Eigen-frequencies of whispering gallery modes of disk dielectric resonators: a dimensional quantization method

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Abstract. A method for calculation of the resonant frequencies as well as free spectral range was developed for the high-Q whispering gallery mode disk dielectric resonators. The method is based on application of the size quantization to the dispersion equation derived for an infinite dielectric cylinder. The size quantization is caused by the requirement of the phase balance for standing waves in the resonator.

1. Introduction

During the past decade or so a high research interest in the field of microwave photonics is evident. The microwave photonics is an interdisciplinary field comprising microwave electronics, wave guiding and integrated optics, and other research areas [1-3]. One of the key elements of the microwave photonics is an optical resonator with whispering gallery modes (WGMs). We assume that the WGMs in such resonators are modes with a high azimuthal index that are "pressed" to the resonator side surface and have a Q-factor much greater than the lowest types of oscillations [4, 5]. Such optical resonators may have a Q-factor greater than 10¹¹ [6] and also a high stability with respect to the external microwave influences. This makes them very attractive for various practical applications [7-9].

Resonators in the form of balls [10-12], toroidal and disk resonators [13, 14] and resonators of "bottle" form [15] are typical examples of the axisymmetric resonators with WGMs. At present, the planar disk resonators of micrometer thickness compatible with silicon technology [13, 14, 16] are developed. It opens up wide possibilities for their fabrication.

Several analytical methods for estimation of the resonant frequencies of the disk dielectric WGM resonators are already known. Among them are the method of partial regions [4], the effective dielectric permittivity method [17], and the magnetic wall method [18]. These methods are based on the simultaneous solution of two dispersion equations having an implicit form. Therefore, finding WGM resonant spectrum is quite laborious and require considerable computational powers.

The purpose of the present work is to develop a theory for calculation of the resonant frequency spectra of the disk WGM resonators.

2. Theoretical investigation

The theory was developed in two stages. In stage 1, a dispersion equation for electromagnetic waves propagating in dielectric cylinder waveguide of an infinite length was obtained. We began from Maxwell’s equations and constitutive equation taking into account the appropriate boundary
conditions for the electromagnetic field on the side surface. As a result, we obtained the following components of the electric and magnetic fields

\[
E_z = \frac{A}{B} Z_m(k_\perp \rho) e^{-i\beta \cos(m\phi)} \sin(m\phi),
\]

\[
H_z = \frac{D}{C} Z_m(k_\perp \rho) e^{-i\beta \cos(m\phi)} \sin(m\phi),
\]

\[
H_\phi = -\frac{i}{k_\perp} \left( \frac{D}{C} \beta Z_m(k_\perp \rho) + \frac{A}{B} \mu_0 \varepsilon \omega Z_m'(k_\perp \rho) \right) \left( e^{-i\beta \cos(m\phi)} \sin(m\phi) \right),
\]

\[
H_\rho = \frac{i}{k_\perp} \left( \frac{A}{B} \beta Z_m(k_\perp \rho) - \frac{D}{C} \mu_0 \varepsilon \omega Z_m'(k_\perp \rho) \right) \left( e^{-i\beta \cos(m\phi)} \sin(m\phi) \right),
\]

\[
E_\phi = -\frac{i}{k_\perp} \left( \frac{A}{B} \beta Z_m(k_\perp \rho) + \frac{D}{C} \mu_0 \varepsilon \omega Z_m'(k_\perp \rho) \right) \left( e^{-i\beta \cos(m\phi)} \sin(m\phi) \right),
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\]

where \( A, B, C, \) and \( D \) are the amplitude constants determined from the boundary conditions, \( Z_m \) is the cylindrical function of a particular form depending on a range of field definition and the stated problem, \( m \) is an order of the cylindrical function showing quantity of the azimuthal variation of the field, \( \beta \) is a longitudinal wave number, \( k_\perp \) is a transversal wave number, \( \rho \) is a radial coordinate, \( \phi \) is an azimuthal angle, \( \varepsilon \) is a dielectric permittivity, which equals to a dielectric permittivity \( \varepsilon_{\text{res}} \) of the resonator materials for \( \rho \leq R \) and to a dielectric permittivity of the space surrounding the resonator (\( \varepsilon_{\text{air}} \)) for \( \rho \geq R \), where \( R \) is the radius of the resonator.

The transverse wave number is determined by the following expression:

\[
k_\perp = \sqrt{\frac{k_{\text{res}}^2 - \beta^2}{k_{\text{air}}^2 - \beta^2}} \text{ for } \rho \leq R,
\]

\[
k_\perp = \sqrt{\frac{k_{\text{res}}^2 - \beta^2}{k_{\text{air}}^2 - \beta^2}} \text{ for } \rho \geq R,
\]

where \( k_{\text{res}} = \frac{\omega}{c} \sqrt{\varepsilon_{\text{res}}} \) is a wave number in the resonator material and \( k_{\text{air}} = \frac{\omega}{c} \sqrt{\varepsilon_{\text{air}}} \) is a wave number of the space surrounding the resonator.

In solving the spectral problem, two cases can be distinguished: the case of a "thick" and a "thin" disk. When \( \beta > k_{\text{air}} \) we will consider the case of a "thick" disk. For this condition \( k_\perp \) is real inside and outside the resonator and standing waves inside the resonator are described by the Bessel functions, i.e. \( Z_m(k_\perp \rho) = J_m(k_\perp \rho) \). Outside the resonator, the waves are described by the Hankel functions of the first and second kinds, i.e. \( Z_m(k_\perp \rho) = H_m^{(1)}(k_\perp \rho) + H_m^{(2)}(k_\perp \rho) \), which corresponds to divergent and incoming spherical waves. For the chosen harmonic time dependence (\( e^{i\omega t} \)), it is necessary to take the second Hankel function, which satisfies the Sommerfeld radiation condition [5].

Applying the boundary conditions of electrodynamics, in the case of a "thick" disk we obtain the following dispersion equation:
where $k_{\perp} = \sqrt{k_{\mathrm{res}}^2 - \beta^2}$ and $k_{\perp 2} = \sqrt{k_{\mathrm{air}}^2 - \beta^2}$. It is seen from this equation that the transverse oscillations in the "thick" disk has a hybrid nature, namely, it is possible to distinguish HE$_{ml}$ and EH$_{ml}$ modes in the solution.

The condition $k_{\mathrm{air}} < \beta < k_{\mathrm{res}}$ is satisfied for the "thin" disk approximation. Similar to a case of a "thick" disk the standing waves inside the resonator are described by the Bessel functions. The transverse wave number $k_{\perp}$ has imaginary value outside the resonator. Therefore, the waves outside the resonator are described by the modified Bessel functions of the first and second kind. Using the condition of the vanishing field at infinity, we take the modified Bessel function of the second kind as the solution, i.e. the MacDonald function. Below, this function is denoted by $\mathrm{mK}_m$. Applying the boundary conditions of the electrodynamics, we obtain the following dispersion equation for the case of a "thin" disk:

\[
\frac{m^2}{R^2} \left( \frac{1}{k_{\perp 1}^2} - \frac{1}{k_{\perp 2}^2} \right) \left( \frac{\epsilon_{\mathrm{res}}}{k_{\perp 1}^2} - \frac{\epsilon_{\mathrm{air}}}{k_{\perp 2}^2} \right) =
\]

\[
= \left( \frac{\epsilon_{\mathrm{res}} J_m'(k_{\perp 1}R) + \epsilon_{\mathrm{air}} H_m'(k_{\perp 2}R)}{k_{\perp 1} J_m(k_{\perp 1}R) + k_{\perp 2} H_m(k_{\perp 2}R)} \right) \times \left( \frac{1}{k_{\perp 1} J_m(k_{\perp 1}R)} + \frac{1}{k_{\perp 2} H_m(k_{\perp 2}R)} \right),
\]

(3)

As is seen from this equation, the modes in the "thin" disk have a hybrid nature. They are named as HE$_{ml}$ and EH$_{ml}$. Note that for the case $m = 0$ the modes are not hybridized. In this case, Eq. (4) splits into two simpler equations describing the transverse TE$_0l$ and TM$_0l$ modes.

In stage 2, we derive the equation for the resonant frequencies of electromagnetic oscillations for the disk of the finite thickness through quantization of the longitudinal wave number of the infinite cylinder. Applying the phase balance condition, we determine the unknown values of the longitudinal wave number as

\[
\beta_n = \frac{\pi \cdot n - \varphi}{d},
\]

(5)

where $n$ is any integer determining the number of the field maxima along the axial coordinate, $d$ is the thickness of the resonator, $\varphi$ is a fitting parameter having sense of a phase shift appearing from the wave reflection at the media interfaces. We will assume further that $\varphi$ equals zero. Substitution the $\beta_n$ to the dispersion equations (3) and (4) allows one to calculate resonant frequencies and free spectrum range.

It should be noted that the resonant types of oscillations of the dielectric disk cavity and the frequencies corresponding to them are characterized by three indices $m$, $l$ and $n$. The index $m$ appears in Eqs. (3) and (4) and shows the number of wavelengths that fit along the angular coordinate. In other words, this is an azimuthal mode index. The index $l$ characterizes the number of field variations along the radial coordinate. It is determined by the number of solutions of Eqs. (3) and (4), for which there is an infinite set of roots. The index $n$ shows the number of variations of the field along the axis of the disk.

Figure 1 shows the results of the resonant frequency calculation for EH$_{mln}$ modes of a disk with a dielectric permittivity of 2.074 and a diameter of 3 µm for $m$ from 2 to 12 and for $l = 1$. The square, triangular and round symbols in the figure depict the resonant frequencies corresponding to the cases of the disks with thicknesses of 1.2 µm, 3 µm, and 5 µm, respectively. As is clear from the figure, with decreasing the thickness of the resonator its resonant frequencies shift to the region of larger
longitudinal wave numbers, which leads to decreasing the free spectral range and increasing the resonance frequency. With an increase in the index m, an increase in the free spectral range occurs, and for \( m >> 1 \) the resonance frequency spectrum becomes practically equidistant, and the distance between neighboring harmonics tends to the value of the free spectral range of an infinite length rod.

3. Conclusions

Thus, the paper presents a simpler and less laborious method for calculating the resonant frequencies of the high-Q WGM disk dielectric resonators in comparison with the known ones [4, 17, 18]. The influence of the thickness of the disk resonator on its own resonant frequencies is considered.

![Graph showing resonant frequencies](image)

**Figure 1.** Results of calculation of resonant frequencies \( EH_{\text{azl}} \) of a disk dielectric resonator with azimuthal oscillations

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