Visual and kinesthetic alleys formed with rods

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ABSTRACT
Geometries of visual and kinesthetic spaces were estimated by alley experiments. For the visual alley, 24 observers set rods that extended in depth so that they appeared 1) to neither diverge nor converge, 2) to be separated by the same lateral distance, or 3) to be perpendicular to the frontal plane. The separation of rods and the height of the observer’s eyes were varied. Under each instruction, another group of 20 observers set the rods visually at eye level or kinesthetically without seeing the rods. We obtained these findings. First, the rods seen obliquely from above were set more accurately than the rods seen at eye level. Second, the visual settings were parallel to one another for small separation and were convergent to the observer for large separation, whereas the kinesthetic settings were divergent to the observer for the small separation and were convergent to the observer for the large separation. These differences between sense modalities were explained by the location of the egocenter(s) and the sensitivity to direction. Third, the visual or kinesthetic settings did not differ with instructions, suggesting that visual and kinesthetic spaces were Euclidean. Fourth, the visual angle of the near ends of the rods, plotted against that of the far ends, was described by Euclidean geometry, provided that the visual angle is exaggerated. Last, the kinesthetic angle of the near ends of the rods, plotted against that of the far ends, was not described by any simple geometry even when we assumed that the kinesthetic angle is exaggerated.

1 Introduction
This study concerns with apparent parallelism in depth. Experiments on this phenomenon have been termed alley experiments in which observers adjusted two lines of light points extending in depth. Traditionally, there are two criteria on which the observers base their setting in adjusting the lines. One is that the lines appear not to meet, neither forwards nor backwards, if they are extended infinitely. This is called the parallel alley. The other is that the lines appear equal in lateral separation. This is called the equidistant alley, according to conventional usage. With either criterion, the lines to be adjusted should appear symmetric to the median plane.

The study of alley experiments can be traced back to experimental psychology in the early twentieth century (Hillebrand, 1902; Blumenfeld, 1913). Blumenfeld compared the parallel and the equidistant alleys and found that the parallel alleys lay inside the equidistant ones. Subsequently, alley experiments were used by Luneburg (1947, 1950) and Blank (1953, 1978, 1958a, 1958b) to estimate the geometry of visual space. They assumed that visual space is a Riemannian space of constant curvature (i.e., a homogeneous space), and that the geometry of visual space can be determined by relative position of the parallel and the equidistant alleys. If the parallel and the equidistant alleys coincide with one another, the space is Euclidean (i.e., zero curvature); if the parallel alleys lie inside the equidistant alleys, the space is hyperbolic (i.e., negative curvature); if the parallel alleys lie outside the equidistant alleys, the space is elliptic (i.e., positive curvature).

The Luneburg-Blank model has been challenged theoretically and empirically. It is axiomatically defined that, if visual space is homogeneous, it is locally Euclidean and allows free mobility (Blank, 1958a). By making the observers construct right triangles at the horizontal eye level, Foley (1972) examined these axioms and failed to confirm them. From this result, Foley concluded that visual space is not homogeneous, implying that there is no simple geometry to describe visual space. Indow (1991), who believed the Luneburg-Blank model without a great modification, disputed Foley’s view by saying that it is difficult to construct right triangles at the horizontal eye level and by stating that the test of free mobility he used is not sufficient and other interpretations are possible.

Another theoretical problem may come from the criteria of parallelism (Shipley, 1957a, 1957b) or ambiguity of visual parallelism (Pirenne, 1970). Hardy et al. (1953) proposed three criteria for
constructing the parallel alleys: (1) two lines should have the same
direction, appearing parallel to the median plane, (2) two lines should
appear perpendicular to the frontal plane, and (3) two lines should
appear to neither converge nor diverge in the distance. In previous
studies, all criteria of parallelism were not told to observers. Parallel
lines were described just as uncrossed lines (Blumenfeld, 1913; Hardy,
Rand, & Rittler, 1951; Indow, Inoue, & Matsushima, 1962; Indow &
Watanabe, 1984a; Shipley, 1957b; Zajaczkowska, 1956), as lines in the
same direction (Blank, 1953), and as a mixture of same-direction lines
and perpendicular lines Blank, 1958b). Battro et al. (1976) and Ishii
(1965) did not define any operational criterion of parallelism in their
instructions.

Partially due to the various criteria of parallelism, the relative re-
lation of the parallel and the equidistant alleys may remain unsettled,
despite the evidence collected over a hundred years. Zajaczkowska
(1956) and Indow et al. (1962) confirmed Blumenfeld’s results, and the
same relative relation of the alleys held true for moving light points
(Indow & Watanabe, 1984a). However, Hardy et al. (1951) and Ishii
(1965) found that about half of their observers set the parallel alleys
inside the equidistant ones, and that the others set the parallel alleys
outside the equidistant ones. Similar evidence was obtained in large
open fields (Battro et al., 1976). Interestingly, on the frontoparallel
plane the discrepancy between the parallel and the equidistant alleys
was not unequivocal (Indow & Watanabe, 1984b) and the curvature on
that plane was zero (Indow and Watanabe, 1988).

In addition to the criteria of parallelism, viewing attitude (apparent
attitude versus objective attitude) contributed to large individual dif-
ferences in alley experiments. Higashiyama et al. (1990) demonstrated
that (1) the alleys for the apparent instructions were positioned closer
to the median plane than those for the objective instructions, (2) in an
indoor setting, the uncrossed and the equidistant alleys were not dif-
ferent under either the apparent or the objective instructions, and (3) in
an outdoor setting, the uncrossed alleys lay inside the equidistant alleys
under both the apparent and the objective instructions.

Some stimulus variables may also affect the relative position of al-
leys. Probably, as the distance or depth cues are rich, the geometry of
visual space approaches Euclidean geometry, as the physical space in
which we are alive (Gibson, 1950). Although early studies of alley ex-
periments were often done with light points presented on the horizontal
eye level plane in otherwise total darkness, visibility of the floor may be
critical to alley perception, as it is to distance perception (Bian et al.,
2005). If alleys are constructed with long sticks extending in depth on
the visible floor, the uncrossed and the equidistant alleys may be close
together.

The size of the stimulus layout may affect the relative location of al-
leys. If the curvature of visual space is constant for a given layout,
then it has been shown that, as the layout increases in visual angle
(Higashiyama, 1981, 1984) and in depth (Koenderink et al., 2000), the
curvature changes from positive to negative through zero.

Spatial features are perceived not only through vision but also
through kinesthetic senses; this means that visual spatial features are
represented kinesthetically. Visual inclination of a line or plane, for
example, was reproduced by rotating a paddle manually (Feresin and
Agostini, 2007; Feresin et al., 1998; Proffitt et al., 1995) and by tilting
the head (Higashiyama and Murakami, 2015; Li and Durgin, 2009) or
body (Higashiyama and Koga, 1998). After having studied visual alleys
in 1913, Blumenfeld (1936) studied kinesthetic alleys, in which two
threads were anchored to needles placed in front of the blindfolded
observer and equidistant from the median plane and were arranged so
that the threads were felt to be parallel to one another and to the
median plane. Then, the kinesthetic alleys were arranged to be di-
vergent to the observer, which was in contrast with the visual alleys
that were convergent to the observer. However, it is not possible from
these data to estimate the geometry of kinesthetic space, because the
study did not report kinesthetic equidistant alleys.

With the history of alley experiments in mind, in Experiment 1, we
compared the visual uncrossed, equidistant, and perpendicular alleys.
The experiment was conducted, in a well-lit office where people work,
walk, and sit, with two rods extending in depth. This was done because
the lines made by rods seems to be ecologically more suitable in normal
life than those made by a series of light points in a dark room. We
controlled the observer’s viewing attitude under the apparent instruc-
tions. We also examined how the uncrossed, equidistant, and perpen-
dicular alleys interact with stimulus parameters including visibility of
the floor and the lateral separation of rods. In Experiment 2, kinesthetic
uncrossed, equidistant, and perpendicular alleys were constructed and
were compared with the visual alleys arranged separately.

2. Experiment 1

2.1. Method

2.1.1. Observers

The observers were 23 students (3 males and 20 females) and one
professor who were recruited at the University of California at Santa
Barbara. All observers provided their informed consent for voluntary
participation in experiments. The experiments were carried out in ac-
cordance with the Code of Ethics of the World Medical Association –
Declaration of Helsinki and were authorized by the university’s ethical
committee for human experiments. The students received course credit
of psychology for their participation.

2.1.2. Apparatus

The apparatus is illustrated in Fig. 1. There were two rods (1.8 cm
wide × 217 cm long) that were supported by placing one box under the
far ends of the rods and another box under a midway point between the
observer and the far ends. Each far end was loosely fixed with a bolt on
a circular board of about 1-m diameter that was placed on the far box.
The left and right far ends were symmetric with respect to the median
plane and the lateral separations between the far ends were 16, 38, and
70 cm. Each near end of the rods was covered with a ping pong ball of
3.6-cm diameter. The observer was able to grasp the left ball with the
left hand and the right ball with the right hand, move the balls laterally,
and change direction of the rods. The distances of the far and near ends
from the observer were 242.7 cm and 27.7 cm, respectively. The ob-
server’s eyes were at the same level as the rods or were 10 cm above the
rods by changing the height of the chin rest and chair. A scale was
pasted on the board to read the distance from the median plane to the
point (i.e., Y coordinate) at which each rod intersected the scale with an
accuracy of 1 mm.

2.1.3. Procedure

When the observer came to the laboratory, the appropriate height of
the chin rest was set before the experiment started. We then explained
the rod system by reading: “You now see two long rods in front of you.
The far ends of the rods are fixed. You can move the near ends of the
rods horizontally. Please grasp the near end of the right rod with your
right hand and the near end of the left rod with your left hand, and try
moving the rods sideways. I will ask you to adjust the positions of the
rods so that they produce specific percepts. During settings, you must
keep your head facing straight ahead, but you should move your eyes
freely. You must keep your head on the chin rest, and you must not turn
it.”

In each of the tasks that the observer performed, the observer was
asked to set the rods so that they appear in a specified relation. In the
instructions, we emphasized the apparent attitude of viewing:
“Although the way that the rods appear may or may not be the same as
how they are physically, settings should always be based on how the
rods appear to you.”

On each trial, the experimenter set the position of the rods back to
the starting point. The observer was asked to judge which direction the
rods had to be moved, to move the rods in that direction until their
appearance corresponded to the instruction for that task, and not to move back in the other direction. The experimenter noted the y-coordinate of the point at which each rod intersected the scale ($x = 2037$ mm for all trials).

There were three tasks. For each task, there were three separations of the far ends of the rods and for each separation there were two starting points: near ends widely separated and near ends close together. Each starting point was randomly repeated twice. There were also two eye heights: the same level as the rods and 10 cm above the rods. Hence, there were 144 trials ($3$ tasks $\times 2$ heights $\times 3$ separations $\times 4$ settings $\times 2$ lateralities) in all. The three tasks were presented in counterbalanced order with all the trials on one task being completed before starting the trials on the next task. In each task, the order of eye heights was counterbalanced among the observers.

The essential instructions for three tasks were:

1. Uncrossed alleys. The task is to set the rods so that they appear to be symmetric with respect to the median plane and so that they appear to neither diverge or converge, that is, they would not cross no matter how far they might be extended in either direction.
2. Equidistant alleys. The task is to set the rods so that they appear to be symmetric with respect to the median plane and so that they appear to be separated by the same lateral distance all along their length.
3. Perpendicular alleys. Imagine the frontal plane that goes through the centers of two eyes. With the head facing straight ahead, you will barely be able to see objects in this plane. We do not provide any visual cues as to where this plane is. You must imagine where it is. The task is to set the rods so that they appear to be perpendicular to the frontal plane, that is, so that they appear to form right angles with this plane.

After completing all tasks, the experimenter asked each observer whether the rods appeared to be straight or curved or their appearance was different in different conditions. When observer's answer was that their appearance was different in different conditions, the observer was asked to describe how their appearance differed with the condition. The experimenter next told the observer that if visual space can be described by Euclidean geometry, then the three tasks the observer just performed all require the same settings of the rods and then asked whether the observer recognized this requirement in the course of the experiment. When the observer answered yes, the experimenter asked when it was and whether, after the observer had noted this requirement, the observer consciously attempted to make the settings of the rods the same. The observer was also required to report the easiest and most difficult settings of the rods. Finally, the experimenter asked whether the observer thought different settings were made for different tasks. When the observer answered yes, the experimenter asked what task produced the narrowest or the widest settings.

2.2. Results

For each observer, we obtained 144 intersections ($3$ tasks $\times 2$ heights $\times 3$ separations $\times 4$ settings $\times 2$ lateralities) between rods and the scale. Each intersection was represented by cartesian coordinates $p(x, y)$, where $x$ stands for the axis that goes through the midpoints between the eyes and is perpendicular to the frontal plane, and $y$ stands for the axis that goes through the eyes (see Fig. 2).

For each combination of task, separation, height, and laterality, the $y$ values of the widening and narrowing settings were averaged for each observer. The individual slopes for each combination were then determined from the mean $y$ values:

$$\text{slope} = \frac{Y - y}{X - x}$$
and the coordinates of the far end of each rod, and x and y are the coordinates of the point at which the rod intersects the scale.

where \( X \) and \( Y \) are the coordinates of the far end of each rod, and \( x \) and \( y \) are the coordinates of the point at which the rod intersects the scale.

2.2.1. Slope analysis of whole data

Table 1 shows the mean slope and standard deviation (SD) taken over observers for each combination of task, height, laterality, and separation. Statistical tests were conducted by using the individual slopes as scores. A four-way repeated-measure ANOVA revealed that the main effect of task was not significant and the interaction of the task with other factors was not significant.

The separation × laterality interaction was significant, \( F(2, 46) = 247.34, p < .001 \). This interaction is illustrated in the left panel of Fig. 3: for the small and middle separations of rods, the mean slopes for the left and right rods were almost zero (i.e., precise settings), but for the large separation, the difference in the mean slope between the rods (left, 0.066; right, −0.057) was significant, \( F(1, 69) = 157.04, p < .001 \).

The height × separation × laterality interaction was significant, \( F(2, 46) = 6.13, p < .01 \), and the height × laterality interaction was significant, \( F(1, 23) = 31.27, p < .001 \). These interactions are illustrated in the left panels of Fig. 4. Although the mean slopes for the left and right rods were significantly different for the eyes at the same level as the rods (left, 0.024; right, −0.026), \( F(1, 46) = 29.07, p < .001 \), and for the eyes above the rods (left, 0.019; right, −0.016), \( F(1, 46) = 15.11, p < .001 \), the difference in slope was larger for the eyes at the same level as the rods. The mean slopes of the right rod for the large, \( F(1, 138) = 40.27, p < .001 \), middle, \( F(1, 138) = 7.15, p < .01 \), and small separations, \( F(1, 138) = 4.86, p < .05 \), and the mean slopes of the left rod for the large separation, \( F(1, 138) = 11.25, p < .01 \), were significantly different between eye heights. Hence, it is suggested that for the eyes at the same level as the rods, the rods were steeply convergent to the observer.

2.2.2. Slope analysis of limited data

Seventeen observers reported that the rods appeared to be straight at all times. Seven observers reported that the rods appeared to be curved depending on the condition: five observers reported that the middle parts of the rods they finally set appeared narrow like an hourglass, one reported that the middle parts appeared wide like a barrel, another reported that the middle parts appeared to be hanging down. The right panels of Figs. 3 and 4 indicate the results of the observers who perceived the rods to be straight. An ANOVA revealed that the separation × laterality interaction was significant, \( F(2, 32) = 156.27, p < .001 \), and the height × laterality interaction was significant, \( F(1, 16) = 24.69, p < .001 \). The main effect of task was not significant and neither was the interaction of the task with other factors. These results are similar to the results of the analysis of the whole data set.

We also conducted an ANOVA for the data of 15 observers who did not notice that all tasks required the same settings in Euclidean geometry (nine observers), or who, although noticing it, did not consciously make the settings of the rods the same (six observers). An ANOVA indicates that the separation × laterality interaction was significant, \( F(2, 28) = 321.71, p < .001 \), and the height × laterality interaction was significant, \( F(1, 14) = 16.83, p < .01 \), and the height × separation × laterality interaction was significant, \( F(2, 28) = 13.21, p < .001 \). The main effect of the task was not significant and neither was the interaction of the task with other factors. These results were very similar to the results of the analysis of the whole data set.

Table 1 shows the numbers of tasks that were chosen by observers as the easiest and the most difficult tasks. The uncrossed settings were most frequently indicated as the easiest task, and the effect was significant, \( \chi^2(2) = 10.75, p < .05 \), and the perpendicular settings were most frequently indicated as the most difficult task, \( \chi^2(2) = 4.75, 0.05 < p < .10 \), although the effect was not significant.

After completing the tasks, 12 observers reported that the settings would be approximately the same in all tasks, 11 observers reported that they would be different among the tasks, and one observer reported that it was difficult to predict what would happen. Three, three, and five of the observers who predicted different settings reported the uncrossed, equidistant, and perpendicular setting as the narrowest settings, respectively. Similarly, five, three, and five of these observers

Table 1: The mean slope and SD for each combination of task, height, laterality, and separation. \( N = 24 \).

| Height          | At eye level          | Below eye level         |
|-----------------|-----------------------|-------------------------|
| Latencyity      | Left | Right | Left | Right | Left | Right | Left | Right |
|\( S^*(cm) \)    | 16   | 38    | 70   | 16    | 38    | 70    | 16   | 38    | 70    |
| Uncrossed:     |       |       |       |       |       |       |       |       |       |
| Mean            | 0.005 | 0.002 | 0.064 | 0.000 | 0.016 | 0.060 | 0.003 | 0.001 | 0.054 |
| SD              | 0.012 | 0.025 | 0.035 | 0.010 | 0.024 | 0.035 | 0.014 | 0.020 | 0.036 |
| Equidistant:    |       |       |       |       |       |       |       |       |       |
| Mean            | −0.002 | 0.006 | 0.076 | 0.002 | 0.019 | 0.070 | −0.004 | 0.004 | 0.070 |
| SD              | 0.022 | 0.025 | 0.032 | 0.018 | 0.025 | 0.033 | 0.039 | 0.037 | 0.029 |
| Perpendicular: |       |       |       |       |       |       |       |       |       |
| Mean            | −0.007 | 0.000 | 0.071 | 0.011 | 0.016 | 0.064 | −0.009 | 0.001 | 0.062 |
| SD              | 0.040 | 0.038 | 0.037 | 0.036 | 0.035 | 0.038 | 0.040 | 0.034 | 0.035 |
| Combined:      |       |       |       |       |       |       |       |       |       |
| Mean            | −0.001 | 0.002 | 0.070 | 0.004 | 0.017 | 0.065 | −0.004 | 0.001 | 0.062 |

*Separation between the far ends of the rods.*
reported the uncrossed, equidistant, and perpendicular setting as the widest settings, respectively (two observers made two choices). No significant trend was found in distribution of the task reported by the observers as the narrowest or widest settings.

Finally, with the exception of the observers who reported that the same settings were made in two or three tasks, we analyzed the slope data of the nine observers who reported that different settings were made for different tasks. A four-way repeated-measure ANOVA (reported size × height × separation × laterality) revealed that the size × laterality interaction was significant, F(2, 18) = 5.82, p < .05, as shown in Table 3. The difference in mean slope between the left and right rods that were reported as the narrowest settings (0.084) was larger than that between the left and right rods reported as the widest settings (0.050). Thus, the observers correctly understood the direction of the rods that had actually been set by themselves. The interaction of size with other factors was not significant.

2.2.3. Visual-angle analysis

We can find geometric features of visual space by applying the trigonometric functions (Hardy et al., 1953, p. 14) to the settings of the equidistant alleys. If visual space is Euclidean, then we obtain

\[ S = D \left(2\tan\frac{\theta}{2}\right) \tag{1} \]

where \( S \) is the perceived extent between the rods at a distance from the observer, \( D \) is the perceived distance from the observer to the midpoint between the rods, and \( \theta \) is the perceived visual angle formed by the rods.

If the perceived extent between the far ends of the rods \( S_f \) equals the perceived extent between the near ends \( S_n \), then:

\[ D'_{n} \left(2\tan\frac{\theta_{n}}{2}\right) = D_{f} \left(2\tan\frac{\theta_{f}}{2}\right) \tag{2} \]

and

\[ 2\tan\frac{\theta_{n}}{2} = \frac{D'_{f}}{D'_{n}} \left(2\tan\frac{\theta_{f}}{2}\right) \tag{3} \]

where \( D'_{f} \) and \( D'_{n} \) are the perceived distances from the observer to the far and the near ends of the rods, respectively; \( \theta_{f} \) and \( \theta_{n} \) are the perceived visual angles formed by the far ends and by the near ends, respectively.

If the visual angle is perceived correctly, then we obtain

\[ \theta' = \theta \tag{4} \]

where \( \theta' \) is the visual angle formed by rods.

From Eqs. (3) and (4), we obtain

\[ 2\tan\frac{\theta_{n}}{2} = \frac{D'_{f}}{D'_{n}} \left(2\tan\frac{\theta_{f}}{2}\right) \tag{5} \]

To predict \( \theta_{n} \) from \( \theta_{f} \) by Eq. (5), we need the estimates of \( D'_{f} \) and \( D'_{n} \). Since egocentric distance is judged correctly in a well-lit office (Higashiyama and Shimono, 1994), if \( D'_{f} \) is the reference (\( D'_{f} = 1 \)), then \( D'_{n} = D_{n}/D_{f} = 1.13 \), where \( D_{f} \) and \( D_{n} \) are the physical distances from the observer to the far and near ends of the rods, respectively.

Equation (5) predicts that \( 2\tan\frac{\theta_{n}}{2} \) should be proportional to \( 2\tan\frac{\theta_{f}}{2} \) with a slope of 8.85 (\( = 1/0.113 \)). However, it may be useful to plot the mean \( \theta_{n} \) taken over tasks against \( \theta_{f} \) as shown in the top left corner of Fig. 5. It seems that the predicted curve well fitted to the settings for the small and middle separations but deviated from the settings for the large separation. The root-mean-square error (RMSE) was 16.4°.

By carrying out analogous reasoning for hyperbolic geometry, we obtain

\[ 2\tan\frac{\theta_{n}}{2} = \frac{\sinh D'_{f}}{\sinh D'_{n}} \left(2\tan\frac{\theta_{f}}{2}\right) \tag{6} \]

where \( \sinh D' \) is the hyperbolic sine of \( D' \).

For elliptic geometry, we obtain

\[ 2\tan\frac{\theta_{n}}{2} = \frac{\sin D'_{f}}{\sin D'_{n}} \left(2\tan\frac{\theta_{f}}{2}\right) \tag{7} \]

where \( \sin D' \) is the sine of \( D' \).

The predictions from Eqs. (6) and (7) are shown in the middle and bottom left panels of Fig. 5, respectively. In this figure, we assumed Eq. (4), \( D'_{f} = 1 \), and \( D'_{n} = 0.113 \) as we did for the Euclidean model. The RMSEs for the hyperbolic and elliptic models were respectively 22.4° and 11.8°. Both models fitted well to the settings for the small separation but deviated from the settings for the large separation.

2.2.4. Visual-angle analysis revised

The results of slope analysis suggested that visual space is Euclidean, whereas the results of visual-angle analysis suggested that visual space is not Euclidean, hyperbolic, nor elliptic. One way to reconcile this discrepancy in results may be to replace Eq. (4) with a
flexible equation. Several studies have indicated that visual angle formed by two points in the horizontal plane is perceived with magnification of about 10% (Foley, 1965), 21% (Higashiyama, 1992, an average of Eqs. (3), (5) and (7) in the paper), or 14 or 25% (Higashiyama, 1996, natural viewings with upright posture). This magnification may be represented by

\[ \theta = Q\beta \]

(8)

where \( Q \) is positive constant.

If Eq. (8) is substituted in Eqs. (3), (6) and (7), then we obtain

\[ 2\tan \frac{Q\theta_n}{2} = \frac{D_f}{D_n} \left( 2\tan \frac{Q\theta}{2} \right) \]

(9)

\[ 2\tan \frac{Q\theta_n}{2} = \frac{\sinh D_f}{\sinh D_n} \left( 2\tan \frac{Q\theta}{2} \right) \]

(10)

and

\[ 2\tan \frac{Q\theta_n}{2} = \frac{\sin D_f}{\sin D_n} \left( 2\tan \frac{Q\theta}{2} \right) \]

(11)

We sought the function of the least RMSE by varying \( Q \) systematically in Eqs. (9)–(11). The results are shown as the predicted curves in the right panels of Fig. 5. The estimated \( Q \)s (and RMSEs) were 1.23

![Graphs showing data for small, middle, and large separations.]

Table 2

| Frequency of each task reported as the easiest or most difficult. \( N = 24 \). | Uncrossed | Equidistant | Perpendicular |
|---|---|---|---|
| Easiest | 14 | 9 | 1 |
| Most difficult | 6 | 5 | 13 |

Table 3

The mean slope and SD of the left or right rod of the settings reported as the narrowest, moderate, or widest. \( N = 9 \).

| Reported size of setting | Narrowest | Moderate | Widest |
|---|---|---|---|
| Latensity | Left | Right | Left | Right | Left | Right |
| Mean | 0.041 | 0.043 | 0.035 | 0.036 | 0.024 | 0.026 |
| SD | 0.041 | 0.037 | 0.043 | 0.038 | 0.037 | 0.033 |

Fig. 4. Mean settings of the rods as a function of height, separation, and laterality. EL stands for the eye level. The left panels are for all observers (\( N = 24 \)); the right panels are for observers who perceived the rods to be straight (\( N = 17 \)).

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(8.9°), 1.35 (7.6°), and 1.10 (10.1°) for the Euclidean, hyperbolic, and elliptic models, respectively. Hence, the RMSE was reduced to about a third for the hyperbolic model, to nearly a half for the Euclidean model, and to about 90% for the elliptic one.

2.3. Discussion

The mean slopes of the rods varied with their separation. For the small separation, the rods were almost parallel with respect to the median plane, but for the large separation, they were convergent to the observer. This may be understood as a deviation from vertical settings with an increase of separation. The slopes of the rods were also affected by visibility of the floor. When the floor was visible, the adjusted separation of the rods was widened, compared to the condition of the invisible floor. This result confirmed the Blumenfeld (1913, p. 324) study, and it may be understood as approaching veridical settings with an increase of depth cues (Bian et al., 2005).

Table 2 indicates that the settings on the perpendicular instructions were difficult, while those on the uncrossed instructions were easy. This was borne out by large SDs for the small and middle separations under the perpendicular instructions, compared to those under the uncrossed instructions. This difference in setting difficulty may be the reason why several studies (Blumenfeld, 1913; Hardy, Rand, & Ritler, 1951; Indow, Inoue, & Matsushima, 1962; Indow & Watanabe, 1984a; Shipley, 1957b) used the uncrossed instructions rather than the same-direction and the perpendicular ones. However, the mean settings of rods did not differ with instructions, in agreement with the Euclidean property of parallelism.

The situation, however, changed when the visual angle formed by the near ends of the rods (the variable) was plotted against the visual angle formed by the far ends (the standard). When it was assumed that the visual angle is perceived correctly, as the early studies assumed (Luneburg, 1947; Blank, 1953; Hardy et al., 1953), the Euclidean, hyperbolic, or elliptic models did not fit well to the settings, in particular, with regards to the large separation. However, when it was assumed that the visual angle is magnified in effective size (Foley et al., 2004) or in perceived size (Foley, 1965; Higashiyama, 1992, 1996), the fit was largely improved in all models.

When the estimated Qs were compared among the models, the Euclidean model seemed more suitable than other models. The estimated Q for the Euclidean model was 1.23, which fell within the magnification range reported in literature (\(Q = 1.10–1.25\)). The estimated Q for the hyperbolic model was 1.34, which was outside the range. The estimated Q for the elliptic model was 1.10, which was just on the margin of the range, but the RMSE for the elliptic model was larger than that for other models.

Even though the Euclidean model was suitable, a large RMSE (8.9°) still remained to be explained, in particular coming from the settings of the rods with the large separation. This error may be due to a tendency of avoiding the large visual angle that was formed by the near ends of the rods. If two same-sized objects (or extents) are seen at different distances in a well-lit condition, then they usually appear same-sized due to the size-constancy mechanism. But if, as in this experiment, the visual angle of the near object is very large because of its close viewing distance, then the observer is so responsive to angular size that the near object may appear larger than the far object. This bias is conspicuous in the painting The Lamentation over the Dead Christ (ca. 1480) by Andrea Mantegna, who modified perceptual skews that were produced by application of rigorous linear perspective in drawing a person lying in depth (Solso, 1994). I think that the Mantegna effect occurs in 3-D layout with an extraordinarily large visual angle of extents.

3. Experiment 2

The alleys can be set kinesthetically. Blumenfeld (1936) required blindfolded observers to set two threads “parallel” to one another and to the median plane and found that the settings were divergent to the observer. In Experiment 2, blindfolded observers set the rods under the uncrossed, equidistant, and perpendicular instructions and then, the same observers, this time with no blindfold, performed the same tasks in a well-lit office. I attempted to explain the differences between the visual and kinesthetic settings by the position of the egocenter(s) and the sensitivity to spatial direction.

3.1. Method

3.1.1. Observers

The observers were 20 students (10 males and 10 females) recruited at Ritsumeikan University. All observers provided their informed consent for voluntary participation in experiments. The experiments were carried out in accordance with the Code of Ethics of the World Medical Association and were authorized by the university’s ethical committee for human experiments. All observers were paid for their participation according to university labor standards.

\footnote{Luneburg (1950) expressed a point in physical space by coordinates \((\gamma, \phi, \theta)\), where \(\gamma, \phi, \) and \(\theta\) are convergence angle, bipolar latitude, and elevation angle of the point, respectively. He also expressed a point in the Euclidean map (i.e., a representative map of visual space) by coordinates \((\rho, \varphi, \theta)\), where \(\rho, \varphi, \) and \(\theta\) are the counterparts of \(\gamma, \phi, \) and \(\theta\), respectively. One coordinate system is related to the other by the mapping functions: \(\rho = f(\gamma), \varphi = \phi \) and \(\theta = \theta\). From the mapping functions, it is derived that physical visual angle equals the perceived visual angle.}
3.1.2. Apparatus
The apparatus closely resembled that used in Experiment 1, except for the far and near ends of the rods, which were respectively 226.5 cm and 9.2 cm away from the frontal plane through the eyes.

3.1.3. Procedure
When the observer came to laboratory, the chin rest was set so that the eyes were at the same horizontal level as the rods. After explaining the rod system, the observer was told that the experiment consisted of visual and kinesthetic tasks, for which a blindfold was needed whenever required by the experimenter. Half the observers first performed the visual tasks and then the kinesthetic tasks, while the other half performed the two tasks in the reverse order. The visual tasks were the same as the three tasks of Experiment 1: uncrossed, equidistant, and perpendicular tasks. The kinesthetic tasks corresponded to the visual ones. The instructions for the kinesthetic tasks were: “Please put on the blindfold and grasp the rods with both hands, the left rod with the left hand, and the right rod with the right hand. Although you cannot see the rods, you feel that the rods are in your hands and extend in depth from there. I will ask you to move the near ends of the rods horizontally and adjust the direction of the rods so that they produce specific percepts. During settings, you must keep your head facing straight ahead. You must keep your head on the chin rest and must not turn it.”

On each trial, the experimenter set the position of the rods back to the starting point. The observer was asked to judge which direction the rods had to be moved, in order for their position to correspond to the instruction for that task, and not to move back the rods in the other direction.

For each task, there were three separations between the far ends of the rods, and for each separation there were two starting points: near ends widely separated and near ends close together. Each starting point was randomly repeated twice. Hence, there were 144 trials (3 tasks × 2 modalities × 3 separations × 4 settings × 2 lateralities) in all. The three tasks were presented in counterbalanced order, with all the trials on one task being completed before starting the trials on the next task.

3.2. Results

3.2.1. Slope analysis
By using the same treatment of data as in Experiment 1, we collapsed the four settings and obtained individual slopes of the rods for combinations of task, modality, separation, and laterality. Table 4 shows the mean slopes and SDs that were taken over observers. We used the individual slopes as scores for statistical tests.

A four-way repeated-measure ANOVA revealed that the main effect of task was not significant. The modality × separation × laterality interaction was significant, $F(2, 38) = 37.99, p < .001$. This interaction is illustrated in Fig. 6: for the large separation, the difference of the mean slopes in the visual settings (left, 0.015; right, −0.021) was not significantly different from that in the kinesthetic settings (left, 0.022; right, −0.019). This suggested that the rods were equally convergent to the observer in both settings. For the middle separation, the difference of the mean slopes in the visual settings (left, −0.006; right, 0.005) were significantly different from that in the kinesthetic settings (left, −0.016; right, 0.024), $F(1, 57) = 10.13, p < .01$. This suggested that the rods were almost parallel to one another in the visual settings and were divergent to the observer in the kinesthetic settings. For the small separation, the difference of the mean slopes in the visual settings (left, −0.006; right, −0.005) were significantly different from that in the kinesthetic settings (left, −0.016; right, 0.024), $F(1, 57) = 53.75, p < .001$. This suggested that the rods were generally divergent to the

![Fig. 6. Mean settings of the rods shown as a function of modality, separation, and laterality. L and R stand for the left and right rods, respectively.](image)

Table 4
The mean slope and SD for each combination of task, modality, laterality, and separation. $N = 20$.

| Modality | Visual | Kinesthetic |
|----------|--------|-------------|
| Laterality | Left | Right | Left | Right |
| $S^*(cm)$ | | | | |
| Uncrossed: | | | | |
| Mean | −0.005 | 0.001 | 0.014 | 0.010 | −0.011 | −0.015 | −0.059 | −0.008 | 0.024 | 0.070 | 0.021 | −0.015 |
| SD | 0.021 | 0.035 | 0.045 | 0.019 | 0.040 | 0.043 | 0.035 | 0.035 | 0.049 | 0.032 | 0.030 | 0.035 |
| Equidistant: | | | | |
| Mean | −0.006 | 0.005 | 0.016 | 0.011 | −0.012 | −0.021 | −0.053 | −0.028 | 0.023 | 0.063 | 0.031 | −0.017 |
| SD | 0.020 | 0.031 | 0.042 | 0.023 | 0.034 | 0.045 | 0.040 | 0.044 | 0.040 | 0.036 | 0.038 | 0.033 |
| Perpendicular: | | | | |
| Mean | −0.006 | −0.005 | 0.014 | 0.007 | 0.008 | −0.026 | −0.055 | −0.012 | 0.020 | 0.058 | 0.018 | −0.025 |
| SD | 0.018 | 0.034 | 0.048 | 0.017 | 0.056 | 0.038 | 0.040 | 0.042 | 0.049 | 0.030 | 0.058 | 0.039 |
| Combined: | | | | |
| Mean | −0.006 | 0.000 | 0.015 | 0.009 | −0.005 | −0.021 | −0.056 | −0.016 | 0.022 | 0.063 | 0.024 | −0.019 |

*Separation between the far ends of the rods.
observed and this trend was salient in the kinesthetic settings.

The modality × laterality interaction was significant, $F(1, 19) = 15.41, p < .001$. Fig. 7 illustrates this interaction. In the visual settings, the mean slope for left rod (0.003) was not significantly different from the mean slope for the right rod ($-0.006$). On the contrary, in the kinesthetic settings, the mean slope for the left rod ($-0.016$) was significantly different from the mean slope for the right rod (0.023), $F(1, 38) = 10.15, p < .01$. Thus, in the visual settings, the rods were generally parallel to one another, whereas in the kinesthetic settings, they were generally divergent to the observer.

The separation × laterality interaction was significant, $F(2, 38) = 91.56, p < .001$. Fig. 8 illustrates this interaction. For large separation, the mean slope of the left rod (0.018) was significantly different from the mean slope of the right rod ($-0.020$), $F(1, 57) = 10.99, p < .01$. For the middle separation, the mean slope of the left rod ($-0.008$) was not significantly different from the mean slope of the right rod (0.009). For the small separation, the mean slope of the left rod ($-0.031$) was significantly different from the mean slope of the right rod (0.036), $F(1, 57) = 33.61, p < .001$. These results suggested that the rods were generally convergent, parallel, and divergent to the observer for the large, middle, and small separations, respectively.

### 3.2.2. Visual and kinesthetic-angle analysis

Fig. 9 shows the mean $\theta_{v}$ plotted against $\theta_{k}$ together with the curves predicted from Eqs. (9)–(11). We assumed that the perceived distance of the far ends was the reference ($D_{f} = 1$) and the perceived distance of the near ends was $D_{n} = D_{o}/D_{f} = .0406$. For each equation, we depicted the curve predicted for the Q equal to 1 and the curve predicted for the Q producing the least RMSE. For the Euclidean model, both curves happened to produce the same RMSE (6.5°); for the hyperbolic model, the RMSE reduced from 6.6° to 3.5° by increasing Q from 1 to 1.05; and for the elliptic model, the RMSE reduced from 12.3° to 10.0° by decreasing Q from 1 to 0.94. We also attempted to predict the kinesthetic settings by varying Q, but we failed to find Q for any model that fell within the reasonable range.

### 3.3. Discussion

Experiment 2 revealed that the mean visual settings were the same regardless of whether the instructions were uncrossed, equidistant, or perpendicular. From the point of parallelism, this result supported the Euclidean model. Furthermore, when the visual angles formed by the near ends of the rods were analyzed by using Eq. (4), the Euclidean model provided the smallest RMSE. However, when these visual angles were analyzed by using Eq. (8), the hyperbolic model provided the smallest RMSE.

The mean kinesthetic settings were the same regardless of whether the instructions were uncrossed, equidistant, or perpendicular. This result supported the Euclidean model. However, when the kinesthetic angles formed by the near ends of the rods were analyzed in the same way as the visual angles, a suitable $Q$ was not estimated for any model.

There were several differences between the visual and kinesthetic settings (Fig. 6). In the visual settings, the left and right rods were almost parallel to one another for the small separation and were convergent to the observer for the large separation. But, in the kinesthetic settings, the rods were divergent to the observer for the large separation. It is also noted that for the small separation, the SDs of the kinesthetic settings were larger than those of the visual settings, but for the large separation, the visual and kinesthetic SDs were almost the same (Table 4).

The differences between the visual and the kinesthetic settings may be due to the number and location of the egocenter that was the origin used in judging direction. In fact, the visual egocenter always lies midway on the interocular axis (Ono, 1979). The kinesthetic egocenter varies with the hand used (Shimoto et al., 2001) and may coincide with the joint (shoulder, elbow, or wrist) around which the limb rotates (Blumenfeld, 1936). If each rod is grasped with the ipsilateral hand as in this study, then there are two kinesthetic egocenters, one at the relevant joint of each limb, and each egocenter is used to determine the kinesthetic direction of the ipsilateral rod.

We assume that, when we set two rods to appear parallel to one another and to the median plane, each rod is apt to align with the line extending from the relevant egocenter to the far end of the rod. It follows that as the separation of the rods increases, the visual settings get more and more convergent to the observer. If, on the other hand, the
kinesthetic egocenter is at each elbow, then the kinesthetic settings also vary with separation of the rods: for the small separation, the setting is divergent to the observer because the separation is smaller than the inter-elbow distance, whereas for the large separation, the setting is convergent to the observer because the separation is larger than the inter-elbow distance (Fig. 10). If it is assumed that when the visual egocenter conflicts with the kinesthetic egocenter(s), the visual egocenter dominates in judging direction, then the present results support
the egocenter hypothesis.

There is another possible interpretation of the difference between the visual and the kinesthetic settings. If we are insensitive to kinesthetic direction to the far ends of the rods, then the kinesthetic settings of the near ends tend to regress to the mean settings of the three separations. This prediction was indeed confirmed in Fig. 6. If we are insensitive to kinesthetic direction, then the kinesthetic settings are even larger in variation than the visual settings. This prediction was confirmed for the small separation, but not for the large separation, where the SDs of the visual and the kinesthetic settings were almost the same (Table 4).

The egocenter and the sensitivity hypothesis are not mutually exclusive. The egocenter hypothesis is about perceptual direction based on the origin(s) of visual and kinesthetic spaces, whereas the sensitivity one is a psychophysical principle of the judgmental process. It is possible that the two processes work together.

4. General discussion

When all data collected in this study were considered, the instructions for constructing visual alleys - uncrossed, equidistant, and perpendicular - did produce the same settings of rods. This tendency was true for the observers who perceived the rod to be straight at all times and was also true for the observers who did not notice that all tasks require the same setting in Euclidean geometry, or who, although they noticed it, did not consciously make the setting of the rods the same. These findings imply that the visual space is Euclidean, consistent with the results of Higashiyama et al. (1990) which showed that the uncrossed and equidistant alleys formed indoors were in agreement within the margin of error. The three instructions also produced the same mean kinesthetic settings, which differed from the mean visual ones. This may be a new finding: if the Luneburg-Blank model is applied to the kinesthetic settings, then the kinesthetic space would also be Euclidean.

When the visual angles formed by the rod ends were analyzed under the assumption that visual angle is perceived correctly, they could not be accounted for by a simple geometry of constant curvature. To account for the difference in results between the analysis on parallelism and the analysis on visual angle, it was assumed that the visual angle is perceptually magnified. Under this assumption, the visual-angle data were interpretable by the Euclidean model with a 23% magnification of visual angle (Experiment 1), or with correct perception of visual angle (Experiment 2). The visual-angle data were also interpretable by the hyperbolic model with a 35% magnification of visual angle (Experiment 1) or with its 5% magnification (Experiment 2). When facing the outcome that both Euclidean and hyperbolic models could account for the visual-angle data, we favored the Euclidean interpretation, because it was compatible with the results of the analysis on parallelism.

Visual settings were not exactly the same for both experiments. The settings for the small and middle separations were nearly parallel to the median plane in both experiments, whereas the settings for the large separation were convergent to the observer in Experiment 1 (see Fig. 3) and were less convergent in Experiment 2 (see Fig. 6). In other words, size constancy (to be accurate, extent-between-rod constancy) broke down only for the large separation in Experiment 1. Why did such a thing occur? It is not easy to answer to this question. Since the physical layout was the same in both experiments, the decrease of size constancy for the large separation in Experiment 1 may be attributable to human factors. A possible tentative factor may be the individual differences in sensitivity to angular size: for unknown reasons, the observers in California might be more susceptible to the Mantegna effect than the observers in Japan.

There are two recent studies that assumed that visual space is described by Euclidean geometry, or its variation, and that there is a transformation between visual space and physical space, which is not as simple as the Luneburg-Blank model assumed as mapping functions. One study by Foley et al. (2004) proposed a general model to predict a visual extent between two points. The model assumed that the theorem of cosines, one of Euclidean theorems, is true in visual space, and that the perceived extent between two points varies with effective size of visual angle formed by the points, as well as with perceived egocentric distances of the points. It was furthermore assumed that the effective size of visual angle is magnified and that the perceived egocentric distance increases with the physical distance, approaching the boundary distance asymptotically. The present results may agree with the Foley et al. model in assuming that visual space is Euclidean and that the visual angle formed by two points is magnified.

Wagner (1985) placed many stakes on an open field and required observers to estimate perceived extents between two stakes, perceived angles subtended at a stake by two other stakes, and perceived area of triangles formed by three stakes. The inter-stake extents in depth appeared smaller than same physical extents in the frontal plane, and the angles facing toward or away from the observer appeared larger than the same angles seen on their sides. To account for the anisotropic properties of perceived extent and angle, he assumed that 1) visual space is an affine transformation of physical space that produces contraction of extent in depth and 2) visual space is still Euclidean after this transformation. The fit of this model to the settings was better than the hyperbolic and elliptic models. Our model agrees with the Wagner study given that visual space is assumed to be Euclidean and a simple transformation is assumed between visual space and physical space.

Two recent studies attempted to determine curvature directly from triangles that were constructed in visual space. Norman et al. (2005) required observers to construct an equilateral triangle, with the observer located at one vertex of the triangle, and with two targets, fixed and movable, located at other vertexes. The observer’s task was to adjust the movable target so that the triangle appeared equilateral. They observed that when a triangle was formed among two targets and the observer, one exocentric interior angle of the triangle decreased as the side of the triangle increased. They suggested that as the triangle increased in area, the curvature changed from positive to negative. Although these two studies relied on the assumption that the interior angles of a triangle are perceived correctly, this assumption is not valid as shown by this study and other existing research (Foley et al., 2004; Li and Durgin, 2016; McCreary, 1965, 1985; Wagner, 1985, 2006). The interior angles obtained by the triangle experiments would have to be calibrated for determination of curvature (Higashiyama, 1984).

In this study, by analyzing the kinesthetic angles formed by the near ends of rods, we found that there was no simple geometric model that fitted well to the settings, even when perceptual enlargement of the kinesthetic angles was assumed. I am not surprised at this result, because there may be two origins in the kinesthetic space, and the conjunction of the origins may determine perceived kinesthetic direction. At the present stage of this study, it is premature to suggest the geometry of kinesthetic space.

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2 Foley et al. concluded that their ‘model is inconsistent with all geometries of the class referred to as metric geometries, including the Riemannian geometries of constant curvature (p. 153).’ But they stated elsewhere (p. 148) that the metric function they used (Eq. (2)) ‘is essentially the equation for the Euclidean distance between two points.’ The author thinks that the latter is suitable interpretation of the model.
5. Conclusions

In summary, the visual alley settings were affected by separation of rods and by height of observer's eyes, but were not affected by the instructions (uncrossed, equidistant, and perpendicular). The kinesthetic alley settings were more drastically affected by separation of rods: the rods were divergent to the observer for the small separation and were convergent to the observer for the large separation, but the kinesthetic alley settings were not affected by the instructions. When the Luneburg-Blank model was applied to these settings, assuming that the visual angle or the kinesthetic angle subtended at observer are perceptually magnified, the visual settings were reasonably described by Euclidean geometry, while the kinesthetic settings were not described by any simple geometry.

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Declaration of Competing Interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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