THE PRIMITIVE EQUATIONS WITH MAGNETIC FIELD APPROXIMATION OF THE 3D MHD EQUATIONS

LILI DU\textsuperscript{1} AND DAN LI\textsuperscript{2}

ABSTRACT. In our earlier work [20], we have shown the global well-posedness of strong solutions to the three-dimensional primitive equations with the magnetic field (PEM) on a thin domain. The heart of this paper is to provide a rigorous justification of the derivation of the PEM as the small aspect ratio limit of the incompressible three-dimensional scaled magnetohydrodynamics (SMHD) equations in the anisotropic horizontal viscosity and magnetic field regime. For the case of $H^1$-initial data case, we prove that global Leray-Hopf weak solutions of the three-dimensional SMHD equation strongly converge to the global strong solutions of the PEM. In the $H^2$-initial data case, the strong solution of the SMHD can be extended to be a global one for small $\varepsilon$. As a consequence, we observe that the global strong solutions of the SMHD strongly converge to the global strong solutions of the PEM. As a byproduct, the convergence rate is of the same order as the aspect ratio parameter.

1. Introduction

1.1. Background and motivation. The magnetohydrodynamics (MHD) system studies the dynamics of electrically conducting fluids under the influence of magnetic fields. There are many examples of conducting fluids, including plasmas, liquid metals, electrolytes, etc. The main idea of magnetohydrodynamics is that conducting fluids can support magnetic fields. More precisely, magnetic fields can induce currents in a moving conducting fluid, creating forces on the fluid and also changing the magnetic fields themselves. The subject of magnetohydrodynamics unites classical fluid dynamics with electrodynamics, and references can be found in [2,5,7,19,21]. Besides their wide physical applications, the global well-posedness of the MHD equations is an active topic in mathematics. The existence and uniqueness results for weak and strong solutions of the 2D MHD equations are well known by Duvaut and Lions in [22]. In general, for the 3D case, it is currently unknown whether the solutions can develop finite-time singularities even if the initial

\textit{2020 Mathematics Subject Classification.} 35Q30, 76W05, 76D05, 35Q86.

\textit{Key words and phrases.} Primitive equations with magnetic field; Anisotropic MHD equation; Strong convergence.
value is sufficiently smooth. Different criteria for regularity in terms of the velocity field, the magnetic field, the pressure, or their derivatives have been proposed (see [10,17,25–27,35,39–45] and references therein). One of the most elegant works is given by He and Xin in [26,27], in which they first realized that the velocity fields played a dominant role in the regularity of the solution to 3D incompressible MHD equations.

Without the magnetic field, the MHD equations reduce to the Navier-Stokes equations. In the context of the geophysical flow concerning the large-scale oceanic dynamics, the vertical scale of the global atmosphere is much smaller than the horizontal one. Based on this relationship, by scaling the incompressible Navier-Stokes equations concerning the aspect ratio parameter and taking the small aspect ratio limit, one formally obtains the primitive equations for the large-scale oceanic dynamics. The global existence of strong solutions for the 2D case was established by Bresch et al. [6] and Temam and Ziane in [38], while the 3D primitive equations have already been known since the breakthrough work by Cao and Titi [9], see also [28–31]. In the last few years, developments concerning the global well-posedness to the anisotropic primitive equations were also made, see [11–16]. The rigorous justification of the primitive equations from the scaled Navier-Stokes equations was studied by Azérad and Guillén in [3]. By relying on the result in [3] to prove the weak convergence, Li and Titi in [33] showed that the Navier-Stokes equations strongly converge to the primitive equations. Furukawa et al. [23] extended the results by Li and Titi, and the convergence result is shown in the much more general $L^p - L^q$ setting. Furukawa, Giga and Kashiwabara [24] showed that the solution to the scaled Navier-Stokes equations with Besov initial data converges to the solution to the primitive equation with the same initial data.

In our paper [20], the anisotropic viscosity and magnetic diffusivity scaling in the horizontal and vertical directions, so that the PEM derived from the scaled MHD equation, as the aspect ratio goes to zero. Moreover, we have shown the global well-posedness of strong solutions to the three-dimensional incompressible PEM without any small assumption on the initial data. Motivated by the rigorous justification of the limiting process, the convergence from Navier-Stokes equations to primitive equations by Li and Titi in [33] in the strong setting. In this paper, we will investigate the rigorously justify the scaled MHD equations convergence strongly to the PEM, globally and uniformly in time, and that the convergence rate is of the same order as the aspect ratio parameter. These will be the main consequences of the studies in this paper.
1.2. Anisotropic MHD equations. In this subsection we consider the three-
dimensional anisotropic MHD equations on an \( \varepsilon \)-dependent thin domain
\[
\Omega_\varepsilon := M \times (-\varepsilon, \varepsilon) \subset \mathbb{R}^3,
\]
where \( \varepsilon > 0 \) is a very small parameter, and \( M = (0, L_1) \times (0, L_2) \), for two positive
constants \( L_1 \) and \( L_2 \) of order \( O(1) \) with respect to \( \varepsilon \).

The incompressible three-dimensional anisotropic MHD system is
\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla p - \mu \Delta_H u - \nu \frac{\partial^2 z}{z} u &= b \cdot \nabla b, \\
\partial_t b + u \cdot \nabla b - \kappa \Delta_H b - \sigma \frac{\partial^2 z}{z} b &= b \cdot \nabla u, \\
\nabla \cdot u &= 0, \quad \nabla \cdot b = 0,
\end{align*}
\]
with
\(-u = (\bar{u}, u_3)\) is the velocity field, where \( \bar{u} = (u_1, u_2) \) is the horizontal velocity field,
\(-b = (\bar{b}, b_3)\) is the magnetic field, where \( \bar{b} = (b_1, b_2) \) is the horizontal magnetic field,
\(-p\) is the pressure,
\(-\mu\) is the horizontal viscous coefficient,
\(-\nu\) is the vertical viscous coefficient,
\(-\kappa\) is the horizontal magnetic diffusivity coefficient,
\(-\sigma\) is the vertical magnetic diffusivity coefficient,
\(-\Delta = \Delta_H + \frac{\partial^2 z}{z} , \) where \( \Delta_H = \partial_x^2 + \partial_y^2 \) is the horizontal Laplacian,
\(-\nabla = (\partial_x, \partial_y, \partial_z)\) is gradient operator,
\(-\nabla \cdot\) is divergence operator.

Similar to the case considered in Azérad-Guillén [3], it is emphasized that the
anisotropic viscosity hypothesis is fundamental for the derivation of the primitive
equations. In this paper, we suppose that \( \mu \) and \( \nu \) have different orders, that is,
\( \mu = O(1) \) and \( \nu = O(\varepsilon^2) \). The orders of magnetic diffusivity coefficients \( \kappa \) and \( \sigma \)
are similar to the viscous coefficients. For the sake of simplicity, we set \( \mu = 1 \) and
\( \nu = \varepsilon^2 \), similarly, \( \kappa = 1 \) and \( \sigma = \varepsilon^2 \).

1.3. Scaled MHD equations. We carry out the following scaling transformation
to the MHD equations (1.1) such that the resulting system is defined on a fixed
domain independent of \( \varepsilon \). To this end, we introduce the new unknowns,
\[
\begin{align*}
u \epsilon (\bar{u}_\epsilon, u_{3,\epsilon}), \quad b_\epsilon &= (\bar{b}_\epsilon, b_{3,\epsilon}), \quad p_\epsilon(x, y, z, t) &= p(x, y, \varepsilon z, t), \\
\bar{u}_\epsilon(x, y, z, t) &= \bar{u}(x, y, \varepsilon z, t) = (u_1(x, y, \varepsilon z, t), u_2(x, y, \varepsilon z, t)), \\
\bar{b}_\epsilon(x, y, z, t) &= \bar{b}(x, y, \varepsilon z, t) = (b_1(x, y, \varepsilon z, t), b_2(x, y, \varepsilon z, t)), \\
\text{and} \quad u_{3,\epsilon}(x, y, z, t) &= \frac{1}{\varepsilon} u_3(x, y, \varepsilon z, t), \quad b_{3,\epsilon}(x, y, z, t) = \frac{1}{\varepsilon} b_3(x, y, \varepsilon z, t).
\end{align*}
\]
For any \((x, y, z) \in \Omega := M \times (-1, 1)\) and \(t \in [0, \infty)\), then \(u_\varepsilon, b_\varepsilon\) and \(p_\varepsilon\) satisfy the following incompressible scaled MHD equations (SMHD)

\[
\begin{aligned}
\partial_t \tilde{u}_\varepsilon + u_\varepsilon \cdot \nabla \tilde{u}_\varepsilon + \nabla_H p_\varepsilon - b_\varepsilon \cdot \nabla \tilde{b}_\varepsilon - \Delta \tilde{u}_\varepsilon &= 0, \\
\varepsilon^2 (\partial_t \tilde{u}_{3,\varepsilon} + u_\varepsilon \cdot \nabla \tilde{u}_{3,\varepsilon} - \Delta \tilde{u}_{3,\varepsilon} - b_\varepsilon \cdot \nabla \tilde{b}_{3,\varepsilon}) + \partial_z p_\varepsilon &= 0, \\
\partial_t \tilde{b}_\varepsilon + u_\varepsilon \cdot \nabla \tilde{b}_\varepsilon - \Delta \tilde{b}_\varepsilon - b_\varepsilon \cdot \nabla \tilde{u}_\varepsilon &= 0, \\
\varepsilon^2 (\partial_t \tilde{b}_{3,\varepsilon} + u_\varepsilon \cdot \nabla \tilde{b}_{3,\varepsilon} - \Delta \tilde{b}_{3,\varepsilon} - b_\varepsilon \cdot \nabla \tilde{u}_{3,\varepsilon}) &= 0, \\
\nabla \cdot u_\varepsilon &= 0, \\
\nabla \cdot b_\varepsilon &= 0,
\end{aligned}
\]

with the following initial conditions

\[
\begin{aligned}
(\tilde{u}_\varepsilon, u_{3,\varepsilon})_{|t=0} &= (\tilde{u}_{\varepsilon,0}, u_{3,\varepsilon,0}), \\
(\tilde{b}_\varepsilon, b_{3,\varepsilon})_{|t=0} &= (\tilde{b}_{\varepsilon,0}, b_{3,\varepsilon,0}).
\end{aligned}
\]

The above equations (1.2) are defined in the fixed domain \(\Omega\). Throughout this paper, we set \(\nabla_H\) to denote \((\partial_x, \partial_y)\). In addition, we equip the system (1.2) with the following periodic boundary conditions,

\[
\begin{aligned}
\tilde{u}_\varepsilon(x, y, z - 1, t) &= \tilde{u}_\varepsilon(x + L_1, y + L_2, z + 1, t), \\
u_{3,\varepsilon}(x, y, z - 1, t) &= u_{3,\varepsilon}(x + L_1, y + L_2, z + 1, t), \\
\tilde{b}_\varepsilon(x, y, z - 1, t) &= \tilde{b}_\varepsilon(x + L_1, y + L_2, z + 1, t), \\
b_{3,\varepsilon}(x, y, z - 1, t) &= b_{3,\varepsilon}(x + L_1, y + L_2, z + 1, t),
\end{aligned}
\]

and

\[
\begin{aligned}
p_\varepsilon(x, y, z - 1, t) &= p_\varepsilon(x + L_1, y + L_2, z + 1, t). \tag{1.8}
\end{aligned}
\]

Furthermore, for simplicity, we suppose that the space of periodic functions with respect to \(z\) with the following symmetry

\[
\begin{aligned}
\tilde{u}_\varepsilon(x, y, z, t) &= \tilde{u}_\varepsilon(x, y, -z, t), \\
u_{3,\varepsilon}(x, y, z, t) &= -u_{3,\varepsilon}(x, y, -z, t), \\
p_\varepsilon(x, y, z, t) &= -p_\varepsilon(x, y, -z, t), \\
\tilde{b}_\varepsilon(x, y, z, t) &= \tilde{b}_\varepsilon(x, y, -z, t), \tag{1.9}
\end{aligned}
\]

and

\[
\begin{aligned}
b_{3,\varepsilon}(x, y, z, t) &= -b_{3,\varepsilon}(x, y, -z, t). \tag{1.10}
\end{aligned}
\]

Note that the dynamics of SMHD preserve these symmetry conditions. In other words, they are automatically satisfied as long as they are satisfied initially. So in this article, without further mention, we always assume that the initial horizontal velocity and magnetic field \((\tilde{u}_{0,\varepsilon}, \tilde{b}_{0,\varepsilon})\) satisfy that

\[
\tilde{u}_{0,\varepsilon}, \tilde{b}_{0,\varepsilon} \text{ are periodic in } x, y, z, \text{ and are even in } z.
\]
By the classic theory see, e.g., [22], for any initial data \((u_0, b_0) \in L^2(\Omega)\), there is global weak solution \((u, b)\) to the SMHD equation (1.2), subject to (1.3)-(1.8), here the weak solutions is defined as follows.

**Definition 1.1.** A weak solution \((u, b)\) of the SMHD (1.2) is called Leray-Hopf weak solution, if \((u, b) \in C_w(0, \infty); L^2(\Omega) \cap L^2(0, \infty), H^1(\Omega)\), where the subscript \(w\) means weakly continuous and \(L^2(\Omega)\) denotes the space consisting of all divergence-free functions in \(L^2(\Omega)\) and

\[
\|\tilde{u}(t)\|_2^2 + \|\tilde{b}(t)\|_2^2 + \varepsilon^2\|u_3(t)\|_2^2 + \varepsilon^2\|b_3(t)\|_2^2 + 2\int_0^t \left(\|\nabla\tilde{u}\|_2^2 + \|\nabla\tilde{b}\|_2^2 + \varepsilon^2\|\nabla u_3\|_2^2 + \varepsilon^2\|\nabla b_3\|_2^2\right) ds 
\leq \|u_0\|_2^2 + \|b_0\|_2^2 + \varepsilon^2\|u_{30}\|_2^2 + \varepsilon^2\|b_{30}\|_2^2,
\]

for a.e. \(t \in [0, \infty)\).

Moreover, the following integral identity holds

\[
\int_Q \left[ -(\tilde{u} \cdot \partial_t \varphi_H + \varepsilon^2 u_3 \partial_t \varphi_3) - (\tilde{b} \cdot \partial_t \psi_H + \varepsilon^2 b_3 \partial_t \psi_3) + (u \cdot \nabla)\tilde{u} \cdot \varphi_H 
- (b \cdot \nabla)\tilde{b} \cdot \varphi_H + \varepsilon^2 u \cdot \nabla u_3 \varphi_3 - \varepsilon^2 b \cdot \nabla b_3 \varphi_3 + (u \cdot \nabla)\tilde{b} \cdot \psi_H - (b \cdot \nabla)\tilde{u} \cdot \psi_H 
+ \varepsilon^2 u \cdot \nabla b_3 \varphi_3 - \varepsilon^2 b \cdot \nabla u_3 \varphi_3 + \nabla \tilde{u} : \nabla \varphi_H + \varepsilon^2 \nabla u_3 \cdot \nabla \varphi_3 + \nabla \tilde{b} : \nabla \psi_H 
+ \varepsilon^2 \nabla b_3 \cdot \nabla \psi_3 \right] dx dy dz dt
\]

\[= \int_{\Omega} (\tilde{u}_0 \cdot \varphi_H(\cdot, 0) + \varepsilon^2 u_{30} \varphi_3(\cdot, 0)) dx dy dz + \int_{\Omega} (\tilde{b}_0 \cdot \psi_H(\cdot, 0) + \varepsilon^2 b_{30} \psi_3(\cdot, 0)) dx dy dz,
\]

where \(Q := \Omega \times (0, \infty)\), for any space periodic functions \(\varphi = (\varphi_H, \varphi_3)\) and \(\psi = (\psi_H, \psi_3)\), with \(\varphi_H = (\varphi_1, \varphi_2)\) and \(\psi_H = (\psi_1, \psi_2)\), the divergence-free test functions \(\varphi\) and \(\psi\) satisfy \(\varphi \in C_0^\infty(\Omega \times [0, \infty))\) and \(\psi \in C_0^\infty(\Omega \times [0, \infty))\).

### 1.4. Primitive equations with magnetic field.

By taking the limit as \(\varepsilon \to 0\) in the SMHD (1.2), it is straightforward to obtain the following primitive equations with magnetic field (PEM)

\[
\begin{cases}
\partial_t \tilde{u} + u \cdot \nabla \tilde{u} - \Delta \tilde{u} - b \cdot \nabla \tilde{b} + \nabla H p = 0, \\
\partial_z p = 0, \\
\partial_t \tilde{b} + u \cdot \nabla \tilde{b} - \Delta \tilde{b} - b \cdot \nabla \tilde{u} = 0, \\
\nabla H \cdot \tilde{u} + \partial_z u_3 = 0, \\
\nabla H \cdot \tilde{b} + \partial_z b_3 = 0.
\end{cases}
\]

Recalling that we consider the periodic initial-boundary value problem to the SMHD equations (1.2), it is clear that one should impose the same boundary conditions and symmetry conditions on the corresponding limiting system (1.13). However, one only needs to impose the initial condition on the horizontal velocity.
field and magnetic field. Since $u_{3,0}$ and $b_{3,0}$ are odd in $z$, we have $u_{3,0}(x, y, 0) = b_{3,0}(x, y, 0) = 0$. Then, $(u_{3,0}, b_{3,0})$ can be determined uniquely by the incompressibility conditions as

$$u_{3,0}(x, y, z) = -\int_0^z \nabla_H \cdot \tilde{u}_0(x, y, \xi) \, d\xi,$$

(1.14)

and

$$b_{3,0}(x, y, z) = -\int_0^z \nabla_H \cdot \tilde{b}_0(x, y, \xi) \, d\xi.$$

(1.15)

Similarly, $(u_3, b_3)$ can also be determined uniquely by the incompressibility conditions as

$$u_3(x, y, z, t) = -\int_0^z \nabla_H \cdot \tilde{u}(x, y, \xi, t) \, d\xi,$$

(1.16)

and

$$b_3(x, y, z, t) = -\int_0^z \nabla_H \cdot \tilde{b}(x, y, \xi, t) \, d\xi.$$

(1.17)

Due to these facts concerning the solutions to (1.13), we only solve the horizontal components $(\tilde{u}, \tilde{b})$, and the vertical components $(u_3, b_3)$ are uniquely determined by (1.16) and (1.17). Throughout this paper, all the velocities and the magnetic field encountered in this paper are of average zero. We will not review this fact in the rest of this article, before using the Poincaré inequality.

1.5. Main ideas of the construction. The main results in this paper are concerned with the strong convergence from the SMHD equations to the PEM, as the aspect ratio parameter goes to zero. For the first result, given $(\tilde{u}_0, \tilde{b}_0) \in H^1(\Omega)$, we prove that global Leray-Hopf weak solutions of the three-dimensional SMHD equation strongly converge with the global strong solutions of the PEM. More precisely, we prove the strong convergence

$$(\tilde{u}_\varepsilon, \varepsilon u_{3,\varepsilon}, \tilde{b}_\varepsilon, \varepsilon b_{3,\varepsilon}) \to (\tilde{u}, 0, \tilde{b}, 0) \text{ in } L^\infty(0, \infty; L^2(\Omega)).$$

On the other hand, given the initial data $(\tilde{u}_0, \tilde{b}_0) \in H^2(\Omega)$, the strong solution $(\tilde{u}_\varepsilon, \varepsilon u_{3,\varepsilon}, \tilde{b}_\varepsilon, \varepsilon b_{3,\varepsilon})$ of the SMHD can be extended to be a global one for small $\varepsilon$. As a consequence, we observe that the global strong solutions of SMHD strongly converge to the global strong solutions of the PEM, that is

$$(\tilde{u}_\varepsilon, \varepsilon u_{3,\varepsilon}, \tilde{b}_\varepsilon, \varepsilon b_{3,\varepsilon}) \to (\tilde{u}, 0, \tilde{b}, 0) \text{ in } L^\infty(0, \infty; H^1(\Omega)),$$

and the converge rate of two regimes are the order $O(\varepsilon)$.

We now make some comments on the analysis of this paper. The treatments on the estimates of the difference function $(U_\varepsilon, B_\varepsilon) = (u_\varepsilon - u, b_\varepsilon - b)$ are different in the proofs of the first and second results. For the case of the first result, since
(\tilde{u}_\varepsilon, u_{3,\varepsilon}, \tilde{b}_\varepsilon, b_{3,\varepsilon})$ is the Leray-Hopf weak solution, the energy estimates cannot be directly used for the system of difference between the SMHD and the PEM, as it is usually done for strong solutions. Instead, we adopt a similar approach by Serrin in [36] (see also Bardos et al. [4] and the reference therein), showing the weak-strong uniqueness of the Navier-Stokes equations. The difference is that in our article, the “strong solutions” is played by the solutions of PEM, while the “weak solutions” is played by the solutions of SMHD. Now, we explain in more detail how we perform the global Leray-Hopf weak solutions of SMHD strong convergence to the global strong solutions of the PEM,

\begin{enumerate}
  \item Using $(\tilde{u}, u_3, \tilde{b}, b_3)$ as the testing functions for the SMHD, we get one equality,
  \item testing the PEM by $(\tilde{u}_\varepsilon, \tilde{b}_\varepsilon)$, it follows from integration by parts, we have one equality,
  \item applying the PEM by $(\tilde{u}, \tilde{b})$, we obtain the basic energy identity of the PEM,
  \item Recalling the definition of Leray-Hopf weak solution to the SMHD, we have
\end{enumerate}

Adding (3) and (4), then subtracting from (1) and (2), we obtain a new integral inequality. Appropriately manipulating this formulas, we get the desired \textit{a priori} estimates for $(U_\varepsilon, B_\varepsilon)$.

For the second case of strong convergence, one gets the desired global in time estimates on $(U_\varepsilon, B_\varepsilon)$ by using standard energy estimates. $(U_\varepsilon, B_\varepsilon)$ denotes the difference between $u_\varepsilon$, $u$, $b_\varepsilon$ and $b$, namely,

$$U_\varepsilon = (\tilde{U}_\varepsilon, U_{3,\varepsilon}), \quad \tilde{U}_\varepsilon = \tilde{u}_\varepsilon - \tilde{u}, \quad U_{3,\varepsilon} = u_{3,\varepsilon} - u_3,$$

and $$B_\varepsilon = (\tilde{B}_\varepsilon, B_{3,\varepsilon}), \quad \tilde{B}_\varepsilon = \tilde{b}_\varepsilon - \tilde{b}, \quad B_{3,\varepsilon} = b_{3,\varepsilon} - b_3,$$

then, one can easily verify that $U_\varepsilon = (\tilde{U}_\varepsilon, U_{3,\varepsilon})$ and $B_\varepsilon = (\tilde{B}_\varepsilon, B_{3,\varepsilon})$ satisfy the following system

\[
\begin{align*}
\partial_t \tilde{U}_\varepsilon - (U_\varepsilon \cdot \nabla) \tilde{U}_\varepsilon - \Delta \tilde{U}_\varepsilon + \nabla P_\varepsilon + (u \cdot \nabla) \tilde{U}_\varepsilon + (U_\varepsilon \cdot \nabla) \tilde{u} & = -b \cdot \nabla B_{3,\varepsilon} - B_\varepsilon \cdot \nabla b - \nabla B \cdot \nabla \tilde{u} - B_\varepsilon \cdot \nabla b - B_\varepsilon \cdot \nabla \tilde{u} - B_3 \cdot \nabla b - B_3 \cdot \nabla \tilde{u} = 0, \\
\varepsilon^2 (\partial_t U_{3,\varepsilon} + U_\varepsilon \cdot \nabla U_{3,\varepsilon} - \Delta U_{3,\varepsilon} + U_\varepsilon \cdot \nabla u_3 + u \cdot \nabla U_{3,\varepsilon} - B_\varepsilon \cdot \nabla B_{3,\varepsilon} & = 0, \\
-b \cdot \nabla B_{3,\varepsilon} - B_\varepsilon \cdot \nabla b_3) + \partial_x \varepsilon^2 (\partial_t u_3 + u \cdot \nabla u_3 - \Delta u_3 - b \cdot \nabla b_3) & = 0, \\
\partial_t B_\varepsilon + U_\varepsilon \cdot \nabla B_\varepsilon - \Delta B_\varepsilon + u \cdot \nabla B_\varepsilon + U_\varepsilon \cdot \nabla b_\varepsilon - B_\varepsilon \cdot \nabla \tilde{u} - B_\varepsilon \cdot \nabla \tilde{u} & = 0, \\
\varepsilon^2 (\partial_t B_{3,\varepsilon} + U_\varepsilon \cdot \nabla B_{3,\varepsilon} + u \cdot \nabla B_{3,\varepsilon} + U_\varepsilon \cdot \nabla b_3 - \Delta B_{3,\varepsilon} - B_\varepsilon \cdot \nabla U_{3,\varepsilon} & = 0, \\
-b \cdot \nabla U_{3,\varepsilon} - B_\varepsilon \cdot \nabla u_3 & = -\varepsilon^2 (\partial_t b_3 + u \cdot \nabla b_3 - \Delta b_3 - b \cdot \nabla u_3), \\
\nabla H \cdot \tilde{U}_\varepsilon + \partial_x U_{3,\varepsilon} & = 0, \quad \nabla H \cdot B_\varepsilon + \partial_x B_{3,\varepsilon} = 0.
\end{align*}
\]  

(1.18)
We recommend some new thoughts described in the following process. At first, since the initial value of \((\hat{U}_\varepsilon, U_3, \hat{B}_\varepsilon, B_3)\) disappears, and there is a small coefficient \(\varepsilon^2\) in the front of the “outside forcing” terms on the right-hand side of the (1.18)_2 and (1.18)_4, we can perform the energy approach and take advantage of the small parameters to obtain the required \textit{a priori} estimate on \((\hat{U}_\varepsilon, U_3, \hat{B}_\varepsilon, B_3)\). Moreover, the strong solution \((\tilde{u}_\varepsilon, u_3, \tilde{b}_\varepsilon, b_3, \varepsilon)\) of the SMHD can be developed to the global solution, for small \(\varepsilon\). Second, the information of \((U_{3, \varepsilon}, B_{3, \varepsilon})\) that comes from equations (1.18)_2 and (1.18)_4 is always related to the parameter \(\varepsilon\), which will ultimately go to zero. That is to say, the equations (1.18)_2 and (1.18)_4 have not provided the information of \(\varepsilon\)-independent of \((U_{3, \varepsilon}, B_{3, \varepsilon})\). After the attainment of the desired \textit{a priori} estimates, the strong convergence follows instantly.

In our previous work [20], we showed the global existence of the strong solution and uniqueness (regularity) to the three-dimensional incompressible PEM without any small assumption on the initial data. More precisely, there exists a unique strong solution globally in time for any given \(H^2\)-smooth initial data. As mentioned in the comments, the global well-posedness of strong solutions to the PEM plays a fundamental role proving the strong convergence of the small aspect ratio limit of the SMHD to the PEM. The main results of [20] are stated as follows.

**Proposition 1.2.** (see [20, Remark 1.2]). If the initial data \((\hat{u}_0, \hat{b}_0)\) belong to \(H^1(\Omega)\), then there exists a unique global strong solution to the PEM (1.13), which satisfies \((\bar{b}, \bar{u}) \in L^\infty([0, \infty); H^1(\Omega)) \cap L^2([0, \infty); H^2(\Omega)), (\partial_t \bar{b}, \partial_t \bar{u}) \in L^2([0, \infty); L^2(\Omega))\).

**Proposition 1.3.** (see [20, Theorem 1.1]). Suppose that \((\tilde{u}_0, \tilde{b}_0) \in H^2(\Omega)\), then there exists a unique global strong solution \((\bar{u}, \bar{b}) \in L^\infty([0, \infty); H^2(\Omega)) \cap L^2([0, \infty); H^4(\Omega))\) of the PEM (1.13), subject to the boundary and initial conditions (1.3)-(1.11). Moreover, we have the following estimate,

\[
\sup_{0 \leq t < \infty} \|\bar{u}\|^2_{H^2}(t) + \sup_{0 \leq t < \infty} \|\bar{b}\|^2_{H^2}(t) + \int_0^\infty (\|\nabla \bar{u}\|^2_{H^2} + \|\partial_t \bar{u}\|^2_{H^1}) \, dt
\]

\[
+ \int_0^\infty (\|\nabla \bar{b}\|^2_{H^2} + \|\partial_t \bar{b}\|^2_{H^1}) \, dt \leq C,
\]

for a constant \(C\) depending only on \(\|\bar{u}_0\|_{H^2}, \|\bar{b}_0\|_{H^2}, L_1\) and \(L_2\).

1.6. **The structure of this paper.** The remainder of this paper is organized as follows: Section 2 is dedicated to the basic notations and some Ladyzhenskaya-type inequalities. Section 3, this section is devoted to the case of \((\tilde{u}_0, \tilde{b}_0) \in H^1(\Omega)\), we prove that global Leray-Hopf weak solutions of the three-dimensional SMHD equations strongly converge to the global strong solutions of the PEM. In Section 4, for the \((\tilde{u}_0, \tilde{b}_0) \in H^2(\Omega)\) case, the strong solution of the SMHD can be extended to
be a global one, for small ε. Moreover, we observe that the global strong solutions of the SMHD strong converge to the global strong solution of the PEM, and the convergence rate is the same order as the aspect ratio parameter.

2. Preliminaries

In this section, we introduce the notations and some Ladyzhenskaya type inequalities for some kinds of three dimensional integrals, which will be frequently used in the rest of this paper.

Notation 2.1. For \( q \in [1, \infty] \), we will denote the Lebesgue spaces on the domain \( \Omega \) by \( L^q = L^q(\Omega) \). For simplicity of notation we will use \( \| \cdot \|_q \) and \( \| \cdot \|_{q,M} \) instead of \( L^q(\Omega) \) and \( L^q(M) \). For \( s \in \mathbb{N} \) the space \( H^s(\Omega) \) consists of \( f \in L^2(\Omega) \) such that \( \nabla^\alpha f \in L^2(\Omega) \) for \( |\alpha| \leq s \) endowed with the norm
\[
\| f \|_{H^s(\Omega)} = \left( \sum_{|\alpha| \leq s} \| \nabla^\alpha f \|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.
\]

Next, we state some Ladyzhenskaya-type inequalities for some kinds of three dimensional integrals.

Lemma 2.2. (see [8, Lemma 2.1] ). The following inequalities hold true
\[
\int_M \left( \int_{-1}^1 \alpha(x,y,z)dz \right) \left( \int_{-1}^1 \beta(x,y,z)\gamma(x,y,z)dz \right) dxdy \leq C \|\alpha\|_2^{\frac{1}{2}} \left( \|\alpha\|_2^{\frac{1}{2}} + \|\nabla_H \alpha\|_2^{\frac{1}{2}} \right) \|\beta\|_2 \|\gamma\|_2 \left( \|\gamma\|_2^{\frac{1}{2}} + \|\nabla_H \gamma\|_2^{\frac{1}{2}} \right),
\]
and
\[
\int_M \left( \int_{-1}^1 \alpha(x,y,z)dz \right) \left( \int_{-1}^1 \beta(x,y,z)\gamma(x,y,z)dz \right) dxdy \leq C \|\alpha\|_2 \|\beta\|_2 \left( \|\beta\|_2^{\frac{1}{2}} + \|\nabla_H \beta\|_2^{\frac{1}{2}} \right) \|\gamma\|_2 \left( \|\gamma\|_2^{\frac{1}{2}} + \|\nabla_H \gamma\|_2^{\frac{1}{2}} \right),
\]
for any \( \alpha, \beta, \gamma \) such that the right-hand sides make sense and are finite, where \( C \) is a positive constant depending only on \( L_1 \) and \( L_2 \).

Lemma 2.3. (see [33, Lemma 2.2] ). Let \( \varphi = (\varphi_1, \varphi_2, \varphi_3) \), \( \phi \) and \( \psi \) be periodic functions with domain \( \Omega \). Assume that \( \varphi \in H^1(\Omega) \), with \( \nabla \cdot \varphi = 0 \) in \( \Omega \), \( \int_\Omega \varphi dxdydz = 0 \), and \( \varphi_3|_{z=0} = 0 \), \( \nabla \phi \in H^1(\Omega) \) and \( \psi \in L^2(\Omega) \). \( \varphi_H = (\varphi_1, \varphi_2) \) denotes the horizontal components of the function \( \varphi \). Then, the following estimate holds
\[
\left| \int_\Omega (\varphi \cdot \nabla \phi) \psi dxdydz \right| \leq C \|\nabla \varphi_H\|_2^{\frac{1}{2}} \|\Delta \varphi_H\|_2^{\frac{1}{2}} \|\nabla \phi\|_2^{\frac{1}{2}} \|\Delta \phi\|_2^{\frac{1}{2}} \|\psi\|_2,
\]
where \( C \) is a positive constant depending only on \( L_1 \) and \( L_2 \).
3. Strong convergence I: the $H^1$ initial data case

This section is devoted to the strong convergence of the SMHD to the PEM with initial data $(\tilde{u}_0, \tilde{b}_0) \in H^1(\Omega)$. The main results are stated as follows.

**Theorem 3.1.** Given a periodic function $(\tilde{u}_0, \tilde{b}_0) \in H^1(\Omega)$, such that

$$\nabla_H \cdot \left( \int_{-1}^1 \tilde{u}_0(x, y, z) \, dz \right) = 0, \quad \int_{\Omega} \tilde{u}_0(x, y, z) \, dx \, dy \, dz = 0,$$

and

$$\nabla_H \cdot \left( \int_{-1}^1 \tilde{b}_0(x, y, z) \, dz \right) = 0, \quad \int_{\Omega} \tilde{b}_0(x, y, z) \, dx \, dy \, dz = 0.$$

Let $(\tilde{u}_\varepsilon, u_{3,\varepsilon}, \tilde{b}_\varepsilon, b_{3,\varepsilon})$ be an arbitrary Leray-Hopf weak solution to the SMHD, $(\bar{u}, u, \bar{b}, b)$ be the unique global strong solution to the PEM, subject to (1.3)-(1.11). We denote

$$(\tilde{U}_\varepsilon, U_{3,\varepsilon}) = (\tilde{u}_\varepsilon - \bar{u}, u_{3,\varepsilon} - u_3), \quad (\tilde{B}_\varepsilon, B_{3,\varepsilon}) = (\tilde{b}_\varepsilon - \bar{b}, b_{3,\varepsilon} - b_3).$$

Therefore, the following a priori estimate holds

$$\sup_{0 \leq t < \infty} \left( \|\tilde{U}_\varepsilon\|_2^2 + \varepsilon^2 \|U_{3,\varepsilon}\|_2^2 + \|\tilde{B}_\varepsilon\|_2^2 + \varepsilon^2 \|B_{3,\varepsilon}\|_2^2 \right)(t)$$

$$+ \int_0^\infty \left( \|\nabla \tilde{U}_\varepsilon\|_2^2 + \varepsilon^2 \|\nabla U_{3,\varepsilon}\|_2^2 + \|\nabla \tilde{B}\|_2^2 + \varepsilon^2 \|\nabla B_{3,\varepsilon}\|_2^2 \right) \, ds$$

$$\leq C \varepsilon^2 \left( \|\tilde{u}_0\|_2^2 + \|\tilde{b}_0\|_2^2 + \varepsilon^2 \|u_{3,0}\|_2^2 + \varepsilon^2 \|b_{3,0}\|_2^2 + 1 \right),$$

for any $\varepsilon > 0$, where $C$ is a positive constant depending only on $\|\tilde{u}_0\|_{H^1}, \|\tilde{b}_0\|_{H^1}, L_1$ and $L_2$. As a direct consequence, we get

$$(\tilde{u}_\varepsilon, \varepsilon u_{3,\varepsilon}, \tilde{b}_\varepsilon, \varepsilon b_{3,\varepsilon}) \to (\bar{u}, 0, \bar{b}, 0), \quad \text{in } L^\infty(0, \infty; L^2(\Omega)),$$

$$(\nabla \tilde{u}_\varepsilon, \varepsilon \nabla u_{3,\varepsilon}, \nabla \tilde{b}_\varepsilon, \varepsilon \nabla b_{3,\varepsilon}) \to (\nabla \bar{u}, 0, \nabla \bar{b}, 0, b_3), \quad \text{in } L^2(0, \infty; L^2(\Omega)),$$

and the convergence rate is of the order $O(\varepsilon)$.

**Remark 3.2.** The assumptions $\int_{\Omega} \tilde{u}_0 \, dx \, dy \, dz = 0$ and $\int_{\Omega} \tilde{b}_0 \, dx \, dy \, dz = 0$ are imposed only for the simplicity of the proof, and the same result still holds for the general case. One can follow the proof presented in this paper, and establish the relevant a priori estimates on $(\tilde{u}_\varepsilon - \overline{u}_{0,\varepsilon}), (\tilde{b}_\varepsilon - \overline{b}_{0,\varepsilon}), (\bar{u} - \overline{u}_{0,\varepsilon})$ and $(\bar{b} - \overline{b}_{0,\varepsilon})$, instead of $(\tilde{u}_\varepsilon, \tilde{b}_\varepsilon, \bar{u}, \bar{b})$ themselves, where $\overline{u}_{0,\varepsilon} = \int_{\Omega} \tilde{u}_0 \, dx \, dy \, dz$ and $\overline{b}_{0,\varepsilon} = \int_{\Omega} \tilde{b}_0 \, dx \, dy \, dz$. 
The definition of weak solutions to the SMHD, the following integral identity
\[ \nabla_H \cdot \left( \int_{-1}^{1} \tilde{u}_0(x, y, z) \, dz \right) = 0 \quad \text{and} \quad \nabla_H \cdot \left( \int_{-1}^{1} \tilde{b}_0(x, y, z) \, dz \right) = 0, \] (3.1)
for all \((x, y) \in M\). Using Proposition 1.2, there is a unique global strong solution \((u, b)\) to the PEM. Following Definition 1.1, there is a global weak solution \((u_\varepsilon, b_\varepsilon)\) to the SMHD (1.2).

Next, we perform the global Leray-Hopf weak solutions of SMHD strong convergence to the global strong solutions of the PEM. As a preparation, we need the following proposition, which is fundamentally obtained by testing the SMHD against \((\tilde{u}, u_3)\) and \((\tilde{b}, b_3)\).

**Proposition 3.3.** Let \((\tilde{u}_\varepsilon, u_{3,\varepsilon}, \tilde{b}_\varepsilon, b_{3,\varepsilon})\) be the solution of SMHD, while the \((\tilde{u}, u_3, \tilde{b}, b_3)\) be the solution of PEM, with \((\tilde{u}_0, b_0) \in H^1(\Omega)\) satisfying (3.1), (1.14) and (1.15), the integral equality holds
\[
- \frac{\varepsilon^2}{2} \| u_3(t_0) \|_2^2 - \frac{\varepsilon^2}{2} \| b_3(t_0) \|_2^2 + \int_{Q_{t_0}} (-\tilde{u}_\varepsilon \cdot \partial_t \tilde{u} - \tilde{b}_\varepsilon \cdot \partial_t \tilde{b} + \nabla \tilde{u}_\varepsilon : \nabla \tilde{u} + \varepsilon^2 \nu u_{3,\varepsilon} \cdot \nabla u_3 \\
+ \nabla \tilde{b}_\varepsilon : \nabla \tilde{b} + \varepsilon^2 \nu b_{3,\varepsilon} \cdot \nabla b_3) \, dx \, dy \, dz \, dt + \left( \int_{\Omega} (\tilde{u}_\varepsilon \cdot \tilde{u} + \tilde{b}_\varepsilon \cdot \tilde{b} + \varepsilon^2 u_3 u_{3,\varepsilon} + \varepsilon^2 b_3 b_{3,\varepsilon}) \, dx \, dy \right)(t)
\]
\[= \frac{\varepsilon^2}{2} \| u_{3,0} \|_2^2 + \frac{\varepsilon^2}{2} \| b_{3,0} \|_2^2 + \| \tilde{u}_0 \|_2^2 + \| \tilde{b}_0 \|_2^2 + \varepsilon^2 \int_{Q_{t_0}} \left( \int_{\Omega} \partial_t \tilde{u} \, d\xi \right) \cdot \nabla U_{3,\varepsilon} \, dx \, dy \, dz \\
+ \varepsilon^2 \int_{Q_{t_0}} \left( \int_{\Omega} \partial_t \tilde{b} \, d\xi \right) \cdot \nabla B_{3,\varepsilon} \, dx \, dy \, dz - \int_{Q_{t_0}} (u_{3,\varepsilon} \cdot \nabla) \tilde{u}_\varepsilon \cdot \tilde{u} - (b_{3,\varepsilon} \cdot \nabla) \tilde{b}_\varepsilon \cdot \tilde{b} \\right)
+ \varepsilon^2 \rho \cdot \nabla u_{3,\varepsilon} u_3 - \varepsilon^2 b_{3,\varepsilon} \cdot \nabla b_{3,\varepsilon} b_3 \]
\[= \frac{\varepsilon^2}{2} \| u_{3,0} \|_2^2 + \frac{\varepsilon^2}{2} \| b_{3,0} \|_2^2 + \| \tilde{u}_0 \|_2^2 + \| \tilde{b}_0 \|_2^2 + \varepsilon^2 \int_{Q_{t_0}} \left( \int_{\Omega} \partial_t \tilde{u} \, d\xi \right) \cdot \nabla U_{3,\varepsilon} \, dx \, dy \, dz \\
+ \varepsilon^2 \int_{Q_{t_0}} \left( \int_{\Omega} \partial_t \tilde{b} \, d\xi \right) \cdot \nabla B_{3,\varepsilon} \, dx \, dy \, dz - \int_{Q_{t_0}} (u_{3,\varepsilon} \cdot \nabla) \tilde{u}_\varepsilon \cdot \tilde{u} - (b_{3,\varepsilon} \cdot \nabla) \tilde{b}_\varepsilon \cdot \tilde{b} \\right)
+ \varepsilon^2 \rho \cdot \nabla u_{3,\varepsilon} u_3 - \varepsilon^2 b_{3,\varepsilon} \cdot \nabla b_{3,\varepsilon} b_3 \]
\[= \frac{\varepsilon^2}{2} \| u_{3,0} \|_2^2 + \frac{\varepsilon^2}{2} \| b_{3,0} \|_2^2 + \| \tilde{u}_0 \|_2^2 + \| \tilde{b}_0 \|_2^2 + \varepsilon^2 \int_{Q_{t_0}} \left( \int_{\Omega} \partial_t \tilde{u} \, d\xi \right) \cdot \nabla U_{3,\varepsilon} \, dx \, dy \, dz \\
+ \varepsilon^2 \int_{Q_{t_0}} \left( \int_{\Omega} \partial_t \tilde{b} \, d\xi \right) \cdot \nabla B_{3,\varepsilon} \, dx \, dy \, dz - \int_{Q_{t_0}} (u_{3,\varepsilon} \cdot \nabla) \tilde{u}_\varepsilon \cdot \tilde{u} - (b_{3,\varepsilon} \cdot \nabla) \tilde{b}_\varepsilon \cdot \tilde{b} \\right)
+ \varepsilon^2 \rho \cdot \nabla u_{3,\varepsilon} u_3 - \varepsilon^2 b_{3,\varepsilon} \cdot \nabla b_{3,\varepsilon} b_3 \]
for any \(t_0 \in [0, \infty)\), where \(Q_{t_0} = \Omega \times (0, t_0)\).

**Proof.** The definition of weak solutions to the SMHD, the following integral identity holds
\[
\int_{Q_{t_0}} \left[ - (\tilde{u}_\varepsilon \cdot \partial_t \varphi_H + \varepsilon^2 u_{3,\varepsilon} \partial_t \varphi_3) - (\tilde{b}_\varepsilon \cdot \partial_t \psi_H + \varepsilon^2 b_{3,\varepsilon} \partial_t \psi_3) + (u_{3,\varepsilon} \cdot \nabla) \tilde{u}_\varepsilon \cdot \varphi_H \\
- (b_{3,\varepsilon} \cdot \nabla) \tilde{b}_\varepsilon \cdot \varphi_H + \varepsilon^2 u_3 \cdot \nabla u_{3,\varepsilon} \varphi_3 - \varepsilon^2 b_3 \cdot \nabla b_{3,\varepsilon} \varphi_3 + (u_{3,\varepsilon} \cdot \nabla) \tilde{b}_\varepsilon \cdot \psi_H - (b_{3,\varepsilon} \cdot \nabla) \tilde{b}_\varepsilon \cdot \psi_H \\
+ \varepsilon^2 u_{3,\varepsilon} \cdot \nabla b_{3,\varepsilon} \psi_3 - \varepsilon^2 b_{3,\varepsilon} \cdot \nabla u_{3,\varepsilon} \psi_3 + \nabla \tilde{u}_\varepsilon \cdot \nabla \varphi_H + \varepsilon^2 \nabla u_{3,\varepsilon} \cdot \nabla \varphi_3 + \nabla \tilde{b}_\varepsilon \cdot \nabla \psi_H \\
+ \varepsilon^2 \nabla b_{3,\varepsilon} \cdot \nabla \psi_3 \right] \, dx \, dy \, dz \, dt
\]
\[
\int_{\Omega} (\tilde{u}_0 \cdot \varphi_H (\cdot, 0) + \varepsilon^2 u_{3,0} \varphi_3 (\cdot, 0)) \, dx \, dy \, dz + \int_{\Omega} (\tilde{b}_0 \cdot \psi_H (\cdot, 0) + \varepsilon^2 b_{3,0} \psi_3 (\cdot, 0)) \, dx \, dy \, dz,
\]
for any periodic function \( \varphi = (\varphi_H, \varphi_3) \) and \( \psi = (\psi_H, \psi_3) \), with \( \varphi_H = (\varphi_1, \varphi_2) \) and \( \psi_H = (\psi_1, \psi_2) \), the divergence-free test functions \( \varphi \) and \( \psi \) satisfy \( \varphi \in C^0_0(\Omega \times [0, \infty)) \) and \( \psi \in C^\infty(\Omega \times [0, \infty)) \), for any \( t_0 \in [0, \infty) \), where \( Q_{t_0} = \Omega \times (0, t_0) \).

Let \( \chi(t) \in C^\infty((0, \infty)) \), with \( 0 \leq \chi(t) \leq 1 \), and \( \chi(0) = 1 \), and set \( \varphi = (\tilde{u}, u_3) \chi(t) \), \( \psi = (\tilde{b}, b_3) \chi(t) \), we note that with the density parameter, we can choose \( \varphi \) and \( \psi \) as the testing function in the integral identity above, with modifying the terms
\[
\int_{Q_{t_0}} u_{3,\varepsilon} \partial_t \varphi_3 \, dx \, dy \, dz \, dt \text{ and } \int_{Q_{t_0}} b_{3,\varepsilon} \partial_t \psi_3 \, dx \, dy \, dz \, dt
\]
as
\[
\int_{Q_{t_0}} u_{3,\varepsilon} \partial_t (u_3 \chi) \, dx \, dy \, dz \, dt = \int_0^\infty \langle \partial_t (u_3 \chi), u_{3,\varepsilon} \rangle_{H^{-1} \times H^1} \, dt,
\]
and
\[
\int_{Q_{t_0}} b_{3,\varepsilon} \partial_t (b_3 \chi) \, dx \, dy \, dz \, dt = \int_0^\infty \langle \partial_t (b_3 \chi), b_{3,\varepsilon} \rangle_{H^{-1} \times H^1} \, dt.
\]
By taking \( \varphi = (\tilde{u}, u_3) \chi \), \( \psi = (\tilde{b}, b_3) \chi \) as a testing function, we get the following integral identity
\[
\int_{Q_{t_0}} \left[ \left( \frac{\varepsilon}{\varepsilon^2} \nabla \chi \cdot \nabla \tilde{u} + \chi \cdot \nabla \tilde{u} \right) \partial_t \tilde{u} - b_\varepsilon \cdot \nabla \tilde{b} + \nabla \tilde{u} + \varepsilon^2 \nabla u_{3,\varepsilon} \cdot \nabla u_3 + \nabla b_\varepsilon \cdot \nabla \tilde{b} \\
+ \varepsilon^2 \nabla b_{3,\varepsilon} \cdot \nabla b_3 \right] \, dx \, dy \, dz \, dt = \int_0^\infty \langle \partial_t (u_3 \chi), u_{3,\varepsilon} \rangle_{H^{-1} \times H^1} \, dt
\]
\[
- \varepsilon^2 \int_0^\infty \langle \partial_t (b_3 \chi), b_{3,\varepsilon} \rangle_{H^{-1} \times H^1} \, dt
\]
\[
= - \int_{Q_{t_0}} \left[ (u_\varepsilon \cdot \nabla) \tilde{u} + (b_\varepsilon \cdot \nabla) \tilde{b} - (b_\varepsilon \cdot \nabla) \tilde{b} + \varepsilon^2 u_\varepsilon \cdot \nabla u_{3,\varepsilon} u_3 - \varepsilon^2 b_\varepsilon \cdot \nabla b_{3,\varepsilon} b_3 + u_\varepsilon \cdot \nabla \tilde{b} + b_\varepsilon \cdot \nabla \tilde{b} \\
- b_\varepsilon \cdot \nabla \tilde{u} - b_\varepsilon \cdot \nabla b_3 - b_\varepsilon \cdot \nabla b_{3,\varepsilon} b_3 - \varepsilon^2 b_\varepsilon \cdot \nabla u_{3,\varepsilon} b_3 \right] \chi \, dx \, dy \, dz \, dt
\]
\[
+ ||\tilde{u}_0||^2 + \varepsilon^2 ||u_{3,0}||^2 \]
\[
+ ||\tilde{b}_0||^2 + \varepsilon^2 ||b_{3,0}||^2.
\]
We rewrite the terms \( \int_0^\infty \langle \partial_t (u_3 \chi), u_{3,\varepsilon} \rangle_{H^{-1} \times H^1} \, dt \) and \( \int_0^\infty \langle \partial_t (b_3 \chi), b_{3,\varepsilon} \rangle_{H^{-1} \times H^1} \, dt \) as
\[
\int_0^\infty \langle \partial_t (u_3 \chi), u_{3,\varepsilon} \rangle_{H^{-1} \times H^1} \, dt = \int_0^\infty \langle \partial_t u_3, u_{3,\varepsilon} \rangle_{H^{-1} \times H^1} \chi \, dt + \int_{Q_{t_0}} u_{3,\varepsilon} \chi' \, dx \, dy \, dz \, dt,
\]
and
\[
\int_0^\infty \langle \partial_t (b_3 \chi), b_{3,\varepsilon} \rangle_{H^{-1} \times H^1} \, dt = \int_0^\infty \langle \partial_t b_3, b_{3,\varepsilon} \rangle_{H^{-1} \times H^1} \chi \, dt + \int_{Q_{t_0}} b_{3,\varepsilon} \chi' \, dx \, dy \, dz \, dt.
\]
Substituting in the previous identity gives that

\[
\begin{align*}
\int_{Q_{t_0}} & (-\tilde{u}_\varepsilon \cdot \nabla \tilde{u}_\varepsilon - \tilde{b}_\varepsilon \cdot \partial_t \tilde{b} + \nabla \tilde{u}_\varepsilon \cdot \nabla \tilde{u} + \varepsilon^2 \nabla u_{3,\varepsilon} \cdot \nabla u_3 + \nabla \tilde{b}_\varepsilon \cdot \nabla \tilde{b} \\
& + \varepsilon^2 \nabla b_{3,\varepsilon} \cdot \nabla b_3 ) \chi(t) \, dx dy dz dt \\
& - \varepsilon^2 \int_0^\infty \langle \partial_t u_3, u_{3,\varepsilon} \rangle_{H^{-1} \times H^1} \chi dt - \varepsilon^2 \int_0^\infty \langle \partial_t b_3, b_{3,\varepsilon} \rangle_{H^{-1} \times H^1} \chi dt \\
& - \int_{Q_{t_0}} (\tilde{u}_\varepsilon \cdot \tilde{u} + \tilde{b}_\varepsilon \cdot \tilde{b} + \varepsilon^2 u_{3,\varepsilon} u_3 + \varepsilon^2 b_{3,\varepsilon} b_3 ) \chi' \, dx dy dz dt \\
& = - \int_{Q_{t_0}} \left[ (u_{\varepsilon} \cdot \nabla) \tilde{u}_\varepsilon \cdot \tilde{u} - (b_{\varepsilon} \cdot \nabla) \tilde{b}_\varepsilon \cdot \tilde{b} + \varepsilon^2 u_{3,\varepsilon} u_3 - \varepsilon^2 b_{3,\varepsilon} b_3 \right] \chi \, dx dy dz dt \\
& + |\tilde{u}_0|^2 + \varepsilon^2 \| u_{3,0} \|_2^2 + \| \tilde{b}_0 \|_2^2 + \varepsilon^2 \| b_{3,0} \|_2^2,
\end{align*}
\]

for any $\chi \in C_0^\infty([0, \infty))$, with $0 \leq \chi \leq 1$ and $\chi(0) = 1$.

Choose $\chi_\delta \in C_0^\infty([0, t_0])$, such that $\chi_\delta \equiv 1$ on $[0, t_0 - \delta]$, $0 \leq \chi_\delta \leq 1$ on $[t_0 - \delta, t_0]$ and $|\chi_\delta'| \leq \frac{2}{\delta}$ on $[0, t_0]$, where $t_0 \in (0, \infty)$, and $\delta \in (0, t_0)$ is a small positive number. As $\delta \to 0$, we have

\[
\int_{Q_{t_0}} (\tilde{u}_\varepsilon \cdot \tilde{u} + \tilde{b}_\varepsilon \cdot \tilde{b} + \varepsilon^2 u_{3,\varepsilon} u_3 + \varepsilon^2 b_{3,\varepsilon} b_3 ) \chi_\delta' \, dx dy dz dt \\
\to - \left( \int_{\Omega} (\tilde{u}_\varepsilon \cdot \tilde{u} + \tilde{b}_\varepsilon \cdot \tilde{b} + \varepsilon^2 u_{3,\varepsilon} u_3 + \varepsilon^2 b_{3,\varepsilon} b_3 ) \, dx dy dz \right)(t_0),
\]

\[
\int_0^\infty \langle \partial_t u_3, u_{3,\varepsilon} \rangle_{H^{-1} \times H^1} \chi_\delta dt \to \int_0^{t_0} \langle \partial_t u_3, u_{3,\varepsilon} \rangle_{H^{-1} \times H^1} dt,
\]

and

\[
\int_0^\infty \langle \partial_t b_3, b_{3,\varepsilon} \rangle_{H^{-1} \times H^1} \chi_\delta dt \to \int_0^{t_0} \langle \partial_t b_3, b_{3,\varepsilon} \rangle_{H^{-1} \times H^1} dt.
\]

The effectiveness of (3.5) and (3.6) comes from the dominant convergence theorem for the integrals,

\[
\langle \partial_t u_3, u_{3,\varepsilon} \rangle = - \left\langle \nabla_{H} \cdot \left( \int_0^z \partial_t \tilde{u} \, d\xi \right), u_{3,\varepsilon} \right\rangle = \int_{\Omega} \left( \int_0^z \partial_t \tilde{u} \, d\xi \right) \cdot \nabla_{H} u_{3,\varepsilon} \, dx dy dz,
\]

and

\[
\langle \partial_t b_3, b_{3,\varepsilon} \rangle = - \left\langle \nabla_{H} \cdot \left( \int_0^z \partial_t \tilde{b} \, d\xi \right), b_{3,\varepsilon} \right\rangle = \int_{\Omega} \left( \int_0^z \partial_t \tilde{b} \, d\xi \right) \cdot \nabla_{H} b_{3,\varepsilon} \, dx dy dz,
\]
which implies \( \langle \partial_t u_3, u_{3,\varepsilon} \rangle \in L^1((0, t_0)) \) and \( \langle \partial_t b_3, b_{3,\varepsilon} \rangle \in L^1((0, t_0)) \). Here, for brevity, we have got rid of the subscript \( H^{-1} \times H^1 \). While for (3.4), we define

\[
h(t) := \left( \int_\Omega (\tilde{u}_\varepsilon \cdot \tilde{u} + \tilde{b}_\varepsilon \cdot \tilde{b} + \varepsilon^2 u_3 u_{3,\varepsilon} + \varepsilon^2 b_3 b_{3,\varepsilon}) \, dx dy dz \right)(t).
\]

It is equivalent to show \( \int_{t_0-\delta}^{t_0} h(t) \chi_\delta' \, dt \to -h(t_0) \), as \( \delta \to 0 \).

Recalling the regularities that \( (u_\varepsilon, b_\varepsilon) \in C_w([0, \infty); L^2(\Omega)) \) and \( (\tilde{u}, \tilde{b}) \in C([0, \infty); H^1(\Omega)) \), hence one has \( (u_3, b_3) \in C([0, \infty); L^2(\Omega)) \), so \( h \) is a continuous function on \( [0, \infty) \). For any \( \sigma > 0 \), there is a positive number \( \rho \), such that \( |h(t) - h(t_0)| \leq \sigma \), and any \( t \in [t_0 - \rho, t_0] \). At present, for any \( \delta \in (0, \rho) \), recalling that \( \chi_\delta \equiv 1 \) on \( [0, t_0 - \delta] \), \( \chi_\delta(t_0) = 0 \), and \( |\chi_\delta'| \leq \frac{2}{\delta} \), on \( [0, \infty) \), we have

\[
\left| \int_{t_0-\delta}^{t_0} h(t) \chi_\delta'(t) \, dt + h(t_0) \right| = \left| \int_{t_0-\delta}^{t_0} (h(t) - h(t_0)) \chi_\delta'(t) \, dt \right|
\leq \int_{t_0-\delta}^{t_0} |h(t) - h(t_0)||\chi_\delta'(t)| \, dt
\leq 2\sigma,
\]

which gives (3.4).

Combining \( u_3 = -\int_0^z \nabla_H \tilde{u} \, d\xi \) and recalling the regularities that \( u_3 \in L^2_{loc}([0, \infty); H^1(\Omega)) \) and \( \partial_t u_3 \in L^2_{loc}([0, \infty); H^{-1}(\Omega)) \), we get

\[
\langle \partial_t u_3, u_{3,\varepsilon} \rangle = \langle \partial_t u_3, u_{3,\varepsilon} - u_3 \rangle + \langle \partial_t u_3, u_3 \rangle
= \left( -\nabla_H \cdot \left( \int_0^z \partial_t \tilde{u} \, d\xi \right), u_{3,\varepsilon} - u_3 \right) + \langle \partial_t u_3, u_3 \rangle
= \int_\Omega \left( \int_0^z \partial_t \tilde{u} \, d\xi \right) \cdot \nabla_H U_{3,\varepsilon} \, dx dy dz + \frac{1}{2} \frac{d}{dt} \|u_3\|_2^2.
\]

Here we use the Lions-Magenes Lemma [37, see, e.g., pages 260-261], we deduce

\[
\int_0^{t_0} \langle \partial_t u_3, u_{3,\varepsilon} \rangle \, dt = \int_{Q_{t_0}} \left( \int_0^z \partial_t \tilde{u} \, d\xi \right) \cdot \nabla_H U_{3,\varepsilon} \, dx dy dz dt + \frac{1}{2} (\|u_{3,0}\|_2^2 - \|u_3(0)\|_2^2).
\]

Similarly, we obtain

\[
\int_0^{t_0} \langle \partial_t b_3, b_{3,\varepsilon} \rangle \, dt = \int_{Q_{t_0}} \left( \int_0^z \partial_t \tilde{b} \, d\xi \right) \cdot \nabla_H B_{3,\varepsilon} \, dx dy dz dt + \frac{1}{2} (\|b_{3,0}\|_2^2 - \|b_3(0)\|_2^2).
\]

Because of the above equality and (3.4)-(3.6), one can select \( \chi = \chi_\delta \) in (3.3), as in the preceding paragraph, taking \( \delta \to 0 \) gives that

\[
- \frac{\varepsilon^2}{2} \|u_3(t_0)\|_2^2 - \frac{\varepsilon^2}{2} \|b_3(t_0)\|_2^2 + \int_{Q_{t_0}} (\tilde{u}_\varepsilon \cdot \partial_t \tilde{u} - \tilde{b}_\varepsilon \cdot \partial_t \tilde{b} + \nabla \tilde{u}_\varepsilon : \nabla \tilde{u} + \varepsilon^2 \nabla u_{3,\varepsilon} : \nabla u_3
\]
The Primitive Equations with Magnetic Field Approximation of the 3D MHD Equations

\[ + \nabla \tilde{b}_\varepsilon : \nabla \tilde{b} + \varepsilon^2 \nabla b_{3,\varepsilon} : \nabla b_3 \] \, dx\,dy\,dz\,dt + \left( \int_{\Omega} \tilde{u}_\varepsilon \cdot \tilde{u} + \tilde{b}_\varepsilon \cdot \tilde{b} + \varepsilon^2 u_3 u_{3,\varepsilon} \right) \, dx\,dy\,dz\,dt

\[ + \varepsilon^2 b_{3,\varepsilon} : dx\,dy\,dz \right) (t_0) \]

\[ = \frac{\varepsilon^2}{2} \| u_{0,3,\varepsilon} \|_2^2 + \frac{\varepsilon^2}{2} \| b_{3,0} \|_2^2 + \| \tilde{u}_0 \|_2^2 + \| \tilde{b}_0 \|_2^2 + \varepsilon^2 \int_{Q_{t_0}} \left( \int_0^t \partial_t \tilde{u} \, d\xi \right) : \nabla_H U_{3,\varepsilon} \, dx\,dy\,dz\,dt \]

\[ + \varepsilon^2 \int_{Q_{t_0}} \left( \int_0^t \partial_t \tilde{b} \, d\xi \right) : \nabla_H B_{3,\varepsilon} \, dx\,dy\,dz\,dt - \int_{Q_{t_0}} \left[ (u_\varepsilon \cdot \nabla) \tilde{u}_\varepsilon \cdot \tilde{u} - (b_\varepsilon \cdot \nabla) \tilde{b}_\varepsilon \cdot \tilde{b} \right] + \varepsilon^2 u_\varepsilon \cdot \nabla u_{3,\varepsilon} u_3 + u_\varepsilon \cdot \nabla \tilde{b}_\varepsilon \cdot \tilde{b} - b_\varepsilon \cdot \nabla \tilde{u}_\varepsilon \cdot \tilde{b} + \varepsilon^2 \tilde{u}_\varepsilon \cdot \nabla b_{3,\varepsilon} b_3 \]

\[ - \varepsilon^2 b_\varepsilon : \nabla u_{3,\varepsilon} b_3 \] \, dx\,dy\,dz\,dt, \quad (3.7)

for any \( t_0 \in [0, \infty) \). This completes the proof. \( \square \)

Now, we can estimate the difference between \( (\tilde{u}_\varepsilon, u_{3,\varepsilon}, \tilde{b}_\varepsilon, b_{3,\varepsilon}) \) and \( (\tilde{u}, u_3, \tilde{b}, b_3) \).

**Proposition 3.4.** Under the same conditions as in Proposition 3.3, we denote \((\tilde{U}_\varepsilon, U_{3,\varepsilon}) := (\tilde{u}_\varepsilon - \tilde{u}, u_{3,\varepsilon} - u_3)\) and \((\tilde{B}_\varepsilon, B_{3,\varepsilon}) := (\tilde{b}_\varepsilon - \tilde{b}, b_{3,\varepsilon} - b_3)\), the following estimate holds

\[
\sup_{0 \leq t < \infty} \left( \| \tilde{U}_\varepsilon \|_2^2 + \| \tilde{B}_\varepsilon \|_2^2 + \varepsilon^2 \| U_{3,\varepsilon} \|_2^2 + \varepsilon^2 \| B_{3,\varepsilon} \|_2^2 \right) + \int_0^\infty \left( \| \nabla \tilde{U}_\varepsilon \|_2^2 + \| \nabla \tilde{B}_\varepsilon \|_2^2 \right) dt
\]

\[ + \varepsilon^2 \| \nabla U_{3,\varepsilon} \|_2^2 + \varepsilon^2 \| \nabla B_{3,\varepsilon} \|_2^2 \right) \, ds \]

\[ \leq \varepsilon^2 C \left( \| u_0 \|_2^2 + \| b_0 \|_2^2 + \varepsilon^2 \| u_{3,0} \|_2^2 + \varepsilon^2 \| b_{3,0} \|_2^2 + 1 \right)^2, \]

where \( C \) is a positive constants depending only on \( \| u_0 \|_{H^1}, \| b_0 \|_{H^1}, L_1 \) and \( L_2 \).

**Proof.** Multiplying (1.13) \( _1 \) and (1.13) \( _3 \) by \( \tilde{u}_\varepsilon \) and \( \tilde{b}_\varepsilon \), respectively, and taking the \( L^2 \) norm on \( Q_{t_0} \), if follows from integration by parts that

\[
\int_{Q_{t_0}} \left( \partial_t \tilde{u} \cdot \tilde{u}_\varepsilon + \partial_t \tilde{b} \cdot \tilde{b}_\varepsilon + \nabla \tilde{u} : \nabla \tilde{u}_\varepsilon + \nabla \tilde{b} : \nabla \tilde{b}_\varepsilon \right) \, dx\,dy\,dz\,dt
\]

\[ = - \int_{Q_{t_0}} u \cdot \nabla \tilde{u} \cdot \tilde{u}_\varepsilon \, dx\,dy\,dz\,dt + \int_{Q_{t_0}} b \cdot \nabla \tilde{b} \cdot \tilde{b}_\varepsilon \, dx\,dy\,dz\,dt \]

\[ - \int_{Q_{t_0}} u \cdot \nabla \tilde{b} \cdot \tilde{b}_\varepsilon \, dx\,dy\,dz\,dt + \int_{Q_{t_0}} b \cdot \nabla \tilde{u} \cdot \tilde{b}_\varepsilon \, dx\,dy\,dz\,dt, \quad (3.8)\]

for any \( t_0 \in [0, \infty) \), applying the first equation and the third equation of (1.13) by \( \tilde{u} \) and \( \tilde{b} \), respectively, integrating the resultant over \( Q_{t_0} \), it follows from integration by parts that

\[
\frac{1}{2} \| \tilde{u}(t_0) \|_2^2 + \frac{1}{2} \| \tilde{b}(t_0) \|_2^2 + \int_0^{t_0} \| \nabla \tilde{u} \|_2^2 \, dt + \int_0^{t_0} \| \nabla \tilde{b} \|_2^2 \, dt
\]

\[ = \frac{1}{2} \| \tilde{u}_0 \|_2^2 + \frac{1}{2} \| \tilde{b}_0 \|_2^2, \quad (3.9)\]
for any $t_0 \in [0, \infty)$. Recalling the definition of Leray-Hopf weak solutions to the SMHD, we have

\[
\begin{align*}
\frac{1}{2} \left( \| \tilde{u}_\varepsilon(t_0) \|_2^2 + \| \tilde{b}(t_0) \|_2^2 + \varepsilon^2 \| u_{3, \varepsilon}(t_0) \|_2^2 + \varepsilon^2 \| b_{3, \varepsilon}(t_0) \|_2^2 \right) \\
+ \int_0^{t_0} \left( \| \nabla \tilde{u}_\varepsilon \|_2^2 + \| \nabla \tilde{b}_\varepsilon \|_2^2 + \varepsilon^2 \| \nabla u_{3, \varepsilon} \|_2^2 + \varepsilon^2 \| \nabla b_{3, \varepsilon} \|_2^2 \right) \, ds \\
\leq \frac{1}{2} \left( \| \tilde{u}_0 \|_2^2 + \| \tilde{b}_0 \|_2^2 + \varepsilon^2 \| u_{3, 0} \|_2^2 + \varepsilon^2 \| b_{3, 0} \|_2^2 \right), \tag{3.10}
\end{align*}
\]

for a.e. $t_0 \in [0, \infty)$.

Adding (3.9) and (3.10), then subtracting from the (3.2) (choose $t = t_0$ there) and (3.8), we get

\[
\begin{align*}
\frac{1}{2} \left( \| \tilde{U}_\varepsilon \|_2^2 + \| \tilde{B}_\varepsilon \|_2^2 + \varepsilon^2 \| U_{3, \varepsilon} \|_2^2 + \varepsilon^2 \| B_{3, \varepsilon} \|_2^2 \right) & (t_0) \\
+ \int_0^{t_0} \left( \| \nabla \tilde{U}_\varepsilon \|_2^2 + \| \nabla \tilde{B}_\varepsilon \|_2^2 + \varepsilon^2 \| \nabla U_{3, \varepsilon} \|_2^2 + \varepsilon^2 \| \nabla B_{3, \varepsilon} \|_2^2 \right) \, dt \\
\leq & -\varepsilon^2 \int_{Q_{t_0}} \left[ \left( \int_0^z \partial_t \tilde{u} \, d\xi \right) \cdot \nabla H U_{3, \varepsilon} + \nabla u_3 \cdot \nabla U_{3, \varepsilon} \right] \, dx \, dy \, dz \\
& -\varepsilon^2 \int_{Q_{t_0}} \left[ \left( \int_0^z \partial_t \tilde{b} \, d\xi \right) \cdot \nabla H B_{3, \varepsilon} + \nabla b_3 \cdot \nabla B_{3, \varepsilon} \right] \, dx \, dy \, dz \\
& + \int_{Q_{t_0}} [(u_\varepsilon \cdot \nabla) \tilde{u}_\varepsilon \cdot \tilde{u} + (u \cdot \nabla) \tilde{u} \cdot \tilde{u}_\varepsilon] \, dx \, dy \, dz dt - \int_{Q_{t_0}} [(b_\varepsilon \cdot \nabla) \tilde{b}_\varepsilon \cdot \tilde{b} + (b \cdot \nabla) \tilde{b} \cdot \tilde{b}_\varepsilon] \, dx \, dy \, dz dt \\
& - \int_{Q_{t_0}} [(b_\varepsilon \cdot \nabla) \tilde{u}_\varepsilon \cdot \tilde{b} + (b \cdot \nabla) \tilde{u} \cdot \tilde{b}_\varepsilon] \, dx \, dy \, dz dt + \int_{Q_{t_0}} [(u_\varepsilon \cdot \nabla) \tilde{b}_\varepsilon \cdot \tilde{b} + (u \cdot \nabla) \tilde{b} \cdot \tilde{b}_\varepsilon] \, dx \, dy \, dz dt \\
& + \varepsilon^2 \int_{Q_{t_0}} u_\varepsilon \cdot \nabla u_{3, \varepsilon} u_3 \, dx \, dy \, dz dt - \varepsilon^2 \int_{Q_{t_0}} b_\varepsilon \cdot \nabla b_{3, \varepsilon} u_3 \, dx \, dy \, dz dt \\
& - \varepsilon^2 \int_{Q_{t_0}} b_\varepsilon \cdot \nabla u_{3, \varepsilon} b_3 \, dx \, dy \, dz dt + \varepsilon^2 \int_{Q_{t_0}} u_\varepsilon \cdot \nabla b_{3, \varepsilon} b_3 \, dx \, dy \, dz dt \\
:= & I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 + I_{10}, \tag{3.11}
\end{align*}
\]

for a.e. $t_0 \in [0, \infty)$. Using the Hölder, Cauchy-Schwarz inequalities and Proposition 1.2, we infer that

\[
I_1 \leq \varepsilon^2 \left( \| \partial_t \tilde{u} \|_{L^2(Q_{t_0})} + \| \nabla u_3 \|_{L^2(Q_{t_0})} \| \nabla U_{3, \varepsilon} \|_{L^2(Q_{t_0})} \right)
\leq \frac{\varepsilon^2}{6} \| \nabla U_{3, \varepsilon} \|_{L^2(Q_{t_0})}^2 + C(\| \tilde{u}_0 \|_{H^1, L^1, L^2}) \varepsilon^2,
\]
similarly, we have

\[
I_2 \leq \varepsilon^2 \left( \| \partial_t \tilde{b} \|_{L^2(Q_{t_0})} + \| \nabla b_3 \|_{L^2(Q_{t_0})} \| \nabla B_{3, \varepsilon} \|_{L^2(Q_{t_0})} \right)
\]
\[ \leq \frac{\varepsilon^2}{6} \|
abla B_{3,\varepsilon}\|_{L^2(Q_{t_0})}^2 + C(\|\tilde{b}_0\|_{H^1}, L_1, L_2)\varepsilon^2. \]

Next, we are going to estimate \( I_3 \). Due to the incompressibility conditions, it follows from integration by parts that

\[
I_3 = \int_{Q_{t_0}} \left[ (u_{\varepsilon} \cdot \nabla) \tilde{u}_{\varepsilon} \cdot \tilde{u} + (u \cdot \nabla) \tilde{u}_{\varepsilon} \cdot \tilde{u}_{\varepsilon} \right] \, dx \, dy \, dz \, dt
\]

\[
= \int_{Q_{t_0}} \left[ (u_{\varepsilon} \cdot \nabla) \tilde{u}_{\varepsilon} \cdot \tilde{u} - (u \cdot \nabla) \tilde{u}_{\varepsilon} \cdot \tilde{u}_{\varepsilon} \right] \, dx \, dy \, dz \, dt
\]

\[
= \int_{Q_{t_0}} \left[ (u_{\varepsilon} - u) \cdot \nabla \tilde{u}_{\varepsilon} \cdot \tilde{u} \right] \, dx \, dy \, dz \, dt
\]

\[
= \int_{Q_{t_0}} \left[ (\tilde{U}_{\varepsilon} \cdot \nabla H) \tilde{U}_{\varepsilon} \cdot \tilde{u} \right] \, dx \, dy \, dz \, dt + \int_{Q_{t_0}} U_{3,\varepsilon} \partial_z \tilde{U}_{\varepsilon} \cdot \tilde{u} \, dx \, dy \, dz \, dt,
\]

\[ := I_{31} + I_{32}. \]

Taking advantage of the Hölder, Sobolev and Young inequalities, we have

\[
I_{31} = \int_{Q_{t_0}} [(\tilde{U}_{\varepsilon} \cdot \nabla H) \tilde{U}_{\varepsilon} \cdot \tilde{u}] \, dx \, dy \, dz \, dt
\]

\[
\leq C \int_{0}^{t_0} \|\tilde{U}_{\varepsilon}\|_3 \|\nabla \tilde{U}_{\varepsilon}\|_2 \|\tilde{u}\|_6 \, dt
\]

\[
\leq C \int_{0}^{t_0} \|\tilde{U}_{\varepsilon}\|_{\frac{3}{2}} \|\nabla \tilde{U}_{\varepsilon}\|_{\frac{3}{2}} \|\nabla \tilde{u}\|_2 \, dt
\]

\[
\leq \frac{1}{16} \|\nabla \tilde{U}_{\varepsilon}\|_{L^2(Q_{t_0})}^2 + C \int_{0}^{t_0} \|\nabla \tilde{u}\|_2^4 \|\tilde{U}_{\varepsilon}\|_2^2 \, dt.
\]

Integration by parts yields

\[
I_{32} = \int_{Q_{t_0}} U_{3,\varepsilon} \partial_z \tilde{U}_{\varepsilon} \cdot \tilde{u} \, dx \, dy \, dz \, dt
\]

\[
= -\int_{Q_{t_0}} \left[ \partial_z U_{3,\varepsilon} \tilde{U}_{\varepsilon} \cdot \tilde{u} + U_{3,\varepsilon} \tilde{U}_{\varepsilon} \cdot \partial_z \tilde{u} \right] \, dx \, dy \, dz \, dt
\]

\[
= \int_{Q_{t_0}} \left[ (\nabla H \cdot \tilde{U}_{\varepsilon}) \tilde{U}_{\varepsilon} \cdot \tilde{u} - U_{3,\varepsilon} \tilde{U}_{\varepsilon} \cdot \partial_z \tilde{u} \right] \, dx \, dy \, dz \, dt
\]

\[ := I_{321} + I_{322}. \]

The same arguments as for \( I_{31} \) yield

\[
I_{321} = \int_{Q_{t_0}} (\nabla H \cdot \tilde{U}_{\varepsilon})(\tilde{U}_{\varepsilon} \cdot \tilde{u}) \, dx \, dy \, dz \, dt
\]
Applying Lemma 2.2, the Poincaré and Young inequalities, we arrive at

\[
\leq \frac{1}{16} \|\nabla \tilde{U}_\varepsilon\|_{L^2(Q_{t_0})}^2 + C \int_0^{t_0} \|\nabla \tilde{u}\|_2^4 \|\tilde{U}_\varepsilon\|_2^2 dt.
\]

Applying Lemma 2.2, the Poincaré and Young inequalities, we arrive that

\[
I_{322} = - \int_{Q_{t_0}} U_{3,\varepsilon} \tilde{U}_\varepsilon \cdot \partial_z \tilde{u} \, dx \, dy \, dz \, dt
\]

\[
= \int_0^{t_0} \int_0^z \left( \int_0^x \nabla_H \tilde{U}_\varepsilon \, d\xi \right) (\tilde{U}_\varepsilon \cdot \partial_z \tilde{u}) \, dx \, dy \, dz \, dt
\]

\[
\leq \int_0^{t_0} \int_M \left( \int_{-1}^1 |\nabla_H \tilde{U}_\varepsilon| \, dz \right) \left( \int_{-1}^1 |\tilde{U}_\varepsilon||\partial_z \tilde{u}| \, dz \right) \, dx \, dy \, dt
\]

\[
\leq C \int_0^{t_0} \|\nabla \tilde{U}_\varepsilon\|_2 \|\tilde{U}_\varepsilon\|_2 \|\nabla \tilde{u}\|_2 \|\Delta \tilde{u}\|_2 dt
\]

\[
\leq \frac{1}{16} \|\nabla \tilde{U}_\varepsilon\|_{L^2(Q_{t_0})}^2 + C \int_0^{t_0} \|\nabla \tilde{u}\|_2^2 \|\Delta \tilde{u}\|_2^2 \|\tilde{U}_\varepsilon\|_2^2 dt.
\]

Thanks to the estimates for \(I_{31}\), \(I_{321}\) and \(I_{322}\), we can bound \(I_3\) as

\[
I_3 \leq \frac{3}{16} \|\nabla \tilde{U}_\varepsilon\|_{L^2(Q_{t_0})}^2 + C \int_0^{t_0} \|\nabla \tilde{u}\|_2^2 \|\Delta \tilde{u}\|_2^2 \|\tilde{U}_\varepsilon\|_2^2 dt.
\]

For the sake of simplicity, we sum up the following two terms, then

\[
I_4 + I_5 = - \int_{Q_{t_0}} [b_\varepsilon \cdot \nabla \tilde{b}_\varepsilon \cdot \tilde{u} + b \cdot \nabla \tilde{b} \cdot \tilde{u}_\varepsilon] \, dx \, dy \, dz \, dt
\]

\[
- \int_{Q_{t_0}} [b_\varepsilon \cdot \nabla \tilde{u}_\varepsilon \cdot \tilde{b} + b \cdot \nabla \tilde{u} \cdot \tilde{b}_\varepsilon] \, dx \, dy \, dz \, dt
\]

\[
= - \int_{Q_{t_0}} [b_\varepsilon \cdot \nabla \tilde{b}_\varepsilon \cdot \tilde{u} + b \cdot \nabla \tilde{b} \cdot \tilde{u}_\varepsilon] \, dx \, dy \, dz \, dt
\]

\[
- \int_{Q_{t_0}} [b_\varepsilon \cdot \nabla \tilde{u}_\varepsilon \cdot \tilde{b} + b \cdot \nabla \tilde{u} \cdot \tilde{b}_\varepsilon] \, dx \, dy \, dz \, dt
\]

\[
= - \int_{Q_{t_0}} [b_\varepsilon \cdot \nabla \tilde{b}_\varepsilon \cdot \tilde{u} - b \cdot \nabla \tilde{b}_\varepsilon \cdot \tilde{u}] \, dx \, dy \, dz \, dt
\]

\[
- \int_{Q_{t_0}} [b_\varepsilon \cdot \nabla \tilde{u}_\varepsilon \cdot \tilde{b} - b \cdot \nabla \tilde{u}_\varepsilon \cdot \tilde{b}] \, dx \, dy \, dz \, dt
\]

\[
:= A + B.
\]

To bound \(A\), we decompose it into two pieces

\[
A = - \int_{Q_{t_0}} [b_\varepsilon \cdot \nabla \tilde{b}_\varepsilon \cdot \tilde{u} - b \cdot \nabla \tilde{b}_\varepsilon \cdot \tilde{u}] \, dx \, dy \, dz \, dt
\]

\[
= - \int_{Q_{t_0}} [(b_\varepsilon - b) \cdot \nabla \tilde{b}_\varepsilon \cdot \tilde{u}] \, dx \, dy \, dz \, dt
\]

\[
= - \int_{Q_{t_0}} B_\varepsilon \cdot \nabla \tilde{b}_\varepsilon \cdot \tilde{u} \, dx \, dy \, dz \, dt
\]

\[
= - \int_{Q_{t_0}} \tilde{B}_\varepsilon \cdot \nabla_H \tilde{b}_\varepsilon \cdot \tilde{u} \, dx \, dy \, dz \, dt
\]

\[
- \int_{Q_{t_0}} B_{3,\varepsilon} \partial_z \tilde{b}_\varepsilon \cdot \tilde{u} \, dx \, dy \, dz \, dt
\]

\[
:= A_1 + A_2.
\]
So that the first part of $A$ can be estimated

$$A_1 = - \int_{Q_{t_0}} \tilde{B}_\varepsilon \cdot \nabla_H \tilde{b}_\varepsilon \cdot \tilde{u} \, dx dy dz dt$$

$$\leq \int_0^{t_0} \|\tilde{B}_\varepsilon\|_6 \|\nabla_H \tilde{b}_\varepsilon\|_2 \|\tilde{u}\|_3 \, dt$$

$$\leq \int_0^{t_0} \|\nabla \tilde{B}_\varepsilon\|_2 \|\nabla \tilde{b}_\varepsilon\|_2 \|\tilde{u}\|_2 \|\nabla \tilde{u}\|_2 \, dt$$

$$\leq \frac{1}{22} \|\nabla \tilde{B}_\varepsilon\|_{L^2(Q_{t_0})}^2 + C \int_0^{t_0} \|\nabla \tilde{b}_\varepsilon\|_2^2 \|\tilde{u}\|_2 \|\nabla \tilde{u}\|_2 \, dt$$

Applying integration by parts gives that

$$A_2 = - \int_{Q_{t_0}} B_{3,\varepsilon}(\partial_z \tilde{b}_\varepsilon \cdot \tilde{u}) \, dx dy dz dt$$

$$= \int_{Q_{t_0}} \partial_z B_{3,\varepsilon}(\tilde{b}_\varepsilon \cdot \tilde{u}) \, dx dy dz dt + \int_{Q_{t_0}} B_{3,\varepsilon} \tilde{b}_\varepsilon \cdot \partial_z \tilde{u} \, dx dy dz dt$$

$$= - \int_{Q_{t_0}} (\nabla_H \cdot \tilde{B}_\varepsilon) \tilde{b}_\varepsilon \cdot \tilde{u} \, dx dy dz dt + \int_{Q_{t_0}} B_{3,\varepsilon} \tilde{b}_\varepsilon \cdot \partial_z \tilde{u} \, dx dy dz dt$$

$$:= A_{21} + A_{22}.$$ 

The first part $A_{21}$ can be estimated as follows

$$A_{21} = - \int_{Q_{t_0}} (\nabla_H \cdot \tilde{B}_\varepsilon) \tilde{b}_\varepsilon \cdot \tilde{u} \, dx dy dz dt$$

$$\leq \int_0^{t_0} \|\nabla_H \cdot \tilde{B}_\varepsilon\|_2 \|\tilde{b}_\varepsilon\|_6 \|\tilde{u}\|_3 \, dt$$

$$\leq C \int_0^{t_0} \|\nabla \tilde{B}_\varepsilon\|_2 \|\nabla \tilde{b}_\varepsilon\|_2 \|\tilde{u}\|_2 \|\nabla \tilde{u}\|_2 \, dt$$

$$\leq \frac{1}{22} \|\nabla \tilde{B}_\varepsilon\|_{L^2(Q_{t_0})}^2 + C \int_0^{t_0} \|\nabla \tilde{b}_\varepsilon\|_2^2 \|\tilde{u}\|_2 \|\nabla \tilde{u}\|_2 \, dt$$

$$\leq \frac{1}{22} \|\nabla \tilde{B}_\varepsilon\|_{L^2(Q_{t_0})}^2 + C \int_0^{t_0} \|\nabla \tilde{b}_\varepsilon\|_2^2 (\|\tilde{u}\|_2^2 + \|\nabla \tilde{u}\|_2^2) \, dt$$

$$\leq \frac{1}{22} \|\nabla \tilde{B}_\varepsilon\|_{L^2(Q_{t_0})}^2 + C \int_0^{t_0} \|\nabla \tilde{b}_\varepsilon\|_2^2 \|\tilde{u}\|_2^2 \, dt + C \int_0^{t_0} \|\nabla \tilde{b}_\varepsilon\|_2^2 \|\nabla \tilde{u}\|_2^2 \, dt.$$
For the second part $A_{22}$, we use Lemma 2.2, the Poincaré and Young inequalities and obtain

$$A_{22} = \int_{Q_{t_0}} \tilde{B}_{3, \varepsilon} \tilde{b}_{\varepsilon} \cdot \partial_z \tilde{u} \, dx \, dy \, dz \, dt$$

$$= \int_0^{t_0} \int_{Q_{t_0}} \left( \int_0^z \nabla H \cdot \tilde{B}_{\varepsilon} \, d\xi \right) \tilde{b}_{\varepsilon} \cdot \partial_z \tilde{u} \, dx \, dy \, dz \, dt$$

$$\leq \int_0^{t_0} \int_M \left( \int_{-1}^1 |\nabla H \tilde{B}_{\varepsilon}| \, dz \right) \left( \int_{-1}^1 |\tilde{b}_{\varepsilon}||\partial_z \tilde{u}| \, dz \right) \, dx \, dy \, dt$$

$$\leq \int_0^{t_0} \|\nabla H \tilde{B}_{\varepsilon}\|_2 \|\tilde{b}_{\varepsilon}\|_2^3 \|\nabla \tilde{b}_{\varepsilon}\|_2 \|\nabla \tilde{u}\|_2 \|\Delta \tilde{u}\|_2 \, dt$$

$$\leq \frac{1}{22} \|\nabla \tilde{B}_{\varepsilon}\|_{L^2(Q_{t_0})}^2 + C \int_0^{t_0} \|\tilde{b}_{\varepsilon}\|_2 \|\nabla \tilde{b}_{\varepsilon}\|_2 \|\nabla \tilde{u}\|_2 \|\Delta \tilde{u}\|_2 \, dt$$

$$\leq \frac{1}{22} \|\nabla \tilde{B}_{\varepsilon}\|_{L^2(Q_{t_0})}^2 + C \int_0^{t_0} (\|\tilde{b}_{\varepsilon}\|_2^2 \|\nabla \tilde{b}_{\varepsilon}\|_2^2 + \|\nabla \tilde{u}\|_2^2 \|\Delta \tilde{u}\|_2^2) \, dt.$$

To deal with $B$, we break it down, one has

$$B = -\int_{Q_{t_0}} [b_{\varepsilon} \cdot \nabla \tilde{u}_{\varepsilon} \cdot \tilde{b} - b \cdot \nabla \tilde{u}_{\varepsilon} \cdot \tilde{b}] \, dx \, dy \, dz \, dt$$

$$= -\int_{Q_{t_0}} [(b_{\varepsilon} - b) \cdot \nabla \tilde{u}_{\varepsilon} \cdot \tilde{b}] \, dx \, dy \, dz \, dt$$

$$= -\int_{Q_{t_0}} B_{\varepsilon} \cdot \nabla \tilde{u}_{\varepsilon} \cdot \tilde{b} \, dx \, dy \, dz \, dt$$

$$= -\int_{Q_{t_0}} \tilde{B}_{\varepsilon} \cdot \nabla H \tilde{u}_{\varepsilon} \cdot \tilde{b} \, dx \, dy \, dz \, dt - \int_{Q_{t_0}} \tilde{B}_{3, \varepsilon} \partial_z \tilde{u}_{\varepsilon} \cdot \tilde{b} \, dx \, dy \, dz \, dt$$

$$:= B_1 + B_2.$$

Along the similar argument for the estimate of $A$,

$$B_1 = -\int_{Q_{t_0}} \tilde{B}_{\varepsilon} \cdot \nabla H \tilde{u}_{\varepsilon} \cdot \tilde{b} \, dx \, dy \, dz \, dt$$

$$\leq \int_0^{t_0} \|\tilde{B}_{\varepsilon}\|_6 \|\nabla H \tilde{u}_{\varepsilon}\|_2 \|\tilde{b}\|_3 \, dt$$

$$\leq \int_0^{t_0} \|\nabla \tilde{B}_{\varepsilon}\|_2 \|\nabla \tilde{u}_{\varepsilon}\|_2 \|\tilde{b}\|_2^{1/2} \|\nabla \tilde{b}\|_2^{1/2} \, dt$$

$$\leq \frac{1}{22} \|\nabla \tilde{B}_{\varepsilon}\|_{L^2(Q_{t_0})}^2 + C \int_0^{t_0} \|\nabla \tilde{u}_{\varepsilon}\|_2 \|\tilde{b}\|_2 \|\nabla \tilde{b}\|_2 \, dt$$

$$\leq \frac{1}{22} \|\nabla \tilde{B}_{\varepsilon}\|_{L^2(Q_{t_0})}^2 + C \int_0^{t_0} \|\nabla \tilde{u}_{\varepsilon}\|_2^2 (|\tilde{b}|_2^2 + |\nabla \tilde{b}|_2^2) \, dt$$
we deduce

\begin{align*}
\leq & \frac{1}{22} \| \nabla \tilde{B}_\varepsilon \|_{L^2(Q_t)}^2 + C \int_0^t \| \nabla \tilde{u}_\varepsilon \|_2 \| \tilde{b} \|_2^2 \, dt + C \int_0^t \| \nabla \tilde{u}_\varepsilon \|_2 \| \nabla \tilde{b} \|_2^2 \, dt. \\
\end{align*}

It follows from integration by parts that

\begin{align*}
B_2 = & - \int_{Q_t} \tilde{B}_{3,\varepsilon} \partial_z \tilde{u}_\varepsilon \cdot \tilde{b} \, dx \, dy \, dz \\
= & \int_{Q_t} \partial_z \tilde{B}_{3,\varepsilon} \tilde{u}_\varepsilon \cdot \tilde{b} \, dx \, dy \, dz + \int_{Q_t} \tilde{B}_{3,\varepsilon} \tilde{u}_\varepsilon \cdot \partial_z \tilde{b} \, dx \, dy \, dz \\
= & - \int_{Q_t} (\nabla H \cdot \tilde{B}_\varepsilon) \tilde{u}_\varepsilon \cdot \tilde{b} \, dx \, dy \, dz + \int_{Q_t} \tilde{B}_{3,\varepsilon} \tilde{u}_\varepsilon \cdot \partial_z \tilde{b} \, dx \, dy \, dz \\
:= & B_{21} + B_{22}.
\end{align*}

The estimate for $B_{21}$ is given as follows. By the Hölder and Young inequalities, we deduce

\begin{align*}
B_{21} = & - \int_{Q_t} (\nabla H \cdot \tilde{B}_\varepsilon) \tilde{u}_\varepsilon \cdot \tilde{b} \, dx \, dy \, dz \\
\leq & \int_0^t \| \nabla H \cdot \tilde{B}_\varepsilon \|_2 \| \tilde{u}_\varepsilon \|_6 \| \tilde{b} \|_3 \, dt \\
\leq & C \int_0^t \| \nabla \tilde{B}_\varepsilon \|_2 \| \nabla \tilde{u}_\varepsilon \|_2 \| \tilde{b} \|_2 \frac{1}{2} \| \nabla \tilde{b} \|_2 \frac{1}{2} \, dt \\
\leq & \frac{1}{22} \| \nabla \tilde{B}_\varepsilon \|_{L^2(Q_t)}^2 + C \int_0^t \| \nabla \tilde{u}_\varepsilon \|_2 \| \tilde{b} \|_2 \| \nabla \tilde{b} \|_2 \, dt \\
\leq & \frac{1}{22} \| \nabla \tilde{B}_\varepsilon \|_{L^2(Q_t)}^2 + C \int_0^t \| \nabla \tilde{u}_\varepsilon \|_2 (\| \tilde{b} \|_2^2 + \| \nabla \tilde{b} \|_2^2) \, dt \\
\leq & \frac{1}{22} \| \nabla \tilde{B}_\varepsilon \|_{L^2(Q_t)}^2 + C \int_0^t \| \nabla \tilde{u}_\varepsilon \|_2 \| \tilde{b} \|_2 \, dt + C \int_0^t \| \nabla \tilde{u}_\varepsilon \|_2 \| \nabla \tilde{b} \|_2 \, dt.
\end{align*}

For the second term of $B_2$, taking advantage of Lemma 2.2, the Poincaré and Young inequalities that

\begin{align*}
B_{22} = & \int_{Q_t} \tilde{B}_{3,\varepsilon} \tilde{u}_\varepsilon \cdot \partial_z \tilde{b} \, dx \, dy \, dz \\
= & \int_0^t \int_{\Omega} \left( \int_{0}^{\xi} \nabla H \cdot \tilde{B}_\varepsilon \, d\xi \right) \tilde{u}_\varepsilon \partial_z \tilde{b} \, dx \, dy \, dz \\
= & \int_0^t \int_{\Omega} \left( \int_{-1}^{1} |\nabla H \tilde{B}_\varepsilon| \, dz \right) \left( \int_{-1}^{1} |\tilde{u}_\varepsilon| |\partial_z \tilde{b}| \, dz \right) \, dx \, dy \, dz \\
\leq & \int_0^t \| \nabla H \tilde{B}_\varepsilon \|_{2} \| \tilde{u}_\varepsilon \|_2 \| \nabla \tilde{u}_\varepsilon \|_2^\frac{1}{2} \| \nabla \tilde{b} \|_2^\frac{1}{2} \| \Delta \tilde{b} \|_2^\frac{1}{2} \, dt \\
\leq & \frac{1}{22} \| \nabla \tilde{B}_\varepsilon \|_{L^2(Q_t)}^2 + C \int_0^t \| \tilde{u}_\varepsilon \|_2 \| \nabla \tilde{u}_\varepsilon \|_2 \| \nabla \tilde{b} \|_2 \| \Delta \tilde{b} \|_2 \, dt
\end{align*}
$$\leq \frac{1}{22} \| \nabla \tilde{B}_\varepsilon \|_{L^2(Q_{t_0})}^2 + C \int_{t_0}^0 (\| \tilde{u}_\varepsilon \|_2^2 \| \nabla \tilde{u}_\varepsilon \|_2^2 + \| \nabla \tilde{b} \|_2^2 \| \Delta \tilde{b} \|_2^2) \, dt.$$  

To deal with $I_6$, we break it down

$$I_6 = \int_{Q_{t_0}} [((u_\varepsilon \cdot \nabla) \tilde{b}_\varepsilon \cdot \tilde{b} + (u \cdot \nabla) \tilde{b} \cdot \tilde{b}) \, dx \, dy \, dz \, dt$$

$$= \int_{Q_{t_0}} [((u_\varepsilon \cdot \nabla) \tilde{b}_\varepsilon \cdot \tilde{b} - (u \cdot \nabla) \tilde{b}_\varepsilon \cdot \tilde{b}) \, dx \, dy \, dz \, dt$$

$$= \int_{Q_{t_0}} [(u_\varepsilon - u) \cdot \nabla] \tilde{b}_\varepsilon \cdot \tilde{b} \, dx \, dy \, dz \, dt$$

$$= \int_{Q_{t_0}} [U_\varepsilon \cdot \nabla] \tilde{b}_\varepsilon \cdot \tilde{b} \, dx \, dy \, dz \, dt$$

$$= \int_{Q_{t_0}} (U_\varepsilon \cdot \nabla H) \tilde{b}_\varepsilon \cdot \tilde{b} \, dx \, dy \, dz \, dt + \int_{Q_{t_0}} U_{3,\varepsilon} \partial_z \tilde{b}_\varepsilon \cdot \tilde{b} \, dx \, dy \, dz \, dt.$$  

$$:= I_{61} + I_{62}.$$  

It follows from the Sobolev and Young inequalities that

$$I_{61} = \int_{Q_{t_0}} (\tilde{U}_\varepsilon \cdot \nabla H) \tilde{b}_\varepsilon \cdot \tilde{b} \, dx \, dy \, dz \, dt$$

$$\leq \int_0^{t_0} \| \tilde{U}_\varepsilon \|_3 \| \nabla H \tilde{b}_\varepsilon \|_2 \| \tilde{b} \|_6 \, dt$$

$$\leq \int_0^{t_0} \| \tilde{U}_\varepsilon \|_2^\frac{3}{2} \| \nabla \tilde{U}_\varepsilon \|_2^{\frac{3}{2}} \| \nabla H \tilde{b}_\varepsilon \|_2 \| \nabla \tilde{b} \|_2 \, dt$$

$$\leq \frac{1}{16} \| \nabla \tilde{U}_\varepsilon \|_2^2 + \frac{1}{22} \| \nabla \tilde{b}_\varepsilon \|_2^2 + C \int_0^{t_0} \| \nabla \tilde{b} \|_2^4 \| \tilde{U}_\varepsilon \|_2^2 \, dt,$$

for $I_{62}$, we further decompose it into two pieces

$$I_{62} = \int_{Q_{t_0}} U_{3,\varepsilon} \partial_z \tilde{b}_\varepsilon \cdot \tilde{b} \, dx \, dy \, dz \, dt$$

$$= -\int_{Q_{t_0}} [\partial_z U_{3,\varepsilon} \tilde{b}_\varepsilon \cdot \tilde{b} + U_{3,\varepsilon} \tilde{b}_\varepsilon \cdot \partial_z \tilde{b}] \, dx \, dy \, dz \, dt$$

$$= \int_{Q_{t_0}} [\nabla H \cdot \tilde{U}_\varepsilon \tilde{b}_\varepsilon \cdot \tilde{b} - U_{3,\varepsilon} \tilde{b}_\varepsilon \cdot \partial_z \tilde{b}] \, dx \, dy \, dz \, dt$$

$$:= I_{621} + I_{622}.$$  

By the Hölder and Young inequalities, we deduce

$$I_{621} = \int_{Q_{t_0}} \nabla H \cdot \tilde{U}_\varepsilon \tilde{b}_\varepsilon \cdot \tilde{b} \, dx \, dy \, dz \, dt$$

$$\leq \int_0^{t_0} \| \nabla H \tilde{U}_\varepsilon \|_2 \| \tilde{b}_\varepsilon \|_3 \| \tilde{b} \|_6 \, dt$$
\[ \leq C \int_0^{t_0} \| \nabla_H \tilde{U}_\varepsilon \|_2 \| \tilde{B}_\varepsilon \|_2^\frac{3}{2} \| \tilde{\nabla} \tilde{b} \|_2 \, dt \]
\[ \leq \frac{1}{16} \| \nabla \tilde{U}_\varepsilon \|_2^2 + \frac{1}{22} \| \nabla \tilde{B}_\varepsilon \|_2^2 + C \int_0^{t_0} \| \nabla \tilde{b} \|_2 \| \tilde{B}_\varepsilon \|_2^2 \, dt. \]

A Similar argument to that for \( B_{22} \), we have

\[ I_{622} = - \int_{Q_{t_0}} U_{3,\varepsilon} \tilde{B}_\varepsilon \cdot \partial_\varepsilon \tilde{b} \, dxdydzdt \]
\[ = \int_0^{t_0} \int_\Omega \left( \int_0^z \nabla_H \cdot \tilde{U}_\varepsilon \, d\xi \right) (\tilde{B}_\varepsilon \cdot \partial_\varepsilon \tilde{b}) \, dxdydzdt \]
\[ \leq \int_0^{t_0} \int_M \left( \int_{-1}^1 |\nabla_H \tilde{U}_\varepsilon| \, dz \right) \left( \int_{-1}^1 |\tilde{B}_\varepsilon| |\partial_\varepsilon \tilde{b}| \, dz \right) \, dxdydt \]
\[ \leq C \int_0^{t_0} \| \nabla_H \tilde{U}_\varepsilon \|_2 \| \tilde{B}_\varepsilon \|_2^{\frac{3}{2}} \| \nabla \tilde{b} \|_2 \| \Delta \tilde{b} \|_2^{\frac{1}{2}} \, dt \]
\[ \leq \frac{1}{16} \| \nabla \tilde{U}_\varepsilon \|_2^2 + \frac{1}{22} \| \nabla \tilde{B}_\varepsilon \|_2^2 + C \int_0^{t_0} \| \nabla \tilde{b} \|_2 \| \Delta \tilde{b} \|_2 \| \tilde{B}_\varepsilon \|_2 \, dt. \]

Using the incompressibility conditions, we deduce that

\[ I_7 = \varepsilon^2 \int_{Q_{t_0}} u_\varepsilon \cdot \nabla u_{3,\varepsilon} u_3 \, dxdydzdt \]
\[ = \varepsilon^2 \int_{Q_{t_0}} u_\varepsilon \cdot \nabla U_{3,\varepsilon} u_3 \, dxdydzdt \]
\[ = \varepsilon^2 \int_{Q_{t_0}} [\tilde{u}_\varepsilon \cdot \nabla_H U_{3,\varepsilon} - u_{3,\varepsilon} \nabla_H \cdot \tilde{U}_\varepsilon] u_3 \, dxdydzdt \]
\[ \leq \varepsilon^2 \int_0^{t_0} \int_M \left( \int_{-1}^1 |\tilde{u}_\varepsilon| |\nabla_H U_{3,\varepsilon}| + |u_{3,\varepsilon}| |\nabla_H \tilde{U}_\varepsilon| \, dz \right) \left( \int_{-1}^1 |\nabla_H \tilde{u}| \, dz \right) \, dxdydt, \]

applying Lemma 2.2, the Poincaré and Young inequalities that

\[ I_7 = C \varepsilon^2 \int_0^{t_0} \left( \| \tilde{u}_\varepsilon \|_2^{\frac{3}{2}} \| \nabla \tilde{u}_\varepsilon \|_2 \| \nabla U_{3,\varepsilon} \|_2 + \| u_{3,\varepsilon} \|_2^{\frac{3}{2}} \| \nabla \tilde{u}_\varepsilon \|_2 \| \nabla U_{3,\varepsilon} \|_2 \| \Delta \tilde{u} \|_2^{\frac{1}{2}} \right) \, dt \]
\[ \leq \frac{1}{16} \| \nabla \tilde{U}_\varepsilon \|_{L^2(Q_{t_0})}^2 + \frac{1}{6} \| \nabla U_{3,\varepsilon} \|_{L^2(Q_{t_0})}^2 \]
\[ + C \varepsilon^2 \int_0^{t_0} \left( \| \tilde{u}_\varepsilon \|_2^{\frac{3}{2}} \| \nabla \tilde{u}_\varepsilon \|_2 + \| \nabla \tilde{u}_\varepsilon \|_2 \| \Delta \tilde{u} \|_2^{\frac{1}{2}} + \varepsilon^2 \| u_{3,\varepsilon} \|_2^{\frac{3}{2}} \| \nabla u_{3,\varepsilon} \|_2 \right) \, dt, \]

from which, recalling (3.10) and by Proposition 1.2, we have

\[ I_7 \leq \frac{1}{16} \| \nabla \tilde{U}_\varepsilon \|_{L^2(Q_{t_0})}^2 + \frac{1}{6} \varepsilon^2 \| \nabla U_{3,\varepsilon} \|_{L^2(Q_{t_0})}^2 + C \varepsilon^2 \left( \| \tilde{u}_0 \|_2^2 + \varepsilon^2 \| u_{3,0} \|_2^2 \right) + C \| \tilde{u}_0 \|_{H^1, L_1, L_2} \]

For \( I_8 + I_9 \), applying the divergence free conditions gives that

\[ I_8 + I_9 = -\varepsilon^2 \int_{Q_{t_0}} b_\varepsilon \cdot \nabla b_{3,\varepsilon} u_3 \, dxdydzdt + \varepsilon^2 \int_{Q_{t_0}} b_\varepsilon \cdot \nabla b_3 u_3 \, dxdydzdt \]
Employing Lemma 2.2, the Hölder and Young inequalities we infer that

\[ I_{s1} + I_{s2} = -\varepsilon^2 \int_{Q_{t0}} (b_\varepsilon \cdot \nabla b_{3,\varepsilon} u_3 - b_\varepsilon \cdot \nabla b_3 u_3) \, dx \, dy \, dz \, dt \]

\[ = -\varepsilon^2 \int_{Q_{t0}} (b_\varepsilon \cdot \nabla_B B_{3,\varepsilon} - b_{3,\varepsilon} \nabla_B \cdot \bar{B}_\varepsilon) u_3 \, dx \, dy \, dz \, dt \]

\[ \leq \varepsilon^2 \int_0^{t_0} \int_M \left( \int_{t_1}^t \left( |\bar{b}_\varepsilon| |\nabla B_{3,\varepsilon}| + |b_{3,\varepsilon}| |\nabla B_{3,\varepsilon}| \right) \, dz \right) \left( \int_{t_1}^t |\nabla_H \tilde{u}| \, dz \right) \, dx \, dy \, dz \, dt, \]

\[ \leq C \varepsilon^2 \int_0^{t_0} \left( |\bar{b}_\varepsilon| |\bar{b}_\varepsilon|^2 + |\nabla \tilde{u}| |\nabla \tilde{u}| + \varepsilon^2 |b_{3,\varepsilon}| |\nabla b_{3,\varepsilon}|^2 \right) \, dt \]

\[ + \frac{1}{22} |\nabla \bar{B}_\varepsilon|^2_{L^2(Q_{t0})} + \frac{1}{6} \varepsilon^2 |\nabla B_{3,\varepsilon}|^2_{L^2(Q_{t0})} \]

\[ \leq \frac{1}{22} |\nabla \bar{B}_\varepsilon|^2_{L^2(Q_{t0})} + \frac{1}{6} \varepsilon^2 |\nabla B_{3,\varepsilon}|^2_{L^2(Q_{t0})} \]

\[ + C \varepsilon^2 \left( |\bar{b}_0|^2 + \varepsilon^2 |b_{3,0}|^2 \right)^2 + C(|\tilde{u}_0|_{H^1}, L_1, L_2) \].

A similar argument to that for \( I_{s1} + I_{s2} \), yields

\[ I_{91} + I_{92} = -\varepsilon^2 \int_{Q_{t0}} (b_\varepsilon \cdot \nabla u_{3,\varepsilon} b_3 - b_\varepsilon \cdot \nabla u_3 b_3) \, dx \, dy \, dz \, dt \]

\[ = -\varepsilon^2 \int_{Q_{t0}} (b_\varepsilon \cdot \nabla U_{3,\varepsilon}) b_3 \, dx \, dy \, dz \, dt \]

\[ = -\varepsilon^2 \int_{Q_{t0}} [\bar{b}_\varepsilon \cdot \nabla_H U_{3,\varepsilon} - b_{3,\varepsilon} \nabla_H \cdot \bar{U}_\varepsilon] b_3 \, dx \, dy \, dz \, dt \]

\[ \leq \varepsilon^2 \int_0^{t_0} \int_M \left( \int_{t_1}^t |\bar{b}_\varepsilon| |\nabla H U_{3,\varepsilon}| + |b_{3,\varepsilon}| |\nabla H \bar{U}_\varepsilon| \, dz \right) \left( \int_{t_1}^t |\nabla_H \bar{b}| \, dz \right) \, dx \, dy \, dz \, dt \]
\[ I_{10} = \varepsilon^2 \int_{Q_{t_0}} u_\varepsilon \cdot \nabla b_{3,\varepsilon} b_3 \, dx dy dz dt \]

\[ \leq \varepsilon^2 \int_{Q_{t_0}} u_\varepsilon \cdot \nabla b_{3,\varepsilon} b_3 \, dx dy dz dt - \varepsilon^2 \int_{Q_{t_0}} u_\varepsilon \cdot \nabla b_3 b_3 \, dx dy dz dt \]

\[ \leq \varepsilon^2 \int_{Q_{t_0}} [\tilde{u}_\varepsilon \cdot \nabla H B_{3,\varepsilon} - u_{3,\varepsilon} \nabla H \cdot \tilde{B}_\varepsilon] b_3 \, dx dy dz dt \]

\[ \leq \varepsilon^2 \int_0^{t_0} \int_M \left( \int_{-1}^{1} (|\tilde{u}_\varepsilon| |\nabla H B_{3,\varepsilon}| + |u_{3,\varepsilon}| |\nabla H \tilde{B}_\varepsilon|) \, dz \right) \left( \int_{-1}^{1} |\nabla H b_3| \, dz \right) \, dx dy dz dt, \]

which is further bounded through the Lemma 2.2, the Poincaré and Young inequalities.

\[ I_{10} \leq C \varepsilon^2 \int_0^{t_0} \left( |\tilde{u}_\varepsilon|^2 \frac{1}{2} ||\nabla \tilde{u}_\varepsilon||_2 + |u_{3,\varepsilon}|^2 \frac{1}{2} ||\nabla u_{3,\varepsilon}\tilde{B}_\varepsilon||_2 \right) \, dt \]

\[ \leq \frac{1}{2} \varepsilon^2 ||\nabla B_{3,\varepsilon}||^2_{L^2(Q_{t_0})} + \frac{1}{22} ||\nabla H \tilde{B}_\varepsilon||^2_{L^2(Q_{t_0})} \]

\[ \leq \frac{1}{2} \varepsilon^2 ||\nabla H B_{3,\varepsilon}||^2_{L^2(Q_{t_0})} + \frac{1}{22} ||\nabla B_{3,\varepsilon}||^2_{L^2(Q_{t_0})} + \varepsilon^2 \left( \left( |\tilde{u}_0|^2 \varepsilon + |u_{3,0}|^2 \varepsilon \right)^2 + C(|\tilde{b}_0||H^1, L_1, L_2) \right). \]

In light of the estimates of \( I_1 - I_{10} \) into (3.11) yields

\[ g(t) := (|\tilde{U}_\varepsilon|^2 + |\tilde{B}_\varepsilon|^2 + \varepsilon^2 |U_{3,\varepsilon}|^2 + \varepsilon^2 |B_{3,\varepsilon}|^2) (t) \]

\[ + \int_0^t \left( ||\nabla \tilde{U}_\varepsilon||_2^2 + ||\nabla \tilde{B}_\varepsilon||_2^2 + \varepsilon^2 ||\nabla U_{3,\varepsilon}||_2^2 + \varepsilon^2 ||\nabla B_{3,\varepsilon}||_2^2 \right) ds \]

\[ \leq C \varepsilon^2 \left( |\tilde{u}_0|^2 \varepsilon + |\tilde{b}_0|^2 \varepsilon + \varepsilon^2 |u_{3,0}|^2 \varepsilon + \varepsilon^2 |b_{3,0}|^2 + 1 \right)^2 + C(\tilde{u}_0, \tilde{b}_0, H^1, L_1, L_2) \]
where applying the Gronwall inequality, and using Proposition 1.2, we get

\[ G' (t) = C \left( \| \nabla \tilde{u} \|_2^2 \| \Delta \tilde{u} \|_2^2 + \| \nabla \tilde{b} \|_2^2 \| \Delta \tilde{b} \|_2^2 \right) (\| \tilde{U}_\varepsilon \|_2^2 + \| \tilde{B}_\varepsilon \|_2^2) \]

for a.e. \( t \in [0, \infty) \). Therefore, we have

\[ G' (t) \leq C \left( \| \nabla \tilde{u} \|_2^2 \| \Delta \tilde{u} \|_2^2 + \| \nabla \tilde{b} \|_2^2 \| \Delta \tilde{b} \|_2^2 \right) g(t) \]

applying the Gronwall inequality, and using Proposition 1.2, we get

\[ g(t) \leq C \left( \| u_0 \|_{H^1}, \| b_0 \|_{H^1}, L_1, L_2 \right) (\| \tilde{u}_0 \|_2^2 + \| \tilde{b}_0 \|_2^2 + \varepsilon^2 \| u_{3,0} \|_2^2 + \varepsilon^2 \| b_{3,0} \|_2^2 + 1)^2, \]

where \( C \) is a positive constants depending only on \( \| \tilde{u}_0 \|_{H^1}, \| \tilde{b}_0 \|_{H^1}, L_1 \) and \( L_2 \). This completes the proof. \( \square \)

4. Strong convergence II: the \( H^2 \) initial data case

In this section, we deal with the strong convergence of the SMHD to the PEM, with the initial data \( (\tilde{u}_0, \tilde{b}_0) \in H^2(\Omega) \), as the aspect ratio parameter \( \varepsilon \) goes to zero. In order to keep the nation simple, we remove the subscript index \( \varepsilon \) of \( (\tilde{U}_\varepsilon, U_{3,\varepsilon}, \tilde{B}_\varepsilon, B_{3,\varepsilon}) \) from the below, in other words, we replace \( (\tilde{U}_\varepsilon, U_{3,\varepsilon}, \tilde{B}_\varepsilon, B_{3,\varepsilon}) \) with \( (\tilde{U}, U_3, \tilde{B}, B_3) \). We have the following results.

**Theorem 4.1.** Given a periodic function \( (\tilde{u}_0, \tilde{b}_0) \in H^2(\Omega) \), such that

\[ \nabla_H \cdot \left( \int_{-1}^{1} \tilde{u}_0(x, y, z) \, dz \right) = 0, \quad \int_{\Omega} \tilde{u}_0(x, y, z) \, dxdydz = 0, \]

and

\[ \nabla_H \cdot \left( \int_{-1}^{1} \tilde{b}(x, y, z) \, dz \right) = 0, \quad \int_{\Omega} \tilde{b}(x, y, z) \, dxdydz = 0. \]

Let \( (\tilde{u}_\varepsilon, u_{3,\varepsilon}, \tilde{b}_\varepsilon, b_{3,\varepsilon}) \) be the unique local (in time) strong solution to the SMHD and \( (\tilde{u}, u_3, \tilde{b}, b_3) \) be the unique global strong solution to the PEM, subject to (1.3)-(1.11). Denote

\[ (\tilde{U}, U_3) = (\tilde{u}_\varepsilon - \tilde{u}, u_{3,\varepsilon} - u_3), \quad (\tilde{B}, B_3) = (\tilde{b}_\varepsilon - \tilde{b}, b_{3,\varepsilon} - b_3). \]

Then, there is a positive constant \( \varepsilon_0 \) depending only on \( \| \tilde{u}_0 \|_{H^2}, \| \tilde{b}_0 \|_{H^2}, L_1 \) and \( L_2 \), such that, for any \( \varepsilon \in (0, \varepsilon_0) \), the strong solution \( (\tilde{u}_\varepsilon, u_{3,\varepsilon}, \tilde{b}_\varepsilon, b_{3,\varepsilon}) \) of the SMHD exists globally in time, and the following estimate holds

\[ \sup_{0 \leq t < \infty} (\| \tilde{U} \|_{H^1}^2 + \varepsilon^2 \| U_3 \|_{H^1}^2 + \| \tilde{B} \|_{H^1}^2 + \varepsilon^2 \| B_3 \|_{H^1}^2) (t) \]
As a consequence, we have the following strong convergence,

\[
+ \int_0^\infty (\|\nabla \tilde{U}\|_{H^1}^2 + \varepsilon^2 \|\nabla U_3\|_{H^1}^2 + \|\nabla \tilde{B}\|_{H^1}^2 + \varepsilon^2 \|\nabla B_3\|_{H^1}^2) \, dt \\
\leq C(\|\tilde{u}_0\|_{H^2}, \|\tilde{b}_0\|_{H^2}, L_1, L_2)\varepsilon^2.
\]

Remark 4.2. Theorem 3.1 and Theorem 4.1 deal with the strong convergence of solutions of SMHD to the PEM is global and uniform in time, and they converge in the same order. In addition, smoothing the initial data is a strong norm in which convergence occurs.

Remark 4.3. Generally, if \((\tilde{u}_0, \tilde{b}_0) \in H^k\), with \(k \geq 2\), then one can show that

\[
\sup_{0 \leq t < \infty} (\|\tilde{U}\|_{H^{k-1}}^2 + \varepsilon^2 \|U_3\|_{H^{k-1}}^2 + \|\tilde{B}\|_{H^{k-1}}^2 + \varepsilon^2 \|B_3\|_{H^{k-1}}^2)(t) \\
+ \int_0^\infty (\|\nabla \tilde{U}\|_{H^{k-1}}^2 + \varepsilon^2 \|\nabla U_3\|_{H^{k-1}}^2 + \|\nabla \tilde{B}\|_{H^{k-1}}^2 + \varepsilon^2 \|\nabla B_3\|_{H^{k-1}}^2) \, dt \\
\leq C(\|\tilde{u}_0\|_{H^k}, \|\tilde{b}_0\|_{H^k}, L_1, L_2)\varepsilon^2,
\]

moreover, we have the following strong convergence,

\[
(\tilde{u}_\varepsilon, \varepsilon u_{3,\varepsilon}, \tilde{b}_\varepsilon, \varepsilon b_{3,\varepsilon}) \rightarrow (\tilde{u}, 0, \tilde{b}, 0), \ \text{in} \ L^\infty(0, \infty; H^{k-1}(\Omega)),
\]

\[
(\nabla \tilde{u}_\varepsilon, \varepsilon \nabla u_{3,\varepsilon}, \nabla \tilde{b}_\varepsilon, \varepsilon \nabla b_{3,\varepsilon}) \rightarrow (\nabla \tilde{u}, 0, u_{3,\varepsilon}, \nabla \tilde{b}, 0, b_{3,\varepsilon}), \ \text{in} \ L^2(0, \infty; H^{k-1}(\Omega)),
\]

\[
(\varepsilon u_{3,\varepsilon}, b_{3,\varepsilon}) \rightarrow (u_{3,\varepsilon}, b_{3,\varepsilon}), \ \text{in} \ L^\infty(0, \infty; H^{k-2}(\Omega)),
\]

and the convergence rate is of the order \(O(\varepsilon)\).

This can be achieved by making higher energy estimates for the difference system (1.18).

Remark 4.4. The smoothing effect is observed in the SMHD and the PEM to the unique strong solutions, we can also prove that in this theorem (rather than theorem 3.1), the strong convergence to a stronger norm always starts from the initial time, in particular, \((\tilde{u}_\varepsilon, u_{3,\varepsilon}) \rightarrow (\tilde{u}, u_{3,\varepsilon})\) and \((\tilde{b}_\varepsilon, b_{3,\varepsilon}) \rightarrow (\tilde{b}, b_{3,\varepsilon})\) in \(C^k(\overline{\Omega} \times (T, \infty))\), for any given time \(T > 0\) and integer \(k \geq 0\).
In this section, we give the proof of Theorem 4.1. Let \((\tilde{u}_0, \tilde{b}_0) \in H^2(\Omega)\), and assume that
\[
\nabla_H \cdot \left( \int_{-1}^{1} \tilde{u}_0(x,y,z) \, dz \right) = 0 \quad \text{and} \quad \nabla_H \cdot \left( \int_{-1}^{1} \tilde{b}_0(x,y,z) \, dz \right) = 0,
\]
for all \((x,y) \in M\). Set \(u_0 = (\tilde{u}_0, u_{3,0})\), \(b_0 = (\tilde{b}_0, b_{3,0})\), \(\nabla \cdot u_0 = 0\) and \(\nabla \cdot b_0 = 0\), with \(u_{3,0}\) and \(b_{3,0}\) given by (1.16) and (1.17), then \((u_0, b_0) \in H^1(\Omega)\). Through the same argument as the standard MHD equations, see, e.g., [22], it can be proved that there is a unique local (in time) strong solution \((\tilde{u}_\varepsilon, u_{3,\varepsilon}, \tilde{b}_\varepsilon, b_{3,\varepsilon})\) to the SMHD subject to (1.3)-(1.11). \(T_\varepsilon^*\) represents the maximal existence time of the strong solution. Because of the smoothing effect of SMHD on the unique strong solution, it can be proved that strong solution \((\tilde{u}_\varepsilon, u_{3,\varepsilon}, \tilde{b}_\varepsilon, b_{3,\varepsilon})\) is smooth over the time interval \((0, T_\varepsilon^*)\), recalling that \((\tilde{u}, u_{3,\varepsilon}, \tilde{b}, b_{3,\varepsilon})\) is smooth away from the initial time, the same as \((\tilde{U}, U_3, \tilde{B}, B_3)\). This guarantees the validity of the argument in the following proof.

We need to make \textit{a priori} estimates of \((\tilde{U}_\varepsilon, U_{3,\varepsilon}, \tilde{B}_\varepsilon, B_{3,\varepsilon})\). We begin with the basic energy estimate described in the following proposition.

**Proposition 4.5.** (Basic \(L^2\) energy estimate). The following \textit{basic energy estimate} holds
\[
\sup_{0 \leq s \leq t} \left( \|\tilde{U}\|_2^2 + \varepsilon^2 \|U_3\|_2^2 + \|\tilde{B}\|_2^2 + \varepsilon^2 \|B_3\|_2^2 \right)
\]
\[
+ \int_0^t \left( \|\nabla \tilde{U}\|_2^2 + \varepsilon^2 \|\nabla U_3\|_2^2 + \|\nabla \tilde{B}\|_2^2 + \varepsilon^2 \|\nabla B_3\|_2^2 \right) \, ds
\]
\[
\leq C \varepsilon^2 \left( \|\tilde{u}_0\|_2^2 + \|\tilde{b}_0\|_2^2 + \varepsilon^2 \|u_{3,0}\|_2^2 + \varepsilon^2 \|b_{3,0}\|_2^2 + 1 \right)^2,
\]
for any \(t \in [0, T_\varepsilon^*)\), where \(C\) is a constant depending on \(\|\tilde{u}\|_{H^1}, \|\tilde{b}\|_{H^1}, L_1\) and \(L_2\).

**Proof.** This is a direct consequence of Proposition 3.4. \(\square\)

The following proposition is the first order energy estimate.

**Proposition 4.6.** (\(H^1\) energy estimates) There is a constant \(\delta_0 > 0\) depending only on \(L_1\) and \(L_2\), so that, we have the following estimate
\[
\sup_{0 \leq s \leq t} \left( \|\nabla \tilde{U}\|_2^2 + \varepsilon^2 \|\nabla U_3\|_2^2 + \|\nabla \tilde{B}\|_2^2 + \varepsilon^2 \|\nabla B_3\|_2^2 \right)
\]
\[
+ \int_0^t \left( \|\Delta \tilde{U}\|_2^2 + \varepsilon^2 \|\Delta U_3\|_2^2 + \|\Delta \tilde{B}\|_2^2 + \varepsilon^2 \|\Delta B_3\|_2^2 \right) \, ds
\]
\[
\leq 2C_1 \varepsilon \epsilon \int_0^t \left( \|\Delta \tilde{u}\|_2^2 \|\nabla \Delta \tilde{u}\|_2^2 + \|\Delta \tilde{b}\|_2^2 \|\nabla \Delta \tilde{b}\|_2^2 \right) \, ds
\]
\[
\times \int_0^t \left( 1 + \|\Delta \tilde{u}\|_2^2 + \|\Delta \tilde{b}\|_2^2 \right) \left( \|\nabla \partial_t \tilde{u}\|_2^2 + \|\nabla \partial_t \tilde{b}\|_2^2 + \|\nabla \Delta \tilde{u}\|_2^2 + \|\nabla \Delta \tilde{b}\|_2^2 \right) \, ds.
\]
for any $t \in [0, T^* \epsilon)$, provided that
\[
\sup_{0 \leq s \leq t} (\|\nabla \tilde{U}\|_2^2 + \epsilon^2 \|\nabla U_3\|_2^2 + \|\nabla \tilde{B}\|_2^2 + \epsilon^2 \|\nabla B_3\|_2^2) \leq \delta_0^2,
\]
where $C$ is a positive constant depending only on $L_1$ and $L_2$.

Proof. Multiplying the equations (1.18)$_1$, (1.18)$_2$, (1.18)$_3$ and (1.18)$_4$ by $-\Delta \tilde{U}$, $-\Delta U_3$, $-\Delta \tilde{B}$ and $-\Delta B_3$, respectively, integrating the result over $\Omega$, then it follows from integration by parts that
\[
\frac{1}{2} \frac{d}{dt} (\|\nabla \tilde{U}\|_2^2 + \epsilon^2 \|\nabla U_3\|_2^2 + \|\nabla \tilde{B}\|_2^2 + \epsilon^2 \|\nabla B_3\|_2^2)
= \int_\Omega \left[ (U \cdot \nabla) \tilde{U} + (u \cdot \nabla) \tilde{U} + (U \cdot \nabla) \tilde{u} \right] \cdot \Delta \tilde{U} \, dx dy dz
- \int_\Omega \left[ (B \cdot \nabla) \tilde{B} + (b \cdot \nabla) \tilde{B} + (B \cdot \nabla) \tilde{b} \right] \cdot \Delta \tilde{U} \, dx dy dz
+ \epsilon^2 \int_\Omega (U \cdot \nabla U_3 + u \cdot \nabla U_3 + U \cdot \nabla u_3) \Delta U_3 \, dx dy dz
- \epsilon^2 \int_\Omega (B \cdot \nabla B_3 + b \cdot \nabla B_3 + B \cdot \nabla b_3) \Delta U_3 \, dx dy dz
+ \epsilon^2 \int_\Omega (\partial_i u_3 + u \cdot \nabla u_3 - \Delta u_3 - b \cdot \nabla b_3) \Delta U_3 \, dx dy dz
+ \int_\Omega \left[ (U \cdot \nabla) \tilde{B} + (u \cdot \nabla) \tilde{B} + (U \cdot \nabla) \tilde{b} \right] \cdot \Delta \tilde{B} \, dx dy dz
- \int_\Omega \left[ (B \cdot \nabla) \tilde{U} + (b \cdot \nabla) \tilde{U} + (B \cdot \nabla) \tilde{b} \right] \cdot \Delta \tilde{B} \, dx dy dz
+ \epsilon^2 \int_\Omega [U \cdot \nabla B_3 + u \cdot \nabla B_3 + U \cdot \nabla b_3] \Delta B_3 \, dx dy dz
- \epsilon^2 \int_\Omega [B \cdot \nabla U_3 + b \cdot \nabla U_3 + B \cdot \nabla u_3] \Delta B_3 \, dx dy dz
+ \epsilon^2 \int_\Omega (\partial_i b_3 + u \cdot \nabla b_3 - \Delta b_3 - b \cdot \nabla u_3) \Delta B_3 \, dx dy dz
:= J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8 + J_9 + J_{10}. \tag{4.2}
\]

Let us estimate the ten terms appearing above. First, by Lemma 2.3, it follows from the Young and Poincaré inequalities that
\[
J_1 = \int_\Omega \left[ (U \cdot \nabla) \tilde{U} + (u \cdot \nabla) \tilde{U} + (U \cdot \nabla) \tilde{u} \right] \cdot \Delta \tilde{U} \, dx dy dz
\leq C (\|\nabla \tilde{U}\|_2 \|\Delta \tilde{U}\|_2 + \|\nabla \tilde{u}\|_2 \|\Delta \tilde{u}\|_2 + \|\nabla U\|_2 \|\Delta U\|_2) \|\Delta \tilde{U}\|_2
\]
using the Poincaré inequality, we get
\[
\leq \frac{1}{15} \|\Delta \tilde{U}\|_2^2 + C(\|\nabla \tilde{U}\|_2^2 \|\Delta \tilde{U}\|_2^2 + \|\nabla \tilde{u}\|_2^2 \|\Delta \tilde{u}\|_2^2 \|\nabla \tilde{U}\|_2^2)
\]
\[
\leq \frac{1}{15} \|\Delta \tilde{U}\|_2^2 + C(\|\nabla \tilde{U}\|_2^2 \|\Delta \tilde{U}\|_2^2 + \|\Delta \tilde{u}\|_2^2 \|\nabla \Delta \tilde{u}\|_2^2 \|\nabla \tilde{U}\|_2^2).
\]
A similar argument to that for $J_1$, yields
\[
J_2 = \int_\Omega [(B \cdot \nabla)\tilde{B} + (b \cdot \nabla)\tilde{B} + (B \cdot \nabla)\tilde{b}] \cdot \Delta \tilde{U} \, dx dy dz
\]
\[
\leq C(\|\nabla \tilde{B}\|_2 \|\Delta \tilde{B}\|_2 + \|\nabla \tilde{b}\|_2^{\frac{3}{2}} \|\Delta \tilde{b}\|_2^{\frac{3}{2}} \|\nabla \tilde{B}\|_2^{\frac{3}{2}} \|\Delta \tilde{B}\|_2^{\frac{3}{2}}) \|\Delta \tilde{U}\|_2
\]
\[
\leq \frac{1}{15} \|\Delta \tilde{U}\|_2^2 + \frac{1}{10} \|\Delta \tilde{B}\|_2^2 + C(\|\nabla \tilde{B}\|_2^2 \|\Delta \tilde{B}\|_2^2 + \|\Delta \tilde{b}\|_2^2 \|\nabla \Delta \tilde{b}\|_2^2 \|\nabla \tilde{B}\|_2^2).
\]
Using the fact that
\[
\|\nabla u_3\|_2 \leq C\|\Delta \tilde{u}\|_2, \quad \|\Delta u_3\|_2 \leq C\|\nabla \Delta \tilde{u}\|_2,
\]
which can be easily verified by recalling $u_3(x, y, z, t) = -\int_0^z \nabla_H \cdot \tilde{u}(x, y, \xi, t) \, d\xi$ and using the Poincaré inequality, we get
\[
J_3 = \varepsilon^2 \int_\Omega [(U \cdot \nabla)U_3 + (u \cdot \nabla)U_3 + (U \cdot \nabla)u_3] \Delta U_3 \, dx dy dz
\]
\[
\leq C\varepsilon^2 \left[ (\|\nabla \tilde{U}\|_2^2 \|\Delta \tilde{U}\|_2^2 + \|\nabla \tilde{u}\|_2^2 \|\Delta \tilde{u}\|_2^2) \|\nabla U_3\|_2^2 \|\Delta U_3\|_2^2
\right.
\]
\[
\left. + \|\nabla \tilde{U}\|_2^2 \|\Delta \tilde{U}\|_2^2 \|\nabla u_3\|_2^2 \|\Delta u_3\|_2^2 \right] \|\Delta U_3\|_2
\]
\[
\leq \frac{1}{15} \|\Delta \tilde{U}\|_2^2 + \frac{1}{10} \varepsilon^2 \|\Delta U_3\|_2^2 + C(\|\nabla \tilde{U}\|_2^2 \|\Delta \tilde{U}\|_2^2 + \varepsilon^4 \|\nabla U_3\|_2^2 \|\Delta U_3\|_2^2)
\]
\[
+ C\varepsilon^2 \|\nabla \tilde{u}\|_2^2 \|\Delta \tilde{u}\|_2^2 \|\nabla U_3\|_2^2 + C\varepsilon^4 \|\nabla u_3\|_2^2 \|\Delta u_3\|_2^2 \|\nabla \tilde{U}\|_2^2
\]
\[
\leq \frac{1}{15} \|\Delta \tilde{U}\|_2^2 + \frac{1}{10} \varepsilon^2 \|\Delta U_3\|_2^2 + C(\|\nabla \tilde{U}\|_2^2 + \varepsilon^2 \|\nabla U_3\|_2^2)
\]
\[
\times \left[ \|\Delta \tilde{U}\|_2^2 + \varepsilon^2 \|\Delta U_3\|_2^2 + (1 + \varepsilon^4) \|\Delta \tilde{u}\|_2^2 \|\nabla \Delta \tilde{u}\|_2^2 \right].
\]
Using again Lemma 2.3 and Hölder, Poincaré and Young inequalities give that
\[
J_4 = \varepsilon^2 \int_\Omega (B \cdot \nabla B_3 + b \cdot \nabla B_3 + B \cdot \nabla b_3) \Delta U_3 \, dx dy dz
\]
\[
\leq C\varepsilon^2 \left[ (\|\nabla \tilde{B}\|_2^2 \|\Delta \tilde{B}\|_2^2 + \|\nabla \tilde{b}\|_2^2 \|\Delta \tilde{b}\|_2^2) \|\nabla B_3\|_2^2 \|\Delta B_3\|_2^2
\right.
\]
\[
\left. + \|\nabla \tilde{B}\|_2^2 \|\Delta \tilde{B}\|_2^2 \|\nabla b_3\|_2^2 \|\Delta b_3\|_2^2 \right] \|\Delta U_3\|_2
\]
\[
\leq \frac{1}{10} \varepsilon^2 \|\Delta U_3\|_2^2 + \frac{1}{10} \|\Delta \tilde{B}\|_2^2 + \frac{1}{10} \varepsilon^2 \|\Delta B_3\|_2^2 + C(\|\nabla \tilde{B}\|_2^2 \|\Delta \tilde{B}\|_2^2 + \varepsilon^4 \|\nabla B_3\|_2^2 \|\Delta B_3\|_2^2)
\]
\[
+ C\varepsilon^2 \|\nabla \tilde{b}\|_2^2 \|\Delta \tilde{b}\|_2^2 \|\nabla B_3\|_2^2 + C\varepsilon^4 \|\nabla b_3\|_2^2 \|\Delta b_3\|_2^2 \|\nabla \tilde{B}\|_2^2
\]
\[
\leq \frac{1}{10} \varepsilon^2 \|\Delta U_3\|_2^2 + \frac{1}{10} \|\Delta \tilde{B}\|_2^2 + \frac{1}{10} \varepsilon^2 \|\Delta B_3\|_2^2
\]
+ \varepsilon^2 \| \Delta \tilde{b} \|_2^2 \left( \| \nabla \tilde{b} \|_2^2 + \| \nabla \tilde{u} \|_2^2 \right),

where in the last step, we have used the fact that

\| \nabla b_3 \|_2 \leq C \| \Delta \tilde{b} \|_2, \quad \| \Delta b_3 \|_2 \leq C \| \nabla \Delta \tilde{b} \|_2.

For \( J_5 \), we have

\[
J_5 = \varepsilon^2 \int_\Omega (\partial_t u_3 + u \cdot \nabla u_3 - \Delta u_3 - b \cdot \nabla b_3) \Delta U_3 \, dx \, dy \, dz
\]

\[
\leq \varepsilon^2 (\| \partial_t u_3 \|_2 + \| \Delta u_3 \|_2) \| \Delta U_3 \|_2 + \varepsilon^2 \| \nabla \tilde{u} \|_2^2 \| \Delta \tilde{u} \|_2^2 + \| \nabla u_3 \|_2^2 \| \Delta u_3 \|_2^2 + \| \nabla b_3 \|_2^2 \| \Delta b_3 \|_2^2)
\]

By the Hölder and Young inequalities, we deduce

\[
J_6 = \int_\Omega \left[ (U \cdot \nabla) \tilde{B} + (u \cdot \nabla) \tilde{B} + (U \cdot \nabla) \tilde{b} \right] \cdot \Delta \tilde{B} \, dx \, dy \, dz
\]

\[
\leq C \left( \| \nabla \tilde{U} \|_2^\frac{3}{2} \| \nabla \tilde{B} \|_2^\frac{1}{2} \| \Delta \tilde{B} \|_2^\frac{1}{2} + \| \nabla \tilde{u} \|_2^\frac{3}{2} \| \Delta \tilde{u} \|_2^\frac{1}{2} \| \nabla \tilde{B} \|_2^\frac{1}{2} \| \Delta \tilde{B} \|_2^\frac{1}{2} \right)
\]

\[
\leq \frac{1}{10} \| \Delta \tilde{B} \|_2^2 + C \left( \| \nabla \tilde{U} \|_2^\frac{3}{2} \| \Delta \tilde{U} \|_2^\frac{1}{2} + \| \nabla \tilde{B} \|_2^\frac{3}{2} \| \Delta \tilde{B} \|_2^\frac{1}{2} \right)
\]

\[
\leq \frac{1}{10} \| \Delta \tilde{B} \|_2^2 + \frac{1}{15} \| \Delta \tilde{U} \|_2^2 + C \left( \| \nabla \tilde{U} \|_2^\frac{3}{2} \| \Delta \tilde{U} \|_2^\frac{1}{2} + \| \nabla \tilde{B} \|_2^\frac{3}{2} \| \Delta \tilde{B} \|_2^\frac{1}{2} \right)
\]

We can estimate \( J_7 \) as follows

\[
J_7 = \int_\Omega \left[ (B \cdot \nabla) \tilde{U} + (b \cdot \nabla) \tilde{U} + (B \cdot \nabla) \tilde{u} \right] \cdot \Delta \tilde{B} \, dx \, dy \, dz
\]

\[
= \int_\Omega \left( \| \nabla \tilde{B} \|_2^\frac{1}{2} \| \Delta \tilde{B} \|_2^\frac{1}{2} \| \nabla \tilde{U} \|_2^\frac{1}{2} \| \Delta \tilde{U} \|_2^\frac{1}{2} \right)
\]

\[
\leq \frac{1}{10} \| \Delta \tilde{B} \|_2^2 + \frac{1}{15} \| \Delta \tilde{U} \|_2^2 + C \left( \| \nabla \tilde{B} \|_2^\frac{3}{2} \| \Delta \tilde{B} \|_2^\frac{1}{2} + \| \nabla \tilde{U} \|_2^\frac{3}{2} \| \Delta \tilde{U} \|_2^\frac{1}{2} \right)
\]

\[
\leq \frac{1}{10} \| \Delta \tilde{B} \|_2^2 + \frac{1}{15} \| \Delta \tilde{U} \|_2^2 + C \left( \| \nabla \tilde{B} \|_2^\frac{3}{2} \| \Delta \tilde{B} \|_2^\frac{1}{2} + \| \nabla \tilde{U} \|_2^\frac{3}{2} \| \Delta \tilde{U} \|_2^\frac{1}{2} \right)
\]
Applying the Lemma 2.3 and Young inequalities once again, one has

\[
J_8 = \varepsilon^2 \int_{\Omega} \left[ (U \cdot \nabla) B_3 + (u \cdot \nabla) b_3 \right] \Delta B_3 \, dx \, dy \, dz
\]

\[
\leq C \varepsilon^2 \left[ \left( \| \nabla U \|_2^\frac{3}{2} \| \Delta U \|_2^\frac{1}{2} + \| \nabla \tilde{u} \|_2^\frac{1}{2} \| \Delta \tilde{u} \|_2^\frac{1}{2} \right) \| \nabla b_3 \|_2^\frac{3}{2} \| \Delta b_3 \|_2^\frac{1}{2} \right] \| \Delta B_3 \|_2^\frac{1}{2}
\]

\[
+ \| \nabla \tilde{U} \|_2^\frac{1}{2} \| \Delta \tilde{U} \|_2^\frac{1}{2} \| \nabla b_3 \|_2^\frac{3}{2} \| \Delta b_3 \|_2^\frac{1}{2} \| \Delta B_3 \|_2
\]

\[
\leq \frac{1}{10} \varepsilon^2 \| \Delta B_3 \|_2^2 + \frac{1}{15} \| \Delta \tilde{U} \|_2^2 + C \left( \| \nabla U \|_2^2 + \varepsilon^2 \| \nabla b_3 \|_2^2 \right)
\]

\[
\times \left( \| \Delta U \|_2^2 + \varepsilon^2 \| \Delta B_3 \|_2^2 + \| \Delta \tilde{u} \|_2^2 \| \Delta \tilde{u} \|_2^2 + \varepsilon^4 \| \Delta \tilde{b} \|_2^2 \| \Delta \tilde{b} \|_2^2 \right),
\]

moreover

\[
J_9 = \varepsilon^2 \int_{\Omega} \left[ (B \cdot \nabla) U_3 + b \cdot \nabla U_3 + B \cdot \nabla u_3 \right] \Delta B_3 \, dx \, dy \, dz
\]

\[
\leq C \varepsilon^2 \left[ \left( \| \nabla \tilde{B} \|_2^\frac{3}{2} \| \Delta \tilde{B} \|_2^\frac{1}{2} + \| \nabla \tilde{b} \|_2^\frac{3}{2} \| \Delta \tilde{b} \|_2^\frac{1}{2} \right) \| \nabla u_3 \|_2^\frac{1}{2} \| \Delta u_3 \|_2^\frac{1}{2} \right]
\]

\[
+ \| \nabla \tilde{B} \|_2^\frac{1}{2} \| \Delta \tilde{B} \|_2^\frac{1}{2} \| \nabla u_3 \|_2^\frac{1}{2} \| \Delta u_3 \|_2^\frac{1}{2} \| \Delta B_3 \|_2
\]

\[
\leq \frac{1}{10} \varepsilon^2 \| \Delta B_3 \|_2^2 + \frac{1}{10} \varepsilon^2 \| \Delta U_3 \|_2^2 + C \left( \| \nabla \tilde{B} \|_2^2 \| \Delta \tilde{B} \|_2^2 + \varepsilon^4 \| \nabla U_3 \|_2^2 \| \Delta U_3 \|_2^2 \right)
\]

\[
+ \varepsilon^2 \| \nabla \tilde{U} \|_2^\frac{1}{2} \| \Delta \tilde{U} \|_2^\frac{1}{2} \| \nabla U_3 \|_2^\frac{1}{2} \| \Delta U_3 \|_2^\frac{1}{2} + C \varepsilon^4 \| \nabla u_3 \|_2^2 \| \Delta u_3 \|_2^2 \| \nabla \tilde{B} \|_2^2 \right)
\]

\[
\leq \frac{1}{10} \varepsilon^2 \| \Delta B_3 \|_2^2 + \frac{1}{10} \varepsilon^2 \| \Delta U_3 \|_2^2 + C \left( \| \nabla \tilde{B} \|_2^2 \| \Delta \tilde{B} \|_2^2 + \varepsilon^2 \| \nabla U_3 \|_2^2 \right)
\]

\[
\times \left( \| \Delta U \|_2^2 + \varepsilon^2 \| \Delta u_3 \|_2^2 \| \Delta \tilde{u} \|_2^2 + \varepsilon^2 \| \Delta \tilde{b} \|_2^2 \| \Delta \tilde{b} \|_2^2 \right).
\]

For the last term \( J_{10} \), we get that

\[
J_{10} = \varepsilon^2 \int_{\Omega} \left( \partial_t b_3 + u \cdot \nabla b_3 - \Delta b_3 - b \cdot \nabla u_3 \right) \Delta B_3 \, dx \, dy \, dz
\]

\[
\leq \varepsilon^2 \left( \| \partial_t b_3 \|_2 + \| \Delta b_3 \|_2 \| \Delta B_3 \|_2 \right) + C \varepsilon^2 \left( \| \nabla \tilde{u} \|_2^\frac{1}{2} \| \Delta \tilde{u} \|_2^\frac{1}{2} \| \nabla b_3 \|_2^\frac{1}{2} \| \Delta b_3 \|_2^\frac{1}{2} \| \Delta B_3 \|_2^\frac{1}{2} \right)
\]

\[
+ C \varepsilon^2 \| \nabla \tilde{b} \|_2^\frac{1}{2} \| \Delta \tilde{b} \|_2^\frac{1}{2} \| \nabla u_3 \|_2^\frac{1}{2} \| \Delta u_3 \|_2^\frac{1}{2} \| \Delta B_3 \|_2^\frac{1}{2} \right]
\]

\[
\leq \frac{1}{10} \varepsilon^2 \| \Delta B_3 \|_2^2 + C \varepsilon^2 \left( \| \partial_t b_3 \|_2^2 + \| \Delta b_3 \|_2^2 + \| \nabla \tilde{u} \|_2^2 \| \Delta \tilde{u} \|_2^2 \right)
\]

\[
+ \| \nabla b_3 \|_2^2 \| \Delta b_3 \|_2^2 + \| \nabla \tilde{b} \|_2^2 \| \Delta \tilde{b} \|_2^2 + \| \nabla u_3 \|_2^2 \| \Delta u_3 \|_2^2 \right)
\]

\[
\leq \frac{1}{10} \varepsilon^2 \| \Delta B_3 \|_2^2 + C \varepsilon^2 \left( \| \nabla \partial_t \tilde{b} \|_2^2 + \| \nabla \Delta \tilde{b} \|_2^2 + \| \Delta \tilde{u} \|_2^2 \| \Delta \tilde{u} \|_2^2 + \| \Delta \tilde{b} \|_2^2 \| \Delta \tilde{b} \|_2^2 \right).
\]
In view of the estimates of $J_1 - J_{10}$, we derive from (4.2) the differential inequality

$$
\frac{1}{2} \frac{d}{dt} \left( \|\nabla \bar{U}\|^2 + \varepsilon^2 \|\nabla U_3\|^2 + \|\nabla \bar{B}\|^2 + \varepsilon^2 \|\nabla B_3\|^2 \right) + \frac{3}{5} \left( \|\Delta \bar{U}\|^2 + \varepsilon^2 \|\Delta U_3\|^2 + \|\Delta \bar{B}\|^2 + \varepsilon^2 \|\Delta B_3\|^2 \right)
\leq C_1 \left( \|\nabla \bar{U}\|^2 + \varepsilon^2 \|\nabla U_3\|^2 + \|\nabla \bar{B}\|^2 + \varepsilon^2 \|\nabla B_3\|^2 \right)
\left( \|\Delta \bar{u}\|^2 + \|\Delta \tilde{b}\|^2 \right) + \left( 1 + \varepsilon^4 \right) \left( \|\Delta \bar{u}\|^2 + \|\Delta \tilde{b}\|^2 \right) \left( \|\nabla U_3\|^2 + \|\nabla B_3\|^2 \right)
\end{align*}

By the assumption $\sup_{0 \leq s \leq t} \left( \|\nabla \bar{U}\|^2 + \varepsilon^2 \|\nabla U_3\|^2 + \|\nabla \bar{B}\|^2 + \varepsilon^2 \|\nabla B_3\|^2 \right) \leq \delta_0^2$, choosing

$$
\delta_0 = \sqrt{\frac{1}{10C_1}},
$$

it follows from the above inequality that

$$
\frac{d}{dt} \left( \|\nabla \bar{U}\|^2 + \varepsilon^2 \|\nabla U_3\|^2 + \|\nabla \bar{B}\|^2 + \varepsilon^2 \|\nabla B_3\|^2 \right) + \|\Delta \bar{U}\|^2 + \varepsilon^2 \|\Delta U_3\|^2 + \|\Delta \bar{B}\|^2 + \varepsilon^2 \|\Delta B_3\|^2
\leq 2C_1 (1 + \varepsilon^4) \left( \|\Delta \bar{u}\|^2 + \|\Delta \tilde{b}\|^2 \right) \left( \|\nabla U_3\|^2 + \|\nabla B_3\|^2 \right)
\end{align*}

recalling $(\bar{U}, U_3)|_{t=0} = 0$ and $(\bar{B}, B_3)|_{t=0} = 0$, it follows from the Gronwall inequality that

$$
\sup_{0 \leq s \leq t} \left( \|\nabla \bar{U}\|^2 + \varepsilon^2 \|\nabla U_3\|^2 + \|\nabla \bar{B}\|^2 + \varepsilon^2 \|\nabla B_3\|^2 \right)
\leq 2C_1 \varepsilon^2 e^{2C_1(1 + \varepsilon^4) \int_0^t \left( \|\Delta \bar{u}\|^2 + \|\Delta \tilde{b}\|^2 \right) ds}
\end{align*}

proving the conclusion. \hfill \Box

Thanks to Propositions 4.5-4.6 and Proposition 1.3, we can prove the following results.

**Proposition 4.7.** There is a positive constant $\varepsilon_0$ that only depends on $\|\bar{u}_0\|_{H^2}$, $\|\tilde{b}_0\|_{H^2}$, $L_1$ and $L_2$, so that for any $\varepsilon \in (0, \varepsilon_0)$, there is a unique global strong solution $(u_\varepsilon, b_\varepsilon)$ to the SMHD, subject to (1.3)-(1.11). In addition, we have the following estimate

\begin{align*}
\sup_{0 \leq s \leq \infty} \left( \|\bar{U}\|^2 + \varepsilon^2 \|U_3\|^2 + \|\bar{B}\|^2 + \varepsilon^2 \|B_3\|^2 \right)
\leq C \varepsilon^2,
\end{align*}
where $C$ is a positive constant depending only on $\|\bar{u}_0\|_{H^2}, \|\bar{b}_0\|_{H^2}, L_1$ and $L_2$.

**Proof.** Denote $T^*_\varepsilon$ be the maximal existence time of the strong solutions $(\bar{u}_\varepsilon, u_{3,\varepsilon}, \bar{b}_\varepsilon, b_{3,\varepsilon})$ to the SMHD, subject to the conditions (1.3)-(1.11). According to Proposition 1.2 and Proposition 4.5, we get the estimate

$$
\sup_{0 \leq s \leq T^*_\varepsilon} (\|\bar{U}\|_2^2 + \varepsilon^2 \|U_3\|_2^2 + \|\bar{B}\|_2^2 + \varepsilon^2 \|B_3\|_2^2)
+ \int_0^{T^*_\varepsilon} (\|\nabla \bar{U}\|_2^2 + \varepsilon^2 \|\nabla U_3\|_2^2 + \|\nabla \bar{B}\|_2^2 + \varepsilon^2 \|\nabla B_3\|_2^2) \, ds
\leq R_1 \varepsilon^2,
$$

(4.3)

where $R_1$ is a positive constant depending only on $\|\bar{u}_0\|_{H^1}, \|\bar{b}_0\|_{H^1}, L_1$ and $L_2$. Let $\delta_0$ be a constant depending only on $L_1$ and $L_2$ in Proposition 4.6. We define

$$
t^*_\varepsilon := \sup \left\{ t \in (0, T^*_\varepsilon) \mid \sup_{0 \leq s \leq t} (\|\nabla \bar{U}\|_2^2 + \|\nabla \bar{B}\|_2^2 + \varepsilon^2 \|\nabla U_3\|_2^2 + \|\nabla B_3\|_2^2) \leq \delta_0^2 \right\}.
$$

On the basis of Proposition 4.6 and Proposition 1.3, for any $t \in [0, t^*_\varepsilon)$, the following estimate holds

$$
\sup_{0 \leq s \leq t} (\|\nabla \bar{U}\|_2^2 + \|\nabla \bar{B}\|_2^2 + \varepsilon^2 \|\nabla U_3\|_2^2 + \|\nabla B_3\|_2^2)
+ \int_0^t (\|\Delta \bar{U}\|_2^2 + \varepsilon^2 \|\Delta U_3\|_2^2 + \|\Delta \bar{B}\|_2^2 + \varepsilon^2 \|\Delta B_3\|_2^2) \, ds
\leq R_2 \varepsilon^2,
$$

(4.4)

where $R_2$ is a positive constant depending only on $\|\bar{u}_0\|_{H^2}, \|\bar{b}_0\|_{H^2}, L_1$ and $L_2$. Setting $\varepsilon_0 = \delta_0 \sqrt{\frac{1}{2R_2}}$, for any $\varepsilon \in (0, \varepsilon_0)$, so the inequality above implies that

$$
\sup_{0 \leq s \leq t} (\|\nabla \bar{U}\|_2^2 + \|\nabla \bar{B}\|_2^2 + \varepsilon^2 \|\nabla U_3\|_2^2 + \|\nabla B_3\|_2^2)
+ \int_0^t (\|\Delta \bar{U}\|_2^2 + \varepsilon^2 \|\Delta U_3\|_2^2 + \|\Delta \bar{B}\|_2^2 + \varepsilon^2 \|\Delta B_3\|_2^2) \, ds
\leq \frac{\delta_0^2}{2},
$$

and for any $t_0 \in [0, t^*_\varepsilon)$, especially when it gives

$$
\sup_{0 \leq s < t_0^*_\varepsilon} (\|\nabla \bar{U}\|_2^2 + \|\nabla \bar{B}\|_2^2 + \varepsilon^2 \|\nabla U_3\|_2^2 + \|\nabla B_3\|_2^2) \leq \frac{\delta_0^2}{2}.
$$

So that, according to the definition of $t^*_\varepsilon$, we must have $t^*_\varepsilon = T^*_\varepsilon$. On account of this, it is obvious that (4.4) holds for any $t \in [0, T^*_\varepsilon)$.

We assert that it must have $T^*_\varepsilon = \infty$. Assume in contradiction that $T^*_\varepsilon < \infty$, then, recalling that (4.4) is true for any $t \in [0, T^*_\varepsilon)$, by the local well-posedness
result of the SMHD, we can extend the strong solution \((u_\varepsilon, b_\varepsilon)\) beyond \(T_\varepsilon\), which contradicts to the definition of \(T_\varepsilon\). Therefore, combining (4.3) and (4.4) to draw the conclusion. \(\square\)

On account of Proposition 4.7, one can give the proof of Theorem 4.1 in the following.

**Proof of Theorem 4.1.** Let \(\varepsilon_0\) depend only on \(\|	ilde{u}_0\|_{H^2}, \|	ilde{b}_0\|_{H^2}, L_1, L_2\) in Proposition 4.7. According to Proposition 4.7, for any \(\varepsilon \in (0, \varepsilon_0)\), there is a unique global strong solution \((u_\varepsilon, b_\varepsilon)\) to the SMHD, subject to the conditions (1.3)-(1.11). In addition, we have the following estimate

\[
\sup_{0 \leq s < \infty} (\|	ilde{U}\|_{H^1}^2 + \varepsilon^2 \|U_3\|_{H^1}^2 + \|	ilde{B}\|_{H^1}^2 + \varepsilon^2 \|B_3\|_{H^1}^2)(t)
+ \int_0^\infty (\|
abla \tilde{U}\|_{H^1}^2 + \varepsilon^2 \|
abla U_3\|_{H^1}^2 + \|
abla \tilde{B}\|_{H^1}^2 + \varepsilon^2 \|
abla B_3\|_{H^1}^2)
\leq C(\|	ilde{u}_0\|_{H^2}, \|	ilde{b}_0\|_{H^2}, L_1, L_2)\varepsilon^2,
\]

where \((\tilde{U}, U_3) = (\tilde{u}_\varepsilon, u_{3,\varepsilon}) - (\tilde{u}, u_3)\) and \((\tilde{B}, B_3) = (\tilde{b}_\varepsilon, b_{3,\varepsilon}) - (\tilde{b}, b_3)\). This proves the estimate stated in the theorem, of which the strong convergence is only a direct deduction of this estimate. This completes the proof of Theorem 4.1. \(\square\)

**Acknowledgments**

This work is supported by the National Natural Science Foundation of China grant 11971331, 12125102, and Sichuan Youth Science and Technology Foundation 2021 JDTD0024.

**Declarations**

**Competing interests** The authors have no relevant financial or non-financial interests to disclose.

**Funding** The work of L. Du and D. Li is funded by the National Natural Science Foundation of China (grant 11971331, 12125102), and Sichuan Youth Science and Technology Foundation (2021 JDTD0024).

**Data availability statement** Our manuscript has no associated data.

**References**

[1] R. Adams, *Sobolev Space*, Academic Press, New York, 1975.
[2] H. Alfvén, Existence of electromagnetic-hydrodynamic waves, *Nature*, **150** (1942) 405-406.
[3] P. Azérad, F. Guillén-Gonzélez, Mathematical justification of the hydrostatic approximation in the primitive equations of geophysical fluid dynamics, *SIAM J. Math. Annl.*, 33 (2001) 847-859.
[4] C. Bardos, M. Filho, D. Niu, H. Lopes, E. Titi, Stability of two-dimensional viscous incompressible flows under three-dimensional perturbations and inviscid symmetry breaking, *SIAM J. Math. Anal.*, 45 (2013) 1871-1885.
[5] D. Biskamp, Nonlinear Magnetohydrodynamics, Cambridge University Press, 1993.
[6] D. Bresch, A. Kazhikhov, J. Lemoine, On the two-dimensional hydrostatic Navier-Stokes equations, *SIAM J. Math. Anal.*, 36 (2004/05) 796-814.
[7] H. Cabannes, Theoretical Magnetohydrodynamics, Academic Press, New York, London, 1970.
[8] C. Cao, E. Titi, Global well-posedness and finite-dimensional global attractor for a 3D planetary geostrophic viscous model, *Commun. Pure Appl. Math.*, 56 (2003) 198-233.
[9] C. Cao, E. Titi, Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics, *Ann. Math.*, 166 (2007) 245-267.
[10] C. Cao, J. Wu, Two regularity criteria for the 3D MHD equations, *J. Differential Equations.*, 248 (2010) 2263-2274.
[11] C. Cao, E. Titi, Global well-posedness of the 3D primitive equations with partial vertical turbulence mixing heat diffusion, *Commun. Math. Phys.*, 310 (2) (2012) 537-568.
[12] C. Cao, J. Li, E. Titi, Local and global well-posedness of strong solutions to the 3D primitive equations with vertical eddy diffusivity, *Arch. Ration. Mech. Anal.*, 214 (2014) 35-76.
[13] C. Cao, J. Li, E. Titi, Global well-posedness of strong solutions to the 3D primitive equations with horizontal eddy diffusivity, *J. Differential Equations.*, 257 (2014) 4108-4132.
[14] C. Cao, J. Li, E. Titi, Global well-posedness of the three-dimensional primitive equations with only horizontal viscosity and diffusion, *Comm. Pure Appl. Math.*, 69 (2016) 1492-1531.
[15] C. Cao, J. Li, E. Titi, Strong solutions to the 3D primitive equations with only horizontal dissipation: near $H^1$ initial data, *J. Funct. Anal.*, 272 (2017) 4606-4641.
[16] C. Cao, J. Li, E. Titi, Global well-posedness of the 3D primitive equations with horizontal viscosity and vertical diffusivity, *Phys. D.*, 412 (2020), 132606, 25 pp.
[17] Q. Chen, C. Miao, Z. Zhang, On the regularity criterion of weak solution for the 3D viscous magnetohydrodynamics equations, *Comm. Math. Phys.*, 284 (2008) 919-930.
[18] P. Constantin, C. Foias, *Navier-Stokes Equations*, The University of Chicago Press, Chicago, IL, 1988.
[19] P. Davidson, An Introduction to Magnetohydrodynamics, Cambridge University Press, Cambridge, England, 2001.
[20] L. Du, D. Li, Global well-posedness of the 3D primitive equations with magnetic field, arXiv: 2206. 06005.
[21] B. Ducomet, E. Feireisl, The equations of magnetohydrodynamics: on the interaction between matter and radiation in the evolution of gaseous stars, *Commun. Math. Phys.*, 226 (2006) 595-629.
[22] G. Duvaut, J. Lions, Inequations en thermoelasticiete et magnetohydrodynamique, *Arch. Rational. Mech. Anal.*, 46 (1972) 241-279.
[23] K. Furukawa, Y. Giga, M. Hieber, A. Hussein, T. Kashiwabara, M. Wrona, Rigorous justification of the hydrostatic approximation for the primitive equations by scaled Navier-Stokes equations, *Nonlinearity*, 33 (2020) 6502-6516.
K. Furukawa, Y. Giga, T. Kashiwabara, The hydrostatic approximation for the primitive equations by the scaled Navier-Stokes equations under the non-slip boundary condition, *J. Evol. Equ.*, 21 (2021) 3331-3373.

A. Hasegawa, Self-organization processed in continuous media, *Adv. Phys.*, 34 (1985) 1-42.

C. He, Z. Xin, Partial regularity of suitable weak solutions to the incompressible magnetohydrodynamic equations, *J. Funct. Anal.*, 227 (2005) 113-152.

C. He, Z. Xin, On the regularity of weak solutions to the magnetohydrodynamic equations, *J. Differential Equations*, 213 (2005) 235-254.

M. Hieber, T. Kashiwabara, Global strong well-posedness of the three dimensional primitive equations in $L^p$-spaces, *Arch. Ration. Mech. Anal.*, 221 (2016) 1077-1115.

M. Hieber, A. Hussein, T. Kashiwabara, Global strong $L^p$ well-posedness of the 3D primitive equations with heat and salinity diffusion, *J. Differential Equations*, 261 (2016) 6950-6981.

G. Kobelkov, Existence of a solution in the large for the 3D large-scale ocean dynamics equations, *C. R. Math. Acad. Sci. Paris*, 343 (2006) 283-286.

I. Kukavica, M. Ziane, On the regularity of the primitive equations of the ocean, *Nonlinearity*, 20 (2007) 2739-2753.

O. Ladyzhenskaya, *The Boundary Value Problems of Mathematical Physics.*, Springer-Verlag, New York, 1985.

J. Li, E. Titi, The primitive equations as the small aspect ratio limit of the Navier-Stokes equations: rigorous justification of the hydrostatic approximation, *J. Math. Pures Appl.*, 124 (2019) 30-58.

J. Li, E. Titi, G. Yuan, The primitive equations approximation of the anisotropic horizontally viscous 3D Navier-Stokes equations, *J. Differential Equations*, 306 (2022) 492-524.

H. Politano, A. Pouquet, P. Sulem, Current and vorticity dynamics in three dimensional magnetohydrodynamic turbulence, *Phys. Plasmas.*, 2 (1995) 2931-2939.

J. Serrin, The initial value problem for the Navier-Stokes equations, in: R. E. Langer (Ed.), Nonlinear Problems, University of Wisconsin Press, Madison, 1963, pp. 69-98.

R. Temam, Navier-Stokes Equations: Theory and Numerical Analysis, revised edition, Studies in Mathematics and its Applications, vol. 2, North-Holland Publishing Co., Amsterdam-New York, 1979.

R. Temam, M. Ziane, Some mathematical problems in geophysical fluid dynamics, *Handbook of mathematical fluid dynamics*, Vol. III, 535-657, North-Holland, Amsterdam, 2004.

J. Wu, Bounds and new approaches for the 3D MHD equations, *J. Nonlinear Sci.*, 12 (2002) 395-413.

J. Wu, Regularity results for weak solutions of the 3D MHD equations, *Discrete Contin. Dyn. Syst.*, 10 (2004) 543-556.

J. Wu, Regularity criteria for the generalized MHD equations, *Commun. Partial Differ. Equ.*, 33 (2008) 285-306.

J. Wu, Global regularity for a class of generalized magnetohydrodynamic equations, *J. Math. Fluid Mech.*, 13 (2011) 295-305.

J. Wu, Y. Zhu, Global solutions of 3D incompressible MHD system with mixed partial dissipation and magnetic diffusion near an equilibrium, *Adv. Math.*, 377 (2021) 107466.

W. Yang, Q. Jiu, J. Wu, The 3D incompressible magnetohydrodynamic equations with fractional partial dissipation, *J. Differential Equations*, 266 (2019) 630-652.
[45] K. Yamazaki, Regularity criteria of MHD systems involving one velocity and one current density component, *J. Math. Fluid Mech.*, 16 (2014) 551-570.

1 Department of Mathematics, Sichuan University, Chengdu 610064, P.R. China
*Email address:* dulili@scu.edu.cn

2 Department of Mathematics, Sichuan University, Chengdu 610064, P.R. China
*Email address:* dandy0219@hotmail.com