Locating Dominating Sets in local tournaments

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Abstract

A dominating set in a directed graph is a set of vertices $S$ such that all the vertices that do not belong to $S$ have an in-neighbour in $S$. A locating set $S$ is a set of vertices such that all the vertices that do not belong to $S$ are characterized uniquely by the in-neighbours they have in $S$, i.e. for every two vertices $u$ and $v$ that are not in $S$, there exists a vertex $s \in S$ that dominates exactly one of them. The size of a smallest set of a directed graph $D$ which is both locating and dominating is denoted by $\gamma^{LD}(D)$. Foucaud, Heydarshahi and Parreau proved that any twin-free digraph $D$ satisfies $\gamma^{LD}(D) \leq \frac{n}{5} + 1$ but conjectured that this bound can be lowered to $\frac{n}{2}$. The conjecture is still open. They also proved that if $D$ is a tournament, i.e. a directed graph where there is one arc between every pair of vertices, then $\gamma^{LD}(D) \leq \lceil \frac{n}{2} \rceil$.

The main result of this paper is the generalization of this bound to connected local tournaments, i.e. connected digraphs where the in- and out-neighbourhoods of every vertex induce a tournament. We also prove $\gamma^{LD}(D) \leq \frac{n}{2}$ for all quasi-twin-free digraphs $D$ that admit a supervising vertex (a vertex from which any vertex is reachable). This class of digraphs generalizes twin-free acyclic graphs, the most general class for which this bound was known.

1 Introduction

In this paper we consider loopless and finite digraphs. Our terminology is consistent with [2]. Especially, we refer the reader to this book for further information about the classes of digraphs we consider. The order of a digraph is its number of vertices. A digraph $D$ is simple if there is at most one arc between two
vertices $x$ and $y$. A digraph $D$ is called connected, if for every vertices $x$ and $y$ of $D$ there exists a (non necessarily directed) path from $x$ to $y$. A digraph $D$ is called strongly connected (or strong for short), if for every vertices $x$ and $y$ of $D$, there exists a directed path from $x$ to $y$ and a directed path from $y$ to $x$.

Let $x$, $y$ be distinct vertices of a digraph $D$. If there is an arc from $x$ to $y$, we say that $x$ dominates $y$ and denote it by $x \rightarrow y$. We say that $y$ is an out-neighbor of $x$, and that $x$ is an in-neighbor of $y$. The open in-neighborhood and the open out-neighborhood of a vertex $v$, denoted by $N^-(v)$ and $N^+(v)$ respectively, are the set of in-neighbors of $v$ and the set of out-neighbors of $v$, respectively. A source is a vertex with no in-neighbors, and a sink is a vertex with no out-neighbors. We use the notations $d^-(v) = |N^-(v)|$ and $d^+(v) = |N^+(v)|$ to denote the in-degree and the out-degree of $v$. Further, the closed in-neighborhood of $v$ is $N^-[v] = N^-(v) \cup \{v\}$ and the closed out-neighborhood of $v$ is $N^+[v] = N^+(v) \cup \{v\}$. Two vertices are called twins if $N^-(x) = N^-(y)$ or if $N^-[x] = N^-[y]$, and quasi twins if $N^-(x) = N^-[y]$ or $N^-(y) = N^-[x]$.

A digraph is twin-free (resp. quasi-twin-free) if it contains no twins (resp. no twins nor quasi-twins). A dominating set $S$ of a digraph $D$ is a set of vertices such that any vertex not in $S$ is dominated by a vertex in $S$. The smallest size of a dominating set of $D$ is called its domination number, and is denoted by $\gamma(D)$. A locating set $S$ of $D$ is a subset of vertices such that for every pair of vertices $x$ and $y$ not in $S$, there exists a vertex $s$ in $S$ that dominates exactly one vertex among $x$ and $y$. The smallest size of a locating set of $D$ is denoted by $\gamma^L(D)$. A locating-dominating set of $D$ is a set of vertices that is both locating and dominating. The minimum size of a locating-dominating set of $D$ is called the location-dominating number, denoted by $\gamma^{LD}(D)$. Note that at most one vertex of $D$ is not dominated by a locating set of $D$, ensuring that:

$$\gamma^L(D) \leq \gamma^{LD}(D) \leq \gamma^L(D) + 1$$

Locating-dominating sets in digraphs have been introduced in [3] [7] and further studied in [5] where some upper bounds on the location-dominination number of digraphs have been proved. If $D$ contains at least one edge, then for any vertex $x$ that is not isolated, the set $V(D) \setminus \{x\}$ is locating-dominating and thus, if $D$ has order $n$, then $\gamma^{LD}(D) \leq n - 1$. This bound is tight and a complete characterization of the digraphs reaching it is given in [5]. However, all these digraphs contain many pairs of twins. The authors of [5] showed that any twin-free digraph $D$ of order $n$ satisfies $\gamma^{LD}(D) \leq \frac{4n}{3} + 1$. Note that similar questions were first considered in the non-oriented case for which it is conjectured that any twin-free graph has a locating-dominating set of size $\frac{n}{2}$ [1] [8].

The authors of [5] lowered the general upper bound $\frac{4n}{3} + 1$ for special cases. A digraph $D$ is a tournament if there is exactly one arc between every pair of distinct vertices of $D$. In particular, it is proven in [5] that for a tournament $D$, the upper bound on the location-dominination number can be lowered to $\gamma^{LD}(D) \leq \lceil \frac{2n}{3} \rceil$. We extend this last result to a larger class of digraphs. A digraph $D$ is a local tournament if it is simple and if the in-neighborhood and
the out-neighbourhood of every vertex of $D$ induce tournaments. The main result of the current paper is the following theorem.

**Theorem 1.** A connected local tournament $D$ of order $n$ satisfies $\gamma^{LD}(D) \leq \lceil \frac{n}{2} \rceil$.

Note that an edgeless graph of order $n$ is a local tournament for which $\gamma^{LD}(D) = n - 1$. So if one removes the ”connected” hypothesis in Theorem 1 then the conclusion does not hold.

In [5], the authors asked if the general upper bound $\frac{4n}{3} + 1$ for twin-free graphs can be lowered to $\frac{2n}{3}$, for which some tight constructions are known (see Figure 1 for the tight strongly connected example given in [5]). They proved that it is the case for quasi-twin-free acyclic digraphs. We extend these results by proving that if a quasi-twin-free digraph $D$ contains a supervising vertex, that is a vertex from which there exists a directed path to all the other vertices of the graph, then $\gamma^{LD}(D) \leq \frac{2n}{3}$. In particular, this bound is true (and asymptotically tight) for quasi-twin-free strongly connected digraphs and quasi-twin-free local-in semi-complete digraphs (i.e. digraphs, not necessarily simple, where two in-neighbours of a common vertex are connected).

![Figure 1: A strongly connected twin-free and quasi-twin-free digraph of order $n$ with location-domination number $\frac{2(n-2)}{3}$, with a locating-dominating set in gray.](image)

After giving some preliminary results on local tournaments in Section 2 we prove Theorem 1 in two steps in Sections 3 and 4. In Section 5 we prove the general upper bound for digraphs that contain a supervising vertex.
2 Preliminaries

In order to prove the upper bound on the location-domination number of local tournaments, we start by exposing some useful properties of local tournaments.

Lemma 2. A connected local tournament is twin-free.

Proof. Consider a connected local tournament $D$ and suppose by contradiction that there exist two distinct vertices $x, y$ of $D$ such that $x, y$ are twins.

Suppose first that $x, y$ have some in-neighbours, then they have a common in-neighbour $z$. Since $x, y$ are both in the out-neighbourhood of $z$, there is an arc between them, a contradiction.

Suppose now that $x, y$ have no in-neighbours. Since $D$ is connected, we can consider a shortest (non necessarily directed) path $P = v_1, \ldots, v_k$ between $x, y$, such that $v_1 = x$ and $v_k = y$. Note that $k \geq 3$. By assumption, we have $v_1 \rightarrow v_2$ and $v_k \rightarrow v_{k-1}$. Thus there exists $1 \leq i < k$ such that $v_{i-1} \rightarrow v_i$ and $v_{i+1} \rightarrow v_i$. So $v_{i-1}$ and $v_{i+1}$ are both in the in-neighbourhood of $v_i$ and thus there is an arc between them, contradicting the fact that $P$ is a shortest path.

The structure of local tournaments has been studied in [1]. In particular, the authors introduce the round decomposition, which will be crucial in our proofs. Therefore, let us have a closer look at the construction of such a decomposition.

We call a digraph $D$ on $r$ vertices round if its vertices can be labelled $v_1, \ldots, v_r$ in such a way that $N^-(v_i) = \{v_{i-d-(v_i)}, \ldots, v_{i-1}\}$ and $N^+(v_i) = \{v_{i+1}, \ldots, v_{i+d+(v_i)}\}$ (indices are understood modulo $r$). One can easily check that a simple round digraph is a local tournament.

Let $R$ be a digraph on $r$ vertices and $L_1, \ldots, L_r$ be a collection of $r$ digraphs. Then $R[L_1, \ldots, L_r]$ is the digraph obtained from $R$ by replacing each vertex $v_i$ of $R$ with $L_i$, and adding an arc from every vertex of $L_i$ to every vertex of $L_j$ if and only if $v_i \rightarrow v_j$ is an arc of $R$. A local tournament $D$ is said to be roundable if it can be written as $R[T_1, \ldots, T_r]$ where $R$ is a round simple digraph on $r \geq 2$ vertices, and $T_1, \ldots, T_r$ are (non empty) strong tournaments. The decomposition $R[T_1, \ldots, T_r]$ is called a round decomposition of $D$.

Given a round decomposition $R[T_1, \ldots, T_r]$ of a roundable local tournament $D$, if there is an arc between a vertex in $T_i$ and a vertex in $T_j$, then all the arcs between $T_i$ and $T_j$ are present in $D$, and we write $T_i \Rightarrow T_j$. If $T_i \Rightarrow T_j$ and $i < j$, then for all $k \in \{i+1, \ldots, j-1\}$, we have $T_i \Rightarrow T_k$ and $T_k \Rightarrow T_j$. If $T_i \Rightarrow T_j$ and $j < i$, then for all $k \in \{1, \ldots, j-1\} \cup \{i+1, \ldots, s\}$, we have $T_i \Rightarrow T_k$ and $T_k \Rightarrow T_j$.

Suppose that $D$ is a roundable connected local tournament that is not strongly connected. Then, as proved in [1], there is a unique round decomposition $R[T_1, \ldots, T_r]$ of $D$ such that $T_i \Rightarrow T_j$ only happens if $i < j$. We often consider this particular round decomposition in this case, that we call the canonical round decomposition of $D$. When $D$ is a roundable strong local tournament, there is a unique round decomposition up to cyclic permutation.
3 Roundable local tournaments

This section is devoted to the study of roundable connected local tournaments. Consider a roundable connected local tournament $D$ and a round decomposition $R[T_1, \ldots, T_r]$ of $D$ that is canonical if $D$ is not strong.

Lemma 3. For $1 \leq i < r$, we have $T_i \Rightarrow T_{i+1}$.

Proof. Suppose by contradiction that there exists $1 \leq i < r$ such that we don’t have $T_i \Rightarrow T_{i+1}$. Then there is no arc $v_i \to v_{i+1}$ in $R$ and so by definition of round, we have $d^+(v_i) = 0$. For $1 \leq k \leq i < \ell \leq r$, there is no arc $v_k \to v_{\ell}$ as such an arc implies an arc $v_i \to v_{\ell}$. Since $v_i$ is a sink, the digraph $D$ is not strong and the decomposition is canonical. So there is no edge between $V(T_1) \cup \cdots \cup V(T_i)$ and $V(T_{i+1}) \cup \cdots \cup V(T_r)$, contradicting the fact that $D$ is connected. \[\square\]

We define the sequence of integers $(i_k)_{0 \leq k \leq t}$ as follows (see Figure 2). Let $i_0 = 0$. If $i_k$ is defined and $i_k < r$, then we define $i_{k+1}$ as the greatest integer $i_k + 1 < i_{k+1} \leq r$ such that $T_{i_k+1} \Rightarrow T_{i_{k+1}}$ (note that this integer exists by Lemma 3). This procedure stops when some $i_t$ is defined equal to $r$. For $1 \leq k \leq t$, let $D_k = D[T_{i_k+1} \cup \cdots \cup T_r]$. Note that $D_k$ is a tournament for all $k$.

![Figure 2: Illustration of the definition of the subgraphs $D_k$. The dashed arc means that there is no arc between the subgraphs, whereas thick arcs means that there are all the arcs between the two subgraphs.](image)

Step by step, from $k = t$ to $k = 1$, we define a set $S$ of vertices of $D$ that will be our candidate for a locating-dominating set of $D$. We consider the following three different cases:

- **Case 1:** $k = t$, or $k < t$ and $S \cap V(D_{k+1})$ is assumed to be a dominating set of $D_{k+1}$.

  If $|V(D_k)|$ is even, we add to $S$ a minimum locating-dominating set of $D_k$, otherwise we add to $S$ a minimum locating set of $D_k$.

- **Case 2:** $k < t$, the set $S \cap V(D_{k+1})$ is not assumed to be a dominating set of $D_{k+1}$, and $|V(T_{i_k})| = 1$.

  Let $v$ be the unique vertex of $T_{i_k}$ and $D_k' = D_k \setminus \{v\}$. We add $v$ to $S$. Moreover we add to $S$ a minimum locating-dominating set of $D_k'$ when
$|V(D'_k)|$ is even, or a minimum locating set of $D'_k$ when $|V(D'_k)|$ is odd. (Note that the set of $D_k$ that is added to $S$, including $v$, is a dominating set of $D_k$ when $|V(D'_k)|$ is odd.)

- **Case 3:** $k < t$, the set $S \cap V(D_{k+1})$ is not assumed to be a dominating set of $D_{k+1}$, and $|V(T_{i_k})| > 1$.

We add to $S$ a minimum locating-dominating set of $D_k$.

Note that in all the three cases we add to $S$ a set of vertices of $D_k$ that is a locating set of $D_k$, even in Case 2 since a locating set of $D'_k$ plus $v$ forms a locating set of $D_k$.

In the next lemmas we prove some properties of the set $S$.

**Lemma 4.** For $1 \leq i \leq r$, the set $S \cap V(T_i)$ is a locating set of $T_i$, in particular if $T_i$ has at least 2 vertices, then $S \cap V(T_i) \neq \emptyset$.

**Proof.** Let $1 \leq i \leq r$. If $V(T_i) \setminus S$ has size 1, then $S \cap V(T_i)$ is a locating set of $T_i$ (even if empty) and the conclusion holds. So we can assume that $V(T_i) \setminus S$ has size at least 2. Let $x, y$ be two distinct vertices of $V(T_i) \setminus S$. Since $S$ is a locating set of $D_k$, there exists a vertex $s$ in $S \cap V(D_k)$ that dominates exactly one vertex among $x$ and $y$. Since $R[T_1, \ldots, T_r]$ is a round decomposition, all the vertices of $D_k$ not in $T_i$ dominates either all or none of the vertices of $T_i$. So $s \in V(T_i)$. Therefore $S \cap V(T_i)$ is a locating set of $T_i$ and $S \cap V(T_i) \neq \emptyset$. \qed

**Lemma 5.** For $1 \leq k \leq t$, if $S \cap V(D_k)$ is a dominating set of $D_k$, then $S \cap V(T_{i_{k-1}}) \neq \emptyset$. Otherwise, there is exactly one vertex of $D_k$ that is not dominated by $S \cap V(D_k)$ and this vertex is in $V(T_{i_{k-1}})$.

**Proof.** Let $1 \leq k \leq t$. By definition of $i_k$, all the vertices of $T_{i_{k-1}+1}$ dominates $D_k \setminus V(T_{i_{k-1}+1})$, so there is no arc from $V(D_k) \setminus V(T_{i_{k-1}+1})$ to $V(T_{i_{k-1}+1})$ (since $R$ is simple). Thus, if $S \cap V(D_k)$ is a dominating set of $D_k$, then $S$ must contains a vertex of $V(T_{i_{k-1}+1})$.

Suppose now that there exists a vertex of $D_k$ that is not dominated by $S \cap V(D_k)$. Since $S \cap V(D_k)$ is a locating set of $D_k$, there is at most one vertex that is not dominated by $S \cap V(D_k)$, and thus exactly one such vertex $v$. If there exists a vertex $u$ in $S \cap V(T_{i_{k-1}+1})$, then $u$ dominates $D_k \setminus V(T_{i_{k-1}+1})$, so $v \in V(T_{i_{k-1}+1})$. If there is no vertices in $S \cap V(T_{i_{k-1}+1})$, then the vertices of $T_{i_{k-1}+1}$ are not dominated by $S \cap V(D_k)$, so $V(T_{i_{k-1}+1}) = \{v\}$. \qed

**Lemma 6.** There is at most one vertex of $D$ that is not dominated by $S$, and if it exists, it is a vertex of $T_1$.

**Proof.** Assume by contradiction that there exists a vertex $x$ of $D$ that is not dominated by $S$ and $x$ is not in $D_1$. Let $2 \leq k \leq t$ such that $x \in D_k$. By Lemma 4 vertex $x$ is in $T_{i_{k-1}+1}$. If $T_{i_{k-1}}$ contains a unique vertex $v$, then, at step $k-1$ of the construction of $S$ we are in Case 2 and $v$ is added to $S$. If $T_{i_{k-1}}$ contains at least two vertices, then, by Lemma 4 it contains a vertex of $S$. In both cases, $S \cap V(T_{i_{k-1}}) \neq \emptyset$. By Lemma 3 we have $T_{i_{k-1}} \Rightarrow T_{i_{k-1}+1}$, so $x$ is dominated by $S$, a contradiction.
Thus all the vertices that are not dominated by \( S \) are in \( D_1 \). By Lemma 5 there is at most one such vertex and if it exists it is a vertex of \( T_i \).

**Lemma 7.** Suppose that there exists a pair \( \{ x, y \} \) of vertices of \( V(D) \setminus S \) not located by \( S \). Then, \( D \) is strongly connected, and one of \( x, y \) is in \( T_1 \) and is the only vertex of \( D_1 \) not dominated by \( S \cap V(D_1) \).

**Proof.** Let \( 1 \leq k \leq t, 1 \leq \ell \leq t \) such that \( x \in V(D_k) \) and \( y \in V(D_\ell) \). Let \( 1 \leq i \leq r, 1 \leq j \leq r \) such that \( x \in V(T_i) \) and \( y \in V(T_j) \). Recall that for all \( 1 \leq m \leq t \), \( S \cap V(D_m) \) is a locating set of \( D_m \), so \( k \neq \ell \) and \( i \neq j \). Since a local tournament is simple, we can assume, w.l.o.g., that \( y \) does not dominate \( x \).

Suppose by contradiction that there exists \( s \in S \cap V(D_\ell) \) that dominates \( y \). Let \( m \) be such that \( s \in T_m \). Since \( s, y \) are vertices of \( D_\ell \), we have \( 1 \leq m \leq j \leq r \).

Since \( k \neq \ell \), we know that \( T_m, T_j \) and \( T_i \) appear in this order along the cyclic order \( T_1, \ldots, T_r \) (with maybe \( m = j \)). Since \( y \) does not dominate \( x \), the vertex \( s \) does not dominate \( x \) by definition of round. Thus \( s \) separates \( x \) and \( y \), a contradiction. So \( S \cap V(D_\ell) \) does not dominate \( y \) and by Lemma 5 \( y \) is the only vertex of \( D_\ell \) that is not dominated by \( S \cap V(D_\ell) \) and \( y \in T_{i_{\ell-1}+1} \).

Suppose by contradiction that \( \ell > 1 \). We use an argument that is similar to the proof of Lemma 6. If \( T_{i_{\ell-1}} \) contains a unique vertex \( v \), then, at step \( \ell - 1 \) of the construction of \( S \) we are in Case 2 and \( v \) is added to \( S \). If \( T_{i_{\ell-1}} \) contains at least two vertices, then, by Lemma 4 it contains a vertex of \( S \). In both cases, \( S \cap V(T_{i_{\ell-1}}) \neq \emptyset \). Let \( s' \in S \cap V(T_{i_{\ell-1}}) \). By Lemma 3 we have \( T_{i_{\ell-1}} \Rightarrow T_{i_{\ell-1}+1} \), so \( s' \) dominates \( y \). Since \( x \) and \( y \) are not located by \( S \), vertex \( s' \) dominates \( x \). Hence, \( T_{i_{\ell-1}}, T_{i_{\ell-1}+1}, T_i \) appear in this order along the cyclic order \( T_1, \ldots, T_r \) (with maybe \( i_{\ell-1} = i \)). Since \( x \) is dominated by \( s' \) and not by \( y \), we have \( i_{\ell-1} = i \). Thus both \( x \) and \( s' \) are in \( V(T_{i_k}) \) and \( y \) is in \( V(T_{i_{k+1}}) \). At step \( k \) of the construction of \( S \), we are in Case 3, and so \( S \cap V(D_k) \) is a dominating set of \( D_k \). By Lemma 5 the set \( S \) contains a vertex of \( V(T_{i_{k-1}+1}) \). By definition of \( i_k \), this vertex dominates \( x \in V(T_{i_k}) \) but not \( y \in V(T_{i_{k+1}}) \), a contradiction. So \( \ell = 1 \), \( y \in T_1 \) and \( y \) is the only vertex of \( D_1 \) that is not dominated by \( S \cap V(D_1) \).

Suppose by contradiction that \( S \) is not strongly connected, then the round decomposition is canonical and there is no vertex of \( D \setminus V(D_1) \) that can dominate \( y \). So \( y \) is not dominated by \( S \) and by Lemma 5 vertex \( x \) is dominated by \( S \), a contradiction. So \( S \) is strongly connected.

We now need to slightly modify \( S \) in a particular case. If \( S \cap V(D_1) \) is not a dominating set of \( D_1 \), then by Lemma 5 let \( z \in T_1 \) be the unique vertex of \( D_1 \) not dominated by \( S \cap V(D_1) \). In this case, let \( S^+ = S \cup \{ z \} \), otherwise, let \( S^+ = S \). We are now able to prove the locating-dominating property of \( S^+ \).

**Lemma 8.** The set \( S^+ \) is a locating-dominating set of \( D \).

**Proof.** Suppose first that \( S \cap V(D_1) \) is a dominating set of \( D_1 \). Then \( S^+ \) is a dominating set of \( D \) by Lemma 5 and a locating set of \( D \) by Lemma 7. Suppose now that \( S \cap D_1 \) is not a dominating set. Recall that \( z \) is the only vertex of \( D_1 \) not dominated by \( S \cap D_1 \). By Lemma 5 there is a unique vertex that is
not dominated by $S$ and this vertex is in $T_1$, so this vertex is $z$. So $S^+$ is a dominating set of $D$. By Lemma 7, vertex $z$ is in all the pairs of vertices of $V(D) \setminus S$ not located by $S$. Thus $S^+$ is locating. 

By Lemma 8 we have defined a locating-dominating set of $D$. We now have to bound its size. For that purpose we use the following theorem from [5]:

**Theorem 9** ([5]). A tournament $D$ of order $n$ satisfies $\gamma^{LD}(D) \leq \left\lceil \frac{n}{2} \right\rceil$ and $\gamma^L(D) \leq \left\lfloor \frac{n}{2} \right\rfloor$.

We use Theorem 9 to first bound the size of $S$ and then of $S^+$. At each step $k$ of the construction of $S$, since $D_k$ is a tournament, the theorem gives an upper bound on the size of the set that is added to $S$:

**Lemma 10.** For $1 \leq k \leq t$, the set of vertices of $D_k$ that is added to $S$ at step $k$ of the construction of $S$ has size at most $\left\lfloor |V(D_k)|/2 \right\rfloor$ when it is assumed to be a locating-dominating set of $D_k$ and size at most $\left\lceil |V(D_k)|/2 \right\rceil$ when it is assumed to be a locating set of $D_k$.

**Proof.** Let $1 \leq k \leq t$. The lemma is clear by Theorem 9 if we are in Cases 1 or 3 at step $k$ of the construction of $S$. Consider now that we are in Case 2. If $|V(D'_k)|$ is even (i.e. $|V(D_k)|$ is odd), then $S \cap V(D_k)$ is composed of a minimum locating-dominating set of $D'_k$ plus the unique vertex of $T_{2k}$.

Thus, by Theorem 9, $S \cap V(D_k)$ is a locating-dominating set of $D_k$ of size at most $\left\lfloor (|V(D_k)| - 1)/2 \right\rfloor + 1 = \left\lfloor |V(D_k)|/2 \right\rfloor$. If $|V(D'_k)|$ is odd (i.e. $|V(D_k)|$ is even), then $S \cap V(D_k)$ is composed of a minimum locating set of $D'_k$ plus the unique vertex of $T_{2k}$. Thus, similarly, $S \cap V(D_k)$ is a locating set of $D_k$ of size at most $\left\lfloor (|V(D_k)| - 1)/2 \right\rfloor + 1 = \left\lceil |V(D_k)|/2 \right\rceil$.

For every $1 \leq k \leq t$, let $n_k = |V(D_k)| + \cdots + |V(D_t)|$. In the following lemma, we bound the size of a minimum locating-dominating set of a subgraph of $D$.

**Lemma 11.** For $1 \leq k \leq t$, the size of $S \cap (V(D_k) \cup \cdots \cup V(D_t))$ is at most $\left\lceil \frac{2n_k}{n_{k+1}} \right\rceil$.

**Proof.** We prove the lemma by induction on $k$ from $t$ down to 1.

At step $t$ of the construction of $S$, we are in Case 1. If $|V(D_t)|$ is even, then $S \cap V(D_t)$ is a minimum locating-dominating set of $D_t$ of size at most $\left\lfloor |V(D_t)|/2 \right\rfloor = \left\lfloor n_t/2 \right\rfloor$ by Theorem 9. If $|V(D_t)|$ is odd, then $S \cap V(D_t)$ is a minimum locating set of $D_t$ of size at most $\left\lfloor n_t/2 \right\rfloor$ by Theorem 9.

Let us now fix $1 \leq k < t$. We assume that the lemma holds for all $j$ with $k+1 \leq j \leq t$ and we prove that it holds for $k$. If $V(D_k)$ has even size or $S \cap V(D_k)$ is not assumed to dominate $D_k$, then by Lemma 11, $|S \cap V(D_k)| \leq \left\lfloor |V(D_k)|/2 \right\rfloor$ and it follows that the lemma holds for $k$. So we can now suppose that $V(D_k)$ has odd size and $S \cap V(D_k)$ is assumed to dominate $D_k$. Thus, we are in Case 2 or 3 at step $k$ and $S \cap V(D_{k+1})$ is not assumed to dominate $D_{k+1}$. Then, at step $k+1$, we are either in Case 1 with $V(D_{k+1})$ has odd size, or in Case 2 with $V(D_{k+1})$ has even size, $k+1 < t$ and $S \cap V(D_{k+1})$ is not assumed to
dominate $D_{k+2}$. In the second case, again it means that at step $k + 2$ we are in the same situation, i.e. we are either in Case 1 with $V(D_{k+2})$ has odd size, or in Case 2 with $V(D_{k+2})$ has even size, $k + 2 < t$ and $S \cap V(D_{k+3})$ is not assumed to dominate $D_{k+3}$. We can repeat this argument until we actually encounter a step $\ell$ of the construction of $S$ such that we are in Case 1 at step $\ell$ with $V(D_{\ell})$ has odd size (we assume $\ell$ is the first such step from $k+1$ towards $t$). Note that for $k + 1 \leq j \leq \ell$, $S \cap V(D_j)$ is not assumed to dominate $D_j$. To summarize, we have:

- $1 \leq k < \ell \leq t$
- $V(D_k)$ and $V(D_\ell)$ have odd sizes
- $V(D_j)$ has even size for $k + 1 \leq j \leq \ell - 1$,
- $S \cap V(D_k)$ is assumed to dominate $D_k$
- $S \cap V(D_j)$ is not assumed to dominate $D_j$ for $k + 1 \leq j \leq \ell$.

Thus, by Theorem 9 we obtain the following:

\[
|S \cap (V(D_1) \cup \ldots \cup V(D_\ell))| \\
\leq (|V(D_k)| + 1)/2 + |V(D_{k+1}) \cup \ldots \cup V(D_{\ell-1})|/2 + (|V(D_\ell)| - 1)/2 \\
\leq \lceil |V(D_k) \cup \ldots \cup V(D_\ell)|/2 \rceil
\]

If $\ell = t$, then the lemma holds for $k$ and we are done. If $\ell < t$, then we use the induction hypothesis that the lemma holds for $\ell + 1$ to conclude. \hfill \Box

A corollary of Lemma 11 is that the size of $S$ is at most $\lceil n/2 \rceil$. We are now able to bound $|S^+|$.

**Lemma 12.** The size of $S^+$ is at most $\lceil n/2 \rceil$.

*Proof.* By Lemma 11, we have $|S| \leq \lceil n/2 \rceil$. Thus if $S = S^+$, then the lemma holds. Thus we can assume that $S \neq S^+$, i.e. $S \cap V(D_1)$ is not a dominating set of $D_1$ and $S^+ = S \cup \{z\}$. If $n$ is odd, then $|S^+| = |S| + 1 \leq \lceil n/2 \rceil + 1 = \lceil n/2 \rceil$ and we are done. So we can assume that $n$ is even.

By Lemma 11 the size of $S \cap V(D_1)$ is at most $\lceil |V(D_1)|/2 \rceil$

By construction of $S$, at step 1, we are either in Case 1 with $|V(D_1)|$ is odd, or in Case 2 with $|V(D_1)|$ is even, $1 < t$ and $S \cap V(D_2)$ is not assumed to dominate $D_2$. We consider this two different cases below.

Consider first that we are in Case 1. Since $|V(D_1)|$ is odd and $n$ is even, we have $1 < t$ and \(n_2 = n - |V(D_1)|\) is odd. By Lemma 11 the size of $S \cap (V(D_2) \cup \ldots \cup V(D_\ell))$ is at most $\lceil n_2/2 \rceil = (n_2 - 1)/2$. Moreover $S \cap V(D_1)$ has size at most $\lceil |V(D_1)|/2 \rceil = (|V(D_1)| - 1)/2$. So in total $S$ has size at most $n/2 - 1$. Therefore $S^+$ has size at most $n/2$ and we are done.

Consider now that we are in Case 2 of the construction at step 1. As in the proof of Claim 11 let $D_\ell$ be the first step from 2 towards $t$ such that we are in Case 1 at step $\ell$ of the construction of $S$ with $V(D_\ell)$ has odd size. Then, we have:
• 2 ≤ ℓ ≤ t
• V(Dj) has even size for 1 ≤ j ≤ ℓ − 1,
• V(Dℓ) has odd size
• S ∩ V(Dj) is not assumed to dominate Dj for 1 ≤ j ≤ ℓ,

Thus, by Theorem 9, we obtain the following:

|S ∩ (V(D1) ∪ ⋯ ∪ V(Dℓ))| ≤ |V(D1) ∪ ⋯ ∪ V(Dℓ−1)|/2 + (|V(Dℓ)| − 1)/2

Since n is even and |V(D1) ∪ ⋯ ∪ V(Dℓ)| is odd, we have ℓ < t and nℓ+1
is odd. By Lemma 11 the size of S ∩ (V(Dℓ+1) ∪ ⋯ ∪ V(Dt)) is at most
|(nℓ+1)/2| = (nℓ+1 − 1)/2. So again S has size at most n/2 − 1 and S⁺ has size
at most n/2.

By combining the results of this section we obtain the following:

Lemma 13. The set S⁺ is a locating dominating set of D of size at most \lfloor n/2 \rfloor.
Moreover, if D is not strongly connected, then S is a locating set of D of size
at most \lfloor n/2 \rfloor.

Proof. The first part is a consequence of Lemmas 8 and 12. The second part is
a consequence of Lemmas 7 and 11.

As a corollary, this proves Theorem 1 for connected roundable local tournaments:

Lemma 14. A connected roundable local tournament D of order n satisfies
γLD(D) ≤ \lceil n/2 \rceil. Moreover if D is not strongly connected, then γL(D) ≤ \lfloor n/2 \rfloor.

4 Non-roundable local tournaments

In the previous section, we proved the upper bound γLD(D) ≤ \lceil n/2 \rceil when D is
a connected roundable local tournament. By Theorem 5 this result is also true
for tournaments. In order to prove Theorem 6 we can now restrict ourselves to
connected local tournaments which are not tournaments and not roundable.

Consider a connected local tournament D that is not a tournament and not
roundable. By [1, Corollary 3.2], every connected local tournament that is not
strong is roundable, so D is strongly connected. Then, by [1, Corollary 3.2,
Lemmas 3.4 and 3.5], there exists a set of vertices X of V(D) such that:
• D \setminus X is not strongly connected and X is minimal for this property
• D \setminus X is a connected local tournament that is not a tournament
• D[X] is a tournament
Let $R[T_1, \ldots, T_r]$ be the canonical round decomposition of $D \setminus X$, then $r \geq 3$ and there are all the arcs from $V(T_r)$ to $X$, and from $X$ to $V(T_1)$.

Let $Y = V(D) \setminus X$ and $Z = Y \setminus V(T_r)$. See Figure 3 for an illustration of the decomposition of $D$. Let $D_1 = D[Y]$ and $D_2 = D[Z]$. For $i \in \{1, 2\}$, let $n_i$ be the order of $D_i$. Note that since $r \geq 3$, the digraph $D_2$ is also a connected local tournament that is not strong with canonical round decomposition $R[T_1, \ldots, T_{r-1}]$. So we can apply on $D_1$ and $D_2$ the method and results of Section 3. For $i \in \{1, 2\}$, let $S_i$ be the set that is defined on $D_i$ exactly as $S$ is defined on $D$ in Section 3. By Lemma 13, the set $S_i$ is a locating set of $D_i$ of size $\lfloor n_i/2 \rfloor$. We will moreover need some particular properties obtained in Section 3, namely Lemmas 4 and 6.

Using these definitions of $S_1$ and $S_2$ we are now able to define a set of vertices $S$ of $D$ that will be our candidate for a locating-dominating set of $D$. We consider the following four cases (see Figure 4 for an illustration):

- **Case 1:** $|V(T_r)| = 1$ and $|X| = 1$
  Let $S$ be the union of $V(T_r)$ and a minimum locating-dominating set of $D_2$.

- **Case 2:** $|V(T_r)| = 1$ and $|X| > 1$
  Let $S$ be the union of $V(T_r)$, $S_2$, and a minimum locating set of $D[X]$.

- **Case 3:** $|V(T_r)| > 1$ and $|X| = 1$
  Let $S$ be the union of $S_1$ and $X$.

- **Case 4:** $|V(T_r)| > 1$ and $|X| > 1$
  Let $S$ be the union of $S_1$ and a minimum locating-dominating set of $D[X]$.

**Lemma 15.** The set $S$ is a locating-dominating set of $D$.

**Proof.** *(Case 1)* Let $x$ be the only vertex of $X$ and $t$ be the only vertex of $T_r$. The set $S$ contains a locating-dominating set of $D_2$ so all the vertices of $Y \setminus S$ are dominated by $S$. Moreover $x$ is dominated by $t$. So the set $S$ is a dominating set of $D$. 

---

**Figure 3:** Decomposition of non-roundable local tournaments
Every pair of vertices of $Y \setminus S$ is separated by $S$. Since the round decomposition is canonical, there is no arc from $t$ to $Z$, so $t$ separates $x$ from any vertex of $Y \setminus S$. Hence $S$ is a locating set of $D$.

(Case 2) As previously, let $t$ be the only vertex of $T_r$. By Lemma $\text{[6]}$ applied on $S_2$, there is at most one vertex of $Z$ that is not dominated by $S_2$, and if it exists, it is a vertex of $T_1$. However, by assumption, $|X| > 1$, so a locating set of $D[X]$ contains at least one vertex of $X$. Thus $S$ contains at least one vertex $s$ of $X$. Since there are all the arcs from $X$ to $V(T_1)$, all the vertices of $T_1$ are dominated by $s$. All the vertices of $X$ are dominated by $t$. So $S$ is a dominating set of $D$.

Every pair of vertices of $X \setminus S$ and every pair of vertices of $Z \setminus S$ are separated by $S$. There are all the arcs from $t$ to $X$ and no arc from $t$ to $Y \setminus \{t\}$. Hence, $t$ separates the vertices of $X$ from the vertices of $Z$, and $S$ is a locating set of $D$.

(Case 3) Let $x$ be the only vertex of $X$. By Lemma $\text{[6]}$ applied on $S_1$, there is at most one vertex of $Y$ that is not dominated by $S_1$, and if it exists, it is a vertex of $T_1$. Since $x \in S$ and there are all the arcs between $x$ and $T_1$, then every vertex of $T_1$ is dominated by $x$. Hence $S$ is a dominating set of $D$.

By definition of $S$, all the pairs of vertices of $Y \setminus S$ are separated by $S$ and $x \in S$. So $S$ is a locating set of $D$.

(Case 4) Note that, since $S$ contains a locating set of $X$ and $|X| > 1$, the set $S$ contains at least one vertex of $X$. Then the proof that $S$ is a dominating set of $D$ is exactly the same as in Case 3.
By definition of $S$, every pair of vertices of $X \setminus S$ and every pair of vertices of $Y \setminus S$ are separated by $S$. Now consider $x \in X \setminus S$ and $y \in Y \setminus S$ and let us show that $x$ and $y$ are separated by $S$. We consider the following two cases:

- **Case A :** $y \in T_r$
  Since $S$ contains a dominating set of $D[X]$, there is a vertex $x'$ of $X \cap S$ that dominates $x$ in $D$. Since there are all the arcs from $V(T_s)$ to $X$, there is an arc from $y$ to $x'$. Since we are considering a simple digraph, there is no arc from $x'$ to $y$, so $x'$ separates $x$ and $y$.

- **Case B :** $y \notin T_r$
  By Lemma 4 applied on $S_1$, there exists a vertex $y'$ in $S \cap T_r$. Since the round decomposition is canonical, there is no arc from $y'$ to $y$. Plus, there are all the arcs from $V(T_r)$ to $X$, and there is an arc from $y'$ to $x$. Hence $y'$ separates $x$ and $y$.

Therefore $S$ is a locating-dominating set of $D$.

**Lemma 16.** The size of $S$ is at most $\lceil \frac{n}{2} \rceil$.

**Proof.** (Case 1) By Lemma 14, a minimum locating-dominating set of $D_2$ has size at most $\lfloor \frac{n}{2} \rfloor$. So $S$ has size at most $\lfloor \frac{n}{2} \rfloor + 1 = \lceil \frac{n}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil$.

(Case 2) By Lemma 13, the size of $S_2$ is at most $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{n-1-|X|}{2} \rfloor$. By Theorem 9, a minimum locating set of $D[X]$ has size at most $\lceil \frac{|X|}{2} \rceil$. So $S$ has size at most $\lfloor \frac{n-1-|X|}{2} \rfloor + \lceil \frac{|X|}{2} \rceil + 1 \leq \lfloor \frac{n}{2} \rfloor$ (check all the parity cases for the last inequality).

(Case 3) By Lemma 13, the size of $S_1$ is at most $\lfloor \frac{n}{2} \rfloor$. So $S$ has size at most $\lfloor \frac{n}{2} \rfloor + 1 \leq \lfloor \frac{n}{2} \rfloor$.

(Case 4) By Lemma 13, the size of $S_1$ is at most $\lfloor \frac{n}{2} \rfloor$. By Theorem 9, a minimum locating-dominating set of $D[X]$ has size at most $\lceil \frac{|X|}{2} \rceil$. So $S$ has size at most $\lfloor \frac{n-1-|X|}{2} \rfloor + \lceil \frac{|X|}{2} \rceil \leq \lfloor \frac{n}{2} \rfloor$ (check all the parity cases for the last inequality).

By combining the results of this section we obtain the following:

**Lemma 17.** A connected local tournament $D$ of order $n$ that is not roundable satisfies $\gamma^{LD}(D) \leq \lceil \frac{n}{2} \rceil$.

**Proof.** By Lemmas 15 and 16, the set $S$ is a locating dominating set of $D$ of size at most $\lceil n/2 \rceil$.

Theorem 1 is a direct consequence of Lemmas 14 and 17.
5 Supervising vertex

From now on, we consider digraphs that are not necessarily simple. In a digraph $D$, a supervising vertex is a vertex $s$ of $D$ such that, for any vertex $v$, there exists a directed path from $s$ to $v$. In this section we prove the following theorem.

**Theorem 18.** Let $D$ be a twin-free digraph on $n$ vertices containing a supervising vertex, then $\gamma^{LD}(D) \leq \frac{3n}{4}$. Moreover, if $D$ is quasi-twin-free, then $\gamma^{LD}(D) \leq \frac{2n}{3}$.

To prove Theorem 18, we will adapt the method used in [4] and [5] to prove general upper bounds on $\gamma^{LD}$. Let $S$ be a set of vertices of a digraph $D$. The $S$-partition of $D$, denoted $\mathcal{P}_S$, is the partition of $V(D) \setminus S$ where two vertices are in the same part if and only if they have the same set of in-neighbours in $S$.

We have the following lemma.

**Lemma 19.** Let $D$ be a twin-free digraph on $n$ vertices and $S$ a dominating set of $D$ such that $|\mathcal{P}_S| \geq |S| - 1$. Then, $\gamma^{LD}(D) \leq \frac{3n}{4}$. Moreover, if $D$ is quasi-twin-free, then $\gamma^{LD}(D) \leq \frac{2n}{3}$.

This result is proved in [5] when $|\mathcal{P}_S| \geq |S| - 1$ (Theorem 8 for $x = 1$). The proof can be adapted if we only have $|\mathcal{P}_S| \geq |S| - 1$.

**Proof.** Let $\mathcal{P}_S = P_1 \cup \cdots \cup P_{n_1} \cup Q_1 \cup \cdots \cup Q_{n_2}$, where $P_1, \ldots, P_{n_1}$ are the parts of size 1 and $Q_1, \ldots, Q_{n_2}$ are the parts of size at least 2.

If $n_2 = 0$, then $S$ is a locating-dominating set of $D$. Since all the parts of $\mathcal{P}_S$ have size 1, $|V(D)| = |S| + |\mathcal{P}_S| \geq 2|S| - 1$. Thus $|S| \leq \frac{n + 1}{2}$ and we are done. Thus in the following we assume that $n_2 > 0$.

We assume that $S$ is maximal with the property that $\mathcal{P}_S$ has at least $|S| - 1$ parts (this is ensured by adding vertices to $S$ while this property holds).

Now, let $X_1 = S \cup P_1 \cup \cdots \cup P_{n_1}$. We have the following property:

**Claim 20.** Two vertices in $V(D) \setminus X_1$ are located by $X_1$, unless they form a pair of quasi-twins.

**Proof of claim.** If two vertices are in different parts of $\mathcal{P}_S$, they are located by some vertices in $S$. Thus, by contradiction, let $q_1$ and $q_2$ be two vertices of $V(D) \setminus X_1$ belonging to some part $Q_i$ of $\mathcal{P}_S$ that are not quasi-twins but are not located by $X_1$. Since $D$ is twin-free, there is a vertex $q_3$ in $V(D) \setminus S$ that can locate $q_1$ and $q_2$: without loss of generality $q_3$ is an in-neighbour of $q_1$ but not $q_2$. By our assumption $q_3 \notin X_1$. Now, consider $S' = S \cup \{q_3\}$, and the corresponding $S'$-partition $\mathcal{P}_{S'}$ of $V(D) \setminus S'$. Since $q_3 \in \bigcup_i Q_i$, any part of $\mathcal{P}_{S'}$ still correspond to some part in $\mathcal{P}_S$. But $Q_i$ has been split into two parts so $\mathcal{P}_{S'}$ has at least one more part than $\mathcal{P}_S$, and thus $|\mathcal{P}_{S'}| \geq |S'| - 1$. This contradicts the choice of $S$, which we assumed to be maximal with this property. (c)

Since $X_1$ is a dominating set, Claim 20 shows that in the absence of quasi-twins, $X_1$ is locating-dominating. Next claim is proved in [5] Claim 8.B] to deal with quasi-twins.
Claim 21. Any two pairs of quasi-twins in $V(D) \setminus X_1$ are disjoint.

For each pair of quasi-twins in $V(D) \setminus X_1$, we add one of the vertices of the pair in $X_1$. By Claims 20 and 21, the resulting set $X'_1$ is a locating-dominating set and has size at most $|S| + n_1 + (n - |S| - n_1)/2 = (n + |S| + n_1)/2$.

Consider now the set $X_2$ of size $n - n_1 - n_2$ consisting of $V(D)$ without one vertex from each part of $P_S$. Then all the vertices of $V(D) \setminus X_2$ are located and dominated by $S$ and thus $X_2$ is a locating dominating set.

Assume now that $D$ has no quasi-twins. Then $X_1$ and $X_2$ are two locating-dominating sets of $D$. If $|X_2| \leq 2n/3$ we are done. Thus we assume that $|X_2| > 2n/3$ which means that $|P_S| < n/3$. Therefore,

$$|X_1| = |S| + n_1$$
$$\leq |P_S| + n_1 + 1$$
$$\leq |P_S| + (n_1 + n_2) \quad \text{since } n_2 \geq 1$$
$$\leq 2|P_S|$$
$$\leq 2n/3$$

and we are done.

If $D$ has some quasi-twins, we use the locating-dominating sets $X'_1$ and $X_2$.
Again, if $|X_2| \leq 3n/4$, we are done. So, assume that $|X_2| > 3n/4$. Then, $|P_S| < n/4$.

Therefore,

$$|X'_1| = \frac{|S| + n + n_1}{2}$$
$$\leq \frac{|P_S| + 1 + n + n_1}{2}$$
$$\leq \frac{|P_S| + n + n_1 + n_2}{2}$$
$$\leq |P_S| + \frac{n}{2}$$
$$\leq \frac{3n}{4}$$

and we are done.

To apply Lemma 19, we prove that such a set $S$ exists when there is a supervising vertex.

Lemma 22. If a digraph $D$ contains a supervising vertex $s$, then, there exists a dominating set $S$ such that $|P_S| \geq |S| - 1$.

Proof. Consider a supervising vertex $s$ of $D$. For $i \geq 0$, let $V_i$ be the set of vertices of $D$ such that the shortest directed path from $s$ to $v$ has length $i$. 

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Let $k$ be the smallest integer such that $V_{k+1}$ is empty. Since $s$ is supervising, $V_0, \ldots, V_k$ form a partition of $V(D)$ where the vertices are sorted according to their distance from $s$. If $k = 0$, then $D$ contains $s$ as a unique vertex and the set $S = \emptyset$ satisfies the lemma. So from now we assume that $k > 0$.

We build the set $S$ by the following method. Let $S$ be a set of vertices of $V_{k-1}$ that dominates $V_k$ and that is minimal for this property. Then, step by step, for $i = 1$ to $k-1$, we assume that $S$ is a set of vertices of $V_{k-1} \cup \ldots \cup V_{k-i}$ that dominates $V_k \cup \ldots \cup V_{k-i+1}$ and we add to $S$ a set of vertices of $V_{k-i-1}$ that dominates $V_{k-i} \setminus S$ and that is minimal for this property. We continue the process until $i = k - 1$.

At each step $i$, by minimality of the choosen set, when a vertex $v$ of $V_{k-i-1}$ is added to $S$ one can choose a vertex $f(v)$ in $V_{k-i} \setminus S$ whose in-neighbourhood in $V_{k-i-1} \cap S$ is exactly $v$. By doing so, at the end of the procedure, each vertex of $f(S)$ has different in-neighbours in $S$. So $|P_S| \geq |S|$.

Finally if $s$ is not already dominated by $S$, then we add it to $S$ so that $S$ is a dominating set of $V(D)$. This might increase the cardinality of $S$ by 1 and in the end $S$ is a dominating set such that $|P_S| \geq |S| - 1$.

Then Theorem [13] is direct consequence of Lemmas [19] and [22].

Since all the vertices in a strongly connected digraph are supervising, we have the following corollary:

**Corollary 23.** Let $D$ be a twin-free strongly connected digraph on $n$ vertices, then $\gamma_{LD}(D) \leq \frac{3n}{4}$. Moreover, if $D$ is quasi-twin-free, then $\gamma_{LD}(D) \leq \frac{2n}{3}$.

Note that the second bound is asymptotically tight (see Figure 11). A digraph is called semicomplete if there is at least one arc between every pair of vertices. A digraph $D$ is locally in-semicomplete if the in-neighbourhood of every vertex $x$ of $D$ induces a semicomplete digraph. Note that semicomplete digraphs are a generalization of tournaments and thus, locally in-semicolonate digraphs naturally generalize local tournaments.

**Lemma 24.** A locally in-semicolonate digraph has a supervising vertex.

**Proof.** Suppose by contradiction, that there exists a locally in-semicolonate digraph $D$ with no supervising vertex. Consider a vertex of $D$, such that the set $S$ of vertices $v$ of $V(D)$ for which there exists a directed path from $s$ to $v$ has maximum size, i.e. $s$ is supervising a set of vertices of maximum size. Note that $s \in S$ since $s$ forms a directed path of length zero from itself to itself. Since $D$ contains no supervising vertices, we know that $V(D) \setminus S$ is nonempty. Plus, all the edges between $V(D) \setminus S$ and $S$ are oriented from $V(D) \setminus S$ to $S$. Since $D$ is connected, there exists at least such an edge from a vertex $u$ of $V(D) \setminus S$ to a vertex of $S$. Recall that $u$ has no in-neighbour in $S$. Let $v$ be a vertex of $S$ that is an out-neighbour of $u$ and such that the length of a shortest directed path $P$ from $s$ to $v$ has minimum length. By the choice of $v$, no vertex of $P$ distinct from $v$ is an out-neighbour of $u$. Note that $v$ is distinct from $s$, since otherwise $u$ is supervising more vertices than $s$, contradicting the choice of $s$. Let $w$ be the in-neighbour of $v$ along $P$. Both $w$ and $u$ are in the in-neighbourhood of $v$.
Hence, by in-semicompleteness of $D$, there must be an arc between them, which is a contradiction.

A consequence of Lemma 24 and Theorem 18 is the following corollary.

**Corollary 25.** Let $D$ be a twin-free locally in-semi-complete digraph on $n$ vertices, then $\gamma^L_D(D) \leq \frac{2n}{3}$. Moreover, if $D$ is quasi-twin-free, then $\gamma^L_D(D) \leq \frac{2n}{3}$.

Consider the digraph $D$ obtained from a vertex whose out-neighbourhood is made of $k$ disjoint oriented triangles (see Figure 5). On this example, $D$ is a quasi-twin-free locally in-semi-complete digraph on $3k+1$ vertices. Note that $\gamma^L_D(D) = 2k$ and $\frac{2n}{3} = 2k + 1$. So we are at distance 1 from the bound given by Corollary 25 and thus asymptotically tight.

![Figure 5: Disjoint oriented triangles forming the out-neighbourhood of an extra vertex. A locating dominating set of minimum size is given by gray vertices.](image)

Note that a similar method cannot be applied for locally out-semi-complete digraphs (digraphs where all the out-neighbourhoods are semi-complete). Indeed, there are twin-free locally out-semi-complete digraphs for which the minimum dominating set has size $2(n - 1)/3$ (see for example the reverse of the digraph of Figure 5). Thus, there is no dominating set $S$ such that $|P_S| \geq |S| - 1$ and Theorem 19 cannot be applied.

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