Problematic solutions to the Landau-Lifshitz equation

W. E. Baylis and J. Huschilt

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Abstract
A critical look at the Landau-Lifshitz equation, which has been recently advocated as an “exact” relativistic classical equation for the motion of a point charge with radiation reaction, demonstrates that it generally does not conserve energy-momentum. Its failure is dramatic in the one-dimensional case of a stepped electric field, where it predicts discontinuous velocity and thus infinite radiation. The Lorentz-Dirac equation, on the other hand, in spite of its preacceleration over distances comparable to the classical electron radius, does not display such problems.

1 Introduction
Well-known problems of the Lorentz-Dirac (LD) equation, such as runaway solutions, preacceleration, nonuniqueness, have led to proposals of modified classical equations for the relativistic motion of a point charge in electromagnetic fields, including radiation reaction. Recently Rohrlieh has asserted that forms of the Landau-Lifshitz (LL) equation represent the exact equation and should replace the flawed one (LD) derived by Dirac. The purpose of this letter is to point out that solutions to the LL equation are not generally consistent with Maxwell’s equations for the radiation. The result is a fairly simple to show but does not seem to have attracted much attention.

The LD equation for the motion of a point charge $e$ of proper velocity $u$ in an external electromagnetic field $F$ can be written in the index-free formulation based on Clifford’s geometric algebra of physical space as

$$m\dot{u} = e \langle (F + F_{\text{self}}) u \rangle_{\mathbb{R}},$$

where the first term on the right $f = e \langle Fu \rangle_{\mathbb{R}} = e \left[ Fu + (Fu)^\dagger \right]/2$ is the covariant Lorentz force, and $F_{\text{self}} = m\tau_e d(\dot{u}\bar{u})/d\tau$ is identified as the effective field of self interaction. We use SI units with $c = 1$, dots indicate derivatives with respect to proper time, and $\tau_e$ is two-thirds the time for light to cross the classical electron radius:

$$\tau_e = \frac{2 \frac{K e^2}{m}}{3} \simeq 6.266 \times 10^{-24} \text{ s}$$

(2)
with \( K = (4\pi\varepsilon_0)^{-1} \).

Equation (1) can be expanded in components to give the standard tensor-component form \( m\dot{u}^\mu = e (F^{\mu\nu} + F^{\mu\nu}_{\text{self}}) u_\nu \), where \( F^{\mu\nu}_{\text{self}} = m\tau e \frac{d}{d\tau} \left( \dot{u}^{[\mu} u^{\nu]} \right) \). The summation convention is adopted, the brackets \( \cdot \cdot \cdot \) indicate the antisymmetric part, and the metric tensor is \( \eta_{\mu\nu} = \text{diag}(1,-1,-1,-1) \). However, the component-free algebraic formulation is cleaner and offers additional computational tools. In it, paravectors (scalars plus vectors) represent spacetime vectors. For example \( u = \gamma + u = u^\mu e_\mu \) is the proper velocity with time component \( \gamma \equiv \gamma e^0 \) and spatial part \( u \).

An overbar indicates the Clifford conjugate \( \bar{u} = \gamma - u \) and the Lorentz-invariant square norm is \( u\bar{u} = \gamma^2 - u^2 = u^\mu u^\nu e_\mu \bar{e}_\nu \). Since \( u \) is a unit paravector, \( u\bar{u} = 1 \), and the proper acceleration \( \ddot{u} \) is orthogonal to \( u \): \( \langle \dot{u}\bar{u} \rangle_S = 0 \). As a consequence, \( \dot{u}\bar{u} \) is a biparavector (a spacetime plane, represented by a complex vector).

The expansion of \( F_{\text{self}} \) gives

\[
F_{\text{self}} = m\tau e \frac{d}{d\tau} \left( \dot{u} \bar{u} \right) = m\tau e \ddot{u} - P
\]

where the Lorentz-invariant Larmor power \( P = -m\tau e \dot{u} \bar{u} \) is seen from Maxwell’s equations \( \partial \mathbf{F} = \mu_0 \mathbf{J} \) to be the power radiated by the accelerating point charge. It is easily seen that the LD equation (1) conserves energy and momentum with the radiation field between any two points on the world line of the charge where the acceleration \( \dot{u} \) is the same.

The LL equation (2) is obtained by replacing \( m\dot{u} \) in the radiation term in (1) by the Lorentz force \( \mathbf{f} \)

\[
m\dot{u}_{\text{LL}} = f + \tau e \left\langle \frac{d}{d\tau} \left( \dot{f} \bar{u} \right) u \right\rangle_R
= f + \tau e \left( \bar{f} + \left\langle \mathbf{f} \bar{u} \right\rangle_S u \right),
\]

where the second line follows from the reality of \( f \) and its orthogonality with \( u \). This is the equation given by Ford and O’Connell [1], by Spohn [2], and by Rohrlich [3] in his (4a). It is easily verified that \( \dot{u} \) here remains orthogonal to \( u \). As Rohrlich noted, the last term in (1b) dictates that the Larmor radiation term \( P \) is replaced in energy-momentum conservation for the LL equation by

\[
P_{\text{LL}} = -\tau e \langle \mathbf{f} \bar{u} \rangle_S .
\]

However, this can differ from the Larmor power \( P \), and it is \( P \) that is given by Maxwell’s equations.

The LL equation is the first term in an iterative expansion of the LD equation in powers of \( \tau e \):

\[
m\dot{u}^{(n+1)} = f + m\tau e \left\langle \frac{d}{d\tau} \left( \dot{u}^{(n)} \bar{u} \right) u \right\rangle_R
\]

(6)
where \( \dot{u}^{(n)} \) is the \( n \)th-order approximation of \( \dot{u} \)

\[
\dot{u}^{(0)} = f. 
\]  

(7)

The lowest-order difference between the proper accelerations of the LD and LL equations is the second-order term \( \ddot{u} - \ddot{u}^{(1)} \approx \tau_e^2 \left( \ddot{u} + \ddot{u} \dddot{u} \right) \).

Rohrlich also gives an alternative form [his (4b)], obtained by replacing \( \dot{f} = e \langle \ddot{F} u \rangle_R + e \langle F \dddot{u} \rangle_R \) with

\[
\dot{f}_R = e \langle \ddot{u} \dddot{u} \rangle_S + \frac{e}{m} \langle F \dddot{u} \rangle_R \]  

(8)

\[
= e \langle \dddot{F} u \rangle_R + \frac{1}{m} \langle F f \rangle_R. \]  

(9)

The difference \( \dot{f}_R - \dot{f}_R \) is first order in \( \tau_e \).

2 A simple example

As a simple example by which to compare solutions of the equations, consider one-dimensional motion in a pure electric field \( \mathbf{F} = \mathbf{E} = E \mathbf{e} \), where \( \mathbf{e} \) is a fixed unit vector. Express the proper velocity in terms of the rapidity \( \omega \) as

\[
u = \exp (w \mathbf{e}) = \cosh w + \mathbf{e} \sinh w.
\]

Then \( \dot{u} = \dot{w} \mathbf{e} \) and the Lorentz force is \( f = e \mathbf{E} \mathbf{u} \). The LD equation (1) becomes

\[
\dot{w} = \frac{e E}{m} + \tau_e \dot{w},
\]

(10)

whereas the LL equation (4) has the form

\[
\dot{w}_{LL} = \frac{e}{m} \left( E + \tau_e \dot{E} \right).
\]

(11)

The LL equation is the first-order iteration of the equation

\[
\dot{u}^{(n+1)} = \frac{e E}{m} + \tau_e \dot{u}^{(n)}
\]

\[
\dot{u}^{(0)} = \frac{e E}{m}
\]

\[
\dot{u}^{(1)} = \frac{e E}{m} + \tau_e \dot{u}^{(0)} = \frac{e}{m} \left( E + \tau_e \dot{E} \right)
\]

\[
\dot{u}^{(n+1)} = \frac{e}{m} \sum_{k=0}^{n} \tau_e^k \frac{d^k}{dt^k} E \frac{1}{m} \left( 1 - \tau_e \frac{d}{dt} \right)^{-1} E
\]

(12)

In the limit \( n \to \infty \), the iterative solution is seen to approach the LD solution. To lowest order, the difference between the LD and LL equations is the second-order term \( \dot{w} - \dot{w}^{(1)} \approx \left( e/m \right) \tau_e^2 \dot{E} \), the power difference is \( P_R - P \approx -\left( e^2/m \right) \tau_e^2 \ddot{E} \), and \( \dot{f} - \dot{f}_R \approx \left( e^2/m \right) \tau_e \dddot{E} \), none of which generally vanishes.

3
Ford and O’Connell (1993) derive analytical solutions of the LL equation (11) for motion of a charge through an electric field in the shape of a step:

\[
E(x) = \begin{cases} 
0, & x < 0 \\
E_0, & 0 < x < L \\
0, & L < x
\end{cases}
\]

(13)

They also show that these are the smooth limit of numerical solutions for a smooth rise and fall of the field. Let \( \tau = 0 \) be the proper time that the charge enters the field from the left \( (x = 0) \) and \( \tau = \tau_1 \) the proper time that it exits at \( x = L \). Integration of (11) gives

\[
w_{LL} = \begin{cases} 
w_0, & \tau < 0 \\
w_0 + \varepsilon + \varepsilon \tau / \tau_e, & 0 < \tau < \tau_1 \\
w_2 = w_0 + \varepsilon \tau_1 / \tau_e, & \tau_1 < \tau
\end{cases}
\]

(14)

where

\[
\varepsilon = \frac{eE_0 \tau_e}{m} \equiv \frac{\tau_e}{L}.
\]

(15)

Note that \( w_{LL} \) is discontinuous: it jumps by \( \varepsilon \) as the charge enters the field at \( \tau = 0 \) and then by \(-\varepsilon\) as the charge leaves at \( \tau = \tau_1 \). Consequently, the acceleration has infinite spikes as the charge enters and leaves the field. In terms of Dirac delta functions \( \delta(\tau) \) and Heaviside step functions \( \theta(\tau) \),

\[
\dot{w}_{LL} = \varepsilon [\delta(\tau) - \delta(\tau - \tau_1)] + \frac{eE_0}{m} \theta(\tau) \theta(\tau_1 - \tau).
\]

(16)

Although \( \dot{w} \) and consequently \( \dot{u} \) are infinite at \( \tau = 0, \tau_1 \), they are integrable. However, the Larmor radiation, proportional to \(-uu' = \dot{w}^2\), is not. Thus, according to Maxwell’s equations for the field of a point charge, infinite energy is radiated from the discontinuities. The distance \( x \) traveled in the region \( 0 < x < L \) is related to \( \tau \) by integration

\[
x(\tau) = \int_0^\tau \sinh w(\tau') \, d\tau' = \frac{L}{\alpha} [\cosh (w_1 + \varepsilon \tau / \tau_e) - \cosh w_1]
\]

(17)

with \( w_1 = w_0 + \varepsilon \). In particular, \( x(\tau_1) = L \). The energy gain of the charge is \( m(\gamma_2 - \gamma_0) \), where \( \gamma_j = \cosh w_j \) and \( w_2 = \cosh^{-1}(\gamma_1 + \alpha + \varepsilon) \). To second order in \( \varepsilon \),

\[
\gamma_2 - \gamma_0 = (\gamma_1 + \alpha) \cosh \varepsilon - \sqrt{(\gamma_1 + \alpha)^2 - 1 \sinh \varepsilon - \gamma_0}
\]

(18)

\[
\cong \alpha + \varepsilon [u_0 - u_2^{(0)}] + \varepsilon^2 \left[ \gamma_0 + \frac{\alpha}{2} \frac{(\gamma_0 + \alpha) u_0}{u_2^{(0)}} \right]
\]

(19)

with \( u_2^{(0)} = \sqrt{(\gamma_0 + \alpha)^2 - 1} \).
Let’s compare this to solutions of the LD equation \( \text{(10)} \). In numerical solutions, runaways are avoided by integrating backward in time\( \text{[12]} \). In analytical solutions, they are avoided by assuming \( \lim_{\tau \to \infty} \tau E(\tau) = 0 \) and putting \( \dot{w}(\infty) = 0 \). This gives the usual integral form that includes brief periods of preacceleration:

\[
\dot{w}(\tau) = \frac{e}{m} \int_0^\infty ds E(\tau + \tau_s) e^{-s}
\]

\[
= \begin{cases} 
(e/m) E_0 e^{\tau_\epsilon/\tau_e} (1 - e^{-\tau_1/\tau_e}), & \tau < 0 \\
(e/m) E_0 (1 - e^{(\tau - \tau_1)/\tau_e}), & 0 < \tau < \tau_1 \\
0, & \tau_1 < \tau 
\end{cases} \quad (20)
\]

A further integration gives

\[
w(\tau) = \begin{cases} 
 w_0 + \varepsilon e^{\tau/\tau_e} (1 - e^{-\tau_1/\tau_e}), & \tau < 0 \\
w_0 + \varepsilon [1 + \tau/\tau_e - e^{(\tau - \tau_1)/\tau_e}], & 0 < \tau < \tau_1 \\
w_2 = w_0 + \varepsilon \tau_1/\tau_e, & \tau_1 < \tau 
\end{cases} \quad (21)
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0, & \tau_1 < \tau 
\end{cases} \quad (20)
\]

\[
w(\tau) = \begin{cases} 
 w_0 + \varepsilon e^{\tau/\tau_e} (1 - e^{-\tau_1/\tau_e}), & \tau < 0 \\
w_0 + \varepsilon [1 + \tau/\tau_e - e^{(\tau - \tau_1)/\tau_e}], & 0 < \tau < \tau_1 \\
w_2 = w_0 + \varepsilon \tau_1/\tau_e, & \tau_1 < \tau 
\end{cases} \quad (21)
\]

Figure 1: Rapidity \( w \) in the case of a stepped field. Solid line: the LL solution; dotted line: the LD solution. Note that \( w \) is continuous in the LD solution by virtue of the preacceleration over times of about \( \tau_\epsilon \), but in the LL case, where there is no preacceleration, \( w \) is discontinuous. (The size of \( \tau_\epsilon \) is greatly magnified for clarity.)

3 Discussion

Derivations of the LD equation \( \text{[1]} \) generally assume an expansion \( u \) in powers of the time difference corresponding to the effective size of the charge. The limit of vanishing size is then taken, traditionally with mass renormalization, although such renormalization can be avoided by taking specified combinations of the self
One cannot expect the proper velocity to be an analytic function of position in regions where the field itself is discontinuous. However, discontinuous fields are simply idealizations convenient for finding analytic solutions. Solutions of both the LD and LL equations can be found numerically for more realistic field configurations, and they approach the analytic solutions in the appropriate limit.

Rohrlich claims to have derived the LL equation as an exact classical equation for the point charge. However, his derivation, like most others, makes the substitution of an approximate expression from the Lorentz-force equation with the justification that the radiation term is small. He then claims that because higher-order derivatives of the velocity than second disappear in Dirac’s derivation when the limit of vanishing charge radius is taken, one should also be able to ignore corresponding derivatives in the field. This approach appears to argue more forcefully for the correctness of the LD equation, which as seen above conflicts with the LL equation. Rohrlich also claims that the LL equation has been obtained in a rigorous mathematical argument by Spohn, but Spohn obtains his critical surface perturbatively and does not claim it to be exact to all orders of $\tau_e$.

The LL equation differs from the LD equation only in second order in $\tau_e$ and its solutions to realistic problems are practically indistinguishable from those of the LD equation since $\tau_e$ is orders of magnitude smaller than the smallest measurable time interval. Nevertheless, as seen above, it is inconsistent with Maxwell’s equations for the radiation of a point charge and this inconsistency is dramatic in the case of rectilinear motion through a stepped field. This is in contrast to the LD equation, which is consistent.

Yaghjian has proposed a different “correction” to the LD equation. (Most of his book discusses a model of the electron as a spherical insulator of finite radius with a fixed surface charge, but the last section discusses the limit of vanishing radius to find the motion of a point charge.) He argues that the radiation terms do not act until the field is turned on and consequently should be multiplied by a scalar function that approaches a step function in the limit of a point charge. This eliminates preacceleration. Although his formulation does not explicitly treat other abrupt changes in the field, for consistency we assume that the sudden drop in the stepped field has no effect on the radiation terms until $\tau_1$ when the charge leaves the field. However, this prescription when applied to the stepped field gives precisely the motion of the charge without any radiation reaction. It is therefore also inconsistent with energy-momentum conservation and Maxwell’s equation.

As frequently pointed out, the problems of the LD equation occur at distance scales well below the Compton wavelength, where quantum effects become important. Its breakdown in the description of real particles at such scales is therefore not surprising. Attempts appear so far unsuccessful to find an alternative classical equation of motion for the point charge that is free from problems and consistent with energy-momentum conservation and Maxwell’s equations.
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