STABILITY RESULTS OF A SINGULAR LOCAL INTERACTION ELASTIC/VISCOELASTIC COUPLED WAVE EQUATIONS WITH TIME DELAY

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Abstract. The purpose of this paper is to investigate the stabilization of a locally coupled wave equations with non smooth localized viscoelastic damping of Kelvin-Voigt type and localized time delay. Using a general criteria of Arendt-Batty, we show the strong stability of our system in the absence of the compactness of the resolvent. Finally, using frequency domain approach combined with the multiplier method, we prove a polynomial energy decay rate of order $t^{-1}$.

1. Introduction.

1.1. Description of the paper. In this paper, we investigate the stability of coupled wave equations with singular localized viscoelastic damping of Kelvin-Voigt type and localized time delay. More precisely, we consider the following system:

$$
\begin{align*}
&u_{tt} - [au_x + b(x)(\kappa_1 u_{tx} + \kappa_2 u_{tx}(x, t-\tau))]_x + c(x)yt = 0, \\
y_{tt} - y_{xx} - c(x)u_t = 0, \\
u(0, t) = u(L, t) = y(0, t) = y(L, t) = 0, \\
(u(x, 0), u_t(x, 0)) = (u_0(x), u_1(x)), \\
y(x, 0), y_t(x, 0) = (y_0(x), y_1(x)), \\
u_t(x, t) = f_0(x, t),
\end{align*}
$$

(1.1)

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where \( L, \tau, a \) and \( \kappa_1 \) are positive real numbers, \( \kappa_2 \) is a non-zero real number and \((u_0, u_1, y_0, y_1, f_0)\) belongs to a suitable space. We suppose that there exists \( 0 < \alpha < \beta < \gamma < L \) and a non-zero constant \( c_0 \), such that

\[
b(x) = \begin{cases} 
1, & x \in (0, \beta), \\
0, & x \in (\beta, L),
\end{cases} \quad \text{and} \quad 
c(x) = \begin{cases} 
c_0, & x \in (\alpha, \gamma), \\
0, & x \in (0, \alpha) \cup (\gamma, L).
\end{cases}
\]

The Figure 1 describes system (1.1)

Figure 1. Local Kelvin-Voigt damping and local time delay feedback.

System (1.1) consists of two wave equations with only one singular viscoelastic damping acting on the first equation, the second one is indirectly damped via a singular coupling between the two equations. The notion of indirect damping mechanisms has been introduced by Russell in [48] and since then, it has attracted the attention of many authors (see for instance [4, 5, 6, 9, 16, 1, 36, 53]). The study of such systems is also motivated by several physical considerations like Timoshenko and Bresse systems (see for instance [2, 3, 39, 41]). In fact, there are few results concerning the stability of coupled wave equations with local Kelvin-Voigt damping without time delay, especially in the absence of smoothness of the damping and coupling coefficients (see Subsection 1.2.1). The last motivates our interest to study the stabilization of system (1.1) in the present paper.

1.2. Previous literature. The wave is created when a vibrating source disturbs the medium. In order to restrain those vibrations, several damping can be added such as Kelvin-Voigt damping which is originated from the extension or compression of the vibrating particles. This damping is a viscoelastic structure having properties of both elasticity and viscosity. In the recent years, many researchers showed interest in problems involving this kind of damping where different types of stability, depend on the smoothness of the damping coefficients, has been showed (see [7, 8, 26, 27, 30, 34, 37, 44, 47]). However, time delays have been used in several applications such as in physical, chemical, biological, thermal phenomenas not only depend on the present state but also on some past occurrences (see [23, 32]) . In the last years, the control of partial differential equations with time delays have become popular among scientists. In many cases the time delay induce some instabilities see [17, 19, 20, 22]. However, let us recall briefly some systems of wave equations with Kelvin-Voigt damping and time delay represented in previous literature.
1.2.1. Coupled wave equations with Kelvin-Voigt damping and without time delay. In 2020, Hayek et al. in [29] studied the stabilization of a multi-dimensional system of weakly coupled wave equations with one or two locally Kelvin-Voigt damping and non-smooth coefficient at the interface. They established different stability results. In 2021, Hassine and Souayeh in [28] studied the behavior of a system with coupled wave equations with a partial Kelvin-Voigt damping, by considering the following system:

\[
\begin{align*}
    u_{tt} - (u_x + b_2(x)u_{tx})_x + v_t &= 0, & (x, t) &\in (-1, 1) \times (0, \infty), \\
    v_{tt} - cv_{xx} - u_t &= 0, & (x, t) &\in (-1, 1) \times (0, \infty), \\
    u(0, t) &= v(0, t) = 0, u(1, t) = v(1, t) = 0, & t &> 0, \\
    u(x, 0) &= u_0(x), u_t(x, 0) = u_1(x), & x &\in (-1, 1), \\
    v(x, 0) &= v_0(x), v_t(x, 0) = v_1(x), & x &\in (-1, 1),
\end{align*}
\]

where \(c > 0\) and \(b_2 \in L^\infty(-1, 1)\) is a non-negative function. They assumed that the damping coefficient is piecewise function in particular they supposed that \(b_2(x) = dI_{[0,1]}(x)\), where \(d\) is a strictly positive constant. So, they took the damping coefficient to be near the boundary with a global coupling coefficient. They showed the lack of exponential stability and that the semigroup loses speed and it decays polynomially with a rate as \(t^{-\lambda}\). In 2021, Akil, Issa and Wehbe in [50] studied the localized coupled wave equations, by considering the following system:

\[
\begin{align*}
    u_{tt} - (au_x + b(x)u_{tx})_x + c(x)y_t &= 0, & (x, t) &\in (0, L) \times (0, \infty), \\
    y_{tt} - y_{xx} - c(x)u_t &= 0, & (x, t) &\in (0, L) \times (0, \infty), \\
    u(0, t) &= u(L, t) = y(0, t) = y(L, t) = 0, & t &> 0, \\
    (u(x, 0), u_t(x, 0)) &= (u_0(x), u_1(x)), & x &\in (0, L), \\
    (y(x, 0), y_t(x, 0)) &= (y_0(x), y_1(x)), & x &\in (0, L),
\end{align*}
\]

where

\[
    b(x) = \begin{cases} 
    b_0, & x \in (\alpha_1, \alpha_3), \\
    0, & \text{otherwise}
\end{cases} \quad \text{and} \quad c(x) = \begin{cases} 
    c_0, & x \in (\alpha_2, \alpha_4), \\
    0, & \text{otherwise}
\end{cases}
\]

where \(a > 0\), \(b_0 > 0\), \(c_0 > 0\) and \(0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < L\). They generalized the results of Hassine and Souayeh in [28] by establishing a polynomial decay rate of type \(t^{-\lambda}\).

1.2.2. Wave equations with time delay and without Kelvin-Voigt damping. The delay equations of hyperbolic type is given by

\[
    u_{tt} - \Delta u(x, t - \tau) = 0.
\]
at the boundary
\[
\begin{aligned}
&u_t(x, t) - \Delta u(x, t) = 0, \quad (x, t) \in \Omega \times (0, \infty), \\
u(x, t) = 0, \quad (x, t) \in \Gamma_D \times (0, \infty), \\
\frac{\partial u}{\partial \nu}(x, t) = -\mu_1 u_t(x, t) - \mu_2 u_t(x, t - \tau), \quad (x, t) \in \Gamma_N \times (0, \infty), \\
u(x, t) = (u_0(x), u_1(x)), \quad x \in \Omega, \\
u_t(x, t) = f_0(x, t), \quad (x, t) \in \Gamma_N \times (-\tau, 0).
\end{aligned}
\]

The second case concerns a wave equation with an internal feedback and a delayed velocity term (i.e. an internal delay) and a mixed Dirichlet-Neumann boundary condition
\[
\begin{aligned}
&u_{tt} - \Delta u + \mu_1 u_t + \mu_2 u_t(x, t - \tau) = 0, \quad (x, t) \in \Omega \times (0, \infty), \\
u(x, t) = 0, \quad (x, t) \in \Gamma_D \times (0, \infty), \\
\frac{\partial u}{\partial \nu}(x, t) = 0, \quad (x, t) \in \Gamma_N \times (0, \infty), \\
u(x, t) = (u_0(x), u_1(x)), \quad x \in \Omega, \\
u_t(x, t) = f_0(x, t), \quad (x, t) \in \Omega \times (-\tau, 0),
\end{aligned}
\]

where \( \Omega \) is an open bounded domain of \( \mathbb{R}^N \) with a boundary \( \Gamma \) of class \( C^2 \) and \( \Gamma = \Gamma_D \cup \Gamma_N \), such that \( \Gamma_D \cap \Gamma_N = \emptyset \). Under the assumption \( \mu_2 < \mu_1 \), an exponential decay achieved for the both systems (1.4)-(1.5). If this assumption does not hold, they found a sequences of delays \( \{\tau_k\}_k, \tau_k \to 0 \), for which the corresponding solutions have increasing energy. Furthermore, we refer to [14] for the Problem (1.5) in more general abstract setting. In 2010, Ammari et al. (see [10]) studied the wave equation with interior delay damping and dissipative undelayed boundary condition in an open domain \( \Omega \) of \( \mathbb{R}^N, N \geq 2 \). The system is described by:
\[
\begin{aligned}
&u_{tt} - \Delta u + \mu_1 u_t + \mu_2 u_t(x, t - \tau) = 0, \quad (x, t) \in \Omega \times (0, \infty), \\
u(x, t) = 0, \quad (x, t) \in \Gamma_0 \times (0, \infty), \\
\frac{\partial u}{\partial \nu}(x, t) = 0, \quad (x, t) \in \Gamma_1 \times (0, \infty), \\
u(x, t) = (u_0(x), u_1(x)), \quad x \in \Omega, \\
u_t(x, t) = f_0(x, t), \quad (x, t) \in \Omega \times (-\tau, 0),
\end{aligned}
\]

where \( \tau > 0, a > 0 \) and \( \kappa > 0 \). Under the condition that \( \Gamma_1 \) satisfies the \( \Gamma \)-condition introduced in [33], they proved that system (1.6) is uniformly asymptotically stable whenever the delay coefficient is sufficiently small.

In 2012, Pignotti in [46] considered the wave equation with internal distributed time delay and local damping in a bounded and smooth domain \( \Omega \subset \mathbb{R}^N, N \geq 1 \). The considered system is given by the following:
\[
\begin{aligned}
&u_{tt} - \Delta u + a\chi_\omega u_t + \kappa u_t(x, t - \tau) = 0, \quad (x, t) \in \Omega \times (0, \infty), \\
u(x, t) = 0, \quad (x, t) \in \Gamma \times (0, \infty), \\
u(x, t) = (u_0(x), u_1(x)), \quad x \in \Omega, \\
u_t(x, t) = f_0(x, t), \quad (x, t) \in \Omega \times (-\tau, 0),
\end{aligned}
\]

where \( \kappa \in \mathbb{R}, \tau > 0, a > 0 \) and \( \omega \) is the intersection between an open neighborhood of the set \( \Gamma_0 = \{ x \in \Gamma : (x - x_0) \cdot \nu(x) > 0 \} \) and \( \Omega \). Moreover, \( \chi_\omega \) is the characteristic function of \( \omega \). We remark that the damping is localized and it acts on a neighborhood of a part of \( \Omega \). She showed an exponential stability results if the coefficients of the delay terms satisfy the following assumption \( |\kappa| < \kappa_0 < a \).
Several researches were done on wave equation with time delay acting on the boundary see ([20, 18, 52, 25, 24, 49, 51]) and different type of stability has been proved.

1.2.3. Wave equations with Kelvin-Voigt damping and time delay. In 2016, Messaoudi et al. in [40] considered the stabilization of the following wave equation with strong time delay

\[
\begin{cases}
  u_{tt} - \Delta u - \mu_1 \Delta u_t - \mu_2 \Delta u_t(x,t-\tau) = 0, & (x,t) \in \Omega \times (0,\infty), \\
  u(x,t) = 0, & (x,t) \in \Gamma \times (0,\infty), \\
  (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), & x \in \Omega, \\
  u_t(x,t) = f_0(x,t), & (x,t) \in \Omega \times (-\tau,0),
\end{cases}
\]

where \( \mu_1 > 0 \) and \( \mu_2 \) is a non zero real number. Under the assumption that \( |\mu_2| < \mu_1 \), they obtained an exponential stability result. In 2015, Nicaise et al. in [43] studied the multidimensional wave equation with localized Kelvin-Voigt damping and mixed boundary condition with time delay

\[
\begin{cases}
  u_{tt}(x,t) - \Delta u(x,t) - \text{div}(a(x)\nabla u_t) = 0, & (x,t) \in \Omega \times (0,\infty), \\
  u(x,t) = 0, & (x,t) \in \Gamma_0 \times (0,\infty), \\
  \frac{\partial u}{\partial \nu}(x,t) = -a(x) \frac{\partial u_t}{\partial \nu}(x,t) - \kappa u_t(x,t-\tau), & (x,t) \in \Gamma_1 \times (0,\infty), \\
  (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), & x \in \Omega, \\
  u_t(x,t) = f_0(x,t), & (x,t) \in \Gamma_1 \times (-\tau,0),
\end{cases}
\]

Under appropriate conditions on the coefficients, a global exponential decay rate is obtained. In 2015, Ammari and al. in [11] considered the stabilization problem for an abstract equation with delay and a Kelvin-Voigt damping. The system is given by the following:

\[
\begin{cases}
  u_{tt} - c_1 \Delta u - c_2 \Delta u_t(x,t-\tau) - d_1 \Delta u_t - d_2 \Delta u_t(x,t-\tau) = 0, & (x,t) \in \Omega \times (0,\infty), \\
  u(x,t) = 0, & (x,t) \in \Gamma_0 \times (0,\infty), \\
  \frac{\partial u}{\partial \nu}(x,t) = 0, & (x,t) \in \Gamma_1 \times (0,\infty), \\
  (u(x,0), u_t(x,0)) = (u_0(x), u_1(x)), & x \in \Omega, \\
  u(x,t) = f_0(x,t), & (x,t) \in \Omega \times (-\tau,0).
\end{cases}
\]

Thus, to the best of our knowledge, it seems to us that there is no result in the existing literature concerning the case of coupled wave equations with localized Kelvin-Voigt damping and localized time delay, especially in the absence of smoothness of the damping and coupling coefficients. The goal of the present paper is to fill this gap by studying the stability of system (1.1).
This paper is organized as follows: In Section 2, we prove the well-posedness of our system by using semigroup approach. In Section 3, by using a general criteria of Arendt-Batty, we show the strong stability of our system in the absence of the compactness of the resolvent. Next, in Section 4, by using frequency domain approach combining with a specific multiplier method, we prove a polynomial energy decay rate of order \( t^{-1} \).

2. Well-posedness of the system. In this section, we will establish the well-posedness of system (1.1) by using semigroup approach. For this aim, as in [42], we introduce the following auxiliary change of variable

\[
\eta(x, \rho, t) := u_t(x, t - \rho \tau), \quad x \in (0, \beta), \quad \rho \in (0, 1), \quad t > 0. \tag{2.1}
\]

Then, system (1.1) becomes

\[
\begin{aligned}
&u_{tt} - (S_b(u, u_t, \eta))_x + c(x)u_t = 0, \quad (x, t) \in (0, L) \times (0, \infty), \\
y_{tt} - y_{xx} - c(x)u_t = 0, \quad (x, t) \in (0, L) \times (0, \infty), \\
\tau \eta_t (x, \rho, t) + \eta(x, \rho, t) = 0, \quad (x, \rho, t) \in (0, \beta) \times (0, 1) \times (0, \infty),
\end{aligned} \tag{2.2}
\]

where \( S_b(u, u_t, \eta) := au_x + b(x)(\kappa_1 u_{tx} + \kappa_2 u_{tx} (x, t - \tau)) \). Moreover, from the definition of \( b(\cdot) \), we have

\[
S_b(u, u_t, \eta) = \begin{cases}
S_1(u, u_t, \eta) := au_x + \kappa_1 u_{tx} + \kappa_2 \eta_t (\cdot, 1, t), & x \in (0, \beta), \\
au_x, & x \in (\beta, L).
\end{cases} \tag{2.5}
\]

With the following boundary conditions

\[
\begin{cases}
u(0, t) = u(L, t) = y(0, t) = y(L, t) = 0, & t > 0, \\
\eta(0, \rho, t) = 0, & (\rho, t) \in (0, 1) \times (0, \infty),
\end{cases} \tag{2.6}
\]

and the following initial conditions

\[
\begin{aligned}
u(x, 0) = u_0(x), & \quad u_t(x, 0) = u_1(x), & x \in (0, L), \\
y(x, 0) = y_0(x), & \quad y_t(x, 0) = y_1(x), & x \in (0, L), \\
\eta(x, \rho, 0) = f_0(x, -\rho \tau), & \quad (x, \rho) \in (0, \beta) \times (0, 1).
\end{aligned} \tag{2.7}
\]

The energy of system (2.2)-(2.7) is given by

\[
E(t) = E_1(t) + E_2(t) + E_3(t), \tag{2.8}
\]

where

\[
\begin{aligned}
E_1(t) &= \frac{1}{2} \int_0^L \left( |u_t|^2 + a |u_x|^2 \right) dx, \\
E_2(t) &= \frac{1}{2} \int_0^L \left( |y_t|^2 + |y_x|^2 \right) dx \quad \text{and} \\
E_3(t) &= \frac{\tau |\kappa_2|}{2} \int_0^1 \int_0^\beta |\eta_{tx}(\cdot, \rho, t)|^2 d\rho dx.
\end{aligned}
\]

Lemma 2.1. Let \( U = (u, u_t, y, y_t, \eta) \) be a regular solution of system (2.2)-(2.7). Then, the energy \( E(t) \) satisfies the following estimation

\[
\frac{d}{dt} E(t) \leq - (\kappa_1 - |\kappa_2|) \int_0^\beta |u_{tx}|^2 dx. \tag{2.9}
\]

Proof. First, multiplying (2.2) by \( \overline{\eta_t} \), integrating over \( (0, L) \), using integration by parts with (2.6), then taking the real part, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_0^L |u_t|^2 dx + \Re \left\{ \int_0^L S_b(u, u_t, \eta) \overline{\eta_{tx}} dx \right\} + \Re \left\{ \int_0^L c(\cdot) y_t \overline{\eta_t} dx \right\} = 0.
\]
From the above equation and the definition of $S_b(u, u_1, \eta)$ and $c(\cdot)$, we deduce that
\[
\frac{d}{dt} E_1(t) = -\kappa_1 \int_0^\beta |u_{tx}|^2 dx - \Re \left\{ \kappa_2 \int_0^\beta \eta_x(\cdot, 1, t) \overline{u_{tx}} dx \right\} - \Re \left\{ c \int_0^\gamma y_t \eta dx \right\}.
\]

Using Young’s inequality in the above equation, we get
\[
\frac{d}{dt} E_1(t) \leq - \left( \kappa_1 - \frac{\kappa_2}{2} \right) \int_0^\beta |u_{tx}|^2 dx + \frac{\kappa_2}{2} \int_0^\beta |\eta_x(\cdot, 1, t)|^2 dx \]
\[\quad - \Re \left\{ c \int_0^\gamma y_t \eta dx \right\}.
\]

Now, multiplying (2.3) by $\overline{y_t}$, integrating over $(0, L)$, using the definition of $c(\cdot)$, then taking the real part, we get
\[
\frac{d}{dt} E_2(t) = \Re \left\{ c \int_0^\gamma u_t \overline{y_t} dx \right\}.
\]

Deriving (2.4) with respect to $x$, we obtain
\[
\tau \eta_{tx}(\cdot, \rho, t) + \eta_{\rho t}(\cdot, \rho, t) = 0.
\]

Multiplying (2.12) by $|\kappa_2| \overline{\eta}(\cdot, \rho, t)$, integrating over $(0, \beta) \times (0, 1)$, using the fact that $\eta_x(\cdot, 0, t) = u_{tx}$, then taking the real part, we get
\[
\frac{d}{dt} E_3(t) = - \frac{|\kappa_2|}{2} \int_0^\beta \left( |\eta_x(\cdot, 1, t)|^2 - |\eta_x(\cdot, 0, t)|^2 \right) dx \]
\[\quad = - \frac{|\kappa_2|}{2} \int_0^\beta \left( |\eta_x(\cdot, 1, t)|^2 - |u_{tx}|^2 \right) dx.
\]

Finally, adding (2.10), (2.11) and (2.13), we obtain (2.9). The proof is thus complete. \(\square\)

In the sequel, the assumption on $\kappa_1$ and $\kappa_2$ will ensure that
\[
\kappa_1 > 0, \quad \kappa_2 \in \mathbb{R}^+ \quad \text{and} \quad |\kappa_2| < \kappa_1.
\]

Under the hypothesis (H) and from Lemma 2.1, the system (2.2)-(2.7) is dissipative in the sense that its energy is non-increasing with respect to time (i.e. $E'(t) \leq 0$). Let us define the Hilbert space $\mathcal{H}$ by
\[
\mathcal{H} := \left( H_0^1(0, L) \times L^2(0, L) \right)^2 \times \mathcal{W},
\]

where
\[
\mathcal{W} := L^2((0, 1); H_0^1(0, \beta)) \quad \text{and} \quad H_0^1(0, \beta) := \{ \overline{\eta} \in H^1(0, \beta) | \overline{\eta}(0) = 0 \}.
\]

The space $\mathcal{W}$ is an Hilbert space of $H_0^1(0, \beta)$-valued functions on $(0, 1)$, equipped with the following inner product
\[
(\eta^1, \eta^2)_\mathcal{W} := \int_0^\beta \int_0^1 \eta^1_x \overline{\eta^2_x} d\rho dx, \quad \forall \eta^1, \eta^2 \in \mathcal{W}.
\]

The Hilbert space $\mathcal{H}$ is equipped with the following inner product
\[
(U, U^1)_\mathcal{H} = \int_0^L \left( a u_{tx} u_{tx}^T + v v^T + y_{tx} y_{tx}^T + z z^T \right) dx \]
\[\quad + \tau |\kappa_2| \int_0^\beta \int_0^1 \eta_x(\cdot, \rho) \overline{\eta}_{\rho x}(\cdot, \rho) d\rho dx,
\]

(2.14)
where \( U = (u, v, y, z, \eta(\cdot, \rho))^\top \), \( U_1 = (u^1, v^1, y^1, z^1, \eta^1(\cdot, \rho))^\top \in \mathcal{H} \). Now, we define the linear unbounded operator \( A : D(A) \subset \mathcal{H} \mapsto \mathcal{H} \) by:

\[
D(A) = \left\{ \begin{array}{l}
U = (u, v, y, z, \eta(\cdot, \rho))^\top \in \mathcal{H} \\
\quad \text{such that } \eta(\cdot, 0) = v(\cdot) \text{ in } (0, \beta)
\end{array} \right\},
\]

and

\[
A \left( \begin{array}{c}
u \\
v \\
\eta(\cdot, \rho)
\end{array} \right) = \left( \begin{array}{c}
v \\
\eta(\cdot, \rho)
\end{array} \right),
\]

(2.15)

for all \( U = (u, v, y, z, \eta(\cdot, \rho))^\top \in D(A) \).

Now, if \( U = (u, u_t, y, y_t, \eta(\cdot, \rho))^\top \), then system (2.2)-(2.7) can be written as the following first order evolution equation

\[
U_t = AU, \quad U(0) = U_0,
\]

(2.16)

where \( U_0 = (u_0, u_1, y_0, y_1, f_0(\cdot, -\rho t))^\top \in \mathcal{H} \).

**Remark 2.1.** The linear unbounded operator \( A \) is injective (i.e. \( \text{ker}(A) = \{0\} \)). Indeed, if \( U \in D(A) \) such that \( AU = 0 \), then \( v = z = \eta_\rho(\cdot, \rho) = 0 \) and since \( \eta(\cdot, 0) = v(\cdot) \), we get \( \eta(\cdot, \rho) = 0 \). Consequently, \( S_b(u, v, \eta)_x = au_{xx} = 0 \) and \( y_{xx} = 0 \). Now, since \( u(0) = u(L) = y(0) = y(L) = 0 \), then \( u = y = 0 \). Thus, \( U = (u, v, y, z, \eta(\cdot, \rho))^\top = 0 \). \( \square \)

**Proposition 2.1.** Under the hypothesis (H), the unbounded linear operator \( A \) is \( m \)-dissipative in the energy space \( \mathcal{H} \).

**Proof.** For all \( U = (u, v, y, z, \eta(\cdot, \rho))^\top \in D(A) \), from (2.14) and (2.15), we have

\[
\Re(\langle AU, U \rangle_{\mathcal{H}}) = \Re \left\{ \int_0^L \kappa v_x^2 dx + \Re \left\{ \int_0^L \left( S_b(u, v, \eta)_x \right)^2 + \Re \left( \int_0^L \eta_x^2 dx \right) \right\} \right\},
\]

Using integration by parts to the second and fourth terms in the above equation, then using the definition of \( S_b(u, v, \eta) \) and the fact that \( U \in D(A) \), we get

\[
\Re(\langle AU, U \rangle_{\mathcal{H}}) = -\kappa_1 \int_0^\beta |v_x|^2 dx - \Re \left\{ \kappa_2 \int_0^\beta \eta_x(\cdot, 1) \eta(\cdot, \rho)d\rho dx \right\}
\]

the fact that \( \eta(\cdot, 0) = v(\cdot) \) in \( (0, \beta) \), implies that

\[
\Re(\langle AU, U \rangle_{\mathcal{H}}) = - \left( \kappa_1 - \frac{|\kappa_2|}{2} \right) \int_0^\beta |v_x|^2 dx
\]

Using Young’s inequality in the above equation and the hypothesis (H), we obtain

\[
\Re(\langle AU, U \rangle_{\mathcal{H}}) \leq - (\kappa_1 - |\kappa_2|) \int_0^\beta |v_x|^2 dx \leq 0,
\]

(2.17)
which implies that $\mathcal{A}$ is dissipative. Now, let us prove that $\mathcal{A}$ is maximal. For this aim, let $F = (f^1, f^2, f^3, f^4, f^5(\cdot, \rho))^\top \in \mathcal{H}$, we look for $U = (u, v, y, z, \eta(\cdot, \rho))^\top \in D(\mathcal{A})$ unique solution of

$$-\mathcal{A}U = F. \quad (2.18)$$

Equivalently, we have the following system

$$-v = f^1, \quad (2.19)$$

$$-(S_b(u, v, \eta))_x + c(\cdot) z = f^2, \quad (2.20)$$

$$-z = f^3, \quad (2.21)$$

$$-y_{xx} - c(\cdot) v = f^4, \quad (2.22)$$

$$\tau^{-1} \eta_{\rho}(\cdot, \rho) = f^5(\cdot, \rho), \quad (2.23)$$

with the following boundary conditions

$$u(0) = u(L) = y(0) = y(L) = 0, \quad \eta(0, \rho) = 0 \quad \text{and} \quad \eta(\cdot, 0) = v(\cdot) \text{ in } (0, \beta). \quad (2.24)$$

From (2.19), (2.23) and (2.24), we get

$$\eta(x, \rho) = \tau \int_0^\rho f^5(x, s)ds - f^1, \quad (x, \rho) \in (0, \beta) \times (0, 1). \quad (2.25)$$

Since, $f^1 \in H^1_0(0, L)$ and $f^5(\cdot, \rho) \in \mathcal{W}$. Then, from (2.23) and (2.25), we get $\eta_{\rho}(\cdot, \rho), \eta(\cdot, \rho) \in \mathcal{W}$. Now, see the definition of $S_b(u, v, \eta)$, substituting (2.19), (2.21) and (2.25) in (2.20) and (2.22), we get the following system

$$\left[ S_b \left( u, f^1, \tau \int_0^\rho f^5(x, s)ds - f^1 \right) \right] + c(\cdot) f^3 = - f^2, \quad (2.26)$$

$$y_{xx} - c(\cdot) f^4 = - f^4, \quad (2.27)$$

$$u(0) = u(L) = y(0) = y(L) = 0, \quad (2.28)$$

where

$$S_b \left( u, -f^1, \tau \int_0^\rho f^5(x, s)ds - f^1 \right) = \begin{cases} a u_x - (\kappa_1 + \kappa_2) f^1_x + \tau \kappa_2 \int_0^\rho f^5_x(\cdot, s)ds, & x \in (0, \beta), \\
                        a u_x, & x \in (\beta, L). \end{cases}$$

Let $(\phi, \psi) \in H^1_0(0, L) \times H^1_0(0, L)$. Multiplying (2.26) and (2.27) by $\overline{\phi}$ and $\overline{\psi}$ respectively, integrating over $(0, L)$, then using formal integrations by parts, we obtain

$$a \int_0^L u_x \overline{\phi}_x dx = \int_0^L f^2 \overline{\phi} dx + c_0 \int_\alpha^\gamma f^3 \overline{\phi} dx + (\kappa_1 + \kappa_2) \int_0^\beta f^1_\rho \overline{\phi}_x dx$$

$$- \tau \kappa_2 \int_0^\beta \left( \int_0^\rho f^5_x(\cdot, s)ds \right) \overline{\phi}_x dx \quad (2.29)$$

and

$$\int_0^L y_x \overline{\psi}_x dx = \int_0^L f^4 \overline{\psi} dx - c_0 \int_\alpha^\gamma f^1 \overline{\psi} dx.$$ \quad (2.30)

Adding (2.29) and (2.30), we obtain

$$B((u, y), (\phi, \psi)) = \mathcal{L}(\phi, \psi), \quad \forall (\phi, \psi) \in H^1_0(0, L) \times H^1_0(0, L), \quad (2.31)$$
where
\[ B((u, y), (\phi, \psi)) = a \int_0^L u_x \phi_x \, dx + \int_0^L y_x \psi_x \, dx \]
and
\[ L(\phi, \psi) = \int_0^L \left( f^2 \phi + f^4 \psi \right) \, dx + c_0 \int_0^\gamma \left( f^3 \phi - f^1 \psi \right) \, dx \]
\[ - \tau \kappa_2 \int_0^\beta \left( \int_0^1 f_\phi^3 (\cdot, s) \, ds \right) \phi_x \, dx + (\kappa_1 + \kappa_2) \int_0^\beta f_\phi^3 \phi_x \, dx. \]

It is easy to see that, \( B \) is a sesquilinear, continuous and coercive form on \((H^1_0(0, L) \times H^1_0(0, L))^2\) and \( L \) is a linear and continuous form on \((H^1_0(0, L) \times H^1_0(0, L))^2\). Then, it follows by Lax-Milgram theorem that (2.31) admits a unique solution \((u, y) \in H^1_0(0, L) \times H^1_0(0, L))\). By using the classical elliptic regularity, we deduce that system (2.26)-(2.28) admits a unique solution \((u, y) \in H^1_0(0, L) \times (H^2(0, L) \cap H^1_0(0, L))\) such that \((S_b(u, v, \eta))x \in L^2(0, L)\) and since \(\text{ker}(A) = \{0\}\) (see Remark 2.1), we get \(U = (u, -f^1, y, -f^3, \tau \int_0^\rho f^5 (\cdot, s) \, ds - f^1) \in D(A)\) is a unique solution of (2.18).

Then, \( A \) is an isomorphism and since \(\rho(A)\) is open set of \(\mathbb{C}\) (see Theorem 6.7 (Chapter III) in [31]), we easily get \(R(\lambda I - A) = \mathcal{H}\) for a sufficiently small \(\lambda > 0\).

This, together with the dissipativeness of \( A \), imply that \( D(A) \) is dense in \(\mathcal{H}\) and that \( A \) is m-dissipative in \(\mathcal{H}\) (see Theorems 4.5, 4.6 in [45]). The proof is thus complete.

According to Lumer-Phillips theorem (see [45]), Proposition 2.1 implies that the operator \( A \) generates a \(C_0\)-semigroup of contractions \(e^{tA}\) in \(\mathcal{H}\) which gives the well-posedness of (2.16). Then, we have the following result:

**Theorem 2.1.** Under the hypothesis (H), for all \(U_0 \in \mathcal{H}\), system (2.16) admits a unique weak solution
\[ U(x, \rho, t) = e^{tA}U_0(x, \rho) \in C^0(\mathbb{R}^+, \mathcal{H}). \]
Moreover, if \(U_0 \in D(A)\), then system (2.16) admits a unique strong solution
\[ U(x, \rho, t) = e^{tA}U_0(x, \rho) \in C^0(\mathbb{R}^+, D(A)) \cap C^1(\mathbb{R}^+, \mathcal{H}). \]

3. **Strong stability.** In this section, we will prove the strong stability of system (2.2)-(2.7). The main result of this section is the following theorem.

**Theorem 3.1.** Assume that (H) is true. Then, the \(C_0\)-semigroup of contraction \((e^{tA})_{t \geq 0}\) is strongly stable in \(\mathcal{H}\); i.e., for all \(U_0 \in \mathcal{H}\), the solution of (2.16) satisfies
\[ \lim_{t \to +\infty} \|e^{tA}U_0\|_\mathcal{H} = 0. \]

According to Theorem A.1 in the appendix, to prove Theorem 3.1, we need to prove that the operator \( A \) has no pure imaginary eigenvalues and \(\sigma(A) \cap i\mathbb{R}\) is countable. The proof of Theorem 3.1 will be achieved from the following proposition.

**Proposition 3.1.** Under the hypothesis (H), we have
\[ i\mathbb{R} \subset \rho(A). \]

We will prove Proposition 3.1 by contradiction argument. Remark that, it has been proved in Proposition 2.1 that \(0 \in \rho(A)\). Now, suppose that (3.1) is false,
then there exists $\omega \in \mathbb{R}^+$ such that $i\omega \notin \rho(A)$. According to Remark A.1 in the appendix, let \(\{(\lambda^n, U^n) := (u^n, v^n, y^n, z^n, \eta^n(\cdot, \rho))^\top\}_{n \geq 1} \subset \mathbb{R}^* \times D(A)\), with
\[
\lambda^n \to \omega \text{ as } n \to \infty \quad \text{and} \quad |\lambda^n| < |\omega|
\] (3.2)
and
\[
\|U^n\|_H = \|(u^n, v^n, y^n, z^n, \eta^n(\cdot, \rho))^\top\|_H = 1,
\] (3.3)
such that
\[
(i\lambda^n I - A)U^n = F^n := (f^{1,n}, f^{2,n}, f^{3,n}, f^{4,n}, f^{5,n}(\cdot, \rho))^\top \to 0 \quad \text{in} \quad H.
\] (3.4)
Equivalently, we have
\[
\begin{align*}
i\lambda^n u^n - v^n &= f^{1,n} \to 0 \quad \text{in} \quad H^1_0(0, L), \quad (3.5) \\
i\lambda^n v^n - (S_b(u^n, v^n, \eta^n))_x + c(\cdot)z^n &= f^{2,n} \to 0 \quad \text{in} \quad L^2(0, L), \\
i\lambda^n y^n - z^n &= f^{3,n} \to 0 \quad \text{in} \quad H^1_0(0, L), \\
i\lambda^n z^n - y^n_{xx} - c(\cdot)v^n &= f^{4,n} \to 0 \quad \text{in} \quad L^2(0, L), \\
i\lambda^n \eta^n(\cdot, \rho) + \tau^{-1}\eta^n_{x}(\cdot, \rho) &= f^{5,n}(\cdot, \rho) \to 0 \quad \text{in} \quad \mathcal{W}.
\end{align*}
\] (3.6)

Then, we will prove condition (3.1) by finding a contradiction with (3.3) such as \(\|U^n\|_H \to 0\). The proof of proposition 3.1 has been divided into several Lemmas.

**Lemma 3.1.** Under the hypothesis \((H)\), the solution \(U^n = (u^n, v^n, y^n, z^n, \eta^n(\cdot, \rho))^\top \in D(A)\) of system (3.5)-(3.9) satisfies the following limits
\[
\begin{align*}
\lim_{n \to \infty} \int_0^\beta |v^n|^2 dx &= 0, \quad (3.10) \\
\lim_{n \to \infty} \int_0^\beta |v^n|^2 dx &= 0, \quad (3.11) \\
\lim_{n \to \infty} \int_0^\beta |u^n|^2 dx &= 0, \quad (3.12) \\
\lim_{n \to \infty} \int_0^\beta \int_0^{\beta} |\eta^n_x(\cdot, \rho)|^2 d\rho dx &= 0, \quad (3.13) \\
\lim_{n \to \infty} \int_0^\beta |\eta^n(\cdot, 1)|^2 dx &= 0, \quad (3.14) \\
\lim_{n \to \infty} \int_0^\beta |S_1(u^n, v^n, \eta^n)|^2 dx &= 0. \quad (3.15)
\end{align*}
\]

**Proof.** First, taking the inner product of (3.4) with \(U^n\) in \(H\) and using (2.17) with the help of hypothesis \((H)\), we obtain
\[
\int_0^\beta |v^n|^2 dx \leq -\frac{1}{\kappa_1 - |\nu_2|} \Re(AU^n, U^n)_H = \frac{1}{\kappa_1 - |\nu_2|} \Re(F^n, U^n)_H \leq \frac{1}{\kappa_1 - |\nu_2|} \|F^n\|_H \|U^n\|_H.
\] (3.16)
Passing to the limit in (3.16), then using the fact that \(\|U^n\|_H = 1\) and \(\|F^n\|_H \to 0\), we obtain (3.10). Now, since \(v^n \in H^1_0(0, L)\), then it follows from Poincaré inequality that there exists a constant \(C_p > 0\) such that
\[
\|v^n\|_{L^2(0, \beta)} \leq C_p \|v^n\|_{L^2(0, \beta)}. \quad (3.17)
\]
Thus, from (3.10) and (3.17), we obtain (3.11). Next, from (3.5) and the fact that \( \int_0^\beta |f_x^{1,n}|^2 \, dx \leq \frac{4}{3} \| F_n \|^2_{\mathcal{H}} \), we deduce that
\[
\int_0^\beta |u_x^n|^2 \, dx \leq \frac{2}{(\lambda n)^2} \int_0^\beta |v_x^n|^2 \, dx + \frac{2}{\lambda n^2} \int_0^\beta |f_x^{1,n}|^2 \, dx \\
\leq \frac{2}{(\lambda n)^2} \int_0^\beta |v_x^n|^2 \, dx + \frac{2}{a(\lambda n)^2} \| F_n \|^2_{\mathcal{H}}.
\]
Passing to the limit in (3.18), then using (3.2), (3.10) and the fact that \( \| F_n \|_{\mathcal{H}} \to 0 \), we obtain (3.12). Moreover, from (3.9) and the fact that \( \eta^n(\alpha, 0) = v^n(\cdot) \in (0, \beta) \), we deduce that
\[
\eta^n(x, \rho) = v^n e^{-i\lambda^n \rho} + \tau \int_0^\rho e^{i\lambda^n \rho(s-\rho)} f_x^{5,n}(x, s) ds, \quad (x, \rho) \in (0, \beta) \times (0, 1).
\]
From (3.19), and the fact that \( \rho \in (0, 1) \) and \( \int_0^\beta \int_0^1 |f_x^{5,n}(\cdot, s)|^2 ds \, dx \leq \frac{1}{\tau|\kappa_2|^2} \| F_n \|^2_{\mathcal{H}} \), we obtain
\[
\int_0^\beta \int_0^1 |\eta_x^n(\cdot, \rho)|^2 d\rho \, dx \\
\leq 2 \int_0^\beta |v_x^n|^2 \, dx + 2\tau^2 \int_0^\beta \int_0^1 \rho |f_x^{5,n}(\cdot, s)|^2 ds \, d\rho \, dx \\
\leq 2 \int_0^\beta |v_x^n|^2 \, dx + 2\tau^2 \int_0^\beta \int_0^1 \rho |f_x^{5,n}(\cdot, s)|^2 ds \, d\rho \, dx \\
= 2 \int_0^\beta |v_x^n|^2 \, dx + 2\tau^2 \int_0^1 |f_x^{5,n}(\cdot, s)|^2 ds \\
\leq 2 \int_0^\beta |v_x^n|^2 \, dx + \tau|\kappa_2|^{-1} \| F_n \|^2_{\mathcal{H}}.
\]
Passing to the limit in the above inequality, then using (3.10) and the fact that \( \| F_n \|_{\mathcal{H}} \to 0 \), we obtain (3.13). On the other hand, from (3.19), we have
\[
\eta_x^n(1, \cdot) = v^n e^{-i\lambda^n \tau} + \tau \int_0^1 e^{i\lambda^n \tau(s-1)} f_x^{5,n}(\cdot, s) ds,
\]
consequently, by using the same argument as proof of (3.13), we obtain (3.14). Next, it is clear to see that
\[
\int_0^\beta |S_1(u^n, v^n, \eta^n)|^2 \, dx = \int_0^\beta |au_x^n + \kappa_1 v_x^n + \kappa_2 \eta_x^n(\cdot, 1)|^2 \, dx \\
\leq 3a^2 \int_0^\beta |u_x^n|^2 \, dx + 3\kappa_1^2 \int_0^\beta |v_x^n|^2 \, dx + 3\kappa_2^2 \int_0^\beta |\eta_x^n(\cdot, 1)|^2 \, dx.
\]
Finally, passing to the limit in the above inequality, then using (3.10), (3.12) and (3.14), we obtain (3.15). The proof is thus complete. \( \square \)

Now, we fix a function \( g \in C^1([\alpha, \beta]) \) such that
\[
g(\alpha) = -g(\beta) = 1 \quad \text{and} \quad \max_{x \in [\alpha, \beta]} |g(x)| = M_g \quad \text{and} \quad \max_{x \in [\alpha, \beta]} |g'(x)| = M_g'.
\]

**Remark 3.1.** To prove the existence of a function \( g \), we need to find an example. For this aim, we can take \( g(x) = 1 + \frac{2(\alpha - x)}{\beta - \alpha} \), then \( g \in C^1([\alpha, \beta]) \), \( g(\alpha) = -g(\beta) = 1 \), \( M_g = 1 \) and \( M_g' = \frac{2}{\beta - \alpha} \). Also, we can take \( g(x) = \cos\left(\frac{(\alpha - x)\pi}{\alpha - \beta}\right) \). \( \square \)
Lemma 3.2. Under the hypothesis \((H)\), the solution \(U^n = (u^n, v^n, y^n, z^n, \eta^n(\cdot, \rho))\) \(\in D(A)\) of system \((3.5)-(3.9)\) satisfies the following inequalities

\[
|z^n(\beta)|^2 + |z^n(\alpha)|^2 \leq M_g' \int_\alpha^\beta |z^n|^2 \, dx + 2|\lambda^n|M_g \left( \int_\alpha^\beta |y^n|^2 \, dx \right)^{\frac{1}{2}} + 2M_g\|F^n\|_H,
\]

and

\[
|y^n(\beta)|^2 + |y^n(\alpha)|^2 \leq M_g' \int_\alpha^\beta |y^n|^2 \, dx + 2(|\lambda^n| + c_0)M_g \left( \int_\alpha^\beta |y^n|^2 \, dx \right)^{\frac{1}{2}} + 2M_g\|F^n\|_H,
\]

where \(\alpha, \beta\) are the limits of system \((3.5)-(3.9)\) satisfies the following inequalities

\[
\lim_{n \to \infty} |v^n(\alpha)| = 0 \quad \text{and} \quad \lim_{n \to \infty} |v^n(\beta)| = 0,
\]

\[
\lim_{n \to \infty} |(S_1(u^n, v^n, \eta^n)) (\alpha)| = 0 \quad \text{and} \quad \lim_{n \to \infty} |(S_1(u^n, v^n, \eta^n)) (\beta^-)| = 0.
\]

Proof. First, from \((3.7)\), we deduce that

\[
i\lambda^n y^n - z^n = f.^3_n.
\]

Multiplying \((3.25)\) and \((3.8)\) by \(2g\bar{z^n}\) and \(2g\bar{y^n}_z\) respectively, integrating over \((\alpha, \beta)\), using the definition of \(c(\cdot)\), then taking the real part, we get

\[
\Re \left\{ 2i\lambda^n \int_\alpha^\beta g y^n_z \bar{z^n} \, dx \right\} - \int_\alpha^\beta g |z^n|^2 \, dx = \Re \left\{ 2 \int_\alpha^\beta g f^3_n \bar{z^n} \, dx \right\} \quad \text{(3.26)}
\]

and

\[
\Re \left\{ 2i\lambda^n \int_\alpha^\beta g y^n_z \bar{y^n_z} \, dx \right\} - \int_\alpha^\beta g |y^n|^2 \, dx - \Re \left\{ 2c_0 \int_\alpha^\beta g v^n \bar{y^n_z} \, dx \right\}
\]

\[
= \Re \left\{ 2 \int_\alpha^\beta g f^4_n \bar{y^n_z} \, dx \right\}. \quad \text{(3.27)}
\]

Using integration by parts in \((3.26)\) and \((3.27)\), we obtain

\[
\left[ -g |z^n|^2 \right]_\alpha^\beta = - \int_\alpha^\beta g' |z^n|^2 \, dx - \Re \left\{ 2i\lambda^n \int_\alpha^\beta g y^n_z \bar{z^n} \, dx \right\} + \Re \left\{ 2 \int_\alpha^\beta g f^3_n \bar{z^n} \, dx \right\}
\]

and

\[
\left[ -g |y^n_z|^2 \right]_\alpha^\beta = - \int_\alpha^\beta g' |y^n_z|^2 \, dx - \Re \left\{ 2i\lambda^n \int_\alpha^\beta g z^n \bar{y^n_z} \, dx \right\} + \Re \left\{ 2c_0 \int_\alpha^\beta g v^n \bar{y^n_z} \, dx \right\}
\]

\[
+ \Re \left\{ 2 \int_\alpha^\beta g f^4_n \bar{y^n_z} \, dx \right\}.
\]

Using the definition of \(g\) and Cauchy-Schwarz inequality in the above equations, we obtain

\[
|z^n(\beta)|^2 + |z^n(\alpha)|^2 \leq M_g' \int_\alpha^\beta |z^n|^2 \, dx + 2|\lambda^n|M_g \left( \int_\alpha^\beta |y^n|^2 \, dx \right)^{\frac{1}{2}} \left( \int_\alpha^\beta |z^n|^2 \, dx \right)^{\frac{1}{2}} + 2M_g \left( \int_\alpha^\beta |f^3_n|^2 \, dx \right)^{\frac{1}{2}} \left( \int_\alpha^\beta |z^n|^2 \, dx \right)^{\frac{1}{2}}
\]

and

\[
|y^n(\beta)|^2 + |y^n(\alpha)|^2 \leq M_g' \int_\alpha^\beta |y^n|^2 \, dx + 2(|\lambda^n| + c_0)M_g \left( \int_\alpha^\beta |y^n|^2 \, dx \right)^{\frac{1}{2}} + 2M_g \left( \int_\alpha^\beta |f^4_n|^2 \, dx \right)^{\frac{1}{2}} \left( \int_\alpha^\beta |z^n|^2 \, dx \right)^{\frac{1}{2}}.
\]
\[
|y_{\alpha}^n(\beta)|^2 + |y_{\alpha}^n(\alpha)|^2 \leq M_{\beta} \int_{\alpha}^{\beta} |y_{x}^n|^2 dx + 2|\lambda^n|M_{\beta} \left( \int_{\alpha}^{\beta} |y_{x}^n|^2 dx \right) \left( \int_{\alpha}^{\beta} |z^n|^2 dx \right)^{\frac{1}{2}} + 2|c_0|M_{\beta} \left( \int_{\alpha}^{\beta} |y_{x}^n|^2 dx \right) \left( \int_{\alpha}^{\beta} |v^n|^2 dx \right)^{\frac{1}{2}} + 2M_{\beta} \left( \int_{\alpha}^{\beta} |f_{z^n}|^2 dx \right) \left( \int_{\alpha}^{\beta} |y_{x}^n|^2 dx \right). \]

Therefore, from the above inequalities and the fact that \( \int_{\alpha}^{\beta} |\xi^n|^2 dx \leq \frac{2}{H} \|U^n\|^2 \_H = 1 \) with \( \xi^n_1 \in \{v^n, y_{x}^n, z^n\} \) and \( \int_{\alpha}^{\beta} |\xi^n|^2 dx \leq \int_{\alpha}^{\beta} |\xi^n_2|^2 dx \leq \frac{2}{H} \|F^n\|^2 \_H \) with \( \xi^n_2 \in \{f_{z^n}, f_{z^n}\} \), we obtain (3.21) and (3.22). On the other hand, from (3.5), we deduce that

\[
i\lambda^n u^n_x - v^n_x = f_{x^n}. \quad (3.28)
\]

Multiplying (3.28) and (3.6) by \( 2g_0v^n \) and \( 2gS_1(u^n, v^n, \eta^n) \) respectively, integrating over \((\alpha, \beta)\), using the definition of \( c(\cdot) \) and \( S_0(u^n, v^n, \eta^n) \), then taking the real part, we get

\[
\Re \left\{ 2i\lambda^n \int_{\alpha}^{\beta} g v^n v^n \overline{\eta^n} \, dx \right\} = \int_{\alpha}^{\beta} g|v^n|^2 \, dx = \Re \left\{ 2 \int_{\alpha}^{\beta} g f_{x^n} \overline{\eta^n} \, dx \right\} \quad (3.29)
\]

and

\[
\Re \left\{ 2i\lambda^n \int_{\alpha}^{\beta} g v^n \overline{S_1(u^n, v^n, \eta^n)} \, dx \right\} = \int_{\alpha}^{\beta} g \left( |S_1(u^n, v^n, \eta^n)|^2 \right) \, dx + \Re \left\{ 2c_0 \int_{\alpha}^{\beta} g z^n \overline{S_1(u^n, v^n, \eta^n)} \, dx \right\} \quad (3.30)
\]

Using integration by parts in (3.29) and (3.30), we get

\[
\left[ -g |v^n|^2 \right]_{\alpha}^{\beta} = - \int_{\alpha}^{\beta} g'|v^n|^2 \, dx - \Re \left\{ 2i\lambda^n \int_{\alpha}^{\beta} g v^n v^n \overline{\eta^n} \, dx \right\} + \Re \left\{ 2 \int_{\alpha}^{\beta} g f_{x^n} \overline{\eta^n} \, dx \right\}
\]

and

\[
\left[ -g |S_1(u^n, v^n, \eta^n)|^2 \right]_{\alpha}^{\beta} = - \int_{\alpha}^{\beta} g'|S_1(u^n, v^n, \eta^n)|^2 \, dx - \Re \left\{ 2i\lambda^n \int_{\alpha}^{\beta} g v^n \overline{S_1(u^n, v^n, \eta^n)} \, dx \right\} - \Re \left\{ 2c_0 \int_{\alpha}^{\beta} g z^n \overline{S_1(u^n, v^n, \eta^n)} \, dx \right\} + \Re \left\{ 2 \int_{\alpha}^{\beta} g f_{x^n} \overline{S_1(u^n, v^n, \eta^n)} \, dx \right\}.
\]

Using the definition of \( g \) and Cauchy-Schwarz inequality in the above equations, then using the fact that

\[
\begin{align*}
\int_{\alpha}^{\beta} |z^n|^2 \, dx &\leq \int_{0}^{L} |z^n|^2 \, dx \leq \|U^n\|^2 \_H = 1, \\
\int_{\alpha}^{\beta} |f_{x^n}^{1, n}|^2 \, dx &\leq \int_{0}^{L} |f_{x^n}^{1, n}|^2 \, dx \leq \frac{1}{a} \|F^n\|^2 \_H, \\
\int_{\alpha}^{\beta} |f_{x^n}^{2, n}|^2 \, dx &\leq \int_{0}^{L} |f_{x^n}^{2, n}|^2 \, dx \leq \|F^n\|^2 \_H,
\end{align*}
\]

we have
Proof. Under the hypothesis (H)\cite{1}, the solution $U^n = (u^n, v^n, \eta^n) \in D(A)$ of system (3.5)-(3.8) satisfies the following limits
\begin{equation}
\lim_{n \to \infty} \int_{\alpha}^{\beta} |z^n|^2 dx = 0 \quad \text{and} \quad \lim_{n \to \infty} \int_{\alpha}^{\beta} |y^n_x|^2 dx = 0.
\end{equation}

**Remark 3.2.** From (3.2), (3.21), (3.22), and the fact that $\|U^n\|_H = 1$ and $\|F^n\|_H \to 0$, we obtain (3.23) and (3.24). The proof is thus complete.

**Lemma 3.3.** Under the hypothesis (H), the solution $U^n = (u^n, v^n, \eta^n) \in D(A)$ of system (3.5)-(3.8) satisfies the following limits
\begin{equation}
\lim_{n \to \infty} \int_{\alpha}^{\beta} |z^n|^2 dx = 0 \quad \text{and} \quad \lim_{n \to \infty} \int_{\alpha}^{\beta} |y^n_x|^2 dx = 0.
\end{equation}

**Proof.** First, multiplying (3.6) by $\overline{\eta}$, integrating over $(\alpha, \beta)$, using the definition of $c(\cdot)$ and $S_0(u^n, v^n, \eta^n)$, then taking the real part, we get
\begin{equation}
\begin{aligned}
\Re \left\{ \lambda^n \int_{\alpha}^{\beta} v^n \overline{\eta} dx \right\} - \Re \left\{ \int_{\alpha}^{\beta} (S_1(u^n, v^n, \eta^n))_x \overline{\eta} dx \right\} + c_0 \int_{\alpha}^{\beta} |z^n|^2 dx \\
= \Re \left\{ \int_{\alpha}^{\beta} f^{2,n} \overline{\eta} dx \right\}.
\end{aligned}
\end{equation}

From (3.7), we deduce that
\begin{equation}
\overline{\eta}_x = -i \lambda^n \overline{y^n_x} - f^{2,n} \overline{\eta}.
\end{equation}

Using integration by parts to the second term in (3.34), then using (3.35), we get
\begin{equation}
c_0 \int_{\alpha}^{\beta} |z^n|^2 dx
\end{equation}
\begin{equation}
= \Re \left\{ \lambda^n \int_{\alpha}^{\beta} S_1(u^n, v^n, \eta^n) \overline{\eta} dx \right\} + \Re \left\{ \int_{\alpha}^{\beta} S_1(u^n, v^n, \eta^n) f^{3,n} dx \right\}
\end{equation}
\begin{equation}
+ \Re \left\{ (S_1(u^n, v^n, \eta^n) \overline{\eta}\big|_{\alpha}^{\beta} \right\} + \Re \left\{ \int_{\alpha}^{\beta} f^{2,n} \overline{\eta} dx \right\} - \Re \left\{ i \lambda^n \int_{\alpha}^{\beta} v^n \overline{\eta} dx \right\}.
\end{equation}
Using Cauchy-Schwarz inequality in the above equation and the fact that \( \int_0^1 |\xi^n_1|^2 \, dx \leq \int_0^1 |\xi^n_2|^2 \, dx \leq \|U^n\|_H^2 = 1 \) with \( \xi^n_1 \in \{g^n, z^n\} \) and \( \int_0^1 |\xi^n_2|^2 \, dx \leq \int_0^1 |\xi^n_2|^2 \, dx \leq \|F^n\|_H^2 \), we obtain

\[
\left| \int_0^1 z^n \, dx \right| \leq (|\lambda^n| + \|F^n\|_H) \left( \int_0^1 |S_1(u^n, v^n, \eta^n)|^2 \, dx \right)^{\frac{1}{2}} + |\lambda^n| \left( \int_0^1 |v^n|^2 \, dx \right)^{\frac{1}{2}} + |(S_1(u^n, v^n, \eta^n))(\beta)| |z^n(\beta)| + |(S_1(u^n, v^n, \eta^n))(\alpha)| |z^n(\alpha)| + \|F^n\|_H. \tag{3.37}
\]

Passing to the limit in the above inequality, then using (3.2), (3.32), (3.24), Lemma 3.1 and the fact that \( \|F^n\|_H \to 0 \), we obtain the first limit in (3.33). On the other hand, multiplying (3.8) by \(-\overline{\lambda^n}(\lambda^n)^{-1}\), integrating over \((\alpha, \beta)\), using the definition of \( c(\cdot) \), then taking the imaginary part, we get

\[
- \int_\alpha^\beta |z^n|^2 \, dx + 3 \left\{ (\lambda^n)^{-1} \int_\alpha^\beta y_{xxz}^n \, dx \right\} + 3 \left\{ c_0(\lambda^n)^{-1} \int_\alpha^\beta v^n \overline{z^n} \, dx \right\} = - 3 \left\{ (\lambda^n)^{-1} \int_\alpha^\beta f^{4, n, \overline{z^n}} \, dx \right\}.
\]

Using integration by parts to the second term in the above equation, then using (3.35), we obtain

\[
\int_\alpha^\beta |y^n_{x_3}|^2 \, dx = \int_\alpha^\beta |z^n|^2 \, dx + 3 \left\{ (\lambda^n)^{-1} \int_\alpha^\beta \overline{y^n_{x_3}} y^n_{x_3} \, dx \right\} - 3 \left\{ (\lambda^n)^{-1} |y^n_{x_3}(\beta)| \right\} - 3 \left\{ c_0(\lambda^n)^{-1} \int_\alpha^\beta v^n \overline{z^n} \, dx \right\} + 3 \left\{ (\lambda^n)^{-1} \int_\alpha^\beta f^{4, n, \overline{z^n}} \, dx \right\}.
\]

Using Cauchy-Schwarz inequality in the above equation and the fact that \( \|U^n\|_H = 1 \), we get

\[
\int_\alpha^\beta |y^n_{x_3}|^2 \, dx \leq \int_\alpha^\beta |z^n|^2 \, dx + c_0|\lambda^n|^{-1} \left( \int_\alpha^\beta |v^n|^2 \, dx \right)^{\frac{1}{2}} + 2|\lambda^n|^{-1} \|F^n\|_H + |\lambda^n|^{-1} |y^n_{x_3}(\beta)||z^n(\beta)| + |\lambda|^{-1} |y^n_{x_3}(\alpha)||z^n(\alpha)|. \tag{3.38}
\]

Now, passing to the limit in (3.21), then using (3.2), the first limit in (3.33) and the fact that \( \|F^n\|_H \to 0 \), we get

\[
\lim_{n \to \infty} |z^n(\alpha)| = 0 \quad \text{and} \quad \lim_{n \to \infty} |z^n(\beta)| = 0. \tag{3.39}
\]

Finally, passing to the limit in (3.38), then using (3.2), (3.11), (3.32), the first limit in (3.33), (3.39), and the fact that \( \|F^n\|_H \to 0 \), we obtain the second limit in (3.33). The proof is thus complete.

\[\square\]

**Lemma 3.4.** Under the hypothesis (H), the solution \( U^n = (u^n, v^n, y^n, z^n, \eta^n(\cdot, \rho))^\top \in D(A) \) of system (3.5)-(3.9) satisfies the following estimations

\[
\lim_{n \to \infty} |u^n(\beta)|^2 = 0 \quad \text{and} \quad \lim_{n \to \infty} |y^n(\beta)|^2 = 0, \tag{3.40}
\]

\[
\lim_{n \to \infty} |u^n(\beta^+)|^2 = 0 \quad \text{and} \quad \lim_{n \to \infty} |y^n(\beta)|^2 = 0. \tag{3.41}
\]
\[ \lim_{n \to \infty} \left( \int_{\beta} y^n |x|^2 \, dx + \int_{\beta} y^n |x|^2 \, dx + \int_{\beta} |y^n|^2 \, dx + \int_{\beta} |y^n|^2 \, dx \right) = 0, \quad (3.42) \]

\[ \lim_{n \to \infty} \int_{\beta} |y^n|^2 \, dx = 0 \quad \text{and} \quad \lim_{n \to \infty} \int_{\beta} |z^n|^2 \, dx = 0. \quad (3.43) \]

**Proof.** First, from (3.5) and (3.7), we get

\[ |u^n(\beta)|^2 \leq 2(\lambda^n)^{-2} |v^n(\beta)|^2 + 2(\lambda^n)^{-2} |f^{\lambda}(\beta)|^2 \]

and

\[ |y^n(\beta)|^2 \leq 2(\lambda^n)^{-2} |z^n(\beta)|^2 + 2(\lambda^n)^{-2} |f^{\lambda}(\beta)|^2. \]

Using the fact that \( |f^{\lambda}(\beta)|^2 \leq \beta \int_{0}^{\beta} |f^{\lambda}(\beta)|^2 \, dx \leq \frac{\beta}{\sigma} \|F^n\|^2_H \) and \( |f^{\lambda}(\beta)|^2 \leq \beta \int_{0}^{\beta} |f^{\lambda}(\beta)|^2 \, dx \leq \beta \|F^n\|^2_H \), in the above inequalities, we obtain

\[ |u^n(\beta)|^2 \leq 2(\lambda^n)^{-2} |v^n(\beta)|^2 + 2\beta^{-1}(\lambda^n)^{-2} \|F^n\|^2_H \]

and

\[ |y^n(\beta)|^2 \leq 2(\lambda^n)^{-2} |z^n(\beta)|^2 + 2\beta(\lambda^n)^{-2} \|F^n\|^2_H. \]

Passing to the limit in the above inequalities, then using (3.2), (3.23), (3.39) and the fact that \( \|F^n\|_H \to 0 \), we obtain (3.40). Secondly, since \( S_0(u^n, v^n, \eta^n) \in H^1(0, L) \subset C([0, L]) \), then we deduce that

\[ \left| (S_1(u^n, v^n, \eta^n)) (\beta^-) \right|^2 = |au^n_x(\beta^+)|^2. \quad (3.44) \]

Thus, from (3.24) and (3.44), we obtain the first limit in (3.41). Moreover, passing to the limit in inequality (3.22), then using (3.2), the second limit in (3.33) and the fact that \( \|F^n\|_H \to 0 \), we obtain the second limit in (3.41). On the other hand, (3.5)-(3.8) can be written in \( (\beta, \gamma) \) as the following form

\[ (\lambda^n)^2 u^n + au^n_x - i\lambda^n c_0 y^n = G^{1,n} \quad \text{in} \quad (\beta, \gamma), \quad (3.45) \]

\[ (\lambda^n)^2 y^n + y^n_{xx} + i\lambda^n c_0 u^n = G^{2,n} \quad \text{in} \quad (\beta, \gamma), \quad (3.46) \]

where

\[ G^{1,n} = -f^{2,n} - i\lambda^n f^{1,n} - c_0 f^{3,n} \quad \text{and} \quad G^{2,n} = -f^{4,n} - i\lambda^n f^{3,n} + c_0 f^{1,n}. \quad (3.47) \]

Let \( V^n = (u^n, u^n_x, y^n, y^n_x)^T \), then (3.45)-(3.46) can be written as the following

\[ V^n = B^n V^n + G^n, \quad (3.48) \]

where

\[
B^n = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-a^{-1}(\lambda^n)^2 & 0 & a^{-1}i\lambda^n c_0 & 0 \\
0 & 0 & 0 & 1 \\
-i\lambda^n c_0 & 0 & -(\lambda^n)^2 & 0 \\
\end{pmatrix} = (b_{ij})_{1 \leq i, j \leq 4} \quad \text{and} \quad G^n = \begin{pmatrix}
0 & a^{-1}G^{1,n} \\
0 & 0 \\
G^{2,n} & 0 \\
\end{pmatrix}.
\]

The solution of the differential equation (3.48) is given by

\[ V^n(x) = e^{B^n(x-\beta)} V^n(\beta^+) + \int_{\beta}^{x} e^{B^n(s-\beta)} G^n(s) \, ds, \quad (3.49) \]

where \( e^{B^n(x-\beta)} = (c_{ij})_{1 \leq i, j \leq 4} \) and \( e^{B^n(s-\beta)} = (d_{ij})_{1 \leq i, j \leq 4} \) are denoted by the exponential of the matrices \( B^n(x-\beta) \) and \( B^n(s-x) \) respectively. Now, from (3.2), the entries \( b_{ij} \) are bounded for all \( 1 \leq i, j \leq 4 \) and consequently, the entries
\(b_{ij}\) \((x - \beta)\) and \(b_{ij}\) \((s - x)\) are bounded. In addition, from the definition of the exponential of a square matrix, we obtain
\[
e^{B^n\zeta} = \sum_{k=0}^{\infty} \frac{(B^n\zeta)^k}{k!} \quad \text{for} \quad \zeta \in \{x - \beta, s - x\}.
\]

Therefore, the entries \(c_{ij}\) and \(d_{ij}\) are also bounded for all \(1 \leq i, j \leq 4\) and consequently, \(e^{B^n(x - \beta)}\) and \(e^{B^n(s - x)}\) are two bounded matrices. From (3.40) and (3.41), we directly obtain
\[
V^n(\beta^+) \to 0 \quad \text{in} \quad (L^2(\beta, \gamma))^4, \quad \text{as} \quad n \to \infty. \quad (3.50)
\]

Moreover, from (3.47), we deduce that
\[
\int_{\beta}^{\gamma} |G^{1,n}|^2 dx \leq 3 \int_{\beta}^{\gamma} |f^{2,n}|^2 dx + 3(\lambda^n)^2 \int_{\beta}^{\gamma} |f^{1,n}|^2 dx + 3c_0^2 \int_{\beta}^{\gamma} |f^{3,n}|^2 dx \quad (3.51)
\]
and
\[
\int_{\beta}^{\gamma} |G^{2,n}|^2 dx \leq 3 \int_{\beta}^{\gamma} |f^{4,n}|^2 dx + 3(\lambda^n)^2 \int_{\beta}^{\gamma} |f^{3,n}|^2 dx + 3c_0^2 \int_{\beta}^{\gamma} |f^{1,n}|^2 dx. \quad (3.52)
\]

Now, since \(f^{1,n}, f^{3,n} \in H_1^1(0, L)\), then it follows from Poincaré inequality that there exist two constants \(C_1 > 0\) and \(C_2 > 0\) such that
\[
\|f^{1,n}\|_{L^2(0, L)} \leq C_1 \|f^{1,n}\|_{L^2(0, L)} \quad \text{and} \quad \|f^{3,n}\|_{L^2(0, L)} \leq C_2 \|f^{3,n}\|_{L^2(0, L)}. \quad (3.53)
\]

Consequently, from (3.51), (3.52) and (3.53), we get
\[
\int_{\beta}^{\gamma} |G^{1,n}|^2 dx \leq 3 \left(1 + a^{-1}(\lambda^n C_1)^2 \right) \|F^n\|^2_{\mathcal{H}}. \quad (3.54)
\]
and
\[
\int_{\beta}^{\gamma} |G^{2,n}|^2 dx \leq 3 \left(1 + (\lambda^n C_1)^2 \right) \|F^n\|^2_{\mathcal{H}}. \quad (3.55)
\]

Hence, from (3.2), (3.54), (3.55) and the fact that \(\|F^n\|_{\mathcal{H}} \to 0\), we obtain
\[
G^n \to 0 \quad \text{in} \quad (L^2(\beta, \gamma))^4, \quad \text{as} \quad n \to \infty. \quad (3.56)
\]

Therefore, from (3.49), (3.50), (3.56) and as \(e^{B^n(x - \beta)}\) and \(e^{B^n(s - x)}\) are two bounded matrices, we get \(V^n \to 0\) in (3.2), (3.42) and consequently, we obtain (3.42). Next, from (3.5), (3.7) and (3.53), we deduce that
\[
\int_{\beta}^{\gamma} |v^n|^2 dx \leq 2(\lambda^n)^2 \int_{\beta}^{\gamma} |u^n|^2 dx + 2 \int_{\beta}^{\gamma} |f^{1,n}|^2 dx \leq 2(\lambda^n)^2 \int_{\beta}^{\gamma} |u^n|^2 dx + \frac{2C_1}{a} \|F^n\|^2_{\mathcal{H}},
\]
\[
\int_{\beta}^{\gamma} |z^n|^2 dx \leq 2(\lambda^n)^2 \int_{\beta}^{\gamma} |y^n|^2 dx + 2 \int_{\beta}^{\gamma} |f^{3,n}|^2 dx \leq 2(\lambda^n)^2 \int_{\beta}^{\gamma} |y^n|^2 dx + 2C_2 \|F^n\|^2_{\mathcal{H}}.
\]

Finally, passing to the limit in the above inequalities, then using (3.2), (3.42) and the fact that \(\|F^n\|_{\mathcal{H}} \to 0\), we obtain (3.43). The proof is thus complete. \(\Box \)

**Lemma 3.5.** Let \(h \in C^1([0, L])\) be a function. Under the hypothesis (H), the solution \(U^n = (u^n, v^n, y^n, z^n, \eta^n(\cdot, \rho))\) \(\in D(A)\) of system (3.5)-(3.9) satisfies the
following equality

\[
\int_0^L h'(\frac{1}{a}|S_b(u^n, v^n, \eta^n)|^2 + |y^n|^2 + |z^n|^2 + |y_x^n|^2)\,dx \\
- \left[ h\left(\frac{1}{a}|S_b(u^n, v^n, \eta^n)|^2 + |y_x^n|^2\right)\right]_0^L - \Re\left\{2\int_0^L c(\cdot)h v^n y_x^n\,dx\right\} \\
+ \Re\left\{\frac{2i\lambda^n}{a} \int_0^L c(\cdot)h z^n S_b(u^n, v^n, \eta^n)\,dx\right\} \\
= \Re\left\{2\int_0^L h f_1^n v^n\,dx\right\} + \Re\left\{\frac{2}{a} \int_0^L h f_2^n S_b(u^n, v^n, \eta^n)\,dx\right\} \\
+ \Re\left\{2\int_0^L h f_3^n z^n\,dx\right\} + \Re\left\{2\int_0^L h f_4^n y_x^n\,dx\right\}.
\]

Proof. First, multiplying (3.6) and (3.8) by \(2a^{-1}h S_b(u^n, v^n, \eta^n)\) and \(2h y_x^n\) respectively, integrating over \((0, L)\), then taking the real part, we get

\[
\Re\left\{\frac{2i\lambda^n}{a} \int_0^L h v^n S_b(u^n, v^n, \eta^n)\,dx\right\} - \frac{1}{a} \int_0^L h\left(|S_b(u^n, v^n, \eta^n)|^2\right)_x\,dx \\
+ \Re\left\{\frac{2}{a} \int_0^L c(\cdot)h z^n S_b(u^n, v^n, \eta^n)\,dx\right\} \\
= \Re\left\{2\int_0^L h f_1^n v^n\,dx\right\} + \Re\left\{\frac{2}{a} \int_0^L h f_2^n S_b(u^n, v^n, \eta^n)\,dx\right\} \\
+ \Re\left\{2\int_0^L h f_3^n z^n\,dx\right\} + \Re\left\{2\int_0^L h f_4^n y_x^n\,dx\right\}.
\] (3.57)

and

\[
\Re\left\{2i\lambda^n \int_0^L h z^n y_x^n\,dx\right\} - \int_0^L h\left(|y_x^n|^2\right)_x\,dx - \Re\left\{2\int_0^L c(\cdot)h v^n y_x^n\,dx\right\} \\
= \Re\left\{2\int_0^L h f_4^n y_x^n\,dx\right\}.
\] (3.58)

From (3.5) and (3.7), we deduce that

\[
i\lambda^n u_x^n = -v_x^n - f_1^n, \\
i\lambda^n y_x^n = -z_x^n - f_3^n.
\] (3.59)

Consequently, from (3.59) and the definition \(S_b(u^n, v^n, \eta^n)\), we have

\[
i\lambda^n S_b(u^n, v^n, \eta^n) \\
= \begin{cases} 
-a\left(\overline{v_x^n} + \frac{f_1^n}{i\lambda^n}\right) + i\lambda^n \left(\kappa_1 \overline{v_x^n} + \kappa_2 \overline{\eta_x^n}(\cdot, 1)\right), & x \in (0, \beta), \\
-a\left(\overline{v_x^n} + \frac{f_3^n}{i\lambda^n}\right), & x \in (\beta, L).
\end{cases}
\] (3.60)
Substituting (3.61) and (3.60) in (3.57) and (3.58) respectively, we obtain

\[- \int_{0}^{L} h \left( |v_n|^2 + \frac{1}{a} |S_b(u^n, v^n, \eta^n)|^2 \right) \, dx \]
\[+ \Re \left\{ \frac{2i\lambda}{a} \int_{0}^{\beta} \nu h u_n^{\beta} \left( \kappa_1 \nu_x^{\beta} + \kappa_2 \nu_x^{\beta} \right) \, dx \right\} + \Re \left\{ \frac{2}{a} \int_{0}^{L} c(\cdot) h z_n \bar{S}_b(u^n, v^n, \eta^n) \, dx \right\} \]
\[= \Re \left\{ \frac{2}{a} \int_{0}^{L} h f_z^2 v_n \, dx \right\} + \Re \left\{ \frac{2}{a} \int_{0}^{L} h f_z^2 \bar{S}_b(u^n, v^n, \eta^n) \, dx \right\} \]

and

\[- \int_{0}^{L} h \left( |z_n|^2 + |y_n|^2 \right) \, dx - \Re \left\{ \frac{2}{a} \int_{0}^{L} c(\cdot) h v_n \bar{y}_n \, dx \right\} \]
\[= \Re \left\{ \frac{2}{a} \int_{0}^{L} h f_z^4 \bar{y}_n \, dx \right\} + \Re \left\{ \frac{2}{a} \int_{0}^{L} h f_z^4 z_n \, dx \right\} . \]

Finally, adding the above equations, then using integration by parts and the fact that \( v^n(0) = v^n(L) = 0 \) and \( z^n(0) = z^n(L) = 0 \), we obtain the desired result. The proof is thus complete.

Now, we fix the cut-off functions \( \chi_1, \chi_2 \in C^1([0, L]) \) (see Figure 2) such that

\[0 \leq \chi_1(x) \leq 1, \quad 0 \leq \chi_2(x) \leq 1, \text{ for all } x \in [0, L] \]

and

\[\chi_1(x) = \begin{cases} 1 & \text{if } x \in [0, \alpha], \\ 0 & \text{if } x \in [\beta, \beta], \end{cases} \quad \text{and} \quad \chi_2(x) = \begin{cases} 0 & \text{if } x \in [0, \beta], \\ 1 & \text{if } x \in [\gamma, L], \end{cases} \]

and set \( \max_{x \in [0, L]} |\chi_1'(x)| = M_{\chi_1} \) and \( \max_{x \in [0, L]} |\chi_2'(x)| = M_{\chi_2} \).

![Figure 2. Geometric description of the functions \( \chi_1 \) and \( \chi_2 \).](image)

**Lemma 3.6.** Under the hypothesis \( (H) \), the solution \( U^n = (u^n, v^n, y^n, z^n, \eta^n(\cdot, \rho))^T \in D(A) \) of system (3.5)-(3.9) satisfies the following limits

\[\lim_{n \to \infty} \left( \int_{0}^{\alpha} |y_x^n|^2 \, dx + \int_{0}^{\alpha} |z^n|^2 \, dx \right) = 0, \quad (3.62)\]
\[\lim_{n \to \infty} \left( a \int_{\beta}^{L} |v_x^n|^2 \, dx + \int_{\gamma}^{L} |v^n|^2 \, dx + \int_{\gamma}^{L} |y_x^n|^2 \, dx + \int_{\gamma}^{L} |z^n|^2 \, dx \right) = 0. \quad (3.63)\]

**Proof.** First, using the result of Lemma 3.5 with \( h = x \chi_1 \), then using the definition of \( c(\cdot), S_b(u^n, v^n, \eta^n) \) and \( \chi_1 \), we get

\[\int_{0}^{\alpha} |y_x^n|^2 \, dx + \int_{0}^{\alpha} |z^n|^2 \, dx \]
\[= - \int_0^\alpha |v| dx - a^{-1} \int_0^\alpha |S_1(u^n, v^n, \eta^n)|^2 dx \]

\[- \int_\alpha^\beta (\chi_1 + \chi_2) \left( a^{-1} |S_1(u^n, v^n, \eta^n)|^2 + |v|^2 + |y_x|^2 + |z_x|^2 \right) dx \]

\[- \Re \left\{ \frac{2c_0}{a} \int_\alpha^\beta x \chi_1 z_x S_1(u^n, v^n, \eta^n) dx \right\} + \Re \left\{ \frac{2c_0}{a} \int_\alpha^\beta x \chi_1 v^n u_x dx \right\} \]

\[- \Re \left\{ \frac{2i\lambda}{a} \int_\alpha^\beta x \chi_1 v^n (\kappa_1 v_x + \kappa_2 \eta_x)(\cdot, 1) dx \right\} + \Re \left\{ \frac{2}{a} \int_\alpha^\beta x \chi_1 f^{2,n} S_1(u^n, v^n, \eta^n) dx \right\} \]

\[+ \Re \left\{ 2 \int_0^L x \chi_1 \left( \frac{f^1, n}{f_x} u^n + \frac{f^4, n}{f_x} z^n + f^{4, n} \frac{y_z}{v} \right) dx \right\}. \]

Using Cauchy-Schwarz inequality in the above equation and the fact that \(\|U^n\|_{H} = 1\), we obtain

\[\int_0^\alpha |y_x^n|^2 dx + \int_0^\alpha |z_x^n|^2 dx \leq \int_0^\alpha |v|^2 dx + a^{-1} \int_0^\alpha |S_1(u^n, v^n, \eta^n)|^2 dx \]

\[+ (1 + \beta M_{\chi_1}) \int_\alpha^\beta \left( a^{-1} |S_1(u^n, v^n, \eta^n)|^2 + |v|^2 + |z_x|^2 + |y_x|^2 \right) dx \]

\[+ \frac{2|c_0|\beta}{a} \left( \int_\alpha^\beta |S_1(u^n, v^n, \eta^n)|^2 dx \right)^{\frac{1}{2}} \]

\[+ \frac{2|\beta|\lambda}{a} \left[ \kappa_1 \left( \int_0^\beta |v_x^n|^2 dx \right)^{\frac{1}{2}} + \kappa_2 \left( \int_0^\beta |\eta_x^n(\cdot, 1)|^2 dx \right)^{\frac{1}{2}} \right] \]

\[+ \frac{2|\beta|}{a} \left( \int_0^\beta |S_1(u^n, v^n, \eta^n)|^2 dx \right)^{\frac{1}{2}} \|F^n\|_{H} + 2L \left( \frac{1}{\sqrt{a}} + 2 \right) \|F^n\|_{H}. \]

Passing to the limit in the above inequality, then using (3.2), Lemmas 3.1, 3.3 and the fact that \(\|F^n\|_{H} \to 0\), we obtain (3.62). On the other hand, using the result of Lemma 3.5 with \(h = (x - L)\chi_2\), then using Cauchy-Schwarz inequality and the fact that \(\|U^n\|_{H} = 1\), we get

\[a \int_\gamma^L |u_x^n|^2 dx + \int_\gamma^L |v^n|^2 dx + \int_\gamma^L |y_x^n|^2 dx + \int_\gamma^L |z_x^n|^2 dx \leq (1 + (L - \beta) M_{\chi_2}) \int_\gamma^\beta \left( a|u_x^n|^2 + |v|^2 + |y_x^n|^2 + |z_x|^2 \right) dx \]

\[+ 2|c_0|(L - \beta) \left( \int_\beta^\gamma |v^n|^2 dx \right)^{\frac{1}{2}} \left( \int_\beta^\gamma |y_x^n|^2 dx \right)^{\frac{1}{2}} \]

\[+ 2|c_0|(L - \beta) \left( \int_\beta^\gamma |z_x^n|^2 dx \right)^{\frac{1}{2}} \left( \int_\beta^\gamma |u_x^n|^2 dx \right)^{\frac{1}{2}} + 4L \left( \frac{1}{\sqrt{a}} + 1 \right) \|F^n\|_{H}. \]

Finally, passing to the limit in the above inequality, then using Lemma 3.4 and the fact that \(\|F^n\|_{H} \to 0\), we obtain (3.63). The proof is thus complete. \(\square\)
Proof of Proposition 3.1. From Lemmas 3.1-3.6, we obtain \( \|U^n\|_H \to 0 \) as \( n \to \infty \) which contradicts \( \|U^n\|_H = 1 \). Thus, (3.1) is holds true. The proof is thus complete. \( \square \)

Proof of Theorem 3.1. From proposition 3.1, we have \( i\mathbb{R} \subset \rho(\mathcal{A}) \) and consequently \( \sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset \). Therefore, according to Theorem A.1 in the appendix, we get that the \( C_0 \)-semigroup of contraction \( (e^{t\mathcal{A}})_{t \geq 0} \) is strongly stable. The proof is thus complete. \( \square \)

4. Polynomial stability. In this section, we will prove the polynomial stability of system (2.2)-(2.7). The main result of this section is the following theorem.

Theorem 4.1. Under the hypothesis (H), for all \( U_0 \in D(\mathcal{A}) \), there exists a constant \( C > 0 \) independent of \( U_0 \) such that the energy of system (2.2)-(2.7) satisfies the following estimation

\[
E(t) \leq \frac{C}{t} \|U_0\|_{D(\mathcal{A})}^2, \quad \forall t > 0.
\]

According to Theorem A.2 in the appendix, to prove Theorem 4.1, we still need to prove the following two conditions

\[
i\mathbb{R} \subset \rho(\mathcal{A}) \quad (4.1)
\]

and

\[
\limsup_{|\lambda| \to \infty} \frac{1}{|\lambda|^2} \|((i\lambda I - \mathcal{A})^{-1})\|_{\mathcal{L}(\mathcal{H})} < \infty. \quad (4.2)
\]

From Proposition 3.1, we obtain condition (4.1). Next, we will prove condition (4.2) by a contradiction argument. For this purpose, suppose that (4.2) is false, then there exists \( \{((\lambda^n, U^n := (u^n, v^n, y^n, z^n, \eta^n(\cdot, \rho))^\top)\}_{n \geq 1} \subset \mathbb{R}^* \times D(\mathcal{A}) \) with

\[
|\lambda^n| \to \infty \quad \text{and} \quad \|U^n\|_{\mathcal{H}} = \|((u^n, v^n, y^n, z^n, \eta^n(\cdot, \rho))^\top)\|_{\mathcal{H}} = 1, \quad (4.3)
\]

such that

\[
(\lambda^n)^2 (i\lambda^n I - \mathcal{A})U^n = F^n := (f_1^{n, 1}, f_2^{n, 2}, f_3^{n, 3}, f_4^{n, 4}, f_5^{n, 5} \cdot \rho)^\top \to 0 \quad \text{in} \quad \mathcal{H}. \quad (4.4)
\]

For simplicity, we drop the index \( n \). Equivalently, from (4.4), we have

\[
i\lambda u - v = \lambda^{-2} f_1, \quad f_1 \to 0 \quad \text{in} \quad H^1_0(0, L), \quad (4.5)
\]

\[
i\lambda v - (S_0(u, v, \eta))_x + c(\cdot) z = \lambda^{-2} f_2, \quad f_2 \to 0 \quad \text{in} \quad L^2(0, L), \quad (4.6)
\]

\[
i\lambda y - z = \lambda^{-2} f_3, \quad f_3 \to 0 \quad \text{in} \quad H^1_0(0, L), \quad (4.7)
\]

\[
i\lambda z - y_{xx} - c(\cdot) v = \lambda^{-2} f_4, \quad f_4 \to 0 \quad \text{in} \quad L^2(0, L), \quad (4.8)
\]

\[
i\lambda \eta(\cdot, \rho) + \tau^{-1} \eta_{\rho}(\cdot, \rho) = \lambda^{-2} f_5(\cdot, \rho), \quad f_5(\cdot, \rho) \to 0 \quad \text{in} \quad W. \quad (4.9)
\]

Here we will check the condition (4.2) by finding a contradiction with (4.3) such as \( \|U\|_{\mathcal{H}} = o(1) \). For clarity, we divide the proof into several Lemmas.

Lemma 4.1. Under the hypothesis (H), the solution \( U = (u, v, y, z, \eta(\cdot, \rho))^\top \in D(\mathcal{A}) \) of system (4.5)-(4.9) satisfies the following estimations

\[
\int_0^\beta |v_x|^2 dx = o(\lambda^{-2}), \quad (4.10)
\]

\[
\int_0^\beta |u_x|^2 dx = o(\lambda^{-4}), \quad (4.11)
\]
Thus, from (4.10), (4.17) and the fact that \( \eta \),

\[
\int_0^\beta \int_0^1 |\eta_x(\cdot, \rho)|^2 d\rho dx = o(\lambda^{-2}), \tag{4.12}
\]

\[
\int_0^\beta |\eta_x(\cdot, 1)|^2 dx = o(\lambda^{-2}), \tag{4.13}
\]

\[
\int_0^\beta |S_1(u, v, \eta)|^2 dx = o(\lambda^{-2}). \tag{4.14}
\]

**Proof.** First, taking the inner product of (4.4) with \( U \) in \( \mathcal{H} \) and using (2.17) with the help of hypothesis (H), we obtain

\[
\int_0^\beta |v_x|^2 dx \leq \frac{1}{\kappa_1 - |\kappa_2|} \Re(AU, U)_{\mathcal{H}} = \frac{\lambda^{-2}}{\kappa_1 - |\kappa_2|} \Re(F, U)_{\mathcal{H}} \leq \frac{\lambda^{-2}}{\kappa_1 - |\kappa_2|} \|F\| \|U\|_{\mathcal{H}}.
\]

Thus, from the above inequality and the fact that \( \|F\|_{\mathcal{H}} = o(1) \) and \( \|U\|_{\mathcal{H}} = 1 \), we obtain (4.10). Now, from (4.5), we deduce that

\[
\int_0^\beta |f_x|^2 dx \leq 2\lambda^{-2} \int_0^\beta |v_x|^2 dx + 2\lambda^{-4} \int_0^\beta |f_x|^2 dx
\]

\[
\leq 2\lambda^{-2} \int_0^\beta |v_x|^2 dx + 2\lambda^{-4} \int_0^L |f_x|^2 dx. \tag{4.15}
\]

Therefore, from (4.10), (4.15) and the fact that \( \|f_x\|_{L^2(0, L)} = o(1) \), we obtain (4.11).

Next, from (4.9) and the fact that \( \eta(\cdot, 0) = v(\cdot) \), we get

\[
\eta(x, \rho) = ve^{-i\lambda\tau \rho} + \tau \lambda^{-2} \int_0^\rho e^{i\lambda\tau(s-\rho)} f^5(x, s) ds, \quad (x, \rho) \in (0, \beta) \times (0, 1). \tag{4.16}
\]

From (4.16), we deduce that

\[
\int_0^\beta \int_0^1 |\eta_x(\cdot, \rho)|^2 d\rho dx \leq 2 \int_0^\beta |v_x|^2 dx + \tau^2 \lambda^{-4} \int_0^\beta \int_0^1 |f_x^5(\cdot, s)|^2 ds dx. \tag{4.17}
\]

Thus, from (4.10), (4.17) and the fact that \( f^5(\cdot, \rho) \to 0 \) in \( \mathcal{W} \), we obtain (4.12). On the other hand, from (4.16), we have

\[
\eta_x(\cdot, 1) = ve^{-i\lambda\tau} + \tau \lambda^{-2} \int_0^1 e^{i\lambda\tau(s-1)} f_x^5(\cdot, s) ds,
\]

consequently, similar to the previous proof, we obtain (4.13). Next, it is clear to see that

\[
\int_0^\beta |S_1(u, v, \eta)|^2 dx = \int_0^\beta |au_x + \kappa_1 v_x + \kappa_2 \eta_x(\cdot, 1)|^2 dx
\]

\[
\leq 3\kappa_2^2 \int_0^\beta |u_x|^2 dx + 3\kappa_1^2 \int_0^\beta |v_x|^2 dx + 3\kappa_2^2 \int_0^\beta |\eta_x(\cdot, 1)|^2 dx.
\]

Finally, from (4.10), (4.11), (4.13) and the above inequality, we obtain (4.14). The proof is thus complete.

**Lemma 4.2.** Let \( 0 < \varepsilon < \min\left(\frac{\alpha}{2}, \frac{\beta - \alpha}{4}\right) \). Under the hypothesis (H), the solution \( U = (u, v, y, z, \eta(\cdot, \rho))^T \in D(A) \) of system (4.5)-(4.9) satisfies the following estimation

\[
\int_\varepsilon^{\beta - \varepsilon} |v|^2 dx = o(1). \tag{4.18}
\]
Proof. First, we fix a cut-off function $\theta_1 \in C^1([0,L])$ (see Figure 3) such that $0 \leq \theta_1(x) \leq 1$, for all $x \in [0, L]$ and

$$\theta_1(x) = \begin{cases} 1 & \text{if } x \in [\varepsilon, \beta - \varepsilon], \\ 0 & \text{if } x \in \{0\} \cup [\beta, L], \end{cases}$$

and set

$$\max_{x \in [0,L]} |\theta'_1(x)| = M\theta'_1.$$

Multiplying (4.6) by $\lambda^{-1} \theta_1 \varpi$, integrating over $(0, L)$, then taking the imaginary part, we obtain

$$\int_0^L \theta_1 |v|^2 dx - \Im \left\{ \frac{1}{\lambda} \int_0^L \theta_1 (S_b(u,v,\eta))_x \varpi dx \right\} + \Im \left\{ \frac{1}{\lambda} \int_0^L c(\cdot) \theta_1 z \varpi dx \right\}$$

$$= \Im \left\{ \frac{1}{\lambda^2} \int_0^L \theta_1 f^2 \varpi dx \right\}.$$  \hspace{1cm} (4.19)

Using integration by parts in the above equation and the fact that $v(0) = v(L) = 0$, we get

$$\int_0^L \theta_1 |v|^2 dx = -\Im \left\{ \frac{1}{\lambda} \int_0^L (\theta'_1 \varpi + \theta_1 \varpi_x) S_b(u,v,\eta) dx \right\}$$

$$- \Im \left\{ \frac{1}{\lambda} \int_0^L c(\cdot) \theta_1 z \varpi dx \right\} + \Im \left\{ \frac{1}{\lambda^3} \int_0^L \theta_1 f^2 \varpi dx \right\}.$$  \hspace{1cm} (4.19)

Using the definition of $c(\cdot)$, $S_b(u,v,\eta)$ and $\theta_1$, then using Cauchy-Schwarz inequality, we obtain

$$\left| \Im \left\{ \frac{1}{\lambda} \int_0^L (\theta'_1 \varpi + \theta_1 \varpi_x) S_b(u,v,\eta) dx \right\} \right|$$

$$= \left| \Im \left\{ \frac{1}{\lambda} \int_0^\beta (\theta'_1 \varpi + \theta_1 \varpi_x) S_1(u,v,\eta) dx \right\} \right|$$

$$\leq \frac{1}{|\lambda|} \left[ M\theta'_1 \left( \int_0^\beta |v|^2 dx \right)^{\frac{1}{2}} + \left( \int_0^\beta |v_x|^2 dx \right)^{\frac{1}{2}} \right] \left( \int_0^\beta |S_1(u,v,\eta)|^2 dx \right)^{\frac{1}{2}}$$  \hspace{1cm} (4.20)
and
\[
\left| \frac{1}{\lambda} \int_0^L c(\cdot) \theta_1 z \, dx \right| = \left| \frac{c_0}{\lambda} \int_\alpha^\beta \theta_1 z \, dx \right| \leq \left| \frac{c_0}{\lambda} \right| \left( \int_\alpha^\beta |z|^2 \, dx \right)^\frac{1}{2} \left( \int_\alpha^\beta |v|^2 \, dx \right)^\frac{1}{2}.
\] (4.21)

From the above inequalities, Lemma 4.1 and the fact that \(v\) and \(z\) are uniformly bounded in \(L^2(0, L)\), we obtain
\[
\begin{cases}
-\Im \left\{ \frac{1}{\lambda} \int_0^L (\theta_1^\# \nu + \theta_1^\# \pi) S_b(u, v, \eta) \, dx \right\} = o(1) \\
-\Im \left\{ \frac{1}{\lambda} \int_0^L c(\cdot) \theta_1 z \, dx \right\} = O(1) \implies o(1).
\end{cases}
\] (4.22)

Inserting (4.22) in (4.19), then using the fact that \(v\) is uniformly bounded in \(L^2(0, L)\) and \(\|f\|_{L^2(0, L)} = o(1)\), we obtain
\[
\int_0^L \theta_1 |v|^2 \, dx = o(1).
\]

Finally, from the above estimation and the definition of \(\theta_1\), we obtain (4.18). The proof is thus complete.

**Lemma 4.3.** Let \(0 < \varepsilon < \min\left(\frac{\beta - \alpha}{2}, \frac{\beta - \alpha}{4}\right)\). Under the hypothesis (H), the solution \(U = (u, v, y, z, \eta, \rho)^T \in D(A)\) of system (4.5)-(4.9) satisfies the following estimations
\[
\int_{\alpha - 2\varepsilon}^{\beta - 2\varepsilon} |z|^2 \, dx = o(1) \quad \text{and} \quad \int_{\alpha + \varepsilon}^{\beta - 3\varepsilon} |y_{xx}|^2 \, dx = o(1).
\] (4.23)

**Proof.** First, we fix a cut-off function \(\theta_2 \in C^1([0, L])\) (see figure 3) such that \(0 \leq \theta_2(x) \leq 1\), for all \(x \in [0, L]\) and
\[
\theta_2(x) = \begin{cases} 
0 & \text{if } x \in [0, \varepsilon] \cup [\beta - \varepsilon, L], \\
1 & \text{if } x \in [2\varepsilon, \beta - 2\varepsilon],
\end{cases}
\]
and set
\[
\max_{x \in [0, L]} |\theta_2'(x)| = M_{\theta_2}'.
\]

Multiplying (4.6) and (4.8) by \(\theta_2 \pi\) and \(\theta_2^\# \pi\) respectively, integrating over \((0, L)\), then taking the real part, we obtain
\[
\Re \left\{ i \lambda \int_0^L \theta_2 \pi \, dx \right\} - \Re \left\{ \int_0^L \theta_2(S_b(u, v, \eta)) \pi \, dx \right\} + \int_0^L c(\cdot) \theta_2 |z|^2 \, dx
\]
\[
= \Re \left\{ \frac{1}{\lambda^2} \int_0^L \theta_2 f^2 \pi \, dx \right\}
\] (4.24)

and
\[
\Re \left\{ i \lambda \int_0^L \theta_2 z \, dx \right\} - \Re \left\{ \int_0^L \theta_2 y_{xx} \pi \, dx \right\} - \int_0^L c(\cdot) \theta_2 |v|^2 \, dx
\]
\[
= \Re \left\{ \frac{1}{\lambda^2} \int_0^L \theta_2 f^4 \pi \, dx \right\}.
\] (4.25)
Adding (4.24) and (4.25), then using integration by parts and the fact that \( v(0) = v(L) = 0 \) and \( z(0) = z(L) = 0 \), we get

\[
\int_0^L c(\cdot) \theta_2 |z|^2 \, dx = \int_0^L c(\cdot) \theta_2 |v|^2 \, dx - \Re \left\{ \int_0^L (\theta_2' \bar{z} + \theta_2 \bar{\bar{z}}) S_0(u, v, \eta) \, dx \right\} - \Re \left\{ \int_0^L (\theta_2' \bar{v} + \theta_2 \bar{\bar{v}}) y_x \, dx \right\} + \Re \left\{ \frac{1}{\lambda^2} \int_0^L \theta_2 (f^2 \bar{z} + f^4 \bar{\bar{z}}) \, dx \right\}. \tag{4.26}
\]

From (4.7), we deduce that

\[
\bar{\bar{z}} = -i \lambda y_x - \lambda^2 f_x^4. \tag{4.27}
\]

Using (4.27) and the definition of \( S_0(u, v, \eta) \) and \( \theta_2 \), then using Cauchy-Schwarz inequality, we obtain

\[
\left| \Re \left\{ \int_0^L (\theta_2' \bar{z} + \theta_2 \bar{\bar{z}}) S_0(u, v, \eta) \, dx \right\} \right| = \left| \Re \left\{ \int_0^L \left[ \theta_2' \bar{z} + \theta_2 \left( -i \lambda y_x - \frac{1}{\lambda^2} f_x^4 \right) \right] S_1(u, v, \eta) \, dx \right\} \right|
\leq M_{\theta_2} \left( \int_0^L \bar{\bar{z}}^2 \, dx \right)^{\frac{1}{2}} + |\lambda| \left( \int_0^L |y_x|^2 \, dx \right)^{\frac{1}{2}} + \frac{1}{\lambda^2} \left( \int_0^L f_x^4 \, dx \right)^{\frac{1}{2}} \left( \int_0^L |S_1(u, v, \eta)|^2 \, dx \right)^{\frac{1}{2}}
\]

and

\[
\left| \Re \left\{ \int_0^L (\theta_2' \bar{v} + \theta_2 \bar{\bar{v}}) y_x \, dx \right\} \right| = \left| \Re \left\{ \int_0^L \left[ \theta_2' \bar{v} + \theta_2 \left( -i \lambda y_x - \frac{1}{\lambda^2} f_x^4 \right) \right] y_x \, dx \right\} \right|
\leq M_{\theta_2} \left( \int_0^L |v|^2 \, dx \right)^{\frac{1}{2}} + \left( \int_0^L |y_x|^2 \, dx \right)^{\frac{1}{2}} \left( \int_0^L |S_1(u, v, \eta)|^2 \, dx \right)^{\frac{1}{2}}.
\]

From the above inequalities, Lemmas 4.1, 4.2 and the fact that \( y_x, z \) are uniformly bounded in \( L^2(0, L) \) and \( \| f_x^4 \|_{L^2(0, L)} = o(1) \), we obtain

\[
\begin{align*}
- \Re \left\{ \int_0^L (\theta_2' \bar{z} + \theta_2 \bar{\bar{z}}) S_0(u, v, \eta) \, dx \right\} = o(1) \\
- \Re \left\{ \int_0^L (\theta_2' \bar{v} + \theta_2 \bar{\bar{v}}) y_x \, dx \right\} = o(1). \tag{4.28}
\end{align*}
\]

Inserting (4.28) in (4.26), then using the fact that \( v, z \) are uniformly bounded in \( L^2(0, L) \) and \( \| f^4 \|_{L^2(0, L)} = o(1) \), \( \| f^4 \|_{L^2(0, L)} = o(1) \), we obtain

\[
\int_0^L c(\cdot) \theta_2 |z|^2 \, dx = \int_0^L c(\cdot) \theta_2 |v|^2 \, dx + o(1).
\]
Therefore, from the above estimation, Lemma 4.2 and the definition of \( c(\cdot) \) and \( \theta_2 \), we obtain the first estimation in (4.23). On the other hand, let us fix a cut-off function \( \theta_3 \in C^1([0, L]) \) (see Figure 3) such that \( 0 \leq \theta_3(x) \leq 1 \), for all \( x \in [0, L] \) and
\[
\theta_3(x) = \begin{cases} 
0 & \text{if } x \in [0, \alpha] \cup [\beta - 2\varepsilon, L], \\
1 & \text{if } x \in [\alpha + \varepsilon, \beta - 3\varepsilon],
\end{cases}
\]
Now, multiplying (4.8) by \(-\lambda^{-1}\theta_3\), integrating over \((0, L)\), then taking the imaginary part, we obtain
\[
- \int_0^L \theta_3|z|^2dx + 3 \left\{ \frac{1}{\lambda} \int_0^L \theta_3v_{xx}zdx \right\} + \Im \left\{ \frac{1}{\lambda^3} \int_0^L c(\cdot) \theta_3vzdx \right\} = -3 \left\{ \frac{1}{\lambda^3} \int_0^L \theta_3f^4zdx \right\}.
\] (4.29)
Using integration by parts in the above equation and the fact that \( z(0) = z(L) = 0 \), then using (4.27), we get
\[
\int_0^L \theta_3|y_x|^2dx = \int_0^L \theta_3|z|^2dx + 3 \left\{ \frac{1}{\lambda} \int_0^L \theta_3y_xzdx \right\}
\] (4.30)
\[
- \Im \left\{ \frac{1}{\lambda} \int_0^L c(\cdot) \theta_3vzdx \right\} = -3 \left\{ \frac{1}{\lambda^3} \int_0^L \theta_3(f^2y_x + f^4z)dx \right\}.
\]
From the definition of \( c(\cdot) \) and \( \theta_3 \), the first estimation of (4.23), and the fact that \( v \) and \( y_x \) are uniformly bounded in \( L^2(0, L) \), we obtain
\[
\left\{ \Im \left\{ \frac{1}{\lambda} \int_0^L \theta_3' y_xzdx \right\} \right\} = \Im \left\{ \frac{1}{\lambda} \int_0^{\beta-2\varepsilon} \theta_3' y_xzdx \right\} = \frac{o(1)}{|\lambda|},
\]
(4.31)
\[
-3 \left\{ \frac{1}{\lambda} \int_0^L c(\cdot) \theta_3vzdx \right\} = -3 \left\{ c_0 \frac{c_0}{\lambda} \int_0^{\beta-2\varepsilon} \theta_3vzdx \right\} = \frac{o(1)}{|\lambda|}.
\]
Inserting (4.31) in (4.30), then using the fact that \( y_x, z \) are uniformly bounded in \( L^2(0, L) \) and \( \|f_x^2\|_{L^2(0, L)} = o(1) \), \( \|f^4\|_{L^2(0, L)} = o(1) \), we get
\[
\int_0^L \theta_3|y_x|^2dx = \int_0^L \theta_3|z|^2dx + o(|\lambda|^{-1}).
\]
Finally, from the above estimation, the first estimation of (4.23) and the definition of \( \theta_3 \), we obtain the second estimation in (4.23). The proof is thus complete. □

**Lemma 4.4.** \( 0 < \varepsilon < \min\left(\frac{\alpha}{7}, \frac{\beta - \alpha}{4}\right) \). Under the hypothesis (H), the solution \( U = (u, v, y, z, \eta, \rho)^\top \in D(A) \) of system (4.5)-(4.9) satisfies the following estimations
\[
|v(\gamma)|^2 + |v(\beta - 3\varepsilon)|^2 + a|u_x(\gamma)|^2 + a^{-1} \left| (S_1(u, v, \eta)) (\beta - 3\varepsilon) \right|^2 = O(1),
\] (4.32)
\[
|z(\gamma)|^2 + |z(\beta - 3\varepsilon)|^2 + |y_x(\gamma)|^2 + |y_x(\beta - 3\varepsilon)|^2 = O(1).
\] (4.33)

**Proof.** First, we fix a function \( g_2 \in C^1([\beta - 3\varepsilon, \gamma]) \) such that
\[
g_2(\beta - 3\varepsilon) = -g_2(\gamma) = 1
\]
and set
\[
\max_{x \in [\beta - 3\varepsilon, \gamma]} |g_2(x)| = M_{g_2} \quad \text{and} \quad \max_{x \in [\beta - 3\varepsilon, \gamma]} |g_2'(x)| = M_{g_2'}.
\]
From (4.5), we deduce that

\[ i\lambda u_x - v_x = \lambda^{-2} f^1_x. \]  \hspace{1cm} (4.34)

Multiplying (4.34) and (4.6) by \(2g_x \overline{v}\) and \(2a^{-1} g_2 \overline{S_\eta(u, v, \eta)}\) respectively, integrating over \((\beta - 3\varepsilon, \gamma)\), using the definition of \(c(\cdot)\) and \(S_\eta(u, v, \eta)\), then taking the real part, we obtain

\[ \Re \left\{ 2i\lambda \int_{\beta-3\varepsilon}^{\gamma} g_2 u_x \overline{v} dx \right\} - \Re \left\{ 2\lambda^{-2} \int_{\beta-3\varepsilon}^{\gamma} g_2 f^1_x \overline{v} dx \right\} \]

and

\[ \Re \left\{ 2i\lambda \int_{\beta-3\varepsilon}^{\gamma} g_2 v \overline{u} dx \right\} + \Re \left\{ \frac{2i\lambda}{a} \int_{\beta-3\varepsilon}^{\gamma} g_2 v (\kappa_1 \overline{u} + \kappa_2 \overline{v}(\cdot, 1)) dx \right\} \]

\[ - \frac{1}{a} \int_{\beta-3\varepsilon}^{\gamma} g_2 \left( |S_1(u, v, \eta)|^2 \right) dx - a \int_{\beta}^{\gamma} g_2 \left( |u_x|^2 \right) dx \]

\[ + \Re \left\{ \frac{2c_0}{a} \int_{\beta-3\varepsilon}^{\gamma} g_2 z \overline{S_1(u, v, \eta)} dx \right\} + \Re \left\{ 2c_0 \int_{\beta}^{\gamma} g_2 z \overline{S_1(u, v, \eta)} dx \right\} \]

\[ \Re \left\{ \frac{2}{a \lambda^2} \int_{\beta-3\varepsilon}^{\gamma} g_2 f^2 \overline{S_1(u, v, \eta)} dx \right\} + \Re \left\{ \frac{2}{\lambda^2} \int_{\beta}^{\gamma} g_2 f^2 \overline{S_1(u, v, \eta)} dx \right\}. \]  \hspace{1cm} (4.35)

Adding the above equations, then using integration by parts, we get

\[ -g_2 |v|^2 \bigg|_{\beta-3\varepsilon}^{\gamma} + \left[ -\frac{1}{a} g_2 |S_1(u, v, \eta)|^2 \bigg|_{\beta-3\varepsilon}^{\gamma} + \left[ -a g_2 |u_x|^2 \bigg|_{\beta}^{\gamma} \right. \]

\[ = \int_{\beta-3\varepsilon}^{\gamma} g_2 |v|^2 dx - a^{-1} \int_{\beta-3\varepsilon}^{\gamma} g_2 |S_1(u, v, \eta)|^2 dx - a \int_{\beta}^{\gamma} g_2 |u_x|^2 dx \]

\[- \Re \left\{ \frac{2i\lambda}{a} \int_{\beta-3\varepsilon}^{\gamma} g_2 v (\kappa_1 \overline{u} + \kappa_2 \overline{v}(\cdot, 1)) dx \right\} - \Re \left\{ \frac{2c_0}{a} \int_{\beta-3\varepsilon}^{\gamma} g_2 z \overline{S_1(u, v, \eta)} dx \right\} \]

\[- \Re \left\{ 2c_0 \int_{\beta}^{\gamma} g_2 z \overline{S_1(u, v, \eta)} dx \right\} + \Re \left\{ \frac{2}{a \lambda^2} \int_{\beta-3\varepsilon}^{\gamma} g_2 f^2 \overline{S_1(u, v, \eta)} dx \right\} \]

Using the definition of \(g_2\) and Cauchy-Schwarz inequality in the above equation, we obtain

\[ |v(\gamma)|^2 + |v(\beta - 3\varepsilon)|^2 + a |u_x(\gamma)|^2 + a^{-1} |(S_1(u, v, \eta)) (\beta - 3\varepsilon)|^2 + K(\beta) \]

\[ \leq M g_2 \left[ \int_{\beta-3\varepsilon}^{\gamma} |v|^2 dx \right] + a^{-1} \int_{\beta-3\varepsilon}^{\gamma} |S_1(u, v, \eta)|^2 dx + a \int_{\beta}^{\gamma} |u_x|^2 dx \]

\[ + \frac{2\lambda}{a} M g_2 \left[ \kappa_1 \left( \int_{\beta-3\varepsilon}^{\gamma} |u_x|^2 dx \right)^{\frac{1}{2}} + |\kappa_2| \left( \int_{\beta-3\varepsilon}^{\gamma} |\eta_x(\cdot, 1)|^2 dx \right)^{\frac{1}{2}} \right] \left( \int_{\beta-3\varepsilon}^{\gamma} |v|^2 dx \right)^{\frac{1}{2}} \]
\[ + \frac{2|c_0|M_{g_2}}{a} \left( \int_{\beta-3\varepsilon}^{\gamma} |S_1(u, v, \eta)|^2 dx \right)^\frac{1}{4} \left( \int_{\beta-3\varepsilon}^{\gamma} |z|^2 dx \right)^\frac{1}{4} + 2|c_0|M_{g_2} \left( \int_{\beta}^{\gamma} |z|^2 dx \right)^\frac{1}{2} \left( \int_{\beta}^{\gamma} |u_x|^2 dx \right)^\frac{1}{2} + \frac{2M_{g_2}}{\lambda^2} \left( \int_{\beta-3\varepsilon}^{\gamma} |f_1|^2 dx \right)^\frac{1}{4} \left( \int_{\beta-3\varepsilon}^{\gamma} v|^2 dx \right)^\frac{1}{4} + \frac{2M_{g_2}}{a\lambda^2} \left( \int_{\beta}^{\gamma} |f|^2 dx \right)^\frac{1}{2} \left( \int_{\beta}^{\gamma} |S_1(u, v, \eta)|^2 dx \right)^\frac{1}{2} + \frac{2M_{g_2}}{\lambda^2} \left( \int_{\beta}^{\gamma} |f|^2 dx \right)^\frac{1}{2} \left( \int_{\beta}^{\gamma} |u_x|^2 dx \right)^\frac{1}{2}. \] 

where \( K(\beta) = g_2(\beta)(a|u_x(\beta+)^2-a^{-1}||S_1(u, v, \eta))(|\beta+y|^2) \). Moreover, since \( S_0(u, v, \eta) \in H^1(0, L) \subset C([0, L]), \) then we obtain
\[ |(S_1(u, v, \eta))(\beta+y)|^2 = |au_x(\beta+y)^2 | \text{ and consequently } K(\beta) = 0. \] 

Inserting (4.37) in the above inequality, then using Lemma 4.1 and the fact that \( u_x, v, z \) are uniformly bounded in \( L^2(0, L) \) and \( \|f_1\|_{L^2(0, L)} = o(1), \|f\|_{L^2(0, L)} = o(1) \), we obtain (4.32). Next, from (4.7), we deduce that
\[ i\lambda y_x - z_x = \lambda^{-2} f_3. \] 

Multiplying (4.38) and (4.8) by \( 2g_2 \gamma \) and \( 2g_2 \gamma \) respectively, integrating over \((\beta-3\varepsilon, \gamma)\), using the definition of \( c(\cdot) \), then taking the real part, we obtain
\[ \Re\left\{ 2i\lambda \int_{\beta-3\varepsilon}^{\gamma} g_2 y_x \gamma dx \right\} - \int_{\beta-3\varepsilon}^{\gamma} g_2 \left( |z|^2 \right) dx = \Re\left\{ \frac{2}{\lambda^2} \int_{\beta-3\varepsilon}^{\gamma} g_2 f_3 \gamma dx \right\} \] 

and
\[ \Re\left\{ 2i\lambda \int_{\beta-3\varepsilon}^{\gamma} g_2 z_y \gamma dx \right\} - \int_{\beta-3\varepsilon}^{\gamma} g_2 \left( |y_x|^2 \right) dx - \Re\left\{ 2c_0 \int_{\beta-3\varepsilon}^{\gamma} g_2 v \gamma dx \right\} \]
\[ = \Re\left\{ \frac{2}{\lambda^2} \int_{\beta-3\varepsilon}^{\gamma} g_2 f \gamma y dx \right\}. \] 

Adding (4.39) and (4.40), then using integration by parts, we obtain
\[ \left[ -g_2 \left( |z|^2 + |y_x|^2 \right) \right]_{\beta-3\varepsilon}^{\gamma} \]
\[ [3.5mm] = - \int_{\beta-3\varepsilon}^{\gamma} g_2'(|z|^2 + |y_x|^2) dx + \Re\left\{ 2c_0 \int_{\beta-3\varepsilon}^{\gamma} g_2 v \gamma dx \right\} \]
\[ [3.5mm] + \Re\left\{ 2\lambda^{-2} \int_{\beta-3\varepsilon}^{\gamma} g_2 f_3 \gamma dx \right\} + \Re\left\{ 2\lambda^{-2} \int_{\beta-3\varepsilon}^{\gamma} g_2 f \gamma y dx \right\}. \] 

Using the definition of \( g_2 \) and Cauchy-Schwarz inequality in the above equation, we obtain
\[ |z(\gamma)|^2 + |z(\beta-3\varepsilon)|^2 + |y_x(\gamma)|^2 + |y_x(\beta-3\varepsilon)|^2 \]
\[ \leq M_{g_2} \int_{\beta-3\varepsilon}^{\gamma} \left( |z|^2 + |y_x|^2 \right) dx + 2|c_0|M_{g_2} \left( \int_{\beta-3\varepsilon}^{\gamma} |v|^2 dx \right)^\frac{1}{2} \left( \int_{\beta-3\varepsilon}^{\gamma} |y_x|^2 dx \right)^\frac{1}{2} \]
Finally, from the above inequality, the fact that \( v, y_x, z \) are uniformly bounded in \( L^2(0,L) \) and \( \|f^3_x\|_{L^2(0,L)} = o(1), \|f^4_x\|_{L^2(0,L)} = o(1) \), we obtain (4.33). The proof is thus complete.

\[ \text{(4.42)} \]

**Lemma 4.5.** Let \( h_2 \in C^1([0,L]) \) be a function. Under the hypothesis \((H)\), the solution \( U = (u,v,y,z,\eta)^\top \in D(A) \) of system \((4.5)-(4.9)\) satisfies the following equality

\[
\begin{align*}
\int_0^L h_2' \left( a^{-1}|S_b(u,v,\eta)|^2 + |v|^2 + |z|^2 + |y_x|^2 \right) dx - \left[ h_2 \left( a^{-1}|S_b(u,v,\eta)|^2 + |y_x|^2 \right) \right]_0^L \\
- \Re \left\{ 2 \int_0^L c(\cdot) h_2 v y_x dx \right\} + \Re \left\{ \frac{2}{a} \int_0^L c(\cdot) h_2 z S_b(u,v,\eta) dx \right\} \\
+ \Re \left\{ \frac{2i\lambda}{a} \int_0^\beta h_2 v^n (\kappa_1 \tau_x + \kappa_2 \bar{\tau}_x (\cdot, 1)) dx \right\}
\end{align*}
\]

\[
= \Re \left\{ \frac{2}{\lambda^2} \int_0^L h_2 f \bar{f}_x^2 dx \right\} + \Re \left\{ \frac{2}{\lambda^2} \int_0^L h_2 f^2 \bar{S}_b(u,v,\eta) dx \right\} \\
+ \Re \left\{ \frac{2}{\lambda^2} \int_0^L h_2 f^4 \bar{y}_x dx \right\}.
\]

**Proof.** See the proof of Lemma 3.5.

Let \( 0 < \varepsilon < \min(\frac{\alpha}{2}, \frac{\beta - \alpha}{4}) \), we fix the cut-off functions \( \theta_4, \theta_5 \in C^1([0,L]) \) (see Figure 4) such that \( 0 \leq \theta_4(x) \leq 1, 0 \leq \theta_5(x) \leq 1, \) for all \( x \in [0,L] \) and

\[
\theta_4(x) = \begin{cases} 
1 & \text{if } x \in [0,\alpha + \varepsilon], \\
0 & \text{if } x \in [\beta - 3\varepsilon, L],
\end{cases} \quad \text{and} \quad \theta_5(x) = \begin{cases} 
0 & \text{if } x \in [0,\alpha + \varepsilon], \\
1 & \text{if } x \in [\beta - 3\varepsilon, L],
\end{cases}
\]

**Figure 4.** Geometric description of the functions \( \theta_4 \) and \( \theta_5 \).
Lemma 4.6. Let $0 < \varepsilon < \min\left(\frac{\alpha}{2}, \frac{\beta - \alpha}{4}\right)$. Under the hypothesis (H), the solution $U = (u, v, y, z, \eta(\cdot, \rho))^T \in D(A)$ of system (4.5)-(4.9) satisfies the following estimations

\begin{align*}
\int_0^{\alpha + \varepsilon} |v|^2 dx + \int_0^{\alpha + \varepsilon} |y_x|^2 dx + \int_0^{\alpha + \varepsilon} |z|^2 dx &= o(1), \quad (4.43) \\
a \int_0^L |u_x|^2 dx + \int_0^L |v|^2 dx + \int_0^L |y_x|^2 dx + \int_0^L |z|^2 dx &= o(1). \quad (4.44)
\end{align*}

Proof. First, using the result of Lemma 4.5 with $h_2 = x\theta_4$, we obtain

\begin{align*}
\int_0^{\alpha + \varepsilon} |v|^2 dx + \int_0^{\alpha + \varepsilon} |y_x|^2 dx + \int_0^{\alpha + \varepsilon} |z|^2 dx \\
= - a^{-1} \int_0^{\alpha + \varepsilon} |S_1(u, v, \eta)|^2 dx - \int_0^{\beta - 3\varepsilon} (\theta_4 + x\theta_4^2) \left( \frac{1}{a} |S_1(u, v, \eta)|^2 + |v|^2 + |y_x|^2 + |z|^2 \right) dx \\
+ \Re \left\{ 2 \int_0^L xc(\cdot)\theta_4 v S_b dx \right\} \\
- \Re \left\{ \frac{2}{a} \int_0^L xc(\cdot)\theta_4^2 S_b(u, v, \eta) dx \right\} - \Re \left\{ \frac{2i\lambda}{a} \int_0^\beta x\theta_4 v (\kappa_1 v + \kappa_2 S_b(\cdot, 1)) dx \right\} \\
+ \Re \left\{ \frac{2}{a} \int_0^L x\theta_4 f S_b dx \right\} + \Re \left\{ \frac{2}{a} \int_0^L x\theta_4 f^2 S_b dx \right\} \\
+ \Re \left\{ \frac{2}{a} \int_0^L x\theta_4 f^2 S_b dx \right\} + \Re \left\{ \frac{2}{a} \int_0^L x\theta_4 f^4 S_b dx \right\}. \quad (4.45)
\end{align*}

From the above equation, Lemmas 4.1-4.3 and the fact that $v, y_x, z$ are uniformly bounded in $L^2(0, L)$ and $\|f_x\|_{L^2(0, L)} = o(1), \|f^3\|_{L^2(0, L)} = o(1), \|f^4\|_{L^2(0, L)} = o(1)$, we obtain

\begin{align*}
\int_0^{\alpha + \varepsilon} |v|^2 dx + \int_0^{\alpha + \varepsilon} |y_x|^2 dx + \int_0^{\alpha + \varepsilon} |z|^2 dx \\
= \Re \left\{ 2 \int_0^L xc(\cdot)\theta_4 v S_b dx \right\} \\
- \Re \left\{ \frac{2}{a} \int_0^L xc(\cdot)\theta_4^2 S_b(u, v, \eta) dx \right\} + \Re \left\{ \frac{2}{a} \int_0^\beta x\theta_4 v (\kappa_1 v + \kappa_2 S_b(\cdot, 1)) dx \right\} \\
- \Re \left\{ \frac{2i\lambda}{a} \int_0^\beta x\theta_4 v (\kappa_1 v + \kappa_2 S_b(\cdot, 1)) dx \right\} + o(1). \quad (4.46)
\end{align*}
Using the definition of $c(\cdot)$, $S_0(u,v,\eta)$ and $\theta_4$, then using Cauchy-Schwarz inequality, we obtain

\[
\begin{align*}
\Re \left\{ 2 \int_0^L x c(\cdot) \theta_4 \sqrt[3]{v} dx \right\} &= \Re \left\{ 2 c_2 \int_0^{\frac{3-3\varepsilon}{\alpha}} \theta_4 \sqrt[3]{v} dx \right\} \\
&\leq 2 |c_2| \left( \int_0^{\frac{3-3\varepsilon}{\alpha}} |v|^2 dx \right)^{\frac{1}{2}} \left( \int_0^{\frac{3-3\varepsilon}{\alpha}} |y_x|^2 dx \right)^{\frac{1}{2}}, \\
\Re \left\{ \frac{2}{a} \int_0^L x c(\cdot) \theta_4 z \Re S_0(u,v,\eta) dx \right\} &= \Re \left\{ \frac{2 c_2}{a} \int_0^{\frac{3-3\varepsilon}{\alpha}} \theta_4 z \Re S_1(u,v,\eta) dx \right\} \\
&\leq 2 c_2 \left( \int_0^{\frac{3-3\varepsilon}{\alpha}} |z|^2 dx \right)^{\frac{1}{2}} \left( \int_0^{\frac{3-3\varepsilon}{\alpha}} |S_1(u,v,\eta)|^2 dx \right)^{\frac{1}{2}}, \\
\Re \left\{ \frac{2}{a \lambda^2} \int_0^L x \theta_4 f^2 S_0(u,v,\eta) dx \right\} &= \Re \left\{ \frac{2 c_2}{a \lambda^2} \int_0^{\frac{3-3\varepsilon}{\alpha}} \theta_4 f^2 S_1(u,v,\eta) dx \right\} \\
&\leq \frac{2(\beta - 3\varepsilon)}{a \lambda^2} \left( \int_0^{\frac{3-3\varepsilon}{\alpha}} |f|^2 dx \right)^{\frac{1}{2}} \left( \int_0^{\frac{3-3\varepsilon}{\alpha}} |S_1(u,v,\eta)|^2 dx \right)^{\frac{1}{2}}, \\
\Re \left\{ \frac{2 i \lambda}{a} \int_0^L x \theta_4 v (\kappa_1 v_x + \kappa_2 \eta_x) dx \right\} &= \Re \left\{ \frac{2 i \lambda}{a} \int_0^{\frac{3-3\varepsilon}{\alpha}} x \theta_4 v (\kappa_1 v_x + \kappa_2 \eta_x) dx \right\} \\
&\leq 2 \left( \int_0^{\frac{3-3\varepsilon}{\alpha}} |v|^2 dx \right)^{\frac{1}{2}} \left( \int_0^{\frac{3-3\varepsilon}{\alpha}} |v_x|^2 dx \right)^{\frac{1}{2}}.
\end{align*}
\]

From the above inequalities, Lemmas 4.1-4.3 and the fact that $v, y_x$ are uniformly bounded in $L^2(0,L)$ and $\|f^2\|_{L^2(0,L)} = o(1)$, we obtain

\[
\begin{align*}
\Re \left\{ 2 \int_0^L x c(\cdot) \theta_4 \sqrt[3]{v} dx \right\} &= o(1), \\
-\Re \left\{ \frac{2}{a} \int_0^L x c(\cdot) \theta_4 z \Re S_0(u,v,\eta) dx \right\} &= o(|\lambda|^{-1}), \\
\Re \left\{ \frac{2}{a \lambda^2} \int_0^L x \theta_4 f^2 S_0(u,v,\eta) dx \right\} &= o(|\lambda|^{-3}), \\
-\Re \left\{ \frac{2 i \lambda}{a} \int_0^L x \theta_4 v (\kappa_1 v_x + \kappa_2 \eta_x) dx \right\} &= o(1).
\end{align*}
\]

Therefore, by inserting (4.47) in (4.46), we obtain (4.43). On the other hand, using the result of Lemma 4.5 with $h = (x - L)\theta_5$, then using the definition of
$S_0(u, v, \eta)$ and $\theta_5$. Lemmas 4.1-4.3, and the fact that $u_x$, $v$, $y_x$, $z$ are uniformly bounded in $L^2(0, L)$ and $\|f_1^1\|_{L^2(0, L)} = o(1)$, $\|f_2^2\|_{L^2(0, L)} = o(1)$, $\|f_2^3\|_{L^2(0, L)} = o(1)$, we obtain

$$a \int_{\beta}^{L} |u_x|^2 dx + \int_{\beta-3\varepsilon}^{L} |v|^2 dx + \int_{\beta-3\varepsilon}^{L} |y_x|^2 dx + \int_{\beta-3\varepsilon}^{L} |z|^2 dx = I + o(1), \quad (4.48)$$

where

$$I := \Re \left\{ 2 \int_{0}^{L} (x - L)c(\cdot)\theta_5 v\overline{v} dx \right\} - \Re \left\{ 2a^{-1} \int_{0}^{L} (x - L)c(\cdot)\theta_5 zS_0(u, v, \eta) dx \right\}.$$ 

Moreover, using the definition of $c(\cdot)$, $S_0(u, v, \eta)$ and $\theta_5$, we get

$$I = \Re \left\{ 2c_0 \int_{\beta-3\varepsilon}^{\beta-3\varepsilon} (x - L)\theta_5 v\overline{v} dx \right\} + \Re \left\{ 2c_0 \int_{\beta-3\varepsilon}^{\gamma} (x - L)v\overline{v} dx \right\} - \Re \left\{ 2c_0 \int_{\beta-3\varepsilon}^{\gamma} (x - L)z\overline{v} dx \right\} - \Re \left\{ 2c_0 \int_{\beta-3\varepsilon}^{\gamma} (x - L)z\overline{v} dx \right\}.$$ 

Using Cauchy-Schwarz inequality, Lemmas 4.1-4.3 and the fact that $z$ is uniformly bounded in $L^2(0, L)$, we obtain

$$\begin{aligned}
&\Re \left\{ 2c_0 \int_{\beta-3\varepsilon}^{\gamma} (x - L)\theta_5 v\overline{v} dx \right\} = o(1), \\
-\Re \left\{ 2c_0 \int_{\beta-3\varepsilon}^{\gamma} (x - L)\theta_5 z\overline{v} dx \right\} = o(|\lambda|^{-1}), \\
-\Re \left\{ 2c_0 \int_{\beta-3\varepsilon}^{\gamma} (x - L)z\overline{v} dx \right\} = o(|\lambda|^{-1}).
\end{aligned} \quad (4.49)$$

Inserting (4.49) in the above equation, we get

$$I = \Re \left\{ 2c_0 \int_{\beta-3\varepsilon}^{\gamma} (x - L)v\overline{v} dx \right\} - \Re \left\{ 2c_0 \int_{\beta-3\varepsilon}^{\gamma} (x - L)z\overline{v} dx \right\} + o(1). \quad (4.50)$$

From (4.5) and (4.7), we deduce that

$$\overline{v} = i\lambda^{-1}\overline{v} + i\lambda^{-\beta}\overline{f}_2 \quad \text{and} \quad \overline{v} = i\lambda^{-1}\overline{v} + i\lambda^{-\beta}\overline{f}_2. \quad (4.51)$$

Substituting (4.51) in (4.50), then using the fact that $v$, $z$ are uniformly bounded in $L^2(0, L)$ and $\|f_1^1\|_{L^2(0, L)} = o(1)$, $\|f_2^3\|_{L^2(0, L)} = o(1)$, we obtain

$$I = \Re \left\{ 2c_0 i \int_{\beta-3\varepsilon}^{\gamma} (x - L)v\overline{v} dx \right\} - \Re \left\{ 2c_0 i \int_{\beta-3\varepsilon}^{\gamma} (x - L)z\overline{v} dx \right\} + o(1).$$

Using integration by parts to the second term in the above equation, we obtain

$$I = \Re \left\{ 2c_0 i \int_{\beta-3\varepsilon}^{\gamma} z\overline{v} dx \right\} - \Re \left\{ 2c_0 i \int_{\beta-3\varepsilon}^{\gamma} [(x - L) z\overline{v}]_{\beta-3\varepsilon} \right\} + o(1). \quad (4.52)$$

Furthermore, by using Cauchy-Schwarz inequality, we get

$$\left| \Re \left\{ 2c_0 i \int_{\beta-3\varepsilon}^{\gamma} z\overline{v} dx \right\} \right| \leq 2|c_0| |\lambda|^{-1} \left( \int_{\beta-3\varepsilon}^{\gamma} |z|^2 dx \right)^{\frac{1}{2}} \left( \int_{\beta-3\varepsilon}^{\gamma} |v|^2 dx \right)^{\frac{1}{2}}. \quad (4.53)$$
and
\[
\Re \left\{ \frac{2ic_0}{\lambda} [(x-L)z\bar{\nu}]_{\beta-3\varepsilon}^\gamma \right\} \\
\leq 2|c_0||\lambda|^{-1} [(L-\gamma)|z(\gamma)||v(\gamma)| + (L-\beta + 3\varepsilon)|z(\beta-3\varepsilon)||v(\beta-3\varepsilon)|].
\] (4.54)

From Lemma 4.4, we deduce that
\[
|v(\beta-3\varepsilon)| = O(1), \ |v(\gamma)| = O(1), \ |z(\beta-3\varepsilon)| = O(1) \text{ and } |z(\gamma)| = O(1).
\] (4.55)
Using the fact that \(v, z\) are uniformly bounded in \(L^2(0,L)\) in (4.53) and inserting (4.55) in (4.54), we obtain
\[
\begin{cases}
\Re \left\{ \frac{2ic_0}{\lambda} \int_{\beta-3\varepsilon}^{\gamma} z\bar{\nu}dx \right\} = O \left( |\lambda|^{-1} \right) = o(1), \\
−\Re \left\{ \frac{2ic_0}{\lambda} [(x-L)z\bar{\nu}]_{\beta-3\varepsilon}^\gamma \right\} = O \left( |\lambda|^{-1} \right) = o(1).
\end{cases}
\] (4.56)
Inserting (4.56) in (4.52), we get
\[
\mathcal{I} = o(1).
\] (4.57)
Finally, inserting (4.57) in (4.48), we obtain (4.44). The proof is thus complete. \(\Box\)

**Proof of Theorem 4.1.** The proof of Theorem 4.1 is divided into three steps. **Step 1.** From Lemmas 4.1-4.3, we obtain
\[
\begin{cases}
\int_0^\beta |u_x|^2dx = o(1), \\
\int_0^\beta |\eta_x(\cdot, \rho)|^2d\rho dx = o \left( \frac{1}{\lambda^2} \right), \\
\int_0^{\beta-2\varepsilon} |v|^2dx = o(1), \\
\int_\alpha^{\beta} |z|^2dx = o(1), \\
\int_{\alpha+\varepsilon}^\beta |y_x|^2dx = o(1).
\end{cases}
\]
**Step 2.** From Lemma 4.6 and **Step 1,** we deduce that
\[
\begin{cases}
\int_0^\alpha |v|^2dx = o(1), \\
\int_0^{\alpha+\varepsilon} |y_x|^2dx = o(1), \\
\int_\alpha^\beta |z|^2dx = o(1), \\
\int_\beta^{\alpha+\varepsilon} |y_x|^2dx = o(1), \\
\int_\beta^L |u_x|^2dx = o(1), \\
\int_\beta^{\beta-\varepsilon} |v|^2dx = o(1), \\
\int_\beta^{\beta-3\varepsilon} |y_x|^2dx = o(1), \\
\int_\beta^{\beta-2\varepsilon} |z|^2dx = o(1).
\end{cases}
\]
**Step 3.** According to **Step 1** and **Step 2,** we obtain \(\|U\|_H = o(1)\) in \((0,L)\), which contradicts (4.3). Thus, (4.2) is holds true. Next, since conditions (4.1) and (4.2) are proved, then according to Theorem A.2 in the appendix, the proof of Theorem 4.1 is achieved. The proof is thus complete. \(\Box\)

5. **Conclusion.** We have studied the stabilization of a locally coupled wave equations with non smooth localized viscoelastic damping of Kelvin-Voigt type and localized time delay. We proved the strong stability of the system by using Arendt-Batty criteria. Finally, we established a polynomial energy decay rate of order \(t^{-1}\).

**Appendix A. Some notions and stability theorems.** In order to make this paper more self-contained, we recall in this short appendix some notions and stability results used in this work.

**Definition A.1.** Assume that \(A\) is the generator of \(C_0\)-semigroup of contractions \((e^{tA})_{t \geq 0}\) on a Hilbert space \(H\). The \(C_0\)-semigroup \((e^{tA})_{t \geq 0}\) is said to be
(1) Strongly stable if
\[ \lim_{t \to +\infty} \|e^{tA}x_0\|_H = 0, \quad \forall x_0 \in H. \]

(2) Exponentially (or uniformly) stable if there exists two positive constants \(M\) and \(\varepsilon\) such that
\[ \|e^{tA}x_0\|_H \leq Me^{-\varepsilon t}\|x_0\|_H, \quad \forall t > 0, \quad \forall x_0 \in H. \]

(3) Polynomially stable if there exists two positive constants \(C\) and \(\alpha\) such that
\[ \|e^{tA}x_0\|_H \leq Ct^{-\alpha}\|x_0\|_H, \quad \forall t > 0, \quad \forall x_0 \in D(A). \]

For proving the strong stability of the \(C_0\)-semigroup \((e^{tA})_{t \geq 0}\), we will recall the result obtained by Arendt and Batty in [12].

**Theorem A.1** (Arendt and Batty in [12]). Assume that \(A\) is the generator of a \(C_0\)–semigroup of contractions \((e^{tA})_{t \geq 0}\) on a Hilbert space \(H\). If \(A\) has no pure imaginary eigenvalues and \(\sigma(A) \cap i\mathbb{R}\) is countable, where \(\sigma(A)\) denotes the spectrum of \(A\), then the \(C_0\)-semigroup \((e^{tA})_{t \geq 0}\) is strongly stable.

There exist a second classical method based on Arendt and Batty theorem and the contradiction argument (see page 25 in [38]).

**Remark A.1.** Assume that the unbounded linear operator \(A : D(A) \subset H \rightarrow H\) is the generator of a \(C_0\)-semigroup of contractions \((e^{tA})_{t \geq 0}\) on a Hilbert space \(H\) and suppose that \(0 \in \rho(A)\). According to (page 25 in [38]), in order to prove that
\[ i\mathbb{R} \equiv \{i\lambda \mid \lambda \in \mathbb{R}\} \subseteq \rho(A), \quad (A.1) \]
we need the following steps:

(i) It follows from the fact that \(0 \in \rho(A)\) and the contraction mapping theorem that for any real number \(\lambda\) with \(|\lambda| < \|A^{-1}\|^{-1}\), the operator \(i\lambda I - A = A(i\lambda A^{-1} - I)\) is invertible. Furthermore, \(\|(i\lambda I - A)^{-1}\|\) is a continuous function of \(\lambda\) in the interval \((-\|A^{-1}\|^{-1}, \|A^{-1}\|^{-1})\).

(ii) If \(\sup \{\|(i\lambda I - A)^{-1}\| \mid |\lambda| < \|A^{-1}\|^{-1}\} = M < \infty\), then by the contraction mapping theorem, the operator \(i\lambda I - A = (i\lambda_0 I - A)(I + i(\lambda - \lambda_0)(i\lambda_0 I - A)^{-1})\) with \(|\lambda_0| < \|A^{-1}\|^{-1}\) is invertible for \(|\lambda - \lambda_0| < M^{-1}\). It turns out that by choosing \(|\lambda_0|\) as close to \(\|A^{-1}\|^{-1}\) as we can, we conclude that \(\{\lambda \mid |\lambda| < \|A^{-1}\|^{-1} + M^{-1}\} \subseteq \rho(A)\) and \(\|(i\lambda I - A)^{-1}\|\) is a continuous function of \(\lambda\) in the interval \((-\|A^{-1}\|^{-1} - M^{-1}, \|A^{-1}\|^{-1} + M^{-1})\).

(iii) Thus it follows from the argument in (ii) that if \((A.1)\) is false, then there is \(\omega \in \mathbb{R}\) with \(\|A^{-1}\|^{-1} \leq |\omega| < \infty\) such that \(\{i\lambda \mid |\lambda| < |\omega|\} \subseteq \rho(A)\) and \(\sup \{(i\lambda I - A)^{-1} \mid |\lambda| < |\omega|\} = \infty\). It turns out that there exists a sequence \((\lambda_n, U_n)\) \(n \geq 1 \subset \mathbb{R} \times D(A)\), with \(\lambda_n \to \omega\) as \(n \to \infty\), \(|\lambda_n| < |\omega|\) and \(\|U_n\|_H = 1\), such that
\[ (i\lambda_n I - A)U_n = F_n \to 0 \text{ in } H, \quad \text{as } n \to \infty. \]

Then, we will prove \((A.1)\) by finding a contradiction with \(\|U_n\|_H = 1\) such as \(\|U_n\|_H \to 0\).

Concerning the characterization of polynomial stability stability of a \(C_0\)-semigroup of contraction \((e^{tA})_{t \geq 0}\), we rely on the following result due to Bormiche and Tomilov [15] (see also [13, 35]).
Theorem A.2. Assume that $A$ is the generator of a strongly continuous semigroup of contractions $(e^{tA})_{t \geq 0}$ on $\mathcal{H}$. If $i\mathbb{R} \subset \rho(A)$, then for a fixed $\ell > 0$ the following conditions are equivalent

$$\limsup_{|\lambda| \to \infty} \frac{1}{|\lambda|^\ell} \left\| (i\lambda I - A)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < \infty,$$  \hfill (A.2)

$$\|e^{tA}U_0\|^2_H \leq C t^\ell \|U_0\|^2_{D(A)}, \quad \forall t > 0, \; U_0 \in D(A), \; \text{for some } C > 0.$$  \hfill (A.3)

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