COMPLEMENTARITY, MEASUREMENT AND INFORMATION
IN INTERFERENCE EXPERIMENTS

G. Bimonte and R. Musto

Dipartimento di Scienze Fisiche, Università di Napoli, Federico II
Complesso Universitario MSA, via Cintia, I-80126, Napoli, Italy;
INFN, Sezione di Napoli, Napoli, ITALY.
e-mail: bimonte,musto@napoli.infn.it

Abstract

Different criteria (Shannon’s entropy, Bayes’ average cost, Dürr’s normalized rms spread) have been introduced to measure the "which-way" information present in interference experiments where, due to non-orthogonality of the detector states, the path determination is incomplete. For each of these criteria, we determine the optimal measurement to be carried on the detectors, in order to read out the maximum which-way information. We show that, while in two-beam experiments, the optimal measurement is always provided by an observable involving the detector only, in multibeam experiments, with equally populated beams and two-state detectors, this is the case only for the Dürr criterion, as the other two require the introduction of an ancillary quantum system, as part of the read-out apparatus.

1 Introduction

The debate on double-slit interference experiments, with photons or matter particles, and on the possibility of detecting, as proposed by Einstein, "which-way" individual particles are taking, helped to shape the basic concept of complementarity in quantum mechanics. According to this early discussion, Young interference experiments were showing the wave nature of both radiation and matter and any attempt to exhibit their, complementary, particle nature, by detecting which path each an individual quantum was travelling, was regarded as implying a disturbance capable of destroying the interference pattern. * It was, however, much later
noticed that "in Einstein's version of the double-slit experiment, one can retain a surprisingly strong interference pattern by not insisting on a 100% reliable determination of the slit through which each photon passes"[2].

More recently this problem has been thoroughly investigated both from a theoretical and an experimental point of view, by proposing gedanken-experiments or actually performing them, in which the quantum unitary evolution of both the system and the detector is completely under control. In many cases care is taken of having the detectors acting on internal degrees of freedom, so that they do not disturb directly the centre of mass motion.

As it is well known the partial loss of contrast of the interference fringes, their modification or total disappearance, find a complete quantum mechanical description in terms of the entanglement between the interfering particles and the detectors. To be more precise, the unitary evolution describing the interaction of the system with the detectors leads to the entangled state

$$|\Psi(t)\rangle = |\psi_1(t)\rangle \otimes |\chi_1\rangle + |\psi_2(t)\rangle \otimes |\chi_2\rangle,$$  \tag{1.1}

where $|\psi_i\rangle$, $i = 1, 2$, denote the states of the beams going through slits 1 and 2, respectively, while $|\chi_i\rangle$, $i = 1, 2$ are the (normalized) detector final states, and $t$ is any time after the system has left the detection region. The structure of the interference fringes may be read off the probability density on the screen:

$$|<x|\Psi(t_1)>|^2 = |<x|\psi_1(t_1)>|^2 + |<x|\psi_2(t_1)>|^2 + 2Re\{<\psi_1(t_1)|x><x|\psi_2(t_1)>|\chi_1\rangle\langle\chi_2|\}.$$

\tag{1.2}

Depending on the value of $<\chi_1|\chi_2>$ there is a continuum between the extreme cases of no which-way detection ($|\chi_1\rangle = |\chi_2\rangle$), where the wave nature is exhibited by interference fringes with maximum contrast, and perfect which-way detection ($<\chi_1|\chi_2> = 0$), where the interference fringes disappear. For example, in the experimental realization[3] of Feynman’s gedanken-experiment[4], the states $|\chi_i\rangle$ describe the scattered photon needed to detect whether the atom (rather than the electron, as in the original discussion) has passed through slit 1 or 2 and the quantity $<\chi_1|\chi_2>$ can be varied by changing the spatial separation between the interfering paths at the point of scattering. In the experimental setup proposed in[5] the which-way detection is performed by micro-maser cavities inserted on the beams of previously exited atoms. Atomic decay in one of the cavities provides a which-way information whose predictability depends on the initial state of the cavities. However we should point out that the detector needs not be a separate physical system: the which-way information may indeed be stored in some internal degrees of freedom of the interfering particles, as it happens in neutron interference experiments[6], where the spin of the neutron in one of the beams is rotated with respect to the original common direction. Notice that, in each of these examples, the structure of the interference fringes, as it is clear from Eq. (1.2), depends on the entanglement of the system with the apparatus, from which a "which-way" information may be eventually recovered by means of an appropriate measurement, and not on the fact of actually performing it. Eq. (1.1) describes only a premeasurement. Therefore the actual measurement
relative to the "which-way" information may be arbitrarily delayed. As Schrödinger puts it, in his "general confession" [6], motivated by the appearance of the Einstein, Podolsky, Rosen paper [7], "entanglement of predictions" goes "back to the fact that the two bodies at some earlier time formed in a true sense one system, that is were interacting, and have left behind traces on each other".

Furthermore, it should be stressed that, apart from the extreme case in which $< \chi_1 | \chi_2 > = 0$, no measurement can provide full information on the way that an individual quantum has taken. One is actually dealing with a problem in quantum detection theory, that is, in statistical decision theory. In order to decide what measurement should be carried out to extract the best possible which-way information, it is necessary to spell out a strategy in which an a priori evaluation criterion is given.

In the pioneering work of Wootters and Zurek, [2], Shannon’s definition of information entropy [9] was taken as a quantitative measure of the gain in "which-way" information obtained by actually performing a measurement on the detector state. In this framework evidence was produced that "the more clearly we wish to observe the wave nature…the most information we must give up about its particle properties". Following this suggestion, Englert [10], by using a different criterion for evaluating the available information, was able to establish, for equally populated beams, a complementarity relationship between the distinguishability, that gives a quantitative estimate of the ways, and the visibility that measures the quality of the interference fringes:

$$\mathcal{D}^2 + \mathcal{V}^2 \leq 1,$$

with equality sign holding if the detector is prepared in a pure state. As usual $\mathcal{V}$ is defined in terms of the maximum and minimum intensity of the fringes ($I_M$ and $I_m$), $\mathcal{V} = (I_M - I_m)/(I_M + I_m)$. $\mathcal{D}$ is simply related to the optimum average Bayes’s cost $\bar{C}_{opt}$, traditionally used in decision theory, by the relation $\mathcal{D} = 1 - 2 \bar{C}_{opt}$.

New problems arise in going from the case of two beams to a multibeam interference process. As shown by Dür [11], the complementarity relationship [Eq. (1.3)] still holds when the visibility and the distinguishability are taken to be, the first as the, properly normalized, deviation of the fringes intensity from its mean value, and the second, following an alternative notion of entropy introduced in Ref. [12], as the maximum average rms spread of the a posteriori probabilities for the different paths (see Sec. 2).

The purpose of this paper is to examine an interesting physical aspect of the problem, that seems to have been overlooked, so far, and it is the following: once a specific criterion to measure the which-way information is chosen, what is the actual measurement that has to be performed on the detectors, in order to extract the optimum information? The usual attitude

\footnote{In Ref. [10] the distinguishability is expressed in terms of the optimum likelihood $L_{opt}$ for "guessing the way right". This optimum likelihood is one minus the optimum average Bayes cost $\bar{C}_{opt}$}
to address this question, is to consider the set $\mathcal{A}_D$ of all observables $A$, relative to the detector, and to search, among them, for the observable that delivers most information. However, it is known from quantum detection theory \cite{13, 14}, that the amount of information that can be obtained in this way does not represent, in general, the absolute maximum. Sometimes, it is possible to do a better job by introducing, in addition to the detector, an ancilla, namely an auxiliary quantum system, neither interacting with the detector, nor having any correlation with it. Despite the fact that the detector and the ancilla are, under all respects, independent systems, it may happen that a larger amount of information can be obtained, by measuring an observable relative to the combined system. In connection with this issue, we point out that, even if the quantity $\mathcal{D}$ appearing in Eq. (1.3) is usually defined in relation with $\mathcal{A}_D$, the proofs leading to Eq. (1.3), say in Refs. \cite{10, 11}, remain valid if one includes the observables for the system formed by the detector and the ancilla together. It follows that the quantity $\mathcal{D}$ really refers to all possible detector+ancilla systems.

Since the need for an ancilla seems to us a source of undesirable complication for the read out apparatus, it would be interesting to know under what circumstances the ancilla is really required. In particular, it would be interesting to know if there exist criteria to measure the which-way information, such that the optimal measurement turns out to be an ordinary observable relative to the detector, and the inclusion of an ancilla does not lead to any improvement. We show that, in the case of two-beams interference experiments, with either one of the two proposed measures of information, the optimal measurement does not involve an ancilla. On the contrary, in the case of multibeam experiments, it is only with the criterion introduced in Ref. \cite{11} that the ancilla is unnecessary, while it is required for the other two criteria, in general. It is interesting to notice that the criterion for which ordinary measurements are good enough is the one that leads to the complementarity relation given by Eq. (1.3). Finally, let us notice that, while inspired by the problem of complementarity in interference experiments, our work is a contribution to the difficult problem of optimization in quantum decision theory.

The paper is organized as follows. In Sec. 2 the quantum detection problem for non-mutually orthogonal detector states is presented and the notion of ancilla is introduced. We review a fundamental theorem by Neumark, stating that measurements involving an ancilla in the enlarged detector-ancilla Hilbert space, can be equivalently described by means of positive operator-valued measures (POVM) on the detector’s Hilbert space, generalizing the ordinary projection-valued measures (PVM), that describe measurements not involving the ancilla. We then list the conditions that must be satisfied by any function, for it to be a good measure of the amount of information provided by a POVM. The different choices present in the literature for such a function are considered, and the resulting optimization problems are studied in Sec. 3, for the case of two beams, and in Sec. 4 for multibeam interferometers. Some of the proofs are postponed to an Appendix. Final remarks and a discussion of perspectives close the paper.
2 The quantum decision problem.

We consider a \( n \)-beam interference experiment: a single beam of identical microscopic systems, like photons, electrons, neutrons, atoms etc. (generically referred to as particles), is divided into \( n \) spatially separated beams by some sort of beam-splitter, like a screen with \( n \) slits. The \( n \) beams are then recombined on a screen, and the interference figure is observed. It is assumed that the intensity of the beam is adjusted so that only one particle at a time passes through the interferometer, and that the populations \( \zeta_i \) of each of the \( n \) beams can be adjusted at will.

We imagine now that a detector, designed to provide which-way information on individual particles passing through the interferometer, is placed along the trajectories of the beams. It is assumed that the detector also can be treated as a quantum system, and that the system-detector interaction gives rise to some unitary process. The detector will serve as which-way detector if, once prepared in some fixed state \(| \chi_0 \rangle\), it is brought by the interaction with the particles into a new state, that depends on the beam occupied by the particle. In formulae, this amounts to requiring that, after the interaction, the state of the particle-detector system is the following entangled state, generalizing [Eq. (1.1)]:

\[
\sum_{i=1}^{n} c_i |\psi_i \rangle \otimes |\chi_i \rangle .
\]

(2.1)

Here, \(|\psi_i \rangle\) denote the normalized particles wave-functions for the individual beams, while \(|\chi_i \rangle\) are \( n \) normalized (but not necessarily orthogonal!) states of the which-way detectors. We define the detector’s Hilbert space \( \mathcal{H}_D \) as the linear span of the states \(|\chi_i \rangle\):

\[
\mathcal{H}_D := \text{span}\{|\chi_i \rangle, \ i = 1, \ldots, n\} .
\]

(2.2)

(Of course, it may very well happen that the set of all possible states of the detector, as a physical system, is actually larger than \( \mathcal{H}_D \).) In concrete experiments \(|\chi_i \rangle\) may in fact be internal states of the particles themselves, in which case \(|\psi_i \rangle\) denotes the space-part of the particles wavefunction. We assume that the amplitudes \( c_i \) are known in advance, such that the weights \( \zeta_i = |c_i|^2 \) give the a priori probabilities for a particle to pass through the \( i \)-th slit. The state [Eq. (2.1)] describes a situation in which there is complete correlation between the beams and the internal states of the detector, such that, if the detector is found to be in the state \(|\chi_i \rangle\), one can tell with certainty that the particle passed through the \( i \)-th slit. Thus the problem of determining the trajectory of the particle reduces to the following one: after the passage of each particle, is there a way to decide in which of the \( n \) states \(|\chi_i \rangle\) the detector was left? If the states \(|\chi_i \rangle\) are orthogonal to each other, the answer is obviously yes. Indeed, if we let \( \mathcal{A}_D \) the set of all hermitean operators in \( \mathcal{H}_D \), we can surely find in \( \mathcal{A}_D \) an observable \( A \), such that:

\[
A |\chi_i \rangle = \lambda_i |\chi_i \rangle , \quad \lambda_i \neq \lambda_j \quad \text{for} \ i \neq j .
\]

(2.3)

If \( A \) is measured, and the result \( \lambda_i \) is found, one can infer with certainty that the detector was in the state \(|\chi_i \rangle\). If, however, the states \(|\chi_i \rangle\) are not orthogonal to each other, for no choices...
of $A$ one can fulfil Eq. (2.3); whichever $A$ one picks, there will be at least one eigenvector of $A$, having a non-zero projection onto more than one state $|\chi_i\rangle$. Therefore, when the corresponding eigenvalue is obtained as the result of a measurement, no unique detector-state can be inferred, and only probabilistic judgments can be made. Under such circumstances, the best one can do is to select the observable that provides as much information as possible, on the average, namely after many repetitions of the experiment. Of course, this presupposes the choice a definite criterion to measure the average amount $\bar{F}(A)$ of which-way information delivered by a certain observable $A$ (the properties of $\bar{F}(A)$, and the various choices proposed so far for this quantity are discussed later in this Section). After this choice is made, the distinguishability $D$ of the trajectories is usually related to the supremum, $F_D$, of $\bar{F}(A)$, over $A_D$.

It may now come as a surprise to notice, as pointed out in the Introduction, that the quantity $F_D$ does not always represent the absolute maximum information that is actually available. Indeed, it is an intriguing feature of the quantum detection problem, for non orthogonal states, that a larger amount of information on the state of the detector can be obtained by considering the detector in combination with an auxiliary quantum system, called ancilla [13, 14]. The ancilla does not interact with the detector, and is prepared in a fixed known state $|\phi_0\rangle\in \mathcal{H}_{aux}$, such that the combined system is in one of the $n$ uncorrelated states $|\chi_i\rangle\otimes|\phi_0\rangle$, belonging to the total Hilbert space $\mathcal{H}_{tot} = \mathcal{H}_D \otimes \mathcal{H}_{aux}$. Let now $A_{tot}$ the set of all hermitean operators in $\mathcal{H}_{tot}$ and $F_{tot}$ the supremum of $\bar{F}(A)$ over $A_{tot}$. Surprisingly enough, even if the detector and the ancilla are uncorrelated, it may happen that $F_{tot} > F_D$, showing that the inclusion of an ancilla may improve the amount of which-way information that can be read-out from the detectors.

Since the state of the ancilla is fixed once and for all, it is possible though to express the probabilities of the possible outcomes resulting from the measurement of any observable $A_{tot}$ in $\mathcal{H}_{tot}$, in terms of quantities defined directly in $\mathcal{H}_D$. We let $P_\mu$, $\mu = 1, \ldots, N$ the orthogonal decomposition of the identity in $\mathcal{H}_{tot}$, relative to $A_{tot}$ (we consider for simplicity an observable with a finite number $N$ of distinct outcomes). Then, the probability $P_{i\mu}$ that the outcome $\mu$ is observed, in the state $|\chi_i\rangle\otimes|\phi_0\rangle$ is given by the well known formula:

$$P_{i\mu} = \text{Tr} \left[ P_\mu (\rho_i \otimes \rho_{aux}) \right]$$

(2.4)

where $\rho_i = |\chi_i\rangle\langle\chi_i|$ and $\rho_{aux} = |\phi_0\rangle\langle\phi_0|$. If the trace is performed in two steps, first on the ancillary Hilbert space and then on $\mathcal{H}_D$, we can rewrite the above expression as

$$P_{i\mu} = \text{Tr} \left[ A_\mu \rho_i \right] ,$$

(2.5)

where

$$A_\mu = \text{Tr}_{aux} [ P_\mu (1 \otimes \rho_{aux}) ] ,$$

(2.6)

and $\text{Tr}_{aux}$ denotes the partial trace over the ancilla Hilbert space. The hermitean operators $A_\mu$ belong to $A_D$, and it is easy to check that they are positive definite, and that they provide
a decomposition of the identity on $\mathcal{H}_D$:

$$\sum_{\mu} A_\mu = 1, \quad \text{on } \mathcal{H}_D \quad (2.7)$$

However, in general, they are not projection operators, neither they commute with each other. We point out also that the number $N$ of different outcomes needs not be the same as neither the number $n$ of detector-states, nor the dimensionality of $\mathcal{H}_D$. The collection $\{A_\mu\}$ of operators constitutes an example of a positive operator-valued measure (POVM) in $\mathcal{H}_D$. More generally $[13, 14]$, a POVM is a map that associates to every (Borel) subset $\Delta$ of the real line $R$, a non-negative (self-adjoint) operator $\Pi(\Delta)$, such that:

i) the empty set $\emptyset$ is mapped to zero;

ii) the entire real line is mapped to the identity operator:

iii) the union of any number of disjoint sets is mapped to the sum of the corresponding operators.

The probability $P(\Delta)$ for the outcome to be in the set $\Delta$ is given by the following expression, generalizing equation (2.5):

$$P(\Delta) = \text{Tr} [\rho \Pi(\Delta)] . \quad (2.8)$$

The axioms i), ii) and iii) listed above ensure the consistency of the above probabilistic interpretation. POVM’s thus represent a generalization of the projection-valued measures (PVM), usually considered in Quantum Mechanics, and it is a theorem due to Neumark $[16]$, that all POVM’s on $\mathcal{H}_D$ can be realized by means of an appropriate ancillary system, in the way sketched above. Since any quantum system not interacting with the detector can play the rôle of the ancilla, this theorem implies that every POVM can be realized by an experimental procedure falling within the usual framework of Quantum Mechanics. Thus, in order to determine what is the maximum amount of which-way information that can obtained by observing the detector, we should maximize $\bar{F}$ over the set of all POVM’s in $\mathcal{H}_D$, and not just over the set of all PVM’s.

It is time now to define precisely the average which-way information $\bar{F}$ delivered by a POVM. For any POVM $\{A_\mu, \mu = 1, \ldots, N\}$ (we shall always consider POVM with a finite number $N$ of different outcomes, in what follows), consider the a posteriori probabilities $Q_{i\mu}$ for observing the $\mu$–th outcome, when the detector is in the state $|\chi_i\rangle$. According to Bayes’ formula:

$$Q_{i\mu} = \frac{\zeta_i P_{i\mu}}{q_\mu} , \quad (2.9)$$

where $q_\mu$ is the a priori probability for the occurrence of the outcome $\mu$:

$$q_\mu = \sum_i \zeta_i P_{i\mu} . \quad (2.10)$$

In order to measure the amount of which-way information, that is gained if the $\mu$-th outcome is observed, we consider the quantity $F_\mu = F(Q_\mu)$, where $\bar{Q}_\mu = (Q_{1\mu}, \ldots, Q_{n\mu})$ and $F$ is some function. It is reasonable to require from $F$ the following properties:
(1) $F$ should be invariant under any permutation of its $n$ arguments.

(2) $F$ should reach its absolute minimum when its $N$ arguments are all equal to $1/N$ (which corresponds to complete lack of information on the detector state);

(3) $F$ should reach its absolute maximum when any of its arguments is equal to one, while all the others are equal to zero (which on the contrary corresponds to certain knowledge of the detector state);

(4) $F$ should be convex, i.e. for any $\lambda \in [0, 1]$ it should hold:

$$F(\lambda \vec{Q}' + (1 - \lambda) \vec{Q}'') \leq \lambda F(\vec{Q}') + (1 - \lambda) F(\vec{Q}'').$$

Equation (2.11) than states that the test $A_\lambda$ cannot carry more information than the weighted sum of the informations obtained from $A'$ and $A''$, separately.

The intuitive meaning of this condition is clear if we interpret $\vec{Q}'$ and $\vec{Q}''$ as giving the a posteriori probabilities of $n$ alternative hypothesis, for two distinct tests $A'$ and $A''$. For any $\lambda \in [0, 1]$, we can consider the combination $A_\lambda$ of the tests $A'$ and $A''$, which consists in performing randomly either $A'$ or $A''$, with relative probabilities $\lambda$ and $1 - \lambda$, respectively. The overall average information delivered by the POVM is defined as the average $\bar{F}$ of the numbers $F_\mu$, over all possible outcomes, weighted with the a priori probabilities $q_\mu$:

$$\bar{F} := \sum_\mu q_\mu F_\mu.$$ (2.12)

The optimization problem consists in searching for the POVM which maximizes $\bar{F}$. Notice that, among the unknowns, we have to consider also the number $N$ of elements of the POVM. Of course, the solution depends on the choice of the function $F$, above. Over the past years, several different choices have been adopted. For example, as we said in the Introduction, the authors of Refs. [2, 14, 15] consider the negative of Shannon’s entropy $H$, which corresponds to taking:

$$F_\mu = -H_\mu := \sum_i Q_{i\mu} \log Q_{i\mu}.$$ (2.13)

References [10, 13] use the negative of Bayes’ cost function $C$:

$$F_\mu = -C_\mu := - \sum_{i \neq j(\mu)} Q_{ij\mu} = Q_{j(\mu)\mu} - 1,$$ (2.14)

where, for each $\mu$, $j(\mu)$ is any index such that $Q_{j(\mu)\mu} = \max\{Q_{1\mu}, \ldots, Q_{n\mu}\}$. Finally, more recently, Dür [11] considered the normalized rms spread $K$:

$$F_\mu = K_\mu := \left[ \frac{n}{n - 1} \sum_i \left( Q_{i\mu} - \frac{1}{n} \right)^2 \right]^{1/2}.$$ (2.15)

$F$ is said to be strictly convex, if the equality sign in Eq. (2.11) holds if and only if the vectors $\vec{Q}'_\mu$ and $\vec{Q}''_\mu$ coincide.
When $n = 2$, it is easy to check that $K_\mu = 1 - 2C_\mu$, and thus the two criteria (2.14) and (2.15) are inequivalent only for more than two beams. Notice also that, while Shannon’s entropy and the rms spread are strictly convex, the Bayes cost function is only convex.

Solving the optimization problem is a difficult task, and so far no general solution is known. However, partial results are available. For POVM’s consisting of a finite number of elements, by using the convexity of the function $F$, it is easy to show \cite{15} that the optimal POVM can be chosen to consist of rank one operators, namely:

$$A_\mu = |\phi_\mu \rangle \langle \phi_\mu| , \quad (2.16)$$

where $\|\phi_\mu\| \leq 1$. Moreover, if $\mathcal{H}_D$ is finite dimensional and $d$ is its dimension, it has been shown \cite{15} that the number $N$ of elements of the optimal POVM can be taken to satisfy:

$$d \leq N \leq d^2 . \quad (2.17)$$

3 Two-beam interferometers.

In this short Section, we consider a two-beam interferometer. For such a case, as pointed out in the previous Section, the criterion using the Bayes cost function [Eq. (2.14)] turns out to be equivalent to that based on the rms spreads [Eq. (2.15)]. The quantum detection problem, with the Bayes cost function as measure of information, is studied at length in Ref.\cite{13}. There, it is shown that, for any number $n$ of linearly independent states $|\chi_i \rangle$ and arbitrary a priori probabilities $\zeta_i$, the optimal measurement is always a PVM. Since, in two-beam interferometers, the detector states $|\chi_1 \rangle$ and $|\chi_2 \rangle$ must be distinct, for any path discrimination to be possible, they are necessarily linearly independent and thus it follows, from the quoted result, that the optimal measurement is a PVM.

To our knowledge, there is no published proof that the optimal measurement is a PVM, even when one uses Shannon’s entropy, as a measure of the which-way information. We have proven it, in the special case of equally populated beams, $\zeta_i = 1/2$. The rather elaborate proof can be found in the Appendix. When the populations $\zeta_i$ are different, we have not been able to work out an analytical proof, but a number of numerical simulations performed for various choices of the populations, seem to indicate that the optimal measurement is a PVM also in this general case.

In conclusion, it appears that for two-beam interferometers, both with Bayes’s cost or with Shannon’s information as measures of which-way information, ordinary PVM’s can read out the maximum which-way information from the detectors, and recourse to ancillas is superfluous. In fact, it turns out that the optimal PVM is the same, for both criteria (see Eq. (7.12) in the Appendix).
4 Multi-beams interferometers.

In this Section we study the case of multi-beam interferometers, with \( n > 2 \) beams. We make the simplifying assumption that \( \mathcal{H}_D \) is two-dimensional. This case is actually realized in experiments using beams of spin-half particles or photons, if the path information is stored in the internal states of the interfering particles. A further simplifying assumption that we make is that the beams are equally populated: \( \zeta_i = 1/n \).

\( \mathcal{H}_D \) is isomorphic to \( \mathbb{C}^2 \), the set of all pairs of complex numbers. As it is well known, rays of \( \mathbb{C}^2 \) can be put in one-to-one correspondence with unit three-vectors \( \hat{n}_i = (n_x, n_y, n_z) \), via the map:

\[
\frac{1 + \hat{n}_i \cdot \vec{\sigma}}{2} |\chi\rangle = |\chi\rangle ,
\]

where \( \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) is a set of Pauli matrices. Thus, assigning \( n \) pure states \( |\chi_i\rangle \) amounts to picking \( n \) unit vectors \( \hat{n}_i \) in \( \mathbb{R}^3 \). Whether the optimal test is a PVM or rather a POVM, now depends on the choice of the function \( F \). Below, we consider in detail the three choices for \( F \), Eqs. (2.13), (2.14) and (2.15), so far considered in the literature.

a) \( F \) is the negative of Shannon’s entropy \( H \) [Eq. (2.13)]. For three or more beams, it is known that the optimal test, in general, is not a PVM but rather a POVM. For example, for three states \( \hat{n}_1, \hat{n}_2 \) and \( \hat{n}_3 \) forming angles of 120° with each other and such that \( \sum_{i=1}^{3} \hat{n}_i = 0 \), it has been shown [14] that the optimal test is provided by the following POVM with three elements:

\[
A_i = \frac{1}{3}(1 - \hat{n}_i \cdot \vec{\sigma})
\]

b) \( F \) is the negative of Bayes’ cost function \( C \) [Eq. (2.14)]. Here too, the optimal test is not a PVM, but a POVM. An example is again provided by the set of three symmetric pure states considered under case (a) above. It is shown in [13] that the optimal POVM is given this time by the following POVM with three elements:

\[
A_i = \frac{1}{3}(1 + \hat{n}_i \cdot \vec{\sigma})
\]

Notice that the above POVM is not the same as [Eq. (1.2)], which is an example of the fact that the solution of the optimization problem depends on the choice of \( F \).

c) \( F \) is given by the rms spread \( K \) [Eq. (2.15)]. Remarkably enough, we can show that, for any number \( n \) of equally populated beams, the optimal test is always a PVM. This is in sharp contrast with what happens for the two other choices of \( F \) previously considered. To prove this claim, consider an optimal POVM, \( A = \{A_\mu; \mu = 1, \ldots N\} \). We know, from Sec. 2, that the operators \( A_\mu \) must be of the form (2.16). Using Eq. (1.1), we can write:

\[
A_\mu = \alpha_\mu (1 + \hat{m}_\mu \cdot \vec{\sigma})
\]

where \( \hat{m}_\mu \) are \( N \) unit three-vectors, and \( \alpha_\mu \) are \( N \) positive numbers. The condition for a POVM, \( \sum_\mu A_\mu = 1 \), is then equivalent to:

\[
\sum_\mu \alpha_\mu = 1 , \quad \sum_\mu \alpha_\mu \hat{m}_\mu = 0
\]
In view of Eq. (4.4), we find:
\[ P_{i\mu} := \langle \chi_i | A_\mu | \chi_i \rangle = \alpha_\mu (1 + \hat{m}_\mu \cdot \hat{n}_i) . \] (4.6)

Using this equation, we compute Eq. (2.10) as:
\[ q_\mu = \alpha_\mu (1 + \hat{m}_\mu \cdot \sum_i \zeta_i \hat{n}_i) . \] (4.7)

In order to evaluate the average information \( \bar{F}(A) \) of \( A \), it is convenient to rewrite the quantities \( q_\mu K_\mu \) as
\[ q_\mu K_\mu = \alpha_\mu \sqrt{\frac{n}{n-1} \left\{ -\frac{q_\mu^2}{n} + \sum_{i=1}^{n} \zeta_i^2 P_{ij}^2 \right\}}^{1/2} . \] (4.8)

Upon using Eqs. (4.6) and (4.7) into the above formula, we obtain, after a little algebra:
\[ q_\mu K_\mu = \alpha_\mu \sqrt{\frac{n}{n-1} \left\{ -\frac{1}{n} \left[ 1 + (\hat{m}_\mu \cdot \sum_i \zeta_i \hat{n}_i)^2 \right] + \sum_i \zeta_i^2 \left[ 1 + (\hat{m}_\mu \cdot \hat{n}_i)^2 \right] + 2 \hat{m}_\mu \cdot \sum_i \zeta_i \left( \zeta_i - \frac{1}{n} \right) \hat{n}_i \right\}}^{1/2} . \] (4.9)

We observe now that, for equally populated beams, \( \zeta_i = 1/n \), the last sum in the above equation vanishes, and the expression for \( q_\mu K_\mu \) becomes invariant under the exchange of \( \hat{m}_\mu \) with \( -\hat{m}_\mu \).

Consider now the POVM \( B = \{ B^+_\mu, B^-_\mu; \mu = 1, \ldots, N \} \), consisting of \( 2N \) elements, such that:
\[ B^+_\mu = \frac{1}{2} A_\mu \, , \quad B^-_\mu = \frac{1}{2} \alpha_\mu (1 - \hat{m}_\mu \cdot \vec{\sigma}) \] (4.10)

Of course, \( q^{(+)}_{\mu} K^{(+)}_{\mu} = q_{\mu} K_{\mu} / 2 \), while the invariance of \( q_{\mu} K_{\mu} \) implies \( q^{(-)}_{\mu} K^{(-)}_{\mu} = q^{(+)}_{\mu} K^{(+)}_{\mu} \).

It follows that the average informations for \( A \) and \( B \) are equal to each other, \( \bar{F}(A) = \bar{F}(B) \).

Now, for each value of \( \mu \), the pair of operators \( B^\pm_\mu / \alpha_\mu = (1 \pm \hat{m}_\mu \cdot \vec{\sigma}) / 2 \) constitutes a PVM, and thus the POVM \( B \) can be regarded as a collection of \( N \) PVM’S, each taken with a non-negative weight \( \alpha_\mu \). But then \( \bar{F}(B) \), being equal to the average of the amounts of information provided by \( N \) PVM’S, cannot be higher than the maximum information \( F_D \) delivered by a PVM. Therefore, we have proven that \( F(A) = F(B) \leq F_D \), which shows that the optimum measurement can always be effected by a means of PVM.

We then see that, in the multibeam case, only with Dürr’s measure of information one can dispose of the ancilla, at least for equally populated beams.

5 Conclusions

When, in an interference experiment, the which-way detector states are not mutually orthogonal, one has an incomplete knowledge of the path followed by the interfering particles. One is then faced with the problem of reading out, in an optimum way, the information stored in the detectors. The best measurement to be performed depends, in a crucial way, on the criterion used to measure the information. This is a problem in quantum decision theory, and our paper
is a contribution to the task of identifying the optimum quantum test, for which no general solution is known so far.

We have shown that for the two beams case, both by using Shannon entropy or Bayes cost function as measures of information, the best test to be performed is given by an ordinary projection valued measurement in the detector’s Hilbert space. Actually, it turns out that both criteria identify the same measurement. In the multibeam case only Dürr’s normalized rms spread criterion leads to a PVM, while the other two lead to a POVM. Notice that in the case of three coplanar symmetric beam states one ends up with two different POVM’s: the one relative to Bayes cost [Eq. (4.3)], allows every time to pick one beam as the most probable one, while the POVM determined by Shannon entropy, allows to exclude one of the three beams as impossible.

We see, then, that in the multibeam case Dürr’s criterion seems to be favoured for two different reason. First of all, it allows to derive a quantitative complementarity relation, as the one given by Eq. (1.3). Second, it allows to work with ordinary quantum mechanical measurements, and to ignore generalized POVM’s, involving an ancillary system. A possible relationship of these two features seems worth studying. This may be related to the fact that, as has been recently shown [17], there are problems in extending the mathematical definition of complementarity to a POVM.

Our results are of limited generality in two respects: first, in the multibeam case they refer to two-state detectors, second, we always considered equally populated beams. For what concern the latter problem, we may add that we have gathered substantial numerical evidence that our results may extend to arbitrarily populated beams. However we lack at the moment an analytic proof. The former limitation seems more difficult to overcome. Fortunately, however, the case we have treated is physically interesting, for it includes many experimental setups in which the “which-way” detection exploits some two-states internal degrees of freedom of the interfering particles.

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7 Appendix

In this Appendix, we prove the following
Theorem: for a two-beams interferometer with equally populated beams, when one uses the negative of Shannon’s entropy to measure the which-way information, the optimal measurement is provided by a PVM (precisely described in Eq. (7.12) below).

More precisely, let $|\chi_+>$ and $|\chi_->$ be the detector states, for the two beams. We exclude the trivial case, when $|\chi_+>$ and $|\chi_->$ are proportional, because then no path-reconstruction would be possible. Therefore, $\mathcal{H}_D$ is two-dimensional and we can represent vectors in $\mathcal{H}_D$ by unit three vectors, according to Eq. (4.1). We loose no generality if we assume that the unit vectors $\hat{n}_+$ and $\hat{n}_-$, associated to $|\chi_+>$ and $|\chi_->$ respectively, have the expressions:

$$\hat{n}_+ = (\sin \theta, 0, \cos \theta), \quad \hat{n}_- = (-\sin \theta, 0, \cos \theta), \quad (7.11)$$

With this parametrization for the states $|\chi_+>$ and $|\chi_->$, our theorem states that, if the which-way information is measured by the negative of Shannon’s entropy $H$, the optimal measurement is provided by the PVM $A$ with elements:

$$A_+ = \frac{1}{2}(1 + \sigma_x), \quad A_- = \frac{1}{2}(1 - \sigma_x). \quad (7.12)$$

Before giving the proof of this Theorem, it is useful to prove first the following

Lemma: consider, in $C^2$, $n$ states $|\xi_i>$, with coplanar vectors $\hat{n}_i$, and arbitrary populations $\zeta_i$. Then, the optimal POVM has elements $A_\mu$ of the form [Eq. (4.4)], with all the vectors $\hat{n}_\mu$ lying in the same plane containing the vectors $\hat{n}_i$.

The proof of the lemma is as follows. Let $B$ be an optimal POVM. Then we know, from the theorems quoted in Sec. 2, that its elements must have rank-one and so are of the form given in Eq. (4.4). Moreover, they must satisfy the POVM conditions given by Eqs. (4.3). Suppose now that some of the vectors $\hat{n}_\mu$ do not belong to the plane containing the vectors $\hat{n}_i$, which we assume to be the $xz$ plane. We show below how to construct a new POVM $A \equiv \{A_\nu \mid \nu = 1, \ldots, N + p\}$, providing not less information than $B$, and such that the vectors $\hat{n}_\nu^{(A)}$ all belong to the $xz$ plane. The first step in the construction of $A$ consists in symmetrizing $B$ with respect to the $xz$ plane. The symmetrization is done by replacing each element $B_\mu$ of $B$, not lying in the $xz$ plane, by the pair $(B'_\mu, B''_\mu)$, where $B'_\mu = B_\mu/2$, and $B''_\mu$ has the same weight as $B'_\mu$, while its vector $\hat{n}_\mu^{(B)\nu}$ is the symmetric of $\hat{n}_\mu^{(B)}$ with respect to the $xz$ plane.

It is easy to verify that the symmetrization preserves the conditions for a POVM [Eqs. (4.5)]. Since all the vectors $\hat{n}_i$ belong by assumption to the $xz$ plane, we see, from Eq. (4.6), that the probabilities $P_{i\mu}$ actually depend only on the projections of the vectors $\hat{n}_\mu^{(B)}$ in the plane $xz$. This implies, at is easy to check, that symmetrization with respect to the $xz$ plane does not change the information $\mathcal{F}$. We assume therefore that $B$ has been preliminarily symmetrized in this way. Now we show that we can replace, one after the other, each pair of symmetric elements $(B'_\mu, B''_\mu)$ by another pair of operators, whose vectors lie in the $xz$ plane, without reducing the information provided by the POVM. Consider for example the pair $(B'_p, B''_p)$. We construct the unique pair of unit vectors $\hat{u}_p$ and $\hat{v}_p$, lying the $xz$ plane, and such that:

$$\hat{u}_p + \hat{v}_p = 2(m_p^{(B)x} \hat{i} + m_p^{(B)z} \hat{k}), \quad (7.13)$$

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where \( \hat{i} \) and \( \hat{j} \) are the directions of the \( x \) and \( z \) axis, respectively. Notice that \( \hat{u}_p \neq \hat{v}_p \). Consider now the collection of operators obtained by replacing the pair \((B'_p, B''_p)\) with the pair \((A'_p, A''_p)\) such that:

\[
A'_p = \alpha_p^{(B)}(1 + \hat{u}_p \cdot \vec{\sigma}) , \quad A''_p = \alpha_p^{(B)}(1 + \hat{v}_p \cdot \vec{\sigma}) .
\]

(7.14)

It is clear, in view of Eqs. (7.13), that the new collection of \( N + p \) operators still forms a resolution of the identity, and thus represents a POVM. Equations (7.13) also imply:

\[
P_{ip}^{(B)_r} = P_{ip}^{(B)_n} = \alpha_p(1 + m_p^{(B)x} n_i^x + m_p^{(B)z} n_i^z) = \\
= \frac{1}{2}\alpha_p(1 + u_p^x n_i^x + u_p^z n_i^z) + \frac{1}{2}\alpha_p(1 + v_p^x n_i^x + v_p^z n_i^z) = \frac{1}{2}(P_{ip}^{(A)_r} + P_{ip}^{(A)_n}) ,
\]

(7.15)

Now, define \( \lambda'_p := q_p^{(A)_r}/(2q_p^{(B)}) \), and \( \lambda''_p := q_p^{(A)_n}/(2q_p^{(B)}) \), where \( q_p^{(B)} := q_p^{(B)} = q_p^{(B)} \). Since \( q_p^{(A)_r} + q_p^{(A)_n} = 2q_p^{(B)} \), we have \( \lambda'_p + \lambda''_p = 1 \). It is easy to verify, using Eqs. (2.11) and (2.10), that:

\[
Q_{ip}^{(B)_r} = Q_{ip}^{(B)_n} = \lambda'_p Q_{ip}^{(A)_r} + \lambda''_p Q_{ip}^{(A)_n} ,
\]

(7.16)

But then, the convexity of \( F \) implies:

\[
d_{ip}^{(B)} F(B)(Q_{ip}^{(B)}) + q_p^{(B)} F(B)(Q_p^{(B)}) = 2q_p^{(B)} F(B)(Q_p^{(B)}) = \\
= 2q_p^{(B)} F(\lambda'_p Q_{ip}^{(A)_r} + \lambda''_p Q_{ip}^{(A)_n}) \leq 2q_p^{(B)}[\lambda'_p F(Q_{ip}^{(A)_r}) + \lambda''_p F(Q_{ip}^{(A)_n})] = \\
= q_p^{(A)} F(A)(Q_{ip}^{(A)_r}) + q_p^{(A)} F(A)(Q_{ip}^{(A)_n}) .
\]

(7.17)

It follows that the new POVM is no worse than \( B \). By repeating this construction \( p \) times, we can obviously eliminate from \( B \) all the \( p \) pairs of elements not lying in the \(xz\) plane, until we get a POVM \( A \), which provides not less information than \( B \), whose elements all lie in the \(xz\) plane. This concludes the proof of the lemma.

We can turn now to the proof of the Theorem, stated at the beginning of this Appendix. The proof consists in showing that the POVM \( A \) in [Eq. (7.13)] provides not less information than any other POVM, \( C \), consisting of more than two elements. By virtue of the lemma just proven, we loose no generality if we assume that the \( N > 2 \) vectors \( m^{(C)}_\mu \) of \( C \) lie in the \(xz\) plane. Our first move is to symmetrize \( C \) with respect to \( z \) axis, by introducing a POVM \( B \), consisting of \( N \) pairs of elements \((B'_\mu, B''_\mu)\), having equal weights, and vectors \( m'_\mu \) and \( m''_\mu \) that are symmetric with respect to the \( z \) axis:

\[
B'_\mu = \frac{1}{2} C_\mu , \quad B''_\mu = \frac{1}{2} \alpha^{(C)}(1 - m'_\mu \sigma_x + m''_\mu \sigma_z) , \quad \mu = 1, \ldots, N .
\]

(7.18)

\( B \) provides as much information as \( C \). Indeed, in view of Eq. (4.6), we find

\[
P_{\pm\mu}^{(C)} = 2 P_{\pm\mu}^{(B)_r} = 2 P_{\pm\mu}^{(B)_n} , \quad \mu = 1, \ldots, N .
\]

(7.19)

The invariance of \( F \) with respect to permutations of its arguments, then ensures that \( \tilde{F}(B) = \tilde{F}(C) \). Thus, we loose no information if we consider a POVM \( B \), that is symmetric with respect to the \( z \) axis. Now we describe a procedure of reduction that, applied to a symmetric POVM
like $B$, gives rise to another symmetric POVM $\tilde{B}$, which contains two elements less than $B$, but nevertheless gives no less information than $B$. The procedure works as follows: we pick at will two pairs of elements of $B$, say $(B'_N, B''_N)$ and $(B'_{N-1}, B''_{N-1})$ and consider the unique pair of symmetric unit vectors $\hat{u}_\pm = \pm u^z \hat{i} + u^\perp \hat{k}$ such that:

$$u^z = \frac{1}{\alpha_N^{(B)} + \alpha_{N-1}^{(B)}} \left( \alpha_N^{(B)} m_N^{(B)z} + \alpha_{N-1}^{(B)} m_{N-1}^{(B)z} \right).$$ (7.20)

Consider the symmetric collection $\tilde{B}$, obtained from $B$ after replacing the four elements $(B'_N, B''_N, B'_{N-1}, B''_{N-1})$ by the pair $(\tilde{B}'_{N-1}, \tilde{B}''_{N-1})$ such that:

$$\tilde{B}'_{N-1} = (\alpha_N^{(B)} + \alpha_{N-1}^{(B)})(1 + \hat{u}_+ \cdot \hat{\sigma}), \quad \tilde{B}''_{N-1} = (\alpha_N^{(B)} + \alpha_{N-1}^{(B)})(1 + \hat{u}_- \cdot \hat{\sigma}).$$ (7.21)

$\tilde{B}$ is still a POVM, as it is easy to verify. Moreover, $\tilde{B}$ provides not less information than $B$, as we now show. Indeed, after some algebra, one finds:

$$\frac{\bar{F}(\tilde{B}) - \bar{F}(B)}{\alpha_N^{(B)} + \alpha_{N-1}^{(B)}} = g(u^z) - \frac{\alpha_N^{(B)}}{\alpha_N^{(B)} + \alpha_{N-1}^{(B)}} g(m_N^{(B)z}) - \frac{\alpha_{N-1}^{(B)}}{\alpha_N^{(B)} + \alpha_{N-1}^{(B)}} g(m_{N-1}^{(B)z}),$$ (7.22)

where the function $g(x)$ has the expression:

$$g(x) = (1 + x \cos \theta) \log(1 + x \cos \theta) +$$

$$- \frac{1}{2} (1 + x \cos \theta + (1 - x^2)^{1/2} \sin \theta) \log \left[ \frac{1}{2} (1 + x \cos \theta + (1 - x^2)^{1/2} \sin \theta) \right] +$$

$$- \frac{1}{2} (1 + x \cos \theta - (1 - x^2)^{1/2} \sin \theta) \log \left[ \frac{1}{2} (1 + x \cos \theta - (1 - x^2)^{1/2} \sin \theta) \right].$$ (7.23)

In view of Eq. (7.20), the r.h.s. of Eq. (7.22) is of the form

$$g(\lambda x_1 + (1 - \lambda) x_2) - \lambda g(x_1) - (1 - \lambda) g(x_2),$$ (7.24)

where $\lambda = \alpha_N^{(B)}/(\alpha_N^{(B)} + \alpha_{N-1}^{(B)})$, while $x_1 = m_N^{(B)z}$ and $x_2 = m_{N-1}^{(B)z}$. It may be checked that, for all values of $\theta$, $g(x)$ is concave, for $x \in [-1,1]$, and so the r.h.s. of Eq. (7.24) is non-negative for any value of $\lambda \in [0,1]$. This implies that the r.h.s. of Eq. (7.22) is non-negative as well, and so $\bar{F}(\tilde{B}) \geq \bar{F}(B)$. After $N - 1$ iterations of this procedure, we end up with a symmetric POVM consisting of two pairs of elements $(B'_1, B''_1)$ and $(B'_2, B''_2)$. But then, the conditions for a POVM, Eqs. (4.15), imply that the quantity between the brackets on the r.h.s. of Eq. (7.24) vanishes, and so Eq. (7.20) gives $u^z = 0$. This means that the last iteration gives rise precisely to the PVM $A$ in Eq. (7.12). By putting everything together, we have shown that $\bar{F}(C) = \bar{F}(B) \leq \bar{F}(\tilde{B}) \ldots \leq \bar{F}(A)$, and this is the required result.

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