Bargmann transform and generalized heat Cauchy problems

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Abstract

In this article we solve explicitly some Cauchy problems of the heat type attached to the generalized real and complex Dirac, Euler and Harmonic oscillator operators. Our principal tool is the Bargmann transform.

Keywords: Bargmann transform, heat Cauchy problem, Dirac operator, Euler operator, harmonic oscillator.

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1 Introduction

The classical Cauchy problem for the heat equation is of the form

\[
\begin{cases}
\Delta u(t, x) = \frac{\partial}{\partial t} u(t, x); & (t, x) \in \mathbb{R}_+^* \times \mathbb{R}^n \\
u(0, x) = u_0(x); & u_0 \in L^2(\mathbb{R}^n)
\end{cases}
\]

(1.1)

where \(\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}\) is the classical Laplacian of \(\mathbb{R}^n\). We can formally replace the operator \(\Delta\) by any partial differential operator and we speak about generalized heat Cauchy problem. Our aim in this paper, is to give a new method to solve explicitly some type of generalized heat Cauchy problems. Using essentially the Bargmann transform as an intertwining operator which intertwines some couple of real and complex differential operators. In this work we compute explicitly the exact solutions of the heat Cauchy problems associated to the generalized complex and real Dirac operators:
\[
\begin{aligned}
D^a U(t, z) &= \frac{\partial}{\partial t} U(t, z); \quad (t, z) \in \mathbb{R}^*_+ \times \mathbb{G} \\
U(0, z) &= U_0(z); \quad U_0 \in F^2_{\mathbb{G}}
\end{aligned}
\] (1.2)

and
\[
\begin{aligned}
d^a_x u(t, x) &= \frac{\partial}{\partial t} u(t, x); \quad (t, x) \in \mathbb{R}^*_+ \times \mathbb{R} \\
u(0, x) &= u_0(x); \quad u_0 \in L^2(\mathbb{R})
\end{aligned}
\] (1.3)

where the generalized real and complex Dirac operators are given by
\[
D^a = \frac{1}{a} \frac{\partial}{\partial z} + \frac{z}{2} \quad \quad d^a_x = \frac{\partial}{\partial x} - ax.
\] (1.4)

We solve also explicitly the following heat Cauchy problems associated to the generalized real and complex Euler operators:
\[
\begin{aligned}
e^a_x v(t, x) &= \frac{\partial}{\partial t} v(t, x); \quad (t, x) \in \mathbb{R}^*_+ \times \mathbb{R} \\
v(0, x) &= v_0(x); \quad v_0 \in L^2(\mathbb{R})
\end{aligned}
\] (1.5)

and
\[
\begin{aligned}
E^a_w Y(t, w) &= \frac{\partial}{\partial t} Y(t, w); \quad (t, w) \in \mathbb{R}^*_+ \times \mathbb{G} \\
Y(0, w) &= Y_0(w); \quad Y_0 \in F^2_{\mathbb{G}}
\end{aligned}
\] (1.6)

where the generalized real and complex Euler operators are given by
\[
E^a_w = -2aw \frac{\partial}{\partial w} - a, \quad e^a_x = ax \frac{\partial}{\partial x}.
\] (1.7)

Finally we find the explicit solutions of the following heat Cauchy problems for the real and complex harmonic oscillators:
\[
\begin{aligned}
h^a_x y(t, x) &= \frac{\partial}{\partial t} y(t, x); \quad (t, x) \in \mathbb{R}^*_+ \times \mathbb{G} \\
y(0, x) &= y_0(x); \quad y_0 \in L^2(\mathbb{R})
\end{aligned}
\] (1.8)

and
\[
\begin{aligned}
H^a_z V(t, z) &= \frac{\partial}{\partial t} V(t, z); \quad (t, z) \in \mathbb{R}^*_+ \times \mathbb{G} \\
V(0, z) &= V_0(z); \quad V_0 \in F^2_{\mathbb{G}}
\end{aligned}
\] (1.9)

where the real and complex harmonic oscillators are given by
\[
H^a_z = \frac{\partial^2}{\partial z^2} - \frac{a^2 z^2}{4} - \frac{a}{2}, \quad h^a_x = \frac{\partial^2}{\partial x^2} - a^2 x^2.
\] (1.10)

And the Fock space \( F^2_{\mathbb{G}} \) is defined in section 2. The main results of this paper are as follow
2 Bargmann transform

V. Bargmann ([1] and [2]) and later G. Folland [7] have introduced an isometry, called Bargmann transform, from the space of square integrable functions $L^2(\mathbb{R})$ to the Fock spaces $F_π^2$ where $F_π^2 = \{ f \in Hol(\mathcal{D}) \cap L^2(\mathcal{D}, e^{-|z|^2}) \}$ as follows:

$$\forall z \in \mathbb{C}, \forall f \in L^2(\mathbb{R}), \quad Bf(z) = 2^{\frac{1}{4}} \int_{-\infty}^{\infty} f(x) e^{2\pi xz - \pi x^2 - \frac{z^2}{2}} \, dx. \tag{2.1}$$

We use in this paper a parametrized forms of the Bargmann and inverse Bargman transforms given by K. Zhu [11] as follows:

$$B_α f(z) = \left( \frac{2α}{π} \right)^{\frac{1}{4}} \int_{-\infty}^{∞} f(x) e^{2αxz - αx^2 - \frac{z^2}{2}} \, dx \tag{2.2}$$

and

$$B_{α}^{-1} f(x) = \left( \frac{2α}{π} \right)^{\frac{1}{4}} \int_{C} f(z) e^{2αxz - αx^2 - \frac{z^2}{2}} \, dλ_α(z) \tag{2.3}$$

An operator $T$ is transported in The Bargmann space $F_α^2$ by the operation $B_α T B_{α}^{-1}$. A number of operators have been thus transported and have been reported in the literature (see [8]). We present some of them in this lemma:

**Lemma 2.1.** For $α > 0$ we have:

1. $B_α \left( x f(x) \right)(z) = \left( \frac{1}{α} \frac{∂}{∂z} + \frac{z}{2} \right) B_α f(z)$

2. $B_α \left( \frac{∂}{∂x} f(x) \right)(z) = \left( \frac{∂}{∂z} - \frac{α}{2} z \right) B_α f(z)$

3. $B_α \left( \left( \frac{∂}{∂x} - ax \right) f \right)(z) = -az B_α f(z)$

4. $B_α \left( \left( \frac{∂}{∂x} + ax \right) f \right)(z) = 2\frac{∂}{∂z} B_α f(z)$

A calculation with polar coordinates shows that $dλ_α$ is a probability measure.

The Fock space $F_α^2$ consists of all entire functions $f$ in $L^2(\mathcal{D}, dλ_α)$. It is easy to show that $F_α^2$ is a closed subspace of $L^2(\mathcal{D}, dλ_α)$. Consequently, $F_α^2$ is a Hilbert space with the following inner product inherited from $L^2(\mathcal{D}, dλ_α)$:

$$< f, g >_α = \int_{C} f(z) g(z) \, dλ_α(z). \tag{2.5}$$
Proof. See Appendix 1 of \[8\] for the proof of 1. and 2. however 3 and 4 are consequences of 1 and 2. \hfill \Box

**Theorem 2.1.** Let $h^a$ and $E^a$ be respectively the real harmonic oscillator and the complex Euler operator given in (1.10) and (1.7) then we have

$$B_{\frac{a}{2}}(h^a f)(z) = E^a(B_{\frac{a}{2}} f)(z) \tag{2.6}$$

Proof. This theorem is a consequence of the formula

$$h^a = \left( \frac{\partial^2}{\partial x^2} - a^2 x^2 \right) = \left( \frac{\partial}{\partial x} - a x \right) \left( \frac{\partial}{\partial x} + a x \right) - a. \tag{2.7}$$

and the lemma 2.1. \hfill \Box

In what follows we give some formulas of Bargmann transforms

**Theorem 2.2.** For all $a > 0$, $b > \frac{a}{2}$ and $s \in \mathbb{R}$ we obtain :

$$B_{\frac{a}{2}} \left( e^{a \frac{x^2}{2}} e^{-b(x-s)^2} \right)(z) = \left( \frac{a}{\pi} \right)^{\frac{1}{4}} \sqrt{\frac{\pi}{b}} e^{as z} e^{\frac{a}{4} \pi (\frac{a}{b} - 1)z^2} \tag{2.8}$$

In particular,

For all $c > 0$, $B_{\frac{a}{2}} \left( e^{-cx^2} \right)(z) = \left( \frac{a}{\pi} \right)^{\frac{1}{4}} \sqrt{\frac{\pi}{c + \frac{a}{2}}} e^{\frac{a}{2}z^2 (\frac{a}{c + a})} \tag{2.9}$

and also

$$B_{\frac{a}{2}} \left( e^{-\frac{a}{2} z^2} \right)(z) = \left( \frac{\pi}{a} \right)^{\frac{1}{4}} \tag{2.10}$$

Proof.

$$B_{\frac{a}{2}} \left( e^{a \frac{x^2}{2}} e^{-b(x-s)^2} \right)(z) = \left( \frac{a}{\pi} \right)^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-b(x-s)^2} e^{ax z - \frac{a x^2}{4}} dx \tag{2.11}$$

$$\left( \frac{a}{\pi} \right)^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-b(x-s - \frac{a x}{2b})^2} e^{as z} e^{\frac{a^2 z^2}{2b}} e^{-\frac{a x^2}{4}} dx \tag{2.12}$$

$$\left( \frac{a}{\pi} \right)^{\frac{1}{4}} \sqrt{\frac{\pi}{b}} e^{as z} e^{\frac{a}{4} \pi (\frac{a}{b} - 1)z^2} \tag{2.13}$$

\hfill \Box
Theorem 2.3. For all function \( f \in F^2_{\mathbb{R}} \) and \( b \in \mathbb{R} \)

\[
B_{\frac{a}{2}} \left( B_{\frac{a}{2}}^{-1} f(x + b) \right) (z) = e^{-\frac{z}{2}b^2} e^{-\frac{a}{2}b^2} f(z + b). \tag{2.14}
\]

Proof. see theorem 8 of [12].

We define here a parametrized form of the Fourier transform:

\[
(F f)(x) = \sqrt{\frac{a}{\pi}} \int_{\mathbb{R}} f(t) e^{iatx} dt \tag{2.15}
\]

Also, for all \( r > 0 \), we define a general form of the Fourier Transform by:

\[
(F_r f)(x) = \sqrt{ar} \frac{\sqrt{\pi}}{\pi} \int_{\mathbb{R}} f(t) e^{iarxt} dt \tag{2.16}
\]

This form will be useful for the next very important result:

Theorem 2.4. For all function \( f \in F^2_{\mathbb{R}} \),

\[
(B_{\frac{a}{2}} F_r B_{\frac{a}{2}}^{-1} f)(z) =
2 \sqrt{\frac{r}{r^2 + 1}} e^{-\frac{z^2}{2} \left( \frac{r^2}{r^2 + 1} \right)} \int_{\mathbb{R}} f(w) e^{\frac{a}{4}r^2 \left( \frac{1}{r^2 + 1} \right)} e^{\frac{iarw}{r^2 + 1}} d\lambda_{\frac{a}{2}}(w) \tag{2.17}
\]

In particular, for \( r = 1 \),

\[
(B_{\frac{a}{2}} F B_{\frac{a}{2}}^{-1} f)(z) = \sqrt{2} f(iz) \tag{2.18}
\]

and then

\[
(B_{\frac{a}{2}} F^{-1} B_{\frac{a}{2}}^{-1} f)(z) = \frac{1}{\sqrt{2}} f(-iz) \tag{2.19}
\]

Proof. Firstly, we have:

\[
(B_{\frac{a}{2}}^{-1} f)(w) = \left( \frac{a}{\pi} \right)^{\frac{1}{4}} \sqrt{\frac{ar}{\pi}} \int_{\mathbb{R}} e^{iarxt} \int_{\mathbb{R}} f(w)e^{arx^2 - \frac{a}{4}x^2} d\lambda_{\frac{a}{2}}(w) dt \tag{2.20}
\]

\[
= \left( \frac{a}{\pi} \right)^{\frac{1}{4}} \sqrt{\frac{ar}{\pi}} \int_{\mathbb{R}} f(w)e^{-\frac{a}{4}w^2} \int_{\mathbb{R}} e^{\frac{a}{4}(t-irx-w)^2} e^{\frac{a}{2}(irx+w)^2} dt d\lambda_{\frac{a}{2}}(w) \tag{2.21}
\]

\[
= \sqrt{\frac{2ar}{(a\pi)^{\frac{1}{4}}}} \int_{\mathbb{R}} f(w)e^{-\frac{a}{4}w^2} e^{\frac{a}{4}(irx+w)^2} d\lambda_{\frac{a}{2}}(w) \tag{2.22}
\]
\[
\sqrt{2\alpha r} e^{-\frac{\alpha^2 r^2}{2}} \int_{\mathbb{R}} f(w) e^{\frac{aw^2}{2}} e^{\alpha i w} d\lambda_{\frac{r}{2}}(w) \quad (2.23)
\]

Thus we obtain:

\[
(B_{\frac{r}{2}} F_r B_{\frac{r}{2}}^{-1} f)(z) =
\]

\[
(\frac{a}{\pi})^{\frac{3}{4}} \sqrt{2\alpha r} \int_{\mathbb{R}} e^{\frac{\alpha xz}{2} + \frac{\alpha^2 x^2}{4} - \frac{aw^2}{2}} e^{\alpha i w} d\lambda_{\frac{r}{2}}(w) \quad (2.24)
\]

\[
= \sqrt{2\alpha r} e^{-\frac{\alpha^2 r^2}{4}} \int_{\mathbb{R}} f(w) e^{\frac{aw^2}{2}} e^{\alpha i w} d\lambda_{\frac{r}{2}}(w) \quad (2.25)
\]

We denote \( c = \frac{a}{2} + \frac{\alpha^2 r}{2} \). Using the change of variable \( t = \sqrt{c} x \) we get:

\[
(B_{\frac{r}{2}} F_r B_{\frac{r}{2}}^{-1} f)(z) =
\]

\[
(\frac{a}{\pi})^{\frac{3}{4}} \sqrt{2\alpha r} \int_{\mathbb{R}} e^{\frac{\alpha xz}{2} + \frac{\alpha^2 x^2}{4} - \frac{aw^2}{2}} e^{\alpha i w} d\lambda_{\frac{r}{2}}(w) \quad (2.26)
\]

In the case \( r = 1 \), \( F_r \) coincide with \( F \) the parametrized Fourier transform and we obtain:

\[
(B_{\frac{1}{2}} F B_{\frac{1}{2}}^{-1} f)(z) = \sqrt{2} \int_{\mathbb{R}} f(w) e^{\frac{aw^2}{2}} d\lambda_{\frac{1}{2}}(w). \quad (2.33)
\]

By the formula of reproducing kernel (see proposition 2.2 of [11]) this function is exactly equal to \( \sqrt{2} f(iz) \). On the other hand,

\[
(B_{\frac{1}{2}} F^{-1} B_{\frac{1}{2}}^{-1}) = (B_{\frac{1}{2}} F B_{\frac{1}{2}}^{-1})^{-1}
\]

which justifies the equation \( (2.19) \).
For \( r > 0 \), we denote \( h_r \) the isomorphism of \( L^2(\mathbb{R}) \) defined by
\[
h_r(f)(x) = f(rx) \quad \text{for all } f \in L^2(\mathbb{R}) \text{ and for all } x \in \mathbb{R}.
\]

**Theorem 2.5.** For all function \( f \in F^2_{\frac{a}{2}}(\mathbb{R}) \):
\[
(B_{\frac{a}{2}} h_r B_{\frac{a}{2}}^{-1}) f(x) = 2 \sqrt{\frac{2r}{r^2 + 1}} e^{-\frac{z^2}{4(r^2 + 1)}} \int_{\mathbb{R}} f(-iw) e^{i\frac{r^2}{4(r^2 + 1)} w} e^{i\frac{w}{r^2 + 1}} d\lambda_{\frac{a}{2}}(w)
\]

**Proof.** A simple calculus give \( \sqrt{r} h_r \circ F = F_r \) and thus \( h_r = \frac{1}{\sqrt{r}} F_r \circ F^{-1} \) then:
\[
(B_{\frac{a}{2}} h_r B_{\frac{a}{2}}^{-1} f)(z) = \frac{1}{\sqrt{r}} (B_{\frac{a}{2}} F_r F^{-1} B_{\frac{a}{2}}^{-1} f)(z)
\]

\[
= \frac{1}{\sqrt{r}} (B_{\frac{a}{2}} F_r B_{\frac{a}{2}}^{-1} B_{\frac{a}{2}} F^{-1} B_{\frac{a}{2}}^{-1} f)(z)
\]

Using formula (2.17) and (2.19) of theorem 2.4 we deduce that:
\[
(B_{\frac{a}{2}} h_r B_{\frac{a}{2}}^{-1} f)(z) = 2 \sqrt{\frac{2r}{r^2 + 1}} e^{-\frac{z^2}{4(r^2 + 1)}} \int_{\mathbb{R}} f(-iw) e^{i\frac{r^2}{4(r^2 + 1)} w} e^{i\frac{w}{r^2 + 1}} d\lambda_{\frac{a}{2}}(w)
\]

\[
\Box
\]

### 3 Heat Cauchy problems for the complex and real Dirac operators

**Theorem 3.1.** The solution of the equation (1.2) is given by the formula:
\[
U(t, z) = e^\frac{zt}{2} e^\frac{t^2}{4a} U_0 \left( z + \frac{t}{a} \right)
\]

**Proof.** Let \( U \) be a solution of (1.2) Then, by applying the inverse of the Bargmann transform \( B_{\frac{a}{2}}^{-1} \), we obtain:
\[
\left\{
\begin{align*}
B_{\frac{a}{2}}^{-1} \left( \frac{1}{a} \frac{\partial}{\partial z} + \frac{z}{2} \right) U(t, z) &= \frac{\partial}{\partial t} B_{\frac{a}{2}}^{-1} (U(t, z)); \quad (t, z) \in \mathbb{R}^+ \times \mathcal{C} \\
B_{\frac{a}{2}}^{-1} (U(0, z)) &= B_{\frac{a}{2}}^{-1} U_0; \quad U_0 \in F^2_{\frac{a}{2}}
\end{align*}
\right.
\]
Using lemma 2.1 we obtain:

\[
\begin{cases}
    x (B_{\frac{1}{a}}^{-1} U)(t, x) = \frac{\partial}{\partial t} (B_{\frac{1}{a}}^{-1} U)(t, x); \quad (t, x) \in \mathbb{R}^*_+ \times \mathbb{R} \\
    (B_{\frac{1}{a}}^{-1} U)(0, x) = (B_{\frac{1}{a}}^{-1} U_0)(x); \quad U_0 \in F_{\frac{1}{a}}^2
\end{cases}
\]

Thus \(B_{\frac{1}{a}}^{-1} U(t, x)\) satisfies the formula

\[
(B_{\frac{1}{a}}^{-1} U)(t, x) = e^{tx} (B_{\frac{1}{a}}^{-1} U_0)(x).
\]

This implies that

\[
U(t, z) = B_{\frac{1}{a}} (e^{tx} B_{\frac{1}{a}}^{-1} U_0)(z)
\]

\[
= \left(\frac{a}{\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} e^{axz - \frac{a}{2} x^2 - \frac{a^2}{8} u} e^{tx} \int_{\mathbb{C}} U_0(v) e^{a(xv - \frac{a}{2} x^2 - \frac{a^2}{4} v^2)} d\lambda_F(v) dx
\]

\[
= \left(\frac{a}{\pi}\right)^{\frac{1}{2}} e^{-\frac{a^2}{4} v^2} e^{-\frac{a}{2} (x + \frac{t}{a})^2} \int_{\mathbb{R}} \int_{\mathbb{C}} U_0(v) e^{a(xv - \frac{a}{2} x^2 - \frac{a^2}{4} v^2)} e^{a(xv - \frac{a}{2} x^2 - \frac{a^2}{4} v^2)} d\lambda_F(v) dx
\]

\[
= e^{\frac{a}{2} t} e^{\frac{a^2}{4} z} B_{\frac{1}{a}} (B_{\frac{1}{a}}^{-1} U_0) \left(z + \frac{t}{a}\right) = e^{\frac{a}{2} t} e^{\frac{a^2}{4} U_0} \left(z + \frac{t}{a}\right)
\]

\[\square\]

**Theorem 3.2.** The solution of the equation (1.3) is given by the formula:

\[
u(t, x) = e^{-axt} e^{-\frac{a^2}{4} u_0}(x + t) \quad (3.2)
\]

Note that the exact formulas for the Poisson and wave Cauchy problems associated to the real Dirac \(D^0\), Euler \(e_a\) and harmonic oscillator \(h_a\) operators are given [9, 10].

**Proof of theorem 3.2** Let \(u\) be a solution of (1.3). Then, by applying the Bargmann transform \(B_{\frac{1}{a}}\), we obtain:

\[
\begin{cases}
    B_{\frac{1}{a}} \left( \left( \frac{\partial}{\partial x} - ax \right) u(t, x) \right) = \frac{\partial}{\partial t} B_{\frac{1}{a}} (u(t, x)); \quad (t, x) \in \mathbb{R}^*_+ \times \mathbb{R} \\
    B_{\frac{1}{a}} (u(0, x)) = B_{\frac{1}{a}} (u_0); \quad u_0 \in L^2(\mathbb{R})
\end{cases}
\]

Using lemma 2.1 we obtain:

\[
\begin{cases}
    -az (B_{\frac{1}{a}} u)(t, z) = \frac{\partial}{\partial t} (B_{\frac{1}{a}} u)(t, z); \quad (t, z) \in \mathbb{R}^*_+ \times \mathbb{C} \\
    (B_{\frac{1}{a}} u)(0, z) = (B_{\frac{1}{a}} u_0)(z); \quad u_0 \in L^2(\mathbb{R})
\end{cases}
\]

8
Then $B_2 u$ satisfies the formula

$$(B_2 u)(t, z) = e^{-azt} (B_2 u_0)(z).$$

This implies that

$$u(t, x) = B^{-1}_2 (e^{-azt} B_2 u_0)(x)$$

By setting $s' = s - t$, we obtain:

$$u(t, x) = B^{-1}_2 \left( e^{azt} \left( B_2 \left( u_0 \left( s' + t \right) e^{-\frac{At}{2} s'^2} e^{-\frac{A}{4} t^2} e^{-as't} e^{-\frac{a}{4} s'^2} e^{-\frac{a}{4} xt} \right) \right) \right)(x) = u_0(x + t) e^{-\frac{A}{4} t^2} e^{-axt}.$$  

\[\square\]

4 Heat Cauchy problems for the complex and real Euler operators

**Theorem 4.1.** The solution of the equation (1.5) is given by the formula:

$$v(t, x) = v_0(e^{at} x)$$  

**Proof.** Let $e$ be the solution of (1.5). We define the function $y(t, x)$ by:

$$y(t, x) = e(t, e^{-at} x).$$

We get easily $\frac{\partial}{\partial t} y(t, x) = 0$ then $y(t, x) = y(0, x) = e(0, x) = e_0(x)$. Then

$$e(t, e^{-at} x) = e_0(x), \text{ for all } (t, x) \in \mathbb{R}^+ \times \mathbb{R}.$$

Consequently, $e(t, x) = e_0(e^{at} x)$ for all $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$.  

\[\square\]

**Theorem 4.2.** The solution of the equation (1.6) is given by the formula:

$$Y(t, w) = e^{-at} Y_0(e^{-2at} w)$$  

**Proof.** The proof of this theorem is Mutatis mutandis the last one.  

\[\square\]
5 Heat Cauchy problems for the real and complex harmonic oscillators

Theorem 5.1. The Cauchy problem (1.8) for the heat equation associated to the harmonic oscillator has the unique solution given by:

\[ y(t, x) = \int_{\mathbb{R}} K_a(x, s, t) y_0(s) \, ds \]

Where \( K_a(x, s, t) = \sqrt{\frac{a}{\pi}} (e^{2at} - e^{-2at})^{-\frac{1}{2}} \exp \left\{ -\frac{a(e^{at}x - e^{-at}s)^2}{e^{2at} - e^{-2at}} + \frac{a}{2}(x^2 - s^2) \right\} \) (5.1)

Note that the heat kernel for the classical real harmonic oscillator has been known for a long time ([3] p.145) and is given by the formula:

\[ K_a(x, s, t) = \sqrt{\frac{a}{\pi}} \frac{1}{\sqrt{\sinh(2at)}} \exp \left\{ -\frac{a}{2}(x^2 + s^2) \coth(2at) + \frac{axs}{\sinh(2at)} \right\} \] (5.2)

but the method used here and the obtained formula are new.

Proof of theorem 5.1 Let \( \gamma \) be a solution of (1.8). Then, by applying the Bargmann transform \( B_{\frac{a}{2}} \), we obtain:

\[
\begin{cases}
B_{\frac{a}{2}}(H^a \gamma(t, x))(z) = \frac{\partial}{\partial t} B_{\frac{a}{2}}(\gamma(t, x))(z); & (t, z) \in \mathbb{R}_+^* \times \mathcal{C} \\
B_{\frac{a}{2}}(\gamma(0, x))(z) = B_{\frac{a}{2}} \gamma_0(Z); & \gamma_0 \in L^2(\mathbb{R})
\end{cases}
\]

Theorem 2.1 assures that:

\[
\begin{cases}
\left( -2az \frac{\partial}{\partial z} - a \right) B_{\frac{a}{2}}(\gamma(t, x))(z) = \frac{\partial}{\partial t} B_{\frac{a}{2}}(\gamma(t, x))(z); & (t, z) \in \mathbb{R}_+^* \times \mathcal{C} \\
B_{\frac{a}{2}}(\gamma(0, x))(z) = B_{\frac{a}{2}}(\gamma_0(z)); & \gamma_0 \in L^2(\mathbb{R})
\end{cases}
\]

Now, theorem 4.2 gives

\[ B_{\frac{a}{2}} \gamma(t, z) = e^{-at} B_{\frac{a}{2}} \gamma_0(e^{-2at}z), \quad \forall (t, z) \in \mathbb{R}_+^* \times \mathcal{C}. \]

So

\[ \gamma(t, x) = B_{\frac{a}{2}}^{-1}(z \mapsto e^{-at} B_{\frac{a}{2}} \gamma_0(e^{-2at}z))(x) \]

\[ = \sqrt{\frac{a}{\pi}} e^{-at} \int_{\mathcal{C}} e^{axz} - \frac{x^2 - z^2}{2} \int_{\mathbb{R}} \gamma_0(s)e^{ase^{-2at}z - \frac{a}{4}e^{-4at}z^2} ds \, d\lambda_{\frac{a}{2}}(z). \]
Using the change of variables $s' = e^{-2at}s$, we get:
\[
\gamma(t, x) = \sqrt{\frac{\alpha}{\pi}} e^{at} \int_{\mathbb{T}} e^{\alpha x^2 - \frac{a}{2} x^2 - \frac{a}{4}z^2} \int_{\mathbb{R}} \gamma_0(e^{2at}s')e^{as'z - \frac{a}{2} e^{4at}s'^2 - \frac{a}{4}e^{-4at}z^2} ds' d\lambda_{\frac{a}{2}}(z)
\]
\[
= \sqrt{\frac{\alpha}{\pi}} e^{at} \int_{\mathbb{R}} \gamma_0(e^{2at}s') e^{-\frac{a}{2} e^{4at}s'^2} \int_{\mathbb{T}} e^{as'z - \frac{a}{2} e^{4at}s'^2 + axz - \frac{a}{2} e^{2at}z^2} d\lambda_{\frac{a}{2}}(z) ds'
\]
\[
= \left( \frac{\alpha}{\pi} \right)^{\frac{1}{2}} e^{at} \int_{\mathbb{R}} \gamma_0(e^{2at}s') e^{-\frac{a}{2} e^{4at}s'^2} B_{\frac{a}{2}}^{-1} \left( z \mapsto e^{as'z - \frac{a}{2} e^{-4at}z^2} \right)(x) ds'
\]
We deduce from relation (2.8) of theorem 2.2 (with $\frac{a}{b} = -1 = e^{-4at}$) that:
\[
B_{\frac{a}{2}}^{-1} \left( z \mapsto e^{as'z - \frac{a}{2} e^{-4at}z^2} \right)(x) = \left( \frac{\pi}{\alpha} \right)^{\frac{1}{2}} \frac{\sqrt{a}}{\sqrt{\pi} \sqrt{1 - e^{-4at}}} e^{\frac{a}{2} e^{4at} - \frac{a}{4} e^{-4at}z^2} e^{\frac{a}{2} e^{2at}z^2}
\]
So we obtain:
\[
\gamma(t, x) = e^{at} e^{\frac{a}{2} x^2} \frac{\sqrt{a}}{\sqrt{\pi} \sqrt{1 - e^{-4at}}} \int_{\mathbb{R}} \gamma_0(s'e^{2at}) e^{-\frac{a}{2} e^{4at}s'^2} e^{\frac{a}{2} e^{4at}e^{2at} - \frac{a}{4} e^{-4at}z^2} \frac{d\lambda_{\frac{a}{2}}(z)}{\sqrt{\pi} \sqrt{1 - e^{-4at}}} \]
Set again $s = e^{2at}s'$ we get:
\[
\gamma(t, x) = e^{-at} e^{\frac{a}{2} x^2} \frac{\sqrt{a}}{\sqrt{\pi} \sqrt{1 - e^{-4at}}} \int_{\mathbb{R}} \gamma_0(s) e^{-\frac{a}{2} e^{4at}s} e^{\frac{a}{2} e^{4at}e^{2at} - \frac{a}{4} e^{-4at}z^2} \frac{d\lambda_{\frac{a}{2}}(z)}{\sqrt{\pi} \sqrt{1 - e^{-4at}}} \]
\[
= \int_{\mathbb{R}} K_a(t, x, s) \gamma_0(s) ds.
\]
\[\square\]

The formula (5.1) obtained in the theorem 5.1 agree with the well known formula given in (5.2).

**Proof.**
\[
K_a(t, x, s) = \frac{\alpha}{\pi} (e^{2at} - e^{-2at})^{-\frac{1}{2}} \exp \left\{-a(e^{at}x - e^{-at}s)^2 e^{2at} - e^{-2at} + a \left( x^2 - s^2 \right) \right\} \]
\[
= \sqrt{\frac{\alpha}{2\pi}} e^{-at} \exp \left\{ -a(e^{2at}x^2 + e^{-2at}s^2 - 2xs) e^{2at} - e^{-2at} + a \left( x^2 - s^2 \right) \right\} \]
\[
= \frac{1}{2\pi} \exp \left\{ -a \left( \frac{2e^{2at}}{e^{2at} - e^{-2at}} - 1 \right) - a \left( \frac{2e^{-2at}}{e^{2at} - e^{-2at}} + 1 \right) \right\} \]
Thus we obtain the formula (5.2).
Theorem 5.2. The Cauchy problem \((1.9)\) for the heat equation associated to the complex harmonic oscillator has the unique solution given by:

\[
V(t, z) = \int_{\mathcal{Q}} J_a(z, w', t) V_0(w') \, d\lambda_{\frac{a}{2}}(w')
\]

Where

\[
J_a(z, w', t) = \frac{2i}{\sqrt{\cosh(at)}} \exp \left\{ \frac{a}{4} (w'^2 - z^2) \tanh(at) + \frac{azw'}{2 \cosh(at)} \right\}
\]

(5.3)

Proof. Firstly, notice that

\[
\left( \frac{\partial}{\partial z} + \frac{az^2}{2} \right) \left( \frac{\partial}{\partial z} - \frac{az^2}{2} \right) = \left( \frac{\partial^2}{\partial z^2} - \frac{a^2z^2}{4} - \frac{a}{2} \right).
\]

Let \(E\) be the solution of \((1.9)\). By applying the inverse of the Bargmann transform \(B_{\frac{a}{2}}^{-1}\), we obtain:

\[
\begin{cases}
B_{\frac{a}{2}}^{-1} \left( \left( \frac{\partial}{\partial z} + \frac{az^2}{2} \right) \left( \frac{\partial}{\partial z} - \frac{az^2}{2} \right) E(t, z) \right) = \frac{\partial}{\partial t} B_{\frac{a}{2}}^{-1}(E(t, z)); & (t, z) \in \mathbb{R}_+^* \times \mathcal{Q} \\
B_{\frac{a}{2}}^{-1}(E(0, z)) = B_{\frac{a}{2}}^{-1}E_0; & E_0 \in F^2_{\frac{a}{2}}
\end{cases}
\]

Using lemma 2.1 we obtain:

\[
\begin{cases}
a x \frac{\partial}{\partial x} (B_{\frac{a}{2}}^{-1}E)(t, x) = \frac{\partial}{\partial t} (B_{\frac{a}{2}}^{-1}E)(t, x); & (t, x) \in \mathbb{R}_+^* \times \mathbb{R} \\
(B_{\frac{a}{2}}^{-1}E)(0, x) = (B_{\frac{a}{2}}^{-1}E_0)(x); & U_0 \in F^2_{\frac{a}{2}}
\end{cases}
\]

So theorem 4.1 assures that

\[
(B_{\frac{a}{2}}^{-1}E)(t, x) = (B_{\frac{a}{2}}^{-1}E_0)(e^{at} x).
\]

This implies that

\[
E(t, z) = B_{\frac{a}{2}} \left( (B_{\frac{a}{2}}^{-1}E_0)(e^{at} x) \right)(z)
\]

By theorem 2.5 we obtain:

\[
E(t, z) = 2 \sqrt{\frac{2r}{r^2 + 1}} e^{-\frac{a^2}{2} \left( \frac{r^2}{r^2 + 1} \right)} \left( f(-iw) e^{\frac{a}{2} \alpha \gamma (1 - \frac{1}{r^2 + 1})} e^{\frac{iax}{r^2 + 1}} d\lambda_{\frac{a}{2}}(w) \right)\text{ where } r = e^{at}
\]

\[
= \frac{2}{\sqrt{\cosh(at)}} e^{-\frac{a^2}{2} \tanh(at)} \left( f(-iw) e^{-\frac{a}{2} \alpha \gamma \tanh(at)} e^{\frac{iax}{2 \cosh(at)}} d\lambda_{\frac{a}{2}}(w) \right).
\]

Using the change of variable \(w' = -iw\) we get

\[
E(t, z) = \int_{\mathcal{Q}} f(w') \frac{2i}{\sqrt{\cosh(at)}} e^{\frac{a}{4} (w'^2 - z^2) \tanh(at)} e^{\frac{iax}{2 \cosh(at)}} d\lambda_{\frac{a}{2}}(w').
\]
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