Poisson Matrix Recovery and Completion

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Abstract—We extend the theory of low-rank matrix recovery and completion to the case when Poisson observations for a linear combination or a subset of the entries of a matrix are available, which arises in various applications with count data. We consider the (now) usual matrix recovery formulation through maximum likelihood with proper constraints on the matrix $M$ of size $d_1 \times d_2$, and establish theoretical upper and lower bounds on the recovery error. Our bounds for matrix completion are nearly optimal up to a factor on the order of $O(\log(d_1 d_2))$. These bounds are obtained by combing techniques for compressed sensing for sparse vectors with Poisson noise and for analyzing low-rank matrices, as well as adapting the arguments used for one-bit matrix completion [1] (although these two problems are different in nature) and the adaptation requires new techniques exploiting properties of the Poisson likelihood function and tackling the difficulties posed by the locally sub-Gaussian characteristic of the Poisson distribution. Our results highlight a few important distinctions of the Poisson case compared to the prior work including having to impose a minimum signal-to-noise requirement on each observed entry and a gap in the upper and lower bounds. We also develop a set of efficient iterative algorithms and demonstrate their good performance on synthetic examples and real data.

Index Terms—low-rank matrix recovery, matrix completion, Poisson noise, estimation, information theoretic bounds

I. INTRODUCTION

Recovering a low-rank matrix $M$ with Poisson observations is a key problem that arises from various real-world applications with count data, such as nuclear medicine, low-dose x-ray imaging [2], network traffic analysis [3], and call center data [4]. There the observations are Poisson counts whose intensities are determined by the matrix (through a subset of its entries or linear combinations of its entries).

Thus far much success has been achieved in solving the matrix completion and recovery problems using nuclear norm minimization, partly inspired by the theory of compressed sensing [5], [6]. It has been shown that when $M$ is low rank, it can be recovered from observations of a subset or a linear combination of its entries (see, e.g. [7]–[15]). Earlier work on matrix completion typically assume that the observations are noiseless, i.e., we may directly observe a subset of entries of $M$. In the real world, however, the observations are noisy, which is the focus of the subsequent work [16]–[21], most of which consider a scenario when the observations are contaminated by Gaussian noise. The theory for low-rank matrix recovery under Poisson noise has been less developed. Moreover, the Poisson problems are quite different from their Gaussian counterpart, since under Poisson noise the variance of the noisy observations is proportional to the signal intensity. Moreover, instead of using $\ell_2$ error for data fit, we need to use a highly non-linear likelihood function.

Recently there has also been work that consider the more general noise models, including noisy 1-bit observations [1], which may be viewed as a case where the observations are Bernoulli random variables whose parameters depend on a underlying low-rank matrix; [22], [23] consider the case where all entries of the low-rank matrix are observed and the observations are Poisson counts of the entries of the underlying matrix, and an upper bound is established (without a lower bound). In the compressed sensing literature, there is a line of research for sparse signal recovery in the presence of Poisson noise [24]–[26] and the corresponding performance bounds. The recently developed SCOPT [27], [28] algorithm can also be used to solve the Poisson compressed sensing of sparse signals but may not be directly applied for Poisson matrix recovery.

In this paper, we extend the theory of low-rank matrix recovery to two related problems with Poisson observations: matrix recovery from compressive measurements, and matrix completion from observations of a subset of its entries. The matrix recovery problem from compressive measurements is formulated as a regularized maximum likelihood estimator with Poisson likelihood. We establish performance bounds by combining techniques for recovering sparse signals under Poisson noise [24] and techniques for establishing bounds in the case of low-rank matrices [29], [30]. Our results demonstrate that as the intensity of the signal increases, the upper bound on the normalized error decays at certain rate depending how well the matrix can be approximated by a low-rank matrix.

The matrix completion problem from partial observations is formulated as a maximum likelihood problem with proper constraints on a matrix $M$ (nuclear norm bound $\|M\|_* \leq \alpha \sqrt{rd_1 d_2}$ for some constant $\alpha$ and bounded entries $\beta \leq M_{ij} \leq \alpha$). We also establish upper and lower bounds on the recovery error, by adapting the arguments used for one-bit matrix completion [1]. The upper and lower bounds nearly match up to a factor on the order of $O(\log(d_1 d_2))$, which shows that the convex relaxation formulation for Poisson matrix completion is nearly optimal. We conjecture that such a gap is inherent to the Poisson problem. Moreover, we also highlight a few important distinctions of Poisson matrix completion compared to the prior work on matrix completion in the absence of noise and with Gaussian noise:

1. Although our arguments are adapted from one-bit matrix

1Note that the formulation differs from the one-bit matrix completion case in that we also require a lower bound on each entry of the matrix. This is consistent with an intuition that the value of each entry can be viewed as the signal-to-noise ratio (SNR) for a Poisson observation, and hence this essentially poses a requirement for the minimum SNR.
completion (where the upper and lower bounds nearly match), in the Poisson case there will be a gap between the upper and lower bounds, possibly due to the fact that Poisson distribution is only locally sub-Gaussian. In our proof, we notice that the arguments based on bounding all moments of the observations, which usually generate tight bounds for prior results with sub-Gaussian observations, do not generate tight bounds here; (2) We will need a lower bound on each matrix entry in the maximum likelihood formulation, which can be viewed as a requirement for the lowest signal-to-noise ratio (since the signal-to-noise ratio (SNR) of a Poisson observation with intensity \( I \) for the lowest signal-to-noise ratio (since the signal-to-noise ratio (SNR) of a Poisson observation with intensity \( I \)). In the following, we assume that the total intensity of \( I \).

We also present a set of efficient algorithms, which can be used for both matrix recovery based on compressive measurements or based on partial observations. These algorithms include the proximal and accelerated proximal gradient descent methods, and the Penalized Maximum Likelihood Singular Value Threshold (PMLSVT) method, which is derived by expanding the likelihood function locally in each iteration and finding an exact solution to the local approximation problem and it results in a simple singular value thresholding procedure [13]. PMLSVT is related to [31]–[33] and can be viewed as a special case where a simple closed form solution for the algorithm exists. Good performance of the proposed algorithm is demonstrated using numerical examples to recover solar flare images and bike sharing count data. We show that the PMLSVT method has much lower complexity than solving the problem directly via semidefinite program and it has fairly good accuracy.

While working on this paper we realize a parallel work [34] which also studies performance bounds for low rank matrix completion with exponential family noise under more general assumptions and using a different approach for proof (Poisson noise is a special case of theirs). Their upper bound for the mean square error (MSE) is on the order of \( O\left(\log(d_1 + d_2)r\max\{d_1, d_2\}/m\right) \) (our upper bound is \( O\left(\log(d_1d_2)/\sqrt{d_1d_2}/m\right) \), and their lower bound is on the order of \( O\left(r\max\{d_1, d_2\}/m\right) \) (versus our lower bound is \( O\left(\sqrt{r(d_1d_2)/m}\right) \). Compared with the more general framework for M-estimator [35], our results are specific to the Poisson case, which may possible be stronger but do not apply generally.

The rest of the paper is organized as follows. Section II sets up the formalism for Poisson matrix completion. Section III presents the matrix recovery based on constrained maximum likelihood and establishes the upper and lower bounds for the recovery accuracy. Section IV presents the PMLSVT algorithm that solves the maximum likelihood approximately and demonstrates its performance on recovering solar flare images and bike sharing count data. All proofs are delegated to the appendix.

The notation in this paper is standard. In particular, \( \mathbb{R}_+ \) denotes the set of positive real numbers; \( \|d\| = \{1, 2, \ldots, d\}; \) \( (x)_+ = \max\{x, 0\} \) for any scalar \( x \); Let \( |x| \) denote the jth element of a vector \( x \); \( \|x\| \) is the indicator function for an event \( \varepsilon \); \( |A| \) denotes the number of elements in a set \( A \); \( \text{diag}(\lambda_i) \) denotes a diagonal matrix with a set of numbers \( \{\lambda_i\} \) on its diagonal; \( \text{I}_{d_1 \times d_2} \) denotes an \( d_1 \times d_2 \) matrix of all ones. Let entries of a matrix \( M \) be denoted by \( M_{ij} \) or \( [M]_{ij} \). For a matrix \( M = [x_1, \ldots, x_n] \), \( \text{vec}(M) = [x_1^\top, \ldots, x_n^\top]^\top \) denote vectorized matrix. Let \( \|M\| \) be the spectral norm which is the largest absolute singular value, \( \|M\|_F = \sqrt{\sum_{i,j} M_{ij}^2} \) be the Frobenius norm, \( \|M\|_p \) be the nuclear norm which is the sum of the singular values, \( \|M\|_{1,1} = \sum_i \sum_j |M_{ij}| \) be the \( L_{1,1} \) norm, and finally \( \|M\|_{\infty} = \max_{ij} |M_{ij}| \) be the infinity norm. Let \( \text{rank}(M) \) denote the rank of a matrix \( M \).

We say that a random variable \( X \) follows Poisson distribution with parameter \( \lambda \) (or \( X \sim \text{Poisson}(\lambda) \)) if its probability mass function \( \mathbb{P}(X = k) = e^{-\lambda} \lambda^k/(k!) \). The inner product for two matrices \( X \) and \( Y \) is denoted by \( \langle X, Y \rangle \triangleq \text{tr}(X^\top Y) \). We also define the KL divergence and Hellinger distance for Poisson distribution as follows: the KL divergence of two Poisson distributions with parameters \( p \) and \( q \), where \( p, q \in \mathbb{R}_+ \), is given by \( D(p\|q) = p \log(p/q) - (p - q) \); the Hellinger distance between two Poisson distributions with parameters \( p \) and \( q \) with \( p, q \in \mathbb{R}_+ \), is given by \( d_H^2(p, q) \triangleq 2 - 2\exp\left(-\frac{1}{2}\left(\sqrt{p} - \sqrt{q}\right)^2\right) \). We further define the average KL divergence and Hellinger distance for entries of two matrices \( P, Q \in \mathbb{R}_{d_1 \times d_2}^+ \), where each entry corresponds to the parameter of a Poisson random variable:

\[
D(P\|Q) = \frac{1}{d_1d_2} \sum_{i,j} D(P_{ij}\|Q_{ij}),
\]

\[
d_H^2(P, Q) = \frac{1}{d_1d_2} \sum_{i,j} d_H^2(P_{ij}, Q_{ij}).
\]

Finally, let \( \mathbb{E}[X] \) denote the expectation of a random variable \( X \).

II. FORMULATION

A. Matrix recovery

Suppose we wish to estimate a matrix \( M \in \mathbb{R}_{d_1 \times d_2}^+ \) consisting of positive entries, using \( N \) Poisson measurements \( y \in \mathbb{Z}_N^+ \) that take the forms of

\[
y \sim \text{Poisson}(\lambda M).
\]

The linear operator \( A : \mathbb{R}_{d_1 \times d_2} \rightarrow \mathbb{R}^N \) models the measuring process of physical devices, e.g., the compressive imaging system given in [2]. The linear sensing operator \( A \) takes the form of

\[
[AM]_i = \langle A_i, M \rangle \triangleq \text{tr}(A_i^\top M),
\]

where \( A_i \in \mathbb{R}_{d_1 \times d_2}^+ \). Define

\[
A \triangleq \begin{bmatrix} \text{vec}(A_1)^\top \\ \vdots \\ \text{vec}(A_N)^\top \end{bmatrix}, \quad f \triangleq \text{vec}(M),
\]

then we can write

\[
AM = Af.
\]

In the following, we assume that the total intensity of \( M \), given by \( f \triangleq \|M\|_{1,1} \), is known a priori. To have physically realizable systems, similar to [24], we also assume that the measurement operator \( A \) satisfies the following constraint: (1) (positivity-preserving) \( M_{ij} \geq 0 \) for all \( i, j \Rightarrow [AM]_{ij} \geq 0 \), for all \( i \); (2) (flux-preserving) \( \sum_{i=1}^N [AM]_i \leq \|M\|_{1,1} \).
The goal is to estimate the signal \( M \in \mathbb{R}^d_+ \times d_2 \) from measurements \( y \in \mathbb{Z}^N \). We consider a regularized maximum-likelihood estimator. Since the probability density function of \( y \) is given by \( p(y|AM) = \prod_{j=1}^N [AM]_{ij} y_i e^{-[AM]_{ij}/y_i} \), the maximum likelihood problem solves the following
\[
\hat{M} \triangleq \arg\max_{M \in \mathbb{F}} \left[ \sum_{j=1}^N y_j \log([AM]_{ij} - [AM]_{ij} - \lambda \text{pen}(M)) \right], \tag{4}
\]
where \( \text{pen}(M) \) is a regularization function, \( \lambda > 0 \) is a regularizing parameter, and \( \mathbb{F} \) is a countable set of feasible estimators
\[
\Gamma \triangleq \{ M_i \in \mathbb{R}^d_+ \times d_2 : \| M_i \|_{1,1} = I, i = 1, 2, \ldots \} \tag{5}
\]
The regularization function satisfies the Kraft inequality [24], [25].
\[
\sum_{\mathbb{F} \in \Gamma} e^{-\text{pen}(M)} \leq 1, \tag{6}
\]
which can be interpreted as a discretized feasible domain version of the general regularized maximum likelihood estimator. The regularization function assigns a small value for a lower rank \( M \) and assigns a large value for a higher rank \( M \). Using Kraft-compliant regularization to prefix codes for estimators is a commonly used technique in constructing estimators [24], [25].

**B. Matrix completion**

A problem related to matrix recovery is the following matrix completion problem based on partially observed noisy entries. Suppose we observe a subset of entries of a matrix \( M \in \mathbb{R}^d_+ \times d_2 \) on the index set \( \Omega \subset [d_1] \times [d_2] \). The indices are randomly selected with \( |\Omega| = m \). In other words, \( \{ (i,j) \in \Omega \} \) are i.i.d. Bernoulli random variables with parameter \( m/(d_1 d_2) \). The observations are Poisson counts of the observed matrix entries
\[
Y_{ij} \sim \text{Poisson}(M_{ij}), \quad \forall (i,j) \in \Omega. \tag{7}
\]
Our goal is to recover \( M \) from the Poisson observations \( \{ Y_{ij} \}_{(i,j) \in \Omega} \).

We make the following assumptions for matrix completion. First, we set an upper bound \( \alpha > 0 \) for the entries of \( M \) to entail the recovery problem is well-posed [19]. This assumption is also reasonable in practice; for instance, \( M \) may represent an image which is usually not too spiky. Second, assume the rank of \( M \) is less than or equal to a positive integer \( r \). The third assumption is characteristic to Poisson matrix completion: we set a lower bound \( \beta > 0 \) for each entry \( M_{ij} \). This entry-wise lower bound is required for our later analysis (so that the cost function is Lipschitz), and it also has an interpretation of a minimum required signal-to-noise ratio (SNR), as the SNR of a Poisson observation with intensity \( I \) is given by \( \sqrt{I} \).

We recover the matrix \( M \) using a regularized maximum likelihood formulation. Note that the log-likelihood function for the Poisson observation model (7) is proportional to
\[
F_{\Omega,Y}(X) = \sum_{(i,j) \in \Omega} Y_{ij} \log X_{ij} - X_{ij}, \tag{8}
\]
where the subscript \( \Omega \) and \( Y \) indicate the random quantities involved in the maximum likelihood function \( F \). Based on previous assumptions, we define a set of candidate estimators
\[
S \triangleq \left\{ X \in \mathbb{R}^d_+ \times d_2 : \| X \|_* \leq \alpha \sqrt{rd_1 d_2}, \beta \leq X_{ij} \leq \alpha, \forall (i,j) \in [d_1] \times [d_2] \right\}. \tag{9}
\]
Here the upper bound on the nuclear norm \( \| M \|_* \) comes from combining the assumptions \( \| M \|_F \leq \alpha \) and rank \( \| M \| \leq r \), since \( \| M \|_* \leq \sqrt{\text{rank}(M)} \| M \|_F \) and \( \| M \|_F \leq \sqrt{d_1 d_2} \| M \|_{\infty} \) lead to \( \| M \|_* \leq \alpha \sqrt{d_1 d_2} \). An estimator \( \hat{M} \) can be obtained by solving the following convex optimization problem:
\[
\hat{M} = \arg\max_{X \in S} F_{\Omega,Y}(X). \tag{10}
\]
Note that the matrix completion problem can also be formulated as a regularized maximum likelihood function problem similar to (4). However, we consider the current formulation for the convenience of drawing a connection, respectively, between the results of Poisson compressed sensing [24] and Poisson low-rank matrix recovery, as well as between the results for Poisson matrix completion and one-bit matrix completion [1].

**III. Performance Bounds**

In the following, we use the squared error
\[
R(M, \hat{M}) \triangleq \| M - \hat{M} \|_F^2, \tag{11}
\]
as a performance metric for both matrix recovery and matrix completion problems.

**A. Matrix recovery**

1) Sensing operator: Let \( Z_i, i = 1, \ldots, N \) denote a \( d_1 \) by \( d_2 \) matrix with entries i.i.d. follow the distribution
\[
[Z_{ij}] = \begin{cases} \sqrt{\frac{1-p}{p}}, & \text{with probability } p; \\ \sqrt{\frac{p}{1-p}}, & \text{with probability } 1-p. \end{cases} \tag{12}
\]
We construct a linear sensing operator \( A \) consisting of a random part and a deterministic part:
\[
A_i \triangleq \sqrt{\frac{p(1-p)}{N}} \hat{A}_i + \frac{1-p}{N} I_{d_1 \times d_2}, \tag{13}
\]
where \( \hat{A}_i \triangleq Z_i/\sqrt{N} \).

It can be verified that this operator \( A \) satisfies the requirements in the previous section. In particular, (1) all entries of \( A_i \) take values of 0 or 1/\( N \); (2) \( A \) satisfies flux preserving: since all entries of \( A_i \) are less than 1/\( N \), for a matrix with positive entries \( [M]_{ij} \geq 0, \)
\[
\| [M] \|_1 = \sum_{i=1}^N \sum_{j=1}^{d_2} [A_i]_{jk} [M]_{jk} \leq \sum_{i=1}^N \sum_{j=1}^{d_2} [M]_{jk} = I; \tag{14}
\]
(3) with probability at least \( 1 - Np^{d_1+d_2} \), every matrix \( A_i \) has at least one non-zero entry. It follows that for a matrix \( M \) such that \( [M]_{ij} \geq c \), we have that
\[
[M]_{ij} = \sum_{j=1}^{d_2} [A_i]_{jk} [M]_{jk} \geq c \sum_{j=1}^{d_2} [A_i]_{jk} \geq c/N. \tag{15}
\]
The random part of the operator satisfies the restrictive isometry property (RIP) [5] in the following sense:

**Theorem 1.** For all $M_1, M_2 \in B_{d_1 \times d_2}^1$, where $B_{d_1 \times d_2}^1 = \{ U \in \mathbb{R}^{d_1 \times d_2} : \| U \|_{1,1} = 1 \}$, there exists absolute constants $c_1, c_2 > 0$, then

$$\| M_1 - M_2 \|_F^2 \leq 4\| \tilde{A}M_1 - \tilde{A}M_2 \|_F^2 + \frac{2c_2^2 \xi_p \log(c_2^4 d_1 d_2 / N)}{N}$$

with probability at least $1 - e^{-c_3 N / \xi_p^4}$, where

$$\xi_p = \left\{ \begin{array}{ll} \sqrt{\frac{3}{2p(1-p)}} & \text{if } p \neq 1/2; \\ 1, & \text{otherwise.} \end{array} \right.$$  

Moreover, there exist constants $c_3, c_4 > 0$, such that for any finite set $T \subseteq S_{d_1 \times d_2}^{d_1 - 1}$, where $S_{d_1 \times d_2}^{d_1 - 1} \equiv \{ M \in \mathbb{R}^{d_1 \times d_2} : \| M \|_F = 1 \}$ is the unit sphere, if $N \geq c_4 \xi_p^4 \log_2 |T|$, then

$$\frac{1}{2} \leq \| \tilde{A}M \|_F \leq \frac{3}{2},$$

for all $M \in T$ holds with probability at least $1 - e^{-c_3 N / \xi_p^4}$.

This theorem follows directly from applying Theorem 1 in [24] to (3) as well as using the fact $\| M \|_F = \| f \|_2$ and $\| M \|_{1,1} = \| f \|_1$.

2) **General matrices:** Next we prove upper bounds of our proposed maximum likelihood regularized estimator (4). The following theorem holds for estimating a general signal $M$ using (4) from Poisson measurement model (1), and it does not require $M$ to be low-rank.

**Theorem 2** (Regret bound). Suppose that all candidate estimator in feasible set (4) satisfies $|\tilde{A}M|_i \geq \frac{1}{\xi_p}$, $M \in \Gamma$, for all $i$ for some constant $c$ in (0, 1). Let $G$ be the collection of all subsets of $\Gamma$, such that $\Gamma_0 \subseteq \Gamma$ with $|\Gamma_0| \leq 2N / c_4 \xi_p^4$. Then with probability at least $1 - d_1 d_2 e^{-c_N K N}$ and $c_3$ and $p$, we have that

$$\frac{1}{T^2} \mathbb{E}[R(M, \tilde{M})] \leq C_{N, p} \min_{G \in \Gamma_0} \min_{M \in \Gamma_0} \left[ \frac{\| \tilde{A}M \|_F}{T} + \lambda \text{pen}(M) \right] + 2c_2^2 \xi_p^4 \log(c_2^4 d_1 d_2 / N)$$

where $C_{N, p} = \max \left\{ \frac{24}{c} \frac{1}{p^{1/(1-p)}} \right\}$.

3) **Nearly low-rank matrices:** Assume the singular value decomposition of a matrix $M \in \mathbb{R}^{d_1 \times d_2}$ is given by $M = U^* \text{diag}(\theta^*) V^*$, where $\theta^* = [\theta_1^*, \ldots, \theta_d^*]$ is a vector consists of singular values, and $|\theta_1^*| \geq |\theta_2^*| \geq \cdots \geq |\theta_d^*|$ with $d = \min\{d_1, d_2\}$. We say that a matrix is nearly low-rank if $\theta$ has few large components and the rest of the components are very small. We further make the assumption that $\theta$ is compressible, i.e., the singular values of $M$ decays geometrically fast: there exists $0 < q < \infty$, $p > 0$, such that $|\theta_j^*| \leq \rho d_j^{-1/q}$, $j = 1, \ldots, d$. For nearly low-rank matrices, the best rank-$l$ approximation to $M$ is constructed as

$$M^{(l)} = U^* \text{diag}(\theta^{(l)}) V^*,$$

where $\theta^{(l)} = [\theta_1^*, \ldots, \theta_l^*, 0, \ldots, 0]^T$. Note that, due to the geometric decay singular value assumption

$$\| M - M^{(l)} \|_F^2 = \| U^* \text{diag}(\theta^* - \theta^{(l)}) V^* \|_F^2$$

$$= \| \theta^* - \theta^{(l)} \|_2^2 \leq l^2 C \rho^2 l^{-2/q + 1},$$

then we can obtain the following result for nearly low-rank matrix $M$ as a consequence of Theorem 2.

**Theorem 3** (Regret bound for nearly low-rank matrices). Assume $M \in \mathbb{R}^{d_1 \times d_2}$ is nearly low-rank, $|M|_{ij} \geq c / N$ for some positive constant $c \in (0, 1)$. Then there exists a finite set of candidate estimators $\Gamma$, and a regularization function that satisfies the Kraft inequality such that

$$\frac{1}{T^2} \mathbb{E}[R(M, \tilde{M})] \leq$$

$$O(N) \min_{1 \leq i \leq l_i} \left[ \left( \frac{1}{T} \right)^{-\frac{r}{2} + 1} + \frac{164}{d^2} + \frac{(d_1 + d_2 + 3) \lambda \log d}{2T} \right]$$

$$+ O(\log(d_1 d_2) / N),$$

where $d = \min\{d_1, d_2\}$ and $l_* = 2N / [c_4 \xi_p^4 (d_1 + d_2 + 3) \log d]$.

B. **Matrix completion**

In the following, we establish an upper bound and an information theoretic lower bound on the mean squared error for the estimator in (10) and show they nearly match up to a logarithmic factor.

**Theorem 4** (Upper bound). Assume $M \in \mathcal{S}$, rank$(M) = r$, $\Omega$ is chosen at random following our sampling model with $\mathbb{E}[\Omega] = m$, and $\tilde{M}$ is the solution to (10). Then with a probability exceeding $(1 - C/(d_1 d_2))$, we have

$$\frac{1}{d_1 d_2} R(M, \tilde{M}) \leq C' \left( \frac{8\alpha T}{1 - e^{-T}} \right) \left( \frac{\alpha \sqrt{T}}{\beta} \right).$$

$$\left( \alpha (e^2 - 2) + 3 \log(d_1 d_2) \right) \frac{d_1 + d_2}{m} \sqrt{1 + \frac{(d_1 + d_2) \log(d_1 d_2)}{m}}.$$

If $m \geq (d_1 + d_2) \log(d_1 d_2)$ then (17) simplifies to

$$\frac{1}{d_1 d_2} R(M, \tilde{M}) \leq \sqrt{2} C' \left( \frac{8\alpha T}{1 - e^{-T}} \right) \left( \frac{\alpha \sqrt{T}}{\beta} \right).$$

$$\left( \alpha (e^2 - 2) + 3 \log(d_1 d_2) \right) \frac{d_1 + d_2}{m}.$$

Above, $T, C', C$ are absolute constants.

The proof of Theorem 4 is an extension of the ingenious arguments for one-bit matrix completion [1]. The extension for Poisson case here is nontrivial for various aforementioned reasons (notably the non sub-Gaussian and only locally sub-Gaussian nature of the Poisson observations). An outline of our proof is as follows. First, we establish an upper bound for the Kullback-Leibler (KL) divergence $D(M \| X)$ for any $M, X \in \mathcal{S}$ by applying Lemma 5 given in the appendix. Second, we find
an upper bound for the Hellinger distance $d_H^2(M, \tilde{M})$ using the fact that the KL divergence can be bounded from below by the Hellinger distance. Finally, we bound the mean squared error in Lemma 6 via the Hellinger distance.

**Remark 1.** Fixing $d_1, d_2, m, \alpha$ and $\beta$, the upper bound in Theorem 4 increases as $r$ increases. This is consistent with the intuition that our method is better at dealing with approximately low-rank matrices (than with nearly full rank matrices). On the other hand, fixing $d_1, d_2, \alpha, \beta$ and $r$, the upper bound decreases as $m$ increases, which is also consistent with our intuition that $M$ is supposed to be recovered more accurately with more observations.

**Remark 2.** In the upper bound (17), the mean-square-error per entry can be arbitrarily small, in the sense that the upper bound goes to zero as $d_1$ and $d_2$ go to infinity when the number of the measurements $m = \mathcal{O}(d_1 + d_2) \log^3(d_1d_2)$ (or $d_1d_2$) for $\delta > 2$ when $r$ is fixed, or for $\delta > 3$ when $r$ is sublinear on the order of $\mathcal{O}(1)$.

The following theorem establishes an information theoretic lower bound and demonstrates that there exists an $M \in S$ such that any recovery method cannot achieve a mean square error per entry less than the order of $\mathcal{O}(\sqrt{r \max\{d_1, d_2\}/m})$.

**Theorem 5 (Lower bound).** Fix $\alpha, \beta, d_1,$ and $d_2$ to be such that $\alpha, d_1, d_2 \geq 1$, $\alpha \geq 2\beta$, and $\alpha^2r \max\{d_1, d_2\} \geq C_0$. Let $\Omega$ be any subset of $[d_1] \times [d_2]$ with cardinality $m$. Consider any algorithm which, for any $M \in S$, returns an estimator $\hat{M}$. Then there exists $M \in \hat{S}$ such that with probability at least $3/4$,

$$\frac{1}{d_1d_2} R(M, \hat{M}) \geq \min \left\{ C_1, C_2\alpha^{3/2} \sqrt{r \max\{d_1, d_2\}/m} \right\} \tag{19}$$

as long as the right-hand side of (19) exceeds $r \alpha^2/\min\{d_1, d_2\}$, where $C_0, C_1, C_2$ are absolute constants.

Similar to [1], [36], the proof of Theorem 5 relies on information theoretic arguments outlined as follows. First we find a set of matrices $\chi \in \hat{S}$ so that the distance between any $X^{(i)}, X^{(j)} \in \chi$, identified as $\|X^{(i)} - X^{(j)}\|_{\infty}$, is sufficiently large. Then, for any $X \in \hat{S}$ and the recovered $\hat{X}$, if we assume that they are sufficiently close to each other with high probability, then we can claim that $X$ is the element in the set $S$ that is closest to $\hat{X}$. Finally, by applying a generalized Fano’s inequality for KL divergence, we claim that the probability for the event that $X$ is the matrix in set $S$ closest to $\hat{X}$ must be small, which leads to a contraction and hence proves our lower bound.

**Remark 3.** The assumptions in Theorem 5 can be achieved, for example, by the following construction. First, choose an $\alpha$ such that $\alpha \geq \max\{1, 2\beta\}$, and then choose an $\alpha \geq 4$. Then, for $d_1$ (or $d_2$) sufficiently large, the conditions that $\alpha^2r \max\{d_1, d_2\} \geq C_0$ and the right-hand side of (19) exceeds $r \alpha^2/\min\{d_1, d_2\}$ are met. Since $r \leq \mathcal{O}(\min\{d_1, d_2\}/\alpha^2)$, $M \in \hat{S}$, what has been chosen is approximately low-rank. In other words, no matter how large $r$ is, we can always find $d_1$ (or $d_2$) large enough so that the assumptions in Theorem 5 are satisfied and thus there exist an $M$ which can not be recovered with arbitrarily small error by any method.

**Remark 4.** When $m \geq (d_1 + d_2) \log(d_1d_2)$ and $m = \mathcal{O}(r(d_1 + d_2) \log^3(d_1d_2))$ with $\delta > 2$, the ratio between the upper bound in (18) and the lower bound in (19) is on the order of $\mathcal{O}(\log(d_1d_2))$. Hence, the lower bound matches the upper bound up to a logarithmic factor.

Our formulation and results for Poisson matrix completion are inspired by one-bit matrix completion [1], yet with several important distinctions. In one-bit matrix completion, the value of each observation $Y_{ij}$ is binary-valued and hence bounded; whereas in our problem, each observation is a Poisson random variable which is unbounded and, hence, the arguments involve bounding measurements have to be changed. In particular, we need to bound $\max_{ij} Y_{ij}$ when $Y_{ij}$ is a Poisson random variable with intensity $M_{ij}$, and the Poisson likelihood function is non Lipschitz (due to a bad point when $M_{ij}$ tends to zero), hence we need to introduce a lower bound on each entry of the matrix $M_{ij}$, which can be interpreted as the lowest required SNR. Other distinctions also include analysis taking into account of the property of the Poisson likelihood function, and using Kullback-Leibler (KL) divergence as well as Hellinger distance that are different from those for the Bernoulli random variable as used in [1].

### IV. Algorithms

In the following, we use nuclear norm in the regularization function in (4). Then, the optimization problem formulated in (4) and (10) both belong to semidefinite program (SDP) as they are nuclear norm minimization problems with convex feasible domains. Hence, we may solve it, for example, via the interior-point method [37]. Although the interior-point method may return an exact solution to (10), it does not scale well with the dimensions of the matrix $d_1$ and $d_2$ as the complexity of solving SDP is $O(d_1^3 + d_1^2d_2 + d_1d_2^3)$.

In this section we develop efficient algorithms that can solve both problems faster than the interior point methods, including the proximal gradient method, the composite gradient descent method, and the Penalized Maximum Likelihood Singular Value Threshold (PMLSVT) algorithm. Even if the convergence analysis of the former method can be generalized easily, the latter method is computationally more preferable under our assumptions. Another possible algorithm that we do not cover here is the non-monotone spectral projected-gradient algorithm [38] and details can be found in [1].

#### A. Generic methods

Here we only focus on solving the matrix completion problem (10) by proximal-gradient method; the matrix recovery problem (4) can be solved similarly as stated at the end of this subsection. First, rewrite $S$ in (9) as the intersection of two closed and convex sets in $\mathbb{R}^{d_1 \times d_2}$:

$$\Gamma_1 \triangleq \{ M \in \mathbb{R}^{d_1 \times d_2} : \beta \leq M_{ij} \leq \alpha, \forall (i, j) \in [d_1] \times [d_2] \},$$

and

$$\Gamma_2 \triangleq \{ M \in \mathbb{R}^{d_1 \times d_2} : \| M \|_* \leq r \sqrt{d_1d_2} \},$$
where the first set is a nuclear norm ball and the second set is a box. Let \( f(M) \triangleq F_{0,Y}(M) \) be the negative log-likelihood function. Then optimization problem (10) is equivalent to

\[
\hat{M} = \arg \min_{M \in \mathbb{R}^{d_1 \times d_2}} f(M).
\]

(21)

Noticing that the search space \( S = \Gamma_2 \cap \Gamma_1 \) is closed and convex and \( f(M) \) is a convex function, we can use proximal gradient methods to solve (21). Let \( I_r(M) \) be an indicator function that takes value zero if \( M \in \Gamma \) and value \( \infty \) if \( M \notin \Gamma \). Then problem (21) is equivalent to

\[
\hat{M} = \arg \min_{M \in \mathbb{R}^{d_1 \times d_2}} f(M) + I_{\Gamma_2 \cap \Gamma_1}(M).
\]

(22)

To guarantee the convergence of proximal gradient method, we need the Lipschitz constant \( L > 0 \), which satisfies

\[
\| \nabla f(X) - \nabla f(Y) \|_F \leq L \| X - Y \|_F, \quad \forall X, Y \in S,
\]

(23)

and hence \( L = \alpha/\beta^2 \) by the definition of our problem. Define the orthogonal projection of \( Y \) onto \( \Gamma \) as

\[
\Pi_\Gamma(Y) = \arg \min_{X \in \Gamma} \| X - Y \|_F^2.
\]

1) Proximal gradient for Poisson matrix completion: Initialize the algorithm by \( [M_0]_{ij} = Y_{ij} \) for \( (i, j) \in \Omega \) and \( [M_0]_{ij} = (\alpha + \beta)/2 \) otherwise. Then the iterations are performed as follows

\[
M_k = \Pi_S(M_{k-1} - (1/L) \nabla f(M_{k-1})).
\]

(24)

This algorithm has linear convergence rate, as established in the following:

Lemma 1. Let \( \{M_k\} \) be the sequence generated by (24). Then for any \( k > 1 \), we have

\[
f(M_k) - f(\hat{M}) \leq \frac{L \| M_0 - \hat{M} \|_F^2}{2k}.
\]

2) Accelerated proximal gradient for Poisson matrix completion: Although proximal gradient can be implemented easily, its converges slowly when the Lipschitz constant \( L \) is large. In such scenarios, we may use Nesterov’s accelerated method [39]. With the same initialization as above, we perform the following two projections at the \( k \)th iteration:

\[
M_k = \Pi_S(Z_{k-1} - (1/L) \nabla f(Z_{k-1})),
\]

\[
Z_k = M_k + ((k-1)/(k+2))(M_k - M_{k-1}).
\]

(25)

(26)

Nesterov’s accelerated method has faster convergence, as stated in the following:

Lemma 2. Let \( \{M_k\} \) be the sequence generated by (25). Then for any \( k > 1 \), we have

\[
f(M_k) - f(\hat{M}) \leq \frac{2L \| M_0 - \hat{M} \|_F^2}{(k+1)^2}.
\]

3) Alternating projection: To use the above two methods, we need to specify how to perform projection onto the search space \( S \). Since \( S \) is an intersection of two convex sets, we may use alternating projection to compute a sequence that converges to this intersection of \( \Gamma_2 \) and \( \Gamma_1 \). Assume that \( U_0 \) is the matrix needed to be projected onto \( S \). Specifically, we can run the following two steps alternatively at the \( j \)th iteration: \( V_j = \Pi_{\Gamma_2}(U_{j-1}) \) and \( U_j = \Pi_{\Gamma_1}(V_j) \), until \( \| V_j - U_j \|_F \) is less than a user-specified error tolerance. Alternating projection algorithm is efficient if there exist some closed forms for projection onto the convex sets. Projection onto the box constraint \( \Gamma_1 \) is quite simple: \( \Pi_{\Gamma_1}(Y)_{ij} \) assumes value \( \beta \) if \( Y_{ij} < \beta \) and assumes value \( \alpha \) if \( Y_{ij} > \alpha \) and otherwise maintains the same value \( Y_{ij} \) if \( \beta \leq Y_{ij} \leq \alpha \). Projection onto \( \Gamma_2 \), the nuclear norm ball, can be achieved by projecting the vector of singular values onto a \( \ell_1 \) norm ball through scaling [13] [40].

To solve (4), similarly, we can assume \( M \) to be approximately rank \( r \) and each entry of \( M \) ranges from \( \beta = c/N \) to \( \alpha = I \) in (20) based on the assumptions for Theorem 3. Similar steps can be applied hereafter if we consider an additional convex set

\[
\Gamma_0 = \{ M \in \mathbb{R}^{d_1 \times d_2} : \| M \|_{1,1} = I \}.
\]

(26)

B. Penalized maximum likelihood singular value threshold (PMLSVT)

For our numerical examples, we use the following algorithm tailored to solving the Poisson matrix recovery and completion, similar to the fast iterative shrinkage-thresholding algorithm (FISTA) [41] and its extension to matrix case with Frobenius error [31].

Similar to the construction in [20] and [32], using \( \lambda_0 \) and \( \lambda_1 \) as regularizing parameters and the convex sets \( \Gamma_0 \) and \( \Gamma_1 \) defined before in (26) and (20), we may rewrite (4) and (10) as

\[
\hat{M} = \arg \min_{M \in \Gamma_1} f_0(M) + \lambda_0 \| M \|_s, \quad i = 0, 1,
\]

(27)

respectively, where \( f_0(M) = -\log p(y|M) \) and \( f_1(M) = -F_{0,Y}(M) \).

The PMLSVT algorithm can be derived as follows (similar to [31]). For simplicity, we denote \( f(M) \) for the \( f_0(M) \) or \( f_1(M) \). In the \( k \)th iteration, we may form a Taylor expansion of \( f(M) \) around \( M_{k-1} \) while keeping up to second term and then solve

\[
M_k = \arg \min_{M \in \Gamma_2} \left[ Q_{t_k}(M, M_{k-1}) + \lambda \| M \|_s \right],
\]

(28)

with

\[
Q_{t_k}(M, M_{k-1}) \triangleq f(M_{k-1}) + \langle M - M_{k-1}, \nabla f(M_{k-1}) \rangle + \frac{t_k}{2} \| M - M_{k-1} \|_F^2,
\]

(29)

where \( \nabla f \) is the gradient of \( f \), \( t_k \) is the reciprocal of the step size in the \( k \)th iteration, which we will specify later. By dropping and introducing terms independent of \( M \) whenever needed (more details can be found in [42]), (28) is equivalent to

\[
M_k = \arg \min_{M} \left[ \frac{1}{2} \| M - \left( M_{k-1} - \frac{1}{t_k} \nabla f(M_{k-1}) \right) \|_F^2 + \lambda \| M \|_s \right].
\]

(30)

Using a theorem proved in [13], we may show that the exact solution to (30) is given by a form of Singular Value
Thresholding (SVT):
\[ M_k = D_{\lambda/t_k} \left( M_{k-1} - \frac{1}{t_k} \nabla f(M_{k-1}) \right), \]  
(31)

where \( D_\tau(S) \triangleq \text{diag}((\sigma_i - \tau)_+) \). The argument is as follows. Consider the following problem
\[ \min_{Y \in \mathbb{R}^{d_1 \times d_2}} \left\{ \frac{1}{2} \| Y - X \|_F^2 + \tau \| Y \|_* \right\}, \]  
(32)

where \( X \in \mathbb{R}^{d_1 \times d_2} \) is given and \( \tau \) is the regularization parameter. For a matrix \( X \in \mathbb{R}^{d_1 \times d_2} \) with rank \( r \), let its singular value decomposition be \( X = U \Sigma V^T \), where \( U \in \mathbb{R}^{d_1 \times r} \), \( V \in \mathbb{R}^{d_2 \times r} \), \( \Sigma = \text{diag}((\sigma_i), i = 1, 2, \ldots, r) \), and \( \sigma_i \) is a singular value of the matrix \( X \). For each \( \tau \geq 0 \), define the singular value thresholding operator as:
\[ D_\tau(X) \triangleq UD_\tau(\Sigma)V^T, \]  
(33)

The solution to (32) is given by (31) according to the following theorem:

**Theorem 6** (Theorem 2.1 in [13]). For each \( \tau \geq 0 \), and \( X \in \mathbb{R}^{d_1 \times d_2} \):
\[ D_\tau(X) = \arg \min_{Y \in \mathbb{R}^{d_1 \times d_2}} \left\{ \frac{1}{2} \| Y - X \|_F^2 + \tau \| Y \|_* \right\}. \]  
(34)

The PMLSVT algorithm is summarized in Algorithm 1. The initialization of matrix recovery problem is suggested by [43] and that of matrix completion problem is to choose an arbitrary element in the set \( \Gamma_1 \). For a matrix \( M \), define a projection of \( M \) onto \( \Gamma_0 \) as follows:
\[ \mathcal{P}(M) = \frac{I}{\| M_+ \|_{1,1}} M_+, \]  

where \((i,j)\)th entry of \( M_+ \) is \( \max\{M_{ij}, 0\} \). In the algorithm description, \( t \) is the reciprocal of the step size, \( \eta > 1 \) is a scale parameter to change the step size, and \( K \) is the maximum number of iterations, which is user specified: a larger \( K \) leads to more accurate solution, and a small \( K \) obtains the coarse solution quickly. If the cost function value does not decrease, the step size is shortened to change the singular values more conservatively. The algorithm terminates when the absolute difference in the cost function values between two consecutive iterations is less than 0.5/\( K \). Convergence of the PMLSVT algorithm cannot be readily established; however, Lemma 1 and Lemma 2 above may shed some light on this.

At each iteration, the complexity of PMLSVT (Algorithm 1) is on the order of \( O(d_1^2d_2 + d_2^2) \) (which comes from performing singular value decomposition). This is much lower than the complexity of solving an SDP, which is on the order of \( O(d_1^4 + d_1d_2^3 + d_1^2d_2^2) \). In particular, for a \( d \)-by-\( d \) matrix, PMLSVT algorithm has a complexity \( O(d^3) \), which is lower than solving an SDP (\( O(d^4) \)). One may also use an approximate SVD method [44] and a better choice for step sizes [41] to accelerate PMLSVT.

---

**Algorithm 1** PMLSVT for Poisson matrix recovery and completion

1: Initialize: The maximum number of iterations \( K \), parameters \( \eta \) and \( t \). \( M_0 = \mathcal{P}(\sum_{i=1}^N y_i A_i) \) \{matrix recovery\}
2: \[ |M_0|_{ij} = Y_{ij} \] for \((i,j) \in \Omega\) and \[ |M_0|_{ij} = (\alpha + \beta)/2 \] otherwise \{matrix completion\}
3: \[ C = M_{k-1} - (1/t) \nabla f(M_{k-1}) \]
4: \[ C = UDV^T \] \{singular value decomposition\}
5: \[ D' = \text{diag}((\text{diag}(D) - \lambda/t)_+) \]
6: \[ M_k = \mathcal{P}(UDV^T) \] \{matrix recovery\}
7: \[ M_k = \mathcal{P}(UDV^T) \] \{matrix completion\}
8: \[ f(M_k) > Q_L(M_k, M_{k-1}) \text{ then } t = \eta t, \text{ go to 4.} \]
9: \[ \text{end for} \]

---

**V. Numerical examples**

**A. Synthetic example based on solar flare image**

1) **Matrix recovery**: In this section, we demonstrate the performance of PMLSVT on synthesized examples based on real solar flare images captured by the NASA SDO satellite (see [45] for detailed explanations). We break the image of size 48-by-48 in Fig. 1 (a) into 8-by-8 patches, and collect the vectorized patches into a 64-by-36 matrix. Such a matrix is well approximated by a low-rank matrix, as demonstrated in Fig. 1 (b). The intensity of the image is \( I = 3.27 \times 10^6 \). To vary SNR, we scale the image intensity by \( \rho \geq 1 \). The parameters for the PMLSVT algorithm are \( t = 10^{-5} \), \( \eta = 1.1 \), and \( K = 2500 \). As illustrated in Fig. 1 (c) and (d) (for fixed \( N = 1000 \) and \( \lambda = 0.002 \), \( \rho = 2 \) and \( \rho = 7 \) respectively), and in Fig. 2, the larger the \( \rho \) (and hence the higher the SNR) the lower the normalized error.

![Fig. 1: Original, low-rank approximation to solar flare image, and recovered images from compressive measurements when the intensity of the underlying signal is scaled by \( \rho \).](image-url)

(c) recovered, \( \rho = 2 \). (d) recovered, \( \rho = 7 \).

In Fig. 3, we run the PMLSVT algorithm and solve the
problems as SDP using CVX\(^2\) with various number of measurements, respectively, when fixing \(\rho = 4\) and \(\lambda = 0.002\). Fig. 3 demonstrates that more measurements lead to more smaller normalized error, as expected. Also, since it is an approximate algorithm, PMLSVT algorithm is less accurate than SDP; however, the increase in the normalized error of PMLSVT algorithm relative to that of SDP is at most 4.89\%.

As shown in Table 1, the CPU running time of PMLSVT algorithm is much smaller than that of solving SDP by CVX.

Fig. 3: Matrix recovery from compressive measurements: normalized error \(R(M, \hat{M})/I^2\) versus the number of measurements \(N\) when \(\rho = 4\) and \(\lambda = 0.002\), for solutions obtained using CVX and PMLSVT, respectively.

| \(N\)  | CVX   | PMLSVT |
|-------|-------|--------|
| 500   | 725   | 172    |
| 750   | 1146  | 232    |
| 1000  | 1510  | 378    |
| 1250  | 2059  | 490    |
| 1500  | 2769  | 642    |

Table I: CPU time (in seconds) of solving SDP by using CVX and PMLSVT algorithms, with \(\rho = 4\) and \(\lambda = 0.002\) with \(N\) measurements.

\(\mathbb{E}[\Omega] = m\). Let \(p \triangleq m/(d_1d_2)\), then we observe (100\(p\)% of entries, and let \(t = 10^{-4}\) and \(\eta = 1.1\) in the PMLSV algorithm. Fig. 5 demonstrates the recovery result when 80\%, 50\% and 30\% of the image are observed. The results show that our algorithm can recover the original image accurately when 50\% or above of the image entries are observed. In the case of only 30\% of the image entries are observed, our algorithm still captures the main features in the image. The running time of the PMLSVT algorithm on a laptop with 2.40Hz dual-core CPU and 8GB RAM for all three examples are less than 1.2 seconds.

2) Matrix completion: We demonstrate the good performance of PMLSVT in matrix completion for recovering the same solar flare image as in the previous section using partial observations. Suppose entries are observed using our sampling model with \(\lambda = 0.1\) and no more than 2000 iterations, where the elapsed times are 1.176595, 1.110226 and 1.097281 seconds, respectively.

\(^2\)http://cvxr.com/cvx/
B. Bike sharing count data

To demonstrate the performance of our algorithm on real data, we consider the bike sharing data set\(^3\), which consists of 17379 bike sharing counts aggregated on hourly basis between the years 2011 and 2012 in Capital bikeshare system with the corresponding weather and seasonal information. We collect counting of 24 hours over 105 Saturdays into a 24-by-105 matrix \(M\), and the resulted matrix is nearly low-rank. Assuming that only a fraction of the entries of this matrix is known (each entry is observed with probability 0.5 and, hence, roughly half of the entries are observed), and that the counting numbers follow Poisson distributions with unknown intensities. We aim at recover the unknown intensities, i.e., filling the missing data and performing denoising. We use PMLSVT with the following parameters: \(\alpha = 1000, \beta = 1, t = 10^{-4}, \eta = 1\), \(K = 4000\) and \(\lambda = 100\). In this case there is no “ground truth” for the intensities, and it is hard to measure the accuracy of recovered matrix. Instead, we are interested in identifying interesting patterns in the recovered results. As shown in Fig. 6 (b), there are two clear increases in the counting numbers after the 17th and the 63th Saturday, which may not be easily identified from the original data in Fig. 6 (a) with missing data and Poisson randomness.

![Fig. 6: Bike sharing count data: (a): observed matrix \(M\) with 50% missing entries; (b): recovered matrix with \(\lambda = 100\) and \(K = 4000\).](http://archive.ics.uci.edu/ml/datasets/Bike+Sharing+Dataset [46].)

VI. CONCLUSIONS

In this paper, we have studied matrix recovery and completion problem when the data are Poisson random counts. We considered a maximum likelihood formulation with constrained nuclear norm of the matrix and entries of the matrix, and presented upper and lower bounds for the proposed estimators. We also developed a set of new algorithms, and in particular the efficient the Poisson noise Maximal Likelihood Singular Value Thresholding (PMLSV) algorithm. We have demonstrated its accuracy and efficiency compared with the semi-definite program (SDP) and tested on real data examples of solar flare images and bike sharing data.

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\(^3\)The data can be downloaded at http://archive.ics.uci.edu/ml/datasets/Bike+Sharing+Dataset [46].
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APPENDIX

In the following, proofs for Lemma 1 and Lemma 2 use results from [39].

Proof of Lemma 1: As is well known, proximal mapping of a $Y \in \mathcal{S}$ associated with a closed convex function $h$ is given by

$$\text{prox}_{th}(Y) \triangleq \arg \min_X \left( t \cdot h(X) + \frac{1}{2} \|X - Y\|_F^2 \right),$$

where $t > 0$ is a multiplier. Define for each $M \in \mathcal{S}$ that

$$G_t(M) \triangleq \frac{1}{t} \left( M - \text{prox}_{th}(M - t\nabla f(M)) \right),$$

then we can know by the characterization of subgradient that

$$G_t(M) - \nabla f(M) \in \partial h(M), \tag{35}$$

where $\partial h(M)$ is the subdifferential of $h$ at $M$. Noticing that $M - G_t(M) \in \mathcal{S}$, then from Lemma 9 we have

$$f(M - tG_t(M)) \leq f(M) - \langle \nabla f(M), tG_t(M) \rangle + \frac{t}{2} \|G_t(M)\|_F^2, \tag{36}$$

for all $0 \leq t \leq 1/L$. In our case, $h(M) = I_{\mathcal{S}}(M)$. Defining $g(M) \triangleq f(M) + h(M)$, combining (35) and (36) and using the fact that $f$ and $h$ are convex functions, we have for any $Z \in \mathcal{S}$

$$g(M - tG_t(M)) \leq g(Z) + \langle G_t(M), M - Z \rangle - \frac{t}{2} \|G_t(M)\|_F^2. \tag{37}$$

Taking $Z = \hat{M}$ in (37), then we have for any $k \geq 0$

$$g(M_{k+1}) - g(\hat{M}) \leq \langle G_t(M_k), M_{k+1} - \hat{M} \rangle - \frac{t}{2} \|G_t(M_k)\|_F^2 \tag{38}$$

where we use the fact that $(M, M) = \|M\|_F^2$. By taking $Z = M_k$ in (37) we know that $f(M_{k+1}) < f(M_k)$ for any $k \geq 0$, so we have by also taking $t = 1/L$

$$g(M_k) - g(\hat{M}) \leq \frac{1}{k} \sum_{i=0}^{k-1} \left( g(M_{i+1}) - g(\hat{M}) \right) \leq \frac{L}{2k} \sum_{i=0}^{k-1} \left( \|M_i - \hat{M}\|_F^2 - \|M_{i+1} - \hat{M}\|_F^2 \right) \leq \frac{L\|M_0 - \hat{M}\|_F^2}{2k}. \tag{39}$$

Finally, the theorem is proved by noticing that $h(M_k) = h(\hat{M}) = 0$ for any $k \geq 0$.

Proof of Lemma 2: Define $V_0 = M_0$ and for any $k \geq 1$,

$$a_k \triangleq \frac{2}{k+1}, \quad V_k \triangleq M_{k+1} - \frac{1}{a_k} (M_k - M_{k-1}).$$

Setting $t = 1/L$, then by noticing that

$$M_k = Z_{k-1} - tG_t(Z_{k-1}),$$

we can rewrite $V_k$ as

$$V_k = V_{k-1} - \frac{t}{a_k} G_t(Z_{k-1}).$$

Taking $Z = M_{k-1}$ and $Z = \hat{M}$ in (37) and make convex combination we have

$$g(M_k) \leq (1 - a_k)g(M_{k-1}) + a_k g(\hat{M}) + a_k \langle G_t(Z_{k-1}), V_{k-1} - \hat{M} \rangle - \frac{t}{2} \|G_t(Z_{k-1})\|_F^2 \tag{40}$$

After rearranging terms,

$$\frac{1}{a_k^2} g(M_k) - g(\hat{M}) \leq \frac{1}{2t} \|V_{k-1} - \hat{M}\|_F^2 \leq \frac{1}{2t} \|M_0 - \hat{M}\|_F^2, \tag{41}$$

which proves the theorem.

Lemma 3 (Covering number for low-rank matrices, Lemma 4.3.1. in [47]). Let $S_\epsilon = \{X \in \mathbb{R}^{d_1 \times d_2} : \text{rank}(X) \leq r, \|X\|_F = 1\}$, then there exists an $\epsilon$-net $S_\epsilon \subset S_r$, with respect to Frobenious norm, i.e., for any $V \in S$, there exists $V_0 \in S_\epsilon$ such that

$$\|V_0 - V\|_F \leq \epsilon.$$

Moreover,

$$|S_\epsilon| \leq \left( \frac{4}{\epsilon} \right)^{(d_1 + d_2 + 1)r} \tag{42}$$

Proof of Theorem 3: In order to use results in Theorem 2, we first construct a suitable finite class of estimators $\Gamma$ and set the regularization function $\Phi(M)$ such that it encourages low-rank $M$ and satisfies Kraft inequality. Note that we can write

$$M^{(l)} = \|\theta^{(l)}\|_2 \hat{M}^{(l)}, \tag{43}$$

with

$$\hat{M}^{(l)} = U^* \text{diag}\{\|\theta^{(l)}\|_2\} V^* \top, \quad \|\hat{M}^{(l)}\|_F = 1.$$ 

Due to intensity constraint, for $M \in \Gamma$, $\|M\|_{1,1} = \sum_i \sum_j M_{ij} = 1$, we have that

$$\|\theta^{(l)}\|_2 \leq \|\theta^{(l)}\|_2 = \sum_{i=1}^d \theta_i^2 = \sqrt{\text{tr}(M^\top M)} \tag{44}$$

Using the parameterization in (43), we can code $M^{(l)}$ using the following steps: (1) quantize $\|\theta^{(l)}\|$ into one of $\sqrt{d}$ bins that uniformly divide the interval $[-I, I]$, and call the result of the quantization $r_q$. To encode $r_q$ requires $\frac{1}{2} \log d$ bits; (2)
quantize $\hat{M}^{(t)}$: since $\|\hat{M}^{(t)}\|_F = 1$, using Lemma 4, we can form a $c$-net meaning that we can form a set of quantizers for $\hat{M}^{(t)}$ so that for every $\hat{M}^{(t)}$ there is a corresponding $M_q^{(t)} \in S_q$, such that $\|\hat{M}^{(t)} - M_q^{(t)}\|_F \leq 9/\sqrt{d}$, and $|S_q| = d^{(\frac{d_1 + d_2 + 1}{2})}$. To encode the elements in $S_q$, we need $\frac{1}{2}(d_1 + d_2 + 1)n \log d$ bits; (3) finally, to encode $l$, the rank of $\hat{M}^{(t)}$, we need $\log d$ bits at most. Let $M_q^{(t)} \triangleq r_q \hat{M}^{(t)}$.

It can be verified that this above quantization scheme with comes up with a set of approximations for $M^{(t)}$ and a prefix code for $\hat{M}^{(t)}$. The average code length for $\hat{M}^{(t)}$ is upper bounded by $\frac{1}{2} \log d + \frac{1}{2}(d_1 + d_2 + 1) n \log d$ bits.

Finally, to ensure total intensity constraint and that each element of $M$ is greater than $c$, we project $M_q^{(t)}$ onto the feasible set

$$C \triangleq \{g \in \mathbb{R}^{d_1 \times d_2} : M_{ij} \geq c \text{ and } \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} M_{ij} = 1 \}.$$ 

and use $P_{C}(M_q^{(t)})$ as a candidate estimator in $\Gamma$, where $P_{C}$ is a projector operator onto the set $C$. Using the above construction for $\Gamma$, the complexity of $M \in \Gamma$ satisfies

$$\text{pen}(M) \leq \frac{3}{2} \log d + \frac{1}{2}(d_1 + d_2 + 1)n \log d.$$ 

Now, given $M = U^* \text{diag}(\theta^*) V^* \Sigma$, and let $M^{(t)}$ be its best rank-$l$ approximation, $M_q^{(t)}$ be the quantized version of $M^{(t)}$, for which we have

$$\|M^{(t)} - M_q^{(t)}\|_F^2 = \|\|\|\theta^t\|\Sigma M^{(t)} - r_q \Sigma M^{(t)} - r_q \Sigma M_q^{(t)}\|\Sigma\|\Sigma\|_F^2$$

$$\leq 2\|\|\|\theta^t\|\Sigma M^{(t)} - r_q \Sigma M^{(t)}\|\Sigma\|\Sigma\|_F^2 + 2\|\|\Sigma M^{(t)} - r_q \Sigma M_q^{(t)}\|\Sigma\|\Sigma\|_F^2$$

$$= 2\|\|\|\theta^t\|\Sigma M^{(t)}\|\Sigma\|\Sigma\|_F^2 + 2\|\|\Sigma M^{(t)} - M_q^{(t)}\|\Sigma\|\Sigma\|_F^2$$

$$\leq \frac{12}{d} l^2 + \frac{12}{d} l^{2} \frac{81}{d} = \frac{164l^2}{d}$$

since $\|\|\|\theta^t\|\Sigma M^{(t)} - r_q \Sigma M^{(t)}\|\Sigma\|\Sigma\|_F^2 \leq 9/\sqrt{d}$, and $\|M^{(t)}\|_F^2 = 1$. Hence, we can bound the distance for candidate estimator in $\Gamma$ with rank $k$:

$$\|M - P_C(M_q^{(t)})\|_F^2 \leq \|M - M_q^{(t)}\|_F^2$$

$$= \|M - M^{(t)} - M^{(t)} - M_q^{(t)}\|_F^2$$

$$\leq 2\|\|\|\theta^t\|\Sigma M^{(t)} - M_q^{(t)}\|\Sigma\|\Sigma\|_F^2$$

This is due to Pythagorean identity.

Given each $1 \leq l \leq d$, let $\hat{\Gamma}_l \subseteq \Gamma$ be the set of all elements in $\Gamma$ that the corresponding elements before projecting down to $C$ are rank $l$. Then $\log |\hat{\Gamma}_l| < \frac{1}{2}(d_1 + d_2 + 3)\log d$. Therefore $\hat{\Gamma}_l \in \mathcal{G}$ (of Theorem 2), whenever $\frac{1}{2}(d_1 + d_2 + 3)\log d \leq N/(c_4 \xi_p)$, i.e.,

$$l \leq l_*, \quad \text{where } l_* \triangleq \frac{2N}{c_4 \xi_p (d_1 + d_2 + 3) \log d}.$$ 

Finally, using Theorem 2, for these approximately low-rank matrices,

$$E[R(M, \hat{M})] \leq C_{N,p} \min_{1 \leq l \leq l_*} \left( \frac{1}{l^2} \|M - P_C(M_q^{(t)})\|_F^2 \right)$$

$$+ \frac{2\text{pen}(M_q^{(t)})}{l} + 2\frac{c_2 \xi_p^4}{l} \log(\frac{2\xi_p^4 d_1 d_2}{N})$$

$$\leq \mathcal{O}(N) \min_{1 \leq l \leq l_*} \left[ \frac{|t - 2\alpha + 164}{d} \right] + \mathcal{O}(\frac{\log(d_1 d_2)}{N^2})$$

In the following, Lemma 4 is used in proving Lemma 5, and Lemma 7 corresponds to Lemma 3 in [1].

**Lemma 4 (Tail bound for Poisson).** For $Y \sim \text{Poisson}(\lambda)$ with $\lambda \leq \alpha$, $P(Y - \lambda \geq t) \leq e^{-t}$, for all $t \geq t_0$ where $t_0 \triangleq \alpha(e^2 - 3)$.

**Proof:** For any $\theta \geq 0$, we have

$$P(\theta Y \geq \theta (t + \lambda)) = P(\exp(\theta Y) \geq \exp(\theta (t + \lambda)))$$

$$\leq \exp(-\theta (t + \lambda)) \cdot \text{exp}(-\theta (t + \lambda) \exp(\lambda(e^\theta - 1)),$$

where we have used Markov’s inequality and the moment generating function for Poisson random variable above. Now let $\theta = 2$, we have

$$\exp(t) \cdot P(Y - \lambda \geq t) \leq \exp(-t + \lambda(e^2 - 3)).$$

Given $t_0 \triangleq \alpha(e^2 - 3)$, then for all $t \geq t_0$, we have $\exp(t) \cdot \mathcal{P}(Y - \lambda \geq t) \leq 1$. It follows that $\mathcal{P}(Y - \lambda \geq t) \leq e^{-t}$ when $t \geq t_0 \geq \lambda(e^2 - 3)$.

**Lemma 5.** Let $F_{\Omega,Y}(X)$ be the likelihood function defined in (8) and $S$ be the set defined in (9), then

$$\mathcal{P} \left( \sup_{X \in S} |F_{\Omega,Y}(X) - EF_{\Omega,Y}(X)| \leq C'(\alpha \sqrt{r} / \beta) (\alpha(e^2 - 2) + 3 \log(d_1 d_2)) \right) \leq \frac{C}{d_1 d_2},$$

where $C'$ and $C$ are absolute positive constants and the probability and the expectation are both over $\Omega$ and $Y$.

**Proof:** In order to prove the lemma, we let $\epsilon_{ij}$ are i.i.d. Rademacher random variables. In the following derivation, the first inequality is due the Radamacher symmetrization argument (Lemma 6.3 in [48]) and the second inequality is due to the power mean inequality: $(a + b)^h \leq 2^{h-1}(a^h + b^h)$ if $a, b \geq 0$, $h \geq 1$. Then we have

$$\mathcal{E} \left( \sup_{X \in S} \left| F_{\Omega,Y}(X) - EF_{\Omega,Y}(X) \right| |X \right) \leq 2^{h-1} \mathcal{E} \left( \sup_{X \in S} \sum_{i,j} \epsilon_{ij} |X_{ij} - X_{ij}|^h \right)$$

$$\leq 2^{h-1} \mathcal{E} \left( \sup_{X \in S} \sum_{i,j} \epsilon_{ij} |X_{ij} - X_{ij}|^h \right).$$
where $E$ denotes the matrix with entries given by $\epsilon_{ij}$, $\Delta_\Omega$ denotes the indicator matrix for $\Omega$ and $\circ$ denotes the Hadamard product.

Similarly, the second term of (46) can be bounded as follows:

\begin{align*}
&2^{h-1}E \left[ \sup_{X \in S} \left| \sum_{i,j} \epsilon_{ij} \mathbb{I}_{i,j} \right|^h \right] \\
\leq &2^{h-1}E \left[ \sup_{X \in S} \left| \sum_{i,j} \epsilon_{ij} \mathbb{I}_{i,j} Y_{ij} \right|^h \right] \\
&+ 2^{h-1} \left( \sup_{X \in S} \left| \sum_{i,j} \epsilon_{ij} \mathbb{I}_{i,j} X_{ij} \right|^h \right) \\
= &2^{h-1}E \left[ \sup_{X \in S} \left| \sum_{i,j} \epsilon_{ij} \mathbb{I}_{i,j} \right|^h \right] \\
&+ 2^{h-1}E \left[ \sup_{X \in S} \left| \sum_{i,j} \epsilon_{ij} \mathbb{I}_{i,j} X_{ij} \right|^h \right],
\end{align*}

(46)

where the expectation are over both $\Omega$ and $Y$.

In the following, we will use contraction principle to further bound the first term of (46). We let $\phi(t) = -\beta \log (t + 1)$. We know $\phi(0) = 0$ and $|\phi(t)| = |\beta/(t + 1)|$, so $|\phi(t)| \leq 1$ if $t \geq \beta - 1$. Setting $Z = X - 1_{d_1 \times d_2}$, then we have $Z_{ij} \geq \beta - 1, \forall (i,j) \in [d_1] \times [d_2]$ and $|Z_{ij}| \leq \alpha \sqrt{d_1 d_2 + d_1 d_2}$ by triangle inequality. Therefore, $\phi(Z_{ij})$ is a contraction and it vanishes at 0. By Theorem 4.12 in [48] and using the fact that $|A, B| \leq \|A\| \|B\|$, we have

\begin{align*}
2^{h-1}E \left[ \sup_{X \in S} \left| \sum_{i,j} \epsilon_{ij} \mathbb{I}_{i,j} \right|^h \right] \\
\leq &2^{h-1}E \left[ \max_{i,j} Y_{ij} \right] \left| \sup_{X \in S} \left| \sum_{i,j} \epsilon_{ij} \mathbb{I}_{i,j} \right|^h \right] \\
= &2^{h-1}E \left[ \max_{i,j} Y_{ij} \right] \left| \sup_{X \in S} \left| \sum_{i,j} \epsilon_{ij} \mathbb{I}_{i,j} \right|^h \right] \\
\leq &2^{h-1} \left( \frac{2}{\beta} \right)^h E \left[ \max_{i,j} Y_{ij} \right] \left| \sup_{X \in S} \left| \sum_{i,j} \epsilon_{ij} \mathbb{I}_{i,j} \right|^h \right] \\
= &2^{h-1} \left( \frac{2}{\beta} \right)^h E \left[ \max_{i,j} Y_{ij} \right] \left| \sup_{X \in S} \left| \sum_{i,j} \epsilon_{ij} \mathbb{I}_{i,j} \right|^h \right] \\
\leq &2^{h-1} \left( \frac{2}{\beta} \right)^h E \left[ \max_{i,j} Y_{ij} \right] \left| \sum_{X \in S} \left| \sum_{i,j} \epsilon_{ij} \mathbb{I}_{i,j} \right|^h \right] \\
&\leq 2^{h-1} \left( \frac{2}{\beta} \right)^h \left( \alpha \sqrt{\beta} + 1 \right)^h E \left[ \max_{i,j} Y_{ij} \right] E \left[ \|E \circ \Delta_\Omega\|^h \right],
\end{align*}

(47)

Plugging (47) and (48) into (46), we have

\begin{align*}
&\mathbb{E} \left[ \sup_{X \in S} |F_{\Omega,Y}(X) - \mathbb{E} F_{\Omega,Y}(X)|^h \right] \\
\leq &2^{h-1} \left( \alpha \sqrt{\beta} + 1 \right)^h \left( \sqrt{d_1 d_2} \right)^h E \left[ \|E \circ \Delta_\Omega\|^h \right].
\end{align*}

(49)

To bound $\mathbb{E} \left[ \|E \circ \Delta_\Omega\|^h \right]$, we can use the result from [1] if we take $h = \log(d_1 d_2) \geq 1$:

\begin{align*}
&\mathbb{E} \left[ \|E \circ \Delta_\Omega\|^h \right] \\
\leq &C_0 \left( 2(1 + \sqrt{\beta}) \right)^h \left( \frac{m(d_1 + d_2) + d_1 d_2 \log(d_1 d_2)}{d_1 d_2} \right)^h
\end{align*}

for some constant $C_0$. Therefore, the only term we need to bound is $\mathbb{E} \left[ \max_{i,j} Y_{ij} \right]$.

From Lemma 4, if $t \geq t_0$, then for any $(i,j) \in [d_1] \times [d_2]$, the following inequality holds since $t_0 > \alpha$:

\begin{align*}
\mathbb{P}(\|Y_{ij} - M_{ij}\| \geq t) &= \mathbb{P}(Y_{ij} \geq M_{ij} + t) + \mathbb{P}(Y_{ij} \leq M_{ij} - t) \\
&\leq \exp(-t) + 0 \\
&= \mathbb{P}(W_{ij} \geq t),
\end{align*}

(50)

where $W_{ij}$'s are independent standard exponential random variables.

Below we use the fact that for any positive random variable
Above, firstly we use triangle inequality and power mean inequality, then along with independence, we use (50) in the third inequality. By standard computations for exponential random variables,

\[ E \left[ \max_{i,j} W_{ij}^h \right] \leq 2h! + \log^h(d_1d_2). \]  

(52)

Thus, we have

\[ E \left[ \max_{i,j} Y_{ij}^h \right] \leq 2^{2h-1} \left( \alpha^h + (t_0)^h + 2h! + \log^h(d_1d_2) \right). \]  

(53)

Therefore, combining (53) and (49), we have

\[
\begin{align*}
E \left[ \sup_{X \in S} |F_{\Omega,Y}(X) - E F_{\Omega,Y}(X)|^h \right] & \leq 2^{2h-1} \left( \alpha \sqrt{\tau} + 1 \right)^h \left( \frac{2}{\beta} \right)^h \left( \alpha^h + (t_0)^h + 2h! + \log^h(d_1d_2) \right), \\
& \leq 16 \left( \alpha \sqrt{\tau} + 1 \right) \left( \sqrt{d_1d_2} \right) E \left[ \left\| E \circ \Delta \Omega \right\|^h \right]^{\frac{1}{h}}. \\
& \leq 16 \left( \alpha \sqrt{\tau} + 1 \right) \left( \sqrt{d_1d_2} \right) E \left[ \left\| E \circ \Delta \Omega \right\|^h \right]^{\frac{1}{h}}. \\
& \leq \frac{2}{\beta} \left( \alpha + t_0 + 2h! + \log(d_1d_2) \right) \\
& \leq 16 \frac{2}{\beta} \left( \alpha \sqrt{\tau} + 1 \right) \left( \frac{\alpha^2 - 2}{\sqrt{\beta}} \right) \left( \frac{\alpha(e^2 - 2) + 3\log(d_1d_2)}{\sqrt{m(d_1 + d_2) + d_1d_2\log(d_1d_2)}} \right). \\
& \leq \frac{2}{\beta} \left( \alpha \sqrt{\tau} + 1 \right) \left( \sqrt{d_1d_2} \right) E \left[ \left\| E \circ \Delta \Omega \right\|^h \right]^{\frac{1}{h}}. \\
& \leq 16 \left( \alpha \sqrt{\tau} + 1 \right) \left( \frac{\alpha \sqrt{\tau}}{\beta} \right) \left( \frac{\alpha(e^2 - 2) + 3\log(d_1d_2)}{\sqrt{m(d_1 + d_2) + d_1d_2\log(d_1d_2)}} \right). \\
& \leq 16 \left( \alpha \sqrt{\tau} + 1 \right) \left( \frac{\alpha \sqrt{\tau}}{\beta} \right) \left( \frac{\alpha(e^2 - 2) + 3\log(d_1d_2)}{\sqrt{m(d_1 + d_2) + d_1d_2\log(d_1d_2)}} \right). \\
& \leq 16 \left( \alpha \sqrt{\tau} + 1 \right) \left( \frac{\alpha \sqrt{\tau}}{\beta} \right) \left( \frac{\alpha(e^2 - 2) + 3\log(d_1d_2)}{\sqrt{m(d_1 + d_2) + d_1d_2\log(d_1d_2)}} \right). \\
& \leq 16 \left( \alpha \sqrt{\tau} + 1 \right) \left( \frac{\alpha \sqrt{\tau}}{\beta} \right) \left( \frac{\alpha(e^2 - 2) + 3\log(d_1d_2)}{\sqrt{m(d_1 + d_2) + d_1d_2\log(d_1d_2)}} \right).
\end{align*}
\]

(54)

Moreover when \( C' \geq 128(1 + \sqrt{6}) \epsilon \),

\[ C_0 \left( \frac{128(1 + \sqrt{6})^{\log(d_1d_2)}}{C'} \right) \leq C_0 \frac{d_1d_2}{d_1d_2}. \]

Therefore we can use Markov inequality to see that

\[ \mathbb{P} \left\{ \sup_{X \in S} |F_{\Omega,Y}(X) - E F_{\Omega,Y}(X)| \geq C' \left( \alpha \sqrt{\tau}/\beta \right) \left( \alpha(e^2 - 2) + 3\log(d_1d_2) \right) \cdot \left( \sqrt{m(d_1 + d_2) + d_1d_2\log(d_1d_2)} \right) \right\} \]

\[ = \mathbb{P} \left\{ \sup_{X \in S} |F_{\Omega,Y}(X) - E F_{\Omega,Y}(X)|^h \geq \left( C' \left( \alpha \sqrt{\tau}/\beta \right) \left( \alpha(e^2 - 2) + 3\log(d_1d_2) \right) \cdot \left( \sqrt{m(d_1 + d_2) + d_1d_2\log(d_1d_2)} \right) \right)^h \right\} \]

\[ \leq \mathbb{E} \left[ \sup_{X \in S} |F_{\Omega,Y}(X) - E F_{\Omega,Y}(X)|^h \right] \]

\[ \leq \frac{C'}{d_1d_2}, \]

where \( C' \geq 128(1 + \sqrt{6}) \epsilon \) and \( C \) are absolute constants.

\[ \text{Lemma 6. Let } \beta \leq M_{ij}, \hat{M}_{ij} \leq \alpha, \forall (i,j) \in [d_1] \times [d_2], \text{ then} \]

\[ d_1^2(M, \hat{M}) \geq \frac{1 - e^{-T}}{4\alpha T} \left\| M - \hat{M} \right\|_F^2, \]

where \( T = \frac{1}{\alpha} \left( \alpha - \beta \right)^2 \).

\[ \text{Proof: Assuming } x \text{ is any entry in } M \text{ and } y \text{ is any entry in } \hat{M}, \text{ then } \beta \leq x, y \leq \alpha \text{ and } 0 \leq |x - y| \leq \alpha - \beta. \text{ By the mean value theorem there exists an } \xi(x,y) \in [\beta, \alpha] \text{ such that} \]

\[ \frac{1}{2}(\sqrt{x} - \sqrt{y})^2 = \frac{1}{2} \left( \frac{1}{2\sqrt{\xi(x,y)}} (x - y) \right)^2 = \frac{1}{8\xi(x,y)} (x - y)^2 \leq T. \]

The function \( f(z) = 1 - e^{-z} \) is concave in \([0, +\infty)\), so if \( z \in [0, T] \), we may bound it from below with a linear function

\[ 1 - e^{-z} \geq \frac{1 - e^{-T}}{T} z. \]

(56)

Plugging \( z = \frac{1}{2}(\sqrt{x} - \sqrt{y})^2 = \frac{1}{8\xi(x,y)} (x - y)^2 \) in (56), we have

\[ 2 - 2 \exp \left( \frac{1}{2}(\sqrt{x} - \sqrt{y})^2 \right) \geq \frac{1 - e^{-T}}{T} \frac{1}{4\xi(x,y)} (x - y)^2 \]

\[ \geq \frac{1 - e^{-T}}{T} \frac{1}{4\alpha} (x - y)^2. \]

(57)

Note that (57) holds for any \( x \) and \( y \). This concludes the proof.

\[ \text{Lemma 7. Let } H = \{ M : \| M \|_* \leq \alpha \sqrt{rd_1d_2}, \| M \|_\infty \leq \alpha \} \text{ and } \gamma \leq 1 \text{ be such that } \frac{\gamma}{\tau^2} \text{ is an integer. Suppose } r/\gamma^2 \leq d_1, \]

\[ 14 \]
then we may construct a set $\chi \in H$ of size

$$|\chi| \geq \exp \left( \frac{rd_2}{16\gamma^2} \right)$$

with the following properties:

1. For all $X \in \chi$, each entry has $|X_{ij}| = \alpha r$.
2. For all $X^{(i)}, X^{(j)} \in \chi$, $i \neq j$, $\|X^{(i)} - X^{(j)}\|_F^2 > \alpha^2 r^2 d_1 d_2 / 2$.

**Lemma 8.** For $x, y > 0$, $D(x||y) \leq (y - x)^2 / y$.

Proof: First assume $z \leq y$. Let $z = y - x$. Then $z \geq 0$ and $D(x||x + z) = x \log \frac{x + z}{x} + z$. Taking the first derivative of this with respect to $z$, we have $\frac{\partial}{\partial z} D(x||x + z) = \frac{z}{x + z}$. Thus, by Taylor’s theorem, there is some $\xi \in [0, z]$ so that $D(x||y) = D(x||x + z) + \frac{1}{2} z^2$. Since the right-hand-side increases in $\xi$, we may replace $\xi$ with $z$ and obtain $D(x||y) \leq \frac{(y - x)^2}{y}$. For $x > y$, with the similar argument we may conclude that for $z = y - x < 0$ there is some $\xi \in [z, 0]$ so that $D(x||y) = D(x||x + z) + \frac{1}{2} z^2$. Since $z < 0$ and $\xi / (x + \xi)$ decreases in $\xi$, then the right-hand-side is decreasing in $\xi$. We may also replace $\xi$ with $z$ and this proves the lemma.

**Proof of Theorem 4:** Lemma 4, Lemma 5, and Lemma 6 are used in the proof. In the following, the expectation are taken with respect to both $\Omega$ and $\{Y_{ij}\}$. First, note that

$$F_{\Omega,Y}(X) - F_{\Omega,Y}(M) = \sum_{(i,j)\in \Omega} \left[ Y_{ij} \log \left( \frac{X_{ij}}{M_{ij}} \right) - (X_{ij} - M_{ij}) \right]$$

Then for any $X \in S$,

$$\mathbb{E} \left[ F_{\Omega,Y}(X) - F_{\Omega,Y}(M) \right] = \frac{m}{d_1 d_2} \sum_{i,j} \left[ M_{ij} \log \left( \frac{X_{ij}}{M_{ij}} \right) - (X_{ij} - M_{ij}) \right]$$

$$= -\frac{m}{d_1 d_2} \sum_{i,j} \left[ M_{ij} \log \left( \frac{M_{ij}}{X_{ij}} \right) - (M_{ij} - X_{ij}) \right]$$

$$= -\frac{m}{d_1 d_2} \sum_{i,j} D(M_{ij}||X_{ij}) = -mD(M||X).$$

For $M \in S$, we know $\tilde{M} \in S$ and $F_{\Omega,Y}(\tilde{M}) \geq F_{\Omega,Y}(M)$. Thus we write

$$0 \leq F_{\Omega,Y}(\tilde{M}) - F_{\Omega,Y}(M)$$

$$= F_{\Omega,Y}(\tilde{M}) + EF_{\Omega,Y}(\tilde{M}) - EF_{\Omega,Y}(M) + EF_{\Omega,Y}(M) - EF_{\Omega,Y}(M)$$

$$\leq \mathbb{E} \left[ F_{\Omega,Y}(\tilde{M}) - F_{\Omega,Y}(M) \right] + \left| EF_{\Omega,Y}(\tilde{M}) - EF_{\Omega,Y}(M) \right| + \left| EF_{\Omega,Y}(M) - EF_{\Omega,Y}(M) \right|$$

$$\leq -mD(M||\tilde{M}) + 2 \sup_{X \in S} \left| F_{\Omega,Y}(X) - EF_{\Omega,Y}(X) \right|.$$

Applying Lemma 5, we obtain that with probability at least $(1 - C/(d_1 d_2))$

$$0 \leq -mD(M||\tilde{M}) + 2C' \left( \alpha \sqrt{r} / \beta \right) \left( \alpha (e^2 - 2) + 3 \log(d_1 d_2) \right) \cdot \left( \frac{\sqrt{m(d_1 + d_2) + d_1 d_2 \log(d_1 d_2)}}{m} \right).$$

After rearranging terms and applying the fact that $\sqrt{d_1 d_2} \leq d_1 + d_2$, we obtain

$$D(M||\tilde{M}) \leq 2C' \left( \alpha \sqrt{r} / \beta \right) \left( \alpha (e^2 - 2) + 3 \log(d_1 d_2) \right) \cdot \left( \frac{d_1 + d_2}{m} \sqrt{1 + \frac{(d_1 + d_2) \log(d_1 d_2)}{m}} \right). \tag{59}$$

Note that the KL divergence can be bounded below by the Hellinger distance (Chapter 3 in [49]):

$$d_H^2(x, y) \leq D(x||y).$$

Thus from (59), we obtain

$$d_H^2(M, \tilde{M}) \leq 2C' \left( \alpha \sqrt{r} / \beta \right) \left( \alpha (e^2 - 2) + 3 \log(d_1 d_2) \right) \cdot \left( \frac{d_1 + d_2}{m} \sqrt{1 + \frac{(d_1 + d_2) \log(d_1 d_2)}{m}} \right). \tag{60}$$

Finally, Theorem 4 is proved by applying Lemma 6.

**Proof of Theorem 5:**

We will prove by contradiction. Lemma 7 and Lemma 8 are used in the proof. Without loss of generality, assume $d_2 \geq d_1$. Choose $\epsilon > 0$ such that

$$\epsilon^2 = \min \left\{ \frac{1}{256}, C_2 \alpha^{3/2} \sqrt{\frac{rd_2}{m}} \right\},$$

where $C_2$ is an absolute constant that will be specified later. First, choose $\gamma$ such that $\frac{r}{\gamma}$ is an integer and

$$\frac{4\sqrt{2\epsilon}}{\alpha} \leq \gamma \leq \frac{8\epsilon}{\alpha} \leq \frac{1}{2}.$$

We may make such a choice because

$$\frac{\alpha^2 r}{64 \epsilon^2} \leq \frac{r}{\gamma^2} \leq \frac{\alpha^2 r}{32 \epsilon^2}$$

and

$$\frac{\alpha^2 r}{32 \epsilon^2} - \frac{\alpha^2 r}{64 \epsilon^2} = \frac{\alpha^2 r}{64 \epsilon^2} > 4 \alpha^2 r > 1.$$

Furthermore, since we have assumed that $\epsilon^2$ is larger than $Cr_\alpha^2/d_1, r/\gamma^2 \leq d_1$ for an appropriate choice of $C$. Let $\chi_{\alpha', 2}$ be the set defined in Lemma 7, by replacing $\alpha$ with $\alpha / 2$ and with this choice of $\gamma$. Then we can construct a packing set $\chi$ of the same size as $\chi_{\alpha', 2}$ by defining

$$\chi = \left\{ X' + \alpha' \left( 1 - \frac{\gamma}{2} \right) I_{d_1 \times d_2} : X' \in \chi_{\alpha', 2} \right\}.$$

The distance between pairs of elements in $\chi$ is bounded since

$$\|X^{(i)} - X^{(j)}\|^2 \geq \frac{\alpha^2 r^2 d_1 d_2}{4} \geq 4d_1 d_2 \epsilon^2. \tag{61}$$

Define $\alpha' = (1 - \gamma) \alpha$, then every entry of $X \in \chi$ has $X_{ij} \in \{\alpha, \alpha'\}$. Since we have assumed $r \geq 4$, for every $X \in \chi$, we have

$$\|X\|_* = \|X' + \alpha' \left( 1 - \frac{\gamma}{2} \right) I_{d_1 \times d_2} \|_* \leq \|X'\|_* + \alpha \left( 1 - \frac{\gamma}{2} \right) \sqrt{d_1 d_2} \leq \frac{\alpha}{2} \sqrt{rd_1 d_2} + \alpha \sqrt{d_1 d_2} \leq \alpha \sqrt{rd_1 d_2},$$

which is a contradiction.
for some $X' \in \chi_{\alpha/2, \gamma}$. Since the $\gamma$ we choose is less than $1/2$, $\alpha'$ is greater than $\alpha/2$. Therefore, from the assumption that $\beta \leq \alpha/2$, we conclude that $\chi \subset S$.

Now consider an algorithm that for any $X \in S$ returns $\hat{X}$ such that
\[
\frac{1}{d_1 d_2} \|X - \hat{X}\|_F^2 < \epsilon^2 \tag{62}
\]
with probability at least $1/4$. Next, we will show this leads to a contradiction. Let
\[
X^* = \arg\min_{X^{(i)} \in \chi} \|X^{(i)} - \hat{X}\|_F^2,
\]
by the same argument as that in [1], we have $X^* = X$ as long as (62) holds. Using the assumption that (62) holds with probability at least $1/4$, we have
\[
\mathbb{P}(X^* \neq X) \leq \frac{3}{4}, \tag{63}
\]
Using a generalized Fano’s inequality for the KL divergence in [50], we have
\[
\mathbb{P}(X^* \neq X) \geq 1 - \frac{\max_{X^{(n)} \neq X^{(i)}} D(X^{(k)} \| X^{(l)}) + 1}{\log |\chi|}. \tag{64}
\]
Define $D \triangleq D(X^{(k)} \| X^{(l)}) = \sum_{(i,j) \in \Omega} D(X_{ij}^{(k)} \| X_{ij}^{(l)})$. We know that each term in the sum is either 0, $D(\alpha \| \alpha')$, or $D(\alpha' \| \alpha)$. From Lemma 8, since $\alpha' < \alpha$, we have
\[
D \leq \frac{m(\gamma \alpha)^2}{\alpha'} \leq \frac{64m\epsilon^2}{\alpha'}.
\]
Combining (63) and (64), we have that
\[
\frac{1}{4} \leq 1 - \mathbb{P}(X \neq X^*) \leq \frac{D + 1}{\log |\chi|} \leq 16\gamma^2 \left( \frac{64m\epsilon^2}{\alpha'} + \frac{1}{rd_2} \right) \leq 1024\epsilon \left( \frac{64m\epsilon^2}{\alpha'} + \frac{1}{rd_2} \right). \tag{65}
\]
Suppose $64m\epsilon^2 \leq \alpha'$, then with (65), we have
\[
\frac{1}{4} \leq 1024\epsilon^2 \left( \frac{2}{\alpha^2 rd_2} \right),
\]
which implies that $\alpha^2 rd_2 \leq 32$. Then if we set $C_0 > 32$, this leads to a contradiction. Next, suppose $64m\epsilon^2 > \alpha'$, then with (65), we have
\[
\frac{1}{4} < 1024\epsilon^2 \left( \frac{128m^2}{(1 - \gamma)\alpha^3 rd_2} \right).
\]
Since $1 - \gamma > 1/2$, we have
\[
\epsilon^2 \geq \frac{\alpha^{3/2} \sqrt{rd_2}}{1024 m}.
\]
Setting $C_2 \leq 1/4096$, this leads to a contradiction. Therefore, (62) must be incorrect with probability at least $3/4$. This concludes our proof.

\[\text{Lemma 9. If } f \text{ is a closed convex function satisfying Lipschitz condition (23), then for any } X, Y \in S \text{, the following inequality holds:}
\]
\[
f(Y) \leq f(X) + \langle \nabla f(X), Y - X \rangle + \frac{L}{2} \|Y - X\|_F^2.
\]
\[\text{Proof: Let } Z = Y - X, \text{ then we have}
\]
\[
f(Y) = f(X) + \langle \nabla f(X), Z \rangle + \int_0^1 \langle \nabla f(X + tV) - \nabla f(X), Z \rangle dt
\]
\[
\leq f(X) + \langle \nabla f(X), Z \rangle + \int_0^1 Z \bullet Z \|F\|_F dt
\]
\[
= f(X) + \langle \nabla f(X), Y - X \rangle + \frac{L}{2} \|Y - X\|_F^2,
\]
where we use Taylor expansion with integral remainder in the first line, the fact that dual norm of Frobenius norm is itself in the second line and Lipschitz condition in the third line.