ABOUT CLASSES OF BASIC FUNCTIONS
FOR GENERALIZED TRIGONOMETRIC FUNCTIONS

Abstract. This article describes some methods of constructing basic functions with certain differential properties, on which generalized trigonometric functions are given. Functions with certain differential properties are considered, and the construction of such functions through Fourier coefficients with a certain decreasing order; combinations of both methods are also possible. It is shown that the class of basic functions is wide enough to coordinate the choice of these functions for research.

Keywords: generalized periodic functions, generalized trigonometric functions, basic functions, divergent series, Fourier series divergent, convolution.

Introduction

In [1], the classes of generalized trigonometric functions were introduced, which, in contrast to the generalized periodic functions (distributions) of Schwartz [2], are given on the classes of basic, finitely differentiated functions. With this approach, the question of constructing such classes of basic functions is important.
Let's consider this question in more detail.

As you know, the trigonometric series is called an expression

\[ \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kt + b_k \sin kt, \]

where \( a_0, a_k, b_k - real\ numbers \ (k = 1, 2, \ldots). \)

Trigonometric series can both coincide and diverge. If these series coincide, their sum is periodic with the period \( \pi. \)

In many cases, we have to deal with divergent trigonometric series that do not have a sum in the usual sense [3]. Such series often arise in theoretical research, in the formal differentiation of convergent trigonometric series, etc. For example, in the theoretical study of the Fourier series, it is necessary to study a series of types

\[ \frac{1}{2} + \sum_{k=1}^{\infty} \cos kt, \]

which is divergent at each point of the interval \([0, 2\pi].\)

One of the approaches to the study of divergent trigonometric series is the approach in which such series are considered generalized periodic functions of slow growth [2]; such functions are set on the classes of basic functions \( K. \) As the main functions of the class \( K \) periodic, infinitely differentiated functions appear. Thus the functional, which is the result of the action of the generalized periodic function on the basic functions of a class \( K, \) is presented in the form of a convergent numerical series. The practicality of this approach is explained by the fact that the numerical series through which the functionals are fed are generalized periodic functions of slow growth in the class \( K, \) always coinciding in the usual sense. However, in our opinion, this convenience significantly limits the use of divergent series, as the requirement of infinite differentiation of basic functions is quite burdensome.

In [1], another approach to the study of divergent trigonometric series was proposed, in which, as before, such series are considered as generalized periodic functions; about that such functions on classes of the basic functions are set \( K(p); \) in the role of the same functions of the class \( K(p) \) paired periodic functions,
differentiated \( p \) \((p \geq 0)\) times. In contrast to the classical approach, generalized periodic functions are given in the class of basic functions \( K(p) \) called generalized trigonometric functions. As before, the functional that is the result of the generalized trigonometric function on the main functions of the class \( K(p) \) are served as a numerical series; the values of the same parameter are chosen to ensure the convergence of these numerical series. This approach proved to be quite fruitful; so, in particular, it leads to classes of interpolation trigonometric splines. Note that the class of trigonometric splines is quite wide; in particular, it includes a class of periodic simple polynomial splines. With this approach, the question of building classes of basic functions is relevant \( K(p) \).

**Analysis of research and publications**

Divergent series in different years were considered by Euler, Leibniz, D'Alembert, Lagrange, Bernoulli, Abel, Frobenius, Gelder, Poisson, G. Voron, S. Bernstein and many others [3]. When considering such a series, the methods of generalized summation were used.

Another approach to the study of divergent trigonometric series was proposed by Schwartz [2]. In this approach, divergent trigonometric series are considered objects of a new type called generalized periodic functions; such distributions as functionals on classes of infinitely differentiated basic functions are set. This approach was considered in the works of Mikusinsky, Edwards, Nikolsky, and others. In [1], an approach to the study of divergent trigonometric series was proposed. As before, such series are considered generalized periodic functions; however, such functions are specified on the classes of basic functions \( K(\alpha,r) \). As functions of class \( K(\alpha,r) \) there are paired periodic functions that depend on the parameter \( \alpha \) \((\alpha > 0)\) and differentiated \( r \) \((r = 0,1,\ldots)\) times. In contrast to the classical approach, generalized periodic functions, which are given in the class of basic functions \( K(\alpha,r) \), was called generalized trigonometric functions; later, this approach was considered in [4] - [6]

As we have already said, when considering generalized trigonometric
functions, constructing classes of basic functions $K(\alpha, r)$ is essential; this question is considered in this paper.

The goal of the work

Development of methods for constructing classes of finite functions $K(\alpha, r)$, depending on the parameter $\alpha (\alpha > 0)$ and differentiated $r (r = 0, 1, \ldots)$ times that can be used as basic functions in constructing generalized trigonometric functions.

Main part

1. Basic functions with certain differential properties.

1.1. In [4] - [6], in the role of class functions $K(\alpha, r)$ were considered periodically extended with the period $2\pi$, normalized basic polynomials $B_r$ - splines. As is known [7], [8], spline $B_r(h, t)$ of order $r (r = 0, 1, \ldots)$ belongs to the class $C^{r-1}_{[-\pi, \pi]}$ ($r = 1, 2, \ldots$) and has a finite vector $\left[\begin{array}{c}
-(r+1)\frac{h}{2}, (r+1)\frac{h}{2}
\end{array}\right]$, which is determined by the step $h = \frac{2\pi}{N}$ of some uniform grid set on $[-\pi, \pi]$; of course, we believe that the value $N$ large enough compared to the degree of the spline $r$.

Normalization of the spline $B_r(h, t)$ is that at any parameter values $r$ and $h$ there is equality

$$\int_{-\pi}^{\pi} B_r(h, t) dt = 1.$$  \hspace{1cm} (1)

From the finiteness of the spline vector and its normalization, it follows that when $h \to 0$ splines $B_r(h, t)$ form a delta-like sequence at any finite value $r$.

For using $B_r$ - splines as basic functions for constructing generalized trigonometric functions, we need to calculate the Fourier coefficients of these splines. In [9], the following submission was received $B_r$ - splines:

$$B_r(h, t) = \frac{\alpha}{\pi} \left( \frac{1}{2} + \sum_{k=1}^{\infty} \left( \frac{\sin k}{k} \right)^{r+1} \cos kt \right).$$ \hspace{1cm} (2)

where $\sin(x) = \frac{\sin x}{x}$.
It follows from (2) that the Fourier coefficients of the spline \( B_r(h,t) \) up to a constant multiplier are determined by the expression

\[
a_k(r,h) = \text{sinc}\left(k \frac{h}{2}\right)^{r+1}.
\]

(3)

It is clear that when \( h \to 0 \) coefficients \( a_k(r,h) \to 1 \) are at any finite value \( r \), we come to the famous Dirac delta function representation.

Naturally, the question arises whether other types of functions can act as the main functions of the class \( K(\alpha,r) \). The answer to this question is yes; Let’s give some examples of such functions.

1.2. Consider periodic with period \( 2\pi \) function \( T(r,\alpha,t) \), which is determined as follows

\[
T(r,\alpha,t) = \begin{cases} C_r \left\{ \sin \left[ \frac{\pi}{2} \left( 1 - \frac{|t|}{|\alpha|} \right) \right] \right\}^r, & |t| < \alpha; \\
0, & |t| \geq \alpha.
\end{cases}
\]

where \( r, \alpha \) – parameters, \( r = 1, 2, \ldots, \alpha > 0 \), and parameter \( C_r \) determined from the condition of normalization of the function \( T(r,\alpha,t) \), that is, from the condition

\[
\int_{-\pi}^{\pi} T(r,\alpha,t) \, dt = 1.
\]

The function graph \( T(r,\alpha,t) \) with \( r = 2 \) and some parameter values \( \alpha \) are shown in Fig.1.

Fig. 1. Function \( T(r,\alpha,t) \) when \( \alpha = 0.5 \), and \( \alpha = 1 \)
It is easy to see that this function is continuous and has a continuous derivative
$r-1$-st order.

Here are the Fourier coefficients of this function for some parameter values $r$, distinguishing between even and odd values of this parameter.

For the odd ones $r$, $r = 2l + 1$, ($l = 0, 1, \ldots$) we get:

$$\mu_k(2l+1, \alpha) = \frac{(-1)^{l+1} [(2l+1)!]^2 \pi^{2l+1} \cos \alpha k}{(4\alpha^2 k^2 - 1^2 \pi^2)(4\alpha^2 k^2 - 3^2 \pi^2)\cdots(4\alpha^2 k^2 - (2l+1)^2 \pi^2)}.$$

For the even ones $r$, $r = 2l$, ($l = 1, \ldots$) we get:

$$\mu_k(2l+1, \alpha) = \frac{(-1)^l [(l)!]^2 \pi^{2l} \sin \alpha k}{(\alpha^2 k^2 - 1^2 \pi^2)(\alpha^2 k^2 - 2^2 \pi^2)\cdots(\alpha^2 k^2 - l^2 \pi^2)} \cdot \frac{\sin \alpha k}{\alpha k}.$$

It is easy to see that the order of decline of the obtained coefficients $a_r(r, \alpha)$ is consistent with the differential properties of the function $T(r, \alpha, t)$. In addition, when $\alpha \to 0$ coefficients $\mu_r(r, \alpha) \to 1$ at any values $r (r = 1, 2, \ldots)$. From this fact, it follows that when $\alpha \to 0$ of function $T(r, \alpha, t)$ form a delta-like sequence.

1.3. Consider periodic with period $2\pi$ function $P(r, \alpha, t)$ ($r = 1, 2, \ldots$), which is determined as:

$$P(r, \alpha, t) = \begin{cases} C_r \left[ 1 - \left( \frac{t}{\alpha} \right)^2 \right]^r, & |t| \leq \alpha; \\
0, & |t| > \alpha. \end{cases}$$

Where is still the parameter $C_r$ determined by the rationing condition.

Graphs of this function at some values of the parameter $r$ are given in Fig. 2.

It's easy to see that feature $P(r, \alpha, t)$ differentiated $r-1$ times. Here are the Fourier coefficients of this function for some parameter values $r$.

$$r = 1; \quad v_k(1, \alpha) = \frac{3(\sin(\alpha k) - \alpha k \cos(\alpha k))}{\alpha^3 k^3};$$

$$r = 2; \quad v_k(2, \alpha) = \frac{-15((\alpha^2 k^2 - 3)\sin(\alpha k) + 3\alpha k \cos(\alpha k))}{\alpha^5 k^5};$$
The descending order of the obtained coefficients \( \mathcal{G}_k(r, \alpha) \) is consistent with the differential properties of the function \( P(r, \alpha, t) \). In addition, when \( \alpha \to 0 \) coefficients \( v_k(r, \alpha) \) go to 1 at any value of \( r \) \( (r = 1, 2, \ldots) \). As before, from this fact, it follows that when \( \alpha \to 0 \) functions \( P(r, \alpha, t) \) form a delta-like sequence.

2. In paragraph 1 were considered the basic finite functions with certain differential properties. However, as we pointed out in [5], another approach is possible, where the main functions will be set through artificially constructed Fourier coefficients; it is clear that the differential properties of such functions are determined by order of decrease of these coefficients. Consider this approach in more detail.

From the analysis of expression (3), it follows that the main functions of the class \( K(\alpha, r) \) you can consider functions \( \varphi(r, \alpha, t) \), which are given by the expression

\[
\varphi(r, \alpha, t) = \frac{1}{2} + \sum_{k=1}^{\infty} v_k(r, \alpha) \cos k t
\]
where the coefficients \( v_k(r,\alpha) \) look like

\[
v_k(r,\alpha) = \frac{\beta(r,\alpha,k)}{P_{1+r}(\alpha,k)}
\]

where \( P_{1+r}(\alpha,k) \) — some polynomial of degree \( 1+r \) on the variable \( k \) with parameter \( \alpha \), which does not rotate to 0 for any integer \( k \), and function \( \beta(r,\alpha,k) \) is limited in any \( k, r \) and \( \alpha \). It is clear that the function is constructed in this way \( \mu(r,\alpha,t) \in C_{[-\pi,\pi]} \).

To illustrate, consider some of the main functions obtained in this way; in this case, we will distinguish between functions that depend on the parameter \( \alpha \), and functions that do not depend on this parameter.

2.1. The main functions depend on the parameter \( \alpha \).

2.1.1. The function \( \varphi_{1}(r,\alpha,t) \) with Fourier coefficients

\[
v_{1_k}(r,\alpha) = \left(\frac{\sin k\alpha}{k}\right)^{1+r}.
\]

2.1.2. The function \( \varphi_{2}(r,\alpha,t) \) with Fourier coefficients

\[
v_{2_k}(r,\alpha) = \frac{\sin k\alpha}{k^{1+r}}.
\]

2.1.3. The function \( \varphi_{3}(r,\alpha,t) \) with Fourier coefficients

\[
v_{3_k}(r,\alpha) = \left(\frac{\sin k\alpha}{k}\right)^{1+r}.
\]

2.1.4. The function \( \varphi_{4}(r,\alpha,t) \) with Fourier coefficients

\[
v_{4_k}(r,\alpha) = \left(\frac{\text{sign}(\sin k\alpha)}{k}\right)^{1+r}.
\]

Similarly, it is easy to build other basic functions of this type.

2.2. Basic functions that do not depend on the parameter \( \alpha \).

2.2.1. The function \( \varphi_{5}(r,\alpha,t) \) with Fourier coefficients
2.2.2. The function $\phi_6(r, \alpha, t)$ with Fourier coefficients

$$v_6(r, \alpha) = \frac{(-1)^{k+1}}{k^{1+r}}.$$

2.2.3. The function $\phi_7(r, \alpha, t)$ with Fourier coefficients

$$v_7(r, \alpha) = \frac{1}{2} \frac{1 - (-1)^k}{2k^{1+r}}.$$

Similarly, it is easy to build other basic functions of this type.

3. When constructing basic functions, you can also use a mixed approach. For example, the coefficients $v_5(1, \alpha)$ are multiplied by the coefficients $v_5(r)$. It is clear that the new Fourier coefficients obtained in this way are in descending order $2 + 1 + r$; a function constructed using such coefficients belongs to the class $C^{1+r}_{[-\pi, \pi]}$. This approach is convenient to use in cases where, for example, the coefficients $v_5(1, \alpha)$ construct a continuous derivative of order $m$ ($m = 1, 2, \ldots$); then, we receive the function of a class $C^m_{[-\pi, \pi]}$ at consecutive integration.

**Conclusions**

1. It is shown that the class of basic functions is not empty and contains a sufficient number of functions, which allows coordinating the choice of basic functions for research purposes.

2. Some methods of construction of the basic functions of a class are offered $K(\alpha, r)$.

3. Finite functions with certain smoothness properties are considered, and their Fourier coefficients, which have certain decreasing orders, are given.

4. Artificial methods of constructing Fourier coefficients with certain decreasing orders are considered; these coefficients allow you to build functions with certain smoothness properties.

5. A mixed approach to constructing Fourier coefficients is considered, which
combines both of the above approaches.

6. Of course, the choice of basic functions and their comparison require further research.

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