Plaquette-triplon analysis of a trimerized spin-1 Kagomé Heisenberg antiferromagnet

Pratyay Ghosh, Akhilesh Kumar Verma, and Brijesh Kumar
School of Physical Sciences, Jawaharlal Nehru University, New Delhi 110067, India
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A quantum spin-1 Heisenberg model on trimerized Kagomé lattice is studied by doing a low-energy bosonic theory in terms of plaquette-triplons defined on the triangular unit-cells of the lattice. The model considered has an intra-triangle antiferromagnetic exchange interaction, \( J \) (set to 1), and two inter-triangle couplings, \( J' > 0 \) (nearest-neighbor) and \( J'' \) (next-nearest-neighbor; of both signs). The triplon analysis performed on this model investigates the stability of the trimerized singlet ground state (which is exact in the absence of inter-triangle couplings) in the \( J'\)-\( J'' \) plane. It gives a quantum phase diagram that has two gapless antiferromagnetically ordered phases separated by the spin-gapped trimerized singlet phase. The trimerized singlet ground state is found to be stable on \( J'' = 0 \) line (the case of nearest-neighbor interactions only), and on both sides of it for \( J'' \neq 0 \), in an extended region bounded by the critical lines of transition to gapless antiferromagnetic phases. The gapless phase in the negative \( J'' \) region has coplanar 120°-antiferromagnetic order with \( \sqrt{3} \times \sqrt{3} \) structure. In this phase, all the local magnetic moments are of equal length, and the angle between any two of them on a triangle is exactly 120°. The magnetic lattice, in this case, has a bigger unit-cell of three triangles. The other gapless phase, in the positive \( J'' \) region, is found to exhibit a different coplanar antiferromagnetic order with ordering wavevector \( \mathbf{q} = (0, 0) \). Here, the two magnetic moments in a triangle are of same magnitude, but shorter than the third. While the angle between the short magnetic moments is 120° ± \( \delta \), it is 120° between a short and the long one. For \( J'' = J' \), however, these magnitudes become equal and the relative-angles 120°. The magnetic lattice in this phase is obviously same as the underlying trimerized Kagomé lattice.

I. INTRODUCTION

The quantum antiferromagnets on frustrated lattices, with competing interactions, tend to disfavor magnetic ordering, and realize interesting quantum-disordered low temperature phases (ground states) such as the quantum spin liquids, valence-bond-solid states, or the dimer plaquette ordered singlet phases [1–4]. The Kagomé quantum antiferromagnet is an interesting example of a frustrated spin system, in which the frustrated geometry of the Kagomé lattice (a triangular lattice of the corner-sharing triangles) and the quantum fluctuations together present a serious detriment to magnetic ordering in its ground state. For instance, the low temperature properties of \( \text{Cu}_3\text{Zn}(\text{OH})_6\text{Cl}_2 \) [5, 6], \( \text{BaCu}_3\text{V}_2\text{O}_8(\text{OH})_2 \) [7], \([\text{NH}_4]_2[\text{C}_7\text{H}_4\text{N}]\text{[V}_7\text{O}_6\text{F}_{18}] \) [8] and \( \gamma - \text{Cu}_3\text{Mg}(\text{OH})_6\text{Cl}_2 \) [9], which are the realizations of the spin-1/2 Kagomé Heisenberg antiferromagnet (KHA), seem to indicate this. While there is a strong support for the spin-1/2 KHA with nearest-neighbor interactions to have a spin liquid ground state, the hardness of this problem has made it very difficult to settle the debate on the true nature of its ground state [10–24]. Likewise, the relatively lesser studied spin-1 and higher spin KHAs are also not very well understood.

For the spin-1 antiferromagnetic Heisenberg model on Kagomé lattice, which is the problem of interest to us in the present paper, a non-magnetic hexagonal-singlet-solid (HSS) ground state with gapped magnetic excitations was proposed by Hida using the exact diagonalization and cluster expansion methods [25]. A more recent study of the spin-1 KHA model using coupled-cluster method also found a non-magnetic ground state [26]. This problem is currently in a surge of theoretical activity, motivated by the experiments on several spin-1 Kagomé materials [27–31]. While some of these materials do not neatly qualify as spin-1 Kagomé antiferromagnets (due to their ferromagnetic or glassy response), but there are some, e.g. \( m - \text{MPYNN} \cdot \text{Bi}_4 \) [27, 28], \( \text{Na}_2\text{V}_6\text{O}_{11} \) [32], and \( \text{KV}_3\text{Ge}_2\text{O}_9 \) [29], which show frustrated antiferromagnetic behavior with a spin-gap at low temperatures. The most recent numerical investigations using tensor network algorithms [33, 34], exact diagonalization and density matrix renormalization group (DMRG) [35] find a gapped spontaneously trimerized singlet ground state for the quantum spin-1 nearest-neighbor KHA model. There are others which either support the HSS state of Hida [36], or suggest a critical resonating AKLT (Affleck-Kennedy-Lieb-Tasaki) loop ground state [37]. Despite the disagreements, they all point towards a non-magnetic ground state for the spin-1 KHA with nearest-neighbor interaction, which is clearly in contrast with the studies that predicted \( \sqrt{3} \times \sqrt{3} \) antiferromagnetic order in its ground state [38, 39]. The spin-1 Kagomé antiferromagnet with spin anisotropies and biquadratic interaction have also been investigated [38, 40], but the pure Heisenberg case is what concerns us here for the moment.

Motivated by these recent studies on the ground state of spin-1 KHA, we study in this paper a Heisenberg model, described in Sec. II, on trimerized Kagomé lattice.
Our basic idea and the strategy are as follows. Since the Kagomé lattice is a triangular lattice of corner-sharing triangles, we construct an effective theory of spin-1 KHA in terms of the eigenstates of its basic triangular units. This we do by deriving, in Sec. III A, a bosonic representation of the spin-1 operators of a triangular plaquette in terms of its singlet and triplet states. It is like the bond-operator representation of the spins of a dimer [41, 42]. This effective theory, formulated in Sec. III B, allows us to study the stability of the trimerized singlet (TS) ground state with respect to the elementary triplon excitations (dispersing triplets), and to find if there is any antiferromagnetic (AF) order. Unlike the spin-wave analysis, which is a small fluctuation bosonic theory for a given classical magnetic order, this plaquette-triplon theory is formulated with respect to the non-magnetic TS state which is ‘quantum disordered’. It can describe classical order as well as quantum disorder in the ground state.

From the triplon analysis performed in this paper, we find a stable TS ground state for the nearest-neighbor spin-1 KHA, in agreement with recent numerical studies [33–35]. We also find this gapped TS phase over a range of second-neighbor interaction. Eventually, for sufficiently negative second-neighbor interaction, it undergoes a transition to the gapless phase with coplanar 120°-AF order with $\sqrt{3} \times \sqrt{3}$ structure. In this phase, the neighboring magnetic moments lie at 120° angle relative to each other, and the magnetic unit-cell consists of three triangles. This AF order has been known to occur in the KHA model with second and third neighbor interactions for large spins [39, 43]. But here, we find it for spin-1, emerging spontaneously from the quantum disordered TS state. For positive second-neighbor interactions, we find a different coplanar AF order with ordering wavevector $\mathbf{q} = (0, 0)$. In this phase, the magnetic moments on a triangle are of unequal magnitudes (two short and one long), and they form $120° - 2\delta$ (between the short moments), or $120° + \delta$ angle (between the long and a short one). This AF order has not been discussed before in a Kagomé antiferromagnet, but here it emerges spontaneously. We discuss all of these results in detail in Sec IV, and conclude this paper with a summary in Sec V.

II. MODEL

In this paper, we study the following quantum spin-1 Hamiltonian on trimerized Kagomé lattice.

$$\hat{H} = J \sum_{\langle i,j \rangle} \mathbf{\hat{S}}_i \cdot \mathbf{\hat{S}}_j + J' \sum_{\langle i,j \rangle} \mathbf{\hat{S}}_i \cdot \mathbf{\hat{S}}_j + J'' \sum_{\langle\langle i,j \rangle\rangle} \mathbf{\hat{S}}_i \cdot \mathbf{\hat{S}}_j \quad (1)$$

As depicted in Fig. 1, it is a problem of the coupled antiferromagnetic triangles. In this explicitly trimerized Kagomé problem, the exchange interaction, $J$, in the up (red) triangles is taken to be stronger than that in the down (green) triangles, $J'$. It resembles, for instance, the situation in NaV$_6$O$_{11}$, where the Kagomé lattice of vanadium ions trimerizes at low temperatures [32]. We also include second-neighbor interaction, $J''$, for generality. In Eq. (1), $\langle i, j \rangle$ denotes the spin-pairs in the up triangles, $\langle i, j \rangle$ denotes the spin-pairs in the down triangles, and the second-neighbor spin-pairs are denoted as $\langle\langle i, j \rangle\rangle$. We take $J$ and $J'$ to be antiferromagnetic, and allow $J''$ to be positive as well as negative. The $H$ becomes the standard nearest-neighbor KHA for $(J', J'') = (0, 0)$.

A simple limiting case of the $\hat{H}$ of Eq. (1) corresponds to $(J', J'') = (0, 0)$, for which the exact ground state is given by the direct product, $\otimes \prod_{\Delta} |s\rangle$, of the singlet states, $|s\rangle$, of the up-triangles. It also has an energy-gap, $J$, to the triplet excitations. Refer to Appendix A for the eigenstates of spin-1 Heisenberg model on a triangle. Since three spin-1’s uniquely form a singlet, this ideal spin-gapped trimerized singlet ground state is also unique. How the non-zero $J'$ and $J''$ affect this TS ground state is the question that we try to address here by studying its stability against the triplon excitations.

III. TRIPOLON ANALYSIS

Just as in the case of dimerized quantum antiferromagnets, where the bond-operator formalism provides a convenient means to construct an effective low-energy theory [41, 44], here we do a low-energy theory of the trimerized KHA model, $\hat{H}$, in terms of the plaquette-operators defined on the triangular unit-cells. Towards this goal, we derive a bosonic representation for the spin-1 operators on a triangle, and do a simple but useful theory of $\hat{H}$ in terms of these plaquette-operators. This theory would find us the region of stability of the TS state, and identify the magnetic order, if any, in the $J'$-$J''$ plane.
A. Plaquette-Operator Representation of the Spin-1 Operators on a Triangle

The spin-1 Heisenberg model on a triangle, that is \( \hat{H}_\Delta = J (S_1 \cdot S_2 + S_2 \cdot S_3 + S_3 \cdot S_1) \), has a unique singlet eigenstate, \(|s\rangle\), with eigenvalue, \(-3J\). It has three sets of triplets (that is, 9 degenerate states with total-spin equal to \(0\) or \(1\)), \(|t_{\mu\nu}\rangle\), given by \( m = 1, 0, 1 \) (total-S) and \( \nu = 1, 0, 1 \) (chirality). Here, \(-1\) is denoted as \(1\), and the quantum number \(\nu\) comes from the threefold rotational symmetry of \(\hat{H}_\Delta\). The energy of these triplets is \(-2J\). It also has two sets of quintets (10 states with total-spin \(2\)) and a heptet (7 eigenstates with total-spin \(3\)) with energies 0 and \(3J\), respectively. The eigenvalue problem for \(\hat{H}_\Delta\) is worked out in detail in Appendix A.

For \( J > 0 \), the singlet is the ground state of \(\hat{H}_\Delta\), and the triplets form the elementary excitations with energy-gap, \(J\). The quintets, that cost an energy \(3J\) from the ground state, are the next higher excitations. Since they are safely above the triplets, in the simplest approximation, we ignore the quintets and the highest energy heptets in writing a low-energy theory of the trimmerized KHA model, \(\hat{H}\). Thus, we restrict the triangle’s Hilbert space to have the singlet, \(|s\rangle\), and all the 9 triplets, \(|t_{\mu\nu}\rangle\), only. This reduced problem would nevertheless be sufficient to do a basic stability check of the TS ground state.

Like the bond-operator representation, that is so useful to the studies of dimer phases [41, 42, 44], we derive here a plaquette-operator representation for the spin-1 operators on a triangle in the reduced basis, \(|\{|s\rangle, |t_{\mu\nu}\rangle\}\). For this, let us first introduce the singlet and triplet operators, \(\hat{s}^\dagger\) and \(\hat{t}_{\mu\nu}^\dagger\), that are defined as follows.

\[
|s\rangle := \hat{s}^\dagger |0\rangle \quad (2a)
\]

\[
|t_{\mu\nu}\rangle := \hat{t}_{\mu\nu}^\dagger |0\rangle \quad (2b)
\]

Here, \(\hat{s}^\dagger\) are \(t_{\mu\nu}^\dagger\) bosonic creation operators in a Fock space with vacuum, \(|0\rangle\). The projection of the infinite dimensional Fock space onto the 10-dimensional Hilbert space spanned by \(|s\rangle\) and \(|t_{\mu\nu}\rangle\) is done by the following constraint on the number of these bosons.

\[
\hat{s}^\dagger \hat{s} + \sum_{\mu,\nu} \hat{t}_{\mu\nu}^\dagger \hat{t}_{\mu\nu} = 1 \quad (3)
\]

In this reduced space, the local Hamiltonian can now be written as: \(\hat{H}_\Delta \approx -3J \hat{s}^\dagger \hat{s} - 2J \sum_{\mu,\nu} \hat{t}_{\mu\nu}^\dagger \hat{t}_{\mu\nu}\).

Next we represent the spin-1 operators of a triangle in terms of the singlet and triplet plaquette-operators introduced above. Please refer to Appendix B for more details on it. Below we write this representation in an approximate form that is simple and useful.

\[
S_{j, z} \approx \hat{s} \sqrt{3} \left\{ c_j \hat{Q}_{z1} - s_j \hat{Q}_{z1} \right\} \quad (4a)
\]

\[
S_{j, \alpha} \approx \frac{2N}{\sqrt{3}} \left\{ c_{j-1} \hat{Q}_{\alpha1} - s_{j-1} \hat{Q}_{\alpha1} \right\} \quad \text{for } \alpha = x, y \quad (4b)
\]

Here, \(j = 1, 2, 3\) are the spins of a triangle (see Fig. 1 for spin labels), and \(\alpha = x, y, z\) denote the components of the spin-1 operators. Moreover, \(c_j = \cos \left(\frac{2\pi j}{3}\right)\) and \(s_j = \sin \left(\frac{2\pi j}{3}\right)\). The ‘coordinate’ operators \(\hat{Q}_{\nu\alpha}\) for \(\nu = 1, \bar{1}\) and \((\alpha, \bar{\alpha})\) are defined as: \(\hat{Q}_{\nu\alpha} = (\hat{t}_{\nu\alpha}^\dagger + \hat{t}_{\nu\bar{\alpha}}^\dagger) / \sqrt{2}\), where \(\hat{t}_{\nu\alpha} = (i - 1) \hat{t}_{\nu\alpha}^\dagger \hat{t}_{\nu\alpha}^\dagger \hat{t}_{\nu\alpha}^\dagger \hat{t}_{\nu\alpha}^\dagger / \sqrt{2}\) and \(\hat{t}_{\nu\alpha} = i \hat{t}_{\nu\alpha}^\dagger \hat{t}_{\nu\alpha}^\dagger / \sqrt{2}\). Likewise, we define the conjugate ‘momentum’ operators, \(\hat{P}_{\nu\alpha}\).

\[
\hat{Q}_{\nu\alpha} = i \delta_{\nu\alpha} \hat{Q}_{\bar{\nu} \bar{\alpha}} \quad \text{and} \quad \hat{P}_{\nu\alpha} = 2 \hat{t}_{\nu\alpha}^\dagger \hat{t}_{\nu\alpha}^\dagger + 1. \quad (5)
\]

B. Plaquette-Triplon Mean-Field Theory

Now we turn to the model, \(\hat{H}\), defined in Sec. II. In the plaquette-operator representation of Eqs. (4), it takes the following effective form in the momentum space.

\[
\hat{H}_t = c_0 N + \sum_{k} \sum_{\alpha=x,y,z} \left\{ \lambda \left( \hat{t}_{\alpha\mu}^\dagger \hat{t}_{\beta\mu} \right) + \frac{1}{2} \right\}
\]

\[
+ \frac{1}{2 \sqrt{N}} \left[ \lambda \left( \hat{P}_{\alpha\mu}^\dagger \hat{P}_{\beta\mu} \right) + \hat{Q}_{\alpha\mu}^\dagger \hat{Q}_{\beta\mu} \right] \right\} \quad (6)
\]

Here, \(N\) is the total number of triangular unit-cells in the trimmerized Kagomé lattice, \(c_0 = -\hat{s}^\dagger J + \lambda \hat{s}^2 - 2J - \frac{1}{2} \lambda\), and \(\lambda\) is the Lagrange multiplier that is introduced to satisfy the local constraint, \(\hat{s}^\dagger \hat{s} + \sum_{\alpha=x,y,z} \hat{t}_{\alpha\mu}^\dagger \hat{t}_{\alpha\mu}^\dagger = 1\), on average. Moreover,

\[
\hat{Q}_{\alpha\mu}(k) = \left[ \hat{Q}_{\alpha1}(k) \right] \quad \text{and} \quad \hat{P}_{\alpha\mu}(k) = \left[ \hat{P}_{\alpha1}(k) \right]
\]

where \(\hat{Q}_{\alpha1}(k)\) and \(\hat{Q}_{\alpha1}(k)\) are the Fourier components of \(\hat{Q}_{\alpha\mu}(k)\). That is, \(\hat{Q}_{\alpha\mu}(k) = \frac{1}{\sqrt{N}} \sum_{r} e^{i k \cdot r} \hat{Q}_{\alpha\mu}(k)\) for \(\nu = 1, \bar{1}\). Here, \(r\) denotes the position vector of the triangular units of the trimmerized Kagomé lattice (see Fig. 1), and \(k\) is the lattice-momentum vector in the first Brillouin zone of the corresponding reciprocal lattice (see Fig. 3).

Likewise, \(\hat{P}_{\alpha\mu}(k) = \frac{1}{\sqrt{N}} \sum_{r} e^{i k \cdot r} \hat{P}_{\alpha\mu}(k)\). Since \(\hat{Q}_{\alpha\mu}(r)\) and \(\hat{P}_{\alpha\mu}(r)\) are Hermitian, therefore, \(\hat{Q}_{\alpha\mu}^\dagger(k) = \hat{Q}_{\alpha\mu}(-k)\) and \(\hat{P}_{\alpha\mu}^\dagger(k) = \hat{P}_{\alpha\mu}(-k)\). Moreover, \(\left[ \hat{Q}_{\alpha\mu}(k) \right], \left[ \hat{P}_{\alpha\mu}(k) \right] =
The unitary matrix \( U \) utilized by making a unitary rotation of \( \hat{\alpha} \) to the new coordinates, \( \hat{\alpha} \), on \( \mathbb{K} \) lattice (see Fig. 1), and a \( \theta \) where \( \eta^* \alpha \kappa \) stay decoupled and local, the effective triplon model, \( \hat{t}_\alpha \), and described by \( Q_\alpha(k) \) and \( \bar{Q}_\alpha(k) \), and coupled via

\[
V_{\alpha,k} = \left[ \begin{array}{c}
l - 2s^2\epsilon_{\alpha\alpha} & s^2\eta_{\alpha,k} \\
l - 2s^2\epsilon_{\alpha\alpha} & \lambda - 2s^2\epsilon_{\alpha\alpha} \end{array} \right].
\] (7)

The \( V_{\alpha,k} \) is a Hermitian matrix, with \( \eta^* \alpha \kappa \) as the complex conjugate of \( \eta_{\alpha,k} \). The \( \epsilon_{\alpha\kappa} \) and \( \eta_{\alpha,k} \) are given below.

\[
\epsilon_{\alpha\kappa} = \epsilon_{\gamma\kappa} \] (9a)

\[
\eta_{\alpha,k} = \eta_{\gamma,k} \] (9b)

\[
\epsilon_{\alpha\kappa} = \epsilon_{\gamma\kappa} \] (9b)

In the diagonal form, the \( \hat{H}_t \) can be written as follows.

\[
\hat{H}_t = E_0 N + \sum_k \sum_{\alpha=x,y,z} \left\{ \lambda \left[ \epsilon_{\alpha\kappa}(k) t_{\alpha\kappa}(k) + \frac{1}{2} \right] \right. \\
+ \sum_{\mu=\pm} E_{\mu\kappa}(k) \left[ \epsilon_{\mu\kappa}(k) t_{\mu\kappa}(k) + \frac{1}{2} \right] \} \] (13)

Here, \( t_{\alpha\kappa}(k) = \sqrt{\frac{E_{\alpha\kappa}(k)}{2\lambda N}} \hat{Q}(\alpha\kappa)(k) + i \sqrt{\frac{\lambda}{2\beta_{\alpha\kappa}(k)}} \hat{P}(\alpha\kappa)(k) \) are the renormalized triplon operators, and

\[
E_{\alpha\kappa}(k) = \sqrt{\lambda(\lambda - 2s^2\epsilon_{\alpha\kappa})} \] (14)

It is a function of two unknown mean-field parameters, \( \lambda \) and \( s^2 \). We determine them by minimizing \( E_g \). That is, \( \partial_x E_g = 0 \) and \( \partial_y e_g = 0 \), which gives us the following mean-field equations.

\[
\frac{s^2}{4} = 4 - \frac{1}{2N} \sum_{k} \sum_{\alpha=x,y,z} \sum_{\mu=\pm} \lambda - \frac{s^2\epsilon_{\alpha\kappa}}{E_{\alpha\kappa}} \] (16a)

\[
\lambda = J + \frac{\lambda}{2N} \sum_{k} \sum_{\alpha=x,y,z} \sum_{\mu=\pm} \frac{\epsilon_{\alpha\kappa}}{E_{\alpha\kappa}} \] (16b)

The self-consistent solution of the above equations gives the physical values of \( \lambda \) and \( s^2 \).

This formulation offers two distinct physical solutions based on whether the triplon dispersions are gapped or gapless. The \( \hat{H}_t \) has nine triplon dispersions. The three \( t_{\alpha\kappa}(k) \)’s have flat dispersions at \( \lambda = 0 \). There, then, are six non-trivial \( E_{\alpha\kappa}(k) \). Note that \( E_{\alpha\kappa}(k) \) is exactly same as \( E_{\gamma\kappa}(k) \), but they are different from \( E_{\alpha\kappa}(k) \). When the minimum of the lowest of these dispersions in the Brillouin zone is strictly greater than zero, it means there is an energy gap that protects the TS ground state against triplon excitations. We surely expect this to happen when \( J' \) and \( J'' \) are near about zero. In the gapped TS phase, Eqs. (16) are applicable in the form as given.

However, as the inter-triangle couplings grow stronger, the triplon gap may close at some point \( q \) in the Brillouin zone for strong enough \( J' \) or \( J'' \). That is, \( E_{\alpha\kappa}(q) = 0 \), for some lower triplon branches. If it happens, then the corresponding, \( k = q \), terms in Eqs. (16) will become singular, giving rise to triplon condensation described by the condensate density, \( n_c \), now a third unknown. But we also have a third equation, which is the condition of gaplessness, in addition to Eqs. (16) (to be revised for \( n_c \)). From our calculations (described in the next section), we
either get $E_{\alpha\mu,q} = 0$ at $q = (0, 0)$ for $\alpha = x, y$ and $\mu = \pm$ in a region for $J'' > 0$, or $E_{\alpha\mu,q} = 0$ for $\alpha = x, y$ at $q = (\frac{\pi}{3}, \frac{\pi}{3})$ in another region for $J'' < 0$. The other dispersions are always gapped (see Figs. 3, 6 and 9).

The revised equations for the gapless case of $q = (0, 0)$ can be written as follows.

$$\lambda = 2s^2\xi_{\alpha\mu,q} \quad \text{(same for } \alpha = x, y \text{ and } \mu = \pm) \quad (17a)$$

$$\bar{s}^2 = 4 - n_c - \frac{1}{2N} \sum_{k \neq q} \sum_{\alpha=x,y} \sum_{\mu=\pm} \frac{\lambda - \bar{s}^2\xi_{\alpha\mu,k}}{E_{\alpha\mu,k}} \quad (17b)$$

$$n_c = \bar{s}^2 \left(1 - \frac{J}{\lambda}\right) - \frac{\bar{s}^2}{2N} \sum_{k \neq q} \sum_{\alpha=x,y} \sum_{\mu=\pm} \frac{\xi_{\alpha\mu,k}}{E_{\alpha\mu,k}} \quad (17c)$$

The equation for $\lambda$ here follows directly from the zero gap condition, $E_{\alpha\mu,q} = 0$. The other two equations are derived from Eqs. (16) by introducing the condensate density, $n_c$, for the singular terms in the summations. That is, $n_c \equiv \frac{1}{2N} \sum_{\alpha=x,y} \sum_{\mu=\pm} (\lambda - \bar{s}^2\xi_{\alpha\mu,q})/E_{\alpha\mu,q}$, in the present case. Likewise, in the other gapless phase with $q = (\frac{\pi}{3}, \frac{\pi}{3})$, the following equations would apply.

$$\lambda = 2s^2\xi_{\alpha\mu,q} \quad \text{(same for } \alpha = x, y \text{)} \quad (18a)$$

$$\bar{s}^2 = 4 - n_c - \frac{1}{2N} \sum_{k \neq q} \sum_{\alpha=x,y} \frac{\lambda - \bar{s}^2\xi_{\alpha\mu,k}}{E_{\alpha\mu,k}} \quad (18b)$$

$$n_c = \bar{s}^2 \left(1 - \frac{J}{\lambda}\right) - \frac{\bar{s}^2}{2N} \sum_{k \neq q} \sum_{\alpha=x,y} \frac{\xi_{\alpha\mu,k}}{E_{\alpha\mu,k}} \quad (18c)$$

Here, $n_c \equiv \frac{1}{2N} \sum_{\alpha=x,y} (\lambda - \bar{s}^2\xi_{\alpha\mu,q})/E_{\alpha\mu,q}$. Physically, a non-zero $n_c$ and $q$ account for the AF order with ordering wavevector $q$ in the ground state. Equations (20) and (21) in Sec. IV B describe the magnetic moments in terms of $n_c$ and $q$ in the two ordered phases.

IV. RESULTS AND DISCUSSION

To determine the ground state properties of $\hat{H}$ within the triplon mean-field theory, we numerically solve the self-consistent equations derived in the previous section. In our calculations, we set $J = 1$ and take $0 \leq J' \leq 1$. We keep the second-neighbor coupling, $J''$, small, but allow it to take both positive and negative values ($|J''| \lesssim 0.6$).

### A. Gapped Trimerized Singlet Phase

According to the theory presented in the last section, the energy gap to triplon excitations decides if the ground state is non-magnetic (TS) or magnetically ordered. As the trivial case of independent triangles is surely gapped, a region around $(J', J'') = (0, 0)$ is also expected to be so. We identify this region of gapped TS phase by following the change in the triplon gap, $\Delta_t$, by gradually increasing the inter-triangle couplings, $J'$ and $J''$. If and when the gap closes, it marks the quantum phase transition to an AF ordered phase. First we discuss this in the nearest-neighbor interaction model for $J'' = 0$, and then in the full model including $J''$.

1. $J'' = 0$

In this case, $J'$ is the only interaction variable. We calculate the triplon gap, $\Delta_t$, by solving Eqs. (16) for $\lambda$ and $\bar{s}^2$ for different values of $J'$ between 0 and 1. Figure 2 presents the calculated values of $\lambda$, $\bar{s}^2$ and $\Delta_t$ as a function of $J'$. At $J' = 0$, it gives $\bar{s}^2 = 1$ and $\Delta_t = 1$, which is exact for the independent triangles. A notable feature of this data is the non-zero triplon gap in the entire range of $J'$ between 0 and 1. Although $\Delta_t$ first decreases as $J'$ increases from 0, but after a while, it turns upwards and keeps growing. It is an interesting result which states that, for $J'' = 0$, the non-magnetic TS ground state is stable against triplon excitations, and it $\textit{adiabatically}$ extends all the way up to $J' = 1$, starting from exact case at $J' = 0$. This result clearly favors the

![FIG. 2. The singlet weight, $\bar{s}^2$, the Lagrange multiplier, $\lambda$, and the triplon gap, $\Delta_t$, calculated from the self-consistent Eqs. (16) for $J'' = 0$ (the nearest-neighbor case of $\hat{H}$).](image)
recent claims of a gapped trimerized singlet ground state for the nearest-neighbor spin-1 KHA model [33–35].

To see where the gap, \( \Delta_t \), comes from in the Brillouin zone, we plot the dispersions in Fig. 3. Of the nine triplon dispersions given in Eq. (13), the two flat dispersions, \( E_{x-k} = E_{y-k} = \sqrt{\lambda (\lambda - 2s^2J')} \), are the lowest. Hence, the triplon gap in this case is, \( \Delta_t = \sqrt{\lambda (\lambda - 2s^2J')} \). Two other dispersions, \( E_{\alpha+k} \) (for \( \alpha = x, y \)), also become degenerate with the lowest ones at \( k = (0,0) \), the \( \Gamma \) point. Moreover, \( E_{z-k} \) and \( \lambda \) (for three \( \nu = 0 \) branches), are also flat. But they not important for the discussion here, as they are not the lowest in energy.

2. \( J'' \neq 0 \)

While the non-magnetic TS phase is stable for the nearest-neighbor case of \( \hat{H} \), it would be nice to know how the second-neighbor interaction, \( J'' \), affects it, or if it generates any magnetic order in the ground state. In the classical KHA problem, even an infinitesimal amount of \( J'' \) causes ordering [43].

For different fixed values of \( J' \), we solve Eqs. (16) with \( J'' \) varying from 0 to \( \pm 0.6 \), and follow the triplon gap. We find that a non-zero \( J'' \) makes the flat modes, \( E_{\alpha+k} \), dispersive, which reduces the gap, and can even close it altogether. For \( J'' < 0 \), the triplon gap always closes at some non-zero critical value of \( J'' \). The gap also closes for \( J'' > 0 \), but only when \( J' > 0.144 \). This is, if \( J' \) is too small, then the ground state stays gapped for any positive \( J'' \). By scanning the \( J'-J'' \) plane for the critical points where the triplon gap vanishes, we compute the boundaries of the TS phase. For instance, the triplon gap for \( J' = 1 \) vanishes at \( J'' = -0.245 \), or at \( J'' = 0.186 \). The quantum phase diagram thus generated is shown in Fig. 4. Clearly, the case of positive \( J'' \) is more frustrated, as it favors the TS phase more than the negative \( J'' \).

We identify the gapless phase for positive \( J'' \) with wavevector \( \mathbf{q} = (0,0) \), the \( \Gamma \) point, at which the triplon gap vanishes. In the other gapless phase for negative \( J'' \), the gap closes at the \( K \)-point in Brillouin zone, that is \( \mathbf{q} = (\pi/3, \pi/\sqrt{3}) \). See Figs. 6 and 9 for the dispersions in the two phases. As mentioned before, these gapless phases exhibit magnetic order through Bose condensation of triplons with their respective \( \mathbf{q} \)'s as the ordering wavevectors. The precise forms of the magnetic orders in the two gapless phases are described below.

B. Antiferromagnetically Ordered Phases

We calculate the properties of the gapless phases from Eqs. (17) and (18). These equations enable the computation of triplon condensate density, \( n_c \), in addition to giving us the dispersions (and \( s^2 \)).

The knowledge of \( n_c \) is of great physical significance. Together with \( \mathbf{q} \), it determines the magnetic order in a gapless phase. A non-zero \( n_c \) implies spontaneous triplon 'displacements', \( \langle Q_{\alpha \mu}(\mathbf{r}) \rangle \), which through the plaquette-operator representation given in Eqs. (4), determine the
local magnetic moments, \( \langle \mathbf{S}_j(r) \rangle \), on Kagomé lattice. Since the triplons with dispersions \( E_{2\mu,k} \) do not condense (as they are gapped; see Fig. 6 and 9), we get \( \langle \hat{Q}_{2\mu}(r) \rangle = 0 \). However, the condensation for \( \alpha = x, y \) at \( q \) gives the following non-zero displacements.

\[
\langle \hat{Q}_{x1} \rangle = \sqrt{2n_c} \sin (q \cdot r), \quad \langle \hat{Q}_{y1} \rangle = \sqrt{2n_c} \cos (q \cdot r)
\]

(19a)

\[
\langle \hat{Q}_{y1} \rangle = \sqrt{2n_c} \cos (q \cdot r), \quad \langle \hat{Q}_{y1} \rangle = -\sqrt{2n_c} \sin (q \cdot r)
\]

(19b)

Here, \( n_{c1} \) and \( n_{c3} \) are the condensate densities for \( \nu = 1, 3 \) (that are same for \( \alpha = x, y \)). Hence, \( n_c = 2(n_{c1} + n_{c3}) \). Since \( n_{c1} \) and \( n_{c3} \) can in general be different, we define a parameter \( \zeta = n_{c1}/n_{c3} \). In terms of \( n_c \) and \( \zeta \), we can write \( n_{c1} = \frac{n_c}{2(1+\zeta)} \) and \( n_{c3} = \frac{n_c}{2(1+\zeta)} \).

By putting these displacements into the plaquette-operator representation for spins, we get the following general form of the magnetic moments.

\[
\mathbf{m}_j(r) = m_j(\cos [\varphi_j - q \cdot r], \sin [\varphi_j - q \cdot r], 0)
\]

(20)

Here, \( \mathbf{m}_j(r) = \langle \mathbf{S}_j(r) \rangle \) is the magnetic moment due to \( j^{th} \) spin in the triangular unit-cell at position \( r \), with three components, \( m_{j,x}(r) = m_j \cos [\varphi_j - q \cdot r] \), \( m_{j,y}(r) = m_j \sin [\varphi_j - q \cdot r] \) and \( m_{j,z}(r) = 0 \). These moments are obviously coplanar. Their amplitudes, \( m_j \), and the angles, \( \varphi_j \), are given below for \( j = 1, 2, 3 \).

\[
(m_1, \varphi_1) = \left(2\bar{s}_1, \frac{\pi}{2} \right)
\]

(21a)

\[
(m_2, \varphi_2) = \left(\frac{\sqrt{1+3\zeta}}{2}, \varphi_1 + \frac{2\pi}{3} + \delta \right)
\]

(21b)

\[
(m_3, \varphi_3) = \left(m_2, \varphi_1 - \frac{2\pi}{3} + \delta \right)
\]

(21c)

Here, \( \delta = \tan^{-1}\left[\sqrt{3}(1-\sqrt{\zeta})/(1+3\sqrt{\zeta})\right] \). The \( \mathbf{m}_j(r) \)'s on every triangle exactly add up to zero, as it should be in an antiferromagnetic phase. From these general considerations, now we turn to the specific cases.


1. **Coplanar AF order with \( q = (0, 0) \)**

From Eqs. (17), applicable to the phase with Goldstone mode at \( q = (0, 0) \), we calculate \( \lambda, \bar{s}^2 \) and \( n_c \). In Fig. (5), we plot \( n_c \) as a function of \( J'' \) for fixed values of \( J' \), alongside \( \Delta_\lambda \) of the gapped phase. It shows a quantum phase transition characterized by the triplon gap that goes to zero continuously at the critical point, and \( n_c \) that grows continuously starting from zero at the critical point. The \( \bar{s}^2 \) and \( \lambda \) also show a kink-like behavior across the transition. In Fig. 6, we show the triplon dispersions, of which, the four dispersions with \( \alpha = x, y \) and \( \mu = \pm \) go to zero linearly in \( |k| \) at \( q = (0, 0) \), the \( \Gamma \) point.

We infer the magnetic order in this phase from Eqs. (20) and (21). Its salient features are as follows.

Firstly, the magnetic moments are independent of \( r \), obviously because \( q = (0, 0) \). That is, the \( \mathbf{m}_j(r) \)'s in all the unit-cells look identical, as shown in Fig. 7.

Secondly, the magnetic moments within a unit-cell are not oriented at 120\(^\circ\) angle with respect to each other. That is, \( \varphi_j(\alpha, \beta) = 120^\circ + \delta \) and \( \varphi_j - \varphi_j = 120^\circ - 2\delta \), where \( \delta \) is non-zero except when \( \zeta = 1 \). Their magnitudes, \( m_j \)'s, are also unequal. These features are clearly at variance with what is known from the semiclassical analysis of the KHA problem for large spins [39, 43]. But then ours is a calculation with reference to the quantum disordered TS state, with no presumptions of any magnetic order. Here, this magnetic order has emerged spontaneously through triplon dynamics. For instance, in

---

**FIG. 5.** The triplon gap, \( \Delta_\lambda \), and the condensate density, \( n_c \), vs. \( J'' \). Together, they characterize the quantum phase transition from the gapped TS phase to the \( q = (0, 0) \) AF ordered phase for \( H \) of Eq. (1). Inset: \( \lambda \) and \( \bar{s}^2 \) vs. \( J'' \).

**FIG. 6.** The triplon dispersions [as given in Eq. (13)] in the gapless AF phase with Goldstone mode at \( q = (0, 0) \). In this phase, the four dispersions, \( E_{\alpha,\mu,k} \) for \( \alpha = x, y \) and \( \mu = \pm \), go to zero linearly in \( |k| \) at the \( \Gamma \) point.
FIG. 7. The coplanar antiferromagnetic order with \( q = (0, 0) \).
Here, the magnetic moments, denoted as \( \tilde{A}, \tilde{B} \) and \( \tilde{C} \), are arranged identically in all the unit-cells (red triangles). The angle between \( \tilde{A} \) and \( \tilde{B} \) is \( 120^\circ + \delta \), which is same as the angle between \( \tilde{A} \) and \( \tilde{C} \). The magnitude of \( \tilde{B} \) is equal to that of \( \tilde{C} \), but shorter than that of \( \tilde{A} \). Refer to Eqs. (21) for details.

FIG. 8. The coplanar antiferromagnetic order with \( q = (0, 0) \).

FIG. 9. The energy dispersions of the triplon excitations in the gapless AF phase with Goldstone mode at \( q = (\pi/3, \pi/\sqrt{3}) \). Here, two degenerate dispersions, \( E_{x-k} \) and \( E_{y-k} \), go to zero linearly at \( q = (\pi/3, \pi/\sqrt{3}) \), the \( K \) point, as shown in Fig. 9.

FIG. 8. The triplon gap, \( \Delta_t \), and the condensate density, \( n_c \), describing the quantum phase transition from the gapped TS phase to the gapless phase with Goldstone mode at \( q = (\pi/3, \pi/\sqrt{3}) \). Inset: \( \lambda \) and \( s^2 \) vs. \( J'' \).

FIG. 9. The energy dispersions of the triplon excitations in the gapless AF phase with Goldstone mode at \( q = (\pi/3, \pi/\sqrt{3}) \). Here, two degenerate dispersions, \( E_{x-k} \) and \( E_{y-k} \), go to zero linearly at \( q = (\pi/3, \pi/\sqrt{3}) \), the \( K \) point, as shown in Fig. 9.

2. Coplanar 120°-AF order with \( \sqrt{3} \times \sqrt{3} \) structure

The solutions of Eqs. (18) determine the nature of the gapless phase for negative \( J'' \). The condensate density, \( n_c \), and other quantities, calculated as a function of \( J'' \) for fixed \( J' \), are plotted in Fig. 8. Here again, we see a continuous rise of \( n_c \) starting from zero at the critical point. Moreover, the dispersions, \( E_{\alpha-k} \) (for \( \alpha = x, y \)), go to zero linearly at \( q = (\pi/3, \pi/\sqrt{3}) \), the \( K \) point, as shown in Fig. 9.

near the critical point for \( J' = 1 \) to \( 0^\circ \) at \( J'' = J' \) line.

The other gapless phase for negative \( J'' \) (to be discussed next), the same triplon analysis gives us the perfect 120°-AF order with \( \sqrt{3} \times \sqrt{3} \) structure, as known from semiclassical analysis. Hence, the AF order that we have got here, for \( q = (0, 0) \) phase, looks like a genuine finding. In fact, we realize that the semiclassical analysis would miss this order completely, because there the local moments in the reference state are given to be of same magnitudes, which leaves their relative-angles with no choice, but to be 120° (on AF triangles). In our triplon analysis, all of this is decided for itself by the triplon dynamics. We neither fix their magnitudes nor the angles from outside.

This brings us to the last feature of note that is about \( \zeta \), which effects the deviation of the moments from 120° orientations through \( \delta \), and makes their magnitudes unequal. The \( \zeta \) in this phase originates from the difference in the slopes, \( v_+ \) and \( v_- \), of \( E_{\alpha+k} \) and \( E_{\alpha-k} \) at \( q = (0, 0) \), which is clearly visible in Fig. 6. More precisely, it is \( \zeta = \frac{v_-}{v_+} = \sqrt{\frac{J' + 2 J''}{J' + 2 J'' + J'}} \). In general, \( \zeta < 1 \), and therefore \( \delta \neq 0 \). However, when \( J'' = J' \), then \( \zeta = 1 \) and \( \delta = 0^\circ \). Only in this case, the \( q = (0, 0) \) phase has perfect 120°-AF order of the moments of equal magnitudes. In the quantum phase diagram shown in Fig. 4, this special case is highlighted by labelling the dashed line for \( J'' = J' \). On either side of this line, \( \zeta < 1 \) and \( \delta \neq 0 \). For example, \( \delta \) varies continuously from about 9°
moments will rotate by \(120^\circ\) relative to each other. These deviations are found to arise from the difference in the triplon velocities at \(q = (0, 0)\) [see Figs. 6 and 7, and Eqs. (21)], and depend on \(J^\prime\) and \(J^\prime\). Only at \(J^\prime = J^\prime\), it forms a perfect \(120^\circ\)-AF order. This interesting coplanar AF order for positive \(J^\prime\) is a new find with a scope for further investigations.

V. SUMMARY

We now conclude by summarizing the main points. Motivated by the current interests in the spin-1 Kagomé quantum antiferromagnets, we have studied a spin-1 Heisenberg model, the \(\tilde{H}\) of Eq. (1), on trimerized Kagomé lattice (see Fig. 1). The \(\tilde{H}\) is a problem of coupled antiferromagnetic triangles (trimers), which in the absence of inter-triangle couplings, \(J^\prime\) and \(J^\prime\), trivially realizes the exact TS (trimerized singlet) ground state with zero local magnetic moments and a finite energy gap to triplet excitations. Here, we have studied the stability of this TS ground state, and its transition to ordered phases, as a function of \(J^\prime\) and \(J^\prime\). This we have done by deriving a bosonic plaquette-operator representation for spin-1 operators in terms of the singlet and triplet eigenstates of a triangle [see Eqs. (4) and the Appendices A and B], and then writing an effective triplon model, \(\tilde{H}\), of Eq. (5), for the \(\tilde{H}\) with reference to the TS state. The notable outcomes of this triplon analysis are as follows.

For \(J^\prime = 0\), that is, in the nearest-neighbor case of \(H\), the TS ground state is found to be always gapped and hence stable against triplon excitations. It smoothly extends right up to \(J^\prime = 1\) (the un-trimerized model), in agreement with the recent numerical findings of the same in the nearest-neighbor spin-1 Kagomé Heisenberg antiferromagnetic model [33–35]. The TS phase is also found to be stable over a range of \(J''\), before undergoing transition to two gapless ordered AF (antiferromagnetic) phases, one with ordering wavevector \(q = (0, 0)\) for positive \(J''\), and the other with \(q = (\pi/3, \pi/\sqrt{3})\) for negative \(J''\). The quantum phase diagram obtained from these calculations is presented in Fig. 4. The magnetic order in the phase with Goldstone modes at \(q = (\pi/3, \pi/\sqrt{3})\) is the familiar coplanar \(120^\circ\)-AF order with \(\sqrt{3} \times \sqrt{3}\) structure (see Fig. 10). In the other AF phase with \(q = (0, 0)\), the magnetic moments are coplanar, but of unequal magnitudes and deviate from \(120^\circ\) angle relative to each other. These deviations are found to arise from the difference in the triplon velocities at \(q = (0, 0)\) [see Figs. 6 and 7, and Eqs. (21)], and depend on \(J^\prime\) and \(J^\prime\). Only at \(J^\prime = J^\prime\), it forms a perfect \(120^\circ\)-AF order. This interesting coplanar AF order for positive \(J''\) is a new find with a scope for further investigations.
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Appendix A: Eigenstates of the spin-1 Heisenberg problem on a triangle

Here, we compute the eigenstates of the Heisenberg model of three spin-1’s, given below.

\[ \hat{H}_{\Delta} = \left( \hat{S}_1 \cdot \hat{S}_2 + \hat{S}_2 \cdot \hat{S}_3 + \hat{S}_3 \cdot \hat{S}_1 \right) \]  \hspace{1cm} (A1)

The \( \hat{H}_{\Delta} \) has spin-rotation symmetry due to which the total-spin, \( S = \hat{S}_1 + \hat{S}_2 + \hat{S}_3 \), is conserved. Its eigenstates would, therefore, be the total-spin eigenstates given by the total-spin quantum number, \( S = 0, 1, 2, 3 \), and the total-\( S^2 \), \( m = \pm 3, \pm 2, \pm 1, 0 \). It also has a discrete threefold rotational symmetry that leads to an additional conserved quantum number, \( \nu = \pm 1, 0 \), describing three discrete rotations, \( \omega^\nu \), of the triangle. Here, \( \omega = e^{i2\pi/3} \) is the cube root of unity. Together the two symmetries make it possible to exactly determine the eigenstates and eigenvalues of \( \hat{H}_{\Delta} \).

We denote the product states of three spin-1’s as \( |m_1m_2m_3\rangle \) in the \( S_z \) basis, where \( m_j = 1, 0, 1 \) are the eigenvalues of the spin operators, \( \hat{S}_{j,z} \) for \( j = 1, 2, 3 \). Here, \( \bar{m} \) denotes ‘\(-m\)’. We use this notation for negative \( m \)'s. First, we sectorize these states according to their total-\( S_z \) quantum number, \( m = m_1 + m_2 + m_3 \), as given in Table I. Next, we reorganize the states in the \( m \)-sectors according to the quantum number, \( \nu \), of the discrete threefold rotational symmetry. The basis states in terms of \( m \) and \( \nu \) are given in Table II. Since \( m \) and \( \nu \) are conserved with respect to \( \hat{H}_{\Delta} \), the states of different \( m,\nu \) subspaces do not mix under \( \hat{H}_{\Delta} \). This greatly reduces the eigenvalue problem. We finally write the \( \hat{H}_{\Delta} \) as matrix in each \( m,\nu \) subspace independently, and solve the corresponding eigenvalue problem. The eigenstates \( |S, m; \nu\rangle \) of \( \hat{H}_{\Delta} \) thus found are written below.

TABLE I. The basis states of \( \hat{H}_{\Delta} \) according to their total-\( S_z \) quantum number, \( m \). The states for negative values of \( m \) can be obtained from the positive states by doing \( S_z \) inversion operation, that is, \( 1 \leftrightarrow \bar{1} \) and \( 0 \leftrightarrow 0 \).

| \( m \) | \( |m_1m_2m_3\rangle \) |
|-------|-----------------|
| 3     | \[|111\rangle\]  |
| 2     | \[|110\rangle, |101\rangle, |011\rangle\] |
| 1     | \[|100\rangle, |010\rangle, |001\rangle, |111\rangle, |111\rangle, |111\rangle\] |
| 0     | \[|000\rangle, |110\rangle, |101\rangle, |011\rangle, |110\rangle, |011\rangle\] |

TABLE II. The basis states of \( \hat{H}_{\Delta} \) in terms of the quantum number, \( \nu \), of threefold rotation. To obtain negative-\( m \) states, do the \( S_z \) inversion operation on the positive-\( m \) states.

| \( m \) | \( \nu \) | Basis states |
|-------|-------|---------------|
| 3     | 0     | \[|111\rangle\] |
| 2     | 0     | \[\frac{1}{\sqrt{3}}\left(|110\rangle + |101\rangle + |011\rangle\right)\] |
|       | 1     | \[\frac{1}{\sqrt{3}}\left(|110\rangle + \omega|101\rangle + \omega^2|011\rangle\right)\] |
|       | -1    | \[\frac{1}{\sqrt{3}}\left(|110\rangle + \omega^2|101\rangle + \omega|011\rangle\right)\] |
| 1     | 0     | \[\frac{1}{\sqrt{3}}\left(|000\rangle + |010\rangle + |001\rangle\right)\] |
|       | 1     | \[\frac{1}{\sqrt{3}}\left(|100\rangle + \omega|010\rangle + \omega^2|001\rangle\right)\] |
|       | -1    | \[\frac{1}{\sqrt{3}}\left(|111\rangle + \omega^2|111\rangle + \omega|111\rangle\right)\] |

\textbf{Heptets:} These are unique \( S = 3 \) and \( \nu = 0 \) eigenstates of \( \hat{H}_{\Delta} \) with eigenvalue 3\( J \).

\begin{align*}
|3, 3; 0\rangle &= |111\rangle \hspace{1cm} (A2a) \\
|3, 2; 0\rangle &= \frac{1}{\sqrt{3}}\left(|110\rangle + |101\rangle + |011\rangle\right) \hspace{1cm} (A2b) \\
|3, 1; 0\rangle &= \frac{1}{\sqrt{15}}\left[2\left(|100\rangle + |010\rangle + |001\rangle\right) \\
&\quad + \left(|111\rangle + |111\rangle + |111\rangle\right)\right] \hspace{1cm} (A2c) \\
|3, 0; 0\rangle &= \frac{1}{\sqrt{10}}\left[2(|000\rangle + |100\rangle + |110\rangle + |010\rangle + |001\rangle\right. \\
&\quad \left. + |111\rangle + |101\rangle + |111\rangle\right] \hspace{1cm} (A2d)
\end{align*}

The negative-\( m \) eigenstates, \( |S, m; \nu\rangle \), can be obtained by changing \( |m_1m_2m_3\rangle \) to \( |\bar{m}_1\bar{m}_2\bar{m}_3\rangle \) in the corresponding positive-\( m \) eigenstates, \(|S, m; \nu\rangle\). For example, \(|3, 3\rangle = |111\rangle\), and likewise for other negative-\( m \) eigenstates.

\textbf{Quintets:} These are \( S = 2 \) eigenstates with eigenvalue equals to 0. Here we get two different sets of quintets, one each for \( \nu = 1 \) and \( \bar{1} \), as written below.

\begin{align*}
|2, 2; \nu\rangle &= \frac{1}{\sqrt{3}}\left(|110\rangle + \omega^\nu|101\rangle + \omega^\nu|011\rangle\right) \hspace{1cm} (A3a) \\
|2, 1; \nu\rangle &= \frac{1}{\sqrt{6}}\left[\left(|100\rangle + \omega^\nu|010\rangle + \omega^\nu|001\rangle\right) \\
&\quad - \left(|111\rangle + \omega^\nu|111\rangle + \omega^\nu|111\rangle\right)\right] \hspace{1cm} (A3b) \\
|2, 0; \nu\rangle &= \frac{1}{\sqrt{6}}\left[\left(|010\rangle + \omega^\nu|101\rangle + \omega^\nu|110\rangle\right) \\
&\quad + \left(|110\rangle + \omega^\nu|011\rangle + \omega^\nu|101\rangle\right)\right] \hspace{1cm} (A3c)
\end{align*}
Here, $\bar{\nu} = -\nu$, and the negative-$m$ states can be obtained by doing the $S_z$-inversion (1 ↔ $I$) of the above states.

**Triplets:** Next, we have three sets of triplets, each one for $\nu = 0, 1$ and $1$, with eigenvalue, $-2J$. Thus, $\hat{H}_\Delta$ has 9 degenerate $S = 1$ eigenstates, written below. Here, we denote the triplet states $|1, m; \nu \rangle$ as $|t_{m\nu} \rangle$. This slightly different notation is introduced to facilitate a convenient notation for the plaquette-operator representation (in the reduced space of the triplets and the singlet), as used in the main text this paper (see Sec. III A).

\[(A4a)\]

\[
|t_{m\nu} \rangle = \frac{1}{\sqrt{6}} \left\{ \left[ (|101\rangle + |01\rangle + |1\rangle) \right] \right.
+ \left[ (|1\rangle + |0\rangle + |10\rangle) \right] \right.
\]

\[(A4b)\]

\[
|t_{m\nu} \rangle = \frac{1}{\sqrt{6}} \left\{ \left[ (|01\rangle + \omega^\nu |10\rangle + \omega^\nu |01\rangle) \right] \right.
+ \left[ (|0\rangle + \omega^\nu |1\rangle + \omega^\nu |0\rangle) \right] \right.
\]

\[(A4c)\]

\[
|t_{m\nu} \rangle = \frac{1}{\sqrt{6}} \left\{ \left[ (|m0\rangle + |0m0\rangle + |00m\rangle) \right] \right.
- 2 \left[ (|\bar{m}m\rangle + |mm\bar{m}\rangle + |mm\rangle) \right] \right.
\]

\[(A4d)\]

**Singlet:** Finally, we write the only singlet eigenstate, that is $|0, 0, 0 \rangle$, of $\hat{H}_\Delta$. Here, it is denoted as $|s\rangle$, and has an eigenvalue of $-3J$.

\[
|s\rangle = \frac{1}{\sqrt{6}} \left\{ (|110\rangle - |110\rangle + |10\rangle - |01\rangle + |01\rangle - |01\rangle) \right\}
\]

\[(A5)\]

For an antiferromagnetic $\hat{H}_\Delta$, that is $J > 0$, the singlet at $-3J$ is the lowest energy eigenstate. The triplets at $-2J$ are the lowest excited states, while the quintets and the heptet sit further up at the higher energies.

**Appendix B: Plaquette-operator representation of the spin-1 operators of a triangle**

We now derive the plaquette-operator representation of the spin-1 operators of an antiferromagnetic triangle in its reduced 10-dimensional basis, $\{|s\rangle, |t_{m\nu}\rangle\}$. Here, $|s\rangle$ is the singlet state and $|t_{m\nu}\rangle$’s are 9 degenerate triplets given in Eq. (A5) and (A4), respectively. This is the minimal basis that can be used to discuss the low-energy dynamics of the trimerized Kagomé model, $\hat{H}$ of Eq. (1).

The operators $S_{jz}$ and $S_{j+}$ are the $z$-component and the raising operator, respectively, of the $j$th spin on a triangle, where $j = 1, 2, 3$ (see Fig. 1 for the spin labels). Let us denote the 10 basis states as $|b_i\rangle$, where the integer $l$ runs from 1 to 10. Now define, $|b_1\rangle = |s\rangle$, and $|b_{l(m, \nu)}\rangle = |t_{m\nu}\rangle$ for $l(m, \nu) = 3m + \nu + 6$, where $m, \nu = 1, 0, 1$. In this notation, $S_{jz} = \sum_{l, \nu} |b_i\rangle \hat{M}_{jz}^{l, \nu} |b_i\rangle$ and $S_{j+} = \sum_{l, \nu} |b_i\rangle \hat{M}_{j+}^{l, \nu} |b_i\rangle$. Here, the matrix elements are, $\hat{M}_{jz}^{l, \nu} = |b_i| S_{jz} |b_i\rangle$, and likewise for $\hat{M}_{j+}^{l, \nu}$. Now we define the bosonic operators, $b_i$, such that $|b_i\rangle := b_i \langle 0|$. These ‘plaquette-operators’ (corresponding to the eigenstates of a triangular plaquette) live in a Fock space with vacuum, $|0\rangle$, and satisfy the constraint, $\sum_i b_i^\dagger b_i = 1$. We now write the plaquette-operator representation of the spin-1 operators on a triangle as:

\[
S_{jz} = \sum_{l, \nu} \hat{M}_{jz}^{l, \nu} b^\dagger_i b_i \quad \text{and} \quad S_{j+} = \sum_{l, \nu} \hat{M}_{j+}^{l, \nu} b^\dagger_i b_i, \tag{B1}
\]

where the matrices $\hat{M}_{jz}$ and $\hat{M}_{j+}$ are given below.

The general representation in Eq. (B1) is the basis of a more simplified plaquette-operator representation given in Eqs. (4) that we have used for doing triplon analysis in the main text. There, we have approximated $\hat{s}$ by a mean-field, $\bar{s}$, and completely neglected the triplet-only terms in Eq. (B1). This latter approximation amounts to ignoring triplon-triplon interactions, akin to ignoring the interaction between the spin-waves in the simplest spin-wave analysis.

\[
\hat{M}_i^- = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
\hat{M}_i^+ = \begin{pmatrix}
0 & -i\sqrt{\frac{3}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & i\sqrt{\frac{3}{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(B2)
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