Shortest Reconfiguration of Perfect Matchings via Alternating Cycles

Takehiro Ito* Tohoku University, Japan takehiro@ecei.tohoku.ac.jp
Naonori Kakimura† Keio University, Japan kakimura@math.keio.ac.jp
Naoyuki Kamiyama‡ Kyushu University, and JST, PRESTO, Japan kamiyama@imi.kyushu-u.ac.jp
Yusuke Kobayashi§ Kyoto University, Japan yusuke@kurims.kyoto-u.ac.jp
Yoshio Okamoto¶ University of Electro-Communications, and RIKEN Center for Advanced Intelligence Project, Japan okamotoy@uec.ac.jp

Abstract

Motivated by adjacency in perfect matching polytopes, we study the shortest reconfiguration problem of perfect matchings via alternating cycles. Namely, we want to find a shortest sequence of perfect matchings which transforms one given perfect matching to another given perfect matching such that the symmetric difference of each pair of consecutive perfect matchings is a single cycle. The problem is equivalent to the combinatorial shortest path problem in perfect matching polytopes. We prove that the problem is \textsc{NP}-hard even when a given graph is planar or bipartite, but it can be solved in polynomial time when the graph is outerplanar.

1 Introduction

Combinatorial reconfiguration is a fundamental research subject that sheds light on solution spaces of combinatorial (search) problems, and connects various concepts such as optimization, counting, enumeration, and sampling. In its general form, combinatorial reconfiguration is concerned with properties of the configuration space of a combinatorial problem. The configuration space of a combinatorial problem is often represented as a graph, but its size is usually exponential in the instance size. Thus, algorithmic problems on combinatorial reconfiguration are not trivial, and require novel tools for resolution. For recent surveys, see [27, 15].

Two basic questions have been encountered in the study of combinatorial reconfiguration. The first question asks the existence of a path between two given solutions in the configuration space, namely the reachability of the two solutions. The second question asks the shortest length of a path between two given solutions, if it exists. The second question is usually referred to as a shortest reconfiguration problem.

*Partially supported by JST CREST Grant Number JPMJCR1402, and JSPS KAKENHI Grant Numbers JP18H04091 and JP19K11814, Japan.
†Supported by JSPS KAKENHI Grant Numbers JP17K00028 and JP18H05291.
‡Partially supported by JSPS KAKENHI Grant Number JPMJPR1753, Japan.
§Partly supported by JSPS KAKENHI Grant Numbers JP16K16010, JP17K19960, and JP18H05291, Japan.
¶Partially supported by JSPS KAKENHI Grant Number 15K00009 and JST CREST Grant Number JPMJCR1402, and Kayamori Foundation of Informational Science Advancement.
In this paper, we focus on reconfiguration problems of matchings, namely sets of independent edges. There are several ways of defining the configuration space for matchings, and some of them have already been studied in the literature [16, 19, 14, 6, 4]. We will explain them in Section 1.1.

We study yet another configuration space for matchings, which we call the alternating path/cycle model. The model is motivated by adjacency in matching polytopes, which we will see soon. In the model, we are given an undirected and unweighted graph $G$, and also an integer $k \geq 0$. The vertex set of the configuration space consists of the matchings in $G$ of size at least $k$. Two matchings $M$ and $N$ in $G$ are adjacent in the configuration space if and only if their symmetric difference $M \triangle N := (M \cup N) \setminus (M \cap N)$ is a single path or cycle. In particular, we are interested in the case where $k = \lfloor |V(G)|/2 \rfloor$, namely the reconfiguration of perfect matchings. In that case, the model is simplified to the alternating cycle model since $M \triangle N$ cannot have a path. See Figure 1 as an example.

The reachability of two perfect matchings is trivial under the alternating cycle model: the answer is always yes. This is because the symmetric difference of two perfect matchings always consists of vertex-disjoint cycles. Therefore, we focus on the shortest perfect matching reconfiguration under the alternating cycle model.

1.1 Related Work

Other Configuration Spaces for Matchings As mentioned, reconfiguration problems of matchings have already been studied under different models [16, 19, 14, 6, 4]. These models chose more elementary changes as the adjacency on the configuration space. Then, the situation changes drastically under such models: even the reachability of two matchings is not guaranteed.

Matching reconfiguration was initiated by the work of Ito et al. [16]. They proposed the token addition/removal model of reconfiguration, in which we are also given an integer $k \geq 0$, and the vertex set of the configuration space consists of the matchings of size at least $k$. Two matchings $M$ and $N$ are adjacent if and only if they differ in only one edge. Ito et al. [16] proved that the reachability of two given matchings can be checked in polynomial time.

Another model of reconfiguration is token jumping, introduced by Kamiński et al. [19]. In the token jumping model, we are also given an integer $k \geq 0$, and the vertex set of the configuration space consists of the matchings of size exactly $k$. Two matchings $M$ and $N$ are adjacent if and

\[ \text{Precisely, their model is defined in a slightly different way, but it is essentially the same as this definition.} \]
only if they differ in only two edges. Kamiński et al. [19, Theorem 1] proved that the token jumping model is equivalent to the token addition/removal model when $|M| = |N|$. Thus, using the result by Ito et al. [16], the reachability can be checked in polynomial time also under the token jumping model [19, Corollary 2].

On the other hand, the shortest matching reconfiguration is known to be hard. Gupta et al. [14] and Bousquet et al. [6] independently proved that the problem is NP-hard under the token jumping model. Then, the problem is also NP-hard under the token addition/removal model, because the shortest lengths are preserved under the two models [19, Theorem 1].

Recently, Bonamy et al. [4] studied the reachability of two perfect matchings under a model close to ours, namely the alternating cycle model restricted to length four. In the model, two perfect matchings $M$ and $N$ are adjacent if and only if their symmetric difference $M \triangle N$ is a cycle of length four. Then, the answer to the reachability is not always yes, and Bonamy et al. [4] proved that the reachability problem is PSPACE-complete under this restricted model.

**Relation to Matching Polytopes** Our alternating cycle model (without any restriction of cycle length) for the perfect matching reconfiguration is natural when we see the connection with the simplex methods for linear optimization, or combinatorial shortest paths of the graphs of convex polytopes.

In the combinatorial shortest path problem of a convex polytope $P$, explicitly or implicitly, and two vertices $v, w$ of $P$. Then, we want to find a shortest sequence $u_0, u_1, \ldots, u_t$ of vertices of $P$ such that $u_0 = v, u_t = w$ and $u_iu_{i+1}$ forms an edge of $P$ for every $i = 0, 1, \ldots, t - 1$. Often, we are only interested in the length of such a shortest sequence, and we are also interested in the maximum shortest path length among all pairs of vertices, which is known as the combinatorial diameter of the polytope $P$. The combinatorial diameter of a polytope has attracted much attention in the optimization community from the motivation of better understanding of simplex methods. Simplex methods for linear optimization start at a vertex of the feasible region, follow edges, and arrive at an optimal vertex. Therefore, the combinatorial diameter dictates the best-case behavior of such methods. The famous Hirsch conjecture states that every $d$-dimensional convex polytope with $n$ facets has the combinatorial diameter at most $n - d$. This has been disproved by Santos [33], and the current best upper bound of $(n - d)\log d O(d/\log d)$ for the combinatorial diameter was given by Sukegawa [34]. On the other hand, for the 0/1-polytopes (i.e., polytopes in which the coordinates of all vertices belong to $\{0, 1\}$), the Hirsch conjecture holds [26].

The shortest perfect matching reconfiguration under the alternating cycle model can be seen as the combinatorial shortest path problem of a perfect matching polytope. The perfect matching polytope of a graph $G$ is defined as follows. The polytope lives in $\mathbb{R}^{E(G)}$, namely each coordinate corresponds to an edge of $G$. Each vertex $v$ of the polytope corresponds to a perfect matching $M$ of $G$ as $v_e = 1$ if $e \in M$ and $v_e = 0$ if $e \notin M$. The polytope is defined as the convex hull of those vertices. It is known that two vertices $u, v$ of the perfect matching polytope form an edge if and only if the corresponding perfect matchings $M, N$ have the property that $M \triangle N$ contains only one cycle [9]. This means that the graph of the perfect matching polytope is exactly the configuration space for perfect matchings under the alternating cycle model.

**Further Related Work** As mentioned before, the matching reconfiguration has been studied by several authors [16, 19, 14, 6, 4]. Extension to $k$-matchings has been considered, too [25, 17].

Shortest reconfiguration has attracted considerable attention. Starting from an old work on the 15-puzzle [31], we see the work on pancake sorting [8], triangulations of point sets [21, 29] and simple polygons [2] under flip distances, and also independent set reconfigurations [35], satisfiability reconfiguration [24], coloring reconfiguration [18], token swapping problems [37, 23, 38, 5, 36, 20]. A tantalizing open problem is to determine the complexity of computing the rotation distance of two rooted binary trees (or equivalently the flip distance of two triangulations.
of a convex polygon, or the combinatorial shortest path of an associahedron).

The computational aspect of the combinatorial shortest path problem on convex polytopes is not well investigated. It is known that the combinatorial diameter is hard to determine [11] even for fractional matching polytopes [32]. In the literature, we find many papers on the adjacency of convex polytopes arising from combinatorial optimization problems [13, 22, 3, 10]. Among others, Papadimitriou [28] proved that determining whether two given vertices are adjacent in a traveling salesman polytope is coNP-complete. This implies that computing the combinatorial shortest path between two vertices of a traveling salesman polytope is NP-hard. However, to the best of the authors’ knowledge, all known combinatorial polytopes with such adjacency hardness stem from NP-hard combinatorial optimization problems and the associated polytopes have exponentially many facets. We also point out the work on a randomized algorithm to compute a combinatorial “short” path [7].

1.2 Our Contribution

To the best of the authors’ knowledge, known results under different models do not have direct relations to our alternating cycle model, because their configuration spaces are different. In this paper, we thus investigate the polynomial-time solvability of the shortest perfect matching reconfiguration under the alternating cycle model. The results of our paper are two-fold.

1. The shortest perfect matching reconfiguration under the alternating cycle model can be solved in polynomial time if the input graph is outerplanar.

2. The shortest perfect matching reconfiguration under the alternating cycle model is NP-hard even when the input graph is planar or bipartite.

Since outerplanar graphs form a natural and fundamental subclass of planar graphs, our results exhibit a tractability border among planar graphs.

The hardness result for bipartite graphs implies that the computation of a combinatorial shortest path in a convex polytope is NP-hard even when an inequality description is explicitly given. This is because a polynomial-size inequality description of the perfect matching polytope can be explicitly written down from a given bipartite graph.

We point out that the hardness results have been independently obtained by Aichholzer et al. [1]. Indeed, they proved the hardness for planar bipartite graphs (i.e., an input graph is planar and bipartite).

Technical Key Points Compared to recent algorithmic developments on reachability problems, only a few polynomial-time solvable cases are known for shortest reconfiguration problems. We now explain two technical key points, especially for algorithmic results on shortest reconfiguration problems.

The first point is the symmetric difference of two given solutions. Under several known models (not only for matchings) that employ elementary changes as the adjacency on the configuration space, the symmetric difference gives a (good) lower bound on the shortest reconfiguration. This is because any reconfiguration sequence (i.e., a path in the configuration space) between two given solutions must touch all elements in their symmetric difference at least once. For example, in Figure 1, the symmetric difference of two perfect matchings \( M \) and \( N \) consists of 16 edges and hence it gives the lower bound of \( 16/4 = 4 \) under the alternating cycle model restricted to length 4 [4]. In such a case, if we can find a reconfiguration sequence touching only the elements in the symmetric difference (e.g., the sequence \( (M, M'_1, M'_2, M'_3, N) \) in Figure 1), then it is automatically the shortest under that model. However, this useful property does not hold under our alternating cycle model, because the length of an alternating cycle for reconfiguration is not fixed.
The second point is the characterization of *unhappy moves* that touch elements contained commonly in two given solutions. For example, the shortest reconfiguration sequence \((M, M_1, M_2, N)\) in Figure 1 has an unhappy move, since it touches the edge in \(M \cap N\) twice. In general, analyzing a shortest reconfiguration becomes much more difficult if such unhappy moves are required. A well-known example is the (generalized) 15-puzzle [31] in which the reachability can be determined in polynomial time, while the shortest reconfiguration is NP-hard. As illustrated in Figure 1, the shortest perfect matching reconfiguration requires unhappy moves even for outerplanar graphs, and hence we need to characterize the unhappy moves to develop a polynomial-time algorithm.

2 Problem Definition

In this paper, a graph always refers to an undirected graph that might have parallel edges and does not have loops. For a graph \(G\), we denote by \(V(G)\) and \(E(G)\) the vertex set and edge set of \(G\), respectively. An edge subset \(M \subseteq E\) is called a *matching* in \(G\) if no two edges in \(M\) share the end vertices. A matching \(M\) is *perfect* if \(|M| = |V(G)|/2\).

A graph is *planar* if it can be drawn on the plane without edge crossing. Such a drawing is called a *plane* drawing of the planar graph. A *face* of a plane drawing is a maximal region of the plane that contains no point used in the drawing. There is a unique unbounded face, which is called the *outer face*. A planar graph is *outerplanar* if it has an outerplane drawing, i.e., a plane drawing in which all vertices are incident to the outer face.

For a matching \(M\) in a graph \(G\), a cycle \(C\) in \(G\) is called *\(M\)-alternating* if edges in \(M\) and \(E(G) \setminus M\) alternate in \(C\). We identify a cycle with its edge set to simplify the notation. We say that two perfect matchings \(M\) and \(N\) are *reachable* (under the alternating cycle model) if there exists a sequence \(\langle M_0, M_1, \ldots, M_t \rangle\) of perfect matchings in \(G\) such that

(i) \(M_0 = M\) and \(M_t = N\);

(ii) \(M_i = M_{i-1} \triangle C_i\) for some \(M_{i-1}\)-alternating cycle \(C_i\) for each \(i = 1, \ldots, t\).

Such a sequence is called a *reconfiguration sequence* between \(M\) and \(N\), and its *length* is defined as \(t\).

For two perfect matchings \(M\) and \(N\), the subgraph \(M \triangle N\) consists of disjoint \(M\)-alternating cycles \(C_1, \ldots, C_t\). Thus it is clear that \(M\) and \(N\) are always reachable for any two perfect matchings \(M\) and \(N\) by setting \(M_i = M_{i-1} \triangle C_i\) for \(i = 1, \ldots, t\). In this paper, we are interested in finding a *shortest* reconfiguration sequence of perfect matchings. That is, the problem is defined as follows:

**SHORTEST PERFECT MATCHING RECONFIGURATION**

**Input:** A graph \(G\) and two perfect matchings \(M\) and \(N\) in \(G\).

**Find:** A shortest reconfiguration sequence between \(M\) and \(N\).

We denote by a tuple \(I = (G, M, N)\) an instance of SHORTEST PERFECT MATCHING RECONFIGURATION. Also, we denote by \(OPT(I)\) the length of a shortest reconfiguration sequence of an instance \(I\). We note that it may happen that \(OPT(I)\) is much shorter than the number of disjoint \(M\)-alternating cycles in \(M \triangle N\) (see Figure 1).

3 Polynomial-Time Algorithm for Outerplanar Graphs

In this section, we prove that there exists a polynomial-time algorithm for SHORTEST PERFECT MATCHING RECONFIGURATION on an outerplanar graph, as follows.

**Theorem 3.1.** SHORTEST PERFECT MATCHING RECONFIGURATION on outerplanar graphs \(G\) can be solved in \(O(|V(G)|^5)\) time.
We give such an algorithm in this section. Let \( I = (G, M, N) \) be an instance of the problem such that \( G = (V, E) \) is an outerplanar graph. We first observe that it suffices to consider the case when \( G \) is 2-connected.

**Lemma 3.2.** Let \( I = (G, M, N) \) be an instance of Shortest Perfect Matching Reconfiguration, and \( G_1, \ldots, G_p \) be the 2-connected components of \( G \). Furthermore, for every \( i = 1, \ldots, p \), let \( I_i = (G_i, M \cap E(G_i), N \cap E(G_i)) \) be an instance of Shortest Perfect Matching Reconfiguration. Then, \( \text{OPT}(I) = \sum_{i=1}^p \text{OPT}(I_i) \).

**Proof.** Let \( G_1, \ldots, G_p \) be 2-connected components in \( G \). Then, since any \( M' \)-alternating cycle is contained in some \( G_i \) for a perfect matching \( M' \) of \( G \), it suffices to solve the problem for each \( G_i \). Specifically, it holds that \( \text{OPT}(I) = \sum_{i=1}^p \text{OPT}(I_i) \), where \( I_i = (G_i, M \cap E(G_i), N \cap E(G_i)) \). \( \square \)

Since the 2-connected components of a graph can be found in linear time, the reduction to 2-connected outerplanar graphs can be done in linear time, too.

We fix an outerplane drawing of a given 2-connected outerplanar graph \( G \), and identify \( G \) with the drawing for the sake of convenience. We denote by \( C_{\text{out}} \) the outer face boundary. Then \( C_{\text{out}} \) is a simple cycle since \( G \) is 2-connected. We denote the set of the inner edges of \( G \) by \( E_{\text{in}} = E \setminus C_{\text{out}} \). In other words, \( E_{\text{in}} \) is the set of chords of \( C_{\text{out}} \).

### 3.1 Technical Highlight

As mentioned in Introduction, there are two technical key points to develop a polynomial-time algorithm for Shortest Perfect Matching Reconfiguration: a lower bound on the length of a shortest reconfiguration sequence, and the characterization of unhappy moves. We here explain our ideas roughly, and will give detailed descriptions in the next subsections.

Since \( G \) is planar, we can define its “dual-like” graph \( G^* \). Then, \( G^* \) forms a tree since \( G \) is outerplanar and 2-connected. (The definition of \( G^* \) will be given in Section 3.2, and an example is given in Figure 2.) We make a correspondence between an edge in \( G^* \) and a set of edges in \( G \). Then, we will define the length \( \ell(e^*) \) of each edge \( e^* \) in \( G^* \) so that it represents the “gap” between \( M \) and \( N \) when we are restricted to the edges in the corresponding set of \( e^* \). It is important to notice that any cycle \( C \) in \( G \) corresponds to a subtree of \( G^* \), and vice versa. Indeed, we focus on a cut \( C^* \) of \( G^* \) clipping the subtree from \( G^* \), that is, the set of edges in \( G^* \) leaving the subtree.

If we apply an \( M \)-alternating cycle \( C \) to a perfect matching \( M \) of \( G \), then it changes lengths \( \ell(e^*) \) of the edges \( e^* \) in the corresponding cut \( C^* \).

For our algorithm, we need a (good) lower bound for the length of a shortest reconfiguration sequence between two given perfect matchings \( M \) and \( N \). Recall that \( |M \triangle N| \) does not give a good lower bound under the alternating cycle model. This is because we can take a cycle of an arbitrary (non-fixed) length, and hence \( |M \triangle N| \) can decrease drastically by only a single alternating cycle. Furthermore, no matter how we define the length \( \ell(e^*) \) of each edge \( e^* \) in \( G^* \), the total length of all edges in \( G^* \) does not give a good lower bound. This is because a cycle \( C \) of non-fixed length in \( G \) may correspond to a cut \( C^* \) having many edges in \( G^* \), and hence it can change the total length drastically. Our key idea is to focus on the total length of each path in \( G^* \), that is, we take the diameter of \( G^* \) (with respect to length \( \ell \)) as a lower bound. Then, because \( G^* \) is a tree, any path in \( G^* \) can contain at most two edges from the corresponding cut \( C^* \). Therefore, regardless of the cycle length, the diameter of \( G^* \) can be changed by only these two edges. By carefully setting the length \( \ell(e^*) \) as in (1), we will prove that the diameter of \( G^* \) is not only a lower bound, but indeed gives the shortest length under the assumption that \( E_{\text{in}} \cap M \cap N \) is empty. Therefore, the real difficulty arises when \( E_{\text{in}} \cap M \cap N \) is not empty.

In the latter case, we will characterize the unhappy moves. Assume that we know the set \( F \subseteq E_{\text{in}} \cap M \cap N \) of chords that are not touched in a shortest reconfiguration sequence between \( M \) and \( N \); in other words, all chords in \( (E_{\text{in}} \cap M \cap N) \setminus F \) must be touched for unhappy moves in that sequence. Then, we subdivide a given outerplanar graph \( G \) into subgraphs \( G_1, \ldots, G_{|F|+1} \).
along the chords in $F$. Notice that each edge in $F$ appears on the outer face boundaries in two of these subgraphs. Furthermore, each chord $e$ in these subgraphs satisfies $e \in (E_{in} \cap M \cap N) \setminus F$ if $e \in M \cap N$. Therefore, all chords in these subgraphs are touched for unhappy moves as long as they are in $M \cap N$. Under this assumption, we will prove that the diameter of $G_i^*$ gives the shortest length of a reconfiguration sequence between $M \cap E(G_i)$ and $N \cap E(G_i)$. Thus, we can solve the problem in polynomial time if we know $F$ which yields a shortest reconfiguration sequence between $M$ and $N$. Finally, to find such a set $F$ of chords, we construct a polynomial-time algorithm which employs a dynamic programming method along the tree $G^*$.

3.2 Preliminaries: Constructing a Dual Graph

Let $I = (G,M,N)$ be an instance of SHORTEST PERFECT MATCHING RECONFIGURATION such that $G$ is a 2-connected outerplanar graph. Since $G$ is planar, we can define the dual of $G$. In fact, we here construct a graph $G^*$ obtained from the dual by applying a slight modification as follows. The construction is illustrated in Figure 2. Let $V^*$ be the set of faces (without the outer face) of $G$. For a face $v^* \in V^*$, let $E_{in}^*$ be the set of edges around $v^*$. We denote the set of faces touching the outer face by $U^*$, i.e., $U^* = \{v^* \in V^* \mid E_{in}^* \cap C_{out} \neq \emptyset\}$. We make a copy of $U^*$, denoted by $\tilde{U}^*$. We set the vertex set of $G^*$ to be $V^* \cup \tilde{U}^*$. For $v^*, w^*$ in $V^*$, an edge $v^*w^*$ in $G^*$ exists if and only if the faces $v^*$ and $w^*$ share an edge in $E_{in}$, i.e., $|E_{in}^* \cap E_{in}^*| = 1$. Also $G^*$ has an edge between $u^*$ and $\tilde{u}^*$ for every $u^* \in U^*$. Thus the edge set of $G^*$ is given by

$$E(G^*) = \{v^*w^* \mid v^*, w^* \in V^*, |E_{in}^* \cap E_{in}^*| = 1\} \cup \{u^*\tilde{u}^* \mid u^* \in U^*\}.$$ 

The first part is denoted by $E_{in}^*$, and the second part is denoted by $\tilde{E}^*$. We observe that $G^*$ is a tree, since $G$ is 2-connected and outerplanar. A face of $G$ that touches only one face (other than the outer face) is called a leaf in $G^* - U^*$. We note that there is a one-to-one correspondence between edges in $E_{in}$ of $G$ and $E_{in}^*$ of $G^*$. For an edge subset $F \subseteq E_{in}$, $F^*$ denotes the corresponding edge subset in $G^*$, that is, $F^* = \{e^* \in E_{in}^* \mid e \in F\}$. Conversely, for an edge subset $F^* \subseteq E(G^*)$, $F$ denotes the corresponding edge subset in $E_{in}$, that is, $F = \{e \in E_{in} \mid e^* \in F^* \cap E_{in}^*\}$. We extend this correspondence to $\tilde{E}^*$, that is, $u^*\tilde{u}^* \in \tilde{E}^*$ corresponds to the edge set $E_{in} \cap C_{out}$ for $u^* \in U^*$, and vice versa.

It follows from the duality that there is a relationship between a cut in $G^*$ and a cycle in $G$. Suppose that we are given a cycle $C$ ($\neq C_{out}$) in $G$. Then, since $G$ is outerplanar, the cycle $C$ surrounds a set $X^*$ of faces such that $X^*$ does not have the outer face. The set $X^*$ induces a connected graph (subtree) in $G^*$, and the set of edges leaving from $X^*$ yields a cut $C^* = \{e^* = v^*w^* \mid v^* \in X^*, w^* \in V(G^*) \setminus X^*\}$. Conversely, let $X^* \subseteq V^*$ be a vertex subset of $G^*$ such that the subgraph induced by $X^*$ is connected. Then the set of edges leaving from $X^*$ yields a cut $C^*$ in $G^*$, which corresponds to a cycle in $G$.

We classify faces in $U^*$ into two groups. For a face $u^* \in U^*$, the edge set $E_{in} \cap C_{out}$ forms a family $P_{u^*}$ of disjoint paths. Since $M$ and $N$ are perfect matchings, each path $P$ in $P_{u^*}$ is both $M$-alternating and $N$-alternating. In addition, $P$ satisfies either
(i) \( E(P) \subseteq M \triangle N \), or
(ii) \( (M \triangle N) \cap E(P) = \emptyset \).

Furthermore, we observe that either (i) holds for every path \( P \) in \( P_u \), or (ii) holds for every path \( P \) in \( P_u \). Indeed, since \( M \triangle N \) consists of disjoint cycles, if some path \( P \) in \( P_u \) satisfies (i), then \( P \) is included in a cycle \( C \) in \( M \triangle N \) that separates \( u \) from the outer face. Since the other paths in \( P_u \) touch the outer face, they are on \( C \). Thus every path satisfies (i), which shows the observation. We divide \( U^* \) into two groups \( U_1^* \) and \( U_2^* \) where each face in \( U_1^* \) satisfies (i) for every path, while each face in \( U_2^* \) satisfies (ii) for every path.

For an edge \( e^* \) in \( E(G^*) \), we define the length \( \ell(e^*) \) to be

\[
\ell(e^*) = \begin{cases} 
|M \cap \{e\}| + |N \cap \{e\}| & \text{if } e^* \in E_{in}^*; \\
1 & \text{if } e^* \in E^* \text{ is from } U_1^*; \\
0 & \text{if } e^* \in E^* \text{ is from } U_2^*.
\end{cases}
\]

See Figure 2 for an example. Let \( \ell(u^*, v^*) \) be the length of the (unique) path from \( u^* \) to \( v^* \) in \( G^* \). We define the gap between \( M \) and \( N \) in the graph \( G \) as the diameter of \( G^* \), that is, we define

\[
\text{gap}(I) = \max\{\ell(u^*, v^*) \mid u^*, v^* \in V(G^*)\}.
\]

This value is simply denoted by \( \text{gap}(M, N) \) if \( G \) is clear from the context.

### 3.3 Characterization for the Disjoint Case

Let \( I = (G, M, N) \) be an instance of Shortest Perfect Matching Reconfiguration such that \( G \) is a 2-connected outerplanar graph. In this subsection, we show that if \( E_{in} \cap M \cap N \) is empty, we can characterize the optimal value with \( \text{gap}(I) \), which leads to a simple polynomial-time algorithm for this case. We note that if \( E_{in} \cap M \cap N \) is empty, then no edge in \( E_{in} \) belongs to both \( M \) and \( N \), and hence \( \ell(e^*) \) can only take 0 or 1; in addition, \( \text{gap}(M, N) = 0 \) if \( M = N \).

**Lemma 3.3.** It holds that \( \text{gap}(M, N) \) is even.

**Proof.** Consider a path \( P^* \) whose length is equal to \( \text{gap}(M, N) \) in \( G^* \). We may assume that the end vertices of \( P^* \) are in \( \tilde{U}^* \), as otherwise we can extend the path to some vertex in \( \tilde{U}^* \) without decreasing the length. Let \( u, v \in \tilde{U}^* \) be the end vertices of \( P^* \). This means that the faces \( u \) and \( v \) touch the outer face. Take arbitrary edges \( e_u \in E_u \cap C_{out} \) and \( e_v \in E_v \cap C_{out} \). Then \( (P \cap E_{in}) \cup \{e_u, e_v\} \) forms a cut \( C \) in \( G \) by the duality. By the definition of \( \ell \), for \( w \in \{u, v\} \), it holds that \( \ell(w, \tilde{w}) = 0 \) if and only if \( |M \cap \{e_w\}| = |N \cap \{e_w\}| \). Hence the parity of \( \sum_{e^* \in E(P^*)} \ell(e^*) \) is the same as that of \( |M \cap C| + |N \cap C| \). Since \( M \) and \( N \) are perfect matchings, the parities of \( |M \cap C| \) and \( |N \cap C| \) are the same. Therefore, \( |M \cap C| + |N \cap C| \) is even, and thus \( \text{gap}(M, N) \) is also even.

A main theorem of this subsection is to give a characterization of the optimal value with \( \text{gap}(M, N) \).

**Theorem 3.4.** Let \( I = (G, M, N) \) be an instance of Shortest Perfect Matching Reconfiguration such that \( G \) is a 2-connected outerplanar graph. If \( E_{in} \cap M \cap N \) is empty, then it holds that \( \text{OPT}(I) = \text{gap}(M, N)/2 \).

**Proof.** To show the theorem, we first prove the following claim.

**Claim 1.** For any \( M \)-alternating cycle \( C \), it holds that

\[
\text{gap}(M, N) \leq \text{gap}(M \triangle C, N) + 2.
\]
Proof of Claim 1. By the duality, the cycle $C$ in $G$ corresponds to a cut $C^*$ in $G^*$ such that the inside is connected. Such a cut intersects with any path in $G^*$ at most twice as $G^*$ is a tree, and only the intersected edges can change the length by one. Therefore, the distance can be decreased by at most 2.

Consider a shortest reconfiguration sequence $\langle M_0, M_1, \ldots, M_t \rangle$ from $M_0 = M$ to $M_t = N$. Then, $t = OPT(I)$. For each $i = 1, \ldots, t$, it then holds that $\text{gap}(M_{i-1}, N) \leq \text{gap}(M_i, N) + 2$. By repeatedly applying the above inequalities, we obtain

$$\text{gap}(M, N) = \text{gap}(M_0, N) \leq \text{gap}(M_t, N) + 2t = 2t = 2OPT(I)$$

since $\text{gap}(M_t, N) = 0$. Hence it holds that $OPT(I) \geq \text{gap}(M, N)/2$.

It remains to show that $OPT(I) \leq \text{gap}(M, N)/2$. We prove the following claim.

**Claim 2.** There exists an $M$-alternating cycle $C$ such that

$$\text{gap}(M, N) = \text{gap}(M \triangle C, N) + 2. \quad (2)$$

**Proof of Claim 2.** We prove the claim by induction on the number of edges.

We first observe that we may assume that $E_{in} \setminus (M \cup N) = \emptyset$. Otherwise, we can just delete all the edges in $E_{in} \setminus (M \cup N)$, and apply the induction to find an $M$-alternating cycle $C$ that satisfies (2) for the modified graph. Since the deletion does not change the gap, $C$ is a desired cycle in $G$ as well. Therefore, we may assume that all the edges in $E_{in}$ have length 1.

In addition, we may assume that any leaf $u^*$ in $G^* - \tilde{U}^*$ belongs to $U_1^*$. In other words, $M$ and $N$ are distinct in $E_{in} \setminus C_{out}$. Indeed, suppose that there exists a leaf $u^*$ in $U_2^*$. Then $\ell(u^*, \tilde{u}^*) = 0$. Since any chord is in either $M$ or $N$ by the above observation and the assumption that $E_{in} \cap M \cap N = \emptyset$, $\ell(u^*, v^*) = 1$, where $v^*$ is the unique neighbor to $u^*$ in $G^* - \tilde{U}^*$. We delete $E_{u^*} \setminus E_{in}$ from $G$, $M$, and $N$, and then delete all the isolated vertices. We denote the obtained graph by $G'$. This corresponds to deleting the face $u^*$ with $\tilde{u}^*$ from $G^*$, and adding $v^*$ to $G^*$ if necessary. We can see that, in the modified graph $(G')^*$, we have $\ell(v^*, \tilde{v}^*) = 1$, as $E_{in} \cap E_{v^*}$ is in either $M$ or $N$. Hence this deletion preserves $\text{gap}(M, N)$. We then apply the induction to $G'$ to find an $M$-alternating cycle $C$ that satisfies (2). This cycle is a desired one in $G$. Thus we may assume that any leaf $u^*$ in $G^* - \tilde{U}^*$ belongs to $U_1^*$.

Since $\text{gap}(M, N)$ is even by Lemma 3.3, we have $\text{gap}(M, N) = 2d$ for some positive integer $d$. Then there exists a vertex $r^* \in V^*$ of $G^*$ such that, for every $v^* \in V(G^*)$, the $r^*-v^*$ path has length at most $d$. Let $X^* \subseteq V^*$ be a minimal vertex subset of $G^*$ such that

- $r^* \in X^*$
- the subgraph induced by $X^*$ is connected in $G^*$
- the cut $C^* = \{ e^* = u^*v^* \mid u^* \in X^*, v^* \in V(G^*) \setminus X^* \}$ has only edges in $M^* \cup \tilde{E}^*$.

Such $X^*$ always exists as $V^*$ satisfies all the conditions. The cut $C^*$ corresponds to a cycle $C$ in $G$. An example is given in Figure 3.
We claim that $C$ is $M$-alternating. Assume not. Then there exist two consecutive edges $e = vw, e' = vw$ in $C$ such that $e, e' \notin M$, which implies that $e, e' \in C_{out}$ as $E(C^*) \subseteq M^* \cup E^*$. Since $M$ is a perfect matching, the vertex $v$ is incident to another edge $f$ in $M$. Since $G$ is 2-connected and outerplanar, there exists a path $P$ from $v$ to $C$ using the edge $f$ that is internally disjoint from $C$. If $P$ has more than one edges, then $P$ ends with $u$ or $w$, since $G$ is outerplanar, which contradicts that $e, e' \in C_{out}$. Hence $P$ has only one edge. However, this implies that $C$ has a chord in $M$, which contradicts that $C$ was chosen to be minimal. Thus $C$ is an $M$-alternating cycle.

Consider taking $M \cap C$. Let $\ell'$ be the length defined by (1) with $M \triangle C$ and $N$. It follows that, for an edge $e^* \in E(G^*)$,

$$\ell'(e^*) = \begin{cases} \ell(e^*) & \text{if } e^* \notin C^*; \\ 1 - \ell(e^*) & \text{if } e^* \in C^*. \end{cases}$$

We will show that, for any vertex $\tilde{v}^*$ in $\tilde{U}^*$, we have $\ell'(r^*, \tilde{v}^*) \leq d - 1$. This proves the claim, as, for any two vertices $\tilde{u}^*, \tilde{v}^*$ in $\tilde{U}^*$, it holds that

$$\ell'(\tilde{u}^*, \tilde{v}^*) \leq \ell'(r^*, \tilde{u}^*) + \ell'(r^*, \tilde{v}^*) \leq 2d - 2.$$

Since $r^* \in X^*$ and no vertex in $\tilde{U}^*$ is in $X^*$, the $r^*-$$\tilde{v}^*$ path $P$ intersects with $C^*$ exactly once. Hence the length of $P$ is changed by one by taking $M \triangle C$. So, if $\ell(r^*, \tilde{v}^*) \leq d - 2$, then $\ell'(r^*, \tilde{v}^*) \leq d - 1$. Thus it suffices to consider the case when $\ell(r^*, \tilde{v}^*) \geq d - 1$, i.e., $\ell(r^*, \tilde{v}^*) = d - 1$ or $d$.

Assume that $\ell(v^*, \tilde{v}^*) = 0$, which implies that $v^* \in U_2^*$ and hence $v^*$ is not a leaf in $G^* - \tilde{U}^*$. In this case, there exists a leaf $u^*$ in $G^* - \tilde{U}^*$ such that $\ell(r^*, u^*) \geq \ell(r^*, v^*) + 1$. Since $u^* \in U_1^*$, we obtain

$$\ell(r^*, \tilde{u}^*) = \ell(r^*, u^*) + 1 \geq \ell(r^*, v^*) + 2 = \ell(r^*, \tilde{v}^*) + 2 \geq d + 1,$$

which is a contradiction.

Thus, we may assume that $\ell(v^*, \tilde{v}^*) = 1$. If the $r^*-$$\tilde{v}^*$ path $P$ intersects $C^* \cap M^*$, then the intersected cut edge has length 1, and hence we see that $\ell'(r^*, \tilde{v}^*) = \ell(r^*, \tilde{v}^*) - 1 \leq d - 1$. Otherwise, that is, if $P$ intersects with $C^* \cap \tilde{E}^*$, then the intersected cut edge is $(v^*, \tilde{v}^*)$, and hence $\ell'(r^*, \tilde{v}^*) = \ell(r^*, \tilde{v}^*) - 1 \leq d - 1$. Thus, $\ell'(r^*, \tilde{v}^*) \leq d - 1$ in each case.

For a perfect matching $M_{i-1}$ in $G$, it follows from Claim 2 that there exists an $M_{i-1}$-alternating cycle $C_i$ such that $\text{gap}(M_{i-1}, N) = \text{gap}(M_{i-1} \triangle C_i, N) + 2$. Define $M_i = M_{i-1} \triangle C_i$, and repeat finding an alternating cycle satisfying the above equation. The repetition ends when $\text{gap}(M_i, N) = 0$, which means that $M_i = N$ when $E_{in} \cap M \cap N$ is empty. The number of repetitions is equal to $\text{gap}(M, N)/2$, and therefore, we have $\text{OPT}(I) \leq \text{gap}(M, N)/2$. Thus the proof is complete.

### 3.4 General Case

Let $I = (G, M, N)$ be an instance of Shortest Perfect Matching Reconfiguration such that $G$ is a 2-connected outerplanar graph. Define $E_{in}' = E_{in} \cap M \cap N$. In this subsection, we deal with the general case, that is, $E_{in}'$ is not necessarily empty. Then, there is a case when changing an edge in $E_{in}'$ reduces the number of reconfiguration steps as in Figure 1. We call such a move an unhappy move. The key idea of our algorithm is to detect a set of edges necessary for unhappy moves.

Since $G$ is outerplanar and 2-connected, any $F \subseteq E_{in}'$ divides the inner region of $C_{out}$ into $|F| + 1$ parts $R_1, \ldots, R_{|F|+1}$. For each $i = 1, \ldots, |F| + 1$, let $G_i$ be the subgraph of $G$ consisting of all the vertices and the edges in $R_i$ and its boundary. Thus, each edge $e \in F$ appears on the outer face boundaries in two of these subgraphs. See Figure 4. Let $G_F = \{G_1, \ldots, G_{|F|+1}\}$. Note that each graph in $G_F$ is outerplanar and 2-connected. For each $H \in G_F$, let $I_H = (H, M \cap E(H), N \cap E(H))$. We now show the following theorem.
Theorem 3.5. \( \text{OPT}(I) = \frac{1}{2} \min_{F \subseteq E'_{\text{in}}} \sum_{H \in G_F} \text{gap}(I_H). \)

Proof. Let \((M_0, M_1, \ldots, M_t)\) be a shortest reconfiguration sequence from \(M_0 = M\) to \(M_t = N\). We denote by \(C_i\) the \(M_{i-1}\)-alternating cycle with \(M_i = M_{i-1} \triangle C_i\). Define
\[
F_{\text{opt}} = \{ e \in E'_{\text{in}} \mid e \notin C_i, \forall i \},
\]
which is the set of edges in \(E'_{\text{in}}\) that do not touch in the shortest reconfiguration sequence. Then \(C_i\) is contained in some \(H \in G_{F_{\text{opt}}}\), and can be used to obtain a reconfiguration sequence from \(M \cap E(H)\) to \(N \cap E(H)\) in \(H\). Therefore, we have
\[
\text{OPT}(I) = \sum_{H \in G_{F_{\text{opt}}}} \text{OPT}(I_H). \tag{3}
\]

We can also see that
\[
\text{OPT}(I) \leq \sum_{H \in G_F} \text{OPT}(I_H) \tag{4}
\]
for any \(F \subseteq E'_{\text{in}}\).

To evaluate \(\text{OPT}(I_H)\) for \(H \in G_F\), we slightly modify the instance \(I_H\) by replacing every inner edge of \(H\) contained in \(M \cap N\) by two parallel edges each in \(M\) and \(N\), respectively. The obtained graph is denoted by \(H'\), and the corresponding instance is denoted by \(I_{H'}\). Since a reconfiguration sequence for \(I_{H'}\) can be converted to one for \(I_H\), it holds that \(\text{OPT}(I_H) \leq \text{OPT}(I_{H'})\), and hence
\[
\text{OPT}(I) \leq \sum_{H \in G_F} \text{OPT}(I_H) \leq \sum_{H \in G_F} \text{OPT}(I_{H'}) \tag{5}
\]
holds for any \(F \subseteq E'_{\text{in}}\) by (4). Moreover, by the definition of \(F_{\text{opt}}\), there exists an index \(i\) such that \(e \in C_i\) for any \(e \in E'_{\text{in}} \setminus F_{\text{opt}}\). Therefore, for \(H \in G_{F_{\text{opt}}}\), the shortest reconfiguration sequence for \(I_H\) can be converted to a reconfiguration sequence for \(I_{H'}\). Thus, \(\text{OPT}(I_H) \geq \text{OPT}(I_{H'})\) holds for \(H \in G_{F_{\text{opt}}}\), and hence
\[
\text{OPT}(I) = \sum_{H \in G_{F_{\text{opt}}}} \text{OPT}(I_H) \geq \sum_{H \in G_{F_{\text{opt}}}} \text{OPT}(I_{H'}) \tag{6}
\]
by (3). By (5) and (6), we obtain
\[
\text{OPT}(I) = \min_{F \subseteq E'_{\text{in}}} \sum_{H \in G_F} \text{OPT}(I_{H'}), \tag{7}
\]
and \(F_{\text{opt}}\) is a minimizer of the right-hand side.

By (7) and Theorem 3.4, we obtain
\[
\text{OPT}(I) = \frac{1}{2} \min_{F \subseteq E'_{\text{in}}} \sum_{H \in G_F} \text{gap}(I_{H'}), \tag{8}
\]
because each $I_H'$ satisfies the condition in Theorem 3.4. Since $(H')^*$ is obtained from $H^*$ by subdividing some edges of length two into two edges of length one, the diameter of $(H')^*$ is equal to that of $H^*$, that is, $\text{gap}(I_H') = \text{gap}(I_H)$. Therefore, we obtain the theorem by (8).

As an example, we apply this theorem to the instance in Figure 2. See Figure 4(c). If $F$ consists of only the right thick edge in Figure 2(c), then $G_F$ consists two graphs $G_1$ and $G_2$ such that $\text{gap}(I_{G_1}) = 6$ and $\text{gap}(I_{G_2}) = 2$. Since we can check that such $F$ attains the minimum in the right-hand side of Theorem 3.5, we obtain $\text{OPT}(I) = 4$ by Theorem 3.5.

In order to compute the value in Theorem 3.5 efficiently, we reduce the problem to \textsc{Min-Sum Diameter Decomposition}, whose definition will be given later.

For $F \subseteq E'_T$, let $F^*$ be the edge subset of $E'_I$ corresponding to $F$, and let $G_F = \{G_1, \ldots, G_{|F|+1}\}$. Then, $G^* - F^*$ consists of $|F| + 1$ components $T_1, T_2, \ldots, T_{|F|+1}$ such that $T_i$ coincides with $G^*_i$ (except for the difference of edges of length zero) for $i = 1, \ldots, |F| + 1$. In particular, for each $i$, we have $\text{gap}(I_{G_i}) = \max\{\ell(u^*, v^*) : u^*, v^* \in V(T_i)\}$, where $\ell$ is the length function on $E(G^*)$ defined by the instance $I = (G, M, N)$. We call $\max\{\ell(u^*, v^*) : u^*, v^* \in V(T_i)\}$ the diameter of $T_i$, which is denoted by $\text{diam}_i(T_i)$. Then, Theorem 3.5 shows that

$$\text{OPT}(I) = \frac{1}{2} \min_{F \subseteq E'_I} \sum_{i=1}^{|F|+1} \text{diam}_i(T_i).$$

Therefore, we can compute $\text{OPT}(I)$ by solving the following problem in which $T = G^*$ and $E_0 = (E'_I)^*$.

**MIN-SUM DIAMETER DECOMPOSITION**

**Input:** A tree $T$, an edge subset $E_0 \subseteq E(T)$, and a length function $\ell : E(T) \to \mathbb{Z}_{\geq 0}$.

**Find:** An edge set $F \subseteq E_0$ that minimizes $\sum_T \text{diam}_i(T)$, where the sum is taken over all the components $T'$ of $T - F$.

In the subsequent subsection, we show that \textsc{Min-Sum Diameter Decomposition} can be solved in time polynomial in $|V(T)|$ and $L := \sum_{e \in E(T)} \ell(e)$.

**Theorem 3.6.** \textsc{Min-Sum Diameter Decomposition} can be solved in $O(|V(T)|L^4)$ time, where $L := \sum_{e \in E(T)} \ell(e)$.

Since (9) shows that Shortest Perfect Matching Reconfiguration on outerplanar graphs is reduced to \textsc{Min-Sum Diameter Decomposition} in which $L = O(|V(T)|)$, we obtain Theorem 3.1.

**3.5 Algorithm for Min-Sum Diameter Decomposition**

The remaining task is to show Theorem 3.6, that is, to give an algorithm for \textsc{Min-Sum Diameter Decomposition} that runs in $O(|V(T)|L^4)$ time. For this purpose, we adopt a dynamic programming approach.

We choose an arbitrary vertex $r$ of a given tree $T$, and regard $T$ as a rooted tree with the root $r$. For each vertex $v$ of $T$, we denote by $T_v$ the subtree of $T$ which is rooted at $v$ and is induced by all descendants of $v$ in $T$. (See Figure 5(a).) Thus, $T = T_r$ for the root $r$. Let $w_1, w_2, \ldots, w_q$ be the children of $v$, ordered arbitrarily. For each $j \in \{1, 2, \ldots, q\}$, we denote by $T_v^j$ the subtree of $T$ induced by $\{v\} \cup V(T_{w_1}) \cup V(T_{w_2}) \cup \cdots \cup V(T_{w_q})$. For example, in Figure 5(b), the subtree $T_v^j$ is surrounded by a thick dotted rectangle. For notational convenience, we denote by $T_v^0$ the tree consisting of a single vertex $v$. Then, $T_v = T_v^0$ for each leaf $v$ of $T$. Our algorithm computes and extends partial solutions for subtrees $T_v^j$ from the leaves to the root $r$ of $T$ by keeping the information required for computing (the sum of) diameters of a partial solution.
We now consider the case where updating.

Group (b) exist only when separators of $T_v$.

Notice that we have computed $f$ of $T_{v_i}$ in the whole tree $T$, (b) subtree $T_{v_i}$ in $T_v$, and (c) an $(x, y, z)$-separator of $T_{v_i}$.

We now define partial solutions for subtrees. For a subtree $T_{v_i}$ and an edge subset $F' \subseteq E_0 \cap E(T_{v_i})$, the frontier for $F'$ is the component (subtree) in $T_{v_i} - F'$ that contains the root $v$ of $T_{v_i}$. We sometimes call it the $v$-frontier for $F'$ to emphasize the root $v$. For three integers $x, y, z \in \{0, 1, \ldots, L\}$, the edge subset $F'$ is called an $(x, y, z)$-separator of $T_{v_i}$ if the following three conditions hold. (See also Figure 5(c).)

- $x = \max\{e(v, u) \mid u \in V(T_{F'})\}$, where $T_{F'}$ is the v-frontier for $F'$. That is, the longest path from $v$ to a vertex in $T_{F'}$ is of length $x$.
- $y = \text{diam}_F(T_{F'})$, that is, $y$ denotes the diameter of the v-frontier $T_{F'}$ for $F'$.
- $z = \sum_{T_{v_i}} \text{diam}_i(T')$, where the sum is taken over all the components $T'$ of $(T - F') \setminus T_{F'}$.

Note that $x \leq y$ always holds for an $(x, y, z)$-separator of $T_{v_i}$. We then define the following function: for a subtree $T_{v_i}$ and two integers $x, y \in \{0, 1, \ldots, L\}$, we let

$$f(T_{v_i}; x, y) = \min \left\{ z \mid T_{v_i} \text{ has an } (x, y, z)\text{-separator} \right\}.$$ 

Note that $f(T_{v_i}; x, y)$ is defined as $+\infty$ if $T_{v_i}$ does not have an $(x, y, z)$-separator for any $z \in \{0, 1, \ldots, L\}$. Then, the optimal objective value to MIN-SUM DIAMETER DECOMPOSITION can be computed as $\min\{y + f(T; x, y) \mid x, y \in \{0, 1, \ldots, L\}\}$.

For a given tree $T$, our algorithm computes $f(T_{v_i}; x, y)$ for all possible triplets $(T_{v_i}, x, y)$ from the leaves to the root $r$ of $T$, as follows.

**Initialization.** We first compute $f(T_{v_i}; x, y)$ for all vertices $v \in V(T)$ (including internal vertices in $T$). Recall that $T_{v_i}$ consists of a single vertex $v$. Therefore, we have

$$f(T_{v_i}; x, y) = \begin{cases} 0 & \text{if } x = y = 0; \\ +\infty & \text{otherwise.} \end{cases}$$

Notice that we have computed $f(T_r; x, y)$ for all leaves $v$ of $T$, since $T_v = T_{v_i}$ if $v$ is a leaf.

**Update.** We now consider the case where $j \geq 1$. To compute $f(T_{v_i}; x, y)$, we classify $(x, y, z)$-separators of $T_{v_i}$ into the following two groups (a) and (b). Note that $(x, y, z)$-separators of Group (b) exist only when $vw_j \in E_0$. 

![Figure 5](image-url)
Figure 6: \((x, y, z)\)-separators of a subtree \(T_v^j\), and their restrictions to subtrees \(T_v^{j-1}\) and \(T_{w_j}\).

(a) The vertices \(v\) and \(w_j\) are contained in the same component. (See also Figure 6(a).)

In this case, the edge \(vw_j\) is not deleted, and the \(v\)-frontier for an \((x, y, z)\)-separator of \(T_v^j\) contains both \(v\) and \(w_j\). Therefore, we can obtain the \(v\)-frontier for an \((x, y, z)\)-separator of \(T_v^j\) by merging the \(v\)-frontier for some \((x', y', z')\)-separator of \(T_v^{j-1}\) with the \(w_j\)-frontier for some \((x'', y'', z'')\)-separator of \(T_{w_j}\). Thus, we define

\[
f^a(T_v^j; x, y) := \min \left\{ f(T_v^{j-1}; x', y') + f(T_{w_j}; x'', y'') \right\},
\]

where the minimum is taken over all integers \(x', y', x'', y'' \in \{0, 1, \ldots, L\}\) such that \(x = \max\{x', x'' + \ell(vw_j)\}\) and \(y = \max\{y', y''\}\).

(b) The vertices \(v\) and \(w_j\) are contained in different components. (See also Figure 6(b).)

In this case, the edge \(vw_j\) is deleted, and hence this case happens only when \(vw_j \in E_0\). Then, the \(v\)-frontier for an \((x, y, z)\)-separator of \(T_v^j\) is the \(v\)-frontier for some \((x', y', z')\)-separator of \(T_v^{j-1}\). Note that \(w_j\) is contained in a non-frontier component for the \((x, y, z)\)-separator of \(T_v^j\), but the component forms the \(w_j\)-frontier for some \((x'', y'', z'')\)-separator of \(T_{w_j}\), as illustrated in Figure 6(b). Thus, we need to take the diameter of the \(w_j\)-frontier into account when we compute \(f(T_v^j; x, y)\) from \(f(T_v^{j-1}; x', y')\) and \(f(T_{w_j}; x'', y'')\). Therefore, we define

\[
f^b(T_v^j; x, y) := \min \left\{ f(T_v^{j-1}; x', y') + f(T_{w_j}; x'', y'') + y'' \right\},
\]

where the minimum is taken over all integers \(x', y', x'', y'' \in \{0, 1, \ldots, L\}\) such that \(x = x'\) and \(y = y'\).

Then, we can compute \(f(T_v^j; x, y)\) as follows:

\[
f(T_v^j; x, y) = \begin{cases} 
\min \left\{ f^a(T_v^j; x, y), f^b(T_v^j; x, y) \right\} & \text{if } vw_j \in E_0; \\
\min \left\{ f^a(T_v^j; x, y) \right\} & \text{otherwise.}
\end{cases}
\]

Since \(x', y', x'', y'' \in \{0, 1, \ldots, L\}\), this update can be done in \(O(L^4)\) time for each subtree \(T_v^j\). The number of subtrees \(T_v^j\) is equal to \(|V(T)| + |E(T)| = 2|V(T)| - 1\). Therefore, this algorithm runs in \(O(|V(T)|L^4)\) time in total.

Note that we can easily modify the algorithm so that we obtain not only the optimal value but also an optimal solution. This completes the proof of Theorem 3.6.

We remark here that the algorithm can be modified so that the running time is bounded by a polynomial in \(|V(T)|\) by replacing the domain \(\{0, 1, \ldots, L\}\) of \(x\) and \(y\) with \(D := \{\ell(u, v) | u, v \in V(T)\}\). This modification is valid, because \(f(T_v^j; x, y) = +\infty\) unless \(x, y \in D\). Since \(|D| = O(|V(T)|^2)\), the modified algorithm runs in \(O(|V(T)||D|^4) = O(|V(T)|^5)\) time. Note that, although this bound is polynomial only in \(|V(T)|\), it is worse than \(O(|V(T)|L^4)\) when \(L = O(|V(T)|)\).
4 NP-Hardness for Planar Graphs and Bipartite Graphs

In this section, we prove that Shortest Perfect Matching Reconfiguration is NP-hard even when the input graph is planar or bipartite.

**Theorem 4.1.** Shortest Perfect Matching Reconfiguration is NP-hard even for planar graphs of maximum degree three.

We reduce the Hamiltonian Cycle Problem problem, which is known to be NP-complete even when a given graph is 3-regular and planar [12].

**Hamiltonian Cycle Problem**

**Input:** A 3-regular planar graph \( H = (V, E) \)

**Question:** Decide whether \( H \) has a Hamiltonian cycle, i.e., a cycle that goes through all the vertices exactly once.

**Proof.** Let \( H \) be a 3-regular planar graph, which is an instance of Hamiltonian Cycle Problem. For each vertex \( v \in V(H) \), we define a 8-vertex graph \( D_v \) (see also the top right in Figure 7):

\[
V(D_v) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\},
\]

\[
E(D_v) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7, v_3v_7, v_5v_8\}.
\]

We construct an instance \( I = (G, M, N) \) of our problem as follows. (See Figure 7 as an example.) We subdivide each edge \( e = uw \) in \( H \) twice, and the obtained vertices are denoted by \( u_e \) and \( v_e \), where \( u_e \) is closer to \( u \). Then, for each vertex \( v \in V(H) \), we replace \( v \) with the graph \( D_v \), and connect \( v \) to \( v_{(1)} \) and \( v_{(2)} \), \( v_{(3)} \) to \( v_{(2)} \) and \( v_{(3)} \), where \( e_v^{(1)} \), \( e_v^{(2)} \), \( e_v^{(3)} \) are edges incident to \( v \) and the order follows the planar embedding of \( H \). Let \( E_v = \{v_{(1)}v_{e_v^{(1)}}, v_{(2)}v_{e_v^{(2)}}, v_{(3)}v_{e_v^{(3)}}\} \). The resulting graph is denoted by \( G \), i.e., \( G \) is defined as follows:

\[
V(G) = \bigcup_{v \in V(H)} V(D_v) \cup \bigcup_{e = uv \in E(H)} \{u_e, v_e\},
\]

\[
E(G) = \left( \bigcup_{v \in V(H)} E(D_v) \cup E_v \right) \cup \{u_ev_e \mid e \in E(H)\}.
\]

It follows that \( G \) is a planar graph of maximum degree three. Furthermore, we define initial and target perfect matchings \( M \) and \( N \) in \( G \), respectively, to be

\[
M = \{v_1v_2, v_3v_4, v_5v_7, v_6v_8 \mid v \in V(H)\} \cup \{u_ev_e \mid e \in E(H)\},
\]

\[
N = \{v_1v_4, v_2v_3, v_5v_7, v_6v_8 \mid v \in V(H)\} \cup \{u_ev_e \mid e \in E(H)\}.
\]

This completes the construction of our corresponding instance \( I = (G, M, N) \). The construction can be done in polynomial time.

We then give the following claims. Recall that \( t^* \) is the length of a shortest reconfiguration sequence for the constructed instance \( I \).

**Claim 3.** It holds that \( t^* \geq 2 \).

**Proof of Claim 3.** We observe that, if \( t^* = 1 \), then \( M \triangle N \) must consist of one \( M \)-alternating cycle, but it is not true for our instance \( I \). Thus the length of a reconfiguration sequence is at least two. \( \square \)

We remark that \( G \) has an \( M \)-alternating path from \( v_e^{(x)} \) to \( v_e^{(y)} \) for any \( x, y \in \{1, 2, 3\} \) with \( x \neq y \). This implies that, for a cycle \( C \) in \( H \), there exists a corresponding \( M \)-alternating cycle \( C' \) in \( G \) such that it goes through vertices of \( D_v \) for every \( v \in V(C) \) and edges \( u_ev_e \) for every \( e \in E(C) \).
Figure 7: Reduction for planar graphs of maximum degree three. Top left: a yes instance $H$ of HAMILTONIAN CYCLE PROBLEM with a green Hamiltonian cycle. Top right: the constructed fragment $D_v$. Bottom left: The initial perfect matching $M$ (red). Bottom middle: The target perfect matching $N$ (blue). Bottom right: The perfect matching obtained as $M \triangle C$, where $C$ corresponds to the Hamiltonian cycle of $H$.

**Claim 4.** If $H$ has a Hamiltonian cycle $C$, then it holds that $t^* = 2$.

*Proof of Claim 4.* We see that $G$ has an $M$-alternating cycle $C'$, corresponding to $C$ of $H$, that has one edge $v_3v_4$ of $C_v$ for each $v \in V(C)$. Then $M' = M \triangle C'$ is a perfect matching. In a similar way, $G$ has an $M'$-alternating cycle $C''$, corresponding to $C$, that uses three edges $v_3v_2$, $v_2v_1$, and $v_1v_4$ of $C_v$ for each $v \in V(C)$. Then $M' \triangle C''$ is equal to $N$. Thus we can find a reconfiguration sequence of length two, which is shortest by Claim 3.

The 4-cycle formed by $v_1, v_2, v_3, v_4$ is denoted by $C_v$.

**Claim 5.** If $t^* = 2$, then $H$ has a Hamiltonian cycle.

*Proof of Claim 5.* We denote by $\langle M, M', N \rangle$ a shortest reconfiguration sequence of $I$. Let $C = M \triangle M'$. We may assume that $C$ is not $C_v$ for any $v \in H$, as $t^* = 2$. We will prove that the edge subset $F = \{ e \in E(H) \mid u, v \in V(C) \}$ forms a Hamiltonian cycle in $H$. We denote $W_C$ by the set of vertices in $H$ used in $F$. Let $\overline{W_C} = V(H) \setminus W_C$. Since $M' \cap C_v$ and $N \cap C_v$ are distinct for $v \in \overline{W_C}$, the symmetric difference $M' \triangle N$ has at least $|\overline{W_C}|$ disjoint $M'$-alternating cycles. Moreover, for a vertex $v \in W_C$, we see that $M' \cap C_v = \{v_1, v_2\}$ and $N \cap C_v = \{v_1v_4, v_2v_3\}$, that are distinct. Hence $M' \triangle N$ has at least one $M'$-alternating cycle disjoint from $\bigcup_{v \in \overline{W_C}} V(D_v)$. Therefore, we have at least $|\overline{W_C}| + 1$ disjoint $M'$-alternating cycles. However, $M' \triangle N$ must consist of one cycle (see Claim 3), implying that $\overline{W_C} = \emptyset$. This means that $C$ goes through $C_v$ for every $v$, and hence $C'$ is a Hamiltonian cycle in $H$. Thus the claim holds.

Therefore, it follows that $H$ has a Hamiltonian cycle if and only if $t^* = 2$. This completes the proof of Theorem 4.1.

The hardness for bipartite graphs of maximum degree at most three can be obtained with a similar proof.

**Theorem 4.2.** Shortest Perfect Matching Reconfiguration is NP-hard even for bipartite graphs of maximum degree three.

We reduce the directed Hamiltonian cycle problem, which is known to be NP-complete even if digraphs have the maximum in-degree two and the maximum out-degree two [30].
Figure 8: Reduction for bipartite graphs of maximum degree three. Top left: a yes instance \( H \) of Directed Hamiltonian Cycle Problem with a green directed Hamiltonian cycle. Top middle: the constructed fragment \( D_v \). Bottom left: The initial perfect matching \( M \) (red). Bottom middle: The target perfect matching \( N \) (blue). Bottom right: The perfect matching obtained as \( M \triangle C \) where \( C \) corresponds to the directed Hamiltonian cycle of \( H \).

**Directed Hamiltonian Cycle Problem**

**Input:** A digraph \( H = (V, E) \)

**Question:** Decide whether \( H \) has a directed Hamiltonian cycle, i.e., a directed cycle that goes through all the vertices exactly once.

**Proof.** Let \( H \) be a digraph, which is an instance of the directed Hamiltonian cycle problem. We assume that \( |V(H)| \geq 3 \); otherwise the problem is trivial. For each vertex \( v \in V(H) \), we define a 6-vertex graph \( D_v \) (see the top right in Figure 8):

\[
V(D_v) = \{v^+, v^-, v_1, v_2, v_3, v_4, v_5, v_6\},
\]

\[
E(D_v) = \{v^+v_1, v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_2, v_5v_6, v_6v^-\}.
\]

The cycle of length four formed by \( v_2, v_3, v_4, v_5 \) is denoted by \( C_v \).

We construct an instance \( I = (G, M, N) \) of our problem as follows. The vertex set and the edge set of \( G \) are defined as

\[
V(G) = \bigcup_{v \in V(H)} V(D_v),
\]

\[
E(G) = \bigcup_{v \in V(H)} E(D_v) \cup \{u^-v^+ \mid uv \in E(H)\},
\]

respectively. Namely, for each directed edge from \( u \) to \( v \) in \( H \), we add an undirected edge to \( G \) between \( u^- \) and \( v^+ \). This finishes the construction of \( G \). Note that \( G \) is bipartite and its maximum degree is at most three as both the maximum in-degree and the maximum out-degree of \( H \) are at most two. Let \( M \) and \( N \) be defined as

\[
M = \bigcup_{v \in V(H)} \{v^+v_1, v_2v_3, v_4v_5, v_6v^-\},
\]

\[
N = \bigcup_{v \in V(H)} \{v^+v_1, v_2v_5, v_3v_4, v_6v^-\}.
\]

Refer to Figure 8 for the illustration. Let \( t^* \) be the length of a shortest reconfiguration sequence for \( I \).
Claim 6. It holds that \( t^* \geq 2 \).

Proof. If \( t^* = 1 \), then \( M \triangle N \) must consist of one \( M \)-alternating cycle, but this is not the case for our instance \( I \). Thus, the length of a reconfiguration sequence is at least two.

Claim 7. If \( H \) has a directed Hamiltonian cycle \( C \), then it holds that \( t^* = 2 \).

Proof. We see that \( G \) has an \( M \)-alternating cycle \( C' \), corresponding to \( C \) of \( H \), that has four edges \( v^+v_1, v_2v_3, v_4v_5, v_6v^- \) of \( D_v \) for each \( v \in V(C) \). Then \( M' = M \triangle C' \) is a perfect matching. In a similar way, \( G \) has an \( M' \)-alternating cycle \( C'' \), corresponding to \( C \), that uses three edges \( v^+v_1, v_2v_5, \) and \( v_6v^- \) of \( C_v \) for each \( v \in V(C) \). Then \( M' \triangle C'' \) is equal to \( N \). Thus we can find a reconfiguration sequence of length two, which is the shortest by Claim 6.

Claim 8. If \( t^* = 2 \), then \( H \) has a directed Hamiltonian cycle.

Proof. Let \( \{ M, M', N \} \) be a shortest reconfiguration sequence of \( I \). Let \( C = M \triangle M' \). We may assume that \( C \) is not \( C_v \) for any \( v \in H \), as \( t^* = 2 \). We will prove that the edge subset \( F = \{ uv \in E(H) \mid u^+v^- \in C \} \) forms a Hamiltonian cycle in \( H \). We denote \( W_C \) by the set of vertices in \( H \) used in \( F \). Let \( \overline{W_C} = V(H) \setminus W_C \). Since \( M' \cap C_v \) and \( N \cap C_v \) are distinct for \( v \in \overline{W_C} \), the symmetric difference \( M' \triangle N \) has at least \( |W_C| \) disjoint \( M' \)-alternating cycles. Moreover, for a vertex \( v \in W_C \), we see that \( M' \cap C_v = \{ v_3v_4 \} \) and \( N \cap C_v = \{ v_3v_4, v_2v_5 \} \), that are distinct. Hence \( M' \triangle N \) has at least one \( M' \)-alternating cycle disjoint from \( \bigcup_{v \in \overline{W_C}} V(D_v) \). Therefore, we have at least \( |W_C| + 1 \) disjoint \( M' \)-alternating cycles. However, \( M' \triangle N \) must consist of one cycle (see Claim 6), implying that \( \overline{W_C} = \emptyset \). This means that \( C \) goes through \( C_v \) for every \( v \), and hence \( C' \) is a Hamiltonian cycle in \( H \). Thus the claim holds.

Therefore, it follows that \( H \) has a directed Hamiltonian cycle if and only if \( t^* = 2 \). This completes the proof.

Note that the reduction does not produce a planar graph even when the input digraph has a planar underlying graph. The example in Figure 8 contains a \( K_5 \)-minor.

The proofs actually show that Shortest Perfect Matching Reconfiguration is \( \text{NP} \)-hard to approximate within a factor of less than \( 3/2 \).

5 Conclusion

In this paper, we studied the shortest reconfiguration problem of perfect matchings under the alternating cycle model, which is equivalent to the combinatorial shortest path problem on perfect matching polytopes. We prove that the problem can be solved in polynomial time for outerplanar graphs, but it is \( \text{NP} \)-hard, and even \( \text{APX} \)-hard for planar graphs and bipartite graphs.

Several questions remain unsolved. For polynomial-time solvability, our algorithm runs only for outerplanar graphs, and it looks difficult to extend the algorithm to other graph classes. A next step would be to try \( k \)-outerplanar graphs for fixed \( k \geq 2 \).

One way to tackle \( \text{NP} \)-hard cases is approximation. We only know the \( \text{NP} \)-hardness of \( 3/2 \)-approximation. We believe the existence of a polynomial-time constant-factor approximation. Note that we do not obtain a constant-factor approximation by flipping alternating cycles in the symmetric difference of two given perfect matchings one by one.

This paper was mainly concerned with reconfiguration of perfect matchings. Alternatively, we may consider reconfiguration of maximum matchings, or maximum-weight matchings. In those cases, we need to adopt the alternating path/cycle model. Then, the question is related to the combinatorial shortest path problem on faces of matching polytopes. Note that the perfect matching polytope is also a face of the matching polytope. Therefore, the study on maximum-weight matchings will be a generalization of this paper.
To the best of the authors’ knowledge, the combinatorial shortest path problem of 0/1-polytopes has not been well investigated while the adjacency in 0/1-polytopes has been extensively studied in the literature. This paper opens up a new perspective for the study of combinatorial and computational aspects of polytopes, and connects them with the study of combinatorial reconfiguration.

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