Lorarithmic violation of scaling in strongly anisotropic turbulent transfer of a passive vector field

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Inertial-range asymptotic behavior of a vector (e.g., magnetic) field, passively advected by a strongly anisotropic turbulent flow, is studied by means of the field theoretic renormalization group and the operator product expansion. The advecting velocity field is Gaussian, not correlated in time, with the pair correlation function of the form $\propto \delta(t - t')/k_d^{d-1+\xi}$, where $k_d = |k_d|$ and $k_d$ is the component of the wave vector, perpendicular to the distinguished direction ('direction of the flow') – the $d$-dimensional generalization of the ensemble introduced by Avellaneda and Majda [Commun. Math. Phys. 131: 381 (1990)]. The stochastic advection-diffusion equation for the transverse (divergence-free) vector field includes, as special cases, the kinematic dynamo model for magnetohydrodynamic turbulence and the linearized Navier–Stokes equation. In contrast to the well known isotropic Kraichnan’s model, where various correlation functions exhibit anomalous scaling behavior with infinite sets of anomalous exponents, here the dependence on the integral turbulence scale $L$ has a logarithmic behavior: instead of power-like corrections to ordinary scaling, determined by naive (canonical) dimensions, the anomalies manifest themselves as polynomials of logarithms of $L$. The key point is that the matrices of scaling dimensions of the relevant families of composite operators appear nilpotent and cannot be diagonalized. The detailed proof of this fact is given for correlation functions of arbitrary order.

Keywords: anomalous scaling, passive vector advection, magnetohydrodynamic turbulence, renormalization group

I. INTRODUCTION

Much attention has been attracted to the problem of intermittency and anomalous scaling in developed magnetohydrodynamic (MHD) turbulence; see, e.g., [1]–[8] and references therein. It has long been known that in the so-called Alfvénic regime, the MHD turbulence demonstrates the behavior, similar to that of the usual fully developed fluid turbulence: cascade of energy from the infrared range towards smaller scales, where the dissipation effects dominate, and self-similar (scaling) behavior of the energy spectra in the intermediate (inertial) range. Moreover, intermittent character of the fluctuations in the MHD turbulence is much strongly pronounced than in ordinary turbulent fluids.

The solar wind provides a kind of an appropriate “wind tunnel” in which different approaches and models of the MHD turbulence can be tested [8]. In solar flares, highly energetic and anisotropic large-scale motions coexist with small-scale coherent structures, finally responsible for the dissipation. Thus modelling the way how the energy is redistributed, transferred along the spectra and eventually dissipated is a difficult task. The intermittency strongly modifies the scaling behavior of the higher-order correlation functions, leading to anomalous scaling, described by infinite sets of independent “anomalous exponents.”

A simplified description of the situation was proposed in [2]: the large-scale field $B_0^0 = n_i B_i^0$ dominates the dynamics in the distinguished direction $n$, while the activity in the perpendicular plane is described as nearly two-dimensional. This picture allows for precise numerical simulations, which show that turbulent fluctuations organize in rare coherent structures separated by narrow current sheets. On the other hand, the observations and simulations show that the scaling behavior in the solar wind is closer to the anomalous scaling in the three-dimensional fully developed hydrodynamic turbulence, rather than to simple Iroshnikov–Kraichnan scaling suggested by two-dimensional picture with the inverse energy cascade; see, e.g., the discussion in [3]. Thus further analysis of more realistic three-dimensional models is welcome.

Two main simplifications of the full-scale model are possible here. First, the magnetic field can be taken passive, that is, not to affect the dynamics of the velocity field. This approximation is valid when the gradients of the magnetic fields are not too large. What is more, the renormalization group analysis shows that such a “kinematic regime” can indeed describe the possible infrared (IR) behavior of the full-scale model [3].

Second, description of the fluid turbulence remains itself a difficult task. Once the feedback of the magnetic field is neglected, the velocity can be modelled by statistical ensembles with prescribed properties.

In spite of their relative simplicity, the models of passive fields, advected by such “synthetic” velocity ensembles, reproduce many of the anomalous features of gen-
A most powerful method to study the anomalous scaling in various statistical models of turbulent advection is provided by the field theoretic renormalization group (RG) and operator product expansion (OPE); see the monographs [29, 30] and references therein. In the RG+OPE scenario [12], anomalous scaling appears as a consequence of the existence in the model of composite fields (‘composite operators’ in the quantum-field terminology) with negative scaling dimensions; see [31] for a review and the references. In a number of papers [19, 21, 22, 23, 24], the RG+OPE approach was applied to the case of passive vector (magnetic) fields in Kraichnan’s ensemble, and to its generalizations (large-scale anisotropy, compressibility, finite correlation time, non-Gaussianity, more general form of the nonlinearity). Explicit analytical expressions were derived for the anomalous exponents to the first [19, 20] and the second [22, 23] orders in ξ. For the pair correlation function of the magnetic field, exact results were obtained within the zero-mode approach [15, 17].

In this paper, we apply the RG+OPE approach to the inertial-range behavior of strongly anisotropic MHD turbulence within the framework of a simplified model, where the magnetic field is passive and the velocity field is modelled by a Gaussian ensemble with prescribed statistics. Our model differs from the conventional Kazantsev–Kraichnan kinematic dynamo model in two respects:

(1) It involves a general relative coefficient $\mathcal{A}$ between the stretching and the advecting terms in the equation for the vector field. Inclusion of this coefficient makes the model non-local in space and requires the introduction of a pressure-like nonlocal term into the equation. The generalized model allows one to study the effects of pressure and includes, as special cases, three models that are interesting on their own: the kinematic MHD model with $\mathcal{A} = 1$ (where the pressure effects disappear), the linearized Navier–Stokes equation with $\mathcal{A} = -1$, and the passive vector ‘admixture’ with $\mathcal{A} = 0$ [24–27].

(2) Second, we focus on the effects of strong anisotropy and chose the Gaussian velocity ensemble as follows: the velocity field is oriented along a fixed direction $\mathbf{n}$ (‘orientation of a large-scale flare’ in the context of the solar corona dynamics) and depends only on the coordinates in the subspace orthogonal to $\mathbf{n}$. In the momentum space, its correlation function is chosen in the form: $\langle vv \rangle \propto \delta(t - t') k_{\perp}^{-d+1-\xi}$, where $k_{\perp} = |\mathbf{k}_{\perp}|$ and $\mathbf{k}_{\perp}$ is the component of the momentum (wave number) $\mathbf{k}$ perpendicular to $\mathbf{n}$. This model can be viewed as a $d$-dimensional generalization of the strongly anisotropic velocity ensemble introduced in [32] in connection with the turbulent diffusion problem and further studied and generalized in a number of papers [33–43]. The model is strongly anisotropic in the sense that, in contrast to previous RG+OPE studies of anisotropic passive advection [44–46], it does not include parameters that could be tuned to make the velocity statistics isotropic, and hence it does not include the isotropic Kraichnan’s model as a special case.

The problem of anomalous scaling in the higher-order correlation function of a scalar field, advected by such a velocity ensemble, was studied, by the RG+OPE techniques, in Ref. [45]. It was shown that, in sharp contrast to the isotropic Kraichnan’s model and its numerous descendants, the correlation functions show no anomalous scaling and have finite limits when the integral turbulence scale tends to infinity. It should be stressed, that such a simple behavior has a rather exotic origin: it results from mixing of families of relevant composite operators, responsible for the IR behavior of a given correlation function. One can say that for typical models the “normal” behaviour is what is normally called the “anomalous” one.

The main result of the present paper is that the inertial-range behavior of vector fields advected by such an ensemble is even more exotic: instead of power-like anomalies, there are logarithmic corrections to ordinary scaling, determined by naive (canonical) dimensions. The key point is that the matrices of scaling dimensions (“critical dimensions” in the terminology of the theory of critical state) of the relevant families of composite operators appear nilpotent and cannot be diagonalized. They can only be brought to Jordan form; hence the logarithms.

It should be stressed that huge families of mixing composite operators are not unfrequent in field theoretic models, see, e.g., Ref. [47], where a set of 2500 operators was encountered in a model of passive vector advection. But usually the corresponding matrices, although not symmetric, appear diagonalizable and have real eigenvalues. The exceptions are known but rare: some mod-
els of dense polymers, sandpiles, diners and percolation; see Refs. [48] and references therein. Furthermore, as a rule, the logarithmic behavior is postulated considerably without a definite Lagrangean field theoretic model, as a hypothetical continuum limit of discrete evolution models. The model presented in our paper provides an example of a renormalizable field theoretic model, where the existence of logarithmic corrections can be proven exactly: although the formulation of the model is rather cumbersome, it turns out that the IR behavior is determined completely by the one-loop approximation of the renormalization group.

To avoid possible misunderstanding, it should be stressed that our “large infrared logarithms” have little to do with the “large ultraviolet logarithms,” known for the \( \phi^4 \) model, quantum electrodynamics, and in models of strong interactions (all in \( d = 4 \)). In the model under consideration, the IR logarithms appear from the highly nontrivial mixing of the relevant composite operators.

The paper is organized as follows.

In sec. [III] we give a detailed description of the model. In sec. [IV] we present the field theoretic formulation of the model, establish its renormalizability, and derive the explicit expression for the self-energy function in the Dyson equation, given exactly by the one-loop approximation. In sec. [V] explicit exact expressions for the renormalization constants, RG functions (anomalous dimensions and \( \beta \) functions) are presented. It is shown that, in a certain range of model parameters, the RG equations possess an IR attractive fixed point that governs the IR asymptotic behavior of the correlation functions. The corresponding differential equations of IR scaling are derived, with the exactly known critical dimensions.

In sec. [VI] the families of composite operators that give the leading contributions in the OPE are introduced and their renormalization is discussed. It is shown that the corresponding renormalization matrices are given exactly by the one-loop approximation. In sec. [VII] explicit expressions for the matrices of renormalization, anomalous dimensions, and critical dimensions, are presented. It turns out that the matrices of critical dimensions cannot be diagonalized. They can be brought to Jordan form with known diagonal elements. As a result, the dependence of the operator mean values on the integral turbulence scale is given by known powers, corrected by polynomials of logarithms.

In sec. [VIII] the IR behavior of the pair correlation functions of the composite operators is discussed. The problem is that, since the matrices of critical dimensions cannot be diagonalized, those correlation functions are described by sets of coupled (“entangled”) differential equations. As a result, their dependence of the separation also involves polynomials of logarithms.

Eventually, in sec. [IX] the solutions of the RG equations for the mean values and correlation functions of the operators are combined with the corresponding OPE’s to give resulting expressions for the inertial-range asymptotic behavior of the pair correlation functions. They involve two types of large logarithms, where the separation enters with the typical ultraviolet and infrared scales (dissipation scale and integral scale). Sec. [X] is reserved for conclusions.

Appendix A contains detailed discussion of the propagator matrix in the field theoretic formulation of our model. It is important that the Green (response) function of the scalar field involves a cumbersome term which does not contribute to the divergent parts of the diagrams and thus in fact can be dropped. Appendix B contains the proof of the fact that the matrices of critical dimensions for all the relevant families of composite operators are nilpotent. Explicit expressions are presented for the matrices that bring them to Jordan form. Although the proof looks rather technical, the statement plays the central role in our analysis of the IR behavior, and we decided to include it in the full form.

II. DESCRIPTION OF THE MODEL

The turbulent advection of a passive scalar field \( \theta(x) \equiv \theta(t,x) \) is described by the stochastic equation

\[
\nabla_t \theta = \nu_0 \partial^2 \theta + f, \quad \nabla_t \equiv \partial_t + v_i \partial_i, \tag{2.1}
\]

where \( \theta(x) \) is the scalar field, \( x \equiv \{t, x\}, \partial_t \equiv \partial/\partial t, \partial_i \equiv \partial/\partial x_i, \nu_0 \) is the molecular diffusivity coefficient, \( \partial^2 \) is the Laplace operator, \( v(x) \equiv \{v_i(x)\} \) is the transverse (owing to the incompressibility) velocity field, and \( f \equiv f(x) \) is an artificial Gaussian scalar noise with zero mean and correlation function

\[
\langle f(x)f(x') \rangle = \delta(t-t') C(r/L), \quad r = x - x'. \tag{2.2}
\]

The parameter \( L \) is an integral scale related to the noise, and \( C(r/L) \) is some function decaying at \( L \to \infty \).

In the more realistic formulation, the field \( v(x) \) satisfies the Navier–Stokes (NS) equation. In the rapid-change model it obeys a Gaussian distribution with zero mean and correlation function

\[
\langle v_i(x)v_j(x') \rangle = \delta(t-t') \int_{k>m} \frac{dk}{(2\pi)^d} P_{ij}(k) \times
\]

\[
\times D_0 \frac{1}{k^{d+\xi}} e^{-k^2}, \tag{2.3}
\]

where \( P_{ij}(k) = \delta_{ij} - k_i k_j / k^2 \) is the transverse projector, \( k \equiv |k|, D_0 > 0 \) is an amplitude factor, \( d \) is the dimensionality of the \( x \) space and \( 0 < \xi < 2 \) is a parameter with the real (“Kolmogorov”) value \( \xi = 4/3 \).

The problem, formulated in equations (2.1)-(2.3), allows for some modifications and generalizations to more complex physical situations. For example, scalar diffusion equation (2.1) can be changed to the vector kinematic magnetohydrodynamics (MHD) equation, describing, for example, the evolution of the fluctuating part \( \theta \equiv \theta(x) \) of the magnetic field in the presence of a mean
component $\bm{\theta}$, which is supposed to be varying on a very large scale:

$$\partial_t \theta_i + \partial_k (v_k \theta_i - v_i \theta_k) = \nu_0 \partial^2 \theta_i + f_i, \quad (2.4)$$

where both $v$ and $\theta$ are divergence-free (‘solenoidal’) vector fields: $\partial_i v_i = \partial_i \theta_i = 0$. They describe also the linearization of the Navier-Stokes equation around the rapid-change background velocity field, which gives the same expression with a different sign in the vertex term:

$$\partial_t \theta_i + \partial_k (v_k \theta_i - v_i \theta_k) + \partial_k \mathcal{P} = \nu_0 \partial^2 \theta_i + f_i. \quad (2.5)$$

The pressure term $\partial \mathcal{P}$ is needed to make the dynamics (2.5) consistent with the transversality conditions $\partial_i \theta_i = 0$ and $\partial_i v_i = 0$.

Both the equations (2.4) and (2.5) can be unified by introducing of a new parameter denoted by $A_0$:

$$\partial_t \theta_i + \partial_k (v_k \theta_i - A_0 v_i \theta_k) + \partial_i \mathcal{P} = \nu_0 \partial^2 \theta_i + f_i. \quad (2.6)$$

Another interesting case is provided by the choice of $A_0 = 0$. Without the stretching term $\partial_i (v_i \theta_k)$ the model acquires additional symmetry under translations $\theta \rightarrow \theta + \text{const}$. This task has to be studied separately, see Ref. [17].

The new parameter requires a new renormalization constant $Z_A$, which can be nontrivial [24]. The pressure can be expressed as the solution of the Poisson equation

$$\partial^2 \mathcal{P} = (A_0 - 1) \partial_i v_k \partial_k \theta_i, \quad (2.7)$$

so that it vanishes for the local magnetic case.

Of course, if we choose the advection equation like (2.4), (2.5), or (2.6), we have to modify the correlation function (2.2). The random external force $f$ in the right hand side (RHS) of the equations also becomes a vector, its statistics is also assumed to be Gaussian, with zero mean and prescribed correlation function of the form:

$$\langle f_i(t, \mathbf{x}) f_k(t', \mathbf{x}') \rangle = \delta(t - t') C_{ik}(r/L). \quad (2.8)$$

Like in equation (2.2), here $r = \mathbf{x} - \mathbf{x}'$, $r = |r|$, the parameter $L \equiv M^{-1}$ is the external turbulence scale connected to the stirring, and $C_{ik}$ is a dimensionless function finite at $r/L \rightarrow 0$ and rapidly decaying for $r/L \rightarrow \infty$.

In the real problem, the velocity field $\mathbf{v}(x)$ satisfies the NS equation, probably with additional terms that describe the feedback of the advected field $\bm{\theta}(x)$ on the velocity field. The framework of most papers is the kinematic problem, where the reaction of the field $\bm{\theta}(x)$ on the velocity field $\mathbf{v}(x)$ is neglected. They assume that at the initial stages $\bm{\theta}(x)$ is weak and does not affect the motion of the conducting fluid. In this case it can be simulated by statistical ensemble with prescribed statistics.

Here we choose the field $\mathbf{v}$ to be strongly anisotropic, namely having a preferred direction $\mathbf{n}$:

$$\mathbf{v}(t, \mathbf{x}) = \mathbf{n} \cdot \mathbf{v}(t, \mathbf{x}_\perp). \quad (2.9)$$

It is assumed to be Gaussian, strongly anisotropic (see [24]), homogeneous, white-in-time, with zero mean and a correlation function

$$\langle v_i(t, \mathbf{x}) v_k(t', \mathbf{x}') \rangle = \gamma_{0i} \gamma_{0k} \langle v(t, \mathbf{x}_\perp) v(t', \mathbf{x}_\perp') \rangle, \quad (2.10)$$

where

$$\langle v(t, \mathbf{x}_\perp) v(t', \mathbf{x}_\perp') \rangle = \delta(t - t') \frac{d}{2\pi}^{d/2} e^{i(k x - k' x')} D_v(k). \quad (2.11)$$

with some function $D_v(k)$, for which we choose

$$D_v(k) = 2 \pi \delta(k_\parallel) D_0 \frac{1}{k_{\perp}^{d-1+\xi}}. \quad (2.12)$$

Like in equation (2.20), here $d$ is the dimensionality of the $\mathbf{x}$ space, $k_\perp \equiv |\mathbf{k}_\perp|$, $1/m$ is another turbulence scale, related to the stirring, the exponent $\xi$ plays the role of the RG expansion parameter and $D_0 > 0$ is an amplitude factor. The power law (2.12) is suggested by the experimental data of the turbulence spectra. To summarize, we will consider the anisotropic vector model, described by the equations (2.6)–(2.14). However, these equations should be generalized by introducing one new dimensionless constant $\gamma_0$, which breaks the $SO_d$ symmetry of the Laplace operator to $SO_{d-1} \otimes Z_2$: $\partial^2 \rightarrow \partial^2_\perp + f_0 \partial^2_\parallel$.

Then the stochastic equation (2.6) takes on the form

$$\partial_t \theta_i + \partial_k (v_k \theta_i - A_0 v_i \theta_k) + \partial_i \mathcal{P} = \nu_0 (\partial^2_\perp + f_0 \partial^2_\parallel) \theta_i + f_i. \quad (2.13)$$

The relations

$$D_0/\nu_0 f_0 = \gamma_0 \equiv A^\xi \quad (2.14)$$

define the coupling constant $\gamma_0$, which plays the part of the expansion parameter in the ordinary perturbation theory, and the characteristic ultraviolet (UV) momentum scale $\Lambda$.

This completes formulation of the model.

III. FIELD THEORETIC FORMULATION OF THE MODEL, UV DIVERGENCES AND DYSON EQUATION FOR THE PAIR CORRELATION FUNCTIONS

A. Field theoretic formulation

The stochastic problem (2.6)–(2.14) is equivalent to the field theoretic model of the set of three fields $\Phi \equiv \{\bm{\theta}, \bm{\theta}', \mathbf{v}\}$ with the action functional

$$S(\Phi) = -\frac{1}{2} \gamma_0 D_{\mathbf{v}}^{-1} v_k - \frac{1}{2} \theta'_k D_0 \theta_k + \theta_k \left[ -\partial_i \theta_k - (v_i \partial_i) \theta_k + A_0 (\partial_i \partial_i) v_k + \nu_0 (\partial^2_\perp + f_0 \partial^2_\parallel) \theta_k \right]. \quad (3.1)$$
Here all the terms (omitting the first one) represent the De Dominicis–Janssen action for the stochastic problem \[2.8\], \[2.13\] at fixed \(v\), and the first term represents the Gaussian averaging over \(v\). Furthermore, \(D_{\theta}\) and \(D_v\) are the correlators \[2.8\] and \[2.10\] respectively, the needed integrations over \(x = (t, x)\) and summations over the vector indices are implied.

The formulation \[3.1\] means that statistical averages of random quantities in the stochastic problem \[2.10\] coincide with functional averages with weight \(\exp S(\Phi)\). The model \[3.1\] corresponds to a standard Feynman diagrammatic technique with the triple vertex \(-(v_i \partial_t)\theta_k + A_0 (\theta_i \partial_k) v_k\) and the three bare propagators: \(\langle \theta_i \theta'_k \rangle_0\), \(\langle \theta_i \theta_k \rangle_0\) and \(\langle v_i v_k \rangle_0\) (the line \(\langle \theta'_i \theta'_k \rangle\) is absent). In the frequency-momentum representation the triple vertex is

\[V_{cab} = i \delta_{bc} p^a_0 - i A \delta_{ac} p^0_0\] (3.2)

(where \(p^a\) is the momentum of the field \(\theta\)), and diagrammatic notation for it is represented in the Figure (III A).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{The triple vertex.}
\end{figure}

The three aforementioned propagators, namely \(\langle \theta_i \theta'_k \rangle_0\), \(\langle \theta_i \theta_k \rangle_0\) and \(\langle v_i v_k \rangle_0\), are represented in the diagrams by slashed, straight and wavy lines, respectively:

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{fig2.png}
\caption{Diagrammatic notation for \(\langle \theta_i \theta'_k \rangle_0\).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{fig3.png}
\caption{Diagrammatic notation for \(\langle \theta_i \theta_k \rangle_0\).}
\end{figure}

Here the slashed end of the solid line corresponds to the field \(\theta'\), the end without a slash corresponds to the field \(\theta\). The line \(\langle v_i v_k \rangle_0\) in the diagrams corresponds to the correlation function \[2.10\].
related to the directions perpendicular and parallel to the vector $\mathbf{n}$: namely, one has to introduce two independent momentum canonical dimensions $d_F^p$ and $d_F^v$ so that $[F] \sim [T]^{-d_F^p}[L_\perp]^{-d_F^\perp}[L_\parallel]^{-d_F^\parallel}$, where $L_\perp$ and $L_\parallel$ are (independent) length scales in the corresponding subspaces. In the present (vector) model, however, we have an additional condition of the transversality of the fields $\theta$ and $\theta^\prime$: both $\partial_\theta \theta^\prime = 0$ and $\partial_\theta \theta^\prime = 0$, which forbids the existence of two independent length scales. In particular, it means that the constant $f_0$, first introduced in (2.13), is dimensionless.

Thus the asymptotic proportionality has the form (3.6), and the dimensions are found from the obvious normalization conditions $d_F^k = -d_S^k = 1$, $d_F^\perp = -d_S^\perp = 0$, $d_F^v = d_S^v = 0$, and from the requirement that each term of the action functional (3.1) be dimensionless (with respect to the two independent dimensions separately). Based on $d_F^p$ and $d_F^v$, one can introduce the total canonical dimension $d_F = d_F^p + 2d_F^v$ (in the free theory, $\partial_\theta \propto \partial_\theta^\perp \propto \partial_\theta^v$), which plays in the theory of renormalization of dynamic models the same role as the conventional (momentum) dimension does in static problems.

The canonical dimensions for the model (3.1) are given in Table I including renormalized parameters, which will be introduced a bit later. From Table I it follows that our model is logarithmic (the coupling constant $g_0 \sim [L]^{-\xi}$ is dimensionless) at $\xi = 0$, so that the UV divergences manifest themselves as poles in $\xi$ in the Green functions.

The total canonical dimension of an arbitrary 1-irreducible Green function $\Gamma = \langle \Phi \Phi \rangle$ is given by the relation

$$d_\Gamma = d + 2 - \sum N_\phi d_\phi = d + 2 - N_\theta d_\theta - N_\nu d_\nu - N_\omega d_\omega.$$  

(3.7)

Here $N_\phi = \{N_\theta, N_\nu, N_\omega\}$ are the numbers of corresponding fields entering the function $\Gamma$, and the summation over all types of the fields in (3.7) and analogous formulas below is always implied.

Superficial UV divergences, whose removal requires counterterms, can be present only in those functions $\Gamma$ for which the ‘formal index of divergence’ $d_\Gamma$ is a non-negative integer. Dimensional analysis should be augmented by the following observations:

(1) In any dynamical model of type (3.1), 1-irreducible diagrams with $N_\nu = 0$ contain closed circuits of retarded propagators (3.5a) and therefore vanish.

(2) For any 1-irreducible Green function $N_\nu - N_\theta = 2N_0$, where $N_0 \geq 0$ is the total number of the bare propagators $\langle \theta \theta \rangle$ entering into any of its diagrams. This fact is easily checked for any given function: it is illustrated by the function with $N_\nu = N_\theta = 1$ and $N_0 = 0$, see Fig. (III C). Clearly, no diagrams with $N_0 < 0$ can be drawn. Therefore, the difference $N_\nu - N_\theta$ is an even non-negative integer for any nonvanishing function.

(3) Using the transversality condition of the fields $\theta$ and $\nu$ we can move one derivative from the vertex $-\theta'_k(v_i \partial_\nu)\theta_k + A_0 \theta'_k(\theta \partial_\nu)\nu_k$ onto the field $\theta'_k$. Therefore, in any 1-irreducible diagram it is always possible to move the derivative onto any of the external ‘tails’ $\theta_k$ or $\theta'_k$, which reduces the real index of divergence: $d_\Gamma = d_\Gamma - N_\theta - N_\nu$. The fields $\theta_k$, $\theta'_k$ enter into the counterterms only in the form of derivatives $\partial_\theta \theta_k$, $\partial_\nu \theta'_k$.

From Table I and (3.7) we find:

$$d_\Gamma = d + 2 - (d + 1)N_\theta + N_\theta - N_\nu$$  

(3.8)

and

$$d_\Gamma^v = (d + 2)(1 - N_\theta) - N_\nu.$$  

(3.9)

From these expressions we conclude that, for any $d$, superficial divergences can be present only in the 1-irreducible functions $\langle \theta \theta \theta \rangle$ with $N_\nu = 1$ and arbitrary $N_\theta$, for which $d_\Gamma = 2$, $d_\Gamma^v = 0$. However, all the functions with $N_\theta > N_\nu$ vanish (see above) and do not require counterterms. We are left with the only superficially divergent function $\langle \theta \theta \rangle$.

C. Dyson equation for the pair correlation functions

Let us denote $\Gamma_{\alpha\beta} = \langle \theta_{\alpha} \theta_{\beta} \rangle_{1-ir}$. For these functions the Dyson equation has the form

$$\Gamma_{\alpha\beta} = -i\omega \cdot \delta_{\alpha\beta} + \nu_0 p_1 \cdot \delta_{\alpha\beta} + \nu_0 f_0 \cdot (p n) \cdot \delta_{\alpha\beta} - \Sigma_{\alpha\beta},$$  

(3.10)

where $\Sigma_{\alpha\beta}$ is self-energy operator, and in the diagrammatic notation it is represented in Figure (III C).

\[ \begin{array}{c}
\sum_{\alpha\beta} = \underbrace{\alpha}_a \underbrace{\beta}_b \underbrace{\gamma}_c + \ldots \\
\end{array} \]

FIG. 5. Diagrammatic notation for $\Sigma_{\alpha\beta}$.

Here the ellipse stands for the 2-, 3- and other N-loop diagrams. The characteristic feature of all the rapid-change models (2.11) with retarded bare propagators (3.5a) is that all the skeleton multiloop diagrams entering into the self-energy operator contain closed circuits of such retarded propagators and therefore vanish. Thus the self-energy operators in (3.10) are exactly given by the one-loop approximation.

To calculate this diagram let us start with its index structure:

$$J_{\alpha\beta} = V_{\alpha ab}(p)V_{cd\beta}(p + k)P_{bd}(p + k)n_an_c,$$  

(3.11)

where $V_{ijk}(p)$ is the triple vertex (4.2); the Greek letters $\alpha, \beta$ and the Roman letters $a, d$ denote the vector indices of the propagators (2.10), (3.3), (3.4) with the implied summation over repeated indices. Note that we need to
calculate only the divergent part of the diagram, i.e., only the term, proportional to $p^2$. There are some remarks, which can simplify our calculations:

(1) Because of our choice of $D_\omega$ (see (2.12)), namely its proportionality to $\delta(k_\parallel)$, all terms, which are also proportional to $k_\parallel$, vanish after the integration over the momentum $k$.

\[
\Sigma_{\alpha\beta} \propto J_{\alpha\beta} = \delta_{\alpha\beta} \cdot (\mathbf{pn})^2 + (\mathbf{A} - 1) \cdot (\mathbf{pn})^2 \cdot \frac{k_\parallel^2}{k^2} = \mathcal{A}(A - 1) \cdot (\mathbf{pk}) \cdot \frac{(\mathbf{pn})k_jn_\perp}{k^2} \cdot k_\perp^d + \xi.
\]

Now we have to integrate this expression over the $d$-dimensional momentum $k$ with the factor $i^2/(2\pi)^d D_\omega(k)$ with $D_\omega(k)$ from (2.12), and over the frequency $\omega$ with the factors $1/2\pi$ and $1/(-i\omega + \nu_0[k^2_\perp + f_0k^2_\parallel])$:

\[
\Sigma_{\alpha\beta} = i^2 \cdot \int \frac{d\omega}{2\pi} \frac{1}{(-i\omega + \nu_0[k^2_\perp + f_0k^2_\parallel])} \times \int \frac{dk}{(2\pi)^d} \cdot (\mathbf{pn})^2 \cdot k_\perp^d + \xi.
\]

The integration over the frequency $\omega$ is simple due to the following definition of the Heaviside step function at coincided times:

\[
\Theta(t - t') = \frac{1}{2}
\]

at $t = t'$, which is justified by the fact that the correlation function is always symmetric in its arguments. Thus

\[
\int \frac{d\omega}{2\pi} \frac{1}{(-i\omega + \nu_0[k^2_\perp + f_0k^2_\parallel])} = \frac{1}{2}.
\]

The integration over $k$ is performed by averaging over the angles, and then integrating over the modulus $k \equiv |k|:

\[
\int dk \cdot f(k) = S_{d-1} \cdot \int_0^{\infty} dk \cdot (f(k)),
\]

where $S_{d-1}$ is the surface area of the unit sphere in the $d - 1$-dimensional space.

Now we need to average the expressions $(\mathbf{nk})_\perp k_\parallel^d/k^2$ and $(\mathbf{nk})_\parallel k_\perp^d/k^2$ over the angles. Let us begin with the first structure. Note that $k_i$ and $k_j$ are two independent components of the vector $k$, therefore it is obvious that

\[
\langle k_i'k_j' \rangle \propto \delta_{ij}.
\]

This means that $(\mathbf{nk})_\parallel k_\perp^d/k^2$ is proportional to the same matrix with the difference in one element – the average of absent elements $\langle k_\parallel k_\parallel \rangle$ is zero, so the first element is $a_{11} = 0$. Therefore, it is proportional to the transverse projector $P_{ij}(\mathbf{n}) = \delta_{ij} - n_i n_j$ for the unit vector $\mathbf{n}$:

\[
\langle k_i'k_j' \rangle \propto \mathbf{P}_{ij}(\mathbf{n}).
\]

To replace the symbol of proportionality with the symbol of equality, we have to consider the transverse projector for the vector $k_\perp$ and its averaged value:

\[
\langle P_{ij}(k_\perp) \rangle = \mathbf{P}_{ij}(\mathbf{n}) = d - 1.
\]

On the other hand, using (3.18), we obtain

\[
\langle P_{ij}(k_\perp) \rangle = d - \frac{k_i^+ k_j^-}{k_\perp^d} = d - C \cdot (d - 1).
\]

From the equations (3.19) and (3.20) one can obtain the sought-for constant $C = 1/(d - 1)$. Therefore

\[
\frac{k_i^+ k_j^-}{k_\perp^d} = \mathbf{P}_{ij}(\mathbf{n}) = \frac{d - 1}{d - 1}.
\]

Using the same trick for second expression $(\mathbf{nk})_\parallel k_\perp^d/k^2$ we obtain

\[
\langle (\mathbf{nk})_\parallel k_\perp^d/k^2 \rangle = - \frac{n_\parallel}{d - 1}.
\]
Now using (3.12), (3.16), (3.21) and (3.22) we perform the integration in (3.13) with the result

\[
\Sigma_{\alpha\beta} = -\frac{1}{2} D_0 \int_{(2\pi)^{d-1}} \frac{e^{i\nu\cdot(pD)}}{d-1} \cdot \left[ \delta_{\alpha\beta} \cdot (pD)^2 + (A - 1) \cdot (pD)^2 \cdot \frac{P_{\alpha\beta}(n)}{d - 1} - A(A - 1) \cdot (pD)^2 \cdot \frac{n_{\alpha}n_{\beta}}{d - 1} \right] \cdot \int_{m} dk_{\perp} \cdot k_{\perp}^{1+\xi} =
\]

\[
= -\frac{1}{2} D_0 \cdot C_{d-1} \cdot \left[ \frac{d - 2 + A}{2(d - 1)} \cdot \delta_{\alpha\beta} + \frac{(A - 1)^2}{2(d - 1)} \cdot n_{\alpha}n_{\beta} \right] \cdot (pD)^2 \cdot \frac{m^{-\xi}}{\xi},
\]

where \( C_{d-1} \equiv S_{d-1}/(2\pi)^{d-1} \) and \( D_0 \) is defined in (2.14).

**IV. FIXED POINTS AND CRITICAL DIMENSIONS**

**A. Renormalization, RG equations and fixed points**

As already written in (3.10), the Dyson equation has the form

\[
\Gamma_{2}^{\alpha\beta} = -i\omega \cdot \delta_{\alpha\beta} + v_0 p_\perp \cdot \delta_{\alpha\beta} + v_0 f_0 \cdot (pD)^2 \cdot \delta_{\alpha\beta} + D_0 \cdot \left[ \frac{d - 2 + A}{2(d - 1)} \cdot \delta_{\alpha\beta} + \frac{(A - 1)^2}{2(d - 1)} \cdot n_{\alpha}n_{\beta} \right] \cdot C_{d-1} \cdot (pD)^2 \cdot \frac{m^{-\xi}}{\xi},
\]

(4.1)

Substituting \( \Sigma_{ab} \) from (3.23) gives:

\[
\Gamma_{2}^{\alpha\beta} = -i\omega \cdot \delta_{\alpha\beta} + v_0 p_\perp \cdot \delta_{\alpha\beta} + v_0 f_0 \cdot (pD)^2 \cdot \delta_{\alpha\beta} + D_0 \cdot \left[ \frac{d - 2 + A}{2(d - 1)} \cdot \delta_{\alpha\beta} + \frac{(A - 1)^2}{2(d - 1)} \cdot n_{\alpha}n_{\beta} \right] \cdot C_{d-1} \cdot (pD)^2 \cdot \frac{m^{-\xi}}{\xi},
\]

(4.2)

From the analysis of Dyson equation it follows that:

1. No renormalization of the parameters \( v_0 \) and \( A_0 \) is required (there is no such counterterms), i.e.,

\[
Z_v = 1, \quad Z_A = 1.
\]

2. As a result of our calculation of the self-energy diagram, the structures \( \delta_{12} \) and \( n_{12} \) have independent differences. Therefore we cannot eliminate the divergences by renormalization of the only parameter \( f_0 \), so that additional (dimensionless) parameter \( u_0 \) is to be introduced. Then the true Dyson equation has the form:

\[
\Gamma_{2}^{\alpha\beta} = -i\omega + v_0 p_\perp \cdot \delta_{\alpha\beta} + v_0 f_0 \cdot (pD)^2 \cdot \delta_{\alpha\beta} + v_0 f_0 u_0 \cdot (pD)^2 \cdot n_{\alpha}n_{\beta} + D_0 \cdot \left[ \frac{d - 2 + A}{2(d - 1)} \cdot \delta_{\alpha\beta} + \frac{(A - 1)^2}{2(d - 1)} \cdot n_{\alpha}n_{\beta} \right] \cdot C_{d-1} \cdot (pD)^2 \cdot \frac{m^{-\xi}}{\xi},
\]

(4.3)

Therefore, using (2.24), (4.3) and defining \( g \equiv \tilde{g} \cdot C_{d-1} \), for the other parameters we can write

\[
f_0 = fZ_f, \quad u_0 = uZ_u, \quad g_0 = g\mu^{\xi}Z_g, \quad Z_g = Z_f^{-1}.
\]

(4.4)

Here \( \mu \) is the reference mass (the additional free parameter of the renormalized theory) in the minimal subtraction scheme (MS), which we always use in what follows, \( g \), \( u \) and \( f \) are renormalized analogous of the bare parameters \( g_0 \), \( u_0 \) and \( f_0 \), and \( Z_i = Z_i(g, \xi, d) \) are the renormalization constants. Their relation in (4.5) results from the absence of renormalization of the contribution with \( D_{e}^{-1} \) in (3.1), so that \( D_0 \equiv g_0 v_0 f_0 = g\mu^{\xi}v f \). No renormalization of the fields and the “mass” \( m \) is needed, i.e., \( Z_\Phi = 1 \) for all \( \Phi \) and \( m_0 = m \). The renormalized action functional has the form

\[
S_R(\Phi) = \frac{1}{2} \phi_k' D_{\theta} \phi_k' - \frac{1}{2} n_i D_{v}^{-1} v_k +
\]
where the function $D_v$ from (4.12) is expressed in renormalized parameters using (4.5).

Now let us introduce the $\beta$ function and the anomalous dimensions $\gamma$ — important RG functions, which determine the asymptotic behavior of the sought-for quantities. The basic RG equation for a multiplicatively renormalizable quantity (correlation function, composite operator, etc.) is the consequence of operating with $\overline{D}_\mu$ on the relation $F = Z_F F_R$, where $\overline{D}_\mu$ denotes the differential operation $\mu \partial_\mu$ for fixed set of bare parameters $c_0 = \{g_0, \nu_0, f_0, u_0, A_0\}$. Consequently, the RG equation has the form

$\left[D_{RG} + \gamma_F\right] F_R = 0$, \hspace{1cm} (4.7)

where $\gamma_F$ is the anomalous dimension of $F$ and $D_{RG} = D_\mu + \beta \partial_\mu \gamma_f \frac{D_f}{D_u} - \gamma_u \frac{D_u}{D_u}$. Here and below $D_x \equiv x \partial_x$ for any variable $x$, and the RG functions are defined as

$\beta_g \equiv \overline{D}_\mu g = g \cdot \left[-\xi - \gamma_g(g)\right]$, \hspace{1cm} (4.8a)

$\beta_u \equiv \overline{D}_\mu u = -u \gamma_u(g, u)$, \hspace{1cm} (4.8b)

$\gamma_F \equiv \overline{D}_\mu \ln Z_F = \beta_g \partial_\mu \ln Z_F$ for any $Z_F$. \hspace{1cm} (4.8c)

The relations between $\beta$ and $\gamma$ in (4.8a) and (4.8b) result from their definitions along with the second and third relations in (4.5).

Substituting $D_\mu$ from (2.14) and using the Dyson equation (4.3), one obtains the renormalization constant $Z_f$ ($f_0 = f \cdot Z_f$) and the anomalous dimension $\gamma_f$ for the parameter $f_0$ that splits the Laplace operator:

$Z_f = 1 - \frac{d - 2 + A}{2(d - 1)} \cdot \frac{g}{\xi} + O(g^2)$, \hspace{1cm} \(4.9\)

$\gamma_f = \frac{d - 2 + A}{2(d - 1)} \cdot \gamma_g$. \hspace{1cm} (4.10)

Now we have to renormalize another (new) constant $u_0$ in such a way, that the expression

$g_0 f_0 u_0 \left[1 + \frac{(A - 1)^2}{2(d - 1)} \cdot \frac{1}{u_0} \right] \cdot n_\alpha n_\beta \cdot (\mathbf{n})^2 \cdot \frac{m^{-\xi}}{\xi}$ \hspace{1cm} (4.11)

be UV finite to the first order in $g$. Therefore,

$Z_u \cdot Z_f = 1 - \frac{(A - 1)^2}{2(d - 1)} \cdot \frac{g}{u} \cdot \frac{1}{\xi} + O(g^2)$; \hspace{1cm} (4.12)

$\gamma_u + \gamma_f = \frac{(A - 1)^2}{2(d - 1)} \cdot \frac{1}{u} \cdot g$. \hspace{1cm} (4.13)

with the previously known constant $\gamma_f$ (see 4.10).

Moreover, we obtain from equation (4.5) that for the coupling constant $g$

$Z_g \cdot Z_f = 1$, \hspace{1cm} (4.14)

so that

$\gamma_g = -\gamma_f = -\frac{d - 2 + A}{2(d - 1)} \cdot g$. \hspace{1cm} (4.15)

One of the basic RG statements is that the leading term of IR-asymptotic is at fixed point $g^*, u^*$, such that

$\beta_u = 0, \quad \partial_u \beta_u > 0; \quad \beta_g = 0, \quad \partial_g \beta_g > 0$. \hspace{1cm} (4.16)

For the coupling constant $g$ this equations along with (4.15) gives

$\beta_g = g(-\xi + \gamma_f) = 0$, \hspace{1cm} (4.17)

and the fixed point is

$g^* = \frac{2(d - 1)}{d - 2 + A} \cdot \xi, \quad \partial_g \beta_g(g^*) = \xi > 0$. \hspace{1cm} (4.18)

The $\beta$-function and the fixed point for second parameter $u$ are

$\beta_u = -u \gamma_u = g \cdot \frac{1}{2(d - 1)} \left[(d - 2 + A) \cdot u - (A - 1)^2\right]$, \hspace{1cm} \(4.19\)

so that

$u^* = \frac{(A - 1)^2}{d - 2 + A}, \quad \partial_u \beta_u(u^*) = \frac{d - 2 + A}{2(d - 1)} \cdot g^*$. \hspace{1cm} (4.20)

Therefore, the system possesses fixed point $u^*, g^*$ only if $g^* > 0$, i.e.,

$d - 2 + A > 0$. \hspace{1cm} (4.21)

This fact implies that the correlation functions of the model (3.2) in the IR region ($\Lambda \gg 1, mr \sim 1$) exhibit scaling behavior up to the logarithmic function. The corresponding critical dimensions $\Delta [F] \equiv \Delta _F$ can be calculated exactly.
B. Critical dimensions

In the leading order of the IR asymptotic behavior the Green functions satisfy the RG equation with the substitution \( g \to g_s, u \to u_s \), which gives

\[
[D_\mu - \gamma_f^* D_f - \gamma_u^* D_u + \gamma_G^*] G^R(c, \mu, \ldots) = 0. \tag{4.22}
\]

Canonical scale invariance is expressed by the relations:

\[
\left[ \sum_{\alpha} d_{\alpha}^k D_{\alpha} - d_G^k \right] G^R = 0,
\tag{4.23}
\]

where \( \alpha \equiv \{ t, x, \mu, \nu, m, M, u, f, \mathcal{A}, g \} \) is the set of all arguments of \( G^R \), \( (t, x) \) is the set of all times and coordinates, and \( d_{\alpha}^k \) and \( d_G^k \) are the canonical dimensions of \( G^R \) and \( \alpha \). Substituting the needed dimensions from Table I into (4.23), we obtain:

\[
[D_\mu + D_m + D_M - 2D_\nu - D_\chi - d_G^4] G^R = 0, \tag{4.24a}
\]

\[
[D_\nu - D_\chi - d_G^4] G^R = 0. \tag{4.24b}
\]

The equations of the type (4.22) and (4.23) describe the scaling behavior of the function \( G^R \) upon the dilation of a part of its parameters. A parameter is dilated if the corresponding derivative enters the equation, otherwise it is kept fixed. We are interested in the IR scaling behavior, in which all the IR relevant parameters (coordinates \( x \), times \( t \) and integral scales \( M \) and \( m \) are dilated, while the irrelevant parameters, related to the UV scale (diffusivity coefficient \( \nu \) and the renormalization mass \( \mu \) are fixed. Thus we combine the equations (4.22) and (4.23) such that the derivatives with respect to the IR irrelevant parameters \( \mu \) and \( \nu \) be eliminated, and obtain the desired equation of critical IR scaling for the model:

\[
[D_\mu - \Delta_t D_t + \Delta_m D_m + \Delta_M D_M + \\
+ \Delta_f D_f + \Delta_u D_u - \Delta G] G^R = 0, \tag{4.25}
\]

where

\[
\Delta_t = - \Delta_\nu = -2, \quad \Delta_m = \Delta_M = 1, \\
\Delta_f = \gamma_f^*, \quad \Delta_u = \gamma_u^* 
\tag{4.26}
\]

and

\[
\Delta[G] \equiv \Delta_G = d_G^4 + 2 d_G^2 + \gamma_G^* \tag{4.27}
\]

are the corresponding critical dimensions.

In particular, for any correlation function \( G^R = \langle \Phi \ldots \Phi \rangle \) of the fields \( \Phi \) we have \( \Delta_G = N_\Phi \Delta_\Phi \), with the summation over all fields \( \Phi \) entering into \( G^R \), namely

\[
\Delta_G = \sum_{\Phi} N_\Phi \Delta_\Phi = N_\Phi d_\Phi + N_\nu d_\nu + N_\mu d_\mu. \tag{4.28}
\]

Since in the model \( \Phi \) the fields themselves are not renormalized (i.e., \( \gamma_\Phi = 0 \) for all \( \Phi \), see Section IV), using (4.27) we conclude, that the critical dimensions of the fields \( \Phi = \{ v, \theta, \theta' \} \) are the same as their canonical dimensions, presented in the Table I. Namely,

\[
\Delta_v = 1, \quad \Delta_\theta = -1, \quad \Delta_{\theta'} = d + 1. \tag{4.29}
\]

It is the specific feature of this model, which distinguishes it from both the isotropic Kraichnan’s vector model \( \Phi \) (in which \( \gamma_\nu \neq 0 \) and anisotropic Kraichnan’s scalar model \( \Phi \) (in which the Laplacian splitting parameter \( f \) is not dimensionless).

V. RENORMALIZATION OF COMPOSITE OPERATORS

A. General scheme

From now on, we will consider composite operators of the form

\[
F_{N_p, m} = (\theta, \theta_i)^p \left( n_i \theta_i \right)^{2m}, \quad N = 2(p + m). \tag{5.1}
\]

They are renormalized multiplicatively, \( F_{N_p} = Z_{N_p} F_{N_p}^R \), and the renormalization constants \( Z_{N_p} = Z_{N_p}(g, \xi, \delta) \) are determined by the requirement that the 1-irreducible correlation function

\[
\langle F_{N_p}^R(x) \theta(x_1) \ldots \theta(x_n) \rangle_{1-ir} =
\]

\[
Z_{N_p}^{-1}(F_{N_p}(x) \theta(x_1) \ldots \theta(x_n))_{1-ir} \equiv Z_{N_p}^{-1}(F_{N_p}(x; x_1, \ldots, x_n)) \tag{5.2}
\]

be UV finite in renormalized theory, that is, have no poles in \( \xi \) when expressed in renormalized variables (4.5). This is equivalent to the UV finiteness of the product \( Z_{N_p}^{-1} \Gamma_{N_p}(x; \theta) \), in which

\[
\Gamma_{N_p}(x; \theta) = \frac{1}{n!} \int dx_1 \ldots \int dx_n \Gamma_{N_p}(x; x_1, \ldots, x_n) \times
\]

\[
\times \theta(x_1) \ldots \theta(x_n) \tag{5.3}
\]
is a functional of the field $\theta(x)$. The contribution of a specific diagram into the functional $\Gamma_{NP}$ in (5.3) for any composite operator $F_{NP}$ is represented in the form

$$\Gamma_{NP} = V_{\alpha \beta \ldots} I_{\alpha \beta \ldots}^{ab} \theta_a \theta_b \ldots, \quad (5.4)$$

where $V_{\alpha \beta \ldots}$ is the vertex factor, $I_{\alpha \beta \ldots}^{ab}$ is the “internal block” of the diagram with free indices, and the product $\theta_a \theta_b \ldots$ corresponds to external lines.

According to the general rules of the universal diagrammatic technique (see, e.g., [30]), for any composite operator $F(x)$ built of the fields $\theta$, the vertex $V_{\alpha \beta \ldots}$ in (5.4) with $k \geq 0$ attached lines corresponds to the vertex factor

$$V_{NP}^k(x; x_1, \ldots, x_k) \equiv \delta^k F_{NP}(x)/\delta \theta(x_1) \ldots \delta \theta(x_k). \quad (5.5)$$

The arguments $x_1 \ldots x_k$ of the quantity (5.5) are contracted with the arguments of the upper ends of the lines $\theta \theta'$ that are attached to the vertex.

### B. One-loop diagram

Now let us turn to calculate the internal block $I_{\alpha \beta \ldots}^{ab}$ in the notation (5.4), namely the diagrams themselves. The one-loop diagram is represented in Figure (V B).

![FIG. 6. The one-loop contribution to the generating functional.](image)

The index structure of this diagram is

$$Y_{\alpha \beta} = V_{\gamma \alpha}(k) V_{\beta \gamma}(-k) \cdot P_{\alpha \gamma}(k) P_{\beta \gamma}(k) \cdot n_a n_z = -A^2 \cdot n_x P_{\gamma \alpha}(k) \cdot n_z P_{\beta \gamma}(k) \cdot k_a k_b, \quad (5.6)$$

where the letters $i, j, x$ and $z$ denote internal indices of the diagram itself. Then we have to integrate $Y_{\alpha \beta}^{ab}$ over the frequency and momentum with the factors like (3.3), namely

$$I_{\alpha \beta}^{ab} = \int \frac{dk}{(2\pi)^d} \frac{1}{-i \omega + \nu k^2 + \nu f k_\parallel^2} \cdot \frac{1}{i \omega + \nu k^2 + \nu f k_\parallel^2} \times \frac{\delta(k_\perp)}{k_\perp^{d-1+\xi}} \cdot D_0 \cdot Y_{\alpha \beta}^{ab}. \quad (5.7)$$

Using (5.2-4) for averaging over the angles, one obtains the following result:

$$I_{\alpha \beta}^{ab} = \frac{A^2}{2^d} \cdot D_0 \int \frac{dk}{(2\pi)^d} \frac{1}{k_\perp^{d-1+\xi}} \cdot \frac{k_\perp^{d-1+\xi}}{k_\perp^{d-1+\xi}} \cdot n_a n_b = \frac{A^2}{2 (d-1)} \cdot P_{ab}(n) \cdot n_a n_b \cdot g \cdot m^{-\xi}. \quad (5.8)$$

### C. Multi-loop diagrams

Any multiloop diagram contains a part with the structure, represented in the Figure (V C).

![FIG. 7. Fragment of arbitrary multiloop diagram.](image)

As a consequence the integral, corresponding to the divergent part of the diagram, contains as a factor the following expression:

$$I_0 = \delta(k_\parallel)\delta(q_\parallel)n_\alpha V_{\beta \alpha \gamma}(k + q)P_{\beta \gamma}(k), \quad (5.9)$$

where $V$ is the vertex (3.2), and the $\delta$-functions appear from velocity correlator (2.10). Since $I_0$ is proportional to the sum of $k_\parallel$ and $q_\parallel$ with some coefficients, after integration with the $\delta$-functions all these diagrams become equal to zero.

There are also multiloop diagrams of the “sand clock” type, represented by products of simpler diagrams. They
contain only higher-order poles in $\xi$ and, in the MS scheme, do not contribute to the anomalous dimensions.

Therefore (and it is another special feature of this model) the one-loop approximation gives us the exact answer.

VI. CRITICAL DIMENSIONS OF COMPOSITE OPERATORS

A. Anomalous dimensions

The objects of interest are the correlation functions $G_{F_iF_j} = \langle F_i F_j \rangle$, and according to the solution of the RG equation we need to know the asymptotic behavior of the operators $F_i$ themselves. Now let us find it: we will consider the operator

$$F_{N,p,m} = (\theta^i \theta_i)^p (n_s \theta_s)^{2m},$$

where $N = 2(p + m)$ is the total number of fields $\theta$, appearing in the operator.

According to (5.4), (5.5) and the exact answer (5.8) for the diagrams, the symbol convolution of the functional $\Gamma$ is

$$\Gamma \propto i^2 \delta_{\alpha} \delta_{\beta} F_{N,p,m} \cdot n_{\alpha} n_{\beta} \cdot P_{ab}(n) \theta_a \theta_b =$$

$$= 2m(2m - 1) \cdot F_{N,p+1,m-1} + (2p + 8pm - 2m(2m - 1)) \cdot F_{N,p,m} +$$

$$+ (4p(p - 1) - 2p - 8pm) \cdot F_{N,p-1,m+1} - 4p(p - 1) \cdot F_{N,p-2,m+2}. \quad (6.2)$$

Equation (6.2) shows that the composite operators are mixed in renormalization, that is, an UV finite renormalized operator $F_R$ has the form

$$F_R = F + \text{counterterms},$$

where the contribution of the counterterms is a linear combination of $F$ itself and other unrenormalized operators with the same total number $N$ of the fields, which “admix” to $F$ in renormalization.

Let $F \equiv \{F_i\}$ be a closed set of operators with the same quantity of fields $\theta$, i.e., with the same number $N$, which mix only with each other in renormalization. The renormalization matrix $\hat{Z}_F \equiv \{Z_{ik}\}$ and the matrix of anomalous dimensions $\hat{\gamma}_F \equiv \{\gamma_{ik}\}$ for this set are given by

$$F_i = \sum_k Z_{ik} F^R_k, \quad \hat{\gamma}_F = \hat{Z}_F^{-1} D_{\mu} \hat{Z}_F. \quad (6.3)$$

The scale invariance and the RG equation for the operator $F_{N,p}$ give us the corresponding matrix of critical dimensions $\Delta_F \equiv \{\Delta_{ik}\}$ in the form similar to the expression, in which $d_F^k, d_F^p$ and $d_F$ are understood as the diagonal matrices of canonical dimensions of the operators in question (with the diagonal elements equal to sums of corresponding dimensions of all fields and derivatives constituting $F$) and $\hat{\gamma}^* = \hat{\gamma}(g^*, u^*)$ is the matrix at the fixed point.

In this notation and in the MS scheme the renormalization matrix $\hat{Z}$ has the form

$$\hat{Z} = \hat{E} + \hat{A}, \quad (6.4)$$

where $\hat{E}$ is a diagonal matrix and each element of the matrix $\hat{A}$ has the form

$$A_{ik} = a_{ik} \cdot \frac{g}{\epsilon}. \quad (6.5)$$

To solve the RG equation we have to use the eigenvalue decomposition of the matrix $\hat{\gamma}$, therefore the critical dimensions of the set $F \equiv \{F_i\}$ are given by the eigenvalues of the matrix $\Delta_{ik}$. In fact this means, that we change the set of operators $\{F^R\}$ to the set of “basis” operators $\{\tilde{F}^R_i\}$ that possess definite critical dimensions and have the form

$$\tilde{F}^R_i = U_{ip} F^R_p, \quad (6.6)$$
where the matrix $U_{lp}$ is such that $\Delta_F = U_{lp}^{-1} \Delta_F U_F$ is diagonal (or has the Jordan form).

As the renormalization matrix $\hat{Z}$ has the form (6.4), the matrix of anomalous dimensions $\hat{\gamma}$ has the form

$$\gamma_{ik} = -a_{ik} \cdot g$$ \hspace{1cm} (6.7)

with the coefficients $a_{ik}$ from (6.5). Combining (6.2)–(6.7) and taking into account the scalar factor, not written in (6.2), but present in (5.8), together with the fact, that the symmetrical coefficient for this one-loop diagram is 1/2, one can obtain the following for the matrix of anomalous dimensions $\hat{\gamma}$:

$$\gamma_{N,p+1} = -\frac{A^2 \cdot f}{4(d-1)} \cdot 2m(2m-1) \cdot g;$$ \hspace{1cm} (6.8a)

$$\gamma_{N,p} = -\frac{A^2 \cdot f}{4(d-1)} \cdot (2p + 8pm - 2m(2m-1)) \cdot g;$$ \hspace{1cm} (6.8b)

$$\gamma_{N,p-1} = -\frac{A^2 \cdot f}{4(d-1)} \cdot (4p(p-1) - 2p - 8pm) \cdot g;$$ \hspace{1cm} (6.8c)

$$\gamma_{N,p-2} = -\frac{A^2 \cdot f}{4(d-1)} \cdot (-4p(p-1)) \cdot g.$$ \hspace{1cm} (6.8d)

Substituting the value of the fixed point $g^* = \frac{2(d-1)}{d-2+A} \xi$ (see (4.18)) gives

$$\gamma_{N,p+1} = -\frac{A^2 \cdot f}{2(d-2+A)} \cdot 2m(2m-1) \cdot \xi;$$ \hspace{1cm} (6.9a)

$$\gamma_{N,p} = -\frac{A^2 \cdot f}{2(d-2+A)} \cdot (2p + 8pm - 2m(2m-1)) \cdot \xi;$$ \hspace{1cm} (6.9b)

$$\gamma_{N,p-1} = -\frac{A^2 \cdot f}{2(d-2+A)} \cdot (4p(p-1) - 2p - 8pm) \cdot \xi;$$ \hspace{1cm} (6.9c)

$$\gamma_{N,p-2} = -\frac{A^2 \cdot f}{2(d-2+A)} \cdot (-4p(p-1)) \cdot \xi.$$ \hspace{1cm} (6.9d)

Therefore the critical dimensions matrix for the operator $F_{N,p}$ has the form

$$\Delta_{Np,Np'} = -2(p + m) \cdot \delta_{pp'} + \hat{\gamma}_{Np,Np'},$$ \hspace{1cm} (6.10)

where $-2(p + m)$ is its canonical dimension, $\delta_{pp'}$ is Kronecker’s $\delta$ symbol and $\hat{\gamma}_{Np,Np'}$ is the value of anomalous dimension matrix at the fixed point.

### B. Critical dimension matrix and its eigenvalue decomposition

Let us find the eigenvalues of the critical dimensions matrix $\Delta_{Np,Np'}$. As a consequence of (6.8), it is a four diagonal for any $N$; moreover it has one line under the main diagonal and two lines above the main diagonal. Therefore the inversion of the matrix and its eigenvalue decomposition are nontrivial tasks.

According to (6.2), the closed set $F = \{F_i\}$ of operators, which mix only with each other in renormalization, consists only of operators with the same total quantity of fields $\theta$, i.e., with the same number $N$. So, let us define the vector $\mathbf{F}$ as

$$\mathbf{F} = \left( \begin{array}{c} (\theta_1 \theta_1)^N \\
(n_1 \theta_1)^N \\
\vdots \\
(n_s \theta_s)^N \end{array} \right).$$ \hspace{1cm} (6.11)

Therefore the relation between the set of unrenormalized operators $\{F\}$ and the set of renormalized operators $\{FR\}$, namely $F_i = Z_{ik}F_{R_k}$, takes on the form
It is important that in this notation the line of the matrix $\tilde{Z}$ corresponds to the original unrenormalized operator, and that the power $p$ of the operator $F_{N,p}$ decreases from right to left.

Let us denote the common factor in (6.9) as $y$, i.e.,

$$y = -\frac{A^2 \cdot f}{2(d - 2 + A)} \cdot \xi,$$  \hspace{1cm} (6.13)

and construct from (6.9), (6.10) and (6.12) the matrix of critical dimensions for several sets of operators. For example, for the set with $N = 2$ we have

$$N = 2: \quad \Delta_{Np,Np'} = \begin{pmatrix} -2 + 2y & -2y \\ 2y & -2 - 2y \end{pmatrix}, \hspace{1cm} (6.14)$$

and the eigenvalues are $\lambda = \{-2; -2\}$; for the set with $N = 8$ we have

$$N = 8: \quad \Delta_{Np,Np'} = \begin{pmatrix} -8 + 8y & 40y & -48y & 0 & 0 \\ 2y & -8 + 28y & -6y & -24y & 0 \\ 0 & 12y & -8 + 24y & -28y & -8y \\ 0 & 0 & 30y & -8 - 4y & -26y \\ 0 & 0 & 0 & 56y & -8 - 56y \end{pmatrix}, \hspace{1cm} (6.15)$$

and the eigenvalues are $\lambda = \{-8; -8; -8; -8; -8\}$; and so on. This fact remains true for any set of operators with arbitrary number $N$. This statement is strictly proven in Appendix [4].

In other words, for any $N$, the matrix of anomalous dimensions (6.9) is nilpotent, and the matrix of critical dimensions (6.10) is degenerate with all the eigenvalues equal to $N$:

$$\lambda_1 = \ldots = \lambda_{N/2+1} = -2(p + m) = -N. \hspace{1cm} (6.16)$$

Therefore the matrix of critical dimensions (6.10) is not diagonalizable, but can be brought to a Jordan form, i.e., $
\Delta_F = U_F \Delta_F U_F^{-1}$, and for the matrix $\Delta_F$ we can write

$$\langle F^R \rangle \propto M^{\tilde{\Delta}_F} \cdot \Phi \left( f \frac{1}{M^p} \right) \cdot C_0, \hspace{1cm} (6.19)$$

which along with the dilute lattice considerations gives
Therefore, after the convolution with the initial-data vector $C_0$ and up to the dimensional factor, the asymptotic form of the mean value of the operators $\tilde{F}^R$ is

\[
\langle \tilde{F}^R \rangle \propto \nu_0^{\Delta F} \cdot M^{-N} \cdot (M/\mu)^{\tilde{\Delta}_F} \cdot \Phi \left( \frac{f}{M^\xi} \right) \cdot C_0, \tag{6.20}
\]

where $\Phi$ is some unknown function of the dimensionless argument, $\gamma F = \xi$ (see equations (4.11) and (4.18)), $\tilde{F}^R$ is a vector built from the “basis” operators (6.6) that possess definite critical dimensions, $C_0$ is some constant vector (“initial data”) and $\tilde{\Delta}_F$ is the matrix of critical dimensions from (6.17). Using the nilpotency of the matrix $\gamma F$ (see (6.17) and Appendix B) and passing to the dimensionless argument $M/\mu$, where $\mu$ is the reference mass, equation (6.20) changes to

\[
\langle \tilde{F}^R \rangle \propto \nu_0^{\Delta F} \cdot M^{-N} \cdot (M/\mu)^{\tilde{\Delta}_F} \cdot \Phi \left( \frac{f}{M^\xi} \right) \cdot C_0, \tag{6.21}
\]

where $-N = -2(p + m)$ is the only eigenvalue of the matrix $\tilde{\Delta}_F$.

Since the matrix $\tilde{\Delta}_F$ in (6.21) has a Jordan form with the only degenerate eigenvalue $\lambda_0 = -2(p + m)$, then the value of a certain scalar function $\mathcal{F}$ with the matrix argument $\tilde{\Delta}_F$ is given by the matrix $\mathcal{F}(\tilde{\Delta}_F)$:

\[
\mathcal{F}(\tilde{\Delta}_F) = \begin{pmatrix}
\mathcal{F}(\lambda_0) & \mathcal{F}'(\lambda_0) / \lambda & \ldots & \mathcal{F}^{(n-1)}(\lambda_0) / \lambda^{n-1} \\
0 & \mathcal{F}(\lambda_0) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & \mathcal{F}(\lambda_0)
\end{pmatrix} \tag{6.22}
\]

If the function $\mathcal{F}$ is chosen as $(M/\mu)^{\tilde{\Delta}_F}$, the logarithms $\ln(M/\mu)$ will appear in the sought-for asymptotic expression:

\[
(M/\mu)^{\tilde{\Delta}_F} = \begin{pmatrix}
(M/\mu)^\lambda & (M/\mu)^\lambda \cdot \ln(M/\mu) & \ldots & (M/\mu)^\lambda \cdot \ln(M/\mu)^{n-1} \\
0 & (M/\mu)^\lambda & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & (M/\mu)^\lambda 
\end{pmatrix}. \tag{6.23}
\]

The correlator $G$ is also multiplicatively renormalizable and, as a consequence, it satisfies the differential RG equation (4.20)–(4.23), which describes the IR scaling behavior. But, due to the mixing condition of the operators $F_{N,p}$ themselves, the solution of this equation for the function $G$ is more involved.

Since the correlator $G$ is a function of $x = r_1 - r_2$, $m$, $M$ and $f$, the differential operator $D_{RG}$ in this case reduces to the form

\[
D_{RG} = -D_r + D_m + D_M + \gamma F D_f. \tag{7.2}
\]

Applying it to the correlator $G$ and denoting $F_{N_1,p_1}$ as $F_1$ and $F_{N_2,p_2}$ as $F_2$ (we recall, that $N_1$ may not be equal to $N_2$, i.e., operators $F_1$ and $F_2$ may belong to different renormalization sets), we obtain the differential equation

\[
D_{RG} G_{ik} = \Delta_{is} G_{sk} + \Delta_{ks} G_{is}, \tag{7.3}
\]

where $G_{ij} = \langle F_i F_j \rangle$, $\Delta_{ij}$ is the critical dimension of the correlator $G_{ij}$, and the summation over repeated indices is implied. Note that due to the difference of the numbers $N_1$ and $N_2$ in the initial operators $F_{N_1,p}$ in (7.1), the matrices $\Delta_{is}$ and $\Delta_{ks}$ in (7.3) can have different dimensions.

Let us now consider the operators $\tilde{G}_{ik}$ instead of $G_{ik}$, namely, the correlation functions of operators $\tilde{F}$ (see (6.6)) that possess definite critical dimensions:

\[
\tilde{G} = \langle F_{N_1,p_1} F_{N_2,p_2} \rangle. \tag{7.1}
\]
\[ \bar{G}_{ik} = \left\langle \bar{F}_i \bar{F}_k \right\rangle. \]  
(7.4)

A few remarks follow about numbering and indices i and k in the definition (7.4):

1. The initial operator \( F \) is defined in (6.11), namely

\[ F_{N,p,m} = (\theta_a \theta_a)^p (n_s \theta_s)^{2m}. \]
(7.5)

2. Since at renormalization the operators, which can mix together, have the same number \( N \) (see (6.2)), therefore for fixed \( N \) we may define a vector \( \bar{F} \) (6.11), namely

\[ \bar{F} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_{N/2+1} \end{pmatrix}, \]
(7.6)

with the matrix \( U_{ip} \) such that the matrix of critical dimensions \( \Delta_F = U_F^{-1} \Delta_F U_F \) is a Jordan matrix (see Section [113] and has the form (6.18).

Therefore the operator \( \bar{F}_i \) in the definition of the correlation function (7.4) is not arbitrary, but is constructed using (6.7) as a linear combination of the operators \( F_i \), whose numbering is strictly defined in (7.6).

The correlation function \( \bar{G}_{ik} \) satisfies the differential equation in the form (7.3), but with Jordan matrices \( \Delta_{ik} \):

\[ \mathcal{D}_{RG} \bar{G}_{ik} = \tilde{\Delta}_{is} \bar{G}_{sk} + \tilde{\Delta}_{ks} \bar{G}_{is}. \]
(7.8)

If the operator \( \bar{F}_i \), entering into the correlator \( \bar{G}_{ik} \), belongs to the set with number \( N_1 \), and the operator \( \bar{F}_k \) belongs to the set with number \( N_2 \), the expression (7.8) is in fact a system of \( (N_1/2+1) \times (N_2/2+1) \) nonseparable (due to nondiagonal, but Jordan form of matrices \( \tilde{\Delta}_{ik} \)) differential equations.

The matrices \( \tilde{\Delta}_{is} \) and \( \tilde{\Delta}_{ks} \) in (7.8) have the form

\[ \tilde{\Delta}_F = \begin{pmatrix} \lambda_1(2) & 1 & 0 & \ldots & 0 \\ 0 & \lambda_1(2) & 1 & \vdots & \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & 1 & \lambda_1(2) \end{pmatrix}, \]
(7.9)

where \( \lambda_1 = -N_1 \) and \( \lambda_2 = -N_2 \) (see Appendix B).

Taking into account the expression (7.9) it is obvious, that if the both operators \( \bar{F}_i \) and \( \bar{F}_k \) are not “the last from the end”, i.e., if \( i \neq N_1/2 + 1 \) and \( k \neq N_2/2 + 1 \), then each of the terms in (7.8) has two contributions – one is the function \( \bar{G}_{ik} \) with coefficient \( \lambda_1(2) \) and the other is either the function \( \bar{G}_{ik+1,k} \) for the first term or the function \( \bar{G}_{ik,k+1} \) for the second term, both having coefficients 1. If one of the operators \( \bar{F}_i \) and \( \bar{F}_k \) is “the last from the end”, i.e., if \( i \) or \( k \) is equal to \( N_1/2 + 1 \), then this contribution will be reduced to the only term \( \bar{G}_{ik} \) with the coefficient \( \lambda_1(2) \).

As a consequence, there is only one differential equation with one term in the RHS, namely

\[ \mathcal{D}_{RG} \bar{G}_{N_1/2+1 N_2/2+1} = (\lambda_1 + \lambda_2) \cdot \bar{G}_{N_1/2+1 N_2/2+1}. \]
(7.10)

Its solution up to a dimensional factor is

\[ \bar{G}_0^R = \bar{G}_{N_1/2+1 N_2/2+1} \propto (\Lambda r)^{-\lambda_1 - \lambda_2} \cdot \Phi (M_r, m_r, f r^\xi), \]
(7.11)

where \( \Lambda \) is the characteristic UV momentum scale, first introduced in (2.13).

Then, if \( i = N_1/2 + 1 \) and \( k = N_2/2 + 1 \) or if \( i = N_1/2 \) and \( k = N_2/2 + 1 \), i.e., if \( k = (N_1 + N_2)/2 + 1 \), we have two equations of type

\[ \mathcal{D}_{RG} \bar{G}_1^R = (\lambda_1 + \lambda_2) \cdot \bar{G}_1^R + \bar{G}_0^R, \]
(7.12)

which involves the already known function \( \bar{G}_0^R \) in the RHS. Its solution contains a power factor and a polynomial of a logarithm, i.e., up to a dimensional factor it is

\[ \bar{G}_1^R \propto (\Lambda r)^{-\lambda_1 - \lambda_2} \cdot P_1 [\ln \Lambda r] \cdot \Phi (M_r, m_r, f r^\xi), \]
(7.13)

where \( P_1 [\ln \Lambda r] \) is a first degree polynomial of the argument \( \ln \Lambda r \). Using (7.11) and (7.13) we may write, that the asymptotic behavior of the sum \( \bar{G}_0^R + \bar{G}_1^R \) is the same as that of the function \( \bar{G}_1^R \) itself:

\[ \bar{G}_0^R + \bar{G}_1^R \propto (\Lambda r)^{-\lambda_1 - \lambda_2} \cdot P_1 [\ln \Lambda r] \cdot \Phi (M_r, m_r, f r^\xi). \]
(7.14)

Then, if \( k = (N_1 + N_2)/2 \), we have three expressions, which in the RHS involve the function \( \bar{G}_2 \) that is already known from expression (7.13), and may also involve the function \( \bar{G}_0 \), that is also known:

\[ \mathcal{D}_{RG} \bar{G}_2^R = (\lambda_1 + \lambda_2) \cdot \bar{G}_2^R + \bar{G}_1^R. \]
(7.15)

Its solution contains a second degree polynomial with \( \ln \Lambda r \) as an argument, i.e.,

\[ \bar{G}_2^R \propto (\Lambda r)^{-\lambda_1 - \lambda_2} \cdot P_2 [\ln \Lambda r] \cdot \Phi (M_r, m_r, f r^\xi). \]
(7.16)
The procedure is similar for the next functions. It is obvious that the number of equations, which contain in the RHS a function that is known from the previous step, increases for \((N_1+N_2)/2+2 \leq i+k \leq (N_1+N_2)/4+1\) and decreases if \((N_1+N_2)/4+1 \leq i+k \leq 2\). As a consequence, in this system there is only one function, namely that with \(i+k = 2\), whose asymptotic behavior contains a polynomial of the maximal power of the logarithm:

\[
\tilde{R}_{11}^R \propto (Ar)^{-\lambda_1-\lambda_2} \cdot P_{(N_1+N_2)/2}[\ln Ar] \cdot \Phi(Mr, mr, fr^\xi).
\]

Finally, expressions like \((7.11)\), \((7.13)\) and \((7.17)\) give the asymptotic behavior of any function \(\tilde{G}_{ik}^R\).

In order to obtain the asymptotic behavior of the correlation functions of the initial operators “without tilde,” we have to use the expression \((7.7)\). The inverse matrix \(U^{-1}\) has the form

\[
U^{-1}_F = \begin{pmatrix}
0 & \ldots & \ldots & 0 & \hat{u}_{1n} \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \hat{u}_{2n} \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \hat{u}_{n-1} & \ldots & \hat{u}_{n,n-2} & \hat{u}_{nm} \\
\hat{u}_{n1} & \ldots & \hat{u}_{n,n-2} & \hat{u}_{n,n-1} & \hat{u}_{nn} \\
\end{pmatrix}
\]

\(\hat{u}_{nm} = \hat{u}_{n,n-1} = 0 \) and so on. Equations \((7.22)\)–\((7.24)\) show that the expression for any function \(\tilde{G}_{ik}^R\) contains in the RHS the function \(G_{N_1/1+N_2/1+1}^R\) with different coefficients \((\hat{u}_{a,b} \neq \hat{u}_{a+1,b} \text{ and } \hat{u}_{a,b} \neq \hat{u}_{a+1,b} \text{ for all } a, b)\), therefore the expression for any function \(G_{ik}^R\) contains in the RHS the function \(\tilde{G}_{ik}^R\). This fact together with the expression \((7.17)\) gives the sought-for asymptotic behavior of the pair correlator function of the initial operators from the set \(\{F\}\):

\[
G_{ik}^R \propto \tilde{G}_{ik}^R \propto (Ar)^{-\lambda_1-\lambda_2} \cdot P_{(N_1+N_2)/2}[\ln Ar] \cdot \Phi(Mr, mr, fr^\xi) \quad \forall i, k.
\]
The inertial range with a certain scaling function \( P \) where \( \tilde{\omega} \sim \) and \( \Phi \) is a function of three dimensionless arguments. Its asymptotic behavior is studied using the OPE.

### VIII. OPERATOR PRODUCT EXPANSION AND ANOMALOUS SCALING

Representations (7.26) for any scaling functions \( \Phi(Mr, mr, fr^\xi) \) describe the behavior of the correlation functions for \( \hat{\lambda} \gg 1 \) and any fixed value of \( Mr \). The inertial range \( l \ll r \ll L \) corresponds to the additional condition \( Mr \ll 1 \). The form of the functions \( \Phi(Mr) \) is not determined by the RG equations themselves; in analogy with the theory of critical phenomena, its behavior for \( Mr \to 0 \) is studied using the well-known Wilson operator product expansion (OPE).

According to the OPE, the equal-time product \( F_1(x')F_2(x'') \) of two renormalized operators for \( x' \equiv (x' + x'')/2 \equiv \text{const} \) and \( r \equiv x' - x'' \to 0 \) has the representation

\[
F_1(x')F_2(x'') = \sum \hat{F}_r(r)\tilde{F}(t,x), \tag{8.1}
\]

where the functions \( \hat{F}_r \) are coefficients regular in \( M^2 \) and \( \tilde{F} \) are all possible renormalized local composite operators allowed by symmetry (more precisely, see below).

Without loss of generality, it can be assumed that the expansion is made in the basis operators \( \tilde{F} \) of the type \( [\ln \Lambda r] \), i.e., those having definite critical dimensions \( \Delta_\tilde{F} \). The renormalized correlator \( \langle F_1(x)F_2(x') \rangle \) is obtained by averaging (8.1) with the weight \( \exp S_R \) with the renormalized action (11.6). The quantities \( \langle \tilde{F} \rangle \propto (Mr)^\Delta_\tilde{F} \) appear on the right hand side. Their asymptotic behavior for \( Mr \to 0 \) is found from the corresponding RG equations and has the form

\[
\langle \tilde{F} \rangle \propto (Mr)^{\tilde{\Delta}_\tilde{F}}, \tag{8.2}
\]

where \( \tilde{\Delta}_\tilde{F} \) is a Jordan matrix (6.17) and \( (Mr)^{\tilde{\Delta}_\tilde{F}} \) is a matrix of type (6.23).

Note that due to the form of the differential operator \( D_{RG} \) (7.2), the solution of the equation (7.8) implies the substitution \( \mu r = 1 \), i.e., the matrix \((M/\mu)^{\tilde{\Delta}_\tilde{F}}\), written in (6.23), is replaced by the matrix \((Mr)^{\tilde{\Delta}_\tilde{F}}\).

From the operator product expansion (8.1) we therefore find the following expression for the scaling function \( \Phi(Mr, mr, fr^\xi) \) in the representation (7.26) of the correlator \( \langle F_1(x)F_2(x') \rangle \):

\[
\Phi(Mr) = \sum \Delta_\alpha A_\alpha (Mr)^{\tilde{\Delta}_\alpha}, \quad Mr \ll 1, \tag{8.3}
\]

where the coefficients \( A_\alpha = A_\alpha(Mr) \), coming from the Wilson coefficients \( C_\alpha \) in (8.1), are regular in \((Mr)^2\).

Here and below we do not distinguish the two IR scales \( M \) and \( m \), first introduced in (2.8) and (2.11), and \( \Phi \) (8.3) is a matrix of type (6.17).

In general, the operators entering into the OPE are those which appear in the corresponding Taylor expansions, and also all possible operators that admit to them in renormalization [29, 30]. From (6.24) it is clear, that the main contribution to the sum (8.3) is given by the operator \( \tilde{F}_1 \), which possesses maximal singularity. Therefore, combining the RG representation (7.26) with the OPE representation (8.3) gives the desired asymptotic expression for the pair correlation function \( G \) (7.1) in the inertial range:

\[
G = \langle F_{N_1,p_1} F_{N_2,p_2} \rangle \propto \nu^{\hat{d}_G} \cdot [\ln \Lambda r] \cdot \tilde{P}_{(N_1+N_2)/2} \cdot \tilde{\Phi}(fr^\xi), \tag{8.4}
\]

where the leading term is

\[
G \propto \nu^{\hat{d}_G} \cdot [\ln \Lambda r]^{(N_1+N_2)/2} \cdot [\ln Mr]^{(N_1+N_2)/2} \cdot \tilde{\Phi}(fr^\xi) \tag{8.5}
\]

with a certain scaling function \( \tilde{\Phi}(fr^\xi) \), restricted in the inertial range \( l \ll r \ll L \).

### IX. CONCLUSION

We applied the field theoretic renormalization group and the operator product expansion to the analysis of the inertial-range asymptotic behavior of a divergence-free vector field, passively advected by strongly anisotropic random flow. The advecting velocity field was taken Gaussian, not correlated in time, with the given pair correlation function described by the expressions (2.10)–(2.12). This ensemble can be viewed as the d-dimensional generalization of the ensemble introduced in [32] in the context of passive scalar problem. Following [24], we included into the stochastic advection-diffusion equation (2.6) an additional arbitrary parameter \( A \), so that the resulting model involves, as special cases, the kinematic dynamo model for magnetohydrodynamic turbulence, the
linearized Navier–Stokes equation and the case of passive vector “impurity.”

In contrast to the famous Kraichnan’s rapid-change model, where the correlation functions exhibit anomalous scaling behavior with infinite sets of anomalous exponents, here the dependence on the integral turbulence scale $L$ demonstrates a logarithmic character: the anomalies manifest themselves as polynomials of logarithms of $L/r$, where $r$ is the separation argument. The inertial-range asymptotic expressions for various correlation functions are summarized in expressions (8.4) and (8.5).

The key point is that the matrices of scaling dimensions of the relevant families of composite fields (operators) appear nilpotent and cannot be diagonalized and can only be brought to Jordan form; hence the logarithms. The detailed technical proof of this fact is given. However, we cannot give yet a clear physical interpretation of a logarithmic violation of scaling behavior.

The possibility of logarithmic dependence of various correlation functions on the integral scale $L$ and the separation $r$ should be taken into account in analysis of experimental data. Of course, it is desirable to analyze the inertial-range behavior of more realistic models, in particular, to introduce finite correlation time to the correlation function of the velocity field. This work is in progress.

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**Appendix A: Propagator matrix**

In Section [1] we introduced a new parameter $u_0$ and modified the Dyson equation (see [4.4]), so it took on the form

$$
\Gamma_2^{\alpha\beta} = -i\omega \delta_{\alpha\beta} + \nu_0 p^2 \cdot \delta_{\alpha\beta} + \nu_0 f_0 \cdot (p u_0) \cdot (p u_0) - \Sigma_{\alpha\beta}.
$$

(A1)

Let us denote the index structure of (A1) as $M_{\alpha\beta}$, i.e.,

$$
M_{\alpha\beta} = -i\omega \delta_{\alpha\beta} + \nu_0 p^2 \cdot \delta_{\alpha\beta} +
\nu_0 f_0 \cdot (p u_0) \cdot (p u_0) \cdot n_{\alpha} n_{\beta} - \Sigma_{\alpha\beta}.
$$

(A2)

and, according to the general rule, in order to find the true bare propagator $\langle \theta \theta^\prime \rangle_0$, we have to find the inverse matrix $M^{-1}_{\alpha\beta}$. At this point some remarks follow:

(1) Since we are dealing with transverse divergence-free fields $\theta$ and $\theta^\prime$ and now consider self-energy diagram with those fields $\theta$ and $\theta^\prime$ on the tails, we need to know not the plain inverse matrix $M^{-1}_{\alpha\beta}$, but the inverse matrix $N^{-1}_{ij}(p) = [P_{\alpha\beta}(p)M_{\alpha\beta}(p)P_{\beta\gamma}(p)]^{-1}$.

(2) The unity operator in the transverse space is the transverse projector, so the basic equality for $N_{ij}(p)$ is

$$
N_{ij}(p) \cdot N^{-1}_{jk}(p) = P_{ik}(p),
$$

(A3)

where $N^{-1}_{jk}(p)$ is the sought-for propagator matrix.

Let us calculate the convolution of $M_{ij}$ with the transverse projectors, i.e., the matrix $N$:

$$
N_{ij}(p) = P_{\alpha\beta}(p)M_{\alpha\beta}(p)P_{\beta\gamma}(p) =
\begin{align*}
-\nu_0 P_{ij}(p) + \nu_0 p^2 \cdot P_{ij}(p) + \\
+ \nu_0 f_0 (pu_0)^2 \cdot P_{ij}(p) + \nu_0 f_0 u_0 (pu_0)^2 \cdot n_{i} \cdot n_{j}
\end{align*}
$$

(A4)

where $\chi$ and $\chi'$ are the coefficients in the index structures $P_{ij}(p)$ and $n_{i} \cdot n_{j}$ and the unit vector $\hat{n}_k$ is

$$
\hat{n}_k = P_{mk}(p)n_m = n_k - p||p_k/p^2.
$$

(A5)

Now, according to (A3), let us find the inverse matrix $N^{-1}_{jk}$. It has the same index structure as $N_{jk}$, but with other coefficients denoted as $x$ and $y$. Therefore

$$
\langle x \cdot P_{ij}(p) + \chi \cdot \hat{n}_i \cdot \hat{n}_j \rangle \cdot \langle x \cdot P_{jk}(p) + y \cdot \hat{n}_j \cdot \hat{n}_k \rangle = P_{ik}(p).
$$

(A6)

From this relation we find:

$$
x = 1/\chi;
$$

(A7a)

$$
y = -\chi/\chi(\chi + \chi \sin^2 \kappa),
$$

(A7b)

where $\kappa$ is the angle between the vectors $\mathbf{n}$ and $\mathbf{p}$. Therefore, for the true propagator function of the fields $\langle \theta_0 \theta^\prime_k \rangle_0$ we obtain

$$
\langle \theta_0 \theta^\prime_k \rangle_0 = x \cdot P_{jk}(p) + y \cdot \hat{n}_j \cdot \hat{n}_k
$$

(A8)

with known coefficients $x$ and $y.$
which coincides with (3.13) after redefinition of the variable $y$ one can write

$$y = -\mathcal{Y}/\mathcal{X}e^{\mathcal{Y}^2/2} \propto (\pm n)^2
\frac{1}{(-i\omega + \eta_1)(-i\omega + \eta_2)},$$

so that the integral of the expression (A10) over $\omega$ converges (without requiring any extension of function definitions like (6.15)) and does not contribute in all of the above expressions.

This means that the only contributing term in this propagator is $x \cdot P_{jk}(p)$, and the final form for it is

$$\langle \theta_j \theta_k \rangle_0 = \frac{P_{jk}(p)}{\pm i\omega + \eta_0 p^0 + \eta_0 f_0 p^1},$$

which coincides with (3.13) after redefinition of the variable $p$ as $k$. Note that the contribution with $y$ must be taken into account in calculation of the convergent parts of those diagrams, but we do not need them in our consideration.

### Appendix B: The nilpotency of the anomalous dimension matrix

#### 1. Definitions and aims

In this section we will prove the nilpotency of the matrix $\gamma_F^*$ from (6.10) and, as a consequence, the Jordan form of the critical dimension matrix $\Delta_{Np,Np'}$ from (6.10). Let us recall some definitions and facts from the Sections VI A and VI B.

Let us define the vector $\mathbf{F}$ as in (6.11), namely

$$\mathbf{F} = \begin{pmatrix} (\theta_1 \theta_1)^N \\ \vdots \\ (\theta_{n-1} \theta_{n-1})^{N-2} \\ (\theta_n \theta_n)^N \end{pmatrix},$$

the relation $F_i = Z_i F^R_{ik}$ between the set of unrenormalized operators $\{F\}$ and the set of renormalized operators $\{F^R\}$ takes the form

$$\Delta_{Np,Np'} = -2(p + m) \cdot \delta_{pp'} + \gamma_{Np,Np'}^*.$$

Here $-2(p + m)$ is its canonical dimension, $\delta_{pp'}$ is Kronecker’s $\delta$-symbol and $\gamma_{Np,Np'}^*$ is the value of anomalous dimension matrix at the fixed point.

The aim is to prove the nilpotency of the matrix $\gamma_F^*$ from (B1.4) and the Jordan form of the matrix $\Delta_{Np,Np'}$ from (B1.5). We will present the explicit expression for the diagonalizing matrix $U_N$ that brings the matrix $\Delta_F$ to the Jordan form $\Delta_F$ by the transformation

$$\Delta_F = U_N \Delta_F U_N^{-1}.\quad (B6)$$

As the number $N$ in (B1.1) may be arbitrary, the dimension of the matrix $\tilde{Z}_F$ in equation (B2) and, as a consequence, of the matrices $\gamma_F$ and $U_N$, namely $(N/2 + 1) \times (N/2 + 1)$, also may be arbitrary. This means, that the expression

$\begin{pmatrix} (\theta_1 \theta_1)^N \\ \vdots \\ (\theta_{n-1} \theta_{n-1})^{N-2} \\ (\theta_n \theta_n)^N \end{pmatrix}$

Note that in the matrix $\tilde{Z}$ the power $p$ of the operator $F_{Np}$ decreases from the right to the left.

\[\mathbf{F} = \begin{pmatrix} (\theta_1 \theta_1)^N \\ \vdots \\ (\theta_{n-1} \theta_{n-1})^{N-2} \\ (\theta_n \theta_n)^N \end{pmatrix}, \text{ where } a_{ij} = (\theta_1 \theta_1)^{i-1} (\theta_{n-1} \theta_{n-1})^{N-j} (\theta_n \theta_n)^{N-(n-1)}.\]

According to (6.9), the elements of the matrix of anomalous dimensions $\gamma_F = \tilde{Z}_F \cdot \mathcal{D}_\mu \tilde{Z}\tilde{F}$ at the fixed point $g^*$ are

\[
\begin{align*}
\gamma_{Np+1} &= 2m(2m-1) \cdot y; \\
\gamma_{Np} &= (2p + 8pm - 2m(2m - 1)) \cdot y; \\
\gamma_{Np-1} &= (4p(p-1) - 2p - 8pm) \cdot y; \\
\gamma_{Np-2} &= (-4p(p-1)) \cdot y,
\end{align*}
\]

and the critical dimension matrix for the operators $F_{Np}$ has the form

\[
\Delta_{Np,Np'} = -2(p + m) \cdot \delta_{pp'} + \gamma_{Np,Np'}^*.\quad (B5)
\]
gives us the algorithm to construct the matrix \( \hat{\gamma}_F \) for the set of initial operators \( \{ F \} \) with arbitrary \( N \) – simply it gives the value of each matrix element. And the difficulty and the fascination of this task is to find an algorithm for constructing the diagonalizing matrix \( U_N \), applicable to an arbitrary number \( N \), or, equivalently, to the matrix \( \hat{\gamma}_F \) with arbitrary dimension. Note, that if the matrix \( \Delta_F \) was diagonalizable, the diagonalizing matrix \( U_N \) would be unique for each fixed number \( N \), but since the matrix \( \Delta_F \) has the Jordan form, the diagonalizing matrix \( U_N \) is not unique for any fixed number \( N \). Therefore, we will show one of the possible forms of the matrix \( U_N \), which “diagonalizes” the matrix \( \hat{\gamma}_F \) and thus solves our problem. Since each element of the matrix \( \hat{\gamma}_F \) is a multiple to the scalar number \( y \), the nilpotency of the matrix \( \hat{\gamma}_F \) is equivalent to the nilpotency of the matrix \( \hat{\gamma}_F \) itself, where \( y \cdot \hat{\gamma}_F = \hat{\gamma}_F \).

2. Motivation and idea

Let us write the \( 3 \times 3 \) (\( N = 4 \)) matrix \( \hat{\gamma}_F \) denoted as \( A_4 \):

\[
A_4 = \begin{pmatrix} 4 & 4 & 8 \\ 2 & 8 & -10 \\ 0 & 12 & -12 \end{pmatrix}.
\]

(B7)

It is nilpotent, its eigenvalues are

\[
\lambda_1 = \lambda_2 = \lambda_3 = 0.
\]

(B8)

The matrix \( U_4 \), which “diagonalizes” the matrix \( A_4 \), is built from the eigenvectors of the matrix \( A_4 \). Find them:

\[
V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \quad V_2 = \begin{pmatrix} 1/6 \\ 1/12 \\ 0 \end{pmatrix}; \quad V_3 = \begin{pmatrix} 1/24 \\ 0 \\ 0 \end{pmatrix}.
\]

(B9)

Note that the eigenvectors are determined by the condition \((A_4 - \lambda I)V_{i+1} = V_i\), which has a unique solution up to an additional constant.

Thus the matrix \( U_4 \) takes the form

\[
U_4 = \begin{pmatrix} 1 & 1/6 & 1/24 \\ 1 & 1/12 & 0 \\ 1 & 0 & 0 \end{pmatrix}
\]

and \( \hat{\gamma}_F U_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \).

(B10)

Here one can notice an interesting property: the product \( \hat{\gamma}_F \cdot U_4 \) is the same as the matrix \( U_4 \), but with all columns shifted by one position to the right, namely

\[
A_1 \cdot U_4 = \begin{pmatrix} 0 & 1 & 1/6 \\ 0 & 1 & 1/12 \\ 0 & 1 & 0 \end{pmatrix}.
\]

(B11)

Now, if we multiply the matrix \( U_4^{-1} \) by the product \( A_4 \cdot U_4 \), it brings \( A_4 \) to the Jordan form:

\[
U_4^{-1} \cdot \begin{pmatrix} 1 & 1/6 & 1/24 \\ 1 & 1/12 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

(B12)

but

\[
U_4^{-1} \cdot \begin{pmatrix} 0 & 1 & 1/6 \\ 0 & 1 & 1/12 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
\]

(B13)

The feature \( \Delta_F \) is not characteristic only for specific matrices, but is a common rule. For any \( M \times M \) nondegenerate matrix

\[
\hat{M} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}
\]

(B14)

the product of \( \hat{M}^{-1} \) and \( \hat{M} \), where \( \hat{M} \) is the matrix \( \hat{M} \) with all columns shifted by one position to the right and with all elements of the first column being equal to zero, is a matrix of Jordan form:

\[
\hat{M}^{-1} \cdot \hat{M} = \hat{M}^{-1} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & a_{nn-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix}.
\]

(B15)

Here empty space denotes the elements, which are equal to zero. The expression \( \Delta_F \) is obvious. Multiplying \( \hat{M}^{-1} \) with the first empty column gives us an empty column in the RHS. Multiplying \( \hat{M}^{-1} \) with the other columns with numbers \( 2, \ldots, n \) gives us the unity matrix, which however starts not from the cell 11, but from the cell 12 – i.e., the “unity” matrix with nonzero terms not on the main diagonal, but on the diagonal above it.

Thus the idea is to find such a matrix \( U_N \) with \( \det U_N \neq 0 \), that makes the product \( A_N \cdot U_N \) equivalent to matrix \( U_N \) itself, but with its columns shifted as \( i \to i+1 \), and with the elements of the first column equal to zero. If we find it, our problem will be solved:

\[
U_N^{-1} \cdot [A_N \cdot U_N] = \begin{pmatrix} 0 & 1 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix}.
\]

(B16)

3. Explicit form of the matrix \( U_N \)

The next step is to understand the explicit form of the matrix \( U_N \). To this end, let us write the \( 10 \times 10 \) matrix
\( \hat{\epsilon}_p \) for \( N = 18 \), denoted as \( A_{18} \), and the diagonalizing matrix \( U_{18} \) (which is found by direct calculation):

\[
A_{18} = \begin{pmatrix}
18 & 270 & -288 \\
2 & 78 & 144 & -224 \\
12 & 114 & 42 & -168 \\
30 & 126 & -36 & -120 \\
56 & 114 & -90 & -80 \\
90 & 78 & -120 & -48 \\
132 & 18 & -126 & -24 \\
182 & -66 & -108 & -8 \\
240 & -174 & -66 & -
\end{pmatrix}, \quad \text{(B17)}
\]

\[
C_{18} = \begin{pmatrix}
1/306 & 36/73440 & 84/3IV \cdot 182 \\
1/8306 & 28/73440 & 56/3IV \cdot 182 \\
1/7306 & 21/73440 & 35/3IV \cdot 182 \\
1/6306 & 15/73440 & 20/3IV \cdot 182 \\
1/5306 & 10/73440 & 15/3IV \cdot 182 \\
1/4306 & 6/73440 & 1/3IV \cdot 182 \\
1/3306 & 3/73440 & 1/3IV \cdot 182 \\
1/2306 & 1/73440 & 1/ \\
1/1306 & & 1
\end{pmatrix}, \quad \text{(B18)}
\]

Here the Roman figures denote the denominators from previous columns, i.e., \( III = 73440 \), \( IV = 73440 \cdot 182 = 13366080 \), etc. As all denominators in one column are identical, the symbol “/” denotes division of the numerator by the denominator, written in the first element of the column.

From the explicit expression \( \text{(B18)} \) it is obvious, that the denominators of the elements of matrix \( U_{18} \) are products of the elements from the diagonal below the main diagonal of the matrix \( A_{18} \) from \( \text{(B17)} \), and the numerators are the elements from Pascal’s triangle, namely \( \binom{n}{k} \), where \( n \) is the number of the row (with numeration going from bottom up) and \( k \) is the number of the column (with numeration going from the left to the right).

Here \( \binom{n}{k} \equiv C_n^k \) is the number of \( k \) combinations from the set of \( n \) elements.

This is the conjecture, which we have to prove: the matrix, constructed by the described rules is the sought-for matrix \( U_N \) for any dimension of initial matrix \( A_N \) (i.e., for the family of operators with any \( N \)).

One remark follows, which will be useful later: since in notation \( \text{(B2)} \) each row of matrix \( A \) corresponds to an operator with fixed number \( p \), thus each element \( \binom{p}{k} \) is actually \( \binom{p}{k} \), where \( C \) is the number of the column (starting from zero).

4. The proof of our assumptions

The proof is divided into several steps: first, we will prove the reliability of the first two columns of the matrix, then the reliability of the three lower diagonals. Finally, we will prove it for all the other elements.

a. The first column (\( C=0 \))

From expressions \( \text{(B4)} \) it follows, that \( \sum_i \gamma_{N,p+i} = 0 \). This is the reason why in the case when the first column of matrix \( U_N \) is \( \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \), the first column of matrix \( A \cdot U \) is \( \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \).

b. The second column (\( C=1 \))

Now to have the base for further steps, we need to prove our conjecture for the second column of matrix \( U_N \), which is the first nontrivial column.
The latest element in the second column (in \[\text{B18}\]) it is \(1/306\) is the element, which is determined by the last element of the diagonal, located below the main diagonal of matrix \(A\). In \[\text{B17}\]) it is equal to \(-306\). Since this element is located on the aforementioned diagonal, it is formed by the condition \[\text{B14a}\]. From the word “the latest” it follows, that this element corresponds to the operator with \(N\) and terms (corresponding to the transition with \(\gamma_N\)). This follows from the requirement that the sum of the two elements in the second column is to check an identity for the first element. This element corresponds to the operator with \(N\) and starting from 1. From equation \[\text{B25}\]) it follows, that this element is located on the aforementioned diagonal, \(X\) and therefore the required element of the diagonal, located below the main diagonal, \(X\) is equal to \(1\). From expression \[\text{B21}\]) it follows, that

\[
X = \frac{k + 3}{N(N - 1)}. \quad \text{(B26)}
\]

Having identified all the elements, all we have to do in the second column is to check an identity for the first element. This element corresponds to the operator with \(m = 0\), \(p = N/2\), therefore from expressions \[\text{B4}\]) it follows, that the equivalent of \[\text{B25}\]) for it is

\[
\left| \frac{\text{B27}\) \right|_{p=N/2} = 1. \quad \text{(B27)}
\]

2\(m(m - 1) \cdot X + [2p + 8pm - 2m(2m - 1)] \frac{k + 2}{N(N - 1)} + [4p(p - 1) - 2p - 8pm] \frac{k + 1}{N(N - 1)} - 4p(p - 1) \frac{k}{N(N - 1)} = 1, \quad \text{(B25)}
\]

This follows from the requirement that the sum of the two terms (corresponding to the transition with \(\gamma_{N,p+1}^{*}\) and \(\gamma_{N,p}^{*}\) \[\text{B14a}\]) be equal to 1, and from the observation, that these elements correspond to the operator with \(p = 1\). From expression \[\text{B21}\]) it follows, that

\[
X = \frac{2}{N(N - 1)}. \quad \text{(B22)}
\]

The element, that is the third from the end, is governed by the sum of three terms, constructed like expressions \[\text{B19}\) and \[\text{B21}\]. As we go one position up, the parameters for the operator \((\theta_i \theta_j)\)(\(n_s \theta_s\))\(2m\) become \(p = 2\) and \(2m = N - 4\). So,

\[
(N - 4)(N - 5) \cdot X + \left[4 + 8(N - 4) - (N - 4)(N - 5)\right] \frac{2}{N(N - 1)} + \left[8 - 4(N - 4)\right] \frac{1}{N(N - 1)} = 1, \quad \text{(B23)}
\]

and hence

\[
X = \frac{3}{N(N - 1)}. \quad \text{(B24)}
\]

Expressions \[\text{B19}\), \[\text{B21}\) and \[\text{B23}\) are constructed from different number of terms, therefore they need to be considered separately. Another distinguished element is the first element, \(9/306\) in expression \[\text{B18}\) – for this element we have to verify an identity. We will come back to it later. Since we know \[\text{B20}\), \[\text{B22}\) and \[\text{B21}\), we may write for all other elements (which are always governed by expressions with \textit{four} terms)

\[
2m(m - 1) \cdot X + [2p + 8pm - 2m(2m - 1)] \frac{k + 2}{N(N - 1)} + [4p(p - 1) - 2p - 8pm] \frac{k + 1}{N(N - 1)} - 4p(p - 1) \frac{k}{N(N - 1)} = 1, \quad \text{(B25)}
\]

This is the aforementioned identity for the first element, and it appears to be true.

This is all we needed to prove for the elements of the second column: from the expressions \[\text{B20}\), \[\text{B22}\) and \[\text{B21}\) it follows, that all elements have the same denominators, namely \(N(N - 1)\), and their numerators are equal to \(k\), where \(k = 1\) corresponds to the second element from the end. All these elements are obtained from the requirement, that by multiplying the diagonalizing matrix \(U_N\) with the matrix of critical dimensions \(A_N\) one obtains the matrix \(U_N\), but with all columns shifted by one position to the right. Furthermore, the previous column is constructed from all the elements, which are equal to 1 (see Appendix \[\text{B4}\)).

Moreover, all the formulas \[\text{B20}\), \[\text{B22}\) and \[\text{B21}\) and
can be unified using combinations: since we are dealing with a column with \( C = 1 \), therefore

\[
X = \frac{1}{N(N - 1)} \cdot \binom{p}{1}. \tag{B28}
\]

c. The three lower diagonals

Now we know the form of the elements from the first and the second columns. This is our starting point for proving the form of all the other elements in a matrix with finite, but arbitrary number of columns. This will be done in two steps: first we will prove this for the three lowest diagonals, and then, in the next section, we will consider all the other elements. The three lowest diagonals are considered separately since the equations which determine the elements in these diagonals contain different number of terms. This situation is similar to the case in the previous section, in which we considered the first three expressions, \( \text{(B19), (B21) and (B23)} \), separately from the general expression \( \text{(B25)} \).

First let us consider the lowest diagonal. The product of each element from it with the corresponding element of the matrix \( A_N \) has to result in the element with the same position of the previous column of matrix \( U_N \).

At this point we know all the elements from the two first columns. Hence we may start from the last element of the already known column with \( C = 1 \) and then describe the sequence of all other elements from the lowest
diagonal.

The rule, which these elements are governed by, is

\[
X \cdot \gamma_{N,p+1} = Y. \tag{B29}
\]

Here \( X \) and \( Y \) denote the elements of the diagonal in question, but \( Y \) is already known element from the column with \( C_Y = i \) and \( X \) is a sought-for element from the column with \( C_X = i + 1 \).

According to expression \( \text{(B31a)} \), the element \( \gamma_{N,p+1}^* \) is equal to \( 2m(2m - 1) \). Let us start from the elements of the two first columns, i.e., \( C_Y = 0 \) and \( C_X = 1 \). In this case \( Y = 1 \) (see Appendix \( \text{[B4a]} \) and \( 2m = N \). Therefore,

\[
X = \frac{1}{N(N - 1)}. \tag{B30}
\]

The vertical position of each following element in the diagonal is higher than of the previous, therefore the number \( 2m \) decreases from \( N \) (column with \( C = 1 \)) to 2 (the latest column). From the expression \( \text{(B29)} \) it then follows, that the sequence of the elements in this (the lowest) diagonal is

\[
\frac{1}{N(N - 1)}; \frac{1}{N(N - 1)(N - 2)(N - 3)}; \ldots \frac{1}{N(N - 1)(N - 2)(N - 3) \cdot \ldots \cdot 2 \cdot 1}. \tag{B31}
\]

For the elements of the second from the bottom diagonal the equation like \( \text{(B29)} \) takes the form

\[
X \cdot \gamma_{N,p+1} = Y. \tag{B32}
\]

where \( X \) is the sought-for element and \( \gamma_{N,p+1}^*, \gamma_{N,p}^* \) are defined in \( \text{[B4a] and (B4b)} \). The numerator of the RHS follows from the explicit form of those equations. For example, if \( C = 1 \), the sought-for element corresponds to the operator with \( p = 2 \), and accordingly to \( \text{(B22)} \), the RHS is equal to \( 2/N(N - 1) \). The solution of this equation, namely expression \( \text{(B33)} \), is proportional to \( p+1 \) and is the starting point for the next element of the diagonal, which corresponds to the operator with \( p = 3 \); etc. Note, that the RHS in the \( \text{(B32)} \) is a known, but not a sought-for quantity. From the expression \( \text{(B32)} \) it follows, that

\[
X = \frac{p + 1}{N(N - 1) \ldots (N - 2p + 1)}. \tag{B33}
\]

in agreement with \( \text{(B34)} \). Moreover, as we investigate the elements from the second from the bottom diagonal, the numerator in \( \text{(B33)} \) may be written as

\[
p + 1 = \binom{p + 1}{p}. \tag{B34}
\]

For the elements of the third from the bottom diagonal the corresponding expression similar to \( \text{(B29) and (B32)} \) is
\[ X \cdot \gamma_{N,p+1} + \gamma_{N,p+1} + \frac{p}{N(N-1) \ldots (N-2p+4)(N-2p+3)} \cdot \gamma_{N,p} + \frac{1}{N(N-1) \ldots (N-2p+4)(N-2p+3)} \cdot \gamma_{N,p-1} = \]

\[ = \frac{\alpha}{N(N-1) \ldots (N-2p+6)(N-2p+5)}. \quad (B35) \]

where \( X \) is the sought-for element and \( \gamma_{N,p+1} \), \( \gamma_{N,p} \), \( \gamma_{N,p-1} \) are defined in (B33), (B37), which are also combinations. In addition,

\[ \alpha = 3 + \sum_{n=3}^{p-1} n = \frac{1}{2} p(p-1). \quad (B36) \]

From expressions (B35) and (B36) it follows, that

\[ X = \frac{1}{N(N-1) \ldots (N-2p+3)} \cdot \frac{1}{N(N-1) \ldots (N-2p+3)} \cdot \gamma_{N,p+1} + \gamma_{N,p} + \frac{1}{N(N-1) \ldots (N-2p+4)(N-2p+3)} \cdot \gamma_{N,p-1} = \]

\[ = \frac{\alpha}{N(N-1) \ldots (N-2p+6)(N-2p+5)}. \quad (B37) \]

which also may be written using combinations, namely

\[ X = \frac{1}{N(N-1) \ldots (N-2p+3)} \cdot \frac{1}{N(N-1) \ldots (N-2p+3)} \cdot \gamma_{N,p+1} + \gamma_{N,p} + \frac{1}{N(N-1) \ldots (N-2p+4)(N-2p+3)} \cdot \gamma_{N,p-1} = \]

\[ = \frac{\alpha}{N(N-1) \ldots (N-2p+6)(N-2p+5)}. \quad (B38) \]

So, we know at this point all the elements from the first two columns and three lowest diagonals, which satisfy the general requirement that the product of \( A_N \cdot U_N \) be the matrix \( U_N \) with all columns, shifted by one position to the right.

\[ (N-2p)(N-2p-1) \cdot X + \frac{2p+4p(N-2p) - (N-2p)(N-2p-1)}{N(N-1) \ldots (N-2C+1)} \cdot C_{L+2} + \]

\[ + \frac{4p(p-1) - 2p - 4p(N-2p)}{N(N-1) \ldots (N-2C+1)} \cdot C_{L+1} + \frac{-4p(p-1)}{N(N-1) \ldots (N-2C+1)} \cdot C_L = \frac{(C-1)_{L+3}}{N(N-1) \ldots (N-2C+3)}. \quad (B40) \]

where \( X \) is the sought-for element. Now we have to verify two hypotheses:

1. The denominator of the element \( X \) is the product \( N(N-1) \ldots (N-2C+1) \).
2. The numerator of the element \( X \), denoted as \( C_{L+3} \), is the corresponding combination. Note that we know all the elements of the three lowest diagonals (see (B31), (B33) and (B37)), which are also combinations.

Therefore we want to check, whether the equation

\[ \frac{(N-2p)(N-2p-1)}{(N-2C+2)(N-2C+1)} \cdot C_{L+3} + \frac{2p+4p(N-2p) - (N-2p)(N-2p-1)}{(N-2C+2)(N-2C+1)} \cdot C_{L+2} + \]

is true if \( X \) satisfies the above written conditions, i.e., if

\[ X = \frac{C_{L+3}}{N(N-1) \ldots (N-2C+1)} \quad (B41) \]

and all \( C \) in (B40) are some combinations.

To verify that let us substitute (B41) into (B40). We find, that

\[ d. \ All \ other \ elements \]

Let us do some redesignation, which is implied only in this subsection. Let us denote the number of the column as \( C \), moreover, \( C = 1 \) corresponds to the first nontrivial column (“second” in the notation of previous sections). Let \( C_L \) be an element from column \( C \) with position \( L \).

The hypothesis is that \( C_{L} = \binom{C}{L} \) for all \( C, L \). Then, we will use trivial relations for the combinations, namely

\[ (\begin{array}{c} L \\ C \end{array}) = \binom{L}{C-1} \cdot \frac{L+1-C}{C}; \quad (B39a) \]

\[ (\begin{array}{c} L+C \\ C \end{array}) = \binom{L+C-1}{C-1} \cdot \frac{L+C}{1+C}; \quad (B39b) \]

\[ (\begin{array}{c} L-1 \\ C \end{array}) = \binom{L-1}{C} \cdot \frac{L-C}{L-C}; \quad (B39c) \]

where \( C \) denotes the number of the column and \( L \) denotes the number of the row (starts also from \( L = 0 \) and with numeration going from bottom up). Expressions (B39a)–(B39c) allow us to move in the horizontal, diagonal and vertical directions in the matrix \( U_N \).

The basic equation in the general case is
\[ + \frac{4p(p - 1) - 2p - 4p(N - 2p)}{(N - 2C + 2)(N - 2C + 1)} \cdot C_{L+1} + \frac{[-4p(p - 1)]}{(N - 2C + 2)(N - 2C + 1)} \cdot C_L = (C - 1)_{L+3}. \] (B42)

Then, let us express \( C_{L+3}, C_{L+2}, C_{L+1} \) and \((C - 1)_{L+3}\) through \( C_L \) using expressions (B39):

\[
C_{L+3} = \frac{(L + 2 + C)(L + 1 + C)(L + C)}{(L + 2)(L + 1)L} \cdot C_L; \tag{B43a}
\]

\[
C_{L+2} = \frac{(L + 1 + C)(L + C)}{(L + 1)L} \cdot C_L; \tag{B43b}
\]

\[
C_{L+1} = \frac{(L + C)}{L} \cdot C_L; \tag{B43c}
\]

\[
(C - 1)_{L+3} = \frac{C(L + 1 + C)(L + C)}{(L + 2)(L + 1)L} \cdot C_L. \tag{B43d}
\]

Substituting (B43) into (B42) gives us an expression without the arbitrary number \( C_L \), namely

\[
\frac{(N - 2p)(N - 2p - 1)}{(N - 2C + 2)(N - 2C + 1)} \cdot \frac{(L + 2 + C)(L + 1 + C)(L + C)}{(L + 2)(L + 1)L} + \frac{2p + 4p(N - 2p) - (N - 2p)(N - 2p - 1)}{(N - 2C + 2)(N - 2C + 1)} \cdot \frac{(L + 1 + C)(L + C)}{(L + 1)L} + \frac{4p(p - 1) - 2p - 4p(N - 2p)}{(N - 2C + 2)(N - 2C + 1)} \cdot \frac{(L + C)}{L} + \frac{[-4p(p - 1)]}{(N - 2C + 2)(N - 2C + 1)} \cdot \frac{C(L + 1 + C)(L + C)}{(L + 2)(L + 1)L}. \tag{B44}
\]

Moreover, it is obvious that the numbers \( L, C \) and \( p \) are not independent: there is a relation between them, namely

\[ 1 + L + C = p. \tag{B45} \]

Using (B45) one may check, that the expression (B44) is true, i.e., it is an identity. This means that our conjecture (B41) is true!

5. Summary

In sections B4a – B4d we have proven the hypothesis, that there exists a matrix of special type, that brings our matrix of critical dimensions to Jordan form. We presented the explicit form of the “diagonalizing” matrix \( U_N \) for any fixed dimension \( N \). As a consequence, the critical dimension matrix (6.10) is degenerate and the asymptotic behavior of the mean value of some quantities does not just have a power behavior, but is a product of a power with a polynomial of a logarithm.

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