Research Article

Some Properties of a Sequence Similar to Generalized Euler Numbers

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Received 27 December 2012; Accepted 7 February 2013

Academic Editors: W. F. Klostermeyer, T. Prellberg, S. Rim, and W. F. Smyth

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We introduce the sequence \( \{U_n(x)\} \) given by generating function
\[
\frac{1}{(e^t + e^{-t} - 1)^x} = \sum_{n=0}^{\infty} U_n(x) \frac{t^n}{n!} \quad (|t| < (1/3)\pi, 1^x := 1)
\]
and establish some explicit formulas for the sequence \( \{U_n(x)\} \). Several identities involving the sequence \( \{U_n(x)\} \), Stirling numbers, Euler polynomials, and the central factorial numbers are also presented.

1. Introduction and Definitions

For a real or complex parameter \( \alpha \), the generalized Euler polynomials \( E_n^{(\alpha)}(x) \) are defined by the following generating function (see [1–4])
\[
\left( \frac{2}{e^t + 1} \right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!} \quad (|t| < \pi, 1^{\alpha} := 1).
\]
Obviously, we have
\[
E_n^{(1)}(x) = E_n(x) \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}),
\]
in terms of the classical Euler polynomials \( E_n(x) \), \( \mathbb{N} \) being the set of positive integers. The classical Euler numbers \( E_n \) are given by the following:
\[
E_n = 2^n E_n \left( \frac{1}{2} \right) \quad (n \in \mathbb{N}_0).
\]
The so-called the generalized Euler numbers \( E_{2n}^{(x)} \) are defined by (see [3, 5])
\[
\left( \frac{2}{e^t + e^{-t}} \right)^x = \sum_{n=0}^{\infty} E_{2n}^{(x)} \frac{t^{2n}}{(2n)!} \quad (|t| < \pi, 1^x := 1).
\]
In fact, \( E_{2n}^{(k)} \) \((k \in \mathbb{Z})\) are the Euler numbers of order \( k \), \( \mathbb{Z} \) being the set of integers. The numbers \( E_{2n}^{(1)} = E_{2n} \) are the ordinary Euler numbers.

Zhi-Hong Sun introduces the sequence \( \{U_n\} \) similar to Euler numbers as follows (see [6, 7]):
\[
U_0 = 1, \quad U_n = -2 \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k} U_{n-2k}, \quad (n \geq 1),
\]
where (and in what follows) \( \lfloor x \rfloor \) is the greatest integer not exceeding \( x \).

Clearly, \( U_{2n-1} = 0 \) for \( n \geq 1 \). The first few values of \( U_{2n} \) are shown below
\[
U_2 = -2, \quad U_4 = 22, \quad U_6 = -602, \quad U_8 = 30742, \quad U_{10} = -2523002, \quad U_{12} = 303692662.
\]

The sequence \( \{U_n\} \) is related to the classical Bernoulli polynomials \( B_n(x) \) (see [8–11]) and the classical Euler polynomials \( E_n(x) \). Zhi-Hong Sun gets the generating function of
\{U_n\} and deduces many identities involving \{U_n\}. As example, (see [6]),

$$\frac{1}{e^{t} + e^{-t} - 1} = \sum_{n=0}^{\infty} U_n \frac{t^n}{n!}$$

and deduces many identities involving \{U_n\}. As example, (see [6]),

$$1 = e^t + e^{-t} - 1 = \sum_{n=0}^{\infty} U_n \frac{t^n}{n!}$$

Similarly, we can define the generalized sequence \{U^{(x)}_n\}. For a real or complex parameter \(x\), the generalized sequence \{U^{(x)}_n\} is defined by the following generating function:

$$\left( e^x + e^{-x} - 2 \right)^x = \sum_{n=0}^{\infty} U^{(x)}_n \frac{t^n}{n!}$$

Obviously,

$$U^{(x)}_0 = 1, \quad U^{(1)}_n = U_n \quad (n \in \mathbb{N}).$$

By using (10), we can obtain

$$U^{(k)}_n = n! \sum_{\nu_1 + \cdots + \nu_k = n} U_{\nu_1} \cdots U_{\nu_k} \left( \prod_{\nu_i \in \mathbb{N}} \frac{1}{\nu_i!} \right) \quad (k \in \mathbb{N}).$$

We now return to the Stirling numbers \(s(n, k)\) of the first kind, which are usually defined by (see [2, 5, 8, 11, 12])

$$x(x - 1)(x - 2) \cdots (x - n + 1) = \sum_{k=0}^{n} s(n, k) x^k$$

or by the following generating function:

$$\left( \log(1 + x) \right)^k = k! \sum_{n=k}^{\infty} s(n, k) \frac{x^n}{n!}.$$

It follows from (13) or (14) that

$$s(n, k) = s(n - 1, k - 1) - (n - 1) s(n - 1, k) \quad (n \in \mathbb{N}),$$

and that

$$s(n, 0) = 0 \quad (n \in \mathbb{N}), \quad s(n, n) = 1 \quad (n \in \mathbb{N}),$$

$$s(n, 1) = (-1)^{n-1} (n - 1)! \quad (n \in \mathbb{N}),$$

$$s(n, k) = 0 \quad (k > n \text{ or } k < 0).$$

The central factorial numbers \(T(n, k)\) are given by the following expansion formula (see [3, 5, 13]):

$$x^n = \sum_{k=0}^{n} T(n, k) x \left( x - 1 \right)^k \left( x - 2 \right)^k \cdots \left( x - (k - 1) \right)^k$$

or by means of the generating function

$$(e^x + e^{-x} - 2)^k = (2k)! \sum_{n=k}^{\infty} \frac{T(n, k)}{(2n)!} x^{2n}.$$  

It follows from (17) or (18) that

$$T(n, k) = T(n - 1, k - 1) + k^2 T(n - 1, k),$$

with

$$T(0, 0) = 1, \quad T(n, 0) = 0 \quad (n \in \mathbb{N}),$$

$$T(n, 1) = 1 \quad (n \in \mathbb{N}).$$

We also find from (18) that

$$T(n, 2) = \frac{1}{4} \left( 4^{n-1} - 1 \right),$$

$$T(n, 3) = \frac{9^n}{360} - \frac{4^n}{60} + \frac{1}{24} \quad (n \in \mathbb{N}).$$

The main purpose of this paper is to prove some formulas for the generalized sequence \{U^{(x)}_n\} and \(E_n(x)\). Some identities involving the sequence \{U^{(x)}_n\}, Stirling numbers \(s(n, k)\), and the central factorial numbers \(T(n, k)\) are deduced.

2. Main Results

**Theorem 1.** Let \(n \geq k \quad (n, k \in \mathbb{N})\) and

$$q(n, k) = (-1)^{k} \sum_{j=0}^{k} \frac{(2j)!}{j!} T(n, j) s(j, k).$$

Then,

$$U^{(x)}_{2n} = \sum_{k=1}^{n} q(n, k) x^k.$$  

**Remark 2.** By (15), (19), (20), and Theorem 1, we know that \(U^{(x)}_{2n}\) is a polynomial of \(x\) with integral coefficients. For example, by setting \(n = 1, 2, 3, 4\) in Theorem 1, we get

$$U^{(x)}_{2} = -2x, \quad U^{(x)}_{4} = 10x + 12x^2, \quad U^{(x)}_{6} = -182x - 300x^2 - 120x^3,$$

$$U^{(x)}_{8} = 6970x + 13692x^2 + 8400x^3 + 1680x^4.$$ 

Taking \(x = 1\) in Theorem 1, we can obtain the following.

**Corollary 3.** Let \(n \in \mathbb{N}\). Then,

$$U_{2n} = \sum_{j=0}^{n} (-1)^{j} (2j)! T(n, j).$$
Corollary 4. Let \( n \in \mathbb{N} \). Then,

\[
U_{2n} \equiv -2 \pmod{24}, \\
U_{2n} \equiv -2 + 24T(n, 2) \pmod{720}, \\
U_{2n} \equiv -2 + 24T(n, 2) - 720T(n, 3) \pmod{40320}.
\]

Theorem 5. Let \( n \geq k \) \((n, k \in \mathbb{N})\). Then,

\[
U_{2n} = \sum_{k=1}^{n} q(n, k), \\
U_{2n} = 2 \sum_{k=1}^{[n/2]} q(n, 2k) - 2 \\
= 2 \sum_{k=1}^{[(n-1)/2]} q(n, 2k + 1) + 2.
\]  

Theorem 6. Let \( n \geq k \) \((n, k \in \mathbb{N})\). Suppose also that \( q(n, k) \) is defined by (22). Then,

\[
k!q(n, k) = \frac{(2n)!}{3^{2n-k}} \times \sum_{v_1, \ldots, v_{2n} \in \mathbb{N}} \left( E_{2v_1 - 1}(0) - E_{2v_2 - 1} \left( \frac{2}{3} \right) \right) \\
\cdots \left( E_{2v_{2n} - 1}(0) - E_{2v_{2n} - 1} \left( \frac{2}{3} \right) \right) \\
\times ((2v_1)! \cdots (2v_{2n})!)^{-1}.
\]

Theorem 7. Let \( n \in \mathbb{N} \). Then,

\[
-2 \sum_{k=0}^{n-1} \binom{2n-1}{2k} U_{2k} = 3^{2n-1} \left( E_{2n-1}(0) - E_{2n-1} \left( \frac{2}{3} \right) \right). \tag{29}
\]

Theorem 8. Let \( n \in \mathbb{N} \). Then,

\[
U_{n+1} = \sum_{k=0}^{n-1} \binom{n}{k} \left( 1 - 2^{n-k} \right) U_{k+1} - 2^{n-k} U_k. \tag{30}
\]

Theorem 9. Let \( n \in \mathbb{N}_0 \). Then,

\[
\sum_{n=0}^{\infty} \frac{1}{(n+1)!} U_n = \frac{1}{\sqrt{3}} \log \frac{2\sqrt{3} - 1}{2 - \sqrt{3}} e - 5 + 3\sqrt{3}. \tag{31}
\]

3. Proofs of Theorems

Proof of Theorem 1. By (10), (13), and (18), we have

\[
\sum_{n=0}^{\infty} U^{(x)}_{2n} \frac{t^{2n}}{(2n)!} = \left( \frac{1}{e^t + e^{-t} - 1} \right)^x
\]

\[
= \sum_{j=0}^{\infty} (-1)^j \left( x + j - 1 \right) (e^t + e^{-t} - 2)^j T(n,j)
\]

\[
= \sum_{j=0}^{\infty} (-1)^j \left( x + j - 1 \right) (2)^j \sum_{n=0}^{\infty} s(n,j) T(n,j) \frac{t^{2n}}{(2n)!}
\]

\[
= \sum_{j=0}^{\infty} \sum_{n=0}^{2n} (-1)^j (2)^j \left( x + j - 1 \right) T(n,j) \frac{t^{2n}}{(2n)!}
\]

which readily yields

\[
U_{2n}^{(x)} = \sum_{j=0}^{n} (-1)^j (2)^j \left( x + j - 1 \right) T(n,j)
\]

\[
= \sum_{j=0}^{n} (-1)^j (2)^j T(n,j) \frac{1}{j!} (x+1) \cdots (x+j-1)
\]

\[
= \sum_{k=0}^{n} q(n,k) \frac{1}{k!} x^k.
\]

This completes the proof of Theorem 1.

Proof of Theorem 5. By (10), we have

\[
\sum_{n=0}^{\infty} U^{(x)}_{2n} \frac{t^{2n}}{(2n)!} = e^t + e^{-t} - 1 = 2 \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} - 1,
\]

and \( U^{(x)}_0 = 1 \), thus

\[
\sum_{n=1}^{\infty} U^{(x)}_{2n} \frac{t^{2n}}{(2n)!} = e^t + e^{-t} - 1 = 2 \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}.
\]

By Theorem 1 and comparing the coefficient of \( t^{2n}/(2n)! \) on both sides of (35), we get

\[
\sum_{k=1}^{n} q(n,k) (-1)^k = U^{(-1)}_{2n} = 2.
\]
Again, by taking \( x = 1 \) in Theorem 1, we have

\[
\sum_{k=1}^{n} q(n,k) = U_{2n}.
\]  
(37)

By (36) and (37), we immediately obtain (27). This completes the proof of Theorem 5.

**Proof of Theorem 6.** By applying Theorem 1, we have

\[
k! q(n,k) = \frac{d^k}{dx^k} \left|_{x=-1} \right. \left[ e^x \right]^{2n}.
\]  
(38)

On the other hand, it follows from (10) that

\[
\sum_{n=k}^{\infty} \frac{d^k}{dx^k} \left|_{x=-1} \right. \left[ e^x \right]^{2n} = \left( \log \left( \frac{1}{e^x + e^{-x} - 1} \right) \right)^k.
\]  
(39)

By using (38) and (39), we find that

\[
k! \sum_{n=k}^{\infty} q(n,k) \frac{t^{2n}}{(2n)!} = \left( \log \left( \frac{1}{e^x + e^{-x} - 1} \right) \right)^k.
\]  
(40)

We now note that

\[
\frac{d}{dt} \left\{ \log \left( \frac{1}{e^x + e^{-x} - 1} \right) \right\}
= \frac{e^{-x} - e^x}{e^x + e^{-x} - 1}
= \frac{e^{-x} - e^x}{2} \left( \frac{2e^x}{e^{3x} + 1} + \frac{2e^{-x}}{e^{-3x} + 1} \right)
= \frac{1}{2} \left( \left( \frac{2}{e^{3x} + 1} - \frac{2}{e^{-3x} + 1} \right) - \left( \frac{2e^{2x}}{e^{3x} + 1} - \frac{2e^{-2x}}{e^{-3x} + 1} \right) \right)
= \frac{1}{2} \left( \sum_{n=0}^{\infty} E_n \left( \frac{2}{3} \right)^n - \sum_{n=0}^{\infty} E_n \left( -\frac{3}{2} \right)^n \right)
= \frac{1}{2} \left( \sum_{n=0}^{\infty} E_n \frac{2}{3}^n - \sum_{n=0}^{\infty} E_n \frac{3}{2}^n \right)
= \sum_{n=0}^{\infty} \frac{3^n}{n!} \left( E_{2n+1}(0) - E_{2n+1}(\frac{2}{3}) \right) \frac{t^{2n}}{(2n+1)!}.
\]  
(41)

Hence,

\[
\frac{1}{e^x + e^{-x} - 1} = \sum_{n=0}^{\infty} \frac{3^{2n+1}}{n!} \left( E_{2n+1}(0) - E_{2n+1}(\frac{2}{3}) \right) \frac{t^{2n+2}}{(2n+2)!}
= \sum_{n=0}^{\infty} \frac{3^{2n-1}}{n!} \left( E_{2n-1}(0) - E_{2n-1}(\frac{2}{3}) \right) \frac{t^{2n}}{(2n)!}.
\]  
(42)

yields

\[
k! \sum_{n=k}^{\infty} q(n,k) \frac{t^{2n}}{(2n)!} = \left( \sum_{n=1}^{\infty} \frac{3^{2n-1}}{n!} \left( E_{2n-1}(0) - E_{2n-1}(\frac{2}{3}) \right) \frac{t^{2n}}{(2n)!} \right)^k
= \sum_{n=k}^{\infty} \frac{t^{2n}}{(2n)!} \left( 2^n - 1 \right) \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{n=0}^{\infty} \frac{t^n}{n!}
= \sum_{n=k}^{\infty} \frac{t^{2n}}{(2n)!} \left( 2^n - 1 \right) \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{n=0}^{\infty} \frac{t^n}{n!}.
\]  
(43)

Comparing the coefficient of \( t^{2n}/(2n)! \) on both sides of (43), we immediately get (28). This completes the proof of Theorem 6.

**Proof of Theorem 7.** Consider

\[
\frac{d}{dt} \left\{ \log \left( \frac{1}{e^x + e^{-x} - 1} \right) \right\} = \frac{e^{-x} - e^x}{e^x + e^{-x} - 1}
= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left( -2 \right)^n \left( \frac{2}{3} \right)^n U_{2n} \frac{t^{2n}}{(2n+1)!}
= -2 \sum_{n=0}^{\infty} \frac{1}{2k} \sum_{k=0}^{n} \left( \frac{2n+1}{2k} \right) U_{2k} \frac{t^{2n+1}}{(2n+1)!}.
\]  
(44)

Thus,

\[
\log \left( \frac{1}{e^x + e^{-x} - 1} \right) = -2 \sum_{n=0}^{\infty} \frac{1}{2k} \sum_{k=0}^{n} \left( \frac{2n+1}{2k} \right) \frac{t^{2n+1}}{(2n+1)!}.
\]  
(45)

By (42) and (45) we obtain (29). This completes the proof of Theorem 7.

**Proof of Theorem 8.** By using (7), we have

\[
\sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} = \frac{e^{-t} - e^t}{(e^t + e^{-t} - 1)^2}.
\]  
(46)

Thus

\[
\left( e^t - e^{-1} \right) \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} = \left( 1 - e^{-t} \right) \sum_{n=0}^{\infty} \frac{t^n}{n!},
= \sum_{n=0}^{\infty} \frac{(t^n-1) t^n}{n!} \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{n=0}^{\infty} \frac{t^n}{n!}.
\]  
(47)
That is,
\begin{equation}
\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \left(2^{n-k} - 1\right) \frac{t^n}{n!} + \sum_{n=0}^{\infty} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} - \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} U_k \frac{t^n}{n!}.
\end{equation}

Comparing the coefficient of $t^n/n!$ on both sides of (48), we get the following:
\begin{equation}
U_{n+1} - U_n = \sum_{k=0}^{n} \binom{n}{k} \left(\left(1 - 2^{n-k}\right)U_{k+1} - 2^{n-k}U_k\right).
\end{equation}

By (49) we immediately obtain (30). This completes the proof of Theorem 8.

**Proof of Theorem 9.** By integrating (7) with respect to $t$ from 0 to 1, we have
\begin{equation}
\sum_{n=0}^{\infty} \frac{1}{(n+1)!} U_n = \int_{0}^{1} \frac{1}{\left\{e^t + e^{-t} - 1\right\}} dt = \int_{0}^{1} \frac{1}{e^{2t} - e^{-2t} + 1} de^{t} = \int_{1}^{e} \frac{1}{x^2 - x + 1} dx.
\end{equation}

By (50) and \(\int (1/(ax^2 + bx + c))dx = (1/\sqrt{b^2 - 4ac}) \log|(2ax + b - \sqrt{b^2 - 4ac})/(2ax + b + \sqrt{b^2 - 4ac})| + c \) (c is constant), we have (31). This completes the proof of Theorem 9.

**Acknowledgments**

This work is partly supported by the Social Science Foundation (no. 2012YB03) of Huizhou University and the Key Discipline Foundation (no. JG2011019) of Huizhou University.

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