On the existence of solutions to the relativistic Euler equations in two spacetime dimensions with a vacuum boundary

Todd A Oliynyk
School of Mathematical Sciences, Monash University, VIC 3800, Australia
E-mail: todd.oliynyk@monash.edu

Received 10 July 2011, in final form 12 April 2012
Published 13 July 2012
Online at stacks.iop.org/CQG/29/155013

Abstract
We prove the existence of a wide class of solutions to the isentropic relativistic Euler equations in two spacetime dimensions with an equation of state of the form $p = K \rho^2$ that have a fluid vacuum boundary. Near the fluid vacuum boundary, the sound speeds for these solutions are monotonically decreasing, approaching zero where the density vanishes. Moreover, the fluid acceleration is finite and bounded away from zero as the fluid vacuum boundary is approached. The existence results of this paper also generalize in a straightforward manner to equations of state of the form $p = K \rho^{\gamma}$ with $\gamma > 0$.

PACS number: 47.75.+f

1. Introduction

Letting $g = g_{\mu\nu} \, dx^\mu \, dx^\nu$, $\mu, \nu = 0, 1$, denote a flat Lorentz metric of signature $(+, -)$ on a two-dimensional manifold $M \subset \mathbb{R}^2$, the relativistic Euler equations are
\[
\nabla_\mu T^{\mu\nu} = 0
\]
where
\[
T^{\mu\nu} = (\rho + p) v^\mu v^\nu - pg^{\mu\nu} \quad \text{and} \quad g_{\mu\nu} v^\mu v^\nu = 1.
\]

Here, $v^\mu$ is the fluid two-velocity, $\rho$ the proper energy density of the fluid and $p$ the pressure. Projecting (1.1) into the subspaces parallel and orthogonal to $v^\mu$ yields the following well-known form of the Euler equations
\[
v^\mu \nabla_\mu \rho + (\rho + p) \nabla_\mu v^\mu = 0, \quad \text{(1.2)}
\]
\[
(\rho + p) v^\mu \nabla_\mu v^\nu - h^{\mu\nu} \nabla_\mu p = 0 \quad \text{(1.3)}
\]
where

\[ h_{\mu\nu} = g_{\mu\nu} - v_\mu v_\nu \quad (v_\mu = g_{\mu\nu} v^\nu) \]  

(1.4)

is the induced negative definite metric on the subspace orthogonal to \( v^\mu \). In this paper, we will be primarily concerned with fluids with an insentropic equation of state of the form

\[ p = K \rho^{\gamma}, \quad K, \gamma > 0, \]  

(1.5)

and in particular \( \gamma = 1 \).

The main aim of this paper is to construct solutions to the Euler equations that contain a vacuum region; that is, solutions for which \( \rho \) vanishes, and the fluid vacuum boundary is located at \( \rho = 0 \). The problem of existence of solutions to the Euler equations with a vacuum region for equations of state for which the pressure and proper energy density simultaneously vanish has been studied by a number of authors. The main difficulty in establishing existence is due to the fact that the Euler equations (1.2)–(1.3) become degenerate as the fluid vacuum boundary is approached. This is due to the vanishing of the quantity \( \rho + p \) at the fluid vacuum boundary. Because of this, standard techniques from hyperbolic PDE theory do not apply. The first general existence result where this problem was overcome is in [16]. In this work, the existence of gravitating, non-relativistic fluid bodies with compactly supported densities in four spacetime dimensions was established (see [18] and [15] for the relativistic case); the technique employed involved a special choice of variables to regularize the Euler equations, and works in all dimensions with or without coupling to gravity. However, the type of fluid solutions obtained by this method have, when coupled to gravity, freely falling boundaries, and hence, do not include static or expanding fluid bodies. This is because the fluid acceleration

\[ a^\mu = v^\mu \nabla_\mu v^\nu \]  

(1.6)

of the solutions from [16] vanish at the fluid vacuum boundary, and so there is no outward force to counteract the gravitational force which leads to collapse of the body. Consequently, to have a class of solutions which include the full physical range including static and expanding fluid bodies, it is necessary to establish the existence of solutions to the Euler equations for which the acceleration (1.6) is non-zero at the fluid vacuum boundary. Due to the finite propagation speed of the Euler equations, it is enough to consider the existence problem in a neighborhood of the fluid vacuum boundary. Away from the boundary where \( \rho + p > 0 \), standard symmetric hyperbolic techniques can be used.

In two spacetime dimensions, the existence of solutions to the non-relativistic Euler equations with non-zero acceleration at the fluid vacuum boundary has been previously
established in the remarkable articles [5, 11]. In these articles, existence is proved using non-
standard energy estimates combined with suitable approximation techniques. The arguments
used are technical, involved, highly original and quite different from one another. We also note
that the results in [5, 11] have been recently extended to four spacetime dimensions [6, 12]
(see also [4]). At the moment, the relationship between the solutions of this paper and those
of [5, 11] is not clear because the solutions presented here are relativistic while those from
[5, 11] are non-relativistic. In order to understand the relationship, it would be necessary to
take the non-relativistic (i.e. \( c \rightarrow \infty \)) limit of our solutions. We will not do this here, but leave
the analysis of this limit to a separate article.

Overview of the paper. In section 2, we review the Frauendiener–Walton formulation of
the isentropic Euler equations, and we use this formulation to show, in section 3, that the
Euler equations are equivalent to the existence of a suitably normalized, commuting set of
vector fields. This commuting set of vector fields allows us to introduce adapted coordinates
for the Lorentzian metric. As detailed in section 4.1, the flatness of the Lorentzian metric
then implies the equivalence of the Euler equations with a nonlinear scalar wave equation.
Conformal coordinates are introduced in section 4.2 which reduce, as described in section 4.3,
the nonlinear wave equation to a linear one. In section 4.4, it is shown that the existence of
conformal coordinates is equivalent to the existence of suitable solutions for a linear wave
equation with singular coefficients. An existence and regularity theory for this type of wave
equation is developed in the appendix. The main result of this paper is contained in section 4.5
where the existence of solutions to the Euler equations with non-zero acceleration at the
vacuum boundary is established, see theorem 4.8 for a precise statement. In section 5, a class
of exact solutions with non-zero acceleration at the vacuum boundary are described.

2. The Frauendiener–Walton formulation of the Euler equations

In [7, 21], Frauendiener and Walton independently showed that the isentropic Euler equations
for a perfect fluid with an equation of state of the form \( p = p(\rho) \) can be written as:

\[
A_{\mu\nu}^\gamma \nabla_\gamma w^\nu = 0
\]

where \( w^\nu \) is a timelike vector field with norm

\[
w^2 = w_\nu w^\nu > 0 \quad (w_\mu = g_{\mu\nu} w^\nu)
\]

and

\[
A_{\mu\nu}^\gamma = \left( 3 + \frac{1}{s^2} \right) \frac{w_\mu w_\nu}{w^2 - w^\gamma w_\gamma} - \delta_\mu^\gamma w_\nu - \delta_\nu^\gamma w_\mu - w^\gamma g_{\mu\nu}
\]

We will refer to these equations as the Euler–Frauendiener–Walton (EFW) equations.

In the Frauendiener–Walton formulation, \( s^2 \) is a function of

\[
\xi = \frac{1}{w}
\]

where

\[
w = \sqrt{w^2}.
\]

An explicit formula for \( s^2 \) can be calculated in the following fashion (see [7] for more details).
First, the pressure \( p = p(\xi) \) is determined implicitly by the equation

\[
\xi = \xi_0 \Upsilon(p(\xi))
\]

1 It is important to note that Frauendiener and Walton use opposite signature conventions for the metric \( g \) and different
notation for the fluid variables. In this paper, we use the notation and signature conventions of Frauendiener [7].
where
\[ \Upsilon(p) = \exp \left( \int_{p_0}^{p} \frac{d\tilde{p}}{\rho(\tilde{p}) + \tilde{p}} \right) \] (2.4)
is the Lichnerowicz index of the fluid. From this, \( s^2 \) can be calculated using the formula
\[ \frac{1}{s^2} = \left( \frac{\xi f'(\xi)}{f(\xi)} - 3 \right) \] (2.5)
where
\[ f(\xi) = \xi^3 p'(\xi). \]
Additionally, the proper energy density \( \rho \) and two-velocity \( v^\mu \) can be recovered from
\[ \rho = f(\xi) - p(\xi) \quad \text{and} \quad v^\mu = \xi w^\mu. \] (2.6)
As shown in [7] and also [21], the triple \( \{ \rho, p, v^\mu \} \) determined from (2.1), (2.3) and (2.6) satisfy the relativistic Euler equations (1.2)–(1.3).

For the equations of states (1.5), it is possible to explicitly determine the functional form of \( s^2 = s^2(\xi) \). To see this, we observe that the Lichnerowicz index (2.4) is given by
\[ \Upsilon(p) = \exp \left( \int_{p_0}^{p} \frac{d\tilde{p}}{\rho(\tilde{p}) + \tilde{p}} \right) \left( 1 + \frac{K}{\rho^{\frac{1}{\gamma}}} \right)^{\gamma+1}. \]
From this expression, it is clear
\[ p(\xi) = \frac{1}{K^{\gamma}} \left( \left( \frac{\xi}{\xi_0} \right)^{\frac{1}{\gamma}} - 1 \right)^{\gamma+1} \]
solves (2.3), and hence determines the pressure as a function of \( \xi \). Without loss of generality, we set \( \xi_0 = 1 \) which gives
\[ p(\xi) = \frac{1}{K^{\gamma}} (\xi^{\frac{1}{\gamma}} - 1)^{\gamma+1}. \]
Substituting this expression into (2.5) then yields
\[ s^2 = \frac{\gamma}{\gamma} \left( \frac{1}{\xi} - 1 \right). \] (2.7)
For latter use, we note that by using (1.3) the fluid acceleration can be written as
\[ a^\nu = \frac{h^{\mu\nu}}{\rho + \tilde{p}} \nabla_\mu p. \]
Since this vector is orthogonal to \( v^\mu \), i.e. \( v_\nu a^\nu = 0 \), we can use the metric \( h_{\mu\nu} \) to calculate the length of \( a^\nu \) via the formula
\[ |a|_h := \sqrt{-h_{\mu\nu} a^\mu a^\nu} = \sqrt{\frac{-h^{\mu\nu}}{(\rho + \tilde{p})^2} \nabla_\mu p \nabla_\nu p}. \] (2.8)
Recalling that the square of the sound speed is given by
\[ s^2 = \frac{dp}{d\rho} = \frac{K(\gamma + 1)}{\rho^\frac{1}{\gamma}}, \] (2.9)
we can write (2.8) as
\[ |a|_h = \sqrt{\frac{\gamma^2 h^{\mu\nu}}{(1 + K \rho^{\frac{1}{\gamma}})^2} \nabla_\mu s^2 \nabla_\nu s^2} = \frac{\gamma}{(1 + K \rho^{\frac{1}{\gamma}})} |ds^2|_h. \] (2.10)
This shows that the acceleration will be non-zero at the fluid vacuum boundary if and only if \( |ds^2|_h > 0 \) there, since \( \rho \) vanishes at the boundary. It is also clear from this formula that the fluid acceleration can only be non-zero at the fluid vacuum boundary if the gradient of \( s^2 \) does not vanish there. Since the proper energy density, and hence the sound speed, is decreasing to zero as the fluid vacuum boundary is approached, it follows that for such solutions \( s^2 \) must be monotonically decreasing in the neighborhood of the boundary.
3. A frame formulation for the EFW equations

Since the metric $g = g_{\mu\nu} \, dx^\mu \, dx^\nu$ is flat, we begin by introducing global Minkowskian coordinates\(^2\) $(x^\mu) = (x^0, x^1)$ for which

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hspace{1cm} (3.1)

With this choice of coordinates, the Euler equations (1.2)–(1.3) can be written explicitly in terms of the proper energy density $\rho(x^0, x^1)$ and a function $v(x^0, x^1)$ satisfying

$$\begin{align*}
\frac{1}{\sqrt{1 - v^2}} \partial_0 \rho + \frac{v}{\sqrt{1 - v^2}} \partial_1 \rho + (\rho + K \rho \frac{1}{\sqrt{1 - v^2}}) \left( \frac{1}{\sqrt{1 - v^2}} \partial_0 \left( \frac{v}{\sqrt{1 - v^2}} \right) + \partial_1 \left( \frac{v}{\sqrt{1 - v^2}} \right) \right) &= 0 \\
(\rho + K \rho \frac{1}{\sqrt{1 - v^2}}) \left( \frac{1}{\sqrt{1 - v^2}} \partial_0 \left( \frac{v}{\sqrt{1 - v^2}} \right) + \frac{v}{\sqrt{1 - v^2}} \partial_1 \left( \frac{v}{\sqrt{1 - v^2}} \right) \right) &+ K \gamma + 1 \gamma^\frac{1}{\gamma} \frac{1}{\sqrt{1 - v^2}} \partial_0 \rho + \left( 1 + \frac{v^2}{1 - v^2} \right) \partial_1 \rho &= 0 
\end{align*}$$

with the two-velocity given by

$$(u^\mu) = \frac{1}{\sqrt{1 - v^2(1, v(0, x^1))}} (1, v(0, x^1)).$$ \hspace{1cm} (3.2)

Using (2.7) and (2.9), a short calculation shows that

$$w = \frac{1}{(1 + K \rho \frac{1}{\sqrt{1 - v^2}})^{\gamma + 1}}$$

from which we obtain

$$(u^\mu) = \frac{1}{(1 + K \rho(1, v(0, x^1)) \frac{1}{\sqrt{1 - v(0, x^1)}^2}) (1, v(0, x^1))}$$ \hspace{1cm} (3.3)

by (3.2) and (2.6). Letting

$$A^0 = \begin{pmatrix}
\left[ \left( 3 + \frac{1}{s^2} \right) \frac{(w^0)^2}{w^2} - 3 \right] w^0 \\
- \left( 3 + \frac{1}{s^2} \right) \frac{(w^0)^2}{w^2} + 1 \end{pmatrix} + \left[ \left( 3 + \frac{1}{s^2} \right) \frac{(w^0)^2}{w^2} + 1 \right] w^1
\end{pmatrix}
$$

and

$$A^1 = \begin{pmatrix}
\left[ \left( 3 + \frac{1}{s^2} \right) \frac{(w^0)^2}{w^2} - 1 \right] w^1 \\
- \left( 3 + \frac{1}{s^2} \right) \frac{(w^0)^2}{w^2} + 1 \end{pmatrix} w^0 - \left[ \left( 3 + \frac{1}{s^2} \right) \frac{(w^0)^2}{w^2} + 1 \right] w^0
\end{pmatrix},
$$

where

$$w^2 = (w^0)^2 - (w^1)^2$$

and (see (2.7))

$$s^2 = \gamma + 1 \left( \frac{1}{(w^2)^{\gamma + 1}} - 1 \right),$$

\(^2\) We note that this step of introducing a global Minkowskian set of coordinates is not necessary. We include it in order to make the following arguments more accessible to readers who may not be familiar with a more abstract approach to fixing coordinates. A more abstract approach becomes particularly important when dealing with non-flat metrics that arise when coupling the fluid to gravity.
we can also write the EFW equations (2.1) explicitly as
\[ A^0 \partial_0 (w^0) + A^1 \partial_1 (w^1) = 0. \]

In order to derive a formulation of the Euler equations that is suitable to analyze the limits \( s^2 \searrow 0 \), we use a particular frame formulation of the EFW equations. First, we set
\[ e^0 = e^\mu_0 \partial_\mu = w = w^\mu \partial_\mu \]
where
\[ \partial_\mu = \frac{\partial}{\partial x^\mu}, \]
and choose
\[ e^1 = e_0^\mu \partial_\mu \]
orthogonal to \( e^0 \) so that the frame metric
\[ g_{ij} = g(e_0, e_0) \]
satisfies
\[ g_{0j} = g_{10} = 0 \]
and
\[ g_{00} = g(e_0, e_0) = w^2. \]

At the moment, we leave the length of \( e^1 \) unspecified. The freedom to fix the length of \( e^1 \) will be used below. From (3.1) and (3.3), it is clear that
\[ (e^\mu_1) = q(x^0, x^1)(v(x^0, x^1), 1) \]
where \( q(x^0, x^1) \) is a function to be determined.

Next, we denote the coframe by
\[ \theta^i = \theta^i_\mu dx^\mu \quad (\theta^i_\mu e^\mu_j = \delta^i_j), \]
and let \( \omega^k_{ij} \) denote the connection coefficients so that
\[ \nabla_r e^r_j = \omega^k_{ij} e^j_k. \]

We also define the connection 1-forms \( \omega^k_{ij} \) in the standard fashion
\[ \omega^k_{ij} = \omega^k_\mu \theta^i_\mu \theta^j_\nu. \]
and set
\[ \omega_{kj} = g_{kl} \omega^l_{ij} = \omega_{jk} \theta^i \]
where
\[ \omega_{jk} = g_{kl} \omega^l_{ij} = \omega_{kj} = g(\nabla_r e^r_j, e^r_k). \]

Letting
\[ A^{ijk} = g^{lp} g^{qi} e^r_p e^s_q \theta^l_r \theta^i_s A_{rs}^{kl}, \]
a short calculation using (2.2), (3.4), (3.5) and (3.6) shows that
\[ A^{ijk} = \left( 3 + \frac{1}{s^2} \right) \frac{\delta^i_0 \delta^j_1 \delta^k_1}{g_{00}} - \delta^i_0 g^{jk} - g^{ij} \delta^k_0 - g^{ij} \delta^k_0. \]
Moreover, it follows from (3.4) and (3.8) that
\[ \omega_{k0} = e^i_k \nabla_\mu w^\mu e^j_f g_{ij}. \]
Together (3.9), (3.10) and the invertibility of $g^{ij}$ and $e^i_{\mu}$ show that the EFW equations (2.1) are equivalent to
\[
A^{jk} \omega_{k0} = 0. \tag{3.11}
\]
Using (2.7), (3.5), (3.6) and (3.9), the $i = 0$ and $i = 1$ components of (3.11) are
\[
\omega_{000} - s^2 \frac{g_{00}}{g_{11}} \omega_{110} = 0 \quad \text{and} \quad \omega_{100} + \omega_{010} = 0, \tag{3.12}
\]
respectively, where
\[
s^2 = \frac{\gamma + 1}{\gamma} \left( \frac{1}{g_{00}^{\frac{1}{\gamma-1}}} - 1 \right). \tag{3.13}
\]
Also, due to the connection $\omega_{ij}$ being metric, the frame metric satisfies (see [2, chapter V, section B])
\[
dg_{jk} = \omega_{jk} + \omega_{kj},
\]
or equivalently
\[
e_1(g_{jk}) = \omega_{jk} + \omega_{kj}, \tag{3.14}
\]
and it follows immediately from (3.5) that
\[
\omega_{001} + \omega_{010} = 0. \tag{3.15}
\]
Since the connection $\omega_{ij}$ is torsion free\(^3\), it satisfies the following Cartan structure equation (see [2, chapter V, section B]):
\[
d\theta^i + \omega_{ij} \land \theta^j = 0,
\]
or equivalently
\[
[e_0, e_1] = (\omega_0^{\ i} - \omega_1^{\ i}) e_k.
\]
We rewrite this as follows:
\[
[e_0, e_1] = -\frac{1}{g_{00}} (\omega_{010} + \omega_{100}) e_0 + \frac{1}{g_{11}} (\omega_{011} - \omega_{110}) e_1 \quad \text{by (3.5) and (3.15)}
\]
\[
= \frac{1}{g_{11}} \left( \omega_{011} - \frac{g_{11}}{s^2 g_{00}} \omega_{000} \right) e_1 \quad \text{by (3.12)}
\]
\[
= \frac{1}{2g_{11}} \left( e_0 (g_{11}) - \frac{g_{11}}{s^2 g_{00}} e_0 (g_{00}) \right) e_1. \quad \text{by (3.14).} \tag{3.16}
\]
Defining a function by
\[
F(\xi) = \frac{1}{\xi^{\frac{1}{\gamma-1}} \left( \frac{1}{\xi^{\frac{1}{\gamma-1}}} - 1 \right)},
\]
a short calculation shows that $F(\xi)$ satisfies
\[
F'(\xi) = \frac{F(\xi)}{2\xi^{\frac{1}{\gamma-1}} \left( \frac{1}{\xi^{\frac{1}{\gamma-1}}} - 1 \right)},
\]
and hence
\[
F'(g_{00}) = \frac{F(g_{00})}{2g_{00} s^2}
\]
\(^3\) This follows by virtue of $\omega_{ij}$ being the Levi-Civita connection for the metric.
by (3.13). But this implies that
\[ e_0 \ln(F(g_{00}))^2 = \frac{1}{s^2 g_{00}} e_0(g_{00}), \]
which we can, in turn, use to write (3.16) as
\[ [e_0, e_1] = \frac{1}{2} e_0 \left( \ln \left( \frac{-g_{11}}{F(g_{00})^2} \right) \right) e_1. \]  
(3.17)
We now use the freedom to fix the length of \( e_1 \) by setting
\[ g_{11} = -F(g_{00})^2. \]  
(3.18)
By (3.17), we arrive at the equivalence of the EFW equations with the vanishing of the following Lie bracket
\[ [e_0, e_1] = 0. \]  
(3.19)
We also note that if we set
\[ u^{2\gamma} = \frac{1}{F(g_{00})^2}, \]  
(3.20)
then (3.20) can be solved for \( g_{00} \) to give
\[ u^2 = g_{00} = (1 - u)^{2(\gamma + 1)}, \]  
(3.21)
which allows us, using (2.7), to write the square of the sound speed as
\[ s^2 = \gamma + \frac{1}{\gamma} \frac{u}{1 - u}. \]  
(3.22)
Using this and the normalization condition (3.18), we see from (3.7) that
\[ q = \frac{1}{\sqrt{1 - v^2 u^{2\gamma}}} = \frac{1}{\sqrt{1 - v^2 \left( \frac{x^1}{x^{1'}} \right)^\gamma}}, \]
and hence, that
\[ (e_1^\mu) = \frac{(1 + K\rho(x^0, x^1)^{1/2})^{\gamma'}}{K\gamma \rho(x^0, x^1)\sqrt{1 - v(x^0, x^1)^2}} \left( v(x^0, x^1), 1 \right), \]
by (2.9).
To summarize, the main result of this section is that the EFW equations are equivalent to the vanishing of the Lie bracket (3.19), which in components reads
\[ e_0^\mu \partial_\mu e_1^\nu - e_1^\rho \partial_\rho e_0^\nu = 0 \]
where the frame components \( e_\mu^\nu \) are given by
\[ (e_0^\mu) = \frac{1}{(1 + K\rho(x^0, x^1)^{1/2})^{\gamma + 1/2}} \sqrt{1 - v(x^0, x^1)^2} \left( v(x^0, x^1), 1 \right), \]  
(3.23)
\[ (e_1^\mu) = \frac{(1 + K\rho(x^0, x^1)^{1/2})^{\gamma'}}{K\gamma \rho(x^0, x^1)\sqrt{1 - v(x^0, x^1)^2}} \left( v(x^0, x^1), 1 \right). \]  
(3.24)
4. Existence of solutions to the Euler equations

4.1. A wave equation formulation of the EFW equations

The vanishing of the Lie bracket (3.19) implies the existence of coordinates which, at least locally, trivialize the vector fields $e_0$ and $e_1$. To construct these coordinates, we let

$$\mathcal{F}_{i,t}(\tilde{x}^0, \tilde{x}^1) = (\mathcal{F}_{i,t}^0(\tilde{x}^0, \tilde{x}^1), \mathcal{F}_{i,t}^1(\tilde{x}^0, \tilde{x}^1)), \quad i = 0, 1,$$

denote the flow maps of the vector fields $e_i$ ($i = 0, 1$), that is, $\mathcal{F}_{i,t}$ is the unique solution to the initial value problem

$$\frac{d}{dt} \mathcal{F}_{i,t}^\mu (\tilde{x}^0, \tilde{x}^1) = e_i^\mu (\mathcal{F}_{i,t} (\tilde{x}^0, \tilde{x}^1)), \quad \mu = 0, 1,$$

where the $e_i^\mu$ are given explicitly by the formulas (3.23)–(3.24). By translating, if necessary, we can assume the origin $(\tilde{x}^0, \tilde{x}^1) = (0, 0)$ is in the domain on which the vector fields $e_i^\mu$ are defined. We then introduce a change of coordinates by the formula

$$(\tilde{x}^0, \tilde{x}^1) = \Psi(\bar{x}^0, \bar{x}^1) = \mathcal{F}_{0, \sqrt{\rho}} \circ \mathcal{F}_{1, e^{-c}}(0, 0)$$

(4.1)

where $c$ is a constant$^4$.

Since the vector fields $w = e_0$ and $e_1$ commute, if we define

$$\hat{w} = \hat{e}_0 = \Psi^* e_0 \quad \text{and} \quad \hat{e}_1 = \Psi^* e_1,$$

then

$$\hat{w} = \hat{e}_0 = \sqrt{\frac{\gamma + 1}{\gamma}} \hat{\partial}_0 \quad \text{and} \quad \hat{e}_1 = \hat{\partial}_1$$

(4.2)

where

$$\hat{\partial}_\mu = \frac{\partial}{\partial \bar{x}^\mu}.$$

As we show below (see remark 4.7), the vacuum boundary where $\rho$ vanishes is contained in the set $\bar{x}^1 = 0$. Since the two-velocity $\hat{v}^\mu$ is given by $\hat{v}^\mu = (\hat{w})^{-1} \hat{w}^\mu$, equation (4.2) shows how the vacuum boundary moves with the fluid as expected.

Next, defining

$$\hat{g} = \Psi^* g = \hat{g}_{\mu \nu} d\bar{x}^\mu d\bar{x}^\nu \quad \text{and} \quad \hat{u} = \Psi^* u,$$

it follows directly from (4.2), (3.18), (3.20) and (3.21) that$^5$

$$\hat{g} = \frac{\gamma (1 - \hat{u})^{2(\gamma + 1)}}{\gamma + 1} d\bar{x}^0 d\bar{x}^0 - \frac{1}{\hat{u}^{2\gamma}} d\bar{x}^1 d\bar{x}^1.$$  

(4.3)

Computing the Ricci scalar of this metric, we find that

$$\hat{R} = \frac{\hat{u}^\rho}{G(\hat{u})^{1/2}} \left[ 2(\gamma + 1) \hat{\partial}_0 \left( \frac{1}{\hat{u}^{\gamma + 1} G(\hat{u})^{1/2}} \hat{\partial}_0 \hat{u} \right) + \hat{\partial}_1 \left( \frac{\hat{u}^\rho G(\hat{u})}{G(\hat{u})^{1/2}} \hat{\partial}_1 \hat{u} \right) \right]$$  

(4.4)

where

$$G(\hat{u}) = (1 - \hat{u})^{2(\gamma + 1)}.$$

$^4$ The constant $c$ is chosen so that the point $\lim_{t \to \infty} \mathcal{F}_{i,t} \circ \mathcal{F}_{1, e^{-c}}(0, 0)$ lies on the vacuum boundary where $\rho = 0$.

$^5$ The metric $\hat{g} = \Psi^* g$ is just the original Minkowski metric $g$ (see (3.1)) expressed in the $(\bar{x}^\mu)$ coordinates defined by (4.1).
Remark 4.1. To extend the analysis to \( \gamma > 0 \), the appropriate \( \hat{z} \) variable that replaces (4.5) can be obtained by solving the initial value problem

\[
\frac{d\hat{z}}{d\hat{u}} = \left( -\frac{G'(\hat{u})}{2(\gamma + 1)\hat{u}G(\hat{u})} \right)^{1/2} : \hat{z}(0) = 0
\]

for \( \hat{u} \geq 0 \). A solution to this initial value problem yields the identity

\[
\frac{1}{\hat{u}^{\gamma+1}G(\hat{u})^{1/2}} \frac{d\hat{z}}{d\hat{u}} = \left( -\frac{\hat{u}''G'(\hat{u})}{2(\gamma + 1)G(\hat{u})^{1/2}} \frac{d\hat{z}}{d\hat{u}} \right)^{-1},
\]

which allows a similar analysis as in the \( \gamma = 1 \) case to be used.

In terms of the \( \hat{z} \) variable, the metric (4.3) and the Ricci scalar (4.4) become

\[
\hat{g} = \frac{(1 + \cos(\hat{z}))^4}{32} \text{d}x^0 \text{d}x^0 - \frac{4}{(1 - \cos(\hat{z}))^2} \text{d}x^1 \text{d}x^1
\]

and

\[
\hat{R} = \frac{(1 - \cos(\hat{z}))}{2(1 + \cos(\hat{z}))^2} \left[ \hat{\partial}_0 \left( \frac{8}{\sin(\hat{z})^3} \hat{\partial}_0 \hat{z} \right) - \hat{\partial}_1 \left( \frac{\sin(\hat{z})^3}{8} \hat{\partial}_1 \hat{z} \right) \right],
\]

respectively. But the metric (4.7) is flat\(^6\), and so (4.8) and the above arguments show that the EFW equations (2.2) are equivalent to the wave equation

\[
\hat{\partial}_0 \left( \frac{8}{\sin(\hat{z})^3} \hat{\partial}_0 \hat{z} \right) - \hat{\partial}_1 \left( \frac{\sin(\hat{z})^3}{8} \hat{\partial}_1 \hat{z} \right) = 0.
\]

Remark 4.2. It is well known that in \( 1 + 1 \) dimensions\(^7\) the Euler equations can be reduced to a quasi-linear scalar wave equations of the form (see [3] or [19] for details)

\[
\nabla_\mu (H(|\nabla \varphi|^2)\nabla^\mu \varphi) = 0 \quad (|\nabla \varphi|^2 = g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi)
\]

where \( H(\cdot) \) is a particular function determined by the equation of state, and the proper energy density \( \rho \) and two-velocity \( v^\mu \) can be recovered via the formulas

\[
|\nabla \varphi|^2 = \exp \left( \int_0^p \frac{dp}{\rho(p) + p} \right) \quad \text{and} \quad v^\mu = \frac{1}{|\nabla \varphi|} g^{\mu\nu} \partial_\nu \varphi.
\]

In light of this, it is perhaps not surprising that we are able to reduce the Euler equations to a quasi-linear scalar wave equation. What is new here is that proper energy density is a function of the scalar \( \hat{z} \), and it vanishes where \( \hat{z} \) vanishes. As will be shown below, this makes the wave equation (4.9) particularly well suited for analyzing the vacuum boundary problem.

\(^6\) We recall that the Ricci scalar is a coordinate invariant which can be stated as \( \hat{R} = R \circ \Psi \). Since \( g \) is the Minkowski metric and consequently flat, its Ricci scalar vanishes. By the coordinate invariance, the Ricci scalar of \( \hat{g} \) must also vanish.

\(^7\) In fact, this is true for irrotational fluids in any dimension.
Defining an auxiliary Lorentzian metric \( \hat{\lambda} \) by

\[
\hat{\lambda} = \hat{\lambda}_{\mu
u} \, d\hat{x}^\mu \, d\hat{x}^\nu = \frac{\sin(\hat{z})^3}{8} \, d\hat{x}^0 \, d\hat{x}^0 - \frac{8}{\sin(\hat{z})^3} \, d\hat{x}^3 \, d\hat{x}^3.
\]

(4.10)

equation (4.9) is easily seen to be equivalent to

\[
\Box \hat{\lambda} = \frac{1}{\sqrt{-\hat{\lambda}}} \partial_\mu (\sqrt{-\hat{\lambda}} \hat{\lambda}^{\mu\nu} \partial_\nu \hat{\lambda}) = 0
\]

(4.11)

where

\[
\sqrt{-\hat{\lambda}} = \sqrt{-\det(\hat{\lambda}_{\mu\nu})}
\]

and \( \hat{\lambda}^{\mu\nu} = (\hat{\lambda}_{\mu\nu})^{-1} \).

4.2. Conformal coordinates

We introduce a second change of coordinates

\[
(\hat{x}^0, \hat{x}^1) = \bar{\Psi}(\bar{x}^0, \bar{x}^1) = (\bar{\psi}(\bar{x}^0, \bar{x}^1), \bar{\phi}(\bar{x}^0, \bar{x}^1))
\]

(4.12)

and let

\[
\bar{z} = \bar{\Psi}^* \hat{z}.
\]

(4.13)

Pulling back the metric (4.10) using the diffeomorphism (4.12), we find, using (4.13), that

\[
\bar{\lambda} = \Phi^* \hat{\lambda} = \bar{\lambda}_{\mu\nu} \, d\bar{x}^\mu \, d\bar{x}^\nu
\]

(4.14)

where

\[
(\bar{\lambda}_{\mu\nu}) = \begin{pmatrix}
\mu (\bar{\partial}_0 \bar{\psi})^2 - \frac{1}{\mu} (\bar{\partial}_0 \bar{\phi})^2 & \mu (\bar{\partial}_0 \bar{\psi} \bar{\partial}_1 \bar{\phi}) - \frac{1}{\mu} (\bar{\partial}_0 \bar{\phi} \bar{\partial}_1 \bar{\psi}) \\
\mu (\bar{\partial}_0 \bar{\psi} \bar{\partial}_1 \bar{\phi}) - \frac{1}{\mu} (\bar{\partial}_0 \bar{\phi} \bar{\partial}_1 \bar{\psi}) & \mu (\bar{\partial}_1 \bar{\psi})^2 - \frac{1}{\mu} (\bar{\partial}_1 \bar{\phi})^2
\end{pmatrix},
\]

\[
\bar{\mu} = \frac{\sin^3(\bar{z})}{8}
\]

and

\[
\bar{\partial}_\mu = \frac{\partial}{\partial \bar{x}^\mu}.
\]

We fix the choice of coordinates by requiring that the metric (4.14) is conformal to the Minkowski metric in these coordinates. We accomplish this by demanding that \( \bar{\phi} \) and \( \bar{\psi} \) satisfy

\[
\bar{\mu} \bar{\partial}_0 \bar{\psi} = \bar{\partial}_1 \bar{\phi} \quad \text{and} \quad \bar{\partial}_0 \bar{\phi} = \bar{\mu} \bar{\partial}_1 \bar{\psi}.
\]

(4.15)

With this choice, the metric (4.14) can be written as

\[
\bar{\lambda} = \bar{\Omega} \bar{m}
\]

(4.16)

where

\[
\bar{m} = d\bar{x}^0 \, d\bar{x}^0 - d\bar{x}^3 \, d\bar{x}^3
\]

(4.17)

and

\[
\bar{\Omega} = \frac{((\bar{\partial}_1 \bar{\phi})^2 - (\bar{\partial}_0 \bar{\phi})^2)}{\bar{\mu}}.
\]

Also, from the Jacobian of the transformation (4.12)

\[
J_{\bar{\psi}} = \begin{pmatrix}
\bar{\partial}_0 \bar{\psi} & \bar{\partial}_1 \bar{\psi} \\
\bar{\partial}_0 \bar{\phi} & \bar{\partial}_1 \bar{\phi}
\end{pmatrix},
\]

(4.18)

we see that

\[
\det J_{\bar{\psi}} = \bar{\Omega}.
\]

(4.19)

\footnote{The metric \( \hat{\lambda}_{\mu\nu} \) is conformal to the acoustical metric \( \hat{a}_{\mu\nu} = \hat{s}^2 \hat{v}_\mu \hat{v}_\nu + \hat{h}_{\mu\nu} \) with the relation between the two given by \( \hat{a}_{\mu\nu} = \frac{w_{\mu\nu}(\hat{z})}{\sin(\hat{z})^2} \hat{\lambda}_{\mu\nu} \).}
4.3. The conformal wave equation

By the conformal invariance of the wave equation in two dimensions\(^9\), and the invertibility of the coordinate transformation (4.12) on regions where \(\det J^\circ\) does not vanish, it follows from (4.16) and (4.19) that the wave equation (4.11) is equivalent to

\[
\Box_{\text{\text{\tiny\text{\eta}}}} \bar{z} = \bar{\partial}_0 \bar{z} - \bar{\partial}_1 \bar{z} = 0. \tag{4.20}
\]

We will solve this system on the spacetime region

\[
U = \{(\bar{x}^0, \bar{x}^1)|\bar{x}^0, \bar{x}^1 > 0\}
\]

with the boundary condition

\[
\bar{z}|_\Gamma = 0
\]

where

\[
\Gamma = \{(\bar{x}^0, 0)|\bar{x}^0 > 0\}.
\]

Since the vanishing of \(\bar{z}\) implies the vanishing of the proper energy density \(\bar{\rho}\) (see (2.9), (3.22), (4.5) and (4.6)), the boundary condition will ensure the vacuum boundary lies in \(\Gamma\).

We let

\[
\Sigma = \{(0, \bar{x}^1)|\bar{x}^1 > 0\}
\]

denote the initial hypersurface, and we define the following related sets

\[
U_{T, \delta} = \{(\bar{x}^0, \bar{x}^1)|0 < \bar{x}^0 < T \text{ and } 0 < \bar{x}^1 < \delta\},
\]

\[
\Gamma_T = \{(\bar{x}^0, 0)|0 < \bar{x}^0 < T\}
\]

and

\[
\Sigma_\delta = \{(0, \bar{x}^1)|0 < \bar{x}^1 < \delta\}.
\]

**Proposition 4.3.** Suppose \(k \in \mathbb{N}, s > 1/2 + k, \bar{z}_0 \in H^s(\Sigma), \bar{z}_1 \in H^{s+1}(\Sigma)\) and the \(\bar{z}_\ell = \bar{\partial}_{\bar{x}^1}^\ell \bar{z}_{\ell-2}, \quad \ell = 2, 3, \ldots, s,\)

satisfy the compatibility conditions

\[
\bar{z}_\ell|_{\bar{x}^1=0} = 0, \quad \ell = 0, 1, \ldots, s,
\]

where \(\Sigma\) and \(\Gamma\) meet. Then

(i) there exists a unique solution \(\bar{z} \in C^k(U)\) to the initial boundary value problem

\[
\Box_{\text{\text{\tiny\text{\eta}}}} \bar{z} = 0, \tag{4.21}
\]

\[
\bar{z}|_\Gamma = 0, \tag{4.22}
\]

\[
\bar{z}|_{\Sigma} = \bar{z}_0, \tag{4.23}
\]

\[
\partial_\eta \bar{z}|_{\Sigma} = \bar{z}_1, \tag{4.24}
\]

\(^9\) The conformal invariance follows directly from the identity \(\Box_{\text{\text{\tiny\text{\eta}}}} \bar{z} = \frac{1}{\text{\text{\eta}}} \Box_{\text{\text{\eta}}} \bar{z}\). See appendix D of [20] for a derivation of this identity.
(ii) for any $T, \delta > 0$,
\[ \tilde{z} \in C^k(U_{T,\delta}) \cap W^{k,\infty}(U_{T,\delta}), \]
and
(iii) if $0 < 2 \kappa < \tilde{z}_t|_{\Gamma} \leq \beta$ and $|\tilde{z}_t|_{\Sigma_i} < \kappa / 4$ for some $\kappa, \delta, \beta > 0$, then there exists a $T_0$ such that
\[ (\tilde{z}_t)^2 (\tilde{z}_t)^2)_{|U_{T_0,\delta}} \geq \kappa^2 / 4 > 0, \]
\[ 0 < \kappa \leq \tilde{z}_t|_{U_{T_0,\delta}} \leq 2 \beta, \]
and
\[ \kappa \tilde{x}^1 \leq \tilde{z}(\tilde{x}^0, \tilde{x}^1) \leq 2 \beta \tilde{x}^1 \]
for all $(\tilde{x}^0, \tilde{x}^1) \in U_{T_0,\delta}$.

**Proof.** Statements (i) and (ii) follows from standard linear hyperbolic theory for initial boundary value problems. For example, see theorems 3–6 in section 7.2 of [1]. The first two bounds from statement (iii) follow from the bounds on the initial data and the continuous differentiability of $\tilde{z}(\tilde{x}^0, \tilde{x}^1)$. The last bound in (iii) follows from integrating the second with respect to $\tilde{x}^1$ while using the fact that $\tilde{z}(\tilde{x}^1, 0) = 0$. \hfill \Box

**Remark 4.4.**

(i) Clearly, we can explicitly solve the initial boundary value problem (4.21)–(4.22) using the well-known formula
\[
\tilde{z}(\tilde{x}^0, \tilde{x}^1) = \begin{cases} 
\frac{1}{2} & \left( \tilde{z}_0(\tilde{x}^0 + \tilde{x}^1) - \tilde{z}_0(\tilde{x}^0 - \tilde{x}^1) + \int_{\tilde{x}^1}^{\tilde{x}^1} \tilde{z}_1(\xi) \, d\xi \right) \quad \tilde{x}^0 < \tilde{x}^1, \ 0 < \tilde{x}^1 \\
\frac{1}{2} & \left( \tilde{z}_0(\tilde{x}^0 + \tilde{x}^1) + \tilde{z}_0(\tilde{x}^1 - \tilde{x}^0) + \int_{\tilde{x}^1}^{\tilde{x}^1} \tilde{z}_1(\xi) \, d\xi \right) \quad \tilde{x}^1 < \tilde{x}^0, \ 0 < \tilde{x}^1 
\end{cases}
\]
(4.25)

The point of proposition 4.3 is that it extends in the obvious manner if we replace $U$ by an open set that is bounded by a timelike hypersurface $\Gamma$ and a spacelike hypersurface $\Sigma$. In this situation, there is no equivalent to the simple formula (4.25) for solutions to the initial boundary value problem (4.21)–(4.22). As we shall see below, the only property of $U$ that we need is
\[ [\xi \tilde{x} = (\xi \tilde{x}^0, \xi \tilde{x}^1) \mid \tilde{x} \in U \text{ and } 0 < \xi < 1] \subset U. \]

In fact, this can be weakened to the existence of a $\epsilon > 0$ small enough so that
\[ (\xi \tilde{x} = (\xi \tilde{x}^0, \xi \tilde{x}^1) \mid \tilde{x} \in U \cap B_{\epsilon}(0) \text{ and } 0 < \xi < \epsilon] \subset U, \]
where $B_{\epsilon}(0) = \{ \tilde{x} \in \mathbb{R}^2 \mid |\tilde{x}| = \sqrt{(\tilde{x}^0)^2 + (\tilde{x}^1)^2} < \epsilon \}$.

(ii) It is not difficult to show that the conditions imposed on the initial data in proposition 4.3 are satisfied for a wide class of initial data. For example, initial data of the form
\[ \tilde{z}_0 = cy + p(y), \quad \tilde{z}_1 = q(y) \quad (c = \text{const} > 0) \]
where
\[ (i, a) \]
\[ p(y) = o(y^\gamma) \text{ and } q(y) = o(y^{\gamma - 2}) \text{ as } y \searrow 0 \]


We recall that the Hodge dual operator on one forms is defined by
\[
\bar{\omega} = \star \bar{\omega}
\]
evaluating \(z\) and \(\bar{\omega}\), we can ensure that
\[
\partial_0 z = \bar{\partial}_0 \bar{z}
\]
if \(s\) is an even integer, and
\[
p(y) = o(y^{-1}) \quad \text{and} \quad q(y) = o(y^{-1}) \quad \text{as} \quad y \searrow 0
\]
if \(s\) is an odd integer,
satisfy all the conditions on the initial data in proposition 4.3.

4.4. Fixing the conformal coordinates

Letting \(\ast_m\) denote the Hodge dual operator\(^{10}\) of the metric (4.17), we can write (4.20) as \(\ast_d(\ast d\bar{z}) = 0\) which implies that the one form \(\ast d\bar{z}\) is closed. Consequently, we get by the proof of the Poincaré lemma (see chapter V section 4, theorem 4.1 of [14]) that
\[
z(\bar{x}) = \int_0^1 \bar{\nabla}_x \bar{z}(\bar{\xi}\bar{x})\bar{x}^0 + \bar{\nabla}_x \bar{z}(\bar{\xi}\bar{x})\bar{x}^1 \, d\xi + z_0 \quad (z_0 \in \mathbb{R})
\]
satisfies
\[
dz = \ast d\bar{z},
\]
or equivalently
\[
\bar{\nabla}_0 z = \bar{\nabla}_x \bar{z} \quad \text{and} \quad \bar{\nabla}_1 z = \bar{\nabla}_x \bar{z}.
\]
Evaluating \(z\) at \(x^0 = 0\), we see that
\[
z(0, \bar{x}^1) = \int_0^1 \bar{z}_1(0, \bar{\xi}\bar{x}^1)\bar{x}^1 \, d\xi + z_0.
\]
Since \(\bar{z}_1 = o(\bar{x}^1)\) as \(\bar{x}^1 \searrow 0\) (see remark 4.4(ii)), it is clear that by choosing \(z_0\) to be
\[
z_0 = -\min \left\{ 0, \inf_{0 < \bar{x}^1 < \delta} \int_0^1 \bar{z}_1(0, \bar{\xi}\bar{x}^1)\bar{x}^1 \, d\xi \right\}
\]
we can ensure that
\[
0 \leq z(0, \bar{x}^1) \leq \delta \quad \text{for} \quad 0 < \bar{x}^1 < \delta.
\]
It then follows from integrating the first equation of (4.28) and the bounds on \(\partial_1 \bar{z}\) from proposition 4.3(iii) that
\[
k \bar{x}^0 \leq z(\bar{x}^0, \bar{x}^1) \leq 2\bar{\mu} \bar{x}^0 + \delta
\]
for all \((\bar{x}^0, \bar{x}^1) \in U_{\bar{\mu}, \bar{\delta}}\).

Since \(dz \wedge d\bar{z} = (\bar{\nabla}_0 \bar{z})^2 - (\bar{\nabla}_x \bar{z})^2\) \(d\bar{x}^0 \wedge d\bar{x}^1\), the non-vanishing of \((\bar{\nabla}_1 \bar{z})^2 - (\bar{\nabla}_x \bar{z})^2\) on \(U_{\bar{\mu}, \bar{\delta}}\) (see proposition 4.3) shows that \(d_z, d\bar{z}\) forms a basis for the space of one forms at every point of \(U_{\bar{\mu}, \bar{\delta}}\). This allows us to look for solutions to (4.15) that are of the form
\[
\bar{\phi} = \phi(z, \bar{z}) \quad \text{and} \quad \bar{\psi} = \psi(z, \bar{z}).
\]
To see this, we get from (4.27) and (4.30) that
\[
\ast_m d\bar{\phi} = \bar{\partial}_x \bar{z} d\bar{z} + \bar{\partial}_1 \bar{z} dz
\]
and
\[
d\bar{\psi} = \partial_1 \bar{\psi} dz + \partial_2 \bar{\psi} d\bar{z}.
\]
\(^{10}\)We recall that the Hodge dual operator on one forms is defined by \(\ast_d \omega = \sqrt{|\det(\bar{\mu})\epsilon_{\bar{\omega}}|} \epsilon_{\bar{\omega}} d\bar{x}^\mu\) where \(|\bar{\mu}| = -\det(\bar{\mu})\) and \(\epsilon_{\bar{\omega}}\) is the completely antisymmetric symbol. In particular, this implies that \(\ast_d d\bar{x}^\mu = d\bar{x}^\mu\) and \(\ast_d d\bar{x}^1 = d\bar{x}^0\).
Writing (4.15) as
\[ \mu(z) \, d\hat{\psi} = *_{\mu} \, d\hat{\phi} \] (4.33)
where
\[ \mu(\xi) = \frac{\sin^3(\xi)}{8}, \]
we see from (4.31), (4.32) and (4.33) that \( \psi \) and \( \phi \) satisfy
\[ \mu(\bar{z}) \partial_z \psi = \partial_{\bar{z}} \phi \quad \text{and} \quad \mu(\bar{z}) \partial_{\bar{z}} \psi = \partial_z \phi. \] (4.34)
From these equations, we then get the wave equation
\[ \partial_z^2 \phi - \mu(\bar{z}) \partial_{\bar{z}} \left( \frac{1}{\mu(\bar{z})} \partial_z \phi \right) = 0 \] (4.35)
for \( \phi \).

**Proposition 4.5.** Suppose the initial data \( (\phi|_{z=0}, \partial_z \phi|_{z=0}) \) for the wave equation (4.35) satisfies\(^{11}\)
\[ \left( \phi \left( \sqrt{\mu(z)} \right) \bigg|_{z=0}, \partial_z \left( \phi \left( \sqrt{\mu(z)} \right) \bigg|_{z=0} \right) \right) \in \mathcal{H}^6 \times \mathcal{H}^5, \] (4.36)
\[ \frac{1}{\mu(\bar{z})} \partial_{\bar{z}} \phi(0, \bar{z}) \geq c > 0 \] (4.37)
and
\[ \partial_z \phi(0, \bar{z}) \geq 0 \] (4.38)
for all \( 0 < \bar{z} < \pi/2 \). Then there exists a unique solution \( \phi(z, \bar{z}) \) to (4.35) that can be written as
\[ \phi(z, \bar{z}) = \tilde{\phi}(z, \bar{z}) \] (4.39)
where \( \tilde{\phi}(t, \xi) \) satisfies
\[ \tilde{\phi} \in C^1([0, \infty), C^{1,1/2}(0, (\pi/2)^4)) \cap \bigcap_{j=0}^5 C^j([0, \infty), C^{5-j}(0, (\pi/2)^4)) \] (4.40)
and
\[ |\partial_t \tilde{\phi}(t, \xi)| + |\tilde{\phi}(t, \xi)| \lesssim \xi. \] (4.41)
Moreover, there exists a \( \tau_0 \) such that
\[ \partial_z \tilde{\phi}(t, \xi) \geq c/2 \] (4.42)
for all \( (t, \xi) \in [0, \tau_0] \times (0, \pi/2) \).

**Proof.** This is just a restatement of theorem A.4 from the section A.3 of the appendix. □

Integrating the first equation in (4.34) with respect to \( z \) while using the second equation of (4.34) to fix the undetermined function of integration, we find the following expression
\[ \psi(z, \bar{z}) = \int_0^z \frac{1}{\mu(\zeta)} \partial_z \phi(z, \bar{z}) \, dz + \int_0^\bar{z} \frac{1}{\mu(\zeta)} \partial_{\bar{z}} \phi(0, \zeta) \, d\zeta \] (4.43)
for \( \psi \). Using (4.39), we can write \( \psi \) as
\[ \psi(z, \bar{z}) = \frac{32\bar{z}^3}{\sin^3(\bar{z})} \int_0^z \partial_z \tilde{\phi}(z, \bar{z}) \, dz + \int_0^\bar{z} \frac{8}{\sin^3(\zeta)} \partial_{\bar{z}} \phi(0, \zeta) \, d\zeta. \] (4.44)

\(^{11}\) The spaces \( \mathcal{H}^k \) are defined in section A.2 of the appendix.
Lemma 4.6. Suppose $k = 3$, $\tilde{z}$ is the solution to the wave equation (4.20) from proposition 4.3. $z$ is a defined by (4.26), $\phi$ and $\bar{\phi}$ are the maps from proposition 4.5 and $\Psi$ is given by (4.44). For then $T_0$ and $\delta_0$ small enough, the change of coordinates map (4.12)

$$\Psi : U_{T_0, \delta} \rightarrow \mathbb{R}^3 : (\tilde{x}) \mapsto (\Psi(\tilde{x}), \bar{\phi}(\tilde{x})))$$

is well defined and of class $C^3$ on $U_{T_0, \delta}$, and the Jacobian matrix is given by the formula

$$J_{\Psi}(\tilde{x}) = \begin{pmatrix} \frac{\partial \phi(z, \tilde{z})}{\partial z} & \frac{\partial \phi(z, \tilde{z})}{\partial \tilde{x}} \\ \frac{\partial \phi(\tilde{z}, \tilde{z})}{\partial z} & \frac{\partial \phi(\tilde{z}, \tilde{z})}{\partial \tilde{x}} \end{pmatrix} \left( \begin{array}{c} \bar{\partial}_1 z \\ \bar{\partial}_{\tilde{x}} \end{array} \right)$$

and satisfies

$$\text{det} J_{\Psi}|_{U_{T_0, \delta}} > 0.$$

Furthermore,

$$\Psi(\tilde{x}^0, 0) = \left( \begin{array}{c} 32 \int_0^{\beta T_0 + \delta} \int_0^\tau \partial \tilde{\phi}(t, 0) \partial \tilde{z} \right).$$

Proof. By proposition 4.3.(iii) and (4.29), we have that

$$(z(\tilde{x}), \tilde{z}(\tilde{x})) \in [0, 2\beta T_0 + \delta) \times [0, 2\beta \delta)$$

for all $\tilde{x} \in U_{T_0, \delta}$. Choosing $\delta$ and $T_0$ small enough, it is clear that we can arrange that

$$(z(\tilde{x}), \tilde{z}(\tilde{x})) \in [0, \tau_0) \times (0, 2\beta \delta) \subset (0, \tau_0) \times (0, \pi/2)$$

for all $\tilde{x} \in U_{T_0, \delta}$. As a consequence, the change of coordinates map (see (4.12))

$$\tilde{x} = \bar{\Psi}(x, \bar{\phi}(\tilde{x})) := (\psi(z(x), \tilde{z}(x)), \phi(x, \tilde{z}(x)))$$

is well defined for all $\tilde{x} \in U_{T_0, \delta}$.

A short calculation using (4.18) and (4.46) shows that

$$J_{\Psi} = \begin{pmatrix} \partial_1 \psi(z, \tilde{z}) & \partial_{\tilde{x}} \psi(z, \tilde{z}) \\ \partial_1 \phi(z, \tilde{z}) & \partial_{\tilde{x}} \phi(z, \tilde{z}) \end{pmatrix}.$$

Using (4.28) and (4.34), we can write this as

$$J_{\Psi} = \begin{pmatrix} \frac{\partial \phi(z, \tilde{z})}{\partial z} & \frac{\partial \phi(z, \tilde{z})}{\partial \tilde{x}} \\ \frac{\partial \phi(\tilde{z}, \tilde{z})}{\partial z} & \frac{\partial \phi(\tilde{z}, \tilde{z})}{\partial \tilde{x}} \end{pmatrix} \left( \begin{array}{c} \bar{\partial}_1 z \\ \bar{\partial}_{\tilde{x}} \end{array} \right).$$

(4.47)

Taking the determinant gives

$$\text{det}(J_{\Psi}) = \mu(\tilde{z}) \left( \left( \frac{\partial \phi(z, \tilde{z})}{\partial \tilde{z}} \right)^2 - \left( \frac{\partial \phi(z, \tilde{z})}{\partial z} \mu(\tilde{z}) \right)^2 \right).$$

It follows from (4.39) and (4.42) that there exists a positive constant $C$ such that

$$\frac{\partial \phi(z, \tilde{z})}{\mu(\tilde{z})} = \frac{4\epsilon^3}{\mu(\tilde{z})} \geq C > 0$$

for all $(z, \tilde{z}) \in [0, \tau_0) \times (0, \pi/2)$. This, in turn, implies via (4.45) that

$$\frac{\partial \phi(z, \tilde{z})}{\mu(\tilde{z})} = \frac{4\epsilon^3}{\mu(\tilde{z})} \frac{\partial \tilde{\phi}(z, \tilde{z}, \tilde{x})}{\mu(\tilde{z})} \geq C > 0$$

for all $\tilde{x} \in U_{T_0, \delta}$.

Next, we observe that

$$\left| \frac{\partial \phi(z, \tilde{z})}{\mu(\tilde{z})} \right| = \left| \frac{\partial \tilde{\phi}(z, \tilde{z})}{\mu(\tilde{z})} \right| \lesssim \frac{\epsilon^3}{\mu(\tilde{z})}.$$
for all \((\bar{z}, \tilde{z}) \in [0, \bar{t}) \times (0, \pi/2)\) by (4.39) and (4.41). This inequality, with the help of (4.45) and the fact that \(\lim_{\tilde{z} \to 0} \tilde{z}^4/\mu(\tilde{z}) = 0\), shows that, by choosing \(\delta\) small enough, we can arrange that
\[
\left| \frac{\partial_t \phi(z(\bar{\bar{\bar{z}}}), \tilde{z}(\bar{\bar{\bar{z}}}))}{\mu(\tilde{z}(\bar{\bar{\bar{z}}}))} \right| \leq \frac{C}{\sqrt{2}} \tag{4.49}
\]
for all \(\bar{\bar{\bar{z}}} \in U_{T_0, \delta}\).

The two inequalities (4.48) and (4.49) together with the bound on \((\tilde{a}_1 \tilde{z})^2 - (\tilde{a}_b \tilde{z})^2\) from proposition 4.3.(iii) and the formula (4.47) show that
\[
J_{\bar{\phi}}|_{\tilde{\nu}_{T_0}} > 0.
\]

From (4.39) and (4.41), we see that
\[
|\phi(z, \bar{z})| = |\tilde{\phi}(z, \tilde{z})| \lesssim |\tilde{z}|^4.
\]

Since \(\tilde{z}(\bar{z}, 0) = 0\), it follows that
\[
\phi(z(\bar{z}, 0), \bar{z}(\bar{z}, 0)) = 0. \tag{4.50}
\]

Using (4.41) again, we have that
\[
\left| \int_0^{\tilde{z}} \frac{1}{\mu(\xi)} \partial_x \phi(0, \xi^4) \, d\xi \right| \leq \int_0^{\tilde{z}} \xi^4 \, d\xi \lesssim \tilde{z}^2.
\]

This together with (4.26) and (4.44) shows that
\[
\psi(z(\bar{z}, 0), \bar{z}(\bar{z}, 0)) = 32 \int_0^{\tilde{z}} \int_0^{\bar{z}} \tilde{a}_1 \tilde{z}(\xi, \bar{z}, 0) \, d\xi + \gamma_0 \frac{\partial_t \tilde{\phi}(t, 0)}{\partial_t \phi}(t, 0) \, dt. \tag{4.51}
\]

From (4.50) and (4.51), we see that
\[
\tilde{\Psi}(\bar{z}, 0) = \left( 32 \int_0^{\tilde{z}} \int_0^{\bar{z}} \tilde{a}_1 \tilde{z}(\xi, \bar{z}, 0) \, d\xi + \gamma_0 \frac{\partial_t \tilde{\phi}(t, 0)}{\partial_t \phi}(t, 0) \right).
\]

**Remark 4.7.** From lemma 4.6, it follows that the vacuum boundary \(\Gamma_{T_0}\) in the coordinates \((\bar{z}, \tilde{z})\) is given by
\[
\tilde{\Psi}(\bar{\nu}_{T_0}) = \left\{ \left( 32 \int_0^{\tilde{z}} \int_0^{\bar{z}} \tilde{a}_1 \tilde{z}(\xi, \bar{z}, 0) \, d\xi + \gamma_0 \frac{\partial_t \tilde{\phi}(t, 0)}{\partial_t \phi}(t, 0) \right) \left| 0 \leq \bar{z} < T_0 \right. \right\}.
\]

We also note that the bounds on \(\tilde{a}_1 \tilde{z}\) and \(\partial_t \tilde{\phi}(t, \xi)\) from proposition 4.3.(iii) and (4.42), respectively, imply that the map
\[
\bar{z} \mapsto 32 \int_0^{\tilde{z}} \int_0^{\bar{z}} \tilde{a}_1 \tilde{z}(\xi, \bar{z}, 0) \, d\xi + \gamma_0 \frac{\partial_t \tilde{\phi}(t, 0)}{\partial_t \phi}(t, 0) \, dt
\]
is strictly increasing. This shows that \(\tilde{\Psi}(\bar{\nu}_{T_0})\) is just a reparameterization of \(\Gamma_{T_0}\), and that
\[
\tilde{\Psi}(\bar{\nu}_{T_0}) = \left( \tilde{z}(1, 0) \left| 0 \leq \tilde{z} < \int_0^{T_0} \tilde{a}_1 \tilde{z}(T_0, 0) \, d\xi + \gamma_0 \frac{\partial_t \tilde{\phi}(t, 0)}{\partial_t \phi}(t, 0) \right. \right).
\]

This is consistent with our assertion from section 4.1 that the vacuum boundary in the \((\bar{z}, \tilde{z})\) coordinates is contained in the line \(\tilde{z} = 0\). This fact combined with \(\hat{w} = \sqrt{\hat{\gamma} + 1} \hat{a}_0\) (see (4.2)) shows that the vacuum boundary moves with the fluid as noted previously.
4.5. Solutions with non-zero acceleration at the vacuum boundary

With the validity of the coordinate transformation (4.12) established, we now turn to showing that the maps \([z, \bar{z}, \psi, \phi]\) determine a solution to the EFW equations that have non-zero acceleration at the boundary. We begin by letting

\[
\bar{g} = \bar{g}_\mu^\nu \mathrm{d}\bar{x}^\mu \mathrm{d}\bar{x}^\nu \quad \text{and} \quad \bar{w} = \bar{w}_\mu \partial_\mu
\]
denote the metric and the Frauendiener–Walton vector field. By (4.2) and (4.7), the coordinate components of \(\bar{g}\) and \(\bar{w}\) are given by

\[
(\bar{g}_{\mu\nu}) = J^2_{\bar{\psi}} \begin{pmatrix}
\frac{(1 + \cos(\bar{z}))^4}{32} & 0 \\
0 & \frac{4}{(1 - \cos(\bar{z}))^2}
\end{pmatrix} J_{\bar{\psi}}
\]

and

\[
(\bar{w}^\mu) = J^{-1}_{\bar{\psi}} \begin{pmatrix}
\sqrt{2} \\
0
\end{pmatrix}
\]

where (see lemma 4.6)

\[
J_{\bar{\psi}} = \frac{1}{K} \begin{pmatrix}
\partial_{\bar{z}}\phi(z, \bar{z}) & \partial_{\bar{z}}\phi(z, \bar{z}) \\
\mu(\bar{z}) & \mu(\bar{z})
\end{pmatrix} \begin{pmatrix}
\partial_1 \bar{z} \\
\partial_0 \bar{z}
\end{pmatrix}.
\]

We also observe that the norm \(\bar{w}^2\), the square of the sound speed \(\bar{s}^2\) and the proper energy density \(\bar{\rho}\) are easily computed to be

\[
\bar{w}^2 = \frac{(1 - \cos(\bar{z}))^4}{16}, \quad \bar{s}^2 = 2 \frac{(1 - \cos(\bar{z})}{1 + \cos(\bar{z})},
\]

and

\[
\bar{\rho} = \frac{1}{K} \frac{1 - \cos(\bar{z})}{1 + \cos(\bar{z})}
\]

using the formulas (2.9), (3.21), (3.22) and (4.6). From (4.53) and (4.55), we then obtain the following formula for the fluid two-velocity

\[
(\bar{u}^\mu) = \frac{4}{1 - \cos(\bar{z})^2} J^{-1}_{\bar{\psi}} \begin{pmatrix}
\sqrt{2} \\
0
\end{pmatrix}.
\]

The analysis contained in sections 3, 4.1–4.4 guarantee that the pair \([\bar{g}_{\mu\nu}, \bar{u}^\mu]\) defined by the formulas (4.52)–(4.54) determine a \(C^2\) solution to the EFW equations (2.1) on the spacetime region \(U_{T_{0,\delta}}\). This, in turn, shows that \([\bar{g}_{\mu\nu}, \bar{u}^\mu, \bar{\rho}]\), with \(\bar{\rho}\) and \(\bar{u}^\mu\) given by the formulas (4.57) and (4.58), is a \(C^2\) solution to the Euler equations (1.2)–(1.3) on \(U_{T_{0,\delta}}\).

With existence established, we are left with calculating the norm of the fluid acceleration at the boundary. We begin by observing that

\[
\bar{g} = \frac{(1 + \cos(\bar{z}))^4}{32} \mathrm{d}\psi \mathrm{d}\psi - \frac{4}{(1 - \cos(\bar{z}))^2} \mathrm{d}\phi \mathrm{d}\phi
\]

and

\[
\bar{w}^\psi = \frac{\sqrt{2}(1 + \cos(\bar{z}))^4}{32} \mathrm{d}\psi
\]
where $\tilde{\tilde{\omega}}^{\mu} = \bar{g}_{\mu \nu} \tilde{\tilde{\omega}}^{\nu}$. Using (4.34), we see that

$$d\psi = \frac{\partial \phi}{\mu(z)} dz + \frac{\partial \phi}{2z \mu(z)} dz^2,$$

(4.61)

and hence by (4.39), that

$$d\psi = \frac{32 c^3}{\sin^3(z)} \partial_\phi \bar{\phi}(z, z^2) dz + \frac{4 c^3}{\sin^3(z)} \bar{\phi}(z, z^2) dz^2.$$

(4.62)

Also, similar calculations show that

$$\frac{1}{z^2} d\phi = \frac{\partial \phi(z, z^2)}{z^2} dz + 2 \partial_\phi \bar{\phi}(z, z^4) dz^2.$$

(4.63)

Next, we introduce the dual basis

$$\tilde{\tilde{0}}^0 = dz \quad \text{and} \quad \tilde{\tilde{0}}^1 = dz^2.$$

(4.64)

As we shall see, the components of the metric and Frauendiener–Walton covector field with respect to this frame have finite limits at the vacuum boundary even though some of the $(x^2)$ components of the $(x^0)$ coordinates components are singular there. It is worthwhile noting that since this basis arises from the coordinates\(^{12}\) $(z, \bar{z}^2)$, we could have introduced yet one more coordinate transformation to investigate the regularity of the fields $(\bar{g}, \bar{\omega})$ near the vacuum boundary. However, it is simpler just to work with the basis (4.64) without introducing another explicit coordinate transformation.

Writing the metric $\bar{g}$ and co-vector field $\bar{\omega}$ as

$$\bar{g} = \bar{g}_{ij} \bar{e}^i \bar{e}^j \quad \text{and} \quad \bar{\omega} = \bar{\omega}_i \bar{e}^i,$$

a straightforward calculation using (4.59)–(4.64) shows that

$$(\bar{g}_{ij}|_{\gamma_0}) = \left( \begin{array}{cc} 512 \partial_\phi \bar{\phi}(z(\bar{z}^0, 0), 0)^2 & 64 \partial_\phi \bar{\phi}(z(\bar{z}^0, 0), 0) \partial_\phi^2 \bar{\phi}(z(\bar{z}^0, 0), 0) \\ 64 \partial_\phi \bar{\phi}(z(\bar{z}^0, 0), 0) \partial_\phi^2 \bar{\phi}(z(\bar{z}^0, 0), 0) & -32 \partial_\phi \bar{\phi}(z(\bar{z}^0, 0), 0)^2 + 8 \partial_\phi^2 \bar{\phi}(z(\bar{z}^0, 0), 0)^2 \end{array} \right)$$

(4.65)

and

$$(\bar{\omega}_i|_{\gamma_0}) = \frac{1}{\sqrt{2}} \left( 32 \partial_\phi \bar{\phi}(z(\bar{z}^0, 0), 0) \quad 4 \partial_\phi^2 \bar{\phi}(z(\bar{z}^0, 0), 0) \right).$$

(4.66)

From (4.65), we find that

$$(\bar{g}^{ij}|_{\gamma_0}) = \left( \begin{array}{cc} 4(\partial_\phi \bar{\phi}(z(\bar{z}^0, 0), 0)^2 - (\partial_\phi^2 \bar{\phi}(z(\bar{z}^0, 0), 0))^2 & 2048(\partial_\phi \bar{\phi}(z(\bar{z}^0, 0), 0))^4 \\ 2048(\partial_\phi \bar{\phi}(z(\bar{z}^0, 0), 0))^4 & 256(\partial_\phi \bar{\phi}(z(\bar{z}^0, 0), 0))^2 \end{array} \right).$$

Using this and (4.66), we get that

$$(\bar{\omega}^i|_{\gamma_0}) = \left( \begin{array}{c} \sqrt{2} \\ 0 \end{array} \right).$$

\(^{12}\) Recall that it was shown in section 4.4 that $(z, \bar{z})$ satisfies $dz \wedge d\bar{z} = (\partial_\phi \bar{\phi}(z(\bar{z}^0, 0), 0))^2 dz^2 \wedge d\bar{z}^3$ with $(\partial_\phi \bar{\phi}(z(\bar{z}^0, 0), 0))^2$ non-vanishing on $U_{\gamma_0}$ and $z$. This shows that $(z, \bar{z})$ define a coordinate system and it follows that $(z, \bar{z}^2)$ does also.
which we can use to write the frame components of $\hat{h}$ (see (1.4)) as

$$
(\hat{h}^\mu|_{r_0}) = \begin{pmatrix}
\frac{(\partial^2_\mu \phi(z(x^0, 0), 0))^2}{2048(\partial_\mu \phi(z(x^0, 0), 0))^4} & \frac{-\partial_\mu^2 \phi(z(x^0, 0), 0)}{256(\partial_\mu \phi(z(x^0, 0), 0))^5} \\
\frac{-\partial^2_\mu \phi(z(x^0, 0), 0)^3}{256(\partial_\mu \phi(z(x^0, 0), 0))^5} & \frac{1}{32(\partial_\mu \phi(z(x^0, 0), 0))^3}
\end{pmatrix}.
$$

(4.67)

Letting
d\xi^2 = d\xi^\mu d\xi^\nu,

we get from (4.56) that
d\xi^2 = \begin{pmatrix}
0 & \frac{2}{\bar{\zeta}}(1 + \cos(\bar{\zeta}))^2
\end{pmatrix}.

From this, we see that

$$
(\text{d}\xi_i^2|_{r_0}) = \begin{pmatrix}
0 & 1
\end{pmatrix},
$$

which, with the help of (4.67), shows that
d\xi_i^2|_{r_0} = \frac{1}{128(\partial_\mu \phi(z(x^0, 0), 0))^2}.

This and (2.10) allows us to conclude that the norm of the fluid acceleration at the vacuum boundary is non-zero and is given by the formula

$$
|\d_i|\big|_{r_0} = \frac{1}{\sqrt{128(\partial_\mu \phi(z(x^0, 0), 0))}}.
$$

(4.68)

We summarize the above results in the following theorem.

**Theorem 4.8.** Suppose $k = 3$, $\bar{\zeta}$ is the solution to the wave equation (4.20) from proposition 4.3, $\bar{z}$ is defined by (4.26), $\phi$ and $\bar{\phi}$ are the maps from proposition 4.5 and $\bar{\psi}$ is given (4.44). Then for $T_0$ and $\delta_0$ small enough, the triple $\{\bar{g}_{\mu\nu}, \bar{\rho}, \bar{v}^\mu\}$ determined by

$$(\bar{g}_{\mu\nu}) = J_{\bar{\psi}} \begin{pmatrix}
(1 + \cos(\bar{\zeta}))^4 & 0 \\
0 & 4
\end{pmatrix} J_{\bar{\psi}}^{-1},$$

$$\bar{\rho} = \frac{1}{K} \frac{1 - \cos(\bar{\zeta})}{1 + \cos(\bar{\zeta})},$$

and

$$(\bar{v}^\mu) = \frac{4}{(1 - \cos(\bar{\zeta}))^2} J_{\bar{\psi}}^{-1} \begin{pmatrix}
\sqrt{2} \\
0
\end{pmatrix},$$

where

$$J_{\bar{\psi}} = \begin{pmatrix}
\frac{\partial_\mu \phi(z, \bar{\zeta})}{\partial_\mu \phi(z, \bar{\zeta})} & \frac{\partial_\mu \phi(z, \bar{\zeta})}{\partial_\mu \phi(z, \bar{\zeta})} \\
\frac{\partial_\mu \phi(z, \bar{\zeta})}{\partial_\mu \phi(z, \bar{\zeta})} & \frac{\partial_\mu \phi(z, \bar{\zeta})}{\partial_\mu \phi(z, \bar{\zeta})}
\end{pmatrix} \begin{pmatrix}
\bar{\partial}_1 \bar{z} \\
\bar{\partial}_0 \bar{z}
\end{pmatrix}.$$

defines a $C^2$ solution of the Euler equations (1.2)–(1.3) on the spacetime region $\mathcal{U}_{T_0, \delta}$. Moreover, the fluid acceleration $|\d_i|\big|_{r_0}$ is non-zero on the vacuum boundary and given by the formula

$$
|\d_i|\big|_{r_0} = \frac{1}{\sqrt{128(\partial_\mu \phi(z(x^0, 0), 0))}}
$$

for $0 \leq \bar{x}^0 < T_0$.

13 Here, $\bar{\rho}$, $\bar{v}^\mu$ and $\bar{g}_{\mu\nu}$ are given in the $(\bar{v}^\mu)$ coordinates with the Jacobian matrix $J_{\bar{\psi}}$ arising from the transformation from the $(\bar{v}^\mu)$ to the $(\bar{v}^\mu)$ coordinates. In the $(\bar{v}^\mu)$ coordinates, $\bar{\rho}$, $\bar{v}^\mu$ and $\bar{g}_{\mu\nu}$ are determined by (2.6), (4.2) and (4.7).
5. Exact solutions

In this section, we show that it is possible to determine certain classes of solutions to the Euler equations with non-zero acceleration at the boundary that are exact in the sense they are determined up to quadrature. First, we note that \( z \) and \( \bar{z} \) given by the formulas (4.25) and (4.26), respectively, are determined up to integrals. Next, by inspection, we observe that

\[
\phi(z, \bar{z}) = (c_1 + c_2z) \left( \frac{1}{12} - \frac{1}{8} \cos(\bar{z}) + \frac{1}{24} \cos^3(\bar{z}) \right)
\]

(5.1)
satisfies the wave equation (4.35), and in fact, satisfies the conditions (4.36)–(4.38). In this case, the map \( \psi(t, \xi) \) is given by

\[
\psi(t, \xi) = (c_1 + c_2t) \left( \frac{1}{12} - \frac{1}{8} \cos(\xi^{1/4}) + \frac{1}{24} \cos^3(\xi^{1/4}) \right),
\]

(5.2)
and it is not difficult, using the power series expansion for \( \cos(x) \), to see that \( \phi(t, \xi) \) admits the following expansion about \( \xi = 0 \):

\[
\phi(t, \xi) = (c_1 + c_2t) \left( \frac{1}{12} \xi - \frac{1}{24} \xi^{3/2} + \frac{1}{1440} \xi^3 \right) + O(\xi^{5/2}).
\]

(5.3)
Next, using (4.43), it follows from (5.1) that the map \( \psi \) is given by

\[
\psi(z, \bar{z}) = c_2 \left( \frac{1}{3} + \frac{1}{3} \ln(2) - \frac{\cos(\bar{z})}{3 \sin^2(\bar{z})} + \frac{\ln(\csc(\bar{z}) - \cot(\bar{z}))}{3} + \frac{1}{2 \sin^2(\bar{z})} \right.
\]

\[
- \frac{\cot^2(\bar{z})}{6} - \frac{\ln(\sin(\bar{z}))}{3} \bigg) + \left( c_1 z + \frac{c_2}{2} z^2 \right).
\]

(5.4)
Together, \( \{z(\bar{z}), \bar{z}(\bar{z})\} \) determined by (4.25) and (4.26), and \( \{\phi, \psi, \psi\} \) determined by (5.1), (5.2) and (5.4) specify completely a solution to the Euler equations via the formulas in theorem 4.8. Moreover, we see from (4.68) and (5.3) that the acceleration at the vacuum boundary for these solutions is given by the formula

\[
|\tilde{a}|_{|\gamma_0} = \frac{32}{\sqrt{128(c_1 + c_2z(\tau^0, 0))}},
\]

and in particular, is constant if \( c_2 = 0 \) and time varying otherwise.

Acknowledgments

This work was partially supported by the ARC grant DP1094582 and a MRA grant. I thank Bernd Schmidt for helpful suggestions and comments. Part of this work was completed while visiting the Albert–Einstein–Institute (AEI). I thank the Institute for its hospitality and for supporting this research.

Appendix. The singular wave equation \( \tilde{\alpha}^2 \Phi + H \Phi = 0 \)

Our goal in this appendix is to prove the existence and regularity of solutions to the singular, linear wave equation (4.35). To simplify notation, we will set \( (z, \bar{z}) = (t, x) \) as we will be thinking of \( z \) and \( \bar{z} \) as a time and space coordinate, respectively. Letting

\[
\Phi(t, x) = \frac{\phi(t, x)}{\sqrt{\mu(x)}},
\]

we see from (4.35) that \( \Phi \) satisfies the equation

\[
\tilde{\alpha}^2 \Phi + H \Phi = 0
\]

(A.1)
where \( H \) is the operator

\[
H \Phi = -\sqrt{\mu(x)} \frac{d}{dx} \left( \frac{1}{\mu(x)} \frac{d}{dx} \left( \sqrt{\mu(x)} \Phi \right) \right).
\]

(A.2)
A.1. The Friedrichs extension of $H$

First, we observe that a simple integration by parts argument shows that $H$ defined (A.2) is symmetric on the domain $C^\infty_0(0, \pi/2)$ with respect to the standard $L^2$ inner product

$$\langle \Phi_1 | \Phi_2 \rangle_{L^2} = \int_0^{\pi/2} \Phi_1(x) \Phi_2(x) \, dx.$$ 

We also observe, again by integration by parts, that

$$H = L^\dagger L,$$

where

$$L(\Phi) = \frac{1}{\sqrt{\mu(x)}} \frac{d}{dx} \left( \sqrt{\mu(x)} \Phi \right)$$

and $\dagger$ is the adjoint given explicitly by

$$L^\dagger \Phi = -\sqrt{\mu(x)} \frac{d}{dx} \left( \frac{1}{\sqrt{\mu(x)}} \Phi \right).$$

This shows that $H$ is a non-negative symmetric operator and the quadratic form

$$q_H(\Phi_1, \Phi_2) = \langle \Phi_1 | H \Phi_2 \rangle$$

associated to $H$ satisfies

$$q_H(\Phi_1, \Phi_2) = \langle L \Phi_1 | L \Phi_2 \rangle,$$

and in particular,

$$q_H(\Phi, \Phi) = \| L \Phi \|^2_{L^2}.$$ 

Defining the norm

$$\| \Phi \|^2_{H^1} = \| L \Phi \|^2_{L^2} + \| \Phi \|^2_{L^2}$$

we let

$$H^1 = C^\infty_0(0, \pi/2)$$

denote the completion of $C^\infty_0(0, \pi/2)$ with respect to the norm (A.5). Then by theorem X.23 of [17], the self-adjoint Friedrichs extension of $H$, which we also denote by $H$, exists and is defined on a dense domain

$$D(H) \subset H^1 \subset L^2(0, \pi/2).$$

A.2. Existence

In order to discuss existence for the wave equation, we need to define the norms

$$\| \Phi \|^2_{H^k} = \sum_{j=0}^{k} \| H^{j/2} \Phi \|^2_{L^2} \quad (H^{j/2} := (H^{1/2})^j)$$

and the spaces

$$H^k = \bigcap_{j=0}^{k} D(H^{j/2}) \subset L^2(0, \pi/2)$$

where $H^{1/2}$ is the positive square root of $H$.

The following theorem, which guarantees existence and uniqueness, follows from well-known results existence and uniqueness results for abstract wave equations. See [9, 10] for details.
Theorem A.1. Suppose $k \in \mathbb{N}_0$ and $(\Phi_0, \Phi_1) \in \mathcal{H}^{k+1} \times \mathcal{H}^k$. Then there exists a unique solution $\Phi \in \cap_{j=0}^{k+1} C^j([0, \infty), \mathcal{H}^{k+1-j})$ to the initial value problem

$$
\frac{d^2 \Phi}{dt^2} + H\Phi = 0, \\
\left(\Phi|_{t=0}, \frac{d\Phi}{dt}|_{t=0}\right) = (\Phi_0, \Phi_1).
$$

A.3. Regularity near the boundary

Away from $x = 0$, the norms $\| \cdot \|_{\mathcal{H}^s}$ are equivalent to the standard Sobolev norms, and hence, any solution to (A.1) in $\mathcal{H}^k$ will also lie in $H^k_{\text{loc}}(0, \pi/2)$. However, the norms $\| \cdot \|_{\mathcal{H}^s}$ are not uniformly equivalent to the standard Sobolev norms as $x$ approaches zero. Consequently, some work is needed to determine the regularity of functions lying in these spaces for small $x$. Our main tools to establish the regularity will be the following.

(i) Sobolev’s inequality. Suppose $sp < 1$, $s \in \mathbb{R}$ and $p \in (1, \infty)$. Then

$$
\|\Phi\|_{L^{p(1-sp)}(a,b)} \lesssim \|\Phi\|_{H^{s,p}(a,b)} \tag{A.8}
$$

for all $\Phi \in H^{s,p}(a,b)$. Here, $H^{s,p}(a,b)$ denotes the fractional Sobolev spaces, which coincide with standard ones for $s \in \mathbb{N}$. We employ the usual notation $H^s(a,b) = H^{s,2}(a,b)$, and we note that for $0 < s \leq 1$ the fractional norm $\|\Phi\|_{H^{s,p}(a,b)}$ can be written as

$$
\|\Phi\|_{H^s(a,b)}^2 = \|\Phi\|_{L^2(a,b)}^2 + \int_a^b \int_a^b \frac{|\Phi(x) - \Phi(y)|^2}{|x-y|^{1+2s}} \, dx \, dy. \tag{A.9}
$$

(ii) Morrey’s inequality. Suppose $p \in (1, \infty]$. Then

$$
\|\Phi\|_{C^{0,1-s/p}(a,b)} \lesssim \|\Phi\|_{H^{s,1}(a,b)} \tag{A.10}
$$

for all $\Phi \in H^{1,p}(a,b)$.

(iii) Fractional order weighted Hardy’s inequality. For $0 \leq d < 1$, the following inequality

$$
\|\Phi\|_{H^{1-d}(0,b)} \lesssim \|\Phi\|_{L^2(0,b)} + \left\| x^{-d} \frac{d\Phi}{dx} \right\|_{L^2(0,b)} \tag{A.11}
$$

follows directly from the fractional order weighted Hardy’s inequality [13, theorem 5.3] and the definition of the fractional norm (A.9). We note that this type of Hardy inequality was also used in the existence results of [5, 6, 11, 12].

To begin, we introduce a new coordinate

$$
\xi = \int_0^x \mu(s) \, ds \tag{A.12}
$$

and let

$$
\xi_0 = \int_0^{\pi/2} \mu(s) \, ds.
$$

We use the notation $L^2_\xi$ to denote the $L^2$ space with respect to the coordinate $\xi$ on the interval $(0, \xi_0)$, or equivalently, the measure

$$
d\xi = \mu(x) \, dx \tag{A.13}
$$

on the interval $(0, \pi/2)$. We use the following notation for the $L^2$ norm:

$$
\|\Phi\|_{L^2_\xi} = \int_0^{\xi_0} \Phi(\xi) \, d\xi.
$$
We will also use the notation $H_{\xi}^2$ when referring to the $L^2$ Sobolev spaces with respect to the variable $\xi$, or equivalently, with the $L^2$ spaces defined using the differential operator

$$\frac{d}{d\xi} = \frac{1}{\mu(x)} \frac{d}{dx}$$ \hspace{1cm} (A.14)

and the measure (A.13). In the following, we use the same notation to denote a function whether thought of as a function of $x$ or a function of $\xi$ whenever this does not lead to an ambiguity.

Since $\mu(x)$ is analytic, $\mu(x) > 0$ and $\mu(x) = O(x^3)$, it follows that

$$\xi \sim x^4$$ \hspace{1cm} (A.15)

for $x$ near zero, or equivalently

$$x \sim \xi^{1/4}$$

for $\xi$ near zero. Thus, in particular,

$$\mu \sim \xi^{3/4}$$ \hspace{1cm} (A.16)

for $\xi$ near zero.

Lemma A.2. Suppose $\Phi \in \mathcal{H}^2$. Then

$$\left\| \sqrt{\mu/\Phi} \right\|_{L^2_\xi}^2 + \left\| d \left( \sqrt{\mu} \Phi \right) / d\xi \right\|_{L^2_\xi}^2 + \left\| \xi^{3/4} \frac{d^2}{d\xi^2} \left( \sqrt{\mu} \Phi \right) \right\|_{L^2_\xi}^2 \lesssim \|\Phi\|_{L^2_\xi}^2.$$ \hspace{1cm} (A.20)

Proof. First, we observe that the identities

$$|\Phi|^2 \, dx = \left| \frac{\Phi}{\sqrt{\mu}} \right|^2 \, d\xi,$$ \hspace{1cm} (A.17)

$$|L\Phi|^2 \, dx = \left| \frac{d}{d\xi} \left( \sqrt{\mu} \Phi \right) \right|^2 \, d\xi$$ \hspace{1cm} (A.18)

and

$$|H\Phi|^2 \, dx = \mu^2 \left| \frac{d^2}{d\xi^2} \left( \sqrt{\mu} \Phi \right) \right|^2 \, d\xi$$ \hspace{1cm} (A.19)

follow directly from the definitions (A.2), (A.3), (A.12)–(A.14). Integrating (A.17)–(A.19) then gives

$$\left\| \sqrt{\mu/\Phi} \right\|_{L^2_\xi}^2 + \left\| d \left( \sqrt{\mu} \Phi \right) / d\xi \right\|_{L^2_\xi}^2 + \left\| \xi^{3/4} \frac{d^2}{d\xi^2} \left( \sqrt{\mu} \Phi \right) \right\|_{L^2_\xi}^2 \lesssim \|\Phi\|_{L^2_\xi}^2 + \|L\Phi\|_{L^2_\xi}^2 + \|H\Phi\|_{L^2_\xi}^2.$$ \hspace{1cm} (A.20)

Since

$$\|L\Phi\|_{L^2_\xi}^2 = \|H^{1/2}\Phi\|_{L^2_\xi}^2,$$

by (A.4), the proof follows from (A.20). \hfill \Box

Lemma A.3. Suppose $\Phi \in \mathcal{H}^4$. Then

$$\left\| \sqrt{\mu/\Phi} \right\|_{L^{1,2}_\xi} \lesssim \|\Phi\|_{\mathcal{H}^4},$$

and

$$|\sqrt{\mu}(\xi)\Phi(\xi)| \lesssim \|\Phi\|_{\mathcal{H}^4 \xi}.$$
Proof. First, we observe that
\[
\left\| \frac{\mu^2}{\xi^{3/4}} \frac{d^2}{d\xi^2} \left( \sqrt{\mu} \Phi \right) \right\|_{L_2^1} \lesssim \left\| \frac{\mu^2}{\xi^{3/4}} \frac{d^2}{d\xi^2} \left( \sqrt{\mu} \Phi \right) \right\|_{L_2^1} \quad \text{(by (A.16))}
\]
\[
\lesssim \| \Phi \|_{H^4} \quad \text{(by lemma A.2).} \tag{A.21}
\]
Next, we see from (A.2) that
\[
|H^2 \Phi|^2 \, dx = \mu^2 \left| \frac{d^2}{d\xi^2} \left( \mu^2 \frac{d^2}{d\xi^2} \left( \sqrt{\mu} \Phi \right) \right) \right|^2 \, d\xi,
\]
and so upon integrating, we find, after using (A.16), that
\[
\left\| \frac{\mu^2}{\xi^{3/4}} \frac{d^2}{d\xi^2} \left( \sqrt{\mu} \Phi \right) \right\|_{L_2^1} \lesssim \| \Phi \|_{H^4}^2. \tag{A.22}
\]
Setting
\[
f = \mu^2 \frac{d^2}{d\xi^2} \left( \sqrt{\mu} \Phi \right),
\]
the two inequalities (A.21) and (A.22) show that
\[
\left\| \frac{1}{\xi^{3/4}} f \right\|_{L_2^1}^2 + \left\| \frac{\xi^{3/4}}{d^2} \frac{d^2}{d\xi^2} f \right\|_{L_2^1}^2 \lesssim \| \Phi \|_{H^4}^2. \tag{A.23}
\]
Letting
\[
f_\lambda(\xi) = f(\lambda \xi),
\]
we compute
\[
\left| \sup_{\lambda/2 < \xi < \lambda} |f(\xi)| \right|^2 = \left| \sup_{1/2^{m+1} < \lambda < 1} |f_\lambda(\xi)| \right|^2
\]
\[
\lesssim \| f_\lambda \|_{H^1(1,2)}^2 \quad \text{(by (A.10))}
\]
\[
\lesssim \int_{1/2}^1 |f_\lambda(\xi)|^2 \, d\xi + \int_{1/2}^1 \left| \frac{d^2 f_\lambda}{d\xi^2} \right|^2 \, d\xi
\]
\[
\lesssim \int_{1/2}^1 \frac{1}{\xi^{3/2}} |f(\lambda \xi)|^2 \, d\xi + \int_{1/2}^1 \xi^{3/4} \lambda^4 \left| \frac{d^2 f}{d\xi^2}(\lambda \xi) \right|^2 \, d\xi
\]
\[
= \lambda^{1/2} \int_{\frac{1}{\lambda^2} \xi^{3/2}}^{\lambda} \frac{1}{\xi^{3/2}} |f(\xi)|^2 \, d\xi + \lambda^{3/2} \int_{\frac{1}{\lambda^2} \xi^{3/2}}^{\lambda} \xi^{3/4} \lambda^4 \left| \frac{d^2 f}{d\xi^2}(\xi) \right|^2 \, d\xi
\]
\[
\lesssim \lambda^{1/2} \| \Phi \|_{H^4}^2,
\]
where in deriving the last inequality, we used (A.23). From this, we conclude that
\[
|f(\xi)| \lesssim \| \Phi \|_{H^1(1,2)} |\xi|^{1/4}. \tag{A.24}
\]
In particular, this shows that
\[
\lim_{\xi \to 0} f(\xi) = 0. \tag{A.25}
\]
Next, we see that from (A.23) and the inequality (A.11) that
\[
\| f \|_{H^2} \lesssim \| \Phi \|_{H^4}.
\]
But since
\[
\| f \|_{H^2} \lesssim \| f_\lambda \|_{H^4} \lesssim \| f \|_{H^4},
\]
we have
\[
\| f \|_{H^2} \lesssim \| f \|_{H^4}.
\]
by (A.8) and (A.10), we see that
\[ \|f\|_{C^{1/4}} \lesssim \|\Phi\|_{\mathcal{H}^{\epsilon}}. \]

This together with (A.25) shows that
\[ |f(\xi)| \lesssim \|\Phi\|_{\mathcal{H}^{\epsilon}} |\xi|^{3/4}, \]
which in turn implies that
\[ \left\| \xi^1 \frac{d^2}{d\xi^2} (\sqrt{\mu} \Phi) \right\|_{L^2_x} \lesssim \|\Phi\|_{\mathcal{H}^{\epsilon}} \frac{|\xi|^{3/2}}{\mu^{1/2}}, \]
for any \( \epsilon > 0 \). Integrating, we find that
\[ \left\| \xi^1 \frac{d}{d\xi} \sqrt{\mu} \Phi (\xi) \right\|_{L^2_x} \lesssim \|\Phi\|_{\mathcal{H}^{\epsilon}}. \]

Applying Morrey’s inequality (A.10), we arrive at
\[ \left\| \xi^1 \frac{d}{d\xi} \sqrt{\mu} \Phi (\xi) \right\|_{L^1_x} \lesssim \|\Phi\|_{\mathcal{H}^{\epsilon}}. \]

Next, we observe that
\[ \left\| \xi^1 \frac{d}{d\xi} \sqrt{\mu} \Phi (\xi) \right\|_{L^1_x} + \left\| \xi^{3/4} \frac{d^2}{d\xi^2} (\sqrt{\mu} \Phi) \right\|_{L^2_x} \lesssim \|\Phi\|_{\mathcal{H}^{\epsilon}} \frac{\xi^{3/2}}{\mu^{1/2}}, \]
by lemma A.2 and (A.16). The same argument used to derive (A.24) from (A.23) shows that
\[ |\sqrt{\mu}(\xi) \Phi (\xi)| \lesssim \|\Phi\|_{\mathcal{H}^{\epsilon}} \xi^{1/4}. \]

Together, lemma A.2, the inequality (A.26) and (A.11) show that
\[ \left\| \sqrt{\mu} \Phi \right\|_{H^1_{1/4+\epsilon}} \lesssim \|\Phi\|_{\mathcal{H}^{\epsilon}}, \]
and hence that
\[ \|\sqrt{\mu} \Phi\|_{L^2_x} \lesssim \|\Phi\|_{\mathcal{H}^{\epsilon}}. \]

by Morrey’s inequality (A.10). Finally, since
\[ \lim_{\xi \to 0} \sqrt{\mu}(\xi) \Phi (\xi) = 0 \]
by (A.27), we get from (A.28) that
\[ |\sqrt{\mu}(\xi) \Phi (\xi)| \lesssim \|\Phi\|_{\mathcal{H}^{\epsilon}} \xi. \]

\[ \square \]

**Theorem A.4.** Suppose \( (\Phi_0, \Phi_1) \in \mathcal{H}^6 \times \mathcal{H}^5 \) and \( \Phi \in \bigcap_{j=0}^{6-j} C^j ([0, \infty), \mathcal{H}^{6-j}) \) is the solution to the wave equation (A.1) from theorem A.1. Then there exists a map \( \tilde{\phi} (t, \xi) \) that satisfies the following:

(i)
\[ \tilde{\phi} \in C^1 ([0, \infty), C^{1/2} (0, (\pi/2)^4)) \cap \bigcap_{j=0}^{5} C^j ([0, \infty), C^{5-j} (0, (\pi/2)^4)). \]
(ii) for any \( t \in (0, \infty) \),
\[
|\partial_t \tilde{\phi}(t, \xi)| + |\tilde{\phi}(t, \xi)| \lesssim \xi
\]
for \( 0 < \xi < (\pi/2)^4 \) and
\[(iii)\]
\[
\sqrt{\mu(x)}\Phi(t, x) = \tilde{\phi}(t, x^4)
\]
for all \((t, x) \in (0, \infty) \times (0, \pi/2)\).

Moreover, if there exists a positive constant \( c \) such that
\[
\frac{1}{\mu(x)} \partial_x \left( \sqrt{\mu(x)}\Phi(0, x) \right) \geq c > 0
\]
for all \( x \in (0, \pi/2) \), then there exists a \( T > 0 \) such that
\[
\partial_\xi \tilde{\phi}(t, \xi) > 0
\]
for all \((t, x) \in (0, T) \times (0, (\pi/2)^4)\).

\textbf{Proof.} Statements (i)–(iii) follows directly from the regularity statement \( \Phi \in \cap_{j=0}^6 C^j([0, \infty), H^{6-j}) \), lemma (A.3) and that fact that \( C^4 \subset H^{6+1} \), which follows from Morrey’s inequality and the inclusion \( H^4 \subset H^4_{1,\infty}(0, \pi) \). Note that we are also using the fact that the variable \( \xi \) is uniformly equivalent to \( x^4 \), see (A.15).

For the final statement, suppose that
\[
\frac{1}{\mu(x)} \partial_x \left( \sqrt{\mu(x)}\Phi(0, x) \right) \geq c > 0
\]
for \( x \in (0, \pi/2) \). Then differentiating \( \sqrt{\mu(x)}\Phi(t, x) = \tilde{\phi}(t, x^4) \) gives
\[
\frac{1}{\mu(x)} \partial_x \left( \sqrt{\mu(x)}\Phi(t, x) \right) = \frac{4x^3}{\mu(x)} \partial_\xi \tilde{\phi}(t, x^4).
\] (A.30)

Since there exists a non-zero constant \( C \) such that
\[
0 < \frac{1}{C} \leq \frac{4x^3}{\mu(x)} \leq C
\]
for \( 0 < x < \pi/2 \), it follows from (A.29) and (A.30) that
\[
\partial_\xi \tilde{\phi}(0, x^4) \geq \frac{c}{C} > 0
\]
for all \( x \in (0, \pi/2) \), or equivalently
\[
\partial_\xi \tilde{\phi}(0, \xi) \geq \frac{c}{C} > 0
\]
for all \( \xi \in (0, (\pi/2)^4) \). From the continuity of \( \partial_\xi \tilde{\phi}(t, \xi) \), we see that there exists a \( T > 0 \) such that
\[
\partial_\xi \tilde{\phi}(t, \xi) \geq \frac{c}{2C} > 0
\]
for all \((t, \xi) \in [0, T) \times (0, (\pi/2)^4)\). \(\square\)
References

[1] Evans L C 1998 Partial Differential Equations (Providence, RI: American Mathematical Society)
[2] Choquet-Bruhat Y, De Witt-Morette C and Dillard-Bleick M 1996 Analysis, Manifolds, and Physics: Part I. Basics revised edn (Amsterdam: North-Holland)
[3] Christodoulou D 2007 The Formation of Shocks in 3-Dimensional Fluids (Zurich: European Mathematical Society)
[4] Coutand D, Lindblad H and Shkoller S 2010 A priori estimates for the free-boundary 3-D compressible Euler equations in physical vacuum Commun. Math. Phys. 296 559–87
[5] Coutand D and Shkoller S 2011 Well-posedness in smooth function spaces for the moving-boundary 1-D compressible Euler equations in physical vacuum Commun. Pure Appl. Math. 64 328–66
[6] Coutand D and Shkoller S 2012 Well-posedness in smooth function spaces for the moving-boundary 3-D compressible Euler equations in physical vacuum Arch. Rational Mech. Anal. at press (arXiv:1003.4721)
[7] Frauendiener J 2003 A note on the relativistic Euler equations Class. Quantum Grav. 20 L193–6
[8] Friedrich H 1998 Evolution equations for gravitating ideal fluid bodies in general relativity Phys. Rev. D 57 2317–22
[9] Goldstein J A 1985 Semigroups of Linear Operators and Applications (Oxford: Oxford University Press)
[10] Goldstein J A and Wacker M 2003 The energy space and norm growth for abstract wave equations Appl. Math. Lett. 16 767–72
[11] Jang J and Masmoudi N 2009 Well-posedness for compressible Euler with physical vacuum singularity Commun. Pure Appl. Math. 62 1327–85
[12] Jang J and Masmoudi N 2010 Well-posedness of compressible Euler equations in a physical vacuum arXiv:1005.4441
[13] Kufner A and Persson L E 2003 Weighted Inequalities of Hardy Type (Singapore: World Scientific)
[14] Lang S 1999 Fundamentals of Differential Geometry (Berlin: Springer)
[15] LeFloch P G and Ukai S 2009 A symmetrization of the relativistic Euler equations in several spatial variables Kinetic and Related Models 2 275–92
[16] Makino T 1986 On a local existence theorem for the evolution equation of gaseous stars Patterns and Waves ed T Nishida, M Mimura and H Fujii (Amsterdam: North-Holland)
[17] Reed M and Simon B 1975 Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness (New York: Academic)
[18] Rendall A D 1992 The initial value problem for a class of general relativistic fluid bodies J. Math. Phys. 33 1047–53
[19] Visser M and Molina-Par´ı C 2010 Acoustic geometry for general relativistic barotropic irrotational fluid flow New J. Phys. 12 095014
[20] Wald R M 1984 General Relativity (Chicago, IL: University of Chicago Press)
[21] Walton R A 2005 A symmetric hyperbolic structure for isentropic relativistic perfect fluids Hous. J. Math. 31 145–60