Asymptotic Behaviour and Cyclic Properties of Weighted Shifts on Directed Trees

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Abstract

In this paper we investigate a new class of operators called weighted shifts on directed trees introduced recently in [JJS3]. This class is a natural generalization of the so called weighted bilateral, unilateral and backward shift operators. In the first part of the paper we calculate the asymptotic limit and the isometric asymptote of a contractive weighted shift on a directed tree and that of the adjoint. Then we use the asymptotic behaviour and similarity properties to deal with cyclicity. We also show that a weighted backward shift operator is cyclic if and only if there is at most one zero weight.

1 Introduction

In [JJS3] the authors study a new class of (not necessarily bounded) linear operators acting on a Hilbert space which is a natural generalization of the so called weighted bilateral, unilateral or backward shift operators ([Sh] is a very thorough paper on weighted shifts). These usual shift operators are the favourite classes of test operators for an operator theorist. The authors of [JJS3] were interested for example in hyponormality, co-hyponormality, subnormality, complete hyperexpansivity e.t.c. Many of their examples for these properties are simpler than those previously found while investigating other classes of operators. In [JJS1] and [JJS2] the authors continued the study of weighted shifts on directed trees and constructed a closed non-hyponormal operator which generates Stieltjes moment sequences and investigated normal extensions of weighted shifts on directed trees. In [BJJS1] and [BJJS2] the authors provided a criterion for subnormality of both the bounded and the unbounded case. This explains why it is worth working with this kind of operators. In this paper we are studying weighted shifts on directed trees in a contraction theoretical view. We will consider bounded (and mainly contractive) weighted shifts on directed trees and investigate the asymptotic behaviour and cyclicity of them.

1.1 Directed graphs and directed trees

We recall the definitions from [JJS3]. The pair \( G = (V, E) \) is a directed graph if \( V \) is an arbitrary set and \( E \subseteq V \times V \setminus \{(v, v) : v \in V \} \). We call every element of

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This research was realized in the frames of TÁMOP 4.2.4. A/2-11-1-2012-0001 "National Excellence Program - Elaborating and operating an inland student and researcher personal support system". The project was subsidized by the European Union and co-financed by the European Social Fund.

The author was also supported by the "Lendület" Program (LP2012-46/2012) of the Hungarian Academy of Sciences.

AMS Subject Classification Numbers: 47A16, 47B37.

Keywords: Weighted Shifts on Directed Trees, Cyclic Properties, Asymptotic limit, Contraction.
$V$ and $E$ a vertex and a (directed) edge of $\mathcal{G}$, respectively. From the definition of $E$, one can see that there are no loopedges. We say that $\mathcal{G}$ is connected if for any two distinct vertices $u, v \in V$ there exists an undirected path between them, i.e. there are finitely many vertices: $u = v_0, v_1, \ldots, v_n = v \in V, n \in \mathbb{N}(= \{1, 2, \ldots\})$ such that $(v_{j-1}, v_j)$ or $(v_j, v_{j-1}) \in E$ for every $1 \leq j \leq n$. The finite sequence of distinct vertices $v_0, v_1, \ldots, v_n \in V, n \in \mathbb{N}$ is called a (directed) circuit if $(v_{j-1}, v_j) \in E$ for all $1 \leq j \leq n$ and $(v_n, v_0) \in E$. The directed graph $T = (V, E)$ is a directed tree if the following three conditions are satisfied:

(i) $T$ is connected,

(ii) for each vertex $v$ there exists at most one other vertex $u$ fulfilling the condition that $(u, v) \in E$, and

(iii) $T$ has no circuit.

From now on $T$ always denotes a directed tree. In the directed tree a vertex $v$ is called a child of $u \in V$ if $(u, v) \in E$. The set of all children of $u$ is denoted by $\text{Chi}_T(u) = \text{Chi}(u)$. Conversely, if for a given vertex $v$ we can find a vertex $u$ such that $(u, v) \in E$ (in this case this is a unique vertex), then we say that $v$ has a parent and it is $u$. We denote $u$ by $\text{par}_T(v) = \text{par}(v)$. We will also use the notation $\text{par}_T^k(v) = \text{par}(\ldots (\text{par}(v)) \ldots)$ if it makes sense, and $\text{par}_T^0$ will be the identity map.

If a vertex is not a child of any other vertex, then we call it a root of $T$. Proposition 2.1.1. of [JJS3] ensures that a directed tree is either rootless or has a unique root. We will denote this unique root by $\text{root}_T = \text{root}$, if it exists.

A subgraph of a directed tree which is itself a directed tree is called a subtree. We will use the notation $V^0 = V \setminus \{\text{root}\}$. If a vertex has no children, then we call it a leaf, and $T$ is leafless if it has no leaves. The set of all leaves of $T$ will be denoted by $\text{Lea}(T)$. Given a subset $W \subseteq V$ of vertices, we put $\text{Chi}(W) = \bigcup_{v \in W} \text{Chi}(v)$, $\text{Chi}^2(W) = W$, $\text{Chi}^{n+1}(W) = \text{Chi}(\text{Chi}^n(W))$ for all $n \in \mathbb{N}$ and $\text{Des}_T(W) = \text{Des}(W) = \bigcup_{n=0}^\infty \text{Chi}^n(W)$, where $\text{Des}(W)$ is called the descendants of the subset $W$, and if $W = \{u\}$, then we simply write $\text{Des}(u)$.

Now let us introduce the notion of the $n$th generation of a vertex. If $n \in \mathbb{Z}_+ (= \mathbb{N} \cup \{0\})$, then the set $\text{Gen}_n,\, T(u) = \text{Gen}_n(u) = \bigcup_{j=0}^n \text{Chi}^j(\text{par}^j(u))$ is called the $n$th generation of $u$ (i.e. we can go up at most $n$ levels and then down the same amount of levels) and $\text{Gen}_T(u) = \text{Gen}(u) = \bigcup_{n=0}^\infty \text{Gen}_n(u)$ is the (whole) generation or the level of $u$. From the equation

$$V = \bigcup_{n=0}^\infty \text{Des}(\text{par}^n(u))$$

(see Proposition 2.1.6 in [JJS3]), one can easily see that the different levels can be indexed by the integer numbers (or with a subset of the integers) in such a way that if a vertex $v$ is in the $k$th level, then the children of $v$ are in the $(k + 1)$th level and $\text{par}(v)$ is in the $(k - 1)$th level if $\text{par}(v)$ makes sense.

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1.2 Bounded weighted shifts on directed trees

The complex Hilbert space $\ell^2(V)$ is the space of all square summable complex functions on $V$ with the standard inner product

$$\langle f, g \rangle = \sum_{u \in V} f(u)g(u), \quad f, g \in \ell^2.$$

For $u \in V$ we define $e_u(v) = \delta_{u,v} \in \ell^2(V)$, where $\delta_{u,v}$ is the Kronecker-delta function. Obviously the set $\{e_u : u \in V\}$ is an orthonormal basis. We will refer to $\ell^2(W)$ as the subspace (i.e. closed linear manifold) $\vee\{e_w : w \in W\}$ for any subset $W \subseteq V$.

Let $\lambda = \{\lambda_v : v \in V^o\} \subseteq \mathbb{C}$ be a set of weights satisfying

$$\sup \left\{ \sqrt{\sum_{v \in \text{Chi}(u)} |\lambda_v|^2} : u \in V \right\} < \infty.$$

Then the weighted shift on the directed tree $T$ is the operator defined by

$$S_\lambda : \ell^2(V) \to \ell^2(V), \quad e_u \mapsto \sum_{v \in \text{Chi}(u)} \lambda_v e_v.$$

By Proposition 3.1.8. of [JJS3], it is a bounded linear operator with norm $\|S_\lambda\| = \sup \left\{ \sqrt{\sum_{v \in \text{Chi}(u)} |\lambda_v|^2} : u \in V \right\}$.

We will consider only bounded weighted shifts on directed trees, especially contractions (i.e. $\|S_\lambda\| \leq 1$) in certain parts of the paper (mainly in the first half). We recall that every $S_\lambda$ is unitarily equivalent to $S_\lambda' = \{(\lambda_v : v \in V^o) \subseteq [0, \infty)\}$. Moreover, the unitary operator $U$ with $S_\lambda' = US_\lambda U^*$ can be chosen such that $e_u$ is an eigen-vector of $U$ for every $u \in V^o$. It is also proposed that if a weight $\lambda_u$ is zero, then the weighted shift on this directed tree is a direct sum of two other weighted shifts on directed trees. These are Theorem 3.2.1 and Proposition 3.1.6. in [JJS3]. In view of these facts, this article will exclusively consider weighted shifts on directed trees with strictly positive weights.

The boundedness and the condition about weights together imply that every vertex has countably many children. Thus, by (1), $\ell^2(V)$ is separable.

1.3 Asymptotic behaviour

Let $\mathcal{H}$ be a complex Hilbert space and let us denote the C*-algebra of bounded linear operators on it by $\mathcal{B}(\mathcal{H})$. If $T \in \mathcal{B}(\mathcal{H})$ is a contraction, then the sequences $\{T^nT^*\}_{n=1}^{\infty}$ and $\{T^nT^*n\}_{n=1}^{\infty}$ of positive contractions are decreasing, so they have unique limits in the strong operator topology (SOT):

$$A = AT = \lim_{n \to \infty} T^nT^* \quad \text{and} \quad A_* = AT^* = \lim_{n \to \infty} T^nT^*n.$$

The operator $A$ is the asymptotic limit of $T$ and $A_*$ is the asymptotic limit of the adjoint $T^*$.

The vector $h \in \mathcal{H}$ is stable for the contraction $T \in \mathcal{B}(\mathcal{H})$ if the orbit of $h$ converges to 0, i.e. $\lim_{n \to \infty} \|T^nh\| = 0$ or equivalently $h \in \mathcal{N}(AT)$ (the
nullspace of $A_T$). The set $\mathcal{N}(A_T)$ of all stable vectors is usually denoted by $\mathcal{H}_0 = \mathcal{H}_0(T)$ and called the stable subspace of $T$. The commutant of $T$ is the set of those operators $C$ which commute with $T$, and will be denoted by $(T)'$. We recall that the stable subspace is hyperinvariant for $T$ (i.e. invariant for every operator $C \in (T)'$), in fact, this can be verified easily.

The contractions can be classified according to the asymptotic behaviour of their iterates and the iterates of their adjoints. Namely, $T$ is stable or of class $C_0$ when $\mathcal{H}_0(T) = \mathcal{H}$, in notation: $T \in C_0(\mathcal{H})$. If the stable subspace consists only of the null vector, then $T$ is of class $C_1$, or $T \in C_1(\mathcal{H})$. In the case when $T^* \in C_i(\mathcal{H})$ ($i = 0$ or $1$), we say that $T$ is of class $C_{i+}$. Finally, the class $C_{i+}(\mathcal{H})$ stands for the intersection $C_i(\mathcal{H}) \cap C_j(\mathcal{H})$.

Let $\mathcal{R}(A)$ stand for the range of the operator $A$ and by $\{\ldots\}^-$ we mean the closure of the corresponding set. We recall that the operator $X \in \mathcal{B}(\mathcal{H}, \mathcal{R}(A_T)^-) = \mathcal{H}_0^\perp$, $Xh = A_1^{1/2}h$ acts as an intertwining mapping in a canonical realization of the so called isometric asymptote of the contraction $T$. This and the unitary asymptote are very efficient tools in the theory of Hilbert space contractions. Here we only give the specific realization but we note that there is a more general setting. There exists a unique isometry $U = U_T \in \mathcal{B}(\mathcal{R}(A_T)^-)$ such that $XT = UX$ holds. The pair $(X, U)$ is the isometric asymptote of $T$. For a detailed study of isometric and unitary asymptotes, including other useful realizations (e.g. with the *-residual part of the minimal unitary dilatation of $T$), we refer to Chapter IX. in [NFBK] and [KT]. (We notice that in many papers about unitary asymptotes, $X$ is denoted by $X_+$ and the intertwining mapping of the unitary asymptote is denoted by $X$).

There are many applications for the isometric (and unitary) asymptotes. For instance, it can be proven whether some contractions have cyclic vectors or not. This will be explained in the next section. An other application is to obtain non-trivial hyperinvariant subspaces. In fact, if $T \notin C_{10}(\mathcal{H}) \cup C_{11}(\mathcal{H}) \cup C_{11}(\mathcal{H})$, then $\mathcal{H}_0(T)$ or $\mathcal{H}_0(T^*)^\perp$ is a non-trivial hyperinvariant subspace. Sz.-Nagy and Foias proved that any $C_{11}$ contraction has a non-trivial hyperinvariant subspace (see Proposition II.3.5. (iii) in [NFBK]). Thus the hyperinvariant subspace problem has an affirmative answer when the contraction $T \notin C_{00}(\mathcal{H}) \cup C_{10}(\mathcal{H}) \cup C_{10}(\mathcal{H})$. Note that the $C_{00}(\mathcal{H})$ is the general case since if $\|T\| < 1$, then $T \in C_{00}(\mathcal{H})$. In the $C_{10}(\mathcal{H})$ case (and hence for the $C_{01}(\mathcal{H})$ case) there are many results, for these see the above references.

In the next section we show how the isometric asymptote can be used for inferring cyclic properties. Section 3 and 4 is a technical part of the paper devoted to calculating the asymptotic limits $A$ and $A_+$ and the isometric asymptotes $U$ and $U_+$ of the contractive $S_\lambda$ and $S_\lambda^*$, respectively. After that in Section 5 we prove a cyclicity theorem for the weighted backward shift operator with countable multiplicity. Finally in Section 6-8 we investigate cyclic properties of weighted shifts on directed trees and their adjoints, using some similarity results and the results of Section 3-4.
2 Cyclic properties of contractions using the isometric asymptote

This section is devoted to the explanation of a proof considering a contraction $T$ and investigating whether it is cyclic or not using its asymptotic behaviour. First we give the definitions of cyclicity and hypercyclicity. An operator $T \in B(\mathcal{H})$ is cyclic if there is a vector such that

$$\mathcal{H}_{T,f} = \mathcal{H}_f := \{ T^n h : n \in \mathbb{Z}_+ \} = \{ p(T) h : p \in \mathcal{P}_\mathbb{C} \}^- = \mathcal{H},$$

where $\mathcal{P}_\mathbb{C}$ denotes the set of all complex polynomials. Such an $h \in \mathcal{H}$ vector is called a cyclic vector for $T$.

The vector $h \in \mathcal{H}$ is hypercyclic for $T$ if

$$\{ T^n h : n \in \mathbb{Z}_+ \}^- = \mathcal{H}.$$ 

Then the operator $T$ is hypercyclic.

If $T$ is cyclic and has dense range, then $h$ is cyclic if and only if $T h$ is cyclic. This and a consequence are stated in the next lemma for Hilbert spaces, but we note that in Banach spaces the proof would be the same. This proves, in that case, that the set of cyclic vectors span the whole space. In fact, this is always true. See [Ge] for an elementary proof.

**Lemma 1.** (i) If a dense range operator $T$ has a cyclic vector $f$, then $T f$ is also a cyclic vector.

(ii) If $T$ is a cyclic operator which has dense range and $N \in B(\mathbb{C}^n)$ is a cyclic nilpotent operator ($n \in \mathbb{N}$), i.e. a 0-Jordan block, then $T \oplus N$ is also cyclic.

**Proof.** (i) The set $\{ p(T) f : p \in \mathcal{P}_\mathbb{C} \}$ is dense. Since $\mathcal{R}(T)$ is also dense, a dense subset has dense image under $T$, so $\{ T p(T) f = p(T) T f : p \in \mathcal{P}_\mathbb{C} \}$ is also dense.

(ii) Let us take a cyclic vector $f \in \mathcal{H}$ for $T$ and a cyclic vector $e \in \mathbb{C}^n$ for $N$. We will show that $f \oplus e$ is cyclic for the orthogonal sum $T \oplus N$. Of course $\vee \{ T^k f \oplus N^k e = T^k f \oplus 0 : k \geq n \} = \mathcal{H}$. Thus $0 \oplus N^j e \in \vee \{ T^k f \oplus N^k e : k \in \mathbb{Z}_+ \}$ for every $0 \leq j < n$, and hence $f \oplus e$ is a cyclic vector.

The previous and the next lemma will be used many times throughout this paper.

**Lemma 2.** (i) if $T, Q, Y \in B(\mathcal{H})$, $Y$ has dense range, $YT = QY$ holds and $f$ is cyclic (or hypercyclic, resp.) for $T$, then $Yf$ is cyclic (or hypercyclic, resp.) for $Q$.

(ii) if $T \in C_1(\mathcal{H})$ is a contraction and the isometric asymptote $U$ has no cyclic vectors, then neither has $T$,

(iii) if $T \in C_1(\mathcal{H})$ is a contraction and the adjoint of the isometric asymptote $U^*$ has a cyclic vector $g$, then $A^{1/2} g$ is cyclic for $U^*$,

(iv) if $T \in C_1(\mathcal{H})$ is a contraction and the adjoint of the isometric asymptote $U^*_*$ has a cyclic vector $g$, then $A^{1/2} g$ is cyclic for $T$. 

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Proof. (i): Of course $Yp(T) = p(Q)Y$ holds for all $p \in \mathcal{P}_C$. Let us assume that $f$ is cyclic for $T$, i.e. $\{p(T)f : p \in \mathcal{P}_C\}$ is dense in $\mathcal{H}$. Then $\{Yp(T)f : p \in \mathcal{P}_C\}$ is also dense since $Y$ has a dense range, which means that $Yf$ is cyclic for $Q$. The hypercyclic case is very similar.

The other three points are special cases of (i). □

There is a remarkable consequence of the previous lemma.

**Corollary 3.** Suppose that the operator $T$ is hypercyclic. Then $T/\|T\| \notin C_1(\mathcal{H})$.

Proof. Obviously $T \neq 0$. Assume that $T/\|T\| \in C_1(\mathcal{H})$ and let us fix a hypercyclic vector $f \in \mathcal{H}$ for $T$. Since $A^{1/2}U^k f = \|T\|^k U^k A^{1/2} f$, $\{\|T\|^k U^k A^{1/2} f\}_{k=0}^\infty$ should be dense in $\mathcal{H}$. But $\|\|T\|^k U^k A^{1/2} f\|$ is bounded or bounded from below, which is a contradiction. □

Easy to see that for the adjoint of a contractive weighted bilateral shift: $S^*_w e_k = w_k e_{k-1}$, $(0 < |w_k| \leq 1$, the asymptotic limit is defined by $A_* e_k = \left( \prod_{j \leq k} |w_j|^2 \right) e_k$.

This means that $S^*_w$ is stable or $S^*_w \in C_1(\ell^2(V))$. If $S^*_w \in C_1(\ell^2(\mathbb{Z}))$, then the isometric asymptote $U_* \in B(\ell^2(\mathbb{Z}))$ of $S^*_w$ is the simple bilateral backward shift: $U_* e_k = e_{k-1}$. Since $U_*^*$ is cyclic, this means that all contractive $C_1$ bilateral shifts are cyclic. Such bilateral shifts exist that have no cyclic vectors and the first example was given by B. Beauzamy in [Be]. In Proposition 42. in [Sh] sufficient conditions can be found for cyclicity and non-cyclicity. But there is no characterization for cyclic bilateral shifts which is a little bit surprising since for other cyclic type properties there are such characterizations (for example hyper- or supercyclicity can be found in [Sa1] and [Sa2]). Therefore it is a challenging problem to give this characterization for cyclicity.

### 3 Asymptotic limits of contractions on directed trees

First we calculate the powers of $S^*_w$ and their adjoints. For empty sums we mean zero and for empty products 1.

**Proposition 4.** For the weighted shift $S^*_w$ on the directed tree $T$ (with strictly positive weights) and for any $n \in \mathbb{N}, u \in \mathcal{V}$ the following hold:

$$S^*_w^n e_u = \sum_{v \in \text{Chi}^n(u)} \prod_{j=0}^{n-1} \lambda_{\text{par}^j(v)} \cdot e_v,$$

$$S^{*n}_w e_u = \begin{cases} \prod_{j=0}^{n-1} \lambda_{\text{par}^j(u)} \cdot e_{\text{par}^n(u)}, & \text{if } \text{par}^n(u) \text{ makes sense}, \\ 0, & \text{otherwise} \end{cases}$$
Proof. For $n = 1$ the first equation holds by definition and the second equation is point (ii) of Proposition 3.4.1. of [JJS3]. Now we will use induction, therefore suppose that we have already proven the first equality for $n = 1, 2, \ldots N$ with some $N \in \mathbb{N}$. We prove it for $n = N + 1$. We have

$$S_{\lambda}^{N+1} e_u = S_{\lambda}^{N} (S_{\lambda}^{N} e_u) = \sum_{v \in \text{Chi}^N(u)} \prod_{j=0}^{N-1} \lambda_{\text{par}^j(v)} \cdot S_{\lambda} e_v$$

$$= \sum_{v \in \text{Chi}^N(u)} \left[ \prod_{j=0}^{N-1} \lambda_{\text{par}^j(v)} \cdot \sum_{w \in \text{Chi}^N(v)} \lambda_w e_w \right]$$

$$= \sum_{v \in \text{Chi}^N(u)} \sum_{w \in \text{Chi}^N(v)} \prod_{j=0}^{N-1} \lambda_{\text{par}^j(w)} \cdot \lambda_w e_w = \sum_{v \in \text{Chi}^N(u)} \sum_{w \in \text{Chi}^N(v)} \prod_{j=0}^{N} \lambda_{\text{par}^j(w)} \cdot e_w,$$

where in the second equality we used the boundedness of $S_{\lambda}$. The last sum is obviously conditionally convergent, but since $\text{Chi}(v) \cap \text{Chi}(v') = \emptyset$ if $v \neq v'$ (see Proposition 2.1.2. of [JJS3]), we have

$$\sum_{v \in \text{Chi}^N(u)} \left[ \prod_{j=0}^{N} \lambda_{\text{par}^j(w)} \cdot \sum_{w \in \text{Chi}^N(v)} e_w \right] \leq \sum_{v \in \text{Chi}^N(u)} \prod_{j=0}^{N} \lambda_{\text{par}^j(w)}^2 e_w.$$}

Therefore it is also absolutely convergent, so we get

$$S_{\lambda}^{N+1} e_u = \sum_{w \in \text{Chi}^{N+1}(u)} \prod_{j=0}^{N} \lambda_{\text{par}^j(w)} \cdot e_w.$$

The other equation can be obtained in a similar way. \hfill \Box

Theorem 5. Let $S_{\lambda}$ be a weighted shift on $T$ which is a contraction. Then every $e_u$ is an eigen-vector for $A$

$$A e_u = \alpha_u e_u \quad \forall u \in V,$$

with the corresponding eigen-values: $\alpha_u = \lim_{n \to \infty} \sum_{v \in \text{Chi}^n(u)} \prod_{j=0}^{n-1} \lambda_{\text{par}^j(v)}^2$.

Proof. For any $n \in \mathbb{N}$ and $u \in V$, since the asymptotic limit exists

$$S_{\lambda}^n S_{\lambda} e_u = \sum_{v \in \text{Chi}^n(u)} \prod_{j=0}^{n-1} \lambda_{\text{par}^j(v)} \cdot S_{\lambda} e_v = \sum_{v \in \text{Chi}^n(u)} \prod_{j=0}^{n-1} \lambda_{\text{par}^j(v)}^2 \cdot e_u \to \alpha_u e_u,$$

which means $A e_u = \alpha_u e_u$. \hfill \Box

Next we obtain some properties of the structure of the orthogonal complement of the stable subspace of $S_{\lambda}$. Since $A$ is a diagonal operator, there exists a set $V' \subset V$ with the properties that $H_0 = \ell^2(V \setminus V')$ and $H_0^* = \ell^2(V')$ hold.
Proposition 6. The following implications are valid for every contractive weighted shift $S_{\lambda}$ on $\mathcal{T}$ and for every vertex $u \in V$:

(i) if $e_u \in \mathcal{H}_0$, then $\ell^2(\text{Des}(u)) \subseteq \mathcal{H}_0$ (i.e. $u \notin V' \implies \text{Des}(u) \subseteq V \setminus V'$),

(ii) $e_u \in \mathcal{H}_0$ if and only if $\ell^2(\text{Chi}(u)) \subseteq \mathcal{H}_0$ (i.e. $u \notin V' \iff \text{Chi}(u) \subseteq V \setminus V'$); this is fulfilled in the special case when $u$ is a leaf,

(iii) if $e_u \in H^+_0$, then $e_{\text{par}^+(u)} \in H^+_0$ for every $k \in \mathbb{Z}_+$ (i.e. $u \in V' \implies \text{par}^+(u) \in V'$, $\forall k \in \mathbb{Z}_+$),

(iv) the subgraph $\mathcal{T}' = (V', E' = E \cap (V' \times V'))$ is a leafless subtree,

(v) if $\mathcal{T}$ has no root, neither has $\mathcal{T}'$, and

(vi) if $\mathcal{T}$ has a root, then either $S_{\lambda} \subseteq C_0(\ell^2(V))$ or $\text{root}_{\mathcal{T}} = \text{root}_{\mathcal{T}'}$.

Proof. The fact that $\mathcal{H}_0$ is invariant for $S_{\lambda}$ and that the weights are strictly positive implies (i).

The sufficiency in (ii) is a part of (i). On the other hand, suppose that $\ell^2(\text{Chi}(u)) \subseteq \mathcal{H}_0$. Then

$$\alpha_u = \lim_{n \to \infty} \sum_{v \in \text{Chi}(u)} \lambda_v^2 \prod_{j=0}^{n-1} \lambda_{\text{par}^j(v)}^2 = \sum_{v \in \text{Chi}(u)} \lambda_v^2 \alpha_v = 0,$$

since $\sum_{v \in \text{Chi}(u)} \lambda_v^2 \sum_{w \in \text{Chi}^{n-1}(v)} \prod_{j=0}^{n-2} \lambda_{\text{par}^j(w)}^2 \leq \sum_{v \in \text{Chi}(u)} \lambda_v^2 \leq 1$ for all $n \in \mathbb{N}$ and $\sum_{w \in \text{Chi}^{n-1}(v)} \prod_{j=0}^{n-2} \lambda_{\text{par}^j(w)}^2 \leq \alpha_v$. This proves the necessity in (ii).

Point (iii) follows from (ii) immediately.

We have to check three conditions for $\mathcal{T}'$ to be a subtree. Two of them is obvious since they were also true in $\mathcal{T}$. In order to see the connectedness of $\mathcal{T}'$, two distinct $u', v' \in V'$ are taken. Since $V = \bigcup_{i=0}^{\infty} \text{Des}_{\mathcal{T}}(\text{par}^i_{\mathcal{T}}(u'))$, $\text{par}^i_{\mathcal{T}}(u') = \text{par}^i_{\mathcal{T}'}(v')$ holds with some $k, l \in \mathbb{Z}_+$. Then (iii) gives $\text{par}^i_{\mathcal{T}}(u') = \text{par}^i_{\mathcal{T}'}(v') \in V'$ for every $i \leq k$ and $j \leq l$, which provides an undirected path in $\mathcal{T}'$ connecting $u'$ and $v'$.

Finally via (ii) it is trivial that $\mathcal{T}'$ is leafless, and the last two points immediately follow from (iii). \qed

In view of (v)-(vi), we have $\text{par}_{\mathcal{T}}(u') = \text{par}_{\mathcal{T}'}(u')$ for any $u' \in V'$, so we will simply write $\text{par}(u')$ in this case as well.

Let us take an arbitrary leafless subtree $\mathcal{T}' = (V', E')$ of $\mathcal{T}$ with the properties that if $\mathcal{T}$ is rootless, then $\mathcal{T}'$ is also rootless, and if $\mathcal{T}$ has a root, then $\mathcal{T}'$ has the same root. It is trivial that $\ell^2(V \setminus V')$ is invariant for $S_{\lambda}$. In the special case when it is the stable subspace, it is also hyperinvariant. Is it hyperinvariant for all weighted shift that is defined on $\mathcal{T}$? The answer is negative as we will see from the next example.

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Example 7. Let \( V = \mathbb{Z}_+ \cup \{ k' : k \in \mathbb{N} \} \) and \( E = \{ (n, n+1) : n \in \mathbb{Z}_+ \} \cup \{(k', (k+1)') : k \in \mathbb{N} \} \cup \{(0, V)\} \). This defines a directed tree \( \tilde{T} = (V, E) \). Set all of the weights to be equal to 1, then \( S_\lambda \) is a bounded weighted shift on the directed tree \( \tilde{T} \). The unitary operator defined by the following equations: \( U e_0 = e_0 \), \( U e_k = e_{k'} \), \( U e_{k'} = e_k \) for every \( k \in \mathbb{N} \), obviously commutes with \( S_\lambda \). But it is easy to see that \( \ell^1(\mathbb{N}) \) is not invariant for \( U \), hence it is not hyperinvariant for \( S_\lambda \).

Now we identify the asymptotic limit \( A_* \) of the adjoint \( S_\lambda^* \). The stable subspace of \( S_\lambda^* \) will be denoted by \( \mathcal{H}_0^* \). Since the weights are in the interval \((0, 1] \) any infinite product is unconditionally convergent.

Theorem 8. If \( S_\lambda \) is a contractive weighted shift on the directed tree \( T \), then the following two points are satisfied:

(i) If \( T \) has a root, then \( S_\lambda \in C_0(\ell^2(V)) \).

(ii) If \( T \) is rootless, then \( \mathcal{H}_0^* = \vee \{ h_u : u \in V \} \) where

\[
  h_u = \sum_{v \in \text{Gen}(u)} \prod_{j=0}^\infty \lambda_{\text{par}^j(v)} \cdot e_v \in \ell^2(V).
\]

The equality \( h_u = h_v \) holds if \( v \in \text{Gen}(u) \) and the vectors \( h_u \) are eigenvectors:

\[
  A_* h_u = a_u h_u \quad \forall u \in V
\]

with the corresponding eigen-values

\[
  a_u = \| h_u \|^2 = \sum_{v \in \text{Gen}(u)} \prod_{j=0}^\infty \lambda_{\text{par}^j(v)}^2.
\]

So, every level has one such \( h_u \). Moreover, if \( h_u \) is not zero for a vertex \( u \), then it is not zero for every \( u \in V \).

Proof. The first statement is clear, so we deal with only (ii). We have

\[
  \sum_{v \in \text{Gen}(u)} \prod_{j=0}^\infty \lambda_{\text{par}^j(v)}^2 \leq \sum_{v \in \text{Gen}(u)} \prod_{j=0}^{n-1} \lambda_{\text{par}^j(v)}^2 = \| S_\lambda e_{\text{par}^n(u)} \|^2 \leq 1,
\]

for all \( n \in \mathbb{N} \). Indeed, \( 0 \leq \sum_{v \in \text{Gen}(u)} \prod_{j=0}^\infty \lambda_{\text{par}^j(v)}^2 \leq 1 \) which means that \( h_u \) is actually a vector of \( \ell^2(V) \). For \( n \in \mathbb{N} \) we have

\[
  S_\lambda^n S_\lambda^* e_u = \prod_{j=0}^{n-1} \lambda_{\text{par}^j(u)} \cdot S_\lambda^n e_{\text{par}^n(u)} = \prod_{j=0}^{n-1} \lambda_{\text{par}^j(u)} \sum_{v \in \text{Gen}(u)} \prod_{j=0}^{n-1} \lambda_{\text{par}^j(v)} \cdot e_v.
\]

Since \( \lim_{n \to \infty} S_\lambda^n S_\lambda^* e_u = A_* e_u \),

\[
  \langle A_* e_u, e_v \rangle = \begin{cases} \prod_{j=0}^\infty \lambda_{\text{par}^j(u)} & \text{if } v \in \text{Gen}(u), \\ 0 & \text{otherwise} \end{cases}
\]

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which yields
\[ A_v e_v = \prod_{j=0}^{\infty} \lambda_{\text{par}(u)} \sum_{v \in \text{Gen}(u)} \prod_{j=0}^{\infty} \lambda_{\text{par}(v)} \cdot e_v = \prod_{j=0}^{\infty} \lambda_{\text{par}(u)} \cdot h_u. \]

Now we get
\[ A_v h_u = \sum_{v \in \text{Gen}(u)} \prod_{j=0}^{\infty} \lambda_{\text{par}(v)} \cdot A_v e_v = \left( \sum_{v \in \text{Gen}(u)} \prod_{j=0}^{\infty} \lambda_{\text{par}(v)} \right) h_u = 0, \]

since \( \text{Gen}(u) = \text{Gen}(v) \) and thus \( h_u = h_v \).

To conclude the relation \( H_0^+ \subseteq \langle \{u : u \in V\} \rangle \), fix a vector \( h \in \ell^2(\text{Gen}(u)) \), \( h \perp h_u \). Using the notation \( \eta_v = \langle h, e_v \rangle \) \( (v \in \text{Gen}(u)) \):
\[ A_v h = \sum_{v \in \text{Gen}(u)} \eta_v A_v e_v = \left( \sum_{v \in \text{Gen}(u)} \prod_{j=0}^{\infty} \lambda_{\text{par}(v)} \right) h_u = 0, \]
by the orthogonality of \( h \) and \( h_u \). Therefore the equation \( H_0^+ = \langle \{u : u \in V\} \rangle \) is trivial since \( h_u = 0 \) if and only if \( a_u = 0 \).

Finally let us suppose that \( h_u = 0 \) holds for a vertex \( u \in V \). Then \( \ell^2(\text{Gen}(u)) \subseteq H_0^+ \), and since \( H_0^+ \) is invariant for \( S_{\lambda} \), \( \ell^2(\text{Gen}(\text{par}(u))) \subseteq H_0^+ \) for every \( k \in \mathbb{N} \). If we set a \( \tilde{w} \in \text{Chi}(u) \), then
\[ a_{\tilde{w}} = \sum_{\tilde{w} \in \text{Gen}(u)} \prod_{j=0}^{\infty} \lambda_{\text{par}(w)} = \sum_{v \in \text{Gen}(u)} \sum_{w \in \text{Chi}(u)} \prod_{j=0}^{\infty} \lambda_{\text{par}(w)} \]
\[ = \sum_{v \in \text{Gen}(u)} \prod_{j=0}^{\infty} \lambda_{\text{par}(v)} \sum_{w \in \text{Chi}(u)} \lambda_{\text{par}(w)} = \sum_{v \in \text{Gen}(u)} \prod_{j=0}^{\infty} |\langle h_u, e_v \rangle|^2 \sum_{w \in \text{Chi}(u)} \lambda_{\text{par}(w)} = 0. \]

So \( \ell^2(\text{Chi}(\text{Gen}(u))) \subseteq H_0^+ \). By induction, we get \( H_0^+ = \ell^2(V) \). \( \square \)

4 Isometric asymptotes

In this section we want to identify the isometric asymptote of \( S_{\lambda} \). We call the vertex \( u \) a branching vertex if \( |\text{Chi}(u)| > 1 \) and the set of all branching vertices is denoted by \( V_\lambda \). The number
\[ \text{Br}(\mathcal{T}) = \sum_{u \in V_\lambda} (|\text{Chi}(u)| - 1) \]

is the branching index of \( \mathcal{T} \). From (ii) of Proposition 3.5.1 of [JJS3] we have
\[ \dim(\mathcal{R}(S_{\lambda})^+) = \begin{cases} 1 + \text{Br}(\mathcal{T}) & \text{if } \mathcal{T} \text{ has a root,} \\ \text{Br}(\mathcal{T}) & \text{if } \mathcal{T} \text{ has no root,} \end{cases} \] (2)

In Proposition 6 we used the notation \( \mathcal{T}' = (V', E') \) for the subtree such that \( \ell^2(V') = H_0^+ \). We will write \( S \in B(\ell^2(\mathbb{Z}))^+ \) and \( S^+ \in B(\ell^2(\mathbb{Z}^+)) \) for the simple bilateral and unilateral shift operators (with multiplicity one), i.e.: \( Sc_n = c_{n+1} \),
Because of the condition

Theorem 9. For such a weighted shift \( S_\lambda \) on \( \mathcal{T} \) that is a contraction and \( S_\lambda \notin C_0(\ell^2(V)) \), the isometric asymptote \( U \in \mathcal{B}(\ell^2(V')) \) is a weighted shift on the subtree \( T' = (V', E') \): \( U = S_{\beta_v} \), with weights \( \beta_v = \frac{\lambda_v \sqrt{\alpha_v}}{\sqrt{\alpha_{\text{par}(v')}}} \), \( v' \in (V')^o \).

This isometry is unitarily equivalent to one of the following:

(i) \( \sum_{j=1}^{\text{Br}(T')} + S^+ \), if \( T \) has a root,

(ii) \( \sum_{j=1}^{\text{Br}(T')} \oplus S^+ \), if \( T \) has no root and \( U \) is a c.n.u. isometry, i.e.:

\[
\sum_{v' \in \text{Gen}_{\text{par}}(v')} \prod_{j=0}^{\infty} \beta_{\text{par}^{j}(v')} = 0 \text{ for some (and then for every) } u' \in V',
\]

(iii) \( S \oplus \sum_{j=1}^{\text{Br}(T')} \oplus S^+ \), if \( T \) has no root and \( U \) is not a c.n.u. isometry.

Proof. Because of the condition \( S_\lambda \notin C_0(\ell^2(V)) \), \( V' \neq \emptyset \). For any \( u' \in V' \)

\[
Ue_{u'} = \frac{1}{\sqrt{\alpha_{u'}}} \cdot U A^{1/2} e_{u'} = \frac{1}{\sqrt{\alpha_{u'}}} \cdot A^{1/2} S_\lambda e_{u'}
\]

\[
= \frac{1}{\sqrt{\alpha_{u'}}} \cdot \sum_{v' \in \text{Chi}_{\text{par}}(u')} \lambda_v \cdot A^{1/2} e_{v'} = \sum_{v' \in \text{Chi}_{\text{par}}(u')} \frac{\lambda_v \sqrt{\alpha_{u'}}}{\sqrt{\alpha_{v'}}} 
\cdot \cdot \cdot
\]

This establishes that \( U \) is a weighted shift on \( T' \) with weights \( \beta_{u'} = \frac{\lambda_v \sqrt{\alpha_{u'}}}{\sqrt{\alpha_{v'}}} \)

\((v' \in (V')^o)\).

First, suppose that \( T \) has a root. Then \( T' \) has the same root as \( T \). But contractive weighted shifts on a directed tree which has a root are of class \( C_0 \), so in this case \( U \) is a unilateral shift. Since the co-rank of \( U \) is \( \text{Br}(T') + 1 \), we infer that \( U \) and \( \sum_{j=1}^{\text{Br}(T')} + S^+ \) are unitarily equivalent.

Second, assume that \( T \) has no root and \( U \) is a c.n.u. isometry. The isometry \( U \) is c.n.u. if and only if \( \sum_{v' \in \text{Gen}_{\text{par}}(v')} \prod_{j=0}^{\infty} \beta_{\text{par}^{j}(v')} = 0 \) for some (and then for every) \( u' \in V' \), by Theorem 8. Again, the co-rank of \( U \) is \( \text{Br}(T') \), and therefore \( U \) is unitarily equivalent to \( \sum_{j=1}^{\text{Br}(T')} \oplus S^+ \).

Finally, let us suppose that \( T \) has no root and \( \sum_{v' \in \text{Gen}_{\text{par}}(v')} \prod_{j=0}^{\infty} \beta_{\text{par}^{j}(v')} > 0 \) for every \( u' \in V' \). Then the unitary part of \( U \) acting on

\[
(\mathcal{H}_0(U))^+ = \left\{ (0 \neq) k_{u'} = \sum_{v' \in \text{Gen}_{\text{par}}(u')} \prod_{j=0}^{\infty} \beta_{\text{par}^{j}(v')} \cdot e_{v'} : u' \in V' \right\}.
\]

Set \( u' \in V' \), then

\[
Uk_{u'} = \sum_{v' \in \text{Gen}_{\text{par}}(u')} \prod_{j=0}^{\infty} \beta_{\text{par}^{j}(v')} \cdot U e_{v'}
\]
is clearly an isometry with 
\[ v' \in \text{Gen}_{\mathcal{T}}(u') \] 
and 
\[ w' \in \text{Chi}_{\mathcal{T}}(v') \] 
for some \( \tilde{w}' \in \text{Chi}_{\mathcal{T}}(u') \). Therefore we get that \( U[H^2_0(U)] \) is a simple bilateral shift. Since the co-rank of \( U \) is \( \text{Br}(\mathcal{T}') \), we get that \( U \) is unitarily equivalent to 
\[ S \oplus \sum_{j=1}^{\text{Br}(\mathcal{T}')} S^+ \].

Remark 10. (i) From the theorem above we can get the unitary asymptote of \( S_u \). In fact, \( W, A^{1/2} \), where \( W \) is the minimal unitary dilation of the isometry \( U \). It is easy to see that this minimal unitary dilation is a bilateral shift with multiplicity \( \text{Br}(\mathcal{T}') \) or \( \text{Br}(\mathcal{T}') + 1 \).

(ii/a) If the directed tree \( \mathcal{T} \) has a root, then any isometric weighted shift on \( \mathcal{T} \) is of class \( C_0 \), i.e.: a unilateral shift with multiplicity \( \text{Br}(\mathcal{T}) \).

(ii/b) In general if we have an isometric weighted shift \( U \) on a directed tree, then the set-up of the tree doesn’t tell us whether \( U \) is a c.n.u. isometry or not. To see this take a rootless binary tree, i.e.: \( \mathcal{T} \) is not a c.n.u. isometry. Indeed

\[ \sum_{v \in \text{Gen}_{\mathcal{T}}(u)} \prod_{j=0}^{\infty} \beta_{\text{par}^j(v)} \cdot \beta_{w'} e_{w'} = k e_{\tilde{w}'} \]

for some \( \tilde{w}' \in \text{Chi}_{\mathcal{T}}(u') \). Therefore we get that \( U[H^2_0(U)] \) is a simple bilateral shift. Since the co-rank of \( U \) is \( \text{Br}(\mathcal{T}') \), we get that \( U \) is unitarily equivalent to 
\[ S \oplus \sum_{j=1}^{\text{Br}(\mathcal{T}')} S^+ \].

(ii/c) Suppose that the rootless directed tree \( \mathcal{T} \) has a vertex \( u \in V \) which has the following property

\[ V = (\text{Des}(u)) \bigcup \left( \bigcup_{k=1}^{\infty} \{ \text{par}^k(u) \} \right). \]

If we take an isometric weighted shift on \( \mathcal{T} \) with weights \( \{ \beta_v : v \in V^0 \} \), then \( S_{\beta} \) is not a c.n.u. isometry. Indeed, \( \beta_{\text{par}^k(u)} = 1 \) for every \( k \in \mathbb{Z}_+ \), and thus

\[ \sum_{v \in \text{Gen}(\text{par}^k(u))} \prod_{j=0}^{\infty} \beta_{\text{par}^j(v)} = 1 > 0. \]

The above points show that two unitarily equivalent weighted shifts on directed trees can be defined on a very different directed tree. Now we turn to the calculation of the isometric asymptote of the adjoint \( S_{\beta}^* \). We calculate the unique isometry \( U_* \in \mathcal{B}((H^2_0)^*) \) with the intertwining property 
\[ A_s^{1/2} S_{\beta}^* = U_* A_s^{1/2} \]

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Theorem 11. Suppose that the contractive weighted shift \( S_\lambda \) on \( \mathcal{T} \) is not in the class \( C_0 \). Then \( \mathcal{T} \) has no root and the isometry \( U_* \) acts as follows:

\[
U_* h_u = \frac{\sqrt{a_u}}{\sqrt{a_{\text{par}(u)}}} \cdot h_{\text{par}(u)},
\]

where \( h_u \neq 0 \) for every \( u \in V \). As a matter of fact, \( U_* \) is a simple unilateral shift if there is a last level (i.e. \( \text{Chi}(\text{Gen}(u)) = \emptyset \) for some \( u \in V \)), and it is a bilateral shift elsewhere.

Proof. If \( S_\lambda \notin C_0(\ell^2(V)) \), then \( h_u \neq 0 \) and \( a_u \neq 0 \) \( \forall \) \( u \in V \). For a \( u \in V \):

\[
U_* \frac{1}{\sqrt{a_u}} h_u = \frac{1}{a_u} U_* A_*^{1/2} h_u = \frac{1}{a_u} A_*^{1/2} S_\lambda^* h_u
\]

\[
= \frac{1}{a_u} \sum_{v \in \text{Gen}(u)} \prod_{j=0}^{\infty} \lambda_{\text{par}^j(v)} A_*^{1/2} S_\lambda^* e_v = \frac{1}{a_u} \sum_{v \in \text{Gen}(u)} \prod_{j=0}^{\infty} \lambda_{\text{par}^j(v)} \lambda_u A_*^{1/2} e_{\text{par}(v)}
\]

\[
= \frac{1}{a_u} \sum_{v \in \text{Gen}(u)} \prod_{j=0}^{\infty} \lambda_{\text{par}^j(v)} \lambda_u \frac{\langle e_{\text{par}(v)}, h_{\text{par}(v)} \rangle}{\| h_{\text{par}(v)} \|^2} A_*^{1/2} h_{\text{par}(v)}
\]

\[
= \frac{1}{a_u \sqrt{a_{\text{par}(u)}}} \left( \sum_{v \in \text{Gen}(u)} \prod_{j=0}^{\infty} \lambda_{\text{par}^j(v)}^2 \right) h_{\text{par}(u)} = \frac{1}{\sqrt{a_{\text{par}(u)}}} h_{\text{par}(u)}.
\]

One can easily see the unitary equivalence with the simple uni- or bilateral shift.

At the end of this section we obtain a characterization for those contractive weighted shifts on directed trees that are similar to isometries or co-isometries.

Corollary 12. Consider the contraction \( S_\lambda \) which is a weighted shift on a directed tree. Then the followings hold:

(i) \( S_\lambda \) is similar to an isometry if and only if \( \inf \{ \alpha_u : u \in V \} > 0 \).

(ii) \( S_\lambda \) is similar to a co-isometry if and only if it is a bilateral weighted shift with \( \prod_{j=\infty}^{\infty} \lambda_j > 0 \), or it is a weighted backward shift with \( \prod_{j=\infty}^{0} \lambda_j > 0 \). Then it is similar to the simple bilateral or the simple unilateral shift operator, respectively.

Proof. (i) is a simple consequence of Proposition 3.8. in [Ku] and Theorem 5 and 9.

(ii) The similarity to a co-isometry implies the similarity of \( S_\lambda^* \) to an isometry. Then by Theorem 8 we have that \( \mathcal{T} \) has no root and \( |\text{Chi}(u)| \leq 1 \) for every \( u \in V \).

First, suppose that \( \mathcal{T} \) has no leaves. Then clearly \( S_\lambda \) is a weighted bilateral shift. By Proposition 3.8. in [Ku] we have \( \prod_{j=\infty}^{\infty} \lambda_j > 0 \) and therefore \( S_\lambda \) is similar to \( S \).

Second, assume that \( \mathcal{T} \) has a leaf. Then it has a unique leaf and trivially \( S_\lambda^* \) is a weighted unilateral shift. Again by Proposition 3.8. in [Ku] we have \( \prod_{j=0}^{\infty} \lambda_{-j} > 0 \) and that \( S_\lambda^* \) is similar to \( S^+ \).

We notice that a contractive weighted shift on a directed tree is similar to a unitary operator if and only if it is a bilateral shift of class \( C_{11} \). This is a simple consequence of the previous corollary.
5 Cyclicity of backward shifts

The aim of this section is to prove that a backward unilateral shift of countable multiplicity is cyclic exactly when it has at most one zero weight. In the article [Gu], written in Chinese, there is a proof for the case when the multiplicity is one, but the author of this paper was unable to read it due to the lack of proper translation. The reader can consider the forthcoming theorems as generalizations of that result.

First, in the next theorem we deal with the injective case. For technical reasons we consider contractive shifts, but this can be assumed without loss of generality. This proof was motivated by the solution of Problem 160 in [Ha].

Theorem 13. Suppose that \( \{ e_{j,k} : j \in J, k \in \mathbb{Z}^+ \} \) is an orthonormal basis in the Hilbert space \( H \) where \( J \neq \emptyset \) is a countable set and \( \{ w_{j,k} : j \in J, k \in \mathbb{Z}^+ \} \subset (0,1] \) is a set of weights. Consider the following backward shift:

\[
B e_{j,k} = \begin{cases} 
0 & \text{if } k = 0 \\
w_{j,k-1} e_{j,k-1} & \text{otherwise}
\end{cases}
\]

Then there exists a cyclic vector \( f \) for \( B \).

Moreover, if there is a vector \( g \in \cap_{n=1}^{\infty} \mathcal{R}(B^n) \) such that for every fixed \( j \in J \), \( \langle g, e_{j,k} \rangle \neq 0 \) is fulfilled for infinitely many \( k \in \mathbb{Z}^+ \), than there is a cyclic vector \( f \) from the linear manifold \( \cap_{n=1}^{\infty} \mathcal{R}(B^n) \).

Proof. Take a vector of the following form

\[
f = \sum_{l=1}^{\infty} \xi_{j_l,k_l} \cdot e_{j_l,k_l} \in H,
\]

where \( \xi_{j_l,k_l} > 0 \) for every \( l \in \mathbb{N} \), \( 0 < k_{l+1} - k_l \not\to \infty \) and for any \( j \in J \) there exist infinitely many \( l \in \mathbb{N} \) which satisfy \( j_l = j \).

Our aim is to modify \( f \) by decreasing its coordinates to be a cyclic vector for \( B \). We have

\[
\frac{1}{\xi_{j_m,k_m} w_{j_m,k_m-1} \cdots w_{j_m,k_m-k}} B^k f
\]

\[
= e_{j_m,k_m-k} + \sum_{l>m} \frac{\xi_{j_l,k_l} w_{j_l,k_l-1} \cdots w_{j_l,k_l-k}}{\xi_{j_m,k_m} w_{j_m,k_m-1} \cdots w_{j_m,k_m-k}} \cdot e_{j_l,k_l-k}
\]

for every \( m \in \mathbb{N}, k_{m-1} < k \leq k_m \), where we set \( k_0 = -1 \). If

\[
\Sigma_m := \max \left\{ \sum_{l>m} \left| \frac{\xi_{j_l,k_l} w_{j_l,k_l-1} \cdots w_{j_l,k_l-k}}{\xi_{j_m,k_m} w_{j_m,k_m-1} \cdots w_{j_m,k_m-k}} \right|^2 : k_{m-1} < k \leq k_m \right\} \leq \frac{1}{2^m}
\]

would be satisfied, then \( e_{j,k} \in \mathcal{H}_B f \) would hold for every \( j \in J, k \in \mathbb{Z}^+ \) and thus \( f \) would be a cyclic vector for \( B \). Now we do the modification on \( f \) such that (4) will hold. Obviously

\[
\sum_{l>m} \left| \frac{\xi_{j_l,k_l} w_{j_l,k_l-1} \cdots w_{j_l,k_l-k}}{\xi_{j_m,k_m} w_{j_m,k_m-1} \cdots w_{j_m,k_m-k}} \right|^2
\]
Therefore \( f \) only finitely many times.

Theorem 15. Corollary 14. For the sufficiency we may consider the operator \( B \) has at most one zero weights. 

Proof. For the sufficiency we consider the operator \( B \) where \( B \in B(H) \), is a backward shift operator with strictly positive weights and \( N \) is a cyclic nilpotent operator acting on \( C^n \) (for some \( n \in \mathbb{N} \)) (i.e. a Jordan block with zero diagonals), because this is unitarily equivalent to the operator mentioned in the statement of the theorem. Since \( B \) has a dense range and it is cyclic, (ii) of Lemma 1 gives us what we wanted.

For the necessity we notice that if the co-dimension of \( R(B) \) is greater then one, then the operator cannot be cyclic.

This provides the characterization for the cyclicity of backward shifts.
In this section we deal with the cyclic properties of $S_\lambda$. From equation (2) we infer that if $T$ has a root and $\text{Br}(T) > 0$, or $T$ is rootless and $\text{Br}(T) > 1$, then the weighted shift on $T$ has no cyclic vectors, since in this case the co-rank of $S_\lambda$ is greater than 1.

When $\text{Br}(T) = 0$, $S_\lambda$ is either a cyclic nilpotent operator acting on a finite dimensional space or a weighted bilateral, unilateral or backward shift. We have dealt with the backward unilateral case. A unilateral shift is always cyclic. For the bilateral shift both can happen, as it was mentioned at the end of Section 2.

So the only interesting pure weighted shift on a directed tree case is when $\text{Br}(T) = 1$ and $T$ has no root. Then there is exactly one vertex which has exactly two children, and the other vertices are leaves or they have exactly one child. In fact, either there is no leaf or there is a unique one or there are exactly two of them (for the case when there is at least one leaf, see Figure 1). We remind the reader that every weight is strictly positive. First we need a similarity lemma.

**Lemma 16.** Consider a directed tree $T$ with the properties $\text{Br}(T) = 1$ and $\text{Lea}(T) \neq \emptyset$, and a bounded weighted shift $S_\lambda$ on $T$. Then $S_\lambda$ is similar to an orthogonal sum $W \oplus N$ where

(i) $W$ is a weighted backward shift and $N$ is a cyclic nilpotent operator acting on a finite dimensional space if $T$ is rootless and $|\text{Lea}(T)| = 2$,

(ii) $W$ is a weighted bilateral shift and $N$ is a cyclic nilpotent operator acting on a finite dimensional space if $T$ is rootless and $|\text{Lea}(T)| = 1$.

**Proof.** Let us denote the branching vertex by 0, $-k := \text{par}_k(0)$ for every $k \in \mathbb{N}$, the children of 0 by 1 and 1', and the child of $k$ (or $k'$ resp.) by $k + 1$ (or $(k + 1)'$ resp.) for every $k \in \mathbb{N}$, if it exists. So the set of vertices is the union of a subset of the integers and a subset of primed natural numbers. Let us assume that $k_0'$ is a leaf of $T$ and if $j_0$ is the other leaf, then $j_0 \geq k_0$ holds.

Let us define the following two subspaces: $\mathcal{E} := \ell^2(\mathbb{Z} \cap V)$ and $\mathcal{E}' := \mathcal{E}^\perp = \ell^2(\{1', 2', 3' \ldots \} \cap V)$. Clearly, the second subspace is finite dimensional. Set the vectors $g_k := (\prod_{j=1}^{k} \frac{1}{\lambda_j}) e_k - (\prod_{j=1}^{k} \frac{1}{\lambda_j'}) e_{k'}$ for every $2 \leq k \leq k_0$. Now we
define two operators on these subspaces as follows:

\[ W : E \to E, \quad e_n \mapsto \begin{cases} \lambda_n e_{n+1} & \text{if } n \text{ is not a leaf} \\ 0 & \text{if } n \text{ is a leaf} \end{cases}, \]

which is a weighted bilateral shift, and

\[ N : E' \to E', \quad e_{k'} \mapsto \begin{cases} \frac{\|g_k\|}{\|g_{k+1}\|} e_{(k+1)'} & \text{if } k < k_0 \\ 0 & \text{if } k = k_0. \end{cases} \]

These are clearly bounded operators. We state that the following operator \( X \) is invertible and \( X(W \oplus N)^* = S^*_\Delta X \):\[ X : \ell^2(V) \to \ell^2(V), \quad e_{k'} \mapsto \frac{1}{\|g_k\|} g_k, \quad e_n \mapsto e_n. \]

The boundedness of \( X \) is trivial, and since \( \lor \{e_k, e_{k'}\} \) is invariant if \( 1 \leq k \leq k_0 \) and \( e_n \) is an eigenvector if \( k < 1 \) or \( k_0 < n \) (\( \leq j_0 \) if there are two leaves), the invertibility is also obvious. The following equations show that \( X(W \oplus N)^* = S^*_\Delta X \) also holds:

\[
X(W \oplus N)^* e_n = X \lambda_n e_{n-1} = \lambda_n e_{n-1} = S^*_\Delta e_n = S^*_\Delta X e_n \quad (n \in V \cap \mathbb{Z})
\]

\[
X(W \oplus N)^* e_{1'} = 0 = S^*_\Delta \left( \frac{1}{\|g_1\|} g_1 \right) = S^*_\Delta X e_{1'}
\]

\[
X(W \oplus N)^* e_{k'} = X \frac{\|g_{k-1}\|}{\|g_k\|} e_{(k-1)'} = \frac{1}{\|g_k\|} g_{k-1}
\]

\[
= S^*_\Delta \left( \frac{1}{\|g_k\|} g_k \right) = S^*_\Delta X e_{k'} \quad (2 \leq k \leq k_0).
\]

This ends our proof. \( \square \)

The next theorem is a consequence of the above lemma and Theorem 15, so we omit the proof.

**Theorem 17.** If the directed tree \( T \) has no root, \( \text{Br}(T) = 1 \) and \( |\text{Lea}(T)| = 2 \), then every bounded weighted shift on \( T \) is cyclic.

Let \( T \) denote the complex unit circle, \( m \) the normalized Lebesgue measure on it and \( L^2 \) the Lebesgue space \( L^2 = L^2(T) \). The simple bilateral shift \( S \) can be represented as a multiplication operator by the identity function \( \chi(\zeta) = \zeta \) on \( L^2 \). It is a known fact that \( g \in L^2 \) is cyclic for \( S \) if and only if \( g(\zeta) \neq 0 \) a.e. \( \zeta \in \mathbb{T} \) and \( \int_\mathbb{T} \log |g| dm = -\infty \). From Lemma 1 it is obvious that \( g \) is cyclic if and only if \( Sg = \chi g \) is cyclic, but this can be obtained from the previous characterization as well.

In the next theorem, we characterise cyclicity in the case when there is a unique leaf in \( T \).

**Theorem 18.** Suppose \( T \) has a unique leaf. A weighted shift \( S_\Delta \) on \( T \) is cyclic if and only if the bilateral shift \( W \) with weights \( \{\lambda_n\}_{n=-\infty}^\infty \) is cyclic. In particular, if \( S_\Delta \notin C_0(\ell^2(V)) \), then \( S_\Delta \) is cyclic.
Proof. By (ii) of Lemma 16, $S_\lambda$ is similar to $W \oplus N$. If $W$ has no cyclic vectors then obviously neither has $S_\lambda$. If $W$ is cyclic then by Lemma 1 we can obviously see that $S_\lambda$ has a cyclic vector. Since $C_1$ bilateral shifts are cyclic, the other statement follows immediately. This ends the proof.

The simple unilateral shift $S^+$ can also be represented as a multiplication operator by $\chi$, but on the Hardy space $H^2 = H^2(\mathbb{T})$. Next we prove that the orthogonal sum $S \oplus S^+$ has no cyclic vectors. This needs only elementary Hardy space techniques.

**Proposition 19.** The operator $S \oplus S^+$ has no cyclic vectors.

**Proof.** Suppose that $f \oplus g \in L^2 \oplus H^2$ is a cyclic vector, and let us denote the orthogonal projection onto $L^2 \oplus \{0\}$ and onto $\{0\} \oplus H^2$ with $P_1$ and $P_2$, respectively. Then $\vee \{\chi^n f: n \in \mathbb{Z}_+\} = P_1(\vee \{\chi^n f \oplus \chi^n g: n \in \mathbb{Z}_+\})$ is dense in $L^2$ i.e.: $f$ is cyclic for $S$, and similarly we get that $g$ is cyclic for $S^+$ too. This implies that $f(\zeta) \neq 0$ for a.e. $\zeta \in \mathbb{T}$ and $g$ is an outer function. We state that $0 \oplus g \notin (L^2 \oplus H^2)/g$. To see this consider an arbitrary complex polynomial $p$. Then

$$\| (pf) \oplus (pg) - 0 \oplus g \|^2 = \int_{\mathbb{T}} |pf|^2 + |p-1)g|^2 dm.$$

One of the sets $A = p^{-1}(\{z \in \mathbb{C}: \text{Re} z < 1/2\})$ or $A^c = p^{-1}(\{z \in \mathbb{C}: \text{Re} z \geq 1/2\})$ has Lebesgue measure at least 1/2. If $m(A) \geq 1/2$, then

$$\| (pf) \oplus (pg) - 0 \oplus g \|^2 \geq \int_A |p-1)g|^2 dm = \int_A |g|^2 / dm \geq \frac{1}{4} \inf \left\{ \int_E |g|^2 dm : E \in \mathcal{L}, m(E) \geq 1/2 \right\} > 0.$$

Similarly if $m(B) \geq 1/2$, then

$$\| (pf) \oplus (pg) - 0 \oplus g \|^2 \geq \frac{1}{4} \inf \left\{ \int_E |f|^2 dm : E \in \mathcal{L}, m(E) \geq 1/2 \right\} > 0.$$

Thus we get that $S \oplus S^+$ has no cyclic vectors.

Now we are able to prove a non-cyclicity theorem for the class of $C_1$ contractions that are weighted shifts on directed trees. In order to do this we will use the notion of the isometric asymptote.

**Theorem 20.** Suppose that $T$ is rootless and $\text{Br}(T) = 1$. If the weighted shift $S_\lambda$ on $T$ is of class $C_1$, then it has no cyclic vectors.

**Proof.** By Theorem 9, the isometric asymptote $U$ of $S_\lambda$ is unitarily equivalent to the orthogonal sum $S \oplus S^+$ which has no cyclic vectors. This implies - together with Lemma 2 - that neither has $S_\lambda$.

On the other hand, the contrary may happen. This will be shown later in the last section.
7 Cyclicity of $S^*_\lambda$

Now we are interested in giving necessary conditions for $S^*_\lambda$ to be cyclic. Let us denote the operator $S^+ + \cdots + S^+_k$ by $S^+_k$ ($k \in \mathbb{N}$) and the orthogonal sum of $\aleph_0$ piece of $S^+$ by $S^+_{\aleph_0}$.

**Theorem 21.** The operator $S \oplus (S_k^+)^*$ is cyclic for every $k \in \mathbb{N}$.

**Proof.** The method is the following: we intertwine $S \oplus S_k^+$ and $S$ with an injective operator $X \in \mathcal{B}(L^2 \oplus H^2, L^2)$: $SX = X(S \oplus S_k^+)$. Then taking the adjoint of both sides in the equation: $(S^* \oplus (S_k^+)^*)X^* = X^*S^*$, $X^*$ has dense range, $S^*$ is cyclic, and this implies the cyclicity of $S^* \oplus (S_k^+)^*$ for any $k \in \mathbb{N} \cup \{\aleph_0\}$ by (i) of Lemma 2, which is unitary equivalent to $S \oplus (S_k^+)^*$.

For the $k = 1$ case the definition of the operator $X$ is the following:

$$X : L^2 \oplus H^2 \to L^2, \quad f \oplus g \mapsto f \varphi + g,$$

where $\varphi \in L^\infty$, $\varphi(\zeta) \neq 0$ for a.e. $\zeta \in \mathbb{T}$ and $\int_{\mathbb{T}} \log |\varphi(\zeta)| d\zeta = -\infty$. An easy estimate shows that $X \in \mathcal{B}(L^2 \oplus H^2, L^2)$. Assume that $0 = f \varphi + g$. If $f = 0$ ($g = 0$, resp.), then $g = 0$ ($f = 0$, resp.) follows immediately. On the other hand, taking logarithms of the absolute values and integrating over $\mathbb{T}$ we get

$$-\infty < \int_{\mathbb{T}} |g| dm = \int_{\mathbb{T}} \log |f| + \log |\varphi| dm \leq \int_{\mathbb{T}} |f| dm + \int_{\mathbb{T}} \log |\varphi| dm = -\infty,$$

which is a contradiction. Therefore $X$ is injective. The equation $SX = X(S \oplus S^+)$ is trivial, therefore $S \oplus (S^+)^*$ is cyclic.

Now let us turn to the case when $k > 1$. We will work with induction, so let us suppose that we have already proven the cyclicity of $S \oplus (S_k^+)^*$ for a $k \geq 2$.

Consider the following operator

$$Y : L^2 \oplus H^2 \oplus \cdots \oplus H^2 \to L^2 \oplus H^2 \oplus \cdots \oplus H^2,$$

$$f \oplus g_1 \oplus \cdots \oplus g_k \mapsto (f \varphi + g_1) \oplus g_2 \oplus \cdots \oplus g_k,$$

with the same $\varphi \in L^\infty$ as in the definition of $X$. Obviously $Y$ is bounded, linear and injective, and we have $Y(S \oplus S_k^+) = (S \oplus S_k^+)^*Y$. This proves that $S \oplus (S_k^+)^*$ is also cyclic. \hfill \Box

Of course, now a question arises naturally. It seems that the previous thoughtline does not work for the $\aleph_0$ case.

**Question 22.** Is the operator $S \oplus (S_\alpha^+)^*$ cyclic?

If $S_\lambda$ is of class $C_1$, then in some cases we can prove that $S_\lambda^*$ is cyclic.

**Theorem 23.** The followings are valid:

(i) If $T$ has a root and the contractive weighted shift $S_\lambda$ on $T$ is of class $C_1$, then $S_\lambda^*$ is cyclic.

(ii) If $T$ is rootless, $\text{Br}(T) < \infty$ and the weighted shift contraction $S_\lambda$ on $T$ is of class $C_1$, then $S_\lambda^*$ is cyclic.
Proof. Obviously $T$ is leafless in both cases. The isometric asymptote $(A^{1/2}, U)$ of $S_{\lambda}$ is just taken. Since $U^*$ is cyclic, thus $S_{\lambda}^*$ is also cyclic by Lemma 2, Theorem 9 and Theorem 15.

If the previous question had a positive answer, then in the previous theorem we would only have to assume that $S_{\lambda} \in C_1(\ell^2(V))$.

8 Similarity of $S_{\lambda}$ to the orthogonal sum of a bi- and a unilateral shift operator

In the last section we examine the case when a weighted shift on $\tilde{T}$ (defined in Example 7) is similar to an orthogonal sum of a bi- and a unilateral shift operator. Then we show that there is a weighted shift on $\tilde{T}$ that is cyclic and an other for which $S_{\lambda}^*$ has no cyclic vectors. The weights are $\lambda = \{\lambda_n: n \in \mathbb{Z}\} \cup \{\lambda_k: k \in \mathbb{N}\}$.

We define an other bounded operator $\tilde{W} \in \ell^2(V)$ by the following equations:

$$\tilde{W}e_n = w_{n+1}e_{n+1}, \quad \tilde{W}e_k' = w_{(k+1)'e_{(k+1)'}} \quad (n \in \mathbb{Z}, k \geq 2)$$

where the weights $\{w_n: n \in \mathbb{Z}\} \cup \{w_k: k \in \mathbb{N} \setminus \{1\}\}$ are bounded. Obviously $\tilde{W}$ is an orthogonal sum of a bi- and a unilateral weighted shift.

Our aim is to find out whether there exists a $\tilde{W}$ such that it is similar to $S_{\lambda}^*$. In order to do this, we will try to find a bounded, invertible $X \in B(\ell^2)$ which intertwines $S_{\lambda}$ with $\tilde{W}$: $XS_{\lambda} = \tilde{WX}$. However, it is easier to examine the adjoint equation: $S_{\lambda}^*X^* = X^*\tilde{W}^*$. We will use the following notations

$$g_k = \prod_{j=1}^{k} \frac{1}{\lambda_j} \cdot e_k - \prod_{j=1}^{k} \frac{1}{\lambda_j'} \cdot e_{k'} \quad (k \in \mathbb{N}).$$

It is easy to see that $S_{\lambda}^*g_k = g_{k-1}$ if $k > 1$ and $0$ if $k = 1$, and that $g_k \perp g_l$ if $k \neq l$. We also define the following subspaces

$$\mathcal{E} = \{e_k: k \in \mathbb{Z}\}, \quad \mathcal{E}' = \{e_k: k \in \mathbb{N}\}, \quad \mathcal{G} = \{g_k: k \in \mathbb{N}\}.$$

First, we need a lemma. For technical reasons, we assume that $S_{\lambda}$ is contractive.

Lemma 24. The following two conditions are equivalent for the contractive $S_{\lambda}$:

(i) the positive sequence $\{\prod_{j=1}^{k} \frac{\lambda_j}{\lambda_j'}: k \in \mathbb{N}\}$ is bounded,

(ii) $\ell^2(V) = \mathcal{E} + \mathcal{G}$ (direct sum).

Proof. (i)⇒(ii): It is obvious that $\mathcal{E} \cap \mathcal{G} = \{0\}$, so we only have to prove the equation $\ell^2(V) = \mathcal{E} + \mathcal{G}$. To do this take an arbitrary vector $x \in \ell^2(V)$ and suppose that $x = e + g$ for some vectors $e \in \mathcal{E}$ and $g \in \mathcal{G}$. With the following notations

$$\xi_{k'} = \langle x, e_{k'} \rangle, \quad \xi_n = \langle x, e_n \rangle, \quad \mu_k = \langle g, e_k \rangle, \quad \nu_n = \langle e, e_n \rangle \quad (k \in \mathbb{N}, n \in \mathbb{Z})$$

and the following equations

$$\xi_{k'} = \xi_{k'+1} + \mu_{k'}, \quad \xi_n = \xi_{n+1} + \nu_n \quad (n \in \mathbb{Z})$$

we observe that

$$\xi_k = \xi_{k+1} + \mu_k + \nu_k \quad (k \in \mathbb{N})$$

which gives us the desired result.
we have the equations
\[ \xi_{k'} = \langle g, e_{k'} \rangle = -\frac{\mu_k}{\|g_k\| \cdot \lambda_{k'} \ldots \lambda_{k'}}, \]
\[ \xi_k = \langle e, e_k \rangle + \langle g, e_k \rangle = \nu_k + \frac{\mu_k}{\|g_k\| \cdot \lambda_{1} \ldots \lambda_{k}}, \quad (k \in \mathbb{N}) \]
and
\[ \xi_m = \nu_m \quad (m \in \mathbb{Z}_+). \]

From them we infer that
\[ \mu_k = -\xi_{k'} \|g_k\| \cdot \lambda_{1} \ldots \lambda_{k'}, \quad \nu_k = \xi_k + \frac{\lambda_{1} \ldots \lambda_{k'}}{\lambda_{1} \ldots \lambda_{k}} \xi_{k'}, \quad (k \in \mathbb{N}). \]

So there exists an \( e \in \mathcal{E} \) and \( g \in \mathcal{G} \) such that \( x = e + g \) if and only if
\[ \sum_{k=1}^{\infty} |\mu_k|^2 / \|g_k\|^2 < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \nu_k^2 < \infty. \]

The first inequality always holds, since
\[ \sum_{k=1}^{\infty} |\xi_{k'}|^2 \|g_k\|^2 \leq \sum_{k=1}^{\infty} |\xi_{k'}|^2 \leq \|x\|^2 < \infty. \]

The second holds if and only if
\[ \sum_{k=1}^{\infty} \left( \frac{\lambda_{1} \ldots \lambda_{k'}}{\lambda_{1} \ldots \lambda_{k}} \right)^2 \|g_k\|^2 < \infty, \]
and this holds if the sequence \( \{ \prod_{j=1}^{k} \frac{\lambda_{j'}}{\lambda_j} : k \in \mathbb{N} \} \) is bounded.

(ii)\( \Rightarrow \) (i): On the contrary, if this sequence is not bounded, then there exists a vector \( x \in \ell^2(V) \) such that this sum is not finite. In fact, if
\[ \prod_{j=1}^{k_m} \frac{\lambda_{j'}}{\lambda_j} > m \quad (\forall \ m \in \mathbb{N}), \]
then let \( \xi_{k_m} = m^{-3/2} \) and with this choice we have
\[ \sum_{k=1}^{\infty} \left( \frac{\lambda_{1} \ldots \lambda_{k'}}{\lambda_{1} \ldots \lambda_{k}} \right)^2 \|g_k\|^2 > \sum_{m=1}^{\infty} \frac{1}{m} = \infty. \]

Now, we are able to prove a similarity theorem. The operator \( T_1 \in B(H) \) is a quasiaffine transform of \( T_2 \in B(K) \) if there exists a quasiaffinity (i.e.: which is injective and has a dense range) \( X \in B(H, K) \) such that \( XT_1 = T_2 X \).

**Theorem 25.** Let \( S_\Lambda \) be a weighted shift contraction on the directed tree \( \tilde{T} \) and set
\[ w_n = \lambda_n, \quad (n \in \mathbb{Z}), \quad w_{k'} = \frac{\|g_{k-1}\|}{\|g_k\|}, \quad (k > 1). \]

The following two points hold:

(i) \( S_\Lambda \) is always a quasiaffine transform of \( \tilde{W} \).
(ii) If \( \{ \prod_{j=1}^k \lambda_j : k \in \mathbb{N} \} \) is bounded, then the weighted shift \( S_\lambda \) is similar to \( \widetilde{W} \).

Proof. Since \( S_\lambda \) is a contraction, \( w_k = \frac{\|g_k - 1\|}{\|g_k\|} \leq 1 \) and hence \( \widetilde{W} \) is bounded. We will define \( X^* \) by the equations

\[
X^*e_{k'} = \frac{1}{\|g_k\|} g_k, \quad X^*e_n = e_n \quad (k \in \mathbb{N}, n \in \mathbb{Z}).
\]

The operator \( X^* \) is bounded and quasiaffine because \( \vee \{e_k, e_{k'}\} \) is invariant for \( X \) for every \( k \in \mathbb{N} \), and \( X^* \vee \{e_k, e_{k'}\} \) has norm less than or equal to 2. The next equations show that \( X^* \) intertwines \( \widetilde{W}^* \) with \( S_\lambda^* \):

\[
S_\lambda^*X^*e_n = S_\lambda^*e_n = \lambda_n e_{n-1} = \lambda_n X^*e_{n-1} = X^*\widetilde{W}^*e_n, \quad (n \in \mathbb{Z})
\]

\[
S_\lambda^*X^*e_{k'} = \frac{1}{\|g_k\|} S_\lambda^*g_k = \begin{cases} 0 & k = 1 \\ \frac{1}{\|g_k\|} g_{k-1} & k > 1 \end{cases} = X^*\widetilde{W}^*e_{k'}. \quad (k \in \mathbb{N})
\]

This proves that \( S_\lambda \) is a quasiaffine transform of \( \widetilde{W} \).

If \( \mathcal{E} + \mathcal{G} = \ell^2(\tilde{V}) \), then since \( X^*\mathcal{E}' \in \mathcal{B}(\mathcal{E}', \mathcal{G}) \) and \( X^*\mathcal{E} \in \mathcal{B}(\mathcal{E}, \mathcal{E}) \) are unitary transformations, \( X^* \) is an invertible bounded operator. This proves the similarity.

\[ \square \]

Corollary 26. If \( S_\lambda \notin C_0(\ell^2(V)) \) which is defined on \( \tilde{T} \), then it is similar to an orthogonal sum of a weighted bi- and a weighted unilateral shift.

Proof. The condition \( S_\lambda \notin C_0(\ell^2(V)) \) is equivalent to the strict positivity of \( \prod_{j=1}^\infty \lambda_j \) or \( \prod_{j=1}^\infty \lambda_j' \). By interchanging \( \lambda_j \) and \( \lambda_j' \) for every \( j \in \mathbb{N} \), if necessary, we can assume that the first one is not zero. Then the sequence \( \{ \prod_{j=1}^k \lambda_j' : k \in \mathbb{N} \} \) is obviously bounded. From the previous theorem we can see the similarity.

\[ \square \]

In the last theorem we show that a weighted shift on \( \tilde{T} \) can be cyclic.

Theorem 27. There is a weighted shift on \( \tilde{T} \) which is cyclic.
Proof. Take an $S_\lambda$ such that is similar to a $\tilde{W}$ and the bilateral summand of $\tilde{W}$ is hypercyclic. By decreasing $|\lambda_k|$-s, we may also assume that the unilateral summand is contractive. Take a hypercyclic vector $f \in E$ for the bilateral summand. We will show that $f \oplus e_1'$ is cyclic for $\tilde{W}$ and therefore $S_\lambda$ is cyclic.

First, let us take an arbitrary vector $e \in E$, then there is a subsequence such that $\frac{1}{k} \tilde{W}^k e \to e$. Therefore $\frac{1}{k} \tilde{W}^k (f \oplus e_1') \to e$ also holds, since the unilateral summand is a contraction.

Second, fix an $n \in \mathbb{Z}_+$. Our aim is to prove that $\tilde{W}^n e_1' \in \vee\{\tilde{W}^k(f \oplus e_1') : k \in \mathbb{Z}_+\}$. Since $\{\tilde{W}^k f : k > n\}$ is dense in $E$, there is a subsequence $\{\tilde{W}^k f\}_{k=1}^\infty$ with the property $\frac{1}{k} \tilde{W}^k f \to \tilde{W}^n f$. This implies that $\frac{1}{k} \tilde{W}^k (f \oplus e_1') \to \tilde{W}^n f$ and hence $\tilde{W}^n e_1' \in \vee\{\tilde{W}^k(f \oplus e_1') : k \in \mathbb{Z}_+\}$. This proves that $f \oplus e_1'$ is cyclic for $\tilde{W}$.

Finally, we note that if we take a weighted shift on a directed tree which is similar to a $\tilde{W}$ and the adjoint of the bilateral summand has no cyclic vectors, then obviously we get an $S_\lambda$ on $\tilde{T}$ such that $S_\lambda^*$ has no cyclic vectors.

Acknowledgement. The author is very grateful to professor László Kérchy for some useful suggestions which helped the author to improve the paper.

References

[Be] B. Beauzamy, A Weighted Bilateral Shift with no Cyclic Vector, J. Operator Theory, 4 (1980), 287–288.

[BJJS1] P. Budzyński, Z. J. Jabłonski, I. B. Jung and J. Stochel, Unbounded Subnormal Weighted Shifts on Directed Trees, J. Math. Anal. Appl. 394 (2012), no. 2, 819–834.

[BJJS2] P. Budzyński, Z. J. Jabłonski, I. B. Jung and J. Stochel, Unbounded Subnormal Weighted Shifts on Directed Trees II., J. Math. Anal. Appl. 398 (2013), no. 2, 600–608.

[Ge] L. Gehér, Cyclic Vectors of a Cyclic Operator Span the Space, Proc. Amer. Math. Soc., 33, (1972), 109–110.

[Gu] Z. Guang Hua, On Cyclic Vectors of Backward Weighted Shifts. (Chinese. English summary), J. Math. Res. Exposition, 4 (1984), no. 3, 1–6.

[Ha] P. R. Halmos, A Hilbert Space Problem Book, Second Edition, Springer Verlag, 1982.

[JJS1] Z. J. Jabłonski, I. B. Jung and J. Stochel, A Non-hyponormal Operator Generating Stieltjes Moment Sequences, J. Funct. Anal. 262 (2012), no. 9, 3946–3980.

[JJS2] Z. J. Jabłonski, I. B. Jung and J. Stochel, Normal Extensions Escape from the Class of Weighted Shifts on Directed Trees, Complex Anal. Oper. Theory 7 (2013), no. 2, 409–419.
[JJS3] Z. J. Jablonski, I. B. Jung and J. Stochel, Weighted Shifts on Directed Trees, Memoirs of the American Mathematical Society, Number 1017, 2012.

[KT] L. Kérchy and V. Totik, Compression of Quasianalytic Spectral Sets of Cyclic Contractions, J. Funct. Anal. 263 (2012), no. 9, 2754–2769.

[Ku] C. S. Kubrusly, An Introduction to Models and Decompositions in Operator Theory, Birkhäuser, 1997.

[NFBK] B. Sz.-Nagy, C. Foias, H. Bercovici and L. Kérchy, Harmonic Analysis of Operators on Hilbert Space, Second Edition, Springer, 2010.

[Sa1] H. N. Salas, Hypercyclic Weighted Shifts, Trans. Amer. Math. Soc. 347 (1995), 993–1004.

[Sa2] H. N. Salas, Supercyclicity and Weighted Shifts, Studia Math. 135 (1999), no. 1, 55–74.

[Sh] A. L. Shields, Weighted Shift Operators and Analytic Function Theory, Topics in Operator Theory, Math. Surveys 13, Amer. Math. Soc., Providence, R. I., 1974, 49–128.

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