Non-Abelian vortices and monopoles in $SO(N)$ theories

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Abstract:

Non-Abelian BPS vortex solutions are constructed in $\mathcal{N} = 2$ theories with gauge groups $SO(N) \times U(1)$. The model has $N_f$ flavors of chiral multiplets in the vector representation of $SO(N)$, and we consider a color-flavor locked vacuum in which the gauge symmetry is completely broken, leaving a global $SO(N)_{C+F}$ diagonal symmetry unbroken. Individual vortices break this symmetry, acquiring continuous non-Abelian orientational moduli. By embedding this model in high-energy theories with a hierarchical symmetry breaking pattern such as $SO(N+2) \rightarrow SO(N) \times U(1) \rightarrow 1$, the correspondence between non-Abelian monopoles and vortices can be established through homotopy maps and flux matching, generalizing the known results in $SU(N)$ theories. We find some interesting hints about the dual (non-Abelian) transformation properties among the monopoles.

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1 Introduction

Recently some significant steps have been made in understanding the non-Abelian monopoles occurring in spontaneously broken gauge field theories. The basic observation is that the regular ‘t Hooft-Polyakov-like magnetic monopoles occurring in a system

\[ G \xrightarrow{v_1} H, \]

where \( H \) is a non-Abelian “unbroken” gauge group, are not objects which transform among themselves under the unbroken group \( H \), but which transform, if any, under the magnetic dual of \( H \), namely \( \tilde{H} \). As field transformation groups, \( H \) and \( \tilde{H} \) are relatively non-local, thus a local transformation in the magnetic group \( \tilde{H} \) would look like a non-local transformation in the electric theory. Although this was implicit in the work by Goddard-Nuyts-Olive and others, the lack of the concrete knowledge on how \( \tilde{H} \) acts on semiclassical monopoles has led to long-standing puzzles and apparent difficulties.

Detailed study of gauge theories with \( \mathcal{N} = 1 \) or \( \mathcal{N} = 2 \) supersymmetry and quark multiplets, on the other hand, shows that light monopoles transforming as multiplets of non-Abelian magnetic gauge group \( \tilde{H} \) do occur quite regularly in full quantum systems. They occur under certain conditions, e.g., that there is a sufficiently large exact flavor symmetry group in the underlying theory, which dresses the monopoles with flavor quantum numbers, preventing them from interacting too strongly. Also, the symmetry requirement (i.e. the symmetry of the low-energy effective theory describing the light monopoles be the correct symmetry of the underlying theory) seems to play an important role in determining the low-energy degrees of freedom in each system. There are subtle, but perfectly clear, logical reasons behind these quantum mechanical realizations of dual gauge symmetries in supersymmetric models. Since there are free parameters in these supersymmetric theories which allow us to move from the fully dynamical regime to semiclassical regions, without qualitatively changing any physics, it must be possible to understand these light degrees of freedom in terms of more familiar soliton-like objects, e.g., semiclassical monopoles.

This line of thought has led us to study the system (1.1), in a regime of hierarchically broken gauge symmetries

\[ G \xrightarrow{v_1} H \xrightarrow{v_2} \mathbb{1}, \quad v_1 \gg v_2, \]

namely, in a phase in which the “unbroken” \( H \) gauge system is completely broken at much lower energies (Higgs phase), so that one expects – based on the standard electromagnetic duality argument – the \( \tilde{H} \) system to be in confinement phase. The “elementary monopoles” confined by the confining strings in \( \tilde{H} \) theory should look like ‘t Hooft-Polyakov monopoles embedded in a larger picture where their magnetic fluxes are frisked away by a magnetic vortex of the \( H \) theory in Higgs phase.

Indeed, in the context of softly broken \( \mathcal{N} = 2 \) models, this kind of systems can be realized concretely, by tuning certain free parameters in the models, typically, by taking the bare quark masses \( m \) (which fix the adjoint scalar VEVs, \( \langle \phi \rangle = v_1 \sim m \)) much larger than the bare adjoint scalar mass \( \mu \) (which sets the scale for the squark VEVs, \( \langle q \rangle = v_2 \sim \sqrt{\mu m} \)). In a high-energy approximation, where \( v_2 \) is negligible, one has a system (1.1), with a set of ’t Hooft-Polyakov monopoles. In the class of supersymmetric models considered, these monopoles are BPS, and their (semiclassical) properties are well understood. In the low-energy approximation (where the massive monopoles are integrated out and \( v_1 \) is regarded as infinitely large) one has the \( H \) theory in Higgs phase, with BPS vortices whose properties can also be studied in great detail.

When the full theory is considered, with “small” corrections which involves factors of \( \frac{v_2}{v_1} \), there is an important qualitative change to be taken into account at the two sides of the mass scales (high-energy and low-energy). Neither monopoles of the high-energy approximation nor the vortices of
the low-energy theory, are BPS saturated any longer. They are no longer topologically stable. This indeed follows from the fact that $\pi_2(G)$ is trivial for any Lie group (no regular monopoles if $H$ is completely broken) or if $\pi_1(G) = 1$ (there cannot be vortices). If $\pi_1(G) \neq 1$ there may be some stable vortices left, but still there will be much fewer stable vortices as compared to what is expected in the low-energy theory (which “sees” only $\pi_1(H)$). As the two effective theories must be, in some sense, good approximations as long as $\frac{v_1}{v} \ll 1$, one faces an apparent paradox.

The resolution of this paradox is both natural and useful. The regular monopoles are actually sources (or sinks) of the vortices seen as stable solitons in the low-energy theory; vice versa, the vortices “which should not be there” in the full theory, simply end at a regular monopole. They both disappear from the spectrum of the respective effective theories. This connection, however, establishes one-to-one correspondence between a regular monopole solution of the high-energy theory and the appropriate vortex of the low-energy theory. As the vortex moduli and non-Abelian transformation properties among the vortices, really depend on the exact global symmetry of the full theory (and its breaking by the solitons), such a correspondence provides us with a precious hint about the nature of the non-Abelian monopoles. In other words, the idea is to make use of the better understood non-Abelian vortices to infer precise conclusions about the non-Abelian monopoles, by-passing the difficulties associated with the latter as mentioned earlier.

A quantitative formulation of these ideas requires a concrete knowledge of the vortex moduli space and the transformation properties among the vortices [16, 17, 18]. This problem has been largely clarified, thanks to our generally improved understanding of non-Abelian vortices [19, 20, 21, 22, 23], and in particular to the technique of the “moduli matrix” [24], especially in the context of $SU(N)$ gauge theories. Also, some puzzles related to the systems with symmetry breaking $SO(2N) \rightarrow U(N)$, or $SO(2N) \rightarrow U(r) \times U(1)^{N-r}$, have found natural solutions [9].

In this article, we wish to extend these analyses to the cases involving vortices of $SO(N)$ theories. In [25] the first attempts have been made in this direction, where softly broken $\mathcal{N} = 2$ models with $SO(N)$ gauge groups and with a set of quark matter in the vector representation, have been analyzed. In the case of $SO(2N+3)$ theory broken to $SO(2N+1) \times U(1)$ (with the latter completely broken at lower energies) one observes some hints how the dual, $USp(2N)$ group, might emerge. In the model considered in [25], however, the construction of the system in which the gauge symmetry is completely broken, leaving a maximum exact color-flavor symmetry (the color-flavor locking), required an ad hoc addition of an $\mathcal{N} = 1$ superpotential, in contrast to $SU(N)$ theories where, due to the vacuum alignment with bare quark masses familiar from $\mathcal{N} = 1$ SQCD, the color-flavor locked vacuum appears quite automatically.

In this article we therefore turn to a slightly different class of $SO(N)$ models. The underlying theory is an $SO(N+2)$ gauge theory with matter hypermultiplets in the adjoint representation, with the gauge group broken partially at a mass scale $v_1$. The analysis is slightly more complicated than the models considered in [25], but in the present model the color-flavor locked vacua occur naturally. Also, these models have a richer spectrum of vortices and monopoles than in the case of [25], providing us with a finer testing ground for duality and confinement.

At scales much lower than $v_1$, the model reduces to an $SO(N) \times U(1)$ theory with quarks in the vector representation. Non-Abelian vortices arising in the color-flavor locked vacuum of this theory transform non-trivially under the $SO(N)_{C+F}$ symmetry. We are interested in their role in the dynamics of gauge theories, but these solitons also play a role in cosmology and condensed matter physics, so the results of sections 3 and 4 of this paper could be of more general interest (for example they can be useful for cosmic strings, see [29]).

In section 2 of this article, we present the high-energy model with gauge group $SO(2N+2)$. In section 3 we study its low-energy effective theory and present the vortex solutions. In section 4 we
study the model with gauge group $SO(2N + 3)$. Finally, in section 5 we discuss the correspondence between monopoles and vortices.

2 The model

We shall first discuss the $SO(2N + 2)$ theory; the case of $SO(2N + 3)$ group will be considered separately later. We wish to study the properties of monopoles and vortices occurring in the system

$$SO(2N + 2) \overset{\text{m}}{\longrightarrow} SO(2N) \times U(1) \overset{\text{m}}{\longrightarrow} 1 .$$

To study the consequences of such a breaking, we take a concrete example of an $\mathcal{N} = 2$ supersymmetric theory with gauge group $SO(2N+2)$ and $N_f$ matter hypermultiplets in the adjoint representation. All the matter fields have a common mass $m$, so the theory has a global $U(N_f)$ flavor symmetry. We also add a small superpotential term $\mu \phi^2$ in the Lagrangian, which breaks softly $\mathcal{N} = 2$ to $\mathcal{N} = 1$.

For the purpose of considering hierarchical symmetry breaking (2.1), we take $m \gg \mu$.

The theory is infrared-free for $N_f > 1$, but one may consider it as an effective low-energy theory of some underlying theory, valid at mass scales below a given ultraviolet cutoff. In any case, our analysis will focus on the questions how the properties of the semiclassical monopoles arising from the intermediate-scale can be understood through the moduli of the non-Abelian vortices arising when the low-energy, $SO(2N)$ theory is put in the Higgs phase.

The superpotential of the theory has the form,

$$W = \sqrt{2} \sum_A \text{Tr} \, \tilde{\zeta}_A [\phi, \zeta_A] + m \sum_A \text{Tr} \, \tilde{\zeta}_A \zeta_A + \frac{\mu}{2} \text{Tr} \, \phi^2 .$$

In order to minimize the misunderstanding, we use here the notation of $\zeta_A$, $\tilde{\zeta}_A$ for the quark hypermultiplets in the adjoint representation of the high-energy gauge group $SO(2N+2)$ (or $SO(2N+3)$), with $A = 1, 2, \ldots, N_f$ standing for the flavor index. We shall reserve the symbols $q_A$, $\tilde{q}_A$ for the light supermultiplets of the low-energy theory, which transform as the vector representation of the gauge group $SO(2N)$ (or $SO(2N + 1)$). The vacuum equations for this theory therefore take the form

$$\begin{align*}
[\phi, \phi^\dagger] &= 0 , \\
\sum_A [\zeta_A, \zeta_A^\dagger] &= \sum_A [\tilde{\zeta}_A, \tilde{\zeta}_A^\dagger] , \\
\sqrt{2} [\zeta_A, \tilde{\zeta}_A] + \mu \phi &= 0 , \\
\sqrt{2} [\phi, \zeta_A] + m \zeta_A &= 0 , \\
-\sqrt{2} [\phi, \tilde{\zeta}_A] + m \tilde{\zeta}_A &= 0 .
\end{align*}$$

We shall choose a vacuum in which $\phi$ takes the vacuum expectation value (VEV)

$$\langle \phi \rangle = \begin{pmatrix} 0 & -iv & 0 & \cdots & 0 \\ iv & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} .$$
which breaks $SO(2N + 2)$ to $SO(2N) \times U(1)$ and is consistent with Eq. (2.4).

We are interested in the Higgs phase of the theory. In order for the $SO(2N) \times U(1)$ symmetry to be broken at energies much lower than $v_1 \equiv v$, we have to find non-vanishing VEVs of the squarks which satisfy Eqs. (2.7), (2.8). This means that $v \sim O(m)$. The magnitude of squark VEVs is then fixed by Eq. (2.6) to be of the order of $(\mu m)^{1/2} \ll m$ and defining $v_2 \equiv |\langle q \rangle| = O(\sqrt{\mu m})$ we obtain the hierarchical breaking of the gauge group (2.11). The D-term condition (2.6) can be satisfied by the ansatz

$$\zeta = \tilde{\zeta}^\dagger.$$  

(2.10)

One must also determine the components of the fields $\zeta, \tilde{\zeta}$ which do not get a mass of the order of $O(v) \equiv O(m)$. We see from Eq. (2.3) that the light squarks are precisely those for which Eqs. (2.7), (2.8) are satisfied non-trivially, i.e., by non-vanishing “eigenvectors” $\zeta, \tilde{\zeta}$. The conditions (2.7), (2.8) require that the light components correspond to the generators of $SO(2N)$ which are lowering and raising operators for $\langle \varphi \rangle$. This condition implies also

$$v = \frac{m}{\sqrt{2}}.$$  

(2.11)

To find the light components of $\zeta, \tilde{\zeta}$, we note that for a single flavor, Eqs. (2.6)-(2.8) together have the form of an $su(2)$ or $so(3)$ algebra, $T_1, T_2, T_3$,

$$\phi \propto T_3 , \quad \zeta_A \propto T_+ = T_1 - iT_2 , \quad \tilde{\zeta}_A \propto T_- = T_1 + iT_2 ,$$  

(2.12)

with appropriate constants.

The simplest way to proceed is to consider the various $SO(3)$ subgroups, $SO(3)_{12j}$, lying in the $(12j)$ three-dimensional subspaces $(j = 3, 4, 5, \ldots)$, with

$$T_3 = H^{(0)} = -i \Sigma_{12} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{12j} ,$$  

$$T_- = T_1 - iT_2 = L_{j,-} \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{pmatrix}_{12j} , \quad T_+ = T_1 + iT_2 = L_{j,+}^\dagger .$$  

(2.13)

(2.14)

The light fields which remain massless can then be expanded as

$$\zeta_A(x) = \sum_{j=3,4,5,\ldots} \frac{1}{2} q_{jA}(x) L_{j,-} , \quad \tilde{\zeta}_A(x) = \sum_{j=3,4,5,\ldots} \frac{1}{2} \tilde{q}_{jA}(x) L_{j,+} .$$  

(2.15)

for each flavor $A = 1, 2, \ldots, N_f$. Written as a full $SO(2N)$ matrix, $L_{j,-}$ looks like

$$L_{j,-} = \begin{pmatrix} 0 & 0 & \ldots & 1 & \ldots \\ 0 & 0 & \ldots & -i & \vdots \\ \vdots & \ddots & \vdots \\ -1 & i & \ddots & \vdots \\ \vdots & \ldots & \ldots & 0 \end{pmatrix} , \quad L_{j,+} = L_{j,-}^\dagger .$$  

(2.16)

In $L_{j,-}$ the only non-zero elements (1 and $-i$) in the first two rows appear in the $(2 + j)$-th column; the only two non-zero elements in the first two columns ($-1$ and $i$) appear in the $(2 + j)$-th row.
An alternative way to find the combinations which do not get mass from \( \langle \phi \rangle \) is to use the independent \( SU(2) \) subgroups contained in various \( SO(4) \) subgroups living in the subspaces \( (1, 2, j, j+1) \), \( j = 3, 5, \ldots, 2N-1 \). As is well known, the \( so(4) \) algebra factorizes into two commuting \( su(2) \) algebras,

\[
so(4) \sim su(2) \times su(2) ,
\]

where for instance for \( SO(4)_{1234} \) one has

\[
\begin{align*}
S_1 &= -\frac{i}{2}(\Sigma_{23} + \Sigma_{41}) , & S_2 &= -\frac{i}{2}(\Sigma_{31} + \Sigma_{42}) , & S_3 &= -\frac{i}{2}(\Sigma_{12} + \Sigma_{43}) , \\
\hat{S}_1 &= -\frac{i}{2}(\Sigma_{23} - \Sigma_{41}) , & \hat{S}_2 &= -\frac{i}{2}(\Sigma_{31} - \Sigma_{42}) , & \hat{S}_3 &= -\frac{i}{2}(\Sigma_{12} - \Sigma_{43}) ,
\end{align*}
\]

where

\[
\Sigma_{23} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{23},
\]

is (up to a phase) the rotation generator in the 23 plane, etc.

Since

\[
\frac{\sqrt{2}}{m} \langle \phi \rangle = H^{(0)} = -i \Sigma_{14} = S_3 + \hat{S}_3 ,
\]

it follows from the standard \( su(2) \) algebra that both \( S_- = S_1 - iS_2 \) and \( \hat{S}_- = \hat{S}_1 - i\hat{S}_2 \) satisfy the relation,

\[
\begin{align*}
\left[ \frac{\sqrt{2}}{m} \langle \phi \rangle , S_- \right] &= -S_- , & \left[ \frac{\sqrt{2}}{m} \langle \phi \rangle , \hat{S}_- \right] &= -\hat{S}_- .
\end{align*}
\]

(2.21)

One can choose the two combinations

\[
L_- = S_- + \hat{S}_- ; \quad L'_- = S_- - \hat{S}_- ,
\]

which satisfy the required relation,

\[
\begin{align*}
\left[ \frac{\sqrt{2}}{m} \langle \phi \rangle , L_- \right] &= -L_- , & \left[ \frac{\sqrt{2}}{m} \langle \phi \rangle , L'_- \right] &= -L'_- .
\end{align*}
\]

(2.23)

These constructions can be done in all \( su(2) \) subalgebras living in \( SO(4)_{1,2,j,j+1}, j = 3, 5, \ldots, 2N-1 \).

Explicitly, \( S_{j-} \), \( \hat{S}_{j-} \), and \( L_{j-} \), \( L'_{j-} \) have the form \( j = 3, 5, \ldots \)

\[
S_{j-} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & i \\ 0 & 0 & -i & 1 \\ -1 & i & 0 & 0 \\ -i & -1 & 0 & 0 \end{pmatrix}_{(1,2,j,j+1)} , \quad \hat{S}_{j-} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & -i & -1 \\ -1 & i & 0 & 0 \\ i & 1 & 0 & 0 \end{pmatrix}_{(1,2,j,j+1)} ;
\]

(2.24)

\[
L_{j-} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -i & 0 \\ -1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{(1,2,j,j+1)} , \quad L'_{j-} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -i & -1 & 0 & 0 \end{pmatrix}_{(1,2,j,j+1)} .
\]

(2.25)

Clearly, one can write

\[
L'_{j-} = i L_{j+1,-} ;
\]

and use the first of Eq. (2.23) to define \( L_{j-} \) for all \( j = 3, 4, 5, \ldots, j \) even or odd. With this definition, \( L_{j-} \) coincide with those introduced in Eq. (2.14) by using various \( SO(3) \) subgroups.
Eqs. (2.21), (2.22), (2.23) show that the light fields (those which do not get mass of order $m$) are the ones appearing in the expansion (2.17). Alternatively, the basis of light fields can be taken as

$$
\zeta_A(x) = \frac{1}{\sqrt{2}} \sum_{i=3,5,\ldots} \left[ Q_{iA}(x) S_{i,-} + \tilde{Q}_{iA}(x) \tilde{S}_{i,-} \right], \quad \tilde{\zeta}_A = \frac{1}{\sqrt{2}} \sum_{i=3,5,\ldots} \left[ \tilde{Q}_{iA}(x) S_{i,+} + \tilde{\tilde{Q}}_{iA}(x) \tilde{S}_{i,+} \right].
$$

The relation between the $q_{iA}(x)$ and $Q_{iA}(x)$ fields is ($i = 3, 5, \ldots$):

$$
Q_{iA}(x) = q_{iA}(x) + i q_{i+1A}(x) \sqrt{2} \quad ; \quad \tilde{Q}_{iA}(x) = q_{iA}(x) - i q_{i+1A}(x) \sqrt{2} = Q_{i+1A}(x). \tag{2.28}
$$

All other components get a mass of order $m$. There are thus precisely $2N$ light quark fields (color components) $q_{iA}$ ($i = 1, 2, \ldots, 2N$) for each flavor. These are the light hypermultiplets of the theory.

Each of the two bases $\{q_{iA}\}$ or $\{Q_{iA}\}$ has some advantages. Clearly the basis $q_{iA}$ ($i = 1, 2, \ldots, 2N$) corresponds to the usual basis of the fundamental (vector) representation of the $SO(M)$ group ($M = 2N$), appearing in the decomposition of an adjoint representation of $SO(M + 2)$ into the irreps of $SO(M)$:

$$
\frac{(M + 2)(M + 1)}{2} = \frac{M(M - 1)}{2} \oplus M \oplus M \oplus 1. \tag{2.29}
$$

The low-energy effective Lagrangian can be most easily written down in terms of these fields, and the symmetry property of the vacuum is manifest here.

On the other hand, the basis $(Q_{jA}, \tilde{Q}_{jA})$, $j = 3, 5, 7, \ldots$, is made of pairs of eigenstates of the $(a \equiv (j - 1)/2)$-th Cartan subalgebra generator,

$$
H^{(a)} = -i \Sigma_{j,j+1} = S_{j,3} - \tilde{S}_{j,3}, \quad a = \frac{j - 1}{2} = 1, 2, \ldots, N, \tag{2.30}
$$

(see Eqs. (2.15), (2.19), (2.20)), with eigenvalues $\pm 1$, so that the vortex equations can be better formulated, and the symmetry maintained by individual vortex solutions can be seen explicitly in this basis. $Q_{iA}$ ($i = 3, 5, \ldots$), form an $\mathbf{N}$ of $SU(N) \subset SO(2N)$; $\tilde{Q}_{iA}$, ($i = 3, 5, \ldots$), form an $\tilde{\mathbf{N}}$. In other words, it represents the decomposition of a $2\mathbf{N}$ of $SO(2N)$ into $\mathbf{N} + \tilde{\mathbf{N}}$ of $SU(N) \subset SO(2N)$. The change of basis from the vector basis ($q$) and $U(N)$ basis ($Q, \tilde{Q}$) is discussed more extensively in Appendix A.

### 3 Vortices in the $SO(2N) \times U(1)$ theory

#### 3.1 The vacuum and BPS vortices

The low-energy Lagrangian for the theory with gauge group $SO(2N) \times U(1)$ and squarks $q_A, \tilde{q}_A$ in the fundamental representation of $SO(2N)$ is

$$
\mathcal{L} = -\frac{1}{4 g_1^2} F_{\mu \nu}^0 F_{\mu \nu}^0 - \frac{1}{4 g_2^2} F_{\mu \nu}^b F_{\mu \nu}^b + |D_\mu q_A|^2 + |\bar{D}_\mu \tilde{q}_A|^2 \tag{3.1}
$$

$$
- \frac{g_2^2 N}{2} \left| q_A^T b q_A - \tilde{q}_A^T b \tilde{q}_A \right|^2 - 2 g_2^2 \left| q_A^T \tilde{q}_A \right|^2 - \frac{g_1^2}{2} \left| q_A \right|^2 \left| q_A \right|^2 + \frac{\mu m}{\sqrt{2}} \left| q_A \right|^2 + \cdots
$$

where the dots denote higher orders in $\mu / m$ and terms involving $\delta \phi = \phi - \langle \phi \rangle$. Note that to this order, the only modification is a Fayet-Iliopoulos term which does not break $\mathcal{N} = 2$ SUSY. The covariant derivative acts as

$$
\bar{D}_\mu q_A = \partial_\mu q_A - i A_\mu^0 q_A - i A_\mu^b b q_A, \tag{3.2}
$$
where $t^a$ is normalized as
\[ \text{Tr} \left( t^a \right)^2 = 1, \] (3.3)
and
\[ t^a = \frac{1}{\sqrt{2}} H^{(a)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_{2a+1,2a+2}, \] (3.4)
where $H^{(a)}$ is the $a$-th Cartan generator of $SO(2N)$, $a = 1, 2, \ldots, N$, which we take simply as
\[ H^{(a)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_{2a+1,2a+2}. \] (3.5)

As we have seen already, each light field carries unit charge with respect to $H^{(0)}$; the pair $(Q_{A,j}, \tilde{Q}_{A,j})$, $j = 3, 5, 7, \ldots$, furthermore carries the charge $\pm 1$ with respect to $H^{(a)}$ ($a = (j - 1)/2$) and zero charge with respect to other Cartan generators.

Let us define
\[ \xi = \frac{\mu m}{2}, \] (3.6)
which is the only relevant dimensional parameter in the Lagrangian. We set $N_f = 2N$, which is enough for our purposes.

By writing $q_{iA}, \tilde{q}_{Ai}$ as color-flavor mixed matrices $q$, $\tilde{q}$, the vacuum equations are now cast into the form
\[ \text{Tr} (qq^\dagger) = \text{Tr} (\tilde{q}^{\dagger}\tilde{q}), \] (3.7)
\[ qq^\dagger - (qq^\dagger)^T = \tilde{q}^{\dagger}\tilde{q} - (\tilde{q}^{\dagger}\tilde{q})^T, \] (3.8)
\[ \text{Tr} (q\tilde{q}) = \xi, \] (3.9)
\[ \text{Tr} (t^bq\tilde{q}) = 0. \] (3.10)

The vacuum we choose to study is characterized by the color-flavor locked phase
\[ \langle q_{A,j} \rangle = \langle \tilde{q}^{\dagger}_{A,j} \rangle = \delta_{A,j} v_2, \quad v_2 = \sqrt{\frac{\xi}{2N}}, \] (3.11)
or
\[ \langle q \rangle = \langle \tilde{q}^{\dagger} \rangle = v_2 \mathbb{1} = v_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \] (3.12)
which clearly satisfies all the equations above. The gauge ($O$) and flavor ($U$) transformations act on them as
\[ q \rightarrow O q U^T, \quad \tilde{q} \rightarrow U^{\ast} \tilde{q} O^T \quad O \in SO(2N) \times U(1), \quad U \in U(2N), \] (3.13)
the gauge group is completely broken, while a global $SO(2N)_{C+F} \times U(1)_{C+F}$ group ($U = O$) is left unbroken.

When looking for vortex solutions, one suppresses time and $z$ dependence of the fields and retains only the component $F_{xy}$ of the field strength. The vortex tension can be cast in the Bogomol'nyi

\[ ^3 \text{Higher } N_f \text{ are interesting because of semilocal vortex configurations arising in these theories. These solutions will be discussed elsewhere.} \]
where we have used the ansatz (3.15). The tension for a BPS solution is

\[ T = \int d^2x \left\{ \left| \frac{1}{2 g_{2N}} F_{ij}^b + g_{2N} \varepsilon_{ij} \tilde{q}_A q_A \right|^2 + \left| \frac{1}{2 g_1} F_{ij}^0 + g_1 \varepsilon_{ij} (\tilde{q}_A q_A - \xi) \right|^2 \right\} \]

The terms with the square brackets in the last line of Eq. (3.14) automatically vanish with the ansatz (20), thus we shall use this ansatz for the vortex configurations. The resulting BPS equations are

\[ \frac{1}{2 g_1} F_{ij}^0 + \eta g_1 \varepsilon_{ij} \left( \tilde{q}_A q_A - \xi \right) = 0 , \quad \eta = \pm 1 , \quad \varepsilon_{ij} = \pm 1 , \]

where we have used the ansatz (3.15). The tension for a BPS solution is

\[ T = \eta \int d^2x \varepsilon_{ij} \xi F_{ij}^0 . \]

To obtain a solution of these equations, we need an ansatz for the squark fields. It is convenient to perform a \( U(2N)_F \) transformation (2.28), where the vacuum takes the block-diagonal form

\[ \langle Q \rangle = \langle \tilde{Q} \rangle = \sqrt{\frac{\xi}{2N}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ i & -i & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & i & -i & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} , \]

In this basis, the ansatz is:

\[ A_i = h_a (r) t^a \varepsilon_{ij} \frac{r_j}{r^2} ; \quad t^0 = \frac{1}{\sqrt{2}} , \quad t^a = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \]

\[ Q(r, \vartheta) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\varphi^+_1} & e^{i\varphi^-_1} & 0 & 0 & \cdots \\ i e^{i\varphi^+_1} & - i e^{i\varphi^-_1} & 0 & 0 & \cdots \\ 0 & 0 & e^{i\varphi^+_2} & e^{i\varphi^-_2} & \cdots \\ 0 & 0 & i e^{i\varphi^+_2} & -i e^{i\varphi^-_2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} , \]

where \( t^a \)s are the generators of the Cartan subalgebra of \( SO(2N) \). The conditions for the fields at \( r \to \infty \) are fixed by the requirement of finite energy configurations:

\[ \varphi_a^\pm (\infty) = \sqrt{\frac{\xi}{2N}} , \]

\[ n^a_\pm = n^{(0)} \mp n^{(a)} , \quad n^{(0)} = \frac{1}{\sqrt{2}} h_0 (\infty) ; \quad n^{(a)} = \frac{1}{\sqrt{2}} h_a (\infty) , \]
where \(n^{(0)}\) and \(n^{(a)}\) are the winding numbers with respect to the \(U(1)\) and to the \(a\)-th Cartan \(U(1) \in SO(2N)\) defined in Eq. (3.3).

Clearly
\[N_0 \equiv n^+_a + n^-_a = 2n^{(0)},\] (3.25)
is independent of \(a\). The regularity of the fields requires that the \(Q_A\)s come back to their original value after a \(2\pi\) rotation, and this yields the quantization condition,
\[n^\pm_a \in \mathbb{Z}, \quad \forall a,\] (3.26)
implying that the \(U(1)\) winding numbers \(n^{(0)}\) and \(n^{(a)}\) are quantized in half integer units, consistently with considerations based on the fundamental groups (see Appendix B and below).

We need only the information contained in Eqs. (3.21), (3.24) to evaluate the tension for a BPS solution:
\[T = 2\eta\xi \lim_{r \to \infty} \int d\theta r A_\theta^0(r) = 2\sqrt{2}\pi \eta \xi h_0(\infty) = 2\pi \eta \xi N_0 = 2\pi \xi |N_0| .\] (3.27)
The last equality comes from the requirement for the tension to be positive, so \(\eta = \text{sign}(N_0)\). Note that the tension depends only on \(|N_0|\), which is twice the \(U(1)\) winding.

From the BPS equations we obtain the differential equations for the profile functions \(h_0, h_a, \varphi^\pm_a\):
\[
\begin{align*}
\frac{dh_0}{dr} &= -2\sqrt{2}\eta g_1^2 r \left( \sum_a (|\varphi^+_a|^2 + |\varphi^-_a|^2) - \xi \right), \\
\frac{dh_a}{dr} &= 2\sqrt{2}\eta g_2^2 N r \left( |\varphi^+_a|^2 - |\varphi^-_a|^2 \right), \\
\frac{d\varphi^\pm_a}{dr} &= \eta \left( n^+_a - h_0 \mp h_a \sqrt{2} \right) \frac{\varphi^\pm_a}{r}.
\end{align*}
\] (3.28, 3.29, 3.30)

In order to cast them in a simple form, we define \(f_0 = h_0 - \frac{N_0}{\sqrt{2}}\) and \(f_a = h_a + \frac{n^+_a - n^-_a}{\sqrt{2}}\) and obtain
\[
\begin{align*}
\frac{df_0}{dr} &= -2\sqrt{2}\eta g_1^2 r \left( \sum_a (|\varphi^+_a|^2 + |\varphi^-_a|^2) - \xi \right), \\
\frac{df_a}{dr} &= 2\sqrt{2}\eta g_2^2 2N r \left( |\varphi^+_a|^2 - |\varphi^-_a|^2 \right), \\
\frac{d\varphi^\pm_a}{dr} &= -\eta \left( f_0 + f_a \right) \frac{\varphi^\pm_a}{r}.
\end{align*}
\] (3.31, 3.32, 3.33)

The boundary conditions at \(r \to \infty\) are
\[\varphi^\pm_a(\infty) = \sqrt{\frac{\xi}{2N}}, \quad f_0(\infty) = f_a(\infty) = 0,\] (3.34)

There are also regularity conditions at \(r = 0\) for the gauge fields \(h_0(0) = h_a(0) = 0\) which are
\[f_0(0) = -\frac{N_0}{\sqrt{2}}, \quad f_a(0) = \frac{n^+_a - n^-_a}{\sqrt{2}},\] (3.35)
Solving Eq. (3.33) for small \(r\) with the conditions (3.35), we obtain \(\varphi^\pm_a \sim r^{n^\pm_a} \eta\). To avoid a singular behavior for these profile functions we need
\[\text{sign} \left( n^\pm_a \right) = \eta,\] (3.36)
Figure 1: Numerically integrated minimum vortex solution with $N_0 = 1$, where we have taken the couplings to be $4g_1^2 = 4g_{2N}^2 = 1$. ($\chi_i \equiv \sqrt{2} f_i$).

Figure 2: Numerically integrated minimum vortex solution with $N_0 = 1$, where we have taken the couplings to be $4g_1^2 = 1$ and $4g_{2N}^2 = 2$ for the left panel and $4g_1^2 = 2$ and $4g_{2N}^2 = 1$ for the right panel. ($\chi_i \equiv \sqrt{2} f_i$).

This condition is consistent with $\eta = \text{sign}(N_0)$. With this condition there are no singularities at $r = 0$ and the equations (3.31), (3.32), (3.33) can be solved numerically with boundary conditions (3.34), (3.35).

The profile functions for the simplest vortex $N_0 = 1$, $n_1^+ = 1$, $n_1^- = 0$ in the $SO(2) \times U(1)$ theory are shown in Figure 1, 2. The profile functions $(f_0, f_a, \varphi_1^+, \varphi_1^-)$ for the minimal vortex $N_0 = 1$, $n_i^+ = 1$, $n_i^- = 0$ in the $SO(2N) \times U(1)$ theory can be obtained by rescaling $g_{2N}^2 \rightarrow g_{2N}^2/N$ and then taking all $\varphi_a^\pm$ equal to the profile functions shown above rescaled by a factor $1/\sqrt{N}$. Similarly, solutions corresponding to the exchange $(n_a^+, n_a^-) = (1, 0) \leftrightarrow (0, 1)$ can be obtained by exchanging $f_a \leftrightarrow -f_a$ and $\varphi_a^+ \leftrightarrow \varphi_a^-$. The typical length scale of the profile functions is $1/\sqrt{\xi}$, which is the only dimensional parameter in the Bogomol’nyi equations.
To study the space of solutions of the BPS equations we have obtained above, it is convenient to rewrite the ansatz \[ \hat{q}(r, \vartheta) = q(r, \vartheta) = \left( \begin{array}{ccc} M_1(r, \vartheta) & 0 & 0 \\ 0 & M_2(r, \vartheta) & 0 \\ 0 & 0 & M_3(r, \vartheta) \end{array} \right), \] (3.37)

\[ M_a(r, \vartheta) = \frac{1}{2} \left( \begin{array}{ccc} e^{i n_a^+ \vartheta} \varphi_a^+(r) + e^{i n_a^- \vartheta} \varphi_a^-(r) & -i \left( e^{i n_a^+ \vartheta} \varphi_a^+(r) - e^{i n_a^- \vartheta} \varphi_a^-(r) \right) \\ i \left( e^{i n_a^+ \vartheta} \varphi_a^+(r) - e^{i n_a^- \vartheta} \varphi_a^-(r) \right) & e^{i n_a^+ \vartheta} \varphi_a^+(r) + e^{i n_a^- \vartheta} \varphi_a^-(r) \end{array} \right). \]

In this basis the action of the \( SO(2N)_C \) isometries on squark fields is simply \( q' = O q O^T \). The first observation is that if \( \hat{q}(r, \vartheta) \) is a solution to the BPS equations, \( O \hat{q}(r, \vartheta) O^T \) is also a solution. Note also that these solutions are physically distinct because they are related by a global symmetry. In this way, from a single solution of the form (3.22), we can obtain a whole continuous \( SO(2N) \) orbit of solutions. Any given vortex solution is a point in the moduli space and \( SO(2N)_C \) acts as an isometry on this space.

From Eqs. (3.24) and (3.36), we see that regular solutions are described by a set of \( 2N + 1 \) integers \( N_0, n_a^\pm \) which satisfy the following conditions:

\[ n_a^+ + n_a^- = N_0, \quad \forall a , \]

\[ \text{sign}(n_a^+) = \text{sign}(n_a^-) = \text{sign}(N_0), \quad \forall a , \] (3.38) (3.39)

where \( N_0 \in \mathbb{Z} \) is related to the winding around the \( U(1) \) and is the only parameter of the solution which enters the tension \( T = 2\pi \xi |N_0| \).

Let us study the solutions with the minimum tension. Minimal vortices have \( N_0 = \pm 1 \) and \( T = 2\pi \xi \). Note that solutions with \( N_0 < 0 \) can be obtained by taking the complex conjugate of solutions with \( N_0 > 0 \), so from now on we will consider only solutions with positive \( N_0 \). These vortices can be divided into two groups, the first has \( 2^{N-1} \) representative (basis) vortices which are

\[ N_0 = 1, \quad \left( \begin{array}{ccc} n_1^+ \\ n_2^+ \\ \vdots \\ n_{N+1}^+ \\ n_N^+ \end{array} \right), \quad \left( \begin{array}{ccc} n_1^- \\ n_2^- \\ \vdots \\ n_{N+1}^- \\ n_N^- \end{array} \right) = \left( \begin{array}{ccc} 1 & 0 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 1 & 0 \end{array} \right) \times \left( \begin{array}{ccc} 0 & 1 \\ 0 & 1 \\ \vdots & \vdots \\ 1 & 0 \\ 1 & 0 \end{array} \right), \] \[ \ldots \] (3.40)

which all have an even number of \( n_i^- \)'s equal to 1; and the second set is represented by \( 2^{N-1} \) vortices, characterized by the integers

\[ N_0 = 1, \quad \left( \begin{array}{ccc} n_1^+ \\ n_2^+ \\ \vdots \\ n_{N+1}^+ \\ n_N^+ \end{array} \right), \quad \left( \begin{array}{ccc} n_1^- \\ n_2^- \\ \vdots \\ n_{N+1}^- \\ n_N^- \end{array} \right) = \left( \begin{array}{ccc} 1 & 0 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 1 & 0 \end{array} \right) \times \left( \begin{array}{ccc} 0 & 1 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{array} \right), \] \[ \ldots \] (3.41)

with an odd number of \( n_i^- \)'s equal to 1.
These two sets belong to two distinct orbits of $SO(2N)_{C+F}$. To see this one must study the way they transform under $SO(2N)_{C+F}$. Consider for instance the case of $N = 2$: the $SO(4)_{C+F}$ transformations 

\[
\begin{pmatrix}
\sigma_3 & 0 \\
0 & \sigma_3
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\] 

exchange $(n^+_1, n^+_2) \leftrightarrow (n^-_1, n^-_2)$ and $(n^+_1, n^-_2) \leftrightarrow (n^-_1, n^+_2)$, respectively.

In the general $SO(2N)$ case, two solutions differing by the exchange $(n^+_i, n^-_j) \leftrightarrow (n^-_i, n^+_j)$ or $(n^+_i, n^-_i) \leftrightarrow (n^-_i, n^-_i)$ for some $i, j$, therefore belong to the same orbit of $SO(2N)_{C+F}$. The vortices in the set (3.40) belong to a continuously degenerate set of minimal vortices; the set (3.41) form the “basis” of another, degenerate set. The two sets do not mix under the $SO(2N)$ transformations.

In order to see better what these two sets might represent, and to see how each vortex transforms under $SO(2N)_{C+F}$, let us assign the two “states”, $|\uparrow\rangle_j$, $|\downarrow\rangle_j$ of a $j$-th $(\frac{1}{2})$ spin, $j = 1, 2, \ldots , N$, to the pair of vortex winding numbers $(n^+_j, n^-_j) = (0, 1), (1, 0)$. Each of the $2^N$ minimum vortices (Eqs. (3.40), (3.41)) can then be represented by the $2^N$ spin state,

\[
|s_1 \otimes s_2 \otimes \cdots \otimes s_N\rangle , \quad |s_j\rangle = |\uparrow\rangle = (0, 1) , \quad \text{or} \quad |\downarrow\rangle = (1, 0) .
\] 

(3.42)

For instance the first vortex of Eq. (3.40) corresponds to the state, $|\downarrow \downarrow \cdots \downarrow\rangle$.

Introduce now the “gamma matrices” as direct products of $N$ Pauli matrices acting as

\[
\gamma_j \equiv \tau_3 \otimes \cdots \otimes \tau_3 \otimes \tau_1 \otimes \cdots \otimes 1 , \quad (j = 1, 2, \ldots , N) ;
\]

(3.43)

\[
\gamma_{N+j} \equiv \tau_3 \otimes \cdots \otimes \tau_3 \otimes \tau_2 \otimes 1 \otimes \cdots \otimes 1 , \quad (j = 1, 2, \ldots , N) .
\] 

(3.44)

$\gamma_k$, $k = 1, 2, \ldots , 2N$ satisfy the Clifford algebra

\[
\{\gamma_i, \gamma_j\} = 2 \eta_{ij} , \quad i, j = 1, 2, \ldots , 2N ,
\]

and the $SO(2N)$ generators can accordingly be constructed by $\Sigma_{ij} = \frac{1}{4i}[\gamma_i, \gamma_j]$. $SO(2N)$ transformations (including finite transformations) among the vortex solutions can thus be represented by the transformations among the $N$-spin states, (3.42).

As each of $\Sigma_{ij}$ ($i \neq j$) flips exactly two spins, the two sets (3.40) and (3.41) clearly belong to two distinct orbits of $SO(2N)$. In fact, a “chirality” operator

\[
\Gamma_5 \equiv P \prod_{j=1}^{2N} \gamma_j , \quad \{\Gamma_5, \gamma_j\} = 0 , \quad j = 1, 2, \ldots , 2N ,
\]

(3.45)

anticommutes with all $\gamma_j$’s, where $P = 1$ ($N$ even) or $P = i$ ($N$ odd), hence commutes with $SO(2N)$. The two sets Eq. (3.40), Eq. (3.41) of minimal vortices thus are seen to transform as two spinor representations of definite chirality, 1 and $-1$, respectively (with multiplicity $2^{N-1}$ each).

Every minimal solution is invariant under a $U(N)$ group embedded in $SO(2N)_{C+F}$. This can be seen from the form of the first solution in (3.40) in the basis (3.37):

\[
q_{(1)} = f_+(r, \vartheta) \begin{pmatrix} 1 \\
\vdots \\
1 \end{pmatrix} + f_-(r, \vartheta) \begin{pmatrix} \sigma_2 \\
\vdots \\
\sigma_2 \end{pmatrix} .
\]

(3.46)

This solution is invariant under the subgroup $U(N) \subset SO(2N)$ acting as $U q_{(1)} U^T$, where $U \in U(N)$ commutes with the second matrix in (3.46).
In the $N$-spin state representation above, the vortex \(3.46\) corresponds to the state with all spins down, \(\downarrow\ldots\downarrow\). In order to see how the $N$-spin states transform under $SU(N) \subset SO(2N)$, construct the creation and annihilation operators

\[
a_j = \frac{1}{2}(\gamma_j - i \gamma_{N+j}) \quad ; \quad a_j^\dagger = \frac{1}{2}(\gamma_j + i \gamma_{N+j}) ,
\]
satisfying the algebra,

\[
\{a_j, a_k\} = \{a_j^\dagger, a_k^\dagger\} = 0 , \quad \{a_j, a_k^\dagger\} = \delta_{jk} .
\]

$SU(N)$ generators acting on the spinor representation, can be constructed as \[27\]

\[
T^a = \sum_{j,k} a_j^\dagger (t^a)_{jk} a_k ,
\]

where $t^a$ are the standard $N \times N$ $SU(N)$ generators in the fundamental representation. The state \(\downarrow\ldots\downarrow\) is clearly annihilated by all $T^a$, as it is annihilated by all

\[
a_k = \tau_3 \otimes \cdots \otimes \tau_3 \otimes \tau_- \otimes 1 \otimes \cdots \otimes 1 , \quad k = 1, 2, \ldots N :
\]

thus, the vortex \(3.46\) leaves $U(N)$ invariant.

All other solutions can be obtained as $Rq_{(1)} R^T$ with $R \in O(2N)$, so each solution is invariant under an appropriate $U(N)$ subgroup $RU R^T$. This means that the moduli space contains two copies of the coset space

\[
\mathcal{M} = SO(2N)/U(N) . \tag{3.47}
\]

The points in each coset space transform according to a spinor representation of definite chirality, each with dimension $2^{N-1}$. When discussing the topological properties of vortices, we will see that these disconnected parts correspond to different elements of the homotopy group.

Vortices of higher windings are described by $N_0 > 1$. In the simplest non-minimal case, the vortices are described by:

\[
N_0 = 2 , \quad \begin{pmatrix} 2 & 0 \\ 2 & 0 \\ \vdots & \vdots \\ 2 & 0 \\ 2 & 0 \end{pmatrix} , \quad \begin{pmatrix} 2 & 0 \\ 2 & 0 \\ \vdots & \vdots \\ 2 & 0 \\ 2 & 0 \end{pmatrix} , \quad \begin{pmatrix} 2 & 0 \\ 2 & 0 \\ \vdots & \vdots \\ 2 & 0 \\ 2 & 0 \end{pmatrix} , \quad \begin{pmatrix} 2 & 0 \\ 2 & 0 \\ \vdots & \vdots \\ 2 & 0 \\ 2 & 0 \end{pmatrix} , \quad \begin{pmatrix} 2 & 0 \\ 2 & 0 \\ \vdots & \vdots \\ 2 & 0 \\ 2 & 0 \end{pmatrix} . \tag{3.48}
\]

These orbits correspond to parts of the moduli space whose structure corresponds to the coset spaces $SO(2N)/U(N - k) \times SO(2k)$, where $k$ is the number of $(1,1)$ pairs. Analogously vortices with $N_0 \geq 3$ can be constructed.

The argument that the minimum vortices transform as two spinor representations implies that the $N_0 = 2$ vortices \(3.48\) transform as various irreducible antisymmetric tensor representations of $SO(2N)_{C+F}$, appearing in the decomposition of products of two spinor representations: e.g.

\[
2^{N-1} \otimes 2^{N-1} \text{ or } 2^{N-1} \otimes \overline{2^{N-1}} , \tag{3.49}
\]

Although all these vortices are degenerate in the semi-classical approximation, non-BPS corrections will lift the degeneracy, leaving only the degeneracy among the vortices transforming as an irreducible multiplet of the group $SO(2N)_{C+F}$. For instance the last vortex $n_\alpha^T = n_\alpha = 1$, for all $a$, carries only the unit $U(1)$ winding and is a singlet, the second last vortex and analogous ones belong to a $2N$, and so on.
Figure 3: Numerically integrated minimum vortex solution with \( N_0 = 2 \), where we have taken the couplings to be \( 4g_1^2 = 4g_{2N}^2 = 1 \). In the left panel we have shown the element \((n^+, n^-) = (1, 1)\) and in the right panel the element \((n^+, n^-) = (2, 0)\). The dependence of the couplings turns out to be similar to the case of the minimal vortex \((n^+, n^-) = (1, 0)\). \((\chi_i \equiv \sqrt{2} f_i)\).

Due to the fact that the tension depends only on \( N_0 = 2 n^{(0)} \) (twice the \( U(1) \) winding) the degeneracy pattern of the vortices does not simply reflect the homotopy map which relates the vortices to the massive monopoles. The monopole-vortex correspondence will be discussed in Section 5 below.

The profile functions \((f_0, f_a, \varphi_+^a, \varphi_-^a)\) for the simplest non-minimal vortex, \( N_0 = 2 \) are illustrated in Figure 3. In the figure is just considered the two simplest elements \((n^+, n^-) = (1, 1)\) and \((n^+, n^-) = (2, 0)\). Adding elements of the same type corresponds just to a rescaling of the coupling \( g_2^2 \) and of the functions \( \varphi^\pm_a \) as in the minimal vortex case \((N_0 = 1)\). Adding elements of different types \(((2, 0)\) or \((1, 1)\)) does not induce new behavior.

4 Vortices in \( SO(2N + 1) \) theories

Consider now the case of a theory with symmetry breaking

\[
SO(2N + 3) \xrightarrow{v_1} SO(2N + 1) \times U(1) \xrightarrow{v_2} 1 .
\]

The fields which remain massless after the first symmetry breaking can be found exactly as in the even \( SO \) theories by use of various \( SO(3) \) groups, leading to Eq. (2.15), with \( A = 1, 2, \ldots, N_f \) where we now take \( N_f = 2N + 1 \). The light quarks can get color-flavor locked VEVs as in Eq. (3.12), leading to a vacuum with global \( SO(2N + 1)_{C+F} \) symmetry.

The ansatz (3.37) must be modified as follows

\[
q(r, \vartheta) = \begin{pmatrix}
M_1(r, \vartheta) & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
\vdots & \cdots & M_N(r, \vartheta) & 0 \\
0 & \cdots & 0 & e^{i\hat{n}\vartheta}\hat{\varphi}(r)
\end{pmatrix},
\]

introducing a new integer \( \hat{n} \) and a new profile function \( \hat{\varphi}(r) \). The equation (3.31) becomes

\[
\frac{df_0}{dr} = -2\sqrt{2} n g_1^2 r \left( \sum_a (|\varphi_+^a|^2 + |\varphi_-^a|^2) + |\hat{\varphi}|^2 - \xi \right),
\]
Figure 4: Numerically integrated minimum vortex solution of the $SO(2N + 1)$ theory, with $N_0 = 2$ and we take the couplings to be $4g_1^2 = 4g_{2N}^2 = 1$. In the left panel we have $(n_1^+, n_1^-) = (1, 1)$ and in the right panel $(n_1^+, n_1^-) = (2, 0)$. The dependence of the couplings turns out to be analogous to the case of the $(n_1^+, n_1^-) = (1, 0)$ vortex. $(\chi_i = \sqrt{2}f_i)$.

while the condition of finite energy gives

$$\hat{\phi}(\infty) = \sqrt{\frac{\xi}{2N + 1}},$$

$$\hat{n} = \frac{h_0(\infty)}{\sqrt{2}} = \frac{N_0}{2},$$

and the equation for $\hat{\phi}(r)$ is

$$\frac{d\hat{\phi}}{dr} = \eta \left( \hat{n} - \frac{h_0}{\sqrt{2}} \right) \frac{\hat{\phi}}{r} = -\eta \frac{f_0}{\sqrt{2}} \frac{\hat{\phi}}{r}. \tag{4.6}$$

Note that the condition $(4.5)$ fixes $\hat{n}$ in terms of $N_0$: as $\hat{n}$ must be an integer, this theory contains only vortices with even $N_0$. This can be traced to the different structure of the gauge groups. In fact, $SO(2N + 3)$ has no center, so the pattern of symmetry breaking is

$$SO(2N + 3) \rightarrow SO(2N + 1) \times U(1) \rightarrow 1,$$ \tag{4.7}

and there are no vortices with half-integer winding around the $U(1)$, or around any other Cartan $U(1)$ subgroups.

The vortices are classified by the same integers $n_a^\pm$ as before, but now there are $SO(2N + 1)_{C+F}$ transformations which exchange $n_a^+ \leftrightarrow n_a^-$ singly. The minimal vortices are labeled by

$$(n_a^+, n_a^-) = \begin{pmatrix} 2 & 0 \\ 2 & 0 \\ \vdots & \vdots \\ 2 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ \vdots & \vdots \\ 0 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \hat{n} = 1. \tag{4.8}$$

The moduli space contains subspaces corresponding to these orbits, whose structure is that of the coset spaces $SO(2N + 1)/ (U(N - k) \times SO(2k + 1))$ where $k$ is the number of $(1, 1)$ pairs.

The vortex profile functions are shown in Figure 4.
5 Monopoles, vortices, topology and confinement

5.1 Homotopy map

The multiplicity of vortex solutions depends on the particular topology of the symmetry-breaking pattern of our model.

Usually, in systems with a gauge Lie group $G$ and a symmetry-breaking pattern

$$G \xrightarrow{v_1} H \xrightarrow{v_2} \mathbb{1},$$

there are:

- Stable Dirac monopoles, classified by $\pi_1(G)$;
- Regular monopoles, classified by $\pi_2(G/H)$; topologically stable only in the limit $v_2 \to 0$;
- Vortices, classified by $\pi_1(H)$; if they correspond to a non-trivial element of $\pi_1(G)$, they are topologically stable; otherwise they are topologically stable only in the limit $v_1 \to \infty$.

Monopoles and vortices are related by the topological correspondence \[8\]

$$\pi_2(G/H) = \pi_1(H)/\pi_1(G),$$

so regular monopoles correspond to vortices which are trivial with respect to $\pi_1(G)$, while vortices which are non-trivial with respect to $\pi_1(G)$ correspond to Dirac monopoles.

In our theories of type $D_N$, however, the center $C_G = \mathbb{Z}_2$ acts trivially on all fields and the breaking pattern is

$$G \xrightarrow{v_1} H \xrightarrow{v_2} C_G,$$

and the topological relation (5.2) is not directly useful. In fact, vortices are classified by $\pi_1(H/C_G)$, which is a richer homotopy group than $\pi_1(H) \sim \pi_2(G/H) \times \pi_1(G)$. In our example the relevant group is

$$\pi_1\left(\frac{SO(2N) \times U(1)}{\mathbb{Z}_2}\right) = \mathbb{Z} \times \mathbb{Z}_2.$$

The failure of (5.2) would mean that the correspondence between monopoles and vortices is lost.

Actually, it is better to formulate the problem as follows. The theory contains only fields in the adjoint representation, so we can neglect the center $C_G$ from the beginning and consider the gauge group as $G' = G/C_G$. In our example, the gauge group of the high-energy theory can be taken as $G' = SO(2N + 2)/\mathbb{Z}_2$, broken to $H' = (SO(2N) \times U(1))/\mathbb{Z}_2$ at scale $v_1$ and then completely broken at scale $v_2$:

$$G' \xrightarrow{v_1} H' \xrightarrow{v_2} \mathbb{1},$$

instead of Eq. (5.3). Then the relation (5.2) reads

$$\pi_2\left(\frac{SO(2N + 2)}{SO(2N) \times U(1)}\right) = \frac{\pi_1\left(\frac{SO(2N) \times U(1)}{\mathbb{Z}_2}\right)}{\pi_1\left(\frac{SO(2N + 2)}{\mathbb{Z}_2}\right)}.$$

Regular monopoles are classified by the same homotopy group as before, because

$$\frac{SO(2N + 2)/\mathbb{Z}_2}{(SO(2N) \times U(1))/\mathbb{Z}_2} = \frac{SO(2N + 2)}{SO(2N) \times U(1)},$$

(5.7)
while for Dirac monopoles the situation is different: the relevant homotopy group is not $\pi_1\left(\text{SO}(2N + 2)\right)$, but the larger group $\pi_1\left(\text{SO}(2N + 2)/\mathbb{Z}_2\right)$ (see Appendix B)

$$\pi_1\left(\frac{\text{SO}(4J)}{\mathbb{Z}_2}\right) = \mathbb{Z}_2 \times \mathbb{Z}_2,$$

while

$$\pi_1\left(\frac{\text{SO}(4J + 2)}{\mathbb{Z}_2}\right) = \mathbb{Z}_4,$$

so that the Dirac monopoles have quantized $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$ charges.

This means that the theory has a larger set of monopoles, and the correspondence between monopoles and vortices (which confine them) is rather subtle

In appendix B we briefly review the structure of the homotopy groups which are relevant for this analysis.

Finally, for the groups of type $B_N$, the situation is slightly simpler as there is no non-trivial center. The non-trivial element of $\pi_1\left(\text{SO}(2N + 3)\right) = \mathbb{Z}_2$ represents the (unique type of) Dirac monopoles; the elements of $\pi_1\left(\text{SO}(2N + 1) \times \text{U}(1)\right) = \mathbb{Z}_2 \times \mathbb{Z}_2$ label the vortices of the low-energy theory. The vortices whose (non-trivial) winding in the group $\text{SO}(2N + 1) \times \text{U}(1)$ corresponds to a contractible loop in the parent theory, confine the regular monopoles.

### 5.2 Flux matching

To establish the matching between regular GNO monopoles and low-energy vortices, we use the topological correspondence discussed in the previous section. Dirac monopoles are classified by $\pi_1\left(\frac{\text{SO}(2N + 2)}{\mathbb{Z}_2}\right)$ or by $\pi_1\left(\text{SO}(2N + 3)\right)$ depending on the gauge group, but regular monopoles are classified by $\pi_2\left(\frac{\text{SO}(2N + 2)}{\text{SO}(2N) \times \text{U}(1)}\right)$ or by $\pi_2\left(\frac{\text{SO}(2N + 3)}{\text{SO}(2N + 1) \times \text{U}(1)}\right)$, i.e. homotopically non-trivial paths in the low-energy gauge group, which are trivial in the high-energy gauge group. Regular monopoles can be sources for the vortices corresponding to these paths.

The vortices of the lowest tension which satisfy this requirement are those with $N_0 = \pm 2$ and $\sum \left(n^+_a - n^-_a\right)/2$ odd, so vortices corresponding to minimal GNO monopoles belong to the $\text{SO}(2N)_{C+F}$ orbits classified by (3.48) with an odd number of $(\pm 2, 0)$ pairs.

For a better understanding of this correspondence, we can also use flux matching between vortices and monopoles [22]. There are $2N$ GNO monopoles obtained by different embeddings of broken $\text{SU}(2) \subset \text{SO}(4)$ in $\text{SO}(2N + 2)$. In a gauge where $\phi$ is constant, their fluxes are

$$\int_{S^2} d^2 \bar{S} \cdot \vec{B}^a t^a = 2\sqrt{2}\pi(t_0 \pm t_i),$$

where $t_0 \pm t_i$ is the unbroken generator of the broken $\text{SU}(2)$ subgroup. In the same gauge, the flux of a vortex is

$$\int_{S^2} d^2 x \vec{B}^a t^a = -N_0 \sqrt{2}\pi t_0 + \left(n^+_j - n^-_j\right) \sqrt{2}\pi t_j,$$

so the fluxes agree for $N_0 = -2, n^+_j - n^-_j = \pm 2\delta_{ij}$. The antimonopoles correspond to the opposite sign $N_0 = 2$.

*Note that the Lagrangian and fields for the two theories with gauge group $\text{SO}(2N + 2)$ and $\text{SO}(2N + 2)/\mathbb{Z}_2$ are the same. The set of vortices is the same for both theories and has a topological correspondence with the larger set of monopoles.
5.3 Monopole confinement: the $SO(2N)$ theory

We have now all the tools needed to analyze the duality in the $SO$ theories at hand. The general scheme for mapping the monopoles and vortices has been set up in Section 5.1. An important point to keep in mind is that, while the vortex tension depends only on the $U(1)$ flux in our particular model (Eq. (3.27)), the classification of vortices according to the first homotopy group reflects the other Cartan charges (w windings in $SO(2N)$ or $SO(2N + 1)$). It is necessary to keep track of these to see how the vortices in the low-energy theory are associated with the monopoles of the high-energy system.

First consider the theories of type $D_N$, with the symmetry breaking

$$SO(2N + 2) \xrightarrow{v_1} SO(2N) \times U(1) \xrightarrow{v_2} 1.$$  

(5.12)

studied in detail in the preceding sections. The vortices with minimum winding, $N_0 = 1$, of Eqs. (3.40), (3.41), correspond to the minimum non-trivial element of $\pi_1((SO(2N) \times U(1))/\mathbb{Z}_2)$, which represent also the minimal elements of $\pi_1(SO(2N + 2)/\mathbb{Z}_2)$. This last fact means that they are stable in the full theory. They would confine Dirac monopoles of the minimum charge in the underlying theory, $1$ of $\mathbb{Z}_4$ or $(1,0)$ or $(0,1)$ of $\mathbb{Z}_2 \times \mathbb{Z}_2$, see Appendix B.2.

Consider now the vortices Eq. (3.48) with $N_0 = 2$. As the fundamental group of the underlying theory is given by either Eq. (5.8) or Eq. (5.9), some of the vortices will correspond to non-contractible loops in the underlying gauge group: they would be related to the Dirac monopoles and not to the regular monopoles. Indeed, consider the last of Eq. (3.48):

$$\begin{pmatrix} n^{-}_{a} \\ n^{+}_{a} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$  

(5.13)

It is characterized by the windings $n^{(0)} = 1$, $n^{(a)} = 0$ for all $a$. Thus it is an ANO vortex of the $U(1)$ theory, with no flux in the $SO(2N)$ part. It corresponds to a $2\pi$ rotation in $(12)$ plane in the original $SO(2N + 2)$ group – the path $P$ in Appendix B.1: it is to be associated with a Dirac monopole of charge 2.

The vortices of the type

$$\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 2 & 1 & 1 & \cdots & 1 \end{pmatrix},$$  

(5.14)

and analogous ones (with $(2,0)$ or $(0,2)$ appearing in different positions) are characterized by the two $U(1)$ windings only: a flux $n^{(0)} = 1$ and one of the Cartan flux of $SO(2N)$, e.g., $n^{(1)} = 1$ ($n^{(a)} = 0$, $a \neq 1$). They correspond to a simultaneous $2\pi$ rotations in $(12)$ and in $(34)$ planes in the gauge group and it represents a contractible loop in the high-energy gauge group. They confine regular monopoles, as can be seen also by the flux matching argument discussed in section 5.2.

Part of the continuous moduli of these vortex solutions include

$$SO(2N)/U(1) \times SO(2N - 2),$$  

(5.15)

as the individual soliton breaks $SO(2N)_{C+F}$ symmetry of the system. This space corresponds to the complex quadric surface $Q^{2N-2}(C)$. As these vortices are not elementary but composite of the minimal vortices, determining their correct moduli space structure is not a simple task.

Nevertheless, there are some indications that these correspond to a vector representation $2N$ of $SO(2N)_{C+F}$, appearing in the decomposition of the product of two spinor representations, Eq. (3.49). In fact, the vortex Eq. (5.14) arises as a product

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix};$$  

(5.16)
i.e., a product of two spinors of the same chirality if $N$ is odd; vice versa, of spinors of opposite chirality if $N$ is even. This corresponds precisely to the known decomposition rules in $SO(4m + 2)$ and $SO(4m)$ groups (see e.g., [27], Eq. (23.40)).

In order to establish that these vortices indeed transform under the $SO(2N)_{C + F}$ as a $2N$ one needs to construct the moduli matrix [24] for these, and study explicitly how the points in the moduli space transform. This problem will be studied elsewhere.

It is interesting to note that there seems to be a relation between the transformation properties of monopoles under the dual GNO group $\tilde{SO}(2N)$ and the transformation properties of the corresponding vortices under the $SO(2N)_{C + F}$ group. In fact, vortices transforming as a vector of $SO(2N)_{C + F}$ have precisely the net magnetic flux of regular monopoles in $2N$ of $SO(2N)$, as classified by the GNO criterion.

Other vortices in Eq. (3.48) correspond to various Dirac (singular) or regular monopoles in different representations of $SO(2N)_{C + F}$.

### 5.4 Monopole confinement: the $SO(2N + 1)$ theory

In the $B_N$ theories with the symmetry breaking

$$SO(2N + 3) \xrightarrow{\pi_3} SO(2N + 1) \times U(1) \xrightarrow{\nu_2} 1,$$

the minimal vortices of the low-energy theory have $N_0 = 2$. Reflecting the difference of $\pi_1$ group of the underlying theory as compared to the $D_N$ cases ($\mathbb{Z}_2$ as compared to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$), the $N_0 = 1$ vortices (with half winding in $U(1)$ and $SO(2N)$) are absent here.

The minimal vortices (4.38) again correspond to different homotopic types and to various $SO(2N + 1)$ representations. The vortex

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & \ldots & 1 \end{pmatrix}, \quad \hat{n} = 1,$$

has the $U(1)$ charge $n^{(0)} = 1$ and no charge with respect to $SO(2N + 1)$. It is associated to the non-trivial element of $\pi_1(SO(2N + 3)) = \mathbb{Z}_2$: it is stable in the full theory. Its flux would match that of a Dirac monopole. This is a singlet of $SO(2N + 1)_{C + F}$ (its moduli space consists of a point).

Consider instead the vortices

$$\begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 2 & 1 & 1 & \ldots & 1 \end{pmatrix}, \quad \hat{n} = 1,$$

and analogous ones, having the winding numbers $n^{(0)} = 1$, $n^{(a)} = \pm 1$, $n^{(b)} = 0$, $b \neq a$, and $\hat{n} = 1$. These would correspond to regular monopoles which, according to GNO classification, are supposed to belong to a $2N$ representation of the dual group $USp(2N)$. Again, though it is not a trivial task to establish that these vortices do transform as $2N$ of such a group, there are some hints they indeed do so. It is crucial that the symmetry group (broken by individual soliton vortices) is $SO(2N + 1)$: it is in fact possible to identify the $2N$ generators constructed out of those of $SO(2N + 1)$, that transform them appropriately (Appendix). Secondly, the flux matching argument of Section 5.2 do connect these vortices to the minimum, regular monopoles appearing in the semiclassical analysis. As in the $D_N$ theories these observations should be considered at best as a modest hint that dual group structure as suggested by the monopole-vortex correspondence is consistent with the GNO conjecture.
In this paper we have explicitly constructed BPS, non-Abelian vortices of a class of $SO(N) \times U(1)$ gauge theories in the Higgs phase. The models considered here can be regarded as the bosonic part of softly broken $\mathcal{N} = 2$ gauge theories with $N_f$ quark matter fields. The vortices considered here represent non-trivial generalizations of the non-Abelian vortices in $U(N)$ models widely studied in recent literature.

The systems are constructed so that they arise as low-energy approximations to theories in which gauge symmetry suffers from a hierarchical breaking

$SO(N + 2) \xrightarrow{v_1} SO(N) \times U(1) \xrightarrow{v_2} 1$, \quad v_1 \gg v_2 \tag{6.1}$

leaving an exact, unbroken global ($SO(N) \times U(1))_{C+F}$ symmetry. Even though the low-energy $SO(N) \times U(1)$ model with symmetry breaking

$SO(N) \times U(1) \xrightarrow{v_2} 1 \tag{6.2}$

can be studied on its own right, without ever referring to the high-energy $SO(N + 2)$ theory, consideration of the system with hierarchical symmetry breaking is interesting as it forces us to try (and hopefully allows us) to understand the properties of the non-Abelian monopoles in the high-energy approximate system with $SO(N + 2) \xrightarrow{v_1} SO(N) \times U(1)$ and their confinement by the vortices – language adequate in the dual variables – from the properties of the vortices via homotopy map and symmetry argument. Note that in this argument, the fact that the monopoles in the high-energy theory and the vortices in the low-energy theory are both almost BPS but not exactly so, is of fundamental importance [9, 28].

In the models based on $SU(N)$ gauge symmetry, the efforts along this line of thought seem to be starting to give fruits, giving some hints on the nature of non-Abelian duality and confinement. Although the results of this paper are a only a small step toward a better and systematic understanding of these questions in a more general class of gauge systems, they provide a concrete starting point for further studies.

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A \quad SO(2N), USp(2N), SO(2N + 1)

The change of basis to the one where a vector multiplet $2N$ of $SO(2N)$ naturally breaks to $N + \bar{N}$ under $U(N)$, is given by (see Eq. (2.28))

$$
\begin{pmatrix}
\hat{Q}_3 \\
\vdots \\
\hat{Q}_{2N+1} \\
-iQ_3 \\
\vdots \\
-iQ_{2N+1}
\end{pmatrix} =
\begin{pmatrix}
1/\sqrt{2} & -i1/\sqrt{2} \\
-i1/\sqrt{2} & 1/\sqrt{2}
\end{pmatrix}
\begin{pmatrix}
q_3 \\
\vdots \\
q_{2N+1} \\
q_4 \\
\vdots \\
q_{2N+2}
\end{pmatrix}.
$$

(A.1)

The $SO(2N)$ generators,

$$
\begin{pmatrix}
E & F \\
-iF & D
\end{pmatrix},
$$

(A.2)

where $D$, $E$, $F$ are all pure imaginary $N \times N$ matrices, with the constraints $^tE = -E$, $^tD = -D$, are accordingly transformed as

$$
\begin{pmatrix}
1/\sqrt{2} & -i/\sqrt{2} \\
-i/\sqrt{2} & 1/\sqrt{2}
\end{pmatrix}
\begin{pmatrix}
E & F \\
-iF & D
\end{pmatrix}
\begin{pmatrix}
1/\sqrt{2} & i/\sqrt{2} \\
i/\sqrt{2} & 1/\sqrt{2}
\end{pmatrix}
= \frac{1}{2}
\begin{pmatrix}
(E + D) + i(F + ^tF) & i(E - D) + (F - ^tF) \\
-i(E - D) + (F - ^tF) & (E + D) - i(F + ^tF)
\end{pmatrix}.
$$

(A.3)

Since both $E$, $D$ are anti-symmetric, $(E + D)$ in the 1st block is the most general anti-symmetric imaginary matrix, while $i(F + ^tF)$ is the most general symmetric real matrix. Their sum gives the most general $N \times N$ hermitian matrix, which corresponds to generators of $U(N)$. In other words, the subgroup $U(N) \subset SO(2N)$ is generated by those elements with $E = D$, $F = ^tF$.

On the other hand, the generators of $USp(2N)$ group have the form

$$
\begin{pmatrix}
B & A \\
C & -^tB
\end{pmatrix},
$$

(A.4)

with the constraints, $^tA = A$, $^tC = C$, $A^* = C$, $B^t = B$. The fact that $A$ is symmetric while the non-diagonal blocks in Eq. (A.3) are antisymmetric, means that there is no further overlap between the two groups, that is, the maximal common subgroup between $SO(2N)$ and $USp(2N)$ is $U(N)$.
It is possible to get a hint on how $USp(2N)$ groups can appear as transformation group of the vortices. In order to see transformations among the vortices $(Q, \bar{Q})$ under which the latter could transform as $2N$, it is necessary to embed the system in a larger group, such as $SO(2N + 1)$ model considered in Section 3. The idea is to build a map between the $SO(2N + 1)$ generators (antisymmetric matrices) and the $USp(2N)$ generators which have the form, Eq. (A.4). The $i$th $SO(4) \sim SU(2) \times SU(2)$ subgroup is generated by (with a simplified notation $(1, 2, 3, 4) \equiv (1, 2i + 1, 2i + 2)$)

$$T_{1}^{\pm} = -\frac{i}{2} (\Sigma_{23} \pm \Sigma_{41}) , \quad T_{2}^{\pm} = -\frac{i}{2} (\Sigma_{31} \pm \Sigma_{42}) , \quad T_{3}^{\pm} = -\frac{i}{2} (\Sigma_{12} \pm \Sigma_{43}) .$$

The two vortices living in this $SO(4)$ group are taken to be $i$-th and $(N + i)$-th components of the fundamental representation of $USp(2N)$. The pairs can be transformed to each other by rotations in the $(2i + 2, 2N + 3)$ plane (⊂ $SO(2N + 1)$), thus

$$A_{i,i} = -i \Sigma_{2i+2,2N+3} .$$

On the other hand, the two vortices associated with subgroups $T_{T}^{\pm}$ living in the $(1, 2, 2i + 1, 2i + 2)$ subspace and those living in the $(1, 2, 2j + 1, 2j + 2)$ subspace, $j \neq i$, are transformed into each other by rotations in the $(2i + 1, 2i + 2, 2j + 1, 2j + 2)$ space: they transform in $SO(2N)$ (in the subspace $i = 3, 4, \ldots , 2N + 2$). We have already seen that they actually do transform as a pair of $U(N)$ representations, in the basis Eq. (A.1). As the $U(N)$ elements are generated by the $SO(2N)$ infinitesimal transformations with $E = D, F = tF$, one finds the map,

$$B_{i,j} = -i (\Sigma_{2i,2j} + \Sigma_{2i+1,2j+1}) + (\Sigma_{2i,2j+1} - \Sigma_{2i+1,2j}) .$$

Non-diagonal elements $A_{ij}, i \neq j$, can be generated by commuting the actions of (A.6) and (A.7).

## B Fundamental groups

Let’s briefly discuss the (first) homotopy groups relevant to us:

### B.1 $SO(2N + 2)$

There is only one non-trivial closed path $P$ in this case, the rotation from 0 to $2\pi$ around any axis. The rotation from 0 to $4\pi$ is homotopically equivalent to the trivial path, so $P^{2} = 1$ and the homotopy group is

$$\pi_{1}(SO(2N + 2)) = \mathbb{Z}_{2} ,$$

### B.2 $SO(2N + 2)/\mathbb{Z}_{2}$

Actually, in the model discussed in this paper, all the fields are in the adjoint representation of $SO(2N + 2)$: the gauge group effectively corresponds to $SO(2N + 2)$ modulo identification $-1 = 1$. The path $P$ is again non-trivial, but now there are also two inequivalent closed paths $P_{+}$ and $P_{-}$ going from 1 to $-1$, defined as $P_{+}P_{-}^{-1} = P$. Explicitly, they can be taken as simultaneous rotations

\footnote{This correspondence can be applied equally well to the minimal regular monopoles constructed semi-classically, and has been discussed in this context in [25].}

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in $N + 1$ planes

\[ P_+ : \quad e^{i\beta_{12}\Sigma_{12}} \prod_{i=3,5,...,N-1} e^{i\beta_{i+1}\Sigma_{i,i+1}} ; \quad \beta_{12} : 0 \to \pi , \quad \beta_{i,i+1} : 0 \to \pi . \quad \text{(B.2)} \]

\[ P_- : \quad e^{i\beta_{12}\Sigma_{12}} \prod_{i=3,5,...,N-1} e^{i\beta_{i+1}\Sigma_{i,i+1}} ; \quad \beta_{12} : 0 \to -\pi , \quad \beta_{i,i+1} : 0 \to \pi . \quad \text{(B.3)} \]

When $N + 1$ is even, $P_+^2 = P_-^2 = 1$ and $P_+P_- = P$. The homotopy group is generated by $P_+P_- = 1$.

When $N + 1$ is odd, $P_+^2 = P_-^2 = P$ and $P_+P_- = 1$, so the homotopy group is generated by $P_+$ only, and is of cyclic order four

\[ \pi_1 \left( \frac{SO(4N)}{\mathbb{Z}_2} \right) = \mathbb{Z}_4 . \quad \text{(B.5)} \]

B.3 \quad $SO(2N) \times U(1)/\mathbb{Z}_2$

After the symmetry breaking at the higher mass scale $v_1$, the theory reduces to an $(SO(2N) \times U(1))/\mathbb{Z}_2$ theory. The division by $\mathbb{Z}_2$ corresponds to the identification $(-1, -1) = (1, 1)$, inherited from the underlying theory. From the point of view of the low-energy effective theory, it is due to the fact that all the light matter fields $q_{A,j}, \tilde{q}_{A,j}$ are in the vector representation of $SO(2N)$ but they carry at the same time the unit charge with respect to $U(1)$.

The non-trivial paths of $SO(2N) \times U(1)$ are combinations of $Q$ (a $2\pi$ rotation in any plane in $SO(2N)$) and the paths $R_n$ winding $n$ times around the $U(1)$. The simplest non-trivial closed paths that arise after the $\mathbb{Z}_2$ quotient are $P_{+,\frac{1}{2}}, P_{+,-\frac{1}{2}}, P_{-,\frac{1}{2}}, P_{-,-\frac{1}{2}}$ going from $(1, 1)$ to $(-1, -1)$ with a half winding around $U(1)$. By taking $U(1)$ to act in the $(12)$ plane, $SO(2N)$ in the $(34 \ldots N)$ space, they can be explicitly chosen as simultaneous rotations in $(12), (34), (56) \ldots$ planes

\[ e^{i\gamma_{12}\Sigma_{12}} e^{i\beta_{34}\Sigma_{34}} \prod_{i=5,7,...,N-1} e^{i\beta_{i+1}\Sigma_{i,i+1}} ; \quad \text{(B.6)} \]

with

\[ P_{+,\frac{1}{2}} : \quad \gamma_{12} : 0 \to \pi , \quad \beta_{34} : 0 \to \pi , \quad \beta_{i,i+1} : 0 \to \pi . \quad \text{(B.7)} \]

\[ P_{+,-\frac{1}{2}} : \quad \gamma_{12} : 0 \to -\pi , \quad \beta_{34} : 0 \to -\pi , \quad \beta_{i,i+1} : 0 \to \pi . \quad \text{(B.8)} \]

\[ P_{-,\frac{1}{2}} : \quad \gamma_{12} : 0 \to \pi , \quad \beta_{34} : 0 \to -\pi , \quad \beta_{i,i+1} : 0 \to \pi . \quad \text{(B.9)} \]

\[ P_{-,-\frac{1}{2}} : \quad \gamma_{12} : 0 \to -\pi , \quad \beta_{34} : 0 \to -\pi , \quad \beta_{i,i+1} : 0 \to \pi . \quad \text{(B.10)} \]

Note that $P_{+,\frac{1}{2}}$ and $P_{+,-\frac{1}{2}}$ correspond respectively to the $P_+$ and $P_-$ paths in the $SO(2N+2)$ theory.

When $N$ is even, $P_{+,a}P_{+,b} = P_{-,a}P_{-,b} = R_{a+b}$ and $P_{+,a}P_{-,b} = Q R_{a+b}$, so every group element can be written as $(P_{+,1/2})^k Q^\delta$ with $k \in \mathbb{Z}$, $\delta = \{0, 1\}$. The homotopy group is

\[ \pi_1 \left( \frac{SO(2N) \times U(1)}{\mathbb{Z}_2} \right) = \mathbb{Z} \times \mathbb{Z}_2 , \quad N \text{ even} , \quad \text{(B.11)} \]

When $N$ is odd, $P_{+,a}P_{+,b} = P_{-,a}P_{-,b} = Q R_{a+b}$ and $P_{+,a}P_{-,b} = R_{a+b}$, and every group element can again be written as $(P_{+,1/2})^k Q^\delta$ with $k \in \mathbb{Z}$, $\delta = \{0, 1\}$, as in the $N$ even case. The homotopy group is

\[ \pi_1 \left( \frac{SO(2N) \times U(1)}{\mathbb{Z}_2} \right) = \mathbb{Z} \times \mathbb{Z}_2 , \quad N \text{ odd} , \quad \text{(B.12)} \]
Even though the homotopy group is the same for the two cases ($N$ even or odd), its embedding in $\pi_1 (SO(2N) \times U(1)) = \mathbb{Z} \times \mathbb{Z}_2$ is different: $R_n$ corresponds to $k = 2n, \delta = 0$ for $N$ even and to $k = 2n, \delta = 1$ for $N$ odd. In other words

$$R_1 = (P_{+,1/2})^2 Q, \quad (N \text{ odd}) ; \quad R_1 = (P_{+,1/2})^2 \quad (N \text{ even}).$$  \hfill (B.13)

**B.4 Relation between the smallest elements of the high-energy and low-energy fundamental groups**

There are simple relations among the smallest elements of the groups $\pi_1 \left( \frac{SO(2N+2)}{\mathbb{Z}_2} \right)$ and $\pi_1 \left( \frac{SO(2N) \times U(1)}{\mathbb{Z}_2} \right)$. From the above explicit constructions one sees that

$$P_+ = P_{+,1/2} ; \quad P_- = P_{+,1/2} = R_{-1} P_{+,1/2} ;$$  \hfill (B.14)

and by using Eq. (B.13), one has

$$P_{+,1/2} = \begin{cases} (P_{+,1/2})^{-1} Q, & \text{odd } N , \\ (P_{+,1/2})^{-1}, & \text{even } N . \end{cases}$$  \hfill (B.15)

**B.5 \text{SO}(2N+3)**

The fundamental group is $\mathbb{Z}_2$ as in the $SO(2N+2)$ cases, and the smallest closed path being

$$P : \quad e^{i\beta_{ij}\Sigma_{ij}} : \quad \beta_{ij} = 0 \rightarrow 2\pi ,$$  \hfill (B.16)

in any plane $(ij)$. $P^2 = 1$ and the homotopy group is

$$\pi_1 (SO(2N+3)) = \mathbb{Z}_2 .$$  \hfill (B.17)

**B.6 \text{SO}(2N+1) \times U(1)**

At the mass scales below $v_1$ the theory reduces to an $SO(2N+1) \times U(1)$ theory with matter in the fundamental representation, $q$ and $\tilde{q}$ carrying charges $\pm 1$ with respect to $U(1)$. The fundamental group is

$$\pi_1 (SO(2N+1) \times U(1)) = \mathbb{Z}_2 \times \mathbb{Z} ,$$  \hfill (B.18)

where $\mathbb{Z}$ represents the number of winding (charge) in the $U(1)$ part and $\mathbb{Z}_2$ a $2\pi$ rotation in any plane in $SO(2N+1)$.

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