A higher-dimensional Contou-Carrère symbol: local theory

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Abstract. We construct a higher-dimensional Contou-Carrère symbol and we study some of its fundamental properties. The higher-dimensional Contou-Carrère symbol is defined by means of the boundary map for $K$-groups. We prove its universal property. We provide an explicit formula for the higher-dimensional Contou-Carrère symbol over $\mathbb{Q}$ and we prove the integrality of this formula. We also study its relation with the higher-dimensional Witt pairing.

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§1. Introduction

We start by recalling the one-dimensional situation, that is, the case of Laurent series in one variable. For any commutative associative unital ring $A$, Contou-Carrère [1], [2] introduced a remarkable bilinear antisymmetric pairing between the groups of invertible elements in the ring $A((t))$ of Laurent series over $A$

$$A((t))^* \times A((t))^* \longrightarrow A^*, \quad (1)$$

which is functorial with respect to $A$. The pairing given in (1) is now called the Contou-Carrère symbol. We will also call it the one-dimensional Contou-Carrère symbol and we will denote it by $CC_1$. Using an observation of Deligne’s, $CC_1$ can be defined with the help of the following analogy with the analytic setting.

Let $X$ be a Riemann surface, $p$ be a point on $X$, and let $f$ and $g$ be two meromorphic functions on $X$. By $(f, g)_p \in \mathbb{C}^*$ denote the tame symbol of $f$ and $g$ at $p$, which is given by the algebraic formula

$$(f, g)_p = (-1)^{\nu_p(f)\nu_p(g)} \left[ \frac{f^{\nu_p(g)}}{g^{\nu_p(f)}} \right](p),$$

where $\nu_p$ denotes the valuation at $p$. Beilinson [3] and Deligne [4] independently discovered an analytic expression for $(f, g)_p$ (see also the detailed exposition in [5], §1). Namely, let $U$ be a complement to a finite set of points in $X$ such that $f$ and $g$ are holomorphic and invertible on $U$. Then, by definition, the functions $f$
and \(g\) are elements of the first Deligne cohomology group \(H^1_D(U, \mathbb{Z}(1)) \simeq H^0(U, \mathcal{O}_U)\) and their cup-product \(f \cup g\) is an element of the second Deligne cohomology group \(H^2_D(U, \mathbb{Z}(2)) \simeq H^1(U, \mathcal{O}_U)\). The last group admits a geometric description as the group of equivalence classes of holomorphic line bundles with holomorphic connection. The point is that the inverse of the tame symbol \((f, g)_p\) is equal to the monodromy of \(f \cup g\) at \(p\), that is, the monodromy along a small circle \(\sigma_p\) described anticlockwise around \(p\). In particular, when \(\nu_p(f) = 0\), then the above interpretation immediately gives the following formula, which is obvious:

\[
(f, g)_p = \exp \left( \frac{1}{2\pi i} \oint_{\sigma_p} \log(f) \frac{dg}{g} \right) = \exp \text{res}_p \left( \log(f) \frac{dg}{g} \right),
\]

where \(\log(f)\) is a branch of the logarithm of \(f\) in a small neighborhood of \(p\) (more general formulae when \(\nu_p(f)\) is an arbitrary integer are given in [3] and in [4], (2.7.2)).

Now we come back to the Contou-Carrère symbol. If \(A\) is a field, then the Contou-Carrère symbol coincides with the tame symbol. For example, the equality \(\text{CC}^1(t, t) = -1\) holds. We shall describe \(\text{CC}^1\) for arbitrary Laurent series. For simplicity, suppose in what follows that \(A\) is not a product of two rings. Then a Laurent series \(f = \sum_{l \geq m} a_l t^l\), \(a_l \in A\), is invertible in \(A((t))\) if and only if there is an integer \(l_0 \geq m\) such that \(a_l \in A^*\) and for any \(l < l_0\) the coefficient \(a_l\) is nilpotent. Denote the integer \(l_0\) by \(\nu(f)\). For all \(a \in A^*\) and \(g \in A((t))^*\) we have \(\text{CC}^1(a, g) = a^{\nu(g)}\). In view of the bilinearity and the antisymmetry property, it remains to define \(\text{CC}^1(f, g)\) when \(\nu(f) = 0\) and the constant term \(a_0\) of \(f\) equals 1, which is the most essential case for the Contou-Carrère symbol. Using an analogy with the analytic formula (2), Deligne [4], §2.9 suggested the following explicit formula for the Contou-Carrère symbol \(\text{CC}^1\) over a \(\mathbb{Q}\)-algebra \(A\):

\[
\text{CC}^1(f, g) = \exp \text{res}_p \left( \log(f) \frac{dg}{g} \right),
\]

where \(f, g \in A((t))^*\), \(\nu(f) = 0\) and \(a_0 = 1\). Here \(\log\) and \(\exp\) are defined by the standard formal series. The log convergence is in an \(A\)-linear topology on \(A((t))\) with the base of open neighborhoods of zero given by \(A\)-submodules \(t^m A[[t]], m \in \mathbb{Z}\) (the ring \(A\) is discrete). It can be shown that the expression under the \(\exp\) is a nilpotent element in \(A\), and so the right-hand side of (3) is a well-defined element in \(A^*\). This defines the Contou-Carrère symbol over \(\mathbb{Q}\)-algebras.

It turns out that formula (3) can be extended by elementary methods to a pairing over all rings, not just \(\mathbb{Q}\)-algebras (see, for instance, [6], §2). In brief, the reason is as follows. Any element in \(A((t))^*\) decomposes uniquely into a product made up of a power of \(t\), an element in \(A^*\) and an infinite product of elements of the form \(1 - u_i t^i\), where \(i \neq 0\), \(u_i \in A\) and \(u_i\) is nilpotent for \(i < 0\) and equals zero when \(i\) is sufficiently large and negative. When \(A\) is a \(\mathbb{Q}\)-algebra, it is easy to deduce from (3) that if \(i > 0\) and \(j < 0\), then \(\text{CC}^1(1 - u_i t^i, 1 - v_j t^j) = (1 - u_i^{j/(i,j)} v_j^{i/(i,j)})^{(i,j)}\), where \((i, j) > 0\) denotes the greatest common divisor of \(i\) and \(j\). This expression does not have nontrivial denominators and makes sense over any ring. In addition, \(\text{CC}^1(1 - u_i t^i, 1 - v_j t^j) = 1\) when \(i\) and \(j\) have the same sign. Thus by bilinearity and the antisymmetry property, we obtain the Contou-Carrère symbol over all rings.
The corresponding explicit formula was given in the Introduction of [7], in the case of Artinian rings. Note that the above discussion also implies that expanding the brackets formally on the right-hand side of (3) one actually obtains a power series with integral coefficients whose variables are coefficients of $f$ and $g$.

Moreover, the extension of $CC_1$ from $\mathbb{Q}$-algebras to all rings is unique. This is a consequence of an easy argument using flatness. Namely, it is because the functor $L\mathbb{G}_m(A) := A((t))^*$ on the category of all rings is represented by an ind-affine scheme, which is ind-flat. This means that $L\mathbb{G}_m$ is a (formal) direct limit of flat affine schemes over $\mathbb{Z}$ (more details are given in [6], §2).

Now we pass to double iterated Laurent series, which are elements of the ring $A((t_1))((t_2))$. It is well known to experts in the field that, usually, the complexity grows drastically with such a generalization because many basic one-dimensional facts fail in the two-dimensional case. Nevertheless, using an analogue of (3), in [6], Definition 3.4, Osipov and Zhu defined a multilinear antisymmetric functorial map

$$CC_2: A((t_1))((t_2))^* \times A((t_1))((t_2))^* \times A((t_1))((t_2))^* \longrightarrow A^*,$$

(4)

where $A$ is a $\mathbb{Q}$-algebra. The key ingredient is the formula

$$CC_2(f, g, h) = \exp \left( \log(f) \frac{dg}{g} \wedge \frac{dh}{h} \right),$$

(5)

where $f, g, h \in A((t_1))((t_2))^*$ and there are some conditions on $f$ so that the series $\log(f)$ converges in the topology of $A((t_1))((t_2))$. Note that this topology is quite different from the topology on $A((t))$. The map $res$ is the two-dimensional residue and is equal to the coefficient by $t_1^{-1}t_2^{-1} dt_1 \wedge dt_2$.

How can we extend (5) to a map over all rings, not just $\mathbb{Q}$-algebras? As in the one-dimensional case, we can show that the value of $CC_2$ on elements of type $1 - u_{i,j}t_1^{i}t_2^{j}$ defined by (5) has no nontrivial denominators and thus makes sense over all rings. However, as was explained in [6], §3.4, this method works for Noetherian rings, for example, but it is unclear how it might work for all rings. The problem arises with the decomposition of an arbitrary element from $A((t_1))((t_2))^*$ into an infinite product of elements of type $1 - u_{i,j}t_1^{i}t_2^{j}$. Because of this, less elementary methods were applied in [6] in order to extend the map $CC_2$ to all rings. To be precise, two alternative approaches were used there: one based on categorical central extensions and higher commutators, and the other based on the boundary map for algebraic $K$-groups

$$\partial_{m+1}: K_{m+1}(A((t))) \longrightarrow K_{m}(A), \quad m \geq 0,$$

where $A$ is any commutative ring. Note that it is important here that the ring $A$ can be nonregular, in particular, it can have nilpotent elements. The construction of the maps $\partial_{m+1}$ is based on ‘the Localization Theorem for projective modules’ proved by Gersten [8] and in a more general case due to Grayson [9]. Kato [10], §2.1 applied this theorem to a particular ring $A((t))$ to obtain the maps $\partial_{m+1}$. In [6], §7 the construction of the maps $\partial_{m+1}$ was extended to arbitrary commutative rings (including non-Noetherian). The maps $\partial_{m+1}$ are functorial with respect to the ring $A$ (see Proposition 7.14 in this paper).
Now, for an arbitrary ring $A$, the map $CC_2$ is defined as the composition of the product of algebraic $K$-groups, the map $\partial_{m+1}$ and the natural map $\det : K_1(A) \to A^*$:

$$A((t_1))((t_2))^* \times A((t_1))((t_2))^* \times A((t_1))((t_2))^* \to K_3(A((t_1))((t_2))) \xrightarrow{\partial_3} K_2(A((t_1))) \xrightarrow{\partial_2} K_1(A) \xrightarrow{\det} A^*.$$  \hfill (6)

It was proved in [6], Theorem 7.2 that the map defined by the explicit formula (5) coincides with the one defined by formula (6) over $\mathbb{Q}$-algebras. To do this the functoriality of the map given by (6) was used together with a careful analysis of geometric properties of the ind-affine scheme that represents the functor $L^2\mathbb{G}_m(A) := A((t_1))((t_2))^*$.

Finally, in [6], Lemma 3.10, it was proved that the extension of $CC_2$ from $\mathbb{Q}$-algebras to all rings is unique. The method used the fact that the ind-scheme $L^2\mathbb{G}_m$ is in the class $\mathcal{E,F}$ (essentially flat; see [6], Definition 3.1 and Lemma 3.5), which is a weaker property than being ind-flat. Note that by contrast with the ind-affine scheme $\mathbb{G}_m$, the ind-affine scheme $L^2\mathbb{G}_m$ is not known to be ind-flat (in particular, schemes in the standard representation of $L^2\mathbb{G}_m$ are not flat; see the discussion after Lemma 3.4 in [6]).

We now come to the higher-dimensional case, that is, to iterated Laurent series in $n$ variables for arbitrary $n \geq 1$. Define the functor $L^n\mathbb{G}_m(A) := A((t_1)) \ldots ((t_n))^*$ and, as usual, put $\mathbb{G}_m(A) := A^*$. From the above point of view it is now natural to define the $n$-dimensional Contou-Carrère symbol $CC_n$ as a multilinear antisymmetric morphism of functors $(L^n\mathbb{G}_m)^{(n+1)} \to \mathbb{G}_m$ given by the formula

$$CC_n : L^n\mathbb{G}_m(A)^{(n+1)} \to K_{n+1}(A((t_1)) \ldots ((t_n))) \xrightarrow{\partial_2 \ldots \partial_{n+1}} K_1(A) \xrightarrow{\det} A^*,$$  \hfill (7)

where the first arrow is the product between algebraic $K$-groups. It is also natural to hope that when $A$ is a $\mathbb{Q}$-algebra, $CC_n$ is given by the following explicit formula, which generalizes (3) and (5):

$$CC_n(f_1, f_2, \ldots, f_{n+1}) = \exp \left( \log(f_1) \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_{n+1}}{f_{n+1}} \right),$$  \hfill (8)

where $f_1, \ldots, f_{n+1} \in L^n\mathbb{G}_m(A)$, $f_1$ satisfies additional conditions so that $\log(f_1)$ is well-defined, and the map $\exp$ is the $n$-dimensional residue. Note that an analytic analogue of (8) was studied by Brylinski and McLaughlin in [11], where it was shown to be related to the product in Deligne cohomology of an $n$-dimensional complex manifold. Establishing equality between formulae (7) and (8) for $\mathbb{Q}$-algebras was one of the main motivations for this paper.

We shall explain the main results of the paper. To do this we first note that since the product of algebraic $K$-groups satisfies the Steinberg relations, the map $CC_n$ defined by (7) factors through the Milnor $K$-group $K_{n+1}^M(\mathbb{G}_m)$. Here, given a commutative ring $B$, its Milnor $K$-group $K_{m}^M(B)$, $m \geq 0$, is defined as the quotient of the group $(\mathbb{B}^*)^m$ by the subgroup generated by the Steinberg relations, that is, by elements $b_1 \otimes \cdots \otimes b_m$ such that $b_i + b_{i+1} = 1$ for some $i$, $1 \leq i \leq m - 1$. Define the functor $L^nK_{n+1}^M(A) := K_{n+1}^M(A((t_1)) \ldots ((t_n)))$.

Our key result is the isomorphism of groups

$$\text{Hom}^{\text{gr}}(L^nK_{n+1}^M, \mathbb{G}_m) \simeq \mathbb{Z},$$
where \( \text{Hom}^{gr} \) denotes the group of all morphisms between the corresponding functors that respect the group structures (see Theorem 8.7). In addition, the morphism \( CC_n \) is the generator of this infinite cyclic group (see Theorem 8.10).

To be explicit, this means the following. Let \( \Phi: (L^n \mathbb{G}_m)^{(n+1)} \to \mathbb{G}_m \) be any multilinear morphism of functors that satisfies the Steinberg relations. Then \( \Phi = (CC_n)^i \) for an integer \( i \in \mathbb{Z} \). As far as we know, this universal property of the \( n \)-dimensional Contou-Carrère symbol is also new in the classical one-dimensional case.

Note that we prove Theorems 8.7 and 8.10 in a more general setting, namely, for restriction of the functors involved to the category of \( R \)-algebras, where \( R \) is an arbitrary ring such that the natural homomorphism \( R \to R \otimes \mathbb{Q} \) is injective.

Moreover, in the course of the proof of Theorem 8.7 we obtain an explicit formula for the map \( CC_n \) over \( \mathbb{Q} \)-algebras, thus proving there is equality between formulae (7) and (8) (see Theorem 8.17). This gives a new meaning to formula (8) as the result of direct calculations of certain canonical maps. (Note that we announced the equality between (7) and (8) in the short note [12], but it did not contain proofs.)

We also show that for any natural number \( N \), the extension of any morphism of functors \( (L^n \mathbb{G}_m)^{\times N} \to (\mathbb{G}_m)^{\times N} \) from \( \mathbb{Q} \)-algebras to all rings is unique (see Theorem 6.12).

We prove that formally expanding the brackets on the right-hand side of (8) yields a power series with integral coefficients whose variables are coefficients of \( f_1, \ldots, f_{n+1} \) (see Theorem 8.34). This gives an explicit expression over any ring for the map defined by (7). Note that this integrality result generalizes the case \( n = 1 \) and is even new in the case \( n = 2 \).

To illustrate our results, we show how the Contou-Carrère symbol \( CC_n \) and its explicit formula (8) lead to a new definition of the \( n \)-dimensional generalization of the Witt pairing (see Proposition 9.3). Note that the \( n \)-dimensional Witt pairing is crucial for Parshin’s explicit construction of the higher local class field theory, see [13].

Our method is based on a study of the geometric properties of ind-affine schemes that represent the functor \( L^n \mathbb{G}_m \) and certain special subgroups of it. To this end we develop a theory of so-called thick ind-cones. A thick ind-cone \( X \) over a ring \( R \) is an ind-closed subscheme in a (possibly, infinite-dimensional) affine space over \( R \) with some additional properties (see Definitions 5.7 and 5.10). Roughly speaking, \( X \) is invariant under homotheties and contains sufficiently many points with nilpotent coordinates. The main feature of a thick ind-cone \( X \) is that regular functions on \( X \) are uniquely expanded as power series in coordinates of the ambient affine space. Thick ind-cones possess many additional nice properties, in particular, the class of thick ind-cones in closed under products and extensions of scalars, which fits our needs perfectly. We could say that the class of thick ind-cones is a suitable replacement of the class of ind-flat ind-affine schemes. We expect a vast range of further applications of this technique to various questions on iterated Laurent series over rings. The notion of a thick ind-cone arose as a modification of the notion of an ind-scheme from the class \( \mathcal{EF} \) used in [6] for two-dimensional iterated Laurent series.

The paper is organized as follows. Section 2 contains notation and terminology, mainly concerning functors, which is used throughout the paper. In §3 we collect...
and prove some general facts about the ring of iterated Laurent series $A((t_1)) \ldots (t_n)$, the topology on it, and its differential forms.

In §4, we define iterated loop groups (see Definition 4.1) and introduce our main object of study, the iterated loop group $L^n G_m$. We construct a decomposition of $L^n G_m$ that generalizes a result of Contou-Carré’s in the case $n = 1$ (see Proposition 4.3). We also introduce special group subfunctors $(L^n G_m)^0$ and $(L^n G_m)^\sharp$ of $L^n G_m$ (see Definition 4.6). Namely, $(L^n G_m)^0$ is the group of iterated Laurent series with trivial valuation, and $(L^n G_m)^\sharp$ is a subgroup of $(L^n G_m)^0$ such that the logarithm is well-defined over $\mathbb{Q}$ on it (see Proposition 4.9).

We develop the theory of thick ind-cones in §5. First we give a short account of ind-affine schemes in §5.1. Then we introduce a ring of power series in infinitely many variables (see Definition 5.1), define algebraic convergence of power series (see Definition 5.2) and consider certain ind-closed subschemes in ind-affine spaces that consist of points with nilpotent coordinates (see Definition 5.5). These ind-closed subschemes are used to define thick ind-cones (see Definitions 5.7 and 5.10). The main properties of thick ind-cones are contained in Propositions 5.11 and 5.17 and Lemmas 5.13 and 5.14. Finally, we discuss the connectedness of ind-schemes over a base ring in §5.5 and the density of an ind-closed subscheme of an ind-affine scheme in §5.6.

In §6 we apply the theory of thick ind-cones to iterated loop groups. We start by introducing a noncommutative product of (strict) ind-sets (see Definition 6.1), which is very useful for working with the representability of iterated loop functors (see Proposition 6.6). Then we prove that the ind-affine schemes that represent the functor $L^n G_m$ and its special subgroups are products of thick-ind cones and ind-flat ind-affine schemes (see Proposition 6.8). This leads to these ind-affine schemes having many nice properties (see Theorems 6.10 and 6.12). We also show that $(L^n G_m)^0$ is connected and that $(L^n G_m)^\sharp$ is dense in $(L^n G_m)^0$ (see Proposition 6.13). We study the characters of $L^n G_m$ over $\mathbb{Q}$ and prove that they commute with their differentials through the exponential map (see Propositions 6.21 and 6.23). Finally, we show that any functorial linear map $\Omega^n A_{((t_1))} \ldots (t_n)) \rightarrow A$ factors through the group of ‘continuous’ differential forms $A((t_1)) \ldots (t_n))dt_1 \wedge \cdots \wedge dt_n$ (see Proposition 6.25).

Section 7 collects auxiliary facts on Milnor $K$-groups and algebraic $K$-groups. Namely, we recall the main result from [14] that establishes an isomorphism between the tangent space $TK_{m+1}^M(A)$ to the Milnor $K$-group and the group of absolute Kähler differential forms $\Omega^m_A$ when $A$ is a ring that contains $1/2$ and has sufficiently many invertible elements (see Theorem 7.6); this is a generalization of a well-known result of Bloch’s [15]. In §7.3 we recall the construction of a boundary map for algebraic $K$-groups, $\partial: K_{m+1}(A((t))) \rightarrow K_m(A)$, and show its functoriality (see Proposition 7.14).

Section 8 contains the main results of the paper, which we described above, and their proofs (see Theorems 8.7, 8.10, 8.17 and 8.34). Before stating the main results, in §8.1 we look at a simpler case of an additive symbol (see Definition 8.1), which allows us to see the main patterns of the arguments related to the Contou-Carrère symbol. The proof of the key result, Theorem 8.7, is contained in §8.3. Briefly, first we pass from a base ring $R$ to the ring $S := R \otimes_{\mathbb{Z}} \mathbb{Q}$ using the theory of thick
ind-cones. Since $S$ is a $\mathbb{Q}$-algebra, we reduce a character of $(L^nK_{n+1}^M)_S$ to its differential. Finally, we apply the description of the tangent space to Milnor $K$-groups in order to obtain both the key result and the explicit formula (8). In order to show the result on the integrality of the explicit formula (8), we introduce a completed version of the Contou-Carrère symbol (see Definition 8.26), which is an interesting object of further study in its own right.

Finally, after a short account of the explicit higher local class field theory in §9.1, we relate the Contou-Carrère symbol $CC_n$ to the $n$-dimensional Witt pairing in §9.2.

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§2. Notation and terminology

By a ring we mean a commutative associative unital ring and the same for algebras over a ring. Throughout the paper, $A$ and $R$ denote arbitrary rings unless something more is specified on the rings $A$ and $R$.

We shall fix some terminology concerning functors. We work with covariant functors from the category of commutative associative unital rings to the category of sets and to the category of Abelian groups. For brevity, we call them functors and group functors, respectively. We usually denote test rings on which we evaluate functors by $A, B, \ldots$.

By a subfunctor $F \subset G$, we mean a morphism of functors $F \to G$ such that for any ring $A$, the corresponding map $F(A) \to G(A)$ is injective. By a group subfunctor $F \subset G$, we mean a subfunctor such that the corresponding morphism $F \to G$ is, in addition, a morphism of group functors.

Given a ring $R$, by a functor over $R$ (a group functor over $R$), we mean a covariant functor from the category of commutative associative unital $R$-algebras to the category of sets (to the category of Abelian groups, respectively). Given a functor $F$, by $F_R$ denote the corresponding functor over $R$ which is the restriction of $F$ to the category of $R$-algebras. We usually denote base rings over which we consider functors by $R, S, \ldots$.

All the functors we consider in this paper satisfy the following property: morphisms from a functor $F$ over a ring $R$ to a functor $G$ over $R$ form a set $\text{Hom}_R(F, G)$. For example, this holds if $F$ and $G$ are representable by (ind-)affine schemes over $R$. If $F$ and $G$ are group functors, then we denote the set of all morphisms of group functors from $F$ to $G$ by $\text{Hom}_R^{gr}(F, G)$. We denote the internal Hom-functor by $\text{Hom}_R^{gr}(F, G)$, that is, $\text{Hom}_R^{gr}(F, G)$ is a group functor over $R$ that sends an $R$-algebra $A$ to the group $\text{Hom}_A^{gr}(F_A, G_A)$.

We also introduce same notation for various products. Let $n$ be a natural number. If $X$ is an object in a category with finite products (for example, $X$ is a set, a scheme or a functor), then we let $X^{\times n}$ denote the $n$th Cartesian power of $X$ (which is a set, a scheme or a functor, respectively). We emphasize that even if $X$ is also a group object (for example, $X$ is a group, a group scheme or a group functor), then $X^{\times n}$ still denotes just an object in the initial category, without a group structure (it is a set, a scheme, or a functor, respectively).
By contrast, if $M$ is an object in an additive category (for example, $M$ is an Abelian group, a module over a ring or a sheaf of Abelian groups), then $M^{\oplus n}$ denotes the $n$-fold direct sum of $M$ with itself (which is an Abelian group, a module over the ring or a sheaf of Abelian groups, respectively). We sometimes abbreviate $M^{\oplus n}$ to $M^n$, in particular, when $M = \mathbb{Z}$.

For Abelian groups $M$ and $N$, by a multilinear map $\varphi: M^n \to N$ we mean a map of sets (according to our notation, $M^n$ is just a set) such that

$$\varphi(m_1, \ldots, m_i, \ldots, m_n) + \varphi(m_1, \ldots, m'_i, \ldots, m_n) = \varphi(m_1, \ldots, m_i + m'_i, \ldots, m_n)$$

for all $m_1, \ldots, m_i, m'_i, \ldots, m_n \in M$, $1 \leq i \leq n$. Note that we will still use the term ‘multilinear’ even if the group law in $M$ or $N$ is denoted multiplicatively; this holds, in particular, for the group of invertible elements of a ring with the group law given by the product of elements.

§ 3. The ring of iterated Laurent series

3.1. Definition. Given a ring $A$, we have the ring $A[[t]]$ of power series and the ring $A((t)) = A[[t]][t^{-1}]$ of Laurent series over $A$ in the formal variable $t$. Explicitly, $A[[t]]$ is the ring of all series of the form $\sum_{0 \leq l \in \mathbb{Z}} a_l t^l$, where $a_l \in A$, and $A((t))$ is the ring of all series of the form $\sum_{m \leq l \in \mathbb{Z}} a_l t^l$, where $m$ can be any integer and $a_l \in A$. For brevity, we put

$$\mathcal{L}(A) := A((t)).$$

Repeating this construction, we obtain the ring of iterated Laurent series over $A$ in the formal variables $t_1, \ldots, t_n$:

$$\mathcal{L}^n(A) := A((t_1)) \ldots ((t_n)).$$

We will give an explicit description of iterated Laurent series. To do this, we introduce the set

$$\Lambda_n := \{ (\lambda_1, \ldots, \lambda_n) \mid \lambda_p : \mathbb{Z}^{n-p} \to \mathbb{Z}, \text{ where } 1 \leq p \leq n-1, \lambda_n \in \mathbb{Z} \}.$$

We stress that $\lambda_p$ is an arbitrary function for each $p$, where $1 \leq p \leq n-1$. Given $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda_n$, we define a set

$$\mathbb{Z}^n_\lambda := \{ (l_1, \ldots, l_n) \in \mathbb{Z}^n \mid l_n \geq \lambda_n, l_{n-1} \geq \lambda_{n-1}(l_n), \ldots, l_1 \geq \lambda_1(l_2, \ldots, l_n) \}.$$

For a multi-index $l = (l_1, \ldots, l_n) \in \mathbb{Z}^n$, we put $t^l := t_1^{l_1} \ldots t_n^{l_n}$. Then $\mathcal{L}^n(A)$ is the ring of all series of the form $\sum_{l \in \mathbb{Z}^n_\lambda} a_l t^l$, where $\lambda \in \Lambda_n$ and $a_l \in A$.

We denote the nilradical of the ring $R$ by $\text{Nil}(R)$, that is, the set of all nilpotent elements of $R$. We will use the following fact about nilpotent iterated Laurent series.

Remark 3.1. Suppose that

$$f = \sum_{l \in \mathbb{Z}^n} a_l t^l = \sum_{i \in \mathbb{Z}} g_i t^i$$

is a nilpotent element in $\mathcal{L}^n(A)$, where $g_i \in \mathcal{L}^{n-1}(A) = A((t_1)) \ldots ((t_{n-1}))$. We can show by induction on $i$ that $g_i \in \text{Nil}(\mathcal{L}^{n-1}(A))$ for all $i \in \mathbb{Z}$. Further, by induction on $n$, we obtain that $a_l \in \text{Nil}(A)$ for all $l \in \mathbb{Z}^n$. 


If $A$ is Noetherian, then the converse is true as well, because $\text{Nil}(A)^N = 0$ for some $N \in \mathbb{N}$. However, in general, the converse is false. For example, take

$$A = \mathbb{Z}[\varepsilon_1, \varepsilon_2, \ldots]/(\varepsilon_1^2, \varepsilon_2^3, \ldots)$$

and consider the power series $f = \sum_{t \geq 1} \varepsilon_t t^l$ in $A[[t]]$. Then all the coefficients of $f$ are nilpotent, but $f$ is not nilpotent (one can show this by taking reductions modulo the primes).

### 3.2. Topology.

We now introduce a topology on the ring of iterated Laurent series $\mathcal{L}^n(A)$ over a ring $A$. This topology is given by iterated direct and inverse limits and is defined inductively as follows. A base of open neighbourhoods of zero in $\mathcal{L}(A) = A((t))$ consists of $A$-submodules $U_m := t^mA[[t]]$, $m \in \mathbb{Z}$. A base of open neighbourhoods of zero in

$$\mathcal{L}^n(A) = A((t_1)) \ldots ((t_n))$$

consists of $A$-submodules

$$U_{m_i}, \{V_j\} := \left( \bigoplus_{j < m} t_j \cdot V_j \right) \oplus t_m \cdot \mathcal{L}^{n-1}(A)[[t_n]], \quad (9)$$

where $m \in \mathbb{Z}$, and for every integer $j$ with $j < m$ the $A$-module $V_j$ is from the base of open neighbourhoods of zero in $\mathcal{L}^{n-1}(A) = A((t_1)) \ldots ((t_{n-1}))$. Now a topology on $\mathcal{L}^n(A)$ is defined uniquely by the condition that the additive group of $\mathcal{L}^n(A)$ be a topological group.

**Remark 3.2.** When $A = \mathbb{F}_q$ is a finite field, the above topology on the higher local field $\mathcal{L}^n(A) = \mathbb{F}_q((t_1)) \ldots ((t_n))$ was introduced by Parshin [13] for constructions in $n$-dimensional local class field theory.

Recall that a **Cauchy sequence** in a topological (Abelian) group is a sequence of elements $\{f_i\}$, $i \in \mathbb{N}$, such that for any open neighbourhood of zero $U$ there is $N \in \mathbb{N}$ that satisfies $f_i - f_j \in U$ for all $i, j \geq N$. Clearly, this property does not depend on the order of elements in the sequence and thus is well-defined for an arbitrary (nonordered) countable set of elements.

**Lemma 3.3.**

1. The topological space $\mathcal{L}^n(A)$ is Hausdorff.
2. For any Cauchy sequence $\{f_i\}$, $i \in \mathbb{N}$, in $\mathcal{L}^n(A)$, there exists $m \in \mathbb{Z}$ such that

$$f_i \in t_m \cdot \mathcal{L}^{n-1}(A)[[t_n]] = t_m \cdot A((t_1)) \ldots ((t_{n-1}))[[t_n]]$$

for all $i \in \mathbb{N}$.
3. Every Cauchy sequence has a limit in $\mathcal{L}^n(A)$.
4. The natural isomorphism of rings

$$\mathcal{L}^{n-1}(A)[[t_n]] \simeq \lim_{\substack{\longrightarrow \atop m \in \mathbb{N}}} \mathcal{L}^{n-1}(A[[t_n]]/(t_m^n))$$

is also an isomorphism of topological groups, where the topology on the left-hand side is restricted from $\mathcal{L}^n(A)$ and the topology on the right-hand side is the inverse limit of the topologies on $\mathcal{L}^{n-1}(B_m)$, where the ring $B_m$ is $A[[t_n]]/(t_m^n)$. 
Proof. (o) This is obvious.

The proofs of items (i) and (ii) are similar to the proofs in [13], §1, Proposition 2.1 and [13], §1, Proposition 2.2, respectively. We give them for the convenience of the reader.

(i) Suppose the converse. Then we obtain a strictly decreasing sequence of negative integers \( \{j_k\}, k \in \mathbb{N} \), such that for each \( k \in \mathbb{N} \), there is a number \( i_k \in \mathbb{N} \) that satisfies

\[
f_{i_k} \in t_{j_k} \cdot \mathcal{L}^{n-1}(A)[[t_n]], \quad f_{i_k} \notin t_{j_k+1} \cdot \mathcal{L}^{n-1}(A)[[t_n]].
\]

Since \( \mathcal{L}^{n-1}(A) \) is Hausdorff, there is an open neighbourhood \( W_k \subset \mathcal{L}^{n-1}(A) \) of zero such that

\[
f_{i_k} \notin t_{j_k} \cdot W_k \oplus t_{j_k+1} \cdot \mathcal{L}^{n-1}(A)[[t_n]].
\]

Now for each \( j < 0 \), let \( V_j := W_k \) if \( j = j_k \) for some \( k \in \mathbb{N} \) and let \( V_j \) be any open neighbourhood of zero in \( \mathcal{L}^{n-1}(A) \) otherwise. Then the subsequence \( \{f_{i_k}\} \) (and hence the sequence \( \{f_i\} \)) does not satisfy the Cauchy condition with respect to the open neighbourhood of zero \( U_0, \{V_j\} \) in \( \mathcal{L}^{n}(A) \), which gives a contradiction.

(ii) The proof is based on item (i) and a trivial induction on \( n \).

(iii) Note that for any \( A \)-module \( M \) we can consider an \( \mathcal{L}^{n}(A) \)-module \( \mathcal{L}^{n}(M) := M((t_1)) \ldots ((t_n)) \) with an analogous topology to that on \( \mathcal{L}^{n}(A) \) given by a formula similar to (9). (We note that \( M \) has the discrete topology.) Induction on \( n \) shows that for all \( A \)-modules \( M \) and \( M' \) the natural isomorphism of \( \mathcal{L}^{n}(A) \)-modules

\[
\mathcal{L}^{n}(M) \oplus \mathcal{L}^{n}(M') \simeq \mathcal{L}^{n}(M \oplus M')
\]

is also an isomorphism of topological groups. In particular, for any \( m \in \mathbb{N} \) we have an isomorphism of topological groups

\[
\mathcal{L}^{n-1}(A)^{\oplus m} \simeq \mathcal{L}^{n-1}(A^{\oplus m}).
\]

Note that for any \( A \)-algebra \( B \) using induction on \( n \) it is easy to show that the base of open neighbourhoods of zero in \( \mathcal{L}^{n-1}(B) \) given by \( B \)-modules using (9) coincides with the base of open neighbourhoods of zero in \( \mathcal{L}^{n-1}(M) \) given by \( A \)-modules. Therefore applying an isomorphism of \( A \)-modules \( A^{\oplus m} \simeq A[[t_n]]/(t_n^m) \) and using the isomorphism of topological groups

\[
\mathcal{L}^{n-1}(A)[[t_n]] \simeq \lim_{m \in \mathbb{N}} \mathcal{L}^{n-1}(A)^{\oplus m}
\]

we complete the proof.

Remark 3.4. Actually, a stronger statement than Lemma 3.3 (ii) holds: the topology on \( \mathcal{L}^{n}(A) \) is complete, that is, every Cauchy net has a limit in \( \mathcal{L}^{n}(A) \), not only any Cauchy sequence. Equivalently, the natural homomorphism

\[
\mathcal{L}^{n}(A) \rightarrow \lim_{(m, \{V_j\})} \mathcal{L}^{n}(A)/U_{m, \{V_j\}}
\]

is an isomorphism. The proof is by induction on \( n \) and uses the fact that inverse limits commute with direct sums in the following sense: suppose that for each
element λ in a set Λ, a directed set \( I_λ \) and an inverse system of Abelian groups \( \{ P_{i}^{λ} \} \), \( i \in I_λ \), are given. Define a directed set \( J := \prod_{λ \in Λ} I_λ \). Then there is an isomorphism of Abelian groups

\[
\bigoplus_{λ \in Λ} \left( \lim_{i \in I_λ} P_{i}^{λ} \right) \simeq \lim_{(i_λ) \in J} \left( \bigoplus_{λ \in Λ} P_{i_λ}^{λ} \right).
\]

The proof of this fact follows the same idea as the proof of Lemma 3.3(i).

Note that \( L^n(A) \) is not a topological ring when \( n \geq 2 \), since for all open \( A \)-submodules \( U \) and \( U' \) in \( L^n(A) \) we have \( U \cdot U' = L^n(A) \) ([13], §1, Remark 1). In fact, for any iterated Laurent series \( f \in L^n(A) \) there is a monomial \( t_i^j t_{n-1}^k \), \( i, j \in \mathbb{Z} \), such that \( t_i^j t_{n-1}^k f \in U \) and \( (t_i^j t_{n-1}^k)^{-1} \in U' \). Even so, the next lemma shows there is a certain compatibility between the topology and the product in \( L^n(A) \).

**Lemma 3.5.** (i) For any element \( f \in L^n(A) \) multiplication by \( f \) is a continuous map from \( L^n(A) \) to itself.

(ii) Given two Cauchy sequences \( \{f_i\}, i \in \mathbb{N}, \) and \( \{g_j\}, j \in \mathbb{N}, \) in \( L^n(A) \), their pairwise product \( \{f_i g_j\}, (i, j) \in \mathbb{N} \times \mathbb{N}, \) is also a Cauchy sequence (with respect to any bijection \( \mathbb{N} \times \mathbb{N} \simeq \mathbb{N} \)) and the limit of the sequence \( \{f_i g_j\} \) equals the product of the limits of \( \{f_i\} \) and \( \{g_j\} \).

**Proof.** We use an argument in the proof of Proposition 2.3 in [13], §1. We use induction on \( n \) to prove both parts. The case \( n = 1 \) is clear for both. To make the induction step, observe that part (i) is evident for \( f = t_i^j, m \in \mathbb{Z} \), that is, multiplication by \( t_i \) is a homeomorphism of the topological group \( L^n(A) \). Now using this fact, we consider each part separately.

(i) By part (i) for \( t_i^m, m \in \mathbb{Z} \), we can suppose that \( f \in L^{n-1}(A)[[t_i]] \). It follows from (9) that a subset of \( L^n(A) \) is open if and only if its intersection with each \( A \)-submodule \( t_i^l \cdot L^{n-1}(A)[[t_i]] \), \( l \in \mathbb{Z} \), is open. Therefore it is enough to prove that multiplication by \( f \) is a continuous map from \( t_i^l \cdot L^{n-1}(A)[[t_i]] \) to itself for all \( l \in \mathbb{Z} \). We can also fix \( l = 0 \) and complete the proof by using Lemma 3.3(iii) and the induction hypothesis.

(ii) By part (i) for \( t_i^m, m \in \mathbb{Z} \), and by Lemma 3.3(i), it is enough to prove a version of part (ii) by the subring

\[
A((t_1)) \ldots ((t_{n-1}))[[t_i]] = L^{n-1}(A)[[t_i]] \subset L^n(A)
\]

with the topology restricted from \( L^n(A) \). By the induction hypothesis and the isomorphism from Lemma 3.3(iii), this completes the proof.

### 3.3. Convergence of power series

By definition, a series \( \sum_{i \geq 0} f_i \) of elements of a topological Abelian group converges if there is a limit of the sequence of partial sums \( \sum_{i=1}^{N} f_i \). By Lemma 3.3(ii), a series converges in \( L^n(A) \) if and only if the sequence of its terms \( \{f_i\}, i \in \mathbb{N}, \) tends to zero, because the base of open neighbourhoods of zero is given by subgroups. If this holds, the result of summation does not depend on the order of summation, nor on the way we represent the series as a double series or a higher iterated series (cf. [17], Ch. 4, §5, Theorem 1). In particular, convergence is well-defined for an infinite sum of countably many elements in \( L^n(A) \).
Remark 3.6. Given two convergent series $\sum_{i \geq 0} f_i$ and $\sum_{j \geq 0} g_j$ in $L^n(A)$, according to Lemma 3.5 (ii), the series $\sum_{(i,j) \in \mathbb{N} \times \mathbb{N}} f_i g_j$ is also convergent (with respect to any bijection $\mathbb{N} \times \mathbb{N} \simeq \mathbb{N}$) and we have the equalities
\[
\left( \sum_{i \geq 0} f_i \right) \cdot \left( \sum_{j \geq 0} g_j \right) = \sum_{i \geq 0} \left( \sum_{j \geq 0} f_i g_j \right) = \sum_{j \geq 0} \left( \sum_{i \geq 0} f_i g_j \right).
\]

Definition 3.7. Define $L^n(A)^{\sharp}$ to be the set of all elements $f \in L^n(A)$ such that the sequence $\{f^i\}, i \in \mathbb{N}$, tends to zero in $L^n(A)$.

It follows from Lemma 3.5 (ii) that if the sequences $\{f^i\}$ and $\{g^i\}$ tend to zero, then $\{(f + g)^i\}$ tends to zero as well. In other words, $L^n(A)^{\sharp}$ is a subgroup of $L^n(A)$. Also, for any power series $\varphi \in A[[x_1, \ldots, x_r]]$ and elements $f_1, \ldots, f_r \in L^n(A)^{\sharp}$, the series $\varphi(f_1, \ldots, f_r)$ converges in $L^n(A)$. Here is a more explicit description of the group $L^n(A)^{\sharp}$ (cf. [18], Lemma 1.1).

Proposition 3.8. For any ring $A$, there is an equality of groups
\[
L^n(A)^{\sharp} = \left\{ \sum_{l \in \mathbb{Z}} a_l t^l \mid \sum_{l \geq 0} a_l t^l \in L^n(A), \sum_{l \leq 0} a_l t^l \in \text{Nil}(L^n(A)) \right\}. \tag{10}
\]

Proof. The proof uses induction on $n$. The base $n = 0$, that is, the case of $A$ with the discrete topology, is clear: $\{f^i\}$ tends to zero in $A$ if and only if $f$ is a nilpotent element in $A$. Now we prove the induction step. It is readily seen that both sides of (10) contain the group $t_n \cdot L^{n-1}(A)[[t_n]]$.

Take an element $f = \sum_{i \leq 0} g_i t_i^n \in L^n(A)^{\sharp}$, where $g_i \in L^{n-1}(A)$. Since $\{f^i\}$ is a Cauchy sequence, using Lemma 3.3 (i), we find that $g_i \in \text{Nil}(L^{n-1}(A))$ for all $i < 0$. Thus the series $\sum_{i < 0} g_i t_i^n$ is nilpotent.

Further, by the induction hypothesis, we have equality between the intersections of the subring $L^{n-1}(A) \subset L^n(A)$ with each side of (10). To prove this, we use Remark 3.1 and the fact that the restriction of the topology from $L^n(A)$ to $L^{n-1}(A)$ coincides with the topology on $L^{n-1}(A)$. This proves that the group $L^n(A)^{\sharp}$ is a subgroup of the group defined by the right-hand side of (10). To see the reverse inclusion we note that $\text{Nil}(L^n(A)) \subset L^n(A)^{\sharp}$.

The proof of the following statement uses Remark 3.6 and copies the proof of Theorem 2 in [17], Ch. 4, § 5 (see also [18], Lemma 1.2).

Lemma 3.9. Let $\varphi \in A[[x_1, \ldots, x_r]]$ and $\psi_i \in A[[y_i]]$, $1 \leq i \leq r$, be power series such that the constant terms of all the $\psi_i$ equal zero. Let $\phi \in A[[y_1, \ldots, y_r]]$ be the formal composition $\varphi(\psi_1, \ldots, \psi_r)$. Then for all elements $f_1, \ldots, f_r \in L^n(A)^{\sharp}$ the equality $\varphi(\psi_1(f_1), \ldots, \psi_r(f_r)) = \phi(f_1, \ldots, f_r)$ holds in $L^n(A)$.

3.4. Differential forms. Consider absolute Kähler differentials $\Omega^1_{L^n(A)}$ for the ring $L^n(A)$. Let $K \subset \Omega^1_{L^n(A)}$ denote the $L^n(A)$-submodule generated by all elements
\[
df - \sum_{i=1}^n \frac{\partial f}{\partial t_i} dt_i,
\]
where $f \in L^n(A)$. In particular, $K$ contains the elements $da$, where $a \in A$. 
Definition 3.10. Define the quotient
\[ \widetilde{\Omega}^1_{\mathcal{L}^n(A)} := \Omega^1_{\mathcal{L}^n(A)}/K. \]

It is easy to show that \( \widetilde{\Omega}^1_{\mathcal{L}^n(A)} \) is a free module over the ring \( \mathcal{L}^n(A) \) with the basis \( dt_1, \ldots, dt_n \). Further, put \( \widetilde{\Omega}^i_{\mathcal{L}^n(A)} := \bigwedge^i \mathcal{L}^n(A) \widetilde{\Omega}^1_{\mathcal{L}^n(A)}. \)

We can check directly that the de Rham differential
\[ d: \Omega^1_{\mathcal{L}^n(A)} \to \Omega^2_{\mathcal{L}^n(A)} \]
sends \( K \) to \( K \wedge \Omega^1_{\mathcal{L}^n(A)} \). Therefore we have a well-defined de Rham differential \( d: \widetilde{\Omega}^i_{\mathcal{L}^n(A)} \to \widetilde{\Omega}^{i+1}_{\mathcal{L}^n(A)} \) for any \( i \geq 0 \) (which we denote by the same letter \( d \)), and, obviously, the following diagram is commutative:

\[
\begin{array}{ccc}
\Omega^i_{\mathcal{L}^n(A)} & \xrightarrow{d} & \Omega^{i+1}_{\mathcal{L}^n(A)} \\
\downarrow & & \downarrow \\
\widetilde{\Omega}^i_{\mathcal{L}^n(A)} & \xrightarrow{d} & \widetilde{\Omega}^{i+1}_{\mathcal{L}^n(A)}
\end{array}
\]

where the vertical arrows are the natural maps.

The topology on \( \mathcal{L}^n(A) \) defined in § 3.2 naturally induces a topology on each free \( \mathcal{L}^n(A) \)-module \( \widetilde{\Omega}^i_{\mathcal{L}^n(A)}, i \geq 0 \). One can easily check that the de Rham differential is continuous with respect to this topology. Therefore for any element \( f \in \mathcal{L}^n(A)^\# \) and any power series \( \varphi \in A[[x]] \), the equality
\[
\frac{\partial \varphi}{\partial x}(f) df = d(\varphi(f)) \tag{11}
\]
holds in \( \widetilde{\Omega}^1_{\mathcal{L}^n(A)} \).

Let
\[
\log(1 + x) := \sum_{i \geq 1} (-1)^{i+1} \frac{x^i}{i}, \quad \exp(x) := \sum_{i \geq 0} \frac{x^i}{i!}
\]
be the usual power series from the ring \( \mathbb{Q}[[x]] \). Let
\[
\text{Li}_2(x) := \sum_{i \geq 1} \frac{x^i}{i^2} \in \mathbb{Q}[[x]]
\]
be the dilogarithm.

The next lemma is a direct consequence of (11).

Lemma 3.11. Let \( A \) be a \( \mathbb{Q} \)-algebra. The following equalities hold in \( \widetilde{\Omega}^1_{\mathcal{L}^n(A)} \).

(i) For any \( f \in \mathcal{L}^n(A)^\# \),
\[
\frac{df}{1 + f} = \frac{d(1 + f)}{1 + f} = d\log(1 + f).
\]
(ii) For any \( f \in \mathcal{L}^n(A) \uplus \cap \mathcal{L}^n(A) \),
\[
\log(1 - f) \frac{df}{f} = d \left( - \text{Li}_2(f) \right).
\]

(iii) For any \( f \in \mathcal{L}^n(A) \uplus \) and series \( \psi \in A[[x]] \), there exists \( \varphi \in A[[x]] \) such that
\[
\log(1 - f) \frac{df}{\psi(f)} = d(\varphi(f)).
\]

There is a homomorphism
\[
\text{res}: \tilde{\Omega}^n_{\mathcal{L}^n(A)} \rightarrow A,
\sum_{l \in \mathbb{Z}^n} a_l t^l \cdot dt_1 \wedge \cdots \wedge dt_n \mapsto a_{-1, \ldots, -1}, \tag{12}
\]
called the residue map. For simplicity, we will also use notation \( \text{res} \) for the map from \( \Omega^n_{\mathcal{L}^n(A)} \) to \( A \) which is the composition of the natural map \( \Omega^n_{\mathcal{L}^n(A)} \rightarrow \tilde{\Omega}^n_{\mathcal{L}^n(A)} \) with the map \( \text{res} \) defined by (12).

For any element \( \eta \in \tilde{\Omega}^{n-1}_{\mathcal{L}^n(A)} \), we have \( \text{res}(d\eta) = 0 \). Moreover we have the following.

**Lemma 3.12.** For any \( \mathbb{Q} \)-algebra \( A \), there is an isomorphism
\[
\text{res}: \tilde{\Omega}^n_{\mathcal{L}^n(A)} / d\tilde{\Omega}^{n-1}_{\mathcal{L}^n(A)} \sim \rightarrow A.
\]

**Proof.** The proof is based on the following observation: \( d \) is continuous and if \( l_i \neq -1 \) for some \( i, \ 1 \leq i \leq n \), then
\[
t^1 \ldots t^n dt_1 \wedge \cdots \wedge dt_n
= d \left( \frac{(-1)^{i-1}}{l_i + 1} t^{l_1} \ldots t_i^{l_i+1} \ldots t^n dt_1 \wedge \cdots \wedge dt_{i-1} \wedge dt_{i+1} \wedge \cdots \wedge dt_n \right).
\]

§ 4. The iterated loop group of \( \mathbb{G}_m \)

**4.1. Iterated loop functors.**

**Definition 4.1.** For a functor \( F \) (see § 2) the loop functor of \( F \) is the functor \( LF \) given by the formula \( LF(A) := F(A((t))) \) for a ring \( A \). We denote the \( n \)-iterated loop functor of \( F \) for a natural number \( n \) by
\[
L^n F := \underbrace{L \ldots L}_{n} F.
\]

If \( F \) is a group functor, then we also call \( LF \) and \( L^n F \) the loop group and the \( n \)-iterated loop group of \( F \), respectively.

Explicitly, we have
\[
L^n F(A) = F(A((t_1)) \ldots ((t_n)))
\]
for a ring \( A \). We have natural morphisms of functors
\[
F \rightarrow LF, \quad F \rightarrow L^n F, \tag{13}
\]
which correspond to the maps
\[
F(A) \longrightarrow F(A((t))), \quad F(A) \longrightarrow F(A((t_1)) \ldots ((t_n))),
\]
given by constant series. For the additive group scheme \( \mathbb{G}_a = \text{Spec}(\mathbb{Z}[x]) \), we see that \( L^n \mathbb{G}_a(A) \) is the additive group of the ring of iterated Laurent series \( \mathcal{L}^n(A) \).

We are interested in the \( n \)-iterated loop group \( L^n \mathbb{G}_m \) of the multiplicative group scheme \( \mathbb{G}_m = \text{Spec}(\mathbb{Z}[x, x^{-1}]) \). To be explicit, \( L^n \mathbb{G}_m(A) \) is the multiplicative group \( \mathcal{L}^n(A)^* \) of invertible elements in the ring \( \mathcal{L}^n(A) \).

**4.2. Decomposition.** In order to describe the \( n \)-iterated loop group \( L^n \mathbb{G}_m \), we consider several group subfunctors in it. First we have an embedding of group functors
\[
\mathbb{G}_m \hookrightarrow L^n \mathbb{G}_m,
\]
given by constant series.

**Definition 4.2.** Let \( \mathbb{Z} \) be the group functor defined as follows: given a ring \( A \), \( \mathbb{Z}(A) \) is the group of locally constant functions on the topological space \( \text{Spec}(A) \) with values in \( \mathbb{Z} \).

In other words, \( \mathbb{Z} \) is the constant sheaf with respect to the Zariski topology associated with the group \( \mathbb{Z} \). Since affine schemes are quasi-compact, given a ring \( A \), an element \( l \in \mathbb{Z}(A) \) determines a decomposition into a finite product of rings
\[
A \simeq \prod_{i=1}^{N} A_i,
\]
and a collection of integers \( l_i \), \( 1 \leq i \leq N \), such that for every \( i \), the restriction of \( l \) to \( \text{Spec}(A_i) \) is the constant function whose value equals \( l_i \).

Analogous results hold for the group functor \( \mathbb{Z}^n \). We have an embedding of group functors
\[
\mathbb{Z}^n \hookrightarrow L^n \mathbb{G}_m
\]
that, given a ring \( A \), sends \( l \in \mathbb{Z}^n(A) \) to \( t^l \), where
\[
t^l := (t_1^{l_{11}}, \ldots, t_1^{l_{1M}}, \ldots, t_n^{l_{n1}}, \ldots, t_n^{l_{nM}}) \in \mathcal{L}^n(A) \simeq \prod_{i=1}^{M} \mathcal{L}(A_i),
\]
and the decomposition \( A \simeq \prod_{i=1}^{M} A_i \) is such that for every \( i \), \( 1 \leq i \leq M \), the restriction of \( l \) to \( \text{Spec}(A_i) \) is the constant function whose value equals \( (l_{1i}, \ldots, l_{ni}) \in \mathbb{Z}^n \).

Further, consider the following *lexicographical order* on \( \mathbb{Z}^n \): we say that \( (l_1, \ldots, l_n) \preceq (l'_1, \ldots, l'_n) \) if and only if either \( l_n < l'_n \) or \( l_n = l'_n \) and \( (l_1, \ldots, l_{n-1}) \preceq (l'_1, \ldots, l'_{n-1}) \). Note that this order is invariant under translations on the group \( \mathbb{Z}^n \).

We abbreviate \((0, \ldots, 0)\) to 0. Define the group functors
\[
\mathbb{V}_{n,+}(A) := \left\{ 1 + \sum_{0 < l \in \mathbb{Z}^n} a_l t^l \left| \sum_{l > 0} a_l t^l \in \mathcal{L}^n(A) \right. \right\}, \quad (16)
\]
\[
\mathbb{V}_{n,-}(A) := \left\{ 1 + \sum_{0 > l \in \mathbb{Z}^n} a_l t^l \left| \sum_{l < 0} a_l t^l \in \text{Nil}(\mathcal{L}^n(A)) \right. \right\}. \quad (17)
\]
The group structure on $\mathbb{V}_{n,+}$ and $\mathbb{V}_{n,-}$ is given by the product of iterated Laurent series and we have embeddings of the group functors

$$\mathbb{V}_{n,+} \hookrightarrow L^n \mathbb{G}_m, \quad \mathbb{V}_{n,-} \hookrightarrow L^n \mathbb{G}_m.$$  

(18)

**Proposition 4.3.** The embeddings (14), (15) and (18) induce an isomorphism of group functors

$$\mathbb{Z}^n \times \mathbb{G}_m \times \mathbb{V}_{n,+} \times \mathbb{V}_{n,-} \simeq L^n \mathbb{G}_m.$$  

(19)

**Proof.** We use induction on $n$. The base $n = 1$ has been proved by Contou-Carrère (see [1], Lemma 1.3 and [2], Lemma 0.8). For the induction step, we consider loop functors of all the functors in formula (19) with $n$ replaced by $n - 1$. Then we use the isomorphism $\mathbb{Z} \simeq L \mathbb{Z}$ induced by the morphisms (13) and proved by Osipov and Zhu in [6], Lemma 3.2, and we use also the isomorphisms

$$\mathbb{V}_{1,+} \times L \mathbb{V}_{n-1,+} \simeq \mathbb{V}_{n,+}, \quad \mathbb{V}_{1,-} \times L \mathbb{V}_{n-1,-} \simeq \mathbb{V}_{n,-}.$$  

(20)

Here, given a ring $A$, elements in $\mathbb{V}_{1,+}(A)$ and $\mathbb{V}_{1,-}(A)$ are Laurent series in $t_1$ with coefficients in $A$ that satisfy conditions from (16) and (17), respectively, with $n = 1$. Similarly, elements in $L \mathbb{V}_{n-1,+}(A)$ and $L \mathbb{V}_{n-1,-}(A)$ are iterated Laurent series in $t_2, \ldots, t_n$ with coefficients in $A((t_1))$ that satisfy the conditions in (16) and (17) with $n$ replaced by $n - 1$ and $A$ replaced by $A((t_1))$. The isomorphisms in (20) can be checked directly with the help of the following obvious fact: given a ring $B$ and subgroups $P, Q \subset B^*$ such that $P \cap Q = \{1\}$ and $P \cdot (Q - 1) = Q - 1$, we have the isomorphism

$$P \times Q \xrightarrow{\sim} P + Q - 1, \quad (p, q) \mapsto p \cdot q = p + p \cdot (q - 1).$$

We apply this, taking $B = \mathcal{L}^n(A)$, $P = \mathbb{V}_{1,+}(A)$, $Q = L \mathbb{V}_{n-1,+}(A)$ and $P = \mathbb{V}_{1,-}(A)$, $Q = L \mathbb{V}_{n-1,-}(A)$.

**Remark 4.4.** For $n = 2$, Proposition 4.3 was proved in [6], §3.1.

The decomposition (19) defines projections

$$\nu: L^n \mathbb{G}_m \longrightarrow \mathbb{Z}^n \quad \text{and} \quad \pi: L^n \mathbb{G}_m \longrightarrow \mathbb{G}_m.$$  

(21)

**Example 4.5.** (i) Suppose that the condition $\nu(f) \in \mathbb{Z}^n \subset \mathbb{Z}^n(A)$ holds for the invertible iterated Laurent series

$$f = \sum_{l \in \mathbb{Z}^n} a_l t^l \in L^n \mathbb{G}_m(A)$$

that is, $\nu(f)$ is a constant function on $\text{Spec}(A)$. For instance, this is true when $\text{Spec}(A)$ is connected. Then $f$ has an invertible coefficient $a_l \in A^*$ for some $l \in \mathbb{Z}^n$ and $\nu(f) \in \mathbb{Z}^n$ is the smallest index of an invertible coefficient with respect to the lexicographical order on $\mathbb{Z}^n$. Moreover, all coefficients of $f$ with smaller indices than $\nu(f)$ are nilpotent elements in $A$.

(ii) If $A = k$ is a field and $n = 1$, then $\nu$ is the natural discrete valuation on the field $\mathcal{L}(A) = k((t))$ and $\pi$ sends a nonzero series to its first nonzero coefficient.

(iii) Let $\varepsilon$ be a formal variable that satisfies $\varepsilon^2 = 0$, take $A = \mathbb{Z}[\varepsilon]$, and consider the invertible Laurent series in $\mathcal{L}(A)$

$$f = (\varepsilon t^{-1} + 1) \cdot (1 + t) = \varepsilon t^{-1} + (1 + \varepsilon) + t.$$  

Then $\nu(f) = 0$ and $\pi(f) = 1$, but the constant term of $f$ is not 1.
4.3. Special subgroups. We have the group functors Nil and $1 + \text{Nil}$ where the group structures are given by the sum and the product of elements in a ring, respectively. In particular, there is an embedding of group functors $(1 + \text{Nil}) \hookrightarrow \mathbb{G}_m$.

Definition 4.6. Define the following group subfunctors of $L^n\mathbb{G}_m$ (see (16), (17) and (21)):

\[
(L^n\mathbb{G}_m)^0 := \text{Ker}(\nu) = \mathbb{G}_m \times \mathbb{V}_{n,+} \times \mathbb{V}_{n,-}, \\
(L^n\mathbb{G}_m)^\sharp := (1 + \text{Nil}) \times \mathbb{V}_{n,+} \times \mathbb{V}_{n,-}.
\]

Evidently, $(L^n\mathbb{G}_m)^\sharp$ is embedded into $(L^n\mathbb{G}_m)^0$, its intersection with $\mathbb{G}_m$ is the group functor $1 + \text{Nil}$, and $(L^n\mathbb{G}_m)^0 = \mathbb{G}_m \cdot (L^n\mathbb{G}_m)^\sharp$. By Proposition 4.3, we have the decompositions

\[
\mathbb{Z}^n \times (\mathbb{G}_m \cdot (L^n\mathbb{G}_m)^\sharp) = \mathbb{Z}^n \times (L^n\mathbb{G}_m)^0 \simeq L^n\mathbb{G}_m.
\]

Here is a more explicit description of the group functors $(L^n\mathbb{G}_m)^0$ and $(L^n\mathbb{G}_m)^\sharp$.

Lemma 4.7. Let $A$ be a ring.

(i) For any element $f \in (L^n\mathbb{G}_m)^0(A)$, the constant term of $f$ is invertible and its class in $A^*/(1 + \text{Nil}(A))$ is equal to the class of $\pi(f) \in A^*$ (cf. Example 4.5 (iii)).

(ii) The group functors satisfy the equality

\[
(L^n\mathbb{G}_m)^0(A) = \left\{ 1 + \sum_{l \in \mathbb{Z}^n} a_lt^l \bigg| 1 + a_0 \in A^*, \sum_{l>0} a_l t^l \in \mathcal{L}^n(A), \sum_{l<0} a_l t^l \in \text{Nil}(\mathcal{L}^n(A)) \right\}.
\]

(iii) The group functors satisfy the equality

\[
(L^n\mathbb{G}_m)^\sharp(A) = \left\{ 1 + \sum_{l \in \mathbb{Z}^n} a_lt^l \bigg| a_0 \in \text{Nil}(A), \sum_{l>0} a_l t^l \in \mathcal{L}^n(A), \sum_{l<0} a_l t^l \in \text{Nil}(\mathcal{L}^n(A)) \right\}.
\]

Proof. (i) We have to expand the brackets in the decomposition (19).

(ii) Note that an iterated Laurent series $\sum_{l \in \mathbb{Z}^n} b_l t^l \in \mathcal{L}^n(A)$ is nilpotent if and only if the constant term $b_0 \in \text{Nil}(A)$ and the iterated Laurent series $\sum_{l>0} b_l t^l$ and $\sum_{l<0} b_l t^l$ are nilpotent (this is easily proved by induction on $n$ with the help of Remark 3.1). Using this, we can show that the left-hand side is contained in the right-hand side. Then we check that the right-hand side is contained in a coset of $(\mathbb{G}_m \times \mathbb{V}_{n,+} \times \mathbb{V}_{n,-})(A)$ in $L^n\mathbb{G}_m(A)$ since, by Proposition 4.3, the subgroup $\mathbb{Z}^n(A)$ is a transversal for the cosets of $(\mathbb{G}_m \times \mathbb{V}_{n,+} \times \mathbb{V}_{n,-})(A)$.

(iii) This follows directly from (i) and (ii).
Definition 4.8. Define the following group subfunctor of $L^n G_a$ (see Definition 3.7 and Proposition 3.8)

$$(L^n G_a)^\sharp(A) := L^n(A)^\sharp = \left\{ \sum_{l \in \mathbb{Z}} a_l t^l \middle| \sum_{l > 0} a_l t^l \in L^n(A), \sum_{l \leq 0} a_l t^l \in \text{Nil}(L^n(A)) \right\}.$$ 

Obviously, there is an isomorphism of functors $(L^n G_m)^\sharp \simeq (L^n G_a)^\sharp$, $f \mapsto f - 1$, which does not respect the group structure. However there is a group isomorphism over $\mathbb{Q}$, given in the following proposition.

Proposition 4.9. There is a well-defined isomorphism of group functors

$$\log: (L^n G_m)^\sharp_{\mathbb{Q}} \congto (L^n G_a)^\sharp_{\mathbb{Q}},$$

whose inverse is given by $\exp$.

Proof. This follows directly from Lemma 3.9, because of the power series equalities

$$\exp(\log(1 + x)) = 1 + x, \quad \log(\exp(x)) = x,$$

$$\log(1 + x + y + xy) = \log(1 + x) + \log(1 + y), \quad \exp(x + y) = \exp(x) \cdot \exp(y).$$

§ 5. Auxiliary results on ind-affine schemes

Let $R$ be a ring.

5.1. Ind-schemes. Recall some general notions related to ind-schemes. Given a category $\mathcal{C}$, we have the category of ind-objects in $\mathcal{C}$ (see [19] and [20]). Explicitly, an ind-object in $\mathcal{C}$ is given by a directed partially ordered set $I$, a collection of objects $C_i$ in $\mathcal{C}$ for all $i \in I$, and a collection of compatible morphisms $C_i \to C_j$ for all $i \leq j$. Such an ind-object is denoted by $\lim_{\longrightarrow} i \in I C_i$.

Ind-objects in the category of schemes are called ind-schemes. An ind-affine scheme is an ind-scheme $X$ which is isomorphic to $\lim_{\longrightarrow} i \in I X_i$ where all the $X_i$, $i \in I$, are affine schemes. Similarly, we have the notions of ind-schemes and ind-affine schemes over a ring $R$.

Given an ind-scheme $X$, we also denote the functor that sends a ring $A$ to the set $X(A) := \text{Hom}(\text{Spec}(A), X)$ by $X$. Note that $X(A)$ equals $\lim_{\longrightarrow} i \in I X_i(A)$ for the ind-scheme $X = \lim_{\longrightarrow} i \in I X_i$. A morphism between ind-schemes is the same as a morphism between the corresponding functors on the category of rings.

For example, the discussion after Definition 4.2 means that the group functor $\mathbb{Z}$ is represented by the ind-affine scheme

$$\lim_{\longrightarrow N \geq 0} \prod_{-N \leq i \leq N} \text{Spec}(\mathbb{Z}).$$

An ind-closed subscheme of a scheme $V$ is an ind-object in the category of closed subschemes of $V$ where the compatible morphisms are embeddings between closed subschemes. Explicitly, an ind-closed subscheme of $V$ is given by a directed partially ordered set $I$ and a collection of closed subschemes $X_i \subset V$ for all $i \in I$ such that $X_i \subset X_j$ if $i \leq j$. In particular, an ind-closed subscheme in an affine scheme is an ind-affine scheme.
An ind-closed subscheme of an ind-scheme $Y \simeq \lim_{j \in J} Y_j$ is an ind-scheme $X \simeq \lim_{i \in I} X_i$ together with a morphism of ind-schemes $X \to Y$ such that for any $i \in I$, there is $j \in J$ and there is a commutative diagram

$$
\begin{array}{ccc}
X_i & \longrightarrow & Y_j \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
$$

where the top horizontal arrow is a closed embedding of schemes and the vertical arrows are the natural morphisms.

In particular, if $Y$ is an ind-closed subscheme of a scheme $V$, then an ind-closed subscheme of $Y$ is the same as an ind-closed subscheme $X \simeq \lim_{i \in I} X_i$ of $V$ such that for any $i \in I$ there is $j \in J$ that satisfies $X_i \subset Y_j$. Equivalently, we have a morphism $X \to Y$ in the category of ind-closed subschemes of $V$.

Given two ind-closed subschemes

$$
X \simeq \lim_{i \in I} X_i \quad \text{and} \quad Y \simeq \lim_{j \in J} Y_j
$$

of a scheme $V$, the intersection $X \cap Y$ is the ind-closed subscheme

$$
\lim_{(i,j) \in I \times J} X_i \cap Y_j
$$

of $V$. Here we take the schematic intersection of two closed subschemes of $V$, that is, the intersection is given by the ideal sheaf generated by the ideal sheaves of the two closed subschemes. More generally, one can also consider the intersection of two ind-closed subschemes in an ind-scheme, though we will not use this.

Given an ind-scheme $X$, the ring of regular functions on $X$ is defined by the formula $\mathcal{O}(X) := \text{Hom}(X, A^1)$. In particular, $V = \text{Spec}(\mathcal{O}(V))$ for an affine scheme $V$. To be explicit, the ind-scheme $X \simeq \lim_{i \in I} X_i$ has an isomorphism $\mathcal{O}(X) \simeq \lim_{i \in I} \mathcal{O}(X_i)$. This isomorphism is well-behaved with respect to morphisms of ind-schemes. A morphism of ind-schemes $\alpha : X \to Y$ induces a homomorphism of rings $\alpha^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. For an ind-scheme $X$ over $R$, the ring $\mathcal{O}(X)$ is naturally an $R$-algebra.

Regular functions on a closed subscheme of an affine space can be represented by polynomials in the coordinates. This is no longer true for an ind-closed subscheme of an affine space. In this case, it is natural to represent regular functions by power series in the coordinates. For example, for the ind-closed subscheme $X = \lim_{d \in \mathbb{N}} \text{Spec}(\mathbb{Z}[x]/(x^d))$ of $A^1 = \text{Spec}(\mathbb{Z}[x])$, we have the isomorphism $\mathcal{O}(X) \simeq \mathbb{Z}[[x]]$. In §5.4, we generalize this for a certain class of ind-closed subschemes of ind-affine spaces.

### 5.2. Algebraic convergence of power series

We introduce some notation concerning polynomials and power series in infinitely many variables. Let $M$ be a (possibly infinite) set. By $R[M]$ denote the algebra of polynomials over $R$ in formal variables $x_m$ that correspond bijectively to elements $m \in M$. We also denote this algebra by $R[x_m; m \in M]$ if we need to specify the formal variables.
Note that there is a bijection between the set of monomials in \(x_m, m \in M\), and the \(d\)th symmetric power \(\text{Sym}^d(M)\) of the set \(M\), that is, with the set of unordered \(d\)-tuples of elements in \(M\). Thus there is an isomorphism

\[
R[M] \simeq \bigoplus_{d \geq 0} R^{\oplus \text{Sym}^d(M)},
\]

where, given a set \(P\), we denote the group of all functions with finite support from \(P\) to \(R\) by \(R^{\oplus P}\).

We say that the affine scheme \(A^M_R := \text{Spec}(R[M])\) is an affine space over \(R\). Thus the formal variables \(x_m, m \in M\), are interpreted as coordinates on the affine space \(A^M_R\). For brevity, we denote \(A^M_R\) by \(A^M\) if it is clear over which ring the affine space is considered. In what follows, we consider affine spaces over \(R\). Given a subset \(M' \subset M\), we have a closed embedding \(A^{M'} \subset A^M\) such that the corresponding epimorphism \(R[M] \to R[M']\) is the identity on \(M'\) and vanishes on \(M \setminus M'\). Let \((M) \subset R[M]\) denote the ideal generated by all \(x_m, m \in M\).

**Definition 5.1.** The algebra of power series over \(R\) in the formal variables \(x_m, m \in M\), is defined by the formula

\[
R[[M]] := \lim_{\leftarrow (M',d)} R[M'/((M')^d],
\]

where \(M'\) runs over all finite subsets of \(M\) and \(d \in \mathbb{N}\).

We also denote this algebra by \(R[[x_m; m \in M]]\). If \(M\) is at most countable, then elements in \(R[[M]]\) are countable sums of pairwise different monomials in \(x_m, m \in M\), with coefficients in \(R\). For example, the infinite sums \(\sum_{m \in M} x_m, \sum_{d \geq 0} x^d\), and \(\sum_{m \in M, d \geq 0} x^d_m\) are power series, while the infinite sum \(x + 2x + 3x + \cdots\) is not. For a general set \(M\), elements in \(R[[M]]\) are transfinite sums of such monomials. In any case, we have the isomorphisms

\[
R[[M]] \simeq \prod_{d \geq 0} \lim_{M'} R^{\oplus \text{Sym}^d(M')} \simeq \prod_{d \geq 0} R^{\text{Sym}^d(M)}, \quad (23)
\]

where, as above, \(M'\) runs over all finite subsets of \(M\) and, for a set \(P\), we denote the group of all functions from \(P\) to \(R\) by \(R^P\). Thus a power series \(\varphi \in R[[M]]\) is decomposed into a countable sum of homogeneous power series: \(\varphi = \sum_{d \geq 0} \varphi_d\), \(\varphi_d \in R^{\text{Sym}^d(M)}\). The support of the power series \(\varphi\) is the set of monomials that have a nonzero coefficient in \(\varphi\). In other words, the support is a subset of \(\bigsqcup_{d \geq 0} \text{Sym}^d(M)\) which corresponds to the nonzero coefficients in \(\varphi\) with respect to the decomposition (23).

**Definition 5.2.** (i) Let \(A\) be an \(R\)-algebra and let \(d \geq 0\) be a natural number. A homogeneous power series \(\varphi_d \in R^{\text{Sym}^d(M)}\) converges algebraically at an \(A\)-point \(p \in A^M(A) = A^M\) if all but finitely many monomials in the support of \(\varphi_d\) vanish at \(p\). If this holds, then \(\varphi_d(p)\) is a well-defined element of the algebra \(A\).

(ii) An arbitrary power series \(\varphi \in R[[M]]\) converges algebraically at an \(A\)-point \(p\) as in item (i) if all its homogeneous components \(\varphi_d, d \geq 0\), converge algebraically.
at $p$ and there exists $d_0 \in \mathbb{N}$ such that $\varphi_d(p) = 0$ when $d \geq d_0$. If this holds, then
\[ \varphi(p) = \sum_{d < d_0} \varphi_d(p) \] is a well-defined element in the algebra $A$.

(iii) A power series $\varphi$ converges algebraically on an ind-closed subscheme $X \subset \mathbb{A}^M$ if $\varphi$ converges algebraically at all its $A$-points for any $R$-algebra $A$. We denote the set of all power series in $R[[M]]$ that converge algebraically on $X$ by $\mathcal{A}(X)$.

**Example 5.3.** (o) Any polynomial converges algebraically at any point.

(i) Any power series converges algebraically at a point with finitely many nonzero coordinates all of which are nilpotent. 

(ii) The series $\sum_{m \in M} x_m$ converges algebraically at a point if and only if all but finitely many coordinates of the point are equal to zero.

(iii) The series $\sum_{d \geq 0} (x_1^d - x_2^d)$ converges algebraically (and, in fact, vanishes) at the point $(1, 1) \in \mathbb{A}^2$, while the series $\sum_{d \geq 0} (-1)^d x_1^d$ does not.

(iv) Given a closed subscheme $V \subset \mathbb{A}^M$, a power series converges algebraically on $V$ if and only if it converges algebraically at the point $p \in \mathbb{A}^M(\mathcal{O}(V))$ that corresponds to the restriction homomorphism $\mathcal{O}(\mathbb{A}^M) \to \mathcal{O}(V)$.

(v) Given an ind-closed subscheme $X = \lim_i X_i$ of $\mathbb{A}^M$, a power series converges algebraically on $X$ if and only if it converges algebraically at all points $p_i \in \mathbb{A}^M(\mathcal{O}(X_i))$ that correspond to the restriction homomorphisms $\mathcal{O}(\mathbb{A}^M) \to \mathcal{O}(X_i)$, $i \in I$.

Given an ind-closed subscheme $X \subset \mathbb{A}^M$, the set $\mathcal{A}(X)$ is an $R$-algebra and we have a canonical homomorphism of $R$-algebras

$$\mathcal{A}(X) \longrightarrow \mathcal{O}(X), \quad (24)$$

given by evaluation of algebraically convergent series at points of $X$. This homomorphism being an isomorphism means that any regular function on $X$ is uniquely expanded as an algebraically convergent power series in $x_m$, $m \in M$. For example, this holds for the affine space $\mathbb{A}^M$.

### 5.3. Ind-affine spaces.

We carry on looking at affine spaces over $R$, as above. We shall work with ind-closed subschemes not only in affine spaces but also in ind-affine spaces. To this end, we need to introduce one more concept. First, recall that an ind-set is an ind-object in the category of sets.

**Definition 5.4.** A strict ind-set is an ind-set $\lim_i E_i$ such that all structure maps $E_i \to E_j$, $i \leq j$, are injective. Given a strict ind-set $E = \lim_i E_i$, we denote the set $\bigcup_{i \in I} E_i$ by $\bar{E}$ and the strict ind-set formed by all finite subsets of $\bar{E}$ by $E_f$.

In particular, any set is also a strict ind-set.

In what follows, $E = \lim_i E_i$ is a strict ind-set. We call the ind-affine scheme $\mathbb{A}^E := \lim_i \mathbb{A}^{E_i}$ an ind-affine space. Clearly, $\mathbb{A}^E$ is an ind-closed subscheme of the affine space $\mathbb{A}^\mathbb{E}$ and there is an isomorphism of $R$-algebras

$$\mathcal{O}(\mathbb{A}^E) \simeq \lim_{i \in I} \bigoplus_{d \geq 0} R \otimes \text{Sym}^d(E_i). \quad (25)$$
Definition 5.5. Given a set \( M \) and a natural number \( d \), define the following closed subscheme of \( \mathbb{A}^M \):
\[
\mathbb{A}^M_{(d)} := \text{Spec}(R[M]/(M)^d).
\]
We also define the following ind-closed subscheme of the ind-affine space \( \mathbb{A}^E \):
\[
\mathbb{A}^E := \lim_{\leftarrow \in I \times \mathbb{N}} \mathbb{A}^E_{(i,d)},
\]
where \( (i,d) \in I \times \mathbb{N} \).

In particular, given a set \( M \) we have \( \mathbb{A}^M_{(d)} = \lim_{\leftarrow \in \mathbb{N}} \mathbb{A}^M_{(d)} \) and there is an isomorphism of \( \mathbb{R} \)-algebras
\[
\mathcal{O}(\mathbb{A}^M_{(d)}) \simeq \prod_{d \geq 0} R^\oplus \text{Sym}^d(M).
\]

Further, we have the equalities
\[
\mathbb{A}^E = \lim_{\leftarrow \in I} \mathbb{A}^E_{(d)} = \lim_{\leftarrow \in \mathbb{N}} \mathbb{A}^E_{(d)},
\]
where for each natural number \( d \), we put \( \mathbb{A}^E_{(d)} := \lim_{\leftarrow \in I} \mathbb{A}^E_{(i,d)} \), and there are isomorphisms of \( \mathbb{R} \)-algebras
\[
\mathcal{O}(\mathbb{A}^E) \simeq \lim_{\leftarrow \in I \times \mathbb{N}} R^\oplus \text{Sym}^d(E_i) \simeq \prod_{d \geq 0} \lim_{\leftarrow \in I} R^\oplus \text{Sym}^d(E_i).
\]

Also, \( \mathbb{A}^{E_f} \) is an ind-closed subscheme of the ind-affine scheme \( \mathbb{A}^E \) and there is an isomorphism of \( \mathbb{R} \)-algebras
\[
\mathcal{O}(\mathbb{A}^{E_f}) \simeq \mathbb{R}[[E]].
\]

To be precise, given an \( \mathbb{R} \)-algebra \( A \), a point \( p \in \mathbb{A}^E(A) = \mathbb{A}^E \) belongs to the ind-closed subscheme \( \mathbb{A}^{E_f} \subset \mathbb{A}^E \) if and only if \( p \) has finitely many nonzero coordinates, all of which are nilpotent.

Remark 5.6. Formulae (25) and (27) imply that \( \mathcal{O}(\mathbb{A}^E) \) and \( \mathcal{O}(\mathbb{A}^E) \) are subalgebras of the algebra of power series \( \mathbb{R}[[E]] \). Moreover, we have \( \mathcal{O}(\mathbb{A}^E) = \mathcal{O}(\mathbb{A}^E) \) and \( \mathcal{O}(\mathbb{A}^E) = \mathcal{O}(\mathbb{A}^E) \), where \( \mathcal{O}(\mathbb{A}^E) \) and \( \mathcal{O}(\mathbb{A}^E) \) are defined by means of the embeddings of \( \mathbb{A}^E \) and \( \mathbb{A}^E \) into the affine space \( \mathbb{A}^E \). Thus the canonical homomorphism (24) is an isomorphism for the ind-closed subschemes \( \mathbb{A}^E \) and \( \mathbb{A}^E \) of the affine space \( \mathbb{A}^E \).

5.4. Thick ind-cones. We continue with our investigation of affine spaces over \( \mathbb{R} \); \( M \) is a set and \( E \) is a strictly ind-set.

Definition 5.7. A closed subscheme \( V \subset \mathbb{A}^M \) is a cone if \( V \) corresponds to a homogeneous ideal in \( R[M] = R[x_m; m \in M] \), where all the \( x_m \) have degree 1. An ind-closed subscheme \( X \subset \mathbb{A}^E \) is an ind-cone if there is an isomorphism \( X \simeq \lim_{\leftarrow \in I} X_i \) such that \( X_i \subset \mathbb{A}^E \) is a cone for all \( i \in I \).
If $X$ is as given above we can show that it is a cone if and only if it is invariant under homotheties, that is, for any $R$-algebra $A$ and an element $a \in A$ the subset $X(A) \subset \hat{A}^E(A) = A^E$ is preserved under multiplication of all coordinates by $a$.

The next lemma contains a crucial property of cones, which is of the utmost importance for the main results of the paper.

**Lemma 5.8.** Let $V \subset \mathbb{A}^M$ be a cone. Then the following hold.

(i) The restriction homomorphism $\theta_V : \mathcal{O}(V) \to \mathcal{O}(V \cap \hat{\mathbb{A}}^M)$ is injective.

(ii) The following commutative diagram is a Cartesian square:

$$
\begin{array}{ccc}
\mathcal{O}(V) \cap \mathcal{O}(\hat{\mathbb{A}}^M) & \longrightarrow & \mathcal{O}(\hat{\mathbb{A}}^M) \\
\downarrow & & \downarrow \\
\mathcal{O}(V) & \longrightarrow & \mathcal{O}(V \cap \hat{\mathbb{A}}^M)
\end{array}
$$

where the intersection in the left-hand upper corner is taken in $R[[M]]$, the intersection in the right-hand bottom corner is taken among ind-closed subschemes of $\mathbb{A}^M$, and $\xi_V$ is the restriction homomorphism.

**Remark 5.9.** To be explicit, Lemma 5.8 (ii) claims that a regular function $\varphi \in \mathcal{O}(\hat{\mathbb{A}}^M)$ viewed as a power series in $R[[M]]$ (see Remark 5.6), converges algebraically on $V$ if and only if the restriction of $\varphi$ to $V \cap \hat{\mathbb{A}}^M$ extends to a regular function on $V$. Moreover, by Lemma 5.8 (i), if it exists, this regular function on $V$ is unique.

**Proof of Lemma 5.8.** (i) Let $I_V \subset R[M]$ be the ideal that defines the closed subscheme $V \subset \mathbb{A}^M$. Since $V$ is a cone, the ring $\mathcal{O}(V) = R[M]/I_V$ is graded, that is, we have a decomposition $\mathcal{O}(V) \simeq \bigoplus_{d \geq 0} A_d$. Therefore, the ring

$$
\mathcal{O}(V \cap \hat{\mathbb{A}}^M) \simeq \lim_{d \in \mathbb{N}} \mathcal{O}(V \cap \hat{\mathbb{A}}^M(d)) \simeq \lim_{d \in \mathbb{N}} R[M]/(I_V + (M)^d)
$$

is isomorphic to $\prod_{d \geq 0} A_d$ and the homomorphism $\theta_V$ is just the natural embedding $\bigoplus_{d \geq 0} A_d \hookrightarrow \prod_{d \geq 0} A_d$.

(ii) Consider a series $\varphi \in \mathcal{O}(\hat{\mathbb{A}}^M) \subset R[[M]]$. Let $p \in \mathbb{A}^M(\mathcal{O}(V))$ be the point that corresponds to the restriction homomorphism $\mathcal{O}(\mathbb{A}^M) \to \mathcal{O}(V)$. By Example 5.3 (iv), $\varphi$ converges algebraically on $V$ if and only if it converge algebraically at $p$. We see that each homogeneous component $\varphi_d \in R^\otimes \text{Sym}^d(M)$ of $\varphi$ converges algebraically at $p$. Hence $\varphi$ converges algebraically at $p$ if and only if $\varphi_d(p) \in A_d$ equals zero for sufficiently large $d$. By part (i), this is equivalent to the fact that $\xi_V(\varphi) = \varphi(p) \in \prod_{d \geq 0} A_d$ is in the image of the ring $\mathcal{O}(V) \simeq \bigoplus_{d \geq 0} A_d$ under the map $\theta_V$. This proves the lemma.

**Definition 5.10.** An ind-closed subscheme $X \subset \mathbb{A}^E$ is thick if $X$ contains the ind-closed subscheme $\hat{\mathbb{A}}^E \subset \mathbb{A}^E$ (see Definition 5.5).

In particular, $\mathbb{A}^E_f$ is thick in $\mathbb{A}^E$ if and only if $E_f \simeq E$.

Now we give the main property of thick ind-cones, which generalizes Remark 5.6.

Proposition 5.11. Let $X \subset \mathbb{A}^E$ be a thick ind-cone. Then the canonical homomorphism $\mathcal{A}(X) \to \mathcal{O}(X)$ is an isomorphism (see (24)), where $\mathcal{A}(X)$ is defined by means of the embedding of $X$ into the affine space $\mathbb{A}^E$.

Proof. We can assume that

$$E \simeq \lim_{i \in I} E_i, \quad X \simeq \lim_{i \in I} X_i$$

and $X_i$ is a cone in the affine space $\mathbb{A}^{E_i}$ for any $i \in I$. For each $i \in I$, applying Lemma 5.8 to the cone $X_i \subset \mathbb{A}^{E_i}$ we get a Cartesian square

$$\begin{array}{ccc}
\mathcal{A}(X_i) \cap \mathcal{O}(\hat{\mathbb{A}}^{E_i}) & \to & \mathcal{O}(\hat{\mathbb{A}}^{E_i}) \\
\downarrow & & \downarrow \\
\mathcal{O}(X_i) & \to & \mathcal{O}(X_i \cap \hat{\mathbb{A}}^{E_i})
\end{array}$$

By (26), we have the isomorphisms

$$\lim_{i \in I} \mathcal{A}(X_i) \cap \mathcal{O}(\hat{\mathbb{A}}^{E_i}) \simeq \mathcal{A}(X) \cap \mathcal{O}(\hat{\mathbb{A}}^E), \quad \lim_{i \in I} \mathcal{O}(\hat{\mathbb{A}}^{E_i}) \simeq \mathcal{O}(\hat{\mathbb{A}}^E).$$

Clearly, $\lim_{i \in I} X_i \cap \hat{\mathbb{A}}^{E_i} \simeq X \cap \hat{\mathbb{A}}^E$. Thus, passing to the inverse limit over $i \in I$ and using the fact that inverse limits are left exact, we obtain a Cartesian square

$$\begin{array}{ccc}
\mathcal{A}(X) \cap \mathcal{O}(\hat{\mathbb{A}}^E) & \to & \mathcal{O}(\hat{\mathbb{A}}^E) \\
\downarrow & & \downarrow \\
\mathcal{O}(X) & \to & \mathcal{O}(X \cap \hat{\mathbb{A}}^E)
\end{array}$$

Since $X$ is thick in $\mathbb{A}^E$, we have $\hat{\mathbb{A}}^E \cap X = \hat{\mathbb{A}}^E$ and the restriction homomorphism $\mathcal{O}(\hat{\mathbb{A}}^E) \to \mathcal{O}(X \cap \hat{\mathbb{A}}^E)$ is an isomorphism. Also, since $\hat{\mathbb{A}}^E \subset X$, we have $\mathcal{A}(X) \cap \mathcal{O}(\hat{\mathbb{A}}^E) = \mathcal{A}(X)$. Since $\mathcal{A}(\hat{\mathbb{A}}^E) = \mathcal{O}(\hat{\mathbb{A}}^E)$ (see Remark 5.6), this implies that $\mathcal{A}(X) \cap \mathcal{O}(\hat{\mathbb{A}}^E) = \mathcal{A}(X)$. Taken together, this proves the proposition.

Remark 5.12. It is easy to check that the composition of the inverse to the isomorphism $\mathcal{A}(X) \sim \mathcal{O}(X)$ in Proposition 5.11 with the embedding $\mathcal{A}(X) \subset R[[E]]$ coincides with the restriction of functions to the ind-closed subscheme $\hat{\mathbb{A}}^{E_1} \subset \hat{\mathbb{A}}^E \subset X$ (see (28)).

We will also use the following simple properties of thick ind-cones.

Lemma 5.13. Let $E_1$, $E_2$ be strict ind-sets and let $X_1 \subset \mathbb{A}^{E_1}$, $X_2 \subset \mathbb{A}^{E_2}$ be ind-closed subschemes of the corresponding ind-affine spaces over $R$. Put

$$E := E_1 \cup E_2, \quad X := X_1 \times X_2 \subset \mathbb{A}^{E_1} \times \mathbb{A}^{E_2} \simeq \mathbb{A}^E.$$

Then $X \subset \mathbb{A}^E$ is thick (is an ind-cone) if and only if $X_1 \subset \mathbb{A}^{E_1}$ and $X_2 \subset \mathbb{A}^{E_2}$ are thick (are ind-cones, respectively).
Proposition 5.17. The following fact is a generalization of Proposition 5.16(i). Let \( X \subset \mathbb{A}^E \) be an ind-flat ind-affine scheme over \( R \). Suppose that for any \( R \)-algebra \( A \) and any point \( p \in X(A) \subset \mathbb{A}^E(A) \) with finitely many nonzero coordinates all of which are nilpotent, the equality \( \alpha(p) = \alpha'(p) \in Y(A) \) holds. Then \( \alpha = \alpha' \).

Proof. For each \( a = 1, 2 \), we have the equalities

\[
X_a = X \cap \mathbb{A}^{E_a}, \quad \hat{\mathbb{A}}^{E_a} = \hat{\mathbb{A}}^E \cap \mathbb{A}^{E_a}
\]

between ind-closed subschemes of \( \mathbb{A}^E \), which proves one implication. To prove the reverse implication, use the isomorphism \( \hat{\mathbb{A}}^{E_1} \times \hat{\mathbb{A}}^{E_2} \cong \hat{\mathbb{A}}^E \).

The proof of the following statement is straightforward.

Lemma 5.14. Let \( X \subset \mathbb{A}^E \) be an ind-closed subscheme and \( R \to S \) a homomorphism of rings. If \( X \) is thick (is an ind-cone) in \( \mathbb{A}^E \), then \( X_S \) is thick (is an ind-cone) in \( \mathbb{A}^E_S \).

Here are some further remarkable properties of functions on thick ind-cones.

Proposition 5.15. Let \( X \subset \mathbb{A}^E \) be a thick ind-cone and let \( \alpha, \alpha' : X \to Y \) be two morphisms to an affine scheme \( Y \) over \( R \). Suppose that for any \( R \)-algebra \( A \) and any point \( p \in X(A) \subset \mathbb{A}^E(A) \) with finitely many nonzero coordinates all of which are nilpotent, the equality \( \alpha(p) = \alpha'(p) \in Y(A) \) holds. Then \( \alpha = \alpha' \).

Proof. Using a closed embedding \( Y \subset \mathbb{A}^M \) for a suitable set \( M \), we can easily reduce the proposition to the case \( Y = \mathbb{A}^1 \). Then the statement follows directly from Proposition 5.11 and Remark 5.12.

Proposition 5.16. Let \( X \subset \mathbb{A}^E \) be a thick ind-cone and let \( R \subset S \) be an embedding of rings. Then the following statements hold.

(i) The natural homomorphism \( \mathcal{O}(X) \to \mathcal{O}(X_S) \) is injective.

(ii) Consider a power series \( \varphi \in S[[E]] \) and a regular function \( \phi \in \mathcal{O}(X) \). Suppose that for any \( S \)-algebra \( A \) and any point \( p \in X(A) \subset \mathbb{A}^E(A) \) with finitely many nonzero coordinates all of which are nilpotent, the equality

\[
\varphi(p) = \phi(p) \in A
\]

holds. Then \( \varphi \) has coefficients in \( R \), that is, \( \varphi \in R[[E]] \). Furthermore, the power series \( \varphi \) converges algebraically on \( X \), that is, \( \varphi \in \mathcal{A}(X) \), and \( \varphi \) is taken to \( \phi \) under the canonical homomorphism \( \mathcal{A}(X) \to \mathcal{O}(X) \).

Proof. (i) This uses Proposition 5.11, Lemma 5.14 and the fact that the natural map from \( \mathcal{A}(X) \subset R[[E]] \) to \( \mathcal{A}(X_S) \subset S[[E]] \) is injective.

(ii) This follows directly from Proposition 5.11 and Remark 5.12.

An ind-affine scheme \( Y \) over \( R \) is ind-flat over \( R \) if there is an isomorphism \( Y \cong \varprojlim_{j \in J} Y_j \) such that \( Y_j \) is a flat affine scheme over \( R \) for any \( j \in J \). The following fact is a generalization of Proposition 5.16 (i).

Proposition 5.17. Let \( X \) be a thick ind-cone in an ind-affine space over \( R \), let \( Y \) be an ind-flat ind-affine scheme over \( R \) and \( Z \) an affine scheme over \( R \). Let \( R \subset S \) be an embedding of rings. Then the natural map

\[
\text{Hom}_R(X \times Y, Z) \longrightarrow \text{Hom}_S((X \times Y)_S, Z_S)
\]

is injective.
Proof. First, using a closed embedding $Z \subset \mathbb{A}^M$ for a suitable set $M$, we can easily reduce the proposition to the case $Z = \mathbb{A}^1$. In other words, it is enough to prove that the natural map

$$\mathcal{O}(X \times Y) \longrightarrow \mathcal{O}((X \times Y)_S)$$

is injective.

Let $Y \simeq \lim\limits_{j \in J} Y_j$ be such that $Y_j$ is a flat affine scheme over $R$ for any $j \in J$. We have canonical isomorphisms

$$\mathcal{O}(X \times Y) \simeq \lim\limits_{j \in J} \mathcal{O}(X \times Y_j) \quad \text{and} \quad \mathcal{O}((X \times Y)_S) \simeq \lim\limits_{j \in J} \mathcal{O}((X \times Y_j)_S).$$

Therefore it is enough to consider the case when $Y$ is a flat affine scheme over $R$, because inverse limits are left exact.

Let $Y \simeq \text{Spec}(A)$, where $A$ is a flat $R$-algebra. Put $B := A \otimes_R S$. Clearly, we have the canonical isomorphisms

$$\mathcal{O}(X \times Y) \simeq \mathcal{O}(X_A) \quad \text{and} \quad \mathcal{O}((X \times Y)_S) \simeq \mathcal{O}(X_B).$$

Since the homomorphism $R \to S$ is injective and $A$ is flat over $R$, the natural homomorphism $A \to B$ is also injective. Thus we complete the proof by applying Lemma 5.14 and Proposition 5.16 (i) to the embedding of rings $A \subset B$.

5.5. Connectedness.

Definition 5.18. An ind-scheme $X$ over $R$ is connected over $R$ if any idempotent in $\mathcal{O}(X)$ is the image of an idempotent in $R$ under the natural homomorphism of rings $R \to \mathcal{O}(X)$. An ind-scheme $X$ over $R$ is absolutely connected over $R$ if for any homomorphism of rings $R \to S$ the ind-scheme $X_S$ is connected over $S$.

Remark 5.19. Idempotents in a ring $A$ correspond bijectively to decompositions of the topological space $\text{Spec}(A)$ into a disjoint union of two closed subsets. Therefore, any surjective homomorphism $A \to B$ such that all elements in the kernel are nilpotent induces a bijection between the sets of idempotents in $A$ and $B$. It follows that an affine scheme $X$ over $R$ is (absolutely) connected over $R$ if and only if $X_{\text{red}}$ is (absolutely) connected over $R_{\text{red}}$, where $R_{\text{red}}$ is the quotient of $R$ by the nilradical $\text{Nil}(R)$ (cf. the proof of [6], Lemma 3.2).

It is easy to check that if $\text{Spec}(R)$ is connected as a topological space, then an ind-scheme $X$ over $R$ is connected over $R$ if and only if any morphism $X \to \text{Spec}(R) \sqcup \text{Spec}(R)$ over $R$ factors through a copy of $\text{Spec}(R)$. If $X$ is a scheme over $R$, then this is also equivalent to $X$ being connected as a topological space.

Example 5.20. (i) The affine line $\mathbb{A}^1$ is absolutely connected over $\mathbb{Z}$. Indeed, by Remark 5.19, it is enough to show that $\mathbb{A}^1_R$ is connected over $R$ for any ring $R$ without nonzero nilpotents. A nonzero idempotent in $\mathcal{O}(\mathbb{A}^1_R)$ corresponds to a polynomial

$$f = a_0 + a_1 x + \cdots + a_d x^d \in R[x]$$

such that $f^2 = f$, where $a_d \neq 0$. If $d > 0$, then $a_0^2 = 0$, which contradicts the condition $a_d \neq 0$. This shows that $\mathbb{A}^1_R$ is connected over $R$. More generally, for any set $M$, the affine space $\mathbb{A}^M$ is absolutely connected over $\mathbb{Z}$ (the proof is similar).

(ii) Similarly to (i), one can show that $\mathbb{G}_m$ is absolutely connected over $\mathbb{Z}$. 
Remark 5.21. Let $X \simeq \lim_{i \in I} X_i$ be an ind-scheme over $R$. Suppose that for any $i \in I$, the scheme $X_i$ is connected over $R$ and the homomorphism of rings $R \to \mathcal{O}(X_i)$ is injective (this holds, for example, if $X_i(R)$ is not empty). Then $X$ is connected over $R$.

The following simple lemma says that ind-cones are absolutely connected. Geometrically speaking, this holds because one can join any point on a cone to its vertex.

\textbf{Lemma 5.22.} Let $X$ be an ind-cone in an ind-affine space over $R$. Then $X$ is absolutely connected over $R$.

\textbf{Proof.} By Lemma 5.14, it is enough to prove that $X$ is connected over $R$. Any cone contains the point $0 := (0, 0, \ldots)$ over any ring. Therefore by Remark 5.21, it is enough to suppose that $X$ is a cone $V$ in an affine space $\mathbb{A}^M$ over $R$, where $M$ is a suitable set. Consider an idempotent $\phi \in \mathcal{O}(V)$. Let $e \in R$ be the value of $\phi$ at the point $0 \in V(R)$. Then $e$ is also an idempotent. We will show that $\phi$ is equal to the image of $e$ under the homomorphism $R \to \mathcal{O}(V)$.

Let $A$ be an $R$-algebra and let $x \in V(A) \subset \mathbb{A}^M(A) = A^M$ be an $A$-point on $V$. Since $V$ is a cone, we have a morphism of schemes over $A$

\[ \gamma: \mathbb{A}^1_A \longrightarrow V_A, \quad a \mapsto a \cdot x, \]

which defines a line on the cone joining $0$ and $x$. By Example 5.20(i), $\mathbb{A}^1_A$ is connected over $A$. Therefore the idempotent $\gamma^*(\phi) \in \mathcal{O}(\mathbb{A}^1_A)$ is the image of an idempotent in $A$ under the homomorphism $A \to \mathcal{O}(\mathbb{A}^1_A)$. In particular, the function $\gamma^*(\phi)$ on $\mathbb{A}^1_A$ is constant and in $A$ we have

\[ \phi(0) = \gamma^*(\phi)(0) = \gamma^*(\phi)(1) = \phi(x). \]

On the other hand, $\phi(0)$ is the image in $A$ of the element $e \in R$ under the homomorphism $R \to A$. Taken together, this proves that $\phi$ is the image of $e$ under the homomorphism $R \to \mathcal{O}(V)$.

Recall that the group functor $\mathbb{Z}$ (see Definition 4.2) is represented by an ind-affine scheme (see §5.1), which is clearly not connected over $\mathbb{Z}$. We will use the following property of connected ind-schemes.

\textbf{Proposition 5.23.} Let $X$ be a connected ind-scheme over $R$ and let $\alpha: X \to \mathbb{Z}_R$ be a morphism of ind-schemes over $R$. Suppose that there exists an $R$-point $x \in X(R)$ such that $\alpha(x) = 0$. Then $\alpha = 0$.

\textbf{Proof.} Let $\phi \in \mathcal{O}(\mathbb{Z}_R)$ be the characteristic function of the zero element, that is, $\phi$ equals one on the connected component $\text{Spec}(\mathbb{Z}) \subset \mathbb{Z}$ which corresponds to $0 \in \mathbb{Z}(R)$ and $\phi$ equals zero on all the other components. In particular, $\phi$ is an idempotent. The idempotent $\alpha^*(\phi) \in \mathcal{O}(X)$ is the image of an idempotent $e \in R$ because $X$ is connected over $R$. The idempotent $e \in R$ defines a decomposition $R \simeq R_1 \times R_2$ such that the image of $\alpha_{R_1}: X_{R_1} \to \mathbb{Z}_{R_1}$ in $\mathbb{Z}_{R_1}$ is equal to zero and the image of $\alpha_{R_2}: X_{R_2} \to \mathbb{Z}_{R_2}$ in $\mathbb{Z}_{R_2}$ is contained in the union of all components in $\mathbb{Z}$ other than the component of the zero element. If $R_2$ is nonzero, then $x$ defines an $R_2$-point $x_2 \in X_{R_2}(R_2)$ such that $\alpha_{R_2}(x_2) = 0$, which gives a contradiction. Therefore, $R = R_1$, $e = 1$, and $\alpha = 0$. 

5.6. Density.

**Definition 5.24.** An ind-closed subscheme $X$ of an ind-affine scheme $Y$ is *dense* if the natural homomorphism of rings $\mathcal{O}(Y) \to \mathcal{O}(X)$ is injective. An ind-closed subscheme $X$ of an ind-affine scheme $Y$ is *absolutely dense* if for any ring $R$, the ind-closed subscheme $X_R$ is dense in the ind-affine scheme $Y_R$.

Note that (absolute) density is not a concept which is relative with respect to the base, that is, (absolute) density does not depend on the base ring over which given ind-schemes are considered.

If $Y$ is an affine scheme, then an ind-closed subscheme $X$ of $Y$ is dense if and only if there is no closed affine subscheme $Z \subset Y$, $Z \neq Y$, such that $X \subset Z$.

**Lemma 5.25.** An ind-closed subscheme $X$ of an ind-affine scheme $Y$ is absolutely dense if and only if for any ind-scheme $T$ and any affine scheme $Z$, the natural map

$$\text{Hom}(Y \times T, Z) \longrightarrow \text{Hom}(X \times T, Z)$$

is injective.

*Proof.* The proof uses ideas similar to those used in the proof of Proposition 5.17.

Suppose that the map (29) is injective for all $T$ and $Z$ as above. Given a ring $R$, put $T = \text{Spec}(R)$ and $Z = \mathbb{A}^1$. Then the isomorphisms $\text{Hom}(X \times T, Z) \simeq \mathcal{O}(X_R)$ and $\text{Hom}(Y \times T, Z) \simeq \mathcal{O}(Y_R)$ imply that $X$ is absolutely dense in $Y$.

We now prove the reverse implication. Using a closed embedding $Z \subset \mathbb{A}^M$ for a set $M$, we can easily reduce the lemma to the case $Z = \mathbb{A}^1$. Let $\alpha, \beta : Y \times T \to \mathbb{A}^1$ be two morphisms of ind-schemes such that $\alpha \neq \beta$. There is a ring $R$ and an $R$-point $x \in T(R)$ such that the compositions of the initial morphisms $\alpha$ and $\beta$ with the embedding $X \subset Y$ are not equal either.

**Example 5.26.** (i) Note that the group functor $\text{Nil}$ is represented by the ind-affine scheme

$$\widehat{\mathbb{A}}^1 \simeq \lim_{\longrightarrow} \text{Spec}(\mathbb{Z}[x]/(x^n)),$$

which is an ind-closed subscheme of the affine space $\mathbb{A}^1 \simeq \text{Spec}(\mathbb{Z}[x])$ representing the group functor $\mathbb{G}_a$. We know that the ind-closed subscheme $\text{Nil}$ is absolutely dense in $\mathbb{G}_a$. In fact, for any ring $R$, the natural homomorphism of $R$-algebras

$$\mathcal{O}((\mathbb{G}_a)_R) \simeq R[x] \longrightarrow \mathcal{O}(\text{Nil}_R) \simeq R[[x]]$$

is injective. Similarly, for any natural number $N$, the ind-closed subscheme $\text{Nil}^{\times N}$ of the affine space $(\mathbb{G}_a)^{\times N} \simeq \mathbb{A}^N$ is absolutely dense.
(ii) The group subfunctor $1 + \text{Nil} \subset \mathbb{G}_m$ is represented by an ind-closed subscheme of $\mathbb{G}_m$ and is absolutely dense in it. Indeed, for any ring $R$, the natural homomorphism of $R$-algebras

$$\mathcal{O}((\mathbb{G}_m)_R) \simeq R[y, y^{-1}] \simeq R[x, (1 + x)^{-1}] \longrightarrow \mathcal{O}((1 + \text{Nil})_R) \simeq R[[x]]$$

is injective, where $y = 1 + x$. Similarly, for any natural number $N$, the ind-closed subscheme $(1 + \text{Nil})^\times N$ is absolutely dense in the affine scheme $(\mathbb{G}_m)^\times N$.

The following proposition generalizes Example 5.26(i).

**Proposition 5.27.** Let $E$ be a strict ind-set and let $X \subset \mathbb{A}^E$ be an ind-closed subscheme of the corresponding ind-affine space. Suppose that $\hat{\mathbb{A}}^{E,f} \subset X$ (see Definitions 5.4 and 5.5). Then $X$ is absolutely dense in $\mathbb{A}^E$.

**Proof.** We have the following embeddings of ind-closed subschemes in $\mathbb{A}^E$:

$$\hat{\mathbb{A}}^{E,f} \subset X \subset \mathbb{A}^E,$$

which induce homomorphisms of rings

$$\mathcal{O}(\mathbb{A}^E) \longrightarrow \mathcal{O}(X) \longrightarrow \mathcal{O}(\hat{\mathbb{A}}^{E,f}) \simeq \mathbb{Z}[[E]].$$

Since the composition of these homomorphisms is injective (see Remark 5.6), the first homomorphism is injective as well, and so $X$ is dense in $\mathbb{A}^E$. A similar argument applies to the embedding $X_R \subset \mathbb{A}^E_R$ over an arbitrary ring $R$.

In particular, Proposition 5.27 implies that any thick ind-closed subscheme in $\mathbb{A}^E$ (see Definition 5.10) is absolutely dense in it.

§ 6. Geometric properties of certain iterated loop groups

6.1. The ordered product of strict ind-sets. We define an operation on strict ind-sets which allows us to deal with the representability of iterated loop functors. Recall that given a strict ind-set $E$, we have the associated set $\overline{E}$ (see Definition 5.4).

**Definition 6.1.** Given strict ind-sets

$$E_1 = \lim_{i_1 \in I_1} E_{1,i_1} \quad \text{and} \quad E_2 = \lim_{i_2 \in I_2} E_{2,i_2}.$$

Their ordered product $E_1 * E_2$ is a strict ind-set given by the formula

$$E_1 * E_2 := \lim_{\lambda \in \Lambda} (E_1 * E_2)_{\lambda},$$

where

$$\Lambda := \{(\lambda_1, \lambda_2) \mid \lambda_1 : \overline{E}_2 \to I_1, \lambda_2 \in I_2\},$$

and for $\lambda = (\lambda_1, \lambda_2) \in \Lambda$ we set

$$(E_1 * E_2)_\lambda := \{(e_1, e_2) \in \overline{E}_1 \times \overline{E}_2 \mid e_2 \in E_{2,\lambda_2}, e_1 \in E_{1,\lambda_1(e_2)}\}.$$
A partial order on $\Lambda$ is defined by $\lambda = (\lambda_1, \lambda_2) \leq \lambda' = (\lambda'_1, \lambda'_2)$ if and only if $\lambda_1 \leq \lambda'_1$ and $\lambda_2 \leq \lambda'_2$, where the order on functions is defined pointwise, that is, $\lambda_1 \leq \lambda'_1$ if and only if $\lambda_1(e_2) \leq \lambda'_1(e_2)$ for all $e_2 \in E_2$.

Note that $\overline{E_1 \times E_2} = \overline{E_1} \times \overline{E_2}$, that is, the strict ind-set $E_1 \ast E_2$ is formed by subsets of the set $\overline{E_1} \times \overline{E_2}$.

**Example 6.2.** (i) Let $M$ be a set and let $E = \lim \limits_{i \in I} E_i$ be a strict ind-set. Then there are canonical isomorphisms of strict ind-sets

$$M \ast E \simeq \lim_{i \in I} M \times E_i \quad \text{and} \quad E \ast M \simeq \lim_{\lambda \in \Lambda} (E \ast M)_{\lambda},$$

where $\Lambda := \{ \lambda : M \to I \}$ and $(E \ast M)_{\lambda} := \{(e, m) \in \overline{E} \times M \mid e \in E_{\lambda(m)}\}$.

(ii) If $E_1$ and $E_2$ are just sets, then by (i), the ordered product $E_1 \ast E_2$ coincides with the Cartesian product $E_1 \times E_2$.

(iii) Denote the one-element set by $U$. Then by (i), for any strict ind-set $E$ there are canonical isomorphisms $U \ast E \simeq E \ast U \simeq E$.

Clearly, the ordered product is functorial with respect to isomorphisms of strict ind-sets. One can check directly that the ordered product $E_1 \ast E_2$ is functorial with respect to arbitrary morphisms of strict ind-sets for the first argument, that is, any morphism of strict ind-sets $E_1 \to E'_1$ defines a morphism of strict ind-sets $E_1 \ast E_2 \to E'_1 \ast E_2$ in a natural way. Also, the ordered product $E \ast M$ between a strict ind-set $E$ and a set $M$ is functorial with respect to injective maps of sets, that is, an injective map of sets $M \to M'$ defines a morphism of strict ind-sets $E \ast M \to E \ast M'$ in a natural way.

**Remark 6.3.** It is easy to check that for strict ind-sets $E_1$ and $E_2$, there is a canonical isomorphism of strict ind-sets

$$E_1 \ast E_2 \simeq \lim_{i_2 \in I_2} E_1 \ast E_{2,i_2},$$

where the limit on the right-hand side is taken in the category of strict ind-sets.

However, the ordered product is not naturally functorial with respect to arbitrary morphisms of strict ind-sets, as the following example shows.

**Example 6.4.** Let $I$ be a directed partially ordered set without a final element (in particular, $I$ is infinite) and let $E = \lim \limits_{i \in I} E_i$ be a strict ind-set such that $E_i \neq E_j$ for $i \neq j$. As above, let $U$ be the one-element set. The unique map of sets $I \to U$ induces a projection $p : \overline{E} \times I \to \overline{E} \times U \simeq \overline{E}$. We claim that the projection $p$ does not extend to a morphism of strict ind-sets from $E \ast I$ to $E \ast U \simeq E$. Indeed, let $\lambda$ be the identity function from $I$ to itself. Then the subset $(E \ast I)_{\lambda}$ of $\overline{E} \times I$ (see Example 6.2 (i)) is equal to

$$\{(e, i) \in \overline{E} \times I \mid e \in E_i\}.$$

In particular, $(E \ast I)_{\lambda}$ maps surjectively to $\overline{E}$ under the projection $p$, while no $E_i$ maps surjectively to $\overline{E}$ for $i \in I$. Therefore there are no morphisms of ind-sets from $E \ast I$ to $E$. 
The ordered product defines a monoidal structure on the category of strict ind-sets with isomorphisms. In other words, given three strict ind-sets $E_1 = \lim_{i_1 \in I_1} E_{1,i_1}, E_2 = \lim_{i_2 \in I_2} E_{2,i_2}$ and $E_3 = \lim_{i_3 \in I_3} E_{3,i_3}$, there is a functorial isomorphism

$$(E_1 \ast E_2) \ast E_3 \simeq E_1 \ast (E_2 \ast E_3)$$

which satisfies the standard coherence conditions (the unit object is the one element set $U$). This holds because both $(E_1 \ast E_2) \ast E_3$ and $E_1 \ast (E_2 \ast E_3)$ are functorially isomorphic to the strict ind-set $\lim_{\delta \in \Delta} (E_1 \ast E_2 \ast E_3)_{\delta}$, where

$$\Delta := \{ (\delta_1, \delta_2, \delta_3) \mid \delta_1 : E_2 \times E_3 \to I_1, \delta_2 : E_3 \to I_2, \delta_3 \in I_3 \},$$

and for $\delta = (\delta_1, \delta_2, \delta_3) \in \Delta$, we set

$$(E_1 \ast E_2 \ast E_3)_{\delta} := \{(e_1, e_2, e_3) \in \overline{E}_1 \times \overline{E}_2 \times \overline{E}_3 \mid e_3 \in E_3, e_2 \in E_2, e_1 \in E_1, \delta_{(e_2, e_3)} \}.$$

In particular, ordered powers of strict ind-sets are well-defined.

Example 6.5. Define the following strict ind-set:

$$D := \lim_{m \in \mathbb{Z}} \mathbb{Z}_{\geq -m},$$

where $\mathbb{Z}_{\geq -m}$ denotes the set of all integers $l$ such that $l \geq -m$. Note that $\overline{D} = \mathbb{Z}$. We use the notation from §3.1. The $n$th ordered power $D^{\star n}$ of $D$ is canonically isomorphic to the strict ind-set $\lim_{\lambda \in \Lambda_n} \mathbb{Z}_\lambda^n$, where a partial order on $\Lambda_n$ is defined as follows:

$$\lambda = (\lambda_1, \ldots, \lambda_n) \leq \lambda' = (\lambda'_1, \ldots, \lambda'_n)$$

if and only if $\lambda_1 \geq \lambda'_1, \ldots, \lambda_n \geq \lambda'_n$ (note the reverse order). Also, we have

$$D^{\star n} = \mathbb{Z}^n.$$

Note that the functor $E \mapsto \overline{E}$ from the category of strict ind-sets with isomorphisms to the category of sets is naturally monoidal, where we are looking at the ordered product of strict ind-sets and the Cartesian product of sets.

We stress that the ordered product is not naturally commutative. More precisely, the above monoidal functor $E \mapsto \overline{E}$ is not symmetric, that is, given strict ind-sets $E_1$ and $E_2$, the canonical bijection $E_1 \times E_2 \simeq E_2 \times E_1$ does not necessarily extend to an isomorphism of strict ind-sets between $E_1 \ast E_2$ and $E_2 \ast E_1$. For instance, this is the case in Example 6.2 (i) with general $M$ and $E$ or in the case $E_1 = E_2 = D$. This noncommutativity motivates the term ‘ordered’.

6.2. Representability of loop functors. We shall now discuss the representability of loop functors (see Definition 4.1). Below, we consider affine spaces over $\mathbb{Z}$.

The functor $LA^1 \simeq LG_a$ is represented by an ind-affine space $A^D$ (see §5.3 and Example 6.5). Now, given a ring $A$, an $A$-point on $A^D$ is given by a set $(a_l)_{l \in \mathbb{Z}}$, where $a_l \in A$ and there exists $m \in \mathbb{Z}$ such that $a_l = 0$ for $l < -m$. The point $(a_l)_{l \in \mathbb{Z}}$ corresponds to the Laurent series $\sum_{l \in \mathbb{Z}} a_l t^l \in \mathcal{L}(A)$.
Proposition 6.6. (i) Given a strict ind-set $E$ (see Definition 5.4), the functor $L\mathbb{A}^E$ is represented by the ind-affine space $\mathbb{A}^{D^{*}E}$ (see Definition 6.1).

(ii) Let $E$ be a strict ind-set and let $X \subset \mathbb{A}^E$ be an ind-closed subscheme. Then the functor $LX$ is represented by an ind-closed subscheme of $\mathbb{A}^{D^{*}E}$ (cf. [21], §1 a).

(iii) In the notation of (ii), if $X$ is thick (is an ind-cone) in $\mathbb{A}^E$, then $LX$ is thick (is an ind-cone) in $\mathbb{A}^{D^{*}E}$ (see Definitions 5.7 and 5.10).

Proof. (i) It is easy to check that the assignment $F \mapsto LF$ commutes with direct limits of functors. Therefore, by Remark 6.3, we can assume that $E$ is a set $M$ and it is enough to produce an isomorphism $L\mathbb{A}^M \simeq \mathbb{A}^{D^{*}M}$. Given a ring $A$, an element in the set $L\mathbb{A}^M(A)$ is the same as a collection $\{f_m\}, m \in M$, of Laurent series in $\mathcal{L}(A)$. Clearly, such a collection defines a function $\lambda: M \to \mathbb{Z}$ and an $A$-point on the affine space $\mathbb{A}^{(D^{*}M)\lambda}$ and vice versa.

(ii) Similarly to (i) we can assume that $X$ is a closed subscheme $V$ of an affine space $\mathbb{A}^M$, where $M$ is a set. Note that $V$ is the fibre over the point $(0,0,0,\ldots)$ of a morphism $\mathbb{A}^M \to \mathbb{A}^{M'}$, where $M'$ is a set of equations defining $V$ in $\mathbb{A}^M$. Since the assignment $F \mapsto LF$ commutes with fibred products of functors, we see by (i) that $LV$ is the fibre over the point $(0,0,0,\ldots)$ of the corresponding morphism between ind-affine spaces $\mathbb{A}^{D^{*}M} \to \mathbb{A}^{D^{*}M'}$. Hence the functor $LV$ is represented by an ind-closed subscheme of $\mathbb{A}^{D^{*}M}$.

(iii) Note that the coefficients of the product $f_1 \cdots f_d$ of the Laurent series $f_1, \ldots, f_d$ are homogeneous polynomials of degree $d$ in the coefficients of the $f_i$. This proves the result on ind-cones in (iii). Further, for any set $M$, we have an embedding $\mathbb{A}^{D^{*}M}_{(d)} \subset L\mathbb{A}^M_{(d)}$ in $\mathbb{A}^{D^{*}M} \simeq L\mathbb{A}^M$, which proves (iii) when $X$ is thick.

Example 6.7. Applying Proposition 6.6(i) iteratively to $\mathbb{A}^1$, we find that $L^n\mathbb{A}^1 \simeq L^n\mathbb{G}_a$ is represented by the ind-affine space $\mathbb{A}^{D^{*}n}$ (see Example 6.5). This can also be shown directly using the isomorphisms

$$D^{*n} \simeq \lim_{\lambda \in \Lambda_n} \mathbb{Z}_\lambda^n \quad \text{and} \quad D^{*n} \simeq \mathbb{Z}^n$$

and the description of iterated Laurent series in $\mathcal{L}^n(A) = L^n\mathbb{G}_a(A)$ for a ring $A$ in terms of the sets $\mathbb{Z}_\lambda^n, \lambda \in \Lambda_n$, given in §3.1.

6.3. Representability of $L^n\mathbb{G}_m$ and of its special subgroups. Next, we will show that the group functor $L^n\mathbb{G}_m$, its group subfunctors $(L^n\mathbb{G}_m)^0$ and $(L^n\mathbb{G}_m)^\sharp$ (see Definition 4.6), and the group functor $(L^n\mathbb{G}_a)^\sharp$ (see Definition 4.8) are represented by ind-affine schemes, which are close to being ind-flat over $\mathbb{Z}$.

Let $\mathbb{Z}_{\geq 0}^n$ be the set of all elements $l \in \mathbb{Z}^n$ such that $l > 0$ (see §4.2 for the lexicographical order on $\mathbb{Z}^n$) and define a strict ind-set $D_{\geq 0}^{*n} := D^{*n} \cap \mathbb{Z}_{\geq 0}^n$ (the strict ind-set $D^{*n}$ is defined in Example 6.5). Similarly, define the sets $\mathbb{Z}_{\leq 0}^n, \mathbb{Z}_{\leq 0}^n, \mathbb{Z}_{\leq 0}^n$ and the strict ind-sets $D_{\geq 0}^{*n}, D_{\leq 0}^{*n}, D_{\leq 0}^{*n}$. Clearly, we have the decompositions

$$\mathbb{Z}^n = \mathbb{Z}_{\geq 0}^n \sqcup \mathbb{Z}_{\leq 0}^n \quad \text{and} \quad D^{*n} = D_{\geq 0}^{*n} \sqcup D_{\leq 0}^{*n}.$$
(ii) The isomorphic functors $(L^n\mathbb{G}_m)^\sharp$ and $(L^n\mathbb{G}_a)^\sharp$ are represented by a thick ind-cone in the ind-affine space $A^{D^n} \simeq L^n\mathbb{G}_a$ (see Example 6.7).

(iii) The functors $(L^n\mathbb{G}_m)^0$ and $L^n\mathbb{G}_m$ are represented by ind-affine schemes of type $X \times X'$, where $X$ is a thick ind-cone in an ind-affine space over $\mathbb{Z}$ and $X'$ is an ind-affine scheme which is ind-flat over $\mathbb{Z}$.

Proof. (i) The assertion for $\mathcal{V}_{n,+}$ follows directly from its definition. We will prove the assertion for $\mathcal{V}_{n,-}$. Recall that the functor $\text{Nil}$ is represented by the ind-closed subscheme of $\mathbb{A}^1$

$$\hat{\mathbb{A}}^1 \simeq \lim_{d \in \mathbb{N}} \text{Spec}(\mathbb{Z}[x]/(xd)),$$

which is definitely a thick ind-cone in $\mathbb{A}^1$. By Proposition 6.6 (iii) and Example 6.7, the functor $L^n\text{Nil}$ is represented by a thick ind-cone in the ind-affine space $L^n\mathbb{A}^1 \simeq A^{D^n}$ over $\mathbb{Z}$.

Further, define $L^n\text{Nil}_{\geq 0}$ as the intersection $L^n\text{Nil} \cap A^{D^n}_0$ between ind-closed subschemes of $A^{D^n} \simeq L^n\mathbb{G}_a$, and similarly for $L^n\text{Nil}_{< 0}$. It follows from Remark 3.1 that there is equality

$$L^n\text{Nil} = L^n\text{Nil}_{\geq 0} \times L^n\text{Nil}_{< 0}$$

between ind-closed subschemes of $A^{D^n} = A^{D^n}_{\geq 0} \times A^{D^n}_{< 0}$. Now the required result follows from Lemma 5.13 and the isomorphism of functors $\mathcal{V}_{n,-} \simeq L^n\text{Nil}_{< 0}$.

(ii) By Definitions 4.6 and 4.8, we have the decompositions

$$(L^n\mathbb{G}_m)^\sharp \simeq \mathcal{V}_{n,-} \times (1 + \text{Nil}) \times \mathcal{V}_{n,+} \quad \text{and} \quad (L^n\mathbb{G}_a)^\sharp \simeq \mathcal{V}_{n,-} \times \text{Nil} \times \mathcal{V}_{n,+}.$$  

Also, we have the decomposition $A^{D^n} = A^{D^n}_{\geq 0} \times \mathbb{A}^1 \times A^{D^n}_{< 0}$, where the $\mathbb{A}^1$ in the middle corresponds to the element $0 \in \mathbb{Z}^n = D^n$. Thus our result follows from (i) and Lemma 5.13.

(iii) By Definition 4.6, we have the decomposition $(L^n\mathbb{G}_m)^0 \simeq \mathcal{V}_{n,-} \times \mathcal{V}_{n,+} \times \mathbb{G}_m$ and by Proposition 4.3, we have the decomposition $L^n\mathbb{G}_m \simeq \mathcal{V}_{n,-} \times \mathcal{V}_{n,+} \times \mathbb{G}_m \times \mathbb{Z}^n$. Clearly, the ind-affine space $\mathcal{V}_{n,+} \simeq A^{D^n}_{\geq 0}$ and the ind-affine scheme $\mathbb{Z}$ (see §5.1) are ind-flat over $\mathbb{Z}$. Hence the ind-affine scheme $\mathcal{V}_{n,+} \times \mathbb{G}_m \times \mathbb{Z}^n$ is also ind-flat over $\mathbb{Z}$, which completes the proof by (i).

Remark 6.9. (i) Propositions 6.8 (i) and 5.11 imply that the ring of regular functions $\mathcal{O}(\mathcal{V}_{n,-}) \simeq \mathcal{A}(\mathcal{V}_{n,-})$ is flat over $\mathbb{Z}$, or, equivalently, is torsion free over $\mathbb{Z}$, because it is a subring of the ring of power series over $\mathbb{Z}$. On the other hand, Proposition 6.8 (i) fixes a presentation of $\mathcal{V}_{n,-}$ as an ind-affine scheme $\lim_{i \in I} X_i$, but the rings $\mathcal{O}(X_i)$ are not flat over $\mathbb{Z}$ (see the discussion after [6], Lemma 3.4); this flatness fails even for the case $n = 1$.

(ii) It is an interesting open question whether the ind-affine scheme $\mathcal{V}_{n,-}$ is ind-flat over $\mathbb{Z}$, that is, whether there is another representation $\lim_{j \in J} Y_j$ of $\mathcal{V}_{n,-}$ as an ind-affine scheme such that all rings $\mathcal{O}(Y_j)$, $j \in J$, are flat over $\mathbb{Z}$ (note that the answer is positive when $n = 1$; see, for instance, [6], §2). It was the lack of an answer to this question that motivated us to develop the theory of thick ind-cones.

Now we obtain a series of important applications by combining Proposition 6.8 with various properties of thick-ind cones from §5. Let $N$ be a natural number. Consider the set $M := \prod_{i=1}^N \mathbb{Z}^n$ and the corresponding set of formal variables $x_{i,j}$,
where $1 \leq i \leq N$, $l \in \mathbb{Z}^n$, which are coordinates on the affine space $\mathbb{A}^M$ (see §5.2). We will denote the ind-affine scheme $((L^n \mathbb{G}_a)^2)^{\times N}$ by $X$ for short. Thus $X$ is an ind-closed subscheme of the affine space $\mathbb{A}^M$ such that the collection of iterated Laurent series $(f_1, \ldots, f_N)$ corresponds to the point $(a_{i,l})$, where $f_i = \sum_{t \in \mathbb{Z}^n} a_{i,l} t^l$ for $1 \leq i \leq N$.

**Theorem 6.10.** Let $R \subset S$ be an embedding of rings. Consider a power series (see Definition 5.1)

$$\varphi \in S[[x_i, l; 1 \leq i \leq N, l \in \mathbb{Z}^N]]$$

and a regular function $\phi \in \mathcal{O}(X_R)$. Suppose that for any $S$-algebra $A$ and any collection of Laurent polynomials (not just series) $f_1, \ldots, f_N$ with nilpotent coefficients in $A$, the equality

$$\varphi(f_1, \ldots, f_N) = \phi(f_1, \ldots, f_N) \in A$$

holds. Then $\varphi$ has coefficients in $R$, converges algebraically on $X_R$ (see Definition 5.2 (iii)) and is taken to $\phi$ under the canonical homomorphism $\mathcal{O}(X_R) \to \mathcal{O}(X_R)$ (see formula (24)).

**Proof.** We define a strict ind-set $E := \coprod_{i=1}^{N} D^*$. Then, by Proposition 6.8 (ii) and Lemma 5.13, $X$ is a thick ind-cone in $A^E$. It follows from Lemma 5.14 that $X_R$ is a thick ind-cone in $A^E$. Thus we complete the proof using Proposition 5.16 (ii).

**Remark 6.11.** Combining Propositions 6.8 (ii) and 5.15, we can also obtain a higher-dimensional generalization of [6], Proposition 3.7.

**Theorem 6.12.** Let $R \subset S$ be an embedding of rings, let $Y$ be an ind-flat ind-affine scheme over $R$, $Z$ an affine scheme over $R$, and let $N$ be a natural number. Then the natural map

$$\text{Hom}_R((L^n \mathbb{G}_m)^{\times N} \times Y, Z) \to \text{Hom}_S((L^n \mathbb{G}_m)^{\times N} \times Y_S, Z_S)$$

is injective.

**Proof.** By Proposition 6.8 (iii) and Lemma 5.13, the functor $(L^n \mathbb{G}_m)^{\times N}$ is represented by an ind-affine scheme of type $X \times X'$, where $X$ is a thick ind-cone in an ind-affine space over $Z$ and $X'$ is an ind-flat ind-affine scheme over $Z$. By Lemma 5.14, the functor $(L^n \mathbb{G}_m)^{\times N}_R$ is represented by a similar ind-affine scheme over $R$. Now, applying Proposition 5.17 completes the proof.

Here are some further geometric properties of $L^n \mathbb{G}_a$ and the special subgroups of $L^n \mathbb{G}_m$.

**Proposition 6.13.** Let $N$ be a natural number.

(i) The ind-closed subscheme $((L^n \mathbb{G}_a)^2)^{\times N} \subset (L^n \mathbb{G}_a)^{\times N}$ is absolutely dense.

(ii) The ind-scheme $((L^n \mathbb{G}_a)^2)^{\times N}$ is absolutely connected over $\mathbb{Z}$.

(iii) The ind-closed subscheme $((L^n \mathbb{G}_m)^2)^{\times N} \subset ((L^n \mathbb{G}_m)^0)^{\times N}$ is absolutely dense.

(iv) The ind-schemes $((L^n \mathbb{G}_m)^2)^{\times N}$ and $((L^n \mathbb{G}_m)^0)^{\times N}$ are absolutely connected over $\mathbb{Z}$.

**Proof.** (i) The ind-closed subscheme $(L^n \mathbb{G}_a)^2$ in $L^n \mathbb{G}_a \simeq \mathbb{A}^{D^*}$ contains the ind-closed subscheme $\mathbb{A}^{(D^*)^I}$ (see Definition 5.4) because $\mathbb{A}^{(D^*)^I}$ corresponds to Laurent
polynomials with nilpotent coefficients. Thus, by Proposition 5.27, \((L^n \mathbb{G}_a)\) is absolutely dense in \((L^n \mathbb{G}_a)\). The case of arbitrary \(N\) is treated similarly.

(ii) Clearly, the ind-closed subscheme \((L^n \mathbb{G}_a)\) in the ind-affine space \(L^n \mathbb{G}_a \simeq A^{D^n} \mathbb{G}_a\) is an ind-cone (see the discussion after Definition 5.7 or Proposition 6.8(ii)). We conclude the proof using Lemma 5.22.

(iii) This follows if we combine Definition 4.6 with Example 5.26(ii) and Lemma 5.25.

(iv) By (iii), it is enough to prove that \(((L^n \mathbb{G}_m)\times N)\) is absolutely connected over \(\mathbb{Z}\). This follows from (ii) and the isomorphism of ind-schemes \((L^n \mathbb{G}_m)\simeq (L^n \mathbb{G}_a)\).

Remark 6.14. All the concepts in §5.5 and 5.6 make sense when ind-schemes are replaced by just functors (see §2). Moreover all the statements in §5.5 and 5.6 still hold in this generality. Thus, in fact, Proposition 6.13 is based neither on representability of iterated loop groups, nor on the theory of thick ind-cones.

6.4. The tangent space to \(L^n \mathbb{G}_m\).

Next, we discuss the tangent space at the neutral element of the group ind-scheme \(L^n \mathbb{G}_m\). In what follows we will also use tangent spaces to group functors, so we give a more general definition. In what follows, let \(\varepsilon\) be a formal variable that satisfies \(\varepsilon^2 = 0\).

Definition 6.15. Given a group functor \(F\) over a ring \(R\) (see §2), the tangent space \(T F\) to \(F\) is the group functor over \(R\) defined by the formula

\[ T F(A) := \ker(F(A[\varepsilon]) \to F(A)), \]

where \(A\) is an \(R\)-algebra.

In particular, given a morphism of group functors \(\Phi: F' \to F\) over \(R\), we have its differential \(T \Phi: T F' \to T F\), which is also a morphism of group functors over \(R\).

Example 6.16. (i) There is an isomorphism of group functors \(T \mathbb{G}_a \xrightarrow{\sim} \mathbb{G}_a, a \varepsilon \mapsto a\), where \(a\) is an element of a ring \(A\).

(ii) There is an isomorphism of group functors \(T \mathbb{G}_m \xrightarrow{\sim} \mathbb{G}_a, 1 + a \varepsilon \mapsto a\), where \(a\) is an element of a ring \(A\).

Lemma 6.17. (i) For any group functor \(F\) there is a canonical isomorphism of group functors \(T L F \simeq L T F\).

(ii) There are the isomorphisms of group functors

\[ T(L^n \mathbb{G}_m)^\sharp \simeq T(L^n \mathbb{G}_m)^0 \simeq TL^n \mathbb{G}_m \simeq L^n \mathbb{G}_a. \]

(iii) There are the isomorphisms of group functors

\[ T(L^n \mathbb{G}_a)^\sharp \simeq T(L^n \mathbb{G}_a) \simeq L^n \mathbb{G}_a. \]

Proof. Let \(A\) be a ring.

(i) We have an isomorphism of rings \(A[\varepsilon][(t)] \simeq A((t))[\varepsilon]\) (cf. Lemma 3.3(iii)). This implies the following sequence of isomorphisms of groups

\[ T L F(A) \simeq \ker(L F(A[\varepsilon]) \to L F(A)) \simeq \ker(F(A[\varepsilon][(t)]) \to F(A((t)))) \]

\[ \simeq \ker(F(A[(t)][\varepsilon]) \to F(A((t)))) \simeq L T F(A). \]
(ii) Let \( f \in TL^n \mathbb{G}_m(A) \). Explicitly, we have that

\[
f \in \text{Ker}(L^n \mathbb{G}_m(A[\varepsilon]) \to L^n \mathbb{G}_m(A)).
\]

Therefore all the coefficients of \( f \) are nilpotent except for the constant term, which belongs to \( 1 + A \varepsilon \subset A[\varepsilon]^* \). Lemma 4.7 (iii) implies that \( f \in (L^n \mathbb{G}_m)^2(A[\varepsilon]). \)

This proves the first and the second isomorphisms in (ii). To establish the third isomorphism, use (i) and the isomorphism of group functors \( T \mathbb{G}_m \simeq \mathbb{G}_a \) (see Example 6.16(ii)).

(iii) The proof is similar to (ii).

Remark 6.18. The first and the second isomorphisms in Lemma 6.17(ii) correspond to the fact that \( (L^n \mathbb{G}_m)^2 \) is absolutely dense in \( (L^n \mathbb{G}_m)^0 \) (see Proposition 6.13(iii)) and also that the quotient \( L^n \mathbb{G}_m/(L^n \mathbb{G}_m)^0 \simeq \mathbb{Z}^n \) is discrete (see formula (22) in §4.3), respectively. The first isomorphism in Lemma 6.17(ii) corresponds to the fact that \( (L^n \mathbb{G}_a)^2 \) is absolutely dense in \( L^n \mathbb{G}_a \) (see Proposition 6.13(i)).

6.5. Characters of \((L^n \mathbb{G}_m)^0\).

Definition 6.19. Given a group functor \( F \) over a ring \( R \), the group of characters of \( F \) is defined by the formula \( X(F) := \text{Hom}^*_R(F, (\mathbb{G}_m)_R) \) and the group of linear functionals on \( F \) is defined by the formula \( X^+(F) := \text{Hom}^*_R(F, (\mathbb{G}_a)_R). \)

Recall the following well-known facts.

Lemma 6.20. (i) For any ring \( R \) of zero characteristic, there is an isomorphism of groups \( R \xrightarrow{\sim} X^+((\mathbb{G}_m)_R), r \mapsto (a \mapsto ar) \), where \( a \) is an element of an \( R \)-algebra \( A \).

(ii) For any ring \( R \), there is an isomorphism of groups \( \mathbb{Z}(R) \xrightarrow{\sim} X((\mathbb{G}_m)_R) \), \( i \mapsto (a \mapsto a^i) \), where \( a \) is an invertible element of an \( R \)-algebra \( A \).

(iii) For any \( \mathbb{Q} \)-algebra \( R \), there is an isomorphism of groups \( \text{Nil}(R) \xrightarrow{\sim} X((\mathbb{G}_a)_R) \), \( r \mapsto (a \mapsto \exp(ar)) \), where \( a \) is an element of an \( R \)-algebra \( A \). The inverse isomorphism sends a character \( \chi : (\mathbb{G}_a)_R \to (\mathbb{G}_m)_R \) to its differential (see (i))

\[
T\chi \in \text{Hom}^*_R(T((\mathbb{G}_a)_R), T(\mathbb{G}_m)_R) \simeq \text{Hom}^*_R((\mathbb{G}_a)_R, (\mathbb{G}_a)_R) \simeq R,
\]

which is necessarily a nilpotent element in \( R \).

Proof. (i) A linear functional \( (\mathbb{G}_a)_R \to (\mathbb{G}_a)_R \) is given by a polynomial \( f \in R[x] \) such that \( f(0) = 0 \) and \( f(x + y) = f(x) + f(y) \). Applying the binomial theorem completes the proof.

(ii) A character \( (\mathbb{G}_m)_R \to (\mathbb{G}_m)_R \) is given by a Laurent polynomial \( f = \sum_i a_i x^i \) in \( R[x, x^{-1}]^* \) such that \( f(1) = 1 \) and \( f(x) \cdot f(y) = f(x \cdot y) \). These identities immediately imply that the coefficients \( a_i \) are orthogonal idempotents in \( R \) and their sum is equal to 1.

(iii) A character \( (\mathbb{G}_a)_R \to (\mathbb{G}_m)_R \) is given by a polynomial \( f \in R[x]^* \) such that \( f(0) = 1 \) and \( f(x) \cdot f(y) = f(x + y) \). Applying log, we get the equality \( \log(f)(x) + \log(f)(y) = \log(f)(x + y) \) between power series in \( R[[x]] \). Similarly to (i), we obtain that \( \log(f)(x) = rx \) for an element \( r \in R \). Since \( f = \exp(\log(f)) \) is a polynomial, we see that \( r \) is nilpotent.
Proposition 6.21. Let $R$ be a $\mathbb{Q}$-algebra and let $\chi \in X((L^n \mathbb{G}_m)^{\sharp}_R)$ be a character. Then the subfunctor (see Lemma 6.17(ii))

$$(L^n \mathbb{G}_a)^{\sharp}_R \subset (L^n \mathbb{G}_a)_R \simeq T(L^n \mathbb{G}_m)^{\sharp}_R$$

is taken by $T\chi$ to the subfunctor

$$(T\chi)\Nil_R \subset (\mathbb{G}_a)_R \simeq T(\mathbb{G}_m)_R$$

and the following diagram commutes (see Proposition 4.9 for the map $\exp$):

$$\begin{array}{ccc}
(L^n \mathbb{G}_a)^{\sharp}_R & \xrightarrow{T\chi} & \Nil_R \\
\exp & & \exp \\
(L^n \mathbb{G}_m)^{\sharp}_R & \xrightarrow{\chi} & (\mathbb{G}_m)_R
\end{array}$$

Proof. Let $A$ be an $R$-algebra. Consider an iterated Laurent series $g \in (L^n \mathbb{G}_a)^{\sharp}(A)$. We obtain a character

$$\kappa: (\mathbb{G}_a)_A \xrightarrow{g} (L^n \mathbb{G}_a)^{\sharp}_A \xrightarrow{\exp} (L^n \mathbb{G}_m)^{\sharp}_A \xrightarrow{\chi} (\mathbb{G}_m)_A,$$

where the first morphism of group ind-schemes sends an element $b$ in an $A$-algebra $B$ to the iterated Laurent series $bg \in \mathcal{L}^n(B)$. Thus $\kappa(b) = \chi(\exp(bg)) \in B^\ast$. We will find the differential $T\kappa \in \Nil(A)$ (see Example 6.16 and Lemma 6.20(i)).

Recall that by Example 6.16 and Lemma 6.17, we have isomorphisms

$$TG_a \simeq \mathbb{G}_a, \quad T(L^n \mathbb{G}_a)^{\sharp} \simeq L^n \mathbb{G}_a, \quad T(L^n \mathbb{G}_m)^{\sharp} \simeq L^n \mathbb{G}_a, \quad TG_m \simeq \mathbb{G}_a.$$

It is easily seen that under these isomorphisms $Tg$ corresponds to $g$ and $T\exp$ corresponds to the identity morphism. Indeed, for any ring $C$ and any iterated Laurent series $h \in L^n \mathbb{G}_a(C)$, we have $\exp(h \varepsilon) = 1 + h \varepsilon$ in $L^n \mathbb{G}_m(C[\varepsilon])$, where $\varepsilon^2 = 0$. It follows that the differential $T\kappa$ corresponds to the element $(T\chi)(g) \in A$. By Lemma 6.20(iii) applied over the ring $A$, we see that the element $(T\chi)(g)$ is nilpotent and that $\kappa(1) = \exp((T\chi)(g))$. On the other hand, $\kappa(1) = \chi(\exp(g))$, which proves the proposition.

Remark 6.22. Proposition 6.21 is a particular case of an algebraic version of the well-known analytic statement that a homomorphism between Lie groups commutes with the differential through the exponential map.

Proposition 6.23. Let $R$ be a $\mathbb{Q}$-algebra. Then the natural homomorphism of groups

$$X((L^n \mathbb{G}_m)^{\ast}_R) \longrightarrow X^+((L^n \mathbb{G}_a)_R), \quad \chi \longmapsto T\chi$$

is injective. In addition, for any character $\chi: (L^n \mathbb{G}_m)_R^0 \to (\mathbb{G}_m)_R$, $R$-algebra $A$, and any element $f \in (L^n \mathbb{G}_m)^{\sharp}(A)$, in $A^\ast$ the equality

$$\chi(f) = \exp(a)$$

holds, where the nilpotent element $a \in \Nil(A)$ is defined by the equality in $A[\varepsilon]^\ast$

$$\chi(1 + \log(f) \varepsilon) = 1 + a\varepsilon.$$
Proof. By Proposition 6.13 (iii) and Lemma 5.25, the restriction map
\[ X((L^n G_m)^0_R) \rightarrow X((L^n G_m)^\sharp_R) \]
is injective. Applying Propositions 4.9 and 6.21 completes the proof.

6.6. Linear functionals on \( L^n \Omega^i \). Define the following morphism of group functors:
\[ \Delta : L^n G_a \rightarrow L^n \Omega^1, \quad f \mapsto df - \sum_{i=1}^n \frac{\partial f}{\partial t_i} dt_i, \]
where \( f \in L^n G_a(A) = \mathcal{L}^n(A) \) for a ring \( A \).

Lemma 6.24. Let \( R \) be a ring, \( \Phi : (L^n \Omega^1)_R \rightarrow A^1_R \) a morphism of functors over \( R \) (which need not respect the group structure), and let \( \phi \in \mathcal{O}((L^n G_a)_R) \) be the composition \( \Phi \circ \Delta_R \). Then for any \( R \)-algebra \( A \) and an iterated Laurent series \( f \in \mathcal{L}^n(A) \), the equality \( \phi(f) = \phi(0) \) is valid.

Proof. We use some notation from §3.1. Recall from Example 6.7 that \( L^n G_a \) is represented by the ind-affine scheme \( \Lambda_n \mathbb{Z}^n \). Let \( \lambda \in \Lambda_n \) be such that \( f \in \Lambda_n \mathbb{Z}^n(A) \), that is, \( f = \sum_{l \in \mathbb{Z}^n} a_l t^l \), where \( a_l \in A \). The restriction of \( \phi \) to the affine space \( (\Lambda_n \mathbb{Z}^n)_R \) over \( R \) is given by a polynomial \( \phi_\lambda \in R[\mathbb{Z}^n_\lambda] \). Let \( T \subset \mathbb{Z}^n_\lambda \) be a finite subset such that the polynomial \( \phi_\lambda \) depends only on variables which correspond to elements in \( T \). Let \( f' = \sum_{l \in \mathbb{Z}^n_\lambda} a'_l t^l \) be a Laurent polynomial such that \( a'_l = a_l \) if \( l \in T \) and \( a'_l = 0 \) if \( l \notin T \). By construction \( \phi(f) = \phi(f') \). On the other hand, it is easy to check that \( \Delta(f') = \Delta(0) = 0 \), and so \( \phi(f') = \phi(0) \). This finishes the proof.

For an integer \( i \geq 0 \), let \( \widetilde{L^n \Omega}^i \) denote the group functor that sends a ring \( A \) to \( \Omega^i_{\mathcal{L}^n(A)} \) (see Definition 3.10). Note that we have a canonical morphism of group functors \( L^n \Omega^i \rightarrow \widetilde{L^n \Omega}^i \).

Proposition 6.25. For any ring \( R \) and integer \( i \geq 0 \), the natural homomorphism of groups
\[ X^+((\widetilde{L^n \Omega}^i)_R) \rightarrow X^+((L^n \Omega^i)_R) \]
is an isomorphism.

Proof. It is sufficient to show that each group functor morphism \( (L^n \Omega^i)_R \rightarrow (G_a)_R \) factors (in a unique way) through the factor \( (L^n \Omega^i)_R \rightarrow (\widetilde{L^n \Omega}^i)_R \). This follows directly from Definition 3.10 and Lemma 6.24.

§ 7. Auxiliary results from algebraic \( K \)-theory

Let \( m \geq 0 \) be an integer.

7.1. Milnor \( K \)-groups and algebraic \( K \)-groups. We will use the following version of Milnor \( K \)-groups for rings.
**Definition 7.1.** The $m$th Milnor $K$-group $K^M_m(A)$ of a ring $A$ is the quotient group of the group $(A^*)^m$ by the subgroup generated by all elements of type

$$a_1 \otimes \cdots \otimes a_i \otimes a \otimes (1-a) \otimes a_{i+3} \otimes \cdots \otimes a_m;$$

these are called *Steinberg relations*.

Notice that in the tensor (30), the multiples $a$ and $1-a$ come one after another. As usual, we denote the class in $K^M_m(A)$ of a tensor $a_1 \otimes \cdots \otimes a_m, a_i \in R^*$ by $\{a_1, \ldots, a_m\}$ and we call it a symbol.

Clearly, $K^M_m$ is a group functor (see §2). Let $\Omega^m$ denote the group functor that sends a ring $A$ to the group of $m$th absolute Kähler differentials $\Omega^m_A$. It is easily checked that there is a morphism of group functors
d

$$d \log: K^M_m \longrightarrow \Omega^m, \quad \{a_1, \ldots, a_m\} \longmapsto \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_m}{a_m}.$$  

We also have algebraic $K$-groups $K_m(A), m \geq 0$, which are functorial with respect to $A$. There are canonical decompositions of group functors

$$K_0 \simeq \mathbb{Z} \times \bar{K}_0, \quad K_1 \simeq \mathbb{G}_m \times SK_1.$$  

Let $\text{rk}: K_0 \to \mathbb{Z}$ and $\text{det}: K_1 \to \mathbb{G}_m$ be the corresponding projections. To be explicit, these are defined by taking the rank of a finitely generated projective module and the determinant of a matrix, respectively.

Loday [22] constructed a graded-commutative product between algebraic $K$-groups, which is functorial with respect to $A$:

$$K_i(A) \otimes \mathbb{Z} K_j(A) \longrightarrow K_{i+j}(A), \quad \alpha \otimes \beta \longmapsto \alpha \cdot \beta, \quad i, j \geq 0.$$  

The Loday product between the group subfunctors $\mathbb{G}_m \subset K_1$ factors through the Milnor $K$-groups (see, for instance, [23], §§1 and 2), that is, we have a morphism of group functors

$$K^M_m \longrightarrow K_m, \quad \{a_1, \ldots, a_m\} \longmapsto a_1 \cdots a_m,$$  

where $a_i \in A^*$ for a ring $A$.

**7.2. The tangent space to Milnor $K$-groups.** We shall state the main result in [14]. First we recall some concepts given there. Let $\varepsilon$ be a formal variable that satisfies $\varepsilon^2 = 0$ (see Definition 6.15 for the tangent space $TF$ to a group functor $F$).

**Example 7.2.** For any ring $A$, we have $d(\varepsilon^2) = 2\varepsilon d\varepsilon = 0$ in $\Omega^1_A[\varepsilon]$ and a calculation shows that there is an isomorphism of $A$-modules

$$T\Omega^{m+1}(A) \simeq (\varepsilon \Omega^m_A) \oplus (d\varepsilon \wedge \Omega^m_A) \oplus (\langle d\varepsilon \wedge \Omega^m_A \rangle/2(\varepsilon d\varepsilon \wedge \Omega^m_A)).$$

(31)

Following a construction of Bloch’s [15], we give the following definition.

**Definition 7.3.** Let

$$\text{Bl}: TK^M_{m+1} \longrightarrow \Omega^m$$

be the composition of the morphism of group functors $T(d \log): TK^M_{m+1} \to T\Omega^{m+1}$ and the projection to the direct summand $\Omega^m \simeq d\varepsilon \wedge \Omega^m$ in the decomposition (31).
For instance, given a collection of invertible elements $a_1, \ldots, a_m \in A^*$ and an element $b \in A$, the map $\text{Bl}$ sends the symbol $\{1 + b \varepsilon, a_1, \ldots, a_m\}$ to the differential form $b \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_m}{a_m}$.

It turns out that the homomorphism of groups $\text{Bl}: TK_M^{m+1}(A) \rightarrow \Omega^m_A$ is an isomorphism when a ring $A$ has sufficiently many invertible elements in the following sense (see Definition 3.1 from Morrow’s paper [24]).

**Definition 7.4.** Given a natural number $k \geq 2$, a ring $A$ is called **weakly $k$-fold stable** if for any collection of elements $a_1, \ldots, a_{k-1} \in A$ there is an invertible element $a \in A^*$ such that the elements $a_1 + a, \ldots, a_{k-1} + a$ are invertible in $A$.

**Remark 7.5.** For any ring $A$, the ring of Laurent series $\mathcal{L}(A) = A((t))$ is weakly $k$-fold stable for all $k \geq 2$. Actually, the invertible element $a$ in Definition 7.4 can be taken in the form $t^i$ for a suitable $i \in \mathbb{Z}$.

The next result is given in [14], Theorem 2.9 (see also Example 6.16 (ii) for the case $m = 0$).

**Theorem 7.6.** Let $A$ be a weakly 5-fold stable ring such that $1/2 \in A$. Then for any $m \geq 0$, the homomorphism $\text{Bl}: TK_M^{m+1}(A) \rightarrow \Omega^m_A$ is an isomorphism.

Combining Remark 7.5, Theorem 7.6, and Lemma 6.17 (i), we obtain the following isomorphism, which is crucial for our main results.

**Proposition 7.7.** There is an isomorphism of group functors

$$(TL^n K_M^{m+1})_{\mathbb{Z}[1/2]} \sim (L^n \Omega^m)_{\mathbb{Z}[1/2]}.$$ 

For any $\mathbb{Z}[1/2]$-algebra $A$, a collection $f_1, \ldots, f_m \in \mathcal{L}(A)^*$ of invertible iterated Laurent series, and an element $g \in \mathcal{L}^n(A)$, this isomorphism sends the symbol $\{1 + g \varepsilon, f_1, \ldots, f_m\}$ to the differential form $g \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_m}{f_m}$.

We will also use the following fact about Milnor $K$-groups, which is proved in [24], Lemma 3.6 using a method given in Nesterenko and Suslin, [25], Lemma 3.2 (see also Lemma 2.2 from the Kerz’s paper [26] and [14], Lemma 3.5).

**Lemma 7.8.** Let $A$ be a weakly 5-fold stable ring. Then for all elements $a, b \in A^*$, the following equalities hold in $K_2^M(A)$:

$$\{a, b\} = -\{b, a\} \quad \text{and} \quad \{a, a\} = \{-1, a\}.$$ 

### 7.3. A boundary map for algebraic $K$-groups.

Let $A$ be an arbitrary ring. We recall the construction of a boundary map (see below for the corresponding references)

$$\partial_{m+1}: K_{m+1}(A((t))) \longrightarrow K_m(A).$$

We show that the map $\partial_{m+1}$ is functorial with respect to $A$, that is, $\partial_{m+1}$ is a morphism of group functors from $L K_{m+1}$ to $K_m$ (see Definition 4.1). Explicitly, this means that for any homomorphism of rings $A \rightarrow B$, the following diagram is commutative:

$$
\begin{array}{ccc}
K_{m+1}(A((t))) & \rightarrow & K_m(A) \\
\downarrow & & \downarrow \\
K_{m+1}(B((t))) & \rightarrow & K_m(B)
\end{array}
$$
where we use the indices $A$ and $B$ to specify the maps $\partial_{m+1}$ for the corresponding rings.

Let $\mathcal{H}(A)$ be the exact category of $A[[t]]$-modules that admit a resolution of length one by finitely generated projective $A[[t]]$-modules and are annihilated by a power of the element $t$. A result of Gersten’s [8] implies the existence of a boundary map $\tilde{\partial}_{m+1}: K_{m+1}(A((t))) \to K_m(\mathcal{H}(A))$. Gersten’s result was later generalized by Grayson [9] (for a more detailed exposition see [23], Theorem 9.1). To show the functoriality of $\tilde{\partial}_{m+1}$ we use the generalization from [9], which we explain below.

**Remark 7.9.** A resolution of a module in $\mathcal{H}(A)$ is an example of a perfect complex of $R[[t]]$-modules with support on the closed subscheme $\text{Spec}(A) \subset \text{Spec}(A[[t]])$ given by the equation $t = 0$. Perfect complexes of this type are crucial in the work of Thomason-Trobaugh [27]. Results from the latter paper are used by Musicantov and Yom Din ([28], §4.1) to compare the boundary map for $K$-groups in low degrees with the valuation, the tame symbol, and the residue map when $A$ is a field.

We introduce some more notation. Given a ring $R$, let $\mathcal{P}(R)$ be the exact category of finitely generated projective $R$-modules and let $\mathcal{P}^1(R)$ be the exact category of $R$-modules that admit a resolution of length one by finitely generated projective $R$-modules. Let $\mathcal{V}(A)$ be the exact category of finitely generated projective $A((t))$-modules that are localizations of finitely generated projective $A[[t]]$-modules. Let $\mathcal{V}^1(A)$ be the exact category of $A((t))$-modules that admit a resolution of length one by $A((t))$-modules in $\mathcal{V}(A)$. By $\text{BQ}$ denote the classifying space of the $Q$-construction of an exact category.

By Quillen’s resolution theorem ([29], §4, Theorem 3, Corollary 1), the embedding of exact categories $\mathcal{V}(A) \to \mathcal{V}^1(A)$ induces a homotopy equivalence $\text{BQ}\mathcal{V}(A) \to \text{BQ}\mathcal{V}^1(A)$, so we obtain isomorphisms of $K$-groups

$$K_m(\mathcal{V}(A)) \simeq K_m(\mathcal{V}^1(A)).$$

Note that all exact sequences in the category $\mathcal{P}(A((t)))$ split. In addition, the subcategory $\mathcal{V}(A)$ is cofinal in $\mathcal{P}(A((t)))$. In other words, any module in $\mathcal{P}(A((t)))$ is a direct summand of a module in $\mathcal{V}(A)$, and also a direct summand of a finitely generated free $A((t))$-module. Therefore by [8], Propositions 1.1 and 1.3, the corresponding map $\text{BQ}\mathcal{V}(A) \to \text{BQ}\mathcal{P}(A((t)))$ is homotopy equivalent to a covering and we obtain isomorphisms of $K$-groups

$$K_m(\mathcal{V}(A)) \simeq K_m(A((t))), \quad m \geq 1.$$  

(The map $K_0(\mathcal{V}(A)) \to K_0(A((t)))$ is injective but not surjective in general.) Finally, consider the following diagram of exact categories:

$$\mathcal{H}(A) \longrightarrow \mathcal{P}^1(A[[t]]) \longrightarrow \mathcal{V}^1(A).$$

(32)

The point is that (32) induces a homotopy fibration

$$\text{BQ}\mathcal{H}(A) \longrightarrow \text{BQ}\mathcal{P}^1(A[[t]]) \longrightarrow \text{BQ}\mathcal{V}^1(A).$$

This statement is essentially the ‘Localization Theorem for projective modules’ in [9] and follows from the beginning and the end of the proof of that theorem.
Consequently we obtain a long exact sequence of homotopy groups and, in particular, we get a boundary map $K_{m+1}(\mathcal{V}^1(A)) \to K_m(\mathcal{H}(A))$. Applying the above isomorphisms between $K$-groups, we obtain the map

$$\tilde{\partial}_{m+1}: K_{m+1}(A((t))) \to K_m(\mathcal{H}(A)).$$

We need the following simple lemma.

**Lemma 7.10.** Given a homomorphism of rings $A \to B$, for any $A[[t]]$-module $M$ in $\mathcal{H}(A)$ we have $\text{Tor}_1^{A[[t]]}(M, B[[t]]) = 0$.

**Proof.** Let

$$0 \to P \xrightarrow{\alpha} Q \to M \to 0$$

be a resolution of $M$ by finitely generated projective $A[[t]]$-modules. We need to show that the map

$$\alpha' := \alpha \otimes_{A[[t]]} B[[t]]: P \otimes_{A[[t]]} B[[t]] \to Q \otimes_{A[[t]]} B[[t]]$$

is injective. Since $M$ is annihilated by a power of $t$, the map $\alpha \otimes_{A[[t]]} A((t))$ is an isomorphism, and so the map $\alpha \otimes_{A[[t]]} B((t)) = \alpha' \otimes_{B[[t]]} B((t))$ is an isomorphism as well. Therefore all elements in the $B[[t]]$-module $\text{Ker}(\alpha')$ are annihilated by some power of $t$. On the other hand, $\text{Ker}(\alpha')$ is a submodule of the projective $B[[t]]$-module $P \otimes_{A[[t]]} B[[t]]$ and $t$ is not a zero divisor in the ring $B[[t]]$. Therefore, $\text{Ker}(\alpha') = 0$.

**Proposition 7.11.** The map

$$\tilde{\partial}_{m+1}: K_{m+1}(A((t))) \to K_m(\mathcal{H}(A))$$

is functorial with respect to the ring $A$.

**Proof.** We use Kato’s idea from the proof of Lemma 2 in [10], §2.1. Observe that the second and third categories in diagram (32) are not functorial with respect to $A$. To overcome this, we define auxiliary exact subcategories. Let $\varphi: A \to B$ be a homomorphism of rings. Let $\mathcal{P}^1_A(A[[t]])$ be the exact category of $A[[t]]$-modules $M$ in $\mathcal{P}^1_A(A[[t]])$ such that $\text{Tor}_1^{A[[t]]}(M, B[[t]]) = 0$. Similarly, let $\mathcal{V}^1_A(A)$ be the exact category of $A((t))$-modules $N$ in $\mathcal{V}^1_A(A)$ such that $\text{Tor}_1^{\mathcal{V}^1_A((t))}(N, B((t))) = 0$. Lemma 7.10 asserts that $\mathcal{H}(A)$ is a subcategory in $\mathcal{P}^1_\varphi(A[[t]])$. Thus we have the following diagram of exact categories with arrows being exact functors:

$$\begin{array}{ccc}
\mathcal{H}(A) & \longrightarrow & \mathcal{P}^1_A(A[[t]]) & \longrightarrow & \mathcal{V}^1_A(A) \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{H}(A) & \longrightarrow & \mathcal{P}^1_\varphi(A[[t]]) & \longrightarrow & \mathcal{V}^1_\varphi(A) \\
\uparrow & & \downarrow & & \downarrow \\
\mathcal{H}(B) & \longrightarrow & \mathcal{P}^1(B[[t]]) & \longrightarrow & \mathcal{V}^1(B)
\end{array}$$

By Quillen’s resolution theorem, the up going arrows in this diagram induce homotopy equivalences between the corresponding BQ-spaces. Therefore by the functoriality of the boundary map in a long exact sequence of homotopy groups, we obtain functoriality for the map $\tilde{\partial}_{m+1}$. 
Now we construct a map from $K_m(\mathcal{H}(A))$ to $K_m(A)$. Osipov and Zhu proved in [6], Proposition 7.1 that any $A[[t]]$-module in the category $\mathcal{H}(A)$ is a finitely generated projective $A$-module and $f$ is a nilpotent endomorphism of $M$. Now, if an $A[[t]]$-module $M$ is in the category $\mathcal{H}(A)$, then, as mentioned above, $M$ is a finitely generated projective $A$-module and, by definition, $t$ acts on $M$ as a nilpotent endomorphism. Conversely, for any pair $(M, f)$ as above, we have an exact sequence of $A[[t]]$-modules

$$0 \longrightarrow M[[t]] \xrightarrow{\alpha} M[[t]] \xrightarrow{\beta} M_f \longrightarrow 0,$$

where $M_f$ is an $A[[t]]$-module $M$ with $t$ acting as $f$ and we put

$$\alpha\left(\sum_{l \geq 0} a_l t^l\right) := \sum_{l \geq 0} (a_{l-1} - f(a_l)) t^l, \quad \beta\left(\sum_{l \geq 0} b_l t^l\right) := \sum_{l \geq 0} f^l(b_l),$$

where we set $a_{-1} := 0$. Now, if $\beta(\sum_{l \geq 0} b_l t^l) = 0$, then $\sum_{l \geq 0} b_l t^l = \alpha(\sum_{l \geq 0} a_l t^l)$, where $a_l = \sum_{j \geq 0} f^j(b_{l+j+1})$ for any integer $l \geq 0$. Since $f$ is nilpotent, the $A[[t]]$-module $M_f$ is in $\mathcal{H}(A)$.

We need the following simple lemma.

**Lemma 7.13.** Let $A \rightarrow B$ be a homomorphism of rings and let $M$ be a module over $A[[t]]$ such that $t^l \cdot M = 0$ for some integer $l \geq 0$. Then the natural homomorphism of $B$-modules $M \otimes_A B \rightarrow M \otimes_A[[t]] B[[t]]$ is an isomorphism.

**Proof.** Define an $A[[t]]$-module $T := A[[t]]/t^l A[[t]]$. Then the lemma holds for $T$, that is, the natural homomorphism of $B$-modules

$$T \otimes_A B \longrightarrow T \otimes_A[[t]] B[[t]]$$

is an isomorphism. In fact, both $B$-modules are isomorphic to the $B$-module $B[[t]]/t^l B[[t]]$. To prove the lemma in the general case, we take the composition of the following isomorphisms of $B$-modules:

$$M \otimes_A B \xrightarrow{\sim} (M \otimes_A[[t]] T) \otimes_A B \xrightarrow{\sim} M \otimes_A[[t]] (T \otimes_A B) \xrightarrow{\sim} M \otimes_A[[t]] (T \otimes_A[[t]] B[[t]]) \xrightarrow{\sim} M \otimes_A[[t]] (M \otimes_A[[t]] T) \otimes_A[[t]] B[[t]] \xrightarrow{\sim} M \otimes_A[[t]] B[[t]].$$

Lemma 7.13 implies directly that the map $I_m$ is functorial with respect to $A$. Finally, define a boundary map for algebraic $K$-groups by the formula

$$\partial_{m+1} := I_m \circ \tilde{\partial}_{m+1} : K_m(A((t))) \longrightarrow K_m(A).$$

(33)

Combining the functoriality of $I_m$ with the functoriality of $\tilde{\partial}_{m+1}$ given by Proposition 7.11, we obtain the following important result.
Proposition 7.14. The boundary map $\partial_{m+1}$ is a morphism of group functors from $LK_{m+1}$ to $K_m$.

We will also need the following special property of the map $\partial_{m+1}$. Let $\lambda_m: K_m(A[[t]]) \to K_m(A((t)))$ denote the map induced by the homomorphism of rings $A[[t]] \to A((t))$. Denote the map $K_m(A[[t]]) \to K_m(A)$ induced by the homomorphism of rings $A[[t]] \to A[[t]]/(t) \simeq A$ by $\alpha \mapsto \bar{\alpha}$. It follows from [10], §2.4, Proposition 5 that for any element $\alpha \in K_m(A[[t]])$ the equality

$$\partial_{m+1}(\lambda_m(\alpha) \cdot t) = \bar{\alpha} \quad (34)$$

holds in $K_m(A)$, where we consider $t$ as an element in $K_1(A((t)))$. In particular, we have that $\partial_1(t) = 1$, where we consider 1 as an element in $K_0(A)$.

§8. The main results

For brevity, we will use the following expression: given two sets $P$ and $Q$, and a map $\alpha: P \to Q$, we say that ‘an element $p \in P$ is uniquely determined by the element $\alpha(p) \in Q$’ if $\alpha^{-1}(\alpha(p)) = \{p\}$. Thus the map $\alpha$ is injective if and only if this condition holds for any element $p \in P$. In the situations that follow, the sets $P$, $Q$ and the map $\alpha$ will be clear from the context and they will not be mentioned explicitly.

8.1. An additive symbol. First we discuss a simpler version of what we shall do later (in §§8.2–8.4) for the Contou-Carrère symbol. The general plan in this simpler case is very similar to that in the case of the Contou-Carrère symbol.

Definition 8.1. An additive symbol $\nu_n$ is the following composition of morphisms of group functors (see formula (33) and Proposition 7.14):

$$\nu_n: L^n K_n^M \longrightarrow L^n K_n \xrightarrow{\partial_1 \cdots \partial_n} K_0 \xrightarrow{\text{rk}} \mathbb{Z}.$$ 

It turns out that all morphisms of group functors from $L^n K_n^M$ to $\mathbb{Z}$ can be described explicitly as follows. The symbol $\{t_1, \ldots, t_n\}$ defines a morphism of group functors

$$\Xi: \mathbb{Z} \longrightarrow L^n K_n^M, \quad i \longmapsto i: \{t_1, \ldots, t_n\}.$$ 

Note that for any ring $R$ there is an isomorphism of groups $\text{Hom}_{R}^{gr}(\mathbb{Z}_R; \mathbb{Z}_R) \simeq \mathbb{Z}(R)$.

Proposition 8.2. For any ring $R$, the natural homomorphism

$$\Xi_R: \text{Hom}_{R}^{gr}((L^n K_n^M)_R; \mathbb{Z}_R) \longrightarrow \mathbb{Z}(R), \quad \Phi \longmapsto \Phi \circ \Xi_R,$$

is an isomorphism.

Before we give a proof of Proposition 8.2, we shall discuss some of its corollaries. To be precise, Proposition 8.2 means the following: suppose that the values of two morphisms of group functors $\Phi, \Phi': (L^n K_n^M)_R \to \mathbb{Z}_R$ coincide at the symbol $\{t_1, \ldots, t_n\}$. Then we have that $\Phi = \Phi'$.

By formula (34), for any ring $A$, there is an equality $\nu_n\{t_1, \ldots, t_n\} = 1$ in $\mathbb{Z}(A)$. Hence there is an equality

$$\Xi^*(\nu_n) = 1 \in \mathbb{Z}(\mathbb{Z}) = \mathbb{Z}. \quad (35)$$
Thus Proposition 8.2 implies the following universal property of the additive symbol \( \nu_n \): for any morphism of group functors \( \Phi: (L^n K^M_n)_R \to \mathbb{Z}_R \), there is a locally constant integer \( i \in \mathbb{Z}(R) \) such that \( \Phi = i \cdot \nu_n \).

Now we will give a proof of Proposition 8.2 based on the absolute connectedness of \((L^n \mathbb{G}_m)^0\) over \( \mathbb{Z} \) (see Proposition 6.13 (iv)).

**Proof of Proposition 8.2.** Formula (35) implies that the homomorphism \( \mathbb{Z}_R^+ \) is surjective. Thus we need to show that \( \mathbb{Z}_R^+ \) is injective. In other words, given a morphism of group functors \( \Phi: (L^n K^M_n)_R \to \mathbb{Z}_R \), we need to show that \( \Phi \) is uniquely determined by the composition \( \Phi \circ \mathbb{Z}_R^+ \).

Consider the following composition of morphisms of functors:

\[
\hat{\Phi}: (L^n \mathbb{G}_m)^0_R \times \mathbb{Z}_R^n \to (L^n K^M_n)_R \to \mathbb{Z}_R
\]

where the first morphism sends a collection of invertible iterated Laurent series \((f_1, \ldots, f_n)\) to the symbol \(\{f_1, \ldots, f_n\}\). Clearly, \( \hat{\Phi} \) is uniquely determined by \( \Phi \).

Actually, \( \hat{\Phi} \) is a morphism of ind-schemes over \( R \). For any \( R \)-algebra \( A \), a collection \( f_2, \ldots, f_n \in (L^n \mathbb{G}_m(A))^0 \) defines a morphism of ind-schemes over \( A \),

\[
\alpha: (L^n \mathbb{G}_m)_A^0 \to \mathbb{Z}_A, \quad f \mapsto \hat{\Phi}(f, f_2, \ldots, f_n).
\]

By Proposition 6.13 (iv), the ind-scheme \((L^n \mathbb{G}_m)^0\) is absolutely connected over \( \mathbb{Z} \). Thus by Proposition 5.23, the morphism \( \alpha \) is equal to zero. Hence the restriction of \( \hat{\Phi} \) to the ind-closed subscheme \((L^n \mathbb{G}_m)^0_R \times (L^n \mathbb{G}_m)^{\times (n-1)}_R \) is also equal to zero.

By Remark 7.5 and Lemma 7.8, the symbols in

\[
L^n K^M_n(A) = K^M_n(\mathcal{L}^n(A))
\]

are antisymmetric. Thus we see that \( \hat{\Phi} \) factors through the composition of morphisms

\[
(L^n \mathbb{G}_m)_R^0 \overset{\nu^n}{\longrightarrow} (L^n \mathbb{Z}_R)_n \overset{\det}{\longrightarrow} \mathbb{Z}_R
\]

because by Definition 4.6, there is an isomorphism \( \nu: L^n \mathbb{G}_m/(L^n \mathbb{G}_m)^0 \sim \mathbb{Z}^n \). Here for any ring \( A \), we consider elements in \( \mathbb{Z}_A^n \) as columns with (locally constant) integral entries and elements in \( \mathbb{Z}_A^n \) as \((n \times n)\)-matrices with (locally constant) integral entries.

We find that \( \hat{\Phi} \) is uniquely determined by its value on the collection \((t_1, \ldots, t_n)\), because this collection is sent to \( 1 \in \mathbb{Z}_R^+(R) \) under the above composition of morphisms (36). This proves the proposition.

Note that in the proof of Proposition 8.2 we have also obtained an explicit formula for the additive symbol \( \nu_n \). Namely, the equality

\[
\nu_n\{f_1, \ldots, f_n\} = \det(\nu(f_1), \ldots, \nu(f_n))
\]

holds for any ring \( A \) and collection \( f_1, \ldots, f_n \in L^n \mathbb{G}_m(A) \) of invertible iterated Laurent series. In fact, the proof of Proposition 8.2 implies that the generator of the cyclic group \( \text{Hom}^R(\mathcal{L}^n K^M_n)_R, \mathbb{Z}_R) \) is given by the right-hand side of (37). On the other hand, as we observed before the proof, this generator is \( \nu_n \).
Actually, we could define the morphism of group functors $\nu_n : L^n K^M_n \to \mathbb{Z}$ explicitly just by using the right-hand side of formula (37), without using the boundary map for $K$-groups and the connectedness of $(L^n \mathbb{G}_m)^0$. To follow this approach, we would have to check the Steinberg relations directly. This is done explicitly as follows.

**Lemma 8.3.** Let $A$ be a ring and let $f_1, \ldots, f_n \in L^n \mathbb{G}_m(A)$ be a collection of invertible iterated Laurent series such that $f_1 + f_2 = 1$. Then $\det(\nu(f_1), \ldots, \nu(f_n)) = 0$.

**Proof.** Changing the ring $A$ if necessary, without loss of generality, we can assume that $\nu(f_1)$ and $\nu(1 - f_1)$ belong to the subgroup $\mathbb{Z}^n \subset \mathbb{Z}(A)^n$. Further, we can assume that $\nu(f_1) \neq 0$ (otherwise, there is nothing to prove). If $\nu(f_1) > 0$, then $\nu(1 - f_1) = 0$ and if $\nu(f_1) < 0$, then $\nu(f_1) = \nu(1 - f_1)$ (see Proposition 4.3 and Example 4.5(i)). This proves the lemma.

We mention one more property of the right-hand side of (37), which we will use later. We have a natural morphism of group functors $\zeta : \mathbb{Z} \to \mathbb{G}_a$. Note that if a ring $A$ has positive characteristic, then the map $\zeta : \mathbb{Z}(A) \to A$ is not injective.

**Proposition 8.4.** For any ring $A$ and a collection of invertible iterated Laurent series $f_1, \ldots, f_n \in L^n \mathbb{G}_m(A)$, the equality

$$\zeta(\det(\nu(f_1), \ldots, \nu(f_n))) = \text{res} \left( \frac{df_1}{f_1} \land \cdots \land \frac{df_n}{f_n} \right)$$

holds in $A$.

**Proof.** Both sides of (38) define morphisms of ind-schemes from $(L^n \mathbb{G}_m)^{\times n}$ to $\mathbb{G}_a$. Therefore by Theorem 6.12 applied to $R = \mathbb{Z}$ and $S = \mathbb{Q}$, it is enough to prove the proposition when $A$ is a $\mathbb{Q}$-algebra, which we will assume from now on.

Both sides of (38) are multilinear alternating maps. Therefore we can assume that the $f_i$ belong to the subgroups of $L^n \mathbb{G}_m(A)$ in the decomposition (22) in §4.3. The case $f_1 \in \mathbb{G}_m(A)$ is trivial. Suppose that $f_1 \in (L^n \mathbb{G}_m)^2(A)$. Then by Definition 4.6, we have $\nu(f_1) = 0$ and thus, $\zeta(\det(\nu(f_1), \ldots, \nu(f_n))) = 0$. On the other hand, using Lemmas 3.11(i) and 3.12 we obtain the equalities

$$\text{res} \left( \frac{df_1}{f_1} \land \cdots \land \frac{df_n}{f_n} \right) = \text{res} \left( d \left( \log(f_1) \frac{df_2}{f_2} \land \cdots \land \frac{df_n}{f_n} \right) \right) = 0.$$

It remains to consider the case when all the $f_i$ lie in the subgroup $\mathbb{Z}^n(A) \subset L^n \mathbb{G}_m(A)$. By multilinearity and the alternating property, it is enough to consider the case $f_1 = t_1, \ldots, f_n = t_n$. Clearly, we have the equalities

$$\zeta(\det(\nu(t_1), \ldots, \nu(t_n))) = 1 = \text{res} \left( \frac{dt_1}{t_1} \land \cdots \land \frac{dt_n}{t_n} \right),$$

which completes the proof.

**Remark 8.5.** When $n = 2$ and $A$ is a $\mathbb{Q}$-algebra, Proposition 8.4 coincides with [6], Lemma 4.1.
8.2. The Contou-Carrère symbol. We now define the main object of our study.

Definition 8.6. The Contou-Carrère symbol $CC_n$ is the following composition of morphisms of group functors (see formula (33) and Proposition 7.14):

$$CC_n : L^n K^M_{n+1} \longrightarrow L^n K_{n+1} \frac{\partial_2 \cdots \partial_{n+1}}{K_1} \longrightarrow \frac{\det}{G_m}. \quad (39)$$

Thus $CC_n$ is a character of the group functor $L^n K^M_{n+1}$. In particular, for any ring $A$, we have a homomorphism of groups, which we denote similarly

$$CC_n : K^M_{n+1}(A((t_1)) \cdots ((t_n))) \longrightarrow A^*.$$ 

It turns out that all characters of $L^n K^M_{n+1}$ can be described explicitly as follows. Define a morphism of group functors

$$\Theta : G_m \longrightarrow L^n K^M_{n+1}, \quad a \longmapsto \{a, t_1, \ldots, t_n\},$$

where $a \in A^*$ for a ring $A$. Recall that for any ring $R$, there is an isomorphism of groups $X((G_m)_R) \simeq \mathbb{Z}(R)$ (see Definition 6.19 and Lemma 6.20(ii)).

A ring $R$ is said to be torsion free over $\mathbb{Z}$ if $R$ has no torsion as a $\mathbb{Z}$-module or, equivalently, if the natural homomorphism of rings $R \rightarrow R \otimes_\mathbb{Z} \mathbb{Q}$ is injective. In particular, if this holds, then the natural homomorphism of rings $\mathbb{Z} \rightarrow R$ is also injective (however, the latter condition is not equivalent to being torsion free over $\mathbb{Z}$). Note that $R$ is torsion free over $\mathbb{Z}$ if and only if $R$ is flat over $\mathbb{Z}$.

Theorem 8.7. Suppose that a ring $R$ is torsion free over $\mathbb{Z}$. Then the natural homomorphism of groups

$$\Theta^*_R : X((L^n K^M_{n+1})_R) \longrightarrow \mathbb{Z}(R), \quad \Phi \longmapsto \Phi \circ \Theta_R,$$

is an isomorphism.

Theorem 8.7 is proved in §8.3 below. Combining Theorem 8.7 with the theory of thick ind-cones (Theorem 6.12), we obtain the following explicit result.

Corollary 8.8. Let

$${\Psi, \Psi'} : (L^n G_m)^{(n+1)}_R \longrightarrow (G_m)_R$$

be multilinear morphisms of functors over a ring $R$. Suppose that the following conditions are satisfied:

(i) there is an $R$-algebra $A_0$ and an element $a_0 \in A_0^*$ such that $a_0$ is not a root of the unit and

$${\Psi}\{a_0, t_1, \ldots, t_n\} = {\Psi'}\{a_0, t_1, \ldots, t_n\};$$

(ii) there is an embedding of rings $R \subset S$ such that $S$ is torsion free over $\mathbb{Z}$ and the morphisms of functors $\Psi_S$ and $\Psi'_S$ satisfy the Steinberg relations, that is, $\Psi_S$ and $\Psi'_S$ are compositions of the natural multilinear morphism of functors $(L^n G_m)^{(n+1)}_S \rightarrow (L^n K^M_{n+1})_S$ with the morphisms of group functors $\Phi_S, \Phi'_S : (L^n K^M_{n+1})_S \rightarrow (G_m)_S$.

Then $\Psi = \Psi'$. 
Proof. Define a morphism of group functors

\[ \Gamma: \mathbb{G}_m \longrightarrow (L^n\mathbb{G}_m)^{\times(n+1)}, \quad a \mapsto (a, t_1, \ldots, t_n), \]

where \( a \in A^* \) for a ring \( A \). Condition (i) implies there is equality between the endomorphisms of \( (\mathbb{G}_m)_R \), namely \( \Psi \circ \Gamma_R = \Psi'_R \circ \Gamma_R \). Using condition (ii) and taking the extension of scalars from \( R \) to \( S \), we obtain the equality

\[ \Phi_S \circ \Theta_S = \Phi'_S \circ \Theta_S. \]

By Theorem 8.7 applied over the ring \( S \), we see that \( \Phi_S = \Phi'_S \). This implies that \( \Psi_S = \Psi'_S \). Finally, by Theorem 6.12, we get the required equality \( \Psi = \Psi' \).

Remark 8.9. By formula (34) in §7.3, for any ring \( A \) and element \( a \in A^* \), the equality \( \text{CC}_n\{a, t_1, \ldots, t_n\} = a \) holds in \( A^* \). Hence

\[ \Theta^*(\text{CC}_n) = 1 \in \mathbb{Z}^*(\mathbb{Z}) = \mathbb{Z}. \]

Theorem 8.7, together with Remark 8.9 imply the following universal property of the Contou-Carrère symbol.

**Theorem 8.10.** Suppose that a ring \( R \) is torsion free over \( \mathbb{Z} \). Then for any morphism of group functors \( \Phi: (L^nK^M_{n+1})_R \longrightarrow (\mathbb{G}_m)_R \), there is a locally constant integer \( i \in \mathbb{Z}(R) \) such that \( \Phi = (\text{CC}_n)^i \).

Next, we comment on the other definitions of the Contou-Carrère symbol. For simplicity, below we consider various multilinear morphisms to \( \mathbb{G}_m \) up to taking the inverse.

Remark 8.11. (i) The classical one-dimensional symbol was constructed by Contou-Carrère himself in [1], as a bilinear morphism of functors \( (L\mathbb{G}_m)^{\times 2} \rightarrow \mathbb{G}_m \) which was not defined explicitly. An explicit formula for the one-dimensional symbol over \( \mathbb{Q} \)-algebras was given in [1]. Another, equivalent, explicit formula over \( \mathbb{Q} \)-algebras was given by Deligne [4], §2.9. Beilinson, Bloch and Esnault constructed a bilinear morphism of functors \( (L\mathbb{G}_m)^{\times 2} \rightarrow \mathbb{G}_m \) in [31], §3.1, using the commutator in a super central extension of the group functor \( L\mathbb{G}_m \). It was shown in [31], Proposition 3.3 that this morphism coincides with the Contou-Carrère symbol. Independently of [31], Anderson and Pablos Romo [7] studied a particular case for Laurent series over Artinian rings in a similar way. It was shown in [6], Theorem 7.2 that the one-dimensional Contou-Carrère symbol is equal to the symbol in Definition 8.6 with \( n = 1 \). This fact can be proved differently using Corollary 8.8 and Remark 8.9. With this approach, one also needs to check that the classical one-dimensional Contou-Carrère symbol satisfies the Steinberg relations for Laurent series over \( \mathbb{Q} \)-algebras. This can be done explicitly as in the proof of Proposition 8.25 below.

(ii) In [6], Definition 3.4 a multilinear morphism of functors \( (L^2\mathbb{G}_m)^{\times 3} \rightarrow (\mathbb{G}_m)_\mathbb{Q} \) on the category of \( \mathbb{Q} \)-algebras was defined using an explicit formula. In addition, in [6], §5.3 a multilinear morphism of functors \( (L^2\mathbb{G}_m)^{\times 3} \rightarrow \mathbb{G}_m \) was defined using a commutator in a central extension of \( L^2\mathbb{G}_m \) by a Picard category. The equality between this morphism and the above morphism over \( \mathbb{Q} \)-algebras was proved in [6],
Theorem 5.9. The equality between these morphisms and the symbol from Definition 8.6 with \( n = 2 \) was proved in [6], Theorem 7.2. Similarly to (i), this equality can be also proved using Corollary 8.8, Remark 8.9 and the Steinberg property from Proposition 8.25 below.

(iii) In [32], Definition 4.9, Braunling, Groechenig, and Wolfson constructed a multilinear morphism of functors
\[
(\cdot, \ldots, \cdot) : (L_n^m)^{(n+1)} \to \mathbb{G}_m
\]
using a commutator in a spectral extension of \( L_n^m \) from their previous work [33]. It follows from this construction that the morphism \((\cdot, \ldots, \cdot)\) satisfies the Steinberg relations. Thus Corollary 8.8 and Remark 8.9 imply that \((\cdot, \ldots, \cdot)\) is equal to the higher Contou-Carrère symbol \( CC_n \) as defined in this paper (see Definition 8.6).

8.3. The proof of Theorem 8.7. In this subsection we prove Theorem 8.7. We will use several auxiliary results. In what follows \( R \) is a ring and
\[
\Phi : (L_n^K_{n+1})_R \longrightarrow (\mathbb{G}_m)_R
\]
is a character. Define the composition of morphisms of functors
\[
\tilde{\Phi} : (L_n^m)^{(n+1)}_R \longrightarrow (L_n^K_{n+1})_R \xrightarrow{\Phi} (\mathbb{G}_m)_R,
\]
where the first morphism sends a collection of invertible iterated Laurent series \((f_1, \ldots, f_{n+1})\) to the symbol \(\{f_1, \ldots, f_{n+1}\}\). In fact, \(\Phi\) is a morphism of ind-affine schemes.

The following result is based on the theory of thick ind-cones (see Proposition 5.17 and Theorem 6.12).

Lemma 8.12. Suppose that a ring \( R \) is torsion free over \( \mathbb{Z} \). Then any character \( \Phi \), as given above, is uniquely determined by the morphism of group functors \( \Phi_S : (L_n^K_{n+1})_S \rightarrow (\mathbb{G}_m)_S \) over \( S \), where \( S := R \otimes \mathbb{Z} \mathbb{Q} \).

Proof. Clearly, \( \Phi \) is uniquely determined by \( \tilde{\Phi} \). Since \( R \) is torsion free over \( \mathbb{Z} \), the natural homomorphism of rings \( R \rightarrow S \) is injective. Therefore, by Theorem 6.12 the morphism of functors \( \tilde{\Phi} \) is uniquely determined by the morphism of functors \( \Phi_S \), and so \( \Phi \) is uniquely determined by \( \Phi_S \).

The following result is based on the antisymmetric property of symbols in Milnor \( K \)-groups of rings of iterated Laurent series (see Remark 7.5 and Lemma 7.8).

Lemma 8.13. Given any ring \( R \), any character \( \Phi \) as above is uniquely determined by the restriction of \( \tilde{\Phi} \) to the ind-closed subscheme
\[
(L_n^m)^0_R \times (L_n^m)^n_R \subset (L_n^m)^{(n+1)}_R.
\]

Proof. By Remark 7.5 and Lemma 7.8 the symbols in \( L_n^K_m(A) = K_m^M(\mathcal{L}^n(A)) \), \( m \geq 2 \), are antisymmetric. Thus the decomposition (22) in §4.3 implies that \( \Phi \) is uniquely determined by the restrictions of \( \tilde{\Phi} \) to the ind-closed subschemes \((L_n^m)^0_R \times (L_n^m)^n_R\) and \((\mathbb{Z}^n)^{(n+1)}_R\) of \((L_n^m)^{(n+1)}_R\).
By multilinearity, we see that the restriction of $\tilde{\Phi}$ to $(\mathbb{Z}^n_R)^{\times(n+1)}$ is uniquely determined by the values of $\Phi$ on the symbols $\{f_1, \ldots, f_{n+1}\}$, where each $f_i$, $1 \leq i \leq n+1$, is equal to some $t_j$, $1 \leq j \leq n$. By Dirichlet’s box principle we may assume that $f_1 = f_2$. Using Remark 7.5 and Lemma 7.8 again, for any element $f \in L^n G_m(A)$ we have $\{f, f\} = \{-1, f\}$ in $K^M_2(\mathcal{L}^n(A))$. Therefore, $\{f_1, f_2, \ldots, f_{n+1}\} = \{-1, f_2, \ldots, f_{n+1}\}$. Since $-1 \in G_m(A) \subset (L^n G_m)^0(A)$, this proves the lemma.

The following result is based on the absolute connectedness of $(L^n G_m)^0$ over $\mathbb{Z}$ (see Proposition 6.13 (iv)).

**Lemma 8.14.** For any ring $R$ and any character $\Phi$ as above, the restriction of $\tilde{\Phi}$ to the ind-closed subscheme $(G_m)_R \times ((L^n G_m)^0_R)^{\times n} \subset (L^n G_m)^{\times(n+1)}_R$ is a constant morphism whose value is equal to $1 \in G_m(R)$.

**Proof.** By multilinearity, the restriction of $\tilde{\Phi}$ to the ind-closed subscheme $(G_m)_R \times ((L^n G_m)^0_R)^{\times n}$ defines a morphism of group ind-schemes

$$\alpha : ((L^n G_m)^0_R)^{\times n} \longrightarrow \text{Hom}^\text{gr}_R((G_m)_R, (G_m)_R) \simeq \mathbb{Z}_R,$$

where $\text{Hom}^\text{gr}$ is the internal Hom-functor for group functors (see §2 and Lemma 6.20 (ii)). By Proposition 6.13 (iv), $(L^n G_m)^0_R$ is a connected ind-scheme over $R$. Thus by Proposition 5.23 we have $\alpha = 0$. This proves the lemma.

Now we are in a position to prove Theorem 8.7.

**Proof of Theorem 8.7.**

*Step 1.* The existence of a functorial boundary map for algebraic $K$-groups (see §7.3) implies that the map $\Theta^*_R$ is surjective. More precisely, this follows from Remark 8.9. Thus we need to show that the map $\Theta^*_R$ is injective, that is, that a morphism of group functors $\Phi : (L^n K^M_{n+1})_R \rightarrow (G_m)_R$ is uniquely determined by the composition $\Phi \circ \Theta_R$.

*Step 2.* By Lemma 8.12, we can assume that $R$ is a $\mathbb{Q}$-algebra, and we will from now on.

Let $A$ be an $R$-algebra and let $f_2, \ldots, f_{n+1} \in L^n G_m(A)$ be a collection of $n$ invertible iterated Laurent series. We obtain a character

$$\chi : (L^n G_m)^0_A \longrightarrow (G_m)_A, \quad f \longmapsto \Phi\{f, f_2, \ldots, f_{n+1}\}.$$ 

By Lemma 8.13, $\Phi$ is uniquely determined by all the characters $\chi$ of this kind.

Since $R$ is a $\mathbb{Q}$-algebra, $A$ is too. Thus a property of the exponential map from Proposition 6.23 implies that each character $\chi$ as above is uniquely determined by its differential $T\chi$. This implies that $\Phi$ is uniquely determined by its differential $T\Phi : (T L^n K^M_{n+1})_R \rightarrow T(G_m)_R$.

Now we use the description of the tangent space to Milnor $K$-groups in terms of differential forms (see §7.2). Let

$$\Psi : (L^n \Omega^n)_R \longrightarrow (G_a)_R$$
be the morphism of group functors that corresponds to the differential $T\Phi$ under the isomorphism from Proposition 7.7 and the isomorphism from Example 6.16(ii) (here we have used the fact that $2$ is invertible in the $\mathbb{Q}$-algebra $R$). From what we said above we see that $\Phi$ is uniquely determined by $\Psi$.

**Step 3.** Using Lemma 8.14, in this step we will show that $\Psi$ equals zero on exact differential forms, that is, that the following morphism of ind-affine schemes

$$\psi: (L^n \mathbb{G}_a)^x_n \longrightarrow (\mathbb{G}_a)_R, \quad (g_1, \ldots, g_n) \longmapsto \Psi(dg_1 \wedge \cdots \wedge dg_n),$$

is equal to zero.

By definition $\psi$ is a regular function in $O((L^n \mathbb{G}_a)^x_n)$. By Proposition 6.13(i), the function $\psi$ is uniquely determined by its restriction to the dense ind-closed subscheme $(L^n \mathbb{G}_a)^x_n \subset (L^n \mathbb{G}_a)^x_n$. Thus it is enough to show that for any $R$-algebra $A$ and elements $g_1, \ldots, g_n \in (L^n \mathbb{G}_a)^x_n(A)$, we have $\Psi(dg_1 \wedge \cdots \wedge dg_n) = 0$.

By Lemma 3.11(i) and Proposition 4.9, for any element $g \in (L^n \mathbb{G}_a)^x(A)$, we have $d^{\exp(g)} = dg$. Therefore the isomorphism from Proposition 7.7 sends the symbol $\{1 + \varepsilon, \exp(g_1), \ldots, \exp(g_n)\}$ to the differential form $dg_1 \wedge \cdots \wedge dg_n$, where $\varepsilon^2 = 0$. Together with Example 6.16(ii), this implies that in the ring $A[\varepsilon]^n$

$$1 + \Psi(dg_1 \wedge \cdots \wedge dg_n) \varepsilon = \Phi\{1 + \varepsilon, \exp(g_1), \ldots, \exp(g_n)\}.$$

Since $1 + \varepsilon \in \mathbb{G}_m(A[\varepsilon])$ and $\exp(g_i) \in (L^n \mathbb{G}_m)^x(A) \subset (L^n \mathbb{G}_m)^y(A[\varepsilon])$ for any $i$, $1 \leq i \leq n$, applying Lemma 8.14, we obtain $\Psi(dg_1 \wedge \cdots \wedge dg_n) = 0$.

**Step 4.** By Proposition 6.25, $\Psi$ factors uniquely through the natural morphism of group functors $(L^n \Omega^n)_R \rightarrow (L^n \Omega^n)_R$. Combining Step 3 with Lemma 3.12, we see that the morphism of group functors $\Psi: (L^n \Omega^n)_R \rightarrow (\mathbb{G}_a)_R$ factors through the residue map $\text{res}: (L^n \Omega^n)_R \rightarrow (\mathbb{G}_a)_R$. In particular, $\Psi$ is uniquely determined by the composition

$$\lambda: (\mathbb{G}_a)_R \longrightarrow (L^n \Omega^n)_R \longrightarrow \Psi(\mathbb{G}_a)_R,$$

where the first morphism takes an element $a \in A$ to the differential form $a d\frac{t_1}{t_n} \wedge \cdots \wedge d\frac{t_i}{t_n}$ for an $R$-algebra $A$.

The isomorphism from Proposition 7.7 takes the symbol $\{1 + a \varepsilon, t_1, \ldots, t_n\}$ to the differential form $a d\frac{t_1}{t_n} \wedge \cdots \wedge d\frac{t_i}{t_n}$. This implies that $\lambda = T(\Phi \circ \Theta_R)$, which completes the proof of the theorem.

### 8.4. An explicit formula

In essence, in the course of the proof of Theorem 8.7 we have obtained an explicit formula for the Contou-Carrère symbol $\text{CC}_n: K_{n+1}^M(\mathcal{L}^n(A)) \rightarrow A^*$ when $A$ is a $\mathbb{Q}$-algebra. To give this explicit formula, we introduce the following auxiliary map.

**Proposition 8.15.** There is a unique multilinear antisymmetric (or, equivalently, symmetric) map

$$\text{sgn}: (\mathbb{Z}^n)^{\times(n+1)} \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

such that for any $l \in \mathbb{Z}^n$ and $r \in (\mathbb{Z}^n)^{\times(n-1)}$, we have $\text{sgn}(l, l, r) \equiv \det(l, r) \pmod{2}$.

**Proof.** The above properties define the values of the map $\text{sgn}$ uniquely on the elements of the standard basis in $(\mathbb{Z}^n)^{\otimes(n+1)}$, by Dirichlet’s box principle. This means
the map \( \text{sgn} \) is unique. Existence follows from the following explicit formula due to Vostokov and Fesenko, which comes from [34], Introduction:

\[
\text{sgn}(l_1, \ldots, l_{n+1}) \equiv \sum_{1 \leq i < j \leq n+1} \det(l_1, \ldots, \hat{l}_i, \ldots, \hat{l}_j, \ldots, l_{n+1}, l_i \cdot l_j) \pmod{2}, \tag{40}
\]

where \( l_i \cdot l_j \) is the coordinate-wise product of the elements \( l_i \) and \( l_j \) in \( \mathbb{Z}^n \) and the hat symbol corresponds to omitting an element. One easily checks that this map satisfies all the conditions of the proposition.

Remark 8.16. Another explicit formula for the map \( \text{sgn} \) is given by Khovanskii [35] (see also [36], Proposition 11 for the case \( n = 2 \)):

\[
\text{sgn}(l_1, \ldots, l_{n+1}) \equiv 1 + \sum_{i=1}^{n+1} \det(l_1, \ldots, \hat{l}_i, \ldots, l_{n+1}) + \prod_{i=1}^{n+1} (1 + \det(l_1, \ldots, \hat{l}_i, \ldots, l_{n+1})) \pmod{2}.
\]

Equality between this formula and (40) is proved in [35], §1.

Now we give an explicit formula for the Contou-Carrère symbol over \( \mathbb{Q} \)-algebras.

**Theorem 8.17.** Let \( A \) be a ring. The map

\[
L^n \mathbb{G}_m(A)^{(n+1)} \longrightarrow \mathbb{G}_m(A), \quad (f_1, \ldots, f_{n+1}) \longmapsto \text{CC}_n \{f_1, \ldots, f_{n+1}\},
\]

is a multilinear antisymmetric map that satisfies the following conditions.

(i) If \( A \) is a \( \mathbb{Q} \)-algebra and \( f_1 \in (L^n \mathbb{G}_m)^2(A) \), then

\[
\text{CC}_n \{f_1, f_2, \ldots, f_{n+1}\} = \exp \left( \log(f_1) \left. \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_{n+1}}{f_{n+1}} \right|_1 \right), \tag{41}
\]

where the series \( \exp \) is applied to a nilpotent element of the ring \( A \).

(ii) If \( f_1 \in A^{*} \), then (cf. (37))

\[
\text{CC}_n \{f_1, f_2, \ldots, f_{n+1}\} = f_1^{\det(\nu(f_2), \ldots, \nu(f_{n+1}))}. \tag{42}
\]

(iii) If \( f_1 = t_{l_1}^{1}, \ldots, f_{n+1} = t_{l_{n+1}}^{1} \), where \( l_i \in \mathbb{Z}^n(A), 1 \leq i \leq n+1 \), then

\[
\text{CC}_n \{f_1, \ldots, f_{n+1}\} = (-1)^{\text{sgn}(l_1, \ldots, l_{n+1})}. \tag{43}
\]

**Proof.** Obviously, the map \( \text{CC}_n \) is multilinear. It is antisymmetric because of the antisymmetric property for symbols in the Milnor \( K \)-group \( K_{n+1}^{\mathcal{M}}(\mathcal{L}^n(A)) \) (see Remark 7.5 and Lemma 7.8).

(i) We will refer to the steps in the proof of Theorem 8.7. Let \( \varepsilon \) be a formal variable that satisfies \( \varepsilon^2 = 0 \). Step 2 and Proposition 6.23 together imply that \( \text{CC}_n \{f_1, \ldots, f_n\} = \exp(a) \), where \( a \in \text{Nil}(A) \) is a nilpotent element that satisfies \( \text{CC}_n \{1 + \log(f_1) \varepsilon, f_2, \ldots, f_n\} = 1 + a \varepsilon \). The isomorphism in Proposition 7.7
takendo symbol \( \{1 + \log(f_1) \varepsilon, f_2, \ldots, f_n\} \) to the differential form \( \log(f_1) \frac{df_2}{f_2} \land \cdots \land \frac{df_{n+1}}{f_{n+1}} \). Therefore,

\[
CC_n \{1 + \log(f_1) \varepsilon, f_2, \ldots, f_n\} = 1 + \Psi \left( \log(f_1) \frac{df_2}{f_2} \land \cdots \land \frac{df_{n+1}}{f_{n+1}} \right) \varepsilon,
\]

where \( \Psi: (L^n \Omega)_Q \to (G_a)_Q \) corresponds to the differential \( T(CC_n)_Q \) of \( (CC_n)_Q \) similarly to the way it corresponds to \( T\Phi \) in Step 2. Finally, Step 4 and Remark 8.9 imply that

\[
\Psi \left( \log(f_1) \frac{df_2}{f_2} \land \cdots \land \frac{df_{n+1}}{f_{n+1}} \right) = \mathrm{res} \left( \log(f_1) \frac{df_2}{f_2} \land \cdots \land \frac{df_{n+1}}{f_{n+1}} \right).
\]

(ii) The Contou-Carrère symbol defines a morphism of group functors (see §2 and Lemma 6.20(ii))

\[
L^n K^M_n \longrightarrow \text{Hom}^{gr}(G_m, G_m) \simeq \mathbb{Z},
\]

\[
\{f_2, \ldots, f_{n+1}\} \longmapsto (a \mapsto CC_n\{a, f_2, \ldots, f_{n+1}\}),
\]

where \( f_2, \ldots, f_{n+1} \in L^nG_m(R) \) for a ring \( R \) and \( a \in A^* \) for an \( R \)-algebra \( A \). We complete the proof by applying Proposition 8.2 and using the discussion after its proof and Remark 8.9.

(iii) The restriction of \( CC_n \) to \( (\mathbb{Z}^n)^{(n+1)} \subset (L^nG_m)^{(n+1)} \) is a multilinear antisymmetric morphism from \( (\mathbb{Z}^n)^{(n+1)} \) to \( G_m \). It follows from Dirichlet’s box principle that it takes values in \( \mathbb{Z}/2\mathbb{Z} \subset G_m \). For a ring \( A \), let \( l_1, \ldots, l_{n+1} \in \mathbb{Z}^n(A) \) be such that \( l_1 = l_2 \). By Remark 7.5 and Lemma 7.8, it follows that in \( K^M_{n+1}(\mathcal{L}^n(A)) \)

\[
\{t^{l_1}, t^{l_2}, \ldots, t^{l_{n+1}}\} = \{-1, t^{l_2}, \ldots, t^{l_{n+1}}\}.
\]

Combining this with item (ii), we see that

\[
CC_n\{t^{l_1}, t^{l_2}, \ldots, t^{l_{n+1}}\} = (-1)^{\det(l_2, \ldots, l_{n+1})}.
\]

Applying Proposition 8.15 completes the proof.

Remark 8.18. For \( n = 1 \), the right-hand sides of formulæ (41)–(43) in Theorem 8.17 appeared in [4], §2.9. (More precisely, a formula in [4], §2.9 gives an inverse map to the one given by (41)–(43).) In addition, another, equivalent, formula appeared in [2], §6 and (with a misprint) in [1].

Here are some examples where we calculate the Contou-Carrère symbol. These are based on the explicit formula from Theorem 8.17 and on some descent arguments from \( \mathbb{Q} \) to \( \mathbb{Z} \).

Proposition 8.19. Let \( A \) be a ring and \( \varepsilon \) a formal variable that satisfies \( \varepsilon^2 = 0 \). Define a ring \( B := A[\varepsilon] \). Then for any collection of invertible iterated Laurent series \( f_1, \ldots, f_n \in L^nG_m(A) = \mathcal{L}^n(A)^* \) and an iterated Laurent series \( g \in \mathcal{L}^n(A) \), in \( B^* \) the equality

\[
CC_n\{1 + g\varepsilon, f_1, \ldots, f_n\} = 1 + \mathrm{res} \left( g \frac{df_1}{f_1} \land \cdots \land \frac{df_n}{f_n} \right) \varepsilon
\]

holds, where the symbol \( CC_n \) is applied to elements of the group \( K^M_{n+1}(\mathcal{L}^n(B)) \).
Proof. Both sides of (44) define morphisms of ind-schemes from \((L^n \mathbb{G}_m)^{\times n} \times L^n \mathbb{G}_a\) to \((\mathbb{G}_a)^{\times 2}\) given by the coefficients of powers of \(\varepsilon\) in the ring \(B = A[\varepsilon]\). It is easy to check that, by Theorem 8.17(i), the values of both sides of (44) coincide when \(A\) is a \(\mathbb{Q}\)-algebra. By Example 6.7, the functor \(L^n \mathbb{G}_a\) is represented by an ind-affine space, which is an ind-flat ind-affine scheme over \(\mathbb{Z}\). Therefore by Theorem 6.12 applied to \(R = \mathbb{Z}, S = \mathbb{Q}\) and \(Y = L^n \mathbb{G}_a\), the above two morphisms coincide.

Proposition 8.20. Let \(A\) be a ring and \(\eta\) a formal variable that satisfies \(\eta^{n+2} = 0\). Define a ring \(B := A[\eta]\). Then for any collection \(g_1, \ldots, g_{n+1} \in \mathcal{L}^n(A)\) of iterated Laurent series, in \(B^*\) the equality

\[
CC_n\{1 + g_1 \eta, \ldots, 1 + g_{n+1} \eta\} = 1 + \text{res}(g_1 dg_2 \land \cdots \land dg_{n+1})\eta^{n+1}
\]  

(45)

holds, where the symbol \(CC_n\) is applied to elements of the group \(K_{n+1}^M(\mathcal{L}^n(B))\).

Proof. Both sides of formula (45) define morphisms of ind-schemes \((L^n \mathbb{G}_a)^{n+1} \rightarrow (\mathbb{G}_a)^{(n+2)}\) given by the coefficients of powers of \(\eta\) in the ring \(B = A[\eta]\). One easily checks that, by Theorem 8.17(i), the values of both sides of (45) coincide when \(A\) is a \(\mathbb{Q}\)-algebra. By Proposition 6.6(i), the functor \((L^n \mathbb{G}_a)^{n+1}\) is represented by an ind-affine space, which is an ind-flat ind-scheme over \(\mathbb{Z}\). Thus the natural homomorphism \(\mathcal{O}(L^n A^{n+1}) \rightarrow \mathcal{O}((L^n A^{n+1})_{\mathbb{Q}})\) is injective. This proves the proposition.

Remark 8.21. Let \(A\) be a \(\mathbb{Q}\)-algebra. The decomposition (22) implies that formulae (41)–(43) uniquely define the multilinear antisymmetric map \(L^n \mathbb{G}_m(A)^{(n+1)} \rightarrow A^*\). Thus we could define a Contou-Carrère symbol over \(\mathbb{Q}\)-algebras explicitly just by using the right-hand sides of formulae (41)–(43) and not using the boundary map for \(K\)-groups and the geometric properties of the iterated loop group \(L^n \mathbb{G}_m\) and its special subgroups. If we follow this approach, we need to check directly that the Contou-Carrère symbol is well-defined, that is, that the map given by (41)–(43) is well-defined and that this map satisfies the Steinberg relations. We do this in Propositions 8.22 and 8.25, respectively, using only the decompositions of the group functor \(L^n \mathbb{G}_m\) given in § 4. In order to avoid confusion, we denote the explicitly defined Contou-Carrère symbol by \(\text{CC}^\text{ex}_n\).

Proposition 8.22. For any \(\mathbb{Q}\)-algebra \(A\), there is a unique multilinear antisymmetric map

\[
\text{CC}^\text{ex}_n : L^n \mathbb{G}_m(A)^{(n+1)} \longrightarrow A^*,
\]

such that (41)–(43) are satisfied, where \(CC_n\) is replaced by \(\text{CC}^\text{ex}_n\). The map \(\text{CC}^\text{ex}_n\) is functorial with respect to \(A\).

Proof. As we explained in Remark 8.21, the uniqueness follows from the decomposition (22) in § 4.3. Thus we need to show that the map \(\text{CC}^\text{ex}_n\) is well-defined, which we do in the following steps.

Step 1. First we check that the right-hand side of formula (41) is well-defined. By Proposition 4.9, the map log in (41) is well-defined. Thus we need to prove that the value of the expression \(\text{res}(\ldots)\) in (41) belongs to \(\text{Nil}(A)\), so that the expression \(\text{res}(\ldots)\) makes sense.
The expression \( \text{res}(\ldots) \) defines a multilinear map

\[
\Psi_n : (L^n \mathbb{G}_m)^{\sharp}(A) \times L^n \mathbb{G}_m(A)^{\times n} \longrightarrow A,
\]

\[
(f_1, f_2, \ldots, f_{n+1}) \longmapsto \text{res}\left( \log(f_1) \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_{n+1}}{f_{n+1}} \right).
\]

Using Definition 4.6, which defines the group \((L^n \mathbb{G}_m)^{\sharp}(A)\) as a decomposition, and the decomposition (19) of the group \(L^n \mathbb{G}_m(A)\) from Proposition 4.3, we see that we can assume that all \(f_i, 1 \leq i \leq n + 1\), belong to the subgroups in the above decompositions.

Suppose first that for some \(i, 1 \leq i \leq n + 1\), we have \(f_i \in (1 + \text{Nil})(A) \times \mathbb{V}_{n - i}(A)\). Then by Remark 3.1, the iterated Laurent series \(\log(f_i) \in L^n \mathbb{G}_m(A)\) has nilpotent coefficients and therefore by Lemma 3.11 (i) we have \(\Psi_n(f_1, \ldots, f_{n+1}) \in \text{Nil}(A)\).

Now suppose that \(f_1\) belongs to the subgroup \(\mathbb{V}_{n+1}(A)\) and that each \(f_i, 2 \leq i \leq n + 1\), belongs either to the subgroup \(G(A) : = \mathbb{G}_m(A) \times \mathbb{V}_{n+1}(A) \times \mathbb{Z}^{n-1}(A)\), where \(\mathbb{Z}^{n-1}(A)\) consists of elements \(l^1_1 \cdots l^{n-1}_i\), or to the subgroup \(\mathbb{Z}(A) = (t_1)\mathbb{Z}(A)\), which consists of elements \(l_1^i\), where \(l_i \in \mathbb{Z}(A), 1 \leq i \leq n\). We show by induction on \(n\) that \(\Psi_n(f_1, \ldots, f_{n+1}) = 0\). The case \(n = 0\), where we put \(\mathbb{V}_{0+1}(A) = \{1\}\) by definition is obvious.

If all \(f_i, 2 \leq i \leq n + 1\), belong to \(G(A)\), then

\[
\log(f_1) \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_{n+1}}{f_{n+1}} \in A((t_1)) \cdots ((t_{n-1}))[t_n] dt_1 \wedge \cdots \wedge dt_n.
\]

Hence, \(\Psi_n(f_1, \ldots, f_{n+1}) = 0\).

Suppose that some \(f_i, 2 \leq i \leq n + 1\), belongs to \(\mathbb{Z}(A) = (t_1)\mathbb{Z}(A)\). Using multilinearity and the alternating property of the wedge product of differential forms, we can suppose, in addition, that \(f_{n+1} = t_n\) and that all \(f_i, 2 \leq i \leq n\), belong to \(G(A)\), because otherwise we have \(\Psi_n(f_1, \ldots, f_{n+1}) = 0\) again.

Under above assumptions we have

\[
\Psi_n(f_1, \ldots, f_n, t_n) = \Psi_{n-1}(\tilde{f}_1, \ldots, \tilde{f}_n),
\]

where the map \(\Psi_{n-1}\) is defined for collections of invertible elements of the ring \(A((t_1)) \cdots ((t_{n-1}))\) and \(f \mapsto \tilde{f}\) denotes the natural homomorphism from \(A((t_1)) \cdots ((t_{n-1}))[t_n]\) to \(A((t_1)) \cdots ((t_{n-1})).\) We can check directly that the homomorphism \(f \mapsto \tilde{f}\) sends \(\mathbb{V}_{n+1}(A)\) to \(\mathbb{V}_{n-1+1}(A)\), where \(n \geq 1\) and, as we said above, \(\mathbb{V}_{0+1}(A) = \{1\}\). Hence, by the inductive hypothesis \(\Psi_{n-1}(\tilde{f}_1, \ldots, \tilde{f}_n) = 0\).

Thus we obtain a well-defined multilinear map

\[
\text{CC}_n^{\text{ex}} : (L^n \mathbb{G}_m)^{\sharp}(A) \times L^n \mathbb{G}_m(A)^{\times n} \longrightarrow A^*,
\]

\[
(f_1, f_2, \ldots, f_{n+1}) \longmapsto \exp \text{res}\left( \log(f_1) \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_{n+1}}{f_{n+1}} \right).
\]

**Step 2.** By Lemma 4.7 (iii), we see that the right-hand sides of (41) and (42) overlap when \(f_1 \in (1 + \text{Nil})(A)\). We will show that in this case their values coincide. Now,
by Proposition 8.4 we have
\[
\exp \left( \log(f_1) \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_{n+1}}{f_{n+1}} \right) = \exp \left( \log(f_1) \left( \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_{n+1}}{f_{n+1}} \right) \right)
\]
\[
= \exp \left( \log(f_1) \cdot \det(\nu(f_2), \ldots, \nu(f_{n+1})) \right) = f_1^{\det(\nu(f_2), \ldots, \nu(f_{n+1}))}.
\]

Note that, in fact, we have only used Proposition 8.4 for \(\mathbb{Q}\)-algebras and in this case Proposition 8.4 is proved explicitly without the theory of thick ind-cones (see the proof of Proposition 8.4).

By the decomposition (22) we obtain that formulae (41) and (42) give a well-defined multilinear map
\[
\text{CC}^\text{ex}_n : (L^n\mathbb{G}_m)\otimes (A) \times \prod_{i=2}^{n+1} L^n\mathbb{G}_m(A) \to A^*.
\] (46)

**Step 3.** Now we check the antisymmetric property of the map in formula (46).

Formulae (41) and (42) are antisymmetric with respect to \(f_2, \ldots, f_{n+1}\) because the wedge product of differential forms and the determinant are alternating.

Further, (41) is antisymmetric with respect to \(f_1, f_2\). This follows because, if \(f_1, f_2 \in (L^n\mathbb{G}_m)^2(A)\), then by Lemmas 3.11 (i) and 3.12
\[
0 = \text{res} \left( \frac{d}{df_3} \left( \log(f_1) \log(f_2) \frac{df_3}{f_3} \wedge \cdots \wedge \frac{df_{n+1}}{f_{n+1}} \right) \right)
\]
\[
= \text{res} \left( \log(f_1) \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \wedge \cdots \wedge \frac{df_{n+1}}{f_{n+1}} \right) + \text{res} \left( \log(f_2) \frac{df_1}{f_1} \wedge \frac{df_3}{f_3} \wedge \cdots \wedge \frac{df_{n+1}}{f_{n+1}} \right).
\]

Finally, formula (42) is also antisymmetric with respect to \(f_1, f_2\) since, if \(f_1, f_2 \in A^*\), then
\[
f_1^{\det(\nu(f_2), \nu(f_3), \ldots, \nu(f_{n+1}))} = f_2^{\det(\nu(f_1), \nu(f_3), \ldots, \nu(f_{n+1}))} = 1,
\]
because \(\nu(f_2) = \nu(f_1) = 0\).

**Step 4.** Formula (43) is antisymmetric by Proposition 8.15. Combining this with Step 3 and the decomposition (22), we obtain a well-defined multilinear antisymmetric map \(\text{CC}^\text{ex}_n : L^n\mathbb{G}_m(A)^{\times (n+1)} \to A^*\). Clearly, this map is functorial with respect to the \(\mathbb{Q}\)-algebra \(A\) because the decomposition (19) is functorial.

**Remark 8.23.** For \(n = 2\), Proposition 8.22 was proved in [6], §3.3.

Now we show that the map \(\text{CC}^\text{ex}_n\) in Proposition 8.22 satisfies the Steinberg relations. First we prove an auxiliary partial result.

**Lemma 8.24.** Let \(A\) be a \(\mathbb{Q}\)-algebra. For any collection \(f_1, f_3, \ldots, f_{n+1}\), of \(n\) elements of the group \(L^n\mathbb{G}_m(A)\)
\[
\text{CC}^\text{ex}_n(f_1, -f_1, f_3, \ldots, f_{n+1}) = 1.
\] (47)

**Proof.** Clearly, the left-hand side of (47) is multilinear in \(f_3, \ldots, f_{n+1}\). The antisymmetry of the map \(\text{CC}^\text{ex}_n\) implies that the left-hand side is also linear in \(f_1\).
Indeed, for any collection \( f_1, f_1', f_3, \ldots, f_{n+1} \) of elements in \( L^n \mathbb{G}_m(A) \) we have the equalities

\[
\begin{align*}
\text{CC}_n^{\text{ex}}(f_1 f_1', -f_1 f_1', f_3, \ldots, f_{n+1}) \\
= \text{CC}_n^{\text{ex}}(f_1, -f_1, f_3, \ldots, f_{n+1}) \text{CC}_n^{\text{ex}}(f_1', f_3, \ldots, f_{n+1}) \\
\times \text{CC}_n^{\text{ex}}(f_1', f_1, f_3, \ldots, f_{n+1}) \text{CC}_n^{\text{ex}}(f_1, -f_1', f_3, \ldots, f_{n+1}).
\end{align*}
\]

Thus we can use the decomposition (22).

Suppose that \( f_i \in (L^n \mathbb{G}_m)^0(A) \) for some \( i = 1, 3, \ldots, n+1 \). If \( f_i \in \mathbb{G}_m(A) \), then the antisymmetric property and (42) imply that \( \text{CC}_n^{\text{ex}}(f_1, -f_1, f_3, \ldots, f_{n+1}) = 1 \). Now suppose that \( f_i \in (L^n \mathbb{G}_m)^{\sharp}(A) \). If \( 3 \leq i \leq n+1 \), then the antisymmetric property and (41) imply that \( \text{CC}_n^{\text{ex}}(f_1, -f_1, f_3, \ldots, f_{n+1}) = 1 \). If \( i = 1 \), that is, \( f_1 \in (L^n \mathbb{G}_m)^{\sharp}(A) \), then \( f_1 = 1 - h \), where \( h \in \mathcal{L}^n(A)^{\sharp} \). Then by (41)

\[
\begin{align*}
\text{CC}_n^{\text{ex}}(f_1, -f_1, f_3, \ldots, f_{n+1}) \\
= \exp \left( \log(1 - h) \frac{dh}{-1 + h} \wedge \frac{df_3}{f_3} \wedge \cdots \wedge \frac{df_{n+1}}{f_{n+1}} \right) \\
= \exp \left( d \left( \varphi(h) \frac{df_3}{f_3} \wedge \cdots \wedge \frac{df_{n+1}}{f_{n+1}} \right) \right) = 1,
\end{align*}
\]

where the power series \( \varphi \in A[[x]] \) exists by Lemma 3.11 (iii) and the last equality follows from Lemma 3.12.

It remains to consider the case when all the \( f_i \) belong to \( \mathbb{Z}(A)^n \). By multilinearity and the antisymmetric property

\[
\text{CC}_n^{\text{ex}}(f_1, -f_1, \ldots, f_{n+1}) = \text{CC}_n^{\text{ex}}(-1, f_1, \ldots, f_{n+1})^{-1} \text{CC}_n^{\text{ex}}(f_1, f_1, \ldots, f_{n+1}).
\]

Thus we complete the proof using (42), (43) and Proposition 8.15.

The next proposition and the antisymmetric property of the map \( \text{CC}_n^{\text{ex}} \) imply that \( \text{CC}_n^{\text{ex}} \) factors through the Milnor K-group, that is, we have a well-defined homomorphism of groups

\[
\text{CC}_n^{\text{ex}} : K^M_{n+1}(A((t_1)) \ldots ((t_n))) \longrightarrow A^*,
\]

which is functorial with respect to a \( \mathbb{Q} \)-algebra \( A \).

**Proposition 8.25.** Let \( A \) be a \( \mathbb{Q} \)-algebra and let \( f_1, \ldots, f_n \in L^n \mathbb{G}_m(A) \) be a collection of invertible iterated Laurent series such that \( f_1 + f_2 = 1 \). Then \( \text{CC}_n^{\text{ex}}(f_1, \ldots, f_{n+1}) = 1 \).

**Proof.** The proof is similar to the proof of the case \( n = 2 \) in [6], Proposition 4.2. For the sake of completeness we briefly repeat it here for arbitrary \( n \) (using different notation from [6]).

Changing the \( \mathbb{Q} \)-algebra \( A \) if necessary, we can assume without loss of generality that \( \nu(f_1) \) and \( \nu(1 - f_1) \) belong to the subgroup \( \mathbb{Z}^n \subset \mathbb{Z}(A)^n \). Consider the following cases.
First suppose that \( \nu(f_1) > 0 \). Then by Proposition 4.3 and Example 4.5 (i), we have \( 1 - f_1 \in (\mathbb{L}^n \mathbb{G}_m)^2(A) \). By Lemmas 3.11 (ii) and 3.12

\[
\text{CC}^\text{ex}_n(f_1, 1 - f_1, f_3, \ldots, f_{n+1}) = \text{CC}^\text{ex}_n(1 - f_1, f_1, f_3, \ldots, f_{n+1})^{-1}
\]

\[
= \exp \text{res} \left( - \log(1 - f_1) \frac{df_1}{f_1} \wedge \frac{df_3}{f_3} \wedge \cdots \wedge \frac{df_{n+1}}{f_{n+1}} \right)
\]

\[
= \exp \text{res} \left( d \left( \text{Li}_2(f_1) \frac{df_3}{f_3} \wedge \cdots \wedge \frac{df_{n+1}}{f_{n+1}} \right) \right) = 1.
\]

Now suppose that \( \nu(f_1) = 0 \). Then by Proposition 4.3 and Example 4.5 (i), we have \( \nu(1 - f_1) \geq 0 \). If \( \nu(1 - f_1) > 0 \), then transposing \( f_1 \) and \( 1 - f_1 \) and using the antisymmetric property of the map \( \text{CC}^\text{ex}_n \), we can reduce it to the previous case. Therefore we can suppose that \( \nu(1 - f_1) = 0 \). By decomposition (22) in §4.3, we have \( f_1 = a(1 - h) \), where \( a \in A^\ast \), \( 1 - a \in A^\ast \), and \( h \in \mathbb{L}^n(A)^2 \). By (42)

\[
\text{CC}^\text{ex}_n(a, 1 - f_1, f_3, \ldots, f_{n+1}) = a^{\text{det}(\nu(1 - f_1), \nu(f_3), \ldots, \nu(f_{n+1}))} = 1.
\]

By (41)

\[
\text{CC}^\text{ex}_n(1 - h, 1 - f_1, f_3, \ldots, f_{n+1})
\]

\[
= \exp \text{res} \left( \log(1 - h) \frac{dh}{a - 1 - 1 + h} \wedge \frac{df_3}{f_3} \wedge \cdots \wedge \frac{df_{n+1}}{f_{n+1}} \right),
\]

because \( 1 - f_1 = 1 - a + ah \). By Lemmas 3.11 (iii) and 3.12, the last expression equals one.

Finally, suppose that \( \nu(f_1) < 0 \). By multilinearity

\[
\text{CC}^\text{ex}_n(f_1, 1 - f_1, f_3, \ldots, f_{n+1})
\]

\[
= \text{CC}^\text{ex}_n(f_1, -f_1, f_3, \ldots, f_{n+1}) \text{CC}^\text{ex}_n(f_1^{-1}, 1 - f_1^{-1}, f_3, \ldots, f_{n+1})^{-1}.
\]

Since \( \nu(f_1^{-1}) > 0 \), the last expression equals one by Lemma 8.24 and the first case considered above.

**8.5. A completed Contou-Carrère symbol.** Now let \( A = \lim \limits_{\leftarrow i \in I} A_i \) be a pro-ring, that is, a pro-object in the category of rings. Recall that a pro-object is the dual concept to an ind-object and the category of pro-rings is anti-equivalent to the category of ind-affine schemes (see §5.1). Let \( \hat{A} := \lim \limits_{\leftarrow i \in I} A_i \) denote the corresponding inverse limit of rings. It is easy to check that there is a natural isomorphism of groups \( \hat{A}^\ast \sim \lim \limits_{\leftarrow i \in I} A_i^\ast \).

We have a pro-ring \( \lim \limits_{\leftarrow i \in I} \mathbb{L}^n(A_i) \). Let \( \mathcal{L}_i^n(A_i) := \lim \limits_{\leftarrow i \in I} \mathcal{L}_i^n(A_i) \) be the corresponding inverse limit of rings.

**Definition 8.26.** The **completed Contou-Carrère symbol** \( \hat{\text{CC}}_n \) is the following composition of homomorphisms of groups:

\[
\hat{\text{CC}}_n : K_{n+1}^M(\mathcal{L}_i^n(A_i)) \longrightarrow \lim \limits_{\leftarrow i \in I} K_{n+1}^M(\mathcal{L}_i^n(A_i)) \longrightarrow \lim \limits_{\leftarrow i \in I} A_i^\ast \sim \hat{A}^\ast,
\]

where the second homomorphism is the inverse limit of the Contou-Carrère symbols \( \text{CC}_n : K_{n+1}^M(\mathcal{L}_i^n(A_i)) \longrightarrow A_i^\ast \) taken over the rings \( A_i, i \in I \).
Clearly, the homomorphism \( \hat{\text{CC}}_n \) is functorial with respect to the pro-ring \( A \).

**Example 8.27.** (i) Let \( R \) be a ring and let

\[
A = \lim_{d \in \mathbb{N}}' R[x]/(x^d).
\]

Then \( \hat{A} \simeq R[[x]] \). For each \( d \in \mathbb{N} \) we have an isomorphism \( \mathcal{L}^n(R[x]/(x^d)) \simeq \mathcal{L}^n(R)[[x]]/(x^d) \) (cf. Lemma 3.3 (iii)). Hence we have an isomorphism

\[
\hat{\mathcal{L}}^n(A) \simeq \mathcal{L}^n(R)[[x]] = R((t_1)) \cdots ((t_n))[x],
\]

and we obtain a symbol \( \hat{\text{CC}}_n : K_{n+1}^M(R((t_1)) \cdots ((t_n))[[M]]) \to R[[M]]^* \).

(ii) More generally, let \( M \) be a set and let

\[
A = \lim_{(M',d)}' R[M']/((M')^d),
\]

where \( M' \) runs over all finite subsets in \( M \) and \( d \in \mathbb{N} \) (see § 5.2). Then \( \hat{A} \simeq R[[M]] \) (see Definition 5.1), \( \hat{\mathcal{L}}^n(A) \simeq R((t_1)) \cdots ((t_n))[[M]] \), and we obtain a symbol \( \hat{\text{CC}}_n : K_{n+1}^M(R((t_1)) \cdots ((t_n))[M]) \to R[[M]]^* \).

(iii) Let \( A = \lim_{d \in \mathbb{N}}' \mathbb{Z}/(p^d) \), where \( p \) is a prime number. Then \( \hat{A} \simeq \mathbb{Z}_p \) is the ring of \( p \)-adic integers and the ring

\[
\hat{\mathcal{L}}^n(A) \simeq \mathbb{Z}_p \{ \{ t_1 \} \} \cdots \{ \{ t_n \} \}
\]

consists of infinite series \( \sum_{l \in \mathbb{Z}^n} a_l l! \), where \( a_l \in \mathbb{Z}_p \) and for any \( d \in \mathbb{N} \), we can find \( \lambda \in \Lambda_n \) (see § 3.1) such that if \( l \notin \mathbb{Z}_\lambda \), then \( a_l \in p^d \mathbb{Z}_p \). We obtain a symbol \( \hat{\text{CC}}_n : K_{n+1}^M(\mathbb{Z}_p \{ \{ t_1 \} \} \cdots \{ \{ t_n \} \}) \to \mathbb{Z}_p^* \).

The following fact a direct consequence of Theorem 8.17 (i).

**Proposition 8.28.** Suppose that all the \( A_i, i \in I \), are \( \mathbb{Q} \)-algebras and consider a collection \( f_1, \ldots, f_{n+1} \in \hat{\mathcal{L}}^n(A)^* \). Suppose that

\[
f_{1} = \lim_{i \in I} f_{1,i} \in \lim_{i \in I} (L^n \mathbb{G}_{m})^\sharp(A_i) \subset \hat{\mathcal{L}}^n(A)^*.
\]

Then in \( \hat{A}^* \) the equality

\[
\hat{\text{CC}}_n \{ f_1, \ldots, f_{n+1} \} = \exp \left( \log(f_1) \frac{df_2}{f_2} \wedge \cdots \wedge \frac{df_{n+1}}{f_{n+1}} \right)
\]

holds, where \( \log(f_1) = \lim_{i \in I} \log(f_{1,i}) \) is a well-defined element of the ring \( \hat{\mathcal{L}}^n(A) = \lim_{i \in I} \mathcal{L}^n(A_i) \), the residue map is the homomorphism

\[
\text{res}: \hat{\mathcal{L}}^n(A) dt_1 \wedge \cdots \wedge dt_n \longrightarrow \hat{A},
\]

the series \( \exp \) is applied to an element of \( \lim_{i \in I} \text{Nil}(A_i) \subset \hat{A} \), and each differential form \( df_i \), \( 1 \leq i \leq n \), belongs to the inverse limit

\[
\lim_{i \in I} \Omega^1_{\mathcal{L}^n(A_i)} = \bigoplus_{j=1}^n \hat{\mathcal{L}}^n(A) dt_j.
\]
For instance, for a pro-ring $A = \lim_{d \to \infty} R[x]/(x^d)$, as in Example 8.27 (i), the residue $\text{res} : R((t_1)) \ldots (t_n))[[x]] dt_1 \wedge \cdots \wedge dt_n \to R[[x]]$ in Proposition 8.28 is taken with respect to the variables $t_1, \ldots , t_n$. Also,

$$\lim_{i \in I} \text{Nil}(A_i) = xR[[x]].$$

in this case.

**Remark 8.29.** It is natural to expect that an extension of the completed Contou-Carrère symbol in Example 8.27 (i) exists as a homomorphism from $K_{n+1}^M(R((t_1)) \ldots (t_n)(x)))$ to $R((x))^*$. Such a new symbol would be a morphism of group functors $L_{n+1}^n K_{n+1}^M \to L_{G_m}^n$. Note that it would definitely differ from the morphism $L(C^n) : L_{n+1}^n K_{n+1}^M \to L_{G_m}^n$ as the latter morphism vanishes on the group $K_{n+1}^M(R((t_1)) \ldots (t_n)(x)))$.

**Remark 8.30.** A particular case of the completed Contou-Carrère symbol $K_2^M(R((t))(x))) \to R((x))^*$ when $R$ is a field of characteristic zero was defined by Osipov in [37], Theorem 2. Osipov looked at a surface fibred over a curve and the symbol was constructed as a local direct image map related to a flag on the surface, given by a point and the fibre through it. Another definition of a completed symbol in a particular case was given by Kato [38]. The completed Contou-Carrère symbol and its relation to the Kato symbol were studied by Asakura (see [39], §4), Pál (see [40], §§3 and 4), Liu (see [41], §3), and Chinburg, Pappas and Taylor (see [42], §§3e and 3f).

The following lemma follows immediately from the construction of the completed Contou-Carrère symbol.

**Lemma 8.31.** The completed Contou-Carrère symbol

$$\widehat{\text{CC}}_n : K_{n+1}^M(L_n^1(A)) \to \hat{A}^*$$

sends the kernel

$$\text{Ker}(K_{n+1}^M(L_n^1(A)) \to K_{n+1}^M(L_n^1(A_i)))$$

to the kernel

$$\text{Ker}({\hat{A}^* \to A_i^*})$$

for each $i \in I$.

8.6. Integrality and the convergence of the explicit formula. The aim of this subsection is to show that the right-hand side of (41) is expressed by some universal power series with integral coefficients. The variables in these universal power series are coefficients of iterated Laurent series to which the Contou-Carrère symbol is applied.

Let $q, 0 \leq q \leq n$, be an integer and let $1 \leq j_1 < \cdots < j_q \leq n$ be a collection of $q$ integers (for $q = 0$, the collection is empty). Put $p := n + 1 - q$. Thus, we have $1 \leq p \leq n + 1$. Consider countably many formal variables $x_{i,l}$, where $1 \leq i \leq p$ and $l \in \mathbb{Z}^n$. Given a ring $R$, for brevity, we set $R[[x_{i,l}]]$ to be the ring of power
series $R[[x_{i,l}; 1 \leq i \leq p, l \in \mathbb{Z}^n]]$ in these formal variables (see Definition 5.1). Consider infinite series

$$f_i := 1 + \sum_{l \in \mathbb{Z}^n} x_{i,l} t^l, \quad 1 \leq i \leq p,$$

which are well-defined elements in $\mathbb{Z}((t_1)) \ldots ((t_n))[[x_{i,l}]]$ (see Example 8.27 (ii)). Moreover, for each $i, 1 \leq i \leq p$,

$$f_i \in \lim_{(M', d)} (L^n \mathbb{G}_m)^d \langle \mathbb{Z}[M']/(M')^d \rangle,$$

where $M'$ runs over all finite subsets in $\{1, \ldots, p\} \times \mathbb{Z}^n$ and $d \in \mathbb{N}$. In particular, $f_i \in \mathbb{Z}((t_1)) \ldots ((t_n))[[x_{i,l}]]^*$. Define the following power series (see Definition 8.26):

$$\varphi_{n,j_1,\ldots,j_q} := \widehat{\text{CC}}_n \{f_1, \ldots, f_p, t_{j_1}, \ldots, t_{j_q}\} \in \mathbb{Z}[[x_{i,l}]]^*.$$

Lemma 8.31 implies that the constant term of the power series $\varphi_{n,j_1,\ldots,j_q}$ is equal to one.

Consider the embeddings of rings

$$\mathbb{Z}[[x_{i,l}]] \subset \mathbb{Q}[[x_{i,l}]], \quad \mathbb{Z}((t_1)) \ldots ((t_n))[[x_{i,l}]] \subset \mathbb{Q}((t_1)) \ldots ((t_n))[[x_{i,l}]].$$

It turns out that the series $\varphi_{n,j_1,\ldots,j_q}$ viewed as an element of $\mathbb{Q}[[x_{i,l}]]^*$ can be constructed explicitly as follows. In view of the functoriality of the completed Contou-Carrère symbol $\widehat{\text{CC}}_n$, Proposition 8.28 implies that in $\mathbb{Q}[[x_{i,l}]]^*$

$$\varphi_{n,j_1,\ldots,j_q} = \exp \left( \log(f_1) \left( \frac{df_2}{f_2} \wedge \ldots \wedge \frac{df_p}{f_p} \wedge \frac{dt_{j_1}}{t_{j_1}} \wedge \ldots \wedge \frac{dt_{j_q}}{t_{j_q}} \right) \right).$$

(48)

To be explicit, the series $\varphi_{n,j_1,\ldots,j_q}$ is obtained by formally expanding the brackets on the right-hand side of formula (48). More precisely, for every finite subset $M' \subset \{1, \ldots, p\} \times \mathbb{Z}^n$ and $d \in \mathbb{N}$, by $f \mapsto \bar{f}$ denote the natural homomorphism of rings

$$\mathbb{Q}((t_1)) \ldots ((t_n))[[x_{i,l}]] \to \mathbb{Q}((t_1)) \ldots ((t_n))[[M']] / (M')^d.$$

Consider the right-hand side of (48) over the ring

$$\mathbb{Q}((t_1)) \ldots ((t_n))[[M']] / (M')^d.$$

The series

$$\log(\bar{f}_1) = \sum_{m \geq 1} (-1)^{m+1} \left( \frac{\bar{f}_1 - 1}{m} \right)$$

is a finite sum in the ring $\mathbb{Q}((t_1)) \ldots ((t_n))[[M']] / (M')^d$, the expression $\exp(\ldots)$ is an element of the ideal

$$(M') \cdot \mathbb{Q}((t_1)) \ldots ((t_n))[[M']] / (M')^d,$$

and the right-hand side of formula (48) applied over the ring

$$\mathbb{Q}((t_1)) \ldots ((t_n))[[M']] / (M')^d.$$
gives a well-defined element
\[ \varphi_{n,j_1,\ldots,j_q}^{(M',d)} \in \mathbb{Q}[M']/\langle M' \rangle^d. \]
Passing to the limit over all \((M',d)\), we obtain the series
\[ \varphi_{n,j_1,\ldots,j_q} = \lim_{\langle M',d \rangle} \varphi_{n,j_1,\ldots,j_q}^{(M',d)}. \]

Using the first construction of the series \(\varphi_{n,j_1,\ldots,j_q}\), we see that formally expanding the brackets on the right-hand side of (48) gives a series with integral coefficients (cf. [4], §2.9). Thus we deduce this integrality from the existence of the Contou-Carrére symbol for all rings, not just \(\mathbb{Q}\)-algebras, that is, from the existence of the boundary map for algebraic \(K\)-groups (see §7.3).

**Example 8.32.** Let
\[ f = 1 + \sum_{l \in \mathbb{Z}^n} x_l t^l \in \mathbb{Z}((t_1)) \ldots ((t_n))[[x_l; l \in \mathbb{Z}^n]]. \]
Then the series \(\varphi_{n,1,\ldots,n} \in \mathbb{Z}[[x_l]]\) is obtained by formally expanding the brackets in the expression
\[ \exp \left( \log(f) \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} \right) = \exp \left( \sum_{m \geq 1} (-1)^{m+1} \frac{[(f-1)^m]_0}{m} \right), \]
where \([g]_0\) denotes the constant term of an iterated Laurent series \(g\). Kontsevich also proved the integrality of this series (see [16], p.2, Step1) by a combinatorial method.

The following lemma follows directly from the construction of the completed Contou-Carrére symbol (see also Example 5.3 (i)).

**Lemma 8.33.** Let \(A\) be a ring and let \(g_1, \ldots, g_p\) be Laurent polynomials (not just series) with nilpotent coefficients in \(A\). Then in \(A^*\)
\[ \text{CC}_n \{1 + g_1, \ldots, 1 + g_p, t_{j_1}, \ldots, t_{j_q}\} = \varphi_{n,j_1,\ldots,j_q}(g_1, \ldots, g_p). \]

Combining Lemma 8.33 with Theorem 6.10, we obtain the following result (see also Definition 5.2 (iii)).

**Theorem 8.34.** The integral power series \(\varphi_{n,j_1,\ldots,j_q}\) converges algebraically on the ind-closed subscheme \((L^n G_a)^P \times \mathbb{A}^{\mathbb{Z}^n})^P\) and for any ring \(A\) and a collection \(g_1, \ldots, g_p \in (L^n G_a)^P(A)\) the equality
\[ \text{CC}_n \{1 + g_1, \ldots, 1 + g_p, t_{j_1}, \ldots, t_{j_q}\} = \varphi_{n,j_1,\ldots,j_q}(g_1, \ldots, g_p) \]
holds in \(A^*\).

Let \(x_{i,l}\) be of weight \(l \in \mathbb{Z}^n\). The weight of a monomial in \(x_{i,l}\) is defined in a natural way. For example, the weight of \(x_{i,l}^d\) is \(dl\) and the weight of \(x_{i,l} x_{l',l'}\) is \(l + l'\). A weight-homogeneous power series in \(\mathbb{Q}[[x_{i,l}]]\) is an infinite sum of monomials of
the same weight, and this is also called the weight of this series. For example, the polynomial $x_{i,l}^2 + x_{i',2l}$ is weight homogeneous of weight $2l$. Let us say that a series $\sum_{l \in \mathbb{Z}^n} \varphi_l t^l \in \mathbb{Q}((t_1)) \ldots ((t_n))[x_{i,l}]$ is weighted if $\varphi_l$ is a weight homogeneous power series of weight $l$ for each $l \in \mathbb{Z}^n$. We will use similar terminology for elements of $\mathbb{Q}((t_1)) \ldots ((t_n))[M'/(M')^d]$, where $M'$ and $d$ are as above. The following result is proved for the case $n = 1$ in [4], §2.9.

**Proposition 8.35.** The power series $\varphi_{n,j_1,\ldots,j_q}$ is weight homogeneous of weight zero.

**Proof.** It is easy to check that sums and products of weighted series in $\mathbb{Q}((t_1)) \ldots ((t_n))[x_{i,l}]$ are weighted. Further, if

$$f = \sum_{l \in \mathbb{Z}^n} \varphi_l t^l \in \mathbb{Q}((t_1)) \ldots ((t_n))[x_{i,l}]$$

is weighted, then $df = \sum_{i=1}^n g_i \frac{dt_i}{t_i}$, where $g_i$, $1 \leq i \leq n$, are also weighted. Indeed, $g_i = \sum_{l \in \mathbb{Z}^n} l_i \varphi_l t^l$ for each $1 \leq i \leq n$. Finally, if

$$f \in \mathbb{Q}((t_1)) \ldots ((t_n))[x_{i,l}]$$

is weighted, then res$(f \frac{dt_1}{t_1} \wedge \ldots \wedge \frac{dt_n}{t_n})$ is a weight homogeneous power series in $\mathbb{Q}[[x_{i,l}]]$ of weight zero, as it is the constant term of $f$.

All this implies that for each finite subset $M' \subset \{1,\ldots,p\} \times \mathbb{Z}^n$ and $d \in \mathbb{N}$, the element $\varphi_{n,j_1,\ldots,j_q}^{(M',d)} \in \mathbb{Q}[M']/(M')^d$ is weight-homogeneous of weight zero. Now we pass to the limit over all $(M',d)$.

§ 9. The relation to higher local class field theory

In this section, we explain how the $n$-dimensional Contou-Carrère symbol $CC_n$ leads to the local reciprocity map in the explicit higher local class field theory for $n$-dimensional local fields of positive characteristic as constructed by Parshin in [13]. (For another approach to higher local class field theory see Kato’s papers [43] and [10].)

Proposition 9.3 below is also of independent interest. It describes a bilinear pairing of group functors $L^nK_n^M \times L^nW_S \to W_S$, where $W_S$ is the group functor of Witt vectors that depends on a set of positive integers $S$ which is divisor closed.

**9.1. Unramified, Kummer and Artin-Schreier-Witt extensions.** Let $\mathbb{F}_q$ be a finite field of characteristic $p$ and let $\mathbb{F}_q$ be its algebraic closure. Higher local class field theory describes the Galois group of the maximal Abelian extension $K_{ab}$ of the $n$-dimensional local field $K := \mathbb{F}_q((t_1)) \ldots ((t_n))$. There is a local reciprocity map

$$K_n^M(K) \to \text{Gal}(K_{ab}/K).$$

The image of the local reciprocity map is dense with respect to the profinite topology on the right-hand side, see [13], §4, Theorem 1 (1). The kernel is also described in topological terms there and it is described in algebraic terms by Fesenko in [44], Introduction as follows: the kernel is the subgroup of all divisible elements in $K_n^M(K)$. 
Let $K^{nr}$ be the maximal Abelian unramified extension of $K$. Explicitly, we have $K^{nr} = \mathbb{F}_q((t_1)) \ldots ((t_n))$ and there is an isomorphism of profinite groups
\[ \text{Gal}(K^{nr}/K) \xrightarrow{\sim} \hat{\mathbb{Z}}. \]

Let $K^{\text{Km}}$ be the maximal Kummer extension of $K$, that is, the maximal Abelian extension of exponent $q-1$. Explicitly, we have $K^{\text{Km}} = \mathbb{F}_{q^2}((\sqrt[2]{t_1})) \ldots ((\sqrt[2]{t_n}))$ and this extension is of finite degree over $K$. By Kummer theory, there is an isomorphism of finite groups
\[ \text{Gal}(K^{\text{Km}}/K) \xrightarrow{\sim} \text{Hom}(K^*/(K^*)^{q-1}, \mathbb{F}_q^*). \]

Finally, let $K^{\text{ASW}}$ be the maximal Artin-Schreier-Witt extension of $K$, that is, the maximal Abelian $p$-extension. The degree of $K^{\text{ASW}}$ over $K$ is infinite. By Artin-Schreier-Witt theory, for each $r \geq 1$, there is an isomorphism of pro-$p$-groups
\[ \text{Gal}(K^{\text{ASW}}/K)/p^r \xrightarrow{\sim} \text{Hom}(W_{p^r}(K)/(Fr - 1)W_{p^r}(K), \mathbb{Z}/p^r\mathbb{Z}), \]
where $W_{p^r}$ denotes the group of $r$-truncated $p$-Witt vectors (see §9.2 below) and $Fr$ denotes the Frobenius homomorphism for Witt vectors.

The maximal Abelian extension $K^{\text{ab}}$ is the composite $K^{nr} . K^{\text{Km}} . K^{\text{ASW}}$. In order to construct the local reciprocity map, define three homomorphisms from $K^n_{\text{M}}(K)$ to the above Galois groups and then check the compatibility between these homomorphisms. These three homomorphisms are defined in the proof of Theorem 1 in [13], §4. Below we explain how they are constructed with the help of the Contou-Carrère symbol $CC_n$.

The homomorphism $K^n_{\text{M}}(K) \to \text{Gal}(K^{nr}/K)$ is the composition of the homomorphism $c_K: K^n_{\text{M}}(K) \to \mathbb{Z}$ defined in [13], §3, Definition 2 and the natural embedding $\mathbb{Z} \to \hat{\mathbb{Z}}$. It follows from [13], §3, Proposition 2 that $c_K = \nu_n$ (see Definition 8.1).

The homomorphism $K^n_{\text{M}}(K) \to \text{Gal}(K^{\text{Km}}/K)$ is induced by a bilinear pairing $K^n_{\text{M}}(K) \times K^* \to \mathbb{F}_q^*$, which is defined as the composition of the product between Milnor $K$-groups and a homomorphism
\[ K^n_{n+1}(K) \to \mathbb{F}_q^*, \quad \{f_1, \ldots, f_{n+1}\} \longmapsto (f_1, \ldots, f_{n+1})_K/\mathbb{F}_q, \]
defined in [13], §3, Definition 2. It follows from [13], §2, Proposition 2 and [13], §3, Proposition 2 that this homomorphism coincides with the Contou-Carrère symbol $CC_n: K^n_{n+1}(\mathbb{L}^n_{\text{m}}(\mathbb{F}_q)) \to \mathbb{F}_q^*$ over the field $\mathbb{F}_q$.

The homomorphism $K^n_{\text{M}}(K) \to \text{Gal}(K^{\text{ASW}}/K)$ is induced by a collection of bilinear pairings $K^n_{\text{M}}(K) \times W_{p^r}(K) \to \mathbb{Z}/p^r\mathbb{Z}$, $r \geq 1$. For each $r \geq 1$, the latter pairing is the composition of a bilinear pairing
\[ K^n_{\text{M}}(K) \times W_{p^r}(K) \to W_{p^r}(\mathbb{F}_q), \quad (49) \]
constructed explicitly in [13], §3, Definition 5 and the trace map $W_{p^r}(\mathbb{F}_q) \to W_{p^r}(\mathbb{F}_p) \simeq \mathbb{Z}/p^r\mathbb{Z}$. The pairing (49) is called the Witt pairing and was defined by Witt [45] for the case $n = 1$. We will show in §9.2 (see Proposition 9.3 and the discussion after it) how to construct the Witt pairing with the help of the Contou-Carrère symbol $CC_n$. Notice that to do this we need to consider the
Contou-Carrère symbol not over fields, but over Artinian rings with nontrivial nilpotents. This relation between the Witt pairing and the Contou-Carrère symbol was discovered in [7], §4.3 for the case $n = 1$. The case $n = 2$ was considered in [6], §8.

In fact, relation between the Contou-Carrère symbol and local class field theory was the main motivation for constructing this symbol in [1] in the case $n = 1$.

9.2. The generalized Witt pairing. First let us recall some basic facts on Witt vectors (see, for instance, [46], Lecture 26). Let $A$ be a ring and let $S$ be a set of positive integers such that $S$ is divisor closed. The set of Witt vectors $W_S(A)$ is defined by the formula

$$W_S(A) := \{ w = (w_i) \mid i \in S, w_i \in A \}.$$ 

The ghost (or auxiliary) coordinates $w(i)$ of a Witt vector $w \in W_S(A)$ are defined by the following polynomials in $w_i$ with integral coefficients:

$$w(i) := \sum_{d|d} d w_i/d, \quad i \in S.$$ 

Note that, conversely, the coordinates $w_i$ are expressed as polynomials in the ghost coordinates $w(i)$ with rational coefficients which have nontrivial denominators.

By definition, addition and multiplication in the ghost coordinates $w(i)$ of Witt vectors are coordinate-wise. Surprisingly, the corresponding addition and multiplication in the coordinates $w_i$ are given by polynomials in $w_i$ with integral coefficients (cf. §8.6). Thus $W_S$ is a ring functor, that is, a functor from the category of rings to the category of rings. For brevity, we put $W(A) := W_S(A)$. For a prime $p$ and an integer $r \geq 1$, the ring $W_{p^r}(A) := W_{\{1,p,\ldots,p^r-1\}}(A)$ is called the ring of $r$-truncated $p$-Witt vectors.

Remark 9.1. Let $S' \subset S$ be an embedding of subsets of $\mathbb{N}$ and let $S$ and $S'$ be divisor closed. Then the natural projection $W_S \to W_{S'}$ is a morphism of ring functors. In the ghost coordinates this homomorphism is also the natural projection.

We will use only the additive structure on Witt vectors, thus we will consider $W_S$ as a group functor (see §2). Note that if $S = \mathbb{N}$ and $A$ is a $\mathbb{Q}$-algebra, then for any Witt vector $w \in W(A)$ the equality

$$- \log \prod_{i \geq 1} (1 - w_i x^i) = \sum_{i \geq 1} w(i) \frac{x^i}{i}$$

holds in $A[[x]]$. It follows that for any ring $A$ (not only for a $\mathbb{Q}$-algebra), there is a functorial isomorphism of groups

$$\Upsilon: W(A) \sim 1 + xA[[x]], \quad w = (w_i) \mapsto \prod_{i \geq 1} (1 - w_i x^i),$$

where the group structure on $1 + xA[[x]]$ is given by the product of power series. In particular, there is a functorial embedding of groups $W(A) \hookrightarrow A[[x]]^*$. 

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Consider now the composition of maps

\[ K_n^M(\mathcal{L}(A)) \times W(\mathcal{L}(A)) \longrightarrow K_n^M(\mathcal{L}(A)[[x]]) \times \mathcal{L}(A)[[x]]^* \times K_n^M(\mathcal{L}(A)[[x]]) \longrightarrow K_{n+1}^M(\mathcal{L}(A)[[x]]) \longrightarrow A[[x]]^*, \]

where the first map is induced by the natural homomorphism of rings \( \mathcal{L}(A) \to \mathcal{L}(A)[[x]] \) and the isomorphism of groups \( \Upsilon: W(\mathcal{L}(A)) \isom 1 + x\mathcal{L}(A)[[x]] \), the second map is the transposition, the third map is given by the product between Milnor \( \mathcal{K} \)-groups, and the last map is the completed Contou-Carrère symbol \( \widehat{\mathcal{C}}_n \) (see Definition 8.26 and Example 8.27 (i)). Lemma 8.31 implies that the image of this composition is contained in the subgroup \( 1 + xA[[x]] \simeq W(A) \) in \( A[[x]]^* \). Thus we obtain a bilinear pairing

\[ K_n^M(\mathcal{L}(A)) \times W(\mathcal{L}(A)) \longrightarrow W(A), \]

which is functorial with respect to the ring \( A \). Let \( \{f_1, \ldots, f_n\}, f_i \in \mathcal{L}(A)^* \), be a symbol in \( K_n^M(\mathcal{L}(A)) \) and let \( \{g_1, g_2, \ldots\}, g_i \in \mathcal{L}(A) \), be a Witt vector in \( W(\mathcal{L}(A)) \). Following Witt [45], we denote the pairing of \( \{f_1, \ldots, f_n\} \) and \( \{g_1, g_2, \ldots\} \) by \( (f_1, \ldots, f_n \mid g_1, g_2, \ldots) \).

**Lemma 9.2.** If \( A \) is a \( \mathbb{Q} \)-algebra, then for any symbol \( \{f_1, \ldots, f_n\} \) in \( K_n^M(\mathcal{L}(A)) \), any Witt vector \( \{g_1, g_2, \ldots\} \) in \( W(\mathcal{L}(A)) \), and any \( i \in \mathbb{N} \), the \( i \)th ghost coordinate satisfies

\[ (f_1, \ldots, f_n \mid g_1, g_2, \ldots)(i) = \text{res}\left( g(i) \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_n}{f_n} \right) \]  \hfill (51)

in \( A \).

**Proof.** Using Proposition 8.28 and (50) we obtain

\[- \log \Upsilon(f_1, \ldots, f_n \mid g_1, g_2, \ldots) = - \log \widehat{\mathcal{C}}_n \{\Upsilon(g_1, g_2, \ldots), f_1, \ldots, f_n\} \]

\[ = - \text{res}\left( \log \left( \prod_{i \geq 1} (1 - g_i x^i) \right) \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_n}{f_n} \right) \]

\[ = \text{res}\left( \sum_{i \geq 1} g(i) \frac{x^i}{i} \right) \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_n}{f_n} = \sum_{i \geq 1} \text{res}\left( g(i) \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_n}{f_n} \right) \frac{x^i}{i}. \]

Now let \( S \) be a divisor closed subset in \( \mathbb{N} \). Define a group functor \( U_S \) by the formula

\[ U_S := \text{Ker}(W \longrightarrow W_S). \]

**Proposition 9.3.** There is a bilinear morphism of functors

\[ L^n K_n^M \times L^n W_S \longrightarrow W_S, \]  \hfill (52)

given in the ghost coordinates by formula (51).

**Proof.** It is enough to prove that for any ring \( A \), any symbol \( \{f_1, \ldots, f_n\} \) in \( K_n^M(\mathcal{L}(A)) \) and any Witt vector \( \{g_1, g_2, \ldots\} \) in \( U_S(\mathcal{L}(A)) \) the pairing \( (f_1, \ldots, f_n \mid g_1, g_2, \ldots) \) belongs to the subgroup \( U_S(A) \). If \( A \) is a \( \mathbb{Q} \)-algebra, then this follows directly from Remark 9.1 and Lemma 9.2.
In general, we need to show that the composition of morphisms of functors
\[(L^n \mathbb{G}_m)^{\times n} \times L^n U_S \longrightarrow L^n K_n^M \times L^n W \longrightarrow W \longrightarrow W_S\]
is equal to zero. Clearly, the functor $U_S$ is represented by the affine space $\mathbb{A}^{N_{\mathbb{A}^S}}$. Hence by Proposition 6.6 (i), $L^n U_S$ is represented by an ind-affine space, which is an ind-flat ind-affine scheme over $\mathbb{Z}$. Thus we complete the proof by applying Theorem 6.12 to the embedding of rings $\mathbb{Z} \subset \mathbb{Q}$.

**Remark 9.4.** Let $S' \subset S$ be an embedding of subsets in $\mathbb{N}$ such that $S$ and $S'$ are divisor closed. Then the pairings (52) for $S$ and $S'$ commute with each other under the natural projections $W_S \rightarrow W_{S'}$ and $L^n W_S \rightarrow L^n W_{S'}$ (see Remark 9.1). This follows directly from the proof of Proposition 9.3.

When $S = \{1, \ldots, p^{r-1}\}$, $r \geq 1$, the pairing (52) evaluated at $A = \mathbb{F}_q$ is equal to the pairing from [13], §3, Definition 5 (this follows from the functoriality of the pairing (52)).

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