A FEW RESULTS ABOUT VARIATIONS OF LOCAL COHOMOLOGY

M. AZEEM KHADAM AND PETER SCHENZEL

Abstract. Let \( q \) denote an ideal of a local ring \((A, m)\). For a system of elements \( \underline{a} = a_1, \ldots, a_t \) such that \( a_i \in q^{e_i}, i = 1, \ldots, t \), and \( n \in \mathbb{Z} \) we investigate a subcomplex resp. a factor complex of the \( \check{C}ech \) complex \( \hat{C}_{\underline{a}} \otimes_A M \) for a finitely generated \( A \)-module \( M \). We start with the inspection of these cohomology modules that approximate in a certain sense the local cohomology modules \( H^i_\underline{a}(M) \) for all \( i \in \mathbb{N} \). In the case of an \( m \)-primary ideal \( \underline{a} A \) we prove the Artinianness of these cohomology modules and characterize the last non-vanishing among them.

1. Introduction

Let \((A, m, k)\) denote a local ring. Let \( q \subset A \) be an ideal and \( \underline{a} = a_1, \ldots, a_t \) denote a system of elements of \( A \) such that \( a_i \in q^{e_i}, i = 1, \ldots, t \). For a finitely generated \( A \)-module \( M \) and an integer \( n \in \mathbb{Z} \) we define a complex \( K_{\bullet}(\underline{a}, q; M; n) \) as the subcomplex of the Koszul complex \( K_{\bullet}(\underline{a}; M) \), where \( K_i(\underline{a}, q; M; n) = \oplus_{1 \leq j_1 < \cdots < j_t \leq n} q^{a_j - e_j} M \) is included in \( K_i(\underline{a}; M) \) and the boundary map is defined as the restriction of the maps in the Koszul complex. In other words, \( K_{\bullet}(\underline{a}, q; M; n) \) is the \( n \)-th graded component of the Koszul complex \( K_{\bullet}(aT^n; R\underline{a}(q)) \). Here \( R_M(q) \) denotes the Rees module of \( M \) with respect to \( q \) and \( aT^n = a_1T^{c_1}, \ldots, a_tT^{c_t} \), where \( a_i T^{c_i}, i = 1, \ldots, t \), is the element \( a_i T^{c_i} \in R_A(q) \), the associated element of degree \( c_i \) in the Rees ring \( R_A(q) = \oplus_{n \geq 0} q^nT^n \) (see Section 3 for more details). We denote this complex by \( K_{\bullet}(\underline{a}, q; M; n) \). The cokernel of this embedding provides a complex \( L_{\bullet}(\underline{a}, q; M; n) \). In the case that \( \underline{a} A, q \) are \( m \)-primary ideals and \( t = \dim M \) the Euler characteristic of \( \mathcal{L}_{\bullet}(\underline{a}, q; M; n) \) gives for all \( n \gg 0 \) the value \( c_1 \cdots c_t e_0(q; A) \), where \( e_0(q; A) \) denotes the Hilbert-Samuel multiplicity of \( q \) (see Section 5 for the details).

A similar construction based on the co-Koszul complex provides complexes \( K^\bullet(\underline{a}, q; M; n) \) and \( L^\bullet(\underline{a}, q; M; n) \) (see Section 4). It follows that \( \{K^\bullet(\underline{a}^k, q; M; n)\}_{k \geq 1} \) with \( \underline{a}^k = \underline{a}_1 \cdots \underline{a}_t \) forms a direct system of complexes and its direct limit \( \hat{C}(\underline{a}, q; M; n) \) is a subcomplex of the \( \check{C}ech \) complex \( \hat{C}_{\underline{a}} \otimes_A M \) with the factor complex \( \hat{L}^\bullet(\underline{a}, q; M; n) \) and their cohomology modules \( \check{H}^i(\underline{a}, q; M; n) \) and \( \check{L}^i(\underline{a}, q; M; n) \) resp. Therefore there is a long exact cohomology sequence

\[ \ldots \rightarrow \check{H}^i(\underline{a}, q; M; n) \rightarrow H^i_\underline{a}(M) \rightarrow \check{L}^i(\underline{a}, q; M; n) \rightarrow \ldots \]

Since \( \check{H}^i(\underline{a}, q; M; n) = 0 \) for \( i > t \), there is an epimorphism \( H^i_\underline{a}(M) \rightarrow \check{L}^i(\underline{a}, q; M; n) \). So the non-vanishing of \( \check{L}^i(\underline{a}, q, M; n) \) for some \( n \in \mathbb{N} \) is an obstruction for the vanishing of \( H^i_\underline{a}(M) \). The main results of the present manuscript is a contribution to the study of \( \check{H}^i(\underline{a}, q; M; n) \) and \( \check{L}^i(\underline{a}, q; M; n) \). In particular we prove the following results:

Theorem 1.1. With the previous notation suppose that \( \underline{a} A \) is an \( m \)-primary ideal.

(a) \( \check{H}^i(\underline{a}, q; M; n) \) and \( \check{L}^i(\underline{a}, q; M; n) \) are Artinian \( A \)-modules for all \( i, n \in \mathbb{N} \).

(b) \( \check{H}^i(\underline{a}, q; M; n) = 0 \) for all \( n \gg 0 \).

(c) \( \check{L}^i(\underline{a}, q; M; n) \neq 0 \) for all \( n \gg 0 \).

2010 Mathematics Subject Classification. Primary: 13D45; Secondary: 13D40.
Key words and phrases. Koszul complex, \( \check{C}ech \) complex, local cohomology, multiplicity.

The first named author is grateful to DAAD for the support this research under grant number 91524811.
For the proof of (a) we refer to 7.4. The claim of (b) is a particular case of 8.3 and (c) is shown in 9.5. Besides of these results there are statements about the structure of the generalized Koszul homology and co-homology modules and their Euler characteristics. For induction arguments we provide some exact sequences (see Section 7). Of a particular interest is a generalization of the notion of superficial sequences and the condition (∗) used in 8.3. A further investigation about the vanishing and the rigidity of the generalized Koszul and co-Koszul complexes is in preparation.

In the special case of \( q = aA \) for a system of elements \( a = a_1, \ldots, a_t \) of \( A \) we have the following:

**Corollary 1.2.** Let \( a = a_1, \ldots, a_t \) denote a system of elements of \( A \). For a finitely generated \( A \)-module \( M \) there are isomorphisms

\[
H^i_{\underline{a}}(M) \cong \hat{L}^i(a, q; M; n)
\]

for all \( n \gg 0 \) and all \( i \geq 0 \).

This follows by view of 6.5. That is, in a certain sense the cohomology \( \hat{L}^i(a, q; M; n) \) provides some additional structure on the usual local cohomology modules.

As a source for basic notions in Commutative Algebra we refer to [1] or [8]. For results on homological algebra we refer to [9] and [13]. The local cohomology is developed in [6] (see also [10]). For a system of elements \( \underline{a} \) we write \( H^i_{\underline{a}}(\cdot) \), \( i \in \mathbb{N} \), for the local cohomology modules with respect to the ideal generated by \( \underline{a} \) (see [6]).

### 2. Preliminaries

First let us fix the notations we will use in the following. For the basics on \( \mathbb{N} \)-graded structures we refer e.g. to [5].

**Notation 2.1.**

(A) We denote by \( A \) a commutative Noetherian ring with \( 0 \neq 1 \). For an ideal \( q \subset A \). An \( A \)-module is denoted by \( M \). Mostly we consider \( M \) as finitely generated.

(B) We consider the Rees and form rings of \( A \) with respect to \( q \) by

\[
R_A(q) = \oplus_{n \geq 0} q^n T^n \subset A[T] \quad \text{and} \quad G_A(q) = \oplus_{n \geq 0} q^n / q^{n+1}.
\]

Here \( T \) denotes an indeterminate over \( A \). Both rings are naturally \( \mathbb{N} \)-graded. For an \( A \)-module \( M \) we define the Rees and form modules in the corresponding way by

\[
R_M(q) = \oplus_{n \geq 0} q^n M T^n \subset M[T] \quad \text{and} \quad G_M(q) = \oplus_{n \geq 0} q^n M / q^{n+1} M.
\]

Note that \( R_M(q) \) is a graded \( R_A(q) \)-module and \( G_M(q) \) is a graded \( G_A(q) \)-module. Note that \( R_A(q) \) and \( G_A(q) \) are both Noetherian rings. In case \( M \) is a finitely generated \( A \)-module then \( R_M(q) \) resp. \( G_M(q) \) is finitely generated over \( R_A(q) \) resp. \( G_A(q) \).

(C) There are the following two short exact sequences of graded modules

\[
0 \rightarrow R_M(q)^+[1] \rightarrow R_M(q) \rightarrow G_M(q) \rightarrow 0 \quad \text{and} \quad 0 \rightarrow R_M(q)_+ \rightarrow R_M(q) \rightarrow M \rightarrow 0,
\]

where \( R_M(q)_+ = \oplus_{n \geq 0} q^n M T^n \).

(D) Let \( m \in M \) and \( m \in q^c M \setminus q^{c+1} M \). Then we define \( m^* := m + q^{c+1} M \in [G_M(q)]_c \). If \( m \in \cap_{n \geq 1} q^n M \), then we write \( m^* = 0 \). \( m^* \) is called the initial element of \( m \) in \( G_M(q) \) and \( c \) is called the initial degree of \( m \). Here \([X]_n, n \in \mathbb{Z}\), denotes the \( n \)-th graded component of an \( \mathbb{N} \)-graded module \( X \).

For these and related results we refer to [5] and [12]. Another feature for the investigations will be the use of Koszul complexes.
Remark 2.2. (Koszul complex.) (A) Let \( a = a_1, \ldots, a_t \) denote a system of elements of the ring \( A \). The Koszul complex \( K_*(a; A) \) is defined as follows: Let \( F \) denote a free \( A \)-module with basis \( e_1, \ldots, e_t \). Then \( K_i(a; A) = \bigwedge^i F \) for \( i = 1, \ldots, t \). A basis of \( K_i(a; A) \) is given by the wedge products \( e_{j_1} \wedge \ldots \wedge e_{j_i} \), for \( 1 \leq j_1 < \ldots < j_i \leq t \). The boundary homomorphism \( d_i : K_i(a; A) \to K_{i-1}(a; A) \) is defined by
\[
d_{j_1 \ldots j_i} : e_{j_1} \wedge \ldots \wedge e_{j_i} \mapsto \sum_{k=1}^i (-1)^{k+i} a_{j_k} e_{j_1} \wedge \ldots \wedge \hat{e}_{j_k} \wedge \ldots \wedge e_{j_i}
\]
on the free generators \( e_{j_1} \wedge \ldots \wedge e_{j_i} \).
(B) Another way of the construction of \( K_*(a; A) \) is inductively by the mapping cone. To this end let \( X \) denote a complex of \( A \)-modules. Let \( a \in A \) denote an element of \( A \). The multiplication by \( a \) on each \( A \)-module \( X_i, i \in \mathbb{Z} \), induces a morphism of complexes \( m_a : X \to X \). We define \( K_*(a; X) \) as the mapping cone \( Mc(m_a) \). Then we define inductively
\[
K_*(a_1, \ldots, a_t; A) = K_(a_t, K_*(a_1, \ldots, a_{t-1}; A)).
\]
It is easily seen that
\[
K_*(a; A) \cong K_*(a_1; A) \otimes_A \cdots \otimes_A K_*(a_t; A).
\]
Therefore it follows that \( K_*(a; A) \cong K_*(a; A) \), where \( a_\sigma = a_{\sigma(1)}, \ldots, a_{\sigma(t)} \) with a permutation \( \sigma \) on \( t \) letters. For an \( A \)-complex \( X \) we define \( K_*(a; X) = K_*(a; A) \otimes_A X \). We write \( H_i(a; X), i \in \mathbb{Z} \), for the \( i \)-th homology of \( K_*(a; X) \). A short exact sequence of \( A \)-complexes \( 0 \to X' \to X \to X'' \to 0 \) induces a long exact homology sequence for the Koszul homology
\[
\cdots \to H_i(a; X') \to H_i(a; X) \to H_i(a; X'') \to H_{i-1}(a; X') \to \cdots.
\]
Let \( a \) as above a system of \( t \) elements in \( A \) and \( b \in A \). Then the mapping cone construction provides a long exact homology sequence
\[
\cdots \to H_i(a; X) \to H_i(a; b; X) \to H_{i-1}(a; X) \to H_{i-1}(a; X) \to \cdots,
\]
where the homomorphism \( H_i(a; X) \to H_{i-1}(a; X) \) is multiplication by \(( -1)^ib \).

(C) Let \( (\cdot)^* = \text{Hom}_A(\cdot, A) \) the duality functor. Then we consider the co-Koszul complex \( K^*(a; A) \) defined by \( \text{Hom}_A(K_*(a; A), A) = (K_*(a; A))^* \). Therefore the homomorphism
\[
K^i(a; A) \to K^{i+1}(a; A)
\]
is induced by \( \text{Hom}_A(d_{j_1 \ldots j_i}, A) \) on the dual basis \( \bigwedge^i F^* \). It follows that \( K_*(a; A) \) and \( K^*(a; A) \) are isomorphic, that is, the Koszul complex is self dual. Let \( X \) denote an \( A \)-complex. Then
\[
K^*(a; X) \cong \text{Hom}_A(K_*(a; A), X) \text{ and } K^*(a; X) \cong K^*(a; A) \otimes_A X.
\]
We denote by \( H^i(a; X), i \in \mathbb{Z} \), the \( i \)-th cohomology of \( K^*(a; X) \). We have isomorphisms \( H_i(a; X) \cong H^{1-i}(a; X) \) for all \( i \in \mathbb{Z} \). Moreover \( aH_i(a; X) = 0 \) for all \( i \in \mathbb{Z} \).

For the proof of the last statement we recall the following well-known argument.

Lemma 2.3. Let \( X \) denote a complex of \( A \)-modules. Let \( a \in A \) denote an element. Then \( aH_i(a; X) = 0 \) for all \( i \in \mathbb{Z} \).

Proof. By the construction of \( K_*(a; X) \) there is a short exact sequence of complexes
\[
0 \to X \to K_*(a; X) \to X[-1] \to 0,
\]
where \( X[-1] \) is the complex \( X \) shifted the degrees by \(-1 \). The differential \( \partial \) on \( K_i(a; X) = X_{i-1} \oplus X_i \) is given by \( \partial(x, y) = (d_{i-1}(x), d_i(y) + (-1)^i y) \). Suppose that \( \partial(x, y) = 0 \). That is \( d_{i-1}(x) = 0 \) and \( d_i(y) = (-1)^iax \) and therefore \( a(x, y) = \partial_{i+1}((-1)^i y, 0) \in \text{Im} \partial_{i+1} \). This proves \( aH_i(a; X) = 0 \).
In fact we shall use a slight modification of the above argument in further arguments.

Remark 2.4. (Čech complex.) (A) For a system of elements \( \underline{a} = a_1, \ldots, a_t \) and an integer \( n \geq 1 \) we denote by \( \underline{a}^n \) the system of elements \( a_1^n, \ldots, a_t^n \). Then \( \{K_\bullet(\underline{a}^n; A)\}_{n \geq 1} \) forms an inverse system of complexes and \( \{K_\bullet(\underline{a}^n; A)\}_{n \geq 1} \) forms a direct system of complexes. In both cases the maps

\[
K_\bullet(\underline{a}^m; A) \to K_\bullet(\underline{a}^n; A) \quad \text{resp.} \quad K_\bullet(\underline{a}^n; A) \to K_\bullet(\underline{a}^m; A)
\]

for \( m \geq n \) are the naturally induced homomorphisms. Here we focus on the direct system \( \{K_\bullet(\underline{a}^n; A)\}_{n \geq 1} \). We start with a sequence consisting of a single element \( a \in A \). So there is the following commutative diagram

\[
\begin{array}{ccc}
K_\bullet(\underline{a}^n; A) : & 0 \to A & \xrightarrow{a^n} A \to 0 \\
& \downarrow & \downarrow \quad \quad \quad \downarrow a^{m-n} \\
K_\bullet(\underline{a}^m; A) : & 0 \to A & \xrightarrow{a^m} A \to 0
\end{array}
\]

for \( m \geq n \). Its direct limit gives a complex \( \check{C}_\underline{a} : 0 \to A \to A_0 \to 0 \), where \( A_0 \) denotes the localization of \( A \) with respect to the multiplicatively closed set \( \{a^n | n \geq 0\} \) with the homomorphism \( A \to A_0, r \mapsto r/1 \). To this end recall the fact that \( \lim_{\longrightarrow} \{A, a\} \cong A_0 \), where the direct system is given by \( A \xrightarrow{a} A \).

(B) In general we define the Čech complex of a system of elements \( \underline{a} = a_1, \ldots, a_t \) by \( \check{C}_\underline{a} = \lim K_\bullet(\underline{a}^n; A) \). By elementary properties of direct limit and tensor products it follows that \( \check{C}_\underline{a} \cong \hat{C}_{a_1} \otimes_A \cdots \otimes_A \hat{C}_{a_t} \). As a consequence there is the description

\[
\check{C}_\underline{a} : 0 \to \hat{C}_a^0 \to \cdots \to \hat{C}_a^t \to \cdots \to \hat{C}_a^t \to 0 \quad \text{with} \quad \hat{C}_a^i = \oplus_{1 \leq j_1 < \ldots < j_i \leq t} A_{a_{j_1} \ldots a_{j_i}}
\]

where the differential \( d^i : \hat{C}_a^i \to \hat{C}_a^{i+1} \) is given at the component \( A_{a_{j_1} \ldots a_{j_i}} \to A_{a_{j_1} \ldots a_{j_i} a_{j_{i+1}}} \) by \((-1)^{k+1}\) times the natural map \( A_{a_{j_1} \ldots a_{j_i} a_{j_{i+1}}} \to A_{a_{j_1} \ldots a_{j_i} a_{j_{i+1}}} \) if \( \{j_1, \ldots, j_i\} \neq \{j_1, \ldots, j_k, \ldots, j_{i+1}\} \) and zero otherwise. For an \( A \)-complex \( X \) we write \( \check{C}_\underline{a}(X) = \check{C}_\underline{a} \otimes_A X \).

(C) The importance of the Čech complex is its relation to the local cohomology. Namely let \( q = (a_1, \ldots, a_t)A \) denote the ideal generated by the sequence \( \underline{a} \). The local cohomology \( H_q^i(M) \) of an \( A \)-module \( M \) is defined as the \( i \)-th right derived functor \( H_q^i(M) \) of the section functor \( \Gamma_q(M) = \{m \in M | \text{Supp}_AM \subseteq V(q)\} \). Then there are natural isomorphisms

\[
H_q^i(M) \cong H^i(\check{C}_\underline{a} \otimes_A M) \cong \lim_{\longrightarrow} H^i(\underline{a}^n; M)
\]

for an \( A \)-module \( M \) and any \( i \in \mathbb{Z} \). As a consequence it follows that \( H^i(\check{C}_\underline{a} \otimes_A M) \) depends only on the radical \( \text{Rad} \underline{a} R \).

For the details of the previous results we refer to [4] and [10].

3. THE CONSTRUCTION OF COMPLEXES

First we fix notations for this section. As above let \( A \) denote a commutative Noetherian ring and \( q \subseteq A \). Let \( \underline{a} = a_1, \ldots, a_t \) denote a system of elements of \( A \). Suppose that \( a_i \in q^{c_i} \) for some integers \( c_i \in \mathbb{N} \) for \( i = 1, \ldots, t \). Let \( M \) denote a finitely generated \( A \)-module.

Notation 3.1. Let \( n \) denote an integer. We define a complex \( K_r(\underline{a}, q, M; n) \) in the following way:

(a) For \( 0 \leq i \leq t \) put \( K_i(\underline{a}, q, M; n) = \oplus_{1 \leq j_1 < \ldots < j_i \leq q} q^{n-c_{j_1} - \ldots - c_{j_i}} M \) and \( K_i(\underline{a}, q, M; n) = 0 \) for \( i > t \) or \( i < 0 \).

(b) The boundary map \( K_i(\underline{a}, q, M; n) \to K_{i-1}(\underline{a}, q, M; n) \) is defined by maps on each of the direct summands \( q^{n-c_{j_1} - \ldots - c_{j_i}} M \). On \( q^{n-c_{j_1} - \ldots - c_{j_i}} M \) it is the map given by \( d_{j_1, \ldots, j_i} \otimes 1_M \) restricted to \( q^{n-c_{j_1} - \ldots - c_{j_i}} M \), where \( d_{j_1, \ldots, j_i} \) denotes the homomorphism as defined in 2.2.
It is clear that the image of the map is contained in \( \oplus_{1 \leq j_1 < \ldots < j_{h-1} \leq n} a^{e_{j_1} - \cdots - e_{j_{h-1}} - 1} M \). Clearly
it is a boundary homomorphism. By the construction it follows that \( K_\bullet(\underline{a}, q, M; n) \) is a sub
complex of the Koszul complex \( K_\bullet(\underline{a}, M) \) for each \( n \in \mathbb{N} \).

Another way for the construction is the following.

**Remark 3.2.** Let \( R_A(q) \) and \( R_M(q) \) denote the Rees ring and the Rees module. For \( a_i, i = 1, \ldots, t \), we consider \( a_iT^{c_i} \in [R_A(q)]_{c_i} \). Then we have the system \( a_iT^{c_i} = a_1T^{c_1}, \ldots, a_tT^{c_t} \) of elements of \( R_A(q) \). Note that \( \deg a_iT^{c_i} = c_i, i = 1, \ldots, t \). Then we may consider the Koszul complex \( K_\bullet(aT^{c}; R_M(q)) \). It is easily seen that the degree \( n \)-component \( [K_\bullet(aT^{c}; R_M(q))]_n \) of \( K_\bullet(aT^{c}; R_M(q)) \) is the complex \( K_\bullet(\underline{a}, q, M; n) \) as introduced in 3.1. We write \( H_i(\underline{a}, q, M; n) \) for the \( i \)-th homology of \( K_\bullet(\underline{a}, q, M; n) \) for \( i \in \mathbb{Z} \).

We come now to the definition of one of the main subjects of the paper.

**Definition 3.3.** With the previous notation we define \( L_\bullet(\underline{a}, q, M; n) \) the quotient of the embedding \( K_\bullet(\underline{a}, q, M; n) \to K_\bullet(\underline{a}, M) \). That is there is a short exact sequence of complexes
\[
0 \to K_\bullet(\underline{a}, q, M; n) \to K_\bullet(\underline{a}, M) \to L_\bullet(\underline{a}, q, M; n) \to 0.
\]
Note that \( L_\bullet(\underline{a}, q, M; n) \cong \oplus_{1 \leq j_1 < \ldots < j_{h-1} \leq n} a_n^{e_{j_1} - \cdots - e_{j_{h-1}} - 1} M \). The boundary maps are those induced by the Koszul complex. We write \( L_i(\underline{a}, q, M; n) \) for the \( i \)-th homology of \( L_\bullet(\underline{a}, q, M; n) \) and any \( i \in \mathbb{Z} \).

For a construction by mapping cones we need the following technical result. For a morphism \( f : X \to Y \) we write \( C(f) \) for the mapping cone of \( f \).

**Lemma 3.4.** With the previous notation let \( b \in q^t \) an element. The multiplication map by \( b \) induces the following morphisms
\[
m_b(K) : K_\bullet(\underline{a}, q, M; n - d) \to K_\bullet(\underline{a}, q, M; n) \text{ and } m_b(L) : L_\bullet(\underline{a}, q, M; n - d) \to L_\bullet(\underline{a}, q, M; n)
\]
of complexes. They induce isomorphism of complexes
\[
C(m_b(K)) \cong K_\bullet(\underline{a}, b, q, M; n) \text{ and } C(m_b(L)) \cong L_\bullet(\underline{a}, b, q, M; n).
\]

**Proof.** The proof follows easily by the structure of the complexes and the mapping cone construction. \(\square\)

We begin with a few properties of the previous complexes.

**Theorem 3.5.** Let \( \underline{a} = a_1, \ldots, a_t \) denote a system of elements of \( A \), \( q \subset A \) an ideal and \( M \) a finitely generated \( A \)-module. Let \( n \in \mathbb{N} \) denote an integer.

(a) \( H_i(\underline{a}, q, M; n) \cong H_i(\underline{a}_\sigma, q, M; n) \) and \( L_i(\underline{a}, q, M; n) \cong L_i(\underline{a}_\sigma, q, M; n) \) for all \( i \in \mathbb{Z} \) and any \( \sigma \), a permutation on \( t \) letters.

(b) \( aH_i(\underline{a}, q, M; n) = 0 \) and \( aL_i(\underline{a}, q, M; n) = 0 \) for all \( i \in \mathbb{Z} \).

(c) \( H_i(\underline{a}, q, M; n) \) and \( L_i(\underline{a}, q, M; n) \) are finitely generated \( A/\underline{a}A \)-modules for all \( i \in \mathbb{Z} \).

**Proof.** The statement in (a) follows by virtue of the short exact sequence of complexes in 3.3 and the long exact homology sequence. Note that the homology of Koszul complexes is isomorphic under permutations.

The claim in (c) is a consequence of (b) since the homology modules \( H_i(\underline{a}, q, M; n) \) and \( L_i(\underline{a}, q, M; n) \) are finitely generated \( A \)-modules.

For the proof of (b) we follow the mapping cone construction of 3.4 with the arguments of 2.3. To this end let \( K_n = K_\bullet(\underline{a}, q, M; n) \) and \( C = C(m_b(K)) = K_\bullet(\underline{a}, b, q, M; n) \). Then there is a short exact sequence of complexes
\[
0 \to K_n \to C \to K_{n-d}[-1] \to 0.
\]
The differential \( \partial_i \) on \((x, y) \in C_i = (K_{n-d})_{i-1} \oplus (K_n)_{i}\) is given by

\[
\partial_i(x, y) = (d_{i-1}(x), d_i(y) + (-1)^{i-1}bx).
\]

Suppose that \( \partial_i(x, y) = 0 \), i.e., \( d_{i-1}(x) = 0 \) and \( d_i(y) = (-1)^i bx \). Then

\[
(y, 0) \in (K_{n-d})_i \oplus (K_n)_{i+1} = C_{i+1}
\]

and therefore \( \partial_{i+1}((y, 0)) = b(x, y) \). That is \( bh_i(C) = 0 \) for all \( i \in \mathbb{Z} \).

In order to show the claim in (b) we use the previous argument. So let us consider \( K_\bullet(a, q, M; n) = C(m_n(K_\bullet(a', q, M; n))) \), where \( a' = a_1, \ldots, a_{t-1} \). The previous argument shows \( H_i(K_\bullet(a', q, M; n)) = 0 \) for all \( i \in \mathbb{Z} \). By view of (a) this finishes the proof in the case of \( H_i(a, q, M; n) \).

For the proof of \( aL_i(a, q, M; n) = 0 \) we follow the same arguments. Instead of the injection \((K_n)_i \subseteq (K_{n-d})_i\) we use the surjection \((\mathcal{L}_n)_i \rightarrow (\mathcal{L}_{n-d})_i\), where \( \mathcal{L}_n = \mathcal{L}_\bullet(a, q, M; n) \). We skip the details here. \( \square \)

\section{4. The construction of co-complexes}

With the notations of the previous section we shall define the co-complex version of the complexes above. This is based on the Koszul co-complex.

\begin{notation}
For an integer \( n \in \mathbb{N} \) we define a complex \( K_\bullet(a, q, M; n) \) similar to the construction in 3.1 by the use of the Koszul co-complex. By view of Remark 3.2 we define it also as the \( n \)-th graded component of \( K_\bullet(aT_c; R_M(q)) \). We may identify \( K^i(aT_c; R_M(q)) \) with \( \oplus_{1 \leq j_1 < \ldots < j_i \leq q^0 + c_1 + \ldots + c_i} M \). It follows that \( K^i(aT_c; R_M(q)) \) is a subcomplex of \( K_\bullet(a, M) \). We define \( \mathcal{L}_\bullet(a, q, M; n) \) as the quotient of this embedding. That is, there is a short exact sequence of complexes

\[
0 \rightarrow K_\bullet(a, q, M; n) \rightarrow K_\bullet(a, M) \rightarrow \mathcal{L}_\bullet(a, q, M; n) \rightarrow 0.
\]

We may identify \( \mathcal{L}_i(a, q, M; n) \) with \( \oplus_{0 \leq i \leq n} M/q^{n+c_1 + \ldots + c_i} M \). We denote by \( H_i(a, q, M; n) \) resp. \( L_i(a, q, M; n) \) the \( i \)-th cohomology of \( K_\bullet(a, q, M; n) \) resp. \( \mathcal{L}_\bullet(a, q, M; n) \) for any \( i \in \mathbb{Z} \).

For an iteration we need the following technical result. For a morphism \( f : X \rightarrow Y \) of co-complexes we write \( D(f) \) for the co-mapping cone of \( f \).

\begin{lemma}
With the previous notation let \( b \in q \) an element. The multiplication map by \( b \) induces the following morphisms

\[
m_b(K) : K_\bullet(a, q, M; n) \rightarrow K_\bullet(a, q, M; n + d) \quad \text{and} \quad m_b(\mathcal{L}) : \mathcal{L}_\bullet(a, q, M; n) \rightarrow \mathcal{L}_\bullet(a, q, M; n + d)
\]

of complexes. They induce isomorphism of complexes

\[
D(m_b(K)) \cong K_\bullet(a, b, q, M; n) \quad \text{and} \quad D(m_b(\mathcal{L})) \cong \mathcal{L}_\bullet(a, b, q, M; n).
\]

\end{lemma}

\begin{proof}
This is easy by reading of the definitions. \( \square \)

\end{proof}

\begin{theorem}
Let \( a = a_1, \ldots, a_t \) denote a system of elements of \( A \), \( q \subset A \) an ideal and \( M \) a finitely generated \( A \)-module. Let \( n \in \mathbb{N} \) denote an integer.

(a) \( H_i(a, q, M; n) \cong H^i(a, q, M; n) \) and \( L_i(a, q, M; n) \cong L^i(a, q, M; n) \) for all \( i \in \mathbb{Z} \) and any \( \sigma \), a permutation on \( t \) letters.
(b) \( aH_i(a, q, M; n) = 0 \) and \( aL_i(a, q, M; n) = 0 \) for all \( i \in \mathbb{Z} \).
(c) \( H_i(a, q, M; n) \) and \( L_i(a, q, M; n) \) are finitely generated \( A/aA \)-modules for all \( i \in \mathbb{Z} \).

\end{theorem}

\begin{proof}
The arguments in the proof are a repetition of those of the proof of Theorem 3.5 with cohomology instead of homology. We skip the details here. \( \square \)

\end{proof}
5. Euler characteristics

Let $A$ denote a commutative ring. Let $X$ denote a complex of $A$-modules.

**Definition 5.1.** Let $X : 0 \to X_n \to \ldots \to X_1 \to X_0 \to 0$ denote a bounded complex of $A$-modules. Suppose that $H_i(X), i = 0, 1, \ldots, n$, is an $A$-module of finite length. Then

$$
\chi_A(X) = \sum_{i=0}^{n} (-1)^i \ell_A(H_i(X))
$$

is called the Euler characteristic of $X$.

We collect a few well known facts about Euler characteristics.

**Lemma 5.2.** Let $A$ denote a Noetherian commutative ring.

(a) Let $0 \to X' \to X \to X'' \to 0$ denote a short exact sequence of complexes such that all the homology modules are of finite length. Then $\chi_A(X) = \chi_A(X') + \chi_A(X'')$.

(b) Suppose $X : 0 \to X_n \to \ldots \to X_1 \to X_0 \to 0$ is a bounded complex such that $X_i, i = 0, \ldots, n$, is of finite length. Then $\chi_A(X) = \sum_{i=0}^{n} (-1)^i \ell_A(X_i)$

**Proof.** The statement in (a) follows by the long exact cohomology sequence derived by $0 \to X' \to X \to X'' \to 0$. The second statement might be proved by induction on $n$, the length of the complex $X$. □

As an application we get the following result about multiplicities, originally shown by [11] and [2].

**Proposition 5.3.** Let $(A, \mathfrak{m})$ be a local ring and $a_1, \ldots, a_d \in \mathfrak{m}$ a system of parameters for $M$, a finitely generated $A$-module. Then

$$
\chi_A(a; M) = e_0(a; M),
$$

where $\chi_A(a; M) = \chi_A(K_\bullet(a; M))$ and $e_0(a; M)$ denotes the Hilbert-Samuel multiplicity.

**Proof.** Let $\underline{a} = a_1, \ldots, a_d$ the system of parameters and $\underline{a}A = q$. We choose $c_i = 1, i = 1, \ldots, d$. Then the short exact sequence of $3.3$ has the following form

$$
0 \to K_\bullet(\underline{a}, \underline{a}; M; n) \to K_\bullet(\underline{a}; M) \to L_\bullet(\underline{a}, \underline{a}; M; n) \to 0.
$$

All of the three complexes have homology modules of finite length and therefore $\chi_A(a; M) = \chi_A(K_\bullet(\underline{a}, \underline{a}; M; n)) + \chi_A(L_\bullet(\underline{a}, \underline{a}; M; n))$ for all $n \in \mathbb{N}$.

First we show that $\chi_A(K_\bullet(\underline{a}, \underline{a}; M; n)) = 0$ for all $n \gg 0$. To this end recall that $H_i(K_\bullet(\underline{a}, \underline{a}; M; n)) = [H_i(a\underline{T}; R_M(\underline{a}))]_n$. We know that $a\underline{T}(H_i(a\underline{T}; R_M(\underline{a}))) = 0$ for all $i = 0, \ldots, d$. Therefore $H_i(a\underline{T}; R_M(\underline{a}))$ is a finitely generated module over $R_A(a)/a\underline{T}RA(a) = A/aA$. This implies that $[H_i(a\underline{T}; R_M(\underline{a})))]_n = H_i(a\underline{a}, M; n) = 0$ for all $n \gg 0$. That is $\chi_A(K_\bullet(\underline{a}, \underline{a}; M; n)) = 0$ for all $n \gg 0$.

By view of Lemma 5.2 (b) we get $\chi_A(L_\bullet(\underline{a}, \underline{a}; M; n)) = \sum_{i=0}^{d} (-1)^i \ell_A(M/a^{n-i}M)$. For $n \gg 0$ the length $\ell_A(M/a^nM)$ is given by the Hilbert polynomial $e_0(a; M)(d+n) + \ldots + e_d(a; M)$. Therefore, it follows that $\chi_A(L_\bullet(\underline{a}, \underline{a}; M; n)) = e_0(a; M)$. This completes the argument. □

The more general situation of a system of parameters $\underline{a} = a_1, \ldots, a_d$ of a finitely generated $A$-module $M$ of a local ring $(A, \mathfrak{m})$ and an ideal $q \supset \underline{a}$ with $a_i \in q^i$, $i = 1, \ldots, d$ is investigated in the following.

**Proposition 5.4.** With the previous notation we have the equality

$$
e_0(a; M) = c_1 \cdots c_d e_0(q; M) + \chi_A(K_\bullet(\underline{a}, q; M; n))
$$

for all $n \gg 0$. In particular, for all $n \gg 0$ the Euler characteristic $\chi_A(K_\bullet(\underline{a}, q; M; n))$ is a constant.
Proof. The proof follows by the inspection of the short exact sequence of complexes
\[ 0 \to K_\bullet(\mathfrak{a}, q, M; n) \to K_\bullet(\mathfrak{a}, M) \to L_\bullet(\mathfrak{a}, q, M; n) \to 0 \]
of 3.3. By view of Proposition 5.3 we have \( e_0(\mathfrak{a}, M) \) for the Euler characteristic of the complex in the middle. For the Euler characteristic on the right we get (see 5.2 (b))
\[ \chi_A(L_\bullet(\mathfrak{a}, q, M; n)) = \sum_{i=0}^{d} (-1)^i \sum_{1 \leq j_1 < \ldots < j_i \leq d} \ell_A(M/q^{n+\epsilon_j_1-\ldots-\epsilon_j_i}M), \]
which gives the first summand in the above formula (see also [3] for the details in the case of \( M = A \)). This finally proves the claim. \( \square \)

For several reasons it would be interesting to have an answer to the following problem.

Problem 5.5. With the notation of Proposition 5.4 it would be of some interest to give an interpretation of \( \chi_A(\mathfrak{a}, q, M) := \chi_A(K_\bullet(\mathfrak{a}, q, M; n)) \) for large \( n \gg 0 \) independently of \( n \). By a slight modification of an argument given in [3] it follows that \( \chi_A(\mathfrak{a}, q, M) \geq 0 \).

6. The modified Čech complexes

Before we shall be concerned with the construction of our complexes we need a technical lemma. To this end let \( A \) denote a commutative Noetherian ring and let \( \mathfrak{a} \subset A \) be an ideal. Let \( M \) denote a finitely generated \( A \)-module. Let \( a \in \mathfrak{a}^e \) denote an element. Then for each integer \( n \in \mathbb{N} \) the multiplication by \( a \) induces a map
\[ q^n M \to q^{n+e} M, \ m \mapsto am. \]
By iterating this map there is a direct system \( \{q^{n+ek}M, a\} \), where \( q^{n+ek}M \to q^{n+(k+1)e}M \) is the multiplication by \( a \in \mathfrak{a}^e \).

Lemma 6.1. Let \( n \in \mathbb{N} \) be an integer. With the previous notation there is an isomorphism
\[ \lim_k \{q^{n+ek}M, a\} \cong q^n M[\mathfrak{a}^e / a], \]
where \( M[\mathfrak{a}^e / a] \subseteq M_a \) consists of all elements of the form \( m_0/1 + q_1 m_1/a + \ldots + q_t m_t/a^t \) for some \( t \geq 0 \) and elements \( m_0, \ldots, m_t \in M \) and \( q_i \in \mathfrak{a}^e, i = 1, \ldots, t. \)

Proof. For the proof we use the direct system \( \{M, a\} \) with its direct limit \( \lim_k \{M, a\} \cong M_a. \) The injection \( q^{n+ek}M \to M \) provides an injection of direct systems \( \{q^{n+ek}M, a\} \to \{M, a\}. \) By the definition of the direct limit there is a commutative diagram with exact rows
\[
\begin{array}{cccccc}
0 & \to & \oplus q^{n+ek}M & \to & \oplus q^{n+ek}M & \to & \lim_k \{q^{n+ek}M, a\} & \to 0 \\
& & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & \oplus M & \to & \oplus M & \to & M_a & \to 0 \\
\end{array}
\]
where the vertical maps are injections. Let \( x \in \lim_k \{q^{n+ek}M, a\}. \) Then it may be written as \( x = q_0 m_0 / 1 + q_1 m_1/a + \ldots + q_t m_t/a^t \) with \( m_i \in M \) and \( q_i \in \mathfrak{a}^e \) for \( i = 0, \ldots, t. \) Then \( x = 1/a^t(q_0 a^t m_0 + q_1 a^{t-1} m_1 + \ldots + q_t m_t) \in (q^n/a^t)(\mathfrak{a}^e, a^t)M. \)

Because of \( a \in \mathfrak{a}^e \) it follows \( x \in (q^n)((\mathfrak{a}^e / a)^t)M \subseteq q^n M[\mathfrak{a}^e / a] \). The reverse containment follows the same line of reasoning. \( \square \)

With these preparations we are ready to define certain modifications of the Čech complexes. As above let \( \mathfrak{a} = a_1, \ldots, a_t \) denote a system of elements with the previous notations.

Notation 6.2. (A) Let \( n \in \mathbb{N} \) denote an integer. We define a complex \( \tilde{C}_\bullet(\mathfrak{a}, q, M; n) \) in the following way:
(a) For $0 \leq i \leq t$ put
\[ \check{C}^i(\underline{a}, q, M; n) = \oplus_{1 \leq j_1 < \cdots < j_i \leq q} a^n M[q^{c_{j_1} + \cdots + c_{j_i}}/(a_{j_1} \cdots a_{j_i})] \subseteq \oplus_{1 \leq j_1 < \cdots < j_i \leq t} M_{a_{j_1} \cdots a_{j_i}} \]
and \( \check{C}^i(\underline{a}, q, M; n) = 0 \) for \( i > t \) or \( i < 0 \).

(b) The boundary map \( \check{C}^i(\underline{a}, q, M; n) \to \check{C}^{i+1}(\underline{a}, q, M; n) \) is the restriction of the boundary map \( \check{C}^i_A \otimes M \to \check{C}^{i+1}_A \otimes M \).

Note that the restriction is really a boundary map on \( \check{C}^*(\underline{a}, q, M; n) \). We write \( H^i(\underline{a}, q, M; n) \) for the \( i \)-th cohomology of \( \check{C}^*(\underline{a}, q, M; n) \) for all integers \( i \in \mathbb{Z} \) and \( n \in \mathbb{N} \).

(B) By the construction it is clear that
\[ \check{C}^*(\underline{a}, q, M; n) \to \check{C}^* \otimes_A M \]
is a subcomplex. We write \( \check{L}^*(\underline{a}, q, M; n) \) for the quotient. Whence there is a short exact sequence of complexes
\[ 0 \to \check{C}^*(\underline{a}, q, M; n) \to \check{C}^*_A \otimes M \to \check{L}^*(\underline{a}, q, M; n) \to 0. \]

We write \( \check{L}^i(\underline{a}, q, M; n) \) for the \( i \)-th cohomology of \( \check{L}^*(\underline{a}, q, M; n) \) for all \( i \in \mathbb{Z} \) and \( n \in \mathbb{N} \).

Next we relate the construction more closely to the Čech complexes. To this end we put \( a^k = a_{i_1} \cdots a_{i_k} \) in \( A \) and \( a^{i T(c_{i_k})} = a^{k T(c_{i_k})} = a_{(i_{k-1}+c_{i_k})} \cdots a_{(i_1+c_{i_k})} \) in \( R_A(q) \).

**Proposition 6.3.** With the previous notation there are isomorphisms of complexes
\[ \check{C}^*(\underline{a}, q, M; n) \cong \operatorname{lim}_{k} K^*(\underline{a}^k, q, M; n) \quad \text{and} \quad \check{L}^*(\underline{a}, q, M; n) \cong \operatorname{lim}_{k} L^*(\underline{a}^k, q, M; n) \]
for all \( n \in \mathbb{N} \).

**Proof.** By the definitions (see 3.1) there are isomorphisms
\[ K^*(\underline{a}^k T(c_{i_k}), R_M(q)) \cong K^*(\underline{a}^k, q, M; n) \]
for all \( k \geq 1 \). These isomorphisms are compatible with the homomorphisms of the corresponding direct systems. By passing to the direct limit there are isomorphisms
\[ \check{C}_{a^T(c_{i_k})}(R_M(q)) \cong \operatorname{lim}_{k} K^*(\underline{a}^k, q, M; n). \]

By an inspection of \( K^*(\underline{a}^k, q, M; n) \) as a subcomplex of \( K^*(\underline{a}^k; M) \) (see 3.1) it follows by virtue of 6.1 that \( \operatorname{lim}_{k} K^*(\underline{a}^k, q, M; n) \cong \check{C}^*(\underline{a}, q, M; n) \). To this end recall that
\[ q^{n+k(c_{j_1} + \cdots + c_{j_k})} M \to q^{n+(k+1)(c_{j_1} + \cdots + c_{j_k})} M \]
is the multiplication by \( a_{j_1} \cdots a_{j_k} \) on the \((j_1, \ldots, j_k)\)-component on \( K^*(\underline{a}^k, q, M; n) \) for all \( 1 \leq j_1 < \cdots < j_k \leq t \).

By virtue of the notation in 3.1 there is a short exact sequence of complexes
\[ 0 \to K^*(\underline{a}^k T(c_{i_k}), R_M(q)) \to K^*(\underline{a}^k; M) \to L^*(\underline{a}^k, q, M; n) \to 0 \]
for all \( n \geq 1 \). By passing to the direct limit the first isomorphism provides the second one by the definitions.

As an application we show that the cohomology \( \check{L}^i(\underline{a}, q, M; n) \) depends only upon the radical \( \operatorname{Rad} \underline{a} \) of the ideal \( \underline{a}A \).
Lemma 6.4. Let \( a = a_1, \ldots, a_k \) and \( b = b_1, \ldots, b_l \) be two sequences of elements of \( A \). Suppose that \( a_i \in q^i, i = 1, \ldots, k \) and \( b_j \in q^{dj}, j = 1, \ldots, l \). If \( \text{Rad}_a A = \text{Rad}_b R \), then there are isomorphisms

\[
\mathcal{C}^*(a, q; M; n) \cong \mathcal{C}^*(b, q, M; n) \text{ and } \mathcal{L}^*(a, q, M; n) \cong \mathcal{L}^*(b, q, M; n)
\]

for all \( n \in \mathbb{N} \).

Proof. First put \( \overline{b}_T = b_1 T^{d_1}, \ldots, b_l T^{d_l} \). Then we claim that \( \text{Rad}_a T^n R_A(q) = \text{Rad}_b T^n R_A(q) \).

Let \( b \in \text{Rad}_a A \), that is \( b \in q^i \) and \( b^n = \sum_{i=1}^{k} a_i r_i \). That is \( (b T^n)^m = \sum_{i=1}^{k} a_i T^{c_i}, r_i T^{m e_i - c_i} \in a T^n R_A(q) \). The reverse inclusion can be proved similarly.

By passing to the direct limit of the short exact sequence at the end of the proof of 6.3 provides a short exact sequence of complexes

\[
0 \to \mathcal{C}_a T^n (R_M(q)) \to \mathcal{C}_a (M) \to \mathcal{L}^*(a, q, M; n) \to 0.
\]

The Čech complexes are isomorphic for ideals equal up to the radical. By construction we have \( \mathcal{C}_a T^n (R_M(q)) \cong \mathcal{C}^*(a, q, M; n) \), which proves the first isomorphism. Since \( \mathcal{C}_a \otimes_A M \cong \mathcal{C}_a \otimes_A M \) the previous sequence proves the second isomorphism of the statement.
Lemma 7.2. With the previous notation let \( a \in A \) denote an \( M \)-regular element with \( a \in q^c \backslash q^{c+1} \) such that \( a^* = a + q^{c+1} \in G_A(q) \) has the property that \( a^* \not\in \mathfrak{P} \) for all \( \mathfrak{P} \in \text{Ass}_M(q) \) with \( \mathfrak{P} \not\supseteq G_A(q)_+ \). Put \( \mathcal{M} = \oplus_{n \geq 0} aM \cap q^n M/aq^{n-c}M \). Then there is a short exact sequence

\[
0 \to \mathcal{M} \to H^0_{qT_c}(R_M(q)/(aT^c)R_M(q)) \to H^0_{qT_c}(R_{M/aM}(q)) \to 0
\]

and isomorphisms \( H^i_{qT_c}(R_M(q)/(aT^c)R_M(q)) \cong H^i_{qT_c}(R_{M/aM}(q)) \) for all \( i > 0 \).

Proof. By the choice of \( a^* \) it follows that

\[
0 : G_M(q) a^* = \oplus_{n \geq 0} (q^{n+c+1}M :_M a) \cap q^n M/q^{n+1}M
\]

has support contained in \( V(G_A(q)_+) \). It is a finitely generated \( A/q \)-module and \( 0 : G_M(q) a^* \mid_n = 0 \) for \( n \gg 0 \). Because \( a \) is \( M \)-regular the Artin-Rees Lemma implies that there is an integer \( \ell \) such that \( q^{n+c}M :_M a = q^n M \) for all \( n \geq \ell \). As a consequence we get

\[
aM \cap q^n M = aq^{n-c}M \text{ for all } n \gg 0.
\]

By inspecting the corresponding graded pieces there is a short exact sequence of graded \( R_A(q) \)-modules

\[
0 \to \mathcal{M} \to R_M(q)/(aT^c)R_M(q) \to R_{M/aM}(q) \to 0.
\]

Since \( \mathcal{M} \) is a finitely generated \( A/q \)-module, it is of \( aT^c \)-torsion and the long exact cohomology sequence provides the short exact sequence as well as the isomorphisms of the statement. \( \square \)

Next we want to prove that our cohomologies allow to mod out the annihilator of a finitely generated module. To be more precise.

Lemma 7.3. Let \( \underline{a}, q, M \) as before with \( I = \text{Ann}_A M \) and \( \tilde{A} = A/I \). Then there are isomorphisms

\[
\tilde{H}^i(\underline{a}, q, M; n) \cong \tilde{H}^i(\underline{a}, q, \tilde{A}, M; n) \quad \text{and} \quad \tilde{L}^i(\underline{a}, q, M; n) \cong \tilde{L}^i(\underline{a}, q, \tilde{A}, M; n)
\]

for all \( i, n \in \mathbb{N} \), where \( \underline{a} \) denotes the sequence formed by \( a \) in \( \tilde{A} \).

Proof. First of all note that \( a_i + I \in q^c + I \) for \( i = 1, \ldots, t \) as elements in \( \tilde{A} \). By base change of local cohomology we have isomorphisms

\[
H^i_{\tilde{q}T_c}(R_M(q)) \cong H^i_{\tilde{q}T_c}(R_{M/q}(\tilde{A})) \quad \text{and} \quad H^i_{\underline{a}A}(M) \cong H^i_{\underline{a}\tilde{A}}(M)
\]

for all \( i \in \mathbb{N} \). Because of

\[
H^i_{\tilde{q}T_c}(R_M(q)) n \cong \tilde{H}^i(\underline{a}, q, M; n) \quad \text{and} \quad H^i_{\tilde{q}T_c}(R_{M/q}(\tilde{A})) n \cong \tilde{H}^i(\underline{a}, q, \tilde{A}, M; n)
\]

this proves the first family of isomorphisms. The second one follows by view of the exact sequence at the end of the proof of 6.4. \( \square \)

We prove now the first structural result on a certain cohomology.

Theorem 7.4. With the previous notation suppose that \( (A, \mathfrak{m}) \) is a local ring and \( \underline{a}A \) is an \( \mathfrak{m} \)-primary ideal. Then

\[
\tilde{H}^i(\underline{a}, q, M; n) \quad \text{and} \quad \tilde{L}^i(\underline{a}, q, M; n)
\]

are Artinian \( A \)-modules for all \( i, n \in \mathbb{N} \).

Proof. We start with the proof for \( \tilde{H}^i(\underline{a}, q, M; n) \cong H^i_{\tilde{q}T_c}(R_M(q)) n \). We show that \( H^i_{\tilde{q}T_c}(R_M(q)) n \) is Artinian for all finitely generated \( A \)-modules \( M \) and all \( n \in \mathbb{N} \) by an induction on \( i \).

For \( i = 0 \) we have that \( H^0_{\tilde{q}T_c}(R_M(q)) \cong \mathcal{N} \) with \( \mathcal{N}_n \cong (0 : M(\underline{a})) \cap q^n M \). As an \( A \)-module of finite length \( \mathcal{N}_n \) is an Artinian \( A \)-module for all \( n \in \mathbb{N} \).

So let \( i > 0 \). By view of 7.1 we may assume that \( \mathcal{N} = 0 \). Therefore we may choose an \( M \)-regular element \( a \) satisfying the requirements of the element \( a \) in 7.2. This can be done by
prime avoidance since \( \text{Ass } G_M(q) \nsubseteq V(G_A(q)_+) \) because otherwise \( \dim G_M(q) = \dim M = 0 \).

By induction hypothesis (see 7.2) it follows that \( H^i_{\mathfrak{a}T^c}(R_M(q)/(aT^c)R_M(q))_n \) is Artinian for all \( n \in \mathbb{N} \). In order to prove that \( H^i_{\mathfrak{a}T^c}(R_M(q))_n \) is an Artinian \( A \)-module we have to show

1. \( \text{Supp}_A H^i_{\mathfrak{a}T^c}(R_M(q))_n \subseteq \{ m \} \)
2. \( \dim \text{Hom}_A(k, H^i_{\mathfrak{a}T^c}(R_M(q))_n) < \infty \).

The first result is easily seen since \( \check{C}_{\mathfrak{a}T^c}(R_M(q))_n \otimes_A A_p \) is exact for any non-maximal prime ideal \( p \). For the proof of (2) we use the short exact sequence

\[
0 \to R_M(q)(-c) \overset{aT^c}{\to} R_M(q) \to R_M(q)/(aT^c)R_M(q) \to 0.
\]

By applying local cohomology and restriction to degree \( n \) it implies a surjection

\[
H^{i-1}_{\mathfrak{a}T^c}(R_M(q)/(aT^c)R_M(q))_n \to \text{Hom}_A(A/aA, [H^i_{\mathfrak{a}T^c}(R_M(q))]_{n-c}) \to 0.
\]

That is, \( \text{Hom}_A(A/aA, [H^i_{\mathfrak{a}T^c}(R_M(q))]_{n-c}) \) is an Artinian \( A \)-module and by adjointness

\[
\dim \text{Hom}_A(k, [H^i_{\mathfrak{a}T^c}(R_M(q))]_{n-c}) = \dim \text{Hom}_A(k, \text{Hom}_A(A/aA, [H^i_{\mathfrak{a}T^c}(R_M(q))]_{n-c})) < \infty.
\]

Therefore \( [H^i_{\mathfrak{a}T^c}(R_M(q))]_n \) is an Artinian \( A \)-module for all \( n \in \mathbb{N} \). This completes the inductive step.

For the Artinianess of \( \check{L}^i(\mathfrak{a}, q, M; n) \) we first note that the local cohomology modules \( H^i_{\mathfrak{a}A}(M) \) are Artinian \( M \)-modules. Then the claim follows by virtue of the long exact sequence

\[
\ldots \to \check{H}^i(\mathfrak{a}, q, M; n) \to H^i_{\mathfrak{a}A}(M) \to \check{L}^i(\mathfrak{a}, q, M; n) \to \ldots
\]

as follows from the short exact sequence of the corresponding Čech complexes. \( \square \)

8. Vanishing results

Let \( \mathfrak{a} = a_1, \ldots, a_t \) denote a system of elements of \( A \). Let \( q \subset A \) be an ideal and let \( M \) be a finitely generated \( A \)-module. Then we have the following technical lemma.

**Lemma 8.1.** With the previous notation let \( \mathfrak{a} \in q^t \). Then the following conditions are equivalent:

(i) \( a^* \notin \mathfrak{P} \) for all \( \mathfrak{P} \in \text{Ass } G_M(q) \setminus V(G_A(q)_+) \).

(ii) There are integers \( l, k \) such that \( (q^{n+l}M :_M a) \cap q^n M = q^n M \) for all \( n \geq k \).

**Proof.** First note that \( 0 : G_M(q) a^* = \oplus_{n \geq 0} (q^{n+1+l}M :_M a) \cap q^n M/q^{n+1} M \). Its support is contained in \( V(G_M(q)_+) \) if and only if it is a finitely generated \( G_A(q)_+/q^n M \). Whence the equivalence of (i) and (ii) follows easily. \( \square \)

Note that in case of \( c = 1 \) in the above statement the element \( a \in A \) is called a superficial element of \( q \) with respect to \( M \) see [12, Section 8.5] for more information.

**Corollary 8.2.** With the notation of 8.1 suppose that \( a^* \notin \mathfrak{P} \) for all \( \mathfrak{P} \in \text{Ass } G_M(q) \setminus V(G_A(q)_+) \). Then \( a^* \notin \mathfrak{P} \) for all \( \mathfrak{P} \in \text{Ass } G_M/\mathfrak{a}G_M(q) \setminus V(G_A(q)_+) \)

**Proof.** By virtue of the equivalent conditions in 8.1 it will be enough to show that

\[
(q^{n+2c}M, 0 :_M a) \cap (q^n M, 0 :_M a) = (q^n M, 0 :_M a) \text{ for all } n \geq k.
\]
Let $m$ be an element of the left side. Then $am \in q^{n+2c}M :_M a \cap q^tM = q^{n+c}M$ as follows by the assumption. Therefore
\[
m \in (q^{n+c}M :_M a) \cap (q^tM, 0 :_M a) = (q^{n+c}M :_M a \cap q^tM, 0 :_M a) = (q^nM, 0 :_M a)
\]
for all $n \geq k$. Since the opposite inclusion is trivial the result follows by 8.1.

In the next we investigate the following condition for a system of elements $\underline{a} = a_1, \ldots, a_t$ with $a_i \in q^t$, $i = 1, \ldots, t$, namely
\[(\ast) \quad a_i^* \notin \mathfrak{P}
\]
for all $\mathfrak{P} \in \text{Ass} G_M/(a_1, \ldots, a_{i-1})M(q) \setminus V(G_A(q_+))$ for $i = 1, \ldots, t$.

In the case of $c_1 = \ldots = c_t = 1$ the condition $(\ast)$ is that of a superficial sequence of $q$ with respect to $M$ (see [12]).

**Theorem 8.3.** We fix the previous notation. Suppose that $\underline{a} = a_1, \ldots, a_t$ is a system of parameters satisfies the condition $(\ast)$. Then
\[
\check{H}^l(\underline{a}, q, M; n) = H^l_{\underline{a}T^c}(R_M(q))_n = 0
\]
for all $n \gg 0$.

**Proof.** First consider the following short exact sequence
\[
0 \to N \to R_M(q) \to R_{M/0:a_1M}(q) \to 0,
\]
where $N = 0 : R_M(q) \langle a_1T^c \rangle$. Recall that $[0 : R_M(q) \langle a_1T^c \rangle]_n = (0 : M \langle a_1 \rangle) \cap q^nM$ for all $n \geq 0$. By base change on the local cohomology it follows that $H^l_{\underline{a}T^c}(N) = 0$. Note that $N$ is annihilated by a power of $(a_1T^c)$. By the long exact cohomology sequence it follows that
\[
H^l_{\underline{a}T^c}(R_M(q)) \cong H^l_{\underline{a}T^c}(R_{M/0:a_1M}(q)).
\]

Therefore, we may assume that $a_1$ is $M$-regular as follows by an iterated use of 8.2 because of $0 : M \langle a_1 \rangle = 0 : M a_1^*$ for a certain large integer $l$.

If $t = 1$, then $\dim A M = 1$ and $M/a_1M$ is of finite length. Then $[R_M(q)/(a_1T^c)]_n = q^nM/a_1q^{n-c_1}M$. By the assumption and since $a \in A$ is $M$-regular it follows that $a_1M \cap q^nM = a_1q^{n-c_1}M$ for all $n \gg 0$. Since $M/a_1M$ is of finite length we get that $q^nM \subset a_1M$ for all $n \gg 0$.

Now we proceed by induction on $t$. If $t = 1$ the short exact sequence in the statement of 7.2 yields $[H^l_{\underline{a}T^c}(R_M(q))/(a_1T^c)]_n = 0$ for all $n \gg 0$. By induction hypothesis this holds for $t > 1$ by view of the isomorphism in the statement in 7.2. Now we show $[H^l_{\underline{a}T^c}(R_M(q))]_n = 0$ for $n \gg 0$. To this end recall the short exact sequence
\[
0 \to R_M(q)(-c_1) \to R_M(q) \to R_M(q)/(aT^c)R_M(q) \to 0.
\]

By the vanishing of $[H^{l-1}_{\underline{a}T^c}(R_M(q)/(a_1T^c)]_n = 0$ for $n \gg 0$ it implies the bijectivity
\[
H^l_{\underline{a}T^c}(R_M(q))(-c_1) \cong H^l_{\underline{a}T^c}(R_M(q))
\]
in large degrees. Assume now there is an $0 \neq r \in [H^l_{\underline{a}T^c}(R_M(q))]_n$ for a large $n$. Because $\text{Supp} H^l_{\underline{a}T^c}(R_M(q)) \subseteq V(aT^c)$ there is an integer $m$ such that $(a_1T^c)^m \cdot r = 0$. By the bijectivity this implies that $r = 0$, a contradiction. This completes the inductive step.

As a consequence of the previous result we have the following corollary.

**Corollary 8.4.** We fix the previous notation. Suppose that $\underline{a} = a_1, \ldots, a_t$ is a system of parameters satisfies the condition $(\ast)$. Then
\[
H^l_{\underline{a}}(M) \cong \check{L}^l(\underline{a}, q, M; n)
\]
for all $n \gg 0$. In particular, if in addition $H^l_{\underline{a}}(M) \neq 0$, then $\check{L}^l(\underline{a}, q, M; n) \neq 0$ for all $n \gg 0$. 

Proof. The short exact sequence of complexes as shown in 6.3 provides an exact sequence

\[ H^l_\mathfrak{A}_{\mathfrak{M}}(R_M(q))_n \to H^l_\mathfrak{A}(M) \to \tilde{L}^l(\mathfrak{a}, q, M; n) \to 0. \]

By virtue of Theorem 8.3 the claim follows. \qed

9. Non Vanishing

In this section let \( \mathfrak{a} = a_1, \ldots, a_t \) be a system of elements of a local ring \((A, \mathfrak{m}, k)\) that generates an \(m\)-primary ideal. Then \(q\) is also an \(m\)-primary ideal. Let \(M\) denote a finitely generated \(A\)-module. By view of 7.3 we may assume that \(\dim A = \dim M\). By view of 6.4 we may assume that \(\mathfrak{a}\) is a system of parameters of \(M\). Then it follows that \(\tilde{L}^i(\mathfrak{a}, q, M; n) = 0\) for \(i > \dim M\) and all \(n \in \mathbb{Z}\). In this section we will investigate \(\tilde{L}^t(\mathfrak{a}, q, M; n)\) for \(t = \dim M\).

Definition 9.1. Let \(a \in \mathfrak{q}^t\). For an integer \(n \geq 0\) it follows

\[ q^n(q^c)^lM :_M a^l \subseteq q^n(q^c)^{l+1}M :_M a^{l+1} \]

for all \(l \geq 0\). We denote the stable value by \((q^nM)^{tA}\). Then \(q^nM \subseteq (q^nM)^{tA}\) and \(0 :_M \langle a \rangle \subseteq (q^nM)^{tA}\) for all \(n \geq 1\). Moreover it is easily seen that \(q^nM[q^n/a] \cap M = (q^nM)^{tA}\) for all \(n \geq 1\).

Then we have the following result.

Proposition 9.2. With the previous notation the following conditions are equivalent:

(i) \((qM)^{tA} = M\).
(ii) \((q^kM)^{tA} = M\) for all \(k \geq 1\).
(iii) \((q^kM)^{tA} = M\) for some \(k \geq 1\).

Proof. If (i) is satisfied, then \(a^lM \subseteq q^{l+cl}M\) for an integer \(l \geq 1\). This implies \(a^kM \subseteq q^k(q^c)^lM\) for all \(k \geq 1\). This yields \(M \subseteq q^k(q^c)^lM :_M a^k \subseteq (q^kM)^{tA}\) which proves (ii). Since (ii) \(\implies\) (iii) is trivially true we have to show (iii) \(\implies\) (i). Let \(m \in M\) denote an arbitrary element. Then

\[ a^lm \in q^l(q^c)^lM \subseteq q^l(q^c)^lM \]

for a certain \(l \in \mathbb{N}\). That is \(m \in (qM)^{tA}\) and the proof is finished. \qed

Lemma 9.3. With the previous notation the following conditions are equivalent:

(i) \(\tilde{L}^1(a, q, M; n) = 0\).
(ii) \(\tilde{L}^1(a, q, M; n) = 0\) for all \(n \geq 1\).
(iii) \(\tilde{L}^1(a, q, M; n) = 0\) for some \(n \geq 1\).
(iv) \((qM)^{tA} = M\).

Proof. By the definition we have that \(\tilde{L}^1(a, q, M; n) \cong \varprojlim \{M/(a^k, q^{n+k}c)M, a\}\), where the maps in the direct system are given by multiplication by \(a\). Then the direct limit is zero if and only if for each \(k \geq 1\) there is an integer \(l\) such that \(a^lM \subseteq (a^{k+l}M, q^{n+k+l}c)M\). This holds if and only if \(M \subseteq a^kM, q^{n+k}c(q^c)^lM :_M a^l\) and by Nakayama Lemma if and only if \(M = q^{n+k}c(q^c)^lM :_M a^l\). But this is equivalent to \(M = (q^{n+k}M)^{tA}\). Then the equivalence of all the conditions of the statement is a consequence of 9.2. \qed

Remark 9.4. With the previous notation it is not clear to us when \((qM)^{tA}\) is a proper submodule of \(M\). Suppose that \(a^* \notin \mathfrak{P}\) for all \(\mathfrak{P} \in \text{Ass } G_M(q) \setminus \{V(G_A(q)_+\} \) and that \(a\) is \(M\)-regular. Then \(q^{n+c}M :_M a = q^nM\) for all \(n \gg 0\). This implies that \(q^n(q^c)^lM :_M a^l = q^nM\) for all \(n \gg 0\). That is \(q^nM)^{tA} = q^nM\) for all \(n \gg 0\) and \(\tilde{L}^1(a, q, M; n) \neq 0\) for all \(n \geq 1\) (see 9.2). Moreover, if \((q^nM)^{tA} = q^nM\) for all \(n \gg 0\) then it follows also \(0 :_M a = 0\) and \(q^{n+c}M :_M a = q^nM\) for all \(n \gg 0\) as easily seen.

Now we are ready to prove the non-vanishing result.
Theorem 9.5. With the previous notation let \( a = a_1, \ldots, a_t \subset q \) denote a system of parameters of \( M \). Then \( \check{L}_t(a, q, M; n) \neq 0 \) for all \( n \gg 0 \).

Proof. By view of 7.3 we may assume that \( \text{Ann} M = 0 \). Then \( \dim G_M(q) = t \). Inductively we choose a system of elements \( b = b_1, \ldots, b_t \) such that \( b \) satisfies the condition \((\star)\) above. Then \( b \) is a system of parameters of \( M \) and \( \check{L}_t(a, q, M; n) \cong \check{L}_t(b, q, M; n) \) (see 6.4). Then by 8.4 it follows that \( 0 \neq H_t^b(M) \cong \check{L}_t(a, q, M; n) \) for all \( n \gg 0 \). Note that the non-vanishing of \( H_t^b(M) \) follows by virtue of [6] since \( \dim M = t \) and \( b \) generates an \( m \)-primary ideal. \( \square \)

References

[1] M. F. Atiyah, I. G. Macdonald: 'Introduction to commutative algebra', Addison-Wesley Publ. Co., Reading, 1969.
[2] M. Auslander, D. A Buchsbaum: Codimension and Multiplicity, Ann. Math. 68 (1958), 625-657.
[3] E. Boña, P. Schenzel: Local Bézout estimates and multiplicities of parameter and primary ideals, Preprint.
[4] W. Bruns, J. Herzog: 'Cohen-Macaulay rings', Cambridge Stud. in Advanced Math., Vol. 39, Camb. Univ. Press, 1993.
[5] S. Goto, K.-I. Watanabe: On graded rings, I, J. Math. Soc. Japan 30 (1978), 179-213.
[6] A. Grothendieck: 'Local cohomology', Notes by R. Hartshorne, Lect. Notes in Math., 20, Springer, 1966.
[7] R. Hartshorne: Algebraic Geometry', Graduate Texts in Math., Vol. 52, Springer-Verlag, 1977.
[8] M. Reid: 'Undergraduate Commutative Algebra, London Math. Soc. Students Texts, Vol. 29, Camb. Univ. Press, 1995.
[9] J. J. Rotman: 'An Introduction to Homological Algebra', Academic Press, 1979.
[10] P. Schenzel: On the use of local cohomology in algebra and geometry. In: Six Lectures in Commutative Algebra, Proceed. Summer School on Commutative Algebra at Centre de Recerca Matemàtica, (Ed.: J. Elias, J. M. Giral, R. M. Miró-Roig, S. Zarzuela), Progress in Math. Vol. 166, Birkhäuser, 1998, pp. 241-292.
[11] J.-P. Serre: Algèbre Locale – Multiplicités. Lect. Notes in Math., Vol. 11, Trois. Édt., Springer, 1975.
[12] I. Swanson, C. Huneke: Integral closure of Ideals, Rings, and Modules. London Math. Soc. Lect. Note Ser., Vol. 336, Cambridge Univ. Press, 2006.
[13] C. Weibel: 'An introduction to homological algebra', Cambridge Univ. Press, 1994.

Abdus Salam School of Mathematical Sciences, GCU, Lahore Pakistan
E-mail address: azeekhadam@gmail.com

Martin-Luther-Universität Halle-Wittenberg, Institut für Informatik, D — 06 099 Halle (Saale), Germany
E-mail address: peter.schenzel@informatik.uni-halle.de