Semi-parametric resampling with extremes

Thomas Opitz\textsuperscript{a}, Denis Allard\textsuperscript{a}, Grégoire Mariethoz\textsuperscript{b},
\textsuperscript{a}Biostatistics and Spatial Processes (BioSP), INRA PACA, 84914 Avignon, France
\textsuperscript{b}Faculté des géosciences et de l’environnement, UNIL, Lausanne, Switzerland

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Abstract Nonparametric resampling methods such as Direct Sampling are powerful tools to simulate new datasets preserving important data features such as spatial patterns from observed or analogue datasets while using only minimal assumptions. However, such methods cannot generate extreme events beyond the observed range of data values. We here propose using tools from extreme value theory for stochastic processes to extrapolate observed data towards yet unobserved high quantiles. Original data are first enriched with new values in the tail region, and then classical resampling algorithms are applied to enriched data. In a first approach to enrichment that we label “naive resampling”, we generate an independent sample of the marginal distribution while keeping the rank order of the observed data. We point out inaccuracies of this approach around the most extreme values, and therefore develop a second approach that works for datasets with many replicates. It is based on the asymptotic representation of extreme events through two stochastically independent components: a magnitude variable, and a profile field describing spatial variation. To generate enriched data, we fix a target range of return levels of the magnitude variable, and we resample magnitudes constrained to this range. We then use the second approach to generate heatwave scenarios of yet unobserved magnitude over France, based on daily temperature reanalysis training data for the years 2010 to 2016.

Keywords: Direct Sampling; Extreme event; Heatwave; Pareto process; Threshold exceedance.

1 Introduction

Nonparametric resampling procedures for multidimensional data, including Analogue Methods (Lorenz, 1969; Yiou, 2014) or geostatistical approaches based on training images (Mariethoz and Caers, 2014), have become powerful tools for problems such as filling in missing data or generating new data scenarios. The general context of such applications is that training data are available, often stemming from observations of very high dimension, such as images. Although not being a model in the strict sense, the training data can be used as nonparametric models used for gener-
ating new data, which should carry the same features as the original data. Common applications include the generation of inputs for impact models and sensitivity analyses, or the use of the newly generated data for bootstrap-like estimation procedures of application-relevant parameters in the original data.

A strong restriction of purely nonparametric procedures, which make minimal assumptions on the data structure, is their inability to generate new data points with values beyond the range of observed values, especially in climate-related applications where measuring the impact of extreme events and new records is crucial. Moreover, when tails are rather heavy in the data distribution, then the observed data points close to the extremes typically have higher inter-point spacing as opposed to the dense scattering in the center of the distribution (de Haan and Ferreira, 2007), which calls for modifications to the simulation algorithm for generating new values lying between those near-extreme points in the lower and upper tails. In addition, the dependence structure at high quantiles can be different from that in the body of the distribution, and functional extreme value theory (i.e., the theory of extremes of stochastic processes) provides appropriate dependence characterizations and statistical tools (Ferreira and de Haan, 2014; Thibaud and Opitz, 2015; Davison and Huser, 2015).

In this contribution, we focus on a family of resampling algorithms called multiple-point statistics (or MPS), which aim at generating realizations of a spatial phenomenon (where “space” can also refer to “time”) based on the spatial dependencies observed in a given training dataset. The approach is based on nonparametric resampling of spatial patterns, which can be small patches or scattered values. As such, it has the advantage of reproducing very complex dependencies observed in the data, given that those dependencies are sufficiently represented in the training set. While these methods were initially designed to simulate categorical attributes (Guardiano and Srivastava, 1993; Strebelle, 2002), in recent years they have been extended to the simulation of continuous variables (Mariethoz et al., 2010; Zhang et al., 2006; Kalantari and Abdollahifard, 2016). In this context, the question of how to include values in the simulations that are outside the range of the training data has become acute in the practice of continuous-variable multiple-point simulations. This naturally extends to handling the occurrence of extreme values, especially when the simulation domain considered is much larger or spanning longer return intervals than the training domain.

Typically, nonparametric resampling shows good performance for reproducing data characteristics such as correlation and marginal distributions in simulated data when we consider the “central” part of the data distribution where training data are densely scattered. However, resampling extreme quantiles is more challenging since it is simply impossible to sample quantiles beyond the observed range of data when no assumptions are made on how to extrapolate data towards more extreme values. To address this issue, we propose in this work a lifting mechanism that enriches the training data such that they have more extreme values than only those observed in a limited dataset, while accounting for the dependence structure at these high quantiles. Our approach uses three main ingredients: 1) the knowledge or an estimation of the univariate marginal probability distribution $F$, especially of its tail behavior characterized by values $F(x)$ for high quantiles $x$ as $F(x)$ gets close to 1 (here, by default, we consider the upper tail); 2) minimal but robust assumptions on the dependence structure between values of
the stochastic process at high quantiles; 3) non-parametric resampling techniques, in our case Direct Sampling (Mariethoz et al., 2010). Our approach thus combines nonparametric methods able to account for complex dependence structures with a theoretically founded parametric model to properly account for univariate extremes. Previous approaches to lifting observed extreme episodes, using extreme-value theory similar to our method but without further resampling steps, have been proposed in Ferreira and de Haan (2014); Chailan et al. (2017); Palacios-Rodriguez et al. (2019).

We assume that training data have been generated by a stationary stochastic process \( \{X_i, i \in \mathcal{I}\} \), where the index set \( \mathcal{I} \) could refer to positions in time, space or space-time, such as regularly spaced observation times, the spatial grid of a spatial random field or a space-time grid. The assumption of stationarity is without loss of generality if appropriate pretransformations are available for nonstationary data, see the example of heatwave simulations over France in Section 6. To report results and formulas where the specific structure of the support is not important, we will assume (without loss of generality) that \( \mathcal{I} = \{1, 2, \ldots, n\} \). The theoretical results from extreme value theory underpinning the method that we propose do not require data on a regular grid, but the standard setup of nonparametric resampling techniques uses gridded data. We will write \( X_{ij}, i \in \mathcal{I}, j = 1, \ldots, m \), if observations at point \( i \) are repeated \( m \) times. Specifically, the case where values for \( j_1 \neq j_2 \) are stochastically independent is relevant.

The remainder of this paper is organized as follows. In Section 2 we present a univariate model of the marginal distribution with minimal assumptions, yet able to model accurately the bulk and the tail of the distribution. In Section 3 we present a first, straightforward approach for resampling extremes, which will prove to show inaccuracies close to the highest quantiles. For this reason we call this approach the naive approach. Based on fundamental results exposed in a short survey of extreme value theory for stochastic processes in Section 4, we develop a procedure for lifting observed extreme episodes in training data to more extreme quantiles, where we exploit the property of threshold stability arising in extreme value limits. This data enrichment step is the foundation for our second, more sophisticated resampling algorithm presented in Section 5. We illustrate this second approach in Section 6 for simulating heatwaves of unprecedented magnitude in France, using gridded daily reanalysis data provided by the French weather service (Météo France) for the 2010-2016 period as training sample. A discussion of our approach with an outlook to follow-up work concludes the paper in Section 7.

2 Modeling and estimating the marginal distribution

For data that are away from extremes, or whose accurate modeling may be of minor importance (e.g., low values in many contexts), the empirical distribution of training data is usually sufficiently accurate for resampling. Especially in the central region of the distribution, data are typically very dense and provide a good coverage of possible values. Nevertheless, to improve on the empirical distribution by allowing for generating new values between already observed ones, we propose to generate new values based on a kernel density estimator of the univariate density function of data. This also allows decreasing data dimension to reduce numerical computations and memory requirements, provided that the estimated density has a lower-dimensional numerical representation.
than the original, potentially very high-dimensional data, which is usually the case. Therefore, we can use the empirical distribution or the kernel density estimation below a high threshold \( u \). Kernel density estimators for spatial processes are consistent under extremely general conditions (Hallin et al., 2004), but since this point is out of the scope of this work, we do not address this aspect any further. In what follows, the kernel estimate of the pdf will be computed using a standard setting. Above this threshold, the decreasing availability of observations may lead to problematic artefacts, even more in the presence of correlation-induced clustering of values in space, as detailed in Section 3.2. Appropriate extrapolation beyond the observed maximum is thus uncertain when using standard kernels.

To generate new values beyond the observed range of data, and to avoid problems with resampling within the range of observations but close to the observed extremes where empirical coverage is only sparse, we need a model for the univariate distribution close to and beyond the extremes. Univariate extreme value theory provides the asymptotically motivated generalized Pareto distribution (GPD) as the theoretical limit distribution of positive threshold exceedances \( Y = X - u \mid X > u \) above a high threshold \( u \) (Davison and Huser, 2015). Its survival function is

\[
P(Y > y) = \begin{cases} 
(1 + \frac{\xi y}{\sigma})^{1/\xi} & \xi \neq 0, \\
\exp(-y/\sigma) & \xi = 0,
\end{cases}
\]

where \((x)_+ = \max\{0, x\}\), \( \sigma > 0 \), and the support for \( y \) is such that the right-hand side is not superior to 1. We denote the density of the GPD by \( f_{\text{GP}}(y \mid \sigma, \xi) \). The scale parameter \( (\sigma_u > 0) \) and the shape parameter \( (\xi_u \in \mathbb{R}) \) have to be estimated from the data. The tail of the univariate distribution is thus fully characterized by these two parameters and the probability \( p_u = P(X > u) \) of exceedance over the threshold \( u \). In our model, we require continuity of the estimated density \( \hat{f} \) of data at the threshold \( u \); see Carreau and Bengio (2009) and Scarrott and MacDonald (2012) (especially their Section 6.3) for closely related approaches of such ”piecing together”, and we therefore determine the parameter \( \sigma_u \) based on the kernel density estimate. This leaves the shape parameter \( \xi_u \) to be estimated by using one of a large variety of tail index estimators proposed in extreme value theory (Scarrott and MacDonald, 2012), for instance the moment estimator of Dekkers et al. (1989), known to be quite robust and accurate in almost all of practically relevant cases, or the likelihood estimator, or the Hill estimator in the heavy-tailed case with \( \xi_u > 0 \); see de Haan and Ferreira (2007) for a comparative discussion of estimators. The parameters of the GPD and of the kernel density estimate may further vary according to covariates or position in space. For notational and conceptual simplicity, we here describe the procedure only for the stationary setting. In summary, we proceed as described in Algorithm 1 below for estimating the univariate model.

Figure 1 illustrates the tail model and its estimation for data sampled from the Gaussian distribution or the log-Gaussian distribution. We assume that the true distribution is unknown, as it is usually the case for real data modeling. Data consist of a single replication \((m = 1)\) of a standard Gaussian random field on the unit square \([0, 1]^2\) with exponential covariance characterized by a scale parameter 0.03, simulated on a regular 250×250 grid. We consider \( p_u \in \{0.1, 0.01, 0.002\} \). The two left columns in Figure 1 show results for the data with Gaussian margins, while the
Algorithm 1 Estimation of the univariate probability density function

**Require:** $p_u$, a small probability of threshold exceedance, e.g. $p_u \in \{0.01, 0.05, 0.1\}$

1: Calculate a kernel density estimation $\hat{f}(x)$ of data.
2: Set $u$ such that $\int_u^\infty \hat{f}(x) \, dx = p_u$.
3: Set $\hat{\sigma}_u = p_u/\hat{f}(u)$.
4: Estimate $\hat{\xi}_u$ using an appropriate tail index estimator.
5: Define the univariate density model $\hat{f}$ for data:

$$
\hat{f}(x) = \begin{cases} 
\hat{f}(x), & x \leq u, \\
p_u f_{\text{GP}}(x | \hat{\sigma}_u, \hat{\xi}_u), & x > u.
\end{cases}
$$

6: return $\hat{f}(x)$.

Two right columns correspond to log-Gaussian margins, obtained by exponentiating the Gaussian observations. For better readability of the displays in these log-Gaussian case, their abscissas are given on log-scale. In all cases, the fitted tail model performs well, as can be seen from comparison with the histogram of observations. A complicating circumstance is that the Gaussian and the log-Gaussian distributions represent cases where the GPD is only an asymptotic approximation and not exact. Nevertheless, we can see that its fit provides an accurate tail model without making more specific assumptions on the marginal distributions, even at a relatively low threshold such as the one associated to $p_u = 0.1$. We here used the moment estimator of Dekkers et al. (1989) for the tail index. For $p_u \in \{0.1, 0.01, 0.002\}$, estimates of $\xi_u$ were $-0.08$, $-0.02$, $0.04$ respectively in the Gaussian case, and $0.33$, $0.34$, $0.38$ respectively in the log-Gaussian case.

3 Naive resampling of extremes

3.1 Naive resampling algorithm

This section develops a first data enrichment technique that is based on two components only: 1) a model to sample unobserved values from the marginal distribution, which requires a univariate model of $F$ estimated on the training data under minimal assumptions; 2) a mechanism, such as a multiple-point simulation algorithm, to generate a process of the same size as the training data while preserving the spatial, temporal or spatio-temporal dependence between ranks. In this procedure, the ranks of the training data are considered as the dependence model. We label this approach naive lifting since the independent sampling in step 1 may lead to certain inaccuracies, especially when the spacings between the highest-ranking values in simulated data become too large. We will discuss this issue in the following Subsection 3.2.

Given data $x_i$ indexed by $i \in I = \{1, \ldots, n\}$, we use the notation $r[i]$ to refer to the rank of $x_i$ among $x_1, \ldots, x_n$, and we write $r^{-1}[k]$ for the index $i$ of the $k$-th ranking value in $x_1, \ldots, x_n$. Algorithm 2 summarizes the procedure for naive resampling.
Figure 1: Univariate tail model $\hat{f}$ for Gaussian (columns 1,2) and log-Gaussian (columns 3,4; with log-scale for abscissas) data. Columns 2 and 4 show a zoom into the tail region. Exceedance probabilities are $p_u = 0.1$ (top), 0.01 (middle) and 0.002 (bottom). Other graphical elements are the threshold position (red), the kernel density (blue), the generalized Pareto density $f_{GP}$ (orange), and the true density (green).
Algorithm 2 Naive resampling

1: Generate a vector of ranks $\tilde{r}[i], i = 1, \ldots, n$, of the same size, $n$, as the training data and reproducing the dependence patterns (spatial, temporal or spatio-temporal) of the training data ranks $r[i], i = 1, \ldots, n$. Nonparametric resampling techniques such as MPS algorithms can be used.

2: Estimate the univariate density function $\hat{f}$ according to Algorithm 1.

3: Sample $n$ new data values $y_1, \ldots, y_n \sim \hat{f}$.

4: Put new data values in non decreasing order: $\tilde{y}_1 \leq \tilde{y}_2 \leq \ldots \leq \tilde{y}_n$.

5: Define new data $\tilde{x}_1, \ldots, \tilde{x}_n$ such that

$$\tilde{x}_{\tilde{r}[k]} = \tilde{y}_k, \quad k = 1, \ldots, n.$$ 

6: return $(\tilde{x}_1, \ldots, \tilde{x}_n)$.

3.2 Limitations of naive resampling

For ease of presentation we here suppose that the stationary marginal distribution of data is standard exponential. This assumption is without loss of generality since the marginal distribution is assumed to be known, and results regarding quantiles of the exponential distribution can be reformulated for the corresponding quantiles of any other continuous distribution. Therefore, data can always be transformed to standard exponential scale using the probability integral transform. We use the index $(i)$ to refer to the $i$-th ranking value in a vector of values. The third step of the naive approach in Algorithm 2 consists in sampling $n$ independent and identically distributed (i.i.d.) standard exponential random variables, and sorting them from lowest to highest value $Y(1), \ldots, Y(n)$ in Step 4. According to Rényi’s theorem (de Haan and Ferreira, 2007, p. 37) we have the following equality in law:

$$Y(i) = \frac{\sum_{l=1}^{i} \frac{Z_k}{n-l+1}}{Z_i \sim \text{Exp}(1), \quad k = 1, \ldots, n,}$$

see also Beirlant et al. (2006, Section 4.4). Several conclusions can be drawn for the naive resampling procedure. First, if the data process is mixing (i.e., it has long-range independence in the observation window), simulated quantiles will correspond well to original quantiles in the central part of the distribution where data are dense, owing to the law of large numbers. However, simulated quantiles can differ substantially from data quantiles close to the extremes. Spacings between $Y(i)$ and $Y(i+1)$ are relatively large when $i$ is close to $n$. For instance, as per Equation (3) the final spacing $Y(n) - Y(n-1)$ follows a standard exponential distribution, which means that it has the same distribution as the original data $Y_i$. For other choices of distributions, such spacings may be smaller, but there still arises a big difference in the probability levels of the corresponding quantiles. Another consequence of this behavior is that Algorithm 2 produces very strong spatial variability in the pixels having a value close to the maximum value $Y(n)$. Indeed, even if a training image shows relatively smooth behavior around the pixel containing the maximum value, naive resampling will tend to produce a very rugged surface around the maximum in simulated images.
Figure 2: Left: values close to the global maximum in the original transformed Gaussian random field with exponential margins, with the global maximum in the center of this subgrid. Right: naive resampling with identical spatial rank pattern.

To illustrate our point, we simulated a transformed Gaussian random field with exponential marginals on a $120 \times 120$ grid covering the unit square. The covariance function is exponential with range equal to $1/8 = 0.125$. Here, the dependence range is relatively small with respect to the size of the domain, and we therefore are in the setting of a single realization with weak dependence at moderate to large distances. Due to the process being mixing, we expect the empirical distribution of values to be close to the theoretical marginal distribution. Unreported results for the setting with replicated spatial fields show very similar behavior in this simulation experiment. We apply the naive resampling approach following Algorithm 2. In order to facilitate visual comparison, the spatial pattern of ranks, generated in Step 1 of the algorithm, remains identical to the one in the training data. The naive approach therefore consists in simply drawing independent standard exponential variables, one for each grid cell, and then re-attributing them to grid cells by matching ranks of training and simulation samples. The left panel of Figure 3 represents a $17 \times 17$ subgrid centered at the maximum of the original field. The right panel shows a simulation obtained with the naive approach. It is clearly visible that the ranks are preserved, but is also obvious that the simulated quantiles are much (unrealistically) higher: the maximum of the new image is above 10, while it was close to 5 in the original image. Furthermore, the surface around the maximum is much less continuous.

As already mentioned above, these conclusions remain valid for any marginal distribution, theoretical or empirical.

A workaround for taking into account the spatial clustering of high order statistics $Y_{(i)}$ when $i$ is close to $n$ could be to model the cluster structure explicitly. Various techniques for such cluster modeling could be developed, such as adding dependence between exponential variables $Z_i$ in Equation (3), but we think that there is no simple parametric choice with good theoretical motivation that avoids rather arbitrary modeling assumptions. Instead, we take advantage of extreme value theory for stochastic processes, which suggests techniques for extrapolating values...
of dependent variables beyond the observed range while avoiding any parametric assumptions on
the dependence structure of extremes.

4 Lifting based on functional extreme value theory

4.1 Scale-profile decomposition of Pareto processes

From now on, we adopt the following convention: random variables and random functions are
denoted with upper case letters. We use lower case letters for scalars, dummy variables as well as
for datasets and realizations of random values and random functions. Extreme value theory for
stochastic processes provides an asymptotic decomposition

\[ Y_d = S \eta \]

of extreme events \( Y_d \) into a scale variable \( S \) and a normalized profile process \( \eta \) for certain choices of marginal distribution in
data, with scale and profile being stochastically independent (Ferreira and de Haan, 2014; Thibaud
and Opitz, 2015; Dombry and Ribatet, 2015; Opitz et al., 2015; de Fondeville and Davison, 2018;
Engelke et al., 2019a; Palacios-Rodriguez et al., 2019). The processes presenting such factorization
of scale and profile are known as Pareto processes. While it is easy to generate new realizations
of the scaling variable \( S \), whose distribution is parameter-free, it is more intricate to provide new
realizations of the profile process \( \eta \) owing to the inherent dependence structure.

Recall that the training data are modeled as a stationary stochastic process \( X_i, i \in I \), with
\( X_i \sim F \). While the upper section of the univariate distribution \( F \) can be conveniently modeled
by the GPD as explained in Section 2, our focus is now only on dependence of different values
\( X_i \). It is useful to abstract away from the specific shape of \( F \). Theory is most easily presented by
assuming a normalized distribution function \( F^* \) with non-negative support and standard Pareto
tails \( 1 - F^*(x) = 1/x \) for large \( x \), which is at the origin of the notion of Pareto processes. We define
normalized data with standard Pareto tails through the transformation

\[ X_i^P = 1/(1 - F(X_i)), \tag{4} \]

which establishes the standard Pareto distribution \( F^* \) for \( X_i^P \sim F^* \) if \( F \) is continuous. It is easy
to verify that \( X_i^P \) possesses a standard Pareto tail, since for \( x \in [1, +\infty) \),

\[ P(X_i^P > x) = P((1 - F(X_i))^{-1} > x) = P(1 - F(X_i) < 1/x) = P(U < 1/x) = 1/x; \]

where \( U \) denotes a uniform random variable on the interval \((0, 1)\).

The fundamental assumption in extreme value theory of stochastic processes is the functional
maximum domain of attraction condition, which we outline here, while technical details can be
consulted in the literature (e.g., de Haan and Ferreira, 2007, and the above-listed references).

Given \( m \) independent replicates \( X_{ij}, i \in I, j = 1, \ldots, m \), of the stochastic process \( X_I \) defined
over a nonempty compact set \( I \subset \mathbb{R}^d \), we assume that deterministic normalizing sequences \( \alpha_{im} \)
(for scale) and \( \beta_{im} \) (for location) exist such that the componentwise maximum converges to a
nondegenerate limit process, that is,

\[ \max_{j=1,\ldots,m} \frac{X_{ij} - \beta_{im}}{\alpha_{im}} \to Z_i, \quad i \in I, \quad m \to \infty, \tag{5} \]
where convergence takes place in an appropriately defined function space, such as the space of continuous functions over $\mathcal{I}$.

In equivalence to the convergence of the dependence structure of extremes under the domain of attraction condition (5), extreme value theory states a certain convergence of the normalized data $X^P_I$ when a summary functional $r$ (also termed aggregation functional, risk functional, or loss functional) exceeds a high threshold $u^P$, which tends towards infinity. Therefore, we fix a homogeneous function

$$r : [0, \infty)^I \to [0, \infty), \quad r(ax) = ar(x), \quad a > 0, \quad (6)$$

that is continuous at 0, where 0 represents a dataset with index running through $\mathcal{I}$ that has value 0 everywhere. For instance, $r$ could be one of max, min, sum, mean, the value at a fixed site $s_0$, the median, or the order statistics $x(i)$ for fixed $i$, but we exclude the trivial case where $r \equiv 0$. The choice of $r$ is usually driven by the type of extreme event that one wants to consider. We are now ready to define $r$-Pareto processes.

**Theorem 1 (Dombry and Ribatet (2015)).** Let the distribution of $X_I$ be in the functional maximum domain of attraction as described in Equation (5), and denote by $X^P_I$ the dataset after normalization to standard Pareto tails using transformation (4). Let $r$ be a summary functional as defined in Equation (6) and denote $R^P_I = r(X^P_I)$. Then,

$$\text{Conditional on } R^P_I \geq q, \quad \frac{X^P_I}{q} \to Y^P_I, \quad \text{as } q \to \infty, \quad (7)$$

where the limit $Y^P_I = \{Y_i, \ i \in \mathcal{I}\}$ is an $r$-Pareto process.

Any $r$-Pareto process factorizes into a standard-Pareto-distributed scale variable $S^P = r(Y^P_I)$, independent of the profile process $\eta^P_I = Y^P_I/S^P$ satisfying $r(\eta^P_I) = 1$:

$$Y^P_I \overset{d}{=} S^P \eta^P_I, \quad S^P \perp \eta^P_I, \quad (8)$$

where $\perp$ means "independent of".

The result holds for any summary function $r$ as defined above. A converse result states dependence structure convergence of maxima in (5) if convergence (7) holds with $r$ chosen as the maximum (Dombry and Ribatet, 2015). The existence of the Pareto process limit in (7) requires a certain regularity of the limit process, such as having realizations that are continuous in space in the case where $\mathcal{I}$ is a subset of $\mathbb{R}^2$. This implies in particular that the data must be asymptotically dependent at small distances, that is, we need a positive limit value of the conditional exceedance probability

$$\lim_{u \to \infty} P(X^P_i > u \mid X^P_j > u) > 0 \quad (9)$$

if $i$ and $j$ are very close in space. For spatial data, an interpretation of this condition is that the spatial extent of clusters of high values should remain comparable when looking at increasingly high quantile levels. For environmental and meteorological variables, this may not always be the case as shown in the recent literature (Davison et al., 2013; Opitz, 2016; Huser et al., 2017; Bacro et al.,
2019), but the asymptotic framework still provides a useful approximation in practice. Moreover, the assumption of asymptotic dependence can be considered as realistic when very extreme events tend to impact large spatial areas simultaneously, for instance heatwaves or thunderstorms (e.g., de Fondeville and Davison, 2020).

The summary variable $R_I$ for original data is known to satisfy

$$q P(R_I > q) \sim \theta_r, \quad q \to \infty,$$

where $\theta_r \geq 0$ is the $r$-extremal coefficient (Engelke et al., 2019a). Its theoretical value is known beforehand in some cases, such as for the mean aggregation where $\theta_r = 1$, but in other cases it depends on the (unknown) extremal dependence structure. The value $\theta_r = 0$, indicating a situation where very extreme aggregated events $R_I$ do not occur, can arise only for very specific dependence structures in $X_I$ and specific choices of $r$ but is usually not of practical importance. Often, $\theta_r$ is not known beforehand, but we want to fix a threshold $q$ corresponding to a specific exceedance probability $p = P(R_I > q)$ by assuming equality in the approximation (10). Then, we have to estimate $\theta_r$ from the data. This step is described later in Section 4.3.

### 4.2 Lifting mechanism

According to Equation (7), the distribution of the process $qY^P_I$ provides a good approximation to the distribution of extreme events in $X^P_I$ if $q$ is high and $R_I > q$. Given $q$ and $\eta^P_I$ and given a realization $x_I$, we can generate a new dataset satisfying $r(x^P_I) > \tilde{q}$ with a threshold $\tilde{q} > 0$, possibly different from $q$, by generating a new scale $s^P$ according to the standard Pareto distribution. The new scale could either be drawn randomly according to the Pareto distribution, or it could be the Pareto quantile associated to a given probability. Because $s^P$ leads to a new realization, we opt for lower case notations here. We then set $\tilde{x}^P_I = \tilde{q}s^P\eta^P_I$. In this case, if $r(x^P_I) = q$ and $\eta^P_I = x^P_I/r(x^P_I)$, we have transformed an event with return level $q$ of the summary $r$ into an event with return level higher than $\tilde{q}$. From Equation (10), we further see that a return level of $q$ corresponds approximately to a return period of $q/\theta_r$. For a concrete example of the lifting mechanism, consider a rank summary $r(x^P_I) = x^P_{(i)}$ with fixed rank $i \in I = \{1, 2, \ldots, n\}$, which includes the minimum, the median and the maximum as special cases. Then, the new sample can be written $\tilde{q}s^P x^P_{(i)}/x^P_{(i)}$.

The normalization to a Pareto scale is natural for establishing limit results in extreme-value theory (Klüppelberg and Resnick, 2008), but is much less used in “classical” statistical analyses. For describing further our resampling approach, we reformulate the above theoretical results by switching to a uniform scale $X^U = F(X) - 1$ on the interval $[-1, 0]$, which may lend itself to easier visual analysis and interpretation, especially when readers are familiar with the copula literature (Joe, 2014). Since $X^U = F(X) - 1 = -1/X^P$, the threshold on this scale is $u = -1/q < 0$ where $q$ is the threshold on the Pareto scale, $S = -1/S^P$ has uniform distribution on $[-1, 0]$, and $\eta^P_I = 1/\eta^P_I$. The convergence (7) now becomes:

$$\text{Conditional on } - R^{-1}_I \geq u, \quad - \frac{X^U_I}{u} \to Y_I, \quad \text{as } u \uparrow 0.$$  

(11)
As before, we have

\[ Y = S \eta, \quad S \perp \eta. \quad (12) \]

We here skip superscripts \( U \) for notational simplicity since we will by default assume the uniform scale \( U \) in the following. We will make repeated use of the variable \( -R_I^{-1} \) and define the notation

\[ V = -R_I^{-1} = -\frac{1}{r(X^P)} = -\frac{1}{r(-1/X^U)}, \]

and we use lower-case notation \( v \) when referring to its observed value. Algorithm 3 below summarizes the steps to generate new \( r \)-Pareto processes by adopting the uniform scale given by Equations (11) and (12), based on threshold exceedances of the summary functional \( r \) in data, where the summary functional has return level constrained to the interval \([u_1, u_2]\) with \( u_1 \leq u_2 \leq 0 \) and \( u_1 < 0 \) if \( u_2 = 0 \). Specifically, \( u_1 = u_2 \) corresponds to a fixed return level, and \( 0 < u_1 < u_2 = 0 \) to threshold exceedances of the return level \( u_1 \).

**Algorithm 3 Lifting with uniform margins**

**Require:** Data \( x_{I,j}^U, j = 1, \ldots, m \), on uniform marginal scale in \([-1,0]\)

**Require:** Summary threshold \( u < 0 \) for extracting extreme events

**Require:** Thresholds \( u_1, u_2 \) for lifting where \( u_1 \leq u_2 \leq 0 \), and \( u_1 < 0 \) if \( u_2 = 0 \)

**Require:** Event \( x_{I,j}^U = x_{I,j_0}^U \) exceeding the threshold such that \( v = -1/r(-1/x_{I,j_0}^U) > u \)

1: Generate a new scale \( \tilde{U} \sim U(u_1, u_2) \).
2: Compute the lifted simulation \( \tilde{x}_I^U = \tilde{U}x_I^U/v \).
3: **return** \( \tilde{x}_I^U \).

### 4.3 Estimation of exceedance probabilities of the summary functional

With replications \( x_{I,j}, j = 1, \ldots, m \), we can estimate the \( r \)-extremal coefficient \( \theta_r \) in \( P(V > u) \sim -u\theta_r \) with \( u < 0 \). A simple empirical estimator is given by

\[ \hat{\theta}_r = -(um)^{-1} \sum_{j=1}^{m} 1( V_j > u), \quad V_j = -1/r(-1/x_{I,j}). \]

If events \( j = 1, \ldots, m \) are independent and identically distributed, this estimator follows a binomial distribution with success probability \( P(V > u) \) and rescaled by the factor \( -(um)^{-1} \). When simulating new data with \( V > u \), we then assume that the probability of observing such events is \( -u\hat{\theta}_r \).

### 4.4 Post-processing non extreme observations

We denote by \( x_{I,j_0}^U \) one of the events exceeding the summary threshold, such that

\[ v = -1/r(-1/x_{I,j_0}^U) > u. \]

An important step after lifting observed profile processes using Algorithm 3 is the post-processing of small or moderately large values, for which the lifting step may create artefacts. Specifically,
we may obtain a lifted value \( \tilde{x}_i < -1 \) outside of the support of the uniform data distribution over \([-1, 0]\), corresponding to a situation that can occur if \( \tilde{u} < v = -1/r(-1/x_{I,j0}^U) \) (downlifting). In other cases, the lifted dataset may satisfy \( \min_{i \in I} \tilde{x}_{ij0} > \varepsilon > -1 \) with a relatively large interval \([-1, \varepsilon]\) not covered by the lifted data, corresponding to a situation where \( \tilde{u} > v \) (uplifting). For this reason, we do not directly apply the lifting step to data values below the marginal threshold \( u_{\text{marg}} = \tilde{F}(u^F) - 1 \), where \( u^F \) is the marginal threshold defined on the original scale of data and \( \tilde{F} \) is the estimated marginal distribution; see Algorithm 1. To set the value \( u_{\text{marg}} \) below which post-processing is applied, it would also be possible to replace \( u^F \) by another, not too low threshold.

For deriving our postprocessing procedure, we consider the dataset lifted according to Algorithm 3 as a mixture of two populations. The first population consists of the part of the training dataset \( x_{I,j}^U \) where values are below \( u_{\text{marg}} \), and we exclude these values from the lifting transformation in step 2 in Algorithm 3. This first population corresponds thus to a uniform distribution on \([-1, u_{\text{marg}}]\) with probability mass \( 1 + u_{\text{marg}} \). The second population corresponds to the part uplifted according to the new scale variable \( \tilde{U} \) generated in step 1 of Algorithm 3. The second population is thus associated with a uniform distribution on \([u_{\text{marg}} \tilde{U}/v, 0]\) and probability mass \( -u_{\text{marg}} \). At this stage, the probability density of the newly generated dataset on the interval \((\min(u_{\text{marg}}, u_{\text{marg}} \tilde{U}/v), \max(u_{\text{marg}}, u_{\text{marg}} \tilde{U}/v))\) is either equal to 0 in the case of uplifting (i.e., \( \tilde{U} > v \)), or equal to \( 1 + v/\tilde{U} \) in the case of downlifting (i.e., \( \tilde{u} < v \)). Without corrections, simulated images resulting from this modification of Algorithm 3 would show artificial discontinuities when moving from areas belonging to the bulk of the distribution to areas corresponding to lifted values. We therefore propose to post-process the first population, i.e., the values \( \tilde{x}_{ij0}^U \) with \( x_{ij0}^U < u^F, i \in I \).

Algorithm 4 below summarizes this modified lifting procedure.

5 Resampling dependent extremes

5.1 Replicated data setting

Statistical analyses in classical extreme value theory are typically applied to datasets \( x_{ij}, i \in I, j = 1, \ldots, m \) with many replications \( m \), such as spatial snapshots taken at regular time steps. In such studies, replications are assumed to be independent, or to correspond to the realization of a stochastic process that is mixing, such that the dependence between distant subsamples of the process becomes very weak, and negligible in practice. Specifically, if \( I \) is given as a domain in 2D space and \( j = 1, 2 \ldots \) are time steps, we assume that stochastic dependence between the processes \( X_{I,j1} \) and \( X_{I,j2} \) ultimately vanishes for increasing time lags \( |j_2 - j_1| \). Then, the extreme events correspond to the replicates satisfying a threshold exceedance criterion of the summary functional \( r \) with a high enough threshold, such that the asymptotic properties stated in Theorem 1 can be expected to provide a good description of the data distribution.
Algorithm 4 Lifting with uniform margins and postprocessing
Requirements are identical to those listed in Algorithm 3.
1: Set \( u^{\text{marg}} = \hat{F}(u^F) - 1 \) (or some other not too low quantile).
2: Generate a new scale \( \tilde{U} \sim \mathcal{U}(u_1, u_2) \).
3: Set the scaling factor \( s = \tilde{U} / v \).
4: Compute the lifted simulation \( \tilde{x}_i^U \):
   for \( (i \in I) \) do
   5: if \( \tilde{x}_i^U > u^{\text{marg}} \) then
   6: Set \( \tilde{x}_i^U = sx_i^U \).
   7: else
   8: Set \( \tilde{x}_i^U = -1 + \frac{1 + s u^{\text{marg}}}{1 + u^{\text{marg}}} (1 + x_i^U) \).
   9: end if
10: end for
11: return \( \tilde{x}_i^U \).

In other situations, for example relating to geology and geosciences (Chilès and Delfiner, 2012), there is no time replicate. If the spatial domain \( I \) is very large enough in comparison to the correlation range, one could divide \( I \) into a large number \( m \) of blocks of same size and proceed as above. In cases where we do not observe any useful replication structure in data, it would be awkward to apply the Algorithm 3, which is based on the large-sample assumptions underpinning Theorem 1. We exclude such cases in the following, since they would require strong assumptions that are difficult to validate in practice. For the remainder of this paper, we assume that the training dataset comprises a large number of replicates \( m \), at least several hundreds.

5.2 Resampling algorithm for dependent extremes

In our approach to enrichment of extreme data, we sample from the distribution of the scale variable \( S \) in Equation (12) and generate new, potentially more extreme events by combining resampled univariate scales with observed profile processes. In cases where the univariate distribution \( F \) is known beforehand, this mechanism is fully nonparametric, except for the \( r \)-extremal coefficient \( \theta_r \) in Equation (10), which we have to estimate in some cases.

We now summarize the main steps of the full procedure for resampling with extremes in the Algorithm 5. We first extract \( m' \) extreme events, then lift them to generate \( m'' \) events, and finally generate \( m''' \) simulations with nonparametric resampling. Simulations can be generated on a support \( I' \) that is different from the original support \( I \). We distinguish two cases: In the first case, we aim to perform simulation for a fixed return level \( \tilde{v} \) of \( V \) such that the return level \( \tilde{v} \) of the lifted dataset should be (approximately) equal to \( \tilde{v} \). In this case, we set \( m''' = m' \) since each extreme event is lifted exactly once to the target return level. In the second case, we want to produce simulations for which the summary functional \( \tilde{v} \) has values within an interval \([u_1, u_2] \); see Algorithm 3. A special case of the interval constraint arises with exceedances of a summary threshold \( u_1 \) such that \( u_2 = 0 \) and \( \tilde{v} \geq u_1 \). In the second case we can generate any number \( m''' \) of
lifted events using Algorithm 3.

In principle, the approach of nonparametric resampling of the profile process, followed by a lifting step of the resampled profile, would also be statistically sound and produce realistic simulations if the constraints on the profile process are preserved. However, explicitly conserving these constraints would require sophisticated extensions of the standard nonparametric resampling techniques, and we therefore prefer the procedure of “lifting first – resampling second”. Note that the lifting step may strongly amplify small perturbations in the profile process when lifting towards very high quantiles, which is our reason to prefer performing nonparametric resampling after having lifted the original data.

It is not easy to provide precise but general rules on how to fix the numbers $m'$, $m''$ and $m'''$. The $m'$ extreme events used as a training sample for data enrichment should correspond to relatively high quantiles of the summary functional $r(X_U^I)$ where the asymptotics leading to the scale-profile decomposition in Equation (12) kick in, and $m'$ should be large enough to provide a training sample that is representative of the extremal dependence patterns in the data-generating process. If we lift to a specific target return value (i.e., $u_1 = u_2$), then $m'' = m'$, and each extreme training event is lifted exactly once. The strategy of setting $m'' = m'''$ usually makes sense, such that the extreme event magnitudes in a dataset enriched with $m''$ images of lifted extremes will be representative for a simulated dataset with $m'''$ extreme episodes. We adopt this approach in our data application in Section 6.

To increase flexibility of our procedure, we further allow for nonstationary margins $X_{ij} \sim F_i$ by estimating the density $\hat{f}_i$ of $F_i$ in Algorithm 1 from a sample of data collected at index $i$ or close to index $i$. By applying a nonparametric resampling method of our choice to the resulting standardized data $x_U^I$, we bypass the intricate handling of nonstationary margins during this resampling step.

Application-specific checks using hold-out data to support validation of the model, such as those used in the data application, are recommended and can help to confirm sound behavior of the extrapolation mechanism. We refer to Palacios-Rodriguez et al. (2019) for a deeper discussion.

6 Application to heatwaves in France

The heatwave that hit France and other parts of Europe in 2019 from June 23rd to June 30th was one of the most extreme heatwaves ever recorded in Europe. June 27th is the hottest day recorded for June over the period 1900-2019 in France with a daily average temperature of 27.9 °C, which is 8.6 °C “above normal”, see Figure 3. We remind the interested reader that the “normal” is, by definition, the 30-year average, recomputed every 10 years. Specifically, it is here the 30 daily average temperatures measured on 27th of June, from 1981 to 2010. The temperature maxima on 26th to 29th of June reached an exceptional level over a large part of Europe, equating or exceeding the value of 35°C. In France, these afternoons were the hottest since the heatwave of August 2003, and a new absolute record was established for 23% of the weather stations maintained by Météo France. It resulted in an excess of 1435 deaths according to Santé Publique France. Up to 10,000 hectares of vineyards were destroyed in the sole department of Hérault where the heatwave peaked at 45.9°C.
Algorithm 5 Resampling dependent extremes

Require: Training data $x_{I,j}$, $j = 1, \ldots, m$, on original scale

Require: Summary functional $r$

Require: Marginal thresholds $u_i^F$, $i \in I$

Require: Summary threshold $u < 0$ for extracting extreme events from uniform data on $[-1,0]$ using $r$

Require: Thresholds $u_1, u_2$ for lifting, where $u_1 \leq u_2 \leq 0$, and $u_1 < 0$ if $u_2 = 0$

1: Estimate the marginal distributions $\hat{F}_i$ using Algorithm 1 with marginal thresholds $u_i^F$, $i \in I$.

2: Standardize data to uniform scale $x_{I,j}^U = \hat{F}_i(x_{ij}) - 1$, $i \in I$, $j = 1, \ldots, m$.

3: Extract extreme events $x_{I,j}^U$, $\ell = 1, \ldots, m'$, satisfying $v_{j\ell} = -1/r \left( -1/x_{I,j\ell}^U \right) > u$.

4: Generate a sample $\tilde{x}_{I,j}^U$, $j = 1, \ldots, m''$, of lifted extreme events using Algorithm 4.

5: Perform nonparametric resampling based on the enriched data to produce $m''$ simulations $\tilde{x}_{ij}^U$, $i \in I'$, $j = 1, \ldots, m''$.

6: Backtransform resampled data to the original marginal scale:

$$\tilde{x}_{ij} = \hat{F}_i^{-1} \left( 1 + \tilde{x}_{ij}^U \right), \quad i \in I', \quad j = 1, \ldots, m''.$$

This exceptional event illustrates the need for simulation techniques able to simulate new extreme fields of unobserved magnitude, which can then be fed to process-based impact models to obtain reliable projections for such extreme events of unprecedented magnitude.

Based on the analysis of 7 years of daily maximum temperatures, our aim is to simulate new heatwaves corresponding to a given return period. In order to focus on the simulation approach, we make the simplifying assumption of temporal stationarity over years, i.e., we deliberately ignore the presence of trends due to climate change and long-term periodic climate cycles.

6.1 Data preprocessing and standardization

We consider SAFRAN reanalysis data over France and part of Switzerland. SAFRAN combines ECMWF global reanalysis archives and all available surface observations from the climatological database of Météo-France to produce high temporal and spatial resolution (6 hours, 8 km) for several meteorological variables, including temperature, humidity, wind speed and cloudiness (Vidal et al., 2010). Here, we selected daily Maximum Temperature for the months of June to September from 2010 to 2016.

Data are preprocessed in the following way. For each $8 \times 8$ km$^2$ SAFRAN grid cell $i$, we first fit a marginal probability density function $\hat{f}_i$ with kernel density estimate in the bulk and GPD tail as described in Section 2 and in Algorithm 1, using a threshold corresponding to the local 95% quantile. The tail indices $\xi(i)$ have been estimated through the maximum likelihood approach with a restriction to non-negative values since power-law tails are not realistic for temperatures in mainland France. Then, using the fitted marginal distribution, the daily maximum temperatures are transformed into $[-1,0]$ uniforms by applying $X_{i,t}^U = F_i(X_t) - 1$, where $i$ is the cell index ranging from 1 to 9892, and $t$ is the day index ranging from 1 to 854. Figure 4 reports the estimated shape
Figure 3: Extreme maximum temperature (in °C; left display) and temperature anomaly (in °C; right display) over Europe, June 23rd to 29th of 2019. Illustration provided by the US National Oceanic and Atmospheric Administration (NOAA).

parameter $\xi_u(i)$ and the estimated scale parameter $\sigma_u(i)$, where $u(i)$ is the local threshold, and maximum likelihood estimation was applied. As summary statistics $r$ for a given day $t$ we use the median value over all cells, i.e.,

$$r(t) = \text{med}\{X_u(\cdot, t)\}.$$

6.2 Data enrichment and resampling

We now illustrate how we can create new extreme events with values beyond the range of observed ones based on the analysis of the dataset presented above. We select the 6 most extreme events, corresponding to the 6 highest values of $r(t)$. In addition, we impose that these events should be separated by 2 days at least in order to prevent selecting highly correlated events. The selected events occurred on 27/06/2011, 19/08/2012, 03/07/2015, 16/07/2015, 19/07/2016, 27/08/2016. They are depicted in Figure 5. The amplitude and the spatial extent of zones with high values are quite variable for these events. For example, the second and most extreme event is characterized by a very large area with very high temperatures.

Figure 6 represents the most extreme event in the original scale (left display) and on the transformed, uniform scale (right display). On the original scale, the heat wave seems to be concentrated in the center of the country, whilst the ocean shore and most notably the Alps seem to have been spared. The transformed uniform scale tells a different story. Thanks to the use of local transformations, we see that the heat wave extends over the whole country with the exception of ocean shores. Most notably, the Alps experience very high temperatures with respect to local distributions.

We chose to simulate new events with a return period of 10 years. Considering that there are
Figure 4: Estimated shape ($\hat{\xi}_u$, left) and scale ($\hat{\sigma}_u$, right) parameters of the Generalized Pareto tail corresponding to the exceedance probability $p_u = 0.05$.

Figure 5: The six most extreme daily events pertaining to different heat episodes. From left to right: 27/06/2011, 19/08/2012, 03/07/2015 (top); 16/07/2015, 19/07/2016, 27/08/2016 (bottom).
122 summer days (corresponding to the period of June 1st to Sept 30th) per year, we therefore set the lower bound \( u_1 = -\hat{\theta}_r / 1220 \) for the simulated median summaries \( r(t) \), and we fix the upper bound to \( u_2 = 0 \). There may be non-negligible day-to-day dependence in extreme temperature events. We point out that this does not modify the daily return period associated to a return level with on average one exceedance of the level during the period, but the exceedances may arrive in a temporally clustered way, such that a relatively relatively low proportion of periods contain several daily events while a relatively high proportion of periods contain no event. The estimate \( \hat{\theta}_r \) of the r-extremal coefficient was obtained following the method described in Section 4.3 where we used the empirical 95%-quantile of the median summaries \( r(t) \) as threshold. In a first stage, the 6 selected episodes are modified through uplifting using Algorithm 4, which results in a new training dataset of uplifted fields with ”uniform” values in the range \([-1, 0]\). In a second stage, a nonparametric resampling algorithm is run on this uplifted dataset, and is able to generate as many as desired new spatial versions of those. We here opt for the Direct Sampling algorithm (DS, Mariethoz et al., 2010) on the transformed scale. As discussed in the introduction, DS computes a distance between the simulated spatial patterns and observed ones. The definition of this distance is rather flexible and allows for covariates. Here, in order to account for the nonstationarity of correlation structure across the French territory, we further introduce a 10% weight on the spatial coordinates, leaving a 90% weight for the spatial pattern. We parameterize DS such as to consider spatial patterns consisting of 20 values. In a third stage, the resampled ”uniform” scores are transformed back to the original scale using the locally estimated probability distributions. Figure 7 shows 6 new realizations of extreme events. One can see that the general pattern is reproduced thanks to the local modelling of the marginal distributions, and that there appears an interesting variety of local patterns among these realizations. Our approach thus takes into account quite naturally the highly nonstationary marginal distributions across the French territory. Figure 8 shows the mean and standard deviation fields computed over 100 realizations of new extreme heatwaves generated.
through our approach. The standard deviation map shows clearly that the ocean shore has a much higher variability during heatwaves than inland zones or the Mediterranean shore, due to the fact that these regions are less likely to be hit by heatwaves striking the large zones of the inland territory.

6.3 Validation

Can simulated data, obtained by data enrichment and nonparametric resampling, realistically reproduce important data features? Validation of extremal data features and of extrapolation performance is notoriously intricate owing to a too small data base. Here, we study this question by considering the 30 most extreme days retained after temporal declustering, of which the 10 least extreme events are used for generating 250 simulations, and the 20 most extreme events are kept as validation data for comparing their summary statistics against the simulations. In the data enrichment step, we have lifted the 10 training days to return levels of the median \( r(t) \) of standardized data larger than \(-0.08\), corresponding to an approximate lower bound for the 20 validation days, and further corresponding to a return period of approximately 17 days within the four-month summer period. Figure 9 shows a histogram of the distribution of summary statistics (mean, median, interquartile range, minimum, maximum, range = maximum – minimum) calculated for the 250 simulations, and values of the 20 most extreme observed events. In all cases, the 20 validation values are covered by the histogram bars of the simulations. The simulations tend to show stronger variability (e.g., for minima, maxima and ranges), and slightly lower values for central tendencies.
(median, mean), but we have to keep in mind that the number of 20 validation days is small in comparison to the number of simulations. When taking into account the heatwave during the end of June 2019, some of the seeming extrapolation biases are mitigated. To date, the SAFRAN product is not yet available for this heatwave, but the maximum temperature of 45.9 degrees observed over the Météo France station network on June 27th of 2019 suggests that the maximum SAFRAN value for this event can be expected to lie in the far right tail of the maxima of resampling simulations reported in Figure 9, where none of the 2010-2016 events occurred. Overall, we consider these validation plots as satisfactory, especially since we are in a setting of complete extrapolation where none of the training data fall within the range of the target return levels of the summary functional 𝑟.

7 Conclusion

We have extended nonparametric resampling techniques to allow for realistic extrapolation of extreme values beyond the observed range of data. An illustration of this approach was given through the generation of artificial, very extreme heat wave scenarios for mainland France. Especially with high-dimensional gridded data, it may be preferable to avoid using generative statistical models whose practical usefulness may be limited by strong assumptions on the structure of data, onerous model fitting procedures and the resulting compromises with respect to models’ complexity and realism. When data are only weakly dependent in the extremes (i.e., with small spatial range of extremal dependence, in particular in the case of asymptotic independence), our naive uplifting technique may work well, although the effective sample size in training data (i.e., the number of effectively independent variables) may increase strongly between training and resampling data and create some artefacts for extreme quantiles. By working with replicated training data structures (e.g., a time series of spatial snapshots) and using mild assumptions from extreme value theory
Figure 9: Validation of summary statistics of simulated lifted extreme episodes. Top row (left to right): median, mean, interquartile range; bottom row (left to right): minimum, maximum, difference maximum – minimum. Histograms show summary values for 250 simulations using Direct Sampling on enriched data. Blue lines show values of the 20 most extreme observed events, held out for validation.

of stochastic processes, we have bypassed the shortcomings of naive resampling discussed in Section 3.2. In some cases, the data transformation steps (marginal transformations, scale-profile decomposition, post-processing of small values) may be based on rather strong assumptions or involve considerable uncertainty, for instance if we have only relatively few replicates, if the spatial range of dependence of extremes is very short, or if data seem to be asymptotically independent such that we have limit 0 in the conditional exceedance probability (9). In such cases, the lifting procedure should be carefully validated. Extensions to include temporal dependence in the simulated data are possible, for example by replacing single snapshots with a small series of consecutive snapshots, such that short-range temporal dependence is preserved.

For clarity of presentation, we have concentrated on accurate data extrapolation in the upper tail, but our results also apply to lower tails modulo switching tails, for instance by considering \(-X_I\) instead of \(X_I\), and only minor adjustments would be necessary to adapt our algorithms to consider joint extrapolation in both upper and lower tails.

The most challenging case remains the situation of training data without independent replication, where asymptotic extreme value theory does not apply. In principle, the lifting procedure using marginal standardization and the scale-profile decomposition suggested by Pareto processes remains applicable, but some rather strong assumptions must be made on the validity of asymptotic theory, and it may be difficult to determine return periods for resampled data with a lifting step.

A more general approach to remove some of the limitations of the methods in this paper could come from the conditional extremes framework (Heffernan and Tawn, 2004), which provides theory
and models based on how values evolve around and away from a fixed reference point, conditional to observing a very high value at this point. To apply this theory, one could initiate the simulation dataset with a new high value for a fixed reference point, either by fixing the value deterministically or by sampling it using an appropriate target distribution, and then one would fill the rest of the simulation in accordance with theoretical constraints. However, it remains to be explored how the semi-parametric asymptotic representations of conditional extremes could be estimated under moderate assumptions and then be transformed into appropriate and simple algorithmic steps during the resampling procedure.

Our procedure heavily relies on a representation with two stochastically independent components: a univariate magnitude variable, from which it is easy to sample, and a profile process, for which classical resampling techniques can be applied. Similar decompositions arise with many other, nonasymptotic stochastic processes, and they offer a much larger pool of models for flexibly representing extremal dependence (e.g., Huser et al., 2017; Engelke et al., 2019b). We envisage follow-up work to this paper to explore how our algorithms could be adapted to such representations. However, a major difficulty with such alternative constructions would arise if the summary functional \( r \) is parametrized through parameters of the dependence structure of the process, which then impedes direct application of nonparametric procedures. For an example, consider random vectors following an elliptically contoured distribution, such as the multivariate Gaussian. Then there exists a scale-profile decomposition with the summary functional defined as the Mahalonobis norm of the inverse of the covariance matrix (or of the dispersion matrix if covariance is not defined), but the need to estimate the covariance structure compromises nonparametric resampling and requires further modifications.

References

Bacro, J.-N., Gaetan, C., Opitz, T., and Toulemonde, G. (2019). Hierarchical space-time modeling of asymptotically independent exceedances with an application to precipitation data. *Journal of the American Statistical Association*, pages 1–26.

Beirlant, J., Goegebeur, Y., Segers, J., and Teugels, J. L. (2006). *Statistics of extremes: theory and applications*. John Wiley & Sons.

Carreau, J. and Bengio, Y. (2009). A hybrid Pareto model for asymmetric fat-tailed data: the univariate case. *Extremes*, 12(1):53–76.

Chailan, R., Toulemonde, G., Bacro, J.-N., et al. (2017). A semiparametric method to simulate bivariate space–time extremes. *The Annals of Applied Statistics*, 11(3):1403–1428.

Chilès, J.-P. and Delfiner, P. (2012). *Geostatistics: Modeling Spatial Uncertainty, Second Edition*. John Wiley & Sons.

Davison, A. C. and Huser, R. (2015). Statistics of extremes. *Annual Review of Statistics and its Application*, 2:203–235.
Davison, A. C., Huser, R., and Thibaud, E. (2013). Geostatistics of dependent and asymptotically independent extremes. *Mathematical Geosciences*, 45(5):511–529.

de Fondeville, R. and Davison, A. C. (2018). High-dimensional peaks-over-threshold inference. *Biometrika*, 105(3):575–592.

de Fondeville, R. and Davison, A. C. (2020). Functional peaks-over-threshold analysis. *arXiv preprint arXiv:2002.02711*.

de Haan, L. and Ferreira, A. (2007). *Extreme value theory: an introduction*. Springer Science & Business Media.

Dekkers, A. L., Einmahl, J. H., and de Haan, L. (1989). A moment estimator for the index of an extreme-value distribution. *The Annals of Statistics*, pages 1833–1855.

Dombry, C. and Ribatet, M. (2015). Functional regular variations, Pareto processes and peaks over threshold. *Statistics and Its Interface*, 8(1):9–17.

Engelke, S., De Fondeville, R., and Oesting, M. (2019a). Extremal behaviour of aggregated data with an application to downscaling. *Biometrika*, 106(1):127–144.

Engelke, S., Opitz, T., and Wadsworth, J. (2019b). Extremal dependence of random scale constructions. *Extremes*. doi:10.1007/s10687-019-00353-3.

Ferreira, A. and de Haan, L. (2014). The generalized Pareto process; with a view towards application and simulation. *Bernoulli*, 20(4):1717–1737.

Guardiano, F. and Srivastava, M. (1993). *Multivariate geostatistics: Beyond bivariate moments*, volume 1, pages 133–144. Kluwer Academic, Dordrecht.

Hallin, M., Lu, Z., and Tran, L. T. (2004). Kernel density estimation for spatial processes: the $l_1$ theory. *Journal of Multivariate Analysis*, 88(1):61–75.

Heffernan, J. E. and Tawn, J. A. (2004). A conditional approach for multivariate extreme values (with discussion). *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 66(3):497–546.

Huser, R., Opitz, T., and Thibaud, E. (2017). Bridging asymptotic independence and dependence in spatial extremes using Gaussian scale mixtures. *Spatial Statistics*, 21:166–186.

Joe, H. (2014). *Dependence modeling with copulas*. CRC Press.

Kalantari, S. and Abdollahifard, M. J. (2016). Optimization-based multiple-point geostatistics: A sparse way. *Computers and Geosciences*, 95:85–98.

Klüppelberg, C. and Resnick, S. I. (2008). The Pareto copula, aggregation of risks, and the emperor’s socks. *Journal of Applied Probability*, 45(1):67–84.
Lorenz, E. N. (1969). Atmospheric predictability as revealed by naturally occurring analogues. *Journal of the Atmospheric sciences*, 26(4):636–646.

Mariethoz, G. and Caers, J. (2014). *Multiple-point geostatistics: stochastic modeling with training images*. John Wiley & Sons.

Mariethoz, G., Renard, P., and Straubhaar, J. (2010). The Direct Sampling method to perform multiple-point geostatistical simulations. *Water Resources Research*, 46(11).

Opitz, T. (2016). Modeling asymptotically independent spatial extremes based on Laplace random fields. *Spatial Statistics*, 16:1–18.

Opitz, T., Bacro, J.-N., and Ribereau, P. (2015). The spectrogram: A threshold-based inferential tool for extremes of stochastic processes. *Electronic Journal of Statistics*, 9(1):842–868.

Palacios-Rodriguez, F., Toulemonde, G., Carreau, J., and Opitz, T. (2019). Generalized Pareto processes for simulating space-time extreme events: an application to precipitation reanalyses. *Submitted*.

Scarrott, C. and MacDonald, A. (2012). A review of extreme value threshold estimation and uncertainty quantification. *REVSTAT—Statistical Journal*, 10(1):33–60.

Strebelle, S. (2002). Conditional simulation of complex geological structures using multiple-point statistics. *Mathematical Geology*, 34(1):1–22.

Thibaud, E. and Opitz, T. (2015). Efficient inference and simulation for elliptical Pareto processes. *Biometrika*, 102(4):855–870.

Vidal, J.-P., Martin, E., Franchistéguy, L., Baillon, M., and Soubeyroux, J.-M. (2010). A 50-year high-resolution atmospheric reanalysis over France with the SAFRAN system. *International Journal of Climatology*, 30(11):1627–1644.

Yiou, P. (2014). Anawege: a weather generator based on analogues of atmospheric circulation. *Geoscientific Model Development*, 7(2):531–543.

Zhang, T., Switzer, P., and Journel, A. (2006). Filter-based classification of training image patterns for spatial simulation. *Mathematical Geology*, 38(1):63–80.