ON MATRIX WREATH PRODUCTS OF ALGEBRAS

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Abstract. We introduce a new construction of matrix wreath products of algebras that is similar to the construction of wreath products of groups introduced by L. Kaloujnine and M. Krasner [17]. We then illustrate its usefulness by proving embedding theorems into finitely generated algebras and constructing nil algebras with prescribed Gelfand-Kirillov dimension.

1. Matrix Wreath Products

Let $F$ be a field and let $A, B$ be two associative $F$-algebras. Let $\text{Lin}(A, B)$ denote the vector space of all $F$-linear transformations $A \to B$.

We will define multiplication on $\text{Lin}(B, B \otimes_F A)$. Let $f, g \in \text{Lin}(B, B \otimes_F A)$. For an arbitrary element $b \in B$, let $g(b) = \sum b_i \otimes a_i$, where $a_i \in A$ and $b_i \in B$. Let $f(b_i) = \sum_{j} b_{ij} \otimes a_{ij}$, where $a_{ij} \in A$ and $b_{ij} \in B$. Define

$$(fg)(b) = \sum_{i,j} b_{ij} \otimes a_{ij}.$$ 

We define a structure of a $B$-bimodule on $\text{Lin}(B, B \otimes_F A)$. For an arbitrary element $b \in B$ and a linear transformation $f : B \to B \otimes_F A$, we define linear transformations $fb$ and $bf$ via

$$(fb)(b') = f(bb') \quad \text{and} \quad (bf)(b') = (b \otimes 1)f(b'), \quad b' \in B.$$ 

In other words, if $f(b') = \sum_i b_i \otimes a_i$ then $(bf)(b') = \sum_i bb_i \otimes a_i$. Now consider the semidirect sum

$$A \wr B = B + \text{Lin}(B, B \otimes_F A)$$

that extends multiplication on $B$ and on $\text{Lin}(B, B \otimes_F A)$.

Theorem 1. $A \wr B$ is an associative algebra.
Choose a basis \( \{ b_i \}_{i \in I} \) of the algebra \( B \). For a linear transformation \( f : B \to B \otimes_F A \), let
\[
f(b_j) = \sum_i b_i \otimes a_{ij}.
\]
Consider the \( I \times I \) matrix \( A_f = (a_{ij})_{I \times I} \). Each column of this matrix contains only finitely many nonzero entries.

Let \( M_{I \times I}(A) \) denote the algebra of \( I \times I \) matrices over \( A \) having finitely many nonzero entries in each column. Then \( f \mapsto A_f, f \in \text{Lin}(B, B \otimes_F A) \), is an isomorphism \( \text{Lin}(B, B \otimes_F A) \to M_{I \times I}(A) \).

The wreath product \( G_1 \wr G_2 \) of two groups \( G_1 \) and \( G_2 \) embeds in the multiplicative group of the matrix wreath product \( F G_1 \wr FG_2 \) of group algebras.

Indeed, let \( \text{Fun}(G_2, G_1) \) be the group of mappings from \( G_2 \) to \( G_1 \) with pointwise multiplication: \( (fg)(a) = f(a)g(a) \) for all \( f, g \in \text{Fun}(G_2, G_1) \) with \( a \in G_2 \). Then \( G_1 \wr G_2 \) is the semidirect product of \( G_2 \text{Fun}(G_2, G_1) \) with \( (b^{-1}fb)(a) = f(ba) \) for arbitrary elements \( a, b \in G_2 \).

For a mapping \( f : G_2 \to G_1 \), consider the “diagonal” linear transformation \( \text{diag}(f) : g \mapsto g \otimes f(g) \) for \( g \in G_2 \). The mappings \( b \mapsto b \) and \( f \mapsto \text{diag}(f) \) for \( b \in G_2 \) and \( f \in \text{Fun}(G_2, G_1) \) extend to an embedding of \( G_1 \wr G_2 \) into the multiplicative group \( F G_1 \wr FG_2 \).

If \( B_M \) is a left module over the algebra \( B \), then we can define
\[
A \wr_M B = B + \text{Lin}(M, M \otimes_F A).
\]

Different constructions of wreath products of Lie algebras were introduced by A. L. Smel’kin [27] and V. Petrogradsky, Y. Razmyslov, E. Shishkin [24] and L. Bartholdi [3].

In what follows, we will always assume that the algebra \( B \) is finitely generated, infinite dimensional, and, moreover, \( \{ b \in B : \dim bB < \infty \} = (0) \).

Along with the algebra of matrices \( M_{I \times I}(A) \), we will consider two important subalgebras:

1. \( M_\infty(A) \) that consists of \( I \times I \) matrices having finitely many nonzero entries, and
2. the subalgebra \( S(A, B) \) that consists of matrices having finitely many nonzero rows. In the language of linear transformations \( \varphi : B \to B \otimes_F A \), the subalgebra \( S(A, B) \) consists of such \( \varphi \) for which there exists a finite dimensional subspace \( V \subset B \) with \( \varphi(B) \subset V \otimes_F A \).

Clearly \( M_\infty(A) \subset S(A, B) \).

**Theorem 2.** Let \( M_\infty(A) \subseteq S \subseteq S(A, B) \) be a subalgebra such that \( BS + SB \subseteq S \). Then

1. the algebra \( B + S \) is prime if and only if the algebra \( A \) is prime, and
2. the algebra \( B + S \) is (left) primitive if and only if the algebra \( A \) is primitive.

We say that a linear transformation \( \gamma : B \to A \) is a generating linear transformation if \( \gamma(B) \) generates the algebra \( A \). Suppose that \( 1 \in B \). Let \( \gamma : B \to A \) be a generating linear transformation. Consider the element
\[
c_\gamma : b \mapsto 1 \otimes \gamma(b) \in B \otimes_F A.
\]
Consider the subalgebra \( (B, c_\gamma) \) generated in \( A \wr B \) by \( B \) and the element \( c_\gamma \).

For an element \( a \in A \) and two indices \( i, j \in I \), let \( e_{ij}(a) \) denote the matrix whose \((i, j)\)-entry is \( a \) and all other entries are equal to zero. For a fixed element
u ∈ A, we consider also the subalgebra \( \langle B, c_\gamma, e_{11}(u) \rangle \). Clearly, \( \langle B, c_\gamma, e_{11}(u) \rangle \) lies in \( B + S(A, B) \). If \( u = 1 \), then \( M_\infty(A) \subseteq \langle B, c_\gamma, e_{11}(1) \rangle \).

Since we always assume that the algebra \( B \) is finitely generated, the algebras \( \langle B, c_\gamma \rangle \), \( \langle B, c_\gamma, e_{11}(u) \rangle \) are finitely generated as well. Our immediate goal now is to estimate growth of these algebras.

We start with some general definitions. Consider an \( F \)-algebra \( R \) generated by a finite dimensional subspace \( V \). Let

\[
V^n = \text{span}_F \{ v_1 \cdots v_k \mid k \leq n, v_i \in V, 1 \leq i \leq k \}.
\]

Then \( \dim_F V^n < \infty \) and \( R \) is the union of the ascending chain \( V^1 \subseteq V^2 \subseteq \cdots \).

The function \( g(V, n) = \dim_F V^n \) is called the growth function of the algebra \( R \) that corresponds to the generating subspace \( V \).

Let \( \mathbb{N} \) denote the set of positive integers. Given two functions \( f, g : \mathbb{N} \to [1, \infty) \), we say that \( f \preceq g \) (\( f \) is asymptotically less than or equal to \( g \)) if there exists a constant \( c \in \mathbb{N} \) such that \( f(n) \leq cg(cn) \) for all \( n \in \mathbb{N} \). If \( f \preceq g \) and \( g \preceq f \), then \( f \) and \( g \) are said to be asymptotically equivalent, i.e., \( f \sim g \). If \( V \) and \( W \) are finite dimensional generating subspaces of \( A \), then \( g(V, n) \sim g(W, n) \). We will denote the class of equivalence of \( g(V, n) \) as \( g_A \).

A finitely generated algebra \( R \) has polynomially bounded growth if there exists \( \alpha > 0 \) such that \( g_R(n) \preceq n^\alpha \). Then

\[
\text{GKdim}(R) = \inf \{ \alpha > 0 \mid g_R(n) \preceq n^\alpha \}
\]

is called the Gelfand-Kirillov dimension of \( R \). If the growth of \( R \) is not polynomially bounded, then we let \( \text{GKdim}(R) = \infty \). If the algebra \( R \) is not finitely generated, then the Gelfand-Kirillov dimension of \( R \) is defined as the supremum of Gelfand-Kirillov dimensions of all finitely generated subalgebras of \( R \).

For \( n \geq 1 \), consider the vector space

\[
W_n = \sum_{i_1 + \cdots + i_r \leq n} \gamma(V^{i_1} \cdots \gamma(V^{i_r}),
\]

and let \( A = \bigcup_{n \geq 1} W_n \). Clearly, \( \dim_F W_n < \infty \) and \( W_1 \subseteq W_2 \subseteq \cdots \subseteq A \). Denote \( w_\gamma(n) = \dim_F W_n \). We call \( w_\gamma(n) \) the growth function of the linear transformation \( \gamma \).

A linear transformation \( \gamma : B \to A \) is said to be dense if for arbitrary linearly independent elements \( b_1, \ldots, b_n \in B \) and an arbitrary nonzero element \( a \in A \), there exists an element \( b \in B \) such that \( \gamma(b, b) = 0 \), \( 1 \leq i \leq n - 1 \), and \( a\gamma(b, b) \neq 0 \).

**Theorem 3.**

1. \( g_{(B, c_\gamma, e_{11}(u))} \preceq g_B^2(n)w_\gamma(n) \).
2. If the generating linear transformation \( \gamma \) is dense, then \( g_{(B, c_\gamma)}(n) \sim g_B^2(n)w_\gamma(n) \).

2. Embedding Theorems

G. Higman, H. Neumann, and B. H. Neumann [15] proved that every countable group embeds in a finitely generated group. The papers [4], [23], [25], and [30] show that some important properties can be inherited by these embeddings. Much of this work relies on wreath products of groups.

Following [15], A. I. Malcev [21] showed that every countable dimensional associative algebra over a field is embeddable in a finitely generated algebra, and
A. I. Shirshov [26] showed that every countable dimensional Lie algebra is embeddable in a finitely generated Lie algebra.

Let $A$ be an associative algebra, and let $I$ be a countable set. As above, we consider the algebra $M_\infty(A)$ of $I \times I$ matrices having finitely many nonzero entries. Clearly, the algebra $A$ is embeddable in $M_\infty(A)$ in many ways. We say that an algebra $A$ is $M_\infty$-embeddable in an algebra $B$ if there exists an embedding $\phi: M_\infty(A) \to B$. We say that $A$ is $M_\infty$-embeddable in $B$ as a (left, right) ideal if the image of $\phi$ is a (left, right) ideal in $B$.

Observe that [1] extended the theorem of Malcev in the following way: every countable dimensional associative algebra over a field is $M_\infty$-embeddable in a finitely generated algebra as an ideal.

3. Radical Algebras

S. Amitsur [2] asked if a finitely generated algebra can have a non-nil Jacobson radical. The first examples of such algebras were constructed by K. Beidar [9]. J. Bell [6] constructed examples having finite Gelfand-Kirillov dimension. Finally, L. Bartholdi and A. Smoktunowicz [29] constructed a finitely generated Jacobson radical non-nil algebra of Gelfand-Kirillov dimension two.

Theorem 4. An arbitrary countable dimensional Jacobson radical algebra is embeddable in a finitely generated Jacobson radical algebra.

Theorem 5. An arbitrary countable dimensional Jacobson radical algebra of Gelfand-Kirillov dimension $d$ over a countable field is embeddable in a finitely generated Jacobson radical algebra of Gelfand-Kirillov dimension $\leq d + 6$.

We will start with the following lemma.

Lemma 1. For an arbitrary Jacobson radical algebra $A$, there exists a Jacobson radical algebra $\tilde{A}$ and an element $u \in \tilde{A}$, with $u^3 = 0$, such that $A$ is embeddable in the right ideal $u\tilde{A}$ (resp. left ideal $\tilde{A}u$).

Proof. Consider the two dimensional nilpotent algebra $B = Fb + Fb^2$, $b^3 = 0$. Then $\tilde{A} = A \wr B = B + M_2(A)$ is a Jacobson radical algebra and $e_{22}(A) \subseteq b\tilde{A}$. □

Sketch of the proof of Theorem 4. Let $B$ be a finitely generated infinite dimensional nil algebra of E. S. Golod [11]. Let $\tilde{B} = B + F \cdot 1$ be its unital hull. Let $\tilde{A}$ be the Jacobson radical algebra of Lemma 1, $u \in \tilde{A}$, $u^3 = 0$, $A \subseteq \tilde{A}u$. Consider a generating linear transformation $\gamma: \tilde{B} \to \tilde{A}$ and the element $c_\gamma \in \text{Lin}(\tilde{B}, \tilde{B} \otimes F \tilde{A})$. Then the algebra $\langle B, c_\gamma, e_{11}(u) \rangle$ is finitely generated and Jacobson radical. Hence, the algebra $A$ is embeddable in a finitely generated Jacobson radical algebra $\langle B, c_\gamma, e_{11}(u) \rangle$. □

To prove Theorem 5, we will need the following lemma.

Lemma 2. Let $A$ be a countable dimensional algebra of Gelfand-Kirillov dimension $\leq d$. Let $B$ be an arbitrary finitely generated algebra. Then there exists a generating linear transformation $\gamma: B \to A$ such that $w_\gamma(n) \leq nd + \epsilon_n$ where $\epsilon_n > 0$, $\epsilon_n \to 0$ as $n \to \infty$.

Instead of the Golod nil algebra $B$, we will consider a finitely generated nil algebra $B$ of polynomially bounded growth. Existence of such algebras was established by T. Lenagan and A. Smoktunowicz in [19] under the assumption that the ground field is countable. In [20], T. Lenagan, A. Smoktunowicz, and A. Young refined the
argument of [19] and obtained a finitely generated nil algebra of Gelfand-Kirillov dimension ≤ 3.

Let $A \hookrightarrow \tilde{A}u$, $u^3 = 0$, be the embedding of Lemma 1, and let $B$ be the nil algebra of [20]. Arguing as above, we embed the algebra $A$ in the finitely generated subalgebra $\langle B, c_\gamma, e_{11}(u) \rangle$ of $\tilde{A} \wr \hat{B}$, where $\gamma$ is a generating linear transformation of Lemma 2. By Theorem 3(1), we have

$$g_{B, c_\gamma, e_{11}(u)} \leq g_B(u)^2w_\gamma(u),$$

which implies $\text{GKdim} \langle B, c_\gamma, e_{11}(u) \rangle \leq d + 6$.

4. Nil Algebras

We say that a nil algebra $A$ is stable nil (resp. stable algebraic) if all matrix algebras $M_n(A)$ are nil (resp. algebraic).

**Theorem 6.** An arbitrary stable nil algebra $A$ is embeddable in a finitely generated stable nil algebra. If $\text{GKdim} A = d < \infty$ and the ground field is countable, then $A$ is embeddable in a finitely generated nil algebra of Gelfand-Kirillov dimension $\leq d + 6$.

To use the wreath product constructions as above, we will need a finitely generated infinite dimensional stable nil algebra. Existence of such algebras can be established using methods from E. S. Golod [11] based on Golod-Shafarevich inequalities [12].

More precisely, let $F(x_1, \ldots, x_m)$ be the associative algebra on $m$ free generators, $m \geq 2$. We consider the free algebra without 1, i.e., it consists of formal linear combinations of nonempty words in $x_1, \ldots, x_m$. Assigning degree 1 to all variables $x_1, \ldots, x_m$, we make $F(x_1, \ldots, x_m)$ a graded algebra. The degree $\deg(a)$ of an arbitrary element $a \in F(x_1, \ldots, x_m)$ is defined as the minimal degree of a nonzero homogeneous component of $a$.

Let $R \subset F(x_1, \ldots, x_m)$ be a subset containing finitely many elements of each degree.

**Golod-Shafarevich Condition:** If there exists a number $\frac{1}{m} < t_0 < 1$ such that

$$\sum_{a \in R} t_0^{\deg(a)} < \infty \text{ and } 1 - mt_0 + \sum_{a \in R} t_0^{\deg(a)} < 0,$$

then the algebra $\langle x_1, \ldots, x_m \mid R = (0) \rangle$ presented by the set of generators $x_1, \ldots, x_m$ and the set of relations $R$ is infinite dimensional.

**Lemma 3.** For $m \geq 2$, there exists a subset $R \subset F(x_1, \ldots, x_m)$ satisfying the Golod-Shafarevich Condition and such that the algebra $\langle x_1, \ldots, x_m \mid R = (0) \rangle$ is stable nil.

For a stable nil algebra $A$ and its extension $A \subset \tilde{A}u$, $u^3 = 0$, of Lemma 1 and an algebra $B$ of Lemma 3, the finitely generated algebra $\langle B, c_\gamma, e_{11}(u) \rangle$ is stable nil. It implies the first part of Theorem 6.

Now let $F$ be a countable field, let $B$ be the Lenagan-Smoktunowicz-Young algebra [20], and let $A$ be a countable dimensional stable nil algebra of $\text{GKdim} A \leq d$. Then the algebra $\langle B, c_\gamma, e_{11}(u) \rangle$ is nil and has Gelfand-Kirillov dimension $\leq d + 6$. We do not know if this finitely generated algebra is stable nil.
5. PRIMITIVE ALGEBRAS

I. Kaplansky [18] asked if there exists an infinite dimensional finitely generated algebraic primitive algebra, a particular case of the celebrated Kurosh Problem. Such examples were constructed by J. Bell and L. Small in [7]. Then J. Bell, L. Small, and A. Smoktunowicz [8] constructed finitely generated algebraic primitive algebras of finite Gelfand-Kirillov dimension provided that the ground field is countable.

**Theorem 7.** An arbitrary countable dimensional stable algebraic primitive algebra is $M_\infty$-embeddable as a left ideal in a 2-generated algebraic primitive algebra.

This theorem answers the first part of question 7 from [8].

**Theorem 8.** Let $F$ be a countable field. An arbitrary countable dimensional stable algebraic primitive algebra of Gelfand-Kirillov dimension $\leq d$ is $M_\infty$-embeddable as a left ideal in a finitely generated algebraic primitive algebra of Gelfand-Kirillov dimension $\leq d + 6$.

Without loss of generality, we assume that a countable dimensional stable algebraic algebra $A$ is unital. As above, we start with Golod’s finitely generated infinite dimensional nil algebra $\hat{B}$ and a generating linear transformation $\gamma : \hat{B} \to A$. Then the algebra $\langle \hat{B}, c, e_{11}(1) \rangle$ is primitive by Theorem 2(2) and contains $M_\infty(A)$ as a left ideal.

The same argument with the Lenagan-Smoktunowicz-Young algebra $B$ and a linear transformation of Lemma 2 implies Theorem 8.

6. ALGEBRAS OF LOCALLY SUBEXPONENTIAL GROWTH

Recently, L. Bartholdi and A. Erschler [4] proved that a countable group of locally subexponential growth embeds in a finitely generated group of subexponential growth. We prove the analog of Bartholdi-Erschler theorem for algebras and semigroups and establish some related results.

Given two functions $f, g : \mathbb{N} \to [1, \infty)$, we say that $f$ is weakly asymptotically less than or equal to $g$ if for arbitrary $\alpha > 0$, we have $f \preceq gn^\alpha$ (denoted $f \preceq_w g$).

A function $f$ is subexponential if $\lim_{n \to \infty} \frac{f(n)}{e^{\alpha n}} = 0$ for any $\alpha > 0$. In the seminal paper [14], R. I. Grigorchuk constructed the first example of a group with an intermediate growth function: subexponential but growing faster than any polynomial. Finitely generated associative algebras with intermediate growth functions come as universal enveloping algebras of certain Lie algebras (see [28]).

A not necessarily finitely generated algebra $A$ is of locally subexponential growth if every finitely generated subalgebra of $A$ has a subexponential growth function.

The growth of $A$ is locally (resp. weakly) bounded by a function $f(n)$ if for every finitely generated subalgebra of $A$ its growth function is $\preceq f(n)$ (resp. $\preceq_w f(n)$).

A function $h(n)$ is superlinear if $\frac{h(n)}{n} \to \infty$ as $n \to \infty$.

**Theorem 9.** Let $f(n)$ be an increasing function. Let $A$ be a countable dimensional associative algebra whose growth is locally weakly bounded by $f(n)$. Let $h(n)$ be a superlinear function. Then the algebra $A$ is $M_\infty$-embeddable as a left ideal in a 2-generated algebra whose growth is weakly bounded by $f(h(n))n^2$.
We then use Theorem 9 to derive an analog of the Bartholdi-Erschler Theorem ([4]).

**Theorem 10.** A countable dimensional associative algebra of locally subexponential growth is $M_\infty$-embeddable in a 2-generated algebra of subexponential growth as a left ideal.

The idea of the proofs of Theorems 9 and 10 is the same as in previous sections. We consider the matrix wreath product $A \wr F[t^{-1},t]$ with the algebra $F[t^{-1},t]$ of Laurent polynomials and choose a generating linear transformation $\gamma : F[t^{-1},t] \to A$ with appropriate subexponential growth function $w_\gamma(n)$. The algebra $A$ is then $M_\infty$-embeddable as a left ideal in the finitely generated algebra $C = \langle F[t^{-1},t], c_\gamma, e_{11}(1) \rangle$. By V. T. Markov’s theorem [22], the matrix algebra $M_n(C)$ is 2-generated for a sufficiently large $n$, which yields the result.

Using [28] and Theorem 10, we can prove an embedding theorem for countable dimensional Lie algebra of locally subexponential growth.

**Theorem 11.** Let $F$ be a field of characteristic $\neq 2$. Every countable dimensional Lie $F$-algebra of locally subexponential growth is embeddable in a finitely generated Lie algebra of subexponential growth.

7. **Gelfand-Kirillov Dimension of Nil Algebras**

In this section, we assume that the ground field $F$ is countable. Question 1 from [8] asks if an arbitrary sufficiently big $\alpha \geq 2$ is the Gelfand-Kirillov dimension of some finitely generated nil algebra.

**Theorem 12.** Let $F$ be a countable field. For an arbitrary $d \geq 8$, there exists a finitely generated nil $F$-algebra of Gelfand-Kirillov dimension $d$.

Let $B$ be the finitely generated infinite dimensional algebra of Lenagan-Smoktunowicz-Young [20] with GKdim $B \leq 3$.

For an arbitrary $\alpha \geq 2$, W. Bohro and H. P. Kraft [10] constructed a graded $F$-algebra $R = \sum_{i=1}^{\infty} R_i$, generated by two elements $x, y \in R_1$, such that for any $\epsilon > 0$ we have

$$n^{\alpha-\epsilon} \leq \dim \sum_{i=1}^{n} R_i \leq n^{\alpha+\epsilon}$$

for all sufficiently large $n$.

Using the Bohro-Kraft algebra, we construct a countable dimensional locally nilpotent algebra $A$ and a dense generating linear transformation $\gamma : B \to A$ of growth $w_\gamma(n)$ such that for an arbitrary $0 < \epsilon < \alpha$, we have

$$\left( \frac{n}{\ln n} \right)^{\alpha-\epsilon} \leq w_\gamma(n) \leq n^{\alpha+\epsilon} (\ln(n))^2.$$ 

By Theorem 3, for the finitely generated algebra $C = \langle B, c_\gamma \rangle$, we have $g_\epsilon(n) \sim g_B(n)^2 w_\gamma(n)$, and therefore GKdim($C$) $= 2$ GKdim($B$) + $\alpha$, which implies Theorem 12.

**Question.** Let $g : \mathbb{N} \to \mathbb{N}$ be an increasing function such that $n^2 \leq g(n)$ and $g(m+n) \leq g(m)g(n)$ for all $m, n \in \mathbb{N}$. Is $g(n)$ asymptotically equivalent to the growth function of some finitely generated associative algebra?
Conjecture. For all sufficiently large functions \( g : \mathbb{N} \to \mathbb{N} \), the following assertions are equivalent:

1. \( g \) is asymptotically equivalent to the growth function of some finitely generated associative algebra,
2. \( g \) is asymptotically equivalent to the growth function of some finitely generated primitive algebra,
3. \( g \) is asymptotically equivalent to the growth function of some finitely generated nil algebra.

L. Bartholdi and A. Smoktunowicz [5] proved that if \( g \) is an increasing submultiplicative function such that \( g(Cn) \geq ng(n) \) for some \( C \in \mathbb{N} \) and all \( n \in \mathbb{N} \) then \( g \) is asymptotically equivalent to the growth function of a finitely generated associative algebra. Moreover, B. Greenfeld [13] showed that in this case there exists a finitely generated primitive monomial algebra with the growth function equivalent to \( g \). This partially answers the questions above.

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