TOPOLOGICAL FIELD THEORIES AND HARRISON HOMOLOGY

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Abstract. Tools and arguments developed by Kevin Costello are adapted to families of “Outer Spaces” or spaces of graphs. This allows us to prove a version of Deligne’s conjecture: the Harrison homology associated to a homotopy commutative algebra is naturally a module over a cobordism category of 3-manifolds.

1. Introduction

A recent theorem of Kevin Costello ([5]) illustrates the relationship between homotopy associative algebras ($A_\infty$ algebras) and moduli spaces of Riemann surfaces. See also [21]. It is an exciting addition to a story which began when Deligne conjectured that the action of the homology of configuration spaces on the Hochschild homology of an associative algebra, $\text{HH}^*(A, A)$, lifts to an action defined at the chain level. Deligne’s conjecture was shown to be true (see [24]), but thinking of configuration spaces as moduli of genus 0 surfaces leads to a more general theorem: the chain level action of genus 0 surfaces extends to a natural action by surfaces of all genera. More specifically, the chain complex computing the Hochschild homology of an $A_\infty$ algebra is the object associated to the circle by a 2-dimensional topological field theory.

Costello considers the moduli spaces of open, open-closed and closed Riemann surfaces, and defines three categories related by inclusions,

$$j : \mathcal{O} \hookrightarrow \mathcal{OC} \hookrightarrow \mathcal{C} : i.$$

The categories of modules over the open, open-closed and closed categories are open, open-closed and closed topological field theories. The category of open theories can be identified with the category of cyclic $A_\infty$ algebras. Given such an algebra $A$, the inclusions $i$ and $j$ yield a functor

$$i^* \circ j_* : \mathcal{O}\text{-mod} \to \mathcal{C}\text{-mod},$$

which assigns to $A$ the closed topological field theory $i^* \circ j_*(A)$. This closed theory associates a chain complex, $i^* \circ j_*(A)(S^1)$, to the circle. The homology of this chain complex is isomorphic to Hochschild homology of $A$, $H_*(i^* \circ j_*(A)(S^1)) \cong \text{HH}_*(A, A)$. It follows that the geometrically defined maps in $\mathcal{C}$ determine natural operations on the Hochschild complex of $A$.

Recent work by Hatcher, Vogtmann and Wahl ([15], [16]) suggests that natural choices of open, open-closed and closed categories may be obtained from the classifying spaces
of mapping class groups of doubled handlebodies, or 3-manifolds of the form \( \#^g S^1 \times S^2 \#^e D^3 \#^4 S^1 \times D^2 \). Such classifying spaces can be modelled by spaces of metric graphs. In this paper, the open, open-closed and closed categories are categories of rational chains on these spaces.

We first prove that the category of modules over the open category is equivalent to the category of cyclic \( C_\infty \) algebras. The extension from the open category to the open-closed category yields a Costello-type theorem:

**Theorem 1.1.** There exists a category \( OC \) of 3-manifolds with objects given by boundary spheres \( S \) and tori \( T \). There are open and closed subcategories with inclusions,

\[
j : \mathcal{O} \hookrightarrow OC \hookrightarrow \mathcal{C} : i.
\]

The category of cyclic \( C_\infty \) algebras is equivalent to the category of modules over the category \( \mathcal{O} \). For any \( C_\infty \) algebra \( A \), the homology of the object associated to the torus is isomorphic to the Harrison homology of \( A \).

\[
H_*((i^* \circ j_*)(A))(T) \cong \text{Harrison}_*(A, A)
\]

In particular, the closed category acts naturally on a chain complex computing the Harrison homology of \( A \).

In section 6, the chain complex Torus\((A) = (i^* \circ j_*)(A)(T)\) will be described explicitly. From this description, it will follow that the homology, \( H_*(\text{Torus}(A)) \), agrees with the Harrison homology \( \text{Harrison}_*(A, A) \) of \( A \).

The spaces of graphs appearing in this paper are natural extensions of the “Outer Spaces” which originally appeared in [6]. Connections between Outer Space and the homotopy commutative operad appear in [20] [11] [10] [22]. The construction in this paper can be viewed as a version of Deligne’s conjecture in the “classical limit” corresponding to the homotopy commutative operad in Kontsevich’s “three worlds” [19].

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## 2. Algebra and Operads

The underlying ring in all constructions will be the field of rational numbers. We denote by Top the category of topological spaces, by Group the category of groups and by Kom the category of chain complexes.
2.1. Monoidal Categories. A category $\mathcal{C}$ is *symmetric monoidal* if it is equipped with a bifunctor,
\[- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C},\]
an object $1$ and isomorphisms,
\[
\begin{align*}
(1) \quad (a \otimes b) \otimes c & \cong a \otimes (b \otimes c) \\
(2) \quad 1 \otimes a & \cong a \cong a \otimes 1 \\
(3) \quad a \otimes b & \cong b \otimes a
\end{align*}
\]
satisfying coherence conditions, see [23]. There are monoidal structures on $\text{Top}$, Group and $\text{Kom}$ given by disjoint union, product and tensor product respectively.

A *monoidal* functor $F : \mathcal{C} \to \mathcal{D}$ between symmetric monoidal categories is equipped with maps $F(a) \otimes F(b) \to F(a \otimes b)$ that are natural in both $a$ and $b$, and satisfy associativity and commutativity criteria.

**Definition 2.2.** $(\text{Ob}(\mathcal{C}))$ Every symmetric monoidal category $\mathcal{C}$ has a subcategory $\text{Ob}(\mathcal{C})$ with the same objects. The morphisms of $\text{Ob}(\mathcal{C})$ are generated by permutations of tensors.

\[a \otimes a' \cong a' \otimes a\]

Notice that $\text{Ob}(\text{Ob}(\mathcal{C})) = \text{Ob}(\mathcal{C})$. This notation agrees with [5].

2.3. Differential Graded Categories. All of the categories in this paper will have extra structure in a sense that can be captured by the idea of enrichment. A category $\mathcal{C}$ is *enriched* over a monoidal category $\mathcal{D}$ when, for all $X, Y \in \text{Ob}(\mathcal{C})$,
\[\text{Hom}_{\mathcal{C}}(X, Y) \in \text{Ob}(\mathcal{D})\]
and the composition in $\mathcal{C}$ respects this $\mathcal{D}$ structure:
\[
\circ : \text{Hom}_{\mathcal{C}}(a, b) \otimes \text{Hom}_{\mathcal{C}}(b, c) \to \text{Hom}_{\mathcal{C}}(a, c) \\
\in \text{Hom}_{\mathcal{D}}(\text{Hom}_{\mathcal{C}}(a, b) \otimes \text{Hom}_{\mathcal{C}}(b, c), \text{Hom}_{\mathcal{C}}(a, c)),
\]
for all $a, b, c \in \text{Ob}(\mathcal{C})$. A category $\mathcal{C}$ that satisfies $\text{Hom}_{\mathcal{C}}(X, Y) \in \text{Top}$ will be called a *topological category*. For instance, the sets $\text{Hom}_{\text{Top}}(X, Y)$ can be endowed with the compact open topology.

If $\mathcal{C}$ is enriched over $\mathcal{D}$ and $F : \mathcal{D} \to \mathcal{E}$ is a monoidal functor, then there is a category $F_*(\mathcal{C})$ enriched over $\mathcal{E}$ defined by:
\[
\text{Ob}(F_*(\mathcal{C})) = \text{Ob}(\mathcal{C}) \quad \text{and} \quad F_*(\mathcal{C})(a, b) = F(\mathcal{C}(a, b)).
\]

For example, the functor $B : \text{Group} \to \text{Top}$ giving the classifying space of a group determines the functor $B_*$ which maps categories enriched over Group to categories
enriched over Top. Another important example is $C_*(\mathbb{Q}; \mathbb{Q})$; rational singular chains. If $C$ is a topological category then there is a category $C_*(C; \mathbb{Q})$ defined by:

$$\text{Ob}(C_*(C; \mathbb{Q})) = \text{Ob}(C)$$

$$\text{Hom}_{C_*(C; \mathbb{Q})}(A, B) = C_*(\text{Hom}_{C}(A, B); \mathbb{Q}).$$

A version of this functor will be defined for cellular spaces in section 2.5.1. Categories of the form $C_*(C; \mathbb{Q})$ are examples of differential graded categories.

A differential graded or dg category is a category enriched over Kom. A differential graded symmetric monoidal, or dgsm category, is a symmetric monoidal category which is differential graded. The category Kom is an example of a dgsm category. If $C$ is a dg category then $H_*(C)$ is a category enriched over the category of graded vector spaces.

A dgsm functor or morphism of dgsm categories, $F : C \to D$, is a monoidal functor which respects the differential graded structure. This is a monoidal functor of categories enriched over Kom, as defined above. Two dgsm categories $C$ and $D$ are quasi-isomorphic when there is a dgsm functor $F : C \to D$ such that $H_*(F)$ is full, faithful and induces isomorphisms on objects.

### 2.4. Modules Over Differential Graded Categories.

If $C$ is a dgsm category then a left $C$-mod is a dgsm functor $C \to \text{Kom}$. A right $C$-mod is a dgsm functor $C^{\text{op}} \to \text{Kom}$. Note that as functors, modules must respect the differential graded structure, specifically if

$$F_{a,b} : \text{Hom}_{C}(a, b) \to \text{Hom}_{\text{Kom}}(F(a), F(b))$$

then $d \circ F_{a,b} = F_{a,b} \circ d$ for all $a, b \in \text{Ob}(C)$.

Maps between modules $M$ and $N$ are natural transformations $\phi : M \to N$ of the underlying functors which satisfy the following conditions:

1. All of the elements $\phi(a) \in \text{Hom}(M(a), N(a))$ are chain maps.
2. The natural transformation $\phi$ respects the monoidal structure,

$$M(a) \otimes M(a') \to N(a) \otimes N(a')$$

$$M(a \otimes a') \to N(a \otimes a').$$

The category of left (right) modules over $C$ will be denoted by $C$-mod (mod-$C$). For a functor to be monoidal we only require the existence of a map

$$F(a) \otimes F(b) \to F(a \otimes b)$$
satisfying the axioms described in section 2.1. It is often the case that these maps satisfy stronger conditions. A module is said to be *split* when the monoidal structure maps \( F(a) \otimes F(b) \to F(a \otimes b) \) are isomorphisms and *h-split*, or homologically split, when they are quasi-isomorphisms. For instance, a TQFT in the sense of Atiyah is a split module over the cobordism category.

The usual product of categories extends to one which respects the dgsm structure. If \( C \) and \( D \) are categories then there is a category \( C \otimes D \) defined by

\[
\text{Ob}(C \otimes D) = \text{Ob}(C) \times \text{Ob}(D)
\]

\[
\text{Hom}_{C \otimes D}(a \times c, b \times d) = \text{Hom}_C(a, c) \otimes \text{Hom}_D(b, d).
\]

If \( C \) and \( D \) are differential graded then \( C \otimes D \) is differential graded using the usual tensor product of chain complexes. If \( C \) and \( D \) are monoidal then \( C \otimes D \) is monoidal using \((a \otimes c) \otimes (b \otimes d) = (a \otimes b) \otimes (c \otimes d)\).

If \( C \) and \( D \) are dgsm categories then a \( D - C \) *bimodule* is a dgsm functor from the category \( D \otimes C^{op} \) to Kom.

The following observation will be used to define the bimodule \( OC \) appearing in theorem 4.11 section 4.6.

**Observation.** Every dgsm category \( C \) yields a \( C - C \) bimodule, \( C : C \otimes C^{op} \to \text{Kom} \) given by \( C(x \times y) = \text{Hom}_C(y, x) \).

If \( M \) is a \( D - C \) bimodule and \( N \) is a left \( C \)-mod then there exists a left \( D \)-mod, \( M \otimes_C N \), defined on objects \( b \in \text{Ob}(D) \) by,

\[
(M \otimes_C N)(b) = \bigoplus_{a \in \text{Ob}(C)} M(b, a) \otimes N(a),
\]

modulo the relation, \( \sim \), which makes the diagram below to commute,

\[
\begin{array}{ccc}
M(b, a) \otimes \text{Hom}_C(a', a) \otimes N(a') & \longrightarrow & M(b, a) \otimes N(a) \\
\downarrow & & \downarrow \\
M(b, a') \otimes N(a') & \longrightarrow & (M \otimes_C N)(b).
\end{array}
\]

Explicitly,

\[
f^*(g) \otimes h \sim g \otimes f_*(h),
\]

for \( f \in \text{Hom}_C(a', a), g \in M(b, a) \) and \( h \in N(a') \).

Although dgsm modules do not form a dg category, they do possess a reasonable notion of weak equivalence. A map \( \varphi : M \to M' \) between \( M, M' \in C\)-mod is a *quasi-isomorphism* if \( \varphi_* : H_*(M(a)) \to H_*(M'(a)) \) for all \( a \in \text{Ob}(C) \).
A functor $F : \mathcal{C} \to \mathcal{D}$ between categories of modules is \textit{exact} when it preserves quasi-isomorphisms. Two functors $F, G : \mathcal{C} \to \mathcal{D}$ are \textit{quasi-isomorphic}, $F \simeq G$, when there are natural transformations $\varphi : F \to G$ such that $\varphi(c)$ is a quasi-isomorphism for all $c \in \text{Ob}(\mathcal{C})$. Two categories $\mathcal{C}$ and $\mathcal{D}$ are \textit{isomorphic} or \textit{quasi-equivalent}, $\mathcal{C} \cong \mathcal{D}$ if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $FG \simeq 1$ and $GF \simeq 1$.

A module $M$ is \textit{flat} when the functor $- \otimes M$ is exact. Since most of the constructions to follow will involve considering dgsm categories and their modules up to quasi-isomorphism, strictly speaking, we should be working in a derived category. As such the tensor product $M \otimes_{\mathcal{C}} N = M \otimes_{\mathcal{C}} \text{Bar}_C N$ where $\text{Bar}_C N$ is $N$ tensored with the Bar construction on $\mathcal{C}$. This gives a canonical flat replacement (see [5]).

Any dgsm functor $F : \mathcal{C} \to \mathcal{D}$ between dgsm categories determines a pair of functors between the corresponding categories of modules, $F^* : \mathcal{D}\text{-mod} \to \mathcal{C}\text{-mod}$ and $F_* : \mathcal{C}\text{-mod} \to \mathcal{D}\text{-mod}$. The functor $F^*$ is called restriction and $F_*(M) = \mathcal{D} \otimes_{\mathcal{C}} M$ is called the induction functor. The latter is defined using the tensor product above and the $\mathcal{D} - \mathcal{C}$ bimodule structure on $\mathcal{D}$ inherited from $F$.

\textbf{Theorem 2.5. (5)} If $F : \mathcal{C} \to \mathcal{D}$ is a quasi-isomorphism of dgsm categories, then the induction and restriction functors,

$$\mathbb{L}F_* : \mathcal{C}\text{-mod} \rightleftarrows \mathcal{D}\text{-mod} : F^*$$

are inverse quasi-isomorphisms between the categories of left (right) $\mathcal{C}$ modules and left (right) $\mathcal{D}$ modules respectively.

\textbf{2.5.1. Cellular Chains.} If $X$ is a cellular space then we would like the equivalence $C^\cell_*(X; \mathbb{Q}) \simeq C_*(X; \mathbb{Q})$ to be natural. In order to accomplish this, our chain complexes are defined to be a colimit over all maps from cellular spaces into a given space (see [3]).

A \textit{cellular space} $X$ is a CW complex of finite type; in other words, there are finitely many cells in each dimension. In particular, each cell attaches to only finitely many other cells. If $X^i$ is the $i$-skeleton of $X$ then $f : X \hookrightarrow Y$ is a map of cellular spaces when it is continuous and $f^{-1}(Y^i) = X^i$. Let $\text{Cell} \subset \text{Top}$ be the subcategory of cellular spaces and cellular maps. For any topological space $Y$, define

$$C_*(Y; \mathbb{Q}) = \text{colim}_{X \in \text{Cell} \downarrow Y} C^\cell_*(X; \mathbb{Q})$$

where $\text{Cell} \downarrow Y$ is the over category and $C^\cell_*(-; \mathbb{Q})$ denotes the functor given by taking rational cellular chains. It follows that if $Y$ is a cellular space then the map $C^\cell_*(Y; \mathbb{Q}) \to C_*(Y; \mathbb{Q})$ is natural.

\textbf{2.6. Operads.} After a brief discussion of operads and cyclic operads, we introduce the Bar and Cobar functors and define the associative and associative commutative operads: $A$ and $C$. The operad $C_\infty$ will first be introduced as a quotient of the $A_\infty$ operad. In section [2.7.3] $C_\infty$ will be defined in terms of the Cobar $\circ$ Bar construction.
2.6.1. **Operads.** In what follows operads will be used to encode axioms for various kinds of algebras and to control stratifications of certain spaces of graphs. For more information regarding operads, see [25, 24, 28].

A differential graded *operad* $\mathcal{O}$ is a sequence of chain complexes $\{\mathcal{O}(n)\}_{n=1}^{\infty}$ and composition maps

$$\gamma : \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \to \mathcal{O}(n_1 + \cdots + n_k)$$

together with an action of the symmetric group $\Sigma_n$ on $\mathcal{O}(n)$ and a unit $1 \in \mathcal{O}(1)$. The compositions $\gamma$ are required to be $\Sigma$-equivariant, associative and unital. In all cases to follow, chain complexes will be finite dimensional and $\mathcal{O}(1)$ will be one dimensional.

A map of operads $\varphi : \mathcal{O} \to \mathcal{O}'$ is given by a collection $\{\varphi_n : \mathcal{O}(n) \to \mathcal{O}'(n)\}_{n=1}^{\infty}$ of $\Sigma$-equivariant chain maps that commute with the operad compositions and take units to units. Two operads $\mathcal{O}$ and $\mathcal{O}'$ are *quasi-isomorphic* when there is a map $\varphi : \mathcal{O} \to \mathcal{O}'$, the individual components of which induce isomorphisms on homology.

Cooperads are operads with the arrows, $\gamma$, reversed. There is a completely analogous category of differential graded cooperads, see [8].

Given a chain complex $X$, define the *endomorphism operad*, $\text{End}_X$, by

$$\text{End}_X(n) = \text{Hom}_{\text{Kom}}^*(X^\otimes n, X).$$

Composition is given by composition of chain maps and the action of $\Sigma_n$ is given by permuting the arguments of $f \in \text{End}_X(n)$. A chain complex $X$ is an *algebra over* an operad $\mathcal{O}$ when there is a morphism of operads $\mathcal{O} \to \text{End}_X$.

A differential graded *cyclic operad* is an operad $\mathcal{O} = \{\mathcal{O}(n)\}_{n=1}^{\infty}$ such that the action of $\Sigma_n$ on $\mathcal{O}(n)$ lifts to an action of $\Sigma_{n+1}$ on $\mathcal{O}(n)$. An algebra $X$ over a cyclic operad $\mathcal{O}$ is required to possess a non-degenerate bilinear form which is invariant with respect to the operations of $\mathcal{O}$, see [9].

Operads are usually pictured as rooted trees with vertices labelled by some distinguished collection of symbols. The composition $\gamma$ corresponds to gluing the roots of $k$ such trees to the unrooted edges of a single tree with $k + 1$ boundary edges. A cyclic operad is an operad in which the trees representing operations lack a preferred root. Cyclic operations can be manipulated in the plane, see section 2.8.

2.6.2. **Homotopy operads.** In this section we give explicit models for the operads $\mathcal{C}$, $\mathcal{A}$, $\mathcal{C}_{\infty}$ and $\mathcal{A}_{\infty}$. The usual definition of $\mathcal{C}_{\infty}$ is given as a quotient of $\mathcal{A}_{\infty}$ by the shuffle relations. Since dg operads defined by quotients cannot control moduli spaces, such as those found in section 4, in section 2.7.3, the Cobar $\circ$ Bar functor is introduced in order to remove the shuffle relations.

The *commutative operad* $\mathcal{C} = \{\mathcal{C}(n)\}_{n=1}^{\infty}$ is both the main object of interest and the simplest operad:

$$\mathcal{C}(n) = \mathbb{Q} \quad \text{for all} \quad n \geq 1,$$
concentrated in degree 0. If $X$ is a vector space then $X$ is an algebra over the commutative operad when $X$ is an associative commutative algebra. $C$ extends to a cyclic operad. A cyclic $C$ algebra is an associative commutative algebra $X$ with an inner product $\langle - , - \rangle : X \otimes X \to \mathbb{Q}$ that satisfies, 

$$\langle a \cdot b, c \rangle = \langle a, b \cdot c \rangle.$$ 

In other words, $X$ is a commutative Frobenius algebra.

The $A_\infty$ operad is generated by all possible compositions of $n$-fold operations $m_n$ subject to the relation that

$$\partial m_n(1, \ldots, n) = \sum_{i+j=n+1, i,j \geq 2} \sum_{s=0}^{n-j} (-1)^{j+s(j+1)} m_i(1, \ldots, m_j(s+1, \ldots, s+j+1), \ldots, n),$$

where $m_n(1, \ldots, n)$ is the operation $m_n$ labelled by its $n$ inputs. The degree of $m_n$ is $n-2$.

Elements of the operad $A_\infty$ are usually pictured as rooted trees in the plane in which the $n+1$-valent vertices represent the operation $m_n$. The operation $m_n$ is sometimes represented by a disk with $n$ distinct boundary points and an extra boundary point corresponding to the root. In this case, a composition of the form $m_i(1, \ldots, m_j(\ldots), \ldots, n)$ is represented by two such disks glued together along one of their boundary points.

The homotopy associative commutative or $C_\infty$ operad is usually introduced as a quotient of the $A_\infty$ operad by shuffle relations. The operad $C_\infty$ is the kernel of the map $A_\infty \to L_\infty$ obtained by extending the map $A \to L$ defined by $[a,b] = ab - ba$.

A $(p,q)$-shuffle, $\sigma \in \text{Sh}(p,q)$, is a permutation $\sigma \in \Sigma_{p+q}$ which satisfies,

$$\sigma(1) < \sigma(2) < \ldots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \sigma(p+2) < \ldots < \sigma(p+q).$$

The $C_\infty$ operad is obtained from the $A_\infty$ operad by imposing the relations,

$$\sum_{\sigma \in \text{Sh}(i,n-i)} \text{sgn}(\sigma)m_n(\sigma(1), \ldots, \sigma(n)) = 0,$$

for all $1 < i < n$, where $\text{sgn}(\sigma)$ is the sign of a permutation. For instance, when $k = 2$, the relation becomes,

$$m_2(a,a') - m_2(a',a) = 0.$$

Cyclic $C_\infty$ and $A_\infty$ algebras possess a non-degenerate inner product $\langle - , - \rangle$ which satisfies

$$\langle m_n(x_0, \ldots, x_{n-1}), x_n \rangle = (-1)^{(n+1)|x_0| \sum_{i=1}^{n-1} |x_i|} \langle m_n(x_1, \ldots, x_n), x_0 \rangle.$$

If $M$ is an $n$-manifold then the de Rham complex $\Omega^*(M)$ is an example of a $C_\infty$ algebra. The transfer theorem determines a $C_\infty$ algebra structure on $H^*(M; \mathbb{R})$. If
$M$ is also compact then $H^*(M; \mathbb{R})$ is cyclic; the inner product is the duality pairing, see [12] and [13].

There is a map of operads $\alpha : C_{\infty} \to C$, defined by

$$\alpha(m_2) = m_2 \quad \text{and} \quad \alpha(m_j) = 0 \text{ if } j \neq 2$$

which is a quasi-isomorphism. We’d like to think of $C_{\infty}$ as a free resolution of $C$. Unfortunately, since we have added the shuffle relations, $C_{\infty}$ is not free in the appropriate sense. In order to obtain a dg operad homotopy equivalent to $C$, which is free of relations, we introduce the Cobar $\circ$ Bar functor in section 2.7.3.

2.7. Resolutions of operads. In this section we introduce definitions for graphs and use these definitions to construct the Bar and Cobar functors. The Bar construction is a functor which takes a dg operad $P$ to a dg cooperad $\text{Bar}(P)$, while the Cobar construction is a functor taking a dg cooperad $O$ to a dg operad $\text{Cobar}(O)$. These form an adjunction between the categories of operads and cooperads, the unit of which,

$$\eta_O : O \to \text{Cobar}(\text{Bar}(O))$$

is a quasi-isomorphism of operads.

2.7.1. Graphs. A graph $G$ is a finite set that has been partitioned in two ways: into pairs $e = \{a, b\}$ called edges and into sets $H(v) = \{h_1, h_2, \ldots, h_n\}$ called vertices.

$$G = \coprod_e \{a, b\} = \coprod_v H(v)$$

Denote the set of vertices of $G$ by $V(G)$ and the set of edges of $G$ by $E(G)$. The elements of $G$ will be called half edges. Two half edges $a, b \in G$ meet if $a, b \in H(v)$ for some vertex $v$. Given an edge $e \in E(G)$, the set $e = \{x, y\}$ is the set of half edges associated to $e$ in $G$. For each vertex $v \in V(G)$, the set $H(v)$ is the set of half edges associated to $v$ in $G$. The valence $\text{val}(v)$ of $v \in V(G)$ is the number of half edges or $|H(v)|$. All graphs in this document are required to have vertices $v$ of valence $\text{val}(v) = 1$ or $\text{val}(v) \geq 3$ unless otherwise noted.

Two graphs $G$ and $H$ are isomorphic when there is a bijective set map between half edges $\varphi : H \to G$ that respects the two partitions.

A subgraph $H$ of $G$ is a graph formed by the set of all vertices of $G$ together with some subset of the set of edges of $G$. A cycle of $G$ is a subgraph $C \subset G$ given by an ordered sequence of edges which begin and end at the same vertex.

The boundary $\partial(G)$ of a graph $G$ is the collection of edges that contain a vertex having valence one. An internal edge is an edge not in the boundary while an external edge is not internal.

Let $[n]$ be the set $\{1, \ldots, n\}$. A graph $G$ is boundary labelled if there is a choice of partition $\partial(G) = \text{In}(G) \cup \text{Out}(G)$ of the boundary into a set of incoming and outgoing
Given an edge is given by 2.7.2. Orientations. A notion of boundary labelling for geometric graphs appears in section 4.1.

A geometric graph is a 1-dimensional CW complex. Every graph $G$ has an associated geometric graph, $|G|$, in which the 0-skeleton is given by the vertices $V(G)$ and the 1-skeleton is formed by gluing on 1-cells corresponding to the edges. We may refer to graphs as either combinatorial or geometric when it is necessary to make a distinction.

A graph $G$ is connected when $H_0(|G|) \cong \mathbb{Q}$. A graph $G$ has genus $g$ if $H_1(|G|) \cong \mathbb{Q}^g$. A forest is a graph of genus 0. A tree is a connected forest. A rooted tree is a tree together with a choice of outgoing edge, the rest of the boundary edges being incoming. A tree with a single internal vertex will be called a corolla. An $n$-tree is a tree with $n$ incoming edges.

Given an edge $e \in E(G)$, $e = \{x, y\}$, we can form a new graph $G/e$ by removing $e$ and replacing $H(x)$ and $H(y)$ with $H(x) \cup H(y) - \{x, y\}$. This operation, called edge collapse, is a homotopy equivalence of $|G|$ if $x$ and $y$ are not contained in the same set of half edges $H(v)$. Collapsing a forest $F \subset G$ is called forest collapse.

2.7.2. Orientations. If $V_n$ is a graded vector space then the $j$-fold desuspension $V[j]_*$ is given by $V[j]_i = V_{i+j}$. An orientation of a graded vector space $W$ of dimension $n = \dim(W)$ is a non-zero vector in the exterior algebra $\det(W) = \Lambda^n(W)[-n]$. The dual is defined by $\det(W)^* = \Lambda^n(W)[n]$. If $S$ is a set then we orient $S$ using $\det(S) = \det(\mathbb{Q}(S))$. Two orientations are equivalent when they are positive scalar multiples of each other. An orientation of a graph $G$ is defined to be an element of

$$\det(G) = \det(E(G)) \otimes \det(\text{Out}(G)) \otimes \det(H_0(G)) \otimes \det(H_1(G))^* [O - \chi]$$

where $O$ is the number of outgoing edges and $\chi = \chi(G)$ is the Euler characteristic of $G$. Using this convention, a graph is placed in degree $|E(G)|$. There are maps,

$$\det(G_0) \otimes \det(G_1) \rightarrow \det(G_0 \# G_1) \quad \text{and} \quad \det(G_0 \coprod G_1) \cong \det(G_0) \otimes \det(G_1).$$

2.7.3. The Bar and Cobar constructions. If $S$ is a set and $\mathcal{O}$ is a cyclic dg (co)operad then a labelling of $S$ by $\mathcal{O}$ is defined by the coinvariants trick:

$$\mathcal{O}(S) = (\mathcal{O}(n) \times \text{Bij}([n+1], S))_{\Sigma_{n+1}}$$

where $\text{Bij}([n+1], S)$ is the set of bijections from $S$ to $[n+1] = \{1, \ldots, n+1\}$ and $\Sigma_{n+1}$ acts diagonally. If $T$ is a tree then a labelling of $T$ by $\mathcal{O}$ is determined by assigning to each vertex $v$ an element of $\mathcal{O}(H(v))$,

$$\mathcal{O}(T) = \bigotimes_{v \in V(T)} \mathcal{O}(H(v)).$$
The collapse of an internal edge \( c : T \to T/e \) induces maps of labellings. If we denote by \( e \) the vertex obtained by the edge collapse and by \( v \) and \( w \) the two identified end points then there are maps,

\[
\mathcal{O}(\text{val}(v)) \otimes \mathcal{O}(\text{val}(w)) \to \mathcal{O}(\text{val}(e)) \quad \text{and} \quad \mathcal{P}(\text{val}(e)) \to \mathcal{P}(\text{val}(v)) \otimes \mathcal{P}(\text{val}(w)).
\]

(Recall from \( \text{2.7.1} \) that \( \text{val}(v) \) is the valence of the vertex \( v \).) Tensoring the above with identity yields maps \( c_* : \mathcal{O}(T) \to \mathcal{O}(T/e) \) and \( c^* : \mathcal{P}(T/e) \to \mathcal{P}(T) \). These maps \( c_* \) and \( c^* \) are used to define the Bar and Cobar differentials below.

The Bar construction \( \text{Bar}(\mathcal{O}) \) of a cyclic differential graded operad \( \mathcal{O} \) is the dg cooperad of labelled trees with an edge contracting differential. Explicitly,

\[
\text{Bar}(\mathcal{O})(n) = \bigoplus_{\text{Tree } T \mid |T| = 1} \mathcal{O}(T) \otimes \det(T) \leftarrow \cdots \leftarrow \bigoplus_{\text{Tree } T \mid |T| = n-1} \mathcal{O}(T) \otimes \det(T).
\]

The Cobar construction \( \text{Cobar}(\mathcal{P}) \) of a cyclic differential graded cooperad \( \mathcal{P} \) is the dg operad of labelled trees with an edge expanding differential. Concretely,

\[
\text{Cobar}(\mathcal{P})(n) = \bigoplus_{\text{Tree } T \mid |T| = 1} \mathcal{P}(T) \otimes \det(T)^* \rightarrow \cdots \rightarrow \bigoplus_{\text{Tree } T \mid |T| = n-1} \mathcal{P}(T) \otimes \det(T)^*.
\]

In the formulas above \( |T| \) is the number of internal vertices of \( T \). The complex is graded so that the term spanned by trees with one internal vertex is situated in degree 0. Alternatively, the grading is determined by the orientation, see section \( \text{2.7.2} \).

The differential \( \delta \) either contracts or expands edges. It can be described by its matrix elements, \( (\delta)_{T,T'} \). If \( T' \) is not isomorphic to \( T/e \) for some internal edge \( e \in T \) then the corresponding component of \( \delta \) is set to zero. Otherwise, let \( c : T \to T' \cong T/e \) so that if \( c_* : \mathcal{O}(T) \to \mathcal{O}(T') \) or \( c^* : \mathcal{P}(T') \to \mathcal{P}(T) \) are the maps above then \( \delta \) is given by

\[
(\delta)_{T,T'} = c_* \otimes p_e \quad \text{or} \quad (\delta)_{T',T} = c^* \otimes p^e.
\]

If collapsing the edge \( e \) identifies the vertices \( u \) and \( v \) to a vertex \( e \), then the map of orientations \( p_e : \det(T) \to \det(T') \) is given by,

\[
p_e(y_0 \wedge \cdots \wedge e \wedge \cdots \wedge y_n) = y_0 \wedge \cdots \wedge \hat{e} \wedge \cdots \wedge y_n
\]

and the orientation map \( p^e \), coupled with the expanding differential is defined analogously. In either case, if the operad \( \mathcal{O} \), or cooperad \( \mathcal{P} \), has a non-trivial differential then the total differential is the sum of the differential defined above together with the original differential.

The composition for the operad \( \text{Cobar}(\mathcal{P}) \) is given by grafting boundary edges and eliminating the resulting bivalent vertex. This satisfies the Leibniz rule with respect
to the differential defined above. Notice that Cobar(\(P\)) is generated by \(P\)-labelled corolla.

The Bar and Cobar functors form an adjunction. The counit and unit maps of this adjunction, 

\[
\text{Bar}(\text{Cobar}(\mathcal{P})) \to \mathcal{P} \quad \text{and} \quad \mathcal{O} \to \text{Cobar}(\text{Bar}(\mathcal{O})),
\]

are quasi-isomorphisms, see [8].

2.8. **Relation to Differential Graded Algebra.** The language of differential graded operads and their algebras in section [2.6] is an important special case of the language of differential graded categories and their modules, see section [2.3]. In this section we establish a connection between sections [2.6] and [2.3].

Given a dg operad \(\mathcal{O}\), we can define the enveloping category \(\mathcal{O}^{\#}\) to be the dgsm category generated by one object \(X\) and morphisms generated by \(\text{Hom}_{\mathcal{O}^{\#}}(X \otimes n, X) = \mathcal{O}(n)\)

using the monoidal structure. Pictorially, if operations \(x \in \mathcal{O}(k)\) are represented by trees then \(y \in \text{Hom}_{\mathcal{O}^{\#}}(X \otimes n, X \otimes m)\) is a disjoint union of trees. By construction, the category \(\mathcal{O}^{\#}\) includes factorization isomorphisms,

\[
\theta_{n,m} = \text{Hom}_{\mathcal{O}^{\#}}(X \otimes n, X \otimes m) \cong \bigotimes_{i=1}^{m} \text{Hom}_{\mathcal{O}^{\#}}(X \otimes n_i, X) \quad \text{such that} \quad \sum n_i = n.
\]

Maps of operads induce functors between their associated enveloping categories. The following is an immediate consequence of the above construction.

**Lemma 2.9.** The category of \(\mathcal{O}\)-algebras is equivalent to the category of split left \(\mathcal{O}^{\#}\) modules.

**Proof.** Any functor \(F : \mathcal{O}^{\#} \to \text{Kom}\) identifies the object \(X\) with a chain complex \(F(X)\) and, by split monoidality, identifies the object \(X \otimes m\) with \(F(X) \otimes m\). Consider the action of \(\mathcal{O}^{\#}\) on \(F(X)\). Using the factorization map \(\theta_{n,m}\),

\[
\varphi = \varphi_1 \otimes \cdots \otimes \varphi_m \quad \text{such that} \quad \varphi_i : X \otimes n_i \to X
\]

where \(n = n_1 + n_2 + \cdots + n_m\). Each map \(\varphi_i\) is also an element of \(\mathcal{O}(n_i)\). This identification commutes with the categorical composition of \(\mathcal{O}^{\#}\) and the operadic composition of \(\mathcal{O}\). \(\square\)

Split modules do not behave as well under quasi-isomorphism as h-split modules. The next lemma tells us that, for our purposes, these two notions of split are equivalent.

**Lemma 2.10.** There is an equivalence of categories between the category of h-split left \(\mathcal{O}^{\#}\) modules and the category of split left \(\mathcal{O}^{\#}\) modules.
Proof. The equivalence is determined by a functor $\eta$ from h-split to split modules. If $F$ is an h-split $O^b$ module then define a split module $\eta(F)$ by $\eta(F)(X^{\otimes n}) = F(X^{\otimes n})$.

Since $F$ is h-split there are quasi-isomorphisms $\varphi_{X,j} : \eta(F)(X^{\otimes j}) \rightarrow F(X^{\otimes j})$. We must extend $\eta(F)$ to a functor. Each $m_j \in O(j)$ induces a map, $(m_j)_* : F(X^{\otimes j}) \rightarrow F(X)$. These are natural with respect to the maps $\varphi_{X,j}$. For any $f \in Hom_{O^b}(X^{\otimes m}, X^{\otimes n})$, $f = \theta^{-1}_{n,m}(m_{n1} \otimes \cdots \otimes m_{nk})$.

So the action of $O$ can be extended to an action of $O^b$, giving a unique split $O^b$ module, $\eta(F)$, which is quasi-isomorphic to the h-split $O^b$ module $F$ via $\{\varphi\}$. □

The following lemma allows us to simplify some rather complicated looking operads. See section 2.4 for the definition of quasi-equivalent.

Lemma 2.11. If $O_1$ and $O_2$ are quasi-isomorphic operads then the associated enveloping categories $O_1^b$ and $O_2^b$ are quasi-isomorphic.

$$O_1^b \cong O_2^b$$

In particular, it follows that the associated categories of modules are quasi-equivalent.

$$O_1^b\text{-mod} \cong O_2^b\text{-mod}$$

The statement about modules follows from the lemmas and Theorem 2.5.

Cyclic differential graded operads $O$ also yield dgsm categories $O^b$ with one object $X$ and morphisms generated by

$$\text{Hom}_{O^b}(X^{\otimes n}, X) = O(n),$$

together with cap and cup morphisms corresponding to an invariant inner product and its dual,

$$\langle -, - \rangle \in \text{Hom}_{O^b}(X \otimes X, \mathbb{Q}) \text{ and } \langle -, - \rangle^* \in \text{Hom}_{O^b}(\mathbb{Q}, X \otimes X).$$

These are represented by pictures,

$$\cap \quad \text{and} \quad \cup,$$

which are subject to the S-bend relations:

$$\cap \cup = | = \cup .$$

The addition of caps and cups yields much larger morphism spaces; $\text{Hom}_{O^b}(X^{\otimes n}, X^{\otimes m})$ is now a space of graphs (not a space of trees). Analogues of the previous lemmas hold for $O^b$ after $O$ algebras are replaced by cyclic $O$ algebras.
A differential graded PROP is a symmetric monoidal category which is generated by a single object $x$ and enriched in the category of chain complexes, see [24, 23]. The construction $-\flat$ is a functor from the category of cyclic dg operads to the category of dg PROPs.

Each dg modular operad $\mathcal{M}$ (see [10]) determines a dg PROP $P\mathcal{M}$ where

$$\Hom_{P\mathcal{M}}(x^\otimes n, x^\otimes m) = \oplus_g \mathcal{M}(g, n + m)$$

and the composition is constructed by gluing the corresponding collections of end points using the structure maps,

$$\circ_{ij} : \mathcal{M}(g, m) \otimes \mathcal{M}(g', n) \to \mathcal{M}(g + g', m + n - 2).$$

A cyclic dg operad $\mathcal{O}$ determines a modular operad $\mathcal{MO}$. Some authors refer to $\mathcal{MO}$ as the naïve modular closure of $\mathcal{O}$, see [3] section 3.2. If $\mathcal{O}$ is a cyclic dg operad then the PROP $\mathcal{O}^\flat$ agrees with $P\mathcal{MO}$,

$$\mathcal{O}^\flat \simeq P\mathcal{MO}.$$
\((A \# B, i, m)\). Associativity follows from the local nature of the gluing composition. Identity morphisms are given by thickened surfaces, \(Y \times [0, 1]\).

The category \(\mathcal{N}\) has a symmetric monoidal structure given by disjoint union.

**Definition 3.2.** (\(\mathcal{M}\)) The differential graded cobordism category \(\mathcal{M}\) is the category of singular chains on classifying spaces of mapping class groups of morphisms in \(\mathcal{N}\).

Specifically,

\[
\text{Ob}(\mathcal{M}) = \text{Ob}(\mathcal{N}) \quad \text{and} \quad \text{Hom}_{\mathcal{M}}(X, Y) = C_*(B\Gamma(\text{Hom}_\mathcal{N}(X, Y), \partial); \mathbb{Q}).
\]

We apply these functors to the triplets above in the most straightforward way. If \(M' = (M, i, j)\) is a morphism in \(\mathcal{N}\) then \(\Gamma(M', \partial) = (\Gamma(M, \partial), i, j)\) and gluing of triples in \(\mathcal{N}\) as defined above induces a composition.

Specifically, if \(A' = (A, i, j) \in \text{Hom}_\mathcal{N}(X, Y)\), \(B' = (B, l, m) \in \text{Hom}_\mathcal{N}(Y, Z)\) then given \((\phi, i, j) \in \Gamma(A', \partial)\) and \((\psi, l, m) \in \Gamma(B', \partial)\), by requiring that group elements fix a neighborhood of the boundary it follows that there exists a map \(\psi \# \phi : A \# B \to A \# B\) induced by \((\psi, \phi) : A \coprod B \to A \coprod B\) so that \((\psi \# \phi, i, m)\) is a morphism in \(\text{Hom}_{\Gamma(\mathcal{N}, \partial)}(X, Z)\). The local nature of the gluing implies associativity of the composition.

If \(\Gamma(M', \partial) \in \text{Hom}_{\Gamma(\mathcal{N}, \partial)}(X, Y)\) then we say that \(g \in \Gamma(M', \partial) = (\Gamma(M, \partial), i, j)\) when \(g \in \Gamma(M, \partial)\). Such elements form a group and so the functor \(B\) can be applied to \(\text{Hom}_{\Gamma(\mathcal{N}, \partial)}(X, Y)\). We apply \(C_*(-; \mathbb{Q})\) to these classifying spaces. As discussed in section 2.3 both \(B\) and \(C_*(-; \mathbb{Q})\) are monoidal.

Notice that the category \(\mathcal{N}\) can be recovered as \(H_0(\mathcal{M}; \mathbb{Q})\). We may think of \(\mathcal{M}\) as a choice of chain level representative for \(\mathcal{N}\). Better terminology might be level 0 differential graded cobordisms.

**Definition 3.3.** (TFT) A 3-dimensional topological field theory is an h-split left \(\mathcal{M}\) module.

### 3.4. Open, Open-Closed and Closed Subcategories.

The category \(\mathcal{M}\) appears to be a very complicated object. We will leverage the relationship between several much simpler subcategories of \(\mathcal{M}\): the open category \(\mathcal{O}\), the open-closed category \(\mathcal{OC}\), and the closed category \(\mathcal{C}\).

Let \(\mathcal{S}\) be a collection of compact oriented 3-manifolds with boundary. If \(\langle \mathcal{S} \rangle\) is the subcategory of \(\mathcal{N}\) generated by \(\mathcal{S}\) then a subcategory \(\langle \langle \mathcal{S} \rangle \rangle\) of \(\mathcal{M}\) is generated by \(\mathcal{S}\) when \(\langle \langle \mathcal{S} \rangle \rangle\) is \(C_*(B\Gamma(\langle \langle \mathcal{S} \rangle \rangle, \partial); \mathbb{Q})\).

The categories below will use doubled handle bodies with sphere and torus boundary as generating manifolds. Let,

\[
M_{(g, e, t)} = \#^g S^1 \times S^2 \#^e D^3 \#^t S^1 \times D^2
\]

be the connected sum of \(g\) copies of \(S^1 \times S^2\), \(e\) copies of \(D^3\) and \(t\) copies of \(S^1 \times D^2\). Notice that each \(D^3\) summand introduces a boundary 2-sphere and each \(S^1 \times D^2\)
introduces a boundary torus. The boundary of \( M_{(g,e,t)} \) consists of \( e \) 2-sphere and \( t \) tori.

We will adopt the following vector subscript notation for the remainder of the paper.

**Notation.** \((M_v)\) We write \( M_v \) where \( v = (g, i + j, n + m) \) for a manifold \( M \) of genus \( g \) with labelled boundary consisting of \( i \) incoming spheres, \( n \) incoming tori, \( j \) outgoing spheres and \( m \) outgoing tori. An operation \( \# \) is defined on composable subscripts by gluing the outgoing boundary of \( M_v \) to the incoming boundary of \( M_w \).

\[ M_v \# M_w \cong M_v \# M_w \]

**Definition 3.5.** \((\mathcal{OC})\) The open-closed category \( \mathcal{OC} \) is the subcategory \( \langle \langle \mathcal{S} \rangle \rangle \subset \mathcal{M} \) generated by \( \mathcal{S} = \{M_{(g,e,t)}\} \) such that there is always incoming and outgoing boundary. If \( t = 0 \) then \( e \geq 2 \) and if \( e = 0 \) then \( t \geq 2 \). In particular, when \( t \neq 0 \), we require that there is always an incoming torus. The set \( \mathcal{S} \) is closed under composition.

The open and closed categories are subcategories of the open-closed category.

**Definition 3.6.** \((\mathcal{O} \text{ and } \mathcal{C})\) The open category \( \mathcal{O} \) is defined to be the subcategory of \( \mathcal{OC} \) whose objects are spheres and whose morphisms are generated by the spaces \( M_v \) where \( v = (g, i + j, 0) \). Similarly, the closed category \( \mathcal{C} \) is the subcategory of \( \mathcal{OC} \) whose objects are tori and whose morphisms are generated by the spaces \( M_v \) where \( v = (g, 0, n + m) \), (note \( n \geq 1 \)).

In each case, the composition is induced from gluing along boundaries and identity morphisms are added as above.

**Definition 3.7.** \((\text{open, open-closed, closed TFT})\) An open-closed topological field theory is an \( h \)-split left \( \mathcal{OC} \) module. An open topological field theory is an \( h \)-split left \( \mathcal{O} \) module. A closed topological field theory is an \( h \)-split left \( \mathcal{C} \) module.

### 4. Outer Spaces

In this section we will use the work of Hatcher, Vogtmann and Wahl on spaces of graphs to reduce the categories \( \mathcal{O} \) and \( \mathcal{OC} \) to combinatorial objects. In section 4.6, we show that mapping class groups of the doubled handlebodies \( M_v \) appearing in section 3.4 are rationally equivalent to certain groups associated to graphs. In section 4.12, we construct “Outer Spaces” (see [6, 17]) which model the rational homotopy type of the classifying spaces of these groups. The associated group homology has been studied by Hatcher and Vogtmann ([14]) and is computed by the forested graph complex. In section 4.23, we show that this complex is generated by a version of the \( C_\infty \) operad.

With the idea of “classical degeneration” in mind, it might be more natural to consider the cobordism category of abstract tropical curves \([7, 26]\). What follows is
An extension of tropicalization to families of curves would allow one to define a tropical analogue of conformal field theory and a restriction functor from conformal field theories to tropical conformal field theories. Working with chain complexes would remove the strict dependence of such theories on the underlying geometry. Work on tropical analogues of Gromov-Witten theories, enumerative tropical geometry, suggests the existence of topological tropical field theories. The references above suggest that aspects of the material developed in this paper may coincide with such a theory. The relationship between cyclic $C_\infty$ algebras and the rational homotopy theory of manifolds could allow one to compare tropical degenerations of topological conformal field theories with constructions in manifold theory.

4.1. Homotopy Equivalence Groups. We will now use a construction of Hatcher and Wahl [16] to show that the mapping class group of morphisms in the open, open-closed and closed categories can be identified with automorphism groups of graphs.

A boundary torus or balloon is the geometric graph formed from two edges with both ends of one edge glued to one end of the other. Define the graph $G_v$ to be the geometric graph consisting of a wedge of $g$ circles with $e$ edges and $t$ boundary tori glued to the one base vertex along the ends of their free edges.

The base vertex $x$ of $G_v$ is the 0-cell onto which the first edge is attached. Let $\text{Htpy}(G_v, \partial)$ be the space self-homotopy equivalences of $G_v$ which,

1. fix the $e$ edges pointwise,
2. fix the $t$ loops of the boundary tori pointwise and
3. do not identify the base vertices of any two boundary tori.

Definition 4.2. ($H_v$) Let $H_v = \pi_0 \text{Htpy}(G_v, \partial)$ be the group of path components of the space of self-homotopy equivalences described above.

When we write $v$ as $(g, i + o, a + b)$ we mean that the number of incoming edges $i = |\text{In}(G)|$, outgoing edges $o = |\text{Out}(G)|$, incoming tori $a = |\text{Tin}(G)|$ and outgoing tori $b = |\text{Tout}(G)|$. If $[n]$ is the set $\{1, \ldots, n\}$ then a boundary labelling is a choice of homeomorphisms, $i_H : [|\text{In}(G)|] \times [0, 1] \to \text{In}(G)$ and $o_H : [|\text{Out}(G)|] \times [0, 1] \to \text{Out}(G)$. So that the interval $i \times [0, 1]$ is mapped homeomorphically onto the $i$th incoming or outgoing edge and $i \times 0$ sent to the boundary vertex. For the tori we use the maps,

$$a_H : [|\text{Tin}(G)|] \times [0, 2\pi) \to \text{Tin}(G) \quad \text{and} \quad b_H : [|\text{Tout}(G)|] \times [0, 2\pi) \to \text{Tout}(G),$$
and we require that the points \( a_H(i,0) \) and \( b_H(i,0) \) are the base vertices of the boundary torus. Compare to section 2.7.1.

**Definition 4.3.** \((\mathcal{OCH})\) There is a symmetric monoidal category \(\mathcal{OCH}\) enriched over \(\text{Group}\) with objects generated by the elements \(e\) and \(t\). The object \(e^\otimes n\) represents \(n\) labelled edges and the object \(t^\otimes k\) represents \(k\) boundary tori. The morphisms of \(\mathcal{OCH}\) are self-homotopy equivalences of boundary labelled graphs fixing boundary elements:

\[
\text{Hom}_{\mathcal{OCH}}(e^\otimes i \otimes t^\otimes j, e^\otimes k \otimes t^\otimes l) = \coprod_g H_{(g,i+k,j+l)}.
\]

There are no morphisms between empty objects and we require \(j \geq 1\) when \(l \geq 0\). The composition of \([\varphi] \in H_v\) and \([\psi] \in H_w\) is given by choosing maps \(\varphi : G_v \to G_v\) and \(\psi : G_w \to G_w\) which preserve the boundary labelling in \([\varphi]\) and \([\psi]\) respectively.

The illustration above depicts three graphs: \(G_v, G_w\) and \(G_v \# w\). The graph \(G_v \# w\) is formed by gluing together the graphs \(G_v\) and \(G_w\). The dark lines represent boundary edges which are not involved in the gluing. Using the standard embeddings found in section 4.6, the picture above can be seen to correspond to the gluing of manifolds \(M_v\).

Two homotopy equivalences can be glued to give an equivalence \(\varphi \# \psi : G_v \# w \to G_v \# w\). For any continuous variation of \(\varphi\) or \(\psi\) within their respective path components, the graph \(\varphi \# \psi\) varies continuously within the corresponding path component of \(\text{Htpy}(G_v \# w, \partial)\). The map \((\varphi, \psi) \mapsto \varphi \# \psi\) yields a composition law,

\[
H_v \times H_w \to H_{v \# w}.
\]

This determines the composition law for the category \(\mathcal{OCH}\).

**Definition 4.4.** \((\mathcal{OH})\) The open homotopy category, \(\mathcal{OH}\), is the subcategory of \(\mathcal{OCH}\) associated to graphs without tori.

It follows from the discussion in section 2.3 that there is a monoidal category \(\mathcal{BOCH}\) enriched over \(\text{Top}\). This category has the same objects and its morphism spaces are equal to classifying spaces of the groups defined above. Applying the functor \(C_*(\cdot; \mathbb{Q})\) yields a differential graded category.
Definition 4.5. \((\mathcal{OG}, \mathcal{OCG})\) The open graph category \(\mathcal{OG}\) and the open-closed graph category \(\mathcal{OCG}\) are the categories of rational chains on the classifying categories of the open and open-closed homotopy categories.

\[
\mathcal{OG} = C_*(B\mathcal{OH}; \mathbb{Q}) \quad \text{and} \quad \mathcal{OCG} = C_*(B\mathcal{OCH}; \mathbb{Q})
\]

4.6. A Theorem of Hatcher, Vogtmann and Wahl. The theorem below appears in the papers of Hatcher, Vogtmann and Wahl. It stems from Hatcher’s work on the homotopy type of the diffeomorphism group of \(S^1 \times S^2\) and Vogtmann’s study of Outer Space. The synthesis of these ideas has recently led to homological stability results for 3-manifolds.

The mapping class groups in our construction will differ from those considered in the references above by requiring that group elements fix a regular neighborhood of the boundary (see section 3.4). As such they will be subgroups \(\Gamma(M_v, \partial) \subset \Gamma(M_v)\) generated by the same generators given by Wahl and Jensen minus those which require Dehn twists of the boundary torus. Differences will be noted along the way.

Definition 4.7. \((\Gamma_v)\) The group \(\Gamma_v = \Gamma(M_v, \partial)\) is the mapping class group of the space \(M_v\) considered in section 3.4.

Since \(\pi_1(SO(3)) \cong \mathbb{Z}/2\), the inclusion \(SO(3) \hookrightarrow \text{Diff}(S^2)\) yields a 1-parameter family of diffeomorphisms \(\varphi : S^2 \times I \to S^2\) such that one composition along the second parameter is homotopic to identity. A Dehn twist along a 2-sphere in a 3-manifold is obtained by deleting a regular neighborhood of the sphere and gluing the two boundary components back together along a copy of \(S^2 \times I\) using the map \(\varphi\).

We fix a standard embedding, \(i : G_v \hookrightarrow M_v\), by mapping the end of each boundary edge \(e\) to a boundary sphere, we require each boundary torus of the graph to map to the loop on the longitude of the boundary torus of \(M_v\) and each of the \(g\) loops to be sent to the \(S^1\) component of the corresponding \(S^1 \times S^2\) term. The inclusion \(i\) induces an isomorphism on fundamental groups. Let \(r : M_v \to G_v\) be the retraction onto \(i(G_v)\) in \(M_v\). These maps are canonical up to isotopy, with respect to the decomposition of \(M_v\) into punctured handle bodies.

\[
G_{(2,2,1)} \hookrightarrow M_{(2,2,1)}
\]

The illustration above consists of a graph \(G_{(2,2,1)}\) embedded inside of a 3-manifold \(M_{(2,2,1)}\). The two internal loops inside of this graph travel around the \(S^1 \times S^2\)
summands in the center of the picture. The two boundary edges of the graph are attached to the two boundary 2-spheres and the one boundary torus loop of the graph is attached around the torus component of the 3-manifold.

If \( l \in \text{Diff}(M, \partial) \) is a diffeomorphism, then we obtain a homotopy equivalence,
\[
h(l) = r \circ l \circ i.
\]

This defines a map \( h : \Gamma_v \to H_v \). The key point for us is that \( h \) is a rational isomorphism, see corollary \([4.9]\).

**Theorem 4.8.** *(Hatcher-Vogtmann-Wahl)* The map \( h : \Gamma_v \to H_v \) is an epimorphism and its kernel is isomorphic to a finite direct sum of \( \mathbb{Z}/2 \)'s generated by Dehn twists along spheres.

\[
1 \longrightarrow \bigoplus_k \mathbb{Z}/2 \longrightarrow \Gamma_v \xrightarrow{h} H_v \longrightarrow 1
\]

**Proof.** The reader may compare what follows to theorem 1.1 in \([16]\). In their work Hatcher, Vogtmann and Wahl allow the mapping class groups above to move the boundary while we do not. In our discussion of the difference, we will simplify matters slightly by only discussing the tori. If the number of edges is equal to zero \((e = 0)\) then the full group of graph automorphisms is generated by:

1. \( P_{i,j} \) exchanges \( x_i \) and \( x_j \)
2. \( I_i \) exchanges \( x_i \) and \( x_i^{-1} \)
3. \( (x_i; x_j) \) \( x_i \to x_i x_j \)
4. \( (x_i; y_j) \) \( x_i \to x_i y_j \)
5. \( (x_i^{-1}; y_j) \) \( x_i \to y_j^{-1} x_i \)
6. \( (y_i^\pm; x_j) \) \( y_i \to x_j^{-1} y_i x_j \)
7. \( (y_i^\pm; y_j) \) \( y_i \to y_j^{-1} y_i y_j \).

The \( x_i \) represent generators of \( \pi_1(G_{(g,0,t)}) \) associated to factors of \( S^1 \times S^2 \) and \( y_i \) represent generators of \( \pi_1(G_{(g,0,t)}) \) associated to factors of \( S^1 \times D^2 \).

If we view our 3-manifold as the boundary of a punctured handle body then generators 3-7 above can be represented by handle slides along the curves \( x_i \) and \( y_j \). Handle slides are associated to generators of the automorphism group as follows.

3. The handle \( x_i \) slides over \( x_j \).
4. The handle \( x_i \) slides over \( y_j \).
5. The handle \( x_i^{-1} \) slides over \( y_j \).
6. The torus \( y_i \) slides over the handle \( x_j \).
7. The torus \( y_i \) slides over the torus \( y_j \).

In order to slide a handle or a torus (thought of as a connected sum of \( S^1 \times D^2 \)'s) over a torus, a Dehn twist must be performed. Fixing the boundary kills generators 4, 5 and 7. Since our homotopy groups are defined to fix the loop of the graph contained in the torus, the correspondence is preserved.

\( \square \)

**Corollary 4.9.** The chain complexes \( C_*(B\Gamma_v; \mathbb{Q}) \) and \( C_*(BH_v; \mathbb{Q}) \) are quasi-isomorphic.
Proof. The map $Bh$ induces an equivalence because $B(\mathbb{Z}/2) \simeq \mathbb{R}P^\infty$ and $\mathbb{R}P^\infty$ is rationally contractible.

□

The corollary above implies that the space of morphisms in the categories $\mathcal{O}$ and $\mathcal{OC}$ (section 3.4) are rationally quasi-isomorphic to those of $\mathcal{OG}$ and $\mathcal{OCG}$ respectively (section 4.5). The theorem below follows from the observation that the map inducing this equivalence is compatible with the gluing of open boundaries.

**Theorem 4.10.** The open category $\mathcal{O}$ of section 3.4 is quasi-isomorphic to the open graph category $\mathcal{OG}$, see definition 4.5.

$$\mathcal{O} \cong \mathcal{OG}$$

Proof. The map $h$ as defined above is compatible with gluing the spherical boundary components,

$$\Gamma_v \times \Gamma_w \# \Gamma_{v\#w}$$

(see notation section 3.4). Given $\varphi \in \text{Diff}(M_v, \partial)$ and $\psi \in \text{Diff}(M_w, \partial)$, the action of $\varphi \# \psi$ on $i(G_v) \# i(G_w) \subset M_{v\#w} = M_v \# M_w$ is the same as the action of $\varphi$ on $i(G_v)$ glued to the incoming edges of $\psi$ acting on $i(G_w)$. This is because $\varphi$ and $\psi$ are required to fix a regular neighborhood of the boundary.

The maps $h$ induce a functor $\mathcal{O} \to \mathcal{OG}$. One can choose sections of $h$, $H_v \to \Gamma_v$, so that there is a functor $i : \mathcal{OG} \to \mathcal{O}$. We have $h \circ i = 1$ and $i \circ h \simeq_\mathbb{Q} 1$. □

Recall the notion of the category $\text{Ob}(\mathcal{D})$ associated to a monoidal category $\mathcal{D}$ (see definition 2.2 section 2.1). The category $\mathcal{OC}$ defines an $\text{Ob}(\mathcal{OC}) - \mathcal{O}$ bimodule,

$$\mathcal{OC} : \text{Ob}(\mathcal{OC}) \otimes \mathcal{O}^{\text{op}} \to \text{Kom}$$

via $(e^{\otimes n} \otimes t^{\otimes m}) \otimes o^{\otimes k} \mapsto \text{Hom}(o^{\otimes k}, e^{\otimes n} \otimes t^{\otimes m})$, see also the observation in section 2.4.

The category $\mathcal{OCG}$ (definition 4.5) defines an $\text{Ob}(\mathcal{OCG}) - \mathcal{OG}^{\text{op}}$ bimodule in the same way. In fact, the category $\mathcal{OCG}$ also defines an $\text{Ob}(\mathcal{OC}) - \mathcal{O}$ bimodule because $\text{Ob}(\mathcal{OCG}) = \text{Ob}(\mathcal{OC})$ and theorem 4.10 above implies that $\mathcal{O} \cong \mathcal{OG}$.

We have two $\text{Ob}(\mathcal{OC}) - \mathcal{O}$ bimodules: $\mathcal{OC}$ and $\mathcal{OG}$. Corollary 4.9 shows that these two bimodules are the same.

**Theorem 4.11.** As $\text{Ob}(\mathcal{OC}) - \mathcal{O}^{\text{op}}$ bimodules the categories $\mathcal{OC}$ and $\mathcal{OCG}$ are quasi-isomorphic.
4.12. **Outer Space.** For each $v = (g,e,t)$, we begin by defining a set $L_v$ consisting of labelled graphs. This set will be used to construct a simplicial set $NL_v$. The geometric realization $|NL_v|$ of $NL_v$ will be a classifying space for the group $H_v$. In what follows, all graphs will be boundary labelled and we will consistently write $L_v$ where $v = (g,i + o,a + b)$, see section 4.1.

A graph $G$ is **labelled** when it is paired with a map $ϕ : G_v → G$ which satisfies the following properties.

1. The function $ϕ$ preserves the incoming and outgoing edges and identifies the ends of each of the boundary tori of $G_v$ with circles $G$. By circle we mean cycles with one edge and one vertex.
2. If $x$ is the vertex of $G_v$ then the induced map, $ϕ_* : π_1(G_v, x) → π_1(G, ϕ(x))$ is an isomorphism.

Two labelled graphs $(G, ϕ)$ and $(G', ψ)$ are **equivalent** if there is a graph isomorphism $ρ : G → G'$ so that the diagram below commutes.

$$
\begin{align*}
\pi_1(G, ϕ(x)) & \xrightarrow{ρ_*} \pi_1(G', ψ(x)) \\
\pi_1(G_v, x) & \xrightarrow{ϕ_*} \pi_1(G_v, ψ(x)) \\
\pi_1(G_v, x) & \xrightarrow{ψ_*}
\end{align*}
$$

**Definition 4.13.** $(L_v)$ If $v = (g,e,t)$ then $L_v$ will denote the set of equivalence classes $(G, G_v, ϕ) → G$ of labelled graphs.

The set $L_v$ can be endowed with a simplicial structure in which the faces of simplices are determined by edge collapses (see section 2.7.1). In what follows, we will use the nerve $NL_v$ of $L_v$. A non-degenerate $n$-simplex in $NL_v$ is given by a sequence

$$(G_0, ϕ_0) ⊂ (G_1, ϕ_1) ⊂ \cdots ⊂ (G_n, ϕ_n)$$

where $(G_i, ϕ_i) ∈ L_v$ for each $i = 0, 1, \ldots, n$ and $(G_i, ϕ_i)$ is obtained from $(G_{i+1}, ϕ_{i+1})$ by collapsing one or more edges (while preserving the homotopy type). Equivalently, simplices of the space $NL_v$ are determined by fixing a forest $F_0 ⊂ G$, and a nested sequence of subforests, $F_n ⊂ F_{n-1} ⊂ \cdots ⊂ F_0 ⊂ G$. If $ϕ$ is a labelling of $G = G_n$ then this gives the simplex,

$$(G/F_0, ϕ_0) ⊂ (G/F_1, ϕ_1) ⊂ \cdots ⊂ (G_n/F_n, ϕ_n).$$

The maps, $ϕ_i$, are induced by collapsing edges. In what follows we will require all forests $F ⊂ G$ to

1. include all of the vertices of $G$,
2. include *none* of the incoming or outgoing open boundary edges and
Simplicial face maps are defined by combining collapses and simplicial degeneracy maps are given by inserting identity collapses.

The group $H_v$ acts on the nerve $NL_v$ by changing the labellings. If $f \in H_v$ then $f : L_v \to L_v$ is defined by $f(G, \phi) = (G, \phi \circ f)$ and so $f : NL_v \to NL_v$ acts by

$$(G/F_0, \phi_0) \subset (G/F_1, \phi_1) \subset \cdots \subset (G/F_n, \phi_n)$$

$$\mapsto (G/F_0, \phi_0 \circ f) \subset (G/F_1, \phi_1 \circ f) \subset \cdots \subset (G/F_n, \phi_n \circ f).$$

**Definition 4.14.** $(NL_v, |NL_v|, X_v)$ The geometric realization of $NL_v$ will be denoted by $|NL_v|$ and the quotient $|NL_v|/H_v$ will be denoted by either $X_v$ or $BH_v$, see theorem 4.15 below.

Suppose that $v = (g, e, t)$, if $t = 0$ and $e = 0$ then $X_v$ is called Outer space since the construction is a model for the classifying space of the group of outer automorphisms of the free group $F_g$, see [6]. If $t = 0$ and $e = 1$ then $X_v$ is known as “Aueter space.” Other generalizations, not involving diffeomorphisms that fix the boundary, can be found in [15, 17, 16].

**Theorem 4.15.** $(|NL_v|/H_v$ models $BH_v)$ The action of $H_v = \pi_0 \text{Htpy}(G_v, \partial)$ on the space $|NL_v|$ is properly discontinuous and the stabilizer of any given simplex is a finite group. Moreover, the space $|NL_v|$ is contractible.

**Proof.** The action of $H_v$ is almost free. If $f \in H_v$ then $f(G, \phi) = (G, f \circ \phi) = (G, \phi)$ if and only if $f$ is an isomorphism of the graph $G$. A graph isomorphism is determined by the manner in which it permutes the edges and so the size of the group of graph isomorphisms is bounded above by the group of all permutations on edges.

The proof of contractibility of $|NL_v|$ is a special case of the proof which appears in Wahl and Jensen’s article [17]. □

**Corollary 4.16.** The quotient space $X_v = |NL_v|/H_v$ is a rational model for the classifying space of $H_v$. In particular,

$$C_\ast(BH_v; \mathbb{Q}) \simeq C_\ast(X_v; \mathbb{Q}).$$

There is a geometric interpretation of the space $X_v$. A **metric graph** is a graph together with a fixed length $l(e) \geq 0$ assigned to each internal edge. A metric graph is **balanced** if $\sum_{e \in E(G)} l(e) = 1$. The space $X_v$ is a subdivision of the space of balanced metric graphs homotopy equivalent to the graph $G_v$. For any balanced metric graph $G$, if $e_0, \ldots, e_k$ are its edges then $G$ is uniquely represented by the barycentric coordinates $(l(e_0), \ldots, l(e_k))$ of a $k$ simplex $\Delta$ associated to the topological type of $G$.

The boundary tori are represented by balloons attached to the graphs representing points in the moduli space $X_v$. The length of the edge at the end of each balloon is
fixed. The length of the edge used to attach the balloon to the rest of the graph is allowed to vary and may approach zero providing two distinct base vertices do not touch as a result.

We metrize the graphs in this way because the edge of the balloon corresponding to a torus in a manifold $M_v$ is completely fixed by the action of any $b \in \Gamma(M_v, \partial)$. The edge about the torus in the graph $G_v$, thought of as embedded in $M_v$, does not vary with respect to the action of the mapping class group. The edge that is used to attach the balloon to the rest of the graph is allowed to vary because $b$ may move the boundary torus about inside of $M_v$. Since there are disjoint regular neighborhoods of the boundary tori in the construction of the cobordism category, we can ask for the base vertices of the balloons representing them not to touch.

In contrast, the open edges are given fixed length. When represented as a graph within $M_v$, this length reflects the disjointness of the regular neighborhoods of 2-spheres in the construction of the cobordism category. Allowing these lengths to vary is not necessary and would not add anything to what follows. If we allowed the lengths to vary then it would be necessary for us to consider the scenario in which the collapse of an edge represented a boundary collision as we have done with the tori above.

4.17. **Cellular Stratification by Cubes.** In order to compute the homology of $X_v$, we group simplices that can be obtained from the same forest into a single cell (see [14, 20, 4]). The cells obtained from this construction will be called cubes.

A cube $[G,F,\varphi] \subset |NL_v|$ is obtained by gluing together all the simplices arising from different filtrations of some fixed forest $F \subset G$.

$$[G,F,\varphi] = \coprod_{F_0 \subset \cdots \subset F_m \subset F} (G/F_0 \subset \cdots \subset G/F_{m-1} \subset G/F_m) \times \Delta^m$$

The collection of all cubes $[G,F,\varphi]$ gives $|NL_v|$ the structure of a CW complex called the *forested graph* stratification. Each cube $[G,F,\varphi]$ in $|NL_v|$ is homeomorphic to a $k$-cube $[0,1]^k$, where $k = |E(F)|$. One can define such a homeomorphism by assigning an axis to each edge.

If the graphs $G$ are planar trees then a construction analogous to the one in section 4.12 produces simplicial subdivisions of associahedra. The cubical stratification above yields the cubical decomposition of associahedra in this context, see [2].

The codimension 1 faces of a cube $[G,F,\varphi]$ are given by two operations on graphs.

1. Collapsing an edge. $[G,F,\varphi] \mapsto [G/e,F/e,\varphi]$ for some edge $e \in E(F)$.
2. Removing an edge from the forest. $[G,F,\varphi] \mapsto [G,F-e,\varphi]$ for some edge $e \in E(F)$. 
The two types of faces, (1) and (2), are illustrated in the figure. The group $H_v$ now acts cellularly. The stabilizer of the cube $[G,F,\varphi]$ consists of automorphisms of $G$ that send the forest $F \subset G$ to itself.

Each cube $[G,F,\varphi]$ in $|NL_v|$ descends to a cube $[G,F]$ in the quotient $X_v$. This cube is not necessarily a cell, but an orbi-cell. This follows from identifying the cube in $|NL_v|$ with a cube $C = [0,1]^k$ where each edge of $F$ is associated to an axis. The portion of the cube that descends to $X_v$ is the quotient of $C$ by the stabilizer $\text{Aut}(G,F,\varphi)$. The action of $\text{Aut}(G,F,\varphi)$ on $C$ fixes the origin and permutes the axes so that $C/\text{Aut}(G,F,\varphi)$ is a cone on the quotient of the boundary $\partial C$.

Lemma 4.18. The quotient of an $n$-sphere by a finite linear group $G \subset GL_n(\mathbb{R})$ is $\mathbb{Q}$-homotopic to either an $n$-sphere or an $n$-ball. The latter case holds only when the action includes reflections.

For proof and discussion, see [14]. Those cubes which have symmetries that do not include reflections survive to the quotient.

In $X_v$ the tori are represented by trees containing the base vertex of the balloons.

4.19. Homology. In this section we complete our description of the morphism spaces of $\mathcal{OC}$ and $\mathcal{O}$. For each $v = (g,e,t)$, we define a generalized Cobar construction: an exact functor $\mathcal{G}_v$ from the category of differential graded cooperads to chain complexes. The complexes $\mathcal{G}_v$ will be those that generate the morphism spaces of the enveloping functor $\text{Cobar}(\mathcal{O})^\flat$ defined in 2.8. We will show that $\mathcal{G}_v$ corresponds to the chain complex obtained from the stratification of $X_v$ by cubes defined in the previous section.

4.19.1. From Operads to Graph Complexes. A bonnet is a graph $B(n)$ isomorphic to a corolla with two edges identified.
Let $S_v$ be the set of boundary labelled combinatorial graphs of genus $g + t$ with $e$ boundary edges and $t$ bonnets. A graph $G \in S_v$ differs from $G_v$, pictured in section 4.1, in that $G$ is allowed to possess internal edges of any kind. Given a cyclic dg cooperad $\mathcal{P}$, the generalized Cobar construction, $\mathcal{G}_v(\mathcal{P})$, is the complex consisting of $S_v$ graphs labelled by $\mathcal{P}$ and oriented using the convention described in section 2.7.2.

**Definition 4.20.** ($\mathcal{G}_v(\mathcal{P})$)

$$\mathcal{G}_v(\mathcal{P}) = \bigoplus_{G \in S_v} \mathcal{P}(G) \otimes \det(G)^*$$

The differential $\delta$ expands edges. It can be described by its matrix elements, $(\delta)_{G',G}$, where $G, G' \in S_v$. If $G'$ is not isomorphic to $G/e$ for some collapsible edge $e \in G$ then set $(\delta)_{G',G} = 0$. Otherwise, let $c : G \to G' \cong G/e$ so that if $c^* : \mathcal{P}(G') \to \mathcal{P}(G)$ is the induced map on the labelling then $\delta$ is given by $(\delta)_{G',G} = c^* \otimes p^e$ where $p^e$ is the map induced on the orientation by collapsing the edge. If the cooperad $\mathcal{P}$ has a non-trivial differential then the total differential is the sum of the differential defined above together with the original differential.

The generalized Cobar construction is introduced in order to mediate between the algebraic world of operads and categories, and the topological world obtained the from moduli spaces defined in earlier sections. In particular, the collection $\{\mathcal{G}_v\}$ naturally models the morphisms of the open and open-closed categories introduced in section 3.4. By construction, we have the following identifications,

$$\mathcal{G}_{(0,e,0)}(\mathcal{P}) = \text{Cobar}(\mathcal{P})(e) \quad \text{and} \quad \text{Hom}_{\text{Cobar}(\mathcal{P})}(x^\otimes n, x^\otimes m) = \bigoplus_g \mathcal{G}_{(g,n+m,0)}(\mathcal{P}).$$

**Remark 4.21.** We can use this observation to relate the $\mathcal{G}_v$ to modular operads. In particular, when $t = 0$ the collection $\{\mathcal{G}_v\}$ determine a PROP, Cobar$(\mathcal{P})^\flat$, see section 2.8. By remark 2.12, the PROP Cobar$(\mathcal{P})^\flat$ agrees with $P \mathcal{M} \text{Cobar}(\mathcal{P})$. On the other hand, the Feynman transform (see [10]) of the modular operad associated to a cyclic operad commutes with the Cobar construction (with appropriate twisting), $\mathcal{M} \text{Cobar}(\mathcal{O}^\vee) \cong \text{FMO}$. This yields a relationship between the generalized Cobar construction and the Feynman transform, $\bigoplus_{t=0} \mathcal{G}_v(\mathcal{P}) \cong P \mathcal{M}(\mathcal{P}^\vee)$. 
If $t > 0$ then it is best to think of the collection $\{G_v\}$ as describing the extension of the $t = 0$ case by data coming from the torus boundary; a dg module over the open category. This comment is made more precise in section 6.

**Lemma 4.22.** The functor $G_v$ is exact: if $\varphi : \mathcal{P} \to \mathcal{P}'$ is a quasi-isomorphism of cooperads then the induced map $G_v(\mathcal{P}) \to G_v(\mathcal{P}')$ is a quasi-isomorphism.

This is proven using a spectral sequence argument, see [10] Theorem 5.2 (3).

4.22.1. *Cubical Chains Compute A Double Dual.* Recall from section 4.17 that the complex of cubical chains on $X_v$ is spanned by cubes $[G, F]$ where $G$ is a boundary labelled graph with $t$ cycles representing boundary tori and $F \subset G$ is a forest containing all of the vertices of $G$ and none of the boundary edges. No two vertices of the boundary tori are contained in the same tree of $F$.

The cube $[G, F]$ is oriented by an ordering of the edges of $F$. Lemma 4.18 in the same section implies that the antisymmetry relation $[G, -F] = -[G, F]$ holds.

The differential is given by the sum over ways to remove an edge from a forest and the sum over ways to contract an edge contained in the forest. In either case the cube is oriented by the induced orientation.

$$\partial [G, F] = \sum_{e \in F} [G/e, F/e] + \sum_{e \in F} [G, F - e]$$

Recall that $C$ is the commutative operad defined in section 2.6.2. The cooperad $\text{Bar}(C)$ is the free cooperad on $n$-corolla satisfying the antisymmetry relation (dual to the $L_\infty$ operad). The trees are edge oriented and the differential contracts edges.

Since $\text{Cobar}(\text{Bar}(C))$ is a double complex, while $C_*(X_v)$ is merely a chain complex, we flatten the double grading as follows,

$$\text{Cobar}(\text{Bar}(C))(n)' = \bigoplus_j \text{Cobar}(\text{Bar}(C))(n)_{j,i}.'$$

The differential $d$ remains the sum of the internal differential, which contracts the edges of $\text{Bar}(C)$, and the external differential, which expands compositions.

**Theorem 4.23.** The rational homology of the spaces $X_v$ is computed by $G_v(\text{Bar}(C))$:

$$C^\text{cell}_*(X_v; \mathbb{Q}) \cong G_v(\text{Bar}(C))'.$$

**Proof.** Assuming $t = 0$, by lemma 2.11 it suffices to show that the operad $\text{Cobar}(\text{Bar}(C))$ is isomorphic to the operad with $O(n) = C_*(X_{(0, n+1, 0)}; \mathbb{Q})$. This forms an operad because the cellular composition, theorem 5.1, is independent of this theorem. We will see that as complexes the two are plainly isomorphic:

$$\text{Cobar}(\text{Bar}(C))(n)' \cong C_*(X_{(0, n+1, 0)}; \mathbb{Q}).$$
In degree $j$, the complex $C^e_{cell}(X_{(0,n+1,0)}; \mathbb{Q})$ is spanned by forested trees, $(T, F)$, where the forest $F$ contains $j$ edges and a connected component associated to each internal vertex of $T$.

In bidegree $(j, i)$, the complex $\text{Cobar}(\text{Bar}(C))(n)_{j,i}$ is spanned by unrooted $n$ trees $T$, containing $j = |T|$ internal vertices each of which is labelled by a tree $F_i \in \text{Bar}(C)(H(v))$. The equation $(j, i) = (|T|, \sum_{m=1}^{|T|}(|F_m| - 1))$ holds for bidegrees. Since the second coordinate is the total number of internal edges, $T \otimes F_1 \otimes \cdots \otimes F_j \in \text{Cobar}(\text{Bar}(C))(n)'$, when $T$ is an unrooted $n$ tree labelled by trees $F_i$ whose internal edges total to $i$.

To a forested tree $[T, F]$ with $F = F_1 \cup \cdots \cup F_j$, we associate the tree with internal vertices labelled by the $F_i$. The inverse map is obtained by doing the opposite: inserting forests at vertices.

The two differentials in either complex are the same. Collapsing an edge in a forest corresponds to contracting an edge in a $\text{Bar}(C)$ labelling. Removing an edge in a forest corresponds to inserting an edge in $G_v$ between two $\text{Bar}(C)$ labellings; this is the Cobar differential. See figure 1 in section 4.17.

The two orientation conventions agree. A forested graph $[T, F]$ is oriented by an ordering of the edges in the forest $F$. If $F = \cup_i F_i$ then

$$\det(E(F)) = \bigotimes_i \det(E(F_i)).$$

On the other hand, if a graph $G$ is a tree $T$ with $j$ vertices labelled by forest components $F_1, \ldots, F_j$ then, the convention described in section 2.7.2 tells us that,

$$\det(T \otimes F_1 \otimes \cdots \otimes F_j) = \det(E(T)) \otimes \det(\text{Out}(T)) \bigotimes_{i=1}^j \det(E(F_i)) \otimes \det(\text{Out}(F_i)).$$

In our computation, the number of outgoing edges of $T$ is one. The internal edges of $T$ join the labellings of two separate vertices by forest components $F_i$. One end of each edge of $T$ is an incoming edge of some forest component and the other end is an outgoing edge of some forest component.

The outgoing components of each forest must correspond to internal edges of $T$ except for the one outgoing edge corresponding to the outgoing edge of $T$. Thus there is a bijection between the set $E(T) \coprod \text{Out}(T)$ and $\coprod_i \text{Out}(F_i)$. Taking graded determinants yields the isomorphism,

$$\det(E(T)) \cong \det(E(T)) \otimes \mathbb{Q} \cong \bigotimes_{i=1}^j \det(\text{Out}(F_i)).$$
It follows that $\det(T \otimes F_1 \otimes \cdots \otimes F_j) \cong \otimes_i \det(E(F_i))$ and so the signs in both differentials agree.

If the number of tori is greater than zero then the cells associated to the boundary tori are the trees containing the base vertex of the balloon associated to the torus. These are represented combinatorially by bonnets in $G_v(\text{Bar}(C))$. \hfill \Box

**Corollary 4.24.**

$$\text{Hom}_\mathcal{O}(e^\otimes i, e^\otimes j) \simeq \text{Hom}_\text{Cobar}(\text{Bar}(C))^\flat(e^\otimes i, e^\otimes j)$$

Recall the notion of the category $\text{Ob}(\mathcal{D})$ associated to a monoidal category $\mathcal{D}$ (definition 2.2 section 2.1).

**Corollary 4.25.** The $\text{Ob}(\mathcal{O}C)\mathcal{O}$ bimodule $\mathcal{O}C$ is quasi-isomorphic to the $\text{Ob}(\mathcal{O}C)\mathcal{O}$ bimodule defined by the functor,

$$(e^\otimes n \otimes t^\otimes m) \otimes o^\otimes k \mapsto \prod_g G_{(g,n+k,m)}(\text{Bar}(C)).$$

The corollary follows from the identification, $C_*(BH_v; \mathbb{Q}) \simeq C^\text{cell}_*(X_v; \mathbb{Q})$ and the previous theorem.

5. **The Open Category**

Corollary 4.24 states that morphisms of the category $\mathcal{O}$ are quasi-isomorphic to spaces of graphs. In this section we show that the composition induced from the gluing of 2-spheres in the open category is cellular. This allows us to extend corollary 4.24 from an equivalence of morphism spaces to an equivalence of categories. The combinatorial open category $\text{Cobar}(\text{Bar}(C))^\flat$ is equivalent to the open category $\mathcal{O}$.

Given two boundary labelled composable forested graphs $[G, F]$ and $[G', F']$, form the graph $G \# G'$ by gluing the relevant ends together and eliminating the resulting bivalent vertices. The forests $F$ and $F'$ together form a forest $F \cup F'$ of $G \# G'$, because forests are not permitted to contain boundary edges.

**Theorem 5.1.** The quasi-isomorphisms of 4.24 respect composition.

\[
\begin{array}{ccc}
\text{Hom}_{\mathcal{O}G}(e^\otimes i, e^\otimes j) \otimes \text{Hom}_{\mathcal{O}G}(e^\otimes j, e^\otimes k) & \xrightarrow{\circ} & \text{Hom}_{\mathcal{O}G}(e^\otimes i, e^\otimes k) \\
\varphi_{ij} \otimes \varphi_{jk} & & \varphi_{ik} \\
\text{Hom}_{\text{Cobar}(\text{Bar}(C))^\flat}(e^\otimes i, e^\otimes j) \otimes \text{Hom}_{\text{Cobar}(\text{Bar}(C))^\flat}(e^\otimes j, e^\otimes k) & \xrightarrow{\circ} & \text{Hom}_{\text{Cobar}(\text{Bar}(C))^\flat}(e^\otimes i, e^\otimes k)
\end{array}
\]
**Proof.** We show that composition respects the cube decomposition of the outer spaces. In everything to follow, whenever the subscript \( v = (g, e, t) \) is used, we will assume that \( t = 0 \).

The composition of \( \mathcal{O} \mathcal{G} \) is defined by maps,

\[
\circ : C_*(X_v; \mathbb{Q}) \otimes C_*(X_w; \mathbb{Q}) \to C_*(X_{v\#w}; \mathbb{Q}).
\]

There are \( \mathbb{Q} \)-homotopy equivalences from the space \( BH_v \) to \( X_v = |NL_v|/H_v \). These spaces are stratified by orbi-cells, \( [G, F] \), indexed by forested graphs. The dimension of \( [G, F] \) is the number of edges in \( F \). Residing above each orbi-cell is a collection of honest cells, \( [G, F, \varphi] \) in \( |NL_v| \), which are indexed in the orbit of the action of \( H_v \) by their labellings \( \varphi \), see [4.17].

Given a cell \( [G, F, \varphi] \) of dimension \( n \) in \( |NL_v| \) and a cell \( [G', F', \varphi'] \) of dimension \( m \) in \( |NL_w| \) (representing a pair of composable graphs) there is a composite \( [G\#G', F \cup F', \varphi\#\varphi'] \) of dimension \( n + m \) and a homeomorphism,

\[
[G, F, \varphi] \times [G', F', \varphi'] \to [G\#G', F \cup F', \varphi\#\varphi']
\]

defined by identifying each cell with a cube in \( \mathbb{R}^{E(F)} \) as described in [4.17]. These homeomorphisms together yield a composition,

\[
|NL_v| \times |NL_w| \to |NL_{v\#w}|
\]

which is equivariant with respect to the action of \( H_v \times H_w \) on the left and \( H_{v\#w} \) on the right, using the map \( H_v \times H_w \to H_{v\#w} \) (see definition 4.3). So there is a composition on the quotient. The composition of two cubes \( [G, F] \) and \( [G', F'] \) is determined by the diagram below.

\[
\begin{array}{ccc}
[G, F, \varphi] \times [G', F', \varphi'] & \xrightarrow{\simeq} & [G\#G', F \cup F', \varphi\#\varphi'] \\
\downarrow & & \downarrow \\
[G, F] \times [G', F'] & \longrightarrow & [G\#G', F \cup F']
\end{array}
\]

It can be seen that the differential acts as a derivation with respect to this composition law using the rule in section [4.17].

The theorem above, together with theorem [4.23], implies the following corollary.

**Corollary 5.2.** The category of h-split \( \mathcal{O} \) modules is equivalent to the category of cyclic \( \text{Cobar(Bar(C))} \) algebras. In particular, the category of h-split \( \mathcal{O} \) modules is equivalent to the category of cyclic \( C_\infty \) algebras.
It is possible to restate the result of theorems 5.1 and 4.23 in the language of cyclic operads. Let $M_n = \#^n D^3$ be the 3-manifold obtained by connect summing $n$ copies of the 3-ball, $D^3$, to itself. If we set

$$H_n = C_*(B\Gamma(M_n, \partial); \mathbb{Q})$$

then the collection $\{H_n\}$ form a cyclic dg operad $H$ quasi-isomorphic to Cobar(Bar$(C)$) where $C$ is the commutative operad. The machinery of modular operads implies the following corollary, see [10].

**Corollary 5.3.** Cyclic $C_\infty$ algebras are algebras over the modular closure of the chain operad $H$ defined above.

### 6. Extension and the Torus

Given a cyclic $C_\infty$ algebra $A$, corollary 5.2 shows that $A$ defines an open TFT in the sense of definition 3.7. From section 2.4, the inclusion $i: \mathcal{O} \to \mathcal{OC}$ induces a derived pushforward,

$$\mathbb{L}i_*: \mathcal{O}\text{-mod} \to \mathcal{OC}\text{-mod}.$$ 

Thus any such algebra $A$ determines an open-closed topological field theory $\mathbb{L}i_*(A)$. On the other hand, the inclusion $j: \mathcal{C} \to \mathcal{OC}$ determines a closed TFT, $j^*\mathbb{L}i_*(A)$. A closed TFT is a $\mathcal{C}$ module. The $\mathcal{C}$-mod structure on $j^*\mathbb{L}i_*(A)$ is equivalent to the existence of a natural map,

$$\mathcal{C}(t^{\otimes i}, t^{\otimes j}) \otimes j^*\mathbb{L}i_*(A)(t^{\otimes i}) \to j^*\mathbb{L}i_*(A)(t^{\otimes j}).$$

In this section, we show that the homology of the chain complex associated to the torus object, $j^*\mathbb{L}i_*(t)$, is the Harrison homology of the algebra $A$. This is proven by studying the $\text{Ob}((\mathcal{OC}) - \mathcal{O}$ bimodule $\mathcal{OC}$ used to define the extension $\mathbb{L}i_*$ above.

Recall that the boundary tori in the forested graph stratification of the space $X_v$ are represented by *bonnets*, $B(n)$, see section 4.19.1. The boundary of the trivial bonnet, $B(0)$, is zero. In general, the boundary of the cell associated to the tori derives from the differential in the Cobar construction.

**Theorem 6.1.** The category $\mathcal{OC}$, when considered as an $\text{Ob}((\mathcal{OC}) - \mathcal{O}$ bimodule, is freely generated by the bonnets $B(n)$.

**Proof.** It follows from corollary 4.25 that we can consider $G_v(\text{Bar}(C))$. If $G \in G_{(g,n+k,m)}(\text{Bar}(C))$ is a basis element then $G$ is a $\text{Bar}(C)$ labelled graph with $n$ incoming edges, $k$ outgoing edges and $j$ bonnets. We can absorb any part of the graph $G$ that doesn’t involve the bonnets using the action of $\mathcal{O}$.

We only need to consider $\text{Hom}_{\mathcal{OC}}(o^{\otimes k}, o^{\otimes i} \otimes t^{\otimes j})$ with $i = 0$ and $j = 1$, because incoming edges can be exchanged with outgoing edges and vice versa using the inner product. Multiple bonnets must be composites of tori with respect to the open composition.

What remains is a composite of open graphs with a single copy of $B(n)$.
Let’s unwind the definitions in order to determine the chain complex, Torus\((A)\), associated to the torus object. Recall that,

\[
(\mathcal{O}\mathcal{C} \otimes_\mathcal{O} A)(t) = \bigoplus_j \mathcal{O}\mathcal{C}(t, e^{\otimes j}) \otimes A(e^{\otimes j}) = \bigoplus_j \text{Hom}_{\mathcal{O}\mathcal{C}}(e^{\otimes j}, t) \otimes A^{\otimes j}
\]

modulo the action of \(\mathcal{O}\). This action is determined by the diagram,

\[
\begin{array}{ccc}
\mathcal{O}\mathcal{C}(t, e^{\otimes k}) \otimes \mathcal{O}(e^{\otimes j}, e^{\otimes k}) \otimes A(e^{\otimes j}) & \to & \mathcal{O}\mathcal{C}(t, e^{\otimes j}) \otimes A(e^{\otimes j}) \\
\downarrow & & \downarrow \\
\mathcal{O}\mathcal{C}(t, e^{\otimes k}) \otimes A(e^{\otimes k}) & \to & (\mathcal{O}\mathcal{C} \otimes_\mathcal{O} A)(t).
\end{array}
\]

As a left \(\mathcal{O}\)-mod, each \(f \in \text{Hom}_{\mathcal{O}}(e^{\otimes j}, e^{\otimes k})\) induces a map \(f_* : A^{\otimes j} \to A^{\otimes k}\) and, as a right \(\mathcal{O}\)-mod, each such \(f\) induces a map,

\[
f^* : \text{Hom}_{\mathcal{O}\mathcal{C}}(e^{\otimes k}, t) \to \text{Hom}_{\mathcal{O}\mathcal{C}}(e^{\otimes j}, t),
\]

given by post-composition. If \(g \otimes e^{\otimes k} \in \text{Hom}_{\mathcal{O}\mathcal{C}}(e^{\otimes k}, t) \otimes A^{\otimes k}\) then the diagram above yields the relation,

\[
f^*(g) \otimes e^{\otimes k} \sim g \otimes f_*(e^{\otimes k}).
\]

Now each complex \(\text{Hom}_{\mathcal{O}\mathcal{C}}(e^{\otimes j}, t)\) is quasi-isomorphic to a chain complex of graphs,

\[
\text{Hom}_{\mathcal{O}\mathcal{C}}(e^{\otimes j}, t) \simeq \bigoplus_g \mathcal{G}_{(g,j,1)}(\text{Bar}(C)),
\]

containing one boundary torus and \(j\) edges which, by theorem 6.1, is generated by the bonnets \(B(n)\).

Recall from section 2.6.1 that two cooperads \(A\) and \(B\) can be isomorphic or quasi-isomorphic. Now the observation that \(\text{Bar}(C) \simeq L^*_\infty \simeq L^*\), together with lemma 4.22 implies that we can think of the complex computing the relevant homology as graphs with vertices labelled by trees satisfying the Jacobi (or IHX) relation. Such graphs are \(C_\infty\) graphs: they satisfy the shuffle product relation of section 2.6.2 at each vertex. This can be seen by applying \(\text{Cobar}\) to the linear dual of the short exact sequence \(C \to A \to L\). So each equivalence class of \((\mathcal{O}\mathcal{C} \otimes_\mathcal{O} A)(t)\) under the relation \(\sim\) has a unique representative of the form,

\[
\mathcal{Q}(B(n)) \otimes A^{\otimes n}.
\]
The differential is determined by the internal differential $\delta$ of $A$ and the sum of all possible ways to add an edge to the collection of edges at the vertex of the boundary torus. This can be described pictorially,

\[
\begin{array}{c}
\circ \\
n \\
\end{array} \quad \rightarrow \quad\begin{array}{c}
\circ \\
n-k+1 \\
k \\
\end{array}
\]

The orientation on the right hand side is induced by the left hand side.

In order to describe the object associated to the torus algebraically, we begin by defining $\text{pre-Torus}(A)$.

\[
\text{pre-Torus}(A) = \bigoplus_{j=1}^{\infty} A^{\otimes j}.
\]

There is a map $\pi$ from $\text{pre-Torus}(A)$ onto $(\mathcal{O}C \otimes_\mathcal{O} A)(t)$. The kernel of $\pi$ is spanned by shuffles because the $C_\infty$ operad’s generators, $m_n$, are precisely those which vanish on shuffle products. If the shuffle product of tensors is defined by,

\[
(a_1 \otimes \cdots \otimes a_i) \ast (a_{i+1} \otimes \cdots \otimes a_n) = \sum_{\sigma \in \text{Sh}(i,n-1)} \pm a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)},
\]

then $\ker(\pi)$ is the ideal of $\text{pre-Torus}(A)$ generated by the shuffle products and the object associated to the torus is the chain complex,

\[
\text{Torus}(A) = \text{pre-Torus}(A) / \ker(\pi).
\]

The differential is the sum of the one given by the $A_\infty$ relation,

\[
d(a_1 \otimes \cdots \otimes a_n) = \sum_{i+j=n+1 \atop i,j \geq 2} \sum_{s=0}^{n-j} (-1)^{j+s(j+1)} a_1 \otimes \cdots \otimes m_j(a_{s+1} \otimes \cdots \otimes a_{s+j+1}) \otimes \cdots \otimes a_n
\]

and the internal differential of $A$,

\[
\delta(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n} a_1 \otimes \cdots \otimes \partial(a_i) \otimes \cdots \otimes a_n.
\]

If $A$ is a commutative algebra or differential graded commutative algebra, then the chain complex $\text{Torus}(A)$ agrees with the chain complex computing Harrison homology, see [1].

The theorem below summarizes the above computation.
Theorem 6.2. If $A$ is a cyclic $C_{\infty}$ algebra and $\mathcal{O}C$ is the $\text{Ob}(\mathcal{O}C) - \mathcal{O}$ bimodule of section 4.6 then, after identifying $A$ as an $\mathcal{O}$ module, the extension $\mathcal{O}C \otimes \mathcal{O} A$ associates to the torus object $t \in \text{Ob}(\mathcal{O}C)$ a chain complex, $\text{Torus}(A)$, computing the Harrison homology of $A$.

$$(\mathcal{O}C \otimes \mathcal{O} A)(t) = \text{Torus}(A) \quad \text{and} \quad H_*(\text{Torus}(A)) \cong \text{Harrison}_*(A, A)$$

6.3. Flatness and Exactness. In this section we show that the closed category $\mathcal{C}$ acts on the Harrison complex associated to a torus by theorem 6.2.

For the extension $\mathcal{O}C \otimes \mathcal{O} A$ to be an open-closed field theory in the sense of definition 3.7, we must show that $i_*(A)$ is h-split. In order to describe the complex $i_*(A)(t)$, a simplification can be made,

$$\mathcal{O}C \otimes_{\mathcal{O}}^L A \cong \mathcal{O}C \otimes_{\mathcal{O}} A,$$

by observing that, as an $\text{Ob}(\mathcal{O}C) - \mathcal{O}$ bimodule, the category $\mathcal{O}C$ is flat.

This is true because there is a natural filtration on the bimodule $\mathcal{O}C$ given by the degree of the bonnets. A bonnet with vertex labelled by $m_n$ must come from a cell of underlying dimension $n - 2$. For instance, the bonnet in degree 0, represented by a trivalent graph, must come from the trivial forest (or zero dimensional cube), covering only the base point of the relevant cycle.

Define a filtration $\mathcal{F}$ of $\mathcal{O}C$ so that $\mathcal{F}^0 \mathcal{O}C$ contains the identity elements and the associated graded $\text{Gr}^n \mathcal{O}C$ is precisely the $n$th bonnet $B(n)$. Since $dB(n)$ is a sum of bonnets of lower degree this is a filtration of complexes. There is an induced filtration on $\mathcal{O}C \otimes_{\mathcal{O}} A$ such that the associated graded

$$\text{Gr}^n(\mathcal{O}C \otimes_{\mathcal{O}} A)(e^{\otimes i} \otimes t^{\otimes j})$$

consists of placing the identity factors on the $i$ edges and labelling the $j$ bonnets by elements of $A^{\otimes n}$. Showing that this is true amounts to a computation nearly identical to that of the previous section.

We will exploit the following lemma,

Lemma 6.4. If $\varphi : A \rightarrow A'$ is a map of filtered complexes such that $\varphi_0 : \mathcal{F}^0 A \rightarrow \mathcal{F}^0 A'$ is a quasi-isomorphism and $\varphi_* : \text{Gr}^n A \rightarrow \text{Gr}^n A'$ is a quasi-isomorphism then $\varphi_n : \mathcal{F}^n A \rightarrow \mathcal{F}^n A'$ is a quasi-isomorphism for all $n$. In particular, $\varphi$ is a quasi-isomorphism.

Theorem 6.5. If $A$ is an h-split $\mathcal{O}$ module, then $\mathcal{O}C \otimes_{\mathcal{O}} A$ is an h-split $\text{Ob}(\mathcal{O}C)$ module.

Proof. We must check that the maps,

$$(\mathcal{O}C \otimes_{\mathcal{O}} A)(x) \otimes (\mathcal{O}C \otimes_{\mathcal{O}} A)(y) \rightarrow (\mathcal{O}C \otimes_{\mathcal{O}} A)(x \otimes y)$$
are quasi-isomorphisms. Since this is true in filtration degree 0 it follows by induction if it holds for the associated graded. A collection of \( i \) bonnets labelled by \( A \) tensored with a collection of \( j \) bonnets labelled by \( A \) is quasi-isomorphic to a collection of \( i + j \) bonnets labelled by \( A \).

**Theorem 6.6.** The category \( \mathcal{OC} \) is a flat \( \text{Ob}(\mathcal{OC}) - \mathcal{O} \) bimodule. That is, the functor
\[
i_* : \mathcal{O} - \text{mod} \to \text{Ob}(\mathcal{OC}) - \text{mod}
\]
given by
\[
i_*(A) = \mathcal{OC} \otimes_\mathcal{O} A
\]
is exact.

**Proof.** Given a quasi-isomorphism of \( C_\infty \) algebras \( \varphi : A \to A' \). We must check that the induced map \( \mathcal{OC} \otimes_\mathcal{O} A \to \mathcal{OC} \otimes_\mathcal{O} A' \) is a quasi-isomorphism. Since this is true in filtration degree 0, it follows by induction if it holds for the associated graded. The map
\[
\bar{\varphi} : \text{Gr}^n(\mathcal{OC} \otimes_\mathcal{O} A) \to \text{Gr}^n(\mathcal{OC} \otimes_\mathcal{O} A')
\]
is the map between bonnets labelled by tensor powers of \( A \) and \( A' \). The map \( \bar{\varphi} \) is a quasi-isomorphism because a tensor product of quasi-isomorphisms is a quasi-isomorphism. \( \Box \)

6.7. Deligne’s Conjecture.

**Corollary 6.8.** The category \( \mathcal{C} \) acts on the complex \( \text{Torus}(A) : \)
\[
\text{Hom}_\mathcal{C}(t^{\otimes i}, t^{\otimes j}) \otimes \text{Torus}_*(A)^{\otimes i} \to \text{Torus}_*(A)^{\otimes j}.
\]

**Proof.** If we consider \( A \) as an \( \mathcal{O} \) - mod and \( \mathcal{OC} \) as an \( \mathcal{OC} - \mathcal{O} \) bimodule then we can define an \( \mathcal{OC} \) module associated to \( A \) by \( \mathcal{OC} \otimes_\mathcal{O} A \). If \( i : \mathcal{C} \to \mathcal{OC} \) is the inclusion then \( i^*(\mathcal{OC} \otimes_\mathcal{O} A) \) is a \( \mathcal{C} \) - mod. If \( X(A) = i^*(\mathcal{OC} \otimes_\mathcal{O} A)(t) \) is the chain complex associated to the torus then there is a natural map
\[
\text{Hom}_\mathcal{C}(t^{\otimes i}, t^{\otimes j}) \otimes X(A)^{\otimes i} \to X(A)^{\otimes j}.
\]

Earlier, we considered \( \mathcal{OC} \) as an \( \text{Ob}(\mathcal{OC}) - \mathcal{O} \) bimodule and saw that \( \text{Torus}(A) = j^*(\mathcal{OC} \otimes_\mathcal{O} A) \). On the other hand, the complex associated to the torus is independent of the choice of \( \text{Ob}(\mathcal{OC}) \) verses \( \mathcal{OC} \). So \( X(A) \) is \( \text{Torus}(A) \).

\( \Box \)

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