TROPICAL HOMOLOGY

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Abstract. Given a tropical variety $X$ and two non-negative integers $p$ and $q$ we define a homology group $H_{p,q}(X)$ which is a finite-dimensional vector space over $\mathbb{Q}$. We show that if $X$ is a smooth tropical variety that can be represented as the tropical limit of a 1-parameter family of complex projective varieties, then $\dim H_{p,q}(X)$ coincides with the Hodge number $h^{p,q}$ of a general member of the family.

1. Introduction

1.1. Homology theory for tropical varieties. Tropical varieties are certain finite-dimensional polyhedral complexes enhanced with the tropical structure. This is a geometric structure that can be thought of as a version of an affine structure for polyhedral complexes. For example, the tropical projective space $\mathbb{T} \mathbb{P}^N$ is a smooth projective tropical variety homeomorphic to the $N$-simplex. The restriction of the tropical structure to the relative interior of a $k$-dimensional face $\sigma$ of $\mathbb{T} \mathbb{P}^N$ turns $\sigma$ into $\mathbb{R}^k$ (with the tautological affine structure of $\mathbb{R}^k = \mathbb{Z}^k \otimes \mathbb{R}$). A projective tropical $n$-variety $X$ is a certain $n$-dimensional polyhedral complex in $\mathbb{T} \mathbb{P}^N$.

A tropical structure on $X$ can be used to define a natural coefficient system $\mathbb{Z} \mathcal{F}_p$. This system is not locally constant everywhere, but it is constant on the relative interiors of faces of $X$. Furthermore, it is a constructible cosheaf of abelian groups. The tropical $(p,q)$-homology group $H_{p,q}(X)$ is the $q$-dimensional homology group of $X$ with coefficients in $\mathcal{F}_p = \mathbb{Z} \mathcal{F}_p \otimes \mathbb{Q}$.

An important example of projective tropical varieties is provided by the tropical limit of an algebraic family $Z_w \subset \mathbb{C} \mathbb{P}^N$, $w \in \mathbb{C}$, $t = |w| \to \infty$, of complex projective $n$-dimensional varieties. It may be shown (cf. e.g. the fundamental theorem of tropical geometry of [MS15]) that the sets $\text{Log}_t(Z_w) \subset \mathbb{T} \mathbb{P}^N$ converge to an $n$-dimensional balanced weighted polyhedral complex $X$ in $\mathbb{T} \mathbb{P}^N$. If $X$ is a smooth tropical variety, then for a generic $w$ the complex variety $Z_w$ is smooth. The main result of this paper establishes the equality between $\dim H_{p,q}(X)$ and the Hodge numbers $h^{p,q}(Z_w)$.

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Theorem 1 (Main Theorem). Let $\mathcal{Z} \subset \mathbb{CP}^N \times \mathcal{D}^*$ be a complex analytic one-parameter family of projective varieties over the punctured disc $\mathcal{D}^*$. Assume that $\mathcal{Z}$ admits a tropical limit $X \subset \mathbb{TP}^N$, which is a smooth projective $\mathbb{Q}$-tropical variety (see Definition 11). Then, the dual spaces $\text{Hom}(H_q(X; \mathcal{F}_p), \mathbb{Q})$ to the tropical homology groups $H_q(X; \mathcal{F}_p)$ are naturally isomorphic to the associated graded pieces $W_{2p}/W_{2p-1}$ of the weight filtration in the limiting mixed Hodge structure on $H^{p+q}(\mathcal{Z}_\infty, \mathbb{Q})$, where $\mathcal{Z}_\infty$ is the canonical fiber of the family $\mathcal{Z}$.

Under the assumptions of Theorem 1, the limiting mixed Hodge structure is of Hodge-Tate type. That is, only even associated graded pieces $\text{Gr}^{W_{2p}} H_k(\mathcal{Z}_\infty; \mathbb{Q}) = W_{2p}/W_{2p-1}$ are non-trivial and they have Hodge $(p,p)$-type. Hence, the Hodge numbers $h^{p,q}(\mathcal{Z}_w)$ agree with the dimensions of the spaces $\text{Gr}^{W_{2p}} H^{p+q}(\mathcal{Z}_\infty; \mathbb{Q})$.

Corollary 2. The Hodge numbers $h^{p,q}(\mathcal{Z}_w)$ of a general fiber equal the dimensions of the tropical homology groups $H_q(X; \mathcal{F}_p)$.

Meanwhile $\mathcal{F}_p$ and $\mathcal{F}_p'$ are equally well defined for an arbitrary tropical variety $X$ even if it cannot be presented as the tropical limit of a family of complex varieties. The groups $H_q(X; \mathcal{F}_p)$ and $H_q(X; \mathcal{F}_p')$ give a homology theory in tropical geometry.

We prefer to work with homology rather than with cohomology because the tropical homology have a more transparent geometric meaning. For cohomology theory one can consider the constructible sheaf $\mathcal{F}_p'$ on $X$ whose stalks are dual to the spaces $\mathcal{F}_p$. Differential forms and currents may also be considered on tropical varieties, see [La10]. Note that the work [CLD12] makes use of the pull-backs of such forms to Berkovich spaces. A recent work [JSS15] provides a link between usage of such differential forms and the tropical homology definition considered in this paper.

Our proof of Theorem 1 consists of providing a quasi-isomorphism between the tropical cellular complexes and the dual row complexes of the $E^1$-term of the Steenbrink-Illusie spectral sequence for the limiting mixed Hodge structure (see Theorem 49).

Remark. If $X$ comes as the tropical limit of complex varieties $Z_w$ then the geometric meaning of the tropical coefficients $\mathcal{F}_p$ and $\mathcal{F}_p'$ originates from the Clemens collapse map $\pi : Z_w \to X$, [Cl77]. Namely, the sheaf $\mathcal{F}_p'$ can be identified with the direct image $R^p\pi_* \mathbb{Q}$. Then the Leray spectral sequence which calculates $H^{p+q}(Z_w; \mathbb{Q})$ has the second term $E_2^{p,q} = H^q(X; \mathcal{F}_p)$.

1.2. Tropical Euler characteristics. Each coefficient system $\mathcal{F}_p$ independently gives homology groups $H_q(X; \mathcal{F}_p)$ for all dimensions $q$. The corresponding Euler characteristic

\[ \chi_p(X) = \sum_{q=0}^{n} (-1)^q \dim H_q(X; \mathcal{F}_p) \]

is a basic invariant of the tropical variety $X$ which is especially easy to compute. The corresponding classical invariants

\[ \chi_p(Z_w) = \sum_{q=0}^{n} (-1)^q h^{p,q}(Z_w) \]
were introduced by Hirzebruch \cite{Hi56} in the form of $\chi_y$-genus $\chi_y(Z_w) = \sum_{p=0}^{n} \chi_p(Z_w)y^p$.

Clearly, Theorem 1 implies that

\begin{equation}
\chi_p(X) = \chi_p(Z_w).
\end{equation}

E.g. $\chi_0(X)$ is nothing else but the conventional Euler characteristic of $X$, while $\chi_0(Z_w)$ is the holomorphic Euler characteristic (arithmetic genus) of $Z_w$. As usual for the Euler characteristic, the alternation of signs in (1) provides additional invariance properties.

Since the definition of the limiting Hodge structure, there was developed a way to compute it with the help of a central fiber $Z_0$ of the family $Z$, i.e., through its extension over $\mathcal{D} \supset \mathcal{D}^\ast$, see \cite{PS08} and references therein. Note that the choice of $Z_0$ is not unique, and different choices yield different homology data of $Z_0$.

In the same time, the Kähler manifolds $Z_w$, $w \neq 0$, are symplectomorphic, and thus have the same topological homology groups. One may expect that it should be possible to find such a central fiber $Z_{\text{nearby}}$ (instead of $Z_0$) that $Z_{\text{nearby}}$ is symplectomorphic to $Z_w$, $w \neq 0$. The problem is that (in the case of non-trivial family $Z$) such $Z_{\text{nearby}}$ cannot carry a complex structure in the conventional sense (the holomorphic tangent subbundle $T^{1,0}(Z_{\text{nearby}})$ in $T(Z_{\text{nearby}}) \otimes \mathbb{C}$ cannot stay transversal to $T(Z_{\text{nearby}}) \otimes \mathbb{R}$ due to behavior of $T^{1,0}(Z_w)$ for small $w \neq 0$).

The notion of motivic nearby fiber (see \cite{DL01}, \cite{Bi05} and \cite{PS08}) avoids this problem by defining a class $\psi$ in the Grothendieck ring of varieties so that it should correspond to $Z_{\text{nearby}}$ (would it exist as a variety). The class $\psi$ can be expressed as a certain linear combination of strata of $Z_0$ and does not depend on ambiguity in the choice of $Z_0$ thanks to the alternation of signs in the expression of $\psi$ via $Z_0$.

When we pass from $Z_w$ to the motivic nearby fiber $\psi$ we loose some homological information. For example, $\psi$ is the class of the empty set for the degeneration corresponding to a smooth tropical elliptic curve (or, more generally, a smooth tropical Abelian variety). Under the assumption of Theorem 1, the motivic nearby fiber $\psi$ is a linear combination of the powers of the class $L$ of the affine line, and thus carries the same amount of data as the $E$-polynomial of Deligne-Hodge.

**Corollary 3.** Under the assumptions of Theorem 1 we have

\begin{equation}
E(\psi) = E(Z_\infty) = \sum_{p=0}^{n} \chi_p u^p v^p.
\end{equation}

**Proof.** The second equality of (4) is the combination of the definition of the $E$-polynomial and Theorem 1. The first equality follows from the additivity of $E$-polynomial with the help of the description of central fiber from Proposition 48 and the well-known $E$-polynomial computation for the hyperplane arrangement complements (cf. Theorem 15). For the first equality see also section 11.2.7 of \cite{PS08}. \qed

In general, $E(\psi)$ does not determine individual numbers $h^{p,q}(Z_w) = \dim H^q(X; F_p)$. In the special case of complete intersections, the Lefschetz hyperplane section theorem
determines $h^{p,q}(Z_w)$ for $p + q \neq n$, and thus $E(\psi)$ (which can easily be read from the combinatorial data of $X$) suffices to recover $h^{p,q}(Z_w)$. This observation appeared in [KST16] for the case of hypersurfaces.

**Remark.** Note that the tropical limit of $Z$ introduced in Definition 37 does not require to make any choice for the central fiber $Z_0$ whatsoever. Different choices of $Z_0$ correspond to different triangulations of $X$ while the homology groups $H_q(X; \mathcal{F}_p)$ do not require introduction of such an additional structure.

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2. Tropical varieties and their homology

Main results in this paper concern smooth $\mathbb{Q}$-tropical varieties embedded in some projective space. In this section we adapt definitions of tropical geometry to this special case. The notion of tropical homology can be defined for more general tropical spaces, e.g. singular or non-compact, see [MZh14] but its relation to the (mixed) Hodge structures of singular complex varieties is to be understood.

2.1. Polyhedral complexes in tropical projective space. The tropical affine space $\mathbb{T}^N$ is the topological space $[-\infty, +\infty)^N$ (homeomorphic to the $n$-th power of a half-open interval) enhanced with an integral affine structure defined as follows. We stratify the space $\mathbb{T}^N$ by

$$\mathbb{T}_I^\circ := \{y = (y_1, \ldots, y_N) \in \mathbb{T}^N : y_i = -\infty, i \in I \text{ and } y_i > -\infty, i \notin I\} \cong \mathbb{R}^{N-I},$$

where $I$ runs over subsets of $\{1, \ldots, N\}$. We set $\mathbb{T}_I \cong \mathbb{T}^{N-I}$ to be the closure of $\mathbb{T}_I^\circ$ in $\mathbb{T}^N$. On each $\mathbb{T}_I^\circ$ the integral affine structure is induced from $\mathbb{R}^{N-I}$, and for pairs $I \subset J$ the projection maps $\mathbb{T}_I \to \mathbb{T}_J$ are $\mathbb{Z}$-affine linear. Here and later we use $N - I$ in the exponent to denote the product of $N - |I|$ factors in the complement of the subset $I$ in $\{1, \ldots, N\}$.

Let $B_I$ be a $(k - |I|)$-dimensional ball in $\mathbb{T}_I^\circ$. A $k$-dimensional $I$-ball $B$ in $\mathbb{T}^N$ is the $\epsilon$-neighborhood of $B_I$ in $\mathbb{T}^N$ (using the projection map $\pi_I^\mathbb{T} : \mathbb{T}^N \to \mathbb{T}_I$):

$$B = \{y \in \mathbb{T}^N : \pi_I^\mathbb{T}(y) \in B_I \text{ and } y_i < \log \epsilon, i \in I\}$$

for some $\epsilon > 0$. In particular, an $\emptyset$-ball is just an ordinary $k$-dimensional ball in $\mathbb{R}^N \subset \mathbb{T}^N$. For $I \neq \emptyset$ an $I$-ball is a $k$-dimensional manifold with corners. The boundary $\partial B$ of an $I$-ball $B$ is defined as $B \setminus B_I$, that is, we exclude from its topological boundary all strata at infinity, and then take the closure in $\mathbb{T}^N$.

The tropical projective space $\mathbb{T}^{\mathbb{P}^N}$ can be defined as the quotient of $\mathbb{T}^{N+1} \setminus (-\infty, \ldots, -\infty)$ by the equivalence $(x_0, \ldots, x_N) \sim (x_0 + \lambda, \ldots, x_N + \lambda)$ for any $\lambda \in \mathbb{R}$. In particular, as a topological space, $\mathbb{T}^{\mathbb{P}^N}$ is homeomorphic to an $N$-simplex. Alternatively, $\mathbb{T}^{\mathbb{P}^N}$ can be glued from $N + 1$ affine charts $U^{(i)} = \{x_i \neq -\infty\} \cong \mathbb{T}^N$ with coordinates $y^{(i)}_k = x_k - x_i, i \neq k$. 

Every two charts $U^{(i)}$ and $U^{(j)}$ are identified along $U^{(i)} \cap U^{(j)} \cong T^{N-1} \times \mathbb{R}$ via $y^{(i)}_k = y^{(j)}_k - y^{(j)}_j$ for $k \neq i, j$, and $y^{(i)}_j = -y^{(j)}_j$.

For any subset $I \subset \{0, \ldots, N\}$ we denote by $\mathbb{T}P_I \cong \mathbb{T}P^{N-I}$ and by $\mathbb{T}P_I^o \cong \mathbb{R}^{N-I}$ the closed and open coordinate strata of $\mathbb{T}P^N$, respectively. That is $\mathbb{T}P_I$ is defined by setting $x_i = -\infty$ for $i \in I$, and for $\mathbb{T}P_I^o \subset \mathbb{T}P_I$ we additionally require $x_i \neq -\infty$ for $i \notin I$.

The directions parallel to the $j$-th coordinate in $\mathbb{R}^N \subset T^N$ towards its $-\infty$-value are called divisorial directions. The primitive integral vector along a divisorial direction (pointing towards $-\infty$ as the direction itself) is called a divisorial vector. The positive linear combinations of divisorial vectors in $I \subset \{1, \ldots, N\}$ span the $I$-th divisorial cone in $\mathbb{R}^N$.

The notions of divisorial directions, vectors and cones are well defined for all open strata in $\mathbb{T}P^N$. Indeed, if $\mathbb{T}P^o_I \subset \mathbb{T}P^N$ is such a stratum and $j \notin I$, we can take any chart $U^{(i)}_{j(i)} \cong T^{N-I}$ of $\mathbb{T}P_I$ such that $i \neq j$ and define the $j$-th divisorial direction as above. Clearly, the $j$-th divisorial directions agree in any two such charts. The same can be said about the $J$-th divisorial cones for any subset $J$ (of size not greater than $N-I-1$) disjoint from $I$.

Recall that an $n$-dimensional polyhedral complex $Y^o \subset \mathbb{R}^N$ with rational slopes is a finite union of $n$-dimensional convex polyhedral domains called facets. Each facet is the intersection of a finite number of half-spaces of the form $mx \leq a$, where $x \in \mathbb{R}^N$, $a \in \mathbb{R}$, $m \in \mathbb{Z}^N$. The intersection of any number of facets is required to be their common face.

**Lemma 4.** The closure in $\mathbb{T}P^N$ of an $n$-dimensional polyhedral complex $Y^o \subset \mathbb{R}^N$ with rational slopes intersects each stratum $\mathbb{T}P^o_I$ in a subset which supports a polyhedral complex of dimension $\leq (n-1)$ with rational slopes.

**Proof.** It is enough to show that the closure in $T^N$ of a polyhedral domain $D \subset \mathbb{R}^N$ intersects a stratum $T^o_I$ in a polyhedral domain $D_I$ of smaller dimension. Let $D_I$ be the image of $D$ under the projection along the divisorial directions in $I$. Clearly, $D_I$ is a polyhedral domain in $T^o_I$.

Consider the intersection of the $I$-th divisorial cone in $\mathbb{R}^N$ with the asymptotic cone of $D$ (that is, the cone formed by the vectors $v \in \mathbb{R}^N$ with the property: $x \in D$ implies that $x + av \in D$ for any $a \geq 0$). Then, observe that $D_I = D_j$ if this intersection contains a ray (in that case the dimension of $D_I$ is smaller than $n$), and $D_I$ is empty otherwise. \hfill \Box

An $n$-dimensional polyhedral complex $Y^o \subset \mathbb{R}^N$ with rational slopes is called weighted if the facets are equipped with non-negative integers. Recall the balancing condition: for every face $\Delta$ of codimension 1 the weighted sum of primitive (relatively to $\Delta$) integer outward tangent vectors in the facets incident to $\Delta$ should be parallel to $\Delta$.

**Definition 5.** A (weighted) polyhedral complex $Y \subset \mathbb{T}P^N$ of dimension at most $n$ is a finite union of (weighted) polyhedral complexes $Y_I$ of dimensions $\leq n$ in $\mathbb{T}P^o_I$, where $I$ runs over the subsets of $\{0, \ldots, N\}$, such that for any pair $I \subset J$ the intersection of the closure of any face of $Y_I$ with $\mathbb{T}P^o_J$ is a face of $Y_J$. The complex $Y$ is (pure) $n$-dimensional if any point in $Y$ lies in the closure of some $n$-dimensional face. It is balanced if all $n$-dimensional complexes in the union satisfy the balancing condition.
By a face of a polyhedral complex \( Y \subset \mathbb{T}^N \) we mean the closure in \( Y \) of a face of a complex from the union. Any polyhedral complex \( Y \) can be considered weighted by setting all weights equal to one. By default, we mean this situation unless other weights are explicitly prescribed.

2.2. Smooth projective tropical varieties. Now we define a more restrictive class of polyhedral complexes in \( \mathbb{T}^N \). A convex regular \( \mathbb{Q} \)-polyhedral domain \( D \) in \( \mathbb{T}^N \) is the intersection of a finite collection of half-spaces \( H_k \) of the form

\[
H_k = \{ x \in \mathbb{T}^N \mid mx \leq a \} \subset \mathbb{T}^N
\]

for some \( m \in \mathbb{Z}^N \) and \( a \in \mathbb{Q} \). Here, we assume that if some component \( m_i \) of \( m \) is negative, the corresponding component \( x_i \) of \( x \) can not take the value \(-\infty\). So that \( H_k \) only contains points \( x \in \mathbb{T}^N \) for which the scalar product \( mx \) is well-defined. The following statement is immediate.

**Lemma 6.** Let \( D \subset \mathbb{T}^N \) be a non-empty convex regular \( \mathbb{Q} \)-polyhedral domain defined by the inequalities \( m^{(r)} x \leq a^{(r)} \). Let \( I \) be a subset of \( \{1, \ldots, N\} \). Then \( D_I := D \cap \mathbb{T}_I \) is non-empty if and only if \( m_i^{(r)} \geq 0 \) for all \( r \) and all \( i \in I \). \( \square \)

The boundary \( \partial H_k \) of a half-space \( H_k \) is given by the equation \( mx = a \). A mobile face \( E \) of \( D \) is the intersection of \( D \) with the boundaries of some of its defining half-spaces given by (6). The adjective mobile stands here to distinguish such faces from more general faces which we define below and which are allowed to have support in \( \mathbb{T}^N \setminus \mathbb{R}^N \), i.e., to be disjoint from \( \mathbb{R}^N \subset \mathbb{T}^N \). (Such faces disjoint from \( \mathbb{R}^N \subset \mathbb{T}^N \) have reduced mobility and are called sedentary.)

The dimension of a convex regular \( \mathbb{Q} \)-polyhedral domain \( D \) is its dimension as a topological manifold (possibly with boundary). Observe that for each non-empty mobile face \( E \) of \( D \) the intersection \( E^\circ = E \cap \mathbb{R}^N \) is non-empty. Each mobile face of \( D \) is a convex regular \( \mathbb{Q} \)-polyhedral domain itself.

**Definition 7.** An \( n \)-dimensional regular \( \mathbb{Q} \)-polyhedral complex \( Y = \bigcup D \subset \mathbb{T}^N \) is the union of a finite collection of \( n \)-dimensional convex regular \( \mathbb{Q} \)-polyhedral domains \( D \), called the facets of \( Y \), subject to the following property: for any collection \( \{D_j\} \) of facets, their intersection \( \bigcap D_j \) is a mobile face of each facet \( D_j \). Such intersections are called the mobile faces of \( Y \).

For mobile faces \( E \) of \( Y \) and subsets \( I \subset \{1, \ldots, N\} \), it is convenient to treat the intersections \( E \cap \mathbb{T}_I \) also as faces (at infinity) of \( Y \).

**Definition 8.** Let \( E \) be a mobile face of \( Y \), and let \( I \) be a subset of \( \{1, \ldots, N\} \). We say that

\[
E_I = E \cap \mathbb{T}_I
\]

is a face of \( Y \). The sedentarity of the face \( E_I \) is \( s = |I| \), while its refined sedentarity is \( I \). We call the face \( E \) a parent of \( E_I \). The poset \( \Pi(E) \) of faces \( E_I \), where \( I \) runs over all subsets \( I \subset \{1, \ldots, N\} \), is called the family of \( E \).
Clearly, the mobile faces are the faces of sedentarity 0. A mobile face $E$ of $Y$ such that $E_I$ is non-empty for some subset $I \subset \{1, \ldots, N\}$ contains a ray along the $j$-th divisorial direction for each $j \in I$ (cf. Lemma [6]). In this case we say that this is a **divisorial direction of the face** $E$, see Figure 1. By convexity this also imply that $E$ contains the entire $I$-th divisorial cone. (Note that a more general polyhedral complex considered in Section 2.1 may have a face whose closure intersects a stratum $\mathbb{T}_I$, but the face does not contain the $I$-th divisorial cone.) Sedentary faces of $Y$ are mobile when considered in the respective strata of $\mathbb{T}^N$, and as such also have divisorial directions defined.

The following lemma describes the geometry of a regular $\mathbb{Q}$-polyhedral complex $Y$ near its sedentary faces.

**Lemma 9.** Let $Y$ be a regular $\mathbb{Q}$-polyhedral complex in $\mathbb{T}^N$. Let $I$ be a subset of $\{1, \ldots, N\}$ such that $Y_I := Y \cap \mathbb{T}_I$ is non-empty. Then, $Y_I$ is a regular $\mathbb{Q}$-polyhedral complex in $\mathbb{T}_I$. Moreover, its regular neighborhood

$$Y^\epsilon_I := \{y \in Y : y_i < \log \epsilon, i \in I\},$$

for sufficiently small $\epsilon > 0$, splits as the product

$$Y^\epsilon_I = Y_I \times \mathbb{T}^I_\epsilon,$$

where $\mathbb{T}^I_\epsilon := \{x_i < \log \epsilon, i \in I\} \subset \mathbb{T}^I$.

**Proof.** Let $E$ be a mobile face of $Y$ such that $E_I := E \cap \mathbb{T}_I$ is non-empty. By Lemma [6] the defining inequalities (6) for $E$ must have $m_i \geq 0$ for all $i \in I$. Furthermore, to define the face $E_I \subset \mathbb{T}_I$ one can take those inequalities for $E$ that have $m_i = 0$ for all $i \in I$. Each of the remaining inequalities for $E$ have $m_i > 0$ for at least one $i \in I$, and hence they are satisfied for sufficiently small $y_i, i \in I$.

For a pair of mobile faces $E, F$ of $Y$ we have $(E \cap F)_I = E_I \cap F_I$ (both are defined by plugging $y_i = -\infty, i \in I$, into the union of the defining inequalities for $E$ and $F$). Thus, the statement for a regular neighborhood of $Y_I$ follows from the statement for each individual mobile face of $Y$. \qed
Let $E \subset Y$ be a mobile face of $Y$, and let $x$ be a point in the relative interior of $E$. Consider the tangent cone $T_x Y$ to $Y$ at $x$, and denote by $\Sigma_E$ the quotient of $T_x Y$ by the linear span $L_E$ of $E \cap \mathbb{R}^N$. This quotient is a $\mathbb{Q}$-polyhedral fan in the vector space $\mathbb{R}^N/L_E$, i.e., a polyhedral complex with rational slopes which has a cone structure in $\mathbb{R}^N/L_E$. The fan $\Sigma_E$ is called relative fan of $Y$ at $E$.

Finally, we recall the tropical notion of smoothness. A matroid $M = (M, r)$ is a finite set $M$ together with a rank function $r : 2^M \to \mathbb{Z}_{\geq 0}$ such that we have the inequalities $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ and $r(A) \leq |A|$, for any subsets $A, B \subset M$, as well as the inequality $r(A) \leq r(B)$ whenever $A \subset B$. Subsets $A \subset M$ such that $r(A) = |A|$ (respectively, $r(A) < |A|$) are called independent (respectively, dependent). Subsets $F \subset M$ such that $r(A) > r(F)$ for any $A \supseteq F$ are called flats of $M$ of rank $r(F)$. Matroid $M$ is loopless if $r(A) = 0$ implies $A = \emptyset$.

The Bergman fan of a loopless matroid $M$ is a $\mathbb{Q}$-polyhedral fan $\Sigma_M \subset \mathbb{R}^{|M|-1}$ constructed as follows (see [AK06]). Choose $|M|$ integer vectors $e_j \subset \mathbb{Z}^{|M|-1}$ such that $\sum_{j \in M} e_j = 0$ and any $|M| - 1$ of these vectors form a basis of $\mathbb{Z}^{|M|-1}$. To any flat $F \subset M$ we associate a vector

$$e_F := \sum_{j \in F} e_j \in \mathbb{R}^{|M|-1}.$$ 

E.g., $e_M = e_\emptyset = 0$, but $e_F \neq 0$ for any other (proper) flat $F$. To any flag of flats $F_{i_1} \subset \cdots \subset F_{i_k}$ we associate the convex cone generated by $e_{F_{i_j}}$. We define $\Sigma_M$ to be the union of such cones, which is, clearly, an $(r(M) - 1)$-dimensional rational simplicial fan. The fan $\Sigma_M$ is called the Bergman fan of $M$.

**Definition 10.** A regular $\mathbb{Q}$-polyhedral complex in $\mathbb{T}^N$ is smooth at a mobile face $E \subset Y$ if the relative fan $\Sigma_E$ has the same support as the Bergman fan $\Sigma_M$ for some loopless matroid $M$ (recall that the support of a fan is the union of its cones). A regular $\mathbb{Q}$-polyhedral complex $Y \subset \mathbb{T}^N$ is called smooth, if it is smooth at all its mobile faces.

Note that since all Bergman fans are balanced (cf. [AK06]), a smooth regular $\mathbb{Q}$-polyhedral complex $Y$ is automatically balanced. That is, every mobile face of $Y$ of codimension 1 has a balanced relative fan: the sum of the outward primitive integer vectors along its rays is zero.

**Definition 11.** A closed subset $X \subset \mathbb{T}\mathbb{P}^N$ is a smooth regular projective $\mathbb{Q}$-tropical variety if on every affine chart $U^{(i)} \cong \mathbb{T}^N \subset \mathbb{T}\mathbb{P}^N$ it restricts to a smooth regular $\mathbb{Q}$-polyhedral complex $X^{(i)}$ in $\mathbb{T}^N$.

By definition, the faces of $X$ are the closures of the faces of $X^{(i)}$ in $X$. A face $\Delta$ of $X$ is determined by its relative interior which coincides with the relative interiors for all non-empty restrictions to the complexes $X^{(i)}$.

For convenience we make the following additional assumption: each face $\Delta$ of $X$ lies entirely in at least one chart $X^{(i)}$. For instance, we avoid considering $X = \mathbb{T}\mathbb{P}^N$ as just
one face, it needs some subdivision. We can always subdivide $X$ in order to achieve this requirement.

Sedentarity in $\mathbb{T}P^N$ is inherited from the projection map $\mathbb{T}P^{N+1} \setminus \{0\} \to \mathbb{T}P^N$. Recall that for any face $\Delta$ of a smooth regular projective $\mathbb{Q}$-tropical variety $X \subset \mathbb{T}P^N$ the divisorial directions are intrinsically defined. The splitting of the boundary neighborhoods $X_f^i = X_f \times \mathbb{T}^i$, however, is not canonical, it depends on the chart.

We introduce some convenient notations and terminology. We write $\Delta' \prec_j^s \Delta$ (and $\Delta \succ_j^s \Delta'$) when $\Delta'$ is a face of $\Delta$ of codimension $j$ and cosedentarity $s$ (that is, the sedentarities of $\Delta'$ and $\Delta$ differ by $s$). We omit the superscript $s$ in case $s = 0$ and simply write $\Delta' \prec_j \Delta$. A face $\Delta$ of $X$ is called infinite if it has a subface of higher sedentarity. Otherwise, $\Delta$ is called finite (the sedentarity of $\Delta$ may be positive). The star of a face $\Delta$ of $X$ is the poset formed by the faces of $X$ that contain $\Delta$ and have the same sedentarity as $\Delta$.

Lemmas 6 and 9 imply the following statement.

**Proposition 12.** Let $X$ be a smooth regular projective $\mathbb{Q}$-tropical variety, and let $\Delta_0$ be a mobile face of $X$.

1. Any face of $X$ belongs to a single family.
2. The face $\Delta_0$ contains a unique subface of maximal sedentarity; denote this surface by $\Delta_J$, where $J$ is its refined sedentarity. The face $\Delta_J$ is finite.
3. The family $\Pi(\Delta_0)$ is the rank $|J|$ lattice (under $\prec_J^0$) isomorphic to the lattice of all subsets of $J$.
4. The asymptotic cone of $\Delta_0$ coincides with the divisorial cone of $\Delta_0$. The divisorial directions of $\Delta_0$ are indexed by the elements of $J$.
5. All faces in the family $\Pi(\Delta_0)$ have isomorphic stars. A relation $\Delta' \prec_j \Delta''$ for any two faces of $X$ also holds for their parents: $\Delta_0' \prec_j \Delta_0''$. In addition, a relation $\Delta_0' \prec_j \Delta_0''$ among mobile faces gives rise to an injection of $\Pi(\Delta_0')$ into $\Pi(\Delta_0'')$. □

### 2.3. Local homology and Orlik-Solomon algebra.

Let $\Sigma = \bigcup \sigma \subset \mathbb{R}^N = \mathbb{Z}^N \otimes \mathbb{R}$ be a $\mathbb{Q}$-polyhedral fan. For each cone $\sigma \subset \Sigma$, we denote by $\langle \sigma \rangle_\mathbb{Z}$ the integral lattice in the vector subspace linearly spanned by $\sigma$.

**Definition 13.** The homology group $\mathbb{Z}F_k(\Sigma)$ is the subgroup of $\wedge^k \mathbb{Z}^N$ generated by the elements $v_1 \wedge \cdots \wedge v_k$, where all $v_1, \ldots, v_k \in \langle \sigma \rangle_\mathbb{Z}$ for some cone $\sigma \in \Sigma$. It is important that all $k$ vectors $v_i$ come from the same cone. The cohomology is the dual group $\mathbb{Z}F^k(\Sigma) := \text{Hom}(\mathbb{Z}F_k(\Sigma), \mathbb{Z})$, which is the quotient of $\wedge^k (\mathbb{Z}^N)^*$ by $(\mathbb{Z}F_k(\Sigma))^\perp$.

It is not hard to see (cf. [Zh13]) that the cohomology groups form a graded algebra $\mathbb{Z}F^*(\Sigma)$ over $\mathbb{Z}$ under the wedge product in $\wedge^k (\mathbb{Z}^N)^*$.

We restrict our attention to the case where $\Sigma$ is the Bergman fan $\Sigma_M$ associated to a loopless matroid $M$. On the other hand, to any loopless matroid $M$ one can also associate its Orlik-Solomon algebra $OS(M)$ as follows (see, e.g. [OT92]).

Let $W$ be a rank $N + 1$ free abelian group generated by elements $f_0, \ldots, f_N$, where $|M| = N + 1$. Then, $OS^*(M) := \wedge^* W / \mathcal{I}^*$, where the Orlik-Solomon ideal $\mathcal{I}$ is generated
by the elements

\[ \partial(f_{i_0} \wedge f_{i_1} \wedge \cdots \wedge f_{i_k}) := \sum_{s=0}^{k} (-1)^s f_{i_0} \wedge \cdots \hat{f}_{i_s} \cdots \wedge f_{i_k}, \]

for all dependent subsets \( I = \{i_0, i_1, \ldots, i_k\} \) of the matroid \( M \).

More relevant for us is the projective Orlik-Solomon algebra \( \text{OS}^\bullet_0(M) \), which is the following modification of \( \text{OS}(M) \). Let \( W_0 \) be the subgroup of \( W \) generated by all differences \( f_i - f_j \). Then, we set \( \text{OS}^\bullet_0(M) := \wedge^\bullet W_0 / I^\bullet_0 \), where \( I_0 = I \cap \wedge^\bullet W_0 \) is the restriction of \( I \) to the subalgebra \( \wedge^\bullet W_0 \subset \wedge^\bullet W \).

**Theorem 14** ([Zh13]). There is a canonical isomorphism \( \mathbb{Z} \mathcal{F}^\bullet(\Sigma_M) \cong \text{OS}^\bullet_0(M) \) of graded \( \mathbb{Z} \)-algebras.

Note that the cohomology groups depend only on the support of a polyhedral fan. Thus, if two matroids \( M_1 \) and \( M_2 \) have Bergman fans with the same support, the above theorem shows that the two matroids have isomorphic Orlik-Solomon algebras.

The main application for us will be when \( M \) is realizable by a hyperplane arrangement in \( \mathbb{C}P^n \). Let \( Y \) denote the complement of the arrangement. Then, it is well known (cf., e.g. [OT92]) that the projective Orlik-Solomon algebra calculates cohomology of \( Y \). This leads to the following corollary.

**Theorem 15.** There is a canonical isomorphism \( \mathbb{Z} \mathcal{F}_k(\Sigma_M) \cong H_k(Y; \mathbb{Z}) \).

2.4. Tropical homology, the cellular version. Let \( X \subset \mathbb{T} \mathbb{P}^N \) be a smooth regular projective \( \mathbb{Q} \)-tropical variety. The polyhedral decomposition of \( X \) into faces gives it a natural cell structure.

Let \( x \in X \) be a point in the relative interior of a face \( \Delta_x \) of sedentarity \( I \) in \( X \). We define \( \Sigma(x) \), the fan at \( x \), to be the cone in \( \mathbb{T}^\circ_I \cong \mathbb{R}^{N-|I|} \) consisting of vectors \( u \in \mathbb{T}^\circ_I \) such that \( x + \epsilon u \in X \cap \mathbb{T}^\circ_I \) for a sufficiently small \( \epsilon > 0 \) (depending on \( u \)).

**Definition 16.** We define the coefficient groups \( \mathcal{F}_k(x) \) and \( \mathcal{F}^k(x) \) to be \( \mathbb{Z} \mathcal{F}_k(\Sigma(x)) \otimes \mathbb{Q} \) and \( \mathbb{Z} \mathcal{F}^k(\Sigma(x)) \otimes \mathbb{Q} \), respectively.

Note that the groups \( \mathcal{F}_k(x) \) and \( \mathcal{F}_k(y) \) are canonically identified by translation if \( x \) and \( y \) belong to the relative interior of the same face \( \Delta \) of \( X \). Thus, we can use the notation \( \mathcal{F}_k(\Delta) \). We can also consider the relative coefficient groups \( \tilde{\mathcal{F}}_k(\Delta) \) defined as \( \mathbb{Z} \mathcal{F}_k(\Sigma_\Delta) \otimes \mathbb{Q} \), where \( \Sigma_\Delta \) is the relative fan at \( \Delta \) (see Section 2.2) if \( \Delta \) is mobile; if \( \Delta \) is sedentary, then \( \Sigma_\Delta \) is the relative fan at the parent mobile face of \( \Delta \) in some affine chart containing the relative interior of \( \Delta \).

If for two points \( x, y \) we have \( \Delta_x \succ \Delta_y \) then there are natural homomorphisms

\[ \iota : \mathcal{F}_k(x) \to \mathcal{F}_k(y). \]

To define the maps \((7)\) we take an affine chart \( U^{(i)} \ni y \). If \( I(y) = I(x) \), then any face adjacent to \( x \) is contained in some face adjacent to \( y \) and the inclusion induces the required
map. If \( I(y) \neq I(x) \) (note that we must have \( I(y) \supset I(x) \)), then the required map is given by the projection along the divisorial directions indexed by \( I(y) \setminus I(x) \).

For a pair of adjacent faces \( \Delta \prec \Delta' \), the map (7) and its dual can be rewritten as

\[
\iota : \mathcal{F}_k(\Delta') \to \mathcal{F}_k(\Delta), \quad \iota^* : \mathcal{F}^k(\Delta) \to \mathcal{F}^k(\Delta').
\]

This allows us to define a complex \( C_\bullet(X; \mathcal{F}_p) \), where

\[
C_q(X; \mathcal{F}_p) = \oplus \mathcal{F}_p(\Delta).
\]

Here, the direct sum is taken over all \( q \)-dimensional faces of \( X \). We can write a chain in \( C_q(X; \mathcal{F}_p) \) as \( \sum \beta_\Delta \Delta \). The boundary map

\[
\partial : C_q(X; \mathcal{F}_p) \to C_{q-1}(X; \mathcal{F}_p)
\]

is the usual cellular boundary combined with the maps \( \iota \) in (8) for any pair of faces \( \Delta \succ 1 \Delta' \). The groups

\[
H_q(X; \mathcal{F}_p) = H_q(C_\bullet(X; \mathcal{F}_p), \partial)
\]

are called the (cellular) tropical \((p, q)\)-homology groups.

We can consider the dual cochain complex \( C^\bullet(X; \mathcal{F}^p) \) of linear functionals on faces \( \Delta \) of \( X \) with values in \( \mathcal{F}^p(\Delta) \) and define the differential \( \delta \) as the usual coboundary combined with the maps \( \iota^* \) in (8). This defines the (cellular) tropical \((p, q)\)-cohomology groups

\[
H^q(X; \mathcal{F}^p) = H^q(C^\bullet(X; \mathcal{F}^p), \delta).
\]

2.5. Other homology theories. We may interpret \( \mathcal{F}_k(x) \) as a system of coefficients suitable to define singular homology groups on \( X \). Namely, we consider finite formal sums

\[
\sum \beta_\sigma \sigma,
\]

where each \( \sigma : \Delta^q \to X \) is a singular \( q \)-simplex which has image in a single face of \( X \) and is such that for each relatively open face \( \Delta' \) of \( \Delta^q \) the image \( \sigma(\Delta') \) is contained in a single face of \( X \). We say that \( \tau = \sigma|_{\Delta'} \) is a face of \( \sigma \). Here \( \beta_\sigma \in \mathcal{F}_k(\Delta) \), where the relative interior of \( \Delta \) contains the image of the relative interior of \( \Delta^q \).

These chains form a complex \( C^\bullet^{sing}(X; \mathcal{F}_k) \) with the differential \( \partial \) given by the standard singular differential combined with the maps \( \iota \) in (8). The elements of \( C^\bullet^{sing}(X; \mathcal{F}_k) \) are called tropical chains. The groups

\[
H_{p,q}(X) = H_q(C^\bullet^{sing}(X; \mathcal{F}_k), \partial)
\]

are called the singular tropical \((p, q)\)-homology groups.

One can also consider Čech version of tropical (co)homology thinking of the coefficients \( \mathcal{F}_k \) and \( \mathcal{F}^k \) as constructible cosheaves and sheaves, respectively, on \( X \). We refer the reader for details to [MZh14].

For the rest of the paper we stick to the cellular version of tropical homology.
3. Complex degenerations and tropical limit

In this section we present a connection between complex and tropical geometry. Tropical varieties appear as certain limits of (scaled sequences of) complex varieties. First, we define the coarse limit as a topological subspace of $\mathbb{T}^N$. Then, we establish the polyhedrality of this limit and put weights on its facets in order to obtain a more refined version.

3.1. Coarse tropical limit.

**Definition 17.** A scaled sequence is a set $A$ together with a scaling map $t: A \to \mathbb{R}$ which is unbounded from above. A scaled subsequence is a subset $A' \subset A$ with the induced scaling (which is still required to be unbounded).

We often drop the word “scaled”. By saying “$\alpha \in A$ is large” we mean that $t(\alpha) \in \mathbb{R}$ is large. Also, sometimes, we write $\alpha > \alpha'$ instead of $t(\alpha) > t(\alpha')$.

**Example 18.** Here are some examples of scaled sequences:

1. conventional sequences $A = \mathbb{N}$ with the inclusion $t: \mathbb{N} \to \mathbb{R}$;
2. $A = \mathbb{R}_{>0}$ with the inclusion $t: \mathbb{R}_{>0} \to \mathbb{R}$;
3. the punctured disc (the most relevant scaled sequence for us) $D^* = \{z \in \mathbb{C} : 0 < |z| < 1 \}$ with the scaling $t(z) = |z|^{-1}$.

Consider a scaled sequence $X_\alpha \subset \mathbb{C}P^N, \alpha \in A$, of projective algebraic varieties. From now on we always assume that $t(\alpha) > 1$ for any $\alpha \in A$. We have the map

$$\text{Log}_{t_\alpha}: \mathbb{C}P^N \to \mathbb{T}^N$$

defined by $(z_0 : \cdots : z_N) \mapsto (\log_{t_\alpha} |z_0| : \cdots : \log_{t_\alpha} |z_N|)$. Note that the map is well defined since the $(N+1)$-tuples of coordinates in $\mathbb{C}P^N$ equivalent under multiplication by a nonzero scalar are mapped to $(N+1)$-tuples equivalent under addition of a scalar in $\mathbb{T}^N$. Also the map respects the coordinate stratifications of $\mathbb{C}P^N$ and $\mathbb{T}^N$, that is, each stratum $\mathbb{C}P^N_I \subset \mathbb{C}P^N$ is sent to the stratum $\mathbb{T}^N_I \subset \mathbb{T}^N$. The set $A_\alpha = \text{Log}_{t_\alpha}(X_\alpha) \subset \mathbb{T}^N$ is called the amoeba of $X_\alpha$, cf. [GKZ94].

**Definition 19** (Coarse tropical limit in $\mathbb{T}^N$). We say that a closed subset $Y \subset \mathbb{T}^N$ is the coarse tropical limit of the scaled sequence $X_\alpha \subset \mathbb{C}P^N$ if the amoebas $A_\alpha \subset \mathbb{T}^N$ converge to $Y \subset \mathbb{T}^N$ in the Hausdorff sense.

This means that if we choose a metric $d$ compatible with the topology on $\mathbb{T}^N$, then the Hausdorff distance

$$\max\{\sup_{x \in A_\alpha} d(x, Y), \sup_{y \in Y} d(A_\alpha, y)\}$$

between $A_\alpha$ and $Y$ tends to 0 as $t_\alpha \to \infty$. Note that if the Hausdorff distance between two closed subsets $Y, Y' \subset \mathbb{T}^N$ is 0 then $Y = Y'$. That is, if a limit exists, it is unique.
Example 20. Since $\mathbb{T}^n$ is compact, any sequence of points $c_\alpha \in \mathbb{CP}^n$ has a subsequence which has a course tropical limit.

Definition 21. We say that a polyhedral complex $Z \subset \mathbb{T}^n$ of dimension at most $n$ is an attractor for a sequence of $n$-dimensional varieties $X_\alpha \subset \mathbb{CP}^n$ if it contains all accumulation points of the amoebas $A_\alpha$.

3.2. Weights on facets of an attractor. Let $Z \subset \mathbb{T}^n$ be an attractor for a sequence of $n$-dimensional varieties $X_\alpha \subset \mathbb{CP}^n$ of degree $d$. Denote by $Z^\epsilon$ a small tubular $\epsilon$-neighborhood of $Z$. We assume that $\alpha$ is sufficiently large, so that the amoebas $A_\alpha$ sit entirely in $Z^\epsilon$.

Fix a subset $I \subset \{0, 1, \ldots, n\}$, and let $M$ be an $(N-n)$-dimensional cooriented $I$-ball in $\mathbb{T}^N$ such that $\partial M \cap Z^\epsilon = \emptyset$ (recall that $\partial M$ means the closure in $\mathbb{T}^N$ of the mobile part of the boundary of $M$). We define $K_{\alpha}^n_M \in H_n((\mathbb{C}^\times)^{N-I} \times \mathbb{C}^I; \mathbb{Z}) = \Lambda^n(\mathbb{Z}^N/\mathbb{Z}^I)$ to be the intersection class of a small perturbation of $\text{Log}_{\alpha}^{-1}(M)$ with $X_\alpha \cap ((\mathbb{C}^\times)^{N-I} \times \mathbb{C}^I)$.

Lemma 22. Let $M_{s}$, $s \in [0, 1]$, be a continuous family of $(N-n)$-dimensional $I$-balls such that $M_s \cap Z^\epsilon = \emptyset$ for each $s \in [0, 1]$. Then, $K_{\alpha}^{\alpha} = K_{\alpha}^{M_0}$. \hfill \Box

We now introduce two special types of $(N-n)$-dimensional $I$-balls in $\mathbb{T}^N$. First, let $F$ be an oriented $n$-dimensional facet of $Z$ with the relative interior $F^0 \subset \mathbb{T}^n_0$. A membrane $M^t_F \subset \mathbb{T}^N$ is a small $(N-n)$-dimensional $I$-ball, such that $\partial M^t_F \cap Z^\epsilon = \emptyset$ and $M^t_F$ intersects $Z$ in a single point $x \in F^0$. An orientation of $F$ induces a coorientation of $M^t_F$. This defines $K_{\alpha}^n_{M^t_F} \in H_n((\mathbb{C}^\times)^{N-I} \times \mathbb{C}^I; \mathbb{Z}) = \Lambda^n(\mathbb{Z}^N/\mathbb{Z}^I)$. Membranes through $F^0$ allow us to associate to $F$ a class $K^\alpha_F \in \Lambda^n(\mathbb{Z}^N/\mathbb{Z}^I)$ (cf. Lemma 22).

Another type comes from choosing a cooriented $(N-n-|I|)$-dimensional rational linear subspace $L_I$ in $\mathbb{RP}^N_{\epsilon} \cong \mathbb{R}^{N-I}$. Let $L$ denote the pullback of $L_I$ to $\mathbb{TP}^N$ via the projection $\pi_I^\epsilon: \mathbb{TP}^N \rightarrow \mathbb{TP}^I$. An $(N-n)$-dimensional $I$-ball $M_{L_I}$ parallel to $L$ with $\partial M_{L_I} \cap Z^\epsilon = \emptyset$ is called an $L_I$-crepe, or simply, a crepe.

Let $y \in Z_I$ be a point such that its $\epsilon$-neighborhood in $\mathbb{T}^n$ is disjoint from all $Z_J$ for $J \nsubseteq I$, and let $L_I$ be transversal (in $\mathbb{TP}^N_\epsilon$) to each $n$-dimensional face $F$ of $Z_I$ such that $y$ belongs to the closure of $F$. Then, through any point in the $\epsilon$-neighborhood of $y$ one can trace an $L_I$-crepe.

Now let $M_{L_I}$ be an $L_I$-crepe, and let $L$ be the product of $\mathbb{C}^I$ with the $(N-n-|I|)$-dimensional subgroup of $(\mathbb{C}^\times)^{N-I}$ corresponding to $L_I$ in $\mathbb{RP}^{N-I}$. Then, to any point $z \in \text{Log}^{-1}_{\alpha}(M_{L_I})$ we can associate a complex $(N-n)$-dimensional submanifold $L_z \subset (\mathbb{C}^\times)^{N-I} \times \mathbb{C}^I$, which is the intersection of $\text{Log}^{-1}_{\alpha}(M_{L_I})$ with the translate $L'$ of $L$ such that $L'$ contains the point $z$.

The following calculation is a direct consequence of the Poincaré duality.

Lemma 23. Let $L_I \subset \mathbb{RP}^{N-I}$ be a cooriented $(N-n-|I|)$-dimensional rational linear subspace, and let $M_{L_I}$ be an $L_I$-crepe. Then, the intersection number of $X_\alpha$ with $L_z$ equals $K_{\alpha}^{M_{L_I}} \wedge \text{Vol}_{L_I} / \text{Vol}_{\mathbb{Z}^{N-I}}$ for any $z \in \text{Log}^{-1}_{\alpha}(M_{L_I})$.

\hfill \Box
A very useful class of \( I \)-balls is provided by the intersection of the above two types. Let \( F \) be an oriented \( n \)-dimensional facet of \( Z \) with the relative interior \( F^0 \subset \mathbb{R}^n \), and let \( L_I \) be an \((N-n-|I|)\)-dimensional rational linear subspace in \( \mathbb{R}^{N-I} \). If \( L_I \) is integrally transversal to \( F \), we call a small crepe \( M_{I,J}^\epsilon \) through a point \( x \in F^0 \) a linear membrane. By Lemma 22 any linear membrane through \( F^0 \) calculates the class \( K_F^\epsilon \).

**Lemma 24.** There exists \( T_F \in \mathbb{R} \) such that for any \( \alpha \) with \( t_\alpha > T_F \) the class \( K_F^\alpha \) is a non-negative integer multiple of \( \text{Vol}_F \).

**Proof.** We consider the case \( F^0 \subset \mathbb{R}^N \). The sedentary case is similar.

Let us fix a basis \( e_1, \ldots, e_N \subset \mathbb{Z}^N \) such that \( \text{Vol}_F = e_1 \wedge \cdots \wedge e_n \). For every map of sets

\[
h : \{n+1, \ldots, N\} \to \{0,1,\ldots,n\},
\]

we consider the subspace \( L(h) \) spanned by the vectors \( e_{n+1} + e_{h(n+1)}, \ldots, e_N + e_{h(N)} \) oriented such that \( \text{Vol}_{L(h)} = (e_{n+1} + e_{h(n+1)}) \wedge \cdots \wedge (e_N + e_{h(N)}) \). We set \( e_0 = 0 \); in particular, \( \text{Vol}_{L(0)} = e_{n+1} \wedge \cdots \wedge e_N \). Note that \( \text{Vol}_F \wedge \text{Vol}_{L(h)} = 1 \) for any \( h \) (here we tacitly divide by the volume element \( e_1 \wedge \cdots \wedge e_N \)), that is, all \( L(h) \) are integrally transversal to \( F \). It is easy to see that the collection of all \((n+1)^{N-n}\) primitive polyvectors \( \{\text{Vol}_{L(h)}\} \) generate \( \Lambda^{N-n}(\mathbb{Z}^N) \).

We assume that \( \epsilon \) is small enough so that every \( L(h) \) is parallel to a linear membrane through some point of \( F^0 \). We write the class

\[
K_F^\alpha = \sum_{|J|=n} c_J^\alpha e_J
\]

in the basis of primitive polyvectors \( e_J = \wedge_{i \in J} e_i \in \Lambda^n(\mathbb{Z}^N) \), where \( J \) runs over the increasing length \( n \) sequences in \( \{1, \ldots, N\} \).

Since the polyvectors \( \{\text{Vol}_{L(h)}\} \) generate \( \Lambda^{N-n}(\mathbb{Z}^N) \), the coefficients \( c_J^\alpha \) can be calculated by taking products \( K_F^\alpha \wedge \text{Vol}_{L(h)} \), that is, according to Lemma 23 by locally intersecting \( X_\alpha \) with complex manifolds \( L(h) \). Thus, all \( c_J^\alpha \) are uniformly (for all large \( \alpha \)) bounded by some constant (which depends on the degree of varieties \( X_\alpha \) and the basis \( e_1, \ldots, e_N \)).

Now let \( w \in \Lambda^{N-n}(\mathbb{Z}^N) \) be any primitive polyvector such that \( \text{Vol}_F \wedge w = 1 \). We assume that \( \epsilon \) is small enough so that there is a linear membrane parallel to \( w \) and passing through some point of \( F^0 \). Then, since complex manifolds intersect non-negatively, we must have \( K_F^\alpha \wedge w \geq 0 \). In particular, wedging \( K_F^\alpha \) with \( w = \text{Vol}_{L(0)} \) gives \( c_{J_0}^\alpha \geq 0 \), where \( J_0 = \{1, \ldots, n\} \). We will show that \( c_J^\alpha = 0 \) for all \( J \neq J_0 \).

Given \( J \), consider two disjoint ordered subsets of \( \{1, \ldots, N\} \):

\[
\{i_1, \ldots, i_k\} = J \setminus J_0, \quad \{j_1, \ldots, j_k\} = J_0 \setminus J,
\]

and the following primitive polyvector

\[
w_J = \pm \wedge_{\ell=1}^k (e_{i_\ell} \pm C e_{j_\ell}) \wedge e_{\{n+1, \ldots, N\} \setminus J} \in \Lambda^{N-n}(\mathbb{Z}^N),
\]
where $C$ is an integer. Then, $\text{Vol}_F \wedge w_J = 1$ (with the right choice of the $\pm$ sign in front). On the other hand,

$$K_F^\alpha \wedge w_J = \pm C^k c_j^\alpha + \sum_{j' \neq j} \pm C^{<k} c_{j'}^\alpha.$$ 

By appropriate choice of the $\pm$ signs and large enough $C$ this can be made negative unless $c_j^\alpha = 0$ (recall that all $c_j^\alpha$ are uniformly bounded). \hfill $\Box$

**Definition 25.** For $\alpha$ with $t_\alpha > T_F$, the number $w_\alpha(F) := \frac{K_F^\alpha}{\text{Vol}_F} \in \mathbb{Z}_{\geq 0}$ is called the weight of the facet $F$.

When we write $w_\alpha(F)$ we assume that $t_\alpha > T_F$. The weights $w_\alpha(F)$ have a priori bounds in terms of the degree of $X_\alpha$. One also observes that $w_\alpha(F)$ does not depend on the choice of orientation of $F$ because the orientation affects signs of both $K_F^\alpha$ and $\text{Vol}_F$. If $Z' \subset \mathbb{TP}^N$ is another attractor, and $F$ is a facet of both, then for large $\alpha$ the weights $w_\alpha(F)$, considering $F$ as a face of either $Z$ or $Z'$, are the same.

**Proposition 26.** Let $Z \subset \mathbb{TP}^N$ be an attractor for a sequence of $n$-dimensional varieties $X_\alpha \subset \mathbb{CP}^N$ of degree $d$.

1. Let $F$ be an $n$-dimensional facet of $Z$ with weights $w_\alpha(F)$. If for any $T \in \mathbb{R}$ there exists an index $\alpha$ with $t_\alpha > T$ and $w_\alpha(F) > 0$, then all points of $F$ are accumulation points of the sequence $X_\alpha$.

2. Conversely, if $x \in Z$ is an accumulation point of the sequence of $X_\alpha$, then for any $T \in \mathbb{R}$ there exists an index $\alpha$ with $t_\alpha > T$ and an $n$-facet $F$ of $Z$ such that $x \in F$ and $w_\alpha(F) > 0$.

**Proof.** (1) If $w_\alpha(F) > 0$, then $K_F^\alpha \neq 0$. This means that any membrane through a point in $F^c$ intersects $A_\alpha$. Thus, any point of $F$ is an accumulation point of the sequence $X_\alpha$.

(2) Let $x \in Z_I$ be an accumulation point. First, assume $x \in \mathbb{R}^N$. Passing to a subsequence, we pick $z_\alpha \in X_\alpha$ such that the sequence $x_\alpha = \log_{t_\alpha}(z_\alpha)$ converges to $x$.

Choose a small crepe $M_{L,x}$ passing through $x$ and consider $L$-crepes through $x_\alpha$ which are small translates of $M_{L,x}$. All these translates define classes $K_{M_{L,x}}^\alpha$ (independent of $x_\alpha$). By using positivity of intersection of complex varieties $X_\alpha$ and $L_{z_\alpha}$ we see that all classes $K_{M_{L,x}}^\alpha$ are non-zero.

Perturbing $M_{L,x}$ we can move off all points of its intersection with $Z$ into the interiors of facets $F_j$ adjacent to $x$, so that we have

$$K_{M_{L,x}}^\alpha = \sum_j K_{F_j}^\alpha,$$

and thus at least one of the facets $F_j$ must have nonzero weight.

We modify $Z$ by removing all faces that are not contained in $n$-facets with non-zero weight and then proceed by induction on $I \subset \{0, \ldots, N\}$ by applying the above argument to the modified attractor. \hfill $\Box$
3.3. Tropical limit for hypersurfaces. Tropical polynomials are the analogs of the classical polynomials where the addition $x + y$ is replaced by $\max\{x, y\}$ and the multiplication $xy$ is replaced by the sum $x + y$. For further analogies and details see, e.g. [Mi06]. A useful observation in this “tropical arithmetics” is the following inequality:

\[
\max_{j \in S}\{\ell_j\} \leq \log_i \sum_{j \in S} \ell_j^t \leq \max_{j \in S}\{\ell_j\} + \log_i |S|
\]

for any finite set $S$, any tropical numbers $\ell_j \in \mathbb{T}$ (where $j \in S$), and any $t > 1$.

Fix the dimension $N$, and let

\[
\Delta^Z_d := \{m \in \mathbb{Z}^{N+1} : m_0 + \cdots + m_N = d \text{ and } m_i \geq 0\}
\]

be the set of integral points of the $N$-dimensional simplex $\Delta_d$ of size $d$ in $\mathbb{R}^{N+1} \supset \mathbb{Z}^{N+1}$. For any function $a : \Delta^Z_d \to \mathbb{T}$, not identically equal to $-\infty$, we define the degree $d$ homogeneous tropical polynomial $P_a : \mathbb{T}^{N+1} \to \mathbb{T}$ as the Legendre transform of the function $-a$:

\[
P_a := \max_{m \in \Delta^Z_d}\{mx + a(m)\}.
\]

Note that since all components of $m$ are non-negative, the expressions $mx + a(m)$ do make sense for all $x \in \mathbb{T}^{N+1}$.

The restriction to any open stratum $\mathbb{T}^o_I \subset \mathbb{T}^{N+1}$ of a tropical polynomial $P_a$ is a convex piecewise linear function $P_{a,I}$. The set of points $x \in \mathbb{T}^{N+1} = \bigcup \mathbb{T}^o_I$ such that $P_{a,I}$ is not smooth at $x$ or equal to $-\infty$ (where $I$ is the refined sedentarity of $x$) is an $N$-dimensional polyhedral complex in $\mathbb{T}^{N+1}$ invariant under the diagonal translation by $\mathbb{R}$. Thus, it descends to an $(N-1)$-dimensional polyhedral complex $V_a$ in $\mathbb{T}^N$. The finite part $V^o_a \subset \mathbb{R}^N$ of this complex is dual to a subdivision of the Newton polytope $\Delta_a$ of $P_a$; this subdivision is given by the upper convex hull of the graph of $-a$.

We assign the weights to the facets of $V_a$ as follows. On the facets of the closure of $V^o_a$ in $\mathbb{T}^N$ the weights are given by the integral length of the gradient change of $P_a$, which is the same as the integral length of the dual edge in the subdivision (induced by $-a$) of the Newton polytope $\Delta_a$. The weight on each boundary divisor $\mathbb{T}^o_{(i)}$ is given by the integral distance from the Newton polytope $\Delta_a$ to the facet $m_i = 0$ in the simplex $\Delta_d$. We discard from $V_a$ the boundary components of weight 0.

**Proposition 27.** The hypersurface $V_a$ associated to a homogeneous tropical polynomial $P_a : \mathbb{T}^{N+1} \to \mathbb{T}$ is a weighted balanced polyhedral complex of dimension $N - 1$ in $\mathbb{T}^N$.

Conversely, any $(N-1)$-dimensional weighted balanced polyhedral complex $Y \subset \mathbb{T}^N$ may be presented as $V_a$ for some homogeneous polynomial $P_a : \mathbb{T}^{N+1} \to \mathbb{T}$.

**Proof.** The statement that $V_a$ is a polyhedral complex is clear from the discussion above. Namely, the intersection of $V_a$ with any $\mathbb{T}^o_I$ is of one of the following three types:

1. the hypersurface defined by the function $a$ restricted to the $I$-th face of $\Delta^Z_d$,
2. or empty, if $a$ has only one finite (not $-\infty$) value on the $I$-th face of $\Delta^Z_d$,
3. or the entire stratum $\mathbb{T}^o_I$, if all values of $a$ are $-\infty$ on the $I$-th face of $\Delta^Z_d$.
The balancing condition needs to be checked only for \( V_a \), where it is a straightforward property of the Legendre transform (cf., e.g., Propositions 2.2 and 2.4 of [Mi00], or Theorem 3.15 of [Mi05]).

Conversely, given a weighted balanced \((N - 1)\)-dimensional polyhedral complex \( Y \subset \mathbb{T}\mathbb{P}^N \), consider the cylinder, invariant under the diagonal translations, over \( Y^\circ := Y \cap \mathbb{R}^N \subset \mathbb{R}^N \) in \( \mathbb{T}^{N+1} \). This is a balanced polyhedral complex of codimension 1. Hence, by Propositions 2.2 and 2.4 of [Mi00] quoted above, it is given as the corner locus of a homogeneous tropical polynomial \( P_a^\circ \) defined uniquely up to an affine linear function. The closure \( Y^\circ \) of \( Y \) in \( \mathbb{T}\mathbb{P}^N \) is a polyhedral complex by Lemma 4. Then, by parallel shifting of the support of \( a^\circ \), it is easy to achieve that this support lies in the positive octant and its integral distances to the coordinate hyperplanes are the given weights of the corresponding boundary divisors \( \mathbb{T}\mathbb{P}_{\{i\}} \) in \( Y \).

**Definition 28.** The degree of an \((N - 1)\)-dimensional weighted balanced polyhedral complex \( Y \subset \mathbb{T}\mathbb{P}^N \) is the degree of a tropical polynomial \( P_a \) such that \( V_a \) coincides with \( Y \).

An important observation is that if we add a constant to the function \( a : \Delta_d^\mathbb{Z} \to \mathbb{T} \) the tropical polynomial \( P_a \) changes by adding (the same) constant. That is, both \( V_a \) and the Newton polytope \( \Delta_a \) remain the same. Hence, the tropical hypersurface \( V_a \) is well defined for \( a \in \mathbb{T}\mathbb{P}_{\Delta_d} \).

Also given \( P_a \), there is still some freedom in choosing the function \( a : \Delta_d^\mathbb{Z} \to \mathbb{T} \) as long as the upper convex hull of the graph of \( -a \) induces the same subdivision of the Newton polytope \( \Delta_a \). One can always take the “minimal” representative of \( a \) by setting \( a(m) = -\infty \) for all non vertices of the induced subdivision of \( \Delta_a \). This reduces the ambiguity in \( P_a \) to an additive constant, cf. Theorem 3.15 of [Mi05].

Consider a scaled sequence of complex hypersurfaces \( X_\alpha \subset \mathbb{C}\mathbb{P}^N \) of degree \( d \). The main result of this subsection is that the sequence \( X_\alpha \) has a coarsely convergent subsequence and its tropical limit \( Y \subset \mathbb{T}\mathbb{P}^N \) is a polyhedral complex.

For an interior point \( x \in \mathbb{R}^N \subset \mathbb{T}\mathbb{P}^N \) we consider its preimage \( T_x \in \mathbb{C}^{N+1} \setminus \{0\} \) under the composition of the two maps:

\[
\mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{C}\mathbb{P}^N \to \mathbb{T}\mathbb{P}^N.
\]

The first map is the quotient by \( \mathbb{C}^* \), and the second is the \( \text{Log}_{st_a} \).

**Definition 29 (FPT00).** For a hypersurface \( X_\alpha \subset \mathbb{C}\mathbb{P}^N \) let \( f_\alpha \) be its defining polynomial. Let \( x \in \mathbb{R}^N \subset \mathbb{T}\mathbb{P}^N \) be a point outside the amoeba \( A_\alpha \). We define the functional \( \text{ind}_\alpha(x) \) on the space of loops in \( T_x \) as follows. For a loop \( \gamma \subset T_x \) we set

\[
\text{ind}_\alpha(x) : \gamma \mapsto \frac{1}{2\pi i} \int_\gamma d\log f_\alpha.
\]

**Remark.** One may think of the value \( \text{ind}_\alpha(x) \) on the loop \( \gamma \) as the linking number of \( \gamma \) and the affine cone \( \hat{X}_\alpha \) over \( X_\alpha \) in \( \mathbb{C}^{N+1} \). If \( \mu \subset \mathbb{C}\mathbb{P}^N \) is a holomorphic disc with boundary \( \partial \mu = \gamma \), then the value of \( \text{ind}_\alpha(x) \) on \( \gamma \) is the intersection number of \( \mu \) with \( \hat{X}_\alpha \).
There are a couple of immediate observations: \( \text{ind}_\alpha(x) \) depends only on the homology class of the loop, and it is a locally constant function of \( x \). For convenience, we extend the map \( \text{ind}_\alpha \) to the boundary strata in \( \ell \) by continuity. Note that \( T_x \) is isomorphic to \( \mathbb{C}^* \times (S^1)^N \), and for any \( x \in \mathbb{R}^N \) the cohomology \( H^1(T_x; \mathbb{Z}) \cong \mathbb{Z}^{N+1} \) can be naturally identified with the lattice of the Laurent monomials in the coordinates on \( \mathbb{C}^{N+1} \). Thus, \( \text{ind}_\alpha \) maps connected components of \( \ell \setminus A_\alpha \) to this lattice.

**Proposition 30** ([FPT00]). Let \( X_\alpha \subset \mathbb{C}P^N \) be a degree \( d \) hypersurface. The map \( \text{ind}_\alpha \) is injective on the components of \( \ell \setminus A_\alpha \) and its image lies in \( \Delta^Z \subset \mathbb{Z}^{N+1} \).

Here is the dictionary between the weights \( w_\alpha(F) \) and the indices \( \text{ind}_\alpha \) for hypersurfaces.

**Proposition 31.** Let \( X_\alpha \subset \mathbb{C}P^N \) be a scaled sequence of hypersurfaces which coarsely converges to \( Y \subset \ell \).

- \( \text{If } F \subset Y \text{ is a facet separating two components } \sigma, \sigma' \text{ of } \ell \setminus Y, \text{ then the weight } w_\alpha(F) \text{ is equal to the integral length of the vector } \text{ind}_\alpha(\sigma) - \text{ind}_\alpha(\sigma'). \)
- \( \text{If } F \text{ is a coordinate hyperplane } \ell_{(j)} \subset Y, \text{ then } w_\alpha(F) \text{ is equal to } (\text{ind}_\alpha)_j(\sigma), \text{ the } j \text{-th component of } \text{ind}_\alpha(\sigma) \in \Delta^Z, \text{ where } \sigma \text{ is an adjacent component of } \ell \setminus Y. \)

**Proof.** Let \( F \subset Y \) be a facet with \( F^o \subset \mathbb{R}^N \) which separates two components \( \sigma, \sigma' \) of \( \ell \setminus Y \). Let \( L \subset \mathbb{R}^N \) be a line which intersects \( F \) integrally transversally at a point \( x \in F^o \), and let \( M_{F,L} \subset L \) be a small interval around \( x \). Denote by \( \hat{F} \subset \mathbb{R}^{N+1} \) the corresponding facet of the cylinder over \( Y \). Let \( \hat{L} \) be a line integrally transversal to \( \hat{F} \) which projects to \( L \), and let \( \hat{M}_{F,L} \subset \hat{L} \) be the corresponding lift of the interval \( M_{F,L}. \)

Then, by Lemma 23, the weight \( w_\alpha(F) \) can be computed as the local intersection number of \( X_\alpha \) and the complex subgroup \( L \cong \mathbb{C}^* \subset (\mathbb{C}^*)^N \) corresponding to \( L \). This equals the number of zeros of the defining polynomial \( f_\alpha \) in the annulus \( \hat{L} \cap \text{Log}_t^{-1}(\hat{M}_{F,L}) \subset \hat{L} \cong \mathbb{C}^* \), bounded by two loops \( \gamma_1, \gamma_2 \) in the homology class \( \langle \hat{L} \rangle \in \mathbb{Z}^{N+1} \). Thus, by Cauchy’s formula this number is

\[
\frac{1}{2\pi i} \int_{\gamma_1 \cup (-\gamma_2)} d \log f_\alpha,
\]

which also can be computed as the value of \( \text{ind}_\alpha(x_1) - \text{ind}_\alpha(x_2) \) on \( \langle \hat{L} \rangle \), where \( x_1 = \text{Log}_t(\gamma_1), x_2 = \text{Log}_t(\gamma_2) \). Since \( \hat{L} \) intersects \( \hat{F} \) integrally transversally, this is precisely the length of the vector \( \text{ind}_\alpha(\sigma) - \text{ind}_\alpha(\sigma') \).

For the coordinate hyperplane \( F = \ell_{(j)} \subset \ell \), let \( x = (x_0 \ldots x_N) \) be a point in a component of \( \ell \setminus Y \) adjacent to it. Then, for \( \alpha \) sufficiently large, the value of \( (\text{ind}_\alpha)_j(x) \) can be calculated by Cauchy’s formula as the intersection number of the affine cone \( \hat{X}_\alpha \) over \( X_\alpha \) in \( \mathbb{C}^{N+1} \) with the disk

\[ \{(z_0 \ldots z_N) \in \mathbb{C}^{N+1} : z_i = t^{x_i}_{\alpha}, i \neq j, \ |z_j| < t^{x_j}_{\alpha}\}. \]

On the other hand, by Lemma 23 the weight \( w_\alpha(F) \) can be calculated as the intersection number of \( X_\alpha \) with the disk

\[ \{(z_0 \ldots z_N) \in \mathbb{C}P^N : z_i = t^{x_i}_{\alpha}, i \neq j, \ |z_j| < t^{x_j}_{\alpha}\}. \]
These two calculations clearly agree.

**Definition 32.** We say that a scaled sequence of hypersurfaces \( X_\alpha \subset \mathbb{CP}^N \) tropically converges to a closed set \( Y \subset \mathbb{TP}^N \) if the following conditions hold:

- \( Y \) is the coarse tropical limit of \( X_\alpha \);
- for every \( x \in \mathbb{TP}^N \setminus Y \) the sequence \( \text{ind}_\alpha(x) \in \mathbb{Z}^{N+1} \) is eventually constant (that is, constant for sufficiently large \( \alpha \)).

By Proposition 30 the degrees of \( X_\alpha \) in a tropically convergent sequence finally stabilize. The second condition makes sense since a point \( x \in \mathbb{TP}^N \setminus Y \) misses all amoebas \( A_\alpha \) for sufficiently large \( \alpha \in A \). In this case we define \( \text{ind}(x) = \text{ind}_\alpha(x) \), where \( \alpha \) is taken to be sufficiently large. It is clear that the function \( \text{ind} : \mathbb{TP}^N \setminus Y \to \mathbb{Z}^{N+1} \) is locally constant.

**Lemma 33.** Let \( Y \) be the coarse tropical limit of a scaled sequence of hypersurfaces \( X_\alpha \subset \mathbb{CP}^N \) of degree \( d \). Then, there is a scaled subsequence \( A' \subset A \) such that \( X_\alpha, \alpha \in A' \), tropically converges to \( Y \).

**Proof.** The set \( \text{ind}(\mathbb{TP}^N \setminus Y) \subset \Delta_\delta^d \) is finite. To get a converging subsequence we pass to a subsequence of constant \( \text{ind}_\alpha \) for each connected component of \( \mathbb{TP}^N \setminus Y \). □

There is another natural way to pass to a tropical limit of the sequence of complex hypersurfaces \( X_\alpha \subset \mathbb{CP}^N \) of degree \( d \). The coefficients of the defining polynomials for \( X_\alpha \) generate the sequence of points \( c_\alpha \in \mathbb{CP}^{\Delta_\delta} \). Its coarse tropical limit \( c \in \mathbb{TP}^N \) (which always exists after passing to a subsequence, cf. Example 20) defines a tropical hypersurface \( V_c \subset \mathbb{TP}^N \).

**Proposition 34.** Let \( X_\alpha \subset \mathbb{CP}^N \) be a scaled sequence of hypersurfaces of degree \( d \).

- If \( c \in \mathbb{TP}^{\Delta_\delta} \) is the coarse tropical limit of the corresponding sequence of coefficients \( c_\alpha \in \mathbb{CP}^{\Delta_\delta} \), then the sequence \( X_\alpha \) tropically converges to (the support of) \( V_c \subset \mathbb{TP}^N \) in the sense of Definition 32.

Conversely, if \( Y \subset \mathbb{TP}^N \) is the coarse tropical limit of the sequence \( X_\alpha \), then \( Y \) is the support of a tropical hypersurface \( V_c \) of degree \( d \), where \( c \in \mathbb{TP}^{\Delta_\delta} \) is an accumulation point of the sequence \( c_\alpha \in \mathbb{CP}^{\Delta_\delta} \).

**Proof.** Suppose that \( x \in \mathbb{TP}^N \setminus V_c \). Then, there exists \( m \in \Delta_\delta^d \) such that \( mx + c(m) > m'x + c(m') \) for all \( m' \neq m \). Then, \( |t_\alpha^{m+\epsilon(m)}| > \sum_{m' \neq m} t_\alpha^{m'+\epsilon(m')} \) for sufficiently large \( t_\alpha \), which, in turn, by (10) means

\[
|e_\alpha^{m}z^m| > \sum_{m' \neq m} e_\alpha^{m'}z^{m'}
\]

for all \( z \in \text{Log}_{t_\alpha}^{-1}(x) \). This implies that \( \text{ind}_\alpha(x) = m \) for all sufficiently large \( t_\alpha \) (cf. [19]).

Furthermore, (12) holds simultaneously for all \( x \in \mathbb{TP}^N \) outside of an \( \epsilon \)-neighborhood of \( V_c \) in \( \mathbb{TP}^N \) by (10) (with different values of \( m \) for different components of \( \mathbb{TP}^N \setminus V_c \)). Thus \( \lim_{\alpha \in A}(\sup_{x \in A_\alpha} d(x, V_c)) = 0 \).
Now we want to show that \( \lim_{\alpha \in A} (\sup_{y \in V_c} d(A_\alpha, y)) = 0 \). By compactness of \( V_c \), it suffices to see that for each open set \( U \subset \mathbb{TP}^N \) with \( U \cap V_c \neq \emptyset \) there exists \( t' > 0 \) such that \( A_\alpha \cap U \neq \emptyset \) for each \( \alpha \) with \( t_\alpha > t' \). Moreover, it is sufficient to prove this for such \( U \) that \( U \cap \mathbb{R}^N \) is convex. Since \( V_c \) is a tropical hypersurface, such \( U \) must intersect at least two components of \( \mathbb{TP}^N \setminus V_c \) with different values of ind. Let \( y_1, y_2 \in U \) be two points with \( \text{ind}(y_1) \neq \text{ind}(y_2) \). The interval connecting \( y_1 \) and \( y_2 \) must intersect \( A_\alpha \) for large \( \alpha \). Therefore, we conclude that \( X_\alpha \) tropically converges to \( V_c \).

Conversely, suppose that \( Y \subset \mathbb{TP}^N \) is the coarse tropical limit of the sequence \( X_\alpha \). Consider an accumulation point \( c \) of the coefficient sequence \( c_\alpha \in \mathbb{TP}\Delta_I^\circ \) (which must exist by compactness of \( \mathbb{TP}\Delta_I^\circ \)). Let \( A' \) be a scaled subsequence of \( A \) such that \( \lim_{\alpha \in A'} c_\alpha = c \). We have already proved that \( X_\alpha \) must converge to \( V_c \) for the subsequence \( A' \). Hence \( V_c = Y \). \( \square \)

**Corollary 35** (Compactness theorem for hypersurfaces). Let \( X_\alpha \subset \mathbb{CP}^N \) be a scaled sequence of hypersurfaces of degree \( d \). Then, \( X_\alpha \) has a subsequence which tropically converges to a tropical hypersurface \( Y \subset \mathbb{TP}^N \) of degree \( d \).

**Proof.** Choose an accumulation point of the coefficient sequence \( c_\alpha \in \mathbb{TP}\Delta_I^\circ \) and pass to a subsequence converging to this accumulation point. By Proposition 34 the tropical limit \( Y \) of the hypersurfaces \( X_\alpha \) is represented as a tropical hypersurface of degree \( d \). \( \square \)

3.4. **Compactness theorem.** In this subsection, we prove that a scaled sequence of \( n \)-dimensional varieties of universally bounded degrees in \( \mathbb{CP}^N \) has a scaled subsequence which tropically converges to a balanced weighted \( n \)-dimensional polyhedral complex in \( \mathbb{TP}^N \).

**Lemma 36.** Let \( X_\alpha \subset \mathbb{CP}^N \) be a scaled sequence of \( n \)-dimensional varieties of degree \( d \). Then, \( X_\alpha \) has a subsequence which possesses an attractor \( Z \subset \mathbb{TP}^N \).

**Proof.** We realize \( Z \) as the intersection of several tropical hypersurfaces. In the line of proof we may need to pass to subsequences.

Let \( \mathbb{CP}_I \subset \mathbb{CP}^N \) be a coordinate projective subspace, and let \( \pi_I : \mathbb{CP}^N \to \mathbb{CP}_I \) be the projection from the dual coordinate projective subspace \( \mathbb{CP}_I \) (where \( I = \{0, \ldots, N\} \setminus I \)). It is a rational map not defined at \( \mathbb{CP}_I \). For a closed subset \( X \subset \mathbb{CP}^N \) we denote by \( \pi_I(X) \) its image, that is the closure of \( \pi_I(X \setminus \mathbb{CP}_I) \) in \( \mathbb{CP}_I \). Similarly, we can consider \( \pi_I^N : \mathbb{TP}^N \to \mathbb{TP}_I \), the tropical version of the projection.

Define \( I \) to be the set of subsets \( I \subset \{0, \ldots, N\} \) such that \( \pi_I(X_\alpha) \subset \mathbb{CP}_I \) are hypersurfaces for all large \( \alpha \). Generically \( I \) consists of the subsets with \( N - n - 1 \) elements. However, it may happen that for some \( I \) with \( |I| = N - n - 1 \) the images \( \pi_I(X_\alpha) \subset \mathbb{CP}_I \) are of codimension higher than \( 1 \) for all large \( \alpha \). Then, we will need to pass to a further projection to \( \mathbb{CP}_{I'} \) with \( I' \supset I \) so that \( \pi_{I'}(X_\alpha) \) are hypersurfaces in \( \mathbb{CP}_{I'} \) (after passing to a subsequence) for all large \( \alpha \).

For each \( I \in \mathcal{I} \), the degrees of \( \pi_I(X_\alpha) \subset \mathbb{CP}_I \) are bounded by \( d = \deg(X_\alpha) \), and hence so are the degrees of the cones \( \pi_I^{-1}(\pi_I(X_\alpha)) \). Thus, after passing to a subsequence we may assume that for all \( I \in \mathcal{I} \) the degrees \( d_I \) of the hypersurfaces \( \pi_I^{-1}(\pi_I(X_\alpha)) \) are fixed. For
each $I \in \mathcal{I}$, by Corollary 35, there exists a tropical limit $Z_I$ of $\pi_I^{-1}(\pi_I(X_\alpha))$ which is a tropical hypersurface of degree $d_I$.

Now observe that $X_\alpha \subset \pi_I^{-1}(\pi_I(X_\alpha))$, hence

$$\text{Log}_{t_\alpha}(X_\alpha) \subset \pi_I^{-1}(\pi_I(X_\alpha)) = (\pi_I^T)^{-1}(\text{Log}_{t_\alpha}(\pi_I(X_\alpha))).$$

This means that $Y$ is contained in any $Z_I$ for $I \in \mathcal{I}$. We define

$$Z := \bigcap_{I \in \mathcal{I}} Z_I.$$

Then $Z \supset Y$, and we claim that $Z$ is the union of polyhedral complexes in $\mathbb{T}_\mathbb{P}^N$ of dimension not greater than $n$. For this note that each hypersurface $Z_I$ is a polyhedral complex. The intersection of the hypersurfaces restricted to some stratum $\mathbb{T}_\mathbb{P}^*_{\ell}$ is given by the intersection of the restrictions of those hypersurfaces to $\mathbb{T}_\mathbb{P}^0_{\ell}$. That shows the face incidence property of Definition 5.

Finally we show that $Z$ cannot contain faces of dimension greater than $n$. Indeed, suppose $F$ is such a face. Then, there is a subset $I' \subset \{0, \ldots, N\}$ of cardinality $N - n - 1$ such that $\pi_{I'}(F)$ is full dimensional. Such $I'$ is a subset of some $I \in \mathcal{I}$ and the projection $\pi_I^T(F)$ is still full dimensional. This contradicts that $F$ is inside the hypersurface $Z_I$.

**Definition 37.** A scaled sequence of $n$-dimensional varieties $X_\alpha \subset \mathbb{C}\mathbb{P}^N$ tropically converges to a weighted $n$-dimensional polyhedral complex $Y \subset \mathbb{T}_\mathbb{P}^N$ if the two following conditions hold:

- $Y \subset \mathbb{T}_\mathbb{P}^N$ is a coarse tropical limit of $X_\alpha$;
- for any facet $F \subset Y$ its weights $w_\alpha(F)$ are equal to $w(F)$ for sufficiently large $\alpha$.

**Proposition 38.** A sequence of hypersurfaces $X_\alpha \subset \mathbb{C}\mathbb{P}^N$ tropically converges to $Y \subset \mathbb{T}_\mathbb{P}^N$ in the sense of Definition 32 if and only if it does in the sense of Definition 37.

**Proof.** Let $Y \subset \mathbb{T}_\mathbb{P}^N$ be the coarse tropical limit of hypersurfaces $X_\alpha$ (required by both Definitions 32 and Definition 37). By Proposition 26 it is an $(N-1)$-dimensional polyhedral complex in $\mathbb{T}_\mathbb{P}^N$. The statement follows from Proposition 31.

**Theorem 39** (Compactness). Suppose that $X_\alpha \subset \mathbb{C}\mathbb{P}^N$, $\alpha \in A$, is a scaled sequence of $n$-dimensional varieties of universally bounded degrees. Then, there exists a scaled subsequence $A' \subset A$ such that $X_\alpha$, $\alpha \in A'$, tropically converges to a (non-empty) balanced weighted $n$-dimensional polyhedral complex in $\mathbb{T}_\mathbb{P}^N$.

**Remark.** The so-called non-Archimedean amoebas, i.e. images of non-Archimedean varieties under coordinatewise valuation maps, may be viewed as counterparts of tropical limits in the world of non-Archimedean algebraic geometry. Polyhedrality properties of non-Archimedean amoebas were discovered by Bieri and Groves [BG84], see also the thesis of Speyer [Sp05].

**Proof.** By Lemma 36 we may assume (after passing to a subsequence) that any accumulation point of $A_\alpha$ is contained in an attractor $Z$. The weights $w_F^\alpha$ of the $n$-facets of $Z$ are
priori bounded in terms of the degrees of $X_\alpha$. Thus, by passing to a subsequence we can assume that these weights stabilize for large $\alpha$. Let $Y$ be the union of $n$-facets of $Z$ with positive weights, which is a (pure) $n$-dimensional polyhedral complex.

Proposition 26 implies that the set of accumulation points of $\mathcal{A}_\alpha$ coincides with $Y$, that is, $Y$ is non-empty and is the coarse tropical limit of $X_\alpha$. Moreover, $Y$ is the tropical limit of $X_\alpha$ in the sense of Definition 37 as the weights of the facets of $Y$ stabilize.

We now show that $Y$ is balanced. Let $E$ be an $(n-1)$-dimensional face of $Y$ with the relative interior $E^\circ$ contained in some $\mathbb{T}_I\mathbb{P}$. Let $F_1, \ldots, F_m$ be the $n$-facets of $Y$ adjacent to $E$. Choose a small $(N-n+1)$-dimensional $I$-ball $M_E$ passing through a point in $E^\circ$ such that $\partial M$ splits as the union of membranes $M_{F_1}, \ldots, M_{F_m}$ passing through $F_1^\circ, \ldots, F_m^\circ$ respectively, with consistent coorientations.

Note that $\text{Log}_{\alpha^{-1}}(M_E) \cap X_\alpha$ is a singular $(n+1)$-chain in $(\mathbb{C}^\times)^{N-I} \times \mathbb{C}^I$ whose boundary is the union of $\text{Log}_{\alpha^{-1}}(M_{F_i}) \cap X_\alpha$, $i = 1, \ldots, m$. Thus,

\begin{equation}
\sum_{j=1}^{m} K_{F_j}^\alpha = 0,
\end{equation}

where all classes are taken in $H_n((\mathbb{C}^\times)^{N-I} \times \mathbb{C}^I; \mathbb{Z}) = \Lambda^n(\mathbb{Z}^N/\mathbb{Z}^I)$. The classes of facets which are not in $\mathbb{T}_I\mathbb{P}$ vanish in $\Lambda^n(\mathbb{Z}^N/\mathbb{Z}^I)$. Thus, (13) is precisely the balancing condition for the facets in $\mathbb{T}_I\mathbb{P}$.

\[ \square \]

3.5. Degree of the tropical limit. First, we recall the definition of the degree of a balanced weighted polyhedral complex $Y \subset \mathbb{T}_I\mathbb{P}^N$.

For a point $x \in \mathbb{R}^N \subset \mathbb{T}_I\mathbb{P}^N$, we consider $L = L^k(x)$, the fan-like linear tropical $k$-subspace constructed as follows. Take the $N+1$ divisorial rays $R_0, \ldots, R_N$ from $x$, that is, the projections of the negative coordinate rays at (any lift of) $x$ in $\mathbb{T}^{N+1} \setminus \{\infty\}$ to $\mathbb{T}_I\mathbb{P}^N$. The $k$-dimensional polyhedral complex $L^k(x) \subset \mathbb{T}_I\mathbb{P}^N$ is the closure in $\mathbb{T}_I\mathbb{P}^N$ of the union of the convex cones in $\mathbb{R}^N$ spanned by all possible collection of $k$ rays from $R_0, \ldots, R_N$ in $\mathbb{R}^N$.

The polyhedral complex $L^k(x)$ is a smooth fan with the vertex $x$. For an underlying matroid one can take $M = \{0, \ldots, N\}$ with dependent subsets $I \subset M$, for all $|I| > k + 2$. Note that $L^k_I(x) := L^k(x) \cap \mathbb{T}_I\mathbb{P}$, the restriction of $L^k(x)$ to the coordinate subspace $\mathbb{T}_I\mathbb{P} \subset \mathbb{T}_I\mathbb{P}^N$ is again a fan-like tropical linear subspace, but of dimension $k - |I|$.

We say that $L^k(x)$ intersects $Y$ transversally at $y \in \mathbb{T}_I\mathbb{P}$ if $y$ belongs to the relative interior of a facet $F$ of $Y$ and the relative interior of a facet $G \subset L^k_I(x)$. In this case, $k + \dim Y = N$, and we define the local tropical intersection number $t_y(Y, L)$ as the index of the sublattice in $\mathbb{Z}^{N-I}$ generated by the lattices $F_Z$ and $G_Z$ of integral vectors in the faces $F \subset Y$ and $G \subset L$ times the weight $w(F)$.

**Lemma 40.** Let $Y \subset \mathbb{T}_I\mathbb{P}^N$ be an $n$-dimensional weighted balanced polyhedral complex. For generic $x \in \mathbb{R}^N$ the linear subspace $L = L^{N-n}(x)$ intersects $Y$ transversally at finitely many points. The total intersection number $d = \sum_{y \in L \cap Y} t_y(Y, L)$ is independent of $x$.

**Proof.** The set of $x$ such that $L = L^{N-n}(x)$ intersects $Y$ not transversally is a finite union of hypersurfaces. It happen when some facet of $Y$ meets a codimension one face of $L$ and
vice versa. In each of the two cases the independence of the local intersection number under slight displacements of \( x \) is a direct consequence of the balancing property of \( L \) and \( Y \).

**Definition 41.** We say that \( d \) is the degree of \( Y \subset \mathbb{T} \mathbb{P}^N \).

**Remark.** If \( Y \) is a tropical hypersurface, then its degree is the degree of a defining tropical polynomial.

**Example 42.** If \( Y \) is the closure in \( \mathbb{T} \mathbb{P}^N \) of the Bergman fan \( \Sigma_M \subset \mathbb{R}^N \), then the degree of \( Y \) is equal to 1. To see this, we choose a subset \( I = \{i_1, \ldots, i_n\} \subset M \) of \( n \) independent elements. For \( L = L^{N-n}(x) \) we choose its vertex \( x \in \mathbb{R}^N \) very close to a corner in the coordinate stratum \( \mathbb{T} \mathbb{P}_I \subset \mathbb{T} \mathbb{P}^N \). Then, the tropical intersection of \( \Sigma_M \) with this \( L \) is just one point \( y \in \mathbb{R}^N \) with multiplicity 1. This can be seen by successively reducing the matroid by removing elements in \( I \): each time the rank of the reduced matroid decreases by 1, finally leading to a matroid of rank 1, whose Bergman fan is just one point.

We now turn to comparing the degree of the complex varieties \( X_\alpha \) in a scaled sequence with the degree of its tropical limit. First, notice that it is easy to construct a sequence of linear \( k \)-dimensional subspaces \( L_t \subset \mathbb{C} \mathbb{P}^N \), \( t \in \mathbb{R}_{>0} \) which tropically converges to a given \( L^k(x) \). For instance, one can take the intersection of \( N-k \) generic hyperplanes \( H^{(j)} = \{ \sum_{i=0}^N c_i^{(j)}(t) z_i = 0 \} \subset \mathbb{C} \mathbb{P}^N \) such that for every \( j \) the sequence of the coefficients \( \{ c_i^{(j)}(t) \} \) tropically converges to \(-x\), that is \( \lim_{t \to \infty} \log_t |c_i^{(j)}(t)| = -x_i \), \( j = 1, \ldots, N-k \).

**Proposition 43.** Let \( Y \) be the tropical limit of a sequence \( X_\alpha \subset \mathbb{C} \mathbb{P}^N \), and let \( L \) be the tropical limit of a sequence of linear \((N-n)\)-subspaces \( L_t \subset \mathbb{C} \mathbb{P}^N \). Let \( y \in \bar{Y} \cap L \) be a point of transversal intersection of \( Y \) and \( L \), and let \( U \supset y \) be a small open neighborhood. Then, for sufficiently large \( \alpha \) the intersection of \( X_\alpha \) and \( L_{t_\alpha} \) in \( \text{Log}_{t_\alpha}^{-1}(U) \) is equal to \( \nu_y(Y, L) \).

**Proof.** We consider the case \( y \in \mathbb{R}^N \), that is, \( y \) is the interior point of the facets \( F \subset Y \) and \( G \subset L \). The sedentary case is similar.

For large \( t_\alpha \), the intersection of \( X_\alpha \) and \( L_{t_\alpha} \) in \( \text{Log}_{t_\alpha}^{-1}(U) \) is calculated by the Poincaré duality as the comparison of \( [\mathcal{M}_F^\alpha] \wedge \text{Vol}_G \) against the volume element in the lattice \( \mathbb{Z}^N \) (cf. Section 3.2). By Lemma 24, this is equal to the coefficient of \( w(F) \text{Vol}_F \wedge \text{Vol}_G \) at the primitive element of \( \Lambda^N \mathbb{Z}^N \), which is, by definition, \( \nu_y(Y, L) \).

As an immediate corollary we obtain the following statement.

**Corollary 44.** Let \( Y \) be the tropical limit of a sequence \( X_\alpha \subset \mathbb{C} \mathbb{P}^N \). Then, the degree of \( Y \subset \mathbb{TP}^N \) equals the degree of \( X_\alpha \) for sufficiently large \( \alpha \).

### 4. Degeneration of 1-parameter projective family and the limiting mixed Hodge structure

Let \( Z \subset \mathbb{C} \mathbb{P}^N \times \mathcal{D}^* \) be a complex analytic one-parameter family of projective varieties of dimension \( n \) over a punctured disc \( \mathcal{D}^* \). That is, \( Z \) is locally given by zeros of finitely many analytic functions \( F_j(x,w) \), such that for each fixed value of \( w = w_0 \) the collection \( \{ F_j(x,w_0) \} \) defines a projective subvariety \( Z_{w_0} \subset \mathbb{C} \mathbb{P}^N \).
We can consider \( Z \) as a scaled sequence over the punctured disc with the scaling \( t = |w|^{-1} \) (cf. (3) in Example 18). Suppose \( X \subset \TP^N \) is the tropical limit of \( Z \) which we assume to be a smooth projective \( \Q \)-tropical variety.

**Remark.** The assumption is not unreasonable. One can show that in the case of algebraic family \( X \) the limit always exists and is a projective \( \Q \)-tropical variety. For \( X \), an algebraic family of degree \( d \) hypersurfaces in \( \CP^N \), there is a simple criterion for smoothness of the tropical limit \( X \). In this case, we may assume (after a rescaling and passing to a subsequence) that the coefficients tropically converge to a function \( a : \Delta^Z_d \to \Z \), well-defined up to a constant. The convex hull of the overgraph of \(-a\) gives a polyhedral decomposition of \( \Delta^Z_d \). Then, \( X \) is smooth if and only if this decomposition is a unimodular triangulation.

We adopt the classical construction of Mumford [KKMS73] of a simple normal crossing model for \( Z \).

### 4.1. Unimodular triangulation of \( X \).

Let \( E \) be a real \( N \)-dimensional affine space, and let \( L \subset E \) be a lattice of rank \( N \). A \( k \)-dimensional simplex \( S \subset E \) is called \( L \)-primitive if its vertices \( v_0, \ldots, v_k \) are in \( L \), and the vectors \( v_1 - v_0, \ldots, v_k - v_0 \) generate the intersection of \( L \) with the affine span of \( S \). A cone with a vertex \( v_0 \) generated by \( k \) linearly independent vectors \( u_1, \ldots, u_k \) in \( E \) is the set \( C = \{ v_0 + \sum a_i u_i \mid a_i \geq 0 \} \subset E \). Such a cone \( C \subset E \) is said to be \( L \)-primitive if \( v_0 \in L \) and the vectors \( u_1, \ldots, u_k \) can be chosen in such a way that \( v_0, v_0 + u_1, \ldots, v_0 + u_k \) are the vertices of a \( k \)-dimensional \( L \)-primitive simplex. An \( n \)-dimensional convex polyhedron \( \Delta' \subset E \) is called \( L \)-primitive if it can be represented as Minkowski sum \( S + C \), where \( S \) is an \( L \)-primitive simplex of dimension \( 0 \leq k \leq n \), and \( C \) is an \( L \)-primitive cone of dimension \( n - k \).

A finite polyhedral subdivision of a convex polyhedron \( \Delta' \subset E \) is called convex, if there exists a piecewise-linear convex function \( \Phi : \Delta' \to \R \) whose domains of linearity coincide with the polyhedra of the subdivision. A finite polyhedral subcomplex of \( E \) is called \( c \)-extendable if it is a subcomplex of a convex polyhedral subdivision of \( E \).

Let \( X \subset \TP^N \) be an \( n \)-dimensional smooth projective \( \Q \)-tropical variety, and let \( L \subset \R^N \subset \TP^N \) be a lattice such that \( \Z^N \supset L \). Put \( X^o = X \cap \R^N \). We say that \( X \) admits a unimodular triangulation with respect to \( L \) if there exists a finite polyhedral subdivision \( \tau \) of \( X^o \) such that

- each \( n \)-dimensional polyhedron of \( \tau \) is \( L \)-primitive,
- the asymptotic cone of each element of \( \tau \) is generated by some (but not all) vectors of the set \( -e_1, \ldots, -e_N, e_1 + \ldots + e_N \), where \( e_1, \ldots, e_N \) form the standard basis of \( \R^N \).

If such a subdivision \( \tau \) can be chosen \( c \)-extendable, we say that \( X \) admits a \( c \)-extendable unimodular triangulation.

**Proposition 45.** Let \( X \subset \TP^N \) be an \( n \)-dimensional smooth projective \( \Q \)-tropical variety. Then, there exists a positive integer \( m \) such that \( X \) admits a \( c \)-extendable unimodular triangulation with respect to the lattice \( \frac{1}{m} \Z^N \).
Lemma 46. There exists a tropical hypersurface $\mathcal{H}_X \subset \mathbb{TP}^N$ such that
- $X$ admits a subdivision $X'$ which is a subcomplex of $\mathcal{H}_X$ (that is, each face of $X'$ is a face of $\mathcal{H}_X$),
- the vertices of $\mathcal{H}_X^o \cap \mathbb{R}^N$ belong to $\frac{1}{m}\mathbb{Z}^N$ for a certain positive integer $m$,
- the asymptotic cone of each element of $\mathcal{H}_X$ is generated by some (but not all) vectors of the set $-e_1, \ldots, -e_N, e_1 + \ldots + e_N$.

Proof. Lemma holds tautologically if $n = \dim X = N - 1$, so assume that $n < N - 1$. Define $\mathcal{I}$ to be the set of subsets $I \subset \{1, \ldots, N\}$ such that $\pi_I^{-1}(X) \subset \mathbb{TP}_I$ are hypersurfaces (see Section 3.2 for the definition of the projections $\pi_I$). The set $\mathcal{I}$ is non-empty; moreover, for any $I \in \mathcal{I}$ put $H_I = (\pi_I^T)^{-1}(\pi_I^T(X))$, and denote by $f_I(x_0, \ldots, x_N)$ a tropical polynomial defining the hypersurface $H_I \subset \mathbb{TP}^N$. For each $I \in \mathcal{I}$ put
\[
\tilde{f}_I(x_0, \ldots, x_N) = \max\{f_I(x_0, \ldots, x_N), \max_{i \in I}\epsilon_i + d_i x_i\},
\]
where $d_I$ is the degree of $f_I$, and $\epsilon_i, i \in I$, are negative integer numbers (with sufficiently big absolute values) such that the hypersurface $\tilde{H}_I \subset \mathbb{TP}^N$ defined by $\tilde{f}_I$ satisfies the following property: $X \subset H_I \cap \tilde{H}_I$. We have $X \subset \bigcap_{I \in \mathcal{I}} \tilde{H}_I$. Put
\[
\mathcal{H}_X = \bigcup_{I \in \mathcal{I}} \tilde{H}_I.
\]
Since the asymptotic cone of each element of $X$ is generated by some vectors of the set $-e_1, \ldots, -e_N, e_1 + \ldots + e_N$ (see Proposition 12), the same is true for the elements of $\mathcal{H}_X$. The coordinates of any vertex of $\mathcal{H}_X$ are solutions of systems of linear equations with rational coefficients. Thus, the vertices of $\mathcal{H}_X^o = \mathcal{H}_X \cap \mathbb{R}^N$ belong to $\frac{1}{m}\mathbb{Z}^N$ for a certain positive integer $m$. A tropical hypersurface $\mathcal{H}_X$ comes with a natural polyhedral subdivision where each face is given by the subset of the tropical monomials taking the maximal value at its relative interior. The intersections of faces of $X$ with the faces of $\mathcal{H}_X$ provide a subdivision $X'$ of $X$. It remains to show that each face of $X'$ is a face of $\mathcal{H}_X$.

Assume that an $n$-dimensional face $F$ of $X'$ has a non-empty intersection with the interior of a face $G \subset \mathcal{H}_X$ of dimension $n' > n$. Since $F$ is contained in the support of $H_I$ for any $I \in \mathcal{I}$, the face $G$ is contained in a face (of dimension at least $n'$) of $H_I$ for any $I \in \mathcal{I}$. Pick a set $I \in \mathcal{I}$ such that the image $\pi_I^T(G)$ of $G$ is full-dimensional in $\mathbb{TP}_I$. Since $\pi_I^T(F)$ has a non-empty intersection with the interior of $\pi_I^T(G)$, the face $G$ is not a face of $H_I$. \qed

Proof of Proposition 45. Let $\mathcal{H}_X \subset \mathbb{TP}^N$ be a tropical hypersurface provided by Lemma 46. The closures of the connected components of the complement of $\mathcal{H}_X$ in $\mathbb{R}^N$ are $N$-dimensional convex polyhedral domains which form a polyhedral complex. We denote this complex by $\mathcal{A}_X$. A tropical polynomial defining the hypersurface $\mathcal{H}_X$ provides a piecewise-linear convex function $f_X : \mathbb{R}^N \to \mathbb{R}$ whose domains of linearity coincide with the $N$-dimensional polyhedral domains of $\mathcal{A}_X$. Denote by $\mathcal{B}_X$ the polyhedral complex formed by all bounded polyhedral domains of $\mathcal{A}_X$. Adding to $f_X$ an appropriate piecewise-linear convex function,
one can assume that the support $S_X$ of $B_X$ is a simplex with vertices in $\frac{1}{m}\mathbb{Z}^N$ and with the outward normal directions of facets given by the vectors

$$-e_1, \ldots, -e_N, e_1 + \ldots + e_N,$$

and each unbounded $N$-dimensional polyhedron in $A_X$ is the Minkowski sum of a bounded polyhedron of dimension $0 \leq k \leq N$ in the boundary of $S_X$ and an $(N-k)$-dimensional cone generated by some vectors of the set $-e_1, \ldots, -e_N, e_1 + \ldots + e_N$. According to [KKMS73], there exists a positive integer $\ell$ such that $B_X$ admits a unimodular triangulation $\tau_X$ with respect to $\frac{1}{\ell m}\mathbb{Z}^N$. Moreover, the triangulation $\tau_X$ can be chosen in such a way that there exists a continuous function $\Phi_X : S_X \to \mathbb{R}$ such that the restriction of $\Phi_X$ to any $N$-dimensional polyhedral domain $\delta$ of $B_X$ is a piecewise-linear convex function whose domains of linearity coincide with $N$-dimensional simplices of $\tau_X$ which are contained in $\delta$. Extend the function $\Phi_X$ to $\mathbb{R}^N$ in the following way: for any point $x \in \mathbb{R}^N \setminus S_X$, consider the point $x' \in S_X$ which is the closest one to $x$ (with respect to the standard Euclidean distance in $\mathbb{R}^N$) among the points of $S_X$, and put $\Phi_X(x) = \Phi_X(x')$. The function $\Phi_X : \mathbb{R}^N \to \mathbb{R}$ is continuous, and its restriction to any $N$-dimensional convex polyhedral domain $\delta$ in $A_X$ is a piecewise-linear convex function. For a sufficiently small positive number $\varepsilon$, the function $f_X + \varepsilon \Phi_X$ defines a convex polyhedral subdivision of $\mathbb{R}^N$, and this subdivision is a unimodular triangulation with respect to $\frac{1}{\ell m}\mathbb{Z}^N$. This unimodular triangulation provides a unimodular triangulation with respect to $\frac{1}{\ell m}\mathbb{Z}^N$ for each subcomplex of $\mathcal{H}_X$. Thus, $X$ admits a $c$-extendable unimodular triangulation with respect to $\frac{1}{\ell m}\mathbb{Z}^N$. 

4.2. Construction of the central fiber. Applying Proposition 45 (and performing a base change) we can assume that $X$ is unimodularly triangulated.

We put the mobile part $X^o = X \cap \mathbb{R}^N$ of the tropical variety $X$ in $\mathbb{R}^N \times \{1\} \subset \mathbb{R}^N \times \mathbb{R}_{\geq 0}$ and take the cone from the origin over it. The closure $\Sigma_X$ of this cone in $\mathbb{R}^N \times \mathbb{R}_{\geq 0}$ is a rational polyhedral (non-complete) fan of dimension $n + 1$ in $\mathbb{R}^{N+1}$ with unimodular faces. Thus, it defines a smooth (non-compact) toric variety $P_{\Sigma_X}$.

The fan $\Sigma_X$ maps to $\mathbb{R}_{\geq 0}$ along the last coordinate, thus the toric variety $P_{\Sigma_X}$ naturally maps to $\mathbb{C}$. The intersection of $\Sigma_X$ with the hyperplane $\mathbb{R}^N \times \{0\}$ coincide with the asymptotic fan $A(X)$ of $X$. The faces $A_I$ of $A(X)$ correspond to nonempty sedentary strata $X_I$ of $X$. We denote the corresponding toric boundary strata of $P_{\Sigma_X}$ by $D_I$. In particular, the rays $A_i, i = 0, \ldots, N$ of $A(X)$ are along the divisorial vectors of $\mathbb{T}\mathbb{P}^N$.

The general fiber $P_\xi$ of the map $P_{\Sigma_X} \to \mathbb{C}$ is the toric variety associated to the asymptotic fan $A(X)$, which is a subfan of the standard fan for $\mathbb{C}\mathbb{P}^N$. Thus, $P_\xi$ is naturally isomorphic to $\mathbb{C}\mathbb{P}^N$ with some coordinate strata removed.

The central fiber $P_0$ is a normal crossing divisor in $P_{\Sigma_X}$ whose components $D_\nu$ are toric varieties associated to the rays of $\Sigma_X$ through the mobile vertices $\nu$ of $X$. More generally, the toric strata of $P_0$ are labelled by the mobile faces of $X$: if $\Delta$ is a mobile face of $X$ which is the Minkowski sum of the simplex spanned by vertices $\nu_0, \ldots, \nu_k$ and the cone spanned by the divisorial vectors indexed by $I$, then the orbit $O_\Delta \cong (\mathbb{C}^*)^{N-m-|I|}$ is the maximal torus in the intersection $D_\Delta := D_{\nu_0} \cap \cdots \cap D_{\nu_k} \cap D_I$. 


Observe that $P_{\Sigma_X}$ is quasi-projective. The polarization is given by a piecewise-linear function provided by Proposition 45. In particular, $P_{\Sigma_X}$ has a Kähler structure.

Notice that the removed coordinate strata from $\mathbb{C}P^N$ do not meet the fibers $Z_w \subset \mathcal{Z}$. Hence our family $\mathcal{Z}$ is embedded in $P_{\Sigma_X}$. We denote its closure by $\bar{\mathcal{Z}}$. It is now a proper family over the full disk $D \ni 0$. Let $Z \subset \bar{\mathcal{Z}}$ denote the fiber over 0. The mobile faces of $X$ label the intersections of the family $\bar{\mathcal{Z}}$ with the toric strata of $P_0$: for a mobile face $\Delta$ of $X$ we let $Z_\Delta := \mathcal{Z} \cap D_\Delta \subset Z$ and $Z^{\circ}_\Delta := \mathcal{Z} \cap O_\Delta \subset Z$.

Lemma 47. Let $\Delta$ be a mobile $k$-face of $X$. Then, there is a compactification of $O_\Delta$ to the projective space $\mathbb{C}P^{N-k}$, such that the closure of $Z^{\circ}_\Delta$ is a linear subspace in $\mathbb{C}P^{N-k}$. In particular, $Z^{\circ}_\Delta \subset O_\Delta$ is isomorphic to the complement of the hyperplane arrangement corresponding to an underlying matroid of the relative fan $\Sigma_\Delta$.

Proof. It is helpful to have a geometric picture. One can consider the logarithmic amoeba $\mathcal{A}$ of the affine part of the family $\mathcal{Z} \cap (\mathbb{C}^*)^N \times D^*$, namely the image under the map

$$\text{Log} : (\mathbb{C}^*)^N \times D^* \to \mathbb{R}^N \times \mathbb{R}_{\geq 0}, \quad (z_1, \ldots, z_N, w) \mapsto (\log |z_1|, \ldots, \log |z_N|, -\log |w|).$$

After shrinking by $\log t$ as $t \to \infty$ this amoeba in the limit coincides with the fan $\Sigma_X$. Adding the central fiber to $\mathcal{Z}$ results in adding divisors at $w = 0$.

We take an $(N-k)$-ball in the horizontal plane ($t = \text{const}$) integrally transversal to the cone over $\Delta$ and intersect it with the amoeba $\mathcal{A}$. We define $\mathcal{A}_\Delta$ as the limit of this intersection as $t \to \infty$ and the size of the ball grows linearly with $t$.

Let $M$ be a matroid whose Bergman fan $\Sigma^0_M \subset \mathbb{R}^{N-k}$ has the same support as $\Sigma_\Delta$. The vectors $e_0, \ldots, e_{N-k} \in \mathbb{R}^{N-k}$ corresponding to the elements of $\{M\}$ define the compactification of $\mathbb{R}^{N-k}$ into the tropical projective space $\mathbb{T}P^{N-k}$ and the compactification of $O_\Delta$ to the projective space $\mathbb{C}P^{N-k}$. These two compactifications are compatible with the Log map.

According to the geometric picture above we can view the closure $Y_M \subset \mathbb{T}P^{N-k}$ of the fan $\Sigma^0_M \subset \mathbb{R}^{N-k}$ as the tropical limit of the constant family $Z_M$, where $Z_M$ is the closure of $Z^{\circ}_\Delta$ in $\mathbb{C}P^{N-k}$ (it depends on the matroid $M$ and may differs from the original closure $Z^{\circ}_\Delta$). The Bergman fan has the tropical degree 1, cf. Example 42. Hence, by Corollary 44 the degree of the subvariety $Z_M \subset \mathbb{C}P^{N-k}$ is also 1, or in other words, $Z_M$ is a linear subspace in $\mathbb{C}P^{N-k}$. Removing the coordinate hyperplanes leads to the second statement of the lemma. \[\Box\]

We summarize the properties of our model needed later for the proof of Theorem 1.

Proposition 48. The family $\bar{\mathcal{Z}}$ is a smooth Kähler manifold (after, perhaps, restricting values of $w$ to a smaller disk). The central fiber $Z$ is a simple normal crossing divisor in $\bar{\mathcal{Z}}$ and the correspondence $\{\Delta\} \leftrightarrow \{Z_\Delta\}$ has the following properties:

1. The subcomplex of $X$ formed by the finite mobile faces is identified with the Clemens complex of $\bar{\mathcal{Z}}$.
2. The infinite mobile faces of $X$ label the intersections of components of $Z$ with the toric strata $D_1$, and all these intersections are also simple normal crossings.
(3) For any mobile face \( \Delta \subset X \) the relatively open part \( Z_\Delta \) of \( Z \) is the complement of a hyperplane arrangement such that the relative fan \( \Sigma_\Delta \) at \( \Delta \) is the Bergman fan of the corresponding matroid.

**Proof.** Note that \( \bar{Z} \) is smooth at the points of the central fiber \( Z \) and is transversal to every \( D_\Delta \) since it is locally defined by linear (degree 1) equations by Lemma 47.

Smoothness is an open condition and hence extends to some neighborhood of \( Z \). In the toric variety \( P_{\Sigma X} \) the boundary divisor \( \bigcup D_\nu \cup \bigcup D_i \) is simple normal crossing. Then, since \( \bar{Z} \) intersect each \( D_\Delta \) transversally, the divisor \( Z \) is also simple normal crossing. The Kähler structure on \( \bar{Z} \) is induced from \( P_{\Sigma X} \). □

### 4.3. Limiting mixed Hodge structure.

We continue assuming that \( X \) is unimodularly triangulated. Denote by \( X^{(k)} \) the collection of finite mobile \( k \)-faces of \( X \), and let \( Z^{(k)} = \bigcup_{\Delta \in X^{(k)}} Z_\Delta \) be the disjoint union of the \( k \)-intersections of components in \( Z \).

The Steenbrink-Illusie spectral sequence (cf. [PS08], Ch. 11) associated to \( Z \) has the first term

\[
E^{r,k}_{1} = \bigoplus_{l \geq \max\{0,r\}} H^{k+r-2l}(Z^{(2l-r)}; \mathbb{Q})[r-l],
\]

where \([r]\) means the \( r \)-th Tate twist.

The odd rows in \( E_1 \) are zero, we disregard them. Then, we dualize and make shifts of indices in the even rows, so that the relabeled term now reads

\[
\tilde{E}^{1}_{q,p} := \text{Hom}(E^{q-p,2p}_{1}; \mathbb{Q}) = \bigoplus_{l=0}^{\min\{p,q\}} H_{2l}(Z^{(p-2l+q)}; \mathbb{Q})[p-l].
\]

Here is the beginning of the new \( \tilde{E}^{1} \) term (where \( H_{2l}(k)[r] \) means \( H_{2l}(Z^{(k)}; \mathbb{Q})[r] \)):

\[
\begin{align*}
H_0(2)[2] & \xleftarrow{d} H_0(3)[2] \xleftarrow{d} H_0(4)[2] \\
\oplus H_2(1)[1] & \oplus H_2(2)[1] \oplus H_4(0)
\end{align*}
\]

\[
\begin{align*}
H_0(1)[1] & \xleftarrow{d} H_0(2)[1] \xleftarrow{d} H_0(3)[1] \xleftarrow{d} H_0(4)[1] \\
\oplus H_2(0) & \oplus H_2(1) \oplus H_2(2)
\end{align*}
\]

\[
\begin{align*}
H_0(0) & \xleftarrow{d} H_0(1) \xleftarrow{d} H_0(2) \xleftarrow{d} H_0(3) \xleftarrow{d} H_0(4)
\end{align*}
\]

The differential \( d = i_* + \text{Gys} \) consists of the pushforward map \( i_* \) and the Gysin map (see [PS08]):

\[
i_* : H_{2l}(k)[r] \to H_{2l}(k-1)[r]
\]

\[
\text{Gys} : H_{2l}(k)[r] \to H_{2l-2}(k+1)[r+1].
\]
Theorem 49. For each $p$ the Steenbrink-Illusie row complex $(\tilde{E}_{•,p}^1, d)$ is quasi-isomorphic to the tropical cellular chain complex $C_•(X; F_p)$. In particular, $\tilde{E}_{q,p}^2 \cong H_q(X; F_p)$.

The proof of Theorem 49 is presented in the next section.

Proof of Theorem 5 and Corollary 5. The Steenbrink-Illusie spectral sequence degenerates at $E_2$ abutting to cohomology of the canonical fiber $Z_\infty$ with the monodromy weight filtration (cf. [PS08], Ch. 11). Thus, Theorem 5 follows from Theorem 49.

Since all closed strata in $Z$ are composed of complements of hyperplanes all even cohomology groups $H^{2i}(Z_\Delta; \mathbb{Q})$ are of $(i, i)$-type (cf. e.g. [DKh78]). Cohomology in odd degrees vanish. In particular, the MHS is of Hodge-Tate type: in the weight filtration only even associated graded pieces are non-trivial and each contains only Hodge $(p, p)$-type. Thus, the weight filtration calculates the Hodge numbers of the canonical fiber (and hence also of a smooth fiber). That is, $h^{p,q}(Z_w) = \dim E_{q-p,2p}^2 = \dim \tilde{E}_{q,p}^2 = \dim H_q(X; F_p)$. \hfill \Box

5. Proof of Theorem 49

The proof goes as follows. For each $p$ we introduce a double complex $(K^{(p)}_•, \partial, \delta)$ and calculate homology of the total complex $(K^{(p)}_•, \partial + \delta)$ in two ways. First, we take the $\delta$-homology, and recover the tropical cellular chain complex $C_•(X; F_p)$ (see Proposition 53). Second, we introduce a filtration $F_m$ on $(K^{(p)}_•, \partial + \delta)$ such that the first term of the resulting spectral sequence is a single row which coincides with the Steenbrink-Illusie complex $(\tilde{E}^1_{•,p}, i_* + \text{Gys})$ (see Proposition 56). In the notations below

\begin{equation}
\tilde{E}_{m,p}^1 = \bigoplus_{l=0}^{\min\{m,p\}} H_{2l}(\Delta),
\end{equation}

where $\Delta$ runs over finite mobile $(p - 2l + m)$-dimensional faces of $X$, and we disregard the Tate twist. Theorem 49 then follows.

5.1. Notations. From now on we assume that all faces of $X$ are oriented. Recall that to any mobile face $\Delta$ of $X$ we can associate $Z_\Delta$, the closed subset of $Z$ which is the intersection of the corresponding components of $Z$ and some toric divisors in $P_{2X}$. Recall also our notation $\Delta \succ_j \Delta'$ (and $\Delta' \succ_j \Delta$) when $\Delta$ is a face of $\Delta'$ of codimension $j$ and cosedentarity $s$. We omit the superscript $s$ in case $s = 0$. Here are some more notations.

- If $\Delta$ is a mobile face of $X$, we set $H_{2l}(\Delta) := H_{2l}(Z_\Delta; \mathbb{Q})$.
- For $\Delta' \succ_1 \Delta$, a consistently oriented pair of mobile faces, $i_* : H_{2l}(\Delta') \to H_{2l}(\Delta)$ is the pushforward map, and Gys : $H_{2l}(\Delta) \to H_{2l-2}(\Delta')$ is the Gysin map.
- If $\Delta$ is sedentary, we set $H_{2l}(\Delta) := H_{2l}(\Delta_0)$, where $\Delta_0$ is the parent of $\Delta$.
- For any face $\Delta$ of $X$ we set $W_r(\Delta) := \wedge^r \mathbb{Q}(\Delta)$ to be the space of rational $r$-polyvectors in the linear span of $\Delta$.

We extend the meaning of the pushforward and the Gysin maps for sedentary faces. If $\Delta' \succ_1 \Delta$ then the pushforward $i_* : H_{2l}(\Delta') \to H_{2l}(\Delta)$ is the same as $i_* : H_{2l}(\Delta_0) \to H_{2l}(\Delta_0)$ for their parents (under the identifications $H_{2l}(\Delta) = H_{2l}(\Delta_0)$ and $H_{2l}(\Delta') = H_{2l}(\Delta_0'))$. And similar for the Gysin map Gys : $H_{2l}(\Delta) \to H_{2l-2}(\Delta')$. 
If $\Delta' \succ 1 \Delta$, then $i^*: H_{2l}(\Delta') \to H_{2l}(\Delta)$ is the identity (both groups equal $H_{2l}(\Delta_0)$). The Gysin map $\text{Gys}: H_{2l}(\Delta) \to H_{2l-2}(\Delta')$ in this case is zero.

For a pair $\Delta' \succ_{j+s} \Delta$ we define the residue map

$$\text{res}_{\Delta' \succ \Delta} : W_r(\Delta') \to W_{r-j}(\Delta)$$

as follows. Let $\Delta''$ be the smallest face between $\Delta$ and $\Delta'$ of the same sedimentary as $\Delta'$. Then, there is a canonical primitive covolume $j$-form $\Omega_{\Delta' \succ \Delta''}$ in $\Delta'$ which vanishes on polyvectors divisible by vectors in $\Delta''$. Now for a polyvector $w \in W_r(\Delta')$ we define its residue $\text{res}_{\Delta' \succ \Delta}(w) \in W_{r-j}(\Delta)$ to be the projection to $W_{r-j}(\Delta)$ of the evaluation of $w$ on $\Omega_{\Delta' \succ \Delta''}$ (It is zero if $r < j$).

The most important cases are $\Delta' \succ 1 \Delta$ and $\Delta' \succ 1 \Delta$. In the first case $\text{res}(w)$ is the evaluation of $w$ on the canonical linear from which defines the facet $\Delta \prec \Delta'$. In the second case $\text{res}(w)$ is just the projection.

Let $\Delta$ be a $k$-dimensional face of $X$, maybe infinite and sedimentary. It will be convenient to treat vertices and divisorial vectors which span $\Delta$ on an equal footing. Namely, for $\Delta$ we write a sequence $(\nu_0 \nu_1 \ldots \nu_k)$, where $\nu_0$ is a vertex and each $\nu_j \neq 0$ may denote a vertex or a divisorial vector of $\Delta$.

The orientation of $\Delta = (\nu_0 \nu_1 \ldots \nu_k)$ is encoded in the sign order of the sequence. To get consistent signs in the Gysin and pushforward maps in the Steenbrink-Illusie differential we use the following convention. For the Gysin map we add a new vertex (or a divisorial vector) at the end of the old sequence. For the pushforward map we remove the last element (after reordering the sequence if needed).

**Lemma 50.** For a class $\beta \in H_{2l}(\Delta)$ we denote by $\beta^q$ its image in $H_{2l-2}(\Delta \nu_q)$ under the Gysin map, and by $\beta_j$ its image in $H_{2l}(\Delta \setminus \nu_j)$ under the pushforward map. Then

$$\begin{align*}
(\beta_j)_i &= (\beta_i)_j, & (\beta^q)_r &= (\beta^q)_r, \\
(\beta_j)^q &= (\beta^q)_j, & \sum_q \beta^q_j + \sum_j \beta^q_j &= 0,
\end{align*}$$

where the sums in the last identity are over all $\nu_j \in \Delta$ and all $\nu_q \in \text{Link}(\Delta)$. (Here and later $\nu_q \in \text{Link}(\Delta)$ means $(\Delta \nu_q) \succ 1 \Delta$).

**Proof.** The first two identities follow from writing $i_*^2 = 0$ and $\text{Gys}^2 = 0$ in components.

The other two follow from the (anti-)commutative diagram $i_*, \text{Gys} + \text{Gys} i_* = 0$:

$$\begin{align*}
H_{2l}(\nu_0 \ldots \nu_k) &\xrightarrow{i_*} \sum_j H_{2l}(\nu_0 \ldots \hat{\nu}_j \ldots \nu_k) \\
\sum_q H_{2l-2}(\nu_0 \ldots \nu_k \nu_q) &\xrightarrow{i_*} H_{2l-2}(\nu_0 \ldots \nu_k) \oplus \sum_{j,q} H_{2l-2}(\nu_0 \ldots \hat{\nu}_j \ldots \nu_k \nu_q)
\end{align*}$$

\(16\)
In $H_{2l-2}(v_0 \ldots v_j \ldots v_k \nu_q)$ we have $(\beta_j)^q - (\beta^q)_j = 0$ (with our sign convention). The last identity

$$
\sum_q \beta^q_j + \sum_j \beta^j_q = 0
$$

takes place in $H_{2l-2}(v_0 \ldots v_k)$. The meanings of $\beta^j_q$ and $\beta^q_j$ are unambiguous: they only makes sense in one order: $\beta^j_q = (\beta_j)^q_j$ and $\beta^q_j = (\beta^q)_q$.

\[ \square \]

5.2. **The double complex** $(K^{(p)}_{\bullet, \bullet}, \partial, \delta)$. For any face $\Delta \in X$ we set

$$A^{(p)}_j(\Delta) = \bigoplus_{\Delta' \in \text{Star}(\Delta)} H_{2l}(\Delta') \otimes W_{p-l}(\Delta') / \sim$$

where the equivalence is defined as follows. For a pair of simplices $\Delta'' \prec_j \Delta'$ we have two maps $i_*$ and $\iota$ (the inclusion of polyvectors) going into opposite directions:

$$
\begin{array}{ccc}
H_{2l}(\Delta'') \otimes W_{p-l}(\Delta'') & \xrightarrow{i_*} & H_{2l}(\Delta') \otimes W_{p-l}(\Delta') \\
\downarrow & & \downarrow \\
H_{2l}(\Delta') \otimes W_{p-l}(\Delta') & \xrightarrow{\iota} & H_{2l}(\Delta'') \otimes W_{p-l}(\Delta'')
\end{array}
$$

We identify elements $\beta \otimes \iota(w) \in H_{2l}(\Delta') \otimes W_{p-l}(\Delta')$ and $i_*(\beta) \otimes w \in H_{2l}(\Delta'') \otimes W_{p-l}(\Delta'')$.

**Remark.** To generate the space $A^{(p)}_j(\Delta)$ it is sufficient to consider $\Delta' \succ_j \Delta$ with $j$ between 0 and $p - l$.

Now we set $K^{(p)}_{k,l} := \bigoplus_{\dim \Delta = k} A^{(p)}_l(\Delta)$, and define two differentials

$$\partial : K^{(p)}_{k,l} \to K^{(p)}_{k-1,l}, \quad \delta : K^{(p)}_{k,l} \to K^{(p)}_{k,l-1}$$

as follows.

The horizontal differential $\partial$ acts essentially just as the boundary map on the cell complex $X$. Namely, if $\Delta'' \prec_1 \Delta$ is a consistently oriented pair, then $\partial : A^{(p)}_l(\Delta) \to A^{(p)}_l(\Delta'')$ acts on representatives $\beta \otimes w \in H_{2l}(\Delta') \otimes W_{p-l}(\Delta')$ as the identity. Clearly, it respects the equivalence.

If $\Delta'' \prec_1 \Delta$, then by Proposition 12 for any face $\Delta' \in \text{Star}(\Delta)$ there is a corresponding face $\Delta''' \in \text{Star}(\Delta'')$ in the same family as $\Delta'$. That is, $H_{2l}(\Delta''') = H_{2l}(\Delta')$, and we let $\pi : W_{p-l}(\Delta') \to W_{p-l}(\Delta''')$ be the natural projection. Then, we define the image of $\beta \otimes w$ in $A^{(p)}_l(\Delta'')$ to be $\beta \otimes \pi(w)$. Since inclusions $\iota$ for the spaces $W_{p-l}(\bullet)$ commute with the projections $\pi$, any equivalence between representatives $\beta \otimes w$ in $A^{(p)}_l(\Delta)$ also holds for their images in $A^{(p)}_l(\Delta'')$.

We define the vertical differential $\delta : A^{(p)}_l(\Delta) \to A^{(p)}_{l+1}(\Delta)$ by a certain combination of the pushforward $i_*$ and the Gysin maps. Namely, given $\beta \otimes w \in H_{2l}(\Delta') \otimes W_{p-l}(\Delta')$ we
first choose a representing sequence \((\nu_0\nu_1 \ldots \nu_k)\) for \(\Delta'\) (in particular, a reference vertex \(\nu_0 \in \Delta'\)) and then set

\[
(17) \quad \delta(\beta \otimes w) := \sum_{\nu_q \in \text{Link}(\Delta')} \beta^q \otimes (\nu_0q \wedge w) + \sum_{\nu_j \in \Delta'} \beta^j \otimes (\nu_0j \wedge w).
\]

Here, \(\nu_0q\) means the vector \((\nu_0\nu_q)\) if \(\nu_q\) is a vertex, and \(\nu_0q = \nu_q\) if \(\nu_q\) is a divisorial vector, and similar for \(\nu_j\). Note that the first sum is in \(\sum_q H_{2l-2}(\Delta'\nu_q) \otimes W_{p-l+1}(\Delta'\nu_q)\) and the second sum is in \(H_{2l-2}(\Delta') \otimes W_{p-l+1}(\Delta')\), but both represent elements in \(A_{l-1}^{(p)}(\Delta)\).

**Lemma 51.** The map \(\delta : A_j^{(p)}(\Delta) \to A_{j-1}^{(p)}(\Delta)\) is well defined.

**Proof.** There are two things to check: (1) it is independent of the choice of the representing sequence \(\Delta' = (\nu_0\nu_1 \ldots \nu_k)\), of which only independence of the choice of the reference vertex \(\nu_0 \in \Delta'\) is non-trivial, and (2) it is defined on the equivalence class of \(\beta \otimes w\). Since the definition (17) is linear in \(w\) we can let \(w = 1\).

For (1) let \(\nu_0 = \nu_1 \in \Delta'\) be a new reference vertex \((\nu_1\neq\text{not a divisorial vector})\). Then

\[
\sum_{\nu_q \in \text{Link}(\Delta')} \beta^q \otimes \nu_0q + \sum_{\nu_j \in \Delta'} \beta^j \otimes \nu_0j - \sum_{\nu_q \in \text{Link}(\Delta')} \beta^q \otimes \nu_q - \sum_{\nu_j \in \Delta'} \beta^j \otimes \nu_j
\]

\[
= \sum_{\nu_q \in \text{Link}(\Delta')} \beta^q \otimes \nu_0q + \sum_{\nu_j \in \Delta'} \beta^j \otimes \nu_01
\]

\[
\sim \sum_{\nu_q \in \text{Link}(\Delta')} \beta^q \otimes \nu_0q + \sum_{\nu_j \in \Delta'} \beta^j \otimes \nu_01,
\]

which is 0 by the last identity in Lemma 50.

For (2) pick an equivalent representative \(\beta_r \in H_{2l}(\Delta' \backslash \nu_r)\) (assuming, of course, \(\Delta' \backslash \nu_r \succ \Delta\)) and compare

\[
\delta \beta = \sum_{\nu_q \in \text{Link}(\Delta')} \beta^q \otimes \nu_0q + \sum_{\nu_j \in \Delta', \ j \neq r} \beta^j \otimes \nu_0j + \beta^r \otimes \nu_0r,
\]

with

\[
\delta(\beta_r) = \sum_{\nu_q \in \text{Link}(\Delta' \backslash \nu_r)} \beta^q_r \otimes \nu_0q + \beta^r \otimes \nu_0r + \sum_{\nu_j \in \Delta' \backslash \nu_r} (\beta_j^r)^i \otimes \nu_0j.
\]

First, observe that in the first sum of the latter expression the Gysin image of those \(\nu_q \in \text{Link}(\Delta' \backslash \nu_r)\) which are not in \(\text{Link}(\Delta')\) must be zero by the third identity in Lemma 50. Furthermore, we have

\[
(\beta_r^j)^i = (\beta_{rj})^i = -(\beta_{jr})^i = -(\beta_{jr})^i = ((\beta_j)^i)_r = (\beta_j^i)_r.
\]

Then, the two expressions are manifestly equivalent. \(\square\)

**Lemma 52.** The triple \((K^{(p)}_\bullet, d, \delta)\) is a double complex.
Proof. The horizontal differential $\partial$ is the standard boundary map on the simplicial complex. The fact that $\partial$ commutes with $\delta$ is clear from the definitions. Finally, $\delta^2 = 0$ follows immediately from the second and the third commutativity identities in Lemma [50]. \hfill $\square$

5.3. Filtration on $(K^{(p)}_k, \delta)$ by columns.

Proposition 53. The pair $(K^{(p)}_k, \delta)$ is a resolution of $C_k(X; F_p)$ and the resulting homology complex $(H_0(K^{(p)}_k, \delta), \partial)$ is isomorphic to $C_*(X; F_p)$.

To prove the proposition we introduce an increasing filtration $F^\Delta$ on the spaces $A^{(p)}_t(\Delta)$:

\begin{equation}
0 \subseteq F^\Delta_{-p+t} \subseteq \cdots \subseteq F^\Delta_0 = A^{(p)}_t(\Delta),
\end{equation}

where $F^\Delta_\gamma A^{(p)}_t(\Delta)$ consists of elements which can be represented by $\beta \otimes w \in H_{2l}(\Delta') \otimes W_{p-l}(\Delta')$ with $\Delta' \succ_{(p-l-r)} \Delta$. Or, equivalently, $\beta \otimes w$ is in $F^\Delta_\gamma A^{(p)}_t(\Delta)$ if $w$ belongs to the ideal generated by $\iota(W_r(\Delta))$. In particular, the filtration on $A^{(p)}_0(\Delta)$ induces one on the coefficient groups $F_p(\Delta)$.

Lemma 54. The associated graded groups $\text{Gr}^{F^\Delta} A^{(p)}_t(\Delta) = F^\Delta_\gamma A^{(p)}_t(\Delta)/F^\Delta_{\gamma-1} A^{(p)}_t(\Delta)$ can be naturally identified with $\bigoplus_{\Delta' \succ_{(p-r-1)} \Delta} H_{2l}(\Delta') \otimes W_r(\Delta)$. The graded pieces $\text{Gr}^{F^\Delta} F_p(\Delta)$ can be naturally identified with $F_p(\Delta) \otimes W_r(\Delta)$, where $F_p(\Delta)$ are the relative coefficient groups at $\Delta$ (cf. Section [2.4]).

Proof. Consider an element in $A^{(p)}_t(\Delta)$ which is represented by $\beta \otimes w$ with $w \in W_{p-l}(\Delta')$ for some $\Delta' \succ \Delta$. This means that this element is in $F^\Delta_{-p+j+t} A^{(p)}_t(\Delta)$. The residue map

$$\text{res}_{\Delta' \succ \Delta} : W_{p-l}(\Delta') \rightarrow W_{p-l-j}(\Delta)$$

is surjective, and $\text{res}_{\Delta' \succ \Delta}(w)$ vanishes if and only if $w \in W_{p-l}(\Delta')$ lies in the ideal generated by $\iota(W_{p-l+j+1}(\Delta'))$, that is, if and only if $\beta \otimes w \in F^\Delta_{-p+j+l} A^{(p)}_t(\Delta)$. Thus, the residue map

$$\beta \otimes w \mapsto \beta \otimes \text{res}_{\Delta' \succ \Delta}(w)$$

provides the desired isomorphism

$$F^\Delta_{-\gamma} A^{(p)}_t(\Delta)/F^\Delta_{-\gamma-1} A^{(p)}_t(\Delta) \cong \bigoplus_{\Delta' \succ_{(p-r-1)} \Delta} H_{2l}(\Delta') \otimes W_r(\Delta).$$

Finally, the induced filtration on $F_p(\Delta)$ is given by considering the polyvectors from $W_{p-l}(\Delta')$ as elements in the space $\wedge^p \mathbb{Q}^N$ of ambient polyvectors. The last statement of the Lemma follows. \hfill $\square$

Proof of Proposition 53. The horizontal differentials $\partial$ on $A^{(p)}_0(\Delta)$ and the cell boundary maps on $F_p(\Delta)$ commute with the augmentation maps $A^{(p)}_0(\Delta) \rightarrow F_p(\Delta)$. Thus, it suffices to check that for each $\Delta$ the augmented complex $A^{(p)}_*(\Delta) \rightarrow F_p(\Delta) \rightarrow 0$ is exact. It is enough to show exactness on the associated graded level with respect to the filtration.
The filtration $F^\Delta$ respects the vertical differential $\delta$, since $\delta$ raises the degree of a polyvector $w$ by 1, but puts it in a face of at most one dimension higher. It also respects the augmentation map $A_{(p)}(\Delta) \to F_p(\Delta)$, and hence defines a filtration on the augmented complex.

By Lemma 54 for each $r$ the associated graded complex $\text{Gr}_{F^\Delta} A_*(\Delta) \to \text{Gr}_{F^\Delta} F_p(\Delta) \to 0$ has the form

$$H_{2p-2r}(\Delta) \otimes W_r(\Delta) \to \bigoplus_{\Delta' \succ \Delta} H_{2p-2r-2}(\Delta') \otimes W_r(\Delta) \to \ldots$$

$$\to \bigoplus_{\Delta' \succ p-r} H_0(\Delta') \otimes W_r(\Delta) \to \mathcal{F}_{p-r}(\Delta) \otimes W_r(\Delta) \to 0,$$

where the differential is $W_r(\Delta)$-linear. From the definition of the vertical map $\delta$ in the double complex we can identify this differential on the homology groups as the Gysin map.

Notice that if $\Delta$ is sedentary with a parent $\Delta_0$, then there is an obvious map $\text{Gr}_{F^\Delta} A_*(\Delta_0) \to \text{Gr}_{F^\Delta} A_*(\Delta)$, which is the identity on the corresponding homology factors $H_2(\Delta_0') = H_2(\Delta')$ tensored with the projection $W_r(\Delta_0) \to W_r(\Delta)$. Taking into account $\mathcal{F}_{p-r}(\Delta_0) = \mathcal{F}_{p-r}(\Delta)$ this projection extends to the augmentation $\mathcal{F}_{p-r}(\Delta_0) \otimes W_r(\Delta_0) \to \mathcal{F}_{p-r}(\Delta) \otimes W_r(\Delta)$.

Thus, we are reduced to proving for each $p$ the exactness of the complex

$$\bigoplus_{\Delta' \succ p-\Delta} H_{2p}(\Delta') \to \mathcal{F}_p(\Delta) \to 0$$

for mobile faces $\Delta$ of $X$.

Recall Deligne’s spectral sequence [De71] which calculates the weight filtration of the mixed Hodge structures for smooth quasi-projective varieties $Y$. In our case, $Y = Z_{\Delta}$ is the relatively open part of $Y^{(0)} = Z_{\Delta}$. Let $Y^{(j)}$ denote the disjoint union of the intersections $Z_{\Delta'}$ for $\Delta' \succ j \Delta$. Then, we have $H_{2l}(Y^{(j)}; \mathbb{Q}) = \bigoplus_{\Delta' \succ j} H_{2l}(\Delta')$.

On the other hand, by construction $Y$ is (cf. Proposition 48) the complement of a hyperplane arrangement in $\mathbb{P}^{n-k}$ and its cohomology $H^p(Y; \mathbb{Q})$ is isomorphic to $\mathcal{F}^p(\Delta)$ by Theorem 15.

The $E_1$ term of the Deligne’s MHS spectral sequence looks as follows (in our case all odd rows are zero):

$$
\begin{align*}
H^0(Y^{(2)}) & \to H^2(Y^{(1)}) \to H^4(Y^{(0)}) \\
H^1(Y^{(1)}) & \to H^3(Y^{(0)}) \\
H^0(Y^{(1)}) & \to H^2(Y^{(0)}) \\
& \to H^1(Y^{(0)}) \\
& \to H^0(Y^{(0)})
\end{align*}
$$
The differential is the Gysin map, and the spectral sequence degenerates at $E_2$. The MHS weight filtration on the diagonals in $E_2 = E_{\infty}$ is given by the rows. The MHS structure on $H^p(Y; \mathbb{C})$ is pure of type $(p, p)$ (cf. [Sh93]). Thus, in $E_2$ the only non-zero terms will be the groups $\mathcal{F}_p^m(\Delta)$ in the left most entries of the even rows. Dualizing all groups and inverting arrows in the $2p$-th row we get the desired exact complex

$$0 \leftarrow \mathcal{F}_p^m(\Delta) \leftarrow H_0(Y^{(p)}) \leftarrow \cdots \leftarrow H_{2p-2}(Y^{(1)}) \leftarrow H_{2p}(Y^{(0)}).$$

\[\square\]

### 5.4. Another filtration on the total complex $K^{(p)}_*$ of $K_{\bullet\bullet}$.

The new increasing filtration $F_m$ on $K^{(p)}_*$ is, in fact, the old filtration $F^\Delta$ on the spaces $A_\bullet^{(p)}(\Delta)$ with some degree shifts depending on $\Delta$. Namely, if $\Delta$ has dimension $k$ and sedentarity $s$ we set $F_m A_t^{(p)}(\Delta) = F_{m-k-l-s} A_t^{(p)}(\Delta)$. We will see below that both $\delta$ and $\partial$ respect the filtration.

**Example 55.** We illustrate first few terms in $\text{Gr}^F_m K^{(p)}_*$ for $p = 2$:

$$\begin{array}{cccccc}
H_4(0)W_0(0)_2 & \leftarrow & H_4(1)W_0(1)_3 & \leftarrow & H_4(2)W_0(2)_4 & \leftarrow & H_4(3)W_0(3)_5 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_2(1)W_0(0)_1 & \leftarrow & H_2(2)W_0(1)_2 & \leftarrow & H_2(3)W_0(2)_3 & \leftarrow & H_2(4)W_0(3)_4 \\
& \oplus H_2(1)W_1(1)_1 & & \oplus H_2(2)W_1(2)_2 & & \oplus H_2(3)W_1(3)_3 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_0(2)W_0(0)_0 & \leftarrow & H_0(3)W_0(1)_1 & \leftarrow & H_0(4)W_0(2)_2 & \leftarrow & H_0(5)W_0(3)_3 \\
& \oplus H_0(2)W_1(1)_0 & & \oplus H_0(3)W_1(2)_1 & & \oplus H_0(4)W_1(3)_2 \\
\end{array}$$

The notation $H_{2l}(k')W_r(k)$ means the direct sum of terms $H_{2l}(\Delta') \otimes W_r(\Delta)$ running over all incident pairs $\Delta' \succ \Delta$ of dimensions $k'$ and $k$ respectively and equal sedentarity. The subscript indicates the grading index $m$, assuming $0$ sedentarity.

The $E^0$-term in the associated spectral sequence is

$$E^0_{m,r} = \text{Gr}^F_m K^{(p)}_r = \bigoplus_{l=0}^{\infty} \bigoplus_{\Delta' \succ \Delta} H_{2l}(\Delta') \otimes W_{r+s}(\Delta),$$

where $\Delta' \succ \Delta$ run over all incident pairs of dimensions $(p - 2l + m - s)$ and $(r - l + m)$, respectively, and $s$ is their sedentarity. We can replace $\Delta'$ in $H_{2l}(\Delta')$ by its parent $\Delta_0$, whose dimension is $(p - 2l + m)$. Note that the terms in the $l$-sum are zero unless $l \leq p$ (original bound in $A^{(p)}_\bullet(\Delta)$) and $l \leq m$ (follows from $r + s \leq r - l + m$).

We calculate the differential $d_0 = \partial_0 + \delta_0$. Pick an element in $\text{Gr}^F_m A^{(p)}_\bullet(\Delta)$ represented by $\beta \otimes w \in H_{2l}(\Delta') \otimes W_{r+l}(\Delta')$. Note that $\delta$ lowers $l$ by 1 and it does not change anything else. Hence, it always lowers the total degree by 1 and will appear only in the $E^1$ term. That is, $\delta_0$ vanishes and $\delta_1$ is the Gysin map (cf. the proof of Proposition [53]).
Let us analyze the effect of the horizontal map $\partial : \mathcal{A}^{(p)}_l(\Delta) \to \mathcal{A}^{(p)}_l(\Delta'')$. Note that $\partial$ lowers the dimension of $\Delta$ by 1. On the other hand, it can raise the sedentarity by 1 or it can lower $r$ by 1 in $F^\Delta_r$ filtration. In the latter case $\partial_0$ will be zero, and $\partial$ will only contribute to $\partial_1$ in the $E^1$ term. We give some details.

If $\Delta'' \prec^1 \Delta$, then consider the corresponding face $\Delta'' \prec^1 \Delta'$ with $H_2l(\Delta'') = H_2l(\Delta') = H_2l(\Delta_0')$ and the commutative diagram of the residue maps:

$$
\begin{array}{ccc}
\beta \otimes \pi(w) & \xleftarrow{\partial = \text{id} \otimes \pi} & \beta \otimes w \\
\downarrow & & \downarrow \\
\beta \otimes \text{res}_{\Delta'' \prec \Delta'}(\pi(w)) & \xleftarrow{\partial_0} & \beta \otimes \text{res}_{\Delta'' \prec \Delta}(w)
\end{array}
$$

Since $\text{res}_{\Delta'' \prec \Delta'}(\pi(w)) = \text{res}_{\Delta'' \prec \Delta'}(w) = \pi(\text{res}_{\Delta'' \prec \Delta}(w))$, we see that on the associated graded level $\partial$ acts $H_2l(\Delta_0')$-linearly and as the projection $\pi$ on $W_{r+s}(\Delta)$:

$$
\partial_0 = \text{id} \otimes \pi : H_2l(\Delta_0') \otimes W_{r+s}(\Delta) \to H_2l(\Delta_0') \otimes W_{r+s}(\Delta'').
$$

If $\Delta'' \prec^1 \Delta$, the commutative diagram of the residue maps

$$
\begin{array}{ccc}
\beta \otimes w & \xleftarrow{\partial = \text{id}} & \beta \otimes w \\
\text{id} \otimes \text{res} & & \text{id} \otimes \text{res} \\
\beta \otimes \text{res}_{\Delta'' \prec \Delta'}(w) & \xleftarrow{\partial_0} & \beta \otimes \text{res}_{\Delta'' \prec \Delta}(w)
\end{array}
$$

shows that $\partial_0$ acts as before by the identity on $H_2l(\Delta')$, and by the residue map on $W_{r+s}(\Delta)$:

$$
\partial_0 = \text{id} \otimes \text{res}_{\Delta'' \prec \Delta'} : H_2l(\Delta_0') \otimes W_{r+s}(\Delta) \to H_2l(\Delta_0') \otimes W_{r+s-1}(\Delta'').
$$

**Proposition 56.** The $E^1$ term of spectral sequence associated to the filtration $F$ on $K^{(p)}_\bullet$ is a single 0-th row which coincides with the Steenbrink-Illusie complex [15].

To calculate the homology of the complex $(\Gr^F_m K^{(p)}_\bullet, \partial_0)$ we need a bit of Koszul-type linear algebra. Let $\Delta_0'$ be a mobile face of $X$. Consider the residue complex

$$
\mathbb{W}_\bullet(\Delta_0') = \bigoplus W_{\bullet+s}(\Delta),
$$

where $\Delta$ runs over all faces of $\Delta_0'$ (including sedentary ones, here $s$ is the sedentarity of $\Delta$). The differential is the residue map

$$
\text{res}_{\Delta'' \prec \Delta'} : W_r(\Delta) \to W_{r-1+s}(\Delta'')
$$

for pairs $\Delta \prec^1 \Delta''$. The residue complex splits into the direct sum of complexes $\mathbb{W}^q_\bullet(\Delta_0')$ according to the degree $q = \dim(\Delta) + \text{sed}(\Delta) - r$ of its terms $W_r(\Delta)$ (the residue map preserves the $q$-degree).

**Lemma 57.** The residue complex $\mathbb{W}_\bullet(\Delta_0')$ is exact if $\Delta_0'$ is an infinite face. If $\Delta_0'$ is finite, then $\mathbb{W}^q_\bullet(\Delta_0')$ has homology $H_0 = \mathbb{Q}$ and $H_{>0} = 0$ for each $q = 0, \ldots, \dim \Delta_0'$. 
Proof. Recall that any face of $X$ is the product $\Delta' = \Delta \times \square$, where $\Delta$ is the unimodular simplex, and $\square$ is the unimodular cone (compactified by the sedendary faces). Then, we can write the residue complex as the tensor product (with the usual alternating sign convention) $W_*(\Delta_0') = W_*(\Delta) \otimes W_*(\square)$. The residue complex $W_*(\square)$ is, in turn, the dim($\square$)-tensor power of the three-term residue complex $W_*(\square^1)$ for 1-dimensional infinite face $\square^1$:  
$$W_0(\text{vertex}_1) \leftarrow W_0(\text{ray}) \oplus W_0(\text{vertex}_0) \leftarrow W_1(\text{ray}),$$  
which is acyclic. Thus, we are reduced to the case when $\Delta_0' = \Delta$ is a simplex.

We calculate the homology of $W_*(\Delta)$ by induction on dimension of $\Delta$. The case when $\Delta$ is a point is clear: $H_0 = \mathbb{Q}$ in degree $q = 0$. For the induction step pick $\nu$, a vertex of $\Delta$, and let $\square_\nu$ be the relative cone of $\Delta$ at $\nu$. Then, we have a short exact sequence of complexes

$$0 \rightarrow W_*(\Delta \setminus \nu) \rightarrow W_*(\Delta) \rightarrow W_*(\square_\nu) \rightarrow 0,$$

where $W_*(\Delta \setminus \nu)$ is the residue complex of the face $(\Delta \setminus \nu) \subset \Delta$, and the quotient $W_*(\square_\nu)$ is the residue complex of the cone $\square_\nu$ (without the sedendary faces). The residue complex $W_*(\square_\nu)$ is the dim($\square_\nu$)-tensor power of one-dimensional two-term complex

$$W_0(\text{ray}) \oplus W_0(\text{vertex}) \leftarrow W_1(\text{ray}),$$

which has homology $H_0 = \mathbb{Q}$ in degree $q = 1$ and all other vanish. Thus, $W_*(\square)$ has homology $H_0 = \mathbb{Q}$ in degree $q = \dim(\square_\nu) = \dim(\Delta)$ and all other vanish.

Note that all maps here respect $q$-grading. Then, the induced long exact sequence in homology (which becomes the short exact sequence in $H_0$'s) does the induction step. □

Remark. The proof works over $\mathbb{Z}$. The unimodularity of $\Delta_0'$ is irrelevant over $\mathbb{Q}$.

Proof of Proposition 56. The map $\partial_0$ acts linearly with respect to the $H_2l(\Delta_0')$ factors (after identifying the groups $H_2l(\Delta') = H_2l(\Delta_0')$ within the same family). Thus, we can decompose

$$E^0_{m,*} = Gr^F_m K^{(p)}_{-m} = \bigoplus_{l=0}^\infty \bigoplus H_2l(\Delta_0') \otimes W_{*+s}(\Delta)$$

into direct sum of complexes $H_2l(\Delta_0') \otimes \mathbb{W}_{m-l}(\Delta_0')$ indexed by $(p - 2l + m)$-dimensional mobile faces $\Delta_0'$ of $X$. By Lemma 57 each $\mathbb{W}_{m-l}(\Delta_0')$ is acyclic unless $\Delta_0'$ is a finite face, in which case it is a resolution of $\mathbb{Q}$. Thus, the $E^1$ term consists of the single row $E^1_{m,0}$ which coincides with the Steenbrink-Illusie complex (15)

$$\tilde{E}^1_{m,p} = \bigoplus_{l=0}^{\min(m,p)} H_2l(\Delta_0').$$

It only remains to check that the differential $d_1 = \delta_1 + \partial_1$ agrees with $\text{Gys} + \iota_*$. In the proof of Proposition 53 we already identified $\delta_1$ with the Gysin map: $\text{Gys} : H_2l(Z^{(k)}) \rightarrow H_2l-2(Z^{(k+1)})$. Now we analyze what remains of the horizontal map $\partial : K^{(p)} \rightarrow K^{(p)}_{-1}$ on the $E^1$ level.

Let $\Delta'$ be a finite $(p - 2l - m)$-dimensional mobile face of $X$, and let $\nu \in \Delta'$ be its vertex. Let $\bar{\beta} \in H_2l(\Delta')$ be an element of $E^1_{m,0}$. We want to compute the image $\partial_1\bar{\beta}$ in $H_2l(\Delta'')$ for $\Delta'' = (\Delta' \setminus \nu)$.
First, we represent $\bar{\beta}$ by an element
\[ \beta \otimes 1 \in H_2(\Delta') \otimes W_0(\Delta) \subset E_{m,0}^0 = \text{Gr}_{m}^{F} K_{m}^{p}, \]
where $\Delta$ is an $(m-l)$-dimensional face of $\Delta'$ containing the vertex $\nu$. Next, a lift of $\beta \otimes 1$ to $A_l^{(p)}(\Delta) \subset K_{m-l}^{p}(\Delta') \otimes W_0(\Delta)$ can be represented by $\beta \otimes w \in H_2(\Delta') \otimes W_{p-l}(\Delta')$, where $w$ is a relative volume polyvector for the pair $\Delta' \succ \Delta$. Moreover, we can take this volume polyvector to be in the form $w = \iota(u)$ where $u \in W_{p-l}(\Delta'')$. Then, $\text{res}_{\Delta'' \succ \Delta''}(u) = 1 \in W_0(\Delta'')$, where $\Delta'' = (\Delta \setminus \nu)$.

On the other hand, $\partial(\beta \otimes w) = \beta \otimes w \sim \iota_*(\beta) \otimes u$ in $A_l^{(p)}(\Delta'')$. In $E_{m-1,0}^0$ (i.e., after applying the residue map) this becomes $\iota_*(\beta) \otimes 1 \in H_2(\Delta'') \otimes W_0(\Delta'')$. Thus, $\partial_1$ is the pushforward map $\iota_*$.

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