Generating Black Strings in Higher Dimensions

L. A. López¹, A. Feinstein² and N. Bretón¹

¹ Dpto. de Física, Centro de Investigación y de Estudios Avanzados del I. P. N., Apdo. 14-740, D.F., México.
² Dpto. de Física Teórica, Universidad del País Vasco, Apdo. 644, E-48080, Bilbao, Spain.

Abstract

Starting with a Zipoy-Voorhees line element we construct and study the three parameter family of solutions describing a deformed black string with arbitrary tension.

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I. INTRODUCTION

Associated with string theory and higher dimensional universes, there has been a renewed interest in higher dimensional solutions to Einstein field equations, among the most interesting being black rings [1], KK bubbles as well as black strings.

One simple way to lift the 4D Schwarzschild black hole to a 5D black string is to add an extra flat dimension. In other words it is possible to uniformly extend the 4D black hole with $S^2$ horizon into the fifth dimension producing a hypercylindrical black hole $S^2 \times R$ [2]. A striking fact related to black strings is their generic instability and final fate and whether these end up as black holes or different objects is still not well understood [3], [4]. In this sense it is interesting testing more general black string solutions both static and non-static. Some generalizations are obtained by applying a boost to the string solution, dotating the string with the momentum along the fifth coordinate [5]. In this work we derive and report a generalized black string solution with mass, arbitrary tension and an additional parameter related to deformation from spherical symmetry.

To generate a generalized black string solution we take as seed the Zipoy-Vorhees (ZV) family of solutions and then lift it to 5D by the procedure introduced in [6], [7]. Particular cases are the static black string with arbitrary tension of C. H. Lee [8] and the Gregory-Laflamme black string [2]. Trapped surfaces and horizons, as well as the mass and string tension parameters, are analyzed and discussed.

II. GENERALIZED BLACK STRING

We briefly explain the idea of the generating technique (see details in [7]). Let us consider the 5D vacuum Einstein action,

$$S = -\frac{1}{16\pi G} \int \hat{R} \sqrt{-\hat{g}} d^4 x dy$$  \hspace{1cm} (1)

where the hatted quantities mean the fifth dimensional version of the scalar curvature $\hat{R}$, the metric tensor $\hat{g}$ and the gravitational Newton constant $G$ and $y$ is the fifth coordinate. The equations of motion derived of the variation of action (1) coincide with the Einstein-scalar coupled equations in four dimensions. For a static line element given in Weyl coordinates,
\[ ds^2 = -e^{2U} dt^2 + e^{-2U+2\sigma}(d\rho^2 + dz^2) + \rho^2 e^{-2U} d\phi^2, \tag{2} \]

and a minimally coupled scalar field \( \varphi \), these field equations are given by

\[
0 = U_{,\rho\rho} + \frac{1}{\rho} U_{,\rho} + U_{,zz},
0 = \varphi_{,\rho\rho} + \frac{1}{\rho} \varphi_{,\rho} + \varphi_{,zz},
\sigma_{,\rho} = \rho(U_{,\rho}^2 - U_{,z}^2) + \frac{\rho}{2}(\varphi_{,\rho}^2 - \varphi_{,z}^2),
\sigma_{,z} = 2\rho U_{,\rho} U_{,z} + 2\rho \varphi_{,\rho} \varphi_{,z}. \tag{3} \]

Let us consider the ZV solution \[9\] in Weyl coordinates (for a discussion on Zipoy-Voorhees solution see \[10\]), with the line element (2) and the metric functions given by:

\[
U = \frac{1}{2} \ln \left[ \frac{r_+ + r_- - 2m}{r_+ + r_- + 2m} \right]^2,
\sigma = \frac{1}{2} \ln \left[ \frac{(r_+ + r_-)^2 - 4m^2}{4r_+ r_-} \right]^2,
\]

\[
r_{\pm}^2 = \rho^2 + (z \pm m)^2. \tag{4} \]

ZV solution is characterized by two free parameters \( m \) and \( q \) related to mass and spherical-symmetry deformation, respectively. For \( q = 2 \) we recover Schwarzschild solution while if \( q = 4 \) it represents the Darmois solution \[9\].

Adding the scalar field in order to use the generating algorithm of \[6\],\[7\]

\[
\varphi = \frac{A}{2} \ln \left[ \frac{r_+ + r_- - 2m}{r_+ + r_- + 2m} \right], \tag{5} \]

where the scalar field is a solution of Laplace’s equation \( \varphi_{,\rho\rho} + \frac{1}{\rho} \varphi_{,\rho} + \varphi_{,zz} = 0 \), we obtain the spacetime with a scalar field \( \varphi \), expressed as

\[
ds_{sc}^2 = -e^{2U} dt^2 + e^{-2U+2(\sigma+\sigma_{sc})}(d\rho^2 + dz^2) + \rho^2 e^{-2U} d\phi^2 \tag{6} \]

with

\[
\sigma_{sc} = \frac{A^2}{2} \ln \left[ \frac{(r_+ + r_-)^2 - 4m^2}{4r_+ r_-} \right]. \tag{7} \]
The above solution is an exact solution of the Einstein-scalar equations (3).

Using now the method described in [1], and considering the metric (6) as seed, we lift the solution to five dimensions:

\[ dS_5^2 = e^{-\frac{2\phi}{\sqrt{3}}} (ds_{sc}^2) + e^{\frac{4\phi}{\sqrt{3}}} d\omega^2, \]  

explicitly

\[ dS_5^2 = -H^{\frac{2}{3}} dt^2 + H^{\frac{2}{3}} - \frac{A}{\sqrt{3}} \rho^2 d\phi^2 + H^{\frac{4}{3}} d\omega^2 + H^{-\frac{2}{3}} \sqrt{3} F^{2+ A^2} (d\rho^2 + dz^2), \]  

where

\[ H = \frac{(r_+ + r_- - 2m)}{(r_+ + r_- + 2m)}, \quad F = \frac{(r_+ + r_-)^2 - 4m^2}{4r_+r_-}. \]  

Transforming the metric (9) to the Schwarzschild-like coordinates \((r, \theta)\), with

\[ \rho^2 = \left(1 - \frac{2m}{r}\right) r^2 \sin^2 \theta, \quad z = r \left(1 - \frac{m}{r}\right) \cos \theta, \]  

we obtain:

\[ dS_5^2 = -G^{\frac{2}{3}} dt^2 + G^{\frac{2}{3}} - \frac{A}{\sqrt{3}} r^2 \sin^2 \theta d\phi^2 + G^{\frac{4}{3}} d\omega^2 + \left(G + \frac{m^2 \sin^2 \theta}{r^2}\right)^{1-A^2 - \frac{2}{3}} G^{P r^2} \left(G^{-1} \frac{dr^2}{r^2} + d\theta^2\right), \]  

where \(G = (1 - 2m/r)\) and \(P = A^2 + \frac{q^2}{4} - \frac{A}{\sqrt{3}} - \frac{q}{2}\).

This solution is a generalized 5D black string characterized by three free parameters: \(A\), \(q\) and \(m\). Note that to recover transverse spherical symmetry, we must choose \(A^2 + \frac{q^2}{4} = 1\) in (12). Doing so the term with \(\sin^2 \theta\) does not appear in \(g_{rr}\) and \(g_{\theta\theta}\), obtaining an element proportional to \(r^2 d\Omega:\)

\[ dS_5^2 = -G^{\frac{2}{3}} dt^2 + G^{\frac{2}{3}} - \frac{A}{\sqrt{3}} r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + G^{\frac{4}{3}} d\omega^2 + G^{-\frac{2}{3}} \sqrt{3} dr^2. \]  

The case \(A = 0\) and \(q = 2\) corresponds to the black string [2],

\[ dS_5^2 = - \left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + d\omega^2. \]
In the case \( A^2 = 3/4 \) and \( q = 1 \) solution (12) becomes
\[
dS^2 = [r^2(d\theta^2 + \sin^2 \theta d\phi^2) + G^{-1}dr^2 + Gd\omega^2] - dt^2, \tag{15}
\]
having in square brackets the Schwarzschild line element with the euclidean signature. By analytic continuation of the 2-sphere in (15) to a 2-de Sitter making \((\theta - \pi/2) \mapsto i\tau\) and \(t \mapsto i\psi\) we get the expression of the 5D KK-bubble of nothing [11]
\[
ds^2 = [r^2(-d\tau^2 + \cosh^2 \tau d\phi^2) + G^{-1}dr^2 + Gd\omega^2] + d\psi^2. \tag{16}
\]
Another interesting particular case is the black string with arbitrary tension [8]. Performing in (13) the coordinate transformation
\[
r = \rho \left(1 + \frac{k}{\rho}\right)^2, \quad \text{with} \quad m = 2k, \tag{17}
\]
the metric is transformed into:
\[
dS_5^2 = D^{-\frac{2}{\sqrt{3}}}A [-D^qdt^2 + (1 - k^2/\rho^2)^2D^{-q}(d\rho^2 + \rho^2d\theta^2 + \rho^2\sin^2 \theta d\phi^2)] + D^{4/3}A d\omega^2, \tag{18}
\]
where \( D = (1 - k/\rho)/(1 + k/\rho) \). Parametrizing with \((s,a)\) related to \((A,q)\) by
\[
2A = \frac{2a - 1}{\sqrt{1 - a + a^2}}, \quad q = \frac{3}{\sqrt{3}(1 - a + a^2)}, \quad -\frac{2}{\sqrt{3}}A + q = s, \quad s = \frac{2(2 - a)}{\sqrt{3}(1 - a + a^2)}, \tag{19}
\]
the metric (18) is written as
\[
dS_5^2 = -D^s dt^2 + (1 - k^2/\rho^2)^2D^{-\frac{1+q}{2-a}}(d\rho^2 + \rho^2d\theta^2 + \rho^2\sin^2 \theta d\phi^2) + D^{1-\frac{q}{2-a}}d\omega^2. \tag{20}
\]
In this parametrization the solution has only two free parameters \((k,a)\) and we identify it as the black string with arbitrary tension previously derived in [8] and studied in [12]. Notice that we have given a much simpler derivation of the solution.
FIG. 1: Rod structure of the 4D Zipoy-Vorhees solution

FIG. 2: Rod structure of the 5D black string generated from Zipoy-Vorhees and inserting a finite rod with scalar charge $2A/\sqrt{3}$ in the interval $(-m, m)$ in the fifth dimension. The bold dotted line along $\partial\phi$ is a rod with negative mass density (conical singularity).

III. PROPERTIES OF THE 5D GENERALIZED BLACK STRING

A. Rod-structure

It is illustrative to sketch briefly how the generation method works in terms of the rod-structure \[13], \[14]. The ZV seed solution, Eqs. (2)-(4), is characterized by the rod-structure shown in Fig. 1. It differs from Schwarzschild only in the mass density along the Killing directions: for Schwarzschild the density in the interval $[-m, m]$ along the $\partial_t$ direction is 1, while for ZV the density is $q/2$. In the intervals $[-\infty, -m]$ and $[m, \infty]$ along $\partial_\phi$ the density is also $q/2$ ($q = 2$ for Schwarzschild).

Introducing the scalar field $\varphi$ and then lifting to 5D in terms of the rod structure amounts to introduce into the fifth dimension a rod of density $2A/\sqrt{3}$. The rod structure of the 5D black string, Eqs. (9)-(10), is (shown in Fig. 2) the following:

There is a density $2A/\sqrt{3}$ in the interval $[-m, m]$ along the $\partial_\omega$ direction. A conical
singularity along the $\partial_\phi$ direction is shown as the negative density $-A/\sqrt{3}$ in the interval $[-m, m]$ (bold dotted line); also there is a positive density of $q/2$ in the intervals $[-\infty, m]$ and $[m, \infty]$ along the $\partial_\phi$ direction. The density is $(q/2 - A/\sqrt{3})$ in the interval $[-m, m]$ along the $\partial_t$ direction; to have a physical horizon, if one wants to keep calling the system a black string, the density along the timelike direction should be positive, i.e. $(q/2 - A/\sqrt{3}) > 0$.

B. Apparent horizons and singularities

The invariant that defines trapped or marginally trapped surfaces for the generated solution (12) is analyzed in what follows.

Trapped surfaces $S_{X^a}$ and located horizons corresponding to the spacetime (12), are determined by the scalar $\kappa$ (see [15]). To calculate it, we fix the coordinates $x^a = \{r, t\}$ and denote the local coordinates on the surface $S_{X^a}$ by $x^A = \{\theta, \phi, \omega\}$. The function

$$e^V \equiv \sqrt{\det g_{AB}},$$

$$g_{aA} \equiv g_{aA}dx^a, \quad (a = r, t),$$

$$H_\mu = \delta_a^\mu(V_a - \text{div} g_a)$$

and defining

$$\kappa_{x^a} = -g^{ab}H_bH_c|_{S_{X^a}},$$

the invariant $\kappa$ for the obtained solution amounts to

$$\kappa_{(r,t)} = -g^{rr}V_{,r}^2.$$  (22)

For the solution of our interest,

$$V = \frac{1}{2} \ln \left[ \frac{r^2(A^2 + \frac{q^2}{4} + 1) \sin^2 \theta (1 - \frac{2m}{r})^{1-q+A^2+\frac{q^2}{4}}}{(r^2 - 2mr + m^2 \sin^2 \theta)^{A^2+\frac{q^2}{4} - 1}} \right]$$  (23)

and therefore, $\kappa$ is obtained as

$$\kappa = -\frac{[r - m(1 + \cos \theta)][r - m(1 - \cos \theta)]r^{A^2+\frac{q^2}{4} - 3} [\tilde{V}_r]^2}{r^{A^2+\frac{q^2}{4} + \frac{q^2}{4} + 1} (r - 2m)^{A^2+\frac{q^2}{4} - \frac{q^2}{4} + 1}}$$  (24)

with

$$\tilde{V}_r = 2r^3 - (6 + q)mr^2 + [4 + 2q + (1 + A^2 + \frac{q^2}{4}) \sin^2 \theta]m^2r - (1 + A^2 + \frac{q^2}{4} + q) \sin^2 \theta m^3$$

$$= (r - m)[2r(r - 2m) + m^2 \sin^2 \theta(A^2 + \frac{q^2}{4} + 1)] - mq[r^2 - 2mr + m^2 \sin^2 \theta],$$  (25)
Let us analyze the marginally trapped surfaces defined when \( \kappa = 0 \) as well as the singularities if \( \kappa \) diverges. To determine those values of \( \kappa \) cancelations in the denominator coming from \( \tilde{V}_r \) should be taken into account.

Assuming that \( A > 0 \) and \( q > 0 \), then there exists a singularity at \( r = 0 \) since the exponent of \( r \) in the denominator of (24) is always positive, \( A^2 + \frac{q^2}{4} + \frac{1}{\sqrt{3}} + \frac{2}{3} + 1 > 0 \).

Also there are two marginally trapped surfaces at \( r_{\pm} = m(1 \pm \cos \theta) \), nevertheless these are wrapped or hidden by the naked singularity located at \( r = 2m \). Depending on the values of \( A \) and \( q \), in the product of \( \tilde{V}_r \) and \( g^{rr} \) may be cancelations that define horizons that coincide with the naked singularity \( r = 2m \). Another kind of marginally trapped surfaces that overlap the singularity in some regions, arise from the real root \( r_h(m, A, q, \theta) \) of the third degree polynomial \( \tilde{V}_r \). These surfaces when \( r_h > 2m \) do not cover completely the singularity due to the dependence on \( \theta \). Then the situation when \( r_h > 2m \) is that for some ranges of \( \theta \) the singularity is hidden but coming from other direction the singularity is naked, i.e. the singularity is partially naked (or partially dressed).

For the black string with arbitrary tension \( (A^2 + q^2/4 = 1) \), \( \kappa \) amounts to

\[
\kappa = -\frac{4}{r^2} \left( 1 - \frac{2m}{r} \right)^{\frac{3}{4} + \frac{q}{2} - 2} \left( 1 - \frac{m(1 + q/2)}{r} \right)^2,
\]

(26)

The solution is singular at \( r = 2m \) unless \( q = 2 \ (a = 1/2) \), case in which it possesses a horizon at \( r = 2m \) \( (r = 4k) \). In any other case \( (q < 2) \) there is a naked singularity at \( r = 2m \) that hides horizons at \( r = m(1 + q/2) \). The \( \kappa \) diverges at \( r = 0 \) showing the singularity there.

In the case \( A = 0 \) and \( q = 2 \) that we recover the black string solution \([2]\), it possesses a horizon at \( r = 2m \) as the expression for \( \kappa \) shows,

\[
\kappa = -\left( 1 - \frac{2m}{r} \right) \left( \frac{2}{r} \right)^2
\]

(27)

In this case the naked singularity disappears and instead a horizon is located at \( r = 2m \).

With respect to the analysis of the singularities we have attempted to draw Penrose diagram to dilucidate the global structure of the solutions. However, due to the reduced symmetry of the line element \([12]\) the metric function \( g^{rr} \) does not depend only on \( r \) but also on \( \theta \). Even if one drops the \( \theta \) dependence, by fixing the angle, the integration to be performed involve hypergeometric functions, and the analysis by this method does not
produce a clear picture. On the other hand, the information one obtains by studying the scalar \( \kappa \), is essentially the same that can be obtained from the Kretschmann scalar.

C. Mass and string tension

To characterize the 5D black string we need to determine the mass and tension of the source using the asymptotic expansion of the metric functions \( g_{tt} \) and \( g_{\omega \omega} \) of (12),

\[
-g_{tt} = \left(1 - \frac{2m}{r}\right)^{-\frac{4\sqrt{3} + q}{2}} \\
\approx 1 + \left(\frac{A}{\sqrt{3}} - \frac{q}{2}\right) \frac{2m}{r} + \frac{1}{2} \left(\frac{A}{\sqrt{3}} - \frac{q}{2}\right)^2 \frac{4m^2}{r^2} + \cdots \tag{28}
\]

\[
g_{\omega \omega} = \left(1 - \frac{2m}{r}\right)^{\frac{2A}{\sqrt{3}}} \approx 1 - \frac{2A}{\sqrt{3}} + \frac{2A}{\sqrt{3}} \left(\frac{2A}{\sqrt{3}} - 1\right) \left(\frac{4m^2}{r^2}\right) + \cdots \tag{29}
\]

Comparing with the asymptotic form of metric around a stationary matter source as given in [16], [17]:

\[
g_{tt} \approx -1 + \frac{4G_5M(2 - a)}{3r}, \tag{30}
\]

\[
g_{\omega \omega} \approx 1 + \frac{4G_5M(1 - 2a)}{3r}, \tag{31}
\]

The mass and string tension of the 5D black string (12) are identified then as (we are considering \( G_5 = 1 \))

\[
M = \frac{2qm}{4G_5}, \quad Ma = \tau = \frac{m}{4G_5} \left(\frac{6A}{\sqrt{3}} + q\right). \tag{32}
\]

Therefore when (2) is lifted to 5D, by means of the scalar field \( \varphi(A,r,m) \), the string tension changes in \( \frac{6A}{\sqrt{3}} \), that can modify the tension and even make it vanish by calibrating \( A = -\sqrt{3}q/6 \), however, from the rod-structure we learned that \( A > q/2 \) in order to have a horizon.

In the case \( A = 0 \) and \( q = 2 \) we recover the black string solution [2], with \( \tau = M/2 \). The case \( A = \sqrt{3}/2 \) and \( q = 1 \) corresponds to a black string with \( \tau = 2M \).
IV. SUMMARY

In this paper by equipping first the 4D Zipoy-Vorhees solution with a scalar field, we lifted it to five dimensions; the so generated 5D solution is a black string characterized by three free parameters: mass, tension and deformation parameter. There is a physical spacetime singularity at $r = 0$. Moreover, the horizons do not cover completely the naked singularity at $r = 2m$, except in the case that $r = 2m$ is the horizon itself. Therefore, in 5D the cosmic censorship conjecture does not hold.

The static black string with arbitrary tension is a particular case when there is transversal spherical symmetry. The stability of the 5D generalized black string deserves further analysis, particularly to acquaint how the deformation parameter $A$ can affect the stability of the solution, since it modifies the string tension. We have seen that the three parameter generalizations of the black string solutions are singular. This may well be an indicator that the black strings are unstable in the sense of [3], [4].

Interesting solutions are also obtained by an analytic continuations of the ones obtained in this report. These and other issues will be discussed elsewhere.

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