On the Eisenstein functoriality in cohomology for maximal parabolic subgroups

Laurent Clozel

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Abstract
In his paper, 'On torsion in the cohomology of locally symmetric varieties', Peter Scholze has introduced a new, purely topological method to construct the cohomology classes on arithmetic quotients of symmetric spaces of reductive groups over \( \mathbb{Q} \) originating from the cohomology of the similar quotients of Levi subgroups of maximal parabolic subgroups. We extend this construction beyond the cases he considers, and, in the complex case, to the cohomology of local systems.

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1 Introduction

Let \( G \) be a reductive group defined over \( \mathbb{Q} \), \( A = A_G \) the neutral component of the group of real points in a maximal split central torus of \( G \), \( K_\infty \subset G(\mathbb{R}) \) a maximal compact subgroup. We are interested in the cohomology of the quotients \( \Gamma \backslash X \), where \( X = G(\mathbb{R})/A_G K_\infty \) is the symmetric space, and \( \Gamma \subset G(\mathbb{R}) \) is a congruence subgroup. For our purposes, it will be better to consider the adèlic version, i.e., the quotients

\[
S_K = G(\mathbb{Q})\backslash G(A) / A_G K_\infty K
\]  

(1.1)

where \( A \) denotes the adèles of \( \mathbb{Q} \), \( A_f \) the finite adèles and \( K \subset G(A_f) \) is a compact open subgroup.

The quotient (1.1) is a finite union of spaces \( \Gamma \backslash X \). It is known from the work of Borel and Franke that \( H^\bullet(S_K, \mathbb{C}) \) can be computed by automorphic forms: in fact

\[
H^\bullet(S_K, \mathbb{C}) = H^\bullet(g, K_\infty; A(G_K))
\]

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1 Mathématiques, Université Paris-Saclay, Bâtiment 307, 91405 Orsay Cedex, France
where $\mathcal{A}$ denotes the space of automorphic forms on

$$G_K = G(\mathbb{Q}) \backslash G(\mathbb{A}) / A_{G K},$$

and the $(\mathfrak{g}, K_\infty)$ cohomology is defined by the action of $G(\mathbb{R})$ by right translations. See [3, 5].

There is a decomposition, due to Langlands,

$$\mathcal{A}(G_K) = \bigoplus_P \mathcal{A}_P(G_K)$$

where $P$ runs over the association classes of $\mathbb{Q}$–parabolic subgroups, and $\mathcal{A}_P(G_K)$ is the space obtained from Eisenstein series “induced from $P$”.

When the space of cusp forms of $G$ is understood, this yields the computation of the part relative to $G$:

$$H^\bullet(\mathfrak{g}, K_\infty; \mathcal{A}_G(G_K)) = H^\bullet(\mathfrak{g}, K_\infty; \mathcal{A}_{cusp}(G_K)) = \bigoplus_{\pi} H^\bullet(\mathfrak{g}, K_\infty; \pi)$$

where $\pi$ ranges over the cuspidal summands of $L^2(G_K)$ with non–trivial cohomology. Harder, and then Schwermer, have proposed a program aiming at constructing cohomology classes in $\mathcal{A}_P(G_K)$ for $P \neq G$, obtained as differential form–valued Eisenstein series “induced” from cuspidal cusp forms on $M(\mathbb{Q}) \setminus M(\mathbb{A})$.

Our purpose here is to show that the method is perfectly general when “Eisenstein functoriality” for a maximal parabolic subgroup is considered.

We carry out Scholze’s construction in two cases: in characteristic $\ell$ (or more generally with a local Artin ring of coefficients $k$) for cohomology with trivial coefficients, in Sect. 2. The results are Theorem 2.4, Theorem 2.5. Then, for complex cohomology, in Sect. 3 where we deal (as is natural in this case) with local systems coming from an arbitrary (complex) representation of $G$. The main results are Theorem 3.4, Proposition 3.5, Theorem 3.6. They imply that there exist cohomology classes

1 Scholze does not seem to think that his method was new. It is new and probably yields new results: see Sect. 4.

2 $k$ a Gorenstein ring, for technical reasons (Sect. 2).

3 We refer the reader to the text, as it would be tedious to introduce the relevant notation here.
in $H^\bullet(S_K)$ associated to all classes in the inner cohomology $H^\bullet(S_{K_M}^M)$ for all Levi subgroups $M$ in maximal parabolic subgroups $P = MN$, $S_{K_M}^M$ being the associated adèlic quotient. (Proposition 3.5 does not seem to follow directly from the analytic construction of Eisenstein cohomology when it applies, since we can directly relate classes with $\bar{\mathbb{Q}}$–coefficients.)

In Sect. 4, we compare the results obtained with the (known) consequences of the Harder–Schwermer method. In particular, if $G = GL(n)$, we examine whether the eigenclasses “on $G$” originating from the cuspidal cohomology of $GL(m) \times GL(n) = 2m$ can be obtained by the formation of Eisenstein classes.

Finally, Sect. 5 explores the apparent limit of the method: if $P \subset G$ is not maximal, we explain why the natural generalisation of Scholze’s method fails. (It would be interesting to find a topological argument in this case.) I am indebted to Leslie Saper for confirming the plausibility of a crucial property of the Borel–Serre compactification (Conjecture 5.2.) It is hoped that its proof will appear later.

In conclusion, I emphasise that this is only the natural development of an idea entirely due to Scholze \(^4\). I thank him, and also Avner Ash, Günter Harder, Benjamin Hennion, Lizhen Ji, Colette Moeglin, Benjamin Schraen, Leslie Saper, Jack Thorne and David Vogan, for useful communications. I especially thank Thorne for numerous explanations about [21]. Finally, I thank the referee for numerous remarks which have improved the exposition.

2 Eisenstein cohomology from maximal parabolic subgroups: scholze’s argument

2.1
In this section we will extend to general reductive groups the proof of Scholze [24, V.2] showing the existence of “Eisenstein” cohomology classes (or rather, eigencharacters) originating from the inner cohomology of maximal parabolic subgroups.

The argument, topological, relies on the Borel–Serre compactification. We first fix some notations.

Let $G$ be a connected reductive group of positive rank over $\mathbb{Q}$. We denote by $A_G$ the (topological) neutral component of the set of $\mathbb{R}$–points of the maximal $\mathbb{Q}$–split torus in the center of $G$. Fix a compact–open subgroup $K$ in $G(\mathbb{A}_f)$. We will assume that $K$ is decomposed: $K = \prod K_p$. Following Newton and Thorne [21], we also assume $K$ neat in the strong sense of Pink [23]: let $\rho$ be a faithful representation of $G$ over $\mathbb{Q}$. We fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$ and an embedding $\mathbb{Q} \rightarrow \bar{\mathbb{Q}}$ for all $p$. If $g = (g_p) \in G(\mathbb{A}_f)$, consider for each $p$ the torsion subgroup $\Gamma_p$ of the group generated in $\bar{\mathbb{Q}}_p$ by the eigenvalues of $g_p$. (Thus $\Gamma_p \subset \bar{\mathbb{Q}}_p$.) Then $g$ is neat if $\bigcap \Gamma_p = \{1\}$. A compact open subgroup $K$ is neat if all its elements are neat. If $G$ is a connected linear algebraic group, $H$ a subgroup, $M$ a quotient, of $G$, and $K \subset G(\mathbb{A}_f)$ is neat, then $K \cap H(\mathbb{A}_f)$ and the image of $K$ in $M(\mathbb{A}_f)$ are neat (ibid.). Such a subgroup is neat in the more

\(^4\) Michael Harris points out that similar “topological” arguments were already introduced in [13, 14]. See also [15, Cor. 6.24]. But the closest approach I find in the literature, for the Borel-Serre compactification, is in Harder’s [10] (see Section 1.2.3).
usual sense \((K \cap G(\mathbb{Q}))\) has no element of finite order but 1), and they form a basis for the compact–open subgroups.

Let \(K_{\infty}\) be a maximal compact subgroup of \(G(\mathbb{R})\). We consider

\[
S_K := G(\mathbb{Q}) \backslash G(\mathbb{A})/A_GK_{\infty}K
= G(\mathbb{Q}) \backslash (X_G \times G(\mathbb{A}_f))/K
\]

where \(X_G = G(\mathbb{R})/A_GK_{\infty}\) is the symmetric space of \(G(\mathbb{R})\). \(^5\) (Thus \(S_K\) is a finite union

\[
S_K = \bigsqcup \Gamma_i \backslash X_G
\]

(2.1)

where the \(\Gamma_i\) are congruence subgroups of \(G(\mathbb{Q})\); the quotients are smooth.)

Let \(P = MN \subset G\) be a \(\mathbb{Q}\)–parabolic subgroup, with a Levi decomposition. There are similar quotients associated to \(P\) and \(M\):

\[
S_{KP}^P = P(\mathbb{Q}) \backslash P(\mathbb{A})/A_MK_{\infty,M}K_P,
S_{KM}^M = M(\mathbb{Q}) \backslash M(\mathbb{A})/A_MK_{\infty,M}K_M
\]

where \(K_{\infty,M} \subset M(\mathbb{R})\) is maximal compact, and \(K_P, K_M\) are compact–open in the sets of finite adelic points. If \(K_P = P(\mathbb{A}_f) \cap K\) and \(K_M\) is the projection of \(K_P\), they are, as we saw, neat. In the construction of the Borel-Serre compactification, the choice of a lifting of \(A_MK_{\infty,M}\) to \(P(\mathbb{R})\) has to be made compatibly with the geodesic action \([2]\), see below.

There is a natural projection

\[
\pi_P : S_{KP}^P \to S_{KM}^M
\]

which realises the quotient \(S_{KP}^P\) as a fiber bundle with compact fiber \(N(\mathbb{Q}) \backslash N(\mathbb{A})/K_N\), where \(K_N = K_P \cap N(\mathbb{A}_f)\). Note that the fiber is of the form \(\Gamma_N \backslash N(\mathbb{R})\), \(\Gamma_N \subset N(\mathbb{Q})\) a congruence subgroup. The quotients \(S_{KP}^P, S_{KM}^M\) have “finite” expressions as in (2.1).

We now consider the Borel–Serre compactification \(S_{K}^{BS}\) of \(S_K\). We start with the Borel–Serre “bordification” of \(X_G\). This is a disjoint union

\[
X_G^{BS} = \bigsqcup_P e(P)
\]

where \(P\) runs over the \(\mathbb{Q}\)–parabolic subgroups of \(G\); \(e(G) = X_G\). Let \(\mathcal{P}\) be the set of \(\mathbb{Q}\)–parabolic subgroups, and \(\mathcal{P}'\) the set of proper parabolic subgroups. Then \(\bigsqcup_{P \in \mathcal{P}'} e(P)\) is the boundary, \(\partial X_G^{BS}\), of the manifold with corners \(X_G^{BS}\). For \(P \in \mathcal{P}'\), \(e(P) = A_P \backslash X_G\) where we write \(A_P\) for \(A_M\), and the action of \(A_P\) (“geodesic flow”) on \(X_G\) commutes with the left action of \(P(\mathbb{R})\) \([2]\). Recall that the action of \(A_M\) is defined as follows. We have fixed \(K_{\infty}\) and therefore a Cartan involution \(\theta\) on \(G(\mathbb{R})\).

Let \(M = P/N\) be the Levi quotient of \(P\). There is a unique Levi subgroup \(M(\mathbb{R})\) of

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\(^5\) More precisely, of \(G(\mathbb{R})/A_G\). Note that \(A_G\) depends on the \(\mathbb{Q}\)-structure.
$P(\mathbb{R})$ stable by $\theta$ ([2, Prop.1.8]). Recall that $G(\mathbb{R}) = P(\mathbb{R})K_\infty$. Let $o \in X_G$ be our base point, associated to $K_\infty$. We then let $A_M$ act on $X_G$ by

$$a \circ y = p.a_0.o$$

for $y \in X_G$, $y = p.o$, $a_0$ being the lifting of $a \in A_M$ to the split component of the $\theta$-stable Levi subgroup. See [2, Section 3.2]. This action does not, in fact, depend on the choice of $K_\infty$. It commutes with the left action of $P(\mathbb{R})$ on $X_G$. We simply denote it by $y \mapsto ay$. Moreover, if $h \in G(\mathbb{R})$, the conjugation $\text{Int}(g) : P(\mathbb{R}) \to gP(\mathbb{R})g^{-1}$ induces a left-action of $g$ sending $e(P)$ to $e(gPg^{-1})$, cf. [2, (5) p. 450]. Therefore $G(\mathbb{Q})$ acts on the left on $X_G^{BS}$. Cf [2, Prop. 7.6].

We define

$$S_{BS}^{K} = G(\mathbb{Q}) \backslash (X_G^{BS} \times G(\mathbb{A}_f))/K.$$  \hspace{1cm} (2.2)

Since $G(\mathbb{A}_f)/K$ is discrete, and $G(\mathbb{Q})$ acts without fixed points, it is a manifold with corners. We recall the fundamental property of this compactification: the embedding $S_K \hookrightarrow S_{BS}^{K}$ is a homotopy equivalence. See [2, 8.3] as well as Section 11 there. We can give two descriptions of $S_{BS}^{K}$. First write

$$G(\mathbb{A}_f) = \bigsqcup_{g} G(\mathbb{Q})gK \ (g \in G(\mathbb{A}_f)),$$

the finite union that leads to the expression (2.1). Then (2.2) easily yields

$$S_{BS}^{K} = \bigsqcup_{g} \Gamma_{g} \backslash X^{BS}_G$$ \hspace{1cm} (2.3)

where $\Gamma_{g} = G(\mathbb{Q}) \cap gKg^{-1}$. Thus $S_{BS}^{K}$ is a finite union of “classical” Borel–Serre compactifications. (In particular $S_{BS}^{K}$, with the natural topology coming from (2.2) and the topology of $X_G^{BS}$, is compact.) On the other hand, the set of $\mathbb{Q}$–parabolic subgroups, modulo $G(\mathbb{Q})$, is finite. Let $Q$ be a set of representatives. Then from (2.2) we get

$$S_{BS}^{K} = \bigsqcup_{P \in Q} P(\mathbb{Q}) \backslash (e(P) \times G(\mathbb{A}_f))/K.$$ \hspace{1cm} (2.4)

Each piece is the part of $S_{BS}^{K}$ contributed by the “parabolic subgroups of type $P$”. It can be further decomposed. Denoting it by $S_{BS}^{K}(P)$, we remark that $P(\mathbb{A}_f) \backslash G(\mathbb{A}_f)/K$ is finite since $P(\mathbb{A}_f) \backslash G(\mathbb{A}_f)$ is the set of points of a projective variety with values in $\mathbb{A}_f$, with a transitive action of $G(\mathbb{A}_f)$. Let $\{h\}$ be a set of representatives. Then

$$S_{BS}^{K}(P) = \bigsqcup_{h} P(\mathbb{Q}) \backslash (e(P) \times P(\mathbb{A}_f))/K_P(h),$$ \hspace{1cm} (2.5)

with $K_P(h) = P(\mathbb{A}_f) \cap hKh^{-1}$. In turn we have (by the properties of the geodesic action)

$$e(P) = A_M \backslash X_G$$
$= A_M \backslash (P(\mathbb{R})K_{G,\infty}/K_{G,\infty})$

$= A_M \backslash P(\mathbb{R})/K_{M,\infty}$,

for $M$ $\theta$-stable,

$= P(\mathbb{R})/A_M K_{M,\infty}$

since the action of $A_M$ commutes with $P(\mathbb{R})$, so each piece of (2.5) is of the form

$P(\mathbb{Q}) \backslash P(\mathbb{A})/A_M K_{M,\infty} K_P = S_{K_P}^P$,

with $K_P = K_P(h)$. This is the correct construction of $S_{K_P}^P$, mentioned above. Thus $S_{K_P}^P$ occurs as one of the pieces of the compactification. We note that (2.4)—in particular when passing to the limit for $K \rightarrow 1$—allows one to see the cohomology of $S_{K}^{BS}(P)$—rather, the limit—as an induced representation of $G(\mathbb{A}_f)$. We will use a simple aspect of this.

Recall that $\bigcap_{P \in Q'} S_{K}^{BS}(P)$, where $Q'$ is the set of proper representatives, is the boundary of $S_{K}^{BS}$. Its dimension is $\dim(X_G) - 1$. If $P$ is a $Q$–parabolic subgroup with split component $A_P = A_M$, the dimension of $e(P)$ is $\dim(X_G) - (\dim A_P - \dim A_G)$. In particular, the quotients $S_{K}^{BS}(P)$ for $P$ a maximal $Q$–parabolic subgroup are the open cells of $\partial S_{K}^{BS}$.

2.2

We now consider the cohomology of these spaces. Although the first arguments could be given for any ring of coefficients, we will assume that $k$ is a local Artin ring that is Gorenstein. (We are mostly interested in the case where $k$ is a field; this however allows us to consider, as does Scholze, the case of $k = \mathbb{Z}/\ell^n\mathbb{Z}$ for a prime $\ell$.)

Let $S$ be the (finite) set of primes such that $K_P \subset G(\mathbb{Q}_p)$ is not hyperspecial. We consider the Hecke algebras

$$\mathcal{H}^S(G) = C_c(K^S \backslash G(\mathbb{A}^S_f)/K^S, \mathbb{Z})$$

$$= \bigotimes_{p \notin S} C_c(K_p \backslash G(\mathbb{Q}_p)/K_p, \mathbb{Z})$$

where $K^S = \prod_{p \notin S} K_p$, and the tensor product is restricted. Similarly we have $\mathcal{H}^S(P)$ and $\mathcal{H}^S(M)$.

We can assume that our set $Q$ of representatives of the parabolic subgroups is the set of standard parabolic subgroups, i.e., those containing a fixed, minimal parabolic subgroup $P_0$. Then there is a choice of maximal compact subgroups $K^0_P$ (all primes) such that

$$G(\mathbb{Q}_p) = K^0_P P(\mathbb{Q}_p) = P(\mathbb{Q}_p) K^0_P$$

(all $p$, all $P \in Q$). We assume $K_P = K^0_P$ so chosen for $p \notin S$. In this situation those are natural $\mathbb{Z}_p$–structures on all groups $G$, $P$, $M$, $N$... for $p \notin S$, and we assume
that the measures on all groups $X$ give mass 1 to $X(\mathbb{Z}_p)$. We have natural maps

$$\rho : \mathcal{H}^S(G) \to \mathcal{H}^S(P)$$

given by the restriction of functions, and

$$\lambda : \mathcal{H}^S(P) \to \mathcal{H}^S(M)$$

given by

$$\varphi \mapsto \int_{N(\mathbb{Q}_p)} \varphi(mn)dn := \lambda \varphi(m).$$

They are morphisms of algebras. Note that the “constant term” $\lambda$ is not normalised.

These algebras act naturally on cohomology spaces. For instance, a double coset $K_p g K_p$ in $H^i_p(G)$ acts on $H^i(SBS_K, k)$ by the correspondence

$$S^{BS}(K \cap g K g^{-1}) \to S^{BS}(g^{-1} K g \cap K)$$

$$\downarrow \quad \downarrow$$

$$S^{BS}(K) \quad S^{BS}(K)$$

(2.6)

$R_g$ being right translation by $g \in G(\mathbb{Q}_p)$, using the description (2.2). Moreover, these correspondences preserve the decomposition (2.4).

In particular we see that $\mathcal{H}^S(G)$ acts on $H^*(\partial S^{BS}_K, k)$, compatibly with the map $H^*(S^{BS}_K, k) \to H^*(\partial S^{BS}_K, k)$.

Assume now that $P \in \mathcal{P}$ is maximal. Then $S^{BS}_K(P)$, and in particular its component $S^{BS}_{Kp}$, is open in $\partial S^{BS}_K$. Thus we obtain a map $j^* : H^i(\partial S^{BS}_K, k) \to H^i(S^{BS}_{Kp}, k)$ as well as $j_* : H^i_c(S^{BS}_{Kp}, k) \to H^i(\partial S^{BS}_K, k)$. We also have the projection $\pi : S^{BS}_{Kp} \to S^{M}_{KM}$, whence $\pi^* : H^i_c(S^{M}_{KM}, k) \to H^i_c(S^{BS}_{Kp}, k)$ since the fibres of $\pi$ are compact.

**Lemma 2.1**

(i) The map $j^*$ is equivariant under $\mathcal{H}^S(G), \mathcal{H}^S(G)$ acting naturally on $H^i(\partial S^{BS}_K, k)$ and by composition with $\rho : \mathcal{H}^S(G) \to \mathcal{H}^S(P)$ on $H^i(S^{BS}_{Kp}, k)$.

(ii) The map $j_*$ is equivariant under $\mathcal{H}^S(G), \mathcal{H}^S(G)$ acting naturally on $H^i(\partial S^{BS}_K, k)$ and by composition with $\rho : \mathcal{H}^S(G) \to \mathcal{H}^S(P)$ on $H^i_c(S^{BS}_{Kp}, k)$.

(iii) The map $\pi^*$ is equivariant under $\mathcal{H}^S(P), \mathcal{H}^S(P)$ acting on $H^i_c(S^{M}_{KM}, k)$ by composition with $\lambda$.

**Remark** This is essentially due to Scholze [24, Lemma V.2.3], who does not, however, give a proof. I am greatly indebted to Newton and Thorne for providing this proof.

The first assertion is Proposition 3.8(2) of Newton-Thorne [21].

The third is Proposition 3.4 loc.cit., taking into account their Lemma 3.2. (Newton and Thorne prove it for ordinary cohomology, but the proof is the same.)

We deduce the second assertion from the first. Indeed, $S^{BS}_{Kp}$ as well as the boundary $\partial S^{BS}_K$ are topological manifolds. Since $k$ is a local, Artin Gorenstein ring, Poincaré
duality holds for both these spaces. Moreover, on each of these spaces, Poincaré duality is compatible with the action of the Hecke algebras \[21, Proposition 3.7\].

For simplicity write \(H^i(S_P^P) = H^i(S_{K_p}^P, k)\), and similarly for \(H^i_c\), and \(H^i(\partial S_K^{BS}, k)\). Suppose \(\alpha \in H^i(\partial)\). Then, by (i),

\[j^*(h\alpha) = \rho(h)j^*\alpha\]

for \(h \in \mathcal{H}^S(G)\). Assume \(\gamma \in H^{d-1-i}(\partial), d = \dim(X_G)\). Then, \((, .)_P, (, .)_G\) denoting Poincaré duality on \(S_{K_p}^P\) and \(\partial S_K^{BS}\),

\[(j_*(\rho(h)\beta), \gamma)_G = (\rho(h)\beta, j^*\gamma)_P = (\beta, \rho(h^\vee)j^*\gamma)_P\]

by the compatibility of Poincaré duality with Hecke action, \(h^\vee(g)\) being \(h(g^{-1})\)

\[= (\beta, j^*(h^\vee\gamma))_P = (j_*\beta, h^\vee\gamma)_G = (hj_*\beta, \gamma)_G\]

since obviously \(\rho(h^\vee) = (\rho(h))^\vee\). This implies (ii) since, with our assumptions, Poincaré duality is non-degenerate.

The comparison with the paper of Newton and Thorne calls for the following comments; we refer to it for some notations, which we do not need here.

(1) Newton and Thorne prove stronger statements; for instance in the case of (i) (their Proposition 3.8) concerning the actions of \(\mathcal{H}^S(G)\) and \(\mathcal{H}^S(P)\) on \(\mathcal{R}_\Gamma_{\partial S_K^{BS}k}\), \(\mathcal{R}_{S_K^{P}k}\) seen as elements of suitable derived categories. Our statement is simply obtained by taking instead the cohomology of the relevant complexes.

(2) In our case all the modules figuring in their statements coincide with the trivial \(k\)-module; the corresponding sheaves are the constant \(k\)-sheaves.

(3) Their definition of the action of the Hecke algebras on the cohomology spaces is given in their Sect. 2 (see their Proposition 2.10 and the subsequent comment.) It coincides with the definition we have given, by correspondences (see their Lemma 2.19.)

Consider now the diagram

\[
\begin{array}{ccc}
H^i_c(S_{K_p}^P, k) & \longrightarrow & H^i(\partial S_K^{BS}, k) \\
\downarrow j_* & & \downarrow j^* \\
H^i(S_{K_p}^P, k) & & \\
\end{array}
\]

Set \(j_c = j^*j_* : H^i_c(S_{K_p}^P, k) \to H^i(S_{K_p}^P, k)\). Its image is by definition the compactly supported part of the cohomology, or 'inner' cohomology, denoted by \(H^i_c(S_{K_p}^P, k)\). The diagram yields naturally a surjective map

\[
\text{Im}(j_*) \longrightarrow H^i_c(S_{K_p}^P, k) .
\]  

(2.7)

\[\text{See the Appendix.}\]
By Lemma 2.1, \( \text{Im} j_* \) is a submodule of \( H^i(\partial S^B_K, k) \) for the action of \( \mathcal{H}^S(G) \). The following argument is again due to Scholze. We consider the diagram

\[
\begin{array}{ccc}
H^i_c(S^P_{K_P}, k) & \xrightarrow{\pi^*} & H^i(S^P_{K_P}, k) \\
\uparrow \pi_* & & \downarrow \pi_* \\
H^i_c(S^M_{K_M}, k) & \xrightarrow{j^M_*} & H^i(S^M_{K_M}, k)
\end{array}
\] (2.8)

The map \( \pi_* \) is obtained as follows. We have \( H^\bullet_{SP_K, k} = H^\bullet_{MSM, R^\pi_* k} \). This is associated to the local system \( H^\bullet(F) \) where \( F \sim \Gamma_1 N \setminus N(\mathbb{R}) \) is the fiber of \( \pi \). However \( F \) is covered by the contractible space \( \tilde{F} = N(\mathbb{R}) \), and therefore we get \( am a\mapsto H^\bullet(F) \rightarrow H^\bullet(\tilde{F}) \). The diagram (2.8) clearly commutes. (See [21, Cor. 3.6]).

**Lemma 2.2** \( \pi_* \) is equivariant, \( \mathcal{H}(P) \) acting naturally on \( H^i(S^P_{K_P}, k) \) and via \( \lambda \) on \( H^i(S^M_{K_M}, k) \).

See [21, Prop. 3.4], where this is proved for the spaces of compactly supported cohomology in the left column of (2.8) replaced by the full cohomology spaces.

We obtain the following corollary. The commutativity of the diagram (2.7) implies that the map \( \pi_* : H^i_c(S^P_{K_P}, k) \rightarrow H^i_c(S^M_{K_M}, k) \) is surjective. We finally have :

**Corollary 2.3** There exists a surjective map

\[
\text{Im} j_* \longrightarrow H^i_c(S^M_{K_M}, k),
\]

where \( j_* : H^i_c(S^P_{K_P}, k) \rightarrow H^i(\partial S^B_K, k) \), equivariant for the action of \( \mathcal{H}^S(G) \) on the left–hand side, and its action via \( \lambda \circ \rho \) on the right–hand side.

Finally, consider the long exact sequence in cohomology for the manifold with boundary \( S^B_K \):

\[
\cdots \longrightarrow H^i_c(S_K, k) \longrightarrow H^i(S^B_K, k) \longrightarrow H^i(\partial S^B_K, k) \longrightarrow \\
H^i+c^i(S_K, k) \longrightarrow \cdots
\] (2.9)

The submodule \( j_* H^i_c(S^P_{K_P}, k) := H \) of \( H^i(\partial S_K, k) \) admits a filtration

\[
0 \longrightarrow H' \longrightarrow H \longrightarrow H'' \longrightarrow 0
\]

with \( H' \subset H^i(S^B_K, k) = H^i(S_K, k) \) and \( H'' \subset H^i+c^i(S_K, k) \). We finally have by Cor. 2.3 :

**Theorem 2.4** Each irreducible subquotient of the \( \mathcal{H}^S_G \)-module \( H^i_c(S^M_{K,M}, k) \) is a subquotient of \( H^i+c^i(S_K, k) \) or of \( H^i(S_K, k) \).
We recall that the action of $\mathcal{H}_G^S$ on $H^i_r(S_{KM}^M, k)$ is through the map $\lambda \circ \rho$. In the “classical” case ($k = \mathbb{C}$), this is the map associated to non–normalized induction from $M(\mathbb{A}^S)$ to $G(\mathbb{A}^S)$ (through $P$).

Note that all the cohomology spaces are finite over $k$. Let $m$ be the maximal ideal of $k$, and $\kappa = k/m$. All modules $H$ considered can be filtered:

$$0 = m^r H \subset m^{r-1} H \subset \cdots \subset mH \subset H,$$

the quotients being vector spaces over $\kappa$, and preserved by the Hecke algebras. On each quotient $H_S^G$ acts via the quotient $H_S^G \otimes \kappa$. Denote by $m_H$ a maximal ideal of $H_S^G \otimes \kappa$. We deduce:

**Theorem 2.5** Assume $H^i_r(S_{KM}^M, k)_{m\mathcal{H}} \neq 0$. Then $H^i_{c'}(S_K, k)_{m\mathcal{H}} \neq 0$ or $H^i(S_K, k)_{m\mathcal{H}} \neq 0$.

For a more precise result, relating the images of $\mathcal{H}_M$ and $\mathcal{H}_G$ in $\text{End}(H^i_r(S_{KM}^M, k))$ and $\text{End}(H^i(S_K, k))$, see [24, Cor.5.2.4] as well as [21, Cor. 3.9].

### 3 The complex case: coefficient systems

#### 3.1

We now return to the constructions of Section 2.2, but we consider the case where $k = \mathbb{C}$, introducing coefficient systems. We assume given an algebraic, complex representation of $G, L$; it defines naturally a local system $\mathcal{L}$ on $S_K$. Since $S_K \subset S_{BS}^K$ is a homotopy equivalence, $\mathcal{L}$ extends to $S_{BS}^K$, and then restricts to $S_{BS}^K(P)$ for any $P$.

Recall (2.5) that $S_{BS}^K(P)$ is a union of components of the form

$$P(\mathbb{Q}) \backslash e(P) \times P(\mathbb{A}_f)/K_p(h),$$

$$e(P) = P(\mathbb{R})/A_M K_M,_{\infty}$$

which in turn are a union of quotients

$$e'(P) = \Gamma_p \backslash P(\mathbb{R})/A_M K_M,_{\infty}$$

for congruence subgroups $\Gamma_p$. The quotient $P(\mathbb{R})/A_M K_M,_{\infty} = N(\mathbb{R}) X (M)$ is simply connected.

Moreover [2, Prop. 9.4] $e'(P) \to S_{BS}^K$ induces the natural map $\Gamma_p \to \Gamma$, where $\Gamma$ is one of the groups occurring in the decomposition (2.4) and $S_{BS}^K$ the corresponding component. Therefore $\mathcal{L}$ also defines a local system on $e'(P)$, given by the restriction of $L$ to $\Gamma_p$. We will simply denote by $\mathcal{L}$ the local system obtained on each of the spaces $S_K, S_{BS}^K, S_{BS}^K(P)$...

Returning to Section 2.2, we first recall that the diagram of maps (2.6) again yields naturally a map $H^i(S_K^{BS}, \mathcal{L}) \to H^i(S_K^{BS}, \mathcal{L})$. For simplicity we describe this in the
case of $S_K$. The total space of the vector bundle $L$ on $S_K$ is
\[ G(\mathbb{Q})\backslash (X \times (G(\mathbb{A}_f)/K) \times L), \quad (3.1) \]
$G(\mathbb{Q})$ acting diagonally on the three factors. The only non–obvious map in (2.6) is the effect of $R_g$. Write $K' = K \cap gKg^{-1}$, $K'' = g^{-1}Kg \cap K$. We need a map $L_y \rightarrow L_{R_g y}$, for a point $y \in S_{K'}$. A representative of $L_y$ in the quotient (3.1) is $(x, h) \times L$ where $(x, h) \in S \times G(\mathbb{A}_f)$ is a representative of $y$. The map $R_g$ is $(x, h) \mapsto (x, hg)$. Similarly $L_{R_g y}$ is represented by $(x, hg) \times L$. The map $(x, h, \ell) \mapsto (x, hg, \ell) (\ell \in L)$ descends to the quotient (3.1). (This is of course well-known ; see Harder [11] for a fuller description of the action of the Hecke algebra.)

As before Lemma 2.1, we obtain compatible actions of $H^S(G)$ on $H^*(\partial S_{S_K}^B, \mathcal{L})$ and $H^*(S_{S_K}^{BS}, \mathcal{L})$. For $P$ maximal, we obtain again a map $j^* : H^i_c(S_{S_K}^P, \mathcal{L}) \rightarrow H^i(\partial S_{S_K}^{BS}, \mathcal{L})$.

Lemma 2.1 (i) and (ii) are proved as before. With $H^S(G)$ acting by composition on the cohomology spaces relative to $P$:

Lemma 3.1 (i) The map $j^*$ is equivariant under $H^S(G)$.
(ii) The map $j^*$ is equivariant under $H^S(G)$.

However, part (iii) of Lemma 2.1 is not sufficient in this case, as $\mathcal{L}$ (when non trivial) does not descend to a local system on $S^M_{K_M}$. In general, Harder and Schwermer have described $H^*(S^P_{S_K}, \mathcal{L})$ as a cohomology space for $S^M_{K_M}$. This is given by the degeneracy of the Leray spectral sequence for the fibration $S^P_{S_K} \rightarrow S^M_{K_M}$ by compact nilmanifolds. Let $n$ be the real Lie algebra of $N$. For each $j$, $H^j(n, L)$ is an $M(\mathbb{R})$–module and defines a local system $H^j(n, L)$ on $S^M_K$. Then
\[ H^i(S^P_{S_K}, \mathcal{L}) = \bigoplus_{j+k=i} H^k(S^M_{K_M}, H^j(n, L)). \quad (3.2) \]
This is due to Harder [7] and Schwermer [25]. Note that $H^S(M)$ acts on the right–hand side. Harder and Schwermer prove the isomorphism (3.2) for classical quotients, i.e., for the components of our adèlic quotients.

However these classical quotients (respectively for $P$ and $M$) are in bijection : use strong approximation for the nilpotent group $N$.

Lemma 3.2 (i) The isomorphism (3.2) is true for compactly supported cohomology on both sides.
(ii) It is equivariant (both for ordinary and compactly supported cohomology) under $H^S(P)$, acting on the right via $\lambda$.

Par (i) is clear since it is the (degenerate) Leray spectral sequence, and the fibres are compact. For (ii), we have to consider the proof of Harder and Schwermer.

Let $m^0$ denote the Lie algebra of $M^0$, where $M(\mathbb{R}) = M^0 A_M$ is the Langlands decomposition. Let $p^0 = m^0 \oplus n$. Let $\mathfrak{t}_M$ be the Lie algebra of $K_{M,\infty} = K_{P,\infty}$. As
described by Schwermer [25, p. 49-50], there is, for each \(i, j\), a natural injection

\[
\eta^i : \text{Hom}_{K_{M,\infty}}(\Lambda^j(m^0/kM), C^\infty(M(\mathbb{Q})\setminus M(\mathbb{A})/A_M K_M) \otimes H^i(n, L))
\]
\[
\rightarrow \text{Hom}_{K_{M,\infty}}(\Lambda^{i+j}(p^0/kM), C^\infty(P(\mathbb{Q})\setminus P(\mathbb{A})/A_M K_P) \otimes L).
\] (3.3)

In fact Schwermer proves this statement for classical quotients \(\Gamma_M\setminus \mathbb{H}/A_M, \Gamma_P\setminus \mathbb{H}/A_M\). Again, this implies the adèlic variant.

The map \(\eta^i\) is obtained from the injection

\[
\Lambda^\bullet(m^0/kM)^* \otimes \Lambda^\bullet(n)^* \rightarrow \Lambda^\bullet(p^0/kM)^*
\]

of dual spaces, and from the natural injection

\[
C^\infty(M(\mathbb{Q})\setminus M(\mathbb{A})/A_M K_M) \rightarrow C^\infty(P(\mathbb{Q})\setminus P(\mathbb{A})/A_M K_P)
\]
given by the projection \(P \rightarrow M\). It induces an isomorphism in cohomology, yielding (3.2).

We note that the same injection obtains between the spaces of compactly supported, smooth functions. The proof of (3.3) ([25, Theorem 2.7]) now yields the same result for cohomology with compact support. The spaces of smooth functions on the adèlic groups in (3.2) receive, respectively, an action of \(H^S(M)\) and \(H^S(P)\) by right translations, which then yield the actions on the cohomology of the respective quotients. Therefore the proof of (ii) is completed by the following easy lemma.

**Lemma 3.3** Denote by \(I\) the natural injection \(C^\infty_c(M(\mathbb{Q})\setminus M(\mathbb{A})/A_M K_M) \rightarrow C^\infty_c(P(\mathbb{Q})\setminus P(\mathbb{A})/A_M K_P)\). Then, if \(\varphi \in H^S(P)\) and \(f \in C^\infty_c(M(\mathbb{Q})\setminus M(\mathbb{A})/A_M K_M)\):

\[
I(\lambda(\varphi)f) = \varphi I(f).
\]

We can omit the compact–open subgroups and work with the functions on the full adèlic quotients. Denote by \(pr\) the projection \(P \rightarrow M\) (for these quotients). Then, for \(g \in P(\mathbb{A})\):

\[
\varphi I(f)(g) = \int_{P(\mathbb{A})} I(f)(gg')\varphi(g')dg'
\]
\[
= \int_{P(\mathbb{A})} f(pr(gg'))\varphi(g')dg',
\]

while for \(m \in M(\mathbb{A})\):

\[
\lambda(\varphi)f(m) = \int_{M(\mathbb{A})} f(mm')(\lambda(\varphi)(m')dm'
\]
\[
= \int_{M(\mathbb{A})} f(mm')\varphi(m'n')dm'dn',
\]
\[
I(\lambda(\varphi)f)(g) = \int_{M(\mathcal{A}f)N(\mathcal{A}f)} f(pr(g)m')\varphi(m'n')dm'dn'.
\]

Since \( pr(gg') = pr(g)pr(g') = pr(g)m' \), the property is clear.

3.2

As in Sect. 2, the diagram

\[
H^i_c(S^P_{K_P}, \mathcal{L}) \xrightarrow{j_*} H^i(S^{BS}_K, \mathcal{L}) \xrightarrow{j^*} H^i(S^P_{K_P}, \mathcal{L})
\]

yields a surjective map

\[
\text{Im}(j_*) \longrightarrow H^i_c(S^P_{K_P}, \mathcal{L}).
\]

of \( \mathcal{H}^S(G) \)--modules. However, since (3.2) is true both for \( H^i \) and \( H^i_c \), the right-hand side is

\[
\bigoplus_{j+k=i} H^k(S^M_{K_M}, \mathcal{H}^j_n, L)).
\]

We can now consider the long exact sequence for \( S^{BS}_K \) as in (2.9), to conclude that the subquotients of (3.4), on which, by Lemma 3.2, \( \mathcal{H}^S(G) \) acts via \( \lambda \circ \rho \), occur in \( H^i(S_K, \mathcal{L}) \) or \( H^{i+1}_c(S_K, \mathcal{L}) \).

Rather than by localising at a maximal ideal, we express the result, as usual in the complex case, in terms of characters of the (commutative) Hecke algebra \( \mathcal{H}^S(G, \mathbb{C}) = \mathcal{H}^S(G) \otimes \mathbb{C} \).

Let \( G_K \) denote the adèlic quotient of the group,

\[
G_K = G(\mathbb{Q}) \backslash G(\mathbb{A})/A_G K,
\]

so \( S_K = G_K/K_{G,\infty} \). We then have an inclusion of representations of \( G(\mathbb{R}) \)

\[
L^2_{\text{cusp}}(G_K) \subset L^2_{\text{dis}}(G_K),
\]

the space of cusp–forms inside the discrete spectrum, and consequently for the automorphic forms :

\[
A_{\text{cusp}}(G_K) \subset A_{\text{dis}}^2(G_K).
\]

The image in \( H^\bullet(g, K_\infty; A(G_K)) \) of the \((g, K_\infty)\)--cohomology of these spaces gives the cuspidal and the \( L^2 \)--cohomology (in the following sense : cohomology represented by \( L^2 \) harmonic forms) of \( S_K \); moreover it is known that

\[
H^\bullet_{\text{cusp}}(S_K) \subset H^\bullet_1(S_K) \subset H^\bullet_2(S_K).
\]
where $H^*_c(S_K, L)$ is the $L^2$-cohomology. Cf. Schwermer [27]. With coefficients $L$, we have

$$H^*_c(S_K, L) = H^*(g, K ; \mathcal{A}_{cusp}(G_K) \otimes L).$$

The same applies to $M$. Finally, the full cohomology of $S_K$ is, by Franke’s result [5]:

$$H^*(g, K_\infty ; \mathcal{A}(G_K)),$$

the $(g, K_\infty)$-cohomology of the full space of automorphic forms. The abstract form of our result is then:

**Theorem 3.4** (i) Assume $\chi$ is a character of $\mathcal{H}^S(M)$ occurring non–trivially in $H^k_i(S_{K_m}^M, \mathcal{H}^j(n, L))$ for some values of $k$, $j$; if $i = k + j$, $\chi \circ (\lambda \circ \rho) := \chi'$, a character of $\mathcal{H}^S(G)$, occurs non–trivially in $H^k_i(S_K, L)$ or $H^{i+1}_c(S_K, L)$.

(ii) In particular, if $\chi$ occurs in $H^*_c(S_{K_m}^M, \mathcal{H}^j(n, L)) = H^k(m, K_M; \mathcal{A}_{cusp}(M_{K_M} \otimes L'))$ where $L' = H^j(n, L)$, $\chi'$ occurs in $H^i(S_K, L)$ or $H^{i+1}_c(S_K, L)$.

(Note that by Poincaré duality $H^{i+1}_c(S_K, L)$ is also described by automorphic forms.)

The proof implies more: not only do the eigenspaces for $\mathcal{H}^S(M)$ (or $\mathcal{H}^S(P)$) yield eigenspaces for $\mathcal{H}^S(G)$, but the dimensions are conserved. Precisely, we have the following result. Denote by $M_\chi$ a generalised eigenspace associated to a character $\chi$.

**Proposition 3.5** Let $\chi$ be a character of $\mathcal{H}^S(M)$, and $\chi'$ the associated character of $\mathcal{H}^S(G)$. Then

$$\dim H^i(S_K, L)\chi + \dim H^{i+1}_c(S_K, L)\chi' \geq \sum_{i=j+k} \dim H^k_i(S_{K_m}^M, \mathcal{H}^j(n, L))\chi.$$

### 3.3
Finally, we give the more explicit formulation of the result given by Kostant’s theorem, following again Harder and Schwermer. Assume $L$ is irreducible, with highest weight $\nu \in h^*_C$, where $h_C$ is a Cartan subalgebra of $g_C = \text{Lie}(G(\mathbb{R})/A) \otimes \mathbb{C}$. Let $m_C = \text{Lie}(M(\mathbb{R})/A) \otimes \mathbb{C}$ and $p_C = \text{Lie}(P(\mathbb{R})/A) \otimes \mathbb{C}$. We fix a Borel subalgebra $b_C \subset p_C$; $p_C = m_C \oplus n_C$. We choose a split real form $g_r$ of $g_C$, whence $h_r$, $b_r$, $p_r = m_r + n_r$. (Note that $n_r = n = \text{Lie} N(\mathbb{R})$. ) Let $W = W_G$, $W_M$ be the complex Weyl groups; we assume given on $h_r$ a scalar product invariant by $W$. Let $R_G$, $R^+_G$, $\Delta_G$ be the roots, positive roots, simple roots for $G$; similar notations for $M$. Let $\rho$ be the half–sum of roots for $R^+_G$. Finally, let $W^P$ be the set of Kostant representatives for $W_M \backslash W$:

$$W^P = \{ w \in W \mid w^{-1} \alpha \in R^+_G \forall \alpha \in \Delta_M \},$$

cf. [25, Section 2.3]. We write $L_M^\xi$ for an irreducible complex representation of $M$ with highest weight $\xi$. Then

$$H^j(n, L) = \bigoplus_{w \in W^P} \bigoplus_{\ell(w) = j} L_M^{w(\nu + \rho) - \rho} \quad (3.5)$$
(Kostant). We will use this description to show that (up to twists, see below) all the inner cohomology of $S^M_{KM}$ gives rise to cohomology of $S_K$ with corresponding action of the Hecke algebras.

Assume therefore that $\xi \in h^*_r$ is a positive weight for $M$. (Note that we assume that all weights vanish on $a = \text{Lie} A \subset h_r$. The reader may as well suppose the split component $A$ trivial.) Following (3.5) we want to write

$$\xi + \rho = w(\nu + \rho)$$

(3.6)

with $\nu$ dominant for $G$ and $w \in W^P$. Since $\nu + \rho$ is strictly positive, this implies that $\xi + \rho$ must be regular for the roots of $G$.

Assume this first. We can certainly write

$$\xi + \rho = w\nu'$$

with $w \in W$ and $\nu'$ positive (dominant) for $G$, and regular. Then for $\alpha \in \Delta_M$

$$<\xi + \rho, \alpha> = <w\nu', \alpha> = <\nu', w^{-1}\alpha>$$

is strictly positive, so $w^{-1}\alpha \in R^+_G$, and $w \in W^P$. Furthermore $\nu'$, being regular, verifies $<\nu', \alpha> \geq 1$ for $\alpha \in \Delta_G$, so $\nu' = \nu + \rho$.

In this case we can realise $L^\xi_M$ as a summand of (3.5), with this choice of $\nu$. Setting $j = \ell(w)$, and considering Theorem (3.5 (ii)), we assume that $H^k(S^M_{KM}, L^\xi_M) \neq 0$. Then, with $i = j + k$, the corresponding character of $\mathcal{H}^S(G) \otimes \mathbb{C}$ will occur in $H^i$ or $H^{i+1}_c$ for $S_K$.

Consider now the case where $\xi + \rho$ is not $G$–regular. For simplicity assume $A = A_G = \{1\}$. We certainly have $<\xi + \rho, \alpha > > 0$ for $\alpha \in \Delta_M$. The dual of the 1–dimensional space $a_M \subset h_r$, $a^*_M$, is identified with $X^*(M) \otimes \mathbb{R}$ where $X^*(M) \cong \mathbb{Z}$ is the group of $\mathbb{Q}$–rational characters of $M$. In particular, a character $\mu$ defines an element $T \in a^*_M \subset h^*_r$. The representation $L^\xi_M \otimes \mu$ then has highest weight $\xi + T$. Recall that we have used a scalar product to identify $h_r$ with its dual. Let $\beta \in \Delta_G$ be the unique root not in $\Delta_M$. Then, if we choose $T$ in a suitable half–line, $<\beta, T > > 0$ and therefore $\xi + \rho + T$ is regular. Consequently, the twisted representations $L^\xi_M \otimes \mu$ (and the corresponding local system) verify our previous conditions.

If $\pi = \pi_\infty \otimes \pi_f$ is a representation of $M(\mathbb{A})$ such that

$$H^*(m, K_M; \pi_\infty \otimes L^\xi_M) \neq 0,$$

and occurring in the discrete spectrum, $\pi \otimes |\mu|^{-1}$ has cohomology with coefficients in $L^\xi_M \otimes \mu$ ; obviously the compactly supported cohomology classes correspond. Finally, we can strengthen Theorem 3.4 :

**Theorem 3.6** Assume $H^i_k(S^M_{KM}, L_M) \neq 0$ for any coefficient system, associated to $L_M$.

Let $\chi : \mathcal{H}^S(M) \otimes \mathbb{C} \rightarrow \mathbb{C}$ occur in this space. Then the character $\chi'$ of $\mathcal{H}^S(G) \otimes \mathbb{C}$ associated to an Abelian, unramified twist of $\chi$ occurs in $H^i(S_K, L)$ or $H^{i+1}_c(S_K, L)$, $L$ being constructed as above.
3.4
We now strengthen the results of Section 3.2, by controlling the rationality of the cohomology classes. Note that the local system $\mathcal{L}$, and all the cohomology spaces, are defined over $\mathbb{Q}$. Precisely, we consider $\mathcal{L}$ as a local system of $\mathbb{Q}$-vector spaces. Lemma 3.2 is still true over $\mathbb{Q}$. Indeed, it is given by the Leray spectral sequence for the map $\pi : S_{K_p}^P \to S_{K_M}^M$. The cohomology of the fiber, $\Gamma_N \backslash N(\mathbb{R})$, is, in the complex case, computed using the van Est isomorphism. However, the van Est isomorphism is true over $\mathbb{Q}$. See [16, 22].

We therefore obtain the same exact spectral sequence, now over $\mathbb{Q}$; since it degenerates over $\mathbb{C}$ it does over $\mathbb{Q}$. The local system $H_j^i(S_{K_p}, \mathcal{L})$ is now a local system of $\mathbb{Q}$–vector spaces. We can now prove the following “rational” result. Denote, as before, by $j : S_{K_p}^P \to \partial S_{BS}^K$ the natural injection, and let

$$\delta : H^i(\partial S_{BS}^K, \mathcal{L}) \to H^{i+1}_c(S_K, \mathcal{L})$$

be the connecting homomorphism in (2.9). Finally, denote by $\text{Res} : H^i(S_{K_p}^P, \mathcal{L}) \to H^i(\partial S_{BS}^K, \mathcal{L})$ the natural restriction.

Let $\chi : \mathcal{H}^\mathcal{S}(G, \mathbb{Q}) \to \mathbb{Q}$ be a character, and, for each (finite–dimensional) $\mathcal{H}^\mathcal{S}(G, \mathbb{Q})$–module $M$, let $M_\chi$ be the generalised eigenspace associated to $\chi$.

Finally, recall that $H^i_j(S_{K_p}^P, \mathcal{L}) = \bigoplus_{j+k=i} H^k(S_{K_M}^M, \mathcal{H}^\mathcal{S}(n, L))$; by the previous argument this decomposition is rational (over $\mathbb{Q}$).

Theorem 3.7
Let $\alpha \in H^i_j(S_{K_p}^P, \mathcal{L})$ be a non–zero generalised eigenclass associated to $\chi$, for the map $\mathcal{H}^\mathcal{S}(G) \to \mathcal{H}^\mathcal{S}(P)$; equivalently, $\alpha \in H^i_j(S_{K_M}^M, \mathcal{H}^\mathcal{S}(n, L))$. Then one of the following properties is true:

(i) $\alpha = j^* \text{Res} \beta$
for a (non–zero) class $\beta \in H^i(S_{K_M}^M, \mathcal{L})_\chi = H^i(S_K, \mathcal{L})_\chi$

(ii) $\alpha = j^* \gamma$
for a class $\gamma \in H^i(\partial S_K, \mathcal{L})_\chi$ such that

$$\beta = \delta(\gamma) \in H^{i+1}_c(S_K, \mathcal{L})_\chi \neq 0.$$ 

We repeat that the cohomology is now taken with $\mathbb{Q}$-coefficients.

4 Relation with Eisenstein cohomology

4.1
We now check, at least in a simple case, the compatibility of this construction with the theory of Eisenstein cohomology.

We will consider only the simplest case (and the only perfectly general one) where “Eisenstein classes” have been constructed : this is Schwermer’s Theorem 4.11 in [25]. (Note that Schwermer’s result is not limited to maximal parabolic subgroups.)

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7 This is announced by Schwermer [27, p.15].
Let $L = L^\nu$, the coefficient system for $G$, be given, and consider a cuspidal cohomology class $[\varphi] \in H^*_{cusp}(S_{K_0}^n, \mathcal{L})$. Assume it is associated to a representation $\pi$ of $M^0(\mathbb{R})$ occurring in the cusp forms, where $M(\mathbb{R}) = M^0(\mathbb{R}) \Lambda_M$ is the Langlands decomposition, and, following (3.4) and (3.5), to $w \in W^P$ — a “class of type $(\pi, w)$” in the terminology of Schwermer [25, p. 82]. Schwermer constructs a (differential–form valued) Eisenstein series $E(\varphi, \lambda)$, where $\varphi$ is a harmonic representative of $[\varphi]$, and $\lambda \in \mathfrak{a}_M^* \otimes \mathbb{C}$. 

The form $E(\varphi, \lambda)$ is constructed, by summation over $\Gamma_P \backslash \Gamma$, from a form $\varphi$, on $e'(P) \times A_P$; see Schwermer [25, Section 3.3] If it is holomorphic at the point

$$\lambda_0 = -w(\nu + \rho)|_{\mathfrak{a}_M},$$

this yields a closed form on $S_K$, which represents a non–trivial cohomology class for $G$. Note that since $\pi$ is cuspidal unitary, the domain of convergence for $E(\varphi, \lambda)$ is given [19, p. 86] by

$$< \Re \lambda, \alpha^\vee > > < \rho_G - \rho_M, \alpha^\vee >$$

for each coroot $\alpha^\vee$, $\alpha \in \Delta_G$, occurring in $N_P$. (cf. [25, Section 6.3]).

Now consider the cohomology classes constructed in Sect. 3. We start, as in Theorem 3.5, with a character $\chi$ of $H^S(M) \otimes \mathbb{C}$ occurring in the cohomology $H^*_{cusp}(S_{K_0}^M, \mathcal{L}_M)$. Then (perhaps after an unramified, Abelian twist) it occurs in $H^*(S_K, \mathcal{L})$ for some coefficient system $L$. If $L_M$ is one of the irreducible representations occurring in $H^j(n, L)$ for some $j$ and $L$, the twist is not required.

### 4.2

We now limit ourselves to the case of $GL(n)^9$. Consider the decomposition (1.1)

$$A(G_K) = \bigoplus_P A_P(G_K)$$

and the corresponding decomposition of the cohomology. The contribution of $A_P(G_K)$ to $H^*(S_K)$ is given by the Eisenstein series (and their residues) coming from cuspidal representations of $M(\mathbb{A})$. Consider specifically the cuspidal cohomology $H^i_{cusp}(S_{K_0}^M, \mathbb{C})$. It injects in $H^i(S_{K_0}^M, \mathbb{C})$. In particular, if a character $\chi$ of $H^S(M)$ occurs in $H^i_{cusp}(S_{K_0}^M, \mathbb{C})$, the associated character $\chi'$ of $H^S(G)$ occurs in $H^i(S_K)$ or $H^i_{cusp}(S_K^M, \mathbb{C})$ — in the second case, the character $\tilde{\chi}'$ associated to the dual representation occurs in $H^{d-i-1}(S_K)$.

Assume now $G = GL(n)$, so $P$ has Levi subgroup $GL(n_1) \times GL(n_2)$. By the results of Jacquet and Shalika on strong multiplicity one, $\chi'$ (or $\tilde{\chi}'$) must occur in the cohomology given by the summand of (4.3) associated to $P$.

We can try to construct it using Schwermer’s theorem. Thus let $\omega = \omega_1 \otimes \omega_2$ be a harmonic $i$–form on $S_{K_0}^M$, a product of the arithmetic quotients for $GL(n_1)$ and $GL(n_2)$, that represents our cohomology class. As recalled above (with different notation), we deduce from $\omega$ and $s \in \mathfrak{a}_M^* \otimes \mathbb{C}$ a form $\omega_s$ on $e'(P) \times A_P$ and (in

8 In fact $(\mathfrak{a}_M/\mathfrak{a}_G)^* \otimes \mathbb{C}$; similarly, consider $\mathfrak{a}_M/\mathfrak{a}_G$ in (4.1). All our linear forms are trivial on $\mathfrak{a}_G$.

9 For a detailed study of certain Eisenstein classes when all the Eisenstein series are holomorphic, in the case of $GL(n)$, see Harder-Raghuram [12].
the appropriate range) an Eisenstein class \( E(\omega, s) \). Since the cohomology with trivial coefficients for \( M \) corresponds to \( w = 1 \) in (4.1), we must evaluate the Eisenstein series \( E(\omega_1 \otimes \omega_2, s) \) at the point \( s = -\rho \mid_{a_M} \).

We must recall the relation between Schwermer’s construction of \( E(\omega_1 \otimes \omega_2, s) \) (a differential form) and the usual Eisenstein series, i.e. functions on \( G_K = G(\mathbb{Q}) \backslash G(\mathbb{A})/KA_G \). Write similarly \( M_L = M(\mathbb{Q}) \backslash M(\mathbb{A})/LA_M \); here \( L \) is a compact-open subgroup for \( M \)\(^{10}\); let \( \mathcal{A}(M), \mathcal{A}^0(M) \) denote the space of automorphic forms, cuspidal automorphic forms on \( M(\mathbb{A})/LA_M \). Let \( \mathcal{C}^*(g, K_\infty, \cdot) \) denote the complex computing (\( g, K_\infty \))-cohomology; ditto for \( m_0 = Lie(M(\mathbb{R})/A_M) \otimes \mathbb{C} \). Then the map \( \omega \mapsto \omega_s \) is given, when defined, by a map

\[
E(s) : \mathcal{C}^*(m_0, K_{M,\infty}; \mathcal{A}^0(M_L)) \otimes H^\bullet(n, \mathbb{C}) \rightarrow \mathcal{C}^*(g, K_\infty; \mathcal{A}(G_K)). \tag{4.4}
\]

Cf. Schwermer [25, Section 3.6], [25, p. 69]. We assume that \( \omega \) is associated to a cuspidal representation \( \pi = \pi_1 \otimes \pi_2 \) of \( M(\mathbb{A}) \); thus \( \pi^L \subset \mathcal{A}^0(M_L) \). On the other hand, for \( s = (s_1, s_2) \subset a_M^* \otimes \mathbb{C} \), we can consider the unitarily induced (adelic) representation \( \text{ind}_D^G(\pi \otimes |\det|^s) = I(s) \), where \( \pi \otimes |\det|^s = \pi_1|\det|^{s_1} \otimes \pi_2|\det|^{s_2} \).

When defined, \( f \mapsto E(s, f) \) can be seen as a map \( E(s) : I(s) \rightarrow \mathcal{A}(G_K) \), cf. Arthur [1, p. 254-255]. Then (4.4) can be described as a composite

\[
\mathcal{C}^*(m, K_{M,\infty}; \pi_\infty) \otimes \pi_f^L \rightarrow \mathcal{C}^*(g, K_\infty; I(s)^K) \rightarrow \mathcal{C}^*(g, K_\infty, \mathcal{A}(G_K))
\]

the last map being induced by \( E(s) : I(s)^K \rightarrow \mathcal{A}(G_K) \). The first map is computed at the Archimedean place, and is independent of the map \( I(s)^K \rightarrow \mathcal{A}(G_K) ; \omega \), or rather its harmonic representative, is naturally in \( \mathcal{C}^*(m, K_{M,\infty}; \pi) \); the degree \( j \) in \( H^\bullet(n, \mathbb{C}) \) is equal to 0 since we consider trivial local systems for \( M \) and \( G \). Now the image of \( \omega \) in \( \mathcal{C}^*(g, K_\infty, \mathcal{A}(G_K)) \) is what we denoted above by \( E(\omega_1 \otimes \omega_2, s) \). For the values of \( s \) for which \( E(\omega, s) \) is closed - those specified, with another notation, in Section 4.1, we obtain a cohomology class in \( H^\bullet(g, K_\infty, \mathcal{A}(G_K)) \).

Now assume further that the level is everywhere unramified. We also assume that \( n_1 = n_2 = \frac{n}{2} \). The holomorphy of \( E(\omega_1 \otimes \omega_2, s) \) is governed by its constant term along \( P \). This is given by

\[
E_P(\omega_1 \otimes \omega_2, s) = \omega_1 \otimes \omega_2 + M(s)(\omega_1 \otimes \omega_2) \tag{4.5}
\]

Here \( \omega_1 \otimes \omega_2 \) is seen, as above, as belonging to \( \text{ind}_P^G(\pi \otimes |\det|^s) \) and \( M(s) \omega \) to \( \text{ind}_P^G(\pi \otimes |\det|^{\tilde{s}}) \) where \( \tilde{s} = (s_2, s_1) \). See [1, Lemma 7] or [19, II.1.6, II.1.7].

The operator \( M(s) \) can be decomposed as \( M(s) = N(s)L(s) \) where \( L(s) \) is, according to Langlands, equal to the scalar

\[
\frac{L(\pi_1 \otimes \tilde{\pi}_2, s_1 - s_2)}{L(\pi_1 \otimes \tilde{\pi}_2, s_1 - s_2 + 1)}.
\]

Here the \( L \)-function is the Rankin \( L \)-function.

\(^{10}\) We choose \( K, L \) adapted to our representations.
Now $2\rho|_{a_M} = 2\rho_N$ (where $P = MN$) is equal to $|\det m_1|^{n_1}|\det m_2|^{-n_2}$, so $-\rho|_{a_M}$ corresponds to $s_1 - s_2 = -\frac{n_2}{2}$. Thus the non–trivial part of the constant term is normalized by the $L$-function

$$\frac{L(\pi_1 \otimes \tilde{\pi}_2, -\frac{n_2}{2})}{L(\pi_1 \otimes \tilde{\pi}_2, -\frac{n_2}{2} + 1)} = \frac{\epsilon(\pi_1 \otimes \tilde{\pi}_2, -\frac{n_2}{2})L(\pi_1 \otimes \tilde{\pi}_2, 1 + \frac{n_2}{2})}{\epsilon(\pi_1 \otimes \tilde{\pi}_2, 1 - \frac{n_2}{2})L(\pi_1 \otimes \tilde{\pi}_2, \frac{n_2}{2})}.$$

This is always holomorphic (including for $n = 2$), and non-zero if $n > 2$, which we now assume. Moreover, we see that the evaluation at $-\rho$ (for the usual parametrization of the Eisenstein series, corresponding to unitary induction) corresponds to the unnormalized constant term: $\mathcal{H}_S(G) \rightarrow \mathcal{H}_S(M)$, so this is compatible with our construction.

Since we are evaluating $M(s)$ on unramified functions, the normalised operator $N(s)$ is reduced to its Archimedean factor

$$N_{\infty}(s) : I(s) := \text{ind}_P^G(\pi_1[s_1] \otimes (\pi_2[s_2]) \rightarrow \text{ind}_P^G(\pi_1[s_2] \otimes (\pi_2[s_1]))$$

$(\pi_1, \pi_2$ now denote the Archimedean components; $\pi[t] = \pi \otimes |det|^t$) where induction is normalised. We will simply denote this operator by $N(s)$. We are evaluating at $s_0 = (-\frac{m}{2}, \frac{m}{2})$ where $m = \frac{n_2}{2}$; $N(s)$ is meromorphic, and Schwermer’s construction relies on the holomorphy of $N(s)$ at $s_0$ - or more precisely, on the vector in $\text{ind}_P^G(\pi_1[-\frac{m}{2}] \otimes (\pi_2[\frac{m}{2}]]) = I(s_0)$ deduced from $\omega_1 \otimes \omega_2$.

Assume, for definiteness, that $m$ is even. The representation $I(s)$ is very explicit. We have $\pi_1 = \pi_2 := \pi$ and this representation is the tempered representation of $GL(m)$ having non-trivial cohomology with trivial coefficients. Cf.[4, Section 3.5].

For $t \in \frac{1}{2} \mathbb{N}, t > 0$, let $\delta_t$ be the discrete series representation of $GL(2, \mathbb{R})$ with Langlands parameter given on $\mathbb{C}^\times = W_C$ by

$$z \mapsto ((z/\tilde{z})^{t}, (z/\tilde{z})^{-t}).$$

Then

$$\pi = \delta_{\frac{1}{2}} \times \delta_{\frac{3}{2}} \times \cdots \times \delta_{\frac{m-1}{2}},$$

a tempered representation. (Here we use $\times$ to denote induction by blocks.) Thus $I(s)$ is

$$((\delta_{\frac{1}{2}} \times \delta_{\frac{3}{2}} \times \cdots \times \delta_{\frac{m-1}{2}})[s_1] \times (\delta_{\frac{1}{2}} \times \delta_{\frac{3}{2}} \times \cdots \times \delta_{\frac{m-1}{2}})[s_2].$$

For $s_0 = (\frac{m}{2}, -\frac{m}{2})$, we see that $I(s_0)$ has a unique irreducible quotient, its Langlands quotient $Q := Q(s_0)$; $\pi$ is self-dual, $I(s_0)$ has a unique irreducible submodule.

Now $N(s)N(s_0) = 1$; $N(s)$ is holomorphic at $s_0$; and $N(s)$ is holomorphic as $s_0$ on $Q \subset I(s_0)$. Since the normalisation factor is holomorphic and non-zero, this is equivalent to the same property for the non-normalised operator, say $N'(s)$. The holomorphy of $N'(s_0)$ is part of the construction of the Langlands quotient, which
also implies that $Q$ is the unique irreducible submodule of $I(s_0)$; the holomorphy of $N(s)$ at $s_0$ on $Q \subset I(s_0)$ follows from the functional equation. Indeed, let $v \neq 0$ be an element of $Q \subset I(s_0)$. We can view $v$ as a vector independent of $s$ in the induced representation. We have $N(s)N(\tilde{s})v = v$. The vector $w = N(\tilde{s})v$ is holomorphic and nonzero at $s_0$, for which value it is in $Q$. This implies that $N(s)$ is holomorphic at $s_0$ on $w$ and therefore on $Q$.

However, $N(s)$ is not holomorphic at $s_0$ on the full representation $I(s_0)$. This follows from [20, I.2 Lemme]: the holomorphy of $N(s)$ is equivalent to the irreducibility of $I(s_0)$; however, this representation is not irreducible as a consequence of a theorem of Speh and Vogan [28, Thm. 6.15]. In fact, one has a more complete result:

**Lemma 4.1** Assume $v \in I(\tilde{s}_0)$ does not belong to $Q$. Then $N(s)v$ has a pole at $s_0$.

Since $N(\tilde{s})$ is holomorphic at $\tilde{s}_0$ and the two induced representations are dual, it defines a holomorphic family of invariant hermitian forms on the constant space of the induced representation. By a result of Vogan [29, Theorem 3.8] this form vanishes, at $s_0$, on $\ker N(\tilde{s}_0)$. Suppose $N(s)v$ is holomorphic at $s_0$. Then $N(\tilde{s})N(s)v = v$ is holomorphic. Thus $v = N(s_0)N(\tilde{s}_0)v$ belongs to $Q = \text{Im}N(s_0)$.

Thus the only vectors in the induced representation on which the Eisenstein series is holomorphic are the vectors in $Q$. The infinitesimal character of $I(s_0)$ is equal to that of the trivial representation, and thus all its subquotients are candidates at having non-trivial cohomology. We do not know in which subquotients the form deduced from $\omega_1, \omega_2$ may occur; however, there seems to be no reason that it will always belong to $Q$. However, it seems difficult to compute the cohomology of $Q$; in particular we do not know if one so obtains a space of the dimension given by Proposition 3.5. Thus, even in this simple case, it is possible that we have obtained classes which are not (directly) obtained by Schwermer’s construction. One should also recall that Scholze’s construction, in general, will succeed for all classes in the inner cohomology, which will not always be represented by cusp forms.

### 5 Non–maximal parabolic subgroups

#### 5.1

In this section we explain why Scholze’s construction seems – without a new idea – limited to the case of maximal parabolic subgroups.

So assume $P \subset G$ is an arbitrary parabolic subgroup, contained in a maximal one, $Q$. We consider again the union of faces $S^P_{Kp} \subset \partial S^B_S$. Recall that this is a union of faces $e'(P)$, of the form $\Gamma_P \setminus e(P)$. The argument will concern one face at a time, so we work classically rather than in the adèlic formulation. We have embeddings

$$e'(P) \hookrightarrow \overline{e'(Q)} \subset \partial S^B_S$$

where $S^B_S$ is now a component of $S^B_K$. See [2, Proposition 9.4]. By “smoothing” (see the Appendix to [2]) we can see $\partial S^B_S$ as a smooth variety; $e'(P)$ is then a locally

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11 I leave it to the reader to unravel the definitions of Speh and Vogan for our data.

12 Vogan, personal communication.
closed submanifold. In particular, \( e'(P) \) and \( \partial S_{\Gamma}^{BS} \) are orientable and satisfy Poincaré duality. Let \( j \) denote the embedding \( e'(P) \to \partial S_{\Gamma}^{BS} \). There still exists a direct image morphism

\[
j_* : H^i_c(e'(P)) \to H^{i+c}(\partial S_{\Gamma}^{BS})
\]

where \( c \) denotes the codimension. If \( d \) is the dimension of the symmetric space, \( \partial S_{\Gamma}^{BS} := \partial \) has dimension \( d - 1 \) and \( e'(P) \to (d - a) \), where \( a = \dim(A_p/A_G) \). Thus \( c = a - 1 \). The map \( j_* \) is defined by Poincaré duality:

\[
(j_* \alpha, \beta)_\partial = (\alpha, j^* \beta)_{e'(P)}.
\]

We forego coefficient systems and consider cohomology with complex coefficients. Here \( \alpha \in H^i_c(e'(P)) \), \( \beta \in H^{d-a-i}(\partial) = H^{\dim(\partial) - c - i}(\partial) \). In order to imitate the previous argument, we would need to consider

\[
H^i_c(e'(P)) \to H^{i+c}(\partial) \to H^{i+c}(e'(P)).
\]

However:

**Proposition 5.1** Assume Conjecture 5.2 below. Assume \( P \) is not maximal. Then \( j^* j_* = 0 \).

This will follow from the following conjecture. Assume \( P = P_1 \subset P_2 \subset P_3 \cdots \subset P_a = Q \) is a sequence of parabolic subgroups, with \( a_i+1 = a_i - 1, a_i = \dim(A_{P_i}/A_G) \), and \( Q \) maximal. Consider the associated embeddings \( e'(P_i) \subset e'(P_{i+1}) \).

**Conjecture 5.2** (i) The embedding \( e'(P_i) \to \overline{e'(P_{i+1})} \subset \partial \) is \( C^\infty \)-homotopic to a smooth immersion \( e'(P_i) \to e'(P_{i+1}) \).

(ii) Consider the composed immersion \( j : e'(P) \to \partial \). Then \( j \) is homotopic within \( \partial \), and in fact within \( \overline{e'(Q)} \), to a map (in fact a smooth immersion) \( k \) such that \( \text{Im}(j) \cap \text{Im}(k) = \emptyset \). If \( \omega \subset \text{Im} j \) is compact, we can even assume that the closure of \( \text{Im}(k) \) does not meet \( \omega \).

According to Leslie Saper, this is extremely probable. However a proof has not been obtained yet.

We will construct \( j_* \alpha \) that is “supported on \( \text{Im} j \)”, implying that \( j^*(j_* \alpha) = k^*(j_* \alpha) = 0 \). We can proceed as follows.

Let \( \tilde{\alpha} \) be a closed, compactly supported form on \( e'(P) \) representing \( \alpha \). Assume its support is contained in an open subset \( U \subset e'(P) \) with compact closure. Since \( e'(P) \to \partial \) is obtained from a sequence of immersions in codimension 1, the normal bundle to \( e'(P) \) in \( \partial \) is trivial. Therefore there exists a neighbourhood \( V \) of \( U \) in \( \partial \), with compact closure, and a diffeomorphism \( U \times I^c \sim V, I = [-1, 1] \). We can assume that \( V \) does not meet \( \text{Im} (k) \). Let \( (y_r), r = 1, \ldots c, \) be the coordinates on \( I^c \).

Consider the current on \( V \):

\[
\tilde{\gamma} = \delta_0(y)\tilde{\alpha} \wedge dy_1 \wedge \cdots \wedge dy_c
\]
where \( \int_c \delta_0(y)\varphi(y)dy = \varphi(0) \). It is then easy to see that for any closed form \( \tilde{\beta} \) on \( \partial \), of degree \( d - a - i \),

\[
\int_\partial \tilde{\gamma} \wedge \tilde{\beta} = \int_{e'(P)} \tilde{\alpha} \wedge j^* \tilde{\beta}.
\]

We can approximate \( \tilde{\gamma} \), as a closed current with compact support in \( V \), by closed forms \( \tilde{\theta} \). We obtain cohomology classes arbitrarily close to \( j_*\alpha \); \( j_*H^i_c(e'(P)) \) being finite–dimensional, we see that we can so obtain this whole space. Clearly the forms \( \tilde{\theta} \) verify \( k^*\tilde{\theta} = 0 \); since \( j \) and \( k \) are homotopic this implies \( j^*j_* = 0 \).

**Remark** With a more thorough argument using currents, it may be possible to dispense with the second part of Conjecture 5.2. However we think that Proposition 5.1 should remain true even with an arbitrary system of coefficients - at least a field \( k \), as in Sect. 2, using Verdier duality. In this case the argument using the deformed embedding may be necessary. This is left to the reader.

5.2

Although this may be obvious, we remark that we cannot use our construction inductively to obtain cohomology for \( G \) from the cohomology of a Levi subgroup. Indeed, we had to start with classes in \( H^*_c(S^P_{K'}) \). The “Eisenstein” classes we constructed in \( H^*(S^G_K) \) come from the boundary, cf. (2.9) ; if they occur in \( H^i(S^B_K) \) they do not belong to \( H^*_c \); if they lie in \( H^i+1(S_K) \) they are sent to 0 in \( H^{i+1}(S^B_K) = H^{i+1}(S_K) \).

Now assume \( G \) itself is a Levi subgroup of a maximal parabolic subgroup of a reductive \( \mathbb{Q} \)-group \( H \). In order to obtain a character of the Hecke algebra of \( H \), associated to the character of \( H^S(P) \) (or \( H^S(M) \)) associated to our original class, we would need to start with a class in the inner cohomology of \( S^G_K \); this is not the case. We cannot use “induction by stages” for a chain of parabolic subgroups!

**Appendix : Poincaré duality for small Gorenstein rings of coefficients**

For clarity of notation, we denote here by \( R \) a local Artin ring that is Gorenstein and by \( k = R/m \) its residue field. The following theorem does not seem to figure in the literature, although (in fact for \( R = \mathbb{Z}/N \), \( N \) being prime to the characteristic) it is well–known for the étale cohomology of varieties over algebraically closed fields (e.g. [18, Cor. 11.2]).

**Theorem A1** Assume \( X \) is a manifold of dimension \( d \).

(i) There is a perfect pairing

\[
H^i_c(X, R) \times H^{d-i}(X, R) \rightarrow R
\]

(ii) If \( X \) is compact, this yields a perfect pairing

\[
H^i(X, R) \times H^{d-i}(X, R) \rightarrow R.
\]

(This is used in the proof of Lemma 2.1).
We first recall some properties of such rings.

\[
\text{Ext}^i_R(k, R) \cong k \ (i = 0) \\
= 0 \ (i > 0). \tag{1}
\]

Let inj dim \( M \) denote the injective dimension of a \( R \)--module \( M \). Then

\[
\text{inj dim } M = 0 \iff \text{Ext}^i_R(k, M) = 0 \ (i > 0). \tag{2}
\]

For (1) see [17, Thm 18.1]. For (2), [17, Lemma 1 p. 139]. In particular, in view of (1), (2) applies to \( M = R \); this implies that \( R \) is injective as a \( R \)--module. \( \tag{3} \)

Finally, let \( E(k) \) be the injective envelope of \( k \) as a \( R \)--module.

\[
E(k) \cong R. \tag{4}
\]

This follows from [17, Thm. 18.4] and the fact that \( R \) is an indecomposable \( R \)--module.

Let \( M \) be a finite \( R \)--module. Then (4) implies :

The pairing \( M \times \text{Hom}_R(M, R) \rightarrow R \)

\[
(x, f) \mapsto f(x) \tag{5}
\]

is non–degenerate.

See [17, Thm 18.6].

Now Theorem A1 follows easily from these facts and the next result. (Compare Weibel [30, 3.6.5].

**Theorem A2** Let \( P_\bullet \) be a chain complex of projective \( R \)--modules. Then, for any \( n \), there exists an isomorphism

\[
H^n(\text{Hom}_R(P, R)) \cong \text{Hom}_R(H_n(P), R).
\]

**Proof** Consider the short exact sequences

\[
0 \rightarrow Z_n \rightarrow P_n \rightarrow dP_n \rightarrow 0,
\]

whence

\[
0 \rightarrow \text{Hom}(dP_n, R) \rightarrow \text{Hom}(P_n, R) \rightarrow \text{Hom}(Z_n, R) \rightarrow 0
\]

since \( \text{Ext}^1(dP_n, R) = 0 \) as \( R \) is injective. This defines an exact sequence of (cochain) complexes, with the dual differential. Consider the cohomology. We obtain exact
sequences:

\[ \cdots \to H^{n-1}(\text{Hom}(Z, R)) \to H^n(\text{Hom}(dP, R)) \to H^n(\text{Hom}(P, R)) \to H^n(\text{Hom}(Z, R)) \to H^{n+1}(\text{Hom}(dP, R)) \to \cdots \]

and the differentials for the complexes $Z$ and $dP$ being zero:

\[ \cdots \to \text{Hom}(Z_{n-1}, R) \to \text{Hom}(dP_n, R) \to H^n(\text{Hom}(P, R)) \to \text{Hom}(Z_n, R) \to \text{Hom}(dP_{n+1}, R) \to \cdots \]

However, the exact sequence

\[ 0 \to dP_{n+1} \to Z_n \to H_n(P) \to 0 \tag{6} \]

yields

\[ 0 \to \text{Hom}(H_n(P), R) \to \text{Hom}(Z_n, R) \to \text{Hom}(dP_{n+1}, R) \to 0 \]

since $\text{Ext}^1(H_n(P), R) = 0$; shifting the indices by 1 in (6) we also get

\[ \text{Hom}(Z_{n-1}, R) \to \text{Hom}(dP_n, R) \to 0. \]

Finally, the last long exact sequence implies

\[ H^n(\text{Hom}(P, R)) \approx \text{Hom}(H_n(P), R). \]

It is now easy to derive Theorem A1. We take $P_\bullet = S_\bullet(X)$ be the simplicial chain complex of $X$ with coefficients in $R$; $\text{Hom}(P_\bullet, R)$ is the simplicial cochain complex. We have the isomorphisms

\[ H^i(X, R) \cong \text{Hom}(H_i(X, R), R), \alpha \mapsto h_\alpha \]

\[ D : H^i_c(X, R) \cong H_{d-i}(X, R), \beta \mapsto D\beta, \]

and we set for $\alpha \in H^i, \beta \in H^{d-i}_c$:

\[ <\alpha, \beta> = h_\alpha(D\beta) \in R. \]

Property (5) now implies that the pairing is non-degenerate (in both variables).

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