EINSTEIN MANIFOLDS WITH NONNEGATIVE
ISOTROPIC CURVATURE ARE LOCALLY SYMMETRIC

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1. Introduction

The study of Einstein manifolds has a long history in Riemannian geometry. An important problem, first studied by M. Berger [2], [3], is to classify all Einstein manifolds satisfying a suitable curvature condition. For example, if \((M, g)\) is a compact Einstein manifold of dimension \(n\) whose sectional curvatures lie in the interval \((\frac{3n}{n-1}, 1]\), then \((M, g)\) has constant sectional curvature (see [5], Section 0.33). A famous theorem of S. Tachibana [20] asserts that a compact Einstein manifold with positive curvature operator has constant sectional curvature. Moreover, Tachibana proved that a compact Einstein manifold with nonnegative curvature operator is locally symmetric. M. Gursky and C. LeBrun [11] have obtained interesting results on four-dimensional Einstein manifolds with nonnegative sectional curvature. Another result in this direction was established by D. Yang [21].

We now describe a curvature condition which was introduced by M. Micallef and J.D. Moore [15]. To that end, let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 4\). We say that \((M, g)\) has positive isotropic curvature if

\[
R(e_1, e_3, e_1, e_3) + R(e_1, e_4, e_1, e_4)
+ R(e_2, e_3, e_2, e_3) + R(e_2, e_4, e_2, e_4)
- 2R(e_1, e_2, e_3, e_4) > 0
\]

for all orthonormal four-frames \(\{e_1, e_2, e_3, e_4\} \subset T_pM\). Moreover, we say that \((M, g)\) has nonnegative isotropic curvature if

\[
R(e_1, e_3, e_1, e_3) + R(e_1, e_4, e_1, e_4)
+ R(e_2, e_3, e_2, e_3) + R(e_2, e_4, e_2, e_4)
- 2R(e_1, e_2, e_3, e_4) \geq 0
\]

for all orthonormal four-frames \(\{e_1, e_2, e_3, e_4\} \subset T_pM\). It was shown in [6] that positive isotropic curvature is preserved by the Ricci flow in all dimensions (see also [17]). This fact plays a central role in the proof of the Differentiable Sphere Theorem (cf. [6], [7], [8]).

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M. Micallef and M. Wang showed that a four-dimensional Einstein manifold with nonnegative isotropic curvature is locally symmetric (see [16], Theorem 4.4). In this paper, we extend the results of Micallef and Wang to higher dimensions:

**Theorem 1.** Let $(M, g)$ be a compact Einstein manifold of dimension $n \geq 4$. If $(M, g)$ has positive isotropic curvature, then $(M, g)$ has constant sectional curvature. Moreover, if $(M, g)$ has nonnegative isotropic curvature, then $(M, g)$ is locally symmetric.

We note that H. Seshadri [18] has obtained an interesting partial classification of manifolds with nonnegative isotropic curvature.

We now give an outline of the proof of Theorem 1. Let $(M, g)$ be a compact Einstein manifold with nonnegative isotropic curvature. Moreover, suppose that $(M, g)$ is not locally symmetric. After passing to the universal cover if necessary, we may assume that $M$ is simply connected. We now consider the holonomy group of $(M, g)$.

If $\text{Hol}(M, g) = SO(n)$, then $(M, g)$ has positive isotropic curvature. We then show that $(M, g)$ has constant sectional curvature. The proof uses the maximum principle, as well as an algebraic inequality established in [6].

If $n = 2m \geq 4$ and $\text{Hol}(M, g) = U(m)$, then $(M, g)$ is a Kähler-Einstein manifold with positive orthogonal bisectional curvature. It then follows from work of S. Goldberg and S. Kobayashi [10] that $(M, g)$ is isometric to $\mathbb{CP}^m$ up to scaling.

If $n = 4m \geq 8$ and $\text{Hol}(M, g) = \text{Sp}(m)\cdot\text{Sp}(1)$, then $(M, g)$ is a quaternionic-Kähler manifold. By a theorem of Alekseevskii (cf. [5], Section 14.41), the curvature tensor of $(M, g)$ can be written in the form $R = R_1 + \kappa R_0$, where $R_1$ has the algebraic properties of a hyper-Kähler curvature tensor, $R_0$ is the curvature tensor of $\mathbb{HP}^m$, and $\kappa$ is a constant. Since $(M, g)$ has nonnegative isotropic curvature, we have $R_1(X, JX, X, JX) < \kappa$ for all points $p \in M$ and all unit vectors $X \in T_pM$. Using the maximum principle, we are able to show that $R_1(X, JX, X, JX) \leq 0$ for all points $p \in M$ and all unit vectors $X \in T_pM$. From this, we deduce that $R_1$ vanishes identically. Consequently, the manifold $(M, g)$ is isometric to $\mathbb{HP}^m$ up to scaling. From this, the assertion follows.

M. Berger [4] has shown that every quaternionic-Kähler manifold with positive sectional curvature is isometric to $\mathbb{HP}^m$ up to scaling. C. LeBrun and S. Salamon [14] have conjectured that a quaternionic-Kähler manifold $(M, g)$ with positive scalar curvature is necessarily locally symmetric. The results in this paper imply that no counterexample to the LeBrun-Salamon conjecture can have nonnegative isotropic curvature.

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2. Preliminary results

Let $V$ be a finite-dimensional vector space equipped with an inner product. An algebraic curvature tensor on $V$ is a multi-linear form $R : V \times V \times V \times V \rightarrow \mathbb{R}$ satisfying

$$R(X, Y, Z, W) = -R(Y, X, Z, W) = R(Z, W, X, Y)$$

and

$$R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0$$

for all vectors $X, Y, Z, W \in V$.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $V$. Moreover, suppose that $R$ and $S$ are two algebraic curvature tensors on $V$. We define an algebraic curvature tensor $B(R, S)$ on $V$ by

$$B(R, S)(X, Y, Z, W) = \frac{1}{2} \sum_{p, q=1}^{n} \left[ R(X, Y, e_p, e_q) S(Z, W, e_p, e_q) + R(Z, W, e_p, e_q) S(X, Y, e_p, e_q) \right]$$

$$+ \sum_{p, q=1}^{n} \left[ R(X, e_p, Z, e_q) S(Y, e_p, W, e_q) + R(Y, e_p, W, e_q) S(X, e_p, Z, e_q) \right]$$

$$- \sum_{p, q=1}^{n} \left[ R(X, e_p, W, e_q) S(Y, e_p, Z, e_q) + R(Y, e_p, Z, e_q) S(X, e_p, W, e_q) \right]$$

for all vectors $X, Y, Z, W \in V$. Finally, for each algebraic curvature tensor $R$, we define $Q(R) = B(R, R)$.

The following result is purely algebraic:

**Proposition 2.** Let $V$ be a vector space of dimension $n \geq 4$ which is equipped with an inner product. Let $R$ be an algebraic curvature tensor on $V$ with nonnegative isotropic curvature. Finally, suppose that $\{e_1, e_2, e_3, e_4\}$ is an orthonormal four-frame in $V$ satisfying

$$R(e_1, e_3, e_1, e_3) + R(e_1, e_4, e_1, e_4)$$

$$+ R(e_2, e_3, e_2, e_3) + R(e_2, e_4, e_2, e_4)$$

$$- 2 R(e_1, e_2, e_3, e_4) = 0.$$ 

Then

$$Q(R)(e_1, e_3, e_1, e_3) + Q(R)(e_1, e_4, e_1, e_4)$$

$$+ Q(R)(e_2, e_3, e_2, e_3) + Q(R)(e_2, e_4, e_2, e_4)$$

$$- 2 Q(R)(e_1, e_2, e_3, e_4) \geq 0.$$ 

**Proof.** This was shown in [6] (see Corollary 10 in that paper).

The term $Q(R)$ arises naturally in the evolution equation for the curvature tensor under Ricci flow (cf. [12], [13]). In the special case of Einstein manifolds, we have the following well-known result:
Proposition 3. Let \((M, g)\) be a Riemannian manifold with \(\text{Ric}_g = \rho g\). Then the Riemann curvature tensor of \((M, g)\) satisfies
\[
\Delta R + Q(R) = 2\rho R.
\]

Proof. It follows from Lemma 7.2 in [12] that
\[
(\Delta R)(X, Y, Z, W) + Q(R)(X, Y, Z, W) = (D^2_{X, Y} \text{Ric})(Z, W) - (D^2_{X, Z} \text{Ric})(Y, W)
- (D^2_{Y, W} \text{Ric})(X, Z) - (D^2_{Y, Z} \text{Ric})(X, W)
+ \sum_{k=1}^n \text{Ric}(X, e_k) R(e_k, Y, Z, W) + \sum_{k=1}^n \text{Ric}(Y, e_k) R(X, e_k, Z, W)
\]
for all vector fields \(X, Y, Z, W\). Since \(\text{Ric}_g = \rho g\), we conclude that
\[
(\Delta R)(X, Y, Z, W) + Q(R)(X, Y, Z, W) = 2\rho R(X, Y, Z, W),
\]
as claimed.

Finally, we shall need the following result:

Proposition 4. Let \((M, g)\) be a compact Einstein manifold of dimension \(n \geq 4\) with nonnegative isotropic curvature. Then the set of all orthonormal four-frames \(\{e_1, e_2, e_3, e_4\}\) satisfying
\[
R(e_1, e_3, e_1, e_3) + R(e_1, e_4, e_1, e_4)
+ R(e_2, e_3, e_2, e_3) + R(e_2, e_4, e_2, e_4)
- 2R(e_1, e_2, e_3, e_4) = 0
\]
is invariant under parallel transport.

Proof. Since \((M, g)\) is an Einstein manifold, we have \(\text{Ric}_g = \rho g\) for some constant \(\rho\). Consequently, the metrics \((1 - 2\rho t)g\) form a solution to the Ricci flow with nonnegative isotropic curvature. Hence, the assertion follows from Proposition 8 in [7].

3. Kähler-Einstein manifolds

Let \((M, g)\) be a compact, simply connected Riemannian manifold of dimension \(2m \geq 4\) with holonomy group \(\text{Hol}(M, g) = U(m)\). Then \((M, g)\) is a Kähler manifold. The following theorem was established by S. Goldberg and S. Kobayashi:

Theorem 5 (S. Goldberg and S. Kobayashi [10]). Assume that \((M, g)\) is Einstein. Moreover, suppose that \((M, g)\) has positive orthogonal bisectional curvature; that is,
\[
R(X, JX, Y, JY) > 0
\]
for all points \(p \in M\) and all unit vectors \(X, Y \in T_p M\) satisfying \(g(X, Y) = g(JX, Y) = 0\). Then \((M, g)\) has constant holomorphic sectional curvature.
In [10], this result is stated under the stronger assumption that \((M, g)\) has positive holomorphic bisectional curvature (see [10], Theorem 5). However, the proof in [10] only uses the condition that \((M, g)\) has positive orthogonal bisectional curvature.

The following result is a consequence of Proposition 4 (see also [18]):

**Proposition 6.** Assume that \((M, g)\) is Einstein. If \((M, g)\) has nonnegative isotropic curvature, then \((M, g)\) has positive orthogonal bisectional curvature.

**Proof.** Consider two unit vectors \(X, Y \in T_p M\) satisfying \(g(X, Y) = g(JX, JY) = 0\). Then

\[
\begin{align*}
R(X, Y, X, Y) + & R(X, JY, X, JY) \\
+ & R(JX, Y, JX, Y) + R(JX, JY, JX, JY) \\
= & 2 R(X, JX, JY, JY).
\end{align*}
\]

Since \((M, g)\) has nonnegative isotropic curvature, it follows that

\[
R(X, JX, JY, JY) \geq 0.
\]

It remains to show that \(R(X, JX, JY, JY) \neq 0\). To prove this, we argue by contradiction. Suppose that \(R(X, JX, JY, JY) = 0\). This implies that the four-frame \(\{X, JX, Y, -JY\}\) has zero isotropic curvature. Let us fix a point \(q \in M\) and two unit vectors \(Z, W \in T_q M\) satisfying \(g(Z, W) = g(JZ, W) = 0\). We claim that

\[
R(Z, JZ, W, JW) = 0.
\]

Since \(\text{Hol}(M, g) = U(m)\), we can find a piecewise smooth path \(\gamma : [0, 1] \to M\) such that \(\gamma(0) = p, \gamma(1) = q, P_p X = Z, \) and \(P_p Y = W\). By Proposition 4, the four-frame \(\{P_p X, P_p JX, P_p Y, -P_p JY\}\) has zero isotropic curvature. Consequently, the four-frame \(\{Z, JZ, W, -JW\}\) has zero isotropic curvature. Thus, we conclude that \(R(Z, JZ, W, JW) = 0\), as claimed.

In the next step, we apply the identity (1) to the vectors \(\frac{1}{\sqrt{2}}(Z + W)\) and \(\frac{1}{\sqrt{2}}(Z - W)\). This yields

\[
0 = R(Z + W, JZ + JW, Z - W, JZ - JW) \\
= R(Z, JZ, Z, JZ) + R(W, JW, W, JW) \\
+ 2 R(Z, JZ, W, JW) - 4 R(Z, JW, Z, JW).
\]

Similarly, if we apply the identity (1) to the vectors \(\frac{1}{\sqrt{2}}(Z + JW)\) and \(\frac{1}{\sqrt{2}}(Z - JW)\), then we obtain

\[
0 = R(Z + JW, JZ - W, Z - JW, JZ + W) \\
= R(Z, JZ, Z, JZ) + R(W, JW, W, JW) \\
+ 2 R(Z, JZ, W, JW) - 4 R(Z, W, Z, W).
\]
We now take the arithmetic mean of (2) and (3). This implies

\[(4) \quad R(Z, JZ, Z, JZ) + R(W, JW, W, JW) = 0\]

for all unit vectors \(Z, W \in T_pM\) satisfying \(g(Z, W) = g(JZ, W) = 0\).

It follows from (1) and (4) that the scalar curvature of \((M, g)\) is equal to zero. Since \((M, g)\) has nonnegative isotropic curvature, Proposition 2.5 in [16] implies that the Weyl tensor of \((M, g)\) vanishes. Consequently, \((M, g)\) is flat. This is a contradiction.

Combining Theorem 5 and Proposition 6, we can draw the following conclusion:

**Corollary 7.** Assume that \((M, g)\) is Einstein. If \((M, g)\) has nonnegative isotropic curvature, then \((M, g)\) has constant holomorphic sectional curvature.

### 4. Quaternionic-Kähler manifolds

Throughout this section, we will assume that \((M, g)\) is a compact, simply connected Riemannian manifold of dimension \(4m \geq 8\) with holonomy group \(\text{Hol}(M, g) = \text{Sp}(m) \cdot \text{Sp}(1)\). These assumptions imply that \((M, g)\) is a quaternionic-Kähler manifold. Hence, there exists a subbundle \(G \subset \text{End}(TM)\) of rank 3 with the following properties:

- \(G\) is invariant under parallel transport.
- Given any point \(p \in M\), we can find linear transformations \(I, J, K \in \text{End}(T_pM)\) such that \(I^2 = J^2 = K^2 = IJK = -\text{id}\),

\[g(X, Y) = g(I X, I Y) = g(J X, J Y) = g(K X, K Y)\]

for all vectors \(X, Y \in T_pM\), and

\[G_p = \{aI + bJ + cK \in \text{End}(T_pM) : a, b, c \in \mathbb{R}\}.

For each point \(p \in M\), we define

\[J_p = \{aI + bJ + cK \in \text{End}(T_pM) : a, b, c \in \mathbb{R}, a^2 + b^2 + c^2 = 1\}.

Note that \(J_p \subset G_p\) is a sphere of radius \(\sqrt{4m}\) centered at the origin. In particular, \(J_p\) is independent of the particular choice of \(I, J, K\).

By a theorem of D. Alekseevskii (see [5], Section 14.41), the curvature tensor of \((M, g)\) can be written in the form \(R = R_1 + \kappa R_0\) for some constant \(\kappa\). Here, \(R_1\) is a hyper-Kähler curvature tensor; that is,

\[R_1(X, Y, Z, W) = R_1(X, Y, IZ, IW) = R_1(X, Y, JZ, JW) = R_1(X, Y, KZ, KW)\]
for all vectors $X, Y, Z, W \in T_p M$. Moreover, $R_0$ is defined by

$$4 R_0(X, Y, Z, W) = g(X, Z) g(Y, W) - g(X, W) g(Y, Z)$$

$$+ 2 g(I X, Y) g(I Z, W) + g(I X, Z) g(I Y, W) - g(I X, W) g(I Y, Z)$$

$$+ 2 g(J X, Y) g(J Z, W) + g(J X, Z) g(J Y, W) - g(J X, W) g(J Y, Z)$$

$$+ 2 g(K X, Y) g(K Z, W) + g(K X, Z) g(K Y, W) - g(K X, W) g(K Y, Z)$$

for all vectors $X, Y, Z, W \in T_p M$. Note that this definition is independent of the particular choice of $I, J, K$.

In the next step, we show that $Q(R) = Q(R_1) + \kappa^2 Q(R_0)$. In order to prove this, we need two lemmata:

**Lemma 8.** Fix a point $p \in M$. Let us define an algebraic curvature tensor $S$ on $T_p M$ by

$$S(X, Y, Z, W) = g(X, Z) g(Y, W) - g(X, W) g(Y, Z)$$

for all vectors $X, Y, Z, W \in T_p M$. Then $B(R_1, S) = 0$.

**Proof.** Let $\{e_1, \ldots, e_{4m}\}$ be an orthonormal basis of $T_p M$. Since the Ricci tensor of $R_1$ vanishes, we have

$$\sum_{p,q=1}^{4m} R_1(X, Y, e_p, e_q) S(Z, W, e_p, e_q) = 2 R_1(X, Y, Z, W)$$

and

$$\sum_{p,q=1}^{4m} R_1(X, e_p, Z, e_q) S(Y, e_p, W, e_q) = -R_1(X, W, Z, Y)$$

for all vectors $X, Y, Z, W \in T_p M$. Using the first Bianchi identity, we obtain

$$B(R_1, S)(X, Y, Z, W) = R_1(X, Y, Z, W) + R_1(Z, W, X, Y)$$

$$- R_1(X, W, Z, Y) - R_1(Y, Z, W, X)$$

$$+ R_1(X, Z, W, Y) + R_1(Y, W, Z, X)$$

$$= 0$$

for all vectors $X, Y, Z, W \in T_p M$. This completes the proof.

**Lemma 9.** Fix a point $p \in M$ and an almost complex structure $J \in \mathcal{J}_p$. Let us define an algebraic curvature tensor $S$ on $T_p M$ by

$$S(X, Y, Z, W) = 2 g(J X, Y) g(J Z, W)$$

$$+ g(J X, Z) g(J Y, W) - g(J X, W) g(J Y, Z)$$

for all vectors $X, Y, Z, W \in T_p M$. Then $B(R_1, S) = 0$. 
Proof. Let \( \{e_1, \ldots, e_{4m}\} \) be an orthonormal basis of \( T_p M \). Since \( R_1 \) is a hyper-Kähler curvature tensor, we have
\[
\sum_{p,q=1}^{4m} R_1(X, Y, e_p, e_q) S(Z, W, e_p, e_q) = 2 R_1(X, Y, Z, W)
\]
and
\[
\sum_{p,q=1}^{4m} R_1(X, e_p, Z, e_q) S(Y, e_p, W, e_q) = 2 R_1(X, J Y, Z, J W) + R_1(X, J W, Z, J Y)
\]
for all vectors \( X, Y, Z, W \in T_p M \). This implies
\[
B(R_1, S)(X, Y, Z, W) = R_1(X, Y, Z, W) + R_1(Z, W, X, Y)
\]
\[
+ 2 R_1(X, J Y, Z, J W) + R_1(X, J W, Z, J Y)
\]
\[
+ 2 R_1(Y, J X, W, J Z) + R_1(Y, J Z, W, J X)
\]
\[
- 2 R_1(X, J Y, W, J Z) - R_1(X, J Z, W, J Y)
\]
\[
- 2 R_1(Y, J X, Z, J W) - R_1(Y, J W, Z, J X)
\]
for all vectors \( X, Y, Z, W \in T_p M \). Using the first Bianchi identity, we obtain
\[
B(R_1, S)(X, Y, Z, W) = 2 R_1(X, Y, J Z, J W) + 2 R_1(X, J W, Y, J Z)
\]
\[
- 2 R_1(X, J Y, W, J Z) - 2 R_1(X, J Z, W, J Y)
\]
\[
= 0
\]
for all vectors \( X, Y, Z, W \in T_p M \). From this, the assertion follows.

Proposition 10. We have \( Q(R) = Q(R_1) + \kappa^2 Q(R_0) \).

Proof. Fix a point \( p \in M \). Moreover, let \( I, J, K \in J_p \) be three almost complex structures satisfying \( IJK = -\text{id} \). We define

\[
S_0(X, Y, Z, W) = g(X, Z) g(Y, W) - g(X, W) g(Y, Z),
\]
\[
S_1(X, Y, Z, W) = 2 g(I X, Y) g(I Z, W)
\]
\[
+ g(I X, Z) g(I Y, W) - g(I X, W) g(I Y, Z),
\]
\[
S_2(X, Y, Z, W) = 2 g(J X, Y) g(J Z, W)
\]
\[
+ g(J X, Z) g(J Y, W) - g(J X, W) g(J Y, Z),
\]
\[
S_3(X, Y, Z, W) = 2 g(K X, Y) g(K Z, W)
\]
\[
+ g(K X, Z) g(K Y, W) - g(K X, W) g(K Y, Z)
\]
for all vectors \( X, Y, Z, W \in T_p M \). It follows from Lemma 8 and Lemma 9 that
\[
B(R_1, S_0) = B(R_1, S_1) = B(R_1, S_2) = B(R_1, S_3) = 0.
\]
Since $S_0 + S_1 + S_2 + S_3 = 4R_0$, we conclude that $B(R_1, R_0) = 0$. This implies

$$Q(R) = Q(R_1) + 2\kappa B(R_1, R_0) + \kappa^2 Q(R_0) = Q(R_1) + \kappa^2 Q(R_0),$$

as claimed.

**Proposition 11.** Fix a point $p \in M$ and an almost complex structure $J \in J_p$. Moreover, let $\{e_1, \ldots, e_{4m}\}$ be an orthonormal basis of $T_pM$. Then

$$Q(R_1)(X, JX, X, JX) \leq -2 R_1(X, JX, X, JX)^2 + 2 \sum_{p,q=1}^{4m} R_1(X, JX, e_p, e_q)^2$$

for every unit vector $X \in T_pM$.

**Proof.** By definition of $Q(R_1)$, we have

$$Q(R_1)(X, JX, X, JX) = \sum_{p,q=1}^{4m} R_1(X, JX, e_p, e_q)^2$$

$$+ 2 \sum_{p,q=1}^{4m} R_1(X, e_p, X, e_q) R_1(JX, e_p, JX, e_q)$$

$$- 2 \sum_{p,q=1}^{4m} R_1(X, e_p, JX, e_q) R_1(JX, e_p, X, e_q).$$

From this, we deduce that

$$Q(R_1)(X, JX, X, JX) = \sum_{p,q=1}^{4m} R_1(X, JX, e_p, e_q)^2$$

$$- 4 \sum_{p,q=1}^{4m} R_1(X, e_p, JX, e_q) R_1(JX, e_p, X, e_q).$$

The expression on the right-hand side is independent of the choice of the orthonormal basis $\{e_1, \ldots, e_{4m}\}$. Hence, we may assume without loss of
generality that \(e_1 = X\) and \(e_2 = JX\). This implies

\[
-4 \sum_{p,q=1}^{4m} R_1(X, e_p, JX, e_q) R_1(JX, e_p, X, e_q)
\]

\[
= -4 \sum_{p,q=3}^{4m} R_1(X, e_p, JX, e_q) R_1(JX, e_p, X, e_q)
\]

\[
\leq \sum_{p,q=3}^{4m} (R_1(X, e_p, JX, e_q) - R_1(JX, e_p, X, e_q))^2
\]

\[
= \sum_{p,q=3}^{4m} R_1(X, JX, e_p, e_q)^2
\]

\[
\leq -2 R_1(X, JX, X, JX)^2 + \sum_{p,q=1}^{4m} R_1(X, JX, e_p, e_q)^2.
\]

Putting these facts together, the assertion follows.

**Lemma 12.** Fix a point \(p \in M\) and an almost complex structure \(J \in J_p\).
Suppose that \(X \in T_p M\) is a unit vector with the property that \(R_1(X, JX, X, JX)\)
is maximal. Moreover, let \(Y \in T_p M\) be a unit vector satisfying \(g(X, Y) = g(JX, Y) = 0\).
Then

\[
R_1(X, JX, X, Y) = R_1(X, JX, X, JY) = 0
\]

and

\[
2 R_1(X, JY, X, JY) \leq R_1(X, JX, X, JX).
\]

**Proof.** Since \(R_1(X, JX, X, JX)\) is maximal, we have

\[
(1 + s^2)^{-2} R_1(X + sY, JX + sJY, X + sY, JX + sJY) \leq R_1(X, JX, X, JX)
\]

for all \(s \in \mathbb{R}\). Consequently, we have

\[
\frac{d}{ds} \left( (1 + s^2)^{-2} R_1(X + sY, JX + sJY, X + sY, JX + sJY) \right)_{s=0} = 0
\]

and

\[
\frac{d^2}{ds^2} \left( (1 + s^2)^{-2} R_1(X + sY, JX + sJY, X + sY, JX + sJY) \right)_{s=0} \leq 0.
\]

This implies

\[
R_1(X, JX, X, JY) = 0
\]

and

\[
2 R_1(X, JY, X, JY) \leq R_1(X, JX, X, JX) - R_1(X, JX, Y, JY).
\]

Replacing \(Y\) by \(JY\) yields

\[
R_1(X, JX, X, Y) = 0
\]
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and

\[ 2 R_1(X, Y, X, Y) \leq R_1(X, JX, X, JX) - R_1(X, JX, Y, JY). \]

Putting these facts together, we obtain

\[ R_1(X, JX, Y, JY) = R_1(X, Y, X, Y) + R_1(X, JY, X, JY) \]
\[ \leq R_1(X, JX, X, JX) - R_1(X, JX, Y, JY). \]

From this, the assertion follows.

**Theorem 13.** Assume that \( R_1(X, JX, X, JX) < \kappa \) for every point \( p \in M \), every almost complex structure \( J \in J_p \), and every unit vector \( X \in T_p M \). Then \( R_1 \) vanishes identically.

**Proof.** Note that \( R_1 \) is a hyper-Kähler curvature tensor. Therefore, the Ricci tensor of \( R_1 \) is equal to 0. Using the identity \( R = R_1 + \kappa R_0 \), we obtain

\[ \text{Ric}_g = (m + 2)\kappa g. \]

Hence, Proposition 3 implies that

\[ \Delta R_1 + Q(R) = (2m + 4)\kappa R. \]

By compactness, we can find a point \( p \in M \), an almost complex structure \( J \in J_p \), and a unit vector \( X \in T_p M \) such that \( R_1(X, JX, X, JX) \) is maximal. This implies

\[ (D^2_{v,v} R_1)(X, JX, X, JX) \leq 0 \]

for all vectors \( v \in T_p M \). Taking the trace over \( v \in T_p M \) yields

\[ (\Delta R_1)(X, JX, X, JX) \leq 0. \]

Putting these facts together, we conclude that

\[ Q(R_1)(X, JX, X, JX) \geq (2m + 4)\kappa R_1(X, JX, X, JX). \]

We now analyze the term \( Q(R_1)(X, JX, X, JX) \). For abbreviation, let \( w_1 = X \) and \( w_2 = IX \). We can find vectors \( w_3, \ldots, w_{2m} \in T_p M \) such that \( \{w_1, Jw_1, w_2, Jw_2, \ldots, w_{2m}, Jw_{2m}\} \) is an orthonormal basis of \( T_p M \) and

\[ R_1(X, JX, w_\alpha, w_\beta) = R_1(X, JX, w_\alpha, Jw_\beta) = 0 \]

for \( 3 \leq \alpha < \beta \leq 2m \). It follows from Lemma 12 that

\[ R_1(X, JX, w_\beta) = R_1(X, JX, Jw_\beta) = 0 \]

for \( 2 \leq \beta \leq 2m \). Moreover, we have

\[ R_1(X, JX, X, w_\beta) = R_1(X, JX, X, JIw_\beta) = 0 \]

for \( 3 \leq \beta \leq 2m \). This implies

\[ R_1(X, JX, IX, w_\beta) = R_1(X, JX, IX, Jw_\beta) = 0. \]
for $3 \leq \beta \leq 2m$. Putting these facts together, we conclude that

\[(6) \quad R_1(X, JX, w_\alpha, w_\beta) = R_1(X, JX, w_\alpha, Jw_\beta) = 0\]

for $1 \leq \alpha < \beta \leq 2m$.

Using Lemma 12, we obtain

\[2 R_1(X, JX, w_\alpha, Jw_\alpha) \leq R_1(X, JX, X, JX)\]

and

\[2 R_1(X, JX, Iw_\alpha, JIw_\alpha) \leq R_1(X, JX, X, JX)\]

for $3 \leq \alpha \leq 2m$. The latter inequality implies that

\[-2 R_1(X, JX, w_\alpha, Jw_\alpha) \leq R_1(X, JX, X, JX)\]

for $3 \leq \alpha \leq 2m$. Thus, we conclude that

\[(7) \quad 4 R_1(X, JX, w_\alpha, Jw_\alpha)^2 \leq R_1(X, JX, X, JX)^2\]

for $3 \leq \alpha \leq 2m$.

By Proposition 11, we have

\[Q(R_1)(X, JX, X, JX) \leq -2 R_1(X, JX, X, JX)^2 + \sum_{\alpha, \beta=1}^{2m} R_1(X, JX, w_\alpha, w_\beta)^2 + 4 \sum_{\alpha, \beta=1}^{2m} R_1(X, JX, w_\alpha, Jw_\beta)^2.\]

Using (6) and (7), we obtain

\[Q(R_1)(X, JX, X, JX) \leq -2 R_1(X, JX, X, JX)^2 + 4 \sum_{\alpha=1}^{2m} R_1(X, JX, w_\alpha, Jw_\alpha)^2 + 4 \sum_{\alpha=1}^{2m} R_1(X, JX, w_\alpha, Jw_\alpha)^2 = 6 R_1(X, JX, X, JX)^2 + 4 \sum_{\alpha=3}^{2m} R_1(X, JX, w_\alpha, Jw_\alpha)^2 \leq (2m + 4) R_1(X, JX, X, JX)^2.\]

Combining (5) and (8), we conclude that

\[\kappa R_1(X, JX, X, JX) \leq R_1(X, JX, X, JX)^2.\]

Since $R_1(X, JX, X, JX) < \kappa$, it follows that $R_1(X, JX, X, JX) \leq 0$. Therefore, $R_1$ has nonpositive holomorphic sectional curvature. Since the scalar curvature of $R_1$ is equal to 0, we conclude that $R_1$ vanishes identically.

**Proposition 14.** Assume that $(M, g)$ has nonnegative isotropic curvature. Then $R_1(X, JX, X, JX) < \kappa$ for every point $p \in M$, every almost complex structure $J \in \mathcal{J}_p$, and every unit vector $X \in T_p M$. 

Proof. Fix a point \( p \in M \) and a unit vector \( X \in T_p M \). Moreover, let \( I, J, K \in J_p \) be three almost complex structures satisfying \( IJK = -\text{id} \). For abbreviation, we put \( Y = IX \). Then

\[
R_1(X, Y, X, Y) + R_1(X, JY, X, JY) + R_1(JX, Y, JX, Y) + R_1(JX, JY, JX, JY) = 2 R_1(X, JX, Y, JY).
\]

Moreover, we have

\[
R_0(X, Y, X, Y) = R_0(X, JY, X, JY) = 1, \\
R_0(JX, Y, JX, Y) = R_0(JX, JY, JX, JY) = 1, \\
R_0(X, JX, Y, JY) = 0
\]

by definition of \( R_0 \). Using the identity \( R = R_1 + \kappa R_0 \), we obtain

\[
R(X, Y, X, Y) + R(X, JY, X, JY) + R(JX, Y, JX, Y) + R(JX, JY, JX, JY) + 2 R(X, JX, Y, JY) = 4 (\kappa + R_1(X, JX, Y, JY)) = 4 (\kappa - R_1(X, JX, X, JX)).
\]

Since \((M, g)\) has nonnegative isotropic curvature, it follows that

\[
R_1(X, JX, X, JX) \leq \kappa.
\]

It remains to show that \( R_1(X, JX, X, JX) \neq \kappa \). To prove this, we argue by contradiction. Suppose that \( R_1(X, JX, X, JX) = \kappa \). This implies that the four-frame \( \{X, JX, Y, -JY\} \) has zero isotropic curvature. Given any unit vector \( Z \in T_p M \), we can find a linear isometry \( L : T_p M \to T_p M \) which commutes with \( I, J, K \) and satisfies \( LX = Z \). Since \( \text{Hol}(M, g) = \text{Sp}(m) \cdot \text{Sp}(1) \), there exists a piecewise smooth path \( \gamma : [0, 1] \to M \) such that \( \gamma(0) = \gamma(1) = p \) and \( P_\gamma = L \). By Proposition 4, the four-frame \( \{P_\gamma X, P_\gamma JX, P_\gamma Y, -P_\gamma JY\} \) has zero isotropic curvature. Hence, if we put \( W = IZ \), then the four-frame \( \{Z, JZ, W, -JW\} \) has zero isotropic curvature. Consequently, we have

\[
R_1(Z, JZ, Z, JZ) = \kappa
\]

for all unit vectors \( Z \in T_p M \). Since \( R_1 \) is a hyper-Kähler curvature tensor, we conclude that \( \kappa = 0 \). Hence, Proposition 2.5 in [16] implies that \((M, g)\) is flat. This is a contradiction.

Corollary 15. If \((M, g)\) has nonnegative isotropic curvature, then \( R_1 \) vanishes identically.
5. Proof of the main theorem

In this section, we show that every Einstein manifold with nonnegative isotropic curvature is locally symmetric. To that end, we need the following result:

**Theorem 16.** Let \((M, g)\) be a compact Einstein manifold of dimension \(n \geq 4\). If \((M, g)\) has positive isotropic curvature, then \((M, g)\) has constant sectional curvature.

**Proof.** After rescaling the metric if necessary, we may assume that \(\text{Ric}_g = (n-1) g\). Using Proposition 3, we obtain
\[
\Delta R + Q(R) = 2(n-1) R.
\]

We now define
\[
S_{ijkl} = R_{ijkl} - \kappa (g_{ik} g_{jl} - g_{il} g_{jk}),
\]
where \(\kappa\) is a positive constant. Note that \(S\) is an algebraic curvature tensor. Let \(\kappa\) be the largest constant with the property that \(S\) has nonnegative isotropic curvature. Then there exists a point \(p \in M\) and a four-frame \(\{e_1, e_2, e_3, e_4\} \subset T_p M\) such that
\[
S(e_1, e_3, e_1, e_3) + S(e_1, e_4, e_1, e_4)
+ S(e_2, e_3, e_2, e_3) + S(e_2, e_4, e_2, e_4)
- 2 S(e_1, e_2, e_3, e_4) = 0.
\]

Hence, it follows from Proposition 2 that
\[
Q(S)(e_1, e_3, e_1, e_3) + Q(S)(e_1, e_4, e_1, e_4)
+ Q(S)(e_2, e_3, e_2, e_3) + Q(S)(e_2, e_4, e_2, e_4)
- 2 Q(S)(e_1, e_2, e_3, e_4) \geq 0.
\]

We next observe that
\[
Q(S)_{ijkl} = Q(R)_{ijkl} + 2(n-1) \kappa^2 (g_{ik} g_{jl} - g_{il} g_{jk})
- 2 \kappa (\text{Ric}_{ik} g_{jl} - \text{Ric}_{il} g_{jk} - \text{Ric}_{jk} g_{il} + \text{Ric}_{jl} g_{ik}),
\]
hence
\[
Q(S)_{ijkl} = Q(R)_{ijkl} + 2(n-1) \kappa (\kappa - 2) (g_{ik} g_{jl} - g_{il} g_{jk}).
\]
Substituting this into (9), we obtain
\[
Q(R)(e_1, e_3, e_1, e_3) + Q(R)(e_1, e_4, e_1, e_4)
+ Q(R)(e_2, e_3, e_2, e_3) + Q(R)(e_2, e_4, e_2, e_4)
- 2 Q(R)(e_1, e_2, e_3, e_4) + 8(n-1) \kappa (\kappa - 2) \geq 0.
\]
On the other hand, we have a fold of dimension 1. Since Proposition 17, \(\text{isotropic curvature}\), then \(\text{vanishes. From this, the assertion follows.}\)

\[
\begin{align*}
(D_{v,v}^2 R)(e_1, e_3, e_4) + (D_{v,v}^2 R)(e_1, e_4, e_3) & \\
+ (D_{v,v}^2 R)(e_2, e_3, e_4) + (D_{v,v}^2 R)(e_2, e_4, e_3) & \\
- 2(D_{v,v}^2 R)(e_1, e_2, e_3, e_4) & \geq 0
\end{align*}
\]

for all vectors \(v \in T_p M\). Taking the trace over \(v \in T_p M\) yields

\[
(\Delta R)(e_1, e_3, e_4) + (\Delta R)(e_1, e_4, e_3)
+ (\Delta R)(e_2, e_3, e_4) + (\Delta R)(e_2, e_4, e_3)
- 2(\Delta R)(e_1, e_2, e_3, e_4) \geq 0.
\]

We now add (10) and (11) and divide the result by \(2(n - 1)\). This implies

\[
\begin{align*}
R(e_1, e_3, e_4) + R(e_1, e_4, e_3) & \\
+ R(e_2, e_3, e_4) + R(e_2, e_4, e_3) & \\
- 2R(e_1, e_2, e_3, e_4) + 4\kappa(\kappa - 2) & \geq 0.
\end{align*}
\]

On the other hand, we have

\[
\begin{align*}
R(e_1, e_3, e_3) + R(e_1, e_4, e_4) & \\
+ R(e_2, e_3, e_3) + R(e_2, e_4, e_4) & \\
- 2R(e_1, e_2, e_3, e_4) - 4\kappa & = 0.
\end{align*}
\]

Since \(\kappa\) is positive, it follows that \(\kappa \geq 1\). Therefore, \(S\) has nonnegative isotropic curvature and nonpositive scalar curvature. By Proposition 2.5 in [16], the Weyl tensor of \(S\) vanishes. From this, the assertion follows.

**Proposition 17.** Let \((M, g)\) be a compact, simply connected Einstein manifold of dimension \(n \geq 4\) with \(\text{Hol}(M, g) = SO(n)\). If \((M, g)\) has nonnegative isotropic curvature, then \((M, g)\) has constant sectional curvature.

**Proof.** Suppose that \((M, g)\) does not have constant sectional curvature. By Theorem [16] there exists a point \(p \in M\) and an orthonormal four-frame \(\{e_1, e_2, e_3, e_4\} \subset T_p M\) such that

\[
\begin{align*}
R(e_1, e_3, e_3) + R(e_1, e_4, e_4) & \\
+ R(e_2, e_3, e_3) + R(e_2, e_4, e_4) & \\
- 2R(e_1, e_2, e_3, e_4) & = 0.
\end{align*}
\]

By assumption, the Weyl tensor of \((M, g)\) does not vanish identically. Hence, we can find a point \(q \in M\) and an orthonormal four-frame \(\{v_1, v_2, v_3, v_4\} \subset T_q M\) such that \(R(v_1, v_2, v_3, v_4) \neq 0\). Since \(\text{Hol}(M, g) = SO(n)\), there exists a piecewise smooth path \(\gamma: [0, 1] \to M\) such that \(\gamma(0) = p, \gamma(1) = q\), and

\[
v_1 = P_\gamma e_1, \quad v_2 = P_\gamma e_2, \quad v_3 = P_\gamma e_3, \quad v_4 = \pm P_\gamma e_4.
\]
Without loss of generality, we may assume that $v_4 = P_4 e_4$. (Otherwise, we replace $v_4$ by $-v_4$.) It follows from Proposition 4 that

$$R(v_1, v_3, v_1, v_3) + R(v_1, v_4, v_1, v_4) + R(v_2, v_3, v_2, v_3) + R(v_2, v_4, v_2, v_4) - 2R(v_1, v_2, v_3, v_4) = 0.$$  

(12)

Using analogous arguments, we obtain

$$R(v_1, v_4, v_1, v_4) + R(v_1, v_2, v_1, v_2) + R(v_3, v_4, v_3, v_4) + R(v_3, v_2, v_3, v_2) - 2R(v_1, v_3, v_4, v_2) = 0$$  

(13)

and

$$R(v_1, v_2, v_1, v_2) + R(v_1, v_3, v_1, v_3) + R(v_4, v_2, v_4, v_2) + R(v_4, v_3, v_4, v_3) - 2R(v_1, v_4, v_2, v_3) = 0.$$  

(14)

Since $(M, g)$ has nonnegative isotropic curvature, it follows that

$$R(v_1, v_2, v_3, v_4) \geq 0,$$

$$R(v_1, v_3, v_4, v_2) \geq 0,$$

$$R(v_1, v_4, v_2, v_3) \geq 0.$$  

Using the first Bianchi identity, we conclude that $R(v_1, v_2, v_3, v_4) = 0$. This is a contradiction.

**Proposition 18.** Let $(M, g)$ be a compact, simply connected Einstein manifold of dimension $n \geq 4$ with nonnegative isotropic curvature. Moreover, suppose that $(M, g)$ is irreducible. Then $(M, g)$ is isometric to a symmetric space.

**Proof.** Suppose that $(M, g)$ is not isometric to a symmetric space. By Berger’s holonomy theorem (see e.g. [5], Corollary 10.92), there are four possibilities:

**Case 1:** $\text{Hol}(M, g) = SO(n)$. In this case, Proposition 17 implies that $(M, g)$ has constant sectional curvature. This contradicts the fact that $(M, g)$ is non-symmetric.

**Case 2:** $n = 2m$ and $\text{Hol}(M, g) = U(m)$. In this case, $(M, g)$ is a Kähler manifold. Moreover, by Corollary 7, $(M, g)$ has constant holomorphic sectional curvature. Consequently, $(M, g)$ is isometric to a symmetric space, contrary to our assumption.

**Case 3:** $n = 4m \geq 8$ and $\text{Hol}(M, g) = \text{Sp}(m) \cdot \text{Sp}(1)$. In this case, $(M, g)$ is a quaternionic-Kähler manifold. Moreover, it follows from Corollary 15 that $(M, g)$ is symmetric. This is a contradiction.

**Case 4:** $n = 16$ and $\text{Hol}(M, g) = \text{Spin}(9)$. In this case, a theorem of D. Alekseevskii implies that $(M, g)$ is isometric to a symmetric space (see
Theorem 19. Let \((M, g)\) be a compact Einstein manifold of dimension \(n \geq 4\) with nonnegative isotropic curvature. Then \((M, g)\) is locally symmetric.

Proof. We first consider the case that \((M, g)\) is Ricci flat. In this case, Proposition 2.5 in [16] implies that the Weyl tensor of \((M, g)\) vanishes. Consequently, \((M, g)\) is flat.

It remains to consider the case that \((M, g)\) has positive Einstein constant. By a theorem of DeRham (cf. [5], Theorem 10.43), the universal cover of \((M, g)\) is isometric to a product of the form \(N_1 \times \ldots \times N_j\), where \(N_1, \ldots, N_j\) are compact, simply connected, and irreducible. Since \((M, g)\) is an Einstein manifold, it follows that the factors \(N_1, \ldots, N_j\) are Einstein manifolds. Since \((M, g)\) has positive Einstein constant, the manifolds \(N_1, \ldots, N_j\) are compact by Myers' theorem. By Proposition 18, each of the factors \(N_1, \ldots, N_j\) is isometric to a symmetric space. Consequently, \((M, g)\) is locally symmetric.

We conclude this paper with an analysis of the borderline case in the Micallef-Moore theorem. This result follows from Corollary 15 and results established in [7].

Theorem 20. Let \((M, g_0)\) be a compact, simply connected Riemannian manifold of dimension \(n \geq 4\) which is irreducible and has nonnegative isotropic curvature. Then one of the following statements holds:

(i) \(M\) is homeomorphic to \(S^n\).
(ii) \(n = 2m\) and \((M, g_0)\) is a Kähler manifold.
(iii) \((M, g_0)\) is isometric to a symmetric space.

Proof. Suppose that \((M, g_0)\) is not isometric to a symmetric space. Let \(g(t), t \in [0, T]\), the unique solution of the Ricci flow with initial metric \(g_0\). By continuity, we can find a real number \(\delta \in (0, T)\) such that \((M, g(t))\) is irreducible and non-symmetric for all \(t \in (0, \delta)\). According to Berger’s holonomy theorem (cf. [5], Corollary 10.92), there are four possibilities:

Case 1: There exists a real number \(\tau \in (0, \delta)\) such that \(\text{Hol}(M, g(\tau)) = SO(n)\). In this case, Proposition 8 in [7] implies that \((M, g(\tau))\) has positive isotropic curvature. By a theorem of Micallef and Moore [15], \(M\) is homeomorphic to \(S^n\).

Case 2: \(n = 2m\) and \(\text{Hol}(M, g(t)) = U(m)\) for all \(t \in (0, \delta)\). In this case, \((M, g(t))\) is a Kähler manifold for all \(t \in (0, \delta)\). Since \(g(t) \to g_0\) in \(C^\infty\), it follows that \((M, g_0)\) is a Kähler manifold.

Case 3: \(n = 4m \geq 8\) and \(\text{Hol}(M, g(\tau)) = \text{Sp}(m) \cdot \text{Sp}(1)\) for some real number \(\tau \in (0, \delta)\). In this case, \((M, g(\tau))\) is a quaternionic-Kähler manifold. By Corollary 15 \((M, g(\tau))\) is isometric to a symmetric space. This is a contradiction.

Case 4: \(n = 16\) and \(\text{Hol}(M, g(\tau)) = \text{Spin}(9)\) for some real number \(\tau \in (0, \delta)\). By Alekseevskii’s theorem, \((M, g(\tau))\) is isometric to a symmetric
space (see [1], Corollary 1, or [9], Theorem 8.1). This contradicts the fact that \((M, g(\tau))\) is non-symmetric.

It is possible to strengthen the conclusion in statement (ii) of Theorem 20. To that end, we consider a compact, simply connected Kähler manifold which is irreducible and has nonnegative isotropic curvature. By a result of Seshadri, any such manifold is biholomorphic to complex projective space or isometric to a symmetric space (cf. [18], Theorem 1.2; see also [19]).

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