Evaluate of time delays influence on effectiveness of nonlinear saturation controller. Part I: Modelling

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Abstract. In this paper a numerical model of a strongly nonlinear beam with attached vibration control system is presented. To reduce vibration a saturation controller is applied. Its input and output are time delayed signals. To estimate an influence of time delays on the control effectiveness, two indicators are proposed. The first indicator is quantitative, while the second one – qualitative is based on value of the largest Lyapunov exponent. The numerical procedure for estimating this exponent is developed and tested.

1. Introduction

Vibrations control of mechanical systems are the subject of research presented in many scientific articles. Among the available methods the Nonlinear Saturation Control (NSC) is one of the most interesting. The basic advantage of this algorithm is presented in paper [1]. The authors tested the system where linear plant (cantilever beam) and NSC controller were coupling through a sensor and piezoelectric actuator. The results of considerations are the conditions that must be met to saturation phenomenon occurs. The amplitude of beam vibrations does not increase, it can be constant despite the increase of the excitation force amplitude. In paper [2], the saturation control was used to reduce the vibration of a nonlinear plant. In beam model the geometrical and inertial nonlinearities are included together with nonlinear damping (self-excitation). Then the effectiveness of the NSC algorithm was tested showing the possibility of chaotic motion existence. In both papers [1, 2] the control subsystem is idealized. In real systems properties of sensor, piezoelectric actuator and digital signal processing (DSP) controller can influence on NSC method efficiency. Time delays in the control system are the key elements. In paper [3], the possibility of NSC controller inputs and output delays is included. Consequently, the differential equations of mechanical part and controller are coupled by terms with time delays. However, in time delayed systems the qualitative analysis of motion is more difficult. The basic criterion for the evaluation of motion (regular or irregular) is a value of the largest Lyapunov exponent. To determine this exponent in systems with time delay or discontinuities the synchronization method can be used. Applications of this method are presented in papers [4–6]. The authors present estimation of the largest Lyapunov exponent for Duffing oscillator with dry friction [3], Van der Pol oscillator with time delay [4] and Duffing oscillator with impact [6]. Other authors also used this method. For example, in paper [7], it helped to identify chaotic areas the time delayed Duffing's system.

In this paper the modelling process of system with delayed saturation controller is presented. The idealized model from paper [2] is modified to include time delay phenomenon. The possibility of
quantitative and qualitative analysis requires the development of appropriate procedures. The paper presents proposed algorithms and verification of their correctness.

2. Numerical model

The two degrees of freedom system consists of two subsystems. The first, a mechanical subsystem, models the nonlinear beam. The simplified model is written by ordinary differential equation of motion [2]:

\[
\ddot{x}_1 + \alpha_1 \dot{x}_1 + \alpha_1 x_1^3 + \omega_x^2 x_1 + \beta \dot{x}_1^3 + \delta (x_1 \dot{x}_1^2 + x_1^2 \dot{x}_1) = x_0 \mu \omega^2 \sin(\alpha t) + \gamma_1 x_2^2
\]  

(1)

where \(x_1\) is coordinate which describes the deformation of the beam (relative motion), \(\alpha_1, \alpha_1\) are damping coefficients of Rayleigh's function, \(\omega_1\) is natural frequency of first bending mode, \(\beta\) is parameter using to describe geometrical nonlinearities, \(\delta\) is parameter of inertial nonlinearities. The right side of equation (1) has two terms: the external force and the control force. The beam is excited kinematically \(x_0 \mu \omega^2 \sin(\alpha t)\) with a constant displacement amplitude of shaker armature \(x_0 = \text{constant}\). This kind of excitation generates external force \(x_0 \mu \omega^2 \sin(\omega t)\), where \(\omega\) is frequency of excitation and \(\mu\) is parameter.

The control force depends on \(\gamma_1\) control gain and \(\gamma_2\) controller coordinate (quadratic nonlinearity). Response \(x_2\) of the second electrical subsystem is calculated from:

\[
\ddot{x}_2 + \alpha_2 \dot{x}_2 + \omega_x^2 x_2 = \gamma_2 x_1 \dot{x}_1 x_2
\]  

(2)

where \(\alpha_2\) is linear damping coefficient, \(\omega_2\) is natural frequency of the controller and \(\gamma_2\) is control gain. Values of dimensionless parameters are taken from [2]: \(\alpha_1 = 0.015, \alpha_2 = 0.05, \omega_1 = 3.06309, \beta = 41.4108, \delta = 3.2746, \mu = 0.89663, x_0 = 0.01, \alpha_2 = 0.00306, \omega_2 = \omega_1/2, \gamma_1 = 0.01, \gamma_2 = 2\).

Since, in every real control system delays exist. Therefore, using suggestions proposed in reference [3] equations (1) and (2) is modified to the following form:

\[
\ddot{x}_1 + \alpha_1 \dot{x}_1 + \alpha_1 x_1^3 + \omega_x^2 x_1 + \beta \dot{x}_1^3 + \delta (x_1 \dot{x}_1^2 + x_1^2 \dot{x}_1) = x_0 \mu \omega^2 \sin(\alpha t) + \gamma_1 x_2^2 (t - t_3)^2
\]  

(3)

\[
\ddot{x}_2 + \alpha_2 \dot{x}_2 + \omega_x^2 x_2 = \gamma_2 x_1 (t - t_1) \dot{x}_2 (t - t_2)^2
\]  

(4)

where \(t_1, t_2, t_3\) are time delays. Now, both subsystems are coupled by time delayed terms. Term with \(t_1\) and \(t_2\) is controller input. Whereas term with \(t_3\) is controller output. Parameters \(t_1, t_2, t_3\) can have different values. When \(t_1 = t_2 = t_3 = 0\) the system is without time delays.

Different values of time delays can cause quantitative and qualitative changes in system motion. During numerical analysis two proposed indicators are used. The first indicator \(J_1\) compares the vibration level for the system with and without time delays:

\[
J_1(t_1, t_2, t_3) = \frac{\text{RMS}(x_1)}{\text{RMS}(x_1^*)}
\]  

(5)

where RMS is the root mean square of mechanical response \(x_1\). The indicator \(J_1\) is calculated from time series for steady states from two simulations. First simulation allows to find \(\text{RMS}(x_1)\) when values of \(t_1, t_2, t_3\) are variables of analysis. Whereas \(\text{RMS}(x_1^*)\) is computed from simulation for non-delayed system when \(t_1 = t_2 = t_3 = 0\). The second indicator allows to specify the kind of motion. It can take three values, respectively for periodic, quasi-periodic and chaotic motion. Based on the largest Lyapunov exponent LLE, the indicator \(J_2\) is written in the form:

\[
J_2(t_1, t_2, t_3) = \begin{cases} 
-1 & \text{LLE} < 0 \\
0 & \text{LLE} = 0 \\
1 & \text{LLE} > 0 
\end{cases}
\]  

(6)

However, calculation of the largest Lyapunov exponent for delayed system is problematic. Therefore, in this investigation the synchronization phenomenon is used to determine LLE.
3. Evaluation of largest Lyapunov exponent

In this section the method of estimation LLE is described. Two identical systems should be unidirectionally coupled. Transparent presentation of the procedure requires a reduced of second order differential equations (3-4) to the first order differential equations. New variables $q_1 = x_1$, $q_2 = dx_1/dt$, $q_3 = x_2$ and $q_4 = dx_2/dt$ are used. As a result, equations (3-4) are transformed to form:

$$
\dot{q}_1 = q_2
$$

$$
\dot{q}_2 = \frac{1}{1 + \Delta q_1^2} \left( -\alpha_1 q_2 - \alpha_2 q_2^3 - \omega_1 q_1 - \beta q_2 - \Delta q_1 q_2^2 + x_0 \mu \omega^2 \sin(\omega t) + \gamma_1 q_2 (t - \tau_3)^2 \right)
$$

$$
\dot{q}_3 = q_4
$$

$$
\dot{q}_4 = -\alpha_2 q_4 - \omega_2 q_3 + \gamma_2 q_1 (t - \tau_1) q_3 (t - \tau_2)
$$

Second system is identical, but it is coupled to first system using feedback mechanism:

$$
\dot{Q}_1 = Q_2 + d(Q_1 - q_1)
$$

$$
\dot{Q}_2 = \frac{1}{1 + \Delta Q_1^2} \left( -\alpha_1 Q_2 - \alpha_2 Q_2^3 - \omega_1 Q_1 - \beta Q_2^3 - \Delta Q_1 Q_2^2 + x_0 \mu \omega^2 \sin(\omega t) + \gamma_1 Q_2 (t - \tau_3)^2 + d(Q_2 - q_2) \right)
$$

$$
\dot{Q}_3 = Q_4 + d(Q_3 - q_3)
$$

$$
\dot{Q}_4 = -\alpha_2 Q_4 - \omega_2 Q_3 + \gamma_2 Q_1 (t - \tau_1) Q_3 (t - \tau_2) + d(Q_4 - q_4)
$$

where $d$ is coupling coefficient. At the beginning of the simulation the responses of both systems can not be the same. Therefore, for equations (7–10) following initial conditions $q_{10}$, $q_{20}$, $q_{30}$, $q_{40}$ are used. The saturation controller is special method that requires a non-zero initial conditions to its activation. One of the values $q_{30}$, $q_{40}$ must be different from zero. Whereas for equations (11–14) initial conditions result from the disturbance of the original trajectory $Q_{10} = q_{10} + \epsilon$, $Q_{20} = q_{20} + \epsilon$, $Q_{30} = q_{30} + \epsilon$, $Q_{40} = q_{40} + \epsilon$. Generally, a very small disturbance $\epsilon$ is introduced. The responses of both systems can be synchronized. The ability to synchronize depends on value of $d$.

Numerical model of the system, equations (7–14) is developed in Matlab-Simulink software. Figure 1 and 2 present selected results, i.e. chaotic response. Time series shows that motion is irregular (figure 1). It is confirmed by Poincare map (figure 2), where chaotic attractor is observed. Therefore, the largest Lyapunov exponent should have positive value.

![Figure 1. Time series of beam response; $\omega = \omega_1$, $\tau_1 = \tau_2 = \tau_3 = 0$.](image)
For this case, influence of coupling coefficient $d$ on synchronization phenomenon is tested. Numerical simulations were made for the parameter $d$ changed from -0.01 to 0.01. Figure 3 shows bifurcation diagram, where error $q_1 - Q_1$ versus $d$ is visualized.

On presented bifurcation diagram the characteristic point $d_{\text{max}}$ can be found. Above this value the synchronization error is equal to zero. Based on the literature [4–6], it can be stated that the largest Lyapunov exponent is close to parameter $d_{\text{max}}$. This point can be read from the bifurcation diagram (manual method). However, this is inefficient. Therefore, an automatic procedure was proposed. This procedure allows to find point $d_{\text{max}}$ with accuracy $\pm \Delta d$. In the first step the simulation is performed for $d = 0$. Depending on the obtained error (for example error $= q_1 - Q_1$) two ways of continuing the calculation are possible. If error $< \text{error}_{\text{permissible}}$ then next simulations will be performed for $d_{\text{new}} = d_{\text{old}} - \Delta d$. Simulations will be repeat until error $> \text{error}_{\text{permissible}}$. Last value $d$ when error $< \text{error}_{\text{permissible}}$ is recognized as $d_{\text{max}}$. Its value should be negative. Whereas, if after first simulation error $> \text{error}_{\text{permissible}}$ then next simulations will be performed for $d_{\text{new}} = d_{\text{old}} + \Delta d$. Simulations will be repeat until error $< \text{error}_{\text{permissible}}$. Last value $d$ when error $> \text{error}_{\text{permissible}}$ is recognized as $d_{\text{max}}$. Its value should be positive. After using proposed procedure for case tested in this section was specified that LLE is equal $0.00423 \pm 0.00001$. This value correlates well with the trend shown in bifurcation diagram (figure 3).
4. Application of numerical procedures

In previous sections, the methodology for determining indicators is presented. The usefulness of both indicators was tested for selected case. The analysis was performed for excitation frequency equal to the natural frequency of the beam \( \omega = \omega_1 \) and when all time delays were identical \( \tau_1 = \tau_2 = \tau_3 \). In the figures 4–6 the bifurcation diagram and curves of indicators \( J_1, J_2 \) are shown.

**Figure 4.** Bifurcation diagram of beam deflection versus time delays; \( \omega = \omega_1 \).

**Figure 5.** Characteristic of indicator \( J_1 \) versus time delays; \( \omega = \omega_1 \).

**Figure 6.** Characteristic of indicator \( J_2 \) versus time delays; \( \omega = \omega_1 \).
The influence of time delays on the saturation controller efficiency is determined by curve of indicator $J_1$ (figure 5). For small values of time delays ($\tau_1 = \tau_2 = \tau_3 < 0.006$) the indicator $J_1$ is smaller then 1. It is desirable, because beam vibration is smaller than for system without delays. Then the value of $J_1$ increases and observed trends from $\tau_1 = \tau_2 = \tau_3 = 0.006$ to $\tau_1 = \tau_2 = \tau_3 = 0.06$ and from $\tau_1 = \tau_2 = \tau_3 = 0.06$ to $\tau_1 = \tau_2 = \tau_3 = 0.092$ are linear. In these areas the different inclination of lines pieces is observed. The most unfavourable value of indicator $J_1$ is for $\tau_1 = \tau_2 = \tau_3 = 0.092$. The beam vibrations are 7.75 times higher than for the system without delays. Next, the indicator decreases ($\tau_1 = \tau_2 = \tau_3 > 0.092$). Irregular indicator changes are visible in this area.

Bifurcation diagram (figure 4) suggests that chaotic or quasi-periodic motion can be expected. This prediction is confirmed by curve of indicator $J_2$ (figure 6). For system without delays and for very small time delays chaotic motion occurs. Next, for time delays from $\tau_1 = \tau_2 = \tau_3 = 0.001$ to $\tau_1 = \tau_2 = \tau_3 = 0.092$ the nature of the motion changes to quasi-periodic. In the last area ($\tau_1 = \tau_2 = \tau_3 > 0.092$), the kind of motion changes again to chaotic.

5. Conclusions

In this paper numerical model of nonlinear beam with saturation control is presented. In the control feedback the delayed signals are applied. Indicators for evaluate influence of time delays on controller effectiveness are proposed. Numerical model and all procedures were developed in Matlab-Simulink software. The obtained results confirm that the proposed indicators allow to quantitative and qualitative evaluation the influence of time delays. Its correct verification allows the use of these procedures to perform a more extensive analysis. Selected results are presented in the second part: Part II: Numerical results.

6. References

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Acknowledgments

The research was financed in the framework of the project Lublin University of Technology-Regional Excellence Initiative, funded by the Polish Ministry of Science and Higher Education (contract no. 030/RID/2018/19).