ARITHMETIC IDENTITIES AND CONGRUENCES FOR PARTITION TRIPLES WITH 3-CORES

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Abstract. Let $B_3(n)$ denote the number of partition triples of $n$ where each partition is 3-core. With the help of generating function manipulations, we find several infinite families of arithmetic identities and congruences for $B_3(n)$. Moreover, let $\omega(n)$ denote the number of representations of a nonnegative integer $n$ in the form $x_1^2 + x_2^2 + x_3^2 + 3y_1^2 + 3y_2^2 + 3y_3^2$ with $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{Z}$. We find three arithmetic relations between $B_3(n)$ and $\omega(n)$, such as $\omega(6n + 5) = 4B_3(6n + 4)$.

1. Introduction

A partition of a positive integer $n$ is any nonincreasing sequence of positive integers whose sum is $n$. For example, $7 = 4 + 2 + 1$ and $\lambda = \{4, 2, 1\}$ is a partition of 7. A partition $\lambda$ of $n$ is said to be a $t$-core if it has no hook numbers that are multiples of $t$. We denote by $a_t(n)$ the number of partitions of $n$ that are $t$-cores. For convenience, we use the following notation

$$(a; q)_\infty = \prod_{n=1}^{\infty} (1 - aq^n), \quad f_k = (q^k; q^k)_\infty.$$  

From [11, Eq. (2.1)], the generating function of $a_t(n)$ is given by

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{f_t}{f_1}.$$  

In particular, for $t = 3$, we have

$$\sum_{n=0}^{\infty} a_3(n)q^n = \frac{f_3}{f_1}.$$  

A partition $k$-tuple $(\lambda_1, \lambda_2, \cdots, \lambda_k)$ of $n$ is a $k$-tuple of partitions $\lambda_1, \lambda_2, \cdots, \lambda_k$ such that the sum of all the parts equals $n$. For example, let $\lambda_1 = \{3, 1\}, \lambda_2 = \{1, 1\}, \lambda_3 = \{1\}$. Then $(\lambda_1, \lambda_2, \lambda_3)$ is a partition triple of 7 since $3 + 1 + 1 + 1 + 1 = 7$. A partition $k$-tuple of $n$ with $t$-cores is a partition $k$-tuple $(\lambda_1, \lambda_2, \cdots, \lambda_k)$ of $n$ where each $\lambda_i$ is $t$-core for $i = 1, 2, \cdots, k$.

Let $A_t(n)$ (resp. $B_t(n)$) denote the number of bipartitions (resp. partition triples) of $n$ with $t$-cores. Then the generating functions for $A_t(n)$ and $B_t(n)$ are given by

$$\sum_{n=0}^{\infty} A_t(n)q^n = \frac{f_t^{2t}}{f_1^t}, \quad B_t(n) = \frac{A_t(n)}{A_t(0)}.$$  

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and
\[
\sum_{n=0}^{\infty} B_t(n)q^n = \frac{f_t^3}{f_1^2}
\] (1.1)
respectively.

In 1996, Granville and Ono \[9\] found that
\[
a_3(n) = d_{1,3}(3n + 1) - d_{2,3}(3n + 1),
\] (1.2)
where \(d_{r,3}(n)\) denote the number of divisors of \(n\) congruent to \(r\) modulo 3. Their proof is based on the theory of modular forms.

Baruah and Berndt \[3\] showed that for any nonnegative integer \(n\),
\[
a_3(4n + 1) = a_3(n).
\]

In 2009, Hirschhorn and Sellers \[12\] provided an elementary proof of (1.2) and as corollaries, they proved some arithmetic identities. For example, let \(p \equiv 2 \pmod{3}\) be prime and let \(k\) be a positive even integer. Then, for all \(n \geq 0\),
\[
a_3(p^k n + p^k - 1) = a_3(n).
\]

Let \(u(n)\) denote the number of representations of a nonnegative integer \(n\) in the form \(x^2 + 3y^2\) with \(x, y \in \mathbb{Z}\). By using Ramanujan’s theta function identities, Baruah and Nath \[4\] proved that
\[
u(12n + 4) = 6a_3(n).
\] (1.3)

In 2014, Lin \[13\] discovered some arithmetic identities about \(A_3(n)\). For example, he proved that \(A_3(8n + 6) = 7A_3(2n + 1)\). Let \(v(n)\) denote the number of representations of a nonnegative integer \(n\) in the form \(x_1^2 + x_2^2 + 3y_1^2 + 3y_2^2\) with \(x_1, x_2, y_1, y_2 \in \mathbb{Z}\). Lin showed that
\[
v(6n + 5) = 12A_3(2n + 1).
\] (1.3)

Again, Baruah and Nath \[2\] generalized (1.3) and established three infinite families of arithmetic identities involving \(A_3(n)\). For example, for any integer \(k \geq 1\),
\[
A_3\left(2^{2k+2}n + \frac{2(2^{2k+2} - 1)}{3}\right) = \frac{2^{2k+2} - 1}{3} \cdot A_3(4n + 2) - \frac{2^{2k+2} - 4}{3} \cdot A_3(n).
\]

For more results and details about \(a_3(n)\) and \(A_3(n)\), see \[1, 2, 3, 4, 12, 13\].

Motivated by their work, we study the arithmetic properties of partition triples with 3-cores. By using some identities of \(q\) series, we prove some analogous results. We will show that
\[
B_3(4n + 1) = 3B_3(2n), \quad B_3(3n + 2) = 9B_3(n), \quad \text{and}
\]
\[
B_3(4n + 3) = 3B_3(2n + 1) + 4B_3(n).
\]

From these relations we deduce three infinite families of arithmetic identities as well as some Ramanujan-type congruences involving \(B_3(n)\). For example, we prove two infinite families of congruences for \(B_3(n)\): for \(k \geq 1\) and all \(n \geq 0\),
\[
B_3(2^{k+1}n + 2^k - 1) \equiv 0 \pmod{4^{k+1} + (-1)^k},
\]
\[
B_3(3^k n + 3^k - 1) \equiv 0 \pmod{3^{2k}},
\]
\[
B_3(3^k n + 2 \cdot 3^{k-1} - 1) \equiv 0 \pmod{3^{2k-1}}.
\]
We will also prove that
\[ \sum_{n=0}^{\infty} B_3(6n+4)q^n = 24\frac{f_3^2 f_3^3}{f_1^5}, \]
from which we deduce the following two Ramanujan-type congruences:
\[ B_3(30n+10) \equiv B_3(30n+28) \equiv 0 \pmod{120}. \]

Furthermore, let \( \omega(n) \) denote the number of representations of a nonnegative integer \( n \) in the form
\[ n = x_1^2 + x_2^2 + x_3^2 + 3y_1^2 + 3y_2^2 + 3y_3^2, \quad x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{Z}. \]
We find some interesting arithmetic relations between \( \omega(n) \) and \( B_3(n) \):
\[ \begin{align*}
\omega(6n+5) &= 4B_3(6n+4), \\
\omega(12n+2) &= 12B_3(6n), \\
\omega(12n+10) &= 6B_3(6n+4).
\end{align*} \]

In the final section, we introduce a unified notation \( A_3^{(k)}(n) \) to denote the number of partition \( k \)-tuples of \( n \) wherein each partition is 3-core. We propose two open questions about that whether we can find some analogous results about \( A_3^{(k)}(n) \) for all positive integer \( k \) or not. This will lead to researches in the future.

2. Main Results and Proofs

Setting \( t = 3 \) in (1.1), we obtain that
\[ \sum_{n=0}^{\infty} B_3(n)q^n = \frac{f_3^9}{f_1^7}. \] (2.1)

The following 2-dissection identities will be important in our proofs.

**Lemma 2.1.** We have
\[ \begin{align*}
\frac{f_3^3}{f_1} &= \frac{f_3^2 f_3^2}{f_2^2 f_1} + q\frac{f_3^2}{f_2^2}, \\
\frac{f_3}{f_3} &= \frac{f_3^0 f_3^0}{f_2^2 f_1^2} + 3q\frac{f_3^2 f_6 f_2^2}{f_2^2}, \\
\frac{f_3^3}{f_3} &= \frac{f_3^2}{f_1^2} - 3q\frac{f_2^2 f_3^2}{f_4 f_6}, \\
\frac{f_3}{f_3} &= \frac{f_2 f_1^2 f_1^2}{f_6} - q\frac{f_2^3 f_1^2}{f_1^2 f_6^2}, \\
\frac{1}{f_1} &= \frac{f_1^{14}}{f_2^{12} f_8^2} + 4q\frac{f_2^4 f_3^4}{f_2^{10}}, \\
\frac{1}{f_1} &= \frac{f_1^{28}}{f_2^{24} f_8^2} + 8q\frac{f_1^{16}}{f_2^{12}} + 16q^2\frac{f_1^4 f_8^8}{f_2^{10}}. \end{align*} \] (2.2) \( \cdots \) (2.7)

**Proof.** For the proofs of (2.2)–(2.3), see [14, Eq. (3.75) and Eq. (3.38)]. The proofs of (2.4)–(2.5) can be found in [11]. For a proof of (2.6), see [14, Eq. (2.11)]. (2.7) follows by squaring both sides of (2.6). \( \square \)
Lemma 2.2. We have

\[
\frac{f_2 f_3}{f_1^2} = \frac{f_2 f_3}{f_1^2 f_6} + q f_6^5, \quad (2.8)
\]

\[
\frac{f_2 f_3 f_6}{f_1^4} = \frac{f_2 f_3}{f_1^2} + q f_6^5. \quad (2.9)
\]

Proof. For convenience, we introduce the following notation

\[ [x; q]_\infty = (x; q)_{\infty} (q/x; q)_{\infty}, \]
\[ [a_1, \ldots, a_n; q]_\infty = \prod_{i=1}^{n} [a_i; q]_\infty. \]

Multiplying both sides of (2.8) by \( f_1^4 f_6 \), we know (2.8) is equivalent to

\[
f_2 f_3 f_6 = f_1 f_2 f_3 + q f_1^4 f_6^9. \quad (2.10)
\]

Note that

\[ f_1 = [q, q^2; q^6]_{\infty} (q^3; q^6)_{\infty} (q^6; q^6)_{\infty}, \quad f_2 = [q^2; q^6]_{\infty} (q^6; q^6)_{\infty}, \]
\[ f_3 = (q^3; q^6)_{\infty} (q^6; q^6)_{\infty}, \quad f_6 = (q^6; q^6)_{\infty}. \]

Substituting these expressions into (2.10) and simplifying, we know (2.10) is equivalent to

\[
[q^3; q^6]_{\infty} = [q, q^3, q^3, q^3; q^6]_{\infty} + q [q; q^6]_{\infty}^4. \quad (2.11)
\]

From [3] Exercise 2.16, p. 61, we know

\[
[x\lambda, x, \mu x, \mu x/\lambda, \nu, \nu/\lambda, \mu v, \mu v/\lambda; q]_{\infty} = [x v, x v/\lambda, \lambda v, \lambda v/\lambda; q]_{\infty} + \frac{\mu}{\lambda} [x \mu, x /\mu, \lambda v, \lambda v/\lambda; q]_{\infty}. \quad (2.12)
\]

Taking \((x, \lambda, \mu, \nu, q) \rightarrow (q^3, q, q^2, 1, q^6)\) in (2.12), we have

\[
[q^3, q^3, q^3, q^3; q^6]_{\infty} = [q^3, q^3, q^3, q^3; q^6]_{\infty} + q [q^3, q, q, q; q^6]_{\infty}. \]

Hence (2.11) holds and we complete our proof of (2.8).

From (2.2) and (2.3), we obtain that

\[
\frac{f_2^3 f_4}{f_1^3} = \frac{f_2^3}{f_1^3} \cdot \frac{f_3}{f_3^2} = \frac{f_2^3 f_6^5}{f_1^2 f_3^2} + 3q^2 \frac{f_4 f_6 f_9}{f_2^4} + 4q^3 \frac{f_4 f_6 f_9 f_{12}}{f_2^4}, \quad (2.13)
\]

Multiplying both sides by \( f_2^6 \), we get

\[
\frac{f_2^3 f_4}{f_1^3} = \frac{f_2^3 f_6^5}{f_2^2 f_3^2} + 4q^2 f_4 f_6 f_9 f_{12}/f_2 + 3q^2 f_2 f_4 f_6 f_9 f_{12}/f_2, \quad (2.14)
\]

Applying (2.2), we obtain that

\[
\frac{f_2^3 f_6^5}{f_2^2 f_3^2} = \frac{f_2^3}{f_1^3} \left( \frac{f_3}{f_1^3} \right)^3 = \frac{f_2^3 f_6^5}{f_2^2 f_3^2} + 3q^2 f_4 f_6 f_9 f_{12}/f_2 + 3q^2 f_2 f_4 f_6 f_9 f_{12}/f_2 + q^3 f_2 f_12 f_9 f_{12}/f_3 f_6. \quad (2.15)
\]

Substituting (2.14) and (2.15) into (2.8), we deduce that

\[
\frac{f_2^3 f_6^5 f_{12}}{f_2} = f_6^8 + q^2 \frac{f_3 f_6^9 f_{12}}{f_4 f_6^3}. \]

Replacing \( q^2 \) by \( q \) and then multiplying both sides by \( f_4^2 f_6^3 \), we obtain (2.9). \qed
Theorem 2.1. For any integer \( k \geq 1 \), we have

\[
B_3(2^{k+1}n + 2^k - 1) = \frac{2^{2k+2} + (-1)^k}{5} \cdot B_3(2n), \tag{2.16}
\]

\[
B_3(2^{k+1}n + 2^{k+1} - 1) = \frac{2^{2k+2} + (-1)^k}{5} \cdot B_3(2n + 1) + \frac{2^{2k+2} - 4(-1)^k}{5} \cdot B_3(n). \tag{2.17}
\]

Proof. Substituting (2.2) into (2.1), we obtain that

\[
\sum_{n=0}^{\infty} B_3(n)q^n = \left( \frac{f_3^3f_6^5}{f_2^3f_1^2} + q\frac{f_3^{12}}{f_4^1} \right) \cdot \left( \frac{f_3^9f_6^3}{f_2^2f_1^2} + q\frac{f_3^{12}}{f_4^1} \right) + q\left( \frac{3f_3^5f_6^4f_{12}}{f_2^2} + q^2\frac{f_3^{12}}{f_4^1} \right).
\]

Extracting the terms involving \( q^{2n} \) and \( q^{2n+1} \), respectively, we get

\[
\sum_{n=0}^{\infty} B_3(2n)q^n = f_3^9f_6^3 \cdot f_1^1f_6^1 + 3q^2f_3^2f_6^5f_{12}^{f_2^2} = \frac{f_3^9f_6^3}{f_1^1f_6^1} + q\left( \frac{3f_3^5f_6^4f_{12}}{f_2^2} + q^2\frac{f_3^{12}}{f_4^1} \right), \tag{2.18}
\]

and

\[
\sum_{n=0}^{\infty} B_3(2n + 1)q^n = 3f_3^5f_6^3f_1^2 + q\frac{f_3^{12}}{f_2^2}. \tag{2.19}
\]

By (2.10), we deduce that

\[
\sum_{n=0}^{\infty} B_3(2n + 1)q^n = 3\frac{f_3^9}{f_1^1} + 4q\frac{f_6^{12}}{f_2^3}
\]

\[
= 3\sum_{n=0}^{\infty} B_3(n)q^n + 4\sum_{n=0}^{\infty} B_3(n)q^{2n+1}.
\]

Equating the coefficients of \( q^{2n} \) and \( q^{2n+1} \) on both sides, respectively, we obtain

\[
B_3(4n + 1) = 3B_3(2n) \tag{2.20}
\]

and

\[
B_3(4n + 3) = 3B_3(2n + 1) + 4B_3(n). \tag{2.21}
\]

We are now able to prove (2.16)–(2.17). Note that (2.20) and (2.21) are (2.16) and (2.17), respectively, for \( k = 1 \). Now we prove (2.16). Replacing \( n \) by \( 2n \) in (2.16), we have

\[
B_3(8n + 3) = 3B_3(4n + 1) + 4B_3(2n).
\]

By (2.20), this implies

\[
B_3(8n + 3) = 13B_3(2n),
\]

which is (2.16) for \( k = 2 \). Now the proof of (2.16) follows by mathematical induction.

Next, replacing \( n \) by \( 2n + 1 \) in (2.21), we have

\[
B_3(8n + 7) = 3B_3(4n + 3) + 4B_3(2n + 1).
\]

Employing (2.21) in the above, we deduce that

\[
B_3(8n + 7) = 13B_3(2n + 1) + 12B_3(n),
\]

which is (2.17) for \( k = 2 \). Now the proof of (2.17) can be completed by mathematical induction. \[\square\]
Corollary 2.1. For any integer \( k \geq 1 \), we have
\[
B_3(2^{k+1}n + 2^k - 1) \equiv 0 \pmod{\frac{2^{2k+2} + (-1)^k}{5}}.
\]

Recall that the general Ramanujan’s theta function \( f(a, b) \) is defined by
\[
f(a, b) := \sum_{n=-\infty}^{\infty} a^{(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.
\]

As some special cases, we have (see [6], for example)
\[
\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^n = (-q; q^2)_\infty^2(q^2; q^2)_\infty^2, \quad (2.22)
\]
\[
\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty^2}{(q; q^2)_\infty^2}, \quad (2.23)
\]
and
\[
f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_{\infty^2}. \quad (2.24)
\]

Lemma 2.3. We have
\[
(q; q)_{\infty^3}^3 = P(q^3) - 3q(q^9; q^9)^3, \quad (2.25)
\]
where
\[
P(q) = \sum_{m=-\infty}^{\infty} (-1)^m (6m + 1)q^{m(3m+1)/2} = f(-q)\varphi(q)\varphi(q^3) + 4qf(-q)\psi(q^2)\psi(q^6).
\]  

(2.26)

Proof. By Jacobi’s identity (see [6, Theorem 1.3.9]), we have
\[
(q; q)_{\infty^3}^3 = \sum_{n=0}^{\infty} (-1)^n (2n + 1)q^{n(n+1)/2}.
\]

Note that \( \frac{n(n+1)}{2} \equiv 0 \pmod{3} \) if and only if \( n \equiv 0 \pmod{3} \) or \( n \equiv 2 \pmod{3} \).

And \( \frac{n(n+1)}{2} \equiv 1 \pmod{3} \) if and only if \( n \equiv 1 \pmod{3} \). Hence we have the following

3-dissection identity
\[
(q; q)_{\infty^3}^3 = P(q^3) + qR(q^3).
\]

We have
\[
P(q^3) = \sum_{m=0}^{\infty} (-1)^{3m}(6m + 1)q^{3m(3m+1)/2} + \sum_{m=0}^{\infty} (-1)^{3m+2}(6m + 5)q^{(3m+2)(3m+3)/2}
\]
\[
= \sum_{m=0}^{\infty} (-1)^m(6m + 1)q^{3m(3m+1)/2} - \sum_{m=-\infty}^{\infty} (-1)^m(6m + 1)q^{3m(3m+1)/2}
\]
\[
= \sum_{m=-\infty}^{\infty} (-1)^m(6m + 1)q^{3m(3m+1)/2}.
\]
Replacing $q^3$ by $q$, we obtain

$$P(q) = \sum_{m=-\infty}^{\infty} (-1)^m (6m + 1)q^{m(3m+1)/2}.$$ 

From [10] we know that

$$P(q) = (q; q)_\infty \left( 1 + 6 \sum_{n\geq 0} \left( \frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right) \right).$$

From [6, Theorem 3.7.9] we have

$$1 + 6 \sum_{n\geq 0} \left( \frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right) = \varphi(q)\varphi(q^3) + 4q\psi(q^2)\psi(q^6).$$

Hence (2.26) is proved.

Again, we have

$$qR(q^3) = \sum_{m=0}^{\infty} (-1)^{3m+1}(6m + 3)q^{(3m+1)(3m+2)/2}.$$ 

Dividing both sides by $q$ and replacing $q^3$ by $q$, we deduce that

$$R(q) = -3 \sum_{m=0}^{\infty} (-1)^m (2m + 1)q^{3m(m+1)/2} = -3(q^3; q^3)_\infty^3.$$ 

□

**Lemma 2.4.** We have

$$P(q)^3 - 27q(q^3; q^3)_\infty^9 = \frac{(q; q)_\infty^{12}}{(q^2; q^3)_\infty^3},$$

(2.27)

and

$$\frac{1}{(q; q)_\infty^3} = \frac{(q^9; q^9)_\infty^3}{(q^3; q^3)_\infty^{12}} \left( P(q^3)^2 + 3qP(q^3)(q^9; q^9)_\infty^3 + 9q^2(q^9; q^9)_\infty^6 \right).$$

(2.28)

**Proof.** Let $\omega = e^{2\pi i/3}$. On the one hand, by Lemma 2.3 we have

$$(q; q)_\infty^3(\omega q; \omega q)_\infty^3(\omega^2 q; \omega^2 q)_\infty^3 = \prod_{k=0}^{2} \left( P(q^3) - 3\omega^k q f_0^3 \right) = P(q^3)^3 - 27q^3 f_0^9.$$
On the other hand, by definition we have
\[
\begin{align*}
(q; q)^3_\infty (\omega q; \omega q)^3_\infty (\omega^2 q; \omega^2 q)^3_\infty = \\
\prod_{n=1}^{\infty} (1 - q^n)^3 (1 - \omega^n q^n)^3 (1 - \omega^2^n q^n)^3 \\
= \left( \prod_{3 \mid n} (1 - q^n)(1 - \omega^n q^n)(1 - \omega^2^n q^n) \right)^3 \cdot \prod_{3 \nmid n} (1 - q^{3n})^3 \\
= \prod_{n=1}^{\infty} (1 - q^{3n})^9 \cdot \prod_{n=1}^{\infty} (1 - q^{3n})^3 / \prod_{n=1}^{\infty} (1 - q^{9n})^3 \\
= \frac{(q^3; q^3)^{12}_\infty}{(q^9; q^9)^3_\infty}.
\end{align*}
\]
Hence we deduce that
\[
P(q^3)^3 - 27q^3f_0^9 = \frac{(q^3; q^3)^{12}_\infty}{(q^9; q^9)^3_\infty}.
\]
Replacing \(q^3\) by \(q\) we obtain (2.27).

By (2.25) we have
\[
\begin{align*}
1 \frac{(q; q)^3_\infty}{(q^3; q^3)^3_\infty} &= \frac{2}{\prod_{k=1}^{3} (P(q^3) - 3\omega^k q^3f_0^3)} \\
&= \frac{1}{P(q^3)^3 - 27q^3f_0^9} \cdot \left( P(q^3)^2 + 3qP(q^3)f_3^3 + 9q^2f_9^6 \right) \\
&= f_0^3 f_3^2 \left( P(q^3)^2 + 3qP(q^3)f_3^3 + 9q^2f_9^6 \right),
\end{align*}
\]
where the last equality follows from (2.27).

\begin{proof}
By (2.11) and (2.28) we have
\[
\begin{align*}
\sum_{n=0}^{\infty} B_3(3n)q^n &= P(q)^2 \frac{(q^3; q^3)^3_\infty}{(q; q)^3_\infty}, \\
\sum_{n=0}^{\infty} B_3(3n + 1)q^n &= 3P(q) \frac{(q^3; q^3)^6_\infty}{(q; q)^3_\infty}, \\
\sum_{n=0}^{\infty} B_3(3n + 2)q^n &= 9 \frac{(q^3; q^3)^9_\infty}{(q; q)^3_\infty},
\end{align*}
\]

Extracting the terms involving \(q^{3n}\), \(q^{3n+1}\) and \(q^{3n+2}\), respectively, we get the desired results.
\end{proof}

\begin{theorem}
We have
\[
\begin{align*}
\sum_{n=0}^{\infty} B_3(3n)q^n &= P(q)^2 \frac{(q^3; q^3)^3_\infty}{(q; q)^3_\infty}, \\
\sum_{n=0}^{\infty} B_3(3n + 1)q^n &= 3P(q) \frac{(q^3; q^3)^6_\infty}{(q; q)^3_\infty}, \\
\sum_{n=0}^{\infty} B_3(3n + 2)q^n &= 9 \frac{(q^3; q^3)^9_\infty}{(q; q)^3_\infty},
\end{align*}
\]
\end{theorem}

\begin{proof}
By (2.11) and (2.28) we have
\[
\sum_{n=0}^{\infty} B_3(n)q^n = \frac{(q^3; q^3)^9_\infty}{(q; q)^3_\infty} = \frac{(q^9; q^9)^3_\infty}{(q^3; q^3)^3_\infty} \left( P(q^3)^2 + 3qP(q^3)f_3^3 + 9q^2f_9^6 \right).
\]
Extracting the terms involving \(q^{3n}\), \(q^{3n+1}\) and \(q^{3n+2}\), respectively, we get the desired results.
\end{proof}
Theorem 2.3. For any integer \( k \geq 1 \), we have
\[
B_3(3^k n + 3^k - 1) = 3^{2k} B_3(n).
\]

Proof. From (2.1) and (2.31) we deduce that
\[
B_3(3n + 2) = 9B_3(n).
\]
This proves the theorem for \( k = 1 \). Replacing \( n \) by \( 3n + 2 \) in (2.33), we deduce that
\[
B_3(9n + 8) = 3^2 B_3(3n + 2) = 3^4 B_3(n),
\]
which proves the theorem for \( k = 2 \). The theorem now follows by induction on \( k \).

Corollary 2.2. For any integer \( k \geq 1 \), we have
\[
B_3(3^k n + 3^k - 1) \equiv 0 \pmod{3^{2k}},
\]
\[
B_3(3^k n + 2 \cdot 3^{k-1} - 1) \equiv 0 \pmod{3^{2k-1}}.
\]

Proof. The first congruence follows immediately from Theorem 2.3. By (2.31), we know that \( B_3(3n + 1) \equiv 0 \pmod{3} \), and this proves the second congruence for \( k = 1 \). For \( k \geq 2 \), by Theorem 2.3, we have
\[
B_3(3^k n + 2 \cdot 3^{k-1} - 1) = B_3(3^{k-1}(3n+1)+3^{k-1}-1) = 3^{2k-2} B_3(3n+1) \equiv 0 \pmod{3^{2k-1}}.
\]

Theorem 2.4. We have
\[
\sum_{n=0}^{\infty} B_3(6n) q^n = \frac{f_2^{10} f_3^9}{f_1^2 f_6^2} + 16q \frac{f_2^7 f_3^3}{f_1^4} + 27q \frac{f_2^2 f_3^5 f_5^2}{f_1^6}.
\]
\[
\sum_{n=0}^{\infty} B_3(6n + 4) q^n = 24 \frac{f_2^5 f_3^2}{f_1^8}.
\]

Proof. From (2.22) and (2.23), it is not hard to see that
\[
\varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \quad \psi(q) = \frac{f_2^3}{f_1^4}.
\]
By (2.26), we have
\[
P(q) = \frac{f_3^5 f_1^5}{f_1^2 f_3^2 f_4^2} + 4q \frac{f_1^4 f_3^4 f_1^2}{f_2 f_6}.
\]
Substituting (2.37) into (2.29), we obtain
\[
\sum_{n=0}^{\infty} B_3(3n) q^n = \left( \frac{f_2^{10} f_6^{10}}{f_1^2 f_3^4 f_4^4 f_1^1} + 16q \frac{f_2^3 f_4^4 f_1^2}{f_1^2 f_3^2 f_6^2} \right) + 8q \frac{f_3^4 f_3^4 f_6^4}{f_1^4}.
\]
By (2.3) and (2.5), we have
\[
\frac{1}{f_1^3 f_3} = \left( \frac{f_3}{f_1^3} \right)^2 \cdot f_1 \frac{f_4^{14}}{f_1^2 f_6 f_1^2} + 3q^2 \frac{f_6 f_1^2}{f_3^2 f_2^2} + q \left( 5 \frac{f_1^{10} f_6^2}{f_2^2 f_6^2} - 9q \frac{f_4^2 f_1^4 f_2}{f_6} \right).
\]
Now substituting (2.22), (2.3) and (2.39) into (2.38), and then extracting the terms involving \( q^{2n} \), we obtain
\[
\sum_{n=0}^{\infty} B_3(6n) q^{2n} = \frac{f_6^{9} f_4^{10}}{f_1^2 f_2^{12}} + 16q \frac{f_1^4 f_3^3 f_2}{f_2^4} + 27q^2 \frac{f_2^2 f_3^5 f_2^2}{f_2^2}.
\]
Replacing $q^2$ by $q$ we prove (2.34).

Similarly, substituting (2.37) into (2.30), we obtain

$$\sum_{n=0}^{\infty} B_3(3n+1)q^n = 3\frac{f_2^6 f_3^4 f_6^5}{f_1^5 f_4^4 f_7^2} + 12q\frac{f_3^6 f_4^2 f_5^2}{f_1^5 f_2^2 f_6}. \quad (2.40)$$

By (2.2), we deduce that

$$f_3^6 f_4^2 f_5^2 = \left( f_3^3 f_5^3 \right)^2 = \frac{f_3^6 f_4^4}{f_2^2 f_1^2} + 2q\frac{f_3^6 f_4^2 f_5^2}{f_2^2} + q^2 f_6^2 f_5^2.$$ Substituting this identity and (2.13) into (2.40), and extracting the terms involving $q^{2n+1}$, we obtain

$$\sum_{n=0}^{\infty} B_3(6n+4)q^{2n+1} = 12q\left( \frac{f_3^3 f_5^6}{f_2^2 f_4^2} + \frac{f_3^6 f_4^2 f_5^2}{f_2^2} + q^2 f_6^2 f_5^2 \right).$$

Dividing both sides by $q$ and replacing $q^2$ by $q$, and applying (2.8) we obtain that

$$\sum_{n=0}^{\infty} B_3(6n+4)q^n = 12 \frac{f_3^3 f_5^6}{f_1^5 f_3} \left( f_3^3 f_5^3 + q f_6^8 \right) + 12 \frac{f_3^6 f_4^2 f_5^2}{f_1^5 f_2} = 24 \frac{f_3^6 f_5^3}{f_1^5}.$$

\[\square\]

**Corollary 2.3.** For any integer $n \geq 0$, we have

$$B_3(6n+4) \equiv 0 \pmod{24}.$$ 

**Proof.** This follows from (2.35). \[\square\]

**Theorem 2.5.** For any integer $n \geq 0$, we have

$$B_3(30n+10) \equiv B_3(30n+28) \equiv 0 \pmod{120}.$$ 

**Proof.** Note that for any prime $p \geq 3$, we have

$$\binom{p}{k} = \frac{p}{k} \binom{p-1}{k-1} \equiv 0 \pmod{p}, \quad 1 \leq k \leq p - 1.$$ By the binomial theorem, we have

$$(1-x)^p = 1 - px + \cdots + px^{p-1} - x^p \equiv 1 - x^p \pmod{p}.$$ Hence for any integer $a \geq 1$, we have

$$f_a^p \equiv \prod_{n=1}^{\infty} (1 - q^{an})^p \equiv \prod_{n=1}^{\infty} (1 - q^{apn}) = f_a^{ap} \pmod{p}.$$ From (2.35) we have

$$\sum_{n=0}^{\infty} B_3(6n+4)q^n = 24 \frac{f_3^6 f_5^3}{f_1^5} \equiv 24 \frac{f_3^6}{f_5} \cdot f_2^3 f_3^3 \pmod{120}. \quad (2.41)$$ By Jacobi’s identity, we have

$$f_2^3 f_3^3 = \left( \sum_{k=0}^{\infty} (-1)^k (2k+1)q^{k(k+1)} \right) \left( \sum_{l=0}^{\infty} (-1)^l (2l+1)q^{3l(l+1)/2} \right).$$
Suppose
\[ f_2^3 f_3^3 = \sum_{m=0}^{\infty} c(m)q^m, \]
then
\[ c(m) = \sum_{k(k+1)+3l(l+1)/2=m} (-1)^{k+l}(2k+1)(2l+1). \]

Note that
\[ m = k(k+1) + \frac{3l(l+1)}{2} \iff 8m + 5 = 2(2k+1)^2 + 3(2l+1)^2. \]

For any integer \( x \), we have \( x^2 \equiv 0, 1, 4 \pmod{5} \). If \( m \equiv 1 \) or \( 4 \pmod{5} \), then at least one of \( 2k+1 \) or \( 2l+1 \) must be divisible by 5. Hence we deduce that
\[ c(5n+1) \equiv c(5n+4) \equiv 0 \pmod{5}. \]

By (2.41) we have
\[ \sum_{n=0}^{\infty} B_3(6n+4)q^n \equiv 24 \frac{f_{10}}{f_5} \sum_{m=0}^{\infty} c(m)q^m \pmod{120}. \]

The theorem now follows by comparing the coefficients of \( q^{5n+r} \) (\( r \in \{1, 4\} \)) on both sides of the above identity.

**Theorem 2.6.** Let \( \omega(n) \) denote the number of representations of a nonnegative integer \( n \) in the form \( x_1^2 + x_2^2 + x_3^2 + 3y_1^2 + 3y_2^2 + 3y_3^2 \) with \( x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{Z} \). Then
\[ \omega(6n+5) = 4B_3(6n+4). \]

**Proof.** By [5, p. 49, Corollary (i)] and Jacobi triple product identity [6, Theorem 1.3.3], we can deduce that
\[ \varphi(q) = \varphi(q^9) + 2q \frac{f_6 f_9 f_{36}}{f_3 f_{12} f_{18}}. \]

The generating function of \( \omega(n) \) is given by
\[ \sum_{n=0}^{\infty} \omega(n)q^n = \varphi^3(q)\varphi^3(q^3) = \varphi^3(q^3)\left( \varphi(q^9) + 2q \frac{f_6 f_9 f_{36}}{f_3 f_{12} f_{18}} \right)^3. \] (2.42)

Extracting the terms \( q^{3n+2} \) from (2.42), dividing by \( q^2 \), replacing \( q^3 \) by \( q \), we obtain that
\[ \sum_{n=0}^{\infty} \omega(3n+2)q^n = 12\varphi^3(q)\varphi^3(q^3) \frac{f_2^2 f_3^2 f_2}{f_1^2 f_4^2 f_6} = 12 \frac{f_2^{19} f_6^3}{f_1^3 f_4^8}. \]

By (2.7) we get
\[ \sum_{n=0}^{\infty} \omega(3n+2)q^n = 12 \frac{f_2^{19} f_6^3}{f_1^3} \left( \frac{f_2^{28}}{f_2^{12} f_8^6} + 8q \frac{f_4^{16}}{f_2^{12}} + 16q^2 \frac{f_4^4 f_8^8}{f_2^{20}} \right). \] (2.43)

If we extract the terms involving \( q^{2n+1} \), divide by \( q \) and replace \( q^2 \) by \( q \), we obtain
\[ \sum_{n=0}^{\infty} \omega(6n+5)q^n = 96 \frac{f_2^8 f_3^3}{f_1^5}. \] (2.44)

Comparing (2.44) with (2.35), we complete our proof. □
Theorem 2.7. For any nonnegative integer $n$, we have

$$\omega(12n + 2) = 12B_3(6n).$$

Proof. If we extract the terms involving $q^{2n}$ in (2.43) and then replace $q^2$ by $q$, we get

$$\sum_{n=0}^{\infty} \omega(6n + 2)q^n = 12\left(\frac{20f_2^9f_3^8}{f_1^9f_4^8} + 16q^{3} f_3^1 f_4^8\right).$$  

(2.45)

By (2.3) we obtain

$$\frac{f_3^3}{f_1^9} = \left(\frac{f_4^6 f_6^3}{f_2^4 f_1^9} + 3q f_2^1 f_6 f_1^3 f_1^2 \frac{f_2^2}{f_2^1}\right)^3$$

$$= \left(\frac{f_4^{18} f_6^9}{f_2^2 f_1^9} + 27q^2 f_4^{10} f_6^5 f_2^2\right) + 9q \left(f_1^{14} f_6^{7} f_2^2 f_1^3 f_1^2 + 3q^2 f_4^{16} f_6^3 f_2^2\right).$$

(2.46)

By (2.2) and (2.46) into (2.45), extracting the terms involving $q^{2n}$ and then replacing $q^2$ by $q$, we deduce that

$$\sum_{n=0}^{\infty} \omega(12n + 2)q^n = 12\left(\frac{f_2^{10} f_6^9}{f_1^9 f_4^8} + 16q f_2^3 f_6^5 f_1^4 + 27q^2 f_6^3 f_2^2\right).$$

(2.47)

Comparing (2.34) with (2.47), we deduce that $\omega(12n + 2) = 12B_3(6n)$.

Theorem 2.8. For any nonnegative integer $n$, we have

$$\omega(12n + 10) = 6B_3(6n + 4).$$

Proof. Extracting the terms involving $q^{3n+1}$ in (2.42), dividing both sides by $q$ and replacing $q^3$ by $q$, we get

$$\sum_{n=0}^{\infty} \omega(3n + 1)q^n = 6\varphi^3(q)\varphi(q)^2 \cdot \frac{f_2^3 f_4 f_1^{12}}{f_1^9 f_4 f_6}.$$ 

Substituting (2.36) into the above identity, we get

$$\sum_{n=0}^{\infty} \omega(3n + 1)q^n = 6\frac{f_1^{15}}{f_1^9 f_4} \cdot \frac{f_6^{10}}{f_1^5 f_1^{12}} \cdot \frac{f_2^3 f_4 f_1^{12}}{f_1 f_4 f_6} = 6\frac{f_2^{17} f_6^9}{f_1^1 f_4 f_1^{12}}.$$ 

(2.48)

Substituting (2.3) and (2.7) into (2.48), we obtain

$$\sum_{n=0}^{\infty} \omega(3n + 1)q^n = 6\frac{f_2^{17} f_6^9}{f_1^1 f_4 f_1^{12}} \cdot \frac{1}{f_1} \cdot \frac{f_1}{f_3}$$

$$= 6\frac{f_2^{17} f_6^9}{f_1^1 f_4 f_1^{12}} \cdot \left(\frac{f_4^{20} f_6^9}{f_2^2 f_8^8} + 8q f_4^{16} f_4^{12} f_8^8 + 16q^2 f_4^{14} f_8^8\right) \left(\frac{f_2 f_4 f_1 f_2}{f_6^3} - q^{3} \frac{f_2^3 f_6 f_4}{f_2^2 f_6^3}\right).$$

Extracting the term involving $q^{2n+1}$, dividing by $q$ and then replacing $q^2$ by $q$, we get

$$\sum_{n=0}^{\infty} \omega(6n + 4)q^n = 6\left(8\frac{f_2^3 f_4 f_1}{f_1 f_6} - \frac{f_2^3 f_4 f_6}{f_1^8} f_4 - 16q f_4^8 f_6^3 f_2^3\right).$$

(2.49)

By (2.3) we have

$$\frac{f_2^2}{f_1} = \left(\frac{f_4^6 f_6^3}{f_2^2 f_2 f_1^9} + 3q f_2^1 f_6 f_1^3 f_1^2 \frac{f_2^2}{f_2^1}\right)^2 = \frac{f_4^{12} f_6^6}{f_2^4 f_4 f_1^9} + 6q \frac{f_4^8 f_6^4}{f_2^2 f_2 f_1^9} + 9q^2 \frac{f_4^4 f_6^2 f_1 f_4 f_1^9}{f_2^4 f_4 f_1^9}.$$

(2.50)
Substituting (2.7) and (2.56) into (2.49), extracting the terms involving $q^{2n+1}$, dividing by $q$ and replacing $q^2$ by $q$, we obtain

$$\sum_{n=0}^{\infty} \omega(12n+10)q^n = 144\frac{f_2^8 f_3^3}{f_1^5}.$$  

Comparing this identity with (2.35), we deduce that $\omega(12n+10) = 6B_3(6n+4)$. □

3. CONCLUDING REMARKS

Let $A_3^{(k)}(n)$ denote the number of partition $k$-tuples of $n$ with 3-cores. In particular, $A_3^{(1)}(n) = a_3(n)$, $A_3^{(2)}(n) = A_3(n)$ appeared in the existing literature (see [2, 4, 12, 13]) and $A_3^{(3)}(n) = B_3(n)$ in this paper.

Again, let $\omega^{(k)}(n)$ denote the number of representations of a nonnegative integer $n$ in the form

$$n = x_1^2 + \cdots + x_k^2 + 3(y_1^2 + \cdots + y_k^2), \quad x_i, y_i \in \mathbb{Z}, \quad i = 1, 2, \cdots, k.$$  

It is easy to see that the generating function of $A_3^{(k)}(n)$ and $\omega^{(k)}(n)$ are given by

$$\sum_{n=0}^{\infty} A_3^{(k)}(n)q^n = \frac{f_3^{3k}}{f_1^k}, \quad \text{and} \quad \sum_{n=0}^{\infty} \omega^{(k)}(n)q^n = \varphi^k(q)\varphi^k(q^3)$$

respectively.

From the existing papers and our work, we know many arithmetic identities about $A_3^{(k)}(n)$ for $k = 1, 2, 3$. Meanwhile, we have seen some relations between $A_3^{(k)}(n)$ and $\omega^{(k)}(n)$, such as

$$\omega^{(1)}(12n + 4) = 6A_3^{(1)}(n),$$

$$\omega^{(2)}(6n + 5) = 12A_3^{(2)}(2n + 1),$$

$$\omega^{(3)}(6n + 5) = 4A_3^{(3)}(6n + 4).$$

Based on these facts and observations, we would like to ask the following two questions.

**Question 1.** Can we find some arithmetic identities involving $A_3^{(k)}(n)$ for all $k$?

**Question 2.** Can we find some arithmetic relations between $A_3^{(k)}(n)$ and $\omega^{(k)}(n)$ for all $k$?

To answer these questions, we believe that one may need to develop some new methods and ideas.

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