Rise and fall of laser-intensity effects in spectrally resolved Compton process

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Abstract

The laser intensity dependence of nonlinear Compton scattering is discussed in some detail. For sufficiently hard photons with energy $\omega'$, the spectrally resolved differential cross section $d\sigma/d\omega'|_{\omega'=\text{const}}$ rises from small toward larger laser intensity parameter $\xi$, reaches a maximum, and falls toward the asymptotic strong-field region. Such a rise and fall of a differential observable is to be contrasted with the monotonously increasing laser intensity dependence of the total probability, which is governed by the soft spectral part. We combine that hard-photon yield from Compton scattering with the seeded Breit–Wheeler pair production in a folding model and obtain a rapidly increasing $e^+ e^-$ pair number at $\xi \lesssim 4$. Laser bandwidth effects are quantified in the weak-field limit of the related trident pair production.

1. Introduction

Quantum electro-dynamics (QED) as pillar of the standard model (SM) of particle physics possesses a positive beta-function [1] which makes the running coupling strength $\alpha(s)$ increasingly with increasing energy/momentum scale $s$ [2]. In contrast, quantum chromo-dynamics (QCD) as another SM pillar possesses a negative beta-function due to the non-Abelian gauge group [1], giving rise to the asymptotic freedom, $\lim_{s\to\infty} \alpha_{\text{QCD}}(s) = 0$, i.e. QCD has a truly perturbative limit. In QED, however, $\lim_{s\to 0} \alpha(s) \rightarrow 1/137.0359895(61)$ is not such a strict limit, nevertheless, predictions/calculations of some observables agree with measurements within 13 digits, see [3–5] for some examples. The situation in QED becomes special when considering processes in external (or background) fields: one can resort to the Furry (or bound-state) picture, where the (tree-level) interactions of an elementary charge (e.g. an electron) with the background are accounted for in all orders, and the interactions with the quantized photon field remains perturbatively in powers of $\alpha$. However, the Ritus–Narozhny (RN) conjecture [6–10] argues that the effective coupling becomes $\alpha \chi^{2/3}$, meaning that the Furry picture expansion breaks down at $\alpha \chi^{2/3} > 1$ [11–14] (for the definition of $\chi$ see below) and one enters a genuinely non-perturbative regime. The latter requires adequate calculation procedures, as the lattice regularized approaches, which are standard since many years in QCD, e.g. in evaluations of observables in the soft sector where $\alpha_{\text{QCD}} > 1$, cf [15]. (In QED itself, an analog situation is meet in the Coulomb field of nuclear systems with proton numbers $Z > Z_{\text{crit}} \approx 173$; if $\alpha Z_{\text{crit}} > 1$ the QED vacuum break-down sets in; cf [16] for the actual status of that field).

With respect to increasing laser intensities the quest for the possible break-down of the Furry picture expansion in line with the RN conjecture becomes, besides its principal challenge, also of ‘practical’ interest, whether one can explore experimentally this yet uncharted regime of QED. (For other configurations, e.g. beam–beam interactions, cf [17]). A prerequisite would be to find observables which display the typical dependence $\propto \alpha \chi^{2/3}$, where we denote by $\alpha$ the above quoted fine-structure constant at $s \to 0$. In doing so
we resort here to the lowest-order QED processes, that is nonlinear Compton and nonlinear Breit–Wheeler (BW). Both processes seem to be investigated theoretically in depth in the past, however, enjoy currently repeated re-considerations w.r.t. refinements [18–23], establishing approximation schemes [24–28] to be implemented in simulation codes [29–31] or to use them as building blocks in complex processes [32], with the starting point at trident [33–35].

Beginning with Compton scattering \((p \text{ and } p' \text{ are the in- and out-electron four-momenta, } k \text{ the laser four-momentum, and } k' \text{ the out-photon’s four-momentum, respectively})\) the relevant (Lorentz and gauge invariant) variables of the entrance channel are [36]

(a) The classical intensity parameter of the laser: \(\xi = |e|\mathcal{E}/(m \omega)\), here expressed by quantities in the lab: \(\mathcal{E}\)-electric laser field strength, \(\omega\)-the central laser frequency; \(-|e| \text{ and } m \text{ stand for the electron charge and mass, respectively, and } e^2/4\pi = \alpha^2\).

(b) The available energy squared: \(k \cdot p/m^2 = (s/m^2 - 1)/2\) with \(s\) as Mandelstam variable,

(c) The quantum nonlinearity parameter: \(\chi = k \cdot p/m^2\).

The latter quantity is often considered as the crucial parameter since in some limits it determines solely the probability of certain processes. \(\chi\) plays also a prominent role in the above mentioned discussion of the RN conjecture, where the Furry picture expansion of QED is argued to break down for \(\alpha \chi^{2/3} > 1\). References [37, 38] point out that the large-\(\chi\) limits, facilitated by either large \(\xi\) (the high-intensity limit) or large \(k \cdot p/m^2\) (the high-energy limit), are distinctively different, with implications for approximation schemes in simulation codes.

Figure 1 exhibits a few selected curves \(\chi = \alpha \chi^{2/3} = 1 \text{ const}\) over the \(\xi \text{ vs } k \cdot p/m^2\) landscape to illustrate the current situation w.r.t. facilities where laser beams and electron beams are (or can be) combined. One has to add the options at E-320 at FACET-II/SLAC [39, 40] and electron beams which are laser-accelerated to GeV scales, e.g. [41–43].

Usually, \(\xi = 1\) is said to mark the onset of strong-field effects, and the corresponding processes at \(\xi > 1\) are termed by the attribute ‘nonlinear’. In this respect, the parameters provided by LUXE [44, 45] and FACET-II [39, 40] are interesting: \(\chi = \mathcal{O}(1)\) and above and \(\xi > 1\) as well\(^5\). In the following we consider the LUXE kinematics (see figure 1), \(k \cdot p/m^2 \approx \omega E_e(1 - \cos \Theta) = 0.2078\) in head-on collisions (\(\cos \Theta = -1\)). Our aim is to quantify a simple observable as a function of \(\xi\). To be specific, we select the invariant differential cross section \(d\sigma/du\), where \(u\) is the light-cone momentum-transfer of the in-electron to the

\(^5\) We employ natural units with \(\hbar = c = 1\).

\(^6\) Despite high intensities at XFELs, e.g. \(I \rightarrow 10^{22} \text{ W cm}^{-2}\), the intensity parameter \(\xi = \frac{\mathcal{E} \cdot \omega}{\sqrt{\omega_p} \cdot \text{ W cm}^{-2}}\) [36] is small due to the high frequency, e.g. \(\omega = 1\text{–}25\text{ keV}\).
out-photon, related to light-front momentum-fraction of the out-photon \( s = \frac{u}{1+u} = \frac{k' \cdot \nu}{k' \cdot p} \). (The mapping \( u \mapsto \omega' \) and \( d\sigma/du \mapsto d\sigma/d\nu' \) is discussed in [46, 47].) To make the meaning of the Ritus variable \( u \) more explicit let us mention the relation

\[
    u = \frac{e^{-\xi'}(1 - \cos \Theta')}{1 - e^{-\xi'}(1 - \cos \Theta')},
\]

where \( \nu' \equiv \omega' / m \), and the electron energy \( E_e \) in lab determines the rapidity \( \zeta \) via \( E_e = m \cosh \zeta \) and \( \Theta' \) denotes the polar lab angle of the out-photon. We call \( d\sigma/du \) a spectrally resolved observable.

Our note is organized as follows. In section 2 we briefly recall a few approximations of the laser beam. Section 3 is devoted to an analysis of the invariant differential cross section \( d\sigma/du \) and its dependence on the laser intensity parameter \( \xi \) in nonlinear Compton scattering. That is, we are going up and down on the vertical dashed line with label LUXE in figure 1 around the point \( \chi = 1 \) or \( \xi = 1 \). The discussion section 4 (i) relates the cross section to the probability and (ii) considers a folding model which uses the hard-photon spectrum emerging from Compton scattering as seed for subsequent BW pair production; a brief discussion of (iii) bandwidth effects relevant for sub-threshold trident pair production complements this section. We conclude in section 5. The appendix A recalls a few basic elements of the one-photon Compton process.

2. Laser models

In plane-wave approximation the laser (circular polarization) can be described by the four-potential in axial gauge, \( A = (0, \vec{A}) \), with

\[
    \vec{A} = f(\phi) \left( \vec{a}_x \cos \phi + \vec{a}_y \sin \phi \right), \tag{1}
\]

where \( a_x^2 = a_y^2 = m^2 \xi^2 / \epsilon^2 \); the polarization vectors \( \vec{a}_x \) and \( \vec{a}_y \) are mutually orthogonal. We ignore a possible non-zero value of the carrier envelope phase and focus on symmetric envelope functions \( f(\phi) \) w.r.t. the invariant phase \( \phi = k \cdot x \). One may classify the such a model class as follows.

(a) Laser pulses: \( \lim_{\phi \to \pm \infty} f(\phi) = 0 \),

(b) Monochromatic beam: \( f(\phi) = 1 \),

(c) Constant cross field (ccf): \( \vec{A} = \phi \vec{a}_x \).

The probabilities for the ccf option (c) coincide with certain limits of the plane-wave model (b) [58]; in [47] they are related to the large-\( \xi \) limit. Item (a) could be divided into several further sub-classes, such as (1.1): finite support region of the pulse, i.e. \( f(\phi) > \phi \\text{pulse length} \), \( \phi \\text{pulse length} < \infty \), and (1.2): far-extended support region, i.e. \( \lim_{\phi \to \pm \infty} f(\phi) \to 0 \), together with non-zero carrier envelope phase, asymmetric pulse shape, frequency chirping, polarization gating etc. A specific class is 1.1) with flat-top

Due to the high symmetry, the exact solutions of the Dirac equation in a plane-wave background are in a comfortably compact form enabling an easy processing in evaluations of matrix elements. Without such a high symmetry, much more attempts are required [51–53]. For some useful parameterizations of laser beams, see [54–56] and further citations therein. The relevance of the Fourier-zero mode of (non-)unipolar planar fields is mentioned in [57].

Figure 2. Invariant differential cross section \( d\sigma(u, \xi, k \cdot p / m^2) / du \) as a function of \( u \) for several values of \( \xi, \xi = 0.2 \) (magenta), 0.4 (blue), 0.7 (green) and 1 (black). The dashed curves (label FPA) are for pulses according to equation (1) with \( f(\phi) = 1 / \cosh(\phi/\pi N) \) for \( N = 10 \). The FPA results for a monochromatic laser beam are depicted by solid curves (with the limit \( d\sigma_{3\hbar}/du_{3\hbar} = 2 \pi \epsilon^2 / k' p \) independent of \( \xi \)). The pronounced harmonic structures around the Klein–Nishina (KN) edge \( u_{\text{KN}} \approx 0.416 \) are irrelevant for the subsequent discussion and, therefore, do not need a detailed representation. For \( \chi = \xi k \cdot p / m^2 = 0.2078 \xi \).
This complements figures 1–3 in [47]. One observes that the harmonic structures (which would become more severe for linear polarization, see [47]) fade away at larger values of $\sigma$.

In particular, we consider now $d^{3}$. Numerical results

more involved FP A calculations.

Differential spectra $d\sigma_{\text{FP A}}/du$, equation (2), while dashed (dotted) curves are based on $d\sigma_{\text{large-} \cdot \cdot \cdot }/du$, equation (4) ($d\sigma_{\text{large-} \cdot \cdot \cdot }/du$, equation (6)).

3. Compton: differential invariant cross section

Let us consider the above pulse envelope function $f(\phi) = 1/\cosh(\phi/(\pi N))$ to elucidate the impact of a finite pulse duration and contrast it later on with the monochromatic laser beam model and some approximations thereof. Differential spectra $d\sigma/du$ as a function of $u$ are exhibited in figure 2 in the region $u \leq 3$ for several values of $\xi \leq 1$ for the FPA (dashed curves) and IPA (solid curves) models recalled below. This complements figures 1–3 in [47]. One observes that the harmonic structures (which would become more severe for linear polarization, see [47]) fade away at larger values of $u$ and $\xi$. Therefore, we are going to analyze that region in parameter space. There, IPA results represent reasonably well the trends of the more involved FPA calculations.

3.1. Numerical results

In particular, we consider now $d\sigma(u, \xi, k \cdot p / m^2)/du$ as a function of $\xi$ for several constant values of $u$, $u = 0.5, 1, 2, 4$ and 8, for $k \cdot p / m^2 = 0.2078$—a value which is motivated by the LUXE opportunities [44, 45]. The solid, dotted and dashed curves in figure 3 are based on easily accessible models [58]:

- Monochromatic model (IPA) (cf equations (15) and (16) in [47]):

$$\frac{d\sigma_{\text{IPA}}}{du} = \frac{2\pi c^2}{k \cdot p} \frac{1}{(1 + u)^2} \sum_{n=1}^{\infty} \Theta(u_n - u) F_n(z_n),$$

$$F_n = -\frac{2}{\xi^2} J_n(z_n)$$

$$+ A \left( J_{n+1}(z_n) + J_{n-1}(z_n) - 2J_n^2(z_n) \right)$$

with $A = 1 + \frac{u^2}{2(1+u)}$, $u_n = \frac{2nk}{m(1+\xi)}$ and

$$z_n(u, u_n) = \frac{2n\xi}{\sqrt{1 + \xi^2}} \sqrt{\frac{u}{u_n} \left( 1 - \frac{u}{u_n} \right)},$$

($J_n$’s denote Bessel functions of first kind).
- IPA-large-\( \zeta \) approximation (cf equation (21) in [47]):

\[
\frac{d\sigma_{\text{large-}\zeta}}{du} = -\frac{4\pi\alpha^2}{m^2\xi} \frac{1}{\chi (1 + u)^2} \mathcal{F}_C
\]

\[
\mathcal{F}_C = \int_0^{\infty} dy \Phi(y) + \frac{2}{z} \left[ 1 + \frac{u^2}{2(1 + u)} \right] \Phi'(z)
\]

where \( \Phi(z) \) and \( \Phi'(z) \) stand for the Airy function and its derivative with arguments \( z = (u/\chi)^{2/3} \), where \( \chi = \xi k \cdot p/m^2 \);

- Large-\( \zeta \)-large-\( u \) approximation (cf equation (22) in [47]):

\[
\frac{d\sigma_{\text{FPA}}}{du} = \frac{\alpha^2}{k \cdot p} \frac{1}{(1 + u)^2} \int_0^\infty \frac{d\ell}{\Theta} \left( \frac{i m^2}{2 k \cdot p} - \ell \right) \omega(\ell),
\]

\[
\omega(\ell, z) = \int_0^{2\pi} d\phi \left[ -\frac{2}{\xi^2} |\tilde{Y}(z)|^2 + A \left( |Y_{\ell-1}(z)|^2 + |Y_{\ell+1}(z)|^2 - 2 \text{Re}\tilde{Y}(z)Y\ast(z) \right) \right]
\]

with \( A \) as above, \( z_0(u, \ell) = 2\xi \sqrt{\frac{m^2}{2z_0(1 - \frac{u}{\chi})}} \) and \( u_0 = \frac{2k \cdot p}{m^2} \cdot \). The basic functions \( Y_{\ell}, \tilde{Y}, X_{\ell} \) and their \( \phi \) dependence are spelled out in [47, 62]; they depend crucially on the pulse envelope function \( f(\phi) \).

Due to numerical accuracy reasons in integrating highly oscillating functions, these evaluations are constrained presently to not too large values of \( \xi \). The common basis of these models is sketched in appendix A.

We consider in figure 3 only \( u > u_{\text{KN}} = 2k \cdot p/m^2 \), since at the KN edge the harmonic structures become severe, as seen in figure 2. The striking feature seen in figure 3 is the pronounced bump-like shape, which we coin ‘rise and fall’ of laser intensity effects. At small \( \xi \), the realistic FPA (\( N = 10 \)) results (asterisks) and the IPA model (solid curves) follow the same trends, consistent with figure 2. The large-\( \xi \) and large-\( \zeta \)-large-\( u \) approximations (dashed and dotted curves) are not supposed to apply in that region.

However, they become useful representatives at large \( \xi \).

### 3.2. The rise

Some guidance of the rising parts of the FPA results in figure 3 can be gained by the monochromatic model. Casting equation (3) in the form

\[
F_n = -\frac{2(1 - \Xi^2)}{\Xi^2} f_n^2(\Xi x_n) + A \left( f_{n+1}^2(\Xi x_n) + f_{n-1}^2(\Xi x_n) - 2f_n^2(\Xi x_n) \right)
\]

with \( \Xi^2 = \xi^2/(1 + \xi^2) \) and \( x_n(u) = 2n\sqrt{\frac{m^2}{2z_0(1 - \frac{u}{\chi})}} \) and expanding in powers of \( \Xi \) yields for the first terms

\[
F_1 = \frac{1}{2}(2A - x_1^2) - \frac{x_1^2}{8}(8A - x_1^2 - 4)\Xi^2 + \frac{x_1^4}{384}(90A - 5x_1^2 - 48)\Xi^4 + O(\Xi^6),
\]

\[
F_2 = \frac{x_1^2}{32}(8A - x_1^2)\Xi^2 - \frac{x_1^4}{192}(18A - x_1^2 - 6)\Xi^4 + O(\Xi^6),
\]

\[
F_3 = \frac{x_1^4}{1152}(18A - x_1^2)\Xi^4 + O(\Xi^6).
\]

Due to the Heaviside function in equation (2), the leading-order power in \( \Xi \) depends on the value of \( u \), e.g. for \( u < u_1 \), the series starts with \( O(\Xi^0) \) and the coefficients of higher orders accordingly sum up. Higher values of \( u \) facilitate higher orders of the leading terms, i.e. the rise becomes steeper since the respective leading-order term is \( \propto \Xi^{2|u/u_1|} \). This statement is based on the structures in equations (10)–(12), suggesting \( F_n = \sum_{|\ell| = n+1} F_n^{2|\ell|} \Xi^{2|\ell|} \) for the first terms \( F_n^{2|\ell|} \), and \( \sum_{n=1}^{\infty} \Theta(u_n - u) \) \( F_n = \sum_{n_{\min}} F_n \) with \( n_{\min} = 1 + \lfloor u/u_1 \rfloor \), (\( \lfloor \cdot \rfloor \) is the floor operation.)
The cross section vs probability

4.1. Cross section vs probability

4. Discussion

The cross section from equation (6). The maximum of the curves exhibited in figure 3 at 3.3. The fall formula equation (6) [58] represents a fairly accurate description supposed ξ < at larger values of ξ structures in IP A become severe.) Nevertheless, some estimate of the maximum position is provided by squeezed into a narrow corridor already in the non-asymptotic experiments. In the strong-field asymptotic region it displays a funneling behavior, i.e. the curves are the asymptotic fall is governed by approximations (dotted and dashed curves) deliver results in fair agreement with the IP A results (solid curves) for u > 1. That is, when being interested in the high-energy photon tails, the simple large-ξ asymptotic formula equation (6) [58] represents a fairly accurate description supposed ξ is sufficiently large. Obviously, at ξ < 1 and u < 2, such an approximation fails quantitatively. (In particular, at u < 1 the harmonic structures in IPA become severe.) Nevertheless, some estimate of the maximum position is provided by ξmax ≈ 1/2um²/k · p yielding

\[ \frac{d\sigma}{du}\bigg|_{\xi_{\text{max}}} \approx 2\sqrt{\pi\alpha^2} \frac{k \cdot p \left( \frac{3}{2} \right)}{m^2} e^{-3/2} u^{-3}. \]

The asymptotic fall is governed by

\[ \frac{d\sigma}{du} \propto \frac{1}{m^2} \frac{m^2}{k \cdot p} \xi^{-3/2} u^{-3/2} \exp\left(-\frac{2um^2}{3\xi k \cdot p}\right) \]

from equation (6).

We emphasize the same pattern of rise and fall of \( d\sigma/d\omega'\big|_{\omega'=\text{const}} \), see figure 4, again for sufficiently large values of \( \omega' \). The spectrally resolved differential cross section \( d\sigma/d\omega'\big|_{\omega'=\text{const}} \) is directly accessible in experiments. In the strong-field asymptotic region it displays a funneling behavior, i.e. the curves are squeezed into a narrow corridor already in the non-asymptotic ξ region, in contrast to \( d\sigma/du\big|_{u=\text{const}} \) in figure 3.

4. Discussion

4.1. Cross section vs probability

The cross section \( \sigma \) and Ritus probability rate \( W \) are related as \( W = \frac{d\sigma}{d\omega'}(\xi = 0) \sigma \) with \( \xi_0 \) denoting the energy component of the quasi-momentum of the in-electron. This relation holds true for circular polarization and applies to respective differential quantities too. The different normalization modifies in particular the ξ dependence: the above emphasized ‘rise and fall’ of \( d\sigma/du \) corresponds to a monotonously rising probability \( dW/du \) as a function of ξ. Having in mind Ritus’ remark ‘the cross-sectional concept becomes meaningless’ since, at \( \xi \to \infty \) we have \( \sigma \to 0 \) while \( W \) remains finite [58], we turn in this sub-section to the probability. In doing so we remind the reader of the subtle Ritus notation \( W(\chi) = \frac{1}{2} \int_0^\infty d\psi P(\chi \sin \psi) \) in distinguishing the probabilities \( W \) and \( P \).

We also stress the varying behavior of total (cf appendix A in [65]) and differential probabilities. Equation (6) emerges as a certain limit of the ccf probability [58]

\[ \frac{dP(\chi, u)}{du} = -V(1 + u)^{-2} J_2(z(\chi, u), u), \]
where the prefactor reads \( V = \alpha m e^2 / \pi q_0 \) and \( \mathcal{F}_C \) is defined in equation (5). With respect to the argument \( z = (u/\chi)^{2/3} \) of the Airy functions in equation (5), the \( \chi-u \) plane can be divided into the regions I (where \( u > \chi \)) and II (where \( u < \chi \)). Furthermore, the combination \( 1 + u \) suggests a splitting into \( u < 1 \) and \( u > 1 \) sub-domains. Picking up the leading-order terms in the related series expansions one arrives at

\[
\frac{dP}{du} \bigg|_{u > \chi} = \frac{V}{2\sqrt{\pi}} \chi^{1/2} \exp\left(-\frac{2u}{3\chi}\right) \begin{cases} 
\frac{u^{-1/2}}{2} & \text{for } u \ll 1, \\
\frac{u^{-3/2}}{2} & \text{for } u \gg 1,
\end{cases} \quad (13)
\]

\[
\frac{dP}{du} \bigg|_{u < \chi} = -V\Phi'(0)\chi^{2/3} \begin{cases} 
2u^{-2/3} & \text{for } u \ll 1, \\
u^{-5/3} & \text{for } u \gg 1.
\end{cases} \quad (14)
\]

It is the soft \( u \)-part \( II^c \) of the differential probability in equation (14) which essentially determines the total probability upon \( u \)-integration, i.e. the celebrated result \( P \propto \alpha \chi^{2/3} \) in Risus notation \([58]\) (stated as \( \lim_{\chi \to \infty} \frac{P}{\chi} \propto \alpha \chi^{2/3} m^2 / k \cdot p \) in \([37]\)), while the hard \( u \)-part \( I^c \) in equation (13) is at the origin of equation (6) when converting to cross section. In other words, one has to distinguish the \( \xi \) (or \( \chi \)) dependence either of integrated or differential observables. In relation to the LUXE plans \([44]\) we mention the photon detector developments \([66]\) which should enable in fact the access to the differential spectra.

### 4.2. A secondary process: seeded pair production

Instead transferring these considerations to the BW process per se (cf \([67–74]\) and further citations below), we estimate now the BW pair production seeded by the hard photons from the above Compton (C) process. The following folding model \( C \otimes BW \) is a pure two-step ansatz on the probabilistic level which mimics in a simple manner some part of the trident process \( e^- \rightarrow e^+ + e^- + e^- \) by ignoring (i) the possible off-shell effects and non-transverse components of the intermediate photon (that would be the one-step contribution) and (ii) the anti-symmetrization of the two electrons in the final state (that would be the exchange contribution). Such an ansatz is similar to the one in \([75]\), where however bremsstrahlung \( \otimes BW \) has been analyzed. An analog approach has been elaborated in \([76]\) for di-muon production.

#### 4.2.1. Estimate of the pair number

Specifically, we consider the two-step cascade process where a GeV-Compton photon with energy \( \omega' \) is produced in the first step, and, in the next step, that GeV-photon interacts with the same laser field producing a BW-\( e^+e^- \) pair. We estimate the number of pairs produced in one pulse by

\[
N_{e^+e^-} = \int_0^{\infty} d\omega' \int_0^{T_C} dt \int_0^{T_{BW}} d\Gamma_{BW}(\omega', t) \cdot d\Gamma_{C}(\omega', t),
\]

where \( d\Gamma_{C}(\omega', t) / d\omega' \) is the Compton rate of photons per frequency interval \( d\omega' \) (which corresponds to \( dP / du \) in section 4.1) emerging at time \( t \in [0, T_C] \), and \( \Gamma_{BW}(\omega', t) \) is the rate of BW pairs generated by a probe photon of frequency \( \omega' \) at lab frame time-distance \( t' \in [t, T_{BW}] \). The underlying picture is that of an electron traversing a laser pulse in head-on geometry near to light cone. The passage time of an undisturbed electron would be \( N T_0 / 2, T_0 = 2\pi / \omega' \). Neglecting spatio-temporal variations within the pulse, the final formula becomes

\[
N_{e^+e^-} = F_t \int_0^{E_c} d\omega' N_0 \frac{d\Gamma_C(\omega')}{d\omega'} \cdot \Gamma_{BW}(\omega')
\]

upon the restriction \( \omega' < E_c \) and \( F_t = T_C(T_{BW} - T_C / 2) \). A crucial issue is the choice of the formation time(s) \([80, 81]\). When gluing \( C \otimes BW \) on the amplitude level, such a time appears linearly in the pair rate \([82]\) or quadratically in the net probability via overlap light-cone times (cf \([77, 78]\) for instance). The time-ordered double integral over the C and BW probabilities, yielding the cascade approximation (cf equation (43) in \([78]\)), is analog to our above formula by folding two rates, which facilitates \( F_t \propto T_0^2 \), if \( T_{C,BW} \propto T_0 \). We attach to the Compton rate the number \( N_0 = 6 \times 10^9 \) of primary electrons per bunch.

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8 The value \( \chi = 1 \) is special since the small-\( u \) region of II and the large-\( u \) region of I must be joint directly, \( II^c \otimes I^c \), while for \( \chi < 1 \) the small-\( u \) region II and the small-\( u \) region of I must be joint followed by the large-\( u \) region of I, \( II^c \otimes I^c \otimes I^c \). In the opposite case of \( \chi > 1 \), the small-\( u \) region of II must be joint with the large-\( u \) region of II followed by the large-\( u \) region of I, \( II^c \otimes II^c \otimes I^c \). This distinction becomes also evident when inspecting the differently shaped sections of the continuous curves \( dP / du \) as a function of \( u \) for several values of \( \chi \).

9 For recent work on the formalism, see \([33–35, 77–79]\) which advance earlier investigations \([80, 81]\).

10 In laser pulses such an additional parameter is not needed \([83]\).
Strictly speaking, imposing a finite duration in the monochromatic laser beam model (2) turns it into a flat-top laser pulse model of
with the differential rate \( \frac{d\Gamma_C}{d\omega'} \) of equation (5). Note the rise of \( \xi \) at fixed values of \( \xi = 1 \) as a function of \( \omega' \). Remarkable is the strong impact of increasing \( \xi \) on the BW rate.

For a first numerical estimate at high laser intensities, the following convenient approximations can be deployed:

\[
\frac{d\Gamma_C}{d\omega'} = \frac{\alpha m^2}{\pi E_c} F_C(z, u), \tag{16}
\]

\[
\Gamma_{BW}(\omega') = \frac{\alpha m^2}{\omega'} F_{BW}, \tag{17}
\]

\[
 F_{BW} = \sqrt{\frac{3\pi}{2\kappa}} \exp \left[ -\frac{8}{3\kappa} \left( 1 - \frac{1}{15\xi^2} \right) \right], \tag{18}
\]

where \( u = \kappa/(\chi - \kappa), \chi = (k \cdot p)\xi/m^2, \kappa = (k \cdot k')\xi/m^2, \) and \( F_C(z, u) \) with \( z = (u/\chi)^{2/3} \) is defined in equation (5). Note the rise of \( \Gamma_{BW} \) and the fall of \( d\Gamma_C/d\omega' \) as a function of \( \omega' \) at fixed values of \( \xi \) and \( k \cdot p \) and laser frequency \( \omega' \), see figure 5. Increasing values of \( \xi \) lift \( d\Gamma_C/d\omega' \) somewhat and make it flatter in the intermediate-\( \omega' \) region, thus sharpening the drop at \( \omega' \rightarrow E_c \). Remarkable is the strong impact of increasing \( \xi \) on the BW rate.

For the differential Compton rate, one may replace equation (5) in (16) by the sum over harmonics expressed by Bessel functions [58] to get an improvement of accuracy in the small-\( \omega' \) region:

\[
\frac{d\Gamma_{IPA}^C}{d\omega'} = \frac{d\sigma_{IPA}^C}{d\omega'} \xi^2 m^2 k \cdot p \left\{ - \frac{4f_n^2}{\pi e^2} \sum_{n=1}^{\infty} \frac{\Theta(u_n - u)}{P_n} \right\} \tag{19}
\]

\[
+ \xi^2 \left[ 2 + \frac{u^2}{1 + u} \right] \left( f_{n+1}^2 - f_n^2 - f_n^2 \right) \tag{20}
\]

with \( q_0 = \sqrt{E_c^2 - m^2} + \beta \omega' \), \( \beta_p = \xi^2 m^2 q \cdot k = k \cdot p \), and arguments \( z = \frac{2n}{\sqrt{1 + \xi^2}} u_n \) of the Bessel functions \( J_n \) as well as

\[
u_n = 2n \frac{k \cdot p}{m^2} \frac{1}{1 + \xi^2}, \quad u = \frac{n\omega' - \omega}{\kappa_n - \kappa + \omega'}, \]

\[
\kappa_n = n\omega - \frac{\beta p}{2} + \frac{1}{2} m(1 + \xi^2) e^{-\xi},
\]

\[
P_n = m |n\omega/m - \sinh \xi + \xi^2/2 e^{-\xi}|.
\]

These equations make the dependence \( u(\omega') \) explicit and relate again the differential cross section \( d\sigma/d\omega' \) with the differential rate \( d\Gamma_C/d\omega' \). Since the BW rate is exceedingly small at small \( \omega' \) (see figure 5), improvements of the Compton rate by catching the details of harmonic structures there are less severe for the pair number equation (15').

---

11 Strictly speaking, imposing a finite duration in the monochromatic laser beam model (2) turns it into a flat-top laser pulse model of class (1.1), exemplified in [59] and applied to the nonlinear Compton process.
4.2.2. Nonlinear Breit–Wheeler pair production

Equation (18) of the BW rate, however, is inappropriate at smaller values of $\xi$ [75] and needs improvement. Instead of using the series expansion in Bessel functions [58], a convenient formula is

$$ F_{\text{BW}} = \frac{1}{4\pi} \sum_{n=0}^{\infty} \int_{n/\chi}^{\infty} \frac{du}{u^{3/2}/\sqrt{1-u}} \exp \left\{ -2n(a - \tanh a) \right\} \frac{1 + 2\xi^2(2u - 1)\sinh^2 a}{a \tanh a} $$

with $n_{\text{min}} = 2(1 + \xi^2)/\chi$ and $u_n = n/n_{\text{min}}$. This representation emerges from the large-$n$ approximation of Bessel functions,

$$ J_n(a) \approx \exp \left\{ -n(a - \tanh a) \right\} / \sqrt{2n\pi} \tanh a $$

and $\tanh a = \sqrt{1 - \xi^2}/\eta^2$. In the large-$\xi$ limit, one may replace the summation over $n$ by an integration to arrive, via a double saddle point approximation, at the famous Ritus expression (18), which in turn is a complement of equation (6), see [58].

4.2.3. Numerical results

Numerical results are exhibited in figure 6 for $E_e = 45$ GeV, 17.5 GeV and 8 GeV. One observes a stark rise of $N^{+e^-}$ up to $\xi \approx 4$, which turns for larger values of $\xi$ into a modest rise. (The latter fact seems consistent with the stabilization/saturation phenomenon described in [67].) To quantify that rise one can employ the power law ansatz

$$ N^{+e^-}(\xi, E_e) = N_0^{+e^-}(E_e) \xi^{p(\xi, E_e)}. $$

Note that, by such a quantification of the $\xi$ dependence, one gets rid of the normalization $F_i$. For $E_e = 17.5$ GeV we find $p(\xi \approx 1) \approx 20$ dropping to $p(\xi \approx 20) \approx 2$, see figure 7. Larger values of $E_e$ reduce $p$, e.g. $p(\xi \approx 1)|_{E_e=45}$ GeV $\approx 10$ in agreement with [82], while $p(\xi \approx 1)|_{E_e=8}$ GeV $> 40$. At $\xi \to 20$, a universal value of $p \approx 2$ seems to emerge. The extreme nonlinear sensitivity of the pair number on the laser intensity parameter $\xi$ at $\xi < 10$, and in particular at $\xi \approx 1$, points to the request of a refined and adequately realistic modeling beyond schematic approaches.

4.3. Bandwidth effects in linear trident

The threshold for linear trident, $e^- + \gamma(1.55 \text{ eV}) \rightarrow e^- + e^- + e^+$, is at $E_e = 337$ GeV, i.e. the LUXE kinematics is in the deep sub-threshold regime, where severe multi-photon effects build up the nonlinearity. However, also bandwidth effects can promote pair production in the sub-threshold region [84, 85], even at $\xi \to 0$. The key is the cross section of linear trident $\sigma_{\text{ppT}}(\sqrt{s}, \Delta\phi)$, which depends on the invariant energy $\sqrt{s} = m\sqrt{1+2k \cdot p/\gamma}$ and the pulse duration $\Delta\phi$ for a given laser pulse. The quantity $\sigma_{\text{ppT}}(\sqrt{s}, \Delta\phi)$ is exhibited in figure 8 as a function of $\sqrt{s}$ for several values of $\Delta\phi$. For definiteness, we employ the laser pulse model of class (1.1) with parameterization $\tilde{A} = \tilde{f}_{\text{ppT}}(\phi) \Delta\phi \cos\phi$ and envelope function

$$ f_{\text{ppT}} = \cos^2 \left( \frac{\phi}{2\Delta\phi} \right) \cap (\phi, 2\Delta\phi), $$

i.e. the number of laser-field oscillations within the pulse is $N = \Delta\phi/\pi$. In contrast to the presentation above, we deploy results in this sub-section for linear polarization and the $\cos^2$ envelope.
Figure 7. The power $p$ (defined in equation (22)) as a function of $\xi$ for $E_e = 45$ GeV (magenta dash-dotted curve), 17.5 GeV (blue solid curve) and 8 GeV (red dashed curve). The results exhibited in figure 6 are fitted by equation (22).

Figure 8. Pulsed perturbative trident cross section $\sigma_{ppT}$ as a function of $\sqrt{\hat{s}}/m$ for several pulse lengths $\Delta \phi$. In the IPA limit, i.e. a monochromatic laser beam or $\Delta \phi \to \infty$, the threshold is at $\sqrt{\hat{s}} = 3m$. Above the threshold, the dots depict a few points (cf. table 1 in [86]) from perturbative trident without bandwidth effects. Bandwidth effects enable the pair production in the sub-threshold region $\sqrt{\hat{s}} < 3m$.

We employ the formalism in [79] and its numerical implementation, that is ‘pulsed perturbative QED’ in the spirit of Furry picture QED in a series expansion in powers of $\xi$. Applied to trident, the pulsed perturbative trident (ppT) arises from the diagrams $\square \varepsilon^+ - \square \varepsilon^+$ (double lines: Volkov wave functions, vertical lines: photon propagator) as leading-order term surviving $\xi \to 0$.

The scaled number of pairs is $N_{tot}/N_0^2 = 2\pi \Gamma_{tot}/\pi$, where the probability rate is given by

$$\Gamma_{tot} = \sigma_{ppT} \frac{\omega}{2 \pi \alpha} \int_{-\infty}^{\infty} d\phi \phi^2_{ppT}(\phi). \quad (23)$$

The chosen pulse implies $\int_{-\infty}^{\infty} d\phi \phi^2_{ppT}(\phi) = \frac{1}{2} \Delta \phi$ and $N_{tot}/N_0^2 = \sigma_{ppT} \frac{2m^2 \xi^2}{\pi \alpha} \Delta \phi$. Note the $\xi^2$ dependence from the ‘target density’ already entering in equation (23) (cf [84, 87] for analog relations). This is in contrast to figure 6, where genuine nonlinear effects are at work and mix with a stronger $\xi$ dependence for $C \otimes BW$. The $\xi^2$ dependence is characteristic for pair production by probe photons provided by an ‘external target’, such as in the bremsstrahlung-laser configuration of LUXE, cf [75]. For the long laser pulses used in E-144 [88–90], such bandwidth effects are less severe.

5. Summary

In summary, inspired by the renewed interest in the Ritus–Narozhny conjecture and the new perspectives offered by the experimental capabilities of LUXE and E-320, we recollect a few features of elementary QED processes within the essentially known formalism. In particular, we focus on the $\xi$ dependence. For nonlinear Compton scattering, we point out that, in the non-asymptotic region $\chi = O(1)$, $k \cdot p \lesssim m^2$, the spectrally resolved cross section $d\sigma/d\omega|_{\omega = \text{cone}}$ as a function of the laser intensity parameter $\xi$ displays a
pronounced bump-like shape for $u > u_{\text{GN}}$ (the ‘rise and fall’; analogously decreasing total pair creation rates at large $\xi$ are termed ‘stabilization’ in [67]). This behavior is in stark contrast with the monotonously rising integrated probability $\lim_{\xi \to \infty} P \propto \alpha \chi^{2/3} m^2 / k \cdot p$. That is, in different regions of the phase space, also different sensitivities of cross sections/rates/probabilities on the laser intensity impact can be observed. The soft (small-$u$) part, which determines the integrated cross section/probability, may behave completely different than the hard (large-$u$) contribution. Transferred to certain approximations used in simulation codes such a behavior implies that one should test differentially where the conditions for applicability are ensured.

The hard photons, once produced by Compton process in a laser pulse, act as seeds for secondary processes, most notably the BW process. A folding model of type $C \otimes BW$ on the probabilistic level points to a rapidly increasing rate of $e^+ e^-$ production in the region $\xi \lesssim 4$, when using parameters in reach of the planned LUXE set-up. The actual plans (see figure 2.10 in LUXE CDR [45]) uncover $\xi = 2$ (40 TW, 8 $\mu m$ laser), 6 (40 TW, 3 $\mu m$) and 16 (300 TW, 3 $\mu m$), and E-320 envisages $\xi = 10$. The folding model may be utilized as reference to identify the occurrence of the wanted one-step trident process in this energy-intensity regime. Furthermore, bandwidth effects in the trident process are isolated by considering the weak-field regime $\xi \to 0$.

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**Data availability statement**

The data that support the findings of this study are available upon reasonable request from the authors.

**Appendix A. Basics of nonlinear Compton process**

Following [92] we recall the basics of the underlying formalism of the nonlinear Compton process. Within the Furry picture the lowest-order, tree-level $12$ An analog situation is known in QCD [91]: tree-level diagrams are calculated primarily with constant $\alpha_{\text{QCD}}$. Renormalization improvement means then replacing $\alpha_{\text{QCD}}$ by $\alpha_{\text{QCD}}(s)$ which accounts at the same time for all vertices in the considered diagram by one global scale $s$. Inserting explicitly loop corrections leads finally to scale dependent couplings $\alpha_{\text{QCD}}(s_v)$ specific for each vertex $v$ separately. formulas of the p-polarization $\epsilon$ decay of a laser-dressed electron $e_l$ in the background field (1) $e_l(p) \to e_l(p') + \gamma(k', \epsilon')$ reads with suitable normalizations of the wave functions

$$S_{R} = -ie \int d^4 x J \cdot \epsilon' \exp\{i k' \cdot x\}, \quad (A1)$$

where the current $J_p(x) = \bar{\Psi}_\epsilon \gamma_\mu \Psi_p$ is built by the Volkov wave function $\Psi_p = E_p \mu_p \exp\{-ip \cdot x\} \exp\{-i f_p\}$ (spin indices are suppressed) and its adjoint $\Psi$ with Ritus matrix $E_p = 1 + \frac{k}{f_p} k \cdot A$ and phase function $f_p(\phi) = \frac{\sqrt{2}}{k_p} \int_0^\phi d\phi' [p \cdot A - \xi A \cdot A]$. We employ Feynman’s slash notation and denote scalar products by the dot between four-vectors; $\mu_p$ is the free Dirac bi-spinor. Exploiting the symmetry of the background field, $A(\phi = k \cdot x)$, equation (A1) can be manipulated (cf [92] for details) to arrive at

$$S_{R} = -ie(2\pi)^4 \frac{2}{k} \delta^{(3)}(p - p' - k') M(\ell), \quad (A2)$$

where $\ell \equiv (k' + p' - p) / k = k' \cdot p / k \cdot p'$ accomplishes the balance equation $p + \ell k - p' - k' = 0$. (See [62] for a formulation with

$$S_{R} = -ie(2\pi)^4 \int \frac{df}{2\pi} \delta^{(4)}(p + \ell k - p' - k') M(\ell).$$

$^{12}$ An analog situation is known in QCD [91]: tree-level diagrams are calculated primarily with constant $\alpha_{\text{QCD}}$. Renormalization
Light-cone coordinates are useful here, e.g. \( k_+ = k^0 - k^3, k_+ = k^0 + k^3, k_\perp = (k^1, k^2), \) and \( \overline{k} = (k_+, k_\perp). \) Imposing gauge invariance yields the matrix element

\[
\mathcal{M} = \sum_{i=1}^{3} j_{i}(0) S_{i}(0)
\]

with the pieces of the electron current

\[
\begin{align}
\tilde{j}_{(0)} &= \bar{u}_\gamma \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} u_p, \\
\tilde{j}_{(1,2)} &= \bar{u}_\gamma \left( d_p \gamma^{\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma} \gamma^{\lambda} u_p \right), \\
\tilde{j}_{(3)} &= 4 k \cdot \epsilon^{\mu} d_p \gamma^{\nu} \gamma^{\rho} u_p \gamma^{\lambda} u_{\nu},
\end{align}
\]

which combine to \( \tilde{j}_{(1,2)} = \tilde{j}_{(1,2)} + \alpha_p \tilde{j}_{(0)}, \) \( \tilde{j}_{(3)} = \tilde{j}_{(3)} + \beta \tilde{j}_{(0)}. \) The following abbreviations are used: \( d_p = \frac{m_\gamma}{k \cdot p}, \) \( \alpha_\perp = m \xi \left( \frac{k_\perp}{p} - \frac{\epsilon^{\nu} p^{\nu}}{k \cdot p} \right), \) \( \beta = \frac{m^2 \xi}{k \cdot p} \) as well.

The phase integrals \( S_{i}(0) \) are the remainders of the integration \( dx = d\phi \cdot d^3x_{\perp} / k_\perp \) in equation (A1):

\[
\begin{align}
S_{(1,2)} &= \int_{-\infty}^{\infty} df \{ f \exp \{ i(\ell \pm 1) \phi - i(\epsilon_p(\phi) - \epsilon_p'(\phi)) \} , \\
S_{(3)} &= \int_{-\infty}^{\infty} df^2 \{ f \exp \{ i\phi - i(\epsilon_p(\phi) - \epsilon_p'(\phi)) \} \} [1 + \cos 2\phi].
\end{align}
\]

For a few special non-unipolar (plane-wave) fields and their envelopes \( f(\phi) \), the phase integrals can be processed exactly by analytic means \( [93, 94] \), but in general a numerical evaluation is needed. The very special IPA case \( f(\phi) = 1 \) allows for a simple representation of \( f_p(\phi) \) with subsequent decomposition of \( S_{i}(0) \) into Bessel functions, yielding final expressions as in equations (2) and (20).

For identifying the weak-field limit, \( \xi \to 0 \), it is necessary to recognize in equation (A3) \( f_{(0)} \propto \xi^0 \), and \( \alpha_\perp, f_{(1,2)} \propto \xi^1 \), and \( \beta, f_{(3)} \propto \xi^2 \). Thus, \( \lim_{\xi \to 0} \mathcal{M} = \mathcal{M}_1 \xi + \mathcal{M}_2 \xi^2 + \cdots \). We emphasize the Fourier transform of the pulse envelope, \( \lim_{\xi \to 0} S_{(1,2)} = \int_{-\infty}^{\infty} df \{ f [\exp \{ i(\ell \pm 1) \phi \}, entering \mathcal{M}_1, where only \( S_{(1)} \) with \( \ell \geq 0 \) contributes to the wanted one-photon emission:

\[
\mathcal{M}_1(\ell) = f_{1}(\ell) \int_{-\infty}^{\infty} df \{ f \exp \{ i(\ell - 1) \phi \},
\]

\[
f_{1}(\ell) = \frac{m_\gamma}{2k_\ell} \epsilon_\perp \epsilon_{++} u_p \left[ \frac{\gamma_\mu \gamma_\nu - 2\gamma^\mu \gamma^\nu}{2k_\ell \cdot p} + \frac{\gamma^\mu \gamma^\nu - 2\gamma^\mu \gamma^\nu}{2k_\ell \cdot p} \right] u_p
\]

with \( k_\ell \equiv k \ell. \) For pulses with broad support, \( f(\phi) \to 1 \), within the interval \( \Delta \phi \gg 1 \), one arrives at the standard Compton KN expression in leading order by using \( \xi \to 2\epsilon / m \). The appearance of \( \delta(\ell - 1) \) combined with the definition of \( \ell \) below equation (A2) leads to the famous Compton formula via \( \delta (\omega' - \omega / m(1 - \cos \theta) / \theta^0) \) in the electron’s rest frame by the subsequent phase space integration(s).

The discussion of the large-\( \xi \) limit needs some care in general, cf \( [26, 28] \). Useful limits are obtained for the IPA case \( [58] \) under the side condition \( \xi^2(1 - \xi^2) = \text{const} \) (cf equations (19) and (20)), e.g. \( |\mathcal{M}|^2 \propto \xi \) if \( \xi (\xi / u)^{1/2} \) and

\[
|\mathcal{M}|^2 \propto u^{-1} \exp \left\{ -\frac{2u}{3 \xi} \frac{m^2}{k \cdot p} \right\} \rightarrow \sqrt{\xi}
\]

(cf equations (14) and (13)). At \( \xi \gg 1 \), the \( \xi \) and \( u \) dependencies of resulting probabilities for circular polarization and ccf backgrounds coincide.

These limits are at the heart of the ‘rise and fall’.

The differential emission probability per electron and per laser pulse follows from (A2) by partial integration over the out-phase space,

\[
\frac{dP}{d\omega'} = \frac{e^2 \omega'}{64 \pi^2 k \cdot p} |\mathcal{M}|^2,
\]

\( [13] \) The same reasoning applies in IVC for one-pair emission, i.e. the finite-width Fourier transform of \( f(\phi) \) at \( \Delta \phi < \infty \) enables the sub-threshold pair production \( e_1(p) \rightarrow e_1(p') + e_1(p'') + \bar{e}_1(p'''). \)
and may be transformed to other coordinates, e.g. $u$ or $t$ etc. Having in mind the IPA limit, one should turn to the dimensionless differential rate \[62\]

\[
\frac{d\Gamma_c}{d\omega \, dp^2} = \frac{e^2 \omega}{32\pi^3 q_0 k \cdot p} |M|^2.
\]

(A9)

Spin averaging of the in-electron, and spin summation of the out-electron and summation over the out-photon polarizations leads to $|M|^2$, unless one is interested in polarization effects as in [19, 20, 95]. The above expressions (A3)–(A7) can be further processed for special field envelopes, or $|M|^2$ is numerically accessible, via equation (A3), as mod-squared sum of complex number products provided by equations (A4)–(A7) which need afterward explicit (numerical) spin and polarization summation/averaging to arrive at $|M|^2$.

The cross section is obtained by normalization on the integrated laser photon flux: $d\sigma = d\Gamma_c \frac{d\omega}{|p|}$ with $n_L = \frac{\pi^2 \xi^2 \omega N_L}$, where $N_L = 1$ (IPA, quasi-momentum $q_0$) or $N_L = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega f(\phi)$ (FPA, $q_0 = p_0$). The circularly polarized laser background $1$ is supposed in these relations.

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