Computed tomography imaging using a short pulse source with angular discontinuity

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Abstract. This paper deals with an inverse problem that consists of the attenuation coefficient identification for the non-stationary radiation transfer equation. To solve the problem, we propose to use a pulsed radiation source with an angular discontinuity. We show that the solution to the radiation transfer equation is the sum of a discontinuous ballistic component and a continuous scattered one. The representation allows us to obtain a formula for finding the attenuation coefficient. The uniqueness theorem for the solution of the inverse problem has been proved. Numerical experiments on a digital phantom show that the method proposed improves the reconstruction quality.

1. Introduction
Nowadays, inverse problems of the radiation transfer theory continue to be relevant [1, 2, 3, 4, 5, 6]. Most papers study problems of reconstructing the attenuation coefficient based on the knowledge of ingoing and outgoing the radiation flux. Such a statement of the problem is conventional for tomography. Here, as a rule, models proposed neglect scattering and internal sources. In the case, the attenuation coefficient of a model is an unique unknown parameter determining medium properties. The inverse problems, in which some of the coefficients are determined without evaluating the others, are especially valuable for practice. For example, in [6, 7, 8, 9] are studied the problems of determining the attenuation coefficient with external sources of a special type. The sources allow suppressing the effect of scattering in a irradiated medium and made it possible to detect a ballistic component of the radiation flux. The authors of [6, 7] have proposed to use a special type of a radiation source with a first order discontinuity on the angular variable, which allows one to determine the attenuation coefficient within the framework of a stationary radiation transfer equation. It is proved that the solution of the radiation transfer equation can be represented as the sum of ballistic and scattered components. In this case the ballistic component of the radiation flux has a first order discontinuity due to the angular variable and the scattered component is a continuous function. Using the representations of the solution, the authors express the Radon transform of the attenuation coefficient in terms of jumps in the flux density of the ingoing and outgoing radiation. The recent published works [3, 4] propose a method to reconstruct the attenuation coefficient of the radiation transfer equation from jumps in boundary measurements. The authors use the discontinuity of the solution of a boundary value problem arising from discontinuous in incoming boundary data due to spatial variables. Another method which can discriminate the scattering component in the solution of the non-stationary radiation transfer equation is the reduction of length of the probe pulse. Pulse
radiation sources are successfully used in optical tomography. In the X-ray energy range, creation of ultra-short pulses is associated with more stringent physical and technological limitations [10]. The interest in using pulsed sources is also associated with a reduction in the dose of radiation exposure during x-ray tomography of biological objects. This article generalizes the results obtained in [6, 7] for the case of non-stationary radiation transfer equation. We have obtained a formula for the Radon transform of the attenuation coefficient based on investigation of solution structure for the non-stationary radiation transfer equation. Numerical testing of the algorithm for solving inverse problems on the phantom proposed in [11] was performed. The scattering filtration due to the pulse width and due to the discontinuous source was compared. It is shown that decreasing pulse duration increases quality of the reconstructed images. Best results are obtained using ultra-short pulses. At the same time, the filtration of the scattered field due to the pulse width and due to the discontinuous source was compared. It is shown that decreasing pulse duration increases quality of the reconstructed images. Best results are obtained using ultra-short pulses. At the same time, the filtration of the scattered field due to the different smoothness of the ballistic and scattered radiation components does not require the use of ultra-short pulses [10]. Use of the jumps at boundary conditions together with decrease in the duration of the probe pulse improves reconstruction results.

2. Statement of the inverse problem
Let us consider following integro-differential radiation transfer equation [1]

\[
\left( \frac{1}{v} \frac{\partial}{\partial t} + \omega \cdot \nabla r + \mu(r) \right) I(r, \omega, t) = \sigma(r) \int_{\Omega} p(r, \omega, \omega') I(r, \omega', t) d\omega'.
\]  

(1)

Where \( I(r, \omega, t) \) is the radiation flux density at the point \( r \in G \subset \mathbb{R}^3 \), in the direction \( \omega \in \Omega = \{ \omega \in \mathbb{R}^3 : |\omega| = 1 \} \), at the time moment \( t \in [0, T] \), \( \mu \) is the attenuation factor, \( \sigma \) is the scattering coefficient, \( v \) is the velocity of photons, and \( p \) is the scattering phase function.

We assume that domain \( G \) is bounded, convex, and represented as

\[ G = \bigcup_{i=1}^{N} G_i, \quad G_i \cap G_j = \emptyset, \text{ for } i \neq j, \ N < \infty, \]

where \( G_i \) is a domain with a piece-wise smooth boundary, and let \( G \) satisfies the generalized convexity condition. The surface \( \partial \overline{G} \) will be called the outer boundary of the domain \( G \), and \( \gamma = \partial G \setminus \partial \overline{G} \) the inner boundary of the domain \( G \).

We denote

\[ L_{r,\omega} = \{ r + \omega t : t > 0 \}, \quad d(r, \omega) = \text{mes}_1(L_{r,\omega} \cap \overline{G}). \]

\[ d^-(r, \omega, t) = \min \{d(r, -\omega), vt\}, \quad d^+(r, \omega, t) = \min \{d(r, \omega), v(T-t)\}, \]

\[ \Gamma_{+\omega} = \{z \in \partial \overline{G} : L_{z,\pm\omega} \cap \overline{G} \neq \emptyset\}. \quad \Gamma^\pm = \Gamma_{+\omega} \times \Omega, \quad \Gamma = \Gamma^+ \cup \Gamma^-. \]

For brevity, we will use the following notation:

\[ X = G \times \Omega \times [0, T], \quad X_0 = G \times \Omega \times \{t = 0\}, \quad Y^\pm = \Gamma^\pm \times [0, T], \quad X^- = Y^- \cup X_0. \]

Let us introduce the function

\[ h(z, \omega, t) = \begin{cases} h_0(z, \omega), & \text{for } (z, \omega, t) \in X_0, \\
 h_{\text{ext}}(z, \omega, t), & \text{for } (z, \omega, t) \in Y^-, \end{cases} \]

and adjoin the initial - boundary condition to equation (1):

\[ I|_{X^-} = h(r, \omega, t). \]  

(2)

We consider the inverse problem of finding function \( \mu \) from (1), (2) and additional condition

\[ I(z, \omega, t) = H(z, \omega, t), \quad (z, \omega, t) \in Y^+, \]

(3)

if \( h, H, \omega \) are given.
3. Smoothness properties of direct problem solution

Consider the space \( C_b(X) \) of functions continuous and bounded on \( X \). It is assumed that the functions \( \mu(r), \sigma(r), p(r, \omega, \omega') \) are non-negative and \( \mu \geq \text{const} > 0 \), \( \mu, \sigma \in C_b(G) \). Phase function \( p(r, \omega, \omega') \in C_b(G \times [-1, 1]) \) satisfies the normalization condition

\[
\int_\Omega p(r, \omega, \omega') d\omega' = 1,
\]

for all \( r \in G \).

We will say that a function \( f \) belongs to \( D(X) \) if the following conditions hold true:

1) for all \( (r, \omega, t) \in X \) function \( f(r + \tau \omega, \omega, t + \tau/v) \) is absolutely continuous in \( \tau, \tau \in [-d^-(r, \omega, t), d^+(r, \omega, t)] \);
2) \( f(r - d^-(r, \omega, t) \omega, t - d^-(r, \omega, t)/v) = 0; \)
3) \( \left( \frac{1}{v} \frac{\partial}{\partial t} + \omega \cdot \nabla_r + \mu \right) f = \frac{\partial}{\partial \tau} f(r + \tau \omega, \omega, t + \tau/v) \bigg|_{\tau=0} \in C_b(X) \);

We introduce operators by the formulas

\[
\mathcal{L}f = \left( \frac{1}{v} \frac{\partial}{\partial t} + \omega \cdot \nabla_r \right) f + \mu f, \quad Sf = \sigma(r) \int_\Omega p(r, \omega, \omega') f(r, \omega', t) d\omega'.
\]

We proved that \( \mathcal{L} : D(X) \to C_b(X) \) and \( S : C_b(G \times (\Omega \setminus \Omega_0) \times [0, T]) \to C_b(X) \), for any arbitrary \( \Omega_0 \subset \Omega \) of measure zero.

Linear set \( D(X) \) with norm \( ||f||_D = ||(\mathcal{L}f)/\mu|| \) is Banach space, and continuous embedding \( D(X) \subset C_b(X) \) fulfills.

Denote the jump of the function \( h \) with respect to the variable \( \omega \), at \( \omega \to \omega^0 = (\omega_1, \omega_2, 0) \), by \( [h](\xi, \omega^0, t) = \lim_{\varepsilon \to 0} (h(\xi, (\omega_1, \omega_2, \varepsilon), t) - h(\xi, (\omega_1, \omega_2, -\varepsilon), t)) \). The theorem has been proved.

**Theorem 1.** If the following conditions are satisfied,

\[
[h](\xi, \omega^0, t) \neq 0, \quad (\xi, \omega^0, t) \in Y^-,
\]

\[
\max_{r \in G} \frac{\mu_s(r)}{\mu(r)} \leq 1,
\]

then there exists a unique solution of problem (1),(2) which is representable in the form \( I = f + I_0 \), where

\[
I_0(r, \omega, t) = h(r - d^-(r, \omega, t) \omega, t - d^-(r, \omega, t)/v) \exp \left( - \int_0^\tau \mu(r - \tau \omega) d\tau \right)
\]

(6)

and \( f \in D(X) \) satisfies the equation \( \mathcal{L}f = Sf + SI_0 \).

4. Inverse problem solution

The function \( f \) in Theorem 1 is interpreted as a scattered radiation flux density and is continuous due to embedding \( D(X) \subset C_b(X) \). Ballistic component \( I_0 \) on the contrary may contains jumps in the variable \( \omega \) in \( X \). Such structure of the radiation transfer equation solution is basis of the method for solving inverse problem.

For boundary points \( (\eta, \omega^0, t) \in Y^+ \) satisfying to constraints \( d(\eta, -\omega^0) \leq vt \leq vT - d(\eta, -\omega^0) \), we get

\[
[H](\eta, \omega^0, t) = [f](\eta, \omega^0, t) +
\]
The mean CT values were calculated for each of the five test materials and were compared with reference values. The radiation source has the form.

$$h(r, \omega, t) = \chi(\omega_3)I_0 \exp \left\{ -4 \ln 2 \frac{(t-t_0)^2}{T_p} \right\},$$

$$[70x777]$$

Radon transforms (8) and (10) has been inverted by the filtered back-projection algorithm. Numerical experiments were performed on digital phantom proposed in [11]. The phantom is a cylinder 10cm in height and diameter. It consists of five individual sections to assess image quality parameters in terms of CT value uniformity and linearity, image noise and contrast, 3D resolution, and artifact behavior.

Our investigation has been focused on the first phantom section designed to test CT values uniformity and linearity [12, 13]. This characteristic is one of the most important for reliable and reproducible treatment planning [13]. The section is a cylinder 100 mm in diameter and 17.5 mm in height, made of water equivalent plastic (HU = 0). It contains four cylindrical inserts 13 mm in diameter and 17.5 mm in height with different CT values (−1000HU, 30HU, −30HU, 100HU). The mean CT values were calculated for each of the five test materials and were compared with reference values. The radiation source has the form.

$$h(r, \omega, t) = \chi(\omega_3)I_0 \exp \left\{ -4 \ln 2 \frac{(t-t_0)^2}{T_p} \right\},$$

$$[70x777]$$
Figure 1. Phantom reconstruction depends on probe pulse durations: 3 ps, 30 ps, 300 ps, 3000 ps. Reconstructions according to formula (10) are on the left, and according to formula (8) on the right.
where $I_0$ is the pulse amplitude, $t_0$ is the time moment corresponding to the maximum signal power $t_p$ is the full width at half-maximum (FWHM) pulse duration and $\chi(\omega_3)$ is Heaviside function. In the experiments, the following values were used: $I_0 = 32000$, $t_0 = 3$ ns, and values of $t_p$ were equals to 3 ps, 30 ps, 300 ps or 3000 ps.

Phantom reconstructions with respect to various probe pulse duration are shown in Figure 1. The figure on the left shows phantom reconstruction using formula (10). The right side of figure 1 presents results of implementation of scattering filtration based on formula (8). Irradiation by short pulses allows to filter a scattered component in the outgoing radiation. As a result, phantom irradiation by short pulses produces good quality reconstruction without implementation of signal processing according to formula (8). In general, the error in reconstructed CT values is about one obtained in [11]. Note that even in this good case, using formula (8) gives a better correspondence between the reference and reconstructed CT values.

Within the framework of conventional tomography formula (10), increasing the pulse width reduces the quality of the reconstruction. Pulse duration of 3000 ps, results in a huge difference between the reconstructed and reference CT values. Visually, it is also difficult to locate inclusions. The application of our algorithm based on formula (8), on the contrary, gives good results. Right series of images show good quality reconstruction at all pulse durations. However, it is can be seen clearly, reconstruction quality decreases with increasing pulse duration. In order to get the best reconstruction quality, it is recommended to supplement a pulsed radiation source with a jump in angular intensity.

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References
[1] Bal G 2009 Inverse transport theory and applications Inverse Problems 25 053001
[2] Volkov N P 2016 Solvability of certain inverse problems for the non-stationary kinetic transport equation Comput. Math. Math. Phys 56 1598—1603
[3] Kawagoe D and Chen I K 2018 Propagation of Boundary-Induced Discontinuity in Stationary Radiative Transfer Journal of Statistical Physics 170 127—140
[4] Chen I K and Kawagoe D 2019 Propagation of boundary-induced discontinuity in stationary radiative transfer and its application to the optical tomography Inverse Problems and Imaging 13 337—351
[5] Bellassoued M and Boughnaja Y 2019 An inverse problem for the linear Boltzmann equation with a time-dependent coefficient Inverse Problems 35 085003
[6] Anikonov D S and Prokhorov I V 1992 Determining a coefficient of the transport equation with energetic and angular singularities of external radiation Dokl. Akad. Nauk SSSR 327(2) 205–207
[7] Anikonov D S, Prokhorov I V and Kovalenok A E 1993 Investigation of scattering and absorbing media by the methods of X-ray tomography Journal of Inverse and Ill-Posed Problems 1(4) 259—282
[8] Anikonov D S, Kovalenok A E and Prokhorov I V 2002 Transport Equation and Tomography (Boston-Utrecht: VSP)
[9] Kovalenok A E and Prokhorov I V 2008 Numerical solution of the inverse problem for the polarized-radiation transfer equation Numerical Analysis and Applications 1(1) 46—57
[10] Fetisov G V 2020 X-ray diffraction methods for structural diagnostics of materials: progress and achievements Physics-Uspekhi 63 2—32
[11] Steiding C, Kolditz D and Kalender W A 2014 A quality assurance framework for the fully automated and objective evaluation of image quality in cone-beam computed tomography Medical Physics 41 031901
[12] Kalender W A 2011 Computed Tomography: Fundamentals, System Technology, Image Quality, Applications (Erlangen : Publicis)
[13] Mah P, Reeves T E and McDavid W D 2010 Deriving Hounsfield units using grey levels in cone beam computed tomography Dentomaxillofac. Radiol. 39 323—335