Self-dual Einstein spaces and the general heavenly equation. Eigenfunctions as coordinates

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Abstract

Eigenfunctions are shown to constitute privileged coordinates of self-dual Einstein spaces with the underlying governing equation being revealed as the general heavenly equation. The formalism developed here may be used to link algorithmically a variety of known heavenly equations. In particular, the classical connection between Plebański’s first and second heavenly equations is retrieved and interpreted in terms of eigenfunctions. In addition, connections with travelling wave reductions of the recently introduced TED equation which constitutes a \( 4 + 4 \)-dimensional integrable generalisation of the general heavenly equation are found. These are obtained by means of (partial) Legendre transformations. As a particular application, we prove that a large class of self-dual Einstein spaces governed by a compatible system of dispersionless Hirota equations is genuinely four-dimensional in that the (generic) metrics do not admit any (proper or non-proper) conformal Killing vectors. This generalises the known link between a particular class of self-dual Einstein spaces and the dispersionless Hirota equation encoding three-dimensional Einstein–Weyl geometries.

Keywords: self-dual Einstein space, eigenfunction, heavenly equation, Hirota equation, Killing vector, Legendre transformation

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1. Introduction

A variety of important solution generation techniques applied to Einstein’s field equations have their origin in (or may be interpreted in terms of) integrable systems theory (see [1, 2] and references therein). Eigenfunctions play a key role in integrable systems theory since they encode integrable systems via the compatibility of the linear systems (Lax pairs) which they satisfy. Eigenfunctions, in turn, obey partial differential equations (eigenfunction equations) which are intimately related to the important Miura-type transformations. The latter provide a link between integrable hierarchies and their associated modified versions such as the Korteweg–de Vries (KdV) and modified KdV hierarchies. Eigenfunctions are the key ingredient in Darboux-type transformations which lie at the heart of solution generation techniques (Bäcklund transformations) and their associated superposition principles (permutability theorems) for solutions of integrable systems. Eigenfunctions also encode conserved quantities associated with conservation laws hidden in integrable systems. Details on the above subjects and corresponding references may be found in, e.g. [2–6].

Hierarchies of 1 + 1-dimensional modified integrable equations are known to admit integrable counterparts which are related by reciprocal transformations. A geometrically important example is the integrable nonlinear Schrödinger equation, the modified version of which is given by the Heisenberg spin equation which, in turn, gives rise to the reciprocally related loop soliton equation (see, e.g. [2] and references therein). In the context of integrable systems, the analogue of reciprocal transformations for 2 + 1-dimensional hierarchies has been shown to involve eigenfunctions which, importantly, constitute some of the new independent variables in the ‘reciprocally’ related hierarchies. A prime example of the link between the associated three types of hierarchies of 2 + 1-dimensional integrable equations is provided by the connection between the Kadomtsev–Petviashvili (KP) hierarchy, its modified (mKP) hierarchy and the reciprocally related 2+1-dimensional Dym hierarchy. In this connection, the reader may wish to consult [7] for details and references.

In this paper, we present a self-contained first application to general relativity of a general scheme to be discussed elsewhere which, in any dimension, employs eigenfunctions of dispersionless integrable systems as independent variables in the corresponding privileged equivalent systems. Even though dispersionless integrable systems have been studied extensively (see, e.g. [8, 9] and references therein) since the pioneering work of Zakharov and Shabat [10], the interpretation of eigenfunctions as independent variables has not been at the forefront of these investigations. Here, we demonstrate that any four eigenfunctions of the self-dual Einstein equations [11] corresponding to four distinct spectral parameters give rise to a unique representation of self-dual Einstein spaces in terms of a potential which depends on the eigenfunctions playing the role of coordinates. Remarkably, the associated underlying dispersionless equation turns out to be the general heavenly equation introduced in [12, 13] as proven in section 4. It is noted in passing that particular coefficients of the Laurent expansions of eigenfunctions associated with the self-dual Einstein equations have been identified in [12, 14] as the independent variables in Plebański’s important first and second heavenly equations [15] governing self-dual Einstein spaces. An invariant definition of the above-mentioned key potential is presented in section 5.

It turns out that the assumption of one or two pairs of coinciding spectral parameters is also admissible. This is discussed in sections 7 and 8 respectively. In this manner, on the one hand, we retrieve the connection between the first heavenly equation and the Husain–Park equation [16] and, on the other hand, we demonstrate that the classical connection between Plebański’s two heavenly equations [15] may be viewed as a particular application of the algorithm developed in this paper. In section 12, the classification in terms of (0, 1 or 2) pair(s) of coinciding
spectral parameters is then shown to go hand in hand with travelling wave reductions of the recently introduced TED equation which constitutes a $4 + 4$-dimensional integrable extension of the general heavenly equation [17]. The connection between these two a priori unrelated subjects is provided by (partial) Legendre transformations. In particular, in section 6, the latter is shown to leave invariant the general heavenly equation. The investigation of Legendre transformations is motivated by the important observation that the gradient of the potential satisfying the general heavenly equation is, in fact, composed of another four eigenfunctions associated with the same spectral parameters.

As a further application of the algorithm presented here, we show that the general heavenly equation may be specialised to a system of four compatible dispersionless Hirota equations [18] by matching the eigenfunction and scaling symmetries of the general heavenly equation [17, 19]. It turns out that the self-dual Einstein metrics obtained in this manner generalise those associated with the three-dimensional Einstein–Weyl geometries known to be governed by the dispersionless Hirota equation [20] as discussed in section 9. In general, in section 10, the metrics generated by generic solutions of the dispersionless Hirota system are proven to be genuinely four-dimensional in the sense that no conformal Killing vectors (including homothetic Killing vectors and Killing vectors) exist. The key to the proof of this property is the derivation of the action of the above-mentioned eigenfunction symmetry constraint on the first heavenly equation, leading to a decomposition into three compatible differential equations. This is achieved in section 11 by exploiting the connection between the general heavenly equation and Plebański’s first heavenly equation derived in section 8.

2. The equations governing self-dual Einstein spaces

Self-dual Einstein spaces constitute four-dimensional manifolds equipped with a metric which are characterised by a self-dual Riemann tensor. Self-duality implies that the Ricci tensor $R_{ik}$ vanishes and, hence, Einstein’s vacuum equations $R_{ik} = 0$ are indeed satisfied [11]. Remarkably, the equations governing self-dual Einstein spaces have been shown to be equivalent to the self-dual Yang–Mills equations with four translational symmetries and the gauge group of volume preserving diffeomorphisms [21]. The latter are encoded in the commutativity of two four-dimensional vector fields $X$ and $Y$ which are linear in a (spectral) parameter $\lambda$ and divergence free with respect to a volume form [4]. Specifically, consider two $\lambda$-dependent commuting vector fields

$$X(\lambda) = A_1 + \lambda A_2, \quad Y(\lambda) = A_3 + \lambda A_4$$

(1)

so that the commutativity condition $[X, Y] = 0$ valid for all $\lambda$ is equivalent to

$$[A_1, A_3] = 0, \quad [A_2, A_4] = 0, \quad [A_1, A_4] + [A_2, A_3] = 0.$$  

(2)

The latter constitute partial differential equations for the $\lambda$-independent coefficients $A^i_\alpha$ of the vector fields

$$A_\alpha = A^i_\alpha \partial_i$$

(3)

which are assumed to be linearly independent, where we have adopted Einstein’s summation convention over repeated indices. In addition, we assume that the vector fields $X$ and $Y$ are divergence free for all $\lambda$ with respect to a $\lambda$-independent volume form

$$\text{vol}_4 = f \, dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$$

(4)
encoded in a function \( f(x') \) so that

\[
    f \, \text{div} A_\alpha = \partial_\nu (f A^\nu_\alpha) = 0.
\]

(5)

It is recalled that, in general, the above volume form does not coincide with the natural volume form induced by the self-dual Einstein metric which may be constructed from the vector fields \( A_\alpha \). The commutativity of \( X \) and \( Y \) implies the compatibility of the Lax pair [4]

\[
    X \Phi = 0, \quad Y \Phi = 0,
\]

(6)

that is, due to Frobenius’ theorem, the linear partial differential equation (6) for a function \( \Phi(x_i; \lambda) \) may be solved simultaneously. As is customary, we refer to \( \Phi \) as an eigenfunction associated with the pair of vector fields \( X \) and \( Y \). As indicated, \( \Phi \) also depends on \( \lambda \) since \( X \) and \( Y \) depend on \( \lambda \).

Different choices of the coordinates \( x_i \) lead to different but equivalent forms of the self-dual Einstein equations represented by the system (2) and (5). For instance, Plebański’s celebrated first heavenly equation [15]

\[
    \Omega_{x^1 x^3} \Omega_{x^2 x^4} - \Omega_{x^2 x^3} \Omega_{x^1 x^4} = 1
\]

(7)

for a function \( \Omega(x') \), which was, in fact, already recorded in 1936 in the context of ‘wave geometry’ [22], corresponds to the choice (see, e.g. [23])

\[
    A_1 = -\partial_{x^3}, \quad A_2 = \Omega_{x^1 x^3} \partial_{x^2} - \Omega_{x^2 x^3} \partial_{x^1}, \\
    A_3 = -\partial_{x^4}, \quad A_4 = \Omega_{x^1 x^4} \partial_{x^2} - \Omega_{x^2 x^4} \partial_{x^1}.
\]

(8)

Here, subscripts denote partial derivatives. The latter vector fields are indeed seen to be divergence free with respect to the function \( f = 1 \) and one may verify that the commutator relations (2) modulo the first heavenly equation (7) are satisfied. The corresponding metric reads [15]

\[
    g = 2 \Omega_{x^1 x^3} \, dx^1 \, dx^3 + 2 \Omega_{x^2 x^3} \, dx^2 \, dx^3 + 2 \Omega_{x^1 x^4} \, dx^1 \, dx^4 + 2 \Omega_{x^2 x^4} \, dx^2 \, dx^4,
\]

(9)

for which the Ricci tensor \( R_{ik} \) may be shown to vanish. It is noted that the signature of real self-dual Einstein spaces is known to be either Riemannian or neutral. In the case of Plebański’s first heavenly equation, neutral signature is associated with real independent variables \( x_i \), while in the Riemannian situation, the independent variables constitute two pairs of complex conjugates. In both cases, \( \Omega \) is real. The following analysis is insensitive to the signature in the sense that all variables could be real or complex and subject to the canonical reality constraints in the Riemannian case.

### 3. Eigenfunctions as privileged coordinates

Within the general setting (1)–(6) of self-dual Einstein spaces, we now consider four (functionally independent) eigenfunctions \( \phi^i \) corresponding to four distinct parameters \( \lambda_i \), that is,

\[
    X(\lambda_i) \phi^i = 0, \quad Y(\lambda_i) \phi^i = 0, \quad i = 1, 2, 3, 4
\]

(10)

(no summation over the index \( i \) as explained below). The important cases of one pair and two pairs of coinciding parameters are dealt with in sections 7 and 8 respectively. In terms of the new coordinates

\[
    y^i = \phi^i,
\]

(11)
the vector fields $X$ and $Y$ are represented by

$$
X = (X^i) \partial y_i = (\lambda - \lambda_i)(A_2 \phi^i) \partial y_i
$$

and

$$
Y = (Y^i) \partial y_i = (\lambda - \lambda_i)(A_4 \phi^i) \partial y_i
$$

(12)

and the volume form becomes

$$
\text{vol}_4 = \tilde{f} \, dy^1 \wedge dy^2 \wedge dy^3 \wedge dy^4, \quad \tilde{f} = \frac{f}{J}, \quad J = \det \left( \frac{\partial y^i}{\partial x^k} \right).
$$

(13)

It is emphasized that the notation $X$ and $Y$ always refers to $X(\lambda)$ and $Y(\lambda)$. If $X$ and $Y$ are evaluated at $\lambda_i$ then this is explicitly indicated as in (10). Moreover, throughout this paper, indices on the parameters $\lambda_i$ are not taken into account when Einstein’s summation convention is applied. Thus, expressions involving two repeated indices such as $\lambda^i a^i$ in which one index is attached to $\lambda$ do not denote a sum but expressions such as

$$
\lambda_i a_i b^i = \sum_{i=1}^{4} \lambda_i a_i b_i, \quad (14)
$$

encode summation over the index $i$. In particular, Einstein’s summation convention applies to the vector field representation (12). The latter naturally leads to the introduction of the four vector fields

$$
\tilde{X}^{(i)} = (A_4 \phi^i) X - (A_2 \phi^i) Y.
$$

(15)

Remarkably, even though the vector fields $\tilde{X}^{(i)}$ constitute linear combinations of the vector fields $X$ and $Y$ with non-constant coefficients, the following theorem obtains.

**Theorem 3.1.** The vector fields $\tilde{X}^{(i)}$ are divergence free.

**Proof.** The proof of this theorem is based on the general (obvious) fact that if two commuting vector fields $X$ and $Y$ are divergence free then the vector field $(Yg)X - (Xg)Y$ is likewise divergence free for any function $g$. Hence, if we set $g = \phi^i$ then

$$
0 = \text{div}[(Y \phi^i)X - (X \phi^i)Y] = (\lambda - \lambda_i)\text{div}[(A_4 \phi^i)X - (A_2 \phi^i)Y]
$$

(16)

since $(A_1 + \lambda_i A_2) \phi^i = 0$ and $(A_3 + \lambda_i A_4) \phi^i = 0$. \hfill \Box

In terms of components, the vector fields $\tilde{X}^{(i)}$ adopt the form

$$
\tilde{X}^{(i)} = (\lambda - \lambda_k) \tilde{A}^{ik} \partial y_k, \quad \tilde{A}^{ik} = (A_4 \phi^i)(A_2 \phi^k) - (A_2 \phi^i)(A_4 \phi^k).
$$

(17)

It is observed that the coefficients of the skew-symmetric matrix $\tilde{A}$ may be regarded as Plücker coordinates $[24]$ of a line

$$
A_4 \left( \begin{array}{c} \phi^1 \\ \phi^2 \\ \phi^3 \\ \phi^4 \\ \phi^5 \\ \phi^6 \\ \phi^7 \\ \phi^8 \\ \phi^9 \\ \phi^{10} \end{array} \right) \wedge A_2 \left( \begin{array}{c} \phi^1 \\ \phi^2 \\ \phi^3 \\ \phi^4 \end{array} \right), \quad (18)
$$

in a three-dimensional projective space represented in terms of homogeneous coordinates. Accordingly, the original Lax pair (6) may equivalently be formulated as the set of four equations

$$
\tilde{f} \tilde{X}^{(i)} \Phi = A^{ik} D_k \Phi = 0, \quad A^{ik} = \tilde{f} \tilde{A}^{ik}, \quad D_k = (\lambda - \lambda_k) \partial y_k.
$$

(19)
since rank $A = \text{rank } \tilde{A} = 2$. The latter is reflected by the Plücker relation
\[ \text{pf}(A) = A^{12}A^{34} + A^{23}A^{14} + A^{31}A^{24} = 0, \] (20)
where pf$(A)$ denotes the Pfaffian of $A$. The relevance of this observation is revealed below. In particular, the significance of the scaling of the matrix $\tilde{A}$ is explained.

4. The general heavenly equation

We are now in a position to present the key theorem of this paper.

**Theorem 4.1.** Let $\phi^i, i = 1, 2, 3, 4$ be four eigenfunctions associated with any vector field representation $X(x; \lambda), Y(x; \lambda)$ of self-dual Einstein spaces and distinct parameters $\lambda_i$. Then, in terms of the independent variables $y^i = \phi^i$, the self-dual Einstein equations are transformed into the general heavenly equation
\[ (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4) \Theta_{y^1y^2} \Theta_{y^3y^4} + (\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4) \Theta_{y^2y^3} \Theta_{y^1y^4} + (\lambda_3 - \lambda_1)(\lambda_2 - \lambda_4) \Theta_{y^3y^1} \Theta_{y^2y^4} = 0, \] (21)
for some potential $\Theta$ encoded in the divergence-free vector fields $X$ and $Y$.

**Remark 4.1.** The differential equation (21), which may be formulated as
\[ c_1 \Theta_{y^1y^2} \Theta_{y^3y^4} + c_2 \Theta_{y^2y^3} \Theta_{y^1y^4} + c_3 \Theta_{y^3y^1} \Theta_{y^2y^4} = 0, \quad c_1 + c_2 + c_3 = 0, \] (22)
was originally derived as the continuum limit of the permutability theorem for both the classical Tzitzéica equation of affine differential geometry and its integrable discretisation [12, 13, 25]. In this context, it was also observed that this equation constitutes yet another avatar of the self-dual Einstein equations. In connection with the classification of integrable symplectic Monge–Ampère equations, it was rediscovered and termed general heavenly equation [23].

The above theorem demonstrates that the general heavenly equation is privileged in that the associated coordinates play the role of eigenfunctions regardless of the concrete realisation of the vector fields $A_i$.

In order to verify the above theorem, we begin by noting that the scaling of the matrix $\tilde{A}$, leading to the matrix $A$ as defined by (19)2, has been introduced for convenience so that the vanishing divergence of the vector fields $\tilde{X}^{(i)}$ simplifies to the system of equations
\[ \partial_{y^i}A^{ik} = 0, \quad \lambda_k \partial_{y^i}A^{ik} = 0. \] (23)

In terms of the skew-symmetric matrix $\omega$ defined by
\[ A^{lm} = \epsilon^{l[m} \omega_{n]} , \] (24)
where $\epsilon^{l[m}$ denotes the totally anti-symmetric Levi–Civita symbol, these adopt the form
\[ \partial_{y^i} \omega_{ij} = 0, \quad \lambda_i \partial_{y^i} \omega_{ij} = 0 \] (25)
and may be regarded as the integrability conditions for the existence of a potential $\Theta$ defined by
\[ \Theta_{y^i} = \frac{\omega_{ik}}{\lambda_k - \lambda_i}. \] (26)
Here, the square brackets indicate total anti-symmetrisation. Accordingly, the solution of (25) is parametrised in terms of $\Theta$ according to

$$\omega_{ik} = (\lambda_k - \lambda_i) \Theta_{yiyk}.$$  

Now, the linear system (19)1 may be formulated as

$$
\begin{pmatrix}
0 & \omega_{34} & \omega_{42} & \omega_{23} \\
\omega_{43} & 0 & \omega_{14} & \omega_{31} \\
\omega_{24} & \omega_{41} & 0 & \omega_{12} \\
\omega_{32} & \omega_{13} & \omega_{21} & 0
\end{pmatrix}
\begin{pmatrix}
D_1 \Phi \\
D_2 \Phi \\
D_3 \Phi \\
D_4 \Phi
\end{pmatrix} = 0
$$

with the associated rank 2 condition $pf(A) = 0$ expressed as

$$pf(\omega) = \omega_{12} \omega_{34} + \omega_{23} \omega_{14} + \omega_{31} \omega_{24} = 0.$$  

The parametrisation (27) therefore leads to the general heavenly equation (21) with, for instance,

$$\tilde{X}^{(3)} \Phi = 0, \quad \tilde{X}^{(4)} \Phi = 0$$

constituting its standard Lax pair [12]. Here, it should be emphasised that the scaled vector fields

$$\tilde{X}^{(3)} = \frac{\omega_{24}}{\omega_{12}} D_1 + \frac{\omega_{41}}{\omega_{12}} D_2 + D_4, \quad \tilde{X}^{(4)} = \frac{\omega_{32}}{\omega_{21}} D_1 + \frac{\omega_{13}}{\omega_{21}} D_2 + D_3$$

commute and are divergence free with respect to the volume form

$$vol_4 = \Theta_{yiyj} dy^i \wedge dy^j \wedge dy^k \wedge dy^l$$

as originally observed in [12]. Accordingly, the general heavenly equation may be regarded as an ‘invariant’ form of the self-dual Einstein equations since neither the eigenfunctions $\phi$ nor the potential $\Theta$ are affected by coordinate transformations $x^i \rightarrow f^i(x^k)$. In particular, any of the known ‘heavenly’ equations governing self-dual Einstein spaces (see, e.g. [23, 26] and references therein) may be mapped to the general heavenly equation by means of the algorithm presented in this section.

### 5. Invariant definition of the potential $\Theta$

In order to interpret the potential $\Theta$ within the original self-dual Einstein setting of section 2 associated with the vector fields $A_{\alpha}(x^i)$, we first make the following important observation.

**Remark 5.1.** The dependent variable $\Theta$ also encodes eigenfunctions in that the quantities

$$\psi_i = \Theta_{yiyj}$$

form another set of eigenfunctions corresponding to the parameters $\lambda_i$. Indeed, since

$$D_k \psi_i|_{\lambda = \lambda_k} = (\lambda_i - \lambda_k) \Theta_{yiyj} = \omega_{ijk}, \quad i \neq k,$$

three equations of the linear system (28) for $\Phi = \psi_i$ and $\lambda = \lambda_i$ are identically satisfied, while the remaining equation turns out to be $pf(\omega) = 0$. It is remarked that this implies that the general
solution of the linear system at $\lambda = \lambda_i$ is an arbitrary function of the particular eigenfunctions $\phi^i = y^i$ and $\psi_i = \Theta_{\phi^i}$.

The following theorem provides the basis of this section.

**Theorem 5.1.** Let $\phi^i, i = 1, \ldots, 4$ be four eigenfunctions of the self-dual Einstein equations for distinct parameters $\lambda_i$. Then, the two-forms $\Omega_k, k = 1, \ldots, 4$ defined by

$$\Omega_k = \frac{1}{4} \epsilon_{ijkl} A_{lm}^{\lambda_k - \lambda_l} d\phi^i \wedge d\phi^j,$$

where

$$A_{lm}^{\lambda} = \int J \tilde{A}_{lm}^{\lambda},$$

$$\tilde{A}_{lm}^{\lambda} = (A_4 \phi^l)(A_2 \phi^m) - (A_2 \phi^l)(A_4 \phi^m), \quad J = \det \left( \frac{\partial \phi^j}{\partial x^i} \right),$$

have the following properties:

$$d\Omega_k = 0, \quad \Omega_k \wedge \Omega_k = 0, \quad \Omega_k \wedge d\phi^k = 0, \quad \sum_{k=1}^4 \Omega_k = 0.$$  \hfill (37)

Here, the underbar indicates that there is no summation over the corresponding index.

**Proof.** Since the algebraic properties (37) are evident, it remains to show that $d\Omega_k = 0$. In terms of the coordinates $y^i$, the latter is given by

$$\epsilon_{ijkl} A_{lm}^{\lambda} \frac{\partial \phi^l}{\partial y^j} d\phi^i \wedge dy^j \wedge dy^k = 0.$$  \hfill (38)

Hence, for any fixed $k$, the contribution of the terms proportional to $dy^{m_0} \wedge dy^{n_0} \wedge dy^k$ reads

$$\frac{\partial \phi^l}{\partial y^j} A_{lm}^{\lambda} + \frac{\partial \phi^l}{\partial y^j} A_{lm}^{\lambda} = 0,$$

wherein the indices $k, l_0, m_0, n_0$ are fixed and distinct. On clearing the denominators of the above, this may be formulated as

$$(\lambda_k - \lambda_m) \frac{\partial \phi^l}{\partial y^j} A_{lm}^{\lambda} = 0$$  \hfill (40)

since the terms for $m = k$ and $m = l_0$ in the above sum vanish identically. Finally, the vanishing divergence conditions (23) imply that the relations (40) are indeed satisfied and, hence, the two-forms $\Omega_k$ are closed. $\square$

The above theorem encodes the existence of a potential $\Theta$ which coincides with that derived in section 4 in connection with the parametrisation of the skew-symmetric matrix $A$ in terms of the coordinates $y^i$.

**Theorem 5.2.** There exist functions $\psi_k, k = 1, \ldots, 4$ such that

$$\Omega_k = d\psi_k \wedge d\phi^k.$$  \hfill (41)
These constitute eigenfunctions for $\lambda = \lambda_k$ and give rise to the existence of a potential $\Theta$ via

$$d\Theta = \psi_k d\phi^k.$$  \hspace{1cm} (42)

**Proof.** By virtue of Darboux’s theorem, the properties (37)\_1,2 imply that the two-forms $\Omega_k$ may be written as exterior products of pairs of differentials. Property (37)\_3 then shows that $\Omega_k$ is of the form (41) for some function $\psi_k$. The compatibility condition $d\psi_k \wedge d\phi^k = 0$ associated with (42) guaranteeing the existence of the potential $\Theta$ is satisfied since

$$d\psi_k \wedge d\phi^k = \sum_{k=1}^{4} \Omega_k = 0 \hspace{1cm} (43)$$

by virtue of (37)\_4. Furthermore, since, in this case,

$$\Theta_{\psi k} = \psi_k, \hspace{1cm} (44)$$

comparison of (35) and (41) results in

$$\Theta_{\psi i j} = \frac{1}{4} e_{i j} \lambda_i \lambda_j A_{i j} A_{i j}$$ \hspace{1cm} (45)

which coincides with (26) by virtue of (24) so that $\psi_k = \Theta_{\psi k}$ indeed constitutes an eigenfunction for $\lambda = \lambda_k$. It is emphasised that, on use of the pair (35) and (41) regarded as a definition of the functions $\psi_k$, one may also directly show that $X(\lambda_k) \psi_k = Y(\lambda_k) \psi_k = 0.$ \hspace{1cm} \Box

6. A Legendre transformation

It is evident that theorem 4.1 applied to the general heavenly equation

$$(\mu_1 - \mu_2)(\mu_3 - \mu_4)H_{i j} H_{k l} - (\mu_1 - \mu_2)(\mu_3 - \mu_4)H_{k l} H_{i j} = 0$$ \hspace{1cm} (46)

provides an invariance of the general heavenly equation since it maps (46) to the general heavenly equation (21). In general, the associated spectral parameters $\lambda_i$ do not have to be the four parameters $\mu_i$ in the general heavenly equation (46) but if we do make this special choice then the corresponding eigenfunctions are given by

$$\phi' = f'(x^i, H_{ij})$$ \hspace{1cm} (47)

as pointed out in the previous section. In particular, the choice

$$\phi' = H_{ij}$$ \hspace{1cm} (48)

is admissible. This raises the question as to whether the quantities $x^i$ and $x_1 := H_{ij}$ play symmetric roles in the general heavenly equation (46). In order to demonstrate that this is the case, we observe that the general heavenly equation may be formulated as

$$(\mu_2 - \mu_3)(\mu_1 - \mu_4)(H_{i j} H_{k l} - H_{i j} H_{k l})$$
$$+ (\mu_3 - \mu_1)(\mu_2 - \mu_4)(H_{i j} H_{k l} - H_{i j} H_{k l}) = 0$$ \hspace{1cm} (49)
and state the following theorem.

**Theorem 6.1.** The general heavenly equation in the form

\[
(\mu_1 - \mu_4)(\mu_2 - \mu_3)dx_1 \wedge dx_3 \wedge dx^1 \wedge dx^3 \\
+ (\mu_1 - \mu_3)(\mu_2 - \mu_4)dx_1 \wedge dx_4 \wedge dx^1 \wedge dx^4 = 0, \quad dH = x_i dx^i
\]

is invariant under the Legendre transformation

\[\mathcal{H}(x^i) \rightarrow \tilde{\mathcal{H}}(x^i) = x_i x_i - \mathcal{H}.\] (51)

**Proof.** In terms of the new variables \(\tilde{\mathcal{H}}, x_i\), relation (50) becomes

\[d\tilde{\mathcal{H}} = x_i dx_i]\ (52)

which proves the invariance since the remaining relation (50) is symmetric in the upper and lower indices.

\(\square\)

It is remarked that the general heavenly equation is also invariant under a ‘partial’ Legendre transformation which interchanges any chosen number of corresponding variables carrying upper and lower indices, that is, the summation over \(i\) in (51) may be restricted to any subset of \(\{1, \ldots, 4\}\) with \(\tilde{\mathcal{H}}\) depending on the associated appropriate variables. Partial Legendre transformations are further discussed in section 12.

7. A pair of coinciding spectral parameters. The Husain–Park equation

For any fixed spectral parameter \(\lambda\), the general eigenfunction is a function of two functionally independent particular eigenfunctions. Hence, instead of demanding that all parameters \(\lambda_i\) be distinct, it is also admissible to choose up to two pairs of coinciding spectral parameters. In this section, we consider the case of one pair of coinciding parameters, that is, \(\lambda_1 = \lambda_2\) without loss of generality. Thus, the task is to find the analogue of the parametrisation (27) which resolves the vanishing divergence conditions (25). To this end, we observe that the defining equation (26) for the potential \(\Theta\) are still meaningful and compatible as long as \((1, 2) \neq (i, k) \neq (2, 1)\). Accordingly, we may adopt the parametrisation of the generic case, that is,

\[\omega_{ik} = (\lambda_k - \lambda_i)\Theta_{y_i y_k}\] (53)

with the coefficients \(\omega_{12}\) and \(\omega_{21} = -\omega_{12}\) being excluded. The latter are determined by the remaining vanishing divergence conditions which read

\[\partial_y \omega_{12} = 0, \quad \partial_y \omega_{12} = 0\] (54)

so that the vanishing Pfaffian condition (29) becomes

\[f_{12} \Theta_{y_i y_k} + \Theta_{y_i y_k} \Theta_{y_i y_k} - \Theta_{y_i y_k} \Theta_{y_k y_k} = 0,\] (55)

where

\[f_{12} = \frac{\lambda_1 - \lambda_2}{(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)} \omega_{12}(y^1, y^2).\] (56)

Hence, on application of an appropriate member of the class of coordinate transformations \((y^1, y^2) \rightarrow f_{12}(y^1, y^2)\), the Husain–Park equation corresponding to \(f_{12} = 1\) is obtained (see [16]...
and references therein). It is noted that this is consistent with the fact that any function of two eigenfunctions corresponding to the same spectral parameter constitutes another eigenfunction so that it is a priori known that the differential equation (55) must be invariant under this class of coordinate transformations. Furthermore, it should be emphasised that rather than considering the coinciding pair \( \lambda_1 = \lambda_2 \) and applying the above algorithm, one may apply a confluence limit of the type

\[
\Theta \rightarrow \Theta + \frac{g_{12}(y^1, y^2)}{\epsilon}, \quad \lambda_2 = \lambda_1 + \epsilon, \quad \epsilon \rightarrow 0
\]

directly to the general heavenly equation (21) to derive the Husain–Park equation in the ‘gauge-invariant’ form (55).

Before we present concrete examples of how the above procedure may be applied, it is worth noting that the vector fields \( X(\lambda_1) \) and \( Y(\lambda_1) \) are tangent to the coordinate surfaces \((y^1, y^2) = \text{const}\) since

\[
X(\lambda_1) y^i = 0, \quad Y(\lambda_1) y^i = 0, \quad i = 1, 2.
\]

Hence, the commutativity of \( X(\lambda_1) \) and \( Y(\lambda_1) \) guarantees the existence of a natural coordinate system \((z^1, z^2, y^1, y^2)\) defined by the additional relations

\[
X(\lambda_1) z^1 = 1, \quad X(\lambda_1) z^2 = 0, \quad Y(\lambda_1) z^1 = 0, \quad Y(\lambda_1) z^2 = 1
\]

in terms of which these two vector fields are ‘straight’, that is,

\[
X(\lambda_1) = \partial_{z^1}, \quad Y(\lambda_1) = \partial_{z^2}.
\]

It is observed that even though, for any fixed spectral parameter, say, \( \lambda_1 \), the corresponding eigenfunction is an arbitrary function of two functionally independent particular eigenfunctions, the kernel of each of the vector fields \( X(\lambda_1) \) and \( Y(\lambda_1) \) is encoded in three functionally independent solutions of \( X(\lambda_1) z^X = 0 \) and \( Y(\lambda_1) z^Y = 0 \) respectively. In fact, the relations (58) and (59) show that, in the above situation, \( z^X = F(z^2, y^1, y^2) \) and \( z^Y = G(z^1, y^1, y^2) \). Hence, the introduction of the coordinate system \((z^1, z^2, y^1, y^2)\) constitutes a natural way of getting around the fact that it is impossible to have three or four coinciding spectral parameters in the formalism presented in this paper. The general relationship between the latter and the classical theory of ‘straightening’ commuting vector fields using ‘eigenfunctions’ (integrals) [27] is currently under investigation.

7.1. Application to the general heavenly equation

As a first illustration, we now show explicitly how the general heavenly equation (46) may be mapped to the Husain–Park equation (55). Since it is natural to select the parameters

\[
\lambda_1 = \lambda_2 = \mu_2, \quad \lambda_3 = \mu_3, \quad \lambda_4 = \mu_4,
\]

we may make the choice

\[
\phi^1 = H z^1, \quad \phi^2 = x^2, \quad \phi^3 = x^3, \quad \phi^4 = x^4.
\]

If we consider the case \( \mu_1 \rightarrow \infty \) and \( \mu_2 = 0 \) without loss of generality then the Lax pair for the general heavenly equation in the form

\[
(\mu_3 - \mu_4) H_{x^1,x^2} H_{x^3,x^4} = \mu_3 H_{x^2,x^3} H_{x^4} - \mu_4 H_{x^1,x^3} H_{x^4}
\]
is given by \[12\]

\[
\Phi_\alpha = \frac{1}{(\lambda - \mu_\alpha)H_{\alpha \beta \gamma}}{(\lambda H_{\alpha \beta \gamma} \Phi_\beta - \mu_\alpha H_{\alpha \beta \gamma} \Phi_\gamma)}
\]

\[
\Phi_\beta = \frac{1}{(\lambda - \mu_\beta)H_{\beta \gamma \delta}}{(\lambda H_{\beta \gamma \delta} \Phi_\gamma - \mu_\beta H_{\beta \gamma \delta} \Phi_\delta)}
\]

so that

\[
A_2 = \partial \Phi_\alpha - \frac{H_{\alpha \beta \gamma}}{H_{\beta \gamma \delta}} \partial \Phi_\beta, \quad A_4 = \partial \Phi_\beta - \frac{H_{\beta \gamma \delta}}{H_{\alpha \beta \gamma}} \partial \Phi_\alpha.
\]

The latter vector fields are divergence free with respect to \(f = H_{\alpha \beta \gamma}\) as pointed out at the end of section 4. Accordingly, the entries (36) of the skew-symmetric matrix \(A = \Lambda\) (by virtue of \(J = H_{\alpha \beta \gamma}\)) read

\[
A^{12} = \frac{H_{\alpha \beta \gamma}H_{\gamma \delta \epsilon} - H_{\alpha \beta \gamma}H_{\delta \epsilon \gamma}}{H_{\beta \gamma \delta}}, \quad A^{13} = H_{\alpha \beta \gamma}H_{\gamma \delta \epsilon} - \frac{H_{\alpha \beta \gamma}}{H_{\beta \gamma \delta}}H_{\delta \epsilon \gamma}^2
\]

\[
A^{41} = H_{\alpha \beta \gamma} = \frac{H_{\alpha \beta \gamma}H_{\gamma \delta \epsilon} - H_{\alpha \beta \gamma}H_{\delta \epsilon \gamma}}{H_{\beta \gamma \delta}}, \quad A^{32} = \frac{H_{\alpha \beta \gamma}}{H_{\beta \gamma \delta}}H_{\delta \epsilon \gamma}^2 - A^{24} = \frac{H_{\alpha \beta \gamma}}{H_{\beta \gamma \delta}}H_{\delta \epsilon \gamma}^2 \quad A^{43} = 1.
\]

On use of the identity

\[
\partial \Phi_\alpha = H_{\alpha \beta \gamma} \partial \Phi_\beta,
\]

we therefore conclude that the connection between the potentials \(H\) and \(\Theta\) encoded in (45) may be formulated as

\[
\Theta_\alpha = -\frac{1}{\mu_\alpha} \frac{\partial \Phi_\alpha}{H_{\alpha \beta \gamma}}, \quad \Theta_\beta = -\frac{1}{\mu_\beta} \frac{\partial \Phi_\beta}{H_{\alpha \beta \gamma}}
\]

and similar expressions for the remaining mixed derivatives of \(\Theta\) except for \(\Theta_\beta^\alpha\). Hence, integration leads, without loss of generality, to the first-order relations

\[
\Theta_\beta = -\frac{1}{\mu_\beta} \frac{H_{\alpha \beta \gamma}}{2 \mu_\alpha}, \quad \Theta_\beta = -\frac{1}{\mu_\beta} \frac{H_{\alpha \beta \gamma}}{2 \mu_\beta}.
\]

One may now directly verify that the above pair is compatible modulo the general heavenly equation (63) and \(\Theta\) indeed satisfies the Husain–Park equation (55) with \(f_{12} = (\mu_\beta^{-1} - \mu_\alpha^{-1})/2\).

### 7.2. Application to Plebański’s first heavenly equation

The connection between Plebański’s first heavenly equation and the (elliptic) Husain–Park equation has been established explicitly in [16]. Here, we demonstrate how this connection may be found algorithmically using our formalism. It is recalled (see section 2) that the standard Lax pair for the first Plebański equation

\[
\Omega_{\alpha \beta \gamma}^1 \Omega_{\beta \gamma \delta}^1 - \Omega_{\alpha \beta \gamma}^2 \Omega_{\alpha \beta \gamma}^2 = 1
\]

reads

\[
\Phi_{\alpha} = \lambda (\Omega_{\alpha \beta \gamma}^1 \Phi_{\beta} - \Omega_{\alpha \beta \gamma}^2 \Phi_{\gamma}) = \lambda A_2 \Phi
\]

\[
\Phi_{\beta} = \lambda (\Omega_{\alpha \beta \gamma}^1 \Phi_{\beta} - \Omega_{\alpha \beta \gamma}^2 \Phi_{\gamma}) = \lambda A_4 \Phi.
\]
For a non-vanishing spectral parameter $\lambda$, this is equivalent to the pair
\[
\Phi_{i,1} = \lambda^{-1}(\Omega_{i,1}\Phi_{i,3} - \Omega_{i,3}\Phi_{i,1})
\]
\[
\Phi_{i,2} = \lambda^{-1}(\Omega_{i,2}\Phi_{i,3} - \Omega_{i,3}\Phi_{i,2}).
\]
(72)

In the following, the most convenient specialisation of the parameters $\lambda_i$ and associated eigenfunctions $\phi^i$ for $i = 1, 2$ is given by
\[
\lambda_1 = \lambda_2 = 0, \quad \phi^1 = x^1, \quad \phi^2 = x^2
\]
(73)

with the remaining eigenfunctions $\phi^3$ and $\phi^4$ corresponding to the parameters $\lambda_3$ and $\lambda_4$ being arbitrary. Accordingly, we obtain
\[
\tilde{A}^{31} = \phi^3_{x^2}, \quad \tilde{A}^{41} = \phi^4_{x^2}, \quad \tilde{A}^{23} = \phi^3_{x^1}, \quad \tilde{A}^{24} = \phi^4_{x^1}.
\]
(74)

Moreover, inversion of the identities
\[
\partial_{x^3} = \phi^3_{x^3}\partial_{x^3} + \phi^3_{x^4}\partial_{x^4}, \quad \partial_{x^4} = \phi^4_{x^3}\partial_{x^3} + \phi^4_{x^4}\partial_{x^4}
\]
(75)

yields
\[
\partial_{x^3} = \frac{\phi^3_{x^3}\partial_{x^3} - \phi^3_{x^4}\partial_{x^4}}{J}, \quad \partial_{x^4} = \frac{\phi^4_{x^3}\partial_{x^3} - \phi^4_{x^4}\partial_{x^4}}{J}
\]
(76)

with $J = \phi^3_{x^3}\phi^4_{x^4} - \phi^3_{x^4}\phi^4_{x^3}$ so that the relations (74)$_{1,2,3,4}$ become
\[
\tilde{A}^{23} = \frac{J}{\lambda_3}\partial_{x^3}\Omega_{x^1}, \quad \tilde{A}^{13} = \frac{J}{\lambda_3}\partial_{x^3}\Omega_{x^2}
\]
\[
\tilde{A}^{42} = \frac{J}{\lambda_4}\partial_{x^4}\Omega_{x^1}, \quad \tilde{A}^{14} = \frac{J}{\lambda_4}\partial_{x^4}\Omega_{x^2}
\]
(77)

by virtue of the Lax pair (72). Now, since $f = 1$ so that $A^{lm} = \tilde{A}^{lm}/J$, four of the relations (45) may be integrated to obtain
\[
\Theta_{i,1} = \frac{1}{2\lambda_3}\Omega_{x^1} + p(x^1, x^2), \quad \Theta_{i,2} = \frac{1}{2\lambda_4}\Omega_{x^2} + q(x^1, x^2)
\]
(78)

where $p(x^1, x^2) = p(x^1, x^2)$ and $q(x^1, x^2) = q(x^1, x^2)$ are functions of integration. By construction, the above pair must be compatible modulo the first Plebański equation (70) and the Lax pair (71) for $\phi^3$ and $\phi^4$ corresponding to the parameters $\lambda_3$ and $\lambda_4$. Indeed, cross-differentiation produces the relation
\[
\lambda_3\lambda_4(p_{x^2} - q_{x^1}) = \lambda_3 + \lambda_4.
\]
(79)

Hence, without loss of generality, we may choose
\[
p = \frac{\lambda_3 + \lambda_4}{2\lambda_3\lambda_4} x^2, \quad q = \frac{\lambda_3 + \lambda_4}{2\lambda_3\lambda_4} x^1.
\]
(80)

Finally, the remaining compatible relation (45)$_{i=3,4}$, namely
\[
\Theta_{i,3,4} = \frac{1}{2J(\lambda_3 - \lambda_4)},
\]
(81)
guarantees that $\Theta$ is a solution of the Husain–Park equation (55) for $f_{12} = (\lambda_3 - \lambda_4)/2\lambda_3^2 \lambda_4^2$.

It is evident that the choice $\lambda_4 = -\lambda_3$ is privileged since, in this case, the pair (78) simplifies to

$$\Theta_{y_1} = -\frac{1}{2\lambda_3^2} \Omega_{y_1}, \quad \Theta_{y_2} = -\frac{1}{2\lambda_4^2} \Omega_{y_2}. \quad (82)$$

The latter represents the analogue of the relations derived in [16] for the elliptic Husain–Park equation. Indeed, if one sets aside the ‘normalisation’ (81) then $\Theta$ defined by the compatible pair (82) constitutes a solution of the Husain–Park equation (55) modulo a suitable gauge transformation of the form $\Theta \rightarrow \Theta + f_{34}(y^3, y^4)$.

8. Two pairs of coinciding spectral parameters. Plebański’s first heavenly equation

Here, we consider the case of two pairs of coinciding parameters, say, $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$. This case can be dealt with in the same manner as before, leading to the parametrisation

$$\omega_{ik} = (\lambda_k - \lambda_i) \Theta_{y_i y_k}, \quad \omega_{12} = \omega_{12}(y^1, y^2), \quad \omega_{34} = \omega_{34}(y^3, y^4), \quad (83)$$

wherein $(i, k) \notin \{(1, 2), (2, 1), (3, 4), (4, 3)\}$. The Pfaffian condition (29) then becomes

$$f_{12} f_{34} + \Theta_{y_1 y_3} \Theta_{y_2 y_4} - \Theta_{y_3 y_1} \Theta_{y_2 y_4} = 0 \quad (84)$$

with

$$f_{12} = \frac{\omega_{12}(y^1, y^2)}{\lambda_1 - \lambda_3}, \quad f_{34} = \frac{\omega_{34}(y^3, y^4)}{\lambda_1 - \lambda_3}, \quad (85)$$

which, on application of a suitable coordinate transformation of the form $(y^1, y^2) \rightarrow f_{12}(y^1, y^2)$ and $(y^3, y^4) \rightarrow f_{34}(y^3, y^4)$, constitutes Plebański’s first heavenly equation corresponding to $f_{12} = f_{34} = 1$. Once again, it is remarked in passing that a confluence limit of the type

$$\Theta \rightarrow \Theta + g_{12}(y^1, y^2) + g_{34}(y^3, y^4) \quad \lambda_2 = \lambda_1 + \epsilon, \quad \lambda_4 = \lambda_3 + \epsilon, \quad \epsilon \rightarrow 0 \quad (86)$$

directly reduces the general heavenly equation (21) to the first heavenly equation in the ‘gauge-invariant’ form (84).

As in the previous section, one may now map any of the known heavenly equations to Plebański’s first heavenly equation. For instance, application of our formalism to the Husain–Park equation results in the ‘inverse’ of the transformation from the first heavenly equation to the Husain–Park equation (cf [16]) derived in the previous section. Here, we focus on the application to Plebański’s second heavenly equation and the general heavenly equation.

8.1. Application to Plebański’s second heavenly equation

The classical link [15, 23] between Plebański’s second heavenly equation

$$\Lambda_{x_1 x_3} + \Lambda_{x_2 x_4} + \Lambda_{y_1 y_4} \Lambda_{y_2 y_3} - \Lambda_{x_1 y_2}^2 = 0 \quad (87)$$

and the first heavenly equation may be formulated in terms of the two-form

$$\bar{\Omega} = (d x^1 - \Lambda_{x_1 x_2} d x^3 + \Lambda_{x_1 x_3} d x^4) \wedge (d x^2 + \Lambda_{x_2 x_3} d x^3 - \Lambda_{x_1 x_2} d x^4) \quad (88)$$

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which has the properties
\[ d\hat{\Omega} = 0, \quad \hat{\Omega} \wedge \hat{\Omega} = 0. \] (89)

Accordingly, Darboux’s theorem guarantees the existence of functions \( y^1 \) and \( y^2 \) such that
\[ \hat{\Omega} = dy^1 \wedge dy^2. \] (90)

Comparison with (88) shows that there exist expansions of the form
\[
\begin{align*}
dx^1 - \Lambda_{x^1 x^2} dx^3 + \Lambda_{x^1 x^2} dx^4 &= u_{13} dy^1 + u_{23} dy^2, \\
dx^2 + \Lambda_{x^1 x^2} dx^3 - \Lambda_{x^1 x^2} dx^4 &= u_{14} dy^1 + u_{24} dy^2
\end{align*}
\] (91)
for some functions \( u_{13}, u_{24}, u_{23} \) and \( u_{14} \) subject to
\[ u_{13}u_{24} - u_{23}u_{14} = 1. \] (92)

If we now regard \( y^1, y^2 \) and \( x^3, y^4 \) as independent variables then the expansions (91) imply that there exists a potential \( \Theta \) defined according to
\[ x^1 = \Theta_{y^1}, \quad x^2 = \Theta_{y^2}. \] (93)

Hence, (91) gives rise to the parametrisation
\[ u_{ik} = \Theta_{y_{ik}} \] (94)
which, in turn, reveals that the algebraic relation (92) encodes Plebański’s first heavenly equation
\[ \Theta_{y^1 y^2} \Theta_{y^3 y^4} - \Theta_{y^2 y^3} \Theta_{y^1 y^4} = 1. \] (95)

The connection with the present formalism is now uncovered by investigating the nature of the coordinates \( y^i \). Thus, if we solve (91) for \( dy^1 \) and \( dy^2 \) then we obtain
\[
\begin{align*}
dy^1 &= u_{23}(dx^4 - \Lambda_{x^1 x^2} dx^3 + \Lambda_{x^1 x^2} dx^3 - u_{23}(dx^4 + \Lambda_{x^1 x^2} dx^3 - \Lambda_{x^1 x^2} dx^4) - u_{23}(dx^4 + \Lambda_{x^1 x^2} dx^4 - \Lambda_{x^1 x^2} dx^4)) \\
dy^2 &= u_{13}(dx^2 + \Lambda_{x^1 x^2} dx^3 - \Lambda_{x^1 x^2} dx^4 - u_{14}(dx^1 - \Lambda_{x^1 x^2} dx^3 + \Lambda_{x^1 x^2} dx^4).
\end{align*}
\] (96)

The latter implies that
\[ u_{13} = x^3_{y^2}, \quad u_{24} = x^2_{y^1}, \quad u_{23} = -x^3_{y^1}, \quad u_{14} = -x^2_{y^1} \] (97)
so that (96) reduces to
\[
\begin{align*}
y^i_{y^3} &= \Lambda_{x^1 x^2} y^i_{y^2} - \Lambda_{x^1 x^2} y^i_{y^1}, \\
y^i_{y^4} &= \Lambda_{x^1 x^2} y^i_{y^2} - \Lambda_{x^1 x^2} y^i_{y^1}, \\
i &= 1, 2.
\end{align*}
\] (98)

Hence, comparison with the Lax pair [23]
\[
\begin{align*}
X(\lambda)\Phi &= 0, \\
Y(\lambda)\Phi &= 0,
\end{align*}
\] (99)
for the second heavenly equation (87) shows that the pairs \( y^1, y^2 \) and \( y^3, y^4 \) constitute eigenfunctions for \( \lambda = 0 \) and \( \lambda \to \infty \) respectively.
It is evident that our formalism is directly applicable even if one or two parameters $\lambda_i$ vanish or, due to the symmetry $\lambda \to \lambda^{-1}$, one or two parameters tend to infinity. If vanishing and infinite parameters are simultaneously present then the situation is more subtle but it is easy to verify that the formalism also applies \textit{mutatis mutandis} in this case. In the current situation, the second-order relations between the potential $\Theta$ and the original quantities associated with the second heavenly equation are obtained by eliminating $\theta_{ik}$ between (94) and (97). Accordingly, the link between the two heavenly equations presented in Plebański’s pioneering work [15] may be regarded as a particular application of the scheme presented here.

### 8.2. Application to the general heavenly equation

Since the connection between the general heavenly equation and the first Plebański equation constitutes the basis of section 11 which, in turn, justifies the reasoning employed in section 10, we now derive the differential relations between the potentials satisfying those two equations. Thus, in order to map the general heavenly equation (63) to the first heavenly equation, it is convenient to choose the eigenfunctions

$$\phi^1 = H_{x^3}, \quad \phi^2 = x^3, \quad \phi^3 = H_{x^4}, \quad \phi^4 = x^4$$

(100)

corresponding to the parameters

$$\lambda_1 = \lambda_2 = \mu_3, \quad \lambda_3 = \lambda_4 = \mu_4.$$  

(101)

Then, evaluation of (36) for the vector fields $A_2$ and $A_4$ given by (65) results in

$$\tilde{A}^{13} = \left( H_{x^3}x^4 - \frac{H_{x^3}x^4 H_{x^2}x^3}{H_{x^2}x^3} \right) \left( H_{x^3}x^4 - \frac{H_{x^3}x^3 H_{x^2}x^4}{H_{x^2}x^3} \right)$$

$$- \left( H_{x^3}x^3 - \frac{H_{x^3}x^3 H_{x^2}x^3}{H_{x^2}x^3} \right) \left( H_{x^3}x^4 - \frac{H_{x^3}x^4 H_{x^2}x^4}{H_{x^2}x^3} \right)$$

(102)

$$\tilde{A}^{41} = \left( H_{x^3}x^3 - \frac{H_{x^3}x^3 H_{x^2}x^3}{H_{x^2}x^3} \right), \quad \tilde{A}^{42} = 1$$

$$\tilde{A}^{32} = \left( H_{x^3}x^4 - \frac{H_{x^3}x^4 H_{x^2}x^4}{H_{x^2}x^3} \right).$$

By construction, the mixed derivatives

$$\Theta_{y^3 y^3} = \frac{1}{2} \frac{A^{42}}{\mu_4 - \mu_3}, \quad \Theta_{y^3 y^4} = \frac{1}{2} \frac{A^{31}}{\mu_4 - \mu_3}$$

$$\Theta_{y^3 y^3} = \frac{1}{2} \frac{A^{14}}{\mu_4 - \mu_3}, \quad \Theta_{y^3 y^4} = \frac{1}{2} \frac{A^{23}}{\mu_4 - \mu_3},$$

(103)

where

$$A^{ik} = \int f \tilde{A}^{ik}, \quad f = H_{x^3}x^2, \quad J = H_{x^2}x^3 H_{x^1}x^4 - H_{x^1}x^3 H_{x^2}x^4,$$

(104)

are compatible modulo the general heavenly equation (63) and the potential $\Theta$ obeys Plebański’s first heavenly equation in the form

$$\Theta_{y^3 y^3} \Theta_{y^3 y^4} - \Theta_{y^3 y^4} \Theta_{y^3 y^3} = \sigma, \quad \sigma = -\frac{1}{4} \frac{\mu_3 \mu_4}{(\mu_4 - \mu_3)^2}.$$

(105)
as required.

As indicated above, the relations (103) provide a key link in the remaining discussion. In order to motivate the content of sections 10 and 11, we first review an important known fact in the context of the general heavenly equation. Before we do so, we may summarise the key results of sections 4, 7 and 8 in the following table.

| # of pairs of coinciding parameters \( \lambda_i \) | Canonical form of the self-dual Einstein equations |
|-------------------------------------------------|--------------------------------------------------|
| 0                                               | General heavenly equation                         |
| 1                                               | Husain–Park equation                             |
| 2                                               | First heavenly equation                           |

9. The dispersionless Hirota equation: Einstein–Weyl geometry

The Jones–Tod procedure [28] provides a connection between four-dimensional spacetimes with anti-self-dual Weyl tensor and a conformal Killing vector and three-dimensional Einstein–Weyl geometries. In particular, in [20], it has been shown how the dispersionless Hirota equation (see, e.g. [18]) simultaneously gives rise to Einstein–Weyl geometries and a particular class of (anti-)self-dual Einstein spaces. Here, we discuss this observation in connection with the general heavenly equation with a view to the generalisation presented in section 10.

The metric of self-dual Einstein spaces governed by the general heavenly equation

\[
(\mu_3 - \mu_4)H_{x_1 x_2}H_{x_3 x_4} = \mu_3 H_{x_2 x_3}H_{x_1 x_4} - \mu_4 H_{x_1 x_3}H_{x_2 x_4}
\]

(106)

with associated Lax representation

\[
\Phi_{x^3} = \frac{1}{(\lambda - \mu_3)H_{x_1 x_2}}(\lambda H_{x_1 x_3} \Phi_{x^2} - \mu_3 H_{x_2 x_3} \Phi_{x_1})
\]

(107)

\[
\Phi_{x^4} = \frac{1}{(\lambda - \mu_4)H_{x_1 x_2}}(\lambda H_{x_1 x_4} \Phi_{x^2} - \mu_4 H_{x_2 x_4} \Phi_{x_1})
\]

is known to be given by [12]

\[
g = q^{-1} \left[ H_{x_1 x_2}H_{x_3 x_4}dx^1 dx^2 + H_{x_1 x_2}(H_{x_3 x_4} + H_{x_4 x_3})dx^1 dx^2 + \cdots \right]
\]

(108)

where \( g = f_{1234} \) and

\[
f_{iklm} = H_{x^i x^k}H_{x^l x^m} - H_{x^i x^l}H_{x^k x^m}
\]

(109)

for distinct indices \( i, k, l, m \). Indeed, one may directly verify that \( R_{ik} = 0 \) modulo the general heavenly equation (106). It is noted that there exist only three essentially different quantities \( f_{iklm} \) and their ratios are constant due to the structure of the general heavenly equation. Hence, up to an irrelevant constant factor, the metric (108) is completely symmetric in the indices.

We now single out a coordinate, say, \( x^4 \) and split the metric (108) into a ‘three-dimensional’ metric and a ‘perfect square’ according to

\[
g = -\frac{f_{1234}}{4} h + \frac{H_{x^1 x^2}H_{x^3 x^4}}{f_{1234}}(dx^4 + \eta)^2
\]

(110)
where
\[
\begin{align*}
h &= \frac{H_{\alpha^1 \alpha^4}}{H_{\alpha^2 \alpha^3} H_{\alpha^4 \alpha^6}} (dx^1)^2 + \alpha^2 \frac{H_{\alpha^2 \alpha^4}}{H_{\alpha^1 \alpha^6} H_{\alpha^3 \alpha^6}} (dx^2)^2 + \beta^2 \frac{H_{\alpha^3 \alpha^4}}{H_{\alpha^1 \alpha^6} H_{\alpha^2 \alpha^6}} (dx^3)^2 \\
&\quad + \frac{2\alpha}{H_{\alpha^2 \alpha^4}} dx^1 dx^2 + \frac{2\beta}{H_{\alpha^3 \alpha^4}} dx^1 dx^3 - \frac{2\alpha\beta}{H_{\alpha^1 \alpha^6}} dx^2 dx^3
\end{align*}
\]
(111)
with the constants
\[
\alpha = \frac{f_{1234}}{f_{1234}}, \quad \beta = \frac{f_{1432}}{f_{1234}} (112)
\]
and
\[
\begin{align*}
\eta &= \frac{1}{2} \left( \frac{H_{\alpha^1 \alpha^2}}{H_{\alpha^1 \alpha^6}} + \frac{H_{\alpha^1 \alpha^3}}{H_{\alpha^1 \alpha^6}} \right) dx^1 + \frac{1}{2} \left( \frac{H_{\alpha^2 \alpha^4}}{H_{\alpha^1 \alpha^6}} + \frac{H_{\alpha^3 \alpha^4}}{H_{\alpha^1 \alpha^6}} \right) dx^2 \\
&\quad + \frac{1}{2} \left( \frac{H_{\alpha^3 \alpha^2}}{H_{\alpha^1 \alpha^6}} + \frac{H_{\alpha^2 \alpha^3}}{H_{\alpha^1 \alpha^6}} \right) dx^3.
\end{align*}
\]
Furthermore, we consider the admissible reduction
\[
H = f(x^4) \omega(x^1, x^2, x^3)
\]
(114)
which specialises the general heavenly equation (106) to the dispersionless Hirota equation
\[
(\mu_3 - \mu_4) \omega^1 \omega^2 \omega^3 - \mu_4 \omega^2 \omega^1 \omega^3 + \mu_4 \omega^3 \omega^2 = 0.
\]
(115)
Since, the dependence of the metric (108) on \(x^4\) is now merely encoded in the overall factor \(f(x^4)\), it is evident that the reduction (114) leads to self-dual Einstein spaces admitting a homothetic Killing vector. Moreover, we obtain
\[
\begin{align*}
h &\sim \frac{\omega^1}{\omega^2 \omega^3} (dx^1)^2 + \alpha^2 \frac{\omega^2}{\omega^2 \omega^3} (dx^2)^2 + \beta^2 \frac{\omega^3}{\omega^2 \omega^3} (dx^3)^2 \\
&\quad + \frac{2\alpha}{\omega^3} dx^1 dx^2 + \frac{2\beta}{\omega^3} dx^1 dx^3 - \frac{2\alpha\beta}{\omega^3} dx^2 dx^3.
\end{align*}
\]
(116)
up to an irrelevant factor depending on \(x^4\), which is precisely the metric governing the Einstein–Weyl geometry associated with the dispersionless Hirota equation [20].

10. A dispersionless Hirota system. Self-dual Einstein spaces not admitting conformal Killing vectors

A non-trivial reduction of the general heavenly equation is obtained by matching its scaling symmetry \(H \partial H\) with the symmetry \(\Phi \partial H\). The fact that any eigenfunction of the general heavenly equation constitutes a symmetry of the general heavenly equation was first observed in [19] and extends to its 4 + 4-dimensional version (TED equation) [17] as further discussed in section 12. Thus, if we set \(\Phi = H\) then the Lax pair (107) gives rise to the pair of dispersionless Hirota equations
\[
\begin{align*}
(\lambda - \mu_3) H_{\alpha^3} H_{\alpha^1 \alpha^2} - \lambda H_{\alpha^2} H_{\alpha^1 \alpha^3} + \mu_3 H_{\alpha^1 \alpha^4} H_{\alpha^2 \alpha^3} &= 0 \\
(\lambda - \mu_4) H_{\alpha} H_{\alpha^1 \alpha^2} - \lambda H_{\alpha^2} H_{\alpha^1 \alpha^4} + \mu_4 H_{\alpha^1 \alpha^4} H_{\alpha^2 \alpha^4} &= 0
\end{align*}
\]
(117)
which is, by construction, compatible with the general heavenly equation (106). Moreover, the pair of dispersionless Hirota equations

\[
(\mu_3 - \mu_4) H_{x^1} H_{x^3} + (\lambda - \mu_3) H_{x^1} H_{x^4} - (\lambda - \mu_4) H_{x^2} H_{x^3} = 0
\]

\[
\lambda(\mu_3 - \mu_4) H_{x^2} H_{x^3} + \mu_4(\lambda - \mu_3) H_{x^2} H_{x^4} - \mu_3(\lambda - \mu_4) H_{x^1} H_{x^2} = 0
\]

is an algebraic consequence of the dispersionless Hirota equation (117) and the general heavenly equation. In fact, any three of the four dispersionless Hirota equations imply the fourth and the general heavenly equation. It is noted that the dispersionless Hirota system is completely symmetric in its indices if the symmetry in the parameters \(\mu_1 = \infty, \mu_2 = 0, \mu_3, \mu_4\) is restored by means of a suitable fractional linear transformation of the parameters, leading to the fully symmetric form (46) of the general heavenly equation. The compatibility of copies of dispersionless Hirota equations was first observed in [18] and is a direct consequence of the multi-dimensional consistency of the general heavenly equation [29] or, more generally, the 4 + 4-dimensional TED equation [17]. It is also emphasised that the above dispersionless Hirota system is invariant under \(H \to F(H)\). Even though this invariance may be proven directly, it is a consequence of the fact that any function of an eigenfunction of the general heavenly equation constitutes another eigenfunction.

**Remark 10.1.** If we impose the constraint (114) and make the choice \(\lambda = \mu_4\) then the dispersionless Hirota system (117) and (118) reduces to the dispersionless Hirota equation (115).

Hence, remarkably, the Einstein–Weyl geometry associated with the dispersionless Hirota equation is captured as a special case by the class of self-dual Einstein spaces governed by the eigenfunction symmetry reduction leading to the dispersionless Hirota system. In fact, the dispersionless Hirota system specialises to the dispersionless Hirota equation if any of the four constraints

\[
H = f(x^i) \omega(x^k, x^l, x^m), \quad \lambda = \mu_i,
\]

where the indices \(i, k, l, m\) are distinct, is imposed. Accordingly, the Einstein–Weyl geometry discussed in section 9 is encoded in four different ways in the self-dual Einstein spaces examined in this section so that, in this sense, the algebraic multi-dimensional consistency of the dispersionless Hirota equation has its geometric counterpart in the ‘multi-dimensional consistency’ of its associated Einstein–Weyl geometry. The exact nature of this geometric property is the subject of ongoing research.

### 10.1. The symmetry \(H \to F(H)\)

In the preceding, it has been demonstrated that the four-dimensional general heavenly equation admits a decomposition into four compatible three-dimensional dispersionless Hirota equations (of which only three are independent). It is therefore natural to inquire as to whether the solutions of the general heavenly equation obtained in this manner are genuinely four-dimensional in the sense that the corresponding self-dual Einstein spaces do not admit conformal Killing symmetries. Furthermore, it is desirable to show that the symmetry \(H \to F(H)\) really acts non-trivially on the metric. At first glance, this appears to be likely due to the appearance of the function \(F\) in the transformed metric. We begin by addressing the latter problem.

We first observe that

\[
H = z^1 + f(z^2),\quad z^1 = \sum_{i=1}^4 x^i,\quad z^2 = \sum_{i=1}^4 \alpha_i x^i
\]
constitutes a trivial solution of the general heavenly equation (106) and, in order for it to satisfy
the pair (117) of dispersionless Hirota equations, the constants \( \alpha_i \) and \( \mu_k \) must be related by
\[
\mu_k = \lambda \frac{\alpha_1 (\alpha_2 - \alpha_4)}{\alpha_2 (\alpha_1 - \alpha_2)}, \quad k = 3, 4.
\]
(121)

However, this solution does not correspond to a viable metric since \( f_{1234} = 0 \). This may be
rectified by boosting the solution using the symmetry \( H \rightarrow F(H) \) to obtain
\[
H = F(z^1 + f(z^2)).
\]
(122)

Indeed, the condition for non-vanishing \( f_{1234} \) is given by
\[
(\alpha_2 - \alpha_3)(\alpha_1 - \alpha_4)F''(u)F''(v) \neq 0.
\]

For example, the solution
\[
H = e^{z_1} \cosh z_2,
\]
(123)
corresponding to \( F(u) = e^u, f(v) = \ln \cosh v, \) falls into this category, provided that \( \alpha_2 \neq \alpha_3 \)
and \( \alpha_1 = \alpha_4 \) being excluded due to (121). This solution generates a flat metric if, for instance,
\( \alpha_1 = -\alpha_2 = 1 \). More generally, for this particular choice of the function \( f \) and the constants \( \alpha_i \)
but arbitrary function \( F \) so that \( H = F(z^1 + \ln \cosh z^2) \), the components of the Riemann
tensor which are not identically zero turn out to be proportional to
\[
R_{iklm} \sim \alpha_3 \alpha_4 \left( |F'(u)|^3 F''(u) F''''(u) - 3[F''(u)]^2 \right),
\]
(124)
where \( u = z^1 + \ln \cosh z^2 \). Thus, for a four-parameter family of functions \( F \), which includes
\( F(u) = e^u \), the metric is flat but, generically, the invariance \( H \rightarrow F(H) \) maps the flat metric
associated with the solution (123) to a non-flat metric. Here, we exclude the case \( \alpha_3 \alpha_4 = 0 \).
This proves that this invariance of the dispersionless Hirota system is non-trivial at the level of
the metric.

10.2. Non-existence of conformal Killing vectors

In this section, we prove the following theorem.

Theorem 10.1. The generic metric of self-dual Einstein spaces governed by the disper-
sionless Hirota system (117) and (118) does not admit conformal Killing vectors (including
homothetic Killing vectors and Killing vectors).

It turns out convenient to examine the above problem in the setting of Plebański’s first
heavenly equation
\[
\Theta_{y^1 y^3} \Theta_{y^2 y^4} - \Theta_{y^1 y^4} \Theta_{y^2 y^3} = 1
\]
(125)
with associated Lax pair
\[
\Phi_{y^3} = \lambda (\Theta_{y^1 y^3} \Phi_{y^2} - \Theta_{y^2 y^3} \Phi_{y^1}),
\]
\[
\Phi_{y^4} = \lambda (\Theta_{y^1 y^4} \Phi_{y^2} - \Theta_{y^2 y^4} \Phi_{y^1}).
\]
(126)

In section 8, it has been demonstrated how the current formalism may be used to establish
the link between the general and first heavenly equations. Extension of this link to the Lax
pairs is readily shown to lead to the above standard Lax pair with \( \lambda \) being related to \( \lambda \) by a
fractional linear transformation (cf (154)). In section 11, this link is exploited to reveal how
the decomposition of the general heavenly equation into the dispersionless Hirota system acts
on the first heavenly equation. Remarkably, this decomposition corresponds to the assumption that
\[ \Phi = \Theta - y^i \Theta_{,i} - y^4 \Theta_{,4} \]  
(127)
constitutes an eigenfunction of the first heavenly equation. Insertion of \( \Phi \) as given by (127) into the Lax pair (126) leads to two constraints on \( \Theta \) which are compatible with the Plebański equation (125). This is discussed in section 11 in more detail. In order to understand the nature of these constraints, we first briefly examine the symmetries of the first heavenly equation.

10.2.1. Symmetries of the first heavenly equation. Here, we consider symmetries of the first Plebański equation, that is, flows
\[ \Theta_s = \Delta \]  
(128)
which leave invariant the first heavenly equation (125). Thus, differentiation of the latter with respect to the symmetry parameter \( s \) produces the linear partial differential equation
\[ \Theta_{,4} \Delta_{,1} + \Theta_{,2} \Delta_{,3} - \Theta_{,1} \Delta_{,2} - \Theta_{,3} \Delta_{,4} = 0 \]  
(129)
for the function \( \Delta \). Any solution of this differential equation gives rise to a symmetry of the first heavenly equation. For instance, it is evident that the latter is invariant under the scaling \( \Theta \to \epsilon \Theta, y^i \to \epsilon^2 y^i \) for any fixed \( i \). This corresponds to
\[ \Delta = \Theta - 2y^i \Theta_{,i} \]  
(130)
One may directly verify that this is indeed a solution of the differential equation (129). Another symmetry is given by
\[ \Delta = \Phi \]  
(131)
which may be seen immediately by cross-differentiating the Lax pair (126), leading to
\[ \Theta_{,4} \Phi_{,1} + \Theta_{,2} \Phi_{,3} - \Theta_{,1} \Phi_{,2} - \Theta_{,3} \Phi_{,4} = 0, \]  
(132)
provided that \( \lambda \neq 0 \). This symmetry has been recorded in [30] in connection with the notion of partner symmetries and is also a direct consequence of the eigenfunction symmetry of the 4+4-dimensional TED equation. We conclude that the admissible constraint (127) corresponds to a symmetry reduction of the first heavenly equation associated with an appropriate linear combination of the symmetries (130) and (131).

Less obvious symmetries are provided by conservation laws associated with the first heavenly equation. Thus, the compatibility condition associated with the pair
\[ \phi^a_{,4} = \Theta_{,4} \Theta_{,4} - \Theta_{,1} \Theta_{,1} + 2y^2 \]  
\[ \phi^a_{,1} = \Theta_{,1} \Theta_{,4} - \Theta_{,3} \Theta_{,3} \]  
(133)
is precisely the first heavenly equation (125) so that the existence of the potential \( \phi^a \) is guaranteed. For reasons of symmetry, the existence of a potential \( \phi^b \) defined according to
\[ \phi^b_{,4} = \Theta_{,4} \Theta_{,4} - \Theta_{,1} \Theta_{,1} - 2y^2 \]  
\[ \phi^b_{,1} = \Theta_{,1} \Theta_{,4} - \Theta_{,3} \Theta_{,3} \]  
(134)
is likewise guaranteed. Differentiation of (133) and (134) with respect to \(y^3, y^4\) and \(y^1, y^2\) respectively then shows that the quantities

\[
\Delta^a = \phi^a, \quad \Delta^b = \phi^b
\]

satisfy the differential equation (129) and therefore constitute symmetries of the first heavenly equation.

### 10.2.2. The homothetic Killing equations.

We will now investigate the existence of homothetic Killing vectors of the class of metrics associated with solutions of the dispersionless Hirota system and then address the (non-)admittance of more general proper conformal Killing vectors. It is recalled (section 2) that, in terms of solutions of the first heavenly equation (125), the metric of self-dual Einstein spaces adopts the form

\[
g = 2\Theta_{1,2} \, dy^1 \, dy^3 + 2\Theta_{2,3} \, dy^2 \, dy^3 + 2\Theta_{3,1} \, dy^1 \, dy^4 + 2\Theta_{1,3} \, dy^2 \, dy^4.
\]

(136)

A vector field \(V^a\) constitutes a homothetic Killing vector field if its covariant representation \(\nabla_i V^a = \chi g_{ik}\)

wherein \(\nabla_i\) denotes the covariant derivative, the round brackets indicate the standard symmetrised sum and \(\chi\) is a constant. If the latter vanishes then \(V^a\) is a Killing vector giving rise to an isometry, while \(\chi \neq 0\) corresponds to a proper homothetic Killing vector. Based on a spinor formulation, it has been demonstrated in [31] that the above Killing equations may be completely resolved. In fact, in the present formulation, one may directly integrate this set of linear partial differential equations to obtain

\[
V_1 = c^a (\Theta_{1,2} \Theta_{2,3} - \Theta_{1,3} \Theta_{2,2}) - a(y^3, y^4) \Theta_{1,2} - b(y^3, y^4) \Theta_{1,3},
\]

\[
V_2 = c^b (\Theta_{1,3} \Theta_{2,3} - \Theta_{1,2} \Theta_{3,3}) - a(y^3, y^4) \Theta_{1,3} - b(y^3, y^4) \Theta_{1,2},
\]

\[
V_3 = c^c (\Theta_{2,1} \Theta_{1,3} - \Theta_{2,3} \Theta_{1,1}) + c(y^1, y^2) \Theta_{2,1} + d(y^1, y^2) \Theta_{2,3},
\]

\[
V_4 = c^d (\Theta_{1,2} \Theta_{3,3} - \Theta_{1,3} \Theta_{2,2}) + c(y^1, y^2) \Theta_{3,3} + d(y^1, y^2) \Theta_{3,1},
\]

(138)

where \(c^a, c^b\) are constants, so that the components of the homothetic Killing vector are given by

\[
V^1 = c^b \Theta_{1,2} + c(y^1, y^2), \quad V^2 = -c^b \Theta_{1,3} + d(y^3, y^4),
\]

\[
V^3 = c^c \Theta_{2,1} + c(y^1, y^2), \quad V^4 = -c^c \Theta_{3,1} - b(y^3, y^4).
\]

(139)

The Killing equations reduce to the ‘master equation’ (in the terminology of [31])

\[
\Delta^b = 0,
\]

(140)

where

\[
\Delta^b = c^a \phi^a + a(y^3, y^4) \Theta_{1,2} + b(y^3, y^4) \Theta_{1,3}
\]

\[
- c^b \phi^b - c(y^1, y^2) \Theta_{2,1} - d(y^1, y^2) \Theta_{2,3} + 2\chi \Theta
\]

(141)
and $\phi^a$ and $\phi^b$ are the potentials defined by (133) and (134) respectively. Moreover, the functions $a, b, c, d$ are constrained by the linear equation

$$a \psi + b \phi + 4 \chi - c \chi - d \varphi = 0.$$  \hspace{1cm} (142)

In fact, the latter constraint constitutes the trace $\nabla \psi V_\chi = 4 \chi$ of the Killing equation (137). It may be solved explicitly since the dependence of the pairs $a, b$ and $c, d$ on different variables implies the separation

$$a \psi + b \phi + 2 \chi - \mu = 0, \quad c \phi + d \varphi - 2 \chi - \mu = 0,$$  \hspace{1cm} (143)

where $\mu$ is an arbitrary constant. In summary, the first heavenly equation (125) admits a homothetic Killing vector if and only if it is constrained by the (non-local) condition (140).

It turns out that the constraint (140) is compatible with the first heavenly equation (125). For instance, if $c^a = c^b = 0$ then $\Delta^b = 0$ constitutes a first-order constraint which may be shown to lead to a three-dimensional reduction of the first heavenly equation. If $c^a c^b \neq 0$ then the necessary conditions $\Delta_{1,1,1}^b = \Delta_{1,1,2}^b = \Delta_{1,2,2}^b = \Delta_{1,2,4}^b = 0$ lead to four third-order differential constraints which are compatible with the first heavenly equation. Conversely, if those four constraints are satisfied then the functions of integration in the definitions (133) and (134) of the potentials $\phi^a$ and $\phi^b$ respectively may be chosen in such a manner that the non-local condition $\Delta^b = 0$ is satisfied. The reason for the compatibility of the master equation $\Delta^b = 0$ is readily revealed by examining the structure of $\Delta^b$ as given by (141). Indeed, the discussion in the previous subsection has revealed that $\phi^a$ and $\phi^b$ are symmetries of the first heavenly equation and if the functions $a, b, c, d$ are suitable multiples of $y^a, y^b, y^c, y^d$ respectively then $\Delta^b \equiv c^a c^b = 0$ also constitutes a symmetry. In fact, in general, the condition (142) is exactly the condition which guarantees that $\Delta^b$ represents a symmetry of the first heavenly equation, that is, $\Delta^b$ satisfies the symmetry condition (129), thereby justifying the notation $\Delta^b$. Accordingly, the master equation $\Delta^b = 0$ is nothing but a symmetry reduction of the first heavenly equation.

In order to address the question as to whether the decomposition of the general heavenly equation into the dispersionless Hirota system corresponds to the assumption of a homothetic Killing vector, it is now required to determine whether the symmetry constraint (127) on the first heavenly equation constitutes a special case of the symmetry constraint $\Delta^b = 0$. To this end, we first observe (as mentioned earlier, cf section 11.2) that insertion of $\Phi$ as given by (127) into the Lax pair (126) leads to two second-order constraints on the first heavenly equation. These constraints together with the first heavenly equation may then be formulated as a system of the type

$$\Theta_{\ell^1, \ell^2} = F, \quad \Theta_{\ell^1, \ell^3} = G, \quad \Theta_{\ell^2, \ell^3} = H, \quad F, G, H \in S.$$  \hspace{1cm} (144)

Here, the exact form of the functions $F, G$ and $H$ is not important. The key property of the above system is that these functions are contained in $S$ which denotes the set of functions depending on $\Theta, \Theta_{\ell^1}, \Theta_{\ell^2}$ and their derivatives of any order with respect to $y^2$ and $y^3$. The functions of this set may also depend explicitly on the independent variables $y^1$. By construction, the associated compatibility conditions are satisfied which, in turn, implies that, generically, the solution of the triple (144) is determined by the Cauchy data

$$\Theta = f_0(y^2, y^3), \quad \Theta_{\ell^1} = f_1(y^2, y^3), \quad \Theta_{\ell^2} = f_3(y^2, y^3) \quad \text{at} \quad (y^1, y^3) = (y^0, y^3_0).$$  \hspace{1cm} (145)
On the other hand, differentiation of \( \Delta^h \) and use of (133) and (134) evidently yields \( \Delta_{\lambda^2,\mu^4}^h \in S \). Hence, \( \Delta_{\lambda^2,\mu^4}^h = 0 \) evaluated at \((y^1, y^3) = (y^1_0, y^3_0)\) constitutes a differential constraint on the Cauchy data \( f_\ell(y^1_0, y^3_0) \). Accordingly, generically, the constrained Plebański system (144), which, as stated earlier, is equivalent to the dispersionless Hirota system, does not give rise to a homothetic Killing vector. Finally, it is well known \cite{32} that any metric satisfying Einstein’s vacuum equations \( R_{\ell} = 0 \) which admits a proper conformal Killing vector also possesses a Killing vector. It is recalled that \( V^\ell \) constitutes a proper conformal Killing vector if it satisfies Killing’s equations (137) for a non-constant function \( \chi \). This completes the proof of theorem 10.1.

11. Decomposition of the first heavenly equation

In the previous section, it has been demonstrated that the dispersionless Hirota system (117) and (118) gives rise to self-dual Einstein spaces which, generically, do not admit any conformal Killing vectors. Since the dispersionless Hirota system has been obtained as a symmetry reduction of the general heavenly equation, it should be investigated whether the corresponding symmetry constraint may be formulated in an invariant manner so that it may be applied to any of the known heavenly equations. The results of this investigation will be presented elsewhere. Here, we briefly present the action of this symmetry constraint on the first heavenly equation since it has been exploited in the previous section.

11.1. A first integral

In section 8.2, it has been shown that the potentials \( H \) and \( \Theta \) obeying the general heavenly equation (63) and the first heavenly equation (105) respectively are linked by the second-order relations (102)–(104). It turns out that an appropriate extension of these relations admits a first integral if the solutions of the general heavenly equation are restricted to the class of solutions satisfying the dispersionless Hirota system

\[
\begin{align*}
(\lambda - \mu_3)H_3^xH_{1,4}^x - \lambda H_3^xH_{1,4}^x + \mu_3 H_3^xH_{2,4}^x &= 0 \\
(\lambda - \mu_4)H_4^xH_{1,2}^x - \lambda H_4^xH_{1,2}^x + \mu_4 H_4^xH_{2,4}^x &= 0 \\
(\mu_3 - \mu_4)H_3^xH_{3,4}^x + (\lambda - \mu_3)H_3^xH_{1,4}^x - (\lambda - \mu_4)H_4^xH_{3,4}^x &= 0 \\
\lambda(\mu_3 - \mu_4)H_3^xH_{3,4}^x + \mu_4(\lambda - \mu_3)H_3^xH_{3,4}^x - \mu_3(\lambda - \mu_4)H_4^xH_{3,4}^x &= 0.
\end{align*}
\]

(146)

Specifically, if we differentiate the ansatz

\[
\kappa H = \Theta - y^1\Theta_{y^1} - y^3\Theta_{y^3}
\]

with respect to \( y^i \), where \( \kappa \) is a constant, then we obtain the additional second derivatives

\[
\begin{align*}
\Theta_{y^1y^1} &= -\frac{\kappa H_{y^1} + y^1\Theta_{y^1y^1}}{y^1}, & \Theta_{y^2y^2} &= \frac{\Theta_{y^2} - \kappa H_{y^2} - y^3\Theta_{y^3y^3}}{y^3} \\
\Theta_{y^3y^3} &= -\frac{\kappa H_{y^3} + y^1\Theta_{y^1y^3}}{y^3}, & \Theta_{y^4y^4} &= \frac{\Theta_{y^4} - \kappa H_{y^4} - y^1\Theta_{y^1y^4}}{y^3}.
\end{align*}
\]

(147)

It is evident that all terms of the right-hand sides of these relations except for \( \Theta_{y^2} \) and \( \Theta_{y^4} \) may be expressed entirely in terms of the potential \( H \) and the associated independent variables \( x^i \).
One may now directly verify (using computer algebra) that these relations are compatible with (103) modulo the dispersionless Hirota system (146) provided that

\[ \kappa = -\frac{1}{2} \frac{\lambda}{(\lambda - \mu_3)(\lambda - \mu_4)}. \]  

(149)

Hence, taking into account the invariance \( H \to H + \text{const} \) of the general heavenly equation, the ansatz (147) is equivalent to the system (148) and may be interpreted as a first integral of the extended system (103) and (148).

11.2. Decomposition

In order to motivate the first-order link (147) between the potentials \( H \) and \( \Theta \) in the case of the restricted class of solutions of the general heavenly equation governed by the dispersionless Hirota system, we note that since the latter is a result of the imposition of a symmetry constraint involving the eigenfunction and a scaling symmetry of the general heavenly equation, one expects to find a similar result if one formulates this symmetry constraint in terms of Plebański’s first heavenly equation. In the previous section, we have demonstrated that the symmetry constraint

\[ \Phi = \Theta - y^1\Theta_{y^1} - y^3\Theta_{y^3}, \]  

(150)

on the first heavenly equation is admissible, where \( \Phi \) obeys the Lax pair

\[ \Phi_{y^1} = \bar{\lambda}(\Theta_{y^1,y^3}\Phi_{y^2} - \Theta_{y^2,y^3}\Phi_{y^1}), \]  
\[ \Phi_{y^3} = \bar{\lambda}(\Theta_{y^3,y^3}\Phi_{y^2} - \Theta_{y^2,y^3}\Phi_{y^1}). \]  

(151)

for the first heavenly equation which, in the current context, is of the form (105). Now, taking into account that \( H \) constitutes an eigenfunction if the general heavenly equation is specialised to the dispersionless Hirota system, the identification of the two eigenfunctions \( \Phi \) and \( \kappa H \) leads to the ansatz (147).

Insertion of \( \Phi \) as given by (150) into the Lax pair (151) leads to

\[ \Theta_{y^1,y^1} = \frac{(y^1\Theta_{y^1,y^2} - \Theta_{y^2,y^1})\Theta_{y^1,y^4} - (y^1\Theta_{y^1,y^4} + y^3\Theta_{y^3,y^4} - \Theta_{y^2,y^4})\bar{\lambda}^{-1} - \sigma y^3}{y^1\Theta_{y^1,y^4}}, \]  
\[ \Theta_{y^2,y^3} = \frac{(y^3\Theta_{y^3,y^4} - \Theta_{y^4,y^3})\Theta_{y^2,y^4} + (y^3\Theta_{y^3,y^4} + y^3\Theta_{y^3,y^3} - \Theta_{y^4,y^3})\sigma \bar{\lambda} - \sigma y^3}{y^3\Theta_{y^2,y^4}}. \]  

(152)

where we have exploited the Plebański equation (105) formulated as

\[ \Theta_{y^1,y^1} = \frac{\Theta_{y^1,y^1}\Theta_{y^1,y^4} + \sigma}{\Theta_{y^2,y^4}}. \]  

(153)

By construction, the system (152) and (153) is compatible. Finally, it may be verified that this system is satisfied if \( \Theta \) is related to the potential \( H \) via the system (103) and (148) provided that

\[ \bar{\lambda} = \frac{2(\mu_3 - \mu_4)(\lambda - \mu_3)}{\mu_3(\lambda - \mu_4)}. \]  

(154)
Hence, the decomposition (152) and (153) of Plebański’s first heavenly equation (105) constitutes an incarnation of the decomposition of the general heavenly equation into the dispersionless Hirota system (146). It is noted that (154) may also be formulated as

\[-(\sigma \lambda)^{-1} = 2 \frac{(\mu_3 - \mu_4)^2}{\mu_4(\lambda - \mu_3)}\]  

which highlights the symmetry of the pair (152).

12. Partial Legendre transformations. The TED equation

In section 6, it has been demonstrated that the general heavenly equation is invariant under a Legendre transformation. The analysis in the previous section naturally leads to the consideration of partial Legendre transformations applied to Plebański’s first heavenly equation and, by extension, to the Husain–Park equation.

12.1. A partial Legendre transform of the first heavenly equation

The constraint (150) suggests considering the partial Legendre transformation

\[(\Theta; y^1, y^2, y^3, y^4) \rightarrow (\bar{\Theta}; y_1, y_2, y_3, y^4),\] (156)

where

\[\bar{\Theta} = \Theta - y^i \Theta_{ji} - y^j \Theta_{ji}, \quad y_i = \Theta_{ji},\] (157)

so that it is natural to examine the first heavenly equation (105) in the form

\[dy_3 \wedge dy_4 \wedge dy^3 \wedge dy^4 = \sigma dy^1 \wedge dy^2 \wedge dy^3 \wedge dy^4.\] (158)

In terms of the new potential \(\bar{\Theta}\), the definition of the variables \(y_i\) formulated as

\[d\Theta = y_i dy^i\] (159)

implies that

\[y^1 = -\bar{\Theta}_{y_1}, \quad y_2 = \bar{\Theta}_{y_2}, \quad y^3 = -\bar{\Theta}_{y_3}, \quad y_4 = \bar{\Theta}_{y_4},\] (160)

leading to the partial Legendre transform

\[\bar{\Theta}_{y_1} \bar{\Theta}_{y_2} y^4 - \bar{\Theta}_{y_3} y^4 = \sigma \left( \bar{\Theta}_{y_1} \bar{\Theta}_{y_3} - \bar{\Theta}_{y_1}^2 \right)\] (161)

of the first heavenly equation. This form of the self-dual Einstein equations together with its Legendre-type connection with Plebański’s first heavenly equation has been recorded in [30]. In the current context, the constraint (150) shows that the counterpart of the decomposition of the general heavenly equation into the dispersionless Hirota system is the decomposition of the heavenly equation (161) into a system analogous to the system (152) and (153) which is generated by matching the eigenfunction and the potential \(\Theta\), that is,

\[\Phi = \bar{\Theta}\] (162)

as in the case of the general heavenly equation.
12.2. Connection with the $4 + 4$-dimensional TED equation

It turns out that the above observation is not a coincidence. As indicated in the preceding, the TED equation constitutes a $4 + 4$-dimensional integrable generalisation of the general heavenly equation and, in fact, exists in $2n + 2n$ dimensions [17]. The TED equation

$$
(\Theta y_{1,2} - \Theta y_{1,3})(\Theta y_{3,4} - \Theta y_{2,3})
+ (\Theta y_{2,3} - \Theta y_{3,4})(\Theta y_{4,1} - \Theta y_{1,4})
+ (\Theta y_{3,4} - \Theta y_{4,1})(\Theta y_{1,2} - \Theta y_{2,1}) = 0
$$

(163)

is multi-dimensionally consistent [17] and encodes many (if not all) known heavenly equations and also, for instance, the six-dimensional second heavenly equation (see, e.g. [23] and references therein). In particular, the travelling wave reduction

$$
\Theta z_i = \lambda_i \Theta y_i
$$

(164)

leads to the general heavenly equation

$$(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)\Theta y_{1,2}\Theta y_{3,4}
+ (\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4)\Theta y_{2,3}\Theta y_{4,1}
+ (\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3)\Theta y_{3,4}\Theta y_{1,2} = 0.
$$

(165)

Moreover, it has been shown that any eigenfunction $\Phi$ obeying the associated Lax pair constitutes a symmetry of the TED equation so that matching this eigenfunction symmetry with the scaling symmetry of the TED equation encapsulated in the constraint $\Phi = \Theta$ leads to a higher-dimensional integrable extension of the dispersionless Hirota system consisting of four compatible $3 + 3$-dimensional generalised dispersionless Hirota equations, namely

$$
(\Theta y_{1,2} - \Theta y_{1,3})(\Theta y_{2,4} - \lambda \Theta y_{3,4})
+ (\Theta y_{2,3} - \Theta y_{2,4})(\Theta y_{3,1} - \lambda \Theta y_{4,1})
+ (\Theta y_{3,4} - \Theta y_{3,1})(\Theta y_{1,2} - \lambda \Theta y_{2,1}) = 0.
$$

(166)

Here, the indices $i, k, l \in \{1, 2, 3, 4\}$ are distinct. Indeed, in the travelling wave reduction (164), the fully symmetric avatar of the dispersionless Hirota system is obtained. One may also directly verify that the TED equation (163) is an algebraic consequence of the generalised dispersionless Hirota system (166).

Another travelling wave reduction of the TED equation generated by

$$
\Theta y_1 = \lambda_1 \Theta y_1, \quad \Theta y_2 = \lambda_1 \Theta y_2 + \nu_2 \Theta y_3,
\Theta y_3 = \lambda_2 \Theta y_3, \quad \Theta y_4 = \lambda_3 \Theta y_4 + \nu_4 \Theta y_3
$$

(167)

reads

$$(\lambda_3 - \lambda_1)^2(\Theta y_{1,3}\Theta y_{2,4} - \Theta y_{2,3}\Theta y_{1,4}) = \nu_2 \nu_4(\Theta y_{1,2}\Theta y_{3,4} - \Theta y_{3,2}\Theta y_{1,4})
$$

(168)

which is exactly of the form (161) (with a slightly different labelling of the dependent and independent variables). Its decomposition into a system of compatible equations via the symmetry
constraint $\Phi = \Theta$ is obtained by imposing the travelling wave constraints (167) on the generalised dispersionless Hirota system (166). This explains why the heavenly equation (161) admits the (symmetry) constraint (162). It is important to note that the travelling wave constraints (167) constitute an extension of the travelling wave constraints (164) subject to the choice $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$. The latter is precisely the specialisation employed in section 8 which has led to Plebani’s first heavenly equation. In fact, this suggests that one should also consider the ‘intermediate’ case given by

\[
\begin{align*}
\Theta_{z1} &= \lambda_1 \Theta_{y1}, \\
\Theta_{z2} &= \lambda_1 \Theta_{y2} + \nu_2 \Theta_{y1}, \\
\Theta_{z3} &= \lambda_3 \Theta_{y3}, \\
\Theta_{z4} &= \lambda_4 \Theta_{y4}
\end{align*}
\]  

(169)

and corresponding to the choice $\lambda_1 = \lambda_2$ in the above-mentioned sense, leading to the reduction

\[
\Theta_{y1} \Theta_{y34} - \Theta_{y2} \Theta_{y134} = \tilde{\sigma}(\Theta_{y1} \Theta_{y3} - \Theta_{y2} \Theta_{y4})
\]

(170)

of the TED equation. Indeed, application of the partial Legendre transformation

\[
(\Theta; y^1, y^2, y^3, y^4) \rightarrow (\tilde{\Theta}; y_1, y^2, y^3, y^4),
\]

(171)

with

\[
\tilde{\Theta} = \Theta - y^i \Theta_{yi}, \quad d\Theta = y_i \, dy^i
\]

(172)

is readily shown to produce the Husain–Park equation

\[
\tilde{\sigma}(\tilde{\Theta}_{y134} \tilde{\Theta}_{y2} - \tilde{\Theta}_{y234} \tilde{\Theta}_{y1}).
\]

(173)

Hence, we conclude that the three sets of travelling wave constraints (164), (167) and (169) of the TED equation correspond to the three canonical choices of the parameters $\lambda_i$ in the current formalism with the associated reductions (165), (168) and (170) being linked to the general heavenly equation, Husain–Park equation and first heavenly equation by the respective (partial) Legendre transformation.

In summary, a link between travelling wave reductions of the TED equation and the classification within the present formalism with respect to the number of pairs of coinciding spectral parameters has been established. This highlights, once again, the significance of the TED equation. In this connection, it is interesting to recall [17] that, as mentioned at the beginning of this section, the Husain–Park and first heavenly equations may also be obtained directly from the TED equation by imposing appropriate constraints.

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