Noether symmetry approach in non-minimal derivative coupling gravity

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ABSTRACT: In this work, we examine solutions of the system of equations obtained by applying the Noether gauge symmetry (NGS) and its conserved quantity for the standard general relativity (GR) and the non-minimal derivative coupling (NMDC) cosmological model. We discover two salient features of the solutions. The first one is $a(t) \propto t^{1/3}$ for a kinetic-dominant phase which may emerge before inflationary period at very early time. The second one is a new form of scalar field $\phi(t)$ governed by the exponential cosmological solution for GR and NMDC $\phi_{GR}(t) = \sqrt{c_1 + c_2 t + c_3 e^{-\lambda t}}$ and $\phi_{NMDC}(t) = \sqrt{c_1 + c_2 e^{-\lambda_1 t} + c_3 e^{-\lambda_2 t}}$, respectively.

KEYWORDS: Noether Gauge Symmetry, non-minimal derivative coupling
1 Introduction

Astrophysical observations including Type Ia Supernovae [1, 2], cosmic microwave background (CMB) radiation [3–9], large scale structure [10], baryon acoustic oscillations (BAO) [11] as well as weak lensing [12] make a strong evidence that the expansion of the universe is presently accelerating. In spite its successes, the so-called Lambda cold dark matter (ΛCDM) [13] is plagued by the cosmological problem [14] and the coincident problem [15]. The phase of late-time cosmic acceleration receives lots of attention. However, the introduction of the so-called “dark energy (DE)” in the context of conventional general relativity is one of promising explanations. Additionally, another possible scenario is to engineer Einstein gravity on the large-scale methodology. There were some reviewed articles published so far regarding the mentioned issues, see for example [16–19] and references therein. However, very little is known about the DE sector of the universe and it possesses one of the unsolved problems in physics.

Alternative paradigms by engineering the Einstein field equations either in the geometric part or in the stress-energy tensor are widely accepted to explain effects of dark ingredients [20]. The $f(R)$ theories of gravity deserves as one of the simplest modifications to the standard general relativity. Here the Lagrangian density of $f$ is an arbitrary function of the scalar curvature $R$ [21, 22]. It is worth noting that there were rigorous reviews on $f(R)$ theories [23, 24] as well as on Born–Infeld inspired modifications of gravity [25]. In ref.[26], the authors investigated the cosmological implications of the modified theories of gravity on inflation, bounce and late-time evolution. Apart from these modified theories of gravity, theories of non-minimal derivative coupling to gravity attract much attention of theoretical and phenomenological points of view, see, e.g., [27–38]. More specifically, their applications on inflation and its consequences were proposed by a clump of authors [39–46].
The important role of the Noether symmetry in cosmology has received increasing attention within decades in order to select the viable models [47]. Conserved quantities of the system, as well as unknown functions, can be determined with the help of the Noether symmetry approach. More specifically, by using the Noether symmetry, we can obtain the exact solutions. The Noether symmetry approach has been applied to study various cosmological scenarios so far including nonlocal $f(T)$ gravity [48–50], viable mimetic $f(R)$ and $f(R, T)$ theories [51], $f(R)$ cosmology [52], the cosmological alpha-attractors [53], $f(G)$ theory [54]. Moreover, the exact solutions for potential functions, scalar field and the scale factors in the Bianchi models have been investigated in Refs.[55–57] and the solutions of the field equations of $f(R)$ gravity are investigated in static cylindrically symmetric space-time using the Noether symmetry technique [58].

The second kind of Noether symmetry approach for cosmological studies in the literature is the so-called Noether gauge symmetry (NGS) approach [59–61]. It is a more generalization of the conventional one. Very recently, the authors of Ref.[62] have discussed the NGS approach for the Eddington-inspired Born–Infeld theory. In the present work, we study a formal framework of the non-minimal derivative coupling (NMDC) gravity scenario through the NGS approach and present a detailed calculation of the point-like Lagrangian. The point-like Lagrangian of the Einstein-Hilbert action including the non-minimal derivative coupling (NMDC) sector are examined with spatially flat FLRW spacetime in which matters in such a universe only has a scalar field and a matter field. The latter model is expected to quantify to what extend the field kinetic term affects the evolution of the universe.

This paper is organized as follows: We will start by making a short recap of a formal framework of NMDC gravity and derive the point-like Lagrangian for underlying theory in Section 2. In Section 3, we study a Hessian matrix and quantify the Euler-Lagrange equations and Hamiltonian equations of the Einstein (GR) and the NMDC universes. In Section 4, the NGS approach for the GR and NMDC is discussed. We discuss exact cosmological solutions of both theories based with the help of the Noether symmetries of point-like Lagrangian. Finally, we conclude our findings in the last section.

2 Non-minimal-derivative coupling gravity

The non-minimal derivative coupling (NMDC) gravity model is a special case of the fifth term of Horndeski Lagrangian density that is given as follows:

$$L_5 = G_5(\phi, X)G^\mu\nu\partial_\mu\partial_\nu\phi,$$

where $X = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$. If we set $G_5(\phi, X) = -\phi/(M^2)$ where $M$ is the a new energy scale in the theory. Performing an integration by parts, we obtain

$$L_{5, \text{NMDC}} = \partial_\mu\left[(\phi, X)G^{\mu\nu}\partial_\nu\phi\right] - (\partial_\nu\phi)\partial_\mu G_5(\phi)G^{\mu\nu},$$

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$$= \frac{1}{M^2} G^{\mu\nu}\partial_\nu\phi\partial_\mu\phi.$$
The appearance of minus sign in front of $\frac{1}{M^2}\dot{G}^{\mu\nu}$ is due to the avoidance of ghosts in the scalar field sector, i.e., $\ddot{\phi} + 2V(\phi) > -\frac{1}{M^2}\dot{G}_{00}\ddot{\phi}^2 = -3(\frac{1}{\kappa})^2\dot{\phi}^2$. Hence, the potential term dominates over the NMDC term \[27\]. The NMDC Lagrangian was first proposed in \[27\] and the action is given by

$$S_{NMDC}(g) = \int d^4x \sqrt{-g} \left[ R - (\epsilon g_{\mu\nu} + \kappa G_{\mu\nu})\phi^\mu \phi^\nu - 2V(\phi) \right] + S_m(g_{\mu\nu}, \Psi), \quad (2.3)$$

where $\kappa \equiv M^{-2}$ is a NMDC free parameter that has dimension of $[M_P^{-2}]$, $S_m(g_{\mu\nu}, \Psi)$ denotes the matter field action. The spatially flat FLRW metric can be written as

$$ds^2_g = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a(t)^2dx^2, \quad (2.4)$$

where $d^3x$ is the spatial volume after a proper compactification for spatial flat section. $\epsilon = +1$ and $\epsilon = -1$ denote an ordinary scalar field and phantom scalar field, respectively. The point-like Lagrangian can be extracted from Eq.\[2.3\] to yield

$$\mathcal{L}_{NMDC}(a, \phi, \dot{a}, \dot{\phi}) = -6a\ddot{a}^2 + \left( \epsilon a^3 \dot{\phi}^2 - 3\kappa a^2 \dot{\phi}^2 \right) - 2a^3V(\phi) - 2\rho_m(a)a^3. \quad (2.5)$$

It is easy to see that the number of configuration space ($n$) (or the minisuperspace) is equal to 2 because of the appearance of variables $\{a(t), \phi(t)\}$ in $\mathcal{L}_{NMDC}$.

### 3 Hessian matrix, EL equations & NMDC universe

The configuration space variables and their time derivative of both gravity models are $q^i = \{a, \phi\}$ and $\dot{q}^i = \frac{dq_i}{dt} = \{\dot{a}, \dot{\phi}\}$. The Hessian matrix for GR by using the point-like Lagrangian, $\mathcal{L}_{GR} = -6a\ddot{a}^2 + \epsilon a^3 \dot{\phi}^2 - 2a^3V(\phi) - 2\rho_m(a)a^3$, can be expressed as

$$[W_{ij}]_{GR} = \begin{bmatrix} \frac{\partial^2 L}{\partial \dot{a}^2} & \frac{\partial^2 L}{\partial \dot{a} \partial \phi} \\ \frac{\partial^2 L}{\partial \phi \partial \dot{a}} & \frac{\partial^2 L}{\partial \phi^2} \end{bmatrix} = \begin{bmatrix} -12a & 0 \\ 0 & 2\epsilon a^3 \end{bmatrix}. \quad (3.1)$$

The determinant of the Hessian matrix of the GR Lagrangian is $\det[W_{ij,GR}] = -24\epsilon a^4 \neq 0$. and the Hessian matrix for NMDC can be expressed as

$$[W_{ij}]_{NMDC} = \begin{bmatrix} \frac{\partial^2 L}{\partial \dot{a}^2} & \frac{\partial^2 L}{\partial \dot{a} \partial \phi} \\ \frac{\partial^2 L}{\partial \phi \partial \dot{a}} & \frac{\partial^2 L}{\partial \phi^2} \end{bmatrix} = \begin{bmatrix} -12a - 6\kappa a^2 & -6\kappa a \dot{\phi}^2 \\ -6\kappa a \dot{\phi}^2 & 2\epsilon a^3 - 6\kappa a^2 \end{bmatrix}. \quad (3.2)$$

The determinant of the Hessian matrix of NMDC Lagrangian are $\det[W_{ij,NMDC}] = -24\epsilon a^4 + 72\kappa a^2 \dot{\phi}^2 - 12\kappa a^4 \dot{\phi}^2 \neq 0$. Without the contribution of NMDC free parameter ($\kappa$), this can be clearly reduced to the parameters derived in GR case. Mathematically, the fact that the determinant of the Hessian matrix is not equal to zero is called a regular or a non-degenerate Lagrangian. It is important to note that a key concept of a gauge field theory is the general solution of the equations of motion that contains arbitrary functions of time and the canonical variables are not all independent but relate to each other by the constraint
The energy function that is a constant of motions is given by equation in GR case as follows:

\[
H = \frac{\partial L}{\partial \dot{a}} \frac{\partial \dot{a}}{\partial \dot{a}} + \frac{\partial L}{\partial \dot{\phi}} \dot{\phi} - L,
\]

where \( p_i \) is the canonical momenta via the Legendre transformation and Lagrangian as follows:

\[
H_{\text{GR}} = \frac{\partial L_{\text{GR}}}{\partial \dot{q}_i} \dot{q}_i - L_{\text{GR}} = \frac{\partial L_{\text{GR}}}{\partial \dot{a}} \dot{a} + \frac{\partial L_{\text{GR}}}{\partial \dot{\phi}} \dot{\phi} - L_{\text{GR}},
\]

\[
= p_a \dot{a} + p_\phi \dot{\phi} - L_{\text{GR}},
\]

\[
= \left(-12a \dot{a} - 3\epsilon k a^2 \dot{\phi}^2\right) \dot{a} - 2\epsilon a^3 \dot{\phi} - L_{\text{GR}} = 0,
\]

\[
= -6a \dot{a}^2 + \epsilon a^3 \dot{\phi}^2 + 2a^3 V(\phi) + 2\rho_m(a)a^3 = 0.
\]

The energy function that is a constant of motions is given by \( E_L = \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = p_i \dot{q}_i - L = H \) where \( p_i \) is the canonical momenta. This condition can be rearranged to give the Friedmann equation in GR case as follows:

\[
3H_{\text{GR}}^2 = \rho_m(a) + \frac{\epsilon}{2} \dot{\phi}^2 + V(\phi).
\]

The Hamiltonian constraint equation for NMDC model can be found similarly as follows:

\[
H_{\text{NMDC}} = \frac{\partial L_{\text{NMDC}}}{\partial \dot{q}_i} \dot{q}_i - L_{\text{NMDC}} = \frac{\partial L_{\text{NMDC}}}{\partial \dot{a}} \dot{a} + \frac{\partial L_{\text{NMDC}}}{\partial \dot{\phi}} \dot{\phi} - L_{\text{NMDC}},
\]

\[
= p_a \dot{a} + p_\phi \dot{\phi} - L_{\text{NMDC}},
\]

\[
= \left(-12a \dot{a} - 6\epsilon k a^2 \dot{\phi}^2\right) \dot{a} - 3\epsilon k a^2 \dot{\phi}^2 + 2\epsilon a^3 \dot{\phi} - L_{\text{NMDC}} = 0,
\]

\[
= -6a \dot{a}^2 - 9\epsilon k a^2 \dot{\phi}^2 + \epsilon a^3 \dot{\phi}^2 + 2a^3 V(\phi) + 2\rho_m(a)a^3 = 0.
\]

This condition can be rearranged to give the modified Friedmann equation\[64\] as

\[
3H^2 = \rho_m(a) + \frac{\epsilon}{2} \dot{\phi}^2 + V(\phi) - \frac{9}{2}\epsilon \kappa H^2 \dot{\phi}^2.
\]

It’s easy to notice that it brings up the Friedmann equation for GR when \( \kappa = 0 \). The total NMDC Hamiltonian \( (H_{\text{NMDC}}) \) can be used to evaluate an evolution invoking the Hamiltonian equations of motion as follows:

\[
p_a = \frac{\partial L_{\text{NMDC}}}{\partial \dot{a}} = \dot{a} \left[-12a - 6\epsilon k a \dot{\phi}^2\right],
\]

\[
p_\phi = \frac{\partial L_{\text{NMDC}}}{\partial \dot{\phi}} = \dot{\phi} \left[2\epsilon a^3 - 6\epsilon k a^2\right]
\]

\[
\dot{a} = \frac{\partial H_{\text{NMDC}}}{\partial p_a} = \frac{-p_a}{12a + 6\epsilon k a \dot{\phi}^2}
\]

\[
\dot{\phi} = \frac{\partial H_{\text{NMDC}}}{\partial p_\phi} = \frac{p_\phi}{2\epsilon a^3 - 6\epsilon k a^2}
\]

\[
p_a = -\frac{\partial H_{\text{NMDC}}}{\partial a} = 6a^2 + 9\epsilon k a^2 \dot{\phi}^2 - 3\epsilon a^2 \dot{\phi}^2 - 6a^2 V(\phi) - 2\rho_m(a)a^3 - 6\rho_m(a)a^2
\]

\[
\dot{p}_a = -\frac{\partial H_{\text{NMDC}}}{\partial \dot{a}} = -2a^3 V'(\phi).
\]
In order to obtain the dynamic solutions, we have to calculate the Euler-Lagrange equations for \( a(t) \) and \( \phi(t) \) which gives the same results as Eq.(2)-(4) of Ref \([64]\) as shown below:

\[
6\dot{a}^2 - 3\epsilon a^2 \dot{\phi}^2 + 3\kappa \dot{a}^2 \phi^2 + 6a^2 V(\phi) + 2\rho'_m(a)a^3 + 6\rho_m(a)a^2 - 12\ddot{a}^2 - 12a\dddot{a} - 6\kappa a^2 \dot{\phi}^2 \\
-6\kappa a\dddot{\phi}^2 - 12\kappa a\ddot{a}\dot{\phi} = 0. \tag{3.13}
\]

Rearranging, this shows that

\[
3\dot{H}^2 + 2\dot{H} = -\frac{1}{2} \left[ \epsilon + \kappa (2\dot{H} + 3H^2 + 4H \frac{\dot{\phi}}{\phi}) \right] + V(\phi) + \frac{\rho'_m(a)a}{3} + \rho_m, \tag{3.14}
\]

where the fluid equation \([65, 66]\), \( \rho'(a) \equiv \frac{d\rho_m}{da} = -3(\rho_m + \rho_m)/a \), has been used to get the last line of Eq.(3.14).

The modified Klien-Gordon equation for NMDC is directly calculated from the Euler-Lagrange equation for scalar field. Thus this gives

\[
2a^3 \dot{V}_\phi + 6a^2 \epsilon \dot{\phi} + 2a^3 \dot{\phi} - 6\kappa a^2 \phi - 12\kappa a\ddot{a}\dot{\phi} - 6\kappa a^2 \dddot{\phi} = 0. \tag{3.15}
\]

Having used the relation \( \frac{\ddot{a}}{a} = \dot{H} + H^2 \), it can be shown that

\[
\epsilon(\ddot{\phi} + 3H \dot{\phi}) - 3\kappa \left( H^2 \ddot{\phi} + 2H \dot{H} \dot{\phi} + 3H^3 \dot{\phi} \right) = -V_\phi. \tag{3.16}
\]

4 Noether gauge symmetries

In this section, we employ the Noether gauge symmetries to figure out exact solutions of the systems both in the standard GR cosmology and in the NMDC universe.

4.1 Standard GR cosmology

The Lagrangian for GR is written as

\[
\mathcal{L}_{GR} = \sqrt{-g}R + \mathcal{L}_\phi + \mathcal{L}_m, \tag{4.1}
\]

where the second term is the scalar field Lagrangian and the last term is the matter field one. We can derive the point-like Lagrangian of GR to obtain

\[
\mathcal{L}_{GR} = -6a\ddot{a}^2 + \epsilon a^3 \dot{\phi}^2 - 2a^3 V(\phi) - 2a^3 \rho_m. \tag{4.2}
\]

The approach of Noether gauge symmetry can be applied to Eq.(2.5) aiming to specify cosmological functions of the standard GR and NMDC gravity. A vector field in this approach can be written as

\[
X_{NGS} = \tau \frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial a} + \phi \frac{\partial}{\partial \phi}. \tag{4.3}
\]
The first prolongation of NGS reads
\[ X_{NGS}^{[1]} = X_{NGS} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\phi} \frac{\partial}{\partial \dot{\phi}}, \] (4.4)
where the undetermined parameter \( \tau \) is a function of \( \{t, a, \phi\} \). The time derivative for \( \alpha(t, a, \phi) \) and \( \varphi(t, a, \phi) \) are defined as
\[
\dot{\alpha}(t, a, \phi) = D_t \alpha - \dot{a} D_t \tau, \\
\dot{\varphi}(t, \phi) = D_t \varphi - \dot{\phi} D_t \tau.
\] (4.5)
Here \( D_t \) is the operator of a total differentiation with respect to \( t \), i.e.
\[ D_t = \frac{\partial}{\partial t} + \dot{a} \frac{\partial}{\partial a} + \dot{\phi} \frac{\partial}{\partial \phi}. \] (4.6)
The vector field \( X_{NGS} \) is a NGS of a Lagrangian \( L(t, a, \phi, \dot{a}, \dot{\phi}) \), if there exists a gauge function \( B(t, a, \phi) \) which obeys the Rund-Trautmann identity, see [67–69] for explicit derivation
\[ X_{NGS}^{[1]} L + L D_t \tau = D_t B. \] (4.7)
For NSG without gauge term, i.e. \( B = 0 \), it requires that \( \tau = 0 \). Therefore Eq. (4.7) is reduced to \( L_{X_{NGS}^{[1]}} L = 0 \) that is the condition for Noether symmetry [61]. The Noether gauge condition yields
\[ X_{NGS}^{[1]} L_{GR} + L_{GR} D_t \tau = D_t B \] (4.8)
Using the Noether gauge condition to \( L_{GR} \), we have 24 terms which can be expressed as follows:
\[
D_t B = -6a^2 V(\phi) \alpha - 6a^2 \alpha \rho_m(a) - 6a \dot{\alpha}^2 - 2a^3 \varphi V'(\phi) + 3e a^2 \alpha \dot{\phi}^2 - 2a^3 \alpha \rho_m'(a) \\
-12a \dot{a} \alpha - 2a^3 V(\phi) \tau_t - 2a^3 \rho_m(a) \tau_t + 6a \dot{a}^2 \tau_t - e a^3 \dot{\phi}^2 \tau_t + 2e a^3 \dot{\varphi} \tau_t - 12a \dot{a} \dot{\phi} \alpha \phi \\
-2a^3 V(\phi) \dot{\phi} \tau_{a \phi} - 2a^3 \rho_m(a) \dot{\phi} \tau_{\phi} + 6a \dot{a} \dot{\phi} \tau_{a \phi} - e a^3 \dot{\varphi}^2 \tau_{\phi} + 2e a^3 \dot{\varphi} \tau_{\phi} - 12a \dot{a} \dot{\phi} \alpha \phi \\
-2a^3 V(\phi) \dot{a} \tau_{a} - 2a^3 \rho_m(a) \dot{a} \tau_{a} + 6a \dot{a} \dot{\alpha} \tau_{a} - e a^3 \dot{\alpha} \dot{\phi}^2 \tau_{\phi} + 2e a^3 \dot{\alpha} \dot{\varphi} \tau_{\phi}. \] (4.9)
After separation of monomials and polynomials in term of \( \dot{a}^2, \dot{\phi}^2, \dot{a}, \dot{\phi}, \dot{a} \dot{\phi}, \dot{\phi}^3, a^3 \) and \( \dot{a} \dot{\phi}^2 \), the constraints equations and the PDEs can be given as
\[
\varphi_a = \alpha_\phi = \tau_\phi = \tau_a = 0 \] (4.10)
\[
\alpha + 2a \alpha_a - a \tau_t = 0 \] (4.11)
\[
3 \alpha + 2a \varphi_\phi - a \tau_t = 0 \] (4.12)
\[
6 \alpha_\phi = e a^2 \varphi_a = 0 \] (4.13)
\[
B_a = -12a \alpha_t \] (4.14)
\[ \alpha(a) \quad \varphi(\phi) \quad \tau(t) \quad \text{add. conditions} \]

| \(c_1 a\) | \(c_2 \phi\) | \(3c_1 t + c_3\) | \(\text{NO}\) |
|---|---|---|---|
| \(c_1 \ln a\) | \(e^{c_2 \phi}\) | \(c_2 t^4 + 3c_1 t + c_5\) | \(\phi(t) = c_3 \ln t, \ c_4 \equiv c_2 c_3\) |
| \(c_1 \ln a\) | \(c_2 \ln \phi\) | \(\int \left(2c_1 \ln a + c_2 \phi + c_3\right) dt\) | \(c_2 = \frac{c_5}{a}(\ln a - c_1)\) |

Table 1. We show possible parameters of the system \(\alpha, \varphi\) and \(\tau\) and their relations.

\[ B_\phi = 2\epsilon a^3 \dot{\varphi}_t \]  
\[ B_t = -\alpha \left[6a^2 V(\phi) + 6a^2 \rho_m(a) + 2a^3 \rho_m'(a)\right] - 2\varphi a^3 V'(\phi) \]
\[ -\tau_t \left[2a^3 V(\phi) + 2a^3 \rho_m(a)\right]. \]  

\[ (4.15) \quad (4.16) \]

Because there has no additional conditions, we are interested in examining only for a linear form of solutions. Using the fact that \(\alpha = c_1\) and \(\varphi = c_2\), this gives \(\tau_t = 3c_1\). The boundary term partly derived from Eq. (4.14) and Eq. (4.15) is expressed as

\[ B_{(\alpha, \phi)} = -6c_1 a^2 \dot{\alpha} + 2c_2 \epsilon a^3 \dot{\phi}. \]  

\[ (4.17) \]

The constant of motion for GR case is given by

\[ \Sigma_{0, \text{GR}} = \alpha \frac{\partial L}{\partial \dot{a}} + \varphi \frac{\partial L}{\partial \dot{\phi}}, \]
\[ = \alpha(-12a\dot{a}) + \varphi(2\epsilon a^3 \dot{\phi}). \]
\[ = -12c_1 a^2 \dot{\alpha} + 2c_2 \epsilon a^3 \dot{\phi}. \]  

\[ (4.18) \]

A first integral or a Noether integral of the system or a conserved quantity associated with \(X\) can be derived using

\[ I(t, q^i, \dot{q}^i) = -\tau(t, q^i) \left(q^i \frac{\partial L}{\partial \dot{q}^i} - L\right) + \eta^i \frac{\partial L}{\partial \dot{q}^i} - B(t, q^i). \]

\[ (4.19) \]

It is easy to see that when setting \(\tau = 0\) or \(\left(q^i \frac{\partial L}{\partial \dot{q}^i} - L\right) = 0\)

\[ I(t, q^i, \dot{q}^i) = \eta^i \frac{\partial L}{\partial \dot{q}^i} - B(t, q^i), \]
\[ = \Sigma_0 - B(t, q^i). \]  

\[ (4.20) \]

where in GR case \(\tau(t) = 3c_1 t + c_3\) and \(\eta^i = \{\alpha(a) = c_1 a, \varphi(\phi) = c_2 \phi\}\). The Lagrangian-related Noether vector \(X\) for GR is

\[ X = \xi(t, q^k) \frac{\partial L}{\partial t} + \eta^i(t, q^k) \frac{\partial}{\partial q^i}. \]  

\[ (4.21) \]

This gives

\[ X_{\text{GR}} = (3c_1 t + c_3) \partial_t + c_1 a \partial_a + c_2 \phi \partial_\phi. \]  

\[ (4.22) \]
The corresponding generators of a Noether point symmetry are

\[ X_1 = \partial_t, \]
\[ X_2 = 3t\partial_t + a\partial_a + \phi\partial_\phi. \]  
(4.23)

The simple commutative algebra of the two symmetry generators is

\[ [X_1, X_2] = 0. \]  
(4.24)

Since it admits the Noether symmetry \( \partial_t \), the first corresponding conserved quantity \( (I_1) \) related to the energy conservation via the time translation symmetry is

\[ I_{1,GR} = E_{LGR} = H_{GR} = 0. \]  
(4.25)

The second Noether integral is

\[
I_{2,GR} = -(3c_1 t + c_3)\left[ -6a\dot{a}^2 + \frac{1}{2}a^3\ddot{\phi}^2 + 2a^3V(\phi) + 2a^3\rho_m \right] - 6c_1a^2\dot{a}.
\]  
(4.26)

Using Eq. (3.3), i.e. \( H_{GR} = 0 \), this gives

\[ I_{2,GR} = (3c_1 t + c_3)H_{GR} - 6c_1a^2\dot{a}, \]
\[ = -6c_1a^2\dot{a}. \]  
(4.27)

That leads to

\[ a(t) = \left( -\frac{I_{2,GR}}{2c_2} \right)^{1/3} t^{1/3}, \]
\[ a(t) = a_0 t^{1/3}. \]  
(4.28)

Comparing to scale factor for single-component universe see section 5.3 of Ref. [65],

\[ a(t) = \left( \frac{t}{t_0} \right)^{2/3w}, \]  
(4.29)

this yields the equation of state parameter for dominance role of kinetic part of the scalar field rather than it potential, i.e. \( \dot{\phi}^2 \gg V(\phi) \)

\[ w = 1, \]  
(4.30)

this indicate the existence a kinetic dominance (KD) phase after the big bang but before the slow roll condition took place. The form of scalar field that corresponds to that scale factor is \( \phi \propto t^{1/6} \) [70].

For \( \alpha(a) = c_1 \ln a \) and \( \varphi(\phi) = c_2 \ln \phi \), this gives \( \alpha_\phi = 0, \varphi_a = \frac{\partial}{\partial a}(c_2 \ln \phi) = 0 \) and
\( c_2 = \frac{\dot{\phi}}{a} (\ln a - c_1) \) as shown as an example in Table (1). The constant of motion and gauge function related to this form of solutions take the form
\[
\Sigma_{\theta, GR(2)} = -12c_1 a \dot{a} \ln a + 2 \epsilon_c 2 a^3 \dot{\phi} \ln \phi,
\]
\[
B_{(a, \phi)} = -12c_1 a \dot{a} + 2 \epsilon_c 2 a^3 \dot{\phi} \ln \phi, 
\] (4.31)
respectively. From Eq. (4.20), we find
\[
\frac{I_{2, GR(2)}}{12c_1} = c_3 = \frac{3a^2}{4} - \frac{1}{2} a^2 \ln a 
\] (4.32)
which leads to a scale factor written of the form
\[
a(t) = \frac{\pm 2\sqrt{|c_3|} t}{\sqrt{\text{production} \ln\left(\frac{2a^2}{\tau}\right)}}. 
\] (4.33)

Here we will not elaborate on this case further.

### 4.2 NMDC universe

For the NMDC universe, the Lagrangian for NMDC gravity is written as
\[
\mathcal{L}_{NMDC} = \sqrt{-g} \left[ R - (\epsilon g_{\mu\nu} + \kappa G_{\mu\nu}) \nabla^\mu \phi \nabla^\nu \phi - 2V(\phi) \right], 
\] (4.34)
in which it can illustrate the point-like Lagrangian of NMDC as
\[
\mathcal{L}_{NMDC} = -6a \dot{a}^2 + \left( \epsilon a^3 \dot{\phi}^2 - 3\kappa a^2 \phi^2 \right) - 2a^3 V(\phi) - 2a^3 \rho_m(a). 
\] (4.35)

The Noether gauge condition in this case yields
\[
X_{NGS}^{[1]} \mathcal{L}_{NMDC} + \mathcal{L}_{NMDC} D_l \tau = D_l B 
\] (4.36)
Using the Noether gauge condition to \( \mathcal{L}_{NMDC} \), this gives 34 terms which can be expressed as follows:
\[
D_l B = -6a^2 V(\phi) \alpha - 6a^2 \alpha \rho_m(a) - 6a^2 \alpha - 2a^3 \phi V'(\phi) + 3\epsilon a^2 \alpha \phi^2 - 3\kappa a^2 \dot{\phi}^2 - 2a^3 \alpha \rho_m \\
-12a \dot{a} \alpha - 6ka \dot{a} \phi^2 \alpha - 2a^3 V(\phi) \tau_\phi - 2a^3 \rho_m(a) \tau_\phi + 6a^2 \tau_\phi - \epsilon a^3 \dot{\phi}^2 \tau_\phi + 9\kappa a^2 \phi^2 \tau_\phi \\
+2\epsilon a^3 \dot{\phi}^2 \tau_\phi - 6\kappa a^2 \dot{\phi}^3 \tau_\phi - 12a \dot{a} \phi \alpha - 6ka \dot{a} \phi^2 \alpha - 3a^3 V(\phi) \dot{\phi} \tau_\phi - 2a^3 \rho_m(a) \dot{\phi} \tau_\phi \\
+6a^2 \dot{\phi} \tau_\phi - \epsilon a^3 \dot{\phi}^2 \tau_\phi + 9\kappa a^2 \phi^2 \tau_\phi + 2\epsilon a^3 \dot{\phi}^2 \phi_\phi - 6\kappa a^2 \phi^2 \phi_\phi - 12a \dot{a} \alpha - 6ka \dot{a} \phi^2 \alpha \\
-2a^3 V(\phi) \dot{\phi} \tau_\phi - 2a^3 \rho_m(a) \dot{\phi} \tau_\phi + 6a \dot{a}^3 \alpha - 6a \dot{a} \phi^2 \alpha - 2a^3 \alpha \rho_m - 9\kappa a^3 \phi^2 \alpha - 2a \dot{a}^3 \phi \dot{\phi} - 6ka \dot{a} \phi^2 \alpha.
\] (4.37)
After separation of polynomials and monomials, we can express each term in a more compact form as follows:

\[
\begin{align*}
\dot{a}^2 &= -6\alpha + 6a\tau_t - 12a\alpha_a = 0, \\
\dot{\phi}^2 &= 3e\alpha - e\alpha^3 + 2e\alpha^3\phi = 0, \\
\dot{a}^2\dot{\phi}^2 &= -3\kappa\alpha + 9\kappa\alpha\tau_t - 6\kappa\alpha\phi - 6\kappa\alpha_a = 0, \\
\dot{\phi}^2 &= -6\kappa\alpha - e\alpha^3\phi = 0, \\
\dot{a}^2\dot{\phi} &= -6\kappa\alpha\phi + 6\kappa\phi = 0, \\
\dot{\phi} &= -12a\alpha + 2e\alpha^3 = 0, \\
\dot{\phi}^3 &= -6\kappa\alpha\phi = 0, \\
\phi^3 &= -e\alpha^3\phi = 0, \\
\dot{a}^2\phi^3 &= 9\kappa\alpha\phi = 0, \\
\dot{a}^2\phi &= 6\kappa\alpha\phi = 0, \\
\dot{\phi}^2 &= 9\kappa\alpha\phi = 0, \\
\dot{a}^2\phi^2 &= -6\kappa\alpha\phi = 0, \\
\dot{\phi} &= 6\kappa\alpha\phi = 0.
\end{align*}
\]

\[
\begin{align*}
\dot{B}_a &= \dot{a} - 12a\alpha\tau_t - 2a^3V(\phi)\tau_a - 2a^3\rho_m(a)\tau_a, \\
\dot{\phi}B &= \dot{\phi} - 2e\alpha^3\tau_t - 2a^3V(\phi)\tau_t - 2a^3\rho_m(a)\tau_\phi, \\
B_t &= -\alpha \left[ 6a^2V(\phi) + 6a^2\rho_m(a) + 2a^3\rho_m'(a) \right] + 2\varphi a^3V'(\phi) \\
&\quad + \tau \left[ 2a^3V(\phi) + 2a^3\rho_m'(a) \right].
\end{align*}
\]

The constraint equations and the PDEs can be given as follows:

\[
\begin{align*}
\phi_a &= \phi_t = \alpha_a = \alpha_t = \tau_a = \tau_\phi = 0. \\
\alpha + 2a\alpha_a - a\tau_t = 0 \\
3\alpha + 2a\alpha_a - a\tau_t = 0 \\
\alpha + 2a\alpha_a + 2a\alpha_a - 3a\tau_t = 0 \\
B_a &= B_\phi = 0
\end{align*}
\]
\[ B_t = -\alpha \left[ 6a^2V(\phi) + 6a^2\rho_m(a) + 2a^3\rho'_m(a) \right] - \varphi \left[ (2a^3V'(\phi)) \right] \]
\[ - \tau_t \left[ 2a^3V(\phi) + 2a^3\rho_m(a) \right]. \tag{4.58} \]

What is different from GR is the non-vanishing of two terms contributing to NMDC, i.e. \( B_a = B_\phi = 0 \). Offsetting this, Eq.\((4.56)\) is modified due to the existence of the NMDC term, i.e.,
\[ \kappa \dot{a}^2 \dot{\phi} \left( 3\alpha + 6a\dot{\phi} + 6a\alpha a - 9a\tau_t \right) = 0. \tag{4.59} \]

Substituting Eq.\((4.54)\) into Eq.\((4.56)\), this yields the relation \( \varphi = \tau_t \). From Eq.\((4.54)\) and Eq.\((4.55)\), this gives
\[ a(t) = \frac{3\alpha}{\varphi} = \frac{3\alpha}{3(2\alpha a - \tau_t)}. \tag{4.60} \]

It should be noted that \( \varphi \neq 0 \) and \( 2\alpha a - \tau_t \neq 0 \). Hence we have one more relation given by
\[ \varphi \phi = 3(2\alpha a - \tau_t). \tag{4.61} \]

Using the fact that \( \varphi = \tau_t \), this leads to the relation of \( \tau_t \), \( \alpha_a \) and \( \varphi \phi \) as follows:
\[ \alpha_a = \frac{2}{3} \tau_t = \frac{2}{3} \varphi, \tag{4.62} \]
\[ c_1 = \frac{2}{3} c_2. \tag{4.63} \]

We find that the relation does not allow for the exponential and Logarithmic forms. Therefore, we assume
\[ \alpha(a) = c_1 a, \tag{4.64} \]
\[ \varphi(\phi) = c_2 \phi, \tag{4.65} \]
this leads to
\[ \tau(t) = c_2 t + c_3, \]
\[ \tau_1 = c_2. \tag{4.66} \]

The constant of motion for NMDC case reads
\[ \Sigma_{0,\text{NMDC}} = \alpha \frac{\partial L}{\partial \dot{a}} + \varphi \frac{\partial L}{\partial \dot{\phi}}, \tag{4.67} \]
\[ = \alpha \left[ -12a\dot{a} - 6ka\dot{a}\phi^2 \right] + \varphi \left[ 2\alpha a^3 \dot{\phi} - 6k\alpha a^2 \phi \right]. \]
\[ = -12c_1 a^2 \dot{a} + 2c_2 a^3 \phi \dot{\phi} - \kappa \left[ 6c_1 a^2 \phi^2 \dot{a} + 6c_2 a^2 \phi^2 \dot{a} \right]. \tag{4.68} \]

A first integral of the system or a conserved quantity associated with \( X_{NGS} \) can be derived from Eq.\((4.19)\) where in NMDC case we have used \( \xi = \tau(t) = c_2 t + c_3 \) and \( \eta' = \{ \alpha(a) = c_1 a, \varphi(\phi) = c_2 \phi \} \). The Lagrangian-related Noether vector \( X \) for NMDC is
\[ X_{\text{NMDC}} = \left( c_2 t + c_3 \right) \partial_t + c_1 a \partial_a + c_2 \phi \partial_\phi. \tag{4.69} \]
The corresponding Noether symmetries are
\[ X_1 = \partial_t, \]
\[ X_2 = t \partial_t + a \partial_a + \phi \partial_\phi. \]  
(4.70)
The simple commutative algebra of the two symmetry generators is
\[ [X_1, X_2] = 0. \]  
(4.71)

\[ I_{1,NMDC} = E_{NMDC} = H_{NMDC} = 0; \]
\[ I_{2,NMDC} = \left( c_2 t + c_3 \right) H_{NMDC} + \Sigma_{NMDC} - B_{NMDC} \]
\[ = -12c_1a^2  + 2c_2a^3\dot{\phi} - \kappa \left[ 6c_1a^2\dot{a} + 6c_2a\dot{a}^2 \right]. \]  
(4.72)

Eq. (4.72) may shed some light on the interplay between \( \phi(t) \) and \( a(t) \). From the fact that
\[ \frac{d\Sigma_{0,NMDC}}{dt} = \frac{dI_{2,NMDC}}{dt} = 0 \]
and the exponential cosmological solution expressed in term of \( a(t) = e^{\frac{1}{2}H_0 t}, \)
\[ \dot{a}(t) = \frac{1}{2}H_0 e^{\frac{1}{2}H_0 t}, \]  
(4.73)

where \( H_0 \) is a constant [71, 72], this leads to an interesting arrangement shown below:
\[ \frac{I_{2,NMDC}}{3c_1} e^{-\frac{3}{2}H_0 t} + \frac{2H_0}{c_1} - c_1 \phi \frac{d\phi}{dt} - \kappa \phi^2 (H_0 + \frac{3}{4}H_0^2) = 0. \]  
(4.75)
The positive solution is
\[ \phi(t) = \frac{\sqrt{c_4 + c_5 e^{-\lambda_1 t} + c_6 e^{-\lambda_2 t}}}{\sqrt{c_7}}. \]  
(4.76)

For simplicity, we have set
\[ c_4 \equiv -\frac{4\kappa H_0}{c_1} (H_0 + \frac{3}{4}H_0^2) - \frac{2\epsilon \lambda_1 H_0}{c_1}, \]
\[ c_5 \equiv \frac{2\kappa I_{2,NMDC}}{3c_1} (H_0 + \frac{3}{4}H_0^2), \]
\[ c_6 \equiv \kappa \epsilon \lambda_1 C[1] (H_0 + \frac{3}{4}H_0^2) + 2\kappa^2 C[1] (H_0 + \frac{3}{4}H_0^2)^2, \]
\[ c_7 \equiv \kappa \left( H_0 + \frac{3}{4}H_0^2 \right) (\epsilon \lambda_1 - 2(H_0 + \frac{3}{4}H_0^2)), \]
\[ \lambda_1 \equiv \frac{3H_0}{2}, \]
\[ \lambda_2 \equiv \frac{2\kappa (H_0 + \frac{3}{4}H_0^2)}{\epsilon}, \]
where \( C[1] \) is arbitrarily constant. We finally consider the case of GR cosmology by applying the same exponential cosmological solution to Eq. (4.78) and this gives
\[ \frac{\Sigma_{0,GR}}{2} e^{-\frac{3}{2}H_0 t} - 3c_1 H_0 - c_2 \epsilon \phi \frac{d\phi}{dt} = 0. \]  
(4.78)
The positive solution of Eq. (4.78) can be simplified into
\[ \phi(t) = \sqrt{c_1 + c_2 t + c_3 e^{-\lambda_1}}, \]  
(4.79)
where \( c_1 = \) arbitrarily constant, \( c_2 = \frac{6c_1 H_0}{c_2^2} \) and \( c_3 = -\frac{2 \Sigma_{0,GR} H_0}{3c_2} \).
5 Conclusion

In the present work, we have considered a formal framework of NMDC gravity and derived the point-like Lagrangian for underlying theory. We have studied a Hessian matrix and quantified the Euler-Lagrange equations and Hamiltonian equations of the Einstein (GR) and the NMDC universes. We further discussed the NGS approach for the GR and NMDC universe. We have studied exact cosmological solutions of both theories with the help of the Noether symmetries of point-like Lagrangian.

We have assumed the linear forms of $\alpha(a)$, $\varphi(\phi)$ and $\tau(t)$ that are compatible with the structure of the standard GR and explained the possible emergence of the kinetic field dominated phase which might take place before the inflationary stage. We discovered that the NGS condition under the structure of NMDC gravity can eliminate the dependence of the variables $a(t)$ and $\phi(t)$ on a gauge function. Consequently, a gauge function is at most dependent on the time variable only. The simplest form of $B(t)$ can be written as $B(t) = c_8 t + c_9$. Interestingly, we have found the new form of the solutions of $\phi(t) = \sqrt{c_1 + c_2 t + c_3 e^{-\lambda t}}$ for GR and $\phi(t) = c_7^{-1/2} \sqrt{c_4 + c_5 e^{-\lambda_1 t} + c_6 e^{-\lambda_2 t}}$ for NMDC universe, respectively. However, it is very interesting to figure out the physical consequences of the found solutions and we leave this for further investigation. Finally, it is hoped that this work will stimulate for further research on seeking other possible solutions of the system equations based on Noether gauge symmetry as shown in Eq.(4.10)-Eq.(4.16) for GR and Eq.(4.53)-Eq.(4.58) for NMDC cosmology.

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