Goldstone Bosons

H. Leutwyler
Institut für theoretische Physik der Universität Bern
Sidlerstr. 5, CH-3012 Bern, Switzerland

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Abstract

The paper concerns the effective field theory methods used to study the low energy structure of systems with a spontaneously broken symmetry. I first explain how the method works in the context of quantum chromodynamics and then discuss a few general aspects, related to the universality of effective theories. In particular, I compare some of the effective field theories used in condensed matter physics with those relevant for particle physics.

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Goldstone bosons occur in many areas of physics. I first discuss the phenomenon in the context of the strong interaction, where the pions play the role of the Goldstone bosons. Later on, I will identify those features which are independent of the particular system under consideration and compare the effective field theory used to analyze the low energy properties of the strong interaction with some of the effective theories encountered in condensed matter physics.

The strong interaction is mediated by the gauge field of colour, which binds the quarks to colour neutral bound states such as the proton or the neutron. The structure of the relevant gauge field theory, quantum chromodynamics, is similar to the one describing the electromagnetic interaction. As it is the case with the coupling between photons and electrons, the interaction of the gluons with the quarks is fully determined by gauge invariance. This implies, in particular, that the various different quark flavours, \( u, d, \ldots \) interact with the gluons in precisely the same manner. As far as the strong interaction is concerned, the only distinction between, say, an \( s \)-quark and a \( c \)-quark is that the mass is different. In this respect, the situation is the same as in electrodynamics, where the interaction of the charged leptons with the photon is also universal, such that the only difference between \( e, \mu \) and \( \tau \) is the mass. As an immediate consequence, the properties of a bound state like the \( \Lambda_s = (uds) \) are identical with those of the \( \Lambda_c = (udc) \), except for the fact that \( m_c \) is larger than \( m_s \).

1 Isospin symmetry

A striking property of the observed pattern of bound states is that they come in nearly degenerate isospin multiplets: \( (p, n) \), \( (\pi^+, \pi^0, \pi^-) \), \( (K^+, K^0) \), \( \ldots \). In fact, the splittings within these multiplets are so small that, for a long time, isospin was taken for an exact symmetry of the strong interaction; the observed small mass difference between neutron and proton or \( K^0 \) and \( K^+ \) was blamed on the electromagnetic interaction. We now know that this picture is incorrect: the bulk of isospin breaking does not originate in the electromagnetic fields, which surround the various particles, but is due to the fact that the \( d \)-quark is somewhat heavier than the \( u \)-quark.

From a theoretical point of view, the quark masses are free parameters — QCD makes sense for any value of \( m_u, m_d, \ldots \) It is perfectly legitimate
to compare the real situation with a theoretical one, where some of the quark masses are given values, which differ from those found in nature. In connection with isospin symmetry, the theoretical limiting case of interest is a fictitious world, with $m_u = m_d$. In this limit, the flavours $u$ and $d$ become indistinguishable. The Hamiltonian acquires an exact symmetry with respect to the transformation

$$
\begin{align*}
  u &\rightarrow \alpha u + \beta d \\
  d &\rightarrow \gamma u + \delta d \\
  V = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}
\end{align*}
$$

provided the $2 \times 2$ matrix $V$ is unitary, $V \in U(2)$. Even for $m_u \neq m_d$, the Hamiltonian of QCD is invariant under a change of phase of the quark fields. The extra symmetry, occurring if the masses of $u$ and $d$ are taken to be the same, is contained in the subgroup SU(2), which results if the phase of the matrix $V$ is subject to the condition $\det V = 1$. The above transformation law states that $u$ and $d$ form an isospin doublet, while the remaining flavours $s, c, \ldots$ are singlets.

In reality, $m_u$ differs from $m_d$. The isospin group SU(2) only represents an approximate symmetry. The piece of the QCD Hamiltonian, which breaks isospin symmetry, may be exhibited by rewriting the mass term of the $u$ and $d$ quarks in the form

$$
m_u \bar{u} u + m_d \bar{d} d = \frac{1}{2}(m_u + m_d)(\bar{u} u + \bar{d} d) + \frac{1}{2}(m_d - m_u)(\bar{d} d - \bar{u} u) .
$$

The remainder of the Hamiltonian is invariant under isospin transformations and the same is true of the operator $\bar{u} u + \bar{d} d$. The QCD Hamiltonian thus consists of an isospin invariant part $\bar{H}_0$ and a symmetry breaking term $\bar{H}_{sb}$, proportional to the mass difference $m_d - m_u$,

$$
H_{\text{QCD}} = \bar{H}_0 + \bar{H}_{sb} , \quad \bar{H}_{sb} = \frac{1}{2}(m_d - m_u) \int d^3 x (\bar{d} d - \bar{u} u) .
$$

The strength of isospin breaking is controlled by the quantity $m_d - m_u$, which plays the role of a symmetry breaking parameter. The fact that the multiplets are nearly degenerate implies that the operator $\bar{H}_{sb}$ only represents a small perturbation — the mass difference $m_d - m_u$ must be very small. QCD thus provides a remarkably simple explanation for the fact that the strong interaction is nearly invariant under isospin rotations: it so happens that the difference between $m_u$ and $m_d$ is small and this is all there is to it.
The symmetry breaking also shows up in the properties of the vector currents, e.g. in those of $\pi\gamma^\mu d$. The integral of the corresponding charge density over space, $I^+ = \int d^3x u^\dagger d$, is the isospin raising operator, converting a $d$-quark into a $u$-quark. The divergence of the current is given by
\[
\partial_\mu (\pi\gamma^\mu d) = i (m_u - m_d) \pi d ,
\]
and only vanishes for $m_u = m_d$, the condition for the charge $I^+$ to be conserved. In the symmetry limit, there are three such conserved charges, the three components of isospin, $\vec{I} = (I^1, I^2, I^3)$. The isospin raising operator considered above is the combination $I^+ = I^1 + i I^2$. Since $\Pi_0$ is invariant under isospin rotations, it conserves isospin,
\[
[\vec{I}, \Pi_0] = 0 .
\]

2 Chiral symmetry

The approximate symmetry of the Hamiltonian explains why the bound states of QCD exhibit a multiplet pattern, but does not account for an observation, which is equally striking and which plays a crucial role in strong interaction physics — the mass gap of the theory, $M_\pi$, is remarkably small. The approximate symmetry, hiding behind this observation, was discovered by Nambu \[\text{[1]}\]. It originates in a phenomenon, which is well-known from neutrino physics: right- and left-handed components of massless fermions do not communicate.

The symmetry, which forbids right-left-transitions, manifests itself in the properties of the axial vector currents, such as $\pi\gamma^\mu\gamma_5 d$. The corresponding continuity equation reads
\[
\partial_\mu (\pi\gamma^\mu\gamma_5 d) = i (m_u + m_d) \pi\gamma_5 d .
\]
While the divergence of the vector current $\pi\gamma^\mu d$ is proportional to the difference $m_u - m_d$, the one of the axial current is proportional to the sum $m_u + m_d$. If the two masses are set equal, the vector current is conserved and the Hamiltonian becomes symmetric with respect to isospin rotations. If they are not only taken equal, but equal to zero, then the axial current is conserved, too, such that the corresponding charge $I_5^+ = \int d^3x d^1\gamma_5 u$ also commutes with the Hamiltonian — QCD acquires an additional symmetry.
The isospin operator $I^+$ converts a $d$-quark into a $u$-quark, irrespective of the helicity. The operator $I_5^+$, however, acts differently on the right- and left-handed components. The sum $\frac{1}{2}(I^+ + I^+_5)$ takes a righthanded $d$-quark into a righthanded $u$-quark, but leaves left-handed ones alone. This implies that, for massless quarks, the Hamiltonian is invariant with respect to chiral transformations: independent isospin rotations of the right- and left-handed components of $u$ and $d$,

$$
\begin{pmatrix} u_R \\ d_R \end{pmatrix} \rightarrow V_R \begin{pmatrix} u_R \\ d_R \end{pmatrix}, \quad \begin{pmatrix} u_L \\ d_L \end{pmatrix} \rightarrow V_L \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad V_R, V_L \in SU(2).
$$

The corresponding symmetry group is the direct product of two separate isospin groups, SU(2)$_R \times$SU(2)$_L$. The symmetry is generated two sets of isospin operators: ordinary isospin, $\vec{I}$ and chiral isospin, $\vec{I}_5$. The particular operator considered above is the linear combination $I^+_5 = I^+_5 + i I^+_5$.

In reality, chiral symmetry is broken, because $m_u$ and $m_d$ do not vanish. As above, the Hamiltonian may be split into a piece which is invariant under the symmetry group of interest and a piece which breaks the symmetry. In the present case, the symmetry breaking part is the full mass term of the $u$ and $d$ quarks,

$$
H_{\text{QCD}} = H_0 + H_{sb}, \quad H_{sb} = \int d^4x (m_u \bar{u} u + m_d \bar{d} d).
$$

The symmetric part conserves ordinary as well as chiral isospin,

$$
[\vec{I}, H_0] = 0, \quad [\vec{I}_5, H_0] = 0.
$$

Note that the symmetry group exclusively acts on $u$ and $d$ — the remaining quarks $s, c, \ldots$ are singlets. The corresponding mass terms $m_s \bar{s}s + m_c \bar{c}c + \ldots$ do not break the symmetry and are included in $H_0$.

3 Spontaneous symmetry breakdown

Much before QCD was discovered, Nambu pointed out that chiral symmetry breaks down spontaneously. The phenomenon plays a crucial role for the properties of the strong interaction at low energy. To discuss it, I return to the theoretical scenario, where $m_u$ and $m_d$ are set equal to zero.
In this framework, isospin is conserved. The isospin group SU(2) represents the prototype of a "manifest" symmetry, with all the consequences known from quantum mechanics: (i) The energy levels form degenerate multiplets. (ii) The operators $\vec{I}$ generate transitions within the multiplets, taking a neutron, e.g., into a proton, $I^+|n\rangle = |p\rangle$. (iii) The ground state is an isospin singlet,

$$I|0\rangle = 0.$$  \hfill (7)

If chiral symmetry was realized in the same manner, the energy levels would occur in degenerate multiplets of the group SU(2)$_R \times$SU(2)$_L$. Since the chiral isospin operators $\vec{I}_5$ carry negative parity, the multiplets would then necessarily contain members of opposite parity. The listings of the Particle Data Group, however, do not show any trace of such a pattern. A particle with the quantum numbers of $I_5^+|n\rangle$ and nearly the same mass as the neutron, e.g., is not observed in nature.

In fact, the symmetry of the Hamiltonian does not ensure that the corresponding eigenstates form multiplets of the symmetry group. In particular, the state with the lowest eigenvalue of the Hamiltonian need not be a singlet. In the case of a magnet, e.g., the Hamiltonian is invariant under rotations of the spin directions, but the ground state fails to be invariant, because the spins are aligned and thereby single out a direction. Whenever the state with the lowest eigenvalue is less symmetric than the Hamiltonian, the symmetry is called "spontaneously broken" or "hidden". Chiral symmetry belongs to this category. For dynamical reasons, the most important state — the vacuum — is symmetric only under ordinary isospin rotations, but does not remain invariant if a chiral rotation is applied,

$$\vec{I}_5 |0\rangle \neq 0.$$  \hfill (8)

Since the Hamiltonian commutes with chiral isospin, the three states $\vec{I}_5 |0\rangle$ have the same energy as the vacuum, $E = 0$. The operators $\vec{I}_5$ do not carry momentum, either, so that the states $\vec{I}_5 |0\rangle$ have $\vec{P} = 0$. This indicates that the spectrum of physical states contains three massless particles. Indeed, the Goldstone theorem [2] rigorously shows that spontaneous symmetry breakdown gives rise to massless particles, "Goldstone bosons". Their quantum numbers are those of the states $\vec{I}_5 |0\rangle$: spin zero, negative parity and $I = 1$.

The three lightest mesons, $\pi^+, \pi^0, \pi^-$, carry precisely these quantum numbers. The chiral isospin operators act like creation or annihilation operators
for pions: Applied to the vacuum, they generate a state containing a pion, 
$I^+_5 |0\rangle = |\pi^+\rangle$. Applied to a neutron, they do not lead to a parity partner, 
but instead yield a state containing a neutron and a pion, 
$I^+_5 |n\rangle = |n\pi^+\rangle$,
  etc.

4 Pion mass

The above discussion concerns the theoretical world, where $u$ and $d$ are assumed to be massless, such that the group $SU(2)_R \times SU(2)_L$ represents an exact symmetry. The Hamiltonian of QCD contains a quark mass term, which breaks the symmetry. To see how this affects the mass of the Goldstone bosons, consider the transition matrix element of the axial current $\bar{\pi} \gamma^\mu \gamma_5 d$, from the vacuum to a one-pion state. Lorentz invariance implies that this matrix element is determined by the pion momentum $p^\mu$, up to a constant,

$$\langle \pi^+(p) | \pi(x) \gamma^\mu \gamma_5 d(x) | 0 \rangle = -ip^\mu \sqrt{2} F_\pi e^{ipx} .$$

The value of the constant is measured in pion decay, $F_\pi \approx 93 \text{ MeV}$. For the divergence $\partial_\mu (\bar{\pi} \gamma^\mu \gamma_5 d)$, this yields an expression proportional to $p^2 = M^2_{\pi^+}$. Denoting the analogous matrix element of the pseudoscalar density by $G_\pi$,

$$\langle \pi^+(p) | \pi(x) \gamma_5 d(x) | 0 \rangle = i\sqrt{2} G_\pi e^{ipx} ,$$

the conservation law (4) thus implies the exact relation

$$M^2_{\pi^+} = (m_u + m_d) \left( G_\pi / F_\pi \right) . \quad (9)$$

The relation confirms that, when the symmetry breaking parameters $m_u, m_d$ are put equal to zero, the pion mass vanishes, independently of the masses of the other quark flavours. The group $SU(2)_R \times SU(2)_L$ then represents a spontaneously broken, exact symmetry, with three strictly massless Goldstone bosons. When the quark masses are turned on, the Goldstone bosons pick up mass: $M_{\pi^+}$ grows in proportion to $\sqrt{m_u + m_d}$. The pions remain light, provided $m_u$ and $m_d$ are small. The quark mass term of the Hamiltonian then amounts to a small perturbation, such that the group $SU(2)_R \times SU(2)_L$ still represents an approximate symmetry, with approximately massless Goldstone bosons.
The decomposition of the QCD Hamiltonian in eq. (3) may be compared with the standard perturbative splitting

\[ H_{\text{QCD}} = H_{\text{free}} + H_{\text{int}} , \]

where the first term describes free quarks and gluons, while the second accounts for their interaction. The corresponding expansion parameter is the coupling constant \( g \). Since QCD is asymptotically free, the effective coupling becomes weak at large momentum transfers — processes which exclusively involve large momenta may indeed be analyzed by treating the interaction as a perturbation. Perturbation theory, however, fails in the low energy domain, where the effective coupling is strong, such that it is not meaningful to truncate the expansion in powers of \( H_{\text{int}} \) after the first few terms. In particular, the structure of the ground state cannot be analyzed in this way, while the above decomposition, which retains the interaction among the quarks and gluons in the ”unperturbed” Hamiltonian \( H_0 \) and only treats \( m_u \) and \( m_d \) as perturbations, is perfectly suitable for that purpose. Note that the character of the perturbation series in powers of \( H_{\text{ab}} \) is quite different from the one in powers of \( H_{\text{int}} \): while the eigenstates of \( H_{\text{free}} \) are known explicitly, this is not the case with \( H_0 \), which still describes a highly nontrivial, interacting system. \( H_0 \) differs from the full Hamiltonian only in one respect: it possesses an exact group of chiral symmetries.

5 Effective field theory

At low energies, the behaviour of scattering amplitudes or current matrix elements can be described in terms of a Taylor series expansion in powers of the momenta. The electromagnetic form factor of the pion, e.g., may be expanded in powers of the momentum transfer \( t \). In this case, the first two Taylor coefficients are related to the total charge of the particle and to the mean square radius of the charge distribution, respectively,

\[ f_{\pi^+}(t) = 1 + \frac{1}{6}\langle r^2 \rangle_{\pi^+} t + O(t^2) . \]

(10)

Scattering lengths and effective ranges are analogous low energy constants occurring in the Taylor series expansion of scattering amplitudes.
The occurrence of light particles gives rise to singularities in the low energy domain, which limit the range of validity of the Taylor series representation. The form factor \( f_\pi(t) \), e.g., contains a branch cut at \( t = 4M_\pi^2 \), such that the formula (10) provides an adequate representation only for \( |t| \ll 4M_\pi^2 \). The problem becomes even more acute if \( m_u \) and \( m_d \) are set equal to zero. The pion mass then disappears, the branch cut sits at \( t = 0 \) and the Taylor series does not work at all. I first discuss the method used in the low energy analysis for this extreme case, returning to the physical situation with \( m_u, m_d \neq 0 \) below.

The reason why the spectrum of QCD with two massless quarks contains three massless bound states is understood: they are the Goldstone bosons of a hidden symmetry. The symmetry, which gives birth to these, at the same time also determines their low energy properties. This makes it possible to explicitly work out the poles and branch cuts generated by the exchange of Goldstone bosons. The remaining singularities are located comparatively far from the origin, the nearest one being due to the \( \rho \)-meson. The result is a modified Taylor series expansion in powers of the momenta, which works, despite the presence of massless particles. In the case of the \( \pi\pi \) scattering amplitude, e.g., the radius of convergence of the modified series is given by \( s = M_\rho^2 \), where \( s \) is the square of the energy in the center of mass system (the first few terms of the series only yield a decent description of the amplitude if \( s \) is smaller than the radius of convergence, say \( s < \frac{1}{2}M_\rho^2 \rightarrow \sqrt{s} < 540 \text{ MeV} \)).

As pointed out by Weinberg [3], the modified expansion may explicitly be constructed by means of an effective field theory, which is referred to as chiral perturbation theory and involves the following ingredients:

(i) The quark and gluon fields of QCD are replaced by a set of pion fields, describing the degrees of freedom of the Goldstone bosons. It is convenient to collect these in a \( 2 \times 2 \) matrix \( U(x) \in \text{SU}(2) \).

(ii) The Lagrangian of QCD is replaced by an effective Lagrangian, which only involves the field \( U(x) \), and its derivatives

\[
\mathcal{L}_{\text{QCD}} \rightarrow \mathcal{L}_{\text{eff}}(U, \partial U, \partial^2 U, \ldots)
\]

(iii) The low energy expansion corresponds to an expansion of the effective Lagrangian, ordered according to the number of derivatives of the field \( U(x) \). Lorentz invariance only permits terms with an even number of derivatives,

\[
\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}^2 + \mathcal{L}_{\text{eff}}^4 + \mathcal{L}_{\text{eff}}^6 + \ldots
\]
Chiral symmetry very strongly constrains the form of the terms occurring in the series. In particular, it excludes momentum independent interaction vertices: Goldstone bosons can only interact if they carry momentum. This property is essential for the consistency of the low energy analysis, which treats the momenta as expansion parameters. The leading contribution involves two derivatives,

\[ \mathcal{L}_{\text{eff}}^2 = \frac{1}{4} F_\pi^2 \text{tr} \{ \partial_\mu U^+ \partial^\mu U \} , \]

and is fully determined by the pion decay constant. At order \( p^4 \), the symmetry permits two independent terms:

\[ \mathcal{L}_{\text{eff}}^4 = \frac{1}{4} l_1 (\text{tr} \{ \partial_\mu U^+ \partial^\mu U \})^2 + \frac{1}{4} l_2 \text{tr} \{ \partial_\mu U^+ \partial_\nu U \} \text{tr} \{ \partial^\mu U^+ \partial^\nu U \} , \]

etc. For most applications, the derivative expansion is needed only to this order.

The most remarkable property of the method is that it does not mutilate the theory under investigation: The effective field theory framework is no more than an efficient machinery, which allows one to work out the modified Taylor series, referred to above. If the effective Lagrangian includes all of the terms permitted by the symmetry, the effective theory is mathematically equivalent to QCD \([3, 4]\). It exclusively exploits the symmetry properties of QCD and involves an infinite number of effective coupling constants, \( F_\pi, l_1, l_2, \ldots \), which represent the Taylor coefficients of the modified expansion.

In QCD, the symmetry, which controls the low energy properties of the Goldstone bosons, is only an approximate one. The constraints imposed by the hidden, approximate symmetry can still be worked out, at the price of expanding the quantities of physical interest in powers of the symmetry breaking parameters \( m_u \) and \( m_d \). The low energy analysis then involves a combined expansion, which treats both, the momenta and the quark masses as small parameters. The effective Lagrangian picks up additional terms, proportional to powers of the quark mass matrix,

\[ m = \begin{pmatrix} m_u \\ m_d \end{pmatrix} \]

\(^1\)In the framework of the effective theory, the anomalies of QCD manifest themselves through an extra contribution, the Wess-Zumino term, which is also of order \( p^4 \) and is proportional to the number of colours.
It is convenient to count $m$ like two powers of momentum, such that the expansion of the effective Lagrangian still starts at $O(p^2)$ and only contains even terms. The leading contribution picks up a term linear in $m$,

$$\mathcal{L}^2_{\text{eff}} = \frac{1}{4} F^2_{\pi} \text{tr}\{ \partial_{\mu} U^+ \partial^{\mu} U \} + \frac{1}{2} F^2_{\pi} B \text{tr}\{ m(U + U^\dagger) \} \ .$$

(13)

Likewise, $\mathcal{L}^4_{\text{eff}}$ receives additional contributions, involving two further effective coupling constants, $l_3, l_4$, etc.

The expression (13) represents a compact summary of the soft pion theorems established in the 1960's: The leading terms in the low energy expansion of the scattering amplitudes and current matrix elements are given by the tree graphs of this Lagrangian. The coupling constant $B$ is needed to account for the symmetry breaking effects generated by the quark masses at leading order. It represents the coefficient of the leading term in the expansion of the pion mass in powers of $m_u$ and $m_d$, $M^2_{\pi} = (m_u + m_d) B + O(m^2)$. According to section 4, the same constant also determines the vacuum-to-pion matrix element of the pseudoscalar density, $G_{\pi} = F_{\pi} B + O(m)$. Furthermore, the relation of Gell-Mann, Oakes and Renner, $F^2_{\pi} M^2_{\pi} = -(m_u + m_d) \langle 0 | \pi u | 0 \rangle + O(m^2)$, which immediately follows from the above expression for the effective Lagrangian, shows that the magnitude of the quark condensate is also related to the value of $B$.

The effective field theory represents an efficient and systematic framework, which allows one to work out the corrections to the soft pion predictions, those arising from the quark masses as well as those from the terms of higher order in the momenta. The evaluation is based on a perturbative expansion of the quantum fluctuations of the effective field. In addition to the tree graphs relevant for the soft pion results, graphs containing vertices from the higher order contributions $\mathcal{L}^4_{\text{eff}}, \mathcal{L}^6_{\text{eff}} \ldots$ and loop graphs contribute. The leading term of the effective Lagrangian describes a nonrenormalizable theory, the "nonlinear $\sigma$-model". The higher order terms in the derivative expansion, however, automatically contain the relevant counter terms. The divergences occurring in the loop graphs merely renormalize the effective coupling constants. The effective theory is a perfectly renormalizable scheme, order by order in the low energy expansion and the results obtained with it are independent of the regularization used.
6 Universality

The properties of the effective theory are governed by the hidden symmetry, which is responsible for the occurrence of Goldstone bosons. In particular, the form of the effective Lagrangian only depends on the symmetry group $G$ of the Hamiltonian and on the subgroup $H \subset G$, under which the ground state is invariant. The Goldstone bosons live on the difference between the two groups, i.e., on the quotient $G/H$. The specific dynamical properties of the underlying theory do not play any role. To discuss the consequences of this observation, I again assume that $G$ is an exact symmetry.

In the case of QCD with two massless quarks, $G = SU(2)_R \times SU(2)_L$ is the group of chiral isospin rotations, while $H = SU(2)$ is the ordinary isospin group. The Higgs model is another example of a theory with spontaneously broken symmetry. It plays a crucial role in the Standard Model, where it describes the generation of mass. The model involves a scalar field $\vec{\phi}$ with four components. The Hamiltonian is invariant under rotations of the vector $\vec{\phi}$, which form the group $G = O(4)$. Since the field picks up a vacuum expectation value, the symmetry is spontaneously broken to the subgroup of those rotations, which leave the vector $\langle 0 | \vec{\phi} | 0 \rangle$ alone, $H = O(3)$. It so happens that these groups are the same as those above, relevant for QCD.

The fact that the symmetries are the same implies that the effective field theories are identical: (i) In either case, there are three Goldstone bosons, described by a matrix field $U(x) \in SU(2)$. (ii) The form of the effective Lagrangian is precisely the same. In particular, the expression

$$\mathcal{L}_{\text{eff}}^2 = \frac{1}{4} F_\pi^2 \text{tr}\{\partial_{\mu} U^+ \partial^\mu U\}$$

is valid in either case. At the level of the effective theory, the only difference between these two physically quite distinct models is that the numerical values of the effective coupling constants are different. In the case of QCD, the one occurring at leading order of the derivative expansion is the pion

\footnote{The structure of the effective Lagrangian rigorously follows from the Ward identities for the Green functions of the currents, which also reveal the occurrence of anomalies. The form of the Ward identities is controlled by the structure of $G$ and $H$ in the infinitesimal neighbourhood of the neutral element. In this sense, the symmetry groups of the two models are the same: $O(4)$ and $O(3)$ are locally isomorphic to $SU(2) \times SU(2)$ and $SU(2)$, respectively.}
decay constant, $F_\pi \simeq 93\,\text{MeV}$, while in the Higgs model, this coupling constant is larger by more than three orders of magnitude, $F_\pi \simeq 250\,\text{GeV}$. At next-to-leading order, the effective coupling constants are also different; in particular, in QCD, the anomaly coefficient is equal to $N_c$, while in the Higgs model, it vanishes.

As an illustration, I compare the condensates of the two theories, which play a role analogous to the spontaneous magnetization $\langle \vec{M} \rangle$ of a ferromagnet (or the staggered magnetization of an antiferromagnet). At low temperatures, the magnetization singles out a direction — the ground state spontaneously breaks the symmetry of the Hamiltonian with respect to rotations. As the system is heated, the spontaneous magnetization decreases, because the thermal disorder acts against the alignment of the spins. If the temperature is high enough, disorder wins, the spontaneous magnetization disappears and rotational symmetry is restored. The temperature at which this happens is the Curie temperature. Quantities, which allow one to distinguish the ordered from the disordered phase are called order parameters. The magnetization is the prototype of such a parameter.

In QCD, the most important order parameter (the one of lowest dimension) is the quark condensate. At nonzero temperatures, the condensate is given by the thermal expectation value

$$\langle \bar{u}u \rangle_T = \frac{\text{Tr}\{ \bar{u}u \exp(-H/kT) \}}{\text{Tr}\{\exp(-H/kT)\}}.$$  

The condensate melts if the temperature is increased. At a critical temperature, somewhere in the range $140\,\text{MeV} < T_c < 180\,\text{MeV}$, the quark condensate disappears and chiral symmetry is restored. The same qualitative behaviour also occurs in the Higgs model, where the expectation value $\langle \phi \rangle_T$ of the scalar field represents the most prominent order parameter.

At low temperatures, the thermal trace is dominated by states of low energy. Massless particles generate contributions which are proportional to powers of the temperature, while massive ones like the $\rho$-meson are suppressed by the corresponding Boltzmann factor, $\exp(-M_\rho/kT)$. In the case of a spontaneously broken symmetry, the massless particles are the Goldstone bosons and their contributions may be worked out by means of effective field theory. For the quark condensate, the calculation has been done [3], up to
and including terms of order $T^6$:

$$\langle u u \rangle_T = \langle 0 | u u | 0 \rangle \left\{ 1 - \frac{T^2}{8F^2_\pi} - \frac{T^4}{384F^4_\pi} - \frac{T^6}{288F^6_\pi} \ln(T_1/T) + O(T^8) \right\} .$$

(14)

The formula is exact — for massless quarks, the temperature scale relevant at low $T$ is the pion decay constant. The additional logarithmic scale $T_1$ occurring at order $T^6$ is determined by the effective coupling constants $l_1, l_2$, which enter the expression (12) for the effective Lagrangian of order $p^4$. Since these are known from the phenomenology of $\pi\pi$ scattering, the value of $T_1$ is also known: $T_1 = 470 \pm 110$ MeV.

Now comes the point I wish to make. The effective Lagrangians relevant for QCD and for the Higgs model are the same. Since the operators of which we are considering the expectation values also transform in the same manner, their low temperature expansions are identical. The above formula thus holds, without any change whatsoever, also for the Higgs condensate,

$$\langle \vec{\phi} \rangle_T = \langle 0 | \vec{\phi} | 0 \rangle \left\{ 1 - \frac{T^2}{8F^2_\pi} - \frac{T^4}{384F^4_\pi} - \frac{T^6}{288F^6_\pi} \ln(T_1/T) + O(T^8) \right\} .$$

In fact, the universal term of order $T^2$ was discovered in the framework of this model, in connection with work on the electroweak phase transition [6].

These examples illustrate the physical nature of effective theories: At long wavelength, the microscopic structure does not play any role. The behaviour only depends on those degrees of freedom, which require little excitation energy. The hidden symmetry, which is responsible for the absence of an energy gap and for the occurrence of Goldstone bosons, at the same time also determines their low energy properties. For this reason, the form of the effective Lagrangian is controlled by the symmetries of the system and is, therefore, universal. The microscopic structure of the underlying theory exclusively manifests itself in the numerical values of the effective coupling constants. The temperature expansion also clearly exhibits the limitations of the method. The truncated series can be trusted only at low temperatures, where the first term represents the dominant contribution. According to the above formula, the quark condensate drops to about half of the vacuum expectation value when the temperature reaches 160 MeV — the formula does not make much sense beyond this point. In particular, the behaviour of
the quark condensate in the vicinity of the chiral phase transition is beyond
the reach of the effective theory discussed here.

7 Nonrelativistic effective Lagrangians

The fact that symmetries may break down spontaneously was discovered in
condensed matter physics. Also, the phenomena associated with the prop-
agation of sound were among the very first to be analyzed in terms of an
effective field theory. The main difference to the situation in particle physics
is that the ground state, which forms, when the number of electrons and
baryons is fixed at a nonzero value, fails to be Lorentz invariant: The rest
frame singles out a preferred frame of reference. The Hamiltonian is invariant
under the Poincaré group $G$, but the ground state is invariant only under a
subgroup thereof, $H_{\text{solid}} \subset H_{\text{fluid}} = H_{\text{gas}} \subset G$. As is well-known, the phonons
may be viewed as Goldstone bosons generated by this spontaneous symme-
try breakdown. Their properties are rather special, however, because they
originate in a space-time symmetry rather than an internal one: The corre-
sponding conserved "currents" are the components of the energy-momentum
tensor $\theta^{\mu \nu}$ and their number is smaller than the dimension of the coset space
$G/H$. I do not elaborate on this further here, but refer to [7–9]. Instead, I add
a few remarks concerning nonrelativistic internal symmetries, emphasizing
the comparison with the relativistic situation.

As an example, I consider the Heisenberg model, where the dynamical
variables form a lattice of spin operators $\vec{s}_n$. The Hamiltonian of the model
reads

$$H = g \sum_{mn} \vec{s}_m \cdot \vec{s}_n,$$

where the sum runs over nearest neighbours. It is invariant under rotations
of the spin directions, generated by

$$\vec{Q} = \sum_n \vec{s}_n.$$

Note that the corresponding group $G = O(3)$ represents an internal sym-
metry, because the space lattice remains put. If the coupling constant $g$ is
positive, the interaction favours an antiparallel alignment of the spins, such
that the model shows the behaviour of an antiferromagnet. For positive cou-
pling, the ground state instead forms a configuration of parallel spins, like
for a ferromagnet. In either case, the ground state singles out a direction and thus spontaneously breaks the symmetry group of the Hamiltonian to the subgroup $H = O(2)$ of the rotations around this direction. In the present case, the Goldstone bosons generated by the spontaneous symmetry breakdown represent spin waves or magnons. The coset space $G/H$ is the unit sphere, such that the effective field is a unit vector $\vec{U}(x)$ and carries two degrees of freedom.

The low energy behaviour of the model may again be analyzed in terms of an effective Lagrangian. Consider first the antiferromagnetic case, $g > 0$, where the relevant order parameter is the staggered magnetization. For a cubic lattice, the leading terms in the derivative expansion of the corresponding effective Lagrangian are given by

$$L_{\text{eff}}^2 = \frac{1}{2} F^2 \{ \partial_i \vec{U} \cdot \partial_i \vec{U} - c^2 \sum_s \partial_s \vec{U} \cdot \partial_s \vec{U} \} ,$$

where the sum extends over the three space directions. The reflection symmetries of the lattice imply that the expression is invariant under space rotations. The constant $c$ is the spin wave velocity, while $F$ is related to the helicity modulus. Evidently, the effective Lagrangian is very similar to the one occurring in QCD or in the Higgs model. A suitable change in scale takes the constant $c$ into the velocity of light: The leading terms in the derivative expansion of the effective Lagrangian relevant for an antiferromagnet are invariant with respect to Lorentz transformations. There is a difference in the structure of the symmetry groups, as we are now dealing with the spontaneous breakdown $O(3) \to O(2)$ rather than $O(4) \to O(3)$. There are two Goldstone bosons instead of three as in QCD. Apart from that, however, the effective Lagrangians are the same. As a consequence, the formula (14) also holds for the staggered magnetization of an antiferromagnet, except that the Clebsch-Gordan coefficients, which accompany the various powers of $T$ are different, because the symmetry groups are not the same. The temperature scale of the melting process is now set by the helicity modulus and is more than eight orders of magnitude smaller than in the case of the quark condensate. Otherwise, the behaviour of the two systems at low temperatures is essentially the same.

Remarkably, the behaviour of a ferromagnet at low energies is quite different. Although the Hamiltonian differs from the preceding case only in the sign of the coupling constant $g$, the corresponding effective Lagrangian is not
the same. The groups involved in the spontaneous symmetry breakdown are identical, such that the Goldstone bosons are again described by a vector field $\vec{U}(x)$ of unit length. The difference in the low energy behaviour arises from the fact that for the antiferromagnet, the mean value $\langle \vec{Q} \rangle$ of the sum over all spins vanishes, while for the ferromagnet, this is not the case: The generators of the symmetry group give rise to an order parameter [8–11].

In the relativistic domain, this cannot happen. The charges of an internal symmetry are integrals over the time components of the corresponding currents. For a Lorentz invariant ground state, currents cannot pick up an expectation value. In the case of a ferromagnet, however, the expectation values of the charges represent the most important order parameter, the spontaneous magnetization,

$$\langle \vec{Q} \rangle = V \langle \vec{M} \rangle .$$

In this perspective, the antiferromagnet is exceptional: The symmetry does not prevent the generators from picking up an expectation value, but it does not ensure that this happens. For an antiferromagnet, the quantity $\langle \vec{Q} \rangle$ happens to vanish, for dynamical reasons.

In the effective Lagrangian, the order parameter $\langle \vec{Q} \rangle$ manifests itself through a topological term, related to the Brower degree. Like the Wess-Zumino term, this contribution is invariant under the symmetry group only up to a total derivative. While the Wess-Zumino term only shows up at higher orders of the low energy expansion, the one relevant for a ferromagnet contributes at leading order and thus profoundly modifies the low energy structure of the system. Although the number of effective fields is the same as in the case of the antiferromagnet, the number of Goldstone particles is different and the dispersion laws are not the same, either: antiferromagnetic magnons possess two polarization states and the dispersion is of the form $\omega(k) \propto |k|$, while for a ferromagnet, only one polarization occurs and $\omega(k) \propto |k|^2$. In a sense, the difference in the low energy structure of a ferromagnet and an antiferromagnet is more pronounced than the one between an antiferromagnet and QCD.
8 Concluding remarks

Spontaneously broken symmetries play an important role, in condensed matter as well as in particle physics. The low energy properties of the Goldstone bosons generated by the symmetry breakdown may be worked out by means of the effective field theory methods, invented in the 1960’s. Since then, the effective Lagrangian technique has been developed into an efficient and mathematically precise tool, used extensively, e.g., to analyze the low energy structure of QCD. Several applications to the strong, electromagnetic and weak interactions of the pseudoscalar mesons were worked out in detail. In particular, rare decays and anomaly driven processes provide sensitive tests of the theory. In addition, the thermal properties of the hadronic phase \[1\], the mass generating sector of the Standard Model \[12\] and finite size effects in models with a spontaneously broken symmetry \[13\] have been analyzed with this method. Much remains to be done in this field, however, also in view of the low energy precision experiments planned at various laboratories. Once lattice simulations of QCD reach the domain, where the long range phenomena associated with the spontaneous breakdown of chiral symmetry become visible, the method should also prove to be an efficient tool to account for the corresponding finite size effects.

In condensed matter physics, spontaneous breakdown occurs for internal as well as space-time symmetries. In the language of the relevant effective Lagrangian, the good old description of the behaviour at long wavelength corresponds to the leading term of the derivative expansion. In the case of the antiferromagnet, the effective Lagrangian proved to be very useful also beyond leading order. For other nonrelativistic systems, such as ferromagnets, the higher order terms, due to the quantum fluctuations of the effective field, yet need to be worked out. Nonrelativistic kinematics is less restrictive than Lorentz invariance and allows the generators of the symmetry to become order parameters. In the effective Lagrangian, these are represented by a term of topological nature, which does not occur in particle physics.

The method has its limitations. In particular, it is useful only at low momenta, small quark masses, weak external magnetic fields, low temperatures and large volumes. The behaviour of the quark condensate in the vicinity of the chiral phase transition, e.g., is beyond the reach of this technique. Another limitation arises from the fact that the quantum fluctuations of the effective field play an important role in the systematic low energy analysis.
These can only be worked out if the dynamics of the effective degrees of freedom may be formulated in terms of a Lagrangian. A phenomenological description of the dissipative effects generated by friction is beyond this framework, because frictional forces cannot be accounted for in terms of a Lagrangian.

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