Dirac Equation for Electrodynamic Model Particles

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Abstract. We set up the Maxwell’s equations and subsequently the classical wave equations for the electromagnetic waves which together with their generating source, an oscillatory charge of zero rest mass in general travelling, make up a particle travelling similarly as the source at velocity $v$ in the field of an external scalar and vector potentials. The direct solutions in constant external field are Doppler-displaced plane waves propagating at the velocity of light $c$; at the de Broglie wavelength scale and expressed in terms of the dynamically equivalent and appropriate geometric mean wave variables, these render as functions identical to the space-time functions of a corresponding Dirac spinor, and in turn identical to de Broglie phase waves previously obtained from explicit superposition. For two spin-half particles of a common set of space-time functions constrained with antisymmetric spin functions as follows the Pauli principle for same charges and as separately indirectly induced based on experiment for opposite charges, the complete wave functions are identical to the Dirac spinor. The back-substitution of the so explicitly determined complete wave functions in the corresponding classical wave equations of the two particles, subjected further to reductions appropriate for the stationary-state particle motion and to rotation invariance when in three dimensions, give a Dirac equation set; the procedure and conclusion are directly extendible to arbitrarily varying potentials by use of the Furious theorem and to particle motions in three dimensions by virtue of the characteristics of de Broglie particle motion. Through the derivation of the Dirac equation, the study hopes to lend insight into the connections between the Dirac wave functions and the electrodynamic components of simple particles under the government by the well established basic laws of electrodynamics.

1. Introduction
P.A.M. Dirac established in [1] a relativistic quantum mechanical wave equation, Dirac equation, for a point electron based on the relativistic energy-momentum relation subjected to Lorentz transformation under rotation. In [1] P.A.M. Dirac also theoretically predicted for the electron the existence of an internal oscillation state, a magnetic moment, and by interpretation of the negative energy solution, an anti-particle state known today as the positron. The Dirac equation has proven to be an accurate equation of motion for (two) spin-half quantum particles at high velocities; most notably, Dirac predicted based on his equation the relativistic intensities of Zeeman components of spectral lines and the frequency differences [1 (1928b)] in exact agreement with experiment. Up to the present however it has remained an open question that, what is waving with the Dirac wave functions, or Dirac spinor, a similar question as for the Schrödinger wave functions and the de Broglie waves? In addition, the Dirac theory meets with a few its own open questions. What is the nature of a Dirac internal oscillation? How are the Dirac space-time functions explicitly connected with the spin orientations, the signs of charges, the signs of the
energies, and in the extreme situation when an electron and position annihilate, the emitted two gamma rays and conversely? What is the symmetry of the total spin of an electron and positron? Also, in the case of an isolated single electron or positron in zero external field where the spin orientation is of no consequence, it would be desirable to have a way to directly write down the corresponding Dirac equation without involving the Pauli matrices. These as well as various other not fully addressed questions relating to fundamental physics seem to consistently point to the inadequacy of the point particle picture of today and the need for a representation of the internal processes of the particles.

Recently, using overall experimental observations as input data the author proposed an internally electrodynamic (IED) particle model [2a,l] or sometimes termed a basic particle formation (BPF) scheme (with coauthor P.-I. Johansson), which states that a simple (basic) particle like an electron and positron, etc., briefly, is constituted of an oscillatory point-like (elementary) charge with a specified sign and a zero rest mass, and the resulting electromagnetic waves in the vacuum. As a broad test of the IED particle model and also as an endeavour of understanding the various puzzles relating to fundamental physics, in terms of solutions for the electrodynamic processes of the model particle with its charge’s sign and total energy as two sole input data, the author has further achieved with coauthor(s) derivations/predictions of a range of basic properties and relations of the simple particles [2a-k] including the relativistic mass, de Broglie wave, de Broglie relations, Schrödinger equation, Einstein energy-mass relation, Newton’s law of gravity and Doebner-Goldin equation, among others. As to the Schrödinger wave function specifically of relevance here, the solution[2a,c] showed that it is the (envelope of the) standing beat wave resulting from superposition of the Doppler-differentiated electromagnetic waves generated by the particle’s travelling charge, that is waving.

As shown in [2a,c-d], the direct solutions for the classical wave equations, derivable from the Maxwell’s equations, for the electromagnetic waves comprising a free particle consist of Doppler-displaced plane waves; these superpose to two opposite-travelling beat waves that represent directly the de Broglie phase waves and in turn the Dirac space-time functions. It therefore is foreseeable that the classical wave equations for the electromagnetic waves here of two given particles would more naturally lead to wave equations of the particles corresponding directly to the Dirac equation in comparison to the Schrödinger equation. We elucidate in this paper a formal procedure which transforms the classical wave equations for the electromagnetic waves of two spin-half particles, of identical space-time functions and tending to approach one another, to the Dirac equation. Through the procedure we show that the Dirac internal oscillation corresponds to the oscillation of the electromagnetic waves at a geometric mean of the respective wave frequencies which in general are Doppler-displaced owing to source motion. And we elucidate the explicit relationships between the internal electromagnetic waves, charges, spins, the centre-of-mass and total wave motions and the associated energies of the particles, pertaining to entities and motions at a sub-quantum mechanical level, under the government of a few established elementary laws of electrodynamics.

2. Wave equations for the electromagnetic waves of particle. Solutions

We consider an IED particle, here an electron or positron, is as its source charge \( q = e \) or \(-e\) travelling at a velocity \( \mathbf{v} \) in \(+z\)-direction for the present along a one-dimensional box of side \( L \) in the vacuum. The charge \( q \) of the particle has an oscillation associated with a total energy \( \varepsilon_q \), which is minimum at \( v = 0 \), denoted by \( \varepsilon_q \); \( \varepsilon_q \) may be endowed e.g. in a pair production in the vacuum. In virtue that it describes the ground state, \( \varepsilon_q \) cannot be dissipated or detached from the charge except in a pair annihilation.

The charge \( q \) of the particle generates owing to its oscillation electromagnetic waves of radiation electric fields \( \mathbf{E}^j \)'s and magnetic fields \( \mathbf{B}^j \)'s described in zero applied potential field by
the Maxwell’s equations as:

\[ \nabla \cdot \mathbf{E}^j = \rho_q^j/\varepsilon_0, \quad \nabla \cdot \mathbf{B}^j = 0, \quad \nabla \times \mathbf{B}^j = \mu_0 \mathbf{j}_q^j + (1/c^2) \partial_t \mathbf{E}^j, \quad \nabla \times \mathbf{E}^j = -\partial_t \mathbf{B}^j. \]  

(1)

Where \( \rho_q^j \) is the density and \( \mathbf{j}_q^j \) the current of the particle’s charge, assuming no other charges and currents present; \( \varepsilon_0 \) is the permittivity and \( \mu_0 \) the permeability of the vacuum, and \( c \) is the velocity of light; \( \partial_t \equiv \partial/\partial t \). Expressing the \( j \)th fields generally by a dimensionless displacement \( \varphi^j \), \( E^j = D\varphi^j \), thus \( B^j = E^j/c = D\varphi^j/c \), with \( D \) a conversion constant, considering regions sufficiently away from the source only so that \( \rho_q^j = \mathbf{j}_q^j = 0 \), and with some otherwise standard algebra of the Maxwell’s equations (1), we obtain the corresponding classical wave equations for the electromagnetic waves \( \varphi^j \)’s

\[ c^2 \nabla^2 \varphi^j = \partial_t^2 \varphi^j, \]  

(2)

with \( \partial_t^2 \equiv \partial^2/\partial t^2 \). In the above, \( j = \dagger \) labels the component wave generated in the direction parallel with \( +\upsilon \), and \( j = \ddagger \) the wave parallel with \( -\upsilon \); within walls there prevail also their reflected components generated by the reflected charge at an earlier time and being at the present time as if generated by a virtual charge travelling in the \(-z\)-direction, labelled by \( j = \text{vir}\dagger \) and \( j = \text{vir}\ddagger \). \( j \) is to distinguish a Doppler effect owing to the source motion to be expressed in (7) below. In Appendix A we outline in relevance to the particle model a few further standard relations of classical and quantum electrodynamics for the electromagnetic waves, and a derivation of the particle’s mass given by the author previously\(^{[2a,e]} \) (with P.-I. Johansson).

To the particle we now apply an electromagnetic force \( \mathbf{F}_q = \mathbf{F}_e + \mathbf{F}_m \), with \( \mathbf{F}_e = -q\nabla \varphi_a \) the Coulomb force in \( z \)-direction and \( \mathbf{F}_m = -q\upsilon \times \nabla \times \mathbf{A}_a \) the Lorentz force due to an external scalar potential \( \varphi_a \) and vector potential \( \mathbf{A}_a \), each expressed in SI units as for all other quantities in this paper. \( \mathbf{F}_m \) may be simplified using the BAC-CAB rule as \( \mathbf{F}_m = -q[\nabla (\upsilon \cdot \mathbf{A}_a) - \mathbf{A}_a (\upsilon \cdot \nabla)] \). In the applications below \( \mathbf{A}_a \) is constant in \( L \) or in each small division in question (see end of Sec. 3), and \( \upsilon \) is constant in \( L \) for the particle being in stationary state and also is parallel with \( \nabla \) and \( z \), so \( \nabla (\upsilon \cdot \mathbf{A}_a) = 0 \) and \( \mathbf{F}_m = q\mathbf{A}_a (\upsilon \cdot \nabla) = q\mathbf{A}_a \upsilon \nabla \). Thus, \( \mathbf{F} = -q\nabla \varphi_a + q\mathbf{A}_a \upsilon \nabla \). The formula of \( \mathbf{F} \) is in the usual usage established for a point particle; so when extending to the extensive IED particle here, \( \mathbf{F} \) apparently directly acts on the particle’s point charge.

We need to map the \( \mathbf{F} \) to a force directly interacting with the internal fields \( E^j, B^j \), or \( \varphi^j \) of the particle. We observe that, in virtue of its form, (2) after multiplying a conversion factor \( a \) of length dimension represents just a classical wave equation for a mechanical wave of a transverse displacement \( a\varphi^j \) propagated in an apparent elastic medium. On grounds of this direct correspondence, but taken as a heuristic means only in this paper (so that we here need not involve the details of this elastic medium), \( \mathbf{F} \) therefore interacts with the internal fields through a force \( \mathbf{F}^j_{\text{med}} \) directly acting on this apparent medium. We can think of the medium to be composed of coupled dipole charges which do not move along the \( z \)-axis but the \( \mathbf{F}^j_{\text{med}} \) propagates across the dipoles at the wave speed \( c \). Viewed in the frame where \( \mathbf{F}^j_{\text{med}} \) is at rest, we then find the dipole charges are travelling at the speed \( c \); so as a first step of the mapping, the Lorentz force on the medium ought to scale as \( \mathbf{F}^j_{\text{med}} = (\pm c/\upsilon) \mathbf{F}_m = \pm q\mathbf{A}_a \upsilon \nabla \) with \( +, - \) for the \( j = \dagger, \ddagger \) waves; thus \( \mathbf{F}^j_{\text{med}} = \mathbf{F}_e + \mathbf{F}^j_{\text{med}} \). Under the actions of the respective forces, the acceleration \( F^j/m^j \) of the particle’s charge of a dynamical mass \( m^j \) (due to the charge’s total motion and equivalently the \( \varphi^j \) motion, see further Appendix A), and that of the medium of a dynamical mass \( m^j_{\varphi} \), \( F^j_{\text{med}}/m^j_{\varphi} \) must equal, i.e. \( \mathbf{F}^j/m^j = \mathbf{F}^j_{\text{med}}/m^j_{\varphi} \). Thus \( \mathbf{F}^j_{\text{med}} = \frac{m^j_{\varphi}}{m^j} \mathbf{F}^j = \frac{m^j_{\varphi}}{m^j} L^j_{\varphi} (\nabla \varphi_a \pm \mathbf{A}_a \upsilon \nabla) \).

By its pure mechanical virtue the force \( \mathbf{F}^j_{\text{med}} \) acting on the continuous medium is nonlocal and will be transmitted uniformly across the medium here along the \( z \)-axis of an effective length \( L^j_{\varphi} \), \( j = \dagger, \ddagger \) (\( L^j_{\varphi} \) winds \( J^j \) loops about \( L \)); thus, using the geometric mean \( L^j_{\varphi} = \sqrt{L^j_{\varphi} L^j_{\varphi}} \), we
have $\nabla \phi_a = \pm (\phi_a / L_\varphi) z$ and $\nabla = \pm 1 / L_\varphi$. With these, putting $\varphi_j^\parallel = \rho_1 L_\varphi^j$ where $\rho_1$ is the mean linear mass density of the medium, writing for conciseness $\nabla \phi_a$ and also the final $F_j^\parallel_{med}$ in scalar forms and keeping the generally arbitrarily oriented $A_a$ in vector form only, $F_j^\parallel_{med}$ becomes

$$F_j^\parallel_{med} = -\rho_1 V_j^\parallel / m_j, \quad V^\parallel = q \phi_a - q A_a c, \quad V^\parallel = -q \phi_a - q A_a c.$$ (3)

$F_j^\parallel_{med}$ can be implemented in (2) by directly establishing the corresponding wave equation for the apparent elastic medium acted by $F_j^\parallel_{med}$. If without $F_j^\parallel_{med}$, the elastic medium would be deformed owing to the disturbance of the oscillation of the source charge alone, by a total displacement $u = a \sum_j \varphi^j$ with $\varphi^j$ given for $\phi_a = A_a = 0$, and be thus subject to a tensile force $F_R = \rho_c c^2$. The applied $F_j^\parallel_{med}$ and $F_R$ add up to a total force acting on the particle through acting directly on the medium

$$F_j^\parallel = F_R - F_j^\parallel_{med} = \rho_1 \left[ c^2 + V_j^\parallel / m_j \right].$$ (4)

Where, the minus sign of $F_j^\parallel_{med}$ is because this force tends to contract the chain. Assuming $\sum \varphi^j$, with $\varphi^j$ now given for a finite $\phi_a, A_a$, is relatively small which in general is the case in practical applications, $F_j^\parallel$ is thus uniform across the $L$. A segment $\Delta L$ of the medium along the box, of mass $\Delta M_\varphi = \Delta \varphi J^j / J^j = \rho_1 \Delta L \approx \rho_1 \Delta z$, will upon deformation be tilted from its equilibrium position $z$-axis an angle $\varphi^j$ and $\varphi^j + \Delta \varphi^j$ at $z$ and $z + \Delta z$. The transverse ($y$-) component force acting on $\Delta M_\varphi$ is $\Delta F_j^\parallel_{nt} = F_j^\parallel_{nt} \langle \sin(\varphi^j + \Delta \varphi^j) - \sin \varphi^j \rangle = F_j^\parallel_{nt} \nabla z (\varphi^j) \Delta z = \rho_1 \left[ c^2 + \frac{1}{m_j} \right] \nabla z^2 \varphi^j \Delta z$. Newton’s second law for the mass $\Delta M_\varphi$ writes $\rho_1 \Delta z \nabla z (\varphi^j) = \Delta F_j^\parallel_{nt}$. The two last equations give the equations of motion, on dividing $a \rho_1 \Delta z$, for per unit length per unit linear mass density of the medium at $z$ or equivalently the classical wave equations for the electromagnetic waves $\varphi^j$'s in the fields of the applied potentials $\phi_a, A_a$:

$$[c^2 + q(\phi_a - A_a c)/m_j] \nabla z^2 \varphi^j = \partial_t^2 \varphi^j, \quad [c^2 - q(\phi_a + A_a c)/m_j] \nabla z^2 \varphi^j = \partial_t^2 \varphi^j.$$ (5)

This for $\phi_a = A_a = 0$ reduces to (2) directly given from the Maxwell’s equations earlier.

Assuming for the present $\phi_a, A_a$ are constant and also $A_a$ is small such that the particle motion effectively deviates not from the linear path, so the solution of (5) consists of plane waves $\varphi^j = \zeta^j \varphi^j$ and $\varphi^j = \zeta^j \varphi^j$ (Figure 1a, solid and dotted curves) generated in $+z$- and $-z$-directions and initially also travelling in these directions at speed $\omega^j / k^j = c$, where

$$f^\parallel = C e^{i[k^j z - \omega^j t + \alpha_a]}, \quad f^\parallel = -C e^{i[-k^j z + \omega^j t - \alpha_a]}.$$ (6)

(Figure 1 a-b, single-dot-dashed and triple-dot-dashed curves), $\zeta = e^{iKz}$, and $C (= 4C_1 / \sqrt{L})$ a normalisation constant. And,

$$k^j = K / (1 - v / c) = \gamma^j K, \quad k^j = K / (1 + v / c) = \gamma^j K \quad \text{and} \quad \omega^j = \gamma^j \Omega, \quad \omega^j = \gamma^j \Omega,$$ (7)

are the source-motion resultant Doppler-displaced wavevectors and angular frequencies; $\gamma^j = 1 / \sqrt{1 - v^2 / c^2}; \quad \gamma^j = 1 / \sqrt{1 + v^2 / c^2}; \quad \gamma^j \Omega = K c$ are values of $k^j, \omega^j$ at $v = 0$. (7) further gives

$$k_d^j = k^j - K = \gamma^j K_d, \quad k_d^j = K - k^j = \gamma^j K_d \quad \text{where} \quad K_d = (v / c) K.$$ (8)

Suppose $v << c$ (yet $v^2 / c^2$ may be large so that dynamically the factor $\gamma$ in (9) below can be different from 1) and accordingly the de Broglie wavelength ($\lambda_d = 2\pi / (\gamma K_d)$) later will be much greater than the electromagnetic wavelength ($\Lambda = \frac{2\pi}{k_d^j}$). So at the scale of $\lambda_d$ the rapid variation of $\zeta$ is to an external observer no different from the constant 1, that is $lim_{v => c} \zeta = 1$. Thus
for each term; this gives the same (for a formal treatment in the case of a Schrödinger system in a varying potential and of a locally one-dimensional motion. The wave equations to be given will formally be otherwise which the plane waves hold true and their sum gives according to the Fourier theorem the total for constant in three dimensions see [2c,j]). We shall thus for simplicity proceed the remainder of treatment or equivalently the.

Restricting to \( \psi < < c \) linear in \( \hbar \) and \( c > > \nu \) future time; and we want similarly for the linear momentum here. We shall thus transform (Appendix B) that the exact solutions \( \psi_j \)'s and in turn the external effective \( f_j \)'s given above placed in the respective wave equations (5) above and (12) below yield exactly the expected relativistic energy-momentum relation.

The Doppler-displaced variables \( k_d^j \)'s, \( \omega_j \)'s in the functions \( f_j \)'s, \( \psi_j \)'s are not single valued and thus are not good dynamical variables of the particle. The respective geometric means

\[
k_d = \sqrt{k_d^j k_d^j} = \gamma K_d, \quad \omega = \sqrt{\omega_j^2} = \gamma \Omega, \quad \text{with } \gamma = \sqrt{\gamma^4 \gamma^4} = 1/\sqrt{1 - \nu^2/c^2}, \tag{9}
\]

are evidently good dynamical variables of particle and also are appropriate in view of the stochastic virtue of the electromagnetic waves (these being distributed functions and having in general random initial phases over a macroscopic time interval. These are also the natural stochastic virtue of the electromagnetic waves (these being distributed functions and having are evidently good dynamical variables of particle and also are appropriate in view of the relativistic energy-momentum relation.

3. Wave equation for total motion of particle

For a single particle or for many particles without regarding the spins, the functions \( \tilde{\psi} \) and \( \tilde{\psi}_\text{vir} \), or equivalently the \( f_r \) and \( f_t \) of (15) below, are seen to be identical to the usual solutions to the Dirac equation, c.f. Appendix C. So their wave equations, originally the (5), evidently must have a direct correspondence with the Dirac equation. The remainder of the task mainly will be to identify a physically justifiable procedure to transform (5) to a form of the Dirac equation under corresponding considerations.

First, similarly as Dirac (or as alternatively but compatibly argued in [2c]) we want the eventual wave functions of the particle, and thus immediately the \( f_j \) or the original \( \varphi_j \) to be linear in \( \hbar \nu \) and thus in \( \partial_t f_j \) here, so that \( f_j \) at any initial time determines its value at any future time; and we want similarly for the linear momentum here. We shall thus transform the second order differential equations (5) to first order ones and in the end take the limit for \( c >> \nu \) as follows. For the \( c^2 \nabla^2 \varphi_j \) terms of (5), starting with the full wave functions \( \varphi_j = \varphi \) we first lower the spatial derivative one order as \( \nabla^2 \varphi_j = \nabla_i [i \gamma^j K \nabla f_j + \varphi \nabla^j f_j]. \) Restricting to \( \nu << c \), we thus can replace \( \nabla \varphi \) by its computed value \( iK \varphi \) and in turn put \( \varphi = \pm 1 \) for each term; this gives

\[
\nabla^2 \varphi_j |_{\varphi = \pm 1} = \left[ -\gamma^j K^2 \varphi_f j + i \gamma^j K \nabla \varphi f_j \right]_{\varphi = \pm 1} = -\gamma^j K^2 \varphi_f j + i \gamma^j K \nabla \varphi f_j, \quad j = \dagger, \ddagger. \tag{10}
\]
For the final expressions we used \( k \) by the particle wave variables, i.e. the wave equations for the electromagnetic waves of the particle expressed by \( k \) and \( \omega \) defined in (9), and the corresponding wave functions; these resemble directly the (opposite-travelling) de Broglie phase waves and are equivalent to the space-time functions of Dirac spinor.

Next, assume \( \phi_a, A_a \) relatively small as typically is true in practical applications, so the resulting force constant (i.e. force per unit displacement) on the particle does not vary across \( L \). We can thus replace the \( \nabla^2 \varphi^j \) in the \( \phi_a, A_a \) terms of (5) by its computed value as \( \nabla^2 \varphi^j|_{\varepsilon=1} = -\gamma^j K^2 f^j \); accordingly

\[
\frac{q(\phi_a - A_a c)}{m^j} \nabla^2 \varphi^j|_{\varepsilon=1} = q(\phi_a + A_a c)\gamma^j \Omega f^j, \quad -\frac{q(\phi_a + A_a c)}{m^j} \nabla^2 \varphi^j|_{\varepsilon=1} = q(\phi_a + A_a c)\gamma^j \Omega f^j \tag{11}
\]

For the final expressions we used \( K_c = \Omega \) as earlier, and \( m^j = \gamma^j M \) and \( M c^2 = h \Omega \) given after (A.1) and (9). Finally, the \( \partial^2 \varphi^j \)'s of (5) lower one order as \( \partial^2 \varphi^j|_{\varepsilon=1} = \gamma^j \Omega \partial_t f^j \), \( \partial^2 \varphi^j|_{\varepsilon=1} = \gamma^j \Omega \partial_t f^j \). Substituting these and equations (10)–(11) in wave equations (5), multiplying the first resulting equation by \(-\frac{hc}{K^2 \gamma^j} \) and the second by \( \frac{hc}{K^2 \gamma^j} \), with \( cK = \Omega \) and \( hK = Mc \) as before and as given after (A.1), we eventually obtain the wave equations for the electromagnetic waves of the particle expressed by \( f^j, f^j \):

\[
[M c^2 + q\phi_a - c(ih \nabla + qA_a)]f^j = ih \partial_t f^j, \quad [-Mc^2 + q\phi_a + c(ih \nabla + qA_a)]f^j = ih \partial_t f^j. \tag{12}
\]

For the particle dynamics in question we want to further transform (12) to be expressed by the particle wave variables, i.e. the \( k_d \) and \( \omega \) defined in (9), and the corresponding wave
functions, the \( f_r, f_\ell \) to be obtained below. We shall below obtain the sought-for functions \( f_r, f_\ell \) through a dynamic equivalence transformation directly from the \( f^1, f^{1'} \); these ought to be and will show to be identical to the \( \tilde{\psi}, \tilde{\psi}^\text{vir} \) obtained in a physically more transparent way earlier. The present approach below will advantageously preserve a direct tractable connection with the original \( f^1, f^{1'} \), thus also \( \varphi^1, \varphi^{1'} \), whose wave equations (12) or (5) give the relativistic energy-momentum relation exactly based on the Doppler equations (7), see (B.2) of Appendix B. What accordingly is in question in the transformation mainly is to maintain an equivalence to the quadratic equation (B.2); this corresponds to the equations \( \frac{\partial f_j}{\partial z} \frac{\partial f_{j'}}{\partial \omega} = \frac{\partial f_{j'}}{\partial z} \frac{\partial f_j}{\partial \omega} \), etc., with \( j, j' = \dagger, \ddagger; \mu, \mu' = r, l, z' = t, z \ (\nu = 0, 3) \); and \( f_\mu \frac{\partial f_{\mu'}}{\partial \omega} = 0 \). The equivalence condition requires in particular the transformed quadratic to be \( \frac{\partial f_j}{\partial z} = k_j^2 \), that is, it has a plus sign in front and the cross-term product of \( \frac{\partial f_j}{\partial \omega} \) with \( M c^2 \) (i.e. the \( \Omega \) discussed after 17) is absent. This can be achieved if we introduce a wavevector being the imaginary of (and thus orthogonal to) \( k_d \):

\[
\bar{k}_d = (\gamma/\imath \gamma)k_d^1, \quad \bar{k}_d = (\gamma/\imath \gamma)k_d^2; \quad \text{thus } k_d^1 k_d^2 = (1/\imath^2)\bar{k}_d^2
\]

(compare \( \bar{k}_d \) with the operator \( p_{\nu, \text{op}} = \hbar/i \nabla \) later). (13) alternatively can be expressed by

\[
(a) : k_d^1(-k_d^2) = \bar{k}_d \bar{k}_d \quad \text{or} \quad (b) : (-k_d^2)k_d^1 = \bar{k}_d \bar{k}_d.
\]

We now first transform the Doppler-differentiated \( f^1(\; k_d^1, \omega^1) \)'s (as the short hand notations of \( f^1(z, t; k_d^1, \omega^1) \)'s which in turn are expanded notations of \( f^1(z, t) \)'s to a pair of mean (wave)-variable functions \( f_\mu(\; k_d^1, \omega^1) \)'s (denoting \( f_\mu(z, t; k_d^1, \omega) \)'s) by, say, satisfying (a) of (14) and ordinarily \( \omega = \sqrt{\omega^1 \omega^1} \) of (9):

\[
f^1(\; k_d^1, \omega^1) \rightarrow f^1_r(\; \bar{k}_d^1, \omega^1) = C_r e^{i[\bar{k}_d^1z - \omega t + \alpha_0]}, \quad f^1(\; k_d^1, \omega^1) \rightarrow f^1_\ell(\; \bar{k}_d^1, \omega^1) = C_\ell e^{i[\bar{k}_d^1z + \omega t + \alpha_0]},
\]

see these functions plotted in Figure 1b. The transformed \( f_r, f_\ell \) indeed are desirably identical functions to the original \( f^1, f^{1'} \) if disregarding the high-order differences between the coefficients \( \gamma^1, \gamma^1 \) and \( \gamma \) in the wave variables and the reversed travel direction of \( f_\ell \) from \( f^1 \). The \( f_r, f_\ell \), being identical functions to the \( \tilde{\psi}, \tilde{\psi}^\text{vir} \) earlier, indeed are therefore the pertinent space-time functions of the particle; these are each functions of the source motion and the total (electromagnetic) wave oscillation and are formally equivalent to de Broglie phase waves; and these in turn are equivalent to Dirac’s space-time functions.

To entail that in the matrix representation later (Sec. 5) a cross-term product \( \Omega \) discussed after (17) is similarly absent, for transformation of the first derivatives we are compelled to satisfy the alternative condition (b) of (14). The use of (14b) and the common \( \omega = \sqrt{\omega^1 \omega^1} \) of (9) first directly leads to the intermediate transformations for the \( f^1 \)'s given in the left column below:

\[
\begin{align*}
 f^1 \bigg|_{k_d^1 - k_d^1} \rightarrow f^1_r \bigg|_{(a1)_{-\alpha_0}} \rightarrow f^1_{\ell}, \quad \nabla f^1 = i k_d^1 f^1 \bigg|_{k_d^1 - k_d^1} \rightarrow i(-\bar{k}_d^1) f_{\ell} = -\nabla f_{\ell}, \\
 f^1 \bigg|_{k_d^1 - k_d^1} \rightarrow f^1_r \bigg|_{(a2)_{-\alpha_0}} - f_r, \quad \nabla f^1 = -i k_d^1 f^1 \bigg|_{k_d^1 - k_d^1} - i\bar{k}_d^1 f_{\ell} = -\nabla f_{\ell}
\end{align*}
\]

where \( f^1_{\ell} = C_\ell e^{i[-\bar{k}_d^1z - \omega t - \alpha_0]}, \quad f^1_r = C_r e^{i[-\bar{k}_d^1z + \omega t + \alpha_0]}, \quad \alpha_0' = -\alpha_0 \). The transformations (a1) and (a2) in (16) to the same \( f_{\ell}, f_r \) as in (15) are on the basis that the latter indeed represent the original \( f^1_r, f^1_r \) in all aspects (having the same phase velocities and wave forms) except the opposite rotating phases on the complex plane and the opposite signs of \( \alpha_0' \) and \( \alpha_0 \) that altogether are dynamically inconsequential. The relations in the left column and the use of (14b)
again then lead to the results in the right column which is actually in question in respect to dynamical equivalence here.

Substituting in wave equations (12) the transformation relations (15) and (16) gives

\[(Mc^2 + q\phi_a)f_r + c(ih\nabla - qA_a)f_\ell = ih\partial_t f_r, \quad (-Mc^2 + q\phi_a)f_\ell + c(ih\nabla + qA_a)f_r = ih\partial_t f_\ell.\] (17)

As a check, placing in (17) the \(f_r, f_\ell\) of (15), multiplying the first and the negative of the second resulting equations, dividing \(f_r, f_\ell\), putting for simplicity \(\phi_a = A_a = 0\) for the problem mainly of concern here, we correctly obtain the same result as (B.2): \(M^2c^4 - \hbar^2k^2c^2 + \varnothing = \hbar^2\omega^2\) where \(k^2 = -k_0^2\) following (13) and (9): \(\varnothing = 0\).

4. Two-particle system: spins, charges, and time-arrows

Consider two spin-half particles 1,2 having identical sets of \(\{f_r, f_\ell\}\)'s tend to occupy the same location \(z\); or more precisely the same region \((0, L)\); suppose these are noninteracting (a finite particle-particle interaction can in principle be included in \(V^j\) and will not affect the general conclusions below). In virtue of the statistical nature of the electromagnetic displacements.

![Figure 2](image_url)

**Figure 2.** Two spin-half IED particles 1 and 2 described by identical, thus mutually symmetric sets of Doppler-displaced space-time functions \(\{\varphi^1, \varphi^1\}'s or effectively \(\{f_r, f_\ell\}'s tend to occupy the same location \(z\); panel (a) shows two electrons and (b) an electron and positron. Particle 1 has a spin \(S\) in \(+z\)-direction, denoted as spin up, \(\alpha(1)\), and is parallel with the generating direction of the \(f_r\) wave; particle 2 has spin \(S\) in \(-z\)-direction, denoted as spin down, \(\beta(2)\), and is parallel with \(f_\ell\). Their opposite counterparts (termed virtual spins) \(\beta(1)\) and \(\alpha(2)\) are parallel with \(f_\ell\) and \(f_r\). Right graphs show the complete wave functions \(\{\psi_\nu(1), \psi_\nu(1)\}\) and \(\{\psi_\nu(2), \psi_\nu(2)\}\) of particles 1 and 2. In panel (a), \(\psi_\nu(1)\) and \(\psi_\nu(2)\) \((\nu = +, -)\) are antisymmetric due to the antisymmetric spin functions and same charges. In (b), these are symmetric (as explicitly shown in the graph) due to antisymmetric spins and also opposite charges \((Q_\nu(2) = -Q_\nu(1))\), the latter of which leads to the radiation electric fields \(E_\nu\)'s are opposite in direction (in the text the \(Q_\nu(2), Q_\nu(1)\) are not explicitly regarded, rendering the \(\psi_\nu(1)\) and \(\psi_\nu(2)\) for the electron and positron to be antisymmetric, thus the same as for two electrons).
and combinatorially of the particles’ center-of mass motions (see an elaborate discussion about this in [2k]), the probability of finding a portion of particles 1 and 2 at locations \( z_1 \) and \( z_2 \) is proportional to the product \( f_{\mu}(z_1, t)f_{\nu}(z_2, t) \). Since these have identical space-time function sets, the corresponding total space-time function is evidently symmetric, thus \( f_{\mu} \), here in the only form \( f_{\mu}(z_1, z_2) = \frac{1}{\sqrt{2}}[f_{\mu}(z_1, t)f_{\ell}(z_2, t) + f_{\ell}(z_2, t)f_{\mu}(z_1, t)] \) to be compatible with the antisymmetric total spin function later.

In the case of two identical electrons (Figure 2, panel a), their spins then need according to Pauli principle be opposite (left graph in the figure) to avoid both particles occupying the same quantum state: the total spin function for this is antisymmetric \( \chi_{\text{Sym}} = \frac{1}{\sqrt{2}}[\alpha(1)\beta(2) - \alpha(2)\beta(1)] \). Apparently it is in general also relevant that we introduce charge functions \( Q_{\nu}(1), Q_{\nu}(2) \) to reflect the sings of charges 1,2; these for the two like charges are trivial identities and lead to a trivial symmetric total charge function, \( Q_{s} \). The above functions together define an antisymmetric two-electron function (Figure 2a, right graph), \( \psi_{s} = f_{s}Q_{s}\chi_{a} \), yielding as expected a total probability that is unchanged when interchanging the two stationary-state identical, indistinguishable (as result of being identically extensively distributed in \( L \)) particles.

In the case of an electron and positron (Figure 2, panel b), the total two-particle function is symmetric (right graph), thus \( \psi_{s} \), as follows from the observational fact that two such particles can approach each other arbitrarily close and, in an extreme case annihilate into “one point” in the vacuum. The total charge function for their two opposite signed charges evidently is antisymmetric, thus \( Q_{s} \). (—If imagine the wave displacement \( \varphi' \) in the medium is executed by a chain of dipole charges then it is immediately clear that the corresponding radiation electric field \( E' \), with \( |E'| \propto \varphi' \), produced by the positive charge is reversed from that by the negative charge, see left graph in Figure 2b.) Placing the two known functions in \( \psi_{s} = \chi_{a}f_{s}\chi_{\text{Sym}} \) gives therefore \( \text{Sym}=\text{antisymmetric} \) and thus \( \chi_{\text{Sym}} = \chi_{a} \) for the electron-positron system.

Our particles contain each the electromagnetic waves \( \varphi', \varphi' \), or effectively \( f_{r}, f_{r} \), generated in the specified \( +z \)- and \( -z \)- directions which assume definite relationships with the spin orientations that turn out in a measurement: Given, say, particle 1 is measured to be spin up, \( \alpha(1) \), with \( f_{r} \) being parallel with it, then its other internal process, \( f_{s} \), is parallel with the opposite of \( \alpha(1) \), denoted by a primed, virtual spin-down state, \( \beta'(1) \). Similarly for particle 2, being then actually spin down, thus its \( f_{r} \) is parallel with \( \beta(2) \), and its \( f_{s} \) is parallel with the opposite of \( \beta(2) \), a primed virtual spin-up state, \( \alpha'(2) \). From the foregoing antisymmetric spin requirement follow the relations for the spin functions:

\[
\alpha' (2) = -\alpha(1), \quad \beta'(1) = -\beta(2); \quad \text{with} \quad \beta' (1) = -\alpha(1), \quad \alpha'(2) = -\beta(2) \tag{18}
\]

for the opposite signs of each particle as is meant by the "virtual" spins. Through (18), the virtual spin vector of particle 1, \( S(1) \) (virtual spin down) and the actual spin vector of particle 2, \( S(2) \) (actual spin down), pointed each in the \( -z \)- direction, are now each represented as scalar quantities with minus signs.

If disregarding the signs of charges explicitly, the electron-electron and the electron-positron are two equivalent systems of identical spin-half particles, each described by the space-time functions \( f_{\mu}(k_{d}, \omega) \)'s with \( \mu = r, \ell \). These joined together with the spin functions give the complete wave functions: \( \psi_{\nu}(n; k_{d}, \omega) = f_{\mu}(k_{d}, \omega)\alpha_{\nu}(n) \) with \( \nu = +, -, n = 1, 2, \alpha_{\nu} = \alpha, \beta, \alpha', \beta' \). Now as a further step to conform our wave equation later to matrix form, we hereafter require that the \( \psi_{\nu}(n) \) functions are elements of a matrix of one column, \( \psi \). The matrix wave equation itself will entail the two desired features discussed after equation (17), that is, (i) the product of \( -k_{d} \) and \( k_{d} \) in the quadratic equation is positive: \( k_{d}^{2} \) (entailed by the situation that these in the matrix form are offdiagonal elements, see (22) or (C.1), and (ii) the total cross-term product \( 0 = 0 \) (entailed by the characteristics that in matrix equation the \( \psi_{\nu}(n) \)'s are explicitly mutually mutually orthogonal). The first of these two features which we have up to now enforced by use
of $k_d$ for $k_d$, which should no longer be used in the matrix form to avoid a dual accounting. The space-time functions accordingly write $f_r(k_d, \omega) = C_r e^{i[k_d z - \omega t]}$, $f_l(k_d, \omega) = C_l e^{i[k_d z + \omega t]}$ with $k_d$ the ordinary scalar quantity and related with $k_d^1, k_d^2$ through (9). Accordingly,

$$\psi_1(1) = \alpha(1) f_r(k_d, \omega) = \alpha(1) C_r e^{i[k_d z - \omega t]}, \quad \psi_1(2) = \beta(2) f_r(k_d, \omega) = \beta(2) C_r e^{i[k_d z - \omega t]}, \quad (19a)$$

$$\psi_1(1) = \beta(1) f_l(k_d, \omega) = \beta(1) C_l e^{i[k_d z + \omega t]}, \quad \psi_1(2) = \alpha(2) f_l(k_d, \omega) = \alpha(2) C_l e^{i[k_d z + \omega t]} \quad (19b)$$

The complete wave functions (19a)–(b) (Figure 2, right graphs) describe two identical particles of opposite oriented actual spins and accordingly opposite virtual spins, and these are identical to the solutions (see equation C.3) for Dirac equation. We can readily check that placing the resulting four equations in the order of spin-up states of particles 1 and 2 first and then spin-down states of particles 1 and 2, we finally obtain a set of four coupled linear first order partial differential equations governing the motions of the two particles in terms of $\psi_\nu(n)$’s:

$$(M c^2 + q_1 A_\nu) \psi_\nu(1) - c(ih \nabla - q_1 A_\nu) \psi_\nu(1) = i\hbar \partial_t \psi_\nu(1) \quad \text{particle 1, spin up}$$

5. Dirac equation

For two identical, spin-half particles of identical sets of space-time functions \{f_r, f_l\}'s each described by wave equations (17) tending to occupy the same location $z$, we shall now express the corresponding wave equations in terms of the complete wave functions of Sec. 4. For particle 1, we thus multiply the first equation of (17) by $f_r$ and the corresponding wave equations in terms of the complete wave functions of Sec. 4. For particle 2, instead we multiply the first equation of (17) by $f_l$. The complete wave functions (19a)–(b) (Figure 2, right graphs) describe two identical particles at any location $z$ in $L$ is not altered by interchanging the locations of the particles (the indistinguishability). We can also check that the same $\psi_\alpha$ is given by the product of the separate total functions: $\psi_\alpha(z_1, z_2) = f_r(z_1, z_2) \chi_\alpha(1, 2)$; notice that once we specified say particle 1 is spin up and 2 spin down, then $f_r(z_1, t) f_r(z_2, t) \beta(1) \alpha'(2)$ and $f_l(z_1, t) f_r(z_2, t) \alpha(1) \beta(2)$ are zero since these do not describe the present reality.

Lastly, the spin-up state of particle 1, $\alpha(1)$, is associated with an effective electromagnetic wave $f_r$ travelling to the right, thus $\partial_t f_r/f_r = -i\omega$, while the spin-up state of particle 2 with $f_l$ travelling to the left, thus $\partial_t f_l/f_l = i\omega$; the latter has as if a reversed time arrow relative to the former. We may introduce the time arrow functions defined for particles 1 and 2 as $\mathcal{T}(1) = 1, \mathcal{T}(2) = -1$, such that the action of these on the time derivatives project the wave propagations to be both in the $+z$-direction: $\mathcal{T}(1) \partial_t \psi_\nu(1) = \partial_t \psi_\nu(1), \mathcal{T}(2) \partial_t \psi_\nu(2) = -\partial_t \psi_\nu(2)$.
\[(Mc^2 - q_2\phi_a)\psi_1(2) + c(i\hbar\nabla + q_2A_a)\psi_2(2) = i\hbar\partial_t\psi_1(2) \quad \text{(particle 2, spin up)}\]
\[(-Mc^2 + q_1\phi_a)\psi_1(1) - c(i\hbar\nabla + q_1A_a)\psi_1(1) = i\hbar\partial_t\psi_1(1) \quad \text{(particle 1, spin down)}\]
\[(-Mc^2 - q_2\phi_a)\psi_2(2) + c(i\hbar\nabla - q_2A_a)\psi_2(2) = i\hbar\partial_t\psi_2(2) \quad \text{(particle 2, spin down)}\]

(22)

From the discussion of Sec. 4 that the \(\psi(n)\)'s and accordingly also their first derivatives are mutually orthogonal, it follows that the linear equations (22) are equivalent to a matrix equation. Supposing specifically the two particles are a positron and an electron and therefore \(q_1 = q, q_2 = -q\), the matrix form of (22) is thus

\[H_{op}\psi = i\hbar\partial_t\psi, \quad \text{where } H_{op} = bMc^2 + q\phi_a + c\alpha(p_{v,op} - qA_a) \quad \text{and } p_{v,op} = -i\hbar\nabla\]

are the relativistic total Hamiltonian and linear momentum operators. And,

\[
b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \quad \text{and}
\]
\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \psi_+ = \begin{pmatrix} \psi_1(1) \\ \psi_2(2) \end{pmatrix}; \quad \psi_- = \begin{pmatrix} \psi_1(1) \\ \psi_2(2) \end{pmatrix}.
\]

(24)

The off-diagonal elements of the matrix \(\sigma_z, \sigma_{z1}(= 1)\) and \(\sigma_{z2}(= -1)\) here correspond to the \(\alpha(1) = 1\) and \(\alpha'(2) = -1\) earlier. We see that, \(\psi\) is equivalent to a Dirac spinor, \(\sigma_z\) the \(z\)-component of Pauli matrices, and as a whole, equation (23) is identical to the Dirac equation for an electron-positron system equivalent to here. For the present case rotation transformation is trivial, so \(\sigma = \sigma_z\hat{x}\).

Suppose more generally the two particles’ spin angular momenta, \(S (= \frac{\hbar}{2}\mathbf{\sigma})\)'s, are along an axis \(\mathbf{n}\) executing in general a precession about the \(z\)-axis at a fixed angle (arccos(\(\frac{S}{S_z}\))). For each particle being in stationary state, its \(S\) (similarly its magnetic moment \(\mathbf{\mu} (= -\mathbf{eS}/m)\)) as a vector quantity when in small rotations about the \(z\)-axis must maintain invariant with respect to its projection on the \(z\)-axis, \(\mathbf{n}\cdot S = \pm \frac{\hbar}{2}\), and is Hermitian. In addition to the antisymmetric condition given by the \(\sigma_z\) of (24) above, an infinitesimal rotation transformation as such needs be unitary. A specific set of transformation matrices having these properties are known to be the Pauli matrices, \(\sigma_x, \sigma_y\) and \(\sigma_z\) of the standard expressions and \(\sigma_z\) as expressed in (24), \(\mathbf{\sigma} = \sigma_x\hat{x} + \sigma_y\hat{y} + \sigma_z\hat{z}\). And, the unitary matrix \(I\hat{z}\) about the \(z\)-axis naturally extends to a unitary matrix about the new \(\mathbf{n}\)-axis in three dimensions, given by \(I = I\hat{x} + I\hat{y} + I\hat{z}\). Substituting in (23) with \(\mathbf{n}\) and \(I\) for \(\sigma_z\) and \(I\) gives a Dirac equation of the same form, now for spins in arbitrary directions.

**Appendix**

**Appendix A. Total energy and inertia of particle wave**

As a general result of classical electrodynamics based on solution to the Maxwell’s equations combined with Lorentz force law, an electromagnetic wave \(\psi\) transmits at the speed of light \(c\) a wave energy \(\varepsilon\) and a linear momentum \(p' = \varepsilon/c\). Here, the amplitudes of \(\varepsilon\), accordingly of \(p'\), \(E'\), \(B'\) and \(\varphi\), etc., are continuous values. Following M. Planck’s discovery of quantum theory in 1901, it has been additionally understood that these quantities are by nature quantized in amplitudes; an electromagnetic wave of frequency \(\omega/2\pi\) has an energy \(\varepsilon = nh\omega\), consisting in general of \(n\) momentum-space quanta, or photons, each of an energy \(h\omega\); and the classical continuous amplitude solutions to these are only approximations when \(n\) is large. In the present problem, in conformity with experiments, especially the pair processes, the electromagnetic wave comprising our electron or positron has a “single energy quantum”, \(n = 1\); so \(\varepsilon = h\omega\). It has been further proven especially through quantum electrodynamics that the Maxwell’s equations, and the subsequent classical wave equation (2) or more generally (5), continue to hold, and
the quantisation of the fields and wave energy etc. is the result of subjecting the canonical displacement and momentum, the \( u(= a \varphi) \) and \( \dot{u} \) here, to the quantum commutation relation \([u, \dot{u}] = i\hbar\). The total wave of our particle of a single energy quantum \( \hbar \omega \) in a one-dimensional box has, following the solution to the Maxwell’s equations earlier [see after (2)], two components, \( \varphi^1 \) and \( \varphi^2 \), with their frequencies being Doppler-displaced to \( \omega^1 \) and \( \omega^2 \) as a result of the source motion as given in (7), which are related to \( \omega \) through (9). For the total wave comprising the particle, the total wave energy \( \varepsilon \) represents therefore a dynamical variable of the particle, here the total energy of the particle.

The electromagnetic waves, \( E^j, B^j \)'s or \( \varphi^j \)'s, rapidly oscillating at frequencies \( (\omega^j/2\pi)'s \), of a geometric mean frequency \( \omega/2\pi \) and wavelength \( \lambda = c/(\omega/2\pi) \) will, when overlooking the oscillation details and speaking of the total dynamical variables \( \varepsilon, p \) only, appear as if two rigid objects, wavetrains, travelling at the speed of light \( c \). In view that their speed of travel, \( c \), is \emph{finite} as contrasted to infinite, the wavetrains have inevitably each \emph{finite} inertial masses, \( m^j \)'s, thus an inertial mass \( m = \sqrt{m^j/m^j} \) for the total wavetrain and hence its resulting particle. This mechanical depiction of the total wave, as a rigid ”wavetrain”, permits us at once to express according to Newtonian mechanics the linear momentum of the wavetrain to be \( p = mc \). Combining this with the classical electrodynamic result \( \varepsilon = pc \) above gives the kinetic energy of the wavetrain \( \varepsilon = mc^2 \), being equivalent to the Einstein’s mass-energy relation. This energy and the Planck energy earlier ought to equal, thus

\[
m = \hbar \omega/c^2; \quad \text{or at } v = 0: M = \hbar \Omega/c^2 \tag{A.1}
\]

with \( M \) the rest mass of the particle. Combining (A.1) with (9) gives \( m = \gamma M \). Combining (A.1) with \( p = mc \) further gives \( mc^2 = (\hbar \omega/c^2)c = \hbar k \) and accordingly \( Mc = \hbar Kc \), with \( \omega = kc \), \( \Omega = Kc \) and \( k = \gamma K \) as earlier.

**Appendix B. Relativistic energy–momentum relation for the electromagnetic waves of particle**

Consider first the simpler case of \( A_a = 0 \). Placing in wave equations (12) with \( f^1, f^1 \) of (6), dividing the resulting first and second equations by \( f^1 \) and \( -f^1 \) and sorting give \( MC^2 + \hbar k^j_d c = \hbar \omega^j - q \phi_a \), \( MC^2 - \hbar k^j_d c = \hbar \omega^j + q \phi_a \). Multiplying gives

\[
M^2 c^4 - \hbar^2 k^j_d k^j_d c^2 + Q = \hbar^2 \omega^j \omega^j - q^2 \phi_a^2 \tag{B.1}
\]

where \( k^j_d k^j_d = \hbar^2 \) and \( \omega^j \omega^j = \omega^2 \) following (9); \( Q = Mc^2 ch(k^j_d - k^j_d), \) with \( k^j_d - k^j_d = 2k_d(\frac{\omega}{c})\gamma \) and \( MC^2 = hKc \), so \( Q = 2\hbar^2 k_d^2 c^2 \). With these, putting \( \hbar k_d = \pm p_v \), \( \hbar \omega = \pm \varepsilon \) where \( p_v, \varepsilon \) are here variables having positive and negative solutions and thus the right hand side of (B.1) reduces as \( \sqrt{[(\hbar \omega - q \phi_a)(\hbar \omega + q \phi_a)]^2 = \sqrt{[(-\varepsilon - q \phi_a)(\varepsilon + q \phi_a)]^2 = (\varepsilon - q \phi_a)^2} \), then (B.1) reduces exactly to \( M^2 c^4 + p_v^2 c^2 = (\varepsilon - q \phi_a)^2 \). This, or this in the more familiar form for \( \phi_a = 0 \),

\[
M^2 c^4 + c^2 p_v^2 = \varepsilon^2 \tag{B.2}
\]

gives just the experimentally widely corroborated relativistic energy–momentum relation. For the more general case of \( A_a \) finite, denoting \( k^j_d' = k^j_d - \frac{2A_a}{\hbar} \), \( k^j_d' = k^j_d + \frac{2A_a}{\hbar} \), the particular feature that (the effective portion of) \( A_a \) is always perpendicular to \( k_d \), leads to \( k_d^2 = k^j_d k^j_d = k^2_d - q^2 A^2_d / h^2 \), or, \( (\pm p_v')^2 \equiv (\pm k_d')^2 = \mp (hk_d - q A_a)(-hk_d - q A_a) = [\mp (p_v - q A_a)]^2 \). (B.2) thus generalises to \( M^2 c^4 + c^2 p_v'^2 = (\varepsilon - q \phi_a)^2 \).
Appendix C. Solution of Dirac equation from the standpoint of particle internal process

We shall here mainly discuss the choice of the solution forms of the Dirac equation from the standpoint of internal processes of the IED particle model for simplicity for spins along z-axis, in an otherwise basically standard procedure. The two equations of (20) or (21) for particle \( n = 1 \) or 2 are coupled in \( \psi_+(n) \) and \( \psi_-(n) \) and can not be solved separately as in Appendix B. We need to solve each two, or more generally the four equations of the Dirac equation (20) together. Let the trial functions be: 

\[
\psi_+(n) = C_{\epsilon n} e^{i[p_\nu \pm q A_n]}, \quad s = +, -, \quad n = 1, 2.
\]

Placing these in (23) and rearranging give

\[
\begin{pmatrix}
\epsilon - Mc^2 - q\phi_a & 0 & -[p_\nu - qA_a]c & 0 \\
0 & \epsilon - Mc^2 - q\phi_a & 0 & 0 \\
-[p_\nu - qA_a]c & 0 & \epsilon + Mc^2 - q\phi_a & 0 \\
0 & [p_\nu - qA_a]c & 0 & \epsilon + Mc^2 - q\phi_a
\end{pmatrix}
\begin{pmatrix}
\psi_+(1) \\
\psi_+(2) \\
\psi_-(1) \\
\psi_-(2)
\end{pmatrix} = 0
\]

(C.1)

(C.1) corresponds to the four linear, homogeneous algebraic equations (C.2) for the \( \psi_{s,j} \)'s as four unknowns below; for these to have nontrivial solutions, the determinant for the matrix of the coefficients of (C.1) needs be zero. This is 

\[
\det = [(\epsilon - q\phi_a)^2 - M^2c^4]^2 - c^4p_\nu^4 = 0,
\]

with \( p'_\nu = p_\nu - qA_a \). This has two degenerate sets of square roots solutions: \( \epsilon - q\phi_a = \pm \sqrt{M^2c^4 + p_\nu^2c^2} \), which being identical to (B.2). In view that each particle has internal processes, we thus naturally assign symmetrically two of the four solutions to particle 1, as 

\[
\epsilon - q\phi_a = \pm \sqrt{M^2c^4 + p_\nu^2c^2},
\]

and the other two for particle 2 as 

\[
\epsilon - q\phi_a = \mp \sqrt{M^2c^4 + p_\nu^2c^2}.
\]

These two distinct sets of square-roots solutions to the algebraic equation above represent two identical (in the manner said earlier) yet distinct particles, like an electron and a positron, which do not transit from one to the other, a point agreeing with reality and having been stressed by P.A.M. Dirac from the very beginning in [1]. These algebraic solutions are in contrast to the usual problem of eigenvalues arising generally from boundary conditions and being each possible states of same particle between which transitions generally can occur.

With \( p_\nu \) and \( \epsilon \) as known parameters, we further solve the four algebraic equations

\[
\begin{align*}
(\epsilon - (Mc^2 + q\phi_0))\psi_+(1) &= p'_\nu c\psi_+(1) & (a) , &
(\epsilon - (Mc^2 + q\phi_0))\psi_+(2) &= -p'_\nu c\psi_+(2) & (b) \\
(\epsilon + (Mc^2 - q\phi_0))\psi_+(1) &= p'_\nu c\psi_+(1) & (c) , &
(\epsilon + (Mc^2 - q\phi_0))\psi_+(2) &= -p'_\nu c\psi_+(2) & (d)
\end{align*}
\]

(C.2)

corresponding to (C.1), for the wave functions. Taking the imaginary of equations (a) and (b) first, multiplying the resulting equation with equation (c) and the second with (d) respectively on opposite sides, substituting with \( \psi^*_{s,j}(1)\psi_+(1) = C_{s,j}^2, \psi^*_{s,j}(1)\psi_+(1) = C_{s,j}^2, \psi_{s,j}^*(2)\psi_-(2) = C_2^2 \) and \( \psi_{s,j}^*(2)\psi_-(2) = C_2^2 \), we get

\[
(\epsilon - (Mc^2 + q\phi_0))C_{s,j}^2 = (\epsilon + Mc^2 - q\phi_a)C_{s,j}^2, \quad (\epsilon - (Mc^2 + q\phi_0))C_2^2 = (\epsilon - Mc^2 - q\phi_a)C_2^2
\]

These have two independent solutions, and in mathematical terms two of the four wave functions can thus be arbitrarily chosen. In view of the IED particle model by which each particle has internal, wave processes consisting of two components in the one-dimensional box, it is natural here that we choose the values for \( C_{s,j} \) and \( C_2 \) symmetrically, in the sense also \( C_2 = -C_{s,j} \), with these in the two equations above the values for \( C_{s,j}^2 \) and \( C_2^2 \) then follow to be uniquely given as

\[
\frac{C_{s,j}}{C_2} = \pm \left( \frac{\epsilon + (Mc^2 + q\phi_0)}{\epsilon - (Mc^2 + q\phi_0)} \right)^{1/2} C, \quad \frac{C_{s,j}}{C_2} = \mp \left( \frac{\epsilon + (Mc^2 + q\phi_0)}{\epsilon - (Mc^2 + q\phi_0)} \right)^{1/2} C
\]

where \( |C_{s,j}C_{s,j}| = C^2, |C_2C_2| = C^2 \). With the above in the trial functions, we get the complete
solution for Dirac equation

\[
\psi = \begin{pmatrix}
\psi_+^(1) \\
\psi_+ ^(2) \\
\psi_- ^(^1) \\
\psi_- ^(^2)
\end{pmatrix} = \begin{pmatrix}
\psi_+(1) \\
\psi_+(2) \\
\psi_-(1) \\
\psi_-(2)
\end{pmatrix} = \begin{pmatrix}
\frac{\sqrt{+MC^2 + q\phi}}{\sqrt{-MC^2 - q\phi}} \cdot Ce^{i(kz - \omega t)} \\
\frac{\sqrt{+MC^2 + q\phi}}{\sqrt{-MC^2 - q\phi}} \cdot Ce^{i(kz + \omega t)} \\
\frac{\sqrt{+MC^2 - q\phi}}{\sqrt{-MC^2 + q\phi}} \cdot Ce^{i(kz - \omega t)} \\
\frac{\sqrt{+MC^2 - q\phi}}{\sqrt{-MC^2 + q\phi}} \cdot Ce^{i(kz + \omega t)}
\end{pmatrix}.
\]  

(C.3)

Agreeing with the wave functions directly based on IED particle model, \(\psi_+ (1), \psi_+ (1)\) are two opposite travelling component waves of particle 1, and \(\psi_+ (2), \psi_+ (2)\) of particle 2; in the meantime, the spin-up component waves of particles 1 and 2, \(\psi_+ (1)\) and \(\psi_+ (2)\) travel in opposite directions and similarly the spin-down component waves.

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