Quantum particle on a Möbius strip, coherent states and projection operators

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Abstract
The coherent states for a quantum particle on a Möbius strip are constructed and their relation to the natural phase space for fermionic fields is shown. The explicit comparison of the obtained states with previous works where the cylinder quantization was used and the spin-1/2 was introduced by hand is given, and the relation between the geometrical phase space, constraints and projection operators is analyzed and discussed.

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1. Introduction

Coherent states (CS) have attracted much attention in many branches of physics [1]. In spite of their importance, the theory of CS when the configuration space has non-trivial topology is far from complete. CS for a quantum particle on a circle [3] and a sphere have been introduced very recently and also in the case of a torus [4, 7]. As well, in all these works the different constructions of the CS for the boson case are practically straightforward, the simple addition by hand of 1/2 to the angular momentum operator $J$ for the fermionic case into the corresponding CS remains obscure and non-natural. The question that naturally arises is: does there exist any geometry for the phase space in which the CS construction leads precisely a fermionic quantization condition? Recently in [9], we demonstrated the positive answer to this question showing that the CS for a quantum particle on the Möbius strip (MS) geometry is a natural candidate to describe fermions exactly as the cylinder geometry for bosons. Then, the purpose of this paper is to analyze deeply this relation between the CS and the geometry of the physical phase space taking into account two important roles played by the CS: as projector operators [10] and as the main link between classical and quantum formulations of a given system [5].
2. Requirements for the coherent states

It is well known that CS provide naturally a close connection between classical and quantum formulations of a given system. A suitable set of requirements for these states is given, in association with a specific Hamiltonian operator \( H \), by

(a) continuity: \( (J', \gamma') \to (J, \gamma) \Rightarrow |J', \gamma'\rangle \to |J, \gamma\rangle \);

(b) resolution of the unity: \( I = \int |J, \gamma\rangle \langle J, \gamma| \, d\mu(J, \gamma) \);

(c) temporal stability: \( e^{-iHt} |J, \gamma\rangle = |J, \gamma + \omega t\rangle, \omega = \text{constant} \);

(d) action identity: \( \langle J, \gamma| H |J, \gamma\rangle = \omega J \).

The first two requirements emphasize the fact that the identity operator may be understood in a restricted sense, namely as a projector onto a finite or infinite subspace. The third requirement ensures that the time evolution of any CS is always a CS. As was shown clearly by Gazeau and Klauder in [4], in this evolution, \( J \) remains constant, while \( \gamma \) increases linearly. These properties are similar to the classical behavior of action-angle variables. If \( J \) and \( \gamma \) denote canonical action-angle variables, they would enter the classical action in the following form:

\[
I = \int_0^T (J\gamma - \omega J) \, dt.
\]

As is easily seen, the classical action can be viewed as the restricted evaluation of the quantum action functional

\[
I = \int_0^T \left[ i \langle J, \gamma| \frac{d}{dt} |J, \gamma\rangle - \langle J, \gamma| H |J, \gamma\rangle \right] \, dt
\]

for different paths \( \{ |J(t), \gamma(t)\rangle : 0 \leq t \leq T \} \) lying in a two-dimensional manifold in the Hilbert space. Thus, the fourth requirement simply codifies the fact that the two coordinates \( (J, \gamma) \) are canonical action-angle variables (it will follow that the kinematical term is \( J\gamma \) as needed [5]). At this point, it seems to be necessary to make the following observations: firstly, the physical meaning of the third requirement is to assert that the path in the Hilbert space represented by \( \{ |J, \gamma + \omega t\rangle : 0 \leq t \leq T \} \) is actually the true quantum temporal for the quantum Hamiltonian \( H \). Then, the restricted quantum action functional in this case is exact, see [1]—Gazeau and references therein; a wider set of variational paths starting at \( |J, \gamma\rangle \) at \( t = 0 \) leads to the same extreme path. Secondly, it is well known that the lack of uniqueness in the possible families of CS corresponding to a given Hamiltonian with discrete spectrum is because the fourth requirement was not taken into account.

3. Geometry of the Möbius band and dynamics

The position of a point into the Möbius strip (MS) geometry can be parameterized as

\[
P_0 = (X_0, Y_0, Z_0), \quad P_1 = (X_0 + X_1, Y_0 + Y_1, Z_0 + Z_1).
\]

The coordinates of \( P_0 \) describe the central cylinder (generated by the invariant fiber of the middle of the weight of the strip)

\[
Z_0 = l, \quad X_0 = R \cos \varphi, \quad Y_0 = R \sin \varphi.
\]

(This is topological invariant of the geometry under study.)

The coordinates of \( P_1 \) (the boundaries of the Möbius band) are of \( P_0 \) (the cylinder) plus

\[
Z_1 = r \cos \theta, \quad X_1 = r \sin \theta \cos \varphi, \quad Y_1 = r \sin \theta \sin \varphi.
\]
The weight of the band is obviously $2r$, then our space of phase is embedded into the torus

\[ X = R \cos \varphi + r \sin \theta \cos \varphi \]
\[ Y = R \sin \varphi + r \sin \theta \sin \varphi \]
\[ Z = l + r \cos \theta. \]

The important point is that the angles are not independent in the case of the Möbius band and are related to the following constraint:

\[ \theta = \frac{\varphi + \pi}{2}. \]

It is very important that this constraint effectively reduces the degree of freedom from the torus to the unoriented surface.

In order to study the dynamics in this non-trivial geometry, we construct the non-relativistic Lagrangian

\[ L = \frac{1}{2} m (X^2 + Y^2 + Z^2) \]

\[ L = \frac{1}{2} \left\{ \varphi^2 \left[ (1 + r \cos(\varphi/2))^2 + \frac{r^2}{4} \right] - r \cos(\varphi/2) Z_0 \varphi + (Z_0)^2 \right\}. \]

From the above expression, the equations of motion are

\[ \frac{\partial L}{\partial \varphi} = \varphi \left[ (1 + r \cos(\varphi/2))^2 + \frac{r^2}{4} \right] - r \cos(\varphi/2) Z_0 \]

\[ \frac{\partial L}{\partial Z_0} = -\frac{r}{2} \cos(\varphi/2) \varphi + Z_0 \]

\[ \frac{\partial L}{\partial \varphi} = \frac{r}{2} \sin(\varphi/2) \varphi \left[ -\varphi (1 + r \cos(\varphi/2)) + \frac{Z}{2} \right] \]

\[ \frac{\partial L}{\partial Z_0} = 0. \]

Taking into account that $Z_0$ is a cyclic coordinate, we have the following constraint:

\[ \left( \frac{\partial L}{\partial Z_0} \right) - \frac{\partial L}{\partial Z_0} = 0 \Rightarrow \frac{\partial L}{\partial Z_0} = -\frac{r}{2} \cos(\varphi/2) \varphi + Z_0; \]

then, looking for the dynamical expressions for $\varphi$,

\[ \frac{\partial L}{\partial \varphi} = \varphi \left[ (1 + r \cos(\varphi/2))^2 + \frac{r^2}{4} \sin^2(\varphi/2) \right] - \frac{r}{2} \cos(\varphi/2) L_0 = J. \]

From the Lagrangian (7), the Hamiltonian is not difficult to obtain

\[ H = p_\varphi \dot{\varphi} + p_{Z_0} \dot{Z_0} - L \]

\[ = \frac{1}{2} \left\{ \varphi^2 \left[ (1 + r \cos(\varphi/2))^2 + \frac{r^2}{4} \right] - r \cos(\varphi/2) Z_0 \varphi + (Z_0)^2 \right\} = L \]

that through the constraint (12) takes the most compact form

\[ H = \frac{1}{2} \left\{ \varphi^2 \left[ (1 + r \cos(\varphi/2))^2 - \frac{r^2}{4} \cos \varphi \right] + L_0^2 \right\}. \]
As usual in the Hamiltonian formulation, it is convenient to introduce
\[ J \equiv \varphi = \left( J + \frac{rL_0}{2} \right)^2 \left[ 1 + r \cos(\varphi/2) \right]^2 + \frac{r^2}{4} \sin^2(\varphi/2) \]  
then, finally expression (15) takes the form
\[ H = \left\{ \frac{\hat{J}}{2} \left[ 1 + r \cos(\varphi/2) \right]^2 - \frac{r^2}{4} \cos \varphi \right\} + L_0^2 \]  
\[ = \frac{1}{2} \left\{ \frac{\hat{J}}{2} \left[ 1 + r \cos(\varphi/2) \right]^2 - \frac{r^2}{4} \cos \varphi \right\} + L_0^2 . \]  
The expressions above involving geometry and dynamics on the MS will be utilized at the quantum level in the following sections.

4. Abstract coherent states

In order to introduce the CS for a quantum particle on the MS geometry, we follow the Barut–Girardello construction [2] and seek the CS as the solution of the eigenvalue equation
\[ X \xi = \xi \xi \]  
with the complex \( \xi \). Similarly to the standard case where the CS \( |z\rangle \) satisfy the eigenvalue equation where \( z \in \mathbb{C} \):
\[ e^{i\alpha} |z\rangle = e^{i\xi} |z\rangle, \]  
where \( \alpha \) is the standard bosonic annihilation operator with \( \hat{q} \) and \( \hat{p} \) the position and momentum operators respectively, then we can define
\[ X := e^{i(\varphi + \hat{J})} . \]  
Taking \( R = 1 \) and inserting (5) into (4), we obtain the parametrization of the band
\[ X = \cos \varphi + r \cos(\varphi/2) \cos \varphi \]
\[ Y = \sin \varphi + r \cos(\varphi/2) \sin \varphi \]
\[ Z = l + r \sin(\varphi/2) . \]  
Taking into account the initial condition and the transformations
\[ X' = e^{-2X} \]
\[ Y' = e^{-2Y} \]
\[ Z' = Z, \]  
we finally have
\[ \xi = e^{-(l+r \sin(\varphi/2)) + i\varphi} \left( 1 + r \cos(\varphi/2) \right) . \]  
Inserting the above expression in the expansion of the CS in the \( j \) basis, we obtain the CS in an explicit form
\[ |\xi\rangle = \sum_{j=-\infty}^{\infty} \xi^{-j} e^{-\frac{\xi}{2} |j\rangle} = \sum_{j=-\infty}^{\infty} e^{i(l+r \sin(\varphi/2)) - i\varphij} e^{-\frac{\xi}{2} |j\rangle} = \sum_{j=-\infty}^{\infty} e^{i(l+r \sin(\varphi/2)) - i\varphij} e^{-\frac{\xi}{2} |j\rangle} . \]
From (24), the fiducial vector is

$$|1\rangle = \sum_{j=-\infty}^{\infty} e^{-\frac{j^2}{2}} |j\rangle;$$  \hspace{3em} (25)$$

then,

$$|\xi\rangle = e^{-(\ln \xi)\hat{J}} |1\rangle.$$  \hspace{3em} (26)$$

(in the expression (25) the sum absolutely converges to a finite value ($\Theta_1(0|e^{-1/2})$) for $j \in \mathbb{R}$). As is easily seen, the fiducial vector $|1\rangle = |0, 0\rangle_r = 0$ in the $(l, \phi)$ parametrization, and this fact permits us to rewrite expression (26) as

$$|l, \phi\rangle = e^{(l+r \sin(\phi/2))-(\ln(1+r \cos(\phi/2))\hat{J})} |0, 0\rangle_r = 0.$$  \hspace{3em} (27)$$

The apparent singularity in (24) corresponding to the case $\xi = 0$ is only for asymptotic values of $(l + r \sin(\phi/2))$. Notice that in (23) the quantity $(1 + r \cos(\phi/2))$ never is zero due to the fact that $0 < r < R$ with $R = 1$. The overlapping and non-orthogonality formulas are explicitly derived from (26)

$$\langle \xi | \eta \rangle = \sum_{j=-\infty}^{\infty} (\xi^* \eta^{-j})e^{-j^2} = \Theta_3 \left( \frac{i}{2\pi} \ln(\xi^* \eta) \bigg| \frac{i}{\pi} \right)$$  \hspace{3em} (28)$$

and

$$\langle l, \phi | h, \psi \rangle = \Theta_3 \left( \frac{i}{2\pi} (\phi - \psi) - \frac{l' + h'}{2} \bigg| \frac{i}{\pi} \right).$$  \hspace{3em} (29)$$

respectively, where we have defined $l'$ and $h'$ in order to have more compact expressions as follows:

$$l' \equiv (l + r \sin(\phi/2)) - \ln(1+r \cos(\phi/2))$$

$$h' \equiv (l + r \sin(\phi/2)) - \ln(1+r \cos(\phi/2)).$$

Finally, the normalization as a function of $\Theta_3$ yields

$$\langle \xi | \xi \rangle = \Theta_3 \left( \frac{i}{\pi} \ln |\xi| \bigg| \frac{i}{\pi} \right)$$  \hspace{3em} (30)$$

$$\langle l, \phi | l, \phi \rangle = \Theta_3 \left( \frac{i}{\pi} \bigg| \frac{i}{\pi} \right).$$  \hspace{3em} (31)$$

5. The physical phase space and the natural quantization

From equations

$$\hat{J}|j\rangle = j|j\rangle$$  \hspace{3em} (32)$$

$$|l, \phi\rangle = \sum_{j=-\infty}^{\infty} e^{j l - i p j} e^{-\frac{j^2}{2}} |j\rangle$$  \hspace{3em} (33)$$

$$\langle j | l, \phi \rangle = e^{j l - i p j} e^{-\frac{j^2}{2}}$$  \hspace{3em} (34)$$

$$\langle l, \phi | l, \phi \rangle = \sum_{j=-\infty}^{\infty} e^{j l} e^{-\frac{j^2}{2}} = \Theta_3 \left( \frac{i l'}{\pi} \bigg| \frac{i}{\pi} \right),$$  \hspace{3em} (35)$$

5
we note that the normalization, which for the cylinder (boson case) does not depend on \( \varphi \),
depends now on \( \varphi \) through \( l' \equiv (l + r \sin(\varphi/2)) - \ln (1 + r \cos(\varphi/2)) \). Also,
\[
\hat{J}(l, \varphi) = \sum_{j=-\infty}^{\infty} e^{j^{\prime} - i j^{\prime} \varphi} e^{-\frac{1}{l} j^{\prime} j};
\]  
(36)
then,
\[
\frac{\langle \xi | \hat{J} | \xi \rangle}{\langle \xi | \xi \rangle} = \frac{\langle l, \varphi | \hat{J} | l, \varphi \rangle}{\langle l, \varphi | l, \varphi \rangle} = \frac{1}{2\Theta_{3}\left(\frac{\nu}{\pi} | \frac{1}{\pi}\right)} \frac{\partial \Theta_{3}\left(\frac{\nu}{\pi} | \frac{1}{\pi}\right)}{\partial l}.
\]  
(37)
Taking into account the identity
\[
\Theta_{3}\left(\frac{\nu}{\pi} | -\frac{1}{\pi}\right) = e^{i\pi \nu^{2}/r} \sqrt{\tau} \Theta_{3}(\nu | \tau),
\]  
(38)
coming from the general formula
\[
\Theta_{3}\left(\frac{\nu}{\pi} | -\frac{1}{\pi}\right) = e^{i\pi \nu^{2}/r} \sqrt{\tau} \Theta_{3}(\nu | \tau),
\]  
(39)
we arrive at the following expression:
\[
\frac{\langle \xi | \hat{J} | \xi \rangle}{\langle \xi | \xi \rangle} = \nu + \frac{1}{2\Theta_{3}(\nu | i\pi)} \frac{\partial \Theta_{3}(\nu | i\pi)}{\partial \nu},
\]  
(40)
which can be expanded using the following identity for the theta functions:
\[
\frac{\partial \Theta_{3}(\nu)}{\partial \nu} = \pi \Theta_{3}(\nu) \left( \sum_{n=1}^{\infty} \frac{2i q^{2n-1} e^{2i\pi \nu}}{1 + q^{2n-1} e^{2i\pi \nu}} - \sum_{n=1}^{\infty} \frac{2i q e^{-2i\pi \nu}}{1 + q e^{-2i\pi \nu}} \right),
\]  
(41)
given explicitly
\[
\frac{\langle \xi | \hat{J} | \xi \rangle}{\langle \xi | \xi \rangle} = \nu + 2\pi \sin(2\nu \pi) \sum_{n=1}^{\infty} \frac{e^{-\pi i (2n-1)}}{(1 + e^{-\pi i (2n-1)})(1 + e^{-\pi i (2n-1)} e^{2i\pi \nu})}.
\]  
(42)
Note the important result coming from the above expression. The fourth condition required for
the CS demands not only \( l \) to be integer or semi-integer (as the case for the circle quantization)
but also that
\[
\varphi = (2k + 1) \pi
\]  
(43)
that leads to a natural quantization similar to the charge quantization in the Dirac monopole.
Precisely this condition on the angle leads the position of the particle in the internal or the
external border of the Möbius band, that for \( r = \frac{1}{2} \) is \( s = \pm \frac{1}{2}, \) to be as requested.
In order to compare our case with the CS constructed in [3], we consider the existence of
the unitary operator \( U \equiv e^{i\varphi} \), such that \( [J, U] = U \); then, \( U | j \rangle = | j + 1 \rangle \) such that the same
average as in the previous case for the \( \hat{J} \) operator is
\[
\frac{\langle \xi | U | \xi \rangle}{\langle \xi | \xi \rangle} = e^{-i e^{i \varphi} \Theta_{3}\left(\frac{\nu}{\pi} | \frac{1}{\pi}\right)} \Theta_{3}\left(\frac{\nu}{\pi} | \frac{1}{\pi}\right)
\]
\[
e^{-i e^{i \varphi} \Theta_{3}(\nu | i\pi)} \Theta_{3}(\nu | i\pi),
\]  
(45)
where in the last equality the relation \( \Theta_{2}(\nu) = e^{i\pi(\nu + \frac{1}{2})} \Theta_{3}(\nu + \frac{\nu}{2}) \) was introduced. Also
as in [3], we can make the relative average for the operator \( U \) in order to eliminate the factor
\( e^{-\frac{1}{\nu}} \); then at first order, expression (45) coincides with the unitary circle. It is clear that
the denominator in the quotient (45), average with respect to the fiducial CS state, plays the role of
centralizing the expression of the numerator. However, the claim that \( U \) is the best candidate for
the position operator is still obscure and requires special analysis that we will give elsewhere
[8].
6. Quantum mechanics in the Möbius strip

The Hamiltonian at the quantum level operates as follows:

\[ \hat{H} |E\rangle = E |E\rangle \quad \text{if} \quad |E\rangle = |j\rangle \rightarrow \]

\[ E = \left\{ \frac{(j + r\cos(\varphi/2))}{\pi} \left[ (1 + r \cos(\varphi/2))^2 - \frac{c^2}{4} \cos \varphi \right] \right\} + L_0^2. \quad (47) \]

Imposing the fourth requirement, namely \( \langle \hat{J} \rangle = l \) for the CS, on the expressions, we have \( \varphi = (2k + 1)\pi \) and expression (47) for the energy takes the form

\[ E = \frac{2j^2}{4 + r^2} + \frac{L_0^2}{2}. \quad (48) \]

From the dynamical expressions given above, it is not difficult to make the following remarks.

1. The Hamiltonian is not a priori \( T \) invariant. The Hamiltonian of the Möbius strip is \( T \) invariant iff \( TL_0 = -L_0 \); the variable conjugate to the external momenta \( l \) changes under \( T \) as \( J \) manifests with this symmetry the full inversion of the motion of the particle on a MS (evidently it is not the case for particle motion on the circle).

2. The distribution of energies is Gaussian: from the Bargmann representation \[ \phi_j(\xi^*) \equiv \langle \xi | E \rangle = (\xi^*)^{-1} e^{-\frac{r^2}{2}}, \]

the distribution of energies is easily found:

\[ \frac{|\langle j | \xi \rangle|^2}{\langle \xi | \xi \rangle} = \frac{|\xi|^{-2} e^{-\frac{r^2}{2}}}{\Theta_3 \left( \frac{1}{2} \ln |\xi| \mid \frac{1}{2} \right)} = \frac{e^{-2rf} e^{-r^2}}{\Theta_3 \left( \frac{1}{2} \mid \frac{1}{2} \right)}. \quad (50) \]

On the other hand, using the approximate relation from the definition of the theta function

\[ \Theta_3 \left( \frac{1}{2} \mid \frac{1}{2} \right) = e^{(r^2)^2} \sqrt{\pi} \left( 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2} \cos(2n' \pi n) \right) \approx e^{(r^2)^2} \sqrt{\pi}, \quad (51) \]

expression (50) can be written as

\[ \frac{|\langle j | \xi \rangle|^2}{\langle \xi | \xi \rangle} \approx \frac{1}{\sqrt{\pi}} e^{-(j-\frac{r^2}{2})}. \quad (52) \]

It is useful to remark here that when \( \varphi = (2k + 1)\pi \) and \( l = l' \), the above equation coincides exactly in the same form with the boson case \[3, 7\], but \( l \) is semi-integer valued.

7. The physical space of phase and the projection method

In order to see how the projection method works in the context of CS quantization, we start from the torus as our quantum phase space. This means that we have, before reduction to the physical phase space via suitable projection operators, 2\( n \) operators: \( \theta, \varphi \) and \( \varphi \):

\[ X = R \cos \varphi + r \sin \theta \cos \varphi \]
\[ Y = R \sin \varphi + r \sin \theta \sin \varphi \]
\[ Z = l + r \cos \theta \quad (53) \]
\[ X = -\dot{\varphi} \sin \varphi (R + r \sin \theta) + r \cos \theta \cos \varphi \dot{\theta} \]
\[ Y = \varphi \cos \varphi (R + r \sin \theta) + r \cos \theta \sin \varphi \dot{\varphi}, \quad Z_0 = l \]

(54)

Then,
\[ L_{\text{torus}} = \frac{m}{2} \left\{ \varphi^2 \left[ (R + r \sin \theta)^2 + \frac{r^2}{4} \right] + (r \dot{\vartheta})^2 - 2r \sin \theta Z_0 \dot{\varphi} + (Z_0)^2 \right\}. \]

(55)

Before we move to the equations of motion of the torus, it is interesting to note that inserting the geometrical constraint (5) into the above expression, the Lagrangian of the torus becomes the Lagrangian (7) for the MS. The Hamiltonian for the torus is easily computed from the following expressions \((m = R = 1)\):
\[ H = p_\varphi \dot{\varphi} + p_\vartheta \dot{\vartheta} + p_\theta \dot{\vartheta} - L \]

(56)

\[ p_\varphi \equiv \frac{\partial L}{\partial \ddot{\varphi}} = \varphi(1 + r \sin \theta)^2 = J_0 \]

(57)

\[ p_\vartheta \equiv \frac{\partial L}{\partial \ddot{\vartheta}} = -r \sin \theta \dot{\varphi} + Z_0 = L_0 \]

(58)

\[ p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}} = r^2 \dot{\vartheta} - r \sin \theta \dot{Z}_0 \]

(59)

\[ H = L_{\text{torus}} = \frac{1}{2} \left\{ \varphi^2 \left[ (1 + r \sin \theta)^2 + \frac{r^2}{4} \right] + (p_\varphi + r \sin \theta L_0)^2 \right\}. \]

(60)

Now, we construct the CS for the torus analogously to that in the previous section for the MS, but in this case the coordinate \(\theta\) is absolutely independent of \(\varphi\). Thus, we assume two ‘cylinder-type’ parametrizations: one for \(0 \leq l \leq \infty\) cylinder with an angular variable \(\varphi\) and the other one with finite \(0 \leq l_2 \leq 2\pi \sin^2 \varphi (R = 1)\):
\[ \xi_{\text{torus}} = e^{-(l+r \cos \theta)+i\varphi} \left( 1 + r \sin \theta \right) e^{-2\pi \sin^2 \varphi + i\theta}, \quad l^2 = k^2 = -1. \]

(61)

We call the above expression the geometrical factorization. From the above expression the physical decomposition for \(|\xi_{\text{torus}}\rangle\) that is useful for our proposal is the following:
\[ |\xi_{\text{torus}}\rangle = \sum_{j,m=-\infty}^{\infty} \xi_{\text{MS}}^{-j} e^{-\frac{j^2}{2}} \xi_{\text{MS}}^{-m} e^{-\frac{m^2}{2}} |j, m\rangle \]

(62)

\[ |\xi_{\text{MS}}\rangle = \sum_{j,m=-\infty}^{\infty} \xi_{\text{MS}}^{-j} e^{-\frac{j^2}{2}} |j, 0\rangle, \]

where we split the part corresponding to the MS of the rest of the toroidal space of phase
\[ \xi_{\text{MS}} = e^{-\left(\frac{1}{2} \ln \sin \varphi + \frac{1}{2} \ln \left( 1 + r \cos \varphi / 2 \right) + i\psi \right)} \]

(63)

and an \(m\) basis was consistently included. This factorization is the physical one.
We already have all ingredients to perform the projection from our toroidal phase space to the physical phase space that we are interested in
\[
\langle \xi_{\text{MS}} \vert \xi'_{\text{MS}} \rangle = \langle \xi_{\text{torus}} \vert \xi_{\text{MS}} \rangle \langle \xi_{\text{MS}} \vert \xi'_{\text{torus}} \rangle = \sum_{j=-\infty}^{\infty} e^{(l'+h')j} e^{-i(\psi-\phi)} e^{-j^2/2} \tag{64}
\]
with, however, \(\langle \xi_{\text{torus}} \rangle \equiv \langle 1_{\text{torus}} \rangle = \sum_{j,m=-\infty}^{\infty} e^{-j^2/2} |j, m\rangle\). It is important to note that we can proceed other time performing the projection from the Möbius geometry to the circle straightforwardly obtaining the CS for the Bose case. Then, the procedure of projections can be synthesized in the following schema:

Torus \(\rightarrow\) Projection Op. \(\rightarrow\) Möbius strip (fermion) \(\rightarrow\) Projection Op. \(\rightarrow\) circle (boson).

Besides the instructive standard procedure given above, where we take advantage of the projection properties of the CS, there exists one powerful method that is based on the universal projector operator
\[
E \left( \theta - \frac{\pi + \psi}{2} \leq \delta \right) = \int_{-\infty}^{\infty} d\lambda e^{-i(\theta+\phi)\lambda} \frac{\sin(\delta^2\lambda)}{\pi\lambda} \tag{65}
\]
that clearly depends only on the constraints, being independent of the specific form of the Hamiltonian or of the form in which we factorize the original ‘big’ phase space. For example, it is well known that the CS defined in [2] are a particular case of the CS defined in [6] by means of a displacement operator. This fact is crucial in order to be consistent when correctly defining the observables of the physical system under consideration, in particular the position operator [7, 8].

8. Concluding remarks

In this work, the coherent states (CS) for the fermions in the Möbius band were constructed and compared with previous works where the cylinder was used and the spin-1/2 was introduced by hand. Using these particularly constructed CS, we have explicitly shown that an unoriented surface such as the Möbius band is the natural phase space for fermionic fields. This is because the symmetry properties of the band and the symmetry of the fermions are closely related: both have the characteristic ‘double covering’ that makes that the symmetry invariance 4\(\pi\) instead of 2\(\pi\) for the bosonic case where the natural phase space is the cylinder. Also, the CS, due to the double role that they play, namely, as projectors [10] and making the connection between classical and quantum formulations [5], are very sensitive to the geometrical framework where they are constructed, given the best description of a given physical system. These important facts permit, as we have also shown here, the reduction from the toroidal phase space to the MS space of phase and lead, due to the wonderful properties of the CS, to a ‘Dirac-like’ quantization.

It will be interesting to construct CS in other geometries and dimensions and to analyze the physical systems that they describe in such cases. This is the main task of future works [8].

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