Online Matching with Set Delay

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Abstract. We initiate the study of online problems with set delay, where the delay cost at any given time is an arbitrary function of the set of pending requests. In particular, we study the online min-cost perfect matching with set delay (MPMD-Set) problem, which generalises the online min-cost perfect matching with delay (MPMD) problem introduced by Emek et al. (STOC 2016). In MPMD, $m$ requests arrive over time in a metric space of $n$ points. When a request arrives the algorithm must choose to either match or delay the request. The goal is to create a perfect matching of all requests while minimising the sum of distances between matched requests, and the total delay costs incurred by each of the requests. In contrast to previous work we study MPMD-Set in the non-clairvoyant setting, where the algorithm does not know the future delay costs. We first show no algorithm is competitive in $n$ or $m$. We then study the natural special case of size-based delay where the delay is a non-decreasing function of the number of unmatched requests. Our main result is the first non-clairvoyant algorithms for online min-cost perfect matching with size-based delay that are competitive in terms of $m$. In fact, these are the first non-clairvoyant algorithms for any variant of MPMD. Furthermore, we prove a lower bound of $\Omega(n)$ for any deterministic algorithm and $\Omega(\log n)$ for any randomised algorithm. These lower bounds also hold for clairvoyant algorithms.

1 Introduction

Studying online problems with delay is a line of work that has recently gained traction in online algorithms (e.g. [3,19,21,22]). In such problems, request arrive over time requiring service. Delaying the service of a request accumulates a delay cost given by a delay function associated with the request. The total cost of a solution is the cost of servicing all requests plus the sum of all delay costs incurred by each request.

We initiate the study of online problems with set delay. In this model, we generalize the notion of delay to one where the instantaneous delay cost at any point in time is determined by an arbitrary monotone non-decreasing function of the set of pending requests, rather than the sum of individual delay functions associated with each request. In particular, we study the online min-cost perfect matching with set delay (MPMD-Set) problem, which generalizes the min-cost perfect matching with delays (MPMD) problem introduced by Emek et al. [19].

In MPMD, $m$ requests arrive over time in a metric space of $n$ points. Upon arrival of a request the algorithm must choose to either match the request, incurring a cost equal to the distance between the two requests, or to delay the request, incurring a cost given by a delay function associated with the request. Prior results for MPMD have mostly focused on each request sharing the same delay function (in particular, linear, concave, and convex) and achieve competitive ratios that solely depend either on $n$ or $m$. Moreover, existing algorithms rely on clairvoyance, where the algorithm has full...
knowledge of future delay costs. Furthermore, existing randomised algorithms rely on
metric embeddings which require knowledge of the metric space in advance.
In this paper, we study the more general MPMD-Set in the least restrictive setting
where the algorithm does not know the metric space in advance and has no knowledge
of future delay costs. We begin by showing that, in contrast to prior results, the MPMD-Set
problem does not admit a deterministic competitive ratio that solely depends on \( n \) or \( m \).

**Theorem 1.** Every deterministic algorithm for MPMD-Set has competitive ratio \( \Omega(D) \),
where \( D \) is the diameter of the metric space.

Our lower bound holds even for simple instances where \( n \) and \( m \) are constants. Thus,
we restrict our attention to designing a competitive solution for the MPMD-Set problem
where the delay cost is a monotone non-decreasing function of the number of unmatched
requests. We call such a delay cost function *size-based* (See Section 2 for a formal
definition). MPMD-Set with size-based delay (MPMD-Size) has natural applications in
practical settings with service-level agreements such as cloud computing.

Our main result is the first competitive algorithms for MPMD-Size, where the
competitive ratio is a function of the number of requests. At the core of our result is a
reduction from MPMD-Size to the well-known Metrical Task System (MTS) problem.
(defined in Section 1.1).

**Theorem 2.** For any \( f(N) \)-competitive algorithm for MTS with \( N \) states, there is an
\( f(N_m) \)-competitive algorithm for MPMD-Size, where \( N_m \sim \left( \frac{m}{e} \right)^{m/2} \frac{e^{\sqrt{m}}}{(4e)^{1/4}} \) is the number
of matchings on \( m \) vertices.

We obtain our main result by applying state-of-the-art algorithms for MTS with
some modifications.

**Corollary 1.** For MPMD-Size, there is an \( O\left( \left( \frac{m}{e} \right)^{m/2} e^{\sqrt{m}} \right) \)-competitive deterministic
algorithm and an \( O(m^4 \log^4 m) \) randomised algorithm.

We emphasise that our algorithms are non-clairvoyant and do not need to know the
metric space in advance. To the best of our knowledge, the only prior works that provide
online algorithms in the non-clairvoyant setting are \([3]\) and \([24]\).

We complement Corollary \([1]\) with the following lower bounds.

**Theorem 3.** Every deterministic algorithm for MPMD-Size has competitive ratio \( \Omega(n) \).

**Theorem 4.** Every randomised algorithm for MPMD-Size has competitive ratio \( \Omega(\log n) \).

### 1.1 Our Techniques

Our main technical contribution is an online reduction from the MPMD-Set problem
to MTS, which constitutes the proof of Theorem \([2]\). The Metrical Task System (MTS)
problem, introduced by Borodin et al. \([14]\), is a cost minimisation problem defined by a
set of states \( S = \{ s_1, s_2, \ldots, s_k \} \) and a cost matrix \( d \) that defines the cost associated with
moving between states. The input is a sequence of tasks \( T = (t_1, \ldots, t_l) \). Each task \( t_j \)
is represented by an \( n \)-dimensional cost vector \( C_j = \{ c_1, c_2, \ldots, c_k \} \) where \( c_i \) defines the

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In these settings, the service level agreement requires the cloud provider to provide a certain
level of service and the provider incurs penalties if the level is not met.
cost of servicing task $t_j$ in state $s_i$. A solution is, for a given input task sequence $T$, a schedule, which is a sequence of states $\sigma = (M_1, M_2, \ldots, M_L)$, where $M_j = s_i$ means that task $j$ is processed in state $s_i$. The aim is to produce a schedule of minimal cost, where the total cost consists of the costs associated with moving states (transition cost), as well as the cost of processing the tasks (processing cost).

We briefly outline the three main parts of the reduction below.

**Step 1: MPMD-Set to MTS** The first part of the reduction transforms an instance of MPMD-Set into an instance of MTS. We achieve this by translating the set of all possible matchings of the requests into the set of input states. The set of input states thus develops over time as more requests arrive. The distance cost can be easily translated into the transition cost between matchings, and the delay cost is translated into the vector cost associated with serving tasks. From here onwards we refer to an instance of MTS that is reduced from MPMD-Set as $MPMD\text{-}Set\text{-}MTS$, and an MTS instance that is reduced from MPMD-Set with size-based delay as $MPMD\text{-}Size\text{-}MTS$.

The main challenge arises from translating the MTS solution back into a matching. In the MPMD-Set problem we are not allowed to remove a match once it has been made, while in the MTS problem, the algorithm may use transitions that correspond to the removal of matches. We therefore define the following property which we need an online schedule to satisfy before we can translate it back into an online matching. Let $\sigma = (M_1, \ldots, M_L)$ be a schedule for MPMD-Set-MTS.

**Definition 1 (Monotone).** We say a schedule $\sigma$ is monotone if $M_i \subseteq M_{i+1}$ for every $i$, i.e. it never removes an edge. We call an algorithm for MPMD-Set-MTS monotone if it always produces a monotone schedule.

A monotone schedule satisfies all requirements of an online matching as it does not use transitions that correspond to the removal of matches. However, we cannot guarantee that an arbitrary scheduling algorithm will produce such a schedule. It remains to prove the existence of a competitive monotone scheduling algorithm. To this end we define a relaxed version of this property as follows. Let $R(M_i)$ be the set of requests that are matched in $M_i$.

**Definition 2 (Sensible).** We consider $\sigma$ to be sensible if $R(M_{i-1}) \subseteq R(M_i)$. In other words, a request can be re-matched to another request but it cannot be unmatched once it has been matched. We call an algorithm for MPMD-Set-MTS sensible if it always produces a sensible schedule.

**Step 2: Converting to Sensible** We show that we can convert, in an online manner, an arbitrary MPMD-Size-MTS solution to a sensible solution at no extra cost. We do this by designing an online algorithm which, given an online sequence of states, produces for each state $M_i$ a corresponding state $M'_i$ such that the resulting schedule produced by the algorithm is sensible. We refer to an algorithm that transforms a given state as a state conversion algorithm. To prove the resulting schedule is of equal or less cost, we use a potential function to keep track of the transition costs incurred by both schedules. For the cost associated with processing the tasks we rely on the fact that, because the delay function satisfies the properties of a size-based delay function, there can be no benefit in un-matching and re-matching a request.
Step 3: Converting to Monotone It remains to show we can convert, in an online manner, every sensible MPMD-Size-MTS solution to a monotone solution at no extra cost. We do this by designing a second online state conversion algorithm which, given an online sequence of states, produces for each state $M'_i$, a corresponding state $M''_i$ such that the resulting schedule produced by the algorithm is monotone. The proof in Section 4.3 works for any delay function. The main technique used by the algorithm is to maintain a monotone matching of the requests that are matched in the previous state by matching the endpoints of vertex-disjoint paths in the symmetric difference between the current and previous state.

Since a monotone MPMD-Size-MTS solution corresponds to a matching of the same cost, the composition of the conversion algorithms in steps 2 and 3 lets us convert every MPMD-Size-MTS solution into an online matching at no extra cost.

Applying MTS algorithms There are two issues that prevent us from applying MTS algorithms directly. First, the cost bounds of all known algorithms for MTS have an additive term that is equal to the diameter of the MTS state space, and the MTS instance created by our reduction has state space with diameter much larger than the optimal. The second issue is that our reduction creates an MTS instance whose state space is constructed online, i.e. the states arrive over time. In Section 4.4 we show how to overcome these issues.

1.2 Related Work

MPMD was introduced by Emek et al. [19] where the delay functions associated with each request are uniform linear. They designed a randomised algorithm that achieves a competitive ratio of $O(\log^2 n + \log \Delta)$, where $n$ is the number of points in the metric space and $\Delta$ is its aspect ratio. Azar et al. [2] used a randomised HST embedding to provide a $O(\log n)$-competitive almost-deterministic algorithm, improving Emek et al.’s bound and removing the dependency on the aspect ratio of the metric space. Furthermore, they provided a lower bound of $\Omega(\sqrt{\log n})$ for any randomised algorithm in the case of linear delay. Liu et al. [1] improved this lower bound to $\Omega(\frac{\log n}{\log \log n})$ and $\Omega(\sqrt{\frac{\log n}{\log \log n}})$ for the bipartite case, which are the best known so far. Liu et al. furthermore adapted the algorithm by Azar et al. to the bipartite setting and improved the analysis of Emek et al.’s algorithm to $O(\log n)$. The next deterministic algorithm for simple metrics was by Emek et al. [20] who proved a competitive ratio of 3 for the simple metric space of 2 points. The first deterministic algorithm for general metric spaces was by Bienkowski et al. [12] and their analysis resulted in a competitive ratio of $O(m^{0.46})$. Bienkowski et al. [11] and Azar et al. [5] concurrently and independently improved this bound to $O(m)$ and $O(m^{0.59})$ respectively, introducing the first linear and sub-linear deterministic solutions to the problem. The algorithms above assumed the delay cost to be given by a uniform linear delay function associated with each individual request.

Liu et al. [25] was the first to consider convex delay functions and demonstrated an interesting gap between the solutions for the case with linear delay and convex delay on a uniform metric space by giving a deterministic asymptotically optimal $O(m)$-competitive algorithm for the uniform metric space.
Azar et al. [7] subsequently considered the problem with concave delay and achieved an $O(1)$-competitive deterministic algorithm for the single point metric space and an $O(\log n)$ randomised algorithm for general metric spaces.

The above algorithms assumed all requests incurred delay in accordance with uniform delay functions and regarded the delay function to be associated with each individual request. Furthermore, all prior solutions to MPMD assumed clairvoyance. To the best of our knowledge, no one has considered the non-clairvoyant generalisation of the problem where the delay function depends on the set of unmatched requests.

Non-clairvoyant algorithms nevertheless have been designed for other online problems such as the Set Cover problem [3,24] and multi-level aggregation [24].

The notion of introducing delay to online problems originated well before it was applied to online metric matching and finds applications in amongst others aggregating messages in computer networks, aggregating orders in supply-chain management, and operating systems. See [3,4,6,8,10,13,16,17,21,22,24,26] for further reading. All problems above define the cost of delay as a function associated with each request. To the best of our knowledge, no online problems with delay have so far defined the cost of delay as an arbitrary function of the set of unmatched requests.

2 Preliminaries

In this section we introduce our notation and give formal definitions for set delay and size-based delay functions, as well as MPMD-Set.

We denote a match between two points by an edge $e$ in the metric space and define the weight of an edge $w(e)$ to be the distance between the two points. For any two matchings $M$ and $M'$, we define the cost of changing from one matching to the other as the total weight of all the edges in the symmetric difference between the two matchings $\sum_{e \in M \oplus M'} w(e)$, which we denote as $c(M \oplus M')$. We divide time into discrete timesteps.

**Definition 3 (Set delay function).** Let $U$ be a set of requests. We define a delay function $f_t : U \rightarrow \mathbb{R}^+$ to be a set delay function if it satisfies the following properties:

- $f_t(\emptyset) = 0$
- $A \subseteq B \Rightarrow f_t(A) \leq f_t(B)$
- For all $\emptyset \neq U \in 2^V$, we have $\sum_{t=0}^{\infty} f_t(U) = \infty$

The last property implies that all requests must eventually be matched.

**Definition 4 (Size-based delay function).** We define a delay function $f_t : U \rightarrow \mathbb{R}^+$ to be size-based if it satisfies all properties of a set delay function and is monotone non-decreasing as a function of the size of the set of requests $U$ for any time $t$.

2.1 Online Min-cost Perfect Matching with Set Delay

MPMD-Set is defined on a metric space $(V, d)$, which consists of a set of points $V$ and distance function $d : V \times V \rightarrow \mathbb{R}^+$. An online input instance over $(V, d)$ is a sequence of requests $R = (r_1, \ldots, r_{2m})$ that arrive at points in the metric space over time. Each $r_k \in R$ has an associated position and arrival time.
We assume that requests only arrive at the start of a timestep. A solution produced by an online matching algorithm is a sequence of matchings $\mathcal{M} = (M_0, \ldots, M_{f_{\text{inal}}})$, where $M_i$ is the matching associated with the $i$th timestep. $\mathcal{M}$ satisfies the following properties:

- $M_0 = \emptyset$
- $M_{f_{\text{inal}}}$ is a perfect matching
- For all $i$, $M_i \subseteq M_{i+1}$

The aim of an online matching algorithm is to produce a sequence of matchings that satisfies the above with minimal cost. An online solution to MPMD-Set incurs two types of cost. The first is the delay cost, which is given by a set delay function of the set of unmatched requests at time $t$.

**Definition 5** ($U_i(M)$). Given a matching $M$, let $U_i(M)$ be the set of unmatched requests in $M$ among the requests that have arrived up to and including time $i$.

The total delay cost incurred by an online matching algorithm is:

$$\sum_{M_i \in \mathcal{M}} f_i(U_i(M_i))$$

The second cost is the distance cost, which can be expressed as the sum of all symmetric differences between the matchings in the sequence $\mathcal{M}$.

$$\sum_{i=0}^{\vert \mathcal{M} \vert - 1} d(M_i, M_{i+1}) = \sum_{i=0}^{\vert \mathcal{M} \vert - 1} c(M_i \oplus M_{i+1})$$

The total cost of solution $\mathcal{M}$ is thus:

$$\text{cost}(\mathcal{M}) = \sum_{i=0}^{\vert \mathcal{M} \vert - 1} (c(M_i \oplus M_{i+1}) + f_i(U_i(M_i)))$$

### 3 A Lower Bound for MPMD-Set

In this section, we prove Theorem 1 by constructing a lower bound instance that results in a competitive ratio of $\Omega(D)$, where $D$ is the diameter of the metric space.

**Proof of Theorem 1**. Consider a four-point metric space (as depicted in figure 1) with three points at distance $\epsilon$ from one another, and the fourth point ($p_4$) at distance $D$ from the other points, where $D$ is the diameter of the metric space.

![Fig. 1. A visualisation of the four-point metric space](image)

We define a request sequence of six requests $R = (r_1, r_2, r_3, r_4, r_5, r_6)$ where the first four requests arrive at time $t = 0$ and the latter two arrive at time $t = 2$. For each...
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At $t = 0$, request $r_1$ reaches its deadline and hence the algorithm will need to match two requests. Since the algorithm is non-clairvoyant, the algorithm has no knowledge of the deadlines of future requests. We therefore assume without loss of generality that it matches $r_1$ to $r_3$ and pays a distance cost of $\epsilon$. At $t = 1$, $r_2$ reaches its deadline and the algorithm is forced to match it to $r_4$ at a distance cost of $D$. At $t = 2$, the final two requests will arrive and instantly reach their deadline. The algorithm will consequently need to match $r_5$ to $r_6$ at a distance cost of $D$. The total cost of ALG is $2D + \epsilon$.

The optimal offline solution OPT is to match $r_1$ to $r_2$ and match locally at $t = 2$ on points $p_3$ and $p_4$. The total cost of OPT is therefore $\epsilon$. This concludes the proof of Theorem 1. □

4 An Online Reduction from MPMD-Set to MTS

In this section, we prove Theorem 2 by defining a reduction from MPMD-Set to MTS. We start by translating an arbitrary instance of MPMD-Set into an instance of MTS in Section 4.1. In Section 4.2, we show that we can transform an arbitrary MPMD-Size-MTS solution into a sensible solution of the same or less cost. We then show in Section 4.3 that for any MPMD-Set-MTS instance, we can transform every sensible solution into a monotone solution at no extra cost. Finally, we use an observation that a monotone schedule directly corresponds to an online matching of equal cost. This completes the proof of Theorem 2. We finish this section with a proof of Corollary 1.

4.1 Translating an instance of MPMD-Set into and instance of MTS

We define the set of internal states of the MTS instance to be the set of all possible matchings of the requests in the metric space associated with the MPMD-Set instance. Without loss of generality, we assume the requests arrive at unique points in the metric space. The cost of moving between two states is defined to be the weight of the symmetric difference between the set of edges in each state $(c(M_i \oplus M_j))$. It is easy to see that this satisfies the necessary requirements for a metric space. Each task in the MTS instance is associated with a given timestep in the MPMD-Set problem. The cost vector associated with each task is, for every possible state, the delay cost accumulated by the set of unmatched requests in the original MPMD-Set instance during that timestep. The total cost of a schedule $\sigma$ can thus be expressed as follows.

\[
\text{cost}(\sigma) = \sum_{i=0}^{\vert \sigma \vert - 1} (c(M_i \oplus M_{i+1}) + f_i(U_i(M_i))).
\]

By construction, the cost associated with processing the tasks represent the delay cost incurred by the requests, while the distance cost is represented by the transition costs associated with moving between states. It remains to prove that we can translate a given schedule produced on the MPMD-Set-MTS instance back into a valid matching.
By definition of the MPMD-Set problem all solutions have the property that once a match has been made, it will never be removed. An MPMD-Set-MTS solution may not necessarily satisfy this property, as the MTS problem definition allows the solution to move between states that correspond to the removal of edges. We can however make the following observation about the MPMD-Set-MTS instance.

**Observation 5.** Every monotone schedule corresponds to an online matching of equal cost.

In the next two subsections, we show how to convert an arbitrary MPMD-Size-MTS solution into a sensible solution, and an arbitrary sensible solution to a monotone one.

### 4.2 Size-based delay functions admit sensible scheduling algorithms

In this subsection, we prove the existence of an online algorithm that converts an arbitrary MPMD-Size-MTS solution into a sensible solution, without incurring any extra cost.

**Lemma 1.** There exists an online algorithm that converts an arbitrary MPMD-Size-MTS solution into a sensible solution of the same or less cost.

**Proof.** To prove Lemma 1, we define an online state conversion algorithm (Sensible-ALG) which, for every state $M_i$ in a schedule $\sigma$, produced by any online scheduling algorithm (OSA), produces a state $M'_i$ such that the cost of the schedule $\sigma' = (M'_0, \ldots, M'_{|\sigma|})$ is at most the cost of the original schedule $\sigma$, and $\sigma'$ is sensible.

To introduce the state conversion algorithm, we require the following definition.

**Definition 6 ($R_i(M)$).** Given a matching $M$, let $R_i(M)$ be the set of matched requests in $M$ among the requests that have arrived up to and including time $i$.

The state conversion algorithm aims to stay as close as possible to the states in the original schedule $\sigma$ while maintaining the main property of a sensible schedule, which is that $R_i(M'_{i-1}) \subseteq R_i(M'_i)$ for every $i \in [|\sigma]|$. Sensible-ALG achieves staying close to the states in $\sigma$ by augmenting along paths that are in the symmetric difference between the previous state produced by Sensible-ALG ($M'_{i-1}$) and the newly arrived state produced by OSA ($M_i$). Sensible-ALG maintains sensibility by only augmenting along $M'_{i-1}$-augmenting paths in the symmetric difference between $M_i$ and $M'_{i-1}$.

We are now ready to formally define Sensible-ALG.

**Description of Algorithm** Our online algorithm takes as input a sequence of states $\sigma = (M_1, \ldots, M_k)$ produced by an online scheduling algorithm, where $\sigma$ satisfies that for all $0 \leq i < |\sigma| - 1$, the symmetric difference $M_i \oplus M_{i+1}$ consists of a single connected component. Note that if the input schedule does not satisfy these properties then we can add intermediate timesteps and states to the schedule such that it satisfies the above property and the cost remains the same. When a new state $M_i$ arrives, the algorithm augments $M'_i$ along every $M'_{i-1}$-augmenting path in the symmetric difference $M'_{i-1} \oplus M_i$. In this online fashion, the algorithm constructs a sequence of states $\sigma' = (M'_1, \ldots, M'_n)$ whose cost is at most the cost of $\sigma$.

Fix a timestep $i$. Since $M_{i-1} \oplus M_i$ is a single component, $M'_{i-1} \oplus M_i$ can have at most one more $M'_{i-1}$-augmenting path than $M'_{i-1} \oplus M_{i-1}$. Moreover, since Sensible-ALG augmented along every $M'_{i-2}$-augmenting path in $M'_{i-2} \oplus M_{i-1}$ in the previous timestep,
there are no \( M'_{i-1} \)-augmenting path in \( M'_{i-1} \oplus M_{i-1} \). Thus, we conclude there is at most one \( M'_{i-1} \)-augmenting path in \( M'_{i-1} \oplus M_{i} \). Let \( P_{i} \) denote the single connected component in \( M'_{i-1} \oplus M_{i} \) and let \( P'_{i} \) denote the single connected component in \( M'_{i-1} \oplus M_{i} \) (note that \( P'_{i} \) can be empty for some \( i \)). Since Sensible-ALG augments along \( P'_{i} \), it follows that 
\[
 c(M'_{i-1} \oplus M'_{i}) = c(P'_{i}).
\]

Since we only ever augment along \( M'_{i-1} \)-augmenting paths in \( M'_{i-1} \oplus M_{i} \), it must be that \( R_{i}(M'_{i-1}) \subseteq R_{i}(M'_{i}) \). We thus conclude that \( \sigma' \) is sensible.

It remains to show that the cost of \( \sigma' \) is at most the cost of \( \sigma \). We start by showing that the transition cost incurred by \( \sigma' \) is at most that incurred by \( \sigma \).

**Lemma 2.** \( \sum_{i=0}^{\lfloor \sigma \rfloor -1} c(M'_{i-1} \oplus M'_{i}) \leq \sum_{i=0}^{\lfloor \sigma \rfloor -1} c(M_{i-1} \oplus M_{i}) \).

**Proof of Lemma 2** We prove this lemma by introducing the potential \( \phi_{i} = c(M_{i} \oplus M'_{i}) \). Note that \( \phi_{i} \) is the distance between the states \( M'_{i} \) and \( M_{i} \). Using the potential, we will prove the cost incurred by the \( i \)th state transition in \( \sigma' \) is at most the cost of the \( i \)th state transition in \( \sigma \).

**Claim.** For all \( i \), we have 
\[
 c(M'_{i-1} \oplus M'_{i}) \leq c(M_{i-1} \oplus M_{i}) - (\phi_{i} - \phi_{i-1}).
\]

**Proof of Claim 2.** We first show that since \( P'_{i} \) is a subset of \( M'_{i-1} \oplus M_{i} \), we can express the transition cost incurred by Sensible-ALG during iteration \( i \) (\( c(P'_{i}) \)) in terms of 
\[
 c(M'_{i-1} \oplus M_{i})
\]
and \( \phi_{i} \). We then use triangle inequality to upper bound the transition cost incurred by OSA during iteration \( i \). Finally, we combine these results to conclude the correctness of the claim.

Since \( M'_{i} = M'_{i-1} \oplus P'_{i} \), it follows that:
\[
 c(M'_{i} \oplus M_{i}) = c((M'_{i-1} \oplus P'_{i}) \oplus M_{i}) = c((M'_{i-1} \oplus M_{i}) \oplus P'_{i}) = c(M'_{i-1} \oplus M_{i}) - c(P'_{i}).
\]
The last equality holds because \( P'_{i} \) is a subset of \( M'_{i-1} \oplus M_{i} \).

The cost of Sensible-ALG during iteration \( i \) can therefore be expressed as follows.
\[
 c(P'_{i}) = c(M'_{i-1} \oplus M_{i}) - c(M'_{i} \oplus M_{i}) = c(M'_{i-1} \oplus M_{i}) - \phi_{i}.
\]

Having expressed the cost of Sensible-ALG during iteration \( i \) in terms of the potential and the symmetric difference \( M'_{i-1} \oplus M_{i} \), we now bound the cost of the original online scheduling algorithm OSA during iteration \( i \). Using triangle inequality, we deduce:
\[
 c(M'_{i-1} \oplus M_{i}) \leq c(M'_{i-1} \oplus M_{i-1}) + c(M_{i-1} \oplus M_{i}) = c(M'_{i-1} \oplus M_{i-1}) + c(P_{i}).
\]

We thus bound the transition cost of OSA during iteration \( i \) as follows:
\[
 c(P_{i}) \geq c(M'_{i-1} \oplus M_{i}) - c(M'_{i-1} \oplus M_{i-1}) = c(M'_{i-1} \oplus M_{i}) - \phi_{i-1}.
\]

Combining Inequalities 1 and 2, we get:
\[
 c(P'_{i}) - c(P_{i}) \leq c(M'_{i-1} \oplus M_{i}) - \phi_{i} - (c(M'_{i-1} \oplus M_{i}) - \phi_{i-1}) = \phi_{i-1} - \phi_{i},
\]
and so 
\[
 c(M'_{i-1} \oplus M'_{i}) \leq c(M_{i-1} \oplus M_{i}) - (\phi_{i} - \phi_{i-1}).
\]

This completes the proof of Claim 2. \( \square \)
Using Claim 4.2, we determine the total transition cost incurred by $\sigma^*$ as follows.

$$\sum_{i=1}^{\lvert \sigma \rvert} c(M'_{i-1} \oplus M_i) \leq \sum_{i=1}^{\lvert \sigma \rvert} c(M_{i-1} \oplus M_i) - \sum_{i=1}^{\lvert \sigma \rvert} (\phi_i - \phi_{i-1})$$

$$= \sum_{i=1}^{\lvert \sigma \rvert} c(M_{i-1} \oplus M_i) - \phi_{\lvert \sigma \rvert} + \phi_0$$

$$\leq \sum_{i=1}^{\lvert \sigma \rvert} c(M_{i-1} \oplus M_i).$$

The last inequality holds because $\phi_0 = 0$ and $\phi_{\lvert \sigma \rvert} \geq 0$. \hfill \square

**Definition 7 (processing cost).** We refer to the cost associated with processing the tasks of an online scheduling algorithm as the processing cost.

Recall that in MPMD-Set-MTS, the processing cost is the delay incurred by the set of unmatched requests in the state. It remains to show that processing cost incurred by $\sigma^*$ is at most that incurred by $\sigma$. To this end we use the following fact.

**Fact 6.** Given two matchings $M$ and $M'$ such that $|R_i(M')| > |R_i(M)|$, there exist at least $\frac{|R_i(M')| - |R_i(M)|}{2}$ vertex-disjoint $M$-augmenting paths in $M \oplus M'$. Furthermore, if $R_i(M) \leq R_i(M')$, then there exist exactly $\frac{|R_i(M')| - |R_i(M)|}{2}$ vertex-disjoint $M$-augmenting paths in $M \oplus M'$ whose endpoints are exactly $R_i(M') \setminus R_i(M)$.

**Lemma 3.** $\sum_{i \in \lvert \sigma \rvert} f_i(U_i(M'_{i-1})) \leq \sum_{i \in \lvert \sigma \rvert} f_i(U_i(M_i)).$

**Proof of Lemma 3.** Since $f_i$ is a size-based delay function, if $|R_i(M')| \geq |R_i(M)|$ (and hence $|U_i(M'_{i-1})| \leq |U_i(M_i)|$) for all $i$, then the lemma must hold. Since the algorithm augments along every $M'_{i-1}$-augmenting path in $M'_{i-1} \oplus M_i$, it follows that $|R_i(M'_{i-1})| \geq |R_i(M_i)|$ for all $i$. \hfill \square

The cost of an MPMD-Set-MTS solution consists of the transitions cost, as well as the processing cost. From Lemmas 2 and 3, we deduce that both the transition cost and the processing cost of $\sigma^*$ is at most that of $\sigma$ produced by any online scheduling algorithm. This concludes the proof of Lemma 1. \hfill \square

### 4.3 Sensible solutions imply monotone solutions

In this subsection, we prove the existence of an online algorithm that converts a sensible MPMD-Size-MTS solution into a monotone solution without incurring any extra cost.

**Lemma 4.** There exists an online algorithm that converts a sensible MPMD-Size-MTS solution into a monotone solution of the same or less cost.

**Proof.** To prove this lemma, we introduce a second online state conversion algorithm Monotone-ALG, which converts any state $M_i$, produced by a sensible scheduling algorithm on MPMD-Size-MTS, into a state $M'_i$ such that the schedule $\sigma^* = (M'_0, ..., M'_k)$ is monotone and of less or equal cost to the original schedule. We start by defining some notation used in the proof.
**Definition 8** (Newly matched request). Given a schedule $\sigma$, a request is newly matched in $M_i$ if it is matched in $M_i$ but unmatched in all prior states $\{M_j\}_{j=0}^{i-1}$ in $\sigma$.

We are now ready to define our algorithm.

**Description of Algorithm** Our online state conversion algorithm Monotone-ALG takes as input an online sequence of states $\sigma = (M_1, M_2, \ldots, M_T)$ produced by a sensible scheduling algorithm and outputs, for each state $M_t \in \sigma$, a state $M'_t$, such that the cost of the schedule $\sigma' = (M'_1, M'_2, \ldots, M'_T)$ is at most that of $\sigma$, and $\sigma'$ is monotone. When a new state $M_t$ arrives, Monotone-ALG computes the symmetric difference between the current state $M_t$ and the previous state $M_{t-1}$ in $\sigma$. By Fact 6, $M_{t-1} \oplus M_t$ contains at least $k = \frac{|R_t(M_t)| - |R_t(M_{t-1})|}{2}$ vertex-disjoint $M_{t-1}$-augmenting paths $P_1, \ldots, P_k$. Furthermore, since $\sigma$ is sensible, all endpoints of the paths are new requests with respect to $M_t$. Monotone-ALG adds, for every path $P_j$, an edge between the endpoints $(u_j, v_j)$ to $M'_t$.

Observe that since the algorithm only adds edges between new requests, it holds that for any $i$, $M'_{t-1} \subseteq M'_i$. Thus, the resulting schedule $\sigma' = (M'_0, \ldots, M'_{|\sigma|})$ is monotone.

It remains to show that the cost of $\sigma'$ is at most the costs of $\sigma$. We start by proving the transition cost incurred by $\sigma'$ is at most the transition cost incurred by $\sigma$.

**Lemma 5.** For every iteration $j$ of the algorithm, $c(M'_{j-1} \oplus M'_j) \leq c(M_{j-1} \oplus M_j)$.

**Proof of Lemma 5** Fix an iteration $j$ of the algorithm. Let $k_j = \frac{|R_t(M_j)| - |R_t(M_{j-1})|}{2}$. Recall that $u_i$ and $v_i$ are endpoints of $P_i$. The transition cost incurred by $\sigma$ is

$$c(M_{j-1} \oplus M_j) \geq \sum_{i=1}^{k_j} c(P_i) \geq \sum_{i=1}^{k_j} c(u_i, v_i) = c(M'_{j-1} \oplus M'_j),$$

where the second inequality is due to triangle inequality. $\square$

We now show that the processing cost incurred by $\sigma'$ is at most that incurred by $\sigma$.

**Lemma 6.** $\sum_{i=0}^{|\sigma|} f_i(U_i(M'_j)) \leq \sum_{i=0}^{|\sigma|} f_i(U_i(M_j))$.

**Proof of Lemma 6** By construction, the cost of processing the $i$th task in any given state is the cost of delaying all unmatched requests in the state at time $i$. We know any set delay function satisfies that for any subsets $A$ and $B$ of requests that have arrived so far, $A \subseteq B$ implies that $f_i(A) \leq f_i(B)$. We consequently introduce the following invariant.

**Invariant 7.** For any $i$, $R_i(M_i) = R_i(M'_i)$. In other words, all requests that are matched in $M_i$ are matched in $M'_i$.

**Proof of Invariant 7** We prove the invariant by induction on $i$.

If $i = 0$, then $M_0 = \emptyset = M'_0$. Hence, $R_i(M_0) = R_i(M'_0)$.

Assume $R_i(M_k) = R_i(M'_k)$ for all $i$ in $\{0..k\}$. Then $R_i(M_{k+1}) \setminus R_i(M_k)$ is simply the set of new requests with respect to iteration $k + 1$. From Fact 6 and the sensibility of $\sigma$, we can deduce that since $R_i(M'_k) = R_i(M_k) \subseteq R_i(M_{k+1})$, there must exist $\frac{R_i(M_{k+1}) - R_i(M'_k)}{2}$ vertex-disjoint paths between all the new requests in $R_i(M_{k+1})$. By construction Monotone-ALG will match all new requests. Therefore, $R_i(M'_{k+1}) = R_i(M_{k+1})$. $\square$
We can now use Invariant 7 to prove the lemma as follows. For all $i$, $R_i(M'_i) = R_i(M_i)$ implies that $U_i(M'_i) = U_i(M_i)$ and thus $f_i(U_i(M'_i)) \leq f_i(U_i(M_i))$. Therefore, $\sum_{i=0}^{\|\sigma\|} f_i(U_i(M'_i)) \leq \sum_{i=0}^{\|\sigma\|} f_i(U_i(M_i))$. This concludes the proof of Lemma 6.

We conclude that the total cost of $\sigma'$ is at most that of $\sigma$ and that $\sigma'$ is monotone, which completes the proof of Lemma 4.

4.4 Applying MTS Algorithms to MPMD-Set-MTS

In this section, we prove Corollary 1. Consider an instance of MPMD-Set with $m$ requests in a metric space of $n$ points and the instance of MPMD-Set-MTS created by applying Theorem 2. Let $N$ be the number of states of the MPMD-Set-MTS instance.

There are two issues that arise when applying MTS algorithms to MPMD-Set-MTS directly. The first issue is that all known MTS algorithms have a cost bound of the form $f(N) \cdot \text{cost}(OPT) + D$ where $OPT$ is the optimal MTS solution and $D$ is the diameter of the MTS state space. Observe that $D$ is equal to the distance between the empty matching and the max-cost perfect matching, i.e. the cost of the max-cost perfect matching. Unfortunately, the cost of the max-cost perfect matching can be much larger than that of the optimal solution. To overcome this, we restrict the MTS solution to only use states whose distance from the initial state is at most $\text{cost}(OPT)$ by setting the costs of the other states to be infinite. This effectively reduces the diameter of the state space to at most $\text{cost}(OPT)$ and thus we get a cost bound of $O(f(N)) \cdot \text{cost}(OPT)$. We can now use the $O(N)$-competitive deterministic algorithm of [14] to obtain our deterministic algorithm for MPMD-Size.

The second issue stems from the fact that the reduction in Theorem 2 creates an MTS instance where the states are arriving over time. This is because the states correspond to matchings of requests and the requests are arriving online. This does not pose a problem for the deterministic $O(N)$-competitive Work Function Algorithm of [14]. However, we cannot directly apply the current-best randomised algorithm for MTS of [15] as it pre-computes a probabilistic embedding of the MTS metric space into a hierarchically separated tree (HST). Instead, we need to use a probabilistic online embedding into a HST together with the $O(\log N)$-competitive randomized algorithm for MTS on HSTs of [15]. Using the online embedding of [23] adds a factor of $O(\log N \log \Phi)$ where $\Phi$ is the ratio of the largest distance to the smallest distance in the MTS state space, i.e. the aspect ratio. However, $\Phi$ can be arbitrarily large. We deal with this by observing that the abstract network design framework of [9] can be extended to apply to MTS [2] using the framework of [9] allows us to reduce the overhead due to the online embedding to $O(\log^3 N)$ for an overall competitive ratio of $O(\log^4 N)$.

5 A Deterministic Lower Bound for MPMD-Size

In this section we lower bound the competitive ratio of any deterministic online matching algorithms for MPMD-Size.

While the definition of abstract network design in [9] only captures the transition cost of MTS, we can extend it to include task processing costs because the latter is not affected by the embedding.
Theorem 8. Every deterministic algorithm for MPMD-Size has competitive ratio $\Omega(n)$, where $n$ is the number of points in the metric space.

Proof. Consider an $n$-point uniform metric space with distance 1 between all points. We fix a deterministic online matching algorithm ALG that will process a request sequence of size $2n - 2$ determined by an adversary. The aim is to force ALG to match requests at two distinct points in the metric space $n - 1$ times by ensuring that each time ALG needs to match two requests there is at most a single unmatched request available at each point in the metric space. We then ensure the optimal solution to the instance is to match requests at distinct points only once by placing two requests at $n - 2$ points in the metric space, and placing only a single request at two points in the metric space. To this end we define the adversary to satisfy the following properties at all times:

1. A new request is only ever placed at a point with no unmatched requests.
2. Every point in the metric space receives at most 2 requests.

The first property of the adversary ensures that at most one unmatched request is available at any point in the metric space at all times and thereby aims to ensure ALG must match requests across two distinct points each time. The second property aims to ensure the optimal solution is to only match requests at distinct points a single time.

Before we define the behaviour of the adversary in more detail, we divide time up into phases and define the delay function in relation to each phase. The first phase starts at $t = 0$ and ends when ALG performs a match. The next phase begins when the previous phase ends, and the last phase ends when ALG has matched the last request in the request sequence. The delay function is the same for all timesteps $t$ within the same phase. For the $i$th phase the delay function is defined as follows.

For all timesteps $t$ in phase $i$, for any subset of requests $S$,

$$f_i(S) = \begin{cases} 0 & |S| \leq (n - i) \\ \infty & |S| = (n - i) + 1 \end{cases}$$

We are now ready to define the behaviour of the adversary in more detail. At time $t = 0$, the first phase begins and the adversary places $n$ requests, one at every point in the metric space. This satisfies all properties of the adversary and invokes an infinite delay cost for ALG, forcing the algorithm to perform a match. Since we have exactly 1 unmatched request at every point, ALG is forced to match requests at distinct points in the metric space and incur a distance cost of 1.

To define the behaviour of the adversary during the remaining phases, we first introduce the following terminology.

Definition 9 (Saturated points). We call a point in the metric space saturated if it has received at least 2 requests so far, and unsaturated otherwise.

Definition 10 (Active points). We call a point in the metric space active if it currently hosts an unmatched request, and inactive otherwise.

For all $i \in \{2 \ldots n - 1\}$, at the start of phase $i$, the adversary will place a request at an inactive unsaturated point. By definition, this satisfies the properties of the adversary
and thereby forces ALG to match two distinct points in the metric space to avoid infinite delay.

Before we analyse the competitive ratio we need to show that the adversary is well-defined. To this end we prove that, regardless of the behaviour of ALG, at the end of each phase \( i \in \{1...n-2\} \), after ALG has performed a match, there always exists an inactive unsaturated point that the adversary can place a request on. We formalise this in the following claim.

Claim. At the end of each phase \( i \) (after ALG has matched), there exist \( i + 1 \) inactive points and at least two of these points are unsaturated.

Proof of Claim\(^5\) We prove the claim by induction on the number of phases \( i \).

The base case is when \( i = 1 \). At the start of phase 1, every point in the metric space receives a single request. At the end of phase 1, ALG is forced to match 2 arbitrary requests at different points in the metric space. Therefore, at the end of phase 1, there will be 2 points in the metric space that are both inactive and unsaturated.

Assume the claim holds for all phases \( i = 1 \) up to \( i = k \). By this assumption, at the start of phase \( k + 1 \) we have \( k + 1 \) points in the metric space that are inactive and at least two of these requests are unsaturated. The adversary can now place a new requests at one of the two unsaturated points, satisfying its current properties, and leaving \( k \) points inactive (one of which is still unsaturated). At the end of phase \( k + 1 \), ALG will be forced to match requests across two distinct points in the metric space again, adding another 2 points to the set of inactive points. By the end of phase \( k + 1 \), there are thus a total of \( k + 2 \) inactive points and at least one of them is unsaturated. It now remains to prove at least two of the inactive points must be unsaturated. Assume for the sake of contradiction that this is not the case. Then all remaining \( k + 1 \) inactive points must be saturated. Since the points are inactive and saturated this means that ALG has matched at least \( 2 \cdot (k + 1) \) points in \( k + 1 \) phases. In every phase ALG matches exactly 2 points. By the end of phase \( k + 1 \), ALG can thus have matched at most \( 2 \cdot (k + 1) \) requests. But the \( k + 1 \) saturated and 1 unsaturated points imply at least \( 2 \cdot (k + 1) + 1 \) requests must have been matched by the end of this phase. This constitutes a contradiction. We conclude that there are \( k + 2 \) inactive points by the end of iteration \( k + 1 \) and that at least 2 of them are unsaturated. By the principle of induction, Claim\(^5\) holds.

We conclude the adversary is well-defined. It remains to analyse the competitive ratio. Because the adversary maintained the first property, it follows that each point in the metric space hosts at most one unmatched request at any time. From this we conclude that ALG incurred a distance cost of 1 during each phase, resulting in a total distance cost of \( n - 1 \). Furthermore, because the adversary maintained the second property and a total of \( 2n - 2 \) request arrived at \( n \) points in the metric space, it follows that two requests arrived at \( n - 2 \) points in the metric space and one request arrived at two points in the metric space. Let us refer to the latter two points as \( v_1 \) and \( v_n \). The optimal solution is to match the requests at \( v_1 \) and \( v_n \) in the first phase and in each consecutive phase, to match the two requests at the same point. Since neither ALG nor OPT incurred any delay cost, the total cost of ALG and OPT are as follows:

\[
\text{Cost}(\text{OPT}) = 1 \\
\text{Cost}(\text{ALG}) = n - 1
\]
From this we conclude our lower bound on the competitive ratio for any deterministic online matching algorithm on MPMD-Size.

6 A Randomised Lower Bound for MPMD-Set with Size-based Delay

In this section, we lower bound the competitive ratio of any randomised online matching algorithm for MPMD-Set.

Theorem 9. Every randomised algorithm for MPMD-Set has competitive ratio $\Omega(\log n)$, where $n$ is the number of points in the metric space.

Proof. Consider an $n$-point uniform metric space with distance 1 between all points. Applying Yao’s principle [27], we define a uniform random distribution over the inputs such that any deterministic online matching algorithm will have expected cost $\Omega(\log n)$. We define the behaviour of the adversary to ensure the expected cost of the optimal solution is always 1. To this end we define the adversary, which will place $2n - 2$ requests, to satisfy the following property at all times:

- Every point in the metric space receives at most 2 requests.

This property ensures the optimal solution is to only match requests at distinct points a single time, regardless of where the requests are placed in the metric space.

Similar to the previous section, we divide time up into phases and define the delay function in relation to each phase. The first phase starts at $t = 0$ and ends when ALG performs a match. The next phase begins when the previous phase ends, and the last phase ends when ALG has matched the last request in the request sequence. The delay function is the same for all timesteps $t$ within the same phase. For the $i$th phase the delay function is defined as follows.

For all timesteps $t$ in phase $i$, for any subset of requests $S$,

$$
\tilde{f}_t(S) = \begin{cases} 
0 & |S| \leq (n - i) \\
\infty & |S| = (n - i) + 1
\end{cases}
$$

(4)

We now define the behaviour of the adversary in more detail. At time $t = 0$, the first phase begins and the adversary places $n$ requests, one at every point in the metric space. This satisfies all properties of the adversary and invokes an infinite delay cost for ALG, forcing the algorithm to perform a match. Since we have exactly 1 unmatched request at every point, ALG is forced to match two requests at distinct points in the metric space, and incur a distance cost of 1. In order to define the behaviour of the adversary on the remaining phases, recall from Section 5 that we define a point to be saturated if it has received 2 requests (and unsaturated otherwise) and active if it hosts an unmatched request (and inactive otherwise).

For all $i \in \{2 \ldots n - 1\}$, at the start of phase $i$, the adversary will drop a request uniformly at random at any of the unsaturated points. By definition, this satisfies the adversary property.
We observe that the expected cost of any deterministic algorithm ALG on the input sequence described above is lower bounded by the expected number of phases in which it is forced to match two requests at distinct points in the metric space.

We thus wish to compute, for each phase $i$, a lower bound on the probability that ALG will have to match two distinct points and thereby incur a distance cost of 1. We note that if the adversary drops a request on an active unsaturated point, ALG can match the two requests at that point without incurring any distance cost. We conclude that ALG will only be forced to match requests at two distinct points in the metric space if the adversary places the request on an inactive unsaturated point. It would thus be helpful to consider the number of unsaturated points the adversary can place a request during phase $i$, as well as how many of these are inactive. To this end we state the following observation followed by a claim regarding the number of unsaturated points, as well as the number of inactive unsaturated points at the end of phase $i$. (Note that Claim 6 is the same claim as we made in Section 5, which requires a new proof due to the change in behaviour of the adversary).

**Observation 10.** At the end of every phase $i$ (after ALG has matched), there exist $n - i + 1$ unsaturated points.

**Claim.** At the end of every phase $i$ (after ALG has matched), there exist $i + 1$ inactive points and at least two of these points are unsaturated.

**Proof of Claim.** We prove the claim by induction on the number of phases $i$.

The base case is when $i = 1$. At the start of phase 1, every point in the metric space receives a single request. At the end of phase 1, ALG is forced to match 2 arbitrary requests at different points in the metric space. Therefore, at the end of phase 1, there will be 2 points in the metric space that are both inactive and unsaturated.

Assume the claim holds for all phases $i = 1$ up to $i = k$. Thus, at the start of phase $k + 1$, we have $k + 1$ points in the metric space that are inactive and at least two of these requests are unsaturated. The adversary can now place a new requests at any of the unsaturated points (both active and inactive).

If it places it on an active unsaturated point, ALG is able to match the two active requests at the same point and incur no distance cost. However, the point will then become inactive (while it was previously active) and thus add 1 to the count of inactive points. This results in $k + 2$ inactive points. Furthermore, since the adversary did not place a request at any of the inactive unsaturated points, we still have at least 2 inactive unsaturated points. We conclude that by the end of phase $k + 1$ we have $k + 2$ inactive points and at least two of these must be unsaturated.

On the other hand, if the adversary manages to place the request on an inactive unsaturated point, this leaves $k$ points inactive (one of which is still unsaturated). At the end of phase $k + 1$, ALG will be forced to match requests across two distinct points in the metric space again, adding another 2 points to the set of inactive points. By the end of phase $k + 1$, there are thus a total of $k + 2$ inactive points and at least one of them is unsaturated. It now remains to prove at least two of the inactive points must be unsaturated. Assume for the sake of contradiction that this is not the case. Then all remaining $k + 1$ inactive points must be saturated. Since the points are inactive and saturated this means that ALG has matched at least $2 \cdot (k + 1)$ points in $k + 1$ phases. In every phase ALG matches exactly 2 points. By the end of phase $k + 1$, ALG can thus
have matched at most $2 \cdot (k + 1)$ requests. But the $k + 1$ saturated and 1 unsaturated points imply at least $2 \cdot (k + 1) + 1$ requests must have been matched by the end of this phase. This constitutes a contradiction. We conclude that there are $k + 2$ inactive points by the end of iteration $k + 1$ and that at least 2 of them are unsaturated. By the principle of induction, Claim 6 holds.

The claim implies that at the end of phase $i$, there exist $n - i + 1$ unsaturated points which the adversary can place a new request at, and at least two of those points are inactive. If the adversary manages to place a request at one of the unsaturated inactive points, this will force ALG to match two distinct points in the metric space and hence incur a distance cost. We define the random variable $b_i$ to be the number of inactive points among the unsaturated points. Conditioned on $b_i$, the probability that ALG will need to match across in phase $i$ can be expressed as $\frac{b_i}{n - i + 1}$. We now use this to lower bound the total expected cost of ALG. Let $X_i$ be the indicator variable that the algorithm pays a distance cost of 1 to match requests at distinct points in phase $i$.

$$E[cost(Alg)] = \sum_{i=1}^{n-1} E[X_i]$$

$$= \sum_{i=1}^{n-1} E[E[X_i | b_i]]$$

$$\geq \sum_{i=1}^{n-1} \frac{E[b_i]}{n - i + 1}$$

$$\geq \sum_{i=1}^{n-1} \frac{2}{n - i + 1}$$

$$= 2 \sum_{k=2}^{n} \frac{1}{k}$$

$$= \Omega(\log n).$$

The second inequality follows from Claim 6 and the last from harmonic series.

Because the adversary only places at most 2 requests at each of the points and places $2n - 2$ requests in total for all inputs in the distribution, this will result in $n - 2$ points that receive exactly 2 requests and 2 points that receive exactly 1 requests. The optimal offline solution is to match the single requests at the two points that received a single request, and to match all other requests locally at the same point. The expected cost of the optimal solution over the distribution of inputs is thus:

$$E[cost(OPT)] = 1.$$ 

From this we conclude the correctness of Theorem 9.

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