HERMITIAN-EINSTEIN CONNECTIONS ON POLYSTABLE PARABOLIC PRINCIPAL HIGGS BUNDLES

INDRANIL BISWAS AND MATTHIAS STEMLER

ABSTRACT. Given a smooth complex projective variety \( X \) and a smooth divisor \( D \) on \( X \), we prove the existence of Hermitian-Einstein connections, with respect to a Poincaré-type metric on \( X \setminus D \), on polystable parabolic principal Higgs bundles with parabolic structure over \( D \), satisfying certain conditions on its restriction to \( D \).

1. Introduction

The Hitchin-Kobayashi correspondence relating the stable vector bundles and the solutions of the Hermitian-Einstein equation has turned out to be extremely useful and important (see [Do87], [UY86], [Si92]). The Hitchin-Kobayashi correspondence has evolved into a general principle finding generalizations to numerous contexts. Here we consider the parabolic Higgs \( G \)-bundles from this point of view.

Parabolic vector bundles on curves were introduced by Seshadri [Se77]. This was generalized to higher dimensional varieties by Maruyama and Yokogawa [MY92]. Motivated by the characterization of principal bundles using Tannakian category theory given by Nori [No76], in [BBN01], parabolic principal bundles were defined. Later ramified principal bundles were defined in [BBN03]; it turned out that there is a natural bijective correspondence between ramified principal bundles and parabolic principal bundles, cf. [BBN03], [Bi06]. Higgs fields on ramified principal bundles were defined in [Bi08].

In [Bq97], Biquard considered vector bundles on a compact Kähler manifold \( (X, \omega_0) \), with parabolic structure over a smooth divisor \( D \), equipped with a Higgs field that has a logarithmic singularity on \( D \). He showed that these data induce certain Higgs bundles (in an adapted sense) on \( D \), which he calls “spécialisés”. In the case of Higgs fields with nilpotent residue on \( D \), these are just the graded pieces of the parabolic filtration equipped with an induced Higgs structure. Given a stable parabolic Higgs bundle such that these induced bundles are polystable and satisfy an additional condition on their slope, he proves the existence of a Hermitian-Einstein metric on \( X \setminus D \) with respect to a Poincaré-type Kähler metric. The Hermitian-Einstein metric is unique up to multiplication by a constant element of \( \mathbb{R}^+ \).

Our aim here is to extend Biquard’s result to the case of parabolic principal Higgs \( G \)-bundles, where \( G \) is a connected reductive linear algebraic group defined over \( \mathbb{C} \). Given such a bundle \((E_G, \theta)\), there is an adjoint parabolic Higgs vector bundle \((\text{ad}(E_G), \text{ad}(\theta))\).

2000 Mathematics Subject Classification. 53C07, 32L05, 14J60.

Key words and phrases. Hermitian-Einstein connection, parabolic Higgs \( G \)-bundle, ramified Higgs \( G \)-bundle.
The Higgs field $\text{ad}(\theta)$ has a nilpotent residue on $D$. This $\text{ad}(\theta)$ induces Higgs fields on the graded pieces $\text{Gr}_\alpha \text{ad}(E_G)$ for the parabolic vector bundle $\text{ad}(E_G)$. The Higgs field on $\text{Gr}_\alpha \text{ad}(E_G)$ induced by $\theta$ will be denoted by $\text{ad}(\theta)_\alpha$.

Let $\psi : E_G \to X$ be the natural projection. The restriction of $\psi$ to $\psi^{-1}(D)$ will be denoted by $\hat{\psi}$. Let $\mathcal{K}$ be the trivial vector bundle over $\psi^{-1}(D)$ with fiber $\text{Lie}(G)$. The group $G$ acts on $\mathcal{K}$ using the adjoint action of $G$ on $\text{Lie}(G)$. Define the invariant direct image $E := (\hat{\psi}^* \mathcal{K})^G$, which is a vector bundle over $D$. The Higgs field $\theta$ defines a Higgs field on $E$, which will be denoted by $\theta'$.

Fix a Kähler form $\omega_0$ on $X$ such that the corresponding class in $H^2(X, \mathbb{R})$ is integral. We obtain the following (see Theorem 12 and Proposition 14):

**Theorem 1.** Let $(E_G, \theta)$ be a parabolic Higgs $G$-bundle on $X$ such that $(E_G, \theta)$ is poly-stable with respect to $\omega_0$, and satisfies the following two conditions:

- The Higgs bundle $(E, \theta')$ on $D$ is polystable, and
- for the graded pieces $(\text{Gr}_\alpha \text{ad}(E_G), \text{ad}(\theta)_\alpha)$ of $(\text{ad}(E_G)|_D, \text{ad}(\theta)|_D)$ the condition
  $$\mu(\text{Gr}_\alpha \text{ad}(E_G)) = -\alpha \deg(N)$$

holds, where degrees are computed using $\omega_0$ and $N$ is the normal bundle to $D$.

Then there is a Hermitian-Einstein connection on $E_G$ over $X \setminus D$ with respect to the Poincaré-type metric.

Conversely, if there is such a Hermitian-Einstein connection satisfying the condition that the induced connection on the adjoint vector bundle $\text{ad}(E_G)|_{X \setminus D}$ lies in the space $\mathcal{A}$ (see (3.3)), then $(E_G, \theta)$ is polystable with respect to $\omega_0$.

## 2. PARABOLIC HIGGS BUNDLES

Let $X$ be a connected smooth complex projective variety of complex dimension $n$, and let $D$ be a smooth reduced effective divisor on $X$. We first recall the definition of a parabolic Higgs vector bundle on $X$ with parabolic structure over $D$.

A parabolic vector bundle $E_\ast$ on $X$ with parabolic divisor $D$ is a holomorphic vector bundle $E$ on $X$ together with a parabolic structure on it, which is given by a decreasing filtration $\{F_\alpha(E)\}_{0 \leq \alpha < 1}$ of holomorphic subbundles of the restriction $E|_D$, which is continuous from the left, satisfying the conditions that $F_0(E) = E|_D$ and $F_1(E) = 0$. The parabolic weights of $E_\ast$ are the numbers $0 \leq \alpha_1 < \cdots < \alpha_l < 1$ such that $F_{\alpha_1 + \varepsilon}(E) \neq F_{\alpha_1}(E)$ for all $\varepsilon > 0$. For later use, we denote the graded pieces of this filtration as

$$\text{Gr}_\alpha E := F_\alpha(E)/F_{\alpha + \varepsilon}(E), \quad \alpha \in \{\alpha_1, \ldots, \alpha_l\}, \quad \varepsilon > 0 \text{ sufficiently small.}$$

Let $\text{ParEnd}(E_\ast)$ be the sheaf of holomorphic sections of $\text{End}(E) = E \otimes E^*$ which preserve the above filtration of $E|_D$. Let $\Omega_X^*(\log D)$ be the vector bundle on $X$ defined by the sheaf
of logarithmic $k$-forms. Note that there is a residue homomorphism
\[ \text{Res}_D : \text{ParEnd}(E) \otimes \Omega^1_X(\log D) \rightarrow \text{ParEnd}(E)|_D \]
defined by the natural residue homomorphism $\Omega^1_X(\log D) \rightarrow \mathcal{O}_D$.

**Definition 2.** A parabolic Higgs vector bundle with parabolic divisor $D$ is a pair $(E, \theta)$ consisting of a parabolic vector bundle $E$ on $X$ with parabolic divisor $D$ and a section $\theta \in H^0(X, \text{ParEnd}(E) \otimes \Omega^1_X(\log D))$, called the Higgs field, such that the following two conditions are satisfied:

- $\theta \wedge \theta \in H^0(X, \text{ParEnd}(E) \otimes \Omega^2_X(\log D))$ vanishes identically, where the multiplication is defined using the Lie algebra structure of the fibers of $\text{End}(E)$, and the exterior product $\Omega^1_X(\log D) \otimes \Omega^1_X(\log D) \rightarrow \Omega^2_X(\log D)$, and
- the residue $\text{Res}_D(\theta)$ is nilpotent with respect to the parabolic filtration in the sense that
  \[ \text{Res}_D(\theta)(F_{\alpha}(E)) \subset F_{\alpha+\varepsilon}(E) \]
  for some $\varepsilon > 0$.

In the following, we will omit the subscript “$*$” in $E$, and denote a parabolic vector bundle by the same symbol as its underlying bundle.

Now we will recall the definitions of ramified Higgs principal bundles and parabolic Higgs principal bundles. For this, let $G$ be a connected reductive linear algebraic group defined over $\mathbb{C}$.

**Definition 3.** A ramified $G$-bundle over $X$ with ramification over $D$ is a smooth complex variety $E_G$ equipped with an algebraic right action of $G$

\[ f : E_G \times G \rightarrow E_G \]
and a surjective algebraic map
\[ \psi : E_G \rightarrow X, \]
such that the following conditions are satisfied:

- $\psi \circ f = \psi \circ p_1$, where $p_1 : E_G \times G \rightarrow E_G$ denotes the natural projection,
- for each point $x \in X$, the action of $G$ on the reduced fiber $\psi^{-1}(x)_{\text{red}}$ is transitive,
- the restriction of $\psi$ to $\psi^{-1}(X \setminus D)$ makes $\psi^{-1}(X \setminus D)$ a principal $G$-bundle over $X \setminus D$, meaning the map $\psi$ is smooth over $\psi^{-1}(X \setminus D)$ and the map to the fiber product
  \[ \psi^{-1}(X \setminus D) \times G \rightarrow \psi^{-1}(X \setminus D) \times_{X \setminus D} \psi^{-1}(X \setminus D) \]
given by $(z, g) \mapsto (z, f(z, g))$ is an isomorphism,
- the reduced inverse image $\psi^{-1}(D)_{\text{red}}$ is a smooth divisor on $E_G$, and
- for each point $z \in \psi^{-1}(D)_{\text{red}}$, the isotropy group $G_z \subset G$ for the action of $G$ on $E_G$ is a finite cyclic group acting faithfully on the quotient line $T_z E_G / T_z(\psi^{-1}(D)_{\text{red}})$. 

Parabolic principal $G$-bundles were defined in [BBN01] as functors from the category of rational $G$-representations to the category of parabolic vector bundles, satisfying certain conditions; this definition was modeled on [No76]. There is a natural bijective correspondence between the ramified principal $G$-bundles with ramification over $D$ and parabolic principal $G$-bundles on $X$ with $D$ as the parabolic divisor ([BBN03], [Bi06]). Let us briefly recall a construction of parabolic principal $G$-bundles from ramified principal $G$-bundles.

Let $E_G$ be a ramified $G$-bundle on $X$ with ramification over $D$. There is a finite (ramified) Galois covering

$$\eta : Y \longrightarrow X$$

such that the normalizer

$$F_G := \overline{E_G \times_X Y}$$

(2.2)

of the fiber product $E_G \times_X Y$ is smooth. Write $\Gamma := \text{Gal}(\eta)$ for the Galois group of $\eta$. Let

$$h : \Gamma \longrightarrow \text{Aut}(Y)$$

(2.3)

be the homomorphism giving the action of $\Gamma$ on $Y$. The projection $F_G \longrightarrow Y$ yields a $\Gamma$-linearized principal $G$-bundle on $Y$ in the following sense:

**Definition 4.** A $\Gamma$-linearized principal $G$-bundle on $Y$ is a principal $G$-bundle

$$\psi : F_G' \longrightarrow Y$$

together with a left action of $\Gamma$ on $F_G'$

$$\rho : \Gamma \times F_G' \longrightarrow F_G'$$

such that the following two conditions are satisfied:

- The actions of $\Gamma$ and $G$ on $F_G'$ commute, and
- $\psi(\rho(\gamma, z)) = h(\gamma)(\psi(z))$ for all $(\gamma, z) \in \Gamma \times F_G'$, and $h$ is defined in (2.3).

Consider $F_G$ constructed in (2.2). Given a finite-dimensional complex $G$-module $V$, there is the associated $\Gamma$-linearized vector bundle $F_G(V) = F_G \times^G V$ on $Y$ with fibers isomorphic to $V$. This $F_G(V)$ in turn corresponds to a parabolic vector bundle on $X$ with $D$ as the parabolic divisor, cf. [Bi97b]; this parabolic vector bundle will be denoted by $E_G(V)$.

The earlier mentioned functor, from the category of rational $G$-representations to the category of parabolic vector bundles, associated to the ramified $G$-bundle $E_G$ sends any $G$-module $V$ to the parabolic vector bundle $E_G(V)$ constructed above.

In the following, we will identify the notions of parabolic and ramified $G$-bundles.

Let $g$ be the Lie algebra of $G$; it is equipped with the adjoint action of $G$. Setting $V = g$, the parabolic vector bundle $E_G(g)$ constructed as above is called the adjoint parabolic vector bundle of $E_G$, and it is denoted by $\text{ad}(E_G)$.

Let $E_G$ be a ramified $G$-bundle over $X$ with ramification over $D$. Let

$$\mathcal{K} \subset T E_G$$

(2.4)
be the holomorphic subbundle defined by the tangent space of the orbits of the action of $G$ on $E_G$; since all the isotropies, for the action of $G$ on $E_G$, are finite groups, $\mathcal{K}$ is indeed a subbundle. Note that $\mathcal{K}$ is identified with the trivial vector bundle over $E_G$ with fiber $\mathfrak{g}$. Let

$$Q := TE_G/\mathcal{K}$$

be the quotient vector bundle. The action of $G$ on $E_G$ induces an action of $G$ on the tangent bundle $TE_G$, which preserves the subbundle $\mathcal{K}$. Therefore, there is an induced action of $G$ on the quotient bundle $Q$. These actions in turn induce a linear action of $G$ on $H^0(E_G, \mathcal{K} \otimes Q^*)$. Combining the exterior algebra structure of $\Lambda Q^*$ and the Lie algebra structure on the fibers of $\mathcal{K} = E_G \times \mathfrak{g}$, one obtains a homomorphism

$$\tau : (\mathcal{K} \otimes Q^*) \otimes (\mathcal{K} \otimes Q^*) \rightarrow \mathcal{K} \otimes \Lambda^2 Q^*.$$ 

For $y \in E_G$, and $a, b \in (\mathcal{K} \otimes Q^*)_y$, the image $\tau(a \otimes b)$ will also be denoted by $a \wedge b$.

**Definition 5.**

1. A **Higgs field** on $E_G$ is a section

$$\theta \in H^0(E_G, \mathcal{K} \otimes Q^*)$$

such that

- $\theta$ is invariant under the action of $G$ on $H^0(E_G, \mathcal{K} \otimes Q^*)$, and
- $\theta \wedge \theta = 0$.

2. A **parabolic Higgs $G$-bundle** is a pair $(E_G, \theta)$ consisting of a parabolic $G$-bundle $E_G$ and a Higgs field $\theta$ on $E_G$.

Now let $H \subset G$ be a Zariski closed subgroup, and let $U \subset X$ be a Zariski open subset. The inverse image $\psi^{-1}(U) \subset E_G$ will be denoted by $E_G|_U$; as before, $\psi$ is the projection of $E_G$ to $X$.

**Definition 6.** A **reduction of structure group of $E_G$ to $H$ over $U$** is a subvariety

$$E_H \subset E_G|_U$$

satisfying the following conditions:

- $E_H$ is preserved by the action of $H$ on $E_G$,
- for each point $x \in U$, the action of $H$ on $\psi^{-1}(x) \cap E_H$ is transitive, and
- for each point $z \in E_H$, the isotropy subgroup $G_z$, for the action of $G$ on $E_G$, is contained in $H$.

Clearly, such an $E_H$ is a ramified $H$-bundle over $U$. Let

$$(2.5) \quad \iota : E_H \rightarrow E_G|_U$$

be a reduction of structure group of $E_G$ to $H$ over $U$. Define the bundles $\mathcal{K}_H$ and $\mathcal{Q}_H$ as before with respect to $E_H$ (in place of $E_G$). Then by [Bis05 (3.8)],

$$\text{Hom}(\mathcal{Q}_H, \mathcal{K}_H) \subset \iota^* \text{Hom}(\mathcal{Q}, \mathcal{K}).$$

Let $\theta \in H^0(E_G, \text{Hom}(\mathcal{Q}, \mathcal{K}))$ be a Higgs field on $E_G$. The projection of $E_G$ to $X$.
**Definition 7.** The reduction $E_H$ in (2.5) is said to be *compatible* with the Higgs field $\theta$ if 
$$\theta|_{E_H} \in H^0(E_H, \text{Hom}(Q_H, K_H)) \subset H^0(E_H, \iota^* \text{Hom}(Q, K)).$$

Fix a very ample line bundle $\zeta$ on $X$. Define the degree $\text{deg}$ (respectively, the parabolic degree $\text{par-deg}$) of a torsion-free coherent sheaf $F$ (respectively, a parabolic vector bundle $E_*$) on $X$ with respect to this polarization $\zeta$.

Fix a basis of $H^0(X, \zeta)$. Using this basis we get an embedding of $X$ in $\mathbb{P}^{N-1}$, where $N = \dim H^0(X, \zeta)$. Let $\omega_0$ be the restriction to $X$ of the Fubini-Study metric on $\mathbb{P}^{N-1}$.

Let $H$ be a parabolic subgroup of $G$. Then $G/H$ is a complete variety, and the quotient map $G \to G/H$ defines a principal $H$-bundle over $G/H$. For any character $\chi$ of $H$, let

$$L_\chi \to G/H$$

be the line bundle associated to this principal $H$-bundle for the character $\chi$. Let $R_u(H)$ be the unipotent radical of $H$ (it is the unique maximal normal unipotent subgroup). The group $H/R_u(H)$ is called the *Levi quotient* of $H$. There are subgroups $L(H) \subset H$ such that the composition $L(H) \hookrightarrow H \to H/R_u(H)$ is an isomorphism. Such a subgroup $L(H)$ is called a *Levi subgroup* of $H$. Any two Levi subgroups of $H$ are conjugate by some element of $H$.

Let $Z_0(G) \subset G$ be the connected component, containing the identity element, of the center of $G$. It is known that $Z_0(G) \subset H$. A character $\chi$ of $H$ which is trivial on $Z_0(G)$ is called *strictly antidominant* if the corresponding line bundle $L_\chi$ over $G/H$ (defined above) is ample.

**Definition 8.** A parabolic Higgs $G$-bundle $(E_G, \theta)$ is called *stable* if for every quadruple $(H, \chi, U, E_H)$, where

- $H \subset G$ is a proper parabolic subgroup,
- $\chi$ is a strictly antidominant character of $H$,
- $U \subset X$ is a non-empty Zariski open subset such that the codimension of $X \setminus U$ is at least two, and
- $E_H \subset E_G|_U$ is a reduction of structure group of $E_G$ to $H$ over $U$ compatible with $\theta$,

the following holds:

$$\text{par-deg}(E_H(\chi)) > 0,$$

where $E_H(\chi)$ is the parabolic line bundle over $U$ associated to the parabolic $H$-bundle $E_H$ for the one-dimensional representation $\chi$ of $H$.

Let $E_G$ be a parabolic $G$-bundle over $X$. A reduction of structure group $E_H \subset E_G$ to some parabolic subgroup $H \subset G$ is called *admissible* if for each character $\chi$ of $H$ which is trivial on $Z_0(G)$, the associated parabolic line bundle $E_H(\chi)$ over $X$ satisfies the following condition:

$$\text{par-deg}(E_H(\chi)) = 0.$$
Definition 9. A parabolic Higgs $G$-bundle $(E_G, \theta)$ is called \textit{polystable} if either $(E_G, \theta)$ is stable, or there is a proper parabolic subgroup $H \subset G$ and a reduction of structure group $E_{L(H)} \subset E_G$ of $E_G$ to a Levi subgroup $L(H) \subset H$ over $X$ such that the following conditions are satisfied:

- The reduction $E_{L(H)} \subset E_G$ is compatible with $\theta$,
- the parabolic Higgs $L(H)$-bundle $(E_{L(H)}, \theta|_{E_{L(H)})}$ is stable (from the first condition it follows that $\theta|_{E_{L(H)}}$ is a Higgs field on $E_{L(H)}$), and
- the reduction of structure group of $E_G$ to $H$, obtained by extending the structure group of $E_{L(H)}$ using the inclusion of $L(H)$ in $H$, is admissible.

3. \textsc{Hermitian-Einstein connection on a parabolic Higgs $G$-bundle}

Let $E_G$ be a parabolic $G$-bundle over $X$. Let

\begin{equation}
0 \to \text{ad}(E_G) \to \text{At}(E_G) \to TX \to 0
\end{equation}

be the Atiyah exact sequence for the $G$-bundle $E_G$ over $X \setminus D$. Recall that a \textit{complex connection} on $E_G$ over $X \setminus D$ is a $C^\infty$ splitting of this exact sequence. Fix a maximal compact subgroup $K \subset G$. A complex connection on $E_G$ over $X \setminus D$ is called \textit{unitary} if it is induced by a connection on a smooth reduction of structure group $E_K$ of $E_G$ to $K$ over $X \setminus D$. Note that (3.1) is a short exact sequence of sheaves of Lie algebras. For a complex unitary connection $\nabla$ on $E_G$ over $X \setminus D$, its \textit{curvature form} $F \in H^0(X \setminus D, \Lambda^{1,1}TX \otimes \text{ad}(E_G))$ measures the obstruction of the splitting of (3.1) defining $\nabla$ to be Lie algebra structure preserving; see [At57] for the details.

For a parabolic Higgs $G$-bundle $(E_G, \theta)$ on $X$, its restriction to $X \setminus D$ is a Higgs $G$-bundle in the usual sense. Given a smooth reduction of structure group $E_K$ of $E_G$ to a maximal compact subgroup $K \subset G$ over $X \setminus D$, the Cartan involution of $g$ with respect to $K$ induces an involution of the adjoint vector bundle $\text{ad}(E_G)$ over $X \setminus D$; this involution of $\text{ad}(E_G)$ will be denoted by $\phi$. Writing $\theta = \sum_i \theta_i dz^i$ in local holomorphic coordinates $z^1, \ldots, z^n$ on $X$ around a point $x \in X \setminus D$, define

$\theta^* := -\sum_i \phi(\theta_i) d\bar{z}^i$.

This definition is clearly independent of the choice of local coordinates.

Let $\mathfrak{z}$ be the center of the Lie algebra $\mathfrak{g}$ of $G$. Since the adjoint action of $G$ on $\mathfrak{z}$ is trivial, an element $\lambda \in \mathfrak{z}$ defines a smooth section of $\text{ad}(E_G)$ over $X \setminus D$, which will also be denoted by $\lambda$.

Definition 10. Let $(E_G, \theta)$ be a parabolic Higgs $G$-bundle on $X$. A complex unitary connection on $E_G$ over $X \setminus D$ is called a \textit{Hermitian-Einstein connection} with respect to
a Kähler metric $\omega$ on $X \setminus D$ and the Higgs field $\theta$, if its curvature form $F$ satisfies the equation

$$\Lambda_\omega(F + [\theta, \theta^*]) = \lambda$$

for some $\lambda \in \mathfrak{g}$, where the operation $[\cdot, \cdot]$ is defined using the exterior product on forms and the Lie algebra structure of the fibers of $\text{ad}(E_G)$.

Note that $\lambda$ in Definition 10 lies in $\mathfrak{g} \cap \text{Lie}(K)$.

In [Bq97], Biquard introduces a Poincaré-type metric on $X \setminus D$ as follows: Let $\sigma$ be the canonical section of the line bundle $O_X(D)$ on $X$ associated to the divisor $D$, meaning $D$ is the zero divisor of $\sigma$. Let $\omega_0$ be the Kähler form on $X$ that we fixed earlier. Choose a Hermitian metric $||\cdot||$ on the fibers of $O_X(D)$. Then

$$(3.2) \quad \omega := T\omega_0 - \sqrt{-1}\partial\bar{\partial} \log \log 2||\sigma||^2$$

defines a Kähler metric on $X \setminus D$ for $T \in \mathbb{R}^+$ large enough.

In [Bq97], Biquard proves the existence of Hermitian-Einstein metrics on stable parabolic Higgs vector bundles under certain additional conditions (see [Bq97, Théorème 8.1]). In his definition of parabolic Higgs vector bundles he does not require the residue of the Higgs field to be nilpotent.

Let $(E, \theta)$ be a parabolic Higgs vector bundle. Consider the graded pieces $\text{Gr}_\alpha E$ in (2.1). Let $\theta_\alpha := \theta|_D : \text{Gr}_\alpha E \longrightarrow (\text{Gr}_\alpha E) \otimes (\Omega^1_X(\log D)|_D)$ be the homomorphism given by $\theta$. Since the residue of $\theta$ is nilpotent with respect to the quasi-parabolic filtration of $E|_D$, the composition

$$\text{Gr}_\alpha E \xrightarrow{\theta_\alpha} (\text{Gr}_\alpha E) \otimes (\Omega^1_X(\log D)|_D) \xrightarrow{\text{Id} \otimes \text{Res}} \text{Gr}_\alpha E \otimes O_D = \text{Gr}_\alpha E$$

vanishes identically. Therefore, $\theta_\alpha \in H^0(D, \text{End}(\text{Gr}_\alpha E) \otimes \Omega^1_D)$. The integrability condition $\theta \wedge \theta = 0$ immediately implies that $\theta_\alpha \wedge \theta_\alpha = 0$. Therefore, $(\text{Gr}_\alpha E, \theta_\alpha)$ is a Higgs vector bundle on $D$.

In [Bq97, pp. 47–48], Biquard uses the parabolic structure of $E$ to construct a background metric on $E$ over $X \setminus D$. Let $\nabla$ be the corresponding Chern connection. He then restricts his attention to connections lying in the space

$$(3.3) \quad \mathcal{A} := \{\nabla + a : a \in \tilde{C}^{1+\theta}_{\delta}(\Omega^1_X \otimes \text{End}(E))\}$$

(see [Bq97, p. 58 and p. 70]), where the Hölder space $\tilde{C}^{1+\theta}_{\delta}(\Omega^1_X \otimes \text{End}(E))$ is defined in [Bq97, pp. 53–54]. Let

$N \longrightarrow D$

be the normal line bundle of the divisor $D$.

With these definitions, Biquard’s theorem can be formulated as follows:

**Theorem 11.** Let $(E, \theta)$ be a stable parabolic Higgs vector bundle on $X$ with parabolic divisor $D$. Assume that all the graded Higgs bundles $(\text{Gr}_\alpha E, \theta_\alpha)$ are polystable and satisfy the condition

$$(3.4) \quad \mu(\text{Gr}_\alpha E) = \text{par}-\mu(E) - \alpha \deg(N)$$
with respect to \( \omega_0 \). Then there is a Hermitian metric \( h \) on \( E \) over \( X \setminus D \), with Chern connection in \( A \), which is Hermitian-Einstein with respect to the Poincaré-type metric \( \omega \), meaning its Chern curvature form \( F \) satisfies
\[
\sqrt{-1} \Lambda_{\omega}(F + [\theta, \theta^*]) = \lambda \cdot \text{id}_E
\]
for some \( \lambda \in \mathbb{R} \).

Such a Hermitian metric is unique up to a constant scalar multiple.

4. Existence of Hermitian-Einstein connection

Let \((E_G, \theta)\) be a ramified Higgs \( G \)-bundle. Let
\[
\psi : E_G \longrightarrow X
\]
be the natural projection. The reduced divisor \( \psi^{-1}(D)_{\text{red}} \) will be denoted by \( \widetilde{D} \). Let
\[
\widehat{\psi} := \psi|_{\widetilde{D}} : \widetilde{D} \longrightarrow D
\]
be the restriction. Consider the subbundle \( \mathcal{K} \) defined in \((2.4)\). The action of the group \( G \) on \( \widetilde{D} \) produces an action of \( G \) on the direct image \( \widehat{\psi}_* \mathcal{K} \longrightarrow D \). Define the invariant part
\[
\mathcal{E} := (\widehat{\psi}_* \mathcal{K})^G \longrightarrow D;
\]
it is a vector bundle over \( D \).

We will give an explicit description of the vector bundle \( \mathcal{E} \). As before, the isotropy subgroup of any \( z \in \widetilde{D} \), for the action of \( G \) on \( \widetilde{D} \), will be denoted by \( G_z \). Let
\[
g_z := g^{G_z} \subset g
\]
be the space of invariants for the adjoint action of \( G_z \). This \( g_z \) is clearly a subalgebra of \( g \). The elements of \( G_z \) are semisimple because \( G_z \) is a finite group. Since \( G_z \) is cyclic, the Lie subalgebra \( g_z \) is reductive (see [Hu95, p. 26, Theorem]). Let \( \mathcal{S} \) be the subbundle of the trivial vector bundle \( \widetilde{D} \times g \longrightarrow \widetilde{D} \) whose fiber over any \( z \in \widetilde{D} \) is the subalgebra \( g_z \). The action of \( G \) on \( \widetilde{D} \) and the adjoint action of \( G \) on \( g \) combine together to define an action of \( G \) on \( \widetilde{D} \times g \); the identification between \( \mathcal{K}|_{\widetilde{D}} \) and \( \widetilde{D} \times g \) commutes with the actions of \( G \). The action of \( G \) on \( \widetilde{D} \times g \) clearly preserves the subbundle \( \mathcal{S} \). We have
\[
\mathcal{E} := (\widehat{\psi}_* \mathcal{K})^G \longrightarrow D; \quad D = \widetilde{D}/G \quad \text{and} \quad \mathcal{E} = \mathcal{S}/G.
\]
That \( \mathcal{S}/G \) is a vector bundle over \( \widetilde{D}/G \) follows from the fact that the isotropy subgroups act trivially on the fibers of \( \mathcal{S} \).

Let \( h \) be any \( G \)-invariant nondegenerate symmetric bilinear form on \( g \). The restriction of \( h \) to the centralizer, in \( g \), of any semisimple element of \( G \) is known to be nondegenerate. From this it follows that the bilinear form induced by \( h \) on the vector bundle \( \mathcal{S} \) in \((4.2)\) is nondegenerate. Since \( h \) is \( G \)-invariant, from \((4.2)\) we conclude that this nondegenerate bilinear form on \( \mathcal{S} \) descends to a nondegenerate bilinear form on \( \mathcal{E} \). This implies that \( \mathcal{E}^* = \mathcal{E} \), in particular, \( \deg(\mathcal{E}) = 0 \) with respect to any polarization on \( D \).
Recall that the fibers of $\mathcal{K}$ are identified with $\mathfrak{g}$. Using this Lie algebra structure of the fibers of $\mathcal{K}$, the Higgs field $\theta$ defines a homomorphism
\[ \mathcal{K} \rightarrow \mathcal{K} \otimes Q^* \rightarrow 0 \]
of vector bundles. On the other hand, over $\tilde{D}$, we have a natural restriction homomorphism
\[ Q^*|_{\tilde{D}} \rightarrow \Omega^1_{\tilde{D}} \]
of vector bundles. Combining these two homomorphisms, we have a homomorphism of vector bundles
\[ \beta : \mathcal{K}|_{\tilde{D}} \rightarrow \mathcal{K}|_{\tilde{D}} \otimes \Omega^1_{\tilde{D}}. \]
The group $G$ acts on both $\mathcal{K}|_{\tilde{D}}$ and $\Omega^1_{\tilde{D}}$. The above homomorphism $\beta$ commutes with the actions of $G$. Therefore, $\beta$ produces a homomorphism
\[ (4.3) \quad \theta' : \mathcal{E} = (\hat{\psi}_*\mathcal{K})^G \rightarrow (\mathcal{K}|_{\tilde{D}} \otimes \Omega^1_{\tilde{D}})^G = \mathcal{E} \otimes \Omega^1_{\tilde{D}}, \]
where $\mathcal{E}$ is defined in (4.1). From the condition $\theta \land \theta = 0$ (see Definition 5) it follows that $\theta'$ is a Higgs field on the vector bundle $\mathcal{E}$.

Consider the adjoint parabolic vector bundle $\text{ad}(E_G)$ for the ramified $G$-bundle $E_G$. The Higgs field $\theta$ produces a Higgs field on the parabolic vector bundle $\text{ad}(E_G)$. This induced Higgs field on $\text{ad}(E_G)$ will be denoted by $\text{ad}(\theta)$.

**Theorem 12.** Let $(E_G, \theta)$ be a parabolic Higgs $G$-bundle on $X$ such that $(E_G, \theta)$ is polystable with respect to the Kähler form $\omega_0$ (see (3.2)), and satisfies the following two conditions:

- The Higgs bundle $(\mathcal{E}, \theta')$ constructed in (4.1) and (4.3) is polystable, and
- for the graded pieces $(\text{Gr}_\alpha \text{ad}(E_G), \text{ad}(\theta)_\alpha)$ of $(\text{ad}(E_G)|_{\tilde{D}}, \text{ad}(\theta)|_{\tilde{D}})$, the condition
\[ (4.4) \quad \mu(\text{Gr}_\alpha \text{ad}(E_G)) = -\alpha \deg(N) \]
holds, where degrees are computed using $\omega_0$ and $N$ is the normal bundle of $D$.

Then there is a Hermitian-Einstein connection on $E_G$ over $X \setminus D$ with respect to the Poincaré-type metric described in Section [5].

**Proof.** We first note that it is enough to prove the theorem under the stronger assumption that the parabolic Higgs $G$-bundle $(E_G, \theta)$ is stable. Indeed, a polystable parabolic Higgs $G$-bundle $(E_G, \theta)$ admits a reduction of structure group $E_{L(P)} \subset E_G$ to a Levi subgroup $L(P)$ of some parabolic subgroup $P$ of $G$ such that the corresponding parabolic Higgs $L(P)$-bundle $(E_{L(P)}, \theta)$ is stable (see Definition 9). The connection on $E_G$ induced by a Hermitian-Einstein connection on $E_{L(P)}$ is again Hermitian-Einstein. Hence it suffices to prove the theorem for $(E_G, \theta)$ stable.

Henceforth, in the proof we assume that $(E_G, \theta)$ is stable.

We will now show that it is enough to prove the theorem under the assumption that $G$ is semisimple.
As before, \( Z_0(G) \subset G \) is the connected component, containing the identity element, of the center of \( G \). The normal subgroup \([G, G] \subset G\) is semisimple, because \( G \) is reductive. We have natural homomorphisms

\[
Z_0(G) \times [G, G] \rightarrow G 

\rightarrow (G/Z_0(G)) \times (G/[G, G]).
\]

Both the homomorphisms are surjective with finite kernel. In particular, both the homomorphisms of Lie algebras are isomorphisms. Let \( \rho : A \rightarrow B \) be a homomorphism of Lie groups such that the corresponding homomorphism of Lie algebras is an isomorphism, let \( E_A \) be a principal \( A \)-bundle, and let \( E_B := E_A \times^\rho B \) be the principal \( B \)-bundle obtained by extending the structure group of \( E_A \) using \( \rho \). Then there is a natural bijective correspondence between the connections on \( E_A \) and the connections on \( E_B \). The curvature of a connection on \( E_B \) is given by the curvature of the corresponding connection on \( E_A \) using the homomorphism of Lie algebras associated to \( \rho \).

Therefore, to prove the theorem for \( G \), it is enough to prove it for \( G/Z_0(G) \) and \( G/[G, G] \) separately. But \( G/[G, G] \) is a product of copies of \( \mathbb{C}^* \), hence in this case the theorem follows immediately from Theorem \( \text{II} \). The group \( G/Z_0(G) \) is semisimple. Hence it is enough to prove the theorem under the assumption that \( G \) is semisimple.

Henceforth, in the proof we assume that \( G \) is semisimple.

Denote by \( \eta : Y \rightarrow X \) the Galois covering with Galois group \( \Gamma := \text{Gal}(\eta) \) and by \( F_G \) the \( \Gamma \)-linearized \( G \)-bundle on \( Y \) corresponding to \( E_G \) as described in Section 2. According to [Bi08, Proposition 4.1], the Higgs field \( \theta \) on \( E_G \) corresponds to a \( \Gamma \)-invariant Higgs field \( \tilde{\theta} \) on \( F_G \). This induces a \( \Gamma \)-invariant Higgs field \( \text{ad}(\tilde{\theta}) \) on the \( \Gamma \)-linearized vector bundle \( \text{ad}(F_G) \). By [Bi97a, Theorem 5.5], this in turn corresponds to a Higgs field \( \text{ad}(\theta) \) on the parabolic vector bundle \( \text{ad}(E_G) \). This way we construct the parabolic Higgs vector bundle \((\text{ad}(E_G), \text{ad}(\theta))\) on \( X \) defined earlier.

The strategy of the proof is to show that the hypotheses of Biquard’s Theorem \( \text{II} \) are satisfied for \((\text{ad}(E_G), \text{ad}(\theta))\) and that the resulting Hermitian-Einstein connection on \( \text{ad}(E_G)|_{X \setminus D} \) is induced by a Hermitian-Einstein connection on \( E_G|_{X \setminus D} \).

First we show that \((\text{ad}(E_G), \text{ad}(\theta))\) is parabolic polystable. Since \((E_G, \theta)\) is stable by hypothesis, it follows as in [Bi08, Lemma 4.2] that \((F_G, \tilde{\theta})\) is \( \Gamma \)-stable. In [AB01] it was shown that if a principal \( G \)-bundle \( E^1_G \) is stable, then its adjoint vector bundle \( \text{ad}(E^1_G) \) is polystable (see [AB01, p. 212, Theorem 2.6]). The proof in [AB01] goes through if \((E^1_G, \theta^1)\) is \( \Gamma \)-stable, and gives that \((\text{ad}(E^1_G), \text{ad}(\theta^1))\) is \( \Gamma \)-polystable. Since the proof goes through verbatim with obvious modifications due to the Higgs field, we refrain from repeating the proof. Therefore, we have \((\text{ad}(F_G), \text{ad}(\tilde{\theta}))\) to be \( \Gamma \)-polystable.

Since \((\text{ad}(F_G), \text{ad}(\tilde{\theta}))\) is \( \Gamma \)-polystable, the parabolic Higgs vector bundle

\[
(\text{ad}(E_G), \text{ad}(\theta))
\]

is parabolic polystable (see [Bi97a, p. 611, Theorem 5.5]).

Let \( M \) be a reductive complex linear algebraic group. The connected component, containing the identity element, of the center of \( M \) will be denoted by \( Z_0(M) \). Let \((E_M, \theta_M)\) be a polystable principal Higgs \( M \)-bundle on a connected complex projective
manifold. If \( V \) is a complex \( M \)-module such that \( Z_0(M) \) acts on \( V \) as scalar multiplications through a character of \( Z_0(M) \), then it is known that the associated Higgs vector bundle \((E_M \times^M V, \theta_V)\) is polystable, where \( \theta_V \) is the Higgs field on the associated vector bundle \( E_M \times^M V \) defined by \( \theta_M \). Indeed, this follows immediately from the fact that \((E_M, \theta_M)\) has a Hermitian-Einstein connection; note that the connection on \((E_M \times^M V, \theta_V)\) induced by a Hermitian-Einstein connection on \((E_M, \theta_M)\) is also Hermitian-Einstein, provided the above condition for the action of \( Z_0(M) \) on \( V \) holds. (See [AB01, p. 227, Theorem 4.10] for the Hermitian-Einstein connection on \((E_M, \theta_M)\).

Since the Higgs bundle \((\mathcal{E}, \theta')\) constructed in (4.1) and (4.3) is given to be polystable, from the above observation it follows that each of the graded pieces \((\text{Gr}_\alpha \text{ad}(E_G), \text{ad}(\theta)_\alpha)\) of \((\text{ad}(E_G), \text{ad}(\theta))\) is polystable.

Since \( G \) is semisimple, the Killing form on its Lie algebra \( g \) is nondegenerate and thus induces an isomorphism \( \text{ad}(F_G) \simeq \text{ad}(F_G)^* \). This implies that \( \text{deg}(\text{ad}(F_G)) = 0 \). By [Bi97b, p. 318, (3.12)], we have

\[
\#\Gamma \cdot \text{par-deg}(\text{ad}(E_G)) = \text{deg}(\text{ad}(F_G)),
\]

and thus \( \text{par-deg}(\text{ad}(E_G)) = 0 \), or equivalently, \( \text{par-\mu}(\text{ad}(E_G)) = 0 \). Consequently, the hypothesis (4.4) on the slopes of the graded pieces implies that the condition (3.4) in Theorem [11] holds for the bundle \( \text{ad}(E_G) \). Therefore, we obtain from Theorem [11] a Hermitian-Einstein metric on \( \text{ad}(E_G) \) over \( X \setminus D \) with respect to the Poincaré-type metric.

Finally we have to show that the corresponding Hermitian-Einstein connection on \( \text{ad}(E_G) \) is induced by a connection on the principal Higgs \( G \)-bundle \( E_G|_{X \setminus D} \); we note that if \( \nabla \) is a connection on \( E_G|_{X \setminus D} \) inducing the Hermitian-Einstein connection on \( \text{ad}(E_G) \), then \( \nabla \) is automatically Hermitian-Einstein.

Let

\[
\Phi \in H^0(X \setminus D, (\text{ad}(E_G) \otimes \text{ad}(E_G))^* \otimes \text{ad}(E_G))
\]

be the section defining the Lie bracket operation on \( \text{ad}(E_G) \). It can be shown that a connection \( \nabla_{\text{ad}} \) on \( \text{ad}(E_G)|_{X \setminus D} \) is induced by a connection of \( E_G|_{X \setminus D} \) if and only if \( \Phi \) is parallel with respect to the connection on \( (\text{ad}(E_G) \otimes \text{ad}(E_G))^* \otimes \text{ad}(E_G) \) induced by \( \nabla_{\text{ad}} \). Indeed, this follows from the fact that \( G \) being semisimple the Lie algebra of the group of Lie algebra preserving automorphisms of \( g \) coincides with \( g \) (see proof of Theorem 3.7 of [AB01]).

Therefore, to complete the proof of the theorem it suffices to show that \( \Phi \) is parallel with respect to the connection on \( (\text{ad}(E_G) \otimes \text{ad}(E_G))^* \otimes \text{ad}(E_G) \) induced by a Hermitian-Einstein connection on \( \text{ad}(E_G) \).

Since \( \text{par-deg}(\text{ad}(E_G)) = 0 \), it follows that

\[
\text{par-deg}((\text{ad}(E_G) \otimes \text{ad}(E_G))^* \otimes \text{ad}(E_G)) = 0.
\]

The connection on \( (\text{ad}(E_G) \otimes \text{ad}(E_G))^* \otimes \text{ad}(E_G) \) induced by the Hermitian-Einstein connection on \( \text{ad}(E_G) \) is also a Hermitian-Einstein connection. Since the Higgs field \( \text{ad}(\theta) \) is induced by the Higgs field \( \theta \) on \( E_G \), it follows that \( \Phi \) is annihilated by the induced
Higgs field on \((\text{ad}(E_G) \otimes \text{ad}(E_G))^* \otimes \text{ad}(E_G)\). Thus the proof of Theorem 12 is completed by Lemma 13.

**Lemma 13.** Let \((E, \theta)\) be a parabolic Higgs vector bundle on \(X \setminus D\) admitting a Hermitian-Einstein connection \(\nabla\) with respect to the Poincaré-type metric. Assume that \((E, \theta)\) is polystable, and par-deg \(E = 0\). Let \(s\) be a holomorphic section of \(E\) such that \(\theta(s) = 0\). Then \(s\) is parallel with respect to \(\nabla\).

**Proof.** Fix a Galois covering \(\eta : Y \to X\) such that there is a \(\Gamma\)-linearized Higgs vector bundle \((V, \varphi)\) on \(Y\) that corresponds to \((E, \theta)\), where \(\Gamma = \text{Gal}(\eta)\). Fix the polarization \(\eta^* \zeta\) on \(Y\), where \(\zeta\) is the polarization on \(X\).

We know that \((V, \varphi)\) is \(\Gamma\)-polystable because \((E, \theta)\) is polystable. Therefore, \((V, \varphi)\) admits a Hermitian-Einstein connection [Si88, p. 978, Theorem 1].

Let \(\tilde{s}\) be the holomorphic section of \(V\) over \(Y\) given by \(\theta\). We note that \(\varphi(\tilde{s}) = 0\) because \(\theta(s) = 0\). We have \(\deg V = 0\) because par-deg \(E = 0\) [Bi97b, p. 318, (3.12)]. Since \((V, \varphi)\) admits a Hermitian-Einstein connection with \(\deg V = 0\), and \(\varphi(\tilde{s}) = 0\), it follows that the holomorphic section \(\tilde{s}\) is flat with respect to the Hermitian-Einstein connection on \((V, \varphi)\) [BS09, p. 548, Lemma 3.4].

If \(s\) vanishes identically, then the lemma is obvious. Assume that \(s\) does not vanish identically. Since \(\tilde{s}\) is flat with respect to the Hermitian-Einstein connection on \((V, \varphi)\), the section \(\tilde{s}\) does not vanish at any point of \(Y\). Let \(L\tilde{s} \subset V\) be the holomorphic line subbundle generated by \(\tilde{s}\). The action of \(\Gamma\) on \(V\) clearly preserves \(L\tilde{s}\). Since \((V, \varphi)\) is \(\Gamma\)-polystable, this implies that there is a \(\Gamma\)-polystable Higgs vector bundle \((V', \varphi')\) such that

\[ (V, \varphi) = (V', \varphi') \oplus (L\tilde{s}, 0) \]

as \(\Gamma\)-linearized Higgs vector bundles.

The above decomposition of the \(\Gamma\)-linearized Higgs vector bundle \((V, \varphi)\) produces a decomposition

\[ (E, \theta) = (E', \theta') \oplus (Ls, 0) \]

of the parabolic Higgs vector bundle; the line subbundle \(Ls\) of \(E\) is generated by \(s\).

The direct sum of the Hermitian-Einstein connections on \((E', \theta')\) and \((Ls, 0)\) is a Hermitian-Einstein connection on \((E, \theta)\). Therefore, from the uniqueness of the Hermitian-Einstein connection (see the second part of Theorem 11) it follows immediately that \(s\) is parallel with respect to the Hermitian-Einstein connection \(\nabla\).

There is also a converse to Theorem 12.

**Proposition 14.** Let \((E_G, \theta)\) be a parabolic Higgs \(G\)-bundle on \(X\). Suppose there is a Hermitian-Einstein connection on \(E_G\) over \(X \setminus D\) with respect to the Poincaré-type metric \(\omega\) such that the induced connection on the adjoint vector bundle \(\text{ad}(E_G)|_{X \setminus D}\) lies in the space \(A\) (see (3.3)). Then \((E_G, \theta)\) is polystable with respect to \(\omega_0\).

**Proof.** By [Br97, Proposition 7.2] we know that the parabolic degree of a parabolic sheaf on \(X\) with respect to \(\omega_0\) coincides with the degree of its restriction to \(X \setminus D\) with respect
to $\omega$, computed using a Hermitian metric with Chern connection in $\mathcal{A}$. Thus the proof in [RS88, pp. 28–29] of the proposition for ordinary principal bundles generalizes to our situation of parabolic Higgs $G$-bundles.

□

References

[AB01] B. Anchouche and I. Biswas: Einstein-Hermitian connections on polystable principal bundles over a compact Kähler manifold, Am. Jour. Math. 123, 207–228 (2001).

[At57] M. F. Atiyah: Complex analytic connections in fibre bundles, Trans. Amer. Math. Soc. 85, 181–207 (1957).

[BBN01] V. Balaji, I. Biswas and D. S. Nagaraj: Principal bundles over projective manifolds with parabolic structure over a divisor, Tohoku Math. Jour. 53, 337–367 (2001).

[BBN03] V. Balaji, I. Biswas and D. S. Nagaraj: Ramified $G$-bundles as parabolic bundles, Jour. Ramanujan Math. Soc. 18, 123–138 (2003).

[Bq97] O. Biquard: Fibres de Higgs et connexions intégrables: Le cas logarithmique (diviseur lisse), Ann. Sci. Éc. Norm. 30, 41–96 (1997).

[Bi97a] I. Biswas: Chern classes for parabolic bundles, Jour. Math. Kyoto Univ. 37, 597–613 (1997).

[Bi97b] I. Biswas: Parabolic bundles as orbifold bundles, Duke Math. Jour. 88, 305–325 (1997).

[Bi06] I. Biswas: Connections on a Parabolic Principal Bundle over a Curve, Can. Jour. Math. 58, 262–281 (2006).

[Bi08] I. Biswas: Parabolic principal Higgs bundles, Jour. Ramanujan Math. Soc. 23, 311–325 (2008).

[BS09] I. Biswas and G. Schumacher: Yang–Mills equation for stable Higgs sheaves, Inter. Jour. Math. 20, 541–556 (2009).

[Do87] S. K. Donaldson: Infinite determinants, stable bundles and curvature, Duke Math. Jour. 54, 231–247 (1987).

[Hu95] J. E. Humphreys: Conjugacy classes in semisimple algebraic groups, Mathematical Surveys and Monographs, 43. American Mathematical Society, Providence, RI, 1995.

[MY92] M. Maruyama and K. Yokogawa: Moduli of parabolic stable sheaves, Math. Ann. 293, 77–99 (1992).

[No76] M. V. Nori: On the representations of the fundamental group, Compos. Math. 33, 29–41 (1976).

[RS88] A. Ramanathan and S. Subramanian: Einstein-Hermitian connections on principal bundles and stability, Jour. Reine Angew. Math. 390, 21–31 (1988).

[Se77] C. S. Seshadri: Moduli of vector bundles on curves with parabolic structures, Bull. Am. Math. Soc. 83, 124–126 (1977).

[Si88] C. T. Simpson: Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization, Jour. Amer. Math. Soc. 1, 867–918 (1988).

[Si92] C. T. Simpson: Higgs bundles and local systems, Inst. Hautes Études Sci. Publ. Math. 75, 5–95 (1992).

[UY86] K. Uhlenbeck and S.-T. Yau: On the existence of Hermitian–Yang–Mills connections on stable vector bundles, Commun. Pure Appl. Math. 39, 257–293 (1986).