Thermodynamics of Rotating Charged Black Branes in Third Order Lovelock Gravity and the Counterterm Method

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Abstract

We generalize the quasilocal definition of the stress energy tensor of Einstein gravity to the case of third order Lovelock gravity, by introducing the surface terms that make the action well-defined. We also introduce the boundary counterterm that removes the divergences of the action and the conserved quantities of the solutions of third order Lovelock gravity with zero curvature boundary at constant $t$ and $r$. Then, we compute the charged rotating solutions of this theory in $n + 1$ dimensions with a complete set of allowed rotation parameters. These charged rotating solutions present black hole solutions with two inner and outer event horizons, extreme black holes or naked singularities provided the parameters of the solutions are chosen suitable. We compute temperature, entropy, charge, electric potential, mass and angular momenta of the black hole solutions, and find that these quantities satisfy the first law of thermodynamics. We find a Smarr-type formula and perform a stability analysis by computing the heat capacity and the determinant of Hessian matrix of mass with respect to its thermodynamic variables in both the canonical and the grand-canonical ensembles, and show that the system is thermally stable. This is commensurate with the fact that there is no Hawking-Page phase transition for black objects with zero curvature horizon.

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I. INTRODUCTION

In four dimensions, the Einstein tensor is the only conserved symmetric tensor that depends on the metric and its derivatives up to second order. However for spacetimes possessing more than four dimensions, as assumed in both string theory and brane world cosmology, this is not the case. In string theory, extra dimensions are a theoretical necessity since superstring theory requires a ten-dimensional spacetime to be consistent from the quantum point of view, while in brane world cosmology matter and gauge interactions are localized on a 3-brane, embedded into a higher dimensional spacetime in which gravity propagates throughout the whole of spacetime. The most natural extension of general relativity in higher dimensional spacetimes with the assumption of Einstein – that the left hand side of the field equations is the most general symmetric conserved tensor containing no more than two derivatives of the metric – is Lovelock theory. Lovelock [1] found the most general symmetric conserved tensor satisfying this property. The resultant tensor is nonlinear in the Riemann tensor and differs from the Einstein tensor only if the spacetime has more than 4 dimensions. Since the Lovelock tensor contains metric derivatives no higher second order, the quantization of the linearized Lovelock theory is ghost-free [2]. The concepts of action and energy-momentum play central roles in gravity. However there is no good local notion of energy for a gravitating system. Quasilocal definitions of energy and conserved quantities for Einstein gravity [3, 4, 5] define a stress energy tensor on the boundary of some region within the spacetime through the use of the well-defined gravitational action of Einstein gravity with the surface term of Gibbons and Hawking [6]. Our first aim in this paper is to generalize the definition of the quasilocal stress energy tensor for computing the conserved quantities of a solution of third order Lovelock gravity with zero curvature boundary. The first step is to find the surface terms for the action of third order Lovelock gravity that make the action well-defined. These surface terms were introduced by Myers in terms of differential forms [7]. The explicit form of the surface terms for second order Lovelock gravity has been written in Ref. [8]. Here, we write down the tensorial form of the surface term for the third order Lovelock gravity, and then obtain the stress energy tensor via the quasilocal formalism. Of course, as in the case of Einstein gravity, the action and conserved quantities diverge when the boundary goes to infinity. We will also introduce a counterterm to deal with these divergences. This is quite straightforward for the cases we
consider in which the boundary is flat. This is because all curvature invariants are zero except for a constant, and so the only possible boundary counterterm is one proportional to the volume of the boundary regardless of the number of dimensions. The coefficient of this volume counterterm is the same for solutions with flat or curved boundary. The issue of determination of boundary counterterms with their coefficients for higher-order Lovelock theories is at this point an open question. Since the Lovelock Lagrangian appears in the low energy limit of string theory, there has in recent years been a renewed interest in Lovelock gravity. In particular, exact static spherically symmetric black hole solutions of the Gauss-Bonnet gravity (quadratic in the Riemann tensor) have been found in Ref. [9], and of the Maxwell-Gauss-Bonnet and Born-Infeld-Gauss-Bonnet models in Ref. [10]. The thermodynamics of the uncharged static spherically black hole solutions has been considered in Ref. [11], of solutions with nontrivial topology and asymptotically de Sitter in Ref. [12] and of charged solutions in Ref. [10, 13]. Very recently NUT charged black hole solutions of Gauss-Bonnet gravity and Gauss-Bonnet-Maxwell gravity were obtained [15]. All of these known solutions in Gauss-Bonnet gravity are static. Not long ago one of us introduced two new classes of rotating solutions of second order Lovelock gravity and investigated their thermodynamics [14], and made the first attempt for finding exact solutions in third order Lovelock gravity with the quartic terms [16]. Our second aim in this paper is to obtain rotating asymptotically anti de Sitter (AdS) black holes of third order Lovelock gravity and investigate their thermodynamics. Apart from their possible relevance to string theory, it is of general interest to explore black holes in generalized gravity theories in order to discover which properties are peculiar to Einstein’s gravity, and which are robust features of all generally covariant theories of gravity. The outline of our paper is as follows. We give a brief review of the field equations of third order Lovelock gravity and the counterterm method for calculating conserved quantities in Sec. III. In Sec. III we introduce the \((n+1)\)-dimensional solutions with a complete set of rotational parameters and investigate their properties. In Sec. IV we obtain mass, angular momentum, entropy, temperature, charge, and electric potential of the \((n+1)\)-dimensional black hole solutions and show that these quantities satisfy the first law of thermodynamics. We also perform a local stability analysis of the black holes in the canonical and grand canonical ensembles. We finish our paper with some concluding remarks.
II. FIELD EQUATIONS

The action of third order Lovelock gravity in the presence of electromagnetic field may be written as

\[ I_G = \frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-g} \left( -2\Lambda + R + \alpha_2 \mathcal{L}_2 + \alpha_3 \mathcal{L}_3 - F_{\mu\nu} F^{\mu\nu} \right) \]  

(1)

where \( \Lambda \) is the cosmological constant, \( \alpha_2 \) and \( \alpha_3 \) are Gauss-Bonnet and third order Lovelock coefficients, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is electromagnetic tensor field and \( A_\mu \) is the vector potential. The first term is the cosmological term, the second term, \( R \), is the Einstein term, the third term is the Gauss-Bonnet Lagrangian given as

\[ \mathcal{L}_2 = R_{\mu\nu\gamma\delta} R^{\mu\nu\gamma\delta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \]  

(2)

and the last term is the third order Lovelock term

\[ \mathcal{L}_3 = 2 R^{\mu\nu\sigma\kappa} R_{\sigma\kappa\rho\tau} R^{\rho\tau}_{\mu\nu} + 8 R^{\mu\nu}_{\sigma\rho} R^{\sigma\kappa}_{\nu\tau} R^{\rho\tau}_{\mu\kappa} + 24 R^{\mu\nu\sigma\kappa} R_{\sigma\kappa\mu\nu} R^\rho_{\mu} + 3 R^{\mu\nu\sigma\kappa} R_{\sigma\kappa\mu\nu} + 24 R^{\mu\nu\sigma\kappa} R_{\sigma\mu} R_{\kappa\nu} + 16 R^{\mu\nu}_{\sigma\nu} R^\sigma_{\mu} - 12 R R^{\mu\nu} R_{\mu\nu} + R^3 \]  

(3)

From a geometric point of view the combination of these terms in seven and eight dimensions is the most general Lagrangian that yields second order field equations, as in the four-dimensional case for which the Einstein-Hilbert action is the most general Lagrangian producing second order field equations, or the five- and six-dimensional cases, for which the Einstein-Gauss-Bonnet Lagrangian is the most general one fulfilling this criterion. Since the third Lovelock term in eq. (1) is an Euler density in six dimensions and has no contribution to the field equations in six or less dimensional spacetimes, we therefore consider \((n+1)\)-dimensional spacetimes with \(n \geq 6\). Varying the action with respect to the metric tensor \( g_{\mu\nu} \) and electromagnetic tensor field \( F_{\mu\nu} \) the equations of gravitation and electromagnetic fields are obtained as:

\[ G^{(1)}_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha_2 G^{(2)}_{\mu\nu} + \alpha_3 G^{(3)}_{\mu\nu} = T_{\mu\nu} \]  

(4)

\[ \nabla_\mu F^{\mu\nu} = 0 \]  

(5)

where \( T_{\mu\nu} = 2 F^{\rho}_{\mu} F_{\rho\nu} - \frac{1}{2} F_{\rho\sigma} F^{\rho\sigma} g_{\mu\nu} \) is the energy-momentum tensor of electromagnetic field, \( G^{(1)}_{\mu\nu} \) is the Einstein tensor, and \( G^{(2)}_{\mu\nu} \) and \( G^{(3)}_{\mu\nu} \) are the second and third order Lovelock tensors given as [17]:

\[ G^{(2)}_{\mu\nu} = 2 (R_{\mu\sigma\kappa\tau} R^\sigma_{\nu\kappa\tau} - 2 R^{\sigma\rho}_{\mu\nu\sigma} R^\tau_{\rho\kappa\tau} - 2 R^{\sigma}_{\mu\sigma} R^\tau_{\nu\kappa} + R R_{\mu\nu}) - \frac{1}{2} \mathcal{L}_2 g_{\mu\nu} \]  

(6)
The derivative of \( \delta g \) principle, since one encounters a total derivative that produces a surface integral involving the derivative of \( \delta g_{\mu\nu} \) normal to the boundary. These normal derivative terms do not vanish by themselves, but are canceled by the variation of the Gibbons-Hawking surface term to be

\[
I^b_G = \frac{1}{8\pi} \int_{\partial M} d^n x \sqrt{-\gamma} K
\]

The main difference between higher derivative gravity and Einstein gravity is that the surface term that renders the variational principle well-behaved is much more complicated. However, the surface terms that make the variational principle well-defined are known for the case of Gauss-Bonnet gravity to be \( I_b^{(1)} + I_b^{(2)} \), where \( I_b^{(2)} \) is

\[
I_b^{(2)} = \frac{1}{8\pi} \int_{\partial M} d^n x \sqrt{-\gamma} \left\{ 2\alpha_2 \left[ J - 2\hat{G}_a^{(1)} K_{\alpha\beta} \right] \right\}
\]

and where \( \gamma_{\mu\nu} \) is induced metric on the boundary, \( K \) is trace of extrinsic curvature of boundary, \( \hat{G}_a^{(1)} \) is the \( n \)-dimensional Einstein tensor of the metric \( \gamma_{ab} \) and \( J \) is the trace of

\[
J_{ab} = \frac{1}{3}(2KK_{ac}K^c_b + K_{cd}K^{cd}K_{ab} - 2K_{ac}K^{cd}K_{db} - K^2 K_{ab})
\]

For the case of third order Lovelock gravity, the surface term that makes the variational principle well defined is \( I_b = I_b^{(1)} + I_b^{(2)} + I_b^{(3)} \), where \( I_b^{(3)} \) is

\[
I_b^{(3)} = \frac{1}{8\pi} \int_{\partial M} d^n x \sqrt{-\gamma} \left\{ 3\alpha_3 (P - 2\hat{G}_a^{(2)} K^{ab} - 12\hat{R}_{a[b}J^{a]} + 2\hat{R}J - 4K^a_{\alpha[a}K^{\alpha b} - 8\hat{R}_{a[b}K^{\alpha c}K^b_c K^{cd}] \right\}
\]

In eq. \( \hat{G}_a^{(2)} \) is the second order Lovelock tensor for the boundary metric \( \gamma_{ab} \), and \( P \) is the trace of

\[
P_{ab} = \frac{1}{5} \left\{ [K^4 - 6K^2 K^{cd}K_{cd} + 8KK_{cd}K^{cd}K_{cd} - 6K_{cd}K^{de}K_{ef}K^{fc} + 3(K_{cd}K^{cd})^2]K_{ab} - (4K^3 - 12K_{cd}K^{cd} + 8K_{de}K^{ef}K^{fd})K_{ac}K^c_b - 24KK_{ac}K^{cd}K_{de}K^e_b + (12K^2 - 12K_{ef}K^{ef})K_{ac}K^{cd}K_{db} + 24KK_{ac}K^{cd}K_{de}K^{ef}K_{bf} \right\}
\]
In general $I_C^b + I_b^{(1)} + I_b^{(2)} + I_b^{(3)}$ is divergent when evaluated on solutions, as is the Hamiltonian and other associated conserved quantities [3, 4, 5]. One way of eliminating these divergences is through the use of background subtraction [3], in which the boundary surface is embedded in another (background) spacetime, and all quasilocal quantities are computed with respect to this background, incorporated into the theory by adding to the action the extrinsic curvature of the embedded surface. Such a procedure causes the resulting physical quantities to depend on the choice of reference background; furthermore, it is not possible in general to embed the boundary surface into a background spacetime. For asymptotically AdS solutions, one can instead deal with these divergences via the counterterm method inspired by AdS/CFT correspondence [18]. This conjecture, which relates the low energy limit of string theory in asymptotically anti de-Sitter spacetime and the quantum field theory on its boundary, has attracted a great deal of attention in recent years. The equivalence between the two formulations means that, at least in principle, one can obtain complete information on one side of the duality by performing computation on the other side. A dictionary translating between different quantities in the bulk gravity theory and their counterparts on the boundary has emerged, including the partition functions of both theories. In the present context this conjecture furnishes a means for calculating the action and conserved quantities intrinsically without reliance on any reference spacetime [19, 20, 21] by adding additional terms on the boundary that are curvature invariants of the induced metric. Although there may exist a very large number of possible invariants one could add in a given dimension, only a finite number of them are nonvanishing as the boundary is taken to infinity. Its many applications include computations of conserved quantities for black holes with rotation, NUT charge, various topologies, rotating black strings with zero curvature horizons and rotating higher genus black branes [22]. Although the counterterm method applies for the case of a specially infinite boundary, it was also employed for the computation of the conserved and thermodynamic quantities in the case of a finite boundary [23]. Extensions to de Sitter spacetime and asymptotically flat spacetimes have also been proposed [24]. All of the work mentioned in the previous paragraph was limited to Einstein gravity. Here we apply the counterterm method to the case of the solutions of the field equations of third order Lovelock gravity. At any given dimension there are only finitely many counterterms that one can write down that do not vanish at infinity. This does not depend upon what the bulk theory is – i.e. whether or not it is Einstein, Gauss-Bonnet, 3rd order Lovelock,
etc. Indeed, for asymptotically (A)dS solutions, the boundary counterterms that cancel divergences in Einstein Gravity should also cancel divergences in 2nd and 3rd order Lovelock gravity. The coefficients will be different and depend on $\Lambda$ and Lovelock coefficients as we will see this for the volume term in the flat boundary case below. Of course these coefficients should reduce to those in Einstein gravity as one may expect. Unfortunately we do not have a rotating solution to either Gauss-Bonnet or 3rd-order Lovelock gravity that does not have a flat boundary at infinity. Consequently we restrict our considerations to counterterms for the flat-boundary case, i.e. $\hat{R}_{abcd}(\gamma) = 0$, for which there exists only one boundary counterterm

$$I_{ct} = \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^n x \sqrt{-\gamma} n - \frac{1}{L}, \quad (13)$$

where $L$ is a scale length factor that depends on $l$, $\alpha_2$ and $\alpha_3$, that must reduce to $l$ as $\alpha_2$ and $\alpha_3$ go to zero. Having the total finite action $I = I_G + I_b^{(1)} + I_b^{(2)} + I_b^{(3)}$, one can use the quasilocal definition to construct a divergence free stress-energy tensor. For the case of manifolds with zero curvature boundary the finite stress energy tensor is

$$T^{ab} = \frac{1}{8\pi} \left\{ (K^{ab} - K \gamma^{ab}) + 2\alpha_2(3J^{ab} - J \gamma^{ab}) \\
+ 3\alpha_3(5P^{ab} - P \gamma^{ab}) + \frac{n - 1}{L} \gamma^{ab} \right\} . \quad (14)$$

The first three terms in eq. (14) result from the variation of the surface action with respect to $\gamma^{ab}$, and the last term is the counterterm that is the variation of $I_{ct}$ with respect to $\gamma^{ab}$. To compute the conserved charges of the spacetime, we choose a spacelike surface $B$ in $\partial\mathcal{M}$ with metric $\sigma_{ij}$, and write the boundary metric in ADM form:

$$\gamma_{ab} dx^a dx^a = -N^2 dt^2 + \sigma_{ij} \left( d\varphi^i + V^i dt \right) \left( d\varphi^j + V^j dt \right), \quad (15)$$

where the coordinates $\varphi^i$ are the angular variables parameterizing the hypersurface of constant $r$ around the origin, and $N$ and $V^i$ are the lapse and shift functions respectively. When there is a Killing vector field $\xi$ on the boundary, then the quasilocal conserved quantities associated with the stress tensors of eq. (14) can be written as

$$Q(\xi) = \int_B d^{n-1} \varphi \sigma n^a \xi^b, \quad (16)$$

where $\sigma$ is the determinant of the metric $\sigma_{ij}$, and $n^a$ is the timelike unit normal vector to the boundary $B$. For boundaries with timelike ($\xi = \partial/\partial t$) and rotational ($\varsigma = \partial/\partial \varphi$) Killing
vector fields, one obtains the quasilocal mass and angular momentum

\[ M = \int_B d^{n-1} \varphi \sqrt{\sigma} T_{ab} n^a \xi^b, \tag{17} \]

\[ J = \int_B d^{n-1} \varphi \sqrt{\sigma} T_{ab} n^a \varsigma^b, \tag{18} \]

provided the surface \( B \) contains the orbits of \( \varsigma \). These quantities are, respectively, the conserved mass and angular momentum of the system enclosed by the boundary \( B \). Note that they will both depend on the location of the boundary \( B \) in the spacetime, although each is independent of the particular choice of foliation \( B \) within the surface \( \partial \mathcal{M} \).

III. \((n + 1)\)-DIMENSIONAL ROTATING SOLUTIONS

As stated before, the third order Lovelock term in eq. (11) is an Euler density in six dimensions and has no contribution to the field equations in spacetimes of dimension six or less. Taking \( n \geq 6 \), we obtain the \((n + 1)\)-dimensional solutions of third order Lovelock gravity with nonvanishing electromagnetic field with \( k \) rotation parameters and investigate their properties. The rotation group in \( n + 1 \) dimensions is \( SO(n) \) and therefore the number of independent rotation parameters is \([n + 1]/2\] , where \([x]\) is the integer part of \( x \). The metric of an \((n + 1)\)-dimensional asymptotically AdS rotating solution with \( k \leq [(n + 1)/2] \) rotation parameters whose constant \((t, r)\) hypersurface has zero curvature may be written as

\[
\begin{align*}
    ds^2 &= -f(r) \left( \Xi dt - \sum_{i=1}^{k} a_i d\phi_i \right)^2 + \frac{r^2}{l^2} \sum_{i=1}^{k} \left( a_i dt - \Xi l^2 d\phi_i \right)^2 \\
    &\quad + \frac{dr^2}{f(r)} - \frac{r^2}{l^2} \sum_{i<j}^{k} (a_i d\phi_j - a_j d\phi_i)^2 + r^2 dX^2,
\end{align*}
\tag{19} \]

where \( \Xi = \sqrt{1 + \sum_i^{k} a_i^2/l^2} \), the angular coordinates are in the range \( 0 \leq \phi_i < 2\pi \) and \( dX^2 \) is the Euclidean metric on the \((n - k - 1)\)-dimensional submanifold with volume \( \Sigma_{n-k-1} \). Using eq. (11), one can show that the vector potential can be written as

\[ A_\mu = \frac{q}{(n-2)r^{n-2}} \left( \Xi \delta_\mu^0 \right) - a_\mu \delta_\mu^i \tag{no sum on i}, \]

where \( q \) is an arbitrary real constant which is related to the charge of the solution. To find the function \( f(r) \), one may use any components of eq. (11). The simplest equation is the \( rr \)
component of these equations which can be written as

\[
180(n-1)\alpha_3 r f^2 - 6(n-1)\alpha_2 f r^3 + \frac{n-1}{2} r^5 \right] f' + \Lambda r^6 \\
+360(n-1)\alpha_3 f^3 - 12(n-1)\alpha_2 f^2 r^2 + (n-1)(n-2)f = -q^2 r^{8-2n}
\]

(21)

where prime denotes the derivative with respect to \(r\). We present the full solution to this equation in the appendix. Here, for simplicity, we consider the solutions of eq. (21) for a restricted version of the \(\alpha_i\)'s given as

\[
\alpha_3 = \frac{\alpha^2}{72(n-2)}, \quad \alpha_2 = \frac{\alpha}{(n-2)(n-3)}
\]

(22)

Equation (21) with condition (22) has one real and two complex solutions that are the complex conjugate of each other. The real solution of eq. (21) with condition (22) is

\[
f(r) = \frac{r^2}{\alpha} \left\{ 1 - \left( 1 + \frac{6\Lambda \alpha}{n(n-1)} + \frac{3\alpha m}{r^n} - \frac{6\alpha q^2}{(n-1)(n-2)r^{2n-2}} \right)^{1/3} \right\}
\]

(23)

where \(m\) is the mass parameter. Although the other components of the field equation (4) are more complicated, one can check that the metric (19) satisfies all the components eq. (4) provided above \(f(r)\) is given by (23). Unlike the solution in Gauss-Bonnet gravity, which has two branches, the solution (23) has only one branch. Indeed, eq. (21) with the above \(\alpha\)'s has the real solution (23) and two complex solutions that are complex conjugates of each other. This feature is the same as the asymptotically AdS solution of Ref. [26], which has a unique anti de Sitter vacuum. These solutions are asymptotically AdS or dS for negative or positive values of \(\Lambda\) respectively. In this paper we are interested in the case of asymptotically AdS solutions, and therefore we put \(\Lambda = -n(n-1)/2l^2\). One can show that the Kretschmann scalar \(R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa}\) diverges at \(r = 0\), and therefore there is a curvature singularity located at \(r = 0\). Seeking possible black hole solutions, we turn to looking for the existence of horizons. As in the case of rotating black hole solutions of Einstein gravity, the above metric given by eqs. (19) and (46) has both Killing and event horizons. The Killing horizon is a null surface whose null generators are tangent to a Killing field. The proof that a stationary black hole event horizon must be a Killing horizon in the four-dimensional Einstein gravity [27] cannot obviously be generalized to higher order gravity. However the result is true for all known static solutions. Although our solution is not static, the Killing vector

\[
\chi = \partial_t + \sum_{i=1}^{k} \Omega_i \partial_{\phi_i},
\]

(24)
is the null generator of the event horizon, where \( k \) denotes the number of rotation parameters. The event horizon is defined by the solution of \( g^{rr} = f(r) = 0 \). For the case of uncharged solutions, there exists only one event horizon located at

\[
r_+ = (ml^2)^{1/n}
\]  
(25)

This feature is different from the case of uncharged spherically symmetric solutions of third order Lovelock gravity with curved horizon \[16\], for which one may have uncharged black holes with two horizons, extreme ones, or a naked singularity. As we demonstrate in the appendix, the general uncharged solution does not have two horizons. The charged solution presents a black hole solution with two inner and outer horizons, provided the mass parameter \( m \) is greater than, \( m_{\text{ext}} \), an extreme black hole for \( m = m_{\text{ext}} \) and a naked singularity otherwise, where \( m_{\text{ext}} \) is

\[
m_{\text{ext}} = \frac{2(n-1)}{(n-2)l^2} \left( \frac{2q^2l^2}{n(n-1)} \right)^{n/2(n-1)}
\]  
(26)

The general charged solution also has only two horizons, as shown in the appendix.

IV. THERMODYNAMICS OF BLACK HOLES

One can obtain the temperature and angular momentum of the event horizon by analytic continuation of the metric. Setting \( t \to i\tau \) and \( a_i \to ia_i \) yields the Euclidean section of (19), whose regularity at \( r = r_+ \) requires that we should identify \( \tau \sim \tau + \beta_+ \) and \( \phi_i \sim \phi_i + \beta_+ \Omega_i \), where \( \beta_+ \) and \( \Omega_i \)'s are the inverse Hawking temperature and the angular velocities of the outer event horizon. One obtains

\[
\beta_+^{-1} = T_+ = \frac{f'(r_+)}{4\pi \Xi} = \frac{n(n-1) - 2l^2q^2r_+^{2(n-1)}}{4\pi (n-1)\Xi l^2} r_+,
\]  
(27)

\[
\Omega_i = \frac{a_i}{\Xi l^2}.
\]  
(28)

Next, we calculate the electric charge of the solutions. To determine the electric field we should consider the projections of the electromagnetic field tensor on special hypersurfaces. The normal to such hypersurfaces is

\[
u^0 = \frac{1}{N}, \quad v^r = 0, \quad v^i = \frac{V^i}{N},
\]  
(29)
and the electric field is $E^\mu = g^{\mu\nu}F_{\mu\nu}$. Then the electric charge per unit volume $V_{n-1}$ can be found by calculating the flux of the electric field at infinity, yielding

$$ Q = \frac{\Xi}{4\pi} q $$

(30)

The electric potential $\Phi$, measured at infinity with respect to the horizon, is defined by

$$ \Phi = A_\mu \chi^\mu \bigg|_{r=\infty} - A_\mu \chi^\mu \bigg|_{r=r_+} $$

(31)

where $\chi$ is the null generator of the horizon given by eq. (24). We find

$$ \Phi = \frac{q}{(n-2)\Xi r^{n-2}} $$

(32)

Conserved quantities associated with the spacetime described by (19) can be obtained via the counterterm method. Using eqs. (17) and (18), the mass and angular momentum will be finite provided

$$ L = \frac{15l^2\sqrt{\alpha(1-\lambda)}}{5l^2 + 9\alpha - l^2\lambda^2 - 4l^2\lambda} $$

$$ \lambda = \left(1 - \frac{3\alpha}{l^2}\right)^{1/3} $$

where we note that $L$ reduces to $l$ as $\alpha$ goes to zero. The mass and angular momentum per unit volume $V_{n-1}$ can then be obtained through the use of eqs. (17) and (18). We find

$$ M = \frac{1}{16\pi} [n\Xi^2 - 1] m $$

(33)

$$ J_i = \frac{1}{16\pi} n\Xi m a_i $$

(34)

Black hole entropy typically satisfies the so-called area law, which states that the entropy of a black hole equals one-quarter of the area of its horizon $28$. This near-universal law applies to all kinds of black holes and black strings in Einstein gravity $29$. However in higher derivative gravity the area law is not satisfied in general $30$. It is known in Lovelock gravity that $31,32$

$$ S = \frac{1}{4} \sum_{k=1}^{[(d-1)/2]} k\alpha_k \int d^{n-1} x \sqrt{\tilde{g}} \tilde{L}_{k-1} $$

(35)

where the integration is done on the $(n-1)$-dimensional spacelike hypersurface of Killing horizon, $\tilde{g}_{\mu\nu}$ is the induced metric on it, $\tilde{g}$ is the determinant of $\tilde{g}_{\mu\nu}$ and $\tilde{L}_k$ is the $k$th order Lovelock Lagrangian of $\tilde{g}_{\mu\nu}$. For the topological class of black holes we are considering, the
horizon curvature is zero. Consequently the area law holds. Denoting the volume of the hypersurface boundary at constant $t$ and $r$ by $V_{n-1} = (2\pi)^k \Sigma_{n-k-1}$, we obtain

$$S = \frac{\Xi}{4} r^{n-1}$$

(36)

for the entropy per unit volume $V_{n-1}$. The entropy can also be obtained through the use of Gibbs-Duhem relation

$$S = \beta (M - \Gamma_i C_i) - I$$

(37)

where $I$ is the finite total action evaluated on the classical solution, and $C_i$ and $\Gamma_i$ are the conserved charges and their associate chemical potentials respectively. For simplicity, we consider the uncharged solutions, for which $C_i = J_i$ and $\Gamma_i = \Omega_i$. Using eqs. (1), (8)-(11) and (13), the finite total action per unit volume $V_{n-1}$ can be calculated as

$$I = -\frac{\beta_+}{16\pi l^2} r^Z$$

(38)

Now using Gibbs-Duhem relation (37) and eqs. (33), (34) and (38), one may confirm that the entropy per unit volume obeys the area law of eq. (36).

A. Energy as a function of entropy, angular momenta and charge

We first obtain the mass as a function of the extensive quantities $S$, $J$ and $Q$. Using the expression for the entropy, the mass, the angular momenta, and the charge given in eqs. (27), (30), (33) and (34), and the fact that $f(r_+) = 0$, one can obtain a Smarr-type formula as

$$M(S,J,Q) = \frac{(nZ - 1)}{nl\sqrt{Z(Z-1)}} J,$$

(39)

where $J^2 = |J|^2 = \sum J_i^2$ and $Z = \Xi^2$ is the positive real root of the following equation

$$(Z-1)^{(n-1)} - \frac{Z}{16S^2} \left\{ \frac{4\pi (n-1)(n-2)l S J}{n[(n-1)(n-2)S^2 + 2\pi^2 Q^2 l^2]} \right\}^{(2n-2)} = 0.$$  

(40)

One may then regard the parameters $S$, $J$ and $Q$ as a complete set of extensive parameters for the mass $M(S,J,Q)$ and define the intensive parameters conjugate to $S$, $J_i$ and $Q$. These quantities are the temperature, the angular velocities, and the electric potential

$$T = \left( \frac{\partial M}{\partial S} \right)_{Q,J}, \quad \Omega_i = \left( \frac{\partial M}{\partial J_i} \right)_{S,Q}, \quad \Phi = \left( \frac{\partial M}{\partial Q} \right)_{S,J}.$$  

(41)
It is a matter of straightforward calculation to show that the intensive quantities calculated by eqs. (39)-(41) are the same as those found earlier in this section. Thus, the thermodynamic quantities calculated in this section satisfy the first law of thermodynamics,

\[ dM = TdS + \sum_{i=1}^{k} \Omega_i dJ_i + \Phi dQ. \] (42)

B. Stability in the canonical and the grand-canonical ensemble

The stability of a thermodynamic system with respect to the small variations of the thermodynamic coordinates, is usually performed by analyzing the behavior of the entropy \( S(M,Q,J) \) near equilibrium. The local stability in any ensemble requires that \( S(M,Q,J) \) be a concave function of its extensive variables or that its Legendre transformation is a convex function of the intensive variables. The stability can also be studied by the behavior of the energy \( M(S,Q,J) \) which should be a convex function of its extensive variable. Thus, the local stability can in principle be carried out by finding the determinant of the Hessian matrix of \( M(S,Q,J) \) with respect to its extensive variables \( X_i, H_{X_i X_j} = \left[ \partial^2 M/\partial X_i \partial X_j \right] \). In our case the entropy \( S \) is a function of the mass, the angular momenta, and the charge. The number of thermodynamic variables depends on the ensemble that is used.

In the canonical ensemble, the charge and the angular momenta are fixed parameters, and therefore the positivity of the heat capacity \( C_{J,Q} = T(\partial S/\partial T)_{J,Q} \) is sufficient to ensure local stability. The heat capacity \( C_{Q,J} \) at constant charge and angular momenta is

\[ C_{Q,J} = \frac{r_+^{(n-1)}}{4\Upsilon} [(n-2)\Xi^2 + 1][(n-1)(n-2)r_+^{2(n-1)} + 2q^2l^2]\{n(n-1)r_+^{2(n-1)} - 2q^2l^2\} \] (43)

where \( \Upsilon \) is

\[
\Upsilon = 4q^4l^4[(3n-6)\Xi^2 - n + 3] - 4(n-1)q^2l^2r_+^{(2n-2)}[(3n-6)\Xi^2 - n^2 + 3] \\
+ n(n-1)^2(n-2)r_+^{(4n-4)}[(n+2)\Xi^2 - (n+1)]
\] (44)

Figure 1 shows the behavior of the heat capacity as a function of the charge parameter. We see that \( C_{Q,J} \) is positive in various dimensions and goes to zero as \( q \) approaches its extreme value. Thus, the \((n+1)\)-dimensional asymptotically AdS charged rotating black brane is locally stable in the canonical ensemble. In the grand-canonical ensemble, after
FIG. 1: $C_{J,Q}$ versus $q$ for $l = 1$, $r_+ = 0.8$, $n = 6$ (bold-line), $n = 7$ (solid-line), and $n = 8$ (dotted-line).

After some algebraic manipulation, we obtain

$$\left| H_{S,Q,J}^M \right| = \frac{128\pi}{n((n-2)\Xi^2 + 1)^2 \Xi^6 r_+^{3n-4}} \left[ \frac{n(n-1)r_+^{2(n-1)} + 2(2n-3)r^2 q^2}{(n-1)(n-2)r_+^{2(n-1)} + 2l^2 q^2} \right]. \quad (45)$$

As one can see from eq. (45), $\left| H_{S,Q,J}^M \right|$ is positive over all phase space. Hence the $(n + 1)$-dimensional asymptotically AdS charged rotating black brane in third order Lovelock gravity is locally stable in the grand-canonical ensemble.

V. CONCLUDING REMARKS

In this paper, first, we introduced the surface terms for the third order Lovelock gravity which make the action well-defined. This is achieved by generalizing the Gibbons-Hawking surface term for Einstein gravity or generalizing the surface terms of Gauss-Bonnet gravity. We generalized the stress energy momentum tensor of Brown and York in Einstein gravity for the third order Lovelock gravity for spacetimes with zero curvature at the boundary. As in the case of Einstein gravity, $I_G$, $I_b^{(1)}$, $I_b^{(2)}$ and $I_b^{(3)}$ of eqs. (1), (8), (9) and (11) are divergent when evaluated on the solutions, as is the Hamiltonian and other associated conserved quantities. We also introduced a counterterm dependent only on the boundary volume, which removed the divergences of the action and conserved quantities of this solution of third
order Lovelock gravity. We also found a new class of rotating solutions, whose hypersurfaces of constant $t$ and $r$ have zero curvature, in third order Lovelock gravity in the presence of cosmological constant and electromagnetic field. These solutions are asymptotically AdS or dS for $\Lambda < 0$ or $\Lambda > 0$ respectively. We obtained solutions for special values of $\alpha_2$ and $\alpha_3$ given in eq. (22) with negative cosmological constant. In the absence of an electromagnetic field, these solutions present black branes with one event horizon. The charged solutions may be interpreted as black brane solutions with two inner and outer event horizons for $m > m_{\text{ext}}$, extreme black holes for $m = m_{\text{ext}}$ or naked singularity for $m < m_{\text{ext}}$, where $m_{\text{ext}}$ is given in eq. (26). We found that the Killing vectors are the null generators of the event horizon, and therefore the event horizon is a Killing horizon for the stationary solution of the third order Lovelock gravity explored in this paper. We computed physical properties of the brane such as the temperature, the angular velocity, the entropy, the electric charge and potential. Finally, we obtained a Smarr-type formula for the mass of the black brane solution as a function of the entropy, the charge and the angular momenta of the black brane and investigated the first law of thermodynamics. We found that the conserved and thermodynamics quantities satisfy the first law of thermodynamics. We also studied the phase behavior of the $(n + 1)$-dimensional charged rotating black branes in third order Lovelock gravity and showed that there is no Hawking-Page phase transition in spite of the charge and angular momenta of the branes. Indeed, we calculated the heat capacity and the determinant of the Hessian matrix of the mass with respect to $S, J$ and $Q$ of the black branes and found that they are positive for all the phase space, which means that the brane is stable for all the allowed values of the metric parameters discussed in Sec. IV. This phase behavior is commensurate with the fact that there is no Hawking-Page transition for a black object whose horizon is diffeomorphic to $\mathbb{R}^p$ and therefore the system is always in the high temperature phase [34].

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VI. APPENDIX

The general solution in $n + 1$ dimensions is

$$f(r) = \frac{b_2 r^2}{b_3 \alpha} \left\{ 1 - \left( \sqrt{\gamma + k^2(r)} + k(r) \right)^{1/3} + b_2 \left( \sqrt{\gamma + k^2(r)} - k(r) \right)^{1/3} \right\}$$  \hspace{1cm} (46)

where

$$\begin{align*}
\alpha_2 &= b_2 \frac{\alpha}{12} \\
\alpha_3 &= b_3 \frac{\alpha^2}{72} \\
\gamma &= \left( \frac{b_3 - b_2}{b_2^3} \right)^3 \\
\lambda &= \frac{1}{2} + \frac{3}{2} \gamma^{1/3} + \frac{3 \alpha \Lambda b_3^2}{n(n - 1)}
\end{align*}$$  \hspace{1cm} (47)

and

$$k(r) = \lambda + 3 \alpha b_3^2 \left( \frac{m}{2r^n} - \frac{q^2}{(n - 1)(n - 2)r^{2(n - 1)}} \right)$$  \hspace{1cm} (48)

The above $f(r)$ reduces to the solution (23) when $b_3 = b_2 = 1$. At large $r$ the function $k$ approaches a constant, and the spacetime is asymptotically de Sitter or anti de Sitter depending on the overall sign of

$$\Lambda_{\text{eff}} = \frac{b_2}{\alpha b_3} \left\{ 1 - \left( \lambda + \sqrt{\gamma + \lambda^2} \right)^{1/3} + b_2^2 \left( -\lambda + \sqrt{\gamma + \lambda^2} \right)^{1/3} \right\}$$

where we have assumed $\alpha > 0$ without loss of generality. First, we investigate the conditions of the reality of $f(r)$. In order to have real $f(r)$ the expression $\gamma + k^2(r)$ should be positive. This occurs for $\gamma > 0$, but for negative $\gamma$, this holds if $k^2(r) > |\gamma|$. The analysis of this case proceeds as follows. 1) If $\lambda > 0$, then $k(r)$ is zero some where and the condition $k^2(r) > |\gamma|$ violated, and therefore the function $f(r)$ is complex near the root(s) of $k(r) = 0$. 2) For $\lambda < 0$ the condition $k^2(r) > |\gamma|$ holds provided

$$m \leq \frac{4(n - 1)}{(n - 2)} \left( \frac{|\lambda| - \sqrt{|\gamma|}}{3 \alpha b_3^2} \right)^{(n-2)/2(n-1)} \left( \frac{q^2}{n(n - 1)} \right)^{n/2(n - 1)}$$

Note that in this case $|\lambda|$ should be larger than $\sqrt{|\gamma|}$. Seeking possible black hole solutions, we turn to looking for the existence of horizons. The roots of the metric function $f(r)$ are located at

$$k(r) = k_0 \equiv \frac{\left( 1 + \sqrt{1 + 4b_2^2 \gamma^3} \right)^6 - 64 \gamma}{16 \left( 1 + \sqrt{1 + 4b_2^2 \gamma^3} \right)^3}$$  \hspace{1cm} (49)

which reduces to the value of $1/2$ when $\gamma = 0$ as in eq. (22). Note that $k_0$ is real if $\gamma^3 > -1/(4b_2^2)$. Thus, there is no black hole solution for $\gamma^3 < -1/(4b_2^2)$. For the case of uncharged solutions, there exists only one event horizon located at

$$r_+ = \left( \frac{m}{2\sigma} \right)^{1/n}, \quad \sigma = \frac{k_0 - \lambda}{3\alpha b_3^2}$$  \hspace{1cm} (50)
provided $\sigma$ is positive. If $\sigma$ is negative or zero then there will be no roots (corresponding to a naked singularity). Note that $\sigma = 1/2l^2$ when $\gamma = 0$ and therefore the above equation reduces to eq. (25). Again this feature is different from the case of uncharged spherically symmetric solutions of third order Lovelock gravity with curved horizon [16], for which one may have uncharged black holes with two horizons, extreme ones, or a naked singularity. The charged solution presents a black hole solution with two inner and outer horizons, provided the mass parameter $m$ is greater than $m_{\text{ext}}$, an extreme black hole for $m = m_{\text{ext}}$ and a naked singularity otherwise, where $m_{\text{ext}}$ is

$$m_{\text{ext}} = \frac{4(n - 1)\sigma}{(n - 2)} \left( \frac{q^2}{n(n - 1)\sigma} \right)^{n/2(n-1)} \quad (51)$$

Again, one may note that for $\gamma = 0$ ($\sigma = 1/2l^2$) eq. (51) reduce to $m_{\text{ext}}$ given in eq. (26).

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