Exact Results for Integrable Asymptotically-free Field Theories

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Abstract

An account is given of a technique for testing the equivalence between an exact factorizable S-matrix and an asymptotically-free Lagrangian field theory in two space-time dimensions. The method provides a way of resolving CDD ambiguities in the S-matrix and it also allows for an exact determination of the physical mass in terms of the Lambda parameter of perturbation theory. The results for various specific examples are summarized.

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1 INTRODUCTION

Integrable models in two space-time dimensions provide a fascinating arena for investigating non-perturbative phenomena in quantum field theory [1, 2, 3]. The integrable theories that we shall focus on here can be divided into three broad classes, but they all have in common the properties of asymptotic freedom and dynamical mass generation which make them akin to realistic models of particle interactions in four dimensions. First, there are bosonic sigma models based on a symmetric space $G/H$. Any model of this sort is classically integrable, but unfortunately anomalies can destroy integrability at the quantum level unless $H$ is simple. Second, there are fermionic models of Gross-Neveu or Thirring type with four-fermion interactions. Finally, there are hybrids of the two previous classes in which fermions are added to bosonic symmetric space models in various ways. In some cases the addition of fermions can cancel the anomalies encountered in the purely bosonic theory, leading to interesting new examples of quantum integrable theories; in particular this is believed to occur for supersymmetric sigma models based on a symmetric space $G/H$. A review of these matters with detailed references can be found in [3].

Exact S-matrices have been proposed for many of the two-dimensional models which are thought to be integrable at the quantum level [3, 13, 14, 15, 16, 17, 18]. These S-matrices describe the scattering of some conjectured set of particles states, and they are postulated on the basis of the specific symmetries of the model in question together with the usual axioms of S-matrix theory and the powerful constraint of factorization. (There may also be additional information available about the theory, such as the existence of bound-states.) However, such S-matrices are always subject to CDD ambiguities [19, 13] which cannot be further constrained by these general considerations. What is needed is some completely non-perturbative way of fixing the CDD ambiguities by testing the proposed equivalence between an S-matrix on the one hand and renormalized Lagrangian perturbation theory on the other. In this paper we shall explain such a programme which can be applied to any of the types of theory mentioned above.

The technique, which was pioneered in [4, 5, 6] following earlier work in [20, 21, 22, 23, 24], can be applied to any integrable model possessing a group $G$ of global symmetries. The particle states in such a theory fall into representations of $G$, and the symmetry is generated by conserved charges in the Lie algebra. The idea is to couple the theory to some particular conserved charge $Q$ by modifying the Hamiltonian from $H$ to $H - hQ$, where $h$ is a coupling constant of mass dimension 1 (so that $Q$ is dimensionless). The corresponding change in the ground-state energy density $\delta \mathcal{E}(h) = \mathcal{E}(h) - \mathcal{E}(0)$ can be computed straightforwardly in perturbation theory. But in the special case of an integrable model, it can also be computed by a very different non-perturbative method starting from the exact S-matrix and using the Thermodynamic Bethe Ansatz (TBA). The comparison of these calculations provides a powerful check that the S-matrix and the Lagrangian really do correspond to one another and it also allows one to extract an exact expression for the mass gap of the model. We now describe more quantitatively how this comparison works.

The result of a perturbative calculation of the ground-state energy density—which is of course
a renormalization group-invariant quantity—is an expansion
\[ \delta \mathcal{E}(h) = h^2 \sum_{j=0}^{\infty} \alpha_j g(h/\Lambda)^{j-1} \]
in ascending powers of the running coupling \( g(h/\Lambda) \) where the \( \alpha_j \) are dimensionless numbers.

The running coupling can be found by integrating the usual beta-function equation
\[ \mu \frac{dg}{d\mu} = \beta(g) = -\sum_{j=1}^{\infty} \beta_j g^{j+1} \]
to obtain
\[ \frac{1}{g(\mu/\Lambda)} = \beta_1 \ln \frac{\mu}{\Lambda} + \beta_2 \ln \ln \frac{\mu}{\Lambda} + \mathcal{O}\left( \ln \ln \frac{\mu}{\Lambda} / \ln \frac{\mu}{\Lambda} \right), \]
where the absence of a constant term in the solution defines the mass scale \( \Lambda \) for the particular renormalization scheme used. In the models we are considering, \( \beta_1 > 0 \) and we expect perturbation theory to be valid in the asymptotic regime where \( \mu \gg \Lambda \). Combining the expressions above we find that the ground-state energy density is given by:
\[ \frac{\delta \mathcal{E}(h)}{h^2} = \alpha_0 \beta_1 \ln \frac{h}{\Lambda} + \alpha_0 \frac{\beta_2}{\beta_1} \ln \ln \frac{h}{\Lambda} + \alpha_1 + \mathcal{O}\left( \ln \ln \frac{h}{\Lambda} / \ln \frac{h}{\Lambda} \right). \quad (1) \]

Notice that if \( \alpha_0 \neq 0 \), there is a classical or tree-level contribution to the ground-state energy density and this quantity is consequently unbounded as \( h \) becomes large. Otherwise, \( \alpha_0 = 0 \) and the leading contribution to the ground-state energy is then the constant term \( \alpha_1 \). We shall have more to say shortly about the important differences between these situations.

Turning now to the TBA calculation: in order to extract the ground-state energy from the S-matrix, one must confront a set of coupled integral equations of Wiener-Hopf type. The special circumstances of interest to us are those in which the temperature is zero, with the coupling to the charge \( Q \) acting like a chemical potential. In general the resulting equations cannot be solved exactly, but it is possible, at least under certain simplifying assumptions, to generate an expansion of the solution in \( h/m \) valid when \( h \gg m \), where \( m \) is some physical mass parameter occurring in the S-matrix. The result is
\[ \frac{\delta \mathcal{E}(h)}{h^2} = \kappa_0 \ln \frac{h}{m} + \kappa_1 \ln \ln \frac{h}{m} + \kappa_2 + \mathcal{O}\left( \ln \ln \frac{h}{m} / \ln \frac{h}{m} \right), \quad (2) \]
where the \( \kappa_j \) are dimensionless numbers. The exact values of the parameters \( \kappa_j \) depend sensitively on the analytic structure of the S-matrix and, in particular, one finds that the presence or absence of CDD factors alters dramatically the result for the ground-state energy obtained in this way.

Equality of the expressions (1) and (2) gives a powerful check that the Lagrangian and S-matrix descriptions are consistent. The greater the accuracy to which we can calculate these expressions, the more conclusive this check will be. Even finding just the first few terms, however, can provide strong evidence in favour of the proposed S-matrix used in the TBA calculation, allowing us to argue that any alteration by CDD factors would destroy the delicate agreement with the perturbative result. In addition, the consistency of (1) and (2) clearly determines the mass gap \( m/\Lambda \), at least if we have calculated the expressions involved to sufficient accuracy.
At this stage it is useful to look more closely at the two possible cases to which we drew attention earlier. If $\alpha_0 \neq 0$, we see that the TBA calculation must reproduce the first two coefficients of the beta-function through the conditions:

$$\alpha_0 \beta_1 = \kappa_0 , \quad \alpha_0 \beta_2 / \beta_1 = \kappa_1 ,$$

providing a highly non-trivial test of the S-matrix. In these circumstances we can also read off the value of the mass gap in the model:

$$\ln\left(\frac{m}{\Lambda}\right) = \frac{(\kappa_2 - \alpha_1)}{\alpha_0} / \frac{(\kappa_2 - \alpha_1)}{\alpha_0 \beta_1}.$$  

We emphasize that these relations are deduced by comparing the terms written explicitly in (1) and (2) and they therefore rely on just a one-loop perturbative calculation of the ground-state energy. If, instead, $\alpha_0 = 0$, we must have $\kappa_0 = \kappa_1 = 0$ and $\kappa_2 = \alpha_1$ for consistency and we are then unable to find the mass-gap by taking the expressions to the order given explicitly in (1) and (2). To extract $m/\Lambda$ in this situation one needs to extend both the TBA and perturbative calculations to higher orders; in fact it is not hard to see that this would involve at least a three-loop perturbative calculation.

The strategy we have just outlined has been applied to several series of integrable models, with results we shall summarize in section 3. The expressions for the exact mass-gaps derived in this way are very useful, because they provide bench-marks for the reliability of other non-perturbative approaches, such as lattice simulations. To minimize the amount of work involved, it is clearly desirable to try to choose $Q$ so as to produce a classical term in the perturbative expression for the ground-state energy (1) because, as we have explained, the mass-gap can then be found from the TBA analysis in conjunction with a one-loop perturbative calculation. It can be shown [12] that such a choice of $Q$ is possible for any bosonic or supersymmetric sigma model based on a symmetric space $G/H$, a result which unifies the treatment of various examples considered previously in [4, 5, 8, 9, 10, 11]. For the purely fermionic Gross-Neveu models, however, it seems that generically the classical contribution vanishes, and so a three-loop calculation is necessary in order to find the mass gap [6, 7].

There is another crucial consideration to be borne in mind when choosing $Q$. In each of the cases considered so far, an important simplifying assumption was made regarding the TBA calculation, namely, that for very particular choices of the charge $Q$, only a small number of particles—in fact those with the largest charge/mass ratio—contribute to the new ground-state. The original one-particle states can be chosen to be eigenvectors of the new Hamiltonian $H - hQ$ with eigenvalues $m_i \cosh \theta_i - h q_i$, where $q_i$ is the charge of the particle labelled by $i$ and $\theta_i$ is its rapidity. If $i = 1$ labels the particle type with the largest charge/mass ratio, then for sufficiently small values of $h$ such that $hq_1 < m_1$ it is not energetically favourable to find any particles in the new ground-state, and so $\delta E(h) = 0$. As $h$ passes the threshold value $m_1/q_1$, it suddenly becomes favourable to fill the original vacuum with particles. For large $h$, it would seem that any arbitrary particle could appear in the new ground-state and hence the TBA calculation would have to keep track of all particles. However, for very particular choices of the charge $Q$ it seems that the particles with largest charge/mass ratio actually repel other particles and so only they appear in the ground state. This assumption greatly simplifies the solution of the TBA equations.
and in most of the existing papers it is taken as a working hypothesis which is vindicated by the consistency of the final results. With more care, it can actually be proven from an analysis of the full TBA equations [12].

Having introduced the general idea behind the technique, we shall discuss in the next section how the TBA equations can be derived under the simplifying assumption explained above. We shall then summarize the known results, concluding with a more detailed example which illustrates the most important points.

2 THE TBA EQUATIONS

In the cases where a single particle-type contributes to the ground state, the TBA analysis is rather straightforward. The elastic scattering of two particles of the same species is described by an S-matrix element which is just a phase $S(\theta)$, where $\theta$ is the rapidity difference of the incoming particles. Since particle number and momenta are conserved on interaction, it makes sense to consider single particle states. In the dilute regime, where the particles are on average well separated, the wavefunction of $N$ particles is built up from these single particle states and has the form

$$\Psi(x_1, \ldots, x_N) = \sum_{Q \in S_N} \Theta(x_Q)\zeta(Q) \exp im(x_1 \sinh \theta_1 + \cdots + x_N \sinh \theta_N)$$

where the sum is taken over all permutations $Q = \{Q_1, \ldots, Q_N\}$ of $\{1, \ldots, N\}$ with

$$\Theta(x_Q) = \begin{cases} 1 & \text{if } x_{Q_1} < x_{Q_2} < \cdots < x_{Q_N} \\ 0 & \text{otherwise} \end{cases}$$

and where $\zeta(Q)$ are numbers defined via the S-matrix so that if $Q$ differs from $Q'$ only by an exchange of two specific elements of the list $Q_i$ and $Q_j$, say, then

$$\zeta(Q') = S(\theta_i - \theta_j)\zeta(Q).$$

This means that the $\zeta(Q)$ are determined up to an overall factor.

We impose periodic boundary conditions on the wavefunctions:

$$\Psi(x_1, \ldots, x_j, \ldots, x_N) = \Psi(x_1, \ldots, x_j + L, \ldots, x_N)$$

which imply that, for each fixed $j$,

$$\exp(imL \sinh \theta_j) \prod_{k \neq j} S(\theta_j - \theta_k) = 1.$$

Taking the logarithm of these equations we have

$$mL \sinh \theta_j - i \sum_{k \neq j} \ln S(\theta_j - \theta_k) = 2\pi n_j \quad (5)$$
where the \( \{ n_j \} \) are a set of \( N \) integers which we assume determine uniquely the set of rapidities \( \{ \theta_j \} \) (this is guaranteed in the dilute regime where the sinh \( \theta_j \) dominates in (5)). It is crucial that in all the integrable models we are considering here the particles behave as fermions in rapidity space. The condition (5) then proves to be the key ingredient when we come to analyse the thermodynamics of such a system.

The thermodynamic limit can be taken in the usual way by letting \( N \to \infty \) and \( L \to \infty \) with \( N/L \) fixed. It now becomes meaningful to talk about the density of one-particle states in rapidity space, \( \varrho(\theta) \), defined by
\[
\frac{dn}{d\theta} = L \varrho(\theta) \frac{d\theta}{2\pi i} \ln S(\theta).
\]
(6)

By differentiating the key condition (5) we deduce
\[
\varrho(\theta) = \frac{m}{2\pi} \cosh \theta + K \ast \sigma(\theta)
\]
(7)

(where \( f \ast g(\theta) = \int_{-\infty}^{\infty} d\theta' f(\theta') g(\theta - \theta') \) for any two functions \( f \) and \( g \)). When \( K(\theta) = 0 \), (7) reduces to the usual expression for the density-of-states in a free theory. But when there is an interaction between the particles, (7) implies that the total density and the density of occupied states are coupled in a complicated manner. Of course for overall consistency we must have \( \sigma(\theta) \leq \varrho(\theta) \), which is guaranteed in the dilute regime since \( \sigma(\theta) \) is then small compared with \( \varrho(\theta) \).

For a macrostate specified by the density \( \sigma(\theta) \), the value of \( H - hQ \) per unit length is
\[
E(\sigma; h) = \int_{-\infty}^{\infty} d\theta \frac{m \cosh \theta - h}{2\pi} \sigma(\theta) .
\]

(8)

(9)

where we assume the charge is normalized to unity on our preferred particle states). To find the ground-state energy density we must minimize \( E(\sigma; h) \) with respect to \( \sigma(\theta) \) subject to the constraint (7). The solution to this variational problem involves filling all the available states of lowest rapidity first, so \( \sigma(\theta) = \varrho(\theta) \) up to some Fermi rapidity \( \pm \theta_F \). It can be shown that this problem can be solved in terms of an energy density \( \epsilon(\theta) \) which satisfies the TBA equation at zero temperature:
\[
\epsilon^+(\theta) + R \ast \epsilon^-(\theta) = m \cosh \theta - h
\]
with the notation
\[
f^\pm(\theta) = \begin{cases} f(\theta) & \text{if } f(\theta) > 0 \\ 0 & \text{otherwise} \end{cases}
\]

The solution to (8) is concave and negative between the values \( \pm \theta_F \), which are fixed by the condition \( \epsilon(\pm \theta_F) = 0 \). The final result for the way in which the ground-state energy density behaves as a function of \( h \) is then given simply by
\[
\delta E(h) = \frac{m}{2\pi} \int_{-\theta_F}^{\theta_F} d\theta \cosh \theta \epsilon(\theta) .
\]

(9)
The problem of finding the ground-state energy density has thus been reduced to solving the Wiener-Hopf integral equation (8). It is not possible to find the solution exactly for arbitrary $h$. But we require the solution only in the asymptotic regime with $h \gg m$ and it is explained in [6] and [8] how to develop such an asymptotic expansion for the solution based on the original approach of [23]. The nature of the expansion depends crucially on whether or not the Fourier transform $\hat{R}(\omega)$ of the kernel $R(\theta)$ defined via

$$R(\theta) = \int_0^\infty \frac{d\omega}{\pi} \cos(\omega \theta) \hat{R}(\omega),$$

vanishes at the origin. Suppose that $\hat{R}(0) = 0$, and that we can decompose $1/\hat{R}(\omega) = G_-(\omega)G_+(-\omega)$ where $G_\pm(\omega)$ are analytic in the upper/lower half planes with $G_+(\omega) = G_-(\omega)$. It can be shown that if $G_+(i\xi)$ has an expansion for small $\xi$ like

$$G_+(i\xi) = \frac{k}{\sqrt{\xi}} \exp(-a\xi \ln(\xi) \left(1 - b\xi + O(\xi^2)\right)) \tag{10}$$

then the ground-state energy density for $h \gg m$ takes the form (2) with

$$\kappa_0 = -k^2/4, \quad \kappa_1/\kappa_0 = a + 1/2, \quad \kappa_2/\kappa_0 = \ln(\sqrt{2\pi}/G_+(i)) - 1 + \ln k + a(\gamma_E - 1 + \ln 8) - b. \tag{11}$$

This matches the expansion in perturbation theory with a classical term in (1). If $\hat{R}(0) \neq 0$, on the other hand, then the asymptotic expansion proceeds slightly differently and the result is an expression of the form (2) with $\kappa_0 = \kappa_1 = 0$. This matches the perturbative result (1) in the case when there is no tree-level contribution.

At this point we should remark that in the regime $h \gg m$ considered above, the Fermi rapidity is large and so the ground-state will contain a large number of particles. It would seem therefore that the system is far from the dilute situation used in our naive derivation of the TBA equations. It is one of the miracles of the TBA that it seems, nevertheless, to be valid for many systems in the deep ultra-violet regime, although the exact reason for this is not understood.

The simple TBA analysis which we have presented also assumed that there is only one particle type that contributes to the new ground-state. In general many particles with different quantum numbers and masses can contribute and the TBA analysis then becomes much more complicated. If the scattering of the particles is purely elastic, the one-particle analysis can be generalized in an obvious way: there is a function $\epsilon_a(\theta)$ for each particle type with mass $m_a$, and the TBA system (8) becomes a matrix equation involving

$$K_{ab}(\theta) = \frac{1}{2\pi i} \frac{d}{d\theta} \ln S_{ab}(\theta),$$

where $S_{ab}(\theta)$ is the S-matrix element between particles $a$ and $b$. If the scattering is not elastic, there are still greater complications, with the net result being that the TBA system involves additional “magnon” degrees of freedom which behave like particles with zero mass. Nevertheless the analysis can sometimes be carried out successfully in this situation too, as we shall see later for a particular example.
3 SUMMARY OF RESULTS

We now summarize the results obtained for several families of integrable models. In each case we shall define the theory by a Lagrangian $L$ displaying a global symmetry group $G$ or $G \times G$, where $G = SU(n)$, $SO(n)$ or $Sp(n)$. To express the results compactly, we introduce the quantity $1/\Delta = n, n - 2$ or $2n + 2$ respectively (this is the dual Coxeter number for $SU(n)$ and $SO(n)$ but twice the dual Coxeter number for $Sp(n)$). The fields in the Lagrangian transform in the defining representation of the symmetry group unless we state otherwise. The mass-gap $m/\Lambda$ will be given for particles which also belong to the defining representation of the symmetry group. The renormalization scheme is $\overline{MS}$ except in one case.

(i) $O(n)$ sigma model \cite{5,4}:

$$L = \frac{1}{2g} \partial_\mu \phi_a \partial^\mu \phi_a$$

where $\phi_a$ is an $n$-component real scalar field obeying $\phi_a \phi_a = 1$. The TBA calculation confirms the S-matrix proposed in \cite{13}, correctly predicting $\beta_1 = 1/2\pi \Delta$ and $\beta_2 = 1/4\pi^2 \Delta$. The mass gap

$$m = \frac{(8/e)^\Delta}{\Gamma(1 + \Delta)} \Lambda_{\overline{MS}}$$

is consistent with the $1/n$ expansion \cite{26}.

(ii) $G \times G$ principal chiral model \cite{8,9}:

$$L = \frac{1}{g} \text{Tr} \left( \partial_\mu U \partial^\mu U^{-1} \right)$$

where $U$ is a $G$-valued field. The TBA calculation with the S-matrices proposed in \cite{16} correctly predicts $\beta_1 = 1/16\pi x \Delta$, $\beta_2 = \beta_1^2/2$ where $x$ is the Dynkin index of the defining representation of $G$ ($x = 1/2$ for $SU(n)$, $Sp(n)$; $x = 1$ for $SO(n)$). The mass gap is

$$m = 2^{(d\Delta + 1/2)} \frac{\sin(\pi \Delta)}{\sqrt{\pi e \Delta}} \Lambda_{\overline{MS}}$$

where $d$ is the dimension of the defining representation of $G$ ($d = n$ for $SU(n)$, $SO(n)$; $d = 2n$ for $Sp(n)$). The mass gap has been measured on the lattice for $G = SU(3)$ and the agreement with the theoretical value is quite accurate \cite{30}.

(iii) $O(n)$ Gross-Neveu model ($n > 4$) \cite{6}:

$$L = \frac{i}{2} \bar{\psi}_a \gamma^\mu \partial_\mu \psi_a + \frac{g}{8} \left( \bar{\psi}_a \psi_a \right)^2$$

where $\psi_a$ is an $n$-component Majorana spinor. The S-matrices proposed in \cite{13,14} are found to be consistent with the TBA/perturbation calculation which correctly predicts $\beta_1 = 1/2\pi \Delta$ and $\beta_2 = -1/4\pi^2 \Delta$. The mass gap is

$$m = \frac{(2e)^\Delta}{\Gamma(1 - \Delta)} \Lambda_{\overline{MS}}$$
which is consistent with the $1/n$ expansion \([27]\). Note that our coupling constant $g$ and mass-scale $\Lambda_{\text{MS}}$ differ slightly from those used in \([3, 27]\).

(iv) SU($n$) chiral Gross-Neveu model \([1]\):

$$\mathcal{L} = \frac{i}{2} \bar{\psi}_a \gamma^\mu \partial_\mu \psi_a + \frac{g}{8} \left\{ \left( \bar{\psi}_a \psi_a \right)^2 - \left( \bar{\psi}_a \gamma^5 \psi_a \right)^2 \right\}$$

where $\psi_a$ is an $n$-component complex spinor. The S-matrix for the massive sector conjectured in \([15]\) is consistent with the TBA/perturbation calculation which correctly predicts $\beta_1 = 1/\pi \Delta$ and $\beta_2 = -1/\pi^2 \Delta$. The mass gap

$$m = \frac{(e/4)^{\Delta / 2}}{\Gamma(1 - \Delta)} \Lambda_{\text{MS}}$$

agrees with the $1/n$ expansion \([1]\). See \([1]\) for full details of the renormalization scheme and note also our slightly different definitions of $g$ and $\Lambda$.

(v) $O(n)$ supersymmetric sigma model ($n > 4$) \([10]\):

$$\mathcal{L} = \frac{1}{2g} \left\{ \left( \partial_\mu \phi_a \right)^2 + i \bar{\psi}_a \gamma^\mu \partial_\mu \psi_a + \frac{1}{4} \left( \bar{\psi}_a \psi_a \right)^2 \right\}$$

where the fields $\phi_a$ and $\psi_a$ are an $n$-component real scalar and spinor satisfying the constraints $\phi_a \phi_a = 1$ and $\phi_a \psi_a = 0$. The model is invariant under $N = 1$ supersymmetry transformations mixing the bosons and fermions. The S-matrix conjectured in \([17]\) is consistent with the TBA/perturbation calculation which correctly predicts $\beta_1 = 1/2\pi \Delta$ and $\beta_2 = 0$. The mass gap

$$m = 2^{2\Delta} \frac{\sin(\pi \Delta)}{\pi \Delta} \Lambda_{\text{MS}}$$

is consistent with the $1/n$ expansion \([28]\).

(vi) SU($n$) or CP$^{n-1}$ supersymmetric sigma model \([14]\):

$$\mathcal{L} = \frac{1}{2g} \left\{ \left| \left( \partial_\mu - A_\mu \right) z_a \right|^2 + i \bar{\psi}_a \gamma^\mu \left( \partial_\mu - A_\mu \right) \psi_a + \frac{1}{4} \left( \bar{\psi}_a \psi_a \right)^2 - \left( \bar{\psi}_a \gamma^5 \psi_a \right)^2 - \left( \bar{\psi}_a \gamma^\mu \psi_a \right)^2 \right\}$$

where $A_\mu = \frac{1}{2} (z_a^* \partial_\mu z_a - z_a \partial_\mu z_a^*)$ and the fields $z_a$ and $\psi_a$ are an $n$-component complex scalar and Dirac spinor satisfying the constraints $z_a^* z_a = 1$ and $z_a^* \psi_a = 0$. The Lagrangian has a local U(1) invariance under which all complex fields transform by a phase and $A_\mu$ transforms as a gauge field. The model is also invariant under $N = 2$ supersymmetry. The S-matrix proposed in \([18]\) correctly predicts $\beta_1 = 1/\pi \Delta$ and $\beta_2 = 0$. The formula for the mass-gap is

$$m = \frac{\sin(\pi \Delta)}{\pi \Delta} \Lambda_{\text{MS}}$$

which is consistent with the $1/n$ expansion \([29]\) and with other non-perturbative approaches specific to $N = 2$ supersymmetric theories \([31]\).
4 DETAILED EXAMPLE: THE SUPERSYMMETRIC \( \mathbb{C}P^{n-1} \) MODEL

Most of the features of the general method we have outlined here are nicely illustrated by the last example, the supersymmetric \( \mathbb{C}P^{n-1} \) model \([32, 33]\), so we now discuss this case in more detail.

To introduce the S-matrix, we assume that in the quantum theory there exist states \( |a,i,\theta\rangle \) representing fundamental particles and states \( |\bar{a},i,\theta\rangle \) representing fundamental antiparticles. Here \( \theta \) is rapidity; \( i = 0, 1 \) distinguishes “bosons” and “fermions” (actually these particles carry fractional statistics \([18, 36]\)); and indices \( a \) and \( \bar{a} \) label the \( n \) and \( \bar{n} \) representations of \( SU(n) \). The fundamental anti-particles can be regarded as bound-states of the fundamental particles, or vice-versa—an example of “nuclear democracy”. There are additional bound states transforming in all the antisymmetric representations of \( SU(n) \).

The integrability of the model \([34]\) implies that the S-matrix factorizes and that all S-matrix elements can be deduced from the two-body ones; furthermore, the S-matrix for any desired set of particles can be obtained from the S-matrix for the fundamental particles. Thus the entire S-matrix is specified by the amplitude

\[
\langle c,k,\theta_2; d,l,\theta_1,\text{out} | a,i,\theta_1; b,j,\theta_2,\text{in} \rangle = S_{N=2}(\theta_1 - \theta_2)^{kl}_{ij} S_{\text{CGN}}(\theta_1 - \theta_2)^{cd}_{ab} \tag{12}
\]

which was first proposed by Köberle and Kurak \([15]\). Following \([35, 36]\), we have written this proposal in a factorized form in which \( S_{\text{CGN}} \) is the S-matrix for the chiral Gross-Neveu model \([15]\), which specifies the scattering of the \( SU(n) \) degrees of freedom, and \( S_{N=2} \) controls the \( N = 2 \) supersymmetric degrees of freedom, as described in \([36]\) (although we have chosen to include the physical strip pole in the CGN part of the S-matrix). Explicit expressions for these factors are given in \([11]\).

The conjectured S-matrix is minimal in the sense that it has the minimum number of poles and zeros on the physical strip (the region \( 0 \leq \text{Im}(\theta) \leq \pi \)) consistent with the requirements of symmetry, the existence of a bound-state and the axioms of S-matrix theory. But this still leaves open the possibility of adding CDD factors to the S-matrix \([19, 13]\): these spoil none of the axioms, they introduce no new poles on the physical strip and they passively respect the bootstrap equations. For our model the CDD ambiguities correspond to multiplying the S-matrix of the fundamental particles by factors of the form

\[
\frac{\sinh \left( \frac{\theta}{2} - \frac{i\pi}{2n} \alpha \right) \sinh \left( \frac{\theta}{2} - \frac{i\pi}{2n} (2 - \alpha) \right)}{\sinh \left( \frac{\theta}{2} + \frac{i\pi}{2n} \alpha \right) \sinh \left( \frac{\theta}{2} + \frac{i\pi}{2n} (2 - \alpha) \right)}, \tag{13}
\]

where \( 0 < \alpha < 2 \). One of the conclusions of our analysis will be that the minimal form is the true S-matrix of the theory, so that all CDD factors are ruled out.

We have emphasized that in following the general method for testing the equivalence between an S-matrix and a Lagrangian, there are at least two important points to be borne in mind when choosing the coupling to a conserved charge \( Q \). First, in order to simplify the analysis of the
TBA system we must choose $Q$ so that the new ground-state consists of a restricted number of particle types. In the models considered in [4, 5, 6, 7, 8, 9] it was possible to find a generator $Q$ such that the new ground-state contained a single particle type. In the present theory, however, we know that the lowest energy states must come in degenerate supersymmetric multiplets, since supersymmetry commutes with the SU($n$) invariance. We might be tempted to make the same choice for $Q$ as in the SU($n$) principle chiral model [8] and the chiral Gross-Neveu model [7]:

$$Q = \text{diag} \left( 1, -\frac{1}{n-1}, \ldots, -\frac{1}{n-1} \right),$$

(14)

for which there will be a single fundamental doublet $|1, j, \theta\rangle$ with the largest charge/mass ratio. But it turns out that this violates our second criterion regarding the choice of $Q$, which is that there should be a classical or tree-level term in the ground-state energy so that we can extract the mass-gap via a one-loop, rather than a higher-loop, calculation. We are therefore motivated to consider an alternative choice:

$$Q = \text{diag} \left( 1, -1, 0, \ldots, 0 \right).$$

(15)

This does indeed lead to the desired tree-level term, but the TBA analysis is complicated by the fact that there are now two fundamental doublets with the largest charge/mass ratio, namely $|1, j, \theta\rangle$ and $|2, j, \theta\rangle$. Despite this complication, the calculation proves tractable with this choice, as we shall describe below.

### 4.1 Perturbation theory calculation

The coupling of the theory to the charge (15) by a change in the Hamiltonian $H \rightarrow H - hQ$ can be achieved by making a corresponding replacement $\partial_0 \rightarrow \partial_0 + i h Q$ in the Lagrangian. Since we are interested in performing a one-loop calculation of the change in the ground-state energy density as a function of $h$, it is enough to expand the resulting Lagrangian to quadratic order in an independent set of fields, and we can drop all terms which are independent of $h$ to this order. By exploiting the local U(1) invariance of the Lagrangian, we can take $z_1$ to be real and we can solve the bosonic constraint $z_αz_α^* = 1$ by writing $z_1 = \{(1 - |π|^2)(\frac{1}{2} + φ)\}^{1/2}$ and $z_2 = e^{iθ}\{(1 - |π|^2)(\frac{1}{2} - φ)\}^{1/2}$ where $π = (z_3, \ldots, z_n)$ and $θ, φ$ are real. The fermionic degrees of freedom and the variable $θ$ decouple to quadratic order and we are left with the expression

$$\mathcal{L}_{1\text{-loop}} = \frac{1}{2g} \left\{ (\partial_μφ)^2 + |∂_μπ|^2 + h^2 - 4h^2φ^2 - h^2|π|^2 \right\}.$$  

We see that there is indeed a tree-level term, as desired.

Using standard dimensional regularization with the $\overline{\text{MS}}$-scheme gives a one-loop expression for the ground-state energy

$$δ\mathcal{E}(h) = -\frac{h^2}{2g} - \frac{h^2}{π} \ln 2 + \frac{h^2n}{4π} \left[ 1 - \ln \frac{h^2}{μ^2} \right].$$
It is important that when we substitute for the running coupling with the known values of the beta-function coefficients

\[ \beta_1 = n/\pi, \quad \beta_2 = 0, \]

the \(\mu\)-dependence must cancel (to leading order) since the quantity we are computing is renormalization group-invariant. From above, the values of the other dimensionless numbers appearing in (1) are, for this model,

\[ \alpha_0 = -1/2, \quad \alpha_1 = (n/4\pi) - (1/\pi) \ln 2. \]

The explicit expression for the ground-state energy is thus, to the required order,

\[ \delta \mathcal{E}(h) = -\frac{h^2 n}{2\pi} \left[ \ln \frac{h}{\Lambda_{\overline{MS}}} - \frac{1}{2} \frac{1}{n} + \frac{2}{n} \ln 2 \right]. \]

4.2 TBA calculation

We must now consider the TBA equations for the model and solve them in the limit \(h \gg m\). Following the hypothesis introduced earlier, we assume that only the multiplets \(|1, j, \theta\rangle\) and \(|\overline{2}, j, \theta\rangle\) contribute to the ground-state. Since the scattering of these multiplets is purely elastic, it is not necessary to perform a diagonalization in the space of SU\((n)\) quantum numbers (although this diagonalization can be done [23]). The remaining difficulty is that the S-matrix for these favoured states is still non-diagonal in the supersymmetric subspace. Fortunately, this problem can be solved—in fact it has been shown by Fendley and Intriligator [36] that it is equivalent to diagonalizing the transfer matrix of the six vertex model at the free fermion point.

The TBA equations involve energy densities \(\epsilon_1(\theta)\) and \(\epsilon_2(\theta)\) for the two supermultiplets and two magnon energy densities \(\xi_0(\theta)\) and \(\xi^\theta(\theta)\) which reflect the non-elastic nature of the scattering amongst the supersymmetric degrees of freedom (two magnons because we are dealing with \(N = 2\) supersymmetry). The resulting TBA equations are, at \(T = 0\),

\[ \epsilon_a(\theta) - \phi_{ab} \ast \epsilon_b^\theta(\theta) - \phi_{al} \ast \xi_l^\theta(\theta) = m \cosh \theta - h, \]
\[ \xi_l(\theta) - \phi_{al} \ast \epsilon_a^\theta(\theta) = 0, \]

where \(a = 1, 2\) and \(l = 0, 0\). In terms of these the ground-state energy density is

\[ \delta \mathcal{E}(h) = \frac{m}{2\pi} \int_{-\infty}^{\infty} d\theta \left[ \epsilon_1^\theta(\theta) + \epsilon_2^\theta(\theta) \right] \cosh \theta. \]

The kernels appearing in (19) are

\[ \phi_{ab}(\theta) = \frac{1}{2\pi i} \frac{d}{d\theta} \ln S_{\text{CGN}}(\theta)^{ba}_{ab}, \]
\[ \phi_{10}(\theta) = \phi_{20}(\theta) = \frac{1}{2\pi \cosh \theta - \cos(\pi/n)}, \]
\[ \phi_{20}(\theta) = \phi_{10}(\theta) = \frac{1}{2\pi \cosh \theta + \cos(\pi/n)}. \]
where $S_{\text{CGN}}(\theta)$ are the appropriate S-matrix elements of the chiral Gross-Neveu model.

To simplify (19) it is important to notice that $\phi_{al}(\theta)$ is a positive kernel, which implies that the magnon variables are given by $\xi^+_l(\theta) = 0$ and $\xi^-_l(\theta) = \phi_{al} * \epsilon^a_\theta(\theta)$. Furthermore, the solution does not distinguish between the values of the favoured SU($n$) quantum numbers and so we have $\epsilon_1(\theta) = \epsilon_2(\theta) \equiv \epsilon(\theta)$. The four equations in (19) then reduce to a single equation for $\epsilon(\theta)$ of the form (8) with kernel

$$R(\theta) = \delta(\theta) - \phi_{11}(\theta) - \phi_{12}(\theta) - [\phi_{10} + \phi_{10}] * [\phi_{10} + \phi_{10}](\theta),$$

and, from (20), each of the densities $\epsilon_1$ and $\epsilon_2$ contributes an amount to the ground-state energy given by (9). We have therefore succeeded in reducing the problem to the relatively simple case which we already know how to handle.

Following the subsequent steps outlined in section 2, we find the Fourier transform of the kernel:

$$\hat{R}(\omega) = \frac{\cosh((\frac{1}{2} - \frac{i}{n})\pi\omega) \sinh(\frac{1}{2}\pi\omega)}{\cosh^2(\frac{1}{2}\pi\omega)} \exp\left\{ \frac{1}{2} \ln n - \frac{1}{2}(1+i\omega) \ln(-i\omega) + i\omega \left[ \ln 2 + \frac{1}{2} + (\frac{1}{2} - \frac{1}{n}) \ln(\frac{1}{2} - \frac{1}{n}) + \frac{1}{n} \ln \frac{1}{n} \right] \right\}$$

This vanishes at the origin and we can decompose it as $1/(G_+^m(\omega)G_-(\omega))$ where $G_\pm(\omega)$ are analytic in the upper/lower half planes and $G_-(\omega) = G_+(-\omega)$. The unique solution is

$$G_+(\omega) = \frac{\Gamma(\frac{1}{2} - \frac{1}{2}i\omega)\Gamma(1 - \frac{1}{n}i\omega)}{\Gamma^2(\frac{1}{2} - \frac{1}{2}i\omega)} \exp\left\{ \frac{1}{2} \ln n - \frac{1}{2} (1+i\omega) \ln(-i\omega) + i\omega \left[ \ln 2 + \frac{1}{2} + (\frac{1}{2} - \frac{1}{n}) \ln(\frac{1}{2} - \frac{1}{n}) + \frac{1}{n} \ln \frac{1}{n} \right] \right\}$$

and we can indeed find an expansion of $G_+(i\xi)$ for small $\xi$ of the form (10) with

$$k = \sqrt{n/\pi}, \quad a = -1/2, \quad b = \frac{1}{2}(1-\gamma_E) - \frac{1}{n} \ln 4n + (\frac{1}{2} - \frac{1}{n}) \ln(\frac{1}{2} - \frac{1}{n}).$$

Substituting in (11) and adding the contributions for the densities $\epsilon_1$ and $\epsilon_2$ produces

$$\kappa_0 = -n/2\pi, \quad \kappa_1 = 0, \quad \kappa_2/\kappa_0 = \ln(n/\pi) + \ln(\sin(\pi/n)) + (2 \ln 2)/n - 1/2, \quad (21)$$

or explicitly, to the required order,

$$\delta\mathcal{E}(h) = -\frac{h^2n}{2\pi} \left[ \ln \frac{h}{m} - \frac{1}{2} + \frac{2}{n} \ln 2 + \ln\left(\frac{n}{\pi}\right) + \ln\sin\left(\frac{\pi}{n}\right) \right]. \quad (22)$$

4.3 Comparison of calculations

By substituting the values (16), (17) and (21) into the general relations (3) and (4)—or simply by comparing the explicit final results (18) and (22)—we see that the TBA calculation correctly reproduces the universal coefficients of the beta-function and that it predicts the value of the mass-gap for the supersymmetric $CP^{n-1}$ model to be $m/\Lambda_{\text{MS}} = (n/\pi) \sin(\pi/n)$, as claimed.
It is instructive to consider what would have happened if we had carried out our analysis using the other charge (14), rather than (15). In that case only the multiplet $|1, j, \theta\rangle$ appears in the ground-state and the resulting TBA equations are simpler in as much as they involve only this single doublet, rather than two doublets. The system can be reduced to a single integral equation in a similar way, but on doing so we find a different kernel:

$$R(\theta) = \delta(\theta) - \phi_{11}(\theta) - \phi_{10}\phi_{10}(\theta) - \phi_{1\bar{0}}\phi_{1\bar{0}}(\theta) .$$

The Fourier transform of this kernel does not vanish at the origin, and so, as explained previously, we would need to go beyond one-loop perturbation theory to carry out a non-trivial test of the S-matrix. This matches precisely the fact that with this different choice of charge the perturbative expansion of the ground-state energy density is also markedly different with no tree-level contribution.

It is also interesting to see for this particular example how the calculation resolves the problem of CDD ambiguities in the S-matrix. An additional CDD factor of the form (13) would alter the kernel appearing in the TBA equation from $\hat{R}(\omega)$ to

$$\hat{R}(\omega) - 2\frac{\cosh((\frac{1}{2} - \frac{1}{n})\pi\omega)\cosh(\frac{1}{n}(\alpha - 1)\pi\omega)}{\cosh(\frac{1}{2}\pi\omega)} .$$

But this expression fails to vanish at the origin and so the agreement with the perturbative result is destroyed.

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