Research on related properties of two types of self-similar networks

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Abstract: Self-similarity is the whole and the part of a complex system, that is, the similarity of structure or property between the part and that part. Studying self-similar networks is helpful for us to better understand the complex networks in the real world. Because the fractal networks have self-similarity, a type of modified Koch network and a type of Austria network were firstly described by this paper, and the exact expressions of degree distribution and aggregation coefficient of the two types of networks were secondly derived, finally, the relationship between the Randic indicator of the two types of network and other invariants is studied.

1 Introduction

Networks are universal in nature and in human society, such as galaxies in nature[1], food chain networks[2-4], protein networks[5], and social networks[6] human society, infectious disease networks[7], traffic networks[8], information and communication networks[9], banking networks[10]. The world of human existence has gradually evolved into a complex network world, the more developed and effective this network, the smaller the world, the more and more human sociality is strengthened. Therefore, the study of complex networks has its important significance.

In 1960, the ER random network evolution model proposed by Hungarian mathematicians ErdEs and Renyi, which was thought to have mathematically pioneered a systematic study of complex network theory, which researchers have been working on for nearly 40 years[11]. At the end of the 20th century, the study of complex network theory entered a new stage of development, and two seminal articles once again set off a boom in the study of complex networks. One of them is the article "collective dynamics of the" small world "network" published by Watts and Strogatz in nature in 1998[12], which proposed a small world network model to describe the transition from a completely regular network to a completely random network. Small-world networks have both clustering characteristics similar to regular networks and smaller average path length similar to random networks, which can describe the nature that most nodes in larger networks have shorter paths. Another one is the article "Emergence of scaling in random networks", which is published in Science in 1999 by Barabási and Albert[13], pointing out that the degree distribution of many real complex networks has power-law form. Since the power law distribution has no obvious characteristic length, this kind of network is called scale-free network. Research shows that many complex networks in the real world are small world or scale-free networks.

Although many network evolution models have been used to analyze and study the potential evolution laws, these studies still ignore some important factors. For example, a connection between computer network nodes. If the connection probability is based on the preferred one, the new nodes will all be connected to the same node, but the real network is not so, but forms different hub-nodes.
This example illustrates that the connection between nodes may be based on some similar properties, and that there is some commonality between the nodes. Therefore, establishing and studying the network evolution model based on similarity is helpful for us to better understand the complex network in the real world.

Self-similarity [14] refers to the similarity of the characteristics of a structure or process from different spatial perspectives or time scales, or the similarity of the local nature or structure of a system or structure with the whole. Its main properties are the same properties at different scales in the system. The characteristic analysis of self-similar networks is to explain the micro-evolution process of network by using the interactive characteristics between network nodes. The fractal and self-similar characteristics of complex networks are some similarities between the whole and part, part and part of complex networks as they evolve into small-world networks. At present, the internal mechanism of self-similar structure and the micro-factors that determine the self-similar of network semblance have become an important part of the complex network evolution model which needs to be studied. It has been concluded that fractal networks are always self-similar, but self-similar networks are not always fractal[15].

The study of self-similar network is beneficial to us to better understand the real complex networks, therefore, in this paper, according to the theory of complex network-related, firstly, the average path length, degree distribution, network diameter and aggregation coefficient index of the fractal network Austria graph and improved koch graph in complex systems are derived and calculated. Then, the two network Randic indicators are given. Finally, the relevant conclusions are drawn.

2. Study on the related nature of the Austria network

2.1 The process of building the Austria network

Let $G = (VG, EG)$ is a finite (multi) graph, where $VG$ is the vertex set and $EG$ is the edge set. $s = |EG|$ is used to denote the number of edges of $G$, $d = dG(v, w)$ is used to denote the distance between points $v$ and $w$ of $G$, and $δ = |VG| - 1$ is used to denote the number of edges of the spanning tree of $G$.

When $t = 0$, the network is a triangle made up of three nodes and three sides, and when $t \geq 1$, $X_t$ is generated by $X_{t-1}$ through the following methods:

A new node is inserted in the middle of each edge in network $X_{t-1}$, and a new node is added to each edge, which is connected with an endpoint and an insertion point of this edge. According to the time step, a self-similar network graph is constructed whose sequence is $X_0, X_1, X_2, X_3$. As shown in Figure 2-1, which is the Austria network graph [16]:

![Figure 1. Austria network diagram from $X_0, X_1, X_2$ to $X_3$](image)

Nextly, we use the related research methods of complex network to deduce and calculate some related properties of Austria network.
2.2 Topological metrics for the Austria network

(1) Degree distribution

The degree distribution of nodes in the Austria network is denoted by the distribution function $P(k)$, which is defined as the probability that a node of any selection has exactly $k$ edges, that is, the ratio of the number of nodes with degree $k$ in the network to the total number of nodes in the network. The degree distribution of nodes is an important statistical feature of the network. The definition is as follows:

The number of edges of Austria network: $E_t = \frac{64}{2} = 32$.

The number of nodes of Austria network: $V_t = 2 + 4 + 12 + 44 = 44$.

Average degree of Austria network: $\bar{k}_t = \frac{2E_t}{V_t} = \frac{64}{2} = 32$.

When $t \to \infty$, $\bar{k}_t \approx 3$.

According to the fractal situation of Austria network, its degree distribution is as follows: $P_t(k)$ values of $k$ from 1 to 5 are calculated as follows:

When $k = 1$, $P_t(1) = \frac{3}{2 \cdot 4^t + 4}$.

When $k = 2$, $P_t(2) = \frac{3 \cdot 4^{t-1} + 3}{2 \cdot 4^t + 4}$.

When $k = 3$, $P_t(3) = \frac{3 \cdot 4^{t-1}}{2 \cdot 4^t + 4}$.

When $k = 4$, $P_t(4) = \frac{4^{t-1} - 1}{2 \cdot 4^t + 4}$.

When $k = 5$, $P_t(5) = \frac{4^{t-1} - 1}{2 \cdot 4^t + 4}$.

(2) Clustering coefficient

The clustering coefficient $C$ of the Austria network is the average value of the clustering coefficient $C_i$ of all nodes, and the clustering coefficient $C_i$ of node $i$ is the ratio of the actual weight of edges of the neighbor of $i$ to the maximum weight of edges of the neighbor of $i$.

According to the generation and evolution model, the calculation results of $C(t)$ are as follows:

$$C(t) = \frac{0.1 + 1 \cdot (4^{t-1} + 1) + \frac{1}{3} \cdot 4^{t-1} + \frac{1}{3} \cdot \frac{1}{3} \cdot (4^{t-1} - 1) + \frac{1}{5} \cdot (4^{t-1} - 1) \cdot V_t}{30 \cdot 4^t + 60} = 68 \cdot 4^{t-1} + 37 \cdot 30 \cdot 4^t + 60$$

When $t \to \infty$, $C(t) \approx \frac{17}{30} = 0.57$.

The clustering coefficient of the Austria network is 0.57, which conforms to the characteristics of the small world network.

(3) Diameter

The diameter of Austria network is the maximum distance of all nodes, which measures the maximum information transmission delay of the network. Its calculation method is as follows:
Define the diameter of time $t$ as $D_t$, the node generated at $t = 0$ is the initial node, and the node generated at the $t$-moment is the new node, and get the diameter of Austria network as follows:

$$D_0 = 1, D_1 = 2, D_2 = 5, D_3 = 11, D_4 = 22, D_5 = 43.$$  

According to the above results, the recurrence formula of the diameter of Austria network can be expressed as follows:

$$D_t = \begin{cases} 
    \frac{1}{2} D_{t-1} + 2', & \text{when } D_{t-1} \text{ are even numbers} \\
    \frac{1}{2} (D_{t-1} + 1) + 2', & \text{when } D_{t-1} \text{ are odd numbers} 
\end{cases}$$

It can be seen from the results that as $t$ increases, $D_t$ tends to increase gradually.

2.3 Randic index of Austria network

The topological structure of molecules determines many properties of molecules, including not only the properties related to the size and shape of molecules, such as molecular volume, boiling point, solubility, refractive index, etc., but also the quantum mechanical properties of molecules, such as energy level, electronic density, etc., which are essentially dependent on the connectivity of atoms. Therefore, it is of great significance to give a quantitative standard (measure) reflecting the basic characteristics of molecular structure, which is usually called topological index. To some extent, these indexes reflect not only the size and shape of molecules, but also the connection mode (connectivity) of their atoms. Among many topological indexes involved in chemical graph theory, Randic index, Hosoya index, Merrifield Simmons index and eigenvalue are common. Randic index reflects the number of branches in chemical species, which was proposed by Randic. In recent years, there have been many studies on the properties of Randic index [17]. In particular, bollos and Erdös prove that all connected graphs have the minimum Randic value [18]. Randic index is closely related to some physical and chemical properties, including boiling point, chromatographic retention value, enthalpy of formation, surface area, etc [17].

Here we mainly discuss Randic index, which are defined as follows:

$$R(G) = \frac{1}{\sum_{uv \in E(G)} d(u) \cdot d(v)}.$$  

According to the above definition, the approximate results of $R_0, R_1, \ldots, R_t$ can be calculated as follows:

$$R_0 = \frac{1}{\sqrt{1 \cdot 1}} = 1,$$

$$R_1 = \frac{1}{\sqrt{1 \cdot 3}} + \frac{2}{\sqrt{3 \cdot 2}} + \frac{1}{\sqrt{2 \cdot 2}} = \frac{1 + \sqrt{2}}{\sqrt{3}} + \frac{1}{2} \approx 1.893;$$

$$R_2 = 4(R_1 - \frac{1}{\sqrt{1 \cdot 3} + \frac{2}{\sqrt{2 \cdot 3}}}) = \frac{1}{\sqrt{1 \cdot 3}} + \frac{2}{\sqrt{2 \cdot 3}} + \frac{3}{\sqrt{3 \cdot 5}} + \frac{2}{\sqrt{2 \cdot 4}} + \frac{2}{\sqrt{3 \cdot 4}} + \frac{2}{\sqrt{2 \cdot 3}};$$

Substitution $R_1$ to $R_2$:

$$R_2 = \frac{2 + 3\sqrt{3}}{\sqrt{3}} + \frac{\sqrt{5} + 3}{\sqrt{5}} + \frac{\sqrt{2}}{2} \approx 2 + 2.422 + 1.407 + 0.707 = 6.236;$$

$$R_3 = 4(R_2 - \frac{1}{\sqrt{1 \cdot 3} + \frac{2}{\sqrt{2 \cdot 3}}}) = \frac{1}{\sqrt{1 \cdot 3}} + \frac{5}{\sqrt{3 \cdot 5}} + \frac{4}{\sqrt{3 \cdot 4}} + \frac{2}{\sqrt{2 \cdot 3}} \approx 22.871;$$
3. Research on the related properties of the improved Koch network

3.1 Improved Koch network generation process

When \( t = 0 \), the network is made up of two nodes and one side, and when \( t \geq 1 \), \( X_t \) is generated by \( X_{t-1} \) through the following methods:

In network \( X_{t-1} \), firstly, an edge is changed to the three-equidistant point spline into four nodes at \( t = 0 \), in which a vertex is added to the second and third nodes. According to the time step, a self-similar network graph is constructed whose sequence is \( X_0, X_1, X_2, X_3 \). As shown in Figure 3-1, which is the improved Koch network graph:

Then, we can know the follows:

The number of edges: \( E_0 = 1, E_1 = 5, E_2 = 25, E_3 = 125, |E_t| = 5^t \).

The number of nodes: \( V_0 = 2, V_1 = 5, V_2 = 20, V_3 = 95, |V_t| = \frac{1}{4}(3 \cdot 5^t + 5) \).

3.2 Topological index of the improved Koch network

(1) Degree distribution

Average degree: \( \bar{k}_t = \frac{2E_t}{V_t} = \frac{8 \cdot 5^t}{3 \cdot 5^t + 5} \), when \( t \to \infty \), \( \bar{k}_t \approx \frac{8}{3} \approx 2.67 \).

Degree distribution: When \( k = 1, 2, 3 \), the values of \( P_t(k) \) are as follows:

When \( k = 1 \), \( P_t(1) = \frac{8}{3 \cdot 5^t + 5} \).

When \( k = 2 \), \( P_t(2) = \frac{5^{t-1} - 1}{3 \cdot 5^t + 5} \).
When \( k = 3 \), \( P_r(3) = \frac{2 \cdot 5 - 2^r}{3 \cdot 5^r + 5} \).

(2) Clustering coefficient

\[
C(t) = \frac{0 \cdot 2 + 1 \cdot \frac{1}{4} (5^r - 1) + \frac{1}{3} \cdot \frac{1}{2} (5^r - 1)}{V_t} = \frac{4 \cdot 5^r}{9 \cdot 5^r + 15},
\]

When \( t \to \infty \), \( C(t) \approx \frac{4}{9} \approx 0.44 \).

(3) Diameter

According to the improved Koch network generation model, define the diameter of time \( t \) as \( D_t \), the node generated at \( t = 0 \) is the initial node, and the node generated at the \( t \)-moment is the new node, and get the diameter of Austria network as follows:

\[
D_0 = 1, D_1 = 3, D_2 = 9, D_3 = 27.
\]

Based on the results above, you can get the following expression:

\[
D_t = 3^t,
\]

When \( t \) is more larger, the diameter grows exponentially in the form of 3.

3.3 Randic index of the improved Koch network

According to the above definition, the approximate results of \( R_0, R_1, \ldots, R_t \) can be calculated as follows:

\[
R_0 = \frac{1}{\sqrt{1 \cdot 1}} = 1,
\]

\[
R_1 = \frac{2}{\sqrt{1 \cdot 3}} + \frac{2}{\sqrt{3 \cdot 3}} + \frac{2}{\sqrt{3 \cdot 2}} = \frac{2 + \sqrt{2}}{\sqrt{3}} + \frac{1}{3} \approx 2.304,
\]

\[
R_2 = 2(R_1 - \frac{1}{\sqrt{1 \cdot 3}}) + 3(R_1 - \frac{2}{\sqrt{1 \cdot 3}}) + \frac{6}{\sqrt{3 \cdot 3}} + \frac{2}{\sqrt{3 \cdot 2}} = 5R_1 + 2 + \frac{\sqrt{2} - 8}{\sqrt{3}},
\]

Substitution \( R_1 \) to \( R_2 \):

\[
R_2 \approx 9.717,
\]

\[
R_3 = 2(R_2 - \frac{1}{\sqrt{1 \cdot 3}}) + 3(R_2 - \frac{2}{\sqrt{1 \cdot 3}}) + \frac{6}{\sqrt{3 \cdot 3}} + \frac{2}{\sqrt{3 \cdot 2}} = 5R_2 - \frac{8}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}} + 2 \approx 48.82,
\]

\[
R_t = 5^{t-1} R_1 - \frac{\sqrt{3}}{2} (5^{t-1} - 1) + \frac{5^{t-1} - 1}{2} = 5^{t-1} \left( \frac{5 + \sqrt{3} + 2\sqrt{6}}{6} \right) + \frac{\sqrt{3} - 1}{2} \approx 5.82 \cdot 5^{t-1}.
\]

4. The research conclusion

In this paper, the production process of the two regular fractal networks, which are the Austria network and the improved Koch network, the characteristics of degree distribution, aggregation coefficient, network diameter and Randic indicators were studied, and the following conclusions were obtained:

(1) Austria network

The number of edges: \( E_t = 4^t \)

The number of nodes: \( V_0 = 2; V_1 = 4; V_2 = 12; V_3 = 44; V_t = 4V_{t-1} - 4 = \frac{2}{3} \cdot 4^t + \frac{4}{3} \)
The average degree: when $t \rightarrow \infty$, $\overline{k} \approx 3$.

Clustering coefficient: when $t \rightarrow \infty$, $C(t) \approx \frac{17}{30} \approx 0.57$.

Diameter: $D_0 = 1; \ D_1 = 2; \ D_2 = 5; \ D_3 = 11; \ D_4 = 22; \ D_5 = 43$

The recurrence formula: $D_t = \begin{cases} \frac{1}{2}D_{t-1} + 2^t, & \text{when } D_{t-1} \text{ are even numbers} \\ \frac{1}{2}(D_{t-1} + 1) + 2^t, & \text{when } D_{t-1} \text{ are odd numbers} \end{cases}$

(2) **The improved Koch network**

The number of edges: $|E_t| = 5^t$.

The number of nodes: $|V_t| = \frac{1}{4}(3 \cdot 5^t + 5)$.

The average degree: when $t \rightarrow \infty$, $\overline{k} \approx \frac{8}{3} \approx 2.67$.

Clustering coefficient: when $t \rightarrow \infty$, $C(t) \approx \frac{4}{9} = 0.44$.

Diameter: $D_0 = 1; \ D_1 = 3; \ D_2 = 9; \ D_3 = 27$

The recurrence formula: $D_t = 3^t$

(3) **The comparison of Randic indicators is as shown in table 4-1:**

| Randic index, $R_i$ | Austria network | Improved Koch network | Comparison results |
|---------------------|-----------------|-----------------------|--------------------|
| $R_0$               | 1               | 1                     | Equal              |
| $R_1$               | 1.893           | 2.304                 | Increase           |
| $R_2$               | 6.236           | 9.717                 | Increase           |
| $R_3$               | 22.871          | 48.82                 | Increase           |
| $R_t$               | $143 \times 4^t - 3$ | $5.82 \times 5^t - 1$ | When $t$ is greater than 3, Koch network is much larger than Austria network |

Both the Austria network and the improved Koch network have a aggregation coefficient greater than one constant (0.57 and 0.44), which is in line with the small world network of about 0.5. With the increase of the number of nodes, the network diameter is linear and exponential, and both have the characteristics of small world network, so that the comparative characteristics of Randic indicators are as shown above. It can be seen that with the increase of $t$, the connectivity of the improved koch network is better than that of the predecessor.

According to the calculation, the diameter expressions of these two types of networks are as follows:
the diameter of Austria network:
\[
D_i = \begin{cases} 
\frac{1}{2} D_{i-1} + 2', & \text{when } D_{i-1} \text{ are even numbers} \\
\frac{1}{2} (D_{i-1} + 1) + 2', & \text{when } D_{i-1} \text{ are odd numbers}
\end{cases}
\]

the diameter of improved Koch network:
\[D_i = 3^i\]

Through the deduction of the above nature, the two network structures have a certain understanding, and the relevant results are obtained, but if there are key edges or key node failures in the whole self-similar fractal structure, the specific evolution behavior and related properties of its changes need to be further studied.

**Acknowledgments**

This work was supported by the National Science Foundation of Qinghai Province of China (2018-ZJ-777)

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