THE UNSTEADY TRANSONIC SMALL DISTURBANCE EQUATION: DATA ON OBLIQUE CURVES

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ABSTRACT. We propose and solve a new problem for the unsteady transonic small disturbance equation. Data are given for the self-similar equation in a fixed, bounded region of similarity space, where on a part of the boundary the equation has degenerate type (a ‘sonic line’) and on the remainder it is elliptic. Previous results on this problem have chosen data so that the solution is constant on the sonic line, but we set up a situation where the solution is not constant on the sonic part of the boundary. The solution we find is Lipschitz up to the boundary. Our solution sets the stage for resolution of some interesting Riemann problems for this equation and for other multidimensional conservation laws.

1. Introduction. The unsteady transonic small disturbance (UTSD) equation has served for some time now as a prototype system of multidimensional conservation laws. Originally developed as an approximation to ideal compressible gas flow in a situation where the dominant flow is in a single direction, the equations, nondimensionalized for mathematical convenience, read

\[ u_t + \left( \frac{u^2}{2} \right)_x + v_y = 0, \]
\[ u_y - v_x = 0. \]

(1)

A brief derivation of this system can be found in [11]. Here \((u, v)\) is the perturbation velocity in the \(x\) and \(y\) directions; these are scaled differently from each other so that a weak nonlinearity in the dominant flow (in the \(x\) direction) balances weak diffraction in the \(y\) direction. The coordinates move with the unperturbed flow, which is why \(t\) is not a time-like variable. With respect to an appropriate hyperbolic direction, the system is strictly hyperbolic and genuinely nonlinear. The nonlinearity of characteristic speeds in the variable \(u\) is similar to the nonlinearity in Burgers’ equation; in fact, this system is sometimes called the ‘two-dimensional
Burgers’ equation”. These properties make (1) a useful model for studying some features of multidimensional compressible flow. In particular, it is used to model reflection of weak shocks (where it gives results that are quantitatively as well as qualitatively correct [11, 20, 22]).

Shock reflection problems have an additional feature of self-similarity: solutions are functions of $(\xi, \eta) = (x/t, y/t)$ alone. In self-similar variables the system becomes

\begin{align}
-\xi u_\xi - \eta u_\eta + \left(\frac{u^2}{2}\right)_\xi + v_\eta &= 0, \\
u_\eta - v_\xi &= 0 .
\end{align}

(2)

It is straightforward to replace (2) by a single second-order equation for $u$,

\begin{align}
(-\xi u_\xi - \eta u_\eta + u u_\xi)_\xi + u u_\eta &= 0,
\end{align}

(3)

and we will take this as our point of departure for what follows.

The equation (3) is hyperbolic or elliptic according as $\xi^2 + \eta^2/4 - u(\xi, \eta) > 0$ or $\xi^2 + \eta^2/4 - u(\xi, \eta) < 0$. Self-similar change of type is due to sub- or supersonic flow in the original equation, and one expects the system to behave somewhat like the equations of steady transonic flow. However, the fact that the sonic locus couples position $(\xi, \eta)$ and velocity $(u, v)$ contrasts with the more familiar situation of steady transonic flow, and this distinction motivates further study of the mathematical properties of (3).

1.1. History. The modern mathematical study of self-similar problems for the UTSD equation began a little under 25 years ago with work of Brio and Hunter [1] and of Tabak and Rosales [19]; shortly after that it was taken up by Čanić, Keyfitz and a number of co-workers [2]-[5], [13]. Early studies focused on problems related to regular shock reflection. In a typical problem, piecewise constant Riemann data, chosen to produce shocks, result in piecewise constant solutions far from the origin in the self-similar variables, where the system is hyperbolic. Change of type typically coincides with interaction between incident and reflected waves. Thus, the interesting questions for (3) typically involve the subsonic region, where the equations are elliptic. When the subsonic flow lies downstream from a reflected shock, the solution is discontinuous across the shock, and the resulting weak solution satisfies the Rankine-Hugoniot conditions, which couple the shock position with the downstream flow. Thus, free boundary problems for self-similar equations arise naturally, and have been considered by a number of researchers; besides the references above, we mention work of Chen and Feldman, for example [6], of S.-X. Chen [7] and of Elling and Liu [8]. The research of these authors involves mainly the transonic potential equation, which has similar behavior. This list is by no means complete.

However, a different type of boundary value problem for (3), in which the equation is degenerate elliptic and changes type, occurs when there is a continuous transition between supersonic and subsonic flow. A simple instance of this, which has been well-studied at this point, occurs in the case of “weak regular reflection”. Immediately behind the reflection point, where a shock hits a wall, say, the flow is supersonic. This flow is constant and is easily found from the shock polar equations. Further downstream the flow interacts with the reflected shock, as described in the previous paragraph, and there is a continuous transition from the constant supersonic flow to a non-uniform subsonic flow across the sonic line.
The nature of this supersonic-to-subsonic transition is interesting. In two early studies [2, 3], Čanić and Keyfitz looked at two possible cases, and found solutions to (3) in a region in which part of the boundary, corresponding to constant data outside, is sonic (so the sonic boundary is given by $\xi + \eta^2/4 = u_0$ for a constant $u_0$); strictly subsonic data are given on the rest of the boundary. There was a qualitative difference between the solutions we found, depending on whether the data on the strictly subsonic boundary satisfied $u > u_0$ or $u < u_0$. In the first case, the solution displays a square-root singularity at the boundary. A moment’s thought reveals that this is exactly how the solution to a linear wave equation, in the same self-similar variables, will behave at a “sonic boundary” – that is to say, on the wave cone. (In [21], this linearization is applied to the UTSD equation.)

And a further small calculation (first carried out in [14]) proves that this solution, coupled with the constant solution on the supersonic side of the boundary, does not yield a weak solution of the original problem. (If the equation is linear, of course, one does obtain a weak solution.) The resolution of this paradox is that in this case the solution must be discontinuous across the sonic line. There will be a transonic shock, whose position is not known a priori but would be found by solving a free boundary problem. On the other hand, if $u < u_0$ on the subsonic boundary, the solution is Lipschitz up to the boundary, and we do have a weak solution of the original problem.

An explanation of this dichotomy in terms of causality is as follows. If one looks at the characteristics of (3), in the self-similar variables, for a constant solution with $u(\xi, \eta) = u_0$, then for a point in the hyperbolic region the forward wave cone is tangent to the sonic parabola $\xi + \eta^2/4 = u_0$. In particular, these characteristics determine what data influence the solution up to the sonic line. However, when the solution is not constant, then the position of the sonic line is known only locally, and data inside the sonic curve for $u_0$ may influence the solution outside the curve, and hence the position of the sonic line itself. When $u < u_0$ everywhere in the subsonic region, the sonic curves for values of $u$ with $u < u_0$ lie inside the parabola $\xi + \eta^2/4 = u_0$ and so the solution is determined, and is $u = u_0$, all the way up to the sonic line; but when $u > u_0$ somewhere inside the original sonic parabola, then this determinacy no longer holds, and a transonic shock may form outside the parabola $\xi + \eta^2/4 = u_0$.

When the supersonic data are not constant, as for example in the case of a rarefaction wave interacting with a wall, the situation is different yet again. Now a standard analysis of the characteristics shows that they are no longer tangent to the sonic line. Furthermore, where the characteristics meet the sonic line they are of course tangent to each other, and one is an ingoing, the other an outgoing, characteristic. In short, the sonic line behaves like a time-like rather than a space-like curve, and there is a two-way interaction between the supersonic and the subsonic solution. Figure 1 gives a sketch of this for the UTSD equation; in this illustration, a rarefaction wave interpolates between two constant states $u = u_0$ and $u = u_1 < u_0$.

Characteristics are tangent to the sonic boundary in parts of the domain where the supersonic solution is constant, but not where the supersonic solution varies. Numerical experiments with this situation, for data that correspond to the rarefaction reflecting from a wall, are reported in [21]. It appears that a shock develops, downstream, and that the subsonic flow influences the supersonic flow, so that the part of the sonic boundary shown as horizontal in Figure 1 is perturbed. Recently
Jegdić and Jegdić [12], assuming that the boundary is fixed, solved the resulting Goursat problem in the hyperbolic region. Our paper provides a complementary perspective, by looking at the elliptic problem in a situation modeled on this illustration, again assuming that the boundary is fixed. Our main result is that the solution to the degenerate elliptic problem is very similar to the situation reported in [3], the case of constant data \( u = u_0 \) on the sonic line with data \( u < u_0 \) on the nondegenerate boundary: A Lipschitz solution to the problem with a fixed boundary exists, and forms a weak solution to the original problem.

1.2. Outline of the Paper. In the next section we set up the problem. We state the precise result in Section 3, and give the proof in Section 4. In Section 5 we comment on the implications for problems such as the one outlined in Figure 1.

2. The Degenerate Elliptic Boundary Value Problem. As indicated in the introduction, there are many prototype problems for the UTSD equation. In this section, we set up the boundary value problem of interest for this paper.

2.1. Reduction to a Standard Form. We introduce so-called parabolic coordinates, with \( \rho = \xi + \eta^2/4 \) replacing \( \xi \). Then (3) becomes

\[
(u - \rho)u_{\rho} + \frac{1}{2}u_{\rho} + u_{\eta\eta} = 0.
\]  

(4)

For convenience in this study we replace \( \rho \) by \(-x\) and \( \eta \) by \( y \) (not to be confused with the original independent variables in (1)) to obtain

\[
(u + x)u_x - \frac{1}{2}u_x + u_{yy} = 0.
\]  

(5)
The flow is sonic at any point \((x_0, y_0)\) where \(u(x_0, y_0) = -x_0\). For a constant solution, the sonic boundary is the line \(x = -u_0\). If the supersonic flow were constant, one could formulate a problem in the subsonic region, which one expects to be \(\{x > x_0\}\), with boundary condition \(u(x_0, y) = -x_0\) and \(u(x, y) = u_1(x, y) > -x\) given at some point(s) \((x, y)\) with \(x > x_0\). For such a problem, \(x = x_0\) represents the position of the sonic line, known a priori. This is the situation that occurs, locally, in the case of weak regular reflection. In weak regular reflection, the downstream flow is also coupled with the free boundary problem for the reflected shock, along which there is a failure of uniform ellipticity. This situation has been dealt with, for example in [4].

To study more general boundary value problems for (5), we here solve the problem of data which are sonic on a given oblique part of the boundary (that is, on a curve on which \(x\) is not constant), and strictly subsonic on the remainder of the boundary.

2.2. **Boundary Conditions on Degenerate Boundaries.** Linear elliptic equations for which ellipticity fails on a portion of the boundary have been well-studied. The text by Olevnik and Radkevich, [17], while now somewhat out of date, gives a comprehensive presentation. The classic condition for determining what parts of the boundary are appropriate for a boundary condition was given by Fichera, [9, 10], and is known as the **Fichera condition**. For a second-order operator, with \((a^{ij})\) positive semi-definite,

\[
L = \sum a^{ij}(x)\partial_i \partial_j + \sum b^k(x)\partial_k + c(x),
\]

and a point \(x\) on the boundary with inward normal \(\nu(x)\), boundary conditions can be prescribed if the Fichera function

\[
b(x) \equiv \sum \left( b^k(x) - \sum \partial_j a^{kj}(x) \right) \nu_k(x) < 0.
\]

In [15] it was noted that if \(L\) is cast in divergence form then \(b\) is precisely the first-order part of the operator, and it was suggested there that this might be a reasonable generalization of the condition to quasilinear operators. This hypothesis is consistent with the example studied in [3], and with the finding that the problem of weak regular reflection for the UTSD equation with constant boundary data on
the sonic line is properly posed in the subsonic region, as was proved in [4]. The “sonic boundary” for that problem, using the coordinates in (5), is \( x = x_0 \), and the domain lies to the right of that boundary. By contrast, if a portion of the boundary had the form \( y = y_0 \), the Fichera condition \( b < 0 \) would be violated for the UTSD equation. (We should emphasize that the inequality in Fichera’s condition is strict.) For the UTSD equation, at least, this is a natural restriction, since, returning to the original space-time coordinates, it can be seen that \( y = y_0 \) is a characteristic surface.

The question left unanswered in previous research is what happens to boundary value problems for (5) if degenerate data are given on an oblique boundary, a curve of the form \( y = Y(x) \) with \( Y' \neq 0 \), with the data \( u(x, Y(x)) + x = 0 \), and a domain lying to the right of this curve. This is the question we answer here.

For simplicity, we take the degenerate portion of the boundary to be a straight line, \( x + y = 0 \). Any other oblique line, \( x + ky = 0 \), can be handled by scaling, and the extension to a smooth curve \( y = Y(x) \) with \( Y' \) bounded would be a technical exercise. Thus in this paper we consider equation (5) in a region \( \Omega \) like that pictured in Figure 2. We assume that \( \Omega \) is bounded and lies to the right of \( \Gamma_2 \), which is a segment of the line \( x + y = 0 \), and that the right boundary of \( \Omega \) is a smooth curve \( \Gamma_1 \) on which smooth data of the form \( u = \tilde{g} \) with \( \tilde{g} + x > 0 \) are given. We will give a further condition on \( \tilde{g} \) later, in equation (10).

3. Construction of a Weak Solution. The key to obtaining the boundedness and continuity of the quasilinear operator is the existence of convenient subsolutions and supersolutions.

3.1. Subsolutions and Supersolutions. We begin by constructing a family of exact solutions to (5).

**Proposition 1.** Let \( L = \max_{\Omega}(x + y) \). Then for any value \( c > (L - \sqrt{L^2 + 4})/2 \), the function

\[
    u_c(x, y) = \frac{(c - 1) \sqrt{1 + c(x + y) - 1}}{c} + y = \frac{(c - 1)(x + y)}{\sqrt{1 + c(x + y)} + 1} + y, \tag{9}
\]

defined for \( x + y \geq 0 \), is a solution of (5) in \( \Omega \), for which \( u(x, -x) + x = 0 \) (that is, (5) is degenerate on the line \( x + y = 0 \)), and for which the equation is strictly elliptic when \( x + y > 0 \).

**Proof.** This is easily verified by differentiation and calculation of \( u_c + x \). We give the details of the construction in Appendix A.

For the remainder of this paper, we consider functions \( \tilde{g} \) on \( \Gamma_1 \), such that for some \( c^+ \) and \( c^- \), with \( c^+ > s^- \) and \( c^- \) satisfying (8),

\[
    u_{c^-} \leq \tilde{g} \leq u_{c^+}. \tag{10}
\]

We begin by extending the function \( \tilde{g} \) to \( \overline{\Omega} \) so that (10) holds and \( \tilde{g} = -x = y \) on \( \Gamma_2 \). In preparation for setting up the weak form of this problem, we reformulate
the problem we want to solve, which is
\[
((u + x)u_x - \frac{1}{2}u)_x + u_{yy} = 0 \quad \text{in} \quad \Omega, \\
u(x, -x) = -x = y \quad \text{on} \quad \Gamma_2, \\
u(x, y) = \tilde{g}(x, y) \quad \text{on} \quad \Gamma_1,
\]
(11)
as a problem for \( h = u - \tilde{g} = u - y - g \) with \( g = \tilde{g} - y \) (the reason for separating out the term \( y \) will become clear later):
\[
- \left( (h + g + x + y)h_x + (g_x - \frac{1}{2}h)_x \right) - h_{yy} = f \quad \text{in} \quad \Omega, \\
h = 0 \quad \text{on} \quad \Gamma_1 \cup \Gamma_2,
\]
(12)
with \( f = ((g + x + y)g_x - \frac{1}{2}g)_x + g_{yy} \).

We work in a weighted Sobolev space, \( X \), with norm defined by
\[
\|u\|_X = \int_\Omega \left( (x + y)u_x^2 + u_y^2 \right) dx dy,
\]
(13)
and take \( X \) to be the completion of \( C_0^\infty(\Omega) \) under this norm. For any \( w \in C_0^1(\Omega) \) and for any \( h \) that satisfies (12), we have
\[
B_g(h, w) = \int_\Omega \left[ (h + g + x + y)h_x + (g_x - \frac{1}{2}h)_x \right] w_x + h_yw_y \, dx \, dy = \int_\Omega fw \, dx \, dy
\]
(14)
for the bivariate form \( B_g \) defined by equation (14). We define the nonlinear operator \( T : X \to X^* \) by
\[
B_g(h, w) = \langle T(h), w \rangle \quad \text{where} \quad \langle u, w \rangle = \int_\Omega uw,
\]
and we say that \( h \in X \) is a weak solution of problem (12) if (14) holds for any \( w \in C_0^1(\Omega) \).

3.2. Main Theorem. Using the framework establishing in Section 3.1, we are now ready to state and prove the main result.

**Theorem 3.1.** For every \( f \in X^* \), there exists \( h \in X \) that solves (12) in the weak sense.

Our tool is the Browder-Minty Theorem [18]:

**Theorem 3.2 (Browder-Minty).** Let \( X \) be a real, reflexive Banach space and let \( T : X \to X^* \) be bounded, continuous, coercive and monotone. Then for any \( f \in X^* \) there exists a solution of the equation \( T(u) = f \).

We recall the definitions.

**Definition 3.3.** A mapping \( T : X \to X^* \) is bounded if it maps bounded sets in \( X \) to bounded sets in \( X^* \), and continuous if for every \( h \in X \) we have
\[
\|T(h) - T(k)\|_{X^*} \to 0 \quad \text{whenever} \quad \|h - k\|_X \to 0.
\]
We say the mapping is monotone if
\[
\langle T(h) - T(k), h - k \rangle \geq 0
\]
for \( h, k \in X \), and it is coercive if
\[
\frac{\langle T(h), h \rangle}{\|h\|} \to \infty \quad \text{as} \quad \|h\| \to \infty.
\]
The operator $T$ fails to be monotone, as did the corresponding operators in [2, 3]. This difficulty is handled by a variant of Theorem 3.2 in which monotonicity is replaced by a weaker condition, pseudo-monotonicity. Theorem 3.2, with this weaker condition, appears in Renardy and Rogers’ text, [18], as Theorem 9.57. In addition, a property called the calculus of variations type is used in this paper to replace pseudo-monotonicity, so the result we apply is actually Corollary 9.60 of [18]. The definitions and explanation are detailed in the next section.

4. Proof of the Theorem. Before beginning the proof, we note the following elementary fact:

**Proposition 2.** For any $h \in X$, $\int_\Omega h^2 \, dx \, dy \leq C \|h\|_X^2$.

**Proof.** Assume $h \in C^1_0(\Omega)$ and write $h(P) = h(x, y) = h(x, y) - h(x_0, y_0)$ where $P_0 = (x_0, y_0) = ((x - y)/2, -(x - y)/2)$ is the point on $\Gamma_2$ such that $P_0P$ is perpendicular to $\Gamma_2$. Now,

$$h(x, y) = \int_{P_0}^P h_t \, dt;$$

therefore,

$$|h(x, y)| \leq \left( \int_{P_0}^P h_t^2 \, dt \right)^{1/2} \left( \int_{P_0}^P \, dt \right)^{1/2},$$

and so

$$\int \int_\Omega h^2 \, dx \, dy \leq \int \int_\Omega \int_{P_0}^P h_t^2 \, dt (x + y) \, dx \, dy \leq C \int \int_\Omega ((x + y)h_x^2 + h_y^2) \, dx \, dy,$$

and since $C^1_0$ is dense in $X$, the proposition is proved.

The operator $T$, as defined, is not bounded (because of the appearance of $h$ in the coefficient of $h_x$ in $B_g$). For a given problem, with $c\pm$ determined, we may modify $T$ by modifying that coefficient to be $u_{c\pm} + x$ outside the range $u_{c-} < u < u_{c+}$; the coefficient becomes

$$u_{c-} + x, \quad \text{if} \quad u < u_{c-},$$

$$h + g + x + y, \quad \text{if} \quad u_{c-} \leq u \leq u_{c+},$$

$$u_{c+} + x, \quad \text{if} \quad u > u_{c+},$$

(recall $u = h + g + y$). A simple argument at the end of the proof will show that this cutoff can be removed, so we do not introduce a new notation for it.

Now we can show that $\langle T(h), w \rangle$ defines a bounded operator for $w \in X^*$.  

**Proposition 3.** For a fixed $h$, $|\langle T(h), w \rangle| \leq C\|w\|_{X^*}$.

**Proof.** We have

$$|\langle T(h), w \rangle| = \int_\Omega \left[ \frac{h + g + x + y}{\sqrt{x + y}} h_x + \frac{h(g_x - \frac{1}{2})}{\sqrt{x + y}} \sqrt{x + y} w_x + h_y w_y \right].$$

Now we use the Cauchy-Schwarz inequality, noting that we can assume $|h + g| \leq C(x + y)$ by our modification in (15) of the operator and the explicit form of $u_{c\pm}$ in (9), so $|h/\sqrt{x + y}| \leq C\sqrt{x + y}$ for a constant $C$. The result follows.

**Proposition 4.** The operator $T$ is continuous.
Proof. We calculate
\[ \langle T(h) - T(k), w \rangle = \int_{\Omega} \left[ (h + g + x + y)h_x + h(g_x - \frac{1}{2})w_x + h_y w_y ight. \]
\[ - [(k + g + x + y)k_x + k(g_x - \frac{1}{2})]w_x - k_y w_y \]
\[ = \int_{\Omega} \left[ (h + g + x + y)(h - k)_x + (h - k)(g_x - \frac{1}{2})]w_x + (h - k)_y w_y \right. \]
\[ + \int_{\Omega} (h - k)k_x. \]
In [3] a similar decomposition of the coefficients of \(B_g\) was used to establish that the symbol of \(T\) is a so-called Nemytskii operator (see [18]) and hence is continuous. The same argument applies here; we omit the details.

**Proposition 5.** The operator \(T\) is coercive.

**Proof.** We have
\[ \langle T(h), h \rangle = \int_{\Omega} \left[ (h + g + x + y)h_x + h(g_x - \frac{1}{2})]h_x + h_y^2 \right. \]
and by assumption
\[ \frac{h + g + x + y}{x + y} \geq \frac{u_x - x}{x + y} = \frac{c^- - 1}{\sqrt{1 + c^- (x + y) + 1}} + 1. \]
A calculation shows that the bound (8) for \(c^-\) gives a positive lower bound for this expression, and this in turn gives a lower bound as \(\|h\| \to \infty:\)
\[ \langle T(h), h \rangle \geq C\|h\|^2_X. \]

4.1. **The Calculus of Variations Type.** In [18], Renardy and Rogers prove that a weaker condition than monotonicity will serve to give the conclusion of Theorem 3.2. They call this condition, which is basically monotonicity in the highest-order derivative terms, pseudo-monotonicity. They show, further, that a simple condition on the operator, that it be of the calculus of variations type is sufficient to give pseudo-monotonicity and hence, combined with boundedness, continuity and coercivity, to give the conclusion of the Browder-Minty theorem. We recall this result.

**Theorem 4.1** (Corollary 9.60 of [18]). Let \(X\) be a real reflexive Banach space and suppose \(T : x \to X^\ast\) is continuous, coercive and of the calculus of variations type. Then for every \(f \in X^\ast\) there exists a solution \(h \in X\) of the equation \(T(h) = f\).

The verification of the calculus of variations type for our problem is similar to that in [3]; we give the definition here for completeness, and because the concept may not be well-known. More detail can be found in [3] and [18].

**Definition 4.2.** A bounded operator \(T\) on a real reflexive Banach space is of the calculus of variations type if it has the representation \(T(h) = \tilde{T}(h, h)\) where \(\tilde{T} : X \times X \to X^\ast\) has the properties
1. For each \(h \in X\), the mapping \(k \mapsto \tilde{T}(h, k)\) is bounded and continuous from \(X\) to \(X^\ast\) and for all \(k\), \(\langle \tilde{T}(h, h) - \tilde{T}(h, k), h - k \rangle \geq 0,\)
2. For each \( k \in X \), the mapping \( h \mapsto \hat{T}(h, k) \) is bounded and continuous from \( X \) to \( X^\ast \).

3. If \( h_j \to h \) weakly in \( X \) and \( \langle \hat{T}(h_j, h_j) - \hat{T}(h_j, \bar{h}), h_j - \bar{h} \rangle \to 0 \), then for every \( k \in X \), \( \hat{T}(h_j, k) \to \hat{T}(\bar{h}, k) \) weakly in \( X^\ast \).

4. If \( h_j \to \bar{h} \) weakly in \( X \) and \( \hat{T}(h_j, k) \to \psi \) weakly in \( X^\ast \), then \( \langle \hat{T}(h_j, k), h_j \rangle \to \langle \psi, \bar{h} \rangle \).

The definition and the following arguments operate in our case by recognizing that the nonlinearity in \( T \) is quadratic, and by separating the quadratic components in a natural way. For our example, then, we define

\[
\langle \hat{T}(h, k), w \rangle = \int_{\Omega} [(h + g + x + y)k_x + h(g_x - \frac{1}{2}y)w_x + k_yw_y].
\]  

(16)

The boundedness and continuity of \( \hat{T} \) follow from the properties we have established for \( T \); the second part of property 1 of Definition 4.2 is now monotonicity in the highest order derivatives of \( T \), which is a consequence of ellipticity (and allows for the elliptic degeneracy). For our example, again, the third and fourth properties of Definition 4.2 amount to compactness in the lower-order terms of \( T \). Specifically, verification of this part of Definition 4.2, for the operator \( \hat{T} \) of equation (16), follows from a straightforward extension of a compactness result proved in [3], whose proof we omit:

**Proposition 6.** If \( h_n \to h \) in \( X \), then \( h_n/\sqrt{x+y} \to h/\sqrt{x+y} \) in \( L^2(\Omega) \).

**Proof of Theorem 3.1.** The modified operator \( T \) satisfies the hypotheses of Theorem 4.1, so there exists a solution for any \( f \in X^\ast \). Furthermore, under our hypotheses on \( \bar{g} \), the functions \( u_{c^+} \) and \( u_{c^-} \) form super- and subsolutions respectively, and hence we can remove the cutoffs. These combine to give a proof of the main theorem. \( \square \)

4.2. **Regularity.** In any compact subset of \( \Omega \), the norm on \( X \) is equivalent to the standard Sobolev norm on \( H^1_0 \), and it follows from the sub- and supersolution bounds on \( h \) that the equation is uniformly elliptic. Hence, if \( \bar{g} \in C^{\alpha} \), interior regularity follows from the usual properties of elliptic equations, and we can obtain Schauder estimates in \( C^{2+\alpha} \) for \( u \). Regularity up to the degenerate boundary holds under certain conditions. Specifically, we have

**Corollary 1.** If the extension of the data, \( \bar{g} \), is differentiable up to the degenerate boundary \( \Gamma_2 \), then \( u \) is Lipschitz up to \( \Gamma_2 \).

**Proof.** We can construct local sub- and supersolutions with \( c^+ \) and \( c^- \) arbitrarily close together. The derivative of \( u_c + x \) (which is a function of \( x + y \)) at \( x + y = 0 \) is \( c + 1/2 \). Coupled with the fact that \( u \) is differentiable in the interior of \( \Omega \), this gives uniform bounds on \( \nabla u \) and we conclude that \( u \) and \( h \) are Lipschitz up to \( \Gamma_2 \). \( \square \)

This is important, because Lipschitz bounds are needed to ensure that the solution can be extended to a weak solution in a neighborhood of \( \Gamma_2 \).

5. **Conclusions.** For a problem with oblique (variable) data, we have found a solution similar to that of the problem in [3], with constant data. Because the proofs of the Browder-Minty theorem and its extension, Theorem 4.1, rely on compactness, they do not permit us to assert uniqueness of the solution. (This was the case, also, in [2] and [3].) Uniqueness will require an additional argument.
We can state that if the hypotheses of Corollary 1 hold, and if the data on \( \Gamma_2 \) extend to a classical hyperbolic solution of (2) to the left of \( \Gamma_2 \), then the composite solution forms a weak solution to (2). For the data to extend, we would need the second component, \( v \), of velocity also to be continuous across \( \Gamma_2 \). This is typically not the case, since, as was mentioned in Section 1.1, a boundary like \( \Gamma_2 \) in our problem forms a time-like curve for the UTSD equation. In this respect, the oblique data curve problem differs significantly from the problem with constant data solved previously in [3]. We expect the resolution of this issue to involve a different type of free boundary problem, in which the position of \( \Gamma_2 \) is a priori unknown and is coupled with the data on both the hyperbolic and the elliptic sides. An example, involving the steady transonic small disturbance equation, was studied in [16], using a completely different method to show that at least some problems of this type have solutions.

Appendix A. Appendix: Calculation of Sub- and Super-solutions. We look for a solution \( u = F(x + y) + y \) to (5), with \( F(0) = 0 \). Let \( s = x + y \); then we obtain the ordinary differential equation in \( s \):

\[
\left( (s + F)F' - \frac{F}{2} \right)' + F'' = 0,
\]

with initial condition \( F(0) = 0 \). Integrating once, we have

\[
(s + F + 1)F' - \frac{F}{2} = A,
\]

for a constant of integration \( A \), and if we let \( F + 1 = G \), we have

\[
(s + G)G' = A + \frac{G - 1}{2} = a + \frac{G}{2}.
\]

We integrate this by writing

\[
\frac{ds}{dG} = \frac{s + G}{a + \frac{G}{2}},
\]

which has the solution, with a second integration constant, \( c \),

\[
s = c \left( a + \frac{G}{2} \right)^2 + 2a - 4 \left( a + \frac{G}{2} \right) = c \left( a + \frac{G}{2} \right)^2 - 2(a + G),
\]

(as is easily checked by differentiating). The condition \( F(0) = 0 \) becomes \( G(0) = 1 \) and gives a relation between the constants:

\[
c = \frac{2(a + 1)}{(a + \frac{1}{2})^2} = \frac{8(a + 1)}{(2a + 1)^2}.
\]

Solving the quadratic equation (18) for \( a + G/2 \) yields

\[
a + \frac{G}{2} = \frac{2 \pm \sqrt{4 + c(s - 2a)}}{c},
\]

whence

\[
G = \frac{4 - 2ac \pm 2\sqrt{4 - 2ac + cs}}{c}.
\]

Using (19), we obtain

\[
4 - 2ac = \frac{4}{(2a + 1)^2} = \frac{1}{(a + \frac{1}{2})^2},
\]
and if we use this in (20), along with (19), we finally obtain
\[ G = 1 \pm \frac{(\alpha - 1)\sqrt{1 + \alpha s}}{\alpha}, \]
where we have let \( 2(\alpha + 1) = \alpha \). For \( u = F + y = G - 1 + y \), we have
\[ u = \frac{\alpha - 1}{\alpha} \left( \sqrt{1 + \alpha(x + y)} - 1 \right) + y, \]
where the boundary condition \( u(x, -x) = -x \) has determined the choice of sign.

It is easy to verify that for any \( \alpha > 0 \) equation (21) gives a solution of equation (5) for which the equation is strictly elliptic, that is \( u + x > 0 \) for all \( (x, y) \) with \( x + y > 0 \). For a given domain \( \Omega \), we can also choose \( \alpha < 0 \), and we obtain a real solution as long as \( \alpha(x + y) > -1 \) or \( \alpha > -1/L \), where \( L = \max_{\Omega}(x + y) \); for the solution to be subsonic (elliptic) in \( \Omega \), we also need
\[ \alpha > \frac{L - \sqrt{L^2 + 4}}{2}. \]

For any \( L \), this is a more stringent bound than \( \alpha > -1/L \), and generates the restriction we employ in Proposition 1. In Section 3.1, we construct both sub- and supersolutions of this form.

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Received xxxx 20xx; revised xxxx 20xx.

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