Transitivity of Commutativity for Second-Order Linear Time-Varying Analog Systems

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Abstract

It is proven that the transitivity property of commutativity is always valid for second-order linear time-varying analog systems whether their initial states are zero or not. Throughout the study, it is assumed that the subsystems considered cannot be obtained from each other by any feed-forward and feedback structure. The results are well validated by MATLAB simulations.

Keywords Differential equations · Initial conditions · Linear time-varying systems · Commutativity · Transitivity

1 Introduction

Second-order differential equations originate in electromagnetic, electrodynamics, transmission lines, communication, circuit and system theory, wave motion and distribution and in many fields of electrics–electronics engineering. They also play a prominent role for modeling problems occurring in mechanical systems, fluid systems, thermal systems and control systems. For example, they are used in cascade-connected and feedback systems to design higher order composite systems for achieving several beneficial properties such as controllability, sensitivity, robustness and design flexibility. When the cascade connection which is an old but still an up-to-date trend in system design [2, 12] is considered, the commutativity concept places an important role to improve different system performances. Since the commutativity of linear time-invariant relaxed systems is straightforward and time-varying systems have found a great deal of applications recently [1, 10], the scope of this paper is focused on commutativity of linear time-varying systems only.
When two systems \( A \) and \( B \) are interconnected one after the other so that the output of the former acts as the input of the latter, it is said that these systems are connected in cascade. If the order of connection in the sequence is not effective on the input–output relation of the cascade connection, then these systems are commutative \[11\], which is the first paper on the subject. In that paper, J. E. Marshall has proven that “for commutativity, either both systems are time-invariant or both systems are time-varying.”

The tutorial papers \[5, 6\] cover almost all the subjects scattered in a great deal of literature about commutativity of linear time-varying analog systems. Yet the subject is not confined into analog systems only; there are interests for the discrete-time systems as well \[4\]. And the commutativity has still been an up-to-date research area \[7, 8\].

Transitivity property of commutativity is first introduced, and some general conditions for it are presented in \[9\], where transitivity of commutativity is fully investigated for first-order systems. On the other hand, no special research has been done on the transitivity property of commutativity for second-order linear time-varying analog systems. This paper completes this vacancy.

In this paper, Sect. 2 summarizes the explicit commutativity conditions for such systems. Section 3 presents some preliminaries that are used in the proofs of Sect. 4 which deals with transitivity property with and without initial conditions. Section 5 includes an example. The paper ends with conclusions which appear in Sect. 6.

### 2 Commutativity Conditions for Second-Order Systems

Let \( A \) be the system described by the second-order linear time-varying differential equation

\[
a_2(t)\ddot{y}_A(t) + a_1(t)\dot{y}_A(t) + a_0(t)y_A(t) = x_A(t); t \geq t_0
\]

with the initial conditions \( y_A(t_0), \dot{y}_A(t_0) \) at the initial time \( t_0 \in R \), where the single (double) dot on the top indicates the first (second)-order derivative with respect to time \( t \in R \); \( x_A(t) \) and \( y_A(t) \) are the input and output of the system, respectively. Since the system is second-order \( a_2(t) \neq 0 \). Further, \( \ddot{a}_2(t), \dot{a}_1(t) \) and \( a_0(t) \) are well-defined continuous functions, that is \( \ddot{a}_2(t), \dot{a}_1(t), a_0(t) \in C[t_0, \infty) \), hence so \( \ddot{a}_2(t), a_2(t), a_1(t) \) are. It is true that System \( A \) has a unique continuous solution \( y_A(t) \) with its first- and second-order derivatives for any continuous input function \( x_A(t) \) \[3\]. Let \( B \) be another second-order linear time-varying system described in a similar way to \( A \).

When the initial conditions are zero, the necessary and sufficient conditions for the commutativity of \( A \) and \( B \) are well known that \[5, 6\]: (i) The coefficients of \( B \) are expressed in terms of those of \( A \) through

\[
\begin{bmatrix}
  b_2(t) \\
  b_1(t) \\
  b_0(t)
\end{bmatrix} = \begin{bmatrix}
  a_2(t) & 0 & 0 \\
  a_1(t) & a_2^{0.5}(t) & 0 \\
  a_0(t) & f_A(t) & 1
\end{bmatrix} \begin{bmatrix}
  k_2 \\
  k_1 \\
  k_0
\end{bmatrix}, \quad \text{where } f_A(t) = \frac{a_2^{-0.5}[2a_1(t) - \dot{a}_2(t)]}{4},
\]

(2a)
and \( k_2, k_1, k_0 \) are some constants with \( k_2 \neq 0 \) for \( B \) is of second order; further, it is assumed that \( B \) cannot be obtained from \( A \) by a constant feed-forward and feedback path gains; hence, \( k_1 \neq 0 \) \[5\]. (ii) The coefficients of \( A \) must satisfy the following equation where \( A_0 \) is a constant:

\[
a_0 - f_A^2a_2^{-0.5}f_A = A_0. \quad \forall t \geq t_0.
\]

(2b)

For commutativity with nonzero initial conditions as well, in addition to the above conditions (i) and (ii), further necessary and sufficient conditions iii) are required;

\[
y_B(t_0) = y_A(t_0) \neq 0; \quad \dot{y}_B(t_0) = \dot{y}_A(t_0),
\]

(3a)

\[
(k_2 + k_0 - 1)^2 = k_1^2(1 - A_0),
\]

(3b)\[
\dot{y}_B(t_0) = -a_2^{-0.5}(t_0) \left[ \frac{k_2 + k_0 - 1}{k_1} + f_A(t_0) \right] y_B(t_0).
\]

(3c)\[
\text{Note that by the nonzero initial condition, it is meant "general values of initial conditions," so one or two of them (but not all) may be zero in special cases. In fact, if the output } y(t_0) \text{ is zero, its derivative needs to be zero due to Eq. (3c); if not, its derivative may or may not be zero depending on whether the term in bracket in (3c) is zero or not.}
\]

3 Inverse Commutativity Conditions for Second-Order Unrelaxed Systems

In the previous section, the necessary and sufficient conditions for the commutativity of \( A \) and \( B \) are expressed dominantly by answering "What are the conditions that must be satisfied by \( B \) to be commutative with \( A \)?" The answer to the reverse question constitutes the inverse commutativity conditions. We express the results by Lemma 1:

**Lemma 1** The necessary and sufficient conditions given in Eqs. (2) and (3) for the commutativity of \( A \) and \( B \) can be expressed by the following formulas:

\[
\begin{bmatrix}
  a_2(t) \\
  a_1(t) \\
  a_0(t)
\end{bmatrix} = \begin{bmatrix}
  b_2(t) & 0 & 0 \\
  b_1(t) & b_2^{0.5}(t) & 0 \\
  b_0(t) & \dot{f}_B(t) & 1
\end{bmatrix} \begin{bmatrix}
  l_2 \\
  l_1 \\
  l_0
\end{bmatrix}, \quad \text{where } f_B(t) = \frac{b_2^{-0.5}[2b_1(t) - \dot{b}_2(t)]}{4},
\]

(4a)

\[
b_0 - f_B^2 - b_2^{0.5}\dot{f}_B = B_0, \quad \forall t \geq t_0,
\]

(4b)

\[
y_A(t_0) = y_B(t_0), \quad \dot{y}_A(t_0) = \dot{y}_B(t_0),
\]

(5a)

\[
(l_2 + l_0 - 1)^2 = l_1^2(1 - B_0),
\]

(5b)

\[
\dot{y}_A(t_0) = -b_2^{-0.5}(t_0) \left[ \frac{l_2 + l_0 - 1}{l_1} + f_B(t_0) \right] y_A(t_0).
\]

(5c)

The proof of the lemma follows just by replacing \( b \rightarrow a, k \rightarrow l, A \rightarrow B \) and \( a \rightarrow b, B \rightarrow A \) in Eqs. (2) and (3); this results in Eqs. (4) and (5), respectively.
Equations (4) and (5) constitute the inverse or dual commutativity conditions to Eqs. (2) and (3), respectively. Further dual equations and dual relations to be used in the proofs of theorems of Sect. 4 are derived as follows:

We solve Eq. (2a) for \(a_2, a_1, a_0\) in terms of \(b_2, b_1, b_0\) and obtain Eq. (4a) where

\[
\left[ l_2 \ l_1 \ l_0 \right]' = \left[ \frac{1}{k_2} \frac{-k_1}{k_2^3} \frac{k_2}{2k_2} - \frac{k_0}{k_2} \right]',
\]

\[
f_B = k_2^{0.5} f_A + \frac{k_1}{2k_2^{0.5}} = l_2^{-0.5} f_A - \frac{l_1}{2l_2}.
\] (6b)

It is worth to write the following dual equations to be used in the sequel;

\[
\left[ k_2 \ k_1 \ k_0 \right]' = \left[ \frac{1}{l_2} \frac{-l_1}{l_2^3} \frac{l_2}{2l_2} - \frac{l_0}{l_2} \right]',
\]

\[
f_A = l_2^{0.5} f_B + \frac{l_1}{2l_2^{0.5}} = k_2^{-0.5} f_B - \frac{k_1}{2k_2}.
\] (7b)

To show (4b) alternatively, we substitute values of \(b_2\) and \(b_0\) from (2a), value of \(f_B\) from (6b) in the left side of (4b), and finally using (2b), we obtain

\[
b_2^{-0.5} (t_0) \left[ \frac{k_2 + k_0 - 1}{k_1} + f_A(t_0) \right] = -b_2^{-0.5} (t_0) \left[ \frac{l_2 + l_0 - 1}{l_1} + f_B(t_0) \right].
\]

Inserting in value of \(b_2\) from (2a) and value of \(f_B(t_0)\) from (6b) yields that

\[
\frac{l_2 + l_0 - 1}{l_1} = k_2^{0.5} \left[ \frac{k_2 + k_0 - 1}{k_1} - \frac{k_1}{2k_2} \right].
\] (9a)

On the other hand, inserting in value of \(a_2\) from (4a) and value of \(f_A(t_0)\) from (7b) yields the dual equation

\[
\frac{k_2 + k_0 - 1}{k_1} = l_2^{0.5} \left[ \frac{l_2 + l_0 - 1}{l_1} - \frac{l_1}{2l_2} \right].
\] (9b)
Using the transformations (6a) and (7a) between $k_i$’s and $l_i$’s, it is straightforward to show that the above relations between $k_i$’s and $l_i$’s are valid.

**Remark** Real systems are considered throughout the paper without the constraint $a_2(t) < 0$ in Eqs. (2) and (3). If $a_2(t) < 0$, to achieve this, the constant $k_1$ in Eqs. (2) and (3) must be chosen purely imaginary if $k_1$ is not zero. Though $f_A$ in Eq. (2a) becomes purely imaginary, all the coefficients of $B$ in Eq. (2a), $A_0$ in Eq. (2b) and $\dot{y}_B(t_0)$ in Eq. (3c) become all real. Accordingly, if $k_2 > 0$, $b_2(t) < 0$, then $l_1$ should be chosen purely imaginary due to Eqs. (4a) to yield a real system $A$. If $k_2 < 0$, $b_2(t) > 0$ due to Eq. (4a) and $l_1$ is real due to Eq. (6a). A detailed investigation of the equations consisting square roots reveals that the theory applies for real systems by choosing some of the nonzero arbitrary constants purely imaginary.

### 4 Transitivity Property of Commutativity

To study the transitivity property of commutativity for second-order linear time-varying systems, consider a third system $C$ of the same type as $A$ and $B$ considered in Sects. 2 and 3. We assume $C$ is commutative with $B$, so similar relations to Eqs. (2) and (3) can be written;

\[
\begin{bmatrix}
  c_2 \\
  c_1 \\
  c_0 
\end{bmatrix} = \begin{bmatrix}
  b_2 & 0 & 0 \\
  b_1 & b_2 \cdot 0.5 & 0 \\
  b_0 & f_B & 1 
\end{bmatrix} \begin{bmatrix}
  m_2 \\
  m_1 \\
  m_0 
\end{bmatrix}, \quad \text{where } f_B = \frac{b_2 \cdot 0.5 (2b_1 - b_2)}{4}; \quad (10a)
\]

\[
b_0 - f^2_B - b_2 \cdot 0.5 f = B_0, \quad \forall t \geq t_0; \quad (10b)
\]

where $m_2 \neq 0$, $m_1 \neq 0$, $B_0$ is a constant. When the initial conditions are nonzero, the following should be satisfied as well;

\[
y_C(t_0) = y_B(t_0) \neq 0, \quad \dot{y}_C(t_0) = \dot{y}_B(t_0), \quad (11a)
\]

\[
(m_2 + m_0 - 1)^2 = m_1^2 (1 - B_0), \quad (11b)
\]

\[
\dot{y}_C(t_0) = -b_2 \cdot 0.5 \left[ \frac{m_2 + m_0 - 1}{m_1} + f_B(t_0) \right] y_C(t_0). \quad (11c)
\]

Considering the inverse commutativity conditions derived for $A$ and $B$ in Sect. 3, some of the inverse commutativity conditions for $B$ and $C$ can be written from Eqs. (4) and (5a) by changing $A \rightarrow B$ and $B \rightarrow C$, and $l_i \rightarrow n_i$. The results are

\[
\begin{bmatrix}
  c_2 \\
  c_1 \\
  c_0 
\end{bmatrix} = \begin{bmatrix}
  n_2 \\
  n_1 \\
  n_0 
\end{bmatrix}, \quad \text{where } f_c = \frac{c_2 \cdot 0.5 (2c_1 - c_2)}{4}; \quad (12a)
\]

\[
c_0 - f^2_c - c_2 \cdot 0.5 f_c = C_0, \quad \forall t \geq t_0; \quad (12b)
\]

\[
y_B(t_0) = y_C(t_0), \quad y'_B(t_0) = y'_C(t_0). \quad (13)
\]
Finally, by the replacements $A \rightarrow B$, $B \rightarrow C$, $k_i \rightarrow m_i$, Eq. (8a, 8b) turns out to be

$$C_0 = m_2 B_0 + m_0 - \frac{m_1^2}{4m_2} \quad \text{or},$$

(14a)

$$B_0 = \frac{1}{m_2} C_0 - \frac{m_0}{m_2} + \frac{m_1^2}{4m_2}.$$  

(14b)

The preliminaries have been ready now for studying the transitivity property of commutativity. Assuming $B$ is commutative with $A$, and $C$ is commutative with $B$, we need to answer that whether $C$ is a commutative pair of $A$. The answer is expressed by the following theorems and their proves.

**Theorem 1** Transitivity property of commutativity for second-order linear time-varying analog systems which cannot be obtained from each other by constant feed-forward and feedback gains is always valid under zero initial conditions.

**Proof** Since it is true by the hypothesis that $B$ is commutative with $A$, Eqs. (2a) and (2b) are valid; since it is true by hypothesis that $C$ is commutative with $B$, Eqs. (10a) and (10b) are also valid. To prove Theorem 1, it should be proven that $C$ is commutative with $A$ under zero initial conditions. Referring to the commutativity conditions for $A$ and $B$ in Eq. (2a) and replacing $B$ by $C$, this proof is done by showing the validity of

$$\begin{bmatrix} c_2 \\ c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} a_2 & 0 & 0 \\ a_1 & d_2^{0.5} & 0 \\ a_0 & f_A & 1 \end{bmatrix} \begin{bmatrix} p_2 \\ p_1 \\ p_0 \end{bmatrix},$$

(15)

where $f_A(t)$ is given as in Eq. (2a), and the coefficients of $A$ already satisfy Eq. (2b) due to the commutativity of $B$ with $A$; further, $p_2$, $p_1$, $p_0$ are some constants to be revealed.

Using Eq. (10a) first and then Eq. (2a), as well as Eq. (6b) (for computing $c_0$), we can express $c_2$, $c_1$, $c_0$ as exactly in the same form as Eq. (15) with the constants $p_2$, $p_1$, $p_0$:

$$\begin{bmatrix} p_2 \\ p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} m_2 k_2 \\ m_2 k_1 + m_1 k_2^{0.5} \\ m_2 k_0 + \frac{m_1 k_1}{2k_2^{0.5}} + m_0 \end{bmatrix}.$$  

(16)

So, the proof is completed.

For the validity of transitivity property for second-order linear time-varying analog systems under nonzero initial conditions, we state the following theorem.

**Theorem 2** Transitivity property of commutativity of systems considered in Theorem 1 is valid for the nonzero initial conditions of the systems as well.

**Proof** The proof is done by showing the commutativity of $C$ with $A$ under nonzero conditions as well. Since $C$ and $A$ are commutative with zero initial conditions, Eq. (15)
and Eq. (2a) are valid as mentioned when proving Theorem 1. To complete the proof, we should show that \( C \) is a commutative pair of \( A \) under nonzero conditions as well. Equations (3a–3c) are satisfied for systems \( C \) (instead of \( B \)) and \( A \). Namely,

\[
y_C(t_0) = y_A(t_0) \neq 0, \quad \dot{y}_C(t_0) = \dot{y}_A(t_0),
\]

\[
(p_2 + p_0 - 1)^2 = p_1^2(1 - A_0),
\]

\[
\dot{y}_C(t_0) = -a_2^{-0.5}(t_0) \left[ \frac{p_2 + p_0 - 1}{p_1} + f_A(t_0) \right] y_C(t_0),
\]

(17a)

(17b)

(17c)

where \( k_i \)’s in Eq. (2a) for system \( B \) are replaced by \( p_i \)’s in Eq. (15) for system \( C \).

Since \((A, B)\) and \((B, C)\) are commutative under nonzero initial conditions by hypothesis, Eqs. (3a) and (13) are satisfied; so it follows that Eq. (17a) are valid.

Since \( B \) and \( C \) are commutative, \( \dot{y}_B(t_0) \) and \( y_B(t_0) \) in the commutativity conditions (3c) can be replaced by \( \dot{y}_C(t_0) \) and \( y_C(t_0) \) due to Eq. (11a); the result is

\[
\dot{y}_C(t_0) = -a_2^{-0.5}(t_0) \left[ \frac{k_2 + k_0 - 1}{k_1} + f_A(t_0) \right] y_C(t_0).
\]

(18)

On the other hand, \( \dot{y}_C(t_0) \) and \( y_C(t_0) \) are related by Eq. (11c). Comparing it with Eq. (18), we write

\[
- b_2^{-0.5}(t_0) \left[ \frac{m_2 + m_0 - 1}{m_1} + f_B(t_0) \right] y_C(t_0) = -a_2^{-0.5}(t_0) \left[ \frac{k_2 + k_0 - 1}{k_1} + f_A(t_0) \right] y_C(t_0).
\]

(19)

Since \((A, B)\) is a commutative pair, substituting the values of \( a_2 \) from Eq. (4a) and \( f_A \) from Eq. (7b) into Eq. (19), making simplifications due to \( b_2(t) \neq 0, y_C(t_0) \neq 0 \), and finally, canceling \( f_B(t_0) \), we result with

\[
\frac{m_2 + m_0 - 1}{m_1} = k_2^{0.5} \left[ \frac{k_2 + k_0 - 1}{k_1} - \frac{k_1}{2k_2} \right],
\]

(20)

which is due to the commutativities of \((A, B)\) and \((B, C)\) under nonzero initial conditions.

Now, to prove Eq. (17b), we proceed as follows: Using Eq. (16), we compute

\[
\frac{p_2 + p_0 - 1}{p_1} = \frac{m_2k_2 + m_2k_0 + \frac{m_kk_1}{2k_2} + m_0 - 1}{m_2k_1 + m_1k_2^{0.5}}.
\]

(21a)

Solving Eq. (20) for \( m_1 \), we have

\[
m_1 = \frac{m_2 + m_0 - 1}{k_2^{0.5} \left( \frac{k_2 + k_0 - 1}{k_1} - \frac{k_1}{2k_2} \right)}.
\]

(21b)
Substituting Eq. (21b) in (21a) and performing a great deal of calculus, we obtain

\[
\frac{p_2 - p_0 - 1}{p_1} = \frac{k_2 + k_0 - 1}{k_1} \frac{m_2(k_2 + k_0) + m_0 - 1 - m_2 \frac{k_1^2}{2k_2}}{\frac{m_2(k_2 + k_0) + m_0 - 1 - m_2 \frac{k_1^2}{2k_2}}{k_1}}. \tag{21c}
\]

Using the equality (21c) in Eq. (3b) directly yields Eq. (17b). On the other hand, when Eq. (21c) is used in Eq. (18), this equation results with the proof of Eq. (17c), so does with the completion of the proof of Theorem 2.

5 Example

To illustrate the results obtained in the previous section and to validate the transitivity by computer simulation, consider the system \(A\) defined by

\[A : \dot{y}_A + (3 + \sin t)\dot{y}_A + \left(3.25 + 0.25 \sin^2 t + 1.5 \sin t + 0.5 \cos t\right)y_A = x_A \quad \tag{22a}\]

for which Eq. (2a) yields \(f_A(t) = 1.5 + 0.5 \sin t\). So \(\dot{f}_A(t) = 0.5 \cos t\), and Eq. (2b) yields that \(A_0 = 1\) which is constant. Choosing \(k_2 = 1, k_1 = -2, k_0 = 0\) in Eq. (2a) yields System \(B\) as the commutative (under zero initial conditions) pair of \(A\) as;

\[B : \dot{y}_B + (1 + \sin t)\dot{y}_B + \left(0.25 + 0.25 \sin^2 t + 0.5 \sin t + 0.5 \cos t\right)y_B = x_B. \quad \tag{22b}\]

We compute \(f_B\) and \(B_0\) from Eq. (22b) by using Eqs. (4a) and (4b); \(f_B = 0.5 + 0.5 \sin t, B_0 = 0\). With the chosen \(k_i\)'s, computed \(A_0\) and \(B_0\), \(f_A\) and \(f_B\), we easily check the validity of Eqs. (6b, 7b, 8).

Considering the requirements for the nonzero initial conditions at \(t_0 = 0\), Eq. (3c) yields \(\dot{y}_B(0) = -1.5y_B(0)\). Hence, for the commutativity of \(A\) and \(B\) under nonzero initial conditions as well, due to Eq. (5a), \(\dot{y}_A(0) = \dot{y}_B(0) = -1.5y_B(0) = -1.5y_A(0)\).

We now consider a third system \(C\) which is commutative with \(B\). Therefore, using Eq. (10a) with \(m_2 = 1, m_1 = 3, m_0 = 3\), we obtain the coefficients of \(C\) so that

\[C : \dot{y}_C + (4 + \sin t)\dot{y}_C + \left(4.75 + 0.25 \sin^2 t + 2 \sin t + 0.5 \cos t\right)y_C = x_C. \quad \tag{22c}\]

Equations (12a) and (12b) yield that \(f_C = 2 + 0.5 \sin t, C_0 = 0.75\). One can easily check that \(C_0\) and \(B_0\) satisfy relations in Eqs. (14a, 14b).

For the commutativity of \(B\) and \(C\) under nonzero initial conditions at time \(t_0 = 0\) as well, Eq. (11) together with Eq. (22b), computed \(f_B\) and chosen values of \(m_i\)'s yields \(\dot{y}_C(0) = \dot{y}_B(0) = -1, 5y_C(0) = -1, 5y_B(0)\).

Considering the transitivity property under nonzero initial conditions, the conditions of Theorem 2 are satisfied. Namely, using the chosen values of \(m_i\)'s and \(k_i\)'s, from Eq. (16), we have \(p_2 = 1, p_1 = 1, p_0 = 0\). And with the computed value of \(A_0 = 1\), Eq. (17b) is satisfied, so is Eq. (17c).
The simulations are done for the interconnection of the above-mentioned systems $A$, $B$, $C$. The initial conditions are taken as $y_A(0) = y_B(0) = y_C(0) = 1$; $\dot{y}_A(0) = \dot{y}_B(0) = \dot{y}_C(0) = -1.5y_A(0) = -1.5y_B(0) = -1, 5y_C(0) = -1.5$. Input is assumed $40\sin (10\pi t)$. It is observed that $AB$, $BA$ yield the same response; $BC$, $CB$ yield the same response, so $CA$, $AC$ yield the same response. These responses are shown in Fig. 1 by $AB = BA$, $BC = CB$, $CA = AC$, respectively. Hence, transitivity property shows up as if $(A, B)$ and $(B, C)$ are commutative pairs so is $(A, C)$.

The second set of simulations are obtained by zero initial conditions, the conditions of Theorem 1 are satisfied by choosing $m_2 = 1$, $m_1 = -1$, $m_0 = 3$, so that $C$ is obtained from $B$ through Eq. (10a) as

$$C : \ddot{y}_C + (2 - \sin t)\dot{y}_C + \left(2.75 + 0.25\sin^2 t + 0.5\cos t\right)y_C = x_C.$$  (23)

Hence, $C$ is commutative with $B$, and together with $B$ being commutative with $A$, the conditions of Theorem 1 are satisfied. So, $C$ is commutative with $A$ as observed in Fig. 2. In this figure, the responses are indicated by $AB = BA$, $BC = CB$, $CA = AC$ which are all obtained by zero initial conditions. Hence, Theorem 1 is validated for $AC$.

Finally, the simulations are performed for arbitrary initial conditions $y_A(0) = 0.4$, $\dot{y}_A(0) = -0.3$, $y_B(0) = 0.2$, $\dot{y}_B(0) = -0.4$, $y_C(0) = -0.5$, $\dot{y}_C(0) = 0.5$. It is observed that $(A, B), (B, C), (C, A)$ are not commutative pairs at all, the plots $AB$, $BA$; $BC$, $CA$; $CA$, $AC$ are shown in Fig. 3, respectively. However, since all systems are asymptot-
Fig. 2 Simulation results obtained with zero initial conditions

Fig. 3 Responses of cascade connection of systems $A$, $B$, $C$
ically stable and the effects of nonzero initial conditions die away as time proceeds, and $A$, $B$, $C$ are pairwise commutative with zero initial conditions, the responses of $AB$ and $BA$, $BC$ and $CB$, $CA$ and $AC$ approach each other with increasing time. That is commutativity property and its transitivity get valid in the steady-state case.

All the simulations are done by MATLAB2010 Simulink Toolbox with fixed time step of 0.02 using ode3b (Bogacki–Shampine) program; the final time is $t = 10$.

6 Conclusions

On the basis of the commutativity conditions for second-order linear time-varying analog systems with nonzero initial conditions, the inverse commutativity conditions are reformulated completely in the form of Lemma 1 by considering the case of nonzero initial conditions. With the obtained results, the transitivity property of commutativity is stated both for relaxed and for unrelaxed cases by Theorems 1 and 2, respectively. Throughout the study, the subsystems considered are assumed not obtainable from each other by any feed-forward and feedback structure, since this excluded case needs special treatments due to special commutativity requirements in case of nonzero initial conditions.

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