Wheeler–DeWitt quantization for point-particles

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Abstract
We present the Hamiltonian formulation of a relativistic point-particle coupled to Einstein gravity and its canonical quantization à la Wheeler–DeWitt. In the resulting quantum theory, the wave functional is a function of the particle coordinates and the 3-metric. It satisfies a particular Hamiltonian and diffeomorphism constraint, together with a Klein–Gordon-type equation. As usual in the Wheeler–DeWitt theory, the wave function is time-independent. This is also reflected in the Klein–Gordon-type equation, where the time derivative is absent. Before considering gravity, we consider the coupling of a particle with electromagnetism, which is treated similarly, but simpler.

Keywords Quantum gravity · Wheeler–DeWitt quantization · Klein Gordon equation

1 Introduction
Canonical quantization of Einstein gravity leads to the Wheeler–DeWitt theory where the state is described by a wave functional $\Psi(h_{ij})$, which is a function of a 3-metric $h_{ij}(x)$ [1]. A common way to introduce matter is through a scalar field, so that the wave functional also is a function of a scalar field $\phi(x)$, i.e., $\Psi = \Psi(h_{ij}, \phi)$. A remarkable feature is that the wave functional turns out to be time-independent. The wave equations are merely constraints on the wave functional: the Hamiltonian and the diffeomorphism constraint. The time-independence of $\Psi$ leads to the question of how time evolution can be accounted for. There are various attempts to answer this question, but there is far from a consensus on the right approach [1–4].

In this paper, we consider the coupling of gravity to point-particles instead of a scalar field. This allows for the study of quantum gravity for a fixed number of particles without having to consider quantum field theory. This system has been studied in the context of the problem of time by Rovelli [5,6] and Pavšič [7,8].
Classically, the history of a point-particle is described by its world-line $X^\mu(\lambda)$, which depends on an arbitrary parameter $\lambda$, while the space-time metric $g_{\mu\nu}(x)$ is a function of space-time $x = (t, x)$. The particle and field are hence parameterized differently, with $X^0(\lambda) = t$. To pass to the Hamiltonian picture, a common temporal parameter for the field and the particle needs to be chosen. This is already familiar from the case of the free particle in Minkowski space-time. In that case, the disadvantage of choosing the parameter $t$ is that one needs to take a square root to express the Hamiltonian in terms of the momenta, resulting in the Hamiltonian $H = \pm \sqrt{p^2 + m^2}$. Choosing the parameter $\lambda$ instead leads to a zero Hamiltonian, but with the constraint $p_\mu p^\mu + m^2 = 0$. Quantization of the latter leads to the Klein–Gordon equation, while the former leads to the positive or negative frequency part of the Klein–Gordon equation.

The situation in the case of gravity is similar. Rovelli [5,6] considered the parameter $t$ and ended up taking a square root. Pavšič seems to use $t$ for the field, while using $\lambda$ for the particle. This does not amount to a standard Hamiltonian formulation and hence makes this quantization of the theory questionable. In this paper, the Hamiltonian formulation is presented using $\lambda$ as the common temporal parameter, together with the corresponding quantum theory. This avoids needing to take the square root and as such generalizes Rovelli’s treatment. The wave function is again time-independent; It does not depend on $X^0$, just as in Rovelli’s treatment, but contrary to the findings of Pavšič. So $X^0$ does not appear as a possible candidate for the evolution parameter (contrary to for example the treatment of dust [9]). It will appear however that Pavšič’s theory (when suitably interpreted) can be derived from ours in a mixed Schrödinger-Heisenberg picture.

To start with, the treatment of point-particles coupled to an electromagnetic field is considered, in Minkowski space-time, which shares many features with the gravitational case.

2 Electrodynamics

To formulate the quantum theory of a point-particle interacting with an electromagnetic field, we start from the classical theory, pass to the Hamiltonian picture, and apply the usual canonical quantization methods [10–14].

Consider a classical point-particle with worldline $X^\mu(\lambda)$ interacting with an electromagnetic field with vector potential $A^\mu = (A_0, A)$. Derivatives with respect to $\lambda$ and $t$ will be denoted by respectively by primes and dots throughout, e.g. $X'^\mu = dX^\mu / d\lambda$. It is also assumed throughout that $X'^0 > 0$. The action is

$$S = S_M + S_F,$$

with

$$S_M = -m \int d\lambda \sqrt{X'^\mu(\lambda)X'^\mu(\lambda)} - e \int d\lambda X'^\mu(\lambda)A_\mu(X(\lambda))$$

1 Units are assumed such that $\hbar = c = 1.$
\[ S_F = -\frac{1}{4} \int d^4 x F_{\mu \nu}(x) F^{\mu \nu}(x), \] (3)

where \( F^{\mu \nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \) is the electromagnetic field tensor. The temporal parameter in the matter action and the field action are different. In the matter action, the parameter \( \lambda \) acts as temporal parameter, while \( t \) does it for the electromagnetic field. In order to pass to a Hamiltonian formulation, a common temporal coordinate is needed. We will first use \( t \) and then \( \lambda \).

To write the matter action in terms of \( t \), the identity \( 1 = \int dt \delta(t - X^0(\lambda)) \) is inserted in the action and an integration over \( \lambda \) is performed. Defining \( \tilde{X}^\mu(t) = X^\mu(\lambda(t)) \), where \( X^0(\lambda(t)) \equiv 1 \), leads to

\[ S_M = -m \int dt \sqrt{1 - \dot{\tilde{X}}^2(t)} + e \int dt \tilde{X}(t) \cdot A(t, \tilde{X}(t)) - e \int dt A_0(t, \tilde{X}(t)). \] (4)

Writing \( S = \int dt L(\tilde{X}, \dot{\tilde{X}}, A, \dot{A}) \), with \( L \) the Lagrangian, the canonical momenta are

\[ \tilde{P} = \frac{\partial L}{\partial \dot{\tilde{X}}} = m \frac{\tilde{X}}{\sqrt{1 - \dot{\tilde{X}}^2}} + eA(\tilde{X}), \] (5)

\[ \Pi_0(x) = \frac{\delta L}{\delta A_0(x)} = 0, \quad \Pi(x) = \frac{\delta L}{\delta A(x)} = \dot{A}(x) + \nabla A_0(x). \] (6)

The canonical Hamiltonian is

\[ H = \tilde{P} \cdot \dot{\tilde{X}} + \int d^3 x \left( \pi_0 \dot{A}_0 + \Pi \cdot \dot{A} \right) - L = \frac{m}{\sqrt{1 - \dot{\tilde{X}}^2}} + eA_0(\tilde{X}) + H_F, \] (7)

where \( H_F \) is the usual Hamiltonian for the free electromagnetic field:

\[ H_F = \int d^3 x \left[ \frac{1}{2} \Pi^2 + \frac{1}{2} (\nabla \times A)^2 + A_0 \nabla \cdot \Pi \right]. \] (8)

As is well known, there are two constraints: the primary constraint

\[ \Pi_0 = 0, \] (9)

and the secondary constraint (which is the Gauss law)

\[ \nabla \cdot \Pi + e\delta(x + \tilde{X}) = 0. \] (10)
The matter part of the Hamiltonian (7) is not yet expressed in terms of the momenta. Using
\[
\frac{m^2}{1 - \tilde{X}(t)^2} = m^2 + (\tilde{\mathbf{P}} - e\mathbf{A}(\tilde{\mathbf{X}}))^2,
\]
we obtain
\[
H = \pm \sqrt{m^2 + (\tilde{\mathbf{P}} - e\mathbf{A}(\tilde{\mathbf{X}}))^2 + eA_0(\tilde{\mathbf{X}}) + H_F},
\]
which involves the choice of a square root.

Quantization in the Schrödinger picture (and dropping the tildes) leads to the following equations for \(\Psi_1(X, A, t)\):
\[
i\partial_t \Psi = \pm \sqrt{m^2 - (\nabla - ie\mathbf{A}(X))^2}\Psi + eA_0(X)\Psi + \hat{H}_F\Psi,
\]
\[
\frac{\delta \Psi}{\delta A_0} = 0, \quad i\nabla \cdot \frac{\delta \Psi}{\delta \mathbf{A}(x)} - e\delta(x - X)\Psi = 0,
\]
where
\[
\hat{H}_F = \frac{1}{2} \int d^3x \left( -\frac{\delta^2}{\delta \mathbf{A}(x)^2} + [\nabla \times \mathbf{A}(x)]^2 \right).
\]
The equations (2) are the constraints (9) and (10) that are imposed as operator constraints. The square root could be eliminated by considering the square of (2) to obtain a Klein–Gordon-like equation. If \(\lambda\) rather than \(t\) is taken as a common temporal parameter then this is indeed the equation that will be obtained, as we will now show.

To use \(\lambda\) as the temporal parameter, \(1 = \int d\lambda \delta(\lambda - \lambda(t))\) is inserted in the field action. Defining \(\tilde{A}(\lambda, x) = A(X^0(\lambda), x)\), so that \(\tilde{A}' = \dot{A}X^0\), the field action can be written as
\[
S_F = \int dt L_F(A, \dot{A}) = \int d\lambda L^*_F(\tilde{A}, \tilde{A}'),
\]
with
\[
L^*_F(\tilde{A}, \tilde{A}') = X^0L_F\left(\tilde{A}, \tilde{A}' / X^0\right).
\]
With \(S = \int d\lambda L^*(X, X', \tilde{A}, \tilde{A}')\), the conjugate momenta are
\[
\tilde{\Pi}_\mu(x) = \frac{\delta L^*}{\delta \dot{A}'(x)} = \frac{\delta L}{\delta A'(x)} = \Pi_\mu(x),
\]
\[ P_\mu = \frac{\partial L^*}{\partial (\dot{X}'^\mu)} = -m \frac{X'_\mu}{\sqrt{\dot{X}'^\mu \dot{X}'^\nu}} - e \tilde{A}_\mu(X) - \delta^0_\mu H_F(\tilde{A}, \tilde{\Pi}). \]  

(19)

So the expressions for the momenta for the field are just the same as before, cf. (6). The particle momentum \( P_0 \) gets a contribution from field Lagrangian which is just minus the field Hamiltonian (8).

The canonical Hamiltonian is

\[ H^* = P_\mu X'^\mu + \int d^3x \tilde{\Pi}_\mu(x) \tilde{A}'^\mu(x) - L^* = 0 \]  

(20)

and is zero because of the reparameterization invariance of the action. There are two primary constraints: \( \chi_1 = \Pi_0 = 0 \) (as before) and

\[ \chi_2 = \left[ P_\mu + e \tilde{A}_\mu(X) + \delta^0_\mu H_F(\tilde{A}, \tilde{\Pi}) \right] \left[ P_\mu + e \tilde{A}_\mu(X) + \delta^0_\mu H_F(\tilde{A}, \tilde{\Pi}) \right] - m^2 = 0. \]  

(21)

There is a secondary constraint which follows from the requirement that the Poisson bracket of the two primary constraints with the total Hamiltonian

\[ H^*_T = H^* + \int d^3x \lambda_1(x) \chi_1(x) + \lambda_2 \chi_2 \]  

(22)

vanishes, which results in \([\chi_1(x), \chi_2]_P = 0\) or

\[ \frac{\delta H_F(\tilde{A}, \tilde{\Pi})}{\delta A_0(x)} + e \delta(x - X) = 0, \]  

(23)

which amounts to the Gauss constraint (10).

Quantization (again dropping the tildes) leads to the following equation for \( \Psi(X, A)\):

\[ \left( \partial_\mu + ie A_\mu(X) + i\delta^0_\mu \tilde{H}_F \right) \left( \partial_\mu + ie A^\mu(X) + i\delta^0_\mu \tilde{H}_F \right) \Psi + m^2 \Psi = 0, \]  

(24)

with \( \tilde{H}_F \) as before in (15), together with the constraints (2). This Klein–Gordon-type equation is just the square of (2), provided \( X^0 \) is identified with \( t \).

So far we have dealt with just a single particle. How to extend the theory to many particles? The theory with the square root Hamiltonian (2) is directly generalized to many particles (2), with now \( \psi = \psi(X_1, \ldots, X_n, A, t)\):

\[ i\partial_t \Psi = \sum_k \left[ \sqrt{m^2 - (\nabla_k - ie A_\mu(X_k))^2 + e A_0(X_k)} \right] \Psi + \tilde{H}_F \Psi, \]  

(25)

\[ \frac{\delta \Psi}{\delta A_0} = 0, \quad i \nabla \cdot \frac{\delta \Psi}{\delta A(x)} - e \sum_k \delta(x - X_k) \Psi = 0. \]  

(26)
We have chosen the positive square root Hamiltonian for each particle. The Klein–Gordon-type equation (24) is not that straightforwardly generalizable to many particles because it is not of the Schrödinger form like (2). One option is to run through the quantization procedure again for many particles, but this seems rather complicated. Another option is to recast the Klein–Gordon-type equation (24) in a Schrödinger form, which could be done using the Kemmer formulation [15] (which actually concerns a Dirac-like equation for spin-0), and which at least in the case of external field is directly extendable to many particles [16]. Yet another option is to use the multi-time picture where the wave function depends on the time-component of each of the particles [17]. We will not pursue this further here.

3 Gravity

The analysis of gravity proceeds completely analogously as that of electromagnetism. The action for a classical point-particle coupled to gravity is

\[ S = S_M + S_G, \]  

(27)

with

\[ S_M = -m \int d\lambda \sqrt{g_{\mu\nu}(X(\lambda))X'^{\mu}(\lambda)X'^{\nu}(\lambda)} \]  

(28)

and

\[ S_G = -\frac{1}{\kappa} \int d^4x \sqrt{-g} R, \]  

(29)

with \( \kappa = 16\pi G \), is the Einstein–Hilbert action.

Before considering the Hamiltonian picture in terms of the temporal parameter \( \lambda \), we recall the usual Hamiltonian formulation for the Einstein–Hilbert action in terms of \( t \) (following the conventions of [18]). It is supposed that space-time can be foliated in terms of space-like hypersurfaces such that the space-time manifold is diffeomorphic to \( \mathbb{R} \times \Sigma \), with \( \Sigma \) a 3-surface. Coordinates \( x^\mu = (t, x) \) are chosen such that the time coordinate \( t \) labels the leaves of the foliation and \( x \) are the coordinates on \( \Sigma \). In terms of these coordinates, the metric and its inverse are written as

\[ g_{\mu\nu} = \begin{pmatrix} N^2 - N_i N^i & -N_i \\ -N_i & -h_{ij} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} \frac{1}{N^2} & -\frac{N^i}{N^2} \\ -\frac{N^i}{N^2} & \frac{N^i N^j}{N^2} - h^{ij} \end{pmatrix}, \]  

(30)

where \( N \) and \( N^i \) are respectively the lapse and the shift vector, and \( h_{ij} \) is the Riemannian metric on \( \Sigma \), with \( h \) its determinant. Spatial indices of \( h_{ij} \) and \( N^i \) and the corresponding momenta are raised and lowered with this spatial metric.
With $L_G$ the gravitational Lagrangian, the canonically conjugate momenta are

$$ \pi^{ij} = \frac{\delta L_G}{\delta \dot{h}_{ij}} = -\frac{1}{\kappa} \sqrt{h} (K^{ij} - K h^{ij}) $$

which is a tensor density of weight $-1$, with

$$ K_{ij} = \frac{1}{2N} (D_i N_j + D_j N_i - \dot{h}_{ij}), \quad K = K_{ij} h^{ij}, $$

the extrinsic curvature, and

$$ \pi = \frac{\delta L_G}{\delta N} = 0, \quad \pi_i = \frac{\delta L_G}{\delta N^i} = 0. $$

The Hamiltonian is

$$ H_G = \int_{\Sigma} d^3 x \left( N \mathcal{H} + N^i \mathcal{H}_i \right), $$

with

$$ \mathcal{H} = \kappa G_{ijkl} \pi^{ij} \pi^{kl} + \mathcal{V}, \quad \mathcal{H}_i = -2 h_{ik} D_j \pi^{jk}, $$

where $G_{ijkl} = (h_{ij} h_{jk} + h_{ik} h_{jk} - h_{ij} h_{kl})/2\sqrt{h}$ is the DeWitt metric, $D_i$ is the covariant derivative corresponding to $h_{ij}$, and $\mathcal{V} = -\sqrt{h} R^{(3)}/\kappa$ is the gravitational potential density. Apart from the primary constraints (33), there are also the secondary constraints

$$ \mathcal{H} = 0, \quad \mathcal{H}_i = 0. $$

In order to use the temporal variable $\lambda$, we introduce $\tilde{g}_{\mu\nu}(\lambda, X) = g_{\mu\nu}(X^0(\lambda), X)$ and similarly for other variables. The total action is

$$ S = \int d\lambda L^*(X, X', \tilde{g}, \tilde{g'}) . $$

The momenta for the metric are the same as before, i.e.,

$$ \tilde{\pi}^{ij} = \frac{\delta L^*}{\delta \dot{h}'_{ij}} = \frac{\delta L_G}{\delta \dot{h}_{ij}} = \pi^{ij}, \quad \tilde{\pi} = \frac{\delta L^*}{\delta \dot{N}} = 0, \quad \tilde{\pi}_i = \frac{\delta L^*}{\delta \dot{N}^i} = 0, $$

and the momentum for the particle becomes (indices of $X^\mu$ and $P_\mu$ are raised and lowered with the metric $\tilde{g}_{\mu\nu}(X)$)

$$ P_\mu = \frac{\partial L^*}{\partial X'^\mu} = -m \frac{X'_\mu}{\sqrt{X'^\nu X'_\nu}} - \delta_\mu^0 \tilde{H}_G, $$

where $\tilde{H}_G$ is the Hamiltonian for the gravitational field.
where

\[ \tilde{H}_G = H_G(\tilde{N}, \tilde{N}^i, \tilde{g}_{ij}, \tilde{\pi}^{ij}). \]  (40)

The canonical Hamiltonian is zero, i.e.,

\[ H^* = P_\mu X'^\mu + \int_\Sigma d^3x \tilde{\pi}^{ij}(x) \tilde{g}'_{ij}(x) - L^* = 0. \]  (41)

There are three primary constraints:

\[ \tilde{\pi} = 0, \quad \tilde{\pi}_i = 0, \]  (42)

\[ \chi = \tilde{g}^{\mu\nu}(X) \left[ P_\mu + \delta_\mu^0 \tilde{H}_G \right] \left[ P_\nu + \delta_\nu^0 \tilde{H}_G \right] - m^2 = 0. \]  (43)

There are two secondary constraints corresponding to \( \delta \chi / \delta \tilde{N} = 0 \) and \( \delta \chi / \delta \tilde{N}_i = 0 \), resulting in

\[ \tilde{\mathcal{H}}(x) = \delta(x - X) \left( \tilde{N}(X) p^0 + \frac{1}{\tilde{N}(X)} \tilde{H}_G \right) \]  (44)

and

\[ \tilde{\mathcal{H}}_i(x) = \delta(x - X) p_i, \]  (45)

where (as in (40)) the tilde on \( \tilde{\mathcal{H}} \) and \( \tilde{\mathcal{H}}_i \) denotes the functions (35) evaluated for fields with tildes. There are no further constraints.

By multiplying (44) with \( \tilde{N}(x) \) and (45) with \( \tilde{N}_i(x) \), integrating over all space, and using \( p^0 = (p_0 - \tilde{N}_i(X) p_i) / \tilde{N}(X)^2 \), it follows that

\[ p_0 = 0. \]  (46)

Hence, another interesting consequence is that

\[ \tilde{N}(x) \tilde{\mathcal{H}}(x) + \tilde{N}_i(x) \tilde{\mathcal{H}}_i(x) = \delta(x - X) \tilde{H}_G. \]  (47)

Summarizing, the Hamiltonian dynamics is completely determined by the constraints (42)–(45). Using (46), they can be simplified to (dropping the tildes):

\[ \pi = 0, \quad \pi_i = 0, \]  (48)

\[ \left[ H_G - N_i(X) p_i \right]^2 - N(X)^2 \left[ h^{ij}(X) p_i p_j + m^2 \right] = 0, \]  (49)

\[ \mathcal{H}(x) = \delta(x - X) \frac{1}{N(X)} \left[ -N_i(X) p_i + H_G \right], \]  (50)

\[ \mathcal{H}_i(x) = \delta(x - X) p_i. \]  (51)
Quantization in the Schrödinger picture leads to the following equations for $\Psi(X, h_{ij})$:

\[ \left[ iN^i(X) \nabla_i + \hat{H}_G \right]^2 \Psi - N \left( \nabla^2 - m^2 \right) \Psi = 0, \quad (52) \]

\[ \hat{\mathcal{H}}(x) \Psi = \delta(x - X) \frac{1}{N} \left[ iN^i(X) \nabla_i + \hat{H}_G \right] \Psi, \quad (53) \]

\[ \hat{\mathcal{H}}_i(x) \Psi = -i \delta(x - X) \nabla_i \Psi, \quad (54) \]

where $\nabla_i$ is the covariant derivative with respect to the metric $h_{ij}(X)$, $\nabla_i \psi = \partial_i \psi$, $\nabla^2 = \nabla_i \nabla^i$ is the Laplacian, and

\[ \hat{H}_G = \int_{\Sigma} d^3x \left( N \hat{\mathcal{H}} + N^i \hat{\mathcal{H}}_i \right), \quad (55) \]

\[ \hat{\mathcal{H}} = -\kappa G_{ijkl} \delta_{hijkl} + V(h, \phi), \quad \hat{\mathcal{H}}_i = 2i h_{ik} D_j \frac{\delta}{\delta h_{jk}}. \quad (56) \]

The wave function $\Psi$ does not depend on $N$, $N^i$, and $X^0$, because of the operator constraints following from (46) and (48). The latter means that the wave functional does not depend on time as is familiar in Wheeler–DeWitt quantization.

The wave functional does not depend on $N$ and $N^i$, but they still appear in the wave equations. Choosing $N = 1$ and $N^i = 0$, results in

\[ \left( \hat{H}_G^2 + \nabla^2 - m^2 \right) \Psi = 0, \quad (57) \]

\[ \hat{\mathcal{H}}(x) \Psi = \delta(x - X) \hat{H}_G \Psi, \quad (58) \]

\[ \hat{\mathcal{H}}_i(x) \Psi = -i \delta(x - X) \nabla_i \Psi. \quad (59) \]

Applying the quantization recipe using $t$ as temporal coordinate leads to the following quantum theory [5,6]:

\[ \hat{\mathcal{H}}(x) \Psi = \pm \delta(x - X) \sqrt{-\nabla^2 + m^2} \Psi, \quad \hat{\mathcal{H}}_i(x) \Psi = -i \delta(x - X) \nabla_i \Psi, \quad (60) \]

with $\Psi(X, h_{ij})$. This theory also follows from (48)–(51) if the square root is taken in (49). As in the case of electromagnetism, the extension to many particles is straightforward, leading to

\[ \hat{\mathcal{H}}(x) \Psi = \sum_k \delta(x - X_k) \sqrt{-\nabla_k^2 + m^2} \Psi, \quad \hat{\mathcal{H}}_i(x) \Psi = -i \sum_k \delta(x - X_k) \nabla_{ki} \Psi, \quad (61) \]

with $k$ the particle label, $\Psi(X_1, \ldots, X_n, h_{ij})$. The extension of the Klein–Gordon form (57)–(59) to many particles requires a bit more effort (though apparently less than in the case of electromagnetism due to absence of the time derivative). The natural generalization seems to be

\[ \text{We have chosen the Laplace-Beltrami operator ordering for the particle but not for gravity [19].} \]
The operators $\hat{O}_k$ can be determined almost completely by self-consistency. Namely, for configurations $(X_1, \ldots, X_N)$ with all the $X_k$ different, we can integrate (63) over a test function $f(x)$ such that $f(X_k) = 1$ and $f(X_l) = 0$ for $l \neq k$, to obtain $\hat{O}_k \Psi = \int d^3x f(x) \hat{H}(x) \Psi$. If not all the $X_k$ are different, the action of the $\hat{O}_k$ could be defined similarly by requiring symmetry. For example, for a configuration with $X_k = X_l$, we can require that $\hat{O}_k \Psi = \hat{O}_l \Psi$. This generalization is consistent with (61), in the sense that (61) follows by taking particular square roots in (62).

The Wheeler–DeWitt quantization of the relativistic particle was considered before by Pavšič [7]. However, in the quantization procedure no common temporal parameter was used. In the definition of the particle momentum the parameter $\lambda$ was used, while in the definition of the field momenta the parameter $t$ was used. As such the quantization deviates from the usual recipe and the resulting quantum theory does not agree with the one presented here. However, the equations obtained by Pavšič follow from ours when interpreted in a partial Heisenberg picture (and as such is reminiscent of the Dirac–Fock–Podolsky formulation of a particle interacting with an electromagnetic field [20]). That is, defining

$$\Phi(X, h_{ij}) = e^{i\hat{H}_G X^0} \Psi(X, h_{ij}), \quad h_{ij}(X) = e^{i\hat{H}_G X^0} h_{ij}(X) e^{-i\hat{H}_G X^0},$$

$$\hat{\mathcal{H}}(x, X^0) = e^{i\hat{H}_G X^0} \hat{\mathcal{H}}(x) e^{-i\hat{H}_G X^0}, \quad \hat{\mathcal{H}}_i(x, X^0) = e^{i\hat{H}_G X^0} \hat{\mathcal{H}}_i(x) e^{-i\hat{H}_G X^0},$$

the Eqs. (57)–(59) reduce to

$$\left( \partial_0^2 - \vec{\nabla}^2 + m^2 \right) \Phi = 0,$$

$$\hat{\mathcal{H}}(x, X^0) \Phi = -i\delta(x - X) \partial_0 \Phi,$$

$$\hat{\mathcal{H}}_i(x, X^0) \Phi = -i\delta(x - X) \vec{\nabla}_i \Phi,$$

where $\vec{\nabla}$ and $\vec{\nabla}^2$ concern the covariant derivative with respect to the metric $h_{ij}(X)$. (Pavšič also chooses a different operator ordering, but that difference is not essential.) For Pavšič the appearance of the time derivative in (68) is interesting the light of the problem of time. However, the proper quantum theory is time-independent.

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