Semidefinite programming and arithmetic circuit evaluation

Sergey P. Tarasov  Mikhail N. Vyalyi

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Abstract

A rational number can be naturally presented by an arithmetic computation (AC): a sequence of elementary arithmetic operations starting from a fixed constant, say 1. The asymptotic complexity issues of such a representation are studied e.g. in [2, 9] in the framework of the algebraic complexity theory over arbitrary field.

Here we study a related problem of the complexity of performing arithmetic operations and computing elementary predicates, e.g. “=” or “⩾”, on rational numbers given by AC.

In the first place, we prove that AC can be efficiently simulated by the exact semidefinite programming (SDP).

Secondly, we give a BPP-algorithm for the equality predicate.

Thirdly, we put ⩾-predicate into the complexity class PSPACE.

We conjecture that ⩾-predicate is hard to compute. This conjecture, if true, would clarify the complexity status of the exact SDP — a well known open problem in the field of mathematical programming.

Keywords: semidefinite programming, complexity, succinct representation.

Algorithmic complexity provides a general framework to analyze complexity of computational problems. It works for many cases and gives results that are important for practical applications. Nevertheless, some basic assumptions of the theory are strange from practical point of view. We are sure that a linear time algorithm running in time $10^{1000}n$ cannot be realized in our Universe. Also, some widely used algorithms have exponential running time in the worst case.

Numerical algorithms are especially in striking disagreement with complexity theory. Probably, the most popular exponential algorithm is the simplex algorithm for linear programming. It is widely used despite the existence of polynomial algorithms that were found after pioneering breakthrough of L.Khachiyan (see, e.g. [8, 7, 13]).

The complexity analysis of the semidefinite programming problem involves even more difficulties. SDP is often considered as tractable due to various approximate algorithms. SDP is a convex optimization problem so the ellipsoid method can be applied to solve it approximately as well as a variety of interior points methods [6, 10, 14, 18]. But there are amazingly few results on complexity of the SDP problem.
Khachiyan and Porkolab [11] found polynomial time algorithm for the SDP problem when the dimension is fixed. They established doubly exponential bounds on the solutions for the general SDP problem and on the discrepancies of infeasible programs.

For our further considerations the most interesting is Ramana’s result [12]. He developed the exact duality theory for SDP. Ramana’s dual program can be constructed in polynomial time. His analogue of Farkas lemma has an immediate complexity-theoretic corollary: a complement to the SDP feasibility problem (SDFP) can be reduced to SDFP itself. It implies that SDFP cannot be NP-complete unless \( \text{NP} = \text{coNP} \). More generally, SDFP should belong to a complexity class that is closed under complement. Examples are P, \( \text{NP} \cap \text{coNP} \), BPP, PSPACE. (For definitions of these classes and other useful information on complexity theory see, e.g., the Sipser’s book [16].)

Here we address the exact SDFP problem. The problem is to check that intersection of the cone of positive semidefinite matrices with some affine subspace of matrices is not empty. The subspace involved is defined by generators that are matrices with rational entries. It is well known that some pathological examples exist for the SDP. For instance, it is possible that feasible program has only doubly exponential solutions and that an infeasible program could have doubly exponentially small discrepancy. These examples show that in some cases a polynomially bounded machine can not simply write a solution to SDFP. Of course, this does not prevent checking the existence of a solution in polynomial time.

Motivated by these examples, we introduce a new problem: comparison of arithmetic computations. It seems to be interesting in its own right and concerns a nonstandard way of representing integers and rationals. The common way to represent integers is to use a positional system. Among positional systems the binary system is the simplest and the most natural. Binary representation of a number \( N \) is a string of length \( \theta(\log N) \). This bound is optimal due to a counting argument. But it is also possible to encode numbers in such a way that some numbers are encoded by very short strings. In this case we speak of a succinct representation of an integer. A natural way for succinct representation of a rational number \( r \) is to use an arithmetic computation or an arithmetic circuit (AC). By definition, AC is a sequence of the elementary arithmetic operations starting from a fixed constant, say 1, and generating \( r \) as output.

To our knowledge, the complexity issues of the AC representation were studied primarily in the asymptotic setting in the framework of the algebraic complexity theory over a field (see, e.g. [15, 9]). Namely, let \( \tau(k) \) be the minimum number of arithmetic operations required to build the integer \( k \) from the constant, say 1, i.e. the length of the minimal AC computing \( k \). A sequence \( x_k \) of integers is said to be “ultimately easy to compute” if there exists another sequence \( a_k \) and polynomial \( p(\cdot) \) such that \( \tau(a_k x_k) \leq p(\log k) \) for all \( k \) (for instance, in can be shown that the sequence \( 2^k \) is ultimately easy to compute). Otherwise the sequence is said to be “ultimately hard to compute”. Counting argument shows that ultimately hard to compute sequences do exist. A central open problem is to show that some explicit sequences, say, \( n! \), \( \lceil (3/2)^n \rceil \) are ultimately hard to compute. For instance, if \( n! \) is ultimately hard to compute then \( \text{P}_C = \text{NP}_C \) over the field of complex numbers [15].

It is worth to mention that the complexity of AC is related to a circuit model of computation for polynomials that was introduced and studied by Valiant [17].
Valiant’s model turns to AC by substitution constants in polynomials. Some interesting applications of this observation are discussed in [9].

In this paper we explicitly treat a related problem of the complexity of performing arithmetic operations and computing elementary predicates, e.g. “=” or “⩾”, on rational numbers represented by AC. An immediate inspection shows that if numbers are represented by AC then to perform an elementary arithmetic operation one should simply merge the AC of the operands in the appropriate way, thus arithmetical operations are easy to perform, despite the evident fact that AC representation is succinct for some numbers (e.g. $2^{2^n}$ can be obtained by repeated squaring). On the other hand, — it is unclear how to efficiently compute elementary predicates “=” or “⩾”, i.e. how to check efficiently whether two AC compute the same value or how to compute the maximal value. In the sequel we denote these predicates by $AC_=$ and $AC_{⩾}$ respectively.

Our main contributions are: we prove that some restricted version of AC can be efficiently simulated by the exact semidefinite programming (SDP) and construct a polynomial reduction of $AC_{⩾}$ to SDPF; we give a BPP-algorithm for the equality predicate; we put $⩾$-predicate into the complexity class $PSPACE$.

Unfortunately, we are unable at the moment to prove any lowerbounds for the $⩾$-predicate. Any nontrivial result in this direction would imply lower bounds for the exact SDP — one of the main open problems in the field of mathematical programming.

The rest of paper is organized as follows. In Section 1 we give a definition of the circuit representation for rationals and state the basic results concerning it. Also, this Section contains technical results about a choice of a basis for AC. In Section 2 we construct a reduction of $AC_{⩾}$ to SDPF. Sections 3, 4 contain proofs of $AC_= \in BPP$ and $AC_{⩾} \in PSPACE$. In the last Section 5 we discuss open questions around the $AC_{⩾}$ and SDPF problems.

1 AC representation of numbers

Let $B$ be a finite collection of functions of type $\mathbb{Q}^k \rightarrow \mathbb{Q}$ ($k$ may vary). It is called a basis. A circuit over basis $B$ is a sequence $S = s_0, s_1, \ldots, s_\ell$ of assignments such that $s_0 := 1$ and for each $i \geq 1$ there exist $j, k < i$ such that $s_i := s_j \ast s_k$ and $\ast \in B$. The size $\ell(S)$ of circuit $S$ is the number $\ell$.

The most natural basis consists of four arithmetic operations $\{+, -, \cdot, /\}$. Circuits over this basis are called arithmetic circuits. Circuits over the basis $\{+, -, \cdot\}$ are called division-free circuits. Monotone (arithmetic) circuits are circuits over the basis $\{+, \cdot\}$.

We will represent rationals by circuits. For each circuit we define the value of the circuit by induction. We will use notation $v(S)$ for the value of circuit $S$. Value of 1 is 1. The value of a circuit $S = s_0, s_1, \ldots, s_\ell$ where $s_\ell = s_j \ast s_k$ is $v(S_j) \ast v(S_k)$. Here $S_i = s_0, s_1, \ldots, s_j$. Note that each prefix of a circuit is a circuit by definition. If some operations can not be performed (e.g. a division by 0) the value of such a circuit is undefined. Let $X = v(S)$. We say that such AC represents $X$.

It is easy to see that arithmetic circuit representation can be much more compressed than the usual binary representation. A circuit $S = 1, s_1, s_2, \ldots, s_\ell$ where $s_1 = 1 + 1$ and $s_{j+1} = s_j \cdot s_j$ for $j \geq 1$ represents $2^{2^{j-1}}$. On the other hand this example is asymptotically optimal. The following statement can be easily verified by induction.
Statement 1. If \( \frac{p}{q} = v(S) \) where \( p, q \) are integers then \( \max\{p, q\} = O(2^\ell) \).

1.1 The complexity of equality and inequality predicates over AC

Implementation of arithmetic operations with rationals represented by arithmetic circuits is straightforward and can be done in linear time. Formally we write

\[
S(X \ast Y) = S(X), \text{tail}(S(Y)), s_{\ell(S(X)) + \ell(S(Y)) - 1},
\]

where \( s_{\ell(S(X)) + \ell(S(Y)) - 1} := s_{\ell(S(X))} * s_{\ell(S(X)) + \ell(S(Y)) - 2} \).

Here \( \text{tail}(S) \) means a circuit \( S \) without the starting 1.

But how difficult may be the computation of the equality predicate and of the inequality predicate? We state these algorithmic problems formally. We always assume that a circuit is represented by a list of triples \((\ast, j, k)\) where \( \ast \in B \) and \( j, k \) are positive integers. The \( n \)th element in the list corresponds to an assignment \( s_n = s_j \ast s_k \). Thus a circuit of size \( \ell \) is written as a \( O(\ell \log \ell) \) binary word.

**Predicate AC= (B).** It is true for a pair of circuits \( S_1, S_2 \) over the basis \( B \) iff \( v(S_1) = v(S_2) \).

**Predicate AC⩾ (B).** It is true for a pair of circuits \( S_1, S_2 \) over the basis \( B \) iff \( v(S_1) ⩾ v(S_2) \).

**Note.** If \( B \) contains division then these predicates are partially defined (value of some circuits can be undefined). Partially defined predicates are called *promise problems*. Most complexity classes can be easily redefined to include promise problems and most results remain true in this, more general, setting. We omit a discussion of promise problems but indicate that their use is safe in our considerations.

We denote the equality and the inequality predicates over the arithmetic basis by \( AC= \) (resp. \( AC⩾ \)).

It is clear that both predicates fall into the complexity class EXPTIME. In fact, we are able to place them into lower levels of computational hierarchy.

**Theorem 1.** \( AC= \in BPP \).

In other words, the equality check can be performed by probabilistic Turing machine in polynomial time.

We are unable to give an exact characterization of the computational complexity of the second predicate. Computing \( AC⩾ \) looks as a computationally hard problem. An obvious way to solve it is to make all calculations indicated in the circuits that form input of the problem. Using binary representation we need exponentially large memory to do it. Using modular arithmetic it is possible to solve \( AC⩾ \) in polynomial memory.

**Theorem 2.** \( AC⩾ \in PSPACE \).

The proof of Theorem 2 uses some constructions from an NC1 algorithm for comparison integers in modular arithmetic [4, 5].
1.2 Equivalent bases

Two bases $B_1, B_2$ are called “$=$” equivalent (respectively, “$\geq$” equivalent) iff the predicates $AC_=(B_1)$ and $AC_=(B_2)$ (respectively, $AC_>(B_1)$ and $AC_>(B_2)$) are mutually polynomially reducible.

Theorem 3. The following bases are “$=$” and “$\geq$” equivalent: arithmetic, division-free, monotone and $\{+, x \mapsto x^2/2\}$.

The last basis in the list is added for technical purposes. It is used in the reduction of the problem $AC_>$ to the feasibility problem for semidefinite programming.

Reductions of all mentioned bases to the arithmetic basis are straightforward. To prove Theorem 3 we establish reductions in the opposite direction.

Note that if a basis contains a subtraction then general predicate $AC_>(B)$ is reducible to its particular case when one of the compared values is zero. Indeed, suppose we are going to compare $v(S_1)$ and $v(S_2)$. We can merge the circuits $S_1, S_2$ into one circuit. This merged circuit contains all assignments of $S_1, S_2$ and ends by the assignment $d := a - b$, where $a$ and $b$ are the last assignments in the circuits $S_1, S_2$. The same argument can be also applied to the predicate $AC_=(B)$.

All reductions described below have similar form. A circuit $S$ over some basis is converted to a circuit $S'$ over another basis using step-by-step substitution of constant-sized groups of assignments instead of each assignment in $S$.

Lemma 1. $AC_>(\{+, -/, \cdot\})$ (resp. $AC_=(\{+, -/, \cdot\})$) is reducible to $AC_>(\{+, -\})$ (resp. $AC_=(\{+, -\})$).

Proof. Informally speaking, the lemma is very simple: we can keep and transform numerators and denominators separately. Below we present a more detailed description of the reduction.

Let $S$ be a circuit of size $\ell$ over the arithmetic basis. We construct a circuit $S'$ of size $O(\ell)$ over division-free basis in the following way.

The circuit $S'$ consists of four sequences of assignments $A^1, A^2, N, D$. For the assignment $s_i := s_j \pm s_k$ in $S$ add the assignments

$$A^1_i := N_j \cdot D_k, \quad A^2_i := D_j \cdot N_k, \quad D_i := D_j \cdot D_k, \quad N_i := A^1_i \pm A^2_i.$$  

Similarly, for the assignment $s_i := s_j \cdot s_k$ in $S$ add the assignments

$$D_i := D_j \cdot D_k, \quad N_i := N_j \cdot N_k,$$

and for the assignment $s_i := s_j/s_k$ in $S$ add the assignments

$$D_i := D_j \cdot N_k, \quad N_i := N_j \cdot D_k.$$

The last assignment in the circuit $S'$ is

$$s'_{N} := N_\ell \cdot D_\ell.$$

It is easy to see that

$$v(S) = \frac{v(N_\ell)}{v(D_\ell)}.$$
So, \( v(S) \geq 0 \) iff \( s'_N \geq 0 \).

Note that due to our assumptions the case \( D_i = 0 \) is impossible. So, the same reduction is valid for the predicate \( AC\).\( \square \)

**Lemma 2.** \( AC_\succ\{(+,-,\cdot)\} \) (resp. \( AC_\equiv\{(+,-,\cdot)\} \)) is reducible to \( AC_\succ\{(+,\cdot)\} \) (resp. \( AC_\equiv\{(+,\cdot)\} \)).

**Proof.** Informally, we do the same trick as above using equalities

\[
\begin{align*}
(A - B) + (C - D) &= (A + C) - (B + D), & (1) \\
(A - B) - (C - D) &= (A + D) - (B + C), & (2) \\
(A - B) \cdot (C - D) &= (A \cdot C + B \cdot D) - (B \cdot C + A \cdot D). & (3)
\end{align*}
\]

Now let consider the details of the reduction.

At first we convert an input \((S_1, S_2)\) of \( AC_\succ\{(+,-,\cdot)\}\) into \((S, 0)\) as it was explained above.

Then we construct two circuits \( L, R \) such that \( v(S) = v(L) - v(R) \). So, \( v(S) \geq 0 \) iff \( v(L) \geq v(R) \) as well as \( v(S) = 0 \) iff \( v(L) = v(R) \). Again, the same reduction will work for both predicates. The size of \( L, R \) will be \( O(s(S)) \) and they will be the circuits over \{+,\cdot\}.

Both circuits \( L \) and \( R \) consist of six sequences of assignments \( L^1, L^2, L^3, R^1, R^2, R^3 \).

For the assignment \( s_j = s_j + s_k \) in \( S \) add assignments to \( R \)

\[
L_i^1 := L^1_j + L^1_k, \quad R_i^1 := R^1_j + R^1_k
\]

(see Eq. (1)).

For the assignment \( s_j = s_j - s_k \) in \( S \) add the assignments to \( R \)

\[
L_i^1 := L^1_j + R^1_k, \quad R_i^1 := R^1_j + L^1_k
\]

(see Eq. (2)).

For the assignment \( s_j = s_j \cdot s_k \) in \( S \) add the assignments to \( R \)

\[
L_i^3 := L^3_j \cdot L^3_k, \quad L_i^2 := R^1_j \cdot R^1_k, \quad L_i^1 := L^2_i + L^3_i, \\
R_i^3 := R^3_j \cdot R^3_k, \quad R_i^2 := L^1_j \cdot R^1_k, \quad R_i^1 := R^1_i + R^3_i
\]

(see Eq. (3)).

The circuit \( L \) has the same structure except \( L_i \) and \( R_i \) assignments (or groups of assignments) are interchanged. By induction, we see that for each \( i \) the value of \( s_i \) is \( v(L_i^1) - v(R_i^1) \). \( \square \)

**Lemma 3.** \( AC_\succ\{(+,\cdot)\} \) (resp. \( AC_\equiv\{(+,\cdot)\} \)) is reducible to \( AC_\succ\{(+, x \mapsto x^2/2)\} \) (resp. \( AC_\equiv\{(+, x \mapsto x^2/2)\} \)).

**Proof.** Let \( S \) be a circuit over the monotone basis. We construct a circuit \( S' \) over the basis \{+, \( x \mapsto x^2/2 \)\} consisting of 10 series of assignments \( P, N, A^t, t \in [1, 8] \) such that for each \( k \) the equality

\[
v(s_k) = v(P_k) - v(N_k)
\]

(4)
holds. The construction is step-by-step substitution as in the above lemmas.

For the assignment \( s_i := s_j + s_k \) in \( S \) add the assignments
\[
P_i := P_j + P_k, \quad N_i := N_j + N_k
\]
and for the assignment \( s_i := s_j \cdot s_k \) add the assignments
\[
A_1^i := P_j + P_k, \quad A_2^i := N_j + N_k, \quad A_3^i := P_j + N_k, \quad A_4^i := N_j + P_k,
\]
\[
A_5^i := (A_1^i)^2/2, \quad A_6^i := (A_2^i)^2/2, \quad A_7^i := (A_3^i)^2/2, \quad A_8^i := (A_4^i)^2/2,
\]
\[
P_i = A_5^i + A_6^i, \quad N_i = A_7^i + A_8^i.
\]

In the latter case the following equations hold
\[
v(P_i) = (v(P_j) + v(P_k))^2/2 + (v(N_j) + v(N_k))^2/2, \quad v(N_i) = (v(P_j) + v(N_k))^2/2 + (v(N_j) + v(P_k))^2/2. \tag{7}
\]

Eq. (4) is verified by induction using Eq. (7)
\[
v(s_i) = (v(P_j) - v(N_j))(v(P_k) - v(N_k)) =
\]
\[
(v(P_j) \cdot v(P_k) + v(N_j) \cdot v(N_k)) - (v(P_j) \cdot v(N_k) + v(N_j) \cdot v(P_k)) =
\]
\[
((v(P_j) + v(P_k))^2/2 + (v(N_j) + v(N_k))^2/2) - ((v(P_j) + v(N_k))^2/2 + (v(N_j) + v(P_k))^2/2) =
\]
\[
v(P_i) - v(N_i). \tag{8}
\]

Now we are able to construct a reduction. Take an instance \( (S_1, S_2) \) of the problem \( AC_{\geq} \{+, \cdot\} \). Convert the circuits \( S_1, S_2 \) into the circuits \( S^1, S^2 \) over the basis \( \{+, x \mapsto x^2/2\} \) as described above. Join them into one circuit \( S \). The circuit \( S' \) is an extension of \( S \) by the assignment \( f' := P_{s_1} + N_{s_2} \). Similarly, the circuit \( S'' \) is an extension of \( S \) by the assignment \( f'' := P_{s_2} + N_{s_1} \). Here \( s_1 \) is a size of \( S_1 \) and \( s_2 \) is a size of \( S_2 \).

A reduction is given by the mapping \( (S_1, S_2) \mapsto (S', S'') \). From Eq. (4) we see that it also works for both types of predicates mentioned in the lemma.

\[\square\]

## 2 Reduction of the problem \( AC_{\geq} \) to SDFP

Linear optimization on the intersection of the cone of positive semidefinite matrices with an affine subspace of matrices is called \emph{semidefinite programming} (SDP). Let denote SDFP the corresponding feasibility problem: to check whether the cone of positive semidefinite matrices has nonempty intersection with affine subspace of subspace.

Semidefinite feasibility problem (SDFP) can be stated in the following form.

\[\textbf{Input:}\ a list \( Q_0, \ldots, Q_m \) of symmetric \( (n \times n) \) matrices with rational entries. Matrices are represented by lists of entries, each entry is represented by a pair (numerator, denominator), integers are given in binary.
Output: ‘yes’ if there exist reals \(x_1, \ldots, x_m\) such that \(Q = Q_0 + \sum_{i=1}^m x_i Q_i\) is a positive semidefinite matrix, otherwise the output is ‘no’.

The proof below uses Ramana’s results on the exact duality theory for SDP [12]. Ramana found a special form of dual program for SDP. It is called the extended Lagrange – Slater dual program (ELSD). Ramana proved that

- ELSD can be constructed from the primal program in polynomial time;
- if both the primal and the dual are feasible then their optimum values are equal.

**Theorem 4.** The problem \(AC \geq (\{+, x \mapsto x^2/2\})\) is reducible to SDFP.

Applying Theorem 3 we have reductions of the problem \(AC \geq\) over all bases discussed in the previous section to SDFP.

**Proof.** The first step is to represent a circuit value as an optimal value of some semidefinite program in the form

\[
t \to \inf, \\
\text{semidefinite conditions on } t \text{ and other variables.} \tag{9}
\]

Let \(S\) be a circuit over the basis \(\{+, x \mapsto x^2/2\}\). We construct a semidefinite program \(P(S)\) such that for each assignment \(s_i\) there is a variable \(x_i\) in \(P(S)\) and a matrix \(M_i\). Matrices \(M_i\) are built as follows

\[
s_i := s_j + s_k \quad \rightarrow \quad M_i = (x_i - x_j - x_k) \\
s_i := s_j^2/2 \quad \rightarrow \quad M_i = \begin{pmatrix} 2x_i & x_j \\ x_j & 1 \end{pmatrix}
\]

Note that in the former case \(M_i \succeq 0\) iff \(x_i \geq x_j + x_k\) and in the latter \(M_i \succeq 0\) iff \(x_i \geq x_j^2/2\).

Let \(\ell\) be a size of \(S\). Consider a SDP

\[
x_\ell \to \inf, \\
M_i \succeq 0 \text{ for each } i, \\
x_0 = 1. \tag{10}
\]

To convert the program (10) to the standard form we replace an equality \(x_0 = 1\) by inequality

\[
\begin{pmatrix} x_0 - 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -x_0 + 1 \end{pmatrix} \succeq 0. \tag{11}
\]

The optimal value of \(P(S)\) is the value of \(S\). Indeed, all operations in the process of circuit evaluation are monotone with respect to each variable. So, any feasible solution of (10) satisfies the condition \(x_\ell \geq v(S)\). On the other hand, \(x_i = v(s_i)\) is a feasible solution.

Take now an instance of \(AC \geq (\{+, x \mapsto x^2/2\})\). It’s input is a pair of circuits \(S^{(1)}, S^{(2)}\). The positive answer in the problem \(AC \geq (\{+, x \mapsto x^2/2\})\) means that
\(v(S^{(1)}) \geq v(S^{(2)})\) which is equivalent to \(-v(S^{(2)}) \geq -v(S^{(1)})\). The value \(-v(S^{(2)})\) is an optimal value for SDP

\[
\begin{align*}
-x^{(2)}_\ell &\to \sup, \\
M^{(2)}_i &\succeq 0, \\
x^{(2)}_0 &= 1.
\end{align*}
\] (12)

To represent \(-v(S^{(1)})\) as the infimum of SDP we use the extended Lagrange–Slater dual program to the program of type (10). In Ramana’s paper ELSD has a mixed form (linear equations are permitted). To convert it to the standard form each linear equation should be replaced by a positive semidefinite condition on a \((2 \times 2)\) matrix as in Eq. (11). After conversion ELSD can be written as follows

\[
\begin{align*}
c(Y) &\to \inf, \\
D &\succeq 0.
\end{align*}
\] (13)

Here \(c(Y)\) is a linear functional on dual variables \(Y\). Thus, the program

\[
\begin{align*}
-x^{(2)}_\ell - c(Y) &\geq 0, \\
M^{(2)}_i &\succeq 0, \\
x^{(2)}_0 &= 1, \\
D &\succeq 0.
\end{align*}
\] (14)

is feasible iff \(-v(S^{(2)}) \geq -v(S^{(1)})\).

This gives a reduction of \(AC \supseteq \{+ : x \mapsto x^2/2\}\) to SDFP. □

### 3 The proof of Theorem 1

The idea is to check the equality modulo random number.

It is easy to compute a remainder of a circuit value modulo ‘short’ number (represented in binary). Addition, multiplication and integer division are made in polynomial time by standard algorithms.

We show that if two circuits have different values then with big enough probability they have different residues modulo random number.

We present a BPP-algorithm for the monotone basis. By theorem 3 we conclude that \(AC = \in BPP\).

**The probabilistic algorithm for AC = \{+, \cdot\}.**

**Input:** a pair \((S_1, S_2)\) where \(S_1, S_2\) are circuits over the monotone basis. Let \(\ell\) be the maximal size of the circuits.

**Step 1.** Set \(B\) to \(2^{2\ell}\).
Step 2. Choose a random integer $m$ from the uniform distribution on the interval $[1, B]$.

Step 3. Compute $r_1 = v(S_1) \mod m$, $r_2 = v(S_2) \mod m$ by making all operations indicated in the circuits $S_1, S_2$ modulo $M$.

Step 4. If $r_1 = r_2$ then output ‘$v(S_1) = v(S_2)$’ else output ‘$v(S_1) \neq v(S_2)$’.

It is clear from the description that the algorithm uses $2\ell$ random bits and runs in polynomial time.

Claim 1. If $v(S_1) = v(S_2)$ then the algorithm outputs ‘$v(S_1) \neq v(S_2)$’ with probability 0.

This is clear.

Claim 2. If $v(S_1) \neq v(S_2)$ then the algorithm outputs ‘$v(S_1) \neq v(S_2)$’ with probability at least $(2\ell)^{-1}$.

Claim 2 is derived from Lemma 4 stated below.

To fit the common definition of the class BPP we need to amplify success probability by the standard procedure of multiple repetitions of the algorithm. Claims show that the gap to be amplified is $\Omega(n^{-1})$ where $n$ is an input size. So, an amplification can be made by $\text{poly}(n)$ repetitions and the resulting algorithm runs in polynomial time.

Now we need a lower bound of the least common multiple of integers taken from an interval $[1, B]$ provided that we take sufficiently many integers.

Lemma 4. Let

$$N_\varepsilon(B) = \min \{\text{lcm}(x_1, \ldots, x_r) : x_i \in [1, B], r > (1 - \varepsilon)B\}.$$ 

Then for all $\varepsilon < (2 \ln B)^{-1}$ we have

$$N_\varepsilon(B) > 2^\Omega(B/\ln B).$$

Proof. Consider a set of integers $R = \{x_1, \ldots, x_r\} \subseteq [1, B]$ such that $|R| > (1 - \varepsilon)B$. Factorize the least common multiple of these integers

$$X = \text{lcm}(x_1, \ldots, x_r) = p_1^{a_1} \cdots p_s^{a_s}.$$

We are going to show that $s = \Omega(B/\ln B)$. Then the lemma will follow from trivial bound $X > 2^s$.

The prime number theorem gives an asymptotic

$$\pi(n) \sim \frac{n}{\ln n}$$

for the prime counting function $\pi(n) = \#\{p : p \leq n, p \text{ prime}\}$.

Let $\hat{R} = [1, B] \setminus \{x_1, \ldots, x_r\}$. At least $\pi(B) - s$ primes in the interval $[1, B]$ belong to the set $\hat{R}$. From an inequality $\pi(B) - s \leq B - r$ we conclude that

$$(1 - \varepsilon)B < r \leq B - \pi(B) + s$$

and $s = \Omega(B/\ln B)$ for $\varepsilon < (2 \ln B)^{-1}$. \qed
To complete the proof of Theorem 1 we derive Claim 2 from Lemma 4.

W.l.o.g. assume that \( v(S_1) > v(S_2) \). From upper bounds of circuit value we have

\[
\Delta = v(S_1) - v(S_2) \leq 2^{1+2^\ell}.
\] (16)

Let \( R \) be a set of integers such that \( m \in R \) iff \( m \in [1, B] \) and

\[
(v(S_1) - v(S_2)) \mod m = 0.
\]

Note that if \( m \notin R \) then the algorithm outputs \( 'v(S_1 \neq v(S_2)' \).

The set \( R \) cannot be large. At first we note that \( \Delta \) is a multiple of \( \text{lcm}_{x \in R} x \).

Suppose that \( \#R > (1 - \varepsilon)B \) for \( \varepsilon = (2^\ell - 1) < (2 \ln B)^{-1} \). We have \( \varepsilon = (2^\ell - 1) = (2 \log_2 B)^{-1} \). Thus, Lemma 4 implies

\[
\Delta \geq \text{lcm}_{x \in R} x > 2^{\Omega(\ln B)} = 2^{\Omega(2^\ell / \ell)} > \Omega(2^{2^{1.5^\ell}})
\]

and we come to the contradiction with (16). This contradiction shows that \( \#R < (1 - (2^\ell - 1)/B) \). Claim 2 follows from this bound.

4 The proof of Theorem 2

We prove Theorem 2 by adjusting an \( \text{NC}^1 \)-algorithm for integer comparison by Davida and Litow [5]. So, we partially reproduce arguments from [4, 5].

We start by some notation. Let \( p_1 = 3, p_2, \ldots, p_m \) be the first \( m \) odd primes. We denote the least nonnegative residue of \( x \) modulo \( n \) by \( [x]_n: x \equiv [x]_n \pmod{n}, 0 \leq [x]_n < n \). For the rest of the section we set \( M = p_1 p_2 \ldots p_m, M_i = M/p_i, \)

\( x_i = [x]_{p_i}, \xi_i = [xM^{-1}]_{p_i} \).

We denote the fractional part \( x - [x] \) of \( x \) by \( \{x\} \).

By the Chinese remainder theorem any integer \( 0 \leq x < M \) is uniquely represented by \( x_i \). For any integer \( x \) the following equality holds

\[
\sum_{i=1}^{m} M_i \cdot \xi_i = \rho(x) \cdot M + [x]_M.
\] (17)

Since \( M_i \xi_i < M \), we have \( 0 \leq \rho(x) < m \). The integer \( \rho(x) \) is called the rank of \( x \) with respect to \( \{p_1, \ldots, p_m\} \).

Dividing both sides of Eq. (17) by \( M \) we get

\[
\sum_{i=1}^{m} \frac{\xi_i}{p_i} = \frac{\rho(x)}{M} + \frac{[x]_M}{M}.
\] (18)

Eq. (18) gives a way to compute rank \( \rho(x) \) by approximating left-hand side. Let choose an integer \( h \) such that \( m/2^h < 1/4 \). For each \( i \) we take a \( (2^{-h}) \)-approximation of \( \xi_i/p_i \), i.e.

\[
\frac{s_i}{2^h} \leq \frac{\xi_i}{p_i} < \frac{s_i + 1}{2^h}, \quad s_i, h \in \mathbb{Z}_+, \quad \text{where} \quad s_i = \left\lfloor \frac{2^h \xi_i}{p_i} \right\rfloor.
\] (19)
Let $\sigma(x) = \sum_{i=1}^{m} s_i 2^{-i}$. Then, by summation of Eqs. (19) for all $i$ we get

$$\{\sigma(x)\} = \rho(x) - [\sigma(x)] + \frac{[x]_M}{M} - \alpha \quad \text{and} \quad 0 \leq \alpha < 1/4. \quad (20)$$

Now we reproduce Lemma 2.3 from [4].

**Proposition 1.** The following holds.

1. If $\{\sigma(x)\} \leq 3/4$ then $\rho(x) = [\sigma(x)]$.

2. If $1/4 \leq [x]_M/M \leq 3/4$ then $\{\sigma(x)\} \leq 3/4$ and $\rho(x) = [\sigma(x)]$.

3. If $[x]_M/M > 1/2$ then $\rho(x) = [\sigma(x)]$.

All these facts are easily derived from Eq. (20). Using them we obtain an analogue of Lemma 2.5 from [4].

**Lemma 5.** Let $k^*(x) = \min(k : k \geq 0 \text{ and } \{\sigma([2^k x]_M)\} \leq 3/4)$. For any $x$ the following holds.

- $k^*(x) < \infty$.
- $k^*(x) = 2^{\text{poly}(m)}$.
- Suppose that $k^*(x) > 0$. Then $[x]_M/M < 1/2$ if $\{[2^{k^*(x)} x]_M\} = 0$.

**Proof.** For $x$ such that $1/4 \leq [x]_M/M \leq 3/4$ we get $k^*(x) = 0$ from Proposition 1.2. So, for this case the lemma is trivial.

Suppose that $[x]_M/M < 1/4$ and $k^*(x) > 0$. Choose $\tilde{k} \geq 1$ such that

$$\frac{1}{2^{k+2}} < \frac{[x]_M}{M} < \frac{1}{2^{k+1}}. \quad (21)$$

Applying Proposition 1.2 we conclude that $\{\sigma([2^{\tilde{k}} x]_M)\} \leq 3/4$. Therefore $k^*(x) \leq \tilde{k} < \infty$. The prime number theorem (15) has an equivalent form

$$p_k \sim k \ln k.$$ 

It gives a bound $M = 2^{\text{poly}(m)}$. Thus $k^*(x) = 2^{\text{poly}(m)}$.

For $k \leq k^*(x) \leq \tilde{k}$ we use the upper bound of (21) to show that $2^{k} [x]_M < M$. It implies that $2^{k^*(x)} [x]_M = 2^{k^*(x)} [x]_M$ is even.

Now suppose that $[x]_M/M > 3/4$ and $k^*(x) > 0$. Let $y = M - [x]_M = [y]_M$.

So, $[y]_M/M < 1/4$ and $[2^{k} y]_M = [-2^{k} x]_M = M - [2^{k} x]_M$. Repeating the above argument we choose $\tilde{k} \geq 1$ such that

$$\frac{1}{2^{k+2}} < \frac{[y]_M}{M} < \frac{1}{2^{k+1}}. \quad (22)$$

Since $[2^{\tilde{k}} x]_M = M - [2^{\tilde{k}} y]_M$, we conclude that $k^*(x) = 2^{\text{poly}(m)}$ and $[2^{k^*(x)} [x]_M]_M = M - 2^{k^*(x)} [y]_M$ is odd. \qed
The algorithm described below compares integers represented by circuits over the division-free basis \{+, -, \cdot\}. As was explained above, in this case comparing values of two circuits is reduced to comparing a circuit value with 0, i.e. to computing the sign of the circuit value.

Let \(S\) be a circuit of size \(\ell\). As it was mentioned above, \(|v(S)| < 2^{2\ell}\). There are \(m = \pi(2^{2\ell}) - 1\) primes from 3 to \(2^{2\ell}\). Due to the prime number theorem
\[
2^{1.5\ell} < m < 2^{2\ell}.
\]
By the Chinese remainder theorem the residues modulo these primes represent uniquely all circuit values for circuits of size \(\ell\). Moreover, \(|v(S)| < M/4\) in these settings.

The algorithm to check \(v(S) > 0\).

Input: \(S\) — a circuit of size \(\ell\) over the division-free basis.

Step 1. Let \(h = 2\ell\).

Step 2. Compute \(k^*(v(S))\).

Step 3. If \(k^*(v(S)) > 0\) go to Step 6.

Step 4. Compute \(a_1 := [v(S)]_2, a_2 := [[v(S)]_M]_2\).

Step 5. If \(a_1 = a_2\) then output ‘yes’ else output ‘no’.

Step 6. Compute \(b := [[2^{k^*(v(S))}v(S)]_M]_2\).

Step 7. If \(b = 0\) then output ‘yes’ else output ‘no’.

Claim 1. The algorithm always gives a correct answer.

Proof. Suppose that the algorithm finishes at Step 5. If \(v(S) > 0\) then \(v(S) = [v(S)]_M\) which implies \([v(S)]_2 = [[v(S)]_M]_2\). If \(v(S) < 0\) then \([v(S)]_2 + [[v(S)]_M]_2 = [M]_2 = 1\). In both cases the algorithm gives a correct answer.

Suppose that the algorithm finishes at Step 7. Note that in this case \(k^*(v(S)) > 0\). So, Lemma 5 can be applied. The algorithm gives a correct answer due to the following observation: \(x > 0\) iff \([x]_M < M/2\) assuming \(0 \leq x < M/2\).

Claim 2. The algorithm runs in polynomial memory.

Proof. Computing \([v(S)]_2\) takes a polynomial time as mentioned above. The algorithm computes \([[2^{k^*(v(S))}]_M]_2\) by using Eq. (17) modulo 2
\[
\sum_{i=1}^{m} \xi_i \equiv \rho(x) + [x]_M \pmod{2}.
\]
(23)
To compute \([x]_M \pmod{2}\) we need to compute \(\xi_i\) and \(\rho(x)\). Computing \(2^{k^*(v(S))} \pmod{n}\) is possible in polynomial time. To compute \(M_i^{-1} \pmod{p_i}\) one can take all primes from 3 to \(2^{2\ell}\) one by one, compute an inverse residue modulo \(p_i\) and multiply it by the current product. This as well as summation of \(\xi_i\) can be done in polynomial memory.
For each \(i\) an \((2^{-2\ell})\)-approximation to \(\xi_i/p_i\) is computed in polynomial time. Summation of these approximations is directly implemented in polynomial memory. So, it is possible to compute \(\lfloor \sigma(2^k v(S)) \rfloor\) and \(\{\sigma(2^k v(S))\}\) in polynomial memory (for \(k = 2^{\operatorname{poly}(\ell)}\)). Starting from \(k = 0\) and incrementing \(k\) until \(k^*(v(S))\) is found may be implemented in polynomial memory. Having all these data, it is easy to compute \(\rho(2^{k^*(v(S))} v(S))\) and \([2^{k^*(v(S))} v(S)]_M\).

\[\Box\]

5 Open questions

The main open question here is the complexity of SDFP. We suggest the problem \(AC_{\geq}\) as a ‘lowerbound’ for SDFP. If it is hard then SDFP is hard too. The maximal result in this direction would be PSPACE-hardness of SDFP. Thus from the complexity viewpoint it is very interesting to put SDFP in PSPACE if possible.

Complexity of \(AC_{\geq}\) itself is the next question. To be a good ‘lowerbound’ it should be hard. But up to our current knowledge nothing prohibits inventing an efficient algorithm for its solution. Such an algorithm would be interesting by other reasons. It would justify using circuit representation for rationals instead of binary system in complexity theoretic questions when time is estimated up to a polynomial slowdown. It might simplify complexity analysis of numeric algorithms very much. So, any definite result about \(AC_{\geq}\) would lead to interesting conclusions.

From algorithmic point of view some intrinsic difficulty of numerical algorithms is related to nonconstructive nature of real numbers. Though some computational models, e.g. BSS model of computation over arbitrary field initially proposed by Smale [2], directly operate with real numbers and thus completely ignore the question of number representation. Presently the relation of BSS model to the common model of algorithmic complexity is not well understood. In particular, in BSS model SDFP falls into the complexity class \(NP_{\mathbb{R}}\) — an analogue of the class NP. Referring to our way of number representation it is natural to ask about the inclusion SDFP \(\in NP^{AC_{\geq}}\). This question is also open.

In the opposite direction, it may be interesting to represent numbers as optimal solutions of SDP. For example, the well known \(n!\) conjecture by Shub and Smale [15] in our setting asks about monotone circuit complexity of \(n!\). It follows from the reduction in Theorem 2 that monotone circuit complexity is lowerbounded by the dimension of a SDP representing \(n!\) as an optimal solution. Is this dimension polylogarithmic on \(n\)?

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