SL(4, R) Embedding for a 3D World Spinor Equation

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Abstract

A generic-curved spacetime Dirac-like equation in 3D is constructed. It has, owing to the $SL(n, R)$ group deunitarizing automorphism, a physically correct unitarity and flat spacetime particle properties. The construction is achieved by embedding $SL(3, R)$ vector operator $X_\mu$, that plays a role of Dirac’s $\gamma_\mu$ matrices, into $SL(4, R)$. Decomposition of the unitary irreducible spinorial $SL(4, R)$ representations gives rise to an explicit form of the infinite $X_\mu$ matrices.

1 Introduction

The Dirac equation turned out to be one of the most successful theoretical achievements of the 20th century physics. It describes the basic matter constituents (both particles and fields), and very significantly, it played a prominent role in ushering the minimal coupling and gauge principle in the field of particle physics. The successes of the standard model based on this equation established beyond doubt.

The objective of this work is to consider a Dirac-like equation describing a spinorial field in a generic-curved spacetime. It turns out that there are some quite nontrivial group theoretic features, related primarily to the spinorial properties of the $SL(n, R) \subset Diff(n, R)$, $n \geq 3$ symmetry groups, that make this quest difficult. For the sake of making a clear parallel as well as understanding important distinctions between the group theoretic structure

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of the flat and generic-curved spacetime spinorial field equations, we recall briefly some relevant basic facts.

The Poincaré relativistic quantum field theory is a field theory obeying Einstein’s principle of relativity, i.e. a field theory invariant w.r.t. the Poincaré symmetry group (cf [1, 2]). The ”representations on states” are representations of the Poincaré group in the Hilbert space of corresponding particle quanta. Unitarity of this Hilbert space is achieved by making use of the unitary irreducible representations of the Poincaré group \( P(4) = T_4 \wedge SO(1, 3) \) (\( T_4 \) and \( SO(1, 3) \) being the translational and the Lorentz subgroups, respectively). Due to the Poincaré group semidirect product nature, the particle states are characterized, besides the quantum numbers of the representation invariants (mass and spin/helicity), by the so-called little group quantum numbers (momentum and spin/helicity projections). The representation space of the little group’s part belonging to the translational subgroup is infinite dimensional due to its noncompactnes (continuous momentum values), while that of the little group’s part belonging to the Lorentz subgroup \( (SO(3) \) for \( m \neq 0 \), i.e. \( SO(2) \subset E(2) \) for \( m = 0 \) ) is finite dimensional. The ”representations on fields” are of the form

\[
(D(a, \Lambda)\Phi_m)(x) = (D(\Lambda))^n_m \Phi_n(\Lambda^{-1}(x - a))
\]

\[
(a, \Lambda) \in T_4 \wedge SO(1, 3),
\]

where \( m, n \) enumerate a basis of the representation space of the field components. The fact that finite-dimensional representations, \( D(\Lambda) \), of the Lorentz subgroup are, due to its noncompactnes, nonunitary is of no physical relevance as long as one, by means of a field equation, prevents propagations of all field components but those corresponding to a unitary representation of the Lorentz subgroup part of the little group. In other words, unitarity is imposed in the Hilbert space of the representations on states only, while the field equations provide for a full Lorentz covariance as well as restrict the field components in such a way that the physical degrees of freedom are as given by the corresponding particle states.

The quantum-mechanical symmetry group \( G_{q,m.} = \{ T(g) \} \) of a given classical symmetry group \( G = \{ g \} \), is given by a set of Hilbert space operators satisfying

\[
T(g_1)T(g_2) = e^{i\omega(g_1, g_2)}T(g_1g_2)
\]

in order to account for the physical Hilbert space ray structure. The nontrivial solutions of \( \omega(g_1, g_2) \) are obtained by making use of the universal covering (topology features), and by finding nonequivalent minimal extensions of the group \( G \) by the \( U(1) \) group (group
deformation features). In the case of a noncompact group, the topology features are determined by its maximal compact subgroup. There are no nontrivial deformations of the Poincaré group, however, there is a double covering of it due to the fact that its maximal compact subgroup $SO(3) \subset SO(1,3)$ is double connected. The quantum-mechanical Poincaré group $P(4)_{q.m.} = \mathcal{P}(4)$ is thus the double covering group of the $P(4)$ group, i.e. $\mathcal{P}(4)/Z_2 \simeq P(4)$, and the novel features are the spinorial particles and fields. The double covering of the maximal compact subgroup $SO(3)$ is the $Spin(3)$ group, that is isomorphic to $SU(2)$, while the double covering of the Lorentz group $SO(1,3)$ is the $Spin(1,3)$ group isomorphic to $SL(2,\mathbb{C})$ (for a detailed account of group-theoretical structure of Poincaré spinors cf [3]).

When passing from special to general relativity, one considers the group of General Coordinate Transformations $GCT = Diff(4,R)$ instead of the Poincaré group. The local (anholonomic) flat (tangent) spacetime tensors (Lorentz group representations) are replaced by the world (holonomic) generic-curved spacetime tensors (homogeneous $Diff(4,R)$ group representations). The $Diff(4,R)$ tensor algebra is basically determined by the tensor calculus of its linear subgroups $SL(4,R) \subset GL(4,R)$, i.e. for finite-component tensors by the tensor calculus of the corresponding compact groups $SU(4) \subset U(4)$ (cf [4, 5, 6]). Note that linear, tensorial representations of $Diff(4,R) \simeq Diff(4,R)/Z_2$ can be both finite-dimensional (nonunitary) and infinite-dimensional (unitary and nonunitary).

The transition from special to general relativity, in the quantum case, is achieved by considering the covering group $\overline{Diff}(4,R)$ of general coordinate transformations instead of the covering of the Poincaré group. This covering is again a double covering, since it is determined by the covering of the $SO(4)$ maximal compact subgroup, which in its turn a double-connected Lie group: $\overline{SO}(4) = Spin(4)$, $SO(4) \simeq Spin(4)/Z_2$. The same double covering result holds for any dimension $D \geq 3$.

Let us consider now the question of the dimensionality of the fundamental (lowest-dimensional) spinorial representations, i.e. the dimensionality of the vector space in which the covering group is defined. The defining space of the $SO(D - p, p)$ groups is $D$ dimensional, while the space of the corresponding covering $Spin(D - p, p)$ groups is $2\left[\frac{D-1}{2}\right]$ dimensional. The defining space dimensionality of the $Diff(D,R)$, $\overline{Diff}(D,R)$ groups is given by the dimensionality of the defining space of the $SL(D,R)$, $\overline{SL}(D,R)$ subgroups, respectively. It turns out that there are no finite-dimensional spinorial repre-
sentations of the $\overline{SL}(D,R)$ groups for $D \geq 3$. The spinorial representations of these groups are infinite dimensional, i.e. the $\overline{Diff}(D,R)$, $n \geq 3$ groups are isomorphic to groups of infinite complex matrices (cf. [7]), and all their linear spinorial representations are infinite dimensional as well.

It follows from the above considerations that a generalization of the Dirac equation to a generic-curved spacetime (not related to any $SO(m,n)$ orthogonal-type group) requires a knowledge of the infinite-component spinorial fields, a construction of the corresponding vector operators generalizing Dirac’s $\gamma$ matrices, as well as physically satisfactory unitarity properties defined by the appropriate particle states little group unitary representations.

One more comment is in order. Dirac’s equation was derived by factorizing the Poincaré second order invariant, $(P^2 - m^2)\psi = 0$. Such an approach cannot be applied in our case due to the fact that $P^2$ is not invariant w.r.t. nonorthogonal type of groups, say $SL(D,R)$ etc.

The standard way of implementing spinors to General Relativity is to consider either nonlinear spinorial $\overline{Diff}(4,R)$ representations w.r.t. its $Spin(1,3) \simeq SL(2,C)$ subgroup, or, what is nowadays customary, to make use of the tetrad formalism and spinorial fields of a tangent flat spacetime. These spinors are spinors of the quantum tangent-spacetime Lorentz group $Spin(1,3)$, however, they transforms as scalars w.r.t. to the group of general coordinate transformations. In contradistinction to the tensorial case, where there are both world and local Lorentz fields that are mutually connected by appropriate tetrad-field combinations, there exist only local Lorentz spinorial fields.

In this work we go beyond the Poincaré invariance and study a Dirac-like equation for infinite-component spinorial field that transforms w.r.t. linear (single-valued) representations of the $\overline{Diff}(4,R) \supset GA(4,R) \supset GL(4,R) \supset SL(4,R) \supset SO(1,3)$ group chain. The Affine $GA(4,R)$ group being a semidirect product, $GA(4,R) = T_4 \wedge GL(4,R)$, of 4-translations and the general linear group. In other words, we consider a first-order wave equation for spinorial field, ”world spinors” [8, 9], in a generic non-Riemannian spacetime of arbitrary torsion and curvature. A flat spacetime version of this equation (constructed below), i.e. the corresponding action, is that to be used in setting up a metric-affine [10, 11] and/or affine [12] gauge theories of gravitational interactions of spinorial matter.

Affine-invariant extensions of the Dirac equation have been considered previously, however, lacking either required physical interpretation or the actual invariance that goes beyond the Lorentz one. Mickelsson [13] has constructed a truly $\overline{Diff}(4,R) \supset GL(4,R)$ covariant extension of the Dirac
equation, however, its physical interpretation is rather unclear. The unitarity problem as well as the questions of $\mathfrak{CL}(4, R)$ irreducible representations content and the physical particle states are not resolved. Cant and Ne’eman [14] found a Dirac-type equation for infinite-component fields of $\mathfrak{SL}(4, R)$, however, this equation does not stretch beyond Lorentz covariance. They use only a subclass of $\mathfrak{SL}(4, R)$ multiplicity-free representations that does not allow for a $\mathfrak{SL}(4, R)$ vector operator and an extension to affine Dirac-like wave equation.

In a recent paper [15], we considered a Dirac-type infinite-component equation from the point of view of building it from physically well-defined Lorentz subgroup components.

The aim of this paper is to provide an explicit construct of a world spinor field equation that satisfies all conditions required by a correct physical interpretation (unitarity, Poincaré particle interpretation for the field components). Owing to the complexity of this task in the $D = 4$ case, as elaborated upon below, we confine in this paper to an explicit construction in the $D = 3$ case, where we make use of the nonmultiplicity-free $\mathfrak{SL}(3, R)$ representations, and consequently achieve a full $\mathfrak{Diff}(3, R)$ covariance. The construction is achieved by embedding the relevant algebraic relations into $D = 4$, and by decomposing a physically motivated $\mathfrak{SL}(4, R)$ spinorial representation to $D = 3$. All expressions of section 2 are valid for any $D = n, n \geq 3$, thus we write them in full generality. In the appendices we present, in an adjusted notation, the group representation results that are essential for this analysis.

2 World spinors

The finite-dimensional world tensor fields in $R^n$ are characterized by the nonunitary irreducible representations of the general linear subgroup $GL(n, R)$ of the Diffeomorphism group $Diff(n, R)$. In the flat-space limit these representations split up into $SO(1, n - 1)$ ($SL(2, C)/Z_2$ for $n = 4$) irreducible pieces. The corresponding particle states are defined in the tangent flat-space only. They are characterized by the unitary irreducible representations of the (inhomogeneous) Poincaré group $P(n) = T_n \wedge SO(1, n - 1)$, and their components are enumerated by the "little" group unitary representations (e.g. $T_{n-1} \otimes SO(n-1)$ for $m \neq 0$). In the generalization to world spinors, the double covering $\mathfrak{SO}(1, n - 1)$ of the $SO(1, n - 1)$ group, that characterizes a Dirac-type fields in $D = n$ dimensions, is enlarged to
the $SL(n, R) \subset \mathcal{GL}(n, R)$ group, while $SA(n, R) = T_n \wedge \overline{SL}(n, R)$ is to replace the Poincaré group itself. Affine "particles" are now characterized by the unitary irreducible representations of the $SA(n, R)$ group, i.e. by the nonlinear unitary representations over an appropriate "little" group (e.g. $T_{n-1} \otimes \overline{SL}(n-1, R) \supset T_{n-1} \otimes SO(n-1)$, for $m \neq 0$).

A mutual particle-field correspondence is achieved by requiring (i) that fields have appropriate mass (Klein-Gordon-like equation condition), and (ii) that the subgroup of the field-defining homogeneous group, that is isomorphic to the homogeneous part of the "little" group, is represented unitarily. Furthermore, one has to project away all little group representations except the one that characterizes the (physical, i.e. propagating) particle states.

A physically correct picture, in the affine case, is obtained by making use of the $SA(n, R)$ group unitary (irreducible) representations for "affine" particles. The affine-particle states itself are characterized by the unitary (irreducible) representations of the $T_{n-1} \otimes \overline{SL}(n-1, R)$ "little" group. The "intrinsic" part of these representations is necessarily infinite dimensional due to noncompactness of the $SL(n, R)$ group. The corresponding affine fields are described by nonunitary infinite-dimensional $SL(n, R)$ representations, that should be unitary when restricted to $\overline{SL}(n-1, R)$, the homogeneous part of the "little" subgroup. Therefore, the first step towards world spinor fields is a construction of infinite-dimensional nonunitary $SL(n, R)$ representations, that are unitary when restricted to the $\overline{SL}(n-1, R)$ subgroup. Each of these fields reduce to an infinite sum of (nonunitary) finite-dimensional $SO(1, n-1)$ fields having the usual relativistic field interpretation.

The deunitarizing automorphism. The unitarity properties, that ensure correct physical interpretation of the affine fields, can be achieved by combining the unitary (irreducible) representations with the so-called "deunitarizing" automorphism of the $\overline{SL}(n, R)$ group [7].

The commutation relations of the $\overline{SL}(n, R)$ generators $Q_{ab}, a, b = 0, 1, \ldots, n - 1$ are

$$[Q_{ab}, Q_{cd}] = i(\eta_{bc}Q_{ad} - \eta_{ad}Q_{cb}),$$

where $\eta_{ab} = \text{diag}(+1, -1, \ldots, -1)$. The important subalgebras are as follows:

(i) $so(1, n - 1)$: The $M_{ab} = Q_{[ab]}$ operators generate the Lorentz-like subgroup $SO(1, n - 1)$ with $J_{ij}$ (angular momentum) and $K_i = M_{0i}$ (the boosts) $i, j = 1, 2, \ldots, n - 1$.

(ii) $so(n)$: The $J_{ij}$ and $N_i = Q_{i0}$ operators generate the maximal compact subgroup $SO(n)$. 

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(iii) \( sl(n-1) \): The \( J_{ij} \) and \( T_{ij} = Q_{\{ij\}} \) operators generate the \( \overline{SL(n-1,R)} \) subgroup - the ”little” group of the massive particle states.

The \( \overline{SL(n,R)} \) commutation relations are invariant under the automorphism,

\[
J_{ij}' = J_{ij}, \quad K_{i}' = iN_{i}, \quad N_{i}' = iK_{i}, \\
T_{ij}' = T_{ij}, \quad T_{00}' = T_{00} (= Q_{00}),
\]

so that \((J_{ij}, iK_{i})\) generate the “new” compact \( \overline{SO(n)}' \) group, while \((J_{ij}, iN_{i})\) generate the “new” noncompact \( \overline{SO(1,n-1)}' \) group.

For the spinorial particle states we start with the basis vectors of the unitary irreducible representations of \( \overline{SL(n,R)}' \), so that the compact subgroup finite multiplets correspond to \( \overline{SO(n)}' \): \((J_{ij}, iK_{i})\) while \( \overline{SO(1,n-1)}' \): \((J_{ij}, iN_{i})\) is represented by unitary infinite-dimensional representations. We now perform the inverse transformation and return to the unprimed \( \overline{SL(n,R)} \) for our physical identification: \( \overline{SL(n,R)} \) is represented nonunitarily, the compact \( \overline{SO(n)} \) is represented by nonunitary infinite representations while the Lorentz group is represented by nonunitary finite representations. These finite-dimensional nonunitary Lorentz group representations are necessary in order to ensure a correct particle interpretation (i.e. boosted proton remain proton). Note that \( \overline{SL(n-1,R)} \), the stability subgroup of \( \overline{SA(n,R)} \), is invariant w.r.t. the deunitarizing automorphism, and thus it remains represented unitarily.

The world spinor fields transform w.r.t. \( \overline{Diff(n,R)} \) as follows

\[
(D(a, \bar{f})\Psi_{M})(x) = (D_{\overline{Diff}_{0}(n,R)}(\bar{f}))_{N}^{M} \Psi_{N}(f^{-1}(x - a));
\]

\((a, \bar{f}) \in T_{n} \wedge \overline{Diff}_{0}(n,R),\)

where \( \overline{Diff}_{0}(n,R) \) is the homogeneous part of \( \overline{Diff}(n,R) \), while \( f \) is the element corresponding to \( \bar{f} \) in \( Diff(n,R) \). The \( \overline{Diff}_{0}(n,R) \) representations can be reduced to direct sum of infinite-dimensional \( \overline{SL(n,R)} \) representations. As a matter of fact, we consider here those representations of \( \overline{Diff}_{0}(n,R) \) that are nonlinearly realized over the maximal linear subgroup \( \overline{SL(n,R)} \).

The affine ”particle” states transform in the following way:

\[
D(a, \bar{s}) \rightarrow e^{ia(sp)}D_{\overline{SL(n,R)}}(L^{-1}(sp)\bar{s}L(p)), \quad (a, \bar{s}) \in T_{n} \wedge \overline{SL(n,R)},
\]

where \( L \in \overline{SL(n,R)}/\overline{SL(n-1,R)} \), and \( p \) is the \( n \)-momentum label. The unitarity properties of various representations in this expression are as described above.
Provided the relevant $\mathcal{SL}(n, R)$ representations are known, one can first define the corresponding general/special Affine spinor fields, $\Psi_A(x)$, in the tangent to $R^n$, and then make use of the infinite-component pseudo-frame fields $E^A_M(x)$, "alephzeroads", that generalize the tetrad fields of $R^4$. Let us define a pseudo-frame $E^A_M(x)$ s.t.

$$\Psi_M(x) = E^A_M(x) \Psi_A(x),$$

where $\Psi_M(x)$ and $\Psi_A(x)$ are the world (holonomic) and local affine (anholonomic) spinor fields, respectively. The $E^A_M(x)$ (and their inverses $E^M_A(x)$) are thus infinite matrices related to the quotient $\text{Diff}_0(n, R)/\mathcal{SL}(n, R)$. Their infinitesimal transformations are

$$\delta E^A_M(x) = i \epsilon^a_b(x) (Q^b_a)^A_B E^B_M(x) + \partial_\mu \epsilon^a_\nu e^\mu_b \{Q^b_a^A_B E^B_M(x),$$

$$\mu, \nu \text{ group parameters of } \mathcal{SL}(n, R) \text{ and } \text{Diff}_0(n, R) \text{ respectively, while } e^a_\nu \text{ are the standard } n\text{-bine frame fields.}$$

The transformation properties of the world spinor fields themselves are given as follows:

$$\delta \Psi^M(x) = i \{\epsilon^a_b(x) E^M_A(x) (Q^b_a)^A_B E^B_N(x) + \xi^\mu [\delta^M_N \partial_\mu + E^M_B(x) \partial_\mu E^B_N(x)]\} \Psi^N(x).$$

The $(Q^b_a)_N^M = E^M_A(x) (Q^b_a)^A_B E^B_N(x)$ is the holonomic form of the $\mathcal{SL}(n, R)$ generators given in terms of the corresponding anholonomic ones. The $(Q^b_a)_N^M$ and $(Q^b_a)_B^A$ act in the spaces of spinor fields $\Psi_M(x)$ and $\Psi_A(x)$, respectively.

The above outlined construction allows one to define a fully $\text{Diff}(n, R)$ covariant Dirac-like wave equation for the corresponding world spinor fields provided a Dirac-like wave equation for the $\mathcal{SL}(n, R)$ group is known. In other words, one can lift up an $\mathcal{SL}(n, R)$ covariant equation of the form

$$(ie^\mu_a (X^a)_A^B \partial_\mu - M) \Psi_B(x) = 0,$$

$$a, \mu \text{ are the spinorial } \mathcal{SL}(n, R) \text{ representation for } \Psi \text{ is given, such that the corresponding representation Hilbert space is invariant w.r.t. } X^a \text{ action. Thus, the crucial step towards a Dirac-like world}$$

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spinor equation is a construction of the vector operator $X^a$ in the space of $SL(n, R)$ spinorial representations.

In the above considerations, we treated curved spacetime as externally defined, and were interested in describing a propagation of a spinorial field $\Psi(x)$ in such a background. One can now pose a question of coupling the matter described by our infinite-component spinorial field to the interactions of both gravitational and nongravitational type. The standard way is provided by localizing the relevant global symmetry groups. The full procedure would be to start by requiring local invariance of an action given in terms of globally covariant fields, and to derive the corresponding interacting field equations with ordinary derivatives replaced by the appropriate covariant ones. Let the global symmetry group be given by $G_{grav} \otimes G_{int}$, where say $G_{grav} = SL(n, R)$, and $G_{int} = SU(3)_c \otimes SU(2)_L \otimes SU(1)_Y$ of the standard model. Localization of this symmetry provides for the gauge potential fields, and the covariant derivative form. In our case, the spinorial equation that describes propagation and appropriate couplings reads symbolically

$$\left( i(X^a)^B_A (e^a_\mu \partial_\mu - iQ^a_{b\mu} \Gamma^a_{b\mu} - ig_{int} \lambda_k A^k_\mu) - M \right) \Psi_B(x) = 0.$$ 

(11)

The gravitational gauge potentials are $e^a_\mu$ and $\Gamma^a_{b\mu}$, $\lambda_k$ stand for all internal group generators, while the corresponding gauge potentials and coupling constants are $A^k_\mu$ and $g_{int}$ respectively.

### 3 $SL(3, R)$ vector operator

In order to illustrate the difficulties one encounters when considering action of an $SL(3, R)$ vector operator $X$ acting in the Hilbert space of field components, i.e. when coupling the vector representation and the representation of some field $\Psi$, let us consider at first the case of finite-dimensional field representations (we leave aside additional difficulties related to the question of auxiliary fields etc).

Let us consider a field $\Psi$ transforming w.r.t. a single irreducible representation $D_{\Psi}(g)$. Algebraic condition necessary for a wave equation construction is that in the reduction of the product of the vector representation $D_X(g)$ and the representation $D_{\Psi}(g)$ one finds either the representation $D_{\Psi}(g)$ itself or its contragradient representation $D^T_{\Psi}(g^{-1})$, thus resulting in a use of $D_{\Psi}(g)$ or $D_{\Psi}(g) \oplus D^T_{\Psi}(g^{-1})$, respectively. In contradistinction to the 3-dimensional Lorentz group $SO(1, 2)$ case, where this condition is always satisfied due to
\[ D^{(1)} \otimes D^{(j)} = D^{(j-1)} \oplus D^{(j)} \oplus D^{(j+2)} \] (\( j \) being the \( SO(1, 2) \) representation label: 0, \( \frac{1}{2} \), 1, ...), the \( SL(3, R) \) case is more complex. Let us denote an \( SL(3, R) \) irreducible representations by a Young tableau \([p, q]\), where \( p \) and \( q \) are respectively, the number of boxes in the first and the second row. The coupling of the vector representation \([1, 0]\) with a generic irreducible representation \([p, q]\), gives in general \([1, 0] \otimes [p, q] = [p+1, q] \oplus [p, q+1] \oplus [p-1, q-1] \).

Taking into account that the contragradient representation to the representation \([p, q]\) is given by \([p, p-q]\), we find that one can satisfy the above necessary algebraic condition only in the special case of a reducible \( SL(3, R) \) representation \([2q+1, q] \oplus [2q+1, q+1]\).

It is well known that one can indeed satisfy the commutation relations

\[ [M_{ab}, X_c] = i(\eta_{ac}X_b - \eta_{bc}X_a). \] (12)

in the Hilbert space of any \( SO(1, 2) \) irreducible representation. However, in order for an \( SO(1, 2) \) vector to be an \( SL(3, R) \) vector as well, it has to satisfy additionally the following commutation relations

\[ [T_{ab}, X_c] = i(\eta_{ac}X_b + \eta_{bc}X_a). \] (13)

This is a much harder task to achieve, and in principle, one can find nontrivial solutions only for particular representation spaces.

Let us turn now to infinite-dimensional representations. The multiplicity free (ladder) unitary (infinite-dimensional) irreducible representations \( D_{SL(3,R)}^{\text{ladd}}(0) \), and \( D_{SL(3,R)}^{\text{ladd}}(1) \), with the \( SO(3) \) subgroup content given by \( \{j\} = \{0, 2, 4, \ldots\} \), and \( \{j\} = \{1, 3, 5, \ldots\} \) respectively, can be viewed as the limiting cases of the series of finite-dimensional representations \([0, 0]\), \([2, 0]\), \([4, 0]\), ..., and \([1, 0]\), \([3, 0]\), \([5, 0]\), ..., respectively. Upon the coupling with the \( SL(3, R) \) vector representation \([1, 0]\), one has \([1, 0] \otimes [2n, 0] \supset [2n+1, 0]\), and \([1, 0] \otimes [2n+1, 0] \supset [2n+2, 0], (n = 0, 1, 2, \ldots) \). It would seem possible, at the first site, to represent the vector operator \( X \) in the Hilbert space of the \( D_{SL(3,R)}^{\text{ladd}}(0) \oplus D_{SL(3,R)}^{\text{ladd}}(1) \) representation. However, the resulting representations obtained after the \( X \) action have different values of the Casimir operators and thus define new (mutually orthogonal) Hilbert spaces.

If one starts, for instance, with the representation space of the scalar representation \([0, 0]\), the vector operator action would produce the space of the vector representation \([1, 0]\) itself, in the next act one gets the spaces of the representations \([2, 0]\) and \([1, 1]\), and so on. Therefore, unless some additional algebraic constraints are imposed, one would end up (independently
of the starting representation) with an infinite-dimensional space consisting of the representation spaces of all $SL(3, R)$ (nonunitary) irreducible representations.

A rather efficient way to impose additional algebraic constraints on the vector operator $X$ consists in embedding it into a non-Abelian Lie-algebraic structure. The minimal semi-simple Lie algebra that contains both the $SL(3, R)$ algebra and the corresponding vector operator $X$ is the $SL(4, R)$ algebra. There are two $SL(3, R)$ vector operators: $A = (A^a, a = 1, 2, 3)$ and $B = (B_a, a = 1, 2, 3)$, in the $SL(4, R)$ algebra that transform w.r.t. $[1, 0]$ and $[1, 1]$ $SL(3, R)$ representations, respectively. Components of each of them mutually commute, while their commutator yields the $SL(3, R)$ generators themselves, i.e.

\[
[A^a, A^b] = 0, \quad [B_a, B_b] = 0, \quad [A^a, B_b] = iQ^a_b. \tag{14}
\]

Now, due to the $SL(4, R)$ algebra constraints, any irreducible representation (or an arbitrary combination of them) of $SL(4, R)$ defines a Hilbert space that is invariant under the action of an $SL(3, R)$ vector operator proportional to $A$ or $B$. As an example, let us consider a ten-dimensional $SL(4, R)$ representation given by the Young tableau $[2, 0, 0]$. This representation reduces to $[2, 0]$, $[1, 0]$ and $[0, 0]$ representations of $SL(3, R)$ that are of dimension 6, 3, and 1, respectively. The action of the $SL(3, R)$ 3-vector $X$ is as follows:

\[
X : \begin{cases}
[0, 0] \rightarrow [1, 0] \\
[1, 0] \rightarrow [2, 0] \\
[2, 0] \rightarrow 0
\end{cases} \tag{15}
\]

where $X : [2, 0] \rightarrow 0$ is due to constraints enforced by the $SL(4, R)$ algebra. In the reduction of the ten-dimensional space to $6 + 3 + 1$, the vector operator $X$ has the following block-matrix form:

\[
X \sim \begin{pmatrix}
0_{(6 \times 6)} & a_{(6 \times 3)} & 0_{(6 \times 1)} \\
b_{(3 \times 6)} & 0_{(3 \times 3)} & a_{(3 \times 1)} \\
0_{(1 \times 6)} & b_{(1 \times 3)} & 0_{(1 \times 1)}
\end{pmatrix} \tag{16}
\]

where the nonzero matrix elements $a_{(m \times n)}$ and $b_{(m \times n)}$ correspond to the $SL(4, R)$ generators $A$ and $B$, respectively.

Let us consider now the $\overline{SL}(3, R)$ spinorial representations, that are necessarily infinite-dimensional. There is a unique multiplicity-free ("ladder")
unitary irreducible representation of the SL(3, R) group, $D^{(ladd)}_{SL(3,R)}(\frac{1}{2})$, that in the reduction w.r.t. its maximal compact subgroup SO(3) yields,

$$D^{(ladd)}_{SL(3,R)}(\frac{1}{2}) \supset D^{(\frac{1}{2})}_{SO(3)} \oplus D^{(\frac{5}{2})}_{SO(3)} \oplus D^{(\frac{9}{2})}_{SO(3)} \oplus \ldots$$

(17)

i.e. it has the following $J$ content: $\{J\} = \{\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \ldots\}$. Owing to the fact that the SO(3) and/or SL(3, R) vector operator can have nontrivial matrix elements only between the SO(3) states such that $\Delta J = 0, \pm 1$, it is obvious (on account of the Wigner-Eckart theorem) that all $X$-operator matrix elements in the Hilbert space of the $D^{(ladd)}_{SL(3,R)}(\frac{1}{2})$ representation vanish. The same holds for the two classes of tensorial ladder unitary irreducible representations $D^{(ladd)}_{SL(3,R)}(0; \sigma_2)$ and $D^{(ladd)}_{SL(3,R)}(1; \sigma_2)$, $\sigma_2 \in \mathbb{R}$, with the $J$ content $\{J\} = \{0, 2, 4, \ldots\}$ and $\{J\} = \{1, 3, 5, \ldots\}$.

Let us consider now the case of SL(3, R) unitary irreducible representations with nontrivial multiplicity w.r.t. its maximal compact subgroup SO(3). An efficient way to construct these representations explicitly is to set up a Hilbert space of square-integrable functions $H = L^2([SO(3) \otimes SO(3)]^d, \kappa)$, over the diagonal subgroup of the two copies of the SO(3) subgroup, with the group action to the right defining the group/representation itself while the group action to the left accounts for the multiplicity. Here, $\kappa$ denotes a kernel of a Hilbert space scalar product, that is generally more singular than the Dirac delta function in order to account for all types of SL(3, R) unitary irreducible representations. Let us make use of the canonical (spherical) basis in this space, i.e. $\sqrt{2J + 1} D^K_M(\alpha, \beta, \gamma)$, where $J$ and $M$ are the representation labels defined by the subgroup chain $SO(3) \supset SO(2)$, while $K$ is the label of the extra copy $SO(2)_L \subset SO(3)_L$ that describes nontrivial multiplicity. Here, $-J \leq K, M \leq +J$, and for each allowed $K$ one has $J \geq K$, i.e. $J = K, K + 1, K + 2, \ldots$.

A generic 3-vector operator $(J = 1)$ in the spherical basis $(\alpha = 0, \pm 1)$ reads:

$$X_\alpha = \mathcal{X}_{(0)} D^{(1)}_{0\alpha}(k) + \mathcal{X}_{(\pm 1)} [D^{(1)}_{+1\alpha}(k) + D^{(1)}_{-1\alpha}(k)], \quad k \in SO(3).$$

(18)

The corresponding matrix elements between the states of two unitary irreducible SL(3, R) representations that are characterized by the labels $\sigma$ and $\delta$ are given as follows:

$$\left< \left( \begin{array}{c} \sigma' \delta' \\ J' \\ K' \ M' \end{array} \right) \left| X_\alpha \right| \left( \begin{array}{c} \sigma \delta \\ J \\ K \ M \end{array} \right) > = (-)^{J'-K'}(-)^{J'-M'} \sqrt{(2J'+1)(2J+1)}$$

12
\[ \times \begin{pmatrix} J' & 1 & J \\ -M' & \alpha & M \end{pmatrix} \begin{pmatrix} J' & 1 & J \\ -K' & 0 & K \end{pmatrix} \]

\[ + \mathcal{X}_{(\pm 1), J'} \begin{pmatrix} J' & 1 & J \\ -K' & 1 & K \end{pmatrix} + \begin{pmatrix} J' & 1 & J \\ -K' & -1 & K \end{pmatrix} \right) \}}. \quad (19) \]

Therefore, the action of a generic \( \overline{SL}(3, R) \) vector operator on the Hilbert space of some nontrivial-multiplicity unitary irreducible representation produces the \( \Delta J = 0, \pm 1 \), as well as the \( \Delta K = 0, \pm 1 \) transitions. Owing to the fact that the states of a unitary irreducible \( \overline{SL}(3, R) \) representation are characterized by the \( \Delta K = 0, \pm 2 \) condition, it is clear that the \( \Delta K = \pm 1 \) transitions due to 3-vector \( X \) take place between the states of mutually inequivalent \( \overline{SL}(3, R) \) representations whose multiplicity is characterized by the \( K \) values of opposite evenness. In analogy to the finite-dimensional (tensorial) representation case, the repeated applications of a vector operator on a given unitary irreducible (spinorial and/or tensorial) \( \overline{SL}(3, R) \) representation would yield, a priori, an infinite set of irreducible representations. Due to an increased mathematical complexity in the case of infinite-dimensional representations, some additional algebraic constraints imposed on the vector operator \( X \) would be even more desirable than in the finite-dimensional case. The most natural option is to embed the \( \overline{SL}(3, R) \) 3-vector \( X \) together with the \( \overline{SL}(3, R) \) algebra itself into the (simple) Lie algebra of the \( \overline{SL}(4, R) \) group. Any spinorial (and/or tensorial) \( \overline{SL}(4, R) \) unitary irreducible representation provides a Hilbert space that can be decomposed w.r.t. \( \overline{SL}(3, R) \) subgroup representations, and most importantly that is invariant under the action of the vector operator \( X \). Moreover, an explicit construction of the starting \( \overline{SL}(4, R) \) representation would provide additionally for an explicit form of \( X \).

4 Embedding into \( \overline{SL}(4, R) \)

The \( \overline{SL}(4, R) \) group, the double covering of the \( SL(4, R) \) group, is a 15-parameter non-compact Lie group, whose defining (spinorial) representation is given in terms of infinite matrices. All spinorial (unitary and nonunitary) representations of \( \overline{SL}(4, R) \) are necessarily infinite-dimensional; the finite-dimensional tensorial representations are nonunitary, while the unitary tensorial representations are infinite-dimensional. The \( \overline{SL}(4, R) \) commutation
relations in the Minkowski space are given by,

\[ [Q_{ab}, Q_{cd}] = i\eta_{bc}Q_{ad} - i\eta_{ad}Q_{cb}. \] (20)

where, \( a, b, c, d \) = 0, 1, 2, 3, and \( \eta_{ab} = \text{diag}(+1, -1, -1, -1) \), while in the Euclidean space they read,

\[ [Q_{ab}, Q_{cd}] = i\delta_{bc}Q_{ad} - i\delta_{ad}Q_{cb}. \] (21)

where, \( a, b, c, d \) = 1, 2, 3, 4, and \( \delta_{ab} = \text{diag}(+1, +1, +1, +1) \)

The relevant subgroup chain reads

\[ \text{SL}(4, \mathbb{R}) \supset \text{SL}(3, \mathbb{R}) \cup \text{SO}(4), \text{SO}(1,3) \supset \text{SO}(3), \text{SO}(1,2). \] (22)

We denote by \( R_{mn}, \ (m, n = 1, 2, 3, 4) \) the 6 compact generators of the maximal compact subgroup \( \text{SO}(4) \) of the \( \text{SL}(4, \mathbb{R}) \) group, and the remaining 9 noncompact generators (of the \( \text{SL}(4, \mathbb{R})/\text{SO}(4) \) coset) by \( Z_{mn} \).

In the \( \text{SO}(4) \cong SU(2) \otimes SU(2) \) spherical basis, the compact operators are \( J_{i}^{(1)} = \frac{1}{2}(\epsilon_{ijk}R_{jk} + R_{i4}) \) and \( J_{i}^{(2)} = \frac{1}{2}(\epsilon_{ijk}R_{jk} - R_{i4}) \), while the noncompact generators we denote by \( Z_{a\beta}, \ (\alpha, \beta = 0, \pm 1) \), and they transform as a \((1, 1)\)-tensor operator w.r.t. \( SU(2) \otimes SU(2) \) group. The minimal set of commutation relations in the spherical basis reads

\[ [J_{0}^{(p)}, J_{\pm}^{(q)}] = \pm\delta_{pq}J_{0}^{(p)}, \quad [J_{+}^{(p)}, J_{-}^{(q)}] = 2\delta_{pq}J_{0}^{(p)}, \quad (p, q = 1, 2), \]
\[ [J_{0}^{(1)}, Z_{a\beta}] = \alpha Z_{a\beta}, \quad [J_{0}^{(1)}, Z_{a\beta}] = \sqrt{2 - \alpha(\pm 1)}Z_{a\beta}, \]
\[ [J_{0}^{(2)}, Z_{a\beta}] = \beta Z_{a\beta}, \quad [J_{0}^{(2)}, Z_{a\beta}] = \sqrt{2 - \beta(\pm 1)}Z_{a\beta}, \]
\[ [Z_{+1}, Z_{-1}] = -(J^{(1)} + J^{(2)}). \] (23)

The \( \text{SO}(3) \) generators are \( J_{i} = \epsilon_{ijk}J_{jk}, \ J_{ij} = R_{ij}, \ (i, j, k = 1, 2, 3) \), while the traceless \( T_{ij} = Z_{ij} \ (i, j = 1, 2, 3) \) define the coset \( \text{SL}(3, \mathbb{R})/\text{SO}(3) \). In the \( \text{SO}(3) \) spherical basis the compact operators are \( J_{0}, \ J_{\pm} \), while the noncompact ones \( T_{\rho}, \ (\rho = 0, \pm 1, \pm 2) \) transform w.r.t. \( \text{SO}(3) \) as a quadrupole operator. The corresponding minimal set of commutation relations reads

\[ [J_{0}, J_{\pm}] = \pm J_{\pm}, \quad [J_{+}, J_{-}] = 2J_{0} \]
\[ [T_{+2}, T_{-2}] = -4J_{0}. \] (24)
There are three (independent) $\mathcal{SO}(3)$ vectors in the algebra of the $\mathfrak{SL}(4, R)$ group. They are: (i) the $\mathcal{SO}(3)$ generators themselves, (ii) $N_i \equiv R_{i4} = Q_{i0} + Q_{0i}$, and (iii) $K_i \equiv Z_{i4} = Q_{i0} - Q_{0i}$. From the latter two, one can form the following linear combinations:

$$A_i = \frac{1}{2}(N_i + K_i) = Q_{i0}, \quad B_i = \frac{1}{2}(N_i - K_i) = Q_{0i}. \quad (25)$$

The commutation relations between $N, K, A,$ and $B$ and the $\mathfrak{SL}(3, R)$ generators read

$$[J_i, N_j] = i\epsilon_{ijk}N_k, \quad [T_{ij}, N_k] = i(\delta_{ik}K_j + \delta_{jk}K_i),$$

$$[J_i, K_j] = i\epsilon_{ijk}K_k, \quad [T_{ij}, K_k] = i(\delta_{ik}N_j + \delta_{jk}N_i),$$

$$[J_i, A_j] = i\epsilon_{ijk}A_k, \quad [T_{ij}, A_k] = i(\delta_{ik}A_j + \delta_{jk}A_i),$$

$$[J_i, B_j] = i\epsilon_{ijk}B_k, \quad [T_{ij}, B_k] = -i(\delta_{ik}B_j + \delta_{jk}B_i). \quad (26)$$

It is clear from these expressions that only $A_i$ and $B_i$ are $\mathfrak{SL}(3, R)$ vectors as well. More precisely, $A$ transforms w.r.t. $\mathfrak{SL}(3, R)$ as the three-dimensional representation $[1, 0]$, while $B$ transforms as its contragradient three-dimensional representation $[1, 1]$.

To summarize, either one of the two nontrivial $\mathfrak{SL}(3, R)$ vector choices

$$X_i \sim A_i, \quad X_i \sim B_i \quad (27)$$

insures that a Dirac-like wave equation $(iX\partial - m)\Psi(x) = 0$ for a (infinite-component) spinorial field is fully $\mathfrak{SL}(3, R)$ covariant. The choices

$$X_i \sim N_i, \quad X_i \sim K_i, \quad (28)$$

would yield wave equations that are Lorentz covariant only, even though the complete $\mathfrak{SL}(3, R)$ acts invariantly in the space of $\Psi(x)$ components. It goes without saying that the correct unitarity properties can be accounted for by making use of the deunitarizing automorphism, as discussed above.

5 Reduction of the $\mathfrak{SL}(4, R)$ multiplicity-free unitary irreducible representations

In the following, we consider the problem of reduction of the unitary irreducible $\mathfrak{SL}(4, R)$ representations into the corresponding $\mathfrak{SL}(3, R)$ subgroup
irreducible representations. Due to complexity of these representations, we confine in this work only to the multiplicity-free representations of the \( SL(4, \mathbb{R}) \) group, that themselves give rise, in the reduction, to nonmultiplicity-free \( SL(3, \mathbb{R}) \) representations. Moreover, in order to maintain a parallelism between infinite-component spinors and tensors, we shall consider here both spinorial and tensorial representations. This parallelism is of interest, for instance, when studying the infinite-component wave equations of Regge-trajectory hadron recurrences [16].

Before proceeding further, let us concentrate on the parity properties of the \( SL(4, \mathbb{R}) \) generators, that can help considerably to simplify the actual decomposition of considered representations. In the 3+1 notation, the \( SL(4, \mathbb{R}) \) generators decompose w.r.t. \( SO(3) \) into one quadrupole \( T(2) \), three vector \( J(1), N(1), K(1) \), and one scalar \( S(0) \sim Q_{00} \) operator.

The action of the parity (space inversion) operator \( P \) on the \( SL(4, \mathbb{R}) \) generators is as follows:

\[
\begin{align*}
P J P^{-1} &= +J, \quad P T P^{-1} = +T, \quad P S P^{-1} = +S \\
P N P^{-1} &= -N, \quad P K P^{-1} = -K.
\end{align*}
\] (29)

The operators \( J \) and \( T \) connect mutually the Hilbert space states of the same parity, i.e. all the states of an \( SL(3, \mathbb{R}) \) irreducible representation have the same parity. For example, the \( J^P \), i.e. the spin, parity content of the unitary irreducible representation \( D^u_{SL(3, \mathbb{R})}(0; \sigma_2, \delta_2) \) is

\[
\{ J^P \} = \{ 0^+, 2^+, 4^+, \ldots; 2^+, 3^+, 4^+, \ldots; 4^+, 5^+, 6^+, \ldots \}. \quad (30)
\]

The repeated action of the \( N \) and \( K \) operators results in the states of alternating parities. For example, the \( J^P \) content of a finite-dimensional \( D(\frac{7}{2}, 2) \) representation that is \( SO(4) \) unitary, i.e. \( SO(1, 3) \) nonunitary reads

\[
\{ J^P \} = \left\{ \frac{3^+}{2}, \frac{5^-}{2}, \frac{7^+}{2}, \frac{9^-}{2}, \frac{11^+}{2} \right\} \quad \text{or} \quad \{ J^P \} = \left\{ \frac{3^-}{2}, \frac{5^+}{2}, \frac{7^-}{2}, \frac{9^+}{2}, \frac{11^-}{2} \right\} \quad (31)
\]

Let us consider at first the reduction of the simplest multiplicity-free unitary irreducible representations of the \( SL(4, \mathbb{R}) \) group, i.e. the ladder ones \( D_{SL(4, \mathbb{R})}^{(ladd)}(0; e_2) \) and \( D_{SL(4, \mathbb{R})}^{(ladd)}(\frac{1}{2}; e_2) \). The \( SL(4, \mathbb{R}) \supset SO(3) \otimes SO(3) \supset SO(3) \) decomposition of the \( D_{SL(4, \mathbb{R})}^{(ladd)}(0; e_2) \) representation, i.e. the \( \{(j_1, j_2)\} \supset \{ J^P \} \)
\[
\{(j_1, j_2)\} = \{(0,0), (1,1), (2,2), \ldots\}
\]
\[
J^P = 0^+ 2^+ 4^+ \ldots 1^- 3^- \ldots 0^+ 2^+ \ldots 1^- \ldots 0^+ \ldots
\]

The action of the noncompact operators \(T\) and \(S\) connects various \(\overline{SO}(3)\) states \(J^P_{(j_1,j_2)}\) (where \((j_1, j_2)\) denotes the "parent" \(\overline{SO}(4)\) state of a given state \(J\)). The irreducible actions of \(T\) are

\[
T : \quad \{0^+_{(0,0)} \leftrightarrow 2^+_{(1,1)} \leftrightarrow 4^+_{(2,2)} \leftrightarrow \ldots\}, \quad \{1^-_{(1,1)} \leftrightarrow 3^-_{(2,2)} \leftrightarrow 5^-_{(3,3)} \leftrightarrow \ldots\},
\]
\[
\{0^+_{(1,1)} \leftrightarrow 2^+_{(2,2)} \leftrightarrow 4^+_{(3,3)} \leftrightarrow \ldots\}, \quad \{1^-_{(2,2)} \leftrightarrow 3^-_{(3,3)} \leftrightarrow 5^-_{(4,4)} \leftrightarrow \ldots\}, \ldots
\]

The irreducible actions of \(S\) are:

\[
S : \quad \{0^+_{(0,0)} \leftrightarrow 0^+_{(1,1)} \leftrightarrow 0^+_{(2,2)} \leftrightarrow \ldots\}, \quad \{1^-_{(1,1)} \leftrightarrow 1^-_{(2,2)} \leftrightarrow 1^-_{(3,3)} \leftrightarrow \ldots\},
\]
\[
\{2^+_{(1,1)} \leftrightarrow 2^+_{(2,2)} \leftrightarrow 2^+_{(3,3)} \leftrightarrow \ldots\}, \quad \{3^-_{(2,2)} \leftrightarrow 3^-_{(3,3)} \leftrightarrow 3^-_{(4,4)} \leftrightarrow \ldots\}, \ldots
\]

Thus, we see that each \((j_1, j_2) = (j, j) \neq (0,0)\) is an "origin" of a pair of \(\overline{SL}(3, R)\) irreducible representations \(D^{\text{add}}_{\overline{SL}(3, R)}(0; \sigma_2)\) and \(D^{\text{add}}_{\overline{SL}(3, R)}(1; \sigma_2)\), while \((j_1, j_2) = (0,0)\) is an "origin" of a single \(\overline{SL}(3, R)\) irreducible representation, \(D^{\text{add}}_{\overline{SL}(3, R)}(0; \sigma_2)\). Symbolically, we write,

\[
D^{\text{add}}_{\overline{SL}(4, R)}(0; e_2) \supset D^{\text{add}}_{\overline{SL}(3, R)}(0; \sigma_2) \bigoplus_{j=1}^{\infty} [D^{\text{add}}_{\overline{SL}(3, R)}(0; \sigma_2(j)) \oplus D^{\text{add}}_{\overline{SL}(3, R)}(1; \sigma_2(j))].
\]

The reduction of the \(D^{\text{add}}_{\overline{SL}(4, R)}(1/2; e_2)\) proceeds analogously.

Let us now consider the reduction of an \(\overline{SL}(4, R)\) unitary irreducible representation that has a nontrivial multiplicity. We shall, for simplicity reasons, illustrate the method in the \(D^{\text{pr}}_{\overline{SL}(4, R)}(0,0; e_2)\) representation case.

The \(\overline{SL}(4, R) \supset \overline{SO}(3) \otimes \overline{SO}(3) \supset \overline{SO}(3)\) decomposition of the \(D^{\text{add}}_{\overline{SL}(4, R)}(0; e_2)\)
representation, i.e. the \( \{(j_1, j_2)\} \supset \{J^P\} \) content reads

\[
\{(j_1, j_2)\} = \{(0, 0); \ (2, 0); \ (1, 1); \ (0, 2); \ (4, 0); \ (3, 1); \ (2, 2), \ldots\}
\]

\[
J^P = \begin{array}{cccccccc}
0^+ & 2^± & 2^± & 4^± & 4^± & 4^+ & \ldots \\
1^- & 3^± & 3^- & \ldots \\
0^+ & 2^± & 2^+ & \ldots \\
1^- & \ldots \\
0^+ & \ldots 
\end{array}
\]

The transitions: \( (j_1 + j_2)_{(j_1, j_2)} \to (j_1 + j_2 \pm 2)_{(j_1, j_2 \pm 1, j_2 \pm 1)} \) are due to the \( T \) operator solely, and thus these states have the same parity. The transitions \( (j_1 + j_2)_{(j_1, j_2)} \to (j_1 + j_2)_{(j_1 \pm 1, j_2 \pm 1)} \) are due to the actions of the linear combinations \( T \pm K \), and thus the resulting states are not the eigenstates of the parity operator \( \mathcal{P} \). The states of definite parity are the symmetric and anti-symmetric combinations of the corresponding states of the \( (j_1, j_2) \) and \( (j_2, j_1) \) multiplets. For example, \( 4^+_{(3,1)+1,3} \equiv 4_{(3,1)} + 4_{(1,3)} \) and \( 4^-_{(3,1)+1,3} \equiv 4_{(3,1)} - 4_{(1,3)} \) are the eigenstates of positive and negative parities, respectively. Owing to the nonvanishing matrix elements of the \( \overrightarrow{SL}(3, R) \) quadrupole operator \( T \) between the \( (j_1 + j_2)_{(j_1, j_2)} \) and \( (j_1 + j_2)_{(j_1 \pm 1, j_2 \pm 1)} \) states, we obtain in the reduction the \( \overrightarrow{SL}(3, R) \) representations with nontrivial multiplicity of \( \overrightarrow{SO}(3) \) subrepresentations. We have now all the information to regroup the \( J^P \) states according to the \( \overrightarrow{SL}(3, R) \) irreducible representations. The lowest-lying \( D^\text{pr}_{\overrightarrow{SL}(4, R)}(0, 0; e_2) \) states organize w.r.t. the \( T \) operator action as follows:

\[
\begin{align*}
&\{0^+_{(0,0)}, 2^+_{(1,1)}, 4^+_{(2,2)}, \ldots; 2^+_{(2,0),(0,2)}, 3^+_{(3,1),(1,3)}, 4^+_{(3,1),(1,3)} \ldots; 4^+_{(4,0),(0,4)}, \ldots\}, \\
&\{0^+_{(1,1)}, 2^+_{(2,2)}, 4^+_{(3,3)}, \ldots; 2^+_{(3,1),(1,3)}, 3^+_{(4,2),(2,4)}, 4^+_{(4,2),(2,4)} \ldots; 4^+_{(5,1),(1,5)}, \ldots\}, \\
&\{1^+_{(1,1)}, 3^-_{(2,2)}, 5^-_{(3,3)}, \ldots; 2^+_{(2,0),(0,2)}, 3^-_{(3,1),(1,3)}, 4^-_{(3,1),(1,3)} \ldots; 4^-_{(4,0),(0,4)}, \ldots\}, \\
&\{0^+_{(2,2)}, 2^+_{(3,3)}, 4^+_{(4,4)}, \ldots; 2^+_{(4,2),(2,4)}, 3^+_{(5,3),(3,5)}, 4^+_{(5,3),(3,5)} \ldots; 4^+_{(6,2),(2,6)}, \ldots\}, \\
&\{1^+_{(2,2)}, 3^-_{(3,3)}, 5^-_{(4,4)}, \ldots; 2^-_{(3,1),(1,3)}, 3^-_{(4,2),(2,4)}, 4^-_{(4,2),(2,4)} \ldots; 4^-_{(5,1),(1,5)}, \ldots\}, \\
&\ldots
\end{align*}
\]

It is seen from these expressions that there is a single \( \overrightarrow{SL}(3, R) \) irreducible representation, \( D^\text{pr}_{\overrightarrow{SL}(3, R)}(0; \sigma_2, \delta_2) \), "originating" from the state \( (j_1, j_2) = (0, 0), \)
while there is a pair of irreducible representations, $D_{SL(3,R)}^{pr}(0;\sigma_2,\delta_2)$ and $D_{SL(3,R)}^{pr}(1;\sigma_2,\delta_2)$ "originating" from each set of states \{(j_1, j_2) \mid j_1 + j_2 = 2j, j = 1, 2, 3, \ldots\}. Symbolically, we write

$$D_{SL(4,R)}^{pr}(0,0;e_2) \supset D_{SL(3,R)}^{pr}(0;\sigma_2,\delta_2) \bigoplus_{j=1}^{\infty} [D_{SL(3,R)}^{pr}(0;\sigma_2(j),\delta_2(j)) \oplus D_{SL(3,R)}^{pr}(1;\sigma_2(j),\delta_2(j))] .$$

The reduction of all other nontrivial-multiplicity representations proceeds analogously.

We list here the results of reductions of all multiplicity free unitary irreducible representations of the $SL(4,R)$ group into the irreducible representations of its $SL(3,R)$ subgroup.

**Principal Series:**

$$D_{SL(4,R)}^{pr}(0,0;e_2) \supset D_{SL(3,R)}^{pr}(0;\sigma_2,\delta_2) \bigoplus_{j=1}^{\infty} [D_{SL(3,R)}^{pr}(0;\sigma_2(j),\delta_2(j)) \oplus D_{SL(3,R)}^{pr}(1;\sigma_2(j),\delta_2(j))] .$$

**Supplementary Series:**

$$D_{SL(4,R)}^{sup}(0,0;e_1) \supset D_{SL(3,R)}^{sup}(0;\sigma_2,\delta_1) \bigoplus_{j=1}^{\infty} [D_{SL(3,R)}^{sup}(0;\sigma_2(j),\delta_1(j)) \oplus D_{SL(3,R)}^{sup}(1;\sigma_2(j),\delta_1(j))] .$$

**Discrete Series:**

$$D_{SL(4,R)}^{disc}(j_0,0) \supset \bigoplus_{j=1}^{\infty} D_{SL(3,R)}^{disc}(j_0;\sigma_2(j),\delta_1(j)) .$$

$$D_{SL(4,R)}^{disc}(0,j_0) \supset \bigoplus_{j=1}^{\infty} D_{SL(3,R)}^{disc}(j_0;\sigma_2(j),\delta_1(j)) .$$
Ladder Series:

\[ D_{SL(4, R)}^{\text{ladd}}(0; e_2) \supset D_{SL(3, R)}^{\text{ladd}}(0; \sigma_2) \]
\[
\bigoplus_{j=1}^{\infty} \left[ D_{SL(3, R)}^{\text{ladd}}(0; \sigma_2(j)) \oplus D_{SL(3, R)}^{\text{ladd}}(1; \sigma_2(j)) \right],
\]

\[ D_{SL(4, R)}^{\text{ladd}}\left(\frac{1}{2}; e_2\right) \supset \bigoplus_{j=1}^{\infty} \left[ D_{SL(3, R)}^{\text{ladd}}(0; \sigma_2(j)) \oplus D_{SL(3, R)}^{\text{ladd}}(1; \sigma_2(j)) \right]\]

6 \text{ \textbf{SL}(3, R) spinorial wave equation}

When embedding \( \text{SL}(3, R) \) into \( \text{SL}(4, R) \), there are, as seen above, two (mutually contragradient) \( \text{SL}(3, R) \) vector candidates, i.e. \( X \sim A = \frac{1}{2}(N + K) \) or \( X \sim B = \frac{1}{2}(N - K) \). The explicit form of the \( N \) operator (in the spherical basis of the \( \text{SO}(4) = SU(2) \otimes SU(2) \) group) is well known, while the embedding approach yields a closed expressions for the \( K \) operator as well.

In particular,

\[ K_\alpha = (-1)^{1-\alpha} i\sqrt{6} \begin{pmatrix} 1 & 1 & 1 \\ -\alpha & \beta & \gamma \end{pmatrix} Z_{\beta \gamma}, \]

where \( \alpha, \beta, \gamma = 0, \pm 1 \), and thus its matrix elements are given explicitly in terms of the \( Z \) ones.

In the \( \begin{pmatrix} J \\ M \end{pmatrix} \) basis of the \( \text{SO}(3) \subset \text{SL}(3, R) \), one has,

\[
\begin{align*}
\langle J' M' | K_\alpha | J M \rangle &= i\sqrt{6} \begin{pmatrix} 1 & 1 & 1 \\ -\alpha & \beta & \gamma \end{pmatrix} Z_{\beta \gamma}, \\
&= i\sqrt{6} \begin{pmatrix} 1 & 1 & 1 \\ -\alpha & \beta & \gamma \end{pmatrix} Z_{\beta \gamma}, \\
&= \sum \left[ (-1)^{1-\alpha} (-1)^{j'_1-m'_1} (-1)^{j'_2-m'_2} \begin{pmatrix} J' & j'_1 & j'_2 \\ -M' & m'_1 & m'_2 \end{pmatrix} \begin{pmatrix} J & j_1 & j_2 \\ -M & m_1 & m_2 \end{pmatrix} \right] \langle j'_1 j'_2 | Z | j_1 j_2 \rangle.
\end{align*}
\]

The sum of the 3-\( j \) symbols in this expression is given in terms of the 9-\( j \)
symbol, and thus, we write
\[
\left\langle \begin{array}{c} J' \\ M' \end{array} \Bigg| K_\alpha \Bigg| \begin{array}{c} J \\ M \end{array} \right\rangle = i \sqrt{6} (-)^{J'-M'} \sqrt{(2J'+1)(2J+1)} \left( \begin{array}{ccc} J' & 1 & J \\ -M' & \alpha & M \end{array} \right) \\
\times \begin{Bmatrix} j'_1 & 1 & j_1 \\ j'_2 & 1 & j_2 \\ J' & 1 & J \end{Bmatrix} <j'_1 j'_2||Z||j_1 j_2>,
\]

(40)

where, \(<j'_1 j'_2||Z||j_1 j_2>\) are the reduced matrix elements of the operator \(Z_{\alpha \beta}\).

Finally, we can write an \(\mathfrak{SL}(3, R)\) covariant wave equation in the form
\[
(iX^\mu \partial_\mu - M) \Psi(x) = 0, \quad (41)
\]
\[
\Psi \sim D^{disc}_{\mathfrak{SL}(4,R)}(j_0, 0), \quad D^{disc}_{\mathfrak{SL}(4,R)}(0, j_0), \quad (42)
\]
\[
X^\mu = \frac{1}{2} (N^\mu + K^\mu) = \frac{1}{2} (J^{(1)\mu} - J^{(2)\mu} + K^\mu), \quad (43)
\]

where \(\mu = 0, 1, 2\), and \(K^0 = K_0, \quad K^1 = -\frac{1}{\sqrt{2}} (K_{+1} - K_{-1}), \quad K^2 = \frac{1}{\sqrt{2}} (K_{+1} + K_{-1})\). The matrix elements of all operators defining the \(\mathfrak{SL}(3, R)\) vector operator \(X^\mu\) in the infinite-component representation of the field \(\Psi(x)\) are explicitly constructed.

7 Appendix A. \(\mathfrak{SL}(3, R)\) unitary irreducible representations

The unitary irreducible representations of the \(\mathfrak{SL}(3, R)\) group \([17]\) are defined in Hilbert spaces which are symmetric homogeneous spaces over certain quotient subgroups \(\mathcal{K}\) of its maximal compact subgroup \(\mathfrak{SO}(3) \simeq SU(2)\). In other words, they are defined in the spaces \(L^2(\mathcal{K})\) of square-integrable functions w.r.t. the invariant measure \(dk\) over \(\mathcal{K}\), i.e.
\[
(f, g) = \int \int_{\mathcal{K} \otimes \mathcal{K}} f^*(k') \kappa(k', k'') g(k'') dk' dk''.
\]

As a matter of fact, in order to account for nontrivial multiplicity of the \(\mathfrak{SO}(3)\) subrepresentations, we work in the space of functions over the diagonal subgroup \([\mathcal{K}_L \otimes \mathcal{K}_R]^d\) corresponding to the left and right group action, respectively. Thus, there is another label, \(K\), that accounts for nontrivial
multiplicity. In order to obtain all representations, one has to consider the most general scalar product of the Hilbert space elements with, in general, a nontrivial kernel \( \kappa \). Furthermore, the irreducibility requirements yield, in general, certain relationships between the representation labels, the corresponding labels of the maximal compact subgroup and the matrix elements of \( \kappa \).

When \( \mathcal{K} = SU(2) \), the representation space basis is \( |J\ K\ M\rangle \). The compact generators matrix elements are the well known ones,

\[
J_0 |J\ K\ M\rangle = M |J\ K\ M\rangle,
\]

\[
J_\pm |J\ K\ M\rangle = \sqrt{J(J + 1) - M(M + 1)} |J\ K\ M\pm 1\rangle
\]

while the matrix elements of the noncompact generators are given by the following expression:

\[
\langle J'\ K'\ M' | T_\rho | J\ K\ M\rangle = -i (-)^{J'-K'} (-)^{J'-M'} \sqrt{(2J'+1)(2J+1)} \left( \begin{array}{ccc} J' & 2 & J \\ -M' & \rho & M \end{array} \right)
\]

\[
\times \left[ \left( \frac{2}{3} (\sigma_1 + i \sigma_2) - \frac{1}{\sqrt{6}} (J'(J'+1) - J(J+1)) \right) \left( \begin{array}{ccc} J' & 2 & J \\ -K' & 0 & K \end{array} \right)
\]

\[
- (\delta_1 + 1 + i \delta_2) \left( \begin{array}{ccc} J' & 2 & J \\ -K' & 2 & K \end{array} \right) - (\delta_1 + 1 + i \delta_2) \left( \begin{array}{ccc} J' & 2 & J \\ -K' & -2 & K \end{array} \right) \right]
\]

where, \( \sigma' = \sigma_1 + i \sigma_2 \), and \( \delta' = \delta_1 + \delta_2 \) (\( \sigma_1, \sigma_2, \delta_1, \delta_2 \in \mathbb{R} \)) are the \( \mathcal{SL}(3, \mathbb{R}) \) representation label. The \( \{J\} \) content, i.e. \( \mathcal{SO}(3) \) subrepresentations, follow the two general rules: (i) \( \Delta K = 0, \pm 1 \), and (ii) for each \( K \neq 0 \), \( J = K, K+1, K+2, \ldots \), while for \( K = 0 \), either \( J = 0, 2, 4, \ldots \) or \( J = 1, 3, 5, \ldots \).

When \( \mathcal{K} = SU(2)/U(1) \), the representation space basis is \( |J\ M\rangle \). The compact generators matrix elements are the well known ones,

\[
J_0 |J\ M\rangle = M |J\ M\rangle,
\]

\[
J_\pm |J\ M\rangle = \sqrt{J(J + 1) - M(M + 1)} |J\ M\pm 1\rangle
\]
while the matrix elements of the noncompact generators are given by the following expression:

\[
\langle \begin{array}{c} J' \\ M' \end{array} | T_\rho | \begin{array}{c} J \\ M \end{array} \rangle = -i(-)^{J'}(-)^{J-M'} \sqrt{(2J'+1)(2J+1)} \left( \begin{array}{ccc} J' & 2 & J \\ -M' & \rho & M \end{array} \right) \\
\times \left( \sqrt{\frac{2}{3}}(\sigma_1 + i\sigma_2 - \frac{1}{\sqrt{6}}(J'(J'+1) - J(J+1)) \right)
\]

where, \(\sigma_1, \sigma_2 \in \mathbb{R}\) are the \(SL(3, \mathbb{R})\) representation labels. The 3-\(j\) symbol \(\left( \begin{array}{ccc} J' & 2 & J \\ 0 & 0 & 0 \end{array} \right)\), with half-integer entries is to be evaluated by taking the explicit expression for the integer case and continuing it to the half-integer one. The \(\{J\}\) content, i.e. \(SO(3)\) subrepresentations, is given by the rule \(\Delta J = 0, \pm 2\).

There are, besides the trivial representation, four series of unitary irreducible representations of the \(SL(3, \mathbb{R})\) group, that are characterized by the representation label, the minimal \(J\) (and when necessary the minimal \(K \geq 0\)) values, and they are defined in Hilbert spaces with the basis vectors corresponding to certain irreducible lattices in the \(J - |K|\) plane, and the scalar products are given in terms of the kernel \(\kappa\), i.e. \(D_{SL(3, \mathbb{R})}(J_K; \sigma, \delta)\)

**Principal Series.**

\[D_{SL(3, \mathbb{R})}^{pr}(0_0; \sigma_2, \delta_2), \quad \{J\} = \{0, 2, 4, \ldots; 2, 3, 4, \ldots; 4, 5, 6, \ldots; \ldots\}\]
\[D_{SL(3, \mathbb{R})}^{pr}(1_0; \sigma_2, \delta_2), \quad \{J\} = \{1, 3, 5, \ldots; 2, 3, 4, \ldots; 4, 5, 6, \ldots; \ldots\}\]
\[D_{SL(3, \mathbb{R})}^{pr}(1_1; \sigma_2, \delta_2), \quad \{J\} = \{1, 2, 3, \ldots; 3, 4, 5, \ldots; 5, 6, 7, \ldots; \ldots\}\]
\[D_{SL(3, \mathbb{R})}^{pr}(1_2; \sigma_2, \delta_2), \quad \{J\} = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots; \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots; \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \ldots; \ldots\}\]

where \(\sigma_2, \delta_2 \in \mathbb{R}\). They are defined in the Hilbert spaces \(H(SU(2), \kappa)\), where \(\kappa = \sqrt{2J+1}, \forall J\).

**Supplementary Series.**

\[D_{SL(3, \mathbb{R})}^{supp}(0_0; \sigma_2, \delta_1), \quad \{J\} = \{0, 2, 4, \ldots; 2, 3, 4, \ldots; 4, 5, 6, \ldots; \ldots\}\]
where $\sigma_2 \in \mathbb{R}$, while $|\delta_1| < \frac{1}{2}$ for integer $J$ ($K = 0$ only), and $|\delta_1| < \frac{1}{2}$ for half integer $J$ ($K = \frac{1}{2}$). They are defined in the Hilbert spaces $H(SU(2), \kappa)$, and the $\kappa$ matrix elements are $\kappa(J; K) = \sqrt{\frac{2J+1}{2J+1} \Gamma(\frac{1}{2}(K+1-\delta_1)) \Gamma(\frac{1}{2}(K+1+\delta_1)) \Gamma(\frac{1}{2}(K+1-\delta_1)) \Gamma(\frac{1}{2}(K+1+\delta_1))}$. 

**Discrete Series.**

\[
D^{\text{disc}}_{\text{SL}(3, \mathbb{R})}(J; \sigma_2, \delta_1), \quad \{J\} = \{J, J + 1, J + 2, \ldots; J + 2, J + 3, J + 4, \ldots; J + 4, J + 5, J + 6, \ldots; \ldots\}
\]

where $J = K = \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots$, and $\delta_1 = 1 - J$. They are defined in the Hilbert spaces $H(SU(2), \kappa)$, and $\kappa(J; K) = \sqrt{2J+1} \Gamma(\frac{1}{2}(K+K)) \Gamma(\frac{1}{2}(K-K))$. 

**Ladder Series.**

\[
D^{\text{ladd}}_{\text{SL}(3, \mathbb{R})}(0; \sigma_2), \quad \{J\} = \{0, 2, 4, \ldots\}
\]

\[
D^{\text{ladd}}_{\text{SL}(3, \mathbb{R})}(0; \sigma_2), \quad \{J\} = \{1, 3, 5, \ldots\}
\]

\[
D^{\text{ladd}}_{\text{SL}(3, \mathbb{R})}(\frac{1}{2}), \quad \{J\} = \{\frac{1}{2}, 5, 9, \ldots\}
\]

where $\sigma_2 \in \mathbb{R}$. They are defined in the Hilbert spaces $H(SU(2)/U(1), \kappa)$, and the $\kappa$ matrix elements are $\kappa(J; K) = \sqrt{2J+1}, \forall J$. Note that, owing to the unitarity requirement, there is a unique spinorial unitary irreducible representation $D^{\text{ladd}}_{\text{SL}(3, \mathbb{R})}(\frac{1}{2})$ corresponding to $\sigma_2 = 0$. 

8 **Appendix B. $\text{SL}(4, \mathbb{R})$ multiplicity-free unitary irreducible representations**

The unitary irreducible representations of the $\text{SL}(4, \mathbb{R})$ group, that are multiplicity-free w.r.t. its maximal compact subgroup $SO(4) \simeq SU(2) \otimes SU(2)$ [18], are defined in Hilbert spaces which are symmetric homogeneous spaces over certain quotient subgroups $K$ of the maximal compact subgroup. In
other words, they are defined in the spaces $L^2(K)$ of square-integrable functions w.r.t. the invariant measure $dk$ over $K$, i.e.

$$(f, g) = \int \int_{K \otimes K} f^*(k') \kappa(k', k'') g(k'') dk' dk'' .$$

In order to obtain all these representations, one has to consider the most general scalar product of the Hilbert space elements with, in general, a non-trivial kernel $\kappa$. Furthermore, the representation irreducibility requirement implies that the $K$ representation eigenvector labels, $j_1, j_2$, which define a basis of the $\overline{SL}(4, R)$ representation Hilbert space, are constrained to belong to certain invariant lattices $L$ of the $(j_1, j_2)$ plane of points. Therefore, we denote the unitary irreducible representation Hilbert spaces symbolically by $H(K, \kappa, L)$.

When $K = [SU(2)/U(1)] \otimes [SU(2)/U(1)]$, the representation space basis is $| j_1 \ j_2 \ m_1 \ m_2 \rangle$. The compact generators matrix elements are the well known ones,

$$
\begin{align*}
J_0^{(1)} & | j_1 \ j_2 \ m_1 \ m_2 \rangle = m_1 | j_1 \ j_2 \ m_1 \ m_2 \rangle, \\
J_0^{(2)} & | j_1 \ j_2 \ m_1 \ m_2 \rangle = m_2 | j_1 \ j_2 \ m_1 \ m_2 \rangle, \\
J_1^{(1)} & | j_1 \ j_2 \ m_1 \ m_2 \rangle = \sqrt{j_1(j_1 + 1) - m_1(m_1 \pm 1)} | j_1 \ j_2 \ m_1 \pm 1 \ m_2 \rangle, \\
J_2^{(1)} & | j_1 \ j_2 \ m_1 \ m_2 \rangle = \sqrt{j_2(j_2 + 1) - m_2(m_2 \pm 1)} | j_1 \ j_2 \ m_1 \ m_2 \mp 1 \rangle,
\end{align*}
$$

while the matrix elements of the noncompact generators are given by the following expression:

$$
\begin{align*}
\langle j'_1 \ j'_2 \ m'_1 \ m'_2 | & Z_{\alpha \beta} | j_1 \ j_2 \ m_1 \ m_2 \rangle = -i(-)^{j'_1 - m'_1}(-)^{j'_2 - m'_2}(-)^{j'_1 + j'_2} \sqrt{(2j'_1 + 1)(2j'_2 + 1)(2j_1 + 1)(2j_2 + 1)} \\
\times & (e - \frac{1}{2}[j'_1(j'_1 + 1) - j_1(j_1 + 1) + j'_2(j'_2 + 1) - j_2(j_2 + 1)]) \\
\times & \left( \begin{array}{c}
 j'_1 \\
 m'_1 \end{array} \right) \left( \begin{array}{c}
 1 \\
 \alpha \end{array} \right) \left( \begin{array}{c}
 j'_2 \\
 m'_2 \end{array} \right) \left( \begin{array}{c}
 1 \\
 \beta \end{array} \right) \left( \begin{array}{ccc}
 j'_1 & 1 & j_1 \\
 0 & 0 & 0 \\
 0 & 0 & 0
\end{array} \right) \left( \begin{array}{ccc}
 j'_2 & 1 & j_2 \\
 0 & 0 & 0 \\
 0 & 0 & 0
\end{array} \right),
\end{align*}
$$
where, \( e = e_1 + ie_2 \), is the \( SL(4,R) \) representation label. The 3-\( j \) symbol 
\[
\begin{pmatrix}
j' & 1 & j \\
0 & 0 & 0
\end{pmatrix},
\]
with half-integer entries is to be evaluated by taking the explicit expression for the integer case and continuing it to the half-integer one. There are, in this case, eight invariant lattices of points in the space of the \( SU(2) \times SU(2) \) representation labels, 
\[
L = \{(j_1,j_2)\}, \quad j_1, j_2 = 0, 1, 2, 3, \ldots
\]
that are characterized by the conditions 
\[
(j_1 + j_2) - (j_{01} + j_{02}) \equiv 0(\text{mod}2), \quad (j_1 - j_2) - (j_{01} - j_{02}) \equiv 0(\text{mod}2),
\]
where \( (j_{01}, j_{02}) \) are the "minimal" \( (j_1, j_2) \) values. This is due to the noncompact generators action allowing only for \( (j_{1}', j_{2}') = (j_1 \pm 1, j_2 \pm 1) \). We define these invariant lattices by their minimal \( (j_1, j_2) \) values, i.e.
\[
L(0,0), \quad L(\frac{1}{2}, 0), \quad L(0,1), \quad L(\frac{1}{2}, \frac{1}{2}), \quad L(0, \frac{1}{2}), \quad L(\frac{1}{2}, \frac{3}{2}), \quad \text{and} \quad L(0, \frac{3}{2}).
\]
When \( \mathcal{K} = [SU(2) \otimes SU(2)]/SU(2) \), one has \( j_1 = j_2 \equiv j \), and the representation space basis is 
\[
\begin{pmatrix}
j \\
m_1 m_2
\end{pmatrix}
\]
The compact generators matrix elements are
\[
\begin{align*}
J^{(1)}_0 \begin{pmatrix}
j \\
m_1 m_2
\end{pmatrix} &= m_1 \begin{pmatrix}
j \\
m_1 m_2
\end{pmatrix}, \\
J^{(2)}_0 \begin{pmatrix}
j \\
m_1 m_2
\end{pmatrix} &= m_2 \begin{pmatrix}
j \\
m_1 m_2
\end{pmatrix}, \\
J^{(1)}_{\pm} \begin{pmatrix}
j \\
m_1 m_2
\end{pmatrix} &= \sqrt{j(j+1) - m_1(m_1 \pm 1)} \begin{pmatrix}
j \\
m_1 \pm 1 m_2
\end{pmatrix}, \\
J^{(2)}_{\pm} \begin{pmatrix}
j \\
m_1 m_2
\end{pmatrix} &= \sqrt{j(j+1) - m_2(m_2 \pm 1)} \begin{pmatrix}
j \\
m_1 \pm 1 m_2
\end{pmatrix},
\end{align*}
\]
while the matrix elements of the noncompact generators are given by the following expression:
\[
\begin{align*}
&\left\langle j' \begin{pmatrix}
m_1' m_2'
\end{pmatrix} \bigg| Z_{\alpha \beta} \bigg| j \begin{pmatrix}
m_1 m_2
\end{pmatrix} \right\rangle \\
&= -i(-)^{j'-m_1'}(-)^{j'-m_2'}\sqrt{(2j'+1)(2j+1)}(e - \frac{1}{2}[j'(j' + 1) - j(j + 1)])
\times \left( \begin{array}{cc}
j' & 1 \\
-m_1' & \alpha
\end{array} \right) \left( \begin{array}{cc}
j' & 1 \\
-m_2' & \beta
\end{array} \right),
\end{align*}
\]
\[26\]
where \( e = e_1 + ie_2 \) is the \( \overline{SL}(4, R) \) representation label. There are, in this case, two invariant lattices:

\[
L(0) = \{(0,0), (1,1), \ldots\} \\
L\left(\frac{1}{2}\right) = \left\{(\frac{1}{2}, \frac{1}{2}), (\frac{3}{2}, \frac{3}{2}), \ldots\right\}. 
\]

There are, besides the trivial representation, four series of multiplicity-free unitary irreducible representations of the \( \overline{SL}(4, R) \) group, that are characterized by the representation label, the minimal \((j_1, j_2)\) values, and they are defined in Hilbert spaces with the basis vectors corresponding to certain irreducible lattices in the \( j_1 - j_2 \) plane, and the scalar products given in terms of the kernel \( \kappa \).

**Principal Series.**

\[
D_{\overline{SL}(4, R)}^{pr}(0, 0; e_2), \quad D_{\overline{SL}(4, R)}^{pr}(1, 0; e_2), \quad e_1 = 0, \quad e_2 \in R.
\]

They are defined in the Hilbert spaces \( H([SU(2)/U(1)] \otimes [SU(2)/U(1)], \kappa, L) \), where \( \kappa(j_1, j_2) = 1, \forall j_1, j_2, \) and the irreducible lattices are, respectively, \( L(0, 0) \) and \( L(1, 0) \).

**Supplementary Series.**

\[
D_{\overline{SL}(4, R)}^{supp}(0, 0; e_1), \quad 0 < |e_1| < 1, \quad e_2 = 0.
\]

They are defined in the Hilbert spaces \( H([SU(2)/U(1)] \otimes [SU(2)/U(1)], \kappa, L) \), where \( \kappa(j_1, j_2) \) is nontrivial and given by

\[
\kappa(j_1, j_2) = \frac{\Gamma(j_1 + j_2 + e_1 + 1)\Gamma(1 - e_1)\Gamma(|j_1 - j_2| + e_1 + 2)\Gamma(2 - e_1)}{\Gamma(j_1 + j_2 - e_1 + 1)\Gamma(1 + e_1)\Gamma(|j_1 - j_2| - e_1 + 2)\Gamma(2 + e_1)^2},
\]

and the irreducible lattice is \( L(0, 0) \).

**Discrete Series.**

\[
D_{\overline{SL}(4, R)}^{disc}(j_0, 0), \quad D_{\overline{SL}(4, R)}^{disc}(0, j_0), \quad e_1 = 1 - j_0, \quad j_0 = \frac{1}{2}, \frac{3}{2}, \ldots, \quad e_2 = 0.
\]

They are defined in the Hilbert spaces \( H([SU(2)/U(1)] \otimes [SU(2)/U(1)], \kappa, L) \), where \( \kappa(j_1, j_2) \) is nontrivial and given by

\[
\kappa(j_1, j_2) = \frac{\Gamma(j_1 + j_2 + e_1 + 1)\Gamma(|j_1 - j_2| + e_1 + 2)}{\Gamma(j_1 + j_2 - e_1 + 1)\Gamma(|j_1 - j_2| - e_1 + 2)}.
\]
and the irreducible lattices are, respectively, \( L(j_0,0|j_1 - j_2 \geq j_0) \subset L(0,0), \ L(\frac{1}{2},0), \ L(1,0), \ L(\frac{3}{2},0) \) and \( L(0,j_0|j_2 - j_1 \geq j_0) \subset L(0,0), \ L(0,\frac{1}{2}), \ L(0,1), \ L(0,\frac{3}{2}). \)

**Ladder Series.**

\[
D_{SL(4,R)}^{ladd}(0;e_2), \quad D_{SL(4,R)}^{ladd}(\frac{1}{2};e_2), \quad e_1 = 0, e_2 \in R.
\]

They are defined in the Hilbert spaces \( H([SU(2) \otimes SU(2)]/SU(2), \kappa,L) \), where \( \kappa(j,j) = 1, \forall j \), and the irreducible lattices are, respectively, \( L(0) \) and \( L(\frac{1}{2}) \).

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**References**

[1] S. Weinberg, "The Quantum Theory of Fields, Vol. 1: Foundations" (Cambridge University Press, Cambridge, 1995).

[2] A. O. Barut and R. Raczka, "Theory of Group Representations and Applications" (Polish Scientific Publishers, Warsaw, 1977).

[3] M. Berg, C. DeWitt-Morette, S. Gwo and E. Kramer, *Rev. Math. Phys.* 13 (2001) 953.

[4] D. B. Lichtenberg, "Unitary Symmetry and Elementary Particles" (Academic Press, New York 1978).

[5] R. Slanski, *Phys. Rep.* 79 (1981) 1.

[6] H. Georgi, "Lie Algebras in Particle Physics" (Addison-Wesley, Reading MA, 1982).

[7] Y. Ne’eman and Dj. Šijački, *Int. J. Mod. Phys.* A 2 (1987) 1655.

[8] Y. Ne’eman and Dj. Šijački, *Phys. Lett.* B 157 (1985) 275.
[9] Dj. Šijački, *Acta Phys. Polonica* **B 29** (1998) 1089.

[10] F.W. Hehl, G.D. Kerlick and P. von der Heyde, *Phys. Lett. B* **63** (1976) 446.

[11] F.W. Hehl, J.D. McCrea, E.W. Mielke and Y. Ne’eman, *Phys. Reports* **258** (1995) 1.

[12] Y. Ne’eman and Dj. Šijački, *Ann. Phys. (N.Y.)* **120** (1979) 292.

[13] J. Mickelsson, *Commun. Math. Phys.* **88** (1983) 551.

[14] A. Cant and Y. Ne’eman, *J. Math. Phys.* **26** (1985) 3180.

[15] I.Kirsch and Dj. Šijački, *Class. Quant. Grav.* **19** (2002) 3157.

[16] Y. Ne’eman and Dj. Šijački, *Phys. Lett. B* **276** (1992) 173.

[17] Dj. Šijački, *J. Math. Phys.* **16** (1975) 298.

[18] Dj. Šijački and Y. Ne’eman, *J. Math. Phys.* **26** (1985) 2475.