A Proof of a Dodecahedron Conjecture for Distance Sets

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Received: 28 September 2020 / Revised: 31 March 2021 / Accepted: 3 April 2021 / Published online: 17 April 2021 © The Author(s), under exclusive licence to Springer Japan KK, part of Springer Nature 2021

Abstract

A finite subset of a Euclidean space is called an \( s \)-distance set if there exist exactly \( s \) values of the Euclidean distances between two distinct points in the set. In this paper, we prove that the maximum cardinality among all 5-distance sets in \( \mathbb{R}^3 \) is 20, and every 5-distance set in \( \mathbb{R}^3 \) with 20 points is similar to the vertex set of a regular dodecahedron.

Keywords Distance sets · Dodecahedron

1 Introduction

For \( X \subset \mathbb{R}^d \), let

\[ A(X) = \{d(x,y) \mid x, y \in X, x \neq y\}, \]

where \( d(x, y) \) is the Euclidean distance between \( x \) and \( y \). We call \( X \) an \( s \)-distance set if \( |A(X)| = s \). Two \( s \)-distance sets are said to be isomorphic if there exists a similar transformation from one to the other. One of the major problems in the theory of distance sets is to determine the maximum cardinality \( g_d(s) \) of \( s \)-distance sets in \( \mathbb{R}^d \) for given \( s \) and \( d \), and classify distance sets in \( \mathbb{R}^d \) with \( g_d(s) \) points up to isomorphism. An \( s \)-distance set \( X \) in \( \mathbb{R}^d \) is said to be optimal if \( |X| = g_d(s) \). Clearly \( g_1(s) = s + 1 \), and the optimal \( s \)-distance set is the set of \( s + 1 \) points on the line whose two consecutive points have an equal interval. For the cases where \( d = 2 \) or \( s = 2 \), \( s \)-distance sets in \( \mathbb{R}^d \) are well studied [1, 2, 9, 10, 13, 14, 18, 19, 22], because

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of their simple structures or the relationship to graphs, see Table 1. For \( d \leq 8 \), the maximum cardinalities \( g_{d}(2) \) are determined, and optimal 2-distance sets in \( \mathbb{R}^{d} \) are classified except for \( d = 8 \) [9, 14]. Moreover, it is known that \( g_{3}(3) = 12 \), \( g_{3}(4) = 13 \) and \( g_{4}(4) = 16 \), and the classification is complete for the three cases [17, 20]. In particular, we recall the classification of optimal \( s \)-distance sets in \( \mathbb{R}^{d} \) for \((d, s) = (2, 4), (3, 3) \) and \((3, 4) \) as in Theorem 1.

**Theorem 1** ([17, 19, 20])

1. Every 9-point 4-distance set in \( \mathbb{R}^{2} \) is isomorphic to the vertices of the regular nonagon or one of the three configurations given in Fig. 1a–c. Moreover, every 8-point 4-distance set in \( \mathbb{R}^{2} \) is isomorphic to the vertices of the regular octagon, the vertices of the regular septagon with its center, Fig. 1d or 8-point subsets of a 9-point 4-distance set.

2. Every 12-point 3-distance set in \( \mathbb{R}^{3} \) is isomorphic to the vertices of the icosahedron.

3. Every 13-point 4-distance set in \( \mathbb{R}^{3} \) is isomorphic to the vertices of the icosahedron with its center point or the vertex set of the cuboctahedron with its center point.

For a 2-distance set \( X \), we consider the graph on \( X \) where two vertices are adjacent if they have the smallest distance in \( X \). We can construct the 2-distance set that has the structure of a given graph [9]. Lisonek [14] gave an algorithm for a stepwise augmentation of representable graphs (adding one vertex per iteration), and classified the optimal 2-distance sets in \( \mathbb{R}^{d} \) for \( d \leq 7 \) by a computer search. Szöllösi and Ostergård [20] extended this algorithm to \( s \)-distance sets and classified optimal \( s \)-distance sets for \((d, s) = (2, 6), (3, 4), (4, 3) \). Indeed, their algorithm is applicable for small \( s \) and \( d \). In the present paper, we add geometrical observations in \( \mathbb{R}^{3} \) to this algorithm, and obtain the main theorem as follows.

**Theorem 2**

Every 20-point 5-distance set in \( \mathbb{R}^{3} \) is isomorphic to the vertices of a regular dodecahedron. In particular, \( g_{3}(5) = 20 \).

This was a long standing open problem [6] as well as the icosahedron conjecture [9]. The icosahedron conjecture was already solved, and the set is the optimal 3-

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**Table 1** Maximum cardinalities of \( s \)-distance sets in \( \mathbb{R}^{d} \)

| \( d \) | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-------|---|---|---|---|---|---|---|
| \( g_{d}(2) \) | 5 | 6 | 10 | 16 | 27 | 29 | 45 |
| \( s \) | 2 | 3 | 4 | 5 | 6 |
| \( g_{2}(s) \) | 5 | 7 | 9 | 12 | 13 |
distance set in $\mathbb{R}^3$ [17, 20] as in Theorem 1 (2). The following theorem plays a key role to prove our main theorem.

**Theorem 3** Every 5-distance set in $\mathbb{R}^3$ with at least 20 points contains an $s$-distance set for some $s \leq 4$ with 8 points.

The main concept to prove Theorem 3 is the diameter graph [7] of a set in $\mathbb{R}^d$. The diameter graph of a set $X$ in $\mathbb{R}^d$ is the graph on $X$ where two vertices are adjacent if the two vertices have the largest distance in $X$. The subset of $X$ corresponding to an independence set of the diameter graph does not have the largest distance in $X$. Thus we can verify the existence of an $s'$-distance subset of an $s$-distance set $X$ with $s' < s$ by the independence number of its diameter graph. The existence of an $s'$-distance set is useful to determine an optimal $s$-distance set in low dimensions [17, 19]. Ramsey numbers or complementary Ramsey numbers [16] are also expected to show the existence of an $s'$-distance subsets of an $s$-distance set. For $\mathbb{R}^3$, if a diameter graph has an odd cycle of length 3 or 5, its independence number is relatively large [7] (see Corollary 1 in the present paper). We consider the minimum value $f(n)$ of the independence numbers among all graphs of order $n$ which contain neither a 3-cycle nor a 5-cycle. The value of $f(n)$ is very useful to estimate the size of an independence set in a diameter graph. We introduce an effective method to determine the value $f(n)$ (Lemma 5).

In Sect. 2, we discuss the distances in a regular dodecahedron and we enumerate the number of 8-point subsets of a regular dodecahedron which are 3- or 4-distance. In Sect. 3, we consider the independence numbers of diameter graphs and prove Theorem 3. The classification of 8-point $s$-distance sets in $\mathbb{R}^3$ for $s \leq 4$ are essentially obtained by Szöllősi and Östergård [20]. In Sect. 4, we introduce their methods, where $s$-distance sets are constructed from $s$-colorings. A map from all pairs of elements of a finite set $X$ to $\{1, \ldots, s\}$ is called an $s$-coloring of $X$. It is possible to classify $s$-distance sets of small points from the classification of $s$-colorings which are representable as $s$-distance sets. Two distinct $s$-distance sets may have the same structure of an $s$-coloring. In Sect. 5, we classify 8-point 3- or 4-distance sets which may be subsets of a 20-point 5-distance set in $\mathbb{R}^3$, and prove Theorem 2.
2 Dodecahedron and its Subsets

Let $G = (V, E)$ be a simple graph, where $V = V(G)$ and $E = E(G)$ are the vertex set and the edge set of $G$, respectively. A subset $W$ of $V(G)$ is an independent set (resp. clique) of $G$ if any two vertices in $W$ are nonadjacent (resp. adjacent). The independence number $\alpha(G)$ (resp. clique number $\omega(G)$) of a graph $G$ is the maximum cardinality among the independent sets (resp. cliques) of $G$. Let $R_i = \{(x, y) \in V \times V \mid b(x, y) = i\}$, where $b$ is the shortest-path distance. The $i$-th distance matrix $A_i$ of $G$ is the matrix indexed by $V$ whose $(x,y)$-entry is $1$ if $(x,y) \in R_i$, and $0$ otherwise. A simple graph $G$ is a distance-regular graph [4, 21] if for any non-negative integers $i, j, k$, the number $p_{ij}^k = |\{z \in V \mid (x,z) \in R_i, (z,y) \in R_j\}|$ is independent of the choice of $(x,y) \in R_k$. The algebra $\mathfrak{A}$ spanned by $\{A_i\}$ over the complex numbers is called the Bose–Mesner algebra of a distance-regular graph. There exists another basis $\{E_i\}$ such that $E_iE_j = \delta_{ij}E_i$, where $\delta_{ij}$ is the Kronecker delta. The matrices $E_i$ are called primitive idempotents, and the matrices are positive semidefinite. The matrices $E_i$ can be interpreted as the Gram matrices of some spherical sets that have the structure of the distance-regular graph, and $E_i$ are called spherical representations of the graph. The following matrices

$$P = (p_i(j))_{j,i} \text{ for } A_i = \sum_j p_i(j)E_j,$$

$$Q = (q_i(j))_{j,i} \text{ for } E_i = \frac{1}{|V|} \sum_j q_i(j)A_j,$$

are called the first and second eigenmatrices, respectively. The entries of $P$ are the eigenvalues of $A_i$, and the entries of $Q$ are the inner products of the spherical representations of $E_i$. The first row $q_i(0)$ of $Q$ is the rank of $E_i$, that is the dimension where the representation $E_i$ exists.

Let $\mathcal{D}_{20}$ be the vertex set of the regular dodecahedron with edge length 1. The set $\mathcal{D}_{20}$ is a 5-distance set, and let $d_1 = 1, d_2, d_3, d_4, d_5$ be the 5-distances of $\mathcal{D}_{20}$ with $1 = d_1 < d_2 < d_3 < d_4 < d_5$. The second-smallest distance $d_2$ is the length of a diagonal line of a face, namely $d_2 = \frac{\sqrt{5} - 1}{\sqrt{2}}$. Since $\mathcal{D}_{20}$ contains the cube with edge length $\tau$, the other distances in the cube are $d_3 = \sqrt{2}\tau$ and $d_5 = \sqrt{3}\tau$. We can calculate $d_4 = \sqrt{3\tau^2 - 1} = \tau + 1$ by Pythagorean theorem. Let $\mathfrak{G}$ be the dodecahedron graph $\mathfrak{G} = (\mathcal{V}, \mathcal{E})$, where $V = \mathcal{D}_{20}$ and $E = \{(x, y) \mid d(x, y) = d_1\}$. The graph $\mathfrak{G}$ is a distance-regular graph, and $d(x, y) = d_i$ if and only if $b(x, y) = i$ for each $i \in \{0, 1, \ldots, 5\}$, where $d_0 = 0$. The second eigenmatrix $Q$ of $\mathfrak{G}$ is

$$Q = \begin{pmatrix}
1 & 3 & 3 & 4 & 4 & 5 \\
1 & \sqrt{5} & -\sqrt{5} & -8/3 & 0 & 5/3 \\
1 & 1 & 1 & 2/3 & -2 & -5/3 \\
1 & -1 & -1 & 2/3 & 2 & -5/3 \\
1 & -\sqrt{5} & \sqrt{5} & -8/3 & 0 & 5/3 \\
1 & -3 & -3 & 4 & -4 & 5
\end{pmatrix}.$$
There are two representations $E_2$ and $E_3$ in the 3-dimensional sphere. Indeed, both $E_2$ and $E_3$ are regular dodecahedrons, and the two graphs of $A_1$ and $A_4$ are isomorphic. Let $\Phi$ be the field automorphism of $\mathbb{Q}(\sqrt{5})$ such that $\Phi(\sqrt{5}) = -\sqrt{5}$ and $\Phi$ fixes all rationals. For a matrix $M = (m_{ij})$ with $m_{ij} \in \mathbb{Q}(\sqrt{5})$, the map $\hat{\Phi}(M)$ is defined by applying $\Phi$ to the entries of $M$, namely $\hat{\Phi}(M) = (\Phi(m_{ij}))$. It follows that

$$\hat{\Phi}(E_2) = 3A_0 + \Phi(\sqrt{5})A_1 + A_2 - A_3 - \Phi(\sqrt{5})A_4 - 3A_5$$

$$= 3A_0 - \sqrt{5}A_1 + A_2 - A_3 + \sqrt{5}A_4 - 3A_5 = E_3.$$

A principal submatrix $T$ of $E_2$ corresponds to a subset of the regular dodecahedron. The matrix $\hat{\Phi}(T)$ is a principal submatrix of $E_3$, and $\hat{\Phi}(T)$ also corresponds to a subset of the regular dodecahedron. The two matrices $T$ and $\hat{\Phi}(T)$ may not be isomorphic as distance sets, but the two colorings of them are equivalent (see Sect. 4 for colorings). This observation gives the following lemma.

**Lemma 1** Let $X$ be a subset of the regular dodecahedron in the unit sphere $S^2$. Let $M$ be the Gram matrix of $X$. Let $\Phi$ be the map defined as above. Then $M$ and $\hat{\Phi}(M)$ are subsets of the regular dodecahedron, and the two colorings of them are equivalent.

Now we discuss 8-point subsets of the regular dodecahedron which have only 3 or 4 distances.

**Lemma 2** There exists a unique 3-distance subset of a regular dodecahedron with 8 points up to isomorphism. The subset is the cube.

**Proof** Let $G$ be the dodecahedron graph with relations $R_i = \{(x, y) \in V \times V \mid d(x, y) = d_i\}$. We define the graphs $G_i = (V, R_i)$ and $G_{ij} = (V, R_i \cup R_j)$. If for given $i$, the independence number $\alpha(G_{ij})$ is less than 8 for each $j \neq i$, then we should take the distance $d_i$ for a 8-point subset. We can determine $\alpha(G) = \alpha(G_1) = \alpha(G_2) = \alpha(G_3) = \alpha(G_4) = \alpha(G_5) = \alpha(G_6)$. This implies $\alpha(G_{i,j}) \leq 8$ and $\alpha(G_{i,j}) \leq 8$ for each $i = 1, 4, 5$. Thus $X$ has both $d_2$ and $d_3$. Moreover, we can determine $\alpha(G_6) = \alpha(G_7) = \alpha(G_8)$, which are calculated by a computer aid. Therefore the distances of $X$ are $d_2$, $d_3$ and $d_5$. The set $X$ corresponding to $\alpha(G)$ is the cube. □

**Lemma 3** There exist exactly 116 of 4-distance subsets of a regular dodecahedron with 8 points up to isomorphism.

**Proof** An 8-point 4-distance subset of a regular dodecahedron does not contain an antipodal pair $\{x, -x\}$, otherwise $X$ is not 4-distance. A regular dodecahedron has only ten antipodal pairs. We choose eight antipodal pairs from the ten pairs, and pick out one point from each antipodal pair, then an 8-point 4-distance set is obtained. Every 8-point 4-distance subset of a regular dodecahedron is obtained by this manner.

First we prove that if two 8-point 4-distance sets $X$ and $Y$ are isomorphic, then $X$ and $Y$ are in the same orbit of the isometry group of a regular dodecahedron. Since the sets $X$ and $Y$ are in the same sphere, there exists an isometry $\sigma$ in the orthogonal
group $O(3)$ such that $X^\sigma = Y$. This implies that $(\pm X)^\sigma = \pm Y$, namely the set of eight antipodal pairs of $\pm X$ are isomorphic to that of $\pm Y$. Since a regular dodecahedron in a given sphere is uniquely determined after one face is fixed, if each set of eight antipodal pairs makes a face of the regular dodecahedron, then $\sigma$ becomes an isometry of the regular dodecahedron. In order to prove that each set of eight antipodal pairs makes a face of the regular dodecahedron, we prove that it is impossible to break all the faces of a regular dodecahedron by removing two antipodal pairs. If we remove an antipodal pair, then six faces are broken. By removing one more antipodal pair, we would like to break the remaining six faces, but it is impossible. Therefore, each set of eight antipodal pairs contains a face of the regular dodecahedron, and $\sigma$ becomes an isometry of the regular dodecahedron.

We can determine the number of the 4-distance sets up to isomorphism by Burnside’s lemma. The isometry group $\text{Aut}(D_{20})$ of a regular dodecahedron is a subgroup of a symmetric group $S_{20}$ on the 20 vertices, which is isomorphic to $A_5 \times C_2$, see [5, Section 3.3.2.3] in details. The vertices are indexed as Fig. 2. Let $N_\sigma$ denote the number of the 4-distance sets fixed by $\sigma \in \text{Aut}(\Gamma)$. For each $\sigma \in \text{Aut}(\Gamma)$, we determine the number $N_\sigma$.

The identity $e$ fixes all the 4-distance sets, namely $N_e = \begin{pmatrix} 10 \\ 2 \end{pmatrix} \cdot 2^8 = 11520$.

The transformations that fix a face of the regular dodecahedron are conjugates of

$$\sigma_1 = (1 \ 2 \ 3 \ 4 \ 5)(6 \ 7 \ 8 \ 9 \ 10)(11 \ 15 \ 14 \ 13 \ 12)(16 \ 20 \ 19 \ 18 \ 17),$$

$\sigma_1^2$, $\sigma_1^3$, or $\sigma_1^4$. The number of such transformations is 24. The size of a subset fixed by $\sigma_1$ is divisible by 5. Thus $N_{\sigma_1} = 0$, and similarly $N_\sigma = 0$ for any transformation $\sigma$ in this case.

The transformations that fix an edge of the regular dodecahedron are conjugates of

$$\sigma_2 = (1 \ 2)(3 \ 6)(5 \ 7)(4 \ 12)(10 \ 11)(8 \ 13)(9 \ 17)(14 \ 16)(15 \ 18)(19 \ 20).$$
The number of such transformations is 15. A 4-distance set fixed by $\sigma_2$ contains one of $\{1, 2\}$ and $\{19, 20\}$, one of $\{3, 6\}$ and $\{15, 18\}$, one of $\{5, 7\}$ and $\{14, 16\}$, and one of $\{4, 12\}$ and $\{9, 17\}$. This implies that $N_{\sigma_2} = 2^4 = 16$, and similarly $N_{\sigma} = 16$ for any transformation $\sigma$ in this case.

The transformations that fix a vertex of the regular dodecahedron are conjugates of

$$\sigma_3 = (2 5 6)(3 10 12)(4 13 7)(8 14 17)(9 18 11)(15 19 16)$$

or $\sigma_3^2$. The number of such transformations is 20. The size of a subset fixed by $\sigma_3$ is congruent to 0 or 1 modulo 3. Thus $N_{\sigma_3} = 0$, and similarly $N_{\sigma} = 0$ for any transformation $\sigma$ in this case.

Let $\tau$ be the transformation such that $\tau(x) = -x$ for any vertex $x$, namely

$$\tau = (1 20)(2 19)(3 18)(4 17)(5 16)(6 15)(7 14)(8 13)(9 12)(10 11).$$

Clearly $N_{\tau} = 0$.

We consider the transformations that are conjugates of

$$\tau\sigma_1 = (1 19 3 17 5 20 2 18 4 16)(6 14 8 12 10 15 7 13 9 11),$$

$\tau\sigma_1^2$, $\tau\sigma_1^3$, or $\tau\sigma_1^4$. The number of such transformations is 24. The size of a subset fixed by $\tau\sigma_1$ is divisible by 10. Thus $N_{\tau\sigma_1} = 0$, and similarly $N_{\sigma} = 0$ for any transformation $\sigma$ in this case.

We consider the transformations that are conjugates of

$$\tau\sigma_2 = (1 19)(2 20)(3 15)(4 9)(5 14)(6 18)(7 16)(12 17).$$

The number of such transformations is 15. A 4-distance set fixed by $\tau\sigma_2$ may contain one of $\{1, 19\}$ and $\{2, 20\}$, one of $\{3, 15\}$ and $\{6, 18\}$, one of $\{5, 14\}$ and $\{7, 16\}$, one of $\{4, 9\}$ and $\{12, 17\}$, one of 8 and 13, or one of 10 and 11. This implies that $N_{\tau\sigma_2} = 2^4 + \left(\frac{4}{3}\right)^2 2^5 = 144$, and similarly $N_{\sigma} = 144$ for any transformation $\sigma$ in this case.

We consider the transformations that are conjugates of

$$\tau\sigma_3 = (1 20)(2 16 6 19 5 15)(3 11 12 18 10 9)(4 8 7 17 13 14)$$

or $\tau\sigma_3^2$. The number of such transformations are 20. A subset fixed by $\tau\sigma_3$ must contain $-x$ for its point $x$. Thus $N_{\tau\sigma_3} = 0$, and similarly $N_{\sigma} = 0$ for any transformation $\sigma$ in this case.

By Burnside’s lemma, the number of 8-point 4-distance subsets of the regular dodecahedron is

$$\frac{1}{|\text{Aut}(G)|} \sum_{\sigma \in \text{Aut}(G)} N_{\sigma} = \frac{1}{120} (1 \cdot 11520 + 24 \cdot 0 + 15 \cdot 16 + 20 \cdot 0 + 1 \cdot 0 + 24 \cdot 0 + 15 \cdot 144 + 20 \cdot 0) = 116.$$
Remark 1  Let $\rho \in S_{20}$ be an isomorphism from the graph $A_1$ to the graph $A_4$. Since the dodecahedron graph is a distance-regular graph, $A_i$ can be expressed by some polynomial $f_i(A_1)$ of degree $i$. Moreover the dodecahedron graph has two structures of distance-regularity, and it satisfies $f_i(A_1) = f_i(A_4)$ for $i = 2, 3, 5$ [8]. From this fact, $\rho \ast A_i = \rho \ast f_i(A_1) = f_i(\rho \ast A_1) = f_i(A_4) = A_i$ for $i = 2, 3, 5$, where $\rho \ast A_i$ is the matrix obtained from the action of $\rho$. Therefore, the map $\Phi$ defined above Lemma 1 is identified with the permutation $\rho$. We may take an isomorphism $\rho$ as

$$\rho = (216)(310)(47)(519)(615)(1118)(1417).$$

Note that the isomorphisms from $A_1$ to $A_4$ are $\rho \cdot \text{Aut}(\Gamma)$. Applying Burnside’s lemma to the group $K = \langle \text{Aut}(\Gamma) \cup \{\rho\} \rangle = \text{Aut}(\Gamma) \cup \rho \cdot \text{Aut}(\Gamma)$ and 8-point 4-distance sets in the regular dodecahedron, we can obtain

$$\frac{1}{|K|} \sum_{\sigma \in K} N_{\sigma} = 63.$$

This implies that the number of 4-colorings on 8 points in the regular dodecahedron is at most 63. We can determine the exact number is 63 by computer calculation. It is not hard to check the equivalence of the colorings among all 8-point 4-distance subsets in the regular dodecahedron.

3 Diameter Graphs and their Independence Numbers

We denote a path and a cycle with $n$ vertices by $P_n$ and $C_n$, respectively. We denote a complete graph of order $n$ by $K_n$. For $X \subset \mathbb{R}^d$, the diameter of $X$ is defined to be the maximum value of $A(X)$. Diameters give us important information when we study distance sets especially in low dimensional space. The diameter graph $DG(X)$ of $X \subset \mathbb{R}^d$ is the graph with $X$ as its vertices and where two vertices $p, q \in X$ are adjacent if $d(p, q)$ is the diameter of $X$. Let $R_n$ be the set of the vertices of a regular $n$-gon. Clearly $DG(R_{2n+1}) = C_{2n+1}$ and $DG(R_{2n}) = n \cdot P_2$. Note that if the independence number $\alpha(DG(X)) = n'$ for an $s$-distance set $X$, then the subset of $X$ corresponding to an independence set of order $n'$ is an $s'$-distance set for some $s' < s$.

For the diameter graphs of sets in $\mathbb{R}^3$, Dol’nikov [7] proved the following theorem. This theorem plays a key role of the proof of Theorem 3.

Theorem 4 (Dol’nikov) Let $G = DG(X)$ be the diameter graph of $X \subset \mathbb{R}^3$. If $G$ contains two cycles with odd lengths, then they have a common vertex.

In particular, we have the following corollary.

Corollary 1  Let $G = DG(X)$ be the diameter graph of $X \subset \mathbb{R}^3$ with $|X| = n$. If $G$ contains an odd cycle $C$ with length $m$, then $\alpha(G) \geq \lceil \frac{n-m}{2} \rceil$.

Proof  If we remove the odd cycle $C$ from $G$, then any odd cycle in $G$ is broken by Theorem 4. This implies $G - C$ is a bipartite graph. Therefore we have $\alpha(G) \geq \alpha(G - C) \geq \lceil \frac{n-m}{2} \rceil$. \qed

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In the remaining of this section, we give a proof of Theorem 3.

By Corollary 1, if the diameter graph $G = DG(X)$ of $X \subset \mathbb{R}^2$ with 20 points contains a 3-cycle or a 5-cycle, then $\alpha(G) \geq 8$. Let $\mathcal{G}_n$ be the set of all graphs of order $n$ which contain neither a 3-cycle nor a 5-cycle. We define

$$f(n) = \min \{ \alpha(G) \mid G \in \mathcal{G}_n \}.$$ 

Since $\alpha(C_n) = \lfloor n/2 \rfloor$, we have $f(n) \leq \lfloor n/2 \rfloor$. For a group $G$ and $S \subset G$, we define the Cayley graph $\text{Cay}(G, S)$ as the graph whose vertex set is $G$ and two vertices $v, w \in G$ are adjacent if $v^{-1}w \in S$. It is easy to see that $\text{Cay}(\mathbb{Z}_{17}, \{\pm 1, \pm 6\})$ contains neither a 3-cycle nor a 5-cycle and

$$\alpha(\text{Cay}(\mathbb{Z}_{17}, \{\pm 1, \pm 6\})) = 7.$$ 

This implies $f(17) \leq 7$.

**Lemma 4** Let $f(n)$ be defined as above. Then $0 \leq f(n + 1) - f(n) \leq 1$ holds.

**Proof** Let $G \in \mathcal{G}_n$ be a graph satisfying $\alpha(G) = f(n)$ and $G'$ be the graph given by adding one isolated vertex to $G$. Then $f(n + 1) \leq \alpha(G') = \alpha(G) + 1 \leq f(n) + 1$. Let $G \in \mathcal{G}_{n+1}$ be a graph satisfying $\alpha(G) = f(n + 1)$ and $H$ be an independent set of $G$ with $|H| = f(n + 1)$. Let $v \in V(G) \setminus H$. Then $H$ is an independent set of $G - \{v\}$, and $\alpha(G - \{v\}) = |H| = f(n + 1)$. Therefore $f(n) \leq \alpha(G - \{v\}) = |H| = f(n + 1)$. \hfill $\Box$

For a vertex $v \in V(G)$, $\Gamma_i(v) = \{w \in V(G) \mid d(v, w) = i\}$, where $d(v, w)$ is the shortest-path distance between $v$ and $w$. We abbreviate $\Gamma(v) = \Gamma_1(v)$. We define $G_i(v)$ as the induced subgraph with respect to $\Gamma_i(v)$ and $k_i(v) = |\Gamma_i(v)|$. Let $m$ be a positive integer. We denote $\Gamma_m^+(v) = \bigcup_{i \geq m} \Gamma_i(v)$ and $k_m^+(v) = |\Gamma_m^+(v)|$. Moreover, we define $G_m^+(v)$ as the induced subgraph of $G$ with respect to $\Gamma_m^+(v)$. Note that we regard $d(v, w) = \infty$ and $w \in \Gamma_m^+(v)$ if there is no path between $v$ and $w$. Then the following degree condition holds.

**Lemma 5** Let $n$ and $t$ be positive integers. Let $G \in \mathcal{G}_n$ and $v \in V(G)$. If $\alpha(G) < t \leq n - k_1(v) + 1$, then

$$k_1(v) + f(n - k_1(v) - t + 1) < t.$$ 

**Proof** Since $G \in \mathcal{G}_n$, $\{v\} \cup \Gamma_2(v)$ is an independent set of $G$. In particular, we have $k_2^+(v) \geq n - k_1(v) - t + 1$ since $1 + k_2(v) \leq \alpha(G) < t$ and $k_2(v) \leq t - 2$. Then

$$t > \alpha(G) \geq k_1(v) + f(k_2^+(v)) \geq k_1(v) + f(n - k_1(v) - t + 1),$$

since $w_1$ and $w_2$ are not adjacent for any $w_1 \in \Gamma_1(v)$ and $w_2 \in \Gamma_3^+(v)$. \hfill $\Box$

For a small integer $n$, we can determine $f(n)$ by using Lemma 5.

**Lemma 6** We have $f(3) = 2, f(5) = 3, f(8) = 4, f(10) = 5, f(13) = 6$ and $f(17) = 7$. 

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Proof Let \( t_n \) be the value of \( f(n) \) in the assertion. The independence numbers of the graphs \( P_3, P_5, C_6, C_{10}, C_{13} \) and \( \text{Cay}(\mathbb{Z}_{17}, \{ \pm 1, \pm 6 \}) \) are \( t_n \) for \( n = 3, 4, 8, 10 \) and 13, respectively. This implies the inequalities \( f(n) \leq t_n \) for each case. It is enough to prove the converse inequalities \( f(n) \geq t_n \). We only prove \( f(17) \geq 7 \) because other inequalities can be proved by a similar way. Suppose that there exists \( G \in \mathcal{G}_{17} \) such that \( \alpha(G) < 7 \). If there exists \( v \in V(G) \) such that \( k_1(v) = 3 \), then \( 7 > 3 + f(8) = 3 + 4 \) from Lemma 5 with \( t = 7 \), which is a contradiction. It is easy to see that we conclude a contradiction for \( k_1(v) \geq 3 \) since \( k_1(v) + f(n - k_1(v) - t + 1) \leq (k_1(v) + 1) + f(n - (k_1(v) + 1) - t + 1) \) holds in general from \( f(n + 1) - f(n) \leq 1 \) in Lemma 4. Therefore \( k_1(v) \leq 2 \) for any \( v \in V(G) \). Then \( G \) is the union of cycles, paths or isolated vertices. Except for an odd cycle, each connected component of \( G \) with \( m \) vertices has an independent set of size \( \lceil m/2 \rceil \). Since \( G \) contains neither a 3-cycle nor a 5-cycle and \( |V(G)| = 17 \), \( G \) contains at most two odd cycles. Then \( \alpha(G) \geq 8 \), which is a contradiction. Therefore \( f(17) \geq 7 \) holds. Then we have \( f(1) = 7 \). \( \square \)

Remark 2 We can determine other \( f(n) \) for small \( n \). For example we have \( f(12) = 5 \) and \( f(16) = 7 \). For \( f(12) = 5 \), it is clear because \( 5 = f(10) \leq f(12) \) and \( \text{Cay}(\mathbb{Z}_{12}, \{ \pm 1, 6 \}) \) has the independence number 5. For \( f(16) = 7 \), it is proved by a similar way to the proof of \( f(17) = 7 \), and an attaining graph is obtained by removing one vertex from \( \text{Cay}(\mathbb{Z}_{17}, \{ \pm 1, \pm 6 \}) \) while maintaining the independence number. However, the values in Lemma 6 are enough to prove Lemmas 7 and 8. 

Lemma 7 Let \( G \) be the diameter graph of \( X \subset \mathbb{R}^3 \) with \( |X| = 20 \). If \( G \) is disconnected, then \( \alpha(G) \geq 8 \).

Proof Since \( G \) is disconnected, there exists a partition \( V = V_1 \cup V_2 \) such that \( v_1 \) and \( v_2 \) are not adjacent for any \( v_1 \in V_1 \) and \( v_2 \in V_2 \). We may assume \( |V_1| \geq 10 \). Let \( n_i = |V_i| \) and \( H_i \) be the induced subgraph of \( G \) with respect to \( V_i \) for \( i = 1, 2 \). Note that we may assume that both \( H_1 \) and \( H_2 \) contain neither a 3-cycle nor a 5-cycle by Corollary 1. If \( 10 \leq n_1 \leq 15 \), then \( \alpha(H_1) \geq f(n_1) \geq f(10) \geq 5 \) and \( \alpha(H_2) \geq f(n_2) \geq f(5) \geq 3 \) by Lemma 6. Then \( \alpha(G) = \alpha(H_1) + \alpha(H_2) \geq 5 + 3 = 8 \). If \( n_1 = 16 \), then \( \alpha(G) \geq f(16) + f(4) \geq f(15) + f(3) = 6 + 2 = 8 \). If \( 17 \leq n_1 \leq 19 \), then \( \alpha(G) \geq f(17) + f(1) \geq 7 + 1 = 8 \). Therefore \( \alpha(G) \geq 8 \). \( \square \)

Lemma 8 Let \( G \) be the diameter graph of \( X \subset \mathbb{R}^3 \) with \( |X| = 20 \). Then \( \alpha(G) \geq 8 \).

Proof Suppose that \( G \) contains a 3-cycle or a 5-cycle. Then \( \alpha(G) \geq 8 \) holds by Corollary 1. Therefore we may assume that \( G \) contains neither a 3-cycle nor a 5-cycle. Let \( v \in V(G) \). Since \( G \) contains neither a 3-cycle nor a 5-cycle, both \( \Gamma_1(v) \) and \( \Gamma_2(v) \cup \{v\} \) are independent sets. Therefore we may assume \( k_1(v) \leq 7 \) and \( k_2(v) \leq 6 \). In particular, we may assume \( k_3^s(v) = 20 - (1 + k_1(v) + k_2(v)) \geq 13 - k_1(v) \). Moreover, we may assume that \( G \) is connected by Lemma 7.

Suppose \( 5 \leq k_1(v) \leq 7 \) for some \( v \in V(G) \). Since \( k_3^s(v) \geq 13 \), \( k_1(v) \geq 6 \), we have \( \alpha(G^s(v)) \geq f(6) \geq f(5) = 3 \). Let \( H \) be an independent set of \( G^s(v) \) with \( |H| = 3 \). Then \( \Gamma_1(v) \cup H \) is an independent set of \( G \). Therefore we have \( \alpha(G) \geq k_1(v) + f(5) \geq 5 + 3 = 8 \). Suppose \( k_1(v) = 4 \) for some \( v \in V(G) \). Since
Let $X = \{p_1, p_2, \ldots, p_n\}$ be an $s$-distance set with $A(X) = \{x_1, x_2, \ldots, x_s\}$. Let $[n] = \{1, 2, \ldots, n\}$ and \(\binom{S}{k}\) = \(\{T \subset S \mid |T| = k\}\) for a finite set $S$. An $s$-distance set with $n$ points is represented by an edge coloring of the complete graph $K_n$ by $s$ colors. We regard an $s$-coloring of the edge set of $K_n$ by a surjection $c : \binom{[n]}{2} \rightarrow [s]$. We define an $s$-coloring $c : \binom{[n]}{2} \rightarrow [s]$ of an $s$-distance set $X$ by a natural manner, namely, $c(\{i, j\}) = k$ where $d(p_i, p_j) = x_k$. Conversely, an $s$-distance set $X$ is called a realization of $c$ if $c$ is a coloring of $X$.

Two $s$-colorings $c_1$ and $c_2$ are said to be equivalent if there exists bijections $g : [s] \rightarrow [s]$ and $h : [n] \rightarrow [n]$ such that $g(c_1(\{h(i), h(j)\})) = c_2(\{i, j\})$ for each $\{i, j\} \in \binom{[n]}{2}$. We define the coloring matrix $C = C(x_1, x_2, \ldots, x_s)$ of the coloring $c$ with respect to $x_1, x_2, \ldots, x_s$ by

$$C_{i,j} = \begin{cases} 0 & \text{if } i = j, \\ x_{c(\{i,j\})} & \text{if } i \neq j. \end{cases}$$

In particular, $C = C(1, 2, \ldots, s)$ is called a normal coloring matrix. A coloring $c$ is often represented as its normal coloring matrix $C$ in this paper. We distinguish them by lowercase letter $c$ and uppercase letter $C$.

For a subset $X = \{p_1, p_2, \ldots, p_n\} \subset \mathbb{R}^d$, we define the squared distance matrix $D = D(X)$ of $X$ by

$$D = (d(p_i, p_j)^2)_{1 \leq i,j \leq n}.$$ 

For an $n \times n$ symmetric matrix $M = (m_{i,j})_{1 \leq i,j \leq n}$, Gram$(M)$ is defined to be the $(n - 1) \times (n - 1)$ symmetric matrix with $(i, j)$ entries

$$m_{i,n} + m_{j,n} - m_{i,j}.$$ 

For $X = \{p_1, p_2, \ldots, p_n\}$, the matrix Gram$(D(X))$ is the Gram matrix of $X$ when $p_n$ is located at the origin.
Theorem 5 ([14, 20]) Let \( M \) be an \((n-1) \times (n-1)\) real symmetric matrix. There exists \( X \) in \( \mathbb{R}^d \) such that \( M \) is equal to the Gram matrix of \( X \) if and only if \( M = (m_{i,j})_{1 \leq i,j \leq n-1} \) satisfies

\[
\begin{cases}
    M \text{ is positive semidefinite}, \\
    \text{rank } M \leq d, \\
    m_{i,j} > 0 \text{ for every } i \in [n-1] \text{ and } \\
    m_{i,j} < \frac{m_{i,i} + m_{j,j}}{2} \text{ for every } 1 \leq i < j \leq n-1.
\end{cases}
\] (1)

An \( s \)-coloring \( c : \binom{[n]}{2} \rightarrow [s] \) is said to be representable in \( \mathbb{R}^d \) if there exists distinct real numbers \( x_1, x_2, \ldots, x_s \) such that \( C(x_1, x_2, \ldots, x_s) \) satisfies (1). To decide if an \( s \)-coloring are not representable, the rank condition in (1) is effective. An \( s \)-coloring \( c : \binom{[n]}{2} \rightarrow [s] \) is said to be quasi representable in \( \mathbb{R}^d \) if there exists distinct complex numbers \( x_1, x_2, \ldots, x_s \) such that \( \text{rank } C(x_1, x_2, \ldots, x_s) \leq d \).

For a square matrix \( M = (m_{i,j})_{1 \leq i,j \leq n} \) and an index set \( T = \{t_1, t_2, \ldots, t_k\} \in \binom{[n]}{k} \), we define a principal submatrix of \( M \) with respect to \( T \) by

\[
\text{sub}(M; T) = (m_{i,j})_{1 \leq i,j \leq k}.
\]

Let

\[
\mathcal{M}_k(M) = \left\{ \text{sub}(M; T) \mid T \in \binom{[n]}{k} \right\}.
\]

We define

\[
r(M) = \max\{k \in [n] \mid \exists S \in \mathcal{M}_k(M), \det S \neq 0\}.
\]

Proposition 1 For a square matrix \( M \), \( r(M) \leq \text{rank } M \) holds. Moreover, \( r(M) = \text{rank } M \) holds if \( M \) is positive semidefinite.

Proof It is well known that \( \text{rank } M \) is the maximum value \( k \) such that there exists a square submatrix \( S \) of size \( k \) in \( M \) with \( \det S \neq 0 \) that may not be principal. This implies \( r(M) \leq \text{rank } M \). Suppose \( M \) is a positive semidefinite matrix of size \( n \). Since \( M \) is positive semidefinite, there exists \( n \times \text{rank } M \) matrix \( N \) such that \( M = NN^\top \) and \( \text{rank } N = \text{rank } M \). For \( T \in \binom{[n]}{k} \), let \( \chi_T \) be the \( n \times n \) diagonal matrix with diagonal entries \( (\chi_T)_{ii} = 1 \) if \( i \in T \), and \( (\chi_T)_{ii} = 0 \) if \( i \notin T \). For a row vector \( x \in \mathbb{R}^n \)
and \( T \in \binom{[n]}{k} \), it follows that \( x(x_T N^T N_T x_T)^T x_T = 0 \) if and only if \( x(x_T N) = 0 \). Thus,
\[
\text{rank } (x_T N N^T x_T) = n - \dim \{ x \in \mathbb{R}^n \mid x(x_T N N^T x_T)^T x_T = 0 \} = n - \dim \{ x \in \mathbb{R}^n \mid x(x_T N) = 0 \} = \text{rank } (x_T N).
\]

For \( T \in \binom{[n]}{k} \), it follows that \( \det(\text{sub}(M; T)) \neq 0 \) if and only if \( \text{rank } (x_T N N^T x_T) \geq k \). This implies that \( r(M) \) is the maximum value \( k \) such that \( \text{rank } (x_T N) = k \), which is \( k = \text{rank } N = \text{rank } M \). □

An \( s \)-coloring \( c : \binom{[n]}{2} \to [s] \) is said to be weakly quasi representable in \( \mathbb{R}^d \) if there exist distinct complex numbers \( x_1, x_2, \ldots, x_s \) such that
\[
r(C(x_1, x_2, \ldots, x_s)) \leq d.
\]

Clearly if an \( s \)-coloring \( c \) is representable in \( \mathbb{R}^3 \), then \( c \) is (weakly) quasi representable in \( \mathbb{R}^3 \). The following proposition is essentially proved by Szőllősi and Östergård [20], but they should take all submatrices \( M \) that may not be principal in their result. Actually, it is enough to use all principal submatrices \( M \) to collect our desired colorings by Proposition 1.

**Proposition 2** An \( s \)-coloring \( c : \binom{[n]}{2} \to [s] \) is weakly quasi representable in \( \mathbb{R}^3 \) if and only if the following system of equations in \( s \) variables
\[
\left\{ \begin{aligned}
det M &= 0 \quad \text{for all } M \in \mathcal{M}_4(\text{Gram}(C(1, x_1, x_2, \ldots, x_{s-1}))), \\
1 + u \prod_{i=1}^{s-1} x_i (x_i - 1) \prod_{1 \leq j < k \leq s-1} (x_j - x_k) &= 0
\end{aligned} \right.
\] (2)
has a complex solution.

## 5 5-Distance Sets Containing 8-Point \( s \)-Distance Sets for \( s \leq 4 \)

By Theorem 3, to classify 20-point 5-distance sets in \( \mathbb{R}^3 \), it is enough to consider 5-distance sets which contain 8-point \( s \)-distance sets in \( \mathbb{R}^3 \) for \( s \leq 4 \). We will prove the following theorem in this section.

**Theorem 6** Let \( Y \) be an \( s \)-distance set in \( \mathbb{R}^3 \) with 8 points for \( 3 \leq s \leq 4 \). Let \( Z \) be a finite subset of \( \mathbb{R}^3 \). If \( Y \cup Z \) is a 5-distance set in \( \mathbb{R}^3 \) with at least 20 points, then \( Y \cup Z \) is isomorphic to a regular dodecahedron.

Note that there exists no 2-distance set with 8 points in \( \mathbb{R}^3 \). In this section, firstly, we consider (weakly) quasi representable \( s \)-colorings \( c \) in \( \mathbb{R}^3 \) instead of \( s \)-distance sets in \( \mathbb{R}^3 \). Then we consider realizations of \( c \) as needed.
Szöllősi and Östergård [20] classified quasi representable \( s \)-colorings in \( \mathbb{R}^3 \) for \( s \leq 4 \), see Table 2, where the values come from Table 6 in [20].

**Lemma 9** (Szöllősi and Östergård [20]) There exist exactly 19 quasi representable 3-colorings in \( \mathbb{R}^3 \) with 8 vertices and exactly 1074 quasi representable 4-colorings in \( \mathbb{R}^3 \) with 8 vertices.

We denote the set of all quasi representable \( s \)-colorings in \( \mathbb{R}^3 \) with \( n \) vertices by \( \mathcal{CG}(n, s) \). By Lemma 9, we have \( |\mathcal{CG}(8, 3)| = 19 \) and \( |\mathcal{CG}(8, 4)| = 1074 \).

Let \( C \in \mathcal{CG}(8, 3) \cup \mathcal{CG}(8, 4) \). We define a graph \( G(C) = (V, E) \) with respect to \( C \) as follows. By a computer search, we find all vectors \( (a_1, a_2, \ldots, a_8) \in [5]^8 \) such that

\[
M = \begin{pmatrix}
0 & a_1 & \cdots & a_8 \\
a_1 & & & \\
& \ddots & & \\
a_8 & & & C
\end{pmatrix}
\]

is a weakly quasi representable \( s \)-colorings in \( \mathbb{R}^3 \) for \( s \leq 5 \) by Proposition 2. In order to check whether \( M \) is weakly quasi representable, first we calculate a Gröbner basis \( \mathcal{B} \) of system (2) for \( C \), see [20] about the manner in details. Then, we calculate a Gröbner basis for the union of \( \mathcal{B} \) and the set of the first equations in (2) for all \( \text{sub}(M; T) \) with \( 1 \in T \), which determine whether \( M \) is weakly quasi representable. Throughout this paper, computer calculations are done with functions of the software Magma [3] and Maple [15]. The calculation of a Gröbner basis sometimes takes much computational cost. It takes at most 5 minutes by a usual PC (Intel Core i5-3470 CPU @ 3.2 GHz) to give all vectors satisfying (3) for almost \( C \in \mathcal{CG}(8, 3) \cup \mathcal{CG}(8, 4) \). For several special graphs, it takes much time but it is about 20 min.

We regard the set of all vectors satisfying (3) as the vertex set \( V \) of the graph \( G(C) \). The largest size of \( V \) is 368 among all \( C \in \mathcal{CG}(8, 3) \cup \mathcal{CG}(8, 4) \), and there are only 8 colorings in \( \mathcal{CG}(8, 3) \cup \mathcal{CG}(8, 4) \) such that \( |V| \geq 100 \). Two vertices \((a_1, a_2, \ldots, a_8), (a'_1, a'_2, \ldots, a'_8) \in V \) are adjacent if there exists \( i \in [5] \) such that

| \( n \) | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|
| \( \# \text{ QRC} \) | 512 | 62095 | 4499 | 1093 | 277 | 59 | 12 | 5 | 2 | 0 |

__Table 2__ Number of quasi representable at most 4-colorings in \( \mathbb{R}^3 \) of \( n \) points

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is a weakly quasi representable $s$-coloring in $\mathbb{R}^3$ for $s \leq 5$. Some special graphs have loops as Lemma 10 below. A weakly quasi representable coloring matrix $M'$ which contains $C$ can be written by

$$M' = \begin{pmatrix} B & A \\ A^\top & C \end{pmatrix}$$

and $A$ may have two identical rows. Thus, we should consider loops in the graph $G(C)$. For positive real numbers $a_1, a_2, \ldots, a_n$ and a subset $X = \{p_1, p_2, \ldots, p_n\} \subset \mathbb{R}^3$ which is not co-linear, there exist at most two points $q \in \mathbb{R}^3$ such that $d(p_i, q) = a_i$ for any $i \in [n]$. In particular, if there exist two points which satisfy this condition, then $X$ is co-planar. By exhaustive computer search, we have the following lemma.

**Lemma 10** Let $C \in CG(8, 3) \cup CG(8, 4)$. $G(C)$ has a loop if and only if a realization of $C$ is isomorphic to one of the following nine 4-distance sets.

(a) the subset with 8 points of a regular nonagon,
(b) a regular octagon,
(c) the six subsets with 8 points of the set in Fig. 1a in Theorem 1,
(d) the set in Fig. 1d in Theorem 1.

Moreover, there exist two loops only for (d) and there is only one loop for other cases.

For $C \in CG(8, 3) \cup CG(8, 4)$, let $\omega^*(C) = \omega(G(C)) + l(G(C))$, where $\omega(G(C))$ is the clique number of $G(C)$ and $l(G(C))$ is the number of loops in $G(C)$. The set of all rows of $A$ in (5) consists a clique in $G(C)$, and the number of rows that appear twice in $A$ is at most $l(G(C))$. Therefore, to prove Theorem 6, it is enough to classify $C \in CG(8, 3) \cup CG(8, 4)$ such that $\omega^*(C) \geq 12$. By exhaustive computer search, we have the following lemma.

**Lemma 11**

(i) There exists a unique coloring $C \in CG(8, 3)$ with $\omega^*(C) \geq 12$, which corresponds to the cube. Moreover, $\omega^*(C) = \omega(G(C)) = 12$ and there exists the unique clique of order 12 for the coloring.

(ii) There exist exactly 63 colorings $C \in CG(8, 4)$ with $\omega^*(C) \geq 12$. Moreover, $\omega^*(C) = \omega(G(C)) = 12$ and there exists the unique clique of order 12 for
each coloring among the 63 colorings.

We classify 8-point $s$-distance sets for $s \leq 4$ which are realizations of quasi representable $s$-colorings in Lemma 11. If a realization $X$ of a coloring $C$ is a subset of the regular dodecahedron, then we have another realization $\tilde{\Phi}(X)$ of $C$ by Lemma 1. The realizations $X$ and $\tilde{\Phi}(X)$ may be isomorphic.

Let $C \in \mathcal{C}G(8, 3)$ be the coloring in Lemma 11 (i) and $X$ be a realization of $C$. Then we have $A(X) = \{1, \sqrt{2}, \sqrt{3}\}$ by solving system (2). Then it is easy to see that $X$ is a cube.

Let

$$C_1 = \begin{pmatrix}
0 & 1 & 2 & 2 & 1 & 3 & 3 & 1 \\
1 & 0 & 1 & 2 & 2 & 3 & 3 & 2 \\
2 & 1 & 0 & 1 & 2 & 1 & 2 & 3 \\
2 & 2 & 1 & 0 & 1 & 2 & 1 & 3 \\
1 & 2 & 2 & 1 & 0 & 3 & 2 & 2 \\
3 & 2 & 1 & 2 & 3 & 0 & 2 & 4 \\
3 & 3 & 2 & 1 & 2 & 2 & 0 & 4 \\
1 & 2 & 3 & 3 & 2 & 4 & 4 & 0
\end{pmatrix}.$$  

Then $C_1 \in \mathcal{C}G(8, 4)$ is a coloring in Lemma 11 (ii). There exist four solutions of system (2) for $C_1$ and there exist exactly four realizations of $C_1$ up to isomorphism. Let

$$W = \{6, 12, 17, 18, 13, 16, 19, 1\}$$  

be a subset of the vertex set of the dodecahedron graph as given in Fig. 2. The shortest-path distance matrix $(d(x, y))_{x, y \in W}$ of $W$ is $C_1$, where $C_1$ is indexed by $W$ using the order of elements in (6). There are four realizations of $C_1$ as follows. Let $Y_1 = \{A_1, A_2, \ldots, A_7\}$, $Y_2 = \{B_1, B_2, \ldots, B_7\}$ and

$$X_i = Y_i \cup \{P_i\}, \quad X'_i = Y_i \cup \{P'_i\} \quad (i = 1, 2),$$

where $P_i$ is the reflection of $P_i$ in the plane $\pi_i$ for $i = 1, 2$, see Fig. 3. Then $X_1, X_2, X'_1$ and $X'_2$ are all the realizations of $C_1$. Both $X_1$ and $X_2$ are subsets of the regular dodecahedron and have the structure of the coloring $C_1$, that shows the situation of Lemma 1. There exist exactly two solutions of system (2) for the other 62 colorings $C \in \mathcal{C}G(8, 4)$.

If $C \in \mathcal{C}G(8, 4)$ is one of the ten colorings obtained from

\begin{align*}
\{2, 7, 9, 10, 13, 16, 17, 18\}, & \quad \{1, 2, 9, 11, 14, 16, 17, 18\}, \quad \{1, 8, 12, 15, 16, 17, 18, 19\}, \\
\{3, 4, 6, 12, 13, 16, 19, 20\}, & \quad \{3, 4, 6, 11, 12, 13, 16, 20\}, \quad \{6, 8, 10, 12, 14, 16, 19, 20\}, \\
\{3, 11, 13, 14, 15, 16, 19, 20\}, & \quad \{1, 12, 13, 15, 16, 17, 18, 19\}, \quad \{2, 9, 11, 13, 14, 15, 16, 20\}, \\
\{1, 12, 14, 15, 16, 17, 18, 19\}, & \quad \{1, 8, 12, 15, 16, 17, 18, 19\},
\end{align*}

by the above manner, then the two realizations of $C$ are isomorphic. Except for the
above 10 colorings and $C_1$, each $C \in \mathcal{G}(8, 4)$ in Lemma 11(ii) has exactly two realizations up to isomorphism. Then we have the following lemma.

**Lemma 12**

(i) **There exists a unique 3-distance set whose coloring is given in Lemma 11 (i).**

(ii) **Among the colorings in Lemma 11(ii), we have the following:**

(a) **There exists exactly one coloring which has four solutions of system (2) and the four realizations corresponding to the solutions are not isomorphic to each other.**

(b) **There exist exactly 52 colorings which have two solutions of system (2) and the two realizations corresponding to the solutions are not isomorphic.**

(c) **There exist exactly 10 colorings which have two solutions of system (2) but the two realizations corresponding to the solutions are isomorphic.**

By Lemma 12, there exist exactly 118 of 4-distance sets in $\mathbb{R}^3$ given from the colorings in Lemma 11. The two sets $X'_1$ and $X'_2$ are not subsets of the regular dodecahedron. By Lemma 3, the remaining 116 4-distance sets should be subsets of the regular dodecahedron. Note that the cube is also a subset of the regular dodecahedron. Let $S$ be the set of the cube and the 116 4-distance subsets. By Lemma 11, $\omega(G(C)) = 12$ for the coloring $C$ obtained from $X \in S$, and the corresponding clique of order 12 is unique. Therefore a 20-point 5-distance set that contains $X \in S$ must be the regular dodecahedron.
In order to prove Theorem 6, we prove that for \( i = 1, 2 \) there is no subset \( Z \subseteq \mathbb{R}^3 \) such that \( |A(X'_i \cup \{ P \})| \leq 5 \) for \( i = 1, 2 \) by using (3). The candidates are \( \{ Q'_1, R'_1, O_1 \} \) for \( X'_1 \) and \( \{ Q'_2, R'_2, O_2 \} \) for \( X'_2 \) in Fig. 3. The points \( Q'_i \) and \( R'_i \) are the reflections of \( Q_i \) and \( R_i \) in the plane \( p_i \), respectively. Moreover, \( O_1 \) (resp. \( O'_1 \)) is the center of the pentagon consisted of \( \{ A_1, A_2, \ldots, A_5 \} \) (resp. \( \{ B_1, B_2, \ldots, B_5 \} \)). Thus the cardinality of a 5-distance set that contains \( X'_i \) is at most 11. Therefore a proof of Theorem 6 is complete.

Finally, we prove Theorem 2.

**Proof of Theorem 2**  By Theorem 3 and \( g_3(2) = 6 \), every 5-distance set in \( \mathbb{R}^3 \) at least 20 points contains an 8-point \( s \)-distance set for some \( s \leq 4 \). Therefore the assertion follows by Theorem 6.

\[ \square \]

**Acknowledgements**  The authors thank Kenta Ozeki for providing information on the papers [11, 12] relating to graphs without two vertex-disjoint odd cycles. Nozaki is supported by JSPS KAKENHI Grant Numbers 19K03445 and 20K03527. Shinohara is supported by JSPS KAKENHI Grant Number 18K0339.6.

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