ELEMENTARY PROOF OF LOGARITHMIC SOBOLEV INEQUALITIES FOR GAUSSIAN CONVOLUTIONS ON $\mathbb{R}$

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Abstract. In a 2013 paper, the author showed that the convolution of a compactly supported measure on the real line with a Gaussian measure satisfies a logarithmic Sobolev inequality (LSI). In a 2014 paper, the author gave bounds for the optimal constants in these LSIs. In this paper, we give a simpler, elementary proof of this result.

1. Introduction

A probability measure $\mu$ on $\mathbb{R}^n$ is said to satisfy a logarithmic Sobolev inequality (LSI) with constant $c \in \mathbb{R}$ if

$$\text{Ent}_\mu(f^2) \leq c \mathcal{E}(f, f)$$

for all locally Lipschitz functions $f : \mathbb{R}^n \to \mathbb{R}_+$, where $\text{Ent}_\mu$, called the entropy functional, is defined as

$$\text{Ent}_\mu(f) := \int f \log f \, d\mu d\mu$$

and $\mathcal{E}(f, f)$, the energy of $f$, is defined as

$$\mathcal{E}(f, f) := \int |\nabla f|^2 d\mu,$$

with $|\nabla f|$ defined as

$$|\nabla f|(x) := \limsup_{y \to x} \frac{|f(x) - f(y)|}{|x - y|}$$

so that $|\nabla f|$ is defined everywhere and coincides with the usual notion of gradient where $f$ is differentiable.

LSIs are a useful tool that have been applied in various areas of mathematics, cf. [1, 2, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20]. In [21], the present author showed that the convolution of a compactly supported measure on $\mathbb{R}$ with a Gaussian measure satisfies a LSI, and an application of this fact to random matrix theory was given. In [22, Thms. 2,3], bounds for the optimal constants in these LSIs were given, and the results were extended to $\mathbb{R}^n$. Those results are stated as Theorems 1 and 2 below. (See [17] for statements about LSIs for convolutions with more general measures).

Theorem 1. Let $\mu$ be a probability measure on $\mathbb{R}$ whose support is contained in an interval of length $2R$, and let $\gamma_\delta$ be the centered Gaussian of variance $\delta > 0$, i.e., $d\gamma_\delta(t) = (2\pi\delta)^{-1/2} \exp(-\frac{t^2}{2\delta}) \, dt$. Then for some absolute constants $K_i$, the optimal log-Sobolev constant $c(\delta)$ for $\mu * \gamma_\delta$ satisfies

$$c(\delta) \leq K_1 \frac{\delta^{3/2} R}{4 R^2 + \delta} \exp \left( \frac{2 R^2}{\delta} \right) + K_2 \left( \sqrt{\delta} + 2R \right)^2.$$

In particular, if $\delta \leq R^2$, then

$$c(\delta) \leq K_3 \frac{\delta^{3/2}}{R} \exp \left( \frac{2 R^2}{\delta} \right).$$

The $K_i$ can be taken in the above inequalities to be $K_1 = 6905, K_2 = 4989, K_3 = 7803$.

Theorem 2. Let $\mu$ be a probability measure on $\mathbb{R}^n$ whose support is contained in a ball of radius $R$, and let $\gamma_\delta$ be the centered Gaussian of variance $\delta$ with $0 < \delta \leq R^2$, i.e., $d\gamma_\delta(x) = (2\pi\delta)^{-n/2} \exp(-\frac{|x|^2}{2\delta}) \, dx$. Then for some absolute constant $K$, the optimal log-Sobolev constant $c(\delta)$ for $\mu * \gamma_\delta$ satisfies

$$c(\delta) \leq K R^2 \exp \left( 20n + \frac{5 R^2}{\delta} \right).$$

$K$ can be taken above to be 289.
Theorem 4 was proved in [22] using the following theorem due to Bobkov and Götze [3] p.25, Thm 5.3:

**Theorem 3** (Bobkov, Götze). Let $\mu$ be a Borel probability measure on $\mathbb{R}$ with distribution function $F(x) = \mu((\infty, x])$. Let $p$ be the density of the absolutely continuous part of $\mu$ with respect to Lebesgue measure, and let $m$ be a median of $\mu$. Let

$$D_0 = \sup_{x < m} \left( F(x) \cdot \log \frac{1}{F(x)} \int_x^m \frac{1}{p(t)} dt \right),$$

$$D_1 = \sup_{x > m} \left( (1 - F(x)) \cdot \log \frac{1}{1 - F(x)} \cdot \int_x^m \frac{1}{p(t)} dt \right),$$

defining $D_0$ and $D_1$ to be zero if $\mu((\infty, m)) = 0$ or $\mu((m, \infty)) = 0$, respectively, and using the convention $0 \cdot \infty = 0$. Then the optimal log Sobolev constant $c$ for $\mu$ satisfies $\frac{1}{150} (D_0 + D_1) \leq c \leq 468(D_0 + D_1)$.

Theorem 5 was proved in [22] using the following theorem due to Cattiaux, Guillin, and Wu [6] Thm. 1.2:

**Theorem 4** (Cattiaux, Guillin, Wu). Let $\mu$ be a probability measure on $\mathbb{R}^n$ with $d\mu(x) = e^{-V(x)} dx$ for some $V \in C^2(\mathbb{R}^n)$. Suppose the following:

1. There exists a constant $K \leq 0$ such that $\text{Hess}(V) \geq K I$.
2. There exists a $W \in C^2(\mathbb{R}^n)$ with $W \geq 1$ and constants $b, c > 0$ such that

$$\Delta W(x) - (\nabla V, \nabla W)(x) \leq (b - c|x|^2)W(x)$$

for all $x \in \mathbb{R}^n$.

Then $\mu$ satisfies a LSI.

The goal of the present paper is to provide an elementary proof of Theorem 4. The result proved is the following:

**Theorem 5.** Let $\mu$ be a probability measure on $\mathbb{R}$ whose support is contained in an interval of length $2R$, and let $\gamma_\delta$ be the centered Gaussian of variance $\delta > 0$, i.e., $d\gamma_\delta(t) = (2\pi\delta)^{-1/2} \exp\left(-\frac{t^2}{2\delta}\right) dt$. Then the optimal log-Sobolev constant $c(\delta)$ for $\mu \ast \gamma_\delta$ satisfies

$$c(\delta) \leq \max \left( 2\delta \exp \left( \frac{4R^2}{\delta} + 4R \sqrt{\delta} + \frac{1}{4} \right), 2\delta \exp \left( \frac{24R^2}{\delta} \right) \right).$$

In particular, if $\delta \leq 16R^2$, we have

$$c(\delta) \leq 2\delta \exp \left( \frac{24R^2}{\delta} \right).$$

The bound in Theorem 5 is worse than the bound in Theorem 4 for small $\delta$, but still has an order of magnitude that is exponential in $R^2/\delta$. (It is shown in [22] Example 21] that one cannot do better than exponential in $R^2/\delta$ for small $\delta$.)

2. Proof of Theorem 5

The proof of Theorem 5 is based on two facts: first, the Gaussian measure $\gamma_1$ of unit variance satisfies a LSI with constant 2. Second, Lipshitz functions preserve LSIs. We give a precise statement of this second fact below.

**Proposition 6.** Let $\mu$ be a measure on $\mathbb{R}$ that satisfies a LSI with constant $c$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be Lipshitz. Then the push-forward measure $T_*\mu$ also satisfies a LSI with constant $c |T|_{\text{Lip}}^2$.

**Proof.** Let $g : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz. Then $g \circ T$ is locally Lipschitz, so by the LSI for $\mu$,

$$\int (g \circ T)^2 \log \frac{(g \circ T)^2}{\int (g \circ T)^2} d\mu \leq c \int |\nabla (g \circ T)|^2 d\mu.$$

But since $T$ is Lipschitz,

$$|\nabla (g \circ T)| \leq (|\nabla g| \circ T)|T|_{\text{Lip}}.$$

So by a change of variables, (1) simply becomes

$$\int g^2 \log \frac{g^2}{\int g^2} dT_*\mu \leq c |T|_{\text{Lip}}^2 \int |\nabla g|^2 dT_*\mu,$$

as desired. □
Lemma 7. For $K$ and $\Lambda$ and $\delta = 1$ (the general case will be handled at the end of the proof by a scaling argument).

Let $F$ and $G$ be the cumulative distribution functions of $\gamma_1$ and $\mu * \gamma_1$, i.e.,

$$F(x) = \int_{-\infty}^{x} p(t) \, dt, \quad G(x) = \int_{-\infty}^{x} q(t) \, dt,$$

where

$$p(t) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) \quad \text{and} \quad q(t) = \int_{-\infty}^{R} p(t - s) \, d\mu(s).$$

Notice that $q$ is smooth and strictly positive, so that $G^{-1} \circ F$ is well-defined and smooth. It is readily seen that $(G^{-1} \circ F)_* (\gamma_1) = \mu * \gamma_1$, so to establish the theorem we simply need to bound the derivative of $G^{-1} \circ F$.

Now

$$(G^{-1} \circ F)'(x) = \frac{1}{G((G^{-1} \circ F)(x))} \cdot F'(x) = \frac{p(x)}{q((G^{-1} \circ F)(x))}.$$

We will bound the above derivative in cases — when $x \geq 2R$, when $-2R \leq x \leq 2R$, and when $x \leq -2R$.

We first consider the case $x \geq 2R$. Define

$$\Lambda(x) = \int_{-R}^{R} e^{xs} \, d\mu(s), \quad K(x) = \log \Lambda(x) + \frac{R}{x}.$$

Note $\Lambda$ and $K$ are smooth for $x \neq 0$.

**Lemma 7.** For $x \geq 2R$,

$$\exp\left(-2R^2 - 2R - \frac{1}{8}\right) \cdot p(x) \leq q(x + K(x)) \leq e^{-R} \cdot p(x).$$

**Proof.** By definition of $q$, $p$, $\Lambda$, and $K$,

$$q(x + K(x)) = \int_{-R}^{R} p(x + K(x) - s) \, d\mu(s) = p(x) \cdot e^{-xK(x)} \int_{-R}^{R} \exp\left(-\frac{(K(x) - s)^2}{2}\right) \cdot e^{xs} \, d\mu(s)$$

$$= \frac{e^{-R} \cdot p(x)}{\Lambda(x)} \int_{-R}^{R} \exp\left(-\frac{(K(x) - s)^2}{2}\right) \cdot e^{xs} \, d\mu(s)$$

$$\leq \frac{e^{-R} \cdot p(x)}{\Lambda(x)} \int_{-R}^{R} e^{xs} \, d\mu(s)$$

$$= e^{-R} \cdot p(x).$$

To get the other inequality, first note that $e^{-Rx} \leq \Lambda(x) \leq e^{Rx}$. (These are just the maximum and minimum values in the integrand defining $\Lambda$.) This implies that $-R + R/x \leq K(x) \leq R + R/x$, so for $-R \leq s \leq R$ and $x \geq 2R$, we have

$$-2R - \frac{R}{x} \leq K(x) - s \leq 2R + \frac{R}{x}$$

so that

$$\exp\left(-\frac{(K(x) - s)^2}{2}\right) \geq \exp\left(-\frac{(2R + R/x)^2}{2}\right) \geq \exp\left(-\frac{(2R + R/(2R))^2}{2}\right) = \exp\left(-2R^2 - R - \frac{1}{8}\right).$$

Therefore

$$q(x + K(x)) = \frac{e^{-R} \cdot p(x)}{\Lambda(x)} \int_{-R}^{R} \exp\left(-\frac{(K(x) - s)^2}{2}\right) \cdot e^{xs} \, d\mu(s) \geq e^{-2R^2 - 2R - \frac{1}{8}} \cdot p(x).$$

$\square$
Lemma 8. \( K'(x) \leq R \) for \( x \geq 2R \).

Proof. Recall that \( e^{-Rx} \leq \Lambda(x) \). (Again, \( e^{-Rx} \) is the minimum value in the integrand defining \( \Lambda \)). We therefore have

\[
K'(x) = \frac{\Lambda'(x)}{x\Lambda(x)} - \frac{\log \Lambda(x)}{x^2} - \frac{R}{x^2} = \frac{\int_{-R}^{0} e^{xs} d\mu(s)}{x\Lambda(x)} - \frac{\log \Lambda(x)}{x^2} - \frac{R}{x^2} \leq \frac{R \int_{-R}^{0} e^{xs} d\mu(s)}{x\Lambda(x)} + \frac{Rx}{x^2} - \frac{R}{x^2} \leq \frac{2R}{x} - \frac{R}{x^2}.
\]

By elementary calculus, the above has a maximum value of \( R \).

Lemma 9. For \( x \geq 2R \),

\[
x - R \leq (G^{-1} \circ F)(x) \leq x + K(x).
\]

Proof. Since \( G \) and \( G^{-1} \) are increasing, the lemma is equivalent to

\[
G(x - R) \leq F(x) \leq G(x + K(x)).
\]

The first inequality follows from the definition of \( G \) and the Fubini-Tonelli Theorem:

\[
G(x - R) = \int_{-\infty}^{x-R} q(t) \, dt = \int_{-\infty}^{x} \int_{-R}^{R} p(t-s) \, d\mu(s) \, dt = \int_{-R}^{R} \int_{-\infty}^{x-R} p(t-s) \, dt \, d\mu(s) = \int_{-R}^{R} \int_{-\infty}^{x-R+s} p(u) \, du \, d\mu(s) \leq \int_{-R}^{R} \int_{-\infty}^{x} p(u) \, du \, d\mu(s) = F(x).
\]

To establish the other inequality, we use Lemmas 1 and 8

\[
1 - G(x + K(x)) = \int_{x+K(x)}^{\infty} q(t) \, dt = \int_{x}^{\infty} q(u + K(u))(1 + K'(u)) \, du \leq \int_{x}^{\infty} p(u) e^{-R}(1 + R) \, du \leq \int_{x}^{\infty} p(u) \, du \leq 1 - F(x),
\]

so that \( F(x) \leq G(x + K(x)) \), as desired.

We are almost ready to bound \((G^{-1} \circ F)'(x)\) for \( x \geq 2R \). The last observation to make is that \( q \) is decreasing on \([R, \infty)\) since

\[
q'(t) = \int_{-R}^{R} p'(t-s) \, d\mu(s) = \int_{-R}^{R} -(t-s)p(t-s) \, d\mu(s) \leq 0 \quad \text{for } t \geq R.
\]

So for \( x \geq 2R \) we have, by lemma 3

\[
q((G^{-1} \circ F)(x)) \geq q(x + K(x)).
\]

Combining this with Lemma 7 we get

\[
(G^{-1} \circ F)'(x) = \frac{p(x)}{q((G^{-1} \circ F)(x))} \leq \frac{p(x)}{q(x + K(x))} \leq \exp\left(2R^2 + 2R + \frac{1}{8}\right)
\]

for \( x \geq 2R \).
In the case where \(-2R \leq x \leq 2R\), first note that for all \(x\),
\[
x - R \leq (G^{-1} \circ F)(x) \leq x + R;
\]
the first inequality above was done in Lemma \(\text{[9]}\) and the second inequality is proven in the same way. So
\[
\sup_{-2R \leq x \leq 2R} (G^{-1} \circ F)'(x) = \sup_{-2R \leq x \leq 2R} \frac{p(x)}{q((G^{-1} \circ F)(x))} \leq \sup_{-R \leq y \leq R} \frac{p(x)}{q(x + y)} = \left( \inf_{-2R \leq x \leq 2R} \frac{q(x + y)}{p(x)} \right)^{-1}.
\]
For convenience, let \(S = \{(x, y) : -2R \leq x \leq 2R, -R \leq y \leq R\}\). Now
\[
\inf_{(x, y) \in S} \frac{q(x + y)}{p(x)} = \inf_{(x, y) \in S} \frac{1}{p(x)} \int_{-R}^{R} p(x + y - s) \, d\mu(s).
\]
Since \(p\) has no local minima, the minimum value of the above integrand occurs at either \(s = R\) or \(s = -R\). Without loss of generality, we assume the minimum is achieved at \(s = R\) (otherwise, we can replace \((x, y)\) with \((-x, -y)\) by symmetry of \(S\) and \(p\)). So
\[
\inf_{(x, y) \in S} \frac{q(x + y)}{p(x)} \geq \inf_{(x, y) \in S} \frac{1}{p(x)} \cdot p(x + y + R).
\]
Elementary calculus shows that the above infimum is equal to \(e^{-12R^2}\) (achieved at \(x = 2R, y = R\)). Therefore
\[
\sup_{-2R \leq x \leq 2R} (G^{-1} \circ F)'(x) = \left( \inf_{(x, y) \in S} \frac{q(x + y)}{p(x)} \right)^{-1} \leq e^{12R^2}.
\]
The case \(x \leq -2R\) is dealt with in the same way as the case \(x \geq 2R\), the analogous statements being:
\[
\exp \left( -2R^2 - 2R - \frac{1}{8} \right) p(x) \leq q(x + K(x)) \leq e^{-R} p(x),
\]
\[
K'(x) \leq R,
\]
\[
x + K(x) \leq (G^{-1} \circ F)(x) \leq x + R,
\]
and \(q\) is increasing for \(x \leq -2R\). The upper bound for \((G^{-1} \circ F)'(x)\) obtained in this case is the same as the one in the case \(x \geq 2R\).
We therefore have
\[
||G^{-1} \circ F||_{\text{Lip}} \leq \max \left( \exp \left( 2R^2 + 2R + \frac{1}{8} \right), e^{12R^2} \right).
\]
So by Proposition \(\text{[9]}\) \(\mu \ast \gamma_1\) satisfies a LSI with constant \(c(1)\) satisfying
\[
c(1) \leq 2||G^{-1} \circ F||_{\text{Lip}}^2 \leq \max \left( 2 \exp \left( 4R^2 + 4R + \frac{1}{4} \right), 2 e^{24R^2} \right).
\]
This proves the theorem for the case \(\delta = 1\).

To establish the theorem for a general \(\delta > 0\), first observe that
\[
\mu \ast \gamma_\delta = (h_\sqrt{\delta})_* \left( ((h_1/\sqrt{\delta}), \mu) \ast \gamma_1 \right),
\]
where \(h_\lambda\) denotes the scaling map with factor \(\lambda\), i.e., \(h_\lambda(x) = \lambda x\). Now \((h_1/\sqrt{\delta}), \mu\) is supported in \([-R/\sqrt{\delta}, R/\sqrt{\delta}]\), so by the case \(\delta = 1\) just proven, \(((h_1/\sqrt{\delta}), \mu) \ast \gamma_1\) satisfies a LSI with constant
\[
\max \left( 2 \exp \left( 4(R/\sqrt{\delta})^2 + 4(R/\sqrt{\delta}) + \frac{1}{4} \right), 2 e^{24(R/\sqrt{\delta})^2} \right).
\]
Finally, since \(||h_1/\sqrt{\delta}||_{\text{Lip}} = \delta\), we have by Proposition \(\text{[9]}\)
\[
c(\delta) \leq \max \left( 2\delta \exp \left( 4R^2 + \frac{4R}{\sqrt{\delta}} + \frac{1}{4} \right), 2\delta \exp \left( \frac{24R^2}{\delta} \right) \right).
\]
In particular, when \(\delta \leq 16R^2\), we have
\[
2\delta \exp \left( \frac{4R^2}{\delta} + \frac{4R}{\sqrt{\delta}} + \frac{1}{4} \right) \leq 2\delta \exp \left( \frac{24R^2}{\delta} \right).
\]
so the above bound on $c(\delta)$ simplifies to

$$c(\delta) \leq 2\delta \exp \left( \frac{24R^2}{\delta} \right).$$

\[ \square \]

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