ELEMENTARY CONSTRUCTION OF MINIMAL FREE RESOLUTIONS OF THE SPECHT IDEALS OF SHAPES \((n-2,2)\) AND \((d,d,1)\)

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Abstract. For a partition \(\lambda\) of \(n \in \mathbb{N}\), let \(I^{\text{Sp}}_{\lambda}\) be the ideal of \(R = K[x_1, \ldots, x_n]\) generated by all Specht polynomials of shape \(\lambda\). We assume that \(\text{char}(K) = 0\). Then \(R/I^{\text{Sp}}_{(n-2,2)}\) is Gorenstein, and \(R/I^{\text{Sp}}_{(d,d,1)}\) is a Cohen-Macaulay ring with a linear free resolution. In this paper, we construct minimal free resolutions of these rings. Berkesch Zamaere, Griffeth, and Sam [2] had already studied minimal free resolutions of \(R/I^{\text{Sp}}_{(n-d,d)}\), which are also Cohen-Macaulay, using highly advanced technique of the representation theory. However we only use the basic theory of Specht modules, and explicitly describe the differential maps.

1. Introduction

For a positive integer \(n\), a partition of \(n\) is a sequence \(\lambda = (\lambda_1, \ldots, \lambda_l)\) of integers with \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 1\) and \(\sum_{i=1}^{l} \lambda_i = n\). A partition \(\lambda\) is frequently represented by its Young diagram. The Young tableau of shape \(\lambda\) is a bijective filling of the Young diagram of \(\lambda\) by the integers in \([n] := \{1, 2, \ldots, n\}\). For example, the following is a tableau of shape \((4, 2, 1)\).

\[
\begin{array}{ccc}
3 & 5 & 1 & 7 \\
6 & 2 \\
4
\end{array}
\]

Let \(\text{Tab}(\lambda)\) be the set of Young tableaux of shape \(\lambda\).

Let \(R = K[x_1, \ldots, x_n]\) be a polynomial ring over a field \(K\), and consider a tableau \(T \in \text{Tab}(\lambda)\) for a partition \(\lambda\) of \(n\). If the \(j\)-th column of \(T\) consists of \(j_1, j_2, \ldots, j_m\) in the order from top to bottom, then

\[
f_T(j) := \prod_{1 \leq s < t \leq m} (x_{j_s} - x_{j_t}) \in R.
\]

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The Specht polynomial $f_T$ of $T$ is given by
\[ f_T := \prod_{j=1}^{\lambda_1} f_T(j). \]

For example, if $T$ is the tableau (11), then $f_T = (x_3 - x_6)(x_3 - x_4)(x_6 - x_4)(x_5 - x_2)$.

The symmetric group $\mathfrak{S}_n$ acts on the vector space spanned by \{ $f_T$ $|$ $T \in \text{Tab}(\lambda)$ \}. As an $\mathfrak{S}_n$-module, this vector space is isomorphic to the Specht module $V_\lambda$, which is very important in the representation theory of symmetric groups (cf. [9]). Here we study the Specht ideal
\[ I_{\lambda}^{\text{Sp}} := ( f_T \mid T \in \text{Tab}(\lambda) ) \]
of $R$. In the previous paper [13], the second author showed the following.

**Theorem 1.1** ([13, Proposition 2.8 and Corollary 4.4]). If $R/I_{\lambda}^{\text{Sp}}$ is Cohen–Macaulay, then one of the following conditions holds.

1. $\lambda = (n - d, 1, \ldots, 1)$,
2. $\lambda = (n - d, d)$,
3. $\lambda = (d, d, 1)$.

If $\text{char}(K) = 0$, the converse is also true.

The case (1) is treated in the joint paper [12] with J. Watanabe, and it is shown that a minimal free resolution of $R/I_{(n-d,1,\ldots,1)}^{\text{Sp}}$ is given by the Eagon-Northcott complex of a “Vandermonde-like” matrix. Since $I_{(n-1,1)}^{\text{Sp}}$ is a linear complete intersection, its free resolution is easy. The second author showed that $R/I_{(n-2,2)}^{\text{Sp}}$ is a 2-dimensional Gorenstein ring ([13, Proposition 5.2]). Similarly, when $\text{char}(K) = 0$, $R/I_{(d,d,1)}^{\text{Sp}}$ is Cohen–Macaulay and has a linear free resolution. In the present paper, we will construct minimal free resolutions of $R/I_{(n-2,2)}^{\text{Sp}}$ and $R/I_{(d,d,1)}^{\text{Sp}}$.

However, after an earlier version was submitted, the authors were informed that minimal free resolutions of $R/I_{(n-d,d)}^{\text{Sp}}$ for $1 \leq d \leq n/2$ had been studied by Berkesch Zamaere, Griffeth, and Sam [2]. More precisely, [2] determined the $\mathfrak{S}_n$-module structure of $\text{Tor}_i^R(K, R/I_{(n-d,d)}^{\text{Sp}})$. (They called $I_{(n-d,d)}^{\text{Sp}}$ the “$(d+1)$-equal ideal”. Of course, this name comes from the decomposition (2.7) below.) However the paper [2] dose not give the differential maps of their resolutions, and uses highly advanced tools of the representation theory (rational Cherednik algebras, Jack polynomials, etc). By contrast, the present paper describes the differential maps explicitly, and uses only the basic theory of Specht modules. Here we do not use results of [2] to make the exposition self-contained.

It is also noteworthy that, recently, many people study monomial ideals in $R$ on which the symmetric group $\mathfrak{S}_n$ naturally acts (cf. [1, 8]). However, their behavior is quite different from that of Specht ideals. For example, $\text{Tor}_i^R(K, R/I_{(n-2,2)}^{\text{Sp}})$ and $\text{Tor}_i^R(K, R/I_{(d,d,1)}^{\text{Sp}})$ are irreducible as $\mathfrak{S}_n$-modules (the same holds for $I_{(n-d,d)}^{\text{Sp}}$ as shown in [2]), but this is far from true for symmetric monomial ideals (cf. [8]).
2. Preliminaries and Backgrounds

In this section, we briefly explain Specht modules and related notions. See [9, Chapter 2] for details. For a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$, we sometimes use “exponential notation”. For example, $(4, 3^2, 2, 1^3)$ means $(4, 3, 3, 2, 1, 1, 1)$. If $\lambda = (\lambda_1, \ldots, \lambda_l)$, then $\text{Tab}(\lambda)$ can be simply written as $\text{Tab}(\lambda_1, \ldots, \lambda_l)$.

We say a tableau $T$ is standard, if all columns (resp. rows) are increasing from top to bottom (resp. from left to right). Let $\text{SYT}(\lambda)$ be the set of standard Young tableaux of shape $\lambda$.

Given any set $A$, let $S(A)$ be the set of all permutations of $A$. Suppose that $T \in \text{Tab}(\lambda)$ has columns $C_1, \ldots, C_k$. Then $C(T) := S(C_1) \times \cdots \times S(C_k)$ is the column-stabilizer of $T$.

For $T, T' \in \text{Tab}(\lambda)$, $T$ and $T'$ are row equivalent, if corresponding rows of $T$ and $T'$ contain the same elements. For $T \in \text{Tab}(\lambda)$, the tabloid $\{T\}$ of $T$ is defined by $\{T\} := \{T' \in \text{Tab}(\lambda) \mid T$ and $T'$ are row equivalent$\}$, and the polytabloid of $T$ is defined by

$$e(T) := \sum_{\pi \in C(T)} \text{sgn}(\pi) \pi \{T\}.$$ 

It is easy to see that $e(T) = \text{sgn}(\sigma) e(\sigma T)$ for $\sigma \in C(T)$.

The vector space $V_\lambda := \sum_{T \in \text{Tab}(\lambda)} K e(T)$ becomes an $\mathfrak{S}_n$-module in the natural way, and it is called the Specht module of $\lambda$. If $\text{char}(K) = 0$, the Specht modules $V_\lambda$ are irreducible, and $V_\lambda$ for partitions $\lambda$ of $n$ form a complete list of irreducible representations of $\mathfrak{S}_n$.

In the previous section, we defined the Specht polynomial $f_T \in R = K[x_1, \ldots, x_n]$. Since $\mathfrak{S}_n$ acts on $R$, the vector subspace

$$\sum_{T \in \text{Tab}(\lambda)} K f_T$$

is also an $\mathfrak{S}_n$-module. Moreover, the map

$$(2.1) \quad V_\lambda \xrightarrow{\cong} \sum_{T \in \text{Tab}(\lambda)} K f_T, \quad e(T) \longmapsto f_T.$$ 

is well-defined, and gives an isomorphism as $\mathfrak{S}_n$-modules.

Note that $\{e(T) \mid T \in \text{Tab}(\lambda)\}$ is linearly dependent, and there are relations called Garnir relations. Its definition for general $\lambda$ becomes long, so we explain it using our examples. See [9, §2.6] for the general case.
For

\[
T = \begin{array}{cccccc}
  a_1 & b_1 & c_1 & c_2 & \cdots & c_{n - i - 1} \\
  a_2 & b_2 &  &  &  & \\
  \vdots &  &  &  &  & \\
  a_{i+1} &  &  &  &  & \\
\end{array} \in \text{Tab}(n - 1 - i, 2, 1^{i-1})
\]

and \(A = \{a_1, \ldots, a_{i + 1}\}, B = \{b_1\}\), set

\[
S_T(A, B) := \left\{ \sigma \in \mathfrak{S}_n \left| \begin{array}{c}
\sigma(i) = i \text{ for } i \notin A \cup B, \\
\sigma(a_1) < \sigma(a_2) < \cdots < \sigma(a_{i + 1}).
\end{array} \right. \right\}
\]

(if there is no danger of confusion, we write \(S(A, B)\) for \(S_T(A, B)\)) and

\[
(2.3) \quad g_{A, B} := \sum_{\sigma \in S(A, B)} \text{sgn}(\sigma) \sigma.
\]

Here we regard \(g_{A, B}\) as an element of the group ring of \(\mathfrak{S}_n\). Then we have

\[
(2.4) \quad g_{A, B} e(T) = \sum_{\sigma \in S(A, B)} \text{sgn}(\sigma) e(\sigma T) = 0
\]

by [9, Proposition 2.6.3].

Next, for \(A = \{a_2, \ldots, a_{i + 1}\}, B = \{b_1, b_2\}\), set

\[
S(A, B) := \left\{ \sigma \in \mathfrak{S}_n \left| \begin{array}{c}
\sigma(i) = i \text{ for } i \notin A \cup B, \\
\sigma(a_2) < \sigma(a_3) < \cdots < \sigma(a_{i + 1}), \sigma(b_1) < \sigma(b_2)
\end{array} \right. \right\}.
\]

Then \(g_{A, B}\) is given in the same way as (2.3), and we have \(g_{A, B} e(T) = 0\) as in (2.4) again. The same is true for \(A = \{b_1, b_2\}, B = \{c_1\}\), and \(A = \{c_i\}, B = \{c_{i+1}\}\). In these cases, \(g_{A, B}\) is called the Garnir element associated with \(A\) and \(B\). It is a classical result that \(\{e(T) \mid T \in \text{SYT}(\lambda)\}\) is a basis of \(V_{\lambda}\) (cf. [9, Theorem 2.6.5]), and that \(\{e(T) \mid T \in \text{SYT}(\lambda)\}\) spans \(V_{\lambda}\) is shown using Garnir relations.

**Example 2.1.** For

\[
T = \begin{array}{cccc}
  2 & 1 & 6 \\
  3 & 5 \\
  4 &
\end{array}
\]

set \(A = \{2, 3, 4\}\) and \(B = \{1\}\), then

\[
g_{A, B} e(T) = e(T) - e\left(\begin{array}{cc}
  1 & 2 \\
  3 & 5
\end{array}\right) - e\left(\begin{array}{cc}
  1 & 3 \\
  4 & 6
\end{array}\right) + e\left(\begin{array}{cc}
  1 & 4 \\
  2 & 5
\end{array}\right) = 0.
\]
Next we consider the following tableau

\[
T = \begin{array}{c|c|c|c}
1 & 2 & 6 \\
3 & 4 & 5 \\
\end{array}
\]

Set \( A = \{4, 5\} \) and \( B = \{2, 3\} \), then

\[
g_{A,B}e(T) = e(T) - e(\begin{array}{c|c|c|c}
1 & 2 & 6 \\
3 & 4 & 5 \\
\end{array}) + e(\begin{array}{c|c|c|c}
1 & 2 & 6 \\
3 & 5 & 4 \\
\end{array}) - e(\begin{array}{c|c|c|c}
1 & 3 & 6 \\
2 & 5 & 4 \\
\end{array})
\]

\[
+ e(\begin{array}{c|c|c|c}
1 & 3 & 6 \\
2 & 4 & 5 \\
\end{array}) + e(\begin{array}{c|c|c|c}
1 & 4 & 6 \\
2 & 5 & 3 \\
\end{array})
\]

\[
= 0.
\]

**Proposition 2.2** (cf. [9, Theorem 2.6.4]). Any linear relations among \( \{e(T) \mid T \in \text{Tab}(\lambda)\} \) is a linear combination of Garnir relations. That is, if

\[
\sum_{i=1}^{m} a_i e(T_i) = 0
\]

in \( V_\lambda \) for \( T_1, \ldots, T_m \in \text{Tab}(\lambda) \) and \( a_i \in K \), then \( \sum_{i=1}^{m} a_i T_i \) (this is a formal sum, and there is no relation among \( T_1, \ldots, T_m \)) is contained in the linear space \( V \) spanned by

\[
\left\{ \sum_{\sigma \in S(A,B)} \text{sgn}(\sigma) \sigma T \mid S(A, B) \text{ gives a Garnir element } g_{A,B} \right\}.
\]

**Proof.** Assume that (2.6) holds. Each \( e(T_i) \) can be rewritten as

\[
e(T_i) = \sum_{T \in \text{SYT}(\lambda)} b_{i,T} e(T)
\]

for some \( b_{i,T} \in K \) using only Garnir relations, see the proof of [9, Theorem 2.6.4]. Hence we have

\[
v_i := T_i - \sum_{T \in \text{SYT}(\lambda)} b_{i,T} T \in V
\]

for each \( i \). Note that

\[
\sum_{i=1}^{m} \sum_{T \in \text{SYT}(\lambda)} a_i b_{i,T} e(T) = \sum_{i=1}^{m} a_i e(T_i) = 0.
\]

However, since \( \{e(T) \mid T \in \text{SYT}(\lambda)\} \) is a basis, we have \( \sum_{i=1}^{m} a_i b_{i,T} = 0 \) for all \( T \in \text{SYT}(\lambda) \). Hence

\[
\sum_{i=1}^{m} a_i T_i = \sum_{i=1}^{m} a_i v_i \in V.
\]

\( \square \)
In the rest of this section, we collect a few remarks on the Specht ideals \( I_{(n-d,d)}^{Sp} \) and \( I_{(d,d,1)}^{Sp} \). First, we have the decomposition

\[
I_{(n-d,d)}^{Sp} = \bigcap_{F \subset \mathbb{F} \atop \#F = d+1} (x_i - x_j \mid i, j \in F),
\]

and the same is true for \( I_{(d,d,1)}^{Sp} \). So these ideals can be seen as special cases of the ideals associated with subspace arrangements (cf. [3, 6]). The second author [13] made much effort to show that \( \sqrt{I_{\lambda}^{Sp}} = I_{\lambda}^{Sp} \) for \( \lambda = (n-d,d) \) \( (d,d,1) \), but it directly follows from [6, Corollary 3.2].

To prove the Cohen–Macaulay-ness of \( I_{(n-d,d)}^{Sp} \) and \( I_{(d,d,1)}^{Sp} \) in characteristic 0, the second author [13] cited a result in [5], which uses the representation theory of rational Cherednik algebras. Recently, McDaniel and Watanabe [7] gave a purely ring theoretic proof. Moreover, in the positive characteristic case, they showed that \( R/I_{(n-d,d)}^{Sp} \) (resp. \( R/I_{(d,d,1)}^{Sp} \)) is Cohen–Macaulay if and only if \( \text{char}(K) \geq d \) (resp. \( \text{char}(K) \geq d+1 \)).

As stated in [2, 10], \( I_{(d,d,1)}^{Sp} \) has a linear free resolution if it is Cohen–Macaulay. Anyway, this is an easy consequence of [11, Theorem 5.3.7].

3. The case \((n-2,2)\): Construction

For \( R/I_{(n-2,2)}^{Sp} \), we define the chain complex

\[
\mathcal{F}^{(n-2,2)}_{\bullet} : 0 \rightarrow F_{n-2} \xrightarrow{\partial_{n-2}} F_{n-3} \xrightarrow{\partial_{n-3}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \rightarrow 0
\]

of graded free \( R \)-modules as follows. Here \( F_0 = R \), \( F_1 = V_{(n-2,2)} \otimes_K R(-2) \),

\[
F_i = V_{(n-1-i,2,1^{i-1})} \otimes_K R(-1-i)
\]

for \( 1 \leq i \leq n-3 \), and \( F_{n-2} = V_{(1^n)} \otimes_K R(-n) \). For \( T \in \text{Tab}(n-2,2) \), set \( \partial_{1}(e(T) \otimes 1) := f_T \in R = F_0 \). To describe \( \partial_i \) for \( 2 \leq i \leq n-3 \), we need the preparation. For the tableau \( T \in \text{Tab}(n-1-i,2,1^{i-1}) \) of \([2,2]\) and \( j \) with
1 ≤ j ≤ i + 1, set

\[
T_j := \begin{array}{cccccc}
    a_1 & b_1 & c_1 & c_2 & \cdots & c_{n-3-i} & a_j \\
    a_2 & b_2 &  &  &  &  & \\
    \vdots &  &  &  &  &  & \\
    a_{j-1} &  &  &  &  &  & \\
    a_{j+1} &  &  &  &  &  & \\
    \vdots &  &  &  &  &  & \\
    a_{i+1} &  &  &  &  &  & \\
\end{array} \in \text{Tab}(n-i,2,1^{i-2}).
\]

Then we set

\[
\partial_i(e(T) \otimes 1) := \sum_{j=1}^{i+1} (-1)^{j-1} e(T_j) \otimes x_{a_j} \in V_{(n-i,2,1^{i-2})} \otimes_K R(-i) = F_{i-1}.
\]

Recall that \(e(\sigma T) = \text{sgn}(\sigma)e(T)\) for \(\sigma \in C(T)\). It is easy to check that the construction of \(\partial_i\) is compatible with this principle, that is, \(\partial_i(e(\sigma T) \otimes 1) = \text{sgn}(\sigma)\partial_i(e(T) \otimes 1)\) holds for \(\sigma \in C(T)\). However, this is not enough. Since \(\{e(T) \mid T \in \text{Tab}(\lambda)\}\) is linearly dependent, the well-definedness of \(\partial_i\) is still non-trivial. We will show this in Theorem 4.2 below.

Finally, we define the differential map

\[
\partial_{n-2} : V_{(1^n)} \otimes_K R(-n) \rightarrow V_{(2,2,1^{n-4})} \otimes_K R(-n+2).
\]

Since \(\dim V_{(1^n)} = 1\), it suffices to define \(\partial_{n-2}(e(T))\) for

\[(3.2) \quad T := \begin{array}{ccc}
    1 & 2 & \vdots \\
    \cdots & \cdots & \cdots \\
    \vdots & \vdots & \vdots \\
    \end{array} \in \text{Tab}(1^n),
\]

and we do not have to care the well-definedness. For \(j,k\) with \(1 ≤ j < k ≤ n\), set

\[
T_{j,k} := \begin{array}{ccc}
    \vdots & j & \vdots \\
    \vdots & \vdots & \vdots \\
\end{array} \in \text{Tab}(2,2,1^{n-4}),
\]
where the first column is the “transpose” of

\[
\begin{array}{cccccccc}
1 & 2 & \cdots & j-1 & j+1 & \cdots & k-1 & k+1 & \cdots & n
\end{array}
\]

Then
\[
\partial_{n-2}(e(T) \otimes 1) := \sum_{1 \leq j < k \leq n} (-1)^{j+k-1} e(T_{j,k}) \otimes x_j x_k \in V_{(2,2,1^{n-4})} \otimes_K R(-n+2) \in F_{n-3}.
\]

We define the \( \mathcal{G}_n \)-module structure on \( F_i = V_{\lambda} \otimes_K R(-j) \) (here \( \lambda \) is a suitable partition of \( n \) and \( j \) is a suitable integer) by \( \sigma(v \otimes f) := \sigma v \otimes \sigma f \) for \( \sigma \in \mathcal{G}_n \). By (2.1), \( \partial_1 \) is an \( \mathcal{G}_n \)-homomorphism. In fact, we have
\[
\partial_1(\sigma(e(T) \otimes g)) = \partial_1(\sigma(e(T)) \otimes \sigma g) = \sigma(f_T \cdot \sigma g) = \sigma(f_T \cdot g) = \sigma(\partial_1(e(T) \otimes g))
\]
for \( T \in \text{Tab}(n-2,2) \) and \( g \in R \). For \( i \) with \( 2 \leq i \leq n-3 \), \( T \in \text{Tab}(n-i,2,1^{i-2}) \) and \( \sigma \in \mathcal{G}_n \), we have \( \sigma(T)_j = \sigma(T_j) \). Hence \( \partial_i(\sigma(e(T) \otimes g)) = \sigma(\partial_i(e(T) \otimes g)) \), that is, \( \partial_i \) are \( \mathcal{G}_n \)-homomorphisms. Similarly, \( \partial_{n-2} \) is also.

**Example 3.1.** Our minimal free resolution \( \mathcal{F}^{(4,2)}_* \) of \( R/I_{(4,2)}^{sp} \) is of the form

\[
0 \longrightarrow V_{\otimes K} R(-6) \overset{\partial_4}{\longrightarrow} V_{\otimes K} R(-4) \overset{\partial_3}{\longrightarrow} V_{\otimes K} R(-3)
\]

(3.3) \[ \overset{\partial_2}{\longrightarrow} V_{\otimes K} R(-2) \overset{\partial_1}{\longrightarrow} R \longrightarrow 0. \]

The differential maps are given by

\[
\partial_4(e(\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}) \otimes 1) = e(\begin{array}{c} 3 \\ 1 \\ 4 \\ 2 \\ 5 \\ 6 \end{array}) \otimes x_1 x_2 - e(\begin{array}{c} 2 \\ 1 \\ 4 \\ 3 \\ 5 \\ 6 \end{array}) \otimes x_1 x_3 + e(\begin{array}{c} 2 \\ 1 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}) \otimes x_1 x_4
\]

\[-e(\begin{array}{c} 2 \\ 1 \\ 3 \\ 5 \\ 4 \\ 6 \end{array}) \otimes x_1 x_5 + e(\begin{array}{c} 2 \\ 1 \\ 3 \\ 6 \\ 4 \\ 5 \end{array}) \otimes x_1 x_6 + e(\begin{array}{c} 1 \\ 2 \\ 4 \\ 3 \\ 5 \\ 6 \end{array}) \otimes x_2 x_3
\]

\[-e(\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}) \otimes x_2 x_4 + e(\begin{array}{c} 1 \\ 2 \\ 3 \\ 5 \\ 4 \\ 6 \end{array}) \otimes x_2 x_5 - e(\begin{array}{c} 1 \\ 2 \\ 3 \\ 6 \\ 4 \\ 5 \end{array}) \otimes x_2 x_6
\]

\[+e(\begin{array}{c} 1 \\ 3 \\ 2 \\ 4 \\ 5 \\ 6 \end{array}) \otimes x_3 x_4 - e(\begin{array}{c} 1 \\ 3 \\ 2 \\ 5 \\ 4 \\ 6 \end{array}) \otimes x_3 x_5 + e(\begin{array}{c} 1 \\ 3 \\ 2 \\ 6 \\ 4 \\ 5 \end{array}) \otimes x_3 x_6
\]

\[+e(\begin{array}{c} 1 \\ 4 \\ 2 \\ 5 \\ 3 \\ 6 \end{array}) \otimes x_4 x_5 - e(\begin{array}{c} 1 \\ 4 \\ 2 \\ 6 \\ 3 \\ 5 \end{array}) \otimes x_4 x_6 + e(\begin{array}{c} 1 \\ 5 \\ 2 \\ 6 \\ 3 \\ 4 \end{array}) \otimes x_5 x_6,
\]
\begin{align*}
\partial_3(e(\begin{array}{c} 3 \\
4 \\
5 \\
6 
\end{array}) \otimes 1) &= e(\begin{array}{c} 1 \\
2 \\
3 \\
4 
\end{array}) \otimes x_3 - e(\begin{array}{c} 3 \\
1 \\
4 \\
5 
\end{array}) \otimes x_4 + e(\begin{array}{c} 3 \\
1 \\
4 \\
5 
\end{array}) \otimes x_5 - e(\begin{array}{c} 3 \\
1 \\
4 \\
5 
\end{array}) \otimes x_6,
\partial_2(e(\begin{array}{c} 1 \\
2 \\
3 \\
5 
\end{array}) \otimes 1) &= e(\begin{array}{c} 5 \\
1 \\
3 \\
6 
\end{array}) \otimes x_4 - e(\begin{array}{c} 5 \\
1 \\
3 \\
6 
\end{array}) \otimes x_5 + e(\begin{array}{c} 5 \\
1 \\
3 \\
6 
\end{array}) \otimes x_6.
\end{align*}

and

\textbf{Theorem 3.2.} If $\text{char}(K) = 0$, the complex $\mathcal{F}^{(n-2,2)}_*$ of (3.1) is a minimal free resolution of $R/I_{(n-2,2)}$.

4. The case $(n-2,2)$: Proof

\textbf{Lemma 4.1.} We have

$$\beta_i(R/I_{(n-2,2)}^{Sp}) = \beta_{i+1}(R/I_{(n-2,2)}^{Sp}) = \dim_K V_{(n-2,2,1^{i-1})}$$

for all $1 \leq i \leq n-3$, and

$$\beta_{n-2}(R/I_{(n-2,2)}^{Sp}) = \beta_{n-2,n}(R/I_{(n-2,2)}^{Sp}) = 1 = \dim_K V_{(1^n)}.$$

\textbf{Proof.} By the hook formula ([9 Theorem 3.10.2]), for all $i$ with $1 \leq i \leq n-3$, we have

$$\dim_K V_{(n-i,2,1^{i-1})} = \frac{n!}{(n-1)(n-(i-1)-2)(n-(i-1)-4)(i-1)!}$$

$$= \frac{n!}{(n-1)(n-i-1)(n-i-3)!} \cdot \frac{i(i+1)!}{(i+1)(n-i-1)!} \cdot \frac{(i-1)!}{(n-1)!}.$$

On the other hand, $R/I_{(n-2,2)}^{Sp}$ is a Gorenstein ring with the Hilbert series

$$\frac{1 + (n-2)t + t^2}{(1-t)^2}$$

by [13 Proposition 5.2] (see its proof for the Hilbert series), and we have

$$\beta_i(R/I_{(n-2,2)}^{Sp}) = \beta_{i+1}(R/I_{(n-2,2)}^{Sp})$$

$$= \binom{n-1}{i+1} \binom{i}{i-1} + \binom{n-1}{i} \binom{n-i-2}{1} - \binom{n-2}{i} \binom{n-2}{1}$$

$$= \frac{(n-1)!i}{(i+1)(n-i-2)!} + \frac{(n-1)!(n-i-2)!}{i!(n-1-i)!} - \frac{(n-2)!(n-2)!}{i!(n-2-i)!}$$

$$= \frac{n!i}{(i+1)(n-i-2)!} + \frac{n!(n-i-2)!}{i!(n-1-i)!} - \frac{n!(n-i-2)!}{i!(n-2-i)!}$$

$$= \frac{n!(n-i-2)i}{(i+1)(n-i-1)!(n-1)}.$$
for all $i$ with $1 \leq i \leq n - 3$. Here we use [11, Proposition 5.3.14] (note that $I_{(n-2,2)}^{Sp}$ is a Gorenstein ideal generated by quadrics and $\text{ht}(I_{(n-2,2)}^{Sp}) = n - 2$). So we get the first equation.

The second one is easy.

**Theorem 4.2.** The maps $\partial_i$ ($1 \leq i \leq n - 3$) defined in the previous section are well-defined.

The proof of this lemma is elementary, but (therefore?) long and technical. By the purpose of the present paper, we do not skip the details. Example 4.3 below, which explain a few details of the proof, must be helpful for better understanding.

Note that an element $\varphi \in V_\lambda \otimes_K R_1$ is uniquely written as $\sum_{i=1}^n v_i \otimes x_i$ ($v_i \in V_\lambda$). We call $v_i \otimes x_i$ the $x_i$-part of $\varphi$.

**Proof.** The well-definedness of $\partial_1$ is nothing other than that of the map (2.1). So we assume that $2 \leq i \leq n - 3$. By Proposition 2.2 it suffices to show that

$$
\sum_{\sigma \in S_T(A,B)} \text{sgn}(\sigma) \partial_i(e(\sigma T) \otimes 1) = 0
$$

for $T \in \text{Tab}(n - 1 - i, 2, 1^{i-1})$. Let $T$ be as in (2.2). Then there are three types of $S_T(A,B)$.

**Case 1.** When $A = \{b_1, b_2\}$ and $B = \{c_1\}$, or $A = \{c_i\}$ and $B = \{c_{i+1}\}$: The left side of (4.1) can be decomposed to the sum of the $x_{a_1}$-part, . . . , and the $x_{a_{i+1}}$-part. Since $g_{A,B}$ is a Garnir element also for $T_j$, the $x_{a_j}$-part of the left side of (4.1) is

$$
\sum_{\sigma \in S(A,B)} (-1)^{j-1} \text{sgn}(\sigma) e((\sigma T)_j) \otimes x_{a_j} = (-1)^{j-1} \sum_{\sigma \in S(A,B)} \text{sgn}(\sigma) e(\sigma(T_j)) \otimes x_{a_j}
$$

$$
= (-1)^{j-1} g_{A,B} e(T_j) \otimes x_{a_j} = 0.
$$

Hence (4.1) holds in this case.

**Case 2.** When $A = \{a_1, \ldots, a_{i+1}\}$ and $B = \{b_1\}$: The left side of (4.1) can be decomposed to the sum of the $x_{a_1}$-part, . . . , the $x_{a_{i+1}}$-part, and the $x_{b_1}$-part. To treat the Garnir relation, we may assume that $b_1 < a_1 < a_2 < \cdots < a_{i+1}$.

Fix an integer $j$ with $1 \leq j \leq i + 1$, and set $A_j := A \setminus \{a_j\}$. Note that

$$
S(A, B) = \{\sigma \in S(A, B) \mid \sigma(a_j) = a_j\} \sqcup \{\sigma \in S(A, B) \mid \sigma(a_{j+1}) = a_j\}
$$

$$
\sqcup \{\sigma \in S(A, B) \mid \sigma(b_1) = a_j\}
$$

For $\sigma \in S(A, B)$, $\sigma(a_j) = a_j$ if and only if $\sigma(b_1) \leq a_{j-1}$. Similarly, $\sigma(a_{j+1}) = a_j$ if and only if $\sigma(b_1) \geq a_{j+1}$.

If $\sigma(a_j) = a_j$, then $\sigma$ also belongs to $S_{T_j}(A_j, B)$, and we have

$$
\{\sigma \in S(A, B) \mid \sigma(a_j) = a_j\} = \{\sigma \in S_{T_j}(A_j, B) \mid \sigma(b_1) \leq a_{j-1}\};
$$

and $\sigma(T_j) = (\sigma T)_j$. (For notational simplicity, we will write $S(A_j, B)$ for $S_{T_j}(A_j, B)$ below.)
If $\sigma(a_{j+1}) = a_j$, then we have $\tau := (\sigma(a_j) a_j) \cdot \sigma \in S(A_j, B)$. In fact,

$$
\tau(k) = \begin{cases} 
a_j & (k = a_j) 
\sigma(a_j) & (k = a_{j+1}) 
\sigma(k) & (k \neq a_j, a_{j+1}),
\end{cases}
$$

and hence $\tau$ only moves elements in $A_j \cup B$, and $\tau(k) < \tau(l)$ for $k, l \in A_j$ with $k < l$. We also have $\tau(b_1) = \sigma(b_1) \geq a_{j+1}$, $\text{sgn}(\tau) = -\text{sgn}(\sigma)$, and

$$
(4.2) \quad \tau(T_j) = (\sigma T)_{j+1}.
$$

(In Example 4.3 (1) below, we will check (4.2) very carefully to get the feeling. From the next paragraph, we will leave similar computations to the reader as easy exercises.) Moreover, the map

$$
f : \{ \sigma \in S(A, B) \mid \sigma(a_{j+1}) = a_j \} \longrightarrow \{ \tau \in S(A_j, B) \mid \tau(b_1) \geq a_{j+1} \}
$$

defined by the above operation is bijective. In fact, the inverse map is given by $S(A_j, B) \ni \tau \longmapsto (a_j \tau(a_{j+1})) \cdot \tau$.

Hence the $x_{a_j}$-part of the left side of (4.1) is

$$
\left( \sum_{\substack{\sigma \in S(A, B) \\
\sigma(a_j) = a_j}} (-1)^{j-1} \text{sgn}(\sigma) e((\sigma T)_j) + \sum_{\substack{\sigma \in S(A, B) \\
\sigma(a_{j+1}) = a_j}} (-1)^j \text{sgn}(\sigma) e((\sigma T)_{j+1}) \right) \otimes x_{a_j}
$$

$$
= \left( \sum_{\substack{\sigma \in S(A_j, B) \\
\sigma(b_1) \leq a_{j-1}}} (-1)^{j-1} \text{sgn}(\sigma) e(\sigma(T_j)) + \sum_{\substack{\tau \in S(A_j, B) \\
\tau(b_1) \geq a_{j+1}}} (-(-1)^j \text{sgn}(\tau) e(\tau(T_j))) \right) \otimes x_{a_j}
$$

$$
= (-1)^{j-1} g_{A_j, B} e(T_j) \otimes x_{a_j}
$$

$$
= 0.
$$

Here the last equality follows from that $g_{A_j, B}$ is a Garnir element for $T_j$.

It remains to check the $x_{b_1}$-part of the left side of (4.1). Consider the tableau $T' := ((a_1 b_1) T)_1$, that is,

$$
T' = \begin{array}{cccccc}
a_2 & a_1 & c_1 & c_2 & \cdots & b_1 \\
a_3 & b_2 \\
\vdots \\
a_{i+1}
\end{array}
$$

Set $A' := A \setminus \{a_1\}$ and $B' := \{a_1\}$, and consider the map

$$
f' : \{ \sigma \in S(A, B) \mid \sigma(a_1) = b_1 \} \ni \sigma \longmapsto (\sigma(b_1) b_1) \sigma \in S_{T'}(A', B').
$$
Of course, we have to check that \( \tau := f'(\sigma) \) actually belongs \( S_{T'}(A', B') \), but it follows from the fact that

\[
\tau(k) = \begin{cases} 
  b_1 & (k = b_1) \\
  \sigma(b_1) & (k = a_1) \\
  \sigma(k) & (k \neq a_1, b_1).
\end{cases}
\]

Moreover, \( f' \) is bijective. In fact, the inverse map is given by \( S_{T'}(A', B') \ni \tau \mapsto (a_1, b_1) \tau \). We also remark that \( \text{sgn}(\sigma) = -\text{sgn}(f'(\sigma)) \) and \( (\sigma T)_1 = f'(\sigma) T' \).

Hence the \( x_{b_1} \)-part of the left side of (4.1) is

\[
\left( \sum_{\sigma \in S(A, B) \atop \sigma(a_1) = b_1} \text{sgn}(\sigma)(\sigma T)_1 \right) \otimes x_{b_1} = -g'_{A', B'}(T') \otimes x_{b_1} = 0.
\]

Below, we will use bijections similar to \( f, f' \) repeatedly. Each time, we will define the bijections explicitly, but we will not check they work well, and leave them as easy exercises.

**Case 3.** When \( A := \{a_2, \ldots, a_{i+1}\} \) and \( B := \{b_1, b_2\} \): Note that the left side of (4.1) can be decomposed to the sum of the \( x_{a_1} \)-part, \( \ldots \), the \( x_{a_{i+1}} \)-part, the \( x_{b_1} \)-part and the \( x_{b_2} \)-part. Fix \( j \) with \( 2 \leq j \leq i + 1 \), and set \( A_j := A \setminus \{a_j\} \). To treat the Garnir relation, we may assume that \( b_1 < b_2 < a_2 < \cdots < a_{i+1} \).

First, we treat the \( x_{a_j} \)-part for \( j \geq 2 \). Set

\[
G_1 := \{ \sigma \in S(A, B) \mid a_j < \sigma(b_1) \},
\]
\[
G_2 := \{ \sigma \in S(A, B) \mid \sigma(b_1) < a_j < \sigma(b_2) \},
\]
\[
G_3 := \{ \sigma \in S(A, B) \mid a_j > \sigma(b_2) \},
\]

and

\[
G'_1 := \{ \sigma \in S(A_j, B) \mid a_j < \sigma(b_1) \},
\]
\[
G'_2 := \{ \sigma \in S(A_j, B) \mid \sigma(b_1) < a_j < \sigma(b_2) \},
\]
\[
G'_3 := \{ \sigma \in S(A_j, B) \mid a_j > \sigma(b_2) \}.
\]

Then we have

\[
\{ \sigma \in S(A, B) \mid \sigma(b_1), \sigma(b_2) \neq a_j \} = G_1 \sqcup G_2 \sqcup G_3
\]

and

\[
S(A_j, B) = G'_1 \sqcup G'_2 \sqcup G'_3.
\]

For \( \sigma \in S(A, B) \), \( \sigma \in G_1 \) if and only if \( \sigma(a_{j+2}) = a_j \). Similarly, if \( \sigma \in G_2 \) (resp. \( \sigma \in G_3 \)), then \( \sigma(a_{j+1}) = a_j \) (resp. \( \sigma(a_j) = a_j \)). Hence we have \( G_3 = G'_3 \). Moreover, we have the following bijections

\[
f_1 : G_1 \ni \sigma \mapsto (a_j \sigma(a_{j+1}) \sigma(a_j)) \sigma \in G'_1
\]
\[
f_2 : G_2 \ni \sigma \mapsto (a_j \sigma(a_j)) \sigma \in G'_2.
\]
Clearly, \( \text{sgn}(f_1(\sigma)) = \text{sgn}(\sigma) \) and \( \text{sgn}(f_2(\sigma)) = -\text{sgn}(\sigma) \). We also remark that \((\sigma T)_{j+2} = f_1(\sigma)(T_j)\) for \(\sigma \in G_1\), \((\sigma T)_{j+1} = f_2(\sigma)(T_j)\) for \(\sigma \in G_2\). For simplicity, set \(\tau := f_k(\sigma)\) for \(\sigma \in G_k\). By these bijections, we see that the \(x_{a_1}\)-part of the left side of (4.1) for \(j \geq 2\) is

\[
(\sum_{\sigma \in G_1} (-1)^{j+2-1} \text{sgn}(\sigma)e((\sigma T)_{j+2}) + \sum_{\sigma \in G_2} (-1)^{j+1-1} \text{sgn}(\sigma)e((\sigma T)_{j+1})
\]

\[
+ \sum_{\sigma \in G_3} (-1)^{j-1} \text{sgn}(\sigma)e((\sigma T)_j) \bigg) \otimes x_{a_j}
\]

\[
= (-1)^{j-1} \left( \sum_{\tau \in G_1^e} \text{sgn}(\tau)e(\tau(T_j)) + \sum_{\tau \in G_2^e} \text{sgn}(\tau)e(\tau(T_j)) \right)
\]

\[
+ \sum_{\sigma \in G_3} \text{sgn}(\sigma)e(\sigma(T_j)) \bigg) \otimes x_{a_j}
\]

\[
= (-1)^{j-1} g_{A_jB}e(T_j) \otimes x_{a_j}
\]

\[
= 0.
\]

Next we check the \(x_{a_1}\)-part. We decompose the \(x_{a_1}\)-part of the left side of (4.1) as follows

\[
(4.3) \quad \left( \sum_{\sigma \in S(A,B)} \text{sgn}(\sigma)e((\sigma T)_1) + \sum_{\sigma \in S(A,B)} \text{sgn}(\sigma)e((\sigma T)_1) \right) \otimes x_{a_1}.
\]

First, we consider the former, that is, the case \(\sigma(b_2) = a_{i+1}\). Set \(\tilde{A} = (A \cup \{b_2\}) \setminus \{a_{i+1}\}, \tilde{B} = \{b_1\}\), and \(\tilde{T} := (a_{i+1} a_1 a_{i-1} a_{i-2} \cdots a_3 a_2 b_2)(T_1)\). Note that

\[
\tilde{T} = \begin{array}{cccccc}
  \cdots & a_1 & c_2 & c_1 & b_1 & b_2 \\
  a_2 & a_{i+1} \\
  \vdots \\
  a_i
\end{array}
\]

We have a bijection

\[
\tilde{f} : \{ \sigma \in S(A,B) \mid \sigma(b_2) = a_{i+1} \} \quad \xrightarrow{\cup} \quad S_{\tilde{T}}(\tilde{A}, \tilde{B})
\]

\[
\sigma \quad \mapsto \quad (a_{i+1} \sigma(a_2) \sigma(a_3) \cdots \sigma(a_{i+1})) \sigma
\]

and can easily check that \((\sigma T)_1 = \tilde{f}(\sigma)\tilde{T}\). For simplicity, set \(\tau := f(\sigma)\). Then we have

\[
\sum_{\sigma \in S(A,B) \atop \sigma(b_2) = a_{i+1}} \text{sgn}(\sigma)e((\sigma T)_1) = (-1)^i \sum_{\tau \in S_{\tilde{T}}(\tilde{A}, \tilde{B})} \text{sgn}(\tau)e(\tau\tilde{T}) = (-1)^i g_{\tilde{A}, \tilde{B}}e(\tilde{T}) = 0.
\]
Next, we consider the case $\sigma(a_{i+1}) = a_{i+1}$ in (4.3). Set $\overline{A} = A \setminus \{a_{i+1}\}$ and

$$\overline{T} := \begin{array}{cccccc}
    a_{i+1} & b_1 & c_1 & c_2 & \cdots & a_1 \\
    a_2 & b_2 \\
    a_3 \\
    \vdots \\
    a_i \\
\end{array}$$

We have

$$\{\sigma \in S(A, B) \mid \sigma(a_{i+1}) = a_{i+1}\} = S_{\overline{T}}(\overline{A}, B),$$

and we have

$$(\sigma T)_1 = (\sigma(a_2) \sigma(a_3) \cdots \sigma(a_i) a_{i+1}) \sigma T$$

for $\sigma \in S(A, B)$ with $\sigma(a_{i+1}) = a_{i+1}$. Hence $e((\sigma T)_1) = (-1)^{i-1} e(\sigma T)$ and

$$\sum_{\sigma \in S(A, B) \atop \sigma(a_{i+1}) = a_{i+1}} \text{sgn}(\sigma) e((\sigma T)_1) = (-1)^{i-1} \sum_{\sigma \in S_{\overline{T}}(\overline{A}, B)} \text{sgn}(\sigma) e(\sigma T)$$

$$= (-1)^{i-1} g_{\overline{A}, \overline{B}} e(\overline{T})$$

$$= 0.$$

It remains to check the $x_{b_1}$-part and the $x_{b_2}$-part of the left side of (4.1). Set $A' := A \setminus \{a_2\}$, $B' := \{b_2, a_2\}$ and $T' := ((b_1 b_2 a_2) T)_2$. Note that

$$T' = \begin{array}{cccccc}
    a_1 & b_2 & c_1 & c_2 & \cdots & b_1 \\
    a_3 & a_2 \\
    \vdots \\
    a_{i+1} \\
\end{array}$$

We have a bijection

$$f' : \{\sigma \in S(A, B) \mid \sigma(a_2) = b_1\} \ni \sigma \mapsto (\sigma(b_2) \sigma(b_1) b_1) \sigma \in S_{T'}(A', B'),$$

and then $(\sigma T)_2 = f'(\sigma)(T')$. Hence the $x_{b_1}$-part of the left side of (4.1) is

$$\left( - \sum_{\sigma \in S(A, B) \atop \sigma(a_2) = b_1} \text{sgn}(\sigma) e((\sigma T)_2) \right) \otimes x_{b_1} = - g_{A', B'} e(T') \otimes x_{b_1} = 0.$$
Set $A'' := A\{a_2\}$, $B'' := \{b_1, a_2\}$ and $T'' := ((b_2 a_2)T)_2$. Note that

$$
T'' = \begin{array}{cccc}
a_1 & b_1 & c_1 & c_2 & \cdots & b_2 \\
a_3 & a_2 \\
\vdots \\
a_{i+1}
\end{array}
$$

Set

$$
L_1 = \{\sigma \in S(A, B) \mid \sigma(a_2) = b_2\},
$$

$$
L_2 = \{\sigma \in S(A, B) \mid \sigma(a_3) = b_2\},
$$

and

$$
L_1' = \{\tau \in S_{T''}(A'', B'') \mid \tau(b_1) = b_1\},
$$

$$
L_2' = \{\tau \in S_{T''}(A'', B'') \mid \tau(a_3) = b_1\}.
$$

If \(\sigma \in L_1\) (resp. \(\sigma \in L_2\)), then \(\sigma(b_1) = b_1\) (resp. \(\sigma(a_2) = b_1\)). We also have

$$
\{\sigma \in S(A, B) \mid \sigma(b_1), \sigma(b_2) \neq b_2\} = L_1 \sqcup L_2
$$

and

$$
S_{T''}(A'', B'') = L_1' \sqcup L_2'.
$$

We have bijections

$$
f''_1 : L_1 \ni \sigma \mapsto (\sigma(b_2) b_2)\sigma \in L_1'
$$

and

$$
f''_2 : L_2 \ni \sigma \mapsto (\sigma(b_2) b_2)\sigma \in L_2'.
$$

Note that \((\sigma T)_2 = f''_1(\sigma)(T'')\) for \(\sigma \in L_1\), and \((\sigma T)_3 = f''_2(\sigma)(T'')\) for \(\sigma \in L_2\). As before, set \(\tau := f''_k(\sigma)\) for \(\sigma \in L_k\). Then the \(x_{b_2}\)-part of the left side of (4.1) is

$$
\left(\sum_{\sigma \in L_1} (-1)^{2-1} \text{sgn}(\sigma)e((\sigma T)_2) + \sum_{\sigma \in L_2} (-1)^{3-1} \text{sgn}(\sigma)e((\sigma T)_3)\right) \otimes x_{b_2}
$$

$$
= \left(\sum_{\tau \in L_1'} \text{sgn}(\tau)e(\tau T'') + \sum_{\tau \in L_2'} \text{sgn}(\tau)e(\tau T'')\right) \otimes x_{b_2}
$$

$$
= g_{A'', B''}(T'') \otimes x_{b_2}.
$$

So we are done. \(\square\)

**Example 4.3.** (1) Here we will check (4.2) step by step. For simplicity, set \(j = 2\) (so \(\sigma(a_3) = a_2\) now). Then we have

$$
\tau(k) = \begin{cases} 
a_2 & \text{if } k = a_2, \\
\sigma(a_2) & \text{if } k = a_3, \\
\sigma(k) & \text{otherwise,}
\end{cases}
$$
$T_2 = \begin{pmatrix} a_1 & b_1 & c_1 & \cdots & a_2 \\ a_3 & b_2 \\ a_4 \\ \vdots \end{pmatrix}$

and

$\tau(T_2) = \begin{pmatrix} \tau(a_1) & \tau(b_1) & \tau(c_1) & \cdots & \tau(a_2) \\ \tau(a_3) & \tau(b_2) \\ \tau(a_4) \\ \vdots \end{pmatrix} = \begin{pmatrix} \sigma(a_1) & \sigma(b_1) & c_1 & \cdots & a_2 \\ \sigma(a_2) & b_2 \\ \sigma(a_4) \\ \vdots \end{pmatrix}$

Since

$\sigma T = \begin{pmatrix} \sigma(a_1) & \sigma(b_1) & c_1 & \cdots \\ \sigma(a_2) & b_2 \\ a_2 \\ \sigma(a_4) \\ \vdots \end{pmatrix}$

we have $\tau(T_2) = (\sigma T)_3$.

(2) For the tableau $T$ of (2.5) in Example 2.1, we will check that the $x_1$-part of

$$\sum_{\sigma \in S(A,B)} \partial_2 (e(T) \otimes 1)$$

is 0 for $A = \{4, 5\}$ and $B = \{2, 3\}$. The $x_1$-part is

$$\left( e\left( \begin{array}{cccc} 4 & 2 & 6 & 1 \\ 5 & 4 & 3 & 1 \end{array} \right) - e\left( \begin{array}{cccc} 3 & 2 & 6 & 1 \\ 5 & 4 & 1 & 1 \end{array} \right) + e\left( \begin{array}{cccc} 3 & 2 & 6 & 1 \\ 4 & 5 & 1 & 1 \end{array} \right) - e\left( \begin{array}{cccc} 2 & 3 & 6 & 1 \\ 4 & 5 & 1 & 1 \end{array} \right) \right) \otimes x_1,$$

but we have

$$e\left( \begin{array}{cccc} 3 & 2 & 6 & 1 \\ 4 & 5 & 1 & 1 \end{array} \right) - e\left( \begin{array}{cccc} 2 & 3 & 6 & 1 \\ 4 & 5 & 1 & 1 \end{array} \right) + e\left( \begin{array}{cccc} 2 & 4 & 6 & 1 \\ 3 & 5 & 1 & 1 \end{array} \right) = 0$$
and
\[ e(\begin{array}{ccc}4 & 2 & 6 \\ 3 & 3 & 1 \\ 5 & 4 & 1 \end{array}) - e(\begin{array}{ccc}3 & 2 & 6 \\ 5 & 4 & 1 \\ 4 & 5 & 1 \end{array}) + e(\begin{array}{ccc}2 & 3 & 6 \\ 5 & 4 & 1 \\ 4 & 5 & 1 \end{array}) = 0. \]

In fact, we can get the former (resp. latter) applying the Garnir element \( g_{A,B} \) for
\( A = \{3, 4\}, B = \{2\} \) (resp. \( A = \{4\}, B = \{2, 3\} \)) to
\[ \begin{array}{ccc}3 & 2 & 6 \\ 4 & 5 & 1 \end{array} \]
(resp. \( \begin{array}{ccc}5 & 2 & 6 \\ 4 & 3 & 1 \end{array} \)).

Recall that
\[ e(\begin{array}{ccc}4 & 2 & 6 \\ 5 & 3 & 1 \end{array}) = -e(\begin{array}{ccc}5 & 2 & 6 \\ 4 & 3 & 1 \end{array}), \]
and the same is true for the related tableaux.

**The proof of Theorem 3.2.** First, we will show that \( F_{(n-2,2)} \) is a chain complex. For the tableau \( T \) of (2.2) and any permutation \( \sigma \) of \( \{c_1, c_2, \ldots\} \), we have \( e(T) = e(\sigma(T)) \). Hence it is easy to see that \( \partial_{i-1} \partial_i = 0 \) holds for \( 3 \leq i \leq n-3 \). For
\[ (4.4) \quad T = \begin{array}{ccc}a_1 & b_1 & c_1 & c_2 & \cdots \\ a_2 & b_2 \\ a_3 \end{array} \]
we have
\[ T_1 = \begin{array}{ccc}a_2 & b_1 & c_1 & c_2 & \cdots \\ a_3 & b_2 \\ a_3 \end{array} \quad T_2 = \begin{array}{ccc}a_1 & b_1 & c_1 & c_2 & \cdots \\ a_3 & b_2 \\ a_3 \end{array} \quad T_3 = \begin{array}{ccc}a_1 & b_1 & c_1 & c_2 & \cdots \\ a_2 & b_2 \\ a_3 \end{array} \]
and
\[ \partial_1 \partial_2 (e(T) \otimes 1) = \partial_1 (e(T_1) \otimes x_{a_1} - e(T_2) \otimes x_{a_2} + e(T_3) \otimes x_{a_3}) \]
\[ = x_{a_1} f_{T_1} - x_{a_2} f_{T_2} + x_{a_3} f_{T_3} \]
\[ = (x_{a_1} (x_{a_2} - x_{a_3}) - x_{a_2} (x_{a_1} - x_{a_3}) + x_{a_3} (x_{a_1} - x_{a_2})) (x_{b_1} - x_{b_2}) \]
\[ = 0, \]
so \( \partial_1 \partial_2 = 0 \). Finally, we will show that \( \partial_{n-3} \partial_{n-2} = 0 \). Let \( T \in \text{Tab}(1^n) \) be the tableau in (3.2). Then \( \partial_{n-3} \partial_{n-2} (e(T) \otimes 1) \) equals
\[ \sum_{1 \leq j < k < l \leq n} (-1)^{j+k+l} \left( e(\begin{array}{ccc}j & k & l \\ \vdots & \vdots & \vdots \\ j & \vdots & k \end{array}) - e(\begin{array}{ccc}j & k & l \\ \vdots & \vdots & \vdots \\ j & \vdots & l \end{array}) + e(\begin{array}{ccc}j & k & l \\ \vdots & \vdots & \vdots \\ l & \vdots & k \end{array}) \right) \otimes x_j x_k x_l, \]
where all of the first columns of the above three tableaux are the “transpose” of
\[ \begin{array}{ccc}1 & 2 & \cdots & j-1 & j+1 & \cdots & k-1 & k+1 & \cdots & l-1 & l+1 & \cdots & n \end{array} \]
However, we have
\[ e(\begin{array}{ccc} j & l \\ k & \end{array}) - e(\begin{array}{ccc} j & k \\ l & \end{array}) + e(\begin{array}{ccc} k & j \\ l & \end{array}) = 0 \]
by the Garnir relation. Hence \( \partial_{n-3}\partial_{n-2}(e(T) \otimes 1) = 0 \), and \( \partial_{n-3}\partial_{n-2} = 0 \). So we have shown that \( F^{(n-2,2)}_\bullet \) is a chain complex. Since \( \text{Im} \partial_1 = I^{\text{Sp}}_{(n-2,2)} \), \( F^{(n-2,2)}_\bullet \) is a subcomplex of a minimal free resolution of \( R/I^{\text{Sp}}_{(n-2,2)} \).

Recall that we regard \( F_i \) as an \( S_n \)-module by \( \sigma(v \otimes f) := \sigma v \otimes \sigma f \in V_\lambda \otimes_{K R} (R(-j)) \), where \( \lambda \) is a suitable partition of \( n \) and \( j \) is a suitable integer uniquely determined by \( i \). In the previous section, we have shown that \( \partial_i : F_i \to F_{i-1} \) is an \( S_n \)-homomorphism. The restriction
\[ [\partial_i]_j : [F_i]_j = V_\lambda \otimes_{K} [R(-j)]_j = V_\lambda \otimes_{K} R_l = [F_{i-1}]_j \]
is also an \( S_n \)-homomorphism, where \( l = 1 \) if \( 2 \leq i \leq n-3 \), and \( l = 2 \) if \( i = 1, n-2 \). Since \( V_\lambda \otimes_{K} K \cong V_\lambda \) is irreducible as an \( S_n \)-module and \( [\partial_i]_j \) is clearly nonzero, we have \( [\partial_i]_j \) is injective. Since \( \mu(\text{Ker} \partial_{i-1}) = \beta_{i,j}(R/I^{\text{Sp}}_{(n-2,2)}) = \dim V_\lambda = \dim_K [\text{Im} \partial_i]_j \) for \( i \geq 2 \) by Lemma 4.1, \( F^{(n-2,2)}_\bullet \) is a (minimal) free resolution of \( R/I^{\text{Sp}}_{(n-2,2)} \). Here \( \mu(\cdot) \) denote the minimal number of generators as an \( R \)-module. \( \square \)

5. The case \((d, d, 1)\): Construction

For \( R/I^{\text{Sp}}_{(d,d,1)} \), we define the chain complex
\[
F^{(d,d,1)}_\bullet : 0 \to F_d \xrightarrow{\partial_d} F_{d-1} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \to 0
\]
of graded free \( R \)-modules as follows. Here \( F_0 = R \) and
\[
F_i = V_{(d,d-i+1,1)} \otimes_{K} R(-d - i - 1)
\]
for \( 1 \leq i \leq d \). As before, set \( \partial_1(e(T) \otimes 1) := f_T \in R = F_0 \). To describe \( \partial_i \) for \( i \geq 2 \), we need the preparation. For

\[
T = \begin{array}{cccccc}
a_1 & b_2 & \cdots & b_{d-i+1} & b_{d-i+2} & \cdots & b_d \\
a_2 & c_2 & \cdots & c_{d-i+1} \\
\vdots \\
a_{i+2}
\end{array}
\]
in Tab\((d, d - i + 1, 1^{i-1})\) and \(j\) with \(1 \leq j \leq i + 2\), set

\[
T_j := \begin{array}{cccccccc}
a_1 & b_2 & \cdots & b_{d-i+1} & b_{d-i+2} & b_{d-i+3} & \cdots & b_d \\
a_2 & c_2 & \cdots & c_{d-i+1} & a_j \\
\vdots \\
a_{j-1} \\
a_{j+1} \\
\vdots \\
\end{array}
\]

in Tab\((d, d - i + 2, 1^{i-1})\). Then we set

\[
\partial_i(e(T) \otimes 1) := \sum_{j=1}^{i+2} \sum_{\sigma \in H} (-1)^{j-1} e(\sigma(T_j)) \otimes x_{a_j} \in V_{(d,d-i+2,1^{i-1})} \otimes_{K} R(-d - i) = F_{i-1}
\]

for \(3 \leq i \leq d - 1\), where \(H\) is the set of permutations of \(\{b_{d-i+2}, b_{d-i+3}, \ldots, b_d\}\) satisfying \(\sigma(b_{d-i+3}) < \sigma(b_{d-i+4}) < \cdots < \sigma(b_d)\), and

\[
\partial_2(e(T) \otimes 1) = \sum_{j=1}^{3} (-1)^{j-1} e(T_j) \otimes x_{a_j} \in V_{(d,d,1^{d+1})} \otimes_{K} R(-d - 2) = F_1
\]

for \(T \in \text{Tab}(d, d - 1, 1, 1)\). That these \(\partial_i\) are well-defined will be shown in Theorem \([6.2]\). We are not sure whether \(\partial_d\) can be defined in the same way, that is, the well-definedness is not clear in this case. However, for our purpose (i.e., to show \(\partial_{d-1} \circ \partial_d = 0\)), it suffices to define \(\partial_d(e(T) \otimes 1)\) for \(T \in \text{SYT}(d, 1^{d+1})\). Since \(\{e(T) \mid T \in \text{SYT}(d, 1^{d+1})\}\) is a basis, we do not have to care the well-definedness. Anyway, we define \(\partial_d(e(T) \otimes 1)\) for \(T \in \text{SYT}(d, 1^{d+1})\) by \([5.3]\).

**Example 5.1.** Our minimal free resolution \(F^{(4,4,1)}_{(4,4,1)}\) of \(R/I_{(4,4,1)}^{Sp}\) is of the form

\[
0 \rightarrow V \otimes_{K} R(-9) \xrightarrow{\partial_2} V \otimes_{K} R(-8) \xrightarrow{\partial_1} V \otimes_{K} R(-7) \xrightarrow{\partial_2} V \otimes_{K} R(-6) \xrightarrow{\partial_1} R \rightarrow 0.
\]

The differential maps are given by
\[ \partial_4(e(\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \otimes 1) = (e(\begin{array}{c} 5 \\ 2 \\ 3 \\ 4 \\ 6 \\ 1 \\ 7 \\ 8 \\ 9 \end{array}) + e(\begin{array}{c} 5 \\ 3 \\ 2 \\ 4 \\ 6 \\ 1 \\ 7 \\ 8 \\ 9 \end{array}) + e(\begin{array}{c} 5 \\ 4 \\ 2 \\ 3 \\ 6 \\ 1 \\ 7 \\ 8 \\ 9 \end{array}) \otimes x_1
\]
\[ - (e(\begin{array}{c} 6 \\ 5 \\ 2 \\ 3 \\ 4 \\ 8 \\ 7 \\ 9 \\ 1 \end{array}) + e(\begin{array}{c} 6 \\ 5 \\ 3 \\ 2 \\ 4 \\ 8 \\ 7 \\ 9 \\ 1 \end{array}) + e(\begin{array}{c} 6 \\ 5 \\ 4 \\ 2 \\ 3 \\ 8 \\ 7 \\ 9 \\ 1 \end{array}) \otimes x_5
\]
\[ \vdots
\]
\[ \vdots
\]
\[ - (e(\begin{array}{c} 6 \\ 5 \\ 2 \\ 3 \\ 4 \\ 8 \\ 7 \\ 9 \\ 1 \end{array}) + e(\begin{array}{c} 6 \\ 5 \\ 3 \\ 2 \\ 4 \\ 8 \\ 7 \\ 9 \\ 1 \end{array}) + e(\begin{array}{c} 6 \\ 5 \\ 4 \\ 2 \\ 3 \\ 8 \\ 7 \\ 9 \\ 1 \end{array}) \otimes x_9
\]

Theorem 5.2. If \( \text{char}(K) = 0 \), the complex \( F^{(d,d,1)}_{(d,d,1)} \) of (5.1) is a minimal free resolution of \( R/I_{(d,d,1)}^{\text{sp}} \).

6. The case \((d, d, 1)\): Proof

Lemma 6.1. For all \( i \) with \( 1 \leq i \leq d \), we have
\[ \beta_{i,i+d+1}(R/I_{(d,d,1)}^{\text{sp}}) = \dim_K V_{(d,d-i+1,1^i)}. \]

Proof. By the hook formula, we have
\[ \dim_K V_{(d,d-i+1,1^i)} = \frac{(2d+1)!}{(d+i+1)d!(d+1)(d-i)!i!}, \]
\[ = \frac{(2d+1)!}{(d+i+1)(d+1)(d-i)!(i-1)!} \]
for all \( i \) with \( 1 \leq i \leq d \).

Since \( I_{(d,d,1)}^{\text{sp}} \) is a Cohen-Macaulay ideal of codimension \( d \) and has a \((d+2)\)-linear resolution, we have
\[ \beta_i(R/I_{(d,d,1)}^{\text{sp}}) = \beta_{i,i+d+1}(R/I_{(d,d,1)}^{\text{sp}}) \]
for all \(i \geq 1\), and [11, Theorem 3.5.17] implies that
\[
\beta_i(R/I_{Sp}^{(d,d,1)}) = \prod_{j=1}^{i-1} \frac{d+1+j}{i-j} \prod_{j=i+1}^{d} \frac{d+1+j}{j-i} 
= \frac{(d+i)!}{(i-1)!(d+1)!} \cdot \frac{(2d+1)!}{(d-i)!(d+i+1)!} 
= \frac{(d+i+1)d+1+j}{(d+i)(d+i+1)!}.
\]
So we are done. \(\square\)

**Theorem 6.2.** The maps \(\partial_i\) defined in the previous section are well-defined.

**Proof.** The well-definedness of \(\partial_1\) is nothing other than that of (2.1), and that of \(\partial_d\) is clear by the construction. So we may assume that \(2 \leq i \leq d-1\). By Proposition [2.2], it suffices to show that
\[
\sum_{\sigma \in S_T(A,B)} \text{sgn}(\sigma) \partial_i(e(\sigma T) \otimes 1)
\]
equals 0 for the tableau \(T\) of (5.2). The cases \(A = \{b_k, c_k\}\) and \(B = \{b_{k+1}\}\) for \(2 \leq k \leq d-i\) are easy. The non-trivial cases are
\[
(1) A = \{a_1, \ldots, a_{i+2}\}, B = \{b_2\}, \\
(2) A = \{a_2, \ldots, a_{i+2}\}, B = \{b_2, c_2\}, \\
(3) A = \{b_{d-i+1}, c_{d-i+1}\}, B = \{b_{d-i+2}\}.
\]
First, we treat the case (1). We may assume that \(b_2 < a_1 < a_2 < \cdots < a_{i+2}\). Fix \(j\) with \(1 \leq j \leq i+2\), and set \(A_j := A \setminus \{a_j\}\). Since \(\tau \in H\) is a permutation of \(\{b_{d-i+2}, b_{d-i+3}, \ldots, b_d\}\), \(\sigma \in S(A_j, B)\) and \(\tau \in H\) are disjoint. Hence we have
\[
(\text{the } x_{a_j}\text{-part of (6.1)}) = (-1)^{i-1} \sum_{\tau \in H} g_{A_j, B} e(\tau(T_j)) \otimes x_{a_j}.
\]
For each \(\tau\), we can show that \(g_{A_j, B} e(\tau(T_j)) = 0\) by an argument similar to the proof of Theorem [4.2]. That (the \(x_{b_2}\)-part of (6.1)) = 0, and the case (2) can be proved in a similar way.

For the case (3), the tableau

| \(a_1\) | \(b_2\) | \(\cdots\) | \(\sigma(b_{d-i+1})\) | \(b_k\) | \(\cdots\) | \(\sigma(b_{d-i+2})\) | \(\cdots\) |
|--------|--------|--------|----------------|--------|--------|----------------|--------|
| \(a_2\) | \(c_2\) | \(\cdots\) | \(\sigma(c_{d-i+1})\) | \(a_j\) |
| \(\vdots\) |
appears in the $x_{a_j}$-part of (6.1) for $d - i + 3 \leq k \leq d$ (here we assume that $j \geq 3$ for notational simplicity). However the above tableau and

\[
\begin{array}{cccccc}
  a_1 & b_2 & \cdots & b_k & \sigma(b_{d-i+1}) & \sigma(b_{d-i+2}) \\
  a_2 & c_2 & \cdots & a_j & \sigma(c_{d-i+1}) \\
  \vdots \\
  & & & & & \\
\end{array}
\]

give the same polytabloid $e(-)$, so the previous argument also works. 

\[\Box\]

**The proof of Theorem 5.2.** First, we will show that $F_{(d,d,1)}$ is a chain complex. It is easy to see that $\partial_{i-1}\partial_i = 0$ for $3 \leq i \leq d$. To show $\partial_1\partial_2 = 0$, consider

\[
T = \begin{array}{cccc}
  a_1 & b_2 & \cdots & b_{d-1} & b_d \\
  a_2 & c_2 & \cdots & c_{d-1} \\
  a_3 & & & & \\
  a_4 & & & & \\
\end{array}
\]

Then we have

\[
T_1 = \begin{array}{cccc}
  a_2 & b_2 & \cdots & b_{d-1} & b_d \\
  a_3 & c_2 & \cdots & c_{d-1} & a_1 \\
  a_4 & & & & \\
\end{array}, \quad T_2 = \begin{array}{cccc}
  a_1 & b_2 & \cdots & b_{d-1} & b_d \\
  a_3 & c_2 & \cdots & c_{d-1} & a_2 \\
  a_4 & & & & \\
\end{array},
\]

\[
T_3 = \begin{array}{cccc}
  a_1 & b_2 & \cdots & b_{d-1} & b_d \\
  a_2 & c_2 & \cdots & c_{d-1} & a_3 \\
  a_4 & & & & \\
\end{array}, \quad T_4 = \begin{array}{cccc}
  a_1 & b_2 & \cdots & b_{d-1} & b_d \\
  a_2 & c_2 & \cdots & c_{d-1} & a_4 \\
  a_3 & & & & \\
\end{array}
\]

in the notation of the previous section, and it holds that

\[
\partial_1\partial_2(e(T) \otimes 1) = x_{a_1}f_{T_1} - x_{a_2}f_{T_2} + x_{a_3}f_{T_3} - x_{a_4}f_{T_4}.
\]

One can show that this equals 0 by a direct computation (note that each part of the right side can be divided by $\prod_{i=2}^{d-1}(x_{b_i} - x_{c_i})$).

\[\Box\]
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