CHAOTIC SADDLES IN A GENERALIZED LORENZ MODEL OF MAGNETOCONECTION

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ABSTRACT

The nonlinear dynamics of a recently derived generalized Lorenz model (Macek and Strumik, Phys. Rev. E 82, 027301, 2010) of magnetoconvection is studied. A bifurcation diagram is constructed as a function of the Rayleigh number where attractors and nonattracting chaotic sets coexist inside a periodic window. The nonattracting chaotic sets, also called chaotic saddles, are responsible for fractal basin boundaries with a fractal dimension near the dimension of the phase space, which causes the presence of very long chaotic transients. It is shown that the chaotic saddles can be used to infer properties of chaotic attractors outside the periodic window, such as their maximum Lyapunov exponent.

Keywords Chaos, Chaotic saddles, Lorenz model, Reduced model, Long chaotic transients

1 Introduction

The study of simple models of thermal convection based on truncated solutions of hydrodynamic equations has been extremely popular since Lorenz’s original work on deterministic nonperiodic flows [17]. The model was originally derived for the study of two-dimensional Rayleigh-Bénard convection, i.e., thermal convection in a plane layer of fluid heated from below and cooled from above [3]. Since then, sets of Lorenz-like equations have been obtained in different contexts, such as lasers [10], dynamos [15] [56], magnetoconvection [14] [19], chemical reactions [24] etc. In the present paper, we explore the generalized Lorenz model introduced by Macek and Strumik [19] in the context of magnetoconvection, where the interaction of an electrically conducting fluid with an imposed magnetic field is considered. The model follows Lorenz’s original derivation of three nonlinear ordinary differential equations, adding a fourth equation for the magnetic field fluctuation and it was shown to exhibit hyperchaos [21] [20]. Although most previous analysis of Lorenz systems (and dynamical systems in general) focus on the asymptotic behaviour, after solutions have converged to an attractor, here we stress the importance of the initial transient dynamics and identify the role of nonattracting chaotic sets.

Nonattracting chaotic sets are subsets of the phase space of a dynamical system where a nonattracting chaotic trajectory can be found, with the corresponding properties of aperiodicity and sensitivity to initial conditions, as measured by a positive Lyapunov exponent [11]. Being nonattracting means that neighbouring trajectories will wonder in the vicinity of a chaotic set for a finite time before they diverge from it, eventually converging to some coexisting attractor in the case of dissipative systems. Thus, the outcome is a transient or decaying chaotic behaviour. Usually, there is a special (fractal) set of initial conditions called stable manifold that converges to the nonattracting chaotic set in forward time, and a set that converges to it in reversed time dynamics, the unstable manifold [11] [80]. For that reason, nonattracting chaotic sets are also known as chaotic saddles, as they lie at the intersection of their stable and unstable manifolds, which are smooth surfaces in the phase space. Chaotic saddles are also known as chaotic repellors and are related to dynamical phenomena like chaotic scattering [18] and fractal basin boundaries [3].
Another important role played by chaotic saddles in dynamical systems is in global bifurcations known as crises [7], where chaotic attractors undergo a sudden change in size or structural stability. In a boundary crisis, a chaotic attractor suddenly disappears, leaving a chaotic saddle in its place [29]; in an interior crisis, a chaotic attractor suddenly enlarges after collision with a chaotic saddle [31]; in a merging crisis, two or more chaotic attractors are united to form a larger attractor, where the former attractors are converted to chaotic saddles [26]. In all types of crises, chaotic saddles affect the observable dynamics through the appearance of chaotic transients and/or crisis-induced intermittency [9, 25, 27]. Due to their close relation to crises, the properties of a chaotic saddle can be used to infer the properties of chaotic attractors generated at crises, such as their Lyapunov exponents and overall topology [31, 26, 33]. In this work, we exemplify this fact in the generalized Lorenz model by comparing the shape of the invariant sets and the value of their maximum Lyapunov exponents for a chaotic saddle and a chaotic attractor in the phase space for different values of the control parameter.

This paper is organized as follows. In section 2, the generalized Lorenz model of magnetoconvection is described; in section 3, the numerical nonlinear analysis of the model is presented; section 4 provides the conclusions.

2 The Generalized Lorenz Model

In this section we summarize the derivation of the generalized Lorenz model of magnetoconvection, which is provided in more details in Macek [20]. Consider the two-dimensional motion of a conducting fluid between two horizontal plates separated by a height $h$ in the $z$ direction, with the temperature at the bottom plate kept higher than the temperature at the top plate and an imposed magnetic field in the horizontal direction ($x$). The evolution of the magnetized fluid is governed by the magnetohydrodynamic (MHD) equations

$$\frac{Dv}{Dt} = -\frac{1}{\rho} \nabla \left( p + \frac{B^2}{2\mu_0} \right) + \frac{(B \cdot \nabla) B}{\mu_0 \rho} + \nu \nabla^2 v + \rho g, \quad (1)$$

$$\frac{DB}{Dt} = (B \cdot \nabla) v + \eta \nabla^2 B, \quad (2)$$

$$\frac{DT}{Dt} = \kappa \nabla^2 T, \quad (3)$$

$$\nabla \cdot B = 0, \quad (4)$$

where $v$ denotes the velocity of the flow, $\rho$ is the mass density, $p$ the pressure, $B$ is the magnetic field, $\mu_0$ is the permeability of vacuum, $\nu$ is the kinematic viscosity, $\eta$ the magnetic resistivity, $\kappa$ the thermal conductivity of the fluid, $T$ is the temperature, $g$ is the constant gravity and $D/Dt = (\partial/\partial t) + \mathbf{v} \cdot \nabla$. Additionally, consider the Boussinesq approximation, where the mass density is considered constant $\rho = \rho_0$, where $\rho_0$ is the density at the lower boundary, except in the buoyancy term ($\rho g$), where $\rho$ is given by $\rho = \rho_0[1 - \beta(T - T_b)]$, where $\beta$ is the constant thermal expansion coefficient and $T_b$ is the temperature at the bottom plate.

By adopting a stream function for the flow velocity, a vector potential for the magnetic field and employing a one-mode Fourier representation, Macek and Strumik [21] obtained the following generalized Lorenz model

$$\dot{x} = -\sigma x + \sigma y - \omega_0 w, \quad (5)$$

$$\dot{y} = -xz + rx - y, \quad (6)$$

$$\dot{z} = xy - bz, \quad (7)$$

$$\dot{w} = \omega_0 x - \sigma_m w, \quad (8)$$

where $x$ is the Fourier coefficient related to the stream function, $y$ and $z$ are the coefficients related to the temperature fluctuation and $w$ is the coefficient related to the magnetic vector potential. The dot denotes derivative with respect to the normalized time $t' = (1 + a^2)\kappa(\pi/h)^2 t$, where $a$ is a parameter associated with the width $(h/a)$ of the convective rolls at the onset of convection. The overdot denotes derivative with respect to the normalized time $t' = (1 + a^2)\kappa(\pi/h)^2 t$, where $a$ is a parameter associated with the width $(h/a)$ of the convective rolls at the onset of convection. The other parameters are $b = 4/(1 + a^2)$, the Prandtl number $\sigma = \nu/\kappa$, the magnetic Prandtl number $\sigma_m = \eta/\kappa$, the normalized Rayleigh number $r = Ra/R_a$, where $Ra = g\beta h^3 \delta T/(\nu \kappa)$ and $R_a = (1 + a^2)^2(\pi^2/a)^2$, and $\omega_0 = v_{A0}/v_0$ is responsible for the strength of the imposed magnetic field, with $v_0 = 16\pi^2\kappa/(abh\mu_0)$ and $v_{A0} = B_0/(\mu_0 \rho_0)^{1/2}$ is the Alfvén speed. When $\omega_0 = 0$ the traditional Lorenz model is obtained.

3 Nonlinear Dynamics Analysis

Equations (5–8) are solved with a fourth-order Runge-Kutta integrator. Just like the original Lorenz system, the generalized model exhibits symmetry under reflection through the $z$ axis, i.e., the system equations are reversible under
We adopt the Poincaré map defined by $x \mapsto -x$, $y \mapsto -y$, $w \mapsto -w$. Figure 1 illustrates this symmetry, where a periodic attractor (red) and its symmetric counterpart (green) are plotted. Due to this property, the nonlinear evolution of an attractor under changes in control parameters is mimicked by its symmetric attractor, with identical bifurcations taking place simultaneously in different parts of the phase space. We refer to both attractors as $A_1$ and $A_2$.

![Figure 1: Symmetry under reflection through the z axis shown in projection to the x,z plane for periodic attractors A1 (red) and A2 (green) for $\sigma = 10$, $b = 8/3$, $\sigma_m = 0.1$, $\omega_0 = 5.95$ and $r = 443$.](image)

We adopt the Poincaré map defined by $x = 0$ with $\dot{x} > 0$, thus a point is plotted everytime a solution crosses the hyperplane $x = 0$ from “left” to “right”. Figure 2(a) shows the bifurcation diagram of the generalized Lorenz model for $\sigma = 10$, $b = 8/3$, $\sigma_m = 0.1$ and $\omega_0 = 5.95$, while $r$ is varied between 430 and 480. This set of parameter values was chosen following Macek and Strumik [21]. For each value of $r$, an initial condition is integrated until the orbit converges to an attractor, after which we start plotting the $z$ component of the Poincaré points in red dots. There is a period-2 periodic window starting with a saddle-node bifurcation (SNB) at $r \approx 455.47$. By reducing $r$, the period-2 attractor undergoes a flip bifurcation at $r \approx 442.35$, where its period in the Poincaré map duplicates, going from period-2 to period-4. As the reduced Rayleigh number $r$ is further decreased, a cascade of period-doubling (flip) bifurcations takes place, leading to a small chaotic attractor localized in two narrow bands. The two green lines inside the window represent the evolution of attractor $A_2$ in parallel with $A_1$. The grey dots represent the transient chaotic behaviour displayed by the trajectories before they converge to either $A_1$ or $A_2$ and were found with the sprinkler method [11]. The chaotic transients are due to a chaotic saddle surrounding the attractors, and we denote this chaotic saddle by $\Lambda_s$. The window ends in a merging crisis (MC) at $r \approx 437.73$, where $A_1$ and $A_2$ simultaneously collide with the surrounding chaotic saddle and the three sets merge, leading to the formation of a large chaotic attractor. To the left of MC as well as to the right of SNB, the trajectories of $A_1$ and $A_2$ are united in a single large chaotic attractor, except in some narrow periodic windows, where the two attractors split again. Figure 2(b) displays the three largest Lyapunov exponents ($\lambda_1 > \lambda_2 > \lambda_3$) of the attracting sets in Fig. 2(a) and is a reproduction of Fig. 2 of Macek and Strumik [21]. Note that $\lambda_1$ suddenly drops to negative values at SNB, rises to positive values near MC and then, suddenly jumps to a much higher value at MC. There is an interval for $r > 454.7$ where hyperchaos is found, with $\lambda_1 > 0$ and $\lambda_2 > 0$, a phenomenon first reported in this system by Macek and Strumik [21].

Inside the periodic window, $A_1$ and $A_2$ attract different sets of initial conditions, defined as basins of attraction. The boundary between these basins is highly complex, as illustrated by Fig. 3 for $r = 453$, where red dots represent initial conditions whose trajectories eventually converge to $A_1$ and green dots represent initial conditions that converge to $A_2$. The complexity is scale invariant, as successive amplifications of a region in the phase space do not simplify the picture, as confirmed by Fig. 4(a), where a two-dimensional slice of the phase space is shown nearby the period-2 attractors $A_1$ (black squares) and $A_2$ (blue circles). These points represent the Poincaré points of the symmetric attracting trajectories shown in Fig. 1 but in a different projection $(w \times y)$. Figure 4(b) shows an enlargement of a region around one of the Poincaré points of $A_1$, where it is clear that the basin only becomes smooth in the close vicinity of the attractor, with an apparently fractal structure otherwise.

The fractal dimension of the basin boundary can be estimated with the aid of the uncertainty exponent [8]. Start with a line connecting two points in the phase space, arbitrarily chosen. Then, randomly choose $N_s = 10000$ initial conditions on this line and determine to which basin of attraction each of them belongs. Next, displace each initial condition by adding a small perturbation $\varepsilon$. A point is considered uncertain if its perturbation converges to a different attractor. Compute the number of uncertain points $N(\varepsilon)$ for different values of $\varepsilon$ and obtain the fraction of uncertain points as...
Figure 2: (a) Bifurcation diagram showing the Poincaré points of state variable $y$ as a function of the control parameter $r$, with $\sigma = 10$, $b = 8/3$, $\sigma_m = 0.1$ and $\omega_0 = 5.95$. The gray area represents the chaotic saddle $\Lambda_s$, red dots represent attractor $A_1$ and green dots represent attractor $A_2$; MC denotes merging crisis and SNB denotes saddle-node bifurcation. (b) The three largest Lyapunov exponents, $\lambda_1$, $\lambda_2$ and $\lambda_3$, for the attractors in (a) as a function of $r$.

Figure 3: Basins of attraction for attractors $A_1$ (red dots) and $A_2$ (green dots) on the Poincaré map ($x = 0$) for $r = 453$. The other parameters are as in Fig. 2.
In fractal basin boundaries, the fraction scales as $f \sim \varepsilon^\alpha$, so $\alpha$ is the slope of the linear relation between $\log(f)$ and $\log(\varepsilon)$. The graph of $\log(f) \times \log(\varepsilon)$ is shown in Fig. 5 from which the slope of the linear regression is $\alpha = 6 \times 10^{-4} \pm 4 \times 10^{-4}$. The dimension of the set of intersecting points of the basin boundary with the 1D line is $d_s = 1 - \alpha$. In the full 3D Poincaré map, the dimension of the basin boundary is $D_s = 3 - \alpha = 2.9994$, a value extremely close to the dimension of the phase space.
The intricacy of the basin boundary and its high fractal dimension result in long chaotic transients before the solutions settle to an attractor. Figure 6 shows a solution with a chaotic behaviour up to \( t_p \approx 13600 \) iterations of the Poincaré map for \( r = 453 \), before the system converges to a period-2 attractor.

As mentioned before, these chaotic transients are due to the presence of a chaotic saddle in the phase space. Inside the periodic window, this chaotic saddle \( \Lambda_s \) is located at the boundary between the basins of attractors \( A_1 \) and \( A_2 \), which coincides with the stable manifold of the chaotic saddle [23, 33]. In order to find \( \Lambda_s \), we first determine the average transient time \( \tau \) of initial conditions in the phase space. We define the lifetime of a trajectory as the time it takes to go from its initial condition until the close vicinity of an attractor. For \( r = 453 \), we define a \( 500 \times 100 \) grid of initial conditions in the \((w \times y)\) plane, with the other state variables fixed. Let \( N_0 \) be the number of initial conditions in the grid and let \( N(t) \) be the number of trajectories from those initial conditions that have not converged to any attractor after \( t \) iterations of the Poincaré map. These trajectories must be near the chaotic saddle \( \Lambda_s \) and, due to its chaotic nature, the probability that the trajectory has not yet escaped from the vicinity of the chaotic saddle on time \( t \) decays exponentially with time [12]

\[
P(t) \sim \exp(-\kappa t),
\]

for some \( t > t_0 \), where \( \kappa \) is the decay rate. Following Hsu et al. [11], Lai and Winslow [15], Sweet and Ott [30], we write \( N(t) \sim N(t)/N_0 \) and \( \kappa = 1/\tau \).

\[
N(t) \sim N_0 \exp(-t/\tau).
\]

Equation (11) defines a linear dependence between \( \log N(t) \) and \( t \), with \(-1/\tau\) as the slope. Thus, the average transient time \( \tau \) can be computed by considering the inverse of the slope of the graph of \( \log(N(t)) \times t \), as shown in Fig. 7. The estimated value from linear regression is \( \tau \approx 1111 \).

Figure 7: Log-linear plot of \( N(t) \), the number of orbits that have not converged to an attractor after \( t_p \) Poincaré map iterations, as a function of \( t_p \) for \( r = 453 \). The other parameters are as in Fig. 2. The slope of the fitted line is \( \beta = -9 \times 10^{-4} \pm 6.5 \times 10^{-6} \), providing an average transient time of \( \tau \approx 1111 \) Poincaré map iterations.
We now describe how the sprinkler method is used to find $\Lambda_s$. First, we find on a grid the set of initial conditions with trajectories that are still chaotic after a long time $t_c$. The value of $t_c$ must be large compared to the average lifetime $\tau$, so we choose $t_c = 3000$. These initial conditions will first approach $\Lambda_s$ through its stable manifold, stay in its vicinity for some time before they depart along the unstable manifold toward an attractor. Thus, if we iterate all selected initial conditions until $t_m = t_c/2$, their trajectories must be very close to $\Lambda_s$. The set of all those points approximate the chaotic saddle. Figure 8(a) depicts the $(w \times y)$ components of the chaotic saddle in the beginning of the periodic window, at $r = 453$, to the left of SNB in Fig. 2(a), when the attractors are periodic. The period-2 attractors are plotted as red ($A_1$) and green ($A_2$) crosses. A comparison with the chaotic attractor to the right of SNB, shown in Figure 2(b) for $r = 455.48$, reveals that the chaotic saddle is formed by a continuation of many of the recurrent points found in the pre-window chaotic attractor, as discussed in Robert and K. T. Alligood [29].

The Lyapunov exponents of the chaotic saddle $\Lambda_s$ can be approximately computed as the Lyapunov exponents of a long chaotic transient. Figure 9 shows the convergence of the first three Lyapunov exponents for a long chaotic transient at $r = 453$. We can compare the value $\lambda_1 \approx 1.5$ of the chaotic saddle with the maximum Lyapunov exponent of the chaotic attractor just to the right of SNB in Fig. 2(b). For $r = 455.58$, $\lambda_1 \approx 1.51$ for the chaotic attractor, a value very close to the maximum Lyapunov exponent of $\Lambda_s$, confirming the relation of continuation between both sets.

The maximum Lyapunov exponent $\lambda_1$ of the chaotic saddle can also be estimated from the average lifetime $\tau$ and from $d_s$, the fractal dimension of the set of intersecting points of a one-dimensional line with the stable manifold of the chaotic saddle. For two-dimensional maps, this relation is given by $\tau = 1/[(1-d_s)\lambda_1]$.

Note that Kantz and Grassberger [12] had previously derived a more general relation for the decay rate as $\kappa = (1-D_1)\lambda_1$, where $D_1$ is the partial information dimension of the chaotic saddle. For a review of this topic, see the chapter 8 of Lai and Tél [16].
For certain higher-dimensional phase spaces, it has been argued that the same relation holds \[15\]. Using the previously computed values of \( \tau \) and \( d_s \) in Eq. (12) for \( r = 453 \), we obtain
\[
\lambda_1 = 1/[(1 - (1 - \alpha))1111].
\] (13)
The error bar in \( \alpha = 6 \times 10^{-4} \pm 4 \times 10^{-4} \) is too large for a precise estimation of \( \lambda_1 \) with Eq. (13). Nonetheless, we observe that using the mean value \( \alpha = 6 \times 10^{-4} \) yields
\[
\lambda_1 = 1/[(1 - 0.9994)1111] \approx 1.5.
\] (14)
The value agrees quite well with the one computed directly from the chaotic transient in Fig. 9 confirming that the chaotic transients are due to the chaotic saddle localized on the fractal basin boundary.

4 Discussion and Conclusions

Transient chaos is still an overlooked topic in most textbooks and theoretical works on dynamical systems, where the tendency is to focus exclusively on the asymptotic dynamics, after trajectories have converged to an attractor. In practice, however, it is usually not possible to proof that an observed behaviour is asymptotic or not, as experimental results can only confirm a certain behaviour up to a finite time scale, as noted by Tél \[33\]. This has strong implications in areas such as transition to turbulence in pipe flows, where long chaotic transients have been observed and it is difficult to find a critical Reynolds number where the system switches from laminar to persistent turbulence and chaotic saddles play a crucial role \[6\]. In this case, the lifetime of the transient chaos follows a supertransient law, i.e., it grows exponentially as a function of the control parameter \[34\], a phenomenon also observed in numerical simulations of Keplerian shear flows in the context of accretion disks \[28, 32\]. For other applications of transient chaos, including intermittency and scattering in leaking systems, see Tél and Lai \[34\], Altmann \[1\] and Tél \[33\].

Our results illustrate the importance of transient chaos in dynamical systems in general and in this magnetoconvection model in particular. The existence of very long chaotic transients as seen in Fig. 6 can mask the true attractors of the system if the time scales considered are smaller than the average transient time. In the case of the Lorenz system, we stress that the presence of a magnetic field causes an increase in the transient lifetime in the range of parameters studied in this paper. It is also worth pointing that the original Lorenz model displayed multistability, with up to three coexisting basins of attraction reported by Yorke and Yorke \[38\], but the basin boundaries are not fractal in those cases \[37\]. We are unaware of the presence of basin boundaries with a fractal dimension close to the dimension of the phase space (as seen in Figs. 3 and 4) in the original Lorenz system. The high fractal dimension is directly related to the long transients reported above, as indicated by Eq. (12).

Regarding the non-dimensional control parameters, they were chosen according to the values commonly employed in the original Lorenz model to focus on the impact of the addition of a magnetic field. The new parameters are \( \omega_0 = 5.95 \), responsible for the intensity of the background magnetic field, and the magnetic Prandtl number \( \sigma_m = 0.1 \), whose values were chosen following Macek and Strumik \[21\]. Although it is a low \( \sigma_m \) value, this value is orders of magnitude higher than what is found in the Earth’s liquid outer core (\( \sigma_m \sim 10^{-6} \)) or in stellar interior (\( 10^{-4} \leq \sigma_m \leq 10^{-6} \)) \[22\]. In fact, considering the relevance of the present work for the study of magnetoconvection, we don’t claim that
Macek’s reduced Lorenz model provides realistic simulations of magnetized convection in stellar or Earth’s interior, where this phenomenon is responsible for the dynamo that maintains the magnetic fields [35]. However, local and global bifurcations, chaotic saddles and transients similar to the ones displayed by this model have also been observed in more realistic direct numerical simulations of three-dimensional Rayleigh-Bénard convection [4]. Thus, the analysis of the reduced model can shed light to the understanding of the complexity present in magnetoconvection and guide future nonlinear analyses of more realistic models.

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References

[1] Altmann, E. G., Portela, J. S. E., and Tél, T. [2013] “Leaking chaotic systems”, Reviews of Modern Physics 85, 869–918.
[2] Battelino, P. M., Grebogi, C., Ott, E., and Yorke, J. A. [1988] “Multiple coexisting attractors, basin boundaries and basic sets”, Physica D 32, 296–305.
[3] Chandrasekhar, S. [1981] Hydrodynamic and hydromagnetic stability (Dover Publications, New York).
[4] Chertovskih, R., Chimanski, E., and Rempel, E. L. [2015] “Route to hyperchaos in Rayleigh–Bénard convection”, Europhysics Letters 112, 14001.
[5] Chian, A. C.-L., Santana, W. M., Rempel, E. L., Borotto, F. A., Hada, T., and Kamide, Y. [2007] “Chaos in driven Alfvén systems: unstable periodic orbits and chaotic saddles”, Nonlinear Processes in Geophysics 14, 17–29.
[6] Eckhardt, B., Schneider, T. M., Hof, B., and Westerweel, J. [2007] “Turbulence Transition in Pipe Flow”, Annual Reviews of Fluid Mechanics 39, 447–468.
[7] Grebogi, C., Ott, E., and Yorke, J. [1982] “Chaotic attractors in crisis”, Physical Review Letters 48, 1507–1510.
[8] Grebogi, C., McDonald, S. W., Ott, E., and Yorke, J. A. [1983] “Final state sensitivity: an obstruction to predictability”, Physics Letters A 99, 415–418.
[9] Grebogi, C., Ott, E., Romeiras, F., and Yorke, J. [1987] “Critical exponents for crisis-induced intermittency”, Physical Review A 36, 5365–5380.
[10] Haken, H. [1975] “Analogy between higher instabilities in fluids and lasers”, Physics Letters A 53, 77–78.
[11] Hsu, G.-H., Ott, E., and Grebogi, C. [1988] “Strange saddles and the dimension of their invariant manifolds”, Physics Letters A 127, 199.
[12] Kantz, H., and Grassberger, P. [1985] “Repellers, semi-attractors, and long-lived chaotic transients”, Physica D 17, 75–76.
[13] Knobloch, E. [1981] “Chaos in the segmented disc dynamo”, Physics Letters A 82, 439–440.
[14] Knobloch, E. [1992] “Heteroclinic bifurcations in a simple model of double-diffusive convection”, Journal of Fluid Mechanics 239, 273–292.
[15] Lai, Y.-C. and Winslow, R. L. [1994] “Geometric properties of the chaotic saddle responsible for supertransients in spatiotemporal chaotic systems”, Physical Review Letters 74, 5208–5211.
[16] Lai, Y.-C., and Tél, T. [2011] “Transient Chaos: Complex Dynamics on Finite-Time Scales”, Springer - New York.
[17] Lorenz, E. N. [1963] “Deterministic nonperiodic flow”, Journal of the Atmospheric Sciences 20, 130–141.
[18] Macau, E. E. and Caldas, I. L. [2002] “Driving trajectories in chaotic scattering”, Physical Review E 65, 026 215.
[19] Macek, M. W. and Strumik, M. [2010] “Model for hydromagnetic convection in a magnetized fluid”, Physical Review E 82, 027 301–1–027 301–4.
[20] Macek, W. M. [2018] “Nonlinear dynamics and complexity in the generalized Lorenz system”, Nonlinear Dynamics 94, 2957–2968.
[21] Macek, W. M. and Strumik, M. [2014] “Hyperchaotic intermittent convection in a magnetized viscous fluid”, Physical Review Letters 112, 074 502.
[22] Mondal, H., Das, A., and Kumar, K. [2018] “Onset of oscillatory Rayleigh-Bénard magnetoconvection with rigid horizontal boundaries”, Physics of Plasmas 25, 012 119.
[23] Péntek, A., Toroczkai, S., Tél, T., Grebogi, C., and Yorke, J. A. [1995] “Fractal boundaries in open hydrodynamical flows: signature of chaotic saddles”, Physical Review E 51, 4076–4088.

[24] Poland, D. [1993] “Cooperative catalysis and chemical chaos: a chemical model for the Lorenz equations”, Physica D 65, 86–99.

[25] Rempel, E. L. and Chian, A. C.-L. [2003] “High-dimensional chaotic saddles in the Kuramoto-Sivashinsky equation”, Physics Letters A 319, 104–109.

[26] Rempel, E. L. and Chian, A. C.-L. [2005] “Intermittency induced by attractor-merging crisis in the Kuramoto-Sivashinsky equation”, Physical Review E 71, 016 203.

[27] Rempel, E. L., Chian, A. C.-L., and Miranda, R. A. [2007] “Chaotic saddles at the onset of intermittent spatiotemporal chaos”, Physical Review E 76, 056 217.

[28] Rempel, E. L., Lesur, G., and Proctor, M. R. E. [2010] “Supertransient Magnetohydrodynamic Turbulence in Keplerian Shear Flows”, Physical Review Letters 105, 044 501.

[29] Robert, C. and K. T. Alligood, E. Ott, J. A. Y. [2000] “Explosion of chaotic sets”, Physica D 144, 44–61.

[30] Sweet, D. and Ott, E. [2000] “Fractal dimension of higher-dimensional chaotic repellors”, Physica D 139, 1–27.

[31] Szabó, K. G. and Tél, T. [1994] “Transient chaos as the backbone of dynamics on strange attractors beyond crisis”, Physics Letters A 196, 173–180.

[32] Teixeira, D. M. and Rempel, E. L. [2018] “Supertransient magnetohydrodynamic turbulence in Keplerian shear flows: The role of the Hall effect”, Europhysics Letters 124, 59 001.

[33] Tél, T. [2015] “The joy of transient chaos”, Chaos 25, 097 619.

[34] Tél, T. and Lai, Y.-C. [2008] “Chaotic transients in spatially extended systems”, Physics Reports 460, 245–275.

[35] Weiss, N. and Proctor, M. R. E. [2014] Magnetoconvection (Cambridge University Press, Cambridge).

[36] Weiss, N. O., Cattaneo, F., and Jones, C. A. [1984] “Periodic and aperiodic dynamo waves”, Geophysical and Astrophysical Fluid Dynamics 30, 305–341.

[37] Xiong, A., Sprott, J. C., Lyu, J., and Wang, X. [2017] “3D Printing — The Basins of Tristability in the Lorenz System”, International Journal of Bifurcation and Chaos 27, 1750 128.

[38] Yorke, J. A. and Yorke, E. D. [1979] “Metastable Chaos: The Transition to Sustained Chaotic Behavior in the Lorenz Model”, Journal of Statistical Physics 21, 263–277.