{NN}(\lambda_{\text{SRG}}) + V^{3N}(\lambda_{\text{SRG}}) + \cdots. \quad (4)$$

Consequently, during the SRG evolution, we should keep many-body terms, as many as possible, even if the starting Hamiltonian does not include the many-body interactions. In this work, three types of Hamiltonians are used. First one, labeled by "NN-only", is obtained by keeping only the $NN$ interaction during the SRG evolution starting without the genuine $3N$ interaction. Second one "$NN+3N$-ind" is obtained by keeping the $NN$ and $3N$ interactions during the SRG evolution starting without the genuine $3N$ interaction. Third one "$NN+3N$-full" is obtained by keeping the $NN$ and $3N$ interactions during the SRG evolution starting with the genuine $3N$ interaction.

To evaluate nuclear root-mean-squared radii, we should transform the radius operator in the same manner as the Hamiltonian:

$$r^2(\alpha) = U^\dagger_{\text{SRG}}(\alpha) r^2 U_{\text{SRG}}(\alpha). \quad (5)$$

The original radius operator is defined as

$$r^2 = r^{2(2)} = \frac{1}{A^2} \sum_{i<j} (r_i - r_j)^2 \quad (6)$$

with the coordinate vector of the $i$-th nucleon $r_i$ and number of nucleon $A$. In the same manner as the Hamiltonian, the many-body radius operator can be induced through the SRG evolution:

$$r^2(\lambda_{\text{SRG}}) = r^{2(2)}(\lambda_{\text{SRG}}) + r^{2(3)}(\lambda_{\text{SRG}}) + \cdots. \quad (7)$$

Following to Refs. [36, 37], we keep up to three-body terms.

To perform many-body calculations, it is numerically efficient to transform to the laboratory frame. Then, our Hamiltonian and radius operators can be rewritten as

$$H = T^{(1)} + [T^{(2)} + V^{NN}(\lambda_{\text{SRG}})] + V^{3N}(\lambda_{\text{SRG}}), \quad (8)$$

$$r^2 = r^{2(1)} + [r^{2(2)}(\lambda_{\text{SRG}}) - r^{2(1)}] + r^{2(3)}(\lambda_{\text{SRG}}). \quad (9)$$

Here, we use $T^{(1)} = \sum_i (1 - 1/A) p_i^2 / 2m$ with the $i$-th nucleon momentum $p_i$ and nucleon mass $m$, $T^{(2)} = - \sum_{i<j} p_i \cdot p_j / m A$, and $r^{2(1)} = (1 - 1/A) \sum_i r_i^2$. Note that $r^{2(1)}$ is chosen so that $r^{2(2)}(\lambda_{\text{SRG}}) - r^{2(1)}$ goes to $- \sum_{i<j} r_i \cdot r_j / A$ in the limit of $\lambda_{\text{SRG}} \rightarrow \infty$. In the second quantization form, they are

$$H = \sum_{a_1 a_2} t_{a_1 a_2} c_{a_2}^\dagger c_{a_2}^0 + \left( \frac{1}{2l^2} \right)^2 \sum_{a_1 a_2 a_3 a_4} t_{a_1 a_2 a_3 a_4}^2 c_{a_1}^\dagger c_{a_2}^\dagger c_{a_4} c_{a_3} + \left( \frac{1}{3l^2} \right)^2 \sum_{a_1 a_2 a_3 a_4 a_5 a_6} t_{a_1 a_2 a_3 a_4 a_5 a_6}^3 c_{a_1}^\dagger c_{a_2}^\dagger c_{a_3}^\dagger c_{a_4} c_{a_5} c_{a_6}. \quad (10)$$

$$r^2 = \sum_{a_1 a_2} r^{2(1)}_{a_1 a_2} c_{a_2}^\dagger c_{a_2}^0 + \left( \frac{1}{2l^2} \right)^2 \sum_{a_1 a_2 a_3 a_4} r^{2(2)}_{a_1 a_2 a_3 a_4} c_{a_1}^\dagger c_{a_2}^\dagger c_{a_4} c_{a_3} + \left( \frac{1}{3l^2} \right)^2 \sum_{a_1 a_2 a_3 a_4 a_5 a_6} r^{2(3)}_{a_1 a_2 a_3 a_4 a_5 a_6} c_{a_1}^\dagger c_{a_2}^\dagger c_{a_3}^\dagger c_{a_4} c_{a_5} c_{a_6}. \quad (11)$$

Here, $c_a$ ($c_a^\dagger$) is the annihilation (creation) operator of the nucleon at the state $a$. In Eqs. (10) and (11), shorthand
notations,
\[ t_{a_1a_2} = \langle a_1 | T^{(1)} | a_2 \rangle, \]  
(12)
\[ \eta^{(2)}_{a_1a_2a_3a_4a_5a_6} = \langle a_1a_2a_3 | V^{NN}(\Lambda_{\text{SRG}}) | a_4a_5a_6 \rangle, \]  
(13)
\[ v^{(3)}_{a_1a_2a_3a_4a_5a_6} = \langle a_1a_2a_3 | V^{NN}(\Lambda_{\text{SRG}}) | a_4a_5a_6 \rangle, \]  
(14)
\[ r^{(1)}_{a_1a_2} = \langle a_1 | \ell^{(1)} | a_2 \rangle, \]  
(15)
\[ r^{(2)}_{a_1a_2a_3a_4a_5a_6} = \langle a_1a_2a_3 | \ell^{(2)}(\Lambda_{\text{SRG}}) - \frac{1}{A-1} \ell^{(1)} | a_4a_5a_6 \rangle, \]  
(16)
\[ r^{(3)}_{a_1a_2a_3a_4a_5a_6} = \langle a_1a_2a_3 | \ell^{(3)}(\Lambda_{\text{SRG}}) | a_4a_5a_6 \rangle, \]  
(17)
are used for one-body-kinetic-term, two-body-interaction, three-body-interaction, one-body-radius, two-body-radius, and three-body-radius matrix elements, respectively. The factor \(1/(A-1)\) in two-body-radius matrix element is due to the normalization when a one-body operator is used as a two-body operator. Because of the computational limitation, however, the direct treatment of the three-body matrix elements is still challenging. Therefore, we follow the recent nuclear \textit{ab initio} studies and introduce the normal-ordered two-body (NO2B) approximation [39, 40]. The key of the approximation is a rearrangement of the three-body term with respect to a reference state \(|\Phi\rangle\). After the rearrangement, the zero-, one-, two-, and three-body pieces show up. In the NO2B approximation, the residual three-body piece is discarded. To apply to the UMOA framework, we take normal order again with respect to the nucleon vacuum state. Then, the Hamiltonian is
\[ H \approx \hat{H}^{(0),\text{NO2B}} + \sum_{a_1a_2} h^{(1),\text{NO2B}}_{a_1a_2} c_{a_1}^\dagger c_{a_2} \]  
\[ + \left( \frac{1}{2!} \right)^2 \sum_{a_1a_2a_3a_4} h^{(2),\text{NO2B}}_{a_1a_2a_3a_4} c_{a_1}^\dagger c_{a_2}^\dagger c_{a_3} c_{a_4} \]  
(18)
with
\[ h^{(0),\text{NO2B}} = \frac{1}{6} \sum_{abc} \eta^{(3)}_{abc} n_{a} n_{b} n_{c}, \]  
(19)
\[ h^{(1),\text{NO2B}}_{a_1a_2} = t_{a_1a_2} - \frac{1}{2} \sum_{bc} \eta^{(3)}_{abc} n_{b} n_{c}, \]  
(20)
\[ h^{(2),\text{NO2B}}_{a_1a_2a_3a_4} = v^{(2)}_{a_1a_2a_3a_4} + \sum_{b} v^{(3)}_{a_1a_2a_3a_4b} n_{b}. \]  
(21)
Here, \(n_{a} = 1\) (\(n_{a} = 0\)) where \(a\) is below (above) the Fermi level. To minimize the effect of the truncated residual three-body piece, the choice of \(|\Phi\rangle\) is crucial. In this work, we use the Hartree-Fock state as \(|\Phi\rangle\). Same as the Hamiltonian, we employ the NO2B approximated radius operator:
\[ r^{2} \approx r^{2(0),\text{NO2B}} + \sum_{a_1a_2} r^{2(1),\text{NO2B}}_{a_1a_2} c_{a_1}^\dagger c_{a_2} \]  
\[ + \left( \frac{1}{2!} \right)^2 \sum_{a_1a_2a_3a_4} r^{2(2),\text{NO2B}}_{a_1a_2a_3a_4} c_{a_1}^\dagger c_{a_2}^\dagger c_{a_3} c_{a_4}. \]  
(22)
with
\[ r^{2(0),\text{NO2B}} = \frac{1}{6} \sum_{abc} r^{2(3)}_{abc} n_{a} n_{b} n_{c}, \]  
(23)
\[ r^{2(1),\text{NO2B}}_{a_1a_2} = \frac{2}{a_1a_2} - \frac{1}{2} \sum_{bc} r^{2(3)}_{a_1bca2bc} n_{b} n_{c}, \]  
(24)
\[ r^{2(2),\text{NO2B}}_{a_1a_2a_3a_4} = r^{2(2)}_{a_1a_2a_3a_4} + \sum_{b} r^{2(3)}_{a_1a_2a_3a_4b} n_{b}. \]  
(25)

III. UNITARY-MODEL-OPERATOR APPROACH

To solve the many-body Schrödinger equation associated with the Hamiltonian Eq. (18), the UMOA [32, 33, 42, 43] is employed in this work. In the UMOA, we construct the effective Hamiltonian with the unitary transformation:
\[ \hat{H} = U^\dagger H U. \]  
(26)
The \(U\) is defined by the product of two exponential operators,
\[ U = e^{S^{(1)}} e^{S^{(2)}}, \]  
(27)
where \(S^{(1)}\) and \(S^{(2)}\) are anti-hermitian one- and two-body correlation operators, respectively. Note that the sole use of \(S^{(1)}\) \((S^{(2)} = 0)\) reduces the UMOA to the Hartree-Fock (HF) theory. The \(S^{(1)}\) and \(S^{(2)}\) are specified by applying iteratively the Okubo-Lee-Suzuki method [44–46] so that \(\hat{H}\) does not induce the one-particle-one-hole and two-particle-two-hole excitations from the reference state \(|\Phi\rangle\). Since the unitary transformation (26) can induce many-body interactions, \(\hat{H}\) can include many-body operators even if the original Hamiltonian is restricted up to the two-body interaction. In actual calculations, we decompose \(\hat{H}\) with the cluster expansion and truncate the effect of the four- and higher-body cluster terms:
\[ \hat{H} \approx \sum_{a_1a_2} \tilde{H}^{(1)}_{a_1a_2} c_{a_1}^\dagger c_{a_2} \]  
\[ + \frac{1}{4} \sum_{a_1a_2a_3a_4} \tilde{H}^{(2)}_{a_1a_2a_3a_4} c_{a_1}^\dagger c_{a_2}^\dagger c_{a_3} c_{a_4} \]  
\[ + \frac{1}{36} \sum_{a_1a_2a_3a_4a_5a_6} \tilde{H}^{(3)}_{a_1a_2a_3a_4a_5a_6} c_{a_1}^\dagger c_{a_2}^\dagger c_{a_3}^\dagger c_{a_4}^\dagger c_{a_5}^\dagger c_{a_6}. \]  
(28)
where \(\tilde{H}^{(1)}_{a_1a_2}, \tilde{H}^{(2)}_{a_1a_2a_3a_4}, \) and \(\tilde{H}^{(3)}_{a_1a_2a_3a_4a_5a_6}\) are the one-, two-, and three-body matrix elements, respectively (see, for example, Ref. [33] for more details). Then, the ground-state energy \(E_{\text{g.s.}}\) can be obtained approximately
TABLE I. Hugenholtz diagrams for the ground-state energy up to the third order. Note that the first order contributions are omitted. The cross and dot indicate the one- and two-body part of Hamiltonian, respectively. The diagram rules are same as in Ref. [41].

| Second order |
| --- |
| S1 |
| S2 |

| Third order |
| --- |
| T1 |
| T2 |
| T3 |
| T4 |
| T5 |
| T6 |
| T7 |
| T8 |
| T9 |
| T10 |
| T11 |
| T12 |
| T13 |
| T14 |

The three-body cluster term contributions compensate the third order of the many-body perturbation theory; $E_{\text{g.s.}} = 1 + \sum_{i=1}^{12} S_i + \sum_{i=1}^{14} T_i - T_{13}$, $E^{1,2\text{BC}}$, $E^{3\text{BC}}$, and $E_{\text{g.s.}}$ are

$$E^{1,2\text{BC}} = E_1 + \sum_{i=1}^{12} S_i + \sum_{i=1}^{14} T_i - T_{13} + \text{(higher order terms)}$$

$$E^{3\text{BC}} = 2T_{13} + T_{14} + \text{(higher order terms)}$$

$$E_{\text{g.s.}} = E_1 + \sum_{i=1}^{12} S_i + \sum_{i=1}^{14} T_i + \text{(higher order terms)}$$

Here, $S_i$ and $T_i$ are the second- and third-order contributions shown in Table I, respectively, and $E_1$ is the first order ground-state energy. At one-plus-two-body cluster level, the third-order diagrams are not completed. The three-body cluster term contributions compensate the third order [49]. Note that $S_1$ and $T_1$ to $T_{11}$ vanishes when the HF basis is employed.

To evaluate the expectation value of the radius operator obtained in Eq. (22), the effective operator $\tilde{r}^2$ is used:

$$\tilde{r}^2 = U^\dagger r^2 U.$$ (36)

Similarly to the Hamiltonian, the unitary transformation of the radius operator induces the many-body terms. However, results examined here are calculated keeping up to two-body terms and does not include any contributions from three- and higher-body terms [33]:

$$\tilde{r}^2 \approx \sum_{a_1 a_2} \tilde{r}_{a_1 a_2}^{2(1)} c_{a_1}^\dagger c_{a_2} + \frac{1}{4} \sum_{a_1 a_2 a_3 a_4} \tilde{r}_{a_1 a_2 a_3 a_4}^{2(2)} c_{a_1}^\dagger c_{a_2}^\dagger c_{a_3} c_{a_4}.$$ (37)

Then, the mean-squared radius $r_{\text{g.s.}}^2$ is approximately evaluated as

$$r_{\text{g.s.}}^2 \approx \sum_a r_{a a}^{2(1)} n_a + \frac{1}{2} \sum_{a b} r_{a b a b}^{2(2)} n_a n_b.$$ (38)

By

$$E_{\text{g.s.}} \approx E^{1,2\text{BC}} + E^{3\text{BC}},$$

$$E^{1,2\text{BC}} = \sum_a \tilde{H}_{a a}^{(1)} n_a + \frac{1}{2} \sum_{ab} \tilde{H}_{a b a b}^{(2)} n_a n_b,$$

$$E^{3\text{BC}} = \frac{1}{6} \sum_{abc} \tilde{H}_{a b a b c c}^{(3)} n_a n_b n_c.$$ (31)
IV. RESULTS AND DISCUSSIONS

In this work, we use the next-to-next-to-next-to-leading order (N^3LO) NN interaction by Entem and Machleidt [51] and local form N^2LO 3N interaction [52] from χEFT. Both two- and three-body SRG evolutions are done in the harmonic-oscillator (HO) space. The two-body interactions are obtained from the N_{max} = 200 space calculations. Here, N_{max} is the boundary of the HO quantum number for the two-body relative coordinate and is N_{max} = max(2n+l) with the radial quantum number n and angular momentum l. Following Ref. [53], the three-body SRG evolution is done in ramp A model space defined in Fig. 3 in Ref [53]. To obtain the three-body matrix element, the frequency conversion technique [53] is used with the parent HO energy \hbar\omega = 35 MeV matrix elements. For N^2LO 3N interaction, we use c_D = -0.2, c_E = 0.098, and \Lambda_3N = 400 MeV/c [38], so as to compare with the other ab initio calculation results. Note that the low-energy constant c_D used here does not fit the ^3H half-life as claimed in the past [54, 55]. The impact of the modification of the 3N force with the c_D that fits ^3H half-life will be discussed in the forthcoming publications. The size of the contributions from induced many-body forces can be estimated from the SRG resolution scale, \lambda_{SRG}, dependence of calculated results. To do so, we employ three SRG resolution scales \lambda_{SRG} = 1.88, 2.0, and 2.24 fm⁻¹. The NO2B approximated Hamiltonian is obtained through the HF calculations at \epsilon_{3max} = 14. Here, \epsilon_{3max} is introduced to handle the three-body matrix element and is \epsilon_{3max} = max(2n_i+l_i+2n_2+l_2+2n_3+l_3) with the single-particle radial quantum number n_i (i = 1, 2, 3) and angular momentum l_i (i = 1, 2, 3). We checked that the changing from \epsilon_{3max} = 12 to \epsilon_{3max} = 14 affects less than 1% of total ground-state energies for nuclei calculated in the present work. UMOA calculations are done in the model space defined by \epsilon_{max} = max(2n_i+l_i) [33].

A. Ground-state energies

Figure 1 shows the convergence property of the ground-state energies for ^4He, ^16O, and ^40Ca calculated with the NN + 3N-full interaction from \chiEFT evolved up to \lambda_{SRG} = 1.88 fm⁻¹. Our calculations are done with varying \hbar\omega and \epsilon_{max} to see the numerical convergence. Note that the final results should not depend on \hbar\omega because the initial Hamiltonian Eq. (1) does not include \hbar\omega. Similar to other ab initio calculations, our ground-state energies show parabolic \hbar\omega-dependence at small \epsilon_{max} and gain with increasing \epsilon_{max}. For all cases examined here, \hbar\omega- and \epsilon_{max}-independent results are obtained \epsilon_{max} = 14. The results with \epsilon_{max} = 14 and \hbar\omega = 25 MeV are used in the following discussion.

To investigate the contributions of the cluster expansion, in Fig. 2, it is illustrated that the comparison among UMOA, Hartree-Fock basis many-body perturbation theory (HF-MBPT), and coupled-cluster method (CCM) energies. In terms of HF-MBPT, the energies E^{1,2BC}_{MBPT}, E^{3BC}_{MBPT}, and E_{g.s.,MBPT} are evaluated as

\begin{align}
E^{1,2BC}_{MBPT} &= E_{HF} + S_2 + T_{12} - T_{13}, \\
E^{3BC}_{MBPT} &= 2T_{13} + T_{14}, \\
E_{g.s.,MBPT} &= E_{HF} + S_2 + T_{12} + T_{13} + T_{14},
\end{align}

with the Hartree-Fock energy E_{HF}. Note that E_{g.s.,MBPT} is the third-order HF-MBPT energy. In the figure, the UMOA and HF-MBPT energies are reasonably close to each other and it can be seen that the main contributions of E^{3BC} are from the third order hole-hole (T_{13}) and particle-hole (T_{14}) ladder diagrams. Also, it is shown that the sum of the higher order terms taken into account in the UMOA is repulsive. Comparing to CCM energies, total UMOA energies (circle) look closer to the CCSD energies (down triangle) than to the CR-CC(2,3) energies (pentagon). The E^{3BC} are −0.71, −3.04, and −7.07 MeV for ^4He, ^16O, and ^40Ca, respectively, and are only a few percent of the total energies. Since the contributions from four- and higher-body cluster terms are expected to be smaller than those from the three-body cluster term, the UMOA results are converged with respect to the cluster expansion. For ^16O, our ground-state energy −127.16 MeV is slightly underbound compared to the experimental energy (−127.62 MeV), while the recent ab initio calculation results show mildly underbound to the experiment, for example, −130.6(1) MeV from in-medium SRG.
TABLE II. Ground-state energies for $^4$He, $^{16}$O, and $^{40}$Ca. All the calculation results are obtained at $\epsilon_{\text{max}} = 14$ and $\hbar \omega = 25$ MeV.

| Nuclide | $\lambda_{\text{SRG}}$ (fm$^{-1}$) | $NN$–only | $NN + 3N$–ind | $NN + 3N$–full | Exp.[50] |
|---------|-----------------|------------|----------------|----------------|---------|
| $^4$He  | 1.88            | −27.94     | −25.19         | −27.81         | −28.30  |
|         | 2.0             | −27.73     | −25.18         | −27.67         |         |
|         | 2.24            | −27.23     | −25.16         | −27.62         |         |
| $^{16}$O| 1.88            | −167.79    | −119.33        | −127.16        |         |
|         | 2.0             | −162.69    | −119.51        | −126.33        | −127.62 |
|         | 2.24            | −152.88    | −119.56        | −124.50        |         |
| $^{40}$Ca| 1.88            | −615.62    | −349.08        | −368.44        |         |
|         | 2.0             | −588.45    | −352.03        | −366.14        | −342.05 |
|         | 2.24            | −536.26    | −355.61        | −360.23        |         |

FIG. 2. (color online) Ground-state energies from the Hartree-Fock basis many-body perturbation theory (HF-MBPT), UMOA, and coupled-cluster method (CCM). The energies of HF-MBPT are calculated with Eqs. (39)-(41). The CCM results are taken from Refs. [18, 56]. The interaction is obtained by SRG evolution of chiral N$^3$LO $NN$ [51] and N$^2$LO $NN$ [6, 39] interactions up to $\lambda_{\text{SRG}} = 1.88$ fm$^{-1}$. For the HF-MBPT and UMOA energies, $E^{\text{CCSD}}$ (square), $E^{\text{ABC}}$ (star), and $E_{\text{gs.}}$ (circle) are shown. For the CCM energies, CCSD (down triangle), triple correction (up triangle), and CR-CC(2,3) energies (pentagon) are shown.

FIG. 3. (color online) Ground-state energies for $^4$He, $^{16}$O, and $^{40}$Ca. All the calculation results are obtained at $\epsilon_{\text{max}} = 14$ and $\hbar \omega = 25$ MeV. The experimental data are taken from Ref. [50].

The explicit treatment of the three-body cluster term seems to be necessary to discuss more precisely the accuracy of the UMOA calculations.

As for calculations with $NN$–only and $NN + 3N$–ind interactions, we observe the similar convergence pattern and find the converged results at $\epsilon_{\text{max}} = 14$ calculations. In Figure 3, the calculated ground-state energies are summarized together with the comparisons to the experimental data. In case of $NN$–only interaction results, as the mass number increases, the ground-state energies show overbinding and $\lambda_{\text{SRG}}$-dependence becomes considerable. By taking the SRG induced 3N interaction into account, the $\lambda_{\text{SRG}}$-dependence is drastically reduced and ground-state energies rise. This $\lambda_{\text{SRG}}$-independence of ground-state energies implies that the contributions from SRG induced four- and many-body interactions are negligible. With the genuine χEFT N$^3$LO 3N interaction, the calculated ground-state energies are comparable to the experimental data for $^4$He and $^{16}$O, while overbinding is seen for $^{40}$Ca. The current choice of the genuine 3N interaction gives 9%, 6%, and 4% attractions for $^4$He, $^{16}$O, and $^{40}$Ca, respectively. The energies presented in Fig. 3 are also displayed in Table II. Our ground-state energies show reasonable agreement with the other ab initio results from the same interaction [14, 16, 29, 39, 40, 57].

Again, this disagreement between our and other ab initio results is same order of the size of the perturbative three-body-cluster contribution and consistent with the accuracy of the UMOA calculations.
FIG. 4. (color online) Expectation values of bare root-mean-squared radius operator for $^4$He, $^{16}$O, and $^{40}$Ca as functions of $\hbar\omega$. Here, $NN + 3N$–full interaction at $\lambda_{\text{SRG}} = 1.88$ fm$^{-1}$ is employed.

B. Root-mean square radii

In the same as the ground-state energy calculations, we calculate the expectation values of the bare root-mean-squared radius operator with the chiral $NN + 3N$–full interaction at $\lambda_{\text{SRG}} = 1.88$ fm$^{-1}$ varying both of $\hbar\omega$ and $\epsilon_{\text{max}}$ to examine the convergence. The results for $^4$He, $^{16}$O, and $^{40}$Ca are illustrated in Fig. 4. As demonstrated in the figure, calculated radii become $\hbar\omega$- and $\epsilon_{\text{max}}$-independent with increasing $\epsilon_{\text{max}}$. At $\hbar\omega = 25$ MeV, we find the converged radii within 0.01 fm for all nuclei calculated here. Note that our converged radius of 2.84 fm for $^{40}$Ca from the interaction evolved up to $\lambda_{\text{SRG}} = 2.0$ fm$^{-1}$ shows reasonable agreement with the SCGF result of 2.89 fm [16] from the same interaction.

We also calculate radii for $^4$He, $^{16}$O, and $^{40}$Ca with the $NN$–only and $NN + 3N$–ind interactions in the same manner as with the $NN + 3N$–full interaction. Then, we find converged results at $\epsilon_{\text{max}} = 14$ and $\hbar\omega = 25$ MeV within 0.01 fm. The results are summarized in Fig. 5 with the comparison to the experimental charge radii [58]. To compare with the experimental charge radii, our charge radii $r_{\text{ch}}$ are evaluated as [59],

$$r_{\text{ch}}^2 = r_{\text{g.s.}}^2 + r_p^2 + \frac{N}{Z} r_n^2 + \frac{3}{4m^2},$$

(42)

Here, we use $r_p = 0.8751(61)$ fm [60], $r_n^2 = -0.1161(22)$ fm$^2$ [60], and $3/4m^2 = 0.033$ fm$^2$, with the averaged nucleon mass $m = 938.919$ MeV/$c^2$. Note that we assume the equivalence of point-proton and point-nucleon distributions in Eq. (42). This assumption would be reasonable because our targets are $N = Z$ stable nuclei. In Fig. 6, the charge radii from $NN$–only interactions are obviously smaller than experimental data, especially for $^{16}$O and $^{40}$Ca and consistent with overbinding ground-state energies from those. The SRG induced three-body operator contributes to spread the nuclear distribution out. Then, the $\lambda_{\text{SRG}}$-dependence is slightly enhanced. With this particular Hamiltonian, the genuine 3$N$ interaction shrinks nuclei and the calculated radii are clearly smaller than the experimental data.

As a possible reason for the calculated small radii, for example, the nuclear interaction can be considered. In fact, the simultaneous reproduction of ground-state energies and radii were accomplished with the chiral EFT N$^3$LO $NN + 3N$ interaction fitted by using some selected data of nuclei up to $A = 25$ [61]. In addition, the saturation property of infinite nuclear matter was reproduced with the combinations of the softened N$^3$LO $NN$ and bare N$^3$LO $3N$ interactions whose low-energy constants are fitted to reproduce data of the few body systems [19]. The simultaneous reproduction of ground-state energies and radii for finite nuclei with such interactions were compensated by the recent ab initio calculations [62]. As another possibility, we can consider amending the treatment of radius operator. In earlier no-core shell model (NCSM) studies [36, 37], the effect of the SRG evolution to several operators were investigated for few-body systems. However, such effects for medium-mass nuclei have not been clarified yet. In this work, we investigate the effect of the SRG evolution to the radius operator.

We calculate the expectation value with the bare, two-
are the results with the NN by solid symbols. The triangles, squares, and circles indicate by thick crosses [50, 58]. The results with the open symbols are from other calculations. In the figure, the ground-state energies per nucleon and charge radii for $^4$He, $^{16}$O, and $^{40}$Ca calculated with the bare, two-body evolved (2B), and three-body evolved (3B) radius operators. The results from both $NN$–3N–ind and $NN$+3N–full are displayed. All the calculation results are obtained at $\varepsilon_{\text{max}} = 14$ and $\hbar \omega = 25$ MeV.

| Nuclide | $\lambda_{\text{SRG}}$ (fm$^{-1}$) | $r_{\text{g.s.}}$ (fm) |
|---------|----------------------|-----------------|
|         |                      | $NN$+3N–ind  | $NN$+3N–full |
|         | Bare  | 2B  | 3B  | Bare  | 2B  | 3B  |
| $^4$He  | 1.88  | 1.48 | 1.47 | 1.46 | 1.42 | 1.40 | 1.39 |
|         | 2.0   | 1.47 | 1.46 | 1.45 | 1.41 | 1.40 | 1.39 |
|         | 2.24  | 1.46 | 1.45 | 1.45 | 1.39 | 1.39 | 1.38 |
| $^{16}$O| 1.88  | 2.32 | 2.29 | 2.26 | 2.28 | 2.24 | 2.22 |
|         | 2.0   | 2.30 | 2.27 | 2.25 | 2.27 | 2.23 | 2.21 |
|         | 2.24  | 2.28 | 2.25 | 2.24 | 2.25 | 2.22 | 2.20 |
| $^{40}$Ca| 1.88  | 2.91 | 2.89 | 2.85 | 2.86 | 2.83 | 2.79 |
|         | 2.0   | 2.89 | 2.86 | 2.83 | 2.84 | 2.82 | 2.79 |
|         | 2.24  | 2.84 | 2.82 | 2.80 | 2.83 | 2.80 | 2.78 |

body evolved, and three-body evolved mean-squared radius operators. In the same as the bare radius operator cases, we check the convergence pattern and find the root-mean-squared radius results converged within 0.01 fm. Evaluated charge radii are illustrated in Fig. 6. Corresponding to Fig. 6, final results for root-mean-squared radii from $NN$+3N–ind and $NN$+3N–full interactions are exhibited in Table III. For all nuclei, as we calculate with higher-body evolved operator, the radii tend to be small and go opposite direction to the data. This behavior is consistent with the earlier NCSM results [36]. Also, similar to the role of the SRG induced three-body interaction, consistently evolved operator moderately reduces the $\lambda_{\text{SRG}}$-dependence of radii. Therefore, it can be concluded that the consistent SRG evolution of the radius operator does not give the significant change compared to the experimental data. This is consistent with the long-range nature of the radius operator [36]. There are some insights about the effect of consistent SRG evolution of radius operator [63]. In this work, however, we do not observe the enhancement of radii discussed in Ref. [63]. The quantitative reproduction of nuclear size is still an open question.

Finally, let us see the saturation plot of the ground-state energies and radii from this work and from other calculations. In Figure 7, the ground-state energies per nucleon and charge radii for $^4$He, $^{16}$O, and $^{40}$Ca with various nuclear interactions are plotted. In the figure, the results obtained in the present work are shown by solid symbols. The triangles, squares, and circles are the results with the $NN$–only, $NN$+3N–ind, and $NN$+3N–full interactions, respectively. For visibility, only the results at $\lambda_{\text{SRG}} = 2.0$ fm$^{-1}$ are marked in the figure. The open symbols are from other calculations [61, 62, 64, 65]. The experimental data are indicated by thick crosses [50, 58]. The results with the $NN$ interactions, $NN$–only (triangle), $NN$+3N–ind (square), CD-Bonn (diamond), and AV18 (left triangle), fail to reproduce the experimental data. The inclusion of 3N interactions, $NN$+3N full (circle) and AV+UIX (right triangle), does not help the calculated radius come close to the experimental data for $^{16}$O and $^{40}$Ca. On the other hand, the other type of chiral $NN$+3N interactions (pentagon and hexagon) show nice agreement with the data. As seen in the figure, calculation results are scattered even if the $NN$ and 3N interactions are used, and further investigations are indispensable how both of the $NN$ and 3N interactions can be determined.

| Nuclide | $\lambda_{\text{SRG}}$ (fm$^{-1}$) | $r_{\text{g.s.}}$ (fm) |
|---------|----------------------|-----------------|
|         |                      | $NN$+3N–ind  | $NN$+3N–full |
| $^4$He  | 1.88  | 1.48 | 1.47 | 1.46 | 1.42 | 1.40 | 1.39 |
|         | 2.0   | 1.47 | 1.46 | 1.45 | 1.41 | 1.40 | 1.39 |
|         | 2.24  | 1.46 | 1.45 | 1.45 | 1.39 | 1.39 | 1.38 |
| $^{16}$O| 1.88  | 2.32 | 2.29 | 2.26 | 2.28 | 2.24 | 2.22 |
|         | 2.0   | 2.30 | 2.27 | 2.25 | 2.27 | 2.23 | 2.21 |
|         | 2.24  | 2.28 | 2.25 | 2.24 | 2.25 | 2.22 | 2.20 |
| $^{40}$Ca| 1.88  | 2.91 | 2.89 | 2.85 | 2.86 | 2.83 | 2.79 |
|         | 2.0   | 2.89 | 2.86 | 2.83 | 2.84 | 2.82 | 2.79 |
|         | 2.24  | 2.84 | 2.82 | 2.80 | 2.83 | 2.80 | 2.78 |

V. CONCLUSION

In the present work, we have calculated the ground-state energies and radii for $^4$He, $^{16}$O, and $^{40}$Ca with the UMOA from $NN$ and 3N interactions based on the $\chi$EFT for the first time. To obtain the computationally tractable Hamiltonian in the UMOA, we employed the SRG evolution and the NO2B approximation.

The resulting ground-state energies and radii agree with the recent ab initio calculation results within a few percent. To discuss the accuracy of the UMOA calculation more precisely, we are going to extend the UMOA framework and directly treat the three-body cluster term beyond the NO2B approximation. The results will be discussed in the future publication.

In addition to expectation values for the bare radius operator, in the present work, we have evaluated those for the two- and three-body SRG evolved radius operators. By taking higher-body evolved operator into account, calculated radii slightly shrink, while the $\lambda_{\text{SRG}}$-dependence of radii is reduced as we keep up to three-body terms. Therefore, it is unlikely to reproduce the
nuclear radii with the interactions employed in this work, even if we continue to include many-body terms induced by SRG evolution. The simultaneous reproduction of the ground-state energies and radii strongly depend on employed nuclear interactions. To specify the proper choice of nuclear interactions, further investigations are needed.

ACKNOWLEDGMENTS

We thank R. Roth for providing us the coupled-cluster calculation results. This work was supported in part by JSPS KAKENHI Grant No. JP16J05707 and by the Program for Leading Graduate Schools, MEXT, Japan. This work was also supported in part by MEXT as "Priority Issue on post-K computer" (Elucidation of the Fundamental Laws and Evolution of the Universe) and JICFuS Projects No. 1p160211 and No. 1p170230 and CNS-RIKEN joint project for large-scale nuclear structure calculations, and the NSERC Grant No. SAPIN-2016-00033. TRIUMF receives federal funding via a contribution agreement with the National Research Council of Canada. A part of numerical calculations were done on Oakforest-PACS, Supercomputing Division, Information Technology Center, the University of Tokyo.

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