Dissipative generators, divisible dynamical maps and Kadison-Schwarz inequality

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We introduce a concept of Kadison-Schwarz divisible dynamical maps. It turns out that it is a natural generalization of the well known CP-divisibility which characterizes quantum Markovian evolution. It is proved that Kadison-Schwarz divisible maps are fully characterized in terms of time-local dissipative generators. Simple qubit evolution illustrates the concept.

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I. INTRODUCTION

Evolution of a quantum system is represented by a family of quantum channels $\Lambda_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ ($t \geq 0$) such that $\Lambda_0 = \text{id}$ (identity map). In what follows $\mathcal{B}(\mathcal{H})$ denotes an algebra of bounded linear operators acting in the Hilbert space $\mathcal{H}$ (actually, in this paper we consider only finite dimensional $\mathcal{H}$). Such family is usually called a dynamical map. For isolated system the dynamical map has a well known structure $\Lambda_t(\rho) = U_t \rho U_t^\dagger$, where $U_t = e^{-it\hat{H}}$ and $\hat{H}$ denotes the Hamiltonian of the (closed) system (we keep $\hbar = 1$). For an open quantum system one often considers a dynamical semigroup governed by the Markovian master equation \cite{1, 2}

$$\dot{\Lambda}_t = \mathcal{L}\Lambda_t,$$

where the generator $\mathcal{L} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is given by the celebrated Gorini-Kossakowski-Sudarshan-Lindblad formula \cite{3, 4}

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_k \gamma_k \left( V_k \rho V_k^\dagger - \frac{1}{2} [V_k^\dagger V_k, \rho] \right),$$

with positive rates $\gamma_k > 0$. This structure guarantees that the solution $\Lambda_t = e^{t\mathcal{L}}$ defines a legitimate dynamical map – completely positive and trace preserving (CPTP). To go beyond dynamical semigroup one considers master equation \cite{1} with time dependent generator $\mathcal{L}_t$. It has exactly the same form as in \cite{2} with time dependent $H(t)$, $V_k(t)$ and $\gamma_k(t)$. The formal solution for $\Lambda_t$ reads

$$\Lambda_t = \mathcal{T}_- \exp \left( \int_0^t \mathcal{L}_s \, ds \right),$$

where $\mathcal{T}_-$ stands for chronological operator. In this case, however, we do not know necessary and sufficient conditions for $\mathcal{L}_t$ which guarantee that \cite{5} is CPTP for all $t > 0$. Time dependent generators are recently analyzed in connection to quantum non-Markovian evolution \cite{6-8}. Recall, that a dynamical map $\Lambda_t$ is called divisible if

$$\Lambda_t = V_{t,s} \Lambda_s, \quad t > s,$$

with a family of ‘propagators’ $V_{t,s} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$. For invertible maps such propagator always exists and it is given by $V_{t,s} = \Lambda_t^{-1} \Lambda_s$. Maps which are not invertible require special treatment \cite{13, 14} (for a recent review of various concepts of divisibility for quantum channels and dynamical maps see \cite{15}). Now, being divisible one calls $\Lambda_t$ P-divisible if $V_{t,s}$ is positive and trace-preserving, and CP-divisible if $V_{t,s}$ is CPTP. Following \cite{9} one calls the evolution represented by $\Lambda_t$, that is, all time-dependent rates satisfy $\gamma_k(t) \geq 0$. Authors of \cite{10} proposed another approach based on distinguishability of quantum states: $\Lambda_t$ is Markovian if for any pair of initial states $\rho$ and $\sigma$ one has

$$\frac{d}{dt} \| \Lambda_t(\rho) - \Lambda_t(\sigma) \|_1 \leq 0,$$

where $\|X\|_1 = \text{Tr}\sqrt{X^\dagger X}$. Interestingly, for invertible maps P-divisibility is equivalent to

$$\frac{d}{dt} \| \Lambda_t(X) \|_1 \leq 0,$$

for all $X^\dagger = X$. Hence, CP-divisibility implies P-divisibility and this implies BLP condition \cite{5}.

In this paper we introduce another notion of divisibility based on the Kadison-Schwarz (KS) inequality. Let us recall that a linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is trace-preserving iff its dual map $\Phi^\dagger$ is unital. $\Phi^\dagger$ is defined by $\text{Tr}(\Phi(X)Y) = \text{Tr}(\Phi^\dagger(Y)X)$ for all $X, Y \in \mathcal{B}(\mathcal{H})$. Hence, if $\Phi(X) = \sum_k \lambda_k A_k X B_k^\dagger$, then $\Phi^\dagger(X) = \sum_k \lambda_k^* B_k^\dagger X A_k$. In particular, if $\Lambda_t(\rho) = U_t \rho U_t^\dagger$ represents Schrödinger evolution of the density operator, then $\Lambda_t^\dagger(X) = U_t^\dagger X U_t$ represents Heisenberg evolution of the observable $X$. Introducing a completely positive map $\Phi(\rho) = \sum_k \gamma_k V_k \rho V_k^\dagger$, the GKSL generator \cite{2} can be rewritten in a compact form as follows
\[ \mathcal{L}(\rho) = -i[H, \rho] + \Phi(X) - \frac{1}{2} (\Phi^2(1), X). \] (7)

Now, a unital map \( \Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \) satisfies Kadison-Schwarz (KS) inequality \([17, 20]\) if

\[ \Phi(XX^\dagger) \geq \Phi(X)\Phi(X^\dagger), \] (8)

for all \( X \in \mathcal{B}(\mathcal{H}) \). We say that dynamical map \( \Lambda_t \) is KS-divisible if the propagator \( V_t^\sharp \) satisfies (8). In this paper we analyze this concept and relate it to P- and CP-divisibility. Interestingly, for invertible maps KS-divisibility is fully controlled by the property of dissipativity [4]. Finally, we illustrate KS-divisibility by simple example of qubit evolution.

**II. CADISON-SCHWARZ MAPS**

Celebrated Cauchy-Schwarz inequality found a lot of applications in mathematics, physics and engineering. It states that for any \( x, y \in \mathcal{H} \) one has \( |\langle x|y \rangle|^2 \leq \langle x|x \rangle \langle y|y \rangle \). Kadison found elegant generalization of this inequality for linear maps in operator algebras \([17, 20]\); a linear map \( \Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \) is positive if for any \( A \geq 0 \) one has \( \Phi(A) \geq 0 \). Equivalently, for any \( X \in \mathcal{B}(\mathcal{H}) \) one has \( \Phi(X^\dagger X) \geq 0 \). A linear map is unital if \( \Phi(1) = 1 \), with \( 1 \) being an identity operator in \( \mathcal{H} \). Now, a unital map \( \Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \) satisfies Kadison-Schwarz (KS) inequality \([17, 19, 20]\) if

\[ \Phi(XX^\dagger) \geq \Phi(X)\Phi(X^\dagger), \] (9)

for all \( X \in \mathcal{B}(\mathcal{H}) \). It immediately follows from \([9]\) that Kadison-Schwarz map is positive. However, the converse needs not be true. An example of a positive unital map which is not KS is provided by the transposition map. Indeed, for \( d = 2 \) taking \( X = |1\rangle\langle 2| \) one finds

\[ T(X^\dagger X) = |1\rangle\langle 1|, \quad T(X^\dagger)T(X) = |2\rangle\langle 2|, \]

and hence \([9]\) is violated. Interestingly, any unital positive map satisfies \([9]\) but for normal operators (i.e. \( X^\dagger X = XX^\dagger \)). Denote by \( M_k(\mathbb{C}) \) linear space of \( k \times k \) complex matrices. Recall, that a linear map \( \Phi \) is \( k \)-positive if the extended map

\[ \text{id}_k : M_k(\mathbb{C}) \otimes \mathcal{B}(\mathcal{H}) \rightarrow M_k(\mathbb{C}) \otimes \mathcal{B}(\mathcal{H}) \]

is positive (\( \text{id}_k \) denotes an identity map). A map which is \( k \)-positive for \( k = 1, 2, \ldots \) is called completely positive (CP). If the dimension of \( \mathcal{H} \) is \( d^2 \), then complete positivity is equivalent to \( d \)-positivity. Among unital maps one has the following hierarchy

\[ \text{CP} = P_d \subset P_{d-1} \subset \ldots \subset P_2 \subset \text{KS} \subset P_1, \] (10)

where \( P_k \) denotes \( k \)-positive unital maps.

If \( \Phi \) and \( \Psi \) are KS, then \( \Phi \Psi \) is KS as well. Moreover, the convex combination \( \lambda \Phi + (1 - \lambda) \Psi \) is again KS \([21]\).

Actually, the concept of unital KS maps may be generalized for maps which are not unital. Consider a map \( \Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \) such that \( \Phi(1) > 0 \), and define

\[ \Psi(X) = \Phi(1)^{-\frac{3}{4}} \Phi(X) \Phi(1)^{-\frac{1}{4}}. \] (11)

Clearly, one has \( \Psi(1) = 1 \). Now, if \( \Psi \) satisfies KS condition \([6]\), then

\[ \Phi(XX^\dagger) \geq \Phi(X)\Phi(1)^{-\frac{3}{4}} \Phi(X^\dagger). \] (12)

**Example 1** Consider a qubit map

\[ \Phi = p_1 \Phi_1 + p_2 \Phi_2 + p_3 \Phi_3, \] (13)

where \( p_1 + p_2 + p_3 = 1 \), and

\[ \Phi_k(X) = \frac{1}{2} (\sigma_k X \sigma_k + X), \]

with \( \sigma_k \) being Pauli matrices. Note, that

\[ \Phi_k(X) = \sum_{\mu=1}^{2} \rho_k^{(\mu)} X \rho_k^{(\mu)}, \]

and \( \rho_k^{(\mu)} \) are eigen-projectors of \( \sigma_k \). It is clear that \( \Phi \) is unital. It is CP iff \( p_k \geq 0 \). Note, that

\[ \Phi(\sigma_k) = p_k \sigma_k, \]

and hence one easily finds that \( \Phi \) is positive iff \( |p_k| \leq 1 \). For example the map

\[ \Phi = \Phi_1 + \Phi_2 - \Phi_3, \]

is positive (but of course not CP). This map is not KS. Indeed, taking \( X = |1\rangle\langle 2| \) one gets

\[ \Phi(XX^\dagger) = |1\rangle\langle 1|, \quad \Phi(X^\dagger)\Phi(X) = |2\rangle\langle 2|. \]

It is shown in the Appendix that if

\[ p_1^2 + p_2^2 + p_3^2 \leq 1 + 2p_1p_2p_3, \] (14)

then \( \Phi \) is KS (cf. Figure 1). Interestingly, the constraint \([14]\) provides good approximation for a set of CP maps (cf. Figure 1). For example all three vertices of the inscribed triangle satisfy \([14]\) with equality.
FIG. 1: Left panel: the convex body satisfying (14). Middle panel: the intersection with the plane \( p_1 + p_2 + p_3 = 1 \). The yellow triangle with vertices \((1, 1, -1), (1, -1, 1)\) and \((-1, 1, 1)\) corresponds to positive maps. Right panel: inscribed triangle of CP maps with vertices \((1, 0, 0), (0, 1, 0)\) and \((0, 0, 1)\).

### III. KS-DIVISIBILITY

It was already observed by Lindblad \([4]\) that \( \Lambda_t^\# = e^{tL^\#} \) is KS iff \( L^\# \) is dissipative, that is,

\[
L^t(X^\dagger X) \geq L^t(X^\dagger)X + X^\dagger L^t(X),
\]

for all \( X \in \mathcal{B}(\mathcal{H}) \). Any dissipative generator has the following structure

\[
L^t(X) = i[H, X] + \Phi(X) - \frac{1}{2}\{\Phi(\mathbb{1}), X\},
\]

where the map \( \Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) satisfies the following condition

\[
\Phi(X^\dagger X) \geq \Phi(X^\dagger)X + X^\dagger \Phi(X) - X^\dagger \Phi(\mathbb{1})X,
\]

that is, it has exactly the same structure as \([7]\) but the CP map \( \Phi \) is replaced by the map satisfying \([17]\) (note that \([7]\) is represented in the Schrödinger picture whereas \([16]\) in the Heisenberg picture). Actually condition \([17]\) is weaker than generalized KS condition \([12]\). One has

**Proposition 1** Any \( \Phi \) satisfying \([12]\) satisfies \([17]\). For the proof see Appendix.

Consider now a dynamical map \( \Lambda_t \) satisfying time-local master equation \( \dot{\Lambda}_t = L_t \Lambda_t \), that is, \( \Lambda_t \) is represented as in \([8]\).

**Theorem 1** If \( \Lambda_t \) is invertible, then

- it is KS-divisible if and only if \( L^t_\# \) is dissipative,
- it is CP-divisible if and only if \( L^t_\# \) is completely dissipative

for all \( t \geq 0 \).

Proof: the existence of \( V_{t,s} \) follows from invertibility of \( \Lambda_t \), that is, \( V_{t,s} = \Lambda_t \Lambda_s^{-1} \). One has

\[
V_{t,s}^\# = \mathcal{T}_- \exp \left( \int_s^t L^\#_\tau d\tau \right),
\]

where now \( \mathcal{T}_- \) stands for anti-chronological operator. Now, if \( L^\#_t \) is dissipative, then \( V_{t,s}^\# \) is unital KS. If \( V_{t,s}^\# \) is KS for any \( t > s \), then for \( \epsilon \to 0^+ \) one has \( V_{t+\epsilon,t}^\# \to e^{\epsilon L^\#_t} \) which implies that \( L^\#_t \) is dissipative. □

### IV. EXAMPLES: KS-DIVISIBLE QUBIT DYNAMICAL MAPS

In this section we consider several simple examples of qubit evolution. In this case the hierarchy \([10]\) reduces to

\[
\text{CP} = \text{P}_2 \subset \text{KS} \subset \text{P}_1,
\]

that is, KS maps interpolate between CP and positive maps.

**Example 2 (Qubit dephasing)** For a qubit dephasing governed by

\[
L_t(\rho) = \gamma(t)(\sigma_3 \rho \sigma_3 - \rho),
\]

\( P_-, \text{KS}-, \) and CP-divisibility coincide and they are equivalent to \( \gamma(t) \geq 0 \). Note, that in this case one has \( L^\#_t = L_t \).

**Example 3 (Amplitude damping channel)** The evolution of amplitude-damped qubit is governed by a single function \( G(t) \)

\[
\Lambda_t(\rho) = \begin{pmatrix} \rho_{11} + (1 - |G(t)|^2)\rho_{22} & G(t)\rho_{12} \\ \rho^{*}_{21}G^*(t) & |G(t)|^2\rho_{22} \end{pmatrix},
\]

where
where the function $G(t)$ depends on the form of the reservoir spectral density $J(\omega)$ \[1\]. This evolution is generated by the following time-local generator

$$\mathcal{L}_t(\rho) = -i[H(t), \rho] + \gamma(t)(\sigma_- \rho \sigma_+ - \frac{1}{2} (\sigma_+ \sigma_-, \rho)),$$

where $\sigma_{\pm}$ are the spin lowering and rising operators, $H(t) = \frac{\omega(t)}{2} \sigma_+ \sigma_-$, together with $\omega(t) = -2\text{Im} \frac{\dot{G}(t)}{G(t)}$, and $\gamma(t) = -2\text{Re} \frac{\dot{G}(t)}{G(t)}$. Again in this case $P$-, $KS$-, and $CP$-divisibility coincide and they are equivalent to $\gamma(t) \geq 0$.

**Example 4 (Pauli channel)** Consider the qubit evolution governed by the following time-local generator

$$\mathcal{L}_t(\rho) = \frac{1}{2} \sum_{k=1}^{3} \gamma_k(t)(\sigma_k \rho \sigma_k - \rho),$$

which leads to the following dynamical map (time-dependent Pauli channel):

$$\Lambda_t(\rho) = \sum_{\alpha=1}^{3} p_{\alpha}(t) \sigma_{\alpha} \rho \sigma_{\alpha},$$

where

$$
\begin{pmatrix}
    p_0(t) \\
    p_1(t) \\
    p_2(t) \\
    p_3(t)
\end{pmatrix} = \frac{1}{4} 
\begin{pmatrix}
    1 & 1 & 1 & 1 \\
    1 & 1 & -1 & -1 \\
    1 & -1 & 1 & -1 \\
    1 & -1 & -1 & 1
\end{pmatrix} 
\begin{pmatrix}
    \lambda_1(t) \\
    \lambda_2(t) \\
    \lambda_3(t)
\end{pmatrix},
$$

and

$$\lambda_i(t) = e^{-\Gamma_i(t)} + \text{cyc.permutations},$$

with $\Gamma_i(t) = \int_0^t \gamma_i(\tau) d\tau$. It was shown \[22\] that $P$-divisibility is equivalent to the following conditions:

$$\gamma_i(t) + \gamma_j(t) \geq 0, \quad i \neq j. \quad (24)$$

Now, it is shown in the Appendix that $KS$-divisibility is equivalent to the following the stronger conditions:

$$\gamma_i(t) + 2\gamma_j(t) \geq 0, \quad i \neq j. \quad (25)$$

In \[22\] authors considered so called eternally non-Markovian evolution corresponding to

$$\gamma_1(t) = \gamma_2(t) = 1, \quad \gamma_3(t) = \tanh t. \quad (26)$$

It gives

$$p_0(t) = \frac{1}{2}(1 + e^{-2t}),$$

$$p_1(t) = p_2(t) = \frac{1}{4}(1 - e^{-2t}),$$

$$p_3(t) = 0.$$ 

Clearly, the map $\Lambda_t$ is $CPTP$ and $P$-divisible. Now, let us consider a simple modification

$$\gamma_1(t) = \gamma_2(t) = 1, \quad \gamma_3(t) = -\frac{1}{2}\tanh t. \quad (27)$$

It gives

$$p_0(t) = \frac{1}{4}(1 + 2e^{-t}\sqrt{\cosh t + e^{-2t}}),$$

$$p_1(t) = p_2(t) = \frac{1}{4}(1 - e^{-2t}),$$

$$p_3(t) = \frac{1}{4}(1 - 2e^{-t}\sqrt{\cosh t + e^{-2t}}).$$

Again, the map $\Lambda_t$ is $CPTP$ due to the fact that $p_\alpha(t) \geq 0$. Indeed, one finds

$$p_3(t) = \frac{e^{-t}}{2}(\cosh t - \sqrt{\cosh t}) \geq 0,$$

due to $\cosh t \geq 1$. Hence, it provides an example of $KS$-divisible qubit evolution since conditions \[25\] are trivially satisfied. Clearly, the evolution governed by \[26\] is $P$-divisible but not $KS$-divisible.

**V. CONCLUSIONS**

In this paper we introduced the concept of $KS$-divisibility which is based on the Kadison-Schwarz inequality \[9\]. This concept interpolates between $CP$-divisibility (often assumed as a definition of Markovianity \[9\]) and $P$-divisibility (which is closely related to the well known notion of information flow \[10\]). Any $CP$-divisible map in $KS$-divisible, and any $KS$-divisible map is $P$-divisible and hence does not display information backflow. For dynamical maps satisfying time-local master equation with time-dependent generator $\mathcal{L}_t$ $KS$-divisibility is fully controlled by the property of the generator $\mathcal{L}_t^f$ (Heisenberg picture), that is, the maps is $KS$-divisible if and only if $\mathcal{L}_t^f$ is dissipative. This concept is illustrated by several examples of qubit evolution. Interestingly for the evolution governed by well known generator $\mathcal{L}_t(\rho) = \frac{1}{2} \sum_k \gamma_k(t)(\sigma_k \rho \sigma_k - \rho)$ we found necessary and sufficient conditions for dissipativity: $\gamma_i(t) + 2\gamma_j(t) \geq 0$ for $i \neq j$. I shows that so called eternally non-Markovian evolution proposed in \[22\] is $P$-divisible but not $KS$-divisible. However, we proposed a simple modification which is again eternally non-Markovian (one of the rate is always negative) but displays $KS$-divisibility. Actually, condition $\gamma_i(t) + 2\gamma_j(t) \geq 0$ for $i \neq j$ has a clear physical interpretation: note that the initial Bloch vector $x = (x_1, x_2, x_3)$ evolves according to

$$x(t) = (\lambda_1(t)x_1, \lambda_2(t)x_2, \lambda_3(t)x_3). \quad (28)$$
where $\lambda_k(t) = \exp(-\int_0^t 1/T_k(\tau)d\tau)$, and the local relaxation times read

$$T_i(t) = \frac{1}{\gamma_j(t) + \gamma_k(t)},$$

for mutually different $i, j, k$. Now, assuming that $\gamma_3(t) < 0$, one finds the following constraints

$$T_1(t), T_2(t) \leq \frac{1}{|\gamma_3(t)|}, \quad T_3(t) \leq \frac{1}{4|\gamma_3(t)|}.$$  

Note, that if the map is only P-divisible one has $T_3(t) \leq \frac{1}{2|\gamma_3(t)|}$ and no additional constraints for $T_1(t)$ and $T_2(t)$. This shows that these two concepts of divisibility have different physical flavour.

It would be interesting to investigate KS-divisibility for higher dimensional systems.

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**Appendix A: Condition [14] for KS map**

We note that every matrix $X \in M_2(\mathbb{C})$ can be written in this basis as $X = w_0 \mathbb{I} + w \cdot \sigma$ with $w_0 \in \mathbb{C}, w = (w_1, w_2, w_3) \in \mathbb{C}^3$, here by $w \cdot \sigma$ we mean the following

$$w \cdot \sigma = w_1 \sigma_1 + w_2 \sigma_2 + w_3 \sigma_3.$$  

One finds

$$\Phi(w_0 \mathbb{I} + w \cdot \sigma) = w_0 \mathbb{I} + T w \cdot \sigma,$$

where the $3 \times 3$ matrix $T_{ij}$ reads $T_{ij} = p_i \delta_{ij}$. Taking into account a result of [21] the KS conditions yields

$$\left(A|w_2 \bar{w}_3 - w_3 \bar{w}_2|^2 + B|w_1 \bar{w}_3 - w_3 \bar{w}_1|^2 + C|w_1 \bar{w}_2 - w_2 \bar{w}_1|^2\right)^{1/2} \leq \alpha|w_1|^2 + \beta|w_2|^2 + \gamma|w_3|^2$$  

(A2)

where

$$\alpha = |1 - p_1^2|, \quad \beta = |1 - p_2^2|, \quad \gamma = |1 - p_3^2|,$$

$$A = |p_1 - p_2 p_3|^2, \quad B = |p_2 - p_1 p_3|^2, \quad C = |p_3 - p_1 p_2|^2.$$

(A3)  

(A4)  

Let us assume $||w|| = 1$. Hence, taking into account $w_1 = r_1 e^{i\alpha_1}, w_2 = r_2 e^{i\alpha_2}, w_3 = r_3 e^{i\alpha_3}$, the inequality (A2) reduces to

$$2\left(A r_1^2 r_3^2 \sin^2 \theta_1 + B r_1^2 r_2^2 \sin^2 \theta_2 + C r_1^2 r_2^2 \sin^2 \theta_3\right)^{1/2} \leq \alpha r_1^2 + \beta r_2^2 + \gamma r_3^2$$  

(A5)

where $\theta_1 + \theta_2 + \theta_3 = 2\pi$. Clearly, the last inequality is satisfied if one has

$$2\left(A r_1^2 r_3^2 + B r_1^2 r_2^2 + C r_1^2 r_2^2\right)^{1/2} \leq \alpha r_1^2 + \beta r_2^2 + \gamma r_3^2$$  

(A6)

Introducing $x := r_1^2, y := r_2^2, z := r_3^2$, the last one is equivalent to

$$2\left(A y z + B x z + C x y\right)^{1/2} \leq \alpha x + \beta y + \gamma z$$  

(A7)

for all $x + y + z = 1, x, y, z \geq 0$. Let us introduce the following function

$$f(x, y) = 2\left(A y (1 - x - y) + B x (1 - x - y) + C x y\right)^{1/2} - \alpha x - \beta y - \gamma (1 - x - y),$$

where the arguments $(x, y)$ satisfy $0 \leq x + y \leq 1$. One can check that this function reaches its maximum on the boundary of $0 \leq x + y \leq 1$. Hence, it is enough to study the following function

$$g(y) = f(0, y) = 2(A y (1 - y))^{1/2} - \beta y - \gamma (1 - y)$$
on the interval \([0, 1]\). One shows that the maximum of \(g(y)\) on the interval \([0, 1]\) is less or equal than 0 if and only if one has \(A \leq \beta \gamma\). Similarly, one finds the other two conditions \(B \leq \alpha \gamma\) and \(C \leq \alpha \beta\). Hence, if

\[
A \leq \beta \gamma, \quad B \leq \alpha \gamma, \quad C \leq \alpha \beta
\]  
(A8)

then \(\Psi\) is KS-operator. Now, taking into account (A3) and \(|q_k| \leq 1\), the last conditions (A8) reduce to

\[
p_1^2 + p_2^2 + p_3^2 \leq 1 + 2p_1p_2p_3.
\]

**Appendix B: proof of Proposition 1**

Denoting \(Y := \Phi(1 I)\) one has

\[
(Y^{-1/2}\Phi(X) - Y^{1/2}X)\dagger(Y^{-1/2}\Phi(X) - Y^{1/2}X) \geq 0
\]

which implies

\[
\Phi(X)\dagger Y^{-1}\Phi(X) \geq \Phi(X)\dagger X + X\dagger\Phi(X) - X\dagger YX.
\]

This together with KS-condition yields the assertion.

**Appendix C: KS Divisibility**

In this section, we show that the generator \(L^\#\) is dissipative which means that the mapping \(\Phi\) in (16) satisfies (17).

\[
L^\#_t(X) = \frac{1}{2} \sum_{k=1}^{3} \gamma_k(t)(\sigma_k X\sigma_k - X) = \Phi_t(X) - \{\Phi_t(1 I), X\},
\]  
(C1)

where \(\Phi_t(X) = \frac{1}{2} \sum_{k=1}^{3} \gamma_k(t)\sigma_k X\sigma_k\). This genetor gives rise to KS-divisible evolution iff

\[
\Phi_t(X\dagger X) \geq \Phi_t(X\dagger)X + X\dagger\Phi_t(X) - X\dagger X\dagger YX.
\]

To simplify notation we skip time-dependence. Note, that \(\Phi(1 I) = \gamma I\), with

\[
\gamma = \frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_3).
\]  
(C3)

Let us observe that the condition (17) can be rewritten as follows:

\[
\Phi(X\dagger X) - \Phi(X)\dagger X - X\dagger\Phi(X) + \gamma X\dagger X \geq 0
\]

So, we have

\[
\Phi(X\dagger X) - \gamma^{-1}\Phi(X\dagger)\Phi(X) + \gamma^{-1}\Phi(X\dagger)\Phi(X) - \Phi(X\dagger)X - X\dagger\Phi(X) + \gamma X\dagger X \geq 0,
\]

and hence

\[
\Phi(X\dagger X) - \gamma^{-1}\Phi(X\dagger)\Phi(X) + [\gamma^{-1/2}\Phi(X) - \gamma^{1/2}X]\dagger[\gamma^{-1/2}\Phi(X) - \gamma^{1/2}X] \geq 0.
\]  
(C4)

Now, to simply analysis without losing generality we put \(\gamma = 1\). We show that if condition (25) is satisfied for any \(t \geq 0\), then \(\Phi\) satisfies

\[
\Phi(X\dagger X) - \Phi(X\dagger)\Phi(X) + [\Phi(X) - X]\dagger[\Phi(X) - X] \geq 0.
\]  
(C5)

One has for \(X = w_0 I + w \cdot \sigma\)
\[ \Phi(X) = w_0 \mathbb{1} + T \mathbf{w} \cdot \sigma, \quad \Phi(X^\dagger) = \overline{w_0} \mathbb{1} + (T \mathbf{w}) \cdot \sigma, \]  
(C6)

where the 3 \times 3 matrix \( T_{ij} \) reads \( T_{ij} = \lambda_i \delta_{ij} \), and the eigenvalues of the map read

\[ \lambda_1 = \frac{1}{2}(\gamma_1 - \gamma_2 - \gamma_3), \quad \lambda_2 = \frac{1}{2}(-\gamma_1 + \gamma_2 - \gamma_3), \quad \lambda_3 = \frac{1}{2}(-\gamma_1 - \gamma_2 + \gamma_3). \]  
(C7)

One finds

\[ X^\dagger X = (|w_0|^2 + \|\mathbf{w}\|^2) \mathbb{1} + (w_0 \overline{w} + \overline{w_0} \mathbf{w} + i \mathbf{w} \times \mathbf{w}) \cdot \sigma, \]  
(C8)

\[ \Phi(X^\dagger X) = (|w_0|^2 + \|\mathbf{w}\|^2) \mathbb{1} + (w_0 T \mathbf{w} + \overline{w_0} T \mathbf{w} + iT(\mathbf{w} \times \mathbf{w})) \cdot \sigma \]  
(C9)

\[ \Phi(X^\dagger)\Phi(X) = (|w_0|^2 + \|T \mathbf{w}\|^2) \mathbb{1} + (w_0 \overline{T \mathbf{w}} + \overline{w_0} T \mathbf{w} + iT \mathbf{w} \times T \mathbf{w}) \cdot \sigma \]  
(C10)

where \( \mathbf{a} \times \mathbf{b} \) stands for the vector product of 3-dimensional vectors. One obtains

\[ \Phi(X^\dagger X) - \Phi(X^\dagger)\Phi(X) = (|w_0|^2 - \|T \mathbf{w}\|^2) \mathbb{1} + i(T(\mathbf{w} \times \mathbf{w}) - T \mathbf{w} \times T \mathbf{w}) \cdot \sigma, \]

and

\[ (\Phi(X) - X)^\dagger (\Phi(X) - X) = ((T \mathbf{w} - \mathbf{w}) \cdot \sigma)((T \mathbf{w} - \mathbf{w}) \cdot \sigma) = \|T \mathbf{w} - \mathbf{w}\|^2 \mathbb{1} + i[T \mathbf{w} - \mathbf{w}, T \mathbf{w} - \mathbf{w}] \cdot \sigma. \]

Hence, we find

\[ \Phi(X^\dagger X) - \Phi(X^\dagger)\Phi(X) + [\Phi(X) - X]^\dagger [\Phi(X) - X] = a \mathbb{1} + \mathbf{b} \cdot \mathbf{s}, \]  
(C11)

with

\[ a = 2\|\mathbf{w}\|^2 - (\langle \mathbf{T w}, \overline{\mathbf{w}} \rangle + \langle \mathbf{w}, \mathbf{T w} \rangle) \]  
(C12)

\[ \mathbf{b} = T(\overline{\mathbf{w}} \times \mathbf{w}) + \overline{\mathbf{w}} \times \mathbf{w} - (T \overline{\mathbf{w}} \times \mathbf{w} + \overline{\mathbf{w}} \times T \mathbf{w}). \]  
(C13)

Now, (C5) is equivalent to

\begin{itemize}
  \item \( a \geq 0 \)
  \item \( a \geq |\mathbf{b}|. \)
\end{itemize}

Using

\[ (T \mathbf{w}, \overline{\mathbf{w}}) = (\mathbf{w}, T \mathbf{w}) = \lambda_1 |w_1|^2 + \lambda_2 |w_2|^2 + \lambda_3 |w_3|^2 \]

one finds

\[ a = 2\left[(1 - \lambda_1)|w_1|^2 + (1 - \lambda_2)|w_2|^2 + (1 - \lambda_3)|w_3|^2\right], \]

and hence taking into account that \( \gamma_1 + \gamma_2 + \gamma_3 = 2 \), one finds

\[ (\gamma_2 + \gamma_3)|w_1|^2 + (\gamma_3 + \gamma_1)|w_2|^2 + (\gamma_1 + \gamma_2)|w_3|^2 \geq 0, \]  
(C14)

which reproduces condition [24]. Hence, condition \( a \geq 0 \) is equivalent to P-divisibility. The second condition \( a \geq |\mathbf{b}| \) provides further restriction which clearly shows that KS-divisibility implies P-divisibility. One finds for the 3-vector \( \mathbf{b} \):

\[ \mathbf{b} = \left( \mu_1(\overline{\mathbf{w}}_2 w_3 - \overline{\mathbf{w}}_3 w_2), \mu_2(\overline{\mathbf{w}}_3 w_1 - \overline{\mathbf{w}}_1 w_3), \mu_3(\overline{\mathbf{w}}_1 w_2 - \overline{\mathbf{w}}_2 w_1) \right) = S(\mathbf{w} \times \mathbf{w}), \]  
(C15)
where the $3 \times 3$ matrix $S$ reads $S_{ij} = \mu_i \delta_{ij}$, with

\begin{align*}
\mu_1 &= 1 + \lambda_1 - \lambda_2 - \lambda_3 = 2\gamma_1, \\
\mu_2 &= 1 - \lambda_1 + \lambda_2 - \lambda_3 = 2\gamma_2, \\
\mu_3 &= 1 - \lambda_1 - \lambda_2 + \lambda_3 = 2\gamma_3,
\end{align*}

where again we took into account $\frac{1}{2}(\gamma_1 + \gamma_2 + \gamma_3) = 1$. The condition $\|b\| \leq a$ reads

\[
\left( \mu_1^2 |\bar{w}_2 w_3 - \bar{w}_3 w_2|^2 + \mu_2^2 |\bar{w}_1 w_3 - \bar{w}_3 w_1|^2 + \mu_3^2 |\bar{w}_1 w_2 - \bar{w}_2 w_1|^2 \right)^{1/2} \leq (1 - \lambda_1)|w_1|^2 + (1 - \lambda_2)|w_2|^2 + (1 - \lambda_3)|w_3|^2.
\]

Introducing the following parametrization: $w_1 = xe^{i\alpha_1}$, $w_2 = ye^{i\alpha_2}$, $w_3 = ze^{i\alpha_3}$, with $x, y, z \geq 0$, it reduces to

\[
2 \left( \mu_1^2 y^2 z^2 \sin^2 \theta_1 + \mu_2^2 x^2 z^2 \sin^2 \theta_2 + \mu_3^2 x^2 y^2 \sin^2 \theta_3 \right)^{1/2} \leq (\gamma_2 + \gamma_3)x^2 + (\gamma_3 + \gamma_1)y^2 + (\gamma_1 + \gamma_2)z^2,
\]

where $\theta_1 + \theta_2 + \theta_3 = 2\pi$. The inequality is clearly satisfied if all $\gamma_k \geq 0$. Now, suppose that $\gamma_3 < 0$ (note, that only one $\gamma_k$ can be negative) and let us consider the worst case scenario maximizing LHS and minimizing RHS of (C17).

Since

\[
\gamma_1 + \gamma_2 \geq \gamma_1 + \gamma_3 \geq 0,
\]

and

\[
\gamma_1 + \gamma_2 \geq \gamma_2 + \gamma_3 \geq 0
\]

let us take $z = 0$ and $\sin \theta_3 = 1$. It leads to

\[
|\gamma_3| \sqrt{xy} \leq \frac{1}{2} \left( (\gamma_2 + \gamma_3)x + (\gamma_1 + \gamma_3)y \right),
\]

which gives $\alpha \geq 1$. Clearly, $\alpha \geq 1$ and $\beta \geq 1$ is sufficient. To show that it is also necessary take $y = 1/x$. It gives

\[
2x \leq \alpha x + \beta/x.
\]

The RHS is minimal for $x = \sqrt{\beta/\alpha}$ and hence

\[
2\sqrt{\beta/\alpha} \leq 2\sqrt{\beta \alpha}
\]

for all $x, y \geq 0$.

Lemma 1 Let $\alpha, \beta > 0$. Condition

\[
\sqrt{xy} \leq \frac{1}{2} (\alpha x + \beta y),
\]

which can be rewritten as

\[
\gamma_i + 2\gamma_3 \geq 0, \quad i = 1, 2.
\]

Let us take $z = 0$ and $\sin \theta_3 = 1$. It leads to

\[
\gamma_1 + \gamma_3 \geq 0.
\]

\[\text{and hence}
\]

\[
\gamma_1 + \gamma_2 \geq \gamma_2 + \gamma_3 \geq 0.
\]

\[
\text{Summarising, we showed that}
\]

\[
\gamma_1 + \gamma_3 \geq 0, \quad \gamma_1 \geq 0.
\]

\[\text{Clearly } \gamma_1 + 2\gamma_3 \geq 0 \text{ and } \gamma_2 + 2\gamma_1 \geq 0.
\]

\[\text{Condition}
\]

\[
\sqrt{xy} \leq \frac{1}{2} (\alpha x + \beta y),
\]

\[\text{Lemma 1 Let } \alpha, \beta > 0. \text{ Condition}
\]

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\]

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\]
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