Quasilinear approach to summation of the WKB series

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It is shown that the quasilinearization method (QLM) sums the WKB series. The method approaches solution of the Riccati equation (obtained by casting the Schrödinger equation in a nonlinear form) by approximating the nonlinear terms by a sequence of the linear ones, and is not based on the existence of a smallness parameter. Each \( p \)-th QLM iterate is expressible in a closed integral form. Its expansion in powers of \( \hbar \) reproduces the structure of the WKB series generating an infinite number of the WKB terms. Coefficients of the first \( 2^p \) terms of the expansion are exact while coefficients of a similar number of the next terms are approximate. The quantization condition in any QLM iteration, including the first, leads to exact energies for many well known physical potentials such as the Coulomb, harmonic oscillator, Pöschl-Teller, Hulthen, Hyleraas, Morse, Eckart etc.

I. INTRODUCTION

The derivation of the WKB solution starts by casting the radial Schrödinger equation into nonlinear Riccati form and solving that equation by expansion in powers of \( \hbar \). It is interesting instead to solve this nonlinear equation with the help of the quasilinearization method (QLM) and compare with the usual WKB series. In addition we prove that the exact quantization condition in any QLM iteration, including the first, leads to exact energies for many well known physical potentials such as the Pöschl-Teller, Hulthen, Hyleraas, Morse, Eckart etc.

The paper is arranged as follows: in the second chapter we expand the first, second and third QLM iterates in the series in powers of \( \hbar \) and solve it by QLM iterations. Then in the third chapter we expand the first, second and third QLM iterates in the series in powers of \( \hbar \) and solve it. Finally in the fourth chapter, we apply our method to equations with the Coulomb, harmonic oscillator, Pöschl-Teller, Hulthen, Hyleraas, Morse and Eckart potentials and prove that the QLM method provides exact energies for both ground and excited states already in the first iteration. Our results, advantages of the quasilinearization method over WKB and its possible future applications are discussed in the final, fifth chapter.

II. QLM APPROACH TO THE SOLUTION OF THE SCHRÖDINGER EQUATION

The usual WKB substitution

\[
\chi(r) = C \exp \left( \lambda \int y(r') dr' \right),
\]

converts the radial Schrödinger equation

\[
\frac{d^2\chi(r)}{dr^2} + \hbar^2 k^2 \chi(r) = 0
\]

to nonlinear Riccati form

\[
\frac{dy(z)}{dz} + (k^2 + y^2) = 0.
\]

Here \( k^2 = E - V - l(l+1)/z^2 \) and \( \lambda = 2m/k^2 \). The proper bound state boundary condition for potentials falling off at \( z \approx z_0 \) is

\[
y(z_0) = \text{const}.
\]

This means that

\[
y'(z_0) = 0,
\]

so that Eq. (3) at \( z \approx z_0 \) reduces to \( k(z_0)^2 + y^2(z_0) = 0 \) or \( y(z_0) = \pm ik(z_0) \). We choose here to define the boundary condition with the plus sign, so that

\[
y(z_0) = ik(z_0).
\]

The quasilinearization of this equation gives a set of recurrence differential equations

\[
\frac{dy_p(z)}{dz} = y_{p-1}^2(z) - 2y_p(z)y_{p-1}(z) - k^2(z).
\]
with the boundary condition deduced from Eq. (5):

\[ y_p(z_0) = ik(z_0). \]  

(7)

The analytic solution of these equations expresses the \( p \)-th iterate \( y_p(z) \) in terms of the previous iterate:

\[ y_p(z) = f_{p-1}(z) - \int_{z_0}^{z} ds \frac{d f_{p-1}(s)}{ds} \exp[-2 \int_{s}^{z} y_{p-1}(t) dt], \]

(8)

where

\[ f_{p-1}(z) = \frac{y_{p-1}^2(z) - k^2(z)}{2y_{p-1}(z)}. \]

Indeed, differentiation of both parts of Eq. (9) leads immediately to Eq. (6) which proves that \( y_p(z) \) is a solution of this equation. The boundary condition (5) is obviously satisfied automatically.

The second term in Eq. (8) could be written as

\[ \int_{z_0}^{z} ds \left( -\frac{d y_{p-1}(s)}{ds} \right) \left( 2y_{p-1}(s) \exp[-2 \int_{s}^{z} y_{p-1}(t) dt] \right), \]

(10)

The second expression in the round brackets in the integrand is the derivative of the exponential there. The integration by parts of this integral therefore gives:

\[
\left[ \left( -\frac{1}{2y_{p-1}(s)} \frac{d f_{p-1}(s)}{ds} \right) \exp[-2 \int_{s}^{z} y_{p-1}(t) dt] \right]_{z_0}^{z} \]

(11)

Since the lower limit in the first term of this expression vanishes in view of Eq. (4), the integration of the second term of Eq. (8) by parts results in:

\[ y_p(z) = f_{p-1}(z) + \left( -\frac{1}{2y_{p-1}(z)} \frac{d f_{p-1}(z)}{dz} \right) - \int_{z_0}^{z} ds \frac{d}{ds} \left( -\frac{d f_{p-1}(s)}{ds} \right) \frac{1}{2y_{p-1}(s)} \exp[-2 \int_{s}^{z} y_{p-1}(t) dt], \]

(12)

The successive integrations by parts of Eq. (12) lead to the series

\[ y_p(z) = \sum_{n=0}^{\infty} \mathcal{L}_n^{(p)}(z) \]

(13)

with \( \mathcal{L}_n^{(p)}(z) \) given by recursive relation

\[ \mathcal{L}_n^{(p)}(z) = \frac{1}{2y_{p-1}(z)} \frac{d}{dz} \left( -\mathcal{L}_{n-1}^{(p)}(z) \right) \]

(14)

and

\[ \mathcal{L}_n^{(0)}(z) = f_{p-1}(z). \]

(15)

Since

\[ \frac{d}{dz} = g \frac{d}{dr}, \quad g = \lambda^{-1} = \frac{h}{\sqrt{2m}}, \]

(16)

Eq. (13) represents the expansion of the \( p \)-th QLM iterate in powers of \( g \), that is in powers of \( h \), which one can compare with the WKB series as will be done in the next section.

For the zeroth iterate \( y_0(z) \) it seems natural to choose the zero WKB approximation that is to set

\[ y_0(z) = ik(z), \]

(17)

which in addition satisfies boundary condition (5). However, one has to be aware that this choice has unphysical turning point singularities. According to the existence theorem for linear differential equations (11), if \( y_{p-1}(z) \) in Eq. (16) is a discontinuous function of \( z \) in a certain interval, then \( y_p(z) \) or its derivatives may also be discontinuous functions in this interval, so consequently the turning point singularities of \( y_0(z) \) may propagate to the next iterates.
first QLM iterate
\[ y_1(z) = ik(z) - i \int_{z_0}^{z} ds k'(s) \exp[-2i \int_{s}^{z} k(t) dt], \]
(18)

Thus the first QLM iterate is expressible in a closed integral form. However, it takes into account, though approximately, infinite number of the WKB terms corresponding to higher powers of \( \hbar \) as it will be shown in the next section. In view of this it is a better approximation than the usual WKB.

III. COMPARISON OF EXPANSIONS OF QLM ITERATES AND WKB SERIES

To obtain the WKB series one has to expand solution \( y \) of the Riccati equation (4) in powers of \( \hbar \). This is easy to do by using Eq. (19) and looking for \( y \) in the form of series in \( g \):
\[ y = \sum_{m=0}^{\infty} g^m Y_m. \]
(19)

Substitution into (3) and equation of terms by the identical powers of \( g \) gives
\[ \frac{dY_{m-1}}{dr} = - \sum_{k=0}^{m} Y_k Y_{m-k}. \]
(20)

This reduces to the recurrence relation
\[ Y_m = \frac{1}{2Y_0} \left( Y'_m + \sum_{k=1}^{m-1} Y_k Y_{m-k} \right). \]
(21)

The derivatives in this and subsequent expressions are in variable \( r \). The zero WKB approximation \( Y_0 \) is given by \( Y_0 = i k \). The subsequent terms \( Y_n \) of the expansion could be obtained from this recurrence relation by use of Mathematica [12].

We present here the WKB expansion (19) up to \( g^7 \) inclusively:

\[
\begin{align*}
  y &= ik - \frac{g k'}{2k} + \frac{i}{8k^3} g^2 (3k'^2 - 2kk'') + \frac{g^3}{8k^5} (6k'^3 - 6kk''' + k''k^{(3)}) + \frac{i}{128k^7} g^4 (-297k'^4 + 396kk'^2 k'' - 52k^2 k''^2 - 80kk'k^{(3)} + 8k^3 k^{(4)}) - \frac{g^5}{32k^9} (306k'^5 - 510kk'^3 k'' + 111k^2 k'^2 k^{(3)} - 3k^2 k' (-48k''^2 + 5kk^{(4)} + k^{(5)})) + k^3 (-24k''k^{(3)} + k^{(5)}) + \frac{i}{1024k^{11}} g^6 (50139k'^6 - 100278kk'^4 k'' + 22704k^2 k'^2 k^{(3)} + 12k^2 k'' (3679k''^2 - 290k^{(4)} + 16k^3 k' (-694k''k^{(3)} + 21k^{(5)})) - 8k^3 (301k'^3 - 80k k'' k^{(4)} + k (-49k^{(3)}^2 + 2kk^{(6)}))) + \frac{g^7}{128k^{13}} (38286k'^7 - 89334kk'^5 k'' + 20721k^2 k'^4 k^{(3)} + k^3 (53724k^2 k''^2 - 3405k^3 k^{(4)} + 3k^3 k'^2 (-5426k'' k^{(3)} + 129k k^{(5)}) + 2k^3 k' (-3528k'^3 + 735k k'' k^{(4)} + 2k (225k^{(3)}^2 - 7kk^{(6)})) + k^4 (1176k'^2 k^{(3)} - 62k k'' k^{(5)})) + \frac{1}{2} (\frac{1}{2} + m) (j - 1) \right)
\end{align*}
\]
(22)

To compare expansion of the first QLM iterate \( y_1 \) in powers of \( \hbar \) with the WKB expansion (22) we have to use, as we have already mentioned in the previous section, Eqs. [13] and [14] together with Eq. [17]. The result up to power \( g^7 \) inclusively is again obtained with the help of Mathematica [12]:
\[ y_1 = i k - \frac{k' g}{2 k} + \frac{i g^2}{4 k^3} (k'^2 - k k'') + \frac{g^3}{8 k^5} (3 k'^3 - 4 k k' k'' + k^2 k^{(3)}) + \frac{g^4}{16 k^7} (-15 k'^4 + 25 k k'^2 k'' - 7 k^2 k' k^{(3)} + k^3 (15 k'' k^{(3)} - k k^{(5)}) - \frac{i g^6}{64 k^{11}} (-945 k^6 + 2205 k k' k''' - 630 k^2 k'^2 k^{(3)} + 14 k^2 k'^2 (-80 k''^2 + 9 k^{(4)} + 2 k^3 k' (175 k'' k^{(3)} - 8 k k^{(5)}) + k^3 (70 k''^3 - 26 k k'' k^{(4)} + k (-15 k^{(3)^2} + k k^{(6)})) + \frac{g^7}{128 k^{13}} (10395 k^7 - 27720 k^5 k''' + 7875 k^2 k'^4 k^{(3)} - 126 k^2 k'^3 k^{(5)} - 784 k k'' k^{(4)} + k (455 k^{(3)^2} - 22 k k^{(6)})) + k^4 (560 k'^2 k^{(3)} - 42 k k'' k^{(5)} + k (-56 k^{(3)^2} + k k^{(7)}))^2 (23)\]

The comparison of expansion of the first QLM iterate in powers of \( h \) and WKB series was originally performed in the works [3, 10]. There it was shown that the expansion reproduces exactly the first two terms and also gives correctly the structure of the WKB series up to the power \( g^3 \) considered in these works, generating series with identical terms, but with different coefficients. Comparison of Eqs. 22 and 23) of the present work shows that this conclusion is true also if higher powers of \( g \) are taken into account.

The computation of the second QLM iterate \( y_2 \) in powers of \( h \) is done by reexpanding the term \( \frac{i}{2} y_1^2 \) in Eq. (13) in the series in powers of \( g \) and keeping the powers up to \( g^7 \) inclusively in this expression as well as in the sum in Eq. (13). The result is given by:

\[ y_2 = i k - \frac{k' g}{2 k} + \frac{i g^2}{8 k^3} (3 k'^2 - 2 k k'') + \frac{g^3}{8 k^5} (6 k'^3 - 6 k k' k'' + k^2 k^{(3)}) + \frac{g^4}{32 k^7} (-74 k'^4 + 99 k k'^2 k'' - 20 k^2 k' k^{(3)} + k^3 (24 k'' k^{(3)} + k k^{(5)}) - \frac{i g^6}{128 k^{11}} (-6186 k^6 + 12446 k k' k''' - 2832 k^2 k'^3 k^{(3)} + 3 k^2 k''^2 (-5503 k''^2 + 435 k k^{(4)} + 2 k^3 k' (694 k'' k^{(3)} - 21 k k^{(5)}) + k^3 (301 k''^3 - 80 k k'' k^{(4)} + k (-49 k^{(3)^2} + 2 k k^{(6)})) + \frac{g^7}{256 k^{13}} (75256 k^7 - 176659 k^5 k''' + 41224 k^2 k'^4 k^{(3)} + 4 k^2 k'^3 (26687 k''^2 - 1700 k k^{(4)} + k k'^2 (-16243 k''^2 k^{(3)} + 387 k k^{(5)} + k^3 k' (-14071 k''^3 + 2940 k k'' k^{(4)} + 8 k (225 k^{(3)^2} - 7 k k^{(6)})) + k^4 (1176 k''^2 k^{(3)} - 62 k k'^2 k^{(5)} + k (-90 k^{(3)^2} + k k^{(7)}))^2 (24)\]

The expansion of \( y_2 \) reproduces exactly already the first four terms of the WKB series. It also gives the proper structure of the other terms of the WKB series, generating series with identical terms which have approximately correct coefficients.

The expansion of \( y_3 \) is obtained in the similar fashion. It reproduces exactly the first eight terms of the WKB series that is all the terms up to the power \( g^7 \) inclusively listed in Eq. 22.

Summing up, we have proved that the expansion of the first, second and third QLM iterates reproduces exactly two, four and eight WKB terms respectively. Since the zero QLM iterate \( y_0 \) was chosen to be equal to the zero WKB approximation \( i k \), one can state that the \( p \)-th QLM iterate contains \( 2^p \) exact terms. In addition expansion of each QLM iterate has the correct structure whose terms are identical to of the WKB series with approximate coefficients.

The \( 2^p \) law is, of course, not accidental. The QLM iterates are quadratically convergent [11, 12, 16], that is the
norm of the difference of the exact solution and the p-th QLM iterate \( \| y - y_p \| \) is proportional to the square of the norm of the differences of the exact solution and the (p-1)-th QLM iterate:

\[
\| y - y_p \| \sim \| y - y_{p-1} \|^2. \tag{25}
\]

Here norm \( \| g \| \) of function \( g(x) \) is a maximum of the function \( g(x) \) on the whole interval of values of \( x \).

Since \( y_0 \) contains just one correct WKB term of power \( g^0 \) and thus \( \| y - y_0 \| \) is proportional to \( g \), one has to expect that \( \| y - y_1 \| \sim g^2 \) and thus \( y_1 \) contains two correct WKB terms of powers \( g^0 \) and \( g^1 \). The difference \( \| y - y_2 \| \sim \| y - y_1 \| \sim g^4 \) so that \( y_2 \) contains four correct WKB terms of powers \( g^0 \), \( g^1 \), \( g^2 \) and \( g^3 \). Finally, the difference \( \| y - y_3 \| \) should be proportional to \( g^5 \), and therefore \( y_3 \) has to contain eight correct terms with powers between \( g^0 \) and \( g^7 \) inclusively. This explains the \( 2^n \) law.

### IV. QLM AND WKB ENERGY CALCULATIONS

The exact quantum mechanical quantization condition for the energy \( E \) has the form:

\[
J = \oint_C y(z) dz = i 2\pi n. \tag{26}
\]

Here \( y(z) \) is the logarithmic derivative of the wave function, given by Eq. (3), \( z = gr, n = 0, 1, 2, \ldots \) counts the number of poles of \( y(z) \) and is the bound state number and the integration is along a path \( C \) in the complex plane encircling the segment of the \( \Re z \) axis between the turning points.

The \( p \)-th QLM iterate \( y_p(z) \), as we have seen, contains in addition to \( 2^p \) exact WKB terms of powers \( g^0, g^1, \ldots, g^{2p-1} \) also an infinite number of structurally correct WKB terms of higher powers of \( g \) with approximate coefficients. One can expect therefore that the quantization condition (26) with \( y(z) \) approximated by any QLM iterate \( y_p(z) \) including the first

\[
J_p = \oint_C y_p(z) dz = i 2\pi n, p = 1, 2, \ldots \tag{27}
\]

gives more accurate energy than the WKB quantization condition

\[
\oint_C k(z) dz = 2\pi (n + \frac{1}{2}), \tag{28}
\]

which is obtained by substituting into exact quantization condition (26) the WKB expansion (22) up to the first power of \( g \sim \hbar \), that is \( y(z) = ik(z) - \frac{\delta k(z)}{\delta x} \) and neglecting all higher powers of \( g \) in the expansion. Indeed, we prove in this section that Eq. (27) leads to exact energies not only for the the Coulomb and harmonic oscillator potentials as it was shown earlier in ref. 11, but for many other well known physical potentials used in molecular and nuclear physics such as the Pöschl-Teller, Hulthen, Hyleraas, Morse, Eckart etc. The WKB quantization condition (26) yields the exact energy only for the first two potentials, but not for the rest of them.

Let us now consider the above mentioned examples:

#### A. Harmonic oscillator

\[ V(x) = \frac{x^2}{2}, -\infty < x < \infty. \]

From now on we will work in the units \( \hbar = m = 1 \) so that from \( z = \lambda x \) follows \( x = \frac{z}{\sqrt{\lambda}} \) and \( V(z) = \frac{z^2}{2} \).

In view of the boundary condition \( y_p(z) \) at the infinity should behave like \( i \sqrt{E - \frac{z^2}{4}} \sim -i \frac{z}{2} + \frac{E}{2} \) where we omitted terms of order \( \frac{1}{z^2} \) and higher. Here we took into account that for bound states the logarithmic derivative at the infinity should be negative. More accurately the pole structure of \( y_p(z) \) at \( z \sim \infty \) could be found by looking for the solution there in the form \( y_p(z) \sim -i \frac{z}{2} + \frac{E}{2} \).

Substituting into the quasilinearized equation (26)

\[
]\frac{dy_p(z)}{dz} = y_{p-1}(z) - 2y_p(z)y_{p-1}(z) - (E - \frac{z^2}{4}) \tag{29}\]

and again neglecting terms of order \( \frac{1}{z^2} \) and higher which does not contribute to the integral yields \( \alpha_p = E - \frac{1}{4} \), so that the pole term in \( y_p(z) \), \( p = 1, 2, \ldots \) is given by \( \frac{E - \frac{1}{4}}{2} \).

The integration in Eq. (27) is counterclockwise along a path \( C \) in the complex plane encircling the segments of the \( \Re z \) axis between the two turning points \( -2\sqrt{E} \) and \( 2\sqrt{E} \). Since the only singularity outside contour \( C \) in the complex plane lays at infinity, the integral (27) could be done by distorting the contour to enclose the pole at \( x = \infty \). The evaluation of the integral yields \( i2\pi (E - \frac{1}{4}) = i2\pi n \) or \( E = n + \frac{1}{2} \), which is the exact equation for the energy levels.

#### B. Spherical harmonic oscillator

\[ V(r) = \frac{r^2}{2}, 0 < r < \infty. \]

This case was already considered in the work 11. We present here somewhat different derivation. Eq. (26) now has the form

\[
\frac{dy_p(z)}{dz} = y_{p-1}(z) - 2y_p(z)y_{p-1}(z) - (E - \frac{z^2}{4} - \frac{\lambda(l+1)}{2}) \tag{30}\]

where \( z = \sqrt{2} \). The pole structure at \( z \sim \infty \) as before will be given by Eq. (30). In order to find the pole at \( z \sim 0 \) we look for \( y_p \) near zero in the form \( \frac{a_p z}{z} + \ldots \)

which terms proportional to \( \frac{1}{z^2}, \frac{1}{z^3} \) etc. are not explicitly displayed since they do not contribute to the integral. The substitution into Eq. (30) yields

\[
a_p(2a_{p-1} - a_p - 1) = l(l+1) \tag{31}\]
When $p$ becomes large we expect that QLM iterates converge to each other and to the exact solution, so for large $p$ one expects $a_{p-1} = a_p$. The solution of Eq. (31) satisfying this condition will be $a_p = (l + 1)$, so the pole term at $z \approx 0$ is $\frac{l+1}{z^2}$. This, of course, agree also with a $z^{l+1}$ behavior of the radial wave function at zero, which leads to the $\frac{l+1}{z^2}$ behavior of its logarithmic derivative. The contour $C$ encloses two positive turning points $z_1$ and $z_2$ defined by the equation

$$E - \frac{z^2}{4} - \frac{i(l+1)}{z^2} = 0$$

and the section $\Re z$ axis between them. Indeed, the physical motion takes place only for positive real $z$. The integrand, as we saw, has the poles at $z = 0$ and $z = \infty$. However, in addition to the poles at the unphysical turning points $\pm z_1$ and $-z_1$. This contribution due to the symmetry of the potential with respect to $z = 0$ will be $-J_p$ since the path of integration around the cut on negative real axis will be clockwise. The contribution of the poles is $2\pi i$ multiplied by $-(l+1)$ and $E - \frac{1}{2}$ respectively, so one obtains $J_p = -J_p + 2\pi i [E - \frac{1}{2} - (l+1)]$. Thus

$$J_p = \pi i (E - \frac{1}{2} - (l+1)) = 2\pi n \text{ or}$$

$$E = 2n + l + \frac{3}{2}$$

which is the exact equation for the energy levels of the three-dimensional oscillator.

C. Coulomb potential $V(r) = -\frac{Z}{r}$, $0 < r < \infty$.

This case was considered before in the work [9]. In our derivation we set, as before, in the previous section $z = \sqrt{2E}$, so $k^2(z) = E + \frac{Z^2}{2 - \frac{(l+1)}{z^2}}$. At large $z$, in view of the boundary condition (32) $y_p(z)$ at the infinity should behave like $i\sqrt{-E} + \frac{Z^2}{2 - \frac{(l+1)}{z^2}} \approx -\sqrt{-E} + \frac{Z^2}{2}$. We took into account that for bound states the logarithmic derivative at the infinity should be negative and omitted terms of order $1/z$ and higher. The residue of the pole at the infinity thus is $-\frac{Z^2}{2E}$. The residue of the pole at $z \approx 0$ is computed as in the previous section and yields the same value $l + 1$.

The contour $C$ encloses two turning points defined by the equation $-\sqrt{-E} + \frac{Z^2}{2 - \frac{(l+1)}{z^2}} = 0$ and the section $\Re z$ axis between them. This equation, unlike equation (33) for the turning points in the previous section which had two positive and two negative real roots, has only two positive real roots $z_1 = \sqrt{\frac{Z^2}{2E} - \frac{(l+1)}{|E|}}$ and $z_2 = \sqrt{\frac{Z^2}{2E} - \frac{(l+1)}{|E|}}$. Therefore unlike the previous case there is no cut along the negative part of the real axis and when $C$ is distorted to enclose the poles at zero and the infinity only contribution of these poles should be taken into account. One obtains $2\pi i \left(-\frac{Z^2}{2E} - (l+1)\right) = i2\pi n$ or $E = -\frac{Z^2}{2(l+1)}$, which are exact energy levels in the Coulomb potential.

D. Cotangent potential

$V(x) = V_0 \cot^2(\frac{\pi x}{a})$, $V_0 > 0$, $0 < x < a$.

Let us introduce a new variable

$$z = \sin^2(\frac{\pi x}{a})$$

so that $x = \frac{a}{2} \arcsin \sqrt{z}$, $dx = \frac{a}{2z^{1/2}} dz$ and $k(z) = \sqrt{2(E + V_0) - \frac{V_0}{z}}$. The QLM equation (6) will now have the form

$$\frac{2\pi}{a} \sqrt{1 - z} \frac{dy_p(z)}{dz} = y_p^2(z) - 2y_p(z)y_{p-1}(z) - k^2(z).$$

One of the singularities of $k^2(z)$ is at $z = 0$. Near this point the equation has a form

$$\frac{2\pi}{a} \sqrt{z} \frac{dy_p(z)}{dz} = y_{p-1}(z) - 2y_p(z)y_{p-1}(z) + \frac{2V_0}{z}.$$

We look for solution of this equation in a form $y_p(z) = \frac{a}{\sqrt{z}}$. Then we obtain for $a_p$ the following recurrence relation:

$$a_p (2a_{p-1} - \frac{\pi}{a}) = a_p^2 (\frac{\pi}{a})^2 \lambda - 1).$$

Here we set $2V_0 = (\frac{\pi}{a})^2 \lambda (\lambda - 1)$ where $\lambda = \frac{\pi}{a} + \sqrt{1 + 2V_0 a^2}$. The solution of this algebraic equation which fulfills the demand that at large $p$ $a_p = a_{p-1}$ is $a_p = \frac{\pi a}{\sqrt{2}}$. The $y_p(z)$ near zero thus has a form $y_p(z) = \frac{a}{\sqrt{z}}$. At $z \approx \infty$ Eq. (35) reduces to \frac{2\pi}{a} \sqrt{z} \frac{dy_p(z)}{dz} = y_{p-1}(z) - 2y_p(z)y_{p-1}(z) - 2(E + V_0)$. Looking for solution in a form $y_p(z) = c_p$ where $c_p$ is some constant, one obtains the recurrent relation for $c_p$, namely $c_{p-1} - 2c_{p}c_{p-1} - 2(E + V_0) = 0$. The solution of this algebraic equation which fulfills the demand that at large $p$ $c_p = c_{p-1}$ is $c_p = \sqrt{-2(E + V_0)}$. The quantization condition (34) in variable $z$ given by Eq. (33) has a form

$$J_p = \frac{a}{\pi} \int_C \frac{y_p(z)}{\sqrt{1 - z}} dz = i2\pi n, p = 1, 2, \ldots$$

where integration is counterclockwise along a path $C$ in the complex plane encircling the cut along the $\Re z$ axis between the zero and $z = \frac{V_0}{2(E + V_0)}$. Since $y_p(z)$ equals to $\frac{\pi \lambda}{a^{1/2}}$ and $\sqrt{-2(E + V_0)}$ at $z \approx 0$ and $z \approx \infty$ respectively,
the integrand in Eq. (33) has poles with residues $\pm \frac{\lambda}{\pi}$ and $\pm \sqrt{2(E + V_0)}$ there. The deformation of the contour to include these poles and computation of their contributions yields

$$J_p = 2\pi i\left(-\frac{a}{\pi}\lambda + \sqrt{2(E + V_0)}\right) = 2\pi in,$$

(39)
or, upon substitution of value of $\lambda$ and inserting $h$ and $m$ from dimensional considerations,

$$E = -V_0 + \frac{\pi a^2}{2m} \left(\frac{\lambda}{2} + 1 + \frac{2mV_0a^2}{\pi^2h^2} + \frac{1}{4}\right).$$

This is the exact equation for the energy levels in the cotangent potential \[14\]. The correspondent WKB expression is different and given by \[14\].

$$E = -V_0 + \frac{\pi a^2}{2ma^2} \left(n + \frac{1}{2} + \frac{2mV_0a^2}{\pi^2h^2} + \frac{1}{4}\right)^2.$$ \[40\]

E. Pöschl-Teller potential hole

$$V(x) = \frac{V_1}{\sinh^2(x)} + \frac{V_2}{\cosh^2(x)}, \quad V_1 > 0, \quad V_2 > 0, \quad 0 < x < \frac{\pi}{2}.$$ \[41\]

This potential is a generalization of the cotangent potential and reduces to it in case $V_1 = V_2 \equiv \frac{\pi}{2}$. Indeed, in this case $V = \frac{V_0}{\sinh^2(x)} = V_0[(\cot^2(x) \frac{\pi}{2}) + 1]$ where $y = 2x$ changes between 0 and $\alpha$. The computation therefore proceeds as in the previous section with a difference that in addition to the poles at the zero and infinity there is also a pole at $z = 1$. Indeed, using again variable $z$ defined by Eq. (41) one writes the Eq. (10) in a form (35) with $k^2(z)$ now given by $2(E - \frac{\lambda}{2} - \frac{1}{2} - \frac{\pi a^2}{h^2}).$

The poles at $z = 0$, $z = 1$ and $z = \infty$ are computed as in the previous section and have respectively the form $\frac{\pi a}{2\lambda_k}, \frac{2mV_0a^2}{\pi^2h^2}$ and $\sqrt{\frac{2E}{k}}$. Here $\lambda_k = \frac{1}{2} + \frac{1}{2} + \frac{2mV_0a^2}{\pi^2h^2}$, so that $2V_k = (\frac{\pi}{2})^2\lambda_k(\lambda_k - 1)$; $k = 1, 2$. The integration in the quantization condition \[27\] is counterclockwise along a path $C$ in the complex plane enclosing the cut along the real axis between the two turning points $z_1, z_2$ given by $z_{1,2} = \frac{1}{\frac{\pi}{2}}[(E + V_1 - V_2) \pm \sqrt{(E + V_1 - V_2)^2 - 4E V_1}]$. The deformation of the contour to include the poles and computation of their contributions yields

$$J_p = 2\pi i\left(-\frac{a}{\pi}\lambda_1 - \frac{a}{\pi}\lambda_2 + \sqrt{2E}\right) = 2\pi in,$$ \[42\]
or, upon substitution of values of $\lambda_k$ and inserting $h$ and $m$ from dimensional considerations,

$$E = \frac{\pi^2a^2}{2ma^2}(2n + 1 + \frac{2mV_0a^2}{\pi^2h^2} + \frac{1}{4} + \frac{2mV_0a^2}{\pi^2h^2} + \frac{1}{4})^2.$$ \[43\]

This is the exact equation for the energy levels in the Pöschl-Teller potential hole \[14\]. The correspondent WKB expression is different and given by

$$E = \frac{\pi^2a^2}{2ma^2}(2n + 1 + \frac{2mV_0a^2}{\pi^2h^2} + \frac{2mV_0a^2}{\pi^2h^2})^2.$$ \[44\]

F. Modified Pöschl-Teller potential

$$V(x) = -\frac{V_0}{\cosh^2(x)}, \quad V_0 > 0, \quad 0 < x < \infty.$$ \[45\]

Setting $z = \cosh^2(x)$ so that $dx = -\frac{a}{2\sinh(x)}dz$ one obtains that the QLM equation (6) now has a form

$$\frac{2\pi}{a} \sqrt{(z - 1)} \frac{dy_p(z)}{dz} = y_p^2(z) - 2y_p(z)y_p - 1 - k^2(z),$$ \[46\]

where $k^2(z)$ is given by $2(|E| + \frac{V_0}{2})$. At $z \approx 0$ this equation has a form

$$\frac{2\pi}{a} \sqrt{-z} \frac{dy_p(z)}{dz} = y_p^2(z) - 2y_p(z)y_p - 1 - \frac{\lambda(\lambda - 1)}{z},$$ \[47\]

where we use the definition of $\lambda = \frac{1}{2} + \frac{1}{2} + \frac{2mV_0a^2}{h^2}$, so that $2V_0 = (\frac{\lambda(\lambda - 1)}{2})$. The solution near zero could be looked for in analogy with previous cases in the form $y_p(z) = \frac{ae}{\sqrt{z}}$. Substitution into \[12\] gives $a_p = a_{p-1} - 2a_p a_{p-1} + 2mV_0a^2$, so that $2V_0 = \frac{\lambda(\lambda - 1)}{2}$. Thus near zero $y_p(z) \approx \frac{ae}{\sqrt{z}}$. The solution at the infinity $y_p(z) \approx \sqrt{2|E|}$ is found in the same way as in the previous two sections. The integration in the quantization condition

$$J_p = a \int_C \frac{y_p(z)}{\sqrt{z} (z - 1)} dz = i2\pi n, \quad p = 1, 2, ...,$$ \[48\]
is counterclockwise along a path $C$ in the complex plane enclosing the cut along the real axis between the two turning points $z_1, z_2$ given by $z_{1,2} = \frac{1}{\frac{\pi}{2}}[(E + V_1 - V_2) \pm \sqrt{(E + V_1 - V_2)^2 - 4E V_1}]$. The deformation of the contour to include the poles and computation of their contributions yields

$$E = -\frac{h^2}{2ma^2}[-(n + \frac{1}{2})^2 + \frac{2mV_0a^2}{h^2} + \frac{1}{4}]^2.$$ \[49\]

This is the exact equation for the energy levels in the Pöschl-Teller potential hole \[14\]. The correspondent WKB expression is different and given by

$$E = -\frac{h^2}{2ma^2}[-(n + \frac{1}{2}) + \frac{2mV_0a^2}{h^2}]^2.$$ \[50\]

G. Hylleraas potential

$$V(x) = -\frac{V_0}{\cosh^2(x)}, \quad V_0 > 0, \quad 0 < r < \infty.$$ \[51\]

This potential, used in the molecular and nuclear physics, was introduced in work \[18\]. The computation proceeds as in previous section with the difference that now the physical motion takes place only for positive real $r$ and the wave function equals zero at the origin. The latter in view of the symmetry of the Hylleraas potential toward exchange $r$ to $-r$ means that the solution of
the present problem is equivalent to considering uneven states in the potential of the previous section. Thus in Eq. (44) \( n \) should be changed to \((2n - 1), n = 1, 2, \ldots\).

Therefore the energy levels in the Hylleraas potential are given by

\[
E = -\frac{\hbar^2}{2ma^2}[-(2n - 1) + \sqrt{\frac{2mv_0da^2}{\hbar^2}} + \frac{1}{4}].
\]

(45)

This coincides with the exact expression \(12\) for the energy levels. The correspondent WKB expression is rather different \(12\):

\[
E = -\frac{\hbar^2}{2ma^2}[-(2n - 1) + \sqrt{\frac{2mv_0da^2}{\hbar^2}}]^2, \quad n=1,2,\ldots
\]

**H. Eckart potential**

\[
V(x) = -A_{\frac{x}{1+e^{-x}}} - B_{\frac{x}{1+e^{-x}}}, \quad B \geq |A|, \quad -\infty < x < \infty.
\]

This potential, introduced in work \(20\), like the Hylleraas potential is also widely used in molecular and nuclear physics. The potential \(V(x) \to -A\) as \(x \to -\infty\) and \(V(x) \to 0\) as \(x \to \infty\). The minimum value \(V_{\text{min}}\) of this potential is \(V_{\text{min}} = -\frac{(A+B)^2}{16B}\); the discrete spectrum of energy lies respectively in the interval \((-\frac{(A+B)^2}{16B}, \text{min}(0,-A))\).

Let us start from calculating the energy levels \(\epsilon = -E\) in the WKB approximation. The WKB quantization condition \(28\) now has the form

\[
\sqrt{\frac{2m}{\hbar}} \int_C \sqrt{\epsilon + A_{\frac{x}{1+e^{-x}}} + B_{\frac{x}{1+e^{-x}}}} \, dx = 2\pi(n + \frac{1}{2}).
\]

(46)

Introducing a new variable \(t = -e^{-\frac{x}{2}}, -\infty < t < 0\) reduces the WKB quantization condition \(10\) to

\[
-\frac{\sqrt{2ma^2}}{\hbar} \int_C \sqrt{-\epsilon - A_{\frac{t}{1-1/t}} - B_{\frac{t}{1-1/t}}} \, dt = 2\pi(n + \frac{1}{2}).
\]

(47)

The contour \(C\) encloses two turning points \(t_{1,2} = \frac{1}{2(\epsilon - A)}(A + B - 2\epsilon) \pm \sqrt{(A + B)^2 - 4\epsilon B}\) defined by the equation \(\epsilon + A_{\frac{t}{1-1/t}} + B_{\frac{t}{1-1/t}} = 0\) and the section \(\Re t\) of the axis between them. When \(C\) is distorted it encloses the poles at zero, at one and the infinity so the contributions of these poles with residues \(\sqrt{-\epsilon}, \sqrt{-B}\) and \(-\sqrt{A - \epsilon}, \text{respectively},\) should be taken into account.

Since the poles at \(t = 0\) and \(t = 1\) are encircled clockwise that is in opposite direction compare with the pole at \(t = \infty\) which is encircled counterclockwise, their contribution enters with opposite signs and equals to \(2\pi i(\sqrt{-\epsilon} + \sqrt{A - \epsilon} - \sqrt{-B})\) which yields to the following equation for the WKB energy levels:

\[
-\frac{\sqrt{2ma^2}}{\hbar}(\sqrt{\epsilon + \sqrt{A - \epsilon} - \sqrt{B}}) = (n + \frac{1}{2})
\]

(48)

This equation coincides with given in work \(13\).

To compute energy levels in the quasilinear approximation we have to use the QLM equation \(43\) which after switching to variable \(t\) has a form

\[
-\frac{t}{a} \frac{dy_p}{dt} = \frac{y_p^2}{2} - 2y_p y_{p-1} + \epsilon + A \frac{t}{1-t} + B \frac{t}{(1-t)^2}
\]

(49)

For convenience of further computations we set here \(\frac{2m}{\hbar}\) equal to unity. The quantization condition \(27\) in variable \(t\) is given by

\[
J_p = -a \int_C \frac{y_p(t)}{t} \, dt = i 2\pi n, \quad p = 1, 2, \ldots
\]

(50)

At the singular point \(t \sim 0\) of the integrand of Eq. \(51\) Eq. \(49\) reduces to

\[
-\frac{t}{a} \frac{dy_p}{dt} = y_p^2 - 2y_p y_{p-1} + \epsilon
\]

(51)

whose solution is \(y_p = c_p\), where \(c_p\) is a constant satisfying the algebraic equations \(c_p^2 - 2c_p c_{p-1} + \epsilon\). Since at large \(p\) we expect \(y_{p-1} \to y_p \to y\), where \(y\) is an exact solution at \(t = 0\), it means, in view of \(c_p\) being a constant, that should be \(c_p = c_{p-1} = c\), that is we are looking for a fixed point solution of Eq. \(51\) which is \(c_p = \sqrt{\epsilon}\). The positive sign before the root is chosen since the first term in expansion of \(y_p(t)\) in the WKB terms is \(ik(t)\). Thus \(y_p(t) \approx ik(t) = \sqrt{\epsilon + A \frac{t}{1-t} + B \frac{t}{(1-t)^2}}\) that is \(y_p(0) \approx \pm \sqrt{\epsilon}\).

Another singular point of the integrand of Eq. \(51\) lays at \(t \sim \infty\), since change of variable \(v = \frac{1}{t}, t \to \infty\) when \(v \to 0\), converts \(y_p(t) \frac{dv}{t}\) into \(-y_p(t) \frac{dv}{v}\) which has pole at \(v = 0\). At \(t \sim \infty\) Eq. \(49\) reduces to

\[
-\frac{t}{a} \frac{dy_p}{dt} = y_p^2 - 2y_p y_{p-1} + \epsilon - A
\]

(52)

The solution of this equation one can look in the form \(y_p = d_p\) where \(d_p\) is a constant which satisfies the algebraic equation \(d_p^2 - 2d_p d_{p-1} + \epsilon = 0\). The fixed point \(d_p = d\) of this equation is \(d = \sqrt{\epsilon - A}\) so that the residue of the integrand of Eq. \(51\) at \(t \sim \infty\) equals \(\sqrt{\epsilon - A}\).

Eq. \(49\) near its singular point \(t = 1\) has a form

\[
-\frac{1}{a} \frac{dy_p}{dt} = y_p^2 - 2y_p y_{p-1} + A \frac{1}{1-t} + B \frac{1}{(1-t)^2}
\]

(53)
Looking for solution of this equation in the form \( y_p = \frac{b_0}{t} \) we obtain for constants \( b_0 \), the recurrence relations \( \frac{1}{a_p} = \frac{b_p^2 - 2b_pb_{p-1} + B}{b_p} \) whose solution at the fixed point \( b_p = b \) of this equation is given by \( b = -\frac{1}{16} (1 - \sqrt{1 + 4a^2B}) \) and \( y_p \sim \frac{1}{b_p} t \) at \( t \sim 1 \). Again, the positive sign before the root is chosen from the same consideration as in the previous paragraphs namely since in zero WKB approximation the residue of \( y_p(t) \) at \( t \sim 1 \) should be of order \( +\sqrt{B} \).

Summing up, the integrand of Eq. (50) has the three residues \( \sqrt{c}, \sqrt{\epsilon - A} \) and \( -\frac{1}{2b} \) \( (1 - \sqrt{1 + 4a^2B}) \) at \( t = 0, t = \infty \) and \( t = 1 \) respectively. After the reinstatement of the factor \( \frac{2m}{\hbar^2} \) and taking into account that poles \( t = 0 \) and \( t = 1 \) are encircled in the opposite direction compare with the pole at the infinity (which is encircled counterclockwise) the expression (50) gives therefore

\[
\sqrt{\frac{2ma^2}{\hbar^2}} \left( -\sqrt{\epsilon - \sqrt{\epsilon - A}} \right) - \frac{1}{2} \left( 1 - \sqrt{1 + \frac{8ma^2B}{\hbar^2}} \right) = n
\]

(54)

This expression coincides with the exact expression for the Eckart potential given in ref. [10] and is different from the WKB quantization condition [48] for the energy obtained earlier in this section.

I. Three dimensional S-wave Eckart potential

\( V(x) = -\lambda \frac{e^{-\frac{a}{x}}}{1 - e^{-\frac{a}{x}}} + \frac{b_0}{(1 - e^{-\frac{a}{x}})^2}, \lambda, b > 0, 0 < r < \infty. \)

This potential is considered in works [21, 22, 23] where exact and WKB expressions for the energy levels were obtained. Introduction of a new variable \( t = e^{-\frac{a}{x}}, 0 < t < 1 \) reduces the WKB quantization condition [23] to

\[
\frac{\sqrt{2ma^2}}{\hbar} \int_C \left( -\sqrt{\epsilon - \frac{t}{1 - t}} - b \frac{t}{(1 - t)^2} \right) dt = 2\pi(n + \frac{1}{2}).
\]

(55)

The integrand has poles at \( t = 0, t = 1 \) and \( t = \infty \) with the residues \( \sqrt{-\epsilon}, \sqrt{-b} \) and \( \sqrt{-\epsilon - \lambda} \) respectively. The calculation of the integral therefore gives [23] \( 2\pi i (\sqrt{-\epsilon} - \sqrt{-b} + \sqrt{-\epsilon - \lambda}) \) so that the WKB energy levels could be computed from expression

\[
\sqrt{\frac{2ma^2}{\hbar^2}} (-\sqrt{\epsilon + \sqrt{\epsilon + \lambda} - \sqrt{b}}) = n + \frac{1}{2}
\]

(56)

On the other side, computation of the energy levels in the quasilinear approximation, following the steps, outlined in previous sections leads to the expression

\[
\sqrt{\frac{2ma^2}{\hbar^2}} (-\sqrt{\epsilon + \sqrt{\epsilon + \lambda} - \sqrt{b}}) - \frac{1}{2} \sqrt{1 + \frac{8mb}{\hbar^2}} = n + \frac{1}{2}
\]

(57)

which is different from the WKB expression (50) and coincides with the exact expression for the energy levels, calculated in work [23].

J. Three dimensional S-wave Hulthen potential

\( V(x) = -\lambda \frac{e^{-\frac{a}{x}}}{1 - e^{-\frac{a}{x}}} - b, \lambda > 0, 0 < r < \infty. \)

This potential is used in the nuclear physics and is a special case of the Eckart potential with \( b = 0 \). The Eqs. (50) and (57) for the WKB and QLM energy levels respectively degenerate to

\[
\sqrt{\frac{2ma^2}{\hbar^2}} (-\sqrt{\epsilon + \sqrt{\epsilon + \lambda}}) = n + \frac{1}{2}
\]

(58)

and

\[
\sqrt{\frac{2ma^2}{\hbar^2}} (-\sqrt{\epsilon + \sqrt{\epsilon + \lambda}} - \frac{1}{2}) = n + \frac{1}{2}
\]

(59)

so that the WKB and QLM energy eigenvalues are explicitly given by somewhat different expressions

\[
E_n = -\frac{\hbar^2}{2m} \frac{(n + \frac{1}{2})^2 - \frac{2ma^2}{\hbar^2}}{4a^2(n + \frac{1}{2})^2}
\]

(60)

and

\[
E_n = \frac{\hbar^2}{2m} \frac{(n + 1)^2 - \frac{2ma^2}{\hbar^2}}{4a^2(n + 1)^2}
\]

(61)

Here \( n = 0, 1, 2, \ldots \) The last expression coincides with the exact expression for the energy levels calculated in works [17, 19].

K. One dimensional Morse potential

\( V(x) = Ae^{-2x^2} - Be^{-\frac{x}{2}}, A, B > 0, -\infty < x < \infty. \)

This potential, introduced in work [24] is heavily used in molecular physics computations to describe interactions between two molecules and for description of vibrations of two-atomic molecules. Introduction of a new variable \( t = e^{-\frac{x}{2}}, 0 < t < \infty \) changes the WKB quantization condition [23] to
The integrand has poles at $t = \infty$ and at $t = 0$ with the residues $\sqrt{-\epsilon}$ and $\frac{B}{2\sqrt{A}}$, respectively. The calculation of the integral therefore gives \[ \int_{C} \sqrt{-\epsilon - \frac{A}{t^2} + \frac{B}{t}} \, dt = 2\pi(n + \frac{1}{2}). \] (62)

The solution of this equation near $t = 0$ has a form

\[ y_p = \frac{b_p}{t} + dp \]

where $a_p$ and $b_p$ are constants satisfying the algebraic equations

\[ b_p^2 - 2b_p dp + A = 0, \]
\[ \frac{b_p}{a} = 2b_p dp - 2(bp dp - b dp) - B. \] (69) (70)

The fixed points $b_p = b_{p-1}$ and $d_p = d_{p-1} = d$ of this equation are $b = \sqrt{A}$ and $d = \frac{B}{2\sqrt{A}}$. From all this the residues of the integrand at $t = \infty$ and $t = 0$ are $\sqrt{\epsilon}$ and $\frac{1}{2a} - \frac{B}{2\sqrt{A}}$, respectively. Taking into account that the pole at $t = 0$ is encircled in the opposite direction compare with the pole at the infinity (which is encircled counterclockwise) the expression \[ (65) \] gives therefore

\[ a(\sqrt{\epsilon} - (\frac{1}{2a} - \frac{B}{2\sqrt{A}})) = n \] (71)

This expression after reinstating the factor $\frac{2m}{\hbar}$ coincides with \[ (67) \] which means that the QLM expression for the energy levels in the Morse potential coincides with the expression \[ (64) \] for the WKB energy levels obtained earlier in this section. It coincides also with the exact expression for the energy levels, see, for example, ref. \[ (22) \]. The coincidence of the exact and the WKB eigenvalues for the Morse potential is consequence of the fact that the introduction of a variable $u = 2\sqrt{2m\hbar} t^{-\frac{1}{2}}, 0 < u < \infty$ reduces the Schrödinger equation with this potential to the radial Coulomb Schrödinger equation \[ (23) \] which, as it is well known, yields exact energy levels also in the WKB approximation.

\[ \sqrt{2ma^2} \int_C \sqrt{-\epsilon - \frac{A}{t^2} + \frac{B}{t}} \, dt = 2\pi(n + \frac{1}{2}). \]

for the energy levels.

To compute energy levels in the quasi-linear approximation one has to use the QLM equation \[ (10) \] which in variable $t$ has a form

\[ \frac{t \, dy_p}{a} \frac{dp}{dt} - y_p^2 - 2y_p y_{p-1} + \epsilon + \frac{A}{t^2} - \frac{B}{t} \]

(65)

We set here and further $\frac{2m}{\hbar}$ equal to unity. The quantization condition \[ (27) \] in variable $t$ is given by

\[ J_p = a \int_C \frac{y_p(t)}{t} \, dt = i 2\pi n, \, p = 1, 2, \ldots \] (66)

At the singular point $t \sim \infty$ of the integrand of Eq. \[ (50) \], Eq. \[ (19) \] reduces to

\[ \frac{t \, dy_p}{a} \frac{dp}{dt} = y_p^2 - 2y_p y_{p-1} + \epsilon \] (67)

whose solution is $y_p = c_p$, where $c_p$ is a constant satisfying the algebraic equations $c_p^2 - 2c_p c_{p-1} + \epsilon$. Since at large $p$ we expect $y_{p-1} \rightarrow y_p \rightarrow y$, where $y$ is an exact solution at $t = 0$, it means, in view of $c_p$ being a constant, that should be $c_p = c_{p-1} = c$, that is we are looking for a fixed point solution of Eq. \[ (11) \] which is $c_p = \sqrt{\epsilon}$. As in previous paragraphs, the positive sign before the root is chosen since the first term in expansion of $y_p(t)$ in the WKB terms is $ik(t)$. Thus $y_p(t) \simeq ik(t) = \sqrt{\epsilon + \frac{A}{t^2} - \frac{B}{t}}$ that is $y_p(\infty) \simeq + \sqrt{\epsilon}$.

Another singular point of the integrand of Eq. \[ (50) \] lays at $t \sim 0$, where Eq. \[ (53) \] reduces to

\[ \frac{t \, dy_p}{a} \frac{dp}{dt} = y_p^2 - 2y_p y_{p-1} + \frac{A}{t^2} - \frac{B}{t} \] (68)

V. CONCLUSION

We have shown that the quasi-linearization method (QLM) which approaches solution of the Riccati-Schrödinger equation by approximating the nonlinear terms by a sequence of the linear ones, and is not based on the existence of a smallness parameter, sums the WKB series.

The advantage of the quasi-linear approach is that each $p$-th QLM iteration is expressible in a closed integral form. We have proved that its expansion in powers of $\hbar$ reproduces the structure of the WKB series generating an infinite number of the WKB term with $2^p$ terms of the expansion reproduced exactly and a similar number approximately. As a result one expects that the exact quantization condition \[ (26) \] with integrand replaced by any QLM iterate \[ (27) \] including the first gives more accurate energy than the WKB quantization condition \[ (23) \] which is obtained by substituting into exact quantization condition of the WKB expansion up to the first power of $\hbar$ and neglecting all higher powers of $\hbar$. We show on many examples that it is indeed so and the approximation by QLM...
iterates leads to exact energies for many well known physical potentials with such as the Coulomb, harmonic oscillator, Pöschl-Teller, Hulthen, Hylernaas, Morse, Eckart etc.

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