Tuning linear response dynamics near the Dirac points
in the bosonic Mott insulator

A. S. Sajna
Faculty of Physics, Adam Mickiewicz University, ul. Umultowska 85, 61-614 Poznań, Poland

Optical lattice systems offer the possibility of creating and tuning Dirac points which are present in the tight-binding lattice dispersions. For example, such a behavior can be achieved in the staggered flux lattice or honeycomb type of lattices. Here we focus on the strongly correlated bosonic dynamics in the vicinity of Dirac points. In particular, we investigate bosonic Mott insulator phase in which quasiparticle excitations have a simple particle-hole interpretation. We show that linear response dynamics around Dirac points, can be significantly engineered at least in two ways: by the type of external perturbation or by changing the lattice properties. The key role is played by the interband transitions. Moreover, we explain that the behavior of these transitions is directly connected to different energy scales of the effective hopping amplitudes for particles and holes. Presented in this work theoretical study about tunability of linear response dynamics near the Dirac points can be directly simulated in the optical lattice systems.

PACS numbers: 03.75.Lm, 05.30.Jp, 03.75.Nt

I. INTRODUCTION

Response of the ultracold atomic systems to the external periodic modulation has been widely investigated experimentally and theoretically within the context of the Bose Hubbard model (BHM) (see, e.g., [1–14]). In particular, periodic modulation protocols help in understanding of strongly correlated bosonic dynamics, e.g. its gapped nature in the bosonic Mott insulator (MI) [1], dynamical conductivity [5], collective Higgs modes [8] or thermal excitations [3, 14]. However so far, these problems are poorly understood within the context of non-trivial lattices which can be generated by optical lattice patterns, e.g. geometrically or by synthetic gauge potentials ([15–17] and literature therein). This non-triviality in dynamics can emerge for example from Dirac points which can be present in the tight binding energy dispersion [16]. Such a problem is especially interesting because the resulting dynamics exhibits much broader types of quasiparticle excitations, e.g. intra and interband transitions. The lattices that show such a dynamics have been very recently studied by Grygiel et al. [18]. They have investigated optical conductivity in the lattice with synthetic gauge potential which correspond to the uniform magnetic field. The Grygiel et al. work has extended the earlier one [19, 22] and in particular they have shown the importance of interband transitions around Dirac points for two-band system with uniform π flux.

In this work, we present that non-trivial dynamics around Dirac points also appears in the broad class of lattices with staggered symmetry. In particular, in comparison to Ref. [18], interband transitions can be also very clearly visible in a much simpler experimentally available optical lattice systems without gauge potential, i.e. in the honeycomb type of lattices [16].

As a main point of our studies we show by analytical and numerical arguments that such interband transitions are very sensitive to the type of experiment made. In particular, we show that these transitions can be engineered at least in two ways: by the type of external perturbation or by changing the lattice properties. In the first case we show that interband transitions can be simply turn off in the isotropic amplitude modulation of the lattice [8, 10] in contrary to the uniaxial phase modulation experiments [23]. In the latter case, interband transitions can be continuously modified by engineering of Dirac like physics in staggered symmetry lattices [16, 24–26]. As an example, we analyze honeycomb type lattices which are able to shift the location of Dirac cones within the Brillouin zone [16]. We also consider the lattice with staggered fluxes for which a tunability of relativistic dynamics can be broadly modified by the steepness of Dirac cones simply observed in the tight-binding dispersion energy [24, 29]. Therefore, we show that engineering of Dirac like physics significantly modifies the linear response dynamics and we explain that this allows changes in the energy absorption rate mostly locally around Dirac points.

Through the paper we focus on the bosonic Mott insulator for which quasiparticle excitations have simple particle-hole interpretation and all peculiar dynamical phenomena appear at frequencies proportional to the bosonic on-site interaction [19, 20].

Manuscript is organized as follows. In Sec. IIA and IIB we shortly introduce the BHM within the coherent path integral framework and we define lattices types which are analyzed in this paper. Next, in Sec. IIC, IID and IIE we discuss the linear response dynamics together with their application to the conductivity and the isotropic energy absorption rate. In Sec. III we summarize our results.
II. METHODS AND RESULTS

A. Model and effective action

We consider strongly correlated lattice bosons which are described by the Bose-Hubbard model [27, 28]. Hamiltonian of this model in the second quantization language has a form

\[ H = H_0 + H_1, \]

\[ H_0 = \frac{U}{2} \sum_i b_i^\dagger b_i^\dagger b_i b_i - \mu \sum_i b_i^\dagger b_i, \]

\[ H_1 = -J \sum_{\langle ij \rangle} b_i^\dagger b_j, \]

where \( b_i \) (\( b_i^\dagger \)) is annihilation (creation) bosonic operator at site \( i \). The parameters \( J, U \) and \( \mu \) correspond to the hopping, interaction and chemical potential energy, respectively. We assume that the sum in Eq. (3) is restricted to the nearest neighbours sites.

Using the coherent state path integral formalism we can obtain the following form of partition function [29]

\[ Z = \int \mathcal{D}b^\dagger \mathcal{D}b e^{-(S_0 + S_1)/\hbar}, \]

\[ S_0 = \sum_i \int_0^{\beta} d\tau \left\{ \dot{b}_i(\tau) \dot{b}_i + \mu \dot{b}_i^\dagger(\tau) b_i(\tau) \right\}, \]

\[ + \frac{U}{2} \dot{b}_i(\tau) \dot{b}_i(\tau) b_i(\tau) - \mu \dot{b}_i^\dagger(\tau) b_i(\tau) \}

\[ S_1 = -J \sum_{\langle ij \rangle} \int_0^{\beta} d\tau \dot{b}_i(\tau) b_j(\tau), \]

In this work, we are interested in the bosonic MI phase which appear for \( J \ll U \) and for integer densities. Therefore we treat \( S_1 \) term in Eq. (6) as a perturbation. One of the powerful method in this regime is based on the strong coupling approach given by Sengupta and Dupuis [30]. This method uses double Hubbard-Stratonovich (HS) transformations which allows for the identification of new HS fields with that from Eqs. (3)-(6). Such an identification is possible because of correlation functions correspondence between the bare fields \( b_i(\tau), b_i(\tau) \) from Eqs. (3)-(6) and the fields after second HS [30]. Then, the effective action takes the form

\[ S_{eff} = -\sum_{ij} \int_0^{\beta} d\tau J_{ij} \dot{b}_i(\tau) b_j(\tau) \]

\[ -\sum_i \int_0^{\beta} d\tau d\tau' \left[ G^{1.c}(\tau' - \tau) \right]^{-1} \dot{b}_i(\tau') b_i(\tau), \]

where the strong coupling expansion is truncated to the second order in the MI phase in which Gaussian fluctuations are good approximation [24]. Moreover, \( \beta \) is the inverse of temperature i.e. \( 1/k_B T \) (\( k_B \) is the Boltzmann constant). Starting from Eq. (7) we set the reduced Planck constant \( \hbar \) to unity for simplicity. Moreover \( G^{1.c}(\tau - \tau') \) is the two-point local Green function which can be defined by its Fourier transform in the Matsubara frequencies domain \( \omega_n \), i.e.

\[ G^{1.c}(i\omega_n) = \frac{n_0 + 1}{i\omega_n + E_{n_0} - E_{n_0 + 1}} - \frac{n_0}{i\omega_n + E_{n_0 - 1} - E_{n_0}}, \]

with the on-site energy \( E_{n_0} = -\mu n_0 + U n_0 (n_0 - 1)/2 \) and \( G^{1.c}(i\omega_n) \) is taken in the zero temperature limit. The Matsubara frequencies have a definition \( \omega_n = 2\pi n/\beta \) and \( n \) is an integer number.

B. Lattices with staggered symmetry

In this work we consider two types of lattices which tight-binding energy dispersions exhibit relativistic behavior. In the first type of lattices, the relativistic energy behavior is introduced by the synthetic gauge field i.e. staggered flux lattice [24, 25] and in the second one by the lattice geometry: honeycomb or brick-wall lattice [16, 26]. Then, the effective action from Eq. (7) in the wave vector basis \( \mathbf{k} = (k_x, k_y) \), can be rewritten to the form

\[ S_{eff} = -\sum_{\mathbf{k}} \int d\tau d\tau' \mathbf{B}_{\mathbf{k}}(\tau') \left[ G^{MI}(\mathbf{k}, \tau' - \tau) \right]^{-1} \mathbf{B}_{\mathbf{k}}(\tau) \]

where

\[ \left[ G^{MI}(\mathbf{k}, \tau' - \tau) \right]^{-1} = -\mathbf{F}(\mathbf{k}) \delta(\tau' - \tau) + \hbar G^{-1}_0(\tau' - \tau), \]

\[ \mathbf{F}(\mathbf{k}) = \begin{bmatrix} 0 & f_{1,2}(\mathbf{k})^{\dagger} \\ f_{1,2}(\mathbf{k}) & 0 \end{bmatrix}, \]

\[ \mathbf{B}_{\mathbf{k}}(\tau) = \begin{bmatrix} b_{A}(\tau) \\ b_{B}(\tau) \end{bmatrix}, \]

and where indices \( A \) and \( B \) label the fields which belong to the corresponding sublattices and moreover we also assume that \( f_{1,2}(\mathbf{k}) = f_{2,1}(\mathbf{k}) \). In the subsequent calculations, we consider three forms of \( f_{1,2}(\mathbf{k}) \) which correspond to the three different lattices exhibiting Dirac points.

- Staggered flux lattice

\[ f^{\text{flux}}_{1,2}(\mathbf{k}) = -2J \left[ e^{i\phi/4} \cos(k_x a) + e^{-i\phi/4} \cos(k_y a) \right], \]

where \( \phi \) is the amplitude of the staggered flux [24, 25].
- Honeycomb lattice
\[ f_{1,2}^{\text{h}}(\mathbf{k}) = -J \left( e^{-ik_xa} + e^{i\frac{\sqrt{3}}{2}k_ya - i\frac{\sqrt{3}}{2}k_ya} + e^{i\frac{\sqrt{3}}{2}k_ya + i\frac{\sqrt{3}}{2}k_ya} \right). \] (14)

- Brick-wall lattice
\[ f_{1,2}^{\text{bw}}(\mathbf{k}) = -J \left( e^{ik_xa} + e^{-ik_xa} + e^{-k_ya} \right). \] (15)

In above equations, \( a \) is the lattice constant.

At the end of this section we also give the form of MI Green function \( G^{\text{MI}} \) (see Eq. (10)) in the Matsubara frequency representation \( \omega_n \) which will be useful in further calculations, i.e.
\[ G^{\text{MI}}(\mathbf{k}, \omega_n) = \begin{bmatrix} G^{\text{MI}}_{1,1}(\mathbf{k}, \omega_n) & G^{\text{MI}}_{1,2}(\mathbf{k}, \omega_n) \\ G^{\text{MI}}_{2,1}(\mathbf{k}, \omega_n) & G^{\text{MI}}_{2,2}(\mathbf{k}, \omega_n) \end{bmatrix}, \] (16)

where diagonal terms \( \alpha = \beta \) are
\[ G^{\text{MI}}_{\alpha,\beta}(\mathbf{k}, \omega_n) \]

Figure 1: (color online) The tight-binding energy dispersion for (a) the honeycomb type of lattices - standard honeycomb and brick-wall lattices, (b) the staggered flux lattice with \( \phi = \pi \) and \( \pi/3 \). The particular paths in the wave vector space are chosen to show existence of Dirac points in the tight-binding dispersion, i.e. points in which dispersions intersect.

Figure 2: (color online) The wave vector \( \mathbf{k} = (k_x, k_y) \) dependence of the quasi-particle and hole energy bands in the MI phase. The staggered flux lattice for \( J/U = 0.043 \) and \( \mu = 0.414 \) is depicted. The arrows show possible transitions in the linear response dynamics considered in this paper. \( uu, ll \) and \( ul, lu \) correspond to the intra and inter-band transitions, respectively.

\[
\Delta_{u/l}(k) = \pm \frac{|f_{1,2}(k)|}{2} - \mu + U \left( n_0 - \frac{1}{2} \right) \pm \frac{1}{2} \Delta_{u/l}(k),
\] (20)

The corresponding \( z_{p,\alpha}(k) \) and \( E_{p,u/l}(k) \) quantities in Eqs. (17)-(18) are

\[
z_{p,\alpha}(k) = \frac{E_{p,\alpha}(k) + \mu + U}{E_{p,\alpha}(k) E_{h,\alpha}(k)},
\] (19)
Important feature of lattices considered here is that they exhibit tunable Dirac points. To show this we plot the tight-binding energy dispersions \(\pm |f_{1,2}(k)|\) of honeycomb and brick-wall lattices in Fig. [1]a and the tight-binding energy dispersions of staggered flux lattice in Fig. [1]b. Corresponding formulas for these dispersions are given in Appendix [IV.A]. Tunability of honeycomb like lattice is obtained by modification of its geometry (standard honeycomb geometry and brick-wall geometry) which cause shift of Dirac points locations in the wave vector space (see Fig. [1]a). This modification also slightly changes the steepness of the relativistic dispersion around Dirac points. Moreover, changing the flux amplitude \(\phi\) in the staggered flux lattice changes also the steepness of relativistic dispersion around Dirac points which is more pronounced than in Fig. [1]a. These effects have direct consequences on the linear response dynamics which will be discussed in the subsequent sections.

Additionally, to show how the excitation spectra of quasiparticles are affected by the Dirac points, we plot MI energy dispersion of staggered flux lattice in Fig. [2] (the analog plot can be given for the honeycomb like lattices). In this Figure, the Dirac points are located in the quasiparticle and hole spectrum at \((\omega > 0)\) (the analog plot can be given for the honeycomb like lattice). In this Figure, the Dirac points are located in the quasiparticle and hole spectrum at \((\omega > 0)\) (the analog plot can be given for the honeycomb like lattice). In this Figure, the Dirac points are located in the quasiparticle and hole spectrum at \((\omega > 0)\) (the analog plot can be given for the honeycomb like lattice). In this Figure, the Dirac points are located in the quasiparticle and hole spectrum at \((\omega > 0)\) (the analog plot can be given for the honeycomb like lattice).

C. Linear response kernel in the MI phase

The linear response kernel for the phase or amplitude modulation perturbation (in the MI phase) can be derived from the following form of the four point correlation function

\[
K(i\omega) = \frac{1}{N} \sum\sum_{k,k'} \int_0^\beta d\tau e^{i\omega \tau} \sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4=1}^2 \gamma_{\lambda_1,\lambda_2}(k) \gamma_{\lambda_3,\lambda_4}(k') \times \langle b^*_\lambda, k b_{\lambda', k} \rangle \langle b_{\lambda', k} b^*_\lambda, k \rangle,
\]

where \(\langle ... \rangle\) is defined over the effective action and we assume translation invariant lattice. Applying Matsubara frequency transformation for \(b, b^*\) fields together with the Wick theorem to the Eq. (22) one get

\[
K(i\omega) = \frac{1}{\beta N} \sum\sum_{k} G^{MI}_{\lambda_1, \lambda_2}(k, \omega_n) G^{MI}_{\lambda_3, \lambda_4}(k, \omega_n + \omega),
\]

in which

\[
\langle b_{\lambda, k} b^*_{\lambda', k+n} \rangle = G^{MI}_{\lambda, \lambda'}(k, \omega_n),
\]

where indices \(\lambda_\mu, \lambda_\nu \in \{1, 2\}\) denotes the matrix element of MI Green function which is 2 by 2 matrix defined in Eq. (10).

Careful explanation of the form \(K(i\omega)\) in Eq. (22) is needed. Firstly it describes the paramagnetic part of linear response in MI phase for which \(\omega > 0\). Secondly the form of \(K(i\omega)\) depend on the form of considered external perturbation. In this work, we consider two quantities: \(\sigma(\omega)\) and function \(S(\omega)\). \(\sigma(\omega)\) is dynamical conductivity and is defined by the current autocorrelation function \(S(\omega)\). \(S(\omega)\) is proportional to the isotropic energy absorption rate and is defined by the kinetic energy autocorrelation function \(|\omega|\). It is important to stress here that \(\sigma(\omega)\) and \(S(\omega)\) are also related with response of the system to the phase and amplitude periodic modulation of optical lattice, respectively. Namely, within notation introduced in Sec. [IIB] the optical conductivity can be obtained from

\[
\sigma(\omega) = \frac{e_{eff}^2}{\omega} \left. K(i\omega) \right|_{i\omega \rightarrow -i\omega + i0^+},
\]

where

\[
\gamma^{1}_{\lambda_1, \lambda_2}(k) = \gamma^{2}_{\lambda_1, \lambda_2}(k) = \partial_{k_z} f_{\lambda_1, \lambda_2}(k),
\]

and \(e_{eff}\) is the effective charge (which in optical lattice can be generated e.g. by the synthetic gauge field). \(i\omega \rightarrow \omega + i0^+\) denotes analytic continuation and \(f_{\lambda_1, \lambda_2}(k)\) is \((\lambda_1, \lambda_2)\) matrix element of \(F(k)\) from Eq. (11). Partial derivative over \(k_z\) in Eq. (26) means that \(xx\) component of the longitudinal conductivity is considered. Form of Eq. (25) agrees with those found in literature [18] [19] [31].

In the case of \(S(\omega)\) one gets

\[
S(\omega) = \text{Im} \left[ K(i\omega)_{i\omega \rightarrow -i\omega + i0^+} \right],
\]

where

\[
\gamma^{1}_{\lambda_1, \lambda_2}(k) = \gamma^{2}_{\lambda_1, \lambda_2}(k) = f_{\lambda_1, \lambda_2}(k).
\]

\(S(\omega)\) has been previously studied in different contexts, e.g. correlated fermions [32], Higgs mode [10] or thermometry in MI phase [14].

Because we are focused on the MI phase, one can give the general form of \(K(i\omega)\) by using Eqs. (23) and (17)-(21), i.e.

\[
K(i\omega) = \frac{1}{\beta N} \sum_{k} \sum_{\alpha, \beta \in \{u, l\}} \Gamma^\gamma_{\alpha, \beta}(k) \left[ \frac{1 - z_{p, \alpha}(k)}{E_{h, \alpha}(k) + i\omega - E_{p, \beta}(k)} \right],
\]

where

\[
\Gamma^\gamma_{uu}(k) = \Gamma^\gamma_{ll}(k) = \left( \gamma^{1,2}_{\lambda_1, \lambda_2}(k) f_{1,2}(k) + \gamma^{2,1}_{\lambda_1, \lambda_2}(k) \tilde{f}_{1,2}(k) \right)^2,
\]

\[
\Gamma^\gamma_{ul}(k) = \Gamma^\gamma_{lu}(k) = -\left( \gamma^{1,2}_{\lambda_1, \lambda_2}(k) f_{1,2}(k) - \gamma^{2,1}_{\lambda_1, \lambda_2}(k) \tilde{f}_{1,2}(k) \right)^2.
\]
and weights $\Gamma_{uu}^\gamma$, $\Gamma_{ll}^\gamma$, $\Gamma_{il}^\gamma$, $\Gamma_{lu}^\gamma$ correspond to the intra and inter-band transitions depicted in Fig. 2 for which more detailed discussion will be given shortly ($\gamma$ index is reserved for conductivity $\Gamma_{cond}^{\alpha\beta}$ and for energy absorption rate $\Gamma_{a,b}^{\rho\phi}$). Moreover, in derivation of Eqs. (29)-(32) we assume a zero temperature limit and for simplicity we identify $\gamma^1$ with $\gamma^2$ i.e. $\gamma^1 = \gamma^2$ which agree with Eqs. (26) and (28).

It is also worth to stress here that Eqs. (29)-(32) are valid for arbitrary form of the staggered symmetry introduced by the hopping amplitude $F(k)$ (see Eq. 11) and therefore these results are not restricted to the lattices considered here (i.e. for the lattices defined by Eqs. (13)-(15)).

D. Dynamical conductivity

To calculate dynamical (longitudinal) conductivity $\sigma(\omega)$ from Eq. (25), analytic continuation of $K(\omega)/\omega$ is needed ($\omega \to \omega + i0^+$). This yields the following form

![Figure 3: (color online) Frequency $\omega$ dependent conductivity $\sigma(\omega)$ (a-c) and $S(\omega)$ (d-f) in the MI phase at zero temperature. The latter quantity is proportional to the isotropic energy absorption rate. The remaining parameters are $\mu/U = 0.414$, $J/U = 0.043$. The names uu, dd, ud, du correspond to transitions depicted in Fig. 2. The bold and dashed gray line gives total response of the system coming from the all transitions summed together (i.e. from intra and interband transitions). $\sigma_q$ denote the quantum unit of conductance defined in Eq. 33.](image)

![Figure 4: (color online) Frequency dependent conductivity in the MI phase at zero temperature. Staggered flux lattice is considered with the flux amplitudes: (a) $\phi = \pi$, (b) $\phi = 2\pi/3$, (c) $\phi = \pi/3$. For clarity, we plot dynamical conductivity with the same absolute detuning of hopping amplitude $J$ from the phase boundary, i.e. $\Delta_c = (J_c - J)/J_c \approx 20\%$ because $J_c$ is $\phi$ dependent function [23, 24]. In particular (a) $J/U \approx 0.049$, $(J/U)_c \approx 0.061$, (b) $J/U \approx 0.040$, $(J/U)_c \approx 0.050$, (c) $J/U \approx 0.036$, $(J/U)_c \approx 0.044$. The bold and dashed gray line gives total response of the system coming from the all transitions summed together (i.e. from intra and interband transitions). The chemical potential is $\mu/U = 0.414$.](image)
of $xx$ component of the longitudinal conductivity
\[
\text{Re} \sigma (\omega) = \sigma_q \frac{2\pi^2}{N} \sum_k \sum_{\alpha,\beta \in \{u,l\}} \Gamma_{\alpha\beta}^{\text{cond}} (k) [1 - z_{p,\alpha} (k)] z_{p,\beta} (k) \frac{E_{h,\alpha} (k) - E_{p,\beta} (k)}{E_{p,\beta} (k) - E_{h,\alpha} (k)} \times \delta (\omega - [E_{p,\beta} (k) - E_{h,\alpha} (k)]),
\] (33)
where we focus on the real part of dynamical conductivity, i.e. $\text{Re} \sigma (\omega)$ and we denote $\sigma_q = \frac{e^2}{\hbar} J_{ff}$ as a quantum unit of conductance. From the form of $\text{Re} \sigma (\omega)$, we see that response of the system appears at the energy difference between quasiparticle ($E_{p,\beta} (k)$) and hole excitations ($E_{h,\alpha} (k)$) through the Dirac delta function $\delta$. Therefore for two band models in the MI phase, one can have four types of excitations. The transitions corresponding to these excitations are schematically drawn in Fig. 9. In general, one can divide such transitions into two classes which have intra ($uu$, $ll$) and interband ($ul$, $lu$) character. The intraband transitions are mostly responsible for the lowest and highest energy excitations and e.g. they can describe remarkable phenomenon like the finite critical conductivity \cite{19, 31, 33, 34}. The interband transitions are responsible for intermediate behavior and in the cases considered in this work, they strongly modify the response around energy scales where Dirac points appear (similar behavior has been very recently reported in the uniform magnetic fields \cite{13} when this work was finalized).

Next, calculating conductivity coefficients $\Gamma_{\alpha\beta}^{\text{cond}}$ in Eqs. (33) by using Eqs. (31)-(32) and (26) one gets
\[
\Gamma_{uu}^{\text{cond}} (k) = \Gamma_{ll}^{\text{cond}} (k) = \left( \partial_k |f_{1,2} (k)| \right)^2, \quad (34)
\]
\[
\Gamma_{ul}^{\text{cond}} (k) = \Gamma_{lu}^{\text{cond}} (k) = - \left( \frac{f_{1,2} (k) \partial_k f_{1,2} (k) - f_{1,2} (k) \partial_k f_{1,2} (k)}{2 |f_{1,2} (k)|} \right)^2, \quad (35)
\]
From the above equations, we see that all four transitions ($uu$, $ll$, $ul$, $lu$) can contribute to the conductivity. However, to consider particular lattice, explicit form of the functions $\Gamma_{\alpha\beta}^{\text{cond}}$ have to be given together with the remaining functions in Eq. (33).

To show how ($uu$, $ll$, $ul$, $lu$) intra and interband transitions behave in the lattices with staggered symmetry, we consider first the honeycomb type of lattices (see Eqs. (14) and (15)). For standard honeycomb lattice one get
\[
\Gamma_{uu}^{\text{cond}} (k) = \Gamma_{ll}^{\text{cond}} (k) = 3J^4 \sin^2 \left( \frac{\sqrt{3}}{2} k_x \right) \times \left( \frac{2 \cos \left( \frac{\sqrt{3}}{2} k_x \right) + \cos \left( \frac{3}{2} k_y \right)}{|f_{1,2} (k)|} \right)^2, \quad (36)
\]
\[
\Gamma_{ul}^{\text{cond}} (k) = \Gamma_{lu}^{\text{cond}} (k) = 3J^4 \sin^2 \left( \frac{\sqrt{3}}{2} k_x \right) \sin^2 \left( \frac{3}{2} k_y \right) \frac{1}{|f_{1,2} (k)|^2}, \quad (37)
\]
and for brick-wall lattice
\[
\Gamma_{uu}^{\text{cond}} (k) = \Gamma_{ll}^{\text{cond}} (k) = 4J^4 \sin^2 (k_x) \times \left( \frac{2 \cos (k_x) + \cos (k_y)}{|f_{1,2} (k)|} \right)^2, \quad (38)
\]
\[
\Gamma_{ul}^{\text{cond}} (k) = \Gamma_{lu}^{\text{cond}} (k) = 4J^4 \sin^2 (k_x) \sin^2 (k_y) \frac{1}{|f_{1,2} (k)|^2}, \quad (39)
\]
where we set lattice constant a to one. It is straightforward to see that the above formulas for honeycomb and brickwall lattice, have a similar form (see also Eqs. (14) and (15)) which result in similar behavior for dynamical conductivity - Figs. 3 a and b. In this Figures one sees that interband transitions ($ul$, $lu$), which appear near the Dirac point, are the most pronounced in comparison to the intraband transitions ($uu$, $ll$) (Dirac point is visible as a vanishing of conductivity at $\omega = U$). Corresponding behavior of significant differences in the intra and interband transition amplitudes have been very recently reported also for the uniform magnetic field \cite{13}. Moreover, shifting of Dirac points in the wave vector space $k$ (Fig. 3 a), change slightly the interband transitions amplitude, i.e. the brick-wall interband transitions have slightly higher amplitude than the corresponding transitions for honeycomb lattice. Therefore, modification of the lattice geometry has a direct consequence on the linear response dynamics near the Dirac points.

We can achieve much more pronounced tuning of the dynamics near the Dirac points if we consider the staggered flux lattice. Before, we show this, let us first write coefficients $\Gamma_{\alpha\beta}^{\text{cond}}$ for this type of lattice:
\[
\Gamma_{uu}^{\text{cond}} (k) = \Gamma_{ll}^{\text{cond}} (k) = 16J^4 \sin^2 (k_x a) \times \left( \frac{\cos (k_x a) + \cos (\phi/2) \cos (k_y a)}{|f_{1,2} (k)|} \right)^2, \quad (40)
\]
\[
\Gamma_{ul}^{\text{cond}} (k) = \Gamma_{lu}^{\text{cond}} (k) = 4J^4 \sin^2 (k_x a) \cos^2 (k_y a) \sin^2 (\phi/2) \frac{1}{|f_{1,2} (k)|^2}. \quad (41)
\]
From above equations, we see that the interband coefficients, i.e. $\Gamma_{ul}^{\text{cond}} (k)$ and $\Gamma_{lu}^{\text{cond}} (k)$, depend on the square power of the sin $(\phi/2)$ function. Therefore, this explains that the interband transitions are the largest for $\phi = \pi$ and they vanish in the limit $\phi \to 0$ (for $\phi = 0$ the square lattice limit is recovered where interband transitions do not exist \cite{19}). We prove this remarkable behavior of interband transitions by plotting dynamical conductivity for the different values of $\phi$ amplitude in Fig. 3. It is important to point out here that such a behavior is directly connected to the steepness of tight-binding dispersion near the Dirac points which is tuned with different $\phi$ amplitude, see Fig. 3 b.
Moreover, it is very instructive to explain why the interband transitions (ul, lu) appear around Dirac points (i.e., around ω = U) (see Figs. 3 a-b). If we look at Fig. 2, one sees that bandwidth of quasiparticle excitations is about two times larger than the quasi-hole band. This can be simplified accounted for the different effective hoppings of particles and holes, e.g., for free bosonic case is simple (n0 + 1)J and n0J, respectively (see, e.g., [33, 34]). Therefore, moving away from the Dirac points which for staggered flux lattice are at (±π/2, ±π/2) (see also Fig. 2), we see that the ul and lu transitions for given k vary slightly from the interaction energy U because of the different effective hoppings of particles and holes. We conclude, that exactly this difference in the effective hoppings, gives rise to the linear response dynamics which concentrates around Dirac points. Corresponding situation can be also found for the honeycomb type of lattices.

E. Isotropic energy absorption rate

In Sec. [11] we have shown how the interband transitions can be engineered by the lattice geometry or staggered flux. Here, we show that such a manipulation of transitions in the strongly correlated bosonic system can be also achieved in much robust way, i.e., by a suitable choice of the external perturbation. Dynamical conductivity considered in the previous section is related to the periodical phase modulation of the optical lattice [5]. Now we show what happen if we consider the periodic amplitude lattice modulation which is directly connected to the function S(ω) (Eq. (27)) [10, 32].

Namely, we consider S(ω) which is proportional to the isotropic energy absorption rate and is defined in Eqs. (27)-(32). After analytical calculation one gets

\[ S(\omega) = \frac{\pi}{N} \sum_{\mathbf{k}} \sum_{\alpha,\beta \in \{u,l\}} \Gamma_{\alpha\beta}^{\text{cond}}(\mathbf{k}) [1 - z_{p,\alpha}(\mathbf{k})] z_{p,\beta}(\mathbf{k}) \times \delta(\omega - |E_{p,\beta}(\mathbf{k}) - E_{h,\alpha}(\mathbf{k})|), \]

where

\[ \Gamma_{uu}^{\text{abs}}(\mathbf{k}) = \Gamma_{ll}^{\text{abs}}(\mathbf{k}) = (|f_{1,2}(\mathbf{k})|^2), \]

\[ \Gamma_{ul}^{\text{abs}}(\mathbf{k}) = \Gamma_{lu}^{\text{abs}}(\mathbf{k}) = 0, \]

and where the \(|f_{1,2}(\mathbf{k})|\) function is the tight-binding energy dispersion of the lattices considered in this work (see, Eqs. [15]-[17]). The result of Eq. (44) is especially interesting. It says that interband transitions (ul, lu) are completely turn off in the isotropic and periodic modulation of the lattice amplitude for the lattices with staggered symmetry introduced by the hopping amplitude (see Eq. [11]). We confirm this analytical result, by plotting Eqs. (42)-(44) in Figs. 3 d, e and f for honeycomb, brick-wall and staggered flux lattice, respectively.

These plots should be compared with the conductivity σ(ω) plots, Figs. 3 a, b and c, for which interband transitions are present (for σ(ω) the interband weights \(\Gamma_{ul}^{\text{cond}}\) and \(\Gamma_{lu}^{\text{cond}}\) do not vanish identically like in S(ω) case, compare Eqs. (35) and (44)).

III. SUMMARY

In this work we have considered linear response dynamics in the MI phase for the lattices with staggered symmetry. This symmetry was introduced by the hopping amplitude and takes into account a broad class of lattice currently realized in the optical lattice systems. We have focused on the lattices which contain Dirac points in their tight-binding energy dispersions and which can allow Dirac like physics engineering.

As the main result of our considerations we have shown that linear response dynamics around Dirac points can be highly tuned by a suitable type of the external perturbation or by modification of the lattice parameters. This shows that interband transitions can be very sensitive to the particular experiment realization and can be a signature for emergence of relativistic dynamics related with Dirac points.

Moreover, we have explained that such a peculiar dynamics around Dirac points is directly connected to the effective hopping energies for holes and particles. This further can provide an indirect proof of the non-equal values of bandwidth for quasiparticle and hole excitations in the MI phase.

It is also important to stress here that presented theoretical framework can be directly applied to the other lattice systems showing staggered lattice symmetry with the two-band lattice model.

Acknowledgments

We are grateful to R. Micnas, T. P. Polak for valuable discussions and carefully reading of the manuscript. This work was supported by the National Science Centre, Poland, project no. 2014/15/N/ST2/03459.

IV. APPENDIX

A. Tight-binding energy dispersions

Tight-binding energy dispersions which are plotted in Fig. 1 have the following forms:
- for honeycomb lattice

\[ |f_{1,2}^{hc}(\mathbf{k})| = \pm J \sqrt{1 + 4 \cos^2 \left( \frac{\sqrt{3}}{2} k_x a \right) + 4 \cos \left( \frac{\sqrt{3}}{2} k_x a \right) \cos \left( \frac{3}{2} k_y a \right)}. \]  

(45)

- for brick-wall lattice

\[ |f_{1,2}^{bw}(\mathbf{k})| = \pm J \sqrt{1 + 4 \cos^2 (k_x a) + 4 \cos (k_x a) \cos (k_y a)}, \]  

(46)

- for staggered flux lattice

\[ |f_{1,2}^{flux}(\mathbf{k})| = \pm 2J \sqrt{\cos^2 (k_x a) + \cos^2 (k_y a) + 2 \cos \left( \frac{\phi}{2} \right) \cos (k_x a) \cos (k_y a)}, \]  

(47)

where \( a \) is the lattice constant.

---

[1] T. Stöferle, H. Moritz, C. Schori, M. Köhl, and T. Esslinger, Phys. Rev. Lett. 92, 130403 (2004).
[2] C. Schori, T. Stöferle, H. Moritz, M. Köhl, and T. Esslinger, Phys. Rev. Lett. 93, 240402 (2004).
[3] A. Reischl, K. P. Schmidt, and G. S. Uhrig, Phys. Rev. A 72, 063609 (2005).
[4] C. Kollath, A. Iucci, T. Giamarchi, W. Hofstetter, and U. Schollwöck, Phys. Rev. Lett. 97, 050402 (2006).
[5] A. Tokuno and T. Giamarchi, Phys. Rev. Lett. 106, 205301 (2011).
[6] J.-W. Huo, F.-C. Zhang, W. Chen, M. Troyer, and U. Schollwöck, Phys. Rev. A 84, 043608 (2011).
[7] M. J. Mark, E. Haller, K. Lauber, J. G. Danzl, A. J. Daley, and H.-C. Nägerl, Phys. Rev. Lett. 107, 175301 (2011).
[8] M. Endres, T. Fukuhara, D. Pekker, M. Cheneau, P. Schauß, C. Gross, E. Demler, S. Kuhr, and I. Bloch, Nature 487, 454 (2012).
[9] M. Łącki, D. Delande, and J. Zakrzewski, Phys. Rev. A 86, 013602 (2012).
[10] L. Pollet and N. Prokof’ev, Phys. Rev. Lett. 109, 010401 (2012).
[11] K. zu Münster, F. Gebhard, S. Ejima, and H. Fehske, Phys. Rev. A 89, 063623 (2014).
[12] H. U. R. Strand, M. Eckstein, and P. Werner, Phys. Rev. A 92, 063602 (2015).
[13] L. Liu, K. Chen, Y. Deng, M. Endres, L. Pollet, and N. Prokof’ev, Phys. Rev. B 92, 174521 (2015).
[14] A. S. Sajna, Phys. Rev. A 94, 043612 (2016).
[15] N. Goldman, G. Juzeliūnas, P. Öhberg, and I. B. Spielman, Reports on Progress in Physics 77, 126401 (2014).
[16] L. Tarruell, D. Greif, T. Uehlinger, G. Jotzu, and T. Esslinger, Nature 483, 302 (2012).
[17] P. Windpassinger and K. Sengstock, Rep. Prog. Phys. (2013).
[18] B. Grygiel, K. Patуча, and T. A. Zaleski, Phys. Rev. B 96, 094520 (2017).
[19] A. S. Sajna, T. P. Polak, and R. Micnas, Phys. Rev. A 89, 023631 (2014).
[20] A. S. Sajna and T. P. Polak, Phys. Rev. A 90, 043603 (2014).
[21] A. Sajna, T. Polak, and R. Micnas, Acta Physica Polonica A 127, 448 (2015).
[22] B. Grygiel, K. Patуча, and T. Zaleski, Acta Physica Polonica A 130, 633 (2016).
[23] A. Iucci, M. A. Cazalilla, A. F. Ho, and T. Giamarchi, Phys. Rev. A 73, 041608 (2006).
[24] L.-K. Lim, A. Hemmerich, and C. M. Smith, Phys. Rev. A 81, 23404 (2010).
[25] L.-K. Lim, C. M. Smith, and A. Hemmerich, Phys. Rev. Lett. 100, 130402 (2008).
[26] D. A. Abanin, T. Kitagawa, I. Bloch, and E. Demler, Phys. Rev. Lett. 110, 165304 (2013).
[27] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Phys. Rev. B 40, 546 (1989).
[28] H. A. Gersh and G. C. Knollman, Phys. Rev. 129, 959 (1963).
[29] H. T. C. Stoof, K. B. Gubbels, and D. B. M. Dickerscheid, Ultracold Quantum Fields (Springer, 2009).
[30] K. Sengupta and N. Dupuis, Phys. Rev. A 71, 033629 (2005).
[31] A. P. Kampf and G. T. Zimanyi, Phys. Rev. B 47, 279 (1993).
[32] C. Kollath, A. Iucci, I. P. McCulloch, and T. Giamarchi, Phys. Rev. A 74, 041604 (2006).
[33] M. P. A. Fisher, G. Grinstein, and S. M. Girvin, Phys. Rev. Lett. 64, 587 (1990).
[34] M.-C. Cha and S. M. Girvin, Phys. Rev. B 49, 9794 (1994).
[35] Y. Yanay and E. J. Mueller, Phys. Rev. A 93, 013622 (2016).
[36] M. Aichhorn, M. Hohenadler, C. Tahan, and P. B. Littlewood, Phys. Rev. Lett. 100, 216401 (2008).