Non-Local Pseudo-Differential Operators

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Abstract

We define, in a consistent way, non-local pseudo-differential operators acting on a space of analytic functionals. These operators include the fractional derivative case. In this context we show how to solve homogeneous and inhomogeneous equations associated with these operators. We also extend the formalism to $d$-dimensional space-time solving, in particular, the fractional Wave and Klein-Gordon equations.
1 Introduction

Interest in non-local field theories has always been present in theoretical physics, associated to several different motivations.

In Ref. [1], J. A. Wheeler and R. P. Feynman considered a description of the interaction between charged particles, where the electromagnetic field does not appear as a dynamical variable (action at distance).

More recently, efforts were made to use non-local theories in connection with the understanding of quark confinement and anomalies [2][3] and in string theories containing non-local vertices [4][5].

Besides the possibility of non-local interactions, a field theory can also display non-local kinetic terms.

Before renormalization theory became well established the possibility was considered of formulating finite theories by means of non-local kinetic lagrangians. A. Pais and G. E. Uhlembeck [6] were one of the first to analyze non-local theories in this context.

The analytic regularization method introduced in Ref. [7], can be thought of as associated to non-local kinetic terms in the lagrangian, specifically to fractional wave and Klein-Gordon equations.

This type of non-locality also arises in effective theories, when integrating over some degrees of freedom in an underlying local field theory [8][9][10].

At a classical level, non-local equations containing arbitrary powers of the D’alambertian ($\Box$) have been studied in Ref. [11]. In this reference a non-trivial relation between the number of dimensions and the power of the operator was established to satisfy the Huygens principle (see Ref. [12]). In particular, in 2+1 dimensions the usual wave equation leads to a Green function that does not satisfy Huygens principle, while a non-local equation with $\Box^{1/2}$, does satisfy this principle (see also Ref. [13]). It is not by chance that the pseudo-differential operator $\Box^{1/2}$ appears also in the process of bosonization in 2 + 1 dimensions. In Refs. [14] and [15] a mapping was established between Dirac’s kinetic term and bosonic terms involving $\Box^{1/2}$. In Ref. [16] it is shown how similar terms appear when 3 + 1 dimensional QED is projected to a physical plane. The kinetic term contains $\Box^{-1/2}$, which in the static limit reproduces correctly the $r^{-1}$ Coulomb potential, instead of the usual logarithmic behavior of 2 + 1 QED. This fact was first noticed in Ref. [12].

Also, in Ref. [17], a fractional generalized Fokker-Planck-Kolmogorov equation is proposed to describe anomalous transport in Hamiltonian sys-
tems, and in Refs. [18] [19] some particular Green functions for fractional diffusion and fractional wave equations were obtained.

In Refs. [13] and [20], solutions to some particular non-local homogeneous equations were proposed on physical grounds.

In this paper, we present a general mathematical approach to deal with pseudo-differential operators which enables us to obtain the whole space of solutions to homogeneous equations associated to them. Therefore, for a given pseudo-differential operator, it is possible to define a general Green function (having a general prescription to avoid the singularities).

In section 2, we define the functional spaces in which we build-up the framework for a correct description of pseudo-differential operators. In these spaces we introduce the representation of analytic functionals as “ultra distributions” [21] [22], a convenient way to handle and operate with the usual Green functions of Quantum Field Theories.

In Section 3 we define local and non-local operators acting on ultra distributions. We write and solve formally, homogeneous and inhomogeneous equations associated to them.

In section 4 some examples are examined, such as rational and power functions of the derivative operator. Solutions of the respective homogeneous equations are given. Also, Green functions are found, and their relations to fractional derivatives of Ref. [8] are exhibited.

In section 5, we extend our developments to space-time, by defining non-local functions of $\Box$. Also, fractional wave and Klein-Gordon equations are introduced, and solutions for the respective homogeneous equations are given. Corresponding Green functions are found.

Finally, section 6 is devoted to a discussion of the developments and results of the paper.

2 Functional Spaces

We will start with the space $\zeta$ of entire analytic test functions $\varphi(k)$, rapidly decreasing in any horizontal band. We will call “ultra-analytic”, any function $\varphi(k) \in \zeta$. They are Fourier transforms of the space $\hat{\zeta}$ of all $C^\infty$ functions $\varphi(x)$, such that $\exp(q |x| D^p \varphi(x))$ is bounded in $R$ for any $q$ and $p$.

In view of the latter property, $\zeta \supset Z$, where $Z$ is the space of Fourier transforms of $K$ ($C^\infty$ functions on a compact set). ([23], ch. 2, §1.1)
The dual of \( \zeta \) is the space \( \zeta' \) of linear functionals defined on \( \zeta \). In \( \zeta' \) we can represent the propagators of a quantum field theory as analytic functionals, with the physical properties that are expected from them [24].

The general form of an analytic functional is:

\[
\psi(\varphi) = \int_{L} \psi(k) \varphi(k) \, dk
\] (1)

The “density” \( \psi(k) \) is an analytic function and the line \( L \) can be deformed as long as no singularity of \( \psi(k) \) is crossed. \( L \) can be an open line or a closed loop.

Not only \( L \) can be deformed without altering \( \psi(\varphi) \). Also, the structure of singularities of the density can be modified. For example, when \( \psi(k) = (k-\tau)^{-1} \), the development: \( (k-\tau)^{-1} = \sum_n \tau^n k^{-n-1} \), allows the pole at \( k = \tau \) to be represented by a series of multipoles located at the origin. This fact is closely related to the expansion of the analytic functional \( \delta(k-\tau) \) as a series of \( \delta^{(n)}(k) \) ([23], ch. 2, §1.4). We can see then, that an analytic functional can be expressed in more than one way.

We are going to use, systematically, the following representation [21][22]:

\[
\psi(\varphi) = \int_{\Gamma} dk \, \psi(k) \varphi(k) \quad (\psi \in \zeta', \ \varphi \in \zeta)
\] (2)

where \( \psi(k) \) is analytic in \( \{ k \in C : |\text{Im } k| > \rho \} \) and \( \psi(k)/k^\rho \) is bounded continuous in \( \{ k \in C : |\text{Im } k| \geq \rho \} \), \( \rho \) depending on \( \psi(k) \), \( \rho \in N \) (\( N \) = set of entire numbers).

The path \( \Gamma \) runs from \(-\infty\) to \(+\infty\) along \( \text{Im } k > \rho \) and from \(+\infty\) to \(-\infty\) along \( \text{Im } k < -\rho \).

If \( \psi(k) \) is an entire function \( a(k) \), then (2) is the zero functional \( (a(k) \) need not be ultra analytic).

\[
\psi_0 = \int_{\Gamma} dk \ a(k) \varphi(k) \equiv 0
\] (3)

Equation (3) tell us that \( \psi(k) \) and \( \psi(k) + a(k) \) represents one and the same functional \( \psi(\varphi) \).

If we take

\[
\psi(k) = \frac{1}{2} \text{Sg}(\text{Im } k)
\] (4)
then
\[
\psi(\varphi) = \frac{1}{2} \int_{\Gamma} dk \ Sg(Im k) \varphi(k)
\]
\[
= \frac{1}{2} \int_{-\infty}^{+\infty} dk \ \varphi(k) - \frac{1}{2} \int_{+\infty}^{-\infty} dk \ \varphi(k)
\]
\[
= \int_{-\infty}^{+\infty} dk \ \varphi(k) = 1(\varphi)
\] (5)
The density given by (4) implies that \(\psi(\varphi)\) is the unit functional.

When we take \(\psi(k) \equiv 0\) for \(Im k > \rho\) (resp. for \(Im k < -\rho\)) the ultra distribution is the retarded (resp. advanced) functional.

Feynman’s causal function can also be represented as as ultra distribution. In fact, the propagator is:
\[
\frac{1}{k_0^2 - k^2 - m^2} = \frac{1}{k_0^2 - \omega^2} = \frac{1}{2\omega} \left( \frac{1}{k_0 - \omega} - \frac{1}{k_0 + \omega} \right), \ \omega > 0
\] (6)
The causal Green function is such that the positive energy pole (at \(k_0 = \omega\)) is propagated by a retarded potential and the negative energy pole propagates in an advanced way. To represent this Green function we take
\[
\psi_F(k) = \frac{1}{2\omega} \frac{1}{k - \omega}, \ \text{for} \ Im k > 0
\]
\[
\psi_F(k) = \frac{1}{2\omega} \frac{1}{k + \omega}, \ \text{for} \ Im k < 0
\] (7)
If we interchange the poles, taking into account the senses of the path \(\Gamma\), we get the anticausal Green function:
\[
\psi_A(k) = -\frac{1}{2\omega} \frac{1}{k + \omega}, \ \text{for} \ Im k > 0
\]
\[
\psi_A(k) = -\frac{1}{2\omega} \frac{1}{k - \omega}, \ \text{for} \ Im k < 0
\] (8)
Wheeler propagator (half causal and half anticausal) is then
\[
\psi_W(k) = \frac{1}{2} \frac{1}{k^2 - \omega^2}, \ \text{for} \ Im k > 0
\]
\[
\psi_W(k) = -\frac{1}{2} \frac{1}{k^2 - \omega^2}, \ \text{for} \ Im k < 0
\]
\[
\psi_W(k) = \frac{1}{2} Sg(Im k) \frac{1}{k^2 - \omega^2}
\] (9)
We shall also need the Fourier transforms of the spaces $\zeta$ and $\zeta'$. For a function $\varphi \in \zeta$, the Fourier transform is:

$$\mathcal{F}\varphi = \hat{\varphi}(x) = \int_{-\infty}^{+\infty} dk \, \varphi(k) e^{-ikx} \quad (10)$$

All functions $\hat{\varphi}(x)$ are $C^\infty$ and $\exp q|x|D^p \hat{\varphi}(x)$ is bounded for any $q$ and $p$. They form the space $\hat{\zeta}$.

A natural definition for the Fourier transforms of the functionals $\psi \in \zeta'$ is:

$$\mathcal{F}\psi(\mathcal{F}\varphi) = \hat{\psi}(\hat{\varphi}) = 2\pi \psi(\varphi) \quad (11)$$

For an ultra distribution represented by (1):

$$2\pi \psi(\varphi) = \hat{\psi}(\hat{\varphi}) = \int_{\Gamma} dk \, \psi(k) \int_{-\infty}^{+\infty} dx \, \hat{\varphi}(x) e^{ikx}$$

$$\hat{\psi}(\hat{\varphi}) = \int_{-\infty}^{+\infty} dx \, \hat{\varphi}(x) \int_{\Gamma} dk \, \psi(k) e^{ikx}$$

So that:

$$\hat{\psi}(\hat{\varphi}) = \int_{-\infty}^{+\infty} dx \, \hat{\varphi}(x) \hat{\varphi}(x) \quad (12)$$

where

$$\hat{\psi}(x) = \int_{\Gamma} dk \, \psi(k) e^{ikx} \quad (13)$$

$\psi(k)$ is a density for $\hat{\varphi}(x) \in \hat{\zeta}$. Of course, if $\psi(k) = a(k)$ (entire analytic), $\hat{\psi}(x) = 0$. So that $\hat{\psi}(x)$ is not altered if we add $a(k)$ to $\psi(k)$.

It should be clear that Eq. (13) is to be understood in the sense of distributions. For example, if $\psi(k)$ is the density for the unit functional (Eq. (1)):

$$\hat{\psi}(x) = \frac{1}{2} \int_{\Gamma} dk \, Sg(Imk) e^{ikx}$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} dk \, e^{ikx} - \frac{1}{2} \int_{+\infty}^{-\infty} dk \, e^{ikx}$$

$$\hat{\psi}(x) = 2\pi \delta(x) \quad (14)$$
3 Pseudo-differential Operators

Our aim is to work properly with some non-local pseudo-differential operators. For this reason we are going to introduce the following operation on the functionals $\hat{\psi}$.

Let us consider a function $f(k)$ such that: $f(k)$ is analytic in $\{k \in \mathbb{C} \mid |\text{Im}k| > \beta\}$ and $f(k)/k^\beta$ is bounded continuous in $\{k \in \mathbb{C} \mid |\text{Im}k| \geq \beta\}$, $\beta$ depends on $f(k)$, $\beta \in \mathbb{N}$.

Then we define

$$f\hat{\psi}(\hat{\phi}) = \int_{-\infty}^{+\infty} \left[ f \left( -i \frac{d}{dx} \right) \hat{\psi}(x) \right] \hat{\phi}(x) \quad (15)$$

where:

$$f(-i \frac{d}{dx}) \hat{\psi}(x) = \int_{\Gamma} dk \ f(k) \psi(k) e^{ikx} \quad (16)$$

and where the path $\Gamma$ runs from $-\infty$ to $+\infty$ along $\text{Im}k > \beta + \rho$ and from $+\infty$ to $-\infty$ along $\text{Im}k < -\beta - \rho$.

We know that the functional $\hat{\psi}(\hat{\phi})$ does not change when we add an arbitrary entire function $a(k)$ to $\psi(k)$. However, such an addition in Eq. (16) gives rise to a new term

$$A(x) = \int_{\Gamma} dk \ f(k) a(k) e^{ikx} \quad (17)$$

When $f(z)$ is an entire function, $A(x) \equiv 0$. For example for polynomial functions of the derivative operator. If $f(z) = z^{-1}$, $f(-i \frac{d}{dx})$ is the inverse of the derivative (an integration). Equation (16) gives a primitive of $\hat{\psi}(x)$. The additional term (17) is:

$$A(x) = \int_{\Gamma} dk \ \frac{a(k)}{k} e^{ikx} = \int_{L} dk \ \frac{a(k)}{k} e^{ikx} \quad (18)$$

where $L$ is a loop around the origin. So that:

$$A(x) = -2\pi i a(0) = \text{arbitrary constant}$$

of course, an integration should give a primitive plus an arbitrary constant.
Analogously, a double (iterated) integration, with \( f(z) = z^{-2} \), gives a primitive plus (17):

\[
A(x) = \int_L dk \frac{a(k)}{k^2} e^{ikx} = \gamma + \delta x
\]

where \( \gamma \) and \( \delta \) are arbitrary constants, as it should be.

It is understandable that a more complex structure of singularities of \( f(z) \) gives rise to a more complicated \( A(x) \).

To solve some pseudo-differential equations, we can work directly with the densities, the null functional having an arbitrary entire function as density. (an arbitrary constant has \( k^{-1}a(k) \) as density, etc.)

For a solution to a homogeneous equation:

\[
f(-i\frac{d}{dx}) \hat{\psi}(x) = 0 \tag{19}
\]

we write

\[
f(k)\psi(k) = a(k)
\]

whose solution is:

\[
\psi(k) = f^{-1}(k)a(k) \tag{20}
\]

then replacing in Eq. (13) we obtain:

\[
\hat{\psi}(x) = \int_{\Gamma} dk f^{-1}(k)a(k)e^{ikx} \tag{21}
\]

In the case where the singularities of \( f^{-1} \) are concentrated on the real axis, the path \( \Gamma \) can be deformed to get:

\[
\hat{\psi}(x) = \int_{-\infty}^{+\infty} dk \left[ f^{-1}(k + i0) - f^{-1}(k - i0) \right] e^{ikx} \tag{22}
\]

To solve the inhomogeneous equation:

\[
f(-i\frac{d}{dx}) \hat{\psi}(x) = \hat{\chi}(x) \tag{23}
\]

where \( \hat{\chi}(x) \) is a given (generalized) function, we write

\[
\begin{align*}
f(k)\psi(k) &= \chi(k) + a(k) \\
\psi(k) &= f^{-1}(k)\chi(k) + f^{-1}(k)a(k) \tag{24}
\end{align*}
\]
where \( \hat{\chi}(x) = \int_{\Gamma} dk \ (\chi(k) + a(k)) \exp ikx \). The last term in Eq. (24) can be recognized as a general solution to the homogeneous equation (19).

If in (23) we choose \( \hat{\chi}(x) = \delta(x) \), we get the equation for the Green function:

\[
f(-i \frac{d}{dx}) \hat{G}(x) = \delta(x)
\]

As a density for Dirac's function is \( \frac{1}{2} \text{Sg}(\text{Im} \ k) \) (cf. (14)), we obtain from (24):

\[
G(k) = \frac{1}{2} f^{-1}(k) \text{Sg}(\text{Im} \ k) + f^{-1}(k)a(k)
\]

Eq. (26) shows that, as usual, the Green function is defined up to a solution to the homogeneous equation. The different choices of \( a(k) \) determine the different prescriptions (advanced, retarded, Feynman, Wheeler, etc.) to avoid the singularities, when evaluating the Fourier transform. In the case where \( a = 0 \) we have:

\[
G(k) = \frac{1}{2} f^{-1}(k) \text{Sg}(\text{Im} \ k) + f^{-1}(k)a(k)
\]

It is interesting to find the Fourier transform of (27):

\[
\hat{G}(x) = \frac{1}{2} \int_{\Gamma} dk \ f^{-1}(k) \text{Sg}(\text{Im} \ k) e^{ikx}
\]

I. e. :

\[
\hat{G}(x) = \frac{1}{2} \int_{-\infty}^{+\infty} dk \ (f^{-1}(k + i0) + f^{-1}(k - i0)) e^{ikx}
\]

\[
\hat{G}(x) = g_{+}(x) + g_{-}(x)
\]

where

\[
g_{\pm} = \frac{1}{2} \int_{-\infty}^{+\infty} dk \ f^{-1}(k \pm i0) e^{ikx}
\]

Let us now consider the composition of two operators \( f_1 \) and \( f_2 \), to give \( f(-i \frac{d}{dx}) = f_1(-i \frac{d}{dx}) \circ f_2(-i \frac{d}{dx}) \). For the densities we have \( f(k) = f_1(k) f_2(k) \) and practically nothing changes in the analysis we have made above. However, if we apply the two operators in succession we have the following situation:
For the equation

\[ f_2(-i\frac{d}{dx}) \left[ f_1(-i\frac{d}{dx}) \hat{\psi} \right] = 0 \]  \tag{30}

the square bracket is a solution of the homogeneous equation for \( f_2 \). Accordingly:

\[ f_1(-i\frac{d}{dx}) \hat{\psi}(x) = \int_{\Gamma} dk \ f_1^{-1}(k) a(k) e^{ikx} \]

Now we have an inhomogeneous equation for \( f_1 \), whose solution is

\[ \hat{\psi}(x) = \int_{\Gamma} dk \ f_1^{-1}(k)f_2^{-1}(k)a(k) e^{ikx} + \int_{\Gamma} dk \ f_1^{-1}(k)b(k) e^{ikx} \]  \tag{31}

If we interchange \( f_1 \) and \( f_2 \) in Eq. (30), the solution (31) changes in an obvious way.

On the other hand, if \( f_1 \) and \( f_2 \) are first composed to give \( f = f_1 \circ f_2 \), the corresponding solution (31) loses its last term.

4 Some Examples

4.1 Rational Functions

We consider first a rational function (see also Ref. [23], Ch. 2, §1.5),

\[ f(-i\frac{d}{dx}) = \frac{P(-i\frac{d}{dx})}{Q(-i\frac{d}{dx})} \]  \tag{32}

\[ P(k) = \prod_{i=1}^{n}(k-k_i) \quad Q(k) = \prod_{j=1}^{n}(k-k'_j) \]  \tag{33}

\( (k_i \neq k'_j \text{ for all } i \text{ and } j) \)

The solution to the homogeneous equation:

\[ \frac{P(-i\frac{d}{dx})}{Q(-i\frac{d}{dx})} \hat{\psi}(x) = 0 \]  \tag{34}
is given by (19) and (20)

\[ \hat{\psi}(x) = \int_\Gamma dk \frac{Q(k)}{P(k)} a(k) e^{ikx} \]  

If we rewrite (34) in the form:

\[ Q^{-1}(-i\frac{d}{dx}) \left[P(-i\frac{d}{dx})\hat{\psi}(x)\right] = 0 \]  

The solution (31) is:

\[ \hat{\psi}(x) = \int_\Gamma dk \frac{Q(k)}{P(k)} a(k) e^{ikx} + \int_\Gamma dk \frac{b(k)}{P(k)} e^{ikx} \]  

The square bracket in (37) is another arbitrary entire function. So that (37) is in fact equivalent to (35).

If now we write the equation:

\[ P(-i\frac{d}{dx}) \left[Q^{-1}(-i\frac{d}{dx})\hat{\psi}(x)\right] = 0 \]  

The solution (31) is:

\[ \hat{\psi}(x) = \frac{1}{P(k)} \left[Q(k) a(k) + b(k)\right] e^{ikx} \]  

But the last integral is zero, as the product \( Qb \exp ikx \) is entire analytic.

The three forms (34), (35) and (36) are completely equivalent. The order of the operators \( P \) and \( Q \) is inmaterial.

For the actual solution of (35), we take into account that

\[ \frac{1}{P(k)} = \frac{1}{\prod(k - k_i)} = \sum_{i=1}^{n} \frac{\alpha_i}{k - k_i} \]  

where \( \alpha_i \) are appropriate constants (For the sake of simplicity we have assumed that all roots are simple).
The only singularities of the integrand in (35) are the poles at \( k = k_i \). The integral can then be evaluated by Cauchy’s theorem:

\[
\hat{\psi}(x) = \sum_{i=1}^{n} \alpha_i \int_{\Gamma} dk \frac{Q(k_i)}{k - k_i} a(k) e^{ikx}
\]

\[
= 2\pi i \sum_{i=1}^{n} \alpha_i Q(k_i) a(k_i) e^{ik_i x}
\]

\[
= \sum_{i=1}^{n} \beta_i e^{ik_i x}
\]  

(40)

where \( \beta_i \) are arbitrary constants. It is easy to check that (40) solves equations (34), (36) and (38).

4.2 Power Functions

For another (less simple) example, let us take the power function:

\[
f(z) = z^\alpha
\]  

(41)

where we introduce a cut along the negative real axis.

For the homogeneous equation

\[
\left( -i \frac{d}{dx} \right)^\alpha \hat{\psi}(x) = 0
\]  

(42)

we have: \( \psi(k) = k^{-\alpha} a(k) \) and

\[
\hat{\psi}(x) = \int_{\Gamma} dk \ k^{-\alpha} a(k) e^{ikx}
\]  

(43)

The integrand is analytic, except on the (negative) real axis. We may write:

\[
\hat{\psi}(x) = \int_{-\infty}^{\infty} dk \ \left[ (k + i0)^{-\alpha} - (k - i0)^{-\alpha} \right] a(k) e^{ikx}
\]  

(44)

But, (Ref. [23], Ch. 1, §3.6)

\[
(k + i0)^{-\alpha} = k^{\alpha}_+ e^{-ik\alpha} k^{-\alpha}_-
\]

\[
(k - i0)^{-\alpha} = k^{\alpha}_- e^{ik\alpha} k^{-\alpha}_+
\]  

(45)
So that:

\[ \hat{\psi}(x) = -2i \sin(\pi\alpha) \int_{-\infty}^{+\infty} dk \, k^{-\alpha} a(k) e^{ikx} \]  

(46)

The equation for a Green function is:

\[ \left( -i \frac{d}{dx} \right)^\alpha \hat{G}(x) = \delta(x) \]

I. e.: (cf. Eq. (4)):

\[ \hat{G}(x) = \frac{1}{2} \int_{\Gamma} dk \, k^{-\alpha} Sg(Im \, k)e^{ikx} \]

(47)

For the Fourier transform we use Ref. \[23\] (Ch. 2. §2.3)

\[ F(k \pm i0)^{-\alpha} = \frac{e^{\pm \frac{i\pi}{2} \alpha}}{\Gamma(\alpha)} x_{\pm}^{\alpha-1} \]

(48)

Then:

\[ \hat{G}(x) = \frac{1}{2\Gamma(\alpha)} \left[ e^{i\frac{\pi}{2} \alpha} x_+^{\alpha-1} + e^{-i\frac{\pi}{2} \alpha} x_-^{\alpha-1} \right] \]

(49)

To see the connection with the fractional derivative, we choose to add to (47), the solution (44) to the homogeneous equation with \( a_k \equiv -1/2 \). The new Green-function is the result of (47) minus one half of (44):

\[ \hat{G}_+(x) = \int_{-\infty}^{+\infty} dk \, (k - i0)^{-\alpha} e^{ikx} = \frac{e^{\frac{i\pi}{2} \alpha}}{\Gamma(\alpha)} x_+^{\alpha-1} \]

(50)

Except of course for the exponential factor, Eq. (50) coincides with the fractional derivative considered in Ref. \[23\] (Ch. 1, §5.5).

If instead of adding (44) with \( a_k \equiv -1/2 \), we add the same function but with \( a_k \equiv +1/2 \) we get:

\[ \hat{G}_-(x) = \frac{e^{-i\frac{\pi}{2} \alpha}}{\Gamma(\alpha)} x_-^{\alpha-1} \]

(51)

which is also a fractional derivative operator.
5 Functions of the D’Alambertian

In this section we are going to examine some non-local functions \( f(\Box) \) of \( \Box = \partial_0^2 - \vec{\partial}^2 \), where \( \vec{\partial}^2 \) is the \((d-1)\)-dimensional laplacian operator.

For \( f(z) \) we will take an analytic function such as \( z^\alpha \) or \( (z + m^2)^\alpha \), with a cut along the negative real axis, running from \(-\infty\) to zero or \(-m^2\), respectively.

The ultra distributions (Eq. (2)) depend now on an \((n-1)\)-dimensional vector \( \vec{k} \) as parameter \( \psi(k_0) \to \psi(k_0, \vec{k}) \).

The Fourier transform of Eq. (13) gives:

\[
\hat{\psi}(k_0, \vec{k}) = \int_{\Gamma} dk_0 \psi(k_0, \vec{k})e^{ik_0x_0} \tag{52}
\]

And, with the usual Fourier transform in the space of the parameters \( \vec{k} \), we obtain:

\[
\hat{\psi}(x) = \hat{\psi}(x_0, \vec{x}) = \int_{-\infty}^{+\infty} d\vec{k} \hat{\psi}(x_0, \vec{k})e^{i\vec{k}\vec{x}} \tag{53}
\]

We define the operation \( f(\Box) \) on \( \hat{\psi}(x) \) by:

\[
f(\Box)\hat{\psi}(x) = \int d\vec{k} f(\partial_0^2 + \vec{k}^2)\hat{\psi}(x_0, \vec{k})e^{-i\vec{k}\vec{x}} \tag{54}
\]

And (Cf. Eq. (10)):

\[
f(\partial_0^2 + \vec{k}^2)\hat{\psi}(x_0, \vec{k}) = \int_{\Gamma} dk_0 f(-k_0 + \vec{k}^2)\psi(k_0, \vec{k})e^{+ik_0x_0} \tag{55}
\]

We can now solve the homogeneous \((n\text{-dimensional})\) equation:

\[
f(\Box)\hat{\psi}(x) = 0 \tag{56}
\]

According to (54), Eq. (56) implies:

\[
f(\partial_0^2 + \vec{k}^2)\hat{\psi}(x_0, \vec{k}) = 0
\]

And, due to (55), we have for the densities:

\[
f(-k_0^2 + \vec{k}^2)\psi(k_0, \vec{k}) = a(k_0^2 + \vec{k}^2)
\]
\[ \psi(k_0, \vec{k}) = f^{-1}(-k_0^2 + \vec{k}^2) a(k_0^2 + \vec{k}^2) \]  (57)

Where \( a(k_0^2 + \vec{k}^2) \) is an arbitrary entire function of \( k_0 \), for any value of \( \vec{k} \).

From (52) and (57) we get:

\[ \hat{\psi}(x_0, \vec{k}) = \int \Gamma dk_0 f^{-1}(-k_0^2 + \vec{k}^2) a(k_0, \vec{k}) e^{ik_0x_0} \]  (58)

The analytic function of \( k_0 \), \( f^{-1}(-k_0^2 + \vec{k}^2) \) presents a cut along the real axis, running from \(-\infty\) to \( k_0 = -\omega \), another twin cut from \( k_0 = +\omega \) to \(+\infty\) \((\omega = +\sqrt{\vec{k}^2 + m^2})\).

The integration in (58) can then be taken along the real axis:

\[ \hat{\psi}(x_0, \vec{x}) = \int_{-\infty}^{+\infty} dk_0 \Delta(f^{-1}) a(k_0, \vec{k}) e^{ik_0x_0} \]  (59)

where \( \Delta \) is the discontinuity at the cuts:

\[ \Delta(f^{-1}) = f^{-1}(-(k_0 + i0)^2 + \vec{k}^2) - f^{-1}(-(k_0 - i0)^2 + \vec{k}^2) \]  (60)

\[ \left\{ \Delta(f^{-1}) = 0, \text{ for } -\omega \leq k_0 \leq \omega, \omega = +\sqrt{\vec{k}^2} \text{ when } m = 0 \right\} \]

For \( f = (-k_0^2 + \vec{k}^2 + m^2)^\alpha = (-k_0^2 + \omega^2)^\alpha \), we have:

\[ \Delta(f^{-1}) = (-k_0^2 + \omega^2 - i0Sg(k_0))^{-\alpha} - (-k_0^2 + \omega^2 + i0Sg(k_0))^{-\alpha} \]

\[ = Sg(k_0) \left[ (-k_0^2 + \omega^2 - i0)^{-\alpha} - (-k_0^2 + \omega^2 + i0)^{-\alpha} \right] \]

And using (63):

\[ \Delta(f^{-1}) = 2i \sin(\pi\alpha) Sg k_0 (-k_0^2 + \omega^2)^{-\alpha} = 2i \sin(\pi\alpha) Sg k_0 (k_0^2 - \omega^2)^{-\alpha} \]  (61)

For \( \alpha = 1 \), Eq. (60) is Klein-Gordon equation. The weight function given by (61), has a pole for this value of \( \alpha \) (Ref. [23], Ch. 3, §3.4). The residue is \( \delta(k_0^2 - \omega^2) = \delta(k^2 - m^2) \). For \( \alpha \to 1 \), Eq. (61) gives \( \Delta \to \delta(k^2 - m^2) \), which is the well-known invariant free wave solution. For a fractional \( \alpha \), the weight function gives a continuum with \( k^2 \geq m^2 \). In this sense we can say that (61) represents free waves corresponding to a continuum of masses \( k^2 = \mu^2 \geq m^2 \). The free wave \( \delta(k^2 - m^2) \) concentrated on \( k^2 = m^2 \), changes for fractional \( \alpha \) into \( (k^2 - m^2)^{-\alpha} \), which is spread from \( k^2 = m^2 \) to \( k^2 \to \infty \) (see also Ref. [24]).
For the fractional wave equation:

\[ \Box^\alpha \hat{\psi}(x) = 0 \] (62)

we can repeat the analysis made above. In this case:

\[ f = (-k_0^2 + \vec{k}^2)^\alpha \]

\[ \Delta(f^{-1}) = (-k_0^2 + \vec{k}^2 - i0Sg(k_0))^{-\alpha} - (-k_0^2 + \vec{k}^2 + i0Sg(k_0))^{-\alpha} \]

\[ = Sg(k_0) \left[ (\alpha^{-\alpha} - (\alpha^{-\alpha}) \right] \]

\[ \Delta(f^{-1}) = 2i\sin(\pi \alpha)Sgk_0^{-\alpha} \] (63)

Again, for \( \alpha \to 1 \), (62) is the usual wave equation and \( \Delta(f^{-1}) \to \delta(k^2) \), which is the elementary solution with support on the surface of the light-cone. When \( \alpha \) is not an integer, Eq. (63) implies that we can find free wave components for any positive mass squared.

Let us now find a Green function for the fractional wave equation:

\[ \Box^\alpha \hat{G}(x) = \delta(x) \] (64)

\[ \Box^\alpha \hat{G}(x) = \int d\vec{k} \ (\partial^2_0 + \vec{k}^2\alpha \hat{G}(x, \vec{k})e^{-i\vec{k}\vec{x}} = \int d\vec{k} \ \delta(x_0)e^{-i\vec{k}\vec{x}} \]

\[ (\partial^2_0 + \vec{k}^2\alpha \hat{G}(x_0, \vec{k}) = \delta(x_0) \]

And we have for the densities:

\[ (-k_0^2 + \vec{k}^2\alpha G(k_0, \vec{k}) = \frac{1}{2}SgImk_0 \]

\[ G(k_0, \vec{k}) = \frac{1}{2} (-k_0^2 + \vec{k}^2)^{-\alpha}SgImk_0 \]

\[ \hat{G}(x_0, \vec{k}) = \frac{1}{2} \int_{-\infty}^{+\infty} dk_0 \ (-k_0^2 + \vec{k}^2)^{-\alpha}SgImk_0 e^{ik_0x_0} \]

So that:

\[ \hat{G}(x_0, \vec{k}) = \frac{1}{2} \int_{-\infty}^{+\infty} dk_0 \left[ (\alpha^{-\alpha} + (\alpha^{-\alpha}) \right] e^{ik_0x_0} \] (65)
The integrand in (65) does not depend on the sign of $k_0$. This Wheeler Green function for the fractional wave equation (64) is an extension (for space-time) of the definition given in eq. (3).

The two terms in the square bracket of (65) are, respectively, the causal and anticausal Green functions for the fractional wave equation. For $\alpha = 1$ they are the usual Feynman propagator and its complex conjugate. The Fourier transforms of those two terms can be found in Ref. [23], Ch 3, §2.6.

Using (45) we can write the square brackets as:

$$\left[ \right] = \left( k^2 \right)^{-\alpha} + e^{i\pi\alpha} \left( k^2 \right)^{-\alpha} + e^{-i\pi\alpha} \left( k^2 \right)^{-\alpha}$$

so,

$$\frac{1}{2} \left[ (-k^2 - i0)^{-\alpha} + (-k^2 + i0)^{-\alpha} \right] = \left( k^2 \right)^{-\alpha} + \cos \pi\alpha \left( k^2 \right)^{-\alpha}$$

(66)

Other Green functions can be found by adding to (65) solutions to the homogeneous equation. Adding together (59) (with (61) and $a = \pm 1/2$) and (65) (with (66)), we get:

$$\hat{G}_{\pm}(x_0, \vec{k}) = \int_{-\infty}^{\infty} dk_0 \left[ \left( k^2 \right)^{-\alpha} + e^{\pm i\pi\alpha Sgk_0 \left( k^2 \right)^{-\alpha}} e^{ik_0x_0} \right]$$

(67)

The two terms of (65) and the two Green functions of (67), coincide with the four Lorentz-invariant propagators for the fractional wave equation found in Ref. [11].

It is possible to repeat this procedure to find Green functions for the fractional Klein-Gordon equation:

$$\left( \Box + m^2 \right)^{\alpha} \hat{G}(x) = \delta(x)$$

(68)

Instead of (65), we now have:

$$\hat{G}(x_0, \vec{k}) = \frac{1}{2} \int_{-\infty}^{+\infty} dk_0 \left[ \left( -k^2 + m^2 - i0 \right)^{-\alpha} + \left( -k^2 + m^2 + i0 \right)^{-\alpha} \right] e^{ik_0x_0}$$

(69)

In this way we obtain the causal and anticausal propagators for the fractional Klein-Gordon equation. The Feynman function coincides with the propagator used in Ref. [7] to regularize the matrix elements of Quantum Electrodynamics.
6 Discussion

The space $\zeta$ of ultra analytic functions, defined in section 4, seems to be an appropriate base of test functions for the development of a framework adapted to the needs of those who want to work with non-local pseudo-differential operators. Its dual space $\zeta'$ contains the propagators appearing in perturbative Quantum Field Theories. It is also flexible enough to allow for them a representation in terms of ultra distributions which is both, simple and general.

The Fourier transformed spaces $\hat{\zeta}$ and $\hat{\zeta}'$, translate the functions of $\zeta$ and $\zeta'$ into functions of coordinates where the pseudo-differential operators are supposed to act. In this way we can define non-local operations on (generalized) functions of the coordinates. These operators are extensions of the ordinary derivative operators, which are now particular cases in a wider scheme.

We have written and solved homogeneous and inhomogeneous equations. In particular, we have found Green functions for some non-local operators.

We have also shown the connection with the operators $\Box^\alpha$, defined and discussed in Ref. 11, and with the fractional derivative of Ref. 23. In these references no attempt was made to solve the respective homogeneous equations. We here exhibited the corresponding solutions. As a consequence, we are able to show that the free solutions of the wave equation, or the Klein-Gordon equation, with their sharp masses, are transformed when the equations are fractional, into a superposition of a continuum of masses with support in the interior of the light cone. Furthermore, the causal propagators for the fractional equations, are seen to coincide with the analytically regularized propagators introduced in Ref. 7. It is then possible to interpret this regularization method, as the matrix elements one would write for a non-local theory having no ultraviolet divergences. The usual infinities appear as poles for the local limit $\alpha \to 1$.

We see then that with the chosen procedures we have convenient tools with which we can handle different pseudo-differential non-local problems. The scheme also provides a base for the quantization of non-local field theories 20.
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