ON THE SECOND BOUNDARY VALUE PROBLEM FOR LAGRANGIAN MEAN CURVATURE FLOW

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Abstract. We consider a fully nonlinear parabolic equation with nonlinear Neumann type boundary condition, and show that the longtime existence and convergence of the flow. Finally we apply this study to the boundary value problem for minimal Lagrangian graphs.

1. INTRODUCTION

Lagrangian mean curvature flow has been studied by many authors since the work of R.P. Thomas and S.T. Yau [1] about mean curvature flow of Lagrangian submanifolds of Calabi-Yau manifolds. Later K. Smoczyk and M.T. Wang obtain the long time existence and convergence of the Lagrangian mean curvature flow in some conditions (cf. [2], [3]). The progress on singularity of Lagrangian mean curvature flow make people have a deeper understanding to Thomas-Yau Conjectures such as J.Y. Chen and J.Y. Li [4], A. Neves [5], [6]. Recently several authors took the equation point of view to study the Lagrangian mean curvature flow such as [7], [8].

Inspired from a parabolic flow leading to the solution of an optimal transport problem [9], we consider the following Lagrangian mean curvature flow with boundary conditions

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \sum_{i=1}^{n} \arctan \lambda_i, \quad t > 0, \quad x \in \Omega, \\
Du(\Omega) &= \hat{\Omega}, \quad t > 0, \\
u &= u_0, \quad t = 0, \quad x \in \Omega.
\end{aligned}
\]

where \( \Omega, \hat{\Omega} \) be strict convex bounded domains with smooth boundary in \( \mathbb{R}^n \),

\[\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\]

be the eigenvalues of \( D^2u = [u_{ij}] \), and \( Du \) be a family of diffeomorphisms from \( \Omega \) to \( \hat{\Omega} \).

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To solve an optimal transportation, J. Kitagawa looked for solutions to the equations

\[
\frac{\partial u}{\partial t} - \ln \det(D^2 u - A(x, Du)) = -\ln B(x, Du) \quad t > 0, \quad x \in \Omega,
\]

\[
Du(\Omega) = \tilde{\Omega}, \quad t > 0,
\]

\[
u = u_0, \quad t = 0, \quad x \in \Omega.
\]

where \(A\) is a matrix value function and \(B\) is a scalar value function defined on the cost function and two measures related to the transportation. Under certain conditions on \(\Omega, \tilde{\Omega}, A, B\) and the initial function, he proved the long time existence to the above flow, and convergence to the solution of the optimal transport problem as \(t \to +\infty\).

In [10], Neumann and second boundary value problems for Hessian and Gauss curvature flows were carefully studied by O.C. Schnurer and K. Smoczyk. They showed that the flow exists for all times and converges eventually to the solution of the prescribed Gauss curvature equation.

Motivated from the above work, our main results concern the long time existence and convergence of the nonlinear parabolic flow (1.1) and then obtain the solution to a boundary value problem for minimal lagrangian graphs [11]. Now we can state our main theorem.

**Theorem 1.1.** Assume that \(\Omega, \tilde{\Omega}\) are bounded, strict convex domains with smooth boundary in \(\mathbb{R}^n\). Then for any given initial function

\[u_0 : \Omega \to \mathbb{R}\]

which is \(C^{2+\alpha}\) strictly convex and satisfies \(Du_0(\Omega) = \tilde{\Omega}\), the strictly convex solution of (1.1) exists for all \(t \geq 0\) and converges smoothly to a function \(u^\infty\) satisfying the second boundary problem

\[
\begin{aligned}
\Sigma_{i=1}^n & \arctan \lambda_i = c, \quad x \in \Omega, \\
Du(\Omega) & = \tilde{\Omega},
\end{aligned}
\]

where \(c\) is a constant determined by \(\Omega, \tilde{\Omega}\) and \(u_0\).

**Remark 1.2.** By the methods in [11], the initial function \(u_0\) can be obtained by considering

\[
\begin{aligned}
\Delta u & = c, \quad x \in \Omega, \\
Du(\Omega) & = \tilde{\Omega}.
\end{aligned}
\]

Here the goal is easier to attack because Laplace equation is simpler than special Lagrangian equation.

It’s well known that (1.2) is special Lagrangian equation with second boundary condition which the solution \((x, Du)\) is a minimal Lagrangian graph in \(\mathbb{R}^n \times \mathbb{R}^n\). Using the ideas of studying fully nonlinear elliptic equations, S. Brendle and M. Warren [11] obtained the existence and uniqueness of the solution to (1.2). As a consequence of Theorem 1.1, we proved the existence result of the minimal Lagrangian submanifolds with the same condition in \(\mathbb{R}^n \times \mathbb{R}^n\).
We outline our proofs as follows. In section 2, we establish the local existence result to the flow (1.1) by the inverse function theory. In section 3, we provide preliminary results which will be used in the proof of the theorem. The techniques used in this section are reflective of those in [12] and [10] to the second boundary value problem for fully nonlinear differential equations, but all of the corresponding a priori estimates to the solution in the current scenario need modification because the structure of (1.1) is unlike Monge-Ampère type. In section 4, we give the proof of our main result.

2. THE SHORT-TIME EXISTENCE OF THE PARABOLIC FLOW

Throughout the following Einstein’s convention of summation over repeated indices will be adopted. Denote

\[ u_i = \frac{\partial u}{\partial x_i}, \quad u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad u_{ijk} = \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k}, \cdots, \]

and

\[ [u^{ij}] = [u_{ij}]^{-1}, \quad F(D^2 u) = \Sigma_{i=1}^n \arctan \lambda_i, \quad F^{ij}(D^2 u) = \frac{\partial F}{\partial u_{ij}}, \quad \Omega_T = \Omega \times (0, T). \]

By the methods on the second boundary value problems for equations of Monge-Ampère type [12], the parabolic boundary condition in (1.1) can be reformulated as

\[ h(Du) = 0, \quad x \in \partial \Omega, \quad t > 0, \]

where \( h \) is a smooth function on \( \tilde{\Omega} \):

\[ \tilde{\Omega} = \{ p \in \mathbb{R}^n | h(p) > 0 \}, \quad |Dh|_{\partial \tilde{\Omega}} = 1. \]

The so called boundary defining function is strictly concave, i.e, \( \exists \theta > 0, \)

\[ \frac{\partial^2 h}{\partial y_i \partial y_j} \xi_i \xi_k \leq -\theta |\xi|^2, \quad \text{for} \quad \forall y = (y_1, y_2, \cdots, y_n) \in \tilde{\Omega}, \quad \xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n. \]

We also give the boundary defining function according to \( \Omega \) (cf.[11]):

\[ \tilde{\Omega} = \{ p \in \mathbb{R}^n | \tilde{h}(p) > 0 \}, \quad |D\tilde{h}|_{\partial \tilde{\Omega}} = 1, \]

\[ \exists \tilde{\theta} > 0, \quad \frac{\partial^2 \tilde{h}}{\partial y_i \partial y_j} \xi_i \xi_k \leq -\tilde{\theta} |\xi|^2, \quad \text{for} \quad \forall y = (y_1, y_2, \cdots, y_n) \in \tilde{\Omega}, \quad \xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n. \]

Thus the parabolic flow is equivalent to the evolution problem:

\[ \begin{aligned}
& \frac{\partial u}{\partial t} = \Sigma_{i=1}^n \arctan \lambda_i, \quad t > 0, \quad x \in \Omega, \\
& h(Du) = 0, \quad t > 0, \quad x \in \partial \Omega, \\
& u = u_0, \quad t = 0, \quad x \in \Omega.
\end{aligned} \tag{2.1} \]

To obtain the short-time existence of classical solution of (2.1) we use an inverse function theorem in Fréchet spaces and the theory of linear parabolic equations for oblique boundary condition.
Lemma 2.1 ([13], Theorem 2). Let $X$ and $Y$ be Banach spaces. Denote $J : X \to Y$ be continuous and Gâteaux-differentiable, with $J(v_0) = w_0$. Assume that the derivative $DJ[v]$ has a right inverse $L[v]$, uniformly bounded in a neighbourhood of $v_0$:

$$\forall \alpha \in Y, \quad DJ[v]L[v] = \alpha$$

$$\|v - v_0\| \leq R \implies \|L[v] - L[v_0]\| \leq m.$$  

For every $w \in Y$ if

$$\|w - w_0\| < \frac{R}{m}$$

then there is some $v$ such that we have:

$$\|v - v_0\| < R,$$

and

$$J(v) = w.$$  

Lemma 2.2 ([14], Theorem 8.8 and 8.9). Assume that $f \in C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega}_T)$ for some $0 < \alpha < 1$, $T > 0$, and $G \in C^1(\partial\Omega \times R)$ such that $\inf_{\partial\Omega}(G_p, \nu) > 0$ where $\nu$ is the inner normal vector of $\partial\Omega$. Let $u_0 \in C^{2+\alpha}(\bar{\Omega})$ be strictly convex and satisfies $G(x, Du_0) = 0$. Then there exists $T_{\text{max}} > 0$ such that we can find an unique solution which is strictly convex in $x$ variable in the class $C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega}_{T_{\text{max}}})$ to the following equations

$$\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u &= f(x, t), \quad T > t > 0, \quad x \in \Omega, \\
G(x, Du) &= 0, \quad T > t > 0, \quad x \in \partial\Omega, \\
u \\
u_0, \quad t = 0, \quad x \in \Omega.
\end{aligned}$$

According to the proof of [12], one can verify the oblique boundary condition.

Lemma 2.3 (J. Urbas[12]).

$u \in C^2(\Omega)$ with $D^2u > 0 \implies \inf_{\partial\Omega} h_{p_k}(Du)\nu_k > 0$ where $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ is the unit inward normal vector of $\partial\Omega$, i.e. $h(Du) = 0$ is strict oblique.

We are now in a position to prove the short-time existence of solution of (2.1) which is equivalent to the problem (1.1).

Proposition 2.4. According to the conditions in Theorem [14] there exists some $T_{\text{max}} > 0$ and $u \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\Omega_{T_{\text{max}}})$ which depend only on $\Omega$, $\Omega$, $u_0$, such that $u$ is a solution of (2.1) and is strictly convex in $x$ variable.

Proof. Denote the Banach spaces

$$X = C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega}_T), \quad Y = C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega}_T) \times C^{1+\alpha, \frac{1+\alpha}{2}}(\partial\Omega \times (0, T)) \times C^{2+\alpha}(\bar{\Omega}),$$

where

$$\|\cdot\|_Y = \|\cdot\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega}_T)} + \|\cdot\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\partial\Omega \times (0, T))} + \|\cdot\|_{C^{2+\alpha}(\bar{\Omega})}.$$  

Define a map

$$J : X \to Y$$

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∀ If we set Then the derivative The strategy is now to use the inverse function theorem to obtain the local existence result. The computation of the Gâteaux derivative shows that:

∀ For each Combining with (2.3), by Lemma 2.1 then it gives the desired results.

Using Lemma 2.2 there exists a unique variable which satisfies the following equations:

For each (f, g, w) ∈ Y, using Lemma 2.2 again there exists a unique v ∈ X satisfying DJ[û](v) = (f, g, w), i.e.,

Then the derivative DJ[û] has a right inverse L[û] and for T = T_max we see that

If we set then one can show that

where C is a constant depending only on the known data. We may apply (2.2) to conclude that ∀ε > 0, ∃T_{max} > 0 to be small enough such that

Thus we obtain

Combining with (2.3), by Lemma 2.1 then it gives the desired results. □
Remark 2.5. By the strong maximum principle, the strictly convex solution to (2.1) is unique.

3. Preliminary results

In this section, the $C^2$ a priori bound is accomplished by proving second derivative estimates on the boundary for solution of parabolic type special lagrangian equation. This treatment is similar to the problems presented in [9], [10] and [12], but requires some modification to accommodate the particular situation. Specifically, Corollary 3.3 is needed in order to drive differential inequalities from which barriers can be used.

For the convenience, we set

\[ g^{ij} \triangleq \left[ \frac{\partial F(D^2 u)}{\partial u_{ij}} \right] = \left[ \delta^{ij} + u_{ik}u_{kj} \right]^{-1}, \quad \beta^k \triangleq \frac{\partial h(Du)}{u_k} = h_p(Du) \]

and let \( \langle \cdot, \cdot \rangle \) be the inner product in \( \mathbb{R}^n \). By Proposition 2.4 and the regularity theory of parabolic equations, we may assume that \( u \) is a strictly convex solution of (2.1) in the class \( C^{2+\alpha,1+\alpha}(\bar{\Omega} \cap C^\infty(\Omega_T)) \) for some \( T = T_{\text{max}} > 0 \).

Lemma 3.1 (\( \dot{u} \)-estimates).
As long as the convex solution to (2.1) exists, the following estimates holds, i.e.

\[ 0 \leq \dot{u} \leq \frac{\partial u}{\partial t} \leq \Theta_0 \triangleq \max_{\bar{\Omega}} F(D^2 u_0). \]

Proof. We use the methods known from Lemma 2.1 in [10].

From (2.1) a direct computation shows that

\[ \frac{\partial \dot{u}}{\partial t} - g^{ij}\partial_i \dot{u} \partial_j \dot{u} = 0. \]

Using the maximum principle we see that

\[ \max_{\partial\Omega_T} \dot{u} = \max_{\partial\Omega_T} \dot{u}. \]

Without loss of generality, we assume that \( \dot{u} \neq \text{constant} \). If \( \exists x \in \partial\Omega, t > 0 \), such that \( \dot{u}(x, t) = \max_{\partial\Omega_T} \dot{u} \). Then we differentiate the boundary condition and obtain

\[ \dot{u}_\beta = \frac{\partial h(Du)}{\partial t} = 0. \]

Since \( \langle \beta, \nu \rangle > 0 \), it is contradict to the Hopf Lemma (cf. [15]) for parabolic equations. So that

\[ \dot{u} \leq \max_{\partial\Omega_T} \dot{u} = \max_{\partial\Omega_T} \dot{u} = \max_{\bar{\Omega}} F(D^2 u_0). \]

On the other hand, \( u \) be convex \( \implies \) \( \min_{\partial \Omega_T} F(D^2 u) \geq 0 \implies \dot{u} = F(D^2 u) \geq 0 \). Putting these facts together, the assertion follows.

Since \( -\frac{\pi}{2} \leq \arctan \lambda_i \leq \frac{\pi}{2} \), \( \arctan \lambda_i = \frac{\pi}{2} \iff \lambda_i = +\infty \). Then \( \Theta_0 < \frac{n\pi}{2} \).
**Lemma 3.2.** Let \((x, t)\) be arbitrary point of \(\Omega_T\), and \(\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\) be the eigenvalues of \(D^2u\) at \(x, t\). Then

\[
0 \leq \lambda_1 \leq \tan\left(\frac{\Theta_0}{n}\right).
\]

**Proof.** It follows from the definition of \(F(D^2u)\) and Lemma 3.1 we see that

\[
n \arctan \lambda_1 \leq \sum_{i=1}^n \arctan \lambda_i = \dot{u} \leq \Theta_0.
\]

Combining with the convexity of \(u\) we obtain

\[
0 \leq \arctan \lambda_1 \leq \frac{\Theta_0}{n}
\]

which yields (3.1). □

Now we can show the operator \(F\) to be uniform ellipticity which will be play an important role in the barrier arguments.

**Corollary 3.3.** For any \((x, t) \in \Omega_T\), we have

\[
\frac{1}{1 + \tan\left(\frac{\Theta_0}{n}\right)^2} \leq \sum_{i=1}^n g^{ii} \leq n.
\]

**Proof.** We observe that

\[
\sum_{i=1}^n g^{ii} = \sum_{i=1}^n \frac{1}{1 + \lambda_i^2}.
\]

By Lemma 3.2 we obtain

\[
\frac{1}{1 + \tan\left(\frac{\Theta_0}{n}\right)^2} \leq \frac{1}{1 + \lambda_1^2} \leq \sum_{i=1}^n g^{ii} = \sum_{i=1}^n \frac{1}{1 + \lambda_i^2} \leq n.
\]

□

Returning to Lemma 2.3, using Corollary 3.3 we can get a uniform positive lower bound for the quantity \(\inf_{\partial \Omega} h_{pk}(Du)\nu_k\) which does not depend on \(t\).

**Lemma 3.4.** As long as the uniformly convex solution to (2.1) exists, the strict oblique estimates can be obtained by

\[
\langle \beta, \nu \rangle \geq \frac{1}{C_1} > 0,
\]

where the constant \(C_1\) is independent of \(t\).

**Proof.** Let \((x_0, t_0) \in \partial \Omega \times [0, T]\) such that

\[
\langle \beta, \nu \rangle(x_0, t_0) = h_{pk}(Du)\nu_k = \min_{\partial \Omega \times [0, T]} \langle \beta, \nu \rangle.
\]

By the computations on [12] it gives

\[
\langle \beta, \nu \rangle = \sqrt{u^{ij}\nu_i\nu_j h_{pk} h_{pl} u_{kl}}.
\]

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Further on, we may assume that $t_0 > 0$ and $\nu(x_0) = (0, 0, \cdots, 1) \triangleq e_n$. As in the proof of Lemma 8.1 in [10], by the convexity of $\Omega$ and its smoothness, we extend $\nu$ smoothly to a tubular neighborhood of $\partial \Omega$ such that in matrix sense

$$D_k \nu_l \equiv \nu_{kl} \leq -\frac{1}{C} \delta_{kl}$$

for some positive constant $C$. One defines

$$v = \langle \beta, \nu \rangle + h(Du).$$

By the above assumptions and the boundary condition, we get

$$v(x_0, t_0) = \min_{\partial \Omega \times [0, T]} v = \min_{\partial \Omega \times [0, T]} \langle \beta, \nu \rangle.$$

In $(x_0, t_0)$, we have

$$0 = v_r = h_{pn} p_k u_{kr} + h_{pk} v_{kr} + h_{pk} u_{kr}, \quad 1 \leq r \leq n - 1,$$

$$0 \leq \dot{v}.$$

We assume that the following key estimates holds which will be proved later,

$$v_n(x_0, t_0) \geq -C,$$

where $C$ is a constant depending only on $\Omega, u_0$ and $h, \tilde{h}$. It’s not hard to check that (3.6) can be rewritten as

$$h_{pn} h_{kn} + h_{pk} \nu_{kn} + h_{pk} u_{kn} \geq -C.$$

Multiplying (3.7) with $h_{pn}$ and (3.5) with $h_{pk}$, respectively, and summing up together we obtain:

$$h_{pk} h_{pi} u_{kl} \geq -C h_{pn} - h_{pk} h_{pi} \nu_{kl} - h_{pk} h_{pn} p_i u_{kl}.$$

By the concavity of $h$, we have

$$-h_{pn} \geq 0, \quad h_{pk} u_{kr} = \frac{\partial h(Du)}{\partial x_r} = 0, \quad h_{pk} u_{kn} = \frac{\partial h(Du)}{\partial x_n} = \frac{\partial h(Du)}{\partial x_n} \geq 0.$$

Substituting this into (3.8) and using (3.4) yields

$$h_{pk} h_{pi} u_{kl} \geq -C h_{pn} + \frac{1}{C} |Dh|^2 = -C h_{pn} + \frac{1}{C}.$$

According to the above last term, we distinguish two cases.

Case (i).

$$-C h_{pn} + \frac{1}{C} \leq 0.$$

Then

$$h_{pk}(Du) \nu_k = h_{pn} \geq \frac{1}{C^2}.$$

It shows that there is a uniform positive lower bound for the quantity $\min_{\partial \Omega \times [0, T]} h_{pk}(Du) \nu_k$.

Case (ii).

$$-C h_{pn}(x_0) + \frac{1}{C} > 0.$$
Then we obtain a positive lower bound for \( h_{pk} h_{pi} u_{kl} \). Introduce the Legendre transformation of \( u \),

\[
y_i = \frac{\partial u}{\partial x_i}, \quad i = 1, 2, \cdots, n, \quad u^*(y_1, \cdots, y_n, t) := \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} - u(x, t).
\]

In terms of \( y_1, \cdots, y_n, u^*(y_1, \cdots, y_n) \), one can easily check that

\[
\frac{\partial^2 u^*}{\partial y_i \partial y_j} = \left[ \frac{\partial^2 u}{\partial x_i \partial x_j} \right]^{-1}.
\]

Since \( \arctan \lambda + \arctan \lambda - 1 = \frac{\pi}{2} \). Then \( u^* \) satisfies

\[
\begin{aligned}
\frac{\partial u^*}{\partial t} - F(D^2 u^*) &= -\frac{n \pi}{2}, \quad T > t > 0, \quad x \in \hat{\Omega}, \\
\tilde{h}(Du^*) &= 0, \quad T > t > 0, \quad x \in \partial \hat{\Omega}, \\
u^* &= u^*_0, \quad t = 0, \quad x \in \hat{\Omega}.
\end{aligned}
\]

where \( \tilde{h} \) is a smooth strictly concave function on \( \Omega \):

\[
\Omega = \{ p \in \mathbb{R}^n | \tilde{h}(p) > 0 \}, \quad |D\tilde{h}|_{|\partial \hat{\Omega}} = 1.
\]

We also define

\[
\tilde{v} = \tilde{\beta}^k \tilde{v}_k + \tilde{h}(Du^*) = \langle \tilde{\beta}, \tilde{v} \rangle + \tilde{h}(Du^*),
\]

where

\[
\tilde{\beta}^k \triangleq \frac{\partial \tilde{h}(Du^*)}{u^*_k} = \tilde{h}_{pk}(Du^*),
\]

and \( \tilde{v} = (\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_n) \) is the inner unit normal vector of \( \partial \hat{\Omega} \). Using the same methods, under the assumption of

\[
\tilde{v}_n(y_0, t_0) \geq -C,
\]

we obtain the lower bounds for \( \tilde{h}_{pk} \tilde{h}_{pi} u_{kl}^* \) or

\[
\tilde{h}_{pk}(Du) u_k = \tilde{h}_{pk}(Du^*) \tilde{v}_k(y_0) = \tilde{h}_p u_n \geq \frac{1}{C^2}.
\]

We see that

\[
\tilde{h}_{pk} \tilde{h}_{pi} u_{kl}^* = \nu_i \nu_j u^{ij}.
\]

Then the claim follows from (3.3) by the positive lower bounds of \( h_{pk} h_{pi} u_{kl} \) and \( \tilde{h}_{pk} \tilde{h}_{pi} u_{kl}^* \).

It remains to prove the key estimates (3.6). The proof of Lemma 8.1 in [10] can also be adapted to here. For convenience of the reader and completeness, we provide the details and arguments below.

Define the linearized operator by

\[
L = g^{ij} \partial_{ij} - \partial_t.
\]

Since \( D^2 \tilde{h} \leq -\tilde{\theta} I \) we obtain

\[(3.10)\]

\[
\tilde{L} \tilde{h} \leq -\tilde{\theta} \sum g^{ii}
\]
On the other hand,

\[ L \nu = h_{pkpnm} \nu^k g^{ij} u_{li} u_{mj} + 2 h_{pkpi} g^{ij} \nu_k u_{li} + h_{pkpi} g^{ij} u_{ij} u_{ki} + h_{pkpi} \nu_k L u_{li} + h_{pk} L u_k. \]

By estimating the first term in the diagonal basis, one yields

\[ | h_{pkpnm} \nu^k g^{ij} u_{li} u_{mj} | \leq C \sum \lambda_i^2 \leq C, \]

where \( C \) is a constant depending only on \( h \) and \( \Omega \). For the same reason, we have

\[ | 2 h_{pkpi} g^{ij} \nu_k u_{li} | \leq C, \quad | h_{pkpi} g^{ij} u_{ij} u_{ki} | \leq C. \]

Now the simple calculation gives

\[ L u_l = L u_k = 0. \]

So there exists a positive constant \( C \) such that

\[ (3.11) \quad | L \nu | \leq C \sum g^{ii}. \]

Here we use Corollary 3.3 and \( C \) depends only on \( h \) and \( \Omega \). For the same reason, we have

\[ | 2 h_{pkpi} g^{ij} \nu_k u_{li} | \leq C, \quad | h_{pkpi} g^{ij} u_{ij} u_{ki} | \leq C. \]

Using the strictly concavity of \( \tilde{h} \) we have

\[ \triangle (C_0 \tilde{h}(x) + A|x - x_0|^2) \leq C (-C_0 \tilde{\theta} + 2A) \sum g^{ii}. \]

Then by choosing the constant \( C_0 \gg A \), we can show that

\[ \triangle (v(x, 0) - v(x_0, t_0) + C_0 \tilde{h}(x) + A|x - x_0|^2) \leq 0. \]

It follows from the maximum principle that we get

\[ (v(x, 0) - v(x_0, t_0) + C_0 \tilde{h}(x) + A|x - x_0|^2) |_{\Omega_\delta} \geq 0. \]

Combining (3.10) with (3.11), letting \( C_0 \) be large enough we obtain

\[ L \Phi \leq (-C_0 \tilde{\theta} + C + 2A) \sum g^{ii} \leq 0. \]
From the above arguments one can verify that $\Phi$ satisfies

$$
\begin{align*}
L\Phi & \leq 0, & (x, t) & \in \Omega_\delta \times [0, T], \\
\Phi & \geq 0, & (x, t) & \in (\partial \Omega_\delta \times [0, T] \cup (\Omega_\delta \times \{ t = 0 \}).
\end{align*}
$$

Using the maximum principle we can deduce that

$$
\Phi \geq 0, & (x, t) \in \Omega_\delta \times [0, T].
$$

Combining it with $\Phi(x_0, t_0) = 0$, we obtain $\Phi_n(x_0, t_0) \geq 0$ which gives the desired estimates \(3.6\) and thus completes the proof of the lemma.

It follows from \(3.11\) that we can state the following result which is similar to Proposition 2.6 in [11].

**Lemma 3.5.** Fix a smooth function $H : \Omega \times \tilde{\Omega} \to \mathbb{R}$ and define $\varphi(x, t) = H(x, Du(x, t))$. Then there holds

$$
|L\varphi| \leq C \sum g^{ii}, \quad (x, t) \in \Omega_T,
$$

where $C$ is a positive constant depending on $h, H, u_0$ and $\Omega$.

We can now proceed to the $C^2$ estimates. The strategy is to bound the interior second derivative firstly.

**Lemma 3.6.** For each $t \in [0, T]$, we have the estimates

$$
\sup_\Omega |D^2u| \leq \max_{\partial \Omega \times [0, T]} |D^2u| + \max_\Omega |D^2u_0|.
$$

**Proof.** Given any unit vector $\xi$, by the concavity of $F$, $u_{\xi \xi}$ satisfies

$$
\partial_t u_{\xi \xi} - g^{ij} \partial_{ij} u_{\xi \xi} = \frac{\partial^2 F}{\partial_{xj} \partial_{xk}} u_{ij \xi} u_{kl \xi} \leq 0.
$$

Combining with the convexity of $u$, and using the maximum principle we obtain

$$
0 \leq |u_{\xi \xi}| = u_{\xi \xi}(x, t) \leq \max_{\partial \Omega_T} u_{\xi \xi}
\leq \max_{\partial \Omega \times [0, T]} |D^2u| + \max_\Omega |D^2u_0|.
$$

Therefore the estimates \(3.13\) is satisfied.

By tangentially differentiating the boundary condition $h(Du) = 0$ we have some second derivative bounds on $\partial \Omega$, i.e.,

$$
u_{\beta \tau} = h_{p_k} (Du) u_{k \tau} = 0.
$$

where $\tau$ denotes a tangential vector. The second order derivative estimates on the boundary is controlled by $u_{\beta \tau}, u_{\beta \beta}, u_{\tau \tau}$.

In the following we give the arguments as in [12]. For $x \in \partial \Omega$, any unit vector $\xi$ can be written in terms of a tangential component $\tau(\xi)$ and a component in the direction $\beta$ by

$$
\xi = \tau(\xi) + \frac{\langle \nu, \xi \rangle}{\langle \beta, \nu \rangle} \beta.
$$
where

$$\tau(\xi) = \xi - \langle \nu, \xi \rangle \nu - \frac{\langle \nu, \xi \rangle}{\langle \beta, \nu \rangle} \beta^T$$

and

$$\beta^T = \beta - \langle \beta, \nu \rangle \nu.$$ Then a simple computation shows that

$$|\tau(\xi)|^2 = 1 - (1 - |\beta^T|^2 \langle \nu, \xi \rangle^2 - 2 \langle \nu, \xi \rangle \langle \beta^T, \xi \rangle \langle \beta, \nu \rangle) \langle \nu, \xi \rangle^2$$

(3.15)

$$\leq 1 + C \langle \nu, \xi \rangle^2 - 2 \langle \nu, \xi \rangle \langle \beta^T, \xi \rangle \langle \beta, \nu \rangle \leq C,$$

where we had used the strict obliqueness (3.2). Let $\tau \triangleq |\tau(\xi)|$. Then by (3.14) and (3.2), we obtain

$$u_{\xi \xi} = |\tau(\xi)|^2 u_{\tau \tau} + |\tau(\xi)| \frac{\langle \nu, \xi \rangle}{\langle \beta, \nu \rangle} u_{\beta \tau} + \frac{\langle \nu, \xi \rangle^2}{\langle \beta, \nu \rangle} u_{\beta \beta}$$

(3.16)

$$= |\tau(\xi)|^2 u_{\tau \tau} + \langle \nu, \xi \rangle^2 \frac{\langle \beta, \nu \rangle}{\langle \beta, \nu \rangle} u_{\beta \beta}$$

$$\leq C(u_{\tau \tau} + u_{\beta \beta}).$$

Along with specifying the boundary condition we can carry out the double derivative estimates in the direction $\beta$.

**Lemma 3.7.** For each $t \in [0, T]$, we have the estimates

$$\max_{\partial \Omega} u_{\beta \beta} \leq C_2$$

where $C_2 > 0$ depending only on $u_0, h, \tilde{h}, \Omega$.

**Proof.** We use the barrier functions for any $x_0 \in \partial \Omega$ and thus consider

$$\Psi \triangleq \pm h(Du) + C_0 \tilde{h} + A|x - x_0|^2.$$ As in the proof of (3.12), we can find the constant $C_0, A$ such that we have

$$\begin{cases} L \Psi \leq 0, & (x, t) \in \Omega_\delta \times [0, T], \\ \Psi \geq 0, & (x, t) \in (\partial \Omega_\delta \times [0, T] \cup (\Omega_\delta \times \{t = 0\}). \end{cases}$$

By the maximum principle we get

$$\Psi \geq 0, \quad (x, t) \in \Omega_\delta \times [0, T].$$

Combining it with $\Psi(x_0, t_0) = 0$ and using Lemma 3.4 we obtain $\Psi_\beta(x_0, t_0) \geq 0$. Then it shows that

$$|u_{\beta \beta}| = \left| \frac{\partial \tilde{h}}{\partial \beta} \right| \leq C_2.$$

We shall obtain here the bounds of double tangential derivative at the boundary.
Lemma 3.8. There exists a constant $C_3 > 0$ depending only on $u_0$, $h$, $\tilde{h}$, $\Omega$ such that

$$\max_{\partial\Omega \times [0, T]} \max_{|r|=1, \langle r, \nu \rangle=0} u_{rr} \leq C_3.$$ 

Proof. Assume that $x_0 \in \partial\Omega$, $t_0 \in [0, T]$ and denotes $\nu = e_n$ to be the inner unit normal of $\partial\Omega$ at $x_0$. Such that

$$\max_{\partial\Omega \times [0, T]} \max_{|r|=1, \langle r, \nu \rangle=0} u_{rr} = u_{11}(x_0, t_0).$$

For any $x \in \partial\Omega$, combining (3.15) with (3.16), we have

$$u_{\xi\xi} = |\tau(\xi)|^2 u_{rr} + \frac{\langle \nu, \xi \rangle^2}{\langle \beta, \nu \rangle} u_{\beta\beta}$$

$$\leq (1 + C\langle \nu, \xi \rangle^2 - 2\langle \nu, \xi \rangle \frac{\langle \beta^T, \xi \rangle}{\langle \beta, \nu \rangle})u_{rr} + \frac{\langle \nu, \xi \rangle^2}{\langle \beta, \nu \rangle} u_{\beta\beta}$$

$$\leq (1 + C\langle \nu, \xi \rangle^2 - 2\langle \nu, \xi \rangle \frac{\langle \beta^T, \xi \rangle}{\langle \beta, \nu \rangle})u_{11}(x_0, x_0) + \frac{\langle \nu, \xi \rangle^2}{\langle \beta, \nu \rangle} u_{\beta\beta}$$

Without loss of generality, we assume that $u_{11}(x_0, t_0) \geq 1$, then by Lemma 3.4 and Lemma 3.7 we get

$$\frac{u_{\xi\xi}}{u_{11}(x_0, t_0)} + 2\langle \nu, \xi \rangle \frac{\langle \beta^T, \xi \rangle}{\langle \beta, \nu \rangle} \leq 1 + C\langle \nu, \xi \rangle^2$$

Let $\xi = e_1$, then we have

$$\frac{u_{11}}{u_{11}(x_0, t_0)} + 2\langle \nu, e_1 \rangle \frac{\langle \beta^T, e_1 \rangle}{\langle \beta, \nu \rangle} \leq 1 + C\langle \nu, e_1 \rangle^2$$

We see that the function

$$w \triangleq A|x - x_0|^2 - \frac{u_{11}}{u_{11}(x_0, t_0)} - 2\langle \nu, e_1 \rangle \frac{\langle \beta^T, e_1 \rangle}{\langle \beta, \nu \rangle} + C\langle \nu, e_1 \rangle^2 + 1$$

satisfies

$$w|_{\partial\Omega \times [0, T]} \geq 0, \quad w(x_0, t_0) = 0.$$ 

As before, by (3.13) we can choose the constant $A$ such that

$$w|_{(\partial B_\delta(x_0) \cap \Omega) \times [0, T]} \geq 0.$$ 

Consider

$$-2\langle \nu, e_1 \rangle \frac{\langle \beta^T, e_1 \rangle}{\langle \beta, \nu \rangle} + C\langle \nu, e_1 \rangle^2 + 1$$

as a known function depending on $x$ and $Du$. Then by Lemma 3.5 we obtain

$$|L(-2\langle \nu, e_1 \rangle \frac{\langle \beta^T, e_1 \rangle}{\langle \beta, \nu \rangle} + C\langle \nu, e_1 \rangle^2 + 1)| \leq C \sum g_{ii}.$$ 

Combining it with the proof of Lemma 3.6 we have

$$Lw \leq C \sum g_{ii}.$$
As in the proof of Lemma 3.7, we consider the function
\[ \Upsilon \triangleq w + C_0 \tilde{h}. \]
A standard barrier argument shows that
\[ \Upsilon_{\beta}(x_0, t_0) \geq 0. \]
By a direct computation we obtain
(3.17) \[ u_{11\beta} \leq C u_{11}(x_0, t_0). \]
On the other hand, differentiating the boundary conditions twice in the direction \( e_1 \) at \((x_0, t_0)\), we have

\[ h_{p_k} u_{k1} + h_{p_k p_l} u_{k1} u_{l1} = 0. \]

The concavity of \( h \) yields
\[ h_{p_k} u_{k1} = -h_{p_k p_l} u_{k1} u_{l1} \geq \hat{C} u_{11}(x_0, t_0)^2. \]
Combining it with \( h_{p_k} u_{k1} = u_{11\beta} \), and using (3.17) we obtain
\[ \hat{C} u_{11}(x_0, t_0)^2 \leq C u_{11}(x_0, t_0) \]
Then we get the bounds for \( u_{11}(x_0, t_0) \) and the desired result follows. \( \square \)

Using Lemma 3.7, 3.8, and combining with (3.16), we obtain the \( C^2 \) a priori bound on the boundary:

**Lemma 3.9.** There exists a constant \( C_4 > 0 \) depending on \( h, \tilde{h} \) and \( u_0, \Omega \) such that
\[ \sup_{\partial \Omega_T} |D^2 u| \leq C_4. \]

Using this and Lemma 3.6, the following conclusion is thus proven:

**Lemma 3.10.** There exists a constant \( C_5 > 0 \) depending on \( h, \tilde{h} \) and \( u_0, \Omega \) such that
\[ \sup_{\Omega_T, |\xi|=1} D_{ij} u \xi_i \xi_j \leq C_5. \]

By the Legendre transformation of \( u \), using (3.9) and repeating the proof of the above lemmas we get the forthcoming result:

**Lemma 3.11.** There exists a constant \( C_6 > 0 \) depending on \( h, \Omega, \tilde{h}, \tilde{\Omega}, u_0 \) such that
(3.18) \[ \frac{1}{C_6} \leq \inf_{\Omega_T, |\xi|=1} D_{ij} u \xi_i \xi_j \leq \sup_{\Omega_T, |\xi|=1} D_{ij} u \xi_i \xi_j \leq C_6. \]
4. PROOF OF MAIN RESULT

**Proof of Theorem 1.1** Now let $u_0$ be a $C^{2+\alpha}$ strictly convex function as in the conditions of Theorem 1.1. Combining Proposition 2.4 with Lemma 3.11, the strictly convex solution of (1.1) exists for all $t \geq 0$ and $\forall T > 0$, $u \in C^{2+\alpha,1+\alpha/2}([0,T])$ which satisfies (3.18). Using the boundary condition, we have

\[
|Du| \leq C_7
\]

where $C_7$ be a constant depending on $\Omega$ and $\tilde{\Omega}$. By Theorem 1.1 in [16] and Schauder estimates for parabolic equations, for any $\bar{\Omega}, \tilde{\Omega}$, we have

\[
\sup_{x \in \bar{\Omega}, t \geq 1} \frac{|D^{2+m}u(x_1, t_1) - D^{2+m}u(x_2, t_2)|}{\max\{|x_1 - x_2|^{\alpha}, |t_1 - t_2|^2\}} \leq C_8
\]

where $C_8$ is a constant depending on the known data and $\text{dist}(\partial \Omega, \tilde{\Omega})$. By Arzelà–Ascoli theorem, a diagonal sequence argument shows that for any \( \{t_k\}_{k=1}^{+\infty} \) with \( \lim_{k \to +\infty} t_k = +\infty \), there exists a subsequence \( \{t_{k_j}\}_{j=1}^{+\infty} \subset \{t_k\}_{k=1}^{+\infty} \) and \( \hat{u} \in C^2(\bar{\Omega}) \cap C^{2+m}(\Omega) \).

Such that

\[
\lim_{j \to +\infty} Du(x, t_{k_j}) = D\hat{u}(x), \quad x \in \tilde{\Omega},
\]

and \( \hat{u} \) satisfies (3.18). Then we get

\[
h(D\hat{u})|_{\partial \Omega} = 0
\]

and

\[
\lim_{j \to +\infty} F(D^2u(x, t_{k_j})) = F(D^2\hat{u}(x)), \quad x \in \Omega.
\]

For each $l$, differentiating the equation (1.1) by $x_l$ yields

\[
\partial_l u_l = g^{ij} \partial_{ij} u_l.
\]

Integrating from 0 to $t$ on both sides we obtain

\[
u_l(x, t) - u_l(x, 0) = \int_0^t g^{ij} \partial_{ij} u_l(x, \sigma) d\sigma.
\]

Combining it with (4.1), (4.2), we have

\[
\lim_{l \to +\infty} g^{ij} \partial_{ij} u_l(x, t) = 0, \quad x \in \Omega.
\]

Using this fact along with (4.2), the following emerges:

\[
g^{ij} \partial_{ij} \hat{u}_l = 0, \quad x \in \Omega, \quad l \in \{1, 2, \cdots, n\}.
\]
Specifically, it is claimed that
\[ F(D^2 u) = C_9, \quad x \in \Omega \]
for some constant \( C_9 \) and it follows from (3.18) that \( C_9 > 0 \). Then the claim of Theorem [14] follows from the above arguments. □

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