Quantum corrected non-thermal radiation spectrum from the tunnelling mechanism

Subenoy Chakraborty, Subhajit Saha and Christian Corda

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1,2Department of Mathematics, Jadavpur University, Kolkata 700032, West Bengal, India

3Scuola Superiore di Studi Universitari e Ricerca "Santa Rita", via San Nicola snc, 81049, San Pietro Infine (CE) Italy.

4Institute for Theoretical Physics and Advanced Mathematics (IFM) Einstein-Galilei, Via Santa Gonda 14, 59100 Prato, Italy

5International Institute for Applicable Mathematics & Information Sciences (IIAMIS), Hyderabad (India) & Udine (Italy)

Email addresses: schakraborty.math@gmail.com, subhajit1729@gmail.com, cordac.galilei@gmail.com

Abstract

Tunnelling mechanism is today considered a popular and widely used method in describing Hawking radiation. However, in relation to black hole (BH) emission, this mechanism is mostly used to obtain the Hawking temperature by comparing the probability of emission of an outgoing particle with the Boltzmann factor. On the other hand, Banerjee and Majhi reformulated the tunnelling framework deriving a black body spectrum through the density matrix for the outgoing modes for both the Bose-Einstein distribution and the Fermi-Dirac distribution. In contrast, Parikh and Wilczek introduced a correction term performing an exact calculation of the action for a tunnelling spherically symmetric particle and, as a result, the probability of emission of an outgoing particle corresponds to a non-strictly thermal radiation spectrum. Recently, one of us (C. Corda) introduced a BH effective state and was able to obtain a non-strictly black body spectrum from the tunnelling mechanism corresponding to the probability of emission of an outgoing particle found by Parikh and Wilczek. The present work introduces the quantum corrected effective temperature and the corresponding quantum corrected effective
metric is written using Hawking’s periodicity arguments. Thus, we obtain further corrections to the non-strictly thermal BH radiation spectrum as the final distributions take into account both the BH dynamical geometry during the emission of the particle and the quantum corrections to the semiclassical Hawking temperature.

**Keywords:** Quantum Tunnelling, Quantum corrected effective temperature, BH information puzzle

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Considering Hawking radiation [1] in the tunnelling approach [2]-[7], the particle creation mechanism caused by the vacuum fluctuations near the BH horizon works as follows. A virtual particle pair is created just inside the horizon and the virtual particle with positive energy can tunnel out the BH horizon as a real particle. Otherwise, the virtual particle pair is created just outside the horizon and the negative energy particle can tunnel inwards. Thus, for both the possibilities, the particle with negative energy is absorbed by the BH and as a result the mass of the BH decreases. The flow of positive energy particles towards infinity is considered as Hawking radiation. Earlier, this approach was limited to obtain only the Hawking temperature through a comparison of the probability of emission of an outgoing particle with the Boltzmann factor rather than the actual radiation spectrum with the correspondent distributions. This problem was formally addressed by Banerjee and Majhi [7]. By a novel formulation of the tunnelling formalism, they were able to directly reproduce the black body spectrum for either bosons or fermions from a BH with standard Hawking temperature. However, considering contributions beyond semiclassical approximation in the tunnelling process, Parikh and Wilczek [2, 3] found a probability of emission compatible with a non-thermal spectrum of the radiation from BH. This non-precisely thermal character of the spectrum is important to resolve the information loss paradox of BH evaporation [8] because arguments that information is lost during hole’s evaporation partially rely on the assumption of strict thermal behavior of the radiation spectrum [8].

The basic difference between the works [2,3] and the work [7] is consideration or non-consideration of the energy conservation. As a result, there will be a dynamical [2,3] or static [7] BH geometry. In fact, due to conservation of energy, in [2,3] the BH horizon contracts during the radiation process which deviates from the perfect black body spectrum. This non-thermal spectrum has profound implications for realizing the underlying quantum gravity theory and suggests interesting approaches to resolve the BH information puzzle [8]. Moreover, the discrete nature of the tunnelling mechanism is characterized by the physical state before the emission of the particle and that after the emission of the particle and, as a result, the radiation spectrum is also discrete [10]. Consequently, particle emission can be interpreted like a quantum transition of frequency $\omega$ between the two discrete states [10]. The tunnelling trajectory is the path joining two distinct classical turning points in the imaginary or complex time [3, 10]. In thermal spectrum, the tunnelling points have zero separation, so there is no clear trajectory because there is no barrier [3, 10]. On the other hand, in non-thermal spectrum, the forbidden region has finite size,
from \( r = r_{\text{initial}} \) to \( r = r_{\text{final}} \) (\( r_{\text{initial}} \) is the radius of the horizon of the BH initially and \( r_{\text{final}} \) is the radius of the horizon of the BH after particle emission) and the particle itself generates a tunnel through the horizon \[3\] \[10\] \[11\].

In Planck units (\( G = c = k_B = \hbar = \frac{1}{4\pi\epsilon_0} = 1 \)), the strictly thermal tunnelling probability is given by \[1\] \[2\] \[3\]

\[
\Gamma \sim \exp \left( -\frac{\omega}{T_H} \right),
\]

(1)

where \( T_H = \frac{1}{8\pi M} \) is the Hawking temperature and \( \omega \) is the energy-frequency of the emitted radiation. However, considering contributions beyond semiclassical approximation and taking into account the conservation of energy, Parikh and Wilczek reformulate the tunnelling probability as \[2\] \[3\]

\[
\Gamma \sim \exp \left[ -\frac{\omega}{T_H} \left( 1 - \frac{\omega}{2M} \right) \right].
\]

(2)

This non-thermal spectrum enables the introduction of an intriguing way to consider the BH dynamical geometry through the BH effective state. In fact, one introduces the effective temperature as \[10\]-\[13\]

\[
T_E(\omega) = \frac{2M}{2M - \omega} T_H = \frac{1}{4\pi(2M - \omega)},
\]

(3)

which permits to rewrite the probability of emission (2) in Boltzmann-Hawking form as \[10\]-\[13\]

\[
\Gamma \sim \exp \left[ -\beta_E(\omega) \omega \right] = \exp \left( -\frac{\omega}{T_E(\omega)} \right),
\]

(4)

where the effective Boltzmann factor takes the form \[10\]-\[13\]

\[
\beta_E(\omega) = \frac{1}{T_E(\omega)}.
\]

(5)

The effective temperature is interpreted as the temperature of a black body that would emit the same total amount of radiation \[10\]-\[13\]. Hence, the effective temperature replaces the Hawking temperature in the equation of the probability of emission. The ratio \( \frac{T_E(\omega)}{T_H} \frac{2M}{2M - \omega} \) stands for the deviation of the radiation spectrum of a BH from the strictly thermal feature \[10\]-\[13\]. It is better to further clarify the definition of effective temperature that has been introduced in BH physics in \[12\] \[13\] for the Schwarzschild BH, in \[19\] for the Kerr BH and in \[20\] for the Reissner-Nordström BH. The probability of emission of Hawking quanta found by Parikh and Wilczek, i.e. eq. (2), shows that the BH does NOT emit like a perfect black body, i.e. it has not a strictly thermal behavior. On the other hand, the temperature in Bose-Einstein and Fermi-Dirac distributions is a perfect black body temperature. Thus, when we have deviations from the strictly thermal behavior, i.e. from the perfect black body, one expects also deviations from Bose-Einstein and Fermi-Dirac distributions. How can one attack
this problem? By analogy with other various fields of Science, also beyond BHs, for example the case of planets and stars. One defines the effective temperature of a body such as a star or planet as the temperature of a black body that would emit the same total amount of electromagnetic radiation \[10, 14\]. The importance of the effective temperature in a star is stressed by the issue that the effective temperature and the bolometric luminosity are the two fundamental physical parameters needed to place a star on the Hertzsprung–Russell diagram. Both effective temperature and bolometric luminosity actually depend on the chemical composition of a star, see again \[10, 14\].

Further, in analogy with the effective temperature, one can define the effective mass and the effective horizon radius as \[10\]-\[13\]

\[
M_E = M - \frac{\omega}{2} \quad \text{and} \quad r_E = 2M_E = 2M - \omega.
\]

Note that these effective quantities are nothing but the average value of the corresponding quantities before (initial) and after (final) the particle emission (i.e., \(M_i = M\), \(M_f = M - \omega\); \(r_i = 2M_i\) and \(r_f = 2M_f\)). Accordingly, \(T_E\) is the inverse of the average value of the inverses of the initial and final Hawking temperatures \[10\]-\[13\]. Hence, there is a discrete character (in time) of the Hawking temperature. Thus, the effective temperature may be interpreted as the Hawking temperature during the emission of the particle \[10\]-\[13\].

Following \[11\] one can use Hawking’s periodicity argument \[11, 23, 24\] to obtain the effective Schwarzschild line element

\[
ds^2_E = -(1 - \frac{2M_E}{r})dt^2 + \frac{dr^2}{1 - \frac{2M_E}{r}} + r^2 (\sin^2 \theta d\phi^2 + d\theta^2),
\]

which takes into account the BH dynamical geometry during the emission of the particle.

Recently, one of us (C. Corda) introduced the above discussed BH effective state \[10\]-\[13\] and was able to obtain a non-strictly black body spectrum from the tunnelling mechanism corresponding to the probability of emission of an outgoing particle found by Parikh and Wilczek \[11\]. The final non-strictly thermal distributions which take into account the BH dynamical geometry are \[10, 11\]

\[
\begin{align*}
<n>_{\text{boson}} &= \frac{1}{\exp[4\pi(2M - \omega)\omega] - 1} \\
<n>_{\text{fermion}} &= \frac{1}{\exp[4\pi(2M - \omega)\omega] + 1}.
\end{align*}
\]

Now, we further modify the effective temperature by incorporating the quantum corrections to the semiclassical Hawking temperature discussed in \[1\]. As a result, the quantum physics of BHs will be further modified. Banerjee and Majhi \[4\] have formulated the quantum corrected Hawking temperature using the Hamilton-Jacobi method \[15\] beyond semiclassical approximation. According to them \[4\], the quantum corrected Hawking temperature (termed as modified
Hawking temperature) is given by

\[ T_H^{(m)} = \left[ 1 + \sum_i \frac{\beta_i}{M^{2i}} \right]^{-1} T_H, \tag{9} \]

where the \( \beta_i \) are dimensionless constant parameters. However, if these parameters are chosen as powers of a single parameter \( \alpha \), then in compact form \[4\]

\[ T_H^{(m)} = \left( 1 - \frac{\alpha}{M^2} \right) T_H. \tag{10} \]

This modified Hawking temperature is very similar in form to the temperature correction in the context of one-loop back reaction effects \[16, 17\] in the spacetime with \( \alpha \) related to the trace anomaly \[18\]. Further, using conformal field theory, if one considers one-loop quantum correction to the surface gravity for Schwarzschild BH then \( \alpha \) has the expression \[4\]

\[ \alpha = -\frac{1}{360\pi} \left( -N_0 - \frac{7}{4} N_{\frac{3}{2}} + 13N_1 + \frac{233}{4} N_{\frac{5}{2}} - 212 N_2 \right), \tag{11} \]

where \( N_s \) denotes the number of field with spin \( s \). Also considering two-loop back reaction effects in the spacetime, the quantum corrected Hawking temperature becomes \[4\]

\[ T_H^{(m)} = \left[ 1 - \frac{\alpha}{M^2} - \frac{\gamma}{M^4} \right] T_H, \tag{12} \]

where second loop contributions are related to the dimensionless parameter \( \gamma \). Thus, it is possible to incorporate higher loop quantum corrections by proper choices of the \( \beta_i \). It should be noted that these correction terms dominate at large distances \[21\].

Using the above mentioned modified Hawking temperature, the modified form of the Boltzmann factor is

\[ \beta^{(m)} = \frac{1}{T_H^{(m)}} = \frac{1}{T_H \left( 1 - \frac{\alpha}{M^2} - \frac{\gamma}{M^4} \right)} = \frac{\beta_H}{\left( 1 - \frac{\alpha}{M^2} - \frac{\gamma}{M^4} \right)}. \tag{13} \]

Thus, the (quantum corrected) modified BH mass has the expression

\[ M^{(m)} = \frac{M}{\left( 1 - \frac{\alpha}{M^2} - \frac{\gamma}{M^4} \right)}. \tag{14} \]

In case of emitted radiation from the BH, the modified Hawking temperature (with quantum correction) becomes

\[ T_H^{(m)} = \frac{1}{8\pi M^{(m)}}. \tag{15} \]

As a result, following \[23\], one can again use Hawking’s periodicity argument \[10, 23, 24\] to obtain the modified Schwarzschild like line element, which takes the form \[10, 11\]

\[(ds_m)^2 = -(1 - \frac{2M^{(m)}}{r})dt^2 + \frac{dr^2}{1 - \frac{2M^{(m)}}{r}} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \tag{16}\]
with modified surface gravity

\[ \kappa^{(m)} = \frac{1}{4M^{(m)}} = \frac{1}{2r^{(m)}} = \frac{(1 - \frac{\omega}{M})}{4M}, \quad (17) \]

Eq. (16) enables the replacement \( M \to M^{(m)} \) and \( T_H \to T_H^{(m)} \) in eqs. (3), (5), and (6). In other words, one can define the (quantum corrected) modified effective temperature

\[ T_E^{(m)}(\omega) \equiv \frac{2M^{(m)}}{2M^{(m)} - \omega} T_H^{(m)} = \frac{1}{4\pi(2M^{(m)} - \omega)}, \quad (18) \]

the (quantum corrected) modified effective Boltzmann factor

\[ \beta_E^{(m)}(\omega) \equiv \frac{1}{T_E^{(m)}(\omega)} \quad (19) \]

and the (quantum corrected) modified effective mass and effective horizon radius

\[ M_E^{(m)} = M^{(m)} - \frac{\omega}{2} \quad \text{and} \quad r_E^{(m)} = 2M_E^{(m)} = 2M^{(m)} - \omega. \quad (20) \]

Following \[11, 23, 24\], one uses again Hawking’s periodicity argument. Then, the euclidean form of the metric will be given by

\[ \left[ ds_E^{(m)} \right]^2 = x^2 \left[ \frac{d\tau}{4M^{(m)}(1 - \frac{\omega}{2M^{(m)}r})} \right]^2 + \left( \frac{r}{r_E^{(m)}} \right)^2 dx^2 + r^2 \sin^2 \theta d\varphi^2 + d\theta^2, \quad (21) \]

which is regular at \( x = 0 \) and \( r = r_E^{(m)} \). \( \tau \) is treated as an angular variable with period \( \beta_E^{(m)}(\omega) \) \[11, 23, 24\]. Replacing the quantity \( \sum \beta_i \frac{r}{2M^{(m)}} \) in \[23\] with the quantity \(-\frac{\omega}{2M^{(m)}}\), if one follows step by step the detailed analysis in \[23\] at the end one easily gets the (quantum corrected) modified effective Schwarzschild line element

\[ \left[ ds_E^{(m)} \right]^2 = -(1 - \frac{2M^{(m)}}{r})dt^2 + \frac{dr^2}{1 - \frac{2M^{(m)}}{r}} + r^2 \sin^2 \theta d\varphi^2 + d\theta^2. \quad (22) \]

One also easily shows that \( r_E^{(m)} \) in eq. (21) is the same as in eq. (20). Thus, the line element (22) takes into account both the BH dynamical geometry during the emission of the particle and the quantum corrections to the semiclassical Hawking temperature.

Starting from the standard Schwarzschild line element, i.e. \[7, 11\]

\[ ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 \sin^2 \theta d\varphi^2 + d\theta^2, \quad (23) \]
the analysis in [7] permitted to write down the (normalized) physical states of the system for bosons and fermions as [7]

$$|\Psi >_{\text{boson}} = (1 - \exp (-8\pi M \omega))^{\frac{1}{2}} \sum_n \exp (-4\pi n M \omega) |n_{\text{out}}^{(L)} > \otimes |n_{\text{out}}^{(R)} >$$

$$|\Psi >_{\text{fermion}} = (1 + \exp (-8\pi M \omega))^{-\frac{1}{2}} \sum_n \exp (-4\pi n M \omega) |n_{\text{out}}^{(L)} > \otimes |n_{\text{out}}^{(R)} >.$$  \hspace{1cm} (24)

Hereafter we focus the analysis only on bosons. In fact, for fermions the analysis is identical [7]. The density matrix operator of the system is [7]

$$\hat{\rho}_{\text{boson}} \equiv |\Psi >_{\text{boson}} < \Psi |_{\text{boson}}$$

$$= (1 - \exp (-8\pi M \omega)) \sum_{n,m} \exp [-4\pi (n + m) M \omega] |n_{\text{out}}^{(L)} > \otimes |n_{\text{out}}^{(R)} > < m_{\text{out}}^{(R)} \otimes < m_{\text{out}}^{(L)} |.$$  \hspace{1cm} (25)

If one traces out the ingoing modes, the density matrix for the outgoing (right) modes reads [7]

$$\hat{\rho}_{\text{boson}}^{(R)} = (1 - \exp (-8\pi M \omega)) \sum_n \exp (-8\pi n M \omega) |n_{\text{out}}^{(R)} > < n_{\text{out}}^{(R)} |.$$  \hspace{1cm} (26)

This implies that the average number of particles detected at infinity is [7]

$$< n >_{\text{boson}} = \text{tr} \left[ \hat{\rho}_{\text{boson}}^{(R)} \right] = \frac{1}{\exp (8\pi M \omega) - 1},$$  \hspace{1cm} (27)

where the trace has been taken over all the eigenstates and the final result has been obtained through a bit of algebra, see [7] for details. The result of eq. (27) is the well known Bose-Einstein distribution. A similar analysis works also for fermions [7], and one easily gets the well known Fermi-Dirac distribution

$$< n >_{\text{fermion}} = \frac{1}{\exp (8\pi M \omega) + 1}.$$  \hspace{1cm} (28)

Both the distributions correspond to a black body spectrum with the Hawking temperature \( T_H = \frac{1}{8\pi M} \). On the other hand, if one follows step by step the analysis in [7], but starting from the (quantum corrected) modified effective Schwarzschild line element (22) at the end obtains the correct physical states for boson and fermions as

$$|\Psi >_{\text{boson}} = (1 - \exp (-8\pi M_E^{(m)} \omega))^{\frac{1}{2}} \sum_n \exp (-4\pi n M_E^{(m)} \omega) |n_{\text{out}}^{(L)} > \otimes |n_{\text{out}}^{(R)} >$$

$$|\Psi >_{\text{fermion}} = (1 + \exp (-8\pi M_E^{(m)} \omega))^{-\frac{1}{2}} \sum_n \exp (-4\pi n M_E^{(m)} \omega) |n_{\text{out}}^{(L)} > \otimes |n_{\text{out}}^{(R)} >.$$  \hspace{1cm} (29)
and the correct distributions as

\[
\langle n \rangle_{\text{boson}} = \frac{1}{\exp(8\pi M_\text{E}^{(m)}\omega)} - 1 = \frac{1}{\exp[4\pi(2M_\text{E}^{(m)} - \omega)\omega] - 1} = \frac{1}{\exp\left[4\pi \left(2 - \frac{\omega}{M_\text{E}^{(m)}\omega}\right)\right] - 1}
\]

\[
\langle n \rangle_{\text{fermion}} = \frac{1}{\exp(8\pi M_\text{E}^{(m)}\omega) + 1} = \frac{1}{\exp[4\pi(2M_\text{E}^{(m)} - \omega)\omega] + 1} = \frac{1}{\exp\left[4\pi \left(2 - \frac{\omega}{M_\text{E}^{(m)}\omega}\right)\right] + 1},
\]

which are not thermal because they take into account both the BH dynamical geometry during the emission of the particle and the quantum corrections to the semiclassical Hawking temperature. We note that setting \(\alpha = \gamma = 0\) in eqs. (30) we find the results in [11], i.e. eqs. (8). In fact, in [11] only the BH dynamical geometry was taken into account. Here, we further improved the analysis by taking into account also the quantum corrections to the semiclassical Hawking temperature.

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