Portfolio Optimization under Small Transaction Costs: a Convex Duality Approach

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Abstract

We consider an investor with constant absolute risk aversion who trades a risky asset with general Itô dynamics, in the presence of small proportional transaction costs. Kallsen and Muhle-Karbe [13] formally derived the leading-order optimal trading policy and the associated welfare impact of transaction costs. In the present paper, we carry out a convex duality approach facilitated by the concept of shadow price processes in order to verify the main results of [13] under well-defined regularity conditions.

Keywords: utility maximization, small transaction costs, duality, shadow price

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1 Introduction

A classical problem of mathematical finance concerns an economic agent who invests in a financial market so as to maximize the expected utility of her terminal wealth. A possible approach to tackle such problems is based on the dual characterization of admissible portfolios with the help of convex analysis. This has been studied mostly in frictionless environments, for instance in [15, 21]. In the context of markets with friction, Cvitanić and Karatzas [5] extended this approach to problems with proportional transaction costs. They rely more or less explicitly on the concept of consistent price systems or shadow price processes, which allow to translate the original problem into a more tractable frictionless one, cf. in particular Loewenstein [17] in this context.

In a recent study, Kallsen and Muhle-Karbe [13] investigate optimal portfolio choice with respect to exponential utility and small transaction costs for general Itô processes. They formally derive a leading-order optimal trading policy and the associated welfare impact. The purpose of the present study is to rigorously prove the main statements of [13] under well-defined regularity conditions. Our approach resembles that of Henderson [7], in the sense that an explicitly known dual control provides us an upper bound to the optimization

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problem. Since this bound coincides to the leading order with the utility of a candidate strategy to the primal problem, the latter must be approximately optimal.

Starting with [22, 25], the problem of optimal investment in the presence of small proportional transaction costs has been studied extensively. For an account of the literature, we refer the reader to [13, 14]. Rigorous derivations of leading-order optimal strategies in related setups are provided e.g. in [10, 2] and in particular [24, 18]. The cited papers carry a strong analytic flavour. [24, 18] make use of the deep theory of homogenization and viscosity solutions. By contrast and as noted above, our more probabilistic approach relies on dual considerations. In particular, the value function as a key object in analytical approaches appears only implicitly here. In fact, even its existence is not obvious if the underlying model fails to be of Markovian structure.

The paper is organized as follows. The market model is introduced in Section 2. Subsequently, we state the main results concerning optimal investment to the leading order. In Section 4, we present two classes of examples, namely the Black-Scholes model and a more general stochastic volatility model. The proofs of the main results are provided in Section 5. The appendix contains the derivation of the frictionless optimizer related to the models of Section 4.

We generally use the notation as in [9]. In particular, \( H \cdot Y := \int_0^T H_t dY_t \) stands for the stochastic integral of \( H \) with respect to \( Y \).

## 2 The market model

We consider the same setup as in [13]: fixing a finite time horizon \( T \in (0, \infty) \), the financial market consists of a riskless asset (bond) with price normalized to 1 and a risky asset (stock) traded with proportional transaction costs. The stock price \( S \) is modelled by a general Itô process

\[
dS_t = b^S_t dt + \sigma^S_t dW_t
\]

defined on a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\). Here, \( W \) is a one-dimensional, standard Brownian motion and \( b, \sigma \) are predictable processes satisfying

\[
\int_0^T \left( |b^S_t| + (\sigma^S_t)^2 \right) dt < \infty \quad \text{a.s.}
\]

Let \( \varepsilon \in (0, 1) \) denote the relative bid-ask spread, i.e., an investor has to pay the higher ask price \((1 + \varepsilon)S\) but only receives the lower bid-price \((1 - \varepsilon)S\) for buying and selling the stock, respectively.

**Definition 2.1.**

1. A **trading strategy** is an \( \mathbb{R}^2 \)-valued predictable process \((\psi^0, \psi)\) of finite variation, where \( \psi^0 \) and \( \psi \) denote the number of shares held in the bank account and in stock at time \( t \), respectively.

2. The **liquidation wealth process** of a trading strategy \((\psi^0, \psi)\) is defined as

\[
X^{\psi,\varepsilon} := \psi^0 + \psi \mathbf{1}_{\{\psi \geq 0\}} (1 - \varepsilon)S + \psi \mathbf{1}_{\{\psi < 0\}} (1 + \varepsilon)S.
\]
3. Writing \( \psi = \psi^\uparrow - \psi^\downarrow \) with increasing predictable processes \( \psi^\uparrow, \psi^\downarrow \) which do not increase at the same time, a trading strategy \( (\psi^0, \psi) \) is called self-financing if

\[
d\psi^0_t = (1 - \varepsilon) d\psi_t^\uparrow - (1 + \varepsilon) S_t d\psi_t^\downarrow,
\]

cf. [11]. For given initial value \( \psi^0_t \), a self-financing trading strategy \( (\psi^0, \psi) \) will be identified with its second component \( \psi \) in the sequel.

4. Given initial wealth \( (x^B, x^S) \in \mathbb{R}^2 \) in the bank account and the stock, respectively, a self-financing trading strategy \( (\psi^0, \psi) \) is said to be admissible for \( (x^B, x^S) \) and written as

\[
\psi \in \mathcal{A}^e(x^B, x^S)
\]

if \( x^B = \psi^0_t, x^S = \psi_0 S_0 \), and if the related liquidation wealth is bounded from below, i.e.,

\[
X_{\psi, e} \geq -K
\]

for some \( K \in \mathbb{R}_+ \).

**Remark 2.2.** The liquidation wealth of a self-financing strategy \( (\psi^0, \psi) \) with \( x^B = \psi^0_t, x^S = \psi_0 S_0 \) can be written as

\[
X_t^{\psi, e} = \psi^0_t + \psi_t \mathbf{1}_{\{\psi \geq 0\}} (1 - \varepsilon) S_t + \psi_t \mathbf{1}_{\{\psi < 0\}} (1 + \varepsilon) S_t
\]

\[
= \psi^0_t + (1 - \varepsilon) S \cdot \psi^\uparrow_t - (1 + \varepsilon) S \cdot \psi^\downarrow_t
\]

\[
+ \psi_t \mathbf{1}_{\{\psi \geq 0\}} (1 - \varepsilon) S_t + \psi_t \mathbf{1}_{\{\psi < 0\}} (1 + \varepsilon) S_t.
\]

(2.1)

If \((1 - \varepsilon) S \cdot \psi^\uparrow = \tilde{S} \cdot \psi^\uparrow \) and \((1 + \varepsilon) S \cdot \psi^\downarrow = \tilde{S} \cdot \psi^\downarrow \) for some Itô process \( \tilde{S} \) with values in \([[(1 - \varepsilon) S, (1 + \varepsilon) S]]\), then (2.1) and integration by parts yield

\[
X_t^{\psi, e} = \psi^0_t - \tilde{S} \cdot \psi_t + \psi_t \tilde{S}_t - \psi_t \mathbf{1}_{\{\psi \geq 0\}} (\tilde{S}_t - (1 - \varepsilon) S_t) - \psi_t \mathbf{1}_{\{\psi < 0\}} (\tilde{S}_t - (1 + \varepsilon) S_t)
\]

\[
= \psi^0_t + \psi_0 \tilde{S}_0 + \psi \cdot \tilde{S}_t - \psi_t \mathbf{1}_{\{\psi \geq 0\}} (\tilde{S}_t - (1 - \varepsilon) S_t) - \psi_t \mathbf{1}_{\{\psi < 0\}} (\tilde{S}_t - (1 + \varepsilon) S_t)
\]

and hence

\[
\left| X_t^{\psi, e} - (x + \psi \cdot \tilde{S}_t) \right| \leq \varepsilon x^S + 2\varepsilon |\psi_t S_t|.
\]

In this setting, we focus on the exponential utility function with constant absolute risk aversion \( p > 0 \):

\[
U(x) := -e^{-px}.
\]

Our optimization problem consists in maximizing the expected utility or, equivalently, the certainty equivalent \( \text{CE}(X_t^{\psi, e}) \) of terminal wealth over all admissible trading strategies \( \psi \) with given initial wealth \( (x^B, x^S) \). As usual, the certainty equivalent of a random payoff \( X \) refers to the deterministic amount with the same utility, i.e.,

\[
\text{CE}(X) := -\frac{1}{p} \ln \mathbb{E}[e^{-px}].
\]
3 Main results

In this section, we present the main theorem of this paper concerning optimal investment to the leading order. To this end, we require that the corresponding frictionless market fulfills some regularity conditions.

**Assumption 3.1.** We suppose that the frictionless price process $S$ allows for an equivalent local martingale measure with finite relative entropy.

Denote the initial wealth before liquidation by $x := x^B + x^S$. According to [6, Theorem 2.1], Assumption 3.1 implies that the minimal entropy (local) martingale measure (MEMM) $Q$ for $S$ exists. By [21, Theorem 2.2 (iv)], there is a predictable, $S$-integrable process $\varphi$ such that $\varphi \cdot S$ is a $Q$-martingale and

$$
\frac{dQ}{dP} = \frac{U'(x + \varphi \cdot S_T)}{y}
$$

with $y := \mathbb{E}[U'(x + \varphi \cdot S_T)]$. Interpreted as number of shares, strategy $\varphi$ is the optimal solution to the frictionless counterpart of the above utility maximization problem.

For any Itô process $X$, we denote by $b^X$ and $c^{X,X}$ its local $Q$-drift and quadratic variation, respectively, i.e.,

$$
dX_t = b^X_t \, dt + dM^X_t, \quad c^{X,X}_t := \frac{d[X,X]_t}{dt},
$$

where $M^X$ is a continuous $Q$-local martingale starting in 0. Similarly, for Itô processes $X$ and $Y$, their local covariation is denoted by

$$
c^{X,Y}_t := \frac{d[X,Y]_t}{dt}.
$$

The local drift rate of Itô process $X$ relative to $P$ rather than $Q$ is written as $b^{X,P}$.

**Assumption 3.2.** We suppose that the frictionless optimizer $\varphi$ and the activity rate

$$
\rho := \frac{c^{\varphi,\varphi}}{c^S S}
$$

are well-defined Itô processes such that $\rho$ never vanishes.

The processes $S$, $\varphi$, $\rho$ and their dynamics depend on the current level of the stock price. In concrete models, the following related normalized processes are easier to work with:

- the stock return process $R := \ln S$,
- the stock holdings $\pi := \varphi S$,
- the normalized activity rate $\eta := \rho S^4$.

**Assumption 3.3.** We assume that

$$
\mathbb{E}_Q \left[ \sup_{t \in [0,T]} |X_t|^n \right] < \infty \quad \text{for any } n \in \mathbb{N} \text{ and any } X \in \mathcal{H},
$$

(3.1)
where
\[ \mathcal{H} := \{ \pi, \eta, \eta^{-1}, b^\pi, b^\eta, c^{R,R}, (c^{R,R})^{-1}, c^{\pi,\pi}, c^{\eta,\eta} \}. \]

Moreover, we suppose
\[ E_Q[\exp(9p\phi \cdot S_T)] < \infty. \quad (3.2) \]

Finally, we assume that \( c^{\pi,\pi}, c^{R,R}, c^{\eta,\eta}, c^{\pi,\eta}, c^{\pi,R}, c^{\eta,R} \) are continuous.

**Theorem 3.4.** Suppose that Assumptions 3.1, 3.2, 3.3 hold. Let
\[ \Delta \phi := \pm \left( \frac{3p}{2} S \epsilon \right)^{1/3} = \pm \left( \frac{3\eta}{2p} \right)^{1/3} S^{-1}. \]

1. There exists a continuous adapted process
\[ \phi^e = \phi + \Delta \phi = \phi^\uparrow - \phi^\downarrow, \]
where \( \Delta \phi \) has values in \( [\Delta \phi^-, \Delta \phi^+] \),
\[ \phi^e_0 = \begin{cases} 
\phi_0 + \Delta \phi^+_0 & \text{if } x^S > (\phi_0 + \Delta \phi^+_0)S_0, \\
\phi_0 + \Delta \phi^-_0 & \text{if } x^S < (\phi_0 + \Delta \phi^-_0)S_0, \\
x^S/S_0 & \text{otherwise,}
\end{cases} \]
and \( \phi^\uparrow, \phi^\downarrow \) are increasing process such that
\[ \phi^\uparrow \text{ increases only on the set } \{ \Delta \phi = \Delta \phi^- \} \subseteq \Omega \times [0, T], \]
\[ \phi^\downarrow \text{ increases only on the set } \{ \Delta \phi = \Delta \phi^+ \} \subseteq \Omega \times [0, T]. \]

2. By slight abuse of notation, we identify \( \phi^e \) with the unique self-financing strategy
\( (\psi^0, \psi) \) that satisfies \( \psi^0_0 = x^B, \psi_0S_0 = x^S, \psi_t = \phi^e_t \text{ for } t \in [0, T] \). Define
\[ \tau^e := \inf \left\{ t \in [0, T] : |X^\phi^e_t - (x + \phi \cdot S_t)| > 1 \text{ or } |X^\phi^e_t| > \epsilon^{-4/3} \right\} \wedge T. \quad (3.3) \]

Then \( P(\tau^e = T) \to 1 \text{ as } \epsilon \to 0. \) Moreover, \( \phi^e \mathbb{1}_{[0, \tau^e]} \) is a utility-maximizing strategy to
the leading order \( O(\epsilon^{2/3}) \), i.e.,
\[ \sup_{\psi \in \mathcal{A}^e(x^B, x^S)} E \left[ U(X^\psi^e_T) \right] = E \left[ U(X^\phi^e_T) \right] + o(\epsilon^{2/3}). \]

(As above, \( \phi^e \mathbb{1}_{[0, \tau^e]} \) here refers to the strategy \( \psi \in \mathcal{A}^e(x^B, x^S) \) with \( \psi_t = \phi^e_t \mathbb{1}_{[0, \tau^e]}(t) \)
for \( t \in (0, T] \).)

3. The optimal certainty equivalent amounts to
\[ \sup_{\psi \in \mathcal{A}^e(x^B, x^S)} CE(X^\psi^e_T) = CE(X^\phi^e_T) + o(\epsilon^{2/3}) \]
\[ = CE(x + \phi \cdot S_T) - \frac{p}{2} E_Q \left[ (\Delta \phi^+)^2 \cdot [S, S]_T \right] + o(\epsilon^{2/3}). \]
PROOF. The proof is split up into several steps given in Section 5. The existence of \( \varphi^\varepsilon \) is linked to the Skorohod problem with time-dependent reflecting barriers (cf. Lemma 5.5). With the help of the shadow price process \( S^\varepsilon \) derived heuristically in [13] (cf. Corollary 5.6), the utility generated by \( \varphi^\varepsilon \) stopped at \( \tau^\varepsilon \) is computed in Lemma 5.10. The optimality of \( \varphi^\varepsilon \) is proved by means of some dual considerations (cf. Lemma 5.13) in conjunction with the conjugate relation (cf. Lemma 5.14). Finally, the proof of the explicit expression for the certainty equivalent loss relies on a random time change and the ergodic property of reflected Brownian motion (cf. Corollary 5.18).

Remark 3.5. Roughly speaking, the assumptions in Theorem 3.4 concern sufficient integrability of the solution to the frictionless utility maximization problem in order to warrant that the maximal expected utility is twice differentiable as a function of \( \varepsilon^{1/3} \). In the subsequent section we verify these assumptions in a general stochastic volatility setup.

From our theorem, the leading-order optimal strategy under transaction costs \( \varphi^\varepsilon \) stays within the random no-trade region \([\varphi + \Delta \varphi^-, \varphi + \Delta \varphi^+] \) around the frictionless optimizer \( \varphi \); and it increases (resp. decreases) only while hitting the lower (resp. upper) bound. In this sense, \( \varphi + \Delta \varphi^+ \) and \( \varphi + \Delta \varphi^- \) correspond to the selling and buying boundary, respectively. At the random time \( \tau^\varepsilon \), the portfolio is liquidated primarily in order to bound losses.

4 Examples

We provide two classes of models for which the frictionless optimizer \( \varphi \) is known explicitly.

4.1 Black-Scholes model

First, we consider the so-called Black-Scholes model

\[
   dS_t = S_t(b dt + \sigma dW_t)
\]

with \( b \in \mathbb{R}, \sigma \in \mathbb{R}_+ \setminus \{0\} \). We show that Assumptions 3.1 3.2 3.3 hold if \( b \neq 0 \).

From Theorem A.1 in the appendix, the frictionless optimal strategy \( \varphi \) satisfies

\[
   \pi_t = \varphi_t S_t = \frac{b}{p \sigma^2} \quad \text{for all } t \in [0, T].
\]

By Itô’s formula, we have

\[
   d\varphi_t = -\frac{b}{p \sigma^2 S_t^2} dS_t + \frac{b}{p \sigma^2 S_t^3} d[S, S],
\]

and hence

\[
   c_t^{\varphi, \varphi} = \frac{b^2}{p^2 \sigma^2 S_t^2}.
\]

This yields

\[
   \rho_t = \frac{b^2}{p^2 \sigma^4 S_t^4}
\]
and
\[ \eta_t = \frac{b^2}{p^2 \sigma^4} \]
for any \( t \in [0, T] \). Therefore, all processes in set \( \mathcal{H} \) as well as \( c^R, \eta, c^\pi, \pi, \) are constant, which in particular yields Condition (3.1). The frictionless optimal terminal gains are of the form
\[ \varphi \cdot S_T = \frac{b}{p \sigma} W^Q_T, \]
where \( W^Q_t = W_t + \frac{b}{\sigma} t, t \in [0, T] \) is a standard Brownian motion under measure \( Q \). Thus Condition (3.2) is satisfied.

The no-trade bounds are obtained from
\[ \Delta \varphi^\pm S = \pm \left( \frac{3b^2 \epsilon^2}{2p^3 \sigma^4} \right)^{1/3} \]
and the certainty equivalent loss due to transaction costs is
\[ \sup_{\psi \in \mathcal{A}(x^B, x^S)} \text{CE}(X_{T}^{\psi, \epsilon}) - \text{CE}(x + \varphi \cdot S_T) = -\left( \frac{9b^4 \epsilon^2}{32 \sigma^2} \right)^{1/3} \frac{T}{p} + o(\epsilon^{2/3}). \]
(4.1) coincides with the formulas in [25, p.319] resp. [3, (3.4)]. The expression in (4.2), on the other hand, is obtained from [3, (3.7, 3.8)] if one uses the equation for \( V_2 \) in [25, p.317]. Note, however, that [3] considers a slightly more involved notion of admissibility.

4.2 Stochastic volatility model

Let us turn to the following stochastic volatility model:
\[ dS_t = S_t (b(Y_t)dt + \sigma(Y_t)dW_t) \]
with continuous functions \( b, \sigma : \mathbb{R} \to \mathbb{R} \) and an Itô process \( Y \) which is independent of the Brownian motion \( W \). The filtration is supposed to be generated by \( W \) and \( Y \).

Proposition 4.1. Suppose that the stochastic volatility model (4.3) is such that

- the coefficients \( b(Y), \sigma(Y) \) are bounded processes that are bounded away from 0,
- the processes
  \[ \pi := \frac{b(Y)}{p \sigma(Y)^2}, \quad \eta := \pi^2 + \frac{c^\pi, \pi}{\sigma(Y)^2}, \]
  are Itô processes with bounded coefficients \( b^\pi, \pi, c^\pi, \pi, b^\eta, \eta, c^\eta, \eta \) and continuous coefficients \( c^\pi, \pi, c^\eta, \eta, c^\pi, \eta \),
- for
  \[ \tilde{Z}_t := \mathbb{E} \left[ \exp \left( -\frac{1}{2} \int_0^T \left( \frac{b(Y_t)}{\sigma(Y_t)} \right)^2 dt \right) \bigg| \mathcal{F}_t \right], \]
  the process \( \tilde{Z} / \tilde{Z}^2 \) is bounded.
Then Assumptions 3.1, 3.2, 3.3 are satisfied. Moreover, \( \pi \) and \( \eta \) are the corresponding stock holdings and normalized activity rate. The no-trade boundary is given by

\[
\Delta \phi \pm S = \pm \left( \frac{3 \eta \varepsilon}{2 \sigma^2} \right)^{1/3}
\]

and the certainty equivalent loss amounts to

\[
\mathbb{E}_Q \left[ \int_0^T \left( \frac{9 p \eta^2}{32} \right)^{1/3} \sigma(Y_t)^2 dt \right] \varepsilon^{2/3} + o(\varepsilon^{2/3}).
\]

**Proof.** Step 1: We show that \([f(S), X] = 0\) for any \( C^2 \)-function \( f \) and any Itô process \( X \) which is \( \sigma(Y) \)-measurable. Indeed, by Itô's formula it suffices to prove that \([W, X] = 0\). Using [19, Theorem II.4], it is easy to show that the martingale part of \( X \) is \( \sigma(Y) \)-measurable. Hence without loss of generality, \( X \) is a local martingale. By localization it suffices to consider the case where \( X \) is a square-integrable martingale. Let \( G \) be the filtration defined in (A.2) in the appendix. Then \( X_t \) is \( \mathcal{F}_0 \)-measurable for any \( t \in [0, T] \) and \( W \) is a Brownian motion relative to both \( \mathcal{F} \) and \( G \). We obtain

\[
\mathbb{E}[W_t X_t | \mathcal{F}_s] = \mathbb{E}[\mathbb{E}[W_t X_t | \mathcal{F}_t] | \mathcal{F}_s] = \mathbb{E}[W_t | \mathcal{F}_t] X_t | \mathcal{F}_s] = W_t X_s
\]

for any \( s < t \). Hence \( WX \) is a martingale, which implies \([W, X] = 0\) as desired.

Step 2: We show that \( \pi, \eta \) in (4.4) coincide with the stock holdings and the normalized activity rate. By Theorem A.1 the frictionless optimizer \( \phi \) satisfies \( \phi_t S_t = \pi_t \) for any \( t \in [0, T] \). From Itô’s formula we get

\[
d\phi_t = -\frac{\pi_t}{S_t^2} dS_t + \frac{\pi_t}{S_t^4} d[S, S]_t + \frac{1}{S_t} d\pi_t + d\left[ \frac{1}{S_t}, \pi_t \right].
\]

Step 1 yields \([S, \pi] = 0\), which implies

\[
c_i^{\phi, \phi} = \frac{\pi_t^2 \sigma(Y_t)^2 + c_i^{\pi, \pi}}{S_t^2}
\]

and

\[
c_i^{\phi, \phi}_{S, S} S_t^4 = \pi_t^2 + \frac{c_i^{\pi, \pi}}{\sigma(Y_t)^2} = \eta_t.
\]

Step 3: Let \( \bar{Z} \) be defined as in (A.3). By Theorem A.1 \( \bar{Z} \bar{Z}/\bar{Z}_0 \) is the density process of the MEMM \( \mathbb{Q} \). For any Itô process \( X \), Girsanov’s theorem implies

\[
b_t^X = b_t^X \bar{Z} + \frac{c_t^{\bar{X}, \bar{Z}}}{\bar{Z}_t \bar{Z}}.
\]
Integration by parts yields
\[
\frac{e^{Z,X}}{ZZ} = \frac{Z,X}{Z} + \frac{Z,X}{Z} \\
\leq \sqrt{\frac{Z,Z}{Z^2}} \sqrt{c^X,X} + \sqrt{\frac{Z,Z}{Z^2}} \sqrt{c^X,X} \\
= \left(\sqrt{\frac{Z,Z}{Z^2}} + \left|\frac{b(Y)}{\sigma(Y)}\right|\right) \sqrt{c^X,X}.
\]

In view of our boundedness assumptions, we conclude that \(b^\pi\) and \(b^\eta\) are bounded. Consequently, all processes in set \(H\) are bounded, which implies Condition (3.1).

Moreover, the frictionless optimal terminal gains are of the form
\[
\varphi \cdot S_T = \int_0^T \frac{b(Y_t)}{p \sigma(Y_t)} dW^Q_t,
\]
where \(W^Q = W + \int_0^t \frac{b(Y_s)}{\sigma(Y_s)} ds\) is a standard Brownian motion under measure \(Q\). If an integrand \(H\) is bounded by \(m \in \mathbb{R}\), we have
\[
E_Q\left[\exp(H \cdot W^Q_T)\right] \leq E_Q\left[\exp\left(H \cdot W^Q_T - \frac{1}{2} \int_0^T H_s^2 ds\right)\right] \exp\left(\frac{1}{2} m^2 T\right) \\
\leq \exp\left(\frac{1}{2} m^2 T\right) < \infty.
\]

Together, we conclude that Condition (3.2) holds. Finally, Step 1 yields that \(c^{\pi,R} = 0\) and \(c^{\eta,R} = 0\), which completes the proof of Assumption 3.3. \(\square\)

### 5 Proof of the main results

As indicated in Section 3, we prove the main theorem in this section. We assume throughout that Assumptions 3.1, 3.2 hold. The idea of our proof can be outlined as follows:

- [13] derives a possibly suboptimal candidate strategy \(\varphi^e\) (in the sense of number of shares of stock) along with a shadow price \(S^e\). This term here refers to a frictionless price process moving within the bid-ask bounds \([ (1 - \epsilon)S_t, (1 + \epsilon)S_t ]\) and such that strategy \(\varphi^e_t\) only buys (resp. sells) stock if the shadow price \(S^e_t\) coincides with the ask price \((1 + \epsilon)S_t\) (resp. bid price \((1 - \epsilon)S_t\)). Evidently, following \(\varphi^e\) in the frictionless market \(S^e\) yields the same wealth process and hence expected utility as in the original market with proportional transaction costs. This expected utility can be computed explicitly to the leading order because both \(\varphi^e\) and \(S^e\) are given in closed form.

- According to [15, 21] dealing with the issue of hedging duality in frictionless markets, the utility maximization problem for \(S^e\) without transaction costs is related to a dual
minimization problem on the set of equivalent local martingale measures. Specifically, the value of the dual problem dominates the expected utility of any admissible trading strategy. In a second step we therefore construct a carefully chosen, explicitly known local martingale measure (identified with its Radon-Nikodym density \( Z^\varepsilon \)) for \( S^\varepsilon \). Since trading in the frictionless market \( S^\varepsilon \) leads to higher profit than in the original market with transaction costs, the Lagrange dual function evaluated at \( Z^\varepsilon \) provides an upper bound to the maximal expected utility in the market with friction. This upper bound can be computed explicitly to the leading order because \( Z^\varepsilon \) is known in closed form.

- In a final step we observe that the suboptimal expected utility of \( \varphi^\varepsilon \) coincides to the leading order with the upper bound above. Hence, we obtain approximate optimality of the candidate strategy.

In the language of [5], \((Z^\varepsilon, Z^\varepsilon S^\varepsilon)\) is a state-price density, which, by duality to the set of self-financing portfolios in the market with friction, provides an upper bound to the expected utility under transaction costs.

Set

\[
\alpha := \frac{p_{c,S,S}}{3 c \varphi \varphi}, \quad \beta := \left( \frac{S}{\alpha} \right)^{1/3} = \Delta \varphi^\varepsilon \left( \frac{2}{\varepsilon} \right)^{1/3}.
\]

We define sets of processes

\[
\mathcal{H}^{b,AS} := \left\{ \beta c^{S,S}, \alpha \beta^2 b \varphi, \beta^2 c^a \varphi, \beta^3 b \alpha, \beta b \beta^2, c^a \beta^2 \varphi \right\},
\]

\[
\mathcal{H}^{c,AS} := \left\{ \alpha \beta^2 c \varphi, \beta^3 c^a \varphi, \beta c \beta^2 \varphi \right\},
\]

\[
\mathcal{H}^{c,AS,\varphi} := \left\{ \alpha \beta^2 c \varphi, \beta^3 c^a \varphi, \beta c \beta^2 \varphi \right\},
\]

\[
\mathcal{G}_1 := \{ S b \varphi \} \cup \mathcal{H}^{c,AS,\varphi} \cup \left\{ \beta b^A : b^A \in \mathcal{H}^{b,AS} \right\},
\]

\[
\mathcal{G}_2 := \left\{ \beta^2 c^{S,S}, S^{c,\varphi} \phi \right\} \cup \left\{ \beta^2 c^{\Delta,\Delta} : c^{\Delta,\Delta} \in \mathcal{H}^{c,AS,\Delta,\Delta} \right\},
\]

\[
\mathcal{G} := \mathcal{G}_1 \cup \mathcal{G}_2.
\]

For the proof of Theorem 3.4, Assumption 3.3 can be replaced by the following two slightly weaker assumptions.

**Assumption 5.1.** We suppose that

\[
\sum_{\varphi \in \mathcal{G}} E_Q \left[ \int_0^T |g_t|^4 dt \right] + E_Q \left[ \sup_{t \in [0,T]} |\varphi_s|^4 + \sup_{t \in [0,T]} |\beta_s S_t|^4 \right] < \infty, \quad (5.1)
\]

\[
\sum_{c^{\Delta,\Delta} \in \mathcal{H}^{c,AS,\Delta,\Delta}} E_Q \left[ \sup_{t \in [0,T]} \left| \frac{c^{\Delta,\Delta}_t}{c^{S,S}_t} \right|^{16} \right] < \infty, \quad (5.2)
\]
\[
\sum_{b,\Delta \in \mathcal{H} \Delta S} E_{Q} \left[ \int_{0}^{T} \frac{|b^\Delta|^{16}}{\sigma^3_t} dt \right] < \infty, \tag{5.3}
\]

\[
E_{Q}[\exp(|9p\varphi \cdot S_T|)] < \infty.
\]

**Assumption 5.2.** The processes \(c^\varphi, c^\psi, c^S, c^\beta, c^\varphi \beta\) are continuous and hence pathwise bounded. Moreover, the processes \(b^\varphi, b^\beta\) are assumed to be pathwise bounded as well.

**Lemma 5.3.** Define

\[
\mathcal{X} := \left\{ (X_t)_{t \in [0,T]} : E_{Q} \left[ \sup_{t \in [0,T]} |X_t|^n \right] < \infty \text{ for any } n \in \mathbb{N} \right\}.
\]

Then

1. For \(X, Y \in \mathcal{X}, c \in \mathbb{R}\), it holds that \(X + Y, XY, cX \in \mathcal{X}\).

2. If \(X \in \mathcal{X}\) and \(f : \mathbb{R} \to \mathbb{R}\) with \(|f(x)| \leq 1 + |x|\) for any \(x \in \mathbb{R}\), then \(f(X) \in \mathcal{X}\).

3. If \(X \in \mathcal{X}\), then

\[
E_{Q}\left[ \int_{0}^{T} |X_t|^n dt \right] < \infty \text{ for any } n \in \mathbb{N}.
\]

**Proof.** This is straightforward. \(\square\)

**Lemma 5.4.** Assumptions 5.1 and 5.2 hold if Assumptions 3.1-3.3 are fulfilled.

**Proof.** This follows from Itô’s formula, straightforward but tedious calculations, and Lemma 5.3. \(\square\)

### 5.1 Existence of shadow price \(S^e\)

**Lemma 5.5.** For

\[
\Delta \varphi_0 := \begin{cases} 
\Delta \varphi_0^+ & \text{if } x^S > (\varphi_0 + \Delta \varphi_0^+)S_0, \\
\Delta \varphi_0^- & \text{if } x^S < (\varphi_0 + \Delta \varphi_0^-)S_0, \\
x^S_0 - \varphi_0 & \text{otherwise},
\end{cases}
\]

there exists a solution \(\Delta \varphi\) to the Skorohod stochastic differential equation (SDE)

\[
d\Delta \varphi_t = -d\varphi_t \tag{5.4}
\]

with reflection at \(\Delta \varphi^-\), \(\Delta \varphi^+\), i.e., there exist a continuous, adapted, \([\Delta \varphi^-, \Delta \varphi^+]-\text{valued}\) process \(\Delta \varphi\) and adapted increasing processes \(\varphi^{\epsilon \uparrow}, \varphi^{\epsilon \downarrow}\) such that

\[
\varphi^{\epsilon \uparrow} \text{ increases only on the set } \{\Delta \varphi = \Delta \varphi^-\} \subseteq \Omega \times [0,T], \tag{5.5}
\]

\[
\varphi^{\epsilon \downarrow} \text{ increases only on the set } \{\Delta \varphi = \Delta \varphi^+\} \subseteq \Omega \times [0,T], \tag{5.6}
\]

and

\[
\Delta \varphi =-\varphi + \varphi^{\epsilon \uparrow} - \varphi^{\epsilon \downarrow}. \tag{5.7}
\]

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Then $\Delta (5.9, 5.10)$.

In particular, and a constant $\Delta$ processes. From (5.5, 5.6, 5.7) we deduce that obviously Lipschitz continuous with respect to $P_\varphi$. Corollary 5.6.

Itô’s formula yields $\Delta S$ is an Itô process with values. Consider the function $\varphi^\pm$ are Lipschitz operators in the sense of [23, Definition 3.1] evaluated at process $\varphi$. The assertion follows now from [23 Theorem 3.3].

**Corollary 5.6.** Let $\Delta \varphi, \varphi^\uparrow, \varphi^\downarrow$ be as in Lemma 5.5 resp. Theorem 3.4. Define

$$\gamma := 3\alpha \beta^2 \left(\frac{\varepsilon}{2}\right)^{2/3}$$

and

$$\Delta \varphi := \alpha \Delta \varphi^3 - \gamma \Delta \varphi.$$  \hspace{1cm} (5.8)

Then $\Delta \varphi$ is an Itô process with values in $[-\varepsilon \varphi, \varepsilon \varphi]$ such that

$\varphi^\uparrow$ increases only on the set \{ $\Delta \varphi = \varepsilon \varphi$ \} $\subseteq \Omega \times [0, T]$, \hspace{1cm} (5.9)

$\varphi^\downarrow$ increases only on the set \{ $\Delta \varphi = -\varepsilon \varphi$ \} $\subseteq \Omega \times [0, T]$. \hspace{1cm} (5.10)

**Proof.** Consider the function $f(x, a, g) := ax^3 - gx$. By Assumption 3.2 $\alpha, \gamma, \varphi$ are Itô processes. From (5.5, 5.6, 5.7) we deduce that

$$d \varphi_t = -d \varphi_t \text{ on } \{ \Delta \varphi \neq \Delta \varphi^\pm \}. \hspace{1cm} (5.11)$$

Since

$$\partial f(\Delta \varphi^\pm, a, \gamma) = 3\alpha(\Delta \varphi^\pm)^2 - \gamma = 0, \hspace{1cm} (5.12)$$

Itô’s formula yields

$$d \Delta \varphi_t = d f(\Delta \varphi, a, \gamma)_t$$

$$= -3\alpha \Delta \varphi_t^2 d \varphi_t + 3\alpha \Delta \varphi_t d[\varphi, \varphi]_t$$

and $f(\Delta \varphi^\pm, a, \gamma) = \mp \varepsilon \varphi$ imply (5.9) (5.10).

The coefficients related to $\Delta \varphi$ can be estimated$^1$ as follows:

$$b^\Delta_t = \begin{cases} p \Delta \varphi_t c_t \varepsilon^{1/3} & \text{if } b^\Delta_t = G^{(1)}_t \varepsilon^{1/3} \end{cases}$$

$$= \begin{cases} \sum_{b^\Delta_t \in \varepsilon^{1/3}} |b^\Delta_t| \varepsilon^{1/3} & \text{if } b^\Delta_t = G^{(2)}_t \varepsilon^{1/3} \end{cases}$$

$$|G^{(1)}_t| \leq \text{cst.} \beta_t c_t \varepsilon^{1/3}, \hspace{1cm} |G^{(2)}_t| \leq \text{cst.} \sum_{b^\Delta_t \in \varepsilon^{1/3}} |b^\Delta_t| \varepsilon^{1/3}, \hspace{1cm} (5.14)$$

$$|c_t | \leq \text{cst.} \sum_{c^\Delta \varphi_t \in \varepsilon^{1/3}} |c_t^\Delta \varphi_t| \varepsilon^{1/3}, \hspace{1cm} (5.15)$$

$^1$Here and in the sequel, inequalities of the form $A \leq \text{cst.} B$ are to be interpreted in the sense that there exists a constant $k \in \mathbb{R}$ which does not depend on $\varepsilon$ and such that $A \leq kB$.
\begin{equation}
|c^t_\Delta S| \leq \text{cst.} \sum_{c^\Delta \in \mathcal{H}^c \Delta S} |c^\Delta | \varepsilon^{4/3}
\end{equation}

and

\begin{equation}
c^\Delta S, \varphi = -pc^S S (\Delta \varphi_t^2 - (\Delta \varphi_t^+)^2) + \Delta \varphi_t \alpha \varphi_t - \Delta \varphi_t \gamma \varphi_t,
\end{equation}

\begin{equation}
|G_t^{(3)}| \leq \text{cst.} \|c^\Delta \varphi_t \varphi| \varepsilon^{2/3}, \quad |G_t^{(4)}| \leq \text{cst.} \sum_{c^\Delta \varphi \in \mathcal{H}^c \Delta S, \varphi \in \{\alpha \beta^2 \varphi \}} |c^\Delta \varphi| \varepsilon. \end{equation}

For \( \Delta \varphi \) and \( \Delta S \) as in Lemma 5.5 and Corollary 5.6 define

\begin{equation}
\varphi^\varepsilon := \varphi + \Delta \varphi, \end{equation}

\begin{equation}
S^\varepsilon := S + \Delta S. \end{equation}

**Remark 5.7.** Due to (5.9), (5.10), trading with \( \varphi^\varepsilon \) at price \( S^\varepsilon \) without friction or at the bid/ask prices \( S(1 \pm \varepsilon) \) generate the same wealth. \( S^\varepsilon \) serves as a proxy to the so-called shadow price process which corresponds to the dual optimizer in the market with transaction costs, cf. the process \( \hat{R}P = \hat{Z}_1 / \hat{Z}_0 \) in [5] Theorem 6.1.

### 5.2 Primal considerations

With the help of the shadow price process \( S^\varepsilon \), we approximate the expected utility generated by the candidate strategy \( \varphi^\varepsilon \). From [19] Section V.2 we recall the \( S^q \)- and \( H^q \)-norms, \( q \in [1, \infty) \), for an Itô process \( X \):

\begin{equation}
\|X\|_{SV(Q)} := \left\| \sup_{t \in [0,T]} |X_t| \right\|_{L^q(Q)},
\end{equation}

\begin{equation}
\|X\|_{H^q(Q)} := \left\| \int_0^T b_t X_t dt + \sqrt{\int_0^T c_t X_t^2 dt} \right\|_{L^q(Q)}.
\end{equation}

To be more precise, [19] requires \( X_0 = 0 \) in the definition of the \( H^q \)-norm.

**Remark 5.8.** Let \( q \in [1, \infty) \). The following inequalities will be useful.

1. Due to [19] Theorem V.2,

\begin{equation}
\|X\|_{SV(Q)} \leq \text{cst.} \|X\|_{H^q(Q)}.
\end{equation}

holds if \( X_0 = 0 \), where the constant does not depend on \( X \).

2. By convexity of the mapping \( x \mapsto |x|^q \) and Jensen’s inequality, we have

\begin{equation}
\left| \sum_{n=1}^N Y_n \right|^q = N^q \left| \sum_{n=1}^N \frac{Y_n}{N} \right|^q \leq N^{q-1} \sum_{n=1}^N \left| Y_n \right|^q
\end{equation}
for any $N \in \mathbb{N}$ and any random variables $Y_1, \ldots, Y_N$. In particular,

$$
\left\| \sum_{n=1}^{N} X^{(n)} \right\|_q^q \leq \text{cst.} \sum_{n=1}^{N} \left\| X^{(n)} \right\|_q^q.
$$

(5.23)

holds for any $N \in \mathbb{N}$, Itô processes $X^{(1)}, \ldots, X^{(N)}$, and $\| \cdot \|_q \in \{ \| \cdot \|_{S^q(Q)}, \| \cdot \|_{H^q(Q)} \}$.

3. For any $q \in [1, \infty)$ and any $g \in L^q([0, T])$, Hölder’s inequality yields

$$
\left( \int_0^T |g(t)|^q dt \right)^{\frac{1}{q}} \leq \|g\|_{L^q([0, T])} \|1\|_q \leq T^{\frac{1}{q} - 1} \int_0^T |g(t)|^q dt
$$

(5.24)

with $\frac{1}{q} + \frac{1}{r} = 1$. Moreover,

$$
\left\| \sqrt{Y} \right\|_{L^q(Q)}^q = \|Y\|_{L^{q/2}(Q)}^q \leq \|Y\|_{L^q(Q)}^q
$$

(5.25)

for any random variable $Y \in L^q(Q)$.

4. Gathering the above inequalities, we obtain

$$
\left\| \int_0^T \left( \sum_{m=1}^{M} X_t^{(m)} \right) dt + \frac{1}{2} \int_0^T \left( \sum_{n=1}^{N} Y_t^{(n)} \right) dt \right\|_{L^q(Q)}^q \leq \text{cst.} \left( \int_0^T \left( \sum_{m=1}^{M} X_t^{(m)} \right) dt \right\|_{L^q(Q)}^q + \int_0^T \left( \sum_{n=1}^{N} Y_t^{(n)} \right) dt \right\|_{L^q(Q)}^q
$$

(5.22)

$$
\left\| \int_0^T \left( \sum_{m=1}^{M} X_t^{(m)} \right) dt \right\|_{L^q(Q)}^q \leq \text{cst.} \left( \int_0^T \left( \sum_{m=1}^{M} X_t^{(m)} \right) dt \right\|_{L^q(Q)}^q + \int_0^T \left( \sum_{n=1}^{N} Y_t^{(n)} \right) dt \right\|_{L^q(Q)}^q
$$

(5.24)

$$
\left\| \sqrt{Y} \right\|_{L^q(Q)}^q \leq \text{cst.} \left( \int_0^T \left( \sum_{m=1}^{M} X_t^{(m)} \right) dt \right\|_{L^q(Q)}^q + \int_0^T \left( \sum_{n=1}^{N} Y_t^{(n)} \right) dt \right\|_{L^q(Q)}^q
$$

(5.25)

$$
\left\| \int_0^T \left( \sum_{m=1}^{M} X_t^{(m)} \right) dt \right\|_{L^q(Q)}^q \leq \text{cst.} \left( \int_0^T \left( \sum_{m=1}^{M} X_t^{(m)} \right) dt \right\|_{L^q(Q)}^q + \int_0^T \left( \sum_{n=1}^{N} Y_t^{(n)} \right) dt \right\|_{L^q(Q)}^q
$$

(5.22)

for any $M, N \in \mathbb{N}$ and processes $X^{(m)}$, $m = 1, \ldots, M$ and $Y^{(n)}$, $n = 1, \ldots, N$.

**Lemma 5.9.** Assume Condition (5.7).

1. For any stopping time $\tau$ we have

$$
\text{E}_Q \left[ X_{t, \tau}^{\phi, \epsilon} - (x + \phi \cdot S_t) \right] = \rho \text{E}_Q \left[ (2\Delta \phi^2 - (\Delta \phi)^2) \cdot [S, S]_\tau \right] + O(\epsilon),
$$

(5.27)

$$
\text{E}_Q \left[ (X_{t, \tau}^{\phi, \epsilon} - (x + \phi \cdot S_t))^2 \right] = \text{E}_Q [\Delta \phi^2 \cdot [S, S]_\tau] + O(\epsilon),
$$

(5.28)

$$
\left\| X_{t, \tau}^{\phi, \epsilon} - (x + \phi \cdot S) \right\|^3_{S^3(Q)} = O(\epsilon).
$$

(5.29)
2. Define stopping times

\[ \tau^{\varepsilon,1} := \inf \left\{ t \in [0,T] : \left| X_t^{\varphi,\varepsilon} - (x + \varphi \cdot S_t) \right| > 1 \right\}, \tag{5.30} \]

\[ \tau^{\varepsilon,2} := \inf \left\{ t \in [0,T] : \left| X_t^{\varphi,\varepsilon} \right| > \varepsilon^{-4/3} \right\}, \tag{5.31} \]

\[ \tau^{\varepsilon} := \tau^{\varepsilon,1} \wedge \tau^{\varepsilon,2} \wedge T. \tag{5.32} \]

Then

\[ Q(\tau^{\varepsilon} < T) = O(\varepsilon^{4/3}) \tag{5.33} \]

and \( \lim_{\varepsilon \to 0} P(\tau^{\varepsilon} < T) = 0. \)

PROOF. 1. Note that

\[ X^{\varphi,\varepsilon} - (x + \varphi \cdot S) = (\varphi \cdot S^{\varepsilon} - \varphi \cdot S) + X^{\varphi,\varepsilon} - (x + \varphi \cdot S^{\varepsilon}) \]

\[ = \Delta \varphi \cdot S + \varphi \cdot \Delta S + \Delta \varphi \cdot S + (X^{\varphi,\varepsilon} - (x + \varphi \cdot S^{\varepsilon})). \tag{5.34} \]

From (5.21, 5.26) and Condition (5.1), we get

\[ \left\| \Delta \varphi \cdot S \right\|_{L^q(Q)} \leq \text{cst.} \] \[ \left\| \Delta \varphi \cdot S \right\|_{H^q(Q)} \leq \text{cst.} \] \[ \sqrt{\mathbb{E}_Q \left[ \int_0^T \left| \beta_t^2 c_t^{S,S} \right|^q dt \right]} \] \[ \varepsilon^q < \infty \tag{5.35} \]

for \( q \leq 4 \). By letting \( q = 2 \) we deduce that \( \Delta \varphi \cdot S \) is a square-integrable \( Q \)-martingale. Thus

\[ \mathbb{E}_Q[\Delta \varphi \cdot S_\tau] = 0, \tag{5.36} \]

\[ \mathbb{E}_Q[(\Delta \varphi \cdot S_\tau)^2] = \mathbb{E}_Q[\Delta \varphi^2 \cdot [S,S]_\tau] \tag{5.37} \]

for any stopping time \( \tau \). Integration by parts yields

\[ \varphi \cdot \Delta S = \varphi \Delta S - \Delta S \cdot \varphi - [\Delta S, \varphi]. \]

By \( |\Delta S_t| \leq \varepsilon |S| \) and Condition (5.1), we have

\[ \left\| \varphi \Delta S \right\|_{L^q(Q)} \leq \text{cst.} \] \[ \left\| \varphi \Delta S \right\|_{H^q(Q)} \leq \text{cst.} \] \[ \mathbb{E}_Q \left[ \int_0^T |S_t b_t^\varphi|^q dt \right] + \sqrt{\mathbb{E}_Q \left[ \int_0^T |S_t^2 c_t^{\varphi,\varphi}|^q dt \right]} \] \[ \varepsilon^q \tag{5.38} \]

for \( q \leq 4 \). From \( |\Delta S_t| \leq \varepsilon |S| \), (5.25), and Condition (5.1), we obtain

\[ \left\| \Delta S \cdot \varphi \right\|_{H^q(Q)} \leq \text{cst.} \] \[ \left\| \Delta S \cdot \varphi \right\|_{H^q(Q)} \leq \text{cst.} \] \[ \mathbb{E}_Q \left[ \int_0^T |S_t b_t^\varphi|^q dt \right] + \sqrt{\mathbb{E}_Q \left[ \int_0^T |S_t^2 c_t^{\varphi,\varphi}|^q dt \right]} \] \[ \varepsilon^q \tag{5.39} \]

for \( q \leq 4 \).
In view of (5.17) (5.18) (5.26), and Condition (5.1),

\[
E_q[\Delta S, \varphi] = -(E_q[(\Delta \varphi_2 - (\Delta \varphi^+_2) \cdot [S,S]^2] + O(\varepsilon),
\]

\[
\|\Delta S, \varphi\|_{H^q}(Q) \leq \text{cst} \sum_{c, \varphi \in H^q \Delta S, \varphi} E_q \left[ \int_0^T |\Delta \varphi| dt \right] e^{\frac{2q}{3}}
\]

(5.41)

holds for any stopping time \( \tau \) and \( q \leq 4 \). For any stopping time \( \tau \) and \( q \leq 4 \), (5.38) (5.41) then yield

\[
E_q[\varphi \cdot \Delta S] = pE_q \left[ (\Delta \varphi^2 - (\Delta \varphi^+_2) \cdot [S,S]^2] + O(\varepsilon),
\right.

\[
\|\varphi \cdot \Delta S\|_{S^q}(Q) = O(e^{2q/3}).
\]

(5.43)

From (5.13) (5.14) (5.16) (5.26), and Condition (5.1), we obtain

\[
E_q[\Delta \varphi \cdot \Delta S] = pE_q \left[ \Delta \varphi^2 \cdot [S,S]^2] + O(\varepsilon),
\right.

\[
\|\Delta \varphi \cdot \Delta S\|_{H^q}(Q) \leq \text{cst} \sum_{b \Delta \varphi \in H^q \Delta S} E_q \left[ \int_0^T |\beta_t b \Delta \varphi| dt \right] e^{\frac{2q}{3}}
\]

(5.45)

for any stopping time \( \tau \) and \( q \leq 4 \). Due to Remark 2.2 and (5.9) (5.10),

\[
\left| X_{t}^{(e)} - (x + \varphi^e \cdot S_t^e) \right| \leq 2|\varphi^e| |S_t^e| e + x^5 e
\]

\[
\leq \text{cst} \left( |\varphi| S_t^e + |\beta_t b_t| + x^5 \right) e
\]

(5.46)

holds for any \( t \in [0, T] \). So by (5.23) and Condition (5.1), we obtain

\[
\left\| X^{(e)} - (x + \varphi^e \cdot S^e) \right\|_{S^q(Q)} \leq \text{cst} E_q \left[ \sup_{t} |\varphi_t S_t^e| q + \sup_{t} |\beta_t | S_t^e| |q + x^5 e \right] e^{q/3}
\]

(5.47)

for \( q \leq 4 \). In view of (5.23), relations (5.34) and (5.36) (5.42) (5.44) (5.47) for \( q = 1 \) imply (5.27). (5.34) and (5.37) (5.43) (5.45) (5.47) for \( q = 2 \) yield (5.28). (5.34) and (5.33) (5.43) (5.45) (5.47) for \( q = 3 \) imply (5.29).

2. (5.30), Markov’s inequality, and (5.23) (5.34) (5.35) (5.43) (5.45) (5.47) yield

\[
Q(\tau^{e,1} < T) \leq Q \left( \sup_{t \in [0, T]} |X_{t}^{(e)} - (x + \varphi \cdot S_t)| > 1 \right)
\]

\[
\leq \left\| X^{(e)} - (x + \varphi \cdot S) \right\|_{S^4(Q)}^4
\]

\[
= O(e^{4/3}).
\]
Condition (3.2) implies that the $Q$-martingale $\varphi \cdot S$ is in fact square-integrable and hence $E_Q[\varphi^2 \cdot [S,S]_T] < \infty$. Moreover, from (5.34), Markov’s inequality, (5.21, 5.24), and (5.26) for $q = 1$, we get

$$Q(\tau^{e,2} < T) \leq Q \left( \sup_{t \in [0,T]} |X_{t}^{\varphi^{e},e}| > \varepsilon^{-4/3} \right)$$

$$\leq \|X^{\varphi^{e}}\|_{S^1(Q)}^4$$

$$\leq \left( \|x + \varphi \cdot S\|_{S^1(Q)} + \|X^{\varphi^{e},e} - (x + \varphi \cdot S)\|_{S^1(Q)} \right) \varepsilon^{4/3}$$

$$\leq \text{cst.} \left[ 1 + \sqrt{E_Q[\varphi^2 \cdot [S,S]_T]} + \|X^{\varphi^{e},e} - (x + \varphi \cdot S)\|_{S^1(Q)} \right] \varepsilon^{4/3}. \leq_{\infty}$$

Therefore, $Q(\tau^{e} < T) \leq Q(\tau^{e,1} < T) + Q(\tau^{e,2} < T) = O(\varepsilon^{4/3}).$ Since $P$ and $Q$ are equivalent, $Q(\tau^{e} < T) \to 1$ implies $P(\tau^{e} < T) \to 1$. \hfill \square

**Lemma 5.10.** Suppose that Conditions (5.7, 3.2) hold. For $\tau^{e}$ as in (3.3) resp. (5.32), we have

$$E \left[ U \left( X_{T}^{\varphi^{e},e} \right) \right] = E[U(x + \varphi \cdot S_T)] - yP E_Q \left[ \left( \Delta \varphi^{+} \right)^2 - \frac{3}{2} \Delta \varphi^2 \right] \cdot [S,S]_T + O(\varepsilon). \tag{5.48}$$

**Proof.** Let $Z$ denote the density process of $Q$. Taylor expansion of $U(x) = -e^{-px}$ yields

$$U \left( X_{T}^{\varphi^{e},e} \right) = U(x + \varphi \cdot S_T)$$

$$+ \frac{1}{2} p Z_T \left( X_{T}^{\varphi^{e},e} - (x + \varphi \cdot S_T) \right)^2 = G^{(5)}$$

$$+ \frac{p^2}{6} Z_T \exp \left( -p \theta \left( X_{T}^{\varphi^{e},e} - (x + \varphi \cdot S_T) \right) \right) \left( X_{T}^{\varphi^{e},e} - (x + \varphi \cdot S_T) \right)^3 = G^{(7)}$$

for some random $\theta \in (0, 1)$. Notice that

$$X_{T}^{\varphi^{e},e} - (x + \varphi \cdot S_T) = \left( X_{T}^{\varphi^{e},e} - (x + \varphi \cdot S_{\varepsilon e}) \right) + (\varphi \cdot S_{\varepsilon e} - \varphi \cdot S_T) \tag{5.49}$$

with

$$\left| X_{T}^{\varphi^{e},e} - (x + \varphi \cdot S_{\varepsilon e}) \right| \leq 1, \tag{5.50}$$

and

$$E_Q \left[ |\varphi \cdot S_{\varepsilon e} - \varphi \cdot S_T|^{2n} \right] \leq \text{cst.} \left( E_Q \left[ |\varphi \cdot S_{\varepsilon e}|^{2n} \right] + E_Q \left[ |\varphi \cdot S_T|^{2n} \right] \right)$$

$$\leq \text{cst.} E_Q \left[ |\varphi \cdot S_T|^{2n} \right] < \infty \tag{5.51}$$
for \( n \in \mathbb{N} \), where the first inequality is due to the fact that \( |\varphi \cdot S|^{2n} \) is a \( Q \)-submartingale. In view of (5.49), Hölder’s inequality, and (5.33, 5.51, 5.50), we have

\[
\begin{align*}
E[G^{(5)}] - yE_Q \left[ X_{\tau^e}^{\varphi, \varepsilon} - (x + \varphi \cdot S_T) \right] &\leq yE_Q \left[ 1_{\{ \tau^e < T \}} |\varphi \cdot S_{\tau^e} - \varphi \cdot S_T | \right] \\
&\leq yQ(\tau^e < T)^{\frac{3}{2}} \sqrt{E_Q \left[ |\varphi \cdot S_{\tau^e} - \varphi \cdot S_T |^4 \right]} \\
&= O(\varepsilon),
\end{align*}
\]

and

\[
\begin{align*}
E[G^{(6)}] - \frac{p}{2} yE_Q \left[ (X_{\tau^e}^{\varphi, \varepsilon} - (x + \varphi \cdot S_T))^2 \right] &\leq \text{cst.} \left( E_Q \left[ 1_{\{ \tau^e < T \}} \varphi \cdot S_{\tau^e} - \varphi \cdot S_T^2 \right] + E_Q \left[ 1_{\{ \tau^e < T \}} \varphi \cdot S_{\tau^e} - \varphi \cdot S_T \right] \right) \\
&\leq \text{cst.} Q(\tau^e < T)^{\frac{3}{2}} \sqrt{E_Q \left[ |\varphi \cdot S_{\tau^e} - \varphi \cdot S_T |^8 \right]} + O(\varepsilon) \\
&= O(\varepsilon).
\end{align*}
\]

From (5.50) and (5.29), we obtain

\[
E[G^{(7)}] = \text{cst.} E_Q \left[ \exp \left( - p \theta (\varphi \cdot S_{\tau^e} - \varphi \cdot S_T) \right) \right] \left| X_{\tau^e}^{\varphi, \varepsilon} - (x + \varphi \cdot S_T) \right|^3 \\
= O(\varepsilon)
\]

where

\[
G^{(8)} = (\sqrt{G^{(9)}})^{\frac{1}{2}}
\]

with

\[
G^{(9)} = \sqrt{E_Q \left[ \exp \left( - 4p \theta (\varphi \cdot S_{\tau^e} - \varphi \cdot S_T) \right) \right] \left| X_{\tau^e}^{\varphi, \varepsilon} - (x + \varphi \cdot S_T) \right|^{12}}.
\]

Again by Hölder’s inequality,

\[
G^{(9)} \leq G^{(10)} = \sqrt[36]{E_Q \left[ \left| X_{\tau^e}^{\varphi, \varepsilon} - (x + \varphi \cdot S_T) \right|^{108}} \right]^{\frac{1}{3}}
\]

with

\[
G^{(10)} = \sqrt[3]{E_Q \left[ \exp \left( - \frac{9}{2} p \theta (\varphi \cdot S_{\tau^e} - \varphi \cdot S_T) \right) \right]}
\]

\[
\leq \sqrt[3]{E_Q \left[ \exp(-9p\theta \varphi \cdot S_{\tau^e}) \right]} \sqrt[3]{E_Q \left[ \exp(9p\theta \varphi \cdot S_T) \right]}.
\]
Since $\phi \cdot S$ is a $Q$-martingale, $\exp(-9p\theta \phi \cdot S)$ is a $Q$-submartingale by Jensen’s inequality. Hence
\[
E_Q[\exp(-9p\theta \phi \cdot S_{t^e})] \leq E_Q[\exp(-9p\theta \phi \cdot S_T)] \quad (3.2)
\]
Therefore
\[
E[G^{(7)}] = O(\varepsilon). \quad (5.54)
\]
Combining (5.52, 5.53, 5.54) with (5.27, 5.28), we obtain
\[
E\left[U(X_{t^e}^{\phi, \varepsilon})\right] - ypE_Q\left[(\Delta_\phi^+)^2 - \frac{3}{2}\Delta^2_\phi \cdot [S, S]_{t^e}\right] + O(\varepsilon).
\]
Moreover, Hölder’s inequality and (5.33) yield
\[
\left|E_Q \left[\int_{t^e}^{T} \left(\Delta_\phi^+\right)^2 - \frac{3}{2}\Delta^2_\phi \right] d[S, S]_t\right|
\leq \text{cst.} Q(\tau^e < T) \frac{4}{3} \sqrt{E_Q [((\Delta_\phi^+)^2 \cdot [S, S]_T]^4]}, \quad (5.55)
\]
where
\[
G^{(11)} \leq \text{cst.} \left[\int_0^T |\beta^2 c\Delta S, S|_t^4 dt\right]^{2/3}. \quad (5.58)
\]
This completes the proof. \hfill \Box

### 5.3 Dual considerations

In order to obtain an approximate upper bound to the maximal expected utility, we construct a dual variable based on Girsanov’s theorem. More specifically, we consider the minimal martingale measure for the appropriately stopped process $S^\varepsilon$ relative to $Q$. This martingale measure turns out to be optimal to the leading order.

Let
\[
\rho^{\varepsilon, 1} := \inf \left\{ t \in [0, T] : \left| \frac{c_t^{\Delta S, S}}{c_t^{S, S}} \right| > \frac{1}{2} \right\} \quad (5.56)
\]
and define $Z^{\varepsilon, Q} := \exp(N^\varepsilon)$ with
\[
N^\varepsilon := -\int_0^t \theta^\varepsilon dS_t - \frac{1}{2} \int_0^t (\theta^\varepsilon)^2 d[S, S]_t, \quad (5.57)
\]
where
\[
\theta^\varepsilon := \frac{b^{S^\varepsilon}}{c^{S^\varepsilon, S}} 1_{[0, \rho^{\varepsilon, 1}]} = \frac{b^{\Delta S}}{c^{S, S} + c^{\Delta S, S}} 1_{[0, \rho^{\varepsilon, 1}]} \cdot (5.58)
\]
Furthermore, let
\[ \rho^{e,2} := \inf \left\{ t \in [0, T] : |Z_t^e - 1| > \frac{1}{2} \right\}, \quad (5.59) \]
\[ \rho^e := \rho^{e,1} \wedge \rho^{e,2} \wedge T \quad (5.60) \]
and define the “stopped” processes
\[ \overline{\phi}^e := \phi^e 1_{[0, \tau^e \wedge \rho^e]}, \quad \Delta \overline{\phi}^e := \Delta \phi^e 1_{[0, \tau^e \wedge \rho^e]}, \]
\[ \overline{S}^e := S \left( 1 + \frac{\Delta S^{\tau^e \wedge \rho^e}}{S^{\tau^e \wedge \rho^e \wedge \rho^e}} \right), \quad \overline{N}^e := (N^e)^{\tau^e \wedge \rho^e}, \quad \overline{Z}^e := (Z^e, Q)^{\tau^e \wedge \rho^e}. \quad (5.61) \]

**Remark 5.11.** By construction, \( \overline{Z}^e \) is a bounded \( Q \)-local martingale and hence a \( Q \)-martingale. If \( Z \) denotes the density process of \( Q \), the process \( Z^e := \overline{Z}^e \) is a \( P \)-martingale. Integration by parts yields that \( \overline{Z}^e \overline{S}^e \) is a \( Q \)-local martingale and hence \( Z^e \overline{S}^e = Z^e \overline{S}^e \) is a \( P \)-local martingale. Consequently, \( \overline{S}^e \) a local martingale under the probability measure with density process \( Z^e \). Therefore, \( Z^e \) corresponds to an equivalent (local) martingale measure for \( \overline{S}^e \). It serves as a dual variable in the frictionless market with shadow price \( \overline{S}^e \).

**Lemma 5.12.**
1. Conditions (5.2) (5.3) imply
\[ E_Q \left[ (Z_T^e - 1)^2 \right] = p^2 E_Q \left[ [\overline{\phi}^2] \cdot [S, S_T] \right] + O(\varepsilon) \quad (5.62) \]
and
\[ E_Q \left[ Z_T^e - 1 \right] = O(\varepsilon). \quad (5.63) \]
2. Condition (5.2) implies
\[ Q(\rho^{e,1} < T) = O(\varepsilon^{4/3}). \quad (5.64) \]
3. Conditions (5.2) (5.3) imply
\[ Q(\rho^{e,2} < T) = O(\varepsilon^{4/3}). \quad (5.65) \]

**Proof.** Note that on \( \{ \rho^e \geq t \} \), (5.13) (5.14) (5.15) (5.56) (5.58) yield
\[ \theta_t^e \sigma_t^S = p \Delta \phi_t \sigma_t^S + \frac{b_t^{\Delta S} \sigma_t^S}{\sigma_t^S + \sigma_t^{\Delta S}} \quad (5.66) \]
\[ |\theta_t^e \sigma_t^S| \leq \text{cst.} \sum_{b \in B} \left| \frac{b_t^B}{\sigma_t^S} \right| e^{1/3}, \quad |p \Delta \phi \sigma^S| \leq \text{cst.} \beta \sigma^S e^{1/3}, \quad (5.67) \]
\[ |G_t^{(12)}| \leq \text{cst.} \sum_{b \in B} \left| \frac{b_t^B}{\sigma_t^S} \right| \left( 1 + \sum_{c \in P, c \in C} \left| \frac{c_t^S}{\sigma_t^S} \right| \right) e^{2/3}, \quad (5.68) \]
From \((5.59)\) and Taylor expansion of \(x \mapsto e^x\), we obtain
\[
\mathcal{Z}^\varepsilon - 1 = -p\Delta \overline{\varepsilon} \cdot S + (\overline{\mathcal{N}}^\varepsilon + p\Delta \overline{\varepsilon} \cdot S) + G^{(13)} \text{ with } |G^{(13)}| \leq \frac{3}{4} |\overline{\mathcal{N}}^\varepsilon|^2. \tag{5.69}
\]

1. From \((5.69), (5.23)\) for \(q = 2, (5.21)\) for \(q = 2, 4, (5.66, 5.67, 5.68, 5.26)\), Cauchy-Schwarz, and Conditions \((5.2, 5.3)\), we deduce that
\[
\mathbb{E}_Q\left[(\mathcal{Z}^\varepsilon_T - 1 + p\Delta \overline{\varepsilon} \cdot S)T\right]^2 \leq \text{cst.} |\overline{\mathcal{N}}^\varepsilon + p\Delta \overline{\varepsilon} \cdot S|_H^2(Q) + \text{cst.} |\overline{\mathcal{N}}^\varepsilon|^4_{H^2(Q)}
\]
\[
\leq \text{cst.} \sum_{m=1,2} \sum_{n=1,2} \left( \sum_{b^\Lambda \in \mathcal{B}_{\mathcal{H},l}} \mathbb{E}_Q \left[ \int_0^T \left| \frac{b^\Lambda}{\sigma^l} \right|^{8m} dt \right] \right) \frac{1}{m} \varepsilon^{4/3}.
\tag{5.70}
\]

Combined with \((5.37)\) and Cauchy-Schwarz’ inequality, this yields \((5.62)\). Analogously, from \((5.69), (5.23)\) for \(q = 3, (5.21)\) for \(q = 3, 6, (5.66, 5.67, 5.26)\), and Condition \((5.3)\), we get
\[
\mathbb{E}_Q\left[(\mathcal{Z}^\varepsilon_T - 1)^3 \right] \leq \text{cst.} |\overline{\mathcal{N}}^\varepsilon|^3_{H^3(Q)} + \text{cst.} |\overline{\mathcal{N}}^\varepsilon|^6_{H^6(Q)}
\]
\[
\leq \text{cst.} \sum_{m=1,2} \sum_{n=1,2} \left( \sum_{b^\Lambda \in \mathcal{B}_{\mathcal{H},l}} \mathbb{E}_Q \left[ \int_0^T \left| \frac{b^\Lambda}{\sigma^l} \right|^{6m} dt \right] \right) \frac{1}{m} \varepsilon.
\]

2. By \((5.56)\), Markov’s inequality, \((5.15), (5.22)\) for \(q = 2\), and Condition \((5.2)\), it holds that
\[
\mathbb{Q}(\rho^{\varepsilon,1} < T) \leq 4 \left\| \frac{\Delta S}{\sigma^l} \right\|_{S^2(Q)}^2 \leq \text{cst.} \sum_{\sigma^\Lambda \in \mathcal{B}_{\mathcal{H},l}} \left\| \frac{\Delta S}{\sigma^l} \right\|_{S^2(Q)}^2 \varepsilon^{4/3}.
\]

3. Notice that
\[
\mathbb{Q}(\rho^{\varepsilon,2} < T) = \mathbb{Q}(\{\rho^{\varepsilon,2} < T\} \cap \{\rho^{\varepsilon,2} < \rho^{\varepsilon,1}\}) + \mathbb{Q}(\rho^{\varepsilon,1} \leq \rho^{\varepsilon,2} < T),
\]
where
\[
\mathbb{Q}(\rho^{\varepsilon,1} \leq \rho^{\varepsilon,2} < T) \leq \mathbb{Q}(\rho^{\varepsilon,1} < T) \leq O(\varepsilon^{4/3}).
\]

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From (5.59), Markov’s inequality, (5.21) for $q = 6$, (5.66, 5.67, 5.26), and Conditions (5.2, 5.3), we deduce that

$$Q\left(\{\rho^{e,2} < T\} \cap \{\rho^{e,1} < \rho^{e,1}\}\right) \leq Q\left(\exists t \leq T : |Z_t^{e} - 1| > \frac{1}{2}\right) \leq Q\left(\exists t \leq T : |N_t^{e} - 1| > \ln \frac{3}{2}\right) \leq \text{cst.} E_Q \left[\sup_{t \in [0,T]} |N_t^{e}|^6\right] \leq \text{cst.} \|N^{e}\|_{H^6(Q)}^6 \leq \text{cst.} \sum_{n=1,2} \left(\sum_{b \in h \neq b^G} \text{E}_Q \left[\int_0^T \frac{b^A}{\sigma^G} \right]^2 dt\right)^{\frac{1}{2n}} \varepsilon^2,$$

which implies (5.65).

Now, let us pass to the convex duality theory. We denote by $\tilde{U}$ the conjugate function of $U$, i.e.,

$$\tilde{U}(y) := \sup_{x \in \mathbb{R}} (U(x) - xy), \quad y \geq 0,$$  \hspace{1cm} (5.71)

which satisfies $-\tilde{U}' = (U')^{-1}$. Since $U(x) = -e^{-px}$, we obtain

$$\tilde{U}'(y) = \frac{1}{p} \ln \frac{y}{p}, \quad \tilde{U}''(y) = \frac{1}{py^2}, \quad \tilde{U}'''(y) = -\frac{1}{py^3}. \hspace{1cm} (5.72)$$

**Lemma 5.13.** Conditions (5.7-5.3) imply

$$E \left[\tilde{U}(yZ_T^e)\right] + xy = E[U(x + \phi \cdot S_T)] - xy, \quad (5.73)$$

**Proof.** By (5.72), Taylor expansion of $\tilde{U}$ yields

$$\tilde{U}(yZ_T^e) = \tilde{U}(yZ_T) + \tilde{U}'(yZ_T)(yZ_T^e - 1) + \frac{1}{2} \tilde{U}''(yZ_T)(yZ_T^e - 1)^2$$

$$- \frac{1}{6p} (1 + \theta(Z_T^e - 1))^{-1} yZ_T^e (Z_T^e - 1)^3$$

for some random $\theta \in (0, 1)$. Due to the optimality of $\phi$ and $yZ_T$ as well as by conjugate relations (cf. [21, Theorem 2.2]), we conclude

$$E[\tilde{U}(yZ_T)] = E[U(x + \phi \cdot S_T)] - xy, \hspace{1cm} (5.74)$$

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\[-\bar{U}'(yZ_T) = x + \phi \cdot S_T,\]
\[\bar{U}''(yZ_T)(yZ_T)^2 = \frac{1}{p}yZ_T,\]
and \(\phi \cdot S\) is a \(Q\)-martingale, i.e.,
\[\mathbb{E}_Q[x + \phi \cdot S_T] = x.\]  
(5.77)

(5.75, 5.77) yield
\[\mathbb{E}_{\bar{\Delta}_1} = y\mathbb{E}_Q[x + \phi \cdot S_T] - xy - y\mathbb{E}_Q[Z^e_T(\phi \cdot S_T)] = 0\]
and
\[-\mathbb{E}_Q[Z^e_T(\phi \cdot S_T)] = -\mathbb{E}_Q[Z^e_T(\bar{\phi}^e \cdot \bar{S}_T)] + \mathbb{E}_Q[\bar{\phi}^e \cdot \bar{S}_T - \phi \cdot S_T] + \mathbb{E}_Q[Z^e_T - 1)(\bar{\phi}^e \cdot \bar{S}_T - \phi \cdot S_T]).\]

\(Z^e_T(\bar{\phi}^e \cdot \bar{S}_T)\) is a \(Q\)-local martingale, cf. Remark 5.11. (5.59, 5.46, 5.31) and Condition (5.1) yield
\[\mathbb{E}_{S^i(Q)} \left\|Z^e(\bar{\phi}^e \cdot \bar{S}_T)\right\| \leq \text{cst.} \left( e^{-4/3} + \mathbb{E}_Q \left[ \sup_{t \in [0,T]} |\phi_t S_t| + \sup_{t \in [0,T]} |\beta_t S_t| \right] \right) < \infty,\]
which implies that \(Z^e(\bar{\phi}^e \cdot \bar{S}_T)\) is a uniformly integrable \(Q\)-martingale and hence
\[\mathbb{E}_Q[Z^e_T(\bar{\phi}^e \cdot \bar{S}_T)] = 0.\]

(5.36, 5.42, 5.44, 5.52) in conjunction with (5.33, 5.64, 5.65) and the argument in (5.52) yield
\[\mathbb{E}_Q[\bar{\phi}^e \cdot \bar{S}_T - \phi \cdot S_T] = p\mathbb{E}_Q \left[ (2\Delta \phi^2 - (\Delta \phi^+)^2) \cdot [S, S]_{\tau \wedge \rho} \right] + O(\epsilon).\]

We have
\[\mathbb{E}_Q \left[ (Z^e_T - 1)(\bar{\phi}^e \cdot \bar{S}_T - \phi \cdot S_T) \right] = p\mathbb{E}_Q \left[ (\Delta \phi^2 - (\Delta \phi^+)^2) \cdot [S, S]_{\tau \wedge \rho} \right] + O(\epsilon).

From (5.37), Cauchy-Schwarz’ inequality and (5.70), we conclude
\[G^{(17)} = -p\mathbb{E}_Q \left[ (\Delta \bar{\phi} \cdot S_T)^2 \right] + \mathbb{E}_Q \left[ (\bar{Z}^e_T - 1 + p\Delta \bar{\phi} \cdot S_T)(\Delta \bar{\phi} \cdot S_T) \right] = -p\mathbb{E}_Q \left[ \Delta \bar{\phi}^2 \cdot [S, S]_T \right] + O(\epsilon).\]
By Cauchy-Schwarz’ inequality, (5.59, 5.62), using (5.43, 5.45, 5.23) for \( q = 2 \), and combining the argument in (5.52) with (5.64, 5.65), it follows that

\[
|G^{(18)}| \leq \text{cst.} \sqrt{E_Q \left[ (Z_T^e - 1)^2 \right]} \sqrt{E_Q \left[ (\phi^e \cdot \Delta S_T)^2 \right]}
+ \text{cst.} E_Q \left[ 1_{\{ \tau^e \wedge \rho^e < T \}} |\phi \cdot S_{\tau^e \wedge \rho^e} - \phi \cdot S_T | \right]
= O(\varepsilon).
\]

Together, we obtain

\[
E \left[ G^{(14)} \right] = yp E_Q \left[ (\Delta \phi^2 - (\Delta \phi^+)^2) \cdot [S, S]_{\tau^e \wedge \rho^e} \right] + O(\varepsilon). \tag{5.78}
\]

(5.76) and (5.62) yield

\[
E \left[ G^{(15)} \right] = \frac{y}{2p} E_Q \left[ (\overline{Z}_T^e - 1)^2 \right]
= \frac{yp}{2} E_Q \left[ \Delta \phi^e \cdot [S, S]_T \right] + O(\varepsilon). \tag{5.79}
\]

By (5.59) and (5.63), we have

\[
\left| E \left[ G^{(16)} \right] \right| \leq \text{cst.} E_Q \left[ |\overline{Z}_T^e - 1|^3 \right] = O(\varepsilon). \tag{5.80}
\]

From (5.74, 5.78, 5.79, 5.80) we obtain

\[
E \left[ \tilde{U} (yZ_T \overline{Z}_T^e) \right] + xy
= E \left[ U (x + \phi \cdot S_T) \right] - yp E_Q \left[ \left( (\Delta \phi^+)^2 - \frac{3}{2} \Delta \phi^2 \right) \cdot [S, S]_{\tau^e \wedge \rho^e} \right] + O(\varepsilon).
\]

Combining this with (5.64, 5.65) and the argument in (5.55), the assertion follows. \( \square \)

### 5.4 Optimality

Having approximated both the primal and dual value of the optimization problem, we are now able to prove the leading-order optimality of the candidate strategy \( \phi^e \).

**Lemma 5.14.** Under Assumption 5.7, we have

\[
\sup_{\psi \in \mathcal{A}_e(x^B, x^S)} E \left[ U (X_t^{\psi, e}) \right] = E \left[ U (X_t^{\phi^e, e}) \right] + O(\varepsilon).
\]

**Proof.** Take an arbitrary admissible trading strategy \( \psi \in \mathcal{A}_e(x^B, x^S) \) and let \( S^e, Z^e \) be as in (5.61). Since \( S^e \) has values in \([ (1 - \varepsilon)S, (1 + \varepsilon)S ] \), we have

\[
x + \psi \cdot S^e \geq X_t^{\psi, e} \geq -K \quad \text{a.s.} \tag{5.81}
\]

for some \( K \in \mathbb{R}_+ \). Recalling Remark 5.11, \( Z^e (x + \psi \cdot S^e) \) is a \( Q \)-local martingale and hence a \( Q \)-supermartingale by (5.81). Therefore

\[
E_Q \left[ Z_T^e (x + \psi \cdot S_T^e) \right] \leq x. \tag{5.82}
\]
Together, we conclude

\[
\begin{align*}
\mathbb{E} \left[ U \left( X_{T}^\psi, \varepsilon \right) \right] & \overset{(5.81)}{\leq} \mathbb{E} \left[ U \left( x + \psi \cdot S_{T} \right) \right] \\
& \overset{(5.71)}{\leq} \mathbb{E} \left[ \tilde{U} \left( yZ_{T} \tilde{Z}_{T}^\varepsilon \right) \right] + y\mathbb{E}_{Q} \left[ \tilde{Z}_{T}^\varepsilon \left( x + \psi \cdot S_{T} \right) \right] \\
& \overset{(5.82)}{\leq} \mathbb{E} \left[ \tilde{U} \left( yZ_{T} \tilde{Z}_{T}^\varepsilon \right) \right] + xy \\
\overset{\text{Lemma } 5.13}{=} & \mathbb{E} \left[ U \left( x + \varphi \cdot S_{T} \right) \right] \\
& - y\mathbb{E}_{Q} \left[ \left( (\Delta \varphi^+)^2 - 2\varepsilon \varphi^2 \right) \cdot [S, S]_{T} \right] + O(\varepsilon) \\
\overset{\text{Lemma } 5.10}{=} & \mathbb{E} \left[ U \left( X_{T}^\varphi, \varepsilon \right) \right] + O(\varepsilon).
\end{align*}
\]

Since \( \varphi^\varepsilon 1_{[0, \tau^\varepsilon]} \in \mathcal{A}^\varepsilon(x^B, x^S) \) by definition, this proves the assertion. \( \square \)

### 5.5 Certainty equivalent loss

In this section we express the minimal loss of utility caused by transaction costs in terms of \( \Delta \varphi^+ \) rather than both \( \Delta \varphi^+ \) and \( \Delta \varphi \), cf. (5.48, 5.73). Throughout this section, we suppose that Assumption 5.2 holds.

Set

\[ q := \frac{\Delta \varphi}{\Delta \varphi^+}. \]

Then \( q \) is a semimartingale reflected to stay between \( \pm 1 \). By Itô’s formula, its dynamics are characterized by

\[ dq_t = b_t^q dt + dM_t^Q + dA_t^+ - dA_t^-, \]

where \( M^Q \) is a continuous \( Q \)-local martingale starting in 0, processes \( A^+ \) and \( A^- \) are increasing processes which grow only on \( \{ q = -1 \} \) and \( \{ q = 1 \} \), respectively, and

\[
\begin{align*}
b_t^q &= -\left( \frac{2}{\varepsilon} \right)^{1/3} \left( \frac{b_t^\varphi}{\beta_t^2} + c_t^{q,1/\beta} \right) - \frac{q_t}{\beta_t} b_t^\beta + \frac{q_t}{\beta_t^2} c_t^\beta b_t^\beta, \\
c_t^{q,q} &= \frac{d[q, q]_t}{dt} = \left( \frac{2}{\varepsilon} \right)^{2/3} c_t^{q,1/\beta} + \left( \frac{16}{\varepsilon} \right)^{1/3} \frac{q_t}{\beta_t} c_t^\beta + \frac{q_t^2}{\beta_t^2} c_t^\beta b_t^\beta.
\end{align*}
\]

Define a stopping time \( \sigma^\varepsilon := \inf \{ t \in [0, T] : |b_t^q| > \frac{1}{\varepsilon} \text{ or } c_t^{q,q} < \varepsilon \} \cap T \) and let \( \overline{\sigma} := q^{\sigma^\varepsilon} \).

#### 5.5.1 Time change

Fix \( t \in [0, T) \). Consider the time change \( (t(\overline{\sigma}))_{\overline{\sigma} \in \mathbb{R}_+} \) defined by

\[ t(\overline{\sigma}) := \inf \{ s \in [t, T] : [q, q]_t - [q, q]_s > \vartheta \} \cap \sigma^\varepsilon. \]

Set

\[ \overline{\vartheta} := [\overline{q}, \overline{q}]_T - [\overline{q}, \overline{q}]_t = \int_t^T c_t^{\overline{q}, \overline{q}} ds. \]
and \( \overline{q}_\theta := \overline{q}_{t(\theta)} \) for \( \theta \in \mathbb{R}_+ \).

Fix \( \omega \in \Omega \). For \( \epsilon > 0 \) small enough we have that \( e^{\theta q}(\omega) \) exceeds \( \epsilon \) on \( [0, T] \). Therefore, the mapping \( \psi \mapsto t(\psi) \) is continuously differentiable on the interval \( (0, \overline{\theta}) \) with derivative \( (c_{t(\theta)})^{-1} \).

**Lemma 5.15.** Recall that Assumption 5.2 is supposed to hold. Setting

\[
\psi^\epsilon := \int_t^{(t+\epsilon^{1/3})\wedge T} c_{\epsilon^2} \, ds,
\]

we have

\[
\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon^{1/3}} \int_t^{(t+\epsilon^{1/3})\wedge T} \left( \psi^\epsilon \right)^2 c_{\epsilon^2} \, ds = \frac{1}{\psi^\epsilon} \int_0^{\psi^\epsilon} \psi^\epsilon \, d\psi \left( \frac{\Delta \varphi^+}{\epsilon^{1/3}} \right)^2 c_{\epsilon^2} \, ds = 0 \quad \text{a.s.} \quad (5.85)
\]

For any \( \omega \in \Omega \) there exists some \( \epsilon_0(\omega), K(\omega), \overline{K}(\omega) > 0 \) such that

\[
K(\omega)\epsilon^{-1/3} \leq \psi^\epsilon(\omega) \leq \overline{K}(\omega)\epsilon^{-1/3}
\]

holds for any \( \epsilon \leq \epsilon_0(\omega) \).

**Proof.** Fix \( \omega \in \Omega \) and consider events

\[
A^{\epsilon,b} := \left\{ \exists t \in [0, T] : |b_t^\epsilon| > \frac{1}{\epsilon} \right\}, \quad A^{\epsilon,c} := \left\{ \exists t \in [0, T] : c_{\epsilon t}^q < \epsilon \right\}.
\]

Since all processes in (5.83) are assumed to have continuous or at least bounded paths, there exists \( C(\omega) < \infty \) such that

\[
\sup_{t \in [0, T]} |b_t^\epsilon(\omega)| \leq C(\omega)\epsilon^{-1/3},
\]

whence \( \omega \notin A^{\epsilon,b} \) for any \( \epsilon \) that is small enough. Similarly, there exists \( c(\omega) > 0 \) such that

\[
\min_{t \in [0, T]} \frac{c_{\epsilon t}^q}{\beta_t}(\omega) > c(\omega)
\]

and hence \( \omega \notin A^{\epsilon,c} \) for any \( \epsilon \) that is small enough. Therefore \( \sigma^\epsilon(\omega) = T \) for \( \epsilon \) small enough, which implies that

\[
\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon^{1/3}} \int_t^{(t+\epsilon^{1/3})\wedge T} \left( \overline{q}_s^2 - \overline{q}_s^2 \right) \left( \Delta \varphi^+ \right)^2 c_{\epsilon^2} \, ds = 0. \quad (5.87)
\]

By continuity of the mapping \( s \mapsto \left( \frac{\Delta \varphi^+}{\epsilon^{1/3}} \right)^2 c_{\epsilon^2} \) at \( t \) and using the mean value theorem, we have

\[
\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon^{1/3}} \int_t^{(t+\epsilon^{1/3})\wedge T} \left( \left( \frac{\Delta \varphi^+}{\epsilon^{1/3}} \right)^2 c_{\epsilon^2} - \left( \frac{\Delta \varphi^+}{\epsilon^{1/3}} \right)^2 c_{\epsilon^2} \right) \, ds = 0. \quad (5.88)
\]
Applying the mean value theorem to the mapping \( t \mapsto t(\vartheta) \), we get
\[
\varepsilon^{1/3} = (t(\vartheta^\varepsilon) - t(0))(\omega) = \left( (\vartheta^\varepsilon_{t(\vartheta^\varepsilon)} - \vartheta^\varepsilon_{t(0)})^{-1} \right)(\omega) \text{ for some } \xi \in [0, \vartheta^\varepsilon(\omega)]
\] (5.89)
for \( \varepsilon \) small enough and
\[
\lim_{\varepsilon \downarrow 0} \left| \frac{1}{\varepsilon^{1/3}} \int_0^{\vartheta^\varepsilon_t} \vartheta^\varepsilon_t \left( (\vartheta^\varepsilon_{t(\vartheta^\varepsilon)} - \vartheta^\varepsilon_{t(0)})^{-1} \right)^{-1} d\vartheta^\varepsilon_t(\omega) = 0.
\]
Change of variables yields
\[
\lim_{\varepsilon \downarrow 0} \left| \frac{1}{\varepsilon^{1/3}} \int_t^{(t+\varepsilon^{1/3})\wedge T} \frac{1}{\vartheta^\varepsilon_t} \left( (\vartheta^\varepsilon_{t(\vartheta^\varepsilon)} - \vartheta^\varepsilon_{t(0)})^{-1} \right)^{-1} d\vartheta^\varepsilon_t(\omega) = 0.
\] (5.90)
Combining (5.87, 5.88, 5.90) yields (5.85). Moreover, (5.86) follows from (5.89, 5.84) and continuity of the coefficients in (5.84). \( \square \)

### 5.5.2 Change of measure

We use the same notation as in Section 5.5.1. From the Dambis-Dubins-Schwarz theorem (cf. [20] Theorems V.1.6, V.1.7), there exists an enlargement \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{Q})\) of the filtered space \((\Omega, \mathcal{F}, (\mathcal{F}_t(\vartheta))_{t \in \mathbb{R}_+}, \mathbb{Q})\) and a standard Brownian motion \(\tilde{W}^Q\) on that space such that \(\tilde{W}^Q = M^Q_{t(\vartheta)} - M^Q_{t(0)}\) for \(\vartheta < \tilde{\vartheta}\).

Since the process \((\tilde{b}_\vartheta)_{\vartheta \in \mathbb{R}_+}\) defined by
\[
\tilde{b}_\vartheta := \frac{b^\vartheta_{t(\vartheta)}}{\vartheta^\varepsilon_{t(\vartheta)}} \mathbf{1}_{\{\vartheta \geq 0\}}(\tilde{\vartheta})
\]
is bounded,
\[
\frac{dQ^\varepsilon}{dQ} = \exp \left( - \int_0^{\vartheta^\varepsilon_t} \tilde{b}_\vartheta d\tilde{W}^Q_\vartheta - \frac{1}{2} \int_0^{\vartheta^\varepsilon_t} \tilde{b}_\vartheta^2 d\vartheta \right)
\]
defines a probability measure on \((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t(\vartheta))_{\vartheta \in \mathbb{R}_+})\) whose Hellinger process \(h(\frac{1}{2}, Q^\varepsilon, Q)\) is given by
\[
h(\frac{1}{2}, Q^\varepsilon, \tilde{Q})_\vartheta = \int_0^{\vartheta^\varepsilon_t \wedge \vartheta^\varepsilon_t} \tilde{b}_\vartheta^2 d\vartheta,
\]
(cf. [9] Theorem IV.1.33]) and such that
\[
\tilde{W}^Q := \tilde{W}^Q + \int_0^{\infty} \mathbf{1}_{\{0, \sigma^\varepsilon \} \{\vartheta \}}(\tilde{\vartheta}) \tilde{b}_\vartheta d\vartheta
\]
is a \(Q^\varepsilon\)-standard Brownian motion. In view of (5.83, 5.84, 5.86), we have
\[
\lim_{\varepsilon \downarrow 0} \int_0^{\infty} \mathbf{1}_{\{0, \sigma^\varepsilon \} \{\vartheta \}}(\tilde{\vartheta}) \tilde{b}_\vartheta^2 d\vartheta = 0 \quad \text{a.s.}
\]
By [9] Theorem V.4.31 and Lemma V.4.3] this implies
\[
\lim_{\varepsilon \downarrow 0} \sup_{A \in \tilde{\mathcal{F}}_\infty} |Q^\varepsilon(A) - \tilde{Q}(A)| = 0.
\] (5.91)
Lemma 5.16. On the probability space \((\Omega, \mathcal{F}, Q)\), we have

\[
\left| \frac{1}{\vartheta^\varepsilon} \int_0^{\vartheta^\varepsilon} \tilde{q}_\sigma^2 d\sigma - \frac{1}{3} \right| \xrightarrow{\varepsilon \downarrow 0} 0 \quad \text{in probability.} \tag{5.92}
\]

PROOF. Let process \(Y\) starting at \(\tilde{q}_0\) be the unique solution to the Skorohod SDE

\[
dY_\vartheta = d\tilde{W}_\vartheta^Q
\]

with reflection at \(\pm 1\) (cf. e.g. [23, Theorem 3.3] for existence and uniqueness). Observe that \(Y\) coincides with \(\tilde{q}\) on \([0, \vartheta^\varepsilon]\). Indeed, according to [8, 10.18], we have

\[
d\tilde{q}_\vartheta = 1_{[0,\vartheta]}(\vartheta) d\tilde{W}_\vartheta^Q + \tilde{b}_\vartheta d\vartheta + dA^+_{\vartheta(\vartheta)} - dA^-_{\vartheta(\vartheta)},
\]

i.e., \(\tilde{q}\) solves the Skorohod SDE on \([0, \vartheta]\], which yields \(Y = \tilde{q}\) on \([0, \vartheta^\varepsilon]\) by uniqueness of the solution to the stopped Skorohod SDE. Note that standard Brownian motion reflected at \(\pm 1\) is a Markov process with uniform stationary distribution, cf. e.g. [4, Appendix 1.5].

Let \(\delta > 0\). Due to [16, Theorem] and in view of (5.86), there exist two constants \(C < \infty\) and \(\zeta < 1\) such that for all \(\varepsilon_0 \in (0, 1), \varepsilon \leq \varepsilon_0, \vartheta \in [0, 1]\) we have

\[
Q^\varepsilon(A^\varepsilon_\vartheta) < C \zeta^\frac{\vartheta}{\varepsilon_0}
\]

for

\[
A^\varepsilon_\vartheta := \left\{ \left| \frac{1}{\vartheta^\varepsilon} \sum_{i=0}^{\lfloor \vartheta^\varepsilon \rfloor} (\tilde{q}_\vartheta^{\varepsilon+i})^2 - \frac{1}{3} \right| > \delta \right\}.
\]

In combination with (5.91) and interpreting \(A^\varepsilon_\vartheta\) naturally as a subset of \(\Omega\), we obtain

\[
\lim_{\varepsilon \downarrow 0} Q(A^\varepsilon_\vartheta) = \lim_{\varepsilon \downarrow 0} \tilde{Q}(A^\varepsilon_\vartheta) = 0.
\]

Fubini’s theorem and dominated convergence yield

\[
\lim_{\varepsilon \downarrow 0} \mathbb{E}_Q \left[ \int_0^1 1_{A^\varepsilon_\vartheta} d\vartheta \right] = \lim_{\varepsilon \downarrow 0} \int_0^1 \mathbb{Q}(A^\varepsilon_\vartheta) d\vartheta = \int_0^1 \lim_{\varepsilon \downarrow 0} \mathbb{Q}(A^\varepsilon_\vartheta) d\vartheta = 0.
\]

Again by dominated convergence we obtain

\[
\mathbb{E}_Q \left[ \int_0^1 \left( \frac{1}{\vartheta^\varepsilon} \sum_{i=0}^{\lfloor \vartheta^\varepsilon \rfloor} (\tilde{q}_\vartheta^{\varepsilon+i})^2 - \frac{1}{3} \right) d\vartheta \right] \leq \mathbb{E}_Q \left[ \int_0^1 \left| \frac{1}{\vartheta^\varepsilon} \sum_{i=0}^{\lfloor \vartheta^\varepsilon \rfloor} (\tilde{q}_\vartheta^{\varepsilon+i})^2 - \frac{1}{3} \right| d\vartheta \right] \to 0
\]

and hence

\[
\int_0^1 \left( \frac{1}{\vartheta^\varepsilon} \sum_{i=0}^{\lfloor \vartheta^\varepsilon \rfloor} (\tilde{q}_\vartheta^{\varepsilon+i})^2 - \frac{1}{3} \right) d\vartheta \to 0 \quad \text{in probability}
\]

for \(\varepsilon \downarrow 0\). Since

\[
\frac{1}{\vartheta^\varepsilon} \int_0^{\vartheta^\varepsilon} (\tilde{q}_\sigma)^2 d\sigma - \int_0^1 \left( \frac{1}{\vartheta^\varepsilon} \sum_{i=0}^{\lfloor \vartheta^\varepsilon \rfloor} (\tilde{q}_\vartheta^{\varepsilon+i})^2 \right) d\vartheta \to 0 \quad \text{a.s.}
\]

as \(\varepsilon \downarrow 0\), the assertion follows. \(\square\)
5.5.3 Asymptotics

Gathering the previous considerations, we are now able to complete our arguments concerning the welfare impact of small transaction costs.

Lemma 5.17. Under Assumptions [5.1], [5.2], we have

\[ E_Q \left[ (\Delta \phi)^2 \cdot [S, S]_T \right] = \frac{1}{3} E_Q \left[ (\Delta \phi^+)^2 \cdot [S, S]_T \right] + o(\varepsilon^{2/3}). \]

Proof. Step 1: Let \( \delta > 0 \) be arbitrary. Fix \( t \in [0, T] \) and define \( \delta^\varepsilon, \tilde{q}^\varepsilon \) as in Section 5.5.1. Let

\[ \Delta X_t^\varepsilon := \frac{1}{\varepsilon^{1/3}} \int_t^{(t+\varepsilon^{1/3}) \wedge T} \frac{\Delta \phi_t^+}{\varepsilon^{1/3}}^2 c_s^S S ds \]

and

\[ A_t^\varepsilon := \left\{ \left| \Delta X_t^\varepsilon - \frac{1}{3} \left( \frac{\Delta \phi_t^+}{\varepsilon^{1/3}} \right)^2 c_t^S S \right| > \delta \right\}. \]

By (5.85) and (5.92), we have

\[ 0 \leq \lim_{\varepsilon \downarrow 0} Q \left( A_t^\varepsilon \right) \leq \lim_{\varepsilon \downarrow 0} Q \left( \left| \Delta X_t^\varepsilon - \frac{1}{3} \left( \frac{\Delta \phi_t^+}{\varepsilon^{1/3}} \right)^2 c_t^S S \right| > \frac{\delta}{2} \right) \]

\[ + \lim_{\varepsilon \downarrow 0} Q \left( \frac{1}{\varepsilon^{1/3}} \int_0^{\varepsilon^1/3} \tilde{q}_0^2 d \theta - \frac{1}{3} \sum_{s} 2^{-1/3} \beta_s c_t^S S > \frac{\delta}{2} \right) = 0. \]

Step 2: Fubini’s theorem, dominated convergence, and Step 1 yield

\[ \lim_{\varepsilon \downarrow 0} E_Q \left[ \int_0^T 1_{A_t^\varepsilon} dt \right] = \lim_{\varepsilon \downarrow 0} \int_0^T Q(A_t^\varepsilon) dt = \int_0^T \lim_{\varepsilon \downarrow 0} Q(A_t^\varepsilon) dt = 0. \] (5.93)

Observe that for any \( t \in [0, T] \),

\[ \Delta X_t^\varepsilon \leq \frac{1}{\varepsilon^{1/3}} \int_t^{(t+\varepsilon^{1/3}) \wedge T} \left( \frac{\Delta \phi_t^+}{\varepsilon^{1/3}} \right)^2 c_s^S S ds =: \Delta X_t^{\varepsilon+}. \]

Using Fubini’s theorem, we conclude

\[ \int_{\varepsilon^{1/3}}^T \left( \frac{\Delta \phi_t^+}{\varepsilon^{1/3}} \right)^2 c_s^S S ds \leq \int_0^T \Delta X_t^{\varepsilon+} dt \leq \int_0^T \left( \frac{\Delta \phi_t^+}{\varepsilon^{1/3}} \right)^2 c_s^S S ds \] (5.94)

and

\[ \int_0^T \Delta X_t^{\varepsilon+} dt \leq \int_0^T \left( \frac{\Delta \phi_t^+}{\varepsilon^{1/3}} \right)^2 c_s^S S ds. \] (5.95)

So by Condition (5.1),

\[ E_Q \left[ \sup_{t \in (0, T)} |\Delta X_t^\varepsilon| dt \right] \leq E_Q \left[ \left( \frac{\Delta \phi^+}{\varepsilon^{1/3}} \right)^2 \cdot [S, S]_T \right] < \infty. \]

The assertion follows from (5.93), (5.94), (5.95) and dominated convergence. \( \square \)
Corollary 5.18. Under Assumptions 5.1, 5.2 we have
\[
\sup_{\psi \in \mathcal{A}_{x}} \mathbb{E} [U(X_T^{\psi, \varepsilon})] = \mathbb{E} [U(x + \varphi \cdot S_T)] - \frac{\gamma p}{2} \mathbb{E}_{Q} \left[ (\Delta \varphi^+)^2 \cdot [S, S]_T \right] + o(\varepsilon^{2/3})
\]
and hence
\[
\sup_{\psi \in \mathcal{A}_{x}} \mathbb{E} (X_T^{\psi, \varepsilon}) = \mathbb{E} (x + \varphi \cdot S_T) - \frac{p}{2} \mathbb{E}_{Q} \left[ (\Delta \varphi^+)^2 \cdot [S, S]_T \right] + o(\varepsilon^{2/3}).
\]

PROOF. The assertion follows from Lemmas 5.10, 5.17, 5.14 and Taylor expansion of \( y \mapsto -\frac{1}{p} \ln(-y) \) at \( \mathbb{E} [U(x + \varphi \cdot S_T)] \).

A Appendix

As an auxiliary result, we determine the explicit solution to the frictionless optimization problem related to the stochastic volatility model in Section 4.2. We proceed analogously as in [12, Theorem 3.1], which deals with power utility.

Theorem A.1. For the stochastic volatility model characterized by (4.3) with bounded \( b(Y) / \sigma(Y) \), the frictionless optimizer \( \varphi \) satisfies
\[
\varphi_t S_t = \frac{b(Y_t)}{p\sigma(Y_t)^2} \quad \text{for all } t \in [0, T].
\] (A.1)

The MEMM \( Q \) has density process
\[
\mathbb{E} \left[ \frac{dQ}{dP} \bigg| \mathcal{F}_t \right] = \frac{\tilde{Z}_t}{Z_0} \exp \left( -\int_0^t \frac{b(Y_s)}{\sigma(Y_s)} dW_s - \frac{1}{2} \int_0^t \left( \frac{b(Y_s)}{\sigma(Y_s)} \right)^2 ds \right), \quad t \in [0, T],
\]
where the process \( \tilde{Z} \) is defined as in (4.5).

PROOF. Step 1: Define filtration \( \mathcal{G} = (\mathcal{G}_t)_{t \in [0, T]} \) by
\[
\mathcal{G}_t := \bigcap_{s > t} \sigma \left( \mathcal{F}_s \cup \sigma((Y_r)_{r \in [0, T]}) \right), \quad t \in [0, T]
\] (A.2)
and let
\[
\varphi := \frac{b(Y)}{p\sigma(Y)^2 S}
\]
in line with (A.1). Moreover, set
\[
Z_t := \exp \left( -\int_0^t \frac{b(Y_s)}{\sigma(Y_s)} dW_s - \frac{1}{2} \int_0^t \left( \frac{b(Y_s)}{\sigma(Y_s)} \right)^2 ds \right), \quad t \in [0, T].
\] (A.3)

By definition of \( \mathcal{G} \), random variable \( \tilde{Z}_T \) is \( \mathcal{G}_0 \)-measurable. Since \( Y \) is independent of \( W \), it follows from [11, Theorem 15.5] that \( W \) is a standard Brownian motion with respect to
as well. Due to boundedness of \( b(Y)/\sigma(Y) \), the local martingale \( Z \) satisfies Novikov’s condition, whence it is a martingale relative to both \( F \) and \( G \). Therefore, we deduce that

\[
Z^G_t := \mathbb{E} \left[ \frac{U'(x + \varphi \cdot S_T)}{U(x + \varphi \cdot S_T)} \bigg| \mathcal{F}_t \right] = \mathbb{E} \left[ p \exp \left( -px - \int_0^T \frac{b(Y_s)}{\sigma(Y_s)} dW_s - \int_0^T \frac{b(Y_s)}{\sigma(Y_s)}^2 ds \right) \bigg| \mathcal{F}_t \right] = \mathbb{E} \left[ pe^{-px}Z_T Z_T \bigg| \mathcal{F}_t \right] = pe^{-px}Z_T Z_t
\]

(A.4)

for any \( t \in [0, T] \). In particular, \( \mathbb{E}[Z^G_T] < \infty \). The normalised \( G \)-martingale \( Z^G \) is the \( G \)-density process of the probability measure \( Q \) with density

\[
\frac{dQ}{dP} := \frac{Z^G_T}{\mathbb{E}[Z^G_T]}
\]

**Step 2:** Let \( \mathcal{Q} \) be the probability measure with density process \( Z \). By Girsanov’s theorem,

\[
W \mathcal{Q} := W + \int_0^T \frac{b(Y_t)}{\sigma(Y_t)} dt
\]

is a standard Brownian motion under measure \( \mathcal{Q} \) relative to both \( F \) and \( G \). Since \( b(Y)/\sigma(Y) \) is bounded,

\[
\varphi \cdot S = \frac{1}{p} \int_0^T \frac{b(Y_s)}{\sigma(Y_s)} dW_s \mathcal{Q}
\]

is a \( \mathcal{Q} \)-martingale with respect to \( G \). Moreover, \( S \) is a \( \mathcal{Q} \)-local martingale relative to both \( F \) and \( G \) because \( dS_t = S_t \sigma(Y_t) dW_t \mathcal{Q} \), cf. [19] Theorem IV.33).

Let \( \psi \) be an admissible strategy in the sense of [21] Definition 1.2, i.e. \( \psi \) is an \( S \)-integrable process such that the related wealth process is uniformly bounded from below. Note that \( \psi \cdot S \) is a \( \mathcal{Q} \)-local martingale which is bounded from below and hence a \( \mathcal{Q} \)-supermartingale. By the generalized Bayes’ formula and in view of (A.4), \( \varphi \cdot S \) is a \( \mathcal{Q} \)-martingale and \( \psi \cdot S \) is a \( \mathcal{Q} \)-supermartingale, both with respect to filtration \( G \). Hence, by concavity of \( U \), we have

\[
\mathbb{E}[U(x + \psi \cdot S_T)] \\
\leq \mathbb{E}[U(x + \psi \cdot S_T)] + \mathbb{E}[U'(x + \psi \cdot S_T)(\psi \cdot S_T - \varphi \cdot S_T)] \\
= \mathbb{E}[U(x + \psi \cdot S_T)] + \mathbb{E}[U'(x + \psi \cdot S_T)] \mathbb{E}_Q[\psi \cdot S_T - \varphi \cdot S_T] \\
\leq 0
\]

**Step 3:** We show that \( U(x + \varphi \cdot S_T) \) lies in the \( L^1 \)-closure of the set

\[
\{ U(x + \psi \cdot S_T) : \psi \text{ is admissible} \}
\]

Indeed, letting

\[
\tau_n := \inf\{ t \in [0, T] : x + \varphi \cdot S_t < -n \},
\]

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we can approximate $\varphi$ by the sequence $(\varphi^{(n)})_{n \in \mathbb{N}}$ defined as $\varphi^{(n)} := \varphi_{1_{[0, t_n]}}$, which fulfills the admissibility requirement in [21, Definition 1.2]. Using Cauchy-Schwarz’ inequality, we obtain

$$
E \left[ \sup_{n \in \mathbb{N}} \left| U(x + \varphi^{(n)} \cdot S_T) \right| \right]
= E \left[ \sup_{n \in \mathbb{N}} \exp \left( -p(x + \varphi^{(n)} \cdot S_T) \right) \right]
\leq E \left[ \sup_{t \in [0, T]} \exp \left( -p(x + \varphi \cdot S_t) \right) \right]
= e^{-px} E \left[ \sup_{t \in [0, T]} \exp \left( -\int_0^t \frac{b(Y_s)}{\sigma(Y_s)} dW_s - \frac{1}{4} \int_0^t \left( \frac{2b(Y_s)}{\sigma(Y_s)} \right)^2 ds \right) \right]
\leq e^{-px} E \left[ \sup_{t \in [0, T]} \exp \left( -\int_0^t \frac{2b(Y_s)}{\sigma(Y_s)} dW_s - \frac{1}{2} \int_0^t \left( \frac{2b(Y_s)}{\sigma(Y_s)} \right)^2 ds \right) \right].
$$

By boundedness of $(2b(Y_t)/\sigma(Y_t))_{t \in [0, T]}$, the process $(M_t)_{t \in [0, T]}$ is an $L^2$-martingale, which implies

$$
\sup_{n \in \mathbb{N}} U(x + \varphi^{(n)} \cdot S_T) \in L^1(\mathbb{P})
$$

by Doob’s quadratic inequality. Dominated convergence yields

$$
\left\| U(x + \varphi^{(n)} \cdot S_T) - U(x + \varphi \cdot S_T) \right\|_{L^1(\mathbb{P})} \to 0 \quad \text{as} \quad n \to \infty.
$$

**Step 4:** By Steps 2 and 3 the payoff $x + \varphi \cdot S_T$ is optimal in the sense of [21, Theorem 2.2(iii)], which implies that $Q$ is the dual optimizer, cf. [21, Equation (42)]. Moreover, we have shown in Step 2 that $\varphi \cdot S$ is a $Q$-martingale with respect to filtration $G$ and hence $F$ as well, which yields that $\varphi$ is the optimal strategy in the sense of [21, Theorem 2.2(iv)].

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