Analytical derivation of DC SQUID response

I I Soloviev$^{1,2,3}$, N V Klenov$^{1,3,4,5}$, A E Schegolev$^4$, S V Bakurskiy$^{1,3}$ and M Yu Kupriyanov$^{1,3,6}$

$^1$Lomonosov Moscow State University Skobeltsyn Institute of Nuclear Physics, 119991, Moscow, Russia
$^2$Lukin Scientific Research Institute of Physical Problems, Zelenograd, 124460, Moscow, Russia
$^3$Moscow Institute of Physics and Technology, State University, Dolgoprudny, Moscow region, Russia
$^4$Physics Department, Moscow State University, 119991, Moscow, Russia
$^5$N. L. Dukhov All-Russia Research Institute of Automatics, 127055, Moscow, Russia
$^6$Solid State Physics Department, Kazan Federal University, 420008, Kazan, Russia

E-mail: isol@phys.msu.ru

Received 19 May 2016, revised 16 June 2016
Accepted for publication 20 June 2016
Published 27 July 2016

Abstract
We consider voltage and current response formation in DC superconducting quantum interference device (SQUID) with overdamped Josephson junctions in resistive and superconducting state in the context of a resistively shunted junction (RSJ) model. For simplicity we neglect the junction capacitance and the noise effect. Explicit expressions for the responses in resistive state were obtained for a SQUID which is symmetrical with respect to bias current injection point. Normalized SQUID inductance $I = 2eI_cL/\hbar$ (where $I_c$ is the critical current of Josephson junction, $L$ is the SQUID inductance, $e$ is the electron charge and $\hbar$ is the Planck constant) was assumed to be within the range $1 \leq l \leq 7$, subsequently expanded up to $l \approx 7$ using two fitting parameters. SQUID current response in the superconducting state was considered for arbitrary value of the inductance. The impact of small technological spread of parameters relevant to low-temperature superconductor (LTS) technology was studied, using a generalization of the developed analytical approach, for the case of a small difference of critical currents and shunt resistances of the Josephson junctions, and inequality of SQUID inductive shoulders for both resistive and superconducting states. Comparison with numerical calculation results shows that developed analytical expressions can be used in practical LTS SQUIDs and SQUID-based circuits design, e.g. large serial SQIF, drastically decreasing the time of simulation.

Keywords: DC SQUID, voltage response, current response, SQIF

(Some figures may appear in colour only in the online journal)

1. Introduction

The superconducting quantum interference device (SQUID) is a basic component of superconductor electronics having numerous applications [1–3]. The DC SQUID, being basically a magnetic flux-to-voltage transformer is used e.g. in highly sensitive magnetometers [4–8], amplifiers [9–11], readout circuits [12] and antennas [13, 14]. These devices are routinely designed for fabrication process utilizing low-temperature superconductors (LTS) and tunnel Josephson junctions, which have become a workhorse of modern superconducting electronics.

The high accuracy of modern LTS fabrication technology allows for the construction of advanced SQUID-based structures with unconventional flux-to-voltage transformation. One example of such structures is superconductor quantum array (SQA) [15] with highly linear voltage response. SQA can be based on superconducting quantum interference filters (SQIFs) resulting from SQUID arrays with unequal loops [16, 17], or bi-SQUID cells [18]. It was argued [19] that SQA can be a basis for active electrically small wideband superconductor antennae. Such antennae could provide significant advances to modern superconducting broadband radio-frequency receiving systems [20]. A variety of structures, like those mentioned above, were proposed and extensively studied both theoretically and experimentally in recent years [13–15, 18, 19, 21–29].
Qualitative understanding of SQUID or SQUID-based structure response [16–18, 29] can be based on an analytical approach, assuming the SQUID inductance to be approximately equal to zero [30, 31]. However, for quantitative estimation of designed LTS circuit characteristics, accounting for a certain real value of the inductance is inevitable.

The main effort in the development of time-averaged analytical dependencies for practical SQUID parameters was made during study of high-temperature superconductor (HTS) SQUIDs [32–37]. High noise level $\Gamma > 0.1$ ($\Gamma = 2\pi k_B T/|I_c\Phi_0|$, where $k_B$ is the Boltzmann constant, $T$ is the temperature $I_c$ is the critical current of Josephson junction, $\Phi_0 = h/2e$ is the flux quantum, $h$ is the Planck constant, $e$ is the electron charge) and high value of SQUID inductance $L > 100 \mu F$ are typical in this case, that results in $\Gamma l \gtrsim 1$ ($l = 2\pi LI_c/\Phi_0$). Developed analytical approaches are valid for $\Gamma l \gtrsim 1$ and based on solution of two-dimensional Fokker–Planck equation [32, 35]. Obtained expression for the voltage-flux function $\tau(\phi)$ of symmetrical SQUID (where $\tau$ is the average voltage across the SQUID, and $\phi = \pi\Phi/\Phi_0$, $\Phi$ is the external magnetic flux) in the frame of these approaches is a simple harmonic function $\tau = a\cos \phi + b$ (where $a$, $b$ are constants), which is not intended for accurate description of the response shape. These methods are used mostly for estimation of the voltage response amplitude and corresponding maximum value of the transfer function. Analysis of the presented results done in [35] shows that these approaches cannot be applied for LTS SQUIDs.

Perturbation analysis developed in the limit of small SQUID inductance $l$ was shown to be well suited for study of some aspects of SQUID dynamics and characteristics like thermal escape problem [38], Shapiro steps [39], persistent current and magnetic susceptibility [40]. However, the study of time-averaged response for practical inductance values still requires numerical calculations [41].

It is interesting to note that an attempt to average SQUID voltage dynamics in the context of another analytical approach, which is somewhat similar to the perturbation analysis, was undertaken even before the works [32, 35] on HTS SQUIDs. In 1983 Peterson and McDonald proposed their voltage and current expressions for DC SQUID with small but finite inductance [42]. In their method the SQUID difference phase $\psi = (\phi_1 - \phi_2)/2$ ($\phi_{1,2}$ are the Josephson phases of the first and the second junction) was expressed as a sum of the normalized external magnetic flux $\phi$ and a small correction $x$, which vanishes with the inductance, $\psi = -\phi_1 + x$. The correction $x$ was sought in a form of Fourier series, and so all final analytical expressions also contain a sum of certain series. This complexity complicates analysis of obtained solutions and their adaptation for more complex practical SQUID-based circuits like a bi-SQUID or a SQIF.

Despite the fact that half a century is gone since the year of the first demonstration of quantum interference between two Josephson junctions connected in parallel by superconducting inductance [43], a shape of DC SQUID response still was not found analytically for practical parameters of the device at low temperature ($T \approx 4.2 \text{ K}$). This statement is confirmed by the fact that practical circuit optimization is always performed in the context of numerical analysis [19, 22, 23, 25, 28, 29], which slows down the design process.

The purpose of this paper is resolution of this long standing problem with the presentation of an analytical study of DC SQUID voltage and current responses on applied magnetic flux in resistive state (see sections 2 and 3 correspondingly), and analysis of the current response in superconducting state (section 3). The analysis includes consideration of the effect of the small technological spread of the SQUID parameters relevant to LTS technology, which is presented in section 4. While the superconducting state is considered for arbitrary value of the inductance, the resistive state is considered first for $l \approx 1$. Analytical voltage-flux and current-flux functions in the resistive state for practical inductances up to $l \approx 7$ are obtained by fitting of numerical data in section 5. We conclude the paper with a discussion of the applicability of the analytical expressions obtained to optimization of practical SQUID-based structures like SQIF. For simplicity we do not account for thermal noise and neglect capacitance of the Josephson junctions. Their effects on the responses of LTS DC SQUID with overdamped junctions having relatively high critical currents ($\Gamma l \lesssim 10^{-3}$) are assumed to be small.

2. SQUID voltage

In this section we consider the voltage response of symmetrical DC SQUID with inductance $l \approx 1$ (figure 1). In the frame of resistively shunted junction (RSJ) model [44] for overdamped Josephson junctions one can write simple equations for the currents $i_{1,2}$ flowing through the SQUID

$$i_{1,2} = \sin \phi_{1,2} + \dot{\phi}_{1,2},$$

where the currents are normalized to the critical current $I_c$ and dot denotes time differentiation with normalized time $\tau = \omega_c t$, $\omega_c = 2\pi I_c R_s/\Phi_0$ is the characteristic frequency, and $R_s$ is the shunt resistance of the junctions. Kirchhoff equations for the SQUID produce the following system of

![Figure 1. Symmetrical DC SQUID fed by bias current $i_b$ and applied magnetic flux $\phi_c$.](image)
differential equations:
\[
\frac{1}{2} \dot{\psi} = - (\psi + \dot{\phi}) - \frac{1}{2} \sin \psi \cos \theta, \quad (1a)
\]
\[
\dot{\theta} = \frac{i_0}{2} - \cos \psi \sin \theta, \quad (1b)
\]
where \( \theta = (\phi_0 + \phi_2)/2 \) is the sum phase, \( i_0 = I_0/L, i_0 \geq 0 \) is the bias current.

Since the inductance is present only in equation (1a), we assume that it primarily affects the difference phase. Following works [18, 42] we consider the difference phase as a sum of slow-varying \( \psi \) and oscillating \( \psi_\pm \) parts. The last one is assumed to be small: \( \psi_\pm \ll 1 \) for shunted junctions and the inductance value \( L \ll 1 \). SQUID voltage response \( \tau = w_\tau \) normalized to \( I, R_\tau \) product (where \( w_\tau \) is the Josephson frequency) is found in the following sequence. First, we obtain the solution \( w_\tau = \frac{2}{L} \theta \) assuming that \( \psi = \psi_\pm \). Then we find \( \psi _\pm \) from (1a) using the determined \( \theta \), which in turn allows us to find the correction \( w_\pm \) and the total Josephson frequency \( w_\tau = w_\pm + w_\mp \).

2.1. Solution for \( \psi = \psi_- \)

The difference phase is equal to its slow-varying part in the limit of vanishing inductance \( l \to 0 \). Equation (1a) in this case gives \( \psi = \psi_- = - \phi_0 \), and an according solution of equation (1b) is well known (see [42] and references therein):
\[
\tan \frac{\theta}{2} = z + \sqrt{1 - z^2} \tan \frac{i_0}{2} \sqrt{1 - z^2},
\]
where \( z = (2/\upsilon_0 \cos \phi_0 \). The Josephson oscillation frequency can be obtained from (2) directly
\[
w_\pm = \frac{i_0}{2} \sqrt{1 - z^2} = \frac{i_0^2}{4} - \cos^2 \phi_0.
\]

It is worth mentioning the known consistency of the expression (3) with the one for a single junction \( w_\pm = \sqrt{\frac{i_0^2}{2} - \frac{i_0^2}{4}} \), so one can treat \( \cos \phi_0 \) as an effective critical current and \( i_0/2 \) as an effective bias current. \( w_\pm \) should be set to zero if \( i_0/2 < |\cos \phi_0| \).

2.2. Solution for \( \psi = \psi_- + \psi_\pm \)

Substitution of the difference phase as the sum \( \psi_- + \psi_\pm \) into (1a) in the limit \( \psi_- \ll 1 \) converts this equation into
\[
\frac{1}{2} \dot{\psi}_\pm = - \dot{\psi}_- - \frac{1}{2} (\sin \psi_- + \cos \psi_- \cos \psi_\pm) \cos \theta.
\]

The solution of equation (4) has a form
\[
\psi_\pm = \frac{l \sin \psi_- [L (\cos \psi_- - \frac{i_0}{2} \sin \theta) - 2 \cos \theta]}{l^2 w_\tau^2 + 4} \quad (5)
\]
Its substitution into (1b) leads to correction of the Josephson frequency
\[
w_\pm = \frac{i_0}{2} - \cos \psi_- \sin \theta + \sin \psi_- \psi_\pm \sin \theta.
\]

```
Figure 2. (a) IV curves of Josephson junction biased by current \( i_0 = i + i_{\text{cir}} \cos (w_\tau \tau) \), where \( i_{\text{cir}} \) corresponds to the amplitude of the circulating current and \( w_\tau \) corresponds to the Josephson frequency of a junction in the SQUID biased at \( \phi_0 = \pi/2 \). The parameters \( i_{\text{cir}}, w_\tau \) are calculated for \( i_0/2 = 1 \) by expressions (10), (12) for \( l = 0, 1 \), and calculated numerically using (1) for \( l = 10 \). Dashed line corresponds to \( i_{\text{cir}} = 0 \). The IV curves are calculated numerically using (1). (b) LR-filter circuit used to analyze the SQUID at \( \phi_0 = \pi/2 \).
```

Explicit form for \( w_\tau \) can be found by time averaging
\[
w_\tau = w_\mp - \frac{\frac{3}{2} w_\tau^2}{l^2 w_\tau^2 + 4} (\frac{i_0}{2} - w_\pm) \tan^2 \psi_-.
\]

This expression describes SQUID voltage response for \( l \leq 1 \).

The first factor of the second term in (7) suggests that decrease of the voltage response comes from filtering properties of the SQUID. This can be understood qualitatively by the example of synchronization of the junctions by circulating current at external flux equal to half flux quantum \( \phi_0 = \pi/2 \).

In general, the circulating current is defined as \( i_{\text{cir}} = (i_1 - i_2)/2 \). According to (1a) in resistive state it is \( i_{\text{cir}} = - 2 \psi_- / l \) and with expression (5) it results to
\[
i_{\text{cir}} = \frac{4}{l^2 w_\tau^2 + 4} \sin \psi_- \sin \theta + \arctan \left( \frac{4}{i_0} \frac{\frac{i_0}{2}}{\sqrt{l^2 w_\tau^2 + 4}} \right).
\]

For the considered applied flux \( \psi_- = - \phi_0 = - \pi/2 \) the sum phase is \( \theta = w_\pm \tau \) and the frequency \( w_\pm = i_0/2 \), as it follows from (2), (3). Assuming that \( l \ll 1 \) the circulating current can be presented as \( i_{\text{cir}} \approx - \cos (i_0/2) \). This means that the total current through each junction in this case is
\[
i_{1,2} = i_0/2 + \cos (\frac{i_0}{2} \tau).
\]

The oscillating part of this current induces Shapiro steps in current-voltage (IV) curve of the junctions, see figure 2(a). Total width of the first step is \( [31] 2 \Delta J_l (I_{\text{circ}}/w_\tau) \) (where \( I_{\text{circ}} = 1 \) is the amplitude of the circulating current in the limit \( l \to 0 \)). Since the middle point of the step lies approximately at the IV-curve of the individual junction, the step starts with the current \( \sqrt{\frac{i_0^2}{2}} + 1 - J_l (2/i_0) \approx i_0/2 \), which is approximately equal to constant current applied to each junction.
Considering an increase of the inductance we note that the junctions are synchronized in antiphase at $\phi_k = \pi/2$. Thus the SQUID can be approximately analyzed as linear circuit presented in figure 2(b), which is a serial connection of two voltage generators and two resistors corresponding to two Josephson junctions, and the inductance that couples them. From expression (9) we deduce that each generator provides harmonic voltage with normalized amplitude equal to unity. Amplitude of the current in such a circuit completely coincides with the amplitude of the considered circulating current (8)

$$i_{\text{circ}}^a = \frac{1}{\sqrt{1 + i^2 w_j^2 / 4}}.$$  \hspace{1cm} (10)

Decrease of the amplitude of the oscillating current flowing through each junction with increase of the inductance leads to decrease of the first Shapiro step width. For fixed constant bias current this, in turn, leads to decrease of the oscillation frequency that shifts the Shapiro step, leading to synchronization of the junctions at lower frequency, see figure 2(a). This means that the decrease of the time-averaged SQUID voltage is caused by filtering of the circulating current, and the junctions are switched at the frequency allowed by LR-relaxation time of the circuit.

Returning to expression (7) we note that at the point $\phi_k = \pi/2$ where $\tan \psi_k$ is infinite the finite value of $w_j$ can be found through the limit for the last two factors of the second term

$$\lim_{\phi_k \to \pi/2} \left( i_b - \frac{i_b^2}{2} \right) \sqrt{1 + \frac{i^2 w_j^2}{4}} \tan^2 \phi_k = \frac{i_b}{i_b}.$$ \hspace{1cm} (11)

Since $w_j(\pi/2) = i_b/2$, the Josephson oscillation frequency at this point is

$$w_j(\pi/2) = \frac{i_b}{2} \left[ 1 - \frac{2I^2}{I^2 i_b^2 + 16} \right].$$ \hspace{1cm} (12)

This gives the amplitude of the voltage response $v_{pp} = w_j(\pi/2) - w_j(0)$

$$v_{pp} = \frac{i_b}{2} \left[ 1 - \frac{2I^2}{I^2 i_b^2 + 16} \right] - \frac{i_b^2}{4} - 1,$$ \hspace{1cm} (13)

where the last term should be set to zero if $i_b/2 < 1$.

It is seen that with the bias current increase both frequencies $w_j(\pi/2), w_j(0)$ tend to $i_b/2$. Since the voltage response appears due to excitation of the circulating current, it vanishes in this case because the circulating current portion of the total current flowing through the junctions becomes small. Since junctions in this limit are biased high above their critical current, time-averaged voltage on the SQUID corresponds to voltage drop on the two shunt resistors connected in parallel.

For the bias current equal to the critical current $i_b = 2$ the voltage response amplitude decreases with the inductance as

$$v_{pp} = 1 - \frac{I^2}{2I^2 + 8}.$$ \hspace{1cm} (14)

With inductance increase this amplitude tends to $v_{pp} \to 0.5$. At the same time, according to (10), the circulating current amplitude tends to zero for $l \to \infty$, and so $v_{pp} \to 0$. Therefore this dependence of the amplitude on the inductance $v_{pp}(l)$ (14) is relevant only in the context of the validity of the proposed approach—i.e., for $l \leq 1$.

### 2.3. Comparison with numerical calculations

Figure 3 shows curves calculated using the presented analytical approach (expressions (7), (13)—solid lines) and using numerical calculations of system (1) (dots). Figure 3(a) presents SQUID voltage responses for the inductance value $l = 1$. It is seen that the curves calculated using both analytical and numerical approaches are well consistent in the wide bias current range $i_b = 1.5 \ldots 2.5$ around the SQUID critical current.

Equation (5) shows that amplitude of the difference phase oscillations is proportional to $\psi_k^a \sim -\sin \phi_k$, and so the inductance mainly affects the middle part of the voltage response. Increase of the inductance leads to increase of $\psi_k^a \sim -H_{\text{circ}}^a/2$ and violation of assumption that $\psi_k \ll 1$.
is seen as small deviation of the analytical curves from the numerical ones in figure 3(b) for \( l > 1 \).

We should note that expression (7) for the voltage response allows one to calculate all related curves like the transfer function \( \partial \Sigma/\partial \phi_e \) or the dynamic resistance \( \partial \Sigma/\partial i_b \), presented in figure 3(c) and its inset respectively.

Figure 3(d) presents the voltage response amplitude dependencies on the bias current and on the inductance. Comparing analytical and numerical data we conclude that within the area of its applicability the presented approach describes SQUID voltage characteristics fairly well.

3. SQUID current

This section is devoted to consideration of symmetrical SQUID current response. The response is first considered in superconducting state for arbitrary SQUID inductance. Consideration of the response in resistive state is conducted for \( l \ll 1 \) using expression (8) for the circulating current obtained in the previous section.

3.1. Superconducting state

Time derivatives of the sum and difference phases \( \dot{\Theta}, \dot{\Psi} \) as well as oscillating part of the difference phase \( \psi_e \), in superconducting state is zero. Expression for the circulating current in this case is given as

\[
i_{\text{circ}} = \sin \psi \cos \Theta,
\]

which can be combined with (1b) to produce

\[
i_{\text{circ}} = \sqrt{\cos^2 \psi - \frac{i_b^2}{4}} \tan \psi.
\]

This expression can be readily used in the case of vanishing inductance \( l \rightarrow 0 \) with \( \psi = -\phi_e \). For zero bias current it simplifies further

\[
i_{\text{circ}} = -\text{sgn}(\cos \phi_e) \sin \phi_e.
\]

For small but nonzero inductances the circulating current was derived in the frame of perturbation analysis, to first [41] and second [40] order in the inductance. However, for the inductance value \( l \approx 1 \) the numerical calculation result becomes inconsistent compared to [40, 41].

To find the circulating current for arbitrary inductance one has to solve an equation \( f(\psi) = 0 \) for the transcendental function

\[
f(\psi) = \frac{l}{2} \sqrt{\cos^2 \psi - \frac{i_b^2}{4}} \tan \psi + \psi + \phi_e
\]

derived from (1a), and substitute the phase difference found into expression (16). In the general case the solution can be found using the following quasianalytical approach.

The method is based on integral definition of root \( x_0 \) of arbitrary continuous function \( f(x) \) having just one zero in the range of interest \([a, b]\):

\[
x_0 = a + \int_a^b \text{H}[\text{sgn}(f(x))] + \text{sgn}(f(x))\text{H}(f(x))\,dx,
\]

where \( \text{H}(x) \) is the Heaviside step function. Since the period of the current modulation corresponds to the range \( \phi_e \in [0, \pi] \), the root of function \( f(\psi) \) has to be sought in the range \( \psi \in [0, -\pi] \). However, function \( f(\psi) \) is undefined inside the region \( |\cos \psi| < i_b/2 \), and can have up to three zeros depending on the parameters \( i_b, l, \phi_e \).

The first obstacle can be overcome by equalizing function \( f(\psi) \) to the mean \( f_m \) of its boundary values inside the region \( |\cos \psi| < i_b/2 \), which is obviously \( f_m = \text{sgn}(\psi)\text{ceil}(|\psi|/\pi)\pi/2 + \phi_e \), where ceil(\( \psi \)) is the ceiling function. In such complementary definition function \( f(\psi) \) is shown in figure 4. It is seen from (18) that \( \phi_e \) just shifts function \( f(\psi) \) along the ordinate axis.

If function \( f(\psi) \) is not monotonic (depending on \( l, i_b \)), the dependence \( \psi_0(\phi_e) \) is hysteretic. In this case the function can have more than one root for a certain \( \phi_e \). To find the proper root, integral (19) should be taken from the starting point \( a \) up to the local extremal point closest to \( a \) \( (\psi_1^\dagger \text{ for } \phi_e \) variation \( \phi_e = 0 \ldots \pi \) and \( \psi_1^\dagger \text{ for } \phi_e = \pi \ldots 0 \), see inset in figure 4) if the function \( f \) at these points \( (f(a) \text{ and } f(\psi_1^\dagger)) \) is of the opposite sign. Otherwise the limits of integration should be left unchanged.

Alteration of the integral upper limit \( b \) can be formalized as follows:

\[
b(\psi_1^\dagger) = b + \text{H}[-f(a)f_1^{\dagger}(\phi_e)](\psi_1^\dagger - b),
\]

where

\[
f_1^{\dagger}(\phi_e) = \begin{cases} f(\psi_1^\dagger); & \phi_e \leq -\pi/2, \\ \pi l/2 - \pi/2 + \phi_e; & \text{otherwise.} \end{cases}
\]

The reason for such definition of \( f_1^{\dagger} \) is the fact that at \( i_b = 0 \) the function \( f \) has form

\[
f(\psi) = \frac{l}{2} \text{sgn}(\cos \psi)\sin \psi + \psi + \phi_e,
\]

Figure 4. Transcendental function \( f(\psi) \) (18) complementary defined inside the region \( |\cos \psi| < i_b/2 \), which implicitly defines the difference phase in superconducting state; \( i_b = 0.2, l = 1, \phi_e = 0.55\pi \). Inset zooms the function nonlinear part forming hysteresis at \( \psi_0(\phi_e) \) and \( i_b(\phi_e) \) dependencies. Dots show local extrema of \( f(\psi) \) function.
which corresponds to the leap at $\psi^\dagger = \psi^\dagger = -\pi/2$. Thus we put $f^\dagger$ equal to the margin values of $f$ function in vicinity of the leap. For $i_b > 0$ coordinates $\psi^\dagger, \psi^\dagger$ can be found using derivative of $f(\psi)$ function

$$\frac{df}{d\psi} = -\frac{f \sin^2 \psi}{\sqrt{\cos^2 \psi - \frac{e^2}{4}}} + \frac{f \sqrt{\cos^2 \psi - \frac{e^2}{4}}}{\cos^2 \psi} + 1 \tag{22}$$

and formula (19)

$$\psi^\dagger = \int_0^{\arccos \frac{e}{2}} H \left[ \frac{\partial f(\phi)}{\partial \phi} \right] d\phi, \tag{23a} \psi^\dagger = -\pi - \psi^\dagger. \tag{23b}$$

Note, that $f(\psi^\dagger) - f(\psi^\dagger)$ defines the width of $\psi_0(\phi_e)$ and $i_{\text{cur}}(\phi_e)$ hysteresis. 

$\psi_0(\phi_e)$ dependencies for both directions of $\phi_e$ variation can be obtained as follows:

$$\psi_0(\phi_e^\dagger) = \int_0^{\pi + \frac{\pi}{2} - \arccos \frac{e}{2}} H[ f(\phi; \phi_e^\dagger) ] d\phi, \tag{24a}$$

$$\psi_0(\phi_e^\dagger) = -\pi + \int_{-\pi}^0 H[ f(\phi; \phi_e^\dagger) ] d\phi. \tag{24b}$$

These dependencies (24) and corresponding circulating currents obtained by expression (16), as well as corresponding data obtained by numerical calculations of system (1) are presented in figure 5. Because of quasi-analytical nature of the presented approach the data obtained using expressions (24), (16) are perfectly consistent with the ones calculated numerically.

Increase of the inductance value clearly leads to enlargement of the hysteresis (see figures 5(a), (c)) since inductance is the amplitude of the non-monotonic term of function $f$. The physical meaning of this fact is as follows. For zero bias current and vanishing inductance $l \to 0$ a fluxon penetrates into the SQUID at $\Phi_e = \Phi_0/2$, which is accompanied by changing of the circulating current flow direction (17). The current flow through a finite inductance produces additional magnetic flux, and so the external flux can be screened without penetration of a fluxon into the SQUID loop up to higher values (figure 5(c)). The inductance value $l = \pi$ corresponds to the case where a fluxon penetrates into the SQUID at $\Phi_e = \Phi_0$. Starting from this inductance value the circuit has two stable states at zero applied flux (with and without a fluxon inside it), and thus the loop is called ‘quantizing’ one.

At the same time, increase of the bias current leads to widening of the range where the $f$ function is initially undefined and decrease of the range where it is non-monotonic. Physically, this means that the bias current flowing through the junctions allows smaller circulating current in the superconducting state. This shrinks the hysteresis (see figures 5(b), (d)) and finally results in formation of the region of resistive state.

3.2. Resistive state

Time averaging of (8) leads to the following expression for the averaged circulating current in resistive state:

$$i_{\text{cur}} = -\frac{2i_b}{\mu_0 l^2} + 4 \left( \frac{i_b}{2} - w_f \right) \tan \psi.. \tag{25}$$

Taking $\psi_\tau = -\phi_e$ we calculated current curves presented in figure 6 by solid lines. Corresponding curves calculated numerically using system (1) are shown by dots.

Zero value of the averaged current at $\phi_e = 0, \pi$ corresponds to an absent or purely harmonic circulating current, according to (8). Nonzero averaged current at $\phi_e \approx \pi/4$ appears due to unharmonic shape of the circulating current.
which is not accurately described with proposed linearization of the equations (1). While consistency for the bias current $i_b = 2$ is only qualitative (figure 6(a)), for higher bias current value it is improved (figure 6(b)). The reason is decrease of oscillating part of the difference phase $\psi_\omega$ (5).

Simplifying expression (25) at the point $\phi_y = \pi/4$, one can estimate the averaged circulating current $i_{ca}^r$.

$$i_{ca}^r = \frac{2(i_b - \sqrt{i_b^2 - 2})}{i_b^2 - 2} + 16.$$ 

(26)

For the bias current value $i_b = \sqrt{6}$ $\approx 2.45$ the amplitude is approximately

$$i_{ca}^r \approx \frac{0.45}{i^2 + 4}.$$ 

(27)

Corresponding curve $i_{ca}^r(l)$ is shown in inset of figure 6(b) with numerical data for comparison.

4. SQUID with small spread of parameters

In this section we study effect of reasonable technological spread of parameters ($\Delta I_c, \Delta R_n$ up to $\pm 20\%$, and arbitrary inequality of SQUID inductive shoulders) on the SQUID responses. Such spread may result in a small asymmetry of the SQUID.

The considered DC SQUID is shown in figure 7. The asymmetry is presented by inequality of the inductive shoulders $l_1 = l_2$, critical currents $i_{c1} \neq i_{c2}$, and shunt resistances $r_{n1} \neq r_{n2}$ of the junctions. We assume below that $|i_{c1} - i_{c2}| \ll 1$, $|r_{n1} - r_{n2}| \ll 1$, and that the bias current is of the order of the SQUID critical current $i_b \approx i_{c1} + i_{c2}$.

4.0.1. Asymmetry of the inductive shoulders. The simplest case of SQUID asymmetry is inequality of the inductances $l_1 = l_2$. If $i_{c1} = i_{c2}$, $r_{n1} = r_{n2}$ then modification appears only in equation (1a). Phase drop on the inductances $l_1i_{b1} - l_2i_{b2}$ rewritten as $(i_b/2)[l_1 - l_2] + (l/2)[i_1 - i_2]$ converts this equation into the following one:

$$\frac{\Sigma l}{2} \psi = - \left( \psi + \phi_y + \frac{i_b \Delta l}{2} \right) - \frac{\Sigma l}{2} \sin \psi \cos \theta,$$

(28)

where $\Sigma l = l_1 + l_2, \Delta l = l_1 - l_2$.

It is seen that this asymmetry can be accounted just by the constant offset $i_b \Delta l/4$ to the external flux $\phi_y$, which appears due to the bias current flow through the part of inductance $\Delta l$. In consideration of asymmetries of the critical currents and shunt resistances we first put $\Delta l = 0$ for simplicity, and then generalize obtained results for arbitrary $\Delta l$.

4.0.2. Asymmetry of the critical currents and shunt resistances. The spread in critical currents of Josephson junctions and shunt resistances affects system (1) noticeably, and these equations can be rewritten as

$$\frac{\Sigma l}{2} \psi = - (\psi + \phi_y) + \frac{l}{2} \frac{i_b \Delta r_n}{\Sigma r_n} - \frac{\Sigma l}{2} \sin \psi \cos \theta + \frac{\Delta \psi}{2} \cos \psi \sin \theta,$$

(29a)

and these equations can be rewritten as

$$\frac{\Sigma l}{2} \psi = - (\psi + \phi_y) + \frac{l}{2} \frac{i_b \Delta r_n}{\Sigma r_n} - \frac{\Sigma l}{2} \sin \psi \cos \theta + \frac{\Delta \psi}{2} \cos \psi \sin \theta.$$

Here $\psi_c = i_c r_n$ is the characteristic voltage, $\Sigma \psi_c = \psi_{c1} + \psi_{c2}, \Delta \psi_c = \psi_{c1} - \psi_{c2}, \Sigma r_n = r_{n1} + r_{n2}, \Delta r_n = r_{n1} - r_{n2}$.

Below we first consider influence of this asymmetry on the voltage response and then describe its effect on the circulating current. The difference phase is again presented as a sum of constant $\psi^r$, and oscillating $\psi^* \psi^*$ parts, where the last one is assumed to be small $\psi^* \approx 1$.

4.1. Voltage on SQUID with small asymmetry

4.1.1. Solution for $\psi = \psi^*$. Even in the absence of the asymmetry of the inductive shoulders asymmetry of the critical currents and shunt resistances still leads to a constant offset of the external flux. This offset can be found by time averaging of equation (29a) at $\phi_y = 0$ and $i_b \approx 2$, assuming that $\Delta i_{c1,2}, \Delta r_n \approx 1$. Since $\psi^r = 0, \cos \theta = 0$, and $\sin \theta = [(i_b/2) - w_1 \ldots / \cos \psi \approx 1$, under the assumptions made the difference phase is

$$\psi^* = - \phi_y - \frac{l}{2} \frac{\Delta i_{c1,2}}{\Sigma i_{c1,2}} - \frac{i_b - \Sigma i_{c1} \Delta r_n}{\Sigma r_n},$$

(30)

where $\Sigma i_{c1} = i_{c1} + i_{c2}, \Delta i_{c1} = i_{c1} - i_{c2}$.

The asymmetries $\Delta i_{c}$ and $\Delta r_n$ provide redistribution of the bias current toward the junction with larger $i_c$ but smaller $r_n$. Corresponding terms enter in (30) with opposite signs accordingly. Effect due to asymmetry of the shunt resistances refers to resistive state that is manifested by prefactor proportional to the bias current deviation from the SQUID critical current $i_b$. 

Figure 7. DC SQUID with asymmetry presented by inequality of the inductances $l_1 = l_2$, critical currents $i_{c1} \neq i_{c2}$, and shunt resistances $r_{n1} \neq r_{n2}$. 

Supercond. Sci. Technol. 29 (2016) 094005 I1 Soloviev et al.
To find the sum phase it is convenient to present equation (29b) in the following form:

$$\dot{\theta} = \frac{i_b}{2} \frac{\psi + \phi_b}{2} \Delta r_n - \frac{1}{4} \left( \psi^2 + \psi \right) - \frac{1}{4} \left( \psi^2 + \psi \right) - \frac{1}{4} \left( \psi^2 + \psi \right) \cos^2 \psi \right) \times \tan \left( \theta + \arctan \left( \frac{\Delta \psi}{\psi \tan \psi} \right) \right). \tag{31}$$

The structure of this equation is quite analogous to that of equation (1b), and so its solution replicates equation (2). The only difference here is the shift of the sum phase:

$$\tan \left( \frac{\theta + \epsilon}{2} \right) = z^* + \sqrt{1 - z^*^2} \tan \left( \frac{\theta + \epsilon}{2} \right), \tag{32}$$

where

$$z^* = \frac{2}{2} \frac{\Delta \psi^2}{\psi \tan \psi},$$

$$\epsilon = \arctan \left[ \frac{\Delta \psi}{\psi \tan \psi} \right].$$

The Josephson frequency similarly follows from equation (32):

$$w_{j_1}^* = \sqrt{\mu_1 / \mu_2} \times \sqrt{\frac{(i_b - i_{c1}) r_{n1} + \psi}{(i_b - i_{c2}) r_{n2} + \psi}} \cos^2 \psi^*, \tag{33}$$

where $w_{j_1}^*$ should be set to zero if the expression under the second square root is negative.

Here the square root of the characteristic voltage product $\nu_{11} \nu_{21}$ is a natural scaling factor for the voltage. Presenting this factor as

$$\sqrt{\nu_{11} \nu_{21}} = \sqrt{\frac{(\Delta \psi^2 - \Delta \psi^2) \Sigma \nu_n^2 - \Delta \psi^2}}{4}, \tag{34}$$

we note that the differences of the critical currents and shunt resistances provide similar contributions.

Considering the first term under the second square root of expression (33) as squared effective bias current divided by the squared SQUID critical current

$$\frac{i_{b_{\text{eff}}}^2}{\Sigma \nu_n^2} = \frac{(i_b - i_{c1}) r_{n1} + \psi}{(i_b - i_{c2}) r_{n2} + \psi} \times \frac{(i_b - i_{c1}) r_{n1} + \psi}{(i_b - i_{c2}) r_{n2} + \psi}, \tag{35}$$

by analogy with expression (3), one can study how this ratio differs from the one for symmetrical case.

For identification of the effect of the critical current difference on this ratio it is convenient to set the difference of the shunt resistances equal to zero $\Delta r_n = 0$ and $\Sigma \nu_n = 2$. In this case the considered ratio is

$$\frac{i_{b_{\text{eff}}}^2}{\Sigma \nu_n^2} = \frac{i_{b1}^2 - \Delta i_c^2}{\Sigma i_c^2 - \Delta i_c^2}, \tag{36}$$

and the squared effective bias current is

$$i_{b_{\text{eff}}}^2 = i_{b1}^2 + \frac{\Delta i_c^2}{\Sigma i_c^2 - \Delta i_c^2} (i_{b2}^2 - \Sigma i_c^2). \tag{37}$$

It is seen that asymmetry of the critical currents $\Delta i_c$ affects both $i_b$ and $\Sigma i_c$ (36). This results in tiny deviation of $i_{b_{\text{eff}}}$ from $i_{b1}$ proportional to the asymmetry in square, and multiplied by deviation of the squares of the bias current from the SQUID critical current (37).

Consideration of shunt resistance difference at $\Delta i_c = 0$, $\Sigma i_c = 2$ leads to the following expression

$$\frac{i_{b_{\text{eff}}}^2}{\Sigma \nu_n^2} = \frac{i_{b1}^2}{4} - \left( \frac{i_b - 2 \Sigma i_c}{4} \right)^2 \Delta i_c^2. \tag{38}$$

Difference of the shunt resistances affects the effective bias current in accordance with the bias current redistribution (see also (30)). In comparison with the effect provided by difference of the critical currents, here the deviation of $i_{b_{\text{eff}}}$ from $i_{b1}$ is smaller (since $i_{b2}^2 - \Sigma i_c^2 > (i_b - \Sigma i_c)^2$ for $i_b > \Sigma i_c$, see (37), (38)).

Obtained results (30), (37), (38) allow us to conclude that at $i_b \approx \Sigma i_c$ the Josephson oscillation frequency is approximately equal to

$$w_{j1}^* \approx \sqrt{\nu_{11} \nu_{21}} \frac{1 - \cos^2 \phi + \frac{\Delta i_c}{2}}{2}, \tag{39}$$

and the asymmetries $\Delta i_c$, $\Delta i_c$ mainly scale the averaged voltage and provide offset to the external flux corresponding to $\Delta i_c$.

Considering effects of asymmetries of the critical currents and shunt resistances on the scaling factor $\sqrt{\nu_{11} \nu_{21}}$, the deviation of $i_{b_{\text{eff}}}$ from $i_{b1}$, and the bias flux offset are illustrated in figure 8.
4.1.2. Solution for $\psi = \psi^r + \psi^s$. Impact of oscillating current circulating in the SQUID can be introduced by oscillating part of the difference phase. Substitution of $\psi = \psi^r + \psi^s$ into equation (29a) leads to

$$
\frac{d}{d\theta} \psi^*_n = -\psi^*_n + \frac{l}{2} \left[ \frac{\Delta I_c}{2} + \frac{\Sigma I_c \Delta r_n}{2 \Sigma r_n} \right] - \frac{l}{\Sigma r_n} \left( \frac{\Sigma V_c}{2} \sin \psi^*_n \cos \theta + \frac{\Delta V_c}{2} \cos \psi^*_n \sin \theta \right).
$$

(40)

It is hard to derive an exact analytical solution of this equation, while approximate solution can be drawn rather easily, assuming $l \ll 1$ and putting $\theta \approx \tilde{\theta} = \psi^r$.:

$$
\psi^*_n = \frac{l}{2} \left[ \frac{\Delta I_c}{2} + \frac{\Sigma I_c \Delta r_n}{2 \Sigma r_n} \right] - \frac{l}{\Sigma r_n} \left( \frac{\Sigma V_c}{2} \sin \psi^*_n \cos \theta + \frac{\Delta V_c}{2} \cos \psi^*_n \sin \theta \right) \\
\times \left[ \frac{\Sigma V_c}{2} \sin \psi^*_n \cos \left( \theta - \arctan \left( \frac{\Sigma V_c}{\Delta V_c} \right) \right) \\
+ \frac{\Delta V_c}{2} \cos \psi^*_n \sin \left( \theta - \arctan \left( \frac{\Sigma V_c}{\Delta V_c} \right) \right) \right].
$$

(41)

To find the correction $w^*_n$ of the Josephson frequency we substitute the found difference phase $\psi = \psi^r + \psi^s$ (30), (41) into equation (29b):

$$
w^*_n = - \frac{\dot{\psi}^r}{I} \Delta r_n + \frac{\Sigma I_c}{2} \sin \psi^r(\psi^r \sin \theta) - \frac{\Delta V_c}{2} \cos \psi^r(\psi^r \cos \theta).
$$

(42)

Since $\psi^r$ is proportional to the inductance (41), the last two terms in (42) represent SQUID LR-filtering of the circulating current. They vanish in the limit $l \to 0$. At the same time, for $l = 0$ the first term is non zero, representing difference of voltage drops of the time-averaged circulating current on the shunt resistors.

Unfortunately, the full expression for $w^*_n$ is rather complicated

$$
w^*_n = - \frac{\dot{\psi}^r}{I} \Delta r_n + \frac{\Sigma I_c}{2} \sin \psi^r(\psi^r \sin \theta) - \frac{\Delta V_c}{2} \cos \psi^r(\psi^r \cos \theta).
$$

(43)

Here the coefficients $K_{1,2,3}$ have following forms

$$
K_1 = \frac{\Sigma V_c}{2} \Delta V_c - 4w^*_n \frac{\Delta r_n}{\Sigma r_n} \left( \frac{\dot{\psi}^r}{2} + w^*_r \right) \\
\times \left( \frac{\dot{\psi}^r}{2} - w^*_r \right) \sin 2\psi^r,
$$

$$
K_2 = \frac{\Sigma V_c}{2} \frac{\Delta V_c}{\Sigma r_n} \frac{1}{2} \frac{l}{\Sigma r_n} + \frac{\Sigma I_c \Delta r_n}{2 \Sigma r_n} \\
- \frac{l}{\Sigma r_n} \left( \frac{\Sigma V_c}{2} \sin \psi^*_n \cos \theta + \frac{\Delta V_c}{2} \cos \psi^*_n \sin \theta \right),
$$

$$
K_3 = \left( \frac{l}{2} \left[ \frac{\Delta I_c}{2} + \frac{\Sigma I_c \Delta r_n}{2 \Sigma r_n} \right] \\
+ \left[ \frac{\Sigma V_c}{2} \sin \psi^*_n \cos \left( \theta - \arctan \left( \frac{\Sigma V_c}{\Delta V_c} \right) \right) \\
+ \frac{\Delta V_c}{2} \cos \psi^*_n \sin \left( \theta - \arctan \left( \frac{\Sigma V_c}{\Delta V_c} \right) \right) \right] \right).
$$

(44)

For symmetrical case this expression takes the form of (14) with according scaling coefficients.

Note, that for $l = 0$ this amplitude (44) is

$$
v^*_pp = \sqrt{V_c v_{c2}} - \frac{\Sigma V_c}{4} \left( 1 - \frac{\Sigma I_c}{2 \Sigma r_n} \right).
$$

(45)

The last term in (45) represents deviation of the voltage response amplitude corresponding to the first term of (42). For zero difference of the shunt resistances $\Delta r_n = 0$ this term is zero. At the same time, for zero difference of the critical currents $\Delta I_c = 0$ (\Delta V_c = \Sigma I_c \Delta r_n/2) this deviation is proportional to $\Delta r_n^2$. In this case the difference of shunt resistances always leads to additional decrease of the voltage response. We will return to consideration of this fact below in the section devoted to time-averaged circulating current.

4.1.3. Generalization to the case of unequal inductive shoulders. Since asymmetry of the inductances $l_1 \neq l_2$ just shifts the bias flux, it can be taken into account by according addition $i_0 \Delta l / 4$ to the expression for the constant difference phase (30)

$$
\psi^*_n = -\phi_c - \frac{i_0 \Delta l}{2} - \frac{\Sigma I_c}{2} \left[ \Delta I_c - \frac{i_0 - \Sigma I_c \Delta r_n}{2 \Sigma r_n} \right]
$$

(46)

and substitution of the total inductance $\Sigma I$ instead of $l$ in equation (40) and subsequent expressions (41)–(44).

Voltage response

$$
\tau^* = w^*_j + w^*_s
$$

(47)

(where $w^*_j$ should be set to zero if $w^*_n = 0$) can be found by combining expressions (33), (43), and (46). The responses calculated from these expressions for a chosen set of parameters $i_0 = 2$, $\Sigma I_c = 1.9$, $\Delta I_c = -0.3$, $\Sigma r_n = 2.05$,
\[ \Delta_{\text{pp}} = 0.35, \Sigma l = 1, \text{and } \Delta l = 0, -0.8. \]

Figure 9(b) illustrates effects of the critical current difference \( \Delta_{\text{ic}} \) at \( \Delta_{\text{pp}} = 0 \) and difference of the shunt resistances \( \Delta_{\text{rn}} \) at \( \Delta_{\text{ic}} = 0 \) on the amplitude of the voltage response \( v_{\text{pp}}^* \) for \( l = \Sigma_{\text{ic}}, \Sigma_{\text{rn}} = 2 \) and \( l = 1 \). Corresponding curves are calculated using (44) and shown by lines. The dependence \( v_{\text{pp}}^* = v_{\text{pp}}(\sqrt{\Sigma_{\text{ic}}/\Sigma_{\text{rn}}}) \), which can be derived from the amplitude \( v_{\text{pp}} \) in the symmetrical case (14) and the voltage scaling factor (34) is represented by dots.

Consistency of the curves indicates that individual effect of the critical current difference reduces to the voltage scaling. For small inductance values \( l \ll 1 \) expression (44) can be approximated by

\[ v_{\text{pp}}^* \approx \sqrt{1 - \frac{\Delta_{\text{ic}}^2}{4}} v_{\text{pp}}. \]

At the same time, the individual effect of difference of the shunt resistances includes additional decrease of \( v_{\text{pp}}^* \) due to time-averaged voltage drop of the circulating current on \( \Delta_{\text{rn}} \) that leads to

\[ v_{\text{pp}}^* \approx \sqrt{1 - \frac{\Delta_{\text{rn}}^2}{4}} v_{\text{pp}} - \frac{\Delta_{\text{rn}}^2 l^2 + 2}{4 l^2 + 4}. \]

Note that decrease of the amplitude \( v_{\text{pp}}^* \) resulting from LR-filtering does not precisely follow \( v_{\text{pp}} \) (14). Therefore obtained expressions (48), (49) are relevant only for \( l \ll 1 \). In general, the circulating current-time dependence, which is purely harmonic in the symmetrical case at \( \psi = -\pi/2 \) is unharmonic at \( \psi = -\pi/2 \) for \( \Delta_{\text{ic}} \neq 0 \). This can be seen from expressions (32), (41). The current-time dependence becomes inclined (at \( \psi = -\pi/2 \)) due to different rates of switching of the junctions \( \psi = \psi_\text{c1} \). Figure 9(c) presents the voltage response amplitude \( v_{\text{pp}}^*(t) \) calculated using expression (44) for the same parameters as the ones taken for figure 9(a) but \( \Delta_{\text{pp}} = 0 \). It is compared with the amplitude \( v_{\text{pp}} \) of symmetrical SQUID (14) versus the total inductance. Instability of the difference \( v_{\text{pp}}^* - v_{\text{pp}} \) illustrates slight changes in the circulating current and conditions of its filtering provided by asymmetry.

4.2. Circulating current in SQUID with small asymmetry

4.2.1. Superconducting state. Effect of the asymmetries on the circulating current in superconducting state \( \psi = 0 \) can be easily found for \( \Delta_{\text{pp}} = 0 \) by combining equations (29a). We consider here only effects of differences of the critical currents since for zero bias current the zero bias current inequality of the inductive shoulders plays no role. The \( f(\psi) \) function (21) obtained from the system (29) has the following form:

\[ f^*(\psi) = g(\psi) \frac{l}{2} \text{sgn}(\cos \psi) \sin \psi + \psi + \phi_0, \]

where

\[ g(\psi) = \frac{\Sigma_{\text{ic}} + \Delta_{\text{ic}} A_l}{\Sigma_{\text{rn}} \sqrt{1 + A_l^2 \tan^2 \psi}} \]

is the factor arising from asymmetry, and

\[ A_l = \frac{\Sigma_{\text{ic}} \Delta_{\text{rn}} - \Delta_{\text{ic}} \Sigma_{\text{rn}}}{\Sigma_{\text{ic}} \Sigma_{\text{rn}} - \Delta_{\text{ic}} \Delta_{\text{rn}}} \]

Constant \( A_l \) is always equal to zero if \( \Delta_{\text{ic}} = 0 \). This means that asymmetry of the shunt resistances itself does not affect the circulating current, as is expected for the superconducting state. In this case the nonlinear term of \( f(\psi) \) (21) is multiplied just by \( g = \Sigma_{\text{ic}}/2 \). Generally, the effect of difference of the shunt resistances on \( g(\psi) \) remains negligible for arbitrary \( \Delta_{\text{ic}} \).

For equal shunt resistances \( \Delta_{\text{rn}} = 0 \) the \( A_l \) constant is \( A_l = -\Delta_{\text{ic}}/\Sigma_{\text{ic}} \) and so the \( g(\psi) \) function is the product of \( 2 i_\text{k1} i_\text{k2} / \Sigma_{\text{ic}} \) and \( \sqrt{1 + (\Delta_{\text{ic}}/\Sigma_{\text{ic}})^2} \tan^2 \psi \). The first factor is of the order of unity for \( i_\text{k1}, i_\text{k2} \approx 1 \). The second factor is positive and less than unity. It has a minimum at \( \psi = \pi/2 + n\pi \) (where \( n \) is integer). These properties of \( g(\psi) \) factors lead to effective smoothing of nonlinear bend of the \( f(\psi) \) function in the vicinity of \( \psi = \pi/2 + n\pi \) (see figure 10(a)). The bend smoothing, in turn, leads to shrinking of hysteresis of the circulating current curve illustrated in figure 10(b). This effect increases with increase of \( \Delta_{\text{ic}} \). It is manifested in a manner somewhat analogous to the nonzero
bias current effect considered in section 3. However, here it results from limitation of the circulating current by smaller critical current of unequal junctions.

The $f(\psi)$, $f^*(\psi)$ functions shown in figure 10(a) are calculated using expressions (21), (50) for the symmetrical case, and for the same parameters of SQUID asymmetry as the ones taken for calculations of the data presented in figure 9(a) respectively. The circulating currents $i_{\text{cir}}$, $i_{\text{cir}}^*$ in figure 10(b) are found using $f$, $f^*$ functions, expressions (16), (24), and derivative of $f^*(\psi)$ function

$$\frac{\partial f^*}{\partial \psi} = g(\psi) \frac{l}{2} \text{sgn}(\cos \psi) [\cos \psi - \sin \psi \tan \psi A_i^2 (1 + \tan^2 \psi)] + 1.$$  

Numerical data shown in figure 10(b) are obtained using system (29).

4.2.2. Resistive state. Time-averaged circulating current in resistive state can be found using expressions (32), (33), (41) and (46):

$$i_{\text{cir}}^* = \frac{\Sigma V_i \Delta V_i - 2 \Sigma V_i^2 \Sigma V_i \sin 2\psi + \Sigma V_i}{2 \Sigma r_n \left( \frac{\Sigma V_i^2 \Sigma V_i^2}{\Sigma r_n^2} + 1 \right) \left( \frac{\Delta V_i^2}{4} + V_i V_i \cos^2 \psi \right)^*} - i_b \frac{\Delta r_n}{2 \Sigma r_n}.$$  

For the symmetrical case this expression converts into (25) with corresponding scaling coefficients. However, contrary to (25) here the time-averaged circulating current, in general, has some offset due to redistribution of the bias current. For the bias current equal to the SQUID critical current $i_b = \Sigma I_i$ the averaged circulating current at $\psi^\pm = 0$ is

$$i_{\text{cir}}^* = \frac{1}{\Sigma r_n} \left( \Delta V_i - \Sigma I_i \Delta r_n \right).$$  

For $\Delta I_i = 0$ the current is $i_{\text{cir}}^* = 0$ as it is expected, but for $\Delta r_n = 0$ the current is $i_{\text{cir}}^* = \Delta I_i / 2$.

Note, that in contrast with (25) the current (54) for $l = 0$ is not zero. At the peak of the voltage response $\psi^\pm = \pi / 2$ and for $i_b = \Sigma I_i$ expression (54) can be simplified if $\Delta I_i$ or $\Delta r_n$ is zero.

For $\Delta I_i = 0$ it is

$$i_{\text{cir}}^* = - \frac{\Delta r_n \Sigma I_i}{2 \Sigma r_n} \left( 1 - \frac{\Sigma r_n}{\Sigma r_n + 2 \sqrt{I_i I_2}} \right).$$  

Remembering that $\Delta V_i = \Sigma I_i \Delta r_n / 2$ and $\Sigma V_i = \Sigma I_i \Sigma r_n / 2$, one can see that (56) is just the second term of the expression (45) for the voltage response amplitude without the factor $\Delta r_n / 2$ arising from the first term of (42). Since the expression in the brackets (56) is always positive, the averaged circulating current in this case decreases the voltage response because it always flows toward the junction with smaller shunt resistor. Comparing (55) and (56) we conclude that the applied flux increases the averaged circulating current induced due to inequality of the shunt resistances.

For $\Delta r_n = 0$ the averaged circulating current is

$$i_{\text{cir}}^* = \frac{\Delta I_i}{2 \Sigma I_i + 2 \sqrt{I_i I_2}}.$$  

Since the second factor in (57) is less than unity, it is seen that the applied flux decreases the averaged circulating current induced due to inequality of the critical currents (compare (55) and (57)).

Figure 11(a) presents the averaged circulating current (54) calculated for $\Delta I_i = 0$, $\Delta r_n = -0.35$ and $\Delta I_i = 0.3$, $\Delta r_n = 0$ at $i_b = \Sigma I_i = 1.9$, $\Sigma r_n = 2.05$, $l = 0$ (solid lines). Corresponding data calculated numerically using system (29) are shown by dots. It is seen that even for small absolute current values $i_{\text{cir}}^*$ the data corresponding to the averaged circulating current in SQUID with asymmetry of the critical currents are perfectly consistent. Although the data for

Figure 10. (a) Transcendental function $f$ for symmetrical SQUID (21), $g(\psi)$ factor affecting its nonlinear term in asymmetrical case, and $f^*(\psi)$ function (50) for asymmetrical SQUID with parameters $\Sigma I_i = 1.9$, $\Delta I_i = -0.3$, $\Sigma r_n = 2.05$, $\Delta r_n = 0.35$, $l = 1$. (b) The circulating currents $i_{\text{cir}}$, $i_{\text{cir}}^*$ corresponding to the functions $f$, $f^*$ presented in panel (a), calculated using expressions (16), (24), (53) (lines). Numerical data obtained using system (29) are shown by dots for comparison. Vertical arrows show direction of variation of functions.
SQUID with asymmetry of the shunt resistances differ noticeably at \( \phi_0 \approx \pi/2 \) resulting from approximate solution for \( \psi^+ \) (41), qualitative behavior is still found correctly for both asymmetries.

Figure 11(b) shows the circulating current calculated for the same set of parameters as the ones taken for figure 9(a) but \( l_b = 2.45 \) \((\Delta l = 0)\), and for SQUID with inverted asymmetry of the critical currents and shunt resistances \((\Delta i_c \rightarrow - \Delta i_c, \Delta r_s \rightarrow - \Delta r_s\) while \(\Sigma i_c, \Sigma r_s\) are held the same). The data obtained by expression (54) are shown by lines, and the ones calculated numerically using (29) are presented by dots. While consistency of the curves remains just qualitative, the offset to the averaged circulating current is found precisely.

5. SQUID with inductance of practical device

The voltage-flux and current-flux functions obtained in sections 2, 3 for small state are valid only for SQUID with small inductance \( l \leq 1 \). In this section we fit numerical data for the SQUID responses by our analytical expressions introducing fitting parameters to expand the frame of validity of our approach to higher values of the inductance. This makes our expressions suitable for the design process of practical SQUIDs and SQUID-based circuits.

5.1. SQUID voltage response

In section 2 it was considered that expression (14) obtained for the voltage response amplitude is relevant only for small values of the inductance \( l \leq 1 \). Since the voltage \( \varphi \) across symmetrical SQUID at \( i_b = 2 \) and \( \phi_0 = 0 \) is zero, the response amplitude is just \( \varphi \) at \( \phi_0 = \pi/2 \). Comparison of dependencies \( \varphi(l) \) calculated using (14) and obtained by numerical calculations of system (1) is shown in figure 12. It is seen that proposed linearization of (1) leads to inaccurate determination of the voltage response amplitude for \( l > 1 \) that limits application of the derived expressions. To make our approach useful in practical circuit design we propose usage of expression (7) for the SQUID voltage response with fitting parameters which can be found from numerical calculations of (1) as follows.
where $l_{\text{fit}}$, $A_1$ are the fitting parameters. While the parameters are defined for arbitrary inductances and bias currents, their expressions are rather complicated

$$A_1 = \frac{(\frac{b}{7})^{0.87}}{(1 - \alpha (\frac{b}{7})^2)(1 + \alpha)(0.91f^{1.4} + 2.26)},$$

where

$$\alpha = \frac{f^{2.32}}{1.4f^{2.39} + 0.31},$$

$$\beta = \frac{\sqrt{2} - 1}{f_{\beta \gamma}} + \frac{32f^2}{12 - 8\sqrt{2}},$$

$$f_{\beta} = \frac{f^{2.84} + 0.69}{6.84f^{3.35} + 6.53},$$

and

$$l_{\text{fit}} = \frac{\sqrt{2} - 1}{f_j} \left(1 + \frac{\text{sgn}(l - 1.3)}{12 - 8\sqrt{2}}\right)^2,$$

$$f_j = \frac{f^{2.03} + 1.2}{5.42f^{2.34} + 9.81}.$$  

Figure 14(a) shows the time-averaged circulating current calculated at different bias currents and inductances using (66) in comparison with corresponding numerical calculations of system (1). The results indicate that the numerical data are fitted by (66) fairly well.
Dependencies of the SQUID current response versus the inductance and versus the bias current calculated by (66) are presented in figures 14(b), (c). It is interesting to note that the time-averaged circulating current has a maximum over parameters \( \phi_c, I, i_b \). Nonzero value of \( i_{cir} \) means that integral over time of the current circulating in the SQUID in the time interval between switchings of the junctions (when a fluxon is passing by the SQUID) is greater than that corresponding to leveling of the currents flowing through the junctions which happens after their switchings.

Obviously, there is no circulating current if the junctions switch simultaneously (\( \phi_c = 0 \)), while if they switch in antiphase (\( \phi_c = \pi/2 \)) the corresponding integrals are equalized. This gives optimum value for \( \phi_c \) maximizing \( |i_{cir}| \) about \( \phi_c = \pi/4 \).

The current \( i_{cir} \) at this applied flux versus both remaining parameters \( I, i_b \) are shown in figure 14(d). Optimum value for the inductance outflows from the fact that for \( I \rightarrow 0 \) the junctions are synchronized inphase, and the circulating current is purely symmetric over the oscillation period. At the same time, for \( I \rightarrow \infty \) the circulating current is negligible due to LR-filtering.

The bias current is taken into consideration through the oscillation frequency \( w_I \). For small bias current the frequency is low, and so the time left after switchings for leveling of the currents flowing through the junctions prevails over the time of fluxon passage, which equalizes the impacts of the integrals. High bias currents mean high frequencies \( w_I \) at which the circulating current is suppressed by LR-filtering as it was considered in section 2.

Certain values of the parameters providing maximum time-averaged circulating current can be found from (66) as \( \{ \phi_c, I, i_b \} = \{ 0.21/\pi, 2.1, 2.56 \} \), at which the current is \( i_{cir} = 0.076 \). The result obtained shows that this current can be safely omitted in the symmetrical case in consideration of more complex circuits e.g. a bi-SQUID as it was done in the work [29].

6. Discussion

Results presented in this paper provide qualitative and quantitative understanding of processes involved in formation of DC SQUID voltage and current responses. Expressions obtained for the voltage-flux and current-flux functions for practical values of SQUID inductance can be used in design of SQUID-based circuits.

One of the difficult problems is modeling of large SQUID arrays. An array containing 2400 Josephson junctions was utilized in recent work [14] as electrically small antenna capable of detecting radio frequencies from distant sources. While shape of the voltage response is one of the most important characteristics of such structures its numerical calculation is quite time consuming. Indeed standard tools for superconductor circuit simulations such as PSCAN practically limit circuit complexity to \( \sim 500 \) junctions, so complex schematics have to be broken into several sub-schematics [45]. In this respect the proposed analytical approach to describe SQUID time-averaged characteristics provides a valuable solution.

To illustrate the applicability of our approach to the design of SQUID arrays we calculate the voltage responses of serial SQIFs with the uniform distribution of inductances of their cells in the range \( l_{SQ} \in [1, 6.8] \). Area of the cells are given as \( a_{SQ} = l_{SQ}^2 \). Curves obtained using presented analytical approach (expression (61)) for SQIF structures with number of cells \( N_{SQ} = 20 \), 2000 are shown by solid lines (the last curve is scaled 1/50). Dots show data for the SQIF with \( N_{SQ} = 20 \) calculated numerically using PSCAN. Analytical curve calculated in zero inductance approximation \( (l_{SQ} = 0) \) for this SQIF with \( N_{SQ} = 20 \) is presented by dotted line.

![Figure 15. Voltage responses of SQIF structures with normal distribution of inductances of their cells in the range \( l_{SQ} \in [1, 6.8] \). Area of the cells are given as \( a_{SQ} = l_{SQ}^2 \). Curves obtained using presented analytical approach (expression (61)) for SQIF structures with number of cells \( N_{SQ} = 20 \), 2000 are shown by solid lines (the last curve is scaled 1/50). Dots show data for the SQIF with \( N_{SQ} = 20 \) calculated numerically using PSCAN. Analytical curve calculated in zero inductance approximation \( (l_{SQ} = 0) \) for this SQIF with \( N_{SQ} = 20 \) is presented by dotted line.](image)

It is seen that data calculated numerically and using our analytical approach are perfectly consistent. At the same time, the curve calculated in zero inductance approximation (for \( l_{SQ} = 0 \)) can provide only a rough estimation of the voltage response amplitude and shape. We got no significant delay in time of calculation of the curve for the SQIF structure with \( N_{SQ} = 2000 \) cells compared to the ones with \( N_{SQ} = 20 \) using our approach. It took about a second on a conventional laptop, while the corresponding time of numerical calculation would be at least three orders greater. This means that our analytical expressions can be readily used for optimization of such complex circuits.

7. Conclusion

In conclusion, we have developed an analytical approach for calculation of DC SQUID voltage and current responses in the resistive state for inductance in the range \( l \lesssim 1 \). Using two
fitting parameters we expanded the context of validity of our approach to practical values of the inductance up to $I \approx 7$. The circulating current in superconducting state was found for arbitrary values of the inductance. We considered the effect provided by technological spread of SQUID parameters relevant to LTS technology, generalizing our approach to a case of slightly different critical currents and shunt resistances of SQUID junctions, and unequal SQUID inductive shoulders. We showed that our analytical expressions can be used for calculation of practical SQUID and serial SQUID array responses, confirming this by comparison with numerical calculation results.

Acknowledgments

Authors are grateful to Matthias Schmelz for careful reading of the paper and valuable suggestions. This work was supported by President of Russian Federation grant MK-5813.2016.2, Ministry of Education and Science of the Russian Federation, grant No 14.Y26.31.0007, RFBR grants No. 15-32-20362-mol_a_ved, 16-29-09515-o, and grant for leading scientific school No. 8168.2016.2. Mikhail Kupriyanov would like to thank support of Program of Competitive Growth at Kazan Federal University.

Development of analytical description of symmetrical SQUID voltage and current responses was performed under support of RFBR grant No. 15-32-20362-mol_a_ved. Study of effect of small technological spread of parameters on SQUID responses was done in the frame of President of Russian Federation grant MK-5813.2016.2. Applicability of developed analytical expressions to practical LTS SQUID-based circuits design was studied under support of RFBR grant No. 16-29-09515-o.f.

References

[1] Clarke J and Braginsky A I 2004 The SQUID Handbook vol 1 (Weinheim: Wiley)
[2] Weinstock H (ed) 1996 SQUID sensors: Fundamentals, Fabrication and Applications (Dordrecht: Kluwer) 1–62
[3] Granata C and Vettoliere A 2016 Nano Superconducting Quantum Interference device: a powerful tool for nanoscale investigations Phys. Rep. 614 1–69
[4] Schöna T, Zakosarenko V, Schmelz M, Stolz R, Anders S, Linzen S, Meyer M and Meyer H-G 2015 A three-axis SQUID-based absolute vector magnetometer Rev. Sci. Instrum 86 105002
[5] Chwala A, Stolz R, Schmelz M, Zakosarenko V, Meyer M and Meyer H-G 2015 SQUID Systems for Geophysical Time Domain Electromagnetics (TEM) at IPHT Jena IEICE Trans. Electron. E98C 167–73
[6] Schöna T, Schmelz M, Zakosarenko V, Stolz R, Meyer M, Anders S, Fritzsch L and Meyer H-G 2013 SQUID-based setup for the absolute measurement of the Earth’s magnetic field Supercond. Sci. Technol. 26 035013
[7] Schmelz M, Stolz R, Zakosarenko V, Schöna T, Anders S, Fritzsch L, Mück M and Meyer H-G 2011 Field-stable SQUID magnetometer with sub-fT Hz$^{-1/2}$ resolution based on sub-micrometer cross-type Josephson tunnel junctions Supercond. Sci. Technol. 24 065009
[8] Schmelz M, Matsui Y, Stolz R, Zakosarenko V, Schöna T, Anders S, Linzen S, Itozaki H and Meyer H-G 2015 Investigation of all niobium nano-SQUIDs based on sub-micrometer cross-type Josephson junctions Supercond. Sci. Technol. 28 015004
[9] Prokopenko G V and Mukhanov O A 2013 Wideband microwave low noise amplifiers based on biSQUID SQUIDs International Superconductive Electronics Conference (ISEC) IEEE 14th ISEC (July 7-11 2013)
[10] Schöna T, Schmelz M, Zakosarenko V, Stolz R, Anders S, Fritzsch L and Meyer H-G 2012 SQIF-based dc SQUID amplifier with intrinsic negative feedback Supercond. Sci. Technol. 25 015005
[11] Kornev V K, Soloviev I I, Klenov N V and Mukhanov O A 2007 Development of sqif-based output broad band amplifier IEEE Trans. Appl. Supercond. 17 569–72
[12] Zakosarenko V et al 2011 Time-domain multiplexed SQUID readout of a bolometer camera for APEX Supercond. Sci. Technol. 24 015011
[13] Kornev V K, Soloviev I I, Sharaﬁev A V, Klenov N V and Mukhanov O A 2013 Active electrically small antenna based on superconducting quantum array IEEE Trans. Appl. Supercond. 23 1800405
[14] de Andrade M C, de Escobar A L, Taylor B J, Berggren S, Häussler C, Oppenländer J and Schopohl N 2001 Nonperiodic macroscopic quantum interference in one—dimensional parallel Josephson junction arrays with unconventional grating structure Phys. Rev. B 63 024511
[15] Häussler C, Oppenländer J and Schopohl N 2001 Nonperiodic flux to voltage conversion of series arrays of dc superconducting quantum interference devices J. Appl. Phys. 89 1875
[16] Kornev V K, Soloviev I I, Klenov N V and Mukhanov O A 2009 Bi-SQUID novel linearization method for dc SQUID voltage response Supercond. Sci. Technol. 22 114011
[17] Kornev V K, Soloviev I I, Klenov N V and Mukhanov O A 2011 Design and experimental evaluation of SQUID arrays with linear voltage response IEEE Trans. Appl. Supercond. 21 394–8
[18] Mukhanov O A et al 2008 Superconductor digital-RF receiver systems IEICE Trans. Electron. E91C 306–17
[19] Kornev V K, Soloviev I I, Klenov N V and Mukhanov O A 2010 Progress in high-linearity multi-element josephson structures Physica C 470 886–9
[20] Kornev V K, Soloviev I I, Klenov N V, Sharaﬁev A V and Mukhanov O A 2011 Linear bi-SQUID arrays for electrically small antennas IEEE Trans. Appl. Supercond. 21 713–6
[21] Longhini P, Berggren A, Palacios A, In V and de Escobar A L 2011 Modeling Non-locally coupled DC SQUID arrays IEEE Trans. Appl. Supercond. 21 391–3
[22] Sharaﬁev A, Soloviev I I, Kornev V, Schmelz M, Stolz R, Zakosarenko V, Anders S and Meyer H-G 2012 Bi-SQUIDs with submicron cross-type Josephson tunnel junctions Supercond. Sci. Technol. 25 045001
[23] Longhini P et al 2012 Voltage response of non-uniform arrays of bi-superconductive quantum interference devices Journ. Appl. Phys. 111 093920
[26] Prokopenko G V, Mukhanov O A, de Escobar A L, Taylor B, de Andrade M C, Berggren S, Longhini P, Palacios A, Nisenoff M and Fagaly R L 2013 DC and RF measurements of serial bi-SQUID arrays IEEE Trans. Appl. Supercond. 23 1400607

[27] Berggren S et al 2013 Development of 2D bi-SQUID arrays with high linearity IEEE Trans. Appl. Supercond. 23 1400208

[28] Wu S M, Cybart S A, Anton S M and Dynes R C 2013 Simulation of Series Arrays of Superconducting Quantum Interference Devices IEEE Trans. Appl. Supercond. 23 1600104

[29] Kornev V K, Sharafiev A V, Soloviev I I and Mukhanov O A 2014 Signal and noise characteristics of bi-SQUID Supercond. Sci. Technol 27 115009

[30] Barone A and Paterno G 1982 Physics and applications of the Josephson effect (New York: Wiley)

[31] Likharev K K 1986 Dynamics of Josephson junctions and circuits (Amsterdam: Gordon and Breach)

[32] Chesca B 1998 Analytical theory of DC SQUIDs operating in the presence of thermal fluctuations J. Low. Temp. Phys. 112 165–96

[33] Chesca B 1999 The effect of thermal noise on the IV curves of high inductance dc SQUIDs in the presence of microwave radiation J. Low. Temp. Phys. 116 167–86

[34] Chesca B 1999 The effect of thermal fluctuations on the operation of DC SQUIDs at 77 K a fundamental analytical approach IEEE Trans. Appl. Supercond. 9 2955–60

[35] Greenberg Ya S 2002 Theory of the voltage-current characteristic of high Tc DC SQUIDs Physica C 371 156–72

[36] Greenberg Ya S 2003 Theory of the voltage-current characteristics of high TC asymmetric DC SQUIDs Physica C 383 354–64

[37] Greenberg Ya S, Novikov I L, Schultze V and Meyer H-G 2005 The influence of the second harmonic in the current-phase relation on the voltage-current characteristic of high Tc DC SQUID Eur. Phys. J. B 44 57–62

[38] Groenbech-Jensen N, Thompson D B, Cirillo M and Cosmelli C 2003 Thermal escape from zero-voltage states in hysteretic superconducting interferometers Phys. Rev. B 67 224505

[39] Romeo F and De Luca R 2004 Effective non-sinusoidal current-phase dependence in conventional d.c. SQUIDs Phys. Lett. A 328 330–4

[40] Torre G and de Luca R 2013 Persistent currents and magnetic susceptibility of two-junction quantum interferometers Results in Physics 3 179–81

[41] De Luca R, Fedullo A and Gasanenko V A 2007 Perturbation analysis of the dynamical behavior of two-junction interferometers Eur. Phys. J. B 58 461–7

[42] Peterson R L and McDonald D G 1983 Voltage and current expressions for a two-junction superconducting interferometer Journ. Appl. Phys. 54 992–6

[43] Jaklevic R C, Lambe J, Silver A H and Mercereau J E 1964 Quantum interference effects in Josephson tunneling Phys. Rev. Lett. 12 159

[44] Stewart W C 1968 Current-voltage characteristics of Josephson junctions Appl. Phys. Lett. 12 277

[45] Mukhanov O A 2015 Recent progress in digital superconducting electronics International Superconductive Electronics Conference (ISEC) IEEE XV ISEC (6-9 July 2015)