General Stress Field for Cracked Orthotropic Plate

Purong Jia, Yongyong Suo and Cheng Jia
School of Mechanics and Civil Engineering & Architecture, Northwestern Polytechnical University, Xi’an 710129, PR China.
Email: prjia@nwpu.edu.cn

Abstract. The solution of singular stresses is discussed for the crack problem of the orthotropic plate. Essential equations are introduced on the basis of linear elastic mechanics and complex variable method. The complex variable functions including the material parameters are fully analyzed for solving the partial derivation equation and to meet the needs of the crack boundary conditions. By constructing new stress functions, the mechanic analysis of the cracked orthotropic plate is carried out, and also the stress boundary problem is resolved. The expressions of the singular stress fields at the crack-tip are determined.

1. Introduction
Nowadays the fracture mechanics of anisotropic materials may really be important and relative to many engineering structures. And the orthotropic plates have been for the base of composite materials in common use [1–3]. The valid method to solve crack tip field problems in anisotropic materials must use the complex analytic function theory. The complex variable theory provides a very powerful tool for the solution of many boundary value problems in the elastic body and the results have been reported [4–6]. But the general solutions of the fracture mechanics for composite materials have not given completely or perfectly. Therefore, it is necessary to make up new and general solutions for the singular stresses of cracked composite plates. Particularly, the study of ordinary crack problem as shown in Figure 1 must be very necessary for the fracture mechanics of composite materials.

Figure 1. Scheme of the plate and coordinates

2. Basic Equations

2.1. Complex Variable Function
To solve the problems of the plane stress boundaries, both rectangular and polar coordinates have proved to be very adequate and useful. In the solution of the partial differential equation and the construction of suitable stress functions, it is very advantageous to use complex variables. We know
that two real numbers $x$ and $y$ form the complex number $z$, $z = x + iy$, with $i$ representing the imaginary number. And often the conjugate complex number $\overline{z} = x - iy$ must be used together. For the convenience of general investigation, another complex variable number $w$ and its conjugate $\overline{w}$ are also be introduced and defined by:

$$w = x + ihy, \quad \overline{w} = x - ihy$$

(1)

Where $h$ is a real arbitrary constant. To aim at giving a precise definition, we suppose the constant $h$ to be positive, $h > 0$. By the definition, we have that: $w\overline{w} = x^2 + h^2y^2$, since $i^2 = -1$ or $i = \sqrt{-1}$. In addition, converting to polar coordinates, as in Figure 1, the complex number can be given as:

$$w = x + ihy = r(\cos \theta + ih \sin \theta)$$

(2)

Next, we consider a complex function, $\Phi(w)$, and $\Phi$ is an analytic function of $w$. Then, we may separate $\Phi$ into real part $P = P(x, y) = \text{Re} \Phi$ and imaginary part $Q = Q(x, y) = \text{Im} \Phi$, as below

$$\Phi = \Phi(w) = P + iQ = P(x, y) + iQ(x, y)$$

(3)

Where, $P$ and $Q$ are real functions. Thus, the partial derivative relations must be obtained by

$$\frac{\partial \Phi}{\partial x} = \frac{d\Phi}{dw} \frac{\partial w}{\partial x} = \Phi' = \frac{\partial P}{\partial x} + i \frac{\partial Q}{\partial x}, \quad \frac{\partial \Phi}{\partial y} = \frac{d\Phi}{dw} \frac{\partial w}{\partial y} = ih\Phi' = \frac{\partial P}{\partial y} + i \frac{\partial Q}{\partial y}$$

(4)

$$\text{Re} \Phi' = \frac{\partial P}{\partial x} = \frac{1}{h} \frac{\partial Q}{\partial y}, \quad \text{Im} \Phi' = -\frac{1}{h} \frac{\partial P}{\partial y}$$

(5)

$$\frac{\partial^2 P}{\partial x^2} = \frac{1}{h} \frac{\partial^2 Q}{\partial x \partial y}, \quad \frac{\partial^2 P}{\partial y^2} = \frac{1}{h^2} \frac{\partial^2 Q}{\partial y^2}, \quad \text{Im} \Phi'^* = -\frac{1}{h} \frac{\partial^2 P}{\partial x \partial y} = -\frac{1}{h^2} \frac{\partial^2 Q}{\partial y^2}$$

(6)

2.2. Partial Derivation Equations

In the absence of the body forces, the stress components of two-dimensional problems can be expressed by a real stress function $F = F(x, y)$ as following:

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y}$$

(7)

The equilibrium equations can be satisfied naturally. For the plane stress problems, the stress function $F$ must be selected reasonably in terms of the boundary conditions. In order to solve two-dimensional elastic problems of orthotropic plate, we suppose the principal elastic directions of the plate coincide with the coordinate directions, and also we let the directions 1, 2 parallel to the axes $x$, $y$, respectively. By using the stress function, the governing equation of the strain compatibility condition can be expressed as follows:

$$\frac{\partial^4 F}{\partial y^4} + 2B \frac{\partial^4 F}{\partial x^2 \partial y^2} + C \frac{\partial^4 F}{\partial x^4} = 0$$

(8)

Where, $B = \frac{E_1}{2G_{12}} - \nu_{12}$, $C = \frac{E_1}{E_2}$ are representative of elasticity coefficients.

Above basic equations have been appeared in hand books and some references. Notice that the key point may be to find the stress function $F$ in terms of stress boundary conditions and to be suitable for the governing equation with four order partial derivation.
3. General Solution

3.1. Stress Function and Solution

The stress function \( F(x, y) \) can be expressed by the real part \( P \) and imaginary part \( Q \) of the complex variable function. Therefore, we must to consider the boundary conditions of a cracked plate shown in Figure 1 and to select some stress functions reasonably, so as to obtain useful expressions for the stresses. By the previous experience, the stress function can be determined in different forms to be written as follows.

3.1.1. Case one

For the first case, we may take the stress function as the below form:

\[
F = A_1P + A_2Q + A_3y \frac{\partial P}{\partial x} + A_4y \frac{\partial Q}{\partial x} + \frac{T}{2} y^2
\]  

(9)

Thereby, some partial derivations can be obtained as:

\[
\frac{\partial F}{\partial x} = A_1 \frac{\partial P}{\partial x} + A_3y \frac{\partial^2 P}{\partial x^2} + A_2 \frac{\partial Q}{\partial x^2} , \quad \frac{\partial F}{\partial y} = A_3h^2 y \frac{\partial^2 P}{\partial x^2} + A_2 \frac{\partial P}{\partial x} - A_4h^2 \frac{\partial^2 Q}{\partial x^2} + Ty
\]

\[
\frac{\partial^2 F}{\partial x^2} = A_1 \frac{\partial^2 P}{\partial x^2} + A_3h^2 y \frac{\partial^3 P}{\partial x^3} + A_2 \frac{\partial^3 Q}{\partial x^3} , \quad \frac{\partial^2 F}{\partial x \partial y} = A_3h^2 y \frac{\partial^3 P}{\partial x^3} + A_2 \frac{\partial^3 P}{\partial x^2} - A_4h^2 y \frac{\partial^3 Q}{\partial x^3}
\]

By using above derivative relations, the governing equation (8) must be transformed into:

\[
(h^4 - 2Bh^2 + C)(A_1 \frac{\partial^4 P}{\partial x^4} + A_3h^2 y \frac{\partial^5 P}{\partial x^5} + A_2 \frac{\partial^5 Q}{\partial x^5}) + 4h(h^2 - B)(A_2 \frac{\partial^4 Q}{\partial x^4} - A_1 \frac{\partial^4 P}{\partial x^4}) = 0
\]  

(10)

Obviously, the coefficients must become zero. Then the constant \( h \) can be determined as:

\[
h^2 = B , \quad h^4 = B^2 = C , \quad h = \frac{E_1}{2G_{12}} - \nu_{12} = \frac{E_1}{E_2}
\]  

(11)

Furthermore, by substituting above partial derivative relations into the expression (7) and utilizing partial derivative equations (5) and (6), the stress components can be expressed by:

\[
\begin{align*}
\sigma_x &= A_1h^2 \text{Re}\Phi^*-A_1h^3y\text{Im}\Phi^*-2A_2h\text{Im}\Phi^*-A_2h^2y\text{Re}\Phi^*+T \\
\sigma_y &= A_1\text{Re}\Phi^*+A_1h^2y\text{Im}\Phi^*+A_2\text{Re}\Phi^* \\
\tau_{xy} &= -A_1h^2y\text{Re}\Phi^* - A_2 \text{Re}\Phi^* + A_2h^2y\text{Im}\Phi^*
\end{align*}
\]  

(12)

3.1.2. Case two

For the second case, we can select the stress and complex variable functions as below:

\[
F = A_1P_1 + A_2P_2 + A_4Q_2 + A_3Q_2 + \frac{T}{2} y^2
\]  

(13)

\[
\Phi_1(w_1) = P_1(x, y) + iQ_1(x, y) , \quad \Phi_2(w_2) = P_2(x, y) + iQ_2(x, y)
\]  

(14)
Where, \( w_1 = x + ih_1 y \), \( w_2 = x + ih_2 y \), \( \text{Re} \Phi_1 = P_1 \), \( \text{Im} \Phi_1 = Q_1 \), \( \text{Re} \Phi_2 = P_2 \), \( \text{Im} \Phi_2 = Q_2 \). And also, the derivative relations can be determined by:

\[
\begin{align*}
\text{Re} \Phi_1' &= \frac{\partial P_1}{\partial x} = \frac{1}{h_1} \frac{\partial Q_1}{\partial y}, & \text{Im} \Phi_1' &= \frac{\partial Q_1}{\partial x} = -\frac{1}{h_1} \frac{\partial P_1}{\partial y} \\
\text{Re} \Phi_2' &= \frac{\partial P_2}{\partial x} = \frac{1}{h_2} \frac{\partial Q_2}{\partial y}, & \text{Im} \Phi_2' &= \frac{\partial Q_2}{\partial x} = -\frac{1}{h_2} \frac{\partial P_2}{\partial y}
\end{align*}
\]

(15)

So the partial derivations of stress function can be obtained easily. In terms of the stress function and derivative relations, the governing equation (8) can become as:

\[
[h_1^4 - 2Bh_2^2 + C][\frac{\partial^4 P_1}{\partial x^4} + A_1 \frac{\partial^4 Q_1}{\partial x^4}] + [h_2^4 - 2Bh_2^2 + C][\frac{\partial^4 P_2}{\partial x^4} + A_2 \frac{\partial^4 Q_2}{\partial x^4}] = 0
\]

(16)

Thus, we have: \( h_1^4 - 2Bh_2^2 + C = 0 \), \( h_2^4 - 2Bh_2^2 + C = 0 \). Then, we can solve both equations for the constants as follows:

\[
\begin{align*}
h_1 &= \sqrt{B + \sqrt{B^2 - C}} = \sqrt{\frac{E_1}{2G_{12}} - v_{12} + \left(\frac{E_1}{2G_{12}} - v_{12}\right)^2 - \frac{E_1}{E_2}} \\
h_2 &= \sqrt{B - \sqrt{B^2 - C}} = \sqrt{\frac{E_1}{2G_{12}} - v_{12} - \left(\frac{E_1}{2G_{12}} - v_{12}\right)^2 - \frac{E_1}{E_2}}
\end{align*}
\]

(17)

Similarly, the stress components can be determined by:

\[
\begin{align*}
\sigma_x &= -h_1^2 A_1 \text{Re} \Phi_1^* - h_2^2 A_2 \text{Re} \Phi_2^* - h_1^2 A_3 \text{Im} \Phi_1^* - h_2^2 A_4 \text{Im} \Phi_2^* + T \\
\sigma_y &= A_1 \text{Re} \Phi_1^* + A_2 \text{Re} \Phi_2^* + A_3 \text{Im} \Phi_1^* + A_4 \text{Im} \Phi_2^* \\
\tau_{xy} &= h_1 A_1 \text{Im} \Phi_1^* + h_2 A_2 \text{Im} \Phi_2^* - h_1 A_3 \text{Re} \Phi_1^* - h_2 A_4 \text{Re} \Phi_2^*
\end{align*}
\]

(18)

3.2. Coordinate Transformation and Complex Function

To consider the polar coordinate system, we introduce another letter \( \beta \) to make the transformation of the coordinate axes. In terms of the polar coordinates, the complex variable can be written as:

\[
w = x + iy = r \cos \theta + ihr \sin \theta = r \lambda^2 (\cos \beta + i \sin \beta) = r \lambda^2 e^{i\beta}
\]

(19)

Where, \( \cos \theta = \lambda^2 \cos \beta \), \( h \sin \theta = \lambda^2 \sin \beta \), \( hy = hr \sin \theta = r \lambda^2 \sin \beta \). Thus we have that:

\[
\tan \beta = h \tan \theta, \quad \cos^2 \theta + h^2 \sin^2 \theta = \lambda^2, \quad \lambda = \frac{1}{\sqrt{\cos^2 \theta + h^2 \sin^2 \theta}}
\]

(20)

To solve the crack problem, the stress function can be selected by:

\[
\begin{align*}
\Phi^* &= \frac{1}{\sqrt{w}} = \frac{1}{\sqrt{r}} \frac{e^{-i\beta/2}}{\lambda} = \frac{1}{\sqrt{r}} \frac{1}{\lambda} \left(\cos \frac{\beta}{2} - i \sin \frac{\beta}{2}\right) \\
\Phi^w &= \frac{-1}{2 \sqrt{w^3}} = -\frac{1}{\sqrt{r}} \frac{1}{2r \lambda^3} \left(\cos \frac{3\beta}{2} - i \sin \frac{3\beta}{2}\right)
\end{align*}
\]

(21)

Hence it can be seen that the solution becomes
\[
\begin{align*}
\text{Re} \Phi^* &= \frac{1}{\sqrt{r}} \frac{1}{\lambda} \cos \frac{\beta}{2} , \\
\text{Im} \Phi^* &= -\frac{1}{\sqrt{r}} \frac{1}{\lambda} \sin \frac{\beta}{2} \\
\text{hy Re} \Phi'^{*} &= -\frac{1}{\sqrt{r}} \frac{1}{2\lambda} \sin \beta \cos \frac{3\beta}{2} = -\frac{1}{\sqrt{r}} \frac{1}{\lambda} \sin \beta \cos \beta \cos \frac{3\beta}{2} \\
\text{hy Im} \Phi'^{*} &= \frac{1}{\sqrt{r}} \frac{1}{2\lambda} \sin \beta \cos \frac{3\beta}{2} = \frac{1}{\sqrt{r}} \frac{1}{\lambda} \sin \beta \cos \beta \sin \frac{3\beta}{2}
\end{align*}
\] (22)

4. Stress Field

On the basis of above complex functions, the stress components can be written in real functions. For the first case, the stress expression (12) can be determined by:

\[
\begin{align*}
\sigma_x &= \frac{A_1 - h^2}{\sqrt{r} \lambda} \cos \frac{\beta}{2} (1 - \sin \frac{\beta}{2} \sin \frac{3\beta}{2}) + \frac{A_2}{\sqrt{r} \lambda} \sin \frac{\beta}{2} (2 + \cos \frac{\beta}{2} \cos \frac{3\beta}{2}) + T \\
\sigma_y &= \frac{A_1 - h^2}{\sqrt{r} \lambda} \cos \frac{\beta}{2} (1 + \sin \frac{\beta}{2} \sin \frac{3\beta}{2}) - \frac{A_2}{\sqrt{r} \lambda} \sin \frac{\beta}{2} (2 - \cos \frac{\beta}{2} \cos \frac{3\beta}{2}) \\
\tau_{xy} &= \frac{A_1 - h^2}{\sqrt{r} \lambda} \sin \frac{\beta}{2} \cos \frac{\beta}{2} \cos \frac{3\beta}{2} - \frac{A_2}{\sqrt{r} \lambda} \cos \frac{\beta}{2} (1 - \sin \frac{\beta}{2} \sin \frac{3\beta}{2})
\end{align*}
\] (23)

In terms of the boundary conditions, the crack surfaces are free. The conditions must be given as:
At \( \theta = \pm \pi \), then \( \beta = \pm \pi \) and \( \sigma_y = \tau_{xy} = 0 \). And also, at \( \theta = 0 \), we have: \( \beta = 0 \), \( \lambda = 1 \).

Therefore, we can determine the arbitrary constants in the following way:

\[
K_I = \lim_{r \to 0} \sqrt{2\pi} \sigma_y \bigg|_{\theta=0} = \sqrt{2\pi} A_1 , \quad K_{II} = \lim_{r \to 0} \sqrt{2\pi} \tau_{xy} \bigg|_{\theta=0} = -\sqrt{2\pi} A_2
\] (24)

Where, \( K_I \) and \( K_{II} \) are called the mode I and mode II stress intensity factors, respectively. Hence, the stress components can be expressed as follows:

\[
\begin{align*}
\sigma_x &= \frac{K_I}{\sqrt{2\pi \lambda}} \cos \frac{\beta}{2} (1 - \sin \frac{\beta}{2} \sin \frac{3\beta}{2}) - \frac{K_{II}}{\sqrt{2\pi \lambda}} \sin \frac{\beta}{2} (2 + \cos \frac{\beta}{2} \cos \frac{3\beta}{2}) + T \\
\sigma_y &= \frac{K_I}{\sqrt{2\pi \lambda}} \cos \frac{\beta}{2} (1 + \sin \frac{\beta}{2} \sin \frac{3\beta}{2}) + \frac{K_{II}}{\sqrt{2\pi \lambda}} \sin \frac{\beta}{2} (2 - \cos \frac{\beta}{2} \cos \frac{3\beta}{2}) \\
\tau_{xy} &= \frac{K_I}{\sqrt{2\pi \lambda}} \sin \frac{\beta}{2} \cos \frac{\beta}{2} \cos \frac{3\beta}{2} + \frac{K_{II}}{\sqrt{2\pi \lambda}} \cos \frac{\beta}{2} (1 - \sin \frac{\beta}{2} \sin \frac{3\beta}{2})
\end{align*}
\] (25)

For the second case, the transformation relation and stress function can be determined by:

\[
\tan \beta_1 = h_1 \tan \theta , \quad \tan \beta_2 = h_2 \tan \theta , \quad \lambda_1 = \frac{h_1 \cos^2 \theta + h_2 \sin^2 \theta}{\sqrt{h_1^2 \cos^2 \theta + h_2^2 \sin^2 \theta}} , \quad \lambda_2 = \frac{h_1 \cos^2 \theta + h_2 \sin^2 \theta}{\sqrt{h_1^2 \cos^2 \theta + h_2^2 \sin^2 \theta}}
\]

\[
\Phi^* = \frac{1}{\sqrt{w_1}} \frac{1}{\lambda_1} (\cos \frac{\beta_1}{2} - i \sin \frac{\beta_1}{2}) , \quad \Phi'^* = \frac{1}{\sqrt{w_2}} \frac{1}{\lambda_2} (\cos \frac{\beta_2}{2} - i \sin \frac{\beta_2}{2})
\]

The stresses and arbitrary constants in equation (18) can be obtained in a similar way to the first case. Therefore, the general stress fields can be given by:
\[
\sigma_x = \frac{K_I}{\sqrt{2\pi h_1 h_2}} \left( \frac{h_1}{\lambda_1} \cos \beta_1 - \frac{h_2}{\lambda_2} \cos \beta_2 \right) + \frac{K_{II}}{\sqrt{2\pi h_1 - h_2}} \left( \frac{h_1^2}{\lambda_1^2} \sin \beta_1 - \frac{h_2^2}{\lambda_2^2} \sin \beta_2 \right) + T
\]
\[
\sigma_y = \frac{K_I}{\sqrt{2\pi h_1 h_2}} \left( \frac{h_1}{\lambda_2} \cos \beta_2 - \frac{h_2}{\lambda_1} \cos \beta_1 \right) + \frac{K_{II}}{\sqrt{2\pi h_1 - h_2}} \left( \frac{h_1}{\lambda_1} \sin \beta_1 - \frac{h_2}{\lambda_2} \sin \beta_2 \right)
\]
\[
\tau_{xy} = \frac{K_I}{\sqrt{2\pi h_1 h_2}} \left( \frac{1}{\lambda_1} \sin \beta_1 - \frac{1}{\lambda_2} \sin \beta_2 \right) + \frac{K_{II}}{\sqrt{2\pi h_1 - h_2}} \left( \frac{h_1}{\lambda_1} \cos \beta_1 - \frac{h_2}{\lambda_2} \cos \beta_2 \right)
\]

In a word, both equations (25) and (26) are the general stress expressions for orthotropic materials.

5. Acknowledgments
The authors are very wishing to acknowledge the financial support by the Natural Science Foundation of China (NSFC Grant No. 51475372).

6. References
[1] Sih G C 1991. Mechanics of fracture initiation and propagation. Kluwer Academic Publishers.
[2] Fallah N, Nikraftar N 2018. Meshless finite volume method for the analysis of fracture problems in orthotropic media. Eng Fract Mech; 204: 46-62.
[3] Jia P, Suo Y, Jia C, Wang Q 2019. Stress analysis of orthotropic wedge loaded on the apex. IOP Conference Series: Materials Science and Engineering, 585: 448-453.
[4] Wang G, Jia P, Suo Y, et al 2019. Elasticity solution of composite material wedge loaded with a concentrated moment. Journal of Materials Science and Chemical Engineering, 7: 77-85.
[5] Fakoor M 2017. Augmented strain energy release rate (ASER): A novel approach for investigation of mixed-mode I/II fracture of composite materials. Eng Fract Mech; 179: 177-189.
[6] Zhang H, Qiao P 2019. A state-based peridynamic model for quantitative elastic and fracture analysis of orthotropic materials. Eng Fract Mech; 206: 147-171.