Coulomb’s law in maximally symmetric spaces

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Abstract

We study the modifications to the Coulomb’s law when the background geometry is a \(n\)-dimensional maximally symmetric space, by using of the \(n\)-dimensional version of the Gauss’ theorem. It is shown that some extra terms are added to the usual expression of the Coulomb electric field due to the curvature of the background space. Also, we consider the problem of existence of magnetic monopoles in such spaces and present analytical expressions for the corresponding magnetic fields and vector potentials. We show that the quantization rule of the magnetic charges (if they exist) would be applicable to our study as well.

Keywords: Coulomb’s law; maximally symmetric space

1 Introduction

Almost all of the physical laws presented in the elementary text books are formulated in a flat Euclidean space. Then, one of the interesting questions which can be arise is that how these laws may be modified if the corresponding physical systems live in a curved space. Maybe the simplest example which can show the effects of the curvature on the behavior of a physical system is the issue of the parallel displacement of a vector in a curved space. As is well-known in a flat space two vectors with the same magnitude which are located in different points are same if they are parallel. This means that in such a space we can parallel transport a vector to everywhere without any change to its magnitude and direction. On the other hand, parallel transporting a vector from one point to another in a non-flat space yields a result that depends on the path taken and specially when the path is a closed loop the vector finally may not coincide with itself. This effect is due to the curvature and therefore the dynamical laws of a point particle (which are the relations between its dynamical vectors) moving in a curved background may show some new aspects which are absent in a flat space, see for example [1] and the references therein.

Here, we would like to look at the electrostatic force law between charged particles. As we know, this interaction is governed by the Coulomb’s law according to which the force between point charges obeys an inverse square rule. Now, the question is that how this law may be corrected if we consider it in a curved space. The first attempts to answer this question return to long ago in the works of E. Fermi [2], where the electric field of a point charge held at rest within a weak gravitational field has been discussed. These efforts were further developed by others via considering the electric phenomena in gravitational fields [3].

In this letter we are going to obtain the modified version of the Coulomb’s law in a \(n\)-dimensional maximally symmetric curved space. For this purpose we shall begin with the \(n\)-dimensional version of the Gauss’ theorem and see how the curvature effects show themselves in the resulting expression for the electric field and its corresponding scalar potential. We show that in the vicinity of charges

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where the effect of curvature is negligible, we recover usual Coulomb’s law. Also, we consider the
problem of existence of the magnetic monopoles in our formalism. As is well-known from the Dirac
theory about these objects [4], an important features of them is that their existence would offer an
quantization rule for the magnitude of electric charges. We shall see that this rule will be preserved
when the background geometry is a maximally symmetric curved space.

2 Geometrical set-up

In this section we briefly review the concept of the maximally symmetric spaces used in the present
work which originally appeared in [5]. As is well-known, the symmetry transformations (isometries)
of a given manifold, i.e., the diffeomorphisms that leave the metric tensor invariant, are characterized
by the existence of the so-called Killing vector fields and the number of such isometries are the
number of independent Killing vectors fields. Standard calculations in differential geometry show
that there can be at most \( \frac{1}{2} n(n+1) \) different independent Killing vector fields in a space of dimension
\( n \). As an example, it is easy to show that the Minkowski space admits 10 Killing vectors, which
is the maximum number possible for a 4-dimensional space-time. Also, in a cosmological situation,
the standard Friedmann-Robertson-Walker models would be maximally symmetric in the sense that
they would admit all the \( \frac{1}{2} n(n+1) \) different Killing vector fields. In general, any \( n \)-dimensional
maximally symmetric manifold should be in the form of \( S^n \) (the \( n \)-dimensional sphere), \( P^n \) (the
\( n \)-dimensional projective space), \( E^n \) (the \( n \)-dimensional Euclidean space) or \( H^n \) (the \( n \)-dimensional
hyperbolic space) [6]. It is worth to note that all of these spaces are homogeneous and isotropic
about all points. This is indeed a generic property of the maximally symmetric spaces and one can
show that a given space is maximally symmetric if and only if it is homogeneous and isotropic. The
study of the curvature properties of such spaces shows that their Riemann-Christoffel tensor takes
the form

\[
R_{ijkl} = K (g_{il}g_{jk} - g_{ik}g_{jl}),
\]

where \( K \) is a constant called the curvature index. Spaces with
this curvature tensor are called spaces of constant curvature and their geometries are quite different
depending on whether the curvature index is positive, negative or zero.

The construction of a \( n \)-dimensional maximally symmetric space may be made by embedding
it in a larger \( (n+1) \)-dimensional flat space. In other word, a maximally symmetric space can be
viewed upon as a hypersurface in a flat \( (n+1) \)-dimensional space. To do this, consider a flat \( (n+1)\)
dimensional space with line element

\[
ds^2 = g_{\mu\nu}dX^\mu dX^\nu, \quad \mu, \nu = 1, 2, \ldots, n+1.
\]

By introducing the constant \( n \times n \) matrix \( c_{ij} \) and also a constant number \( K \), the above line element
may be rewrite as follows

\[
ds^2 = c_{ij} dx^i dx^j + K^{-1} dw^2, \quad i, j = 1, 2, \ldots, n,
\]

in which we used the symbol \( w \) for the coordinate \( X^{n+1} \). Now, let us look at our desired maximally
symmetric space and define it as the following hypersurface

\[
K c_{ij} x^i x^j + w^2 = 1.
\]

Depending on the sign of \( K \) this hypersurface represents the surface of a sphere (if \( K > 0 \)) or a
pseudosphere (if \( K < 0 \)). Upon substitution (3) into relation (2) we obtain

\[
ds^2 = c_{ij} dx^i dx^j + \frac{K(c_{ij} dx^i dx^j)^2}{(1 - K c_{mn} dx^m dx^n)},
\]

which means that the metric functions induced on the hypersurface can then be written in terms of
\( c_{ij} \) and \( K \) as
\[ g_{ij} = c_{ij} + \frac{K(c_{ik}c_{jk}x^i x^k)}{(1 - K c_{mn} dx^m dx^n)}. \] 

Now, it is straightforward to compute the corresponding Riemann-Christoffel tensor with result

\[ R_{ijkl} = K (g_{il} g_{jk} - g_{ik} g_{jl}), \]

which has the desired form for that the underlying geometry to be maximally symmetric. Note that contracting once yields

\[ R_{ij} = K (n - 1) g_{ij}, \]

where \( R_{ij} \) is the Ricci tensor corresponds to the metric \( g_{ij} \). We see that the Ricci tensor is proportional to the metric. Contracting once more gives us the following relation between the Ricci scalar \( R \) and the curvature index

\[ K = \frac{R}{n(n - 1)}. \]

Before going any further, let us consider the structure of an \( n \)-dimensional maximally symmetric space in a special case in which \( c_{ij} = K^{-1} \delta_{ij} \) if \( K \neq 0 \) and \( c_{ij} = \delta_{ij} \) if \( K = 0 \). For this choice the metric (4) takes the form

\[ ds^2 = |K|^{-1} \left( dx_i dx^i + \frac{K|K|^{-1}}{1 - K|K|^{-1}(x_i x^i)} (x_i dx^i)^2 \right), \]

for \( K \neq 0 \) and

\[ ds^2 = dx_i dx^i, \]

for \( K = 0 \). To simplify the above form of the metric, let us introduce a new set of variables

\[ \begin{align*}
  x_1 &= R \cos \theta_1, \\
  x_2 &= R \sin \theta_1 \cos \theta_2, \\
  &\quad \ldots \ldots \ldots \ldots \ldots \ldots , \\
  x_{n-1} &= R \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-2} \cos \theta_{n-1}, \\
  x_n &= R \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-2} \sin \theta_{n-1},
\end{align*} \]

where \( R^2 = x_i x^i \). In terms of these new variables the line element (9) takes the form

\[ ds^2 = |K|^{-1} \left( \frac{dR^2}{1 - kR^2} + R^2 d\Omega_{n-1}^2 \right), \]

where \( k = \frac{K}{|K|} \) and

\[ d\Omega_{n-1}^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \ldots + \sin^2 \theta_1 \sin^2 \theta_2 \ldots \sin^2 \theta_{n-2} d\theta_{n-1}^2. \]

Notice that for \( K \neq 0 \) the parameter \( k \) takes the values \( \pm 1 \) and for \( K = 0 \) we identify \( k = 0 \). In the following we investigate in more detail the geometry of the above space for three cases \( k = 0, +1 \) and \( -1 \).

- Case \( k = 0 \):
  
  In this case from (10) we have
\[ ds^2 = dR^2 + R^2 d\Omega_{n-1}^2 = dx_1^2 + dx_2^2 + \ldots + dx_n^2, \] \[ (14) \]

which represents a \( n \)-dimensional Euclidean space and is called flat space.

- Case \( k = +1 \):

  Taking \( k = +1 \) in (12) yields

  \[ ds^2 = \frac{dr'^2}{1 - |K|r'^2} + r'^2 d\Omega_{n-1}^2, \]
  \[ (15) \]

  in which we have defined \( r' \) by \( R = r'|K|^{1/2} \). Since the coefficient of \( dr'^2 \) becomes singular as \( r' \rightarrow |K|^{-1/2} \), we introduce a new coordinate \( r = \int \frac{dr'}{\sqrt{1 - |K|r'^2}} \) which transforms (15) into the form

  \[ ds^2 = dr^2 + |K|^{-1} \sin^2 \left( \sqrt{|K|r} \right) d\Omega_{n-1}^2. \]
  \[ (16) \]

  This space has a compact topology and we call it as a closed or spherical space \( S^n \). Now, it is clear that the \( (n - 1) \)-hypersurfaces \( r = \text{cons.} \) are \( (n - 1) \)-spheres \( S^{n-1} \) with line element

  \[ d\sigma^2 = |K|^{-1} \sin^2 \left( \sqrt{|K|r} \right) d\Omega_{n-1}^2. \]
  \[ (17) \]

- Case \( k = -1 \):

  Following the same steps as in the previous case but now with the variable \( r = \int \frac{dr'}{\sqrt{1 + |K|r'^2}} \) we are led to the metric of the \( n \)-dimensional hyperbolic space as

  \[ ds^2 = dr^2 + |K|^{-1} \sinh^2 \left( \sqrt{|K|r} \right) d\Omega_{n-1}^2, \]
  \[ (18) \]

  which has obviously an open topology. Again the \( (n - 1) \)-hypersurfaces are specified by

  \[ d\sigma^2 = |K|^{-1} \sinh^2 \left( \sqrt{|K|r} \right) d\Omega_{n-1}^2. \]
  \[ (19) \]

3 Coulomb’s law

As is well-known the interaction between charged particles can be described by the Coulomb’s law according to which any two pair of such particles exert a force on each other. This force is a central force which is proportional to the magnitude of each charge and varies inversely as the square of the mutual distance. Furthermore, this is an attractive force if the charges are unlike and the like charges repel each other. According to these statements the force exerted by point charge \( q \) located at \( r' \) on charge \( q_0 \) located at \( r \) can be written as

\[ \mathbf{E}(r) = \mathbf{F} = \frac{q}{4\pi\epsilon_0 r^2} \hat{r}, \]

where \( \hat{r} \) is the unit vector directed from the charge \( q \) to the second charge \( q_0 \) and \( r = |r - r'| \). Here, \( \mathbf{E} \) is is the electric force per unit charge and usually is called the electric field strength. Now, assume that a field is set up by a system of point charges, then according to the superposition principle the total electric field strength equals the vector sum of the electric fields due to the individual charges. One of the important features of an electrostatic field strength is the equality to zero of its curl at every point of the field, that is, \( \nabla \times \mathbf{E}(r) = 0 \). Therefore, the strength of an electrostatic field can be
represented as the gradient of a scalar function \( \phi(r) : \mathbf{E}(r) = -\nabla \phi(r) \). The function \( \phi(r) \) is known as the potential of an electrostatic field and for a point charge \( q \) is equal to \( \phi(r) = \frac{q}{4\pi\epsilon_0 r} \).

From the general courses of physics we are familiar with Gauss’ theorem which for a field in vacuum can be worded as follows: the flux of the vector field \( \mathbf{E} \) through a closed surface is proportional to the algebraic sum of the charges confined within the surface, that is

\[
\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{q}{\epsilon_0}.
\] (21)

In electrostatics coulomb’s law may be deduced from Gauss’ theorem and vice versa. In this section we are going to obtain a modified version of the Coulomb’s law when the background geometry is a \( n \)-dimensional maximally symmetric space. To do this, we shall use the \( n \)-dimensional version of the Gauss’ theorem [7]. Suppose that the point charge \( q \) is located at the origin of the coordinate system in a \( n \)-dimensional maximally symmetric space, then according to the \( n \)-dimensional Gauss’ theorem the flux through the hypersurface \( S_{n-1} \) is given by

\[
\oint_{S_{n-1}} \mathbf{E}_n \cdot dS_{n-1} = \frac{q}{\epsilon_n},
\] (22)

where \( \epsilon_n \) is the permittivity of the \( n \)-dimensional free space and dimensionally may be related with its three dimensional counterpart \( \epsilon_0 \) as \([\epsilon_n] = [\epsilon_0]/[L^{n-3}]\). Since a maximally symmetric space is homogeneous and isotropic we can assume that \( \mathbf{E}_n = \mathbf{E}_n(r) \). Now, using (22) we have

\[
E_n(r) = \frac{q}{\epsilon_n \oint dS_{n-1}}.
\] (23)

What remain is to compute the area of the hypersurface over which we would like to take the above integral. Using the equality

\[
S_{n-1}(EveryRadius) = (Radius)^{n-1}S_{n-1}(Radius = 1),
\] (24)

and

\[
S_{n-1}(Radius = 1) = \int_0^\pi \sin^{n-2}(n) \sin^{n-3}(n) \cdots \int_0^\pi \sin^{n-2}(n) \int_0^{2\pi} d(\theta_{n-1}) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})},
\] (25)

we obtain the following relations for the area of a \((n-1)\)-hypersurface

\[
S_{n-1} = \begin{cases}
\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \left[ |K|^{-\frac{1}{2}} \sin(|K|r) \right]^{n-1}, & \text{spherical space,} \\
\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \left[ |K|^{-\frac{1}{2}} \sinh(|K|r) \right]^{n-1}, & \text{hyperbolic space.}
\end{cases}
\] (26)

With these results at hand, we can compute the electrostatic field strength related to a point charge placed at the origin of a \( n \)-dimensional maximally symmetric space as

\[
E_n(r) = \begin{cases}
\frac{q\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}\epsilon_n \left[ |K|^{-\frac{1}{2}} \sin(|K|r) \right]^{n-1}}, & \text{spherical space,} \\
\frac{q\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}\epsilon_n \left[ |K|^{-\frac{1}{2}} \sinh(|K|r) \right]^{n}}, & \text{hyperbolic space.}
\end{cases}
\] (27)
Let us focus on expressions (27) which represents the electric field in a maximally symmetric curved space. Their expansions in terms of the $r$ powers results

$$E_n(r) = \frac{\Gamma(n)q}{2\pi^n \epsilon_n r^{n-1}} \left[ 1 \pm \frac{(n-1)|K|}{6} + \frac{(5n-3)(n-1)|K|^2}{360} r^4 + \ldots \right],$$

(28)

where the positive and negative signs correspond to the spherical and hyperbolic spaces respectively. The first term on the right hand side, is the Coulomb’s law in a flat $n$-dimensional space (see [8]) and the other terms show the effects of the curvature. In three dimension the above relation takes the following more familiar form

$$E_3(r) = \frac{q}{4\pi \epsilon_0 r^2} \pm \frac{|K|q}{12 \pi \epsilon_0} + \frac{|K|^2 qr^2}{60 \pi \epsilon_0} + \ldots$$

(29)

This is the modified form of the Coulomb’s law in three-dimensional maximally symmetric spaces which its $r = \text{cons.}$ hypersurfaces have the metric $d\Omega^2 = a^2(d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2)$. Since the curvature index for this hypersurface is $K = 1/a^2$, we get

$$E_3(r) = \frac{q}{4\pi \epsilon_0 r^2} \pm \frac{qr}{12 \pi \epsilon_0 a^2} + \frac{qr^2}{60 \pi \epsilon_0 a^4} + \ldots$$

(30)

As we have mentioned above, the first term is nothing but the usual Coulomb’s law in flat space. The second term although depends on the curvature, does not depend on the distance. It seems that this term shows a global effect of the curvature. Finally, each of the remaining terms denote the local effects of the curvature on the Coulomb’s law. Now, let us to deal with the scalar potential for the electric field calculated above. From the standard definition

$$\phi_n(r) = -\int E_n \cdot dr,$$

(31)

and by using of the relations (27) we arrive at the solution

$$\phi_n(r) = \begin{cases} 
\frac{\Gamma(n)q}{2\pi^n \epsilon_n |K|^{\frac{1}{2}}} \cos(\sqrt{|K|}r) \ 2F_1\left(\frac{1}{2}, \frac{n}{2}; \frac{3}{2}; \cos^2(\sqrt{|K|}r)\right) + C_+, & \text{spherical space,} \\
\frac{(-1)^n \Gamma(n)q}{2\pi^n \epsilon_n |K|^{\frac{1}{2}}} \cosh(\sqrt{|K|}r) \ 2F_1\left(\frac{1}{2}, \frac{n}{2}; \frac{3}{2}; \cosh^2(\sqrt{|K|}r)\right) + C_-, & \text{hyperbolic space,}
\end{cases}$$

(32)

where $2F_1(a, b, c; z)$ is hypergeometric function and $C_\pm$ are some constants. In the case of $n = 3$ where $K = 1/a^2$, the scalar potential for a three dimensional maximally symmetric space takes the form

$$\phi_3(r) = \begin{cases} 
\frac{q}{4\pi \epsilon_0 a} \cot(\frac{r}{a}) + C_+, & \text{spherical space,} \\
\frac{q}{4\pi \epsilon_0 a} \coth(\frac{r}{a}) + C_-, & \text{hyperbolic space.}
\end{cases}$$

(33)

As before the above relation can be expanded in the following way

$$\phi_3(r) = C_\pm + \frac{q}{4\pi \epsilon_0 r} \pm \frac{qr}{12 \pi \epsilon_0 a^2} - \frac{qr^3}{180 \pi \epsilon_0 a^4} + \ldots,$$

(34)

where the upper and lower signs correspond to the spherical and hyperbolic spaces respectively. The discussions on the comparison between different terms of this equation and their corresponding effects are the same as previous, namely similar discussion based on (30) would be applicable to this case as well.
4 Magnetic monopole

In this section we shall deal with the problem of magnetic monopole in a curved maximally symmetric space. This issue is one of the most interesting predictions of quantum mechanics which so far had not been established experimentally. As is well-known from elementary courses in electrodynamics, although there is a strong symmetry between electric field $E$ and magnetic field $B$ in Maxwell equations, but unlike the electric charges their magnetic counterparts known as magnetic monopoles do not enter these equations. Indeed, the source of all observable magnetic fields in the nature is either moving electric charges or magnetic dipoles and there is no evidence for the existence of magnetic monopoles. This means that instead of the relation $\nabla \cdot B = \rho_m$ which is the magnetic version of $\nabla \cdot E = \rho_e$, in the usual way of writing Maxwell equations $\nabla \cdot B$ is equal to zero. As a remark, we would like to emphasize that quantum mechanics does not predict that the magnetic monopole should exist but in a clear way implies that if it exists the magnitude of the magnetic charges should obey a quantization rule [9]. Here, we are going to show the validity of this statement in a curved maximally symmetric background. So, let us suppose that there exist magnetic charge density $\rho_m$. The Maxwell equation would then be $\nabla \cdot B = \rho_m$. Therefore, if a point magnetic charge $e_m$ is placed at the origin of a coordinate system, the magnetic counterpart of the Gauss’ law should be modified as

$$\oint_{S_{n-1}} B_n \cdot dS_{n-1} = e_m. \quad (35)$$

Following the same steps as in the previous section, this equation lead us the magnetic field strength due the magnetic charge $e_m$ as

$$B_n(r) = \begin{cases} \frac{\Gamma(\frac{n}{2})e_m}{2\pi^{\frac{n}{2}} \left[ |K|^{-\frac{1}{2}} \sin(\sqrt{|K|}r) \right]^{n-1}}, & \text{spherical space,} \\ \frac{\Gamma(\frac{n}{2})e_m}{2\pi^{\frac{n}{2}} \left[ |K|^{-\frac{1}{2}} \sinh(\sqrt{|K|}r) \right]^{n-1}}, & \text{hyperbolic space.} \end{cases} \quad (36)$$

Now, an essential question is that how can the magnetic fields (36) be represented by a vector potential $A$ such that $\mathbf{B} = \nabla \times \mathbf{A}$. To deal with this question let us restrict ourselves to the special case $n = 3$ from now on for which the relations (36) take the form

$$B_3(r) = \frac{e_m}{4\pi |K|^{-1} \sin^2 \left( \sqrt{|K|} r \right)} e_r, \quad (37)$$

and

$$B_3(r) = \frac{e_m}{4\pi |K|^{-1} \sinh^2 \left( \sqrt{|K|} r \right)} e_r, \quad (38)$$

where $e_r$ is the local unit vector in $r$-direction. On the other hand, bearing in the mind the general relation

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_3} \left[ \partial_2 (h_3 A_3) - \partial_3 (h_2 A_2) \right] e_1 + \frac{1}{h_1 h_3} \left[ \partial_3 (h_1 A_1) - \partial_1 (h_3 A_3) \right] e_2 + \frac{1}{h_1 h_2} \left[ \partial_1 (h_2 A_2) - \partial_2 (h_1 A_1) \right] e_3, \quad (39)$$

for the curl of a vector field in a curvilinear metric space with scale factors $h_1$, $h_2$ and $h_3$, we can write the following expressions for $\nabla \times \mathbf{A}$ in a three dimensional maximally symmetric space

\[ \text{---} \]
\[ \nabla \times \mathbf{A} = \frac{1}{|K|^{-\frac{1}{2}\sin(\sqrt{|K|\theta})}} \left[ \partial_\theta (\sin \theta A_\theta) - \partial_\varphi A_\varphi \right] e_r \]

\[ + \frac{1}{|K|^{-\frac{1}{2}\sin(\sqrt{|K|\theta})}} \left[ \partial_\varphi A_\varphi - \partial_r \left( |K|^{-\frac{1}{2}\sin(\sqrt{|K|\theta})} \sin \theta A_\varphi \right) \right] e_\theta \]

\[ + \frac{1}{|K|^{-\frac{1}{2}\sin(\sqrt{|K|\theta})}} \left[ \partial_r \left( |K|^{-\frac{1}{2}\sin(\sqrt{|K|\theta})} A_\varphi \right) - \partial_\theta A_\varphi \right] e_\varphi, \quad (40) \]

for spherical space and

\[ \nabla \times \mathbf{A} = \frac{1}{|K|^{-\frac{1}{2}\sinh(\sqrt{|K|\theta})}} \left[ \partial_\theta (\sin \theta A_\theta) - \partial_\varphi A_\varphi \right] e_r \]

\[ + \frac{1}{|K|^{-\frac{1}{2}\sinh(\sqrt{|K|\theta})}} \left[ \partial_\varphi A_\varphi - \partial_r \left( |K|^{-\frac{1}{2}\sinh(\sqrt{|K|\theta})} \sin \theta A_\varphi \right) \right] e_\theta \]

\[ + \frac{1}{|K|^{-\frac{1}{2}\sinh(\sqrt{|K|\theta})}} \left[ \partial_r \left( |K|^{-\frac{1}{2}\sinh(\sqrt{|K|\theta})} A_\varphi \right) - \partial_\theta A_\varphi \right] e_\varphi, \quad (41) \]

for hyperbolic space. Since the magnetic fields (37) and (38) only have the \( r \) component, we suppose that their corresponding vector potentials only have the azimuthal component and suggest the following ansatz

\[ A = \frac{e_m}{4\pi|K|^{-\frac{1}{2}\sin(\sqrt{|K|\theta})}} \frac{1 - \cos \theta}{\sin \theta} e_\varphi, \quad (42) \]

for spherical space and

\[ A = \frac{e_m}{4\pi|K|^{-\frac{1}{2}\sinh(\sqrt{|K|\theta})}} \frac{1 - \cos \theta}{\sin \theta} e_\varphi, \quad (43) \]

for hyperbolic space. A problem related to the above potentials is that they are singular at \( \theta = \pi \). Following [9], to pass this problem, instead of an expression for the vector potential which is valid everywhere we may define two vector potentials \( A^{(I)} \) and \( A^{(II)} \) each of which is valid in one of the regions \( \theta < \pi - \delta \) and \( \theta > \delta \). Therefore, we modify the equations (42) and (43) as

\[ \mathbf{A} = \left\{ \begin{array}{ll}
A^{(I)} &= \frac{e_m}{4\pi|K|^{-\frac{1}{2}\sin(\sqrt{|K|\theta})}} \frac{1 - \cos \theta}{\sin \theta} e_\varphi, \quad \theta < \pi - \delta \\
A^{(II)} &= -\frac{e_m}{4\pi|K|^{-\frac{1}{2}\sin(\sqrt{|K|\theta})}} \frac{1 + \cos \theta}{\sin \theta} e_\varphi, \quad \theta > \delta,
\end{array} \right. \quad (44) \]

for spherical space and

\[ \mathbf{A} = \left\{ \begin{array}{ll}
A^{(I)} &= \frac{e_m}{4\pi|K|^{-\frac{1}{2}\sinh(\sqrt{|K|\theta})}} \frac{1 - \cos \theta}{\sin \theta} e_\varphi, \quad \theta < \pi - \delta \\
A^{(II)} &= -\frac{e_m}{4\pi|K|^{-\frac{1}{2}\sinh(\sqrt{|K|\theta})}} \frac{1 + \cos \theta}{\sin \theta} e_\varphi, \quad \theta > \delta,
\end{array} \right. \quad (45) \]

for hyperbolic space. It is seen that the potentials \( A^{(I)} \) and \( A^{(II)} \) together yield a correct expression for the magnetic field which is valid everywhere. On the other hand, since in the overlap region
δ < θ < π − δ, these two potentials result the same magnetic field, a gauge transformation of the kind $A^{(II)} \rightarrow A^{(I)} + \nabla \Lambda$ may connect them in this region [10]. Noting that

$$A^{(II)} - A^{(I)} = \begin{cases} -\frac{e_m}{4\pi|K|} \frac{e_m}{2 \sin (\sqrt{|K|}|r|)} \sin \theta e_\varphi, & \text{spherical space} \\ -\frac{e_m}{4\pi|K|} \frac{e_m}{2 \sinh (\sqrt{|K|}|r|)} \sin \theta e_\varphi, & \text{hyperbolic space}, \end{cases}$$

(46)

and

$$\nabla \Lambda = \begin{cases} (\partial_r \Lambda_r) e_r + \frac{1}{|K|} \frac{1}{2 \sin (\sqrt{|K|}|r|)} (\partial_\theta \Lambda_\theta) e_\theta + \frac{1}{|K|} \frac{1}{2 \sinh (\sqrt{|K|}|r|)} \sin \theta (\partial_\varphi \Lambda_\varphi) e_\varphi, & \text{spherical space}, \\ (\partial_r \Lambda_r) e_r + \frac{1}{|K|} \frac{1}{2 \sinh (\sqrt{|K|}|r|)} (\partial_\theta \Lambda_\theta) e_\theta + \frac{1}{|K|} \frac{1}{2 \sinh (\sqrt{|K|}|r|)} \sin \theta (\partial_\varphi \Lambda_\varphi) e_\varphi, & \text{hyperbolic space}, \end{cases}$$

(47)

a simple calculation shows that

$$\Lambda = -\frac{e_m}{2\pi} \varphi.$$  

(48)

Now, consider a charged particle of charge $e$ (say an electron) moving in the magnetic field (37) or (38). Since in the overlap region we can use either $A^{(I)}$ or $A^{(II)}$ the corresponding wave functions obey a phase transformation of the kind

$$\Psi^{(II)} = \exp \left( \frac{ie\Lambda}{\hbar c} \right) \Psi^{(I)}$$

(49)

where $\hbar$ and $c$ are the Planck constant and speed of light respectively. If we demand that $\Psi$ should be a single-valued function of $\varphi$, then we are led to

$$e_m = \frac{2\hbar c}{e} n, \quad n = 0, \pm 1, \pm 2, \ldots$$

(50)

which shows the quantization rule of the magnetic charges in a curved maximally symmetric geometry is valid as well.

## 5 Conclusions

In this letter we have studied the modifications to the Coulomb’s law when the electric charges are living in a $n$-dimensional maximally symmetric curved space. We obtained the general form for the metric of such spaces and showed that they may be divided into three classes which are called flat, spherical and hyperbolic spaces. Then, by using of the $n$-dimensional version of the Gauss’ theorem we found analytic expressions for the electric field intensity and scalar potential of a point charge placed at the origin of the coordinate system. Our analysis showed that the corresponding modifications have their origin in the curvature of the background space and the usual inverse square Coulomb field can be recovered in the limit where this curvature tends to zero. We then dealt with the problem of the magnetic monopole in these spaces and through calculating the corresponding magnetic field and vector potential we arrived at the same quantization rule for a magnetic charge as in their usual Dirac theory.
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