Determining the shape of the Universe using discrete sources

G.I. Gomero,*

Instituto de Física Teórica,
Universidade Estadual Paulista,
Rua Pamplona, 145
São Paulo, SP 01405–900, Brazil

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Abstract

Suppose we have identified three clusters of galaxies as being topological copies of the same object. How does this information constrain the possible models for the shape of our Universe? It is shown here that, if our Universe has flat spatial sections, these multiple images can be accommodated within any of the six classes of compact orientable 3-dimensional flat space forms. Moreover, the discovery of two more triples of multiple images in the neighbourhood of the first one, would allow the determination of the topology of the Universe, and in most cases the determination of its size.

1 Introduction

The last two decades have seen a continuously increasing interest in studying cosmological models with multiply connected spatial sections (see [1] and references therein). Since observational cosmology is becoming an increasingly high precision science, it would be of wide interest to develop methods to systematically construct specific candidates for the shape of our Universe in order to analyse whether these models are consistent with observational data.

Since one of the simplest predictions of cosmological models with multiply connected spatial sections is the existence of multiple images of discrete cosmic objects, such as clusters of galaxies,1 the following question immediately arises: Suppose we have identified three clusters of galaxies as being different topological copies of the same object, how does this information constrain the possible models for the shape of our Universe? The initial motivation for this work was the suggestion of Roukema and Edge that the X–ray clusters RXJ 1347.5–1145 and CL 09104+4109 may be topological images of the Coma cluster [4]. Even if these particular clusters turn out not to be topological copies of the same object, the

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*german@ift.unesp.br

1Provided that the scale of compactification is small enough (see [2] and [3]).
suggestion of Roukema and Edge raises an interesting challenge. What if one day a clever astrophysicist discovers three topological copies of the same object?

It is shown here that these (would be) multiple images could be accommodated within any of the six classes of compact orientable 3-dimensional flat space forms. Moreover, and this is the main result of this paper, the discovery of two more triples of multiple images in the neighbourhood of the first one, would be enough to determine the topology of the Universe, and in most cases even its size. Thus, two interesting problems appear now, (i) does our present knowledge of the physics of clusters of galaxies (or alternatively, of quasars) may allow the identification of a triple of multiple images if they actually exist?, and (ii) given that such an identification has been achieved, how easy can other triples of topological copies near the first one be identified? The present paper does not deal with these two problems, however it should be noticed that a recent method proposed by A. Bernui and me in [5] (see also [6]) could be used to test, in a purely geometrical way, the hypothesis that any two given clusters of galaxies are topological copies.

The model building procedure is explained in the next section, while section 3 presents some numerical examples illustrating specific candidates for the shape of our Universe, under the presumed validity of the Roukema–Edge hypothesis. In section 4 it is discussed the main result of this paper: how the topology of space could be determined with the observation of just two more triples of images; and how, in most cases, one could even determine the size of our Universe. Finally, section 5 consists of discussions of the results presented in this letter and suggestions for further research.

2 Model Building

Suppose that three topological copies of the same cluster of galaxies have been identified. Let $C_0$ be the nearest copy from us, $C_1$ and $C_2$ the two other copies, $d_1$ and $d_2$ the distances from $C_0$ to $C_1$ and $C_2$ respectively, and $\theta$ the angle between the geodesic segments $C_0C_1$ and $C_0C_2$. Roukema and Edge [1] have suggested an example of this configuration, the Coma cluster being $C_0$ and the clusters RXJ 1347.5–1145 and CL 09104+4109 being $C_1$ and $C_2$ (or vice versa). The distances of these clusters to Coma are 970 and 960$h^{-1}$ Mpc respectively (for $\Omega_0 = 1$ and $\Lambda = 0$), and the angle between them, with the Coma cluster at the vertex, is $\approx 88^\circ$. Under the assumption that these multiplicity of images were due to two translations of equal length and in orthogonal directions, they constructed FL cosmological models whose compact flat spatial sections of constant time were (i) 3-tori, (ii) manifolds of class $G_2$, or (iii) manifolds of class $G_4$, all of them with square cross sections, and scale along the third direction larger than the depth of the catalogue of X-ray clusters used in the analysis.

Let us consider the possibility that at least one of the clusters $C_i$ is an image of $C_0$ by a screw motion, and do not assume that the distances from $C_0$ to $C_1$ and $C_2$ are equal, nor that they form a right angle (with $C_0$ at the vertex). It is shown in this section that one can accommodate this generic configuration of clusters within any of the six classes of compact orientable 3-dimensional flat space forms, thus providing a plethora of models for the shape of our Universe consistent with the (would be) observational fact that these clusters are in fact the same cluster. Moreover, one could also consider the possibility that one of the clusters $C_i$ is an image of $C_0$ by a glide reflection, thus giving rise to non–orientable manifolds as models for the shape of space. However, these cases will not be considered here since they do not give qualitatively different results, and the corresponding calculations can be done whenever needed.
We will fit the set of multiple images \{G_1 \} in the basis formed by the set \{\beta \} and \{\gamma \} if \{\theta \} is located at \{\alpha \}.

The classification of the Euclidean space forms take the matrix forms $G_p$ for any point $p$. The action is given by $\alpha : p \mapsto Ap + a$, \hspace{1cm} (1)

for any point $p$. The orientation preserving orthogonal transformations that appear in the classification of the Euclidean space forms take the matrix forms

$$ A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, $$

$$ B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}, $$(2)

in the basis formed by the set \{a, b, c\} of linearly independent vectors that appear in Table 1. We will fit the set of multiple images \{C_0, C_1, C_2\} within manifolds of classes $G_2 - G_6$, since the class $G_1$ (the 3–torus) is trivial.

Let us first deal with the classes $G_2 - G_5$. The generators for the corresponding covering groups are $\alpha = (A, a)$, $\beta = (I, b)$ and $\gamma = (I, c)$, with $A = A_1, B, C$ and $D$ for the classes $G_2, G_3, G_4$ and $G_5$ respectively, and $I$ is the identity transformation. For these classes we will consider the following non-trivial configuration: denoting the position of $C_0$ by $p$, $C_1$ is located at $\alpha(p)$ and $C_2$ at $\beta(p)$. The configuration in which $C_2$ is located at $\gamma(p)$ is equivalent to the former, while the configuration in which $C_1$ and $C_2$ are images of $C_0$ by pure translations (strictly possible only in $G_2$, and a convenient approximation in $G_4$ if $\theta \approx 90^\circ$, and the distances of $C_1$ and $C_2$ to $C_0$ are almost equal, as is the case in the Roukema–Edge hypothesis) is equivalent to that of a torus.

For space forms of the classes $G_2 - G_6$ the following facts are easily derivable from the generators of their corresponding covering groups (see \cite{3} for details):

1. The vector $a$ is orthogonal to both $b$ and $c$.

2. The angle between $b$ and $c$ is a free parameter for the class $G_2$, while its value is fixed to be $120^\circ$, $90^\circ$ and $60^\circ$ for the classes $G_4, G_4$ and $G_5$ respectively.

3. Denoting by $|a|$ the length of the vector $a$, and similarly for any other vector, one has that $|b| = |c|$ for the classes $G_3 - G_5$, while both lengths are independent free parameters in the class $G_2$. Moreover, in all classes $G_2 - G_5,$ $|a|$ is an independent free parameter.

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**Table 1:** Diffeomorphism classes of compact orientable 3-dimensional Euclidean space forms. The first row contains Wolf’s notation for each class, and the second gives the generators of the corresponding covering groups.

| Class | $G_1$ | $G_2$ | $G_3$ | $G_4$ | $G_5$ | $G_6$ |
|-------|-------|-------|-------|-------|-------|-------|
| Generators | $a, b, c$ | $(A_1, a), b, c$ | $(B, a), b, c$ | $(C, a), b, c$ | $(D, a), b, c$ | $(A_1, a)$, $(A_2, b + c)$, $(A_2, b - c)$ |
### Table 2

| Class | $\alpha(p)$ | $\delta_\alpha(p)$ | $\delta_\alpha(p) \cos(\alpha, \beta)$ |
|-------|-------------|-------------------|-----------------------------------|
| $G_2$ | $(x + |a|, -y, -z)$ | $\sqrt{|a|^2 + 4(y^2 + z^2)}$ | $-2y$ |
| $G_3$ | $(x + |a|, -\frac{1}{2}y - \frac{\sqrt{3}}{2}z, \frac{\sqrt{3}}{2}y - \frac{1}{2}z)$ | $\sqrt{|a|^2 + 3(y^2 + z^2)}$ | $-\frac{\sqrt{3}}{2}(\sqrt{3}y + z)$ |
| $G_4$ | $(x + |a|, -z, y)$ | $\sqrt{|a|^2 + 2(y^2 + z^2)}$ | $-(y + z)$ |
| $G_5$ | $(x + |a|, \frac{1}{2}y - \frac{\sqrt{3}}{2}z, \frac{\sqrt{3}}{2}y + \frac{1}{2}z)$ | $\sqrt{|a|^2 + (y^2 + z^2)}$ | $-\frac{1}{2}(y + \sqrt{3}z)$ |

Table 2: The second column gives the position of $C_1$ for each class of the manifolds considered in the first column. The third column gives the distance between $C_0$ and $C_1$, and the last one the cosine of the angle between the segments $C_0C_1$ and $C_0C_2$, $\cos(\alpha, \beta)$.

4. Denoting the canonical unitary basis vectors in Euclidean space by $\{i, j, k\}$, one can always write $a = |a|i$, $b = |b|j$ and $c = |c|\cos \varphi j + |c|\sin \varphi k$, for the basis $\{a, b, c\}$, where $\varphi$ is the angle between $b$ and $c$, as established in the item 2.

Writing $p = (x, y, z)$ for the components of the position of $C_0$ in the basis $\{i, j, k\}$, one can easily work out the expressions for the components of the position of $C_1$, $\alpha(p)$, the distance function $\delta_\alpha(p)$, and the cosine of the angle between $C_0C_1$ and $C_0C_2$, $\cos(\alpha, \beta)$.

The resulting expressions are shown in Table 2.

For the configuration we are dealing with, one trivially has $d_2 = \delta_\beta(p) = |b|$, since $\beta$ is a pure translation. More interestingly, from $\delta_\alpha(p) = d_1$ and $\cos(\alpha, \beta) = \cos \theta$, one can partially solve the equations for the components of the position of $C_0$. The resulting expressions are shown in Table 3. Observe that for each class we have two solutions in terms of the free parameter $|a|$. For the classes $G_3 - G_5$ the two solutions are those for which $d_1 \cos \theta$ is given by the fourth column in Table 2.

Two remarks are in order here. First, it is convenient to write down the components of the position of $C_0$ in terms of the parameter $|a|$, because this parameter can be easily determined once two more triples of multiple images, say $\{D_0, D_1, D_2\}$ and $\{E_0, E_1, E_2\}$, in the neighbourhood of $\{C_0, C_1, C_2\}$ have been identified, as shown in Section 4.\(^3\) Once this has been done, the positions of $C_0$, $D_0$ and $E_0$ can be used to predict multiple images of them due to the inverse isometry $\alpha^{-1}$, thus yielding a definitive observational test for the hypothesis.

\(^2\)Note that the origin of a coordinate system is implicitly determined by the axes of rotation of the orthogonal transformations in (2), and can be taken as the centre of the fundamental polyhedron for the corresponding manifold. Moreover, this origin does not necessarily coincide with the position of our galaxy.

\(^3\)Actually, it can be done much more than that. If the topology of the Universe turns out to be of any of the classes $G_2 - G_6$, the triples $\{D_0, D_1, D_2\}$ and $\{E_0, E_1, E_2\}$ would be enough to decide which topology our Universe has, and except in the case of $G_2$ and a configuration in $G_6$, it would be possible to specify completely the parameters of the manifold that models the spatial sections of the spacetime.
of the multiply connectedness of our Universe. Second, note that the \( x \)-coordinate is not constrained by this configuration of topological images. This freedom of the \( x \)-coordinate is a consequence of homogeneity of manifolds of classes \( G_2 \) – \( G_5 \) along the \( x \)-axis. This partial homogeneity is due to the fact that the orthogonal transformations involved in the corresponding covering groups have the \( x \)-axis as their axis of rotation.

We now fit the multiple images \( \{ C_0, C_1, C_2 \} \) within manifolds of class \( G_6 \). The generators for the covering group of a manifold of this class are \( \alpha = (A_1, a) \), \( \beta = (A_2, b + c) \) and \( \mu = (A_2, b - c) \). The vectors \( \{ a, b, c \} \) are mutually orthogonal but their lengths are free parameters. For manifolds of class \( G_6 \) we have two possible configurations, both of them with \( C_0 \) located at \( p \),

1. \( C_1 \) located at \( \alpha(p) \) and \( C_2 \) at \( \beta(p) \), and
2. \( C_1 \) located at \( \beta(p) \) and \( C_2 \) at \( \mu(p) \).

The case in which \( C_1 \) is at \( \alpha(p) \) and \( C_2 \) at \( \mu(p) \) is equivalent to the first configuration.

The expressions for the distances \( \delta_\alpha(p) \), \( \delta_\beta(p) \) and \( \delta_\mu(p) \), and angles \( \cos(\alpha, \beta) \) and \( \cos(\beta, \mu) \) are

\[
\begin{align*}
\delta_\alpha(p) &= \sqrt{|a|^2 + 4(y^2 + z^2)} \\
\delta_\beta(p) &= \sqrt{|b|^2 + 4x^2 + (2z - |c|)^2} \\
\delta_\mu(p) &= \sqrt{|b|^2 + 4x^2 + (2z + |c|)^2}
\end{align*}
\]

and

\[
\begin{align*}
\cos(\alpha, \beta) &= \frac{4z^2 - 2(|a|x + |b|y + |c|z)}{\delta_\alpha(p)\delta_\beta(p)} \\
\cos(\beta, \mu) &= \frac{4x^2 + 4z^2 + |b|^2 - |c|^2}{\delta_\beta(p)\delta_\mu(p)}
\end{align*}
\]
For the first configuration one has $\delta_\alpha(p) = d_1$, $\delta_\beta(p) = d_2$ and $\cos(\alpha, \beta) = \cos \theta$, thus yielding the equations

$$y^2 + z^2 = \frac{1}{4}(d_1^2 - |a|^2)$$
$$4x^2 + (2z - |c|)^2 = d_2^2 - |b|^2$$
$$4z^2 - 2(|a|x + |b|y + |c|z) = d_1 d_2 \cos \theta.$$

This is a system of three quadratic equations with six unknowns, the three coordinates $(x, y, z)$ of the point $p$, and the three coordinates $(|a|, |b|, |c|)$ in the parameter space of the $G_6$ manifold (see [3]). An algebraic solution of these equations for $(x, y, z)$ in terms of $(|a|, |b|, |c|)$, or vice versa, would in general yield higher degree (decoupled) equations for each variable, and thus are not so illuminating. Particular solutions can be obtained by (i) assuming specific values for the parameters $(|a|, |b|, |c|)$, and then calculating numerically the position of $C_0$, or (ii) assuming some particular position for $C_0$, and then calculating the parameters $(|a|, |b|, |c|)$. This second method does not follow the strategy of determining the parameters of the manifold using two more triples of clusters of galaxies (see Section 4), thus it will not be pursued here. The next section presents examples of application of the first method.

Finally, let us examine the second configuration which is simpler. One has $\delta_\beta(p) = d_1$, $\delta_\mu(p) = d_2$ and $\cos(\beta, \mu) = \cos \theta$, thus yielding the equations

$$4x^2 + (2z - |c|)^2 = d_1^2 - |b|^2$$
$$4x^2 + (2z + |c|)^2 = d_2^2 - |b|^2$$
$$4x^2 + 4z^2 + |b|^2 - |c|^2 = d_1 d_2 \cos \theta.$$

These equations can be partially solved giving

$$z = \frac{1}{8|c|} (d_2^2 - d_1^2)$$
$$|c| = \frac{1}{2} \sqrt{d_1^2 + d_2^2 - 2d_1 d_2 \cos \theta}$$
$$x^2 + z^2 = \frac{1}{16} (d_1^2 + d_2^2 + 2d_1 d_2 \cos \theta) - \frac{1}{4} |b|^2.$$

In this case the $y-$ coordinate is not constrained by the configuration of topological images, since the only orthogonal transformation involved in the calculations has the $y-$ axis as its axis of rotation.

### 3 Numerical Examples

Let us now apply the results obtained in the previous section to the proposed multiple images of Roukema and Edge [4], in a FL universe whose matter components are pressureless dust and a cosmological constant. The models presented below are small universes with compactification scales much smaller than the horizon radius, so they may seem to be in conflict with constraints on the topology coming from observations of the CMBR. However, it must be recalled that all current constraints for flat universes hold exclusively for models with (i) toroidal spatial sections [8, 9], or (ii) any flat (compact and orientable) spatial section, but in cosmological models without a dark energy component, and moreover, with
Table 4: Examples of models within classes $\mathcal{G}_3 - \mathcal{G}_5$.

| Class | $|a| \,(h^{-1}\,\text{Mpc})$ | $y \,(h^{-1}\,\text{Mpc})$ | $z \,(h^{-1}\,\text{Mpc})$ |
|-------|-----------------|-----------------|-----------------|
| $\mathcal{G}_3$ | 1156 | $-21.4$ | $-39.2$ | $-32.9$ | $-2.0$ |
| | 1142 | $22.2$ | $82.8$ | $108.5$ | $73.5$ |
| $\mathcal{G}_4$ | 1156 | $-14.9$ | $-45.8$ | $-45.8$ | $-14.9$ |
| | 1142 | $60.7$ | $-121.3$ | $-121.3$ | $60.7$ |
| $\mathcal{G}_5$ | 1156 | $-3.5$ | $-57.1$ | $-67.9$ | $-37.0$ |
| | 1142 | $127.3$ | $-187.9$ | $143.5$ | $38.5$ |

The models within class $\mathcal{G}_5$ do not yield models with a fixed value of $y$; instead, both $y$ and $z$ depend on the parameter $|a|$. In Table 5 we show the values of $y$ and $z$ calculated from Table 3 for $|a| = 1156$ and $1142h^{-1}\,\text{Mpc}$. In this table the first column for each coordinate corresponds to the first solution of Table 3 and the second column for the second solution.

Now we deal with models within class $\mathcal{G}_6$. For the first configuration one obtains from eqs. (5)

\[(2x + |a|)^2 + (2y + |b|)^2 = d_1^2 + d_2^2 - 2d_1d_2 \cos \theta - |c|^2,\]

which implies that

\[|c|^2 \leq d_1^2 + d_2^2 - 2d_1d_2 \cos \theta.\]

Furthermore, from the first and third equations in (5) one also has

\[|a| \leq d_1 \quad \text{and} \quad |b| \leq d_2.\]
A family of simple examples are obtained by taking $|a| = d_1$. In fact, in this case one has

$$y = z = 0, \quad x = -\frac{1}{2}d_2 \cos \theta, \quad |b|^2 + |c|^2 = d_2^2 \sin^2 \theta.$$ 

Thus, taking $|b| = |c|$, one model of a universe with spatial sections of class $\mathcal{G}_6$ that fits the first configuration with the Roukema–Edge hypothesis is

$$|a| = 1158h^{-1} Mpc \quad \text{and} \quad |b| = |c| = 1140.4h^{-1} Mpc,$$

with Coma located at

$$x = -29.9h^{-1} Mpc \quad \text{and} \quad y = z = 0.$$ 

On the other side, for the second configuration one has

$$|c| = 791.6h^{-1} Mpc, \quad z = -5.8h^{-1} Mpc, \quad |b| \leq 834.1h^{-1} Mpc,$$

the last inequality being obtained from the last equation in (7). It is illustrative to give two specific examples as done with the $\mathcal{G}_2$ models.

1. First consider the case when $|b|$ is slightly lower than its maximum value allowed by (9), say $|b| = 834h^{-1} Mpc$, then one has $x = \pm 7h^{-1} Mpc$.

2. Second consider the symmetric case when $|b| = |c| = 791.6h^{-1} Mpc$. In this case $x = \pm 131.5h^{-1} Mpc$.

4 The case of three triples of images

In this section it is shown that the discovery of two additional triples of clusters of galaxies close to $\{C_0, C_1, C_2\}$ would allow the determination of the topology of the universe, and in most cases the determination of its size. Let us denote by $\{D_0, D_1, D_2\}$ and $\{E_0, E_1, E_2\}$ these two additional triples of topological images. Mathematically, to characterize the closeness relation between two triples $\{C_i\}$ and $\{D_i\}$ it suffices the lengths of the geodesic segments $\overline{C_iD_i}$ ($i = 0, 1, 2$) to be the same and smaller than the injectivity radius. Observationally, it is enough that $C_0$ and $D_0$ are two nearby clusters of galaxies, while the distances between $C_i$ and $D_i$ ($i = 1, 2$) are equal (within the observational error bounds) to the distance between $C_0$ and $D_0$.

By parallel transporting the triangle $C_1D_1E_1$ along the geodesic segment $\overline{C_0C_1}$, one obtains two triangles with a common vertex, namely the triangle of nearby clusters $C_0D_0E_0$, and that of transported clusters of $C_1D_1E_1$. It is just a matter of elementary analytic geometry to determine the unique rotation that takes one triangle to the other. Note however that one can easily find also the unique reflection that takes one triangle to the other, if it exists. If the angle of rotation is different from $\pi, 2\pi/3, \pi/2$ or $\pi/3$, then the isometry that takes $C_0$ to $C_1$ is not a screw motion, but a reflection, and the Universe would be spatially non-orientable. On the contrary, if the angle of rotation is either $\pi, 2\pi/3, \pi/2$ or $\pi/3$, then one can think this is not by coincidence, so the Universe would be spatially orientable. In

\footnote{Strictly speaking, this closeness relation is not a necessary condition, but observationally it would be simpler to look for other triples of images in the neighborhood of the first one.}
such a case, if the angle of rotation is different from $\pi$, it uniquely determines to which class
the topology of the Universe belongs, namely $G_3$, $G_4$ or $G_5$ respectively.

Let us restrict our analysis to the orientable case in order to be specific. The deter-
mination of the rotation taking $C_0D_0E_0$ to the parallel transportation of $C_1D_1E_1$ provides
also the direction of the axis of rotation of the screw motion linking $C_0$ with $C_1$. If the
Universe has a topology of class $G_3$, $G_4$ or $G_5$, the translation vector is parallel to this axis,
so elementary geometry can be used to determine the parameter $|a|$ and the position of the
axis. Moreover, the isometry linking $C_0$ with $C_2$ has to be a translation, and a parallel
transport of the triangle $C_2D_2E_2$ to $C_0D_0E_0$ would confirm it. A remarkable fact is that,
if the topology of the Universe has been identified to be of class $G_3$, $G_4$ or $G_5$, the vector $c$
is automatically fixed, and observational searches can be performed to find the topological
images of $C_0$, $D_0$ and $E_0$ due to the isometries $\gamma$ and $\gamma^{-1}$ for validation of the model.

Let us now consider the case when the angle of rotation taking $C_0D_0E_0$ to the parallel
transport of $C_1D_1E_1$ is $\pi$. In this case the topology of the Universe has to be of class $G_2$
or $G_6$. One can decide between these two possibilities by parallel transporting $C_2D_2E_2$ to
$C_0D_0E_0$. If the angle of rotation between these triangles is null, then the isometry linking
$C_0$ to $C_2$ is a translation, and the Universe has topology of class $G_2$. On the other hand,
if the angle of rotation is $\pi$, the Universe has topology of class $G_6$. In the former case one
can proceed as before and determine the length $|a|$ and the position of the axis of the screw
motion. However, since for the class $G_2$, the vector $c$ is a free parameter, its modulus and
direction remain undetermined.

If the topology of the Universe turns out to be of class $G_6$, the multiple images can be fitted
within the two inequivalent configurations described in Section 2. One can decide between
both configurations by just looking at the directions of the axes of rotation, for if they are
orthogonal the first configuration would be the correct one, while if they are parallel the
correct one is the second. Using elementary geometry one can completely determine the three
axes of rotation and translations (thus determining the global shape of space) if the multiple
images fit with the first configuration. On the other hand, with the second configuration
one can determine the vectors $b$ and $c$, and thus the direction of $a$, but it is impossible to
determine the length $|a|$, as could have been anticipated from eqs. (6). However, in this
latter case, one can design effective search procedures to look for multiple images due to
the isometries $\alpha$ and $\alpha^{-1}$, thus providing at least robust constraints for the parameter $|a|$
(see [5]).

5 Discussion and Further Remarks

The work presented in this paper originated with the following problem in the context of
cosmological models with flat spatial sections: Suppose we have identified three clusters of
galaxies as being different topological images of the same object. How do these multiple
images constrain the possible models for the shape of our Universe? A natural extension
of this work would be the study of this problem in the context of universes with non–flat
spatial sections, specifically those with positive curvature (since multiply connected spaces
of negative curvature are very unlikely to have a detectable topology [2]).

It has been shown here that one can accommodate any of the six classes of compact
orientable 3–dimensional flat space forms to fit with any configuration of three topological

5Note however that if there exists a reflection taking one triangle to the other, in order to settle definitely
the orientability of space, it would be necessary to identify a fourth triple of multiple images.
images of a cosmic object. It can be seen from the construction of the models that one could also easily fit any of the non–orientable flat manifolds. Moreover, the main result in this paper is that the identification of two more triples of multiple images of clusters of galaxies, in the neighbourhood of the first one, is enough to completely determine the topology of space, as well as its size in most of the cases.

Even if the primary goal of this paper is not to construct specific candidates for the shape of our universe, but to present a systematic procedure for building such models, it turns out that the illustrative examples constructed by using the Roukema–Edge hypothesis are not in contradiction with current observational data.

In view of these results, it seems of primary importance to state and test hypotheses like that of Roukema and Edge, i.e. that the clusters RXJ 1347.5–1145 and CL 09104+4109 are topological images of the Coma cluster, since the identification of a very small quantity of multiple images is, as has been shown here, enough to determine (or almost determine) the global shape of the universe. The problem of testing this kind of hypothesis can be solved by the Local Noise Correlations (LNC) method proposed in \[5\]. The problem of generating such kind of hypotheses seems to be much harder, although current efforts are being done to find multiple images of our Galaxy \[11\], clusters of galaxies \[12\] and radio-loud AGNs \[13\].

To close this paper, let us stress that there have only been considered here models in which the topological images are related by the generators of the covering groups of the corresponding manifolds. This needs not be the case, for one could also consider other isometries (compositions of the generators) as being the responsible for the multiple images. Thus the list of possible models presented here is not exhaustive.

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