On a problem of Neumann

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Abstract

A conjecture widely attributed to Neumann is that all finite non-desarguesian projective planes contain a Fano subplane. In this note, we show that any finite projective plane of even order which admits an orthogonal polarity contains a Fano subplane. The number of planes of order less than \( n \) previously known to contain a Fano subplane was \( O(\log n) \), whereas the number of planes of order less than \( n \) that our theorem applies to is not bounded above by any polynomial in \( n \).

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1 Introduction

A fundamental question in incidence geometry is about the subplane structure of projective planes. There are relatively few results concerning when a projective plane of order \( k \) is a subplane of a projective plane of order \( n \). Neumann [9] found Fano subplanes in certain Hall planes, which led to the conjecture that every finite non-desarguesian plane contains \( \text{PG}(2,2) \) as a subplane (this conjecture is widely attributed to Neumann, though it does not appear in her work).

Johnson [7] and Fisher and Johnson [4] showed the existence of Fano subplanes in many translation planes. Petrak [10] showed that Figueroa planes contain \( \text{PG}(2,2) \) and Caliskan and Petrak [8] showed that Figueroa planes of odd order contain \( \text{PG}(2,3) \). Caliskan and Moorhouse [2] showed that all Hughes planes contain \( \text{PG}(2,2) \) and that the Hughes plane of order \( q^2 \) contains \( \text{PG}(2,3) \) if \( q \equiv 5 \pmod{6} \). We prove the following.

Theorem 1. Let \( \Pi \) be a finite projective plane of even order which admits an orthogonal polarity. Then \( \Pi \) contains a Fano subplane.

Ganley [5] showed that a finite semifield plane admits an orthogonal polarity if and only if it can be coordinatized by a commutative semifield. A result of Kantor [8] implies that the number of nonisomorphic planes of order \( n \) a power of 2 that can be coordinatized by a commutative semifield is not bounded above by any polynomial in \( n \). Thus, Theorem 1 applies to many projective planes.

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2 Proof of Theorem 1

The proof of Theorem 1 is graph theoretic, and we collect some definitions and results first. Let \( \Pi = (\mathcal{P}, \mathcal{L}, \mathcal{I}) \) be a projective plane of order \( n \). We write \( p \in l \) or say \( p \) is on \( l \) if \( (p, l) \in \mathcal{I} \). Let \( \pi \) be a polarity of \( \Pi \). That is, \( \pi \) maps points to lines and lines to points, \( \pi^2 \) is the identity function, and \( \pi \) respects incidence. Then one may construct the polarity graph \( G^\pi \) as follows.

\[
V(G^\pi) = \mathcal{P} \quad \text{and} \quad p \sim q \text{ if and only if } p \in \pi(q).
\]

That is, the neighborhood of a vertex \( p \) is the line \( \pi(p) \) that \( p \) gets mapped to under the polarity. If \( p \in \pi(p) \), then \( p \) is an absolute point and the vertex \( p \) will have a loop on it. A polarity is orthogonal if exactly \( n+1 \) points are absolute. We note that as neighborhoods in the graph represent lines in the geometry, each vertex in \( G^\pi \) has exactly \( n+1 \) neighbors (if \( v \) is an absolute point, it has exactly \( n \) neighbors other than itself). We provide proofs of the following preliminary observations for completeness.

Lemma 1. Let \( \Pi \) be a projective plane with polarity \( \pi \), and \( G^\pi \) be the associated polarity graph.

(a) For all \( u, v \in V(G^\pi) \), \( u \) and \( v \) have exactly 1 common neighbor.
(b) \( G^\pi \) is \( C_4 \) free.
(c) If \( u \) and \( v \) are two absolute points of \( G^\pi \), then \( u \not\sim v \).
(d) If \( v \in V(G^\pi) \), then the neighborhood of \( v \) induces a graph of maximum degree at most 1.
(e) Let \( e = uv \) be an edge of \( G^\pi \) such that neither \( u \) nor \( v \) is an absolute point. Then \( e \) lies in a unique triangle in \( G^\pi \).

Proof. To prove (a), let \( u \) and \( v \) be an arbitrary pair of vertices in \( V(G^\pi) \). Because \( \Pi \) is a projective plane, \( \pi(u) \) and \( \pi(v) \) meet in a unique point. This point is the unique vertex in the intersection of the neighborhood of \( u \) and the neighborhood of \( v \). (b) and (c) follow from (a).

To prove (d), if there is a vertex of degree at least 2 in the graph induced by the neighborhood of \( v \), then \( G^\pi \) contains a 4-cycle, a contradiction by (b).

Finally, let \( u \sim v \) and neither \( u \) nor \( v \) an absolute point. Then by (a) there is a unique vertex \( w \) adjacent to both \( u \) and \( v \). Now \(uvw \) is the purported triangle, proving (e).

Proof of Theorem 1. We will now assume \( \Pi \) is a projective plane of even order \( n \), that \( \pi \) is an orthogonal polarity, and that \( G^\pi \) is the corresponding polarity graph (including loops). Since \( n \) is even and \( \pi \) is orthogonal, a classical theorem of Baer ([1], see also Theorem 12.6 in [6]) says that the \( n+1 \) absolute points under \( \pi \) all lie on one line. Let \( a_1, \ldots, a_{n+1} \) be the set of absolute points and let \( l \) be the line containing them. Then there is some \( p \in \mathcal{P} \) such that \( \pi(l) = p \). This means that in \( G^\pi \), the neighborhood of \( p \) is exactly the set of points \( \{a_1, \ldots, a_{n+1}\} \). For \( 1 \leq i \leq n+1 \), let \( N_i \) be the neighborhood of \( a_i \). Then by Lemma 1(b) \( N_i \cap N_j = \emptyset \) if \( i \neq j \). Further, counting gives that

\[
V(G^\pi) = p \cup \left( \bigcup_{i=1}^{n+1} a_i \right) \cup \left( \bigcup_{i=1}^{n+1} N_i \right).
\]
Let $ER^o_2$ be the graph on 7 points which is the polarity graph (with loops) of $PG(2, 2)$ under the orthogonal polarity.

**Lemma 2.** If $ER^o_2$ is a subgraph of $G^o_\pi$, then $\Pi$ contains a Fano subplane.

**Proof.** Let $v_1, \ldots, v_7$ be the vertices of a subgraph $ER^o_2$ of $G^o_\pi$. Let $l_i = \pi(v_i)$ for $1 \leq i \leq 7$. Then the lines $l_1, \ldots, l_7$ in $\Pi$ restricted to the points $v_1, \ldots, v_7$ form a point-line incidence structure, and one can check directly that it satisfies the axioms of a projective plane. 

Thus, it suffices to find $ER^o_2$ in $G^o_\pi$. To find $ER^o_2$ it suffices to find distinct $i, j, k$ such that there are $v_i \in N_i$, $v_j \in N_j$, and $v_k \in N_k$ where $v_i v_j v_k$ forms a triangle in $G^o_\pi$, for then the points $p, a_i, a_j, a_k, v_i, v_j, v_k$ yield the subgraph $ER^o_2$. Now note that for all $i$, and for $v \in N_i$, $v$ has exactly $n$ neighbors that are not absolute points. There are $n$ choices for $i$ and $n - 1$ choices for $v \in N_i$. As each edge is counted twice, this yields

$$\frac{n(n - 1)(n + 1)}{2}$$

edges with neither end an absolute point. By Lemma 1.e, there are at least

$$\frac{n^3 - n}{6}$$

triangles in $G^o_\pi$. By Lemma 1.c, there are no triangles incident with $p$, by Lemma 1.b, there are no triangles that have more than one vertex in $N_i$ for any $i$, and by Lemma 1.d, there are at most $\left\lfloor \frac{n - 1}{2} \right\rfloor = \frac{n}{2} - 1$ triangles incident with $a_i$ for each $i$. Therefore, by 1.d, there are at least

$$\frac{n^3 - n}{6} - (n + 1) \left(\frac{n}{2} - 1\right)$$

copies of $ER^o_2$ in $G^o_\pi$. This expression is positive for all even natural numbers $n$. 

3
3 Concluding Remarks

First, we note that the proof of Theorem 1 actually implies that there are $\Omega(n^3)$ copies of $PG(2, 2)$ in any plane satisfying the hypotheses, and echoing Petrak [10], perhaps one could find subplanes of order 4 for $n$ large enough. We also note that it is crucial in the proof that the absolute points form a line. When $n$ is odd, the proof fails (as it must, since our proof does not detect if $\Pi$ is desarguesian or not).

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