Decomposition numbers and canonical bases

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Abstract
We obtain some simple relations between decomposition numbers of quantized Schur algebras at an $n$th root of unity (over a field of characteristic 0). These relations imply that every decomposition number for such an algebra occurs as a decomposition number for some Hecke algebra of type $A$. We prove similar relations between coefficients of the canonical basis of the $q$-deformed Fock space representation of $U_q(\hat{sl}_n)$ introduced in [20]. It follows that these coefficients can all be expressed in terms of those of the global crystal basis of the irreducible sub-representation generated by the vacuum vector.

As a consequence, using works of Ariki and Varagnolo-Vasserot, it is possible to give a new proof of Lusztig’s character formula for the simple $U_v(\hat{sl}_r)$-modules at roots of unity, which does not involve representations of $\hat{sl}_r$ of negative level.

1 Introduction

It is well known that the $p$-modular decomposition numbers for the symmetric groups are equal to certain composition multiplicities of the Weyl modules of the general linear groups over an algebraically closed field of characteristic $p$ (see e.g. [12], p.341, [15], p.317). This follows from the properties of the Schur functor from representations of the general linear groups to representations of the symmetric groups. In other words, every decomposition number for a symmetric group $S_m$ occurs as a decomposition number for the associated Schur algebra $S_m$.

More recently Erdmann [9] has obtained a kind of converse result, namely that the decomposition matrix of $S_m$ is a sub-matrix of the decomposition matrix of the symmetric group $S_{m'}$ of rank $m' = pm + (p-1)(m) \frac{m}{2}$. The proof uses the Frobenius map of the general linear group and the theory of tilting modules.

The first of these results has a quantum analogue given by James [13] and Dipper-James [3]. It states that the decomposition matrix of the Hecke algebra $H_m(v)$ of type $A_{m-1}$ embeds into the decomposition matrix of the $v$-Schur algebra $S_m(v)$. But, as noted in [3], “there is no evident quantization” of Erdmann’s theorem, because the quantum analogue of the Frobenius map does not go from the quantum algebra $U_v(gl_r)$ to itself but from $U_v(gl_r)$ to the classical enveloping algebra $U(gl_r)$ (see also [4], p. 104).

Nevertheless, we will show in the first part of this note that conversely every decomposition number for $S_m(v)$ occurs as a decomposition number for the Hecke algebra $H_{m''}(v)$, where $m'' = m + 2(n-1)(m) \frac{m}{2}$ and $n$ is the multiplicative order of $v^2$. Our formula also
relies on some properties of the tilting modules, but the argument of Erdmann based on the Frobenius map is replaced by a result of Andersen relating injective $U_v(\widehat{sl}_r)$-modules to tilting modules ([1], Prop. 5.8). This result being valid only over a field of characteristic 0, the same restriction applies to our formula. So the question remains open in the so-called mixed case.

In the second part of this note we consider the coefficients of the canonical basis introduced in [20] of the Fock space representation $\mathcal{F}$ of $U_q(\widehat{sl}_n)$. These are polynomials $d_{\lambda,\mu}(q)$ indexed by pairs $(\lambda, \mu)$ of partitions, as the decomposition numbers $d_{\lambda,\mu}$ of the $v$-Schur algebras. The main conjecture of [20] was that
\begin{equation}
 d_{\lambda,\mu}(1) = d_{\lambda,\mu},
 \end{equation}
i.e. the $d_{\lambda,\mu}(q)$ are some $q$-analogues of the decomposition numbers. In [26], Varagnolo and Vasserot proved that the $d_{\lambda,\mu}(q)$ are certain parabolic Kazhdan-Lusztig polynomials for the affine symmetric groups $\widetilde{S}_r$, and thus showed that Eq. (1) results from the so-called Lusztig conjecture on the characters of the simple $U_v(sl_r)$-modules [22]. This last conjecture is proved, due to work of Kazhdan-Lusztig [18] and Kashiwara-Tanisaki [17] involving a certain category of $\widehat{sl}_r$-modules of negative level $-n - r$.

In Section 4 we prove a relation between the $d_{\lambda,\mu}(q)$ which is a $q$-analogue of the relation between decomposition numbers established in the first part. Its proof is based on a theorem of Soergel on Kazhdan-Lusztig polynomials ([24], Theorem 5.1) and on a certain duality satisfied by the canonical basis of $\mathcal{F}$, which was announced in [20] and proved in [21] (Theorem 7.14).

This implies that all the coefficients in the expansion of the canonical basis of $\mathcal{F}$ on its natural basis consisting of the pure $q$-wedges can be expressed in terms of those of the global crystal basis of the simple submodule $L(\Lambda_0)$ generated by the vacuum vector.

It follows from these two sets of relations (between decomposition numbers on one hand, and between coefficients of the canonical basis on the other hand) that it is enough for proving Eq. (1) to verify it in the case where $\mu$ is an $n$-regular partition. This was done by Ariki in [4], using results of Kazhdan-Lusztig and Ginzburg on representations of affine Hecke algebras, and the geometric description by Lusztig of the canonical basis of $U_q(\widehat{sl}_n)$. Therefore, as shown in Section 3, Ariki’s theorem implies Lusztig’s formula for the characters of the simple $U_v(sl_r)$-modules, as well as Soergel’s formula for the characters of the indecomposable tilting $U_v(\widehat{sl}_r)$-modules [25].

Varagnolo and Vasserot have already attempted in [26] to give a proof of Lusztig’s formula based on the level 1 Fock space representation $\mathcal{F}$ of $U_q(\widehat{sl}_n)$. They have extended the action of $U_q(\widehat{sl}_n)$ on $\mathcal{F}$ to an action of the Hall algebra of the cyclic quiver, and have defined in this way, using intersection cohomology, a geometric basis $\mathcal{B}$ of $\mathcal{F}$. They conjectured that $\mathcal{B}$ coincides with the canonical basis of [20], which is defined in an algebraic way, and proved that this equality would amount to a $q$-analogue of Lusztig’s formula. This very interesting conjecture is proved by Schiffmann in the case $n = 2$ [23], but remains open in general.

2 Statement of results

We fix two integers $m, n \geq 2$ and denote by $D_m$ the decomposition matrix of the quantized Schur algebra $S_m(v)$ over a field of characteristic 0, with parameter $v$ such that $v^2$ is a
Here we regard $S_m(v)$ as a suitable quotient of the quantized enveloping algebra $U_v(gl_r)$ for $r \geq m$, as in [14]. The rows and columns of $D_m$ are labelled by the set of partitions of $m$, and we use the notational convention of [14], that is, the entry on row $\lambda$ and column $\mu$ is

$$d_{\lambda,\mu} := [W(\lambda') : L(\mu')]_{S_m(v)},$$

(2)

where $\lambda'$ stands for the partition conjugate to $\lambda$, and $W(\lambda)$, $L(\lambda)$ are respectively the Weyl module and the simple module with highest weight $\lambda$.

We fix some $r \geq 2$ and put $\rho_r = (r-1,r-2,\ldots,1,0)$. A partition $\lambda$ with at most $r$ parts is identified with an $r$-tuple in $\mathbb{N}^r$ in the standard way by appending a tail of 0. Given two such partitions $\lambda$, $\mu$ of $m$, we shall define two partitions $\hat{\mu}$ and $\check{\lambda}$ of $m'' = m + (n-1)r(r-1)$ in the following way. There is a unique decomposition $\mu = \mu^{(0)} + n\mu^{(1)}$ with the partition $\mu^{(0)}$ $n$-restricted, that is, the difference between any two consecutive parts is $< n$. Define

$$\hat{\mu} = 2(n-1)\rho_r + w_0(\mu^{(0)}) + n\mu^{(1)},$$

(3)

$$\check{\lambda} = \lambda + ((n-1)(r-1),\ldots,(n-1)(r-1)),$$

(4)

where for $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_r)$ we set $w_0(\alpha) = (\alpha_r,\alpha_{r-1},\ldots,\alpha_1)$. It is easy to check that the parts of $\hat{\mu}$ are pairwise distinct, and thus $\hat{\mu}$ is always an $n$-regular partition, i.e. no part has multiplicity $\geq n$.

**Theorem 1** For all partitions $\lambda$, $\mu$ of $m$ of length $\leq r$, we have

$$d_{\lambda',\mu'} = d_{\lambda',\check{\lambda}^{\hat{\mu}}}.\tag{5}$$

Since $\hat{\mu}$ is $n$-regular, we see that $d_{\lambda',\check{\lambda}^{\hat{\mu}}}$ is a decomposition number for the Hecke algebra $H_{m''}(v)$, and therefore if $r \geq m$ every decomposition number for $S_m(v)$ occurs as a decomposition number for $H_{m''}(v)$.

Let now $d_{\lambda,\mu}(q)$ be the polynomial defined in [20] (see also [21], Section 7). The affine symmetric group $\tilde{S}_r$ acts on $\mathbb{Z}^r$ via its level $n$ action $\pi_n$ (see [21], Section 2), and the $q$-analogue of the linkage principle states that $d_{\lambda,\mu}(q) \neq 0$ only if $\lambda + \rho_r$ and $\mu + \rho_r$ belong to the same orbit under $\tilde{S}_r$, i.e. the partitions $\lambda$ and $\mu$ have the same $n$-core. Let $\nu$ be the unique point of the orbit $\tilde{S}_r(\mu + \rho_r)$ such that $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_r$ and $\nu_1 - \nu_r \leq n$. The stabilizer of $\nu$ is a standard parabolic subgroup of $\tilde{S}_r$ and we write $\ell_\mu$ for the length of its longest element. We also put $\ell = r(r-1)/2$, the length of the longest element of $\tilde{S}_r$.

**Theorem 2** For all partitions $\lambda$, $\mu$ of $m$ of length $\leq r$, we have

$$d_{\lambda',\mu'}(q) = q^{\ell_\mu} d_{\lambda',\check{\lambda}^{\hat{\mu}}}(q^{-1}).\tag{6}$$

Since $\hat{\mu}$ is $n$-regular, it labels a vector of the global crystal basis of the simple module $L(\Lambda_0)$. Hence every vector

$$G_{\mu'}^+ = \sum_{\lambda} d_{\lambda',\mu'}(q) |\lambda'\rangle$$

(7)

of the canonical basis of the Fock space $\mathcal{F}$ can be easily computed from the corresponding vector $G_{\mu}^+$ of the global basis of $L(\Lambda_0)$.
Example 3 Let $n = 3$ and $\mu = (6,2,1)$, a partition of $m = 9$. The non-zero $d_{\lambda^\prime,\mu^\prime}$ and $d_{\lambda^\prime,\mu^\prime}(q)$ are obtained for $\lambda = (6,2,1)$, $(7,1,1)$, $(6,3)$, $(8,1)$. We can take $r = 3$, so that $m'' = 9 + 2.3.2 = 21$. We have

$$
\mu = \mu^{(0)} + 3\mu^{(1)} = (3,2,1) + 3(1,0,0),
\tilde{\mu} = (1,2,3) + 4(2,1,0) + 3(1,0,0) = (12,6,3).
$$

Hence, taking $\tilde{\lambda} = (10,6,5)$, $(11,5,5)$, $(10,7,4)$, $(12,5,4)$, respectively, we have

$$
d_{\lambda^\prime,(3,2,1,1,1,1)} = d_{\tilde{\lambda},(12,6,3)}. $$

Moreover, $\mu + \rho_r = (8,3,1)$ is regular for the level 3 action of $\tilde{S}_3$, hence $\ell_\mu = 0$ and we have

$$
d_{\lambda^\prime,(3,2,1,1,1,1)}(q) = q^3 d_{\tilde{\lambda},(12,6,3)}(q^{-1}).
$$

3 Proof of Theorem 1

Let $m, r$ be two integers $\geq 2$. Let $U_\nu(gl_r)$ and $U_\nu(sl_r)$ denote Lusztig’s restricted specialization at $\nu$ of the quantum enveloping algebras of $gl_r$ and $sl_r$, respectively. As mentioned above, we regard the $\nu$-Schur algebra $S_{m,r}(\nu)$ as the homomorphic image of $U_\nu(gl_r)$ obtained via its action on the $m$th tensor power of the vector representation, and we put $S_m(\nu) = S_{m,m}(\nu)$ (see [3], [5]). Thus every $S_{m,r}(\nu)$-module becomes a $U_\nu(gl_r)$-module and by restriction a $U_\nu(sl_r)$-module. We shall abuse notation and denote in the same way these various modules. The simple modules and Weyl modules of $S_{m,r}(\nu)$ are labelled by the set of partitions of $m$ with at most $r$ parts, and are denoted by $L(\lambda)$ and $W(\lambda)$. It is known that for such partitions $\lambda, \mu$,

$$
[W(\lambda) : L(\mu)]_{S_{m,r}(\nu)} = [W(\lambda) : L(\mu)]_{S_m(\nu)}.
$$

Hence, it is enough to determine the decomposition numbers $d_{\lambda,\mu}$ as defined in (2).

Let now $\lambda, \mu$ be two partitions of $m$ and fix an integer $r$ greater or equal to the number of parts of both $\lambda$ and $\mu$, for example $r = m$. We have

$$
d_{\lambda,\mu} = [W(\lambda) : L(\mu)]_{S_{m,r}(\nu)} = [W(\lambda) : L(\mu)]_{U_\nu(gl_r)} = [W(\lambda) : L(\mu)]_{U_\nu(sl_r)}.
$$

Let $I(\mu)$ denote the injective hull of the $U_\nu(sl_r)$-module $L(\mu)$. By the reciprocity formula (see [3]) we have

$$
[W(\lambda) : I(\mu)]_{U_\nu(sl_r)} = [I(\mu) : W(\lambda)]_{U_\nu(sl_r)}.
$$

Given a partition $\alpha$ of $m$ of length $\leq r$, let $T(\alpha)$ denote the unique indecomposable tilting $S_{m,r}(\nu)$-module such that $\alpha$ is the maximal partition $\beta$ (for the usual dominance order) for which $[T(\alpha) : L(\beta)] \neq 0$. Consider the $S_{m',r}(\nu)$-module $T(\tilde{\mu})$, where $\tilde{\mu}$ is defined by (3). Then, regarding $T(\tilde{\mu})$ as a $U_\nu(sl_r)$-module, we have by [3], Prop. 5.8 that $T(\tilde{\mu})$ is isomorphic to $I(\mu)$. (In [3] there are some restrictions on $n$, that is, on the multiplicative order of the root of unity $\nu^2$. These restrictions have been later removed in [3], and although the statement of [3], Prop. 5.8 is not given in that paper, the arguments easily carry over.) Thus, noting that the $S_{m,r}(\nu)$-module $W(\lambda)$ and the $S_{m',r}(\nu)$-module $W(\tilde{\lambda})$ become isomorphic when considered as $U_\nu(sl_r)$-modules, we get

$$
d_{\lambda,\mu} = [T(\tilde{\mu}) : W(\lambda)]_{U_\nu(sl_r)} = [T(\tilde{\mu}) : W(\tilde{\lambda})]_{S_{m',r}(\nu)} = [T(\tilde{\mu}) : W(\tilde{\lambda})]_{S_{m',r}(\nu)}.
$$
Finally, by [8], Proposition 8.2 (a), or [8], Proposition 4.15 (ii),

\[ [T(\hat{\mu}) : W(\hat{\lambda})]_{S_{\alpha}(\mu)} = [W(\hat{\lambda}') : L(\hat{\mu}')]_{S_{\alpha}(\mu)} = d_{\hat{\mu},\hat{\lambda}}^- , \]  

(12)

and Theorem 1 is proved.

4 Proof of Theorem 2

We shall use the same notation as in [21]. In particular \( G_r, \tilde{G}_r \) and \( \hat{G}_r \) stand for the symmetric group, the affine symmetric group and the extended affine symmetric group, respectively, and \( H_r, \tilde{H}_r \) and \( \hat{H}_r \) are the corresponding Hecke algebras. The standard generators of \( \hat{G}_r \) are denoted by \( s_0, \ldots, s_r, \) and \( G_r, \tilde{G}_r, \hat{G}_r \) are the subgroups generated by \( s_1, \ldots, s_r \) and \( s_0, \ldots, s_r, \) respectively. We write \( w_0 \) for the longest element of \( G_r. \) For \( x, w \in \hat{G}_r, \) we have the Kazhdan-Lusztig polynomial \( P_{x,w}(q), \) and the inverse Kazhdan-Lusztig polynomial \( Q_{x,w}(q) \) defined via the equations

\[ \sum_{x \in \hat{G}_r} Q_{x,z}(-q)P_{x,w}(q) = \delta_{z,w}, \quad (z, w \in \hat{G}_r). \]  

(13)

The group \( \hat{G}_r \) acts on \( P_r := \mathbb{Z}^r \) via the level \(-n\) action \( \pi_{-n}, \) giving rise to the parabolic Kazhdan-Lusztig polynomials \( P_{\mu,\lambda}^- \) indexed by \( \lambda, \mu \in P_r. \) Let \( Q_{\mu,\lambda}^- \) denote the inverse parabolic Kazhdan-Lusztig polynomial defined by the equations

\[ \sum_{\mu \in P_r} Q_{\mu,\alpha}^- P_{\mu,\lambda}^- = \delta_{\alpha,\lambda}, \quad (\alpha, \lambda \in P_r). \]  

(14)

We recall (see [21], Section 2) that if \( \lambda_i > \lambda_{i+1} \) then

\[ P_{\mu,\lambda}^- = \begin{cases} 0 & \text{if } \mu_i = \mu_{i+1}, \\ qP_{\mu,\lambda}^- & \text{if } \mu_i > \mu_{i+1}. \end{cases} \]  

(15)

Thus if \( \lambda \) is strictly dominant, \( i.e. \lambda \in P_r^{++} := \{ \alpha \in P_r | \alpha_1 > \alpha_2 > \cdots > \alpha_r \}, \) we get that \( P_{\mu,\lambda}^- = 0 \) for \( \mu \notin \tilde{G}_r P_r^{++}, \) \( i.e. \) if the coordinates of \( \mu \) are not pairwise distinct, and otherwise

\[ P_{\beta,\lambda}^-(q) = q^{l(s)} P_{\beta,\lambda}^-(q), \quad (\beta \in P_r^{++}, s \in \tilde{G}_r). \]  

(16)

Let us fix \( \lambda, \alpha \in P_r^{++}. \) Using (14) we obtain

\[ \delta_{\alpha,\lambda} = \sum_{\mu \in P_r} Q_{\mu,\alpha}^- P_{\mu,\lambda}^-. \]  

Hence, setting

\[ r_{\beta,\alpha}(q) := \sum_{s \in \tilde{G}_r} (-q)^{l(s)} Q_{s\beta,\alpha}^-(q), \quad (\alpha, \beta \in P_r^{++}), \]  

(17)

we have obtained
Lemma 4 For $\alpha, \lambda \in P_{r}^{++}$, there holds
\[ \sum_{\beta \in P_{r}^{++}} r_{\beta, \alpha}(-q)P_{\beta, \lambda}^{-}(q) = \delta_{\alpha, \lambda}. \quad (18) \]
\[ \square \]

Now comparing Eq. (18) with [21], Eq. (91) and Corollary 7.15 we get

Proposition 5 Let $\lambda, \mu$ be two partitions of $m$ with at most $r$ parts, and put $\beta = \lambda + \rho_r$, $\alpha = \mu + \rho_r$. Then
\[ r_{\beta, \alpha}(q) = d_{\lambda', \mu'}(q). \]
\[ \square \]

Let $W^{f}$ denote the set of minimal length representatives for the right cosets $\mathfrak{S}_r \backslash \tilde{\mathfrak{S}}_r$. We shall express the polynomials $r_{\beta, \alpha}(q)$ in terms of the
\[ m^{x, w}(q) := \sum_{s \in \mathfrak{S}_r} (-q)^{(w(a)) - \ell(s)} Q_{s x, w}(q), \quad (x, w \in W^{f}). \quad (19) \]

These are the inverse parabolic Kazhdan-Lusztig polynomials coming from the right action of $\tilde{H}_r$ on the parabolic module $1_{q} \otimes_{H_r} \tilde{H}_r$, where $1_{q}$ denotes the 1-dimensional right $H_r$-module given by $1_{q}T_i = q1_{q}$, $(1 \leq i \leq r - 1)$, (see [24], Proposition 3.7).

We need to recall some notation from [24]. Let $k \in \mathbb{Z}^+$. We write $A_{r,k}$ for the fundamental alcove in $P_r$ associated with the level $k$ action $\pi_k$ of $\tilde{\mathfrak{S}}_r$. For $\lambda \in P_r$, we denote by $w(\lambda, k)$ the element of $\tilde{\mathfrak{S}}_r$ of minimal length such that $w(\lambda, k)^{-1}\lambda \in A_{r,k}$.

In [24], Soergel works with $\tilde{\mathfrak{S}}_r$ and $\tilde{H}_r$ instead of $\tilde{\mathfrak{S}}_r$ and $H_r$. Recall that every element $w \in \tilde{\mathfrak{S}}_r$ can be expressed uniquely as $w = \sigma r^a$, $(\sigma \in \tilde{\mathfrak{S}}_r, a \in \mathbb{Z})$. We define a projection of $\tilde{\mathfrak{S}}_r$ onto $\mathfrak{S}_r$ by setting $\underline{w} := \sigma$.

Let $\mu = \sigma \lambda \in P_r$ with $\sigma \in \tilde{\mathfrak{S}}_r$. Then clearly $\sum_{i=1}^{r} \mu_i = \sum_{i=1}^{r} \lambda_i$, hence $\tilde{\mathfrak{S}}_r$ acts on the hyperplane $\sum_{i=1}^{r} \lambda_i = 0$ that we may identify with $P_r := P_r/\mathbb{Z}(1, \ldots, 1)$. We denote by $\lambda \mapsto \underline{\lambda}$ the natural projection $P_r \rightarrow P_r$, and we take
\[ A_{r,n} := \{ \underline{\lambda} \mid \lambda_1 \geq \cdots \geq \lambda_r, \lambda_1 - \lambda_r \leq n \} \]
as fundamental alcove of $\tilde{\mathfrak{S}}_r$ acting on $P_r$ via the action $\underline{x}_n$ induced by $\pi_n$. Let $w(\lambda, n)$ denote the element of $\tilde{\mathfrak{S}}_r$ of minimal length such that $w(\lambda, n)^{-1}\lambda \in A_{r,n}$. Then one has $w(\lambda, n) = w(\lambda, n)$. We shall also use the shorthand notation $w_{\alpha} := w(\alpha, n)$.

Proposition 6 For $\alpha, \beta \in P_{r}^{++}$, there holds
\[ r_{\beta, \alpha}(q) = m^{w_{\beta}, w_{\alpha}}(q). \]

Proof — First we note that if $\alpha \in P_{r}^{++}$ then $w_{\alpha} \in W^{f}$. Indeed, since the alcove $A = w_{\alpha}A_{r,n}$ contains $\underline{a} \in P_{r}^{++}$, it has to be contained in $P_{r}^{+}$, which is equivalent to $w_{\alpha} \in W^{f}$.

Let us rewrite the definition (17) of $r_{\beta, \alpha}$ in terms of the (ordinary) inverse Kazhdan-Lusztig polynomials $Q_{x, w}$. By [24], Proposition 3.7, we have
\[ Q_{\mu, \lambda}^{-} = Q_{w(\mu, -n), w(\lambda, -n)}, \quad (\lambda, \mu \in P_r). \]
We have to relate the level $-n$ and $+n$ actions of $\widetilde{S}_r$ and $\mathfrak{S}_r$ on $P_r$. Let $\tau$ denote the automorphism of $\mathfrak{S}_r$ defined by $\tau^w = \tau^{-1}$ and $s_i^w = s_{-i}$, where $i$ is understood modulo $r$. Then it is easy to check that

$$w(s_{\beta},-n) = (sw_0).w(\beta,n) = sw_0 \cdot w_\beta^s, \quad (\beta \in P_r^{++}, \ s \in \mathfrak{S}_r).$$

Therefore, noting that $\mathfrak{S}_r$ is stable under $\tau$ and $Q_{x^\tau,y^\tau} = Q_{x,y}$, we get for $\alpha, \beta \in P_r^{++}$,

$$r_{\beta,\alpha}(q) = \sum_{s \in \mathfrak{S}_r} (-q)^{\ell(s)}Q_{sw_0w_\beta,w_0w_\alpha}(q) = \sum_{s \in \mathfrak{S}_r} (-q)^{\ell(w_0) - \ell(s')}Q_{sw_\beta,w_0w_\alpha}(q) = m_{w_\beta,w_\alpha}(q).$$

In [24], Soergel considers a map $A \mapsto \tilde{A}$ on the set of $\mathfrak{S}_r$-alcoves. The next lemma relates this operation to the map $\mu \mapsto \tilde{\mu}$ on partitions defined in Section 4.

**Lemma 7** Let $\mu$ be a partition with at most $r$ parts, and put $\alpha := \mu + \rho_r \in P_r^{++}$. Let $\xi$ be the point in $A_{\alpha,r,n}$ congruent to $\alpha$ under $\mathbb{Z}_n$, and set $\hat{A} := w_\alpha A_{\alpha,r,n}$. Then

$$\hat{A} = \tilde{w} A_{\alpha,r,n}$$

where $\tilde{w} \in W^F$ is given by $\tilde{w} = w(\hat{\mu} + \rho_r,n)w_{0,\xi}$, and $w_{0,\xi}$ is the longest element of the stabilizer of $\xi$ in $\tilde{S}_r$.

**Proof** — By definition of $\mu(0)$ and $\mu(1)$ (see Section 2), the translated alcove $B := A - n\mu(1)$ lies in $n\Pi$, where

$$n\Pi = \{A \in P_r \mid 0 \leq \lambda_i - \lambda_{i+1} \leq n, \ i = 1, \ldots, r\}$$

is the fundamental box for the level $n$ action of $\tilde{S}_r$. Hence, by definition of $\tilde{A}$ (see [24]), we can write $\tilde{A} = w_0(B) + 2n\rho_r + n\mu(1)$, which shows that $\tilde{A}$ contains $\hat{\mu} + \rho_r$. Thus, if the stabilizer of $\xi$ is trivial, we are done. Otherwise, since $w_{\alpha}$ is of minimal length among the elements $w \in \tilde{S}_r$ such that $w\xi = \alpha$, we see that $A$ is the lowest alcove adjacent to $\alpha$, and therefore $\hat{A}$ is the highest alcove adjacent to $\hat{\mu} + \rho_r$, which means that $\hat{A} = \tilde{w} A_{\alpha,r,n}$ with $\tilde{w} = w(\hat{\mu} + \rho_r,n)w_{0,\xi}$. $\square$

**Proposition 8** Let $\lambda$, $\mu$ be partitions of $m$ with at most $r$ parts. Put $\beta = \lambda + \rho_r$, $\alpha = \mu + \rho_r$. Then, with the notation of Theorem 4

$$r_{\beta,\alpha}(q) = q^{\ell-\ell_\mu} d_{\lambda,\beta}(q^{-1}).$$

**Proof** — Let $\xi = w_{\alpha}^{-1} \alpha$. For $s \in S_r$, we have

$$Q_{sw_\beta,w_0w_\alpha}(q) = (-q)^{-\ell(w_0,\xi)}Q_{sw_\beta w_0,\xi,w_0w_\alpha}(q)$$

(see [24], proof of Proposition 3.7). Hence, we can rewrite Proposition 8 as

$$r_{\beta,\alpha}(q) = (-q)^{-\ell(w_0,\xi)} m_{w_\beta w_0,\xi,w_\alpha}(q).$$

By [24], Theorem 5.1, and Lemma 7 above, we get

$$m_{w_\beta w_0,\xi,w_\alpha}(q) = q^{\ell(w_0)} m_{w_\beta w_0,\xi,w_0,\xi}(q^{-1}),$$
where for \( x, y \in W_f \),
\[
n_{x,y}(q) = \sum_{s \in \mathcal{S}_r} (-q)^{\ell(s)} P_{s,x,y}(q),
\]
(\cite{24}, Prop. 3.4). On the other hand, the expression of \( d_{\lambda,\mu}(q) \) in terms of Kazhdan-Lusztig polynomials given by Varagnolo-Vasserot may be written as
\[
d_{\lambda,\mu}(q) = \sum_{s \in \mathcal{S}_r} (-q)^{\ell(s)} P_{s,w_{\beta w_0,\xi},w_\alpha w_0,\xi}(q)
\]
(see \cite{21}, Eq. (93)). Hence
\[
d_{\lambda,\mu}(q) = n_{w_{\beta w_0,\xi},w_\alpha w_0,\xi}(q),
\]
and using the fact that \( \lambda = \tilde{\lambda} \), we get
\[
r_{\beta,\alpha}(q) = (-q)^{\ell(w_0) - \ell(w,\xi)} n_{w_{\beta w_0,\xi},w_\alpha w_0,\xi}(q^{-1}) = (-q)^{\ell(w_0) - \ell(w,\xi)} d_{\lambda,\mu}^{-1}(q^{-1}).
\]

Combining Proposition 3 and Proposition 8, we have proved Theorem 2.

5 Lusztig’s and Soergel’s formulas

In \cite{20} (see also \cite{21}), the polynomials \( d_{\lambda,\mu}(q) \) and \( e_{\lambda,\mu}(q) \) were introduced as the coefficients of two canonical bases \( \{ \mathcal{G}_{\lambda}^+ \} \) and \( \{ \mathcal{G}_{\lambda}^- \} \) of the Fock space \( \mathcal{F} \), and it was conjectured that
\[
\begin{aligned}
d_{\lambda,\mu}(1) &= [W(\lambda') : L(\mu')] = d_{\lambda,\mu}, \\
e_{\lambda,\mu}(-1) &= [L(\mu) : W(\lambda)].
\end{aligned}
\]
(The notation of \cite{21} slightly differs from that of \cite{24}, and here we follow \cite{21}.) Let \( S(\lambda) \) denote the Specht module for \( H_m(v) \) corresponding to \( \lambda \). If \( \mu \) is \( n \)-regular, \( S(\mu) \) has a simple head that we denote by \( D(\mu) \), and it is known \cite{3} that
\[
[S(\lambda) : D(\mu)] = d_{\lambda,\mu}.
\]
The subset \( \{ \mathcal{G}_{\lambda}^+ \mid \mu \text{ is } n\text{-regular} \} \) is nothing but Kashiwara’s global crystal basis \cite{16} of \( U^-_q(\widehat{sl}_n) \), and it had been previously conjectured in \cite{19} that for \( \mu \) \( n \)-regular
\[
d_{\lambda,\mu}(1) = [S(\lambda) : D(\mu)].
\]
This conjecture was proved by Ariki \cite{4} (see \cite{10} for a detailed review of this work).

Using Theorem 1 and Theorem 2 we obtain immediately that \( \cite{23} \) implies \( \cite{21} \), that is, Ariki’s theorem yields also a proof of the main conjecture of \( \cite{20} \).

On the other hand, we know that the matrix \([e_{\lambda,\mu}^{-1}(q)]\) is the inverse of \([d_{\lambda,\mu}(q)]\) (see \cite{20} and \cite{21}, Theorem 7.14). Hence \( \cite{21} \) implies \( \cite{23} \).

It was proved by Varagnolo-Vasserot \cite{26} that the \( e_{\lambda,\mu}(q) \) are exactly the parabolic Kazhdan-Lusztig polynomials occurring in Lusztig’s formula for the expression of the character of \( L(\mu) \) in terms of those of the \( W(\lambda) \). Therefore, we see that Lusztig’s formula can be derived from Ariki’s theorem.

Finally, using again the following result of Du-Parshall-Scott and Donkin (see Section 3)
\[
[T(\mu) : W(\lambda)] = [W(\lambda') : L(\mu')] = d_{\lambda,\mu},
\]
and the expression \( \cite{20} \) of \( d_{\lambda,\mu}(q) \) as a parabolic Kazhdan-Lusztig polynomial obtained by Varagnolo-Vasserot (see also Goodman-Wenzl \cite{11}), we recover Soergel’s formula for the character of the tilting module \( T(\mu) \).
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