KINETICALLY CONSTRAINED SPIN MODELS ON TREES

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Abstract.
We analyze kinetically constrained 0-1 spin models (KCSM) on rooted and unrooted trees of finite connectivity. We focus in particular on the class of Friedrichson Andersen models FA-jf and on an oriented version of them. These tree models are particularly relevant in physics literature since some of them undergo an ergodicity breaking transition with the mixed first-second order character of the glass transition. Here we first identify the ergodicity regime and prove that the critical density for FA-jf and OFA-jf models coincide with that of a suitable bootstrap percolation model. Next we prove for the first time positivity of the spectral gap in the whole ergodic regime via a novel argument based on martingales ideas. Finally, we discuss how this new technique can be generalized to analyse KCSM on the regular lattice $\mathbb{Z}^d$.

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1. Introduction

Facilitated or kinetically constrained spin models (KCSM) are interacting particle systems which have been introduced in physics literature [5] [6] to model liquid/glass transition and more generally “glassy dynamics” [17, 11]. They are defined on a locally finite, bounded degree, connected graph $G = (V, E)$ with vertex set $V$ and edge set $E$. Here we will focus on models for which the graph is an infinite rooted or unrooted tree of finite connectivity $k + 1$, which we will denote by $\bar{T}_k$ and $T_k$ respectively. A configuration is given by assigning to each site $x \in V$ its occupation variable $\eta_x \in \{0, 1\}$ which corresponds to an empty or filled site, respectively. The evolution is given by a Markovian stochastic dynamics of Glauber type. Each site waits an independent, mean one, exponential time and then, provided the current configuration around it satisfies an a priori specified constraint, its occupation variable is refreshed to an occupied or to an empty state with probability $p$ or $1 - p$, respectively. For each site $x$ the corresponding constraint does not involve $\eta_x$, thus detailed balance w.r.t. Bernoulli$(p)$ product measure $\mu$ can be easily verified and the latter is an invariant reversible measure for the process.

Among the most studied KCSM we recall FA-jf models [5] for which the constraint requires at least $j$ (which is sometimes called “facilitating parameter”) empty sites among the nearest neighbors. FA-jf models display a feature which is common to all KCSM introduced in physics literature: for each vertex $x$ the constraint impose a maximal number of occupied sites in a proper neighborhood of $x$ in order to allow the moves.

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As a consequence the dynamics becomes slower at higher density and an ergodicity breaking transition may occur at a finite critical density \( p_c < 1 \). This threshold corresponds to the lowest density at which a site belongs with finite probability to a cluster of particles which are mutually and forever blocked due to the constraints (see Section 3).

The FA-jf models on \( \mathbb{Z}^d \) do not display an ergodicity breaking transition at a non trivial critical density, that is \( p_c = 0 \) for \( j > d \) and \( p_c = 1 \) otherwise [3]. On the other hand they do display such a transition on non rooted trees when \( 1 < j < k \)?[7, 10, 14, 9]. Furthermore if \( j \neq k - 1 \) this transition is expected to display a mixed first/second character and to share similar features to the mode coupling transition, a property which makes them particularly interesting from the point of view of the glass transition [14].

Another key feature of KCSM is the existence of blocked configurations, namely configurations with all creation/destruction rates identically equal to zero. This implies the existence of several invariant measures and the occurrence of unusually long mixing times compared to high-temperature Ising models (see Section 7.1 of [3]). Furthermore the constrained dynamics is usually not attractive so that monotonicity arguments valid for e.g. ferromagnetic stochastic Ising models cannot be applied.

Due to the above properties the basic issues concerning the large time behavior of the process are non-trivial. The first rigorous results were derived in [1] for the East model which is defined on \( \mathbb{Z} \) with the constraint requiring the nearest neighbor site to the right to be empty. In [1] it was proven that the spectral gap of East is positive for all \( p < 1 \) and also that it shrinks faster than any polynomial in \( (1 - p) \) as \( p \uparrow 1 \). In [3] positivity of the spectral gap of KCSM inside the ergodicity region (i.e. for \( p < p_c \)) has been proved in much greater generality and (sometimes sharp) bounds for \( p \nearrow p_c \) were established. These results include FA-jf models on any \( \mathbb{Z}^d \) for any choice of the facilitating parameter \( j \) and of the spatial dimension \( d \).

The technique developed in [3] cannot be applied to models on trees because of the exponential growth of the number of vertices and, so far, very few rigorous results have been established. Indeed the only models for which results on the spectral gap are available are: (i) the FA-1f model on \( T^k \) and \( \bar{T}^k \) (actually on a generic connected graph) and (ii) the so-called East model on \( \bar{T}^k \) for which the root is unconstrained while, for any other vertex \( x \), the constraint requires the ancestor of \( x \) to be empty. For these specific models \( p_c = 1 \) and the positivity of the spectral gap has been proven in [2] in the whole ergodicity region and for any choice of the graph connectivity.

Here we will study FA-jf models on \( T^k \) and \( \bar{T}^k \) for \( 1 < j \leq k \) together with a new class of models that we call oriented FA-jf models (OFA-jf). In the OFA-jf model the constraint at \( x \) requires at least \( j \) empty sites among the children of \( x \).

We first prove that the ergodicity treshold \( p_c \) for the FA-jf and OFA-jf models, with the same choice for the parameter \( j \) and the same graph connectivity \( k + 1 \), coincide and it is non trivial (see Theorem 2.3). Then we prove positivity of the spectral gap in the whole ergodicity regime for the oriented OFA-jf models. Finally, by combining the above results together with an appropriate comparison technique, we establish positivity of the spectral gap in the whole ergodicity regime for the FA-jf models. The results concerning the spectral gap can be found in Theorem 2.5. Finally, in the non ergodic
regime, we prove that, for the oriented or non-oriented FA-jf models, the spectral gap shrinks to zero exponentially fast in the system size (see Theorem 2.9).

The new technique devised to study constrained models on trees can be generalized to deal also with KCSM on other graphs. In Section 5 we discuss how one can recover the result of positivity of the spectral gap in the ergodic regime for models on $\mathbb{Z}^d$. We detail in particular the case of the North-East model on $\mathbb{Z}^2$ (Theorem 5.3), a result which was already derived in [3] but with a completely different (and more lengthy) technique.

2. Models and main results

The models we consider are either defined on the infinite regular tree of connectivity $k + 1$, in the sequel denoted by $T^k$, or on the infinite rooted $k$-ary tree $\bar{T}^k$.

In the unrooted case each vertex $x$ has $k + 1$ neighbours, while in the rooted case each vertex different from the root has $k$ children and one ancestor and the root $r$ has only $k$ children. In the sequel we will denote by $V$ the set of vertices of either $T^k$ or of $\bar{T}^k$ whenever no confusion arises, by $N_x$ the set of neighbors of a given vertex $x$ and, in the rooted case, by $K_x$ the set of its children. In the rooted case we denote by $d_x$ the depth of the vertex $x$, i.e. the graph distance between $x$ and the root $r$.

For both oriented and non-oriented models we choose as configuration space the set $\Omega = \{0, 1\}^V$ whose elements will usually be assigned Greek letters. We will often write $\eta_x$ for the value at $x$ of the element $\eta \in \Omega$. With a slight abuse of notation, for any $A \subset V$ and any $\eta, \omega \in \Omega$, we also let $\eta_A$ be the restriction of $\eta$ to $A$ and for any $A, B \subset V$ such that $A \cap B = \emptyset$ we let $\eta_A \cdot \omega_B$ be the configuration which equals $\eta$ on $A$ and $\omega$ on $B$.

Fix now $k \in \mathbb{Z}_+$, a density $p \in [0, 1]$ and a facilitating parameter $j \in [1, \ldots, k]$. Let also $\mu$ be the Bernoulli product measure on $\Omega$ with density $p$.

**Definition 2.1.** The FA-jf and OFA-jf models at density $p$ are continuous time Glauber type Markov processes on $\Omega$, reversible w.r.t. $\mu$, with Markov semigroups $P_t = e^{tL}$ and $\bar{P}_t = e^{t\bar{L}}$ respectively, whose infinitesimal generators $L, \bar{L}$ act on local functions $f : \Omega \mapsto \mathbb{R}$ as follows:

$$L f(\omega) = \sum_{x \in T^k} c_x(\omega) [\mu_x(f) - f(\omega)]$$

$$\bar{L} f(\omega) = \sum_{x \in \bar{T}^k} \bar{c}_x(\omega) [\mu_x(f) - f(\omega)].$$

The functions $c_x, \bar{c}_x$, in the sequel referred to as the constraint at $x$, are defined by

$$c_x(\omega) = \begin{cases} 1 & \text{if } \sum_{y \in N_x} (1 - \omega_y) \geq j \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{c}_x(\omega) = \begin{cases} 1 & \text{if } \sum_{y \in K_x} (1 - \omega_y) \geq j \\ 0 & \text{otherwise} \end{cases}$$

and $\mu_x(f)$ is simply the conditional expectation of $f$ given $\{\omega_y\}_{y \neq x}$.
It is easy to check by standard methods (see e.g. [12]) that the processes are well defined and their generators can be extended to non-positive self-adjoint operators on \( L^2(\mathbb{T}^k, \mu) \) and \( L^2(\bar{T}^k, \mu) \) respectively.

**Remark 2.2.** Both processes can of course be defined also on finite regular trees, rooted or unrooted. In that case however, in order to get an irreducible processes (actually irreducible continuous time Markov chains), the constraints at the leaves must be modified and set to be identically equal to one. In other words the dynamics on the leaves is unconstrained.

### 2.1. Ergodicity

Given \( k \) and \( j \leq k \), it is natural to define (see [3]) a critical density for each model as follows:

\[
p_c = \sup\{ p \in [0, 1] : 0 \text{ is simple eigenvalue of } \mathcal{L} \} \quad (2.5)
\]
\[
\bar{p}_c = \sup\{ p \in [0, 1] : 0 \text{ is simple eigenvalue of } \bar{\mathcal{L}} \} \quad (2.6)
\]

The regime \( p < p_c \) or \( p < \bar{p}_c \) is called the **ergodic region** and we say that an **ergodicity breaking transition** occurs at the critical density. We will first establish the coincidence of the critical threshold for oriented and unoriented models.

**Theorem 2.3.** Let \( g_p(x) := p \sum_{i=k-j+1}^{k} \binom{k}{i} x^i (1-x)^{k-i} \) and let

\[
\tilde{p} := \sup\{ p \in [0, 1] : x = 0 \text{ is the unique fixed point of } g_p(x) \}.
\]

Then \( \tilde{p} < 1 \), \( p_c = \tilde{p}_c = \tilde{p} \) and for any \( p < \tilde{p} \) the value 0 is a simple eigenvalue of the generators \( \mathcal{L} \) and \( \bar{\mathcal{L}} \).

**Remark 2.4.** In the unoriented case the above result has been established in the physics literature in [14]. Thanks to Proposition 3.1 below, the proof of the above result reduces to show that the critical thresholds of two different bootstrap percolation problems coincide. In the unoriented case the critical threshold has been identified in [10].

We then turn to the study of the relaxation to equilibrium in \( L^2(\mu) \) in the ergodic region \( p < p_c \) or \( p < \bar{p}_c \). A key object here is the spectral gap (or inverse of the relaxation time) of the generator \( \mathcal{L} \) (or \( \bar{\mathcal{L}} \)), defined as

\[
gap(\mathcal{L}) := \inf_{f \in \text{Dom}(\mathcal{L}) \setminus \{\text{const}\}} \frac{\mathcal{D}(f)}{\text{Var}(f)} \quad (2.7)
\]

where the Dirichlet form \( \mathcal{D}(f) \) is the quadratic form \( \mathcal{D}(f) = \mu(f, -\mathcal{L} f) \) associated to \( -\mathcal{L} \) and \( \text{Var}(f) \) is the variance of \( f \) w.r.t. \( \mu \). Indeed a positive spectral gap implies that the reversible measure \( \mu \) is mixing for the semigroup \( P_t \) with exponentially decaying correlations,

\[
\int d\mu(\eta) [P_t f(\eta) - \mu(f)]^2 \leq \exp(-2t \gap(\mathcal{L})) \text{Var}(f)
\]

for any \( f \in L^2(\mu) \). The main result of our work is the positivity of the spectral gap in the whole ergodicity region, namely

**Theorem 2.5.** Given \( k, j \in \mathbb{Z}_+ \) with \( j \leq k \), fix \( p < p_c = \bar{p}_c \). Then \( \gap(\mathcal{L}) > 0 \) and \( \gap(\bar{\mathcal{L}}) > 0 \).
Remark 2.6. Exactly as in [3] (see Proposition 2.13 there), in order to prove positivity of the spectral gap for the infinite trees $\mathbb{T}^k$ or $\overline{\mathbb{T}}^k$, it is enough to prove a lower bound on the spectral gap on finite regular subtrees with unconstrained leaves which is uniform in the size of the tree.

Remark 2.7. It is important to observe that, in the oriented case, the above result completes the proof of the exponential decay to equilibrium when $p < p_c$ and the initial distribution is either a Bernoulli product measure with density $p' \neq p$, $p' < p_c$, or it is a $\delta$-measure on a deterministic configuration which does not contain blocked clusters. These results were indeed proven in [16] (see Theorems 4.2 and 4.3) modulo the hypothesis of positivity of the spectral gap in the ergodic region.

It is natural to ask about the behavior of the models outside the ergodic region, i.e. when $p \geq p_c$. Let us first discuss the super-critical case $p > p_c$.

Proposition 2.8. Given $k, j \in \mathbb{N}$ with $j \leq k$, fix $p > p_c$. Then
\[
\lim_{t \to \infty} \int d\mu(\eta)[P_t \eta_r - p]^2 > 0.
\]
and the same inequality holds with $P_t$ instead of $\overline{P}_t$.

In order to state the second result we need few extra notation. Given $L \in \mathbb{Z}_+$ denote by $\mathcal{L}_L$ ($\overline{\mathcal{L}}_L$) the generator of the FA-jf (OFA-jf) model on a subtree of $\mathbb{T}^k$ ($\overline{\mathbb{T}}^k$) of depth $L$ centered (rooted) at $r$ with all the leaves (i.e. vertices with distance $L$ from $r$) unconstrained.

Theorem 2.9. Given $k, j \in \mathbb{N}$ with $j \leq k$, fix $p > p_c$. Then there exists $c, \overline{c} > 0$ such that
\[
e^{-cL} \leq \text{gap}(\mathcal{L}_L) \leq e^{-L/c}
\]
\[
e^{-\overline{c}L} \leq \text{gap}(\overline{\mathcal{L}}_L) \leq e^{-L/\overline{c}}
\]
The critical case $p = p_c$ is much more delicate and a detailed analysis is postponed to future work [15]. We anticipate here that it is possible to show that the spectral gap of a tree of depth $L$ shrinks at least polynomially fast in $L^{-1}$. In the rooted case with $j = k$ one can also proves a converse poly$(1/L)$ lower bound (a much harder task). If $2 \leq j < k$ the analysis becomes much more difficult because of the discontinuous character of the bootstrap percolation transition, i.e. the fact that at $p_c$ with positive probability the root $r$ belongs to an infinite blocked cluster (while this probability is zero if $j = k$).

3. Ergodicity threshold and blocked clusters: proof of Theorem 2.3

In [3] for KCSM on $\mathbb{Z}^d$ we have identified the critical density defined in the previous section with the percolation thresholds for a proper deterministic map, called the bootstrap map. This result can be proven along the same lines for the tree models under consideration.
Fix $k$ and $j \leq k$. For FA-jf models we define the bootstrap map $B : \{0, 1\}^{\mathbb{Z}^k} \to \{0, 1\}^{\mathbb{Z}^k}$ by
\[
B(\eta)_x = 0 \quad \text{if either} \quad \eta_x = 0 \quad \text{or} \quad c_x(\eta) = 1
\]
with $c_x$ defined in (2.3). Analogously we define the map $B$ for OFA-jf models by replacing $c_x$ with $\tilde{c}_x$ of (2.3). Then we denote by $\mu^{(n)}$ the probability measure obtained by iterating $n$-times the map $B$ starting from $\mu$. As $n \to \infty$, $\mu^{(n)}$ converge to a limiting measure $\mu^{(\infty)}$ [18] and it is natural to define bootstrap percolation thresholds $p_{bp}$ as the supremum of the density such that, with probability one, a given vertex $x$ is emptied under the bootstrap procedure,
\[
p_{bp} = \sup\{p \in [0, 1] : \mu^{(\infty)}(\eta_x = 0) = 1\}.
\]
Analogously we can define $\overline{\mu}^{(n)}, \overline{\mu}^{(\infty)}$ and $\overline{p}_{bp}$ in the oriented case. It is easy to verify that in both definitions the bootstrap threshold does not depend on the choice of the vertex $x$. Then the following can be proved following exactly the same lines as Proposition 2.5 of [3]

**Proposition 3.1.** For any $j \leq k$ it holds $p_c = p_{bp}$ and $\overline{p}_c = \overline{p}_{bp}$.

**Proof of Theorem 2.3.** Fix $j \leq k$. We begin by proving that, for any $p$ sufficiently close to 1, there exists an exponentially attracting fixed point $p_\infty \in (0, p)$ of $g_p(x)$. That proves $\tilde{p} \in (0, p)$. For this purpose we compute
\[
\frac{d}{dx} g_p(x) = p \sum_{i=k-j+1}^{k} \binom{k}{i} \left[ i x^{i-1} (1-x)^{k-i} - (k-i) x^i (1-x)^{k-i-1} \right]
\]
\[
= p \left[ \sum_{i=k-j}^{k-1} k \binom{k-1}{i} x^i (1-x)^{k-1-i} - \sum_{i=k-j+1}^{k-1} \binom{k-1}{i} x^i (1-x)^{k-1-i} \right]
\]
\[
= pk \mathbb{P}(N_{x,k} = k-j) > 0
\]
where $N_{x,k} \sim \text{Binom}(k-1, x)$. Next we observe that
\[
g_p(0) = 0, \quad g_p(1) = p
\]
\[
\frac{d}{dx} g_p(x) \bigg|_{x=0} = \begin{cases} 0 & \text{if } j < k \\ pk & \text{if } j = k \end{cases}
\]
and that $g_p(x)$ is increasing in $p$.

If $j = k$ and $p > 1/k$ then the existence of an attracting fixed point in the open interval $(0, p)$ follows at once from (3.2). Thus in this case $\tilde{p} = 1/k$ and at $\tilde{p}$ the only fixed point is the origin.

If instead $j \in [2, k-1]$, $1-p \ll 1$ and $x = 2p - 1$ then
\[
g_p(x) = p - pk \int_{x}^{1} dx' \mathbb{P}(N_{x',k} = k-j) \geq p - pk(k-1)(1-x)^2 > x
\]
which implies the existence of an attracting fixed point $p_\infty(p)$ inside the interval $(2p - 1, p)$. It is easy to check that $p_\infty(p)$ is increasing in $p$ and that, contrary to the case $j = k$, $p_\infty(\tilde{p}) > 0$. 


We now show that \( p_{bp} = \bar{p}_{bp} = \bar{p} \). In the non-oriented case OFA-jf we choose an arbitrary vertex \( x \) together with one of its neighbors \( y \) and we define \( \mathbb{T}_y^k \) to be the rooted \( k \)-ary tree with root at \( y \) and such that \( x \notin \mathbb{T}_y^k \). Then, for any configuration \( \eta \) on \( \mathbb{T}_y^k \), we define the new configuration \( \tilde{\eta} \) to be identically equal to one on \( \mathbb{T}_y^k \) and equal to \( \eta \) elsewhere. Finally we define

\[
p_n := \mu(\eta : B^n(\tilde{\eta})_x = 1).
\]

In the oriented case OFA-jf we instead define

\[
\bar{p}_n := \bar{\mu}(\eta_r = 1).
\]

We claim that \( p_n, \bar{p}_n \) obey the same recursion equations

\[
p_n = g_p(p_{n-1}), \quad \bar{p}_n = g_p(\bar{p}_{n-1}).
\]

Therefore \( p_n = \bar{p}_n \) since for \( n = 0 \) they both coincide with \( p \). Moreover \( \lim_{n \to \infty} p_n = 0 \) \( \iff \) \( p > \bar{p} \) and \( j = k \) or \( \iff p \geq \bar{p} \) and \( j \in [2, k) \).

Let us justify the first claim, \( p_n = g_p(p_{n-1}) \). Clearly

\[
p_n = \mu(\eta : \eta_x = 1; \sum_{z \in \mathbb{N}_x \setminus y} B^{n-1}(\tilde{\eta})_z \geq k - j + 1)
\]

Moreover, if for all \( z \in \mathbb{N}_x \setminus y \) we define \( \eta^{(z)} \) to be identically equal to one in the \( k \)-ary tree rooted at \( x \) and not containing \( z \) and equal to \( \eta \) otherwise, then the event

\[
\{\eta : \eta_x = 1 \text{ and } \sum_{z \in \mathbb{N}_x \setminus y} B^{n-1}(\eta^{(z)})_z \geq k - j + 1\}
\]

coinsides with the event

\[
\{\eta : \eta_x = 1 \text{ and } \sum_{z \in \mathbb{N}_x \setminus y} B^{n-1}(\eta^{(z)})_z \geq k - j + 1\}.
\]

The equality among the two events can be immediately established by noticing that the first event implies that for any \( m \leq n - 1 \) it holds \( B^m(\tilde{\eta})_x = 1 \) thus \( B^m(\tilde{\eta})_z = B^m(\eta^{(z)})_z \) for any \( z \in \mathbb{N}_x \setminus y \). Clearly the events \( \{B^{n-1}(\eta^{(z)})_z = 1\} \) are independent as \( z \) varies in \( \mathbb{N}_x \setminus y \), are independent from \( \{\eta : \eta_x = 1\} \) and each one has probability \( p_{n-1} \). Hence \( p_n = g_p(p_{n-1}) \). Similarly for the oriented case we get \( \bar{p}_n = g_p(\bar{p}_{n-1}) \).

We now proceed to identify the critical points \( p_{bp}, \bar{p}_{bp} \). In the non-oriented case we observe that

\[
\{\eta : B^n(\eta)_x = 1\} = \{\eta : \eta_x = 1 \text{ and } \sum_{z \in \mathbb{N}_x} B^{n-1}(\eta^{(z)})_z \geq k - j + 1\}
\]

where we use the fact that \( B^n(\eta)_x = 1 \) implies that \( B^{n-1}(\eta^{(z)})_z = B^{n-1}(\eta)_z \) for any \( z \in \mathbb{N}_x \). Hence

\[
\mu^{(n)}(\eta_x = 1) = p \sum_{i=k-j+1}^{k+1} \binom{k+1}{i} p_{i-1} (1 - p_{n-1})^{k+1-i}
\]

and \( \lim_{n \to \infty} \mu^{(n)}(\eta_x = 1) = 0 \) \( \iff \lim_{n \to \infty} p_n = 0 \). In the oriented case the same conclusion follows from the definition of \( \bar{p}_n \) and the above proven result \( \bar{p}_n = g_p(\bar{p}_{n-1}) \). Thus we have proven that \( p_{bp} = \bar{p}_{bp} = \bar{p} \). \( \square \)
Remark 3.2. It is obvious that $p_n$ tends to zero at least exponentially fast when $p < p_c$ because in that case $q_p'(0) < 1$.

4. Spectral gap: proofs

In what follows we fix once and for all $j, k \in \mathbb{Z}_+$ with $j \leq k$, together with a density $p \in [0, p_c)$.

Proof of Theorem 2.5. We begin by proving positivity of the spectral gap in the oriented case OFA-jf at density $p$. We first fix some additional notation. We denote by $T := \mathbb{T}_n^k \subset \mathbb{T}_n^k$ the finite $k$-ary tree of depth $n$ rooted at $r$, where $n$ should be thought to be arbitrarily large compared to all other constants. For $x \in T$, $T_x$ will denote the $k$-ary sub-tree of $T$ rooted at $x$ with depth $n - d_x$, where $d_x$ is the depth of $x$. In other words the leaves of $T_x$ are a subset of the leaves of $T$. We also denote the tree $T_x$ stripped off its root $x$ by $\bar{T}_x$.

We write $\text{Var}_T(f)$ for the variance w.r.t. the measure $\mu$ on $T$ of the function $f$, $\text{Var}_x(f)$ for the conditional variance of $f$ w.r.t. the variable $\eta_x$ given all the other variables and $\mu_{T_x}(f)$ for the conditional expectation of $f$ given all the variables outside $T_x$.

Finally, given $\ell \ll n$, we define new auxiliary constraints $c_x^{(\ell)}(\eta) \in \{0, 1\}$, $\eta \in \{0, 1\}^T$ and $x \in T$, as follows. If the depth of $T_x$ is less than $\ell$, $c_x^{(\ell)}(\eta) = 1$. Otherwise we define the constraint to be satisfied, i.e. $c_x^{(\ell)}(\eta) = 1$, iff, by applying at most $\ell$ times the bootstrap map $\bar{T}$ to the configuration $\eta$ which coincides with $\eta$ everywhere with the only possible exception of $x$ where $\eta_x = 1$, the vertex $x$ can be flipped to zero. Clearly the auxiliary constraint and the original one $\bar{c}_x$ coincide if $\ell = 1$.

Remark 4.1. In other words, with the new constraint $c_x^{(\ell)}$, a vertex $x$ is free to flip iff, by a sequence of legal (i.e. satisfying the original constraint $\bar{c}$ in (2.4)) flips in the subtree of depth $\ell$ rooted at $x$ and which never changes the value of the sites at depth $\ell$, at least $j$ of the $k$ children of $x$ can be made vacant. The condition $c_x^{(\ell)}(\eta) \equiv 1$ if the depth of $T_x$ is smaller than $\ell$ can be visualized by imagining the configuration $\eta$ extended to the whole tree $\mathbb{T}_n^k$ by setting it equal to zero on $\mathbb{T}_n^k \setminus T$ and by defining the constraint only through the bootstrap requirement.

In what follows we will first consider an auxiliary long range kinetically constrained model which is defined by the infinitesimal generator (2.2) but with $\bar{c}_x$ substituted by $c_x^{(\ell)}$. We will show that this auxiliary model has a spectral gap which is bounded away from zero uniformly in the depth $n$ of $T$ provided $\ell$ is large enough depending on $p, j, k$. Then we will apply a simple limiting procedures together with standard comparison arguments between the Dirichlet forms with constraints $\bar{c}$ and $c^{(\ell)}$ to complete the proof.

Let $D^{(\ell)}(f)$ denote the new Dirichlet form corresponding to the generator

$$\mathcal{L}^{(\ell)} f(\omega) = \sum_{x \in T} c_x^{(\ell)}(\omega) [\mu_x(f) - f(\omega)]$$
with the auxiliary constraints \( c_x^{(\ell)} \), i.e.

\[
D^{(\ell)}(f) = \frac{1}{2} \sum_{x \in T} \mu \left( c_x^{(\ell)} \Var_{x_x}(f) \right)
\]

Our aim is to establish the so-called Poincaré inequality

\[
\Var(f) \leq \lambda D^{(\ell)}(f), \quad \forall f
\]

for some constant \( \lambda \) independent of the depth \( n \) of the tree \( T \).

**Remark 4.2.** Notice that (4.1) is the natural analog of the renormalized Poincaré inequality in [3] (see formula (5.1) there).

For the reader’s convenience we recall some elementary properties of the variance which will be applied in the sequel. If \( \nu_1, \nu_2 \) are probability measures on two probability spaces \( \Omega_1, \Omega_2 \) and \( \nu = \nu_1 \otimes \nu_2 \) denotes the associate product measure then

\[
\Var_{\nu}(f) \leq \nu (\Var_{\nu_1}(f) + \Var_{\nu_2}(f)) \quad \text{and} \quad \Var_{\nu}(\nu_1(f)) \leq \nu_1 (\Var_{\nu_2}(f))
\]

with self-explanatory notation. Moreover

\[
\Var_{\nu}(f) = \nu_2 (\Var_{\nu_1}(f) + \Var_{\nu_2}(\nu_1(f))
\]

Back to the proof and motivated by [13] we first claim that

\[
\Var(f) \leq \sum_x \mu \left( \Var_x \left( \mu_{T_x}^x(f) \right) \right)
\]

where here and in the following we let \( \Var = \Var_T \) and \( \mu = \mu_T \). To prove the claim we proceed recursively on the depth \( n \) of \( T \). The claim is trivially true for \( n = 0 \). We now assume (4.3) when \( T \) has depth \( n - 1 \) and using the formula for the conditional variance we write

\[
\Var_{\nu}(f) = \mu (\Var(f | \eta_r)) + \Var (\mu(f | \eta_r))
\]

Notice that, given the spin \( \eta_r \) at the root, \( \Var(f | \eta_r) \) is nothing but the variance of \( f \) w.r.t. the product measure on \( T \setminus \{r\} \). Thus

\[
\Var(f | \eta_r) \leq \sum_{y \in K} \mu \left( \Var_{T_y}(f) \mid \eta_r \right)
\]

and

\[
\mu (\Var(f | \eta_r)) \leq \sum_{y \in K} \mu \left( \Var_{T_y}(f) \right).
\]

Each one of the sub-trees \( T_y \) has depth \( n - 1 \) and therefore the inductive assumption implies that

\[
\sum_{y \in K} \mu \left( \Var_{T_y}(f) \right) \leq \sum_{y \in K} \sum_{z \in T_y} \mu \left( \Var_{x_z} \left( \mu_{T_z}^x(f) \right) \right)
\]

\[
= \sum_{x \neq r} \mu \left( \Var_{x_x} \left( \mu_{T_x}^x(f) \right) \right).
\]

By putting together the r.h.s. of (4.5) with the last term in (4.4) we get the claim for depth \( n \).
We now examine a generic term \( \mu \left( \text{Var}_x \left( \mu_{T_x}^z(f) \right) \right) \) in the r.h.s. of (4.3). We write

\[
\mu_{T_x}^z(f) = \mu_{T_x}^z(c_x^{(t)} f) + \mu_{T_x}^z([1 - c_x^{(t)}] f)
\]

so that

\[
\text{Var}_x \left( \mu_{T_x}^z(f) \right) \leq 2 \text{Var}_x \left( \mu_{T_x}^z(c_x^{(t)} f) \right) + 2 \text{Var}_x \left( \mu_{T_x}^z(1 - c_x^{(t)}) f) \right)
\]

Schwartz inequality shows that

\[
\text{Var}_x \left( \mu_{T_x}^z(c_x^{(t)} f) \right) \leq \mu \left( \text{Var}_x \left( c_x^{(t)} f \right) \right) = \mu \left( c_x^{(t)} \text{Var}_x(f) \right)
\]

because \( c_x^{(t)} \) does not depend on the spin at \( x \). Notice that the r.h.s. in (4.7) is just the contribution of the root to the Dirichlet form \( D^{(\ell)}(f) \).

We now turn to the more complicate second term \( \text{Var}_x \left( \mu_{T_x}^z(1 - c_x^{(t)}) f) \right) \).

We write

\[
\text{Var}_x \left( \mu_{T_x}^z(1 - c_x^{(t)}) f) \right) = \text{Var}_x \left( \mu_{T_x}^z(1 - c_x^{(t)})(f - \mu_{T_x}^z(f) + \mu_{T_x}^z(f)) \right) = \text{Var}_x \left( \mu_{T_x}^z(1 - c_x^{(t)}) g) \right)
\]

where we let \( g := f - \mu_{T_x}^z(f) \) and we use the fact that \( \mu_{T_x}^z((1 - c_x^{(t)}) \mu_{T_x}^z(f)) \) does not depend on \( \eta_z \). Recall that the constraint \( c_x^{(t)} \) depends only on the spin configurations in the first \( \ell \) levels below \( x \), in the sequel denoted by \( \Delta_x \). Then

\[
\text{Var}_x \left( \mu_{T_x}^z(1 - c_x^{(t)}) g) \right) \leq \mu_x \left( \mu_{T_x}^z((1 - c_x^{(t)} \mu_{T_x}^z \Delta_x g))^2 \right) \leq \mu_x \left( \mu_{T_x}^z((1 - c_x^{(t)} \mu_{T_x}^z \Delta_x g)^2 \right) = \delta(\ell) \mu_x \left( \mu_{T_x}^z \Delta_x g)^2 \right)
\]

where \( \delta(\ell) := \mu_{T_x}^z((1 - c_x^{(t)}) \) \), we use Cauchy-Schwartz to obtain the second inequality and for the third inequality we use the fact that \( \mu_{T_x}^z((1 - c_x^{(t)}) \) does not depend on \( \eta_z \). Notice that \( \delta(\ell) \) coincides with \( \tilde{p}_\ell / p \) where \( \tilde{p}_\ell \) was defined at the beginning of the proof of theorem 2.3. Next we note that

\[
\mu_x \left( \mu_{T_x}^z \left( \mu_{T_x}^z \Delta_x g) \right)^2 \right) = \mu_{x \cup \Delta_x} \left( \mu_{T_x}^z \Delta_x g) \right)^2 = \text{Var}_{x \cup \Delta_x} \left( \mu_{T_x}^z \Delta_x g \right)
\]

where we used the fact that \( \mu_{x \cup \Delta_x} \left( \mu_{T_x}^z \Delta_x g) = \mu_{T_x}^z \right) \) which is zero by the definition of \( g \). Then by using (4.3), (4.9) and (4.10) we get

\[
\text{Var}_x \left( \mu_{T_x}^z((1 - c_x^{(t)}) g) \right) \leq \delta(\ell) \sum_{z \in x \cup \Delta_x} \mu_{x \cup \Delta_x} \left( \text{Var}_x(\mu_{T_x}^z \mu_{T_x}^z \Delta_x g) \right) \leq \delta(\ell) \sum_{z \in x \cup \Delta_x} \mu_{x \cup \Delta_x} \left( \text{Var}_x(\mu_{T_x}^z g) \right)
\]
where we used the convexity of the variance to obtain the second inequality. In conclusion

$$\sum_x \mu \left( \text{Var}_x \left( \mu_{\tilde{\mu}} (f) \right) \right)$$

$$\leq 2 \sum_{x \in T} \mu \left( \mathcal{C}_x^{(\ell)} \text{Var}_x (f) \right) + 2 \delta(\ell) \sum_{x \in T} \sum_{z \in A_x} \mu \left( \text{Var}_z \left( \mu_{\tilde{\mu}} (f) \right) \right)$$

$$\leq 4D(\ell) (f) + 2(\ell + 1) \delta(\ell) \sum_x \mu \left( \text{Var}_x \left( \mu_{\tilde{\mu}} (f) \right) \right).$$

(4.12)

where the factor $\ell + 1$ accounts for the number of vertices $x$ such that a given vertex $z$ falls inside $\Delta_x$. We now appeal to Remark 3.2 and conclude that for any $p < p_c$ there exists $\ell_0$ (which depends on $p$ and it diverges as $p \uparrow p_c$) such that $(\ell + 1) \delta(\ell) \leq 1/4$ for any $\ell \geq \ell_0$. With this choice the Poincaré inequality (4.1) with $\lambda = 8$ follows uniformly in the depth $n$ of $T$, namely the auxiliary long range model has a positive spectral gap if $\ell \geq \ell_0$.

We are now in a position to conclude the proof in the oriented case. Starting from (4.1) and using path arguments exactly as in section 5 of [3] we conclude that, for any $\ell \geq \ell_0$ we can find a constant $\lambda(\ell, k, j) \geq 1$ independent of $n$ such that

$$\text{Var}(f) \leq \lambda(\ell, k, j) \sum_{x \in T} \mu \left( \bar{c}_x \text{Var}_x (f) \right)$$

where with a slight abuse if notation we denote by $\bar{c}_x$ the function which equals one if $x$ is a leaf of $T$ and otherwise is given by the definition in (2.3). Thus, thanks to Remark 2.6, we can conclude that the spectral gap of the oriented model on the infinite tree $\bar{T}^k$ is bounded from below by $\lambda(\ell, k, j)^{-1}$.

**Remark 4.3.** We observe that the dependence on $p$ of $\lambda(\ell, k, j)$ comes from the fact that $\ell > \ell_0$ and that the critical scale $\ell_0$ depends on $p$ and diverges as $p \uparrow p_c$.

We now examine the non-oriented FA-jf model in the infinite tree $\bar{T}^k$. Choose a vertex $r$ and call it the root. Next we introduce the following auxiliary block dynamics with two blocks $A = \{r\}$ and $B = \bar{T}^k \setminus \{r\}$, reversible w.r.t. the measure $\mu$. The block dynamics goes as follows. With rate one the current configuration inside $B$ is resampled from the equilibrium measure $\mu_B$ while the block $A$ with rate one resample the variable $\eta_r$ iff the constraint at the root is satisfied ($c_r(\eta) = 1$). It is easy prove a Poincaré inequality with some finite constant $\gamma = \gamma(j, k) \geq 1$ for the block dynamics of the form (compare to Proposition 4.4 in [3])

$$\text{Var}(f) \leq \gamma \mu \left( c_r \text{Var}_r (f) + \text{Var}_B (f) \right)$$

(4.13)

Notice that $B$ is the union of $k + 1$ copies of the rooted tree $\bar{T}^k$, $B = \cup_{y \in N_r} \bar{T}^k_y$, and therefore

$$\text{Var}_B (f) \leq \sum_{y \in N_r} \mu_B (\text{Var}_{\bar{T}^k_y} (f))$$

Thanks to the result in the oriented case and using $\bar{c}_x \leq c_x$ we get

$$\text{Var}_{\bar{T}^k_y} (f) \leq \lambda \sum_{x \in \bar{T}^k_y} \mu_{\bar{T}^k_y} (\bar{c}_x \text{Var}_x (f)) \leq \lambda \sum_{x \in \bar{T}^k_y} \mu_{\bar{T}^k_y} (c_x \text{Var}_x (f))$$

(4.14)
where \( \lambda = \lambda(\ell, k, j) \). Thus

\[
\mu(\text{Var}_B(f)) \leq \lambda \sum_{x \in \overline{T}^k \atop x \neq r} \mu(c_x \text{Var}_x(f))
\]  

(4.15)

Inserting (4.15) into (4.13) we conclude that the spectral gap of the FA-jf model is lower bounded by \((\gamma \lambda)^{-1}\). \(\square\)

Proof of Proposition 2.8. If \( p > p_c \) with positive probability the root belongs to an infinite cluster of occupied vertices which is stable upon iterations of the bootstrap map \( \overline{B} \). Clearly any vertex belonging to such a cluster can never change its occupation variable during the dynamics of OFA-jf. Hence the result. The result for the non oriented model can be established via the same lines. \(\square\)

Proof of Theorem 2.9. Fix \( L \) and consider for simplicity only the rooted case, the unrooted one being treated along the same lines. We denote by \( \overline{T}^k_L \) the \( k \)-ary rooted tree of depth \( L \) and root \( r \). We begin by proving the stated upper bound.

Choose as test function \( f \) to be used in the Poincaré inequality

\[
\text{gap}(\mathcal{L}_L) \leq \mathcal{D}_L(f) / \text{Var}(f) \quad \forall \ f \in L^2(\overline{T}^k_L)
\]

the indicator of the event \( A \) that the root is occupied after applying infinitely many times the bootstrap map \( \overline{B} \) of the infinite rooted tree \( \overline{T}^k \) to the configuration obtained from the original one in the finite subtree of depth \( L \) by filling up all the vertices outside the subtree. Equivalently the root is occupied after \( L \) iterations of the bootstrap map. The key fact is that, if \( p > p_c \), then \( \text{Var}(f) > 0 \) uniformly in \( L \).

Next we compute the Dirichlet form \( \mathcal{D}_L(f) \). We first observe that if \( x \) is not a leaf of \( \overline{T}^k_L \), then the corresponding contribution \( \mu(c_x \text{Var}_x(f)) \) to the Dirichlet form vanishes. If not that would imply that one could connect the event \( A \) with its complement by means of a legal flip, i.e. one with \( c_x = 1 \). But that is clearly impossible by the definition of \( A \). If instead \( x \) is a leaf, so that \( c_x \equiv 1 \) (recall that the finite volume model has unconstrained leaves), then

\[
\mu(c_x \text{Var}_x(f)) = 2\mu(\eta \in A; \eta^x \notin A) = 2\mu(x \text{ is pivotal for } A)
\]

The latter probability can be computed explicitly and it is equal to \( \prod_{y \leq x} p_y \) where \( y \leq x \) means that \( y \) is an ancestor of \( x \) and \( p_y \) is the probability that \( y \) is occupied and that exactly \( j-1 \) among those children of \( y \) which are not ancestors of \( x \) are not occupied after \( L - dp - 1 \) iteration of the bootstrap map. Since the probability \( p^{(\alpha)} \) that the root is occupied after \( n \)-iterations of the bootstrap map converges to the largest fixed point \( p_{\infty} \), which moreover is exponentially stable, we get that

\[
\prod_{y \leq x} p_y \leq C \left( p \binom{k-1}{j-1} (1 - p_{\infty})^{j-1} p_{\infty}^{k-j} \right)^L.
\]

so that \( \mathcal{D}_L(f) \leq C \left( kp^{(j-1)} p_{\infty}^{k-j} (1 - p_{\infty})^{j-1} \right)^L \). The proof of the upper bound is complete once we observe that

\[
k p \left( \binom{k-1}{j-1} p_{\infty}^{k-j} (1 - p_{\infty})^{j-1} \right) = \frac{d}{dx} g_p(x) \left|_{x=p_{\infty}} \right. < 1.
\]
We now turn to the lower bound. The proof is based on the same argument used in theorem 2.5 to treat the unoriented case, specific for trees which we now shortly detail. In what follows $\gamma(L)$ will denote the inverse spectral gap for the OFA-$k_f$ model on $\overline{\tau}_L^k$.

By monotonicity of the rates we assume $j = k$ (indeed if $j < k$ the spectral gap of OFA-$j_f$ is larger or equal the spectral gap of OFA-$k_f$, thus it is enough to prove the result in the case $j = k$). Next, as before, consider the auxiliary block dynamics in which: (i) each sub-tree rooted at one of the children of the root with rate one updates at the same time all its vertices by choosing the new configuration from the equilibrium distribution and (ii) with rate one but only if all the children are empty the root $r$ refreshes its occupation variable by sampling from the equilibrium measure.

It is easy to check that, for any $p \in (0, 1)$, the spectral gap of the block dynamics is positive uniformly in $L$. Therefore we can write

$$\text{Var}_L(f) \leq C \mu_{c_r} \left( c_r \text{Var}_r(f) + \sum_{x \in N_r} \text{Var}_{\overline{T}_x}(f) \right)$$

for some $C > 0$. By definition, for each $x \in N_r$, $\text{Var}_{\overline{T}_x}(f) \leq \gamma(L-1)D_{\overline{T}_x}(f)$. Therefore

$$\text{Var}_L(f) \leq C \max(1, \gamma(L-1))D_L(f)$$

which implies $\gamma(L) \leq CL$.

□

5. Generalizations of the technique

In this section we discuss some applications of the technique that we have devised to prove Theorem 2.5. We show in particular that this technique allows to recover the positivity of the spectral gap in the whole ergodicity region for the KCSM on $\mathbb{Z}^d$ which were studied in [3] via a completely different methods. We start by treating explicitly the case of the North-East model and then we will describe how to extend the analysis to more general models [8].

Definition 5.1. The North-East (N-E) model is a KCSM on $\mathbb{Z}^2$ for which the constraint at $x \in \mathbb{Z}^2$ requires the nearest neighbours both in the North and in the East direction to be empty. More precisely it is a continuous time Markov process on $\Omega = \{0, 1\}^{\mathbb{Z}^2}$ with generator $L^{NE}$ defined as in Definition 2.1 but with the sum in the generator now running on the sites of $\mathbb{Z}^2$ and with the function $c_x$ now substituted by $c_x^{NE}$ with

$$c_x^{NE}(\omega) = \begin{cases} 1 & \text{if } \eta_x + \vec{e}_1 = \eta_x + \vec{e}_2 = 0 \\ 0 & \text{otherwise} \end{cases} \quad (5.1)$$

with $\vec{e}_1$ and $\vec{e}_2$ the Euclidean unit vectors on $\mathbb{Z}^2$.

Let us recall some well known properties of the North-East (we refer the reader to [3] section 6.4, where these results have been derived by using the analog of our Proposition 3.1 and via the results on oriented percolation of [18] and [4]). Let $p^{NE}_c$ be the critical density for the North-East model defined as in (2.5) with $L$ substituted by $L^{NE}$. We also define the bootstrap map corresponding to the N-E rules exactly in the same way as for FA-$j_f$ and OFA-$j_f$ models and by $\mu^{(n)}_{NE}$ the measure obtained by iterating $\ell$ times this map starting from $\mu$. Then we let $p^{NE}_\ell$ be the probability that a site is occupied after $\ell$ iterations of the boostrap map, namely

$$p^{NE}_\ell := \mu^{(\ell)}_{NE}(\eta_x = 1).$$
Proposition 5.2. $p_c^{NE} = p_c^o$ where $p_c^o$ is the critical threshold for oriented percolation in $\mathbb{Z}^2$. In particular, thanks to the results on oriented percolation (see [4]), $0 < p_c^{NE} < 1$. Moreover, for any $p < p_c^{NE}$,
\[ \lim_{\ell \to \infty} \ell^2 p_c^{NE} = 0 \]

We will now prove via our technique the following result.

Theorem 5.3. Assume $p < p_c^{NE}$. Then $\text{gap}(L^{NE}) > 0$.

Proof. As for the models on trees, we prove a lower bound on the spectral gap on a finite region with proper boundary conditions which is uniform in the size of the region. Then the result on infinite volume follows by standard methods. The finite region that we consider is the triangle $\Lambda \subset \mathbb{Z}^2$ with a vertex in the origin, a vertex in $L\vec{e}_1$ and a vertex in $L\vec{e}_2$ and we now prove a lower bound which is uniform on $\Lambda$ for the spectral gap of the North-East model on $\Lambda$ with empty boundary conditions. For any $x \in \Lambda$ we let $C_x$ be the subset of $\Lambda$ which includes only the sites belonging to the quadrant with bottom left corner on $x$, namely
\[ C_x := \{ z : z \cdot \vec{e}_1 \geq x \cdot \vec{e}_1 \text{ and } z \cdot \vec{e}_2 \geq x \cdot \vec{e}_2 \} \]
and we let $\hat{C}_x := C_x \setminus x$.

It holds
\[ \text{Var}_\Lambda(f) \leq \sum_x \mu_\Lambda \left( \text{Var}_x \left( \mu_{\hat{C}_x}(f) \right) \right). \quad (5.2) \]

In order to prove this inequality which is the analog of (4.3) one has to follow a slightly different method due to the fact that the geometry is more complicated than in the tree case (in particular given a site $x$ and its neighbours $y$ and $z$ in the north and east direction then $C_y \cap C_z \neq \emptyset$). The inequality (5.2) can instead be easily proven following the lines used to prove inequality (11) in [13] and using the fact that $\mu_\Lambda$ is the Bernoulli product measure.

Then we let $c^{\ell} \eta$ be one iff, by applying at most $\ell$ times the bootstrap map corresponding to the N-E rules with empty boundary conditions on $\Lambda$ to the configuration $\tilde{\eta}$ which equals one at $x$ and equals $\eta$ elsewhere, the vertex $x$ can be flipped to zero. We call again $D^\ell$ the Dirichlet form of the generator corresponding to this choice of the constraints. Then, by using the key inequality (5.2), and via exactly the same lines as in the proof of Theorem 2.5 we obtain
\[ \text{Var}_\Lambda(f) \leq 4D^{(\ell)}(f) + 2(\ell + 1)^2 \delta^{NE}(\ell) \sum_x \mu \left( \text{Var}_x \left( \mu_{\hat{C}_x}(f) \right) \right) \quad (5.3) \]
where $\delta^{NE}(\ell) = p_c^{NE}/p$ and the factor $(\ell + 1)^2$ (instead of $(\ell + 1)$ of (4.13)) is due to geometric reasons: it accounts for the number of vertices $x$ such that a given vertex $z$ falls inside $C_x$. By using the property of $p_c^{NE}$ stated in Proposition 5.2 we therefore conclude that there exists $\ell_0$ such that for any $\ell \geq \ell_0$ it holds $(\ell + 1)^2 \delta^{NE}(\ell) < 1/4$. The rest of the proof is now exactly as that of Theorem 2.5.

Remark 5.4. Via a proper generalization of our technique we can establish the positivity of the spectral gap for all the KCSM covered by Theorem 3.3 of [3]. These include, besides N-E model, some of the KCSM which have been most studied in physics literature, namely
the East model on $\mathbb{Z}$, the Friedrickson Andersen model on $\mathbb{Z}^d$ and the modified basic model on $\mathbb{Z}^d$ (see Section 2.3 of [3] for the definitions). More precisely our technique allows to prove Theorem 4.1 of [3] (in a completely different way), namely to establish the positivity of the spectral gap in a proper regime for a model (the so-called auxiliary general model in [3]) which is nothing but a generalized North-East model in $d$-dimensions. Then the proof of positivity of the spectral gap for each specific model can be completed via the renormalization technique detailed in Section 5 of [3]. We can also along the same lines recover the positivity of the spectral gap for the spiral model, a result which was previously established in [2].

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