Bahadur representations for the bootstrap median absolute deviation and the application to projection depth weighted mean

Qing Liu\textsuperscript{a,b}, Xiaohui Liu\textsuperscript{a,b} 1, Zihao Hu\textsuperscript{a,b}

\textsuperscript{a} School of Statistics, Jiangxi University of Finance and Economics, Nanchang, Jiangxi 330013, China
\textsuperscript{b} Research Center of Applied Statistics, Jiangxi University of Finance and Economics, Nanchang, Jiangxi 330013, China

\textsuperscript{1}Corresponding author’s email: csuliuxh912@gmail.com.
Abstract

Median absolute deviation (hereafter MAD) is known as a robust alternative to the ordinary variance. It has been widely utilized to induce robust statistical inferential procedures. In this paper, we investigate the strong and weak Bahadur representations of its bootstrap counterpart. As a useful application, we utilize the results to derive the weak Bahadur representation of the bootstrap sample projection depth weighted mean—a quite important location estimator depending on MAD.

Key words: Bootstrap MAD; Bootstrap projection depth weighted mean; Bahadur representation

2000 Mathematics Subject Classification Codes: 62F10; 62F40; 62F35

1 Introduction

Let $F$ be the distribution function of $X$. The related median $v = \text{Med}(X)$ is then defined as $F^{-1}(1/2) = \inf \{ x : F(x) \geq 1/2 \}$ which satisfies

$$F(v-) \leq 1/2 \leq F(v).$$

(1)

Suppose $X_1, X_2, \cdots, X_n \overset{iid}{\sim} F$ and let $X_{1:n}, X_{2:n}, \cdots, X_{n:n}$ be the related order statistics. The sample median is usually defined as

$$\text{Med}_n = \frac{X_{\lfloor n+1 \rfloor/2:n} + X_{\lfloor n+2 \rfloor/2:n}}{2},$$

where $\lfloor \cdot \rfloor$ denotes the floor function. In the literature, the sample median is known as its high robustness properties and usually serves as an alternative to the sample mean in the location setting (Small, 1990).

Based on $\text{Med}_n$ above, the sample MAD is defined as

$$\text{MAD}_n = \frac{W_{\lfloor n+1 \rfloor/2:n} + W_{\lfloor n+2 \rfloor/2:n}}{2},$$

(2)

where $W_{i:n}, i = 1, 2, \cdots, n,$ denote the order statistics related to $W_1 = |X_1 - \text{Med}_n|, W_2 = |X_2 - \text{Med}_n|, \cdots, W_n = |X_n - \text{Med}_n|$. Clearly, the population version, say $\xi$, of $\text{MAD}_n$ is the median of the distribution $G$ of $|X - v|$, i.e.,

$$G(y) = P(|X - v| \leq y) = F(v + y) - F(v - y\text{--}), \ y \in \mathbb{R}. \quad (3)$$

Similar to the sample median, $\text{MAD}_n$ is a famous robust scatter measure and hence a desirable alternative to the sample variance when outliers are present (Mazumder and Serfling, 2009). They together are widely used in statistics to construct some statistical inferential procedures, which have high breakdown point robustness. Among them, one famous example is the projection depth studied by Liu (1992); Zuo (2003), which depends on a combination of
one location estimator and one scale estimator with the most commonly used combination being (Med, MAD). Based on the projection depth, a few desirable estimators, as well as some inferential procedures, have been developed in the past decades; see, e.g., Zuo (2003); Zuo et al. (2004); Zuo (2006); Dutta and Ghosh (2012) and references therein for details.

One well-known projection depth based estimator is the projection depth weighted mean, which includes the famous Stahel-Donoho estimator as a special case (Donoho, 1982; Stahel, 1981). It turns out that this estimator enjoys very high efficiency and robustness (Zuo et al., 2004). Especially, it is interesting to find by Zuo (2010) that combining the projection depth weighted mean with the bootstrap procedure, it is possible to construct a confidence interval which is even more optimal than the classical $t$ confidence interval in the sense of having better finite sample performance. Nevertheless, the good property of the bootstrap sample projection depth weighted mean of Zuo (2010) was only confirmed by some simulated examples, having no theoretical argument related to its limit distribution as far to the best of our knowledge. This motivates us to conduct the current research.

To achieve this, we need first to investigate the asymptotic properties of the related bootstrap median and bootstrap MAD. In the literature, it is known that the Bahadur representation, named after Bahadur (1966), is a useful tool to study the asymptotic properties of an estimator, because it provides not only an approximation to the estimator in the form of a sum of independent variables, but also a higher-order remainder from which one can see the convergence rate of the estimator as the sample size $n$ increases. Much attention has been paid to this tool since its introduction; see, e.g. Kiefer (1967); He and Shao (1996); Wu (2005); Wendler (2011) for details. Recently, Mazumder and Serfling (2009) considered the Bahadur representation of the sample MAD, and Zuo (2015) considered the Bahadur representations of the bootstrap sample quantiles.

In view of this, we will first consider the Bahadur representations for the bootstrap sample MAD, and then apply the results to the case of bootstrap sample projection depth weighted mean. Given the random sample $X_1, X_2, \ldots, X_n$ above, let $X_1^*, X_2^*, \ldots, X_n^*$ be the bootstrap sample from its empirical function $F_n$. Hereafter, denote $F^*_n$, Med$_n^*$ and MAD$_n^*$ as the empirical function, median and MAD corresponding to $X_1^*, X_2^*, \ldots, X_n^*$, respectively.

Although the definition of MAD is essentially a quantile of the absolute deviation values, the result of Zuo (2015) cannot be trivially applied directly to the bootstrap sample MAD, as well as the bootstrap sample projection weighted mean, since it involves in the sample median, which depends on all of the bootstrapping observations; see (2).

For simplicity, we introduce some frequently used notations before starting the discussions.
For any random event \( A \), denote \( P^*(A) = P(A|X_1, X_2, \ldots, X_n) \), i.e., the conditional probability. \( \epsilon \) is any given positive constant, whose value may be not the same at different places. The term ‘a.s.’ stands for ‘almost surely’. For any fixed integers \( l \) and \( m \) such that \( \left\lfloor \frac{n}{2} \right\rfloor \geq l \geq 1 \) and \( \left\lfloor \frac{n}{2} \right\rfloor \geq m \geq 1 \), denote \( \hat{v}_{n,l} = X_{\left\lfloor \frac{n+l}{2} \right\rfloor,n} \), \( \hat{\xi}_{n,m,l} = W_{\left\lfloor \frac{n+m}{2} \right\rfloor,n,l} \), with \( W_{1,n,l} \leq \ldots \leq W_{n,n,l} \) the ordered statistics of \( W_{i,l} = |X_i - \hat{v}_{n,l}| \), \( 1 \leq i \leq n \). Their bootstrap counterparts will be denoted by \( \hat{v}^*_{n,l} = X^*_{\left\lfloor \frac{n+l}{2} \right\rfloor,n} \) and \( \hat{\xi}^*_{n,m,l} = W^*_{\left\lfloor \frac{n+m}{2} \right\rfloor,n,l} \) respectively. Without confusion, we assume that all \( l \) and \( m \) are fixed and satisfy \( \left\lfloor \frac{n}{2} \right\rfloor \geq l, m \geq 1 \) in the sequel.

The rest of this paper is organized as follows. Section 2 states the strong Bahadur representation of the bootstrap sample MAD, while its weak Bahadur representation is given in Section 3. Based on these representations and the result of Zuo (2015), we further derive the joint distribution of the bootstrap sample median and MAD in Section 4. As an application, we employ these results to further derive the weak Bahadur representation, as well as the limit distribution, of the bootstrap projection depth weighted mean. Some concluding remarks end this paper.

2 Strong Bahadur representation for the bootstrap MAD

In this section, we consider the strong Bahadur representation for the bootstrap sample MAD under a twice differentiable condition. Similar representations for the sample MAD can be found in Mazumder and Serfling (2009). Before proceeding to the derivation of the main result, we need several preliminary lemmas as follows.

Lemma 1. (Hoeffding; see Serfling (1980)) Let \( Y_1, Y_2, \ldots, Y_n \) be independent random variables satisfying \( P(a \leq X_i \leq b) = 1 \), each \( i \), where \( a < b \). Then for \( t > 0 \),

\[
P \left( \sum_{i=1}^{n} (Y_i - E(Y_i)) \geq nt \right) \leq e^{-2nt^2/(b-a)^2}.
\]

The detailed proof of the Hoeffding inequality was given in Serfling (1980). Relying on it, we are able to show the following useful probability inequalities given in Lemmas 2-3.

Lemma 2. Let \( v = F^{-1}(1/2) \) be the unique solution to (1), then for any \( \epsilon > 0 \), fixed integer \( l \) and sufficiently large \( n \), we have

\[
P \left( |\hat{v}^*_{n,l} - v| > \epsilon \right) \leq 2e^{-\sqrt{2nd^2}},
\]

where \( d_{\epsilon,n} = \min\{a_{0,l},b_{0,l}\} \) with

\[
a_{0,l} := a_0(\epsilon, l) = F \left( v + \frac{\epsilon}{2} + \frac{n+l}{2} - \left\lfloor \frac{n+l}{2} \right\rfloor - 1 \right) / n,
\]

\[
b_{0,l} := b_0(\epsilon, l) = 1 - F \left( v - \frac{\epsilon}{2} - \frac{n+l}{2} + \left\lfloor \frac{n+l}{2} \right\rfloor - 1 \right) / n.
\]
Proof of Lemma 2. By Hoeffding’s inequality, we have

\[
\begin{align*}
\mathbb{P}^\ast(\hat{v}_{n,l}^* > v + \frac{\epsilon}{2}) &= \mathbb{P}^\ast\left(nF_n^\ast(v + \frac{\epsilon}{2}) \leq \left\lfloor \frac{n + l}{2} \right\rfloor - 1 \right) \\
&= \mathbb{P}^\ast\left(\sum_{i=1}^{n} I\left(X_i^* > v + \frac{\epsilon}{2}\right) \geq n - \left(\left\lfloor \frac{n + l}{2} \right\rfloor - 1 \right) \right) \\
&= \mathbb{P}^\ast\left(\sum_{i=1}^{n} I\left(X_i^* > v + \frac{\epsilon}{2}\right) - \sum_{i=1}^{n} (1 - F_n(v + \frac{\epsilon}{2})) \geq nF_n(v + \frac{\epsilon}{2}) - \left(\left\lfloor \frac{n + l}{2} \right\rfloor - 1 \right) \right) \\
&\leq \exp\left\{ -2n\left(F_n(v + \frac{\epsilon}{2}) - \frac{\left\lfloor \frac{n + l}{2} \right\rfloor - 1}{n} \right)^2 \right\}.
\end{align*}
\]

By the Glivenko-Cantelli theorem, \(F_n(v + \frac{\epsilon}{2}) \to F(v + \frac{\epsilon}{2})\), a.s. Thus for sufficiently large \(n\),

\[
F_n(v + \frac{\epsilon}{2}) - \frac{\left\lfloor \frac{n + l}{2} \right\rfloor - 1}{n} > \left(F\left(v + \frac{\epsilon}{2}\right) - \frac{\left\lfloor \frac{n + l}{2} \right\rfloor - 1}{n} \right) / \sqrt{2} > 0, \quad \text{a.s.}
\]

by noting that \(F(v + \frac{\epsilon}{2}) > \frac{1}{2}\) for any \(\epsilon > 0\). Hence

\[
\mathbb{P}(\hat{v}_{n,l}^* > v + \frac{\epsilon}{2}) = \mathbb{E}\left[\mathbb{P}^\ast(\hat{v}_{n,l}^* > v + \frac{\epsilon}{2})\right] \leq e^{-\sqrt{2}n\alpha_0^2(\epsilon,l)}.
\]

Similar discussion leads to

\[
\mathbb{P}(\hat{v}_{n,l}^* < v - \frac{\epsilon}{2}) = \mathbb{E}\left[\mathbb{P}^\ast(\hat{v}_{n,l}^* < v - \frac{\epsilon}{2})\right] \leq e^{-\sqrt{2}n\alpha_0^2(\epsilon,l)}.
\]

Then the conclusion follows. \( \blacksquare \)

Lemma 3. Suppose \(v = F^{-1}(1/2)\) be the unique solution to (1), and \(\xi = G^{-1}(1/2)\) is the unique solution to (3). For fixed \(l, m\) and any \(\epsilon > 0\), when \(n\) is sufficiently large, we have

\[
\mathbb{P}(|\hat{\xi}_{n,m,l}^* - \xi| > \epsilon) \leq 6e^{-\sqrt{2}n\Delta^2_{\epsilon,n}},
\]

where \(\Delta_{\epsilon,n} = \min\{a_{0,l}, b_{0,l}, c_{0,m}, d_{0,m}\}\) with \(a_{0,l}, b_{0,l}\) defined in (4) and (5), and \(c_{0,m}, d_{0,m}\) defined by

\[
\begin{align*}
c_{0,m} := c_{0}(\epsilon,m) &= F(v + \xi + \frac{\epsilon}{2}) - F\left(v - \xi - \frac{\epsilon}{2}\right) - \left(\left\lfloor \frac{n + m}{2} \right\rfloor - 1 \right) / n, \\
d_{0,m} := d_{0}(\epsilon,m) &= \left\lfloor \frac{n + m}{2} \right\rfloor / n - F(v + \xi + \frac{\epsilon}{2}) + F\left(v - \xi - \frac{\epsilon}{2}\right).
\end{align*}
\]
Proof of Lemma 3. Denote by $G_n^*(y)$ the empirical distribution function of $W_{1*}^*, W_{2*}^*, \ldots, W_{n*}^*$. Set $\alpha_n = (\lfloor \frac{n+m}{2} \rfloor - 1)/n$, then we have

$$
P^* \left( \xi_{n,m,l}^* > \xi + \epsilon \right) = P^* \left( W_{\lfloor \frac{n+m}{2} \rfloor}^* > \xi + \epsilon \right)
$$

$$
= P^* (G_n^*(\xi + \epsilon) \leq \alpha_n)
$$

$$
\leq P^* \left( G_n^*(\xi + \epsilon) \leq \alpha_n, |\hat{v}_{n,l}^* - v| \leq \frac{\epsilon}{2} \right) + P^* \left( |\hat{v}_{n,l}^* - v| > \frac{\epsilon}{2} \right)
$$

$$
\leq P^* \left( \sum_{i=1}^{n} I \left( v - \xi - \frac{\epsilon}{2} \leq X_i^* \leq v + \xi + \frac{\epsilon}{2} \right) \leq n\alpha_n \right) + P^* \left( |\hat{v}_{n,l}^* - v| > \frac{\epsilon}{2} \right),
$$

where the last inequality follows from

$$
\left\{ \sum_{i=1}^{n} I(\hat{v}_{n,l}^* - \xi - \epsilon \leq X_i^* \leq \hat{v}_{n,l}^* + \xi + \epsilon) \leq n\alpha_n, v - \frac{\epsilon}{2} \leq \hat{v}_{n,l}^* \leq v + \frac{\epsilon}{2} \right\}
$$

$$
\subset \left\{ \sum_{i=1}^{n} I \left( v - \xi - \frac{\epsilon}{2} \leq X_i^* \leq v + \xi + \frac{\epsilon}{2} \right) \leq n\alpha_n, v - \frac{\epsilon}{2} \leq \hat{v}_{n,l}^* \leq v + \frac{\epsilon}{2} \right\}.
$$

For the first part, by Hoeffding’s inequality, we have

$$
P^* \left( \sum_{i=1}^{n} I \left( v - \xi - \frac{\epsilon}{2} \leq X_i^* \leq v + \xi + \frac{\epsilon}{2} \right) \leq n\alpha_n \right)
$$

$$
= P^* \left( \sum_{i=1}^{n} I \left( v - \xi - \frac{\epsilon}{2} \leq X_i^* \leq v + \xi + \frac{\epsilon}{2} \right) - np_n \leq n(\alpha_n - p_{n1}) \right)
$$

$$
= P^* \left( \sum_{i=1}^{n} (Y_i - E^* Y_i) \geq n(p_{n1} - \alpha_n) \right)
$$

$$
\leq \exp\{-2n(p_{n1} - \alpha_n)^2\},
$$

where $Y_i = 1 - I \left( v - \xi - \frac{\epsilon}{2} \leq X_i^* \leq v + \xi + \frac{\epsilon}{2} \right)$, $i = 1, \ldots, n$ and $p_{n1} = F_n \left( v + \xi + \frac{\epsilon}{2} \right) - F_n \left( v - \xi - \frac{\epsilon}{2} \right)$. Note that $G(\xi + \frac{\epsilon}{2}) = F \left( v + \xi + \frac{\epsilon}{2} \right) - F \left( v - \xi - \frac{\epsilon}{2} \right)$, the Glivenko-Cantelli theorem implies that

$$
p_{n1} \to F \left( v + \xi + \frac{\epsilon}{2} \right) - F \left( v - \xi - \frac{\epsilon}{2} \right) > \frac{1}{2}, \ \text{a.s.}
$$

Then for sufficiently large $n$

$$
p_{n1} - \alpha_n > \frac{F \left( v + \xi + \frac{\epsilon}{2} \right) - F \left( v - \xi - \frac{\epsilon}{2} \right) - (\lfloor \frac{n+m}{2} \rfloor - 1)/n}{\sqrt{2}} > 0, \ \text{a.s.}
$$

It follows from the discussion above and Lemma 2 that

$$
P \left( \xi_{n,m,l}^* > \xi + \epsilon \right) = E \left[ P^* \left( \xi_{n,m,l}^* > \xi + \epsilon \right) \right]
$$

$$
\leq 2e^{-\sqrt{2n}\delta_{\epsilon,n}^2} + E \left[ e^{-2n(p_n - \alpha_n)^2} \right]
$$

$$
\leq 2e^{-\sqrt{2n}\delta_{\epsilon,n}^2} + e^{-\sqrt{2n}\delta_{\epsilon,m}^2}
$$

$$
\leq 3e^{-\sqrt{2n}\Delta_{\epsilon,n}^2}.
$$
Set $\beta_n = \left\lfloor \frac{n+m}{2} \right\rfloor / n$, a similar argument leads to

$$P^*(\hat{\xi}_{n,m,l} < \xi - \epsilon) \leq P^*\left(\sum_{i=1}^n I \left( v - \xi + \frac{\epsilon}{2} \leq X_i \leq v + \xi - \frac{\epsilon}{2} \right) \geq n\beta_n \right) + P^*\left(\mid \hat{v}^*_{n,l} - v \mid > \frac{\epsilon}{2} \right)$$

$$\leq \exp\{-2n(\beta_n - p_{n2})^2\} + 2e^{-\sqrt{2n}\delta_{e,n}},$$

where $p_{n2} = F_n(v + \xi - \frac{\epsilon}{2}) - F_n(v - \xi + \frac{\epsilon}{2})$, and $\beta_n - p_{n2} > 0$ for sufficiently large $n$. Then we have

$$P(\hat{\xi}_{n,m,l} < \xi - \epsilon) = E\left[ P^*(\hat{\xi}_{n,m,l} < \xi - \epsilon) \right] \leq 3e^{-\sqrt{2n}\Delta_{e,n}}.$$

The conclusion has been proved. □

**Lemma 4.** Let $F$ be differentiable at $v$ and $v \pm \xi$, with $F'(v) > 0$ and $G'(\xi) = F'(v - \xi) + F'(v + \xi) > 0$, then for any fixed $l \geq 1$ and $m \geq 1$, we have almost surely

$$|\left(\hat{v}^*_{n,l} - \hat{\xi}_{n,m,l}^*\right) - (v - \xi)| \leq D \frac{(\log n)^{1/2}}{n^{1/2}}$$

and

$$|\left(\hat{v}^*_{n,l} + \hat{\xi}_{n,m,l}^*\right) - (v + \xi)| \leq D \frac{(\log n)^{1/2}}{n^{1/2}}$$

for sufficiently large $n$, where $D = \max\{8/F'(v), 8/G'(\xi)\}$.

**Proof of Lemma 4.** Put $\epsilon_n = D \frac{(\log n)^{1/2}}{n^{1/2}}$. It follows from Lemma 2 and Lemma 3 that, for any fixed $l \geq 1$ and $m \geq 1$,

$$P(|\left(\hat{v}^*_{n,l} + \hat{\xi}_{n,m,l}^*\right) - (v + \xi)| > \epsilon_n) \leq 8 \exp\{-\sqrt{2n}\Delta_{e,n}^2\}.$$

Since $F(v) = 1/2$, we have

$$a_0\left(\frac{\epsilon_n}{2}, l\right) = F\left(v + \frac{\epsilon_n}{4}\right) - \frac{\lfloor n+l \rfloor - 1}{n}$$

$$= F\left(v + \frac{\epsilon_n}{4}\right) - \frac{1}{2} + O\left(\frac{1}{n}\right)$$

$$= F\left(v + \frac{\epsilon_n}{4}\right) - F(v) + O\left(\frac{1}{n}\right)$$

$$= \frac{F'(v)}{4} \epsilon_n + o(\epsilon_n) + O\left(\frac{1}{n}\right)$$

$$> \frac{(\log n)^{1/2}}{n^{1/2}}, \text{ for sufficiently large } n.$$
By similar arguments using \( F(v + \xi) - F(v - \xi) = 1/2 \), we also obtain
\[
c_0 \left( \frac{\epsilon_n}{2}, m \right) > \frac{(\log n)^{1/2}}{n^{1/2}}, \quad \text{for sufficiently large } n
\]
and
\[
d_0 \left( \frac{\epsilon_n}{2}, m \right) > \frac{(\log n)^{1/2}}{n^{1/2}}, \quad \text{for sufficiently large } n.
\]

The conclusion follows from the inequalities above and the Borel-Cantelli lemma.

**Lemma 5.** Let \( F \) be differentiable at \( v \) and twice differentiable at \( v \pm \xi \), with \( F'(v) > 0 \) and \( G'(\xi) = F'(v - \xi) + F'(v + \xi) > 0 \), then for any fixed \( l \geq 1 \) and \( m \geq 1 \), as \( n \to \infty \), we have

\[
H_{n1} = \left| F_n^*(\hat{v}_{n,l}^* - \hat{\xi}_{n,m,l}^*) - F_n^*(v - \xi) - F(v - \xi) \right| = O \left( n^{-3/4} \log n \right), \quad \text{a.s.}
\]
and

\[
H_{n2} = \left| F_n^*(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^*) - F_n^*(v + \xi) - F(v + \xi) \right| = O \left( n^{-3/4} \log n \right), \quad \text{a.s.}
\]

**Proof of Lemma 5.** Denote by \( \theta_p \) the \( p \)-th quantile of \( F \) for \( p \in (0, 1) \). Let \( a_n = \frac{c \log n}{n^{1/2}} \) for some positive constant \( c \), and define

\[
H_{pn}(x) := \left[ F_n^*(x) - F_n^*(\theta_p) \right] - \left[ F(x) - F(\theta_p) \right].
\]

It follows from Lemma 3.7 of Zuo (2015) that

\[
\sup_{|x - \theta_p| < a_n} |H_{pn}(x)| = O \left( n^{-3/4} \log n \right), \quad \text{as } n \to \infty, \quad \text{a.s.}
\]

Let we express \( v - \xi \) as the \( p \)-th quantile of \( F \): \( v - \xi = F^{-1}(p) = \theta_p \), and put \( x_n = \hat{v}_{n,l}^* - \hat{\xi}_{n,m,l}^* \) for any fixed \( l, m \geq 1 \), then Lemma 4 implies

\[
|x_n - \theta_p| \leq D \frac{(\log n)^{1/2}}{n^{1/2}} < a_n, \quad \text{for sufficiently large } n.
\]

Now we have

\[
H_{n1} \leq \sup_{|x - \theta_p| < a_n} |H_{pn}(x)| = O \left( n^{-3/4} \log n \right), \quad \text{a.s.}
\]

Similarly we can obtain

\[
H_{n2} = O \left( n^{-3/4} \log n \right), \quad \text{a.s.}
\]

The proof has been completed. \( \blacksquare \)
Lemma 6. Suppose \( v = F^{-1}(1/2) \) be the unique solution to (1), and \( \xi = G^{-1}(1/2) \) is the unique solution to (3). Then for any fixed \( m \geq 1 \), it holds almost surely that

\[
G^*_n(\hat{\xi}^*_{n,m,l}) = \frac{1}{2} + O\left( \frac{\log n}{n} \right), \quad n \to \infty.
\]

Proof of Lemma 6. For convenience we set \( l = m = 1 \). Recall that \( G^*_n(y) \) is the empirical distribution function of \( W^*_1, W^*_2, \ldots, W^*_n \). Since \( \hat{\xi}^*_n,1,1 = \frac{[n+1]}{2} \) unless there is a tie. If such a tie exists, we have some \( X^*_i = \hat{v}^*_n,1 \pm \hat{\xi}^*_n,1,1 \). It follows from Zuo (2015) that for large \( n \)

\[
\sum_{i=1}^n I\left(X^*_i = \hat{v}^*_n,1 \pm \hat{\xi}^*_n,1,1\right) < 2 \log n, \ a.s.
\]

That is, we have for large \( n \)

\[
nG^*_n(\hat{\xi}^*_n,1,1) \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 2 \log n, \ a.s.
\]

Then almost surely

\[
G^*_n(\hat{\xi}^*_n,1,1) = \frac{1}{2} + O\left( \frac{\log n}{n} \right), \quad n \to \infty.
\]

This completes the proof of this lemma. 

After proving Lemmas 2-6, we now are able to show the following theorem, which states the strong Bahadur representation for \( \hat{\xi}^*_n,m,l \) for any fixed \( l, m \geq 1 \).

Theorem 1. Suppose \( F \) is continuous in neighborhoods of \( v \pm \xi \) and twice differentiable at \( v \) and \( v \pm \xi \), with \( F'(v) > 0 \) and \( G'(\xi) = F'(v - \xi) + F'(v + \xi) > 0, \) then for any fixed \( m \geq 1 \),

\[
\hat{\xi}^*_n,m,l - \xi = \frac{1}{2} - \frac{[F^*_n(v + \xi) - F^*_n(v - \xi)]}{G'(\xi)} + \frac{F'(v + \xi) - F'(v - \xi)}{G'(\xi)} \frac{1}{2} - \frac{F^*_n(v)}{F'(v)} + R_{n1}
\]

with

\[
R_{n1} = O(n^{-3/4} \log n), \ a.s.
\]

Proof of Theorem 1. It follows from Lemma 5 that almost surely, for any fixed \( l, m \geq 1 \),

\[
F(\hat{v}^*_n,l + \hat{\xi}^*_n,m,l) - F(v + \xi) = F^*_n(\hat{v}^*_n,l + \hat{\xi}^*_n,m,l) - F^*_n(v + \xi) + O(n^{-3/4} \log n)
\]

and

\[
F(\hat{v}^*_n,l - \hat{\xi}^*_n,m,l) - F(v - \xi) = F^*_n(\hat{v}^*_n,l - \hat{\xi}^*_n,m,l) - F^*_n(v - \xi) + O\left(n^{-3/4} \log n \right).
\]
Taking the difference yields

$$F(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^*) - F(\hat{v}_{n,l}^* - \hat{\xi}_{n,m,l}^*) - F(v + \xi) + F(v - \xi)$$

$$= F_n^*(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^*) - F_n^*(\hat{v}_{n,l}^* - \hat{\xi}_{n,m,l}^*) - F_n^*(v + \xi) + F_n^*(v - \xi) + O\left(n^{-3/4} \log n\right).$$

On the other hand, using Taylor expansion and Lemma 4, we have as $n \to \infty$

$$F(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^*) - F(v + \xi) = F'(v + \xi)(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^* - v - \xi) + O\left(\frac{\log n}{n}\right)$$

and

$$F(\hat{v}_{n,l}^* - \hat{\xi}_{n,m,l}^*) - F(v - \xi) = F'(v - \xi)(\hat{v}_{n,l}^* - \hat{\xi}_{n,m,l}^* - v + \xi) + O\left(\frac{\log n}{n}\right),$$

which implies

$$F(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^*) - F(\hat{v}_{n,l}^* - \hat{\xi}_{n,m,l}^*) - F(v + \xi) + F(v - \xi)$$

$$= F'(v + \xi)(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^* - v - \xi) - F'(v - \xi)(\hat{v}_{n,l}^* - \hat{\xi}_{n,m,l}^* - v + \xi)$$

$$= F_n^*(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^*) - F_n^*(\hat{v}_{n,l}^* - \hat{\xi}_{n,m,l}^*) - F_n^*(v + \xi) + F_n^*(v - \xi) + O(n^{-3/4} \log n)$$

$$= G_n^*(\hat{\xi}_{n,m,l}^*) - [F_n^*(v + \xi) - F_n^*(v - \xi)] + O(n^{-3/4} \log n)$$

$$= \frac{1}{2} - [F_n^*(v + \xi) - F_n^*(v - \xi)] + O(n^{-3/4} \log n),$$

where the last equality follows from Lemma 6. By noting that

$$F'(v + \xi)(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^* - v - \xi) - F'(v - \xi)(\hat{v}_{n,l}^* - \hat{\xi}_{n,m,l}^* - v + \xi)$$

$$= [F'(v + \xi) - F'(v - \xi)](\hat{v}_{n,l}^* - v) + G'(\xi)(\hat{\xi}_{n,m,l}^* - \xi),$$

we have

$$\hat{\xi}_{n,m,l}^* - \xi = \frac{1}{2} - \frac{[F_n^*(v + \xi) - F_n^*(v - \xi)]}{G'(\xi)}$$

$$+ \frac{F'(v + \xi) - F'(v - \xi)}{G'(\xi)}(\hat{v}_{n,l}^* - v) + O(n^{-3/4} \log n).$$

Finally, taking $p = \frac{1}{2}$ in Theorem 3.9 of Zuo (2015) yields

$$\hat{v}_{n,l}^* = v + \frac{1}{2} - \frac{F_n^*(v)}{F'(v)} + O(n^{-3/4} \log n).$$

The proof is now completed by inserting (9) into (8).
Since Theorem 1 holds for any fixed $l, m \geq 1$, its result is quite general. Following a similar fashion to this theorem, it is easy to check the following theorem, which states the strong Bahadur representation for $\text{MAD}_n^*$. 

**Theorem 2.** Under the conditions of Theorem 1, we have as $n \to \infty$

\[
\text{MAD}_n^* - \xi = \frac{1}{2} - [F_n^*(v + \xi) - F_n^*(v - \xi)] \frac{G'(\xi)}{G'(v)} + \frac{F'(v + \xi) - F'(v - \xi)}{F'(v)} + O(n^{-3/4} \log n), \ a.s.
\]

**Proof of Theorem 2.** Observe that $\text{Med}_n^* = (\hat{\nu}_{n,1}^* + \hat{\nu}_{n,2}^*)/2$, it is easy to verify that the result of Lemma 2 also holds for $\text{Med}_n^*$. Let $\tilde{\xi}_{n,m}^* = \tilde{W}_{i,n}^* |_{n,1}$, where $\tilde{W}_i = \{X_i^* - \text{Med}_n^*, 1 \leq i \leq n\}$ are the ordered statistics of $\tilde{W}_i^* = \{X_i^* - \text{Med}_n^*, 1 \leq i \leq n\}$. Then by the same arguments, the results of Lemma 3-Lemma 6 still hold with $\hat{\nu}_n^*$ and $\tilde{\xi}_{n,m,l}^*$ replaced by $\text{Med}_n^*$ and $\tilde{\xi}_{n,m,l}^*$, respectively. Following the proof of Theorem 1, we have

\[
\tilde{\xi}_{n,m}^* - \xi = \frac{1}{2} - [F_n^*(v + \xi) - F_n^*(v - \xi)] \frac{G'(\xi)}{G'(v)} + \frac{F'(v + \xi) - F'(v - \xi)}{F'(v)} + O(n^{-3/4} \log n), \ a.s.
\]

Hence the conclusion follows by noting that $\text{MAD}_n^* = (\hat{\nu}_{n,1}^* + \hat{\nu}_{n,2}^*)/2$. \[\blacksquare\]

## 3 Weak Bahadur representation for the bootstrap MAD

The strong Bahabar representation is somewhat too strong. In statistics, deriving the weak Bahadur representation may suffice for many practical applications, such as deriving the limit distribution. Hence, in this section, we also consider the weak Bahadur representation of the bootstrap MAD under weaker conditions than Section 2.

To achieve this, we first present some useful preliminary lemmas as follows.

**Lemma 7.** Let $F$ be differentiable at $v$ and $v \pm \xi$, with $F'(v) > 0$ and $G'(\xi) = F'(v - \xi) + F'(v + \xi) > 0$, then for any fixed $l, m \geq 1$, we have as $n \to \infty$

\[
|(\hat{\nu}_{n,l} - \tilde{\xi}_{n,m,l}) - (v - \xi)| = O_p(n^{-1/2}) \quad \text{and} \quad |(\hat{\nu}_{n,l} + \tilde{\xi}_{n,m,l}) - (v + \xi)| = O_p(n^{-1/2}).
\]

**Proof of Lemma 7.** For any $\epsilon > 0$, let $M > \frac{\sqrt{\log(1/\epsilon)}}{\sqrt{2}}$. Put $\epsilon_n = D \frac{M}{n^{1/2}}$, where the constant $D$ is defined in Lemma 4. It can be seen from Lemmas 2-3 that

\[
P(|(\hat{\nu}_{n,l} + \tilde{\xi}_{n,m,l}) - (v + \xi)| > \epsilon_n) \leq 8 \exp\{-\sqrt{2n}\Delta_{\epsilon_n/2,n}^2\}.\]
Similar to Lemma 4, we have
\[ a_0 \left( \frac{\epsilon_n}{2}, l \right) = F \left( v + \frac{\epsilon_n}{4} \right) - \frac{\left| \frac{n+l}{2} \right| - 1}{n} \]
\[ = F \left( v + \frac{\epsilon_n}{4} \right) - F(v) + O \left( \frac{1}{n} \right) \]
\[ = \frac{F'(v)}{4} \epsilon_n + o(\epsilon_n) + O \left( \frac{1}{n} \right) \]
\[ > \frac{M}{n^{1/2}}, \quad \text{for all sufficiently large } n. \]

The same results hold for \( b_0(\frac{\epsilon_n}{2}, l), c_0(\frac{\epsilon_n}{2}, m) \) and \( d_0(\frac{\epsilon_n}{2}, m) \). Now we have
\[ \sqrt{2} n \Delta^2_{\epsilon_n/2, n} \geq \sqrt{2} M^2 \quad \text{for all sufficiently large } n, \]
whence for \( n \) large enough
\[ P(n^{1/2}|(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^*) - (v + \xi)| > DM) \leq e^{-\sqrt{2}M^2} < \epsilon, \]
which implies
\[ |(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^*) - (v + \xi)| = O_p(n^{-1/2}). \]

The rest part can be proved by using the same steps. \[ \square \]

Lemma 8. (Ghosh, 1971) Let \( \{U_n\} \) and \( \{V_n\} \) be sequences of random variables on some probability space \((\Omega, \mathcal{F}, P)\). Suppose that (a) \( V_n = O_p(1), n \to \infty \), and (b) For all \( t \) and all \( \epsilon > 0 \),
\[ \lim_{n \to \infty} P(U_n \geq t + \epsilon, V_n \leq t) = 0 \]
\[ \lim_{n \to \infty} P(U_n \leq t, V_n \geq t + \epsilon) = 0. \] (10)

Then \( U_n - V_n = o_p(1), n \to \infty \).

Lemma 9. Let \( F \) be continuous in the neighborhoods of \( v \pm \xi \), and differentiable at \( v \) and \( v \pm \xi \), with \( F'(v) > 0 \) and \( G'(\xi) = F'(v - \xi) + F'(v + \xi) > 0 \), then as \( n \to \infty \), we have for any fixed \( l, m \geq 1 \)
\[ H_{n1} = |F_n^*(\hat{v}_{n,l}^* - \hat{\xi}_{n,m,l}^*) - F_n^*(v - \xi) - F^*(\hat{v}_{n,l}^* - \hat{\xi}_{n,m,l}^*) + F(v - \xi)| = o_p(n^{-1/2}) \]
and
\[ H_{n2} = |F_n^*(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^*) - F_n^*(v + \xi) - F^*(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}) + F(v + \xi)| = o_p(n^{-1/2}). \]

Proof of Lemma 9. Let
\[ U_n = n^{1/2}[F_n^*(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^*) - F_n^*(v + \xi)] \]
\[ V_n = n^{1/2}[F(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^*) - F(v + \xi)]. \]
By Taylor expansion and Lemma 7,
\[ F(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^*) - F(v + \xi) = O(|\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^* - v - \xi|) = O_p(n^{-1/2}), \ n \to \infty. \]
Thus \( V_n \) satisfies (a) of Lemma 8.

Consider the case \( t > 0 \). Define the right limit as
\[ \beta := \lim_{t \to 0^+} F^{-1}(F(v + \xi) + t/\sqrt{n}). \]
Since \( F^{-1} \) may be not continuous at \( F(v + \xi) \), there are two cases to consider. When \( \beta = v + \xi \), using \( F(x) < p \) if and only if \( x < F^{-1}(p) \), we have
\[
\{ V_n \leq t \} = \left\{ F(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^*) - F(v + \xi) \leq \frac{t}{\sqrt{n}} \right\}
\subset \left\{ F(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^*) < F(v + \xi) + \frac{t + \epsilon/2}{\sqrt{n}} \right\}
= \left\{ \hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^* < F^{-1} \left( F(v + \xi) + \frac{t + \epsilon/2}{\sqrt{n}} \right) \right\}
\subset \left\{ F_n(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^*) \leq F_n(\eta_n(t)) \right\}
\]
where
\[ \eta_n(t) = F^{-1} \left( F(v + \xi) + \frac{t + \epsilon/2}{\sqrt{n}} \right). \]

By (11) and the expressions of \( U_n \) and \( V_n \), we have
\[
P(U_n \geq t + \epsilon, V_n \leq t) \leq P \left( F_n(\eta_n(t)) - F_n^*(v + \xi) \geq \frac{t + \epsilon}{\sqrt{n}} \right). \quad (12)
\]
Since \( F \) is continuous at \( v + \xi \), which implies that \( F(\eta_n(t)) - F(v + \xi) = \frac{t + \epsilon/2}{\sqrt{n}} > 0 \). Then for all \( n \) sufficiently large
\[ p_n := F_n(\eta_n(t)) - F_n(v + \xi) > 0, \ a.s. \]
Then for a sufficiently large \( n \), given \( X_1, X_2, \ldots, X_n \), we have
\[ Z_n^* =: n \left( F_n^*(\eta_n(t)) - F_n^*(v + \xi) \right) \sim \text{Binomial}(n, p_n). \]

By using the Chebyshev inequality, and noting that \( E(p_n) = \frac{t + \epsilon/2}{\sqrt{n}} \), we have
\[
P \left( F_n^*(\eta_n(t)) - F_n^*(v + \xi) \geq \frac{t + \epsilon}{\sqrt{n}} \right)
= E \left( \mathbb{P}_* \left( F_n^*(\eta_n(t)) - F_n^*(v + \xi) \geq \frac{t + \epsilon}{\sqrt{n}} \right) \right)
= E \left[ \mathbb{P}_* \left( Z_n^* - np_n \geq \sqrt{n}(t + \epsilon) - np_n \right) \right]
\leq E \left[ \mathbb{P}_* \left( |Z_n^* - np_n| \geq \frac{\epsilon}{3} \sqrt{n} \right) \right]
\leq E \left[ \frac{9p_n(1 - p_n)}{\epsilon^2} \right] \leq \frac{9(t + \epsilon/2)}{\sqrt{n} \epsilon^2} \to 0, \ n \to \infty.
\]
Returning to (12), the first condition in (b) of Lemma 9 is established for $t > 0$ and $\beta = v + \xi$.

When $t > 0$ and $\beta > v + \xi$, let $\theta$ be any point in the open interval $(v + \xi, \beta)$. As has been proved in Section 2 that $\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^* \to v + \xi$, a.s. which implies $\Pr(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^* > \theta) \to 0$ and

$$\Pr(U_n \geq t + \epsilon, V_n \leq t) = \Pr(U_n \geq t + \epsilon, V_n \leq t, \hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^* \leq \theta) + o(1), n \to \infty.$$  

Since $\eta_n(t) \to \beta > \theta$, then for sufficiently large $n$

$$\{V_n \leq t, \hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^* \leq \theta\} \subset \{F(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^*) < F(v + \xi) + \frac{t + \epsilon/2}{\sqrt{n}}, \hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^* \leq \theta\}$$

$$\subset \{\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^* < F^{-1}(F(v + \xi) + \frac{t + \epsilon/2}{\sqrt{n}}), \hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^* \leq \theta\}$$

$$\subset \{F_n(\hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^*) \leq F_n^*(\theta)\}.$$

Then similar to (12), we have

$$\Pr(U_n \geq t + \epsilon, V_n \leq t, \hat{v}_{n,l}^* + \hat{\xi}_{n,m,l}^* \leq \theta) \leq \Pr\left(F_n^*(\theta) - F_n^*(v + \xi) \geq \frac{t + \epsilon}{\sqrt{n}}\right), \quad (13)$$

Note that by the definition of $\beta$ and $\theta$, almost surely there are no sample in the interval $[v + \xi, \theta]$, hence no bootstrap sample in the same interval. So $F_n^*(\theta) - F_n^*(v + \xi) = 0$, a.s. Hence

$$\Pr\left(F_n^*(\theta) - F_n^*(v + \xi) \geq \frac{t + \epsilon}{\sqrt{n}}\right) = 0.$$  

Thus we establish the first condition in (b) of Lemma 9 for $t > 0$. The case $t \leq 0$ and the second condition of (b) can be proved similarly. That is, we obtain $H_{2n} = o_p(n^{-1/2})$.

The proof of $H_{1n} = o_p(n^{-1/2})$ follows a similar fashion. We omit the details.

Based on Lemma 7 and Lemma 9, we have the following theorem.

**Theorem 3.** Suppose $F$ is continuous in the neighborhoods of $v \pm \xi$, and differentiable at $v$ and $v \pm \xi$, with $F'(v) > 0$ and $G'(\xi) = F'(v - \xi) + F'(v + \xi) > 0$, then as $n \to \infty$

$$\hat{\xi}_{n,m,l}^* - \xi = \frac{1}{2} - \frac{F_n^*(v + \xi) - F_n^*(v - \xi)}{G'(\xi)} + \frac{F'(v + \xi) - F'(v - \xi)}{G'(\xi)} \frac{1}{2} - \frac{F_n^*(v)}{F'(v)} + R_{n2}$$

with

$$R_{n2} = o_p(n^{-1/2}).$$

**Proof of Theorem 3.** Similar to the proof of Theorem 1, it follows from Lemma 6, for any fixed $l, m \geq 1$, that

$$G_n^*(\hat{\xi}_{n,m,l}^*) = \frac{1}{2} + o_p(n^{-1/2}), \quad n \to \infty. \quad (14)$$

Following the same steps as the proof of Theorem 1, Lemma 9, Lemma 7 and (14) yield
\[ \xi_{n,m,l}^* - \xi = \frac{1}{2} - \frac{[F_n^*(v + \xi) - F_n^*(v - \xi)]}{G'(\xi)} \]
\[ + \frac{F'(v + \xi) - F'(v - \xi)}{G'(\xi)} (\hat{\nu}_{n,t}^* - v) + o_p(n^{-1/2}). \]

Note that Lemma 3.4 of Zuo (2015) implies
\[ \hat{\nu}_{n,t}^* = v + \frac{1}{2} - \frac{F_n^*(v)}{F'(v)} + o_p(n^{-1/2}). \]

The proof is now completed by inserting (16) into (15). \( \blacksquare \)

Similar to the proof of Theorem 2, by the same arguments of Lemma 7, Lemma 9 and Theorem 3, we have the following weak Bahadur representation of bootstrap sample MAD.

**Theorem 4.** Under the conditions of Theorem 3, we have as \( n \to \infty \)
\[ \text{MAD}_n^* - \xi = \frac{1}{2} - \frac{[F_n^*(v + \xi) - F_n^*(v - \xi)]}{G'(\xi)} \]
\[ + \frac{F'(v + \xi) - F'(v - \xi)}{G'(\xi)} \frac{1}{2} - \frac{F_n^*(v)}{F'(v)} + o_p(n^{-1/2}). \]

4 Joint asymptotic normality for the bootstrap median and MAD

In this section, we consider the joint asymptotic normality of \((\text{Med}_n^*, \text{MAD}_n^*)\). As in Falk (1997) and Serfling and Mazumder (2009), define \( \alpha = F(v-\xi) + F(v+\xi) \), \( \beta = F'(v-\xi) - F'(v+\xi) \), and \( \gamma = \beta^2 + 4(1-\alpha)\beta F'(v) \). We need the following lemma, which is Proposition A.1 in Wang and Chen (2009).

**Lemma 10.** Let \( \{V_i\} \) be a sequence of random variables, such that for some function \( h \), as \( n \to \infty \), \( h(V_1, \ldots, V_n) \overset{d}{\to} \Theta \), where \( \Theta \) has a distribution function \( H \). If \( \{U_i\} \) is a sequence of random variables such that
\[ P(U_n - h(V_1, \ldots, V_n) \leq s|V_1, \ldots, V_n) \to F(s) \]
almost surely for all \( s \in \mathbb{R} \), where \( F \) is a continuous distribution function, then
\[ P(U_n \leq t) \to (H * F)(t) \]
for all \( t \in \mathbb{R} \), where "*" denotes the convolution operator.

Based on this lemma, we are now able to show the following theorem.
Theorem 5. Suppose $F$ is continuous in the neighborhoods of $v \pm \xi$, and differentiable at $v$ and $v \pm \xi$, with $F'(v) > 0$ and $G'(\xi) = F'(v - \xi) + F'(v + \xi) > 0$, then as $n \to \infty$

\[
\begin{pmatrix}
\sqrt{n}(\text{Med}_n^* - v) \\
\sqrt{n}(\text{MAD}_n^* - \xi)
\end{pmatrix}
\xrightarrow{d}
N \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma \right)
\]

where $\Sigma = (\sigma_{ij})_{2 \times 2}$ with

\[
\begin{align*}
\sigma_{11} &= \frac{1}{2F'(v)^2}, \\
\sigma_{12} &= \sigma_{21} = \frac{1}{2F'(v)G'(\xi)} \left( 1 - 4F(v - \xi) + \frac{\beta}{F'(v)} \right), \\
\sigma_{22} &= \frac{1}{2G'(\xi)^2} \left( 1 + \frac{\gamma}{F'(v)^2} \right).
\end{align*}
\]

Proof of Theorem 5. For every vector $\lambda = (\lambda_1, \lambda_2)^T$ such that $\lambda^T \Sigma \lambda > 0$, it suffice to show

\[
\lambda^T \begin{pmatrix}
\sqrt{n}(\text{Med}_n^* - v) \\
\sqrt{n}(\text{MAD}_n^* - \xi)
\end{pmatrix}
\xrightarrow{d}
N(0, \lambda^T \Sigma \lambda), \quad n \to \infty.
\]

Note that

\[
\lambda^T \begin{pmatrix}
\sqrt{n}(\text{Med}_n^* - v) \\
\sqrt{n}(\text{MAD}_n^* - \xi)
\end{pmatrix} = \lambda^T \begin{pmatrix}
\sqrt{n}(\text{Med}_n^* - \text{Med}_n) \\
\sqrt{n}(\text{MAD}_n^* - \text{MAD}_n)
\end{pmatrix} + \lambda^T \begin{pmatrix}
\sqrt{n}(\text{Med}_n - v) \\
\sqrt{n}(\text{MAD}_n - \xi)
\end{pmatrix}.
\]

It follows from Serfling and Mazumder (2009) that as $n \to \infty$

\[
h(X_1, \ldots, X_n) = \lambda^T \begin{pmatrix}
\sqrt{n}(\text{Med}_n - v) \\
\sqrt{n}(\text{MAD}_n - \xi)
\end{pmatrix}
\xrightarrow{d}
N \left( 0, \frac{1}{2} \lambda^T \Sigma \lambda \right).
\]

By Lemma 10, we need only to show that as $n \to \infty$

\[
\sup_{s \in \mathbb{R}} \left| P^* \left( \frac{\sqrt{n}(\text{Med}_n^* - \text{Med}_n, \text{MAD}_n^* - \text{MAD}_n) \lambda}{\sqrt{\lambda^T \Sigma \lambda / 2}} \leq s \right) - \Phi(s) \right| \to 0, \quad a.s.
\]

(17)

where $\Phi$ is the distribution function of $N(0,1)$. By the weak Bahadur representations of $\text{Med}_n, \text{Med}_n^*, \text{MAD}_n$ and $\text{MAD}_n^*$, we have

\[
\sqrt{n}(\text{Med}_n^* - \text{Med}_n) = \sqrt{n} \frac{F_n(v) - F_n^*(v)}{F'(v)} + o_p(1),
\]

\[
\sqrt{n}(\text{MAD}_n^* - \text{MAD}_n) = \sqrt{n} \frac{F_n(v + \xi) - F_n(v - \xi) - F_n^*(v + \xi) + F_n^*(v - \xi)}{G'(\xi)}
\]

\[
- \frac{\beta}{G'(\xi)} \sqrt{n} \frac{F_n(v) - F_n^*(v)}{F'(v)} + o_p(1).
\]
Then the left of (17) can be expressed as

\[
\sup_{s \in \mathbb{R}} \left| P^* \left( \frac{\sqrt{n} \bar{Y}^*_n - \bar{Y}_n}{\bar{\sigma}_n} \leq \frac{\sqrt{\lambda^T \Sigma \lambda/2}}{\bar{\sigma}_n} s \right) - \Phi(s) \right|
\]

\[
= \sup_{s \in \mathbb{R}} \left| P^* \left( \frac{\sqrt{n} \bar{Y}^*_n - \bar{Y}_n}{\bar{\sigma}_n} \leq s \right) - \Phi \left( \frac{\bar{\sigma}_n}{\sqrt{\lambda^T \Sigma \lambda/2}} s \right) \right|
\]

\[
\leq \sup_{s \in \mathbb{R}} \left| P^* \left( \frac{\sqrt{n} \bar{Y}^*_n - \bar{Y}_n}{\bar{\sigma}_n} \leq s \right) - \Phi(s) \right| + \sup_{s \in \mathbb{R}} \left| \Phi \left( \frac{\bar{\sigma}_n}{\sqrt{\lambda^T \Sigma \lambda/2}} s \right) - \Phi(s) \right|,
\]

where \( \bar{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i^* \) with

\[
Y_i^* = -\frac{\lambda_2}{G'(\xi)} I(v - \xi < X_i^* \leq v + \xi) + \left( \frac{\lambda_2 \beta}{G'(\xi)} - \lambda_1 \right) I(X_i^* \leq v),
\]

and \( \bar{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i = E(Y_i^*|X_1, \ldots, X_n), \) \( \bar{\sigma}_n^2 = \text{var}(Y_i^*|X_1, \ldots, X_n) = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \bar{Y}_n)^2 \) with

\[
Y_i = -\frac{\lambda_2}{G'(\xi)} I(v - \xi < X_i \leq v + \xi) + \left( \frac{\lambda_2 \beta}{G'(\xi)} - \lambda_1 \right) I(X_i \leq v).
\]

Since \( Y_1^*, \ldots, Y_n^* \) are iid random variables given \( X_1, \ldots, X_n \), it follows from Berry-Essen theorem that

\[
\sup_{s \in \mathbb{R}} \left| P^* \left( \frac{\sqrt{n} \bar{Y}^*_n - \bar{Y}_n}{\bar{\sigma}_n} \leq s \right) - \Phi(s) \right| \leq \frac{33}{4} \frac{\sum_{i=1}^{n} |Y_i - \bar{Y}_n|^3}{n^{3/2} \bar{\sigma}_n^3}.
\]

Note that \( Y_i - \bar{Y}_n \) is bounded for \( 1 \leq i \leq n \), and as \( n \to \infty \)

\[
\bar{\sigma}_n \to \sqrt{\lambda^T \Sigma \lambda/2} \quad \text{a.s.}
\]

which implies

\[
\sup_{s \in \mathbb{R}} \left| P^* \left( \frac{\sqrt{n} \bar{Y}^*_n - \bar{Y}_n}{\bar{\sigma}_n} \leq s \right) - \Phi(s) \right| \to 0, \quad \text{a.s.}
\]

In addition, Taylor expansion and (18) yield

\[
\sup_{s \in \mathbb{R}} \left| \Phi \left( \frac{\bar{\sigma}_n}{\sqrt{\lambda^T \Sigma \lambda/2}} s \right) - \Phi(s) \right| \to 0, \quad \text{a.s.}
\]

Now we have proved (17), then the conclusion follows. \( \blacksquare \)

### 5 An application to the bootstrap projection depth weighted mean

In this section, we apply the previous results to obtain the weak Bahadur representation of the bootstrap sample projection depth weighted mean, including the famous Stahel-Donoho location estimator as its special case, described in Zuo (2010).
Following by Zuo et al. (2004), the projection depth weighted mean is defined as

\[ \text{PWM}(F) = \frac{\int_{-\infty}^{\infty} x w(PD(x, F))dF(x)}{\int_{-\infty}^{\infty} w(PD(x, F))dF(x)} \]

where \( w(t) \) is a weight function on \([0, 1] \), \( PD(x, F) = \frac{1}{1+|x-v|/\xi} \) with \( v \) and \( \xi \) standing for the median and MAD, respectively. By replacing \( F \) with \( F_n \) and \( F_n^* \), respectively, we get the sample and bootstrap versions of PWM, i.e.

\[
\text{PWM}(F_n) = \frac{\sum_{i=1}^{n} w_i X_i}{\sum_{i=1}^{n} w_i}, \quad \text{and} \quad \text{PWM}(F_n^*) = \frac{\sum_{i=1}^{n} w_i^* X_i^*}{\sum_{i=1}^{n} w_i^*},
\]

with

\[ w_i = w(PD(X_i, F_n)) = w \left( \frac{1}{1 + |X_i - Med_n|/\text{MAD}_n} \right) \]

and

\[ w_i^* = w(PD(X_i^*, F_n^*)) = w \left( \frac{1}{1 + |X_i^* - Med_n^*|/\text{MAD}_n^*} \right). \]

The follows theorem states the weak Bahadur representation of \( \text{PWM}(F_n^*) \).

**Theorem 6.** Suppose \( F \) is continuous in the neighborhoods of \( v \pm \xi \), and differentiable at \( v \) and \( v \pm \xi \), with \( F'(v) > 0 \) and \( G'(\xi) = F'(v - \xi) + F'(v + \xi) > 0 \), \( w(t) \) is continuously differentiable with \( w(0) = 0 \). Then as \( n \to \infty \), we have

\[
\text{PWM}(F_n^*) - \text{PWM}(F) = \frac{1}{n} \sum_{i=1}^{n} [K(X_i^*) - E[K(X_i^*)]] + o_p \left( \frac{1}{\sqrt{n}} \right),
\]

where

\[ K(x) = \frac{\int_{-\infty}^{\infty} [y - \text{PWM}(F)] w'(PD(y, F)) f(y, x) dF(y) + |x - \text{PWM}(F)| w(PD(x, F))}{\int_{-\infty}^{\infty} w(PD(x, F)) dF(x)} \]

with

\[ f(x, y) = \frac{|x - v|}{(\xi + |x - v|)^2} \left( \frac{1}{2} - I(v - \xi < y \leq v + \xi) \right) \frac{G'(\xi)}{G'(\xi + |x - v| |x - v|)^2} + \left( \frac{|x - v|}{G'(\xi + |x - v| |x - v|)^2} I(v - \xi < y \leq v + \xi) \right) \frac{G'(\xi)}{G'(\xi + |x - v| |x - v|)^2} + \frac{\xi \text{sign}(x - v)}{(\xi + |x - v| |x - v|)^2} \left( \frac{1}{2} - I(y \leq v) \right) \frac{F'(v)}{F'(v)}. \]

**Proof of Theorem 6.** By Theorem 5, it is easy to check that

\[
PD(x, F_n^*) - PD(x, F) = \frac{1}{1 + |x - Med_n^*|/\text{MAD}_n^*} - \frac{1}{1 + |x - v|/\xi}
\]

\[
= \frac{1}{(\text{MAD}_n^* - \xi) |x - v| + \xi (|x - v| - |x - Med_n^*|)} \frac{|x - v|}{(\xi + |x - v| |x - v|)^2} \left( \text{MAD}_n^* - \xi \right) + \frac{\xi \text{sign}(x - v)}{(\xi + |x - v| |x - v|)^2} \left( \text{Med}_n^* - v \right) + o_p \left( \frac{1}{\sqrt{n}} \right). \]
From Theorem 4 and Lemma 3.4 of Zuo (2015), we have

\[
\text{Med}_n^* - v = \frac{1}{n} \sum_{i=1}^{n} \frac{\frac{1}{2} - I(X_i^* \leq v)}{F'(v)} + o_p\left(\frac{1}{\sqrt{n}}\right)
\]

and

\[
\text{MAD}_n^* - \xi = \frac{1}{n} \sum_{i=1}^{n} \frac{\frac{1}{2} - I(v - \xi < X_i^* \leq v + \xi)}{G'(-\xi)} + F'(v + \xi) - F'(v - \xi) \text{Med}_n^* - v + o_p\left(\frac{1}{\sqrt{n}}\right),
\]

which imply that

\[
PD(x, F_n^*) - PD(x, F) = \frac{1}{n} \sum_{i=1}^{n} f(x, X_i^*) + o_p\left(\frac{1}{\sqrt{n}}\right),
\]

where \(f(x, y)\) is defined by (19). Without loss of generality, we assume \(PWM(F) = 0\), then

\[
\sqrt{n} \int_{-\infty}^{\infty} xw(PD(x, F_n^*))dF_n^*(x) = \sqrt{n} \int_{-\infty}^{\infty} xw(PD(x, F_n^*))dF_n^*(x) - \sqrt{n} \int_{-\infty}^{\infty} xw(PD(x, F))dF(x) = \int_{-\infty}^{\infty} xw'(\theta_n^*(x))\{\sqrt{n}[PD(x, F_n^*) - PD(x, F)]\}dF_n^*(x) - \int_{-\infty}^{\infty} xw(PD(x, F))d\{\sqrt{n}[F_n^*(x) - F(x)]\},
\]

where \(\theta_n^*(x)\) is between \(PD(x, F_n^*)\) and \(PD(x, F)\), hence satisfying

\[
\sup_{x \in \mathbb{R}}|\theta_n^*(x) - PD(x, F)| = O_p\left(\frac{1}{\sqrt{n}}\right), \quad n \to \infty.
\]

Note that as \(n \to \infty\)

\[
\sup_{x \in \mathbb{R}}|x[PD(x, F_n^*) - PD(x, F)]| = O_p\left(\frac{1}{\sqrt{n}}\right),
\]

which implies that

\[
\left|\int_{-\infty}^{\infty} x\{w'(\theta_n^*(x)) - w'(PD(x, F))\}\{\sqrt{n}[PD(x, F_n^*) - PD(x, F)]\}dF_n^*(x)\right| = o_p(1)
\]

and

\[
\left|\int_{-\infty}^{\infty} xw'(PD(x, F))\{\sqrt{n}[PD(x, F_n^*) - PD(x, F)]\}d\{F_n^*(x) - F(x)\}\right| = \frac{1}{\sqrt{n}} \left|\int_{-\infty}^{\infty} xw'(PD(x, F))\{\sqrt{n}[PD(x, F_n^*) - PD(x, F)]\}d\{\sqrt{n}[F_n^*(x) - F(x)]\}\right| = o_p(1).
\]
Hence we have as $n \to \infty$

$$\int_{-\infty}^{\infty} x w'(\theta_n^*(x))\{\sqrt{n}[PD(x, F_n^*) - PD(x, F)]\} dF_n(x)$$

$$= \int_{-\infty}^{\infty} x w'(PD(x, F))\{\sqrt{n}[PD(x, F_n^*) - PD(x, F)]\} dF(x) + o_p(1).$$  \hfill (22)

It follows from (22) and Fubini’s theorem that

$$\int_{-\infty}^{\infty} x w'(\theta_n^*(x))\{\sqrt{n}[PD(x, F_n^*) - PD(x, F)]\} dF_n(x)$$

$$= \int_{-\infty}^{\infty} x w'(PD(x, F))\left(\int_{-\infty}^{\infty} f(x, y)d\{\sqrt{n}[F_n^*(y) - F(y)]\}\right) dF(x) + o_p(1)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y w'(PD(y, F))f(y, x)dF(y)d\{\sqrt{n}[F_n^*(x) - F(x)]\} + o_p(1).$$  \hfill (23)

Similarly, we can show that

$$\int_{-\infty}^{\infty} w(PD(x, F_n^*))dF_n(x) = \int_{-\infty}^{\infty} w(PD(x, F))dF(x) + o_p(1).$$  \hfill (24)

Then the desired result follows from (20), (21), (23), (24) and Slutsky’s theorem. \hfill □

**Corollary 1.** Under the conditions of Theorem 6, we have as $n \to \infty$

$$\sqrt{n}(PWM(F_n^*) - PWM(F)) \xrightarrow{d} N(0, \text{var}[K(X)]),$$

where

$$K(x) = \frac{\int_{-\infty}^{\infty}[y - PWM(F)]w'(PD(y, F))f(y, x)dF(y) + [x - PWM(F)]w(PD(x, F))}{\int_{-\infty}^{\infty} w(PD(x, F))dF(x)}$$

with $f(x, y)$ defined in Theorem 6.

**Proof of Corollary 1.** The conclusion follows from Theorem 6, Theorem 3.1 of Zuo et al. (2004), and the arguments in the proof of Theorem 5, and we omit the details. \hfill □

**Remark 1.** To increase the breakdown point robustness of the projection median, Zuo (2003) suggested to use a modified sample MAD, i.e.,

$$\text{MAD}_{nk}^* = \frac{1}{2} \left( W_n^*_{\left\lfloor \frac{n+k}{2} \right\rfloor} + W_n^*_{\left\lfloor \frac{n+k+1}{2} \right\rfloor} \right),$$

for some proper choice of $k = 1, \ldots, n-1$, instead, in the definition of the projection depth. If the weighted mean is defined on this depth, the limit distribution of the related bootstrap estimator can be derived in a similar way to those of Theorem 1 and Theorem 3.
6 Concluding remarks

In this paper, we considered the strong and weak Bahadur representations of the bootstrap MAD, and then used these results to derive the joint limit distribution of the bootstrap sample median and MAD. As an application, we further investigated the weak Bahadur and limit distribution of the bootstrap projection depth weighted mean in one-dimensional space. Being aware that the limit distribution of some inferential estimators/procedures induced from the projection depth, e.g., projection median, is not standard in spaces of dimension greater than 1 due to involving the methodology of projection pursuit. This may hamper their practical applications. Additional bootstrap procedures are needed to obtain the related critical values. We hope the research conducted in the current paper will have the potential to help these studies.

Acknowledgements

Xiaohui Liu’s research was supported by NSF of China (Grant No.11601197, 11461029), China Postdoctoral Science Foundation funded project (2016M600511, 2017T100475), the Postdoctoral Research Project of Jiangxi (2017KY10), NSF of Jiangxi Province (No.20171ACB21030).

References

Bahadur, R. (1966). A note on quantiles in large samples. The Annals of Statistics, 37, 577–580.

Donoho, D.L. (1982). Breakdown properties of multivariate location estimators. Ph.D. Qualifying Paper. Dept. Statistics, Harvard University.

Dutta, S., Ghosh, A. K. (2012). On robust classification using projection depth. Annals of the Institute of Statistical Mathematics, 64(3), 657-676.

Falk, M. (1997). On MAD and comedians. The Annals of the Institute of Statistical Mathematics, 49(4), 615-644.

He, X. and Shao, Q. (1996). A general Bahadur representation of $M$-estimators and its application to linear regression with nonstochastic designs. The Annals of Statistics, 24, 2608-2630.

Kiefer, J. (1967). On Bahadur’s representation of sample quantiles. The Annals of Mathematical Statistics, 38(5), 1323-1342.

Liu, R. Y., 1992. Data depth and multivariate rank tests. In L1-Statistical Analysis and Related Methods (Y. Dodge, ed.), 279-294. North-Holland, Amsterdam.
Mazumder, S., Serfling, R., (2009). Bahadur representations for the median absolute deviation and its modifications. *Statistics and Probability Letters* 79, 1774–1783.

Small, G. (1990). A survey of multidimensional medians. Int. Statist. Rev., 58, 263-277.

Serfling, R. J. (1980). Approximation theorems of mathematical statistics (Vol. 162). John Wiley & Sons.

Serfling, R., Mazumder, S., (2009). Exponential probability inequality and convergence results for the median absolute deviation and its modifications. *Statistics and Probability Letters* 79, 1767–1773.

Stahel, W.A., 1981. Breakdown of covariance estimators. Research Report 31. Fachgruppe für Statistik. ETH, Zürich.

Tukey, J.W., 1975. Mathematics and the picturing of data. In Proceedings of the International Congress of Mathematicians, 523-531. Cana. Math. Congress, Montreal.

Wang, D., Chen, S., (2009). Empirical likelihood for estimating equations with missing values. *The Annals of Statistics* 37, 490–517.

Wendler, M. (2011). Bahadur representation for U-quantiles of dependent data. Journal of Multivariate Analysis, 102(6), 1064-1079.

Wu, W. B. (2005). On the Bahadur representation of sample quantiles for dependent sequences. *The Annals of Statistics*, 33, 1934-1963.

Zhu, Y., Zhang, L., Zhang, Y., (2013). Optimal reinsurance under Haezendonck risk measure. *Statistics and Probability Letters* 83, 1111–1116.

Zuo, Y.J., 2003. Projection based depth functions and associated medians. Ann. Statist. 31, 1460-1490.

Zuo, Y.J. (2006). Multidimensional Trimming Based on Projection Depth. *The Annals of Statistics*, 34, 2211-2251.

Zuo, Y.J., (2010). Is the $t$ confidence interval $X \pm t_{\alpha}(n - 1)s/\sqrt{n}$ optimal? *The American Statistician*, 64, 170-173.

Zuo, Y. (2015). Bahadur representations for bootstrap quantiles. *Metrika*, 78(5), 597-610.

Zuo, Y.J., Cui, H.J., He, X.M. (2004). On the Stahel-Donoho estimators and depth-weighted means for multivariate data. *The Annals of Statistics*, 32, 189-218.