Byzantine Agreement in Polynomial Time with Near-Optimal Resilience*

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Abstract

It has been known since the early 1980s that Byzantine Agreement in the full information, asynchronous model is impossible to solve deterministically against even one crash fault [FLP85], but that it can be solved with probability 1 [Ben83], even against an adversary that controls the scheduling of all messages and corrupts up to \( f < n/3 \) players [Bra87]. The main downside of [Ben83, Bra87] is that they terminate in \( 2^{\Theta(n)} \) rounds in expectation whenever \( f = \Theta(n) \).

King and Saia [KS16, KS18] developed a polynomial protocol (polynomial rounds, polynomial computation) that is resilient to \( f < (1.14 \times 10^{-9})n \) Byzantine faults. The new idea in their protocol is to detect—and blacklist—coalitions of likely-bad players by analyzing the deviations of random variables generated by those players over many rounds.

In this work we design a simple collective coin-flipping protocol such that if any coalition of faulty players repeatedly does not follow protocol, then they will eventually be detected by one of two simple statistical tests. Using this coin-flipping protocol, we solve Byzantine Agreement in a polynomial number of rounds, even in the presence of up to \( f < n/4 \) Byzantine faults. This comes close to the \( f < n/3 \) upper bound on the maximum number of faults [BT85, FLM86, LSP82].

1 Introduction

The field of forensic accounting is concerned with the detection of fraud in financial transactions, or more generally, finding evidence of fraud, malfeasance, or fabrication in data sets. Some examples include detecting faked digital images [BBMT20], suspicious reports of election data [Rou14] and political fundraising [GA17], fraudulent COVID numbers,1 and manipulated economic data [TJ12, Kau19, AMVBJ13] via Newcomb-Benford’s law [KG21], detecting fabricated data sets2 in social science research [Sim13, SSN15], or detecting match-fixing in sumo wrestling [DL02].

Theoretical computer science has a strong tradition of embracing a fundamentally adversarial view of the universe that borders on being outright paranoid. Therefore it is somewhat surprising that TCS is, as a whole, credulous when it comes to adversarial manipulation of data and transactions. In other words, fraud detection does not play a significant part in most algorithm design, even in multi-party models that explicitly posit the existence of malicious parties.

To our knowledge, the only work in TCS that has explicitly adopted a forensic accounting mindset is King and Saia’s [KS16, KS18] breakthrough in Byzantine Agreement in the most challenging model: the full-information (no crypto) asynchronous model against an adaptive adversary. In this problem there are \( n \) players, each with initial input bits in \( \{-1, 1\} \), up to \( f \) of which may fail (i.e., be adaptively corrupted by the adversary) and behave arbitrarily. They must each decide on a bit in \( \{-1, 1\} \) subject to:

**Agreement:** All non-corrupted players decide the same value \( v \).

**Validity:** If all players begin with the same value \( v \), all non-corrupted players decide \( v \).

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1https://theprint.in/opinion/benfords-law-detects-data-fudging-so-we-ran-it-through-indian-states-covid-numbers/673085/

2http://datacolada.org/98
### Table 1: Byzantine Agreement in the full information model against an adaptive adversary.

| Citation                   | Byzantine Faults ($f$) | Expected Rounds / Computation Per Round |
|---------------------------|------------------------|-----------------------------------------|
| Fischer, Lynch, Patterson | $f \geq 1$             | impossible deterministically             |
| [LSP82, BT85, FLM86]      | $f \geq n/3$           | impossible, even with randomization      |
| Ben-Or                    | $f < n/5$              | $\exp(n) / \text{poly}(n)$              |
|                           | $f < O(\sqrt{n})$     | $O(1) / \text{poly}(n)$                |
| Bracha                    | $f < n/3$              | $\exp(n) / \text{poly}(n)$              |
| King & Saia               | $f < n/400$            | $\text{poly}(n) / \exp(n)$              |
|                           | $f < n/(1.14^{-1} \times 10^9)$ | $\text{poly}(n) / \text{poly}(n)$      |
| new                       | $f < n/4$              | $\text{poly}(n) / \text{poly}(n)$      |

See Section 2.1 for details of the model. Prior to King and Saia’s work [KS16, KS18], it was known from Bracha [Bra87] (see also Ben-Or [Ben83]) that the problem could be solved with probability 1 in $2^9(n)$ time in expectation even if $f < n/3$ players fail, that $f < n/3$ cannot be improved [LSP82, BT85, FLM86], and by Fischer, Lynch, and Patterson’s impossibility result [FLP85], that no deterministic protocol exists even against a single crash failure.

King and Saia [KS16] reduce the problem to a certain coin-flipping game, in which all players—good and adversarial—attempt to generate a (global) unbiased coin flip and agree on its outcome. Coin flipping games have been studied extensively under adversarial manipulation (see Section 1.2), but the emphasis is always on bounding the power of the adversarial players to bias the coin flip in their desired direction. King and Saia recognized that the primary long term advantage of the adversary is anonymity. In other words, it can bias the outcome of coin flips at will, in the short term, but its advantage simply evaporates if good players can merely identify who the adversarial players are, by detecting likely fraud via a statistical analysis of their transactions. Good players can blacklist (ignore) the adversarial players, removing their influence over the game. If a sufficient number of fraudulent players are blacklisted, collective coin-flipping by a set of good players becomes easy.

The journal version of King and Saia’s work [KS16] presents two methods for blacklisting players, which leads to different fault tolerance levels. The first protocol has a polynomial round complexity and requires a polynomial amount of local computation; it is claimed to be resilient to $f < (4.25 \times 10^{-7})n$ Byzantine faults. The second protocol is tolerant to $f < n/400$ Byzantine faults, but requires exponential local computation. In response to some issues raised by Melynyk, Wang, and Wattenhofer (see Melynyk’s Ph.D. thesis [Mel20, Ch. 6]), King and Saia [KS18] published a corrigendum, reducing the tolerance of the first protocol to $f < (1.14 \times 10^{-9})n$.

### 1.1 New Results

In this paper we solve Byzantine Agreement in the full-information, asynchronous model against an adaptive adversary, by adopting the same forensic accounting paradigm of King and Saia [KS16]. We design a coin-flipping protocol and two simple statistical tests such that if the Byzantine players continually foil attempts to flip a fair coin, they will be detected in a polynomial number of rounds by at least one of the tests, so long as $f < n/4$. (The tests measure individual deviation in $l_2$ norm and pair-wise correlation.) Our analysis is tight inasmuch as these two particular tests may not detect anything when $f \geq n/4$.

One factor contributing to the low resiliency of King and Saia’s protocols [KS16, KS18] is that two good players may blacklist different sets of players, making it easier for the adversary to induce disagreements on the outcome of the shared coin flip. A technical innovation in our protocol is a method to drastically reduce the level of disagreement between the views of good players. First, we use a fractional blacklisting scheme. Second, to ensure better consistency across good players, we extend King and Saia’s [KS16] Blackboard to an Iterated Blackboard primitive that drastically reduces good players’ disagreements of the historical transaction record by allowing retroactive corrections to the record.
1.2 Related Work

The approach of King and Saia [KS16] was foreshadowed several years earlier by Lewko [Lew11], who showed that protocols broadly similar to Ben-Or and Bracha must take an exponential number of rounds. The key assumption is that messages are taken at face value, without taking into account the identity of the sender, nor the history of the sender’s messages.

Byzantine agreement has been studied in synchronous and asynchronous models, against computationally bounded or unbounded adversaries, and with adaptive or non-adaptive adversaries. (In particular, a special case of the problem that restricts attention to crash failures, called consensus, has been very extensively studied.) We refer the reader to [Asp03, AC08, BJBO98, BOPV06, CVNV11, KS11] for some key results and surveys of the literature. A result that is fairly close to ours is that of Kapron et al. [KKK+10]. They proved that against a non-adaptive adversary (all corruptions made in advance) Byzantine agreement can be solved asynchronously, against \( f < n/(3 + \epsilon) \) faults.

Collective coin flipping has an illustrious history in computer science, as it is a key concept in cryptography, distributed computing, and analysis of boolean functions. The problem was apparently first raised by Blum [Bhu81], who asked how two mutually untrusted parties could flip a shared coin over the telephone. His solution used cryptography. See [Cle86, HT17, MNS16, BOO15, DMM14, HMO18, BHLT21] for some recent work on coin flipping using cryptography.

Ben-Or and Linial [BL85] initiated a study of full information protocols for coin-flipping. The players broadcast messages one-by-one in a specific order, and the final coin flip is a function of these messages. The goal is to minimize the influence of a coalition of \( k \) bad players, which is, roughly speaking, the amount by which they can bias the outcome towards heads or tails. Ben-Or and Linial’s [BL85] protocol limits \( k < \frac{n}{\log n} \) bad players to influence \( O(k/n) \) messages. Saks [Sak89] and Ajtai and Linial [AL93] improved it to \( O(k/n) \) influence with up to \( k = O(n \log n) \) players, and Alon and Naor [AN93] achieved optimum \( O(k/n) \) influence for \( k \) even linear in \( n \). The message size in these protocols is typically more than a single bit. If only single-bit messages are allowed and each player speaks once, the problem is equivalent to bounding the influence of variables in a boolean function [KKL88]. Russel, Saks, and Zuckerman [RSZ02] considered parallel coin-flipping protocols. The proved that any protocol that uses 1-bit messages and is resilient to linear-size coalitions must use \( \Omega(\log^* n) \) rounds.

Aspnes [Asp98] considered a sequential coin-flipping game where \( n \) coins are flipped sequentially and the outcomes are broadcast, but up to \( t \) of these may be suppressed by the adversary. Regardless of which function is used to map the coin-flip sequence to a shared coin, the adversary can bias it whenever \( t = \Omega(\sqrt{n}) \). Very recently Haitner and Karidi-Heller [HK20] resolved the complexity of Ben-Or-Linial-type sequential coin flipping games against an adaptive adversary, that can corrupt players at will, as information is revealed. They proved that any such shared coin can be fixed to a desired outcome with probability \( 1 - o(1) \) by adaptively corrupting \( O(\sqrt{n}) \) parties.

1.3 Organization

In Section 2 we review the model, the reliable broadcast primitive, and Bracha’s Byzantine agreement protocol, and introduce the Iterated Blackboard primitive, which generalizes [KS16, Kim20].

In Section 3 we begin with a simplified iterated coin-flipping game and then proceed to study a more complicated iterated coin-flipping game that can be implemented in the asynchronous distributed model and used within Bracha’s algorithm.

Appendix A contains proofs from Section 2 on reliable broadcast and the iterated blackboard. Appendix B reviews some standard concentration inequalities and other theorems. Appendix C contains some proofs showing that a certain fractional matching algorithm has a Lipschitz property.

2 Preliminaries

2.1 The Model

There are \( n \) processes, \( p_1, \ldots, p_n \), and \( 2n^2 \) message buffers, \( \text{In}_{i \to i} \) and \( \text{Out}_{i \to j} \) for all \( i, j \in [n] \). All processes are initially good (they obey the protocol) and the adversary may dynamically corrupt up to \( f \) processes. A
bad/corrupted process is under complete control of the adversary and may behave arbitrarily. The adversary controls the pace at which progress is made by scheduling two types of events.

- A compute\((i)\) event lets \(p_i\) process all messages in the buffers \(\text{In}_{j \rightarrow i}\), deposit new messages in \(\text{Out}_{i \rightarrow j}\), and change state.
- A deliver\((i, j)\) event removes a message from \(\text{Out}_{i \rightarrow j}\) and moves it to \(\text{In}_{i \rightarrow j}\).

Note that the adversary may choose a malicious order of events, but cannot, for example, misdeliver or forge messages. The adversary must eventually allow some good process to make progress. In particular, we can assume without loss of generality that the scheduling sequence is of the form

\[ A_0, A_1, A_2, \ldots, \]

where each \(A_k\) contains a finite number of events, including either the first compute\((i)\) event for good process \(i\), or the delivery of a message from \(\text{Out}_{i \rightarrow j}\) to \(\text{In}_{i \rightarrow j}\) followed later by compute\((j)\), for some good process \(i\). Each \(A_k\) can contain an arbitrary number of compute events for bad processes, or the delivery of messages sent by bad processes.

The adversary is computationally unbounded and is aware, at all times, of the internal state of all processes. Thus, cryptography is not helpful, but randomness potentially is, since the adversary cannot predict the outcome of future coin flips.

In this model, the communication time or latency is defined w.r.t. a hypothetical execution in which all local computation occurs instantaneously and all messages have latency in \([0, \Delta]\). The latency of the algorithm is \(L\) if all non-corrupt processes finish by time \(L\Delta\). Note that in this hypothetical, \(\Delta\) is unknown and cannot influence the execution of the algorithm.

### 2.1.1 Reliable Broadcast

The goal of Reliable-Broadcast is to simulate a broadcast channel using the underlying point-to-point message passing system. In Byzantine Agreement protocols, each process initiates a series of Reliable-Broadcasts. Call \(m_{p,\ell}\) the \(\ell\)th message broadcast by process \(p\).

**Theorem 1.** If a good process \(p\) initiates the Reliable-Broadcast of \(m_{p,\ell}\), then all good processes \(q\) eventually accept \(m_{p,\ell}\). Now suppose a bad process \(p\) does so and some good \(q\) accepts \(m_{p,\ell}\). Then all other good \(q'\) will eventually accept \(m_{p,\ell}\), and no good \(q'\) will accept any other \(m'_{p,\ell} \neq m_{p,\ell}\). Moreover, all good processes accept \(m_{p,\ell-1}\) before \(m_{p,\ell}\), if \(\ell > 1\).

The property that \(m_{p,\ell}\) is only accepted after \(m_{p,\ell-1}\) is accepted is sometimes called FIFO broadcast. This property is explicitly used in the Iterated-Blackboard algorithm outlined in Section 2.2. See Appendix A.1 for a proof of Theorem 1.

#### Algorithm 1 Reliable-Broadcast\((p, \ell)\)

```plaintext
1: if \(\ell > 1\) then wait until \(m_{p,\ell-1}\) has been accepted.
2: if I am process \(p\) then generate \(m_{p,\ell}\) and send (init, \(m_{p,\ell}\)) to all processes.
3: wait until receipt of one (init, \(m_{p,\ell}\)) message from \(p\), or more than \((n + f)/2\) (echo, \(m_{p,\ell}\)) messages, or
   \(f + 1\) (ready, \(m_{p,\ell}\)) messages.
   send (echo, \(m_{p,\ell}\)) to all processes.
4: wait until the receipt of \((n + f)/2\) (echo, \(m_{p,\ell}\)) messages or \(f + 1\) (ready, \(m_{p,\ell}\)) messages.
   send (ready, \(m_{p,\ell}\)) to all processes.
5: wait until receipt of \(2f + 1\) (ready, \(m_{p,\ell}\)) messages.
   accept \(m_{p,\ell}\).
```

### 2.1.2 Validation and Bracha’s Protocol

Consider a protocol \(\Pi\) of the following form. In each round \(r\), each process reliably broadcasts its state to all processes, waits until it has accepted at least \(n - f\) validated messages from round \(r\), then processes
all validated messages, changes its state, and advances to round \( r + 1 \). A good process validates a round-\( r \) state (message) \( s_{q,r} \) accepted from another process \( q \) only if (i) it has validated the state \( s_{q,r-1} \) of \( q \) at round \( r - 1 \), and (ii) it has accepted \( n - f \) messages that, if they were received by a correct \( q \), would cause it to transition from \( s_{q,r-1} \) to \( s_{q,r} \). The key property of validation (introduced by [Bra87]) is:

**Lemma 2.** A good process \( p \) validates the message of another process \( q \) in an admissible execution \( \alpha \) of \( \Pi \) if and only if there is an execution \( \beta \) of \( \Pi \) in which \( q \) is a good process and the state of every other good process (including \( p \)) is the same in \( \alpha \) and \( \beta \) (with respect to their validated messages).

To recap, reliable broadcast prevents the adversary from sending conflicting messages to different parties (i.e., it is forced to participate as if the communication medium were a broadcast channel) and the validation mechanism forces its internal state transitions to be consistent with the protocol. Its remaining power is limited to (i) substituting deterministic outcomes for coin flips in bad processes, (ii) dynamic corruption of good processes, and (iii) malicious scheduling.

Bracha’s protocol improves the resilience of Ben-Or’s protocol to the optimum \( f < n/3 \). Each process \( p \) initially holds a value \( v_p \in \{-1, 1\} \). It repeats the same steps until it decides a value \( v \in \{-1, 1\} \) (Line 8). As we will see, if some process decides \( v \), all good processes will decide \( v \) in this or the following iteration. Thus, good processes continue to participate in the protocol until all other good processes have executed Line 8. Here \( \text{sgn}(x) = 1 \) if \( x \geq 0 \) and \(-1 \) if \( x < 0 \).

### Algorithm 2 Bracha-Agreement() from the perspective of process \( p \)

**Require:** \( v_p \in \{-1, 1\} \).

1: loop
2: reliably broadcast \( v_p \) and wait until \( n - f \) messages are validated from some processes \( S \).
   - set \( v_p := \text{sgn}(\sum_{q \in S} v_q) \).
3: reliably broadcast \( v_p \) and wait until \( n - f \) messages are validated.
   - if more than \( n/2 \) messages have some value \( v \) then set \( v_p := (\text{dec}, v) \).
4: reliably broadcast \( v_p \) and wait until \( n - f \) messages are validated.
   - let \( x_p \) be the number of \( (\text{dec}, v) \) messages validated by \( p \).
5: \( \text{if } x_p \geq 1 \text{ then} \)
6: set \( v_p := v \).
7: \( \text{if } x_p \geq f + 1 \text{ then} \)
8: decide \( v \).
9: \( \text{if } x_p = 0 \text{ then} \)
10: \( v_p := \text{Coin-Flip}() \). \( \triangleright \text{Returns value in } \{-1, 1\} \).

**Correctness.** Suppose that at the beginning of an iteration, there is a set of at least \( (n + f + 1)/2 \) good processes who agree on a value \( v \in \{-1, 1\} \).\(^3\) It follows that in Line 2, every process hears from at least \( (n + f + 1)/2 - f > (n - f)/2 \) of these good processes, i.e., a strict majority in any set of \( n - f \). Thus, every good process broadcasts \( v \) in Line 3, and due to the validation mechanism, any bad process that wishes to participate in Line 3 also must broadcast \( v \). Thus, every good process \( p \) will eventually validate \( n - f > n/2 \) votes for \( v \) and set \( v_p := (\text{dec}, v) \) indicating it is prepared to decide \( v \) in this iteration. By the same reasoning, every good process \( p \) will set \( x_p := n - f \geq f + 1 \) and decide \( v \) in Line 8.

It is impossible for \( p \) to validate two messages \((\text{dec}, v)\) and \((\text{dec}, v')\) in Line 4 with \( v \neq v' \). To validate such messages, \( p \) would need to receive strictly greater than \( n/2 \) “\( v \)” and “\( v' \)” messages in Line 3, meaning some process successfully broadcast two distinct messages with the same timestamp. By Theorem 1 this is impossible.

Now suppose that in some iteration \( p \) decides \( v \) in Line 8. This means that \( p \) validated \( n - f \) messages in Line 4 and set \( x_p \geq f + 1 \). Every other good process \( q \) must have validated at least \( n - 2f \) of the messages that \( p \) validated, and therefore set \( x_q \geq 1 \), forcing it to set \( v_q := v \) in Line 6. Thus, at the beginning of the next iteration \( n - f \) good processes agree on the value \( v \) and all decide \( v \) (Line 8) in that iteration.\(^4\)

\(^3\)Note that this is always numerically possible since \( (n + f + 1)/2 \leq n - f \) with equality if \( f = (n - 1)/3 \).

\(^4\)Bracha [Bra87] sets the thresholds in Line 5 and 7 to be \( f + 1 \) and \( 2f + 1 \). The idea was to guarantee that if \( x_p \geq f + 1 \)
The preceding paragraphs establish correctness. Turning to efficiency, consider any iteration in which no process decides $v$ in Line 8. We can partition the good population into $G_6$ and $G_{10}$, depending on whether they execute Line 6 (setting $v_p := v$) or Line 10. If a sufficiently large number of calls to Coin-Flip() made by $G_{10}$-processes returns $v$ (specifically, $(n + f + 1)/2 - |G_6|$) then by the argument above, all processes will decide $v$ (Line 8) in the next iteration. Call this happy event $\mathcal{E}$. If $G_6 = \emptyset$ then both values of $v$ are acceptable, which just increases the likelihood of $\mathcal{E}$.

Bracha [Bra87] and Ben-Or [Ben83] implement Coin-Flip by each process privately flipping an independent, unbiased coin. Thus, for any $f < n/3$, $\Pr(\mathcal{E}) \geq 2^{-(n-f-1)}$ and the expected number of iterations is at most $2^{\Theta(n)}$. If there were a mechanism to implement Coin-Flip as a roughly unbiased shared coin (all processes in $G_{10}$ see the same value; see Rabin [Rab83] and Toueg [Tou84]), then $\Pr(\mathcal{E})$ is constant and we only need $O(1)$ iterations in expectation. Efficient collective coin-flipping is therefore the heart of the Byzantine Agreement problem in this model.

### 2.2 The Iterated Blackboard Model

King and Saia [KS16] implemented a Coin-Flip() routine using a blackboard primitive, which weakens the power of the scheduling adversary to give drastically different views to different processes.5 Their blackboard protocol is resilient to $f < n/4$ faults. Kimmett [Kim20] simplified and improved this protocol to tolerate $f < n/3$ faults. In this section, we describe a useful extension of the Kimmett-King-Saia style blackboard that further reduces the kinds of disagreements that good processes can have.

In the original model [KS16, Kim20], a blackboard is an $m \times n$ matrix $\text{BB}$, initially all blank ($\bot$), such that column $\text{BB}(:,i)$ is only written to by process $i$. Via reliable broadcasts, process $i$ attempts to sequentially write non-$\bot$ values to $\text{BB}(r,i)$, $r \in [m]$. The scheduling power of the adversary allows it to control the rate at which different processes write values. Because there could be up to $f$ crash-faults, no process can count on $\text{BB}$ containing more than $n – f$ complete columns (those $i$ for which $\text{BB}(m,i) \neq \bot$). The final $\text{BB}$-matrix may therefore contain up to $f$ partial columns.

The main guarantee of [KS16, Kim20] is that every process $p$ has a mostly accurate view $\text{BB}^{(p)}$ that agrees with the “true” blackboard $\text{BB}$ in all but at most $f$ locations. In particular, the last non-$\bot$ entry of each partial column in $\text{BB}$ may still be $\bot$ in $\text{BB}^{(p)}$. If we were to generate a sequence of blackboards with [KS16, Kim20], the views from two processes could differ by $f$ locations in each blackboard.

An iterated blackboard is an endless series $\text{BB} = (\text{BB}_1, \text{BB}_2, \ldots)$ of $m \times n$ blackboards, such that process $i$ only attempts to write its column in $\text{BB}_t$ once it completes participation in $\text{BB}_{t-1}$. After $p$ regards $\text{BB}_t$ as complete, $p$ obtains a view of the full history $\text{BB}^{(p,t)} = (\text{BB}_1^{(p,t)}, \ldots, \text{BB}_t^{(p,t)})$ that differs from $(\text{BB}_1, \ldots, \text{BB}_t)$ in $f$ locations in total. As a consequence, $\text{BB}^{(p,t-1)}$ may not be identical to the first $t-1$ matrices of $\text{BB}^{(p,t)}$, i.e., $p$ could record “retroactive” updates to previous matrices while it is actively participating in the construction of $\text{BB}_t$.

The following theorem is proved in Appendix A.2.

**Theorem 3.** There is a protocol for $n$ processes to generate an iterated blackboard $\text{BB}$ that is resilient to $f < n/3$ Byzantine failures. For $t \geq 1$, the following properties hold:

1. Upon completion of the matrix $\text{BB}_t$, each column consists of a prefix of non-$\bot$ values and a suffix of all-$\bot$ values. Let $\text{last}(i) = (t',r)$ be the position of the last value written by process $i$, i.e., $\text{BB}^{(p,t)}(r,i) \neq \bot$ and if $t' < t$ then $i$ has not written to any cells of $\text{BB}_t$. When $\text{BB}_t$ is complete, it has at least $n – f$ full columns and up to $f$ partial columns.

2. Once $\text{BB}_t$ is complete, each process $p$ forms a history $\text{BB}^{(p,t)} = (\text{BB}_1^{(p,t)}, \ldots, \text{BB}_t^{(p,t)})$ such that for every $t' \in [t]$, $i \in [n]$, $r \in [m]$,

$$
\text{BB}^{(p,t)}_{t'}(r,i) \begin{cases} 
= \text{BB}^{(p,t)}(r,i) & \text{if } \text{last}(i) \neq (t',r) \\
\in \{\text{BB}^{(p,t)}(r,i), \bot\} & \text{otherwise}
\end{cases}
$$

then at least one good process sent $p$ a $(dec, v)$ message. However, because of the validation mechanism this is not important. A corrupt process can try to send a $(dec, v)$ message but it will not be validated unless $v$ does, in fact, have a strict majority ($> n/2$) of messages sent in Line 3.

5For example, in Line 2 of Bracha-Agreement, the scheduling adversary can show $p$ any $n – f$ messages $S$, and therefore have significant control over the value of $\text{sgn}(\sum_{v \in S} v_p)$. 

6
3. If \( q \) writes any non-\( \perp \) value to \( \text{BB}_{t+1} \), then by the time any process \( p \) fixes \( \text{BB}^{(p,t+1)} \), \( p \) will be aware of \( q \)’s view \( \text{BB}^{(q,t)} \) of the history up to blackboard \( t \).

## 3 Iterated Coin Flipping Games

We begin in Section 3.1 with a simplified coin-flipping game and extend it in Section 3.2 to the real coin-flipping game we use to implement Coin-Flip() in Bracha-Agreement. In the real coin-flipping game we assign weights to the processes, which is a measure of trustworthiness. Section 3.3 explains how the weights are updated and Section 3.4 bounds numerical inconsistencies in different processors views.

### 3.1 A Simplified Game

In this game there are \( n \) players partitioned into \( n - f \) good players \( G \) and \( f = n/(3 + \epsilon) \) bad players \( B \), for some small \( \epsilon > 0 \). The good players are unaware of the partition \((G,B)\). The game is played up to \( T \) times in succession according to the following rules. Let \( t \in [T] \) be the current iteration.

- The adversary privately picks an adversarial direction \( \sigma(t) \in \{-1, 1\} \).
- Each good player \( i \in G \) picks \( X_i(t) \in \{-1, 1\} \) uniformly at random. The bad players see these values then generate their values \( \{X_i(t)\}_{i \in B} \), each in \( \{-1, 1\} \), as they like.
- If the outcome of the coin flip, \( \text{sgn}(\sum_{i \in [n]} X_i(t)) \), is equal to \( \sigma(t) \), the game continues to iteration \( t + 1 \).

From the good players’ perspective, the nominal goal of this game is to eventually achieve the outcome \( \text{sgn}(\sum_{i \in [n]} X_i(t)) \neq \sigma(t) \), but the adversary can easily foil this goal if \( T = \text{poly}(n) \). We consider a secondary goal: namely to identify bad players based solely on the historical data \( \{X_i(t)\}_{i,t} \). This turns out to be a tricky problem, but we can identify a pair of processes, at least one of which is bad, w.h.p.

**Lemma 4.** Suppose the game does not end after \( T \) iterations. If \( T = \tilde{O}((n/\epsilon)^2) \), then the pair \( (i,j) \in [n]^2 \), \( i \neq j \), maximizing

\[
\langle X_i, X_j \rangle = \sum_{t=1}^{T} X_i(t)X_j(t)
\]

has \( B \cap \{i,j\} \neq \emptyset \).

**Proof.** If \( i,j \in G \) are good, by a Chernoff-Hoeffding bound (Theorem 22, Appendix B) \( \langle X_i, X_j \rangle \leq \beta = \tilde{O}(\sqrt{T}) \) with high probability, thus every pair whose inner product exceeds \( \beta \) must contain at least one bad process. We now argue that there exists an \( i^*,j^* \in B \) such that \( \langle X_{i^*}, X_{j^*} \rangle \) exceeds \( \beta \). Observe that

\[
\sum_{(i \neq j) \in B^2} X_i(t)X_j(t) = \left(\sum_{i \in B} X_i(t)\right)^2 - \sum_{i \in B} (X_i(t))^2 = \left(\sum_{i \in B} X_i(t)\right)^2 - f. \tag{1}
\]

Let \( S(t) = \sum_{i \in G} X_i(t) \) be the sum of the good processes in iteration \( t \). The bad players force the sign of the sum to be \( \sigma(t) \), i.e., \( \text{sgn}(S(t)) + \sum_{i \in B} X_i(t)\sigma(t) = 1 \). Thus,

\[
\left(\sum_{i \in B} X_i(t)\right)^2 \geq \begin{cases} (S(t))^2 & \text{if } \text{sgn}(S(t)) \neq \sigma(t) \\ 0 & \text{otherwise} \end{cases} = (\max\{0, -\sigma(t)S(t)\})^2. \tag{2}
\]

Let \( Z(t) = (\max\{0, -\sigma(t)S(t)\})^2 \). By a Chernoff-Hoeffding bound (Theorem 22, Appendix B), w.h.p. \( Z(t) \leq \gamma = \tilde{O}(n) \) for every \( t \). Moreover, since the distribution of \( S(t) \) is symmetric around the origin,

\[
\mathbb{E}[Z(t)] \geq \frac{1}{2} \mathbb{E}[S(t)^2 \mid -\sigma(t)S(t) \geq 0] = \frac{1}{2}(n - f). \tag{3}
\]

\[\text{In the context of Bracha-Agreement, } \sigma \text{ would be } -v, \text{ where } v \text{ is the value set by processes executing Line 6}.\]
Thus, by linearity of expectation and Chernoff-Hoeffding (Theorem 22, Appendix B), we have, w.h.p.,

\[
\sum_{t=1}^{T} Z(t) \geq \frac{1}{2} T(n-f) - \gamma \cdot \tilde{O}(\sqrt{T}) = \frac{1}{2} T(n-f) - \tilde{O}(n\sqrt{T}).
\]

Combining Eqns. (1), (2), and (4), we have, w.h.p.,

\[
\sum_{(i \neq j) \in B^2} \langle X_i, X_j \rangle = \sum_{t \in [T]} \left(\sum_{i \in B} X_i(t)\right)^2 - f \geq \sum_{t \in [T]} (Z(t) - f) \geq \frac{1}{2} T(n - 3f) - \tilde{O}(n\sqrt{T}) = \epsilon nT/2 - \tilde{O}(n\sqrt{T}).
\]

We lower bound the average correlation score within \( B \) by dividing Eqn. (5) by the \( f(f-1) \) distinct pairs \( i, j \in B^2 \). Using the fact that \( f = n/(3+\epsilon) \), we have

\[
\max_{i^*,j^* \in B, i^* \neq j^*} \langle X_i, X_j \rangle \geq \frac{1}{f(f-1)} \left( \epsilon nT/2 - \tilde{O}(n\sqrt{T}) \right) \geq \frac{(3+\epsilon)\epsilon}{2(f-1)} T - \tilde{O}(\sqrt{T}/n).
\]

Note that the \( \tilde{O}(\sqrt{T}/n) \) term is negligible and that \( \frac{(3+\epsilon)\epsilon}{2(f-1)} T \gg \beta = \tilde{O}(\sqrt{T}) \) whenever \( T = \tilde{O}(n/\epsilon^2) \).

### 3.2 The Real Coin-Flipping Game

In this section we describe a protocol for calling Coin-Flip() iteratively in the context of Bracha-Agreement. It is based on a coin-flipping game that differs from the simplified game of Section 3.1 in several respects, most of which stem from the power of the adversarial scheduler to give good players slightly different views of reality. The differences are as follows.

- The bad players are *not* fixed in advance, but may be corrupted at various times.
- Rather than picking \( X_i(t) \in \{-1, 1\} \), the processes generate an iterated blackboard \( BB \) where each write is a value in \( \{-1, 1\} \), chosen uniformly at random if the writing process is good. Each blackboard \( BB_i \) has \( n \) columns and \( m = \Theta(n/\epsilon^2) \) rows. When \( BB_i \) is complete, let \( X_i(t) \) be the sum of all non-⊥ values in column \( BB_i \) of \( i \). Every player’s view of reality is slightly different. \( X_i^{(p)}(t) \) refers to \( p \)'s most up-to-date view of \( X_i(t) \), which is initially the sum of column \( BB_i^{(q)}(\cdot, i) \). By Theorem 3, \( \sum_{t \in [n]} |X_i^{(p)}(t) - X_i(t)| \leq f \) for any \( p, t \).
- Each process \( i \) has a weight \( w_i \in [0, 1] \), initially 1, which is non-increasing over time. At all times, the processes maintain complete agreement on the weights of the actively participating processes, i.e., those who broadcast coin flips. This is accomplished as follows. By Theorem 3(3), if any process \( q \) writes to \( BB_i \), every other process \( p \) learns \( BB_i^{(q,t-1)} \) by the time they finish computing \( BB_i \). Based on the history \( BB_i^{(q,t-1)} \), \( p \) can locally compute the weight vector \( (w_i^{(q)})_{i \in [n]} \) of \( q \). However, due to different views of the history, \( (w_i^{(q)})_{i \in [n]} \) may be slightly different than \( (w_i^{(p)})_{i \in [n]} \). We reconcile this by defining the weight of each participating process based on its own view of history, i.e.

\[
w_i = \begin{cases} w_i^{(i)} & \text{if } w_i^{(i)} > w_{\min} \\ 0 & \text{otherwise}. \end{cases}
\]

In other words, \( w_i \) is drawn from the weight vector computed by process \( i \). Thus, by Theorem 3(3), the weight \( w_i \) of any process participating in \( BB_i \) is common knowledge. (It is fine that the weights of non-participating processes remain uncertain.) For technical reasons, a weight is rounded down to 0 if it is less than a small threshold, \( w_{\min} = \sqrt{n \ln n}/T \), where \( T \) is defined below.

- In iteration \( t \), process \( p \) sets its own output of Coin-Flip() to be \( \text{sgn} \left( \sum_{i \in [n]} w_i X_i^{(p)}(t) \right) \). If this quantity is \(-\sigma(t)\) for every good process \( p \), the game ends “naturally.” (In the next iteration of Bracha’s algorithm, all processes will decide on a common value.)
• The iterations are partitioned into $O(f)$ epochs, each with $T = \Theta(n^2 \ln^3 n/e^2)$ iterations, where the goal of each epoch is to either end the game naturally or gather enough statistical evidence to reduce the weight of some processes before the next epoch begins. This can be seen as fractional blacklisting.

• Because the scheduling adversary can avoid delivering messages from $f$ good processes, the resiliency of the protocol drops to $f = n/(4 + \epsilon)$. Any positive $\epsilon > 0$ suffices, so we can tolerate $f$ as high as $(n - 1)/4$. In some places we simplify calculations by assuming $\epsilon \leq 1/2$.

Throughout $c$ is an arbitrarily large constant. All “with high probability” bounds hold with probability $1 - n^{-\Omega(c)}$. Since each process flips at most $m$ coins in each iteration, by a Chernoff-Hoeffding bound (Theorem 22, Appendix B) we have

$$|X_i(t)| \leq \sqrt{cm \ln n} \overset{\text{def}}{=} X_{\text{max}}$$

holds for all $i, t$, with high probability. To simplify some arguments we will actually enforce this bound deterministically. If $X_i(t)$ is not in the interval $[-X_{\text{max}}, X_{\text{max}}]$, map it to the nearest value of $\pm X_{\text{max}}$.

With high probability, the weight updates always respect Invariant 1, which says that the total weight-reduction of good players is at most the weight reduction of bad players, up to an additive error of $e^2 f/8$. This error term arises from the fact that we are integrating slightly inconsistent weight vectors $(w_i^{(p)})$ for each $p$ to yield $(w_i)$. With the assumption $\epsilon \leq 1/2$, Invariant 1 implies that the total weight of good processes is always $\Omega(n)$.

**Invariant 1.** Let $G$ and $B$ denote the set of good and bad processes at any given time. Then,

$$\sum_{i \in G} (1 - w_i) \leq \sum_{i \in B} (1 - w_i) + e^2 f/8.$$ 

Whereas pairwise correlations alone suffice to detect bad players in the simplified game, the bad players can win the real coin-flipping game without being detected by this particular test. As we will see, this can only be accomplished if $\{X_i(t)\}_{t \in [T]}$ differs significantly from a binomial distribution, for some $i \in B$. Thus, in the real game we measure individual deviations in the $l_2$-norm in addition to pairwise correlations. Define $\text{dev}(i)$ and $\text{corr}(i, j)$ at the end of a particular epoch as below. The iterations of the epoch are indexed by $t \in [T]$ and throughout the epoch the weights $\{w_i\}$ are unchanging.

$$\text{dev}(i) = \sum_{t \in [T]} (w_i X_i(t))^2,$$

$$\text{corr}(i, j) = \sum_{t \in [T]} w_i w_j X_i(t) X_j(t).$$

Naturally each process $p$ estimates these quantities using its view of the historical record; let them be $\text{dev}^{(p)}(i)$ and $\text{corr}^{(p)}(i, j)$.

The Gap Lemma says that if we set the deviation and correlation thresholds $(\alpha_T, \beta_T)$ properly, no good player will exceed its deviation budget, no pairs of good players will exceed their correlation budget, but some bad player or pair involving a bad player will be detected by one of these tests. One subtle point to keep in mind in this section is that random variables that depend on the coins flipped by good players can still be heavily manipulated by the scheduling power of the adversary.\(^7\) See [Mel20, Ch. 6] for further discussion of this issue.

**Lemma 5** (The Gap Lemma). Consider any epoch in which the game does not end, and let $\{w_i\}_{i \in [n]}$ be process weights. Let $G$ and $B$ be the good and bad processes at the end of the epoch. With high probability,

1. Every good $i \in G$ has $\text{dev}(i) \leq w_i^2 \alpha_T$, where $\alpha_T = m(T + \sqrt{T(c \ln n)^3})$.

2. Every pair $i, j \in G$ has $\text{corr}(i, j) \leq w_i w_j \beta_T$, where $\beta_T = m\sqrt{T(c \ln n)^3}$.

---

\(^7\)For example, by Doob’s optional stopping theorem for martingales, it is true that $\mathbb{E}[X_i(t)] = 0$, but not true that the distribution of $X_i(t)$ is symmetric around 0, or that it is close to binomial, or that we can say anything about $X_i(t)$ after conditioning on some natural event, e.g., that it was derived from summing the values in a full column of BB_t(,i).
3. If the weights satisfy Invariant 1 and no processes were added to $B$ in this epoch, then

$$\sum_{i \in B} \max\{0, \text{dev}(i) - w_i^2 \alpha_T\} + \sum_{(i \neq j) \in B^2} \max\{0, \text{corr}(i, j) - w_i w_j \beta_T\} \geq \frac{c}{10} f \alpha_T.$$ 

Proof of The Gap Lemma, Parts 1 and 2. Part 1. Fix a good process $i \in G$ and $t \in [T]$. For $r \in [m]$, let $\delta_r \in \{-1, 0, 1\}$ be the outcome of its $r$th coin-flip, being 0 if the adversary never lets it flip $r$ coins in iteration $t$. Then for any $r < s$, $E[\delta_r \delta_s] = 0$. This clearly holds when $\delta_s = 0$, and if the adversary lets the $s$th flip occur, $E[\delta_r \delta_s | \delta_s \neq 0, \delta_r] = 0$ since $\delta_s \in \{-1, 1\}$ is uniform and independent of $\delta_r$. Therefore, $E[(X_i(t))^2] = E[(\sum_{r=0}^{m} \delta_r)^2] = \sum_{r=0}^{m} E[\delta_r^2] + \sum_{r \neq s} E[\delta_r \delta_s] = \sum_{r=1}^{m} E[\delta_r^2] \leq m$.

Now consider the sequence of random variables $(S_t)_{t \in [0,T]}$ where $S_0 = 0$ and $S_t = S_{t-1} + (X_i(t))^2 - m$. Since $E[S_t | S_{t-1}, \ldots, S_0] \leq S_{t-1}$, $(S_t)$ is a supermartingale. For all $t \in [T]$ we guarantee $|X_i(t)| \leq X_{\text{max}}$, so $|S_t - S_{t-1}| = |(X_i(t))^2 - m| \leq X_{\text{max}}^2$. Hence, by Azuma’s inequality (Theorem 23, Appendix B), $S_T \leq X_{\text{max}}^2 \sqrt{T(c \ln n)}$ with probability $1 - \exp\{-(X_{\text{max}}^2 \sqrt{T(c \ln n)})^2/2T X_{\text{max}}^2\} = 1 - n^{-\Omega(c)}$. Therefore, with high probability, for all $t \in [n]$,

$$\text{dev}(i) = \sum_{t=1}^{T} (w_i^2 X_i(t))^2 = w_i^2 (S_T + T m) \leq w_i^2 (T m + X_{\text{max}}^2 \sqrt{T(c \ln n)}) = w_i^2 m (T + \sqrt{T(c \ln n)^3}) = w_i^2 \alpha_T.$$

Part 2. Fix a $t \in [T]$ and let $\delta_{i,r} \in \{-1, 0, 1\}$ be the outcome of the $r$th coin-flip of $i$ in iteration $t$. By the same argument as above, $E[X_i(t) X_j(t)] = E[\sum_{r=0}^{m} \delta_{i,r} \delta_{j,s}] = \sum_{r,s} E[\delta_{i,r} \delta_{j,s}] = 0$. Now consider the sequence $(S_i)_t \in [0,T]$ where $S_0 = 0$ and $S_t = S_{t-1} + X_i(t) X_j(t)$. It follows that $E[S_i | S_{t-1}, \ldots, S_0] = S_{t-1}$, so $(S_i)$ is a martingale. By assumption, for all $t$, both $|X_i(t)|, |X_j(t)| \leq X_{\text{max}}$. So, $|S_t - S_{t-1}| = |X_i(t)||X_j(t)| \leq X_{\text{max}}^2$. By Azuma’s inequality (Theorem 23, Appendix B), $S_T \leq X_{\text{max}}^2 \sqrt{T(c \ln n)}$ with probability $1 - n^{-\Omega(c)}$. Therefore, with high probability, for all $i, j$,

$$\text{corr}(i, j) = \sum_{t=1}^{T} w_i w_j X_i(t) X_j(t) \leq w_i w_j X_{\text{max}}^2 \sqrt{c \ln n} = w_i w_j \cdot m \sqrt{c \ln n}^3 = w_i w_j \cdot \beta_T.$$ 

Part 3 of the Gap Lemma is proved in Lemmas 6–11. By Invariant 1, the total weight loss of the good players is at most the weight loss of the bad players plus $\epsilon^2 f/8$. Define $\rho$ to be the relative weight loss of the bad players:

$$\rho \geq 0 \text{ is such that } \sum_{i \in B} w_i = (1 - \rho) f.$$ 

Thus, at this moment $\sum_{i \in G} (1 - w_i) \leq \rho f + \epsilon^2 f/8$. Remember that the scheduling adversary can allow the protocol to progress while neglecting to schedule up to $f$ good players. Thus, in Lemma 6 we consider an arbitrary set $G \subset G$ of $n - 2f$ good players.

Lemma 6. If Invariant 1 holds then

1. For any $G \subseteq G$ with $|G| = n - 2f$, $\sum_{i \in G} w_i^2 \geq (1 - \max\{\rho/2, \epsilon/8\})^2 (n - 2f)$.

2. $\sum_{(i \neq j) \in B^2} w_i w_j \leq (1 - \rho)^2 f^2$ and $(1 - \rho)^2 f \leq \sum_{i \in B} w_i^2 \leq (1 - \rho) f$.

Proof. We first claim that, for any real numbers $\bar{w}_1, \ldots, \bar{w}_k \in [0, 1]$, if $\sum_{i=1}^{k} \bar{w}_i = (1 - \hat{\rho})k$ for some $\hat{\rho} \in [0, 1]$, then $(1 - \hat{\rho})^2 k \leq \sum_{i=1}^{k} \bar{w}_i^2 \leq (1 - \hat{\rho}) k$. The lower bound follows from Jensen’s inequality (Theorem 24) and is achieved when all weights are equal. The upper bound follows from the fact that $\bar{w}_i^2 \leq \bar{w}_i$.

Part 1. Note that

$$\sum_{i \in \hat{G}} w_i = n - 2f - \sum_{i \in \hat{G}} (1 - w_i) \geq n - 2f - (\rho + \epsilon^2) f \quad \text{(Invariant 1)}$$

$$= (1 - \rho + \epsilon^2/8)(n - 2f)$$

$$\geq (1 - \max\{\rho/2, \epsilon/8\})(n - 2f)$$

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Thus the relative weight loss from \( \hat{G} \)'s point of view is less than \( \hat{\rho} = \max\{\rho/2, \epsilon/8\} \), and from the first claim of the proof, \( \sum_{i \in \mathcal{G}} w_i^2 \geq (1 - \max\{\rho/2, \epsilon/8\})^2 (n - 2f) \).

**Part 2.** From the first claim of the proof with \( \hat{\rho} = \rho \), we have \( (1 - \rho)^2 f \leq \sum_{i \in B} w_i^2 \leq (1 - \rho) f \). For the other claim,\[
\sum_{(i \neq j) \in B^2} w_i w_j = (\sum_{i \in B} w_i)^2 - \sum_{i \in B} w_i^2 \leq (1 - \rho)^2 f^2 - (1 - \rho)^2 f^2 \leq (1 - \rho)^2 f^2 .
\]

Let us recall a few key facts about the game. Before \( \mathbb{B}_f \) is constructed the adversary commits to its desired direction \( \sigma(t) \). The \( m \times n \) matrix \( \mathbb{B}_f \) is complete when it has \( n - f \) full columns, therefore the adversary must allow at least \( m(n - 2f) \) coins to be flipped by good players. We define \( S_G(t) \) to be the weighted sum of all the coin flips flipped by good players. I.e., if the set \( G \) is stable throughout iteration \( t \) then
\[
S_G(t) = \sum_{i \in G} w_i X_i(t) .
\]

If a process \( i \) were corrupted in the middle of iteration \( t \) then only a prefix of its coin flips would contribute to \( S_G(t) \). If \( \text{sgn}(S_G(t)) = \sigma(t) \) then the adversary is happy. For example, it can just let the sum of the coin flips controlled by corrupted players sum up to zero, which does not look particularly suspicious. However, if \( \text{sgn}(S_G(t)) \neq \sigma(t) \) then the adversary must counteract the good coin flips. Due to disagreements in the state of the blackboard (see Lemma 10), players can disagree about the sum of blackboard entries by up to \( f \), so the adversary may only need to counteract the good players by \( -\sigma(t) S_G(t) \). \textbf{Lemma 7} lower bounds the second moment of this objective.

**Lemma 7.** For all \( t \in [T] \),
\[
\mathbb{E}[(\max\{0, -\sigma(t) S_G(t) - f\})^2] \geq m \left(1 - \max\{\rho/2, \epsilon/8\}\right)^2 (n/2 - f) - \epsilon f/16 .
\]

**Proof.** Let \( S_r, r \geq 0 \), be the weighted sum of the first \( m(n - 2f) + r \) coin flips generated by good players, and \( Z_r = (\max\{0, -\sigma(t) S_G(t) - f\})^2 \) be the objective function for \( S_r \). The adversary can choose to stop letting the good players flip coins at any time after \( m(n - 2f) \), thus \( \mathbb{E}[(\max\{0, -\sigma(t) S_G(t) - f\})^2] = \mathbb{E}[Z_{2mf}] \), which we argue is at least \( \mathbb{E}[Z_0] \). Note that if \( Z_{r-1} = 0 \) then the adversary has achieved the minimum objective and has no interest in further flips, so \( \mathbb{E}[Z_r \mid Z_{r-1} = 0] \geq Z_{r-1} \). If \( Z_{r-1} > 0 \), then were the adversary to allow some \( i \in G \) to flip another coin, we would have \( S_r = S_{r-1} + w_i \delta_i, \delta_i \in \{-1, 1\} \), and
\[
Z_r = \begin{cases}
\sigma(t) S_{r-1} - f + w_i^2 = Z_{r-1} + 2w_i(\sigma(t) S_{r-1} - f) + w_i^2 & \text{with probability } \frac{1}{2} , \\
-\sigma(t) S_{r-1} - f - w_i^2 = Z_{r-1} - 2w_i(\sigma(t) S_{r-1} - f) + w_i^2 & \text{with probability } \frac{1}{2} .
\end{cases}
\]

Thus, \( \mathbb{E}[Z_r \mid Z_{r-1} > 0, |\delta_i| > 0| = Z_{r-1} + w_i^2 \geq Z_{r-1} \), i.e., if the adversary is trying to minimize the objective function \( (\max\{0, -\sigma(t) S_G(t) - f\})^2 \), it will not allow any good coin flips beyond the bare minimum.

To lower bound \( \mathbb{E}[Z_0] \), the analysis above shows that any adversary minimizing this objective will let the player \( i \) with the smallest weight flip the next coin (thereby minimizing \( w_i^2 \)), conditioned on any prior history. Thus, in the worst case the \( n - 2f \) good players with the smallest weights each flip \( m \) coins.

We compute \( \mathbb{E}[Z_0] \) under this strategy. Since \( S_0 = \sum_{i=1}^{n-2f} \sum_{r=1}^m w_i \delta_i, \delta_i \in \{-1, 1\} \) are fair coin flips, \( \Pr(-\sigma(t) S_0 \geq 0) \geq \frac{1}{2} \) by a simple bijection argument \( \delta_i, \delta_i \mapsto -\delta_i, \delta_i \). Hence, \( \mathbb{E}[Z_0] \geq \frac{1}{2} \mathbb{E}[Z_0 \mid -\sigma(t) S_0 \geq 0] \).

Continuing,
\[
\mathbb{E}[Z_0 \mid -\sigma(t) S_0 \geq 0] = \mathbb{E}[(\sigma(t) S_0 - f) \mid -\sigma(t) S_0 \geq 0] + \mathbb{E}[Z_0 - (\sigma(t) S_0 - f)^2 \mid -\sigma(t) S_0 \geq 0] \\
\geq \mathbb{E}[(\sigma(t) S_0 - f)^2 \mid -\sigma(t) S_0 \geq 0 - f^2 \\
\geq \mathbb{E}[(S_0)^2 \mid -\sigma(t) S_0 \geq 0] - 2f \mathbb{E}[-\sigma(t) S_0 \mid -\sigma(t) S_0 \geq 0] \\
= \mathbb{E}[(S_0)^2] - 2f \mathbb{E}[\|S_0\|] .
\]

The first inequality comes from the fact that \( Z_0 \neq (\sigma(t) S_0 - f)^2 \) only when \( -\sigma(t) S_0 \in [0, f) \) (given the conditioning) and in this range is \( -\sigma(t) S_0 - f)^2 \geq -f^2 \). The second inequality comes from expanding \( -\sigma(t) S_0 - f)^2 \), linearity of expectation, and the fact that \( \sigma(t)^2 = 1 \). Since \( \mathbb{E}[\delta_i, \delta_i, \delta_i^r] = 0 \) for \( (i, r) \neq (i^r, r^r),
\[
\mathbb{E}[(S_0)^2] = \mathbb{E}[(\sum_{i, r} w_i \delta_i^r)^2] = \sum_{i, r, i^r, r^r} w_i w_{i^r} \mathbb{E}[\delta_i, \delta_i^r, \delta_i^r, \delta_i^r] = m \sum_{i=1}^{n-2f} w_i^2 .
\]

\[1\]
We bound the expected value of $|S_0|$ as follows

$$E[|S_0|] \leq \sqrt{E[(S_0)^2]} \quad \text{(Equation 1)}$$

and putting it all together we have

$$E[Z_0] \geq \frac{1}{2} E[Z_0 | -\sigma(t)S_0 \geq 0] \quad \text{(Lemma 6)}$$

The last line follows since $m = \Theta(n/\epsilon^2)$.

**Lemma 8.** With high probability, for every $t \in [T]$, $\max\{0, -\sigma(t)S_G(t) - f\}^2 \leq c_m n \ln n$.

**Proof.** The total number of good coin flips is at most $mn$. By a Chernoff-Hoeffding bound (Theorem 22, Appendix B), $S_G(t) \leq \sqrt{c_m n \ln n}$ with high probability and the lemma follows.

**Lemma 9.** With high probability,

$$\sum_{t=1}^{T} \max\{0, -\sigma(t)S_G(t) - f\}^2 \geq m \left[(1 - \max\{\rho/2, \epsilon/8\})^2(n/2 - f) - \epsilon f/16\right] T - n\sqrt{T(c\ln n)^3}. \quad \text{(Equation 2)}$$

**Proof.** Let $\gamma' = (1 - \max\{\rho/2, \epsilon/8\})^2(n/2 - f) - \epsilon f/16$. Consider the sequence of random variables $A_0, A_1, \ldots, A_T$, where $A_0 = 0$ and $A_t = A_{t-1} + \max\{0, -\sigma(t)S_G(t) - f\}^2 - m\gamma$. By Lemma 7, $E[A_t | A_{t-1}, \ldots, A_0] \geq 0$. So, $(A_t)$ is a submartingale. By Lemma 8, with high probability, for all $t \in [T]$, $\max\{0, -\sigma(t)S_G(t) - f\}^2 \leq m\gamma'$, where $\gamma' = cn \ln n$. Assuming this holds, $|A_t - A_{t-1}| \leq m\gamma'$ and, by Azuma’s inequality (Theorem 23, Appendix B), $A_T \leq -m\gamma'\sqrt{Tc\ln n}$ with probability $1 - n^{-\Omega(c)}$. Therefore, with high probability,

$$\sum_{t=1}^{T} \max\{0, -\sigma(t)S_G(t) - f\}^2 = m\gamma T + A_T \geq m(\gamma T - \gamma'\sqrt{Tc\ln n}). \quad \text{(Equation 2)}$$

**Lemma 10.** For every epoch in which no players are corrupted,

$$\sum_{i \in B} \text{dev}(i) + \sum_{(i \neq j) \in B^2} \text{corr}(i, j) \geq \sum_{t=1}^{T} \max\{0, -\sigma(t)S_G(t) - f\}^2.$$

**Proof.** Define $S_B(t)$ to be the sum of coin flips declared by corrupted players. I.e., if $B$ were stable throughout iteration $t$ then $S_B(t) = \sum_{i \in B} w_i X_i(t)$. Then

$$\sum_{t \in [T]} (S_B(t))^2 = \sum_{t \in [T]} \left(\sum_{i \in B} (w_i X_i(t))^2 + \sum_{(i \neq j) \in B^2} w_i w_j X_i(t) X_j(t)\right) = \sum_{i \in B} \text{dev}(i) + \sum_{(i \neq j) \in B^2} \text{corr}(i, j).$$

In iteration $t \in [T]$, the adversary must convince at least one good process $p$ that $\text{sgn} \left(\sum_i w_i X_i^{(p)}(t)\right) = \sigma(t)$. By Theorem 3, $\sum_i |X_i^{(p)}(t) - X_i(t)| \leq f$ and hence the total disagreement between $p$’s weighted sum and the true weighted sum is

$$\sum_i \left|w_i X_i^{(p)}(t) - w_i X_i(t)\right| = \sum_i w_i \left|X_i^{(p)}(t) - X_i(t)\right| \leq f.$$
Thus, if \(-\sigma(t)S_G(t) \geq f\) (the good players sum is in the non-adversarial direction by at least \(f\)) the bad players must correct it by setting \(\sigma(t)S_B(t) \geq -\sigma(t)S_B(t) - f\). Therefore, for any \(S_G(t)\) we must have 

\[ (S_B(t))^2 \geq \max\{0, -\sigma(t)S_G(t) - f\}^2 \]

and the lemma follows. \(\square\)

Recall from Parts 1 and 2 of The Gap Lemma (Lemma 5) that every good player \(i \in G\) has \(dev(i) \leq w_i^2\alpha_T\) and every good pair \((i, j) \in G^2\) has \(corr(i, j) \leq w_iw_j\beta_T\). Lemma 11 lower bounds the excess of the dev / corr-values involving bad players, beyond these allowable thresholds.

**Lemma 11.** In any epoch in which no processes are corrupted, With high probability,

\[
\sum_{i \in B} \max\{0, dev(i) - w_i^2\alpha_T\} + \sum_{(i \neq j) \in B^2} \max\{0, corr(i, j) - w_iw_j\beta_T\} \geq \frac{\epsilon}{16} f\alpha_T.
\]

**Proof.** By Lemma 9 and Lemma 10, with high probability,

\[
\sum_{i \in B} dev(i) + \sum_{(i \neq j) \in B^2} corr(i, j) \geq m \left(\left((1 - \max\{\rho/2, \epsilon/8\})^2(n/2 - f) - \epsilon f/16\right)T - n\sqrt{T(c \ln n)^3}\right). \tag{1}
\]

Recall that \(\alpha_T = m(T + \sqrt{T(c \ln n)^3})\), \(\beta_T = m\sqrt{T(c \ln n)^3}\), and, by Lemma 6, that \(\sum_{i \in B} w_i^2 \leq (1 - \rho)^2 f^2\) and \(\sum_{(i \neq j) \in B^2} w_iw_j \leq (1 - \rho)^2 f^2\). Putting these together we have

\[
\alpha_T \sum_{i \in B} w_i^2 + \beta_T \sum_{(i \neq j) \in B^2} w_iw_j \leq m \left(T + \sqrt{T(c \ln n)^3}\right) \cdot (1 - \rho)f + m\sqrt{T(c \ln n)^3} \cdot (1 - \rho)^2 f^2. \tag{2}
\]

The expression we wish to bound is at least (1) minus (2), namely:

\[
m \left(\left((1 - \max\{\rho/2, \epsilon/8\})^2(n/2 - f) - \epsilon f/16 - (1 - \rho)f\right)T - \left((1 - \rho)f + (1 - \rho)^2 f^2\right)\sqrt{T(c \ln n)^3}\right) \tag{3}
\]

Now depending on the larger value of \(\rho/2\) and \(\epsilon/8\), there are two cases expanding Equation 3.

**Case 1:** \(\rho/2 \leq \epsilon/8\). In this case, we simplify Equation 3 by setting \(\rho = 0\).

\[
(Equation 3) \geq mT \left((1 - \epsilon/8)^2(1 + \epsilon/2)f - \epsilon f/16 - f\right) - f(f + 1)m\sqrt{T(c \ln n)^3}
\]

\[
\geq mT \left[(1 + \epsilon/4 - \epsilon/8)^2f - \epsilon f/16 - f\right] - n^2 m\sqrt{T(c \ln n)^3}
\]

\[
\geq \frac{\epsilon}{8} f mT - n^2 m\sqrt{T(c \ln n)^3} \tag{\epsilon \leq 1/2}
\]

\[
\geq \frac{\epsilon}{16} f\alpha_T.
\]

**Case 2:** \(\rho/2 > \epsilon/8\). In this case, we expand the \((1 - \rho/2)^2\) term and simplify Equation 3 using the identity \(n = (4 + \epsilon)f\).

\[
(Equation 3) \geq mT \left[(n/2 - 2f)(1 - \rho) + (n/2 - f)\rho^2/4 - \epsilon f/16\right] - n^2 m\sqrt{T(c \ln n)^3}
\]

\[
= mTf \left[(\epsilon/2)(1 - \rho) + (1 + \epsilon/2)\rho^2/4 - \epsilon f/16\right] - n^2 m\sqrt{T(c \ln n)^3},
\]

which is minimized when \(\rho = \epsilon/(1 + \epsilon/2)\), hence

\[
\geq mTf \left[\epsilon \left(1 - \frac{\epsilon}{1 + \epsilon/2}\right) + (1 + \epsilon/2)\left(\frac{\epsilon}{1 + \epsilon/2}\right)^2/4 - \epsilon f/16\right] - n^2 m\sqrt{T(c \ln n)^3}
\]

\[
= mTf \left[\frac{7\epsilon}{16} - \frac{\epsilon^2}{4(1 + \epsilon/2)}\right] - n^2 m\sqrt{T(c \ln n)^3},
\]

and since \(T = \Theta(n^2 \ln^3 n/\epsilon^2)\) and \(\alpha_T = m(T + \sqrt{T(c \ln n)^3})\), with \(\epsilon < 1/2\) this is lower bounded by

\[
\geq \frac{\epsilon}{4} f\alpha_T. \tag{\square}
\]

**Remark 1.** We are able to upper bound correlation scores between two good players, and lower bound the average correlation score between two bad players. However, the correlations between good and bad players cannot be usefully limited. This is why Lemma 10 and Lemma 11 only apply to epochs in which no processes are corrupted, since any \(corr(i, j)\) score is difficult to analyze when \(i\) is corrupted halfway through the epoch.
3.3 Weight Updates

When the $T$ iterations of an epoch $k$ are complete, we reduce the weight vector $(w_i)$ in preparation for epoch $k+1$. According to The Gap Lemma, if an individual deviation score $\dev(i)$ is too large, $i$ is bad w.h.p., and if a correlation score $\corr(i,j)$ is too large, $B \cap \{i, j\} \neq \emptyset$ w.h.p., so reducing both $i$ and $j$’s weights by the same amount preserves Invariant 1. With this in mind, Weight-Update (Algorithm 3) constructs a complete, vertex- and edge-capacitated graph $G$ on $[n]$, finds a fractional maximal matching $\mu$ in $G$, then docks the weights of $i$ and $j$ by $\mu(i, j)$, for each edge $(i, j)$.

**Definition 1** (Fractional Maximal Matching). Let $G = (V, E, c_V, c_E)$ be a graph where $c_V : V \rightarrow \mathbb{R}_{\geq 0}$ are vertex capacities and $c_E : E \rightarrow \mathbb{R}_{\geq 0}$ are edge capacities. A function $\mu : E \rightarrow \mathbb{R}_{\geq 0}$ is a feasible fractional matching if $\mu(i, j) \leq c_E(i, j)$ and $\sum_j \mu(i, j) \leq c_V(i)$. It is maximal if it is not strictly dominated by any feasible $\mu'$. The saturation level of $i$ is $\sum_j \mu(i, j)$; it is saturated if this equals $c_V(i)$. An edge $(i, j)$ is saturated if $\mu(i, j) = c_E(i, j)$. (Note that contrary to convention, a self-loop $(i,i)$ only counts once against the capacity of $i$, not twice.)

**Rounding Weights Down.** Recall that if $p$ participates in a blackboard $\BB_p$, that every other process can compute the weight vector computed from $p$’s local view $\BB_{B_{p, t-1}}$ through blackboard $t - 1$. The processes use a unified weight vector in which $w_i$ is derived only from $i$’s local view:

$$w_i = \begin{cases} w_i^{(i)} & \text{if } w_i^{(i)} > w_{\min} = \sqrt{\frac{n \ln n}{\beta}} \\ 0 & \text{otherwise.} \end{cases}$$

As we will see, the maximum pointwise disagreement $|w_i^{(p)} - w_i^{(q)}|$ between processes $p,q$ is at most $w_{\min}$, and as a consequence, if any $p$ thinks $w_i^{(p)} = 0$ then all processes agree that $w_i = 0$.

**Excess Graph.** The excess graph $G = (V, E, c_V, c_E)$ used in Algorithm 3 is a complete undirected graph on $V = [n]$, including self-loops, capacitated as follows:

$$c_V(i) = w_i,$$
$$c_E(i, i) = \frac{16}{\epsilon_f \alpha_T} \cdot \max\{0, \dev(i) - w_i^2 \alpha_T\},$$
$$c_E(i, j) = \frac{16}{\epsilon_f \alpha_T} \cdot 2 \max\{0, \corr(i, j) - w_i w_j \beta_T\},$$

The reason for the coefficient of “2” in the definition of $c_E(i,j)$ is that $(i,j)$ is a single, undirected edge, but it represents two correlation scores $\corr(i, j) = \corr(j, i)$, which were accounted for separately in Lemma 11. By parts 1 and 2 of The Gap Lemma, $c_E(i, j) = 0$ whenever both $i$ and $j$ are good.

The Weight-Update algorithm from the perspective of process $p$ is presented in Algorithm 3. We want to ensure that the fractional matchings computed by good processes are numerically very close to each other, and for this reason, we use a specific maximal matching algorithm called Rising-Tide (Algorithm 4) that has a continuous Lipschitz property, i.e., small perturbations to its input yield bounded perturbations to its output. Other natural maximal matching algorithms such as greedy do not have this property.

3.3.1 Rising Tide Algorithm

The Rising-Tide algorithm initializes $\mu = 0$ and simply simulates the continuous process of increasing all $\mu(i,j)$-values in lockstep, so long as $i,j$, and $(i,j)$ are not saturated. At the moment one becomes saturated, $\mu(i,j)$ is frozen at its current value.

**Lemma 12.** Rising-Tide (Algorithm 4) correctly returns a maximal fractional matching.

**Proof.** Obvious. \hfill $\square$

Recall that $c_V(i)$ is initialized to be the (old) weight $w_i$ and the new weight is set to be $c_V(i) - \sum_j \mu(i, j)$. We are mainly interested in differences in the new weight vector computed by processes that start from slightly different graphs $G, H$. Lemma 13 bounds these output differences in terms of their input differences.
Algorithm 3 Weight-Update from the perspective of process $p$.

Output: Weights $(w_{i,k})_{i \in [n], k \geq 0}$ where $w_{i,k-1}$ refers to the weight $w_i$ after processing epoch $k - 1$, and is used throughout epoch $k$.

1: Set $w_{i,0} \leftarrow 1$ for all $i$. \hfill $\triangleright$ All weights are 1 in epoch 1.
2: for epoch $k = 1, 2, \ldots, K_{\text{max}}$ do \hfill $\triangleright$ $K_{\text{max}}$ = last epoch
3: \hspace{1em} Play the coin flipping game for $T$ iterations with weights $(w_{i,k-1})$ and let $\text{dev}^{(p)}$ and $\text{corr}^{(p)}$ be the resulting deviation and correlation scores known to $p$. Construct the excess graph $G_{k}^{(p)}$ with capacities:
   \[
   c_{V}(i) = w_{i,k-1},
   \]
   \[
   c_{E}(i, i) = \frac{16}{\epsilon f_{\alpha T}} \cdot \max \left\{ 0, \text{dev}^{(p)}(i) - w_{i,k-1}^{2} \alpha T \right\},
   \]
   \[
   c_{E}(i, j) = \frac{16}{\epsilon f_{\alpha T}} \cdot 2 \max \left\{ 0, \text{corr}^{(p)}(i, j) - w_{i,k-1} w_{j,k-1} \beta T \right\}.
   \]
4: $\mu_{k} \leftarrow \text{Rising-Tide}(G_{k})$ \hfill $\triangleright$ A maximal fractional matching
5: For each $i$ set
   \[
   w_{i,k}^{(p)} \leftarrow w_{i,k-1} - \sum_{j} \mu_{k}(i, j).
   \]
6: Once $(w_{i,k}^{(q)})$ are known for $q \in [n]$, set
   \[
   w_{i,k} = \begin{cases}
   w_{i,k}^{(i)} & \text{if } w_{i,k}^{(i)} > w_{\text{min}} \overset{\text{def}}{=} \frac{\sqrt{n \ln n}}{T} \\
   0 & \text{otherwise}.
   \end{cases}
   \]

Lemma 13 (Rising Tide Output). Let $G = (V, E, c_{V}^{G}, c_{E}^{G})$ and $H = (V, E, c_{V}^{H}, c_{E}^{H})$ be two capacitated graphs, which differ by $\eta_{E} = \sum_{i,j} |c_{E}^{G}(i, j) - c_{E}^{H}(i, j)|$ in their edge capacities and $\eta_{V} = \sum_{i} |c_{V}^{G}(i) - c_{V}^{H}(i)|$ in their vertex capacities. Let $\mu_{G}$ and $\mu_{H}$ be the fractional matching computed by Rising-Tide (Algorithm 4) on $G$ and $H$ respectively. Then:
   \[
   \sum_{i} \left| \left( c_{V}^{G}(i) - \sum_{j} \mu_{G}(i, j) \right) - \left( c_{V}^{H}(i) - \sum_{j} \mu_{H}(i, j) \right) \right| \leq \eta_{V} + 2\eta_{E}.
   \]

See Appendix C for proof of Lemma 13.

3.4 Error Accumulation and Reaching Agreement

The maximum number of epochs is $K_{\text{max}} = 2.5f$. Let $k \in [1, K_{\text{max}}]$ be the index of the current epoch, and let $w_{i,k-1}$ be the weights that were used in the execution of Coin-Flip() during epoch $k$. Upon completing epoch $k$, each process $p$ applies Algorithm 3 to update the consensus weight vector $(w_{i,k-1})_{i \in [n]}$ to produce a local weight vector $(w_{i,k}^{(p)})_{i \in [n]}$, and then the consensus weight vector $(w_{i,k})_{i \in [n]}$ used throughout epoch $k + 1$.

Lemma 14 (Maintaining Invariant 1). Suppose for some $\epsilon > 0$ that $n = (4 + \epsilon)f$, $m = \Theta(n/\epsilon^{2})$, and $T = \Theta(n^{2} \log^{2} n / \epsilon^{4})$. At any point in epoch $k \in [1, K_{\text{max}}]$, with high probability,
   \[
   \sum_{i \in G} (1 - w_{i,k-1}) \leq \sum_{i \in B} (1 - w_{i,k-1}) + \frac{\epsilon^{2}}{\sqrt{n}} \cdot (k - 1).
   \]

Proof. We prove by induction on $k$. For the base case $k = 1$ all the weights are 1 so Lemma 14 clearly holds. We will now prove that if the claim holds for $k$, it holds for $k + 1$ as well. Fix any good process $p$. 


Lemma 13

The vector \((w_{i,k}^{(p)})\) is derived from \((w_{i,k-1})\) by deducting at least as much weight from bad processes as from good processes, with high probability, and \((w_{i,k})\) is derived from \((w_{i,k}^{(q)})_{q\in[n],i\in[n]}\) by setting \(w_{i,k} = w_{i,k}^{(i)}\) and rounding down to 0 if it is at most \(w_{\text{min}}\). Thus, by the inductive hypothesis,

\[
\sum_{i\in G} (1 - w_{i,k}^{(p)}) \leq \sum_{i\in B} (1 - w_{i,k}^{(p)}) + \frac{\epsilon^2}{\sqrt{n}} \cdot (k - 1)
\]

Therefore,

\[
\sum_{i\in G} (1 - w_{i,k}) \leq \sum_{i\in B} (1 - w_{i,k}) + \frac{\epsilon^2}{\sqrt{n}} \cdot (k - 1) + \sum_{i\in[n]} |w_{i,k}^{(p)} - w_{i,k}^{(i)}| + w_{\text{min}} n_0,
\]

where \(n_0\) is the number of processes whose weight is rounded down to 0 after epoch \(k\).

Hence, it suffices to show that \(\sum_{i\in[n]} |w_{i,k+1}^{(p)} - w_{i,k+1}^{(i)}| + w_{\text{min}} n_0 \leq \epsilon^2/\sqrt{n}\). By Lemma 13, the computed weight difference between process \(p\) and any process \(q\) can be bounded by twice the sum of all edge capacity differences. According to Algorithm 3, the edge capacities differ due to underlying disagreement on the \(\text{dev}(i)\) and \(\text{corr}(i,j)\) values. Thus,

\[
|w_{i,k}^{(p)} - w_{i,k}^{(q)}| \leq 2 \cdot \frac{16}{\epsilon f \alpha_T} \left( \sum_i |\text{dev}^{(p)}(i) - \text{dev}^{(q)}(i)| + \sum_{i\neq j} |\text{corr}^{(p)}(i,j) - \text{corr}^{(q)}(i,j)| \right)
\]

By Theorem 3, two processes may only disagree in up to \(f\) cells of the blackboards \((BB_1, \ldots, BB_i)\). Since the sum of each column in each blackboard is bounded by \(X_{\text{max}}\), we have \(|\text{dev}^{(p)}(i) - \text{dev}^{(q)}(i)| < 2X_{\text{max}}\) for at most \(f\) values of \(i\), and \(|\text{corr}^{(p)}(i,j) - \text{corr}^{(q)}(i,j)| < 2X_{\text{max}}\) for at most \(nf\) pairs \(i\in B, j\in (G \cup B)\). Continuing,

\[
\leq 2 \cdot \frac{16}{\epsilon f \alpha_T} \left( f \cdot 2X_{\text{max}} + nf \cdot 2X_{\text{max}} \right)
\]

\[
\leq \frac{64(n + 1)X_{\text{max}}}{\epsilon m T}
\]

\[
\leq \frac{\sqrt{n \ln n}}{T}
\]

\[
= w_{\text{min}}
\]

(assuming \(m = \Omega(n/\epsilon^2))\)
Now the inductive step for $k$ holds by noticing that
\[
\sum_{i \in [n]} |w_{i,k}^{(p)} - w_{i,k}^{(i)}| + w_{\min} n_0 \leq 2w_{\min} n \\
\leq \frac{\epsilon^2}{n^{1.3} \log^2 n} \cdot n \\
< \frac{\epsilon^2}{\sqrt{n}}.
\]
(\text{using } T = \Omega(n^2 \log^3 n/\epsilon^2))

Therefore, with $K_{\text{max}} = 2.5f$ we obtain Invariant 1. That is, for any weight vector $(w_i)$ that are used on a blackboard,
\[
\sum_{i \in G} (1 - w_i) \leq \sum_{i \in B} (1 - w_i) + \frac{\epsilon^2}{\sqrt{n}} \cdot 3f
\leq \sum_{i \in B} (1 - w_i) + \frac{1}{8} \epsilon^2 f.
\]
(whenever $n \geq 576$)

Note that Invariant Invariant 1 is also preserved whenever a process is corrupted, transferring it from $G$ to $B$.

The next observation and Lemma 15 shows that the weight of every bad process becomes 0 after running $K_{\text{max}}$ epochs of Weight-Updates without reaching agreement.

Observation 1. For any $i$ and $k$, if there exists process $p$ such that $w_{i,k}^{(p)} = 0$, then $w_{i,k} = 0$.

Proof. In the proof of Lemma 14 it was shown that $|w_{i,k}^{(p)} - w_{i,k}^{(i)}| \leq \sqrt{n \ln n}/T = w_{\min}$, hence if $w_{i,k}^{(p)} = 0$, $w_{i,k}$ is rounded down to 0. See Algorithm 3.

Lemma 15. If agreement has not been reached after $K_{\text{max}} = 2.5f$ epochs, all bad processes have weight 0, with high probability.

Proof. There are at most $f$ epochs in which the adversary corrupts at least one process. We argue below that after all other epochs, in the call to Weight-Update, the total edge capacity of the graph induced by $B$ is at least 1. This implies that in each iteration of Weight-Update, either some $i \in B$ with $c_V(i) = w_i > w_{\min}$ becomes saturated (and thereafter $w_i = 0$ by Observation 1), or the total weight of all processes in $B$ drops by at least 2. The first case can occur at most $f$ times and the second at most $f/2$, hence after $K_{\text{max}} = 2.5f$ epochs, all bad players’ weights are zero, with high probability.

We now prove that the total edge capacity is at least 1. Recall that each edge $(i,j)$, $i \neq j$, represents the two correlation scores $\text{corr}(i,j)$ and $\text{corr}(j,i)$. Hence, by Lemma 11, the sum of edge capacities on $B$ is:
\[
\sum_{(i,j) \in B} c_E(i,j) = \frac{16}{\epsilon f \alpha_T} \left( \sum_{i \in B} \max\{0, \text{dev}(i) - w_{i,k}^2 \alpha_T\} + \sum_{(i \neq j) \in B^2} \max\{0, \text{corr}(i,j) - w_{i,k} w_{j,k} \beta_T\} \right)
\geq \frac{16}{\epsilon f \alpha_T} \left( \frac{\epsilon}{16} \alpha_T \right)
\geq 1.
\]
(by Lemma 5)

Lemma 16. Suppose Invariant 1 holds. In any iteration in which the bad processes have zero weights, the good processes agree on the outcome of the coin flip, with constant probability.

Proof. Let $S = \sum_i w_i X_i(t)$ be the weighted sum of the players. Through its scheduling power, the adversary may still be able to create disagreements between good players on the outcome of the coin-flip if $S \in [-f, f]$. Moreover, good process still possess $\Omega(n)$ total weight by Invariant 1. With constant probability, $|S|$ is larger than its standard deviation, namely $\Theta(\sqrt{mn})$, which is much larger than $f$ as $m = \Omega(n/\epsilon^2)$. Thus, with constant probability all good players agree on the outcome.
Theorem 17. Suppose $n = (4 + \epsilon)f$ where $\epsilon > 0$, $m = \Theta(n/\epsilon^2)$, and $T = \Theta(n^2 \log^3 n/\epsilon^2)$. Using the implementation of Coin-Flip from Section 3, Bracha-Agreement solves Byzantine agreement with probability 1 in the full information, asynchronous model against an adaptive adversary. In expectation the total communication time is $\tilde{O}(n/\epsilon^4)$. The local computation at each process is polynomial in $n$.

Proof. By Lemma 15, after $K_{\text{max}} = 2.5f$ epochs, all bad processes’ weights become zero, with high probability. From then on, by Lemma 16, each iteration of Bracha-Agreement achieves agreement with constant probability. Thus, after one more epoch, all processes reach agreement with high probability. The total communication time (longest chain of dependent messages) is $O((K_{\text{max}} + 1)mT) = \tilde{O}(n/\epsilon^4)$. If, by chance, the processes fail to reach agreement after this much time, they restart the algorithm with all weights $w_i = 1$ and try again. Thus, the algorithm terminates with probability 1.

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A Proofs from Section 2

A.1 Reliable Broadcast: Proof of Theorem 1

**Theorem 1.** If a good process $p$ initiates the **Reliable-Broadcast** of $m_{p,\ell}$, then all good processes $q$ eventually accept $m_{p,\ell}$. Now suppose a bad process $p$ does so and some good $q$ accepts $m_{p,\ell}$. Then all other good $q'$ will eventually accept $m_{p,\ell}$, and no good $q'$ will accept any other $m'_{p,\ell} \neq m_{p,\ell}$. Moreover, all good processes accept $m_{p,\ell-1}$ before $m_{p,\ell}$, if $\ell > 1$. 
Proof. According to the first line of Reliable-Broadcast, no message \( m_{p,t} \) can be accepted until \( m_{p,t-1} \) is accepted, if \( \ell > 1 \). This establishes the FIFO property. The other correctness properties follow from several claims.

We claim that if two good processes \( q, q' \) send (ready, \( m_{p,t} \)) and (ready, \( m'_{p,t} \)), then \( m_{p,t} = m'_{p,t} \). Suppose not. Let \( q \) be the first good process to send a (ready, \( m_{p,t} \)) message, and let \( q' \) be the first good process to send a (ready, \( m'_{p,t} \)), for some \( m'_{p,t} \neq m_{p,t} \). By definition, \( q \) and \( q' \) received strictly more than \((n + f)/2\) (echo, \( m_{p,t} \)) and (echo, \( m'_{p,t} \)) messages, respectively. Thus, at least \( 2 \left( (n + f + 1)/2 \right) - n \geq f + 1 \) processes sent both \( q \) and \( q' \) conflicting (echo, \( \cdot \)) messages, and therefore some good process sent conflicting (echo, \( \cdot \)) messages, which is impossible.

We now claim that if a good process \( q \) accepts \( m_{p,t} \) then every good process eventually accepts \( m_{p,t} \). It follows that \( q \) has already accepted \( m_{p,1}, \ldots, m_{p,t-1} \). By induction, every other good process eventually accepts \( m_{p,t-1} \). Before accepting \( m_{p,t} \), \( q \) received at least \( f + 1 \) (ready, \( m_{p,t} \)) messages from good processes. These \( f + 1 \) messages will eventually be delivered to all \( n - f \geq 2f + 1 \) good processes, causing all to send their own (ready, \( m_{p,t} \)) messages and eventually accept the same value.

If the sender \( p \) is good, then every good process will clearly eventually accept \( m_{p,t} \). Moreover, as a consequence of the claims above, if \( p \) is bad it is impossible for good processes to accept different messages \( m_{p,t} \neq m'_{p,t} \).

**A.2 Iterated Blackboard**

The Iterated-Blackboard algorithm uses the reliable broadcast primitives broadcast and accept (Theorem 1) to construct a series of blackboards \( \text{BB} = (\text{BB}_1, \text{BB}_2, \ldots, \text{BB}_n) \), the columns of which are indexed by process IDs in \([n]\) and the rows of which are indices in \([0, m]\). The blackboard proper consists of rows 1, \ldots, \( m \); the purpose of row zero is to reduce disagreements between the views of good processes. Every process \( p \) maintains \( \text{BB}^{(p)} = (\text{BB}^{(p)}_1, \text{BB}^{(p)}_2, \ldots, \text{BB}^{(p)}_n) \), where \( \text{BB}^{(p)}_i(r, i) \) records the value written by process \( i \) to \( \text{BB}^{(p)}_i(r, i) \) and accepted by process \( p \), or \( \perp \) if no such value has yet been accepted by \( p \). Each process maintains a vector \( \text{last}^{(p)}_i \) indicating the position of the last accepted write from each process, i.e., \( \text{last}^{(p)}_i = (t, r) \) if \( p \) accepted \( i \)’s write to \( \text{BB}^{(p)}_i(r, i) \), but has yet to accept any subsequent writes from \( i \) to \( \text{BB}^{(p)}_i \), nor to \( \text{BB}^{(p)}_{i+1}, \text{BB}^{(p)}_{i+2}, \ldots \).

Algorithm 5 gives the algorithm Iterated-Blackboard(\( t \)) for generating \( \text{BB}^{(t)}_1 \) from the perspective of process \( p \). Process \( p \) may only begin executing it if \( t = 1 \) or if it has finished executing Iterated-Blackboard(\( t-1 \)) and therefore already fixed \( \text{BB}^{(p,t-1)} \) and \( \text{last}^{(p)}_t \).

**Algorithm 5 Iterated-Blackboard(\( t \)) from the perspective of process \( p \)**

1. Set complete(\( t \)) \( \leftarrow \) false and set \( \zeta \leftarrow \text{last}^{(p)}_{t-1} \) if \( t > 1 \) or any dummy value if \( t = 1 \). Broadcast the write \( \text{BB}^{(p)}(0, p) \leftarrow \zeta \).
2. upon validating \( \geq n - f \) accept(\( \text{BB}^{(p)}(m, q) \))’s for \( \geq n - f \) different \( q \) for the first time:
   set complete(\( t \)) \( \leftarrow \) true and \( \text{last}^{(p)}_t \leftarrow \text{last}^{(p)}_i \), then broadcast the vector \( \text{last}^{(p)}_i \).
3. upon validating accept(\( \text{BB}^{(p)}(r, p) \))’s from \( \geq n - f \) different processes for the first time:
   if \( \neg \text{complete}(t) \land (r < m) \) then generate a value \( \zeta \) and broadcast the write \( \text{BB}^{(p)}(r + 1, p) \leftarrow \zeta \).
4. upon validating \( \text{BB}^{(p)}(r, q) \) from process \( q \) for the first time:
   set \( \text{BB}^{(p)}_i(r, q) \leftarrow \text{BB}^{(p)}_i(r, q) \) and \( \text{last}^{(p)}(q) \leftarrow (t, r) \); if \( \neg \text{complete}(t) \) then broadcast accept(\( \text{BB}^{(p)}(r, q) \)).
5. upon validating \( \text{last}^{(q)}_i \) vectors from \( \geq n - f \) different processes \( q \) for the first time:
   set \( \text{last}^{(p)}_i(i) \leftarrow \max_q \{ \text{last}^{(q)}_i(i) \} \) (point-wise maximum, lexicographically).
   At this point \( \text{BB}^{(p,t)} = (\text{BB}^{(p,t)}_1, \ldots, \text{BB}^{(p,t)}_n) \) is fixed as follows:

\[
\text{BB}^{(p,t)}_i(r, i) = \begin{cases} 
\text{BB}^{(p,t)}_i(r, i) & \text{if } (i', r) \leq \text{last}^{(p)}_i(i) \text{ and } r \in [1, m] \\
\perp & \text{otherwise}
\end{cases}
\]
In Line 1, \( p \) broadcasts the write \( \text{BB}_t(0, p) \leftarrow \text{last}_{t-1}^{(p)} \) (or a dummy value if \( t = 1 \)). This serves two purposes: first, to let other processes know that \( p \) has begun \textit{Iterated-Blackboard}(t), and second, to let them know exactly how it fixed the history \( \text{BB}^{(p,t)} \) after \( \text{BB}_{t-1} \) was completed. Once any process accepts this message, if they do not already consider \( \text{BB}_t \) to be \textit{complete}, then they broadcast an \textit{acknowledgement} (Line 4). In general, once process \( p \) receives \( n - f \) acknowledgements for its write to \( \text{BB}_t(r, p) \), if \( r < m \) and \( p \) does not consider \( \text{BB}_t \) complete, it proceeds to generate and broadcast a write to \( \text{BB}_t(r + 1, p) \) (Line 3). A process \( q \) generates an acknowledgement \( \text{ack}(\text{BB}_t(r, p)) \) only if \( q \) accepts \( \text{BB}_t(r, p) \) and does not consider \( \text{BB}_t \) to be complete (Line 4).

An important point is that the conditions of the “upon” statements on Lines 2, 3, 4, and 5 are checked \textit{whenever} process \( p \) accepts a new message, in the order that they are written. In particular, process \( p \) may set \( \text{BB}_t^{(p)}(r, q) \) after it considers \( \text{BB}_t \) to be complete, and even after it has moved on to the execution of \textit{Iterated-Blackboard}(t + 1). (These can be thought of as \textit{retroactive} corrections to \( \text{BB}_t \).) However, the body of each “upon” statement is executed at most once.

Eventually, process \( p \) will see that a set of at least \( n - f \) columns of \( \text{BB}_t \) are full, i.e., for each such column \( q \), it has received at least \( n - f \) acknowledgements for the \( m \)th row of column \( q \). Once this occurs, process \( p \) sets \( \text{complete}(t) \) to true (Line 2). At this point there can still be considerable disagreements between \( p \)'s local view and another process’s local view of the blackboard. To (mostly) resolve this, process \( p \) broadcasts the current state of its last-vector \( \text{last}_{t}^{(p)} = \text{last}_{t-1}^{(p)} \). It then waits to receive \( \text{last}_{t}^{(q)} \) vectors from at least \( n - f \) different \( q \) before it finalizes what it considers to be the last position of each column \( i \) of \( \text{BB} \) at the end of the \( t \)th iteration, i.e., \( \text{max}_{q} \{ \text{last}_{t}^{(q)}(i) \} \) (Line 5).

A critical aspect of the protocol is that \( p \) refrains from participating in any broadcast unless it has validated the message, i.e., accepted messages that are a prerequisite for its existence. Specifically:

1. No process participates in the broadcast of a write to \( \text{BB}_t(r, p) \) unless \(( t, r ) = (1, 0) \) (it’s \( p \)'s first write) or it has accepted the last write from \( p \), which is \( \text{BB}_t(r - 1, p) \) or if \( r = 0 \), some \( \text{BB}_{t-1}(r', p) \). (This is already captured by the FIFO property of Theorem 1 but it is useful to highlight it again.) Moreover, no process participates in the broadcast of a write \( \text{BB}_t(0, p) \leftarrow \text{last}_{t-1}^{(p)} \) unless it has accepted \( n - f \) \( \text{last}_{t-1}^{(p)} \) vectors whose point-wise maxima are exactly \( \text{last}_{t-1}^{(p)} \).

2. No process participates in the broadcast of \( \text{ack}(\text{BB}_t(r, q)) \) until it has accepted \( \text{BB}_t(r, q) \).

3. No process participates in the broadcast of \( \text{BB}_t(r + 1, q) \) until it has accepted acknowledgements for \( \text{BB}_t(r, q) \) from at least \( n - f \) processes.

4. No process participates in the broadcast of \( \text{last}_{t}^{(q)} \) until it has, for all columns \( i \) and \( \text{last}_{t}^{(q)}(i) = (t', r) \) already accepted \( \text{BB}_{t'}(r, i) \). In other words, the process must first accept every blackboard value that \( q \) purports to have accepted.

To emulate this, we modify the \textit{Reliable-Broadcast} implementation (Algorithm 1) so that when process \( p \) receives a message inviting it to participate in a \textit{Reliable-Broadcast} (e.g., an “init” message), it simply delays reacting to the message until the prerequisite conditions are met. In the next lemma, we show that these changes to the broadcast mechanism do not cause any deadlocks.

\textbf{Lemma 18.} If at least \( n - f \) good processes execute \textit{Iterated-Blackboard}(t), then every good process that executes \textit{Iterated-Blackboard}(t) eventually sets \( \text{last}_{t}^{(p)} \) and \( \text{BB}^{(p,t)} \).

\textbf{Proof.} First, we claim that, if any good process considers \( \text{BB}_t \) complete, then every good process that executes \textit{Iterated-Blackboard}(t) eventually considers \( \text{BB}_t \) complete. Indeed, if a good process considers \( \text{BB}_t \) complete, then it must have accepted \( \text{ack}(\text{BB}_t(m, q)) \)s from at least \( n - f \) processes, for at least \( n - f \) values of \( q \) (Line 2). By the properties of reliable broadcast (Theorem 1), every other good process eventually accepts these acknowledgements as well. Thus, every good process that executes \textit{Iterated-Blackboard}(t) eventually considers \( \text{BB}_t \) complete.

Next, we claim that, if at least \( n - f \) good processes consider \( \text{BB}_t \), complete, then every good process \( p \) that executes \textit{Iterated-Blackboard}(t) will eventually set \( \text{last}_{t}^{(p)} \). Indeed, by Line 2, every good process \( q \) that considers \( \text{BB}_t \) complete will broadcast a \( \text{last}_{t}^{(q)} \) vector. By the properties of reliable broadcast, any
blackboard values accepted by \( q \) will eventually be accepted by every good process and, hence, every good process will eventually participate in \( q \)'s broadcast of \( \text{last}_{i}^{(q)} \). Thus, every good process \( p \) eventually accepts \( \text{last}_{i}^{(q)} \) vectors from at least \( n - f \) different processes \( q \) and sets \( \text{last}_{i}^{(p)} \) (Line 5).

Finally, since at least \( n - f \) good processes execute \text{Iterated-Blackboard}(t)\), by our preceding discussion, it suffices to show that at least one such process considers \( \text{BB}_{i} \) complete. Suppose, for a contradiction, that this is not the case. Consider any good process \( p \) that executes \text{Iterated-Blackboard}(t) with the minimum number of writes to its column of \( \text{BB}_{i} \).

Suppose \( p \) writes to row \( m \). Then, by minimality, every good process \( q \) that executes \text{Iterated-Blackboard}(t) writes to row \( m \) in their respective columns, i.e., broadcasts \( \text{BB}_{i}(m, q) \). By the properties of reliable broadcast, the \( n - f \) \( \text{ack}(\text{BB}_{i}(m - 1, q)) \)'s that allow each such process \( q \) to broadcast \( \text{BB}_{i}(m, q) \) will eventually be accepted by every good process. Thus, every good process will eventually accept \( \text{BB}_{i}(m, q) \) and, as they do not consider \( \text{BB}_{i} \) complete by assumption, they will broadcast \( \text{ack}(\text{BB}_{i}(m, q)) \). Therefore, every good process will receive at least \( n - f \) \( \text{ack}(\text{BB}_{i}(m, q)) \) for at least \( n - f \) different \( q \) and, consequently, consider \( \text{BB}_{i} \) complete (Line 2), which is a contradiction.

Now suppose \( p \) writes to row \( r < m \). If \( r > 0 \), then the \( n - f \) \( \text{ack}(\text{BB}_{i}(r - 1, p)) \)'s that allow \( p \) to broadcast \( \text{BB}_{i}(r, p) \) will eventually be accepted by every good process. Hence, every good process will eventually participate in \( p \)'s broadcast of \( \text{BB}_{i}(r, p) \) and accept \( \text{BB}_{i}(r, p) \). Similarly, if \( r = 0 \). Since \( p \) does not write to row \( r + 1 \leq m \), it never accepts \( n - f \) \( \text{ack}(\text{BB}_{i}(r, p)) \)'s. Since at least \( n - f \) good process accept \( \text{BB}_{i}(r, p) \), it follows that at least one such process does not broadcast an \( \text{ack}(\text{BB}_{i}(r, p)) \). By Line 4, this process must have considered \( \text{BB}_{i} \) to be complete by the time it accepts \( \text{BB}_{i}(r, p) \), which is a contradiction.

Therefore, in both cases, we reach the desired contradiction. \( \square \)

Recall that \( p \)'s view of the history after executing \text{Iterated-Blackboard}(t) is \( \text{BB}^{(p, t)} \), defined to be:

\[
\text{BB}^{(p, t)}_{i}(r, i) = \begin{cases} 
\text{BB}^{(p)}_{i}(r, i) & \text{if } (t', r) \leq \text{last}^{(p)}_{i}(i) \text{ and } r \in [1, m] \\
\perp & \text{otherwise}
\end{cases}
\]

In other words, we obtain \( \text{BB}^{(p, t)}_{i} \) by stripping off the zeroth row of every \( \text{BB}^{(p)}_{i} \) matrix and replacing any values in column \( i \) after \( \text{last}^{(p)}_{i}(i) \) with \( \perp \). We emphasize that, in contrast to the local matrix \( \text{BB}^{(p)}_{i} \) of process \( p \), once \( \text{BB}^{(p, t)}_{i} \) is set, it never changes. (In the context of Bracha’s algorithm, \( p \) uses \( \text{BB}^{(p, t)}_{i} \) to decide the outcome of the \( t \)th call to \text{Coin-Flip}()\), but the series of blackboards \( \text{BB}^{(p, :)} \) is used to decide how \( p \) reduces the weight vector \( (w_{i}) \) after the first epoch.)

**Lemma 19.** Suppose at least \( n - f \) good processes execute \text{Iterated-Blackboard}(t) and some good process \( q \) accepts \( \text{ack}(\text{BB}_{i}(r, i)) \) from at least \( n - f \) different processes. Then every good process \( p \) that finishes iteration \( t \) has \( \text{BB}^{(p)}_{i}(r, i) = \text{BB}^{(p, t)}_{i}(r, i) = \text{BB}_{i}(r, i) \).

**Proof.** Since there are at most \( f \) bad processes, \( q \) accepts \( \text{ack}(\text{BB}_{i}(r, i)) \)'s from at least \( n - 2f \) good processes. Thus, \( \text{last}^{(q')}_{i}(i) \geq (t, r) \) holds for a set \( S_{0} \) of at least \( n - 2f \) processes \( q' \). Similarly, when a good process \( p \) finishes iteration \( t \), it has received \( \text{last}^{(q')}_{i}(i) \) vectors from a set \( S_{1} \) of at least \( n - 2f \) good processes. Since \( n > 3f \), \( n - 2f > (n - f) / 2 \) and \( S_{0} \cap S_{1} \) contains at least one common good process. Thus, \( \text{last}^{(p)}_{i}(i) = \max_{q'} \{ \text{last}^{(q')}_{i}(i) \} \geq (t, r) \), meaning that \( p \) will not finish Line 5 until it accepts \( \text{BB}_{i}(r, i) \) (due to validation), recording it in \( \text{BB}^{(p)}_{i} \) and hence \( \text{BB}^{(p, t)}_{i} \). \( \square \)

**Lemma 20.** Suppose that each good process executes \text{Iterated-Blackboard}(1), \ldots, \text{Iterated-Blackboard}(t)\), beginning iteration \( t + 1 \) only after it has executed Line 5 of iteration \( t \). Then, for any two good processes \( p, q \) that finish iteration \( t \), \( \text{BB}^{(p, t)} \) and \( \text{BB}^{(q, t)} \) disagree in at most \( f \) positions in total. If they disagree on the contents of any position, one is \( \perp \).

**Proof.** The properties of reliable broadcast ensures that \( \text{BB}^{(p, t)}_{i}, \text{BB}^{(q, t)}_{i} \) cannot contain distinct non-\( \perp \) values in any position. Therefore we must argue that they differ in at most \( f \) positions. Fix a process \( k \) and let \( \text{BB}_{i}(rk, k) \) be the last of \( k \)'s blackboard writes for which it accepted at least \( n - f \) \( \text{ack}(\text{BB}_{i}(rk, k)) \)’s. By Lemma 19, \( \text{BB}^{(p, t)}_{i}(rk, k) = \text{BB}^{(q, t)}_{i}(rk, k) \). Moreover, due to validation, \( p, q \) have both accepted all of \( k \)'s
Lemma 21. If $BB_{t+1}(1, q) \neq \perp$, then by the time $p$ fixes $BB^{(p,t)}$, it is aware of $q$’s history $BB^{(q,t)}$ through blackboard $t$.

Proof. Before $q$ wrote anything to $BB_{t+1}(1, q)$ it must have written $BB_{t+1}(0, q) \leftarrow \text{last}^{(q)}_t$ and caused $n-f$ acknowledgements $\text{ack}(BB_{t+1}(0, q))$ to be broadcast. By Lemma 19 every process will accept $BB_{t+1}(0, q)$ before fixing $BB^{(p,t+1)}$, and hence be able to reconstruct $BB^{(q,t)}$ from $\text{last}^{(q)}_t$. ∎

In conclusion, we have proved the following theorem:

**Theorem 3.** There is a protocol for $n$ processes to generate an iterated blackboard $BB$ that is resilient to $f < n/3$ Byzantine failures. For $t \geq 1$, the following properties hold:

1. Upon completion of the matrix $BB_t$, each column consists of a prefix of non-$\perp$ values and a suffix of all-$\perp$ values. Let last$(i) = (t', r)$ be the position of the last value written by process $i$, i.e., $BB_{t'}(r, i) \neq \perp$ and if $t' < t$ then $i$ has not written to any cells of $BB_t$. When $BB_t$ is complete, it has at least $n-f$ full columns and up to $f$ partial columns.

2. Once $BB_t$ is complete, each process $p$ forms a history $BB^{(p,t)} = (BB^{(p,t)}_1, \ldots, BB^{(p,t)}_t)$ such that for every $t' \in [t], i \in [n], r \in [m],$

$$BB^{(p,t)}_{t'}(r, i) \begin{cases} = BB_{t'}(r, i) & \text{if last}(i) \neq (t', r) \\ \in \{BB_{t'}(r, i), \perp\} & \text{otherwise} \end{cases}$$

3. If $q$ writes any non-$\perp$ value to $BB_{t+1}$, then by the time any process $p$ fixes $BB^{(p,t+1)}$, $p$ will be aware of $q$’s view $BB^{(q,t)}$ of the history up to blackboard $t$.

In the context of our implementation of Bracha’s algorithm, the purpose of Theorem 3(3) is to ensure that if $q$ writes anything to $BB_{t+1}$ (specifically $BB_{t+1}(1, q) \neq \perp$) that every other process $p$ can determine $w_q$, which is a function of $q$’s history $BB^{(q,t)}$.

### B Concentration Bounds

Random variables $X_1, \ldots, X_n$ are mutually independent (or, just independent) if for any sequence of real numbers $x_1, \ldots, x_n$, the events $\{X_1 \leq x_1\}, \ldots, \{X_n \leq x_n\}$ are independent.

**Theorem 22** (Chernoff-Hoeffding bound). Let $X_1, \ldots, X_n$ be independent random variables and let $X = \sum_{i=1}^n X_i$. If each $X_i \in [a_i, b_i]$ for some $a_i \leq b_i$, then, for all $t \geq 0$,

$$\Pr\{X - E[X] \geq t\}, \Pr\{X - E[X] \leq -t\} \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right).$$

A sequence of random variables $X_0, X_1, \ldots$ is a submartingale if $E[X_t | X_{t-1}, \ldots, X_0] \geq X_{t-1}$ for all $t$; it is a supermartingale if $E[X_t | X_{t-1}, \ldots, X_0] \leq X_{t-1}$; and it is a martingale if it is both a submartingale and a supermartingale.

**Theorem 23** (Azuma’s inequality). Let $X_0, X_1, \ldots$ be a martingale such that, for each $i \geq 1$, $|X_i - X_{i-1}| \leq c_i$ for some $c_i \geq 0$. Then for all $n \geq 1$ and $t \geq 0$,

$$\Pr\{X_n - X_0 \geq t\}, \Pr\{X_n - X_0 \leq -t\} \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^n c_i^2}\right).$$
Moreover, the bound on $\Pr\{X_n - X_0 \geq t\}$ holds when $\{X_t\}_{t\geq 0}$ is a supermartingale while the bound on $\Pr\{X_n - X_0 \leq -t\}$ holds when $\{X_t\}_{t\geq 0}$ is a submartingale.

A function $f : \mathbb{R} \to \mathbb{R}$ is convex if for any $x, y \in \mathbb{R}$ and $t \in [0, 1]$, $f((1-t)x + ty) \leq (1-t)f(x) + tf(y)$.

**Theorem 24** (Jensen’s inequality). Let $f : \mathbb{R} \to \mathbb{R}$ be convex. Then, for any random variable $X$, 
\[
f(E[X]) \leq E[f(X)].
\]

### C Rising Tide Algorithm Proofs

The goal to this entire section is to prove **Lemma 13**, the heart of bounding different perspectives from each process.

#### C.1 Dependency Graphs

We first introduce an idea of a dependency graph that captures the moments when vertices become saturated in **Algorithm 4**. We will then use structural properties of dependency graphs to prove **Lemma 13**.

**Definition 2** (Dependency Graph). Let $D_G$ be a directed graph with the same set of vertices $V(D_G) = V$. Consider the execution of **Algorithm 4** on $G$. For each edge $e = (i, j) \in E$, if at the moment $e$ is removed from the working set $E'$ (Line 7), $i$ (resp. $j$) is saturated, then we include in $D_G$ a directed edge $j \to i$ (resp. $i \to j$). Notice that if both $i$ and $j$ are saturated simultaneously, then $D_G$ includes both edges $i \to j$ and $j \to i$.

We first state a useful continuity property of Rising-Tide, that if we continuously deform the input capacities, the output fractional matching also changes continuously.

**Lemma 25** (The Continuity Lemma). Let $G$ and $H$ be two fractional matching instances where every vertex- and edge-capacity differs by at most $\xi$. Then, for every edge $e$, $|\mu_G(e) - \mu_H(e)| \leq F(n)\xi$ for some function $F$ which depends only on the size of the graph but not on $\xi$.

**Proof.** Without loss of generality we can assume that each edge capacity $c_G^E(i, j) \leq \min\{c_G^E(i), c_G^E(j)\}$ is always bounded by the capacities of its endpoints.

Imagine running Rising-Tide simultaneously on both $G$ and $H$, stopping at the first saturation event that occurs in, say, $G$ but not $H$. (A “saturation event” is the saturation of a vertex or edge with non-zero capacity.) Let $\mu'_G, \mu'_H$ be the fractional matchings at this time and $G', H'$ be the residual graphs, i.e., obtain new capacities by subtracting each $\mu'_G(i, j)$ from $c_G^E(i, j)$, and $c_H^E(i, j)$. The maximum difference in vertex- or edge-capacities between $G', H'$ is $n\xi$. The argument can be applied inductively to $G', H'$, and since there are $O(n^2)$ saturation events, the maximum difference between any capacity (and hence an $\mu$-value) is always bounded by $F(n)\xi$, where $F(n) = n^{O(n^2)}$. □

Note that the magnitude of $F$ is immaterial, so long as it depends only on $n$. **Lemma 25** allows us to make several simplifying assumptions.

A1. First, although we are comparing two graphs $G, H$ with possibly many capacity differences, we can assume w.l.o.g. that they differ in precisely one vertex- or edge-capacity.

A2. Second, we can assume that the dependency graphs for $G$ and $H$ are identical.

A3. Third, we can assume, via infinitesimal perturbations, that no two vertices are saturated simultaneously. In particular, this implies that $D_G$ is acyclic. (See **Lemma 26**.)

**Lemma 26** (Basic Properties between $\mu$ and $D_G$). Assume graph $G$ satisfies assumption (A3). Let $\mu$ be the output of $G$ from **Algorithm 4**. Then:

1. For any two edges $e_1$ and $e_2 \in E$, if $e_1$ gets removed from $E'$ before $e_2$, then $\mu(e_1) < \mu(e_2)$.

2. For each $u \in V$, all edges directed towards $u$ in $D_G$ have the same $\mu$-value.
(3) For any edge \( u \to v \) in \( D_G \) and any edge \((v, w) \in E\), \( \mu(u, v) \geq \mu(v, w) \).

(4) (Monotonic Path Property) For any walk \( u_0 \to u_1 \to u_2 \to \cdots \) on \( D_G \), their \( \mu \) values must be non-increasing. That is, \( \mu(u_0, u_1) \geq \mu(u_1, u_2) \geq \cdots \).

(5) (Directed Acyclic Graph Property) \( D_G \) is a DAG.

Proof. To show (1), it suffices to observe that in Algorithm 4 the fractional matching \( \mu \) grows strictly increasing at each iteration.

To show (2), it suffices to show that for each vertex \( u \in V \) with any two incoming edges \( v \to u \) and \( w \to u \) on \( D_G \), \( \mu(v, u) = \mu(w, u) \). Suppose conversely and without loss of generality \( \mu(v, u) > \mu(w, u) \). By the time \((w, u)\) gets removed from the working set \( E' \), \( u \) is already saturated. However, it is now impossible to increase \( \mu(v, u) \) anymore, contradicting to the assumption that \( \mu(v, u) > \mu(w, u) \).

To show (3), we notice that at the time \((u, v)\) is removed from \( E' \), \( v \) is saturated. At this moment any edge \((v, w) \in E\) incident to \( v \) cannot increase its \( \mu \) value anymore. Hence, \((v, w)\) will be removed from \( E' \) at the same time with \((u, v)\) or prior to the time when \((u, v)\) is removed from \( E' \). Thus, by (1) we have \( \mu(u, v) \geq \mu(v, w) \). (4) follows from (3) directly.

To show (5), assume contradictory that there exists a cycle \( u_0 \to u_1 \to \cdots \to u_0 \) in \( D_G \). By the monotonic path property (4), all edges \( \mu(u_1, u_{i+1}) \) have the same fractional value when they were removed from the working set \( E' \) in the rising tide algorithm. Moreover, by definition of \( D_G \), all vertices are simultaneously saturated, which contradicts (A3).

Henceforth (A3) is assumed to hold in all graphs.

C.2 Proof of Lemma 13

To prove Lemma 13, it suffices to show (via an interpolating argument) that the statement holds whenever (1) exactly one saturated vertex changes capacity but no edge changes capacity, or (2) exactly one saturated edge changes capacity but no vertex changes capacity. Moreover, with the continuity lemma (Lemma 25) it suffices to prove the statements (1) and (2) with the assumption that \( D_G = D_H \).

We start with some observations when there is only one change on the capacities between \( G \) and \( H \).

Lemma 27. Let \( G \) and \( H \) be two input graphs with the same dependency graph \( D := D_G = D_H \). If \((i, j)\) is an edge for which neither \( i \to j \) nor \( j \to i \) appear in \( D \), then \( \mu_G(i, j) = \mu_H(i, j) \).

Proof. From the definition of the dependency graphs, if both \( i \) and \( j \) are not saturated by the time \((i, j)\) gets removed from \( E' \), then by the fact (Lemma 12) that both \( \mu_G \) and \( \mu_H \) are maximal fractional matchings, \( \mu_G(i, j) = c_G^E(i, j) = c_H^E(i, j) = \mu_H(i, j) \).

Suppose graph \( G \) and \( H \) have the same capacities except at some vertex \( s \in V \). Then, Lemma 27 implies that if we run the rising tide algorithm on both instances \( G \) and \( H \), the first moment they differ from each other, is the case where on one graph \( s \) is saturated but on another graph \( s \) is not. In this case, we can think of \( s \) being the source of all the disagreement. Intuitively, if we look at an edge \( e \) where \( \mu_G(e) \neq \mu_H(e) \), we should be able to trace and blame this disagreement to the source of shenanigans.

Lemma 28. Assume that \( G \) and \( H \) differ only in the capacity of one vertex \( s \) and that \( D := D_G = D_H \). Consider any edge \((i, j)\) such that \( \mu_G(i, j) \neq \mu_H(i, j) \). Then, there exists a (possibly empty) paths in the dependency graph \( D \) from \( i \) and \( j \) to \( s \). Moreover, any edge \( e \) on this path satisfies \( \mu_G(e) \neq \mu_H(e) \).

Proof. Without loss of generality, when we consider an edge \((i, j)\) with \( \mu_G(i, j) \neq \mu_H(i, j) \), we may always assume \( j \to i \) appears in \( D_G \). (This edge must exist by Lemma 27.) That is, when \((i, j)\) is removed from the Rising-Tide algorithm that runs on \( G \) it is because \( i \) is saturated. Since \( D_G = D_H \), \( i \) is also saturated by the time when \((i, j)\) gets removed on both instances \( G \) and \( H \). Now we prove this lemma by induction on all edges from the smallest \( \mu_G \) value to the largest \( \mu_G \) value.

Base Case. Suppose \((i, j)\) is one of the edges with the minimum \( \mu_G \)-value such that \( \mu_G(i, j) \neq \mu_H(i, j) \). Since this is the first moment when the algorithm behaves differently, and we assume that \( j \to i \) on \( G \), it
follows that at time \( \mu_G(i, j) \), the vertex \( i \) is saturated in \( G \) but not in \( H \). Moreover, all other edges incident to \( i \) have the same \( \mu_G \)-value at this time. Therefore \( c^G_V(i) < c^H_V(i) \), and hence \( i = s \). There is a trivial path from \( i = s \) to \( s \) and a path from \( j \) to \( s \) via \( j \to i \).

**Inductive Case.** Now let us prove the inductive case. Suppose \( \mu_G(i, j) \neq \mu_H(i, j) \) and when \( (i, j) \) is removed from \( E' \), the vertex \( i \) is saturated. If \( i = s \) then we are done. Otherwise, we have \( c^G_V(i, j) = c^H_V(i, j) \). By Lemma 26 statements (2) and (3) and summing up all fractional matching values around the vertex \( i \), we know that \( \mu_G(i, j) < \mu_G(i, j) \). By Lemma 27 we know that there exists an edge \( (i, j) \) as well. If \( D \) is not saturated then there are no incoming edges to \( s \). Therefore, there exists paths from \( i \) and \( j \) to \( s \) in \( D \) as well.

Now, we prove the simplest version of Lemma 13 where only one vertex capacity is different with the assumption that the dependency graphs are the same.

**Lemma 29.** Assume \( G \) and \( H \) only differ in the capacity of one vertex \( s \), and that \( D := D_G = D_H \). Then, the total differences among the remaining vertex capacities can be bounded by

\[
\sum_i \left( c^G_V(i) - \mu_G(i, j) \right) - \left( c^H_V(i) - \mu_H(i, j) \right) \leq |c^G_V(s) - c^H_V(s)|.
\]

**Proof.** By Lemma 28, all edges that have different fractional matching values form a subgraph \( D_{\text{diff}} \) of \( D \) with \( s \) being the only minimal element. If \( s \) is not saturated then there are no incoming edges to \( s \). By Lemma 28 we know that \( D_{\text{diff}} = \emptyset \implies \mu_G = \mu_H \) and in this case the equality holds for the statement.

Observe that whenever there is an incoming edge to a vertex \( i \) in \( D \), the vertex \( i \) must be saturated. Since we are measuring differences in the remaining vertex capacities, the only place where such disagreement could happen is on all maximal vertices of \( D_{\text{diff}} \). Let \( T \) be the set of maximal vertices, i.e., those without incoming edges.

We prove a certain inequality by induction over all sets \( S \) such that \( S \subseteq V - T \) and \( S \) is downward closed, meaning there is no outgoing edge from \( S \) to \( V - S \). As a consequence \( s \in S \). Let \( \partial S \) be the set of incoming edges from \( V - S \) to \( S \). We will prove that for any coefficients \( \{\nu_{i \to j} \in [-1, 1] \}_{(i \to j) \in \partial S} \) we have

\[
\left| \sum_{(i \to j) \in \partial S} \nu_{i \to j} (\mu_G(i, j) - \mu_H(i, j)) \right| \leq |c^G_V(s) - c^H_V(s)|.
\]

**Base Case.** The minimal downward closed set is \( S = \{s\} \). By Lemma 26 statement (2) all incoming edges have the same \( \mu_G(i, s) - \mu_H(i, s) \) values. That is, all terms in \( \{\mu_G(i, s) - \mu_H(i, s) \} \) are of the same sign and hence claim is true for the base case.

**Inductive Case.** Consider any downward-closed set \( S \subseteq V - T \) with \( |S| \geq 2 \), and let \( \{\nu_{i \to j} \in [-1, 1] \}_{(i \to j) \in \partial S} \) be any set of coefficients on the fringe \( \partial S \). Let \( u \neq s \) be any maximal element in \( S \).

Let \( X_{\text{in}} \) and \( X_{\text{out}} \) be the set of incoming and outgoing edges incident to \( u \). Since \( S \) is downward-closed, we have

\[
\partial S = \partial(S - \{u\}) \cup X_{\text{in}} - X_{\text{out}}.
\]

Now, by Lemma 26 we know that each incoming edge \( (i \to u) \) in \( X_{\text{in}} \) has the same fractional matching value in both \( \mu_G \) and \( \mu_H \). We denote the difference by \( \Delta \equiv \mu_G(i, u) - \mu_H(i, u) \).
Let \( \nu_u = \frac{1}{\nu_{in}} \left( \sum_{(i \rightarrow u) \in X_{in}} \nu_{i \rightarrow u} \right) \in [-1, 1] \) be the average coefficient among all incoming edges. Since \( u \) is saturated, we have
\[
\sum_{(u \rightarrow j) \in X_{out}} \nu_u (\mu_G(u, j) - \mu_H(u, j)) + \sum_{(i \rightarrow u) \in X_{in}} \nu_{i \rightarrow u} (\mu_G(i, u) - \mu_H(i, u)) = \sum_{(u \rightarrow j) \in X_{out}} \nu_u (\mu_G(u, j) - \mu_H(u, j)) + \nu_u |X_{in}| \cdot \Delta \quad \text{(by definition of } \nu_u \text{)}
\]
\[
= \nu_u \left( \sum_{(u \rightarrow j) \in X_{out}} (\mu_G(u, j) - \mu_H(u, j)) + \sum_{(i \rightarrow u) \in X_{in}} (\mu_G(i, u) - \mu_H(i, u)) \right)
= \nu_u (c^G_V(u) - c^H_H(u))
= 0.
\]

Now, by removing \( u \) from \( S \) we have obtained a smaller subset which we can apply induction hypothesis on. Define coefficients \( \{\nu'_{i \rightarrow j}\} \) with \( \nu'_{u \rightarrow j} = -\nu_u \) for all \((u \rightarrow j) \in X_{out}\) and \( \nu'_{i \rightarrow u} = \nu_{i \rightarrow j} \) for all unrelated edges not incident to \( u \). Then, we have
\[
\left| \sum_{(i \rightarrow j) \in \partial S} \nu_{i \rightarrow j} (\mu_G(i, j) - \mu_H(i, j)) \right| \leq \left| \sum_{(i \rightarrow j) \in \partial(S \setminus \{u\})} \nu'_{i \rightarrow j} (\mu_G(i, j) - \mu_H(i, j)) \right| + \nu_u \cdot 0 \quad \text{(vertex } u \text{ is saturated)}
\]
\[
\leq |c^G_V(i) - c^H_H(i)|. \quad \text{(by induction hypothesis)}
\]

By choosing \( S = V \setminus T \) and coefficients \( \nu_{i \rightarrow j} = \text{sgn}(\mu_G(i, j) - \mu_H(i, j)) \) for every edge \((i \rightarrow j) \in \partial S\), we conclude that
\[
\sum_i \left| \left( c^G_V(i) - \sum_j \mu_G(i, j) \right) - \left( c^H_H(i) - \sum_j \mu_H(i, j) \right) \right|
= \sum_{i \neq s} \sum_j \mu_G(i, j) - \sum_j \mu_H(i, j) \quad \text{(for all } i \neq s, c^G_V(i) = c^H_H(i), \text{ and } s \text{ is saturated)}
= \sum_{(i \rightarrow j) \in \partial S} \nu_{i \rightarrow j} (\mu_G(i, j) - \mu_H(i, j)) \quad \text{(use } \nu_{i \rightarrow j} \text{ to remove the absolute operation)}
\]
\[
= \sum_{(i \rightarrow j) \in \partial S} \nu_{i \rightarrow j} (\mu_G(i, j) - \mu_H(i, j)) \quad \text{(this sum is positive)}
\leq |c^G_V(s) - c^H_H(s)|. \]

Similarly, by a reduction to vertices changes, we have the bound for the edge changes.

**Lemma 30.** Assume that \( G \) and \( H \) differ only in the capacity of one edge \((s, t) \in E\). Assume that \( D := D_G = D_H \). Then,
\[
\sum_i \left| \left( c^G_V(i) - \sum_j \mu_G(i, j) \right) - \left( c^H_H(i) - \sum_j \mu_H(i, j) \right) \right| \leq 2|c^G_E(s, t) - c^H_E(s, t)|.
\]

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Proof. The proof is by reduction to Lemma 29. Create $G'$ by subdividing $(s, t)$ into $(s, x), (x, t)$ with $c^G_{E}(s, x) = c^G_{E}(x, t) = \infty$ and $c^G_{V}(x) = 2c^G_{E}(s, t)$. Create $H'$ from $H$ in the same way. Since $D_G = D_H$, the same vertices must be saturated in both, and in particular, among $s, t, (s, t)$, both executions saturate the same element first. If they both saturate $s$ or $t$ first, then the capacity of $(s, t)$ has no influence on the execution and $\mu_G = \mu_H$. If they both saturate $(s, t)$ first, then the executions on $\{G, H\}$ proceed identically to their counterpart executions on $\{G', H'\}$. Note that $G', H'$ differ in one vertex capacity, with $|c^G_{V}(x) - c^H_{V}(x)| = 2|c^G_{E}(s, t) - c^H_{E}(s, t)|$. The lemma then follows from Lemma 29 applied to $G', H'$.

We can now prove Lemma 13.

Proof of Lemma 13. Imagine continuously transforming $(c^G_{V}, c^G_{E})$ into $(c^H_{V}, c^H_{E})$ by modifying one vertex capacity or one edge capacity at a time. In this continuous process there are two types of breakpoints to pay attention to. The first is when we switch from transforming one capacity to another, and the second is when the dependency graph changes. Let $G = G_0, G_1, \ldots, G_k = H$ be the sequence of graphs at these breakpoints. Up to a tie-breaking perturbation, we can assume each pair $(G_i, G_{i+1})$ differ in one edge or vertex capacity, and have the same dependency graph. By Lemma 25 the objective function is continuous in the input, and does not have any discontinuities at breakpoints. By Lemma 29 and Lemma 30 the objective function is bounded by $\sum_i(\eta_V(i) + 2\eta_E(i)) = \eta_V + 2\eta_E$.