IN INVARIANCE OF REGULARITY CONDITIONS UNDER DEFINABLE, LOCALLY LIPSCHITZ, WEAKLY BI-LIPSCHITZ MAPPINGS

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Abstract. In this paper we describe the notion of a weak lipschitzianity of a mapping on a $C^q$ stratification. We also distinguish a class of regularity conditions that are in some sense invariant under definable, locally Lipschitz and weakly bi-Lipschitz homeomorphisms. This class includes the Whitney (B) condition and the Verdier condition.

Introduction

The first goal of this paper is to study the notion of a weakly Lipschitz mapping on a fixed $C^q$ stratification, which generalizes the notion of a Lipschitz function. Section 1 consists of the basic definitions and notation, while in Section 2 we introduce the main idea together with its geometrical interpretation and discuss its fundamental properties. Weakly Lipschitz mappings were used earlier under the name fonctions rugueuses by J.-L. Verdier [Ver], who considered them from a different point of view compared to the present paper.

The second goal of this paper is to distinguish a special class of regularity conditions (Section 3); namely, the conditions which are definable, generic, $C^q$ invariant and having the property of lifting with respect to locally Lipschitz mappings and the property of projection with respect to weakly Lipschitz mappings. In the definable case these properties make the regularity conditions in some sense invariant with respect to definable, locally Lipschitz, weakly bi-Lipschitz homeomorphisms (Theorem 3.15).

In Sections 4 and 5 we prove that the Whitney (B) condition and the Verdier condition belong to the distinguished class.

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1. Preliminaries

We denote by $|\cdot|$ the euclidean norm of $\mathbb{R}^n$, $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. In the whole paper $q$ denotes the class of smoothness of a mapping, so $q \in \mathbb{N}$ or $q \in \{\infty, \omega\}$, unless otherwise indicated. Also we denote by $G_{k,n}$ the Grassmann manifold of $k$ dimensional vector subspaces of $\mathbb{R}^n$. Then $\mathbb{P}_{n-1} = G_{1,n}$ is the real projective space of dimension $n-1$.

Definition 1.1. Let $v \in S^{n-1}$ and $W$ be a nonzero linear subspace of $\mathbb{R}^n$. We put

$$d(v, W) = \inf \{|\sin(v, w)| : w \in W \cap S^{n-1}\},$$

where $\sin(v, w)$ denotes the sine of the angle between the vectors $v$ and $w$. We also put $d(u, W) = 1$ if $W = \{0\}$.

Definition 1.2. For any $P \in G_{k,n}$ and $Q \in G_{l,n}$, we put

$$d(P, Q) = \sup \{d(\lambda; Q) : \lambda \in P \cap S^{n-1}\},$$

when $k > 0$, and $d(P, Q) = 0$, when $k = 0$.

Now we list some elementary properties of the function $d$, leaving the proof to the reader.

Proposition 1.3.

a) $0 \leq d(P, Q) \leq 1$.

b) $d(P, Q) = 0 \iff P \subset Q$.

c) $d(P, Q) = 1 \iff P \cap Q^\perp \neq \{0\}$ (or $d(P, Q) < 1 \iff P \cap Q^\perp = \{0\}$).

d) $d(P, R) \leq d(P, Q) + d(Q, R)$.

e) $d(P, Q) = d(Q, P)$ if $\dim P = \dim Q$.

f) $d$ is a metric on every $G_{k,n}$.

g) $d(\mathbb{R}v, Q) = |v - \pi_Q(v)| = \text{dist}(v, Q) = |\pi_Q(v)| = \sin \left( v, \frac{\pi_Q(v)}{|\pi_Q(v)|} \right) = d \left( \mathbb{R}v, \mathbb{R} \frac{\pi_Q(v)}{|\pi_Q(v)|} \right)$, where $\pi_Q$ denotes the orthogonal projection onto $Q$, $v$ is a unit vector, not orthogonal to $Q$ and $\text{dist}(v, Q) = \inf \{|v - w| : w \in Q\}$.

h) Consider the following metric on $\mathbb{P}_{n-1}$:

$$\tilde{d}(\mathbb{R}v, \mathbb{R}w) = \min\{|u - w|, |u + w|\} \text{ for } u, w \in S^{n-1}.$$ Then we have the following inequalities

$$\frac{1}{\sqrt{2}} \tilde{d}(\mathbb{R}v, \mathbb{R}w) \leq d(\mathbb{R}v, \mathbb{R}w) \leq \tilde{d}(\mathbb{R}v, \mathbb{R}w).$$
i) If $P \subset P'$, then $d(P, Q) \leq d(P', Q)$.

j) If $Q' \subset Q$, then $d(P, Q) \leq d(P, Q')$.

To transform $d$ into a metric $D$ in $G_n = \bigcup_{k=1}^{n-1} G_{k,n}$ (disjoint union), we put

$$D(P, Q) = \max\{d(P, Q), d(Q, P)\}.$$  

Then $G_{k,n}$ are open-closed components in $G_n$ and $D(G_{k,n}, G_{l,n}) = 1$ for $k \neq l$. It is easy to check that

$$|d(P_1, Q_1) - d(P_2, Q_2)| \leq D(P_1, P_2) + D(Q_1, Q_2),$$

hence the function $d$ is continuous.

We will need another function which characterizes the mutual position of two linear subspaces $V$ and $W$ of $\mathbb{R}^n$.

**Definition 1.4.**

$$\delta(V, W) = \inf\{d(v, W) : v \in V \cap S^{n-1}\}$$

if $V \neq \{0\}$, and $\delta(V, W) = 1$ if $V = \{0\}$.

The reader will easily check the following properties

**Proposition 1.5.**

i) $\delta(V, W) = 0 \iff V \cap W \neq \{0\}$.

ii) $\delta(V, W) > 0 \iff V \cap W = \{0\}$.

iii) $\delta(V, W) = 1 \iff V \perp W$.

iv) $\delta(V, W) \leq d(V, W) \leq D(V, W)$ if $V \neq \{0\} \neq W$.

v) $\delta$ is continuous.

**Proposition 1.6.** Let $\Lambda \subset \mathbb{R}^n$ be a $C^q$ submanifold, $f : \Lambda \rightarrow \mathbb{R}^m$ be a $C^q$ mapping. Assume that for each $x \in \text{graph } f|_{\Lambda}$

$$\delta(T_x \text{graph } f|_{\Lambda}, \{0\} \times \mathbb{R}^m) \geq \alpha > 0,$$

where $\alpha$ is a positive constant. Then

i) for any point $x_0 \in \overline{\text{graph } f|_{\Lambda}}$ and for any sequence $\{x_\nu\}_{\nu \in \mathbb{N}} \subset \text{graph } f|_{\Lambda}$ converging to $x_0$ such that the sequence $\{T_{x_\nu} \text{graph } f|_{\Lambda}\}_{\nu \in \mathbb{N}}$ is convergent,

$$\delta \left( \lim_{\nu \rightarrow +\infty} T_{x_\nu} \text{graph } f|_{\Lambda}, \{0\} \times \mathbb{R}^m \right) \geq \alpha > 0.$$
ii) for any $C^q$ submanifold $M \subset \Lambda$, $M \times \mathbb{R}^m$ is transversal to $\text{graph } f|_{\Lambda}$ in $\Lambda \times \mathbb{R}^n$.

Proof. Observe that i) follows from the continuity of $\delta$.

ii) By Proposition 1.5 ii)

$$T_x \text{graph } f|_{\Lambda} \cap \{0\} \times \mathbb{R}^m = \{0\},$$

hence it is enough to observe that $\dim T_x \text{graph } f|_{\Lambda} = \dim \Lambda$.

□

Proposition 1.7. Let $\Lambda \subset \mathbb{R}^n$ be a $C^q$ submanifold and let $f : \Lambda \longrightarrow \mathbb{R}^m$ be a Lipschitz $C^q$ mapping. Then there exists a positive constant $\alpha$ such that

$$\delta(T_x \text{graph } f, \{0\} \times \mathbb{R}^m) \geq \alpha > 0,$$

for each $x \in \text{graph } f$.

Proof. There exists $L > 0$ such that $||d_y f|| \leq L$, for each $y \in \Lambda$. Now let $v \in T_x \text{graph } f$, $|v| = 1$ for some point $x = (y, f(y))$, $y \in \Lambda$. Then there exists a vector $\tilde{v} \in T_y \Lambda$ such that

$$v = (\tilde{v}, d_y f(\tilde{v})).$$

Then we have

$$1 = |v| = \sqrt{|\tilde{v}|^2 + |d_y f(\tilde{v})|^2} \leq \sqrt{1 + L^2} \cdot |\tilde{v}|.$$

Therefore

$$|\tilde{v}| \geq \frac{1}{\sqrt{1 + L^2}} > 0.$$

On the other hand

$$d(v, \{0\} \times \mathbb{R}^n) = |\tilde{v}|.$$

□

Now we recall briefly the notion of a $C^q$ stratification.

Definition 1.8. Let $A$ be a subset of $\mathbb{R}^n$. A $C^q$ stratification of the set $A$ is a (locally) finite family $\mathcal{X}_A$ of connected $C^q$ submanifolds of $\mathbb{R}^n$ (called strata) such that

1) $A = \bigcup \mathcal{X}_A$;
2) if $\Gamma_1, \Gamma_2 \in \mathcal{X}_A$, $\Gamma_1 \neq \Gamma_2$ then $\Gamma_1 \cap \Gamma_2 = \emptyset$;
3) for each $\Gamma \in \mathcal{X}_A$ the set $(\Gamma \setminus \Gamma) \cap A$ is a union of some strata from the family $\mathcal{X}_A$ of dimension $< \dim \Gamma$.

We say that the stratification $\mathcal{X}_A$ is compatible with a family of sets $B_i \subset A$, $i \in I$ if every set $B_i$ is a union of some strata of $\mathcal{X}_A$. 
Actually, we will be interested only in finite stratifications.

**Definition 1.9.** Let $A \subset \mathbb{R}^n$ and let $f : A \rightarrow \mathbb{R}^m$ be a continuous mapping, $\mathfrak{X}_A$ be a $C^q$ stratification of the set $A$ such that $f|_\Gamma$ is of class $C^q$ for all $\Gamma \in \mathfrak{X}_A$. Then by the *induced $C^q$ stratification* of the *graph* $f$, we will mean the following:

$$\mathfrak{X}_{graph f}(\mathfrak{X}_A) = \{graph f|_\Gamma : \Gamma \in \mathfrak{X}_A\}.$$

A natural setting for our results is the theory of o-minimal structures (or more generally geometric categories), as presented in [D] (or [DM]). In the whole paper the adjective *definable* (i.e. definable subset, definable mapping) will refer to any fixed o-minimal structure on the ordered field of real numbers $\mathbb{R}$.

### 2. Weakly Lipschitz mappings

In this section we describe the idea of the weak lipschitzianity of a mapping and list its important properties.

**Definition 2.1.** Let $A$ be a subset of $\mathbb{R}^n$ and let $\mathfrak{X}_A$ be a finite $C^q$ stratification of the set $A$. Consider a mapping $f : A \rightarrow \mathbb{R}^m$.

We say that $f$ is *weakly Lipschitz of class $C^q$ on the stratification $\mathfrak{X}_A$*, if for each stratum $\Gamma \in \mathfrak{X}_A$ the restriction $f|_\Gamma$ is of class $C^q$ and the pair $(f, \mathfrak{X}_A)$ satisfies one of the following equivalent conditions:

a) Whenever $\Lambda, \Gamma \in \mathfrak{X}_A$, $\Gamma \subset \overline{\Lambda} \setminus \Lambda$, $a \in \Gamma$ and $\{a_\nu\}_{\nu \in \mathbb{N}}, \{b_\nu\}_{\nu \in \mathbb{N}}$ are arbitrary sequences such that $a_\nu \in \Gamma, b_\nu \in \Lambda$, for $\nu \in \mathbb{N}$, then

$$a_\nu, b_\nu \rightarrow a \quad (\nu \rightarrow +\infty) \quad \Rightarrow \quad \lim_{\nu \rightarrow +\infty} \sup \frac{|f(a_\nu) - f(b_\nu)|}{|a_\nu - b_\nu|} < +\infty.$$

b) For any stratum $\Gamma \in \mathfrak{X}_A$ and any point $a \in \Gamma$ there exists a neighbourhood $U_a$ of $a$ such that the mapping

$$\psi : (\Gamma \cap U_a) \times ((\Lambda \setminus \Gamma) \cap U_a) \ni (x, y) \mapsto \frac{|f(x) - f(y)|}{|x - y|} \in \mathbb{R}$$

is bounded.

c) For any strata $\Lambda, \Gamma \in \mathfrak{X}_A$, $\Gamma \subset \overline{\Lambda} \setminus \Lambda$ and for any $a \in \Gamma$ there exists a neighbourhood $U_a$ of $a$ such that the mapping

$$\psi : (\Gamma \cap U_a) \times (\Lambda \cap U_a) \ni (x, y) \mapsto \frac{|f(x) - f(y)|}{|x - y|} \in \mathbb{R}$$

is bounded.
d) Whenever $\Lambda, \Gamma \in \mathfrak{X}_{\text{graph}f}(\mathfrak{X}_A)$, $\Gamma \subset \overline{\Lambda} \setminus \Lambda$, $x \in \Gamma$ and $\{x_\nu\}_{\nu \in \mathbb{N}} \subset \Gamma$, $\{y_\nu\}_{\nu \in \mathbb{N}} \subset \Lambda$ are arbitrary sequences convergent to $x$, then

$$d \left( \lim_{\nu \to +\infty} \mathbb{R}(x_\nu - y_\nu), \{0\} \times \mathbb{R}^m \right) > 0.$$ 

e) Whenever $\Lambda, \Gamma \in \mathfrak{X}_{\text{graph}f}(\mathfrak{X}_A)$, $\Gamma \subset \overline{\Lambda} \setminus \Lambda$, $x \in \Gamma$ if $\{x_\nu\}_{\nu \in \mathbb{N}} \subset \Gamma$, $\{y_\nu\}_{\nu \in \mathbb{N}} \subset \Lambda$ are arbitrary sequences convergent to $x$ and there exists a limit

$$L = \lim_{\nu \to +\infty} \mathbb{R}(x_\nu - y_\nu),$$

then $L \cap \{0\} \times \mathbb{R}^m = \{0\}$.

**Proposition 2.2.** The conditions a)-e) from Definition 2.1 are equivalent.

**Proof.** a) $\Rightarrow$ c). Suppose that c) is not satisfied. Then we find some strata $\Gamma, \Lambda \in \mathfrak{X}_A$, $\Gamma \subset \overline{\Lambda} \setminus \Lambda$ and a point $a \in \Gamma$, for which we can find a basis of neighbourhoods $\{U_n\}_{n \in \mathbb{N}}$ and two sequences $\{a_n\}_{n \in \mathbb{N}}$, $\{b_n\}_{n \in \mathbb{N}}$ such that $a_n \in \Gamma \cap U_n$, $b_n \in \Lambda \cap U_n$ and

$$\frac{|f(a_n) - f(b_n)|}{|a_n - b_n|} > n,$$

a contradiction with the assumption.

c) $\Rightarrow$ a), b) $\Leftrightarrow$ c) are trivial.

a) $\Leftrightarrow$ d) $\Leftrightarrow$ e). Since $\Gamma = \text{graph}f|_{\Gamma'}$, $\Lambda = \text{graph}f|_{\Lambda'}$, $x = (a, f(a))$, $x_\nu = (a_\nu, f(a_\nu))$ and $y_\nu = (b_\nu, f(b_\nu))$, when $\Gamma', \Lambda' \in \mathfrak{X}_A$, $a, a_\nu \in \Gamma'$, $b, b_\nu \in \Lambda'$, $a_\nu \to a$ and $b_\nu \to b$ ($\nu \to +\infty$), it is enough to observe that

$$d \left( \mathbb{R}(x_\nu - y_\nu), \{0\} \times \mathbb{R}^m \right) = \frac{|a_\nu - b_\nu|}{|(a_\nu, f(a_\nu)) - (b_\nu, f(b_\nu))|} = \frac{1}{\sqrt{1 + \left( \frac{|f(a_\nu) - f(b_\nu)|}{|a_\nu - b_\nu|} \right)^2}}.$$

\[ \square \]

**Remark 2.3.** If $f : A \to \mathbb{R}^m$ is weakly Lipschitz on a stratification $\mathfrak{X}_A$ of the set $A$, then $f$ is continuous on $A$.

Of course, the weak lipschitzianity is a generalization of the Lipschitz condition. Obviously, we have the following
Proposition 2.4. Let $A \subset \mathbb{R}^n$, $f : A \rightarrow \mathbb{R}^m$ be a locally Lipschitz mapping. Assume that $A$ admits a $C^q$ stratification $\mathcal{X}_A$ such that for all strata $\Gamma \in \mathcal{X}_A$ the map $f|_{\Gamma}$ is of class $C^q$. Then $f$ is weakly Lipschitz of class $C^q$ on the stratification $\mathcal{X}_A$.

By the $C^q$ Cell Decomposition Theorem (see [DM]), we have the following

Corollary 2.5. Let $A \subset \mathbb{R}^n$, $f : A \rightarrow \mathbb{R}^m$ be a definable locally Lipschitz mapping. There exists a definable $C^q$ stratification $\mathcal{X}_A$ of $A$ such that $f$ is weakly Lipschitz of class $C^q$ on the stratification $\mathcal{X}_A$.

The weak lipschitzianity is a much weaker property than the local Lipschitz condition, as it is shown in the examples below.

Example 2.6. Let $A \subset \mathbb{R}^2$, $A = \Lambda \cup \Gamma_1 \cup \Gamma_2$ and $\mathcal{X}_A = \{\Lambda, \Gamma_1, \Gamma_2\}$, where

$\Lambda = \{(x, y) \in \mathbb{R}^2 : x \in (0, 1), \frac{1}{2}x^2 < y < x^2\}$,

$\Gamma_1 = \{(x, y) \in \mathbb{R}^2 : x \in (0, 1), y = \frac{1}{2}x^2\}$,

$\Gamma_2 = \{(0, 0)\}$.

Consider the mapping

$f : A \ni (x, y) \rightarrow (x, \sqrt{y}) \in \mathbb{R}^2$.

Then $f$ is not Lipschitz in any neighbourhood of the point $(0, 0)$, because $\frac{\partial f}{\partial y} = \left(0, \frac{1}{2\sqrt{y}}\right)$ is unbounded. However, $f$ is weakly Lipschitz, because it is locally Lipschitz on $A \setminus \{(0, 0)\}$ and

$$\frac{|f(x, y)|}{|(x, y)|} = \sqrt{\frac{x^2 + y}{x^2 + y^2}} \leq \sqrt{\frac{2x^2}{x^2}} \leq \sqrt{2}.$$ 

Example 2.7. Let $\Lambda = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, $\Gamma = \{(x, y) \in \mathbb{R}^2 : y = 0\}$, $A = \Lambda \cup \Gamma$. Consider the mapping $f : A \rightarrow \mathbb{R}$ defined by the following formula

$$f(x, y) = \begin{cases} \left(\frac{x^2}{y^2} - y^2\right)^2 & 0 < x^2 < y^9, \\ 0 & x^2 \geq y^9 \geq 0. \end{cases}$$

Then $f$ is weakly lipschitzian of class $C^1$ on $\{\Lambda, \Gamma\}$. Indeed, $f$ is $C^1$ on $A \setminus \{(0, 0)\}$, hence it is locally Lipschitz on $A \setminus \{(0, 0)\}$. Moreover, if $(x, y) \in \Lambda$, $(x', 0) \in \Gamma$ and $f(x, y) \neq 0$, then

$$\frac{|f(x, y) - f(x', 0)|}{|(x, y) - (x', 0)|} = \frac{\left(\frac{x^2}{y^2} - y^2\right)^2}{\sqrt{(x - x')^2 + y^2}} \leq \frac{y^4}{y} = y^3.$$
Nevertheless, \( f \) is not a Lipschitz mapping in any neighbourhood of \((0,0)\), because
\[
\left| \frac{\partial f}{\partial x} \left( \frac{y^2}{\sqrt{3}}, y \right) \right| = \frac{8}{3\sqrt{3}} \sqrt{y} \rightarrow +\infty,
\]
if \( y \rightarrow 0 \).

The proofs of the following three propositions are straightforward.

**Proposition 2.8.** Let \( A \subset \mathbb{R}^n \), \( f : A \rightarrow \mathbb{R}^n \) be weakly Lipschitz of class \( C^q \) \((q \geq 1)\) on a \( C^q \) stratification \( \mathcal{X}_A \). Let \( B \subset A \). Then for any \( C^q \) stratification \( \mathcal{X}_B \) of the set \( B \), compatible with \( \mathcal{X}_A \), the mapping \( f \) is weakly Lipschitz of class \( C^q \) on the stratification \( \mathcal{X}_B \).

**Proposition 2.9.** Let \( f : A \rightarrow \mathbb{R}^p \) be a weakly Lipschitz \( C^q \) mapping \((q \geq 1)\) on a \( C^q \) stratification \( \mathcal{X}_A \) of a set \( A \subset \mathbb{R}^n \) and let \( g : B \rightarrow \mathbb{R}^r \) be a weakly Lipschitz \( C^q \) mapping on a \( C^q \) stratification \( \mathcal{X}_B \) of a set \( B \subset \mathbb{R}^p \). Assume that the image under \( f \) of each stratum from \( \mathcal{X}_A \) is contained in some stratum from \( \mathcal{X}_B \) (in particular, \( f(A) \subset B \)). Then \( g \circ f : A \rightarrow \mathbb{R}^r \) is a weakly Lipschitz \( C^q \) mapping on \( \mathcal{X}_A \).

**Remark 2.10.** The last proposition allows to define a category of stratified sets as objects and weakly Lipschitz \( C^q \) mappings \((q \geq 1)\) as morphisms.

**Proposition 2.11.** Let \( f : A \rightarrow \mathbb{R}^m \) and \( g : A \rightarrow \mathbb{R}^p \) be two weakly Lipschitz \( C^q \) mappings on a \( C^q \) stratification \( \mathcal{X}_A \) of a set \( A \subset \mathbb{R}^n \). Then the mapping
\[
(f, g) : A \ni x \mapsto (f(x), g(x)) \in \mathbb{R}^m \times \mathbb{R}^p
\]
is weakly Lipschitz of class \( C^q \) as well.

**Definition 2.12.** For a homeomorphic embedding \( f : A \rightarrow \mathbb{R}^m \) and a \( C^q \) stratification \( \mathcal{X}_A \) of \( A \) such that for any \( \Gamma \in \mathcal{X}_A \) the map \( f|_{\Gamma} \) is a \( C^q \) embedding, we have a natural \( C^q \) stratification of the image \( f(A) \)
\[
f\mathcal{X}_A = \{ f(\Gamma) : \Gamma \in \mathcal{X}_A \}.
\]

This leads to the following definition of a weakly bi-Lipschitz homeomorphism:

**Definition 2.13.** Let \( A \subset \mathbb{R}^n \) be a set, \( f : A \rightarrow \mathbb{R}^m \) be a homeomorphic embedding. Let \( \mathcal{X}_A \) be a \( C^q \) stratification \((q \geq 1)\) of the set \( A \) such that for all \( \Gamma \in \mathcal{X}_A \) the mapping \( f|_{\Gamma} \) is a \( C^q \) embedding.
We say that the mapping $f$ is *weakly bi-Lipschitz of class $C^q$* on the stratification $\mathcal{X}_A$, if $f$ is weakly Lipschitz of class $C^q$ on $\mathcal{X}_A$ and the inverse mapping $f^{-1} : f(A) \to A \subset \mathbb{R}^n$ is weakly Lipschitz of class $C^q$ on the stratification $f\mathcal{X}_A$.

In order to check that the inverse mapping is weakly Lipschitz on $f\mathcal{X}_A$, we can use the following obvious

**Proposition 2.14.** Let $A \subset \mathbb{R}^n$, $f : A \to \mathbb{R}^m$ be a homeomorphic embedding. Let $\mathcal{X}_A$ be a $C^q$ stratification of the set $A$ and assume that for each stratum $\Gamma \in \mathcal{X}_A$ the mapping $f|_\Gamma$ is a $C^q$ embedding.

Then the mapping $f^{-1} : f(A) \to A$ is weakly Lipschitz of class $C^q$ on the stratification $f\mathcal{X}_A$, if and only if it satisfies the following condition

a') for any strata $\Gamma, \Lambda \in \mathcal{X}_A$, $\Gamma \subset \overline{\Lambda} \backslash \Lambda$ and for any point $a \in \Gamma$ if $\{a_\nu\}_{\nu \in \mathbb{N}}$, $\{b_\nu\}_{\nu \in \mathbb{N}}$ are arbitrary sequences such that $a_\nu \in \Gamma, b_\nu \in \Lambda$ for $\nu \in \mathbb{N}$, then

$$a_\nu, b_\nu \to a \ (\nu \to +\infty) \implies \liminf_{\nu \to +\infty} \frac{|f(a_\nu) - f(b_\nu)|}{|a_\nu - b_\nu|} > 0.$$ 

3. **The WL class of regularity conditions**

In this section we describe some class of regularity conditions and we prove that in o-minimal geometry they are in some sense invariant under definable, locally Lipschitz, weakly bi-Lipschitz homeomorphisms. As it is shown in the next sections, this class includes the Whitney (B) condition and the Verdier condition.

From now on we fix on the ordered field $\mathbb{R}$ an o-minimal structure $\mathcal{D}$ admitting definable $C^q$ Cell Decompositions ($q \geq 1$ is also fixed). In the whole paper definable means definable in $\mathcal{D}$.

**Theorem 3.1.** Let $p \in \mathbb{N}$, $p \geq 1$. For any finite family of definable sets $A, B_1, \ldots, B_p \subset A \subset \mathbb{R}^n$, there exists a finite definable $C^q$ stratification of $A$, compatible with $B_i, i = 1, 2, \ldots, p$.

**Proof.** This easily follows from $C^q$ Cell Decomposition Theorem (see [DM], 4.2).

Let $\mathcal{Q}$ be a condition on pairs $(\Lambda, \Gamma)$ at points $x \in \Gamma$, where $\Lambda, \Gamma \subset \mathbb{R}^n$ are $C^q$ submanifolds, $\Gamma \subset \overline{\Lambda} \backslash \Lambda$. Sometimes we will refer to $\mathcal{Q}$ as a regularity condition.
**Definition 3.2.** We say that $Q$ is *local* if for an open neighbourhood $U$ of the point $x \in \Gamma$ the pair $(\Lambda, \Gamma)$ satisfies the condition $Q$ at $x$ if and only if the pair $(\Lambda \cap U, \Gamma \cap U)$ satisfies the condition $Q$ at the point $x$.

We will be considering only local conditions. We adopt the following notation:

- $W^Q(\Lambda, \Gamma) \equiv$ the condition $Q$ is satisfied for the pair $(\Lambda, \Gamma)$ at $x \in \Gamma$.

- $\sim W^Q(\Lambda, \Gamma) \equiv$ the negation of $W^Q(\Lambda, \Gamma)$.

**Definition 3.3.** We say that $Q$ is *definable* if for any definable $C^q$ submanifolds $\Gamma, \Lambda \subset \mathbb{R}^n$ such that $\Gamma \subset \overline{\Lambda} \setminus \Lambda$, the set

$$\{x \in \Gamma : W^Q(\Lambda, \Gamma, x)\}$$

is definable.

**Definition 3.4.** Let $Q$ be a definable condition. We say that $Q$ is *generic* if for any definable $C^q$ submanifolds $\Lambda, \Gamma \subset \mathbb{R}^n$, $\Gamma \subset \overline{\Lambda} \setminus \Lambda$, the set

$$\{x \in \Gamma : \sim W^Q(\Lambda, \Gamma, x)\}$$

is nowhere dense in $\Gamma$.

We are interested in $C^q$ stratifications satisfying a condition $Q$.

**Definition 3.5.** Let $\mathcal{X}_A$ be a $C^q$ stratification of $A$. We say that $\mathcal{X}_A$ is a *stratification with a condition $Q$* (or *$Q$ - stratification*) if for each pair $\Lambda, \Gamma \subset \mathcal{X}_A$

$$\Gamma \subset \overline{\Lambda} \setminus \Lambda \implies W^Q(\Lambda, \Gamma).$$

**Theorem 3.6.** *(Lojasiewicz-Stasica-Wachta)* Let $Q$ be a definable, generic condition. Then for any finite family of definable subsets $\{A_j\}_{j \in I}$ of $\mathbb{R}^n$, there exists a finite definable $C^q$ stratification $\mathcal{X}_{\mathbb{R}^n}$ of $\mathbb{R}^n$ with the condition $Q$, compatible with $\{A_j\}_{j \in I}$.

**Proof.** A procedure was given in [LSW], Prop.2 in the subanalytic case. It suffices to observe that the same argument works in a general definable case. \hfill $\square$

**Corollary 3.7.** Let $Q$ be a definable, generic condition. Given a definable $C^q$ stratification $\mathcal{X}_A$ of a definable set $A \subset \mathbb{R}^n$, there exists a finite definable $C^q$ stratification $\mathcal{X'}_A$ of the set $A$ with the condition $Q$ such that $\mathcal{X'}_A$ is compatible with $\mathcal{X}_A$ and moreover

$$\{\Gamma' \in \mathcal{X}_A : \dim \Gamma' = \dim A\} = \{\Gamma \in \mathcal{X}_A : \dim \Gamma = \dim A\}.$$
Proof. Observe that using the downward induction from [LSW] and correcting a definable $C^q$ stratification to the one satisfying the condition $Q$, all we need to do is to substratify these strata of $\mathcal{X}_A$ that are of dimension $< \dim A$. □

**Definition 3.8.** We say that a condition $Q$ is $C^q$ invariant (or invariant under $C^q$ diffeomorphisms) if for any $C^q$ diffeomorphism $\phi : U \rightarrow W$ of open subsets $U, W \subset \mathbb{R}^n$ and any $C^q$ submanifolds $\Lambda, \Gamma \subset U$ such that $\Gamma \subset \overline{\Lambda} \setminus \Lambda$ and for any point $x \in \Gamma$

$$W^Q(\Lambda, \Gamma, x) \iff W^Q(\phi(\Lambda), \phi(\Gamma), \phi(x)).$$

The class of conditions that we are to describe consists of the conditions that are definable, generic and invariant under definable $C^q$ diffeomorphisms. Additionally, two more features are required.

**Definition 3.9.** We say that a condition $Q$ has a projection property with respect to weakly Lipschitz mappings of class $C^q$ if for any $C^q$ mapping $f : A \rightarrow \mathbb{R}^m$ weakly Lipschitz on a $C^q$ stratification $\mathcal{X}_A$ of a set $A \subset \mathbb{R}^n$, we have

$$\mathcal{X}_{graph f}(\mathcal{X}_A) \text{ is a } Q \text{- stratification} \implies \mathcal{X}_A \text{ is a } Q \text{- stratification}.$$ Notice that this condition is equivalent to the following one:

For any subset $E \subset \mathbb{R}^n \times \mathbb{R}^m$ and its $C^q$ stratification $\mathcal{X}_E$ such that $\pi_1|_E : E \rightarrow \mathbb{R}^n$ is a homeomorphic embedding and for each $\Gamma \in \mathcal{X}_E$, $\pi_1|_\Gamma : \Gamma \rightarrow \mathbb{R}^n$ is a $C^q$ embedding and $(\pi_2|_E) \circ (\pi_1|_E)^{-1}$ is weakly Lipschitz of class $C^q$ on $\pi_1\mathcal{X}_E$, we have

$$\mathcal{X}_E \text{ is a } Q \text{- stratification} \implies \pi_1\mathcal{X}_E \text{ is a } Q \text{- stratification},$$

where $\pi_1 : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $\pi_2 : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ denote natural projections.

**Remark 3.10.** By Propositions 2.9 and 2.11 the condition that the mapping $(\pi_2|_E) \circ (\pi_1|_E)^{-1}$ is weakly Lipschitz of class $C^q$ on $\pi_1\mathcal{X}_E$ is equivalent to the condition that $\pi_1|_E$ is weakly bi-Lipschitz of class $C^q$ on the stratification $\mathcal{X}_E$.

A proof of the following proposition is trivial.

**Proposition 3.11.** Let $Q$ be a condition, having the projection property with respect to weakly Lipschitz mappings of class $C^q$. Let $f : A \rightarrow \mathbb{R}^m$ be weakly Lipschitz of class $C^q$ on a stratification $\mathcal{X}_A$ of a set $A \subset \mathbb{R}^n$. 
Then for any $C^q$ submanifolds $\Gamma, \Lambda \subset \text{graph} f$ such that $\Gamma \subset \overline{A} \setminus \Lambda$, $\dim \Gamma < \dim \Lambda$ and $\{\Lambda, \Gamma\}$ are compatible with the stratification $\mathcal{X}_{\text{graph} f}(\mathcal{X}_A)$

$$W^{\mathcal{Q}}(\Lambda, \Gamma) \implies W^{\mathcal{Q}}(\pi_1(\Lambda), \pi_1(\Gamma)).$$

**Definition 3.12.** We say that a condition $\mathcal{Q}$ has a lifting property with respect to locally Lipschitz mappings of class $C^q$ if for any two $C^q$ submanifolds $\Lambda, \Gamma \subset \mathbb{R}^n$ such that $\Gamma \subset \Lambda \setminus \overline{\Lambda}$, and for any locally Lipschitz mapping $f : \Lambda \cup \Gamma \to \mathbb{R}^m$ such that the restrictions $f|_{\Lambda}$, $f|_{\Gamma}$ are of class $C^q$ and for any $C^q$ submanifolds $M, N \subset \mathbb{R}^n$ such that $N \subset \overline{M} \setminus N$ and $\{M, N\}$ is compatible with $\{\Lambda, \Gamma\}$, we have

$$W^{\mathcal{Q}}(M, N), W^{\mathcal{Q}}(\text{graph} f|_{\Lambda}, \text{graph} f|_{\Gamma}) \implies W^{\mathcal{Q}}(\text{graph} f|_{M}, \text{graph} f|_{N}).$$

Equivalently, $\mathcal{Q}$ has the lifting property with respect to locally Lipschitz mappings of class $C^q$ if for any two $C^q$ submanifolds $\Lambda, \Gamma \subset \mathbb{R}^n \times \mathbb{R}^m$ such that $\Gamma \subset \overline{A} \setminus \Lambda$ and $\pi_1|_{A \cup \Gamma}$ is a homeomorphic embedding, $\pi_1|_{\Lambda}$, $\pi_1|_{\Gamma}$ are $C^q$ embeddings and the mapping $\pi_2|_{A \cup \Gamma} \circ (\pi_1|_{A \cup \Gamma})^{-1} : \pi_1(\Lambda) \cup \pi_1(\Gamma) \to \mathbb{R}^m$ is locally Lipschitz, the following holds true:

- for any $C^q$ submanifolds $M, N \subset \mathbb{R}^n$ such that $N \subset \overline{M} \setminus M$,

  i) $W^{\mathcal{Q}}(M, N), W^{\mathcal{Q}}(\Lambda, \Gamma), M \subset \pi_1(\Lambda), N \subset \pi_1(\Gamma) \implies W^{\mathcal{Q}}((M \times \mathbb{R}^m) \cap \Lambda, (N \times \mathbb{R}^m) \cap \Gamma)$

  ii) $W^{\mathcal{Q}}(M, N), M, N \subset \pi_1(\Lambda) \implies W^{\mathcal{Q}}((M \times \mathbb{R}^m) \cap \Lambda, (N \times \mathbb{R}^m) \cap \Lambda)$

  iii) $W^{\mathcal{Q}}(M, N), M, N \subset \pi_1(\Gamma) \implies W^{\mathcal{Q}}((M \times \mathbb{R}^m) \cap \Gamma, (N \times \mathbb{R}^m) \cap \Gamma)$.

**Remark 3.13.** If a condition $\mathcal{Q}$ is $C^q$ invariant, then the implications ii) and iii) are always satisfied.

**Definition 3.14.** We say that a condition $\mathcal{Q}$ is a $\mathcal{W}L$ condition of class $C^q$ if it is

- definable;
- generic;
- invariant under definable $C^q$ diffeomorphisms;
- has the projection property with respect to weakly Lipschitz mappings of class $C^q$;
- has the lifting property with respect to locally Lipschitz mappings of class $C^q$.

\[\text{Remark}\ 3.13. \text{If a condition } \mathcal{Q} \text{ is } C^q \text{ invariant, then the implications ii) and iii) are always satisfied.}\]

\[\text{Definition}\ 3.14. \text{We say that a condition } \mathcal{Q} \text{ is a } \mathcal{W}L \text{ condition of class } C^q \text{ if it is}\]

- definable;
- generic;
- invariant under definable $C^q$ diffeomorphisms;
- has the projection property with respect to weakly Lipschitz mappings of class $C^q$;
- has the lifting property with respect to locally Lipschitz mappings of class $C^q$.

The inequality $\dim \Gamma < \dim \Lambda$ is required for $\{\Lambda, \Gamma\}$ to be a stratification of $\Lambda \cup \Gamma$. In the definable case the inequality follows from $\Gamma \subset \overline{A} \setminus \Lambda$.
Conditions of type $\mathcal{WL}$ are invariant under definable locally Lipschitz, weakly bi-Lipschitz homeomorphisms in the following sense:

**Theorem 3.15. (Invariance of $\mathcal{WL}$ conditions under definable, locally Lipschitz, weakly bi-Lipschitz homeomorphisms)** Let $Q$ be a $\mathcal{WL}$ condition of class $C^q$. Let $A \subset \mathbb{R}^n$ be a definable set and let $f : A \rightarrow \mathbb{R}^m$ be a definable homeomorphic embedding, weakly bi-Lipschitz of class $C^q$ on a definable $C^q$ stratification $\mathcal{X}_A$ of the set $A$. Assume that for any two submanifolds $\Lambda, \Gamma \in \mathcal{X}_A$ such that $\Gamma \subset \Lambda \setminus \Lambda$, the mapping $f|_{\Lambda \setminus \Lambda}$ is locally Lipschitz.

Then there exists a definable $C^q$ stratification $\mathcal{X}'_A$ of $A$, compatible with $\mathcal{X}_A$ such that

$$\{\Gamma \in \mathcal{X}_A : \dim \Gamma = \dim A\} = \{\Gamma' \in \mathcal{X}'_A : \dim \Gamma' = \dim A\}$$

and such that the condition $Q$ is invariant with respect to the pair $(f, \mathcal{X}'_A)$ in the following sense

for any definable $C^q$ submanifolds $M, N \subset A$ such that $N \subset \overline{M \setminus M}$ and $\{M, N\}$ are compatible with the stratification $\mathcal{X}'_A$

$$W^Q(M, N) \implies W^Q(f(M), f(N)).$$

**Proof.** Consider $\text{graph}f \subset \mathbb{R}^n \times \mathbb{R}^m$ and its definable $C^q$ stratification

$$\mathcal{X}_{\text{graph}f}(\mathcal{X}_A) = \{\text{graph}f|_{\Gamma} : \Gamma \in \mathcal{X}_A\}.$$ 

By Corollary 3.7 we find a definable $C^q$ substratification $\mathcal{X}^Q_{\text{graph}f}(\mathcal{X}_A)$ of the $\text{graph}f$ with the condition $Q$ that is compatible with the family $\mathcal{X}_{\text{graph}f}(\mathcal{X}_A)$ and moreover

$$\{\Gamma' \in \mathcal{X}^Q_{\text{graph}f}(\mathcal{X}_A) : \dim \Gamma' = \dim A\} = \{\Gamma \in \mathcal{X}_{\text{graph}f}(\mathcal{X}_A) : \dim \Gamma = \dim A\}.$$ 

Now we will show that

$$\mathcal{X}'_A = \{\pi_1(\Lambda) : \Lambda \in \mathcal{X}^Q_{\text{graph}f}(\mathcal{X}_A)\}$$

is a required stratification.

Of course $\mathcal{X}'_A$ is really a definable $C^q$ stratification of $A$, compatible with $\mathcal{X}_A$ and such that

$$\{\Gamma \in \mathcal{X}_A : \dim \Gamma = \dim A\} = \{\Lambda \in \mathcal{X}'_A : \dim \Lambda = \dim A\}.$$ 

By the projection property of the condition $Q$ with respect to weakly Lipschitz mappings, $\mathcal{X}'_A$ is a $Q$-stratification.
Observe that for any two strata $\Lambda, \Gamma \in \mathcal{X}_A$, $\Gamma \subset \overline{\Lambda} \setminus \Lambda$, the mapping $f|_{\Lambda \cup \Gamma}$ is still locally Lipschitz, because of the compatibility of $\mathcal{X}'_A$ with $\mathcal{X}_A$.

For the same reason $f \mathcal{X}'_A$ is compatible with $f \mathcal{X}_A$. Therefore, from Proposition 2.8 we get that $f$ is still weakly bi-Lipschitz of class $C^q$ on $\mathcal{X}'_A$.

Consider now two definable $C^q$ submanifolds $M, N$ in $\mathbb{R}^n$, $N \subset \overline{M} \setminus M$ that are compatible with the stratification $\mathcal{X}'_A$ and $\mathcal{W}(M, N)$. Then two cases are possible:

I. There exists a submanifold $\Lambda \in \mathcal{X}'_A$ such that $M, N \subset \Lambda$. As $f|_\Lambda$ is a definable $C^q$ embedding, then by the invariance of the condition $\mathcal{Q}$ under definable $C^q$ diffeomorphisms we get that $\mathcal{W}(f(M), f(N))$.

II. There exist two different $\Lambda, \Gamma \in \mathcal{X}'_A$ such that $N \subset \Gamma$, $M \subset \Lambda$. Then we have $\Gamma \subset \overline{\Lambda} \setminus \Lambda$ and observe that $f|_{\Lambda \cup \Gamma}$ is still locally Lipschitz.

In this case, by the construction of the stratification $\mathcal{X}'_A$ we know that $\mathcal{W}(\text{graph } f|_\Lambda, \text{graph } f|_\Gamma)$, so as the condition $\mathcal{Q}$ has the lifting property under locally Lipschitz mappings of class $C^q$, we get

$$\mathcal{W}(\text{graph } f|_M, \text{graph } f|_N).$$

On the other hand the mapping $f^{-1}$ is weakly Lipschitz on the definable $C^q$ stratification $\{f(\Lambda), f(\Gamma)\}$ and thus on $\{f(M), f(N)\}$. Observe that

$$\Phi(\text{graph } f|_M) = \text{graph } f^{-1}|_{f(M)}, \quad \Phi(\text{graph } f|_N) = \text{graph } f^{-1}|_{f(N)},$$

where $\Phi : \mathbb{R}^n \times \mathbb{R}^m \ni (x, y) \mapsto (y, x) \in \mathbb{R}^m \times \mathbb{R}^n$. Therefore, by the projection property of the condition $\mathcal{Q}$ with respect to weakly Lipschitz mappings we have

$$\mathcal{W}(\pi_2(\text{graph } f|_M), \pi_2(\text{graph } f|_N)).$$

But $\pi_2(\text{graph } f|_M) = f(M)$, $\pi_2(\text{graph } f|_N) = f(N)$, which completes the proof.

\[\square\]

**Remark 3.16.** As a corollary from the proof of Theorem 3.15 we get that the definable $C^q$ stratifications $\mathcal{X}'_A$ and $\mathcal{X}'_{f(A)} = \{f(\Gamma) : \Gamma \in \mathcal{X}'_A\}$ are the $\mathcal{Q}$-stratifications.

4. **The Whitney (B) condition as a $\mathcal{WL}$ condition**

In this section we will prove that the Whitney (B) condition is of type $\mathcal{WL}$ of class $C^q$, $q \geq 1$. 

Definition 4.1. Let $N, M$ be $C^q$ submanifolds of $\mathbb{R}^n$ such that $N \subset M \setminus M$ and let $a \in N$.

We say that the pair of strata $(M, N)$ satisfies the Whitney (B) condition at the point $a$ (notation: $W^B(M, N, a)$) if for any sequences $\{a_\nu\}_{\nu \in \mathbb{N}} \subset N$, $\{b_\nu\}_{\nu \in \mathbb{N}} \subset M$ both converging to the point $a$ and such that the sequence of the secant lines $\{(a_\nu - b_\nu)\}_{\nu \in \mathbb{N}}$ converges to a line $L \subset \mathbb{R}^n$ in $\mathbb{P}_{n-1}$ and the sequence of the tangent spaces $\{T_{b_\nu}M\}_{\nu \in \mathbb{N}}$ converges to a subspace $T \subset \mathbb{R}^n$ in $G_{\dim M, n}$, always $L \subset T$.

When the pair of $C^q$ submanifolds $(M, N)$ satisfies (respectively, does not satisfy) the Whitney (B) condition at a point $a \in N$, we write $W^B(M, N, a)$ (respectively $\sim W^B(M, N, a)$). If for any point $a \in N$ we have $W^B(M, N, a)$, we write $W^B(M, N)$.

Theorem 4.2. The Whitney (B) condition is definable and generic.

Proof. The proof in [TL2] for the structure $(\mathbb{R}, +, \cdot, \exp, (r)_{r \in \mathbb{R}})$ remains valid for arbitrary o-minimal structures on $(\mathbb{R}, <, +, \cdot)$. See also [DM].

□

Definition 4.3. A definable $C^q$ stratification $\mathfrak{X}_A$ of a definable set $A \subset \mathbb{R}^n$ is called a Whitney stratification if for any pair of strata $\Gamma, \Lambda \in \mathfrak{X}_A$ such that $\Gamma \subset \overline{\Lambda} \setminus \Lambda$, the condition $W^B(\Lambda, \Gamma)$ is satisfied.

Remark 4.4. The Whitney (B) condition is invariant under $C^1$ diffeomorphisms (see [Tr]a).

In order to show that the Whitney (B) condition has the projection property with respect to weakly Lipschitz mappings of class $C^q$, it suffices to prove the following theorem.

Theorem 4.5. Let $\Lambda, \Gamma \subset \mathbb{R}^n$ be $C^q$ submanifolds such that $\Gamma \subset \overline{\Lambda} \setminus \Lambda$ and $\dim \Gamma < \dim \Lambda$. Consider a mapping $f : \Lambda \cup \Gamma \rightarrow \mathbb{R}^m$ weakly Lipschitz of class $C^q$ on the stratification $\{\Lambda, \Gamma\}$. Then

$$W^B(\text{graph } f|_\Lambda, \text{graph } f|_\Gamma) \Rightarrow W^B(\Lambda, \Gamma).$$

Proof. Let $a \in \Gamma$ and $\{a_k\}_{k \in \mathbb{N}} \subset \Gamma$, $\{b_k\}_{k \in \mathbb{N}} \subset \Lambda$ be sequences converging to $a$ and such that

$$\mathbb{R}(a_k - b_k) \rightarrow L, \quad T_{b_k}\Lambda \rightarrow T,$$

when $k \rightarrow +\infty$. 

In order to show that the Whitney (B) condition has the projection property with respect to weakly Lipschitz mappings of class $C^q$, it suffices to prove the following theorem.
with some \( L \in \mathbb{P}_{n-1}, T \in \mathbb{G}_{\dim L, n} \). Then,
\[
(a_k, f(a_k)) \longrightarrow (a, f(a)), \quad (b_k, f(b_k)) \longrightarrow (a, f(a)), \quad k \longrightarrow +\infty.
\]
Without loss of generality we may assume that for \( k \longrightarrow +\infty \) we have
\[
\mathbb{R} \left( (a_k, f(a_k)) - (b_k, f(b_k)) \right) \longrightarrow L', \quad T_{(b_k, f(b_k))} \text{graph } f\big|_\Lambda \longrightarrow T'
\]
with some \( L' \in \mathbb{P}_{n+m-1}, T' \in \mathbb{G}_{\dim L, n+m} \). Since \( \mathcal{W}^B(\text{graph } f\big|_\Lambda, \text{graph } f\big|_\Gamma) \),
then \( L' \subset T' \). Because \( f \) is weakly Lipschitz on \( \{\Lambda, \Gamma\} \), thus
\[
\pi_1(L') = L.
\]
On the other hand, as \( f|_\Lambda \) is of class \( C^q \), then for any \( k \in \mathbb{N} \)
\[
\pi_1(T_{(b_k, f(b_k))} \text{graph } f|_\Lambda) = T_{b_k} \Lambda,
\]
hence, by the continuity of \( \pi_1 \) we get the inclusion \( \pi_1(T') \subset T \). Consequently,
\[
L = \pi_1(L') \subset \pi_1(T') \subset T.
\]

Now we deal with the lifting property with respect to locally Lipschitz mappings of class \( C^q \) for the Whitney (B) condition. It will easily follow from a more general transversal intersection theorem for the Whitney (B) condition.

**Theorem 4.6.** Let \( \Lambda_i, \Gamma_i \ (i = 1, 2) \) be two pairs of \( C^q \) submanifolds in \( \mathbb{R}^n \)
such that \( \Gamma_i \subset \overline{\Lambda_i} \setminus \Lambda_i \) and \( \mathcal{W}^B(\Lambda_i, \Gamma_i) \). Assume that \( \Lambda_1 \cap \Lambda_2 \) and \( \Gamma_1 \cap \Gamma_2 \)
are \( C^q \) submanifolds such that \( \Gamma_1 \cap \Gamma_2 \subset \overline{\Lambda_1 \cap \Lambda_2} \) and for each \( x_0 \in \Gamma_1 \cap \Gamma_2 \)
and any sequence \( \{y_{\nu}\}_{\nu \in \mathbb{N}} \subset \Lambda_1 \cap \Lambda_2 \) converging to \( x_0 \), we have
\[
T_{y_{\nu}} \Lambda_i \longrightarrow S_i \quad (i = 1, 2) \quad \Longrightarrow \quad \dim S_1 \cap S_2 = \dim \Lambda_1 \cap \Lambda_2.
\]
Then \( \mathcal{W}^B(\Lambda_1 \cap \Lambda_2, \Gamma_1 \cap \Gamma_2) \).

**Proof.** Let \( x_0 \in \Gamma_1 \cap \Gamma_2 \). Consider two sequences
\[
\{x_n\}_{n \in \mathbb{N}} \subset \Gamma_1 \cap \Gamma_2, \quad \{y_n\}_{n \in \mathbb{N}} \subset \Lambda_1 \cap \Lambda_2,
\]
such that \( x_n, y_n \longrightarrow x_0 \) for \( n \longrightarrow +\infty \). Assume that \( \mathbb{R}(x_n - y_n) \longrightarrow L \) and
that the following sequences
\[
\{T_{y_0} (\Lambda_1 \cap \Lambda_2)\}_{n \in \mathbb{N}}, \quad \{T_{y_n} \Lambda_1\}_{n \in \mathbb{N}}, \quad \{T_{y_n} \Lambda_2\}_{n \in \mathbb{N}}.
\]
\[2\]If for any \( G \in \{\Lambda_1, \Gamma_1\} \) and \( K \in \{\Lambda_2, \Gamma_2\} \) the submanifolds \( G \) and \( K \) are transversal in \( \mathbb{R}^n \), then Theorem 4.7 follows from Lemma 4.2.2 in [Te].
are convergent. Let
\[ T = \lim_{y_n \to x_0} T_{y_n}(\Lambda_1 \cap \Lambda_2). \]
By the assumptions
\[ L \subset \lim_{y_n \to x_0} T_{y_n}(\Lambda_1 \cap \Lambda_2) = T. \]

**Theorem 4.7.** Let \( \Lambda, \Gamma \) be \( C^q \) submanifolds of \( \mathbb{R}^n \) such that \( \Gamma \subset \Lambda \). Let \( f : \Lambda \cup \Gamma \to \mathbb{R}^m \) be a locally Lipschitz mapping such that \( f|_\Lambda, f|_\Gamma \) are of class \( C^q \). Let \( M, N \) be \( C^q \) submanifolds of \( \mathbb{R}^n \) such that \( N \subset M \) and \( \{M, N\} \) are compatible with \( \{\Lambda, \Gamma\} \). Then
\[ W_B(\text{graph}f|_\Lambda, \text{graph}f|_\Gamma) \implies W_B(\text{graph}f|_M, \text{graph}f|_N). \]

**Proof.** If \( M, N \subset \Lambda \) or \( M, N \subset \Gamma \), then \( W_B(\text{graph}f|_M, \text{graph}f|_N) \) holds true, because the Whitney (B) condition is \( C^q \) invariant. Now let \( M \subset \Lambda \), \( N \subset \Gamma \). It is enough to use Theorem 4.6 where
\[ \Lambda_1 = \text{graph}f|_\Lambda, \quad \Gamma_1 = \text{graph}f|_\Gamma, \quad \Lambda_2 = M \times \mathbb{R}^m, \quad \Gamma_2 = N \times \mathbb{R}^m. \]
The last assumption of Theorem 4.6 is fulfilled, because \( f \) is locally Lipschitz (use Prop. 1.6 and 1.7).

**Corollary 4.8.** The Whitney (B) condition is a \( WL \) condition of class \( C^q \).

**Corollary 4.9.** Theorem 3.15 holds true for the Whitney (B) condition.

**Remark 4.10.** A similar theorem holds true in the analytic-geometric category defined in [DM], as the Whitney stratification theorem holds true in this category.

## 5. The Verdier condition as a \( WL \) condition

We start with some preparation.

**Lemma 5.1.** Let \( V \) be a linear subspace of \( \mathbb{R}^n \), \( \mathbb{R}^n = V \oplus V^\perp \). Let \( 0 < \alpha \leq 1 \) be a constant and consider a set
\[ B_\alpha = \{ u \in S^{n-1} : \ d(\mathbb{R}u, V^\perp) \geq \alpha \}. \]
Then there exists \( C_\alpha > 0 \) such that:
- for any \( u \in B_\alpha, w \in B_\alpha \) we have
  \[ d(\mathbb{R}\pi_V(u), \mathbb{R}\pi_V(w)) \leq C_\alpha \cdot d(\mathbb{R}u, \mathbb{R}w), \]
ii) for any two linear subspaces \( L, K \subset \mathbb{R}^n \) such that \( L \cap S^{n-1} \subset B_\alpha \) and \( K \cap S^{n-1} \subset B_\alpha \),
\[
d(\pi_V(L), \pi_V(K)) \leq C_\alpha \cdot d(L, K).
\]

Proof. i). Let \( u, w \in B_\alpha \). Then
\[
|\pi_V(u)| = |u - \pi_V \perp (u)| = d(\mathbb{R}u, V^\perp) \geq \alpha,
\]
\[
|\pi_V(w)| = |w - \pi_V \perp (w)| = d(\mathbb{R}w, V^\perp) \geq \alpha.
\]

Hence
\[
\left|\frac{\pi_V(u)}{|\pi_V(u)|} - \frac{\pi_V(w)}{|\pi_V(w)|}\right| = \left|\frac{\pi_V(u) - \pi_V(w)}{|\pi_V(u)| \cdot |\pi_V(w)|} - \frac{\pi_V(u) - \pi_V(w)}{|\pi_V(u)| \cdot |\pi_V(w)|}\right| = \frac{1}{\alpha} \cdot \left|\pi_V(u) \cdot (|\pi_V(w)| - |\pi_V(w)|) + |\pi_V(u)| \cdot (\pi_V(u) - \pi_V(w))\right|
\]
\[
\leq \frac{1}{\alpha^2} \cdot \left(\left(|\pi_V(u)| \cdot |\pi_V(u)| - |\pi_V(w)|\right) + |\pi_V(u)| \cdot |\pi_V(u) - \pi_V(w)|\right)
\]
\[
\leq \frac{2}{\alpha^2} \left(|\pi_V(u) - \pi_V(w)|\right) = \frac{2}{\alpha^2} |\pi_V(u - w)| \leq \frac{2}{\alpha^2} |u - w|.
\]

Similarly,
\[
\left|\frac{\pi_V(u)}{|\pi_V(u)|} + \frac{\pi_V(w)}{|\pi_V(w)|}\right| = \left|\frac{\pi_V(u)}{|\pi_V(u)|} - \frac{\pi_V(-w)}{|\pi_V(-w)|}\right| \leq \frac{2}{\alpha^2} |u + w|.
\]

Finally,
\[
d(\mathbb{R}\pi_V(u), \mathbb{R}\pi_V(w)) \leq d(\mathbb{R}\pi_V(u), \mathbb{R}\pi_V(w)) \leq \frac{2}{\alpha^2} \cdot d(\mathbb{R}u, \mathbb{R}v)
\]
\[
\leq \frac{\sqrt{2}}{\alpha} d(\mathbb{R}u, \mathbb{R}v).
\]

The ii) is an easy corollary from i).

\[\square\]

We will also need the following definition of the sine of the angle between two linear subspaces.

**Definition 5.2.** Let \( S, K \subset \mathbb{R}^n \) be linear subspaces. Then we define
\[
\lambda(S, K) = \begin{cases} 
\inf\{d(\mathbb{R}u, \mathbb{R}w) : u \in S \cap S^{n-1}, w \in K \cap S^{n-1}, u, w \perp S \cap K\}, & \text{for } S \not\subset K \text{ and } K \not\subset S, \\
0, & \text{for } S \subset K \text{ or } K \subset S.
\end{cases}
\]

**Remark 5.3.** Notice that if \( S \not\subset K \) and \( K \not\subset S \), then
\[
\lambda(S, K) = \delta \left(S \cap (S \cap K)^\perp, K \cap (S \cap K)^\perp\right).
\]

**Proposition 5.4.** Let \( k, l, n \in \mathbb{N} \) and let \( S \in \mathbb{G}_{s,n}, K \in \mathbb{G}_{k,n} \) be such that \( \lambda(S, K) > 0 \). Then
i) for any \( v \in \mathbb{R}^n, |v| = 1 \) we have
\[
d(\mathbb{R}v, S \cap K) \leq \frac{1}{\lambda(S, K)}(d(\mathbb{R}v, S) + d(\mathbb{R}v, K)).
\]

ii) If \( R, L \) are linear subspaces in \( \mathbb{R}^n \), then
\[
d(R \cap L, S \cap K) \leq \frac{1}{\lambda(S, K)}(d(R, S) + d(L, K)).
\]

**Proof.** i). If \( v \in S \cap K \), then the above inequality is satisfied as
\[
d(v, S \cap K) = d(v, S) = d(v, K) = 0.
\]

Assume now that \( v \not\in S \cap K \), which means that \( d(v, S \cap K) > 0 \). Then
\[
\lambda(S, K) \leq d(\mathbb{R}(\pi_K(v) - \pi_{S\cap K}(v)), \mathbb{R}(\pi_S(v) - \pi_{S\cap K}(v)))
\]
\[
\leq d(\mathbb{R}(\pi_K(v) - \pi_{S\cap K}(v)), \mathbb{R}(v - \pi_{S\cap K}(v)))+
\]
\[
d(\mathbb{R}(v - \pi_{S\cap K}(v)), \mathbb{R}(\pi_S(v) - \pi_{S\cap K}(v)))
\]
\[
= \frac{|v - \pi_K(v)|}{|v - \pi_{S\cap K}(v)|} + \frac{|v - \pi_S(v)|}{|v - \pi_{S\cap K}(v)|} = \frac{d(v, K)}{d(v, S \cap K)} + \frac{d(v, S)}{d(v, S \cap K),}
\]
as required.

The assertion ii) follows trivially from i).

**Proposition 5.5.** Let \( p, k, s, n \in \mathbb{N}, p < \min\{k, s\} \). Consider a set
\[
\Sigma = \{(S, K) \in \mathcal{G}_{s,n} \times \mathcal{G}_{k,n} : \dim(S \cap K) = p\}.
\]
Then the mapping
\[
\lambda : \Sigma \ni (S, K) \mapsto \lambda(S, K) \in [0, 1]
\]
is continuous.

**Proof.** The continuity of \( \lambda \) follows easily from the fact that the mapping
\[
\psi : \Sigma \ni (S, K) \mapsto S \cap K \in \mathcal{G}_{p,n}
\]
is continuous at any point \((S_0, K_0) \in \Sigma\), because Proposition 5.4 ii) implies the inequality
\[
d(S \cap K, S_0 \cap K_0) \leq \frac{1}{\lambda(S_0, K_0)}(d(S, S_0) + d(K, K_0))
\]
for any \((S, K) \in \Sigma\).  

□
Proposition 5.6. Let $s, k, p, n \in \mathbb{N}$ and $p < \min\{k, s\}$. Let $\tilde{\Sigma}$ be a compact subset of the set

$$\Sigma = \{(S, K) \in \mathbb{G}_{s,n} \times \mathbb{G}_{k,n} : \dim(S \cap K) = p\}.$$ 

Then

$$\inf\{\lambda(S, K) : (S, K) \in \tilde{\Sigma}\} > 0.$$ 

Proof. Trivial as $\lambda : \Sigma \ni (S, K) \mapsto \lambda(S, K) \in [0, 1]$ is continuous and $\tilde{\Sigma}$ is compact. \(\square\)

Corollary 5.7. Let $s, k, p, n \in \mathbb{N}$, $p < \min\{k, s\}$ and let $\tilde{\Sigma}$ be a compact subset of

$$\Sigma = \{(S, K) \in \mathbb{G}_{s,n} \times \mathbb{G}_{k,n} : \dim(S \cap K) = p\}.$$ 

Then there exists $C > 0$ such that for any linear subspaces $R, L$ of $\mathbb{R}^n$ and for any $(S, K) \in \tilde{\Sigma}$

$$d(R \cap L, S \cap K) \leq C \cdot (d(R, S) + d(L, K)).$$

Proof. By Proposition 5.6 and 5.4 ii) the above inequality holds true with the constant

$$C = \frac{1}{\inf\{\lambda(S, K) : (S, K) \in \tilde{\Sigma}\}}.$$ 

\(\square\)

Definition 5.8. Let $\Lambda, \Gamma$ be $C^2$ submanifolds of $\mathbb{R}^n$, $\Gamma \subset \overline{\Lambda} \setminus \Lambda$. We say that the pair $(\Lambda, \Gamma)$ satisfies the Verdier condition at $x_0 \in \Gamma$ (notation: $\mathcal{W}^V(\Lambda, \Gamma, x_0)$) if there exists an open neighbourhood $U_{x_0}$ of $x_0$ in $\mathbb{R}^n$ and $C_{x_0} > 0$ such that

$$\forall x \in \Gamma \cap U_{x_0} \forall y \in \Lambda \cap U_{x_0} \quad d(T_x\Gamma, T_y\Lambda) \leq C_{x_0} \cdot |x - y|.$$ 

We say that $(\Lambda, \Gamma)$ satisfies the Verdier condition (notation $\mathcal{W}^V(\Lambda, \Gamma)$) if for each $x_0 \in \Gamma$ we have $\mathcal{W}^V(\Lambda, \Gamma, x_0)$.

In 1998 Ta Le Loi proved that

Theorem 5.9. The Verdier condition is definable and generic.

Proof. See [TL1] (compare to [LSW] and [DW]). \(\square\)
Remark 5.10. As it was explained in [Ver], the Verdier condition is invariant under definable $C^\eta$ diffeomorphisms, $q \geq 2$. However, it is not $C^1$ invariant, as was shown in [BT].

Now we prove that the Verdier condition has the projection property with respect to weakly Lipschitz mappings of class $C^q$, where $q \geq 2$.

Theorem 5.11. Let $q \geq 2$ and let $\Lambda$, $\Gamma$ be $C^q$ submanifolds of $\mathbb{R}^n \times \mathbb{R}^m$, $\Gamma \subset \overline{\Lambda} \setminus \Lambda$ and $\dim \Gamma < \dim \Lambda$. Let the mapping $\pi_1|_{\Lambda \cup \Gamma}$ be a homeomorphic embedding such that $\pi_1|_\Lambda$, $\pi_1|_\Gamma$ are $C^q$ embeddings. Assume that the mapping $\pi_2|_{\Lambda \cup \Gamma} \circ (\pi_1|_{\Lambda \cup \Gamma})^{-1}$ is weakly Lipschitz on the $C^q$ stratification $\{\pi_1(\Lambda), \pi_1(\Gamma)\}$.

Then

$$
\mathcal{W}^V(\Lambda, \Gamma) \longrightarrow \mathcal{W}^V(\pi_1(\Lambda), \pi_1(\Gamma)).
$$

Proof. Put $x' = \pi_2|_{\Lambda \cup \Gamma} \circ (\pi_1|_{\Lambda \cup \Gamma})^{-1}(x)$, for any $x \in \pi_1(\Lambda \cup \Gamma)$.

Let $x_0 \in \pi_1(\Gamma)$. After making a suitable $C^2$ change of coordinates in a neighbourhood of $(x_0, x'_0)$ we can assume that $\pi_1(\Gamma) = \mathbb{R}^k \times \{0\}^{n-k}$ and $\Gamma = \pi_1(\Gamma) \times \{0\}^m = \mathbb{R}^k \times \{0\}^{n+m-k} = \mathbb{R}^k$. Then $x' = 0$, for any $x \in \pi_1(\Gamma)$.

There exists an open neighbourhood $U_{(x_0,0)}$ of the point $(x_0,0)$ in $\mathbb{R}^n \times \mathbb{R}^m$ and a constant $C_{(x_0,0)} > 0$ such that

$$
d(T_{(x,0)}\Gamma, T_{(y,y')}\Lambda) \leq C_{(x_0,0)} \cdot |(x,0) - (y,y')|,
$$

for each $(x,0) \in \Gamma \cap U_{(x_0,0)}$ and $(y,y') \in \Lambda \cap U_{(x_0,0)}$.

Since the mapping $\pi_2|_{\Lambda \cup \Gamma} \circ (\pi_1|_{\Lambda \cup \Gamma})^{-1}$ is weakly Lipschitz on the stratification $\{\pi_1(\Lambda), \pi_1(\Gamma)\}$, there is a neighbourhood $U_{x_0}$ of the point $x_0$ in $\mathbb{R}^n$ and a constant $L_{x_0} > 0$ such that

$$
\frac{|0 - y'|}{|x - y|} \leq L_{x_0},
$$

for each $x \in \pi_1(\Gamma) \cap U_{x_0}$ and $y \in \pi_1(\Lambda) \cap U_{x_0}$.

Without loss of generality we may assume that $U_{x_0} = U_{(x_0,0)} \cap (\mathbb{R}^n \times \{0\}^m)$. Then from the above argument, for all points $(x,0) \in \Gamma \cap U_{(x_0,0)}$ and $(y,y') \in \Lambda \cap U_{(x_0,0)}$

$$
d(T_{(x,0)}\Gamma, T_{(y,y')}\Lambda) \leq C_{(x_0,0)} \cdot |(x,0) - (y,y')| \leq C_{(x_0,0)} \cdot \sqrt{1 + (L_{x_0})^2} \cdot |x - y|,
$$

3Again we have to assume that $\{\Lambda, \Gamma\}$ is a $C^q$ stratification of $\Lambda \cup \Gamma$.

4As before $\pi_1 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $\pi_2 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ denote natural projections.
in other words
\[ d(\mathbb{R}^k, T_{(y,y')}(\Lambda)) \leq C(x_0,0) \cdot \sqrt{1 + L_{x_0}^2|x - y|}. \]

Hence, after diminishing \( U_{(x_0,0)} \),
\[ d(\mathbb{R}^k, T_{(y,y')}(\Lambda)) \leq 1 - \alpha, \]
where \( 0 < \alpha < 1 \) and then denoting by \( K_y \) the orthogonal projection of \( \mathbb{R}^k \) onto \( T_{(y,y')}(\Lambda) \), we get a \( k \) dimensional subspace \( K_y \) of \( T_{(y,y')}(\Lambda) \) such that
\[ d(\mathbb{R}^k, K_y) = d(\mathbb{R}^k, T_{(y,y')}(\Lambda)). \]

Consequently, by Lemma 5.1 \((ii)\):
\[ d(T_x \pi_1(\Gamma), T_y \pi_1(\Lambda)) = d(\mathbb{R}^k, \pi_1(T_{(y,y')}(\Lambda))) = \]
\[ d(\pi_1(\mathbb{R}^k \times \{0\}), \pi_1(K_y)) \leq C_\alpha \cdot d(\mathbb{R}^k, T_{(y,y')}(\Lambda)) \leq \tilde{C} \cdot |x - y|, \]
which completes the proof.

\[ \square \]

**Corollary 5.12.** The Verdier condition has the projection property with respect to weakly Lipschitz mappings of class \( C^q \), where \( q \geq 2 \).

It remains to deal with the property of lifting the Verdier condition with respect to locally Lipschitz mappings of class \( C^q \), \( q \geq 2 \). The argument is analogous to that in the case of the Whitney (B) condition.

**Theorem 5.13.** Theorem 4.6 remains true when the Whitney (B) condition is replaced by the Verdier condition, assuming that \( q \geq 2 \).

**Proof.** Let \( x_0 \in \Gamma_1 \cap \Gamma_2 \). There exists a neighbourhood \( U_{x_0} \) of \( x_0 \) and a constant \( C_{x_0} > 0 \) such that for each \( x \in \Gamma_i \cap U_{x_0} \) and each \( y \in \Lambda_i \cap U_{x_0} \)
\[ d(T_x \Gamma_i, T_y \Lambda_i) \leq C_{x_0} \cdot |x - y|, \]
where \( i = 1,2 \).

**Case I.** Assume that \( \dim \Lambda_1 \cap \Lambda_2 = \min\{\dim \Lambda_1, \dim \Lambda_2\} = \dim \Lambda_1 \) (the case, when \( \dim \Lambda_1 \cap \dim \Lambda_2 = \min\{\dim \Lambda_1, \dim \Lambda_2\} = \dim \Lambda_2 \) is similar).
Then \( \Lambda_1 \cap \Lambda_2 \) is open in \( \Lambda_1 \), hence \( T_y(\Lambda_1 \cap \Lambda_2) = T_y \Lambda_1 \) and
\[ d(T_x(\Gamma_1 \cap \Gamma_2), T_y(\Lambda_1 \cap \Lambda_2)) \leq d(T_x \Gamma_1, T_y \Lambda_1) \leq C_{x_0} \cdot |x - y| \]
for each \( x \in \Gamma_1 \cap \Gamma_2 \cap U_{x_0} \) and \( y \in \Lambda_1 \cap \Lambda_2 \cap U_{x_0} \), which completes the proof of the Case.
Case II. Assume now that \( \dim \Lambda_1 \cap \Lambda_2 < \min \{ \dim \Lambda_1, \dim \Lambda_2 \} \). Let
\[
\Sigma = \{(S, K) \in G_{\dim \Lambda_1,n} \times G_{\dim \Lambda_2,n} : \dim(S \cap K) = \dim \Lambda_1 \cap \Lambda_2 \}.
\]
Then by the assumptions, after perhaps diminishing the neighbourhood \( U_{x_0} \) the closure \( \tilde{\Sigma} \) of the set
\[
\{(T_y \Lambda_1, T_y \Lambda_2) \in G_{\dim \Lambda_1,n} \times G_{\dim \Lambda_2,n} : y \in \Lambda_1 \cap \Lambda_2 \cap U_{x_0} \}
\]
is a closed subset of \( \Sigma \). By Corollary 5.7 there exists a constant \( C > 0 \) such that for all \( x \in \Gamma_1 \cap \Gamma_2 \cap U_{x_0} \), \( y \in \Lambda_1 \cap \Lambda_2 \cap U_{x_0} \) we have
\[
d(T_x \Gamma_1 \cap T_x \Gamma_2, T_y \Lambda_1 \cap T_y \Lambda_2) \leq C \cdot (d(T_x \Gamma_1, T_y \Lambda_1) + d(T_x \Gamma_2, T_y \Lambda_2)).
\]
Consequently,
\[
d(T_x (\Gamma_1 \cap \Gamma_2), T_y (\Lambda_1 \cap \Lambda_2)) \leq C \cdot 2C_{x_0} \cdot |x - y|.
\]

**Theorem 5.14.** Theorem 4.7 remains true when the Whitney (B) condition is replaced by the Verdier condition, assuming that \( q \geq 2 \).

**Proof.** It follows from Theorem 5.13 in the same way as Theorem 4.7 follows from Theorem 4.6.

**Corollary 5.15.** The Verdier condition is a \( \mathcal{WL} \) condition of class \( C^q \), \( q \geq 2 \).

**Corollary 5.16.** Theorem 3.15 holds true for the Verdier condition with \( q \geq 2 \).

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**References**

[BT] H. Brodersen, D. Trotman, *Whitney (b) - regularity is weaker than Kuo’s ratio test for real algebraic stratifications*, Math. Scand. 45 (1979), p.27-34.

[D] L. van den Dries, *Tame topology and o-minimal structures*, Cambridge University Press, 1998.

[DM] L. van den Dries, C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. 84 (1996), no. 2, 497-540.
[DW] Z. Denkowska, K. Wachta, *Une construction de la stratification sous-analytique avec la condition (w)*, Bull. Pol. Acad. Sci. Math. 35 (1987), n. 7-8, 401-405.

[LSW] S. Lojasiewicz, J. Stasica, K. Wachta, *Stratifications sous-analytiques. Condition de Verdier*, Bull. Pol. Acad. Sci. Math. 34 (1986), 531-539.

[L] S. Lojasiewicz, *An introduction to complex analytic geometry* Birkhäuser, 1991.

[S] M. Shiota, *Whitney triangulations of semialgebraic sets*, Ann. Polon. Math. 87, 2005, p. 237-246.

[Te] B. Teissier *Variétés Polaires II: Multiplicités polaires, sections planes et conditions de Whitney*, Algebraic Geometry Proceedings, La Rabida 1981.

[TL1] T. Le Loi, *Verdier and strict Thom stratifications in o-minimal structures*, Illinois J. of Math., vol. 42, no. 2 (1998), 347-356.

[TL2] T. Le Loi, *Whitney stratification of sets definable in the structure \( \mathbb{R}_{\exp} \)*, Banach Center Publications, vol. 33 (1996), 401-409.

[Tro] D.J.A. Trotman, *Geometric versions of Whitney regularity for smooth stratifications*, Ann. Sci., Ec.Norm.Sup. 4° serie, t.12, 1979, p. 453-463.

[Ver] J.-L. Verdier, *Stratifications de Whitney et théorème de Bertini-Sard*, Invent. Math. 36 (1976), p.295-312.

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