Model independent reconstruction of cosmological accelerated-decelerated phase

Salvatore Capozziello1,2,3,4, Peter K.S. Dunsby5,6,7 and Orlando Luongo8,9,10*

1 Dipartimento di Fisica E. Pancini, Università di Napoli “Federico II”, Via Cintia, I-80126 Napoli, Italy.
2 Scuola Superiore Meridionale, Largo S. Marcellino 10, I-80138, Napoli, Italy.
3 Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Napoli, Via Cintia, Napoli, Italy.
4 Laboratorio di Theoretical Cosmology, Tomsk State University of Control Systems and Radioelectronics (TUSUR), Tomsk, Russia.
5 Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch 7701, Cape Town, South Africa.
6 Astrophysics, Cosmology and Gravity Centre (ACGC), University of Cape Town, Rondebosch 7701, Cape Town, South Africa.
7 South African Astronomical Observatory, Observatory 7925, Cape Town, South Africa.
8 School of Science and Technology, University of Camerino, Via Madonna delle Carceri 9, 62032, Camerino, Italy.
9 Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo 5, Pisa, 56127, Italy.
10 Institute of Experimental and Theoretical Physics, Al-Farabi Kazakh National University, Almaty 050040, Kazakhstan.

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ABSTRACT

We propose two model independent methods to obtain constraints on the transition and equivalence redshifts $z_{tr}$, $z_{eq}$. In particular, we consider $z_{tr}$ as the onset of cosmic acceleration, whereas $z_{eq}$ the redshift at which the densities of dark energy and pressureless matter are equated. With this prescription, we expand the Hubble and deceleration parameters up to two hierarchical orders and show a linear correlation between transition and equivalence, from which we propose exclusion plots where $z_{eq}$ is not allowed to span. To this end, we discuss how to build up cosmographic expansions in terms of $z_{tr}$ and compute the corresponding observable quantities directly fitting the luminosity and angular distances and the Hubble rate with cosmic data. We make our computations through Monte Carlo fits involving type Ia supernova, baryonic acoustic oscillation and Hubble most recent data catalogs. We show at 1σ confidence level the ΛCDM predictions on $z_{tr}$ and $z_{eq}$ are slightly confirmed, although at 2σ confidence level dark energy expectations cannot be excluded. Finally, we theoretically interpret our outcomes and discuss possible limitations of our overall approach.

1 INTRODUCTION

The standard cosmological model is plagued by a high degree of physical uncertainty due to the existence of dark energy and dark matter whose nature is currently the object of much theoretical debate (Copeland et al. 2006; Capozziello et al. 2013). Arguably the greatest puzzle of the standard cosmological model is the prospect that 75% of universe’s content is made up of dark energy, responsible for the cosmic speed up (Dunsby et al. 2016), whereas dark matter contributes for about the 20% as an elusive component needed to guarantee structure formation1. A complete understanding of both these two constituents represents a real challenge of the the standard cosmological model, hereafter referred to as the ΛCDM paradigm (Steinhardt 2011).

This approach involves treating dark energy as a vacuum energy cosmological constant, Λ, derived from quantum field fluctuations (Weinberg 1989) and interpreted as a fluid that uniformly fills all of space. In the current epoch, the ΛCDM paradigm requires at least pressureless matter, negligible radiation and spatial curvature (Capozziello et al. 2013), with an overall negative pressure, determining the bizarre repulsive gravity (Bull et al. 2016) that accelerates the universe at the transition time. The ΛCDM scenario is plagued by different issues that limit its theoretical interpretation. First, both Λ and matter magnitudes are unexpectedly comparable, leading to a coincidence problem. Second, constraints over Λ turns out to be impressively smaller than the predicted vacuum energy density, providing a fine tuning problem. In turn, these caveats spurred the search for other physical explanations to describe the dynamics of the universe today, leading to a cascade of alternative approaches (Yoo & Watanabe 2012). Recent tensions, in particular, seem to shed light on the need of a more general framework that extends the ΛCDM paradigm (Di Valentino et al. 2020; Capozziello et al. 2020a; Benetti & Capozziello 2019).

Several extensions are possible2, although none of these models represent concrete alternatives to the standard concordance paradigm. In particular, one always identifies two relevant epochs associated with any dark energy models: I) the time at which dark energy and matter have the same density, namely the equivalence

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1 There is a growing need of new dark matter candidates as the most popular ones, e.g., weakly interacting massive particles, axions, and sterile neutrinos have not been detected so far. A possible different perspective on dark matter’s origin is provided by (Luongo & Muccino 2018).

2 Among others, e.g., the addition of dynamical scalar fields with slowly varying potentials, models with interacting dark matter and dark energy components, and modifications to general relativity on large scales have been proposed (Capozziello et al. 2019).

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time, 2) the time of dark energy domination over matter, \( i.e., \) the onset of acceleration - the transition time. One can quantitatively identify the two epochs, i.e., the first epoch starts at an equivalent redshift, \( z_{eq} \), at which the equality between matter, \( \rho_m \), and dark energy, \( \rho_X \), formally holds, namely \( \rho_m(z_{eq}) = \rho_X(z_{eq}) \). The second epoch corresponds to the transition redshift, namely \( z_{tr} \), at which the universe enters a phase of accelerated expansion (Melchiorri et al. 2007; Hawking & Ellis 1973).

Recently it has been argued that such redshifts can be considered as kinematic quantities, in a sense that constraining their values in a model-independent manner provides a way to describe the universe dynamics without postulating the model \( a \) priori (Dunsby & Luongo 2016). Commonly, claims on the dependence of these two quantities on cosmological models have been made in the literature (see for example (Melchiorri et al. 2007) and references therein). To characterise them without postulating the underlying cosmological model can significantly help to discriminate between different dark energy scenarios, in analogy to the strategy of modern cosmography (Aviles et al. 2012; Cattoën & Visser 2007, 2008a,b; Visser 2005).

In this work, we propose two strategies to obtain the transition and equivalence redshifts using the model-independent treatment offered by cosmography. We adopt two procedures, the first makes use of a direct Hubble expansion, whereas the second expands the deceleration parameter. In both the two methods, we assume expansions around the transition redshift, namely \( z \approx z_{tr} \). The corresponding cosmological distances can be reformulated accordingly and this naturally implies to confront our expansions directly with data. We assume two different hierarchies, based on the order at which the expansions are performed. In particular, we first expand up to the jerk and then up to the snap parameters at \( z = z_{tr} \), respectively. Afterwards, we take into account two kinds of fits based on the use of observational Hubble data (OHD) first and then Pantheon type Ia supernovae (SNe Ia), baryonic acoustic oscillations (BAOs) and OHD, for a total of eight fits performed through Markov Chain Monte Carlo (MCMC) simulations by means of a modified free available Wolfram Mathematica code (Arjona et al. 2019), developed with the Metropolis-Hastings algorithm. We discuss multiplicative and additive degeneracies over our coefficients, and we debate about possible underestimated error bars and on the overall \( z_{tr} \) and \( z_{eq} \) accuracy. The corresponding jerk and snap terms at the transition are also obtained. We therefore interpret the feasibility of our numerical results and introduce exclusions regions for \( z_{eq} \). To do so, we provide intervals of validity for the set \( (z_{tr}, z_{eq}) \) and discuss how these terms can discriminate dark energy evolution and its nature, confronting our outcomes with the standard cosmological model.

The paper is structured as follows. In Sect. 2, we present the main features of the transition and equivalence redshifts in view of our cosmographic approach. In Sect. 3, we propose our two treatments adopted throughout this work in order to determine bounds on \( z_{tr} \) and \( z_{eq} \). In Sects. 4 and 5 we first propose the numerical methodology that we adopt for our fits and then we highlight our experimental procedures. We also propose our exclusion regions for \( z_{eq} \). Once our results have been estimated, in Sect. 6 we provide our theoretical interpretation, i.e., we discuss the limitations and consequences of our recipe, together with the theoretical consequences of our findings on dark energy. Finally, in Sect. 7 we present our conclusions and perspectives. Details on cosmographic series, observational data and contour plots are given in Appendices A, B and C, respectively.

## 2 RECONSTRUCTING COSMOGRAPHIC TRANSITION AND EQUIVALENCE REDSHIFTS

In this section, we describe the main features of the transition and equivalence redshifts, \( z_{tr} \) and \( z_{eq} \). We highlight the most important properties of such quantities and fix them for the standard cosmological model, the LCDM paradigm, and for a generic class of dark energy models, where we do not \( a \) priori postulate the underlying dark energy evolution.

To do so, we need to add extra conditions to ensure the consistency of our theory with the cosmological principle. For instance, to correctly embed within it, in a homogeneous and isotropic universe, we take the total pressure as sum of all sub-presatures of each constituents, \( i.e., P = \sum_i P_i \). Thus, at the level of the background cosmology, we neglect spatial curvature and assume pressureless matter, \( P_m = 0 \), immediately providing \( P = P_X \), where \( P_X \) is dark energy’s pressure. In analogy, we take the total density to be \( \rho = \rho_m + \rho_X \).

In addition, one way to work out what sort of onset of cosmic acceleration and equivalence between dark energy and matter we expect, let us examine two more hypotheses. The first relies on the fact \( z_{tr} \) and \( z_{eq} \) are not free but rather they depend upon dark energy’s parameters (Farooq & Ratra 2013; Farooq et al. 2013). Consequently, it is natural to admit a connection between \( z_{eq} \) and \( z_{tr} \). The second is the requirement \( z_{eq} \leq z_{tr} \), since dark energy requires time to equate matter first and then to dominate over it (Farooq et al. 2017; Yu et al. 2018).

Thus, we adopt the homogeneous and isotropic Friedman-Robertson-Walker universe, where the dynamics is dictated by the Friedmann equations

\[
H^2 = \frac{a}{a} = \frac{8\pi G}{3}\rho, \quad \dot{H} + H^2 = -\frac{4\pi G}{3}(\rho + 3P), \tag{1}
\]

in which we made the aforementioned assumption of a spatially flat universe. In this respect, a generic spatially-flat dark energy scenario, with negligible radiation and neutrino contributions, is characterized by a generic evolution of dark energy. In this respect, let us consider a dark energy density, namely \( \rho_X = G f(z) \), with \( G \) a constant with \( f(z) \) an arbitrary-defined dark energy function. Then, to guarantee that \( H(z = 0) = H_0 \) the constant, \( G \) turns out to be \( G = 1 - \Omega_{m,0} \), when \( f(z = 0) \equiv f_0 = 1 \). This choice represents the simplest approach since we are excluding any coupling term between dark matter and dark energy. The largest variety of dark energy models falls within this class of frameworks and well-behaves at different epochs of the universe evolution. In other words, \( f(z) \) is a smooth function that fully determines how dark energy evolves with time, fulfilling the standard experimental evidence for which \( f(\infty) \leq 1 \) in order to guarantee small perturbations at primordial times. Thus, plugging this information in the first Friedmann equation, we provide the Hubble evolution by (Luongo et al. 2015)

\[
\left( \frac{H}{H_0} \right)^2 = \Omega_{m,0}(1 + z)^3 + (1 - \Omega_{m,0}) f(z), \tag{2}
\]

where \( \Omega_{m,0} \equiv \frac{\rho_{m,0}}{\rho_{cr}}, \rho_{cr} \equiv \frac{3H_0^2}{8\pi G} \). Under this assumption, differ-

\( \text{3 The LCDM model depends on six parameters: baryon and cold dark matter densities, the age of the universe, the scalar spectral index, the curvature fluctuation amplitude and reionization optical depth. Our choice dramatically reduces this set to \{H_0, \rho_m}\}. Any dark energy models turn out to be more complicated, depending on a few extra terms.}
entailing with respect to \( z \), one has
\[
2 \frac{1 + q}{1 + z} E^2 = 3 \frac{\Omega_m(z)}{1 + z} + (1 - \Omega_{m,0}) f'(z),
\]
with \( E^2 \equiv \left( \frac{H^2}{H_0^2} \right)^2 \) and \( \Omega_m(z) \equiv \Omega_{m,0}(1 + z)^3 \).

In Eq. (3), we introduced the deceleration parameter by
\[
q \equiv - \frac{a}{H^2} \frac{\ddot{a}}{a} = -1 + \frac{1 + z}{1 + z} \frac{dH}{dz},
\]
where we used \( a = (1 + z)^{-1} \) with \( a(0) = a_0 = 1 \). Note the prime indicates the derivative with respect to \( z \), while the dot refers to time derivative.

### 2.1 Relating transition to equivalence redshifts

The transition redshift defines the onset of cosmic speed up. Since before current time there exists a matter-dominated epoch, in order to ensure dark energy to accelerate the universe today, we require the deceleration parameter to change its sign. The situation is clear: to ensure dark energy to accelerate the universe today, we require
\[
\text{transition redshift defines the onset of cosmic speed up. Since before current time there exists a matter-dominated epoch, in order to ensure dark energy to accelerate the universe today, we require the deceleration parameter to change its sign. The situation is clear.}
\]

Immediately, for the concordance paradigm we compute
\[
q(z_{eq}) = \frac{\Omega_{\Lambda}}{\Omega_{m,0}} - 1.
\]

Confronting Eqs. (6) and (8), or more easily Eqs. (7) and (9), we can combine \( z_{tr} \) with \( z_{eq} \) to formally yield \( z_{tr} = z_{tr}(z_{eq}) \) and arguing de facto that there exists a correlation between transition and equivalence, as we speculated above. The \( \Lambda \)CDM case provides
\[
z_{eq} = 2^{1/2} (1 + z_{eq}) - 1.
\]

With arbitrary dark energy models, it is reasonable to get the same functional behaviour. So, approximating at first order, a linear dependence between \( z_{eq} \) and \( z_{tr} \) can be motivated in analogy to the standard paradigm to give the following ansatz
\[
z_{tr} = \alpha + \beta z_{eq}.
\]

We assume Eq. (11) to be reasonable from now on. Moreover, later in the text we find its validity for generic dark energy models by using our strategies of numerical analysis. For instance, looking at Eqs. (10) and (11), we notice \( \alpha = 2^{1/2} - 1 \) and \( \beta = 2^{1/4} \). The phase space of \( (\alpha, \beta) \) is not arbitrary, since the condition \( z_{tr} < z_{eq} \) holds. So, it appears evident that not only \( \beta > 0 \) but also \( \beta > 1 \). Indeed, since \( z_{tr} \leq 1 \), \( z_{eq} \leq 1 \) and \( \alpha > 1 \), then
\[
\frac{1 + z_{tr}}{1 + z_{eq}} > 1,
\]

implying \( \beta > 1 \) as stated because \( \alpha \) cannot be negative definite by construction, giving the additional requirement \( \alpha > 0 \). These facts influence the experimental outcomes that we are going to describe in the next sections and are compatible with theoretical priors over \( z_{tr} \) and \( z_{eq} \). Once the constraints over \( z_{tr} \) and \( z_{eq} \) are known, it is therefore licit to display exclusion plots where Eq. (11) is no longer valid, displaying the phase space where \( z_{eq} \) does not satisfy \( z_{tr} \geq z_{eq} \).

### 2.2 How to get constraints over \( z_{tr} \) and \( z_{eq} \)

Here, our goal is to come up with a reasonable recipe for finding model independent constraints on \( z_{tr} \) and \( z_{eq} \) in analogy to the cosmographic procedure. When we state “model-independent” we mean without postulating \( f(z) \) \( \text{a priori} \), albeit that a few conditions have been reasonably assumed such as homogeneity and isotropy. For the sake of clearness, the method is therefore not fully independent of the universe description. However, it is much less model dependent than postulating a dark energy function and fit it with data, making it more attractive and useful in order to check departures from the \( \Lambda \)CDM paradigm.

We quoted above the cosmographic procedure that provides constraints on the so-called cosmographic parameters. These represent the set of free terms to evaluate in cosmography (Dunsby & Luongo 2016; Aviles et al. 2012; Visser 2005; Aviles et al. 2017; Gruber & Luongo 2014; Aviles et al. 2014; Luongo & Muccino 2020a; Capozziello et al. 2018). This treatment consists in expanding the scale factor \( a(t) \) around the present day, namely \( t_0 \) and in relating this expansion to observable quantities. To better focus on it, we therefore start with
\[
a(t) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^n a}{dt^n} (t - t_0)^n,
\]
and define the cosmographic series, that for our purposes can be truncated up to the snap parameter, corresponding to fourth order in \((t - t_0)\). The jerk and snap parameters are given by
\[
j = \frac{1}{aH^2} \frac{d^3 a}{dt^3}, \quad s = \frac{1}{aH^3} \frac{d^4 a}{dt^4},
\]
and writing down the corresponding series up to the fourth order in
the cosmic time difference $t - t_0$ as follows
\begin{equation}
    a(t) = 1 - H_0\Delta t - \frac{q_0}{2} H_0^2 \Delta t^2 + \frac{j_0}{6} H_0^3 \Delta t^3 + \frac{s_0}{24} H_0^4 \Delta t^4 + \ldots ,
\end{equation}
we can relate the cosmographic set \(^4\) to \((z_{eq}, z_{eq})\).

Clearly, there is an intrinsic difficulty to work with $z_{eq}$ instead of $z_{tr}$. In fact, $q(z_{eq}) \neq 0$, while we notice that plugging the condition (5) into cosmography would reduce the complexity of our procedure.

In this paper, as we will stress also later in the text, we therefore directly fit the transition redshift using the cosmographic method and then we exclude the regions in which $z_{eq}$ is forbidden. Clearly this represents our choice, essentially based on reducing the overall complexity of our fits by virtue of condition (5).

Thus, for the sake of completeness our main target is not to constrain theoretical models, but rather to bound $z_{tr}$ from fits and exclude $z_{eq}$ consequently, as model independent as possible. Since this would represent a “top-down” approach, attempting to deduce constraints in a model-independent way. As we stressed above, we

In this section, our concern is to determine the transition redshift $z_{tr}$, portraying formal expressions for the cosmographic coefficients become functions of $z_{tr}$.

III. We finally get constraints over $z_{eq}$, portraying formal exclusion plots in which we report the regions where $z_{eq}$ can span by virtue of Eq. (11).

Differently of the standard cosmographic set, namely $H_0, q_0, j_0$ and $s_0$ the corresponding priors are not known \textit{a priori}. So, the possible price we pay is that, after expanded in Taylor series, the new observable quantities of interest could be fully unbounded.

To this end, an intriguing strategy to get hints toward the priors to use, is to somehow furnish a correspondence $j_0 = j_0(z_{tr})$ and $s_0 = s_0(z_{tr})$. This is the philosophy of the second method that we are going to describe below. Henceforth we shall baptize the first strategy as \textit{direct Hubble expansion} (DHE), whereas the second as \textit{direct deceleration parameter expansion} (DDPE).

Following the above considerations, the first treatment concerns the \textit{direct expansion} around $z = z_{eq}$, of the quantities of our interest, while the second requires the deceleration parameter is expanded around $z = z_{eq}$ and then it \textit{directly confronts} the original cosmographic sets with the new one. We describe in detail below the two approaches.

3 THEORETICAL SETUP

In this section, our concern is to determine the transition redshift constraints in a model-independent way. As we stressed above, we directly work with $z_{tr}$ to reduce the complexity by virtue of Eq. (5) and exclude the regions in which $z_{eq}$ is forbidden \textit{a posteriori}. To do so, we intend to follow two distinct strategies, based on the following basic demands.

I. All our formal expansions are built up through cosmological quantities of interest that we directly compare with data, \textit{i.e.} the Hubble rate, luminosity distance, angular distance, and so forth. The expansions are computed around $z_{tr}$, with $q(z_{tr}) = 0$.

II. Every contribution does not threaten to blow up, ruining the stability of the expansion, by fixing appropriate orders of Taylor series. In this respect, we evaluate the new cosmographic series where the cosmographic coefficients become functions of $z_{tr}$.

III. We finally get constraints over $z_{eq}$, portraying formal exclusion plots in which we report the regions where $z_{eq}$ can span by virtue of Eq. (11).

\textbf{3.1 First procedure: DHE method}

We first discuss the direct procedure of expanding around $z = z_{tr}$ the Hubble rate. We immediately get
\begin{equation}
    H = H(z_{tr}) + H'(z_{tr}) (z - z_{tr}) + \frac{1}{2} H''(z_{tr}) (z - z_{tr})^2 + \frac{1}{6} H'''(z_{tr}) (z - z_{tr})^3 + O((z - z_{tr})^4).
\end{equation}

The above expression should be normalized to $H(z = 0) \equiv H_0$ at $z = 0$. At this stage, this additional constraint means that our new cosmographic series rescates all the other coefficients in terms of one of them, \textit{i.e.}, the Hubble rate today. Indeed, choosing to expand $H$ to a given order $n$, it is arguable that only $n - 1$ cosmographic coefficients are effectively independent.

To see this more clearly, we write the connection between the cosmographic series and the Hubble rate (Dunsby & Luongo 2016)
\begin{equation}
    \dot{H} = -H'^2 (1 + q),
\end{equation}
\begin{equation}
    \ddot{H} = H'^3 (j + 3q + 2),
\end{equation}
\begin{equation}
    \dddot{H} = H' [s - 4j - 3q(q + 4) - 6].
\end{equation}
Now, we can use the identity
\begin{equation}
    \frac{dz}{dt} \equiv - (1 + z) H(z),
\end{equation}
and so we immediately get the derivatives of the deceleration parameter with respect to $z$
\begin{equation}
    \frac{dq}{d\zeta} = \frac{j - 2q^2 - q}{1 + \zeta},
\end{equation}
\begin{equation}
    \frac{d^2 q}{d\zeta^2} = \frac{1}{1 + \zeta} \left[ 2 \frac{dq}{d\zeta} (2q + 1) - \frac{dj}{d\zeta} \right].
\end{equation}

So, Eqs. (17) - (19) certify that only $n - 1$ coefficients are really independent. At this stage, since $z \equiv (1 + z)^{-1}$, with $a(z = 0) = a_0 = 1$, we get from Eqs. (17) the following expressions
\begin{equation}
    H''' = \frac{H(3q - 1) - j (3 + 4q)}{(1 + z)^4},
\end{equation}
\begin{equation}
    H'' = \frac{H(j - q^2)}{(1 + z)^2},
\end{equation}
\begin{equation}
    H' = \frac{H(1 + q)}{1 + z},
\end{equation}
that at the transition time reduce to
\begin{equation}
    H'''_{tr} = \frac{-H_{tr}(3j_{tr} + s_{tr})}{(1 + z_{tr})^4},
\end{equation}
\begin{equation}
    H''_{tr} = \frac{H_{tr}j_{tr}}{(1 + z_{tr})^2},
\end{equation}
\begin{equation}
    H'_{tr} = \frac{H_{tr}}{1 + z_{tr}}.
\end{equation}
Thus, we set the following constraints respectively for Eqs. (22) and (23) respectively we get

\[ E^{(3)}(z) \simeq \frac{6 + z(6 + 3j_{tr} z - (3j_{tr} + s_{tr}) s^2) + 12z_{tr} + 3s(4 + s_{tr} z + j_{tr}(-2 + 4z)) z_{tr} + 3(2 + j_{tr} + (2 - 5j_{tr} - s_{tr}) z_{tr} + j_{tr}(3 + 6z_{tr})))}{6 + z_{tr}(12 + z_{tr}(6 + s_{tr} z_{tr} + j_{tr}(3 + 6z_{tr})))} , \]

and

\[ E^{(2)}(z) = H_0 \left[ \frac{2 + j_{tr} z^2 + 2s(1 + z_{tr} - j_{tr} z_{tr}) + z_{tr}(2 + j_{tr} z_{tr})}{2 + z_{tr}(2 + j_{tr} z_{tr})} \right] . \]

Thus, we set the following constraints respectively for Eqs. (22) and (23)

\[ H_{tr} = \frac{6H_0(1 + z_{tr})^3}{6 + z_{tr}(12 + z_{tr}(6 + s_{tr} z_{tr} + j_{tr}(3 + 6z_{tr})))} , \]

\[ H_{tr} = \frac{2H_0(1 + z_{tr})^2}{2 + z_{tr}(2 + j_{tr} z_{tr})} , \]

with the obvious requirement that \( j_{tr} \neq j_{tr} \) and \( s_{tr} \neq s_{tr} \), besides \( q_{tr} = 0 \). However, since we are limiting our analysis to late-time, the corresponding redshifts are confined around \( z \lesssim 2 \). Within this sphere, we can approximately assume any further orders beyond snap to be negligible. Immediately, from Eqs. (22) and (23), we notice new priors on \( j_{tr} \) and \( s_{tr} \) are needful, as we will discuss later in the text.

For the sake of completeness, it is evident the denominators of Eqs. (22) and (23) do not limit the priors to impose, although \( H_0 \) would do, as due to the \( H_0 \) tension.

Finally, it is remarkable to stress a strong multiplicative degeneracy in fitting \( z_{tr} \), with \( j_{tr} \) and \( s_{tr} \). In particular, this would imply a further uncertainty in the fitting procedures.

### 3.2 Second procedure: DDPE method

The second procedure, as stated above, consists of expanding the deceleration parameter in a Taylor series up to a given order around the transition redshift \( z_{tr} \). For practical reasons, we fixed two hierarchical orders in analogy to the DHE, i.e., the first and second orders of Taylor expansions of the deceleration parameter as

\[ q^{(1)}(z_{tr}) \simeq q(z_{tr}) + \frac{dq}{dz} \bigg|_{z_{tr}} (z - z_{tr}) , \]

\[ q^{(1)}(z_{tr}) \simeq q^{(1)}(z_{tr}) + \frac{1}{2} \frac{d^2q}{dz^2} \bigg|_{z_{tr}} (z - z_{tr})^2 . \]

By definition, \( q(z_{tr}) \) vanishes and in the redshift interval \( z < 1 \) we can approximate the value of the deceleration parameter to the value inferred from cosmography today, \( q_{0} \), by simply taking \( z = 0 \) inside Eqs. (25). One can therefore compute the extreme values of \( q \) within the interval \( 0 < z < z_{tr} \), corresponding to a minimum and maximum value of \( q \). In particular, the deceleration parameter is negative definite to guarantee current acceleration. It follows that

\[
\begin{align*}
q_{0, \text{min}} &= -\frac{dq}{dz} \bigg|_{z_{tr}} (z_{tr} - z) \left. \frac{d^2q}{dz^2} \bigg|_{z_{tr}} \right. (z_{tr} - z), \\
q_{0, \text{max}} &= 0, \quad z = z_{tr} ,
\end{align*}
\]

in agreement with the fact that the transition occurs when \( q \) changes sign (passing from a positive to a negative value). The above request of minimum and maximum values for \( q \) leads to small errors in expanding \( q \) around \( z = z_{tr} \).

The procedure to get the Hubble rate that we adopt here for our fits is the following. We plug Eqs. (25) inside the second Friedmann equation of Eqs. (1). Then, known \( q \) we infer \( H \) in function of \( z_{tr} \) and we require \( H(z = 0) = H_0 \) as we did before. It is remarkable to notice that taking the direct expansion of \( H \) around \( z = 0 \), as reported in Appendix A and substituting Eqs. (25) with \( z = 0 \) would be also possible and apparently easier. However, the former strategy leads to worse results in computation. In particular, this technique is not reasonable because of the main caveat: taking \( z = 0 \) Hubble expansion and plugging and expanded \( q \) with \( z = 0 \) would induce an approximation within an approximation. This double approximation is therefore a source of further errors in the numerical computations that we are going to present. Motivated by these reasons, we decide to follow the other strategy to compute the quantities to fit. Hence, we immediately get for order two and three the following normalized Hubble rates

\[ E^{(2)}(j_{tr}, z_{tr}) = \exp \left[ \frac{j_{tr} z_{tr}}{H(z_{tr})} \right] (1 + z) \left. \left( 1 - j_{tr} \right) \right|_{z_{tr}} , \]

\[ E^{(3)}(j_{tr}, z_{tr}) = E^{(2)} \exp \left[ \frac{(j_{tr} + s_{tr})(1 + z_{tr}) - 2s_{tr} z_{tr}}{H(z_{tr})} \right] (1 + z) \left. \left( j_{tr} + s_{tr} \right) \right|_{z_{tr}} (1 + z_{tr}) . \]

### 4 METHODOLOGY

The strategy is to approximate distances making use of the standard formula

\[ r_0 \equiv \int_0^d dz' \frac{dz'}{H(z')} . \]

In particular, to get the luminosity and angular distances, as we report below, the idea is to integrate Eqs. (22) and (23) for the DHE method and Eqs. (27) for the DDPE method in order to fix the cosmographic coefficients evaluated around the transition time.

As above stated, we thus consider for our numerical purposes three observable quantities, i.e., two distances, the luminosity and angular ones, and the Hubble rate, truncated with the above orders. For every quantity since we imposed a vanishing curvature parameter, say \( \Omega_k = 0 \) (Planck Collaboration et al. 2020), we do not care about the likely degeneracy between transition and \( \Omega_k \). In what follows, we summarise the numerical strategy for each of the aforementioned quantities that we handle in our fits. Moreover, we report the main caveats that we can encounter throughout our computation.

#### 4.1 Overcoming bias the truncated series issues

Expanding in terms of \( z_{tr} \) with DHE and DDPE strategies guarantees to get constraints over the free parameters, namely \( z_{tr} \) and \( H_{tr}, j_{tr}, s_{tr} \). Thereafter, one can make comparisons to the measured quantities and accept or reject the realization of our fits. The
former could be due to the truncating series errors and/or to bias affecting the procedure itself.

In case one restarts the procedure with a new realization of the underlying parameters, it is useful to understand how to reduce bias and systematics on our numerical fits, but also the correlations that occur among parameters, i.e., intimately related to the degeneracy among coefficients\(^6\). As the true cosmological bounds over transition is expected to marginally lie around ΛCDM predictions, we expect that a successful numerical set follows this trend in a small neighborhood of its best fits, indicating unreasonable degeneracy.

To overcome the above caveats and removing unwanted degeneracies as discussed above, we propose a new route that uses Hubble’s expansions as input functions inside Eq. (28a). In other words, instead of expanding Eq. (28a) as one commonly does in standard cosmography, we take the above computed Hubble’s expansions for the DHE and DDPE methods and evaluate the corresponding exact integrals from Eq. (28a).

Using this strategy, we see the dispersion and bias turn out to be much smaller than standard cosmography, speeding up de facto the overall numerical computations.

4.2 The luminosity distance and \(z_{tr}\)

We start with the standard definition for the luminosity distance in the case of spatially flat universe. We have

\[
d_L(z) = (1 + z) \int_0^z \frac{dz'}{H(z')} = (1 + z)r_0 .
\] (29)

In particular, Eq. (29) can be used in the following way: we construct the expanded Hubble parameter up to the order that we require. Then we insert the Hubble expansion within Eq. (29), depending on the transition redshift and the cosmographic series evaluated at the transition. This strategy makes it position to understand

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\(^6\) For example, in the spatially-flat ΛCDM model, one encounters a linear correlation between the snap parameter and \(q_0\), namely \(q_0,\Lambda\text{CDM} = -2 - 3q_0,\Lambda\text{CDM} \).
the role played by the two coefficients \( j_{tr} \) and \( s_{tr} \), with analogous degeneracy problems than standard cosmography.

Another intriguing fact to remark is that it could be possible to directly expand Eq. (29), instead of inserting a Hubble expansion inside it. However, we follow the strategy presented in (Aviles et al. 2017), in which it has been argued that the cosmographic expansions with the aforementioned assumptions do not provide a significant increase of degeneracy among cosmographic coefficients, enabling the one-to-one identification:

\[
\hat{d}(z; \Omega_m, \Omega_X) \rightarrow \hat{d}_L(z; \theta),
\]

where \( \theta \) represents the cosmographic set from which \( \hat{d}_L \) is thought to depend on, while \( \Omega_X \) is the generic dark energy density.

The convergence of truncated Hubble rate at a given order jeopardizes the overall analysis and may produce systematics in our computation. To expect a more stable numerical output from our fits, we leave our analyses fully free to vary in the priors. In fact, as jerk and snap parameters change, the transition redshift can severely switch, suggesting that fixing cosmographic parameters in our analyses is not suitable to work with. The marginalization procedure is also dangerous since we do not know a priori how much priors could influence our analysis. In fact, even the choice of priors has been as wide as possible, within a range that possibly does not influence the analysis itself.

It is remarkable that dark energy density does not modify the expected bounds over \( z_{eq} \) and \( z_{eq} \), because the strategy here involved is fully model independent, besides the choice of zero spatial curvature. The only limitation could be to consider that well-motivated dark energy models do not predict transition outside the sphere \( z \leq 1 \). So, under the hypothesis of barotropic dark energy paradigms, we assume to rule out the models that show fits outside the sphere \( z \leq 1 \). More complicated cases are however possible, but weakly supported by other observations. For example, assuming modified and/or extended theories of gravity, providing highly different dark energy model, is disfavoured by current bounds in the sphere \( z \leq 1 \). The Occam razor suggests us to work out at late times, the simplest barotropic dark energy models, instead of invoking more complicated paradigms that would change the series we are handling and modify the expected ranges for the transition redshift.

### 4.3 The angular distance and \( z_{tr} \)

In this case, we start with the standard definition got from BAO, where the angular distance can be described in terms of uncorrelated and correlated data points. We write the distance

\[
d_L(x, z) \equiv r_s \left( \frac{H(x, z)}{cz} \right) \frac{(1+z)^{\frac{3}{2}}}{d_L^2(x, z)},
\]

with

\[
A_s(x, z) \equiv \frac{\sqrt{\Omega_m} H_0 r_s}{c z d_L(x, z)},
\]

where the comoving sound horizon, \( r_s \), depends on baryon drag redshift \( z_d \) and \( x \) is the set of free parameters involved in calculations.

In what follows, we only focus on uncorrelated BAO data points. This choice reduces the data dependence of our fits. Finally, we do not use BAO catalogs alone, but only with SNe Ia and Hubble data to guarantee the less predicted biases from simulations and the robustness of our fits.

### 4.4 The observational Hubble expansions

To concern with OHD data, we directly expand the Hubble rate as we previously highlighted for the DHE and DDPE methods. Every \( H(z) \) expansion formally provides

\[
H(q_0, j_0, s_0) \rightarrow H(z, j_{tr}, s_{tr}),
\]

and then, we take the expansions of \( d_L \) and \( d_A \) and plug the new definitions of \( H \) expanded in the two forms, for the DHE and DDPE methods. Once the functions are got, we experimentally fit our truncated models directly with data.

We combine the three data sets, SNe Ia, BAO and OHD, to get limits over the transition redshift. However, our first fits are prompted using OHD alone. Using a single data set with not so much data points clearly compromises the overall accuracy. For these reasons, for all our computations, we need to study different sets of parameters with fixed orders, leading to a hierarchy in our numerical analyses.
order around
we write
it is possible to infer
equality holds
on
become larger. Thus, since
the deceleration parameter is negative and as the redshift increases tends to
start from our previous prescriptions on
z_
and
z = 0 into Eq. (33), we have
\begin{align}
q(z_{eq}) &= q(z_{eq}) + \frac{dq}{dz_{eq}} (z_{tr} - z_{eq}) = 0, \quad (34a) \\
q(0) &= q(z_{eq}) - \frac{dq}{dz_{eq}} z_{eq} = q_0. \quad (34b)
\end{align}
Our intend is to get an expression that relates
z_{tr}
with
z_{eq}. At a first glance, we can first evaluate, from Eq. (34a), the quantity
\frac{dq}{dz_{eq}} = \frac{q_{eq}}{z_{eq} - z_{tr}}
and then we plug it into Eq. (34b) to get
\begin{equation}
z_{eq} = \frac{q_0 - q_{eq}}{q_0 z_{tr}}. \quad (35)
\end{equation}
The expression in Eq. (35) relates the present value of the deceleration parameter with the equivalence and transition redshifts. In principle, for fixed
q_0,
knowing the intervals in which
q_{eq}
can span, it is possible to infer
z_{eq},
once we measured the allowed values for
z_{tr},
portrayed in Tab. 1.
In addition, it is straightforward to notice that the following equality holds
\begin{equation}
\frac{dq}{dz_{eq}} = \frac{j_{eq} - 2q_{eq}^2 - q_0}{1 + q_{eq}} = \frac{q_{eq}}{z_{eq} - z_{tr}}, \quad (36)
\end{equation}
so that, solving with respect to
z_{tr},
and collecting in terms of
z_{eq},
we write
\begin{equation}
z_{tr} = \frac{q_0 - q_{eq}}{j_{eq} - q_{eq}(1 + 2q_{eq})} + \frac{j_{eq} - 2q_{eq}(1 + q_0)}{j_{eq} - q_{eq}(1 + 2q_{eq})} z_{eq}. \quad (37)
\end{equation}
Thus, confronting Eqs. (11) and (37), we infer
\begin{align}
\alpha &= - \frac{q_{eq}}{j_{eq} - q_{eq}(1 + 2q_{eq})}, \quad (38a) \\
\beta &= \frac{j_{eq} - 2q_{eq}(1 + q_0)}{j_{eq} - q_{eq}(1 + 2q_{eq})}. \quad (38b)
\end{align}
By virtue of the considerations discussed in Sec. 2.1, recalling \( \alpha > 0 \) and \( \beta > 1 \), we stress that the condition \( \alpha > 0 \) implies from Eq. (38a)
\begin{equation}
\alpha = \frac{q_0}{j_{eq} - 2q_{eq}(1 + q_0)} > 1,
\end{equation}
which is larger than \( q_{eq} \) as expected. On the other hand, the condition \( \beta > 1 \) is verified if the inequality \( j_{eq} - 2q_{eq}(1 + q_0) > j_{eq} - q_{eq}(1 + 2q_{eq}) \) holds in Eq. (38b). However, since \( j_{eq} - q_{eq}(1 + 2q_{eq}) > 0 \) because of Eq. (39), we therefore get a more stringent requirement over \( j_{eq} \) from Eq. (38b)
\begin{equation}
\beta > \frac{j_{eq} - q_{eq}(1 + 2q_{eq})}{j_{eq} - q_{eq}(1 + 2q_{eq})} = \frac{q_0}{j_{eq} - 2q_{eq}(1 + q_0)}, \quad (40)
\end{equation}
\begin{equation}
\beta > \frac{q_0}{j_{eq} - 2q_{eq}(1 + q_0)}, \quad (40)
\end{equation}
from which we argue the basic constraint on \( q_{eq} \)
\begin{equation}
-1 < q_{eq} < \frac{1}{7}, \quad (41)
\end{equation}
In particular, the left part of the above equality, namely \( q_{eq} > -1 \), derives from Eq. (40), and it is quite obvious because \( q_{eq} \) cannot exceed a de Sitter phase, say \( q_{eq} < -1 \), for describing the dark energy evolution. The right part of the equality, namely \( q_{eq} < -\frac{1}{7} \), derives from both Eq. (39) in which we made the assumptions \( q_{eq} < 0 \) and \( j_{eq} > 0 \). While the first of the former considerations has been previously discussed, the second one, on the contrary, is a natural request that one assume in cosmography (Capozziello et al. 2020b). In fact, the jerk parameter is thought to be positive-definite to guarantee the deceleration parameter to change its sign. Thus, there is no reason \textit{a priori} to imagine \( j_{eq} \) negative by construction. Moreover, looking at Tab. 1 we got all positive values of \( j_{eq} \), and since \( z_{tr} > z_{eq} \), though \( z_{tr} \approx z_{eq} \), it is improbable to get a negative value for \( j_{eq} \) that rapidly jumps to a positive one at the transition.
Last but not least, since \( q_{eq} > q_0 \) is naturally fulfilled to guarantee the universe speeds up at current time, using Eq. (35), we immediately get a likely exclusion plot for \( z_{eq} \), employing the following bounds over \( q_0, z_{tr} \) and \( q_{eq} \)
\begin{align}
q_0 &\in [-0.7; -0.55], \quad (42a) \\
z_{tr} &\in [0.47; 1.18], \quad (42b) \\
q_{eq} &\in [-0.5; 0], \quad (42c)
\end{align}
and guaranteeing \( z_{tr} > z_{eq} \) as reported in previous sections. Here, the first of the above intervals has been taken from Ref. (Capozziello et al. 2020b), where we single out the Taylor fits on \( q_0 \). These limits are clearly suitable because compatible with the most recent bounds on \( q_0 \), see e.g. (Capozziello et al. 2019; Aviles et al. 2012). The second interval over \( z_{tr} \) is naively got from our results, see Tab. 1, where we considered the smallest and largest \( z_{tr} \), mean values got from our computations. The last interval corresponds to the largest interval in which \( q_{eq} \) can span by virtue of Eqs. (39) - (40) and (41). In fact, since \textit{q(z)} \( q_{eq} > q_0 \), as the cosmic speed up has not started yet at the equivalence, we are forced to assume \textit{q(z) > q_0} because dark energy did not reach the time to dominate over matter. Since \( q_0 \in [-1; 0] \), it is licit to presume \textit{q(z)} \( \in [-0.5; 0] \). Hence, we baptize the \textit{z} \( \textit{z} \textit{eq} \) plot with the name exclusion plot. This plot emphasizes the regions where \( z_{eq} \) is not allowed to vary by means of Eqs. (39), (40) and (41). Thus, we portray the exclusion plot in Figs. 3.
Two additional considerations on \( z_{eq} \) are now needful to clarify the above treatment. The first is that it is more convenient to pass from Eq. (35) instead of Eq. (37) to get \( z_{eq} \) plots, because we do not know \textit{a priori} the value of \( j_{eq} \) that is an extra parameter entering Eq. (37). The second consideration is noticing that it is not convenient to directly fit \( z_{eq} \) following the strategy above developed for \( z_{tr} \). The reason purely lies on statistics, i.e., using \( z_{eq} \) directly in the cosmographic fits leads to a severe degeneracy with \( q_{eq}, j_{eq} \), but also with \( z_{tr} \). So that, it would be much more convenient to infer \( z_{eq} \) bounds, once that the results over \( z_{tr} \) are got from the computation. Similar considerations have been made also in previous sections, when we discussed to work with \( z_{tr} \) in order to reduce complexity.

7 By construction, in the redshift domain in which dark energy dominates, the deceleration parameter is negative and as the redshift increases tends to become larger. Thus, since \( z_{tr} > z_{eq} \), we have \( q_{eq} < q_{tr} = 0 \).

8 For the sake of clearly, we could in principle take a mean \( z_{tr} \), and then take the above interval using propagated error bars. However, the choice (42) enables us to get larger domains for \( z_{tr} \).
In the case of SNe Ia, we computed the Table 1.

In Table 1, the MCMC results at the 68% (95%) confidence level for our different cosmographic techniques from the combination of data sets. $H_0$ values are expressed in km/s/Mpc. The corresponding 1σ and 2σ confidence levels are portrayed in Appendix C, where we report the contour plots of our analyses. The $\chi^2$ values are here not normalized.

| Model     | Datasets                | $H_0$      | $z_{tr}$ | $J_{tr}$ | $s_{tr}$ | $\chi^2$ |
|-----------|-------------------------|------------|----------|----------|----------|----------|
| $DHE(N = 2)$ | SN+OHD+BAO              | 74.879     | 3.010(1.988) | 0.646+0.020(0.031) | 2.544+0.506(0.847) | --       | 1128.97  |
| $DHE(N = 2)$ | OHD                    | 67.659     | 1.082(9.246)  | 0.659+0.371(9.414) | 0.982+0.519(1.375) | --       | 14.7377  |
| $DHE(N = 3)$ | SN+OHD+BAO              | 93.187     | 0.767(1.600)  | 0.656+0.025(0.047) | 2.263+0.083(1.090) | --       | 1658.14  |
| $DHE(N = 3)$ | OHD                    | 76.331     | 3.946(10.083) | 0.473+0.178(0.624) | 2.964+0.578(1.676) | 0.196+1.433(2.526) | 12.9656  |
| $DDPE(N = 1)$ | SN+OHD+BAO              | 75.091     | 0.170(0.647)  | 0.860+0.013(0.289) | 2.160+0.237(0.291) | --       | 1129.01  |
| $DDPE(N = 1)$ | OHD                    | 66.534     | 2.998(6.760)  | 0.700+0.260(0.438) | 0.830+0.303(0.854) | --       | 15.0466  |
| $DDPE(N = 2)$ | SN+OHD+BAO              | 75.092     | 0.204(0.767)  | 1.183+0.029(0.011) | 1.173+0.011(0.036) | 5.826+0.170(0.268) | 1129.04  |
| $DDPE(N = 2)$ | OHD                    | 76.702     | 4.83(10.397)  | 0.489+1.07(0.178) | 2.804+0.47(0.957)  | 17.564+6.040(7.261) | 1129.04  |

Table 1. MCMC results at the 68% (95%) confidence level for our different cosmographic techniques from the combination of data sets. $H_0$ values are expressed in km/s/Mpc. The corresponding 1σ and 2σ confidence levels are portrayed in Appendix C, where we report the contour plots of our analyses. The $\chi^2$ values are here not normalized.

5 STATISTICAL ANALYSES

Our fits utilise low redshift data from different surveys. To perform the numerical procedures, we work out MCMC simulations sampled within the widest possible parameter space over the cosmographic coefficients. The strategy is to modify the freely available Wolfram Mathematica code reported in (Arjona et al. 2019) that makes use of the widely adopted Metropolis-Hastings algorithm. We thus confront our luminosity distance expressed in terms of the new cosmographic set, based on $z_{tr}$ and $z_{eq}$, directly with data, explicitly reporting how to reduce the dependence on statistical distributions through the algorithm itself. The numerical procedure is characterised by minimising the chi squares computed with different data sets. So, we indicate with $x$ the best fit cosmological parameters. The set $x$ clearly minimizes the total $\chi^2$, constructed by means of SNe Ia, BAO and OHD data sets, written as

$$\chi^2_{\text{SN} + \text{BAO} + \text{OHD}} = \chi^2_{\text{SN}} + \chi^2_{\text{BAO}} + \chi^2_{\text{OHD}}.$$  

In particular, we report below the way in which each $\chi^2$ is evaluated.

SNe Ia In the case of SNe Ia, we computed the $\chi^2$ considering the most recent Pantheon sample. Still now, it turns out to be the largest combined sample consisting of 1048 SNe Ia. The SNe Ia data points span within the interval $0.01 < z < 2.3$ (Scolnic et al. 2018) and enable to get the corresponding distance modulus by

$$\mu_{\text{SN}} = m_B - (M - \alpha X_1 + \beta C - \Delta M - \Delta B),$$

that consists in a parametric version of the distance modulus based on $m_B$ and $M$, respectively the $B$-band apparent and absolute magnitudes. In Eq. (44), $X_1$ and $C$ represent the SN light-curve stretch and the colour factors, respectively whereas $\alpha$, $\beta$ and $\Delta M$ show the luminosity stretch and colour and distance correction. Moreover, $\Delta B$ is a distance correction. It is formulated considering the host galaxy mass containing SNe Ia and it is got by predicted biases inferred from simulations.

The error propagation is not affected by the fitting procedure since uncertainties of each SN do not depend on $M$. The form of SN $\chi^2$ easily becomes

$$\chi^2_{\text{SN}} = (\Delta \mu_{\text{SN}} - \langle M \rangle)^T C^{-1} (\Delta \mu_{\text{SN}} - \langle M \rangle),$$

where the module of the vector of residuals takes the form $\Delta \mu_{\text{SN}} \equiv \mu_{\text{SN}} - \mu_{\text{SN}}(x, z)$, while $C$ is the covariance matrix. There we statistical and systematic uncertainties on the light-curve parameters are provided. We assume a flat prior to remove $M$ from our fitting procedure. This process of marginalisation leads to

$$\chi^2_{\text{SN},M} = a + \log \frac{e}{2\pi} - \frac{b^2}{e},$$

where $a \equiv \Delta \mu_{\text{SN}}^2 + C^{-1} \Delta \mu_{\text{SN}}$ and $b \equiv \Delta \mu_{\text{SN}} C^{-1} \Delta \mu_{\text{SN}}$, and $e \equiv 1/C^{-1}$. BAO BAO waves are observed as a peak in the large-scale structure correlation function. This process is produced in the early universe by the sound wave propagation leading to an angular distance measure, providing a comoving volume variation $D_V(x, z)$ at a given $z$, implying the BAO observable for uncorrelated data $d_s(x, z)$ through

$$D_V(x, z) = \frac{c z}{H(x, z)} \left[ \frac{d_L(x, z)}{1 + z} \right]^2, \quad d_s(x, z) = \frac{r_s(z_d)}{D_V(x, z)}.$$  

The MCMC simulations are performed according to the results prompted in (Planck Collaboration et al. 2020), whose best fit values $z_d = 1059.62 \pm 0.31$ and $r_s(z_d) = 147.41 \pm 0.30$, with the BAO points reported in Table 1 in Appendix B, in which the correlated BAO from the WiggleZ data (Blake et al. 2011) have

9 Marginalising over $\alpha$ and $\beta$ has not been performed. Their contributions enter SN uncertainties.

10 Differently from SNe Ia, BAO data are more model dependent as co-moving sound horizon $r_s(z_d)$ depends upon the baryon drag redshift $z_d$. 

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been excluded. This guarantees that we do not assume observable quantities depending upon $\Omega_m$. With this recipe we take

$$\chi^2_{\text{BAO}} = \sum_{i=1}^{N_{\text{BAO}}} \left[ \frac{d_{\text{BAO}}^i - d_{\text{th}}^i(x_i,z_i)}{\sigma_{d_{\text{BAO}}} i} \right]^2 . \quad (48)$$

**OHD** The Hubble points, namely the set got from the differential age method, are determined from reconstructing Hubble sets measured through spectroscopy (Jimenez & Loeb 2002). Measurements of the age difference take $\Delta t$ and $\Delta z$ of couples of passively evolving galaxies. The idea is that these galaxies are formed at the same time. Under this hypothesis, we assume $\Delta z/\Delta t \simeq \frac{dz}{dt}$, leading to an approximate expression for the Hubble rate, whose update sample consists till now of 31 OHD data, see e.g. (Amati et al. 2019).

In particular, data here depend on stellar metallicity estimates, population synthesis models, progenitor biases, and the presence of an underlying young component in massive and passively evolving galaxies. This data set invokes severe systematics that fortunately are limited up to a 3% error rate at intermediate redshifts. Thus, systematic errors, expected in our computations, are safely kept below the statistical ones.

The overall treatment allows to obtain model-independent estimations of the Hubble rate, providing the idea of cosmic chronometers that lead to

$$H_{\text{obs}} = -\frac{1}{(1+z)} \left( \frac{dt}{dz} \right)^{-1} , \quad (49)$$

spanning withing the interval $0 < z < 3$ and showing the corresponding chi square

$$\chi^2_{\text{OHD}} = \sum_{i=1}^{31} \left[ \frac{H_{\text{obs}}(z_i) - H_{\text{obs}}(z_i)}{\sigma_{H}^i} \right]^2 . \quad (50)$$

### 5.1 Error propagation and numerical outcomes

Unfortunately, our method presents a number of limitations, essentially related to the kind of approximations made throughout the analysis. We briefly list such limitations below

1. Direct low fit errors due to the multiplicative degeneracy between $j_1$ and $z_{\text{eff}}$, as consequence of Eq. (28). A way out to face the issue is to marginalize over $j_1$, within a given set of plausible values for it. This procedure, however, does not remove the degeneracy and consequently $z_{\text{eff}}$ values are influenced by $j_1$.

2. Errors due to the truncated cosmographic Hubble series, i.e., we do not know a priori how truncating series influences the fitting error bars. For these reasons we expect statistical differences using distinct hierarchies.

3. The same as above, but for the deceleration parameter, i.e., errors due to the truncated expansions of $q(z)$, consequence of the fact that we truncate it at a given unknown order.

Summing up one can imagine to increase the error bars $\sigma_{\theta}$ considering the relation

$$\sigma_{\theta} = \sqrt{\sum_{\mu} \sigma^2_{\text{exp},\mu} + \sum_{i=1}^{3} \sigma^2_{i} (\nu) ,} \quad (51)$$

where $\sigma_{\text{exp}}$ represents the experimental error directly got from our fits and reported in Tabs. 1 while $\sigma^{(i\nu)}$ are the errors induced by the aforementioned points raised above. Notice that $\sigma \equiv \frac{\delta}{2}$, where $N$ is the number of errors induced by the approximation. Following the treatment of assuming underestimated error bars of Ref. (Verde 2010), we notice enlarging them through Eq. (51) does not change significantly our findings since the errors would increase of about $\sim 10\%$. We thus believe the overall procedure works fairly well and even enlarging the errors does not furnish more information in our analysis.

For our numerical fits, we adopt the priors that follow: $z_{\text{tr}} \in [0; 1]$, $H_0/100\, \text{km}\, \text{s}^{-1}\, \text{Mpc}^{-1} \in [0.5; 1]$, $j_1 \in [0.5; 5]$ and $s_{\text{tr}} \in [-25; 25]$ for all our computations.

### 6 THEORETICAL DISCUSSION AND COMPARISON WITH MODELS

Our two strategies made use of Taylor expansions and hierarchy between coefficients. The overall merit of each method seems to be not roughly equivalent. The contours provide similar results that do not depend significantly from each other, as reported in Appendix C. The DHE method involves $H(z)$ expansions, whereas the DDPE method considers expansions of $q(z)$. In particular, from the results of Tab. 1 we argue the DHE fits indicate that at 1$\sigma$ the transition redshifts are compatible with the standard cosmological model, although they leave open the possibility that dark energy is not under the form of a genuine cosmological constant.

The use of OHD data only with fits up to $s_{\text{tr}}$ is clearly disfavored than combining the three data sets together, SN+BAO+OHD, up to $s_{\text{tr}}$. Concerning OHD alone, the best suite of cosmological results is for $N = 2$, whereas deviations are evident in the case $N = 3$, i.e. $z_{\text{tr}}$ seems to decrease. The reason is that OHD data points are only 31 and then it is more difficult to constrain a further parameter, namely $s_{\text{tr}}$. The prize to pay is enlarging the $H_0$ outcomes and decreasing $z_{\text{tr}}$. The Hubble rate is not so far from current bounds, being in overall agreement with (Planck Collaboration et al. 2020) and (Riess et al. 2018), within the 1$\sigma$ confidence level. The error bars, in fact, are here extremely larger for all the coefficients, for the same reasons we quoted before. Even though the corresponding fits are not ruled out, they are not as feasible as for SN+OHD+BAO.

The jerk parameter is not well constrained, because as one fits up to $s_{\text{tr}}$, it dramatically increases with respect to the ones got up to $j_1$. When the hierarchy is up to $j_1$, it seems to converge to $j_0 \simeq 1$ as the Hubble rate decreases today. This appears in agreement with the $\Lambda$CDM expectations. This would indicate that the larger $H_0$ bounds lead to larger jerk constraints, besides $N = 1$ for the DDPE method with OHD. These strong departures disagree with the $\Lambda$CDM predictions, indicating that dark energy may be favored in framing out the universe dynamics, at least with this kind of approximation. However, again, the problem could lie on the small number of points inside the OHD catalog.

Adding SNe Ia and BAO, the situation seems to change. Here, the DDPE fitting procedure seems to be more stable than DHE. The error bars at 1$\sigma$ confidence level are quite similar than DHE, but the matching with the concordance paradigm is more precise. The constraints over $H_0$ do not mostly favor the Planck expectations, since seem to be in agreement with (Riess et al. 2018).

Besides the case $N = 3$ of the DHE method with SN+OHD+BAO data sets, combining more than one data set appears essential to argue more suitable intervals on $z_{\text{tr}}$, albeit the case $N = 2$ with the same three data sets of the DDPE method surprisingly favors larger $z_{\text{tr}}$. The jerk parameter is again quite large, except for the last case we have cited. Comparing the outcomes
Reconstruction of accelerated-decelerated phase

over \( s_{tr} \), we notice it agrees with a negative value, but severely different than the standard cosmological paradigm. Concluding, even in this case dark energy is not excluded, while an overall agreement with the \( \Lambda CDM \) predictions on \( z_{tr} \) still persists.

Summing up, in both the two methods, we infer the main physical conclusions reported below.

A. The most feasible approximations suggest transition redshift compatible intervals than the \( \Lambda CDM \) predictions, though our findings cannot exclude dark energy at 2\( \sigma \) confidence level.

B. The multiplicative and additive degeneracy among coefficients is mainly responsible for the overall error bars estimated in our fits, while using the OHD catalog favors a hierarchy up to \( j_{tr} \).

C. Error bars are not significantly underestimated and at 1\( \sigma \) we find a slight concordance with the standard cosmological model for \( H_0 \) and \( z_{tr} \). On the other hand, at 2\( \sigma \) confidence levels the trend of compatibility with the standard model is no longer valid.

D. The cosmographic bounds on the jerk and snap coefficients strongly disagree, even within the 1\( \sigma \), with the standard cosmological prescriptions at the transition time.

E. Evaluating the regions in which \( z_{eq} \) spans, within the expectations over \( z_{tr} \), leads to exclusions plots that are compatible with the \( \Lambda CDM \) predictions, but do not exclude, as well as for the transition redshift, a dark energy evolution, different from a pure cosmological constant.

From the above summary, we define the most suitable values for \( H_0 \), \( z_{tr} \), \( j_{tr} \) and \( s_{tr} \) below, taking the averages of each hierarchy. The first results below are for hierarchy 1, i.e., lowest \( \mathcal{N} \).

and for hierarchy 2, i.e., largest \( \mathcal{N} \), we have

\begin{align*}
\text{DHE method} \\
\text{SN + OHD + BAO} \\
\bar{z}_{eq} &\in [0.211; 0.406] , \quad (53a) \\
\text{OHD} \\
\bar{z}_{eq} &\in [0.189; 0.562] , \quad (53b) \\
\text{DDPE method} \\
\text{SN + OHD + BAO} \\
\bar{z}_{eq} &\in [0.364; 0.621] , \quad (53c) \\
\text{OHD} \\
\bar{z}_{eq} &\in [0.204; 0.513] . \quad (53d)
\end{align*}

Clearly our approach does not fully exclude the CPL parametrization and the \( \omega CDM \) paradigm. Moreover the hypothesis of a dark fluid is also plausible as clearly shown above. Such models fully degenerate with respect to the concordance paradigm, but we believe, in view of the Occam razor, the \( \Lambda CDM \) would be the statistically favored one. For the sake of completeness, however, more investigations with additional data sets would be essential to heal the degeneracy among models and to fully conclude which model is effectively the best framework predicted by the set of transition and equivalence redshifts.

7 CONCLUSIONS

In this work, we investigated two possible model independent methods to estimate the transition and equivalence redshifts. The first redshift is associated with the time at which the deceleration parameter changes sign, whereas the second is associated with the equivalence of the matter and dark energy densities. The strategy was based on cosmographic Taylor expansions of the Hubble rate and deceleration parameters respectively. In particular, we first expanded the Hubble rate around \( z_{tr} \) up to two hierarchies and then we worked out the same for the deceleration parameter \( q(z) \), again with two hierarchies. Both the expansions have been evaluated up to the snap parameter at the transition.

The corresponding fits have been performed using \( z_{tr} \), expansions only by virtue of the requirement \( q(z_{tr}) = 0 \), that mostly simplified our computation. We thus involved two types of fits based on OHD data first and then combining SNe Ia with OHD and BAO. We then computed the luminosity distance, the angular distance and the Hubble rate expanded with the aforementioned hierarchies. We evaluated a set of constraints taking free all the coefficients, and adopting MCMC simulations, making use of the Metropolis-Hastings algorithm.

Even though this technique led to promising results, we obtained large systematics that have been discussed throughout the text. Even if we did not underestimate the error bars, the concordance paradigm is inside our results within 1\( \sigma \), but deviations seemed to be more evident at 2\( \sigma \) confidence level. However, even if the corresponding \( \Lambda CDM \) predictions looked plausible at 1\( \sigma \) confidence level only, we believe our treatments could be strongly plagued by the approximation we made and by fixing the hierarchy \( \mathcal{N} \). Consequently, the concordance paradigm results were not
excluded, being in a good agreement at least within the limits we imposed. For the sake of completeness, our approach did not fully exclude that dark energy, within the redshift $z < 1$, could slightly evolve. This fact could be subject of refined analyses that will be carried forward, increasing the fit accuracy or simply adopting a high-redshift cosmographic analysis. To stress how much the concordance paradigm matched our results, we also computed exclusion regions in which $\Delta z_{eq}$ is not allowed to span. We showed such regions are again compatible with the standard paradigm, but still not completely able to exclude an evolving dark energy term that degenerates with the $\Lambda$CDM paradigm. Finally, we noticed that fixing suitable intervals of $(\Delta z_{\text{tr}}, \Delta z_{eq})$ implies we can use them as discriminators for dark energy models. In other words, we can improve our analyses, fixing tighter intervals over $(\Delta z_{\text{tr}}, \Delta z_{eq})$, and investigating de facto whether dark energy is in the form of a pure cosmological constant or weakly evolve.

Thus, future fits are expected to improve the quality of current ones, by adding further orders beyond what is presented here. We therefore expect to improve our treatment by including additional catalogs of data and to confront specific dark energy models with the DHE and DDPE methods.

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DATA AVAILABILITY
SN and BAO data are available in Luongo & Muccino (2021) and Luongo & Muccino (2020b) respectively whereas OHD data are available in Amati et al. (2019).

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Table 1. Numerical values take from uncorrelated BAO points with the corresponding references.

| Survey     | z        | $d_A$       | Ref.                      |
|------------|----------|-------------|---------------------------|
| 6dFGS      | 0.106    | 0.3360 ± 0.0150 | Beutler et al. (2011)     |
| SDSS-DR7   | 0.15     | 0.2239 ± 0.0084 | Ross et al. (2015)        |
| SDSS       | 0.20     | 0.1905 ± 0.0061 | Percival et al. (2010)    |
| SDSS-III   | 0.32     | 0.1181 ± 0.0023 | Anderson et al. (2014)    |
| SDSS       | 0.35     | 0.1097 ± 0.0036 | Percival et al. (2010)    |
| SDSS-III   | 0.57     | 0.0726 ± 0.0007 | Anderson et al. (2014)    |
| SDSS-III   | 2.34     | 0.0320 ± 0.0016 | Delubac et al. (2015)     |
| SDSS-III   | 2.36     | 0.0329 ± 0.0012 | Font-Ribera et al. (2014) |

Appendix A: COSMOGRAPHIC SERIES

The luminosity distance is expanded around $z \approx 0$ as

$$d_L^{(3)}(q_0, j_0, z) = \frac{H_0}{50}\left[z + z^2\left(1 - \frac{q_0}{2}\right) + z^3\left(-\frac{1}{6} + \frac{j_0}{6} + \frac{q_0^2}{2}\right)\right].$$

(1a)

$$d_L^{(4)}(q_0, j_0, z) = d_L(q_0, j_0, z)^{(3)} + z^4\left(1 + \frac{5j_0}{24} - \frac{q_0}{12} + \frac{5j_0q_0}{24} - \frac{5q_0^2}{8} - \frac{5q_0^3}{8} + \frac{s_0}{24}\right).$$

(1b)

The Hubble rate is expanded around $z \approx 0$ by

$$H^{(2)}(q_0, j_0, z) = H_0\left[1 + (1 + q_0)z + \frac{z^2}{2}(j_0 - q_0^2)\right].$$

(2a)

$$H^{(3)}(q_0, j_0, s_0, z) = H^{(2)}(q_0, j_0, z) + z^3\left(3q_0^2 + 3q_0^3 - j_0(3 + 4q_0) - s_0\right).$$

(2b)

The angular distance expansions around $z \approx 0$ read

$$d_A^{(2)}(q_0, j_0, z) = \frac{100r_s}{c^2z}\times$$

$$\times \left[1 + \frac{2}{3}(1 + q_0)z + \frac{1}{36}(-3 + 10j_0 - 6q_0 - 13q_0^2)z^2\right].$$

(3a)

$$d_A^{(3)}(q_0, j_0, s_0, z) = d_A^{(2)}(q_0, j_0, z) +$$

$$z^3\frac{324}{13 - 111j_0 + 39q_0 - 138j_0q_0 + 150q_0^2 + 124q_0^3 - 27s_0).$$

Appendix B: BAO AND OHD DATA POINTS

In this appendix, we list the data points used in the paper, besides SNe Ia. In particular, below we report the BAO and OHD catalogs.

| $z$ | $H \pm \sigma_H$ |
|-----|--------------------|
| 0.0708 | 69.00 ± 19.68 |
| 0.09  | 69.00 ± 12.00 |
| 0.012 | 68.8 ± 26.21 |
| 0.17  | 83.0 ± 8.0  |
| 0.179 | 75.0 ± 4.0  |
| 0.199 | 75.0 ± 5.0  |
| 0.20  | 72.9 ± 29.6 |
| 0.27  | 77.0 ± 14.0 |
| 0.28  | 88.8 ± 36.6 |
| 0.35  | 82.1 ± 4.85 |
| 0.352 | 83.0 ± 14.0 |
| 0.3802 | 83.0 ± 13.5 |
| 0.4   | 95.0 ± 17.0 |
| 0.4004 | 77.0 ± 10.2 |
| 0.4247 | 87.1 ± 11.2 |
| 0.4497 | 92.8 ± 12.9 |
| 0.4783 | 80.9 ± 9.0  |
| 0.48  | 97.0 ± 62.0 |
| 0.593 | 104.0 ± 13.0 |
| 0.68  | 92.0 ± 8.0  |
| 0.781 | 105.0 ± 12.0 |
| 0.875 | 125.0 ± 17.0 |
| 0.88  | 90.0 ± 40.0 |
| 0.9   | 117.0 ± 23.0 |
| 1.037 | 154.0 ± 20.0 |
| 1.3   | 168.0 ± 17.0 |
| 1.363 | 160.0 ± 33.6 |
| 1.43  | 177.0 ± 18.0 |
| 1.53  | 140.0 ± 14.0 |
| 1.75  | 202.0 ± 40.0 |
| 1.895 | 186.5 ± 50.4 |

Appendix C: CONTOUR PLOTS

In this section, we report the contours got from our analyses.

**DHE method**

The contours using the OHD data are got in Figs. 1 and 2.

The contours using SNe Ia with OHD and BAO catalogs are portrayed in Figs. 3 and 4.

**DDPE method**

The contours evaluated using OHD are displayed in Figs. 5 and 6.

The contours evaluated using SNe Ia and OHD with BAO data sets are reported in Figs. 7 and 8.
Figure 1. Contours for the DHE method using the OHD data set and the second order expansion of the Hubble parameter.

Figure 2. Contours for the DHE method using the OHD data set and the third order expansion of the Hubble parameter.

Figure 3. Contours for the DHE method using the SN+OHD+BAO data sets and the second order expansion of the Hubble parameter.
Figure 4. Contours for the DHE method using the SN+OHD+BAO data sets and the third order expansion of the Hubble parameter.

Figure 5. Contours for the DDPE method using the OHD data set and the first order expansion of the deceleration parameter.
Figure 6. Contours for the DDPE method using the OHD data set and the second order expansion of the deceleration parameter.

Figure 7. Contours for the DDPE method using the SN+OHD+BAO data sets and the first order expansion of the deceleration parameter.
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Figure 8. Contours for the DDPE method using the SN+OHD+BAO data sets and the second order expansion of the deceleration parameter.