Lie-algebraic interpretation of the maximal superintegrability and exact solvability of the Coulomb–Rosochatius potential in $n$ dimensions

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Abstract

The potential group method is applied to the $n$-dimensional Coulomb–Rosochatius potential, whose bound states and scattering states are worked out in detail. As far as scattering is concerned, the $S$-matrix elements are computed by the method of intertwining operators and an integral representation is obtained for the scattering amplitude. It is shown that the maximal superintegrability of the system is due to the underlying potential group and that the $2n - 1$ integrals of motion are related to Casimir operators of subgroups.

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1. Introduction

The Coulomb system in arbitrary dimensions is among the well-known and best-studied exactly solvable systems in quantum mechanics [1–13]. It describes the dynamics of charged particles under the influence of the $1/r$ potential. This is what we conventionally mean by the Coulomb system, even if the true $n$-dimensional Coulomb potential satisfying the Poisson equation goes like $1/r^{n-2}$, for $n \geq 3$.

It has been shown that the bound and scattering states of a Coulomb system in $n$ dimensions can be associated with the rotation group $SO(n+1)$ and the Lorentz group $SO(2, n+1)$, respectively [1–4]. In other words, for a fixed energy, these groups appear as invariance groups of the system. These symmetries for $n = 3$ were discovered by Fock [14] and Bargmann [15]. Moreover, it was pointed out in [7] that the dynamical group that yields the positive and negative energy spectra of the Coulomb problem in $n$ dimensions is $SO(2, n+1)$, while the dynamical group in the case $n = 3$ had been discussed by Barut [16].
It is also worth noting that Coulomb and harmonic-oscillator systems are the only spherically symmetric superintegrable systems in arbitrary dimensions.

We recall that in classical mechanics a closed system with \( n \) degrees of freedom is completely integrable if it admits \( n \) integrals of motion (including the Hamiltonian) that are independent and in involution, i.e. the Poisson brackets of any two integrals are zero. The system is called superintegrable if there exist \( q, 1 \leq q \leq n - 1 \), additional independent integrals of motion. The cases \( q = 1 \) and \( q = n - 1 \) correspond to minimal and maximal superintegrability, respectively. In quantum mechanics the definitions of complete integrability and superintegrability are same, but Poisson brackets are replaced by commutators (see [17] for a general review).

In this work, we consider a quantum system characterized by a Hamiltonian of the form

\[
H = \frac{1}{2} p^2 - \frac{\alpha}{r} + \sum_{i=1}^{n} \frac{\beta_i}{2x_i^2}. \tag{1}
\]

We call the system governed by the Hamiltonian given above the Coulomb–Rosochatius system [18]. The motion is confined to the region bounded by the singularity of \( H \) at hyper-planes \( x_i = 0, \ i = 1, 2, \ldots, n \), i.e. hyper-octant (one of the \( 2^n \) regions of the Euclidean space \( \mathbb{E}^n \)). Without loss of generality this can be taken to be the non-negative hyper-octant (i.e. where \( x_i \geq 0, \ i = 1, 2, \ldots, n \)).

The main interest of the Coulomb–Rosochatius system consists in its maximal superintegrability, recently proved for \( n = 3 \) in the classical case [19] (see also [20]). The system had been proved long ago [21, 22] to be quasi-maximally superintegrable for \( n \leq 3 \), i.e. to admit \( 2n - 2 \) operators quadratic in the momenta. For \( n = 3 \), the fifth integral of motion quartic in the momenta was explicitly worked out in [19, 20], thus proving the maximal superintegrability of the classical Coulomb–Rosochatius system. Finally, the maximal superintegrability of the classical system (1) in \( n \)-dimensional spherical, hyperbolic and Euclidean spaces was proved in [23]; here again, one of the \( 2n - 1 \) functionally independent integrals of motion turns out to be quartic in the momenta, while the remaining ones are quadratic.

The main purpose of this work is the complete algebraic solution of bound states and scattering states of the quantum-mechanical Coulomb–Rosochatius system in \( n \) dimensions and the derivation of the \( 2n - 1 \) integrals of motion expected for maximal superintegrability.

Before describing the method of solution of the quantum Coulomb–Rosochatius system in \( n \) dimensions, a few definitions are in order: a Lie group \( G \) is an invariance group for a quantum-mechanical system with the Hamiltonian \( H \) if the latter can be related to a suitable function of a Casimir operator, \( C \), of the group \( G \):

\[
H = f ( C ). \tag{2}
\]

An algebraic derivation of the energy spectrum is possible also when

\[
H = f ( C ) |_{\mathcal{H}}, \tag{3}
\]

where \( \mathcal{H} \) is a subspace of the carrier space. In this case, the group \( G \) describes the same energy states of a family of Hamiltonians \( H \) with different potential strength. This is why \( G \) is called potential group [24]. Such an approach was proposed by Ghirardi [25], who worked it out in detail for the Scarf potential [26]. It is similar to the approach of Olshanetsky and Perelomov [27, 28], where quantum integrable systems are related to the radial part of the Laplace operator on homogeneous spaces, i.e. to the radial part of a second-order Casimir operator of a Lie group. Moreover, it has been shown in [29] that the \( S \) matrix can be associated with an intertwining operator, \( A \), between two Weyl-equivalent representations, \( U^X \) and \( U^\tilde{X} \), of \( G \), i.e. two representations with the same Casimir eigenvalues.
By definition, $A$ satisfies the following equations:

$$AU^\chi (g) = U^\tilde{\chi} (g)A, \ \forall \ g \in G$$ (4)

and

$$AdU^\chi (b) = dU^\tilde{\chi} (b), \ \forall \ b \in g,$$

where $dU^\chi$ and $dU^\tilde{\chi}$ are the corresponding representations of the algebra, $g$, of $G$. Equations (4)–(5) have high restrictive power, determining the intertwining operator up to a constant. The $S$ matrix coincides with the intertwining operator $A$

$$S = A$$

if equation (2) holds, and with the reduction of $A$ to a proper Hilbert subspace $H$

$$S = A|_H$$

if equation (3) holds.

The plan of the paper is as follows: section 2 will describe in full detail the potential group for the Coulomb–Rosochatius system, with a potential specialized to

$$V(r) = -\frac{\gamma}{r} + \sum_{i=1}^{n} \frac{\kappa_i (\kappa_i + 1)}{x_i^2},$$

with subsection 2.1 dedicated to the derivation of bound states and subsection 2.2 to scattering states. Finally, section 3 will be devoted to conclusions and perspectives.

2. Potential group for the Coulomb–Rosochatius system

It is well known that the group $SO(N+1)$ ($SO(N,1)$) has a class of unitary irreducible representations (UIRs) characterized by a number $j$, $j = 0, 1, 2, \ldots$ ($j = -\frac{N-1}{2} + i\rho$, $\rho > 0$), in which the basis vectors are completely labelled by $N-1$ numbers only. This representation can be realized in the Hilbert space spanned by negative-energy (positive-energy) states corresponding to a fixed eigenvalue of the Coulomb Hamiltonian in $N$ dimensions. According to this, one introduces the angular momentum operators $L_{ij}$ and the $N$-dimensional Runge–Lenz vector $A_i$:

$$L_{ij} = \zeta_i \pi_j - \zeta_j \pi_i \quad (i, j = 1, \ldots, N)$$

and

$$A_i = -\frac{1}{2} \sum_{j=1}^{N} (L_{ij} \pi_j + \pi_j L_{ij}) + \frac{\gamma \zeta_i}{\sqrt{\zeta^2}}$$

$$= (\zeta \cdot \pi) \pi_i - \zeta_i \pi^2 - i \frac{N-1}{2} \pi_i + \frac{\gamma \zeta_i}{\sqrt{\zeta^2}} \quad (i = 1, \ldots, N),$$

respectively, where $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_N)$, $\pi = (\pi_1, \pi_2, \ldots, \pi_N)$, $\pi_j = -i \frac{\partial}{\partial \zeta_j}$, $\zeta^2 = \sum_{i=1}^{N} \zeta_i^2$, $\pi^2 = \sum_{i=1}^{N} \pi_i^2$. These operators satisfy the following commutation relations:

$$[L_{ij}, L_{kl}] = i(\delta_{ik} L_{jl} + \delta_{jl} L_{ik} - \delta_{il} L_{jk} - \delta_{jk} L_{il})$$

$$[L_{ij}, A_k] = i(\delta_{ik} A_j - \delta_{jk} A_i)$$

$$[A_i, A_j] = -2i\hbar L_{ij}$$

where $\hbar$ is the Coulomb Hamiltonian in $N$ dimensions

$$\hbar = \frac{1}{2} \pi^2 - \frac{\gamma}{\sqrt{\zeta^2}}$$

(12)
The generators where $d^{4}$

Then, as a result, we obtain the Lie algebra of $SO (N+1)$ (when $ε$ is negative) and the Lie algebra of $SO (N, 1)$ (when $ε$ is positive)

where $μ, ν = 0, 1, 2, \ldots, N$ and

or

The generators $M_{μν}$ act in the eigenspace $H$ of $h$ equipped with the scalar product

where $dξ = dξ_{1} dξ_{2} \cdots dξ_{N}$. A detailed discussion of the $SO (N+1)$ ($SO (N, 1)$) representation generated by operators (14) is given in [1–4].

At this stage we note that, in general, one can define the generators of $SO (N+1)$ ($SO (N, 1)$) (let us call them $\tilde{M}_{μν}$) as follows:

where $λ$ is some non-negative function of $ξ$. Now the generators $\tilde{M}_{μν}$ act in the eigenspace $\tilde{H}$ of $\tilde{h} = λ^{1/2} (ξ) \circ h \circ λ^{-1/2} (ξ)$ equipped with the scalar product

where $dμ (ξ) = λ^{-1/2} (ξ) dξ$ is a quasi-invariant measure on $R^{N}$. The representations acting in $H$ and $\tilde{H}$ are, of course, unitarily equivalent. The unitary mapping $W$ which realizes the equivalence is given by

Although these representations are equivalent from the mathematical viewpoint, they may be related to different physical problems. We prove that the bound states and scattering states of the quantum system (8) are related to the representation of $SO(3n+1)$ and $SO(3n,1)$, respectively, acting in the Hilbert space with the scalar product

where $dμ (ξ) = \prod_{i=1}^{n} (ξ_{2i-2}^{2} + ζ_{2i-1}^{2} + ζ_{2i}^{2})^{-1} dξ$. To this end, we choose $N = 3n$ and $λ (ξ) = \prod_{i=1}^{n} (ξ_{2i-2}^{2} + ζ_{2i-1}^{2} + ζ_{2i}^{2}).$ Then, we introduce the second-order Casimir operator

Since $[L_{ij}, h] = [A_{i}, h] = 0$, we may restrict the above algebra to a subspace where $h$ has a definite eigenvalue $ε$ and define a new set of operators as follows:

$$M_{i,0} = -M_{0,i} = [2ε]^{-1} A_{i}, \quad (i = 1, \ldots, N)$$

$$M_{ij} = L_{ij}, \quad (i, j = 1, \ldots, N).$$

Then, as a result, we obtain the Lie algebra of $SO (N+1)$ (when $ε$ is negative) and the Lie algebra of $SO (N, 1)$ (when $ε$ is positive)

$$[M_{μν}, M_{σλ}] = i(g_{μσ}M_{νλ} + g_{νσ}M_{μλ} - g_{μλ}M_{νσ} - g_{νλ}M_{μσ}).$$

where $g_{μν} = (+, +, \ldots, +, +)$ for $SO (N+1)$

or

$g_{μν} = (-, +, \ldots, +, +)$ for $SO (N, 1)$.

The generators $M_{μν}$ act in the eigenspace $H$ of $h$ equipped with the scalar product

$$\langle φ_{1}, φ_{2} \rangle = \int_{R^{N}} φ_{1}^{*} (ξ) φ_{2} (ξ) dξ, \quad ξ ∈ R^{N},$$

where $dξ = dξ_{1} dξ_{2} \cdots dξ_{N}$. A detailed discussion of the $SO (N+1)$ ($SO (N, 1)$) representation generated by operators (14) is given in [1–4].

At this stage we note that, in general, one can define the generators of $SO (N+1)$ ($SO (N, 1)$) (let us call them $\tilde{M}_{μν}$) as follows:

$$\tilde{M}_{i,0} = -\tilde{M}_{0,i} = [2ε]^{-1} \lambda^{1/2} (ξ) \circ A_{i} \circ \lambda^{-1/2} (ξ)$$

$$\tilde{M}_{ij} = \lambda^{1/2} (ξ) \circ L_{ij} \circ \lambda^{-1/2} (ξ),$$

where $λ$ is some non-negative function of $ξ$. Now the generators $\tilde{M}_{μν}$ act in the eigenspace $\tilde{H}$ of $\tilde{h} = \lambda^{1/2} (ξ) \circ h \circ \lambda^{-1/2} (ξ)$ equipped with the scalar product

$$\langle \tilde{φ}_{1}, \tilde{φ}_{2} \rangle = \int_{R^{N}} \tilde{φ}_{1}^{*} (ξ) \tilde{φ}_{2} (ξ) dμ (ξ), \quad ξ ∈ R^{N},$$

where $dμ (ξ) = λ^{-1/2} (ξ) dξ$ is a quasi-invariant measure on $R^{N}$. The representations acting in $H$ and $\tilde{H}$ are, of course, unitarily equivalent. The unitary mapping $W$ which realizes the equivalence is given by

$$W : \quad φ → \tilde{φ} = \lambda^{1/2} (ξ) φ.$$
and consider the following operator depending on \( \tilde{C} \):
\[
\frac{\gamma^2}{[\tilde{C} + \left( \frac{3n-1}{2} \right)^2]} = \frac{\partial^2}{\partial \xi_1^2} + \cdots + \frac{\partial^2}{\partial \xi_{3n}^2} - 2 \left( \frac{\partial}{\partial \xi_1} + \frac{\partial}{\partial \xi_2} + \frac{\partial}{\partial \xi_3} \right)
\]
\[
- \cdots - \frac{2}{\xi_{3n-2}^2 + \xi_{3n-1}^2 + \xi_{3n}^2} \left( \frac{\partial}{\partial \xi_{3n-2}} + \frac{\partial}{\partial \xi_{3n-1}} + \frac{\partial}{\partial \xi_{3n}} \right) + \frac{2\gamma}{\sqrt{\xi}}.
\]
Decomposing \( R^{3n} \) into the direct sum of three-dimensional subspaces and introducing spherical coordinates in these subspaces

\[
\xi = (x_1 u_1, x_2 u_2, \ldots, x_n u_n),
\]
where \( u_i = (\sin \alpha_i, \sin \beta_i, \cos \beta_i, \cos \alpha_i) \), \( x_i \geq 0 \), \( 0 \leq \alpha_i \leq \pi \), \( 0 \leq \beta_i \leq 2\pi \), \( (i = 1, 2, \ldots, n) \), we have
\[
\frac{\gamma^2}{[\tilde{C} + \left( \frac{3n-1}{2} \right)^2]} = \sum_{i=1}^{n} \left[ \frac{\partial^2}{\partial x_i^2} + \frac{1}{x_i^2} \left( \frac{\partial}{\partial \alpha_i} \sin \alpha_i \frac{\partial}{\partial \alpha_i} + \frac{1}{\sin^2 \alpha_i} \frac{\partial^2}{\partial \beta_i^2} \right) \right] + \frac{2\gamma}{\sqrt{x_i}}
\]
with \( x = (x_1, x_2, \ldots, x_n) \). With this coordinate system the measure \( d\mu (\xi) \) in formula (18) becomes \( d\mu (\xi) = dx \prod_{i=1}^{n} \sin \alpha_i \, d\alpha_i \, d\beta_i \), where \( dx = dx_1 \, dx_2 \cdots dx_n \).

Let \( \mathcal{K}_K, K = (k_1, k_2, \ldots, k_n; \sigma_1, \sigma_2, \ldots, \sigma_n) \), be a subspace of functions \( \hat{\phi} (\xi) = \Psi (x) \prod_{i=1}^{n} Y_{k_i}^{\sigma_i} (\alpha_i, \beta_i) \), with fixed \( K \), where \( Y_{k_i}^{\sigma_i} (\alpha_i, \beta_i) \) are spherical harmonics of degree \( \kappa_i \). Thus, the operator (21) restricted to this subspace becomes
\[
\frac{\gamma^2}{[\tilde{C} + \left( \frac{3n-1}{2} \right)^2]} \bigg|_{\mathcal{K}_K} = \sum_{i=1}^{n} \left[ \frac{\partial^2}{\partial x_i^2} - \frac{k_i (k_i + 1)}{x_i^2} \right] + \frac{2\gamma}{\sqrt{x_i}}.
\]
Hence, the Hamiltonian
\[
H = -\frac{1}{2} \frac{\gamma^2}{2\tilde{C} + \left( \frac{3n-1}{2} \right)^2} \bigg|_{\mathcal{K}_K}
\]
can be described in terms of the potential groups \( SO(3n+1) \) and \( SO(3n, 1) \) since
\[
H = -\frac{\gamma^2}{2[\tilde{C} + \left( \frac{3n-1}{2} \right)^2]} \bigg|_{\mathcal{K}_K}.
\]
Moreover, since the Casimir operators of the groups in the chain
\[
G \supset SO(3n) \supset SO(3n-3) \times SO(3) \supset SO(3n-6) \times SO(3) \times SO(3) \supset \cdots \supset SO(3) \times SO(3) \times \cdots \times SO(3) \supset SO(2) \times SO(2) \times \cdots \times SO(2),
\]
where \( G \) is either \( SO(3n+1) \) or \( SO(3n, 1) \), form a complete set of \( 3n \) commuting operators (including the Casimir operator, \( \tilde{C} \), of \( G \)), i.e. the operators \( \tilde{C}, \tilde{C}^{SO(3n-3p)}, \tilde{C}^{SO(3b)} \), and \( \tilde{C}^{SO(2n)} \), where \( p = 0, 1, 2, \ldots, n-2 \), \( k = 1, 2, \ldots, n \), and
\[
\tilde{C}^{SO(3n-3p)} = \frac{1}{2} \sum_{j=1}^{3n-3p} M_{ij} = -\frac{1}{2} \sum_{j=1}^{3n-3p} \left( x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2 - \sum_{j=1}^{3n-3p} x_j^2
\]
\[
\times \left[ \frac{n-p}{\sin \alpha_j} \frac{\partial \sin \alpha_j}{\partial \alpha_j} + \frac{1}{\sin^2 \alpha_j} \frac{\partial^2}{\partial \beta_j^2} \right] - 2 (n-p) (n-p-1),
\]
 are integrals of motion. These integrals of motion are responsible for the separability of $H$ in spherical coordinates.

The remaining $n-1$ integrals of motion can be related to Casimir operators

$$
\tilde{C}^{SO(3n-3q)} = \frac{1}{2} \sum_{i,j=q+1}^{3n} \tilde{M}_{ij} = -\frac{1}{2} \sum_{i,j=q+1}^{n} \left( \frac{x_i}{x_j} \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2 - \sum_{i=q+1}^{n} x_i^2 \sum_{j=q+1}^{n} \frac{\kappa_j (\kappa_j + 1)}{x_j^2} , \quad p = 0, 1, 2, \ldots, n-2 ,
$$

(28)

$$
\tilde{C}^G = \frac{1}{2} \sum_{\mu, \nu}^{3} \tilde{M}^{\mu\nu} \tilde{M}_{\mu\nu} = -\frac{1}{2\kappa} \left\{ A_1 + 2 \left( \frac{n}{x_i^2} \frac{\partial}{\partial x_i} \sin \alpha_i \frac{\partial}{\partial \alpha_i} + \frac{1}{\sin^2 \alpha_i} \frac{\partial^2}{\partial \beta_i^2} \right) \right\}^2
$$

(29)

$$
= \frac{1}{4\kappa_i^2} \left( p \cdot \frac{x}{x_i} \cdot p - 2\kappa \right) + \frac{1}{4\kappa_i^2} \left( \frac{x_i}{x_j} \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2 - \frac{1}{4\kappa_i^2} \left( \frac{x_i}{x_j} \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2 + 2\kappa
$$

with $A_1 = (x \cdot p) p_1 - x_1 p_2 - i \frac{x_1}{x_i} p_1 + \frac{x_1}{x_i} p_2$, $x = (x_1, x_2, \ldots, x_n)$, $p = (p_1, p_2, \ldots, p_n)$ and $p_j = -\frac{1}{n} \frac{\partial}{\partial x_j}$, of the groups in the chain

$$
G \supset SO(3n-3) \supset G' \supset SO(3n-6) \times SO(3) \times SO(3) \supset \cdots \supset SO(3) \times SO(3) \times \cdots \times SO(3) \supset SO(2) \times SO(2) \times \cdots \times SO(2) ,
$$

where $G'$ is either $SO(4)$ or $SO(3, 1)$. Namely,

$$
J_q = [\tilde{C}^{SO(3n-3q)} + 2(n-q)(n-q-1)] \| \gamma_k
$$

(30)

$$
= -\frac{1}{2} \sum_{i,j=q+1}^{n} \left( \frac{x_i}{x_j} \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2 - \sum_{i=q+1}^{n} x_i^2 \sum_{j=q+1}^{n} \frac{\kappa_j (\kappa_j + 1)}{x_j^2} , \quad q = 1, 2, \ldots, n-2
$$

and

$$
J_{n-1} = -2\kappa \tilde{C}^G - \tilde{C}^{SO(3n)} - 1 \| \gamma_k
$$

(31)

$$
= \left[ A_1 - x_1 \sum_{i=1}^{n} \frac{\kappa_i (\kappa_i + 1)}{x_i^2} \right] \left( \kappa_1 + 1 \right) (p \cdot x) \left( \frac{x}{x_i} \right) + \kappa_1 (\kappa_1 + 1) (p \cdot x) \frac{1}{x_i} (x \cdot p) - \frac{(n-1)(n-3)}{4\kappa_i^2}
$$

are constants of motion, too.
It is worth noting that, since \( J_{n-1} \) is quartic in the momenta, the complete set of mutually commuting operators \( \{ H, J_q, \; q = 1, 2, \ldots, n - 1 \} \) does not specify a separable coordinate system.

2.1. Bound states

The bound-state spectrum is immediately obtained from the eigenvalue of the Casimir operator \( \tilde{C} \) of the potential group \( SO(3n + 1) \), i.e. \( j (j + 3n - 1) \), in the form

\[
E = -\frac{\gamma^2}{2 \left( j + \frac{3n-1}{2} \right)^2},
\]

where \( j \) takes on integer values from \( \kappa_1 + \kappa_2 + \cdots + \kappa_n \) upwards.

The basis functions \( |j; IMK \rangle \) for the Hilbert space \( \mathcal{H} \) can be defined as the common set of eigenfunctions of the Casimir operators of the groups forming the chain (23)

\[
\tilde{C} |j; IMK \rangle = j (j + 3n - 1) |j; IMK \rangle
\]

where \( m_0 \equiv l \) and \( M \) and \( K \) are the collective indexes \( (m_1, m_2, \ldots, m_{n-2}) \) and \( K = (\kappa_1, \kappa_2, \ldots, \kappa_n; \sigma_1, \sigma_2, \ldots, \sigma_n) \), respectively. It is also known that in correspondence with every chain of subgroups in \( SO(N) \) it is possible to define a polyspherical system on \( R^N \) (see chapter 10 of [30]). According to this, we introduce polyspherical coordinates on \( R^m \) by choosing \( x_i \) in (20) as

\[
x_1 = r \sin \theta_{n-1} \sin \theta_{n-2} \cdots \sin \theta_2 \sin \theta_1 \\
x_2 = r \sin \theta_{n-1} \sin \theta_{n-2} \cdots \sin \theta_2 \cos \theta_1 \\
\vdots \\
x_{n-1} = r \sin \theta_{n-1} \cos \theta_{n-2} \\
x_n = r \cos \theta_{n-1}
\]

where \( r \geq 0, \; 0 \leq \theta_i \leq \pi/2 \), for \( i = 1, 2, \ldots, n - 1 \).

By construction

\[
\langle \xi | j; IMK \rangle = \psi(x) \prod_{i=1}^n Y_{\kappa_i}^{\alpha_i} (\alpha_i, \beta_i),
\]

where \( \psi(x) \) is the bound-state wavefunction

\[
\psi(x) = R_{jl}(r) \tilde{Y}_{lM} (\hat{x}), \quad \hat{x} = x/r.
\]

Here, \( R_{jl}(r) \) is the radial part of the wavefunction, while \( \tilde{Y}_{lM} (\hat{x}) \) is the angular part of it. According to (17), \( R_{jl}(r) \) is related to the radial part \( R^{Coul}_{jl}(r) \) of the 3n-dimensional Coulomb wavefunction [5] as

\[
R_{jl}(r) = r^l \overline{R}^{Coul}_{jl}(r) = cu^l e^{-\hat{r}^2/2} L_{j-l}^{2l+3n-2}(u), \quad u = 4\gamma r/(2j + 3n - 1),
\]

where \( L_n^l \) are Laguerre polynomials and

\[
c = (2\gamma)^{-n/2} \left[ j + \frac{1}{2} (3n-1) \right]^{-1/2} (j + l + 1) \left[ \frac{\Gamma (j-l+1)}{2 \Gamma (j+l+3n-1)} \right]^{1/2}.
\]
while $\mathcal{Y}_{LM}(\hat{x})$ are related to the $(3n-1)$-dimensional spherical harmonics $Y_{LM}(\hat{\zeta})$ in the above polynomials as
\[
Y_{LM}(\hat{\zeta}) = \mathcal{Y}_{LM}(\hat{x}) \prod_{i=1}^{n-1} (\sin^{n-i} \theta_{n-i} \cos \theta_{n-i})^{-1} \prod_{i=1}^{n} Y_{\kappa_i}^{\sigma_i}(\alpha_i, \beta_i), \quad \hat{\zeta} = \zeta / r
\] (37)

In order to write $Y_{LM}(\hat{\zeta})$ we make use of a graphical method called the tree method (see section 10.5 of [30]). According to this method polyspherical coordinate systems on the sphere of unit radius can be described by graphs called trees. The trees contain nodes and each node has two edges, which are distinguished as left or right edge. An angle is associated with each node and the sine (cosine) of this angle is associated with the corresponding left (right) edge. The free (upper) ends of the tree are labelled left to right with Cartesian coordinates. Then, the Cartesian coordinate $\xi$ is equal to the product of trigonometrical functions on the edges along the unique path connecting the lowest node with $\xi$. In figure 1, we associate a tree with a polyspherical coordinate system on $S^{n-1}$ given by formulae (20) and (33). Hence, the $(3n-1)$-dimensional spherical harmonics $Y_{LM}(\hat{\zeta})$ corresponding to this tree can be obtained by the rules described in section 10.5.3 of [30]. As a result, we have
\[
\mathcal{Y}_{LM}(\hat{x}) = \chi \prod_{i=1}^{n-2} \sin^{m_i+n-i} \theta_{n-i} \cos^{\kappa_i+1} \theta_{n-i} P_{(m_n+n-(m_i+n-i))/2}^{(\alpha_i, \beta_i)} (\cos 2\theta_{n-i})
\times \sin^{\kappa_{i+1}} \theta_{i} \cos^{\kappa_{i-1}+1} \theta_{i} P_{(m_{i-2}-\kappa_{i-1}+1)/2}^{(\alpha_{i-2}, \beta_{i-2})} (\cos 2\theta_{i}), \quad (38)
\]

where $P_{n}^{(\alpha, \beta)}$ are Jacobi polynomials and $\chi$ is a normalization constant
\[
\chi = \prod_{i=1}^{n-2} \left[ \frac{\Gamma\left(\frac{1}{2}(m_{i-1}+m_i+\kappa_i+3n-3i+1)\right)\Gamma\left(\frac{1}{2}(m_{i-1}-m_i-\kappa_i+3n-3i+1)\right)}{\Gamma\left(\frac{1}{2}(m_{i-1}+m_i-\kappa_i)\right)\Gamma\left(\frac{1}{2}(m_{i-1}-m_i+\kappa_i)\right)} \right]^{\frac{1}{2}}
\times \left[ \frac{\Gamma\left(\frac{1}{2}(m_{i-2}+\kappa_n+\kappa_{n-1}+1)\right)\Gamma\left(\frac{1}{2}(m_{i-2}-\kappa_n-\kappa_{n-1}+2)\right)}{\Gamma\left(\frac{1}{2}(m_{i-2}+\kappa_n-\kappa_{n-1}+1)\right)\Gamma\left(\frac{1}{2}(m_{i-2}-\kappa_n+\kappa_{n-1}+1)\right)} \right]^{\frac{1}{2}}. \quad (39)
\]
2.2. Scattering states

Once the group structure of the problem has been recognized, the associated $S$ matrix can be computed by using matrices that intertwine Weyl-equivalent representations of $SO(3n, 1)$ in the bases corresponding to the reduction (23). We find it expedient to use, for this purpose, equation (4). By realizing the principal series of $SO(3n, 1)$ on suitable Hilbert spaces of appropriate functions, one can derive from (4) the functional relations satisfied by the kernel of the intertwining operator, written in integral form and, consequently, the explicit representation of the matrix elements of the operator itself.

It is known that the most degenerate principal series representations of $SO(N, 1)$ labelled with the quantum number $j = -\frac{N-1}{2} + i\rho$, with $\rho > 0$, can be realized on $L_2(S^{N-1})$ (see section 9.2.1 of [30]):

$$U_j(q)f(\eta) = (\omega_j)^j f(\eta_k), \quad \eta \in S^{N-1},$$

(40)

where

$$\omega_j = \sum_{i=1}^{N} g_{NN}^{-1} \eta_i + g_{NN}, \quad (\eta_k) = \sum_{i=1}^{N} \frac{g_{NN}^{-1} \eta_i}{\sum_{i=1}^{N} g_{NN}^{-1} \eta_i + g_{NN}}.$$

The representations specified by labels $j$ and $1 - N - j$ are Weyl-equivalent.

The operator $A$ defined by

$$(Af)(\eta) = \int K(\eta, \eta')f(\eta') \, d\eta'$$

(41)

intertwines representations $j$ and $1 - N - j$ on the condition that

$$K(\eta_k, \eta'_k) = (\omega_j)^{N-1+j}(\omega_j)^{N-1+j}K(\eta, \eta').$$

(42)

The kernel, $K$, is uniquely determined by equation (42) up to a constant and is given by

$$K(\eta, \eta') = \kappa (1 - \eta \cdot \eta')^{1-N-j},$$

(43)

with

$$\kappa = 2^{-\frac{N-1}{2} + i\rho} \frac{\Gamma \left( \frac{N-1}{2} + i\rho \right)}{\pi^{\frac{N-1}{2}} \Gamma (-i\rho)}.$$  

(44)

Taking into account the fact that the spherical harmonics $Y_{LMK}$ (37) form a basis in $L_2(S^{N-1})$, corresponding to the above reduction, we obtain the following integral representation for the matrix elements of $A$ :

$$\langle j; l'M'K'|A| j; l'MK \rangle = \int K(\eta, \eta')Y_{l'M'K'}^*(\eta')Y_{l'MK}(\eta) \, d\eta \, d\eta'.$$

(45)

Therefore,

$$\langle j; l'M'K'|A| j; l'MK \rangle = A_l \delta_H \delta_{MM} \delta_{KK'},$$

(46)

where

$$A_l = \frac{\Gamma \left( \frac{3n-1}{2} + i\rho + l \right)}{\Gamma \left( \frac{3n-1}{2} - i\rho + l \right)}.$$  

(47)

According to this, we have

$$S(p; p') = \sum_{lM} A_l \tilde{Y}_{l'M}(\hat{p}) \tilde{Y}_{l'M}^*(\hat{p'}).$$

(48)

Thus, the scattering amplitude, $f(p; p')$, is defined by

$$f(p; p') = (-i) \left( \frac{2\pi}{p} \right)^{\frac{N-1}{2}} \sum_{lM} (A_l - 1) \tilde{Y}_{l'M}(\hat{p}) \tilde{Y}_{l'M}^*(\hat{p'}).$$

(49)
We can omit unity in the brackets of formula (49) when \( \hat{p}' \neq \hat{p} \), leaving

\[
f(p; p') = (-i) \left( \frac{2\pi}{p} \right) \frac{2^{i\frac{n-1}{2} + i\rho + l}}{\Gamma(-i\rho)} \sum_{iM} \Gamma\left( \frac{3n-1}{2} + i\rho + l \right) Y_{iM}(\hat{p}) Y_{iM}(\hat{p}').
\]

(50)

Moreover, formulae (37) and the following expansion of the kernel:

\[
(1 - \eta \cdot \eta')^{-\frac{3n-1}{2} - i\rho} = (2\pi)^{\frac{n-1}{2}} 2^{-i\frac{3n-1}{2} + i\rho} \Gamma(-i\rho) \sum_{iM} \Gamma\left( \frac{3n-1}{2} + i\rho + l \right) Y_{iM}(\eta) Y_{iM}(\eta').
\]

(51)

yield an integral representation of the scattering amplitude

\[
f(p; p') = \frac{1}{i\rho \pi^{n/2}} \frac{2^{i\rho} \Gamma\left( \frac{n-1}{2} + i\rho \right)}{\Gamma(-i\rho)} \Delta(\theta_{n-1}, \ldots, \theta_1) \Delta(\theta'_{n-1}, \ldots, \theta'_1)
\]

\[
\times \int_0^{\pi} \cdots \int_0^{\pi} (1 - \hat{p}_1 \hat{p}_1' \cos \alpha_1 - \hat{p}_2 \hat{p}_2' \cos \alpha_2 - \cdots - \hat{p}_n \hat{p}_n' \cos \alpha_n)^{-\frac{3n-1}{2} - i\rho}
\]

\[
\times P_{\epsilon_1}(\cos \alpha_1) P_{\epsilon_2}(\cos \alpha_2) \cdots P_{\epsilon_n}(\cos \alpha_n) \sin \alpha_1 \sin \alpha_2 \cdots \sin \alpha_n \sin \alpha_1 \sin \alpha_2 \cdots \sin \alpha_n d\alpha_1 d\alpha_2 \cdots d\alpha_n,
\]

(52)

where

\[
\Delta(\theta_{n-1}, \ldots, \theta_1) = \prod_{i=1}^{n-1} \sin^{n-i} \theta_{n-i} \cos \theta_{n-i}.
\]

When \( \kappa_i = 0 \), formula (52) simplifies to

\[
f(p; p') = \frac{1}{i\rho \pi^{n/2}} \frac{2^{i\rho} \Gamma\left( \frac{n-1}{2} + i\rho \right)}{\Gamma(-i\rho)} \sum_{\epsilon_i = \pm} \sigma (1 - \epsilon_1 \hat{p}_1 \hat{p}_1' - \epsilon_2 \hat{p}_2 \hat{p}_2' - \cdots - \epsilon_n \hat{p}_n \hat{p}_n')^{-\frac{3n-1}{2} - i\rho}
\]

(53)

with

\[
\sigma = \prod_{i=1}^{n} \epsilon_i.
\]

It is worth noting that the amplitude (53) does not reduce to the Coulomb amplitude \( f_{\text{Coul}} \) in \( n \) dimensions [4]

\[
f_{\text{Coul}}(p; p') = \frac{1}{i\rho \pi^{n/2}} \frac{2^{i\rho} \Gamma\left( \frac{n-1}{2} + i\rho \right)}{\Gamma(-i\rho)} (1 - \hat{p}_1 \hat{p}_1' - \hat{p}_2 \hat{p}_2' - \cdots - \hat{p}_n \hat{p}_n')^{-\frac{3n-1}{2} - i\rho}
\]

(54)

when \( \kappa_i \) is set equal to zero. The reason for this discrepancy lies in the fact that we solve the Schrödinger equation for \( x_i \geq 0 \) \( (i = 1, 2, \ldots, n) \) with boundary conditions

\[
\Psi(x) = 0 \quad \text{for} \quad x_i = 0, \ (i = 1, \ldots, n)
\]

(55)

3. Conclusions and outlook

This work is the latest in a series of papers where the potential group approach and the method of intertwining operators (formulae (2)–(7)) have been applied to non-central extensions of the Coulomb potential. The bound states of the three-dimensional Coulomb potential plus a barrier term with the \( SO(5) \) potential group were studied in [31] and the scattering states of similar potentials with \( SO(5) \) potential group in [32]. A first simultaneous analysis of bound and scattering states of a three-dimensional Coulomb–Rosochatius potential of type (8) was performed in [33], where potential groups \( SO(7) \) for bound states and \( SO(6, 1) \) for scattering states were used. Generally, in the \( n \)-dimensional case for Coulomb–Rosochatius potential one can choose potential groups \( SO(qn + 1) \) and \( SO(qn, 1) \) with \( q = 2, 3, \ldots \). Then, the
potential strengths $\beta_i$ will be related to eigenvalues $k_i (k_i + q - 2)$, $k_i = 0, 1, 2, \ldots$, of Casimir operators of subgroups $SO(q)$, as $\beta_i = (k_i + \frac{q-2}{2})^2 - \frac{1}{4}$. In this work, we have chosen $q = 3$.

The method has proved useful also in deriving the full set of $2n - 1$ constants of motion, related to Casimir invariants of subgroups appearing in the decomposition chains, thus proving the maximal superintegrability of the system.

The potential group approach is quite general and, obviously, not limited to the orthogonal and pseudo-orthogonal symmetries underlying the Coulomb–Rosochatius Hamiltonian. Systems with different symmetries will be studied in future works.

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