AN INTRODUCTION TO HIGGS BUNDLES VIA HARMONIC MAPS

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Abstract. These notes are an extended version of lecture notes prepared for the 3-hour mini-course “An introduction to cyclic Higgs bundles and complex variation of Hodge structures” that the author gave at University of Illinois at Chicago.

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1. Preliminaries and non-abelian Hodge correspondence

Let S be a closed orientable surface of genus g \( \geq 2 \).

Definition 1.1. A connection on a complex vector bundle E over S is a differential operator

\[ D : \Omega^k(S, E) \to \Omega^{k+1}(S, E) \]

satisfying the Leibniz rule: if \( \alpha \in \Omega^p(S, \mathbb{C}) \), \( \sigma \in \Omega^k(S, E) \),

\[ D(\alpha \wedge \sigma) = d\alpha \wedge \sigma + (-1)^p \alpha \wedge \nabla \sigma. \]

Definition 1.2. The curvature of a connection D on E is the operator

\[ F_D = D \circ D : \Omega^k(S, E) \to \Omega^{k+2}(S, E). \]

Fact: \( F_D \) turns out to be \( C^\infty \)-linear, i.e., \( f_D \in \Omega^2(S, \text{End}(E)) \).

Call a connection D flat if \( F_D = 0 \). The holonomy of a flat connection D gives rise to a representation \( \rho : \pi_1(S) \to GL(n, \mathbb{C}) \) and \( (E, D) \simeq (\tilde{S} \times_\rho \mathbb{C}^n, \text{natural connection descends from } d \text{ on } \tilde{S} \times \mathbb{C}^n) \).

\[ \text{References} \]

1 The author acknowledges “The UIC NSF RTG grant DMS-1246844” and L.P. Schaposnik’s UIC Start up fund.
Definition 1.3. (1) A connection is called irreducible if there exists no proper \(D\)-invariant subbundle. (2) A connection is called reductive if it is a direct sum of irreducible connections.

Definition 1.4. A Hermitian metric on a complex vector bundle \(E\) over \(S\) is a family of Hermitian inner products on \(E\), which is \(\mathbb{C}\)-linear in the second variable and conjugate-linear in the first variable.

Definition 1.5. A connection \(D\) is called \(H\)-unitary if for any two sections \(s, t \in \Omega^0(M, E)\),
\[
d(H(s, t)) = H(Ds, t) + H(s, Dt).
\]

Equip \(S\) with a complex structure \(J\), we obtain a Riemann surface \(\Sigma\) and denote by \(K\) its holomorphic cotangent bundle.

Definition 1.6. A holomorphic structure on a complex vector bundle \(E\) over \(\Sigma\) is a differential operator \(\bar{\partial}_E : \Omega^{p,q}(\Sigma, E) \to \Omega^{p,q+1}(\Sigma, E)\) satisfying the Leibniz rule: if \(\alpha \in \Omega^{p,q}(\Sigma, \mathbb{C})\), \(\sigma \in \Omega^{k,l}(\Sigma, E)\),
\[
\bar{\partial}_E(\alpha \wedge \sigma) = (\bar{\partial}\alpha) \wedge \sigma + (-1)^{p+q}\alpha \wedge \bar{\partial}_E\sigma.
\]

Call a section \(\sigma\) of \(E\) holomorphic if \(\bar{\partial}_E\sigma = 0\).

Given any connection \(D\) on \(E\), \(D^{0,1}\) gives a holomorphic structure on \(E\). But given a holomorphic structure on \(E\), there are many connections \(D\) such that \(D^{0,1} = \bar{\partial}_E\).

Theorem 1.7. For a holomorphic vector bundle \(E\) with a Hermitian metric \(H\), there exists a unique connection \(D\), called the Chern connection, such that
(1) \(D^{0,1} = \bar{\partial}_E\);
(2) \(D\) is \(H\)-unitary.

Question 1.8. Suppose we are only given with a flat connection \(D\) on \(E\) over \(\Sigma\), how do we obtain more geometric structures?

For any Hermitian metric \(H\) on \(E\), take \(\Psi_H \in \Omega^1(\Sigma, \text{End}(E))\) such that
\[
H(\Psi_H s, t) = \frac{1}{2}(H(Ds, t) + H(s, Dt) - d(H(s, t))).
\]

So we have

Lemma 1.9. Any connection \(D\) on a Hermitian bundle \((E, H)\) decomposes uniquely as
\[
D = D_H + \Psi_H,
\]

where
(1) \(D_H\) is a \(H\)-unitary connection;
(2) \(\Psi_H \in \Omega^1(\Sigma, \text{End}(E))\) is \(H\)-Hermitian, i.e., \(H(\Psi_H s, t) = H(s, \Psi_H t)\).

We aim to choose a “best” \(H\).

There is a natural Hermitian metric on \(\text{End}(E)\) induced from the Hermitian metric \(H\) on \(E\). Fix any background conformal metric \(g_0\) on \(\Sigma\), it induces a natural pairing on \(\Omega^1(\Sigma, \mathbb{C})\). Therefore we have the pairing \(\langle \Psi_H, \Psi_H \rangle_{g_0}\).

Definition 1.10. A Hermitian metric \(H\) is called harmonic on \((E, D)\) if it is a critical point of the functional \(E(H) = \int_S \langle \Psi_H, \Psi_H \rangle_{g_0} \text{vol}_{g_0}\). Equivalently, \(D_H(\ast \Psi_H) = 0\), where \(\ast\) is the Hodge star.

Theorem 1.11. (Corlette \[6\], Donaldson \[10\]) If \(D\) is an irreducible flat connection, then there exists a unique (up to a scalar multiple) harmonic metric \(H\) on \(E\).
Why use the name “harmonic”? Let us recall the definition of equivariant harmonic maps.

Given a $\pi_1 M$-equivariant map $f : \tilde{M}_1 \to M_2$ between two Riemannian manifolds, then $df \in \Omega^0(\tilde{M}_1, T^* \tilde{M}_1 \otimes f^{-1} TM_2)$ is also $\pi_1 M_1$-equivariant. This implies that

$$ e(f) = \langle df, df >_{T^* \tilde{M}_1 \otimes f^{-1} TM_2} : \tilde{M}_1 \to \mathbb{R} $$

is a $\pi_1 M_1$-invariant function and hence descends to $M_1$. Call $e(f)$ the energy density on $\tilde{M}_1$ and also on $M_1$. The energy $E(f)$ is the integral of $e(f)$ with respect to the volume form of $M_1$.

**Definition 1.12.** The map $f$ is harmonic if it is a critical point of the energy functional $E(f)$.

In dimension 2, by the definition of energy, it only depends on the conformal class of the metric on the domain. So is the harmonic map.

Step 1: The space of Hermitian metrics on $\mathbb{C}^n$ is

$$ N = \{ A \in M_n(\mathbb{C}) | A^t = A, A > 0 \}, $$

the space of positive definite Hermitian matrices. A Hermitian metric on $E = \tilde{S} \times_{\rho} \mathbb{C}^n$ is an equivariant metric $H$ on $\tilde{S} \times \mathbb{C}^n$. Equivalently, it is a map $f : \tilde{S} \to N \subset M_n(\mathbb{C})$ satisfying

$$ f(m) = \rho(\gamma) f(\gamma \cdot m) \rho(\gamma), \quad \forall \gamma \in \pi_1 \tilde{S}, m \in \tilde{S} $$

by $H_m(s, t) = \bar{s}^t f(m)t$, for any two sections $s, t$ of $\tilde{S} \times \mathbb{C}^d$.

Step 2: The open subset $N \subset Herm(\mathbb{C}^n)$ has a structure of a Riemannian manifold. If $X, Y \in T_A N = Herm(\mathbb{C}^n)$,

$$ < X, Y >_A = \text{tr}(A^{-1} X A^{-1} Y). $$

This is the unique $GL(n, \mathbb{C})$-invariant Riemannian metric on $N$ up to scale.

Together with $\Psi_H = -\frac{1}{2} f^{-1} df$ (see Lemma 1.13 and use the same background metric $(S, g_0)$ (conformal to $\Sigma$), we obtain

$$ \frac{1}{4} < df, df > = < \Psi_H, \Psi_H >. $$

We finally see that $H$ being harmonic (minimizing the functional $E(H)$) is equivalent to $f : (\tilde{S}, \bar{g}_0) \to N$ being harmonic (minimizing the energy of $f$).

**Lemma 1.13.**

$$ \Psi_H = -\frac{1}{2} f^{-1} df. $$

The following proof is taken from the lecture note of O. Guichard [11].

**Proof.** Firstly, $f^{-1} df \in \Omega^1(\tilde{S}, End(\mathbb{C}^n))$ is equivariant under the $\pi_1 S$-action. Hence it descends to $S$ and “belongs” to $\Omega^1(S, End(E))$.

For $s, t$ two sections of $\tilde{S} \times \mathbb{C}^n$, by definition, $H_m(s, t) = \bar{s}^t f(m)t$. We obtain

1. $d(H(s, t)) = H(D_H s, t) + H(s, D_H t)$;
2. $d(H(s, t)) = d(\bar{s}^t f t) = d\bar{s}^t \cdot f \cdot t + \bar{s}^t \cdot df \cdot t + \bar{s}^t \cdot f \cdot dt = H(ds, t) + \bar{s}^t \cdot df \cdot t + H(s, dt)$.
3. $d = D_H + \Psi_H$ (lifted version).

Combining (1)(2)(3), we get

$$ H(\Psi_H s, t) + \bar{s}^t df t + H(s, \Psi_H t) = 0 $$

and then $\Psi_H = -\frac{1}{2} f^{-1} df$. \qed

Let’s go one step further.

**Definition 1.14.** A Higgs bundle on $\Sigma$ is a pair $(E, \phi)$ where $E$ is a holomorphic vector bundle and $\phi \in H^0(\Sigma, End(E) \otimes K)$. A $SL(n, \mathbb{C})$-Higgs bundle is a Higgs bundle $(E, \phi)$ satisfying $\det E = \mathcal{O}$ and $tr \phi = 0$. 

3
1.1. From flat bundles to Higgs bundles. Given a harmonic metric $H$ on $(E, D)$, we have

$$D = D_H + \Psi_H \text{ unitary + Herm}$$

$$= D_H^{1,0} + D_H^{0,1} + \Psi_H^{1,0} + \Psi_H^{0,1} \text{ by type of form}$$

In fact, the pair $(D_H^{1,0}, \Psi_H^{1,0})$ obtained from this way is a Higgs bundle. This is because

1. $D_H(*\Psi_H) = 0$ (harmonicity) and
2. $F_D = 0$ (flatness) implies

$$F_D = D_H^2 + \Psi_H \wedge \Psi_H = 0$$

One can check that (1) and (2b) together imply $(D_H)^{0,1}\Psi_H^{1,0} = 0$. Therefore, we may rephrase the theorem of Corlette ad Donaldson as follows.

**Theorem 1.15.** Given $D$ a flat irreducible connection, there exists a unique (up to scalar) Hermitian metric $H$ such that $(E, D_H^{1,0}, \Psi_H^{1,0})$ is a Higgs bundle.

In fact, the above Higgs bundle is stable.

**Definition 1.16.** (1) A Higgs bundle $(E, \phi)$ is stable if for any proper $\phi$-invariant holomorphic subbundle $F$, we have

$$\text{slope}(F) = \frac{\deg F}{\text{rank} F} < \text{slope}(E) = \frac{\deg E}{\text{rank} E}.$$  

(2) A Higgs bundle $(E, \phi)$ is polystable if it is a direct sum of stable Higgs bundles of the same slope.

1.2. From Higgs bundles to flat bundles.

**Theorem 1.17.** (Hitchin [12], Simpson [27]) Let $(E, \bar{\partial}_E, \phi)$ be a stable Higgs bundle of degree 0, then there exists a unique (up to scalar) Hermitian metric $H$ on $E$, called the harmonic metric, such that

$$D = D_H + \phi + \phi^*H$$

is flat, where $D_H$ is the Chern connection uniquely determined by $H$ and $\bar{\partial}_E$ and $\phi^*H$ is the Hermitian adjoint of $\phi$.

$D$ being flat is equivalent to the Hitchin equation

$$F_{D_H} + [\phi, \phi^*H] = 0.$$  

In fact, the connection $D$ here is irreducible.

**Definition 1.18.** (1) The moduli space $\mathcal{M}_{Higgs}$ of $SL(n, \mathbb{C})$-Higgs bundles is the space of gauge equivalent classes of polystable $SL(n, \mathbb{C})$-Higgs bundles.

(2) The moduli space $\mathcal{M}_{flat}(SL(n, \mathbb{C}))$ of flat connections is the space of gauge equivalent classes of reductive flat $SL(n, \mathbb{C})$-connections.

(3) The representation variety $\text{Rep}(\pi_1 S, SL(n, \mathbb{C}))$ is the space of conjugation classes of reductive representations from $\pi_1(S)$ to $SL(n, \mathbb{C})$.

We obtain a 1-1 correspondence

$$N\mathcal{A}H_{\Sigma}: \mathcal{M}_{Higgs}(SL(n, \mathbb{C})) \rightarrow \mathcal{M}_{flat}(SL(n, \mathbb{C})) \cong \text{Rep}(\pi_1 S, SL(n, \mathbb{C}))$$

$$(E, \phi) \rightarrow D \rightarrow \text{the holonomy of } D.$$  

This is called the non-abelian Hodge correspondence.

**Remark 1.19.** One can generalize the non-abelian Hodge correspondence to general reductive Lie groups $G$. In later sections, we’ll directly mention $G$-Higgs bundle without more explanation.
2. Hitchin fibration and $\mathbb{C}^*$-action

There are two important concepts in the moduli space of $SL(n, \mathbb{C})$-Higgs bundles.

2.1. Hitchin fibration and Hitchin section. Given a basis of $SL(n, \mathbb{C})$-invariant homogeneous polynomials $p_i$ of degree $i$ over $s(l(n, \mathbb{C}))$, $2 \leq i \leq n$. The Hitchin fibration is a map from the moduli space of $SL(n, \mathbb{C})$-Higgs bundles over $\Sigma$ to the direct sum of the holomorphic differentials

$$h : M_{\text{Higgs}} \longrightarrow \bigoplus_{j=2}^{n} H^0(\Sigma, K^j) \ni (q_2, q_3, \ldots, q_n).$$

$$(E, \phi) \longmapsto (p_2(\phi), \ldots, p_n(\phi))$$

Note that $p_2(\phi)$ is always a constant multiple of $tr(\phi^2)$.

By choosing an appropriate basis of polynomials, the Hitchin section $s$ of the Hitchin fibration can be defined explicitly as follows. Define

$$s(q_2, q_3, \ldots, q_n) = (E = K^{n-1} \oplus K^{n-2} \oplus \cdots \oplus K^{1-n}, \phi = \begin{pmatrix} 0 & q_2 & q_3 & \cdots & q_n \\ r_1 & 0 & q_2 & \cdots & q_{n-1} \\ r_2 & 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & q_2 & \vdots \\ r_{n-1} & 0 & \cdots & 0 & r_{n-1} \end{pmatrix}),$$

where $r_i = \frac{i(2-n)}{2}$.

Hitchin \cite{13} shows that the Higgs bundles in the image of Hitchin section have holonomy in $SL(n, \mathbb{R})$. Moreover, it is a connected component of the moduli space of $SL(n, \mathbb{R})$-Higgs bundles, called Hitchin component and denoted by $H\text{it}_n$. In later sections, we will use Hitchin component referring to both Higgs bundles and corresponding representations.

Note that the image $s(q_2, 0, \ldots, 0)$ correspond to the Teichmüller locus. Each representation corresponding to $s(q_2, 0, \ldots, 0)$ for some $q_2$ is a Fuchsian representation post composing the unique irreducible representation from $SL(2, \mathbb{R})$ to $SL(n, \mathbb{R})$, called an $n$-Fuchsian representation.

2.2. $\mathbb{C}^*$-action. There is a natural $\mathbb{C}^*$-action on the moduli space of $SL(n, \mathbb{C})$-Higgs bundles:

$$\mathbb{C}^* \times M_{\text{Higgs}} \longrightarrow M_{\text{Higgs}} \quad t \cdot [(E, \phi)] = [(E, t\phi)].$$

3. Cyclic Higgs bundles

Definition 3.1. A cyclic Higgs bundle $(E, \phi)$ over $\Sigma$ is a $SL(n, \mathbb{C})$-Higgs bundle of the form

$$E = L_1 \oplus L_2 \oplus \cdots \oplus L_n, \quad \phi = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{n-1} \end{pmatrix},$$

where $L_i$ are holomorphic line bundles, $\gamma_i \in H^0(\Sigma, L_i^{-1}L_{i+1}K)$ and for $1 \leq i \leq n-1$, $\gamma_i \neq 0$.

A cyclic Higgs bundle in the Hitchin component is of the form

$$E = K^{n-1} \oplus K^{n-2} \oplus \cdots \oplus K^{1-n}, \quad \phi = \begin{pmatrix} r_1 & q_2 & q_3 & \cdots & q_n \\ r_2 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & q_2 & \vdots \\ r_{n-1} & 0 & \cdots & 0 & r_{n-1} \end{pmatrix}.$$
When is a Higgs bundle stable? A sufficient condition of \((E, \phi)\) being stable is that \(\lim_{t \to 0} [t \cdot (E, \phi)]\) is stable. Note that it is not a necessary condition.

Two facts about stability:
1. \(\mathbb{C}^*\)-action preserves stability.
2. Stability is an open condition.

When is a cyclic Higgs bundle stable?

**Proposition 3.2.** Suppose a cyclic Higgs bundle \((E, \phi)\) of the form \(\text{diag}\) satisfies \(\sum_{i=1}^{k} \deg L_{n+1-i} < 0\) for each \(1 \leq k \leq n-1\), then \((E, \phi)\) is stable.

What is good about cyclic Higgs bundles?

**Lemma 3.3.** *(Baraglia [2])* Stable cyclic Higgs bundles have diagonal harmonic metrics.

**Proof.** Consider the gauge transformation \(g = \text{diag}(1, \omega, \cdots, \omega^{n-1})\), where \(\omega = e^{\frac{2\pi}{n}}\). Since

\[
g\phi g^{-1} = \left(\begin{array}{cccc}
\omega \gamma_1 & & & \\
& \omega \gamma_2 & & \\
& & \ddots & \\
& & & \omega \gamma_{n-1}
\end{array}\right) = \omega \cdot \phi,
\]

we have \(g \cdot (E, \phi) = (E, \omega \cdot \phi)\). If \(H\) solves the Hitchin equation of \((E, \phi)\), then \(g \cdot H = (\tilde{g})^{-1} H g^{-1}\) solves the Hitchin equation of \(g \cdot (E, \phi)\). We can see that \(H\) also solves the Hitchin equation of \((E, \omega \phi)\). By the uniqueness of harmonic metrics, \(g \cdot H = H\) and \((\tilde{g})^{-1} H g^{-1} = H\). Hence \(H\) is diagonal. \(\square\)

### 4. Some calculations

#### 4.1. Rewrite the Hitchin equation.

Consider a local coordinate chart \(U \subset \Sigma\) where \(E\) has a local trivialization. With respect to a local holomorphic frame \(e = (e_1, e_2, \ldots, e_n)\) on \(U\), \(D_H = \partial_H + \bar{\partial}\). Denote \(H_{ij} = H(e_i, e_j)\). For two sections \(s = e \cdot \xi, t = e \cdot \eta\), where \(\xi, \eta \in \Omega^0(U, \mathbb{C}^n)\), \(H(s, t) = \xi^i H_{\eta^i}\).

- **The operator \(\partial_H\)?** Assume \(\partial_H e = e \cdot A, A \in \Omega^{1,0}(U, \text{End}(\mathbb{C}^n))\). Note that

\[
\partial(H(s, t)) = H(\bar{\partial}_H s, t) + H(s, \bar{\partial}_H t).
\]

Since the frame \(e\) is holomorphic, we have

\[
\partial(\xi^i H_{\eta^i}) = H(e \cdot \bar{\partial}_H \xi, e \cdot \eta) + H(e \cdot \xi, \partial_H e \cdot \eta + e \cdot \partial \eta) \\
\implies \partial(\xi^i \cdot H \cdot \eta + \xi^i \cdot \partial H \cdot \eta + \xi^i \cdot H \cdot \partial \eta = \bar{\partial} \xi^i \cdot H \cdot \eta + \xi^i \cdot H \cdot \partial \eta + \xi^i \cdot H \cdot \partial \eta.
\]

This implies that \(\partial H = H \cdot A\) and \(A = H^{-1} \partial H\).

- **Curvature \(F_{DH}\)?** Locally

\[
F_{DH} = D^H \circ D^H = (d + A) \circ (d + A) = dA + A \wedge A.
\]

Since \(A = H^{-1} \partial H \in \Omega^{1,0}(U, \text{End}(\mathbb{C}^n))\), \(F_{DH} = \bar{\partial}(H^{-1} \partial H)\).

- **The term \([\phi, \phi^{*H}]\)?** It is \(\phi \wedge \phi^{*H} + \phi^{*H} \wedge \phi\). Note that by definition, \(\phi^{*H}\) is such that \(H(\phi s, t) = H(s, \phi^{*H} t)\). Let’s first write \(\phi\) in terms of the frame \(e\). For \(s = e \cdot \xi\), set \(\hat{\phi}, \hat{\phi}^{*H} \in \Omega^0(U, \text{End}(\mathbb{C}^n))\) such that

\[
\phi(s) = e \cdot \hat{\phi} \xi \cdot dz, \quad \phi^{*H}(s) = e \cdot \hat{\phi}^{*H} \xi \cdot d\bar{z}.
\]
So

\[ H(e \cdot \hat{\phi} \cdot dz, e \cdot \eta) = H(e \cdot \xi, e \cdot \hat{\phi}^* \eta \cdot \bar{d}z) \]

\[ \implies \hat{\phi}^* \cdot H \cdot \eta = \hat{\xi}^* \cdot H \cdot \hat{\phi}^* \eta \]

\[ \implies \hat{\xi}^* H = H \hat{\phi}^* \]

\[ \implies \hat{\phi}^* H = H^{-1} \hat{\phi}^* \]

So in terms of the frame \( e \), \([\phi, \phi^*] = (\hat{\phi}^{-1} \hat{\phi}^* H - H^{-1} \hat{\phi}^* H) dz \wedge d\bar{z} \).

So the Hitchin equation \( F_{\phi^*} + [\phi, \phi^*] = 0 \) is locally

\[ \bar{\partial}(H^{-1} \partial H) + (\hat{\phi}^{-1} \hat{\phi}^* H - H^{-1} \hat{\phi}^* H) dz \wedge d\bar{z} = 0. \]

**Example 4.1.** Given the Higgs bundle \( (E = K^{1/2} \oplus K^{-1/2}, \phi = \begin{pmatrix} 0 & q_2 \\
1 & 0 \end{pmatrix} \) \), where \( q_2 \in H^0(\Sigma, K^2) \).

Clearly it is stable and the Hermitian metric is \( H = \begin{pmatrix} h & 0 \\
0 & h^{-1} \end{pmatrix} \). Choose a local coordinate \( z \) and a local holomorphic frame \( e = (e_1, e_2) \), where \( e_1 = dz^{1/2}, e_2 = dz^{-1/2} \). Since \( \phi(e_1) = e_2 \cdot dz \) and \( \phi(e_2) = e_1 \cdot q_2(z) dz \), we have

\[ \phi(e \cdot \begin{pmatrix} 1 \\
0 \end{pmatrix}) = e \cdot \begin{pmatrix} 0 \\
1 \end{pmatrix} \cdot dz, \quad \phi(e \cdot \begin{pmatrix} 0 \\
1 \end{pmatrix}) = e \cdot \begin{pmatrix} q_2(z) \\
0 \end{pmatrix} \cdot dz. \]

So \( \hat{\phi} = \begin{pmatrix} 0 & q_2(z) \\
1 & 0 \end{pmatrix} \). Let \( h = H(e_1, e_1) \) and then \( h^{-1} = H(e_2, e_2) \). This implies that

\[ \bar{\partial}(H^{-1} \partial H) = \bar{\partial}(\begin{pmatrix} h & 0 \\
0 & h^{-1} \end{pmatrix})^{-1} = \bar{\partial}(\begin{pmatrix} h & 0 \\
0 & h^{-1} \end{pmatrix})^{-1} \cdot dz \wedge d\bar{z}. \]

\[ \bar{\partial}(\begin{pmatrix} h & 0 \\
0 & h^{-1} \end{pmatrix})^{-1} \cdot dz \wedge d\bar{z} = \left( |q_2(z)|^2 h^2 - h^2 \right) \cdot dz \wedge d\bar{z}. \]

The Hitchin equation reduces to a single scalar equation

\[ \partial_z \partial_{\bar{z}} \log h + h^{-2} - |q_2(z)|^2 h^2 = 0. \]

Let \( g_0 = g_0(z) dz \otimes d\bar{z} \) be the Hermitian metric on \( K^{-1} \) so that \( \Delta \log g_0(z) = g_0(z) \), meaning that the corresponding metric on \( \Sigma \) is hyperbolic. We then have

\[ \partial_z \partial_{\bar{z}} \log(hg_0(z)^{1/2}) + h^{-2} - |q_2(z)|^2 h^2 - \frac{1}{2} \partial_z \partial_{\bar{z}} \log g_0(z) = 0. \]

Let \( h = g_0(z)^{-1/2} e^u \), where \( u \) is a smooth function over \( \Sigma \). We have

\[ \partial_z \partial_{\bar{z}} u + g_0(z)^{-1} |q_2(z)|^2 e^{2u} \cdot g_0(z)^{-1} - \frac{1}{2} g_0 = 0. \]

The operator \( \frac{1}{g_0(z)^{1/2}} \partial_z \partial_{\bar{z}} \) and \( |q_2(z)|^2 g_0(z)^{-2} \) do not depend on \( z \), denoted by \( \Delta_{g_0} \) and \( \|q_2\|_{g_0}^2 \) respectively. Hence Equation (2) becomes

\[ \Delta_{g_0} u + e^{-2u} - \|q_2\|_{g_0}^2 e^{2u} - \frac{1}{2} g_0 = 0, \]

which indeed holds globally.

In fact, the function \( e^u \) here is the holomorphic energy density of the harmonic map \( f : \Sigma \to (S, \rho(w)|dw|^2) \). And Equation (3) coincides with the Bochner equation for the holomorphic energy density.
4.2. Harmonic maps in terms of Higgs bundles. Let’s write the harmonic map’s info in terms of Higgs bundle and the harmonic metric $H$. First,

$$\Psi_H = \phi + \phi^*H$$

by decomposing into $(1,0)$ and $(0,1)$-forms. Locally, $\Psi_H = \hat{\phi}dz + \hat{\phi}^*Hz$.

Let the Riemann surface $\Sigma$ be equipped with a background conformal metric $g_0 = g_0(z)|dz|^2$.

Data of harmonic maps are as follows:

(a) Energy density: $e(f) = \frac{1}{2} < df, df > - 2 < \Psi_H, \Psi_H > = 4tr(\hat{\phi}^*H)/g_0(z)$.

(b) Energy: $E(f) = \frac{1}{2} < df, df > = 4i\int_\Sigma tr(\hat{\phi}^*H)dz \wedge d\bar{z}$. This is also the Morse function on the moduli space of Higgs bundles.

(c) Pullback metric $g = f^*g_N$ is

$$g = 4tr(\phi^2) + e(f) \cdot g_0 + 4\overline{tr(\phi^2)}.$$

(d) Hopf differential of $f$ is $g^{2,0} = 4tr(\phi^2)$, which is holomorphic. Note that $f$ is conformal if $H_{op}f(f) = 0$.

(e) Curvature of $f$: For the symmetric space $N$, the sectional curvature of the plane spanned by $Y_1, Y_2 \in T_1N = Herm(C^n)$ is given by

$$K(Y_1, Y_2) = \frac{\text{tr}([Y_1, Y_2]^2)}{<Y_1, Y_1> \cdot <Y_2, Y_2> - |<Y_1, Y_2>|^2} \leq 0.$$

The curvature of the tangent plane $\sigma$ of $f(\Sigma)$,

$$K = K(f_x, f_y) = K(f^{-1}f_x, f^{-1}f_y) = K(\Psi_H(\partial_x), \Psi_H(\partial_y)) = \frac{\text{tr}([\phi + \phi^*H, i(\phi - \phi^*H)]^2)}{\text{tr}(\phi + \phi^*H)^2 \cdot \text{tr}(\phi + \phi^*H) - |\text{tr}(\phi + \phi^*H)(i(\phi - \phi^*H))|^2}$$

If $f$ is conformal, $K(f_x, f_y) = \frac{\text{tr}(\phi, \phi^*H)^2}{(\text{tr}(\phi \phi^*H))^2}$.

5. Selected topics on harmonic maps and minimal surfaces

5.1. Labourie conjecture. For a fixed Riemann surface $\Sigma$, the Hitchin component $Hit_n$ is parametrized by $\bigoplus_{i=2}^n H^0(\Sigma, K^i)$. Consider the map

$$T(S) \times \bigoplus_{i=2}^n H^0(\Sigma, K^i) \rightarrow Hit_n \subset Rep(\pi_1S, PSL(n, \mathbb{R}))$$

$$(\Sigma, (q_3, \ldots, q_n)) \mapsto NAH_{\Sigma}(0, q_3, \ldots, q_n).$$

The left hand side has the same dimension as the right hand side. The map is equivariant with respect to the mapping class group action.

**Question 5.1. Is this map a bijection?**

- Surjective (shown by Labourie [16].)

Labourie [14] showed that Hitchin representations are quasi-isometric embeddings. Using this property, Labourie [16] showed that for a fixed Hitchin representation $\rho$, the function $f_\rho : T(S) \rightarrow \mathbb{R}$ sending each $\Sigma$ to $E(NAH_{\Sigma}(\rho))$ is proper. So the function $f_\rho$ has a critical point. By the classical
results of Sacks-Uhlenbeck [23, 24] and Schoen-Yau [26], the critical Riemann surface $\Sigma$ is such that the corresponding harmonic map is conformal. That is, $tr(\phi^2) = 0$.

- Injective.

This is the hard part, called the “Labourie conjecture”. What do we know so far? Let’s generalize to consider Hitchin representation for real split Lie groups.

(1) $G = SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$, Schoen [25].
(2) $G = SL(3, \mathbb{R})$, independently by Labourie [17], Loftin [21].
(3) $G = SL(2, \mathbb{R}) \times SL(2, \mathbb{R}), SL(3, \mathbb{R}), PSp(4, \mathbb{R}), G_2$, i.e., all rank 2 split real Lie groups, Labourie [18].

The Labourie conjecture holds means that for any Hitchin representation $\rho$, there exists a unique Riemann surface $\Sigma$ such that the $\rho$-equivariant harmonic map $f : \Sigma \to N$ is conformal. That is, there exists a unique $\rho$-equivariant minimal mapping of $\Sigma$ into $N$.

**Remark 5.2.** One can also consider the generalized Labourie conjecture for maximal representations. What do we know in this case?

(1) $G = Sp(4, \mathbb{R})$, Collier [3].
(2) $G = PSp(4, \mathbb{R})$, Alessandrini-Collier [1].
(3) $G = SO(2, n)$, Collier-Tholozan-Toulisse [5].

5.2. **Asymptotics.** We discuss the behavior of the harmonic maps $f_t$ along the $C^\gamma$-family $t \cdot [(E, \phi)] \in M_{Higgs}(SL(n, \mathbb{C}))$ as $t \to \infty$.

**Definition 5.3.** Denote by $D(E, \phi)$ the set of points where the Higgs field $\phi$ fails to have $n$ distinct eigen 1-forms, called the discriminant of the Higgs bundle.

Take a universal covering $\pi : Y \to \Sigma \setminus D(E, \phi)$, we have the decomposition of the Higgs bundle $\pi^*(E, \bar{\partial} E, \phi) = \oplus_{i=1}^n(E_i, \bar{\partial} E_i, \phi_i \cdot \text{id}_{E_i})$, where $\phi_i$ are holomorphic 1-forms. We have $\text{rank } E_i = 1$ and $\phi_i - \phi_j (i \neq j)$ have no zeros. Let $\gamma : [0, 1] \to Y$ be a $C^\infty$-path, we have the expression $\gamma^*(\phi_i) = a_i ds$, where $a_i$ are $C^\infty$-functions on $[0, 1]$.

**Definition 5.4.** The path $\gamma$ is called non-critical if $\Re a_i(s) \neq \Re a_j(s) (i \neq j)$ for any $s \in [0, 1]$. Let’s reorder $a_i(s)$ such that $\Re a_i(s) > \Re a_j(s) (i < j)$. We set $\alpha_i := -\int_0^1 \Re (a_i(s))ds$.

To talk about asymptotics of harmonic maps, we introduce the vector distance of two metrics. Let $V$ be an $n$-dimensional complex vector space. For two Hermitian metrics $h_1, h_2$, we can take a basis $e_1, e_2, \cdots, e_n$ of $V$ which is orthogonal with respect to both $h_1$ and $h_2$. We have the real numbers $k_j (j = 1, 2, \cdots, n)$ determined by $k_j = \log |e_j|_{h_2} - \log |e_j|_{h_1}$. We impose $k_1 \geq k_2 \geq \cdots \geq k_n$. Then we set $d(h_1, h_2) := (k_1, \ldots, k_n) \in \mathbb{R}^n$.

**Conjecture 5.5.** (Hitchin WKB problem, Katzarkov-Noll-Pandit-Simpson [15]) As $t \to \infty$, the harmonic map $f_t$ satisfies for a non-critical path $\gamma : [0, 1] \to Y$,

$$\frac{1}{t} \tilde{d}(f_t(\gamma(0)), f_t(\gamma(1))) \sim 2(\alpha_1, \cdots, \alpha_n).$$

**Definition 5.6.** We call a Higgs bundle $(E, \bar{\partial} E, \phi)$ generically regular semisimple if the set $D(E, \phi)$ is finite.

**Theorem 5.7.** (Mochizuki [22]) Let $(E, \bar{\partial} E, \phi)$ be a stable Higgs bundle of degree 0 on $\Sigma$. Suppose it is generically regular semisimple. If $\gamma : [0, 1] \to Y$ is non-critical, then there exist positive constants $C_0$ and $\epsilon_0$ such that the following holds:

$$\left| \frac{1}{t} \tilde{d}(f_t(\gamma(0)), f_t(\gamma(1))) - 2(\alpha_1, \cdots, \alpha_n) \right| \leq C_0 \exp(-\epsilon_0 t).$$
The constants $C_0$ and $\epsilon_0$ may depend only on $\Sigma, \phi_1, \ldots, \phi_n$ and $\gamma$.

The key estimate is the following theorem on “decoupled Hitchin equation”.

**Theorem 5.8.** (Mochizuki [22]) Under the same assumptions in Theorem 5.8. Then take any neighborhood $N_0$ of $D(E, \phi)$, there exists a constant $C_0 > 0$ and $\epsilon_0 > 0$ such that the following holds on $\Sigma \setminus D(E, \phi)$,

\[
|F_{D(H)}|_{H_t, g_\Sigma} = |t|^2 [\phi, \phi^* H^t]_{H_t, g_\Sigma} \leq C_0 \exp(-\epsilon_0 t).
\]

The constants $C_0, \epsilon_0$ only depend on $(\Sigma, g_\Sigma), N_0$ and $(E, \phi)$. For any local section $s$ of $\text{End}(E) \otimes \Omega^{p,q}(\Sigma)$, $|s|_{H_t, g_\Sigma} : \Sigma \to \mathbb{R}$ is the norm of $s$ with respect to $H_t$ on $\text{End}(E)$ and $g_\Sigma$ on $\Sigma$.

**Remark 5.9.** (1) For cyclic Higgs bundles in the Hitchin component, Theorem 5.7 and 5.8 were first shown in Collier-Li [4].

(2) For those Higgs bundles which are not generically regular semisimple, Conjecture 5.5 is not necessarily true. But the complete description is still open.

### 5.3. Negative Curvature.

**Conjecture 5.10.** Given a Hitchin representation $\rho$ for $\text{PSL}(n, \mathbb{R})$ and a Riemann surface $\Sigma$, the unique $\rho$-equivariant harmonic map $f : \bar{\Sigma} \to N$ satisfies that it is never tangential to a flat in $N$.

For the case $f$ is a minimal mapping, this is the negative curvature conjecture in Dai-Li [7].

Using the expression of curvature form (4), the above conjecture is rephrased as

**Conjecture 5.11.** For $(E, \phi)$ in the Hitchin component for $\text{PSL}(n, \mathbb{R})$,

\[
F_{D(H)} \neq 0, \quad [\phi, \phi^* H^t] \neq 0.
\]

**Remark 5.12.** (1) A direct corollary of Conjecture 5.10 is that the induced metric of minimal surface is strictly negatively curved.

(2) This phenomenon in Equation (6) is opposite to the asymptotical behavior of generically regular semisimple Higgs bundles in which case the Hitchin equation decouples in exponential decay as in Equation (5).

What do we know so far for Conjecture 5.10?

(1) for cyclic Higgs bundles in the Hitchin components, Dai-Li [7].

(2) One can also consider the conjecture for maximal representations. For $\text{Sp}(4, \mathbb{R})$-maximal representations in Gothen components, Dai-Li [8].

### 5.4. Monotonicity along $\mathbb{C}^*$-flow.

Let’s focus on the harmonic maps along the $\mathbb{C}^*$-action.

**Theorem 5.13.** (Hitchin) Along $\mathbb{C}^*$-flow, the Morse function (energy) decreases as $|t|$ decreases.

How about the energy density as a function over $\Sigma$ along $\mathbb{C}^*$-flow?

**Conjecture 5.14.** Along $\mathbb{C}^*$-flow, the energy density $e(f_t)$ of corresponding harmonic maps $f_t$ decreases as a function as $|t|$ decreases.

Dai and Li in [8] showed Conjecture 5.14 holds for stable cyclic Higgs bundles.

A weaker conjecture is about the domination comparing with the limit as $t \to 0$.

**Conjecture 5.15.** For a Higgs bundle $(E, \phi)$ in the moduli space $\mathcal{M}_{\text{Higgs}}$ of $\text{SL}(n, \mathbb{C})$-Higgs bundles, denote by $f([E, \phi])$ the corresponding equivariant harmonic map. Then the energy density satisfies

\[
e(f([E, \phi])) \geq e(f(\lim_{t \to 0} t \cdot [E, \phi])).
\]
In the following Theorem 5.16, we see that Conjecture 5.15 holds for each Higgs bundle in the Hitchin section. Renormalize the metric on $N$ such that in the base $n$-Fuchsian case, the totally geodesic copy of the hyperbolic plane inside $N$ is of curvature $-1$.

**Theorem 5.16.** (Li [19]) Let $\rho$ be a Hitchin representation for $PSL(n, \mathbb{R})$, $g_0$ be a hyperbolic metric on $S$, and $f$ be the unique $\rho$-equivariant harmonic map from $(\tilde{S}, \tilde{g}_0)$ to the symmetric space $N$. Then its energy density $e(f)$ satisfies

$$e(f) \geq 1.$$ 

Moreover, equality holds at one point only if $e(f) \equiv 1$ in which case $\rho$ is the base $n$-Fuchsian representation of $(S, g_0)$.

5.5. **Hitchin fibration.** We discuss the relation between harmonic maps and the Hitchin fibration.

**Conjecture 5.17.** (Dai-Li [8]) Inside each Hitchin fiber, the Hitchin section maximizes the energy density of corresponding harmonic maps.

In fact, even the integral version is still open. A weaker conjecture is as follows.

**Conjecture 5.18.** Inside each Hitchin fiber, the Hitchin section maximizes the energy of corresponding harmonic maps.

The following theorem shows that Conjecture 5.17 holds for Hitchin fiber at $(q_2, 0, \ldots, 0)$.

**Theorem 5.19.** (Li [20]) For any Higgs bundle in Hitchin fiber at $(q_2, 0, \ldots, 0)$, the pullback metric of its corresponding harmonic map is strictly dominated by the one of the Hitchin section $s(q_2, 0, \ldots, 0)$ unless it coincides with $s(q_2, 0, \ldots, 0)$.

The fact that the $n$-Fuchsian point being maximal gives us a pure topological control on representations arising from such Hitchin fibers as shown in Theorem 5.21.

**Definition 5.20.** (1) The translation length of $\gamma \in \pi_1(S)$ with respect to $\rho: \pi_1(S) \to SL(n, \mathbb{C})$ is defined by

$$l_\rho(\gamma) := \inf_{x \in \mathbb{N}} d(x, \rho(\gamma)(x)),$$

where $d(\cdot, \cdot)$ is the distance induced by the Riemannian metric $g_N$ on $N$.

(2) Given two representations $\rho_1, \rho_2: \pi_1(S) \to SL(n, \mathbb{C})$, we call $\rho_2$ is strictly dominated by $\rho_1$, if there exists a positive constant $\lambda < 1$ such that for any non-identity element $\gamma \in \pi_1(S)$, $l_{\rho_2}(\gamma) < \lambda l_{\rho_1}(\gamma)$.

**Theorem 5.21.** (Li [20]) Any representation arises from a polystable $SL(n, \mathbb{C})$-Higgs bundle in the Hitchin fiber at $(q_2, 0, \ldots, 0)$ is strictly dominated by a $n$-Fuchsian representation unless it is a $n$-Fuchsian representation.

**Remark 5.22.** In the case of $SL(2, \mathbb{C})$, Theorem 5.19 and 5.21 were shown by Deroin and Tholozan [9].

**References**

[1] D. Alessandrini, B. Collier, The geometry of maximal components of the $PSp(4,R)$ character variety, to appear in Geometry and Topology.

[2] D. Baraglia, $G_2$ Geometry and integrable system, thesis, arXiv:1002.1767v2, 2010.

[3] B. Collier, Maximal $Sp(4,R)$ surface group representations, minimal surfaces and cyclic surfaces, Geometriae Dedicata 180 (2015), no. 1, 241–285.

[4] B. Collier, Q. Li, Asymptotics of Higgs bundles in the Hitchin component, Adv. Math. 307 (2017), 488–558, MR3590524, Zbl 06670884.
[5] B. Collier, Nicolas Tholozan, Jérémie Toulisse, The geometry of maximal representations of surface groups into $SO(2, n)$, arxiv 1702.08799.
[6] K. Corlette, Flat $G$-bundles with canonical metrics, J. Diff. Geom. 28(1988), no. 3, 361-382, MR965220, Zbl 0676.58007.
[7] S. Dai, Q. Li, Minimal surfaces for Hitchin representations, to appear in Journal of differential geometry.
[8] S. Dai, Q. Li, On Cyclic Higgs bundles, arXiv 1710.10725.
[9] B. Deroin, N. Tholozan, Dominating surface group representations by Fuchsian ones, Int. Math. Res. Not. IMRN 2016, no. 13, 4145–4166, MR3544632.
[10] S. K. Donaldson, Twisted harmonic maps and the self-duality equations, Proc. London Math. Soc. (3) 55 (1987), no. 1, 127–131, MR0887285, Zbl 0634.58007.
[11] O. Guichard, An introduction to the Differential geometry of Flat bundles and of Higgs bundles, http://irma.math.unistra.fr/ guichard/assets/files/intro-bdle-ims.pdf.
[12] N. J. Hitchin, The self-duality equations on a Riemann surface, Proc. London Math. Soc. (3) 55 (1987), no. 1, 59–126, MR0887284, Zbl 0634.53045.
[13] N. J. Hitchin, Lie groups and Teichmüller space, Topology 31 (1992), no. 3, 449–473, MR1174252, Zbl 0769.32008.
[14] F. Labourie, Anosov flows, surface groups and curves in projective space. Invent. Math. 165(2006), 51–114.
[15] L. Katzarkov, A. Noll, P. Pandit, C. Simpson, Harmonic maps to buildings and singular perturbation theory, Comm. Math. Physics 336 (2015), 853–903.
[16] F. Labourie, Cross ratios, Anosov representations and the energy functional on Teichmüller space, Ann. Sci. École. Norm. Sup. (4) 41 (2008), no. 3, 437–469. MR2482204, Zbl 1160.37021.
[17] F. Labourie, Flat projective structures on surfaces and cubic holomorphic differentials, Pure Appl. Math. Q. 3 (2007), 1057–1099. MR 2402597. Zbl 1158.32006.
[18] F. Labourie, Cyclic surfaces and Hitchin components in rank 2, Ann. of Math. (2) 185 (2017), no. 1, 1–58, MR3583351, Zbl 06686583.
[19] Q. Li, Harmonic maps for Hitchin representations, arXiv:1806.06884.
[20] Q. Li, Some properties of Higgs bundles with Higgs fields of certain types, in preparation.
[21] J. Loftin, Affine spheres and convex $\mathbb{R}P^n$-manifolds, Amer. J. Math. 123 (2001), no. 2, 255–274. MR1828223, Zbl 0997.53010.
[22] T. Mochizuki, Asymptotic behaviour of certain families of harmonic bundles on Riemann surfaces, J. Topol. 9 (2016), no. 4, 1021–1073. MR3620459.
[23] J. Sacks, K. Uhlenbeck, The existence of minimal immersions of 2-spheres, Ann. of Math. 113 (1981), 1–24.
[24] J. Sacks, K. Uhlenbeck, Minimal immersions of closed Riemann surfaces, Trans. Amer. Math. Soc. 271 (1982), 639–652.
[25] R. Schoen, The role of harmonic mappings in rigidity and deformation problems, in Complex geometry (Osaka, 1990), Lecture Notes in Pure and Appl. Math. no. 143, Dekker, New York, 1993, pp. 179–200. MR 1201611. Zbl 0806.58013.
[26] R. Schoen, S. T. Yau, Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature, Ann. of Math. 110 (1979), 127–142.
[27] C. Simpson, Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization, J. Amer. Math. Soc. 1 (1988), no. 4, 867–918. MR0944577, Zbl 0669.58008.

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