CONDITIONED OBSERVABLES
IN QUANTUM MECHANICS

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Abstract

This paper presents some of the basic properties of conditioned observables in finite-dimensional quantum mechanics. We begin by defining the sequential product of quantum effects and use this to define the sequential product of two observables. The sequential product is then employed to construct the conditioned observable relative to another observable. We then show that conditioning preserves mixtures and post-process of observables. We consider conditioning among three observables and a complement of an observable. Corresponding to an observable, we define an observable operator in a natural way and show that this mapping also preserves mixtures and post-processing. Finally, we present a method of defining conditioning in terms of self-adjoint operators instead of observables. Although this technique is related to our previous method it is not equivalent.

1 Introduction

Various studies in quantum mechanics are based on the results of a measurement conditioned on the value of a previous measurement. For example, one might want to know the position of a particle when the particle is in a given energy state. Although the conditioning of observables seems to be a useful concept there does not appear to be any systematic investigations concerning it. This article does not develop any deep or penetrating results. Instead, it presents an introduction to a theory of conditioned observables.
Also, we restrict attention to finite-dimensional quantum mechanics. Although this is a strong restriction, it includes the framework of quantum computation and information theory [9, 11]. These are important topics that have attracted great attention in the recent literature.

We begin with the study of quantum effects. These correspond to simple experiments with only two values or outcomes. These values are usually denoted by yes-no (or 1-0). A general effect may be imprecise or fuzzy while a precise effect is called sharp. If \( a \) and \( b \) are effects we define their sequential product \( a \circ b \) which is the effect that describes the experiment in which \( a \) is measured first and then \( b \) is measured second. Because of quantum interference, \( a \) can interfere with the measurement of \( b \), while \( b \) cannot interfere with the measurement of \( a \). We also call \( a \circ b \), the effect \( b \) conditioned on the effect \( a \) and write \( (b | a) = a \circ b \). Upon introducing the concept of a state we can also define a corresponding conditional probability.

Now a general observable \( A \) may have many possible outcomes \( x_1, \ldots, x_n \). If \( a_x \) is the effect that occurs when \( A \) has outcome \( x \) we can think of \( A \) as a set of effects \( A = \{a_x : x = x_i, i = 1, 2, \ldots, n\} \). If \( B = \{b_y : y = y_j, j = 1, 2, \ldots, m\} \) is another observable, we shall show in Section 3 how to combine the effects \( (b_y | a_x) \) to form an observable \( (B | A) \) that describes \( B \) conditioned on \( A \). We show that \( (B | A) \) has a simple form when \( A \) and \( B \) are sharp observables. We also consider multiple conditionings \(( (B | A) | C) \) and \((B | (A | C)) \).

There are two important ways of combining observables called mixtures and post-processing [2, 10]. Section 3 shows that conditioning preserves both of these combination methods. Corresponding to an observable \( A \), we define a self-adjoint operator \( \hat{A} \) called the observable operator. The operator \( \hat{A} \) describes \( A \) in various ways and we show that \( \wedge \) preserves mixtures and post-processing in Section 4. Section 5 discusses a complement of an observable.

Finally, Section 6 considers conditioning from a different point of view. Instead of describing a measurable quantity by an observable, we can describe it by a certain self-adjoint operator. Although this viewpoint is related to our previous work, it is not equivalent to it.

2 Quantum Effects

Let \( \mathcal{L}(H) \) be the set of linear operators on a finite-dimensional complex Hilbert space \( H \). We also denote the set of self-adjoint operators on \( H \) by
$\mathcal{L}_S(H)$ and the zero and unit operators by 0, $I$ respectively. For $S, T \in \mathcal{L}(H)$ we write $S \leq T$ if $\langle \phi, S\phi \rangle \leq \langle \phi, T\phi \rangle$ for all $\phi \in H$. We define the set of effects by

$$\mathcal{E}(H) = \{ a \in \mathcal{L}(H) : 0 \leq a \leq I \}$$

An effect $a$ is said to occur when a yes-no experiment for $a$ has the value yes\cite{1,3,9}. It is well-known that $\mathcal{E}(H) \subseteq \mathcal{L}_S(H)$ \cite{3,9}. For $a \in \mathcal{E}(H)$, we call $a' = I - a \in \mathcal{E}(H)$ the complement of $a$ and view $a'$ as the effect that occurs when the previous yes-no experiment has the value no. Clearly, $0, I \in \mathcal{E}(H)$ and 0 corresponds to the experiment that never occurs (is always no) and $I$ responds to the experiment that always occurs (is always yes). We denote the set of projections on $H$ by $\mathcal{P}(H)$. It is clear that $\mathcal{P}(H) \subseteq \mathcal{E}(H)$ and we call elements of $\mathcal{P}(H)$ sharp effects \cite{5}. A one-dimensional projection $P_\phi = |\phi\rangle \langle \phi| \text{ where } ||\phi|| = 1$ is atomic. If $\phi \in H$, $\phi \neq 0$ we write $\tilde{\phi} = \phi / ||\phi||$. We then have

$$P_\phi = \frac{1}{||\phi||} |\phi\rangle \langle \phi|$$

An effect $\rho \in \mathcal{E}(H)$ is a partial state if the trace $\text{tr}(\rho) \leq 1$ and $\rho$ is a state if $\text{tr}(\rho) = 1$. We denote the set of states by $\mathcal{S}(H)$. If $\rho \in \mathcal{S}(H)$, $a \in \mathcal{E}(H)$ we call $E_\rho(a) = \text{tr}(\rho a)$ the probability that $a$ occurs in the state $\rho$. Of course, $0 \leq E_\rho(a) \leq 1$. If $P_\phi$ is atomic, then $P_\phi \in \mathcal{S}(H)$ and we call $P_\phi$ (and $\phi$) a pure state. We then write

$$E_\phi(a) = E_{P_\phi}(a) = \text{tr}(P_\phi a) = \langle \phi, a\phi \rangle$$

If $\phi$ and $\psi$ are pure states, we call $|\langle \phi, \psi \rangle|^2$ the transition probability from $\phi$ to $\psi$.

We denote the unique positive square root of $a \in \mathcal{E}(H)$ by $a^{1/2}$. For $a, b \in \mathcal{E}(H)$, their sequential product is the effect $a \circ b = a^{1/2}ba^{1/2}$ \cite{6,7,8}. We interpret $a \circ b$ as the effect that results from first measuring $a$ and that $a \circ b = b \circ a$ if and only if $ab = ba$ where $ab$ is the usual operator product \cite{5}. This is interpreted as saying that $a$ and $b$ do not interfere if and only if $a$ and $b$ commute. We also call $a \circ b$ the effect $b$ conditioned on the effect $a$ and write $(b \mid a) = a \circ b$. For short, we sometimes call $(b \mid a)$ the effect $b$ given $a$. We have that $(a \mid a) = a^2$ and $a$ is sharp if and only if $(a \mid a) = a$.

Notice that if $b_1, b_2, b_1 + b_2 \in \mathcal{E}(H)$, then $(b_1 + b_2 \mid a) = (b_1 \mid a) + (b_2 \mid a)$. In particular, $\mathcal{E}(H)$ is convex and if $\lambda_i \geq 0$ with $\sum \lambda_i = 1$, then

$$\left( \sum \lambda_i b_i \mid a \right) = \sum \lambda_i (b_i, a)$$
so \( b \mapsto (b \mid a) \) is a convex function. Of course, \( a \mapsto (b \mid a) \) is not convex in general. Also, for every \( \lambda \in [0,1] \subseteq \mathbb{R} \) we have

\[
(b \mid \lambda a) = (\lambda b \mid a) = \lambda(b \mid a)
\]

Moreover,

\[
\text{tr } [(b \mid a)] = \text{tr } (ba) = \text{tr } (ab) = \text{tr } [(a \mid b)]
\]

If \( \rho \in \mathcal{S}(H) \), \( a \in \mathcal{E}(H) \), since \( \rho \circ a \leq \rho \) we have that

\[
\text{tr } [(\rho \mid a)] = \text{tr } (a \circ \rho) = \text{tr } (\rho \circ a) \leq \text{tr } (\rho) \leq 1
\]

Hence, \((\rho \mid a)\) is a partial state. For \( b \in \mathcal{E}(H) \) we obtain

\[
E_\rho [(b \mid a)] = \text{tr } [\rho(b \mid a)] = \text{tr } [\rho a \circ b] = \text{tr } [(a \circ \rho) b] = \text{tr } [(\rho \mid a) b]
\]

We interpret \( \text{tr } [(\rho \mid a) b] \) as the probability that \( b \) occurs for the partial state \((\rho \mid a)\). If \( E_\rho(a) = \text{tr } (\rho a) \neq 0 \) we can form the state \((\rho \mid a)/\text{tr } (\rho a)\). Then as a function of \( b \)

\[
\widehat{E}_\rho [(b \mid a)] = \frac{E_\rho [(b \mid a)]}{E_\rho (a)}
\]

becomes a probability measure on \( \mathcal{E}(H) \) and we call (2.1) the conditional probability of \( b \) given \( a \).

We now examine some specific examples of \((b \mid a)\). The simplest case is when \( a = P_\phi \) is atomic. We then obtain

\[
(b \mid P_\phi) = P_\phi \circ b = |\phi\rangle \langle \phi| b |\phi\rangle = \langle \phi, b\phi \rangle P_\phi
\]

Hence, \((b \mid P_\phi)\) is \( P_\phi \) attenuated by the probability of \( b \) in the state \( \phi \). If \( b = P_\phi \) is atomic and \( a^{1/2} \phi \neq 0 \) we have that

\[
(P_\phi \mid a) = a \circ P_\phi = a^{1/2} |\phi\rangle \langle \phi| a^{1/2} = |a^{1/2} \phi\rangle \langle a^{1/2} \phi|
\]

\[
= \left| a^{1/2} \phi \right|^2 P_{(a^{1/2} \phi)^*} = \langle \phi, a\phi \rangle P_{(a^{1/2} \phi)^*}
\]

If \( a = P_\phi, b = P_\psi \) are both atomic, we obtain

\[
(P_\psi \mid P_\phi) = P_\phi \circ P_\psi = \langle \phi, P_\psi \phi \rangle P_\phi = |\langle \phi, \psi \rangle|^2 P_\phi
\]

where \(|\langle \phi, \psi \rangle|^2 \) is the transition probability from \( \phi \) to \( \psi \).
More generally, let \( P \in \mathcal{P}(H) \) so \( P \) is a sharp effect. We can then write \( P = \sum P_{\phi_i} \) where \( \phi_i \) are mutually orthogonal. We then have 

\[
(P \mid a) = a \circ P = \sum a \circ P_{\phi_i} = \sum \langle \phi_i, a\phi_i \rangle P_{(a; 1/2 \phi_i)}^\wedge
\]

and 

\[
(b \mid P) = P \circ b = PbP = \sum_{i,j} P_{\phi_i} bP_{\phi_j} = \sum_{i,j} |\phi_i \rangle \langle \phi_i| b |\phi_j \rangle \langle \phi_j|
\]

\[
= \sum_{i,j} \langle \phi_i, b\phi_j \rangle |\phi_i \rangle \langle \phi_j|
\]

### 3 Observables

For a finite set \( \Omega_A \), an observable with value-space \( \Omega_A \) is a subset \( A = \{a_x : x \in \Omega_A\} \) of \( \mathcal{E}(H) \) such that \( \sum_{x \in \Omega_A} a_x = I \). We write \( a_x \) as the effect that occurs when \( A \) has the value \( x \). The condition \( \sum a_x = I \) ensures that \( A \) has one of the values \( x \in \Omega_A \). Observables are also called finite positive operator-valued measures [9, 11]. If an observable \( A \) has only one value, then \( A = \{I\} \) so \( A \) is called trivial. If \( A \) has two values, say yes and no then \( A = \{a, a'\} \) where \( a \in \mathcal{E}(H) \) and \( a \) is the effect that \( A \) has value yes, while \( a' \) is the effect that \( A \) has value no.

If \( a_x \in \mathcal{P}(H) \) for all \( x \in \Omega_A \), we call \( A \) a sharp observable. In this case we have for all \( y \in \Omega_a \) that

\[
a_y + a_y \circ \sum_{x \neq y} a_x = a_y \circ \sum_{x \in \Omega_A} A_x = a_y
\]

Hence, \( \sum_{x \neq y} a_y \circ a_x = 0 \) which implies that \( a_y \circ a_x = 0 \) whenever \( x \neq y \). We conclude that \( a_y a_x = a_x a_y = 0 \) whenever \( x \neq y \). Hence, the effects for a sharp observable commute and are mutually orthogonal which makes them much simpler than unsharp observables. If the effects \( a_x, x \in \Omega_A \), are atoms, we say that the observable \( A \) is atomic. In this case, \( a_x = P_{\phi_x} \) where \( \{\phi_x : x \in \Omega_A\} \) is an orthonormal basis for \( H \). In general, if \( \Omega_A \subseteq \mathbb{R} \) and \( \rho \in \mathcal{S}(H) \), we define the expectation of \( A \) in the state \( \rho \) by

\[
E_{\rho}(A) = \sum x E_{\rho}(a_x) = \sum x \operatorname{tr}(\rho a_x) = \operatorname{tr}(\rho \sum x a_x)
\]
Notice that $\hat{A} = \sum x a_x$ is a self-adjoint operator that we call the observable operator for $A$. This operator has the same expectations as $A$ for every state $\rho \in \mathcal{S}(H)$.

Let $A, B$ be observables with $A = \{a_x : x \in \Omega_A\}$ and $B = \{b_y : y \in \Omega_B\}$. We define their sequential product $A \circ B$ to have value-space $\Omega_A \times \Omega_B$ and

$$A \circ B = \{a_x \circ b_y : (x, y) \in \Omega_A \times \Omega_B\}$$

To show that $A \circ B$ is indeed an observable, we have that

$$\sum_{(x,y)} a_x \circ b_y = \sum_x a_x \circ \left( \sum_y b_y \right) = \sum_x a_x \circ I = \sum_x a_x = I$$

The left-marginal of $A \circ B$ consists of the effects

$$\sum_y a_x \circ b_y = a_x \circ \sum_y b_y = a_x \circ I = a_x$$

so the left-marginal of $A \circ B$ is just $A$. In a similar way the right-marginal of $A \circ B$ consists of the effects

$$\sum_x a_x \circ b_y = \sum_x a_x^{1/2} b_y a_x^{1/2}$$

As before, the right-marginal of $A \circ B$ is an observable but it need not equal $B$ and we denote it by $(B \mid A)$. We thus have that $\Omega_{(B \mid A)} = \Omega_B$ and

$$(B \mid A) = \left\{ \sum_x a_x \circ b_y : y \in \Omega_B \right\} = \left\{ \sum_x (b_y \mid a_x) : y \in \Omega_B \right\}$$

We call $(B \mid A)$ the observable $B$ conditioned on the observable $A$. For short, we call $(B \mid A)$ the observable $B$ given $A$. We denote the effects in $(B \mid A)$ by

$$(B \mid A)_y = \sum_x a_x \circ b_y = \sum_x (b_y \mid a_x)$$

We use $\mathcal{O}(H)$ for the set of observables on $H$.

If $\rho \in \mathcal{S}(H)$ and $A \in \mathcal{O}(H)$ we define the state $\rho$ conditioned on $A$ by

$$(\rho \mid A) = \sum_x a_x^{1/2} \rho a_x^{1/2}$$
Note that \((\rho \mid A) \in \mathcal{S}(H)\) because
\[
\text{tr} (\rho \mid A) = \sum_x \text{tr} \left( a_x^{1/2} \rho a_x^{1/2} \right) = \sum_x \text{tr} (a_x \rho) = \text{tr} \left( \sum_x a_x \rho \right) = \text{tr} (\rho) = 1
\]

The next result gives a duality between states and observables.

**Lemma 3.1.** If \(A, B \in \mathcal{O}(H)\) with \(\Omega_B \subseteq \mathbb{R}\) and \(\rho \in \mathcal{S}(H)\), then
\[
E_{\rho}(B \mid A) = E_{(\rho \mid A)}(B)
\]

**Proof.** We have that
\[
E_{\rho}(B \mid A) = \sum_y y \text{ tr} [\rho(B \mid A)_y] = \sum_y \text{ tr} (\rho \sum_x a_x \circ b_y)
\]
\[
= \sum_{x,y} y \text{ tr} [\rho(a_x \circ b_y)] = \sum_{x,y} y \text{ tr} (\rho a_x^{1/2} b_y a_x^{1/2})
\]
\[
= \sum_{x,y} y \text{ tr} (a_x^{1/2} \rho a_x^{1/2} b_y)
\]
\[
= \sum_y \text{ tr} \left[ \left( \sum_x a_x^{1/2} \rho a_x^{1/2} \right) b_y \right] = \sum_y \text{ tr} \left[ (\rho \mid A) b_y \right]
\]
\[
= E_{(\rho \mid A)}(B)
\]

We now consider \(A \circ B\) and \((B \mid A)\) for special cases \(A, B \in \mathcal{O}(H)\). If \(A\) and \(B\) are atomic with \(a_x = P_{\phi_x}, \ b_y = P_{\psi_y}, \ x \in \Omega_A, \ y \in \Omega_B\) we have that
\[
A \circ B = \left\{ |\langle \phi_x, \psi_y \rangle|^2 P_{\phi_x} : x \in \Omega_A, y \in \Omega_B \right\}
\]
It follows that
\[
(B \mid A)_y = \sum_x |\langle \phi_x, \psi_y \rangle|^2 P_{\phi_x}
\]
If \(A\) is atomic and \(B \in \mathcal{O}(H)\) is arbitrary, we have
\[
A \circ B = \{ \langle \phi_x, b_y \phi_x \rangle P_{\phi_x} : x \in \Omega_A, y \in \Omega_B \}
\]
and
\[
(B \mid A)_y = \sum_x \langle \phi_x, b_y \phi_x \rangle P_{\phi_x}
\]
If $A \in \mathcal{O}(H)$ is arbitrary and $B$ is atomic, we have

$$A \circ B = \left\{ (\psi_y, a_x \psi_y) P_{(a_x \psi_y)^\wedge} : x \in \Omega_A, y \in \Omega_B \right\}$$

and

$$(B \mid A)_y = \sum_x (\psi_y, a_x \psi_y) P_{(a_x \psi_y)^\wedge}$$

Let $B^{(i)} \in \mathcal{O}(H), i = 1, 2, \ldots, n$, with the same value-space $\Omega$ where

$$B^{(i)} = \left\{ b^{(i)}_y : y \in \Omega \right\}$$

For $\lambda_i \in [0,1], i = 1, 2, \ldots, n$, with $\sum \lambda_i = 1$ we define the mixture $\sum \lambda_i B^{(i)} \in \mathcal{O}(H)$ by

$$\sum_{i=1}^{n} \lambda_i B^{(i)} = \left\{ \sum_{i=1}^{n} \lambda_i b^{(i)}_y : y \in \Omega \right\}$$

It is easy to check that $\sum \lambda_i B^{(i)}$ is indeed an observable. Mixtures are an important way of combining observables and have been well-studied [2, 10].

It is convenient to use the notation $\left( \sum_{i=1}^{n} \lambda_i B^{(i)} \right)_y = \sum_{i=1}^{n} \lambda_i b^{(i)}_y$.

Let $\Omega_A, \Omega_B$ be value-spaces and let $\nu = [\nu_{xy}], x \in \Omega_A, y \in \Omega_B$ be a matrix. We call $\nu$ a stochastic matrix if $\nu_{xy} \in [0,1] \subseteq \mathbb{R}$ and $\sum_{y \in \Omega_B} \nu_{xy} = 1$ for all $x \in \Omega_A$. The matrix $\nu$ is called a classical channel and $\nu_{xy}$ gives the probability of a transition from $x$ to $y$ [2, 10]. The condition $\sum_{y \in \Omega_B} \nu_{xy} = 1$ means that $x$ makes a transition to some $y \in \Omega_B$ with probability one.

Now let $A \in \mathcal{O}(H)$ with $A = \{a_x : x \in \Omega_A\}$ and let $\nu$ be a classical channel from $\Omega_A$ to $\Omega_B$. Define $B \in \mathcal{O}(H)$ by $B = \{b_y : y \in \Omega_B\}$ where $b_y = \sum_{x \in \Omega_A} \nu_{xy} a_x$. Now the value-space of $B$ is $\Omega_B$ and $B$ is indeed an observable because

$$\sum_y b_y = \sum_y \sum_x \nu_{xy} a_x = \sum_x \sum_y \nu_{xy} a_x = \sum_x a_x = I$$

We use the notation $B = \nu \cdot A$ and call $B$ a post-processing of $A$ [2, 10]. The next result shows that conditioning preserves mixtures and post-processing.
Theorem 3.2.  (i) \( A \circ \sum \lambda_i B^{(i)} = \sum \lambda_i A \circ B^{(i)} \).  (ii) \( \langle \sum \lambda_i B^{(i)} \mid A \rangle = \sum \lambda_i \langle B^{(i)} \mid A \rangle \).  (iii) If \( C = \{ c_z : z \in \Omega_C \} \) is an observable, then \( (\nu \bullet A \mid C) = \nu \bullet (A \mid C) \).

Proof.  (i) For any \( x \in \Omega_A \) and \( y \in \Omega \) we have that
\[
(A \circ \sum \lambda_i B^{(i)})_{(x,y)} = a_x \circ \left( \sum \lambda_i B^{(i)} \right)_y = a_x \circ \sum \lambda_i b^{(i)}_y = \sum \lambda_i a_x \circ b^{(i)}_y
\]

\[
= \sum \lambda_i (A \circ B^{(i)})_{(x,y)} = \left( \sum \lambda_i A \circ B^{(i)} \right)_{(x,y)}.
\]

The result now follows.  (ii) This follows from (i).  (iii) For all \( y \in \Omega \) we have that
\[
(\nu \bullet A \mid C)_y = \sum_z c_z \circ (\nu \bullet A)_y = \sum_z c_z \circ \left( \sum_x \nu_{xy} a_x \right) = \sum_x \nu_{xy} \sum_z c_z \circ a_x
\]

\[
= \sum_x \nu_{xy} (A \mid C)_x = [\nu \bullet (A \mid C)]_y
\]

The result follows.

We now briefly discuss multiple conditioning.  Letting \( A, B, C \in \mathcal{O}(H) \) we can form the biconditional \( (B \mid A) \mid C \) in which \( C \) is measured first, \( A \) is measured second and \( B \) is measured last.  By definition, we have that
\[
((B \mid A) \mid C) = \left\{ \sum_z c_z \circ (B \mid A)_y : y \in \Omega_B \right\}
\]

\[
= \left\{ \sum_z c_z \circ \left( \sum_x a_x \circ b_y \right) : y \in \Omega_B \right\}
\]

\[
= \left\{ \sum_z c_z \circ (a_x \circ b_y) : y \in \Omega_B \right\}
\]

\[
= \left\{ \sum_{z,y} c^{1/2}_z a^{1/2}_x b_y a^{1/2}_z c^{1/2}_z : y \in \Omega_B \right\}
\]

In particular, if \( A = \{ P_{\alpha_x} \}, C = \{ P_{\beta_z} \} \) are atomic, we have that
\[
((B \mid A) \mid C) = \left\{ \sum_{z,x} |\langle \beta_z, \alpha_x \rangle|^2 \langle \alpha_x, b_y \alpha_x \rangle P_{\beta_z} : y \in \Omega_B \right\}
\]
Because of nonassociativity, the biconditional is different than
\[
(B \mid (A \mid C)) = \left\{ \sum_x (A \mid C)_x \circ b_y : y \in \Omega_B \right\}
\]
\[
= \left\{ \sum_x \left( \sum_z c_z \circ a_x \right) \circ b_y : y \in \Omega_B \right\}
\]
which cannot be simplified further even if $A, C$ are atomic.

4 Observable Operators

We now consider the observable operator $\hat{A} = \sum x a_x$ where $A = \{a_x : x \in \Omega_A\}$ and $\Omega_A \subseteq \mathbb{R}$. In general $\hat{A} \in \mathcal{L}_S(H)$ is not unique. If $A$ is atomic, then $\hat{A}$ is unique, the values $x \in \Omega_A$ are the eigenvalues of $A$ and $a_x$ is the projection for the corresponding eigenvector. If $f$ is a real-valued function $f : \mathbb{R} \to \mathbb{R}$, we define $\hat{f}(\hat{A}) = \sum f(x)a_x$. The reason we use the notation $\hat{f}$ is because $\hat{f}(\hat{A})$ is not the usual function of an operator. For example, if $f(x) = x^2$, then
\[
\hat{f}(\hat{A}) = \sum x^2a_x \neq (\hat{A})^2 = f(\hat{A})
\]
If $A$ happens to be sharp, then we do have $\hat{f}(\hat{A}) = f(\hat{A})$. In general, $\hat{A}$ determines $A$ because for any $a_x \in A$ there exists a polynomial $p_x$ such that $a_x = p_x(\hat{A})$.

If $\nu$ is a classical channel from $\Omega_A$ to $\Omega_B$, we define the function $f_\nu : \Omega_A \to \mathbb{R}$ by $f_\nu(x) = \sum_{y \in \Omega_B} y\nu_{xy}$. If we have another channel $\mu$ from $\Omega_B$ to $\Omega_C$, then the matrix product $\nu \mu$ is a classical channel from $\Omega_A$ to $\Omega_C$. Indeed, we have that
\[
\sum_z (\nu \mu)_{xz} = \sum_z \left( \sum_y \nu_{xy} \mu_{yz} \right) = \sum_y \nu_{xy} \sum_z \mu_{yz} = \sum_y \nu_{xy} = 1
\]
so $\nu \mu$ is stochastic.

Lemma 4.1. If $\nu$ and $\mu$ are classical channels as above and $A = \{a_x : x \in \Omega_A\}$ is an observable, then
\[
\mu \ast (\nu \ast A) = (\nu \mu) \ast A
\]
Proof. For all \( z \in C \) we have that
\[
[\mu (\nu \bullet A)]_z = \sum_y \mu_y (\nu \bullet A)_y = \sum_y \mu_y \sum_x \nu_{xy} a_x = \sum_x \sum_y \nu_{xy} \mu_y a_x \\
= \sum_x (\nu \mu)_{xz} a_x = [(\nu \mu) \bullet A]_z
\]
The result follows \( \square \)

The next result shows that \( \wedge \) preserves post-processing and mixtures.

**Theorem 4.2.** (i) Using the above notation, we have that
\[
(\nu \bullet A) \wedge = \widehat{f}_\nu (A)
\]
(ii) \([\mu (\nu \bullet A)] \wedge = \widehat{f}_{\nu \mu} (A)\). (iii) If \( \sum \lambda_i B^{(i)} \) is a mixture of the observables \( B^{(i)} \), then
\[
\left[ \sum \lambda_i B^{(i)} \right] \wedge = \sum \lambda_i \left[ B^{(i)} \right] \wedge
\]

**Proof.** (i) Since
\[
(\nu \bullet A)^\wedge = \sum_y y (\nu \bullet A)_y = \sum_y y \sum_x \nu_{xy} a_x = \sum_x \left( \sum_y y \nu_{xy} \right) a_x \\
= \sum_x f_\nu (x) a_x = \widehat{f}_\nu (A)
\]
The result follows.

(ii) The result follows from Part (i) and Lemma 4.1.

(iii) Since
\[
\left[ \sum \lambda_i B^{(i)} \right] \wedge = \sum_x x \left[ \sum_i \lambda_i B^{(i)} \right] \wedge = \sum_i \lambda_i \sum_x B^{(i)}_x = \sum \lambda_i \left[ B^{(i)} \right] \wedge
\]
the result follows \( \square \)

If \( A = \{ a(x,y) : (x,y) \in \Omega_A \} \) is an observable with value-space \( \Omega_A \subseteq \mathbb{R}^2 \), we define the observable operator of \( A \) by \( \hat{A} = \sum_{x,y} x y a(x,y) \). If \( a \in \mathcal{E}(H) \) and \( T \in \mathcal{L}(H) \) we use the notation
\[
(T \mid a) = a^{1/2} T a^{1/2} = a \circ T
\]

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Theorem 4.3. If $A = \{a_x : x \in \Omega_A\}$ and $B = \{b_y : y \in \Omega_B\}$ are real-valued observables then (i) $(B \mid A) = \sum_x (\hat{B} \mid a_x)$ and (ii) $(A \circ B) = \sum_x x(\hat{B} \mid a_x) = \sum_x x(a_x \circ \hat{B})$.

Proof. (i) The result follows from

$$(B \mid A) = \sum_y y(B \mid A)_y = \sum_y y \sum_x a_x \circ b_y = \sum_x \left( a_x \circ \sum_y y b_y \right) = \sum_x \left( a_x \circ \hat{B} \right) = \sum_x (\hat{B} \mid a_x)$$

(ii) The result follows from

$$(A \circ B) = \sum_{x,y} x y A \circ B (x,y) = \sum_{x,y} x y a_x \circ b_y = \sum_{x,y} x y a_x^{1/2} b_y a_x^{1/2} = \sum_x x a_x^{1/2} b_y a_x^{1/2} = \sum_x x (\hat{B} \mid a_x) = \sum_x x (a_x \circ \hat{B})$$

Example. The simplest example is the qubit Hilbert space $H = \mathbb{C}^2$ and dichotomic (two-valued) atomic observables $A = \{P_{\phi_1}, P_{\phi_2}\}$, $B = \{P_{\psi_1}, P_{\psi_2}\}$ where $\{\phi_1, \phi_2\}, \{\psi_1, \psi_2\}$ are orthonormal bases for $H$. The sequential product observable becomes

$$A \circ B = \{P_{\phi_1} \circ P_{\psi_1}, P_{\phi_1} \circ P_{\psi_2}, P_{\phi_2} \circ P_{\psi_1}, P_{\phi_2} \circ P_{\psi_2}\}$$

Letting $\Omega_A = \{x_1, x_2\}, \Omega_B = \{y_1, y_2\}$, $B$ conditioned on $A$ is the observable

$$(B \mid A)_{y_i} = (A \circ B)_{(x_1,y_i)} + (A \circ B)_{(x_2,y_i)} = |\langle \phi_1, \psi_i \rangle|^2 P_{\phi_1} + |\langle \phi_2, \psi_i \rangle|^2 P_{\phi_2}$$

for $i = 1, 2$. If $\Omega_A, \Omega_B \subseteq \mathbb{R}$, the observable operators become

$$\hat{A} = x_1 P_{\phi_1} + x_2 P_{\phi_2}, \quad \hat{B} = y_1 P_{\psi_1} + y_2 P_{\psi_2}$$

Applying Theorem [4.3](i) we obtain

$$(B \mid A) = \left( \phi_1, \hat{B} \phi_1 \right) P_{\phi_1} + \left( \phi_2, \hat{B} \phi_2 \right) P_{\phi_2}$$
\[ \begin{align*}
&= \left[ y_1 |\langle \phi_1, \psi_1 \rangle|^2 + y_2 |\langle \phi_1, \psi_2 \rangle|^2 \right] \Phi_1 \\
&\quad + \left[ y_1 |\langle \phi_2, \psi_1 \rangle|^2 + y_2 |\langle \phi_2, \psi_2 \rangle|^2 \right] \Phi_2 \\
&= \left[ y_2 + (y_1 - y_2) |\langle \phi_1, \psi_1 \rangle|^2 \right] \Phi_1 + \left[ y_2 + (y_1 - y_2) |\langle \phi_2, \psi_1 \rangle|^2 \right] \Phi_2 \\
&= \left[ y_2 + (y_1 - y_2) |\langle \phi_1, \psi_1 \rangle|^2 \right] \Phi_1 + \left[ y_2 + (y_1 - y_2) |\langle \phi_2, \psi_1 \rangle|^2 \right] \Phi_2 \\
&= \left[ y_2 + (y_1 - y_2) |\langle \phi_1, \psi_1 \rangle|^2 \right] \Phi_1 + \left[ y_2 + (y_1 - y_2) |\langle \phi_2, \psi_1 \rangle|^2 \right] \Phi_2 \\
&\quad + \left[ y_1 + (y_2 - y_1) |\langle \phi_1, \psi_1 \rangle|^2 \right] \Phi_2 \\
\end{align*} \]

Moreover, applying Theorem 4.3(ii) gives
\[ (A \mid B) \wedge \Phi = x_1 \langle \phi_1, \Phi \rangle \Phi_1 + x_2 \langle \phi_2, \Phi \rangle \Phi_2 \]
\[ = \left[ x_1 y_1 |\langle \phi_1, \psi_1 \rangle|^2 + x_1 y_2 |\langle \phi_1, \psi_2 \rangle|^2 \right] \Phi_1 \\
+ \left[ x_2 y_1 |\langle \phi_2, \psi_1 \rangle|^2 + x_2 y_2 |\langle \phi_2, \psi_2 \rangle|^2 \right] \Phi_2 \\
= x_1 \left[ y_2 + (y_1 - y_2) |\langle \phi_1, \psi_1 \rangle|^2 \right] \Phi_1 \\
+ x_2 \left[ y_2 + (y_1 - y_2) |\langle \phi_2, \psi_1 \rangle|^2 \right] \Phi_2 \\
= x_1 \left[ y_2 + (y_1 - y_2) |\langle \phi_1, \psi_1 \rangle|^2 \right] \Phi_1 \\
+ x_2 \left[ y_1 + (y_2 - y_1) |\langle \phi_1, \psi_1 \rangle|^2 \right] \Phi_2 \]
Proof. For sufficiency we have that
\[
I'_A = \left\{ \frac{1}{n-1} \left( \frac{1}{n} I x \right)' : x \in \Omega_A \right\} = \left\{ \frac{1}{n-1} (I - \frac{1}{n} I x) : x \in \Omega_A \right\} = \left\{ \frac{1}{n} I x : x \in \Omega_A \right\} = I_A
\]

For necessity, if \( A' = A \), we obtain for all \( x \in \Omega_A \) that
\[
a_x = \frac{1}{n-1} a'_x = \frac{1}{n-1} (I - a_x) = \frac{1}{n-1} I - \frac{1}{n-1} a_x
\]
This implies that \( a_x = \frac{1}{n} I \). Hence, \( A = I_A \).

The next result shows that complementation preserves conditioning and mixtures.

**Theorem 5.2.** (i) \( (B | A)' = (B' | A) \) for all \( A, B \in \mathcal{O}(H) \). (ii) If \( \lambda_i \in [0, 1], i = 1, 2, \ldots, m, \sum \lambda_i = 1 \) and \( A_i \in \mathcal{O}(H), i = 1, 2, \ldots, m, \) with the same value-spaces \( \Omega \), then
\[
\left( \sum \lambda_i A_i \right)' = \sum \lambda_i A'_i
\]

**Proof.** (i) The sequential product \( A \circ B' \) becomes
\[
A \circ B' = \left\{ a_x \circ \left( \frac{1}{n-1} b'_y \right) : (x, y) \in \Omega_A \times \Omega_B \right\} = \left\{ \frac{1}{n-1} a_x \circ b'_y : (x, y) \in \Omega_A \times \Omega_B \right\} = \left\{ \frac{1}{n-1} a_x \circ (I - b_y) : (x, y) \in \Omega_A \times \Omega_B \right\} = \left\{ \frac{1}{n-1} (a_x - a_x \circ b_y) : (x, y) \in \Omega_A \times \Omega_B \right\}
\]
Hence,
\[
(B' | A) = \left\{ \frac{1}{n-1} \sum_x (a_x - a_x \circ b_y) : y \in \Omega_B \right\} = \left\{ \frac{1}{n-1} \left( I - \sum_x a_x \circ b_y \right) : y \in \Omega_B \right\} = \left\{ \frac{1}{n-1} \left( \sum_x a_x \circ b_y \right)' : y \in \Omega_B \right\} = (B | A)'
\]
(ii) For $A_i = \{a_{ix}: x \in \Omega\}$ we have that
\[
\left( \sum \lambda_i A_i \right)' = \left\{ \frac{1}{n-1} \left( \sum \lambda_i a_{ix} \right)' : x \in \Omega \right\} = \left\{ \frac{1}{n-1} \left( I - \sum \lambda_i a_{ix} \right) : x \in \Omega \right\}
\]
\[
= \left\{ \frac{1}{n-1} \left( \sum \lambda_i I - \sum \lambda_i a_{ix} \right) : x \in \Omega \right\}
\]
\[
= \left\{ \frac{1}{n-1} \sum \lambda_i (I - a_{ix}) : x \in \Omega \right\} = \left\{ \frac{1}{n-1} \sum \lambda_i a'_{ix} : x \in \Omega \right\}
\]
\[
= \left\{ \sum \lambda_i \frac{1}{n-1} a'_{ix} : x \in \Omega \right\} = \sum \lambda_i A'_i \]
Proof. The statement clearly holds for \( m = 1 \). To show it holds for \( m = 2 \)
we have that
\[
A'' = \left\{ \frac{1}{n-1} \left( \frac{1}{n-1} a_x' \right) : x \in \Omega_A \right\} = \left\{ \frac{1}{n-1} \left( I - \frac{1}{n-1} a_x' \right) : x \in \Omega_A \right\}
\]
\[
= \left\{ \frac{1}{n-1} \left[ I - \frac{1}{n-1} \left( I - a_x \right) \right] : x \in \Omega_A \right\}
\]
\[
= \left\{ \left[ \frac{1}{n-1} - \frac{1}{(n-1)^2} \right] I + \frac{1}{(n-1)^2} a_x : x \in \Omega_A \right\}
\]
\[
= \left\{ \left\{ \frac{(n-2)m}{(n-1)^2} \frac{1}{n} I_x + \frac{1}{(n-1)^2} a_x : x \in \Omega_A \right\}
\]
\[
= \left[ 1 - \frac{1}{(n-1)^2} \right] I_A + \frac{1}{(n-1)^2} A
\]
(5.3)

Proceeding by induction, suppose the result holds for the integer \( m \). If \( m \)
is even, then (5.1) holds. Applying Lemma 5.1 and Theorem 5.2(ii) we conclude that
\[
A^{m+1} = \left[ 1 - \frac{1}{(n-1)^m} \right] I_A + \frac{1}{(n-1)^m} A'
\]
which is (5.2) with \( m \) replaced by \( m + 1 \). Hence, the result holds for \( m + 1 \).
If \( m \) is odd, then (5.2) holds. Again, by Lemma 5.1 and Theorem 5.2(ii) we obtain
\[
A^{m+1} = \left[ 1 - \frac{1}{(n-1)^{m-1}} \right] I_A + \frac{1}{(n-1)^{m-1}} A''
\]
Applying (5.3) we conclude
\[
A^{m+1} = \left[ 1 - \frac{1}{(n-1)^{m+1}} \right] I_A + \frac{1}{(n-1)^{m+1}} \left[ 1 - \frac{1}{(n-1)^2} \right] I_A + \frac{1}{(n-1)^{m+1}} A
\]
\[
= \left[ 1 - \frac{1}{(n-1)^{m+1}} \right] I_A + \frac{1}{(n-1)^{m+1}} A
\]
which is (5.1) with \( m \) replaced by \( m + 1 \). Hence, the result again holds for \( m + 1 \). It follows by induction that the result holds for all \( m \in \mathbb{N} \). \( \square \)

We conclude from Theorem 5.4 that if \( m \) is even, then \( A^m \) is the observable \( A \) with noise content \( \left[ 1 - \frac{1}{(n-1)^m} \right] \) and if \( m \) is odd, then \( A^m \) is the observable \( A' \) with noise content \( \left[ 1 - \frac{1}{(n-1)^{m-1}} \right] \). The dichotomic \( (n = 2) \) case is an exception and we then have that \( A^m = A \) when \( m \) is even and \( A^m = A' \) when \( m \) is odd. Notice that \( A' \) is a special case of a post-processing of \( A \). In fact, \( A' = \nu \cdot A \) where for all \( x, y \in \Omega_A \) we have that
\[
\nu_{xy} = \begin{cases} 
\frac{1}{n-1} & \text{if } x \neq y \\
0 & \text{if } x = y
\end{cases}
\]
6 A Different Viewpoint

We now consider conditioning from another viewpoint. Besides observables, measurable quantities are frequently represented by self-adjoint operators. For $T \in \mathcal{L}_S(H)$, the corresponding spectral observable is given by the unique sharp observable $P = \{P_x\}$ where $T = \sum xP_x$, $P_x \in \mathcal{P}(H)$, $x \in \mathbb{R}$. In this case, the $x$ are the distinct eigenvalues of $T$. Notice that $P$ is a real-valued observable and $\hat{P} = \sum xP_x = T$ so our concepts are consistent. Let $S \in \mathcal{L}_S(H)$, with $S = \sum yQ_y$, $Q_y \in \mathcal{P}(H)$, $y \in \mathbb{R}$ so $Q = \{Q_y\}$ is the spectral observable for $S$. Letting $\Omega_T, \Omega_S$ be the sets of eigenvalues for $T$ and $S$, respectively, we have that

$$Q \circ P = \{Q_y \circ P_x : x \in \Omega_T, y \in \Omega_S\}$$

and $(P \mid Q)_x = \sum_{y \in \Omega_S} Q_y \circ P_x$. We then define the operator $(T \mid S) \in \mathcal{L}(H)$ by $(T \mid S) = (P \mid Q)\wedge$ and call $(T \mid S)$ the operator $T$ conditioned on the operator $S$. We then have that

$$(T \mid S) = \sum_x x(P \mid Q)_x = \sum_x x \sum_y Q_y \circ P_x = \sum_x x \sum_y Q_y P_x Q_y$$

$$= \sum_y Q_y \left( \sum_x xP_x \right) Q_y = \sum_y Q_y TQ_y \quad (6.1)$$

It is interesting to note that $(T \mid S)$ depends on $T$ and $Q_y$, $y \in \Omega_S$, but not on the particular values of $y$.

**Lemma 6.1.** We have that $(T \mid S) = T$ if and only if $ST = TS$.

**Proof.** If $ST = TS$ then it is well-known that $Q_y T = TQ_y$ for all $y \in \Omega_S$. Applying (6.1) gives

$$(T \mid S) = \sum_y Q_y T = T$$

Conversely, suppose that $(T \mid S) = T$. Applying (6.1) again, we obtain

$T = \sum_y Q_y TQ_y$. It follows that

$$Q_y T = Q_y TQ_y = TQ_y$$

for all $y \in \Omega_S$ so that $ST = TS$. Notice that $T \mapsto (T \mid S)$ is a real linear function.
Theorem 6.2. (i) If \( T \geq 0 \), then \( (T \mid S) \geq 0 \). (ii) \( \text{tr } [(T \mid S)] = \text{tr } (T) \). (iii) If \( \rho \in \mathcal{S}(H) \), then \( (\rho \mid S) \in \mathcal{S}(H) \) and
\[
\text{tr } [\rho(T \mid S)] = \text{tr } [(\rho \mid S)T]
\]
Proof. (i) Assume that \( T \geq 0 \) and \( \phi \in H \). Applying (6.1) gives
\[
\langle \phi, (T \mid S)\phi \rangle = \sum_y \langle \phi, Q_y TQ_y \phi \rangle = \sum_y \langle \phi, Q_y \phi \rangle \geq 0
\]
Hence, \( (T \mid S) \geq 0 \). (ii) Again, applying (6.1) gives
\[
\text{tr } [(T \mid S)] = \text{tr } \left( \sum_y Q_y TQ_y \right) = \sum_y \text{tr } (Q_y T) = \text{tr } (T)
\]
(iii) If \( \rho \in \mathcal{S}(H) \), it follows from (i) and (ii) that \( (\rho \mid S) \in \mathcal{S}(H) \). Moreover, it follows from (6.1) that
\[
\text{tr } [\rho(T \mid S)] = \text{tr } \left[ \rho \sum_y Q_y TQ_y \right] = \text{tr } \left[ \sum_y Q_y \rho Q_y T \right]
\]
\[
= \text{tr } [(\rho \mid S)T]
\]
\[
\Box
\]
When \( Q \) is atomic with \( Q = \{ P_\psi \} \), then (6.1) becomes
\[
(T \mid S) = \sum_y \langle \psi_y, T \psi_y \rangle P_\psi = \sum_{x,y} x \langle \psi_y, P_x \psi_y \rangle P_\psi
\]
and when \( P \) is atomic with \( P = \{ P_\phi \} \), then (6.1) becomes
\[
(T \mid S) = \sum_{x,y} x \langle \phi_x, Q_y \phi_x \rangle P_{(Q_y \phi_x)}
\]
When \( P \) and \( Q \) are both atomic as above, then (6.1) gives
\[
(T \mid S) = \sum_{x,y} x |\langle \psi_y, \phi_x \rangle|^2 P_\psi
\]
where \( |\langle \psi_y, \phi_x \rangle|^2 \) is the transition probability from \( \phi_x \) to \( \psi_y \).

Although this technique is related to our previous method, it is not equivalent because the observables are sharp.
References

[1] A. Dvurečenskij and S. Pulmannová, *Difference posets, effects and quantum measurements*, *Int. J. Theor. Phys.* **33**, 819–850 (1994).

[2] S. Filippov, T. Heinosaari and L. Leppäjärvi, Simulability of observables in general probabilistic theories, *Phys. Rev A* **97**, 062102 (2018).

[3] D. J. Foulis and M. K. Bennett, Effect algebras and unsharp quantum logics, *Found. Phys.* **24**, 1325–1346 (1994).

[4] A. Gheondea and S. Gudder, Sequential product of quantum effects, *Proc. Amer. Math. Soc.* **132**, 503–512 (2004).

[5] S. Gudder, Sharp and unsharp quantum effects, *Adv. Appl. Math.* **20**, 169–187 (1998).

[6] S. Gudder and R. Greechie, Sequential products on effect algebras, *Rep. Math. Phys.* **49**, 87–111 (2002).

[7] S. Gudder and F. Latrémoilère, Characterization of the sequential product on quantum effects, *J. Math. Phys.* **49**, 052106 (2008).

[8] S. Gudder and G. Nagy, Sequential quantum measurements, *J. Math. Phys.* **42**, 5212–5222 (2001).

[9] T. Heinosaari and M. Ziman, The Mathematical Language of Quantum Theory, Cambridge University Press, Cambridge, 2012.

[10] M. Oszmaniec, L. Guerini, P. Wittek and A. Acin, Stimulating positive-operator valued measures with projective measurements, *Phys. Rev. Lett.* **119** 190501, (2017).

[11] M. Nielsen and I. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, Cambridge, 2000.