LIMIT PERIODIC UPPER AND LOWER SOLUTIONS IN A GENERIC SENSE

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(Communicated by Rafael de la Llave)

Abstract. The method of upper and lower solutions is a main tool to prove the existence of periodic solutions to periodic differential equations. It is known that, in general, the method does not extend to the almost periodic case. The aim of the present paper is to show that, however, something interesting survives: if the classical assumptions of the method are satisfied, then the expected existence result holds generically in the limit periodic framework.

1. Introduction. In this paper, we are concerned with the second order scalar equation:

\[ u'' + cu' + g(u) = h(t) \]  \hspace{1cm} (1)

where \( g \in C^1(\mathbb{R}) \) is a (possibly nonmonotone) nonlinearity, the constant \( c \) measures the strength of friction and the forcing term \( h \in C(\mathbb{R}) \). We assume that a pair of functions \( \alpha, \beta \in BC^2(\mathbb{R}) \) is given, such that

\[ \alpha \ll \beta \ll \beta'' + c\beta' + g(\beta) \ll \alpha'' + c\alpha' + g(\alpha) \]  \hspace{1cm} (2)

and that moreover \( h \) belongs to the class

\[ H_{\alpha\beta} = \{ h \in BC(\mathbb{R}) : \beta'' + c\beta' + g(\beta) \ll h \ll \alpha'' + c\alpha' + g(\alpha) \} . \]

Here \( \varphi \ll \psi \) stands for a strong version of the standard pointwise ordering, defined by:

\[ \inf_{t \in \mathbb{R}} \{ \psi(t) - \varphi(t) \} > 0 . \]

In other words, \( \alpha \) and \( \beta \) are well ordered lower and upper solutions for the equation (1) in a rather strong sense. As a consequence of these assumptions, equation (1) also admits a well ordered solution, namely belonging to:

\[ \Delta_{\alpha\beta} = \{ u \in BC^2(\mathbb{R}) : \alpha \ll u \ll \beta \} . \]

2010 Mathematics Subject Classification. Primary: 34A34; Secondary: 34C27.

Key words and phrases. Limit periodic function, module, exponential dichotomy, Sard-Smale lemma, contraction mapping principle.

The second author is partially supported by the Key Lab of Random Complex Structures and Data Science, Chinese Academy of Sciences (Grant No. 2008DF173182) and the National Natural Science Foundation of China (Grant No. 11301512 and No. 11671982).

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Here of course $BC(\mathbb{R})$ and $BC^2(\mathbb{R})$ denote spaces of bounded continuous functions, the latter with two bounded derivatives. The proof can be found in the book [5] by De Coster and Habets, where various methods of constructing specific solutions are also described. For instance, some kind of extremal solutions in $\Delta_{\alpha\beta}$ exist and may be obtained via a monotone iteration scheme, starting from $\alpha$ and $\beta$.

In the same book one can find also the proof of a more classical and older result, stating that a similar conclusion holds in the periodic framework, namely when $BC(\mathbb{R})$ and $BC^2(\mathbb{R})$ are replaced by the smaller classes of continuous $T$-periodic functions $P_T(\mathbb{R})$ and $P^2_T(\mathbb{R})$ respectively. It seems then quite natural to expect that the same must be true when we consider the intermediate class of almost periodic functions:

$$P_T(\mathbb{R}) \subset AP(\mathbb{R}) \subset BC(\mathbb{R}).$$

For instance, this certainly happens when $g$ is strictly increasing; see Fink’s book [6] for a proof. In fact, a general result for the almost periodic case was stated for the case $c = 0$ in the paper [12] by Schmitt and Ward, claiming the almost periodicity of the extremal solutions inside $\Delta_{\alpha\beta}$. Later on, however, the proof was discovered not being complete and, after a decade, Ortega and Tarallo published in [11] a counter–example for the case where $c > 0$ and large enough. Though it is not yet available in the literature, it is a common thinking that a counter–example should exist also when $c = 0$.

The aim of this paper is to reverse the perspective, showing that a solvability result in the almost periodic framework is nevertheless possible, though in a generic sense only and for a restricted type of almost periodicity: this is the same idea already worked out in the pair of recent papers [1] and [10], for a different context. The involved functions are those which, among the almost periodic functions, are better approximated by purely periodic ones. They are the so–called limit periodic functions, defined as:

$$LP(\mathbb{R}) = \{h \in AP(\mathbb{R}) : \exists (h_n)_n \subset P(\mathbb{R}) \text{ with } \|h_n \to h\|_{\infty} \to 0\}$$

where of course $P(\mathbb{R})$ stands for the class of the continuous periodic functions of any period. A further restriction is however needed, which actually is already present in the classical periodic result: the working period $T$ is the same for all the involved functions. The standard way to translate this prescription in the general almost periodic framework is by using the so–called module of an almost periodic function $\varphi$, denoted by $\text{mod} (\varphi)$: this is the least additive subgroup of $\mathbb{R}$ which contains all the characteristic exponents of $\varphi$. With this notion, the role of $P_T(\mathbb{R})$ in the limit periodic context is played by the class:

$$LP_\omega(\mathbb{R}) = \{h \in LP(\mathbb{R}) : \text{mod} (h) \subset \omega \mathbb{Q}\}$$

where $\omega > 0$ is any given real number. See Section 2 for further details on the subject. The main result is then the following:

**Theorem 1.1.** Assume that $\alpha, \beta \in LP^2_\omega(\mathbb{R})$ and satisfy condition (2). Then equation (1) has a solution $u \in \Delta_{\alpha\beta} \cap LP_\omega(\mathbb{R})$ for a class of forcing terms $h$’s which is open and dense in $H_{\alpha\beta} \cap LP_\omega(\mathbb{R})$.

The theorem will be proved in Section 6. The restriction to the limit periodic framework is casted into the proof strategy, as in the already mentioned papers [1] and [10]. Roughly speaking, one starts from purely periodic problems showing that most of them are nice in the following sense: periodic solutions exist, which
possess some strong stability property. This is done in Section 4, based on the characterization of the exponential dichotomy for Hill’s equation given in Section 3.

Then the stability property of the solutions to these nice periodic problems, is used to extend the solvability to all the almost periodic problems which are sufficiently near to them. This is a known consequence of the exponential dichotomy, which for instance may be deduced from the arguments in Chapter 8 of Fink’s book [6]. It is also referred and used in [10]: we decided to sketch nevertheless the proof in Section 5, since we are not able to provide an explicit reference.

Look finally at the resulting class of solvable almost periodic problems. They are in the neighborhoods of some nice periodic problem and, while this class certainly is a thin subset of all the possible almost periodic problems. It is not difficult to guess that it is a fat subset of the limit periodic ones: this is the ultimate reason for which Theorem 1.1 refers to limit periodic forcing terms only.

Before concluding the introduction, it is probably worth to stress what our result does not say: being able to generically solve the problem in the limit periodic class, does not mean that the method of upper and lower solutions is generically working in the same class. To be more specific, we don’t know whether the monotone iterative scheme generically produces limit periodic solutions, nor we are able to say that the extremal solutions generically are limit periodic. For a different problem, Ortega obtained a more clear conclusion in [10]. The equation there is frictionless \((c = 0)\) and the nonlinearity has a periodic primitive. These facts enable the author to build up a nonstandard variational setting for the search of almost periodic solutions to (1), when the forcing term \(h\) is also almost periodic and has mean value zero. The conclusion is again generic in the forcing term, but twofold: roughly speaking, for most of the limit periodic \(h’s\) a limit periodic solution does exist though, at the same time, the action functional in unbounded from below.

Notations. Throughout the paper \(T > 0\) and \(\omega > 0\) stand for given period and frequency respectively, and all the considered functions are from \(\mathbb{R}\) to \(\mathbb{R}\). For \(k \in \mathbb{N}\), we denote by \(C^k(\mathbb{R})\) be the space of the \(k\)-times continuously differentiable functions. When these derivatives are also bounded, we use the symbol \(BC^k(\mathbb{R})\).

The latter is a Banach space with respect to the natural norm:

\[
\|f\|_{k, \infty} := \|f\|_{\infty} + \|f'\|_{\infty} + \cdots + \|f^{(k)}\|_{\infty}.
\]

The norm is inherited by all the subsets of \(BC^k(\mathbb{R})\), like for instance the \(AP^k(\mathbb{R})\), \(LP^k_\omega(\mathbb{R})\), \(LP^k_T(\mathbb{R})\) already used in the Introduction.

2. Limit periodic functions. In this section we recall some facts on almost periodic functions, especially on limit periodic functions, which will be used in the following sections. We denote the space of all (Bohr) almost periodic functions by \(AP(\mathbb{R})\). It is well-known that \(AP(\mathbb{R})\) is a Banach algebra endowed with the uniform norm [6]. It is obvious that \(P(\mathbb{R}) \subset AP(\mathbb{R}) \subset BC(\mathbb{R})\) in the topology of the uniform convergence on \(\mathbb{R}\).

Definition 2.1. A function \(f \in BC(\mathbb{R})\) is said to be limit periodic if there exists a sequence of functions \(\{f_n : f_n \in P(\mathbb{R})\}\) such that \(\lim_{n \to \infty} \|f_n - f\|_{\infty} = 0\). The class of all limit periodic functions will be denoted by \(LP(\mathbb{R})\). The typical example is

\[
f(t) = \sum_{n=0}^{\infty} a_n \sin \left(\frac{t}{2^n}\right),
\]
where the coefficients $a_n$ are nonzero and summable. It is easy to check that $L^p(\mathbb{R})$ is closed in $AP(\mathbb{R})$ so that it is a complete metric space. However, $L^p(\mathbb{R})$ is not a Banach space because it is not closed with respect to algebraic operations.

For $\lambda \in \mathbb{R}$ and $f \in AP(\mathbb{R})$, we denote the corresponding Fourier coefficient as:

$$\hat{f}(\lambda) := \lim_{T \to +\infty} \frac{1}{T} \int_{0}^{T} f(t) e^{-i\lambda t} \, dt,$$

and define the set of Fourier exponents of $f$ by:

$$\exp(f) := \{ \lambda \in \mathbb{R} : \hat{f}(\lambda) \neq 0 \}.$$

It is well known that the type of almost periodicity is decided by the least additive subgroup of $\mathbb{R}$ which contains $\exp(f)$: this is the so-called module of $f$, which we denote by $\text{mod}(f)$. For instance $f \in P_T(\mathbb{R})$ if and only if $\text{mod}(f) \subset \frac{2\pi}{T} \mathbb{Z}$, while for the $f$ is given in (3) one has $\text{mod}(f) = \{ \frac{m}{2^n} : m \in \mathbb{Z}, n \in \mathbb{N} \}$. The following characterization of $L^p(\mathbb{R})$ can be found in [1, Proposition 2.7].

**Lemma 2.2.** $f \in L^p(\mathbb{R})$ if and only if there exists $\omega \in \mathbb{R}$ such that $\text{mod}(f) \subset \omega \mathbb{Q}$.

We are especially interested in a special subclass of $L^p(\mathbb{R})$, namely $L^p_\omega(\mathbb{R}) = \{ f \in L^p(\mathbb{R}) : \text{mod}(f) \subset \omega \mathbb{Q} \}$ for given $\omega \in \mathbb{R}$. From the previous lemma it follows that $P_T(\mathbb{R}) \subset L^p_\omega(\mathbb{R})$ if and only if

$$\frac{2\pi}{T} \in \omega \mathbb{Q}.$$

Essentially because of that, the space $L^p_\omega(\mathbb{R})$ has many of the good properties of $P_T(\mathbb{R})$ itself, contrarily to the bigger $L^p(\mathbb{R})$. They are summarized in the next lemma.

**Lemma 2.3.** The space $L^p_\omega(\mathbb{R})$ is a Banach subalgebra of $AP(\mathbb{R})$, whose all trigonometric polynomials are purely periodic.

We recall that, if $g \in C(\mathbb{R})$ and $f \in AP(\mathbb{R})$ then $g \circ f \in AP(\mathbb{R})$ and moreover

$$\text{mod}(g \circ f) \subset \text{mod}(f).$$

See [6, Theorem 4.5] for a proof. As a consequence, the space $L^p_\omega(\mathbb{R})$ is also closed with respect to the same operation.

**Proof of lemma 2.3.** All the statements follow from the following fact: any finite set of periodic functions $f_1, \ldots, f_m \in L^p_\omega(\mathbb{R})$ has a common period. To see that the last claim is true, denote by $T_1, \ldots, T_m$ the involved periods and use Lemma 2.2 to write it in the form:

$$T_k = \frac{2\pi}{\omega} \frac{q_k}{p_k} = \frac{2\pi}{\omega} \frac{Q_k}{P},$$

where the involved numbers are integers and we set $P = p_1 \cdots p_m$. By setting $Q = Q_1 \cdots Q_m$, it is clear that $T = \frac{2\pi}{\omega} \frac{Q}{P}$ is a common multiple of all the $T_k$’s.

Every almost periodic function can be approximated by trigonometric polynomials which respect its module of frequencies. There are many ways to do it but, as far as we are approximating a function in $L^p_\omega(\mathbb{R})$, the previous lemma guarantees that all these ways give rise to periodic approximations. However, in order to approximate conveniently a pair of twice differentiable well ordered upper and lower solutions $\alpha, \beta \in L^p_\omega(\mathbb{R})$, we need to specialize a little bit the approximation.
scheme. The problem is that we need that the periodic approximation is from inside the order interval
\[ \alpha \ll \beta , \]
and that preserves the strict inequality
\[ \alpha'' + c\alpha' + g(\alpha) \gg \beta'' + c\beta' + g(\beta) . \]
Now the first task alone can be easily achieved, but the second one involves derivatives and is more delicate: approximating different derivatives with different polynomials does not give the desired result, and we have to use a scheme that commutes with derivatives. This is done in the next lemma, where achieving the task is casted into the use of the standard \( BC^k(\mathbb{R}) \) norm for the space
\[
LP^k_\omega(\mathbb{R}) = \{ f \in LP_\omega(\mathbb{R}) : f^{(j)} \in LP_\omega(\mathbb{R}) \text{ for every } 1 \leq j \leq k \} .
\]

**Lemma 2.4.** Let \( f \in LP^k_\omega(\mathbb{R}) \). Then there exist \( f_m \in P^k_{\omega} (\mathbb{R}) \) with \( \frac{2\pi}{T_m} \in \omega \mathbb{Q} \) such that
\[
f \ll f_m \quad \|f_m - f\|_{k, \infty} \to 0 .
\]
A specular result clearly holds for the reversed inequality.

**Proof of lemma 2.4.** We prove it only for \( k = 1 \), since the same argument can be repeated for higher derivatives. Assume that \( f \in LP^1_\omega(\mathbb{R}) \) and then, for the moment, let us worry about the second statement only. Notice that \( \exp(f) \) and \( \exp(f') \) have the same rational basis \( \{\omega\} \). Denote the Bochner-Fejér polynomial of \( f \) and \( f' \) separately as:
\[
\sigma^f_m = \sum_{k=-m^2}^{m^2} \left( 1 - \frac{|k|}{m^2} \right) f \left( \frac{k}{m!} \right) e^{i\omega \frac{k}{m} t} ,
\]
and
\[
\sigma^{f'}_m = \sum_{k=-m^2}^{m^2} \left( 1 - \frac{|k|}{m^2} \right) f' \left( \frac{k}{m!} \right) e^{i\omega \frac{k}{m} t} .
\]
By [4, Theorem 1.24] or [7, Theorem in Page 29], we know that
\[
\|\sigma^f_m - f\|_{\infty} + \|\sigma^{f'}_m - f'\|_{\infty} \to 0 .
\]
Since clearly
\[
\hat{f}' \left( \frac{k}{m!} \right) = i\omega \frac{k}{m!} \hat{f} \left( \frac{k}{m!} \right),
\]
we have that
\[
\sigma^{f'}_m = \left( \sigma^f_m \right)',
\]
and we know that \( \sigma^f_m \in P^1_{\omega} (\mathbb{R}) \) with \( \frac{2\pi}{T_m} \in \omega \mathbb{Q} \) because of Lemma 2.3.

To conclude, we have to prove that it is not restrictive to assume \( f_m \gg f \). But this property can be easily achieved first replacing \( f \) with \( f + \frac{2}{m} \) and then using the above result to find \( f_m \) such that:
\[
\left\| \left( f + \frac{2}{m} \right) - f_m \right\|_{k, \infty} \leq \frac{1}{m} .
\]
\[ \square \]
3. Hill’s equation and exponential dichotomy. The topic of the present section is the damped second order equation:

\[ v'' + cv' + a(t)v = 0 \]  

(4)

where \( a \in BC(\mathbb{R}) \) and \( c \in \mathbb{R} \), with the aim of discussing the existence or nonexistence of bounded solutions. As a first order system, the equation takes the form:

\[
\begin{pmatrix}
  v' \\
  w'
\end{pmatrix}
= 
\begin{pmatrix}
  0 & 1 \\
  -a(t) & -c
\end{pmatrix}
\begin{pmatrix}
  v \\
  w
\end{pmatrix}
\]

(5)

and we denote by \( X_a(t) \) its principal matrix solution, namely the only matrix solution with \( X_a(0) = I \). The bounded solutions are clearly unaffected by the change of perspective. Next lemma points out a fine property of \( X_a(t) \) which will be needed in Section 5. To state it, let us write

\[
X_a(t) = 
\begin{pmatrix}
  v_1(t) & v_2(t) \\
  v'_1(t) & v'_2(t)
\end{pmatrix}
\]

by columns.

**Lemma 3.1.** Assume that \( a(t) < 0 \) for all \( t \in \mathbb{R} \) in (4). Then we have

(a) \( v_1(t) > 1, \) for \( t > 0 \).
(b) \( v_2(t) > 0, \) for \( t > 0 \).
(c) \( e^{ct}v'_2(t) > 1, \) for \( t > 0 \).
(d) \( \text{tr}X_a(t) > 1 + e^{-ct}, \) for \( t > 0 \).

**Proof.** To prove (a) start observing that \( v_1 \) is a solution of

\[
\begin{cases}
  (e^{ct}v'_1)^' + a(t)e^{ct}v_1 = 0, \\
  v_1(0) = 1, \quad v'_1(0) = 0.
\end{cases}
\]

(6)

By the initial condition of \( v_1(t) \) at \( t = 0 \) and (4), we have \( v_1''(0) = |a(t)||v_1(0)| > 0 \). This implies that \( v_1(t) > 1 \), when \( t > 0 \) is small enough. If (a) is false, then there exists a \( t^* > 0 \) such that \( v_1(t) > 1 \) for all \( t \in (0, t^*) \). Then by (6), for all \( t \in (0, t^*) \), we have \( e^{ct}v'_1(0) > 0 \) and hence \( e^{ct}v'_1(t) > e^{ct}v'_1(0) = 0 \). This implies that \( v'_1(t) > 0 \) for all \( t \in (0, t^*) \) which contradicts the fact that \( v_1(0) = v_1(t^*) = 1 \).

Concerning (b) notice that \( v_2 \) is a solution of

\[
\begin{cases}
  (e^{ct}v'_2)^' + a(t)e^{ct}v_2 = 0, \\
  v_2(0) = 0, \quad v'_2(0) = 1.
\end{cases}
\]

(7)

By the initial condition of \( v_2(t) \) at \( t = 0 \), we have \( v_2(t) > 0 \), when \( t > 0 \) is small enough. If (b) is false, then there exists a \( t^* > 0 \) such that \( v_2(t^*) = 0 \) and \( v_2(t) > 0 \) for all \( t \in (0, t^*) \). It follows from (7) that \( (e^{ct}v'_2)^' > 0 \) for all \( t \in (0, t^*) \) and hence \( e^{ct}v_2(t) > e^{ct}v_2(0) = 1 \). This implies that \( v_2(t) > 0 \) for all \( t \in (0, t^*) \) which contradicts the fact that \( v_2(0) = y_2(t^*) = 0 \).

To prove (c) use (b) and (7) to deduce \( e^{ct}v'_2(t) > 0 \) for all \( t > 0 \). Then \( e^{ct}v'_2(t) > e^{ct}v'_2(0) = 1 \) for all \( t > 0 \).

Finally, to prove (d), by (a) and (c), we have

\[
w(t) = e^{ct}(v_1(t) - v_2(t)) > 0, \quad \text{for } t > 0.
\]

Thus it implies that

\[
\text{tr}X_a(t) = v_1(t) + v'_2(t) > 1 + e^{-ct}, \quad \text{for } t > 0.
\]
Equation (4) takes the name of Hill’s equation when \( a \in P_T(\mathbb{R}) \). This case has been widely investigated in the literature, especially when \( c = 0 \); see for instance Hill’s book [8]. Independently of the value of \( c \), it is well known that the existence of nontrivial bounded solutions is decided by the spectral properties of the matrix \( X_a(T) \) alone: they do exist if and only if \( \sigma(X_a(T)) \) intersects the complex unit circle, the very special cases \( \pm 1 \in \sigma(X_a(T)) \) corresponding to \( T \)-periodic and \( T \)-antiperiodic solutions. For \( c = 0 \) it is standard and useful fact to re-express this information in terms of the trace of \( X_a(T) \) itself: next lemma does the same for every value of \( c \).

**Lemma 3.2.** Assume that \( a \in P_T(\mathbb{R}) \). Then system (5) has nontrivial \( T \)-periodic or \( T \)-antiperiodic solutions if and only if

\[
\text{tr}X_a(T) = 1 + e^{-cT} \quad \text{or} \quad \text{tr}X_a(T) = -1 - e^{-cT}
\]

respectively. They are the only nontrivial bounded solutions when \( c \neq 0 \), while when \( c = 0 \) there are more of them if and only if

\[
|\text{tr}X_a(T)| < 2.
\]

**Proof.** By Liouville’s formula \( \det X_a(T) = e^{-cT} \) and hence the eigenvalues of \( X_a(T) \) are given by the solutions of the following equation

\[
\rho^2 - (\text{tr}X_a(T))\rho + e^{-cT} = 0.
\]

We distinguish two cases, depending on the sign of discriminant. In the first case

\[
(\text{tr}X_a(T))^2 - 4e^{-cT} \geq 0
\]

the two (possibly coincident) eigenvalues are real

\[
\rho_{\pm} = \frac{1}{2} \left\{ \text{tr}X_a(T) \pm \sqrt{(\text{tr}X_a(T))^2 - 4e^{-cT}} \right\}.
\]

To have modulus one, an eigenvalue must coincide with \( \pm 1 \): bounded solutions then are necessarily \( T \)-periodic or \( T \)-antiperiodic. It can be easily checked that

\[
\rho_+ = 1 \quad \Leftrightarrow \quad \begin{cases} 
    c \geq 0 \\
    \text{tr}X_a(T) = 1 + e^{-cT}
\end{cases}
\]

whereas

\[
\rho_- = 1 \quad \Leftrightarrow \quad \begin{cases} 
    c \leq 0 \\
    \text{tr}X_a(T) = 1 + e^{-cT}
\end{cases}.
\]

Summing up, one has \( 1 \in \sigma(X_a(T)) \) if and only if \( \text{tr}X_a(T) = 1 + e^{-cT} \). A specular result holds for the eigenvalue \( -1 \) by reversing the sign of the trace.

The remaining case is

\[
(\text{tr}X_a(T))^2 - 4e^{-cT} < 0
\]

which corresponds to a pair of complex conjugated eigenvalues

\[
\rho_{\pm} = \frac{1}{2} \left\{ \text{tr}X_a(T) \pm i\sqrt{4e^{-cT} - (\text{tr}X_a(T))^2} \right\}.
\]

In particular, no \( T \)-periodic or \( T \)-antiperiodic solutions can exist. Since moreover

\[
|\rho_\pm| = e^{-cT},
\]

the eigenvalues are off the complex unit circle unless \( c = 0 \). In this last case only, we have bounded (aperiodic) solutions. \( \Box \)
Continuing the exam of the periodic case, let us consider the eigenvalue equation:

\[ v'' + cv' + \left[a(t) + \lambda\right]v = 0 \tag{8} \]

where we look for solutions \( v \in \mathcal{P}_T(\mathbb{R}) \) which are non-trivial. When \( c = 0 \) the equation is self-adjoint and all the eigenvalues are real: see [8] for a detailed description of the spectral properties of (8). The case \( c \neq 0 \) is much less understood but some of the results survive: though complex eigenvalues may appear, the first real eigenvalue makes sense again and has the expected properties.

**Lemma 3.3.** Assume that \( a \in \mathcal{P}_T(\mathbb{R}) \). Then there exists \( \lambda_1 \in \mathbb{R} \) such that equation (8) admits a non-trivial \( T \)-periodic solution \( \varphi_1 \gg 0 \) for \( \lambda = \lambda_1 \), while it has no non-trivial \( T \)-periodic solutions for \( \lambda < \lambda_1 \).

More can be said about the spectral properties, but it is not relevant for our aims. The result is known, but we nevertheless decide to sketch the proof since we are not able to provide an explicit reference for it.

**Proof of lemma 3.3.** Choose \( \lambda_0 \) such that \( \lambda_0 + a(t) < 0 \) for every \( t \). Lemma 3.1 (d) guarantees that:

\[ \text{tr} X_{a+\lambda}(T) > 1 + e^{-cT}, \quad \text{for } \lambda \leq \lambda_0 \tag{9} \]

and hence Lemma 3.2 says that, for the same values of \( \lambda \), equation (8) has no non-trivial \( T \)-periodic solutions. In particular, because of the classical Fredholm Alternative, for every \( z \in \mathcal{P}_T(\mathbb{R}) \) the equation:

\[ -v'' - cv' - \left[a(t) + \lambda_0\right]v = z(t) \]

has a unique solution \( v \in \mathcal{P}_T^2(\mathbb{R}) \) where, as usual, we think of \( R \) as acting \( \mathcal{P}_T(\mathbb{R}) \rightarrow \mathcal{P}_T(\mathbb{R}) \).

With these notations, it is clear that \( v \in \mathcal{P}_T(\mathbb{R}) \) solves (8) for any given real or complex \( \lambda \neq \lambda_0 \) if and only if:

\[ Rv = \mu v \quad \text{where} \quad \mu = \frac{1}{\lambda - \lambda_0}. \tag{10} \]

Notice that, by construction, \( \mu = 0 \) is not an eigenvalue of \( R \). With the very same arguments of the case \( c = 0 \), the operator \( R \) can be proved to be compact and positive. To be more precise, the strong version of the Maximum Principle applies to show that \( R \) is strictly positive, in the sense that:

\[ z \geq 0 \quad \text{and} \quad z \neq 0 \quad \text{imply} \quad Rz \gg 0. \]

See for instance [5, Theorem A-5.1]. If we denote by \( \rho \) the spectral radius of \( R \), then the Krein-Rutman Theorem says that \( \rho > 0 \) and that it is a simple eigenvalue, with an eigenvector \( \varphi_1 \gg 0 \). See for instance [2, Theorem VI.13]. Set now:

\[ \lambda_1 = \lambda_0 + \frac{1}{\rho}. \]

Since the eigenvalues of \( R \) satisfy \( |\mu| \leq \rho \), we deduce from (10) that equation (8) may have non-trivial \( T \)-periodic solutions only when \( |\lambda - \lambda_0| \geq 1/\rho \). To conclude the proof, add the information provided by (9).

Lemma 3.3 will be only used in the proof of Lemma 5.1. Concerning Lemma 3.2, let us notice that the nontrivial bounded solutions are strictly related to the crucial notion of exponential dichotomy. For a general \( a \in BC(\mathbb{R}) \), the system (5) has an exponential dichotomy (in the whole \( \mathbb{R} \)) when there exist a projection \( P \) and positive constants \( \sigma, K \) such that:

\[ \|X_a(t)PX_a^{-1}(s)\| \leq Ke^{-\sigma(t-s)} \quad t \geq s \tag{11} \]
\[ \|X_a(t)(I - P)X_a^{-1}(s)\| \leq Ke^{-\sigma(s-t)} \quad t \leq s. \quad (12) \]

With a little abuse of language, we denote this occurrence also by saying that the coefficient \( a \) gives rise to an exponential dichotomy. We refer to Coppel’s book [3] for an introduction to the subject, including the proofs of the few properties we need. The first one concerns the bounded solutions. It is clear that, if (5) has an exponential dichotomy then it has nontrivial bounded solutions. In general the implication cannot be reversed, but a main point is that this can be done at least in the special case \( a \in P_T(\mathbb{R}) \).

Hence, in this case Lemma 3.2 fully decides whether system (5) has an exponential dichotomy: it happens if and only if
\[ |\text{tr}X_a(T)| > 2 \quad \text{or} \quad |\text{tr}X_a(T)| \neq 1 + e^{-cT} \]
according to the fact that \( c = 0 \) or \( c \neq 0 \) respectively.

The second property we need is roughness of exponential dichotomy. Roughness means that, if system (5) has an exponential dichotomy for a given \( a \in BC(\mathbb{R}) \), then the same is true when we replace it with any another \( b \in BC(\mathbb{R}) \) such that \( \|b - a\|_\infty < \varepsilon \), as long as \( \varepsilon > 0 \) is sufficiently small. We will use this device to propagate the exponential dichotomy from some selected \( a \in P_T(\mathbb{R}) \) to all the nearby aperiodic systems, and in particular the limit periodic ones.

The third and last property concerns the inhomogeneous equation
\[ \begin{pmatrix} v' \\ w' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a(t) & -c \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} g(t) \\ f(t) \end{pmatrix}. \quad (13) \]
It is well known that (5) has an exponential dichotomy if and only if (13) has an unique and bounded solution for every given \( f, g \in BC(\mathbb{R}) \) [3]. This solution at \( t \) can be easily computed as
\[ \int_{-\infty}^{t} X_a(t)PX_a(s)^{-1}F(s) \, ds + \int_{t}^{\infty} X_a(t)(I - P)X_a(s)^{-1}F(s) \, ds \]
where \( F \) stands for the pair \((g, f)\) and \( P \) is the projection involved in the definition (11)–(12) of exponential dichotomy.

In particular, for every \( f \in BC(\mathbb{R}) \) the inhomogeneous equation
\[ v'' + cv' + a(t)v = f(t) \quad (14) \]
has a unique bounded solution \( \varphi_f \) which moreover satisfies:
\[ \|\varphi_f\|_{1,\infty} \leq \frac{2K}{\sigma} \|f\|_\infty. \]
The priori estimate follows directly from the integral representation formula mentioned above and will be crucial in Section 5. It implies that the map defined by the equation (4), that is
\[ L_a v = v'' + cv' + a(t)v, \]
is indeed an isomorphism \( BC^2(\mathbb{R}) \cong BC(\mathbb{R}) \). It is important to notice that, if \( a \in AP(\mathbb{R}) \)
then such isomorphism restricts to an isomorphism \( AP^2(\mathbb{R}) \cong AP(\mathbb{R}) \). To be more precise, if also \( f \in AP(\mathbb{R}) \) then it happens that
\[
\varphi_f \in AP^2(\mathbb{R})
\]
and moreover the following inclusion
\[
\mod(\varphi_f) \subset \mod(a, f)
\]
takes place. See for instance [6, Theorem 7.7] for a proof.

We conclude the section with a comment about the most classical case
\( a, f \in P_T(\mathbb{R}) \).

In this case, it is well known that the existence of bounded solutions to (14) implies that of \( T \)-periodic solutions. In turn, the existence of \( T \)-periodic solutions is governed by the classical Fredholm Alternative: see for instance Hill’s book [8]. As a consequence of the Fredholm Alternative, we have that the following two facts are equivalent:

1. equation (14) has a \( T \)-periodic solution for every \( f \in P_T(\mathbb{R}) \);
2. equation (4) has no nontrivial \( T \)-periodic solution.

If we take \( L_a \) and we restrict it \( P^2_T(\mathbb{R}) \to P_T(\mathbb{R}) \), then (1) means that the restriction is surjective, while (2) is injectivity. In both cases we get an isomorphism \( P^2_T(\mathbb{R}) \cong P_T(\mathbb{R}) \). Another equivalent way to express the same fact by saying that
\[
1 \notin \sigma(X_a(T)) \text{ or, in view of Lemma 3.2}
\]
\[
\operatorname{tr}X_a(T) \neq 1 + e^{-cT}.
\]
This last way is the characterization we will use in Section 5.

4. A Sard-Smale type lemma for the nonlinear periodic case. Let us consider the nonlinear map \( \Phi : BC^2(\mathbb{R}) \to BC(\mathbb{R}) \) defined by:
\[
\Phi(u) := u'' + cu' + g(u)
\]
where we assume that the involved spaces are endowed with the natural norms, that is \( \| \cdot \|_{2, \infty} \) and \( \| \cdot \|_{\infty} \) respectively. This map is related to the nonlinear equation (1), inasmuch \( u \) is a bounded solution of the latter if and only if
\[
\Phi(u) = h.
\]
Since \( g \in C^1(\mathbb{R}) \) we know that \( \Phi \) is continuously (Frechét) differentiable at every \( u \in BC^2(\mathbb{R}) \), the derivative being
\[
\Phi'(u)v = v'' + cv' + g'(u)v
\]
for every \( v \in BC^2(\mathbb{R}) \). Hence, when we set \( a = g'(u) \) condition:
\[
\Phi'(u)v = 0
\]
is equivalent to say that \( v \) is a bounded solution of the linear equation (4).

In this section we are mainly concerned about the action of \( \Phi \) on periodic functions. To this aim, given an arbitrary \( T > 0 \) we introduce the restriction
\[
\Phi_T : P^2_T(\mathbb{R}) \to P_T(\mathbb{R})
\]
where the involved spaces have the inherited norms. Contrarily to \( \Phi \), the restriction \( \Phi_T \) has some good and helpful functional properties. Indeed, it is well known that \( \Phi_T \) is a Fredholm map of index zero and hence, as a consequence of the classical Sard–Smale Lemma, it has a residual subset of regular values.
To describe the result, let us recall that \( u \in P^2_T(\mathbb{R}) \) is a regular point of \( \Phi_T \) when the linear map \( \Phi_T'(u) \) is surjective. A main point is that this is true if and only if \( \Phi_T'(u) \) is injective. As we already said at the end of the previous section, the equivalence is a particular case of the classical Fredholm Alternative, which is also the ultimate responsible for \( \Phi_T \) being a Fredholm map of index zero. See [9, Lemma 2] for a proof.

A given \( h \in P_T(\mathbb{R}) \) is a regular value for \( \Phi_T \) when every \( u \in \Phi_T^{-1}(h) \) is a regular point. Sard–Smale Lemma guarantees that these regular values are a topologically large set, in the sense that they can be written as the intersection of a countable class of open and dense subsets of \( P_T(\mathbb{R}) \); see [13] for a proof.

The Sard-Smale Lemma is a powerful instrument, but a point must be stressed: the conclusion may be empty, since by the very definition all the values which are not attained are regular. Some mechanism is then needed to give full strength to the result. Here the mechanism is the method of lower and upper solutions, which in turn calls for some minor modifications in the lemma. To introduce them, let us consider two functions \( \alpha_T, \beta_T \in P^2_T(\mathbb{R}) \) such that the inequalities
\[
\alpha_T < \beta_T \quad \beta_T'' + c\beta_T' + g(\beta_T) < \alpha_T'' + c\alpha_T' + g(\alpha_T)
\]
hold pointwise in the whole \( \mathbb{R} \), and define:
\[
\Delta^T_{\alpha_T, \beta_T} = \{ u \in P^2_T(\mathbb{R}) : \alpha_T < u < \beta_T \}
\]
and
\[
H^T_{\alpha_T, \beta_T} = \{ h \in P_T(\mathbb{R}) : \beta_T'' + c\beta_T' + g(\beta_T) < h < \alpha_T'' + c\alpha_T' + g(\alpha_T) \}. \]
Notice that, due to the periodicity of the involved functions, all the previous \( < \) are indeed strong inequalities \( \ll \). In particular \( \Delta^T_{\alpha_T, \beta_T} \) and \( H^T_{\alpha_T, \beta_T} \) are open subsets of \( P^2_T(\mathbb{R}) \) and \( P_T(\mathbb{R}) \) respectively. We finally weaken the notion of regular values as follows.

**Definition 4.1.** We denote by \( S^T_{\alpha_T, \beta_T} \) the set of the values \( h \in H^T_{\alpha_T, \beta_T} \) such that every \( u \in P^2_T(\mathbb{R}) \) satisfying
\[
\Phi_T(u) = h \quad \text{and} \quad \alpha_T < u < \beta_T
\]
is a regular point for \( \Phi_T \).

Next lemma says that the set of such values is also topologically large.

**Lemma 4.2.** All the values in \( H^T_{\alpha_T, \beta_T} \) are attained by \( \Phi_T \) at a point of \( \Delta^T_{\alpha_T, \beta_T} \).
Moreover \( S^T_{\alpha_T, \beta_T} \) is open and dense in \( H^T_{\alpha_T, \beta_T} \).

The proof is similar to Theorem 2 in [9]. The same argument appears also in [10], which however uses also some deeper results of [9].

**Proof of lemma 4.2.** The first statement is a standard application of the method of upper and lower solutions: see for instance [5, Theorem III-1.8]. To prove the second statement, start noticing that, since \( H^T_{\alpha_T, \beta_T} \) is open in \( P_T(\mathbb{R}) \), it follows from the Sard–Smale Lemma that the regular values of \( \Phi_T \) meet \( H^T_{\alpha_T, \beta_T} \) in a dense subset. This is clearly true also for \( S^T_{\alpha_T, \beta_T} \), which contains all the regular values in \( H^T_{\alpha_T, \beta_T} \).

It remains to prove that \( S^T_{\alpha_T, \beta_T} \) is open in \( H^T_{\alpha_T, \beta_T} \). Assume by contradiction that there exists a sequence \( h_n \notin S^T_{\alpha_T, \beta_T} \) such that \( \|h_n - h\|_\infty \to 0 \) with \( h \in S^T_{\alpha_T, \beta_T} \).
Since $h \in H_{\alpha_T \beta_T}^T$, which is open, we can assume that $h_n \in H_{\alpha_T \beta_T}^T$ holds for every $n$. Thus there exists $u_n \in \Delta_T^{\alpha_T \beta_T}$ such that

$$u_n'' + cu_n' + g(u_n) = h_n.$$  

Since $\|u_n\|_{\infty}$ and $\|h_n\|_{\infty}$ are bounded, the sequence $\|u_n\|_{2,\infty}$ is bounded as well. Then standard compactness arguments apply to show that, along a subsequence, $\|u_n - u\|_{2,\infty} \to 0$ for some suitable $u \in P_2^T(\mathbb{R})$ satisfying

$$u'' + cu' + g(u) = h.$$  

Of course the large inequality

$$\alpha_T \leq u \leq \beta_T$$

holds pointwise in the whole $\mathbb{R}$. To see that both the inequalities are strict, suppose for instance that $\alpha_T(t_0) = u(t_0)$ for some $t_0$. Then $\alpha_T'(t_0) = u'(t_0)$ and $\alpha_T''(t_0) \leq u''(t_0)$, which is impossible due to $h \in H_{\alpha_T \beta_T}^T$. Thus actually $u \in \Delta_T^{\alpha_T \beta_T}$.

Let us now show that $u$ cannot be a regular point. Since $h_n \notin S_{\alpha'_{T} \beta_{T}}^T$ by construction, it is not restrictive to assume that $u_n$ is not a regular point of $\Phi_T$. Due to the Fredholm Alternative, $\Phi'(u_n)$ cannot be injective. Let $v_n \in P_2^T(\mathbb{R})$ be such that $\|v_n\|_{2,\infty} = 1$ and

$$v_n'' + cv_n' + g'(u_n)v_n = 0.$$  

Standard compactness arguments show that $\|v_n - v\|_{2,\infty} \to 0$ along a subsequence and for some suitable $v \in P_2^T(\mathbb{R})$ satisfying $\|v\|_{2,\infty} = 1$ and $v'' + cv' + g'(u)v = 0$.

Thus $\Phi'(u)$ is not injective and, again due to the Fredholm alternative, is not surjective. In other words, $u$ is not a regular point for $\Phi_T$, contradicting the assumption $h \in S_{\alpha_T \beta_T}^T$ and then concluding the proof.  

\vspace{1em}

5. Perturbing periodic nonlinear equations. To have a control on the nonlinear perturbation, we need solutions with stronger stability properties as follows.

**Lemma 5.1.** Let $\alpha_T, \beta_T \in P_2^T(\mathbb{R})$ and satisfy condition (15). For every $h \in S_{\alpha_T \beta_T}^T$, equation (1) has a solution $u_h \in \Delta_T^{\alpha_T \beta_T}$ and the linearized equation

$$v'' + cv' + g'(u_h)v = 0$$

admits an exponential dichotomy.

**Proof.** We divide this proof into the several steps.

**Step 1.** Due to [5, Proposition III-1.1, Proposition III-1.2], $\alpha_T$ and $\beta_T$ are strict lower and upper solutions. Then by [5, Theorem I-5.6], we know that there exists a minimum solution $u_h \in P_2^T(\mathbb{R})$ to (1) such that $\alpha_T \ll u_h \ll \beta_T$ and every solution $u \in P_2^T(\mathbb{R})$ to (1) with $\alpha_T \leq u \leq \beta_T$ satisfies $u \geq u_h$.

**Step 2.** We claim that the first eigenvalue $\lambda_h$ of

$$\begin{cases}
    v'' + cv' + g'(u_h)v + \lambda v = 0 \\
    v(0) = v(T), \quad v'(0) = v'(T)
\end{cases}$$

satisfies $\lambda_h \geq 0$. We assume by contradiction that $\lambda_h < 0$. Let $\epsilon > 0$ such that $\lambda_h + \epsilon < 0$ and $0 < \delta < \min_{t \in \mathbb{R}} (u_h - \alpha_T) > 0$ such that $g'(u_h + \epsilon) \geq g'(u_h) - \epsilon$ for all $t \in \mathbb{R}$ and $v \in [-\delta, \delta]$. According to Lemma 3.3, let $\varphi_1 > 0$ be the first eigenfunction of (16) with $\|\varphi_1\|_{\infty} = 1$.  

Let us prove that \( \beta_1 = u_h - \frac{1}{2} \varphi_1 \) is a upper solution of
\[
\begin{cases}
  u'' + cu' + g(u) = h , \\
  u(0) = u(T), \ u'(0) = u'(T) .
\end{cases}
\]

In fact, we have
\[
g(u_h - \frac{1}{2} \varphi_1) - g(u_h) = - \int_0^1 g'(u_h - s \frac{1}{2} \varphi_1) \frac{1}{2} \varphi_1 ds \leq - (g'(u_h) - \varepsilon) \frac{1}{2} \varphi_1(t) .
\]

We deduce that
\[
\beta''_1 + c\beta'_1 + g(\beta_1) = u''_h + cu'_h + g(u_h) - \frac{1}{2} (\varphi''_1 + c\varphi'_1)
\]
\[
= h - g(u_h) + g(u_h - \frac{1}{2} \varphi_1) + \frac{1}{2} (g'(u_h) - \varepsilon) \varphi_1 + \frac{1}{2} (\lambda_h + \varepsilon) \varphi_1
\]
\[
\leq h .
\]

Hence we have \( \alpha_T \leq \beta_1 < u_h \). By [5, Theorem I-5.3], this implies that a solution \( u_1 \) with \( \alpha_T \leq u_1 \leq \beta_1 < u_h \) exists, which contradicts the fact that \( u_h \) is the minimum solution in \([\alpha_T, \beta_T]\).

**Step 3.** Since \( h \in S_{\alpha_T, \beta_T}^T \) and \( \alpha_T \ll u_h \ll \beta_T \) by Step 1, then \( u_h \in P^T_h(\mathbb{R}) \) is a regular point for the map \( \Phi_T \). It is equivalent to say that the only \( T \)-periodic solution to
\[
v'' + cv' + g'(u_h)v = 0 \quad (17)
\]
is the trivial one. Then we have \( \lambda_h \neq 0 \). Combining with Step 2, it implies that \( \lambda_h > 0 \).

**Step 4.** Now we claim that the first order system associated to (17), that is,
\[
\begin{pmatrix}
v' \\
w'
\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -g'(u_h) & -c \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}
\]
has an exponential dichotomy. To see this, from Lemma 3.1 (d) and Lemma 3.2, we have that
\[
\text{tr}X_{g'(u_h)}\lambda(T) > 1 + e^{-cT}
\]
for every \( \lambda < \lambda_h \). Then it is true for \( \lambda = 0 \) and the claim follows from Lemma 3.2; see the comments at the end of Section 3.

From now on, \( h \) and \( u_h \) are supposed to be the functions considered in Lemma 5.1. We focus on what happens for \( k \in BC(\mathbb{R}) \) which is near to \( h \). Of course we expect to find a solution \( u \in BC(\mathbb{R}) \) to the equation
\[
u'' + cu' + g(u) = k \quad (18)
\]
which is near to \( u_h \). It is common knowledge that this happens indeed, due to the exponential dichotomy proved in Lemma 5.1. However, though a related result is proved in Chapter 8 of Fink’s book [6], a complete proof seems to be unavailable in the literature. In order to give it hereafter, let us start changing the variable as follows:
\[
z = u - u_h .
\]

The new equation is:
\[
z'' + cz' + g'(u_h(t))z = k(t) - h(t) - R_{u_h}(z, t) \quad (19)
\]
where
\[
R_{u_h}(z, t) = g(u_h(t) + z) - g(u_h(t)) - g'(u_h(t))z .
\]
Due to the characterization of the exponential dichotomy given in Section 2, we know that for every $f \in BC(\mathbb{R})$ the equation
\[ v'' + cv' + g'(u_h(t))v = f(t) \]
admits a unique solution $v \in BC(\mathbb{R})$. Moreover the following prior estimate holds
\[ \|v\|_\infty \leq \frac{1}{L} \|f\|_\infty , \tag{20} \]
where $L = \frac{\sigma}{2K}$ and $K, \sigma$ are the parameters defining the exponential dichotomy.

Given an arbitrary $k \in BC(\mathbb{R})$, consider now the nonlinear operator on $BC(\mathbb{R})$ defined as follows: $v = T_k z$ is the unique solution in $BC(\mathbb{R})$ to the linear differential equation
\[ v'' + cv' + g'(u_h(t))v = k(t) - h(t) - R_{u_h}(z,t) . \]
Next lemma discusses the properties of $T_k$ on some suitable ball
\[ B_\delta = \{ z \in BC(\mathbb{R}) : \|z\|_\infty \leq \delta \} . \]

**Lemma 5.2.** For every $\epsilon < L$, there exists $\delta > 0$ such that $T_k$ is a contraction on $B_\delta$ for every
\[ \|k - h\|_\infty \leq \delta(L - \epsilon) . \]

**Proof.** First introduce
\[ G_u(z,t) = g(u(t) + z) - g'(u(t))z . \]
By the uniform continuity of $g'$ on compact sets, we have that for every $\epsilon > 0$, there exists $\delta > 0$ such that $|z| \leq \delta$ implies
\[ \left| \frac{\partial G_u}{\partial z}(z,t) \right| \leq \epsilon \quad \forall t \in \mathbb{R} . \]
Note that
\[ R_{u_h}(z,t) = G_u(z,t) - G_u(0,t) . \]
Then we have
\[ |R_u(z_1,t) - R_u(z_2,t)| = \left| \frac{\partial G_u}{\partial z}(\xi,t) \right| |z_1 - z_2| , \]
for some suitable $\xi$ between $z_1$ and $z_2$. When $|z_i| \leq \delta$, we have
\[ |R_u(z_1,t) - R_u(z_2,t)| \leq \epsilon |z_1 - z_2| . \tag{21} \]

- **Claim 1.** $T_k$ is a contraction. Indeed, denote $v_i = T_k z_i$, $i = 1,2$. We have
\[ v_i'' + cv_i' + g'(u_h(t))v_i = k(t) - h(t) - R_{u_h}(z_i,t) , \quad i = 1,2. \]
These imply that
\[ (v_1 - v_2)'' + c(v_1 - v_2)' + g'(u_h(t))(v_1 - v_2) = R_{u_h}(z_1,t) - R_{u_h}(z_2,t) . \]
By (20) and (21), we have
\[ \|T_k z_1 - T_k z_2\|_\infty \leq \frac{1}{L} \|R_{u_h}(z_2,t) - R_{u_h}(z_1,t)\|_\infty \leq \frac{\epsilon}{L} \|z_2 - z_1\|_\infty . \]

- **Claim 2.** $T_k$ maps $B_\delta$ to $B_\delta$. Indeed, denote $T_k 0 = v$. Since $R_{u_h}(0,t) \equiv 0$, then we have
\[ v'' + cv' + g'(u_h(t))v = k(t) - h(t) . \]
By (20), we obtain
\[ \|T_k 0\|_\infty \leq \frac{1}{L} \|k - h\|_\infty \leq \frac{1}{L} \delta(L - \epsilon) = \delta(1 - \frac{\epsilon}{L}) . \]
Now we take $z \in B_{\delta}$. By Claim 1, we have
\[ \|T_k z\|_{\infty} \leq \|T_k 0\|_{\infty} + \frac{\varepsilon}{L} \|z\|_{\infty} \leq \delta. \]
\hfill \Box

The following result is a direct result of Lemma 5.2 in the $LP_\omega(\mathbb{R})$ framework.

**Lemma 5.3.** Let $2\pi/T \in \omega \mathbb{Q}$ and $h \in S^{T}_T \alpha_T \beta_T$. Then there exists an open neighborhood $U_h \subset BC(\mathbb{R})$ centered at $h$ such that for each $k \in U_h \cap LP_\omega(\mathbb{R})$, the equation (18) admits a solution $u \in LP^2_\omega(\mathbb{R})$ near to $u_h$ with $\alpha_T \ll u \ll \beta_T$.

**Proof.** Using the notations $\varepsilon$, $h$, $\delta$ and $L$ in Lemma 5.2, we define
\[ U_h = \{ k \in BC(\mathbb{R}) : \|k - h\|_{\infty} < \delta (L - \varepsilon) \}. \]
If $k \in U_h \cap LP_\omega(\mathbb{R})$, then $T_k$ is a contraction on $B_{\delta}$. By [6, Theorem 7.7], we have
\[ \text{mod}(T_k z) \subset \text{mod}(g'(u_h(t)), k(t) = h(t) - R_{u_h}(z, t)) = \text{mod}(u_h, k, h, z). \]
It implies that $T_k$ is a contraction on $B_{\delta} \cap LP_\omega(\mathbb{R})$. Since $B_{\delta} \cap LP_\omega(\mathbb{R})$ is a closed subset of a Banach space, there exists a fixed point $z \in B_{\delta} \cap LP_\omega(\mathbb{R})$ such that $T_k z = z$. That is, $z$ is a solution to (19). It follows that $u = z + u_h$ is a solution in $LP^2_\omega(\mathbb{R})$ to (18). If we choose $\delta$ small enough, then $\alpha_T \ll u_h \ll \beta_T$ implies (22).

6. **Proof of Theorem 1.1.** We are finally ready to conclude the proof of Theorem 1.1 in the Introduction. The aim is to construct an open and dense subset $S \subset H_{\alpha\beta} \cap LP_\omega(\mathbb{R})$ such that, for every $k \in S$, the equation
\[ u'' + cu' + g(u) = k(t) \]  
has a solution $u \in \Delta_{\alpha\beta} \cap LP_\omega(\mathbb{R})$.

To construct the set $S$, we consider all the periods $T > 0$ such that
\[ \frac{2\pi}{T} \in \omega \mathbb{Q} \]
and then all the pairs of $\alpha_T, \beta_T \in P^T_T(\mathbb{R})$ such that
\[ \alpha < \alpha_T \ll \beta_T < \beta \]
\[ \beta_T'' + c\beta_T' + g(\beta_T) \ll \alpha_T'' + c\alpha_T' + g(\alpha_T) \]
and finally all the forcing terms
\[ h_T \in S^{T}_T \alpha_T \beta_T \]
as introduced in Definition 4.1. To every such $h_T$, we associate an open neighborhood $U_{h_T} \subset BC(\mathbb{R})$ as in Lemma 5.3. We set
\[ S := \left\{\bigcup U_{h_T}\right\} \cap H_{\alpha\beta} \cap LP_\omega(\mathbb{R}) \]
where the union runs over $T, \alpha_T, \beta_T$ and $h_T$ as declared just above.

When we take $k \in S$, Lemma 5.3 says that equation (23) admits a solution
\[ u \in \Delta_{\alpha_T \beta_T} \cap LP_\omega(\mathbb{R}) \subset \Delta_{\alpha\beta} \cap LP_\omega(\mathbb{R}). \]
Moreover $S$ is open by construction, though it may be not evident why it is nonempty. In any case, to conclude the proof it is enough to show that $S$ is dense in $H_{\alpha\beta} \cap LP_\omega(\mathbb{R})$. 

LIMIT PERIODIC UPPER AND LOWER SOLUTIONS 307
To this aim, let $\varepsilon > 0$ and consider any $h \in H_{\alpha \beta} \cap LP_\omega (\mathbb{R})$.

By definition of $LP_\omega (\mathbb{R})$, it is always possible to find $T_1 > 0$ and $\tilde{h}_{T_1} \in P_{T_1} (\mathbb{R})$ such that
\[
\frac{2\pi}{T_1} \in \omega \mathbb{Q} \quad \| h - \tilde{h}_{T_1} \|_\infty < \varepsilon .
\] (27)

Use now the full force of Lemma 2.4 to find $T_2, T_3 > 0$ such that
\[
\frac{2\pi}{T_2} \in \omega \mathbb{Q} \quad \frac{2\pi}{T_3} \in \omega \mathbb{Q}
\] together with $\alpha_{T_2} \in P_{T_2}^2 (\mathbb{R})$ and $\beta_{T_3} \in P_{T_3}^2 (\mathbb{R})$ such that
\[
\| \alpha - \alpha_{T_2} \|_{2, \infty} < \varepsilon \quad \alpha \ll \alpha_{T_2} \quad \| \beta - \beta_{T_3} \|_{2, \infty} < \varepsilon \quad \beta \gg \beta_{T_3} .
\] (28)

Because of (27) and (28), it is clear that $T_1, T_2, T_3$ have a common multiple $T$ satisfying (24). We rename
\[
h_{T_1} = h_T \quad \alpha_{T_2} = \alpha_T \quad \beta_{T_3} = \beta_T
\]
and notice that the inequalities (27), (29) and (30) persist unchanged. By taking $\varepsilon$ small enough into them, we get (25) and
\[
\beta_T'' + c\beta_T' + g(\beta_T) \ll \tilde{h}_T \ll \alpha_T'' + c\alpha_T' + g(\alpha_T)
\]
and moreover
\[
h_T \in H_{\alpha \beta} .
\]

In particular, (26) is satisfied. It only remains to use Lemma 4.2 to find a $h_T \in S_{\alpha_T \beta_T}^T \cap H_{\alpha \beta} \subset S$ arbitrarily near to $\tilde{h}_T$ and hence to $h$.

\[\square\]

**Acknowledgments.** We express the sincere thanks to Prof. De Coster for Lemma 5.1.

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Received February 2017; revised July 2017.

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