BMO TEICHMÜLLER SPACES AND THEIR QUOTIENTS
WITH COMPLEX AND METRIC STRUCTURES

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Abstract. The paper presents some recent results on the BMO Teichmüller space, its subspaces and quotient spaces. We first consider the chord-arc curve subspace and prove that every element of the BMO Teichmüller space is represented by its finite composition. Moreover, we show that these BMO Teichmüller spaces have affine foliated structures induced by the VMO Teichmüller space. By which, their quotient spaces have natural complex structures modeled on the quotient Banach space. Then, a complete translation-invariant metric is introduced on the BMO Teichmüller space and is shown to be a continuous Finsler metric in a special case.

1. Introduction

The Teichmüller space is originally a universal classification space of the complex structures on a surface of given quasiconformal type, but according to complex analytic objects we focus on, we can also consider various kinds of Teichmüller spaces. The universal Teichmüller space plays a role of their ambient space, and its intrinsic natures (complex structures and invariant metrics) dominate any included Teichmüller spaces. For instance, the Teichmüller space of a Riemann surface can be represented in the universal Teichmüller space as the fixed point locus of the Fuchsian group. In a different direction to this, Teichmüller spaces in our study are obtained by adding a certain regularity to ingredients of the space. Recently, this type of Teichmüller spaces become more popular as a branch of infinite dimensional Teichmüller theory.

The Bers model of the universal Teichmüller space $T$ is defined by the Schwarzian derivative $S(f|_{D^*})$ of the conformal homeomorphism $f$ of the exterior of the unit disk $D^*$ that is quasiconformal on the unit disk $D$. In this way, $T$ is embedded in a certain Banach space as a bounded domain. The image $\Gamma$ of the unit circle $S$ under $f$ is called a quasicircle. The universal Teichmüller space $T$ can be also characterized as the set of all quasicircles

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up to a Möbius transformations of the Riemann sphere \( \hat{\mathbb{C}} \). Let \( \Omega \) denote the inner domain of \( \Gamma \), and let moreover \( g \) be a Riemann map of \( \mathbb{D} \) onto \( \Omega \). We define the conformal welding homeomorphism \( h \) with respect to \( \Gamma \) by \( h = (g|_S)^{-1} \circ (f|_S) \), which is quasisymmetric. The universal Teichmüller space \( T \) is identified with the group \( QS \) of quasisymmetric self-homeomorphisms \( h \) of \( S \) modulo the group \( \text{Möb}(S) \) of Möbius transformations of \( S \), i.e., \( T = \text{Möb}(S) \backslash QS \). In general, the quasisymmetric homeomorphism \( h \) does not satisfy any regularity conditions such as absolute continuity. As well, the quasicircle \( \Gamma \) might not even be rectifiable. In fact, its Hausdorff dimension, though less than 2, can be arbitrarily close to 2 (see [5, 6, 20]).

The BMO theory has been studied often in the framework of Teichmüller theory. The corresponding subspaces of \( T \) have generally satisfactory characteristics in terms of the quasicircle \( \Gamma \) and the quasisymmetric homeomorphism \( h \) (see [4, 21, 22]). In this paper, we shall continue to study the BMO theory of the universal Teichmüller space, because of its great importance in the application to the harmonic analysis (see [8, 13, 15, 21]) and also of its own interest. We will especially focus on BMO Teichmüller spaces, the subspaces of the universal Teichmüller space \( T \) closely related to BMO functions, Carleson measures and \( A_\infty \) weights. In Section 2, we survey the standard theory of the universal Teichmüller space and BMO Teichmüller spaces.

Basic problems are considered by going back and forth between the quasicircle \( \Gamma \) and the conformal welding homeomorphism \( h \) corresponding to \( \Gamma \). It is known that the set \( QS \) of all strongly quasisymmetric homeomorphisms of \( S \), which correspond to Bishop-Jones quasicircles, forms a partial topological group under the BMO topology; the neighborhood base is given at the identity by using the BMO norm and is distributed at every point \( h \in QS \) by the right translation. It is proved in [25] that its characteristic topological subgroup \( SS \) consists of strongly symmetric homeomorphisms, which correspond precisely to asymptotically smooth curves in the sense of Pommerenke. We consider intermediately the set \( CQS \) of conformal welding homeomorphisms with respect to chord-arc curves \( \Gamma \). In Section 3, we prove that every element of \( QS \) can be represented as a finite composition of elements in \( CQS \) (Theorem 3.3). As a consequence, we see that \( CQS \) does not carry a group structure under the composition (Corollary 3.4).

The Bers embedding of the universal Teichmüller space \( T \) is a map into the Banach space of bounded holomorphic quadratic differentials. Affine foliated structures of \( T \) and the quotient Bers embeddings are induced by its subspaces. This was first investigated by Gardiner and Sullivan [14] for the little subspace \( T_0 \), which consists of the asymptotically conformal elements of \( T \). Later, it was proved that the Bers embedding is compatible with the coset decomposition \( T_0 \backslash T \) and the quotient Banach space. By which the complex structure modeled on the quotient Banach space is provided for \( T_0 \backslash T \) through the quotient Bers embedding. In Sections 4 and 5, we show two examples of affine foliated structures.
of the BMO Teichmüller spaces $T_c = \text{Möb}(\mathbb{S}) \backslash \text{CQS}$ and $T_b = \text{Möb}(\mathbb{S}) \backslash \text{SQS}$ (Corollary 4.3, Theorem 5.1) and the injectivity of their quotient Bers embeddings by $T_v = \text{Möb}(\mathbb{S}) \backslash \text{SS}$ (Corollary 5.2).

In Section 6, a new invariant metric $m_C$ under the right translation is introduced on the BMO Teichmüller space by using the Carleson norm. We call this the Carleson metric. This is shown to be a continuous Finsler metric in the special case for $T_v$ (Theorem 6.2). Moreover, the Carleson metric $m_C$ induces a quotient metric on the quotient BMO Teichmüller space. Then, a list of intended results is presented in this section, following the work on the asymptotic Teichmüller space $T_0 \backslash T$ by Earle, Gardiner and Lakic [9]. In the following Section 7, we show that the Carleson distance induced by $m_C$ is complete in the BMO Teichmüller spaces (Theorem 7.4). We also compare the Carleson distance with the Teichmüller distance and the Kobayashi distance.

One of our motivations to study those structures of BMO Teichmüller spaces is to consider an open problem of the connectivity of the chord-arc curve subspace (see [4]). The topology on this space is induced by the BMO norm of the conformal welding homeomorphisms. The distribution of the chord-arc curve subspace $T_c$ in the BMO Teichmüller space $T_b$ (Theorem 4.1) can translate the problem of the connectivity to the quotient $T_v \backslash T_c$. By introducing the (quotient) Carleson metric in this space, we can investigate a certain convexity of the chord-arc curve subspace to consider the problem.

2. Preliminaries

In this section, we review basic facts on the universal Teichmüller space (see [1, 16, 18]) and the BMO theory of the universal Teichmüller space (see [4, 21, 22]).

The universal Teichmüller space $T$ is defined as the group QS of all quasisymmetric homeomorphisms of the unit circle $\mathbb{S} = \{ z \mid |z| = 1 \}$ modulo the left action of the group Möb($\mathbb{S}$) of all Möbius transformations of $\mathbb{S}$, i.e., $T = \text{Möb}(\mathbb{S}) \backslash \text{QS}$. A topology of $T$ can be defined by quasisymmetry constants of quasisymmetric homeomorphisms. The universal Teichmüller space $T$ can be also defined by using quasiconformal homeomorphisms of the unit disk $\mathbb{D} = \{ z \mid |z| < 1 \}$ with complex dilatations $\mu$ in the space of Beltrami coefficients

$$M(\mathbb{D}) = \{ \mu \in L^\infty(\mathbb{D}) \mid \|\mu\|_\infty < 1 \}.$$ 

Then, $T$ is the quotient space of $M(\mathbb{D})$ under the Teichmüller equivalence. The topology of $T$ coincides with the quotient topology induced by the projection $\pi : M(\mathbb{D}) \to T$.

The universal Teichmüller space $T$ is identified with a domain in the Banach space

$$B(\mathbb{D}^*) = \{ \varphi(z)dz^2 \mid \|\varphi\|_B = \sup_{z \in \mathbb{D}^*} (|z|^2 - 1)^2|\varphi(z)| < \infty \}$$

of bounded holomorphic quadratic differentials on $\mathbb{D}^* = \hat{\mathbb{C}} - \overline{\mathbb{D}}$ under the Bers embedding $\beta : T \to B(\mathbb{D}^*)$. This map is given by the factorization of a map $\Phi : M(\mathbb{D}) \to B(\mathbb{D}^*)$.
by the projection $\pi$, i.e., $\beta \circ \pi = \Phi$. Here, for every $\mu \in M(\mathbb{D})$, $\Phi(\mu)$ is defined by the Schwarzian derivative $\mathcal{S}(f_\mu|_{\mathbb{D}^*})$ of the conformal homeomorphism $f_\mu$ of $\mathbb{D}^*$ that is quasiconformal on $\mathbb{D}$ with the complex dilatation $\mu$. The Bers embedding $\beta : T \to B(\mathbb{D}^*)$ is a homeomorphism onto the image $\beta$ where the supremum is taken over all arcs $I$ of $M(D)$. The operation on the groups $M(\mathbb{D})$ and $T$ is denoted by $\ast$. For every $\mu \in M(\mathbb{D})$, the normalized quasiconformal self-homeomorphism of $\mathbb{D}$ with the complex dilatation $\mu$ (and its quasisymmetric extension to $\mathbb{S}$) is denoted by $f^\mu$. Then, the operation $\ast$ is defined by the relation $f^{\mu_1} \circ f^{\mu_2} = f^{\mu_1 \ast \mu_2}$.

For every $\nu \in M(\mathbb{D})$, the right translation $r_\nu : M(\mathbb{D}) \to M(\mathbb{D})$ is defined by $r_\nu(\mu) = \mu \ast \nu^{-1}$. For every $\tau \in T$, the right translation $R_\tau : T \to T$ is defined by $R_\tau(\sigma) = \sigma \ast \tau^{-1}$. Both $r_\nu$ and $R_\tau$ are biholomorphic automorphisms of $M(\mathbb{D})$ and $T$, respectively. Moreover, for $\tau = \pi(\nu)$, we have $R_\tau \circ \pi = \pi \circ r_\nu$.

A quasisymmetric homeomorphism $h \in QS$ is called strongly quasisymmetric if for any $\varepsilon > 0$ there is some $\delta > 0$ such that for any arc $I \subset \mathbb{S}$ and any Borel set $E \subset I$, $|E| \leq \delta |I|$ implies that $|h(E)| \leq \varepsilon |h(I)|$. It should be noted that each $h$ is absolutely continuous and $\log h'$ is in BMO($\mathbb{S}$). Here, a locally integrable function $\phi$ on $\mathbb{S}$ belongs to BMO($\mathbb{S}$) if

$$\|\phi\|_{\text{BMO}} = \sup_{I \subset \mathbb{S}} \frac{1}{|I|} \int_I \frac{1}{2\pi} \int_I |\phi - \phi_I| \, d\theta < \infty,$$

where the supremum is taken over all arcs $I$ on $\mathbb{S}$, $|I| = \int_I d\theta/2\pi$ is the length of $I$, and $\phi_I$ denotes the average of $\phi$ over $I$. We denote by SQS the group of all strongly quasisymmetric homeomorphisms. We assign the following BMO distance to SQS: $d_{\text{BMO}}(h_1, h_2) = \|\log h_1' - \log h_2'\|_{\text{BMO}}$. The BMO Teichmüller space is defined by $T_b = \text{Mob}(\mathbb{S}) \setminus \text{SQS}$, which is equipped with a topology induced by the BMO distance.

As in the case of the universal Teichmüller space, the BMO Teichmüller space $T_b$ has the corresponding space for Beltrami coefficients. For a simply connected domain $\Omega$ in the Riemann sphere $\hat{\mathbb{C}}$ with $\infty \notin \partial \Omega$, a measure $\lambda = \lambda(z) \, dx \, dy$ on $\Omega$ is called a Carleson measure if

$$\|\lambda\|_c = \sup \left\{ \frac{1}{r} \int_{|z - \zeta| < r, \ z \in \Omega} \lambda(z) \, dx \, dy \mid \zeta \in \partial \Omega, \ 0 < r < \text{diameter}(\partial \Omega) \right\}.$$
is finite. We denote the set of all Carleson measures on \( \Omega \) by \( \text{CM}(\Omega) \). For any \( \mu \in L^\infty(\mathbb{D}) \) and for the Poincaré density \( \rho_D(z) = (1 - |z|^2)^{-1} \) (with curvature constant equal to \(-4\)) on \( \mathbb{D} \), we set
\[
\lambda_\mu(z) dx dy = |\mu(z)|^2 \rho_D(z) dx dy.
\]
Then, a linear subspace \( L = L(\mathbb{D}) \subset L^\infty(\mathbb{D}) \) consisting of all \( \mu \) with \( \lambda_\mu \in \text{CM}(\mathbb{D}) \) is a Banach space with a norm \( \|\mu\|_* = \|\mu\|_\infty + \|\lambda_\mu\|_c^{1/2} \). We define \( M(\mathbb{D}) = L(\mathbb{D}) \cap M(\mathbb{D}) \). This corresponds to \( T_b \) in such a way that \( T_b \) is the image of \( M(\mathbb{D}) \) under the projection \( \pi : M(\mathbb{D}) \to T \) and the topology on \( T_b \) induced from \( M(\mathbb{D}) \) by \( \pi \) coincides with the topology on \( T_b \) induced from the BMO distance.

There is also a subspace of bounded quadratic differentials corresponding to \( T_b \). For \( \varphi \in B(\mathbb{D}^*) \), another norm is given by
\[
\|\varphi\|_B = \|\varphi(z)|^2 \rho_D^{-3}(z) dx dy\|_c^{1/2}
\]
as a Carleson measure on \( \mathbb{D}^* \), where \( \rho_D(z) = (|z|^2 - 1)^{-1} \) is the Poincaré density on \( \mathbb{D}^* \). We consider the Banach space \( B(\mathbb{D}^*) \subset B(\mathbb{D}^*) \) consisting of all such elements \( \varphi \) that \( \|\varphi(z)|^2 \rho_D^{-3}(z) dx dy \in \text{CM}(\mathbb{D}^*) \) equipped with this norm. Then, it was proved in [22, Theorem 5.1] that \( \Phi \) restricted to \( M(\mathbb{D}) \) is a holomorphic split submersion to \( B(\mathbb{D}^*) \) and the Bers embedding \( \beta \) of \( T_b \) is a homeomorphism onto the domain \( \beta(T_b) = \Phi(M(\mathbb{D})) \) in \( B(\mathbb{D}^*) \). In particular, \( T_b \) has a complex structure modeled on \( B(\mathbb{D}^*) \).

3. CHORD-ARC CURVES DO NOT HAVE THE GROUP STRUCTURE

Let \( \Gamma \) be a Jordan curve in the Riemann sphere \( \hat{\mathbb{C}} \), let \( \Omega \) and \( \Omega^* \) denote its inner and outer domains in \( \hat{\mathbb{C}} \), respectively, and let \( g \) and \( f \) be conformal maps of \( \mathbb{D} \) and \( \mathbb{D}^* \) onto \( \Omega \) and \( \Omega^* \), respectively. We define the conformal welding homeomorphism \( h \) with respect to \( \Gamma \) by \( h = (g|_S)^{-1} \circ (f|_S) \).

A rectifiable Jordan curve \( \Gamma \) in the complex plane \( \mathbb{C} \) is called a chord-arc curve if \( l_T(z_1, z_2) \leq K|z_1 - z_2| \) for any \( z_1, z_2 \in \Gamma \), where \( l_T(z_1, z_2) \) denotes the Euclidean length of the shorter arc of \( \Gamma \) between \( z_1 \) and \( z_2 \). The smallest such \( K \) is called the chord-arc constant for \( \Gamma \). It is a well-known fact that a chord-arc curve is the image of \( S \) under a bi-Lipschitz homeomorphism \( f \) of \( \mathbb{C} \). That is, there exists a homeomorphism \( f : \mathbb{C} \to \mathbb{C} \) with a constant \( C \geq 1 \) such that \( f(S) = \Gamma \) and \( C^{-1}|z - w| \leq |f(z) - f(w)| \leq C|z - w| \) for all \( z, w \in \mathbb{C} \). When \( \Gamma \) is a Jordan curve passing through \( \infty \), we may replace the Euclidean distance in the definition above with the spherical distance in order to define \( \Gamma \) to be a chord-arc curve. Bi-Lipschitz homeomorphisms preserve the Hausdorff dimension, and hence Hausdorff dimensions of chord-arc curves are one.

Although chord-arc curves are in a very special class of quasicircles, no characterization has been found in terms of their conformal welding homeomorphisms of \( S \). We denote the set of all these conformal welding homeomorphisms by CQS. It is known that if
If \( h \in \text{CQS} \) then \( h \in \text{SQS} \) (see [4]), that is, \( h \) is strongly quasisymmetric, and in particular, \( \| \log h' \|_{\text{BMO}} < \infty \). Conversely, there exists some constant \( c > 0 \) such that if \( \| \log h' \|_{\text{BMO}} < c \) then \( h \in \text{CQS} \) and the corresponding \( \Gamma \) is a chord-arc curve with the chord-arc constant \( K \) sufficiently close to \( \pi/2 \).

In this section, we prove that every element of \( \text{SQS} \) can be represented as a finite composition of elements in \( \text{CQS} \). As a consequence, we see that \( \text{CQS} \) does not carry a group structure under the composition. We state our results in the framework of Teichmüller theory. The chord-arc curve space is identified with a subspace \( T_c \) of the \( \text{BMO} \) Teichmüller space \( T_b \), which is given by the set \( \text{CQS} \) modulo \( \text{Möb}(S) \), i.e., \( T_c = \text{Möb}(S) \backslash \text{CQS} \subset T_b \). By regarding \( T_c \) as a subset of the group \( (T_b, \ast) \), we can think of the inverse and the composition of elements of \( T_c \).

For the proof of the main result in this section, we first claim that \( T_c \) (or \( \text{CQS} \)) is preserved under the inverse.

**Proposition 3.1.** The inverse element \( \tau^{-1} \) belongs to \( T_c \) for every \( \tau \in T_c \).

**Proof.** If \( h = g^{-1} \circ f \) is the conformal welding homeomorphism corresponding to a chord-arc curve \( \Gamma \), then \( h^{-1} = f^{-1} \circ g = (j \circ f \circ j)^{-1} \circ (j \circ g \circ j) \) is the conformal welding homeomorphism corresponding to \( j(\Gamma) \), where \( j(z) = z^* = \bar{z}^{-1} \) is the standard reflection of \( S \). By bi-Lipschitz continuity of \( j \), \( j(\Gamma) \) is a chord-arc curve. This proves that if \( \tau = [h] \in T_c \) then \( \tau^{-1} \in T_c \). \( \square \)

**Remark 3.2.** Noting that \( \log(h^{-1})' = -\log h' \circ h^{-1} \), we conclude by Jones [15] that \( \| \log(h^{-1})' \|_{\text{BMO}} \leq C\| \log h' \|_{\text{BMO}} \) for some constant \( C > 0 \) depending only on the strongly quasisymmetric constant for \( h \). Thus, the inverse mapping \( T_c \ni \tau = [h] \mapsto \tau^{-1} \in T_c \) is continuous at the origin \( o = [\text{id}] \) under the \( \text{BMO} \) topology. However, this correspondence should not be continuous except at the origin.

We now prove the claim mentioned above as follows.

**Theorem 3.3.** Each element of \( T_b \) can be represented as a finite composition of elements in \( T_c \).

**Proof.** Let \( V \) denote a subset of \( T_b \) consisting of all \( \tau \) for which there exists an open neighborhood \( W \) such that each \( \tau' \in W \) can be represented as finite composition of elements in \( T_c \). Since \( T_b \) is connected, in order to prove that \( V \) coincides with \( T_b \), it suffices to show that \( V \) is nonempty, open, and closed. \( T_c \) is an open subset of \( T_b \) containing the origin \( o = [\text{id}] \). This is essentially shown by Zinsmeister [28] (see also [1]). We see that \( o \in V \), and hence \( V \) is nonempty. By the definition of \( V \), \( V \) is open.

Now we prove that \( V \) is closed. Let \( \{\tau_n\} \subset V \) be a sequence such that \( \tau_n \to \tau \) as \( n \to \infty \). We will show that \( \tau \in V \). Let \( U \) be an open neighborhood of \( o \) in \( T_c \). Then,
\( R_\tau(\tau_n) \in U \subset T_c \) for all sufficiently large \( n \). Hence, there exists an element \( \sigma \in U \subset T_c \) such that \( \tau_n * \tau^{-1} = \sigma \). By Proposition 3.1, we have \( \tau * \tau_n^{-1} = \sigma^{-1} \in T_c \).

We see that \( R_\tau^{-1}(U) \) is a neighborhood of \( \tau \). For each \( \tau' \in R_\tau^{-1}(U) \), we have \( R_\tau(\tau') \in U \). Namely, there exists an element \( \sigma' \in U \subset T_c \) such that \( \tau' * \tau^{-1} = \sigma' \). It follows that 
\[
\tau' = \sigma' * \tau = \sigma' * \sigma^{-1} * \tau_n.
\]
Therefore, we have a neighborhood \( R_\tau^{-1}(U) = \hat{W} \) of \( \tau \) such that each \( \tau' \in \hat{W} \) can be represented as a finite composition of elements in \( T_c \). This completes the proof.

As \( T_c \subset \not\subset T_b \) by definition, we have the following immediate consequence from this theorem.

**Corollary 3.4.** \( T_c \) is not a subgroup of \((T, *)\).

### 4. Foliated structure of the chord-arc curve subspace

We have mentioned that the chord-arc curve subspace \( T_c \) is an open subset of \( T_b \). There is a long-standing open question about whether \( T_c \) is connected or not. For a recent account to a related result, see Astala and González [3]. In this section, we prove a result concerning the distribution of \( T_c \) in \( T_b \).

In the universal Teichmüller space \( T \), there is a closed subspace \( T_0 \) defined by \( T_0 = \pi(M_0(\mathbb{D})) \), where \( M_0(\mathbb{D}) \) is the space of Beltrami coefficients vanishing on the boundary. The subspace \( T_0 \) can be also defined to be \( \text{Mobj}(\mathbb{S}) \backslash \text{Sym} \) by the subgroup \( \text{Sym} \subset \text{QS} \) consisting of symmetric homeomorphisms of \( \mathbb{S} \), which are the boundary extension of asymptotically conformal homeomorphisms of \( \mathbb{D} \) whose complex dilatations belong to \( M_0(\mathbb{D}) \). We denote by \( B_0(\mathbb{D}^*) \) the Banach subspace of \( B(\mathbb{D}^*) \) consisting of all elements \( \varphi \) such that \( \rho_0^2(z)|\varphi(z)| \to 0 \) as \( |z| \to 1^+ \). By the Bers embedding \( \beta : T \to B(\mathbb{D}^*) \), \( T_0 \) is mapped into \( B_0(\mathbb{D}^*) \) and identified with a domain \( \beta(T_0) = \Phi(M_0(\mathbb{D})) \).

Similarly, there is a closed subspace in \( T_b \) that can be given by vanishing Carleson measures on \( \mathbb{D} \). Here, we say that a Carleson measure \( \lambda(z)dx \, d\mu \) on a simply connected domain \( \Omega \) is vanishing if

\[
\lim_{r \to 0} \frac{1}{r} \int_{|z-\zeta|<r, \ z \in \Omega} \lambda(z)dx \, d\mu = 0
\]

uniformly for \( \zeta \in \partial \Omega \). The set of all such vanishing Carleson measures on \( \Omega \) is denoted by \( \text{CM}_0(\Omega) \). Let \( \mathcal{M}_0(\mathbb{D}) \) be the subspace of \( \mathcal{M}(\mathbb{D}) \) consisting of all Beltrami coefficients \( \mu \) such that \( \lambda_\mu(z)dx \, d\mu \in \text{CM}_0(\mathbb{D}) \). Then, \( T_v = \pi(\mathcal{M}_0(\mathbb{D})) \) is a closed subspace of \( T_b \), which is called the VMO Teichmüller space. We denote \( B_0(\mathbb{D}^*) \) by the Banach subspace of \( B(\mathbb{D}^*) \) consisting of all elements \( \varphi \) such that \( |\varphi(z)|^2 \rho_0^2(z)dx \, d\mu \in \text{CM}_0(\mathbb{D}^*) \). Then \( B_0(\mathbb{D}^*) \subset B_0(\mathbb{D}^*) \) by [22, Lemma 4.1]. It is proved in [22, Theorems 4.1, 5.2] that \( \Phi \) maps \( \mathcal{M}_0(\mathbb{D}) \) into \( B_0(\mathbb{D}^*) \) and the Bers embedding \( \beta \) of \( T_v \) is a homeomorphism onto a domain \( \beta(T_v) = \Phi(\mathcal{M}_0(\mathbb{D})) \) in \( B_0(\mathbb{D}^*) \).
The VMO Teichmüller space $T_v$ can be also defined to be $T_v = \text{Möb}(S) \backslash \text{SS}$ by the characteristic topological subgroup \text{SS} of the partial topological group \text{SQS} consisting of all strongly symmetric homeomorphisms. Here, we say that $h \in \text{SQS}$ is strongly symmetric if $\log h' \in \text{VMO}(S)$, where a function $\phi \in \text{BMO}(S)$ belongs to $\text{VMO}(S)$ if

$$
\lim_{|t| \to 0} \frac{1}{|t|} \int_I |\phi - \phi_t| \frac{d\theta}{2\pi} = 0
$$

uniformly. In fact, $\text{VMO}(S)$ is the closed subspace of $\text{BMO}(S)$ which is precisely the closure of the space of all continuous functions on $S$ under the BMO topology. The inclusion relation $\text{SS} \subset \text{Sym}$ is known.

We prove that $T_c$ is distributed in $T_b$ entirely in all directions of $T_v$ in the following sense.

**Theorem 4.1.** For each $\tau \in T_c$, we have $R_{\tau}^{-1}(T_v) \subset T_c$. Hence, $T_c = \bigcup_{|\tau| \in T_v \setminus T_c} R_{\tau}^{-1}(T_v)$. 

*Proof.* For each $\sigma \in T_v$, we will show that $\hat{\sigma} \doteq \sigma \ast \tau$ belongs to $T_c$. Let $g^{-1} \circ f$ and $g_1^{-1} \circ f_1$ be the conformal welding homeomorphisms such that $[g^{-1} \circ f] = \tau$ and $[g_1^{-1} \circ f_1] = \hat{\sigma}$. We set $\Omega = g(\mathbb{D})$, $\Omega^* = f(\mathbb{D}^*)$, and $\Gamma = \partial \Omega = \partial \Omega^*$ which is a chord-arc curve. Similarly, we set $\Omega_1 = g_1(\mathbb{D})$ and $\Omega_1^* = f_1(\mathbb{D}^*)$ with a quasicircle $\Gamma_1 = \partial \Omega_1 = \partial \Omega_1^*$. Let $f^\mu$ and $f^\nu$ be normalized quasiconformal self-homeomorphisms of $\mathbb{D}$ corresponding to $\tau$ and $\sigma$, respectively. Noting that $\sigma \in T_v$, we can assume that the complex dilatation $\mu$ of $f^\mu$ induces a vanishing Carleson measure $\lambda_\mu \in \text{CM}_0(\mathbb{D})$. Then, $g \circ f^\nu$ and $g_1 \circ f^\mu \circ f^\nu$ are quasiconformal extensions of $f$ and $f_1$ to $\mathbb{D}$, respectively.

We define

$$
\hat{f} = \begin{cases} 
    f_1 \circ f^{-1} & \text{on } \Omega^* \\
    (g_1 \circ f^\mu \circ f^\nu) \circ (g \circ f^\nu)^{-1} = g_1 \circ f^\mu \circ g^{-1} & \text{on } \Omega.
\end{cases}
$$

Then, $\hat{f} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is conformal on $\Omega^*$ and asymptotically conformal on $\Omega$ whose complex dilatation $\hat{\mu}$ satisfies $|\hat{\mu}|^2 \rho_\Omega = \lambda_\mu \circ g^{-1}|(g^{-1})'|$ for the Poincaré density $\rho_\Omega$ on $\Omega$. As $\lambda_\mu \in \text{CM}_0(\mathbb{D})$, we have that $|\hat{\mu}|^2 \rho_\Omega \in \text{CM}_0(\Omega)$ by [26 Theorem 3.2].

We decompose $\hat{f}$ into $\hat{f}_0 \circ \hat{f}_1$ as follows. The quasiconformal homeomorphism $\hat{f}_1 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is chosen so that its complex dilatation $\hat{\mu}_1$ coincides with $\hat{\mu}$ on $\Omega - \Omega_0$ for some compact subset $\Omega_0$ of $\Omega$ and zero elsewhere. Then $\hat{f}_0$ is defined to be $\hat{f} \circ \hat{f}_1^{-1}$. We have the following commutative diagram:
Here, the compact subset $\Omega_0 \subset \Omega$ is chosen so that $|\hat{\mu}_0|^2_\rho_{\Omega} \in \text{CM}_0(\Omega)$ has a sufficiently small norm of the Carleson measure. It follows from [26, Lemma 4.1] that $|S(\hat{f}_1)|^2_\rho_{\Omega} \in \text{CM}_0(\Omega^*)$ with a small norm. By [26, Theorem 3.1], we have that

$$|S(\hat{f}_1 \circ f) - S(f)|^2_\rho_{\Omega_0^*} = (|S(\hat{f}_1)|^2_\rho_{\Omega^*} \circ f |f'| \in \text{CM}_0(\mathbb{D}^*),$$

and moreover we see that it can be of a small norm according to that of $|S(\hat{f}_1)|^2_\rho_{\Omega^*}$. Combined with the facts that $\Gamma$ is a chord-arc curve and that the subspace $T_c$ is open, it implies that $\partial \hat{f}_1(\Omega)$ is also a chord-arc curve. Since the complex dilatation $\hat{\mu}_0$ of $\hat{f}_0$ has the compact support $\hat{f}_1(\Omega_0) \subset \hat{f}_1(\Omega)$, we conclude that $\Gamma_1$ is the image of $\partial \hat{f}_1(\Omega)$ under the conformal mapping $\hat{f}_0$ defined on $\hat{\mathbb{C}} - \hat{f}_1(\Omega_0)$, which is bi-Lipschitz in a neighborhood of $\partial \hat{f}_1(\Omega)$. Thus, we see that $\Gamma_1$ is again a chord-arc curve, which implies that $\hat{\sigma} \in T_c$. $\square$

We consider the projection

$$p : T_b = \text{Möb}(\mathbb{S}) \setminus \text{SQS} \to T_c \setminus T_b = \text{SS} \setminus \text{SQS}.$$

The quotient space $T_c \setminus T_b$ is endowed with the quotient topology. We apply this quotient map to the subspace $T_c$. Then, Theorem 4.1 is equivalent to saying that $T_c = p^{-1}(p(T_c))$. Concerning the topology of $T_c$ and $p(T_c)$, we immediately see the following. Noting the fact that $T_c$ is contractible shown in [24], the connectedness problem on $T_c$ can be also passed to this quotient.

**Corollary 4.2.** The quotient space $p(T_c)$ is a proper open subset of $p(T_b)$. If $p(T_c)$ is connected, then so is $T_c$.

The quotient Bers embedding from $T_c \setminus T_b = p(T_c)$ into $\mathcal{B}_0(\mathbb{D}^*) \setminus \mathcal{B}(\mathbb{D}^*)$ is considered in [26, Theorem 2.2] to be well-defined and injective. We also generalize this theorem to the entire space $T_c \setminus T_b = p(T_b)$ in the next section. Combining the claim for $T_c$ with Theorem 4.1 we have the following result naturally.
Corollary 4.3. \( \beta \circ R_{\tau}^{-1}(T_v) = \beta(T_v) \cap \{ \mathcal{B}_0(\mathbb{D}^*) + \beta(\tau) \} \) for every \( \tau \in T_c \).

By this result, we have the decomposition of the Bers embedding as
\[
\beta(T_c) = \bigsqcup_{[\tau] \in T_v \setminus T_c} \beta \circ R_{\tau}^{-1}(T_v) = \bigsqcup_{[\psi] \in \mathcal{B}_0(\mathbb{D}^*) \setminus \mathcal{B}(\mathbb{D})} \beta(T_c) \cap (\mathcal{B}_0(\mathbb{D}^*) + \psi).
\]
Each component \( \beta(T_c) \cap (\mathcal{B}_0(\mathbb{D}^*) + \psi) \) is biholomorphically equivalent to \( T_v \cong \beta(T_v) \). We call this decomposition the affine foliated structure of \( T_c \) induced by \( T_v \).

5. The Quotient Bers Embedding of the BMO Teichmüller Space

In this section, we prove the affine foliated structure of the BMO-Teichmüller space \( T_b \) and the injectivity of the quotient Bers embedding induced by the VMO-Teichmüller space \( T_v \). From this result, we provide the quotient BMO Teichmüller space \( T_v \setminus T_b = p(T_b) \) with a complex structure modeled on the quotient Banach space \( \mathcal{B}_0(\mathbb{D}^*) \setminus \mathcal{B}(\mathbb{D})^* \).

Theorem 5.1. \( \beta \circ R_{\tau}^{-1}(T_v) = \beta(T_v) \cap \{ \mathcal{B}_0(\mathbb{D}^*) + \beta(\tau) \} \) for every \( \tau \in T_b \).

Proof. For every \( \tau \in T_b = \text{Möb}(\mathbb{S}) \setminus \text{SQS} \), let \( f^\nu : \mathbb{D} \to \mathbb{D} \) be a normalized quasiconformal extension of \( \tau \) with complex dilatation \( \nu \in \mathcal{M}(\mathbb{D}) \) (i.e. \( \pi(\nu) = \tau \)) that is bi-Lipschitz under the Poincaré metric on \( \mathbb{D} \) (for instance the Douady-Earle extension of \( \tau \); see [7]), and let \( \psi = \Phi(\nu) \in \mathcal{B}(\mathbb{D}^*) \).

For one inclusion \( \subset \), we divide the arguments into two steps. We first deal with the special case that \( \mu \in \mathcal{M}_0(\mathbb{D}) \) has a compact support. Then, we extend this to the general case by means of an approximation process.

We take a Beltrami coefficient \( \mu \) on \( \mathbb{D} \) with compact support. Clearly, \( \mu \in \mathcal{M}_0(\mathbb{D}) \). We will show that \( \Phi(\mu * \nu) - \Phi(\nu) \in \mathcal{B}_0(\mathbb{D}^*) \). Then, the inclusion \( \subset \) follows from
\[
\Phi(\mu * \nu) - \Phi(\nu) = \beta \circ \pi(\mu * \nu) - \beta \circ \pi(\nu) = \beta \circ R_{\tau}^{-1}(\pi(\mu)) - \beta(\tau).
\]

Let \( f_\nu : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be a quasiconformal homeomorphism with complex dilatation \( \nu \in \mathcal{M}(\mathbb{D}) \) that is conformal on \( \mathbb{D}^* \) with \( S_{f_\nu|_{\mathbb{D}^*}} = \psi \). Let \( f_{\mu * \nu} : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be a quasiconformal homeomorphism with complex dilatation \( \mu * \nu \) on \( \mathbb{D} \) that is conformal on \( \mathbb{D}^* \). Set \( \hat{f} = f_{\mu * \nu} \circ f_\nu^{-1} \). Then, \( \hat{f} \) is a quasiconformal homeomorphism with complex dilatation \( \hat{\mu} \) on \( \Omega = f_\nu(\mathbb{D}) \) with a compact support contained in a Jordan domain \( \Omega_0 \) with \( \overline{\Omega_0} \subset \Omega \), and is conformal on \( \Omega^* = f_\nu(\mathbb{D}^*) \) with
\[
(*) \quad |S(\hat{f})|^2\rho_{\Omega_0}^{-2} = \left| |S(f_{\mu * \nu}) - S(f_\nu)|^2\rho_{\mathbb{D}^*}^{-2}\right| \circ f_\nu^{-1}(|(f_\nu^{-1})'|).
\]

We note that \( \hat{f} \) is conformal on \( \Omega_0^* = \hat{\mathbb{C}} - \overline{\Omega_0} \).

It is known that \( |S(\hat{f})(z)|^2\rho_{\Omega_0^*}^{-2}(z) \leq 12 \) for \( z \in \Omega_0^* \) (see [16, p.67]). Combined with the monotone property \( \rho_{\Omega_0}(z) \leq \rho_{\Omega^*}(z) \) of Poincaré densities, this inequality implies that
there exists a constant $C$ such that
$$|S(\hat{f})(z)|^2 \rho_{\Omega_t}^{-3}(z) \leq 144 \rho_{\Omega_t}^{-4}(z) \rho_{\Omega_t}^{-3}(z) \leq 144 \rho_{\Omega_t}^{-3}(z) \leq C$$
for $z \in \Omega^*$. From which we deduce that $|S(\hat{f})|^2 \rho_{\Omega_t}^{-3} \in C_{M_0}(\Omega^*)$. By (9) and well-definedness of the pull-back operator from $C_{M_0}(\Omega^*)$ into $C_{M_0}(\mathbb{D}^*)$ (see [26, Theorem 3.1]), we have that $|S(f_{\mu_t}) - S(f_\nu)|^2 \rho_{\Omega_t}^{-3} \in C_{M_0}(\mathbb{D}^*)$, which yields that $\Phi(\mu * \nu) - \Phi(\nu) \in B_0(\mathbb{D}^*)$.

For any $\sigma \in T_\nu = \text{M"ob}(\mathbb{S}) \setminus \text{SS}$, the complex dilatation of the Douady-Earle extension of $\sigma$ is denoted by $\mu$. Then, $\mu \in M_0(\mathbb{D})$ by [24, Theorem 3.7] (see also [19]). We take an increasing sequence of positive numbers $r_n < 1 \ (n = 1, 2, \ldots)$ tending to 1. Let $\Delta_n = \mathbb{D}(0, r_n)$, the disk of radius $r_n$ centered at the origin, and let $A_n = \mathbb{D} - \Delta_n$. We define
$$\mu_n = \begin{cases} \mu & \text{on } \overline{\Delta_n} \\ 0 & \text{on } A_n. \end{cases}$$
Then, $\{\mu_n\}$ is a sequence of complex dilatations with compact support such that
$$\|\mu - \mu_n\|_{\ast} = \|\mu - \mu_n\|_{\infty} + \|\lambda_{\mu - \mu_n}\|_c = \|\mu|_{A_n}\|_{\infty} + \|\lambda_{\mu}|_{A_n}\|_c \to 0$$
as $n \to \infty$. Indeed, it was proved in [11] that the complex dilatation of the Douady-Earle extension of a symmetric homeomorphism is in $M_0(\mathbb{D})$. Combined with the inclusion relation $\text{SS} \subset \text{Sym}$, we see that $\mu$ belongs to $M_0(\mathbb{D})$, which yields that the first term tends to 0. By the definition of $M_0(\mathbb{D})$, we have that the second term tends to 0.

Since $f^\nu$ is bi-Lipschitz under the Poincaré metric, $\nu$ induces a biholomorphic automorphism $r^\nu_{\nu} : M(\mathbb{D}) \to M(\mathbb{D})$ (see [22, Remark 5.1]). Then, we have
$$\|r_{\nu}^{-1}(\mu) - r_{\nu}^{-1}(\mu_n)\|_{\ast} = \|\mu * \nu - \mu_n * \nu\|_{\ast} \to 0$$
as $n \to \infty$. The continuity of $\Phi$ yields that
$$\|\Phi(\mu * \nu) - \Phi(\nu) - (\Phi(\mu * \nu) - \Phi(\nu))\|_{B} = \|\Phi(\mu * \nu) - \Phi(\mu * \nu)\|_{B} \to 0$$
as $n \to \infty$. We have proved that $\Phi(\mu_n * \nu) - \Phi(\nu) \in B_0(\mathbb{D}^*)$ in the first step. Then, it follows from the fact that $B_0(\mathbb{D}^*)$ is closed in $B(\mathbb{D}^*)$ that $\Phi(\mu * \nu) - \Phi(\nu) \in B_0(\mathbb{D}^*)$. This proves the inclusion $\subset$.

For the other inclusion $\supset$, it can be proved by using the following claim, which is shown in [17, Proposition 2.2].

**Claim.** Let $f_\nu : \hat{\Omega} \to \hat{\Omega}$ be a quasiconformal homeomorphism with complex dilatation $\nu \in M(\mathbb{D})$ that is bi-Lipschitz between $\mathbb{D}$ and $\Omega = f_\nu(\mathbb{D})$ under their Poincaré metrics, and is conformal on $\mathbb{D}^*$ with $S(f_\nu|_{\mathbb{D}^*}) = \psi$. Then, for every $\varphi \in B_0(\mathbb{D}^*)$, there exists a quasiconformal homeomorphism $\hat{f} : \hat{\Omega} \to \hat{\Omega}$ with complex dilatation $\hat{\mu}$ on $\Omega$ vanishing at the boundary that is conformal on $\Omega^* = f_\nu(\mathbb{D}^*)$ with $S(\hat{f} \circ f_\nu|_{\mathbb{D}^*}) = \varphi + \psi$ such that the
following statements are valid: $\hat{f}$ is decomposed into two quasiconformal homeomorphisms $\hat{f}_0$ and $\hat{f}_1$ of $\hat{\mathbb{C}}$ with $\hat{f} = \hat{f}_0 \circ \hat{f}_1$, where $\hat{f}_1$ is conformal on $\Omega^*$ with $\mathcal{S}(\hat{f}_1 \circ f_{\nu}|_{\Omega^*}) = \varphi_1 + \psi$, satisfying the following properties:

1. the complex dilatation $\hat{\mu}_1$ of $\hat{f}_1$ on $\Omega$ satisfies
   \[ |\hat{\mu}_1 \circ f_{\nu}(z)| \leq \frac{1}{\varepsilon} \rho_{\mathbb{D}^*}^2(z^*)|\varphi_1(z^*)| \quad (z^* = \bar{z}^{-1}) \]
   for some $\varepsilon > 0$ and for every $z \in \mathbb{D}$;
2. the support of the complex dilatation $\mu_0$ of the normalized quasiconformal homeomorphism $f_0 : \mathbb{D} \to \mathbb{D}$, which is conformally conjugate to $\hat{f}_0 : \hat{f}_1(\Omega) \to \hat{f}(\Omega)$, is contained in a compact subset of $\mathbb{D}$;
3. for the complex dilatation $\mu_1$ of the normalized quasiconformal homeomorphism $f_1 : \mathbb{D} \to \mathbb{D}$, which is conformally conjugate to $\hat{f}_1 : \Omega \to \hat{f}_1(\Omega)$, we have
   \[ \varphi - \varphi_1 = \Phi(\mu_0 \ast \mu_1 \ast \nu) - \Phi(\mu_1 \ast \nu). \]

Combining all those maps in the claim above, we have the following commutative diagram, where $g_{\nu}$, $g_1$, and $g$ are the conjugating conformal maps:

\[ \begin{array}{ccc}
\mathbb{D} & \xrightarrow{f_{\nu}} & \mathbb{D} \\
\downarrow{g_{\nu}} & & \downarrow{g_1} \\
\Omega & \xrightarrow{\hat{f}_1} & \hat{f}_1(\Omega) \\
\downarrow{\hat{f}} & & \downarrow{\hat{f}_0} \\
\hat{f}(\Omega) & \xrightarrow{\hat{f}_0 \circ \hat{f}_1} & \hat{f}(\Omega)
\end{array} \]

We take $\varphi \in \mathcal{B}_0(\mathbb{D}^*)$ such that $\varphi + \psi \in \beta(T_b)$. Since $\mathcal{B}_0(\mathbb{D}^*) \subset \mathcal{B}_0(\mathbb{D}^*)$, there is a quasiconformal homeomorphism $\hat{f} : \hat{C} \to \hat{\mathbb{C}}$ conformal on $\Omega^*$ and asymptotically conformal on $\Omega$ such that $\mathcal{S}(\hat{f} \circ f_{\nu}|_{\mathbb{D}^*}) = \varphi + \psi$. According to the claim above, we consider the decomposition $\hat{f} = \hat{f}_0 \circ \hat{f}_1$ together with other maps that appear in it, and apply the properties shown there.

Since $\varphi \in \mathcal{B}_0(\mathbb{D}^*)$, if $\varphi - \varphi_1 \in \mathcal{B}_0(\mathbb{D}^*)$, then $\varphi_1 \in \mathcal{B}_0(\mathbb{D}^*)$. By property (2), $\mu_0$ in particular belongs to $\mathcal{M}_0(\mathbb{D})$, and property (3) asserts that $\varphi - \varphi_1 = \Phi(\mu_0 \ast \mu_1 \ast \nu) - \Phi(\mu_1 \ast \nu)$. By the previous arguments showing the inclusion $\subset$, we see that $\varphi - \varphi_1 \in \mathcal{B}_0(\mathbb{D}^*)$. Hence, $\varphi_1 \in \mathcal{B}_0(\mathbb{D}^*)$.

By property (1), $\varphi_1 \in \mathcal{B}_0(\mathbb{D}^*)$ implies that $\hat{\mu}_1 \circ f_{\nu} \in \mathcal{M}_0(\mathbb{D})$ (see Section 6 for details). By $|\hat{\mu}_1 \circ f_{\nu}| = |\mu_1 \circ f_{\nu}|$, we have $\mu_1 \circ f_{\nu} \in \mathcal{M}_0(\mathbb{D})$. It follows from the bi-Lipschitz
continuity of $f^\nu$ and [24, Proposition 3.5] that $\mu_1 \in M_0(\mathbb{D})$. By property (2), the support of the complex dilatation $\mu_0$ of $f_0$ is contained in a compact subset of $\mathbb{D}$. Hence, we see that the complex dilatation $\mu_f = \mu_0 * \mu_1$ of $f = f_0 \circ f_1$ belongs to $M_0(\mathbb{D})$. Since the complex dilatation of the quasiconformal homeomorphism $\tilde{f} \circ f_\nu$ on $\mathbb{D}$ is $r^{-1}_\nu(\mu_f)$, we have that

$$\varphi + \psi = \Phi(\mu_0 * \mu_1 * \nu) = \Phi \circ r^{-1}_\nu(\mu_f) \in \Phi \circ r^{-1}_\nu(M_0(\mathbb{D})) = \beta \circ R^{-1}_\tau(T_v),$$

which proves the inclusion $\supset$. □

By this theorem, we have the decomposition of the Bers embedding as

$$\beta(T_b) = \bigsqcup_{[\tau] \in T_v \setminus T_b} \beta \circ R^{-1}_\tau(T_v) = \bigsqcup_{[\psi] \in B_0(\mathbb{D}^*) \setminus B(\mathbb{D}^*)} \beta(T_b) \cap (B_0(\mathbb{D}^*) + \psi).$$

Each component $\beta(T_b) \cap (B_0(\mathbb{D}^*) + \psi)$ is biholomorphically equivalent to $T_v \cong \beta(T_v)$. This is the affine foliated structure of $T_b$ induced by $T_v$.

From Theorem 5.1 we also see that the quotient space $T_v \setminus T_b$ can be identified with a domain in the quotient Banach space $B_0(\mathbb{D}^*) \setminus B(\mathbb{D}^*)$.

**Corollary 5.2.** The quotient Bers embedding

$$\tilde{\beta} : T_v \setminus T_b \to B_0(\mathbb{D}^*) \setminus B(\mathbb{D}^*)$$

is well-defined and injective. Moreover, $\tilde{\beta}$ is a homeomorphism of $T_v \setminus T_b$ onto its image. Consequently, $T_v \setminus T_b$ possesses a complex structure such that $\tilde{\beta}$ is a biholomorphic automorphism from $T_v \setminus T_b$ onto its image.

**Proof.** The well-definedness and injectivity of the map

$$\tilde{\beta} : T_v \setminus T_b \to B_0(\mathbb{D}^*) \setminus B(\mathbb{D}^*)$$

are direct consequences from Theorem 5.1.

We will show that the quotient Bers embedding $\tilde{\beta}$ is also a homeomorphism from $T_v \setminus T_b$ onto its image. For the quotient maps $p : T_b \to T_v \setminus T_b$ and $P : B(\mathbb{D}^*) \to B_0(\mathbb{D}^*) \setminus B(\mathbb{D}^*)$, the following commutative diagram holds:

$$\begin{array}{ccc}
T_b & \xrightarrow{\beta} & B(\mathbb{D}^*) \\
\downarrow{p} & & \downarrow{P} \\
T_v \setminus T_b & \xrightarrow{\tilde{\beta}} & B_0(\mathbb{D}^*) \setminus B(\mathbb{D}^*)
\end{array}$$
For an arbitrary open subset $V \subset T_b$, we have
\[ p^{-1}(p(V)) = \bigcup_{\tau \in T_v} R_\tau(V). \]
This shows that $p$ is an open map. In the same way, for an arbitrary open subset $U \subset B(\mathbb{D}^*)$, we have
\[ P^{-1}(P(U)) = \bigcup_{\varphi \in B_0(\mathbb{D}^*)} (U + \varphi). \]
This shows that $P$ is an open map. Moreover, the Bers embedding $\hat{\beta} : T_b \to B(\mathbb{D}^*)$ is a homeomorphism from $T_b$ onto its image. Thus, $\hat{\beta}$ is open and continuous. Combined with the injectivity of $\hat{\beta}$, this implies that $\hat{\beta}$ is a homeomorphism of $T_v \setminus T_b$ onto its image. □

Concerning biholomorphic automorphisms of $p(T_b) = T_v \setminus T_b$ with respect to its complex structure, we have the following. This kind of arguments are well-known in the theory of asymptotic Teichmüller spaces.

**Corollary 5.3.** For every $\tau \in T_b$, the biholomorphic automorphism $R_\tau$ of $T_b$ induces a biholomorphic automorphism $\hat{R}_\tau$ of $p(T_b)$ that satisfies $p \circ R_\tau = \hat{R}_\tau \circ p$.

**Proof.** For each $\sigma \in T_b$, we have that $R_\tau(T_v \ast \sigma) = T_v \ast (\sigma \ast \tau^{-1})$. This shows that the correspondence $[\sigma] \mapsto [\sigma \ast \tau^{-1}]$ is well-defined to be a map $\hat{R}_\tau : p(T_b) \to p(T_b)$ that satisfies $p \circ R_\tau = \hat{R}_\tau \circ p$. By considering the inverse mapping $R_\tau^{-1} = R_{\tau^{-1}}$, we see that $\hat{R}_\tau$ is bijective. In the same way as the proof of Corollary 5.2, $\hat{R}_\tau$ is shown to be a homeomorphism. For the statement, it suffices to prove that $\hat{R}_\tau$ is holomorphic.

We may identify $T_b$ with the domain $\beta(T_b)$ in $B(\mathbb{D}^*)$. The conjugate $\tilde{R}_\varphi = \beta \circ R_\tau \circ \beta^{-1}$ for $\varphi = \beta(\tau)$ is a biholomorphic automorphism of $\beta(T_b) \subset B(\mathbb{D}^*)$. We use its projection $\tilde{R}_\varphi$ to $P(\beta(T_b)) = \tilde{\beta}(p(T_b))$ as a replacement of $\tilde{R}_\tau$, which satisfies $P \circ \tilde{R}_\varphi = \tilde{R}_\varphi \circ P$. Let $\phi_1, \phi_2 \in \beta(T_b)$ with $\phi_1 - \phi_2 \in B_0(\mathbb{D}^*)$ and let $\psi_1, \psi_2 \in B(\mathbb{D}^*)$ with $\psi_1 - \psi_2 \in B_0(\mathbb{D}^*)$. The derivative of $\tilde{R}_\varphi$ satisfies
\[ d_{\phi_1} \tilde{R}_\varphi(\psi_1) = \lim_{t \to 0} \frac{1}{t} (\tilde{R}_\varphi(\phi_1 + t\psi_1) - \tilde{R}_\varphi(\phi_1)), \]
\[ d_{\phi_2} \tilde{R}_\varphi(\psi_2) = \lim_{t \to 0} \frac{1}{t} (\tilde{R}_\varphi(\phi_2 + t\psi_2) - \tilde{R}_\varphi(\phi_2)), \]
where the limits refer to the convergence in the norm. From this, we see that $d_{\phi_1} \tilde{R}_\varphi(\psi_1) - d_{\phi_2} \tilde{R}_\varphi(\psi_2)$ belongs to $B_0(\mathbb{D}^*)$ because $B_0(\mathbb{D}^*)$ is closed and
\[ \{ \tilde{R}_\varphi(\phi_1 + t\psi_1) - \tilde{R}_\varphi(\phi_1) \} - \{ \tilde{R}_\varphi(\phi_2 + t\psi_2) - \tilde{R}_\varphi(\phi_2) \} = \{ \tilde{R}_\varphi(\phi_1 + t\psi_1) - \tilde{R}_\varphi(\phi_2 + t\psi_2) \} - \{ \tilde{R}_\varphi(\phi_1) - \tilde{R}_\varphi(\phi_2) \} \]
This implies that its length $R$ of the right translation $T$ of such metrics although they are in fact invariant under all biholomorphic automorphisms.

Moreover, since we may assume that $\beta_p$licity, the metric is given in the Bers embedding $T$ defined on the tangent bundle of $R$ belongs to $B_0(\mathbb{D}^*)$. Indeed, for every $[\psi] \in B_0(\mathbb{D}^*) \setminus \mathcal{B}(\mathbb{D}^*)$ and every $\varepsilon > 0$, we choose $\psi \in \mathcal{B}(\mathbb{D}^*)$ such that $P(\psi) = [\psi]$ and $\|\psi\| \leq ||[\psi]|| + \varepsilon$. Then,

$$\|A_{[\phi]}^\varphi([\psi])\| = \|P \circ d_\varphi \tilde{R}_\varphi(\psi)\| \leq \|d_\varphi \tilde{R}_\varphi(\psi)\| \leq \|d_\varphi \tilde{R}_\varphi\| \cdot ||\psi|| \leq \|d_\varphi \tilde{R}_\varphi\|(||[\psi]|| + \varepsilon).$$

Moreover, since we may assume that $\|\psi\| \leq 2||[\psi]||$ in the above choice of $\psi$, we have that

$$\|\tilde{R}_\varphi([\phi] + [\psi]) - \tilde{R}_\varphi([\phi]) - A_{[\phi]}^\varphi([\psi])\| \leq \|d_\varphi \tilde{R}_\varphi(\phi + \psi) - P \circ \tilde{R}_\varphi(\phi) - P \circ d_\varphi \tilde{R}_\varphi(\psi)\| \leq \|\tilde{R}_\varphi(\phi + \psi) - \tilde{R}_\varphi(\phi) - d_\varphi \tilde{R}_\varphi(\psi)\| = o(||[\psi]||).$$

This implies that $\tilde{R}_\varphi$ is differentiable at every $[\phi] \in P(\beta(T_b))$ in every direction $[\psi] \in B_0(\mathbb{D}^*) \setminus \mathcal{B}(\mathbb{D}^*)$ with the derivative $d_{[\phi]}^\varphi \tilde{R}_\varphi([\psi]) = A_{[\phi]}^\varphi([\psi])$. \hfill $\square$

6. THE CARLESON METRIC AND ITS QUOTIENT

In this section, we consider translation-invariant metrics on the BMO Teichmüller space $T_b$ and its quotient space $p(T_b) = T_v \setminus T_b$. A translation-invariant metric $m = m(x, v)$ defined on the tangent bundle of $T_b$ satisfies $R_\tau^* m = m$ for every $\tau \in T_b$. Here, the pull-back $R_\tau^* m$ of the metric $m$ by the automorphism $R_\tau$ is given by $R_\tau^* m(x, v) = m(R_\tau(x), d_\tau R_\tau(v))$.

Since $T_b$ is a complex manifold, the Kobayashi and the Carathéodory metrics are examples of such metrics although they are in fact invariant under all biholomorphic automorphisms of $T_b$.

We define the following translation-invariant metric on $T_b$ in a canonical way. For simplicity, the metric is given in the Bers embedding $\beta(T_b)$. As before, we use the conjugate of the right translation $R_\tau$ of $T_b$ for $\tau \in T_b$ by the Bers embedding $\beta$, which is the biholomorphic automorphism $\tilde{R}_\varphi = \beta \circ R_\tau \circ \beta^{-1}$ of $\beta(T_b)$ for $\varphi = \beta(\tau)$. The derivative $d_\varphi \tilde{R}_\varphi : \mathcal{B}(\mathbb{D}^*) \to \mathcal{B}(\mathbb{D}^*)$ at any point $\phi \in \beta(T_b)$ is a bounded linear operator.

**Definition.** A translation-invariant metric $m_C$ at any point $\varphi \in \beta(T_b) \subset \mathcal{B}(\mathbb{D}^*)$ and for any tangent vector $\psi \in \mathcal{B}(\mathbb{D}^*)$ is defined to be $m_C(\varphi, \psi) = \|d_\varphi \tilde{R}_\varphi \psi\|_\mathcal{B}$. We call this metric $m_C$ the Carleson metric on the BMO Teichmüller space $T_b \cong \beta(T_b)$. The pseudo-distance induced by this metric is denoted by $d_C(\cdot, \cdot)$, which we call the Carleson distance.

We note that for a smooth curve $\gamma = \gamma(t)$ in $\beta(T_b) \subset \mathcal{B}(\mathbb{D}^*)$ with parameter $t \in [a, b]$, its length $l_C(\gamma)$ is defined by the upper integral as

$$l_C(\gamma) = \int_a^b m_C(\gamma(t), \dot{\gamma}(t))dt.$$
Then, the Carleson distance $d_C(\varphi_1, \varphi_2)$ is the infimum of $l_C(\gamma)$ taken over all smooth curves $\gamma$ connecting $\varphi_1$ and $\varphi_2$.

Here is a list of intended results on the Carleson metric. Concerning the classical case of the Teichmüller metric, we refer to the work by Earle, Gardiner, and Lakic [9].

1. A similar metric to the Carleson metric $m_C$ is introduced by considering extremal Beltrami coefficients in $M(\mathbb{D})$, which is metrically equivalent to $m_C$.
2. A predual space to $L(\mathbb{D})$ or $B(\mathbb{D}^*)$ is characterized and utilized to consider the metric.
3. Under a certain smoothness of $m_C$, the BMO Teichmüller space $T_b$ is equipped with a Finsler structure.
4. The quotient metric of $m_C$ provides a Finsler structure for $p(T_b)$.
5. The Carleson distance can be compared with the distance induced by the BMO norm.
6. There is also a certain inequality between $m_C$ and the Kobayashi metric on $T_b$.

In this section, we only prove that the Carleson metric restricted to $T_v$ is continuous. In the next section, we prove that $T_b$ is complete with respect to the Carleson distance and certain relations between the Carleson distance and the Teichmüller-Kobayashi distance.

Let $U(r) \subset B(\mathbb{D}^*)$ and $U^\infty(r) \subset B(\mathbb{D}^*)$ denote the open balls of radius $r$ centered at the origin. We set $\delta_0 = 2/L$, where $L$ is the absolute constant satisfying the condition $\|\varphi\|_B \leq L\|\varphi\|_B$ as in [22, Lemma 4.1]. More precisely, [22] handles the case for $\mathbb{D}$, but if we note that $z^4 \varphi(z)$ is a holomorphic function in $\mathbb{D}^*$ for every $\varphi \in B(\mathbb{D}^*)$, the maximum principle justifies its arguments and the constant $L$ can be computed explicitly. Then $U(\delta_0) \subset U^\infty(2) \subset \beta(T)$.

We recall that a holomorphic local section of $\Phi : M(\mathbb{D}) \to B(\mathbb{D}^*)$ at the origin $0 \in B(\mathbb{D}^*)$ can be given explicitly by Ahlfors and Weill [2]. The following form is the adaptation to the unit disk case. For every $\varphi \in U^\infty(2)$, let

$$\sigma(\varphi)(z) = -\frac{1}{2} \rho_{\mathbb{D}^*}(z^*)(zz^*)^2 \varphi(z^*).$$

Then, $\nu(z) = \sigma(\varphi)(z)$ belongs to $M(\mathbb{D})$ and satisfies $\Phi(\nu) = \varphi$. Here, $z^* = 1/\bar{z} = j(z) \in \mathbb{D}^*$ is the reflection of $z \in \mathbb{D}$ with respect to $\mathbb{S}$. Hence, $\sigma : U^\infty(2) \to M(\mathbb{D})$ is a holomorphic local section of $\Phi$ around $0$. Noting that $|\sigma(\varphi)(z)| = \frac{1}{2} |\varphi(z^*)| (|z^*|^2 - 1)^2$, we see that $\|\sigma(\varphi)\|_\infty = \frac{1}{2} \|\varphi\|_B \leq \frac{1}{2} L \|\varphi\|_B$ for every $\varphi \in U(\delta_0)$. Moreover, we have that

$$\lambda_{\sigma(\varphi)} = \frac{1}{4} (|\varphi(j(z))|^2 (|j(z)|^2 - 1)^3) |j_\perp(z)| \, dx \, dy,$$
Proof. It is proved in [23, Lemma 2.5] that for every $\sigma \in U(\delta_0)$ onto its image in $\mathcal{M}(\mathbb{D})$ and the operator norm of $\sigma$ is well-defined from $\sigma(U(\delta_0))$ onto its image in $\mathcal{M}(\mathbb{D})$ and the operator norm of $\sigma$ is bounded by $\frac{1}{2}(L + M)$. By the linearity of $\sigma$, we see that $\|d\sigma\| \leq \frac{1}{2}(L + M)$.

We borrow the following discussion from Takhtajan and Teo [23].

Lemma 6.1. For $\nu = \sigma(\varphi)$, $f^\nu$ and $(f^\nu)^{-1}$ are bi-Lipschitz continuous under the Poincaré metric if $\varphi \in U^\infty(2)$ belongs to a small neighborhood of the origin of $B(\mathbb{D}^*)$, and $\partial f^\nu$ and $\partial (f^\nu)^{-1}$ converge locally uniformly to 1 as $\varphi \to 0$ in $B(\mathbb{D}^*)$.

Proof. It is proved in [23, Lemma 2.5] that for every $\varepsilon > 0$, there exists $0 < \delta < 1$ such that for all $\nu \in \sigma(U^\infty(2))$ with $\|\nu\|_{\infty} < \delta$, we have that
\[
\left| \frac{|\partial f^\nu(z)|^2}{(1 - |f^\nu(z)|^2)^2} - \frac{1}{(1 - |z|^2)^2} \right| \leq \frac{\varepsilon}{(1 - |z|^2)^2}
\]
for every $z \in \mathbb{D}$, and the same inequality holds for $(f^\nu)^{-1}$. Then,
\[
\sqrt{1 - \varepsilon} < \frac{1 - |z|^2}{1 - |f^\nu(z)|^2} |\partial f^\nu(z)| < \sqrt{1 + \varepsilon}.
\]
This proves the first statement. Combined with the fact that $f^\nu$ converges to the identity map uniformly on $\mathbb{D}$ as $\varphi \to 0$, this inequality also proves the second statement. \qed

The continuity of the Carleson metric in the special case is obtained as follows.

Theorem 6.2. The Carleson metric $m_C$ is continuous on the VMO Teichmüller space $T_\nu$.

Proof. By the invariance, the continuity of $m_C$ follows from that at the origin; $m_C(\varphi, \psi) \to m_C(0, \psi_0)$ as $(\varphi, \psi) \to (0, \psi_0)$. Moreover, by
\[
\|d_\varphi \tilde{R}_\varphi(\psi) - \psi_0\|_\mathcal{B} \leq \|d_\varphi \tilde{R}_\varphi(\psi) - \psi\|_\mathcal{B} + \|\psi - \psi_0\|_\mathcal{B},
\]
it suffices to show that for each tangent vector $\psi \in \mathcal{B}_0(\mathbb{D}^*)$, $\|d_\varphi \tilde{R}_\varphi(\psi) - \psi\|_\mathcal{B}$ converges to 0 as $\varphi$ tends to 0 in $\mathcal{B}(\mathbb{D}^*)$. The derivative $d_\varphi \tilde{R}_\varphi$ can be decomposed into $d_\varphi \tilde{R}_\varphi = d_0 \Phi \circ d_\varphi r_\nu \circ d_\varphi \sigma$ for $\nu = \sigma(\varphi)$. Then,
\[
d_\varphi \tilde{R}_\varphi(\psi)(z) = -\frac{6}{\pi} \int_\mathbb{D} \frac{d_\nu r_\nu(\mu)(w)}{(w - z)^4} dudv \quad (z \in \mathbb{D}^*)
\]
for $\mu = d\sigma(\psi)$. We conclude that
\[
|d_{\varphi}R_{\varphi}(\psi)(z) - \psi(z)|^2 \rho_{\varphi}^{-3}(z)
\leq A \int_{\mathbb{D}} \left| d_{\nu}r_{\nu}(\mu)(w) - \mu(w) \right|^2 \frac{(1 - |w|^2)(1 - |\zeta|^2)}{|w - z|^4} dudv
\]
for some absolute constant $A > 0$.

We consider the pull-back of this Carleson measure by the reflection $j$ with respect to $S$. It holds that
\[
|d_{\varphi}R_{\varphi}(\psi)(j(\zeta)) - \psi(j(\zeta))|^2 \rho_{\varphi}^{-3}(j(\zeta))|j(\zeta)|
\leq A \int_{\mathbb{D}} \left| d_{\nu}r_{\nu}(\mu)(w) - \mu(w) \right|^2 \rho_{\nu}(w) \frac{(1 - |w|^2)(1 - |\zeta|^2)}{|w - z|^4} dudv
\]
for $z = j(\zeta)$ ($\zeta \in \mathbb{D}$). We will show that the Carleson norm of $|d_{\nu}r_{\nu}(\mu)(w) - \mu(w)|^2 \rho_{\nu}(w)$ converges to 0 as $\varphi \to 0$ in $B(\mathbb{D}^*)$. Then, by [7, Lemma 11] and [27] (also see [26, Theorem 1.1]), the Carleson norm of $|d_{\varphi}R_{\varphi}(\psi)(z) - \psi(z)|^2 \rho_{\varphi}^{-3}(z)$ converges to 0, which implies that $\|d_{\varphi}R_{\varphi}(\psi) - \psi\|_{B} \to 0$ as $\varphi \to 0$ in $B(\mathbb{D}^*)$.

By computation, we see that
\[
d_{\nu}r_{\nu}(\mu)(w) = \frac{\mu(\zeta)}{1 - |\nu(\zeta)|^2} \frac{\partial f^{\nu}(\zeta)}{\partial f^{\nu}(\zeta)} (j(\zeta))
\]
for $w = f^{\nu}(\zeta)$. Hence,
\[
|d_{\nu}r_{\nu}(\mu)(w) - \mu(w)|^2 \leq 2|d_{\nu}r_{\nu}(\mu)(w)|^2 + 2|\mu(w)|^2
\leq 2 \left( 1 - \|\nu\|_{\infty}^2 \right) \|\mu \circ (f^{\nu})^{-1}(w)\|^2 + 2|\mu(w)|^2.
\]

We note that $\psi \in B_0(\mathbb{D}^*)$ implies $\mu \in M_0(\mathbb{D})$. By [24, Proposition 3.5], we also have $\mu \circ (f^{\nu})^{-1} \in M_0(\mathbb{D})$. Then, for any $\varepsilon > 0$, we can choose some $0 < r < 1$ such that the Carleson norm of $|d_{\nu}r_{\nu}(\mu)(w) - \mu(w)|^2 \chi_{A_r}(w) \rho_{\nu}(w)$ is less than $\varepsilon$. Here, $\chi_{A_r}$ denotes the characteristic function of $A_r = \{ w \mid 1 - r < |w| < 1 \}$.

Moreover, Lemma [6.1] implies that $d_{\nu}r_{\nu}(\mu)(w)$ converges locally uniformly to $\mu(w)$ as $\varphi$ tends to 0. Thus, $|d_{\nu}r_{\nu}(\mu)(w) - \mu(w)|^2 \rho_{\nu}(w) < \varepsilon$ for every $w \in \overline{\sigma_r} = \mathbb{D} - A_r$ if $\varphi$ is sufficiently close to 0 in $B(\mathbb{D}^*)$. In this case, it is easy to see that the Carleson norm of $|d_{\nu}r_{\nu}(\mu)(w) - \mu(w)|^2 \chi_{\overline{\sigma_r}} \rho_{\nu}(w)$ is less than $2\pi \varepsilon$. Therefore, the Carleson norm of $|d_{\nu}r_{\nu}(\mu)(w) - \mu(w)|^2 \rho_{\nu}(w)$ is less than $(2\pi + 1)\varepsilon$ if $\varphi$ is sufficiently close to 0.

By this theorem, we can say that the VMO Teichmüller space $T_v$ has a continuous Finsler structure with the Carleson metric.

We close this section by mentioning the quotient metric on $p(T_b) = T_v \setminus T_b$ induced by $m_C$. We note that $m_C$ is invariant under the group structure of $T_b$ (the transitive group action of $T_b$ is isometric with respect to $m_C$) and the projection $p$ is given by taking the
quotient of the subgroup $T_v \subset T_b$. Then, the quotient metric $\hat{m}_C$ on $p(T_b) \cong \hat{\beta}(p(T_b)) \subset \mathcal{B}_0(\mathbb{D}^*) \setminus \mathcal{B}(\mathbb{D}^*)$ is defined by

$$\hat{m}_C(\varphi, \psi) = \inf \{ m_C(\varphi, \psi) \mid P(\varphi) = \varphi, \ P(\psi) = \psi \}$$

for any $\varphi \in \hat{\beta}(p(T_b))$ and $\psi \in \mathcal{B}_0(\mathbb{D}^*) \setminus \mathcal{B}(\mathbb{D}^*)$. Moreover, we see that $\hat{m}_C$ is invariant under every biholomorphic automorphism $\hat{R}_\tau$ of $p(T_b)$ verified in Corollary 5.3. The pseudo-distance induced by $\hat{m}_C$ on $p(T_b)$ coincides with

$$\hat{d}_C(\varphi_1, \varphi_2) = \inf \{ d_C(\varphi_1, \varphi_2) \mid P(\varphi_1) = \varphi_1, \ P(\varphi_2) = \varphi_2 \},$$

and this is in fact a distance. See [10] and Remark 7.3 in the next section.

### 7. Properties of the Carleson distance

In this section, we prove further properties of the Carleson distance mentioned in the previous section. First, we give the following estimate of the operator norm of the derivative $d_0 \Phi : \mathcal{L}(\mathbb{D}) \to \mathcal{B}(\mathbb{D}^*)$ explicitly. This can be used alternatively in the proof of Theorem 6.2 to show the convergence of the Carleson norm of $|d_\varphi \hat{R}_\varphi(\psi)(z) - \psi(z)|^2 \rho_{\mathbb{D}^*}^3(z)$.

We remark that this explicit estimate is not necessary for other arguments in this section, but might serve as a refinement of the results.

**Proposition 7.1.** $\|d_0 \Phi\| \leq 24$.

**Proof.** The derivative $d_0 \Phi$ can be represented by

$$\varphi(z) = d_0 \Phi(\mu)(z) = -\frac{6}{\pi} \int_{\mathbb{D}} \frac{\mu(w)}{(w-z)^4} dudv \quad (z \in \mathbb{D}^*).$$

Applying the Cauchy-Schwarz inequality and the equation

$$\int_{\mathbb{D}} \frac{dudv}{|w-z|^4} = \pi \rho_{\mathbb{D}^*}^3(z),$$

we have

$$|\varphi(z)|^2 \rho_{\mathbb{D}^*}^3(z) \leq \frac{36(|z|^2-1)}{\pi} \int_{\mathbb{D}} \frac{|\mu(w)|^2}{|w-z|^4} dudv.$$

This shows that for every $\zeta \in \mathbb{S}$,

$$\int_{\Delta(\zeta, r) \cap \mathbb{D}^*} |\varphi(z)|^2 \rho_{\mathbb{D}^*}^3(z) dxdy \leq \frac{36}{\pi} \int_{\Delta(\zeta, r) \cap \mathbb{D}^*} \left( \int_{\mathbb{D} \setminus \Delta(\zeta, 5r/3)} \frac{|\mu(w)|^2}{|w-z|^4} dudv \right) (|z|^2-1) dxdy$$

$$+ \frac{36}{\pi} \int_{\Delta(\zeta, r) \cap \mathbb{D}^*} \left( \int_{\mathbb{D} \setminus \Delta(\zeta, 5r/3)} \frac{|\mu(w)|^2}{|w-z|^4} dudv \right) (|z|^2-1) dxdy,$$

where $\Delta(\zeta, r)$ denotes the disk with center $\zeta$ and radius $r \in (0, 2)$.

For the first term $I_1$ in the right-hand side of the inequality above, we note that $|w-z| \geq 2r/3$ and that $\mathbb{D} \setminus \Delta(\zeta, 5r/3)$ is in the half-space divided by a line passing through a given
Hence, it suffices to consider $d \phi$ at $z \in \Delta(\zeta, r) \cap \mathbb{D}^*$. Moreover, $\Delta(\zeta, r) \cap \mathbb{D}^*$ is included in $S \cap \mathbb{D}^*$, where $S$ is a sector with center 0, radius $1 + r$, and central angle at most $\pi r$. Hence, we have that

$$I_1 \leq \frac{36}{\pi} \|\mu\|_\infty^2 \int_{S \cap \mathbb{D}^*} \left( (|z|^2 - 1) \cdot \frac{1}{2} \int_{\{w : |w-z| > 2r/3\}} \frac{dudv}{|w-z|^4} \right) dx dy$$

$$\leq \frac{81}{2r^2} \|\mu\|_\infty^2 \int_{S \cap \mathbb{D}^*} (|z|^2 - 1) dx dy$$

$$\leq \frac{81}{2r^2} \|\mu\|_\infty^2 \cdot \pi r \int_1^{1+r} (t^2 - 1) dt$$

$$\leq \frac{81\pi}{2} r \left( \frac{r^2}{4} + r + 1 \right) \|\mu\|_\infty^2 \leq 162\pi r \|\mu\|_\infty^2.$$

For the second term $I_2$, similarly we have that

$$I_2 \leq \frac{36}{\pi} \int_{S' \cap \mathbb{D}^*} \left( \int_{S' \cap \mathbb{D}^*} \frac{|z|^2 - 1}{|w-z|^4} dx dy \right) |\mu(w)|^2 dudv$$

$$\leq \frac{36}{\pi} \int_{S' \cap \mathbb{D}^*} \left( \frac{3}{4} \int_{\{z \in |w-z| > 1 - |w|\}} \frac{2 + r}{|w-z|^3} dx dy \right) |\mu(w)|^2 dudv$$

$$\leq 576 \cdot \frac{3}{5} \int_{S' \cap \mathbb{D}^*} \frac{|\mu(w)|^2}{1 - |w|^2} dudv,$$

where $S'$ is a sector with center $w$ and central angle at most $3\pi/2$.

From these estimates, we obtain that

$$\|\varphi\|^2_\mathcal{B} \leq \frac{1}{r}(I_1 + I_2) \leq 162\pi \|\mu\|_\infty^2 + 576 \|\lambda_\mu\|_c \leq 576 \|\mu\|_c^2.$$

Thus, $\|\varphi\|_\mathcal{B} \leq 24 \|\mu\|_\ast$, which implies that $\|d_0\Phi\| \leq 24$. \hfill $\Box$

In the following result, we obtain a locally uniform estimate for the operator norm $\|d_\varphi \tilde{R}_\varphi\|$ when $\varphi \in \beta(T_b)$ is around the origin.

**Proposition 7.2.** The operator norm $\|d_\varphi \tilde{R}_\varphi\|$ of the derivative $d_\varphi \tilde{R}_\varphi : \mathcal{B}(\mathbb{D}^*) \to \mathcal{B}(\mathbb{D}^*)$ at $\varphi$ is uniformly bounded from above and bounded away from zero for every $\varphi$ in an open ball $U(\delta_1) \subset \mathcal{B}(\mathbb{D}^*)$ centered at the origin with some radius $\delta_1 \leq \delta_0$.

**Proof.** For the upper estimate, we decompose $\tilde{R}_\varphi$ into $\tilde{R}_\varphi = \Phi \circ r_\nu \circ \sigma$, where $\nu = \sigma(\varphi) \in \mathcal{M}(\mathbb{D})$. Then $d_\varphi \tilde{R}_\varphi = d_0\Phi \circ d_\nu^{-1} \circ d_\varphi \sigma$. Here, the Ahlfors-Weill section $\sigma$ is linear with $\|d\sigma\| \leq \frac{1}{2}(L + M)$ as before, and $d_0\Phi$ is a bounded linear operator with $\|d_0\Phi\| \leq 24$. Hence, it suffices to consider $d_\nu r_\nu$.

As before, the derivative $d_\nu r_\nu$ in direction $\mu \in \mathcal{L}(\mathbb{D})$ is given by

$$d_\nu r_\nu(\mu)(w) = \frac{\mu(\zeta)}{1 - |\nu(\zeta)|^2} \frac{\partial f(\omega)}{\partial \nu(\zeta)}$$
for \( w = f^{\nu}(\zeta) \). Then,
\[
\frac{|d_\nu r_\nu(\mu)(w)|^2}{1 - |w|^2} \leq \frac{1}{1 - \|\nu\|^2_\infty} \frac{1 - |(f^{\nu})^{-1}(w)|^2}{1 - |w|^2} |\partial(f^{\nu})^{-1}(w)|^{-1} \times \frac{|\mu((f^{\nu})^{-1}(w))|^2}{1 - |(f^{\nu})^{-1}(w)|^2} |\partial(f^{\nu})^{-1}(w)|.
\]
Here, the first factor of the right-hand side of the above inequality is uniformly bounded whenever \( \|\nu\|_\infty \) is less than some positive constant by Lemma 6.1. In particular, for every \( \varphi \in U(\delta_1) \) (\( \nu = \sigma(\varphi) \)) with some \( \delta_1 \leq \delta_0 \), this is uniformly bounded. The second factor of the right-hand side of the above inequality is defined to be the pull-back \(((f^{\nu})^{-1})^* \lambda_\mu\) of the Carleson measure \( \lambda_\mu = \lambda_\mu(\zeta) d\zeta d\eta \) on \( \mathbb{D} \) by \((f^{\nu})^{-1}\).

By Semmes [21, Lemma 4.8] (see also [24, Proposition 3.5]), we see that \( \|(f^{\nu})^{-1})^* \lambda_\mu\| c \leq C \|\nu\|_c \), for some constant \( C \) depending only on the bi-Lipschitz constant of \((f^{\nu})^{-1}\) and the strongly quasisymmetric constants of the boundary extension of \((f^{\nu})^{-1}\). In fact, the former constant depends on \( \|\nu^{-1}\|_\infty \) (\( \leq \|\nu^{-1}\|_* \)) by Lemma 6.1 as we have seen above, and the latter constants depend only on \( \|\nu^{-1}\|_* \) by Fefferman, Kenig, and Pipher [13] (see also [24, Lemma 4.3]). Furthermore, \( \|\nu^{-1}\|_* \) can be estimated in terms of \( \|\nu\|_* \). Therefore, we see that the constant \( C \) is uniformly bounded. This implies that there exists some constant \( \tilde{C} \) such that the Carleson norms satisfy \( \|\lambda_{d_\nu r_\nu(\mu)}\|_c \leq \tilde{C} \|\lambda_\mu\|_c \) for every \( \varphi \in U(\delta_1) \) (\( \nu = \sigma(\varphi) \)). Combined with the fact that \( \|d_\nu r_\nu(\mu)\|_\infty \leq (1 - \|\nu\|^2_\infty)^{-1}\|\nu\|_\infty \), this proves that the operator norm \( \|d_\nu r_\nu\| \) is uniformly bounded.

For the lower estimate of the operator norm \( \|d_\nu \tilde{R}_\varphi\| \), we consider the upper estimate of \( \|d_\nu \tilde{R}_\varphi\| = \|d_\nu \tilde{R}_\varphi\|^{-1} \) by using the decomposition \( \tilde{R}_\varphi^{-1} = \Phi \circ r_{\nu^{-1}} \circ \sigma \). Correspondingly, the derivative is \( d_\nu \tilde{R}_\varphi^{-1} = d_\nu \Phi \circ d_0 r_{\nu^{-1}} \circ d_0 \sigma \). We know that \( |d\sigma| \leq \frac{1}{2}(L + M) \) as before. Moreover, we have the derivative \( d_0 r_{\nu^{-1}} \) in direction \( \mu \in \mathcal{L}(\mathbb{D}) \) as
\[
d_0 r_{\nu^{-1}}(\mu)(w) = \mu(\zeta)(1 - |\nu^{-1}(\zeta)|^2) \frac{\partial(f^{\nu})^{-1}(\zeta)}{\partial(f^{\nu})^{-1}(\zeta)} = \mu(f^{\nu}(w))(1 - |\nu(w)|^2) \frac{\partial f^{\nu}(w)}{\partial f^{\nu}(w)} \quad (w = (f^{\nu})^{-1}(\zeta)).
\]
Then, by a similar argument as before, we can prove that the operator norm \( \|d_0 r_{\nu^{-1}}\| \) is uniformly bounded for every \( \varphi \in U(\delta_1) \) (\( \nu = \sigma(\varphi) \)) by replacing \( \delta_1 \) with a smaller constant if necessary.

The locally uniform boundedness of the operator norm \( \|d_\nu \Phi\| \) is a consequence of the holomorphy of \( \Phi \). This is a general argument but for the completeness, we review it here. The derivative of \( \Phi \) of the second order is the derivative of \( d\Phi : \mathcal{M}(\mathbb{D}) \to L(\mathcal{L}(\mathbb{D}), \mathcal{B}(\mathbb{D}^*)) \) given by the correspondence \( \nu \mapsto d_\nu \Phi \), where \( L(\mathcal{L}(\mathbb{D}), \mathcal{B}(\mathbb{D}^*)) \) is the Banach space of bounded linear operators \( \mathcal{L} \to \mathcal{B}(\mathbb{D}^*) \) with respect to the operator norm. Then, the
property that \( d\Phi \) is differentiable at 0 is equivalent to the existence of a bounded linear operator \( A : \mathcal{L} \to L(\mathcal{L}(\mathbb{D}), \mathcal{B}(\mathbb{D}^*)) \) such that

\[
\|d\nu \Phi - d0 \Phi - A\nu\| = o(\|\nu\|) \quad (\|\nu\| \to 0).
\]

Therefore, \( \|d\nu \Phi\| \leq \|d0 \Phi\| + \|A\|\|\nu\| + o(\|\nu\|) \), which yields the locally uniform boundedness of \( \|d\nu \Phi\| \).

\[\square\]

**Remark 7.3.** To make the arguments in this section precise, we should note here that the estimate of \( \|d\varphi \tilde{R}\varphi\| \) as in Proposition 7.2 guarantees a locally uniform comparison of the metric with the norm of the Banach space. Then, the pseudo-distance induced by the Carleson metric is a distance and it defines the same topology as the original one on \( T_b \).

The completeness of the Carleson distance then follows from this proposition.

**Theorem 7.4.** The Carleson distance \( d_C \) is complete on the BMO Teichmüller space \( T_b \).

**Proof.** For any \( \varphi_0, \varphi_1 \) in \( U(\delta_1) \subset \mathcal{B}(\mathbb{D}^*) \), we choose the segment \( \gamma = \{ t\varphi_1 + (1-t)\varphi_0 \}_{t \in [0,1]} \) connecting \( \varphi_0 \) and \( \varphi_1 \). Then, the Carleson length \( l_C(\gamma) \) of \( \gamma \) is given by

\[
l_C(\gamma) = \int_0^1 m_C(t\varphi_1 + (1-t)\varphi_0, \varphi_1 - \varphi_0) \, dt.
\]

Proposition 7.2 asserts that there is a constant \( K \) such that

\[
\|d_\varphi \tilde{R}_\varphi \|^B \leq K \|\varphi_1 - \varphi_0\|^B.
\]

Thus, we see that \( l_C(\gamma) \leq K \|\varphi_1 - \varphi_0\|^B \). Then, \( (**\) \)

\[
d_C(\beta^{-1}(\varphi_1), \beta^{-1}(\varphi_0)) \leq K \|\varphi_1 - \varphi_0\|^B.
\]

We consider any Cauchy sequence in \( (T_b, d_C) \). It suffices to consider its tail whose diameter can be arbitrary small. As the group of the right translations \( \{ R_\tau \} \) acts isometrically and transitively on \( T_b \), we may assume that the tail of the Cauchy sequence is contained in \( \beta^{-1}(U(\delta_1)) \). From the lower estimate of the derivative as in Proposition 7.2, we see that the Bers embedding of the Cauchy sequence is a Cauchy sequence, which is a convergent sequence with respect to the norm \( \| \cdot \|_B \). Hence, \( (**\) \) implies that the Cauchy sequence also converges with respect to \( d_C \).

\[\square\]

We compare the Teichmüller metric and the Carleson metric. The Teichmüller metric \( m_T \) is given by defining a norm of a tangent vector \( \psi \in B(\mathbb{D}^*) \) at the base point of the universal Teichmüller space \( T \cong \beta(T) \subset B(\mathbb{D}^*) \). The norm of \( \psi \) is the operator norm of the bounded linear functional

\[
H(\psi) : A^1(\mathbb{D}^*) \to \mathbb{C}, \quad \varphi \mapsto \int_{\mathbb{D}^*} \overline{\varphi(z)} \overline{\psi(z)} \rho_{\mathbb{D}^*}^2(z) \, dxdy,
\]
where \( A^1(\mathbb{D}^*) \) is the Banach space of integrable holomorphic quadratic differentials on \( \mathbb{D}^* \). The operator norm \( \|H(\psi)\| \) is comparable with \( \|\psi\|_B \), and clearly \( \|H(\psi)\| \leq \|\psi\|_B \). At any point \( \varphi \in \beta(T) \), the Teichmüller metric is given by \( m_T(\varphi,\psi) = \|H(d_\varphi R_\psi \psi)\| \). The distance induced by this metric is the Teichmüller distance \( d_T \). We consider the restriction of \( d_T \) to the BMO Teichmüller space \( T_b \).

**Remark 7.5.** For a smooth curve \( \gamma = \gamma(t) \) \( (a \leq t \leq b) \) in \( \beta(T_b) \), the Teichmüller length of \( \gamma \) is defined by

\[
    l_T(\gamma) = \int_a^b m_T(\gamma(t),\dot{\gamma}(t)) dt.
\]

Then, the infimum of \( l_T(\gamma) \) taken over all smooth curves in \( T_b \equiv \beta(T_b) \) connecting two points defines an inner distance \( d^i_T \) between them, which clearly satisfies \( d_T \leq d^i_T \).

**Proposition 7.6.** There exists a constant \( L > 0 \) such that \( m_T \leq Lm_C \) on \( T_b \). Hence, \( d_T \leq Ld_C \) on \( T_b \).

**Proof.** It was proved in \[22, Lemma 4.1\] that there is some constant \( L \) such that \( \|\psi\|_B \leq L\|\psi\|_B \) for every \( \psi \in B(\mathbb{D}^*) \). Combined with \( \|H(\psi)\| \leq \|\psi\|_B \), it follows that \( \|H(\psi)\| \leq L\|\psi\|_B \), and then the assertion follows. \( \square \)

It was shown by Fan and Hu \[12\] that the Kobayashi distance \( d_K \) defined on the complex manifold \( T_b \) coincides with the restriction of the Teichmüller distance \( d_T \). In fact, \( d_K = d^i_T = d_T \) on \( T_v \). Then, by Proposition 7.6, we have \( d_K \leq Ld_C \) on the VMO Teichmüller space \( T_v \). However, \( d_T \) and \( d_C \) are not comparable, that is, there is no inequality of the opposite direction either for \( T_b \) or for \( T_v \). This is because the Carleson distance \( d_C \) is complete in \( T_b \) by Theorem 7.4 and so is in the closed subspace \( T_v \), but \( d_T \) is not complete either in \( T_b \) or in \( T_v \). In fact, the closure of \( T_v \) in the universal Teichmüller space \( (T,d_T) \) is \( T_0 \), the little subspace given by vanishing Beltrami coefficients (asymptotically conformal maps), which contains an element not belonging to \( T_b \).

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