NEURAL NETWORKS IN FRÉCHET SPACES

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Abstract. We derive approximation results for continuous functions from a Fréchet space \( X \) into a Banach Space \( \mathcal{Y} \). The approximation results are in the spirit of the well known universal approximation theorem for continuous functions from \( \mathbb{R}^n \) to \( \mathbb{R} \), where approximation is done with (multiayer) neural networks [10, 16, 12, 20]. The approximating functions that we obtain are easy to implement and allow for fast computation and fitting. The resulting neural network architecture is applicable for prediction tasks based on functional data.

1. Introduction

The universal approximation theorem shows that any continuous function from \( \mathbb{R}^n \) to \( \mathbb{R} \) can be approximated arbitrary well with a one layer neural network. More precisely, for a fixed continuous function \( \sigma : \mathbb{R} \to \mathbb{R} \) and \( a \in \mathbb{R}^n, \ell, b \in \mathbb{R} \), a neuron is a function \( \mathcal{N}_{\ell,a,b} \in C(\mathbb{R}^n; \mathbb{R}) \) defined by \( x \mapsto \ell \sigma(a^\top x + b) \). The universal approximation theorem states conditions on the activation function \( \sigma \) such that the linear space of functions generated by the neurons

\[ \mathfrak{A}(\sigma) := \text{span}\{\mathcal{N}_{\ell,a,b} ; \ell, b \in \mathbb{R}, a \in \mathbb{R}^n\} \]

is dense with respect to the topology of uniform convergence on compacts. This means that for every \( f \in C(\mathbb{R}^n; \mathbb{R}) \) and compact subset \( K \subset \mathbb{R}^n \) and a given \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) and \( \ell_i, b_i \in \mathbb{R}, a_i \in \mathbb{R}^n \) for \( i = 1, \ldots, N \) such that

\[ \sup_{x \in K} |f(x) - \sum_{i=1}^{N} \mathcal{N}_{\ell_i,a_i,b_i}(x)| \leq \varepsilon. \]

Possibly the most widely known property of \( \sigma \) that was shown in [10] and [16] to lead to the density of \( \mathfrak{A}(\sigma) \subset C(\mathbb{R}^n; \mathbb{R}) \) is the sigmoid property, which requires \( \sigma \) to be such that \( \lim_{t \to -\infty} \sigma(t) = 1 \) and \( \lim_{t \to +\infty} \sigma(t) = 0 \). This condition has later been relaxed to a boundedness condition [12] and a non-polynomial condition [20]. See also [18] for a unified approach of approximation result for a wide class of network architectures and [8] for an overview of the mathematical results related to deep learning.

In this paper we are concerned with more general functions \( f \in C(\mathcal{X}; \mathcal{Y}) \), where \( \mathcal{X} \) is an \( \mathbb{F} \)-Fréchet space, i.e., a Fréchet space over the field \( \mathbb{F} \) and \( \mathcal{Y} \) an \( \mathbb{F} \)-Banach space. We start with \( \mathcal{Y} = \mathbb{F} \) and in the definition of a neuron, we replace \( a^\top x + b \) by an affine function on \( \mathcal{X} \), the activation function \( \sigma : \mathbb{R} \to \mathbb{R} \) by a function in \( C(\mathcal{X}; \mathcal{X}) \), and the scalar \( \ell \) by a linear form. With \( \langle \cdot, \cdot \rangle \) the canonical pairing between \( \mathcal{X}' \) and \( \mathcal{X} \) (\( \mathcal{X}' \) denoting the topological dual of \( \mathcal{X} \)), for \( \ell \in \mathcal{X}' \), \( A \in \mathcal{L}(\mathcal{X}), b \in \mathcal{X} \) we then define a neuron \( \mathcal{N}_{\ell,A,b} \) by

\[ \mathcal{N}_{\ell,A,b}(x) = \langle \ell, \sigma(Ax + b) \rangle \]

and ask for conditions on \( \sigma : \mathcal{X} \to \mathcal{X} \) that ensure that \( \mathfrak{A}(\sigma) := \text{span}\{\mathcal{N}_{\ell,A,b} ; \ell \in \mathcal{X}', A \in \mathcal{L}(\mathcal{X}), b \in \mathcal{X}\} \) is dense in \( C(\mathcal{X}; \mathbb{F}) \) under some suitable topology.

To indicate the conditions we obtain for \( \sigma \), recall that any map \( \psi \in \mathcal{X}' \) defines a hyperplane by the set of points \( \Psi_0 := \{x \in \mathcal{X} ; \langle \psi, x \rangle = 0\} \). This hyperplane splits the space \( \mathcal{X} \) into the sets \( \Psi_- := \{x \in \mathcal{X} ; \langle \psi, x \rangle < 0\} \) and \( \Psi_+ := \{x \in \mathcal{X} ; \langle \psi, x \rangle > 0\} \). We show that the main property for the activation function to ensure that \( \mathfrak{A}(\sigma) \) is dense in \( C(\mathcal{X}; \mathbb{F}) \) is, informally, that an \( \psi \in \mathcal{X}' \) exists such that the value \( \sigma(x) \) converges, as \( x \) moves away from the hyperplane that is defined by \( \psi \). The limiting values on both sides of the hyperplane need to be different. We provide several simple examples of easy to calculate activation functions with the required property. To our knowledge, this is the first such approximation

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result for functions defined on spaces other than $\mathbb{R}^n$ or subsets thereof (see for example [19] for approximations on manifolds in $\mathbb{R}^n$). In a second step, we extend our results to $f \in C(\mathcal{X}; \mathcal{Y})$, where $\mathcal{Y}$ is an $F$-Banach space.

While such an approximation result might be of interest in its own, from a practical perspective it is not clear how the functions $\mathcal{N}_{\ell,A,b}$, which involve infinite dimensional quantities, can actually be programmed. We therefore address the question of approximating the maps $\mathcal{N}_{\ell,A,b}$ by finite dimensional, easy to calculate quantities. Under the assumption that the Fréchet space $\mathcal{X}$ permits a Schauder basis, we show that such an approximation is possible. The resulting neural network has an architecture similar to classical neural networks, with the exception that the activation function is now multidimensional. It does however still permit for an easy to calculate gradient, which is crucial for training the network via a back-propagation algorithm. Finally, we also derive the approximation property for deep neural networks with a given fixed number of layers.

Possible applications of our results are within the area of machine learning. In particular in the many situations where the input of each sample in the training set is actually a function, our results can be applied. Examples for such situations are numerical solutions of partial differential equations [13, 17, 8, 1], stock price prediction [26], option pricing and hedging [7, 2], and many others. If the function space of the inputs is a Fréchet space with a Schauder basis, this basis provides structural information about the elements. Traditional neural networks must be of very high dimension (large input dimension, large number of neurons) to approximate a function well. The more variability there is in the function, the larger the number of parameters that is needed. Therefore, instead of using a classical network to approximate a function on a grid, our approach allows one to use information in the basis functions instead to capture the structure and get theoretical convergence results. Our approximation thus focuses on features of the function related to the coefficients in the basis expansion.

The outline for the paper is as follows. In Section 2 we derive our first main result Theorem 2.3, which shows that if $\sigma$ has a property called discriminatory property, then $\mathcal{N}(\sigma)$ is dense in $C(\mathcal{X}; \mathcal{F})$. The main technical challenge is then to derive conditions that ensure that a given function $\sigma : \mathcal{X} \rightarrow \mathcal{X}$ is actually discriminatory, which is done in Theorem 2.8. We also provide some first examples of discriminatory functions in this section. We then extend these results in Section 3 to functions $f \in C(\mathcal{X}; \mathcal{Y})$, $\mathcal{Y}$ Banach space. In Section 4 we address the question of finite dimensional approximations to the neural network which can easily be computed and trained. In most generality, only under the assumption that the Fréchet space $\mathcal{X}$ has a Schauder basis, the approximation is covered in Theorem 4.3. In Section 5 we cover the approximation with multi-layered neural networks.

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2. An abstract approximation result

Let $\mathcal{F} \in \{\mathbb{R}, \mathbb{C}\}$, and let $\mathcal{X}$ be an $F$-Fréchet space. Let $(p_k)_{k \in \mathbb{N}}$ be an increasing sequence of seminorms that generates the topology of $\mathcal{X}$. We can then consider a metric $d$ on $\mathcal{X}$ (that generates the same topology) given by

$$d(x, y) := \sum_{k=1}^{\infty} 2^{-k} \frac{p_k(x - y)}{1 + p_k(x - y)},$$

for $x, y \in \mathcal{X}$.

Let us consider $\sigma : \mathcal{X} \rightarrow \mathcal{X}$ continuous function. Let $A : \mathcal{X} \rightarrow \mathcal{X}$ be in $\mathcal{L}(\mathcal{X})$, i.e. a linear and continuous operator, $b \in \mathcal{X}$ and $\ell \in \mathcal{X}'$, where $\mathcal{X}'$ denotes the topological dual of $\mathcal{X}$. Let us consider the following function:

$$\mathcal{N}_{\ell,A,b} : \mathcal{X} \rightarrow \mathcal{F}, \quad \mathcal{N}_{\ell,A,b}(x) := \langle \ell, \sigma(Ax + b) \rangle = \ell(\sigma(Ax + b)), \quad x \in \mathcal{X},$$
where \( \langle \cdot, \cdot \rangle \) is the canonical pairing between \( X' \) and \( X \). We will call such function a neuron. Every neuron \( N_{\ell,A,b} \) is clearly continuous by composition of continuous maps, i.e. \( N_{\ell,A,b} \in C(X;F) \), the space of \( F \)-valued continuous functions on \( X \).

We define
\[
\mathcal{N}(\sigma) := \text{span}\{N_{\ell,A,b}; \ell \in X', \ A \in \mathcal{L}(X), b \in X\},
\]
namely we consider all linear combinations of the form
\[
\sum_{j=1}^{N} \alpha_j N_{\ell_j,A_j,b_j}, \quad \alpha_j \in F, \ N \in \mathbb{N}.
\]
Evidently, \( \mathcal{N}(\sigma) \subset C(X;F) \). The maps \( N_{\ell_1,A_1,b_1}, \ldots, N_{\ell_N,A_N,b_N} \) build a hidden layer with \( N \) neurons.

We endow \( C(X;F) \) with the topology of uniform convergence on compacts. Being \( X \) metrizable, it is clearly Tychonoff, and in particular completely regular. For a given compact subset \( K \subset X \), define
\[
q_K(f) := \sup_{x \in K} |f(x)|, \quad f \in C(X;F).
\]
This is a seminorm on \( C(X;F) \). We consider the topology generated by the family of seminorms \( \{q_K; K \subset X, \text{ compact}\} \), which is the coarsest topology that makes all the seminorms continuous functions on \( C(X;F) \). This is also called the projective topology induced by the maps \( q_K \) for \( K \) compact or the topology of compact subsets. Thus, we obtain a locally convex topology on \( C(X;F) \), namely \( C(X;F) \) is an \( F \)-locally convex space. Conway [8, Proposition 4.1, p. 114] provides us with the following Riesz representation theorem, which we are going to employ in the sequel:

**Proposition 2.1.** If \( \phi : C(X;F) \to F \) is a continuous and linear functional, then there is a compact set \( K \subset X \) and a regular Borel measure \( \mu \) on \( K \) such that \( \phi(f) = \int_K f \, d\mu \) for every \( f \in C(X;F) \). Conversely, each such measure defines an element of \( C(X;F)' \). (Observe en passant that \( |\mu|(K) < \infty \).

We recall that for a locally compact space \( Y \) equipped with its Borel \( \sigma \)-algebra \( \mathcal{B}(Y) \), a positive measure \( \nu \) on \( \mathcal{B}(Y) \) is a regular Borel measure if
1. \( \nu(F) < \infty \) for every \( F \subset Y \) compact,
2. for any \( E \in \mathcal{B}(Y) \), \( \nu(E) = \sup \{\nu(F); F \subset E, F \text{ compact}\} \),
3. for any \( E \in \mathcal{B}(Y) \), \( \nu(E) = \inf \{\nu(U); U \supset E, U \text{ open}\} \).

If \( \nu \) is complex-valued or signed instead, then it is regular if \( |\nu| \) is.

In the following the expression \( (\mu, K) \) will denote a compact subset \( K \subset X \) and a regular \( F \)-valued Borel measure \( \mu \) on \( K \). We say that \( \sigma : X \to X \) continuous is discriminatory if for any fixed pair \( (\mu, K) \)
\[
\int_K \langle \ell, \sigma(Ax + b) \rangle \, d\mu(dx) = 0
\]
for all \( \ell \in X', A \in \mathcal{L}(X), b \in X \) implies that \( \mu = 0 \).

**Remark 2.2.** It would be tempting, albeit more challenging, to establish our universal approximation result (Thm 2.5) in a “global” setting, namely to work directly in the space of bounded continuous functions \( C_b(X;F) \), endowed with the supremum norm (upon imposing suitable boundedness conditions on the non-linearity \( \sigma \)), rather than staying at a “local” level as we are doing now.

The main obstruction that prevented us from employing this approach is explained by the succeeding observation: If we aim at following Cybenko’s blueprint [10] (refer to the proof of Thm. 2.5) to establish our result, then in that case we would be required to work with the space
\[
\text{rba}(X) := \{ \mu : \mathcal{B}(X) \to F; \mu(0) = 0, \text{ finitely additive, finite and regular} \}
\]
which is known to be the dual of \( C_b(X;F) \), i.e. \( C_b(X;F)' = \text{rba}(X) \). As a general fact, dealing with finitely additive measures is definitely more involved, and many standard results from classical measure theory cease to hold. In particular, at this stage it is not clear to us to envisage a suitable set of conditions that the non-linearity \( \sigma \) must satisfy in order to be discriminatory (look at Def 2.7).
Nonetheless, we deem this potential extension of our result to be interesting and worthy to be explored (most likely by deviating completely from Cybenko’s strategy), and we hope to be able to come back to this question in the future.

The following first main result shows the density of \( \mathfrak{M}(\sigma) \) if \( \sigma \) is discriminatory. The result takes inspiration from Cybenko [18] (see also [12, 18] and [20]), where a similar result has been shown for the case \( X = \mathbb{R}^n \). For general \( X \) however, showing that a function \( \sigma : X \to \mathbb{R} \) is actually discriminatory can be involved. But later, in Theorem 2.3, we state conditions that can easily be verified and give rise to a large family of discriminatory functions.

**Theorem 2.3.** Let \( X \) be an \( F \)-Fréchet space, and let \( \sigma : X \to \mathbb{R} \) be continuous such that \( \mathfrak{M}(\sigma) = 0 \). Then \( \mathfrak{M}(\sigma) \) is dense in \( C(X; F) \) when equipped with the projective topology with respect to the seminorms \( q_K \). In other words, given \( f \in C(X; F) \), then, for any compact subset \( K \) of \( X \), and any \( \varepsilon > 0 \), there exists \( \sum_{m=1}^M \alpha_m N_{\varepsilon_m, A_m, b_m} \in \mathfrak{M}(\sigma) \) with suitable \( \alpha_m \in \mathbb{F}, \ell_m \in X', A_m \in \mathcal{L}(X) \) and \( b_m \in X \) such that

\[
\sum_{m=1}^M \alpha_m N_{\varepsilon_m, A_m, b_m} \in \{ g \in C(X; \mathbb{R}) ; q_K(g - f) < \varepsilon \}.
\]

**Proof.** We assume that \( cl(\mathfrak{M}(\sigma)) \subsetneq C(X; \mathbb{F}) \), and observe that \( cl(\mathfrak{M}(\sigma)) \) is clearly still a vector subspace.

We choose \( u_0 \in C(X; \mathbb{F}) \setminus cl(\mathfrak{M}(\sigma)) \). Since the complement of \( cl(\mathfrak{M}(\sigma)) \) is open, we may find \( n \in \mathbb{N}, \) seminorms \( q_{K_1}, \ldots, q_{K_n} \) on \( C(X; \mathbb{F}) \) and \( \varepsilon_1, \ldots, \varepsilon_n > 0 \) such that

\[
U := \bigcap_{j=1}^n \{ u \in C(X; \mathbb{F}) ; q_{K_j}(u - u_0) < \varepsilon_j \} \subset C(X; \mathbb{F}) \setminus cl(\mathfrak{M}(\sigma)).
\]

Clearly \( u_0 \notin U \), \( U \) is convex, open and disjoint from \( cl(\mathfrak{M}(\sigma)) \). From one of the Corollaries of the Hahn-Banach Theorem [22] Thm. 8.5.4 there exists \( \phi \in C(X; \mathbb{F}) \) linear and continuous such that

\[
\phi \big|_{cl(\mathfrak{M}(\sigma))} = 0, \quad \mathfrak{R}(\phi) > 0 \quad \text{on} \quad U.
\]

In particular, \( \phi \) is not identically zero. Then by Proposition 2.1 there exists a compact subset \( K \subset X \) and a regular Borel measure (complex or signed) \( \mu \neq 0 \) on \( K \) such that

\[
\phi(f) = \int_K f(x) \mu(dx), \quad f \in C(X; \mathbb{F}).
\]

In particular, for any \( \ell \in X', A \in \mathcal{L}(X), b \in X \) it holds

\[
\int_K \langle \ell, (A x + b) \rangle \mu(dx) = 0.
\]

But \( \sigma \) was assumed to be discriminatory. Thus we infer \( \mu = 0 \), and this is a contradiction to \( \mathfrak{R}(\phi) > 0 \) on \( U \). We conclude that \( \mathfrak{M}(\sigma) \) is dense in \( C(X; \mathbb{F}) \) with respect to the topology of compact subsets of \( X \). This implies that there exits \( M \) and \( \alpha_m \in \mathbb{F}, \ell_m \in X', A_m \in \mathcal{L}(X) \) and \( b_m \in X \) for \( m = 1, \ldots, M \) such that (23) holds. \( \square \)

**Example 2.4.** Let us give an example of an infinite dimensional compact set. First recall that for \( X \) Banach space, a subset \( S \subset X \) is compact if and only if (i) \( S \) is closed and bounded, (ii) for all \( \varepsilon > 0 \), there exists a finite dimensional subspace \( X_\varepsilon \subset X \) such that for all \( s \in S \), it holds that \( d(s, X_\varepsilon) < \varepsilon \). Let now \( X \) be a separable Hilbert space and let \( (e_k)_{k \in \mathbb{N}} \) be an orthonormal basis for \( X \). Then every \( x \in X \) can be represented as \( x = \sum_{k=1}^{\infty} x_k e_k \) with coefficients \( x_k \in \mathbb{F} \). Let us choose \( (\sigma_k)_{k \in \mathbb{N}} \in \ell^2 \) with \( \sigma_k \geq 0 \) for all \( k \in \mathbb{N} \). Here \( \ell^2 \) denotes the space of square integrable sequences. The set

\[
S := \{ x \in X : |x_k| \leq \sigma_k, \quad \forall k \in \mathbb{N} \}
\]

is then compact. To see this, first observe that \( S \) is clearly bounded. Now, let \( y \in cl(S) \). Then we may find a sequence \( (x(n))_{n \in \mathbb{N}} \) in \( S \) such that \( x(n) \) converges to \( y \). This in particular
means that \( x_k(n) \) converges to \( y_k \) for all \( k \in \mathbb{N} \). But this implies that \( |y_k| \leq \sigma_k \) and hence \( y \in S \) and \( S \) is closed (i.e., (i) holds). Finally, let \( \varepsilon > 0 \), then choose \( N_\varepsilon \in \mathbb{N} \) such that

\[
\sum_{k=N_\varepsilon+1}^{\infty} \sigma_k^2 < \varepsilon^2
\]

and set \( X_\varepsilon := \text{span}\{e_1, \ldots, e_{N_\varepsilon}\} \), which is clearly finite dimensional. For any \( x \in S \) it holds that

\[
\left\| x - \sum_{k=1}^{N_\varepsilon} x_k e_k \right\|^2 = \sum_{k=N_\varepsilon+1}^{\infty} x_k^2 < \varepsilon^2,
\]

which clearly implies that \( d(x, X_\varepsilon) \leq \left\| x - \sum_{k=1}^{N_\varepsilon} x_k e_k \right\| < \varepsilon \) and hence (ii) holds.

For the sequel, we need a boundedness assumption on the activation function \( \sigma \). First, recall that a set \( A \subset \mathcal{X} \) is von Neumann-bounded if for any \( k \in \mathbb{N} \) there exists \( c_k > 0 \) such that \( \sup_{x \in A} p_k(x) \leq c_k \). We assume that the set

\[
(4) \quad \sigma(\mathcal{X}) \subset \mathcal{X}
\]

is von Neumann-bounded.

**Remark 2.5.** We have another concept of metric-boundedness available: A subset \( A \) of a metric space \((\mathcal{X}, d)\) is bounded if there exists \( R > 0 \) such that for all \( x_1, x_2 \in A \) it holds \( d(x_1, x_2) < R \). This concept is not sufficiently stringent, because \( \text{diam} (\mathcal{X}) \leq 1 \) under the metric defined in \((\text{i})\), and thus any subset of \( \mathcal{X} \) is bounded. von Neumann-boundedness is more well-suited when one works with metrizable topological vector spaces.

von Neumann-boundedness is convenient as it shall enable us to interchange limits and integrals, say. Observe that in the case in which \( \mathcal{X} \) is normed, we are back to the classical concept of boundedness.

In view of the von Neumann-boundedness assumption on \( \sigma \), for any \( \ell \in \mathcal{X}', A \in \mathcal{L}(\mathcal{X}) \), \( b \in \mathcal{X} \)

\[
|N_{\ell,A,b}(x)| \leq C_\ell p_{\ell \sigma}(\sigma(A x + b)), \quad x \in \mathcal{X}
\]

for some constant \( C_\ell \geq 0 \) (compare \([22, \text{Thm. 1.1, p. 74}]\)), and thus, for a constant \( C(\ell, \sigma) \) depending on \( \ell \) and \( \sigma \)

\[
|N_{\ell,A,b}(x)| \leq C(\ell, \sigma), \quad x \in \mathcal{X}.
\]

We next investigate under which conditions a non-linear function \( \sigma \) is discriminatory. From now on, we assume that \( \mathbb{F} = \mathbb{R} \), because we need that hyperplanes disconnect the space \( \mathcal{X} \). If \( \mathbb{F} = \mathbb{C} \), this of course, cannot hold.

We now state a condition that ensures that \( \sigma \) is discriminatory. In order to develop some intuition for this condition, first recall that any \( \psi \in \mathcal{X}' \setminus \{0\} \) defines a hyperplane in \( \mathcal{X} \) by the set \( \Psi_0 = \ker(\psi) \). This hyperplane splits \( \mathcal{X} \) between the two half-spaces \( \Psi_+ = \{ x \in \mathcal{X}; \langle \psi, x \rangle > 0 \} \) and \( \Psi_- = \{ x \in \mathcal{X}; \langle \psi, x \rangle < 0 \} \), which lie on either side of the hyperplane. It turns out that measures on \( \mathcal{B}(\mathcal{X}) \cap K \) are fully determined by their values on the half-spaces arising from all shifted hyperplanes. If now \( \sigma \) splits the space \( \mathcal{X} \) in the sense that there exists one particular hyper-plane \( \Psi_0 \) such that on either side of this hyperplane, the function \( \sigma(\lambda x) \) converges as \( \lambda \to \infty \), then this implies that \( \sigma(\lambda x) \) converges pointwise to a function that is constant on both half-spaces separated by \( \Psi_0 \). Integrating this pointwise limit over either of those spaces determines the value of the measure on them. The maps \( A \in \mathcal{L}(\mathcal{X}), b \in \mathcal{X} \) now allow to rotate, shift and project to all possible half-spaces and determine the measure on them.

The following separating property is thus the infinite-dimensional counterpart to the well known sigmoideal property for functions from \( \mathbb{R} \) to \( \mathbb{R} \) (see \([10]\)):

**Definition 2.6.** Separating property: There exist \( \psi \in \mathcal{X}' \setminus \{0\} \) and \( u_+, u_-, u_0 \in \mathcal{X} \) such that either \( u_+ \notin \text{span}\{u_0, u_-\} \) or \( u_- \notin \text{span}\{u_0, u_+\} \) and such that

\[
\begin{aligned}
\lim_{\lambda \to \infty} \sigma(\lambda x) &= u_+, \quad \text{if } x \in \Psi_+ \\
\lim_{\lambda \to \infty} \sigma(\lambda x) &= u_-, \quad \text{if } x \in \Psi_- \\
\lim_{\lambda \to \infty} \sigma(\lambda x) &= u_0, \quad \text{if } x \in \Psi_0
\end{aligned}
\]

(5)
where we have set as above

\[ \Psi_+ = \{ x \in X; \langle \psi, x \rangle > 0 \}, \quad \Psi_- = \{ x \in X; \langle \psi, x \rangle < 0 \} \]

and \( \Psi_0 = \ker(\psi) \).

We point out that as a particular case of the Separating property we may choose \( u_0 = u_- = 0 \) and \( u_+ \neq 0 \) for instance.

**Example 2.7.** We are going to give a construction of a continuous and von Neumann-bounded function \( \sigma : X \to X \) satisfying the Separating property in Definition 2.6, for \( u_+, u_-, u_0 \in X \) such that either \( u_+ \notin \text{span}\{u_0, u_-\} \) or \( u_- \notin \text{span}\{u_0, u_+\} \).

Let us recall this abstract result first: given a metric space \((Z, d)\) and \( \emptyset \neq Y \subset Z \), define

\[ F_\varepsilon(x) := \max(1 - \varepsilon^{-1}d(x, Y)), \quad x \in Z, \varepsilon > 0. \]

Then \( F_\varepsilon \) is Lipschitz continuous, \( F_\varepsilon \in [0, 1] \) and \( F_\varepsilon(x) \to I_Y(x) \) for any \( x \in Z \) as \( \varepsilon \to 0 \).

Consider \( \psi \in X' \setminus \{0\} \) arbitrary. We approximate with this trick the indicator functions \( I_{\{\psi \geq 1\}}, I_{\{\psi < -1\}} \) and \( I_{\{\psi = 0\}} \), obtaining respectively \( F_{\varepsilon,1}, F_{\varepsilon,-1} \) and \( F_{\varepsilon,0} \). The scaling parameter \( \varepsilon \) is chosen small enough such that the supports of these functions do not meet. This is clearly possible. Indeed: suppose first that \( d(\{\psi = 1\}, \{\psi = 0\}) = 0 \). Then we might find \((z_n, y_n) \in \{\psi = 1\} \times \{\psi = 0\}\) such that \( d(z_n, y_n) \to 0 \), namely \( p_k(z_n - y_n) \to 0 \) for any \( k \in \mathbb{N} \). But on the other hand, for some \( j \in \mathbb{N} \) and \( c_j > 0 \)

\[ 1 = |(\psi, z_n) - (\psi, y_n)| \leq c_j p_j(z_n - y_n) \to 0 \]

and thus \( d(\{\psi = 1\}, \{\psi = 0\}) > 0 \). Since \( \supp F_{\varepsilon,1} = \text{cl}(\{\psi \geq 1\}\varepsilon) \) and \( \supp F_{\varepsilon,0} = \text{cl}(\{\psi = 0\}) \varepsilon \) (for an arbitrary subset \( Y \), \( Y_\varepsilon \) denotes its \( \varepsilon \)-neighborhood), for \( \varepsilon < d(\{\psi = 1\}, \{\psi = 0\}) \) we obtain that the supports do not meet. The same holds for the other cases.

Define

\[ \sigma(x) := F_{\varepsilon,1}(x)u_+ + F_{\varepsilon,-1}(x)u_- + F_{\varepsilon,0}(x)u_0, \quad x \in X. \]

Then \( \sigma \) is (Lipschitz)-continuous and von Neumann-bounded, because for any \( k \in \mathbb{N} \) and \( x \in X \) we clearly have

\[ p_k(\sigma(x)) \leq p_k(u_+) + p_k(u_-) + p_k(u_0), \]

and the condition [5] is satisfied.

The following theorem shows that a function \( \sigma \) that satisfies Definition 2.6 is discriminatory from which density of \( \mathfrak{N}(\sigma) \) follows then by Theorem 2.3.

**Theorem 2.8.** Let \( X \) be a real Fréchet space. Let \( \sigma : X \to X \) be continuous, von Neumann-bounded and satisfying the Separating property from Definition 2.6 above. Assume that for a given compact subset \( K \subset X \) and a given regular Borel measure \( \mu \) on \( K \) it holds

\[ \int_K (\ell, \sigma(Ax + b)) \mu(dx) = 0 \]

for all \( \ell \in X', A \in \mathcal{L}(\mathbb{X}), b \in \mathbb{X} \). Then \( \mu = 0 \).

**Proof.** Consider \( \lambda > 0 \). Then for any \( \ell \in X', A \in \mathcal{L}(X), b \in X \) it holds

\[ \int_K (\ell, \sigma(\lambda(Ax + b))) \mu(dx) = 0. \]

Observe that, as \( \lambda \to \infty \), pointwise in \( x \in X \),

\[ \langle \ell, \sigma(\lambda(Ax + b)) \rangle \to \begin{cases} 
(\ell, u_+), & \text{if } Ax + b \in \Psi_+ \\
(\ell, u_-), & \text{if } Ax + b \in \Psi_- \\
(\ell, u_0), & \text{if } Ax + b \in \Psi_0
\end{cases} \]

Since, \( \sigma \) is von Neumann-bounded, then there exists a constant \( C(\ell, \sigma) \) such that

\[ |\langle \ell, \sigma(\lambda(Ax + b)) \rangle| \leq C(\ell, \sigma), \]

uniformly in \( \lambda \) and \( x \). By the Hahn-Jordan decomposition (see [3 Thm. 3.1.1., Cor. 3.1.2]), we can write the measure \( \mu = \mu_1 - \mu_2 \) for two positive measures \( \mu_1, \mu_2 \) on \( K \). This implies that

\[ \int_K (\ell, \sigma(\lambda(Ax + b)))\mu(dx) = \int_K (\ell, \sigma(\lambda(Ax + b)))\mu_1(dx) - \int_K (\ell, \sigma(\lambda(Ax + b)))\mu_2(dx). \]
Since we are integrating on the compact set $K$, and $\mu$ is a regular Borel measure, constants are integrable with respect to $\mu$ on $K$. The same holds then for $\mu_1$ and $\mu_2$.

Therefore, by Lebesgue’s dominated convergence theorem applied to each integrand above, it follows that

$$
(6) \quad \langle \ell, u_+ \rangle \mu[K \cap A^{-1}(\Psi_+ - b)] + \langle \ell, u_- \rangle \mu[K \cap A^{-1}(\Psi_- - b)] + \langle \ell, u_0 \rangle \mu[K \cap A^{-1}(\Psi_0 - b)] = 0
$$

for any $\ell \in \mathcal{X}'$, $A \in \mathcal{L}(\mathcal{X})$, $b \in \mathcal{X}$.

Let us first assume that $u_+ \notin \text{span}\{u_0, u_-\}$. Then by the Hahn-Banach theorem [5, Chap IV, Cor. 3.15] we can choose $\ell \in \mathcal{X}'$ such that $\langle \ell, u_+ \rangle = 1$ and $\langle \ell, u_- \rangle = \langle \ell, u_0 \rangle = 0$. This leads us to conclude from (6) that

$$
\mu[K \cap A^{-1}(\Psi_+ - b)] = 0
$$

for all $A \in \mathcal{L}(\mathcal{X})$, $b \in \mathcal{X}$. Let now $t \in \mathbb{R}$ and $b \in \mathcal{X}$ such that $t = \psi(-b)$. Then, it is immediate to see that

$$
\psi^{-1}(t, \infty)
$$

and thus

$$
\mu[K \cap (\psi \circ A)^{-1}(t, \infty)] = 0
$$

for each $t \in \mathbb{R}$ and $A \in \mathcal{L}(\mathcal{X})$. By Lemma [20] below, we therefore deduce that

$$
(7) \quad \mu[K \cap \gamma^{-1}(t, \infty)] = 0
$$

for each $t \in \mathbb{R}$ and $\gamma \in \mathcal{X}'$. In the case that $u_- \notin \text{span}\{u_0, u_+\}$ instead, a similar line of reasoning leads to conclude that

$$
(8) \quad \mu[K \cap \gamma^{-1}(-\infty, t)] = 0.
$$

Observe in particular that $\mu(K) = 0$. For the sake of convenience, we trivially extend $\mu$ to the whole $\mathcal{X}$, namely

$$
\mu_{\text{ext}}(E) := \mu(K \cap E), \quad E \in \mathcal{B}(\mathcal{X})
$$

and notice that $|\mu_{\text{ext}}|(\mathcal{X}) = |\mu|(K) < \infty$, where $|\mu_{\text{ext}}| = \mu_{\text{ext},1} + \mu_{\text{ext},2}$, and $\mu_{\text{ext}} = \mu_{\text{ext},1} - \mu_{\text{ext},2}$ is the Hahn-Jordan decomposition for the extended measure ($\mu_{\text{ext},1}$ and $\mu_{\text{ext},2}$ are positive finite measures on $\mathcal{B}(\mathcal{X})$). Clearly, then it follows from $\mu(K) = 0$ that $\mu_{\text{ext}}(\mathcal{X}) = 0$. Recall also that $\mathcal{B}(K) = \mathcal{B}(\mathcal{X}) \cap K$.

Because $\mu$ is regular Borel measure, it follows in particular that for every $E \subset K$ and $\varepsilon > 0$, there exists compact $K_\varepsilon \subset K$ such that $|\mu|(E \setminus K_\varepsilon) < \varepsilon$. This property extends to $E \in \mathcal{X}$ for $\mu_{\text{ext}}$ as we may use that $|\mu_{\text{ext}}|(\cdot) = |\mu|(\cdot \cap K)$ and choose $K_\varepsilon \subset E \cap K$ such that $|\mu|(E \setminus K) < \varepsilon$ and it follows that $|\mu_{\text{ext}}|(E \setminus K_\varepsilon) = |\mu_{\text{ext}}|(E \setminus K) \setminus K_\varepsilon + |\mu_{\text{ext}}|(E \cap K_\varepsilon) = |\mu|(E \setminus K) \setminus K_\varepsilon < \varepsilon$. This shows that $\mu_{\text{ext}}$ is a Radon measure in the sense of [5, Def. 7.1.1].

Moreover, (7) or (8) is now telling us that

$$
\mu_{\text{ext}} = 0 \quad \text{on } \sigma(\mathcal{X}') \subset \mathcal{B}(\mathcal{X}),
$$

the sigma-algebra generated by all the elements of $\mathcal{X}'$. We want to show that actually $\mu_{\text{ext}} = 0$ on $\mathcal{B}(\mathcal{X})$ as well. We argue by contradiction and assume there exists $E \in \mathcal{B}(\mathcal{X})$ such that $\mu_{\text{ext}}(E) \neq 0$. In virtue of [5, Prop. 7.12.1] we may find $B \in \sigma(\mathcal{X}')$ such that

$$
|\mu_{\text{ext}}|(E \Delta B) = 0,
$$

namely

$$
\mu_{\text{ext},i}(E \Delta B) = 0, \quad i = 1, 2.
$$

Since $E \Delta B = (E \cup B) \setminus (E \cap B)$ and $\mu_{\text{ext}}$ are positive finite measures, we infer

$$
\mu_{\text{ext},i}(E \cup B) = \mu_{\text{ext},i}(E \cap B), \quad i = 1, 2,
$$

which implies, $i = 1, 2$,

$$
\begin{cases}
\mu_{\text{ext},i}(E) \leq \mu_{\text{ext},i}(E \cup B) = \mu_{\text{ext},i}(E \cap B) \leq \mu_{\text{ext},i}(E) \\
\mu_{\text{ext},i}(B) \leq \mu_{\text{ext},i}(E \cup B) = \mu_{\text{ext},i}(E \cap B) \leq \mu_{\text{ext},i}(B)
\end{cases}
$$

and finally $\mu_{\text{ext},i}(E) = \mu_{\text{ext},i}(B)$ for $i = 1, 2$. Therefore,

$$
0 \neq \mu_{\text{ext}}(E) = \mu_{\text{ext},1}(E) - \mu_{\text{ext},2}(E) = \mu_{\text{ext},1}(B) - \mu_{\text{ext},2}(B) = \mu_{\text{ext}}(B)
$$
and at the same time \( \mu_{ext}(B) = 0 \), because \( B \in \sigma(\mathcal{X}') \). Thus, it must hold \( \mu_{ext} = 0 \) on \( \mathcal{B}(\mathcal{X}) \), and hence, \( \mu = 0 \) on \( \mathcal{B}(\mathcal{K}) \), which concludes the proof.

The next lemma completes the proof of Theorem 2.8. It requires another result, which we show in the subsequent Lemma.

**Lemma 2.9.** Let \( \mathcal{X} \) be a real Fréchet space. Let \( \psi \in \mathcal{X}' \) be not identically zero. Then, for arbitrary \( \gamma \in \mathcal{X}' \), the equation

\[
\gamma = \psi \circ A
\]

is solvable for some \( A \in \mathcal{L}(\mathcal{X}) \).

**Proof.** For arbitrary \( \phi \in \mathcal{X}' \), \( t \in \mathbb{R} \) we write \( \phi_t := \{ x \in \mathcal{X}; \langle \phi, x \rangle = t \} \). Clearly, we can assume \( \gamma \) not identically zero, otherwise the problem is trivial. Therefore, let \( z \in \mathcal{X} \) be such that \( \langle \gamma, z \rangle = 1 \) and \( \langle \psi, z \rangle \neq 0 \). Clearly, such \( z \) exists in view of Lemma 2.10 below. Moreover, let \( w \in \mathcal{X} \) such that \( \langle \psi, w \rangle = 1 \).

Let \( \Psi_0 = \ker(\psi) \) and \( \Gamma_0 = \ker(\gamma) \). We observe that

\[
\mathcal{X} = \Gamma_0 + \langle z \rangle = \Psi_0 + \langle w \rangle
\]

where \( \langle z \rangle = \{ sz; s \in \mathbb{R} \} \subset \mathcal{X} \) and \( \langle w \rangle = \{ su; s \in \mathbb{R} \} \subset \mathcal{X} \). Furthermore, \( \Gamma_0 \cap \langle z \rangle = \{ 0 \} \) and \( \Psi_0 \cap \langle w \rangle = \{ 0 \} \), namely \( \Gamma_0 \) and \( \langle z \rangle \), are algebraic complements. The same holds for \( \Psi_0 \) and \( \langle w \rangle \). Furthermore, \( \Gamma_0 \) and \( \Psi_0 \) are closed by continuity, and have codimension one. By [23] Prop. 3.5., page 22, it follows that \( \Gamma_0 \) and \( \langle z \rangle \) (respectively, \( \Psi_0 \) and \( \langle w \rangle \)) are also topologically complemented.

Therefore, any \( x \in \mathcal{X} \) may be written in a unique way as

\[
x = x_{\Gamma_0} + \gamma(x)z = x_{\Psi_0} + \psi(x)w,
\]

where \( x_{\Gamma_0} \in \Gamma_0, x_{\Psi_0} \in \Psi_0 \). We can therefore define the following projections operators:

\[
\begin{align*}
\Pi_{\Gamma_0} : \mathcal{X} &\to \Gamma_0, \quad x \mapsto x_{\Gamma_0}, \\
\Pi_{\langle z \rangle} : \mathcal{X} &\to \langle z \rangle, \quad x \mapsto \gamma(x)z, \\
\Pi_{\Psi_0} : \mathcal{X} &\to \Psi_0, \quad x \mapsto x_{\Psi_0}, \\
\Pi_{\langle w \rangle} : \mathcal{X} &\to \langle w \rangle, \quad x \mapsto \psi(x)w.
\end{align*}
\]

Since \( \psi, \gamma \) and the identity operator are continuous, it follows that \( \Pi_{\Psi_0}(x) = x - \psi(x)w, \Pi_{\Gamma_0}(x) = x - \gamma(x)z, \Pi_{\langle z \rangle} \) and \( \Pi_{\langle w \rangle} \) are in \( \mathcal{L}(\mathcal{X}) \). Define \( A_0 := \Pi_{\Psi_0} \circ \Pi_{\Gamma_0} + \Pi_{\langle w \rangle} \circ \Pi_{\langle z \rangle} \in \mathcal{L}(\mathcal{X}) \).

Let \( x \in \mathcal{X} \) arbitrary, and write it as \( x = x_{\Gamma_0} + \gamma(x)z \). Write \( z = wz_0 + \psi(z)w \). Then,

\[
A_0x = \Pi_{\Psi_0}x_{\Gamma_0} + \gamma(x)\Pi_{\langle w \rangle}z = \Pi_{\Psi_0}x_{\Gamma_0} + \gamma(x)\psi(z)w,
\]

and

\[
\psi(A_0x) = \gamma(x)\psi(z)\psi(w) = \gamma(x)\psi(z), \quad x \in \mathcal{X}.
\]

But \( \psi(z) \neq 0 \), and thus \( A := \psi(z)^{-1}A_0 \in \mathcal{L}(\mathcal{X}) \) does the job.

Here is the final Lemma settling the proof of Theorem 2.8.

**Lemma 2.10.** Given \( \phi, \psi \in \mathcal{X}' \setminus \{0\} \) there exists \( z \in \mathcal{X} \) such that \( \phi(z) = 1 \) and \( \psi(z) \neq 0 \).

**Proof.** Linearity of \( \phi \) implies that the set \( \Phi_+ \cup \Phi_- \), where \( \Phi_+ = \{ x \in \mathcal{X}; \langle \phi, x \rangle > 0 \} \) and \( \Phi_- = \{ x \in \mathcal{X}; \langle \phi, x \rangle < 0 \} \), is actually dense. To see this, we need to show that each \( x \in \Phi_0 = \ker(\phi) \) can be approximated with a sequence in \( \Phi_+ \cup \Phi_- \). Consider \( u_n = n^{-1}u \in \mathcal{X} \) with some \( u \in \mathcal{X} \) such that \( \phi(u) = 1 \) and define \( x_n = x + u_n \). Then clearly \( x_n \in \Phi_+ \) and \( x_n \to x \) and hence we get that \( \phi(\Phi_+ \cup \Phi_-) = \mathcal{X} \). Suppose that \( \psi \) vanishes on the set \( \Phi_+ \cup \Phi_- \). Again by continuity of \( \psi \) we would get \( \psi = 0 \) identically. Therefore, there must exist \( w \in \Phi_+ \cup \Phi_- \) such that \( \psi(w) \neq 0 \). The element \( z = w/\phi(w) \) does the job.

**Example 2.11.** Let us give some concrete applications of our abstract framework. Let now for the sake of simplicity \( \mathcal{X} \) be a real separable Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and corresponding norm \( \| \cdot \| \). Further, we denote by \( (e_k)_k \) an orthonormal basis for \( \mathcal{X} \). Any \( x \in \mathcal{X} \) may be uniquely written as \( x = \sum_{k \in \mathbb{N}} x_k e_k \), where \( x_k = \langle e_k, x \rangle \).
Consider $\beta_i \in C(\mathbb{R}; \mathbb{R})$, $i = 1, 2, 3$ such that
\[
\begin{align*}
\lim_{\xi \to +\infty} \beta_1(\xi) &= 1, \quad \lim_{\xi \to -\infty} \beta_1(\xi) = -1, \quad \beta_1(0) = 0, \\
\lim_{\xi \to +\infty} \beta_2(\xi) &= 1, \quad \lim_{\xi \to -\infty} \beta_2(\xi) = 1, \quad \beta_2(0) = 1, \\
\lim_{\xi \to +\infty} \beta_3(\xi) &= -1, \quad \lim_{\xi \to -\infty} \beta_3(\xi) = 2, \quad \beta_3(0) = 0,
\end{align*}
\]
and define
\[\sigma(x) = \beta_1(x_1)e_1 + \beta_2(x_2)e_2 + \beta_3(x_1)e_3, \quad x \in \mathfrak{X}.
\]
Evidently, $\sigma \in C(\mathfrak{X}; \mathfrak{X})$; besides, since $\|\sigma(x)\| = \beta_1^2(x_1) + \beta_2^2(x_2) + \beta_3^2(x_1)$, it holds $\sup_x \|\sigma(x)\| < \infty$, because $\beta_1, \beta_2$ and $\beta_3$ are bounded. Thus $\sigma$ is von Neumann-bounded. Consider now the linear bounded functional
\[
\psi(x) := (e_1, x) = x_1, \quad x \in \mathfrak{X}.
\]
Clearly, $\Psi_+ = \{x \in \mathfrak{X}; x_1 > 0\}$, $\Psi_- = \{x \in \mathfrak{X}; x_1 < 0\}$ and $\Psi_0 = \{x \in \mathfrak{X}; x_1 = 0\}$ and, as $\lambda \to \infty$
\[
\sigma(\lambda x) \to \begin{cases} 
  e_1 + e_2 - e_3, & \text{if } x \in \Psi_+ \\
  -e_1 + e_2 + 2e_3, & \text{if } x \in \Psi_- \\
  e_2, & \text{if } x \in \Psi_0
\end{cases}
\]
which are linearly independent. We can therefore apply our results to infer that $\mathfrak{N}(\sigma)$ is dense in $C(\mathfrak{X}; \mathbb{R})$ with respect to the topology of uniform convergence on the compact subsets of $\mathfrak{X}$.

We can even go further. By the comment after Definition 2.3 indeed it is enough to consider a function $\beta \in C(\mathbb{R}; \mathbb{R})$ such that
\[
\lim_{\xi \to +\infty} \beta(\xi) = 1, \quad \lim_{\xi \to -\infty} \beta(\xi) = 0, \quad \beta(0) = 0,
\]
and arbitrary $z \in \mathfrak{X}$ in order to define
\[
\sigma(x) = \beta(\psi(x))z = \beta(x_1)z, \quad x \in \mathfrak{X}
\]
which still enables us to conclude that $\mathfrak{N}(\sigma)$ is dense in $C(\mathfrak{X}; \mathbb{R})$. Example 4.4 below extends this example for more general choices of $\psi$. A natural question now would be to find “optimal” $\beta$ and $z$ such that the convergence of the approximation to the function we want to learn is “fast”.

3. APPROXIMATION FOR GENERAL CODOMAIN

In this section we are going to show that our results can be extended to functions $f \in C(\mathfrak{X}; \mathfrak{Y})$ where $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ is an $F$-Banach space.

As a first step, we need the following simple lemma, which enables us to approximate with our neural network continuous functions from $\mathfrak{X}$ into $\mathbb{R}^d, d \in \mathbb{N}$:

Lemma 3.1. Let $\mathfrak{X}$ be an $F$-Fréchet space, and let $\sigma : \mathfrak{X} \to \mathfrak{X}$ be continuous and discriminatory. Then, given $f \in C(\mathfrak{X}; \mathbb{R}^d)$, a compact subset $K$ of $\mathfrak{X}$, and $\varepsilon > 0$, there exist $N^i = \sum_{m=1}^{M_i} \alpha^i_m N_{\ell^i_m} A^i_{m}, b^i_m \in \mathfrak{N}(\sigma), i = 1, \ldots, d$, with suitable $\alpha^i_m \in \mathbb{R}, \ell^i_m \in \mathfrak{X}^i, A^i_{m} \in \mathcal{L}(\mathfrak{X})$ and $b^i_m \in \mathfrak{X}$ such that
\[
\sup_{x \in K} \|f(x) - (N^1(x), \ldots, N^d(x))\|_{\mathbb{R}^d} < \varepsilon
\]
where for all $\xi \in \mathbb{R}^d$ we have $\|\xi\|_{\mathbb{R}^d} = \sum_{i=1}^{d} \|\xi^i\|$.

Proof. We write $f = (f^1, \ldots, f^d)$ with $f^i \in C(\mathfrak{X}; \mathbb{R}), i = 1, \ldots, d$. Given $K \subset \mathfrak{X}$ and $\varepsilon > 0$, Theorem 2.3 guarantees the existence of $N^i \in \mathfrak{N}(\sigma)$ such that
\[
\sup_{x \in K} \|f^i(x) - N^i(x)\| < \varepsilon/d
\]
and we are done. \hfill $\square$

We are ready to prove the following:
Theorem 3.2. Let $\mathcal{X}$ be an $F$-Fréchet space, and let $\sigma : \mathcal{X} \to \mathcal{X}$ be continuous and discriminatory. Let $(\mathcal{Y}, \|\cdot\|_\mathcal{Y})$ be an $F$-Banach space. Then, given $f \in C(\mathcal{X}; \mathcal{Y})$, a compact subset $K$ of $\mathcal{X}$, and $\varepsilon > 0$, there exist $d \in \mathbb{N}$, $v_1, \ldots, v_d$ linear independent unit vectors of $\mathcal{Y}$, $N^1, \ldots, N^d \in \mathcal{N}(\sigma)$, such that, by defining

$$\mathcal{N}(x) := \sum_{i=1}^{d} N^i(x)v_i, \quad x \in \mathcal{X},$$

it holds

$$\sup_{x \in K} \|f(x) - \mathcal{N}(x)\|_\mathcal{Y} < \varepsilon.$$

Proof. We recall the following general approximation result (see for example [6, Ch. 6.1]): given a topological space $(Z, \tau)$, an $F$-Banach space $(\mathcal{Y}, \|\cdot\|_\mathcal{Y})$ and a continuous map

$$T : Z \to \mathcal{Y}$$

such that $T(Z)$ is relatively compact in $Y$, then, given $\varepsilon > 0$ there exists $T_\varepsilon : Z \to \mathcal{Y}$ continuous, with $T_\varepsilon(Z)$ contained in a finite-dimensional subspace of $\mathcal{Y}$, and such that

$$\|T_\varepsilon(z) - T(z)\|_\mathcal{Y} < \varepsilon, \quad z \in Z.$$

To apply this result in our present setting, we first restrict $f$ to $K$

$$f\big|_K : K \to \mathcal{Y},$$

obtaining a continuous function whose range is compact in $\mathcal{Y}$. Therefore, we may find $f_\varepsilon : K \to \mathcal{Y}$ continuous and such that

1. $f_\varepsilon(K) \subset \text{span\{}v_1, \ldots, v_d\text{}} \subset \mathcal{Y}$ for suitable linear independent elements $v_1, \ldots, v_d$, whose norm we assume to be equal to 1.
2. $\sup_{x \in K} \|f(x) - f_\varepsilon(x)\|_\mathcal{Y} = \sup_{x \in K} \|f\big|_K(x) - f_\varepsilon(x)\|_\mathcal{Y} < \varepsilon/2$.

We set for convenience $V = \text{span\{}v_1, \ldots, v_d\text{}}$, and we write $f_\varepsilon$ as

$$f_\varepsilon(x) = \sum_{i=1}^{d} f_\varepsilon^i(x)v_i, \quad x \in K$$

with suitable $f_\varepsilon^i \in C(K; \mathbb{F})$, $i = 1, \ldots, d$. Being $\mathcal{X}$ metrizable, it is clearly normal. Therefore, by the Tietze extension theorem (since $K$ is closed), there exist $g_\varepsilon^i \in C(\mathcal{X}; \mathbb{F})$ extensions of $f_\varepsilon^i$, $i = 1, \ldots, d$.

We define $g_\varepsilon(x) := \sum_{i=1}^{d} g_\varepsilon^i(x)v_i, \quad x \in \mathcal{X}$. Then $g_\varepsilon \in C(\mathcal{X}; \mathcal{Y})$, $g_\varepsilon(\mathcal{X}) \subset V$ and

$$\sup_{x \in K} \|f(x) - g_\varepsilon(x)\|_\mathcal{Y} < \varepsilon/2.$$

By Lemma 3.1 we may approximate on $K$

$$\mathcal{X} \ni x \mapsto (g_\varepsilon^1(x), \ldots, g_\varepsilon^d(x)) \in \mathbb{F}^d$$

with $(N^1, \ldots, N^d)$ such that

$$\sup_{x \in K} \|(g_\varepsilon^1(x), \ldots, g_\varepsilon^d(x)) - (N^1(x), \ldots, N^d(x))\|_{\mathbb{F}^d} < \varepsilon/2.$$

We define

$$\mathcal{N}(x) := \sum_{i=1}^{d} N^i(x)v_i, \quad x \in \mathcal{X},$$

letting $\mathcal{X}$ be an $F$-Fréchet space. Therefore, we may approximate on $K$.
which has the required property, since we have
\[
\sup_{x \in K} \| f(x) - \mathcal{N}(x) \|_Y \leq \sup_{x \in K} \| f(x) - g_\varepsilon(x) \|_Y + \sup_{x \in K} \| g_\varepsilon(x) - \mathcal{N}(x) \|_Y \\
< \varepsilon/2 + \sup_{x \in K} \sum_{i=1}^d |g_\varepsilon^i(x) - N^i(x)| \cdot ||v_i||_Y \\
= \varepsilon/2 + \sup_{x \in K} \sum_{i=1}^d |g_\varepsilon^i(x) - N^i(x)| \\
= \varepsilon/2 + \sup_{x \in K} \| (g_\varepsilon^1(x), \ldots, g_\varepsilon^d(x)) - (N^1(x), \ldots, N^d(x)) \|_{\mathcal{E}d} \\
< \varepsilon.
\]

\[\square\]

4. Approximation with finite dimensional neural networks

In this section we prove a result that ensures that one can approximate a given abstract neural net arbitrary well via a neural network that is constructed from finite dimensional maps and can thus be trained. Of course, this can only work if we can approximate any given \( x \in \mathfrak{X} \) sufficiently well with a finite dimensional quantity as otherwise we could not even represent \( x \) in a computer. It is therefore plausible that we can derive such results only if some kind of approximation property holds on \( \mathfrak{X} \). This approximation property must ensure that one can approximate the identity map on \( \mathfrak{X} \) by continuous linear maps of finite rank, uniformly on some subset \( K \subset \mathfrak{X} \) of interest. In spaces with a countable Schauder basis \( \{e_n\}_{n \in \mathbb{N}} \), the approximating linear maps are usually the projections \( \Pi_N : \mathfrak{X} \to \text{span}\{e_1, \ldots, e_N\} \). Unfortunately, not every Fréchet space has a Schauder basis as shown in [11]. We refer the reader to [23] Ch. III, Sec. 9 for a discussion of the approximation property and existence of a Schauder basis for Fréchet space, which was an open problem until answered in [11]. Whenever the space \( \mathfrak{X} \) has a Schauder basis, however, we can actually derive an approximation of our abstract neural network with a trainable finite dimensional neural network as we shall see in this section.

To start with, we are first going to work in a Banach space setting. For several applications that we have in mind, as for instance the numerical solution of partial differential equations and Feynmann-Kac formulas, this setting is sufficient. There, the standard structures are indeed Banach spaces.

Let \( \mathfrak{X} \) be a real separable Banach space with norm denoted by \( \| \cdot \| \) that admits a normalized Schauder basis \( \{e_k\}_{k \in \mathbb{N}} \), namely each \( x \in \mathfrak{X} \) has a unique representation \( x = \sum_{k=1}^{\infty} x_k e_k \) and \( \|e_k\| = 1 \) for all \( k \). It follows by [23] Thm. 9.6, p. 115 (look also at its proof) that

\[
\Pi_N : \mathfrak{X} \to \text{span}\{e_1, \ldots, e_N\}, \quad x \mapsto \sum_{k=1}^N x_k e_k, \quad N \in \mathbb{N}
\]

is linear and bounded with \( \sup_{N \in \mathbb{N}} \| \Pi_N \|_{op} \leq C \) for some suitable constant \( C \geq 1 \), and that for any \( K \subset \mathfrak{X} \) compact we have \( \sup_{x \in K} \| x - \Pi_N x \| \to 0 \) as \( N \to \infty \).

While we know by [11] that there exist Banach spaces without a Schauder basis, it is also true that “all usual separable Banach spaces of Analysis admit a Schauder basis” (see [5]). For example for the Banach spaces \( L^p(\mathbb{R}^n) \), where \( 1 \leq p < \infty \), as well as for the Sobolev and Besov spaces, a basis is given by wavelets [25]. See [14] for many more examples.

We assume now that the activation function \( \sigma : \mathfrak{X} \to \mathfrak{X} \) is Lipschitz, namely

\[
\| \sigma(x) - \sigma(y) \| \leq \text{Lip}(\sigma) \| x - y \|, \quad x, y \in \mathfrak{X}.
\]

(10)

where \( 0 \leq \text{Lip}(\sigma) < \infty \). Of course since \( \mathfrak{X} \) is already a metric space, we do not use the metric \( d \) defined in [11], but the one implied by the norm, i.e. \( d(x_1, x_2) = \| x_1 - x_2 \| \).

Observe also that the activation functions in Example [2.11] become Lipschitz as soon as we impose that the \( \beta_i \)'s are Lipschitz. The activation function in Example [2.7] is already Lipschitz. Therefore, this condition does not seem very restrictive.

We are ready to prove:
Proposition 4.1. Let $\mathcal{X}$ be a real separable Banach space that admits a normalized Schauder basis $(e_k)_{k \in \mathbb{N}}$ and let $\sigma$ be Lipschitz. Let $f \in C(\mathcal{X}; \mathbb{R}), \ K \subset \mathcal{X}$ compact and $\varepsilon > 0$. Assume

$$N^\varepsilon(x) = \sum_{j=1}^{M} \langle \ell_j, \sigma(A_j x + b_j) \rangle, \ x \in \mathcal{X}$$

with $\ell_j \in \mathcal{X}', A_j \in \mathcal{L}(\mathcal{X})$ and $b_j \in \mathcal{X}$ such that

$$\sup_{x \in K} |f(x) - N^\varepsilon(x)| < \varepsilon.$$

Fix $\delta > 0$. Then there exists $N_* = N_*(N^\varepsilon, \delta) \in \mathbb{N}$ such that for $N \geq N_*$(11)

$$\sup_{x \in K} \left| f(x) - \sum_{j=1}^{M} \langle \ell_j \circ \Pi_N, \sigma(\Pi_N A_j \Pi_N x + \Pi_N b_j) \rangle \right| < \varepsilon + \delta.$$

Proof. For $j = 1, \ldots, M, N \in \mathbb{N}$ and $x \in K$ we indeed have

$$|\langle \ell_j, \sigma(A_j x + b_j) \rangle - \langle \ell_j \circ \Pi_N, \sigma(\Pi_N A_j \Pi_N x + \Pi_N b_j) \rangle|$$

$$\leq |\langle \ell_j, \sigma(A_j x + b_j) - \Pi_N \sigma(A_j x + b_j) \rangle|$$

$$+ |\langle \ell_j, \Pi_N \sigma(A_j x + b_j) - \Pi_N \sigma(\Pi_N A_j \Pi_N x + \Pi_N b_j) \rangle|$$

$$\leq \|\ell_j\| \|\sigma(A_j x + b_j) - \Pi_N \sigma(A_j x + b_j)\|$$

$$+ \|\ell_j\| C \|\sigma(A_j x + b_j) - \sigma(\Pi_N A_j \Pi_N x + \Pi_N b_j)\|,$$

where in the last line we have used that $\sup_{N \in \mathbb{N}} \|\Pi_N\|_{op} \leq C$. Thus, as far as it concerns the second term, it holds

$$\|\ell_j\| C \|\sigma(A_j x + b_j) - \sigma(\Pi_N A_j \Pi_N x + \Pi_N b_j)\|$$

$$\leq \|\ell_j\| \text{Lip} (\sigma) \|A_j x + b_j - \Pi_N A_j \Pi_N x - \Pi_N b_j\|$$

$$\leq \|\ell_j\| \text{Lip} (\sigma) \left\{ \|A_j x - \Pi_N A_j x\| + \|\Pi_N A_j x - \Pi_N A_j \Pi_N x\| + \|b_j - \Pi_N b_j\| \right\}$$

$$\leq \|\ell_j\| \text{Lip} (\sigma) \left\{ \sup_{x \in K} \|A_j x - \Pi_N A_j x\| + \|b_j - \Pi_N b_j\| \right\}$$

$$= \|\ell_j\| \text{Lip} (\sigma) \left\{ \sup_{y \in A_j K} \|y - \Pi_N y\| + \|b_j - \Pi_N b_j\| \right\}.$$

Setting for convenience $\sigma_j := \sigma(A_j K + b_j)$, and noticing that it is compact, we eventually arrive at

$$|\langle \ell_j, \sigma(A_j x + b_j) \rangle - \langle \ell_j \circ \Pi_N, \sigma(\Pi_N A_j \Pi_N x + \Pi_N b_j) \rangle|$$

$$\leq \|\ell_j\| \text{Lip} (\sigma) \left\{ \sup_{y \in A_j K} \|y - \Pi_N y\| + C \|A_j\|_{op} \sup_{x \in K} \|x - \Pi_N x\| + \|b_j - \Pi_N b_j\| \right\}$$

$$+ \|\ell_j\| \sup_{y \in \sigma_j} \|y - \Pi_N y\|.$$


Observe that $A_j K \subset \mathcal{X}$ is compact. By the approximation property provided by the Schauder basis $(e_k)_{k \in \mathbb{N}}$, we may find $N(j) \in \mathbb{N}$ such that:

$$
\begin{align*}
\sup_{y \in A_j K} \| y - \Pi_N y \| < \frac{\delta}{4M ||e_j||_{Lip(\sigma)}}, \\
\sup_{y \in \sigma} \| y - \Pi_N y \| < \frac{\delta}{4M ||e_j||_{Lip(\sigma)}}, \\
\sup_{x \in K} \| x - \Pi_N x \| < \frac{\delta}{4M ||e_j||_{Lip(\sigma)}}, & \quad \text{if } \| A_j \|_{op} \neq 0
\end{align*}
$$

for all $N \geq N(j)$. With this choice, we then have

$$
\sup \{|\ell_j, \sigma(A_j x + b_j)| - |\ell_j \circ \Pi_N, \sigma(\Pi_N A_j \Pi_N x + \Pi_N b_j)|\} < \delta/M.
$$

Therefore, setting $N_* := \max\{N(1), \ldots, N(M)\}$, we conclude that for all $N \geq N_*$

$$
\sup_{x \in K} \left| f(x) - \sum_{j=1}^{M} \langle \ell_j \circ \Pi_N, \sigma(\Pi_N A_j \Pi_N x + \Pi_N b_j) \rangle \right| < \varepsilon + \delta.
$$

We mention that the function $N^\varepsilon : \mathcal{X} \to \mathbb{R}$, which is required in the proposition above, exists for instance in view of Theorem [23], as soon as we assume that $\sigma$ is discriminatory too.

**Remark 4.2.** The terms appearing in the sum in (17) can now easily be programmed in a computer. We see that for large $N$, it is sufficient to consider the finite dimensional input values $\Pi_N(x)$ instead of $x$, and then successively the restriction of the operators $\Pi_N A_j, \sigma$ and $\ell_j$ to $\text{span}\{e_1, \ldots, e_N\}$ instead of the maps $A_j, \sigma$ and $\ell_j$ for $j = 1, \ldots, M$. The maps $\Pi_N A_j, \sigma$ and $\ell_j$ are finite dimensional when restricted to $\text{span}\{e_1, \ldots, e_N\}$ and the sum above thus resembles a classical neural network. However, instead of the typical one dimensional activation function, the function $\Pi_N \circ \sigma$ restricted to $\text{span}\{e_1, \ldots, e_N\}$ is multidimensional.

With an extra effort it is possible to generalize this result to real separable Fréchet spaces that admit Schauder basis. Examples include for instance the Schwartz space of rapidly decreasing functions, for which a basis is given in terms of Hermite functions [24] and the Hida test function and distribution space [15] Def 2.3.2.]

Let us now see how to do this generalization. Following [21] 28.10, p. 331], a Schauder basis for a real separable Fréchet space is a sequence $(e_k)_{k \in \mathbb{N}} \subset \mathcal{X}$, such that each $x \in \mathcal{X}$ has a unique representation $x = \sum_{k=1}^{\infty} x_k e_k$. As above, we define

$$
\Pi_N : \mathcal{X} \to \text{span}\{e_1, \ldots, e_N\}, \quad x \mapsto \sum_{k=1}^{N} x_k e_k, \quad N \in \mathbb{N}
$$

which is linear and bounded. Still from [21] 28.10, p. 331], we see that for any $j \in \mathbb{N}$ there exists $m \in \mathbb{N}$ and $C > 0$ such that for any $x \in \mathcal{X}$

$$
\sup_{N \in \mathbb{N}} p_j(\Pi_N x) \leq C p_m(x)
$$

Moreover, we can easily see that for any $K \subset \mathcal{X}$ compact and any $j \in \mathbb{N}$ we have

$$
\sup_{x \in K} (x - \Pi_N x) \to 0
$$

as $N \to \infty$. Indeed, following [23] p. 81] and from (12) we see that

$$
\sup_{N \in \mathbb{N}} \sup_{x \in S} p_j(\Pi_N x) \leq C \sup_{x \in S} p_m(x) < \infty
$$

for any $j \in \mathbb{N}$ and $S \subset \mathcal{X}$ with finite cardinality. Trivially, $\sup_{x \in S} p_j(x) < \infty$. We therefore deduce that the subset $\{\Pi_N\}_{N \in \mathbb{N}} \cup \{I\} \subset \mathcal{L}(\mathcal{X})$ is simply bounded, with $I$ being the identity map. By [23] Thm 4.2, p. 83], it is equicontinuous, being $\mathcal{X}$ a Baire space. By [23] Thm 4.5, p. 85] we therefore conclude that we have convergence on all precompact subsets of $\mathcal{X}$.
We are now going to impose the following “graded” Lipschitz condition on the non-linearity $\sigma$:

\begin{equation}
\exists k_0 \in \mathbb{N} : \forall k \geq k_0 \exists C_k \geq 0 : p_k(\sigma(x) - \sigma(y)) \leq C_k p_k(x - y), \quad x, y \in \mathcal{X}.
\end{equation}

Notice that such a map $\sigma$ is automatically continuous.

We are ready to prove:

**Theorem 4.3.** Let $\mathcal{X}$ be a real separable Fréchet space that admits a Schauder basis $(e_k)_{k \in \mathbb{N}}$ and let $\sigma$ satisfy condition \[\text{(12)}\]. Let $f \in C(\mathcal{X}; \mathbb{R})$, $K \subset \mathcal{X}$ compact and $\varepsilon > 0$. Assume

\[N^\varepsilon(x) = \sum_{j=1}^{M} \langle \ell_j, \sigma(A_j x + b_j) \rangle, \quad x \in \mathcal{X}\]

with $\ell_j \in \mathcal{X}'$, $A_j \in \mathcal{L}(\mathcal{X})$ and $b_j \in \mathcal{X}$ such that

\[\sup_{x \in K} |f(x) - N^\varepsilon(x)| < \varepsilon.\]

Fix $\delta > 0$. Then there exists $N_\ast = N_\ast(N^\varepsilon, \delta) \in \mathbb{N}$ such that for $N \geq N_\ast$

\[\sup_{x \in K} \left| f(x) - \sum_{j=1}^{M} \langle \ell_j \circ \Pi_N, \sigma(\Pi_N A_j \Pi_N x + \Pi_N b_j) \rangle \right| < \varepsilon + \delta.\]

**Proof.** For $j = 1, \ldots, M$, $N \in \mathbb{N}$ and $x \in K$ we indeed have, for suitable integers $r(\ell_j)$, $t(\ell_j)$, $m(\ell_j, \sigma)$ and $n(\ell_j, \sigma, A_j)$,

\begin{align*}
|\langle \ell_j, \sigma(A_j x + b_j) \rangle - \langle \ell_j \circ \Pi_N, \sigma(\Pi_N A_j \Pi_N x + \Pi_N b_j) \rangle| &\leq |\langle \ell_j, \sigma(A_j x + b_j) \rangle - \Pi_N \sigma(A_j x + b_j)\rangle| \\
&\quad + |\langle \ell_j, \Pi_N \sigma(A_j x + b_j) \rangle - \Pi_N \sigma(\Pi_N A_j \Pi_N x + \Pi_N b_j)\rangle| \\
&\leq C(\ell_j) p_{t(\ell_j)}(p_j(A_j x + b_j) - \Pi_N \sigma(A_j x + b_j)) \\
&\quad + C(\ell_j) p_{t(\ell_j)}(|\Pi_N \sigma(A_j x + b_j) - \Pi_N \sigma(\Pi_N A_j \Pi_N x + \Pi_N b_j)|) \\
&\leq C(\ell_j) p_{t(\ell_j)}(p_j(A_j x + b_j) - \Pi_N \sigma(A_j x + b_j)) \\
&\quad + C(\ell_j) \sup_{N \in \mathbb{N}} p_{t(\ell_j)}(|\Pi_N \sigma(A_j x + b_j) - \Pi_N \sigma(\Pi_N A_j \Pi_N x + \Pi_N b_j)|) \\
&\leq C(\ell_j) p_{t(\ell_j)}(p_j(A_j x + b_j) - \Pi_N \sigma(A_j x + b_j)) \\
&\quad + C(\ell_j) \sup_{N \in \mathbb{N}} p_{t(\ell_j)}(\sigma(A_j x + b_j) - \Pi_N \sigma(\Pi_N A_j \Pi_N x + \Pi_N b_j))
\end{align*}

where in the last line we have used the fact that the constant $C$ in \[\text{(12)}\] is independent of $N$ and $x$. Therefore, for the second term in the last expression we have

\begin{align*}
C(\ell_j) p_{t(\ell_j)}(p_j(A_j x + b_j) - \Pi_N \sigma(\Pi_N A_j \Pi_N x + \Pi_N b_j)) &\leq C(\ell_j) p_{t(\ell_j)}(p_j(A_j x + b_j) - \Pi_N \sigma(\Pi_N A_j \Pi_N x + \Pi_N b_j)) \\
&\leq C(\ell_j) \sup_{x \in K} |p_j(A_j x + b_j) - \Pi_N \sigma(\Pi_N A_j \Pi_N x + \Pi_N b_j)| \\
&\leq C(\ell_j, \sigma) \left\{ \sup_{x \in K} |p_j(A_j x - \Pi_N A_j x) + p_{t(\ell_j \vee k_0)}(\Pi_N A_j x - \Pi_N A_j \Pi_N x)| \right. \\
&\quad + \left. p_{t(\ell_j \vee k_0)}(b_j - \Pi_N b_j) \right\} \\
&\leq C(\ell_j, \sigma) \left\{ \sup_{x \in K} |p_j(A_j x - \Pi_N A_j x) + C'(\ell_j, \sigma) p_{m(\ell_j, \sigma)}(A_j x - A_j \Pi_N x) \right. \\
&\quad + p_{t(\ell_j \vee k_0)}(b_j - \Pi_N b_j) \right\} \\
&\leq C(\ell_j, \sigma) \left\{ \sup_{x \in K} |p_j(A_j x - \Pi_N A_j x) + C'(\ell_j, \sigma, A_j) p_{n(\ell_j, \sigma, A_j)}(x - \Pi_N x) \right. \\
&\quad + p_{t(\ell_j \vee k_0)}(b_j - \Pi_N b_j) \right\} \\
&\leq C(\ell_j, \sigma) \left\{ \sup_{x \in K} |p_j(A_j x - \Pi_N A_j x) + C'(\ell_j, \sigma, A_j) p_{n(\ell_j, \sigma, A_j)}(x - \Pi_N x) \right. \\
&\quad + p_{t(\ell_j \vee k_0)}(b_j - \Pi_N b_j) \right\} \\
&\leq C(\ell_j, \sigma) \left\{ \sup_{x \in K} |p_j(A_j x - \Pi_N A_j x) + C'(\ell_j, \sigma, A_j) p_{n(\ell_j, \sigma, A_j)}(x - \Pi_N x) \right. \\
&\quad + p_{t(\ell_j \vee k_0)}(b_j - \Pi_N b_j) \right\} \\
&\leq C(\ell_j, \sigma) \left\{ \sup_{x \in K} |p_j(A_j x - \Pi_N A_j x) + C'\langle \ell_j, \sigma, A_j \rangle p_{n(\ell_j, \sigma, A_j)}(x - \Pi_N x) \right. \\
&\quad + \left. p_{t(\ell_j \vee k_0)}(b_j - \Pi_N b_j) \right\}
\end{align*}

\begin{align*}
&\leq C(\ell_j, \sigma) \left\{ \sup_{y \in A_j K} p_{t(\ell_j \vee k_0)}(y - \Pi_N y) \right. \\
&\quad + C'\langle \ell_j, \sigma, A_j \rangle \sup_{x \in K} |p_{n(\ell_j, \sigma, A_j)}(x - \Pi_N x) + p_{t(\ell_j \vee k_0)}(b_j - \Pi_N b_j) \right. \\
&\quad + \left. p_{t(\ell_j \vee k_0)}(b_j - \Pi_N b_j) \right\}
\end{align*}
Observe that $A_j, K \subset X$ is compact. Setting for convenience $\sigma_j := \sigma(A_j K + b_j)$, and noticing that it is compact, we eventually arrive at
\[
|\langle \ell_j, \sigma(A_j x + b_j) \rangle - \langle \ell_j \circ \Pi_N, \sigma(\Pi_N A_j \Pi_N x + \Pi_N b_j) \rangle| 
\leq C(\ell_j, \sigma) \left\{ \sup_{y \in A_j K} p_{\ell_j}(y - \Pi_N y) + \sup_{y \in \sigma_j} p_{\ell_j}(y - \Pi_N y) \right. \\
\left. + C'(\ell_j, \sigma_j) \sup_{x \in K} p_{\ell_j}(x - \Pi_N x) + p_{\ell_j}(b_j - \Pi_N b_j) \right\}.
\]

By the approximation property provided by the Schauder basis $(e_k)_{k \in \mathbb{N}}$, we may find $N(j) \in \mathbb{N}$ such that:
\[
\begin{align*}
\sup_{y \in A_j K} p_{\ell_j}(y - \Pi_N y) &< \frac{\delta}{4MC(\ell_j, \sigma)} \\
\sup_{y \in \sigma_j} p_{\ell_j}(y - \Pi_N y) &< \frac{\delta}{4MC(\ell_j, \sigma)} \\
\sup_{x \in K} p_{\ell_j}(x - \Pi_N x) &< \frac{\delta}{4MC(\ell_j, \sigma)} \\
p_{\ell_j}(b_j - \Pi_N b_j) &< \frac{\delta}{4MC(\ell_j, \sigma)}
\end{align*}
\]
for all $N \geq N(j)$. With this choice, we then have
\[
\sup_{x \in K} |\langle \ell_j, \sigma(A_j x + b_j) \rangle - \langle \ell_j \circ \Pi_N, \sigma(\Pi_N A_j \Pi_N x + \Pi_N b_j) \rangle| < \frac{\delta}{M}.
\]
Therefore, setting $N_* := \max\{N(1), \ldots, N(M)\}$, we conclude that for all $N \geq N_*$
\[
\sup_{x \in K} \left| f(x) - \sum_{j=1}^{M} \langle \ell_j \circ \Pi_N, \sigma(\Pi_N A_j \Pi_N x + \Pi_N b_j) \rangle \right| < \varepsilon + \delta.
\]
\[ \square \]

Again, the required function $N^\varepsilon : X \rightarrow \mathbb{R}$ exists in view of Theorem [2.3]. However, we need to enhance Example [2.11] to show that non-linearities $\sigma$ satisfying condition [13] exist.

**Example 4.4.** Let $X$ be a real Fréchet space (not necessarily admitting a Schauder basis). Consider a function $\beta \in \text{Lip}(\mathbb{R}; \mathbb{R})$ such that
\[
\lim_{\xi \to +\infty} \beta(\xi) = 1, \quad \lim_{\xi \to -\infty} \beta(\xi) = 0, \quad \beta(0) = 0,
\]
and arbitrary $z \in X, z \neq 0$. Let $\psi \in X^* \setminus \{0\}$. Define
\[
\sigma(x) = \beta(\psi(x))z, \quad x \in X.
\]
Evidently, $\sigma$ is continuous and von Neumann-bounded, because for any $j \in \mathbb{N}$
\[
p_j(\sigma(x)) \leq |\beta(\psi(x))| p_j(z) \leq \|\beta\|_\infty p_j(z) < \infty
\]
uniformly in $x \in X$. Furthermore, it is clear that $\sigma$ satisfies [5]. Let us finally check that condition [13] is met. To this aim, let $k \in \mathbb{N}$. We have
\[
p_k(\sigma(x) - \sigma(y)) = |\beta(\psi(x)) - \beta(\psi(y))| p_k(z) \\
\leq \text{Lip}(\beta) p_k(z) |\psi(x) - \psi(y)| \\
\leq \text{Lip}(\beta) p_k(z) C_\psi p_m(\psi)(x - y) \\
:= C(\beta, z, \psi; k) p_m(\psi)(x - y), \quad x, y \in X
\]
for some $m(\psi) \in \mathbb{N}$. Therefore, for any $k \geq m(\psi)$, since the seminorms are non-decreasing, we have
\[
p_k(\sigma(x) - \sigma(y)) \leq C(\beta, z, \psi; k) p_k(x - y), \quad x, y \in X.
\]
5. Multi-layer Neural Networks

In this section we are going to show that results analogous to Theorems 2.8 and 2.9 hold also for multi-layer neural networks with a fixed number \( n > 1 \) of layers. We consider the following \( n \)-layers neural nets

\[
\mathcal{N}_{\ell,A_1,b_1,\ldots,A_n,b_n} : \mathcal{X} \to \mathbb{R}, \quad \mathcal{N}_{\ell,A_1,b_1,\ldots,A_n,b_n}(x) := \langle \ell, (\sigma \circ T_1 \circ \cdots \circ \sigma \circ T_n)(x) \rangle, \quad x \in \mathcal{X},
\]

with \( \ell \in \mathcal{X}', A_1, \ldots, A_n \in \mathcal{L}(\mathcal{X}), b_1, \ldots, b_n \in \mathcal{X}, \sigma : \mathcal{X} \to \mathcal{X} \) continuous, and where we have set

\[
T_j(x) := A_jx + b_j, \quad x \in \mathcal{X}, \quad j = 1, \ldots, n.
\]

Define

\[
\mathfrak{N}(\sigma) := \text{span}\{\mathcal{N}_{\ell,A_1,b_1,\ldots,A_n,b_n}; \ell \in \mathcal{X}', A_1, \ldots, A_n \in \mathcal{L}(\mathcal{X}), b_1, \ldots, b_n \in \mathcal{X}\}.
\]

Before embarking on the proof of the density of \( \mathfrak{N}(\sigma) \), we need to establish the following result, which will turn out to be very fruitful in the sequel.

**Lemma 5.1.** Assume that \( \mathcal{X} \) is a real separable Fréchet space. Let \( \sigma : \mathcal{X} \to \mathcal{X} \) be continuous and satisfying the following condition: there exist \( \psi \in \mathcal{X}' \setminus \{0\} \) and \( 0 \neq u_+ \in \mathcal{X} \) such that

\[
\begin{align*}
\lim_{\lambda \to \infty} \sigma(\lambda x) &= u_+, \quad \text{if } x \in \Psi_+ \\
\lim_{\lambda \to \infty} \sigma(\lambda x) &= 0, \quad \text{if } x \in \Psi_-
\end{align*}
\]

Let \( 0 \neq y \in \mathcal{X} \) be arbitrary. Then there exists \( A \in \mathcal{L}(\mathcal{X}) \) such that \( \sigma( Ay ) \neq 0 \).

**Proof.** We need to distinguish two cases:

1. \( y \in \Psi_0 \).
2. \( y \notin \Psi_0 \).

In the first case, let \( \phi \in \mathcal{X}' : \phi(y) \neq 0 \). By Lemma 2.10 choose \( z \) accordingly, i.e. \( \phi(z) = 1, \psi(z) \neq 0 \). Consider the projection onto \( \Phi_0 = \text{ker}(\phi) \)

\[
\Pi_{\Phi_0} : \mathcal{X} \to \Phi_0, \quad x \mapsto x_{\Phi_0} = x - \phi(x)z,
\]

which we know belongs to \( \mathcal{L}(\mathcal{X}) \). Thus, \( \psi(\Pi_{\Phi_0} y) = -\phi(y)\psi(z) \neq 0 \), namely \( \Pi_{\Phi_0} y \notin \Psi_0 \). If \( \Pi_{\Phi_0} y \in \Psi_+ \), set \( A = \lambda \Pi_{\Phi_0} y \), where \( \lambda > 0 \). Then \( \sigma(\lambda \Pi_{\Phi_0} y) \to u_+ \neq 0 \) as \( \lambda \to \infty \), and therefore for \( \lambda \gg 0 \) we obtain \( \sigma( Ay ) \neq 0 \). If on the other hand \( \Pi_{\Phi_0} y \in \Psi_- \), set \( A = -\lambda \Pi_{\Phi_0} y \) this time, to get the same conclusion, i.e. \( \sigma( Ay ) \neq 0 \).

If \( y \notin \Psi_0 \), then define \( A = \pm \lambda I \) with \( \lambda \gg 0 \), accordingly if \( y \in \Psi_+ \) or \( \Psi_- \). \( \square \)

With this result at hand, we are now ready to prove:

**Proposition 5.2.** Let \( \mathcal{X} \) be a real and separable Fréchet space, and let \( \sigma : \mathcal{X} \to \mathcal{X} \) be von Neumann-bounded and satisfy the conditions of Lemma 5.1 Then \( \mathfrak{N}(\sigma) \) is dense in \( C(\mathcal{X}; \mathbb{R}) \) with respect to the topology of compact subsets of \( \mathcal{X} \).

**Proof.** Evidently, \( \mathfrak{N}(\sigma) \subset C(\mathcal{X}; \mathbb{R}) \). Assume once again that \( cl(\mathfrak{N}(\sigma)) \subsetneq C(\mathcal{X}; \mathbb{R}) \). Then, once again we obtain the following

\[
\int_K \langle \ell, (\sigma \circ T_1 \circ \cdots \circ \sigma \circ T_n)(x) \rangle \mu(dx) = 0,
\]

for all \( \ell \in \mathcal{X}', A_1, \ldots, A_n \in \mathcal{L}(\mathcal{X}), b_1, \ldots, b_n \in \mathcal{X} \).

Observe that \( \sigma(0) = 0 \). Reasoning as in the proof of Proposition 2.8, this time we get that, as \( \lambda \to \infty \), pointwise in \( x \in \mathcal{X} \),

\[
\langle \ell, (\sigma \circ T_1 \circ \cdots \circ \sigma)(\lambda T_n(x)) \rangle \to \begin{cases}
\{ \ell, (\sigma \circ T_1 \circ \cdots \circ \sigma)(T_{n-1}(u_+)) \}, & \text{if } T_n(x) \in \Psi_+ \\
\{ \ell, (\sigma \circ T_1 \circ \cdots \circ \sigma)(b_{n-1}) \}, & \text{otherwise}
\end{cases}
\]

and hence, since \( \sigma \) is von Neumann-bounded, by the dominated convergence theorem (for finite signed measures)

\[
\begin{align*}
\langle \ell, (\sigma \circ T_1 \circ \cdots \circ \sigma)(T_{n-1}(u_+)) \rangle \mu[K \cap T_n^{-1}(\Psi_+)] \\
+ \langle \ell, (\sigma \circ T_1 \circ \cdots \circ \sigma)(b_{n-1}) \rangle \{ \mu[K \cap T_n^{-1}(\Psi_-)] + \mu[K \cap T_n^{-1}(\Psi_0)] \} = 0
\end{align*}
\]
for any $\ell \in \mathcal{X}'$, $A_1, \ldots, A_n \in \mathcal{L}(\mathcal{X})$, $b_1, \ldots, b_n \in \mathcal{X}$.

Choosing $b_1 = b_2 = \cdots = b_{n-1} = 0$ results in

$$\langle \ell, (\sigma \circ A_1 \circ \cdots \circ \sigma \circ A_{n-1})(u_+) \rangle \mu[K \cap T_{n}^{-1}(\Psi_+)] = 0$$

for any $\ell \in \mathcal{X}'$, $A_1, \ldots, A_n \in \mathcal{L}(\mathcal{X})$, and $b_n \in \mathcal{X}$. Define iteratively backward

$$y_{n-1} = \sigma(A_{n-1}u_+),$$
$$y_j = \sigma(A_jy_{j+1}), \quad j = 1, \ldots, n-2,$$

where $A_1, \ldots, A_{n-1} \in \mathcal{L}(\mathcal{X})$ are chosen in such a way that

$$y_{n-1} \neq 0, y_{n-2} \neq 0, \ldots, y_1 \neq 0.$$

This is achievable in virtue of Lemma 5.1. At the last step of the iteration we arrive at

$$\langle \ell, y_1 \rangle \mu[K \cap T_{n}^{-1}(\Psi_+)] = 0$$

for any $\ell \in \mathcal{X}'$, $A_n \in \mathcal{L}(\mathcal{X})$ and $b_n \in \mathcal{X}$, and hence $\mu[K \cap T_{n}^{-1}(\Psi_+)] = 0$, namely

$$\mu[K \cap A^{-1}(\Psi_+ + b)] = 0$$

for any $A \in \mathcal{L}(\mathcal{X})$, $b \in \mathcal{X}$. Following the steps in the proof of Proposition 2.3, we conclude once more that $\mu = 0$ and hence that $\mathfrak{F}(\sigma)$ is dense in $C(\mathcal{X}; \mathbb{R})$. □

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