Alternative approach to the solution of the momentum-space Schrödinger equation for bound states of the \( N \)-dimensional Coulomb problem

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The Schrödinger–Coulomb problem in \( \mathbb{R}^N \), \( N \geq 2 \), is considered in the momentum representation. The adjoint Sturmian eigenvalue problem is discussed first. The resulting radial integral equation is solved with the aid of a symmetric Poisson-type series expansion of the Legendre function of the second kind into products of the Gegenbauer polynomials, established in the 1950’s by Ossicini. A relationship between solutions to the Sturmian problem and to an energy eigenvalue problem is then exploited to find the Coulomb bound-state energy levels in \( \mathbb{R}^N \), together with explicit representations of the associated momentum-space wave functions.

1 Introduction

A closed-form expression for the bound-state hydrogenic wave functions in the momentum representation in \( \mathbb{R}^3 \) was first obtained in 1929 by Podolsky and Pauling [1], who succeeded to carry out a Fourier transform of the position-space Coulomb eigenfunctions, found three years earlier by Schrödinger [2]. Guided by the results of [1], in 1932 Hylleraas [3] derived a second-order partial differential equation obeyed by the momentum-space Coulomb wave functions. In 1933, Elsasser [4] attacked the problem in the spirit of [1], but with the Fourier transform done with the aid of the complex contour-integration technique. A completely different approach to the subject was presented in 1935 by Fock [5]. He showed that the momentum-space Schrödinger–Coulomb wave equation in \( \mathbb{R}^3 \), known to be an integral one, might be transformed into a form which appeared to be identical with an equation satisfied on the unit hypersphere \( S^3 \) by the hyperspherical harmonics. This fact led Fock to the conclusion that for bound states the symmetry group of the quantum-mechanical Coulomb problem in \( \mathbb{R}^3 \) is \( O(4) \). That thread was pursued further a year later by Bargmann [6]. Since then, the problem was revisited and discussed in various aspects by a number of authors. Space limitations allow us to mention here only several representative papers [7–20], as well as relevant reviews and monographs [21–28], where references to other pertinent works may be found in abundance.

In this paper, we present a method which allows one to determine bound-state energy levels and associated Hamiltonian eigenfunctions of the Schrödinger–Coulomb problem in the \( N \)-dimensional (\( N \geq 2 \)) Euclidean momentum space. It is different from other approaches known from the literature. In particular, we make no use of the \( O(N + 1) \) dynamical symmetry specific to the problem nor we refer to the Erdélyi’s generalization [29] (cf. also [30, Sect. 11.4]) of the Funk–Hecke theorem [31,32] concerning the integral equation obeyed by the hyperspherical harmonics (which constituted the basis of the approach presented by Lévy [7] in the case of \( N = 3 \)).

Our reasoning proceeds along the following route. First, we replace the integral energy eigenproblem by a related Sturmian integral eigenvalue equation, in which energy is fixed and the role of the eigenparameter is played by the Coulomb potential strength. Next, we reduce the aforementioned Sturmian integral equation in \( \mathbb{R}^N \) to a Fredholm one on \( \mathbb{R}^+ \), exploiting a recent result from the theory of Gegenbauer polynomials due to Cohl [33]. A kernel in the latter one-dimensional equation contains the Legendre function of the second kind. We solve that integral equation using a Poisson-type expansion of the...
We aim at finding solutions to the above equation which are associated with, yet unknown, negative (bound-state) energy eigenvalues $E$.

3 Solution of the integral equation for the momentum-space Schrödinger–Coulomb Sturmian functions

3.1 Preliminaries

Instead of attacking Eq. (2.5) directly, it is easier to consider first the associated Sturmian eigenproblem

$$\left(\frac{p^2}{2m} - E\right) \Sigma(E, p) = \lambda \cdot \frac{Z e^2}{(4\pi \epsilon_0)^{1/2}} \int_{\mathbb{R}^N} d^N p' \frac{\Phi(p')}{|p' - p|^{N-1}},$$

in which $\lambda = \lambda(E)$ is an eigenvalue, whereas $E < 0$ is a fixed parameter. It is evident that once the $\lambda$-spectrum ($\{\lambda_a\}$) for Eq. (3.1) is determined, the sought negative energy eigenvalues to the original problem (2.5) are roots to the algebraic equations

$$\lambda_a(E) = 1 \quad (E < 0).$$

If $E_{a\beta} < 0$ is a particular solution to Eq. (3.2) (the second subscript at $E$ serves to distinguish between different roots to Eq. (3.2), if there are more such roots than one), the corresponding negative-energy momentum-space Coulomb wave function $\Phi_{a\beta}(p)$ is

$$\Phi_{a\beta}(p) = A \Sigma_{a\gamma}(E_{a\beta}, p),$$

A being an arbitrary non-zero constant (the second subscript at $\Sigma$ appears if the eigenvalue $\lambda_a$ is degenerate).

Before proceeding further, we shall rewrite Eq. (3.1) in a slightly more compact form. To this end, we define

$$q = \sqrt{-2mE}, \quad q_B = \frac{\hbar}{a_B}$$

(here $a_B = (4\pi \epsilon_0)^{1/2}$ is the Bohr radius). Use of Eq. (3.4) transforms Eq. (3.1) into

$$\left(\frac{p^2 + q^2}{2m} - E\right) \Sigma(E, p)$$

$$= \lambda q B \int_{\mathbb{R}^N} d^N p' \frac{\Phi(p')}{|p' - p|^{N-1}}.$$

3.2 Reduction of Eq. (3.5) to the radial form

We try whether Eq. (3.5) has solutions in the form

$$\Sigma_{l\eta}(E, p) = F_l(E, p) Y_{l\eta}(p).$$
Here \( Y^{(N-1)}_{\ell \eta}(n_p) \), with \( \ell \in \mathbb{N} \) and \( \eta \in \{1,2,\ldots,d_i^{(N-1)}\} \), where
\[
d_i^{(N-1)} = (2I+N-2) \frac{(I+N-3)!}{I!(N-2)!},
\]
are the hyperspherical harmonics [25, 26], that are orthogonal on the unit hypersphere \( S^{N-1} \) in the sense of
\[
\int_{S^{N-1}} d^{N-1} n_p \ Y_{\ell \eta}^{(N-1)}(n_p) Y_{\ell' \eta'}^{(N-1)}(n_p) = \delta_{\ell \ell'} \delta_{\eta \eta'}. 
\]
We insert Eq. (3.6) into Eq. (3.5), multiply the resulting equation by \( Y_{\ell \eta}^{(N-1)}(n_p) \), and then sum over \( \eta \). Using the addition theorem [26, Eqs. (3.80), (3.86) and (3.40)]
\[
\sum_{\eta=1}^{N} Y_{\eta}^{(N-1)}(n_p) Y_{\eta'}^{(N-1)}(n_p') = \sum_{q=1}^{N} \sum_{\xi=1}^{N} \frac{C_{\eta}^{(N/2-1)}(n_p \cdot n_p')}{S_{N-1}} \frac{(p^2 + q^2)^{N/2} - n_p \cdot n_p'}{2pp'} \int_{S^{N-1}} d^{N-1} n_p' \ C_{\eta}^{(N/2-1)}(n_p \cdot n_p') 
\]
where \( C_{\eta}^{(N/2-1)}(\xi) \) is the Gegenbauer polynomial and \( S_{N-1} \) is the surface area of \( S^{N-1} \), we obtain
\[
(p^2 + q^2) F_{I}(E, p) 
= \lambda Z q_B \frac{\Gamma(\frac{N-1}{2})}{2^{(N-1)/2} \pi^{(N+1)/2} C_{I}^{(N/2-1)}(1)p^{(N-1)/2}} 
\times \int_0^{\infty} dp' \ p^{(N-1)/2} F_{I}(E, p') 
\times \int_{S^{N-1}} d^{N-1} n_p' \ C_{\eta}^{(N/2-1)}(n_p \cdot n_p') \frac{(p^2 + q^2)^{-N/2} - n_p \cdot n_p'}{2pp'} 
\]
Since the integrand in the angular integral appearing in Eq. (3.10) depends on the orientation of the unit vector \( n_p' \) through the scalar product \( n_p \cdot n_p' \) only, it is evident that the latter integral may be expressed as
\[
\int_{S^{N-1}} d^{N-1} n_p' \ C_{\eta}^{(N/2-1)}(n_p \cdot n_p') \frac{(p^2 + q^2)^{(N-3)/2} - \xi}{2pp'} 
= S_{N-2} \int_{-1}^{1} d\xi \ (1 - \xi^2)^{(N-3)/2} \ C_{\eta}^{(N/2-1)}(\xi) \frac{(p^2 + q^2)^{(N-3)/2} - \xi}{2pp'} 
\]
However, it has been recently shown by Cohl [33] that
\[
\int_{-1}^{1} d\xi \ \eta^{\alpha - 1/2} C_{\mu}^{(\alpha)}(\xi) \eta^{\alpha + 1/2} C_{\nu}^{(\alpha)}(1) 
\times \Gamma(\alpha + \frac{1}{2}) \Gamma(\alpha - \frac{1}{2}) (\xi^2 - 1)^{(\alpha - k)/2} \Omega^{(\alpha - k)/2} = 2^{\alpha + 1/2} C_{\mu}^{(\alpha)}(1) \Omega^{(\alpha - k)/2} 
\]
where \( \Omega^{(\alpha - k)/2}(z) \) is the associated Legendre function of the second kind. It follows from Eq. (3.12) that
\[
\int_{-1}^{1} d\xi \ \eta^{\alpha - 1/2} C_{\mu}^{(\alpha)}(\xi) \eta^{\alpha + 1/2} C_{\nu}^{(\alpha)}(1) \Omega^{(\alpha - k)/2} = 2^{\alpha + 1/2} C_{\mu}^{(\alpha)}(1) \Omega^{(\alpha - k)/2} 
\]
Plugging Eq. (3.14) into Eq. (3.10) and making use of the relation
\[
S_{N-2} = \frac{(N-1)p^{(N-1)/2}}{\Gamma(\frac{N+1}{2})},
\]
we arrive at the following integral equation for the radial function \( F_{I}(E, p) \):
\[
(p^2 + q^2) F_{I}(E, p) = \lambda \frac{2Z q_B}{\pi p^{(N-1)/2}} \int_0^{\infty} dp' \ p^{(N-1)/2} \Omega_{I+1}(N-3)/2 \left( \frac{p^2 + q^2}{2pp'} \right) F_{I}(E, p')
\]
(for \( N = 3 \) and \( \lambda = 1 \), Eq. (3.16) coincides, as it should, with Eq. (8.7) in [36]). After some straightforward rearrangements, this may be further cast into the form
\[
\sqrt{\frac{p^2 + q^2}{p^{(N-1)/2} F_{I}(E, p)}} 
= \lambda \frac{2Z q_B}{\pi} \int_0^{\infty} dp' \ \Omega_{I+1}(N-3)/2 \left( \frac{p^2 + q^2}{2pp'} \right) \sqrt{p^2 + q^2} \sqrt{p^2 + q^2} 
\times \left[ \sqrt{p^2 + q^2} \ p^{(N-1)/2} F_{I}(E, p') \right],
\]
the kernel in the above equation being real and manifestly symmetric.
3.3 Intermezzo on the spectral theory for Fredholm integral equations

Before we attack Eq. (3.17), we have to recall some very basic facts from the spectral theory for integral equations. If the kernel $M(p, p')$ in the homogeneous Fredholm equation of the second kind

$$f(p) = \lambda \int_0^\infty dp' M(p, p') f(p')$$

(3.18)

is real and symmetric, and if its $\lambda$-spectrum (known to be real, in view of the aforementioned assumptions about the kernel) is purely discrete, the following spectral expansion of $M(p, p')$ holds:

$$M(p, p') = \sum_k \frac{f_k(p) f_k(p')}{\lambda_k},$$

(3.19)

provided the eigenfunctions $\{f_k(p)\}$ are orthonormal in the sense of

$$\int_0^\infty dp f_k(p) f_{k'}(p) = \delta_{kk'},$$

(3.20)

This implies that if we cope with some particular equation of the form (3.18), and if, in whatever manner and from whichever premises, we infer that the kernel $M(p, p')$ possesses the series representation

$$M(p, p') = \sum \varepsilon_k g_k(p) g_k(p'),$$

(3.21)

the functions $\{g_k(p)\}$ being orthogonal,

$$\int_0^\infty dp g_k(p) g_{k'}(p) = ||g_k||^2 \delta_{kk'},$$

(3.22)

we may immediately deduce that the eigenvalues for Eq. (3.18) are

$$\lambda_k = \varepsilon_k ||g_k||^{-2},$$

(3.23)

whereas the associated normalized eigenfunctions are

$$f_k(p) = \frac{g_k(p)}{||g_k||} = \sqrt{\varepsilon_k \lambda_k} g_k(p)$$

(3.24)

(observe that $\varepsilon_k^{-1} = \varepsilon_k$).

3.4 Solution of the radial integral equation (3.17)

We return to Eq. (3.17). It is evident that it falls into the category (3.18), with

$$f_l(p) = \sqrt{p^2 + q^2} p^{(N-1)/2} F_l(E, p)$$

(3.25)

(here the subscript $l$ at $f$ corresponds to the one at $F$) and

$$M_l(p, p') = \frac{2Z q_\lambda^2 \Omega_{l+(N-3)/2}}{\pi} \frac{(p^2 + p'^2)^{(N-1)/2}}{\sqrt{p^2 + q^2 \sqrt{p'^2 + q^2}}},$$

(3.26)

An expansion of the kernel (3.26) in the symmetric form (3.21) may be derived from the following Poisson-type identity:

$$\Omega_{\nu} \left( \frac{1 - 2h \xi' + h^2}{(1 - \xi^2)(1 - \xi'^2)} \right)$$

$$= 2^{2\nu+1} \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 2)} \left( 1 - \xi^2 \right)^{\nu+1} \left( 1 - \xi'^2 \right)^{\nu+1}$$

$$\times \sum_{n=0}^\infty \frac{n!}{\Gamma(n + 2\nu + 2)} C_{\nu}^{(v+1)}(\xi) C_{\nu}^{(v+1)}(\xi')$$

(3.27)

discovered several decades ago by Ossicini [34, Eq. (11)] (there is a misprint in the formula provided originally by Ossicini – on its left-hand side $\Gamma(\lambda)$ should be replaced by $\Gamma(\lambda)^2$; moreover, one should be aware that in [37, Eq. (47.6.11)] the formula in question was reprinted with several errors!). For $h = 1$, Eq. (3.27) becomes

$$\Omega_{\nu} \left( \frac{1 - \xi' h}{(1 - \xi^2)(1 - \xi'^2)} \right)$$

$$= 2^{2\nu+1} \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 2)} \left( 1 - \xi^2 \right)^{\nu+1} \left( 1 - \xi'^2 \right)^{\nu+1}$$

$$\times \sum_{n=0}^\infty \frac{n!}{\Gamma(n + 2\nu + 2)} C_{\nu}^{(v+1)}(\xi) C_{\nu}^{(v+1)}(\xi')$$

(3.28)

from which, with the substitutions

$$\xi = \frac{q^2 - p^2}{q^2 + p'^2}, \quad \xi' = \frac{q^2 - p'^2}{q^2 + p^2},$$

(3.29)

we infer that

$$\Omega_{\nu} \left( \frac{p^2 + p'^2}{2pp'} \right)$$

$$= 2^{2\nu+1} \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 2)} \left( \frac{2qp}{q^2 + p'^2} \right)^{\nu+1} \left( \frac{2qp'}{q^2 + p^2} \right)^{\nu+1}$$

$$\times \sum_{n=0}^\infty \frac{n!}{\Gamma(n + 2\nu + 2)} C_{\nu}^{(v+1)} \left( \frac{q^2 - p^2}{q^2 + p'^2} \right) C_{\nu}^{(v+1)} \left( \frac{q^2 - p'^2}{q^2 + p^2} \right)$$

(3.30)

(the particular reason for which we have denoted the summation index as $n$ will become clear shortly). Combining Eqs. (3.26) and (3.30), we see that the kernel in
Eq. (3.17) may be written as

\[ M_l(p, p') = \sum_{n=0}^{\infty} g_{n; l}(p) g_{n; l}(p') \]  

\( \forall n, l \in \mathbb{N} : \varepsilon_{nl} = 1 \)  \hspace{1cm} (3.31)

with

\[ g_{n; l}(p) = \Gamma \left( l + \frac{N-1}{2} \right) \frac{Z q^l n_l!}{\pi(n_l + 2l + N - 2)!} \]

\times \frac{(4qp)^{l+(N-1)/2}}{(q^2 + p^2)^{l+(N-1)/2}} C_{l+n}^{l+(N-1)/2} \left( \frac{q^2 - p^2}{q^2 + p^2} \right). \hspace{1cm} (3.32)

In the next step, we evaluate the integral

\[ \int_0^\infty dp \ g_{n; l}(p) g_{n'; l}(p), \]

in order to verify whether the functions (3.32) form an orthogonal set [as they should, cf. Eq. (3.22)]. Exploiting the known \([38, \text{Eq. (7.313)}]\) integral formula

\[ \int_1^1 d\xi (1 - \xi^2)^{\alpha-1/2} C_{n_l}^{(a)}(\xi) C_{n'_l}^{(a)}(\xi) \]

\[ = \frac{\pi \Gamma(n_l + 2\alpha)}{2^{2\alpha-1} n_l! (n_l + \alpha) \Gamma(\alpha)^2} \delta_{n_l n'_l} \]  \hspace{1cm} (Re \alpha > -\frac{1}{2}, \alpha \neq 0),

we find that

\[ \int_0^\infty dp \ g_{n; l}(p) g_{n'; l}(p) = \frac{Z}{n_l + l + \frac{N-1}{2}} \frac{q^l}{q} \delta_{n_l n'_l}, \hspace{1cm} (3.34) \]

i.e., the functions (3.32) are indeed orthogonal, the norm of \(g_{n; l}(p)\) being

\[ ||g_{n; l}|| = \sqrt{\frac{Z}{n_l + l + \frac{N-1}{2}} \frac{q^l}{q}}. \hspace{1cm} (3.35) \]

Hence, applying Eqs. (3.23) and (3.24) we infer that the spectrum of eigenvalues in Eq. (3.17) is given by

\[ \lambda_{n,l} = \frac{n_l + l + \frac{N-1}{2}}{Z} q \]  \hspace{1cm} (n_l \in \mathbb{N}), \hspace{1cm} (3.36) \]

whereas the associated orthonormal eigenfunctions are

\[ f_{n; l}(E, p) = \Gamma \left( l + \frac{N-1}{2} \right) \sqrt{\frac{q^{n_l!(n_l + l + \frac{N-1}{2})}}{\pi(n_l + 2l + N - 2)!}} \]

\times \frac{(4qp)^{l+(N-1)/2}}{(q^2 + p^2)^{l+(N-1)/2}} C_{l+n}^{l+(N-1)/2} \left( \frac{q^2 - p^2}{q^2 + p^2} \right). \hspace{1cm} (3.37)

Consequently (cf. Eq. (3.25)), the sought radial momentum-space Coulomb Sturmian functions are found to be

\[ F_{n; l}(E, p) = 2^{N-1} \Gamma \left( l + \frac{N-1}{2} \right) \sqrt{\frac{n_l!(n_l + l + \frac{N-1}{2})}{\pi(n_l + 2l + N - 2)!}} \]

\times q^{N/2} \frac{(4qp)^l}{(q^2 + p^2)^{l+(N-1)/2}} C_{l+n}^{l+(N-1)/2} \left( \frac{q^2 - p^2}{q^2 + p^2} \right). \hspace{1cm} (3.38)

(it is seen that \(n_l\) is simply the radial quantum number). They obey the weighted orthonormality relation

\[ \int_0^\infty dp \ p^{N-1} (p^2 + q^2) F_{n; l}(E, p) F_{n'; l}(E, p) = \delta_{n,n'} \hspace{1cm} (3.39) \]

and are standardized to be positive in the immediate vicinity of \(p = 0\).

### 3.5 Some properties of solutions to Eq. (3.5)

Once the radial integral equation (3.16) is solved, we may return to the \(N\)-dimensional problem (3.5).

It follows from Eq. (3.36) that the degree of degeneracy of the eigenvalues in Eq. (3.5) is higher than \(d_l^{(N-1)}\), as \(\lambda_{n;l}\) depends on the quantum numbers \(n_l\) and \(l\) through their sum. To distinguish between different eigenvalues, we introduce the principal quantum number

\[ n = n_l + l + 1, \hspace{1cm} (3.40) \]

in terms of which we have

\[ \lambda_{n,l} \equiv \lambda_n = \frac{n + \frac{N-3}{2}}{Z} q. \hspace{1cm} (3.41) \]

The degree of degeneracy of the particular eigenvalue \(\lambda_n\) obviously is

\[ D_n^{(N)} = \sum_{l=0}^{n-1} d_l^{(N-1)} \hspace{1cm} (3.42) \]

In view of Eq. (3.7), this may be rewritten as

\[ D_n^{(N)} = 2 \sum_{l=0}^{n-1} \left( \frac{l + N - 2}{N - 2} - \sum_{l=0}^{n-1} \right) \left( \frac{l + N - 3}{N - 3} \right). \hspace{1cm} (3.43) \]

The sums in Eq. (3.43) are elementary; we have \([38, \text{Eq. (0.151)}]\)

\[ \sum_{k=0}^{n} \binom{k + m}{m} = \binom{n + m + 1}{m + 1}. \hspace{1cm} (3.44) \]

and consequently

\[ D_n^{(N)} = (2n + N - 3) \frac{(n + N - 3)!}{(n-1)!(N-1)!}. \hspace{1cm} (3.45) \]
Comparison of Eqs. (3.45) and (3.7) shows that it holds that
\[ D_{n}^{(N)} = d^{(N)}_{n} \]  
(3.46)
(The relation in Eq. (3.46) is, of course, a symptom of the \(O(N + 1)\) dynamical symmetry of the \(N\)-dimensional negative-energy Schrödinger–Coulomb problem.)

Degenerate Sturmian eigenfunctions associated with the characteristic numbers (3.41), labeled with the principal rather than the radial quantum number, are
\[ \Sigma_{nl\eta}(E,\mathbf{p}) = F_{n-l-1,1}(E,\mathbf{p})Y_{l\eta}^{(N-1)}(\mathbf{n}_{p}). \]  
(3.47)
It is implied by Eqs. (3.8) and (3.39) that these functions obey the weighted orthonormality relation
\[ \int_{\mathbb{R}^{N}} d^{N}p \left( p^{2} + q^{2} \right) \Sigma_{nl\eta}^{*}(E,\mathbf{p}) \Sigma_{n'l'\eta'}(E,\mathbf{p}) = \delta_{nn'}\delta_{\eta\eta'}\delta_{ll'}. \]  
(3.48)

It remains to consider the completeness problem for the Sturmian set \( \{ \Sigma_{nl\eta}(E,\mathbf{p}) \} \). It is known that the functions \( \{ (1 - \xi^{2})^{\alpha/2 - 1/4}C_{\alpha}^{(n\xi)}(\xi) \} \), with \( \text{Re} \alpha > \frac{1}{2} \), form a complete set in the functional space \( L^{2}((-1,1)) \), the corresponding closure relation being
\[ 2^{2\alpha-1} [\Gamma(\alpha)]^{2} \sum_{n_{\xi}=0}^{\infty} \frac{n_{\xi}!(n_{\xi} + \alpha)}{\Gamma(n_{\xi} + 2\alpha)} C_{\alpha}^{(n\xi)}(\xi') \]
\[ \times \delta(\xi - \xi') \]
\[ \frac{[1 - (1 - \xi^{2})^{(1 - \xi^{2})}]^{\alpha/2 - 1/4}}{(-1 < \xi, \xi' < 1, \text{Re} \alpha > \frac{1}{2})}, \]  
(3.49)
with \( \delta(\xi - \xi') \) denoting the Dirac delta distribution. We set in the above equation \( \alpha = 1 + (N - 1)/2 \), plug Eq. (3.29), combine the result with Eq. (3.38), and then exploit the identity
\[ \delta(\xi - \xi') = \frac{(p^{2} + q^{2})(p'^{2} + q'^{2})}{4\sqrt{pp'q'q'^{2}}} \delta(p - p'), \]  
(3.50)
which follows from Eq. (3.29) and the well-known properties of the Dirac delta. This chain of operations transforms Eq. (3.49) into the closure relation for the radial Sturmians:
\[ \sum_{n_{l}=0}^{\infty} F_{n_{l}l}(E,\mathbf{p})F_{n_{l}l}(E,\mathbf{p}') \]
\[ = \frac{\delta(p - p')}{(pp')^{(N-1)/2} \sqrt{(p^{2} + q^{2})(p'^{2} + q'^{2})}}. \]  
(3.51)
Next, we have the closure relation
\[ \sum_{l=0}^{\infty} \sum_{\eta=1}^{\infty} Y_{l\eta}^{(N-1)}(\mathbf{n}_{p})Y_{l\eta}^{(N-1)*}(\mathbf{n}'_{p}) = \delta^{(N-1)}(\mathbf{n}_{p} - \mathbf{n}'_{p}), \]  
(3.52)
which reflects the fact that the hyperspherical harmonics form a complete set in the functional space \( L^{2}(\mathbb{S}^{N-1}) \). We insert Eqs. (3.51) and (3.52) into the right-hand side of the identity
\[ \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \sum_{\eta=1}^{\infty} \Sigma_{n\eta l}(E,\mathbf{p})\Sigma^{*}_{n\eta l}(E,\mathbf{p}') \]
\[ = \sum_{l=0}^{\infty} \sum_{\eta=1}^{\infty} \delta^{(N-1)}(\mathbf{n}_{p} - \mathbf{n}'_{p}) \]
\[ \times \sum_{n=0}^{\infty} F_{n_{l}l}(E,\mathbf{p})F_{n_{l}l}(E,\mathbf{p}'). \]  
(3.53)
and make use of the obvious fact that
\[ \delta^{(N)}(\mathbf{p} - \mathbf{p}') = \frac{\delta(p - p')\delta^{(N-1)}(\mathbf{n}_{p} - \mathbf{n}'_{p})}{(pp')^{(N-1)/2}}. \]  
(3.54)
This leads us to the symmetric closure relation
\[ \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \sum_{\eta=1}^{\infty} \Sigma_{n\eta l}(E,\mathbf{p})\Sigma^{*}_{n\eta l}(E,\mathbf{p}') \]
\[ = \frac{\delta^{(N)}(\mathbf{p} - \mathbf{p}')}{\sqrt{(p^{2} + q^{2})(p'^{2} + q'^{2})}}. \]  
(3.55)
We have thus proved that the Sturmian functions \( \{ \Sigma_{n\eta l}(E,\mathbf{p}) \} \) form a complete orthonormal set in the functional space \( L^{2}(\mathbb{R}^{N}) \); equivalently, the functions \( \{ \sqrt{p^{2} + q^{2}} \Sigma_{n\eta l}(E,\mathbf{p}) \} \) form a complete orthonormal set in the functional space \( L^{2}(\mathbb{S}^{N-1}) \).

4 Determination of bound-state eigensolutions to the momentum-space Schrödinger–Coulomb wave equation (2.5)

We are now ready to determine bound-state eigensolutions to Eq. (2.5), following the procedure outlined at the beginning of Sect. 3.1. At first, we substitute \( \lambda_{n} \), as given in Eq. (3.41), for \( \lambda_{\alpha}(E) \) in Eq. (3.2), then use Eq. (3.4), and solve the resulting equation for \( E \). For each particular \( n \), there is just one solution to that equation, which we shall denote as \( E_{n} \); it is
\[ E_{n} = -\frac{Z^{2}e^{2}}{2 \left( n + \frac{N-3}{2} \right)^{2} (4\pi\epsilon_{0})a_{\beta}}. \]  
(4.1)
Corresponding eigenfunctions, in accordance with Eqs. (3.3) and (3.6), are
\[ \Phi_{n\eta l}(\mathbf{p}) = \mathcal{F}_{n_{l}l}(\mathbf{p})Y_{l\eta}^{(N-1)}(\mathbf{n}_{p}), \]  
(4.2)
with
\[ F_{n\ell} (p) = AF_{n-\ell-1,\ell} (E_n, p). \] (4.3)

Equation (2.5) shows that \( E_n^{-1} \) is an eigenvalue of the Hermitian (in fact – real and symmetric) kernel
\[ \frac{pp'}{2m} \delta^{(N)}(p-p') - \frac{Ze^2}{(4\pi \epsilon_0)} \frac{\Gamma \left( \frac{N-1}{2} \right)}{2\pi^{(N+1)/2} \hbar} |p-p'|^{-(N-1)}. \]

From this and from Eqs. (4.2) and (3.8), one readily deduces that the functions \( \{ \Phi_{n\ell\eta} (p) \} \) are orthogonal in the sense of
\[ \int_{\mathbb{R}^N} d^N p \Phi_{n\ell\eta}^*(p) \Phi_{n'\ell'\eta'} (p) = \delta_{nn'} \delta_{\ell\ell'} \delta_{\eta\eta'}. \] (4.4)

If, additionally, we require the functions under consideration are normalized to unity under the same scalar product that is used in the above equation, and if we choose the constant \( A \) in Eq. (4.3) to be real, we obtain the constraint
\[ A^2 \int_0^\infty dp \, p^{N-1} |F_{n-\ell-1,\ell} (E_n, p)|^2 = 1. \] (4.5)

To evaluate the integral in Eq. (4.5), we use Eq. (3.38). This yields
\[ \int_0^\infty dp \, p^{N-1} |F_{n-\ell-1,\ell} (E_n, p)|^2 = 2^{2l+N-3} \left[ \frac{\Gamma \left( l + \frac{N-1}{2} \right)}{\pi (n+l+N-3)!} q_n^{-2} \right]^2 \times \int_{-1}^1 d\xi_n \left( 1 - \xi_n^2 \right)^{l+N/2-1} \left( 1 + \xi_n \right) \left[ C_n^{(l+N-1)/2} (\xi_n) \right]^2, \] (4.6)

with
\[ q_n = \frac{Zq_n}{n + \frac{N-2}{2}}. \] (4.7)

It is evident that in the integrand on the right-hand side of Eq. (4.6) one may replace the factor \( 1 + \xi_n \) by 1 without altering the value of the integral. Applying then the formula (3.33) and imposing a further restriction on \( A \) to be positive, we obtain
\[ A = \sqrt{2} q_n. \] (4.8)

Hence, we conclude that the bound-state momentum-space eigenfunctions of the Schrödinger–Coulomb problem in \( \mathbb{R}^N, N \geq 2 \), associated with the energy levels (4.1) and orthonormal in the sense of
\[ \int_{\mathbb{R}^N} d^N p \Phi_{n\ell\eta}^*(p) \Phi_{n'\ell'\eta'} (p) = \delta_{nn'} \delta_{\ell\ell'} \delta_{\eta\eta'}. \] (4.9)

may be chosen in the form
\[ \Phi_{n\ell\eta} (p) = 2^{N-1/2} \Gamma \left( l + \frac{N-1}{2} \right) \sqrt{(n-l-1)! (n+N-3)!} \times q_n^{N/2+1} \left( \frac{4q_n p}{q_n^2 + p^2} \right)^l \rho_{n\eta}(p). \] (4.10)

The degeneracy of the energy level \( E_n \) is, obviously, the same as that of the Sturmian eigenvalue \( \lambda_n \), and is given by Eq. (3.45).

With Eqs. (4.1) and (4.10) being obtained, our task to determine bound-state eigensolutions to the Schrödinger–Coulomb problem in the \( N \)-dimensional Euclidean momentum space is accomplished.

### 5 Conclusions

In this paper, we have proposed a method that enables one to find the Schrödinger–Coulomb bound-state energy levels and associated momentum-space wave functions in \( \mathbb{R}^N, N \geq 2 \). The approach presented above is direct, in the sense that at no place we have had to refer to known solutions of the problem in the position representation. It also seems worthwhile to emphasize that in the course of our reasoning, we have made no use of the \( O(N+1) \) dynamical symmetry of the negative-energy Coulomb problem; this distinguishes our work from the majority of relevant publications.

It is natural to ask whether a similar approach might be developed for solving the Dirac–Coulomb problem in the momentum representation. We plan to make an investigation in that direction in the near future.

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**Key words.** Coulomb problem, momentum representation, integral equations, Gegenbauer polynomials, Legendre functions.

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