On the irrationality measure of the Thue–Morse constant

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(Received 24 September 2017; revised 20 February 2018)

Abstract

We provide a non-trivial measure of irrationality for a class of Mahler numbers defined by infinite products. This class includes the Thue–Morse constant. Among other things, our results imply a generalisation to $[12]$. $^1$

1. Introduction

Let $\xi \in \mathbb{R}$ be an irrational number. Its irrationality exponent $\mu(\xi)$ is defined to be the supremum of all $\mu$ such that the inequality

$$|\xi - \frac{p}{q}| < q^{-\mu}$$

has infinitely many rational solutions $p/q$. This is an important property of a real number since it shows how closely it can be approximated by rational numbers in terms of their denominators. The irrationality exponent can be further refined by the following notion. Let $\psi(q) : \mathbb{R}_\geq \rightarrow \mathbb{R}_\geq$ be a function which tends to zero as $q \rightarrow \infty$. Any function $\psi$ with these properties is referred to as the approximation function. We say that an irrational number $\xi$ is $\psi$-well approximable if the inequality

$$|\xi - \frac{p}{q}| < \psi(q)$$

has infinitely many solutions $p/q \in \mathbb{Q}$. Conversely, we say that $\xi$ is $\psi$-badly approximable if (1.1) has only finitely many solutions. Finally, we say that $\xi$ is badly approximable if it is $c/q$-badly approximable for some positive constant $c > 0$.

If a number $\xi \in \mathbb{R}$ is $\psi$-badly approximable, we also say that $\psi$ is a measure of irrationality of $\xi$.

The notions of $\psi$-bad and $\psi$-well approximability allow us to characterise the Diophantine approximation properties of real numbers in a more refined way than the irrationality exponent. Indeed, the statement $\mu(\xi) = \mu$ is equivalent to saying that for any $\epsilon > 0$, $\xi$ is both $q^{-\mu-\epsilon}$-well approximable and $q^{-\mu+\epsilon}$-badly approximable. On the other hand, $(q^2 \log q)^{-1}$-badly approximable numbers are in general worse approached by rationals when compared...
to \((q^2 \log^2 q)^{-1}\)-badly approximable numbers, even though that both of them have irrationality exponent equal to 2.

**Remark 1.** It is quite easy to verify that for any \(\xi \in \mathbb{R}\) and any \(c \in \mathbb{Q} \setminus \{0\}\), there exists a constant \(K\), which depends on \(\xi\) and \(c\) only, such that, for any approximation function \(\psi\), if the number \(\xi\) is \(\psi\)-badly approximable, then \(c\xi\) is \(K\psi\)-badly approximable, and, similarly, if \(\xi\) is \(\psi\)-well approximable, then \(c\xi\) is \(K\psi\)-well approximable.

A big progress has been made recently in determining Diophantine approximation properties of so called Mahler numbers. Their definition slightly varies in the literature. In this paper, we define Mahler functions and Mahler numbers as follows. An analytic function \(F(z)\) is called a **Mahler function** if it satisfies the functional equation

\[
\sum_{i=0}^{n} P_i(z)F(z^{d^i}) = Q(z),
\]

where \(n\) and \(d\) are positive integers with \(d \geq 2\), \(P_i(z)\), \(Q(z) \in \mathbb{Q}[z]\), \(i = 0, \ldots, n\) and \(P_0(z)P_n(z) \neq 0\). We will only consider those Mahler functions \(F(z)\) which lie in the space \(\mathbb{Q}((z^{-1}))\) of Laurent series. Then, for any \(\alpha \in \mathbb{Q}\) inside the disc of convergence of \(F(z)\), a real number \(F(\alpha)\) is called a **Mahler number**.

One of the classical examples of Mahler numbers is the so called Thue–Morse constant which is defined as follows. Let \(t = (t_0, t_1, \ldots) = (0, 1, 1, 0, 1, 0, \ldots)\) be the Thue–Morse sequence, that is the sequence \((t_n)_{n \in \mathbb{N}_0}\), where \(\mathbb{N}_0 := \mathbb{N} \cup \{0\}\), defined by the recurrence relations \(t_0 = 0\) and for all \(n \in \mathbb{N}_0\)

\[
t_{2n} = t_n,
\]

\[
t_{2n+1} = 1 - t_n.
\]

Then, the Thue–Morse constant \(\tau_{TM}\) is a real number which binary expansion is the Thue–Morse word. In other words,

\[
\tau_{TM} := \sum_{k=0}^{\infty} \frac{t_k}{2^{k+1}}. \tag{1.3}
\]

It is well known that \(\tau_{TM}\) is a Mahler number. Indeed, one can check that \(\tau_{TM}\) is related with the generating function

\[
f_{TM}(z) := \sum_{i=0}^{\infty} (-1)^i z^{-i} \tag{1.4}
\]

by the formula \(\tau_{TM} = \frac{1}{2}(1 - f_{TM}(2)/2).\) At the same time, the function \(f_{TM}(z)\), defined by (1.4), admits the following presentation \([4, \text{section} 13.4]:\)

\[
f_{TM}(z) = \prod_{k=0}^{\infty} \left(1 - z^{-2^k}\right),
\]

and the following functional equation holds:

\[
f_{TM}(z^2) = \frac{z}{z-1}f_{TM}(z). \tag{1.5}
\]

So it is indeed a Mahler function.
Approximation of Mahler numbers by algebraic numbers has been studied within a broad research direction on transcendence and algebraic independence of these numbers. This direction has been initiated by K. Mahler in 1930 [19]. We refer the reader to the monograph [20] for more details on this topic and for a list of further references.

It has to be mentioned that, though some results on approximation by algebraic numbers can be specialised to results on rational approximations, most often they become rather weak in this case. This happens because the results on approximations by algebraic numbers necessarily involve complicated constructions, which results in some loss of precision. A more fundamental reason is that rational numbers enjoy a significantly more regular (and much better understood) distribution in the real line when compared to the algebraic numbers.

The research of sharp approximation properties of Mahler numbers by rational numbers, which are inaccessible as a specialisation of the general results on approximations by algebraic numbers of an arbitrary degree, to the best of our knowledge, was only started at the beginning of the 1990’s with the work of Shallit and van der Poorten [21], where they considered a class of numbers that contains some Mahler numbers, including Fredholm constant \( \sum_{n=0}^{\infty} 10^{-2^n} \), and they proved that all numbers from that class are badly approximable.

The next result on the subject, where the authors are aware of, is due to Adamczewski and Cassaigne. In 2006, they proved [1] that every automatic number (which, according to [8, theorem 1], is a subset of Mahler numbers) has finite irrationality exponent, or, equivalently, every automatic number is not a Liouville number. Later, this result was extended to all Mahler numbers [9]. We also mention here the result by Adamczewski and Rivoal [2], where they showed that some classes of Mahler numbers are \( \psi \)-badly approximable, for various functions \( \psi \) depending on a class under consideration.

The Thue–Morse constant is one of the first Mahler numbers whose irrationality exponent was computed precisely, it was done by Bugeaud in 2011 [12]. This result served as a foundation for several other works, establishing precise values of irrationality exponents for ever wider classes of Mahler numbers, see for example [14, 16, 23].

Bugeaud, Han, Wen and Yao [13] computed the estimates of \( \mu(f(b)) \) for a large class of Mahler functions \( f(z) \), provided that the distribution of indices at which Hankel determinants of \( f(z) \) do not vanish is known (this distribution, in its turn, could be read from the continued fraction of \( f(z) \), see [5, corollary 1]). In many cases, these estimates lead to the precise value of \( \mu(f(b)) \). We will consider this result in more detail in the next subsection. Later, Badziahin [5] provided a continued fraction expansion for the functions of the form

\[
 f(z) = \prod_{t=0}^{\infty} P(z^{-d^t})
\]

where \( d \in \mathbb{N}, d \geq 2 \) and \( P(z) \in \mathbb{Q}[z] \) with \( \deg P < d \). This result, complimented with [13], allow us to find sharp estimates for the values of these functions at integer points.

Despite rather extensive studies on irrationality exponents of Mahler numbers, very little is known about their sharper Diophantine approximation properties. In 2015, Badziahin and Zorin [6] proved that the Thue–Morse constant \( \tau_{TM} \), together with many other values of \( f_{TM}(b), b \in \mathbb{N} \), are not badly approximable. Moreover, they proved
THEOREM BZ. There is an explicit constant $C > 0$ and infinitely many rationals $p/q$ such that

$$\left| \tau_{TM} - \frac{p}{q} \right| < \frac{C}{q^2 \log \log q}.$$ 

In other words, the Thue-Morse constant $\tau_{TM}$ is $C/q^2 \log \log q$-well approximable.

Later, in [7], they extended this result to the values $f_3(b)$, where $b$ is from a certain subset of positive integers, and

$$f_3(z) := \prod_{t=0}^{\infty} (1 - z^{-3}).$$

Khintchine’s Theorem implies that outside of a set of the Lebesgue measure zero, all real numbers are $1/q^2 \log q$-well approximable and $1/q^2 \log^2 q$-badly approximable. Of course, this metric result implies nothing for any particular real number, or countable family of real numbers. However, it sets some expectations on the Diophantine approximaiton properties of real numbers.

Theorem BZ does not provide the well-approximability result for the Thue–Morse constant suggested by Khintchine’s theorem, but it falls rather short to it. At the same time, the bad-approximability side, suggested by Khintchine’s theorem, seems to be hard to establish (or even to approach) in the case of the Thue–Morse constant and related numbers. In this paper we prove that a subclass of Mahler numbers, containing, in particular, the Thue–Morse constant, is $(q \exp(K \sqrt{\log q \log \log q}))^{-2}$-badly approximable for some constant $K > 0$, see Theorem 2 at the end of Subsection 1.1. For the purposes of comparison with Theorem BZ, here we state Theorem 2 for the Thue–Morse constant (in the settings of Theorem 2, the Thue–Morse constant is $f(2)$ where $d = 2$, and $P(x) = 1 - x$).

**THEOREM.** There is a constant $K > 0$ such that

$$\left| \tau_{TM} - \frac{p}{q} \right| > \frac{1}{q^2 \exp \left( K \sqrt{\log q \log \log q} \right)}.$$ 

for all rational numbers $q \geq 10$.

This result is still pretty far from what is suggested by Khintchine’s theorem, however it significantly improves the best result [12] available at this moment, namely, that the irrationality exponent of Thue–Morse constant equals 2.

From a more general perspective, it could be noted that the irrationality measure for the Thue–Morse constant provided in [12] is the same as the irrationality measure of algebraic numbers given by the Roth’s famous theorem. There is a well-known problem to improve this irrationality measure for algebraic numbers of degree at least 3 to anything better than $q^{-2-\varepsilon}$, for any $\varepsilon > 0$. This problem remains open, though there has been a significant progress in this direction (see [10, 11] and references therein).

To the best of our knowledge, there are very few explicit examples of numbers with proven irrationality measure strictly better than one appearing in Roth’s theorem, $q^{-2-\varepsilon}$. In fact, all the examples known to us are rational powers of Euler’s constant, $e^q$ for $q \in \mathbb{Q} \setminus \{0\}$ (indeed, for these numbers the explicit form of continued fraction is known), and the numbers

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1 The authors would like to thank the referee, who pointed out this aspect to them.
specifically constructed by means of continued fractions or lacunary series to have prescribed irrationality measures.

From this point of view, our Theorem 2 provides a new family of numbers with the measure of irrationality strictly better than Roth’s one. To the best of our knowledge, these are the first numbers of this kind for which we do not know an explicit form of the continued fraction and which are defined in a natural way (i.e. they are not constructed specifically to have prescribed Diophantine approximation properties).

1.1. Continued fractions of Laurent series

Consider the set \( \mathbb{Q}((z^{-1})) \) of Laurent series equipped with the standard valuation which is defined as follows: for \( f(z) = \sum_{k=-d}^{\infty} c_k z^{-k} \in \mathbb{Q}((z^{-1})) \), its valuation \( \|f(z)\| \) is the biggest degree \( d \) of \( z \) having non-zero coefficient \( c_{-d} \). For example, for polynomials \( f(z) \) the valuation \( \|f(z)\| \) coincides with their degree. It is well known that in this setting the notion of continued fraction is well defined. In other words, every \( f(z) \in \mathbb{Q}((z^{-1})) \) can be written as

\[
f(z) = [a_0(z), a_1(z), a_2(z), \ldots] = a_0(z) + \frac{1}{a_1(z) + \frac{1}{a_2(z) + \ldots}} = a_0(z) + \sum_{k=1}^{\infty} \frac{1}{a_k(z)},
\]

where \( a_i(z), i \in \mathbb{Z}_{\geq 0} \), are non-zero polynomials with rational coefficients of degree at least 1.

The continued fractions of Laurent series share most of the properties of classical ones [22]. Furthermore, in this setting we have an even stronger version of Legendre theorem:

**Theorem L.** Let \( f(z) \in \mathbb{Q}((z^{-1})) \). Then \( p(z)/q(z) \in \mathbb{Q}(z) \) in a reduced form is a convergent of \( f(z) \) if and only if

\[
\left\| f(z) - \frac{p(z)}{q(z)} \right\| < -2\|q(z)\|.
\]

Its proof can be found in [22]. Moreover, if \( p_k(z)/q_k(z) \) is the \( k \)th convergent of \( f(z) \) in its reduced form, then

\[
\left\| f(z) - \frac{p_k(z)}{q_k(z)} \right\| = -\|q_k(z)\| - \|q_{k+1}(z)\|.
\]  

(1.6)

For a Laurent series \( f(z) \in \mathbb{Q}((z^{-1})) \), consider its value \( f(b) \), where \( b \in \mathbb{N} \) lies within the disc of convergence of \( f \). It is well known that the continued fraction of \( f(b) \) (or indeed of any real number \( x \) encodes, in a pretty straightforward way, approximational properties of this number. At the same time, it is a much subtler question how to read such properties of \( f(b) \) from the continued fraction of \( f(z) \). The problem comes from the fact that after specialisation at \( z = b \), partial quotients of \( f(z) \) become rational, but often not integer numbers, or they may even vanish. Therefore the necessary recombination of partial quotients is often needed to construct the proper continued fraction of \( f(b) \). The problem of this type has been studied in the beautiful article [21]. Despite this complication, in many cases some information on Diophantine approximation properties of \( f(b) \) can be extracted.

In particular, this is the case for Mahler numbers. Bugeaud, Han, Wen and Yao [13] provided the following result that links the continued fraction of \( f(z) \) and the irrationality exponents of values \( f(b), b \in \mathbb{N} \). In fact, they formulated it in terms of Hankel determinants. The present reformulation can be found in [5]:
THEOREM BHWY. Let $d \geq 2$ be an integer and $f(z) = \sum_{n=0}^{\infty} c_n z^{-n}$ converges outside of the unit disk. Suppose that there exist integer polynomials $A(z), B(z), C(z), D(z)$ with $B(0)D(0) \neq 0$ such that

$$f(z) = \frac{A(z^{-1})}{B(z^{-1})} + \frac{C(z^{-1})}{D(z^{-1})} f(z^d). \quad (1.7)$$

Let $b \geq 2$ be an integer such that $B(b^{-d^n})C(b^{-d^n})D(b^{-d^n}) \neq 0$ for all $n \in \mathbb{Z}_{\geq 0}$. Define

$$\rho := \limsup_{k \to \infty} \frac{\deg q_{k+1}(z)}{\deg q_k(z)},$$

where $q_k(z)$ is the denominator of $k$th convergent to $z^{-1} f(z)$. Then $f(b)$ is transcendental and

$$\mu(f(b)) \leq (1 + \rho) \min\{\rho^2, d\}.$$  

The corollary of this theorem is that, as soon as

$$\limsup_{k \to \infty} \frac{\deg q_{k+1}(z)}{\deg q_k(z)} = 1, \quad (1.8)$$

the irrationality exponent of $f(b)$ equals two. Then the natural question arises: can we say anything better on the Diophantine approximation properties of $f(b)$ in the case when the continued fraction of $z^{-1} f(z)$ satisfies a stronger condition than (1.8)? In particular, what if the degrees of all partial quotients $a_k(z)$ are bounded by some absolute constant or even are all linear? Here we answer this question for a subclass of Mahler functions.

The main result of this paper is the following.

THEOREM 2. Let $d \geq 2$ be an integer and

$$f(z) = \prod_{i=0}^{\infty} P(z^{-d^i}), \quad (1.9)$$

where $P(z) \in \mathbb{Z}[z]$ is a polynomial such that $P(1) = 1$ and $\deg P(z) < d$. Assume that the series $f(z)$ is badly approximable (i.e. the degrees of all partial quotients of $f(z)$ are bounded from above by an absolute constant). Then there exists a positive constant $K$ such that for any $b \in \mathbb{Z}$, $|b| \geq 2$, we have either $f(b) = 0$ or $f(b)$ is $q^{-2} \exp(-K \sqrt{\log q \log \log q})$-badly approximable.

Remark 3. The constant $K$ in the statement of Theorem 2 can be made explicit. Indeed, we deduce Theorem 2 from Theorem 11, where the corresponding constant is explicitly computed. Note however that the constant $\gamma$ in the statement of Theorem 11 is not quite the same as the constant $K$ in the statement of Theorem 2. One reason for this is an extra factor 36 in the denominator in Theorem 11. Another reason is that the constant $\gamma$ in the statement of Theorem 11 is computed only for the denominators $q$ large enough.

2. Preliminary information on series $f(z)$.

In the following discussion, we consider a series $f(z)$ which satisfies all the conditions of Theorem 2. Most of these conditions are straightforward to verify, the only non-evident point is to check whether the product function $f(z)$, defined by (1.9), is badly approximable. To address this, one can find a nice criteria in [5, proposition 1]: $f(z)$ is badly approximable...
Consequently, follows:

We denote by the degree of denominator of the \( f(z) \) is precisely \( k \), for all \( k \in \mathbb{N} \).

As shown in [5], it is easier to compute the continued fraction of a slightly modified series

\[
g(z) = z^{-1} f(z).
\]

Since Diophantine approximation properties of numbers \( f(b) \) and \( g(b) = f(b)/b \) essentially coincide, for any \( b \in \mathbb{N} \), we will further focus on the work with the function \( g(z) \). As we assume that \( f(z) \) is a badly approximable function, the function \( g(z) \) defined by (2.1) is also badly approximable. In what follows, we will denote by \( p_k(z)/q_k(z) \) the \( k \)th convergent of \( g(z) \), and then, by [5, proposition 1], we infer that \( \deg q_k(z) = k \).

Write down the polynomial \( P(z) \) in the form

\[
P(z) = 1 + u_1 z + \cdots + u_{d-1} z^{d-1}.
\]

Then \( P(z) \) is defined by the vector \( u = (u_1, \ldots, u_{d-1}) \in \mathbb{Z}^{d-1} \) and, via (1.9) and (2.1), so is \( g(z) \). To emphasise this fact, we will often write \( g(z) \) as \( g_u(z) \).

2.1. Coefficients of the series, convergents and Hankel determinants

We write the Laurent series \( g_u(z) \in \mathbb{Z}[[z^{-1}]] \) in the following form

\[
g_u(z) = \sum_{n=1}^{\infty} c_n z^{-n}.
\]

We denote by \( c_n \) the vector \( (c_1, c_2, \ldots, c_n) \). Naturally, the definition of \( g_u(z) \) via the infinite product (see (1.9) and (2.1)) imposes the upper bound on \( |c_n|, n \in \mathbb{N} \).

**Lemma 4.** The term \( c_n \) satisfies

\[
|c_n| \leq \|u\|_\infty^{l(n)} = \|u\|_{\log d}^{n+1}.
\]

Consequently,

\[
\|c_n\|_\infty \leq \|u\|_{\log d}^{n+1}
\]

**Proof.** Look at two different formulae for \( g_u(z) \):

\[
g_u(z) = z^{-1} \prod_{t=0}^{\infty} (1 + u_1 z^{-d^t} + \cdots + u_{d-1} z^{-(d-1)d^t}) = \sum_{n=1}^{\infty} c_n z^{-n}.
\]

By comparing the right- and the left-hand sides one can notice that \( c_n \) can be computed as follows:

\[
c_n = \prod_{j=0}^{l(n)} u_{d_n, j},
\]

where \( d_0 d_{n,1} \cdots d_{n,l(n)} \) is the \( d \)-ary expansion of the number \( n - 1 \). Here we formally define \( u_0 = 1 \). Equation (2.5) readily implies that \( |c_n| \leq \|u\|^{l(n)} \). Finally, \( l(n) \) is estimated by \( l(n) \leq [\log_d (n - 1)] \leq [\log_d n] \). The last two inequalities clearly imply (2.3), hence (2.4).

Let \( p_k(z)/q_k(z) \) be a convergent of \( g_u(z) \) in its reduced form. Recall that throughout the text we assume that \( f(z) \) is badly approximable, hence \( g_u(z) \) defined by (2.1) is badly approximable, and because of this (and employing [5, proposition 1]) we have

\[
\deg q_k = k.
\]
Because of (2.6), the following upper bounds hold true:

\[ a_{k,k} \neq 0. \quad (2.8) \]

The Hankel matrix is defined as follows:

\[ H_k = H_k(g_u) = \begin{pmatrix} c_1 & c_2 & \cdots & c_k \\ c_2 & c_3 & \cdots & c_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_k & c_{k+1} & \cdots & c_{2k-1} \end{pmatrix}. \]

It is known (see, for example, [5, section 3]) that the convergent in its reduced form with \( \deg q_k(z) = k \) exists if and only if the Hankel matrix \( H_k \) is invertible. Thus in our case we necessarily have that \( H_k(g_u) \) is invertible for any positive integer \( k \).

From (1.6), we have that

\[ \| q_k(z)g_u(z) - p_k(z) \| = -k - 1. \quad (2.9) \]

In other words, the coefficients for \( z^{-1}, \ldots, z^{-k} \) in \( q_k(z)g_u(z) \) are all zero and the coefficient for \( z^{-k-1} \) is not. This suggests a method for computing \( q_k(z) \). One can check that the vector \( a_k = (a_{k,0}, a_{k,1}, \ldots, a_{k,k}) \) is the solution of the matrix equation \( H_{k+1}a_k = c \cdot e_{k+1} \), where \( c \) is a non-zero constant and

\[ e_{k+1} = (0, \ldots, 0, 1)^t. \]

This equation has the unique solution since the matrix \( H_{k+1} \) is invertible. So, we can write the solution vector \( a_k \) as

\[ a_k = c \cdot H_{k+1}^{-1}e_{k+1}. \quad (2.10) \]

In what follows, we will use the norm of the matrix \( \| H \|_\infty \), defined to be the maximum of the absolute values of all its entries. Given a polynomial \( P(z) \) we define its height \( h(P) \) as the maximum of absolute values of its coefficients. In particular, we have

\[ h(p_k(z)) = \| b_k \|_\infty \quad \text{and} \quad h(q_k(z)) = \| a_k \|_\infty. \]

**Lemma 5.** For any \( k \in \mathbb{N} \), the \( k \)th convergent \( p_k(z)/q_k(z) \) of \( g_u(z) \) can be represented by \( p_k(z)/q_k(z) = \widetilde{p}_k(z)/\widetilde{q}_k(z) \), where \( \widetilde{p}_k, \widetilde{q}_k \in \mathbb{Z}[z] \) and

\[ h(\widetilde{q}_k) \leq (\| e_{2k+1} \|_\infty^2 \cdot k)^{k/2}, \quad (2.11) \]

\[ h(\widetilde{p}_k) \leq (\| e_{2k+1} \|_\infty)^{k+1} \cdot k^{(k+2)/2}. \quad (2.12) \]

Consecutively, the following upper bounds hold true:

\[ h(\widetilde{q}_k) \leq \| u \|_\infty^{k \cdot \log_2(2k+1)} \cdot k^{k/2}, \quad (2.13) \]

\[ h(\widetilde{p}_k) \leq \| u \|_\infty^{(k+1) \cdot \log_2(2k+1)} \cdot k^{(k+2)/2}. \quad (2.14) \]

**Proof.** By applying Cramer’s rule to the equation \( H_{k+1}a_k = c \cdot e_{k+1} \) we infer that

\[ a_{k,i} = c \cdot \frac{\Delta_{k+1,i}}{\det H_{k+1}}, \quad i = 0, \ldots, k, \quad (2.15) \]
where $\Delta_{k+1,i}$ denotes the determinant of the matrix $H_{k+1}$ with the $i$th column replaced by $e_{k+1}, i = 1, \ldots, k + 1$. Then we use the Hadamard’s determinant upper bound to derive
\[
|\det H_{k+1}| \leq \|H_{k+1}\|_{\infty}^{k+1} \cdot (k + 1)^{(k+1)/2} = (\|e_{2k+1}\|_{\infty}^2 (k + 1))^{(k+1)/2}.
\] (2.16)

Moreover, by expanding the matrix involved in $\Delta_{k+1,i}$ along the $i$-th column and by using Hadamard’s upper bound again we get
\[
|\Delta_{k+1,i}| \leq \|H_{k+1}\|_{\infty} \cdot k^{k/2} = (\|e_{2k+1}\|_{\infty}^2 \cdot k)^{k/2}, \quad i = 0, \ldots, k.
\]

To define $\tilde{q}(z)$, set $c = \det H_{k+1}$ in (2.15). Then we readily have $
\tilde{q}_k(z) = \sum_{i=0}^k \Delta_{k+1,i} z^i.$ By construction, it has integer coefficients and $h(\tilde{q}_k)$ satisfies (2.11).

Next, from (2.9) we get that the coefficients of $\tilde{p}_k(z)$ coincide with the coefficients for positive powers of $z$ in $\tilde{q}_k(z) g_u(z)$. By expanding the latter product, we get
\[
|b_{k,i}| = \left| \sum_{j=i+1}^k a_{k,j} c_{j-i-1} \right| \leq \|e_{2k+1}\|_{\infty}^{k+1} \cdot k^{(k+2)/2}.
\]

Hence (2.12) is also satisfied.

The upper bounds (2.13) and (2.14) follow from (2.11) and (2.12) respectively by applying Lemma 4.

**Notation 6.** For the sake of convenience, further in this text we will assume that all the convergents to $g_u(z)$ are in the form described in Lemma 5. That is, we will always assume that $p_k(z)$ and $q_k(z)$ have integer coefficients and verify the upper bounds (2.11) and (2.12), as well as (2.13) and (2.14).

For any $k \in \mathbb{N}$ we define a suite of coefficients $(\alpha_{k,i})_{i \geq k}$ by
\[
q_k(z) g_u(z) - p_k(z) =: \sum_{i=k+1}^\infty \alpha_{k,i} z^{-i}.
\] (2.17)

Note that from the equation $H_{k+1} a_k = c \cdot e_{k+1}$ we can get that $\alpha_{k,k+1} = c = \det H_{k+1}$. In particular, it is a non-zero integer.

**Lemma 7.** For any $i, k \in \mathbb{N}, i > k \geq 1$, we have
\[
|\alpha_{k,i}| \leq (k + 1) \|e_{k+i}\|_{\infty}^2 (\|e_{2k+1}\|_{\infty}^2 \cdot k)^{k/2}
\]
\[
\leq (k + 1) \|u\|_{\infty}^{\log_2(k+i)+1} \|u\|_{\infty}^{k \log_2(2k+1) + 1} \cdot k^{k/2}.
\] (2.18)

**Proof.** One can check that $\alpha_{k,i}$ is defined by the formula $\alpha_{k,i} = \sum_{j=0}^k a_{k,j} c_{j+i}$, which in view of (2.11) from Lemma 5 implies the first inequality in (2.18). Then, the second inequality in (2.18) follows by applying Lemma 4.

2.2. *Using functional equation to study Diophantine approximaiton properties*

From (1.9) one can easily get a functional equation for $g_u(z) = z^{-1} f(z)$:
\[
g_u(z) = P^*(z) g_u(z^d), \quad P^*(z) = z^{d-1} P(z^{-1}) = z^{d-1} + u_1 z^{d-2} + \ldots + u_{d-1}.
\] (2.19)
This equation allows us, starting from the convergent \( p_k(z)/q_k(z) \) to \( g_u(z) \), to construct an infinite sequence of convergents \( \left( p_{k,m}(z)/q_{k,m}(z) \right)_{m \in \mathbb{N}_0} \) to \( g_u(z) \) by

\[
q_{k,m}(z) := q_k(z^{d^m}), \quad p_{k,m}(z) := \prod_{t=0}^{m-1} P^t(z^{d^t}) p_k(z^{d^m}).
\] (2.20)

This fact can be checked by substituting the functional equation (2.19) into the condition of Theorem L. The reader can also compare with [5, lemma 3].

By employing (2.19) and (2.17), we find

\[
q_{k,m}(z) g_u(z) - p_{k,m}(z) = \prod_{t=0}^{m-1} P^t(z^{d^t}) \sum_{i=k+1}^{\infty} a_{k,i} z^{-d^m-i}.
\] (2.21)

Consider an integer value \( b \) which satisfies the conditions of Theorem 2. Define

\[
p_{k,m} := p_{k,m}(b), \quad q_{k,m} := q_{k,m}(b),
\] (2.22)
(2.23)

where \( p_{k,m}(z) \) and \( q_{k,m}(z) \) are polynomials defined by (2.20).

Clearly, for any \( k \in \mathbb{N}, m \in \mathbb{N}_0 \) we have \( p_{k,m}, q_{k,m} \in \mathbb{Z} \).

**Lemma 8.** Let \( b, k, m \in \mathbb{N}, b \geq 2 \). Assume

\[
b^{d^m} > 2^{1+\log_d \|u\|_\infty}.
\] (2.24)

Then the integers \( p_{k,m} \) and \( q_{k,m} \) verify

\[
\left| g_u(b) - \frac{p_{k,m}}{q_{k,m}} \right| \leq \frac{2(k + 1)k^{k/2}d^m \|u\|_\infty^{m+(k+1)(\log_d(2k+1)+1)}}{q_{k,m} \cdot b^{d^m+k+1}}.
\] (2.25)

Moreover, under the stronger assumption

\[
b^{d^m} \geq 4(k + 1)k^{k/2} \|u\|_\infty^{(k+1)(\log_d(2k+1)+1)},
\] (2.26)

then

\[
\frac{|g_u(b)|}{4q_{k,m} \cdot b^{d^m-k}} \leq \left| g_u(b) - \frac{p_{k,m}}{q_{k,m}} \right|.
\] (2.27)

**Proof.** Consider Equation (2.21) with substituted \( z := b \):

\[
q_{k,m} g_u(b) - p_{k,m} = \prod_{t=0}^{m-1} P^t(b^{d^t}) \sum_{i=k+1}^{\infty} a_{k,i} b^{-d^m-i}.
\] (2.28)

\[\text{There is a slight abuse of notation in using the same letters \( p_{k,m} \) and \( q_{k,m} \) both for polynomials from \( \mathbb{Z}[z] \) and for their values at \( z = b \). However, we believe that in this particular case such a notation constitutes the best choice. Indeed, the main reason to consider polynomials \( p_{k,m}(z) \) and \( q_{k,m}(z) \) is to define eventually \( p_{k,m} = p_{k,m}(b) \) and \( q_{k,m} = q_{k,m}(b) \), which will play the key role in the further proofs. At the same time, it is easy to distinguish the polynomials \( p_{k,m}(z) \), \( q_{k,m}(z) \) and the corresponding integers \( p_{k,m} \) and \( q_{k,m} \) by the context. Moreover, we will always specify which object we mean and always refer to the polynomials specifying explicitly the variable, that is \( p_{k,m}(z) \), \( q_{k,m}(z) \) and not \( p_{k,m} \) and \( q_{k,m} \).}
Each of the factors in $|P^*(b^d)|$ on the right-hand side of (2.28) can be upper bounded by $d \cdot \|u\|_\infty b^{d(d-1)}$. So, the product on the right-hand side of (2.28) can be estimated by

$$\left| \prod_{t=0}^{m-1} P^*(b^d) \right| \leq d^m \|u\|_\infty^m \cdot b^{dm-1}. \tag{2.29}$$

Further, we estimate the second term on the right-hand side of (2.28) by employing Lemma 7:

$$\left| \sum_{i=k+1}^{\infty} \alpha_{k,i} b^{-d^m-i} \right| \leq (k+1) \|u\|_\infty^{k(\log_d(2k+1)+1)} \cdot k^{k/2} \sum_{i=k+1}^{\infty} \frac{\|u\|_\infty^{\log_d(k+i)+1}}{b^{d^m-i}}. \tag{2.30}$$

The last sum on the right-hand side of (2.30) is bounded from above by

$$\sum_{i=k+1}^{\infty} \|u\|_\infty^{\log_d(2k+1)+1} \frac{(i+1)^{\log_d \|u\|_\infty}}{b^{d^m-i}} \leq \|u\|_\infty^{1+\log_d(2k+1)} \cdot C(b, d, m, \|u\|_\infty), \tag{2.31}$$

where

$$C(b, d, m, \|u\|_\infty) = \sum_{i=0}^{\infty} \frac{(i+1)^{\log_d \|u\|_\infty}}{b^{d^m-i}}. \tag{2.32}$$

Note that for any $i \in \mathbb{Z}$, we have $i + 1 \leq 2^i$. Because of this, assumption (2.24) implies

$$C(b, d, m, \|u\|_\infty) \leq 2. \tag{2.32}$$

Finally, by putting together, (2.28), (2.29), (2.30), (2.31) and (2.32) we get

$$\left| q_{k,m} g_u(b) - p_{k,m} \right| \leq \frac{2(k+1)^{k/2}d^m \|u\|_\infty^{m+(k+1)(\log_d(2k+1)+1)}}{b^{d^m-k+1}}. \tag{2.33}$$

Dividing both sides by $q_{k,m}$ gives (2.25).

To get the lower bound, we first estimate the product in (2.21).

$$\prod_{t=0}^{m-1} P^*(b^d) = b^{d^m-1} \prod_{t=0}^{m-1} P(b^{-d^t}) = b^{d^m} \frac{g_u(b)}{\prod_{t=m}^{\infty} P(b^{-d^t})}. \tag{2.24}$$

By (2.26), the denominator can easily be estimated as

$$\prod_{t=m}^{\infty} P(b^{-d^t}) \leq \prod_{t=m}^{\infty} \left( 1 + \frac{2\|u\|_\infty}{b^{d^t}} \right) < 2. \tag{2.26}$$

Therefore,

$$\prod_{t=0}^{m-1} P^*(b^d) \geq \frac{1}{2} b^{d^m} g_u(b).$$
For the series on the right-hand side of (2.21), we show that the first term dominates this series. Indeed, we have \(|ak_{k,k+1}| \geq 1\) since it is a non-zero integer. Then,

\[
|q_{k,m}g_u(b) - p_{k,m}| = \left| \prod_{i=0}^{m-1} P^i(b^d) \cdot \sum_{i=k+1}^{\infty} a_{k,i}b^{-d^{m-i}} \right| \\
\geq \frac{1}{2} b^{d^{m-1}} |g_u(b)| \left( b^{-d^m(k+1)} - \sum_{i=k+2}^{\infty} |a_{k,i}| b^{-d^{m-i}} \right) \\
\geq \frac{1}{2} b^{d^{m-1}} |g_u(b)| \left( 1 - \sum_{i=k+2}^{\infty} |a_{k,i}| b^{-d^{m-i}} \right) \\
\geq \frac{1}{2} b^{d^{m-1}} |g_u(b)| \left( 1 - \sum_{i=k+2}^{\infty} |a_{k,i}| b^{-d^{m-i}} \right)
\]

(2.30),(2.31)

Recall that by (2.32), we have \(C(b, d, m, \|u\|_\infty) \leq 2\). So, by using assumption (2.26), we finally get

\[
|q_{k,m}g_u(b) - p_{k,m}| \geq \frac{1}{2} b^{d^m} |g_u(b)| |g_u(b)| = \frac{|g_u(b)|^2}{4b^{d^m}}.
\]

Finally, dividing both sides by \(q_{k,m}\) leads to (2.27).

**Lemma 9.** Let \(b, k, m \in \mathbb{N}, k \geq 1\) and let

\[
b^{d^m} \geq 3 \cdot (\|c_{k+1}^2\|_{\infty,k}^2)^{k/2}.
\]

Recall the notations \(a_{k,i}, i = 0, \ldots, k\), for the coefficients of \(q_k, k \in \mathbb{N}\), is defined in (2.7). Then,

\[
\frac{1}{2} |a_{k,k}| \cdot b^{kd^m} \leq q_{k,m} \leq \frac{3}{2} |a_{k,k}| \cdot b^{kd^m}.
\]

**Proof.** The leading term of \(q_{k,m}(z) = a_{k,k}z^{kd^m}\). We know that \(\deg q_k(z) = k\), therefore \(a_{k,k} \neq 0\) and \(a_{k,k}\) is an integer. Recall also that by (2.11) the maximum of the coefficients \(a_{k,i}, i = 0, \ldots, k\), does not exceed \((\|c_{k+1}^2\|_{\infty,k}^2 \cdot k)^{k/2}\). Thus we find, by using assumption (2.34),

\[
\left| \sum_{n=0}^{k-1} a_{k,n} \cdot b^{nd^m} \right| \leq b^{kd^m} \left| \sum_{n=1}^{k} 3^{-n} \right| \leq \frac{1}{2} b^{kd^m}.
\]

We readily infer, by taking into account \(q_{k,m} = a_{k,0} + a_{k,1}b^{d^m} + \cdots + a_{k,k}b^{kd^m}\),

\[
\frac{1}{2} |a_{k,k}| b^{kd^m} \leq |a_{k,k}| b^{kd^m} - \frac{1}{2} b^{kd^m} \leq |q_{k,m}| \leq |a_{k,k}| b^{kd^m} - \frac{1}{2} b^{kd^m} \leq q_{k,m} \leq |a_{k,k}| b^{kd^m} + \frac{1}{2} b^{kd^m} = \frac{3}{2} |a_{k,k}| b^{kd^m}.
\]

This completes the proof of the lemma.

**Proposition 1.** Let \(k \geq 2, m \geq 1\) be integers and assume that (2.26) is satisfied. Then, the integers \(p_{k,m} = p_{k,m}(b)\) and \(q_{k,m} = q_{k,m}(b)\), defined by (2.22) and by (2.23), satisfy

\[
\left| g_u(b) - \frac{p_{k,m}}{q_{k,m}} \right| \leq \frac{3(k + 1)k^{2}d^m\|u\|_{\infty}(m+2k+1)(\log_{2}(2k+1)+1)}{b \cdot q_{k,m}^2},
\]

(2.36)

\[
\frac{|g_u(b)|}{8bq_{k,m}^2} \leq \left| g_u(b) - \frac{p_{k,m}}{q_{k,m}} \right|.
\]

(2.37)

Moreover, if, additionally, \(k\) and \(m\) satisfy

\[
k \cdot d^m \log_2 b - 1 \geq \frac{1}{3} m^2 (\log \|u\|_\infty)^2,
\]

(2.38)
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then

\[ \left| g_u(b) - \frac{p_{k,m}}{q_{k,m}} \right| \leq 3 \cdot 2^C \sqrt{\log_2 q_{k,m} \log_2 q_{k,m}} \cdot \frac{2}{q_{k,m}^2}, \]  

(2.39)

where

\[ C = 2 \sqrt{2} + 2 \sqrt{5} \cdot \log \| u \|_\infty + 2. \]  

(2.40)

**Proof.** From Lemma 9 we have

\[ b^k d^m \geq \frac{2q_{k,m}}{3|a_{k,k}|} \geq \frac{2q_{k,m}}{3\| u \|_\infty^{k(\log_d (2k+1) + 1) + k/2}}. \]

Similarly, by using \(|a_{k,k}| \geq 1\) together with Lemma 9, we get the lower bound

\[ b^k d^m \leq 2q_{k,m}. \]  

(2.41)

These two bounds on \(b^k d^m\) allow to infer the inequalities (2.36) and (2.37) straightforwardly from the corresponding bounds in Lemma 8.

We proceed with the proof of the estimate (2.39). We are going to deduce it as a corollary of (2.36). To this end, we are going to prove, under the assumptions of this proposition, \((k+1)b^k d^m \| u \|_\infty^{m+(k+1)\log_2 q_{k,m}} \leq 2^C \sqrt{\log_2 q_{k,m} \log_2 q_{k,m}}\),

(2.42)

where the constant \(C\) is defined by (2.40). It is easy to verify that (2.36) and (2.42) indeed imply (2.39). Therefore in the remaining part of the proof we will focus on verifying (2.42).

The inequality (2.41) together with condition (2.26) imply

\[ \log_2 q_{k,m} \geq (k - 1) + k \log_2 (k + 1) + \frac{k^2}{2} \log_2 k + k(k + 1)(\log_d (2k + 1) + 1) \log_2 \| u \|_\infty. \]

(2.43)

By taking logarithms again one can derive that \(\log_2 \log_2 q_{k,m} \geq \log_2 k\). Now we compute

\[ \log_2 q_{k,m} \log_2 \log_2 q_{k,m} \geq \frac{k^2}{2} (\log_2 k)^2 > \frac{1}{8} (k \log_2 k + \log_2 (k + 1))^2. \]

(2.44)

The last inequality in (2.44) holds true because \(k \log_2 k > \log_2 (k + 1)\) for \(k \geq 2\).

Another implication of (2.43) is

\[ \log_2 q_{k,m} \log_2 \log_2 q_{k,m} \geq k + 1)(\log_d (2k + 1) + 1) \log_2 k \log_2 \| u \|_\infty. \]

(2.45)

Since for \(d \geq 2\) and \(k \geq 2\) we have \(\log_2 k \geq \frac{1}{4} \log_d (2k + 1) + 1\) and \(k(k + 1) \geq \frac{1}{5}(2k + 1)^2\), therefore we readily infer from (2.45)

\[ \log_2 q_{k,m} \log_2 \log_2 q_{k,m} \geq \frac{1}{20 \log_2 \| u \|_\infty} (2k + 1)^2 (\log_d (2k + 1) + 1)^2 (\log_2 \| u \|_\infty)^2. \]

(2.46)

Next, it follows from (2.41) that

\[ \log_2 q_{k,m} \geq k \cdot d^m \log_2 b - 1. \]

(2.47)

Therefore assumption (2.38) implies that \(\log_2 q_{k,m} \geq \frac{1}{4} m^2 (\log_2 \| u \|_\infty)^2\). At the same time, the assumptions \(k \geq 2\) joint with (2.26) readily imply \(b^k d^m \geq 576\), hence, by adding (2.41), we find \(\log_2 \log_2 q_{k,m} \geq \log_2 \log_2 288 > 3\). So,

\[ \log_2 q_{k,m} \log_2 \log_2 q_{k,m} > m^2 (\log_2 \| u \|_\infty)^2. \]

(2.48)
Also, by these considerations we deduce from (2.47)
\[ \log_2 q_{k,m} \log_2 q_{k,m} > 3d^m > (m \cdot \log_2 d)^2. \] (2.49)

Finally, by taking square root in the both sides of (2.44), (2.46), (2.48) and (2.49) and summing up the results we find
\[ C \sqrt{\log_2 q_{k,m} \log_2 q_{k,m}} \geq \log_2 (k+1) + k \log_2 k + m \log_2 d + (m + (2k + 1)(\log_d(2k + 1) + 1)) \log_2 \|u\|_\infty, \] (2.50)
where the constant $C$ is defined by (2.40). Finally, by taking the exponents base two from both sides of (2.50), we find (2.42), and hence derive (2.39).

Remark 10. Note that the constant $C$ in Proposition 1 is rather far from being optimal. The proof above can be significantly optimised to reduce its value. However that would result in more tedious computations. All one needs to show is the inequality (2.50).

3. Proof of Theorem 2

We will prove the following result.

**Theorem 11.** Let $b \geq 2$. For any $p \in \mathbb{Z}$ and any large enough $q \in \mathbb{N}$ we have
\[ \left| g_u(b) - \frac{p}{q} \right| \geq \frac{|g_u(b)|}{36q^2 \exp(\gamma \sqrt{\log_2 q \log_2 \log_2 q})}, \] (3.1)

where $\gamma = \ln 2 \cdot (2d \tau \log_2 b + 2\tau_0 + 2C)$, $C$ is defined by (2.40) and $\tau = 2 + 3 \log_2 \|u\|_\infty$.

It is easy to see that Theorem 2 is a straightforward corollary of Theorem 11. Indeed, if $f(b)$ from Theorem 2 is not zero then so is $g_u(b)$ and the lower bound (3.1) is satisfied for all large enough $q$, therefore for any $K > \gamma$ the inequality
\[ \left| g_u(b) - \frac{p}{q} \right| < q^{-2} \exp(-K \sqrt{\log q \log \log q}) \]
has only finitely many solutions. By definition, this implies that $g_u(b)$ and in turn $f(b)$ are both $q^{-2} \exp(-K \sqrt{\log q \log \log q})$-badly approximable.

**Proof of Theorem 11.** In this proof, we will use the constant $C$ defined by (2.40). Fix a couple of integers $p$ and $q$. We start with some preliminary calculations and estimates.

Define $x > 2$ to be the unique solution of the following equation
\[ q = \frac{1}{12} \cdot x \cdot 2^{-\frac{1}{4}c \sqrt{\log_2 x \log_2 \log_2 x}}. \] (3.2)

The condition $x > 2$ ensures that both $\log_2 x$ and the double logarithm $\log_2 \log_2 x$ exist and are positive, hence $2^{-C \sqrt{\log_2 x \log_2 \log_2 x}} < 1$ and thus
\[ 12q < x. \] (3.3)

For large enough $q$ we then have
\[ \frac{81}{4} C^2 \log_2 \log_2 x < \log_2 x \]
and therefore
\[ 2^{\frac{3}{2}C \sqrt{\log_2 x \log_2 \log_2 x}} < x^{1/3}. \] (3.4)
From (3·2) and (3·4) we readily infer
\[ x < (12q)^{3/2}. \] (3·5)
Rewrite (3·2) in the following form
\[ x = 12q \cdot 2^{c/2} \sqrt{\log_2 x \log_2 \log_2 x}. \] (3·6)
Then, by applying (3·5) to it, we find that for large enough \( q \),
\[ x < 12q \cdot 2^{c/2} \sqrt{\log_2 q \log_2 \log_2 q}. \] (3·7)
Denote
\[ t := \log_b x. \] (3·8)
For \( q > 12^{-1} b^{-2} d^4 \) inequality (3·3) implies
\[ t > t^2 d^4 \geq 64 \] (3·9)
in the last inequality we took into consideration that \( d \geq 2 \) and \( \tau \geq 2 \). Because of this, \( \sqrt{t} > \log_2 t \) and
\[ d \leq \frac{1}{\tau \log_2 t}. \] (3·10)
Choose an integer \( n \) of the form \( n := k \cdot d^m \) with \( m \in \mathbb{N}, k \in \mathbb{Z} \) such that
\[ t \leq n \leq t + d \tau \log_2 t, \]
\[ \tau \sqrt{t \log_2 t} \leq d^m \leq d \tau \sqrt{t \log_2 t}. \] (3·11)
One can easily check that such \( n \) always exists.
Inequalities (3·10), (3·11) and (3·12) imply
\[ k = \frac{n}{d^m} \leq \frac{t + d \tau \sqrt{t \log_2 t}}{\tau \sqrt{t \log_2 t}} = \frac{1}{\tau \log_2 t} \sqrt{\frac{t}{\log_2 t}} + d \leq \frac{2}{\tau \log_2 t} \sqrt{\frac{t}{\log_2 t}}. \] (3·13)
Then, with help of (3·9), we deduce
\[ k \log_2 k \leq \frac{2}{\tau \log_2 t} \sqrt{\frac{t}{\log_2 t}} \left( \log_2 (2/\tau) + \frac{1}{2} \log_2 t - \frac{1}{2} \log_2 \log_2 t \right) < \frac{1}{\tau \sqrt{t \log_2 t}}, \] (3·14)
and therefore
\[ 2 + \log_2 (k + 1) + \frac{k}{2} \log_2 k \]
\[ + (k + 1)(\log_d (2k + 1) + 1) \log_2 \| \mathbf{u} \|_\infty < \tau \sqrt{t \log_2 t}. \] (3·15)
Indeed, for \( k \leq 3 \) we can verify it by direct computation, and for \( k \geq 4 \) we have \( 2 + \log_2 (k + 1) \leq \frac{3}{2} \log_2 k/2 \) and \( (k + 1)(\log_d (2k + 1) + 1) \leq 3k \log_2 k \). Hence (3·15) readily follows from (3·14) for \( \tau = 2 + 3\| \mathbf{u} \|_\infty \).
By taking the exponent base two of the left-hand side of (3·15) and the exponent base \( b \) of the right-hand side of (3·15), and then using (3·12), we ensure that (2·26) is satisfied. We can also take \( q \) (and, consecutively, \( t \)) large enough so that \( m \), bounded from below by (3·12), satisfies \( d^m \geq |m^2 (\log_2 \| \mathbf{u} \|_\infty)^2 | \), and then necessarily (2·38) is verified. Also, (3·11) and (3·12) imply that, for \( t \) large enough, \( k \geq 2 \).
Hence we have checked all the conditions on $k$ and $m$ from Proposition 1. It implies that the integers $p_{k,m}$ and $q_{k,m}$, defined by (2·22) and (2·23), satisfy inequalities (2·37) and (2·39). Lemma 4 and inequality (2·26) imply the inequality (2·34), so we can use Lemma 9, i.e. we have

$$\frac{1}{2} |a_{k,k}|b^n \leq q_{k,m} \leq \frac{3}{2} |a_{k,k}|b^n.$$  \hfill (3·16)

In the case that

$$\frac{p}{q} = \frac{p_{k,m}}{q_{k,m}},$$

the result (3·1) readily follows from the lower bound (2·37) in Proposition 1.

We proceed with the case

$$\frac{p}{q} \neq \frac{p_{k,m}}{q_{k,m}}.$$ \hfill (3·17)

By the triangle inequality, and then by the upper bound (2·39), we have

$$\left| g_u(b) - \frac{p}{q} \right| \geq \left| \frac{p_{k,m}}{q_{k,m}} - \frac{p}{q} \right| - \left| g_u(b) - \frac{p_{k,m}}{q_{k,m}} \right|$$

$$\geq \frac{1}{q_{k,m}q} - \frac{3 \cdot 2^c \sqrt{\log_2 q_{k,m} \log_2 \log_2 q_{k,m}}}{q_{k,m}^2}. \hfill (3·18)$$

By applying the upper bound in (3·16) complemented with (2·13), we find

$$\log_2 q_{k,m} \leq \log_2 \frac{3}{2} + k/2 \log_2 k + k(\log_d (2k + 1) + 1) \log_2 \|u\|_\infty + n \log_2 b.$$  

Upper bounds (3·13) on $k$ and (3·11) on $n$ ensure that for large enough $q$ we have

$$2^c \sqrt{\log_2 q_{k,m} \log_2 \log_2 q_{k,m}} \leq 2^{\frac{3}{2}c} \sqrt{\log_2 x \log_2 \log_2 x}. \hfill (3·19)$$

The formula (3·8) for $t$ and the lower bound in (3·11) together give $b^n \geq x$. Then, by using the lower bound (3·16), we find

$$q_{k,m} \geq \frac{1}{2} b^n \geq \frac{1}{2} x. \hfill (3·20)$$

By using the estimates (3·19) and (3·20) on the numerator and denominator respectively, and then by substituting the value of $x$ given by (3·6), we find

$$\frac{3 \cdot 2^c \sqrt{\log_2 q_{k,m} \log_2 \log_2 q_{k,m}}}{q_{k,m}^2} \leq \frac{3 \cdot 2^{\frac{3}{2}c} \sqrt{\log_2 x \log_2 \log_2 x}}{\frac{1}{2} x \cdot q_{k,m}} \leq \frac{1}{2q_{k,m}q},$$

hence, recalling (3·18), we find

$$\left| g_u(b) - \frac{p}{q} \right| \geq \frac{1}{2q_{k,m}q}. \hfill (3·21)$$

By inequality (3·16) combined with the upper bound in (3·11) and then (3·8) and (3·7) we get that, for $q$ large enough,

$$q_{k,m} \leq \frac{3}{2} |a_{k,k}|b^n \leq \frac{3}{2} |a_{k,k}|b^{t + d\tau_0 \sqrt{\log^2 t}} \leq 18 |a_{k,k}|q \cdot 2^{(2d\tau_0 \log_2 b + 2C) \sqrt{\log_2 q \log_2 \log_2 q}},$$

where $\tau_0$ is given by (?).
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The bound (2·13) implies
\[ \log_2 |a_{k,k}| \leq \frac{k}{2} \log_2 k + k(\log_d (2k + 1) + 1) \log_2 \|u\|_\infty. \] (3·22)

By comparing the right-hand side of this inequality with the left-hand side in (3·15) we find
\[ |a_{k,k}| \leq 2^{2\tau_0} \sqrt{\log_d q \log_2 q \log_2 q} \]
and then
\[ q_{k,m} \leq 18q \cdot 2^{(2d \tau_0 \log_2 b + 2r + 2C) \sqrt{\log_d q \log_2 q \log_2 q}}. \]

Finally, (3·21) implies
\[ \left| g_u(b) - \frac{p}{q} \right| \geq \frac{1}{36q^2 \cdot 2^{(2d \tau_0 \log_2 b + 2r + 2C) \sqrt{\log_d q \log_2 q \log_2 q}}} . \]

This completes the proof of the theorem.

Acknowledgements. The authors would like to express their gratitude to the referee, whose profound remarks and careful reading of the first version of this paper significantly contributed to the eventual quality of the article.

Evgeniy Zorin acknowledges the support of EPSRC Grant EP/M021858/1.

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