Norm of Hilbert operator on sequence spaces

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Dedicated to Prof. Maryam Mirzakhani who, in spite of a short lifetime, left a long standing impact on mathematics.

Abstract

In this paper, we focus on the problem of finding the norm of Hilbert operator on some sequence spaces. Meanwhile, we obtain several interesting inequalities and inclusions.

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1 Introduction

Let $p \geq 1$ and $\omega$ denote the set of all real-valued sequences. The space $\ell_p$ is the set of all real sequences $x = (x_k) \in \omega$ such that

$$\|x\|_{\ell_p} = \left( \sum_{k=0}^{\infty} |x_k|^p \right)^{1/p} < \infty.$$ 

Hilbert matrix

The Hilbert matrix $H = (h_{j,k})$ was introduced by David Hilbert in 1894 to study a question in approximation theory. The finite and infinite Hilbert matrices are

$$H_{n \times n} = \begin{pmatrix} 1 & 1/2 & \cdots & 1/n \\ 1/2 & 1/3 & \cdots & 1/(n+1) \\ \vdots & \vdots & \ddots & \vdots \\ 1/n & 1/n + 1 & \cdots & 1/2n - 1 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hilbert matrices are frequently used both in mathematics and computational sciences. In image processing, for example, Hilbert matrices are commonly used. Any two-dimensional array of natural numbers in the range $[0, n]$ for all $n \in \mathbb{N}$ can be viewed as a gray-scale digital image.

Cryptography is another example of applications of the Hilbert matrix. Cryptography is the science of using mathematics to encrypt and decrypt data. Cryptography enables you to store sensitive information or transmit it across insecure networks so that it cannot be...
read by anyone except the intended recipient. The objectives of the proposed work are to propose a new cryptographic method based on the special matrix called the Hilbert matrix for authentication and confidentiality and to propose a model for confidentiality and authentication using a combination of symmetric and public cryptosystems. In some studies related to cryptographic methods, the Hilbert matrix is used for authentication and confidentiality [16]. It is well known that the Hilbert matrix is very unstable [15] and so it can be used in security systems.

In this paper we only focus on the infinite version of Hilbert matrix \( H = (h_{j,k}) \), which is defined by

\[
h_{j,k} = \frac{1}{j+k+1} = \begin{pmatrix}
1 & 1/2 & 1/3 & \cdots \\
1/2 & 1/3 & 1/4 & \cdots \\
1/3 & 1/4 & 1/5 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad j, k = 0, 1, \ldots,
\]

and is a bounded operator on \( \ell_p \) with \( \ell_p \)-norm \( \|H\|_{\ell_p} = \Gamma(1/p)\Gamma(1/p^*) = \pi \csc(\pi/p) \) by Theorem 323 in [8].

**Matrix domain** The matrix domain of an infinite matrix \( A \) in a sequence space \( X \) is defined as

\[
A_p = \{ x \in \omega : Ax \in X \},
\]

which is also a sequence space. It is easy to see that, for an invertible matrix \( A \), the matrix domain \( A_p \) is a normed space with \( \|x\|_{A_p} := \|Ax\|_{\ell_p} \). By using matrix domains of special triangle matrices in classical spaces, we can define more general sequence spaces than the space \( \ell_p \).

More recently, the author and some other mathematicians have investigated the problem of finding the norm of operators on several matrix domains [5, 6, 10–14, 17, 20, 21, 23]. Throughout this research, we use the notations \( \|\cdot\|_{A_p,\ell_p} \), \( \|\cdot\|_{\ell_p,A_p} \) and \( \|\cdot\|_{A_p,B_p} \) for the norm of operators from the matrix domain \( A_p \) into sequence space \( \ell_p \), for the norm of operators from \( \ell_p \) into the matrix domain \( A_p \) and for the norm of operators from matrix domain \( A_p \) into the matrix domain \( B_p \), respectively.

**Motivation** Although a variety of research has been done on the finite Hilbert operator, see [1, 4, 9, 22], and a lot of properties of this matrix have been discovered (determinant, inverse, ...) there exists a few information about the infinite version of Hilbert matrix, specially in the area of finding the norm of this operator on sequence spaces. Recently the author [18, 19] has introduced some factorizations for the infinite Hilbert matrix based on the generalized Cesàro matrix and Cesàro and Gamma matrices of order \( n \). Through this study the author has tried to compute the norm of Hilbert operator on several sequence spaces that have not been done before.

## 2 Norm of Hilbert operator on some sequence spaces

In this part of our study, we investigate the problem of finding the norm of well-known Hilbert operator on some sequence spaces. The following lemma plays a key role in finding the norm of operators between matrix domains.
Lemma 2.1 Let $U$ be a bounded operator on $\ell_p$ and $A_p, B_p$ be two matrix domains such that $A_p \simeq \ell_p$.

(i) If $T$ has a factorization of the form $T = UA$, then $T$ is a bounded operator from the matrix domain $A_p$ into $\ell_p$ and

$$\|T\|_{A_p, \ell_p} = \|U\|_{\ell_p}.$$  

(ii) If $BT = UA$, then $T$ is a bounded operator from the matrix domain $A_p$ into $B_p$ and

$$\|T\|_{A_p, B_p} = \|U\|_{\ell_p}.$$  

Proof (i) Since $A_p$ and $\ell_p$ are isomorphic,

$$\|T\|_{A_p, \ell_p} = \sup_{x \in A_p} \frac{\|Tx\|_{\ell_p}}{\|x\|_{A_p}} \|Ax\|_{\ell_p} = \sup_{x \in A_p} \frac{\|UAx\|_{\ell_p}}{\|x\|_{A_p}} \|y\|_{\ell_p} = \|U\|_{\ell_p},$$  

which gives the desired result.  

(ii) Again by isomorphism between $A_p$ and $\ell_p$ we have

$$\|T\|_{A_p, B_p} = \sup_{x \in A_p} \frac{\|Tx\|_{B_p}}{\|x\|_{A_p}} \|Ax\|_{\ell_p} = \sup_{x \in A_p} \frac{\|UAx\|_{\ell_p}}{\|x\|_{A_p}} \|y\|_{\ell_p} = \|U\|_{\ell_p},$$  

which gives the desired result.  

2.1 Norm of Hilbert operator on Cesàro and Copson sequence spaces

Consider the Hausdorff matrix $H^\mu = (h_{j,k})_{j,k=0}^\infty$ with entries of the form

$$h_{j,k} = \begin{cases} \int_0^1 \binom{j}{k} \theta^k (1-\theta)^{j-k} \, d\mu(\theta) & j \geq k, \\ 0 & j < k, \end{cases}$$

where $\mu$ is a probability measure on $[0,1]$. The Hausdorff matrix contains several famous classes of matrices like Cesàro, Gamma, Hölder and Euler matrices. Hardy's formula ([7], Theorem 216) states that the Hausdorff matrix is a bounded operator on $\ell_p$ if and only if

$$\|H^\mu\|_{\ell_p} = \int_0^1 \frac{1}{\theta^{\frac{1}{p}}} \, d\mu(\theta).$$

(2.1)

Cesàro matrix By letting $d\mu(\theta) = n(1-\theta)^{n-1} \, d\theta$ in the definition of the Hausdorff matrix, the Cesàro matrix of order $n$, $C^n = (c^n_{j,k})$, is defined by

$$c^n_{j,k} = \binom{n+j-1}{j-k} \frac{(n+j-1)!}{(n-j)!} \quad 0 \leq k \leq j,$$

$$0 \quad \text{otherwise},$$

(2.2)
which according to Eq. (2.1) has the $\ell_p$-norm

$$\|C^n\|_{\ell_p} = \frac{\Gamma(n + 1)\Gamma(1/p^*)}{\Gamma(n + 1/p^*)},$$

(2.3)

where $p^*$ is the conjugate of $p$ i.e. $\frac{1}{p} + \frac{1}{p^*} = 1$. Note that $C^1 = C$ is the well-known Cesàro matrix

$$c_{jk} = \begin{cases} \frac{1}{j+1} & 0 \leq k \leq j, \\ 0 & \text{otherwise}, \end{cases} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & \cdots \\ 1/3 & 1/3 & 1/3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which has the $\ell_p$-norm $\|C\|_{\ell_p} = p^*$.

Some more examples are

$$C^2 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 2/3 & 1/3 & 0 & \cdots \\ 3/6 & 2/6 & 1/6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad C^3 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 3/4 & 1/4 & 0 & \cdots \\ 6/10 & 3/10 & 1/10 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The Cesàro matrix domain $C^n_p$ is the set of all sequences whose $C^n$-transforms are in the space $\ell_p$; that is,

$$C^n_p = \left\{ x = (x_j) \in \omega : \sum_{j=0}^{\infty} \left| \frac{1}{(n+k)} \sum_{k=0}^{j} \binom{n+j-k-1}{j-k} x_k \right|^p < \infty \right\},$$

which is a Banach space with the norm

$$\|x\|_{C^n_p} = \left( \sum_{j=0}^{\infty} \left| \frac{1}{(n+k)} \sum_{k=0}^{j} \binom{n+j-k-1}{j-k} x_k \right|^p \right)^{1/p}.$$

Transposing the Cesàro matrix of order $n$ results the Copson matrix of order $n$, $C^{nt} = (c_{jk}^{nt})$, which has the entries

$$c_{jk}^{nt} = \begin{cases} \frac{(n+k-j-1)}{(n+k)} & 0 \leq k \leq j, \\ 0 & \text{otherwise}, \end{cases}$$

and the $\ell_p$-norm

$$\|C^{nt}\|_{\ell_p} = \frac{\Gamma(n + 1)\Gamma(1/p)}{\Gamma(n + 1/p)},$$

(2.4)

by the Hellinger–Toeplitz theorem, which is the following.

**Theorem 2.2** ([2], Proposition 7.2) Suppose that $1 < p, q < \infty$. A matrix $A$ maps $\ell_p$ into $\ell_q$ if and only if the transposed matrix, $A^t$, maps $\ell_q^*$ into $\ell_{p^*}$. We then have $\|A\|_{\ell_p, \ell_q} = \|A^t\|_{\ell_q^*, \ell_{p^*}}$. 

For a positive integer $n$, we define the Hilbert matrix of order $n$, $H^n = (h^n_{j,k})$, by

$$h^n_{j,k} = \frac{1}{j+k+n+1} \quad (j,k = 0,1,\ldots). \quad (2.5)$$

Note that, for $n = 0$, $H^0 = H$ is the well-known Hilbert matrix. For more examples:

$$H^1 = \begin{pmatrix} 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ 1/4 & 1/5 & 1/6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad H^2 = \begin{pmatrix} 1/3 & 1/4 & 1/5 & \cdots \\ 1/4 & 1/5 & 1/6 & \cdots \\ 1/5 & 1/6 & 1/7 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$ 

For non-negative integers $n$, $j$ and $k$, let us define the matrix $B^n = (b^n_{j,k})$ by

$$b^n_{j,k} = \frac{(k+1)\cdots(k+n)}{(j+k+1)\cdots(j+k+n+1)}.$$ 

Consider that, for $n = 0$, $B^0 = H$, where $H$ is the Hilbert matrix. 

Note that the matrix $B^n$ has also the representation

$$b^n_{j,k} = \binom{n+k}{k} \beta(j+k+1,n+1) \quad (j,k = 0,1,\ldots),$$

where the $\beta$ function is

$$\beta(m,n) = \int_0^1 z^{m-1}(1-z)^{n-1} \, dz \quad (m,n = 1,2,\ldots).$$

For computing the norm of Hilbert operator on Cesàro and Copson matrix domains we need the following lemma.

**Lemma 2.3** The Hilbert matrix $H$ and the Hilbert matrix of order $n$, $H^n$, have the following factorizations based on the Cesàro matrix of order $n$:

(i) $H = B^n C^n$,

(ii) $H^n = C^n B^n$,

(iii) $C^n H = H^n C^n$,

(iv) $B^n$ is a bounded operator on $\ell_p$ and

$$\left\| B^n \right\|_{\ell_p} = \frac{\Gamma(n+1/p^*)\Gamma(1/p)}{\Gamma(n+1)}.$$

**Proof** (i) By applying the identity $\sum_{j=0}^{\infty} \binom{n+1}{j} z^j = (1-z)^{-n}$ for $|z| < 1$, we deduce that

$$(B^n C^n)_{j,k} = \sum_{i=k}^{\infty} \binom{n+i}{i} \beta(j+i+1,n+1) \frac{\binom{n+i+1}{i+k} \beta(i+1,n+1)}{\binom{n+i}{i}}$$

$$= \sum_{i=0}^{\infty} \binom{n+i-1}{i} \beta(j+i+k+1,n+1)$$

For computing the norm of Hilbert operator on Cesàro and Copson matrix domains we need the following lemma.
\[= \int_0^1 \sum_{i=0}^{\infty} \binom{n+i-1}{i} z^i z^{j+k}(1-z)^n \, dz \]
\[= \int_0^1 z^{j+k} \, dz = \frac{1}{j+k+1} = h_{j,k}.\]

(ii) For convenience, let \( \lambda = \binom{c_{j+k}}{c_j} \). The factorization will be obtained by the following calculations:

\[(C^n B^n)_{i,k} = \sum_{j=0}^{i} \frac{(n+j-1)}{\binom{n+j}{i}} \binom{n+k}{k} \beta(i+k+1, n+1)\]
\[= \lambda \left\{ \binom{n+i-1}{i} \beta(k+1, n+1) + \cdots + \binom{n+1}{i} \beta(i+k-1, n+1) + \frac{(n+1)(i+k-1)!}{(i+k+n-1)(i+k+n+1)} \right\} \]
\[= \lambda \left\{ \binom{n+i-1}{i} \beta(k+1, n+1) + \cdots + \binom{n+2}{i} \beta(i+k-2, n+1) + \frac{(n+2)(i+k-2)!}{2(i+k+n-2)(i+k+n+1)} \right\} \]
\[= \cdots \]
\[= \lambda \left\{ \binom{n+i-1}{i} \beta(k+1, n+1) + \frac{(n+i-1)(k+1)!}{(i-1)(i+k+n+1)!} \right\} \]
\[= \lambda \left\{ \frac{(n+i-1)!(n+k+1)(n+1)}{2!(n+k+1)!} \frac{1}{i+k+n+1} \right\} = \frac{1}{i+k+n+1} = h_{i,k}^n.\]

(iii) This is obvious by parts (i) and (ii). (iv) For computing the \( \ell_p \)-norm of \( B^n \), we introduce a family of matrices, \( B(w) \), \( 0 < w \leq 1 \), given by

\[b(w)_{j,k} = \binom{j+k}{k} w^j (1-w)^{n+k}.\]

Since
\[\sum_{k=0}^{\infty} b(w)_{j,k} = w^j (1-w)^n \sum_{k=0}^{\infty} \binom{j+1+k-1}{k} (1-w)^k = w^j (1-w)^n (1-(1-w))^{-j+1} = \frac{(1-w)^n}{w},\]
and
\[\sum_{j=0}^{\infty} b(w)_{j,k} = (1-w)^{n+k} \sum_{j=0}^{\infty} \binom{k+1+j-1}{j} w^j = (1-w)^{n+k-1},\]
the row sums are all \( \frac{(1-w)^n}{w} \) and the column sums are all \((1 - w)^{n-1}\). Thus Schur’s theorem results in
\[
\|B(w)\|_{\ell_p} \leq (1 - w)^{n-\frac{1}{p}} w^{\frac{1}{p^*}}.
\]

On the other hand,
\[
\int_0^1 b(w)_{jk} dw = \left( \begin{array}{c} j+k \\ k \end{array} \right) \int_0^1 w^j (1-w)^{n-k} dw = \left( \begin{array}{c} j+k \\ k \end{array} \right) \beta(j+1, n+k+1) = \left( \begin{array}{c} n+k \\ k \end{array} \right) \beta(j+k+1, n+1) = b^*_{jk}.
\]

Now
\[
\|B^n\|_{\ell_p} = \left\| \int_0^1 B(w) dw \right\| \leq \int_0^1 \|B(w)\| dw 
\leq \int_0^1 (1-w)^{n-1/p} w^{-1/p^*} dw = \beta(n - 1/p + 1, 1 - 1/p^*) \frac{\Gamma(n+1/p^*) \Gamma(1/p)}{\Gamma(n+1)}.
\]

Also the factorization \( H = B^n C^n \) results \( \|H\|_{\ell_p} \leq \|B^n\|_{\ell_p} \|C^n\|_{\ell_p} \). Therefore
\[
\|B^n\|_{\ell_p} \geq \frac{\|H\|_{\ell_p}}{\|C^n\|_{\ell_p}} \frac{\Gamma(n+1/p^*) \Gamma(1/p)}{\Gamma(n+1)}
\]
which completes the proof.

**Remark 2.4** In parallel, the two Hilbert and Hilbert matrices of order \( n \), \( H \) and \( H^n \), have the following factorizations based on the Copson matrix:

(i) \( H = C^n B^n \),

(ii) \( H^n = B^n C^n \),

(iii) \( HC^n = C^n H^n \),

(iv) \( B^n \) is a bounded operator on \( \ell_p \) and
\[
\|B^n\|_{\ell_p} = \frac{\Gamma(n + 1/p^*) \Gamma(1/p)}{\Gamma(n+1)}.
\]

**Theorem 2.5** Let \( H^n \) be the Hilbert matrix of order \( n \). Then

(i) \( H \) is a bounded operator from \( C^n_p \) into \( \ell_p \) and
\[
\|H\|_{C^n_p, \ell_p} = \frac{\Gamma(n+1/p^*) \Gamma(1/p)}{\Gamma(n+1)}.
\]

(ii) \( H \) is a bounded operator from \( C^n_p \) into \( C^n_p \) and
\[
\|H\|_{C^n_p} = \pi \csc(\pi/p).
\]
(iii) $H^n$ is a bounded operator from $C^n_p$ into $\ell_p$ and

$$
\|H^n\|_{C^n_p, \ell_p} = \frac{\Gamma(n+1/p)\Gamma(1/p^*)}{\Gamma(n+1)}.
$$

(iv) $H^n$ is a bounded operator from $C^n_p$ into $C^n_p$ and

$$
\|H^n\|_{C^n_p} = \pi \csc(\pi/p).
$$
Proof. According to Lemma 2.3, $H = B^n C^n$, hence
\[
\|Hx\|_{\ell_p} = \left\| B^n C^n x \right\|_{\ell_p} \leq \frac{\Gamma(n + 1/p^*) \Gamma(1/p)}{\Gamma(n + 1)} \|C^n x\|_{\ell_p}.
\]
Consider that, for $n = 0$, $C^0 = I$ and we have the Hilbert inequality. Also by applying Lemma 2.3, $H^n = B^n C^n$, hence we have
\[
\|H^n x\|_{\ell_p} = \left\| B^n C^n x \right\|_{\ell_p} \leq \frac{\Gamma(n + 1/p^*) \Gamma(1/p)}{\Gamma(n + 1)} \|C^n x\|_{\ell_p}.
\]
Consider that, for $n = 0$, $C^0 = I$ and we have the Hilbert inequality. □

Let $H^n_p$ be the sequence space associated with the Hilbert matrix $H^n$, which is
\[
H^n_p = \left\{ x = (x_k) \in \omega : \sum_{j=0}^{\infty} \left( \sum_{k=0}^{\infty} x_k \right)^{j+k+n+1} < \infty \right\}
\]
and has the norm
\[
\|x\|_{H^n_p} = \left( \sum_{j=0}^{\infty} \left( \sum_{k=0}^{\infty} x_k \right)^{j+k+n+1} \right)^{1/p^*}.
\]
As another application of Lemma 2.3, we have the following inclusions.

Corollary 2.7 Let $p > 1$. Then
(i) $C^p \subset H_p$,
(ii) $C^{p*} \subset H^n_p$,
(iii) $H^n_p \subset H^{n-1}_p \subset \cdots \subset H_p$.

2.2 Norm of Hilbert operator on the generalized Cesàro matrix domain
Suppose that $N \geq 1$ is a real number. The generalized Cesàro matrix, $C^N = (c^N_{j,k})$, has the $\ell_p$-norm $\|C^N\|_{\ell_p} = p^*$ ([3], Lemma 2.3), and the entries
\[
c^N_{j,k} = \begin{cases} \frac{1}{jN} & 0 \leq k \leq j, \\ 0 & \text{otherwise}, \end{cases}
\]
has the $\ell_p$-norm $\|C^N\|_{\ell_p} = p^*$ ([3], Lemma 2.3), and the entries
\[
c^N_{j,k} = \begin{pmatrix} \frac{1}{N} & 0 & 0 & \cdots \\ \frac{1}{N^2} & \frac{1}{N} & 0 & \cdots \\ \frac{1}{N^3} & \frac{1}{N^2} & \frac{1}{N} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
\]
Note that $C^1$ is the well-known Cesàro matrix $C$. Some more examples are

$$C^2 = \begin{pmatrix} 1/2 & 0 & 0 & \cdots \\ 1/3 & 1/3 & 0 & \cdots \\ 1/4 & 1/4 & 1/4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad C^3 = \begin{pmatrix} 1/3 & 0 & 0 & \cdots \\ 1/4 & 1/4 & 0 & \cdots \\ 1/5 & 1/5 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$  

The matrix domain associated with this matrix is the set

$$C^N_p = \left\{ x = (x_k) \in \omega : \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{x_k}{j+N} \right\}^p < \infty \right\},$$

which has the following norm:

$$\|x\|_{C^N_p} = \left( \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{x_k}{j+N} \right)^{1/p}.$$  

**Theorem 2.8** The Hilbert operator $H$ is a bounded operator from $C^N_p$ into $\ell_p$ and

$$\|H\|_{C^N_p, \ell_p} \leq \frac{\pi N}{p^*} \csc(\pi/p).$$

In particular, the Hilbert operator $H$ is a bounded operator from $C_p$ into $\ell_p$ and $\|H\|_{C_p, \ell_p} = \frac{\pi}{p^*} \csc(\pi/p)$.

**Proof** The author in [18] has proved that the Hilbert matrix admits a factorization of the form $H = B^N C^N$, where $B^N$ is a bounded operator on $\ell_p$ and

$$\frac{\pi}{p^*} \csc(\pi/p) \leq \|B^N\|_{\ell_p} \leq \frac{\pi N}{p^*} \csc(\pi/p).$$

Now, according to Lemma 2.1 we have the result. □

### 2.3 Norm of Hilbert operator on Gamma sequence spaces

By letting $d\mu(\theta) = n^{n-1} d\theta$ in the definition of the Hausdorff matrix, the Gamma matrix of order $n$, $F^n = (\gamma^n_{f,k})$, is

$$\gamma^n_{f,k} = \begin{cases} \binom{n+k-1}{k}^{-1} & 0 \leq k \leq j, \\ 0 & \text{otherwise}, \end{cases}$$

which according to the Hardy formula has the norm

$$\|F^n\|_{\ell_p} = \frac{np}{np-1}. \quad (2.6)$$
Note that $\Gamma^1$ is the well-known Cesàro matrix. Some more examples are

$$\Gamma^2 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/3 & 2/3 & 0 & \cdots \\ 1/6 & 2/6 & 3/6 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \Gamma^3 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/4 & 3/4 & 0 & \cdots \\ 1/10 & 3/10 & 6/10 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$  

The sequence space associated with $\Gamma^n$, is the set

$$\Gamma^n_p = \{ x = (x_k) \in \ell^p : \Gamma^n x \in \ell^p \} = \left\{ x = (x_k) \in \omega : \sum_{j=0}^{\infty} \left| \frac{1}{(n+1)^j} \sum_{k=0}^{j} \binom{n+k-1}{k} x_k \right|^p < \infty \right\},$$

which is called the Gamma space of order $n$.

**Theorem 2.9** The Hilbert operator $H$ is a bounded operator from $\Gamma^n_p$ into $\ell^p$ and

$$\|H\|_{\Gamma^n_p, \ell^p} = \pi \left( 1 - \frac{1}{np} \right) \csc(\pi/p).$$

In particular, the Hilbert operator $H$ is a bounded operator from $C_p$ into $\ell^p$ and $\|H\|_{C_p, \ell^p} = \frac{\pi}{p} \csc(\pi/p)$.

**Proof** The author in [19] has proved that the Hilbert matrix has a factorization of the form $H = S^n \Gamma^n$, where the matrix $S^n$ has the $\ell^p$-norm

$$\|S^n\|_{\ell^p} = \pi \left( 1 - \frac{1}{np} \right) \csc(\pi/p).$$

Now, by applying part (ii) of Lemma 2.1 the proof is obvious. \qed

### 2.4 Norm of Hilbert operator on difference sequence spaces

Let $n \in \mathbb{N}$ and $\Delta^n = (\delta^n_{jk})$ be the forward difference operator of order $n$ with entries

$$\delta^n_{jk} = \begin{cases} (-1)^{k-j} \binom{n}{k-j} & j \leq k \leq n+j, \\
0 & \text{otherwise}. \end{cases}$$

We define the sequence space $\ell_p(\Delta^n)$ as the set $\{ x = (x_k) : \Delta^n x \in \ell^p \}$ or

$$\ell_p(\Delta^n) = \left\{ x = (x_k) : \sum_{j=0}^{\infty} \left| \sum_{k=0}^{n} (-1)^{k-j} \binom{n}{k} x_{k+j} \right|^p < \infty \right\},$$

with semi-norm, $\| \cdot \|_{\ell_p(\Delta^n)}$, which is defined by

$$\|x\|_{\ell_p(\Delta^n)} = \left( \sum_{j=0}^{\infty} \left| \sum_{k=0}^{n} (-1)^{k-j} \binom{n}{k} x_{k+j} \right|^p \right)^{\frac{1}{p}}.$$
Note that this function will not be a norm, since if \( x = (1, 1, 1, \ldots) \) then \( \| x \|_{\ell^p(\Delta^n \mathbb{N})} = 0 \), while \( x \neq 0 \). The definition of backward difference sequence space \( \ell^p(\Delta^n \mathbb{N}) \) is similar to \( \ell^p(\Delta^n \mathbb{N}) \), except \( \| \cdot \|_{\ell^p(\Delta^n \mathbb{N})} \) is a norm.

**Theorem 2.10** The Hilbert matrix \( H \), is a bounded operator from \( \ell^p(\Delta^n \mathbb{N}) \) into \( \ell^p(\Delta^n \mathbb{N}) \) and

\[
\| H \|_{\ell^p(\Delta^n \mathbb{N}), \ell^p(\Delta^n \mathbb{N})} = \frac{\pi}{\sin(\pi/p)}.
\]

**Proof** Let \( A = \Delta^n H \). By using the identity \( \sum_{j=0}^{n} (-1)^j \binom{n}{j} z^j = (1 - z)^n \), we have

\[
a_{i,k} = \sum_{j=0}^{n} b_{i,j} h_{j,k} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{1}{j + k + 1} \\
= \sum_{j=0}^{n} (-1)^j \binom{n}{j} \int_0^1 z^{j+i+k} \, dz \\
= \int_0^1 \sum_{j=0}^{n} (-1)^j \binom{n}{j} z^{j+i+k} \, dz = \int_0^1 z^{i+k}(1 - z)^n \, dz \\
= \beta(i + k + 1, n + 1).
\]

Obviously, \( A \) is a symmetric matrix which implies that \( \Delta^n H = H \Delta^n \). Now by using Lemma 2.1 part (ii), \( \| H \|_{\ell^p(\Delta^n \mathbb{N}), \ell^p(\Delta^n \mathbb{N})} = \| H \|_{\ell^p} = \pi \csc(\pi/p) \).

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