Phase-coherent tunneling through a mesoscopic superconductor coupled to superconducting and normal metal electrodes.

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Phase-coherent diffusive transport through mesoscopic hybrid superconductor/normal metal tunneling structures is investigated. For a \( N-s-S \) two-barrier tunneling system with bulk \( S \) and \( N \) electrodes coupled by a mesoscopic superconducting constriction \( s \), zero-bias conductance and non-linear I-V curves are calculated under the assumption that the dwell times of quasiparticles in the \( s \) region is shorter than inelastic relaxation time. It is shown that the low voltage conductance of this system determined by the Andreev reflection processes may exceed the conductance in the normal state and its value is very sensitive to the weak pairing interaction of electrons in the \( s \) region. We show that even weak pairing electron interaction may result in the significant qualitative and quantitative change of the conductance temperature dependence with respect to the case of structures with the normal mesoscopic region. We calculate the I-V curves and show that they depend on the applied voltage in a non-monotonic way, therefore differential conductance becomes negative with increasing voltage. Such behavior is due to the voltage dependence of the order parameter in the constriction and the phase difference \( \varphi \) between the \( S \) and \( s \) superconductors. It is shown that if the tunneling processes determine the form of the quasiparticle distribution function in the \( s \) superconductor, the phase \( \varphi \) is stationary at arbitrary voltages. For quasiparticle tunneling interferometers in which the mesoscopic superconductor, \( s \), couples the superconductor, \( S \), and the normal metal, \( N \), the zero bias conductance, as a function of the phase difference between the \( S \) electrodes is investigated. It is shown that the amplitude of the conductance oscillations may exceed the conductance of this structure in the normal state.

A. Introduction

Phase-coherent transport in mesoscopic superconductor/normal metal (S/N) systems has been an active area of research during the last decade [1]. The interest in the theoretical investigations was stimulated by impressive technological advances and by experimental activity in studying various properties of small mesoscopic structures [2–10]. Interesting phenomena in mesoscopic systems are due to the importance of both the phase coherence established in the \( s \) constriction by the proximity effect and significant departure of quasiparticles from equilibrium. This is completely true for two-barrier structures \( N-s-S \) with barriers at the interfaces between the \( N \) and \( S \) electrodes connected by a superconducting constriction \( s \) of length \( d \). The dimensions of the constriction transverse to the current direction are assumed to be small in comparison with the London penetration depth in the \( S \) electrode. We consider the system with diffusive transport, i.e. we suppose that the mean free path \( l \) in the \( s \) region is small with respect to the constriction length \( d \). Because of non-conservation of the momentum, the interference of normal electron wave functions related to reflections from the barriers is not essential. Nevertheless the coherence of different (ordinary and Andreev) reflection processes related to the condensate wave function and nonzero order parameter \( \Delta \) in the superconductor \( s \) is very important because the inter-barrier distance \( d \) is supposed to be small in comparison with \( \sqrt{\hbar D/\Delta} \), where \( D = l v_F/3 \) is the diffusion coefficient. In what follows we assume that transparencies of both barriers \( D_{1,2} \), (averaged over momentum directions) are small enough to allow the main contribution to the resistance of the system to be due to the barrier resistances. Tunneling processes determine the dwell times \( \tau_{b1,2} = D_{1,2} v_F/4d \) in the \( s \) region which are supposed to be shorter than the inelastic relaxation time \( \tau_{\text{in}} \) in the superconductor \( s \), so that the following conditions should be fulfilled

\[
\tau_{\text{dif}} \ll \tau_{b1,2} \ll \tau_{\text{in}}
\]

where \( \tau_{\text{dif}} = \hbar/(D/d^2) \) is the diffusion time of quasiparticles through the length \( d \). It is clear that the quantum nature of the tunneling processes becomes more pronounced if the tunneling rates \( \hbar/\tau_{b1,2} \) are comparable with the characteristic scale of the quasiparticle energy,

\[
\hbar/\tau_{b1,2} \sim \Delta.
\]
because under this condition the classical notion of a quasiparticle whose dwell time should be longer than its energy, loses its sense. Nevertheless the Green’s function approach enables one to obtain the quantum kinetic equations as given in [13] which are valid beyond the classical limits, i.e. when the quasiparticle energy is not large compared to the tunneling rate. Note that under the conditions (1) the proximity effect, i.e. the influence of $S$ and $N$ electrodes on the condensate wave function and on the order parameter in the s region, is strong. We also note that unusual features of transport properties are due to the significant role of Andreev reflection processeses in the considered system. These processeses occur in the presence of two potential barriers (at $x = 0$ and $x = d$) and the superconducting order parameter which has a two-step form: $\Delta(x) = \Delta \theta(x) \theta(d - x) + \Delta_S \theta(x - d) \exp(i\varphi)$, where $\varphi$ is the phase difference between the superconductors arising at non-zero voltage, $V$ and $\theta(x)$, is the Heavyside function. As a result, the energy dependent transmission coefficient $D_{\nu}(\Delta, \varphi)$ of quasiparticles with energy $\epsilon < \Delta_S$, that determines the current at low temperatures, appears to be a function strongly dependent on $V$ through the voltage dependence of $\Delta$ and $\varphi$. All these circumstances result in nontrivial features of the quasiparticle phase coherent transport through the $N - s - S$ system which will be investigated in this paper. It should be noted that some of these phenomena have been studied in [14]. We therefore investigate the transport phenomena in more detail with emphasis upon the case of a weak pairing electron interaction in the s region, i.e. when the critical temperature, $T_{\text{c}0}$ of the superconductor, $s$ in the absence of the pair-breaking and proximity effect ($\tau_{01,2} = \infty$), is small in comparison with the critical temperature of the $S$ electrode, $T_{\text{c}S}$. It will be shown that in spite of a small ratio $t_c / T_{\text{c}0}$ the properties of the $N - s - S$ system may radically differ from properties of a two barrier $N - N - S$ structure well studied in the limit $t_c = 0$ [14,21]. We also study the zero bias conductance as a function of the phase difference between the $S$ electrodes in a quasiparticle tunneling interferometer with two superconducting electrodes coupled by a mesoscopic superconductor $s$. It is shown that the amplitude of the conductance oscillation may exceed the conductance of this structure in the normal state.

**B. The N-s-S system**

We consider the $N - s - S$ system shown in Fig.1a. As in Refs. [14,22] we use the approach based on the equations for the quasiclassical Green’s function $\hat{G} = \hat{G}(r, p; \epsilon)$ which is the 4x4 supermatrix(see Ref. [13]),

$$
\hat{G} = \begin{pmatrix} \hat{G}^R & \hat{G}^K \\ \hat{G}^K & \hat{G}^A \end{pmatrix}
$$

consisting of the retarded $\hat{G}^R$, advanced $\hat{G}^A$, and Keldysh $\hat{G}^K$, Green’s functions which are 2x2 matrices in Nambu space. Note that we suppose that a stationary solution realizes and the Green’s functions do not depend on time. This non-obvious assumption is justified by the final result. The matrix $\hat{G}^K$ is related to the matrix distribution function $\hat{f} = f_0 \hat{1} + \hat{f} \hat{\sigma}_z$

$$
\hat{G}^K = \hat{G}^R \hat{f} - \hat{f} \hat{G}^A
$$

(3)

The matrices $\hat{G}^{R,A}$ have the following form

$$
\hat{G}^A = g^A \hat{\sigma}_z + \hat{f}^A, \quad \hat{G}^R = f^R i \hat{\sigma}_y \exp(i\hat{\sigma}_z \chi)
$$

where $\chi$ is the phase of the order parameter $\mu = R(A)$. The current in the system is given by the following relation

$$
I = \frac{\sigma_A}{8} \xi \int d\epsilon (\hat{G}^R \partial_\epsilon \hat{G}^K + \hat{G}^K \partial_\epsilon \hat{G}^A)
$$

(4)

where $A = w_y w_z$ is the cross-section area of the s region. The transverse dimensions $w_{y,z}$ should be small compared to the London penetration depth. Therefore we need to solve a one-dimensional equation in the s region ($0 < x < d$, $x$-axis coincides with the direction of the current), where in the considered diffusive case the matrix $\hat{G} \equiv \hat{G}(x, \epsilon)$ averaged over the momentum direction obeys the equation (see Ref. [13])

$$
D \partial_x (\hat{G} \partial_x \hat{G}) + i[\epsilon \hat{\sigma}_z + \hat{\Delta}, \hat{G}] = 0.
$$

(5)

$\hat{\sigma}_z = \hat{1} \hat{\sigma}_z$ is the Pauli supermatrix, and the order parameter supermatrix in the film is $\hat{\Delta} = \hat{1} \hat{\Delta}$ where

$$
\hat{\Delta} = \begin{pmatrix} 0 & \Delta \\ -\Delta^* & 0 \end{pmatrix}
$$
The order parameter is given by the self-consistency relation which in the framework of the weak coupling theory has the form

$$ \dot{\Delta} = \lambda \int_{0}^{\omega_D} d\epsilon (\dot{f}R - \hat{f} \dot{A}) $$

(6)

where the constant $\lambda$ determines the critical temperature $T_{c0}$ of the superconductor $s$ in the absence of pair-breaking factors and the proximity effect,

$$ T_{c0} = 1.14 \omega_D \exp(-1/\lambda) $$

The matrix $\hat{G}$ obeys the normalization condition

$$ \hat{G}^2 = 1 $$

(7)

In order to solve Eq.(6) we need to take into account the boundary conditions [12] that in the diffusive case reduce to [13] (see also [1]).

$$ D(\hat{G}\partial_x \hat{G})(+0) = \epsilon_{b1} d[\hat{G}(+0), \hat{G}_N], \ D(\hat{G}\partial_x \hat{G})(d - 0) = \epsilon_{b2} d[\hat{G}_S, \hat{G}(d - 0)] $$

(8)

where $\epsilon_{bj} = \rho D/2dR_{bj}$, $R_{b1,2}$ is the interface resistance per unit area at the $N/s$ $(x = 0)$ and $s/S$ $(x = d)$ interfaces, $\hat{G}_{S,N}$ equilibrium Green’s functions in the electrodes, $\rho$ is the normal state specific resistivity of the superconductor $s$. Note that the energies, $\epsilon_{bj}$, are connected with the characteristic dwell times: $\tau_{bj} = h/\epsilon_{bj}$. In terms of the Thouless energy $D/d^2 = E_{Th}$, the conditions [14] may be written

$$ h/\tau_{in} \ll \epsilon_{bj} \ll E_{Th} $$

(9)

Suppose that the length $d$ of the $s$ region is small enough i.e. $d \ll \sqrt{\hbar D/\Delta S}$. Then the solution of Eq.(6) is readily found (see Appendix I). The retarded and advanced Green’s functions are given by

$$ \hat{G}^\mu(\epsilon) = g^\mu(\epsilon)\hat{\sigma}_z + \hat{f}^\mu(\epsilon) = \frac{\epsilon^\mu(\epsilon)\hat{\sigma}_z + [\Delta i\hat{\sigma}_y + i\epsilon_{b2} \hat{f}^\mu(\epsilon)]}{\zeta^\mu(\epsilon)} $$

(10)

where

$$ \zeta^\mu(\epsilon) = [(\epsilon^\mu(\epsilon))^2 - \Delta^2 - 2\Delta i\epsilon_{b2} f_S^\mu \cos \varphi + (\epsilon_{b2} f_S^\mu)^2]^{1/2} $$

$$ \epsilon^R, A(\epsilon) = \epsilon + i\epsilon_{b2} g_{R, A}(\epsilon) \pm i\epsilon_{b1} $$

The Keldysh function is given by Eq.(15) and it is convenient to separate the anomalous part $\hat{G}_a^K$ to give

$$ \hat{G}^K = \hat{G}^R n - n \hat{G}^A + \hat{G}_a^K $$

We have for $\hat{G}_a^K$

$$ \hat{G}_a^K = (\hat{E}_a^K - \hat{G}_a^K \hat{E}_a^K \hat{G}^A) \frac{1}{(\zeta^R + \zeta^A)} $$

(11)

with the anomalous self-energy

$$ \hat{E}_a^K = 2i\epsilon_{b1}\{n-(\epsilon) + \hat{\sigma}_z [n_+(\epsilon) - n_-(\epsilon)]\} $$

(12)

where $n(\epsilon) = \tanh(\epsilon/2T)$,

$$ n_\pm(\epsilon) = [n(\epsilon + eV) \pm n(\epsilon - eV)]/2 $$

(13)

Thus the anomalous part $\hat{G}_a^K$ is determined by $\hat{E}_a^K$ which contains only the self-energy depending on the $N$-electrode Green’s function $\hat{G}^K$. Using Eq.(14) enables one to find the non-equilibrium part of the matrix distribution function

$$ \delta \hat{f} = (f_0 - n) + f \hat{\sigma}_z $$

(14)

which determines the anomalous part of $\hat{G}^K$:
\[ G_a^K = \hat{G}^R f - \delta f \hat{G}^A \]  

(15)

From Eq. (11) we find for the non-equilibrium parts of the distribution functions

\[ f = \frac{1}{4\nu(\zeta^R + \zeta^A)} \text{Tr} E_a^K (1 - \hat{G}^A \hat{G}^R) \]  

(16)

\[ \delta f_0 = \frac{1}{4\nu(\zeta^R + \zeta^A)} \text{Tr} E_a^K (\hat{\sigma}_z - \hat{G}^A \hat{\sigma}_z \hat{G}^R) \]  

(17)

where \( \nu = \text{Re} g^R \) is the density of states, \( \delta f_0 = f_0 - n \). Using Eq. (12) for \( \hat{E}_a^K \) the non-equilibrium part of the distribution functions may be written in the following form

\[ \delta f_0 = a_+ (n_+ - n) + b n_- 

f = a_- n_+ - b (n_+ - n) \]  

(18)

where

\[ a_\pm = \frac{\epsilon_{b1} M_\pm}{2\nu \text{Im} \zeta^R}, \quad b = \frac{\Delta \epsilon_{b1} \epsilon_{b2}}{\nu \text{Im} \zeta^R} \frac{\text{Re} f^R}{|\zeta^R|^4} \sin \varphi \]

\[ M_\pm = 1 - g^R g^A \pm [\Delta^2 - 2\epsilon_{b2} \Delta \text{Im} f^R \cos \varphi + (\epsilon_{b2} |f^R|^2)] \frac{1}{|\zeta^R|^2} \]

We took into account that \( \zeta^A = -(\zeta^R)^* \), \( g^A = -(g^R)^* \). From the self-consistency relation (15) we obtain the following system of equations for \( \Delta \) and \( \varphi \),

\[ \Lambda \Delta = \epsilon_{b2} (\alpha \cos \varphi - \beta_1 \sin \varphi), \]  

(19)

\[ \beta_0 \Delta = \epsilon_{b2} (\alpha \sin \varphi + \beta_1 \cos \varphi), \]  

(20)

where

\[ \Lambda = \ln(T/T_c) - \int_0^\infty d\epsilon \left( f_0(\epsilon) \text{Re} \frac{1}{\zeta^R(\epsilon)} - \frac{n(\epsilon)}{\epsilon} \right) \]

\[ \alpha = -\int_0^\infty df_0(\epsilon) \text{Im} \frac{f^R(\epsilon)}{\zeta^R(\epsilon)} \]

\[ \beta_k = \int_0^\infty df(\epsilon) \text{Im} \frac{k - 1 + k i f^R(\epsilon)}{\zeta^R(\epsilon)}, \quad k = 0, 1. \]

Note that Eq. (20) is the consequence of the current conservation law in the x-direction. Introducing the normalized order parameter \( \delta = \Delta/\epsilon_{b2} \) one can reduce Eqs. (19) and (20) to the equivalent ones

\[ \delta = \sqrt{\frac{\alpha^2 + \beta_1^2}{\Lambda^2 + \beta_0^2}}, \]  

(21)

\[ \exp(i\varphi) = \frac{\alpha \Lambda + \beta_0 \beta_1 + i(\alpha \beta_0 - \Lambda \beta_1)}{\sqrt{\alpha^2 + \beta_1^2} \sqrt{\Lambda^2 + \beta_0^2}} \]  

(22)

From the self-consistency equations (15) and (20) or (21) and (22), it follows that the stationary solution for the order parameter exists at arbitrary \( V \) and transition to the ac Josephson effect (time dependent phase difference \( \varphi \)) does not occur with increasing voltage. In other words the critical current of the \( S/s \) tunnel junction is absent.
in the considered mesoscopic system. Such a situation differs radically from that in a single S/S tunnel junction composed of two bulk superconductors. If at least one of the two superconductors has mesoscopic dimensions, it is important how it is connected with the conductors and non-equilibrium states arising in the presence of the current play a significant role in this case. In our system one of the important aspects of the non-equilibrium state is the quasiparticle charge-imbalance determined by the distribution function $f(\epsilon)$, and, as a consequence, the gauge-invariant potential $\mu = \Phi + (\hbar/2e)\partial_{\chi}$ in the $s$ region, where $\Phi$ is electrical potential and $\chi$ is the order parameter phase. Under the assumption \(\chi = 0\) the solution for the phase difference between the superconductors is stationary for arbitrary voltages. Therefore we can set $\chi = 0$ in the $s$ region so that $\Phi = -(\hbar/2e)\partial_{\varphi} = 0$. In other words the Josephson relation between the frequency (equal to zero) and the voltage drop across the superconducting tunnel junction is violated in the structure under consideration. The current may be calculated at any point $x$ and, for example, at the $N/s$ interface we obtain

$$I = \frac{1}{2eR_N} \int_{-\infty}^{\infty} \frac{d\epsilon}{\epsilon} \left[ F_-(\epsilon) n_-(\epsilon) + F_+(\epsilon) [n_+ (\epsilon) - n(\epsilon)] \right],$$  \hspace{1cm} (23)$$

where

$$F_-(\epsilon) = (1 + r)\nu(\epsilon)[1 - a_-(\epsilon)], \quad F_+(\epsilon) = \nu(\epsilon) b(\epsilon)(1 + r), \quad r = R_{b2}/R_{b1} = \epsilon_{b1}/\epsilon_{b2}.$$  

If the $S$ electrode is a conventional BCS superconductor,

$$g_S^R (\epsilon) = f_S^R (\epsilon) \epsilon/\Delta_S = \epsilon/\sqrt{(\epsilon + i0)^2 - \Delta_S^2}.$$  

Then for $|\epsilon| < \Delta_S$, $b(\epsilon) = 0$, $F_+(\epsilon) = 0$ and at $|eV| < \Delta$ at zero temperature the current reads

$$I = \frac{1}{eR_N} \int_0^V d\epsilon F_-(\epsilon).$$  \hspace{1cm} (24)$$

Note that the function $F_-(\epsilon) = F_-(\epsilon; V)$ depends on voltage through the voltage dependence of $\Delta$ and $\varphi$. It represents the transmission coefficient of the system which determines the efficiency of Andreev reflection processes. Taking into account that $b(\epsilon) = 0$ and assuming $|eV| < \Delta_S$, we find for the non-equilibrium part of the distribution functions

$$f(\epsilon) = a_-(\epsilon) \text{sgn}(eV) \theta(|eV| - |\epsilon|),$$  \hspace{1cm} (25)$$

$$\delta_{\varphi} (\epsilon) = -a_+(\epsilon) \text{sgn}(\epsilon) \theta(|eV| - |\epsilon|).$$  \hspace{1cm} (26)$$

Consider the case of small critical temperatures of the superconductor $s$, $T_{c0}/T_{cS} \ll 1$, and also assume that the following condition is fulfilled

$$\Delta, \epsilon_{b1}, \epsilon_{b2}, eV \ll \Delta_S.$$  \hspace{1cm} (27)$$

In this case Eqs. (21) and (22) for $\Delta$ and $\varphi$ can be simplified and presented in the form (see Appendix II).

$$\delta = \sqrt{\frac{\alpha^2 + \beta_0^2}{\Lambda^2 + \beta_0^2}},$$  \hspace{1cm} (28)$$

$$\cos \varphi + i \sin \varphi = \frac{\alpha \Lambda - \beta_0^2 + i \beta_0 (\alpha + \Lambda)}{\sqrt{\alpha^2 + \beta_0^2} \sqrt{\Lambda^2 + \beta_0^2}}.$$  \hspace{1cm} (29)$$

From Eqs. (24) and (30) we find the current

$$\frac{I}{(\epsilon_{b1} + \epsilon_{b2})/eR_N} = 2\Omega_\varphi \int_0^V du \frac{\nu(u, \Omega_\varphi)}{u^2 + r^2 + \Omega_\varphi + |\xi(u, \Omega_\varphi)|^2}.$$  \hspace{1cm} (30)$$

The I-V curves obtained by numerical calculations results on the basis of Eqs. (28), (29) and (30) are presented in Fig.2. One can see that (as a consequence of the order parameter suppression in the $s$ region) the differential conductance becomes negative with growing voltage.
In general the solution of Eqs. (28), (30) and the current can only be determined numerically because the formulas are rather complicated. Nevertheless the zero-bias conductance \( g_0 = G(0)/G_N \) can be found from Eq. (30), where \( G(V) = dI/dV \). It is given by

\[
g_0(\delta, r) = \frac{(1 + r)(1 + \delta)^2}{\left[r^2 + (1 + \delta)^2\right]^{3/2}}
\]  

(31)

where according to Eqs. (28) and (30) \( \delta = \Delta/\epsilon_{b2} \) is defined by the equation

\[
(\delta + 1) \ln \frac{r + \sqrt{r^2 + (\delta + 1)^2}}{\delta_0} = \ln \frac{4}{\epsilon_{b2}}
\]  

(32)

It follows from Eq. (32) that (under the condition (27)) the proximity effect is strong, i.e. \( \Delta > 0 \). Moreover

\[
\frac{\Delta}{\Delta_0} \to \infty \text{ at } \Delta_0 \to 0 ,
\]

i.e. due to the proximity effect, anomalously big enhancements of the order parameter occur at very weak pairing electron interaction in the s region. Assuming \( \delta << 1 \), one can obtain from (32) that

\[
\frac{\Delta}{\Delta_0} = \frac{1}{\delta_0} \ln \frac{4\Delta_s}{\epsilon_{b2}(r + \sqrt{r^2 + 1})}.
\]

(33)

This expression is valid for very small \( \delta_0 \) which satisfies the condition

\[
\ln \frac{(r + \sqrt{r^2 + 1})}{\delta_0} \gg \ln \frac{4\Delta_s}{\epsilon_{b2}(r + \sqrt{r^2 + 1})}
\]

that is fulfilled provided \( \delta_0 \ll (\epsilon_{b1} + \epsilon_{b2})/\Delta_S^2 \ll 1 \). In particular if \( \epsilon_{b1} + \epsilon_{b2} \sim 10^{-2}\Delta_S \) the requirement \( \delta_0 \ll 10^{-4} \) means that (33) is valid provided \( T_c0 \) is anomalously small, \( T_c0 \ll 10^{-6}T_{c,s} \), then \( \Delta >> 10^4\Delta_0 \). It can be seen from condition (27) that one can ignore the presence of the order parameter in the s region \( (\delta \ll 1) \) only if the pairing interaction in the s region is very weak, i.e. \( T_c0 \ll \epsilon_{b2}(\epsilon_{b1} + \epsilon_{b2})/\Delta_S^2 \). The zero-bias conductance as a function of \( t_c \) is shown in Fig.3 for different parameters \( r = R_{b2}/R_{b1} \) (and \( \epsilon_{b2} = 0.05\Delta_S \)). One can see that the normalized conductance may be both smaller and bigger than unity. In particular from (31) we find that for \( r > 1/\sqrt{2} \) the maximum value of the conductance, \( g \) corresponds to \( \delta = \delta_m \) where \( (1 + \delta_m) = \sqrt{2}r \) and Eq. (31) equals

\[
g_{0,\max} = \frac{2}{3\sqrt{2}}(1 + r).
\]

From Eq. (32) we find that the maximum conductance is realized for the case when the critical temperature \( T_{c0} = T_{c0}^m \), where

\[
T_{c0}^m = 4T_{c,s} \left[ \frac{\epsilon_{b1}(1 + \sqrt{3})}{4T_{c,s}} \right]^{1/(1 - 1/\sqrt{2})}.
\]

(34)

Eq. (34) is applicable for \( r \) satisfying the condition \( T_{c0}^m \ll T_{c,s} \); in particular it is true for \( r > 3\sqrt{3}/2 - 1 \) which corresponds to \( g_{0,\max} > 1 \). At low temperatures \( T \ll \Delta_S \) for zero-bias conductance we find from (23)

\[
g(t) = 2(1 + r)\Omega(t) \int_0^\infty \frac{du}{\cosh^2 u} \frac{\text{Re}(2tu + ir)/\zeta(2tu, \Omega(t))}{(2tu)^2 + r^2 + \Omega(t) + [\zeta(2tu, \Omega(t))]^2}
\]

(35)

where \( t = T/\epsilon_{b2} \), \( \Omega(t) = (1 + \delta(t))^2 \) and \( \zeta(u, \Omega) = [(u + ir)^2 - \Omega]^1/2 \), the function \( \delta(t) \) is defined by the equation

\[
\delta = \frac{\alpha(\Omega, t)}{\Lambda(\Omega, t)}
\]

with
\[ \alpha(\Omega, t) = \ln \frac{4\Delta S}{(\sqrt{\Omega} + r^2 + r)\epsilon_b} - \int_0^\infty du \frac{2}{\exp u + 1} \frac{\Re 1}{\zeta(tu, \Omega)} \]

\[ \Lambda(\Omega, t) = \ln \frac{\sqrt{\Omega} + r^2 + r}{\delta_0} + \int_0^\infty \frac{du}{\cosh^2 u} \ln \frac{|\zeta(2tu, \Omega) + 2ut + ir|}{\sqrt{\Omega} + r^2 + r} \]

The results of numerical calculations on the basis of Eqs.(35), (36) are presented in Figs. 4 and 5. We see that the conductance may be a non-monotonic function of temperature that radically differs from the corresponding dependencies occurring in the case of normal mesoscopic region with \( T_{c0} = 0 \) shown by dashed lines in Figs. 4 and 5. Thus a weak pairing electron interaction results in significant qualitative (for \( r \geq 1 \)) and quantitative changes of the conductance dependence with respect to the case of structure within the normal mesoscopic region. Fig.6 shows that if \( r < 1 \), the conductance may be non-monotonic function of temperature even at \( T_{c0} = 0 \). We see that the pairing electron interaction results in a shift of the position of the conductance maximum to higher temperatures together with an increase in width of the maximum. The latter is due to the slow decrease of the order parameter with increasing temperature.

Consider the case when the resistance of the barrier at the \( N/s \) interface is small enough (\( r \gg 1 \)). To be more exact we suppose that

\[ \epsilon_b \gg \Delta, \epsilon_b \quad (37) \]

i.e. \( r \gg \delta \). In this case the energy gap is absent in the superconductor \( s \) due to a strong pair-breaking effect of the normal electrode. If the condensate Green’s function is small, all the expressions are significantly simplified and from (30) at \( T, eV \ll \Delta_S \) we find for the current

\[ I = \frac{\epsilon_b + \epsilon_b}{e\Lambda} (\delta^2 + 2\delta \cos \varphi + 1) \ln \frac{(\Gamma + ieV/2\pi T)}{\Psi(\Gamma + ieV/2\pi T)} \]

where \( \Psi(z) \) is the digamma-function, \( \Gamma = 1/2 + \epsilon_b/2\pi T, \delta \) and \( \cos \varphi \) are given by Eqs.(28), (29) with

\[ \Lambda = \ln(T/T_{c0}) + \Re \Psi(\Gamma + ieV/2\pi T) - \Psi(1/2), \quad \beta_0 = \Im \Psi(\Gamma + ieV/2\pi T), \]

(39)

\[ \alpha = \Delta_S \int_0^{\Delta S} \frac{d\epsilon}{(\epsilon^2 + \epsilon_b^2)\sqrt{\Delta_S^2 - \epsilon^2}} \]

At zero temperature we obtain from Eqs.(38), (39)

\[ \frac{I}{(\epsilon_b + \epsilon_b)/e\Lambda} = (\delta^2 + 2\delta \cos \varphi + 1) \arctan(v/r) \]

(40)

where Eqs.(39) reduce to

\[ \Lambda = \ln \frac{2\sqrt{v^2 + r^2}}{\delta_0}, \quad \alpha = \ln \frac{2\delta_0}{t_c \sqrt{v^2 + r^2}}, \quad \beta_0 = \arctan(v/r) \]

The I-V curves computed for this case are shown in Fig.7. At small voltages we have \( IR_N = g_0 V \), where the normalized conductance is given by the expression

\[ g_0 = \frac{\ln^2 \frac{4}{r}}{r \ln^2 \frac{2v}{\delta_0}} \]

At large voltages, \( \epsilon_b \ll eV \ll \Delta_S \), the normalized current has the form

\[ \frac{I}{(\epsilon_b + \epsilon_b)/e\Lambda} = \frac{\ln^2(2\Delta_S/eV) + \ln^2(2eV/\Delta_0) + 2\sqrt{\ln^2(2\Delta_S/eV) + 1}[\ln^2(2eV/\Delta_0) + 1] \ln^2(2eV/\Delta_0) + 1}{\ln^2(2eV/\Delta_0) + 1} \]

At \( (2eV)^2 > \Delta_S \Delta_0 \) this function slowly decreases with increasing voltage.
C. Quasiparticle interferometer

Consider a quasiparticle interferometer composed of three tunnel junctions (see Fig. 1b) in which the phase difference \( \varphi \) between two different \( S/s \) interfaces is set by an external magnetic field. A similar system in which \( S \) and \( N \) electrodes were in contact with a normal metal was considered in [16–19]. Suppose that two barriers at the \( S/s \) interfaces are symmetrical with resistances equal to \( R_{b1} \) and the resistance of the barrier at the \( N/s \) interface equals \( R_{b1} \). We again assume that the resistance of the system is determined by the barriers and in the normal state is given by the expression \( R_N = R_{b2}/2 + R_{b1} \). Assuming that the width of the superconductor \( s \) is small, \( W \ll \sqrt{hD/\Delta} \) one can neglect the spatial variation of the Green’s function. Then Eq. (A1) is valid with

\[
\Sigma = \imath \epsilon_{b1} \hat{G}_{S+} + \imath \epsilon_{b2} \hat{G}_{S-} + \epsilon_{b1} \hat{G}_{N},
\]

where the Green’s functions \( \hat{G}_{S\pm} \) correspond to the phases \( \pm \varphi/2 \), \( \epsilon_{b1} = \rho D w_1/2 dW R_{b1} \), \( w_1 = W \) being the width of the \( s \) region and \( w_2 \) is the width of the \( S/s \) interfaces. As in previous cases we find

\[
\hat{G}^\mu(\epsilon) = g^\mu(\epsilon)\hat{\sigma}_2 + \hat{f}^\mu(\epsilon) = \frac{\epsilon^\mu(\epsilon)\hat{\sigma}_2 + \Delta^\mu(\epsilon)i\hat{\sigma}_y}{\zeta^\mu(\epsilon)},
\]

where

\[
\zeta^\mu(\epsilon) = \{(\epsilon^\mu(\epsilon))^2 - [\Delta^\mu(\epsilon)]^2\}^{1/2},
\]

\[
\epsilon^{R,A}(\epsilon) = \epsilon + 2i\epsilon_{b2}\theta^R,A(\epsilon) \pm \epsilon_{b1}, \quad \Delta^\mu(\epsilon) = \Delta + 2i\epsilon_{b2}f^\mu(\epsilon)\cos(\varphi/2).
\]

From the self-consistency relation (8) at zero voltage between the \( S \) and \( N \) electrodes the following system of equations for \( \Delta \) and \( \varphi \) can be found

\[
\Lambda \Delta = \alpha = -2\epsilon_{b2}\cos(\varphi/2) \int_0^\infty d\epsilon \frac{f_S^R(\epsilon)}{\zeta^R(\epsilon)} \frac{\text{Im} f_S^R(\epsilon)}{\zeta^R(\epsilon)}.
\]

where

\[
\Lambda = \ln(T/T_c) - \int_0^\infty de \left( \text{Re} \frac{1}{\zeta^R(\epsilon)} - \frac{1}{\epsilon} \right) n(\epsilon).
\]

At \( T = 0 \) assuming as before that \( \Delta_0, \epsilon_{b1,2} \ll \Delta_S \), we obtain

\[
\Lambda = \ln \frac{\sqrt{\Delta_0^2 + \epsilon_{b2}^2} + \epsilon_{b1}}{\Delta_0},
\]

\[
\alpha = 2\epsilon_{b2}\cos(\varphi/2) \ln \frac{4\Delta_S}{\epsilon_{b1} + \sqrt{\epsilon_{b1}^2 + \Delta_\varphi^2}},
\]

where \( \Delta_\varphi = \Delta + 2\epsilon_{b2}\cos(\varphi/2) \). It is convenient to introduce the function \( \delta_\varphi : \Delta/2\epsilon_{b2} = \delta_\varphi \cos(\varphi/2) \), then from Eqs. (43) and (45) the following equation for \( \delta_\varphi \) can be found

\[
(\delta_\varphi + 1) \ln \frac{(\delta_\varphi + 1)^2 \cos^2(\varphi/2) + r^2 + r}{\delta_0} = \ln \frac{4}{\delta_0},
\]

where \( \delta_0 = \Delta_0/2\epsilon_{b2} \). After calculations similar to those carried out in Refs. [16–19] we obtain for the zero-bias conductance of a symmetrical quasiparticle interferometer at zero temperature

\[
\frac{G(0, \varphi)}{G_N} = \frac{(1 + r)(1 + \delta_\varphi)^2 \cos^2(\varphi/2)}{[r^2 + \cos^2(\varphi/2)(1 + \delta_\varphi)]^{3/2}}.
\]

Note that at \( \varphi = 0 \) Eq. (17) is identical to Eq. (31). Thus the amplitude of the conductance oscillations may exceed \( G_N \) if the ratio \( r \) is large enough. For \( r^2 \gg (\delta + 1)^2 \) (when the energy gap is absent in the \( s \) region), Eq. (17) yields

\[
\frac{G(0, \varphi)}{G_N} = \frac{\ln^2(4/t_c)}{r \ln^2(2r/\delta_0)} \cos^2(\varphi/2).
\]

Since \( 4/t_c \gg 2r/\delta_0 \gg 1, \Delta_S \gg \epsilon_{b1} \) the amplitude of the conductance oscillations appears to be much larger than in the case of interferometer with a normal mesoscopic region [18–19].
D. Conclusion

In conclusion, we have studied phase-coherent diffusive transport through different tunnel structures with $S$ and $N$ electrodes coupled by a mesoscopic superconductor $s$. Our study has centered upon the case of a weak pairing electron interaction in the $s$ region which defines a critical temperature $T_{c0} \ll T_{cS}$. If the dwell time in the $s$ region, determined by the tunneling processes $\tau_b$, is small or comparable with $\hbar/\Delta_0$, the proximity effect is strong, i.e. the order parameter $\Delta \gg \Delta_0 \sim T_{c0}$. As a consequence, the subgap conductance of an $N - s - S$ tunneling structure and of a quasiparticle tunneling interferometer, depends strongly upon the pairing electron interaction in the $s$ region when $T_{c0} \ll T_{cS}$. Depending upon the ratio of the barrier resistances, the value of the subgap conductance, determined by Andreev reflection processes may be both larger and smaller than the conductance of these structures in the normal state. We have shown that even weak pairing electron interaction may result in the significant qualitative (in the case $r \geq 1$) and quantitative change of the conductance temperature dependence with respect to the case of structures with the normal mesoscopic region. The subgap current non-monotonously depends upon the voltage, due to the suppression of the order parameter in the mesoscopic superconductor.

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I. APPENDIX

Integrating Eq. (3) over $x$ and taking into account the boundary conditions, we obtain the following equation for $\hat{G}$ (in the following $\hat{G}$ denotes the function averaged over the length $d$ )

$$ [\hat{E}, \hat{G}] = 0, \quad \text{(A1)} $$

where

$$ \hat{E} = \epsilon \hat{\sigma}_z + \hat{\Delta} + \hat{\Sigma}, $$

$$ \hat{\Sigma} = i \epsilon b_1 \hat{G}_N + i \epsilon b_2 \hat{G}_S \quad \text{(A2)} $$

When writing equations (3) and (A1) we disregarded inelastic collisions due to condition (1). Let the potential of the superconducting electrode be zero and the potential of the normal electrode be equal to $V$, so that

$$ \hat{G}_{R,A}^{N} = \pm \hat{\sigma}_z, \quad \hat{G}_K^{N} = (1 + \hat{\sigma}_z)n(\epsilon + eV) + (1 - \hat{\sigma}_z)n(\epsilon - eV), \quad \text{(A3)} $$

$$ \hat{G}_{R,A}^{S} = g_{R,A}^{S} \hat{\sigma}_z + f_{R,A}^{S}, \quad \hat{G}_K^{S} = (\hat{G}_R^{S} - \hat{G}_A^{S})n(\epsilon). \quad \text{(A4)} $$

When finding the solution of (A1) it is convenient to let the phase of the order parameter in the s layer equal zero and the phase of the superconducting electrode $\phi$ be equal to the phase difference $\varphi$ which arises in the presence of the current. Then from (A1) and (7) we find the expressions for the retarded and advanced Green’s given in Eq.(10). The equation for $\hat{G}_K$ has the form

$$ \hat{E}_R^K \hat{G}_K - \hat{G}_K \hat{E}_A = \hat{G}_R \hat{E}_K - \hat{E}_K \hat{G}_A \quad \text{(A5)} $$

where $\hat{E}_K = \hat{\Sigma}_1^K + \hat{\Sigma}_2^K$. It is useful to take into account that

$$ \hat{E}_R^{A} = \zeta^{R,A} \hat{G}_R^{A} \quad \text{(A6)} $$

Then using Eq.(7), we have

$$ \hat{G}_R \hat{G}_K + \hat{G}_K \hat{G}_A = \hat{0} \quad \text{(A7)} $$

Therefore from (A6) it follows that $\hat{E}_R^K \hat{G}_K - \hat{G}_K \hat{E}_A = (\zeta^R + \zeta^A)\hat{G}_R \hat{G}_K$ and from (A7) we find for the Keldysh functions

$$ \hat{G}_K = (\hat{E}_K - \hat{G}_R \hat{E}_K \hat{G}_A) \frac{1}{(\zeta^R + \zeta^A)} \quad \text{(A8)} $$

II. APPENDIX

Here we present simplified formulas for $\Omega, \alpha$ and $\beta$ by taking into account the following identity which may be readily proved for small energies $\epsilon \ll \Delta_S$:

$$ (2 \nu \text{Im} \zeta^R)(u, \Omega_\varphi) = r \left( \frac{u^2 + r^2 + \Omega_\varphi}{|\zeta(u, \Omega_\varphi)|^2} \right) + r, $$

Using the notations $u = \epsilon/\epsilon_b$, $\zeta(u, \Omega) = [(u + ir)^2 - \Omega]^{1/2}$, $\Omega_\varphi = \delta^2 + 2\delta \cos \varphi + 1$ one can obtain the expressions for $a_\pm$ from (13).

$$ a_+ = 1, \quad \text{(A9)} $$

$$ a_- = 1 - \frac{2\Omega_\varphi}{u^2 + r^2 + \Omega_\varphi + |\zeta(u, \Omega_\varphi)|^2}. \quad \text{(A10)} $$
At zero temperature and $eV \ll \Delta_S$, one has

$$\Lambda = \Lambda_0 - \int_0^{eV} d\epsilon \delta f_0(\epsilon) \text{Re} \frac{1}{\xi R(\epsilon)}, \quad (A11)$$

where

$$\Lambda_0 = \ln \frac{\sqrt{\Omega^2 + r^2} + r}{\delta_0},$$

$$\delta_0 = \frac{\Delta_0}{\epsilon_{b2}}, \quad \Delta_0 = 1.76T_{c0},$$

$\Delta_0$ being the gap of the superconductor $s$ at $T = 0$ in the absence of pair-breaking factors and the proximity effect ($\epsilon_{b2} = 0$). Introducing the normalized voltage $v = Ve/\epsilon_{b2}$, from Eq.$(A11)$ we find expressions for $\alpha$ and $\beta$, (see Eqs.$19$ and $20$).

$$\Lambda = \ln |\zeta(v, \Omega) + v + ir|$$

$$\alpha = \ln \frac{4\delta_0}{|\zeta(v, \Omega) + v + ir| t_c}$$

$$\beta_0 = -\beta_1 = \frac{r}{2} \int_0^v du \frac{M_{-}(u, \Omega)}{\nu(u, \Omega) |\zeta(u, \Omega)|^2}$$
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FIG. 1. The $N/s/S$ system under consideration. (a). The $S/s$ and the $N/s$ interfaces have $R_{b2}$ and $R_{b1}$ barrier resistances respectively. (b). The schematical representation of the Andreev interferometer.

FIG. 2. I-V curves at zero temperature for $r$ less than 1, $\epsilon_{b2}/\Delta_S = 0.05$. 

$eV/\epsilon_{b2}$
FIG. 3. Dependence of the normalised conductance, $g_o$ on $t_c$.

FIG. 4. Temperature dependence of the zero-bias conductance of the $N-s-S$ structure for $t_c = 0$ and $0.05$, $\epsilon_{b2}/\Delta_S = 0.05$: $r = 1, r = 2, r = 3$.

FIG. 5. Temperature dependence of the zero-bias conductance of the $N-s-S$ structure for $r = 1, \epsilon_{b2}/\Delta_S = 0.05$: $t_c = 0.0, 0.01, 0.02, 0.04$. 
FIG. 6. Temperature dependence of the zero-bias conductance of the $N-s-S$ structure for $t_c = 0.0$ and 0.05, $\epsilon_b^2/\Delta_S = 0.05$ : $r = 1/5$, $r = 1/3$.

FIG. 7. I-V curves at zero temperature for $r \gg 1$, $\epsilon_b^2/\Delta_S = 0.05$. 
