THE PICARD GROUP OF THE COMPACTIFIED UNIVERSAL JACOBIAN

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Dedicated to the memory of Torsten Ekedahl, with great admiration.

Abstract. We compute the Picard groups of the universal Jacobian stack and of its compactification over the stack of stable curves. Along the way, we prove some results concerning the gerbe structure of the universal Jacobian stack over its rigidification by the natural action of the multiplicative group and relate this with the existence of generalized Poincaré line bundles. We also compare our results with Kouvidakis-Fontanari computations of the divisor class group of the universal (compactified) Jacobian scheme.

1. Introduction

The Picard group of a given moduli stack carries important informations on the geometry of the moduli problem one is dealing with. Since Mumford’s pioneer work in [Mum65], the subject has been widely developed and nowadays the literature on the computation of the Picard group of moduli stacks is quite vast. Remarkable examples are the Picard groups of the moduli stacks of curves with possible level structures (see [DN89], [Kou91], [Kou93], [KN97], [LS97], [BLS98], [BH10]) and of the moduli stacks of principal bundles over curves (see [DN89], [Kou91], [Kou93], [KN97], [LS97], [BL90], [BH10]).

The aim of this paper is to determine the Picard group of the degree-$d$ universal Jacobian stack $\mathcal{J}_{ac,d,g}$ over the moduli stack $\mathcal{M}_g$ of smooth curves of genus $g$ and of its compactification $\overline{\mathcal{J}}_{ac,d,g}$ over the moduli stack $\overline{\mathcal{M}}_g$ of stable curves of genus $g$, constructed by Caporaso in [Cap94] and [Cap05] and later generalized by the first author in [Mel09].

Let us briefly recall the definitions of the stacks $\mathcal{J}_{ac,d,g}$ and $\overline{\mathcal{J}}_{ac,d,g}$, referring to Section 2 for more details. The degree-$d$ universal Jacobian stack $\mathcal{J}_{ac,d,g}$ is the (Artin) stack whose fiber over a scheme $S$ consists of families of smooth curves $C \to S$ over $S$ endowed with a line bundle $L$ over $C$ of relative degree $d$ over $S$. The stack $\mathcal{J}_{ac,d,g}$ is contained as a dense open substack in the degree-$d$ compactified Jacobian stack $\overline{\mathcal{J}}_{ac,d,g}$, whose fiber over a scheme $S$ consists of families of quasi-stable curves $X \to S$ endowed with a properly balanced line bundle over $X$ of relative degree $d$ over $S$ (see [2.1] for the definitions). The stack $\overline{\mathcal{J}}_{ac,d,g}$ is smooth and irreducible of dimension $4g-4$, and it is endowed with a (forgetful) universally closed surjective morphism $\Phi_d$ to the stack $\overline{\mathcal{M}}_g$ of stable curves.

The stack $\overline{\mathcal{J}}_{ac,d,g}$ is naturally endowed with the structure of a $\mathbb{G}_m$-stack, since the group $\mathbb{G}_m$ naturally injects into the automorphism group of every object $(C \to S, L) \in \overline{\mathcal{J}}_{ac,d,g}(S)$ as multiplication by scalars on $L$. Therefore $\overline{\mathcal{J}}_{ac,d,g}$ becomes a $\mathbb{G}_m$-gerbe over the $\mathbb{G}_m$-rigidification $\mathcal{J}_{d,g} := \mathcal{J}_{ac,d,g} / \mathbb{G}_m$. We call $\nu_d : \overline{\mathcal{J}}_{ac,d,g} \to \mathcal{J}_{d,g}$ the rigidification map. Analogously, $\mathcal{J}_{ac,d,g}$ is a $\mathbb{G}_m$-gerbe over its rigidification $\mathcal{J}_{d,g} := \mathcal{J}_{ac,d,g} / \mathbb{G}_m$ which is an open dense substack of $\mathcal{J}_{d,g}$. The stack $\mathcal{J}_{d,g}$ is smooth and irreducible of dimension $4g-3$, and the morphism $\Phi_d : \overline{\mathcal{J}}_{ac,d,g} \to \overline{\mathcal{M}}_g$ factors through $\Phi_d : \mathcal{J}_{d,g} \to \overline{\mathcal{M}}_g$, which is again a universally closed surjective morphism.

Caporaso’s compactification $\mathcal{J}_{d,g}$ of the universal Jacobian variety $J_{d,g}$ over the moduli scheme $\overline{\mathcal{M}}_g$ of stable curves (see [Cap94]) is an adequate moduli space for $\overline{\mathcal{J}}_{ac,d,g}$ and for $\mathcal{J}_{d,g}$ (in the sense of [Alp2]) and even a good moduli space (in the sense of [Alp1]) if our base field $k$ has characteristic zero. We will call it simply the moduli space for $\mathcal{J}_{ac,d,g}$ and for $J_{d,g}$

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1In the literature, the universal (resp. universal compactified) Jacobian stack is often called the universal (resp. universal compactified) Picard stack and it is denoted by $\mathcal{P}_{ic,d,g}$ (resp. $\mathcal{P}_{ic,d,g}$), see e.g. [Cap05], [Mel09], [BFV11]. Similarly the
The main result of this paper is a description of the Picard groups of the stacks $J_{ac,d,g}$ and $J_{d,g}$, and of their compactifications $\overline{J_{ac,d,g}}$ and $\overline{J_{d,g}}$. Since $J_{ac,d,g} \subset \overline{J_{ac,d,g}}$ and $J_{d,g} \subset \overline{J_{d,g}}$ are open inclusions of smooth stacks, the natural restriction morphisms $\text{Pic}(\overline{J_{ac,d,g}}) \to \text{Pic}(J_{ac,d,g})$ and $\text{Pic}(\overline{J_{d,g}}) \to \text{Pic}(J_{d,g})$ are surjective. Moreover, since $\nu_4$ is a $G_m$-gerbe, the pull-back morphisms $\nu_4^* : \text{Pic}(\overline{J_{d,g}}) \to \text{Pic}(J_{ac,d,g})$ and $\nu_4^* : \text{Pic}(J_{d,g}) \to \text{Pic}(J_{ac,d,g})$ are injective. Therefore, the above Picard groups are related by the following commutative diagram

\[
\begin{array}{ccc}
\text{Pic}(\overline{J_{ac,d,g}}) & \xrightarrow{\nu_4^*} & \text{Pic}(J_{ac,d,g}) \\
\downarrow & & \downarrow \\
\text{Pic}(\overline{J_{d,g}}) & \xrightarrow{\nu_4^*} & \text{Pic}(J_{d,g})
\end{array}
\]

in which the horizontal arrows are surjective and the vertical arrows are injective. We will prove that the four Picard groups of diagram (1.1) are generated by boundary line bundles and tautological line bundles, which we are now going to define.

In Section 3 we describe the irreducible components of the boundary divisor $\overline{J_{ac,d,g}} \setminus J_{ac,d,g}$. Clearly, the boundary of $\overline{J_{ac,d,g}}$ is the pull-back via the morphism $\Phi_d : J_{ac,d,g} \to M_g$ of the boundary of $M_g$. Recall that $\overline{M}_g \setminus M_g = \bigcup_{i=0}^{[g/2]} \delta_i$, where $\delta_0$ is the irreducible divisor whose generic point is an irreducible curve with one node and, for $i = 1, \ldots, [g/2]$, $\delta_i$ is the irreducible divisor whose generic point is the stable curve with two irreducible components of genera $i$ and $g - i$ meeting in one point. In Theorem 3.2 we prove that $\delta_i := \Phi_d^{-1}(\delta_i)$ is irreducible if either $i = 0$ or $i = g/2$ or the number $\frac{g(g^2 - 2g + 1)}{2}$ does not divide $(2i - 1)$ and, otherwise, that $\Phi_d^{-1}(\delta_i)$ is the union of two irreducible divisors, that we call $\tilde{\delta}_i^1$ and $\tilde{\delta}_i^2$ (see Section 3 for the precise description of these two divisors). Since $J_{ac,d,g}$ is a smooth stack, the boundary divisors $\{\tilde{\delta}_i^1, \tilde{\delta}_i^2\}$ are Cartier divisors and therefore they give rise to line bundles on $J_{ac,d,g}$, that we denote by $\{O(\tilde{\delta}_i^1), O(\tilde{\delta}_i^2)\}$, and we call the boundary line bundles of $J_{ac,d,g}$. Note that the irreducible components of the boundary of $J_{d,g}$ are the divisors $\delta_i := \nu_4(\tilde{\delta}_i^1), \tilde{\delta}_i^2 := \nu_4(\tilde{\delta}_i^2)$ and $\delta_i := \nu_4(\tilde{\delta}_i^2)$. The associated line bundles $\{O(\delta_i), O(\delta_i^1), O(\delta_i^2)\}$ are called boundary line bundles of $J_{d,g}$ and clearly we have $\nu_4^* O(\tilde{\delta}_i^1) = O(\delta_i^1), \nu_4^* O(\tilde{\delta}_i^2) = O(\delta_i^2)$ and $\nu_4^* O(\tilde{\delta}_i^2) = O(\delta_i^2)$ (see Corollary 3.3).

In Section 5 we introduce the line bundles $K_{1,0}, K_{0,1}, K_{-1,2}$ and $\Lambda(m,n)$ (for $n, m \in \mathbb{Z}$) on $\overline{J_{ac,d,g}}$, which we call tautological line bundles. The tautological line bundles are defined in terms of the determinant of cohomology $d_\lambda(-)$ and of the Deligne pairing $\langle -,- \rangle_\pi$ applied to the universal family $\pi : \overline{J_{ac,d,g,1}} \to \overline{J_{ac,d,g}}$ (see Section 2.15 for the definition and basic properties of the determinant of cohomology and of the Deligne pairing). More precisely, we define

\[
\begin{align*}
K_{1,0} & := \langle \omega_x, \omega_x \rangle_\pi, \\
K_{0,1} & := \langle \omega_x, L_d \rangle_\pi, \\
K_{-1,2} & := \langle L_d, L_d \rangle_\pi, \\
\Lambda(n,m) & := d_\lambda(\omega_x^n \otimes L_d^m),
\end{align*}
\]

where $\omega_x$ is the relative dualizing sheaf for $\pi$ and $L_d$ is the universal line bundle on $\overline{J_{ac,d,g,1}}$. We define the tautological subgroup $\text{Pic}^{\text{aut}}(\overline{J_{ac,d,g}}) \subseteq \text{Pic}(\overline{J_{ac,d,g}})$ as the subgroup generated by the tautological line bundles together with the boundary line bundles of $\overline{J_{ac,d,g}}$. Similarly, we can restrict the tautological line bundles to $J_{ac,d,g}$ and consider the subgroup $\text{Pic}^{\text{aut}}(J_{ac,d,g}) \subseteq \text{Pic}(J_{ac,d,g})$ generated by them. Moreover, using the pull-back morphism $\nu_4^*$ (see diagram (1.1)), we can define the tautological subgroups $\text{Pic}^{\text{aut}}(J_{d,g}) := (\nu_4^*)^{-1}(\text{Pic}^{\text{aut}}(\overline{J_{d,g}})) \subseteq \text{Pic}(J_{d,g})$ and $\text{Pic}^{\text{aut}}(J_{d,g}) := (\nu_4^*)^{-1}(\text{Pic}^{\text{aut}}(\overline{J_{d,g}})) \subseteq \text{Pic}(J_{d,g})$.

universal (resp. universal compactified) Jacobian scheme is often called the universal (resp. universal compactified) Picard scheme and it is denoted by $P_{d,g}$ (resp. $\overline{P}_{d,g}$), see e.g. [Cap94]. Following [CMKV] and [BMV], we prefer here to use the word universal (resp. universal compactified) Jacobian stack/scheme and consequently the symbols $J_{ac,d,g}, \overline{J_{ac,d,g}}, J_{d,g}$ and $\overline{J_{d,g}}$ for two reasons: (i) the word Jacobian stack/scheme is used only for curves while the word Picard stack/scheme is used also for varieties of higher dimensions and it is more ambiguous; (ii) the expression “the Picard group of the Picard stack/scheme” seems a bit cacophonous.
Pic(\(J_{d,g}\)). Following a strategy due to Mumford in [Mum83], we next apply the Grothendieck-Riemann-Roch theorem to the morphism \(\pi : J_{ac,d,g} \to J_{ac,d,g}\) in order to produce relations among the tautological line bundles. In particular, we prove in Theorem 5.3 that all the tautological line bundles can be expressed in terms of \(\Lambda(1,0), \Lambda(0,1)\) and \(\Lambda(1,1)\). In particular, the tautological subgroup \(\text{Pic}^{\text{taut}}(J_{ac,d,g})\) (resp. \(\text{Pic}^{\text{taut}}(\overline{J_{ac,d,g}})\)) is generated by the three line bundles \(\Lambda(1,0), \Lambda(0,1)\) and \(\Lambda(1,1)\) (resp. and the boundary line bundles).

After these preliminaries, we can now state the main results of this paper, concerning the Picard groups of \(J_{ac,d,g}\) and \(J_{d,g}\) of their compactifications \(\overline{J_{ac,d,g}}\) and \(\overline{J_{d,g}}\). We prove that all the Picard groups in question are free and generated by tautological line bundles and boundary line bundles (if any). More precisely, we have the following.

**Theorem A.** Assume that \(g \geq 3\).

(i) The Picard group of \(J_{ac,d,g}\) is freely generated by \(\Lambda(1,0), \Lambda(0,1)\) and \(\Lambda(1,1)\).

(ii) The Picard group of \(\overline{J_{ac,d,g}}\) is freely generated by the boundary line bundles and the tautological line bundles \(\Lambda(1,0), \Lambda(0,1)\) and \(\Lambda(1,1)\).

**Theorem B.** Assume that \(g \geq 3\).

(i) The Picard group of \(J_{d,g}\) is freely generated by the tautological line bundles \(\Lambda(1,0)\) and

\[
\Xi := \Lambda(0,1)^{\frac{d-g-1}{g}} \otimes \Lambda(1,1)^{\frac{d-g-1}{g}}.
\]

(ii) The Picard group of \(\overline{J_{d,g}}\) is freely generated by the boundary line bundles and the tautological line bundles \(\Lambda(1,0)\) and \(\Xi\).

Let us now sketch the strategy that we use to prove Theorems A and B. Since the stack \(\overline{J_{ac,d,g}}\) is smooth we have a natural exact sequence

\[
\bigoplus_{k, h, g > 0} \langle \mathcal{O}(\delta_1) \rangle \bigoplus_{k, h, g > 0} \langle \mathcal{O}(\delta_1), \mathcal{O}(\delta_2) \rangle \to \text{Pic}(J_{ac,d,g}) \to \text{Pic}(\overline{J_{ac,d,g}}) \to 0.
\]

In Theorem 4.1 we prove that the above exact sequence is also exact on the left, or in other words that the boundary line bundles are linearly independent in the Picard group of \(J_{ac,d,g}\). In order to prove this, we use the same strategy used by Arbarello-Cornalba in [ACS7] to prove the analogous statement for the boundary line bundles of \(\overline{M}_g\); we construct some test curves \(\overline{F}_j \to \overline{J_{ac,d,g}}\) in number equal to the number of boundary line bundles, and prove that the intersection matrix between these test curves \(\overline{F}_j\) and the boundary line bundles of \(\overline{J_{ac,d,g}}\) is non-degenerate. This reduces the proof of Theorem A(ii) to the proof of Theorem B(ii).

Moreover, using the fact that the pull-back morphism \(\nu_d : \overline{J_{d,g}} \to \text{Pic}(\overline{J_{ac,d,g}})\) is injective and it sends the boundary line bundles of \(\overline{J_{d,g}}\) into the boundary line bundles of \(\overline{J_{ac,d,g}}\), we get that also the boundary line bundles of \(\overline{J_{d,g}}\) are linearly independent (see Corollary 4.6), or in other words that we have an exact sequence:

\[
0 \to \bigoplus_{k, h, g > 0} \langle \mathcal{O}(\delta_1) \rangle \bigoplus_{k, h, g > 0} \langle \mathcal{O}(\delta_1), \mathcal{O}(\delta_2) \rangle \to \text{Pic}(\overline{J_{d,g}}) \to \text{Pic}(J_{d,g}) \to 0.
\]

This reduces the proof of Theorem B(ii) to the proof of Theorem B(iii).

The Picard groups of \(J_{ac,d,g}\) and of \(J_{d,g}\) are related via the following exact sequence coming from the Leray spectral sequence for the étale sheaf \(\mathcal{G}_m\) with respect to the rigidification map \(\nu_d : J_{ac,d,g} \to J_{d,g}\) (see 6.1):

\[
0 \to \text{Pic}(J_{d,g}) \xrightarrow{\nu_d^*} \text{Pic}(J_{ac,d,g}) \xrightarrow{\text{res}} \text{Pic}(BG_m) = \text{Hom}(\mathcal{G}_m, \mathcal{G}_m) \cong \mathbb{Z} \xrightarrow{\text{abs}} \text{Br}(J_{d,g}).
\]

The map res is the restriction to the fibers of \(\nu_d\) (which are isomorphic to the classifying stack \(BG_m\) of the multiplicative group \(\mathcal{G}_m\)) and obs sends \(1 \in \mathbb{Z}\) into the class \([\nu_d]\) of the \(\mathcal{G}_m\)-gerbe \(\nu_d : J_{ac,d,g} \to J_{d,g}\) in the cohomological Brauer group \(\text{Br}(J_{d,g}) := H^2_c(J_{d,g}, \mathcal{G}_m)\) of \(J_{d,g}\). In Theorem 6.4 we prove that the order of \([\nu_d]\) is the greatest common divisor \((d + 1 - g, 2g - 2)\). In proving this, we interpret in Proposition 6.6 the order of \([\nu_d]\) as the smallest natural number \(m\) for which there exists an \(m\)-Poincaré line bundle (in the sense of Definition 5.5) on the universal family \(J_{ac,d,g}\) over \(J_{d,g}\). Using Proposition 6.6, Theorem 6.4 follows then from a result of Kouvidakis (see [Kou93, p. 514]). Note also that by combining Theorem 6.4 and Proposition 6.6, we recover the well-known result of Mestrano-Ramanan.
where the map \( \chi_d \) sends a line bundle \( L \in \text{Pic}(\mathcal{J}_{d,g}) \) to the integer \( m \in \mathbb{Z} \) such that the class of \( L \) is equal to \( (2g - 2, d + 1 - g) \cdot m \). Moreover, we compute the values of the map \( \chi_d \) on the generators of the tautological subgroup \( \text{Pic}^\text{taut}(\mathcal{J}_{d,g}) \subseteq \text{Pic}(\mathcal{J}_{d,g}) \) in Lemma 5.1 and deduce that \( \text{Pic}^\text{taut}(\mathcal{J}_{d,g}) \) is isomorphic to \( \Lambda \). In Theorem 7.2 we compute the values of \( \chi_d \) on the generators of the tautological subgroup \( \text{Pic}^\text{taut}(\mathcal{J}_{d,g}) \subseteq \text{Pic}(\mathcal{J}_{d,g}) \) and deduce that \( \chi_d(\text{Pic}^\text{taut}(\mathcal{J}_{d,g})) = \frac{2g - 2}{(2g - 2, d + 1 - g)} \cdot \mathbb{Z} \). From the exact sequence (1.5), we deduce now that \( \text{Pic}^\text{taut}(\mathcal{J}_{d,g}) = \text{Pic}(\mathcal{J}_{d,g}) \) is free of rank two. Theorem B now follows.

In the last Section of the paper, we relate the Picard group of the moduli stack \( \mathcal{J}_{d,g} \) with the divisor class group \( \text{Cl}(\mathcal{J}_{d,g}) \) of its moduli scheme \( \mathcal{J}_{d,g} \), which was computed by Fontanari (see Theorem 5.5) based upon the work of Kouvidakis (Kou91) on the Picard group of the open subscheme \( \mathcal{J}_{d,g}^\alpha \subset \mathcal{J}_{d,g} \) consisting of pairs \( (C, L) \) such that \( C \) does not have non-trivial automorphisms. Fontanari proved in (Kou91) that the boundary of \( \mathcal{J}_{d,g} \) is the union of the irreducible divisors \( \Delta_i := \phi_{d, i}^{-1}(\Delta_i) \) for \( i = 1, \ldots, [g/2] \), where \( \phi_d : \mathcal{J}_{d,g} \to \mathcal{M}_{g/2} \) is the natural map towards the moduli scheme of stable curves of genus \( g/2 \). In Section 8 we prove the following result.

**Theorem C.** The pull-back map \( \Psi_d^* : \text{Cl}(\mathcal{J}_{d,g}) \to \text{Pic}(\mathcal{J}_{d,g}) \) induced by the natural map \( \Psi_d : \mathcal{J}_{d,g} \to \mathcal{J}_{d,g} \) fits into a commutative diagram with exact rows

\[
0 \to \bigoplus_{i=0}^{[g/2]} \mathbb{Z} \cdot \Delta_i \to \text{Cl}(\mathcal{J}_{d,g}) \to \text{Cl}(\mathcal{J}_{d,g}) \to 0,
\]

where the last map is the restriction map and the first map sends each \( \Delta_i \) into its class in \( \text{Cl}(\mathcal{J}_{d,g}) \). The Picard group of \( \mathcal{J}_{d,g} \) and the divisor class group of \( \mathcal{J}_{d,g} \) are related by the pull-back via the natural map \( \Psi_d : \mathcal{J}_{d,g} \to \mathcal{J}_{d,g} \) which induces a map from the exact sequence (1.4) into the exact sequence (1.6). In Section 8 we prove the following result.

(i) the map \( \beta_d \) is an isomorphism;
(ii) the map \( \alpha_d \) satisfies

\[
\alpha_d(\Delta_i) = \begin{cases} 
\mathcal{O}(\Delta_i) & \text{if } k_{d,g} \leq |2i - 1|, \\
\mathcal{O}(\Delta_i) + \mathcal{O}(\overline{\Delta_i}) & \text{if } k_{d,g} |(2i - 1) \text{ and } i \neq g/2, \\
\mathcal{O}(2\Delta_i) & \text{if } k_{d,g} |(2i - 1) \text{ and } i = g/2.
\end{cases}
\]
It is likely that the same techniques used in this paper could lead to the computation of the Picard group of the degree-$d$ compactified universal Jacobian stack $\overline{\mathcal{J}}_{ac,d,n}$ over the stack $\mathcal{M}_g,n$ of $n$-pointed stable curves of genus $g$ constructed in [Mel10] and of the universal vector bundle over $\mathcal{M}_g$ constructed in [Pan96]. We plan to come back to these two problems in a near future.

The paper is organized as follows. In Section 2 we summarize the known properties of the stacks $\overline{\mathcal{J}}_{ac,d,g}$ and $\overline{\mathcal{J}}_{d,g}$, including a description as quotients stacks, as well as the properties of their moduli scheme $\overline{\mathcal{J}}_{d,g}$, including its construction as a GIT quotient (see 2.1 and 2.8). Moreover, we recall some basic facts about the Picard group of the stack $\mathcal{M}_g$ of stable curves of genus $g$ by Harer and Arbarello-Cornalba (see 2.19). In Section 3 we describe the boundary divisors of $\overline{\mathcal{J}}_{ac,d,g}$ and we explain how they are related to the pull-back of the boundary divisors of $\mathcal{M}_g$. In Section 4 we show that the line bundles on $\overline{\mathcal{J}}_{ac,d,g}$ associated to the boundary divisors are linearly independent. In Section 5, we introduce the tautological line bundles on $\overline{\mathcal{J}}_{ac,d,g}$ and we study the relations among them. In Section 6 we compare the Picard groups of $\mathcal{J}_{ac,d,g}$ and of $\mathcal{J}_{d,g}$ using the Leray’s spectral sequence associated to the rigidification map $\nu_d : \overline{\mathcal{J}}_{ac,d,g} \to \overline{\mathcal{J}}_{d,g}$. Moreover, we compute the order of the $\mathbb{C}^*$-gerbe $\nu_d$ in the Brauer group of $\overline{\mathcal{J}}_{d,g}$. In Section 7 we compute the Picard group of $\mathcal{J}_{d,g}$ using the fibration $\Phi_d : \mathcal{J}_{d,g} \to \mathcal{M}_g$. Moreover, in Lemma 7.4, we determine the relation between the line bundle $\Xi$ and the universal theta divisor. In Section 8, we compare the Picard group of $\mathcal{J}_{d,g}$ with the divisor class group of its moduli scheme $\overline{\mathcal{J}}_{d,g}$.

1.1. Relation to algebraic topology. After a preliminary version of this manuscript has been posted on arXiv, J. Ebert and O. Randal-Williams posted on arXiv the manuscript [ERW], which contains, among other beautiful results, Ebert and Randal-Williams compute the analytic Néron-Severi group $\text{NS}$, the topological Picard group $\text{Pic}_{\text{top}}$ and the second cohomology group with integer values $H^2(\cdot, \mathbb{Z})$ of the above two stacks (see [ERW Thm. B, Thm. C]), under the assumption that $g \geq 6$.

**Theorem 1.1** (Ebert, Randal-Williams). Assume that $g \geq 6$. Then
(i) $NS(H^d\pi_g) = \text{Pic}_\text{top}(H^d\pi_g) = H^2(H^d\pi_g, \mathbb{Z})$ is freely generated by $\lambda$, $\kappa_{-1,2}$, and $\zeta := \frac{\kappa_{0,1} - \kappa_{-1,2}}{2}$. 

(ii) $NS(Pic^d_g) = \text{Pic}_\text{top}(H^d\pi_g) = H^2(Pic^d_g, \mathbb{Z})$ is the subgroup of $H^2(H^d\pi_g, \mathbb{Z})$ generated by $\lambda$ and $\eta := \frac{d \kappa_{0,1} + (g-1)\kappa_{-1,2}}{2g - 2, g + d - 1}$.

The diagram (1.1) gives two natural homomorphisms

\begin{align}
(1.9) \\
c_1 : \text{Pic}(\mathcal{F}_{\text{ac}d,g}) &\to H^2(H^d\pi_g, \mathbb{Z}), \\
c_1 : \text{Pic}(\mathcal{F}_{d,g}) &\to H^2(Pic^d_g, \mathbb{Z}).
\end{align}

The next result is obtained by comparing Theorems A(i) and B(i) with Theorem 1.1.

\begin{corollary}
Assume that $g \geq 6$. The homomorphisms of (1.9) are isomorphisms.
\end{corollary}

\begin{proof}
The fact that the first map in (1.9) is an isomorphism follows by comparing Theorem A(i) and Theorem 1.1(i) by mean of the formulas

\begin{align}
(*) \\
c_1(\Lambda(1,0)) &= \lambda, \\
c_1(\Lambda(1,1)) &= \frac{\kappa_{-1,2} - \kappa_{0,1}}{2} = -\zeta, \\
c_1(\Lambda(0,1)) &= \frac{\kappa_{-1,2} + \kappa_{0,1}}{2} + \lambda = \zeta + \kappa_{-1,2} + \lambda,
\end{align}

where the first formula follows from Lemma 5.1 and the last two formulas follow from Theorem 5.3 together with the facts that $c_1(K_{-1,2}) = \kappa_{-1,2}$ and $c_1(K(0,1)) = -\kappa_{0,1}$. Note that the minus sign appearing in this last equality is due to the fact that in defining the classes $\kappa_{i,j} \in H^2(H^d\pi_g, \mathbb{Z})$ (see (1.8)), Ebert and Randal-Williams use the relative tangent sheaf while our definition of the tautological line bundles $K_{i,j} \in \text{Pic}(\mathcal{F}_{\text{ac}d,g})$ uses its dual sheaf, namely the sheaf of relative differentials.

The fact that the second map in (1.9) is an isomorphism follows by comparing Theorem B(i) and Theorem 1.1(ii) using the formula

$$c_1(\Xi) = \frac{(d + g - 1)c_1(\Lambda(0,1)) - (d - g + 1)c_1(\Lambda(1,1))}{(d + g - 1, d - g + 1)} = \eta + \frac{d + g - 1}{(d + g - 1, d - g + 1)}\lambda.$$ \hfill \qed

\end{proof}

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\begin{notations}
1.3. We fix two integers $g \geq 2$ and $d$: $g$ will always denote the genus of the curves and $d$ the degree of the Jacobian varieties. Given two integers $m$ and $n$, we set $(n,m)$ for the greatest common divisor of $n$ and $m$. In particular the greatest common divisor $(2g - 2, d + 1 - g) = (2g - 2, d - 1 + g) = (d + 1 - g, d - 1 + g)$ will appear often in what follows. Similarly the number

\begin{align}
k_{d,g} := \frac{2g - 2}{(2g - 2, d + g - 1)}
\end{align}

will appear repeatedly throughout the paper and hence it deserves a special notation.

1.4. We work over an algebraically closed field $k$ of characteristic 0. All the schemes and stacks we will deal with are of finite type over $k$. The only place where the assumption on the characteristic of $k$ is used is the fact that we use the explicit determination of the Picard group of $\overline{\mathcal{M}}_g$ by Harer and Arbarello-Cornalba (see Theorem 2.21 for the precise statement), which is known to be true only in characteristic zero. However, in positive characteristic, the same statement remains true for the rational Picard groups of $\overline{\mathcal{M}}_g$ by the work of Moriwaki in [Mor01]. Therefore, all our statements hold in positive characteristic for the rational Picard groups.

\end{notations}
1.5. We will often assume, for simplicity, that \( g \geq 3 \). This is the case for two of the main results of this paper, namely Theorems A and B.

The reason for this assumption is that the Picard group of \( \overline{M}_g \) is freely generated by the Hodge line bundle \( \Lambda \) and the boundary line bundles \( \{O(\delta_1), \ldots, O(\delta_g/2)\} \) if \( g \geq 3 \) (see Theorem 2.21) while if \( g = 2 \) then \( \text{Pic}(\overline{M}_g) \) is still generated by \( \Lambda \) and the boundary line bundles but with the relation \( \Lambda^{10} \otimes O(-2\delta_1) = 0 \) (see 2.19). Indeed, all the above mentioned results continue to hold for \( g = 2 \) if we add the relation pull-backed from the relation \( \Lambda^{10} \otimes O(-2\delta_1) = 0 \) in \( \text{Pic}(\overline{M}_2) \) or its image \( \Lambda_0 = 0 \) in \( \text{Pic}(\overline{M}_2) \).

2. Preliminaries

2.1. The stacks \( \mathcal{J}ac_{d,g} \) and \( \mathcal{J}_d \)

Let \( \mathcal{J}ac_{d,g} \) be the universal Jacobian stack over the moduli stack \( \mathcal{M}_g \) of smooth curves of genus \( g \). The fiber of \( \mathcal{J}ac_{d,g} \) over a scheme \( S \) is the groupoid whose objects are families of smooth curves \( C \rightarrow S \) endowed with a line bundle \( L \) over \( C \) of relative degree \( d \) over \( S \) and whose arrows are the obvious isomorphisms. \( \mathcal{J}ac_{d,g} \) is a smooth irreducible (Artin) algebraic stack of dimension \( 4g - 4 \) endowed with a natural forgetful morphism \( \Phi_d : \mathcal{J}ac_{d,g} \rightarrow \mathcal{M}_g \).

The multiplicative group \( \mathbb{G}_m \) naturally injects into the automorphism group of every object \( (C \rightarrow S, \mathcal{L}) \in \mathcal{J}ac_{d,g}(S) \) as multiplication by scalars on \( \mathcal{L} \), endowing \( \mathcal{J}ac_{d,g} \) with the structure of a \( \mathbb{G}_m \)-stack in the sense of [Hof07, Def. 3.1] or, equivalently, with a \( \mathbb{G}_m \)-structure in the sense of [AGV09, Appendix C].

There is a canonical procedure to remove such automorphisms, called \( \mathbb{G}_m \)-rigidification (see [ACV03, Sec. 5], [Rom05, Sec. 5] and [AGV09, Appendix C]). The outcome is a new stack \( \mathcal{J}_{d,g} := \mathcal{J}ac_{d,g} \sslash \mathbb{G}_m \) together with a smooth and surjective map \( \nu_d : \mathcal{J}ac_{d,g} \rightarrow \mathcal{J}_{d,g} \). Indeed, the map \( \nu_d \) makes \( \mathcal{J}ac_{d,g} \) into a gerbe banded by \( \mathbb{G}_m \) (or a \( \mathbb{G}_m \)-gerbe in short) over \( \mathcal{J}_{d,g} \) (we refer to [Gir71] for the theory of gerbes). The forgetful map \( \Phi_d \) factors via \( \nu_d \) and we get a commutative diagram

\[
\begin{array}{ccc}
\mathcal{J}ac_{d,g} & \xrightarrow{\nu_d} & \mathcal{J}_{d,g} \\
\Phi_d \downarrow & & \downarrow \Phi_d \\
\mathcal{M}_g & & \\
\end{array}
\]

The new stack \( \mathcal{J}_{d,g} \) is a smooth, irreducible and separated Deligne-Mumford stack of dimension \( 4g - 3 \) and the map \( \Phi_d \) is representable.

A modular compactification of the stacks \( \mathcal{J}ac_{d,g} \) and \( \mathcal{J}_{d,g} \) was described by Caporaso in [Cap03] for some degrees and later by Melo in [Mel09] for the general case, based upon previous work of Caporaso in [Cap04]. Let us review this compactification.

Definition 2.2. [Cap04, Sec. 3.3] A connected, projective nodal curve \( X \) is said to be quasistable if it is (Deligne-Mumford) semistable and if the exceptional components of \( X \) do not meet. The exceptional locus of \( X \), denoted by \( X_{\text{exc}} \), is the union of the exceptional components of \( X \).

Definition 2.3. [BMV, Def. 3.5 and 3.6] Let \( X \) be a quasistable curve of genus \( g \geq 2 \) and \( L \) a line bundle of degree \( d \) on \( X \).

1. We say that \( L \) (or its multidegree) is properly balanced if
   - for every subcurve \( Z \) of \( X \) the following (“Basic Inequality”) holds
     \[
     m_Z(d) := \frac{dw_Z}{2g - 2} - \frac{k_Z}{2} \leq \deg_Z L \leq \frac{dw_Z}{2g - 2} + \frac{k_Z}{2} := M_Z(d),
     \]
     where \( w_Z := \deg_Z(\omega_X) \) and \( k_Z := g(Z \cap X \setminus Z) \).
   
2. We say that \( L \) (or its multidegree) is strictly balanced if it is properly balanced and if for each proper subcurve \( Z \) of \( X \) such that \( \deg_Z L = m_Z(d) \), the intersection \( Z \cap Z^c \) is contained in the exceptional locus \( X_{\text{exc}} \) of \( X \).

Remark 2.4. It is easy to check that:

1. The basic inequality (2.1) for \( Z \) is equivalent to the one for the complementary subcurve \( Z^c := X \setminus Z \);
2. If \( Z \) is a disjoint union of the subcurves \( Z_1 \) and \( Z_2 \), then the basic inequality (2.1) for \( Z_1 \) and \( Z_2 \) implies the one for \( Z \).
In particular, it is enough to check the basic inequality \((2.11)\) for all subcurves \(S\) such that \(S\) and \(Z\) are connected.

**Definition 2.5.** A stable curve \(X\) is said to be \(d\)-general if and only if every properly balanced line bundle on \(X\) is strictly balanced. We denote by \(\Mc^d_{\text{gen}} \subset \Mc_S\) the open substack whose sections are families of \(d\)-general curves.

For later use, we need to recall the description of the locus \(\Mc_S \setminus \Mc^d_{\text{gen}}\) of \(d\)-special curves, given in [Mel09 Prop. 2.2]. Recall that a vine curve of genus \(g\) and type \((g_1, g_2)\) is a stable curve of genus \(g\) formed by two smooth curves of genus \(g_1\) and \(g_2\) meeting at \(k := g - g_1 - g_2 + 1\) points.

**Proposition 2.6 (Melo).**

(i) A stable curve \(C\) is \(d\)-special (i.e. it belongs to \(\Mc_S \setminus \Mc^d_{\text{gen}}\)) if and only if it is a specialization of a \(d\)-special vine curve.

(ii) A stable vine curve of genus \(g\) and type \((i, g - i - k + 1)\) is \(d\)-special if and only if

\[
kd_g := \frac{2g - 2}{(2g - 2, d - g + 1)}(2i - 2 + k).
\]

Let \(\tilde{\mathcal{J}}\) be the category fibered in groupoids whose fiber over a scheme \(S\) consists of the groupoid whose objects are families of quasistable curves \(\mathcal{C} \to S\) endowed with a line bundle \(L\) of relative degree \(d\), whose restriction to each geometric fiber is properly balanced (we say that \(\mathcal{L}\) is properly balanced), and whose arrows are the obvious isomorphisms. The multiplicative group \(G_m\) injects into the automorphism group of every object \((\mathcal{C} \to S, L) \in \tilde{\mathcal{J}}\) as multiplication by scalars on \(L\). As in the smooth case, the rigidification morphism \(\nu_d : \tilde{\mathcal{J}} \to J_d \subset \tilde{\mathcal{J}}\) factors through a morphism \(\Phi_d : \tilde{\mathcal{J}} \to \tilde{\mathcal{M}}\).

There is a natural morphism of category fibered in groupoids \(\Phi_d : \tilde{\mathcal{J}} \to \tilde{\mathcal{M}}\), obtained by sending \((\mathcal{C} \to S, L) \in \tilde{\mathcal{J}}(S)\) into the stabilization \(\mathcal{C}^u \to S \in \mathcal{M}_S(S)\) of the family of quasi-stable curves \(\mathcal{C} \to S\). Clearly, the morphism \(\Phi_d\) factors through a morphism \(\tilde{\Phi}_d : \tilde{\mathcal{J}} \to \tilde{\mathcal{M}}\).

The following theorem summarizes the known properties of \(\tilde{\mathcal{J}}\) and of \(J_d\), proved in [Cap05] under the assumption that \((d + g - 1, 2g - 2) = 1\) and in [Mel09] for arbitrary \(d\).

**Theorem 2.7 (Caporaso, Melo).**

1. \(\tilde{\mathcal{J}}\) is an irreducible and smooth (Artin) stack of finite type over \(k\) and of dimension \(4g - 4\) (resp. \(4g - 3\). It contains the stack \(\mathcal{J}\) as a dense open substack.

2. The morphism \(\tilde{\Phi}_d : \tilde{\mathcal{J}} \to \tilde{\mathcal{M}}\) (resp. \(\Phi_d : \tilde{\mathcal{J}} \to \tilde{\mathcal{M}}\)) is of finite type, universally closed and surjective. Moreover, the following diagram commutes:

\[
\begin{array}{ccc}
\tilde{\mathcal{J}} & \xrightarrow{\tilde{\nu}_d} & \tilde{\mathcal{M}} \\
\downarrow \Phi_d & & \downarrow \Phi_d \\
\mathcal{J} & \xrightarrow{\nu_d} & \mathcal{M}
\end{array}
\]

3. The following conditions are equivalent:

   (i) \((d + 1 - g, 2g - 2) = 1\);
   (ii) \(\Phi_d\) is separated;
   (iii) \(\Phi_d\) is (strongly) representable;
   (iv) \(\tilde{\Phi}_d\) is separated;
   (v) \(\tilde{\Phi}_d\) is a Deligne-Mumford stack.

4. More generally, for all \(d \in \mathbb{Z}\), (i)-(v) above hold for the restriction

\[
\Phi_d : \tilde{\mathcal{J}}_{d, \text{gen}} := \tilde{\mathcal{J}}_{d, \text{gen}} \times_{\mathcal{M}_{d, \text{gen}}} \mathcal{M}_{d, \text{gen}} \to \tilde{\mathcal{M}}_{d, \text{gen}}.
\]

2.8. \(\tilde{\mathcal{J}}\) and \(\mathcal{J}\) as quotient stacks

In [Cap05] and [Mel09], the stacks \(\tilde{\mathcal{J}}\) and \(\mathcal{J}\) are described as quotient stacks. Let us review this description since we will need it in what follows.
Note that, for every $n \in \mathbb{Z}$, there are isomorphisms
\begin{equation}
\overline{\phi_d^n} : \overline{\text{Jac}_{d,g}} \xrightarrow{\cong} \overline{\text{Jac}_{d+n(2g-2),g}}
\end{equation}
\quad \left( \mathcal{C} \to S, \mathcal{L} \right) \mapsto \left( \mathcal{C} \to S, \mathcal{L} \otimes \omega_{\mathcal{C}/S}^n \right).

Clearly, $\overline{\phi_d^n}$ is an isomorphism of $\mathbb{G}_m$-stacks and therefore, by passing to the $\mathbb{G}_m$-rigidification, it induces an isomorphism $\phi_d^n : \text{Jac}_{d,g} \xrightarrow{\cong} \text{Jac}_{d+n(2g-2),g}$. Therefore, we can (and will) assume that $d > 0$.

Let $r = d - g$ and consider the Hilbert scheme $\text{Hilb}_{d,g}$ parametrizing curves in $\mathbb{P}^r$ of degree $d$ and of arithmetic genus $g$. Consider the action of $PGL(r+1)$ on $\text{Hilb}_{d,g}$ induced by the natural action of $PGL(r+1)$ on $\mathbb{P}^r$ (this action can be naturally linearized by embedding $\text{Hilb}_{d,g}$ in a suitable Grassmannian). Let $H_d$ be the open subset of $\text{Hilb}_{d,g}$ consisting of curves $X \subset \mathbb{P}^r$ that are connected and GIT-semistable. The points of $H_d$ are described by the following

**Theorem 2.9** (Gieseker, Caporaso). Assume that $d \geq 10(2g-2)$. Then a point $[X \subset \mathbb{P}^r] \in \text{Hilb}_{d,g}$ belongs to $H_d$ if and only if the following conditions are satisfied:

(i) $X$ is a quasistable curve;

(ii) $\mathcal{O}_X(1)$ is properly balanced.

Moreover, $H_d$ is smooth and irreducible.

More precisely, Gieseker proved in [Gie82] that the conditions of the above Theorem are necessary for the point $[X \subset \mathbb{P}^r]$ to belong to $H_d$ and Caporaso proved in [Cap04] that they are also sufficient. The above Theorem [2.9] has been recently extended to the case $d > 4(2g-2)$ by Bini-Melo-Viviani (see [BMV] Thm. A), who also observed that the conclusions of the Theorem are false if $d < 4(2g-2)$.

With these notations, we can describe the stacks $\text{Jac}_{d,g}$ and $\text{M}_{d,g}$ as quotient stacks (see [Cap05 Section 5] and [Mel09 Thm. 3.1]) and reinterpret Caporaso’s compactification $\text{M}_{d,g}$ of the universal Jacobian variety over $\text{M}_{g}$ (see [Cap04]) as an adequate moduli space for $\text{Jac}_{d,g}$ and $\text{M}_{d,g}$ (in the sense of Alper [Alp2]) and even a good moduli space (in the sense of Alper [Alp1]) if $\text{char}(k) = 0$.

**Theorem 2.10** (Caporaso, Melo). Fix $d \geq 10(2g-2)$. Then we have:

(i) There are isomorphisms of stacks
\[
\text{Jac}_{d,g} \cong [H_d/\text{GL}(r+1)],
\]
\[
\text{M}_{d,g} \cong [H_d/\text{PGL}(r+1)].
\]

(ii) The GIT-quotient $\text{M}_{d,g} = H_d/\text{PGL}(r+1)$ is an adequate moduli space for $\text{Jac}_{d,g}$ and $\text{M}_{d,g}$ and a good moduli space for $\text{Jac}_{d,g}$ and $\text{M}_{d,g}$ if $\text{char}(k) = 0$. $\text{M}_{d,g}$ is a coarse moduli space for $\text{M}_{d,g}$ if and only if $(d+1-g, 2g-2) = 1$.

(iii) The geometric points of $\text{M}_{d,g}$ are in bijection with the pairs $(X, L)$, where $X$ is a quasistable curve of genus $g$ and $L$ is a strictly balanced line bundle of degree $d$.

The above construction gives the following commutative diagram, which we record for later use:

\begin{equation}
\begin{array}{ccc}
\text{Jac}_{d,g} & \xrightarrow{\phi_d} & \text{M}_{g} \\
\downarrow{\phi_d} \quad \downarrow{\phi_d} \quad \downarrow{\phi_d} & & \downarrow{\phi_d} \\
\text{M}_{d,g} & & \text{M}_{g}
\end{array}
\end{equation}

2.11. **The Picard and the Chow groups of a stack**

In this subsection, we are going to briefly recall the definition and the main properties of the Picard group and of the Chow group of an algebraic stack that we are going to use later. We refer to [Edi] for a nice survey on the subject.

Let $\mathcal{X}$ be an Artin stack of finite type over $k$. The definition of the (functorial) Picard group of $\mathcal{X}$ was introduced by Mumford (see [Mum65] p. 64).

**Definition 2.12** (Mumford). A line bundle $\mathcal{L}$ on $\mathcal{X}$ is the data consisting of a line bundle $\mathcal{L}(f) \in \text{Pic}(\mathcal{S})$ for every morphism $f : S \to \mathcal{X}$ from a scheme $S$ and, for every composition of morphisms $T \xrightarrow{g} S \xrightarrow{f} \mathcal{X}$, an isomorphism $\mathcal{L}(f \circ g) \cong g^* \mathcal{L}(f)$, with the obvious compatibility requirements.

The tensor product of two line bundles $\mathcal{L}$ and $\mathcal{M}$ on $\mathcal{X}$ is the new line bundle $\mathcal{L} \otimes \mathcal{M}$ on $\mathcal{X}$ defined by $(\mathcal{L} \otimes \mathcal{M})(f) := \mathcal{L}(f) \otimes \mathcal{M}(f)$ together with the isomorphisms $(\mathcal{L} \otimes \mathcal{M})(f \circ g) \cong g^*(\mathcal{L} \otimes \mathcal{M})(f)$ induced by those of $\mathcal{L}$ and $\mathcal{M}$.
The abelian group consisting of all the line bundles on $\mathcal{X}$ together with the operation of tensor product is called the Picard group of $\mathcal{X}$ and is denoted by $\text{Pic}(\mathcal{X})$.

If $\mathcal{X}$ is isomorphic to a quotient stack $[X/G]$, where $X$ is a scheme of finite type over $k$ and $G$ is a group scheme of finite type over $k$, then $\text{Pic}(\mathcal{X})$ is isomorphic to the group $\text{Pic}^G(X)$ of $G$-linearized line bundles on $X$ in the sense of [GT65, 1.3] (see e.g. [EG98 Prop. 18]).

The (operational) Chow groups of an Artin stack $\mathcal{X}$ were introduced by Edidin-Graham in [EG98 Sec. 5.3] (see also [Edi, Def. 3.5]), generalizing the definition of the (or bivariant) Chow groups of a scheme (see [Ful Chap. 17]).

**Definition 2.13** (Edidin-Graham). An $i$-th Chow cohomology class $c$ on $\mathcal{X}$ is the data consisting of an element $c(f)$ belonging to the $i$-th operational Chow group $A^i(S)$ for every morphism $f : S \to \mathcal{X}$ from a scheme $S$ and, for every composition of morphisms $T \xrightarrow{g} S \xrightarrow{f} \mathcal{X}$, an isomorphism $c(f \circ g) \cong g^* c(f)$, with the obvious compatibility requirements.

The sum of two $i$-th Chow cohomology classes $c$ and $d$ on $\mathcal{X}$ is the new $i$-th Chow cohomology class $c \oplus d$ on $\mathcal{X}$ defined by $(c \oplus d)(f) := c(f) \oplus d(f)$ together with the isomorphisms $(e \oplus d)(f \circ g) \cong g^* (e \oplus d)(f)$ induced by those of $c$ and $d$.

The abelian group consisting of all the $i$-th Chow cohomology classes on $\mathcal{X}$ together with the operation of sum is called the $i$-th Chow group of $\mathcal{X}$ and is denoted by $A^i(\mathcal{X})$.

If $\mathcal{X}$ is isomorphic to a quotient stack $[X/G]$, where $X$ is a scheme of finite type over $k$ and $G$ is a group scheme of finite type over $k$, then $A^i(\mathcal{X})$ is isomorphic to the $i$-th (operational) equivariant Chow group $A^i_G(X)$ defined by Edidin-Graham in [EG98 Sec. 2.6] (see [EG98 Prop. 19]).

The first Chern class gives an homomorphism

$$c_1 : \text{Pic}(\mathcal{X}) \longrightarrow A^1(\mathcal{X})$$

$$L \mapsto c_1(L)$$

where $c_1(L) \in A^1(\mathcal{X})$ is defined by setting $c_1(L)(f) := c_1(L(f))$ for every morphism $f : S \to \mathcal{X}$ from a scheme $S$.

In the sequel, we will use the following results concerning the Picard group of a smooth quotient stack.

**Fact 2.14** (Edidin-Graham). Let $\mathcal{X} = [X/G]$ where $X$ is a smooth variety and $G$ is an algebraic group acting on $X$.

(i) The first Chern class map $c_1 : \text{Pic}(\mathcal{X}) \to A^1(\mathcal{X})$ is an isomorphism.

In particular, every Weil divisor $D$ on $\mathcal{X}$ is a Cartier divisor and hence it gives rise to a line bundle $\mathcal{O}_X(D)$ on $\mathcal{X}$.

(ii) Given a Weil divisor $D$ of $\mathcal{X}$ with irreducible components $D_i$, there is an exact sequence

$$\bigoplus_i \mathbb{Z} \cdot (\mathcal{O}_X(D_i)) \to \text{Pic}(\mathcal{X}) \to \text{Pic}(\mathcal{X} \setminus D) \to 0.$$

(iii) If $Y$ is a closed substack of $\mathcal{X}$ of codimension greater than 1 then there is an isomorphism

$$\text{Pic}(\mathcal{X}) \cong \text{Pic}(\mathcal{X} \setminus Y).$$

Part (i) follows from [EG98 Cor. 1]. Part (ii) follows from [EG98 Prop. 5]. Part (iii) follows from [EG98 Lemma 2(a)]

By Theorems 2.1(iii) and 2.1(ii), all the properties stated in Fact 2.14 hold for the stacks we will deal with, namely $\mathcal{J}_{ac,d,g}$, $\mathcal{J}_{d,g}$, $\mathcal{J}_{ac,d,g}$ and $\mathcal{J}_{d,g}$. Moreover, it is well-known that the same properties hold true for $\mathcal{M}_g$ and $\mathcal{M}_g$.

**2.15. The determinant of cohomology and the Deligne pairing**

There are two standard methods to produce line bundles on a stack parametrizing nodal curves with some extra-structure (as $\mathcal{J}_{ac,d,g}$), namely the determinant of cohomology (introduced in [KM76]) and the Deligne pairing (introduced in [Del87]). The aim of this subsection is to recall the main properties of these two constructions, following the presentation given in [ACG11 Chap. 13, Sec. 4 and 5].

Let $\pi : X \to S$ be a family of nodal curves, i.e. a proper and flat morphism whose geometric fibers are nodal curves. Given a coherent sheaf $\mathcal{F}$ on $X$ flat over $S$ (e.g. a line bundle on $X$), the determinant of cohomology of $\mathcal{F}$ is a line bundle $d_0(\mathcal{F}) \in \text{Pic}(S)$ defined as it follows. In the special case where $\pi_* (\mathcal{F})$ and $R^1 \pi_* (\mathcal{F})$ are locally free sheaves on $S$, one sets

$$d_0(\mathcal{F}) := \text{det} \pi_* (\mathcal{F}) \otimes (\text{det} R^1 \pi_* (\mathcal{F}))^{-1}.$$
In the general case, one can always find a complex of locally free sheaves \( f : K^0 \to K^1 \) on \( S \) such that \( \ker f = \pi_* (\mathcal{F}) \) and \( \coker f = R^1 \pi_* (\mathcal{F}) \) and then one sets
\[
d_\pi (\mathcal{F}) := \det K^0 \otimes (\det K^1)^{-1}.
\]
The determinant of cohomology satisfies the following properties, whose proof can be found in [ACG11 Chap. 13, Sec. 4].

**Fact 2.16.** Let \( \pi : X \to S \) be a family of nodal curves and let \( \mathcal{F} \) be a coherent sheaf on \( X \) flat over \( S \).

(i) For every exact sequence of coherent sheaves on \( X \) flat over \( S \)
\[
0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0,
\]
there is a canonical isomorphism
\[
d_\pi (\mathcal{F}) \cong d_\pi (\mathcal{E}) \otimes d_\pi (\mathcal{F}).
\]
(ii) If \( \mathcal{F} \) is locally free then there is a canonical isomorphism
\[
d_\pi (\omega_\pi \otimes \mathcal{F}^\vee) \cong d_\pi (\mathcal{F}),
\]
where \( \omega_\pi \) is the relative dualizing sheaf of the family \( \pi \).
(iii) The first Chern class of \( d_\pi (\mathcal{F}) \) is equal to
\[
c_1 (d_\pi (\mathcal{F})) = c_1 (\pi_1 (\mathcal{F})) := c_1 (\pi_* (\mathcal{F})) - c_1 (R^1 \pi_* (\mathcal{F})).
\]
(iv) The formation of the determinant of cohomology is functorial in the following sense: given a Cartesian diagram
\[
\begin{array}{ccc}
X' & \xrightarrow{g} & X \\
\downarrow & & \downarrow \pi \\
S' & \xrightarrow{f} & S
\end{array}
\]
we have a canonical isomorphism
\[
f^* d_\pi (\mathcal{F}) \cong d_{\pi'} (g^* \mathcal{F}).
\]

Given two line bundles \( \mathcal{M} \) and \( \mathcal{L} \) on the total space of a family of nodal curves \( \pi : X \to S \), the Deligne pairing of \( \mathcal{M} \) and \( \mathcal{L} \) is a line bundle \( \langle \mathcal{M}, \mathcal{L} \rangle_\pi \in \text{Pic}(S) \) which can be defined as
\[
\langle \mathcal{M}, \mathcal{L} \rangle_\pi := d_\pi (\mathcal{M} \otimes \mathcal{L}) \otimes d_\pi (\mathcal{M})^{-1} \otimes d_\pi (\mathcal{L})^{-1} \otimes d_\pi (\mathcal{O}_X).
\]
The Deligne pairing satisfies the following properties, whose proof can be found in [ACG11 Chap. 13, Sec. 5].

**Fact 2.17.** Let \( \pi : X \to S \) be a family of nodal curves.

(i) The Deligne pairing is symmetric and bilinear in each factor, namely there are canonical isomorphisms
\[
\langle \mathcal{M}, \mathcal{L} \rangle_\pi \cong \langle \mathcal{L}, \mathcal{M} \rangle_\pi,
\]
\[
\langle \mathcal{M} \otimes \mathcal{M}', \mathcal{L} \rangle_\pi \cong \langle \mathcal{M}, \mathcal{L} \rangle_\pi \otimes \langle \mathcal{M}', \mathcal{L} \rangle_\pi,
\]
\[
\langle \mathcal{M}, \mathcal{L} \otimes \mathcal{L}' \rangle_\pi \cong \langle \mathcal{M}, \mathcal{L} \rangle_\pi \otimes \langle \mathcal{M}, \mathcal{L}' \rangle_\pi,
\]
\[
\langle \mathcal{M}, \mathcal{O}_X \rangle_\pi \cong \langle \mathcal{O}_X, \mathcal{M} \rangle_\pi \cong \mathcal{O}_S.
\]
(ii) The first Chern class of \( \langle \mathcal{M}, \mathcal{L} \rangle_\pi \) is equal to
\[
c_1 (\langle \mathcal{M}, \mathcal{L} \rangle_\pi) = \pi_* (c_1 (\mathcal{M}) \cdot c_1 (\mathcal{L})).
\]
(iii) The formation of the Deligne pairing is functorial in the following sense: given a Cartesian diagram
\[
\begin{array}{ccc}
X' & \xrightarrow{g} & X \\
\downarrow & & \downarrow \pi \\
S' & \xrightarrow{f} & S
\end{array}
\]
we have a canonical isomorphism
\[
f^* \langle \mathcal{M}, \mathcal{L} \rangle_\pi \cong \langle g^* \mathcal{M}, g^* \mathcal{L} \rangle_{\pi'}.
\]

**Remark 2.18.** Since the determinant of cohomology and the Deligne pairing are functorial (see Fact 2.16 and Fact 2.17), we can extend their definition to the case when \( \pi : Y \to X' \) is a representable, proper and flat morphism of Artin stacks whose geometric fibers are nodal curves.
2.19. The Picard group of \( \overline{M}_g \)

In this subsection, we recall the definition of the tautological line bundles on \( \overline{M}_g \) and the computation of the Picard group \( \text{Pic}(\overline{M}_g) \).

The universal family \( \pi : \overline{M}_{g,1} \to \overline{M}_g \) is a representable, proper and flat morphism whose geometric fibers are nodal curves. Using the relative dualizing sheaf \( \omega_\pi \) and the constructions recalled in Section 2.16, we can produce the following tautological line bundles on \( \overline{M}_g \):

\[
\Lambda(n) := d_\pi(\omega_\pi^\otimes n) \quad \text{for any } n \in \mathbb{Z},
\]

\[
K_1 := \langle \omega_\pi, \omega_\pi \rangle_\pi.
\]

The line bundle \( \Lambda(1) \) is called the Hodge line bundle and it is denoted by \( \Lambda \). Using Fact 2.16, it is easily checked that \( \Lambda \) associates to a family of stable curves \( \{ f : \mathcal{C} \to S \} \in \overline{M}_g(S) \) the line bundle

\[
\Lambda(f) = \det f_* (\omega_{\mathcal{C}/S}) \otimes \det (R^1 f_* (\omega_{\mathcal{C}/S}))^{-1} = \bigwedge^g f_* (\omega_{\mathcal{C}/S}) \in \text{Pic}(S).
\]

We will abuse the notation and denote also with \( \Lambda \) the restriction of \( \Lambda \) to \( M_g \) is also denoted by \( \Lambda \). We will apply the same abuse of notation to the other tautological line bundles in (2.6).

Recall that the boundary \( \overline{M}_g \setminus M_g \) decomposes as the union of irreducible divisors \( \delta_i \) for \( i = 0, \ldots, [g/2] \) which are defined as follows: \( \delta_0 \) is the boundary divisor of \( \overline{M}_g \) whose generic point is an irreducible nodal curve of genus \( g \) with one node while, for any \( 1 \leq i \leq [g/2] \), \( \delta_i \) is the boundary divisor of \( \overline{M}_g \) whose generic point is a stable curve formed by two irreducible components of genera \( i \) and \( g - i \) meeting in one point. We will denote by \( \Delta_i \subset \overline{M}_g \) the image of \( \delta_i \subset \overline{M}_g \) via the natural map \( \overline{M}_g \to \overline{M}_g \). We set \( \delta := \sum \delta_i \) and denote by \( \mathcal{O}(\delta) \) the associated line bundle on \( \overline{M}_g \) (see Fact 2.14(i)). Similarly for \( \mathcal{O}(\delta_j) \in \text{Pic}(\overline{M}_g) \).

Mumford showed in [Mum83] that all the tautological line bundles (2.6) on \( \overline{M}_g \) can be expressed in terms of the Hodge line bundle \( \Lambda \) and the line bundle \( \mathcal{O}(\delta) \). More precisely, he proved the following result (see also [ACG11, Chap. 13, Sec. 7] for a proof).

Theorem 2.20 (Mumford). The tautological line bundles on \( \overline{M}_g \) satisfy the following relations

\[
K_1 = \Lambda^{12} \otimes \mathcal{O}(-\delta),
\]

\[
\Lambda(n) = \Lambda^{6n^2 - 6n + 1} \otimes \mathcal{O}
\left( -\left(\frac{n}{2}\right) \delta \right).
\]

The Picard groups of \( \overline{M}_g \) and of \( M_g \) are described by the following theorem proved by Arbarello-Cornalba in [AC87, Thm. 1], based upon a result of Harer [Har83].

Theorem 2.21 (Harer, Arbarello-Cornalba). Assume that \( g \geq 3 \). Then
(i) \( \text{Pic}(M_g) \) is freely generated by \( \Lambda \).
(ii) \( \text{Pic}(\overline{M}_g) \) is freely generated by \( \Lambda, \mathcal{O}(\delta_0), \ldots, \mathcal{O}(\delta_{[g/2]}). \)

If \( g = 2 \), then \( \text{Pic}(M_g) \) (resp. \( \text{Pic}(\overline{M}_g) \)) is still generated by \( \Lambda \) (resp. by \( \Lambda \text{and } \mathcal{O}(\delta_0), \mathcal{O}(\delta_1) \)) but with the extra relation \( \Lambda^{10} = 0 \) (resp. \( \Lambda^{10} \otimes \mathcal{O}(-\delta_0 - 2\delta_1) = 0 \)), see respectively [Vis98] and [Cor07].
we need a local description of the morphism

\( \tilde{\delta} \) (resp. \( \tilde{\delta}^1 \)) is the divisor whose generic point is a pair \((C, L)\) (resp. \((C, L_2)\)), where \(C\) consists of two smooth irreducible curves \(C_1\) and \(C_2\) of genera respectively \(i\) and \(g - i\) meeting in one point, and \(L_1\) and \(L_2\) are line bundles of multidegree

\[
\begin{align*}
(\deg_{C_1}(L_1), \deg_{C_2}(L_1)) &= \left( \frac{2i - 1}{2g - 2}, \frac{1}{2}, \frac{2(g - i) - 1}{2g - 2} \right), \\
(\deg_{C_1}(L_2), \deg_{C_2}(L_2)) &= \left( \frac{2i - 1}{2g - 2} + \frac{1}{2}, \frac{2(g - i) - 1}{2g - 2} - \frac{1}{2} \right).
\end{align*}
\]

(D) If \(g\) is even and \(k_{d,g} \mid (g - 1)\) (i.e. \(d\) is odd), \(\delta_{g/2}\) is the divisor whose generic point is a pair \((C, L)\), where \(C\) is formed by two smooth irreducible curves \(C_1\) and \(C_2\) both of genera \(g/2\) meeting in one point, and \(L\) is a line bundle of multidegree

\[
(\deg_{C_1}(L), \deg_{C_2}(L)) = \left( \frac{d - 1}{2}, \frac{d + 1}{2} \right).
\]

Note that in the above cases (C) and (D), the divisibility condition \(k_{d,g} \mid (2i - 1)\) is equivalent to the condition that \(C = C_1 \cup C_2\) is a \(d\)-special vine curve (see Proposition 2.6), or equivalently to the fact that \(M_{C,d}(d)\) and \(m_{C,d}(d)\) are integers (see Definition 2.3). Moreover, the case (D) is different from the case (C) since in the case (D) the two components \(C_1\) and \(C_2\) have the same genus and hence it is not possible to distinguish “numerically” a line bundle of multidegree \((\deg_{C_1}(L), \deg_{C_2}(L)) = \left( \frac{d - 1}{2}, \frac{d + 1}{2} \right)\) from one of multidegree \((\deg_{C_1}(L), \deg_{C_2}(L)) = \left( \frac{d + 1}{2}, \frac{d - 1}{2} \right)\).

3.1. Notation: Sometimes it is convenient to unify the notation for the cases (A) and (B) and for the cases (C) and (D). For this reason, we always assume that \(k_{d,g} \mid (2 \cdot 0 - 1) = -1\) (even when \(k_{d,g} = 1\)) and we set \(\delta_{g/2}^2 = \delta_{g/2}^2\) if \(g\) is even and \(k_{d,g} \mid (g - 1)\) (i.e. if \(g\) is even and \(d\) is odd).

As usual, we denote by \(\mathcal{O}(\tilde{\delta}_i)\) the line bundle on \(\overline{Jac_{d,g}}\) associated to \(\delta_i\) and similarly for \(\mathcal{O}(\tilde{\delta}^1_i)\) and \(\mathcal{O}(\tilde{\delta}^2_i)\). Using the above notation 3.1, we also set

\[
\tilde{\delta} := \sum_{k_{d,g} \mid (2i - 1)} \tilde{\delta}_i + \sum_{k_{d,g} \mid (2i - 1)} (\tilde{\delta}^1_i + \tilde{\delta}^2_i),
\]

and we denote by \(\mathcal{O}(\tilde{\delta}) = \mathcal{O}(\tilde{\delta}_i)\) is its associated line bundle. Note that, according to Notation 3.1, if \(g\) is even and \(d\) is odd then \(\tilde{\delta}_{g/2} = \tilde{\delta}_{g/2}^\ast\) appears with coefficient two in \(\tilde{\delta}\).

Via the natural forgetful map \(\Phi_d : \overline{Jac_{d,g}} \to \overline{M_g}\), we can relate the boundary divisors of \(\overline{Jac_{d,g}}\) to those of \(\overline{M_g}\) as follows.

Theorem 3.2. (i) The boundary \(\overline{Jac_{d,g}} \setminus \overline{Jac_{d,g}}\) of \(\overline{Jac_{d,g}}\) consists of the irreducible divisors \(\{\tilde{\delta}_i : k_{d,g} \mid (2i - 1)\) or \(i = g/2\) and \(\{\tilde{\delta}^1_i, \tilde{\delta}^2_i : k_{d,g} \mid (2i - 1)\) and \(i < g/2\)\}.

(ii) For any \(0 \leq i \leq g/2\), we have

\[
\Phi_d^\ast \mathcal{O}(\delta_i) = \begin{cases} 
\mathcal{O}(\tilde{\delta}_i) & \text{if } k_{d,g} \mid (2i - 1), \\
\mathcal{O}(\tilde{\delta}^1_i + \tilde{\delta}^2_i) & \text{if } k_{d,g} \mid (2i - 1).
\end{cases}
\]

In particular, \(\Phi_d^\ast \mathcal{O}(\delta) = \mathcal{O}(\tilde{\delta})\).

Proof. By construction we have that \(\overline{Jac_{d,g}} \setminus \overline{Jac_{d,g}} = \overline{\Phi_d^{-1}(M_g \setminus M_g)}\) (see 2.1) and moreover \(\overline{M_g \setminus M_g} = \bigcup_i \delta_i\) (see 2.19). By the Definition 2.3 it is easy to check that we have a set-theoretical equality

\[
\Phi_d^{-1}(\delta_i) = \begin{cases} 
\tilde{\delta}_i & \text{if } k_{d,g} \mid (2i - 1), \\
\tilde{\delta}^1_i \cup \tilde{\delta}^2_i & \text{if } k_{d,g} \mid (2i - 1).
\end{cases}
\]

This proves part (i).

Part (ii) is equivalent to proving that we have a scheme-theoretic equality in (3.2). To achieve that, we need a local description of the morphism \(\Phi_d : \overline{Jac_{d,g}} \to \overline{M_g}\) at a general point \((C, L)\) of \(\delta_i\) or of \(\tilde{\delta}^1_i \cap \tilde{\delta}^2_i\). Recall that locally at \((C, L)\), the morphism \(\Phi_d\) looks like

\[
q : [\Def_{(C, L)} / \Aut(C, L)] \to [\Def_{C^\ast} / \Aut(C^\ast)],
\]

where \(\Def_{C^\ast}\) (resp. \(\Def_{(C, L)}\)) is the minimal universal deformation space of the stabilization \(C^\ast\) of \(C\) (resp. of the pair \((C, L)\)) and \(\Aut(C^\ast)\) (resp. \(\Aut(C, L)\)) is the automorphism group of \(C^\ast\) (resp. the automorphism
group of the pair \((C, L)\)). Using the results on the local structure of \(\mathcal{J}_{ac,d,g}\) given in [BFVII Sec. 2.15], we can describe explicitly the above morphism \(q\) at a general point of \(\delta_i\) or of \(\delta_1^i \cap \delta_2^i\) in the boundary of \(\mathcal{J}_{ac,d,g}\). To this aim, we need to distinguish between the case \(k_{d,g} \not\mid (2i - 1)\) (cases (A) and (B)) and the case \(k_{d,g} \mid (2i - 1)\) (cases (C) and (D)).

Suppose first that \(k_{d,g} \not\mid (2i - 1)\). Consider a general point \((C, L)\) of \(\delta_i\). Since \(C = C^{st}\) is a general element of \(\delta_i\), it is well-known that \(\text{Def}_C = \text{Spf} k[[x_1, \ldots, x_{3g-3}]]\) and

\[
\text{Aut}(C) = \begin{cases} 
\{1\} & \text{if } i \not= 1, \\
\mathbb{Z}/2\mathbb{Z} & \text{if } i = 1,
\end{cases}
\]

where, in the second case, the unique non-trivial automorphism is the elliptic involution on the elliptic tail of \(C\). On the other hand, we have that \(\text{Def}_{(C, L)} = \text{Spf} k[[x_1, \ldots, x_{3g-3}, t_1, \ldots, t_g]]\) and \(\text{Aut}(C, L) = \mathbb{G}_m\) acts trivially on it (see [BFVII Proof of Thm. 1.5, Cases (1) and (2)]), where the coordinates \(x_i\) correspond to the deformation of the curve \(C\) and the coordinates \(t_j\) correspond to the deformation of the line bundle \(L\). The morphism \(q\) is given by the natural equivariant projection \(\text{Def}_{(C, L)} \rightarrow \text{Def}_C\). Moreover, we can choose local coordinates \(x_1, \ldots, x_{3g-3}\) for \(\text{Def}_{(C, L)}\) in such a way that the first coordinate \(x_1\) corresponds to the smoothing of the unique node of \(C\) and, if \(i = 1\), the action of the generator of \(\text{Aut}(C) = \mathbb{Z}/2\mathbb{Z}\) sends \(x_1 \mapsto -x_1\) and fixes the other coordinates. For such a choice of the coordinates, we have that the equation of \(\delta_i\) inside \(\text{Def}_{(C, L)}\) is given by \((x_1 = 0)\) and the equation of \(\tilde{\delta}_i\) inside \(\text{Def}_C\) is given by \((x_1 = 0)\). Since \(q'(x_1) = (x_1)\), we conclude in this case.

Suppose now that \(k_{d,g} \mid (2i - 1)\) (hence that \(i > 0\) by Notation \(3.1\)). If \(i < g/2\) then a general point \((C, L)\) of \(\tilde{\delta}_1^i \cap \tilde{\delta}_2^i\) consists of the two general curves \(C_1\) and \(C_2\) of genera respectively \(i\) and \(g - i\) joined by a rational curve \(R \cong \mathbb{P}^1\). By convention, in the case \(i = g/2\) and \(k_{d,g} \mid (g - 1)\), we set \(\tilde{\delta}_1^i \cap \tilde{\delta}_2^i\) to be the closure of the locus of curves consisting of two smooth curves of genera \(g/2\) joined by a rational curve \(R \cong \mathbb{P}^1\). The stabilization \(C^{st}\) is obtaining by contracting the rational curve \(R\) to a node \(n\) and it will be a general point of \(\delta_i\). As before, we have that \(\text{Def}_{(C, L)} = \text{Spf} k[[x_1, \ldots, x_{3g-3}, t_1, \ldots, t_g]]\), where \(x_1\) can be chosen as the coordinate corresponding to the smoothing of the node \(n\), and \(\text{Aut}(C)\) is as in \((3.3)\).

On the other hand, by [BFVII Proof of Theorem 1.5, Case (3)]\), \(\text{Def}_{(C, L)} = \mathbb{G}_m\), \(\text{Def}_{(C, L)} = \text{Spf} k[[u_1, v_1, x_2, \ldots, x_{3g-3}, t_1, \ldots, t_g]]\) where \(u_1\) corresponds to the node \(C_1 \cap R\) and \(v_1\) corresponds to the node \(C_2 \cap R\). Moreover, the action of \(\mathbb{G}_m\) on \(\text{Def}_{(C, L)}\) is given by \((\lambda, \mu) \cdot (u_1, v_1) = (\lambda u_1, \lambda^{-1} v_1)\) while it is the identity on the other coordinates. The morphism \(q\) is induced by the equivariant morphism \(\text{Def}_{(C, L)} \rightarrow \text{Def}_C\) that, at the level of rings, sends \(x_1\) into \(u_1 \cdot v_1\) and \(x_i\) into \(x_i\) for \(i > 1\). The equation of \(\delta_i\) inside \(\text{Def}_{(C, L)}\) is given by \((x_1 = 0)\) while the equations of \(\tilde{\delta}_1^i\) and \(\tilde{\delta}_2^i\) inside \(\text{Def}_C\) are given by \((u_1 = 0)\) and \((v_1 = 0)\) (note that in the special case \(i = g/2\) and \(k_{d,g} \mid (g - 1)\), the divisor \(\tilde{\delta}_2^i\), even though irreducible, has two branches locally at \((C, L)\), which we call \(\tilde{\delta}_2^i\) and \(\tilde{\delta}_2^i\), whose equations are \((u_1 = 0)\) and \((v_1 = 0)\)). Since \(q'(x_1) = (u_1 \cdot v_1)\), we conclude also in this case.

\[\mathbf{\Box}\]

As a Corollary of the above Theorem \(3.2\), we can determine also the irreducible components of the boundary of \(\mathcal{J}_{d,g}\). We set \(\delta_1^i := \nu_d(\delta_i), \delta_2^i := \nu_d(\delta_1^i)\) and \(\tilde{\delta}_i := \nu_d(\tilde{\delta}_i)\) according to the above Cases (A)–(B), where as usual \(\nu_d : \mathcal{J}_{ac,d,g} \rightarrow \mathcal{J}_{d,g}\) is the rigidification map.

**Corollary 3.3.**

(i) The boundary \(\mathcal{J}_{d,g} \setminus \mathcal{J}_{d,g}\) of \(\mathcal{J}_{d,g}\) consists of the irreducible divisors \(\{\delta_1 : k_{d,g} \not\mid (2i - 1)\ or \ i = g/2\}

and \(\{\delta_1^2, \delta_2^2 : k_{d,g} \mid (2i - 1)\ and \ i < g/2\}\).

(ii) For any \(0 \leq i \leq g/2\), we have

\[
\begin{align*}
\nu_d^j(\delta_1) &= \mathcal{O}(\delta_1) & \text{if } k_{d,g} \not\mid (2i - 1), \\
\nu_d^j(\delta_2) &= \mathcal{O}(\delta_2) & \text{if } k_{d,g} \mid (2i - 1)\ and \ j = 1, 2.
\end{align*}
\]

**Proof.** The Corollary follows straightforwardly from Theorem \(3.2\) and the fact that \(\nu_d : \mathcal{J}_{ac,d,g} \rightarrow \mathcal{J}_{d,g}\) is a \(\mathbb{G}_m\)-gerbe such that \(\nu_d^{-1}(\mathcal{J}_{d,g}) = \mathcal{J}_{ac,d,g}\). \[\mathbf{\Box}\]

4. Independence of the boundary divisors

The aim of this Section is to prove that the line bundles corresponding to the irreducible components of the boundary of \(\mathcal{J}_{ac,d,g}\) are linearly independent in \(\text{Pic}(\mathcal{J}_{ac,d,g})\). More precisely, we will prove the following result.

**
Theorem 4.1. We have an exact sequence

\[
(4.1) \quad 0 \to \bigoplus_{k_d,g \not\equiv \frac{3i-1}{2}, \text{ or } i=g/2} \langle O(\delta_i) \rangle \bigoplus_{k_d,g \not\equiv \frac{2i-1}{2}, \text{ and } i \neq g/2} \langle O(\delta_i^2) \rangle \to \text{Pic}(\mathcal{F}_{ac,d,g}) \to \text{Pic}(\mathcal{F}_{ac,d,g}) \to 0,
\]

where the right map is the natural restriction morphism and the left map is the natural inclusion.

Using Theorem 3.2(i) and Fact 2.14(ii), we have that the exact sequence (4.1) is exact except perhaps to the left. It remains to prove that the map on the left is injective, or in other words that the line bundles associated to the boundary divisors of \( \mathcal{F}_{ac,d,g} \) are linearly independent in \( \text{Pic}(\mathcal{F}_{ac,d,g}) \).

The strategy that we will use to prove this is the same as the one used by Arbarello-Cornalba in [AC87, p. 156-159]. For that reason, we will be using their notations.

**Assumption 4.2.** The degree \( d \) satisfies \( 0 \leq d < 2g-2 \).

**The Family \( \tilde{F} \)**

Start from a general pencil of conics in \( \mathbb{P}^2 \). Blowing up the four base points of the pencil, we get a conic bundle \( \phi : X \to \mathbb{P}^1 \). The four exceptional divisors \( E_1, E_2, E_3, E_4 \subset X \) of the blow-up of \( \mathbb{P}^2 \) are sections of \( \phi \) through the smooth locus of \( \phi \). Note that \( \phi \) will have three singular fibers consisting of two incident lines. Let \( C \) be a fixed irreducible, smooth and projective curve of genus \( g-3 \) and \( p_1, p_2, p_3, p_4 \) four points of \( C \). We construct a surface \( Y \) by setting

\[
Y = \left( X \bigsqcup (C \times \mathbb{P}^1) \right) / \langle E_i \sim \{p_i\} \times \mathbb{P}^1 : i = 1, \ldots, 4 \rangle.
\]

We get a family \( f : Y \to \mathbb{P}^1 \) of stable curves of genus \( g \): the general fiber of \( f \) consists of \( C \) and a smooth conic \( Q \) meeting in \( 4 \) points (see Figure 1 below), while the three special fibers consist of \( C \) and two lines \( R_1 \) and \( R_2 \) such that \(| R_1 \cap R_2 | = 1, | R_1 \cap C | = | R_2 \cap C | = 2 \) (see Figure 2 below).

Choose a line bundle \( L \) of degree \( d \) on \( C \), pull it back to \( C \times \mathbb{P}^1 \) and call it again \( L \). Since \( L \) is trivial when restricted to \( \{p_i\} \times \mathbb{P}^1 \), we can glue it with the trivial line bundle on \( X \) and, thus, we obtain a line bundle \( \mathcal{L} \) on the family \( Y \to \mathbb{P}^1 \) of relative degree \( d \).

**Lemma 4.3.** The line bundle \( \mathcal{L} \) is properly balanced.

**Proof.** Since the property of being properly balanced is an open condition, it is enough to check that \( \mathcal{L} \) is properly balanced on the three special fibers of \( f : Y \to \mathbb{P}^1 \). According to Remark 2.4, it is enough to check the basic inequality for the three subcurves \( R_1 \cup R_2, R_1 \) and \( R_2 \). The balancing condition for \( R_1 \cup R_2 \)

\[
\left| \deg_{R_1\cup R_2}(\mathcal{L}) - \frac{d \cdot 2}{2g-2} \right| \leq \frac{4}{2}.
\]
Figure 2. The three special fibers of $f : Y \to \mathbb{P}^1$

is true because $\deg_{R_1 \cup R_2}(\mathcal{L}) = 0$ and $0 \leq d < 2g - 2$. The balancing condition for each of the subcurves $R_i$ ($i = 1, 2$) is

$$\left| \deg_{R_i}(\mathcal{L}) - \frac{d \cdot 1}{2g - 2} \right| \leq \frac{3}{2}$$

which is satisfied because $\deg_{R_i}(\mathcal{L}) = 0$ and $0 \leq d < 2g - 2$. □

We call $\tilde{F}$ the family $f : Y \to \mathbb{P}^1$ endowed with the line bundle $\mathcal{L}$. Forgetting the line bundle $\mathcal{L}$, we are left with the family $F$ of [AC87, p. 158]. We can compute the degree of the pull-backs of the boundary classes in $\text{Pic}(\mathcal{J}_{ac_d,g})$ to the curve $\tilde{F}$:

$$\begin{align*}
\deg_{\tilde{F}} \mathcal{O}(\tilde{\delta}_0) &= -1, \\
\deg_{\tilde{F}} \mathcal{O}(\tilde{\delta}_i) &= 0 & \text{if } 1 \leq i \text{ and } k_{d,g} \nmid (2i - 1) \text{ or } i = g/2, \\
\deg_{\tilde{F}} \mathcal{O}(\tilde{\delta}_1) &= \deg_{\tilde{F}} \mathcal{O}(\tilde{\delta}_2) = 0 & \text{if } 1 \leq i < g/2 \text{ and } k_{d,g} \nmid (2i - 1).
\end{align*}$$

The first relation follows from the fact that $\deg_{\tilde{F}} \mathcal{O}(\tilde{\delta}_0) = \deg_{\tilde{F}} \mathcal{O}(\delta_0)$ (by using the projection formula) and the relation $\deg_{\tilde{F}} \mathcal{O}(\delta_0) = -1$ proved in [AC87, p. 158]. The last two relations follow by the obvious fact that $\tilde{F}$ does not meet the divisors $\tilde{\delta}_i$ or $\tilde{\delta}_1$ and $\tilde{\delta}_2$ for $i \geq 1$.

The Families $F'_1$ and $F'_2$

We start with the same family of conics $\phi : X \to \mathbb{P}^1$ that we considered in the construction of the family $\tilde{F}$. Let $C$ be a fixed irreducible, smooth and projective curve of genus $g - 3$, $E$ be a fixed irreducible, smooth and projective elliptic curve and take points $p_1 \in E$ and $p_2, p_3, p_4 \in C$. We construct a surface $Z$ by setting

$$Z = \left( X \prod (C \times \mathbb{P}^1) \prod (E \times \mathbb{P}^1) \right) / (E_i \sim \{ p_i \} \times \mathbb{P}^1 : i = 1, \ldots, 4).$$

We get a family $g : Z \to \mathbb{P}^1$ of stable curves of genus $g$: the general fiber of $g$ consists of $C$, $E$ and a smooth conic $Q$ intersecting as in Figure 3. The three special fibers consist of $C$, $E$ and two lines $R_1$ and $R_2$, intersecting as shown in Figure 4.

Figure 3. The general fibers of $g : Z \to \mathbb{P}^1$.  

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We choose two line bundles of degree $d$ and $d - 3$ on $C$, we pull them back to $C \times \mathbb{P}^1$ and call them, respectively, $L_1$ and $L_2$. Similarly, we choose two line bundles of degree 0 and 1 on $E$, we pull them back to $E \times \mathbb{P}^1$ and call them, respectively, $M_1$ and $M_2$. We glue the line bundle $L_1$ (resp. $L_2$) on $C \times \mathbb{P}^1$, the line bundle $M_1$ (resp. $M_2$) on $E \times \mathbb{P}^1$ and the line bundle $\mathcal{O}_X$ (resp. $\omega_{X/\mathbb{P}^1}^{-1}$, the relative anti-canonical bundle of $\phi : X \to \mathbb{P}^1$) on $X$, obtaining a line bundle $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) on $Z$ of relative degree $d$.

**Lemma 4.4.** The line bundle $\mathcal{M}_1$ is properly balanced if $0 \leq d \leq g - 1$. The line bundle $\mathcal{M}_2$ is properly balanced if $g - 1 \leq d < 2g - 2$.

**Proof.** Since the property of being properly balanced is an open condition, it is enough to check that $\mathcal{M}$ is properly balanced on the three special fibers of $g : Z \to \mathbb{P}^1$. By Remark 2.4 it is enough to check the basic inequality for the subcurves $E, C, R_1$ and $R_2 \cup E$. The basic inequality for $C$,

$$\left|\operatorname{deg}_C(\mathcal{M}_i) - \frac{d \cdot (2g - 5)}{2g - 2}\right| \leq \frac{3}{2},$$

is satisfied since $\operatorname{deg}_C(\mathcal{M}_i) = d$ if $0 \leq d \leq g - 1$ and $\operatorname{deg}_C(\mathcal{M}_i) = d - 3$ if $g - 1 \leq d < 2g - 2$. For $E$, we get

$$\left|\operatorname{deg}_E(\mathcal{M}_i) - \frac{d \cdot 1}{2g - 2}\right| \leq \frac{1}{2},$$

which is satisfied since $\operatorname{deg}_E(\mathcal{M}_i) = 0$ if $0 \leq d \leq g - 1$ and $\operatorname{deg}_E(\mathcal{M}_i) = 1$ if $g - 1 \leq d < 2g - 2$. The basic inequality for $R_1$ is

$$\left|\operatorname{deg}_{R_1}(\mathcal{M}_i) - \frac{d \cdot 1}{2g - 2}\right| \leq \frac{3}{2},$$

which is satisfied since $\operatorname{deg}_{R_1}(\mathcal{M}_i) = 0$ if $0 \leq d \leq g - 1$ and $\operatorname{deg}_{R_1}(\mathcal{M}_i) = 1$ if $g - 1 \leq d < 2g - 2$ (note that the relative anti-canonical bundle $\omega_{X/\mathbb{P}^1}^{-1}$ has degree 1 on $R_1$ and $R_2$). Finally, the basic inequality for $R_2 \cup E$

$$\left|\operatorname{deg}_{R_2 \cup E}(\mathcal{M}_i) - \frac{d \cdot 2}{2g - 2}\right| \leq \frac{2}{2} = 1,$$

is satisfied since $\operatorname{deg}_{R_2 \cup E}(\mathcal{M}_i) = 0$ if $0 \leq d \leq g - 1$ and $\operatorname{deg}_{R_2 \cup E}(\mathcal{M}_i) = 2$ if $g - 1 \leq d < 2g - 2$. \hfill \qed

If $0 \leq d \leq g - 1$, we call $\widetilde{F}_1^d$ the family $g : Z \to \mathbb{P}^1$ endowed with the line bundle $\mathcal{M}_1$; if $g - 1 \leq d < 2g - 2$, we call $\widetilde{F}_2^d$ the family $g : Z \to \mathbb{P}^1$ endowed with the line bundle $\mathcal{M}_2$. Both families $\widetilde{F}_1^d$ and $\widetilde{F}_2^d$, when defined, are liftings of the family $F_1^d$ of [AC87, p. 158]. We can compute the degree of the pull-backs of some of the boundary classes in $\operatorname{Pic}(\mathbb{P}^1_{a_{d,g}})$ to the curves $\widetilde{F}_1^d$ and $\widetilde{F}_2^d$, in the ranges of degrees where they are defined (note that $\tilde{\Phi}_{d}^{-1}(\delta_1)$ is the union of two irreducible divisors if and only if $k_{d,g} = 1$, i.e. iff $d = g - 1$):

\begin{equation}
\begin{cases}
\deg_{\widetilde{F}_1^d} \mathcal{O}(\tilde{\delta}_1) = \deg_{\widetilde{F}_2^d} \mathcal{O}(\tilde{\delta}_1) = -1 & \text{if } d \neq g - 1, \\
\deg_{\widetilde{F}_1^d} \mathcal{O}(\tilde{\delta}_1) = \deg_{\widetilde{F}_2^d} \mathcal{O}(\tilde{\delta}_1) = -1 & \text{and } \deg_{\widetilde{F}_1^d} \mathcal{O}(\tilde{\delta}_1) = \deg_{\widetilde{F}_2^d} \mathcal{O}(\tilde{\delta}_1) = 0 & \text{if } d = g - 1, \\
\deg_{\widetilde{F}_1^d} \mathcal{O}(\tilde{\delta}_1) = \deg_{\widetilde{F}_2^d} \mathcal{O}(\tilde{\delta}_1) = 0 & \text{if } 1 < i \text{ and } k_{d,g} j(2i - 1) \text{ or } i = g/2, \\
\deg_{\widetilde{F}_1^d} \mathcal{O}(\tilde{\delta}_1) = \deg_{\widetilde{F}_2^d} \mathcal{O}(\tilde{\delta}_1) = 0 & \text{if } 1 < i < g/2 \text{ and } k_{d,g} j(2i - 1) \text{, for } j = 1, 2.
\end{cases}
\end{equation}
The first relation follows, by using the projection formula, from the relation deg_{F_i} O(\delta_i) = -1 proved in [AC87, p. 159]. The second relation is deduced in a similar way using the projection formula and the (easily checked) fact that \( \tilde{F}_1 \) does not meet \( \tilde{\delta}_1 \) and that \( \tilde{F}_2 \) does not meet \( \tilde{\delta}_1 \). The last two relations follow from the fact that \( \tilde{F}_1 \) and \( \tilde{F}_2 \) do not meet the divisors \( \delta_i \) or \( \tilde{\delta}_1 \) for \( i > 1 \).

**The Families** \( \tilde{F}_{h,1} \) and \( \tilde{F}_{h,2} \) \( (for \ 1 \leq h \leq \frac{g - 2}{2}) \)

Fix irreducible, smooth and projective curves \( C_1, C_2 \) and \( \Gamma \) of genera \( h, g - h - 1 \) and 1, and points \( x_1 \in C_1, x_2 \in C_2 \) and \( \gamma \in \Gamma \). Consider the surfaces \( Y_1 = C_1 \times \Gamma, Y_3 = C_2 \times \Gamma \) and \( Y_2 \) given by the blow-up of \( \Gamma \times \Gamma \) at \( (\gamma, \gamma) \). Let us denote by \( p_2 : Y_2 \to \Gamma \) the map given by composing the blow-down \( Y_2 \to \Gamma \times \Gamma \) with the second projection, and by \( \pi_1 : Y_1 \to \Gamma \) and \( \pi_3 : Y_3 \to \Gamma \) the projections along the second factor. As in [AC87, p. 156], we set (see also Figure 5):

\[
A = \{x_1\} \times \Gamma, \\
B = \{x_2\} \times \Gamma, \\
E = \text{exceptional divisor of the blow-up of } \Gamma \times \Gamma \text{ at } (\gamma, \gamma), \\
\Delta = \text{proper transform of the diagonal in } Y_2, \\
S = \text{proper transform of } \{\gamma\} \times \Gamma \text{ in } Y_2, \\
T = \text{proper transform of } \Gamma \times \{\gamma\} \text{ in } Y_2.
\]

**Figure 5.** Constructing \( f : X \to \Gamma \).

We construct a surface \( X \) by identifying \( S \) with \( A \) and \( \Delta \) with \( B \). The surface \( X \) comes equipped with a projection \( f : X \to \Gamma \). The fibers over all the points \( \gamma' \neq \gamma \) are shown in Figure 6 while the fiber over the point \( \gamma \) is shown in Figure 7.

**Figure 6.** The general fiber of \( f : X \to \Gamma \).

We will first construct several line bundles over the three surfaces \( Y_1, Y_2 \) and \( Y_3 \), and then we will glue them in a suitable way.

Consider the line bundles \( M_i \ (i = 1, \cdots, 4) \) on \( Y_2 \) given by

\[
M_1 := O_{Y_2}, \ M_2 := O_{Y_2}(\Delta), \ M_3 := O_{Y_2}(\Delta + E), \ M_4 := O_{Y_2}(2\Delta + E).
\]
Using that \( \deg E \mathcal{O}(E) = -1 \), we get that the restrictions of \( M_i \) to \( E \) and \( T \) have degrees:

\[
(\deg_E M_i, \deg_T M_i) = \begin{cases} (0, 0) & \text{if } i = 1, \\ (1, 0) & \text{if } i = 2, \\ (0, 1) & \text{if } i = 3, \\ (1, 1) & \text{if } i = 4. \\ \end{cases}
\]

Notice that the diagonal \( \Delta \) of \( \Gamma \times \Gamma \) is such that \( \mathcal{O}_{\Gamma \times \Gamma}(\overline{\Delta}) = \mathcal{O}_\Gamma \) since \( \Gamma \) is an elliptic curve. By applying the projection formula to the blow-up \( Y_2 \to \Gamma \times \Gamma \), we get that \( \mathcal{O}_{Y_2}(\overline{\Delta}) = \mathcal{O}_{\Delta}(\overline{-\gamma}) \). Using this, we can easily compute the restrictions of \( M_i \) to \( S \) and \( \Delta \) (which are canonically isomorphic to \( \Gamma \)):

\[
(M_i)_{|\Delta} = \begin{cases} \mathcal{O}_\Gamma & \text{if } i = 1, 3 \\ \mathcal{O}_\Gamma(-\gamma) & \text{if } i = 2, 4 \end{cases} \quad \text{and} \quad (M_i)_{|S} = \begin{cases} \mathcal{O}_\Gamma & \text{if } i = 1, 2 \\ \mathcal{O}_\Gamma(\gamma) & \text{if } i = 3, 4. \end{cases}
\]

Consider now the integers \( \alpha_1, \alpha_2 \) defined by:

\[
\alpha_1 := \left\lfloor \frac{2(2g - 2h - 3)}{2g - 2} \right\rfloor, \quad \alpha_2 := \left\lceil \frac{2(2g - 2h - 3)}{2g - 2} \right\rceil, \quad \text{if } \frac{2(2g - 2h - 3)}{2g - 2} \equiv \frac{1}{2} \mod \mathbb{Z}
\]

\[
\alpha_1 = \alpha_2 := \text{the unique integer which is closest to } \frac{2(2g - 2h - 3)}{2g - 2}, \text{ otherwise.}
\]

Take two line bundles on \( C_2 \) of degrees \( \alpha_1 \) and \( \alpha_2 \), and call, respectively, \( L_1 \) and \( L_2 \) their pull-backs to \( Y_3 = C_2 \times \Gamma \). We may assume that \( L_1 = L_2 \) if \( \alpha_1 = \alpha_2 \).

Analogously, consider the integers \( \beta_1, \beta_2 \) defined by:

\[
\beta_1 := \left\lfloor \frac{2(2h - 1)}{2g - 2} \right\rfloor, \quad \beta_2 := \left\lceil \frac{2(2h - 1)}{2g - 2} \right\rceil, \quad \text{if } \frac{2(2h - 1)}{2g - 2} \equiv \frac{1}{2} \mod \mathbb{Z}
\]

\[
\beta_1 = \beta_2 := \text{the unique integer which is closest to } \frac{2(2h - 1)}{2g - 2}, \text{ otherwise.}
\]

Consider two line bundles on \( C_1 \) of degrees \( \beta_1 \) and \( \beta_2 \), and call, respectively, \( N_1 \) and \( N_2 \) their pull-back to \( Y_1 = C_1 \times \Gamma \). We may assume that \( N_1 = N_2 \) if \( \beta_1 = \beta_2 \).

We now want to define two (possibly equal) line bundles \( I_1 \) and \( I_2 \) on \( X \), by gluing in a suitable way some of the line bundles on \( Y_1, Y_2 \) and \( Y_3 \), we have just defined. We shall distinguish between several cases:

**CASE A:** \( \frac{d(2g - 2h - 3)}{2g - 2} \neq \frac{1}{2} \mod \mathbb{Z} \) (i.e., \( \alpha_1 = \alpha_2 \)). In this case, we have that

\[
(4.5) \quad \alpha_1 - \frac{1}{2} < \frac{d(2g - 2h - 3)}{2g - 2} < \alpha_1 + \frac{1}{2} \quad \text{and} \quad \beta_1 - \frac{1}{2} < \frac{d(2h - 1)}{2g - 2} \leq \beta_1 + \frac{1}{2}.
\]

**Subcase A1:** \( 0 \leq d \leq g - 1 \). Using the inequalities (4.5), we get that

\[
-1 \leq -1 + \frac{d}{g - 1} = -1 + d - \frac{d(2g - 2h - 3)}{2g - 2} - \frac{d(2h - 1)}{2g - 2} < d - \alpha_1 - \beta_1 <
\]

\[
(4.6) \quad < 1 + d - \frac{d(2g - 2h - 3)}{2g - 2} - \frac{d(2h - 1)}{2g - 2} = 1 + \frac{d}{g - 1} < 2.
\]
If \( d - \alpha_1 - \beta_1 = 0 \) then we define \( \mathcal{I}_2 = \mathcal{I}_2 \) to be equal to the line bundle on \( X \) obtained by gluing \( N_1, M_1 \) and \( L_1 = L_2 \), which is possible since, by (4.3), we have that \( (N_1)_{\mathcal{A}} = \mathcal{O}_\Gamma = (M_1)_{\mathcal{B}} \) and \( (L_1)_{\mathcal{B}} = \mathcal{O}_\Gamma = (M_1)_{\Delta} \).

Otherwise, if \( d - \alpha_1 - \beta_1 = 1 \), then we define \( \mathcal{I}_1 = \mathcal{I}_2 \) to be equal to the line bundle on \( X \) obtained by gluing the sheaves \( N_2, M_1 \) and \( L_1 \otimes \pi_3^* \mathcal{O}_\Gamma(-\gamma) \), which is possible since, by (4.4), we have that 
\[
(N_1)_{\mathcal{A}} = \mathcal{O}_\Gamma = (M_2)_{\mathcal{B}} \text{ and } (L_1 \otimes \pi_3^* \mathcal{O}_\Gamma(-\gamma))_{\mathcal{B}} = \mathcal{O}_\Gamma(-\gamma) = (M_2)_{\Delta}.
\]

Subcase A2: \( g - 1 < d < 2g - 2 \).

Arguing similarly to the above inequality (4.6), we get that \( d - \alpha_1 - \beta_1 = 1, 2 \).

If \( d - \alpha_1 - \beta_1 = 1 \), then we define \( \mathcal{I}_1 = \mathcal{I}_2 \) to be equal to the line bundle on \( X \) obtained by gluing \( N_1 \otimes \pi_1^* \mathcal{O}_\Gamma(\gamma), M_3 \) and \( L_1 \), which is possible since, by (4.3), we have that \( (N_1 \otimes \pi_1^* \mathcal{O}_\Gamma(\gamma))_{\mathcal{A}} = \mathcal{O}_\Gamma(\gamma) = (M_3)_{\mathcal{B}} \) and \( (L_1)_{\mathcal{B}} = \mathcal{O}_\Gamma = (M_3)_{\Delta} \).

If \( d - \alpha_1 - \beta_1 = 2 \), then we define \( \mathcal{I}_1 = \mathcal{I}_2 \) to be equal to the line bundle on \( X \) obtained by gluing \( N_1 \otimes \pi_1^* \mathcal{O}_\Gamma(\gamma), M_4 \) and \( L_1 \otimes \pi_3^* \mathcal{O}_\Gamma(-\gamma) \), which is possible since, by (4.4), we have that \( (N_1 \otimes \pi_1^* \mathcal{O}_\Gamma(\gamma))_{\mathcal{A}} = \mathcal{O}_\Gamma(\gamma) = (M_4)_{\mathcal{B}} \) and \( (L_1 \otimes \pi_3^* \mathcal{O}_\Gamma(-\gamma))_{\mathcal{B}} = \mathcal{O}_\Gamma(-\gamma) = (M_4)_{\Delta} \).

CASE B: \( d - \frac{2(2g - 2\beta_1 - 3)}{2g - 2} = \frac{1}{2} \mod \mathbb{Z} \) (i.e. \( \alpha_1 = \alpha_2 - 1 \)).

In this case, we have that \( d \alpha_1 + \frac{1}{2}(2g - 2\beta_1 - 3) = \alpha_2 - \frac{1}{2}, \beta_1 < \frac{1}{2} < \frac{d(2g - 2\beta_1 - 3)}{2g - 2} \leq \beta_1 + \frac{1}{2} \), and that \( \beta_2 - \frac{1}{2} \leq \frac{2h - 1}{2g - 2} < \beta_2 + \frac{1}{2} \). So, arguing similarly to the above inequality (4.6), we get that
\[
d - \alpha_1 - \beta_2 = \begin{cases} 1 & \text{if } 0 \leq d \leq g - 1, \\ 2 & \text{if } g - 1 \leq d < 2g - 2. \end{cases}
\]

If \( 0 \leq d \leq g - 1 \), we define \( \mathcal{I}_2 \) to be equal to the line bundle on \( X \) obtained by gluing the sheaves \( N_2, M_4 \) and \( L_1 \otimes \pi_3^* \mathcal{O}_\Gamma(-\gamma) \), which is possible since, by (4.3), we have that \( (N_2)_{\mathcal{A}} = \mathcal{O}_\Gamma = (M_2)_{\mathcal{B}} \) and \( (L_1 \otimes \pi_3^* \mathcal{O}_\Gamma(-\gamma))_{\mathcal{B}} = \mathcal{O}_\Gamma(-\gamma) = (M_2)_{\Delta} \).

If \( g - 1 < d < 2g - 2 \), we define \( \mathcal{I}_1 \) to be equal to the line bundle on \( X \) obtained by gluing \( N_2 \otimes \pi_1^* \mathcal{O}_\Gamma(\gamma), M_4 \) and \( L_1 \otimes \pi_3^* \mathcal{O}_\Gamma(-\gamma) \), which is possible since, by (4.3), we have that \( (N_2 \otimes \pi_1^* \mathcal{O}_\Gamma(\gamma))_{\mathcal{A}} = \mathcal{O}_\Gamma(\gamma) = (M_4)_{\mathcal{B}} \) and \( (L_1 \otimes \pi_3^* \mathcal{O}_\Gamma(-\gamma))_{\mathcal{B}} = \mathcal{O}_\Gamma(-\gamma) = (M_4)_{\Delta} \).

Similarly, we get that
\[
d - \alpha_2 - \beta_1 = \begin{cases} 0 & \text{if } 0 \leq d < g - 1, \\ 1 & \text{if } g - 1 \leq d < 2g - 2. \end{cases}
\]

If \( 0 \leq d < g - 1 \), we define \( \mathcal{I}_2 \) to be the line bundle on \( X \) obtained by gluing \( N_1, M_1 \) and \( L_2 \), which is possible since, by (4.3), we have that \( (N_1)_{\mathcal{A}} = \mathcal{O}_\Gamma = (M_1)_{\mathcal{B}} \) and \( (L_1)_{\mathcal{B}} = \mathcal{O}_\Gamma = (M_1)_{\Delta} \).

If \( g - 1 < d < 2g - 2 \), we define \( \mathcal{I}_2 \) to be equal to the line bundle on \( X \) obtained by gluing \( N_1 \otimes \pi_1^* \mathcal{O}_\Gamma(\gamma), M_3 \) and \( L_2 \), which is possible since, by (4.3), we have that \( (N_1 \otimes \pi_1^* \mathcal{O}_\Gamma(\gamma))_{\mathcal{A}} = \mathcal{O}_\Gamma(\gamma) = (M_3)_{\mathcal{B}} \) and \( (L_2)_{\mathcal{B}} = \mathcal{O}_\Gamma = (M_3)_{\Delta} \).

**Lemma 4.5.** The line bundles \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) on \( X \) are properly balanced of relative degree \( d \).

**Proof.** Since the property of being properly balanced is an open condition and the degree is a deformation invariant, it is enough to check that the restriction of \( \mathcal{I}_1 \) to the special fiber \( f^{-1}(\gamma) \) is a properly balanced line bundle of degree \( d \), \( i = 1, 2 \).

Observe that, for \( i = 1, 2, \mathcal{I}_i \) is obtained, in all the above cases, by gluing \( N_j \otimes \pi_i^* \mathcal{O}_\Gamma(\alpha \gamma) \) on \( Y_i \) (for some \( j = 1, 2 \) and \( \alpha = 0, 1) \), \( M_h \) on \( Y_2 \) (for some \( h = 1, 2, 3, 4 \)) and \( L_k \otimes \pi_3^* \mathcal{O}_\Gamma(\beta \gamma) \) on \( Y_3 \) (for some \( k = 1, 2 \) and \( b = -1, 0)) \). Fix such a representation. Then,
\[
\begin{align*}
\deg_{C_i}(\mathcal{I}_i) &= \deg_{\pi_i^{-1}(\gamma)}(N_j), \\
\deg_{E}(\mathcal{I}_i) &= \deg_{E}(M_h), \\
\deg_{\Gamma}(\mathcal{I}_i) &= \deg_{\Gamma}(M_k), \\
\deg_{C_2}(\mathcal{I}_i) &= \deg_{C_2}(L_k).
\end{align*}
\]

Using these facts, the equality \( \deg_{f^{-1}(\gamma)}(\mathcal{I}_i) = \deg_{C_1}(\mathcal{I}_i) + \deg_{E}(\mathcal{I}_i) + \deg_{\Gamma}(\mathcal{I}_i) + \deg_{C_2}(\mathcal{I}_i) \) is easily checked through a case by case inspection. Accordingly to Remark 2.4, in order to check that \( (\mathcal{I}_i)_{f^{-1}(\gamma)} \) is properly balanced, it is enough to check the basic inequalities for the subcurves \( C_1, C_2 \) and \( \Gamma \). The basic inequality for \( C_1 \)
\[
\deg_{C_1}(\mathcal{I}_i) = \frac{d(2h - 1)}{2g - 2} \leq \frac{1}{2}
\]
is satisfied since \( \deg_{\mathcal{C}_1}(\mathcal{I}_1) = \deg_{\pi_1^{-1}(\gamma)}(\mathcal{N}_j) = \beta_1 \) or \( \beta_2 \), which are by definition the closest integers to \( \frac{d(2h-1)}{2g-2} \). The basic inequality for \( \mathcal{C}_2 \)

\[
\left| \deg_{\mathcal{C}_2}(\mathcal{I}_1) - \frac{d(2g - 2h - 3)}{2g - 2} \right| \leq \frac{1}{2}
\]

is satisfied since \( \deg_{\mathcal{C}_2}(\mathcal{I}_1) = \deg_{\pi_2^{-1}(\gamma)}(\mathcal{L}_k) = \alpha_1 \) or \( \alpha_2 \), which are by definition the closest integers to \( \frac{d(2g - 2h - 3)}{2g - 2} \). Finally, the basic inequality for \( \Gamma \subset f^{-1}(\gamma) \)

\[
\left| \deg_{\mathcal{T}}(\mathcal{I}_1) - \frac{d - 1}{2g - 2} \right| \leq \frac{1}{2}
\]

is satisfied if and only

\[
\deg_{\mathcal{T}}(M_k) = \begin{cases} 
0 & \text{if } 0 \leq d \leq g - 1, \\
1 & \text{if } g - 1 \leq d < 2g - 2.
\end{cases}
\]

An easy case by case inspection concludes the proof.

We call \( \widetilde{F}_{h,1} \) the family \( f : X \to \Gamma \) endowed with the line bundle \( \mathcal{I}_1 \) and \( \widetilde{F}_{h,2} \) the family \( f : X \to \Gamma \) endowed with the line bundle \( \mathcal{I}_2 \). Note that \( \widetilde{F}_{h,1} = \widetilde{F}_{h,2} \) if and only if we are in case A, which happens exactly when \( k_{d,g} \equiv 2h + 1 \). Both families \( \widetilde{F}_{h,1} \) and \( \widetilde{F}_{h,2} \) are liftings of the family \( F_h \) of [ACS7, p. 156]. We can compute the degrees of the pull-backs of some of the boundary classes in \( \text{Pic}(\overline{Jac_{d,g}}) \) to the curves \( \widetilde{F}_{h,1} \) and \( \widetilde{F}_{h,2} \):

\[
\begin{align*}
\deg_{\widetilde{F}_{h,1}} \mathcal{O}(\delta_{h+1}) &= -1 \text{ if } k_{d,g} \not| 2h + 1 \text{ or } h + 1 = g/2, \\
\deg_{\widetilde{F}_{h,1}} \mathcal{O}(\delta_1) &= \deg_{\widetilde{F}_{h,2}} \mathcal{O}(\delta_1) = -1 \text{ if } k_{d,g} | 2h + 1 \text{ and } h + 1 \neq g/2, \\
\deg_{\widetilde{F}_{h,1}} \mathcal{O}(\tilde{\delta}_1) &= \deg_{\widetilde{F}_{h,2}} \mathcal{O}(\tilde{\delta}_1) = 0 \text{ if } k_{d,g} | 2h + 1 \text{ and } h + 1 \neq g/2, \\
\deg_{\widetilde{F}_{h,1}} \mathcal{O}(\delta_i) &= 0 \text{ if } h + 1 < i \text{ and } k_{d,g} \not| (2i - 1) \text{ or } i = g/2, \\
\deg_{\widetilde{F}_{h,1}} \mathcal{O}(\tilde{\delta}_i) &= \deg_{\widetilde{F}_{h,2}} \mathcal{O}(\tilde{\delta}_i) = 0 \text{ if } h + 1 < i < g/2 \text{ and } k_{d,g} \not| (2i - 1), \text{ for } j = 1, 2.
\end{align*}
\]

The first relation follows by using the projection formula, from the relation \( \deg_{F_h} \mathcal{O}(\delta_{h+1}) = -1 \) proved in [ACS7, p. 157]. The second and third relations are deduced in a similar way using the projection formula and the (easily checked) fact that \( \widetilde{F}_{h,1} \) does not meet \( \tilde{\delta}_{h+1} \) and \( \widetilde{F}_{h,2} \) does not meet \( \tilde{\delta}_1 \). The last two relations follow from the fact that \( \widetilde{F}_{h,1} \) and \( \widetilde{F}_{h,2} \) do not meet the divisors \( \tilde{\delta}_i \) or \( \tilde{\delta}_1 \) and \( \tilde{\delta}_i \) for \( i > h + 1 \).

With the help of the above, we can finally conclude the proof of our main theorem.

**Proof of Theorem 4.2.** As observed before, it is enough to prove that the line bundles associated to the boundary divisors \( \{ \delta_i : k_{d,g} \not| 2i - 1 \text{ or } i = g/2 \}, \{ \tilde{\delta}_1, \tilde{\delta}_2 : k_{d,g} | 2i - 1 \text{ and } i \neq g/2 \} \) (for \( 0 \leq i \leq g/2 \)) are linearly independent on \( \overline{Jac_{d,g}} \). Suppose there is a linear relation

\[
\mathcal{O} \left( \sum_{k_{d,g} | 2i - 1 \text{ or } i = g/2} a_i \delta_i + \sum_{k_{d,g} | 2i - 1 \text{ and } i \neq g/2} (a_1^{\delta_1} + a_2^{\tilde{\delta}_2}) \right) = \mathcal{O},
\]

in the Picard group of \( \overline{Jac_{d,g}} \). We want to prove that all the above coefficients \( a_i, a_1^{\delta_1} \) and \( a_2^{\tilde{\delta}_2} \) are zero. Pulling back the above relation (4.8) to the curve \( \widetilde{F}_h \to \overline{Jac_{d,g}} \) and using the formulas (4.2), we get that \( a_0 = 0 \). Pulling back (4.8) to the curves \( \widetilde{F}_1 \to \overline{Jac_{d,g}} \) and \( \widetilde{F}_2 \to \overline{Jac_{d,g}} \) (in the range of degrees in which they are defined) and using the formulas (4.3), we get that \( a_1 = 0 \) if \( k_{d,g} \not| 1 \) (i.e. if \( d \neq g - 1 \)) or that \( a_1^{\delta_1} = a_2^{\tilde{\delta}_2} = 0 \) if \( k_{d,g} | 1 \) (i.e. if \( d = g - 1 \)). Finally, by pulling back the relation (4.8) to the families \( \widetilde{F}_{h,1} \to \overline{Jac_{d,g}} \) and \( \widetilde{F}_{h,2} \to \overline{Jac_{d,g}} \) (for any \( 1 \leq h \leq (g - 1)/2 \)) and using the formulas (4.7), we get that \( a_{h+1} = 0 \) if \( k_{d,g} \not| (2h + 1) \) or \( h + 1 = g/2 \) and \( a_1^{\delta_1} = a_2^{\tilde{\delta}_2} = 0 \) if \( k_{d,g} \not| (2h + 1) \) and \( h + 1 \neq g/2 \), which concludes the proof.

As a corollary of the above Theorem 4.1, we can prove that the boundary line bundles of \( \overline{Jac_{d,g}} \) are linearly independent.
Corollary 4.6. We have an exact sequence

\[
0 \rightarrow \bigoplus_{k \quad k \neq \frac{g}{2}} \langle \mathcal{O}(\delta_1^i) \rangle \bigoplus \langle \mathcal{O}(\delta_1^2) \rangle \oplus \langle \mathcal{O}(\delta_2^2) \rangle \rightarrow \text{Pic}(\mathcal{J}_{d,g}) \rightarrow \text{Pic}(\mathcal{J}_{d,g}) \rightarrow 0,
\]

where the right map is the natural restriction morphism and the left map is the natural inclusion.

Proof. As observed before, the only thing to prove is that the above exact is exact on the left, or in other words that the boundary line bundles \{\mathcal{O}(\delta_1^i), \mathcal{O}(\delta_1^2), \mathcal{O}(\delta_2^2)\} are linearly independent in Pic(\mathcal{J}_{d,g}). This follows from Theorem 4.1 using Corollary 3.3(ii) and the fact that the pull-back map \nu_d^*: \text{Pic}(\mathcal{J}_{d,g}) \rightarrow \text{Pic}(\mathcal{J}_{ac_{d,g}}) is injective, as observed in the introduction (see diagram (1.1))..

5. Tautological line bundles

The aim of this section is to introduce some natural line bundles on \mathcal{J}_{ac_{d,g}}, which we call tautological line bundles, and to determine the relations among them.

Let \pi : \mathcal{J}_{ac_{d,g},1} \rightarrow \mathcal{J}_{ac_{d,g}} be the universal family over \mathcal{J}_{ac_{d,g}} (see \cite{Mel10} for a modular description of \mathcal{J}_{ac_{d,g},1}). The stack \mathcal{J}_{ac_{d,g},1} comes equipped with two natural line bundles: the universal line bundle \mathcal{L}_d and the relative dualizing sheaf \omega_\pi. Since \pi is a representable, flat and proper morphism whose geometric fibers are nodal curves, we can apply the formalism of the determinant of cohomology and of the Deligne pairing (see Section 2.15) to produce some natural line bundles on \mathcal{J}_{ac_{d,g}} which we call \emph{tautological} line bundles:

\[
\begin{align*}
K_{1,0} &:= \langle \omega_\pi,\omega_\pi \rangle_{\pi}, \\
K_{0,1} &:= \langle \omega_\pi,\mathcal{L}_d \rangle_{\pi}, \\
K_{-1,2} &:= \langle \mathcal{L}_d,\mathcal{L}_d \rangle_{\pi}, \\
\Lambda(n,m) &= d_\pi^*(\log_\pi \mathcal{O}_d^n \oplus \mathcal{L}_d^m) \quad \text{for } m,n \in \mathbb{Z}.
\end{align*}
\]

By abuse of notation, we use the same notation for the restriction of a tautological class to the open substack \mathcal{J}_{ac_{d,g}}. Using Facts 2.10 and 2.17, the first Chern classes of the above tautological line bundles are given by

\[
\begin{align*}
\kappa_{1,0} &:= c_1(K_{1,0}) = \pi^*(c_1(\omega_\pi)^2), \\
\kappa_{0,1} &:= c_1(K_{0,1}) = \pi^*(c_1(\omega_\pi) \cdot c_1(\mathcal{L}_d)), \\
\kappa_{-1,2} &:= c_1(K_{-1,2}) = \pi^*(c_1(\mathcal{L}_d)^2), \\
\lambda(n,m) &= c_1(\Lambda(n,m)) = c_1(\pi^*(\log_\pi \mathcal{O}_d^n \oplus \mathcal{L}_d^m)) \quad \text{for } n,m \in \mathbb{Z}.
\end{align*}
\]

Note that, if \(k = \mathbb{C}\), the image of the classes \(\kappa_{i,j}\) via the natural map \(A^1(\mathcal{J}_{ac_{d,g}}) \rightarrow H^2(\mathcal{J}_{ac_{d,g}},\mathbb{Z}) \rightarrow H^2(\text{Hol}_k^1,\mathbb{Z})\) are, up to sign, the \(\kappa_{i,j}\) classes that were considered by Erbert and Randal-Williams in \cite{ERW} (see Section 1.1).

The pull-back of the tautological line bundles (2.26) of \(\overline{\mathcal{M}}_g\) via the natural map \(\tilde{\Phi}_d : \mathcal{J}_{ac_{d,g}} \rightarrow \overline{\mathcal{M}}_g\) are again tautological line bundles on \(\mathcal{J}_{ac_{d,g}}\).

Lemma 5.1. We have that

\[
\tilde{\Phi}_d^*(K_1) = K_{1,0},
\]

In particular, \(\lambda(1,0)\) is the pull-back of the Hodge line bundle \(\lambda\) on \(\overline{\mathcal{M}}_g\).

Proof. Consider the diagram

\[
\begin{array}{ccc}
\mathcal{J}_{ac_{d,g},1} & \xrightarrow{\tilde{\Phi}_{d,1}} & \overline{\mathcal{M}}_{g,1} \\
\pi \downarrow & & \downarrow \nu_d \\
\mathcal{J}_{ac_{d,g}} & \xrightarrow{\tilde{\Phi}_d} & \overline{\mathcal{M}}_g
\end{array}
\]

Recall from Section 2.1 that the map \(\tilde{\Phi}_d\) sends an element \((\mathcal{C} \rightarrow S, \mathcal{L}) \in \mathcal{J}_{ac_{d,g}}(S)\) into the stabilization \(\mathcal{C}^{st} \rightarrow S \in \overline{\mathcal{M}}_g(S)\). Now it is well-known that for every quasi-stable (or more generally semistable) curve \(X\) with stabilization morphism \(\psi : X \rightarrow X^{st}\), the pull-back via \(\psi\) induces an isomorphism \(\psi^* : \]
\( H^0(X^\pi, \omega_{X^\pi}) \cong H^0(X, \omega_X) \). Therefore, the relative dualizing sheaves of the families \( \pi \) and \( \overline{\pi} \) are related by
\[
(5.5) \quad \tilde{\Phi}^*_{\pi,1}(\omega_{\pi}) = \omega_{\overline{\pi}}.
\]
We conclude by using the functoriality of the determinant of cohomology (see Fact 2.16) and of the Deligne pairing (see Fact 2.17).

**Definition 5.2.** The tautological subgroup \( \text{Pic}^{\text{taut}}(\overline{\text{Jac}}_{d,g}) \subseteq \text{Pic}(\overline{\text{Jac}}_{d,g}) \) is the subgroup generated by the tautological line bundles (5.1) together with the line bundles associated to the boundary divisors of \( \overline{\text{Jac}}_{d,g} \). The image of \( \text{Pic}^{\text{taut}}(\overline{\text{Jac}}_{d,g}) \subseteq \text{Pic}(\overline{\text{Jac}}_{d,g}) \) via the natural restriction map \( \text{Pic}(\overline{\text{Jac}}_{d,g}) \to \text{Pic}(\text{Jac}_{d,g}) \) is defined to be \( \text{Pic}^{\text{taut}}(\text{Jac}_{d,g}) \).

There are some relations between the tautological line bundles on \( \overline{\text{Jac}}_{d,g} \), as shown in the following.

**Theorem 5.3.** The tautological line bundles on \( \overline{\text{Jac}}_{d,g} \) satisfy the following relations
\[
(i) \quad K_{1,0} = \Lambda(1,0)^{12} \otimes \mathcal{O}(-\delta),
(ii) \quad K_{0,1} = \Lambda(1,1) \otimes \Lambda(0,1)^{-1},
(iii) \quad K_{-1,2} = \Lambda(0,1) \otimes \Lambda(1,1) \otimes \Lambda(1,0)^{-2},
(iv) \quad \Lambda(n, m) = \Lambda(0, 1) \otimes \Lambda(1, 1)^{6n - 6n - m^2 + 1} \otimes \Lambda(1, 0)^{-m^2 + (\frac{m}{2})} \otimes \Lambda(0, 1)^{m^2 + (\frac{m}{2})} \otimes \mathcal{O} \left( -\left( \frac{n}{2} \right) \cdot \delta \right).
\]

**Proof.** Since the first Chern class map \( c_1 : \text{Pic}(\overline{\text{Jac}}_{d,g}) \to A^1(\overline{\text{Jac}}_{d,g}) \) is an isomorphism by Fact 2.16, it is enough to prove the above relations in the Chow group \( A^1(\overline{\text{Jac}}_{d,g}) \).

Following the same strategy as in the proof of Mumford’s relations in Theorem 2.20 (see [ACG11, Chap. 13, Sec. 7]), we apply the Grothendieck-Riemann-Roch Theorem to the morphism \( \pi : \overline{\text{Jac}}_{d,g,1} \to \overline{\text{Jac}}_{d,g} \): (5.6)
\[
\text{ch}(\pi_!(\omega_{\pi}^n \otimes L_{\pi}^n)) = \pi_*(\text{ch}(\omega_{\pi}^n \otimes L_{\pi}^n) \cdot \text{Td}(\Omega_{\pi})^{-1}),
\]
where \( \text{ch} \) denotes the Chern character, \( \text{Td} \) denotes the Todd class and \( \Omega_{\pi} \) is the sheaf of relative Kähler differentials.

Using Fact 2.16 we can compute the degree one part of the left hand side of (5.6):
\[
(5.7) \quad \text{ch}(\pi_!(\omega_{\pi}^n \otimes L_{\pi}^n)) = c_1(\pi_!(\omega_{\pi}^n \otimes L_{\pi}^n)) = c_1(d_x(\omega_{\pi}^n \otimes L_{\pi}^n)) = \lambda(n, m).
\]
Let us now compute the degree one part of the right hand side of (5.6). Note that, as proved in [ACG11, p. 383], we have that \( c_1(\Omega_{\pi}) = c_1(\omega_{\pi}) \) and that \( c_2(\Omega_{\pi}) \) is the class of the nodal locus of the morphism \( \pi \). In particular, we have that
\[
(5.8) \quad \pi_*(c_2(\Omega_{\pi})) = \overline{\delta} \in A^1(\overline{\text{Jac}}_{d,g}),
\]
where \( \overline{\delta} \) is the total boundary divisor (3.1) of \( \overline{\text{Jac}}_{d,g} \). The first three terms of the inverse of the Todd class of \( \Omega_{\pi} \) are equal to
\[
(5.9) \quad \text{Td}(\Omega_{\pi})^{-1} = 1 + \frac{\delta_1(\Omega_{\pi})}{2} + \frac{\delta_2(\Omega_{\pi})}{12} + \ldots + \frac{\delta_1(\omega_{\pi})}{2} + \frac{\delta_1(\omega_{\pi})^2}{12} + \ldots + \frac{\delta_2(\omega_{\pi})}{12} + \ldots
\]
Using the multiplicity of the Chern character, we get
\[
\text{ch}(\omega_{\pi}^n \otimes L_{\pi}^n) = \left( 1 + c_1(\omega_{\pi}) + \frac{c_1(\omega_{\pi})^2}{2} + \ldots \right)^n \cdot \left( 1 + c_1(L_{\pi}) + \frac{c_1(L_{\pi})^2}{2} + \ldots \right) = \left( 1 + nc_1(\omega_{\pi}) + m^2c_1(\omega_{\pi})^2 + \ldots \right) \cdot \left( 1 + mc_1(L_{\pi}) + \frac{m^2c_1(L_{\pi})^2}{2} + \ldots \right) =
\]
\[
(5.10) \quad = 1 + [nc_1(\omega_{\pi}) + mc_1(L_{\pi})] + \left[ \frac{n^2c_1(\omega_{\pi})^2}{2} + nmc_1(\omega_{\pi}) \cdot c_1(L_{\pi}) + \frac{m^2c_1(L_{\pi})^2}{2} \right] + \ldots
\]
Combining (5.9) and (5.10) and using (5.2) together with (5.8), we can compute the degree one part of the right hand side of (5.6)
\[
\pi_*(\text{ch}(\omega_{\pi}^n \otimes L_{\pi}^n) \cdot \text{Td}(\Omega_{\pi})^{-1}) = \pi_*(\text{ch}(\omega_{\pi}^n \otimes L_{\pi}^n) \cdot \text{Td}(\Omega_{\pi})^{-1}_2) =
\]
\[
(5.11) \quad = \frac{6n^2 - 6n + 1}{12} \kappa_{1,0} + \frac{2nm - m}{2} \kappa_{0,1} + \frac{m^2}{2} \kappa_{1,2} + \frac{\overline{\delta}}{12}.
\]
Putting together (5.7) and (5.11), we get the relation

\[(5.12) \quad \lambda(n, m) = \frac{6n^2 - 6n + 1}{12} \kappa_{1,0} + \frac{2nm - m}{2} \kappa_{0,1} + \frac{m^2}{2} \kappa_{-1,2} + \frac{\tilde{\delta}}{12}, \]

Formula (5.12) for \(n = 1\) and \(m = 0\) gives that

\[(*) \quad \lambda(1, 0) = \frac{\kappa_{1,0}}{12} + \frac{\tilde{\delta}}{12}, \]

which proves part (i). By substituting (*) into (5.12), we get

\[(5.13) \quad \lambda(n, m) = (6n^2 - 6n + 1)\lambda(1, 0) + \frac{2nm - m}{2} \kappa_{0,1} + \frac{m^2}{2} \kappa_{-1,2} - \binom{n}{2} \tilde{\delta}. \]

Formula (5.13) for \((n, m) = (0, 1)\) and \((n, m) = (1, 1)\) gives that

\[(***) \begin{cases} 
\kappa_{0,1} = \lambda(1, 0) - \lambda(0, 1), \\
\kappa_{-1,2} = -2\lambda(1, 0) + \lambda(0, 1) + \lambda(1, 1), 
\end{cases} \]

The system of equations (***) is equivalent to the system

\[(****) \begin{cases} 
\lambda(0, 1) = \lambda(1, 0) - \frac{\kappa_{0,1}}{2} + \frac{\kappa_{-1,2}}{2}, \\
\lambda(1, 1) = \lambda(1, 0) + \frac{\kappa_{0,1}}{2} + \frac{\kappa_{-1,2}}{2}. 
\end{cases} \]

The system of equations (****) is equivalent to the system

\[\begin{cases} 
\kappa_{0,1} = \lambda(1, 1) - \lambda(0, 1), \\
\kappa_{-1,2} = -2\lambda(1, 0) + \lambda(0, 1) + \lambda(1, 1), 
\end{cases} \]

which also proves parts (ii) and (iii). Substituting (****) into (5.13), we get the following relation

\[(5.14) \quad \lambda(n, m) = (6n^2 - 6n + 1 - m^2)\lambda(1, 0) + \binom{m + 1}{2} \lambda(0, 1) + \left[ mn + \binom{m}{2} \right] \lambda(1, 1) - \binom{n}{2} \tilde{\delta}, \]

which proves part (iii).

Using Lemma 5.1 it is easy to see that the relations in Theorem 5.3(i) and in Theorem 5.3(ii) with \(m = 0\) are the pull-back to \(\mathcal{J}ac_{d,g}\) of Mumford’s relations among the tautological classes of \(\mathcal{M}_g\) (see Theorem 2.20).

As a direct consequence of Theorem 5.3, we get a set of generators for the tautological subgroup of the Picard group of \(\mathcal{J}ac_{d,g}\) and of \(\mathcal{J}ac_{d,g}\).

**Corollary 5.4.**

(i) The tautological subgroup \(\text{Pic}^\text{taut}(\mathcal{J}ac_{d,g}) \subseteq \text{Pic}(\mathcal{J}ac_{d,g})\) is generated by the boundary divisors and the tautological line bundles \(\Lambda(1, 0), \Lambda(0, 1)\) and \(\Lambda(1, 1)\).

(ii) The tautological subgroup \(\text{Pic}^\text{taut}(\mathcal{J}ac_{d,g}) \subseteq \text{Pic}(\mathcal{J}ac_{d,g})\) is generated by the tautological line bundles \(\Lambda(1, 0), \Lambda(0, 1)\) and \(\Lambda(1, 1)\).

6. Comparing the Picard groups of \(\mathcal{J}ac_{d,g}\) and \(\mathcal{J}_{d,g}\)

The aim of this Section is to study the pull-back map

\[\nu_d^* : \text{Pic}(\mathcal{J}_{d,g}) \to \text{Pic}(\mathcal{J}ac_{d,g})\]

induced by the map \(\nu_d : \mathcal{J}ac_{d,g} \to \mathcal{J}_{d,g}\) (see Section 2.1). To this aim, consider the Leray spectral sequence for the étale sheaf \(\mathbb{G}_m\) with respect to the map \(\nu_d\):

\[E_r^{pq} = H^p_{\text{ét}}(\mathcal{J}_{d,g}, (R^q\nu_d)_*\mathbb{G}_m) \Rightarrow H^{p+q}_{\text{ét}}(\mathcal{J}ac_{d,g}, \mathbb{G}_m).\]

The first terms of the above spectral sequence give rise to the exact sequence

\[0 \to H^1_{\text{ét}}(\mathcal{J}_{d,g}, (R^0\nu_d)_*\mathbb{G}_m) \to H^1_{\text{ét}}(\mathcal{J}ac_{d,g}, \mathbb{G}_m) \to H^0_{\text{ét}}(\mathcal{J}_{d,g}, (R^1\nu_d)_*\mathbb{G}_m) \to H^2_{\text{ét}}(\mathcal{J}_{d,g}, (R^0\nu_d)_*\mathbb{G}_m).\]

Since \(\nu_d\) is a \(\mathbb{G}_m\)-gerbe, we have that \((R^0\nu_d)_*\mathbb{G}_m = \mathbb{G}_m\) and \((R^1\nu_d)_*\mathbb{G}_m = \text{Pic} B\mathbb{G}_m\), where \(\text{Pic} B\mathbb{G}_m\) is canonically identified with the group \((\mathbb{G}_m)^* \cong \mathbb{Z}\) of characters of \(\mathbb{G}_m\). By plugging these isomorphisms into the above long exact sequence, we get the exact sequence

\[(6.1) \quad 0 \to \text{Pic}(\mathcal{J}_{d,g}) \xrightarrow{\nu_d^*} \text{Pic}(\mathcal{J}ac_{d,g}) \xrightarrow{\text{res}} \mathbb{Z} \xrightarrow{\text{ob}} \text{Br}(\mathcal{J}_{d,g}),\]
where the above maps admits the following interpretation (which one can easily check via standard cocycle computations): $\nu_d^*$ is the pull-back map induced by $\nu_d$; $\operatorname{res}$ is the restriction to the fibers of $\nu_d$ (it coincides with the weight map defined in [Hof07, Def. 4.1] and with the character appearing in the decomposition in [Lc08, Prop. 3.1.1.4]) and $\operatorname{ob}$ (the obstruction map) sends $1 \in \mathbb{Z} = (\mathbb{G}_m)^*$ into the class $[\nu_d]$ of the $\mathbb{G}_m$-gerbe $\nu_d$ in the (cohomological) Brauer group $\operatorname{Br}(\mathcal{J}_{d,g}) := H^2_{\text{et}}(\mathcal{J}_{d,g},\mathbb{G}_m)$ (see [Gir71 Chap. IV.3]).

Since $\nu_d^*$ is injective, we can define a tautological subgroup of $\operatorname{Pic}(\mathcal{J}_{d,g})$ by intersecting $\operatorname{Pic}(\mathcal{J}_{d,g})$ (which we identify with its image via $\nu_d^*$) with the tautological subgroup $\operatorname{Pic}^\text{taut}(\mathcal{J}ac_{d,g})$, as follows.

**Definition 6.1.** The tautological subgroup of $\operatorname{Pic}(\mathcal{J}_{d,g})$ is defined as

$$\operatorname{Pic}^\text{taut}(\mathcal{J}_{d,g}) := \operatorname{Pic}^\text{taut}(\mathcal{J}ac_{d,g}) \cap \operatorname{Pic}(\mathcal{J}_{d,g}) \subseteq \operatorname{Pic}(\mathcal{J}ac_{d,g}).$$

In order to compute generators for $\operatorname{Pic}^\text{taut}(\mathcal{J}_{d,g})$, we need first to compute the map $\operatorname{res}$ from (6.1) on the generators of $\operatorname{Pic}^\text{taut}(\mathcal{J}ac_{d,g})$.

**Lemma 6.2.** We have that

$$\begin{align*}
\operatorname{res}(\Lambda(1,0)) &= 0, \\
\operatorname{res}(\Lambda(0,1)) &= d - g + 1, \\
\operatorname{res}(\Lambda(1,1)) &= d + g - 1.
\end{align*}$$

**Proof.** Using the functoriality of the determinant of cohomology (see Fact 2.16), we get that the fiber of $\Lambda(1,0)$ is canonically isomorphic to $\det H^0(C, \omega_C) \otimes \det^{-1} H^1(C, \omega_C)$. Since $\mathbb{G}_m$ acts trivially on $H^0(C, \omega_C)$ and on $H^1(C, \omega_C)$, we get that $\operatorname{res}(\Lambda(1,0)) = 0$.

Similarly, the fiber of $\Lambda(0,1)$ over a point $(C, L) \in \mathcal{J}ac_{d,g}$ is canonically isomorphic to $\det H^0(C, L) \otimes \det^{-1} H^1(C, L)$, and $\mathbb{G}_m$ acts with weight one on the vector spaces $H^0(C, L)$ and $H^1(C, L)$, Riemann-Roch gives that

$$\operatorname{res}(\Lambda(0,1)) = \dim H^0(C, L) - \dim H^1(C, L) = \chi(C, L) = d + 1 - g.$$

Finally, the fiber of $\Lambda(1,1)$ over a point $(C, L) \in \mathcal{J}ac_{d,g}$ is canonically isomorphic to $\det H^0(C, \omega_C \otimes L) \otimes \det^{-1} H^1(C, \omega_C \otimes L)$. Since $\mathbb{G}_m$ acts with weight one on the vector spaces $H^0(C, \omega_C \otimes L)$ and $H^1(C, \omega_C \otimes L)$, Riemann-Roch gives that

$$\operatorname{res}(\Lambda(1,1)) = \dim H^0(C, \omega_C \otimes L) - \dim H^1(C, \omega_C \otimes L) = \chi(C, \omega_C \otimes L) = d + g - 2 + 1 - g = d + 1 + g.$$

Combining the above Lemma 6.2 with Corollary 5.4 we get the following

**Corollary 6.3.**

(i) The image of $\operatorname{Pic}^\text{taut}(\mathcal{J}ac_{d,g})$ via the map $\operatorname{res}$ of (6.1) is the subgroup generated by $(d + g - 1, d - g + 1) = (d + g - 1, 2g - 2)$.

(ii) $\operatorname{Pic}^\text{taut}(\mathcal{J}_{d,g})$ is generated by $\Lambda(1,0)$ and

$$\Xi := \Lambda(0, 1) = (d + g - 1)(d - g + 1) \otimes \Lambda(1, 1) = \frac{d + g - 1}{(d + g - 1)(d - g + 1)}.$$

Corollary 6.3 combined with the exact sequence (6.1) gives that the order of $[\nu_d]$ in the Brauer group $\operatorname{Br}(\mathcal{J}_{d,g})$ is $(d + g - 1, 2g - 2)$. Indeed the following is true:

**Theorem 6.4.** The order of $[\nu_d]$ in $\operatorname{Br}(\mathcal{J}_{d,g})$ is equal to $(d + g - 1, 2g - 2)$.

In order to prove the theorem, we will reinterpret the order of $[\nu_d]$ in terms of the existence of a (generalized) Poincaré bundle.

Consider the universal family $\pi : \mathcal{J}ac_{d,g,1} \to \mathcal{J}ac_{d,g}$. The $\mathbb{G}_m$-rigidification of $\mathcal{J}ac_{d,g,1}$, denoted by $\mathcal{J}_{d,g,1} := \mathcal{J}ac_{d,g,1} / \mathbb{G}_m$, has a natural map $\tilde{\pi} : \mathcal{J}_{d,g,1} \to \mathcal{J}_{d,g}$ which is indeed the universal family over $\mathcal{J}_{d,g}$. However, the universal (or Poincaré) line bundle $\mathcal{L}_d$ on $\mathcal{J}ac_{d,g,1}$ does not necessarily descend to a line bundle on $\mathcal{J}_{d,g,1}$. Instead, it turns out that there always exists on $\mathcal{J}_{d,g,1}$ an $m$-Poincaré line bundle as in the definition below.

**Definition 6.5.** Let $m \in \mathbb{Z}$. An $m$-Poincaré line bundle for $\mathcal{J}_{d,g}$ is a line bundle $\mathcal{L}$ on $\mathcal{J}_{d,g,1}$ such that the restriction of $\mathcal{L}$ to the fiber $\tilde{\pi}^{-1}(C, L) \cong C$ over a geometric point $(C, L)$ of $\mathcal{J}_{d,g}$ is isomorphic to $L^m$.

The above definition generalizes the classical definition of Poincaré line bundle, which corresponds to the case $m = 1$. 

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Proposition 6.6. The order of $[\nu_d]$ in the group $\text{Br}(J_{d,g})$ is equal to the smallest number $m \in \mathbb{N}$ such that there exists an $m$-Poincaré line bundle for $J_{d,g}$.

Proof. In order to prove the statement, we need to introduce some auxiliary stacks. Given $m \in \mathbb{Z}$, consider the stack $\mathcal{J}ac^m_{d,g}$ whose fiber $\mathcal{J}ac^m_{d,g}(S)$ over a scheme $S$ consists of families $C \to S$ of smooth curves of genus $g$ endowed with a line bundle $L$ of relative degree $d$ and whose morphisms between two objects $(C' \to S', L')$ and $(C \to S, L)$ are given by a triple $(g, \phi, \eta)$ where

$$
\begin{array}{c}
C' \xrightarrow{\phi} C \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
Lemma 2), it follows that there is a morphism of groups

\[ \theta: \pi_1(J) \to \pi_1(J_{d,g}) \]

where the first map is the restriction to the fiber \( \Phi \) of the Picard group of \( J \). Since the fiber of \( \Phi \) is a smooth curve without non-trivial automorphisms. A positive answer to Conjecture 6.9 together with Theorem 6.3 would imply that the above map \( \pi_1 \) is an isomorphism for \( g \geq 6 \).

From the above Theorem 6.3, we deduce the following Corollary 6.10.

(i) The image of \( \text{Pic}(J_{d,g}) \) via the map \( \text{res} \) of \( (6.1) \) is the subgroup generated by \( (d + g - 1, 2g - 2) \).

(ii) The pull-back map \( \nu^* \) induces an isomorphism

\[ \nu^*: \text{Pic}(J_{d,g})/\text{Pic}^\text{taut}(J_{d,g}) \cong \text{Pic}(J_{d,g})/\text{Pic}^\text{taut}(J_{d,g}). \]

Proof. Part (i) follows from the exact sequence \( (6.1) \) together with Theorem 6.3.

Part (ii): using Corollary 6.3(i) and part (i), we get the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \to & \text{Pic}(J_{d,g}) & \xrightarrow{\nu^*} & \text{Pic}(J_{d,g}) & \xrightarrow{\text{res}} & \mathbb{Z} \cdot ((d + g - 1, 2g - 2)) & \to & 0 \\
0 & \to & \text{Pic}^\text{taut}(J_{d,g}) & \xrightarrow{\nu^*} & \text{Pic}^\text{taut}(J_{d,g}) & \xrightarrow{\text{res}} & \mathbb{Z} \cdot ((d + g - 1, 2g - 2)) & \to & 0
\end{array}
\]

The conclusion follows from the snake lemma. \( \square \)

7. The Picard group of \( J_{d,g} \)

In this subsection, we will determine the Picard group of the stack \( J_{d,g} \), using a strategy similar to the one used by Kontsevich [Kou91] to determine the Picard group of \( J^d_{d,g} \), the open subset of \( J_{d,g} \) consisting of pairs \((C, L)\) where \( C \) is a smooth curve without non-trivial automorphisms.

Consider the representable morphism \( \Phi_d: J_{d,g} \to \mathcal{M}_g \). Clearly, the fiber of \( \Phi_d \) over \( C \in \mathcal{M}_g \) is the degree-\( d \) Jacobian \( J^d(C) \) of \( C \). Since \( \Phi_d \) has connected fibers, the pull-back map \( \Phi^*_d: \pi_1(\mathcal{M}_g) \to \pi_1(J_{d,g}) \) is injective. The cokernel of \( \Phi^*_d \) is denoted by \( \mathcal{R}(\pi_1(J_{d,g})) \) and is called classically the group of rationally determined line bundles of the family \( J_{d,g} \to \mathcal{M}_g \) (see e. g. [Cil87]). Therefore, we have the following exact sequence

\[ 0 \to \mathcal{R}(\pi_1(J_{d,g})) \to \mathcal{R}(\pi_1(J_{d,g})) \to 0. \]

Since the fiber of \( \Phi_d \) over \( C \in \mathcal{M}_g \) is the degree-\( d \) Jacobian \( J^d(C) \) of \( C \), we have a natural map

\[ \rho_C: \pi_1(J_{d,g}) \to \pi_1(J^d(C)) \to \pi_1(J^d(C)), \]

where the first map is the restriction to the fiber \( \Phi_d^{-1}(C) = J^d(C) \) and the second map is the projection of the Picard group of \( J^d(C) \) onto the Néron-Severi group of \( J^d(C) \), which parametrizes divisors on \( J^d(C) \) up to algebraic equivalence. We will use additive notation for the group law on \( NS(J^d(C)) \).

Consider the theta divisor \( \Theta(C) \subset J^{9g-1}(C) \) and denote by \( \theta_C \in NS(J^{9g-1}(C)) \) its algebraic equivalence class. By choosing an isomorphism \( \iota_M: J^d(C) \cong J^d(C) \) given by sending \( L \in J^d(C) \) into \( L \otimes M \in J^{9g-1}(C) \) for some \( M \in J^{9g-1-d}(C) \), we can pull-back \( \theta_C \) to get a well-defined (i.e., independent of the chosen isomorphism \( \iota_M \)) class in \( NS(J^d(C)) \) which, by a slight abuse of notation, we will still denote by \( \theta_C \). Since, for a very general curve \( C \in \mathcal{M}_g \), \( NS(J^d(C)) \) is generated by \( \theta_C \) (see e. g. [Kou91], Lemma 2), it follows that there is a morphism of groups

\[ \chi_d: \pi_1(J_{d,g}) \to \mathbb{Z}. \]
sending $\mathcal{L} \in \text{Pic} (\mathcal{J}_{d,g})$ to the integer $m$ such that $\rho_C (\mathcal{L}) = m \theta_C$ for every $C \in \mathcal{M}_g$ (see also [Kou91], p. 840). We will need the following two results of Kouvidakis, describing the image and the kernel of the above map $\chi_d$. Actually, Kouvidakis proves these results in [Kou91] for the variety $J^0_{d,g}$, but a close inspection reveals that the same proof works for $\mathcal{J}_{d,g}$.

**Theorem 7.1** (Kouvidakis).

(i) $\ker \chi_d = \text{Im} \Phi_d^*.$

(ii) $\text{Im} \chi_d \subseteq (2g - 2, d + g - 1). Z \subseteq Z.$

Part (i) follows from [Kou91] Thm. 3; part (ii) follows from [Kou91] Formula (*), p. 844. Note that part (i) implies (and it is indeed equivalent to) that the map $\chi_d$ factors as

\[ \chi_d : \text{Pic}(\mathcal{J}_{d,g}) \rightarrow \mathbb{R} \text{Pic}(\mathcal{J}_{d,g}) \rightarrow \mathbb{Z}. \]

We now compute the image of the map $\chi_d$ on the tautological subgroup $\text{Pic}^{\text{taut}}(\mathcal{J}_{d,g})$ of $\text{Pic}(\mathcal{J}_{d,g})$ (see Definition 6.1).

**Theorem 7.2.** We have that

\[ \chi_d (\text{Pic}^{\text{taut}}(\mathcal{J}_{d,g})) = \frac{2g - 2}{(2g - 2, d + g - 1)} \cdot Z \subseteq Z. \]

**Proof.** According to Corollary 6.3, $\text{Pic}^{\text{taut}}(\mathcal{J}_{d,g})$ is generated by the tautological classes $\Lambda (1, 0)$ and $\Xi$. Lemma 5.1 gives that $\Lambda (1, 0) = \Phi_d^* (\Lambda)$; hence clearly $\chi_d (\Lambda (1, 0)) = 0$ (this is the easy inclusion in Theorem 7.1). Therefore, the proof will follow if we show that

\[ \chi_d (\Xi) = \frac{2g - 2}{(2g - 2, d + g - 1)}, \]

or equivalently that

\[ \rho_C (\Xi) = \frac{2g - 2}{(2g - 2, d + g - 1)} \theta_C \]

for any $C \in \mathcal{M}_g$. In order to prove this, consider the following diagram

$$
\begin{array}{ccc}
\mathcal{L}_C & \xrightarrow{id \times \nu_C} & \mathcal{J} \mathcal{a} \mathcal{c}^d(C) \\
\downarrow & & \downarrow \\
\mathcal{J} \mathcal{a} \mathcal{c}^d(C) & \xrightarrow{\pi} & \mathcal{J} \mathcal{a} \mathcal{c}_{d,1}^d(C) \\
\downarrow & & \downarrow \\
\mathcal{J}^d(C) & \xrightarrow{\phi_d} & \mathcal{J}_{d,1}^d(C) \\
\downarrow & & \downarrow \\
C & \xrightarrow{p_2} & \mathcal{M}_g
\end{array}
$$

where the Cartesian square on the left is the fiber of the Cartesian square on the right over the point $C \in \mathcal{M}_g$ and $\mathcal{L}_C$ is the fiber of the universal line bundle $\mathcal{L}_d$ over $C \in \mathcal{M}_g$. In particular, the stack $\mathcal{J} \mathcal{a} \mathcal{c}^d(C)$ is the degree-$d$ Jacobian stack of $C$ (i.e. the stack whose fiber over a scheme $S$ is the groupoid of line bundles on $C \times S$ of relative degree $d$ over $S$) and $\mathcal{L}_C$ is the universal (or Poincaré) line bundle for $\mathcal{J} \mathcal{a} \mathcal{c}^d(C)$.

The map $\nu_C : \mathcal{J} \mathcal{a} \mathcal{c}^d(C) \rightarrow \mathcal{J}^d(C)$ is a $\mathbb{G}_m$-gerbe which is well-known to be trivial, or in other words $\mathcal{J} \mathcal{a} \mathcal{c}^d(C) \cong (\mathcal{J}^d(C) \times \text{BG}_m)$. Therefore, there exists a section $s$ of $\nu_C$ and we can define $\tilde{\mathcal{L}}_C := (id \times s)^* (\mathcal{L}_C)$. By construction, we have that $\tilde{\mathcal{L}}_{|C \times \{M\}} = M$ for any $M \in \mathcal{J}^d(C)$. Any line bundle on $C \times \mathcal{J}^d(C)$ with this property is called a Poincaré line bundle for $\mathcal{J}^d(C)$. Indeed, any Poincaré line bundle for $\mathcal{J}^d(C)$ is isomorphic to $(id \times s)^* (\mathcal{L}_C)$ for a uniquely determined section $s$ of $\nu_C$. Moreover, two Poincaré line bundles for $\mathcal{J}^d(C)$ differ by the tensor product with the pull-back of a line bundle on $\mathcal{J}^d(C)$. Note that for any Poincaré line bundle $\tilde{\mathcal{L}}_C = (id \times s)^* (\mathcal{L}_C)$ for $\mathcal{J}^d(C)$, we have that $(id \times \nu_C)^* (\tilde{\mathcal{L}}_C) = (id \times \nu_C)^* ((id \times s)^* (\mathcal{L}_C)) = \mathcal{L}_C$. 28
Recalling the definition of $\Xi$ from Corollary 6.3(ii) and applying the functoriality of the determinant of cohomology (see Fact 2.16) to the above diagram, we get that

$$\rho c(\Xi) = \frac{d + g - 1}{(d + g - 1, d - g + 1)}[dp_2(\tilde{L}_C)] - \frac{d - g + 1}{(d + g - 1, d - g + 1)}[dp_2(\tilde{L}_C \otimes p_1^*(\omega_C))],$$

where $p_1 : C \to J^d(C)$ denotes the projection onto the first factor and $\tilde{L}_C$ is any Poincaré line bundle for $J^d(C)$. Note that the fact that $\Xi \in \text{Pic}(J_{d,g})$ guarantees that the right hand side of (7.8) is independent of the choice $\tilde{L}_C$.

In order to compute the right hand side of (7.8), we can choose a Poincaré line bundle $\tilde{L}_C$ for $J^d(C)$ that satisfies the following equality.

**Condition (*)**: $[(\tilde{L}_C)|_{p_1^{-1}(r)}] = 0 \in NS(J^d(C))$ for any $r \in C$.

Indeed, since $\tilde{L}_C$ can be seen as a family of line bundles on $J^d(C)$ parametrized by $C$, if condition (*) holds for a certain point $r_0 \in C$ then it holds for all points $r \in C$. However, up to tensoring $\tilde{L}_C$ with the pull-back of a line bundle on $J^d(C)$, we can always assume that $(\tilde{L}_C)|_{p_1^{-1}(r_0)}$ is the trivial line bundle on $J^d(C)$, q.e.d.

With the above condition on $\tilde{L}_C$, we can prove the following two claims.

**Claim 1**: If $\tilde{L}_C$ satisfies condition (*) then

$$[dp_2(\tilde{L}_C \otimes p_1^*(M))] = [dp_2(\tilde{L}_C)] \in NS(J^d(C))$$

for any $M \in J(C)$.

Indeed, write $M = O_C(\gamma - \delta + \epsilon)$ with $\gamma = \sum a_i\gamma_i$ and $\delta = \sum b_j\gamma_j$ effective divisors on $C$. From the exact sequences defining the structure sheaves of $p_1^{-1}(\delta) \subset C \times J^d(C)$ and $p_1^{-1}(\gamma) \subset C \times J^d(C)$, we get

$$0 \to \tilde{L}_C \otimes p_1^*O_C(-\gamma) \to \tilde{L}_C \to (\tilde{L}_C)|_{p_1^{-1}(\gamma)} \to 0,$$

$$0 \to \tilde{L}_C \otimes p_1^*O_C(-\delta) \to \tilde{L}_C \otimes p_1^*(M) \to (\tilde{L}_C)|_{p_1^{-1}(\delta)} \to 0.$$

From the multiplicity of the determinant of cohomology (see Fact 2.16) applied to the above exact sequences, we get

$$[dp_2(\tilde{L}_C \otimes p_1^*(M))) = dp_2((\tilde{L}_C)|_{p_1^{-1}(\delta)} \otimes dp_2((\tilde{L}_C)|_{p_1^{-1}(\gamma)})^{-1} =$$

$$= \bigotimes_j (\tilde{L}_C)|_{p_1^{-1}(r_j)} \bigotimes_i (\tilde{L}_C)|_{p_1^{-1}(r_i)}.$$ 

Claim 1 follows now by condition (*).

**Claim 2**: If $\tilde{L}_C$ satisfies condition (*) then

$$[dp_2(\tilde{L}_C)] = \theta_C \in NS(J^d(C)).$$

Indeed, choose a line bundle $M \in J^{d-\gamma+1}(C)$ and consider the Cartesian diagram

$$\begin{array}{ccc}
(id \times t_M)^*(\tilde{L}_C) & \to & \tilde{L}_C \\
\downarrow & & \downarrow \\
C \times J^{d-1}(C) & \underset{id \times t_M}{\to} & C \times J^d(C) \\
\downarrow p_2 & & \downarrow p_2 \\
J^{d-1}(C) & \underset{t_M}{\to} & J^d(C),
\end{array}$$

where $t_M$ is the map sending $L \in J^{d-1}(C)$ into $L \otimes N \in J^d(C)$. The line bundle $\tilde{L}_C := (id \times t_M)^*(\tilde{L}_C) \otimes p_1^*(M)^{-1}$ is clearly a Poincaré line bundle for $J^{d-1}(C)$ and satisfies condition (*) since $\tilde{L}_C$ satisfies condition (*) by assumption. Therefore, using the functoriality of the determinant of cohomology (see Fact 2.16 and Claim 1, we get the following equality in $NS(J^{d-1}(C))$:

$$[f_M dp_2(\tilde{L}_C)] = [dp_2((id \times t_M)^*(\tilde{L}_C))] = [dp_2(\tilde{L}_C \otimes p_1^*(M))] = [dp_2(\tilde{L}_C)].$$

Claim 2 now follows from the well-known fact that $dp_2(\tilde{L}_C) \in \text{Pic}(J^{d-1}(C))$ is the line bundle associated to the theta divisor $\Theta(C) \subset J^{d-1}(C)$ for any Poincaré line bundle $\tilde{L}_C$ for $J^{d-1}(C)$. 

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Now choosing a Poincaré line bundle \( \tilde{L} \) that satisfies condition (*) formula (7.8) together with Claim 1 and Claim 2 gives that
\[
\rho_C(\Xi) = \frac{d + g - 1}{(d + g - 1, d - g + 1)} \theta_C - \frac{d - g + 1}{(d + g - 1, d - g + 1)} \theta_C =
\frac{2g - 2}{(2g - 2, d + g - 1)} \theta_C,
\]
which proves (7.6).

By combining the above results, we can now prove the main Theorems A and B from the introduction.

**Proof of Theorem B.** Let us first prove Theorem B(i). By combining Theorem 7.1(ii) with Theorem 7.2, we get that
\[
\chi_d(\text{Pic}(\mathcal{J}_{d,g})) = \chi_d(\text{Pic}^{\text{taut}}(\mathcal{J}_{d,g})).
\]
By Theorem 7.1(i), the kernel of \( \chi_d \) is equal to \( \Phi^* \text{Pic}(\mathcal{M}_g) \), which is generated by \( \Lambda(1,0) = \Phi^* \text{Pic} \) by Theorem 2.21 and Lemma 5.1; hence \( \text{Im} \Phi^* \subset \text{Pic}^{\text{taut}}(\mathcal{J}_{d,g}) \). We deduce that
\[
\text{Pic}^{\text{taut}}(\mathcal{J}_{d,g}) = \text{Pic}(\mathcal{J}_{d,g}).
\]
Therefore, \( \text{Pic}(\mathcal{J}_{d,g}) \) is generated by \( \Lambda(1,0) \) and by \( \Xi \) by Corollary 6.3. Consider now the exact sequence (7.4). Combining the factorization of \( \chi_d \) provided by (7.4) with formula (7.5), we get that \( \mathcal{R}\text{Pic}(\mathcal{J}_{d,g}) \) is free of rank one. On the other hand, using Theorem 2.21 (since \( g \geq 3 \) by assumption), we know that \( \text{Pic}(\mathcal{M}_g) \) is free of rank one. Therefore the exact sequence (7.4) gives that \( \text{Pic}(\mathcal{J}_{d,g}) \) is free of rank two, which concludes the proof of part (i).

Theorem B(ii) follows now from part (i) and Corollary 4.6.

**Proof of Theorem A.** Let us first prove Theorem A(i). From (7.10) and Corollary 6.10(ii), we deduce that
\[
\text{Pic}(\mathcal{J}_{ac,d,g}) = \text{Pic}(\mathcal{J}_{ac,d,g}).
\]
Therefore, \( \text{Pic}(\mathcal{J}_{ac,d,g}) \) is generated by \( \Lambda(1,0), \Lambda(0,1) \) and \( \Lambda(1,1) \) by Corollary 5.4. Moreover, the exact sequence (6.1) together with Theorem B(i) implies that \( \text{Pic}(\mathcal{J}_{ac,d,g}) \) is free of rank three. Part (i) is now proved.

Theorem A(ii) follows now from part (i) and Theorem 4.1.

We can now compare our computation of \( \text{Pic}(\mathcal{J}_{d,g}) \) (see Theorem B(ii)) with the computation of \( \text{Pic}(\mathcal{J}_{d,g}^0) \) carried out by Kouvidakis in [Kou91].

**Remark 7.3.** Assume that \( g \geq 3 \). Then the natural map \( \Psi_d : \mathcal{J}_{d,g} \to \mathcal{J}_{d,g}^0 \) is an isomorphism over the open subset \( J_{d,g}^0 \subset \mathcal{J}_{d,g} \) parametrizing pairs \((C,L) \in \mathcal{J}_{d,g}\) such that \( C \) does not have non-trivial automorphisms. In other words, the map \( \Psi_d \) induces an isomorphism
\[
\Psi_d : J_{d,g}^0 := \Psi_d^{-1}(J_{d,g}^0) \cong J_{d,g}^0.
\]
Therefore, we get a natural homomorphism
\[
\psi : \text{Pic}(\mathcal{J}_{d,g}) \to \text{Pic}(\mathcal{J}_{d,g}^0) \xrightarrow{\cong} \text{Pic}(\mathcal{J}_{d,g}^0),
\]
where the first homomorphism is the natural restriction map.

If \( g \geq 4 \), then the codimension of \( J_{d,g} \setminus J_{d,g}^0 \) inside \( \mathcal{J}_{d,g} \) is at least two and hence the map \( \psi \) is an isomorphism by Fact 2.14(iii). Hence Theorem B(ii) recovers [Kou91] Thm. 4. However, this does not hold anymore if \( g = 3 \) since in this case \( J_{d,g} \setminus J_{d,g}^0 \) is a divisor inside \( \mathcal{J}_{d,g} \), namely the pull-back of the hyperelliptic (irreducible) divisor in \( \mathcal{M}_3 \), whose class in \( A^1(\mathcal{M}_g) \) is equal to \( 9\lambda \) (see [HM98, Chap. 3, Sec. E]). Therefore, by Fact 2.14(iii), we get that \( \text{Pic}(\mathcal{J}_{d,g}^0) \cong \text{Pic}(J_{d,g}^0) \) is the quotient of \( \text{Pic}(\mathcal{J}_{d,g}) \) by the relation \( \Lambda(1,0)^0 = 0 \).
7.1. Relation between $\Xi$ and the universal theta divisor. There is a close relationship between the line bundle $\Xi \in \text{Pic}(\mathcal{J}_{d,g}) \subset \text{Pic}(\mathcal{J}ac_{d,g})$ and the universal theta divisor $\Theta \subset \mathcal{J}ac_{g-1,g}$, which is the closed substack parametrizing pairs $(C,L) \in \mathcal{J}ac_{g-1,g}$ such that $h^0(C,L) > 0$. Observe that $\Theta$ naturally descends to a divisor on the rigidification $\mathcal{J}_{g-1,g}$, which we denote by $\overline{\Theta}$ and we call the universal theta divisor on $\mathcal{J}_{g-1,g}$. By construction, the restriction of $\overline{\Theta}$ to any fiber $\Phi_d^{-1}(C) = J^{g-1}(C)$ is isomorphic to the theta divisor $\Theta(C) \subset J^{g-1}(C)$.

Consider first the special case $d = g - 1$. From the definition (6.2) of $\Xi$ and using the definition (5.1) of the tautological line bundles, we get that $\Xi = \Lambda(0, 1) = d_\pi(\mathcal{L}_{g-1})$, where $\mathcal{L}_{g-1}$ is the universal line bundle on the universal family over $\mathcal{J}ac_{g-1,g}$. It is well know that $d_\pi(\mathcal{L}_{g-1})$ is the line bundle associated to the universal theta divisor, or in other words we have that

$$\Xi = \mathcal{O}(\Theta) \quad \text{if} \quad d = g - 1.\quad (7.13)$$

For an arbitrary $d$, we consider the stack $\mathcal{S}^{1/k_d,g}_{d,g}$ of $k_d,g$-spin curves, as usual

$$k_{d,g} = \frac{2g - 2}{(2g - 2, d + 1 - g)}.\quad (7.14)$$

Recall that $\mathcal{S}^{1/k_d,g}_{d,g}$ is the stack whose fiber over a scheme $S$ consists of the groupoid of families of smooth curves $C \to S$ of genus $g$, plus a line bundle $\eta$ on $C$ of relative degree $(d - g + 1, 2g - 2)$ over $S$ endowed with an isomorphism $\eta^{\otimes e_{d,g}} \cong \omega_C/S$. The stack $\mathcal{S}^{1/k_d,g}_{d,g}$ is a smooth Deligne-Mumford stack endowed with a (forgetful) finite and étale map $\mathcal{S}^{1/k_d,g}_{d,g} \to \mathcal{M}_g$ of degree $(2g)^{k_{d,g}}$. We have a diagram

\[
\begin{array}{ccc}
\mathcal{J}ac_{d,g,1} & \xrightarrow{\pi_2} & \mathcal{J}ac_{g-1,g,1} \\
\phantom{\mathcal{J}ac_{d,g}} \downarrow \mathcal{S}^{1/k_d,g}_{d,g} \times_{\mathcal{M}_g} \mathcal{J}ac_{d,g} & \xrightarrow{\pi_1} & \mathcal{J}ac_{g-1,g} \\
\end{array}
\]

where $p_2$ is the projection onto the second factor and $s$ sends the element $(C \to S, \eta, \mathcal{L}) \in \mathcal{S}^{1/k_d,g}_{d,g} \times_{\mathcal{M}_g} \mathcal{J}ac_{d,g}(S)$ into $(C \to S, \mathcal{L} \otimes \eta^{-e_{d,g}}) \in \mathcal{J}ac_{g-1,g}(S)$, where

$$e_{d,g} := \frac{d - g + 1}{(d - g + 1, 2g - 2)}.\quad (7.15)$$

The universal family $\mathcal{F}$ is endowed with a universal line bundle $\mathcal{L}_d$ of relative degree $d$ which is the pulled-back from $\mathcal{J}ac_{d,g,1}$ and a universal spin line bundles $\eta_{k_d,g}$ which is pulled-back from the universal family above $\mathcal{S}^{1/k_d,g}_{d,g}$. By the definition of the morphism $s$, we get that

$$s^*(\mathcal{L}_{g-1}) = \eta_{k_d,g}^{-e_{d,g}} \otimes \mathcal{L}_d.\quad (7.16)$$

The relation between the line bundle $\Xi \in \text{Pic}(\mathcal{J}ac_{d,g})$ and the universal theta divisor $\Theta \subset \mathcal{J}ac_{g-1,g}$ is provided by the following.

**Lemma 7.4.** We have that

$$p_2^*(\Xi) = s^*\mathcal{O}(k_{d,g} \cdot \Theta) \otimes (\eta_{k_d,g})^{-k_{d,g}(k_{d,g}+e_{d,g})/2}.\quad (7.17)$$

**Proof.** By the definition (6.2) of $\Xi$ and the standard properties of the determinant of cohomology (see Fact 2.10), we compute

$$p_2^*(\Xi) = d_\pi(\mathcal{L}_d)_{k_{d,g} - d(g - 1)} \otimes d_\pi(\omega_{\pi} \otimes \mathcal{L}_d)_{k_{d,g} - d(g - 1)} = d_\pi(\mathcal{L}_d)^{k_{d,g} + e_{d,g}} \otimes d_\pi(\eta_{k_d,g} \otimes \mathcal{L}_d)^{-e_{d,g}}.\quad (7.18)$$

Using (7.13) and (7.15) together with standard properties of the determinant of cohomology (see Fact 2.10), we get that

$$s^*(\mathcal{O}(k_{d,g} \cdot \Theta)) = s^*(d_\pi(\mathcal{L}_{g-1})^{k_{d,g}}) = d_{\pi}(\eta_{k_d,g}^{-e_{d,g}} \otimes \mathcal{L}_d)^{k_{d,g}}.\quad (7.19)$$
In order to compare (7.16) and (7.17), we apply the Grothendieck-Riemann-Roch theorem to the sheaf $\eta^R_{d,g} \otimes L_m^g$ on the universal family $\pi : \mathcal{F} \to \mathcal{S}^{1/k_{d,g}}_g \times \mathcal{M}_g \mathcal{J}_{ac,d,g}$. After some easy computations similar to the ones done in the proof of Theorem 5.3 which we leave to the reader, we get that

\[
c_1(d_\pi(\eta^R_{d,g} \otimes L_m^g)) = \frac{6n^2 - 6k_{d,g}n + k_{d,g}^2}{12} + \frac{2mn - k_{d,g}m}{2} c_1(\langle \eta_{d,g} \rangle, L_{\pi}^g) + m^2 c_1(\langle L_d \rangle, \pi).
\]

Using the above formula (7.18), we can compute the difference between the first Chern classes of the line bundles in (7.16) and in (7.17):

\[
c_1(p_1^\ast(\Xi)) - c_1(s^\ast(\mathcal{O}(k_{d,g}, \Theta))) = (k_{d,g} + e_{d,g}) c_1(\langle L_{\pi} \rangle) - e_{d,g} c_1(\langle \eta_{k_{d,g}} \rangle) = \frac{2}{k_{d,g}} c_1(\langle \eta_{d,g} \rangle, \eta_{k_{d,g}}).
\]

The result now follows since $c_1 : \text{Pic}(\mathcal{S}^{1/k_{d,g}}_g \times \mathcal{M}_g \mathcal{J}_{ac,d,g}) \to A^1(\mathcal{S}^{1/k_{d,g}}_g \times \mathcal{M}_g \mathcal{J}_{ac,d,g})$ is an isomorphism (see Fact 2.14(ii)).

Remark 7.5. Using the computation of the Picard group of the moduli stacks of spin curves by Jarvis [Jar01], it can be proved that the pull-back morphism $p_1^\ast : \text{Pic}(\mathcal{J}_{ac,d,g}) \to \text{Pic}(\mathcal{S}^{1/k_{d,g}}_g \times \mathcal{M}_g \mathcal{J}_{ac,d,g})$ is injective. Therefore, Lemma 7.4 uniquely determines the line bundle $\Xi$. However, while the definition 6.2 extends naturally to $\mathcal{J}_{ac,d,g}$, we do not know how to extend the formula of Lemma 7.4 to $\mathcal{J}_{ac,d,g}$.

The problem is that we do not know how to extend the correspondence between $\mathcal{J}_{ac,d,g}$ and $\mathcal{J}_{ac_{g-1},g}$ given in diagram (7.11) to a correspondence between $\mathcal{J}_{ac_{d,g}}$ and $\mathcal{J}_{ac_{g-1},g}$.

8. Relation with the Moduli Space $\mathcal{J}_{d,g}$

The aim of this section is to relate the Picard group of the stack $\mathcal{J}_{d,g}$ with the divisor class group and the rational Picard group of its moduli space $\mathcal{J}_{d,g}$, computed by Fontanari in [Fon05, Thm. 5, Cor. 1], based upon the results of Kouvidakis [Kou01].

Recall that, given a variety $Y$, the divisor class group $\text{Cl}(Y)$ is the group of Weil divisors modulo rational equivalence. If $Y$ is normal, denoting by $Y_{\text{reg}}$ the open subset of regular points of $Y$, then we have that

\[
(8.1) \quad \text{Pic}(Y) \to \text{Cl}(Y) \cong \text{Cl}(Y_{\text{reg}}) \cong \text{Pic}(Y_{\text{reg}}).
\]

Note that $\mathcal{J}_{d,g}$ is a normal variety since it is constructed as the GIT quotient of a non-singular variety, namely $H_d$ (see Theorem 2.10). Moreover it comes equipped with a morphism $\phi_d : \mathcal{J}_{d,g} \to \overline{\mathcal{M}}_g$ into the coarse moduli space of stable curves of genus $g$ (see diagram 2.9).

**Theorem 8.1** (Fontanari). Set $\overline{\Delta}_i := \phi_d^{-1}(\Delta_i) \subset \mathcal{J}_{d,g}$ for $i = 0, \ldots, [g/2]$.

(i) The divisors $\overline{\Delta}_i$ are irreducible and we have an exact sequence

\[
0 \to \bigoplus_{i=0}^{[g/2]} \mathbb{Z} \cdot \overline{\Delta}_i \to \text{Cl}(\mathcal{J}_{d,g}) \to \text{Cl}(\mathcal{J}_{d,g}) \to 0.
\]

(ii) The natural inclusion $\text{Pic}(\mathcal{J}_{d,g}) \to \text{Cl}(\mathcal{J}_{d,g})$ is of finite index, i.e. every Weil divisor on $\mathcal{J}_{d,g}$ is $\mathbb{Q}$-Cartier.

We have therefore a commutative diagram with exact rows:

\[
\begin{array}{ccccccc}
0 & \to & \bigoplus_{i=0}^{[g/2]} \mathbb{Z} \cdot \overline{\Delta}_i & \to & \text{Cl}(\mathcal{J}_{d,g}) & \to & \text{Cl}(\mathcal{J}_{d,g}) & \to & 0 \\
& \alpha_d & & \psi_d^\ast & & \beta_d & & \\
0 & \to & \bigoplus_{k_{d,g}^i \neq 2i-1 \text{ or } i=g/2} (\mathcal{O}(\overline{\Delta}_i)) \oplus \bigoplus_{k_{d,g}^i \neq 2i-1 \text{ and } i \neq g/2} (\mathcal{O}(\overline{\Delta}_i)^2) & \to & \text{Pic}(\mathcal{J}_{d,g}) & \to & \text{Pic}(\mathcal{J}_{d,g}) & \to & 0
\end{array}
\]

where the map $\psi_d^\ast$ is the pull-back map induced by $\psi_d : \mathcal{J}_{d,g} \to \mathcal{J}_{d,g}$. We can now prove Theorem C from the introduction.
Proof of Theorem A. In order to prove part (i) of Theorem A consider the commutative diagram, obtained by pulling back divisors along the two fibrations $J_{d,g} \to M_g$ and $J_{d,g} \to M_g$:
\[
\begin{array}{c}
\text{Cl}(M_g) \\
\downarrow \gamma_d \\
\text{Pic}(M_g) \\
\text{Pic}(J_{d,g}) \\
\text{RPic}(J_{d,g}) \\
\end{array}
\quad
\begin{array}{c}
\text{Cl}(J_{d,g}) \\
\downarrow \delta_d \\
\text{Pic}(J_{d,g}) \\
\text{RPic}(J_{d,g}) \\
0 \\
0 \\
\end{array}
\]

The map $\gamma_d$ is well-known to be an isomorphism (see e.g. [AC87 Prop. 2]). The map $\delta_d$ is an isomorphism since the group of rational determined line bundles $\text{RPic}$ of a fibration is birational on the base (see [Cil87 Lemma 1.3]) and the map $J_{d,g} \to M_g$ is representable. Since the rows of the above diagram are exact, we conclude that $\beta_d$ is an isomorphism, q.e.d.

In order to prove part (ii) of Theorem A, we need a local description of the morphism $\Psi_d : \overline{J}_{d,g} \to \overline{J}_{d,g}$ at the general point of $\Delta_i$. This was carried on in [BFV11 Proof of Thm. 1.5] for the morphism $\nu_d \circ \Psi_d : \overline{J}_{ac,d,g} \to \overline{J}_{d,g}$, but it is very easy to adapt the description in loc. cit. to the morphism $\Psi_d$ (simply by passing to the $\mathbb{G}_m$-rigidification).

If $k_{d,g} \not= (2i - 1)$ (which corresponds to the cases (1) and (2) of loc. cit.) then the morphism $\Psi_d$ is an isomorphism locally at the general point of $\Delta_i$ (see [BFV11 p. 25]). Therefore $\Psi_d^*(\Delta_i) = \mathcal{O}(\delta_i)$.

If $k_{d,g} \not= (2i - 1)$ (which corresponds to the case (3) of loc. cit.) then the morphism $\Psi_d$ looks like (after neglecting trivial coordinates)
\[
\mathcal{X} := \text{Spf } k[[x,y]] \widehat{\otimes} A / \mathbb{G}_m \to X := \text{Spf } k[[x,y]] / \mathbb{G}_m \widehat{\otimes} A = \text{Spf } k[[xy]] \widehat{\otimes} A,
\]
where $A = \text{Spf } k[[y_1, \ldots, y_{g-4}]]$, $\mathbb{G}_m$ acts via $\lambda \cdot (x,y) = (\lambda x, \lambda^{-1} y)$ and trivially on $A$ (see [BFV11 p. 26]). In this local description, the divisor $\Delta_i$ corresponds to the divisor $(x = 0)$ on $\mathcal{X}$ and the divisors $\delta_1^i$ and $\delta_2^i$ correspond to the divisors $(x = 0)$ and $(y = 0)$ on $X$ (note that in the particular case $i = g/2$ and $k_{d,g}(g-1)$, the divisor $\delta_{g/2}$, even though irreducible, locally analytically splits into two components, which we can call $\delta_{g/2}^1$ and $\delta_{g/2}^2$, so that the above description remains valid also in this case). From the explicit form of the map $p$, it is clear that $p^*(xy = 0) = (x = 0) + (y = 0)$, from which we deduce that
\[
\Psi_d^*(\Delta_i) = \begin{cases} 
\mathcal{O}(\delta_i^1 + \delta_i^2) & \text{if } i < g/2, \\
\mathcal{O}(2\delta_{g/2}) & \text{if } i = g/2.
\end{cases}
\]

Part (ii) is now proved. 

\[\square\]

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