Deterministic linear optics quantum computation utilizing linked photon circuits

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We suggest an efficient scheme for quantum computation with linear optical elements utilizing "linked" photon states. The linked states are designed according to the particular quantum circuit one wishes to process. Once a linked-state has been successfully prepared, the computation is pursued deterministically by a sequence of teleportation steps. The present scheme enables a significant reduction of the average number of elementary gates per logical gate to about 20 – 30 CZ/16 gates.

Optical systems have proven to be a very successful tool for implementing quantum information and communication tasks such as quantum cryptography, teleportation, and quantum dense coding. However, when it comes to more complicated protocols, let alone scalable quantum computation, such systems suffer from a major disadvantage – the lack of interaction between photons that is needed for implementation of conditional logic gates. In a recent work Knill, Laflamme and Milburn (KLM) proposed a scheme for quantum computation based on linear optics, demonstrating that this obstacle can be overcome. A two-qubit gate is performed, according to this scheme, in two stages. First a standard ancillary state is prepared and subjected to some gates. This off-line preparation step may succeed with low probability. In the second stage the prepared state can be used to apply a logical gate on the input state by means of the teleportation scheme suggested by Gottesman and Chuang. This second step is again probabilistic, however, the teleportation success probability can be made arbitrarily close to unity, at the cost of a very significant increase of the required number of elementary operations.

In this letter, we suggest a new scheme for quantum computation with linear optical elements. A key point in our scheme is the introduction of a multi-photon entangled "linked" photon state whose structure is dictated by the form of the quantum circuit that one wishes to construct. This state is prepared by employing (KLM-type) non-deterministic gates. However, once the state has been successfully constructed, the remaining computation process, which utilizes a deterministic teleportation scheme, can be ideally completed with unit probability. This simplification results in a dramatic reduction in the required number of elementary gate operation per logical gate.

The teleportation protocol employed here uses an extension of the idea proposed by Popescu and experimentally realized by De Martini et al. In this method a pair of EPR entangled photons is utilized, and both the teleported state and "half" of the EPR state are associated with a single photon. Consequently, linear optics elements are sufficient for implementing a complete Bell-state measurement. We begin by extending this scheme to a chain of photons where each photon is entangled with two nearest neighbors photons. Let us denote by $p_1, p_2, ..., p_{n+1}$ the photons, and consider the following chain-state

$$
|\chi\rangle = (\alpha | \downarrow \rangle_{p_1} + \beta | \leftrightarrow \rangle_{p_1}) | 1 \rangle_{p_1} + | 2 \rangle_{p_1} | \leftrightarrow \rangle_{p_2} + | 3 \rangle_{p_2} | \leftrightarrow \rangle_{p_3} + ... + | 2n-1 \rangle_{p_n} | \leftrightarrow \rangle_{p_{n+1}} + | 2n \rangle_{p_n} | \leftrightarrow \rangle_{p_{n+1}} + | 2n+1 \rangle_{p_{n+1}} \tag{1}
$$

where $| \alpha \rangle = | p_1, p_2, ..., p_{n+1} \rangle$ is an arbitrary polarization state of the first photon in the chain, and $| m \rangle$ and $| \downarrow, \leftrightarrow \rangle$ denote the position and polarization states of the photons, respectively. The chain state, depicted schematically in Fig. 1, manifests pairwise maximal entanglement between position and polarization. The arrows portray the teleportation sequence of the state $|\chi\rangle$ to the last photon in the chain.

FIG. 1. Schematic description of a single chain-state with $n+1$ photons. A rectangle indicates a photon, whose polarization and path degree of freedom are denoted by empty and full circles, respectively. The linking horizontal lines denote the entanglement between position and polarization. The arrows portray the teleportation sequence of the state $|\chi\rangle$ to the last photon in the chain.

The above state can be used to teleport the state $|\chi\rangle_{p_i}$ through the whole chain by means of $N$ separate teleportation steps. In each step polarizing beam splitters (BPS) are utilized for a path-polarization Bell measurement on photon $p_i$. This sends the state $|\chi\rangle$ (after correcting $p_{i+1}$) to the polarization state of the next photon ($p_{i+1}$) in the chain. By $N$ such sequential teleportation steps $|\chi\rangle$ is teleported to the last photon in the chain.

We shall use one such chain to represent the "world line" of a single qubit in quantum circuit. The time step
corresponds here to teleportation steps. Hence for a circuit with \( N \) input qubits we will use \( N \) chains. To include the required gates, we next introduce gates between photons of different chains. Since the teleportation of the input state sends \( \chi \) to the polarization states, we need to apply the gate between the polarization states of the appropriate photons in each chain. However as was shown by Gottesman and Chuang we can first apply the relevant single qubit corrections. Then we teleport the polarization of the first photon in each chain (henceforth referred to as a linked-state) in which each photon in a chain is entangled to another photon of a different chain, and later teleport the input state through the linked-state.

To exemplify this construction, consider the three qubit circuit depicted in Fig. 2, which sends \( \chi_{1,2,3} \rightarrow G_{1,2}G_{1,3}G_{1,2}\chi_{1,2,3} \). We replace this circuit with the linked-state depicted in Fig. 3, that can be constructed as follows. We begin with three chain-states, \( |I\rangle = |p_1, p_2, p_3, p_4\rangle_I \), \( |II\rangle = |p_1, p_2, p_3\rangle_{II} \) and \( |III\rangle = |p_1, p_2\rangle_{III} \). The gates are then applied on the polarization states according to \( |I, II, III\rangle \rightarrow G(I_{p_1}, II_{p_2})G(I_{p_3}, III_{p_2})G(I_{p_2}, II_{p_2})|I, II, III\rangle \), where \( I_{p_2} \) denotes the polarization states of the second photon in the first chain, etc. Next, in order to perform the computation, we introduce the input state \( \chi \) by rotating the polarization of the first photon in each chain (assuming for the time being that \( \chi \) is a non-entangled, known state). Then we teleport \( \chi \) through the linked-state and apply the relevant single qubit corrections.

![FIG. 2. A simple circuit with three qubits and gate operations.](image)

![FIG. 3. A schematic description of the linked state that is needed for generating the quantum circuit in Fig. 2. The vertical lines represent the entanglement that is produced by applying the gates \(- G_{i,j}\).](image)

Our scheme generalizes to any quantum circuit, where the number of links in each chain is determined by the number of gates that are applied on that particular qubit. We next address the preparation process in detail.

**Preparation of linked states.** The off-line part of the computation consists of two basic operations: addition of a new link to each chain, and application of a two-qubit gate between polarization states of different chains. We next show how these two operations may be performed by applying KLM’s non-deterministic conditional phase flip \( CZ \) gates. (We can use either the basic or the improved gates proposed by KLM as well as the improvements suggested in [6]). These gates operate on the path degree of freedom of modes all with identical polarization. This poses no difficulty in our case, because we can easily move the information carried by polarization states back and forth between the path and polarization degrees of freedom by employing polarizing beam splitters and polarization rotation plates.

Consider the construction of a new link to one of the chains (Fig. 4.). To achieve that, we apply a gate between the path degree of freedom of the last photon in the chain and the polarization of an additional photon. Suppose that \( b \) is the last photon in the chain \( |(1)_{a}\rangle [\downarrow]_{b} + |(2)_{a}\rangle [\uparrow]_{b}|(3)_{b}\rangle \). We now add photon \( c \) in a state \((5)_{c} + (6)_{c}\rangle \), in two consecutive steps. First \( PBS_{50} \) the pair \( \{3, 4\}_{b} \) and \( \{5, 6\}_{c} \), followed by a 50/50 beam splitter to 3 and 4. This will take the state of the four modes to \( |3)_{b}[5)_{c} + |4)_{b}[6)_{c} \). In the second step, the polarization of modes \( 3' \) and \( 4' \) is rotated (so it matches the polarization of \( c \)) and same the procedure is repeated for \( \{3', 4'\}_{b} \). A successful sequence of (two) gate operations creates a link between \( b \) and \( c \). Finally, the entanglement is transferred to the required path-polarization form

\[
|1)_{a}[\downarrow]_{b} + |2)_{a}[\uparrow]_{b}|(3)_{b}[\downarrow]_{c} + |4)_{b}[\uparrow]_{c}|(5)_{c} \]  

(2)

Notice that 3, 4 and 3', 4' carry different polarizations, hence we next apply \( CZ \) in two consecutive steps. First between the pair \( \{3, 4\}_{b} \) and \( \{5, 6\}_{c} \), followed by a 50/50 beam splitter to 3 and 4. This will take the state of the four modes to \( |3)_{b}[5)_{c} + |4)_{b}[6)_{c} \). In the second step, the polarization of modes \( 3' \) and \( 4' \) is rotated (so it matches the polarization of \( c \)) and same the procedure is repeated for \( \{3', 4'\}_{b} \). A successful sequence of (two) gate operations creates a link between \( b \) and \( c \). Finally, the entanglement is transferred to the required path-polarization form

\[
\cdot \cdots \cdots ( |1)_{a}[\downarrow]_{b} + |2)_{a}[\uparrow]_{b} |(3)_{b}[\downarrow]_{c} + |4)_{b}[\uparrow]_{c}|(5)_{c} \]  

(3)
Having constructed the relevant links, we next consider a two-qubit gate (Fig. 5). In principle, the gate can be applied after completing the construction of the chains. However, as we apply the gate to the polarization of photons which are entangled to other photons in the chain, this would require the application of four CZ gates for each logic gate. It would therefore be more efficient to apply the gates to the proper photons immediately after these links have been established, before the next links in each chain are produced. As the last photon in each chain is located in a single mode, the operation can be implemented by a single CZ application. Suppose that we want to apply a two-qubit gate between the polarization states of photons $b$ and $d$ in the chain states

$$
\cdots \times (|1\rangle_b |3\rangle_c |5\rangle_b + |2\rangle_a |6\rangle_b)\text{ and }
\cdots \times (|3\rangle_c |6\rangle_d + |4\rangle_c)\text{ (6) }
$$

Employing a PBS for each chain and rotating the polarization of the modes corresponding to the horizontal modes ($5'$ and $6'$) we obtain

$$
\cdots \times |\Downarrow\rangle_b (|1\rangle_a |3\rangle_c |5\rangle_b + |2\rangle_a |5'\rangle_b)\text{ and }
\cdots \times |\Downarrow\rangle_d (|3\rangle_c |6\rangle_d + |4\rangle_c |6'\rangle_b)\text{ (5) }
$$

At this point we can apply the CZ gate which produces a conditional phase flip (other two qubit gates are equivalent up to single-qubit operations). Finally we transfer the entanglement between photons in each pair $\{a, b\}$ and $\{c, d\}$ back to the path polarization form. A successful CZ gate operation this leaves us in the desired state:

$$
( |1\rangle_a |\Downarrow\rangle_b |3\rangle_c |\Downarrow\rangle_d + |1\rangle_a |\Uparrow\rangle_b |4\rangle_c |\Downarrow\rangle_d + |2\rangle_a |\Uparrow\rangle_b |4\rangle_c |\Downarrow\rangle_d) |5\rangle_b |6\rangle_d \text{ (6) }
$$

FIG. 5. Conditional phase flip on two photons of two different chains with a single CZ operation.

Efficient construction of large circuits. A basic building block of the KLM scheme is a conditional phase flip gate that employs interference of input photons and ancillary photons and post-selection. This basic gate operates successfully with probability $1/16$. In our scheme this gate can be used for generating small circuits. However, for long enough quantum circuits the preparation process becomes inefficient, and gates with higher success rate must be employed. In the construction of the overall linked-state we proceed step by step, since a failure in the gate operation in one step might destroy previously constructed links and gates, we require that the combined two stage process of link/gate generation has an average a probability larger than 1/2 to progress. This can be achieved by replacing the basic CZ gates with an improved gate version proposed by KLM (for a recent improvement see [8]). These gates operate through the application of a new type of teleportation protocol based on the $n+1$ point Fourier transform ($F_{n+1}$), which operates successfully with probability $n/(n+1)$. This gate, $CZ_{n^2/(n+1)^2}$, is constructed of two independent $F_{n+1}$-based teleportations and therefore operates successfully with probability $n^2/(n+1)^2$. The application of each $CZ_{n^2/(n+1)^2}$ requires that a special ancillary state of $2n$ photons in $4n$ modes would be prepared in advance (by utilizing basic CZ gates). Thus, the preparation stage of our scheme has two parts. In the first part we prepare independent small-scale ancillary states with which we apply, in the second part, the $CZ_{n^2/(n+1)^2}$ gates to construct the overall linked-state.

The $CZ_{n^2/(n+1)^2}$ fails when either one of the independent teleportation protocols fails. A failure of the teleportation protocol results in the measurement of the teleported qubit. Let us consider the operation of applying a gate between two chains. By inspecting Eq. 6, it is clear that failure in teleporting photon $b$ would leave photon $a$ in either mode 1 or mode 2 breaking the link. In the same way failure in teleporting photon $d$ would break its corresponding link. Clearly, it would be more efficient to apply the two teleportation protocols in a sequence, where the second is applied only if the first has succeeded, eliminating the possibility of breaking two links. Thus, in applying a gate, the probability of success is $p = n^2/(n+1)^2$ while with probability $(1-p)$ one link is broken.

In adding a link to a chain we apply two CZ operations. These are applied to one photon (in four modes) which constitutes the last link of a chain, together with a newly introduced photon. As this new photon is not entangled to any chain, there is no point in wasting an $F_{n+1}$-based teleportation protocol on it. This photon can be prepared as part of the ancilla. Therefore, each of the CZ operations in adding a link will be carried out through the application of a "half" $CZ_{n^2/(n+1)^2}$ gate in which a pair of modes ($3$ and $4$ in Fig. 2) and afterwards $3'$ and $4'$ undergo teleportation, but the other pair ($5$ and $6$) does not. The $CZ$ is applied to this pair of modes together with the components of the ancilla in the first part of the preparation. The ancillary state $\tilde{\phi}$ for one $F_{n+1}$-based teleportation is $\tilde{\phi}_n = \sum_{j=0}^{n} |1\rangle^{j} |0\rangle^{n-j} |0\rangle^{j} |1\rangle^{n-j}$, defining a modified $\tilde{\phi}_n$ as $\tilde{\phi}_n = \sum_{j=0}^{n} (-1)^j |1\rangle^{j} |0\rangle^{n-j} |0\rangle^{j} |1\rangle^{n-j}$ The ancillary state for the two CZ operations would be

$$
|5\rangle |\tilde{\phi}_n|_{1} |\tilde{\phi}_n|_{2} + |6\rangle |\tilde{\phi}_n|_{1} |\tilde{\phi}_n|_{2} \text{ (7) }
$$

where 1 and 2 denote the states used for the teleportation of modes $\{3, 4\}$ and $\{3', 4'\}$ respectively. In this case, the overall process can fail in two ways. If a maximal
number of photons is detected at the outputs of the \( \hat{F}_{n+1} \) operation (thus destroying also the teleported photon), then the previous link is destroyed together with the gate operation that was applied to it. If no photon is detected the entanglement is not completely destroyed and we can still bring the system back to the initial state. Thus, the whole process (the two teleportation protocols) succeeds with probability \( p = n^2/((n+1))^2 \) and in the case of failure we have the same probability \((1-p)/2\) to either destroy the previous link (together with the gate operation) or to remain in the same initial state.

Employing our scheme for every logic gate in the required computation we need to perform six successful \( \hat{F}_{n+1} \) based teleportation protocols, which are equivalent to three \( CZ n^2/((n+1))^2 \) gates (two for adding two links to two different chains and one for applying the gate to those links). \( n = 3 \) is the smallest value for which the step-by-step construction of the linked-state can advance forward with higher probability than moving backwards. In a computer simulation of a construction process for a two qubit linked-state (a two qubit circuit), we obtained the average number of gates per logical gate, of \( \sim 220 \) for a \( CZ_{9/16} \) gate, and \( \sim 15 \) for a \( CZ_{16/25} \) gate. The large average number required for the case of the \( CZ_{9/16} \), results from an overall probability very close to 1/2 to advance forward. This number can be however significantly reduced if we change the overall construction by adding inert links to the chains, i.e. photons on which no gate operation is applied. In this case, for a qubit that takes part in \( n \) two-qubit gates we construct a chain of \( 2n \) links while the gates are applied to every second link. The only purpose of the inert links is to prevent a possible failure from spreading backwards, destroying previously constructed links and gates (if the step of adding a second link fails destructively then only the previous link is destroyed, the previous gate is not affected). Using this type of construction we need five successful gate operations for every logic gate. Applying a computer simulation on a pair of chains we obtain the number of \( \sim 23 \) \( CZ_{9/16} \) applications on average for every logic gate. The gate \( CZ_{4/9} \) can also be used by introducing additional inert links (at least three). When six inert links are added for each logic gate (thus, requiring 13 successful gate operations for each logic gate) we get around \( \sim 220 \) \( CZ_{4/9} \) applications on average per gate.

It should be noted that for large scale linked-states we can start the computation before the completion of the preparation. The computation is then carried out simultaneously with the construction of the linked state, where a ‘safety margin’ (in terms of the number of links) is maintained, keeping the probability for a sequence of failures that could destroy part of the data negligible. This method is more economical in terms of the required quantum storage capacity, as only a part of the complete linked-state is kept at a given instance.

So far we have assumed that the input is received classically. If we receive a quantum state as an input, where each qubit is encoded for example in two modes, then by measuring this qubit together with the path degree of freedom of the photon in the first link in the Bell basis will transfer this qubit to the head of the chain (to the polarization of the second photon as the first was measured). This measurement can be accomplished by applying a \( CZ \) gate together with additional one-qubit operations. Clearly, as this operation involves the actual input, the applied \( CZ \) gate must be one with a very high success rate. However, this operation is carried out only once for each qubit.

In the present scheme each photon carries two qubits. To achieve that we utilized, both polarization and path degree of freedom. This led to a simple path-polarization factorization of the linked state (1). However, our scheme does not require the use of polarization. It has been shown that a linear optical realization exists for any \( N \times N \) unitary matrix \( \begin{bmatrix} 10 \end{bmatrix} \). We can therefore represent the chain state (1) in terms of path degrees of freedom alone, by attributing four possible modes to each photon.

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