Limit Theorems for Default Contagion and Systemic Risk

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Motivation

- Financial institutions are becoming more and more connected to each other.
- The size and diversity of the financial system are also becoming larger and larger.
- This leads to a significant systemic risk.
- **Financial default contagion:** The bankruptcies of some institutions bring loss to its neighbors, might leading new insolvency and propagating through the network.
- Even a small part of institutions' defaults can trigger a large default cascade.
Main Issues

• In many cases, the size of financial network is large.
• Heterogeneity (diversity) is high in the financial network.
• Partial informations available:
  • We do not know the structure of linkages in the network;
  • We know partial characteristics of the institutions: the total assets and liabilities, the total number of out-links and in-links...

Our concerns: Using limit theorems to

• Analyse the network structure at the end of the contagion
• Quantify the systemic risk of the network
• Try to minimize the systemic risk
Overview

Model

Limit theorems

Quantifying Systemic Risk

Targeting Interventions in Financial Networks
• **Interbank liability:** For two financial institutions $i, j \in [n]$, $\ell_{ij} \geq 0$ denotes the cash-amount that bank $i$ owes bank $j$.

• A link from $i$ to $j$ means that there is interbank liability from $i$ to $j$, i.e. $\ell_{ij} > 0$. 
The **capital structure** of institution $i$ in the network:

- **Assets**: Interbank asset $a_i = \sum_{j \in [n]} \ell_{ji}$, external assets $e_i$, cash $h_i$.
- **Liabilities**: Interbank liability $\ell_i = \sum_{j \in [n]} \ell_{ij}$.
- **Shock scenario**: a fraction of loss $\epsilon_i$ in external assets.

**Capital before shock**: $c_i(\epsilon) = e_i + a_i + h_i - \ell_i$.

**Capital after shock**: $c_i(\epsilon) = (1 - \epsilon_i)e_i + a_i + h_i - \ell_i$. 
Default cascade

Shock scenario $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in [0, 1]^n$ for size $n$ network, the set of fundamental defaults:

$$D_0(\epsilon) = \{ i \in [n] : c_i(\epsilon_i) < 0 \}.$$  

Liability recovery rates matrix $R = R_{ij}$, satisfying:

$$h_i + (1 - \epsilon_i)e_i + \sum_{j=1}^{n} R_{ji} \ell_{ji} \geq \sum_{j=1}^{n} R_{ij} \ell_{ij}.$$  

Default cascade: evolution of the defaulted set at $k$ step

$$D_k = D_k(\epsilon, R) = \{ i \in [n] : c_i(\epsilon_i) < \sum_{j \in D_{k-1}} (1 - R_{ji}) \ell_{ji} \}.$$  

$D_k \nearrow$, can not be larger than $[n]$. There is a final set of defaulted institutions $D^*$. 
It is natural to model financial network as a random graph, where all institutions are nodes and they connect to others uniformly at random through directed edges.

- In-degree sequence $d_n^+ = (d_1^+, \ldots, d_n^+)$ and out-degree sequence $d_n^- = (d_1^-, \ldots, d_n^-)$.
- $\sum_{i \in [n]} d_i^+ = \sum_{i \in [n]} d_i^-$.

In the above figure, $(d_1^+, d_1^-) = (3, 3)$, $(d_2^+, d_2^-) = (3, 2)$. 
- A **configuration** is a matching of all out half-edges with all in half-edges.
- The **configuration model** is the random directed multigraph which is uniformly distributed across all possible configurations.

The above figure is a configuration between four nodes with degree 
$$(d_1^+, d_1^-) = (2, 3), (d_2^+, d_2^-) = (3, 1), (d_3^+, d_3^-) = (2, 2), (d_4^+, d_4^-) = (2, 3).$$
Default threshold: For a node $i$, the default threshold is the maximum number of defaulted neighbours that $i$ can tolerate before becoming defaulted, provided that its counterparties default in an order that is uniformly at random.

- Similar to Amini, Cont, Minca 2016, the information regarding assets, liabilities, capital after exogenous shocks and recovery rates (distributions) could all be encoded in a single probability threshold function.
- Each node $i$ has a random threshold $\Theta^{(n)}(i)$ with certain distribution.
- To reduce the dimensionality, consider a classification of financial institutions into a countable (finite or infinite) set of characteristics $\mathcal{X}$.
- For each $x \in \mathcal{X}$, it contains all observable informations for the financial institutions:
  \[ x = (d_x^+, d_x^-, e_x, h_x, \ldots). \]
- Any institutions belongs to one characteristics,
  \[ x_i^{(n)} = (d_i^+, d_i^-, e_i, h_i, \ldots) \in \mathcal{X}. \]
Finding the final solvent institutions

We call a link coming from a defaulted node as infected link. Using the default threshold, the default set $\mathcal{D}_k$ can be identified by

$$\mathcal{D}_k = \left\{ i \in [n] : \sum_{j : j \to i} 1 \{ j \in \mathcal{D}_{k-1} \} \geq \Theta_i \right\},$$

The dynamics of contagion:

- We reveal the infected links one by one.
- We can set the duration between two successive reveals as we want.
Death process of balls and bins

Regard all nodes as bins and all half-edges as balls. We have the following types:

- **Bins**: $D$ (defaulted), $S$ (solvent).
- **Balls**: $H^+$ (healthy in), $H^-$ (healthy out), $I^+$ (infected in), $I^-$ (infected out).

Balls’ death and colouring:

- Initially, all $I^-$ balls white, all $H^+ \cup I^+$ (in) balls alive, and randomly recolor a white ball red.
- From time 0 on, in balls start to die randomly.
- If there are $\ell$ in balls remaining, next random death for an in ball is after a exponential time with mean $1/\ell$;
- we recolor a white ball red randomly at the same moment when an in ball dies.
Death process of balls and bins

• Denote by $W_n(t)$ the number of white balls at time $t$.

• The contagion stops when all infected links are revealed, denote the stopping time by $\tau_n^\star$.

• Infected links $\rightarrow$ White balls $\rightarrow$ Institutions $\rightarrow$ Bins
Reveal an infected link $\rightarrow$ An in ball’s death + Coloring a white ball

• Let $S_{x,\theta,\ell}^{(n)}(t)$ be the number of solvent institutions (bins) with type $x$, threshold $\theta$ and $\ell$ defaulted neighbors at time $t$.

• Let $S_n(t)$ and $D_n(t)$ be the number of solvent (defaulted) institutions et time $t$ respectively,

\[
S_n(t) = \sum_{x \in \mathcal{X}} \sum_{\theta=1}^{d_x^+} \sum_{\ell=0}^{\theta-1} S_{x,\theta,\ell}^{(n)}(t).
\]
LLN of the default contagion

Assumptions

- The institutions in the same characteristic class have the same threshold distribution function independently. Namely, for all \( x \in \mathcal{X} \), \( \theta = 0, 1, \ldots, d^+_x \):

\[
P(\Theta_x^{(n)} = \theta) = q_x^{(n)}(\theta).
\]

- For some probability distribution functions \( \mu \) and \( q_x \) over the set of characteristics \( \mathcal{X} \) and independent of \( n \), we have \( \mu_x^{(n)} \rightarrow \mu_x \) and \( q_x^{(n)}(\theta) \rightarrow q_x(\theta) \) as \( n \rightarrow \infty \), for all \( x \in \mathcal{X} \) and \( \theta = 0, 1, \ldots, d^+_x \).

- As \( n \rightarrow \infty \), the average degree converges:

\[
\lambda^{(n)} := \sum_{x \in \mathcal{X}} d^+_x \mu_x^{(n)} = \sum_{x \in \mathcal{X}} d^-_x \mu_x^{(n)} \rightarrow \lambda := \sum_{x \in \mathcal{X}} d^+_x \mu_x \in (0, \infty).
\]
LLN of the default contagion

**Theorem 1**

Let \( \tau_n \leq \tau_n^* \) be a stopping time such that \( \tau_n \xrightarrow{p} t_0 \) for some \( t_0 > 0 \). Then for all \((x, \theta, \ell)\), we have (as \( n \to \infty \))

\[
\sup_{t \leq \tau_n} \left| \frac{S_{x,\theta,\ell}^{(n)}(t)}{n} - \mu_x q_x(\theta) b \left( d_x^+, 1 - e^{-t}, \ell \right) \right| \xrightarrow{p} 0,
\]

\[
\sup_{t \leq \tau_n} \left| \frac{S_n(t)}{n} - f_S(e^{-t}) \right| \xrightarrow{p} 0,
\]

\[
\sup_{t \leq \tau_n} \left| \frac{D_n(t)}{n} - f_D(e^{-t}) \right| \xrightarrow{p} 0,
\]

\[
\sup_{t \leq \tau_n} \left| \frac{W_n(t)}{n} - f_W(e^{-t}) \right| \xrightarrow{p} 0.
\]
The limit function of $S_n(t)/n$, $D_n(t)/n$ and $W_n(t)/n$ are given by:

$$f_S(e^{-t}) := \sum_{x \in \mathcal{X}} \mu_x \sum_{\theta=1}^{d_x^+} q_x(\theta) \beta(d_x^+, e^{-t}, d_x^+ - \theta + 1), \quad f_D(e^{-t}) = 1 - f_S(e^{-t}),$$

$$f_W(e^{-t}) := \lambda e^{-t} - \sum_{x \in \mathcal{X}} \mu_x d_x^- \sum_{\theta=1}^{d_x^+} q_x(\theta) \beta(d_x^+, e^{-t}, d_x^+ - \theta + 1).$$

where

$$b(d, z, \ell) := \mathbb{P}(\text{Bin}(d, z) = \ell) = \binom{d}{\ell} z^\ell (1 - z)^{d-\ell},$$

$$\beta(d, z, \ell) := \mathbb{P}(\text{Bin}(d, z) \geq \ell) = \sum_{r=\ell}^{d} \binom{d}{r} z^r (1 - z)^{d-r},$$

and Bin$(d, z)$ denotes the binomial distribution with parameters $d$ and $z$.

**In fact**, we obtained LLN for all quantities regarding the network structure, even for the numbers of balls of four different types $H^+, H^-, I^+, I^-$. 

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Define further,

\[ z^* := \sup \{ z \in [0, 1] : f_W(z) = 0 \}. \]

Since the white ball process \( W_n(t) \) control the stopping time of the contagion dynamics and \( f_W(e^{-t}) \) is the limit function of \( W_n(t)/n, z^* \) should be the limit of \( e^{-\tau_n^*} \).

In fact, we show that (as \( n \to \infty \)):

(i) If \( z^* = 0 \) then \( \tau_{n}^* \xrightarrow{p} \infty \).

(ii) If \( z^* \in (0, 1] \) and \( z^* \) is a stable solution, i.e. \( f_W'(z^*) > 0 \), then \( \tau_{n}^* \xrightarrow{p} -\ln z^* \).
LLN of the final structure

Theorem 2
The final fraction of defaulted institutions satisfies:

(i) If \( z^* = 0 \) then asymptotically almost all institutions default during the cascade and
\[
D_n(\tau_n^*) = n - o_p(n).
\]

(ii) If \( z^* \in (0, 1] \) and \( f'_W(z^*) > 0 \), then
\[
\frac{D_n(\tau_n^*)}{n} \xrightarrow{p} f_D(z^*).
\]

Further,
\[
\frac{S_{x,\theta,\ell}(\tau_n^*)}{n} \xrightarrow{p} \mu_x q_x(\theta) b(d_x^+, 1 - z^*, \ell).
\]
Quantifying Systemic Risk

Use some aggregation functions to measure the systemic risk:

- **Number of solvent banks**: $\Gamma_n^\#(t) := S_n(t) = n - D_n(t)$.

- **External wealth**: Let $\bar{\Gamma}_n^\odot$ denotes the total external wealth to society if there is no default in the financial system.

\[
\Gamma_n^\odot(t) := \bar{\Gamma}_n^\odot - \sum_{x \in \mathcal{X}} \bar{L}_x^\odot D_x^{(n)}(t),
\]

where we assume a bounded constant type-dependent societal loss $\bar{L}_x^\odot$ over each defaulted institution.
• **System-wide wealth**: Let $\bar{\Gamma}_n^\diamond$ denote the total wealth in the financial system if there is no default in the system. We define the system-wide aggregation function as

$$\Gamma_n^\diamond(t) := \bar{\Gamma}_n^\diamond - \sum_{x \in \mathcal{X}} \bar{L}_x^\diamond D_x^{(n)}(t) - \sum_{x \in \mathcal{X}} \bar{L}_x^\diamond \sum_{\theta=1}^{d_x^+} \sum_{\ell=1}^{\theta-1} \ell S_{x,\theta,\ell}^{(n)}(t),$$

where we assume a bounded fixed (host institutions’ type-dependent) cost $\bar{L}_x^\diamond$ over each defaulted links. Assume that $\bar{\Gamma}_n^\diamond / n \to \bar{\Gamma}^\diamond$ when the size of network $n \to \infty$.

The corresponding limit function should be:

$$f^\diamond(z) := \bar{\Gamma}^\diamond - \sum_{x \in \mathcal{X}} \bar{L}_x^\diamond f_{D_x}(z) - \sum_{x \in \mathcal{X}} \bar{L}_x^\diamond \sum_{\theta=1}^{d_x^+} \sum_{\ell=1}^{\theta-1} \ell s_{x,\theta,\ell}(z),$$

where $f_{D_x}(z) := \mu_x \left(1 - \sum_{\theta=1}^{d_x^+} q_x(\theta) \beta(d_x^+, e^{-t}, d_x^+ - \theta + 1)\right)$. 

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Theorem 6 (LLN for systemic risk)

Let Assumptions before hold and \( \tau_n \leq \tau^*_n \) be a stopping time such that \( \tau_n \xrightarrow{p} t_0 \) for some \( t_0 > 0 \). Then, as \( n \to \infty \),

\[
\sup_{t \leq \tau_n} \left| \frac{\Gamma_n^{\diamond}(t)}{n} - f^{\diamond}(e^{-t}) \right| \xrightarrow{p} 0.
\]

Further, the final (system-wide) aggregation functions satisfy:

(i) If \( z^* = 0 \) then asymptotically almost all institutions default during the cascade and

\[
\frac{\Gamma_n^{\diamond}(\tau^*_n)}{n} \xrightarrow{p} \bar{\Gamma}^{\diamond} - \sum_{x \in \mathcal{X}} \mu_x \bar{L}^{\diamond}_x.
\]

(ii) If \( z^* \in (0, 1] \) and \( z^* \) is a stable solution, i.e. \( f'_W(z^*) > 0 \), then

\[
\frac{\Gamma_n^{\diamond}(\tau^*_n)}{n} \xrightarrow{p} f^{\diamond}(z^*).
\]
Consider a planner (lender of last resort or government) who seeks to minimize the systemic risk at the beginning of the financial contagion, after an exogenous macroeconomic shock $\epsilon$, subject to a budget constraint.

- The planner only has information regarding the type of each institution and, consequently, the institutions’ threshold distributions.
- The planner’ decision is only based on the type of each institution.
- Intervene an infected link means save an infected link (or remove this link from the financial network).
- These interventions will be type-dependent and at random over all defaulted links leading to the same type institutions.
- Denote by $\alpha_x^{(n)}$ the planner intervention decision on the fraction of the saved links leading to any institution of type $x \in \mathcal{X}$.
- The cost to save an infected link of type $x$ is $C_x$. 
Let us define

\[ f_{W}^{(\alpha)}(z) := \lambda z - \sum_{x \in \mathcal{X}} \mu_{x} d_{x}^{-} \sum_{\theta = 1}^{d_{x}^{+}} q_{x}(\theta) \beta(d_{x}^{+}, \alpha_{x} + (1 - \alpha_{x})z, d_{x}^{+} - \theta + 1), \]

and,

\[ z_{\alpha}^{*} := \sup\{ z \in [0, 1] : f_{W}^{(\alpha)}(z) = 0 \}. \]

• Intuitively, in the intervened network, we have a different probability that a link is infected compared with the original network. Namely, in the original network, the links that come from a defaulted institution is infected with probability 1, while in the intervened one, the probability is \((1 - \alpha_{x}^{(n)})\).

• Save an infected link means that we let an in ball which should have died remain alive.

• each in ball has a probability \( \alpha_{x}^{(n)} + (1 - \alpha_{x}^{(n)})e^{-t} \) to stay alive before time \( t \).
Let Assumptions hold and $\alpha_n \to \alpha$ as $n \to \infty$. If $z^*_\alpha$ is a stable solution,

(i) For all $x \in X, \theta = 1, \ldots, d^+_x$ and $\ell = 0, \ldots, \theta - 1$, the final fraction of solvent institutions with type $x$, threshold $\theta$ and $\ell$ defaulted neighbors under intervention $\alpha_n$ converges to

$$S^{(n)}_{x,\theta,\ell}(\alpha_n) \xrightarrow{n \to \infty} S_{x,\theta,\ell}(z^*_\alpha) := \mu_x q_x(\theta) b (d^+_x, (1 - \alpha_x)(1 - z^*_\alpha), \ell).$$

(ii) The total number of defaulted institutions under intervention $\alpha_n$ converges to:

$$D_n(\alpha_n) \xrightarrow{n \to \infty} f_D(\alpha)(z^*_\alpha) := 1 - \sum_{x \in X} \mu_x \sum_{\theta = 1}^{d^+_x} q_x(\theta) \beta (d^+_x, \alpha_x + (1 - \alpha_x)z^*_\alpha, d^+_x - \theta + 1).$$

(iii) The system-wide wealth under the intervention decision $\alpha_n$ converges to

$$\Gamma_n(\alpha_n) \xrightarrow{n \to \infty} f_\gamma(\alpha)(z^*_\alpha) := \bar{\Gamma} - \sum_{x \in X} \bar{L}_x f_D(\alpha)(z^*_\alpha) - \sum_{x \in X} \sum_{\theta = 1}^{d^+_x} \sum_{\ell = 1}^{\theta - 1} \ell s^{(\alpha)}_{x,\theta,\ell}(z^*_\alpha).$$

(iv) The total cost of interventions $\alpha_n$ for the planner converges to

$$\Phi_n(\alpha_n) \xrightarrow{n \to \infty} \phi(z^*_\alpha) := \sum_{x \in X} \mu_x \alpha_x C_x \sum_{\ell = 1}^{d^+_x} \ell b (d^+_x, 1 - z^*_\alpha, \ell).$$
Planner optimal decision

\[
\max_{\alpha} f^{(\alpha)}_{\hat{\phi}}(z^{\star}_{\alpha}) := \bar{\Gamma}_{\hat{\phi}} - \sum_{x \in \mathcal{X}} \bar{L}^{x}_{\hat{\phi}} f^{(\alpha)}_{D}(z^{\star}_{\alpha}) - \sum_{x \in \mathcal{X}} \bar{L}^{x}_{\hat{\phi}} \sum_{\theta=1}^{D^{+}_{x}} \sum_{\ell=1}^{\theta-1} \ell s^{(\alpha)}_{x,\theta,\ell}(z^{\star}_{\alpha}),
\]

subject to \( \phi(z^{\star}_{\alpha}) := \sum_{x \in \mathcal{X}} \mu_{x} \alpha_{x} C_{x} \sum_{\ell=1}^{D^{+}_{x}} \ell b(d^{+}_{x}, 1 - z^{\star}_{\alpha}, \ell) \leq C, \)

for some budget constraint \( C > 0. \)
Thank you