The complex Toda chains and the simple Lie algebras \{ solutions and large time asymptotics \}

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Abstract

The asymptotic regimes of the N-site complex Toda chain (CTC) with fixed ends related to the classical series of simple Lie algebras are classified. It is shown that the CTC models have much richer variety of asymptotic regimes than the real Toda chain (RTC). Besides asymptotically free propagation (the only possible regime for the RTC), CTC allow bound state regimes, various intermediate regimes when one (or several) group(s) of particles form bound state(s), singular and degenerate solutions. These results can be used e.g., in describing the soliton interactions of the nonlinear Schrodinger equation. Explicit expressions for the solutions in terms of minimal sets of scattering data are proposed for all classical series \( B_r \{ D_r \).
1 Introduction

The Toda chain model \[ \frac{d^2 q_k}{dt^2} = \exp(q_{k+1} - q_k) \exp(q_k - q_{k-1}) \]; (1)
is one of the paradigms of integrable nonlinear chains and lattices. It has been thoroughly studied for a number of initial and boundary conditions, such as:

- xed ends boundary conditions, i.e. \( q = q_{N+1} = 1 \); this will be the case we are interested in;
- in finite chain \( 1 < k < 1 \) with \( \lim_{k \to 1} q_k = 0 \) and \( \lim_{k \to 1} q_k = \text{const}, \) or equivalently, \( \lim_{k \to 1} (q_{k+1} - q_k) = 0; \)
- quasi-periodic boundary conditions \( q_{N} = q_1 + c, \) where \( c = \text{const}. \)

This model appeared rst in describing the oscillations of a one-dimensional crystalline lattice [1]. Since then many other applications became known, see e.g. [2].

The model (1) is directly related to the algebra \( \mathfrak{sl}(N) \), where \( N \) is the number of sites of the chain. Most of the references cited above are devoted to the case when \( q_k \) (\( t \)) are real-valued functions. That is why for de niteness we will call this model real Toda chain (RTC). Other generalizations of the RTC are related to: a) simple Lie algebras; b) an (or Kac-Moody) algebras; c) 2-dimensional generalizations.

Another possibility for generalizing the RTC, which as far as we know has not been investigated, is the complex Toda chain (CTC). We see two main reasons for this:

(i) the generic solutions of the CTC are readily obtained from the ones of RTC by simply making all parameters complex. In fact, technically solving the RTC requires additional eorts in ensuring that the scattering data of the Lax matrix \( L \) are real-valued.

(ii) the CTC has not been known to have physical applications.

Recently however, it was discovered [3,14,15] that the CTC describes the \( N \)-soliton train interaction of the nonlinear Schrodinger (NLS) equation.

\[ \frac{d^2 q_k}{dt^2} = \exp(q_{k+1} - q_k) \exp(q_k - q_{k-1}) \]
in the adiabatic approximation. The variables $q_k(t)$ in (1) are related to the solitons parameters by

$$ q_k = 2 \, \Omega_k + i (2 \, \Omega_k \, \kappa) + \text{const} $$

(2)

where $\kappa$ and $\Omega_k$ characterize the center-of-mass position and the phase of the $k$-th soliton in the chain; $\Omega_0$ and $\kappa_0$ are the average amplitude and velocity of the soliton train. Such soliton trains and their asymptotic behavior appear to be important for the needs of soliton-based fiber optics communications.

In what follows we will view the CTC as a model of "complex" particles characterized by a coordinate and phase which, in analogy with (2), are related to $\text{Re} q_k(t)$ and $\text{Im} q_k(t)$ respectively.

Another reason for the present paper is in the fact, that along with the similarities between the solutions of CTC and RTC, there are also important qualitative differences between the asymptotic properties of these solutions.

The purpose of the present paper is to derive analytically the large-time asymptotics of the solutions to the CTC models:

$$ \frac{d^2 q}{dt^2} X^2 \frac{d^2 q}{dt^2} H_k = X \, H \, e \, (q';) $$

(3)

related to the classical series of simple Lie algebras extending the results of [13, 14]. Here $g$ is the set of simple roots of the algebra $g$, $q(t)$ is a complex-valued function of $t$ taking values in the Cartan subalgebra $h$ of $g$; $H_k$ form a basis in $h$ dual to the orthonormal basis $e_k, f_k, \gamma_k$ in the root space $E^r$ and $r$ is the rank of $g$; $H = E_{k-1} \, (e_k) H_k$. For more details about the structure of the simple Lie algebras, see e.g. [15]. Equation (3) is a particular case of (3) for the $A_r$ series, i.e., for $g = \text{sl}(r+1)$. These results, and especially the ones for the series $A_r$, can be used as a tool for deriving the asymptotic behavior of the $N$-soliton trains from the initial set of soliton parameters [13, 14, 15].

We also specify the minimum sets of scattering data $\mathcal{T}_g$ for $L$ which determine uniquely both $L$ and the solutions of (3) and obtain explicit expressions for the solutions of (3) in terms of $\mathcal{T}_g$.

2 Comparison between RTC and CTC

Since the paper of Moser [13] on the finite nonperiodic real Toda lattice, there have been proposed many methods for solving (3) for the various choices
of the initial and boundary conditions, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and the numerous references therein. The RTC model was also extended to indefinite metric spaces [19] and was shown to possess singular solutions "blowing up" for nine values oft. These models can be viewed as special cases of the CTC in which part of the $a_k$'s defined below are real while the rest are purely imaginary.

As we already mentioned, a number of properties of the CTC are obtained from the corresponding ones of RTC trivially by assuming the corresponding variables complex. We list the four most important of them below:

S1) The Lax representation: There are several equivalent formulations of the Lax representation for (3). Below we will use the "symmetric" one:

$$L(t) = \sum_{k=1}^{N} \left( \varphi_k H_k + a_k (E_k + E_k^*) \right);$$

$$M(t) = \frac{1}{2} \sum_{k=1}^{N} a_k (E_k + E_k^*);$$

where $a_k = \frac{1}{2} \exp \left( (q; q) = 2 \right)$ and $b_k = \frac{1}{2} dq_k dt$ for $g \in \mathfrak{sl}(N)$ we have $a_k = \frac{i}{2} \exp \left( (q_{k+1} - q_k) = 2 \right)$. It is well known that to each root $g \in \mathbb{R}^r$ one can put into correspondence the element $H \in \mathbb{R}^r$. Analogously, to $q(t) = \text{Re} q(t) + i \text{Im} q(t)$ there corresponds the vector $q(t) = \text{Re} q(t) + i \text{Im} q(t)$, whose real and imaginary parts are vectors in $\mathbb{R}^r$.

S2) The integrals of motion in involution are provided by the eigenvalues, $\lambda_k$, of $L$.

S3) The solutions of both the CTC and the RTC are determined by the scattering data for $L_0 L(0)$. When the spectrum of $L_0$ is nondegenerate, i.e. $k \neq j$ for $k \neq j$, then this scattering data consists of

$$T = \ell_1; \ldots; \ell_N \ell_1; \ldots; \ell_N g;$$

where $\ell_k$ are the first components of the corresponding eigenvectors $r_k = V_lk$ of $L_0$ in the typical representation $R(1_1)$ of $g, N = \dim R(1_1)$:

$$L_0 V = V Z; \quad Z = \text{diag} (1; \ldots; N);$$

They are determined (up to an overall sign) by the normalization conditions:

$$\sum_{k=1}^{N} (V_{s k})^2 = (V^{(s)}; V^{(s)}) = 1; \quad s = 1; \ldots; N;$$

\[4\]
S4) Lastly, the eigenvalues of $L_0$ uniquely determine the asymptotic behavior of the solutions; these eigenvalues can be calculated directly from the initial conditions. We will extensively use this fact for the description of the different types of asymptotic behavior.

However, there are important differences between the RTC and CTC, especially the asymptotic behavior of their solutions. Indeed, for the RTC, one has \([3,6]\) that, both the eigenvalues, $\lambda_k$, and the coefficients, $r_k$, are always real-valued. Moreover, one can prove that $\lambda_k \neq \lambda_j$ for $k \neq j$, i.e. no two eigenvalues can be exactly the same. As a direct consequence of this, it follows that the only possible asymptotic behavior in the RTC is an asymptotically separating, free motion of the particles (solitons).

This situation is different for the CTC. Now the eigenvalues $\lambda_k = \lambda_k + i \epsilon_k$, as well as the coefficients $r_k$ become complex. Furthermore, the argument of Moser can not be applied so one can have multiple eigenvalues. The collection of eigenvalues, $\lambda$, still determines the asymptotic behavior of the solutions. In particular, it is $\lambda_k$ that determines the asymptotic velocity of the $k$-th particle (soliton). For simplicity, we assume $\lambda_k \neq \lambda_j$ for $k \neq j$. However, this condition does not necessarily mean that $\lambda_k \neq \lambda_j$. We also assume that the $\lambda_k$'s are ordered as:

$$\lambda_1 < \lambda_2 < \cdots < \lambda_N.$$  \hfill (9)

Once this is done, then for the corresponding set of $N$ particles (train of $N$ solitons), there are three possible general configurations:

- D1) $\lambda_k \neq \lambda_j$ for $k \neq j$. Since the asymptotic velocities are all different, one has the asymptotically separating, free particles (solitons).

- D2) $\lambda_1 = \lambda_2 = \cdots = \lambda_N$. In this case, all $N$ particles (solitons) will move with the same mean asymptotic velocity, and therefore will form a "bound state". The key question now will be the nature of the internal motions in such a bound state.

- D3) One may have also a variety of intermediate situations when only one group (or several groups) of particles move with the same mean asymptotic velocity; then they would form one (or several) bound state(s) and the rest of the particles will have free asymptotic motion.

Obviously the cases D2) and D3) have no analogues in the RTC and physically are qualitatively different from D1). The same is also true for the special degenerate cases, where two or more of the $\lambda_k$'s may become equal.

See \([3, \overline{3}, \overline{6}]\); then $V^T = V^{-1}$. 

and for the singular solutions. These cases will be considered briefly below.

3 Solutions of the CTC

The solutions for the CTC can be formally obtained from the well known ones for RTC by inserting the corresponding complex parameters. We remind the reader of the RTC solution for $g'_{\text{sl}(N)}$, see [3,15,20], and the references therein. Here we express the mass center at the origin by

$$
q^N_{k=1} (t) = 0:
$$

The velocity of the center of mass is given by $\mathbf{tr} \mathbf{L}_0 = \sum_{k=1}^N k = 0$, due to $\mathbf{L}_2 \mathbf{sl}(N)$. Then the solution has the form:

$$
q_k (t) = q_k (0) + \ln \frac{A_k}{A_{k-1}}
$$

where $A_0 = 1$,

$$
A_1 (t) = \sum_{k=1}^N r_k^2 e^{-2k^2 t}
$$

$$
A_k (t) = \sum_{1 < k < k_1 < \ldots < k_N} (r_k r_{k_1} \cdots r_{k_N})^2 W^{(2)} (l_1; l_2; \ldots ; l_k) e^{2(l_1 + \cdots + l_k)t}
$$

and

$$
A_N = W^{(N)} (l_1; l_2; \ldots ; l_N) = e^{2N q_1 (0)}:
$$

By $W (l_1; \ldots ; l_k)$ we denote the Wankel one determinant:

$$
W (l_1; \ldots ; l_k) = \sum_{s+p = k} (2 \sigma_l \sigma_2)_{s+p}:
$$

$$
\sigma_s \sigma_p (l_1; \ldots ; l_k) = (q(t); !_{k}) (q(0); !_{k}) = \ln h_l k \exp (2L_0 t) j_k i_l
$$
where \( k \) are the fundamental weights of \( g \). The fact that the large time asymptotics of \( q_k(t) \) for the \( g' \) \( \text{sl}(N) \) RTC have the form:

\[
\lim_{t \to 1} (q_k(t) \ v_k(t)) = k
\]

(17)

where the asymptotic velocities \( v_k^+ = 2k \) and \( v_k^- = 2N+1-k \) have been derived by Moser [3]. He also evaluated the differences \( N \ k+1 \) which characterizes the interaction in the RTC model.

The minimal set of scattering data for \( g' \) \( \text{sl}(N) \) is obtained from (3) by imposing on \( T \) the restrictions \( \sum_{k=1}^{N} k = 0 \) and

\[
\chi_k^N, k = 1:
\]

(18)

Note that \( \exp(\ q_0(0)) \) is expressed through \( T \) by \( \{14\} \).

For the RTC related to the other classical series of Lie algebras it is known \( \{15,16\} \) that

\[
\lim_{t \to 1} (q(t) \ v(t)) = ;
\]

(19)

\( v \neq 2h, \ 2h \ \text{and} \ v^+ = w_0(v) \), where \( w_0 \) is the element of the Weyl group, which maps the highest weight of each irreducible representation of \( g \) into the lowest weight. The action of \( w_0 \) on the simple roots is well known [18]:

\[
w_0(k) = k;
\]

(20)

\( K = r \ k+1 \) for \( A_r \), \( K = k \ k = 1; \cdots ; r \) for \( B_r \), \( C_r \) and, when \( r \) is even, also for \( D_r \). When \( g' \) \( D_r \) and \( r \) is odd \( K = k \) for \( r \ 2 \) and \( w_0(r-1) = -r \), \( w_0(1) = 1 \).

What we will do below is to: i) specify how minimal sets of scattering data \( T_g \) can be extracted from \( \{9\} \); ii) nd explicit expressions for \( \ k \) ( \( e_k \) ) for each of the classical Lie algebras in terms of \( T_g \).

Like in the \( \text{sl}(N) \) -case \( k \) and \( x_k \) are the eigenvalues and the \( x \) st components of the eigenvectors of \( L_0 \) in the typical representation, namely:

\[
L_0 V = \sum_{k=1}^{x_N} k H_k;
\]

(21)

The requirement that \( L_0 \) (and as consequence, \( L \)) belongs to one of the algebras in the \( B_r \) or \( C_r \) series imposes on \( q_k \) the following natural restrictions:

\[
q_k = q_N \ k+1
\]

(22)
which leads to

\[ a_k = a_{N-k} \quad \text{(23)} \]
\[ b_k = b_{N+1-k} \quad \text{(24)} \]

Thus the solutions for \( g' \) B\(_r\) and C\(_r\) can formally be obtained from the ones for sl(\(N\)) \((11)-(14)\) by imposing on them the involutions \((22)-(24)\). So we have to find out what are the restrictions on \(T\) imposed by \((22)-(24)\); this will provide us with the corresponding minimal set of scattering data \(T_g\), which must obviously contain only \(2r\) parameters. It is easy to find that \(k = k\), so only \(r\) of them are independent. It is not so trivial to derive the corresponding relations which reduce the number of the coefficients \(r_k\). Our analysis shows that:

\[ r_k r_k = \exp \left( q_k(0) \right) w_k; \quad k = 1; \ldots; r \quad \text{(25)} \]

where \(k = N + 1\) and \(w_k\) is expressed in terms of \(k\). Below we provide the explicit formulas for \(w_k\) for each of the classical series B\(_r\), C\(_r\) and D\(_r\).

**B\(_r\)-series:** \(N = 2r + 1\). Note that in this case \(r+1 = 0\), and in addition to \((25)\):

\[ r_{r+1}^2 = \exp \left( q_k(0) \right) \frac{1}{2^{2r}(1 2 \cdots r)^2} \quad \text{(26)} \]

and the expression for \(w_k\) is provided by:

\[ w_k = \frac{1}{4} \sum_{s=1}^{k} \frac{1}{s} \frac{1}{2} \frac{1}{s} \frac{1}{4} \frac{1}{s} \frac{2^r}{s} \frac{1}{k} \frac{2}{s} \frac{1}{k} \frac{2}{s} ; \quad \text{(27)} \]

Inserting \((25)-(27)\) into \((18)\) we obtain a quadratic equation for \(\exp \left( q_k(0) \right)\), so it can be expressed in terms of \(T_g\).

**C\(_r\)-series:** \(N = 2r\). Here

\[ w_k = \frac{1}{4} \sum_{s=1}^{k} \frac{1}{s} \frac{1}{2} \frac{1}{s} \frac{1}{4} \frac{1}{s} \frac{2^r}{s} \frac{1}{k} \frac{2}{s} \frac{1}{k} \frac{2}{s} ; \quad \text{(28)} \]

Like for the B\(_r\)-series \(\exp \left( q_k(0) \right)\) is determined from \((15)\).

**D\(_r\)-series:** \(N = 2r\). Here

\[ w_k = \frac{1}{4} \sum_{s=1}^{k} \frac{1}{s} \frac{1}{2} \frac{1}{s} \frac{1}{4} \frac{1}{s} \frac{2^r}{s} \frac{1}{k} \frac{2}{s} \frac{1}{k} \frac{2}{s} ; \quad \text{(29)} \]
Again \( \exp (q_1(0)) \) is determined from (15) and (20). The derivation of the solution for this series requires some additional efforts. The main problem here is to find explicit parametrization for the right hand sides of (16) for \( k = r - 1 \) and \( r \) in terms of \( r_k \), which characterize the matrix \( V \) in (21) in the typical representation. Skipping the details we just present the result, namely that \( q_k \) with \( k = 1; \ldots ; r - 1 \) are again given by (11) (13) where \( k = k_k \) and \( r_k \) are restricted by (25), (29). For \( k = r \) the solution for \( q_r(t) \) is provided by (11) with

\[
A_r(t) = X_{1 < 2 < \cdots < N} (r_1, r_2, \ldots, r_k)^2 W^2 (l_1; l_2; \ldots, l_r) f_{l_1, \ldots, l_r}^2 e^{2(\lambda_1 + \cdots + \lambda_r)t} \tag{30}
\]

and

\[
f_{l_1, \ldots, l_r} = \frac{1}{2} \left( 1 + \frac{1}{1 + 2 \cdots + r} \right) \tag{31}
\]

The proof of these formulas is based on detailed analysis of the properties of the fundamental representations \( R(l_k) \) of \( g \) and of their tensor products.

We remind that these formulas are valid both for RTC and CTC. In the latter case one should be careful to avoid the subset of singular cases, when one or more of the functions \( A_k(t) \) may develop zeroes for finite values of \( t \), see also the discussion in Section V below.

4 Large time asymptotics

Let us now express large time asymptotics of \( q_k(t) \) in terms of the minimum set of scattering data \( T_g \) and analyze the different types of asymptotic regimes.

As we mentioned above, we view the CTC as a model describing non-trivial scattering of \( N \) complex particles so that \( \text{Re} q_k(t) \) and \( \text{Im} q_k(t) \) correspond to their coordinates and "phases".

D 1) A symptotically free states \( k \notin j \) for \( k \notin j \). It is specified by imposing on \( k = k + i k, k = 1; \ldots ; N \) the so-called sorting condition:

\[
1 < 2 < \cdots < k < N \tag{32}
\]

Now we have to express \( k \) in terms of \( T_g \). Skipping the details we list the results for the classical series of simple Lie algebras \( A_r \{ D_r \).
The corresponding \( T_g \) is formed from \( f_k; \) by imposing \( g^N_{k-1} \) by imposing \( \sum_{k=0}^{N} k = 0 \) and the normalization condition (13). Then

\[
\begin{align*}
    q_k(t) &= q_k(0) + \ln r_k^2 + \ln (2 s \ 2 \ k)^2; \\
    \eta_k &= q_k(0) + \ln r_k^2 + \ln (2 s \ 2 \ k)^2;
\end{align*}
\]

(33)

(34)

where \( k = N + 1 \). 

Now it is easy to calculate the shift of the relative position which is the effect of the particle interaction. Naturally these shifts are also complex-valued. If \( q_k \) correspond to NLS solitons then \( q_k^{\pm} \) will describe the shifts of both the relative positions and phases of the solitons, see formula (33). Note that in the class of regular solutions the particles do not collide, i.e. their trajectories do not intersect. Since we have ordered the particles by their velocities assuming \( 1 < 2 < \cdots < N \), so for \( t \to 1 \) the \( k \)-th particle will move with velocity \( 2 \ k \). For \( t \to 1 \) however we find that due to the interaction now the \( k \)-th particle moves with velocity \( 2 \ N_{k+1} = 2 \ k \). This is the so-called "sorting property" characteristic for the RTC. If we identify the \( k \)-th particle (soliton) by its velocity then its shift of position will be given by the real part of the relation:

\[
q_k^{\pm} = \sum_{j \neq k} \eta_j \ln (2 \ j \ 2 \ k)^2; 
\]

(35)

where \( \eta_j = 1 \) for \( j > k \) and \( \eta_j = 1 \) for \( j < k \). The imaginary part of (33) will provide the shift in the phase of the corresponding particle (soliton). Equation (33) is a natural generalization of the corresponding result of Moser [3] for RTC.

The asymptotics for the \( B_1, C_1 \) and \( D_r \) series have the form (13):

\[
\lim_{t \to 1} q_k(t) = 2 \ k t = k : 
\]

(36)

with

\[
q_k^{\pm} = q_k(0) + \ln r_k^2 + \ln (2 s \ 2 \ k)^2 
\]

(37)

\[
\]
\[ k = q_k(0) + \ln \frac{w_k^2}{r_k} + \ln (2s \times 2k)^2 \] for \( k = 1; \ldots ; r \).

The only exception is for the series \( D_r \) in the case of odd \( r \), \( k = r \) and \( t! \neq 1 \) which reads:

\[ \lim_{t!1} (q_r(t) + 2rt) = r \] (39)

where

\[ r = q_k(0) + \ln r_k^2 + \ln (2r \times 2s)^2 \] (40)

D 2) Bound states: \( 1 = 2 = \ldots = N = 0 \). Now all \( N \) particles will move with the same mean asymptotic velocity for both \( t! = 1 \) and \( t! = 1 \); by Galilean transformation this velocity can always be made 0. The individual velocities of the particles oscillate around the common mean value. In other words we find that all \( N \) particles generically do not separate but form a bound state with a large number of degrees of freedom. The explicit solutions for \( q \) now do not simplify even for \( t! = 1 \). Nevertheless two features are worth noting.

The solutions will be periodic functions of \( t \) provided the ratios \( (k/m) = (k/j) \) are rational numbers for all \( k, m \) and \( j \). In some cases they can also become singular, see the next section.

Important for possible physical applications is the so-called quasi-equidistant regime in which the distances between the neighboring particles, i.e. \( \text{Re}(q_{k+1}(t)) \) oscillate with very small amplitude; with rather good accuracy they could be considered constant. The fact that such regimes are possible and describe adequately the behavior of certain NLS soliton trains is shown in [25].

D 3) Mixed regimes. As we mentioned above there are a number of intermediate cases. Here we start with the case when \( m + 1 \) out of the \( N \) particles form a bound state, i.e.:

\[ 1 < \ldots < k = \ldots = k+m < k+m+1 < \ldots < N \] (41)

and \( \delta \neq j \text{ for } i \neq j \leq k; k+1; \ldots ; k+m \). Skipping the details we present the results for the case \( g' \in \mathbb{R}^N \) and \( m = 1 \), i.e., only two of the particles
form a bound state. Particles with numbers different from \( k, k + 1 \) are free, and for \( k \)-th and \( k + 1 \)-st have
\[
q_{k+1} (t) = q_k (0) + u_k t + q_{k+1} (t) + O (e ^ t); \tag{42}
\]
for \( t \neq 1 \) and
\[
q_{k+2} (t) = q_k (0) + u_k t + q_{k+2} (t) + O (e ^ t); \tag{43}
\]
for \( t = 1 \). Here \( a = 0; 1, u_k = (k + k+1), \)
\[
= m \ln (k \quad k \quad 1 \quad k+2 \quad k+1); \tag{44}
\]
and
\[
k+1 = \sim_k + (1)^g B_k (t) + a \ln (2_k \quad 2_{k+1})^2; \tag{45}
\]
\[
B_k (t) = \ln 2 \cos (\kappa t \quad \kappa); \tag{46}
\]
\[
\sim_k = \ln \theta r_{k+1} \quad r_k \quad (2_s \quad 2_k) (2_s \quad 2_{k+1}) \quad A; \tag{47}
\]
where \( Q_s \quad Q_{k+1} \quad 1 \) and \( Q_{N} \quad Q_{N \quad k+2} \).

5 On the singular and degenerate solutions of CTC

It is only natural that some of the CTC solutions do not enjoy all the good properties of the RTC ones. We mentioned already two of them:

P 1) The elements of \( T_g \) for the RTC are real-valued;

P 2) The eigenvalues \( k \) are pairwise different, i.e. \( k \neq j \) for \( k \neq j \).

An immediate consequence of P 1) and P 2) is the fact that the solutions \( q_k (t) \) of the RTC are regular functions for all finite values of \( t \).

Generalizing to CTC we lose both properties P 1) and P 2). As a result, besides the regular solutions, we obtain also singular and degenerate solutions. Obviously, all solutions leading to the D 1 regime are regular. Even if we assume that property P 2) holds, we may still have singular solutions.
Indeed, from equations (45) \{ (47)\} we see that in the oscillating part of the motion $B_k(t)$ is periodic. If in addition the parameters in $T_g$ are such that $\text{Re} \ k = 0$ in (47) then $B_k(t)$ will develop singularities for finite values oft. The same holds true for the functions $q_k(t)$: there exist submanifolds of $T_g$ for which $q_k(t)$ become singular for finite values oft. This fact has been noted in [19] for RTC on spaces with indefinite metric and (or) with purely imaginary interaction constant. Such RTC can be viewed as particular case of the CTC when one or several of the functions $a_k(t)$ are purely imaginary, while the others remain real.

Let us now examine the degeneration, i.e. the case when the property P2) is violated and some of the eigenvalues of $L_0$ become equal. In [3, 22] it is proved that the spectrum of $L_0$ is simple for real $\mathfrak{sl}(N)$-Toda chain. We state the following generalization:

**Lemma 1** Let us consider $L_0$ for the complex Toda models related to the classical series of simple Lie algebras $A_r$, $B_r$, $C_r$, $D_r$. If $L_0$ does not contain Jordan cells, then its spectrum is simple.

Therefore the degeneration can take place only for CTC models and only provided in the diagonalization of $L_0$ Jordan cells are present. Let us have $g'\mathfrak{sl}(N)$ and let $L_0$ has a 2 2 Jordan cell, $1 = 2 \ldots$. Then $L_0$ has an eigenvector $v^{(1)}$ and an adjoint eigenvector $v^{(2)}$:

$$L_0 v^{(1)} = v^{(1)};$$

$$L_0 v^{(2)} = v^{(2)} + v^{(1)};$$

where $v^{(2)}$ can be expressed as linear combination of $v^{(1)}(\ )$ and its first derivative with respect to $t$. The corresponding $A_k(t)$ besides the standard exponential term will contain also terms of the form $te^{2t}$. More generally, if the degeneracy is of higher order, i.e., $1 = \ldots = m$ then we will need linear combinations of $v^{(1)}(\ )$ and its derivatives with respect to $t$ of order 1, ..., $m$ and $A_k$ will contain terms of the form $te^{2n}t$ with $p = 1; \ldots; m \ldots$ and $n = 1; \ldots; m \in (k; m)$, see also [23].

In particular, if we have complete degeneracy (i.e., all $k$ are equal and equal to zero) the solution of the $\mathfrak{sl}(N)$-CTC is expressed through $A_k$, which are polynomials of degree $k(N - k)$. Their coefficients depend on $N \ldots$
constants $f_k, k = 1;: : :; N$. For example, for $N = 3$ and $1 = 2 = 3 = 0$ we get:

\[ A_1(t) = \frac{1}{2} t^2 + f_1 t + f_2; \quad (50) \]

\[ A_2(t) = \frac{1}{2} t^2 + f_1 t + f_2; \quad (51) \]

and $A_3 = 1$. Obviously, these solutions will be regular if $A_k$ have complex roots and will develop singularities if one (or more) of their roots are real.

6 Conclusions

We have shown that the $N$-site CTC with fixed ends has richer variety of asymptotic regimes than the RTC.

In particular, we showed that the CTC allows solutions in which the complex" particles form regular bound states. This could be important for the applications to soliton interactions, see e.g. [21], where this is done for $N = 3$. Then the problem to determine the initial soliton parameters (i.e., the values of $a_k$ and $b_k$) is reduced to the analysis of the characteristic equation for $L_0$:

\[ \det(L_0) = \sum_{k=0}^{N} p_k \chi^k \quad (52) \]

$p_0 = 1, p_N = 0$ and to the requirement that the eigenvalues of $L_0$ be purely imaginary. This is an algebraic problem which often can be solved analytically. It allows one to determine the set of initial soliton parameters in such a way, that the solitons will not only form a stable bound state, but also will propagate quasi-equidistantly. This type of propagation is of importance for soliton based optical communications [21].

We have studied also the asymptotic regimes for the CTC related to the other simple Lie algebras. We proposed explicit solutions to these models in terms of the minimal sets of scattering data $T_g$. The degenerate and singular solutions of the CTC which have no counterparts in RTC are also briefly analyzed.
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