LOCAL EXISTENCE TO THE CROSS CURVATURE FLOW ON 3-MANIFOLDS WITH BOUNDARY

LI MA, BAIYU LIU

Abstract. In this paper, we use the DeTurck trick to study the short-time existence of solutions to the Dirichlet and Newmann boundary problems of the cross curvature flow on 3-manifolds with boundary.

Mathematics Subject Classification 2000: 53Cxx,35Jxx

Keywords: Cross curvature flow, local existence, Newmann boundary condition

1. Introduction

In this paper, we study the Dirichlet and Newmann boundary problems of the cross curvature flow on compact three-manifolds with boundary. Let \((M, g)\) be a compact three-dimensional Riemannian manifold with boundary \(\partial M\) and with positive sectional curvature or negative sectional curvature. The cross curvature tensor \(c = (c_{ij})\) on the Riemannian manifold \((M^3, g)\) is defined by

\[
c_{ij} = \det E(E^{-1})_{ij} = \frac{1}{2} \mu^{ipq} \mu^{jrs} E_{pr} E_{qs} = \frac{1}{8} \mu^{ipq} \mu^{jrs} R_{ilpq} R_{kjrs},
\]

where \(E_{ij} = R_{ij} - \frac{1}{2} R g_{ij}\) is the Einstein tensor of the metric \(g = (g_{ij})\), \(\det E \approx \frac{\det E_{ij}}{\det g_{ij}}\), and \(\mu^{ijk}\) are the components of the volume form \(d\mu\) with indices raised.

We say that a 1-parameter family of metrics \((g(t))\) on 3-manifold \(M^3\) with the negative sectional curvature is a solution of the cross curvature flow (XCF) if it satisfies

\[
\frac{\partial}{\partial t} g = 2c.
\]

Likewise, for the family \(g(t)\) with the positive sectional curvature, we say that \((M^3, g(t))\) is a solution if

\[
\frac{\partial}{\partial t} g = -2c.
\]

We have two local existence results.

Theorem 1. Let \((M^3, g_0)\) be a compact 3-manifold with boundary. If the sectional curvature of \((M^3, g_0)\) is either negative everywhere or positive everywhere, then there is a unique short time solution \(g(t), t \in [0, \epsilon)\), where
$\epsilon > 0$, to the cross curvature flow (XCF) with
\[ g(x, 0) = g_0(x), \quad x \in M \]
and
\[ g(x, t) = g_0(x), \quad t \in [0, \epsilon) \quad x \in \partial M. \]

**Theorem 2.** Let $(M^3, g_0)$ be a compact 3-manifold with boundary. Given any smooth function $\lambda(t)$ depending only on the time variable $t \in [0, +\infty)$. If the sectional curvature of $(M^3, g_0)$ is either negative everywhere or positive everywhere, then there is a unique short time solution $g(t), t \in [0, \epsilon)$, where $\epsilon > 0$, to the cross curvature flow (XCF) with
\[ g(x, 0) = g_0(x), \quad x \in M \]
and
\[ h_{\alpha\beta}(x, t) = \lambda(t)g_{\alpha\beta}(x, t), \quad \alpha, \beta = 1, 2, \quad x \in \partial M. \]

We shall use the DeTurck trick to prove the above results. Since the arguments for both results are similar, we shall only provide the detail for the proof of Theorem 2.

The short-time existence of solutions to the cross curvature flow on closed 3-manifolds has been obtained by B.Chow and Hamilton [5] by the method of Nash-Moser implicit theorem. A simpler proof for this result has been obtained by Buckland [3] (see also [6]) and Buckland’s argument used DeTurck method [8, 9]. Interesting examples for the cross curvature flows have been studied by X.Cao, Y.Ni, and Laurent Saloff-Coste in [4] and by L.Ma and D.Chen [10]. In the case of curvature flows on manifolds of lower dimensions with boundary, since the literature is huge, we only cite [1, 2, 7, 11] here and one may find more references therein. In [11], Y. Shen has considered the Neumann boundary value problem for the Ricci flow:
\[ \begin{cases} \frac{\partial g}{\partial t} = -2Ric, & \quad x \in M \\ g(x, 0) = g_0, & \quad x \in M \\ h_{\alpha\beta}(x, t) = \lambda g_{\alpha\beta}(x, t), & \quad x \in \partial M, \end{cases} \]
where $h_{\alpha\beta}$ is the second fundamental form of $\partial M$ in $M$ and $\lambda$ is a constant. The short time existence of (5) for such problem has been obtained in [11].

2. Short Time Existence

In this section, we present the proof of Theorem 2 by using the DeTurck trick.

Since the case of positive sectional curvature is similar, we only consider the case of negative sectional curvature, namely
\[ \begin{cases} \frac{\partial g}{\partial t} = 2c, & \quad x \in M \\ g(x, 0) = g_0, & \quad x \in M \\ h_{\alpha\beta}(x, t) = \lambda(t)g_{\alpha\beta}(x, t), & \quad x \in \partial M. \end{cases} \]
We adopt the convention that Latin indices range from 1 to 3, while Greek indices range from 1 to 2.

**Step 1.** Analyze the linearization of (6).

We note that the linearization of (XCF) has been computed by Buckland [3]. For reader’s convenience, we give full details.

If \( \frac{\partial g_{ij}}{\partial s} = v_{ij} \) is a variation of the metric \( g_{ij} \), then

\[
\frac{\partial}{\partial s} R_{ijkl} = \frac{1}{2} \left( \frac{\partial^2 v_{ij}}{\partial x^i \partial x^k} + \frac{\partial^2 v_{ik}}{\partial x^j \partial x^l} - \frac{\partial^2 v_{jl}}{\partial x^i \partial x^k} - \frac{\partial^2 v_{il}}{\partial x^j \partial x^l} \right) + \ldots,
\]

where the dots denote the lower derivatives terms. We normalize such that \( \mu_{123} = \mu^{123} = 1 \) and recall from p.491 of [6] the following:

\[
\mu^{pqk} \mu^{rsl} R_{kjsl} = 2E^{ml} (\delta^p_j \delta^q_m - \delta^p_m \delta^q_j).
\]

Applying (8) to (7), we obtain

\[
\frac{\partial}{\partial s} c_{ij} = \frac{1}{8} \mu^{pqk} \mu^{rsl} \left( \frac{\partial}{\partial s} R_{ilpq} \right) R_{kjsl} + \frac{1}{8} \mu^{pqk} \mu^{rsl} \left( \frac{\partial}{\partial s} R_{kjsl} \right)
\]

\[
= \frac{1}{8} \left( \frac{\partial^2 v_{jp}}{\partial x^i \partial x^q} + \frac{\partial^2 v_{jq}}{\partial x^i \partial x^p} - \frac{\partial^2 v_{ip}}{\partial x^j \partial x^q} - \frac{\partial^2 v_{jq}}{\partial x^i \partial x^p} \right) E^{ml} (\delta^p_j \delta^q_m - \delta^p_m \delta^q_j)
\]

\[
+ \frac{1}{8} \left( \frac{\partial^2 v_{jr}}{\partial x^i \partial x^s} + \frac{\partial^2 v_{js}}{\partial x^i \partial x^r} - \frac{\partial^2 v_{is}}{\partial x^j \partial x^r} - \frac{\partial^2 v_{js}}{\partial x^i \partial x^r} \right) E^{mk} (\delta^i_j \delta^k_m - \delta^i_m \delta^k_j) + \ldots
\]

\[
= \frac{1}{2} E^{ml} \left( \frac{\partial^2 v_{ij}}{\partial x^i \partial x^m} + \frac{\partial^2 v_{jm}}{\partial x^i \partial x^m} - \frac{\partial^2 v_{ij}}{\partial x^j \partial x^m} - \frac{\partial^2 v_{jm}}{\partial x^i \partial x^m} \right) + \ldots.
\]

Hence, the linearization of the map \( X \) which takes \( g \) to \( 2c \) is a second-order partial differential operator. Its symbol is

\[
[\sigma DX (g) (\zeta) v]_{ij} = E^{ml} (\zeta^i v_{jm} + \zeta_j v_{im} - \zeta_i v_{jm} - \zeta_j v_{im} - \zeta_k v_{im}).
\]

Since this is homogenous, we may assume \( \zeta \) has length one and rotate the coordinates so that \( \zeta_1 = 1 \) and \( \zeta_2 = \zeta_3 = 0 \). It follows that

\[
[\sigma DX (g) (\zeta) v]_{ij} = \delta_{i1} \delta_{j1} E^{ml} v_{lm} + v_{ij} E^{11} - \delta_{i1} E^{11} v_{lj} - E^{ml} \delta_{j1} v_{im}.
\]

We then deduce that

\[
\sigma DX (g) (\zeta) = \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{12} & v_{13} \\ v_{12} & v_{13} & v_{12} & v_{13} & v_{12} \\ v_{13} & v_{12} & v_{13} & v_{12} & v_{13} \\ v_{22} & v_{23} & v_{22} & v_{23} & v_{22} \\ v_{33} & v_{23} & v_{33} & v_{23} & v_{33} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & E^{22} & E^{33} & 2E^{23} \\ 0 & 0 & 0 & -E^{12} & 0 & -E^{13} \\ 0 & 0 & 0 & -E^{13} & -E^{12} & 0 \\ 0 & 0 & 0 & E^{11} & 0 & 0 \\ 0 & 0 & 0 & E^{11} & 0 & E^{11} \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{22} \\ v_{33} \end{pmatrix}.
\]

Notice that \( E^{11} > 0 \), when the sectional curvature is negative. Therefore, the eigenvalues of matrix \( \sigma DX (g) (\zeta) \) is nonnegative, which indicates that (XCF) is weakly parabolic.

**Step 2.** Modify the initial boundary value problem (6).
Following the idea from Shen [11], we deduce a modified problem of (XCF).

Let \( \tilde{g} \) be any background metric on \( M \) with connection \( \tilde{\Gamma} \). Define the vector field such that its components are given by

\[
W^k = g^{pq}(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq}).
\]

Now we are going to solve the following modified system:

\[
\begin{cases}
\frac{\partial g}{\partial t} = 2c + L_W g, & x \in M \\
g(x,0) = g_0, & x \in M \\
h_{\alpha\beta}(x,t) = \lambda(t)g_{\alpha\beta}(x,t), & x \in \partial M \\
g_{3\alpha}(x,t) = 0, & x \in \partial M \\
W^3(x,t) = 0, & x \in \partial M.
\end{cases}
\]

For a given point \( x \in \partial M \), we can choose a local coordinate chart around \( x \) such that \( \partial \mathbb{R}^3 \) form basis for \( T_x\partial M \) and \( \frac{\partial}{\partial x^3} \) is transversal to \( T_x\partial M \).

Therefore, one can check that

\[
e_3 = \frac{g^{3i}}{(g^{33})^{\frac{1}{2}}} \frac{\partial}{\partial x^3},
\]

is the unit normal to \( T_x\partial M \) in a small neighborhood of \( x \). The second fundamental form of \( (\partial M, g_{\partial M}) \) in \( (M, g) \) is

\[
h_{\alpha\beta} = -\frac{1}{2} L_{\partial x^3} g_{\alpha\beta} = -\frac{1}{2} \frac{g^{3i}}{(g^{33})^{\frac{1}{2}}} \nabla_i g_{\alpha\beta}.
\]

The given boundary condition \( g_{3\alpha} = 0 \) implies \( g'^{\alpha} = 0 \) and

\[
h_{\alpha\beta} = -\frac{1}{2} \frac{(g^{33})^{\frac{1}{2}}}{(g^{33})^{\frac{1}{2}}} \nabla_3 g_{\alpha\beta}.
\]

Combining the above equation with \( h_{\alpha\beta} = \lambda(t)g_{\alpha\beta} \), we get

\[
\nabla_3 g_{\alpha\beta} = -\frac{2\lambda(t)}{(g^{33})^{\frac{1}{2}}} g_{\alpha\beta}.
\]

By the definition of \( W \), we know

\[
W^3 = \frac{1}{2} g^{ij} g^{3l} (\nabla_j g_{3l} + \nabla_l g_{3j} - \nabla_i g_{lj})
\]

\[
= g^{ij} g^{3l} (\nabla_j g_{3l} - \frac{1}{2} \nabla_l g_{ij}).
\]

The given boundary condition \( g_{3\alpha} = 0 \) implies \( g'^{3\alpha} = 0 \). Thus we have

\[
W^3 = (g^{33})^2 \nabla_3 g_{33} - \frac{1}{2} g^{33} g^{ij} g_{ij}
\]

\[
= \frac{1}{2} (g^{33})^2 \nabla_3 g_{33} - \frac{1}{2} g^{33} g_{3\alpha} \nabla_3 g_{\alpha\beta}.
\]

Hence the condition \( W^3 = 0 \) is equivalent to

\[
\nabla_3 g_{33} = \frac{g^{\alpha\beta} \nabla_3 g_{\alpha\beta}}{(g^{33})^2} = -2\lambda(t) \frac{g_{3\alpha} g_{\alpha\beta}}{(g^{33})^2} = -4\lambda(t) \frac{1}{(g^{33})^2}.
\]
where we have used (13).

We conclude that (12) is equivalent to

$$\begin{align*}
\frac{\partial g}{\partial t} &= 2c + L_W g, \quad x \in M \\
g(x, 0) &= g_0, \quad x \in M \\
\nabla_3 g_{\alpha \beta}(x, t) &= -\frac{2\lambda(t)}{(g^{33})^2} g_{\alpha \beta}(x, t), \quad x \in \partial M \\
g_{3\alpha}(x, t) &= 0, \quad x \in \partial M \\
\nabla_3 g_{33} &= -4\lambda(t) \frac{1}{(g^{33})^2}, \quad x \in \partial M.
\end{align*}$$

(14)

Next, we show that equation (14) has a short time solution, by proving that it is a parabolic equation. The symbol $\sigma DX(g)(\zeta)v$ has been obtained in (9). We now compute the symbol of the operator

$$Y(g) \doteq L_W g.$$ 

Compute

$$Y(g)_{ij} = \nabla_j W_i + \nabla_i W_j$$

$$= g_{ik} g_{pq} \nabla_j \Gamma_{pq} - g_{jk} g_{pq} \nabla_i \Gamma_{pq}$$

$$= \frac{1}{2} g_{ik} g_{pq} \frac{\partial}{\partial x^q} \left( \frac{\partial g_{pl}}{\partial x^i} + \frac{\partial g_{ql}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^i} \right)$$

$$+ \frac{1}{2} g_{jk} g_{pq} \frac{\partial}{\partial x^q} \left( \frac{\partial g_{pl}}{\partial x^j} + \frac{\partial g_{ql}}{\partial x^p} - \frac{\partial g_{pq}}{\partial x^j} \right) + \ldots$$

$$= \frac{1}{2} g_{pq} \left( \frac{\partial^2 g_{pi}}{\partial x^j \partial x^q} + \frac{\partial^2 g_{qi}}{\partial x^j \partial x^p} - \frac{\partial^2 g_{pq}}{\partial x^j \partial x^i} \right) + \frac{1}{2} g_{pq} \left( \frac{\partial^2 g_{pj}}{\partial x^i \partial x^q} + \frac{\partial^2 g_{qj}}{\partial x^i \partial x^p} - \frac{\partial^2 g_{pq}}{\partial x^i \partial x^j} \right) + \ldots$$

It follows that

$$\frac{\partial}{\partial s} Y(g)_{ij} = \frac{1}{2} g_{pq} \left( \frac{\partial^2 v_{pi}}{\partial x^j \partial x^q} + \frac{\partial^2 v_{qi}}{\partial x^j \partial x^p} - \frac{\partial^2 v_{pq}}{\partial x^j \partial x^i} \right) + \frac{1}{2} g_{pq} \left( \frac{\partial^2 v_{pj}}{\partial x^i \partial x^q} + \frac{\partial^2 v_{qj}}{\partial x^i \partial x^p} - \frac{\partial^2 v_{pq}}{\partial x^i \partial x^j} \right) + \ldots$$

$$= g_{pq} \left( \frac{\partial^2 v_{pi}}{\partial x^j \partial x^q} + \frac{\partial^2 v_{qi}}{\partial x^j \partial x^p} - \frac{\partial^2 v_{pq}}{\partial x^j \partial x^i} \right) + \ldots$$

and

$$[\sigma DX(g)(\zeta)v]_{ij} = g_{pq} (\zeta_j \zeta_q v_{pi} + \zeta_i \zeta_q v_{pj} - \zeta_j \zeta_i v_{pq}).$$

Choosing an orthonormal basis and taking $\zeta_1 = 1$, $\zeta_2 = \zeta_3 = 0$, we find that

$$\sigma DX(g)(\zeta) = \begin{pmatrix} v_{11} \\
v_{12} \\
v_{13} \\
v_{22} \\
v_{33} \\
v_{23} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 & v_{11} \\
0 & 1 & 0 & 0 & 0 & 0 & v_{12} \\
0 & 0 & 1 & 0 & 0 & 0 & v_{13} \\
0 & 0 & 0 & 0 & 0 & 0 & v_{22} \\
0 & 0 & 0 & 0 & 0 & 0 & v_{33} \\
0 & 0 & 0 & 0 & 0 & 0 & v_{23} \end{pmatrix}.$$
Adding (10) and (15) together, we finally arrive at

\[
\sigma D(X+Y)(g)(\zeta) = \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{22} \\ v_{33} \\ v_{23} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & E^{22} - 1 & E^{33} - 1 & 2E^{23} \\ 0 & 1 & 0 & -E^{12} & 0 & -E^{13} \\ 0 & 0 & 1 & 0 & -E^{13} & -E^{12} \\ 0 & 0 & 0 & E^{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & E^{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & E^{11} \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{22} \\ v_{33} \\ v_{23} \end{pmatrix},
\]

which implies that (14) is parabolic. Therefore, (14) has a unique short time solution, and so does (12). Let \( g(x, t) \) \((t \in [0, \epsilon])\) be a solution of (12).

**Step3.** Finally, we obtain the solution of (6) from the solution of (12), i.e. \( g(x, t) \) \((t \in [0, \epsilon])\).

Let \( \varphi_t(x) \) be the one-parameter family of diffeomorphism which is determined by:

\[
\begin{aligned}
\frac{\partial}{\partial t} \varphi_t(x) &= W(\varphi(x), t), & x \in M \\
\varphi_0 &= id, & x \in M.
\end{aligned}
\]

Here \( id \) denotes the identity map on \( M \) and vector field \( W \) has components \( W^k \) as defined by (11). Let \( \bar{g}(y, t) = (\varphi_t^{-1})^*(g(x, t)) \).

We claim that \( \bar{g}(y, t) \) is the solution of (XCF) with conditions (3) and (4). First, we compute

\[
\frac{\partial}{\partial t} \bar{g}(y, t) = \frac{\partial}{\partial t}((\varphi_t^{-1})^*(g(x, t)))
\]

\[
= (\varphi_t^{-1})^*(\frac{\partial}{\partial t}g(x, t)) - (\varphi_t^{-1})^*(L_W g)
\]

\[
= (\varphi_t^{-1})^*(2c(g) + L_W g) - (\varphi_t^{-1})^*(L_W g)
\]

\[
= (\varphi_t^{-1})^*(2c(g))
\]

\[
= 2c(\varphi_t^{-1})^*(g)
\]

\[
= 2c(\bar{g}).
\]

Secondly, we check that the boundary condition (4) is satisfied. Let \( \bar{h} \) be the second fundamental form of \( (\partial M, \bar{g} |_{\partial M}) \) in \( (M, \bar{g}) \), then we have

\[
\bar{h}_{\alpha\beta} = ((\varphi_t^{-1})^*(h))_{\alpha\beta} = \lambda(t)\bar{g}_{\alpha\beta}.
\]

Therefore, we have proved that (XCF) with condition (3) and (4) has a unique short time solution.

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