NON-COLLAPSING CONDITION AND SOBOLEV EMBEDDINGS
FOR HAJLÅS-BESOV SPACES

JOAQUIM MARTÍN* AND WALTER A. ORTIZ**

ABSTRACT. In this paper we will focus on understanding the relation between Sobolev embedding theorems for Hajłasz-Besov spaces defined on a doubling metric measure space \((Ω, d, µ)\) and the non-collapsing condition of the measure, i.e.
\[
\inf_{x \in Ω} µ(B(x, 1)) > 0.
\]

We will also obtain embedding results for Hajłasz-Besov spaces whose modulus of smoothness is generated by a rearrangement invariant quasi-norm.

1. INTRODUCTION

In recent years, an important area of researches in metric measure spaces has been focused on understanding the relation between Sobolev embedding theorems and growth properties of the underlying measure (see for example [24], [2], [1], [14], [17], [20], [36], [37] and the references quoted therein). In this work we will continue with this topic for Hajłasz-Besov spaces.

Throughout the paper \((Ω, d, µ)\) denotes a doubling metric measure space, i.e. we assume that the measure \(µ\) satisfies the following doubling condition: there exists a constant \(C_µ > 1\), such that
\[
0 < µ(B(x, 2r)) \leq C_µ µ(B(x, r)) < ∞
\]
for every ball \(B(x, r)\), for all \(x \in Ω\) and \(r > 0\). The smallest constant \(C\) for which (1) is satisfied will be called the doubling constant and will be denoted by \(C_µ\). The above doubling condition is equivalent to that, for any ball \(B\) and any \(λ \in [1, ∞)\),
\[
µ(B(x, λr)) \leq C_µ λ^Q µ(B(x, r))
\]
where \(Q := \log_2 C_µ\) is called the upper dimension of \(Ω\). We will assume that \(µ(\{x\}) = 0\) for all \(x \in Ω\), (see Section 2.1 below).

Our main objective will be to prove the equivalence between the following non-collapsing condition: there exists a constant \(b > 0\) (called the non-collapsing constant) such that,
\[
\inf_{x \in Ω} µ(B(x, 1)) = b,
\]
and general versions of Hajlasz-Besov type embeddings theorems\footnote{Precise definitions and properties concerning all the topics that will appear in this Introduction and throughout the whole paper are contained in Section\ref{sec:2} below.}. Since if $\mu(\Omega) < \infty$, then obviously\footnote{In the Euclidean setting, applying Fubini’s theorem it is easy to verify that $E_p(f, t)$ is equivalent to the classical $L^p$--modulus of smoothness $\omega_p(f, t) = \sup_{|h| \leq t} \|f(x + h) - f(x)\|_{L^p}$ (see \cite{12}), therefore, $B^s_{p,q}(\mathbb{R}^n)$ coincides with the classical definition.} (2) holds, we will always assume in what follows that $\mu(\Omega) = \infty$.

As in the Euclidean case, there are several equivalent ways to define Hajlasz-Besov spaces in the setting of doubling metric measure spaces (see for example \cite{12}, \cite{13}, \cite{16}, \cite{19}, \cite{40}, \cite{41}, \cite{44} and the references therein). In this paper, we will use the approach based on modulus of smoothness for two reasons, (i) it just uses the metric measure structure of $\Omega$, and (ii) its relation with the real interpolation method.

We start, in order not to obscure the simplicity of the arguments, by considering the version of our main result for Hajlasz-Besov whose modulus is generated by $L^p$-spaces, delaying the general version to the last part of this introductory section.

To this end, we shall need some definitions.

Given $0 < p < \infty$, the $L^p$--modulus of smoothness of a measurable function $f$ in $\Omega$, is given by (see \cite{12} and \cite{9})

$$E_p(f, r) := \left( \int_\Omega \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(x) - f(y)|^p \, d\mu(y) \right)^{1/p}, \quad r > 0,$$

and the homogeneous Hajlasz-Besov space $\dot{B}^s_{p,q}(\Omega), 0 < s < 1, 0 < p < \infty, 0 < q \leq \infty$, is defined as the set of measurable functions for which the Besov seminorm

$$\|f\|_{\dot{B}^s_{p,q}(\Omega)} := \begin{cases} \left( \int_0^\infty \left( \frac{E_p(f, t)}{t^s} \right)^q \, \frac{dt}{t} \right)^{1/q}, & 0 < q < \infty, \\ \sup_{t > 0} t^{-s} E_p(f, t), & q = \infty, \end{cases}$$

is finite\footnote{See Section\ref{sec:2.2} below.}.

For measurable functions $f : \Omega \to \mathbb{R}$, the decreasing rearrangement of $f$ is the decreasing function $f^*$ defined on $[0, \infty)$ by\footnote{See Section\ref{sec:2.2} below.}

$$f^*(t) = \inf \{ s \geq 0 : \mu_f(s) \leq t \}, \quad t \geq 0.$$ 

where $\mu_f$ denotes the distribution function of $f$.

Associated to $f^*$, we consider the maximal function $f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds$, and for $0 < \alpha \leq 1$ the $\alpha$--oscillation of $f^*$ defined by

$$O(|f|^\alpha, t) := (|f|^{\alpha^*}(t) - |f|^{\alpha^*}(t)).$$

The $L^p$ version of our main result, is the following:

**Theorem 1.** Let $(\Omega, d, \mu)$ be doubling metric measure space with doubling constant $C_\mu$ an upper dimension $Q$. The following statements are equivalent

(i) Condition\footnote{In the Euclidean setting, applying Fubini’s theorem it is easy to verify that $E_p(f, t)$ is equivalent to the classical $L^p$--modulus of smoothness $\omega_p(f, t) = \sup_{|h| \leq t} \|f(x + h) - f(x)\|_{L^p}$ (see \cite{12}), therefore, $B^s_{p,q}(\mathbb{R}^n)$ coincides with the classical definition.} (3) holds.
(ii) Let $0 < p \leq \infty$, $0 < q \leq \infty$, $0 < s < 1$ and $\alpha = \min(1, p)$, then there is a positive constant $C = C(C_{\mu}, \alpha, q)$ such that for all $f \in L^\alpha + L^\infty$, we have that
\[
\left( \int_0^1 \left[ O(|f|^\alpha, t) \frac{dt}{t} \right]^q \right)^{1/q} \leq C \left( \|f\|_{E_{p,q}(\Omega)} + \|f\|_{L^\alpha + L^\infty} \right).
\]

Remark 2. In the setting of probability metric measure spaces the technique to obtain pointwise estimates of the special differences $O(|f|, t) := f^{**}(t) - f^*(t)$, called the oscillation of $f^*$, in terms of appropriate functionals depending on $f$ and the isoperimetric profile of the space has been developed by M. Milman and J. Martín (see [34], [35], [36]) and provide a considerable simplification in the theory of embeddings of Sobolev spaces.

We now turn to our main objective to present the corresponding version of Theorem 1 for Hajlasz-Besov spaces defined in a more abstract setting. The main idea is to change $L^p$ by more general function spaces in the definition of the modulus of smoothness, and the natural class seems to be the rearrangement invariant quasi-Banach function spaces ($q.r.i$ spaces for short) since, like Lebesgue spaces, $q.r.i$ spaces have the property equimeasurable functions $f, g$ (i.e., such that $f^* = g^*$) have equal quasi-norms (see Section 2.3 below).

Given a $q.r.i$ space $X$ on $\Omega$ and $0 < r < \infty$, the $r-$convexification of $X$ is defined as $X^{(r)} = \{ f : |f|^r \in X \}$ endowed with the following quasi-norm $\|f\|_{X^{(r)}} = \|\|f\|^r\|_X^{1/r}$. It is plain that if $r \geq 1$, then $X^{(r)}$ is a $q.r.i$ space, however for $0 < r < 1$, the functional $\|\cdot\|_{X^{(r)}}$ is not necessarily a norm. Notice that with this definition we have that $(L^p)^{r} = L^{pr}$.

Associated to $X^{(r)}$, we introduce a general variant of the modulus of smoothness in the following way: Let $0 < \alpha \leq 1$ and $f \in L^\alpha(\Omega) + L^\infty(\Omega)$. The $X^{(\alpha)}$-modulus of smoothness $E_{X^{(\alpha)}} : X^{(\alpha)} \times (0, \infty) \to (0, \infty)$ is defined by
\[
E_{X^{(\alpha)}}(f, r) = \|\nabla^\alpha_r f\|_{X^{(\alpha)}},
\]
where
\[
\nabla^\alpha_r f(x) = \left( \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(x) - f(y)|^\alpha \, d\mu(y) \right)^{1/\alpha}, \quad r > 0.
\]

The corresponding homogeneous Hajlasz-Besov space $B^s_{X^{(\alpha)}, q}(\Omega)$, $0 < q \leq \infty$ and $0 < s < 1$, defined by this modulus, consists of all functions $f \in L^\alpha(\Omega) + L^\infty(\Omega)$ for which
\[
\|f\|_{B^s_{X^{(\alpha)}, q}(\Omega)} = \left( \int_0^\infty \left( r^{-s} E_{X^{(\alpha)}}(f, r) \right)^q \frac{dr}{r} \right)^{1/q}
\]
is finite (with the usual modification when $q = \infty$).

\footnote{We shall refer to Banach function spaces that satisfies this property as rearrangement invariant spaces ($r.i. \text{ spaces for short}$).}
Remark 3. If $0 < p \leq 1$, then $E_{(L^1)^p}(f, r) = E_p(f, r)$, but if $1 < p < \infty$, then by Hölder’s inequality, we get

$$E_{L^p}(f, r) = \left( \int_{\Omega} \left( \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(x) - f(y)| d\mu(y) \right)^p d\mu(x) \right)^{1/p}$$

$$\leq \left( \int_{\Omega} \left( \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(x) - f(y)|^p d\mu(y) \right) d\mu(x) \right)^{1/p}$$

$$= E_p(f, r).$$

However, we will see in Remark 15 below that both modulus of smoothness produce the same Hajłasz-Besov spaces.

Our main result is the following:

Theorem 4. Let $(\Omega, d, \mu)$ be doubling metric measure space with doubling constant $C_\mu$ an upper dimension $Q$. The following statements are equivalent.

(i) Condition (2) holds.

(ii) Let $0 < \alpha \leq 1$, $0 < q \leq \infty$ and $0 < s < 1$. Then there is a positive constant $C = C(C_\mu, \alpha, q)$ such that for any r.i. space $X$ on $\Omega$ we have that

$$\left( \int_0^1 \left( O(\|f\|_{C^{s/q}}^q, t)^{1/\alpha} (t \frac{\phi_{X(\alpha)}(t)}{t^{s/q}})^q \frac{dt}{t} \right) \right)^{1/q} \leq C \left( \|f\|_{B_{X(\alpha)}^{s/q}} + \|f\|_{L^\infty + L\infty} \right),$$

where $\phi_{X(\alpha)}$ denotes the fundamental function of $X^{(\alpha)}$ (see Section 2.3 below).

Remark 5. Theorem 7 follows from Remark 3 and Theorem 4 considering $X = L^p$ and $\alpha = 1$ if $p \geq 1$, or $X = L^{1/p}$ and $\alpha = p$ if $p < 1$.

Theorem 4 represents an improvement over the known results for two reasons, first does not require any Poincaré inequality assumption or topological requirement such as being uniformly perfect or being and $RD$–space, etc. (see for example [1], [12], [36], [40] or [41]) secondly it can be applied, following the philosophy used in [11], to q.r.i. spaces. Let us explain this second point in more detail.

Given a r.i. space $X$, the scale $X^{(\alpha)}$, $0 < \alpha \leq 1$, has quasi-Banach spaces, but $(X^{(\alpha)})^{(1/\alpha)} = X$ is Banach. This observation is crucial if we want to apply Theorem 4 to more general scales. The main idea is to require that if $X$ is a q.r.i. space, then on the scale $\{X^{(r)}\}_{1 \leq r < \infty}$ there may be quasi-Banach spaces, but there are also Banach spaces. This fact can be translated into a convexity assumption about the space in the following way:

A q.r.i. space $X$ for which there is $0 < \alpha \leq 1$, such that the functional $\|\cdot\|_{X^{(1/\alpha)}}$ is equivalent to a norm is called an $\alpha$-convex space (see Section 2.3.1 below). There are examples of q.r.i. spaces that are not $\alpha$–convex for any $0 < \alpha \leq 1$ (see [21]) but they are very rare as Grafakos and Kalton said (see [15]) of all practical quasi-Banach rearrangement invariant spaces are $\alpha$–convex for some $0 < \alpha \leq 1$, for this reason throughout the paper we will restrict ourselves with $\alpha$-convex spaces.

The corresponding version of Theorem 4 for $\alpha$-convex spaces reads as follows.

Theorem 6. Let $(\Omega, d, \mu)$ be doubling metric measure space with doubling constant $C_\mu$ an upper dimension $Q$. Let $X$ be an $\alpha$–convex q.r.i space $(0 < \alpha \leq 1)$. The following statements are equivalent.
The non-collapsing condition holds.

(ii) Let \(0 < q \leq \infty\) and \(0 < s < 1\). Then there is a positive constant \(C = C(C_\mu, \alpha, q)\) such that

\[
\left( \int_0^1 \left( O(||f||^\alpha, t)^{1/\alpha}(t) \frac{\phi_X(t)}{t^{s/Q}} \right)^q \frac{dt}{t} \right)^{1/q} \leq C \left( \|f\|_{B^{s,q}_{\lambda,\mu}} + ||f||_{L^n+L^\infty} \right).
\]

**Remark 7.** The formulation given in Theorem 4 and the equivalent one presented in the previous Theorem reflect that there are two different points of view: suppose that we want to get estimates in \(L^{1,\infty}\). The first formulation consists of looking at the q.r.i space \(X = L^{1,\infty}\) which has the property that for any \(0 < \alpha < 1\), \(X^{(1/\alpha)} = L^{1/\alpha, \infty}\) is a r.i. space. This convexity allows us to apply Theorem 6 to \(X\). Alternatively we can start from \(X = L^{1/\alpha, \infty}\) which is a r.i. space and by the first formulation get estimates in \(X^{(p)}\) for all \(0 < p < 1\), and in particular in \(X^{(\alpha)} = L^{1,\infty}\).

The paper is organized as follows. In Section 2, we introduce the notation and the standard assumptions used in the paper, we give the basic definitions and results we will need from the theory or q.r.i spaces, Hajlasz-Sobolev and Hajlasz-Besov spaces. Section 3 is devoted to interpolation, our main result is Theorem 12 where we will need from the theory or q.r.i. spaces, Hajlasz-Sobolev and Hazlas-Besov the standard assumptions used in the paper, we give the basic definitions and results for classical Hajlasz-Sobolev spaces, \(M^{1,p}\) one has that

\[
\hat{B}^{s}_{\lambda,\mu}(\Omega) = (X^{(\alpha)}, M^{1,X^{(\alpha)}})_{s,q},
\]

for \(0 < s < 1\), \(0 < p < \infty\) and \(0 < q \leq \infty\). This result is an extension of Theorem 4.1 where it was proved that for classical Hajlasz-Sobolev spaces \(M^{1,p}\) one has that

\[
\hat{B}^{s}_{p,\nu}(\Omega) = (L^p, M^{1,p})_{s,q}.
\]

Section 4 contains the proofs of our main results, Theorems 4 and 6.

Finally, in Section 5 we will derive embeddings results from inequalities 3 or 4 our main difficulty will be that the left hand side of these inequalities depends neither on the growth of \((||f||^\alpha)^*\) nor on \((||f||^\alpha)^{**}\) but rather on the oscillation, this will solved by showing (see Lemma 21) that there is a weight \(w\) (which depends on \(q\) and \(\alpha\)) such that for all \(f \in L^\alpha + L^\infty\) we get

\[
\left( \int_0^1 \left( f^*(t)w(t) \right)^q \frac{dt}{t} \right)^{1/q} \leq C \left( \left( \int_0^1 \left( O(||f||^\alpha, t)^{1/\alpha}(t) \frac{\phi_X(t)}{t^{s/Q}} \right)^q \frac{dt}{t} \right)^{1/q} + ||f||_{L^\alpha + L^\infty} \right).
\]

We also analyze the role that the upper dimension \(Q\) and the parameter \(\alpha\) play. Let us to explain this second point in more detail. Consider the scale of Lebesgue spaces \(\{L^p\}_{0 < p < \infty}\), taking into account that \(L^p\) is min\(1,p\)–convex and that \(\phi_L(t) = t^{1/p}\) \(\hat{B}^{s}_{p,\nu}(\Omega)\) reads

\[
\left( \int_0^1 \left( O(||f||^{\min(1,p)}, t)^{\min(1,p)}(t) \frac{dt}{t}^{\frac{1}{p} - \frac{1}{s}} \right)^{1/q} \frac{dt}{t} \right) \leq C \left( \|f\|_{B^{s,q}_{\lambda,\mu}} + ||f||_{L^\min(1,p) + L^\infty} \right).
\]

\(^5\text{In case of } p > 1, q > 1 \text{ this was earlier obtained in [12] under some additional assumptions.}\)

\(^6\text{Sobolev-Besov embedding for } B^{s,q}_{\lambda,\mu}(\mathbb{R}^n) \text{ spaces has a different character if } 0 < p < \frac{2}{n}, p = \frac{2}{n} \text{ or } p > \frac{2}{n}.\) In the doubling metric context the counterpart of the dimension } n \text{ is given by the upper dimension } Q.
This inequality implies (see Theorem [24] below)

(i) If \( s < \frac{q}{p} \), then

\[
\left( \int_0^1 \left[ t^{\frac{s}{p}} f^*(t) \right]^q \frac{dt}{t} \right)^{1/q} \leq C \left( \|f\|_{B_{p,q}^*} + \|f\|_{L^{\min(1,p)+L}} \right).
\]

(ii) If \( s = \frac{q}{p} \) and \( \min(1, p) < q < \infty \), then

\[
\left( \int_0^1 \left[ \frac{f^*(t)}{(1 + \ln \frac{t}{t'})^{\frac{1}{\min(1,p)}}} \right]^q \frac{dt}{t} \right)^{1/q} \leq C \left( \|f\|_{B_{p,q}^*} + \|f\|_{L^{\min(1,p)+L}} \right).
\]

(iii) If \( s = \frac{q}{p} \), and \( q \leq \min(1,p) \) or \( s > \frac{q}{p} \), then

\[\|f\|_{L^\infty} \leq C \left( \|f\|_{B_{p,q}^*} + \|f\|_{L^{\min(1,p)+L}} \right).\]

If instead of \( \{L^p\}_{0<p<\infty} \) we consider the scale of weak-Lebesgue spaces \( \{L^{p,\infty}\}_{0<p<\infty} \), then \( \phi_{L^{p,\infty}}(t) = t^{1/p} \), but for \( 0 < p \leq 1 \) the space \( L^{p,\infty} \) is \( \alpha \)-convex only if \( \alpha < p \), thus inequality (4) is now

\[
\left( \int_0^1 \left( O(|f|^\alpha, t) t^{1/\alpha} \right)^q \frac{dt}{t} \right)^{1/q} \leq C \left( \|f\|_{B_{X,q}^*} + \|f\|_{L^{\infty}} \right).
\]

From here we obtain (see Theorem [24] below)

(i) If \( s < \frac{q}{p} \), then

\[
\left( \int_0^1 \left[ t^{\frac{s}{p}} f^*(t) \right]^q \frac{dt}{t} \right)^{1/q} \leq C \left( \|f\|_{B_{L^{p,\infty},q}^*} + \|f\|_{L^{\min(1,p)+L}} \right).
\]

(ii) If \( s = \frac{q}{p} \),

(a) If \( 1 < p \), then

\[
\left( \int_0^1 \left[ \frac{f^*(t)}{(1 + \ln \frac{t}{t'})^{\frac{1}{\min(1,p)}}} \right]^q \frac{dt}{t} \right)^{1/q} \leq C \left( \|f\|_{B_{L^{p,\infty},q}^*} + \|f\|_{L^{\min(1,p)+L}} \right).
\]

(b) If \( 0 < p \leq 1 \), then for any \( 0 < \alpha < p \), we have that

\[
\left( \int_0^1 \left[ \frac{f^*(t)}{(1 + \ln \frac{t}{t'})^{\frac{1}{\min(1,p)}}} \right]^q \frac{dt}{t} \right)^{1/q} \leq C_{\alpha} \left( \|f\|_{B_{L^{p,\infty},q}^*} + \|f\|_{L^{\infty}} \right).
\]

(iii) If \( s = \frac{q}{p} \), and \( q \leq \min(1,p) \) or \( s > \frac{q}{p} \), then

\[\|f\|_{L^\infty} \leq C \left( \|f\|_{B_{L^{p,\infty},q}^*} + \|f\|_{L^{\min(1,p)+L}} \right).\]

**Remark 8.** Notice that \( \phi_{L^p}(t) = \phi_{L^{p,\infty}}(t) \), however embeddings (5) and (6) have different expressions when \( 0 < p \leq 1 \), this is due to the fact that if \( 0 < p \leq 1 \), then \( (L^p)^{(1/p)} = L^1 \) is a Banach spaces but \( (L^{p,\infty})^{(1/p)} = L^{1,\infty} \) not.

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7As far we know this part is new even in the Euclidean case.
Remark 9. The embedding of Besov spaces $B^{s}_{p,q}(\Omega)$, $p > 1, q \geq 1$, into Lorentz spaces was proved in \cite{12} under the assumption that $\Omega$ is $Q$–Ahlfors regular, i.e. if there is a constant $c_Q \geq 1$ such that

$$\frac{r^Q}{c_Q} \leq \mu(B(x,r)) \leq c_Q r^Q$$

and supports a $(1,p)$–Poincaré inequality. For $Q$–Ahlfors regular spaces and $0 < p < \infty$, $0 < q \leq \infty$ was obtained in \cite{19}.

Throughout the paper, we denote by $C$ a positive constant which is independent of the main parameters, but which may vary from line to line. The symbol $f \lesssim g$ means that $f \leq c g$ for some $c > 0$. If $f \lesssim g$ and $g \lesssim f$ we then write $f \simeq g$.

2. Notation and preliminaries

In this section we establish some further notation and background information and we provide more details about metrics spaces and function spaces in which we will be working with.

2.1. Metric spaces. Let $(\Omega, d)$ be a metric space. As usual a ball $B$ in $\Omega$ with a center $x$ and radius $r > 0$ is a set $B = B(x, r) := \{y \in \Omega; d(x, y) < r\}$. Throughout the paper by a metric measure space we mean a triple $(\Omega, d, \mu)$, where $\mu$ is a Borel measure on $(\Omega, d)$ such $0 < \mu(B) < \infty$, for every ball $B$ in $\Omega$, we also assume that $\mu(\Omega) = \infty$ and $\mu(\{x\}) = 0$ for all $x \in \Omega$.

We say that $(\Omega, d, \mu)$ is a doubling metric space, if $\mu$ is a doubling measure on $\Omega$, i.e. $\mu$ satisfies that there exists a constant $C > 1$, such that

$$0 < \mu(B(x, 2r)) \leq C \mu(B(x, r)) < \infty$$

for every ball $B$, for all $x \in \Omega$ and $r > 0$. The smallest constant $C$ for which (7) is satisfied will be called the doubling constant and will be denoted by $C_\mu$. The upper dimension of $\Omega$ defined by

$$Q = \log_2 C_\mu.$$ 

By means of an iteration of condition (7) we get (see for example \cite[Lemma 4.7]{18} for the details)

$$\mu(B(x, r)) \geq \left(\frac{r}{4R}\right)^Q \mu(B(y, R))$$

whenever $x, y \in \Omega$ satisfy $B(x, r) \subset B(y, R)$ and $0 < r \leq R < \infty$.

We say that $(\Omega, d, \mu)$ satisfies the non-collapsing condition if there is a constant $b > 0$, called the non-collapsing constant, such that

$$\inf_{x \in \Omega} \mu(B(x, 1)) = b.$$

2.2. Rearrangements of functions. Let $(\Omega, d, \mu)$ be a doubling metric measure space. For measurable functions $f : \Omega \to \mathbb{R}$, the distribution function of $f$ is given by

$$\mu_f(t) = \mu\{x \in \Omega : |f(x)| > t\} \quad (t > 0).$$

The decreasing rearrangement of $f$ is the function decreasing function $f^*$ defined on $[0, \infty)$ by

$$f^*_\mu(t) = \inf\{s \geq 0 : \mu_f(s) \leq t\}, \quad t \geq 0.$$
The main property if \( f^* \) of that it is equimeasurable with \( f \), i.e.

\[
\mu\{x \in \Omega : |f(x)| > t\} = |\{s \in [0, \infty) : f^*(s) > t\}|
\]

For every positive \( \alpha \) we have \( (|f|^\alpha)^* = (f^*)^\alpha \), and if \( |g| \leq |f| \) \( \mu \)-almost everywhere on \( \Omega \), then \( g^* \leq f^* \). A detailed treatment of rearrangements may be found in Bennett and Sharpley \[7\].

A basic property of rearrangements is the Hardy-Littlewood inequality which tells us that, if \( f \) and \( g \) are two \( \mu \)-measurable functions on \( \Omega \), then

\[
\int_{\Omega} |f(x)g(x)| \, d\mu \leq \int_{0}^{\infty} f^*(t)g^*(t) \, dt.
\]

in particular, for any \( \mu \)-measurable set \( E \subset \Omega \)

\[
\int_{E} |f(x)| \, d\mu \leq \int_{0}^{\mu(E)} f^*(s) \, ds.
\]

Since \( f^* \) is decreasing, the maximal function \( f^{**} \) of \( f^* \), defined by

\[
f^{**}(t) = \frac{1}{t} \int_{0}^{t} f^*(s) \, ds,
\]

is also decreasing, and

\[ f^* \leq f^{**}. \]

Notice that

\[
\frac{\partial}{\partial t} f^{**}(t) = -\left( \frac{f^{**}(t) - f^*(t)}{t} \right).
\]

We single out two subadditivity properties, if \( f \) and \( g \) are two \( \mu \)-measurable functions on \( \Omega \), then for \( t > 0 \)

\[
(f + g)^* (2t) \leq f^*(t) + g^*(t)
\]

and

\[
(f + g)^{**}(t) \leq f^{**}(t) + g^{**}(t).
\]

These facts will be used throughout this paper.

2.3. Background on Rearrangement Invariant Spaces. We recall briefly the basic definitions and conventions we use from the theory of rearrangement invariant spaces and refer the reader to \[17\] and \[27\], for a complete treatment. Let \( (\Omega, d, \mu) \) be a metric measure space. Let \( L^0(\Omega) \) be the space of all real-valued \( \mu \)-measurable functions on \( \Omega \), with the topology of local convergence in measure. A quasi-Banach rearrangement invariant function space \( X = X(\Omega) \) on \( \Omega \) (a q.r.i space) will be a subspace of \( L^0 \) equipped with a quasi-norm \( \|\cdot\|_X \) such that

(i) \( X \) is complete (i.e. a quasi-Banach space) for \( \|\cdot\|_X \).

(ii) The injection \( X \to L^0 \) is continuous.

(iii) If \( E \) is a set of finite measure, then \( \chi_E \in X \).

(iv) If \( 0 \leq f_n \overset{\mu}{\rightarrow} f \) \( \mu \)-a.e. and \( f \in X \) then \( \|f_n\|_X \overset{\mu}{\rightarrow} \|f\|_X \).

(v) \( \|f\|_X = \|g\|_X \) whenever \( f^* = g^* \).
If $\|\cdot\|_X$ is a norm, then we shall refer to $X$ as a \textbf{r.i. space.}

A r.i. space $X(\Omega)$ can be represented by an r.i. space on the interval $(0, \infty)$, with Lebesgue measure, $\bar{X} = \bar{X}(0, \infty)$, such that

$$\|f\|_X = \|f^\ast\|_{\bar{X}},$$

for every $f \in X$. A characterization of the norm $\|\cdot\|_X$ is available (see [7, Theorem 4.10 and subsequent remarks]).

The associated space $X'(\Omega)$ of a r.i. space $X(\Omega)$ is the r.i. space of all measurable functions $h$ for which the r.i. norm given by

$$\|h\|_{X'(\Omega)} = \sup_{g \neq 0} \int_{\Omega} |g(x)h(x)| \, d\mu \|g\|_{X(\Omega)}.$$

Note that by (13), the generalized Hölder inequality

$$\int_{\Omega} |g(x)h(x)| \, d\mu \leq \|g\|_{X(\Omega)} \|h\|_{X'(\Omega)}$$

holds.

An important consequence of (9) and (13) is the Hardy-Littlewood-Pólya principle stating that

$$\int_0^t f^\ast(s) \, ds \leq \int_0^t g^\ast(s) \, ds \quad \forall t > 0 \Rightarrow \|f\|_X \leq \|g\|_X$$

for any r.i. space $X$.

A useful tool, in the study of a q.r.i. space $X$ is the \textbf{fundamental function} of $X$ defined by

$$\phi_X(t) = \|\chi_E\|_X,$$

where $E$ is any measurable subset of $\Omega$ with $\mu(E) = t$. This function is increasing with $\phi_X(0) = 0$. For example if $X = L^p$, then $\phi_{L^p}(t) = t^{1/p}$.

In case that $X$ is a r.i. space, then $\phi_X$ is quasi-concave, that is, $\phi_X(t)/t$ is decreasing. By renorming, if necessary, we can always assume that $\phi_X$ is concave. Moreover,

$$\phi_{X'}(s)\phi_X(s) = s.$$

Associated with an r.i. space $X$ there are some useful Lorentz and Marcinkiewicz spaces, namely the Lorentz and Marcinkiewicz spaces defined by the quasi-norms

$$\|f\|_{\Lambda(X)} = \int_0^\infty f^\ast(t) \, d\phi_X(t).$$

Notice that

$$\phi_{\Lambda(X)}(t) = \phi_{\Lambda_X}(t) = \phi_X(t),$$

and that

$$\Lambda(X) \subset X \subset M(X).$$

Let $r > 0$ and let $X$ be a q.r.i. space on $\Omega$, the $r-$\textbf{convexification} $X^{(r)}$ of $X$, (see [28] and [21]) is defined by

$$X^{(r)} = \{f : |f|^r \in X\}, \quad \|f\|_{X^{(r)}} = \|f|^r\|_X^{1/r}. $$
It follows also that $\phi_X^{(r)}(t) = (\phi_X(t))^{1/r}$. As we pointed in the introduction section, if $X$ is a r.i. space and $r \geq 1$ then, $X^{(r)}$ still is a r.i. space but, in general, for $0 < r < 1$, the space $X^{(r)}$ is not necessarily Banach.

2.3.1. $\alpha$–convex q.r.i. spaces. As mentioned, in the introduction section, we are interested in q.r.i. spaces $X$ with the property that its $r$–convexification is a Banach space for some large power $r$. Here we will give some equivalent characterizations for this class of spaces.

Let $X$ be a q.r.i space $X$, we will say that $X$ is $\alpha$–convex for some $0 < \alpha \leq 1$, it there exists a positive constant $C$ such that for all $f_1, \ldots, f_n \in X$ we have

\begin{equation}
\left\| \left( \sum_{j=1}^{n} |f_j|^{\alpha} \right)^{1/\alpha} \right\|_X \leq C \left( \sum_{j=1}^{n} \|f_j\|_{X}^{\alpha} \right)^{1/\alpha}.
\end{equation}

For $0 < q < \infty$ we say $X$ is $q$–concave if for some $C < \infty$,

\begin{equation}
\left( \sum_{j=1}^{n} \|f_j\|_{X}^{q} \right)^{1/q} \leq C \left( \sum_{j=1}^{n} |f_j|^{q} \right)^{1/q}.
\end{equation}

$X$ is called geometrically convex (see [23]) there exists a positive constant $C$ such that for all $f_1, \ldots, f_n \in X$ we get

\begin{equation}
\left\| |f_1| \cdots |f_n|^{1/n} \right\|_X \leq C \left( \prod_{j=1}^{n} \|f_j\|_{X} \right)^{1/n}.
\end{equation}

Finally, $X$ is said to be $L$–convex (see [22]) if there is $0 < \varepsilon < 1$ so that if $0 \leq f \in X$ with $\|f\|_{X} = 1$ and $0 \leq f_i \leq f$, $i = 1, \ldots, n$, satisfy

$$
\frac{1}{n} \sum_{i=1}^{n} f_i \geq (1 - \varepsilon)f,
$$

then

$$
\max_{1 \leq i \leq n} \|f_i\|_{X} \geq \varepsilon.
$$

Proposition 10. Let $X$ and $Y$ be q.r.i spaces.

(i) If $X$ $\alpha$–convex, then the functional $\|\cdot\|_{X^{(1/\alpha)}}$ is equivalent to a norm.

(ii) The following statements are equivalent:

(a) $X$ is $\alpha$–convex.

(b) $X$ is $L$–convex.

(c) $X$ is geometrically convex.

(iii) If $X$ is $q$–concave for some $0 < q < \infty$, then $X$ is $L$–convex.

(iv) If $X$ and $Y$ are $\alpha$–convex, then spaces $X + Y$ and $X \cap Y$ are $\alpha$–convex q.r.i spaces.

(v) If $X$ is $\alpha$–convex, then

\begin{equation}
L^{\infty}(\Omega) \cap L^{\alpha}(\Omega) \subset X^{(\alpha)}(\Omega) \subset L^{\alpha}(\Omega) + L^{\infty}(\Omega),
\end{equation}

with continuous embedding.
Proof. (i) If $f_1, \ldots, f_n \in X^{(1/\alpha)}$, then $g_j = |f_j|^{1/\alpha} \in X$, thus from (17) it follows
\[
\left\| \sum_{j=1}^{n} |f_j| \right\|_{X^{(1/\alpha)}} = \left\| \left( \sum_{j=1}^{n} |g_j|^{\alpha} \right)^{1/\alpha} \right\|_{X} \leq C^{\alpha} \sum_{j=1}^{n} \|g_j\|_{X}^{\alpha} = C^{\alpha} \sum_{j=1}^{n} \|f_j\|_{X^{1/\alpha}}.
\]

Now, it is easy to show that the functional
\[
\|f\| := \inf \left\{ \sum_{j=1}^{n} \|f_j\|_{X^{(1/\alpha)}} : |f| \leq \left( \sum_{j=1}^{n} |f_j|^{1/\alpha} \right)^{\alpha} : f_1, \ldots, f_n \in X^{(1/\alpha)} \right\}
\]
is a r.i. norm on the space $X^{(1/\alpha)}$, which is equivalent to the original quasi-norm.

(ii) Follows immediately because every $L-$convex q.r.i. space is $\alpha-$convex for some $0 < \alpha \leq 1$ [22, Theorem 2.2]. On the other hand, from [22] and [23] we get that $X$ is $L-$convex if and only if it is geometrically convex.

(iii) Was proved in [22], (iv) in [3, Lemma 3.2] and (v) in [5].

It follows from the previous Proposition that given an $\alpha-$convex q.r.i. space $X$, we have that $Y = X^{(1/\alpha)}$ is equivalent to a r.i. space. Moreover, since obviously
\[
Y^{(\alpha)} = X
\]
we get that all $\alpha-$convex q.r.i. spaces can be obtained as $\alpha-$convexification of r.i. spaces, for this reason throughout the paper we will work with r.i. spaces or with $\alpha-$convexifications of r.i. spaces for some $0 < \alpha \leq 1$.

2.3.2. Examples. We shall see below that all commonly arising examples of q.r.i spaces (e.g. the $L^p$-spaces, Orlicz spaces, Lorentz spaces, Lorentz-Zygmund spaces, generalized Lorentz-Zygmund spaces, Marcinkiewicz spaces, etc.) are $\alpha-$convex.

(i) Classical Lorentz spaces: The spaces $L^{p,q}$ are defined by the function quasi-norm
\[
\|f\|_{L^{p,q}} = \left( \int_0^\infty \left( t^{1/p} f^*(t) \right)^q t^{dt} \right)^{1/q}
\]
when $0 < p, q < \infty$, and
\[
\|f\|_{L^{p,\infty}} = \sup_{0 < t < \infty} t^{1/p} f^*(t).
\]
If $X = L^{p,q}$, then $(L^{p,q})^r = L^{rp,rq}$ which is equivalent to a Banach spaces for $r > \max\{1/p, 1/q\}$.

(ii) Lorentz $\Lambda-$spaces: The Lorentz spaces $\Lambda^q(w)$ are defined by the functional
\[
\|f\|_{\Lambda^q(w)} = \left( \int_0^\infty f^*(t)^q w(t)dt \right)^{1/q}
\]
where $0 < q < \infty$ and $w$ is a weight on $(0, \infty)$. If $w(t) = t^{q/p-1}$, then one obtain $\Lambda^q(w) = L^{p,q}$. If $w(t) = t^{q/p-1} (1 + \log t)^{1/2}$, then $\Lambda^q(w) = L^{p,q}(\log L)^{1/2}$ are the Lorentz-Zygmund spaces (see [22]).
\( t^{q/p-1} (1 + \log^+ \frac{t}{T})^{\beta} (1 + \log^+ \log^+ \frac{t}{T})^\gamma \) then \( \Lambda^q(w) = L^{p,q}(\log L)^\beta (\log \log L)^\gamma \) are the generalized Lorentz-Zygmund spaces.

If \( X = \Lambda^q(w) \) where the weight \( w \) satisfies that there exists large enough \( r > 1 \) such that
\[
\int_t^\infty w(z) \frac{dz}{z^r} \leq C \int_0^t w(z) dz, \quad t > 0,
\]
then (see [42]) \( \Lambda^r(w) \) is equivalent to a Banach space. This combined with the fact that \( (\Lambda^1(w))^{(p)} = \Lambda^p(w) \), implies that \( X^{(r)} \) is equivalent to a r.i. spaces for large \( r \).

(iii) Marcinkiewicz spaces: Let \( \varphi \) be an increasing concave function with \( \varphi(0^+) = 0 \). The Marcinkiewicz space \( M_\varphi \) is defined by the function norm
\[
\|f\|_{M_\varphi} = \sup_{t > 0} \frac{\varphi(t)}{t} \int_0^t f^*(z) dz.
\]
The space \( M_\varphi \) is a r.i. space with fundamental function \( \varphi \). If \( X \) is a r.i. space with fundamental function \( \phi_X \), then \( X \subset M_{\phi_X} \). We also consider the q.r.i space \( \tilde{M}_\varphi \) defined by the functional
\[
\|f\|_{\tilde{M}_\varphi} = \sup_{t > 0} \varphi(t) f^*(t).
\]
Obviously \( M_\varphi \subset \tilde{M}_\varphi \). If
\[
\frac{\varphi(t)}{t} \int_0^t \frac{1}{\varphi(z)} dz \leq C,
\]
then \( M_\varphi = \tilde{M}_\varphi \). Moreover for any \( r > 1 \), \( (\tilde{M}_\varphi)^{(r)} \) is a r.i. space. Indeed, direct computation shows that \( (\tilde{M}_\varphi)^{(r)} = \tilde{M}_{\varphi^{1/r}} \), and since \( \varphi^{1/r} \) is concave and \( \varphi^{1/r}(t)/t^{1/r} \) is decreasing hence (19) holds.

For example if \( X = L^1 \), then \( \phi_X = t \) and \( M_{\phi_X} = L^1 \) but \( \tilde{M}_{\phi_X} = L^{1,\infty} \) that is not normable. However, \( (L^{1,\infty})^{(r)} \) is a Banach space for any \( r > 1 \).

(iv) Orlicz spaces: An Orlicz function is a continuous strictly increasing function \( \Phi : [0, \infty) \to [0, \infty) \) with \( \Phi(0) = 0 \) and satisfying the \( \Delta_2 \)-condition, i.e. \( \Phi(2x) \leq C\Phi(x) \).

If we suppose (Matuszewska and Orlicz [39]) that for some \( p > 0 \)
\[
\inf_{x,y \geq 1} \frac{\Phi(xy)}{\Phi(x)y^p} > 0
\]
then the Orlicz space \( L^\Phi \) is defined by
\[
\|f\|_{L^\Phi} = \inf \left\{ \lambda > 0 : \int_\Omega \Phi \left( \frac{|f(x)|}{\lambda} \right) d\mu(x) \leq 1 \right\},
\]
is a q.r.i. space. \( L^\Phi \) is \( \alpha \)-convex for \( 0 < \alpha < \sup \{ p : \inf_{x,y \geq 1} \frac{\Phi(xy)}{\Phi(x)y^p} > 0 \} \) (see [28]).
2.4. Hajlasz-Sobolev and Hajlasz-Besov spaces built on r.i. spaces. Let 
\((\Omega, d, \mu)\) be doubling metric measure space. Let \(X\) be a r.i. space on \(\Omega\) and 
\(0 < \alpha \leq 1\).

We say that a \(\mu\)-measurable function \(f \in M^{1,X^{(\alpha)}}(\Omega)\), if there exits a nonnegative measurable function \(g \in X^{(\alpha)}\) such that
\[
|f(x) - f(y)| \leq d(x, y)(g(x) + g(y)) \quad \mu \text{-a.e. } x, y \in \Omega.
\]
A function \(g\) satisfying (20) will be called a 1–gradient of \(f\). We denote by \(D(f)\) the collection of all 1–gradients of \(f\). The homogeneous Hajlasz Sobolev space \(M^{1,X^{(\alpha)}}(\Omega)\) consists of measurable functions \(f \in L^\alpha + L^\infty\) for which
\[
\|f\|_{M^{1,X^{(\alpha)}}(\Omega)} = \inf_{g \in D(f)} \|g\|_{X^{(\alpha)}}
\]
is finite.

The Hajlasz-Sobolev space \(M^{1,X^{(\alpha)}}(\Omega)\) is \(M^{1,X^{(\alpha)}}(S) \cap X^{(\alpha)}\) equipped with the quasi-norm
\[
\|f\|_{M^{1,X^{(\alpha)}}(\Omega)} = \|f\|_{X^{(\alpha)}} + \|f\|_{M^{1,X^{(\alpha)}}(\Omega)}.
\]
When \(\alpha = 1\) we will write \(M^{1,X}\) (resp. \(M^{1-X}\)).

Remark 11. The definition formulated above is motivated by the Hajlasz approach to the definition of Sobolev spaces on a metric measure space (see [17] and [18])
where \(M^{1,p}(\Omega)\) was defined as the set of measurable functions \(f\) for which
\[
\|f\|_{M^{1,p}(\Omega)} = \inf_{g \in D(f)} \|g\|_{L^p}
\]
is finite\(^8\). Based on this definition spaces \(M^{1,X^{(r)}}\) appear naturally when replacing the Lebesgue norm by the quasi-norm \(\|\cdot\|_{X^{(r)}}\).

This generalization have been previously considered in some particular cases, for example Tuominen [43] considered the Orlicz case. Recently Heikkinen and Karak in [20] have studied in detail Orlicz-Hajlasz-Sobolev spaces. Costea and Miranda [10] studied the Lorentz case. Malý, in [21] and [30], investigated spaces associated with a general quasi-Banach function lattice. Spaces \(M^{1,Z}(\Omega)\) where \(Z\) is a r.i. space was considered in [34].

Given \(f \in L^\alpha(\Omega) + L^\infty(\Omega)\), we define the \(X^{(\alpha)}\)-modulus of continuity \(E_X^{(\alpha)} : X^{(\alpha)} \times (0, \infty) \to (0, \infty)\) by
\[
E_X^{(\alpha)}(f, r) = \|\nabla_r^{\alpha} f\|_{X^{(\alpha)}},
\]
where
\[
\nabla_r^{\alpha} f(x) = \left(\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(x) - f(y)|^{\alpha} \, d\mu(y)\right)^{1/\alpha}, \quad r > 0.
\]
Let \(0 < q \leq \infty\) and \(0 < s < 1\). The homogeneous Hajlasz-Besov space \(B_{X^{(\alpha)},q}^{s}(\Omega)\) consists of all functions \(f \in L^\alpha(\Omega) + L^\infty(\Omega)\) for which
\[
\|f\|_{B_{X^{(\alpha)},q}^{s}(\Omega)} = \left( \int_0^\infty \left( r^{-s} E_{X^{(\alpha)}}(f, r) \right)^q \frac{dr}{r} \right)^{1/q}
\]
\(^8\)For \(p > 1\), \(M^{1-p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)\) [17], [18], whereas for \(n/(n+1) < p \leq 1\), \(M^{1-p}(\mathbb{R}^n)\) coincides with the Hardy-Sobolev space \(H^{1,p}(\mathbb{R}^n)\) [20 Theorem 1].
is finite (with the usual modification when \( q = \infty \)).

Similarly the Hajłasz-Besov space \( B^s_{X(\alpha),q}(\Omega) \) is \( B^s_{X(\alpha),q}(\Omega) \cap X(\alpha) \) equipped with the quasi-norm
\[
\|f\|_{B^s_{X(\alpha),q}} = \|f\|_{X(\alpha)} + \|f\|_{B^s_{X(\alpha),q}(\Omega)}.
\]

When \( \alpha = 1 \) we write \( B^s_{X,q} (\text{resp. } B^s_{X,q}) \).

3. Interpolation Theorems for Hajłasz-Besov Spaces

In this section, we prove new interpolation for Hajłasz-Besov spaces. Let us start recalling some essential definitions and properties of the real interpolation theory; see, for example, the classical references [7], [4] for the details.

Let \( X_0 \) and \( X_1 \) be (quasi-semi)normed spaces continuously embedded into a topological vector space \( X \). For every \( f \in X_0 + X_1 \) and \( t > 0 \), the \( K \)–functional is
\[
K(f, t; X_0, X_1) = \inf_{f = f_0 + f_1} \left\{ \|f_0\|_{X_0} + t \|f_1\|_{X_1} \right\}.
\]

Let \( 0 < s < 1 \) and \( 0 < q \leq \infty \). The interpolation space \( (X_0, X_1)_{s,q} \) consists of functions \( f \in X_0 + X_1 \) for which
\[
\|f\|_{(X_0, X_1)_{s,q}} = \begin{cases} 
(\int_0^\infty (t^{-s}K(f, t; X_0, X_1))^q \frac{dt}{t})^{1/q}, & \text{if } 0 < q < \infty, \\
\sup_{t>0} t^{-s}K(f, t; X_0, X_1), & \text{if } q = \infty,
\end{cases}
\]
is finite.

Our main result in this section gives us the relation between \( K \)–functional for the couple \( (X(\alpha), M^{1,X(\alpha)}) \) and the \( X(\alpha) \)–modulus of smoothness.

**Theorem 12.** Let \((\Omega, d, \mu)\) be a doubling measure metric space. Let \( X \) be an r.i. space and \( 0 < \alpha \leq 1 \), then that for all \( f \in L^\alpha + L^{\infty} \) and \( t > 0 \),
\[
E_{X(\alpha)}(f, t) \leq K(f, t, X(\alpha), M^{1,X(\alpha)}) \leq \left( \sum_{j=0}^{\infty} 2^{-j\alpha} E_{X(\alpha)}(f, 2^jt)^\alpha \right)^{1/\alpha}
\]
and
\[
K(f, t, X(\alpha), M^{1,X(\alpha)}) \simeq K(f, t, X(\alpha), M^{1,X(\alpha)}) + \min(1, t) \|f\|_{X(\alpha)}.
\]

This result will achieved through an appropriate modification of the proof given in [19] Theorem 4.1 for the couple \((L^p, M^{1,p})\), where the authors obtained a similar expression for \( E_{X(\alpha)} \) considering the modulus \( \mathcal{E}_p \) instead of \( E_{X(\alpha)} \). The key point in our approach will be the next result.

**Lemma 13.** Let \( X \) be an r.i. space and \( 0 < \alpha \leq 1 \). Given \( f \in L^\alpha + L^{\infty} \) and \( r > 0 \) we define
\[
T^\alpha_r f(x) = \left( \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y)|^\alpha \, d\mu(y) \right)^{1/\alpha}.
\]
Then the family of operators \( \{T^\alpha_r\}_{r>0} \) is uniformly bounded on \( X(\alpha) \).

**Proof.** We claim that the family of operators \( \{T^\alpha_r\}_{r>0} \) is uniformly bounded on \( L^{\infty} \) and on \( L^\alpha \). Obviously
\[
\|T^\alpha_r f(x)\|_{L^{\infty}} \leq \|f\|_{L^{\infty}}.
\]
On the other hand, by Fubini’s theorem, we get

$$
\| T^\alpha_r f \|_{L^\alpha} = \int_\Omega \left( \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|^\alpha \, d\mu(y) \right) \, d\mu(x)
$$

Using the doubling property and the fact that $B(x,r) \subset B(y,2r)$ whenever $y \in B(x,r)$, we conclude that

$$
\int_{B(y,r)} \frac{1}{\mu(B(x,r))} \, d\mu(x) \leq C \mu \int_{B(y,r)} \frac{1}{\mu(B(y,2r))} \, d\mu(x)
$$

Thus, for all $r > 0$

$$(24) \quad \| T^\alpha_r f \|_{L^\alpha} \leq C \| f \|_{L^\alpha}.$$

By combining (23) and (24) with the definition of the $K$ functional, we obtain

$$
K(T^\alpha_r f, t^{1/\alpha}, L^\alpha, L^\infty) \lesssim K(f, t^{1/\alpha}, L^\alpha, L^\infty), \quad (t > 0).
$$

Since (see [4, Theorem 5.1])

$$
K(g, t^{1/\alpha}, L^\alpha, L^\infty) \simeq \left( \int_0^t (g^*(s))^{\alpha/\alpha} \, ds \right)^{1/\alpha}
$$

we have

$$
\int_0^t ((T^\alpha_r f)^*(s))^{\alpha/\alpha} \, ds \lesssim \int_0^t (f^*(s))^{\alpha/\alpha} \, ds
$$

which by (15) implies that

$$
\| T^\alpha_r f \|_X \lesssim \| f \|_X,
$$

as we was wished to show.

Proof. (of Theorem 12) We begin with the first inequality of (21). Let $f = g + h$, where $g \in X^{(\alpha)}$, $h \in M^{1, X^{(\alpha)}}$ and let $t > 0$. Taking into account that $0 < \alpha \leq 1$, we get

$$
|g(x) - g(y)|^\alpha \leq |g(x)|^\alpha + |g(y)|^\alpha
$$

thus

$$
\left( \frac{1}{\mu(B(x,t))} \int_{B(x,t)} |g(x) - g(y)|^\alpha \, d\mu(y) \right)^{1/\alpha} \lesssim |g(x)| + T^\alpha_r g(x),
$$

consequently, by Lemma [13]

$$
E^{(\alpha)}_X(g, t) \lesssim \| g \|_{X^{(\alpha)}} + \| T^\alpha_r g \|_{X^{(\alpha)}} \lesssim \| g \|_{X^{(\alpha)}}.
$$
On the other hand, since \( h \in \hat{M}^{1,X(\alpha)} \), by the definition of the 1-gradient, if \( g \in D(h) \cap X(\alpha) \), then

\[
\nabla_\alpha x h(x) = \left( \frac{1}{\mu(B(x,\alpha))} \int_{B(x,\alpha)} |h(x) - h(y)|^\alpha d\mu(y) \right)^{1/\alpha}
\]

\[
\leq \left( \frac{1}{\mu(B(x,t))} \int_{B(x,t)} d(x,y)^\alpha |\varphi(x) + \varphi(y)|^\alpha d\mu(y) \right)^{1/\alpha}
\]

\[
\leq \left( \frac{1}{\mu(B(x,t))} \int_{B(x,t)} d(x,y)^\alpha (\varphi(x) + \varphi(y))^\alpha d\mu(y) \right)^{1/\alpha}
\]

\[
\leq t (\varphi(x) + T_t^\alpha \varphi(x)) ,
\]

hence

\[
E_{X(\alpha)}(h,t) \leq t \| \varphi \|_{X(\alpha)} + t \| T_t^\alpha \varphi \|_{X(\alpha)}
\]

\[
\leq t \| \varphi \|_{X(\alpha)} \text{ (by Lemma 13)}.
\]

In consequence

\[
E_{X(\alpha)}(h,t) \leq t \inf_{\varphi \in D(u)} \| \varphi \|_{X(\alpha)} = t \| h \|_{\hat{M}^{1,X(\alpha)}(\Omega)},
\]

and taking the infimum over all representations of \( f \) in \( X(\alpha) + \hat{M}^{1,X(\alpha)}(\Omega) \), we obtain

\[
E_{X(\alpha)}(f,t) \leq K(f,t,X(\alpha),\hat{M}^{1,X(\alpha)}).
\]

For the converse inequality we will follow the same argument of the proof of [19, Theorem 4.1], with some elementary modifications. For the reader’s convenience, we will include here the main steps. Let \( f \in L^\alpha + L^\infty \) and \( t > 0 \). By a standard covering argument, there is a covering of \( \Omega \) by balls \( B_i = B(x_i,t/6) \), \( i \in \mathbb{N} \), such that \( \sum_i \chi_{2B_i} \leq N \) with the overlap constant \( N > 0 \) depending only on \( C_\mu \).

Let \( \{ \varphi_i \}_{i \in \mathbb{N}} \) be a collection of \( Ct^{-1} \)-Lipschitz functions \( \varphi_i : \Omega \to [0,1] \) such that \( \text{supp} \ \varphi_i \subseteq 2B_i \) and \( \sum_i \varphi_i(x) = 1 \) for all \( x \in \Omega \). Let \( h : \Omega \to \mathbb{R} \) defined by

\[
h(x) = \sum_{i \in \mathbb{N}} m_f(B_i) \varphi_i(x), \quad x \in \Omega
\]

where \( m_f(B_i) \) is the median value\(^9\) of \( f \) in \( B_i \), and let \( g = f - h \), i.e.

\[
g(x) = \sum_{i \in \mathbb{N}} (f(x) - m_f(B_i)) \varphi_i(x) = \sum_{i \in I_x} (f(x) - m_f(B_i)) \varphi_i(x)
\]

where \( I_x = \{ x : x \in 2B_i \} \). The function \( g \) satisfies (see [19, formula (4.5)])

\[
|g(x)| \leq C \left( \frac{1}{\mu(B(x,t))} \int_{B(x,t)} |f(x) - f(z)|^\alpha d\mu(z) \right)^{1/\alpha} = C\nabla_\alpha x f(x).
\]

Therefore

\[
\| g \|_{X(\alpha)} \leq CE_{X(\alpha)}(f,t).
\]

\(^9\)The median value of a measurable function \( u \) on a set \( A \subset \Omega \) is

\[
m_u(A) = \max_{a \in \mathbb{R}} \left\{ \mu(\{ x \in A : u(x) < a \}) \leq \frac{\mu(A)}{2} \right\}.
\]
Next we estimate $h$ in the $\dot{M}^{1,\alpha}(\Omega)$ norm. Let $t > 0$ and let $x, y \in \Omega$. We consider the following two cases.

Case 1: If $d(x, y) \leq t$, then (see [19, page 352])

$$|h(x) - h(y)| \leq C \frac{d(x, y)}{t} \left( \frac{1}{\mu(B(x, 2t))} \int_{B(x, 2t)} |f(x) - f(z)|^\alpha \, d\mu(z) \right)^{1/\alpha}$$

Case 2: Let $d(x, y) > t$. Since

$$|h(x) - h(y)| \leq |f(x) - f(y)| + |g(x)| + |g(y)|,$$

it suffices to estimate the terms on the right side. The assumption $d(x, y) > t$ and (26) imply that

$$|g(x)| \leq C \frac{d(x, y)}{t} \left( \frac{1}{\mu(B(x, t))} \int_{B(x, t)} |f(x) - f(z)|^\alpha \, d\mu(z) \right)^{1/\alpha}$$

and a corresponding upper bound holds for $|g(y)|$.

By [19, page 352] and the doubling property we obtain

$$|f(x) - f(y)| \leq Cd(x, y) \left( f^#_x(x) + f^#_y(y) \right)$$

where

$$f^#_x(x) = \sup_{r \geq t} \frac{1}{r} \left( \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(x) - f(z)|^\alpha \, d\mu(z) \right)^{1/\alpha}.$$ 

Collecting the above estimates, we get, in both cases, that

$$|h(x) - h(y)| \leq Cd(x, y) \left( f^#_x(x) + f^#_y(y) \right).$$

Using the definition of $f^#_x$ and the doubling property, we have

$$f^#_x(x) \leq C \frac{1}{t} \sum_{j=0}^{\infty} 2^{-j} \left( \frac{1}{\mu(B(x, 2^{j+1}t))} \int_{B(x, 2^{j+1}t)} |f(x) - f(z)|^\alpha \, d\mu(z) \right)^{1/\alpha}.$$ 

Since $0 < \alpha \leq 1$, we obtain $^{10}$

$$f^#_x(x) \leq C \frac{1}{t} \sum_{j=0}^{\infty} 2^{-j} \|\Delta^\alpha_{2^{j+1}t} f\|_{X^{(\alpha)}}^\alpha = C \frac{1}{t} \sum_{j=0}^{\infty} 2^{-j} E_{X^{(\alpha)}}(f, 2^{j}t)^\alpha.$$ 

(26) Thus, the required inequality

$$K(f, t, X^{(\alpha)}, \dot{M}^{1,\alpha}(\Omega)) \leq C \left( \sum_{k=0}^{\infty} 2^{-j} E_{X^{(\alpha)}}(f, 2^{j}t)^\alpha \right)^{1/\alpha}$$

is obtained using (25), (26) and the definition of the $K$–functional.

Finally, equivalence (22) follows with the same proof of [19 Theorem 4.1]. $^\square$

$^{10}$ recall that $(\sum_{i\in\mathbb{N}} a_i)^\alpha \leq \sum_{i\in\mathbb{N}} a_i^\alpha$, if $a_i \geq 0$ and $0 < \alpha \leq 1$. 

Having determined the $K$–functional between $X^{(\alpha)}$ and $\dot{M}^{1,\alpha}(\Omega)$ in terms of the $X^{(\alpha)}$–modulus of smoothness, it is now routine to identify the corresponding interpolation spaces.
Corollary 14. Suppose $X$ is a r.i. space and $0 < \alpha < 1$. If $0 < s < 1$ and $0 < q \leq \infty$, then
\[ \dot{B}^s_{X^{(\alpha)},q} = \left( X^{(\alpha)}, M^{1,X^{(\alpha)}} \right)_{s,q}, \]
and
\[ B^s_{X^{(\alpha)},q} = \left( X^{(\alpha)}, M^{1,X^{(\alpha)}} \right)_{s,q}, \]
with equivalent norms.

Remark 15. It follows from the previous result that if $1 \leq p < \infty$, then
\[ \dot{B}^s_{L^p,q} = \left( L^p, \dot{M}^{1-p} \right)_{s,q}, \]
but also, by [19, Theorem 4.1]
\[ B^s_{p,q} = \left( L^p, \dot{M}^{1-p} \right)_{s,q}, \]
thus
\[ \dot{B}^s_{L^p,q} = \dot{B}^s_{p,q}, \]
with equivalent norms.

4. The proofs of Theorems 4 and 6

We will start with a lemma, which is a particular case of [37, Theorem 4], needed in the proof of our main results

Lemma 16. Let $(\Omega, d, \mu)$ be doubling metric measure space with upper dimension $Q$. Assume that there is a constant $c > 0$ such that
\[ \mu(B(x,r)) \geq cr^Q, \quad 0 < r \leq 1, \; x \in \Omega. \]
Let $X$ be a r.i. space on $\Omega$. Let $0 < \alpha \leq 1$, then for all $f \in \dot{M}^{1,X^{(\alpha)}}$ and $g \in D(f)$, we have that
\[ (f^\alpha)^* (t) - (f^\alpha)^* (0) \leq c\alpha^{\alpha/Q} (g^\alpha)^* (t), \quad 0 < t < \frac{c}{2}, \]
where $c = c(\alpha, C_\mu)$ is a constant that just depends on the doubling constant $C_\mu$ and $\alpha.$

Proof. (of Theorem 4) Given $X$ a r.i. space on $\Omega$ and $0 < \alpha \leq 1$, we define
\[ K_\alpha(f, t, X^{(\alpha)}, M^{1,X^{(\alpha)}}) = \inf_{f = f_0 + f_1} \left( \| f_0 \|_{X^{(\alpha)}}^{\alpha} + t^{\alpha} \| f_1 \|_{M^{1,X^{(\alpha)}}} \right)^{1/\alpha}. \]
In what follows, for simplify, we shall write $K_\alpha(f, t, X^{(\alpha)}, M^{1,X^{(\alpha)}}) = K_\alpha(f, t)$ and the classical $K-$functional $K(f, t, X^{(\alpha)}, M^{1,X^{(\alpha)}})$ will be denoted by $K(f, t)$.

(i)→ (ii) Positivity will play a role in the arguments so it will be useful to note for future use that if $\| \cdot \|$ denotes either $\| \cdot \|_{X^{(\alpha)}}$ or $\| \cdot \|_{\dot{M}^{1,X^{(\alpha)}}}$, we have
\[ \| f \| = \| f \|. \]
Notice that if $h \in \dot{M}^{1,X^{(\alpha)}}$ and $\varphi \in D(h)$, then $\| h \| \in \dot{M}^{1,X^{(\alpha)}}$ and $\varphi \in D(\| h \|)$, since
\[ \| h \| (x) - \| h \| (y) \leq \| h(x) - h(y) \| \leq d(x, y) (\varphi(x) + \varphi(y)) \]
hence
\[ \| h \|_{\dot{M}^{1,X^{(\alpha)}}} = \inf_{\varphi \in D(\| h \|)} \| \varphi \|_{X^{(\alpha)}} \leq \inf_{\varphi \in D(h)} \| \varphi \|_{X^{(\alpha)}} = \| h \|_{\dot{M}^{1,X^{(\alpha)}}}. \]
Let $\varepsilon > 0$, and consider any decomposition $f = f - h + h$, with $h \in M^{1, X^{(\alpha)}}$, such that

\begin{equation}
\tag{28}
\left(\|f - h\|_{X^{(\alpha)}}^\alpha + t^\alpha \|h\|_{M^{1, X^{(\alpha)}}}^\alpha\right)^{1/\alpha} \leq K_\alpha (f, t) + \varepsilon.
\end{equation}

Since by (27), $h \in M^{1, X^{(\alpha)}}$ implies that $|h| \in \hat{M}^{1, X^{(\alpha)}}$, this decomposition of $f$ produces the following decomposition of $|f|:$

$$
|f| = |f| - |h| + |h|.
$$

Therefore, by (27) and (28) we have

\begin{equation}
\tag{29}
\inf_{0 \leq h \in \hat{M}^{1, X^{(\alpha)}}} \left(\|f| - |h|\|_{X^{(\alpha)}}^\alpha + t^\alpha \||h||_{M^{1, X^{(\alpha)}}}^\alpha\right)^{1/\alpha} \leq K_\alpha (f, t).
\end{equation}

Consequently

\begin{equation}
\tag{30}
I(t) := ((|f|)^* (t) - (|f|)^* (t)) \\
\leq ((|f| - |h|)^* (t) + (h^\alpha)^* (t) - ((h^\alpha)^* (t) - (|f| - |h|)^* (t))) \\
= ((|f| - |h|)^* (t) + (|f| - |h|)^* (t)) + (h^\alpha)^* (t) - (h^\alpha)^* (2t) \\
\leq 2 ((|f| - |h|)^* (t) + (h^\alpha)^* (t) - (h^\alpha)^* (2t)) \\
= 2 ((|f| - |h|)^* (t) + (h^\alpha)^* (t) - (h^\alpha)^* (2t)) + ((h^\alpha)^* (2t) - (h^\alpha)^* (2t)) \\
= (I) + (II) + (III).
\end{equation}
Therefore, by Hölder’s inequality (14) and (16) we have:

\[(II) = (h^\alpha)^* (t) - (h^\alpha)^* (2t)\]

\[= \int_t^{2t} ((h^\alpha)^* (s) - (h^\alpha)^* (s)) \frac{ds}{s}\]

\[\leq 2t ((h^\alpha)^* (2t) - (h^\alpha)^* (2t)) \int_t^{2t} \frac{ds}{s^2}\]

\[= ((h^\alpha)^* (2t) - (h^\alpha)^* (2t))\]

\[= (III).\]

Inserting this information in (31) we obtain

\[(31) \quad I(t) \leq 2 \left( (||f| - h|^\alpha)^* (t) + (h^\alpha)^* (2t) - (h^\alpha)^* (2t) \right) .\]

On the other hand, inequality (30) with \(R = 1\) and non-collapsing condition (2) imply the following lower bound for the growth of the measure

\[\mu(B(x, r)) \geq \frac{b}{4Q} r^Q, \quad 0 < r \leq 1,\]

which by Lemma (16) implies the existence of a positive constant \(c = c(\alpha, C_\mu) \geq 1\) such that if \(h \in M^1, X^{(\alpha)}\) and \(g \in D(h)\), then

\[(h^\alpha)^* (t) - (h^\alpha)^* (t) \leq ct^{\alpha/Q} (g^\alpha)^* (t), \quad 0 < t < \left(\frac{b}{2}\right) \frac{1}{4Q}.\]

Inserting this information in (31) we have that if \(0 < t < \frac{b}{4Q} \frac{1}{\alpha}\), then

\[(32) \quad I(t) \leq 2 \left( (||f| - h|^\alpha)^* (t) + ct^{\alpha/Q} (g^\alpha)^* (t) \right) \]

\[\leq 2c \left( (||f| - h|^\alpha)^* (t) + t^{\alpha/Q} (g^\alpha)^* (t) \right) \]

\[= 2c (A(t) + B(t)).\]

We now estimate the two terms on the right hand side of (32). For the term \(A(t)\):

Note that for any \(G \in X^{(\alpha)}\),

\[(||G||^\alpha)^* (t) = \frac{1}{t} \int_0^t (||G||^\alpha)^* (s) ds = \frac{1}{t} \int_0^1 (||G||^\alpha)^* (s) \chi_{(0,t)} (s) ds.\]

Therefore, by Hölder’s inequality (14) and (16) we have

\[(33) \quad (||f| - h|^\alpha)^* (t) = \frac{1}{t} \int_0^1 (||f| - h|^\alpha)^* (s) \chi_{(0,t)} (s) ds\]

\[\leq ||(f| - h)^\alpha||_X \frac{\phi_X(t)}{t} \]

\[= ||(f - h)|^\alpha||_{X^{(\alpha)}} \frac{1}{\phi_X(t)}.\]

Similarly, for \(B(t)\) we get

\[(34) \quad B(t) = t^{\alpha/Q} (g^\alpha)^* (t) \leq t^{\alpha/Q} \frac{||g||_{X^{(\alpha)}}}{\phi_X(t)}.\]
Inserting (33) and (34) back in (32) we find that,

\[ I(t) \leq \frac{1}{\phi_X(t)} \cdot 2^c \left( \|(f| - h)\|_{X^{(\alpha)}} + t^{\alpha/2} \|g\|_{X^{(\alpha)}} \right) \]

Thus

\[ I(t)^{1/\alpha} \leq \frac{(2c)^{\frac{1}{\alpha}}}{\phi_X(t)} \left( \|(f| - h)\|_{X^{(\alpha)}} + t^{\alpha/2} \|h\|_{X^{(\alpha)}} \right)^{1/\alpha}. \]

Therefore, by (35)

\[
I(t)^{1/\alpha} \leq \frac{(2c)^{\frac{1}{\alpha}}}{\phi_X(t)} \inf_{0 \leq t \leq \min(I, X^{(\alpha)})} \left( \|(f| - h)\|_{X^{(\alpha)}} + t^{\alpha/2} \|h\|_{X^{(\alpha)}} \right)^{1/\alpha} = (2c)^{\frac{1}{\alpha}} K(f, t^{1/\alpha}) \frac{1}{\phi_X(t)}.
\]

In summary we have proved that

\[ O(|f|^{\alpha}, t)^{1/\alpha} \leq C \frac{K(f, t^{1/\alpha}, X^{(\alpha)}, X^{(\alpha)})}{\phi_X(t)}, \quad 0 < t < \frac{b}{4Q+1}. \]

Let \( T = \min \left( \frac{b}{4Q+1}, 1 \right). \) By (35) we have

\[
I = \int_0^1 \left( O(|f|^{\alpha}, t)^{1/\alpha} \frac{\phi_X(t)}{t^{\alpha/2}} \right)^{\frac{q}{t}} dt - \int_0^T \left( O(|f|^{\alpha}, t)^{1/\alpha} \frac{\phi_X(t)}{t^{\alpha/2}} \right)^{\frac{q}{t}} dt + \int_T^1 \left( O(|f|^{\alpha}, t)^{1/\alpha} \frac{\phi_X(t)}{t^{\alpha/2}} \right)^{\frac{q}{t}} dt \leq C \left( \int_0^T \left( \frac{K(f, t^{1/\alpha})}{t^{\alpha/2}} \right)^{\frac{q}{t}} dt \right) + \int_T^1 \left( \frac{(|f|^{\alpha})^{\star}(t)}{t^{\alpha/2}} \right)^{\frac{q}{t}} dt \leq C \left( \int_0^\infty \left( \frac{K(f, z^{1/\alpha})}{z^{\alpha/2}} \right)^{\frac{q}{z}} dz \right) + \left( |f|^{\alpha} \right) \int_T^1 \left( \frac{\phi_X(t)}{t^{\alpha/2}} \right)^{\frac{q}{t}} dt \leq \int_0^\infty \left( \frac{K(f, z^{1/\alpha})}{z^{\alpha/2}} \right)^{\frac{q}{z}} dz + \|f\|_{L^{\alpha} + L^\infty}^q \leq \|f\|_{L^\infty}^q + \|f\|_{L^\infty}^q,
\]

as we wanted to see.

(ii)\( \rightarrow \) (i) Let \( 0 < s < \min(Q, 1) \). Taking into account (14), by the fundamental Theorem of Calculus, we get

\[
f^{**}(t) = \int_0^1 (f^{**}(z) - f^*(z)) \frac{dz}{z} + f^{**}(1),
\]
consequently, by Fubini's theorem we get

\[
\int_0^1 \frac{f^{**}(t)}{t^{s/Q}} \, dt \leq \int_0^1 \left( \int_t^1 (f^{**}(z) - f^{*}(z)) \frac{dz}{z} \right) \frac{dt}{t^{s/Q}} + \frac{f^{**}(1)}{1 - s/Q} \quad \text{(since } s/Q < 1) \]

\[
= \int_0^1 (f^{**}(t) - f^{*}(t)) \left( \frac{1}{t} \int_t^1 \frac{dz}{z^{s/Q}} \right) \, dt + \frac{f^{**}(1)}{1 - s/Q} \]

\[
= \frac{1}{1 - s/Q} \left( \int_0^1 (f^{**}(t) - f^{*}(t)) \frac{dt}{t^{s/Q}} + f^{*}(1) \right) \]

(36)

\[
= \frac{1}{1 - s/Q} \left( \int_0^1 (f^{**}(t) - f^{*}(t)) \frac{dt}{t^{s/Q}} + \|f\|_{L^1 + L^\infty} \right).
\]

Considering \( X = L^1, \alpha = 1, \) in (3) we obtain

\[
f^{**}(t) - f^{*}(t) \leq C \frac{K(f, t^{1/Q}, L^1, M^{1/1})}{t}, \quad 0 < t < T = \min\left(\frac{b}{4Q+1}, 1\right) \quad \text{(since } \phi_{L^1}(t) = t). \]

Inserting this information into (36) we find that

\[
\int_0^1 \frac{f^{**}(t)}{t^{s/Q}} \, dt \leq \int_0^T \frac{K(f, t^{1/Q}, L^1, M^{1/1})}{t^{s/Q}} \, dt + \int_0^1 (f^{**}(t) - f^{*}(t)) \frac{dt}{t^{s/Q}} + \|f\|_{L^1 + L^\infty} \]

\[
\leq \int_0^\infty \frac{K(f, t^{1/Q}, L^1, M^{1/1})}{z^{s/Q}} \, dt + \int_T^1 f^{**}(t) \frac{dt}{t^{s/Q}} + \|f\|_{L^1 + L^\infty} \]

\[
\leq \int_0^\infty \frac{K(f, t^{1/Q}, L^1, M^{1/1})}{z^{s/Q}} \, dt \quad \text{if } \phi_{L^1}(t) \leq T = \min\{b/4Q+1, 1\} \]

(37)

\[
\leq \|f\|_{B_{1,1}^s(\Omega)} + f^{**}(T) \int_T^1 \frac{dt}{t^{s/Q}} + \|f\|_{L^1 + L^\infty} \quad \text{if } \phi_{L^1}(t) \leq T = \min\{b/4Q+1, 1\}. \]

For a fixed \( x_0 \in \Omega, \) we define the Lipschitz function

\[
u_{x_0}(y) := \begin{cases} 
2 - d(x_0, y) & \text{if } y \in B(x_0, 2) \setminus B(x_0, 1), \\
1 & \text{if } y \in B(x_0, 1), \\
0 & \text{if } y \in \Omega \setminus B(x_0, 2).
\end{cases}
\]

It is easily seen that \( \nu_{x_0}(y) = \chi_{B(x_0, 2)}(y) \in D(u_{x_0}) \)

By Fubini's theorem

\[
E_{L^1}(u_{x_0}, t) \leq \int_\Omega |u_{x_0}(x)| \, d\mu(x) + \int_\Omega \frac{1}{\mu(B(x, t))} \int_{B(x,t)} |u_{x_0}(y)| \, d\mu(y) \, d\mu(x) \]

\[
\leq \|u_{x_0}\|_{L^1} + \int_\Omega |u_{x_0}(y)| \left( \int_{B(y,t)} \frac{1}{\mu(B(x, t))} \, d\mu(x) \right) \, d\mu(y) \]

(38)

\[
\leq \|u_{x_0}\|_{L^1},
\]

the last estimate follows from the doubling property of \( \mu \) and since \( B(y, t) \subset B(x, 2t) \) whenever \( x \in B(y, t). \)
Using that \( g_{x_0} \in D(u_{x_0}) \) and with a similar argument as in \((38)\), we get

\[
E_L^1(u_{x_0}, t) = \int_\Omega \left( \frac{1}{\mu(B(x,t))} \int_{B(x,t)} |u_{x_0}(x) - u_{x_0}(y)| \, d\mu(y) \right) \, d\mu(x)
\]
\[
\leq \int_\Omega \left( \frac{1}{\mu(B(x,t))} \int_{B(x,t)} d(x,y) |g_{x_0}(x) + g_{x_0}(y)| \, d\mu(y) \right) \, d\mu(x)
\]
\[
\leq t \left( \int_\Omega |g_{x_0}(x)| \, d\mu(x) + \frac{1}{\mu(B(x,t))} \int_{B(x,t)} |g_{x_0}(y)| \, d\mu(y) \, d\mu(x) \right)
\]
\[
\leq t \|g_{x_0}\|_{L^1}.
\]

Thus, combining \((38)\) and \((39)\), we see that

\[
E_L^1(u_{x_0}, t) \leq \min(\|u_{x_0}\|_{L^1}, t \|g_{x_0}\|_{L^1})
\]
\[
\leq \min(\mu(B(x_0,2)), t \mu(B(x_0,2)))
\]
\[
\leq \min(1,t) \mu(B(x_0,2)).
\]

Therefore,

\[
\|u_{x_0}\|_{B^{t,1}_1} \leq \mu(B(x_0,2)),
\]

and obviously

\[
\|u_{x_0}\|_{L^{1+L^\infty}} \leq \|u_{x_0}\|_{L^1} \leq \mu(B(x_0,2)).
\]

On the other hand,

\[
\int_0^1 \left( \frac{u_{x_0}}{t^{s/Q}} \right) \geq \int_0^1 \left( \frac{u_{x_0}}{t^{s/Q}} \right) \geq \frac{1}{\mu(B(x_0,1))} \int_0^{1/(\mu(B(x_0,1)))} \frac{dt}{t^{s/Q}} = \frac{(1, \mu(B(x_0,1)))^{1-s/Q}}{1-s/Q}.
\]

The previous computation and \((37)\) implies that

\[
\frac{(\min(1,\mu(B(x_0,1)))^{1-s/Q}}{1-s/Q} \leq 2 \mu(B(x_0,2)) \leq 2C \mu(B(x_0,1))
\]

and from here we conclude that

\[
1 \leq \mu(B(x_0,1))
\]

which completes the proof. \(\square\)

**Remark 17.** Theorem 6 follows from Theorem 4 considering the r.i. spaces \(X^{(1/\alpha)}\).

**Remark 18.** Inequality \((33)\) fits into the research program developed in Martín and Milman where Sobolev-Poincaré inequalities for Lipschitz functions and K-functional are widely considered (for further details, we refer the reader to \([35, 31, 38]\) for further details).

5. **Embedding Theorems for Hajlasz-Besov spaces into q.r.i. spaces**

In this section is devoted to find Sobolev type embedding for \(\dot{B}^{X_{(\alpha,q)}(\Omega)}_{1}\) spaces into a q.r.i. spaces. The proofs make substantial use of the (nonlinear) spaces \(L(\nu, \alpha, q)\) defined to be the set of all functions \(f \in L^\infty \) such that

\[
\|f\|_{L(\nu, \alpha, q)} = \left\{ \left( \int_0^1 \left( O(|f|^{\alpha}, z)^{1/\alpha} \nu(z) \right)^q \, dz \right)^{1/q}, \quad 0 < q < \infty, \right. \\
\left. \left( \sup_{0 < t < 1} (O(|f|^{\alpha}, t)) \nu(t) \right), \quad q = \infty. \right. 
\]

is finite. (Here \(\nu\) is a weight (a positive locally integrable function on \((0, \infty))\).
Before continuing, we note \( \|f\|_{L(v,\alpha,q)} \) depends neither on the growth of \(|f|^\alpha\) nor on \(|f|^\alpha\) but rather on the oscillation, which causes obstacles for applications, to overcome such difficulties we shall need the next results.

**Remark 19.** Given \( q \) a positive number and \( f \) a \( \mu \)-measurable function on \( \Omega \), it follows ready that

\[
\frac{\partial}{\partial t} (|f|^\alpha)^{**} (t) = -q (|f|^\alpha)^{**} (t) q^{-1} O(|f|^\alpha, t) \frac{1}{t},
\]

therefore, by the Fundamental Theorem of Calculus,

\[
(|f|^\alpha)^{**} (t)^q = \int_t^1 \left( (|f|^\alpha)^{**} (t)^{q^{-1}} O(|f|^\alpha, t) \right) \frac{dt}{t} + (|f|^\alpha)^{**} (1)^q.
\]

(40)

\[
\text{In particular, if } 0 < q \leq 1, \text{ in view of } O(|f|^\alpha, t) \leq (|f|^\alpha)^{**} (t), \text{ we get}
\]

\[
(|f|^\alpha)^{**} (t)^q \leq q \int_t^1 \left( O(|f|^\alpha, z) \right)^q \frac{dz}{z} + q \|f\|_{L^\alpha + L^\infty}^q.
\]

(41)

**Lemma 20.** Let \( v \) be a weight, \( 0 < \alpha \leq 1 \) and \( 0 < q \leq \infty \), then the following embedding holds

\[
\|f\|_{L^\infty} \leq \|f\|_{L(v,\alpha,q)} + \|f\|_{L^\alpha + L^\infty},
\]

if \( m_{v,\alpha,q}(0) < \infty \), where

\[
m_{v,\alpha,q}(t) := m(t) := \left\{ \begin{array}{ll}
\int_t^1 \left( \frac{1}{v(z)} \right)^{\frac{\alpha q}{q-\alpha}} \frac{dz}{z} & \text{if } \alpha < q \leq \infty, \\
\int_t^1 \frac{1}{v(z)} \frac{dz}{z} & \text{if } q = \infty, \\
\sup_{z \in (t,1]} v(z) & \text{if } 0 < q \leq \alpha.
\end{array} \right.
\]

**Proof.** It follows from (40) that

\[
\|f\|_{L^\infty}^q = \|f\|_{L(v,\alpha,q)}^q \int_0^1 (O(|f|^\alpha, t) \frac{dt}{t} + \|f\|_{L^\alpha + L^\infty}^q = I + \|f\|_{L^\alpha + L^\infty}.
\]

**Case** \( \alpha < q < \infty \). Since \( \alpha/q > 1 \), by Holder inequality we obtain

\[
I \leq \left( \int_0^1 (O(|f|^\alpha, t)v(t))^{q^{-1}} \frac{dt}{t} \right)^{\frac{\alpha}{q}} \left( \int_0^1 \frac{1}{v(t)} \frac{dz}{t} \right)^{\frac{q-\alpha}{q}},
\]

whence

\[
\|f\|_{L^\infty} \leq \left( m(0) \right)^{\frac{\alpha q}{q-\alpha}} \|f\|_{L(v,\alpha,q)} + \|f\|_{L^\alpha + L^\infty}.
\]

**Case** \( q = \infty \). Obviously

\[
I \leq \sup_{t \in [0,1]} (O(|f|^\alpha, t)v(t)) \int_0^1 \frac{1}{v(t)} \frac{dt}{t},
\]

thus

\[
\|f\|_{L^\infty} \leq m(0) \|f\|_{L(v,\alpha,\infty)} + \|f\|_{L^\alpha + L^\infty}.
\]
Case $q \leq \alpha$. By inequality (41), we get
\[
\|f\|_{L^\infty} = \|f|^{\alpha}\|^q (0) \leq \frac{q}{\alpha} \int_0^1 (O(|f|^\alpha, t)^{1/\alpha} v(t) t^{\frac{q-1}{q}} w(t)^{\frac{q}{2}} + \|f\|_{L^\alpha + L^\infty}^q
\]
\[
\leq \frac{q}{\alpha} \sup_{t \in [0,1]} \left( \frac{1}{v(t)^{\frac{q}{2}}} \right) \|f\|^q_{L^\alpha(v,\alpha,\infty)} + \|f\|^q_{L^\alpha + L^\infty},
\]
hence
\[
\|f\|_{L^\infty} \leq m(0) \|f\|_{L^\alpha(v,\alpha,\infty)} + \|f\|_{L^\alpha + L^\infty},
\]
as we wished to show.

Lemma 21. Let $v$ be a weight. Let $0 < \alpha \leq 1$ and $0 < q \leq \infty$ be such that $m_{v,\alpha,q}(0) = \infty$, then

(i) If $\alpha < q < \infty$, then
\[
\left( \int_0^1 (|f|^{\alpha})^{**} (t)^{1/\alpha} w(z) \frac{dz}{z} \right)^{1/q} \leq \|f\|_{L^\alpha(v,\alpha,\infty)} + \|f\|_{L^\alpha + L^\infty},
\]
where $w$ is defined by
\[
w^q(t) = \frac{\partial}{\partial t} \left( 1 + \int_t^1 \left( \frac{1}{v(z)^{\frac{q}{2}}} \right)^{\frac{\alpha q}{1-q}} d\frac{dz}{z} \right)^{1-\frac{q}{\alpha}}.
\]

(ii) If $q = \infty$, then
\[
\sup_{t \in [0,1]} \frac{(|f|^{\alpha})^{**}(t)^{1/\alpha}}{1 + \int_t^1 \frac{dz}{v(z)^{\frac{q}{2}}} \|f\|_{L^\alpha}} \leq \|f\|_{L^\alpha(v,\alpha,\infty)}
\]

(iii) If $0 < q \leq \alpha$, then for any weight $u$ such that
\[
\int_0^t u^q(z) \frac{dz}{z} \leq v^q(t), \quad 0 < t < 1
\]
we have that
\[
\left( \int_0^1 (|f|^{\alpha})^{**} (t)^{1/\alpha} u(t) \right)^{1/q} \leq \|f\|_{L^\alpha(v,\alpha,\infty)} + \|f\|_{L^\alpha + L^\infty}.
\]

Proof. (i) By (40) we have that
\[
I = \int_0^1 (|f|^{\alpha})^{**} (t)^{q/\alpha} w^q(t) \frac{dt}{t}
\]
\[
= \frac{q}{\alpha} \left( \int_0^1 \int_t^1 (|f|^{\alpha})^{**} (z)^{q/\alpha-1} (O(|f|^{\alpha}, z) \frac{dz}{z}) w^q(t) \frac{dt}{t} \right)
\]
\[
+ (|f|^{\alpha})^{**} (1)^{q/\alpha} \int_0^1 w^q(t) \frac{dt}{t}
\]
\[
= \frac{q}{\alpha} II + (|f|^{\alpha})^{**} (1)^{q/\alpha} \int_0^1 w^q(t) \frac{dt}{t}.
\]
Applying Fubini’s Theorem and Hölder inequality, we obtain

\[
II = \int_0^1 (|f|^\alpha)^{**} (t)^{\frac{\alpha}{2} - 1} (O(|f|^\alpha, t)) \left( \frac{1}{t} \int_0^t w^q(z) \frac{dz}{z} \right) dt
\]

\[
= \int_0^1 (|f|^\alpha)^{**} (t)^{\frac{\alpha}{2} - 1} \left( \frac{w^q(t)}{t} \right)^{1 - \frac{\alpha}{2}} (O(|f|^\alpha, t)) \left( \frac{w^q(t)}{t} \right)^{\frac{\alpha}{2} - 1} \left( \frac{1}{t} \int_0^t w^q(z) \frac{dz}{z} \right) dt
\]

\[
\leq \left( \int_0^1 (|f|^\alpha)^{**} (t) w^q(t) \frac{dt}{t} \right)^{1 - \frac{\alpha}{2}}
\]

\[
\times \left( \int_0^1 (O(|f|^\alpha, t)) \left( \frac{w^q(t)}{t} \right)^{1 - \frac{\alpha}{2}} \left( \frac{1}{t} \int_0^t w^q(z) \frac{dz}{z} \right)^{\frac{\alpha}{2}} dt \right)^{\frac{\alpha}{2}}
\]

(43)

On the other hand, an elementary computation shows that

\[
\left( \frac{w^q(t)}{t} \right)^{1 - \frac{\alpha}{2}} \left( \frac{1}{t} \int_0^t w^q(z) \frac{dz}{z} \right)^{\frac{\alpha}{2}} \approx \frac{v^\alpha(t)}{t},
\]

whence combining (42) and (43) we obtain

\[
\left( \int_0^1 (|f|^\alpha)^{**} (t) w^q(t) \frac{dt}{t} \right)^{\frac{\alpha}{2}} \leq \|f\|_{L(v,\alpha,q)}^\alpha \left( \int_0^1 \frac{w^q(t) dt}{t} \right)^{1 - \frac{\alpha}{2}}.
\]

Finally, since

\[
\frac{(|f|^\alpha)^{**} (1) \frac{\alpha}{2} \int_0^1 w^q(t) \frac{dt}{t}}{\left( \int_0^1 (|f|^\alpha)^{**} (t) \frac{\alpha}{2} w^q(t) \frac{dt}{t} \right)^{1 - \frac{\alpha}{2}}}
\]

\[
\leq \frac{(|f|^\alpha)^{**} (1) \frac{\alpha}{2} \int_0^1 w^q(t) \frac{dt}{t}}{\left( \int_0^1 (|f|^\alpha)^{**} (t) \frac{\alpha}{2} w^q(t) \frac{dt}{t} \right)^{1 - \frac{\alpha}{2}}}
\]

\[
= (|f|^\alpha)^{**} (1) \left( \int_0^1 w^q(t) \frac{dt}{t} \right)^{\frac{\alpha}{2}}
\]

\[
= \|f\|_{L^{\alpha + L^\infty}}^\alpha \left( \int_0^1 w^q(t) \frac{dt}{t} \right)^{\frac{\alpha}{2}}
\]

the desired inequality follows.

(ii) Using (40) we can write

\[
(|f|^\alpha)^{**} (t) = \int_t^1 (O(|f|^\alpha, z)) \frac{dz}{z} + \|f\|_{L^{\alpha + L^\infty}}^\alpha
\]

\[
\leq \sup_{t \in [0,1]} (O(|f|^\alpha, t) v^\alpha(t)) \int_t^1 \frac{1}{v^\alpha(z)} \frac{dz}{z} + \|f\|_{L^{\alpha + L^\infty}}^\alpha
\]

\[
\leq \left( \|f\|_{L(v,\alpha,\infty)}^\alpha + \|f\|_{L^{\alpha + L^\infty}}^\alpha \right) \left( 1 + \int_t^1 \frac{1}{v^\alpha(z)} \frac{dz}{z} \right).
\]

Therefore

\[
\sup_{t \in [0,1]} \frac{(|f|^\alpha)^{**} (t)^{1/\alpha}}{\left( 1 + \int_t^1 \frac{1}{v^\alpha(z)} \frac{dz}{z} \right)^{1/\alpha}} \leq \|f\|_{L(v,\alpha,\infty)}.
\]
Theorem 22. Let \((\Omega, d, \mu)\) be doubling with upper dimension \(Q\) that satisfies the non-collapsing condition. Let \(X\) be a r.i. space on \(\Omega\), \(0 < \alpha \leq 1\), \(0 < s < 1\), \(0 < q \leq \infty\) and consider the function

\[
m_{X,s,\alpha,q} : [0, 1] \to [0, \infty)
\]

defined by

\[
m_{X,s,\alpha,q}(t) := m(t) = \begin{cases} 
\int_{t}^{1} \left( \frac{f^s(t)}{\varphi_{X(q)}(z^{1/\alpha})} \right)^{q/\alpha} \frac{dz}{z} & \text{if } \alpha < q \leq \infty, \\
\int_{t}^{1} \left( \frac{f^s(t)}{\varphi_{X(q)}(z^{1/\alpha})} \right)^{q/\alpha} \frac{dz}{z} & \text{if } q = \infty, \\
\sup_{z \in [t, 1]} \left( \frac{f^s(t)}{\varphi_{X(q)}(z^{1/\alpha})} \right)^{q/\alpha} & \text{if } 0 < q \leq \alpha.
\end{cases}
\]

Then

(i) If \(m(0) < \infty\), then

\[
\|f\|_{L^\infty} \leq \|f\|_{B^s_{\alpha,q}} + \|f\|_{L^{\infty} + L^\infty}.
\]

(ii) If \(m(0) = \infty\), then

(a) If \(\alpha < q < \infty\), then

\[
\left( \int_{0}^{1} f^s(t)\left| \frac{m'(t)}{1 + m(t)} \right|^{q/\alpha} ds \right)^{1/q} \leq \|f\|_{B^s_{\alpha,q}} + \|f\|_{L^{\infty} + L^\infty}.
\]

(b) If \(q = \infty\), then

\[
\sup_{t \in [0,1]} \left( \frac{f^s(t)}{1 + m(t)^{1/\alpha}} \right) \leq \|f\|_{B^s_{\alpha,\infty}} + \|f\|_{L^{\infty} + L^\infty}.
\]

(c) If \(0 < q \leq \alpha\), then for any weight \(u\) such that

\[
\int_{0}^{1} u^q(z) dz \leq u^q(t), \quad 0 < t < 1
\]

there is a positive constant \(C\) such that

\[
\left( \int_{0}^{1} (f^s(t)u(t))^q \frac{dt}{t} \right)^{1/q} \leq C \left( \|f\|_{B^s_{\alpha,q}} + \|f\|_{L^{\infty} + L^\infty} \right).
\]

(iii) By (11) and Fubini’s Theorem we get

\[
I = \int_{0}^{1} \left( |\int_{0}^{t} (\int_{0}^{s} f^s(t)ds)\right)^{q/\alpha} dt \frac{dt}{t} 
\leq \frac{q}{\alpha} \int_{0}^{1} \left( \int_{0}^{t} (\int_{0}^{\alpha} f^s(t)ds)\right)^{q/\alpha} \frac{dz}{z} \int_{0}^{1} \left( \int_{0}^{t} u^q(t)dt \right)^{q/\alpha} \frac{dt}{t} 
= \frac{q}{\alpha} \int_{0}^{1} \left( \int_{0}^{t} u^q(t)dt \right)^{q/\alpha} \frac{dt}{t} 
\leq C \frac{q}{\alpha} \|f\|_{L^{\infty} + L^\infty} + \|f\|_{L^{\infty} + L^\infty} \int_{0}^{1} u^q(t)dt,
\]
which completes the proof.
Proof. Let us write \( K(f,t^{1/Q}) = K(f,t^{1/Q},X^{(\alpha)},\dot{M}^{1},X^{(\alpha)}) \) and \( T = \min(b/4Q+1,1) \).

If \( T \leq 1 \), then

\[
I = \left( \int_0^T \left( O(|f|^\alpha,t)^{1/\alpha} \left( \frac{\phi_X(t)^{1/\alpha}}{t^\alpha} \right) \right)^q \frac{dt}{t} \right)^{1/q}
\]

\[
\leq \left( \int_0^1 \left( O(|f|^\alpha,t)^{1/\alpha} \left( \frac{\phi_X(t)^{1/\alpha}}{t^\alpha} \right) \right)^q \frac{dt}{t} \right)^{1/q} \quad \text{(since } \phi_X(t)^{1/\alpha} = (\phi_X)^{1/\alpha})
\]

\[
\leq \left( \int_0^\infty \left( \frac{K(f,t^{1/Q})}{t^{s/Q}} \right)^q \frac{dt}{t} \right)^{1/q} \quad \text{(by Theorem 4)}
\]

\[
\leq \|f\|_{B^s_{X^{(\alpha)q}}} + \|f\|_{L^{\alpha+L^\infty}}.
\]

If \( T > 1 \), then

\[
I = \left( \int_T^1 \left( O(|f|^\alpha,t)^{1/\alpha} \left( \frac{\phi_X(t)^{1/\alpha}}{t^\alpha} \right) \right)^q \frac{dt}{t} \right)^{1/q}
\]

\[
\leq \left( \int_T^T \left( O(|f|^\alpha,t)^{1/\alpha} \left( \frac{\phi_X(t)^{1/\alpha}}{t^\alpha} \right) \right)^q \frac{dt}{t} \right)^{1/q} + \left( \int_0^1 \left( O(|f|^\alpha,t)^{1/\alpha} \left( \frac{\phi_X(t)^{1/\alpha}}{t^\alpha} \right) \right)^q \frac{dt}{t} \right)^{1/q}
\]

\[
= (A) + (B).
\]

The term \((A)\) is controlled as above, and

\[
(B) = \left( \int_T^1 \left( O(|f|^\alpha,t)^{1/\alpha} \left( \frac{\phi_X(t)^{1/\alpha}}{t^\alpha} \right) \right)^q \frac{dt}{t} \right)^{1/q}
\]

\[
\leq (|f|^\alpha)^{(s)\ast} (T)^{1/\alpha} \left( \int_T^1 \left( \frac{\phi_X(t)^{1/\alpha}}{t^\alpha} \right)^q \frac{dt}{t} \right)^{1/q}
\]

\[
\leq \|f\|_{L^{\alpha+L^\infty}}.
\]

In summary, in both cases we have proved that

\[
\left( \int_0^1 \left( O(|f|^\alpha,t)^{1/\alpha} \left( \frac{\phi_X(t)^{1/\alpha}}{t^\alpha} \right) \right)^q \frac{dt}{t} \right)^{1/q} \leq \|f\|_{B^s_{X^{(\alpha)q}}} + \|f\|_{L^{\alpha+L^\infty}},
\]

therefore, the result follows applying Lemmas 20 and 21 to the weight

\[
v(t) = \frac{t^{\frac{s}{\alpha}}}{\phi_X(t)}
\]

and using that \( f^\ast(t) \leq (|f|^\alpha)^{(s)\ast}(t)^{1/\alpha} \). \( \square \)

Remark 23. It is plain that if we work with Hajłasz-Besov spaces \( B^s_{X^{(\alpha)q}} \) instead of homogeneous Hajłasz-Besov spaces \( B^s_{X^{(\alpha)q}} \), then Theorem 22 remains true, considering in the right hand side of the inequalities the term \( \|f\|_{B^s_{X^{(\alpha)q}}} \) instead of \( \|f\|_{B^s_{X^{(\alpha)q}}} + \|f\|_{L^{\alpha+L^\infty}} \), since by (13) we know that \( \|f\|_{L^{\alpha+L^\infty}} \leq \|f\|_{X^{(\alpha)}} \).
5.0.1. Example: Homogeneous Lorentz-Zygmund-Hajłasz-Besov spaces. Lorentz-Zygmund spaces were introduced in [6], they contain many interesting nontrivial function spaces as Lebesgue spaces, Lorentz spaces or Zygmund classes, which have important applications, mainly in various limiting or critical situations, see for example [5]. In what follows we investigate Hajłasz-Besov embedding results for such spaces.

Given $0 < p < \infty$, $0 < r \leq \infty$ and $\beta \in \mathbb{R}$, the Lorentz-Zygmund space $L^{p,r}(\log L)^\beta$ consists of all $\mu$–measurable functions $f$ on $\Omega$ for which the quasi-norm

$$
\|f\|_{L^{p,r}(\log L)^\beta} = \left\{ \begin{array}{ll}
\left( \int_0^\infty \left[ \frac{t^r}{(1 + \ln^+ \frac{1}{r})^\beta} f^+(t) \right] dt \right)^{1/r} & , \quad 0 < r < \infty, \\
\sup_{t > 0} t^r (1 + \ln^+ \frac{1}{r})^\beta f^+(t) & , \quad r = \infty,
\end{array} \right.
$$

is finite. The fundamental function of $L^{p,r}(\log L)^\beta$ satisfies that

$$
\phi_{L^{p,r}(\log L)^\beta}(t) \equiv t^r (1 + \ln^+ \frac{1}{r})^\beta.
$$

Lorentz-Zygmund spaces are $\alpha$–convex, indent taking into account that if $1 < p \leq \infty$ and $1 \leq r \leq \infty$, the functional $\|\cdot\|_{L^{p,r}(\log L)^\beta}$ is equivalent to a norm, and $L^{1,1}(\log L)^\beta$ is a Banach space if $\beta > 0$, it follows ready that choosing $0 < \alpha \leq 1$ satisfying

$$
\alpha < p \quad \text{if} \quad 0 < p < r \quad \text{with} \quad 0 < p \leq 1, \quad \text{or} \quad 0 < p = r \leq 1 \quad \text{and} \quad \beta < 0,
$$

we have that the $\frac{1}{\alpha}$–convexification of $X = L^{p,r}(\log L)^\beta$ is a Banach space. On the other hand, from (44) and since $(X^{(1/\alpha)})^{(\alpha)} = X$, we obtain that given $0 < s < 1$ and $0 < q \leq \infty$ the function $m_{X^{(1/\alpha)},s,\alpha,q}$ defined in Theorem 22 verifies that

$$
m_{X^{(1/\alpha)},s,\alpha,q}(t) \simeq \left\{ \begin{array}{ll}
\int_1^t \left( \frac{z + \frac{1}{r} \left(1 + \ln^+ \frac{1}{r} \right)^\beta}{(1 + \ln^+ \frac{1}{r})^\beta} \right)^{\frac{\alpha}{s}} \frac{dz}{z} & , \quad 0 < q < \infty, \\
\int_1^t \left( \frac{z + \frac{1}{r} \left(1 + \ln^+ \frac{1}{r} \right)^\beta}{(1 + \ln^+ \frac{1}{r})^\beta} \right)^{\alpha} \frac{dz}{z} & , \quad q = \infty,
\end{array} \right.
\sup_{t \in [1,1)} \left( \frac{z + \frac{1}{r} \left(1 + \ln^+ \frac{1}{r} \right)^\beta}{(1 + \ln^+ \frac{1}{r})^\beta} \right)^{\alpha} \quad \text{if} \quad 0 < q \leq \alpha.
$$

Henceforth we shall assume that $(\Omega, d, \mu)$ is doubling with doubling constant $C_{\mu}$ and upper dimension $Q$ which satisfies the non-collapsing condition.

Theorem 24. Let $X = L^{p,r}(\log L)^\beta$ be a Lorentz-Zygmund space on $\Omega$ $(0 < p < \infty$, $0 < r \leq \infty$, $\beta \in \mathbb{R})$. Let $0 < s < 1$, $0 < q \leq \infty$. The embedding

$$
\|f\|_{L^\infty} \lesssim \|f\|_{L^{p,r}(\log L)^\beta} + \|f\|_{L^{\min(1,p,r),s,\alpha,q}} ,
$$

holds in the following situations:

(i) $s > \frac{Q}{p}$.

(ii) $s = \frac{Q}{p}$ and

$$
\left\{ \begin{array}{ll}
\beta > \frac{1}{\min(1,p,r)} - \frac{1}{q} & , \quad \text{if} \quad \min(1,p,r) < q \leq \infty, \\
\beta \geq 0 & , \quad \text{if} \quad 0 < q \leq \min(1,p,r).
\end{array} \right.
$$
Theorem 25. Let \( X = L^{p,r}(\log L)^{\beta} \) pick \( 0 < \alpha \leq 1 \) satisfying (45) and let \( m_{X^{(1/\alpha)},\alpha,q}(0) < \infty \), which by Theorem 22 implies that (47) holds.

(i) Case \( s > \frac{Q}{p} \). An elementary computation shows that \( m_{X^{(1/\alpha)},\alpha,q}(0) < 0 \).

(ii) Case \( s = \frac{Q}{p} \). Notice that condition (48) implies that \( \beta \geq 0 \), then

(a) If \( \min(1,p,r) = \min(1,r) \), then by (45) \( \alpha = \min(1,r) \). Besides, condition \( \beta > \frac{1}{\min(1,r)} - \frac{1}{q} \) implies \( \frac{\beta \min(1,r) q}{q - \min(1,r)} > 1 \) if \( \min(1,r) < q < \infty \) and \( \beta \min(1,r) > 1 \) if \( q = \infty \), hence \( m_{X^{(1/\alpha)},\alpha,q}(0) < \infty \).

Finally, if \( 0 < q \leq \min(1,r) \) and \( \beta \geq 0 \), then

\[
m_{X^{(1/\alpha)},\alpha,q}(0) = \sup_{z \in [0,1]} \frac{1}{(1 + \ln z)^{\beta}} < \infty.
\]

(b) If \( \min(1,p,r) = p \), then given \( \beta > \frac{1}{p} - \frac{1}{q} \), we select \( \alpha < p \) such that \( \beta > \frac{1}{\alpha} - \frac{1}{q} > \frac{1}{p} - \frac{1}{q} \). This choice of \( \alpha \) implies that \( \frac{\beta q}{q - \min(1,r)} > 1 \) if \( \alpha < q < \infty \) and \( \beta q > 1 \) if \( q = \infty \), therefore \( m_{X^{(1/\alpha)},\alpha,q}(0) < \infty \).

In case that \( p \leq q \leq \infty \), from \( \alpha < p \), if follows that

\[
m_{X^{(1/\alpha)},\alpha,q}(0) = \sup_{z \in [0,1]} \frac{1}{(1 + \ln z)^{\beta}} < \infty
\]

when \( 0 < q \leq \alpha < p \).

\[\square\]

**Theorem 25.** Let \( X = L^{p,r}(\log L)^{\beta} \) be a Lorentz-Zygmund space on \( \Omega \) \((0 < p < \infty, 0 < r \leq \infty, \beta \in \mathbb{R})\). Let \( 0 < s < 1, 0 < q \leq \infty \). The following embedding holds.

(i) If \( s < \frac{Q}{p} \), then

\[
\left( \int_0^1 \left[ t^{\frac{\beta}{p} - \frac{1}{q}} \left( 1 + \ln \frac{1}{t} \right)^{\beta} f^*(t) \right]^q \frac{dt}{t} \right)^{1/q} \leq \| f \|_{B^{s}_{L^{p,r}(\log L)^{\beta},q}} + \| f \|_{L_{\min(1,r),p} + L^\infty}.
\]

and

\[
\sup_{0 < t < 1} t^{\frac{\beta}{p} - \frac{1}{q}} \left( 1 + \ln \frac{1}{t} \right)^{\beta} f^*(t) \leq \| f \|_{B^{s}_{L^{p,r}(\log L)^{\beta},\infty}} + \| f \|_{L_{\min(1,r),p} + L^\infty}.
\]

(ii) If \( s = \frac{Q}{p} \) and

\[
\begin{cases}
\beta \leq \frac{1}{\min(1,p,r)} - \frac{1}{q} & \text{if } \min(1,p,r) < q \leq \infty, \\
\beta < 0 & \text{if } 0 < q \leq \min(1,p,r),
\end{cases}
\]

then

(a) If \( \min(1,r) < p \), or \( \min(1,r) = p \) an \( \beta \geq 0 \), then

(i) If \( \beta = \frac{1}{\min(1,r)} - \frac{1}{q} \) and \( \min(1,r) < q \leq \infty \), then

\[
\left( \int_0^1 \frac{f^*(t)^q}{(1 + \ln \frac{1}{t}) \left( 1 + \ln \left( 1 + \ln \frac{1}{t} \right) \right)^{\min(1,r)}} \frac{dt}{t} \right)^{1/q} \leq \| f \|_{B^{s}_{L^{p,r}(\log L)^{\beta},q}} + \| f \|_{L_{\min(1,r),p} + L^\infty},
\]

if \( \min(1,r) < q \leq \infty \), and

\[
\sup_{t \in [0,1]} \frac{f^*(t)}{(1 + \ln \left( 1 + \ln \frac{1}{t} \right))^2} \leq \| f \|_{B^{s}_{L^{p,r}(\log L)^{\beta},\infty}} + \| f \|_{L_{\min(1,r),p} + L^\infty}.
\]
(ii) If $\beta < \frac{1}{\min(1, r)} - \frac{1}{q}$ and $\min(1, r) < q \leq \infty$, then

$$\left( \int_0^1 \left( \frac{f^*(t)}{1 + \ln \frac{1}{t}} \right)^{\frac{q}{\alpha}} \frac{dt}{t} \right)^{1/q} \leq \|f\|_{L_p,r(\log L)^\beta_{\alpha,q}} + \|f\|_{L^{\min(1, r)+L\infty}},$$

if $\min(1, r) < q < \infty$, and

$$\sup_{t \in [0, 1]} \frac{f^*(t)}{1 + \ln \frac{1}{t}}^{\frac{1}{\min(1, r)} - \beta} \leq \|f\|_{L_p,r(\log L)^\beta_{\alpha,q}} + \|f\|_{L^{\min(1, r)+L\infty}}.$$

(iii) If $\beta < 0$ and $0 < q \leq \min(1, r)$, then

$$\left( \int_0^1 f^*(t)^q \left( 1 + \ln \frac{1}{t} \right)^{\frac{\beta - q - 1}{\alpha}} \frac{dt}{t} \right)^{1/q} \leq \|f\|_{L_p,r(\log L)^\beta_{\alpha,q}} + \|f\|_{L^{\min(1, r)+L\infty}},$$

if $p \leq q < \infty$, and

$$\sup_{t \in [0, 1]} \frac{f^*(t)}{1 + \ln \frac{1}{t}}^{\frac{1}{\min(1, r)} - \beta} \leq C_\alpha \left( \|f\|_{L_p,r(\log L)^\beta_{\alpha,q}} + \|f\|_{L^{\min(1, r)+L\infty}} \right).$$

(b) If $p < \min(1, r)$ or $\min(1, r) = p$ and $\beta < 0$, then for any $\alpha < p$ we have that

(i) If $\beta \leq \frac{1}{p} - \frac{1}{q}$ and $p \leq q \leq \infty$, then there is positive constant $C_\alpha$ that blows up when a tends to $p$, such that

$$\left( \int_0^1 \left( \frac{f^*(t)}{1 + \ln \frac{1}{t}} \right)^{\frac{q}{\alpha}} \frac{dt}{t} \right)^{1/q} \leq C_\alpha \left( \|f\|_{L_p,r(\log L)^\beta_{\alpha,q}} + \|f\|_{L^{\min(1, r)+L\infty}} \right),$$

if $p \leq q < \infty$, and

$$\sup_{t \in [0, 1]} \frac{f^*(t)}{1 + \ln \frac{1}{t}}^{\frac{1}{\min(1, r)} - \beta} \leq C_\alpha \left( \|f\|_{L_p,r(\log L)^\beta_{\alpha,q}} + \|f\|_{L^{\min(1, r)+L\infty}} \right).$$

(ii) If $\beta < 0$ and $0 < q < p$, then

$$\left( \int_0^1 f^*(t)^q \left( 1 + \ln \frac{1}{t} \right)^{\frac{\beta - q - 1}{\alpha}} \frac{dt}{t} \right)^{1/q} \leq \|f\|_{L_p,r(\log L)^\beta_{\alpha,q}} + \|f\|_{L^{\min(1, r)+L\infty}}.$$

**Proof.** Let $0 < \alpha \leq 1$ satisfying (45), $0 < s < 1$ and $0 < q \leq \infty$. Let $m(t) = m_X(1/\alpha,s,q,t)$ the function defined as in (46). It is a matter of a tedious but elementary calculation to verify that if $s < \frac{Q}{p}$, then $m(0) = \infty$ and

$$\left\{ \begin{array}{ll}
\frac{|m'(t)|}{(1+m(t))^{\gamma/\alpha}} \approx (1 + \ln \frac{1}{t})^{\frac{\beta - q}{\alpha}} \frac{1}{t} & \text{if } \alpha < q < \infty, \\
\frac{1}{(1+m(t))^{1/\gamma}} \approx t^{\beta - \frac{q}{\alpha}} (1 + \ln \frac{1}{t})^{\beta} & \text{if } q = \infty, \\
\int_0^t u^q(z)dz \leq u^q(t) & \text{if } 0 < q \leq \alpha,
\end{array} \right.$$}

consequently part 1 of Theorem (22) applies.

In the same way, if $s = \frac{Q}{p}$, then $m(0) = \infty$, and
Thus:

(i) If $\min(1, r) < p$, or $\min(1, r) = p$ and $\beta \geq 0$, then $\alpha = \min(1, r)$ and the part (a) of statement 2 follows by Theorem 22.

(ii) If $p < \min(1, r)$ or $\min(1, r) = p$ and $\beta < 0$, then for any $\alpha < p$, we have that $\beta \leq \frac{1}{p} - \frac{\alpha}{q} < \frac{1}{\alpha} - \frac{1}{q}$, therefore using one more time Theorem 22 we obtain the (b)-part of statement 2.

As a byproduct of the above Theorem, having $\beta = 0$, we obtain the following embedding result for Lorentz-Hajłasz-Besov spaces $B_{L^{p,r},q}^r$.

**Corollary 26.** Let $0 < p < \infty$, $0 < r \leq \infty$, $0 < q \leq \infty$ and $0 < s < 1$. The following assertions are true:

(i) If $s < \frac{q}{p}$, then

$$
\left( \int_0^1 \left[ t^\frac{1}{q} \int f^*(t) \frac{dt}{t} \right]^q \frac{dt}{t} \right)^{1/q} \leq \|f\|_{B_{L^{p,r},q}} + \|f\|_{L^{\min(1,p,r),L^\infty}} ,
$$

if $0 < q < \infty$, and

$$
\sup_{0 < t < 1} t^\frac{1}{q} f^*(t) \leq \|f\|_{B_{L^{p,r},\infty}} + \|f\|_{L^{\min(1,p,r),L^\infty}} .
$$

(ii) If $s = \frac{q}{p}$, then

(a) if $0 < r < p$, then

$$
\left( \int_0^1 \left[ \frac{f^*(t)}{(1 + \ln \frac{1}{t})^{\min(1,r)}} \right]^q \frac{dt}{t} \right)^{1/q} \leq \|f\|_{B_{L^{p,r},q}} + \|f\|_{L^{\min(1,r),L^\infty}} ,
$$

if $\min(1, r) < q < \infty$, and

$$
\sup_{0 < t < 1} \frac{f^*(t)}{(1 + \ln \frac{1}{t})^{\min(1,r)}} \leq \|f\|_{B_{L^{p,r},\infty}} + \|f\|_{L^{\min(1,p),L^\infty}} .
$$

(b) if $1 < p \leq r$, or $1 \leq p = r$, then

$$
\left( \int_0^1 \left[ \frac{f^*(t)}{(1 + \ln \frac{1}{t})} \right]^q \frac{dt}{t} \right)^{1/q} \leq \|f\|_{B_{L^{p,r},q}} + \|f\|_{L^1 + L^\infty} ,
$$

if $1 < q < \infty$, and

$$
\sup_{0 < t < 1} \frac{f^*(t)}{1 + \ln \frac{1}{t}} \leq \|f\|_{B_{L^{p,r},\infty}} + \|f\|_{L^{\min(1,p),L^\infty}} .
$$
(c) if $0 < p \leq r$ and $0 < p \leq 1$, then for any $\alpha < p$:

$$\left( \int_0^1 \left[ \frac{f^*(t)}{(1 + \ln 1/t)^{\bar{s}}} \right]^q \frac{dt}{t} \right)^{1/q} \leq \|f\|_{B^{\alpha}_{L^p,r,q}} + \|f\|_{L^\alpha + L^\infty},$$

and

$$\sup_{0 < t < 1} \frac{f^*(t)}{(1 + \ln 1/t)^{\bar{s}}} \leq \|f\|_{B^{\alpha}_{L^p,r,\infty}} + \|f\|_{L^\alpha + L^\infty}.$$

(iii) If $s > \frac{Q_p}{p}$ or $s = \frac{Q_p}{p}$ and $q < p$, then

$$\|f\|_{L^\infty} \leq \|f\|_{B^{\alpha}_{L^p,r,q}} + \|f\|_{L^\alpha + L^\infty}.$$

Similarly, if $p = r$ then we obtain embeddings for Lorentz-Zygmund-Hajlasz-Besov spaces $B^s_{L^p((\log L)^{\beta})}$, we left the details to the interested reader.

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Department of Mathematics, Universitat Autònoma de Barcelona

Email address: jmartin@mat.uab.cat

Department of Mathematics, Universitat Autònoma de Barcelona

Email address: waortiz@mat.uab.cat