Inhomogeneous charged black hole solutions in asymptotically anti-de Sitter spacetime

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We investigate static inhomogeneous charged planar black hole solutions of the Einstein-Maxwell system in an asymptotically anti-de Sitter (AdS) spacetime. According to the AdS/CFT duality [1], the gravitational theory of a black hole spacetime is dual to a strongly coupled gauge theory at finite temperature. As a fascinating application of the duality, a holographic model of superconductors has been constructed from black hole solutions with charged scalar hair [2]. This indicates that there is a variety of black hole solutions in an asymptotically AdS spacetime, which could be useful to understand strongly correlated condensed matter physics.

Even in the Einstein-Maxwell system, many black hole solutions with various topologies can be constructed in an asymptotically AdS spacetime [3, 4], while a uniqueness theorem has been established for the static black hole solution without charge by restricting the topology to $S^2$ [5, 10]. The planar Reissner-Nordström AdS solution with $R^2$ topology is one of the black hole solutions [5]. It attracts much attention as a holographic model of strongly correlated condensed matter systems because the dual theory lives in a flat 2+1 dimensional spacetime.

When we apply the AdS/CFT duality to condensed matter systems, it is interesting to incorporate the lattice structure into the boundary theory. As it is typical in condensed matter physics, the lattice structure induces a periodic inhomogeneous electric potential. In the holographic theory, such inhomogeneity in the boundary theory corresponds to that of the gauge field in the bulk theory. So, in this paper, we construct inhomogeneous charged static black hole solutions, by perturbing the planar Reissner-Nordström AdS solution, and investigate their geometrical properties.

On the other hand, from the perspective of General Relativity, one of the issues in the bulk spacetime is whether a naked singularity appears. According to the (strong) cosmic censorship hypothesis [11], any singularity should be hidden inside the event horizon and the geometry cannot be extended beyond the Cauchy horizon. However, in the case of the Reissner-Nordström AdS solution, there exists a region where a naked singularity can be observed behind the regular Cauchy horizon. Even though the Cauchy horizon is unstable against dynamical perturbations due to the infinite blueshift [12], the possibility to observe a singularity should still remain because the resulting singularity would be a null weak singularity [13]. In the context of the AdS/CFT duality, the hypothesis was also argued in [14, 15].

Another issue regarding the cosmic censorship hypothesis is whether the zero temperature black hole solutions can have a regular event horizon. In a class of extremal black holes in string theory, it has been shown that the event horizon cannot be smooth even though the Kretschmann scalar curvature invariant $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ is small there [16]. Recently, a similar phenomenon was observed in the Lifshitz spacetime [17]. For the AdS black hole solutions with charged scalar hair, it has been shown that the event horizon cannot be regular in the extremal limit [18]. So, the extremal Reissner-Nordström AdS solution with a regular event horizon seems to be exceptional.

Therefore, in this paper, we will particularly investigate the geometry inside the event horizon of our inhomogeneous charged black hole solutions, both numerically and analytically. Being quite different from the unperturbed Reissner-Nordström AdS solution, the Kretschmann scalar curvature invariant becomes infinitely large toward the Cauchy horizon for any wavelength perturbations. In the extremal case, we show...
that the black hole solutions with long wavelength inhomogeneity cannot have a regular event horizon even though the Kretschmann scalar curvature invariant is small there. This is because the tidal force which an observer freely falling into the black hole experiences diverges towards the horizon. This tidal force is strong in the sense that the shear of any timelike geodesic congruence diverges infinitely. Hence, smooth extension of the geometry beyond the event horizon is impossible.

In the next section, we derive the static perturbed field equations of the planar Reissner-Nordström-AdS black hole solution. In Sec. III we numerically and analytically construct the non-extremal solutions and observe that the curvature blows up towards the Cauchy horizon for any wavelength perturbation. In Sec. IV, we analyze the extremal solutions. Conclusion and discussions are devoted in Sec. V.

II. STATIC PERTURBATIONS OF REISSNER-NORDSTRÖM-ADS BLACK HOLE

We consider the four-dimensional Einstein-Maxwell system in an asymptotically anti-de Sitter spacetime with the action

\[
S = \int d^4x \sqrt{-g} \left( R + \frac{6}{L^2} - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} \right),
\]

where \( L \) is the AdS curvature radius and \( F_{\mu \nu} = 2 \partial_\mu A_\nu \). The field equations are

\[
G_{\mu \nu} = \frac{3}{L^2} g_{\mu \nu} + \frac{1}{2} \left( F_{\mu \alpha} F_{\nu}^{\alpha} - \frac{1}{4} g_{\mu \nu} F^2 \right),
\]

\[
\sqrt{-g} \nabla_\mu F^\mu = \partial_\nu (\sqrt{-g} F^{\mu \nu}) = 0.
\]

The unperturbed plane-symmetric static black hole solution is the Reissner-Nordström-AdS black hole solution:

\[
ds^2 = \bar{g}_{\mu \nu}(u) \, dx^\mu \, dx^\nu,
\]

\[
g(u) := 1 - (1 + c) u^3 + cu^4,
\]

\[
\dot{A}_\mu dx^\mu = \dot{A}(u) du = \frac{2L^2}{r_+^2} (1 - u) \, dt,
\]

where \( Q \) and \( r_+ \) are the charge density and the radius of the black hole. The horizon and the spatial infinity are located at \( u = 1 \) and \( u = 0 \), respectively. Note that \( c \leq 3 \) for all the black hole solutions and the upper bound corresponds to the extremal black hole. According to the AdS/CFT duality, the chemical potential \( \mu \) is given by \( \mu = \lim_{u \to 0} \dot{A}(u) = 2QL^2/r_+^2 \).

We consider static linear perturbations of the solution \( 2.3 \) by adding a small chemical potential with sinusoidal fluctuation in the \( x \)-direction. In the scalar-type static perturbations, we can set \( g_{xx} = g_{yy} \) and \( g_{xx} = 0 \) by a suitable gauge choice \( 2.2 \). Then, we take the metric ansatz

\[
ds^2 = \frac{L^2}{u^2} \left[ - g(u)(1 + \epsilon a(u)e^{iqx}) \, dt^2 + \frac{1 + \epsilon b(u)u e^{iqx}}{g(u)} \, du^2 + \left( 1 + 2 \epsilon F(u) e^{iqx} \right)(dx^2 + dy^2) \right],
\]

with the gauge field

\[
A_\mu dx^\mu = A_i(u, x) \, dt = \left( \dot{A}_i + \frac{\epsilon \nu^2}{Q} B_i(u) e^{iqx} \right) \, dt.
\]

It is noteworthy that the variables \( a, b, \) and \( F \) agree with the gauge invariant quantities adopted in Ref. \( 2.2 \) for the gauge choice \( 2.4 \).

For the electromagnetic perturbations \( 2.6 \), we immediately obtain

\[
a(u) = -b(u)
\]

from the combination of \( xx \) and \( yy \) components of the field equations \( 2.2 \). Let us introduce a variable \( Y \) as

\[
Y(u) := b(u) - 2F(u) = -a(u) - 2F(u).
\]

In terms of the three variables, \( F, Y, \) and \( B_i \), the following four coupled differential equations are derived:

\[
0 = \ddot{Y} + \left( \frac{\dot{g}(u)}{g(u)} - \frac{2}{u} \right) (2F + Y) - \frac{2u^2}{g(u)} B_i, \tag{2.8a}
\]

\[
0 = \frac{u^3}{g(u)} \left( \dot{B}_i - \frac{2B_i}{u} \right) + u \left( 2 - \frac{\dot{g}(u)}{g(u)} \right) \dot{F}
\]

\[
+ \frac{2}{u} F + \left( 1 - \frac{q^2 u}{2g(u)} \right) Y, \tag{2.8b}
\]

\[
0 = \ddot{F} - \frac{q^2}{g(u)} \left( F + \frac{Y}{2} \right), \tag{2.8c}
\]

\[
0 = \ddot{B}_i - 4\epsilon \dot{F} - \frac{q^2}{g(u)} B_i, \tag{2.8d}
\]

where a dot means the derivative with respect to \( u \). The first and second equations \( 2.8a, 2.8b \) correspond to the momentum and the Hamiltonian constraint equations, respectively, when we foliate the spacetime by timelike hypersurfaces homeomorphic to the AdS boundary. These equations are not independent of each other because Eq. \( 2.8a \) is derived from Eqs. \( 2.8b, 2.8c \). Hence, Eqs. \( 2.8 \) admit four linearly-independent mode solutions.

To construct the solutions, we shall impose the following boundary conditions:

- The spacetime is an asymptotically AdS spacetime.
• The solution is regular at the horizon in the sense that the Kretschmann scalar curvature invariant \( \mathcal{K} := R_{\mu
u\rho\sigma}R^{\mu
u\rho\sigma} \) is bounded at the horizon,

\[
|\mathcal{K}(u = 1)| = |R_{\mu
u\rho\sigma}R^{\mu
u\rho\sigma}(u = 1)| < \infty. \tag{2.9}
\]

At the horizon, the gauge invariant part \( \delta \mathcal{K} =: \delta \mathcal{K}_q(u) e^{iqx} \) of the perturbation of the Kretschmann scalar curvature invariant is expressed as

\[
\delta \mathcal{K}_q(1) \simeq \frac{4\epsilon}{L^4} [q^2 \{(3 - 5c)Y + 6(1 - c)F\} - 2(3 - 5c)\{3c - 2F - 2B_t\}] + O(\epsilon^2), \tag{2.10}
\]

for the gauge choice (2.3). Then, using Eqs. (2.5), we can show that the condition (2.9) is satisfied if and only if the variables \( F, Y \), and \( B_t \) are finite at the horizon.

The asymptotic AdS boundary condition requires \( a(u = 0) = b(u = 0) = F(u = 0) = 0 \). As easily checked from Eqs. (2.8), \( a(u = 0) = b(u = 0) = 0 \) if \( F(u = 0) = 0 \). Therefore, the boundary conditions can be described in terms of the variables \( F, Y \), and \( B_t \) as

\[
F(u = 0) = 0, \quad |F(u = 1)|, \quad |Y(u = 1)|, \quad |B_t(u = 1)| < \infty. \tag{2.11a}
\]

As seen later, there are two mode solutions satisfying the condition (2.11b). If a perturbed chemical potential \( \delta \mu(x) = \delta \mu_0 e^{iqx} \) is given at the boundary, a unique solution of Eqs. (2.8) can be derived from the condition (2.11a). In other words, the normalization of the solution is determined by the amplitude \( \delta \mu_0 \) of the inhomogeneous chemical potential. For the discussions below, however, we are concerned with linear perturbations, and hence the amplitude of the perturbation can be renormalized at our disposal, and we will do so in what follows.

The condition (2.11a) guarantees that the dual theory lives in a flat 2+1 dimensional spacetime even though the bulk spacetime is inhomogeneous. In the case of vacuum perturbations, it is impossible to construct an inhomogeneous black hole solution under the boundary conditions, as the uniqueness theorem is established in the vacuum case \(^2\). In the following sections, we analytically and numerically construct inhomogeneous charged black hole solutions satisfying the boundary conditions (2.11).

### III. NON-EXTREME SOLUTIONS

#### A. Analytic solutions in the long wavelength limit

We can construct analytically an inhomogeneous non-extremal charged black hole solution of Eqs. (2.8) in the long wavelength limit \( q \to 0 \). In this subsection, we give the solution as the first few terms in a power series in \( q \). Under the boundary condition (2.11a), we can read off from Eq. (2.5) the behavior of the variables \( F, Y \), and \( B_t \) near the AdS boundary as

\[
F(u) \simeq u - \frac{q^2}{6} u^3, \quad Y(u) \simeq -4u + O(u^3), \quad B_t(u) \simeq \gamma_1 + \gamma_2 u. \tag{3.1}
\]

Here, we normalized \( F \) as \( \lim_{u \to 0} F(u)/u = 1 \) for simplicity.

We can adopt Eqs. (2.5a), (2.5b), and (2.5c) as fundamental equations of the static perturbations which admit four-independent solutions. As easily checked, Eq. (2.10) diverges at the horizon for the solutions that are non-analytic in a neighborhood of the horizon \( u = 1 \). So, the solutions satisfying the regularity condition (2.11b) are analytic in a neighborhood of the horizon. From the analyticity and Eq. (2.8d), we obtain

\[
B_t(1) = 0. \tag{3.2}
\]

Let us expand the variables \( F, Y \), and \( B_t \) as a series in \( q^2 \):

\[
F(u) = u + q^2 F_1(u) + O(q^4), \quad B_t(u) = B_{t0}(u) + q^2 B_{t1}(u) + O(q^4), \quad Y(u) = Y_0(u) + q^2 Y_1(u) + O(q^4). \tag{3.3}
\]

The zeroth and first order equations are

\[
0 = \dot{Y}_0 + \left( \frac{\dot{g}(u)}{g(u)} - \frac{2}{u} \right) (2F_0 + Y_0) - \frac{2u^2}{g(u)} B_{t0}, \tag{3.4a}
\]

\[
0 = \frac{u^3}{g(u)} \left( \ddot{B}_{t0} - \frac{2B_{t0}}{u} \right) + u \left( \frac{2}{u} - \frac{\dot{g}(u)}{g(u)} \right) \dot{F}_0 + \frac{2F_0 + Y_0}{u}, \tag{3.4b}
\]

and

\[
0 = \dot{Y}_1 + \left( \frac{\dot{g}(u)}{g(u)} - \frac{2}{u} \right) (2F_1 + Y_1) - \frac{2u^2}{g(u)} B_{t1}, \tag{3.5a}
\]

\[
0 = \frac{u^3}{g(u)} \left( \ddot{B}_{t1} - \frac{2B_{t1}}{u} \right) + u \left( \frac{2}{u} - \frac{\dot{g}(u)}{g(u)} \right) \dot{F}_1 + \frac{2F_1 + Y_1}{u} - \frac{u}{2g(u)} Y_0, \tag{3.5b}
\]

\[
0 = \dot{F}_1 - \frac{1}{g(u)} \left( F_0 + \frac{Y_0}{2} \right). \tag{3.5c}
\]

The solution of Eqs. (3.4) satisfying the boundary conditions (2.11b) and (3.2) is given by \(^3\)

\[
B_{t0}(u) = (1 - u) \left[ 3 + c - 2c u \right], \quad Y_0(u) = -2u \frac{2 + 2u - (1 + c) u^2}{1 + u + u^2 - c u^3}. \tag{3.6a}
\]

\[\text{Eq. (2.8d) is useful to guess the solution (3.6b).}\]
By similar procedure, we can construct the first order solutions, $F_1$, $B_1$, and $Y_1$. We give the explicit form in the Appendix B. Thus, we obtain the analytic solution up to $O(q^2)$ which satisfy both the asymptotic boundary condition (3.1) and the regularity conditions at the horizon, (2.11b) and (5.2).

B. Numerical solutions and the curvature growth near the Cauchy horizon

In this subsection we numerically construct inhomogeneous charged black hole solutions in an asymptotically AdS spacetime for various wavelengths. Eliminating $F(u)$ from Eqs. (2.3), we obtain two coupled differential equations,

$$g(u) \left( \frac{2g(u)}{u} - \dot{g}(u) \right) \dot{Y}(u)$$

$$- \left( 2g^2(u) - \frac{2g(u)\dot{g}(u)}{u} - 2\ddot{g}(u) + 4cu^2\dot{g}(u) \right) \dot{Y}(u)$$

$$- q^2 \left( \frac{2g(u)}{u} - \dot{g}(u) \right) Y(u) - 24B_t(u) = 0,$$  \hspace{1cm} (3.7a)

$$g(u)\{u\dot{g}(u) - 2g(u)\}^2 \dot{B}_t(u)$$

$$- 4cu^3g(u)(u\dot{g}(u) - 2g(u)) \dot{B}_t(u)$$

$$+ \{ -4(q^2 - 6cu^2)g^2(u) - q^2u^2\dot{g}^2(u) \} \dot{B}_t(u)$$

$$+ 4ug(u)\{ [q^2 - 4cu^2]\ddot{g}(u) + cu(-12 + u^2\ddot{g}(u)) \} \dot{B}_t(u)$$

$$+ 2cq^2u\dot{g}(u)(-2g(u) + u\ddot{g}(u) - 4cu^3)\ddot{Y}(u)$$

$$+ 2c^2u^2\dot{g}(u)(-2g(u) + u\ddot{g}(u))Y(u) = 0.$$  \hspace{1cm} (3.7b)

As mentioned in the previous subsection, the solutions satisfying the regularity conditions (2.11b) and (3.2) are analytic in a neighborhood of the horizon. By imposing analyticity for the variables $Y$ and $B_t$ we obtain $2\dot{Y}(1) = -q^2\dot{Y}(1)/(3-c)$ from Eqs. (3.7). For simplicity, we shall normalize $Y(1)$ as $Y(1) = 3 - c$ for the non-extremal black holes (Recall $c < 3$). Then, the only free parameter at the horizon is $B_t(1)$ for a fixed $q$ and $c$. By scanning through possible values of $B_t(1)$, we numerically find the value $B_t(1)$ satisfying the asymptotic boundary condition (2.11a) for each $q$ and $c$.

For later convenience, we shall introduce a parameter $\xi$ defined by

$$c = \xi + \xi^2 + \xi^3.$$  \hspace{1cm} (3.8)

Then, the Cauchy horizon (inner horizon) radius $u_i$ is represented by $\xi (< 1)$ as $u_i = 1/\xi$. In Figs. 1, 2, we give the numerical results for the short wavelength ($q = 3$) and long wavelength ($q = 0.5$) cases at $\xi = 0.5$, respectively.

We also solve Eqs. (3.7) from the event horizon towards the Cauchy horizon $u_i = 1/\xi$ of the black hole. We numerically find that, for any wave number $q$, $F(u)$ blows up towards the Cauchy horizon, and that the gauge invariant part $\delta K_q(u)$ of the perturbation of the Kretschmann scalar curvature invariant also does. Figs. 3 and 4 show the numerical results for the same parameters as in Figs. 1, 2.

In the long wavelength limit, $q \rightarrow 0$, $\delta K_q(u)$ can be expanded as $\delta K_q(u) = I_0(u) + q^2I_1(u) + O(q^4)$. Substituting the solution constructed in Sec. III into $\delta K_q(u)$, we find that $I_0(u)$ is finite at $u_i = 1/\xi$, but $I_1(u)$ blows up when $u \rightarrow 1/\xi$ as

$$I_1(u) \sim -\frac{24(1 + \xi)^2(1 + \xi^2)(5 - 3\xi)(8 - 3\xi)}{L^2\xi^6(1 + \xi(2 + 3\xi))^2} \ln(1 - \xi u).$$  \hspace{1cm} (3.9)

These numerical and analytical results imply that the inhomogeneous charged black hole geometries terminate
at singularity instead of the appearance of the Cauchy horizon.

Inside the event horizon, the timelike killing vector becomes spacelike. If one compactifies the three spacelike directions, \( t, x, \) and \( y \), the spacetime becomes \( R^1 \times T^3 \) Gowdy universe with two commuting spacelike killing vector fields, \( \partial_t \) and \( \partial_y \). As shown numerically and analytically in the vacuum case \cite{22 24}, such a universe generically terminates at spacelike curvature singularity, supporting the strong cosmic censorship conjecture. These results would not change even if electromagnetic field exists because the dominant energy condition is still satisfied. So, our analysis is consistent with the results \cite{22 24} and we conjecture that spacelike singularity generically appears instead of the appearance of the Cauchy horizon in the inhomogeneous charged black hole solutions.

\section*{IV. EXTREMLAL SOLUTIONS}

In this section, we investigate extremal inhomogeneous solutions of the two coupled equations \cite{3 7}. Introducing new variable \( \Gamma(u) \) as \( \Gamma = B_i(u)/(1 - u) \) and substituting \( c = 3 \), Eqs. \cite{3 7} are rewritten as

\begin{align}
(1 - u)^2(1 + 2u + 3u^2)(1 + u + u^2 + 3u^3)\dot{Y}(u) \\
- 6(1 - u)u^2(6 + 5u + 4u^2 + 3u^3)\dot{Y}(u) \\
- q^2(1 + u + u^2 + 3u^3)Y(1 - u) - 12u\dot{\Gamma}(u) = 0, \quad (4.1a) \\
(1 - u)^2(1 + u + u^2 + 3u^3)(1 + 2u + 3u^2)\Gamma(u) \\
- 2(1 - u)(1 + u + u^2)(1 + u + u^2 + 3u^3) × \\
(1 + 2u + 3u^2)\dot{\Gamma}(u) \\
- [q^2(1 + u + u^2 + 3u^3)^2 \\
+ 6u^2(3 + u + u^2 + u^3)(1 + 2u + 3u^2)]\Gamma(u) \\
+ 3(1 - u)(1 + 2u + 3u^2)\dot{Y}(u) \\
- 3q^2u(1 + u + u^2 + 3u^3)(1 + 2u + 3u^2)Y(u) = 0.
\end{align}

These equations \cite{1 1} can be transformed into four coupled first-order differential equations by introducing two variables, \( P \) and \( Q \) as

\begin{align}
P(u) := (1 - u)\dot{Y}(u), \quad Q(u) := (1 - u)\dot{\Gamma}(u).
\end{align}

Near the event horizon, \( u = 1 \), the four coupled first-order differential equations are represented as a regular matrix form:

\begin{align}
(1 - u) \dot{X} = MX \simeq \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 2 \\
q^2/6 & 1/3 & 0 & 0 \\
q^2/2 & 1 + q^2/6 & 0 & 0
\end{pmatrix} X,
\end{align}

where \( X = (Y, \Gamma, P, Q) \).

To find the indices characterizing the behavior of the solution near \( u = 1 \), we substitute the ansatz \( X_\lambda(u) = (1 - u)^\lambda H(u; \lambda) \) into Eq. \cite{1 1}. Thus, finding the indices \( \lambda \) is equivalent to finding the eigenvalues \( \lambda \) for the matrix \( M \), i. e., \( |M + \lambda I| = 0 \). The four eigenvalues and the corresponding eigenvectors \( H(1; \lambda) \) are given by

\begin{align}
\lambda_{nn} &= -\frac{1}{6} \left[ 3 + \sqrt[3]{15 + 2q^2 - 4\sqrt[3]{3 + q^2}} \right], \\
\lambda_{n} &= -\frac{1}{6} \left[ 3 - \sqrt[3]{15 + 2q^2 - 4\sqrt[3]{3 + q^2}} \right], \quad (4.4a) \\
\lim_{u \to 1} H(u; \lambda) &= (1, G(\lambda), -\lambda, -\lambda G(\lambda)), \quad (4.4b)
\end{align}

where

\begin{align}
G(\lambda) := 3\lambda^2 + 6\lambda - \frac{q^2}{2}.
\end{align}

We can obtain all the variables \( Y, B_i \), and \( F \) from each solution \( X_\lambda \) and Eq. \cite{2 8a}. It is easily checked that the Hamiltonian constraint equation \cite{2 8a} is automatically satisfied for each mode \( H(u; \lambda) \). Thus, the solutions of Eqs. \cite{2 8} are constructed from the four independent mode solutions, \( \{H(u; \lambda)\} \).

Since \( \lambda_{nn} < 0 \) for any \( q \neq 0 \), the mode solutions \( H(u; \lambda_{nn}) \) are “non-normalizable”, i. e., the variables...
and $F$ diverge at the horizon. Substituting the eigenvectors $[4.14b]$ into Eq. (2.10), we can show that the scalar curvature invariant $\hat{R}$ blows up at the horizon for each "non-normalizable" mode solution. So, we must abandon the "non-normalizable" solutions by the regularity condition (2.11a). Since $\lambda_{n\pm} > 0$ for any $q \neq 0$, the other two mode solutions $H(u; \lambda_{n\pm})$ are "normalizable" and Eq. (2.11a) is finite at the horizon. Then, the "normalizable" mode solutions $H(u; \lambda_{n\pm})$ are permitted as a regular solution at the horizon.

To construct an extremal inhomogeneous black hole solution in an asymptotically AdS spacetime, we must impose the asymptotic boundary condition (2.11a). Let us consider the one parameter family of solutions $H(u; p)$ of the equations (4.1) by superposing the two independent mode solutions, $H(u; \lambda_{n\pm})$ as $H(u; p) = H(u; \lambda_{n+}) + pH(u; \lambda_{n-})$ for a fixed $q$. By scanning all possible values of $p$, we find the value $p_0$ satisfying the asymptotic condition (2.11a). We numerically find the solutions $H(u; p_0)$ for various values of $q$. Two typical cases are shown in Fig. 5 ($q = 3$) and Fig. 6 ($q = 8$). The former case represents the solution where the derivatives of the metric functions diverge at the horizon, while the latter case represents a smooth solution at the horizon. As shown below, the singular behavior in the former case induces a curvature singularity at the horizon even though Eq. (2.11a) is satisfied. For each extremal black hole solutions with wave number $q$, $H(u; p_0)$ generically includes the mode solution $H(u, \lambda_{n-})$, as $p_0$ is not generically zero. Since $\lambda_{n-} < \lambda_{n+}$, the singular behavior must be encoded in the the mode solution $H(u, \lambda_{n-})$.

We calculate the curvature component in the frame parallelly propagated along a freely falling observer into the event horizon, for the mode solution $H(u, \lambda_{n-})$. If we take the real part of the metric, the tangential vector $V_0^\mu$ of the timelike geodesic of the freely falling observer within $x = y = 0$ is given by

$$
V_0^u = \dot{u} = \frac{E u^2}{L^2 g(u)} (1 + \epsilon b(u)),
$$

$$
V_0^\alpha = \dot{\alpha} = \sqrt{\frac{2 E^2 u^4}{L^4} - \frac{u^2 g'(u)}{L^2} (1 - \epsilon b(u))},
$$

$$
V_0^x = V_0^y = 0,
$$

within $x = y = 0$ is given by

$$
E_0 = \frac{u}{L \sqrt{g(u)}} \left( 1 + \frac{\epsilon}{2} b(u) \cos qx \right) \partial_t,
$$

$$
E_1 = \frac{u \sqrt{g(u)}}{L} \left( 1 - \frac{\epsilon}{2} b(u) \cos qx \right) \partial_u,
$$

$$
E_2 = \frac{u}{L} (1 - \epsilon F(u) \cos qx) \partial_x,
$$

$$
E_3 = \frac{u}{L} (1 - \epsilon F(u) \cos qx) \partial_y,
$$

a boost parameter $\alpha$ is defined by

$$
V_0 = \cosh \alpha \ E_0 + \sinh \alpha \ E_1.
$$

Some curvature components relevant below are calcu-
where we used Eq. (2.8c) in the second equality of \( R_{ttxx} \). Thus, near the horizon, the curvature component

\[ R_{\mu \nu \alpha \beta} V_0^\mu E_2^\nu V_0^\alpha E_2^\beta \]

in the parallelly propagated frame is calculated as

\[
R_{\mu \nu \alpha \beta} V_0^\mu E_2^\nu V_0^\alpha E_2^\beta = \cosh^2 \alpha R_{\mu \nu \alpha \beta} E_0^\mu E_2^\nu E_0^\alpha E_2^\beta \\
+ \sinh^2 \alpha R_{\mu \nu \alpha \beta} E_1^\mu E_2^\nu E_0^\alpha E_2^\beta \\
\approx \cosh^2 \alpha \left( \frac{\epsilon q^2 u(g(u) - 2u\dot{g}(u))}{L^2} + O((1 - u)^{\lambda_n+1}) \right) \\
\sim \frac{\epsilon E^2 q^2}{L^2} (1 - u)^{\lambda_n-2}. \tag{4.10}
\]

Here, we used \( \cosh^2 \alpha \approx E^2/L^2 g(u) \) near the horizon. Note that this divergence is a peculiar phenomenon for the inhomogeneous solution, as it originates in the curvature component \( R_{ttxx} \) at \( O(\epsilon) \). Thus, the inhomogeneous extremal solution generically includes p. p. curvature singularity \[26\] at the event horizon when

\[ \lambda_n < 2 \text{ i. e., } |q| < \sqrt{36 + 12\sqrt{3}}. \tag{4.11} \]

This implies that we cannot smoothly extend the geometry inside the event horizon even though the Kretschmann scalar curvature invariant remains small.

As shown below, we can see that this curvature singularity is a strong curvature singularity when

\[ \lambda_n < 1 \text{ i. e., } |q| < 2\sqrt{6} \tag{4.12} \]

is satisfied. Let us consider a timelike geodesic congruence with the tangent vector field \( V \) including the timelike geodesic with the tangent vector \( V_0 \). The shear tensor obeys the evolution equation along the timelike congruence as

\[
\frac{d\sigma_{\mu \nu}}{dt} \sim C_{\mu \nu \alpha \beta} V_0^\alpha E_2^\nu V_0^\alpha E_2^\beta \sim (1 - u)^{\lambda_n-2}. \tag{4.13}
\]

So, the shear tensor in the parallelly propagated frame diverges for any timelike geodesic congruence when \[4.12\] is satisfied. In this sense, the p. p. curvature singularity is a strong curvature singularity.

The p. p. curvature singularity has been observed in a class of extremal black holes in string theory \[16, 25\]. This is caused by the divergence of the stress-energy tensor of the dilaton field at the horizon, while in our case, the matter field associated with the gauge field \( \delta F_{\mu \nu} \) does not diverge there. Therefore, the origin of the p. p. curvature singularity is different between the inhomogeneous extremal black hole and the extremal black holes in string theory.

V. CONCLUSION AND DISCUSSIONS

We have investigated four-dimensional inhomogeneous charged black hole solutions in the anti-de Sitter backgrounds in the Einstein-Maxwell system. In the framework of linear perturbations, we have constructed the solutions where the inhomogeneity is induced by a spatially inhomogeneous chemical potential with wave number \( q \). For long wavelength limit, \( q \to 0 \), an analytic solution is constructed, up to \( O(q^2) \). By superposing the solutions with different wave numbers, we can obtain an inhomogeneous charged black hole solution for an arbitrary configuration of the chemical potential.

At the extremal case, p. p. curvature singularity generically appears at the event horizon for the long wavelength perturbations even though the Kretschmann scalar curvature invariant \( R_{\mu \nu \alpha \beta} R_{\mu \nu \alpha \beta} \) remains small. This implies that any freely-falling observer into the inhomogeneous black hole feels infinite tidal force at the event horizon. As the shear of any timelike geodesic congruence of the freely-falling observer diverges infinitely \( (4.12) \), the p. p. curvature singularity is a strong curvature singularity and the geometry cannot be smoothly extended into the inside of the black hole. Quite recently, it has been shown that the Lifshitz spacetime possessing the same p. p. curvature singularity is unstable against quantum corrections in string theory \[27\]. This suggests that the extremal black holes are also generically unstable against the quantum corrections, even though the extremal Reissner-Nordström AdS solution is stable.

The generic appearance of the p. p. curvature singularity would be associated with the inner causal structure of the non-extremal solutions. As shown in Sec. III, the perturbation with any wave length breaks down at

---

5 Along the timelike geodesic with tangent vector \( V_0 \), \( E_2 \) is a basis in the parallelly propagated frame.
the Cauchy horizon and the scalar curvature grows towards the Cauchy horizon. As discussed in Sec. III, this curvature growth suggests that the inner causal structure of the inhomogeneous black hole solution is similar to the one of the Schwarzschild-AdS spacetime (Fig. 7) rather than the unperturbed Reissner-Nordström-AdS spacetime (Fig. 8). Thus, the curvature at the event horizon grows as the Cauchy horizon approaches the event horizon. However, it is not still clear why short wavelength perturbation (large \( q \) case) does not yield the p. p. curvature singularity. It would be interesting to explore whether or not the perturbation of Reissner-Nordström-AdS spacetime with spherical or hyperbolic structure is a key ingredient to understand the energy gap or the band structure. So, it would be interesting to investigate the mechanism of energy dissipation in the periodic charged black hole background.

**FIG. 7:** The causal structure of Schwarzschild-AdS spacetime. \( r_+ \) is the event horizon.

**FIG. 8:** The causal structure of Reissner-Nordström-AdS spacetime. \( r_- \) is the Cauchy horizon.

Finally we refer to the AdS/CFT duality. According to the AdS/CFT duality, a charged black hole solution is dual to the boundary field theory at a finite temperature with a finite chemical potential. An inhomogeneous chemical potential induces an “electric” force in the charged matter of the boundary field theory. In our equilibrium configurations, we can show that this “electric” force balances with the pressure gradient,

\[
\partial_x \langle T^{xx} \rangle = \langle J^t \rangle \mathcal{E}_x , \tag{5.1}
\]

where \( \langle T^{ab} \rangle \) and \( \langle J^a \rangle \) \((a, b = t, x, y)\) are the expectation values of the energy-momentum tensor and the conserved current of the dual field theory, and \( \mathcal{E}_x \) is the “electric” field \( \mathcal{E}_x : = \lim_{u \to 0} F_{tx} \) on the boundary. The equation \((5.1)\) means that the pressure gradient is balanced with the “electric” force (see appendix \[A\] for details). It is one of the reasons why the Einstein-Maxwell system permits such inhomogeneous black hole solutions under the asymptotically AdS boundary condition, while the Einstein vacuum system does not permit them\(^6\). According to the AdS/CFT duality, our black hole solution is dual to the strongly coupled gauge theory under the periodic chemical potential in a flat \( 2+1 \)-dimensional spacetime. In condensed matter physics, such a periodic structure is a key ingredient to understand the energy gap or the band structure. So, it would be interesting to investigate the mechanism of energy dissipation in the periodic charged black hole background.

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**Appendix A: force balance equation**

It is convenient to derive various expectation values of a dual field theory in the framework of ADM-like decomposition in which a spacetime \( M \) is foliated by timelike hypersurfaces homeomorphic to the AdS boundary \( \partial M \):

\[
ds_4^2 = N^2 du^2 + \gamma_{ab} (dx^a + N^a du) (dx^b + N^b du) \tag{A1}
\]

\[
\lim_{u \to 0} \frac{L^2}{u^2} \left[ du^2 + \eta_{ab} dx^a dx^b + O(u^3) \right] ,
\]

Here, the AdS boundary is located at \( u = 0 \) and \( x^a \) are the coordinates on the \( u = \) const. timelike hypersurface \( \partial M_u \). The induced metric \( \gamma_{ab} \) on \( \partial M_u \) is given by \( \gamma_{\mu\nu} : = g_{\mu\nu} - n_\mu n_\nu \), where \( n^\mu \) is the unit normal to \( \partial M_u \) and is chosen to be “outward pointing”.

Using the extrinsic curvature \( K^{\mu\nu} : = \gamma^{\mu\lambda} \nabla_\lambda n^\nu \) of \( \partial M_u \) and the holographic counter-term \( S_{\text{ct}} \), the expectation values \( \langle T^{ab} \rangle \) and \( \langle J^a \rangle \) are expressed by \( \langle T^{ab} \rangle = \lim_{u \to 0} T^{ab} \) and \( \langle J^a \rangle = \lim_{u \to 0} J^a \), respectively, where

\[
T^{ab} = 2 \left( \frac{L}{u} \right)^5 \left( \gamma^{ab} K - K^{ab} + \frac{1}{\sqrt{-\gamma}} \frac{\delta S_{\text{ct}}}{\delta \gamma^{ab}} \right) , \tag{A2}
\]

\[
J^a = \left( \frac{L}{u} \right)^3 F^{a\mu} n_\mu . \tag{A3}
\]

\( S_{\text{ct}} \) must be chosen to cancel divergences that arise as \( \partial M_u \) approaches the AdS boundary \( \partial M \). In the following argument, we need the properties of \( S_{\text{ct}} \) which is invariant under the diffeomorphism on \( \partial M_u \). We may, however, refer to Ref. \cite{23} for the concrete expression of \( S_{\text{ct}} \).

---

\(^6\) If we relax the asymptotically AdS boundary condition, there exists the static inhomogeneous solutions even in the Einstein vacuum system. In that cases, the pressure gradient balances with the external gravitational force on the boundary.
The conservation laws of \( \{ T^{ab} \} \) and \( \{ J^a \} \) are encoded by the constraints in the bulk side,

\[
0 = -2 D_b (\gamma^{ab} K - K^{ab}) + (F^{b \mu} n_\mu) F^{a \mu}, \quad (A4)
\]
\[
0 = -D_a (F^{a \mu} n_\mu), \quad (A5)
\]

where \( D_a \) is the covariant derivative with respect to the induced metric \( \gamma_{ab} \) on \( \partial M_u \).

Since the Christoffel symbol \( \Gamma^c_{bc} [\gamma] \) with respect to \( \gamma_{ab} \) behaves as \( \Gamma^c_{bc} [\gamma] = O(u^3) \), the current conservation law holds as

\[
\partial_a \langle J^a \rangle = \lim_{u \to 0} D_a J^a = \lim_{u \to 0} \left( \frac{L}{u} \right)^3 D_a (F^{a \mu} n_\mu) = 0.
\]

Similarly, we obtain

\[
\partial_b \langle T^{ab} \rangle = \lim_{u \to 0} D_b T^{ab} = 2 \lim_{u \to 0} \left( \frac{L}{u} \right)^5 D_b (\gamma^{ab} K - K^{ab})
\]
\[
= \lim_{u \to 0} \left( \frac{L}{u} \right)^5 (F^{b \mu} n_\mu) F^{a \mu}
\]
\[
= \langle J^b \rangle \lim_{u \to 0} \eta^{ac} F_{cb}, \quad (A6)
\]

where we make use of \( D_b [(1/\sqrt{-\gamma}) (\delta S_{ct} / \delta \gamma_{ab})] = 0 \) because \( S_{ct} \) is invariant under the diffeomorphism on \( \partial M_u \), \( \gamma_{ab} \to \gamma_{ab} - 2D_a \xi_b \). Thus, we get the force-balance equation \( (A3) \) for the static inhomogeneous configurations in the \( x \)-direction.

### Appendix B: Solution at \( O(q^2) \)

In the long wavelength limit, an explicit inhomogeneous charged black hole solution is given, up to \( O(q^2) \), as

\[
B_{11}(u) = \frac{1}{2(1 + 2\xi + 3\xi^2)^3(3 + 2\xi + 3\xi^2)} \times
\]
\[
\left[ 4(1 - u)(1 + 2\xi + 3\xi^2) \times \right.
\]
\[
(3 + 20\xi(1 + \xi + \xi^2) + \xi^2(1 + \xi + \xi^2)^2(25 - 8u))
\]
\[
- 3(1 - \xi^2)^3(1 + \xi^2)(3 + 2\xi + 3\xi^2) \ln \left\{ \frac{(1 - \xi^2)}{3 + 2\xi + 3\xi^2} \right\} \times
\]
\[
+ 3(2 \ln(1 - \xi u) - \ln\{1 + (1 + \xi)u + (1 + \xi + \xi^2)u^2\} \times
\]
\[
(1 + \xi)^2(1 + \xi^2)(1 - u\{1 + \xi + \xi^2 + 3\xi^3 - 2\xi(1 + \xi + \xi^2)\})
\]
\[
+ (1 + \xi)(1 + \xi^2) \arctan \left( \frac{1 + \xi + 2u(1 + \xi + \xi^2)}{\sqrt{1 + 2\xi + 3\xi^2}} \right) \times
\]
\[
(1 + 2\xi + 3\xi^2)^3(3 + 2\xi + 3\xi^2)^3/2\]
\[
\left\{ 3 + u\{3(1 + \xi)(1 + \xi^2) - 2u(1 + \xi + \xi^2)\} \times (2 + 7\xi + 10\xi^2 + 16\xi^3 + 12\xi^4 + 9\xi^5)
\]
\[
+ \xi(25 + 61\xi + 109\xi^2 + 129\xi^3 + 111\xi^4 + 63\xi^5 + 27\xi^6) \right]\}
\]
\[
+ (1 + \xi)(1 + \xi^2) \arctan \left( \frac{1 + \xi}{\sqrt{1 + 2\xi + 3\xi^2}} \right) \times
\]
\[
(1 + 2\xi + 3\xi^2)^3(3 + 2\xi + 3\xi^2)^3/2\]
\[
\left\{ 3 + \xi + \xi^2 + \xi^3 - 3u\{1 + \xi\}(1 + \xi^2) + 2u^2\xi(1 + \xi + \xi^2) \right\} \times (2 + 7\xi + 10\xi^2 + 16\xi^3 + 12\xi^4 + 9\xi^5)
\]
\[
+ (1 + \xi)(1 + \xi^2) \arctan \left( \frac{3 + 3\xi + 2\xi^2}{\sqrt{1 + 2\xi + 3\xi^2}} \right) \times
\]
\[
(1 + 2\xi + 3\xi^2)^3(3 + 2\xi + 3\xi^2)^3/2\]
\[
\left\{ 9 + 48\xi + 100\xi^2 + 176\xi^3 + 198\xi^4 + 176\xi^5 + 100\xi^6 + 48\xi^7 + 9\xi^8 \right\}. \quad (B2)
\]
\[b_1(u) := Y_1(u) + 2F_1(u)\]
\[- = \frac{6u(1 + \xi^2)(1 - \xi u)(1 + \xi^2)(\xi + 2\xi^2 u - (1 + \xi + \xi^2)u^2)}{(1 - u)(1 + 2\xi + 3\xi^2)^3(1 + (1 + \xi + \xi^2)u^2)} \ln(1 - \xi u)
\[- = \frac{6u^3(1 - \xi^2)^3(1 + \xi^2)\ln(1 - \xi)}{(1 - u)(1 - \xi u)(1 + 2\xi + 3\xi^2)^3(1 + (1 + \xi)u + (1 + \xi + \xi^2)u^2)}\]
\[- \times u(1 - u)(1 + 2u - \xi(1 + \xi + \xi^2)u^2)(2 + 7\xi + 10\xi^2 + 16\xi^3 + 12\xi^4 + 9\xi^5)
\[- \times 6(1 + \xi)(1 + \xi^2)u^3 \arctan \left[ \frac{1 + \xi + 2u(1 + \xi + \xi^2)}{\sqrt{3 + 2\xi + 3\xi^2}} \right]
\[- \times 3u^3(1 - \xi^2)^2(1 + \xi^2)\ln(3 + 2\xi + \xi^2)
\[- \times (1 - u)(1 - \xi u)(1 + 2\xi + 3\xi^2)^3(3 + 2\xi + 3\xi^2)^{3/2}(1 + (1 + \xi)u + (1 + \xi + \xi^2)u^2)
\[- \times (2 + 7\xi + 10\xi^2 + 16\xi^3 + 12\xi^4 + 9\xi^5)(1 + (1 + \xi)(1 + \xi^2)u^3 - \xi(1 + \xi + \xi^2)u^4)
\[- + (3 + 25\xi + 61\xi^2 + 109\xi^3 + 129\xi^4 + 111\xi^5 + 63\xi^6 + 27\xi^7)u^2]
\[- 3u^3(1 - \xi^2)^2(1 + \xi^2)\ln(3 + 2\xi + \xi^2)
\[- \times 3\xi(1 + \xi + \xi^2)(24 + 29\xi(1 + \xi + \xi^2))]. \quad (B3)\]

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