Strict convexity and $C^1$ regularity of solutions to generated Jacobian equations in dimension two

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Abstract

We present a proof of strict $g$-convexity in 2D for solutions of generated Jacobian equations with a $g$-Monge–Ampère measure bounded away from 0. Subsequently this implies $C^1$ differentiability in the case of a $g$-Monge–Ampère measure bounded from above. Our proof follows one given by Trudinger and Wang in the Monge–Ampère case. Thus, like theirs, our argument is local and yields a quantitative estimate on the $g$-convexity. As a result our differentiability result is new even in the optimal transport case: we weaken previously required domain convexity conditions. Moreover in the optimal transport case and the Monge–Ampère case our key assumptions, namely A3w and domain convexity, are necessary.

Mathematics Subject Classification 35J96 · 35J66

1 Introduction

Generated Jacobian equations are PDEs of the form

$$\det D_Y(\cdot, u, Du) = \psi(\cdot, u, Du),$$  \hspace{1cm} (1)

where the vector field $Y$ has a particular structure. This class of equations includes the Monge–Ampère equation and the Jacobian equation from optimal transport as special cases. Precise statements concerning the structure of $Y$ are given in Sect. 2. For now we state our purpose is to show, for generalised notions of convexity, that solutions of

$$\det D_Y(\cdot, u, Du) \geq c > 0$$  \hspace{1cm} (2)

on a convex domain $\Omega \subset \mathbb{R}^2$ are strictly convex. By considering an analogue of the Legendre transform we obtain that when (2) is instead a bound from above and $u$ satisfies the second...
boundary value problem

\[ Y(\cdot, u, Du)(\Omega) = \Omega^*, \]  

(3)

for a domain \( \Omega \subset \mathbb{R}^2 \) and a convex domain \( \Omega^* \subset \mathbb{R}^2 \), then \( u \) is \( C^1 \).

The class of \( Y \) we work with, those obtained from a generating function, embraces applications in optimal transport and geometric optics, whilst allowing the use of a large family of techniques developed for the study of the Monge–Ampère equation. Thus these equations, first studied by Trudinger [18], provide a good combination of applicability and tractability. For example, our proof follows the corresponding proof for the Monge–Ampère equation as given by Trudinger and Wang [20, Remark 3.2]. Though there the result is originally due to Alexandrov [1] and Heinz [9].

The work most relevant to ours is that of Figalli and Loeper [5] who dealt with optimal transport, that is when \( Y = Y(\cdot, Du) \). They proved the \( C^1 \) differentiability of solutions in two dimensions under a bound from above on the \( c \)-Monge–Ampère measure. They assumed a uniformly \( c \)-convex source \( \Omega \) and a strictly \( c^* \)-convex target \( \Omega^* \). Thanks largely to the maturity of the relevant convexity theory we are able to reduce these convexity conditions. Our \( C^1 \) result requires no convexity condition on the source and only \( g^* \)-convexity on the target. This condition is necessary even for the Monge–Ampère equation.

Our results are inherently two dimensional—in higher dimensions strict convexity does not hold under only a bound from below. However one of the key applications of generated Jacobian equations (GJE) is geometric optics and this takes place in two dimensions. In higher dimensions a two sided bound on the Monge–Ampère measure is required for strict convexity and differentiability. Here the relevant work is that of Caffarelli [2] for the Monge–Ampère equation, Chen, Wang [3], Figalli, Kim, McCann [4], Guillen, Kitagawa [7], and Vétois [21] for optimal transport, and Guillen, Kitagawa [8] for GJE.

Our plan is to define generating functions, GJE, and the related convexity notions in Sect. 2. Here we also state precisely our main results: Theorem 1 and Corollary 1. In Sect. 3 we state a versatile differential inequality (essentially taken from [18,19]) whose proof we relegate to the “Appendix”. We prove strict convexity in Sect. 4 and conclude with \( C^1 \) differentiability in Sect. 5.

2 Generating functions and GJE

Generated Jacobian equations are equations of the form (1) where \( Y \) derives from a generating function (defined below). This requirement allows us to develop a framework extending convexity theory. The material below is largely due to Trudinger [18] with other presentations in [8,10–12,14,17]. Guillen’s survey article [6] is a good introduction and lists the 2D theory we develop here as an open problem. Throughout it is helpful to keep in mind the cases \( g(x, y, z) = x \cdot y - z \) and \( g(x, y, z) = c(x, y) - z \) where \( c \) is a cost function from optimal transport. In these settings \( g \)-convexity simplifies to standard convexity and the cost convexity of optimal transport respectively.

2.1 Generating functions

The structure of a particular generated Jacobian equation derives from a generating function. We denote this function by \( g \), and require it satisfy the following assumptions.
**A0.** \( g \in C^4(\Gamma) \) where \( \Gamma \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \) is a bounded domain satisfying that the projections

\[
I_{x,y} := \{ z; (x, y, z) \in \Gamma \}
\]

are open intervals. Moreover we assume there are domains \( \Omega, \Omega^* \subset \mathbb{R}^n \) such that for all \( x \in \overline{\Omega}, y \in \overline{\Omega^*} \) we have that \( I_{x,y} \) is nonempty.

**A1.** For all \( (x, u, p) \in \mathcal{U} \) defined as

\[
\mathcal{U} = \{(x, g(x, y, z), g_x(x, y, z)); (x, y, z) \in \Gamma\},
\]

there is a unique \( (x, y, z) \in \Gamma \) such that

\[
g(x, y, z) = u \quad \text{and} \quad g_x(x, y, z) = p.
\]

**A2.** On \( \Gamma \) there holds \( g_z < 0 \) and the matrix \( E \) with entries

\[
E_{ij} := g_{x_i,y_j} - (g_z)^{-1} g_{x_i,z} g_{y_j},
\]

satisfies \( \det E \neq 0 \).

### 2.2 Generated Jacobian equations

Assumption A1 allows us to define mappings \( Y : \mathcal{U} \to \mathbb{R}^n \) and \( Z : \mathcal{U} \to \mathbb{R} \) by requiring they solve

\[
\begin{align*}
g(x, Y(x, u, p), Z(x, u, p)) & = u, \\g_x(x, Y(x, u, p), Z(x, u, p)) & = p.
\end{align*}
\]

The PDE in equation (1) is called a *generated Jacobian equation* provided \( Y \) derives from solving (4) and (5) for some generating function.

Generated Jacobian equations may be rewritten as Monge–Ampère type equations. Suppose \( u \in C^2(\Omega) \) satisfies (1). Then differentiating (4) and (5) evaluated at \( (x, Y(x, u, Du), Z(x, u, Du)) \) yields

\[
Du = g_x + g_y DY + g_z DZ,
\]

and

\[
D^2u = g_{xx} + g_{xy} DY + g_{xz} DZ.
\]

Since the first equation with (5) implies

\[
D_j Z = -\frac{1}{g_z} g_{y_k} D_j Y^k,
\]

the second can be solved for \( DY \) by which we have

\[
DY(\cdot, u, Du) = E^{-1} \left[ D^2u - g_{xx}(Y(\cdot, u, Du), Z(\cdot, u, Du)) \right],
\]

with \( E \) as in A2. Thus \( C^2 \) solutions of (1) solve

\[
\det \left[ D^2u - A(\cdot, u, Du) \right] = B(\cdot, u, Du),
\]

for

\[
A(\cdot, u, Du) = g_{xx}(Y(\cdot, u, Du), Z(\cdot, u, Du)),
\]

\[
B(\cdot, u, Du) = \det E(\cdot, u, Du) \psi(\cdot, u, Du).
\]

This PDE is elliptic when \( D^2u > g_{xx}(Y(\cdot, u, Du), Z(\cdot, u, Du)) \) as matrices. A necessary condition for ellipticity is \( B > 0 \).
2.3 $g$-convex functions

We introduce an analogue of convexity theory in which the generating function plays the role of a supporting hyperplane. We say a function $u : \Omega \to \mathbb{R}$ is $g$-convex provided for all $x_0 \in \Omega$ there is $y_0, z_0$ such that

$$
g (x_0, y_0, z_0) = u (x_0), \quad (9)$$

$$
g (x, y_0, z_0) \leq u (x) \quad \text{for all } x \neq x_0, x \in \Omega. \quad (10)$$

In this case $g(\cdot, y_0, z_0)$ is called a $g$-support at $x_0$. We say $u$ is strictly $g$-convex if the inequality in (10) is strict, that is, any given $g$-support only touches $u$ at a single point. Note when $u$ is differentiable, since $u(\cdot) - g(\cdot, y_0, z_0)$ has a minimum at $x_0$, we have $Du(x_0) = g_u(x_0, y_0, z_0)$.

This combined with (9) is equivalent to (5) and (4) so that $y_0 = Y(x_0, u(x_0), Du(x_0))$ and $z_0 = Z(x_0, u(x_0), Du(x_0))$. Moreover, if $u$ is $C^2$, again since $u(\cdot) - g(\cdot, y_0, z_0)$ has a minimum at $x_0$, we have $D^2u(x_0) \geq g_{xx}(x_0, y_0, z_0)$ and (8) is degenerate elliptic. However when $u$ is not differentiable (9) and (10) may hold for more than one $y_0, z_0$. The set of $y_0$ for which there is $z_0$ such that (9) and (10) hold is denoted $Y_u(x_0)$. Similarly for $Z_u(x_0)$. When both are a singleton we identify these sets with their single element.

2.4 Alexandrov solutions

The mapping $Y_u$ allows us to define a notion of Alexandrov solution. A $g$-convex function $u$ is called an Alexandrov solution of (1) on $\Omega$ provided for every Borel $E \subset \Omega$

$$
|Y_u(E)| = \int_E \psi(x, u, Du) \, dx.
$$

Since $g$-convex functions are locally semi-convex and thus differentiable almost everywhere the integrand on the right-hand side is well defined almost everywhere. We see, via the change of variables formula, that when $u$ is $C^2$ and $Y(\cdot, u, Du)$ is a $C^1$ diffeomorphism Alexandrov solutions are classical solutions. Moreover in the case of inequality (2) the notion of Alexandrov solution is that for every $E \subset \Omega$

$$
|Y_u(E)| \geq c|E|.
$$

Here we call the measure $\mu$ defined on Borel $E \subset \Omega$ by $\mu(E) = |Y_u(E)|$ the $g$-Monge–Ampère measure.

2.5 Domain convexity

With the goal of introducing a notion of domain convexity we introduce a generalisation of line segments. A collection of points $\{x_\theta\}_{\theta \in [0, 1]}$ is called a $g$-segment with respect to $y \in \Omega^*, z \in \cap_{y} I_{x_\theta, y}$ joining $x_0$ to $x_1$ provided

$$
\frac{gy}{gz} (x_\theta, y, z) = \theta \frac{gy}{gz} (x_1, y, z) + (1 - \theta) \frac{gy}{gz} (x_0, y, z).
$$

Using this we say a set $\Omega$ is $g$-convex with respect to $(y, z)$ provided for each $x_0, x_1 \in \Omega$ the above $g$-segment is contained in $\Omega$. The $g$-segment is unique via condition A1* in Sect. 2.7.

If $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}$ we say $\Omega$ is $g$-convex with respect to $A \times B$ if $\Omega$ is $g$-convex with respect to every $(y, z) \in A \times B$. 

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2.6 Conditions for regularity

Domain convexity and a Monge–Ampère measure bounded below is necessary and sufficient for strict convexity in 2D in the Monge–Ampère case. In the optimal transport case we need the now well known A3w condition. This condition was introduced for optimal transport by Ma, Trudinger, and Wang [16] and generalised to GJE by Trudinger. For GJE we use an additional condition A4w (due to Trudinger [18]).

**A3w.** A generating function \( g \) is said to satisfy the condition A3w provided for all \( \xi, \eta \in \mathbb{R}^n \) with \( \xi \cdot \eta = 0 \) and \( (x, u, p) \in U \) there holds

\[
D_{pkp} A_{ij}(x, u, p) \xi_i \xi_j \eta_k \eta_l \geq 0. 
\]

**A4w.** The matrix \( A \) is non-decreasing in \( u \), in the sense that for any \( \xi \in \mathbb{R}^n \)

\[
D_u A_{ij}(x, u, p) \xi_i \xi_j \geq 0.
\]

In light of Lemma 1 (Sect. 3) we regard A3w and A4w as tools for controlling how \( g \)-convex functions separate from their supporting hyperplanes.

2.7 The dual generating function

Strictly convex functions have \( C^1 \) Legendre transforms. This provides a useful technique for proving solutions of Monge–Ampère inequalities are \( C^1 \). The same technique in the \( g \)-convex case requires the \( g^* \)-transform which is defined in terms of the dual generating function. We introduce the dual generating function here and the \( g^* \) transform in Sect. 5. We set

\[
\Gamma^* = \{(x, y, g(x, y, z)); (x, y, z) \in \Gamma\}.
\]

The **dual generating function**, \( g^* \), is the unique function defined on \( \Gamma^* \) by

\[
g(x, y, g^*(x, y, u)) = u. \tag{12}
\]

It follows that if \( (x, y, z) \in \Gamma \) then

\[
g^*(x, y, g(x, y, z)) = z. \tag{13}
\]

Further, if \( u \) is \( g \)-convex and \( y \in Y_u(x_0) \) then the corresponding support is \( g(\cdot, y_0, g^*(x_0, y_0, u(x_0))) \).

We introduce dual conditions on \( g^* \).

**A1** For all \( (y, z, q) \in V \) defined as

\[
V = \left\{ \left( y, g^*(x, y, u), g^*_y(x, y, u) \right); (x, y, u) \in \Gamma^* \right\},
\]

there is a unique \( (x, y, u) \in \Gamma^* \) such that

\[
g^*(x, y, u) = z \quad \text{and} \quad g^*_y(x, y, u) = q.
\]

Moreover we define mapping \( X(y, z, q), U(y, z, q) \) as the unique \( x, u \) that satisfy these equations (cf. (4) and (5)). As remarked in Sect. 2.5 A1 implies uniqueness of the \( g \)-segment between two points. For this note by differentiating (13) we obtain \( g^*_y(x, y, z) = -g^*_y(x, y, g(x, y, z)) \). The right hand side is injective in \( x \) for fixed \( y, z \).

We define dual objects by swapping the roles of \( x \) and \( y \) as well as \( z \) and \( u \). In particular \( g^* \)-convex functions are those defined on \( \mathcal{Q}^* \) with \( g^* \) supports, and for a \( g^* \)-convex \( v \) we have the mappings \( X_v(\cdot), U_v(\cdot) \) which are analogues of \( Y_u, Z_u \). Similarly \( g^* \)-segments are
used to define $g^*$-convex sets. We also define dual conditions $A2^*$, $A3^*$, $A4w^*$. However Trudinger [18] showed the conditions $A2^*$ and $A3w^*$ are satisfied provided $A2$ and $A3w$ are.

### 2.8 Main results

**Theorem 1** Let $g$ be a generating function satisfying $A0$, $A1$, $A2$, $A1^*$, $A3w$ and $A4w$. Let $\Omega \subset \mathbb{R}^2$ and $u : \Omega \to \mathbb{R}$ be a $g$-convex function satisfying for $c > 0$ and all $E \subset \Omega$

$$|Y_u(E)| \geq c|E|. \quad (14)$$

If $\Omega$ is $g$-convex with respect to $Y_u(\Omega) \times Z_u(\Omega)$ then $u$ is strictly $g$-convex.

**Corollary 1** Let $g$ be a generating function satisfying $A0$, $A1$, $A2$, $A1^*$, $A3w$, and $A4w^*$. Let $\Omega \subset \mathbb{R}^2$ and $u : \Omega \to \mathbb{R}$ be a $g^*$-convex function satisfying that for $C > 0$ and all $E \subset \Omega$

$$|Y_u(E)| \leq C|E|, \quad (15)$$

$$Y_u(\Omega) = \Omega^*. \quad (16)$$

If $\Omega^*$ is $g^*$-convex with respect to $\Omega \times u(\Omega)$ then $u \in C^1(\Omega)$.

Note conditions $A4w$ and $A4w^*$ are always satisfied in the optimal transport case. Indeed Corollary 1 implies the $C^1$ differentiability of potentials for the optimal transport problem (and subsequently continuity of the optimal transport map) whenever the right hand side of the associated Monge–Ampère type equation is bounded from above, the target is $c^*$-convex, and the $A3w$ condition is satisfied. These conditions are all known to be necessary. Necessity of the $c^*$-convexity is due to Ma, Trudinger, and Wang [16] whilst necessity of $A3w$ is due to Loeper [15].

The proof of Theorem 1 is given in Sect. 4. Finally the proof of Corollary 1 (which follows from Theorem 1) is given in Sect. 5.

### 3 Main lemma

We make frequent use of a differential inequality for the difference of a $g$-convex function and a $g$-affine function. Similar inequalities are used in [18,19] and the works of Kim and McCann [13, Proposition 4.6] and Guillen and Kitagawa [8, Lemma 9.3].

**Lemma 1** Let $g$ be a generating function satisfying $A0$, $A1$, and $A2$. Let $y_0 \in \Omega^*$, $z_0 \in \cap_{x \in \Omega} I_{x,y}$ be given, $\{x_\theta\}$ a $g$-segment with respect to $y_0, z_0$, and $u$ a $C^2$ $g$-convex function. Then

$$h(\theta) := u(x_\theta) - g(x_\theta, y_0, z_0),$$

satisfies

$$\frac{d^2}{d\theta^2} h(\theta) \geq [D_{ij}u(x_\theta) - g_{ij}(x_\theta, Y_u(x_\theta), Z_u(x_\theta))](x_\theta_i \dot{x}_\theta_j) + D_{pi, pj} A_{ij}(x_\theta_i \dot{x}_\theta_j, \dot{x}_\theta_j, \dot{x}_\theta_i) + D_{pi} A_{ij}(x_\theta_i \dot{x}_\theta_j, \dot{x}_\theta_j) h(\theta) - K|h'(\theta)|,$$

where $K$ depends only on the values of $g$ and its derivatives on $(x_\theta, y_0, z_0)$ and $\dot{x}_\theta = \frac{d}{d\theta} x_\theta$. We’ve used the shorthand $D_k h(\theta) = u_k(x_\theta) - g_k(x_\theta, y_0, z_0)$. The arguments of $A_{ij,u}$ and $D_{pi, pj} A_{ij}$ are given in the proof.
The proof can be found in [18]. For completeness we include a proof in Appendix A. To use A3w in (17) we need to control $D_{p_k p_l} A_{ij} \xi_i \xi_j \eta_k \eta_l$ for arbitrary $\xi, \eta$. We claim if $\xi, \eta \in \mathbb{R}^n$ are arbitrary then A3w implies

$$D_{p_k p_l} A_{ij} \xi_i \xi_j \eta_k \eta_l \geq -K ||\xi|| \eta ||\xi \cdot \eta||. \quad (18)$$

Here $K$ is a non-negative constant depending only on $\|D_{p_k} A\|_{C^0}$. To obtain (18) from A3w first prove it for arbitrary unit vectors $\xi, \eta$ by using A3w with $\xi$ as given and $\eta$ replaced by $\eta - \xi \cdot \eta \xi$. Thus when A3w, A4w are satisfied and $h(\theta)$ satisfies $\eta \geq 0$ we have

$$h'' \geq -K |Du - g_\alpha| |\dot{\xi}_\alpha||h'| \geq -C|h'|,$$

where $C$ depends on sup $|Du|$ and $|\dot{x}_\alpha|$. This inequality appears in [8, Lemma 9.3], [22, pg. 310, pg. 315] and [18,19]. It implies, amongst other things, estimates on $h'$ in terms of sup $|h|$ via the following lemma.

**Lemma 2** Let $h \in C^2((a, b))$ be a function satisfying $h'' \geq -K|h'|$. For $t \in (a, b)$ there holds

$$-C_0 \sup_{[a, t]} |h| \leq h'(t) \leq C_1 \sup_{[t, b]} |h|,$$

where $C_0, C_1$ depend on $t-a, b-t$ respectively and $K$.

**Proof** First note if $h'(\tau) = 0$ at any $\tau \in (a, b)$ then $h'(a) \leq 0$. To see this assume $\tau$ is the infimum of points with $h'(\tau) = 0$. By continuity if $\tau = a$ we are done. Otherwise $h'$ is single signed on $(a, \tau)$ and if $h' < 0$ on this interval then again by continuity we are done. Thus we assume $h' > 0$ on $(a, \tau)$. The inequality $h'' \geq -K|h'|$ implies $\frac{d}{dt} \log(h'(t)) \geq -K$ on $(a, \tau)$. Subsequently for $a < t_1 < t_2 < \tau$ integration gives

$$h'(t_1) \leq e^{K(t_2-t_1)} h'(t_2), \quad (19)$$

and sending $t_1 \to a, t_2 \to \tau$ gives $h'(a) \leq 0$.

Now to prove the inequality in the lemma let’s deal with the upper bound first. We assume $h' > 0$ on $[t, b]$ otherwise the argument just given implies $h'(t) \leq 0$. We obtain (19) for $t_1 = t$ and $t_2 \in (t, b)$. Integrating with respect to $t_2$ from $t$ to $b$ establishes the result. The other inequality follows by applying the same argument to the function $k$ defined by $k(t) := h(-t)$.

\[\square\]

### 4 Strict convexity in 2D

In this section we present the proof of Theorem 1. The proof follows closely the proof of Trudinger and Wang [20, Remark 3.2] who obtained the same result in the Monge–Ampère case. The key ideas of our proof will be more transparent if the reader is familiar with their proof. There are two key steps: First we obtain a quantitative $g$-convexity estimate for $C^2$ solutions of $\det(D^2 u - A(\cdot, u, Du)) \geq c$ (importantly the estimate is independent of bounds on second derivatives). Then we obtain a convexity estimate for Alexandrov solutions via a barrier argument.

**Proof (Theorem 1).** Step 1. Quantitative convexity for $C^2$ solutions
Initially we assume $u$ is $C^2$. Let $g(\cdot, y_0, z_0)$ satisfy $u \geq g(\cdot, y_0, z_0)$ for $y_0 \in Y_u(\Omega), z_0 \in Z_u(\Omega)$. Assume for some $\sigma \geq 0$ there is distinct $x_{-1}, x_1 \in \Omega$ with

$$u(x_{-1}) \leq g(x_{-1}, y_0, z_0) + \sigma, \quad u(x_1) \leq g(x_1, y_0, z_0) + \sigma. \quad (20)$$

Let $\{x_\theta\}_{\theta \in [-1,1]}$ denote the $g$-segment with respect to $y_0, z_0$ that joins $x_{-1}$ to $x_1$ and set

$$h_\sigma(x) = u(x) - g(x, y_0, z_0) - \sigma.$$  

We use the shorthand $h_\sigma(\theta) = h_\sigma(x_\theta)$. Lemma 1 along with A3w and A4w yields $h''_\sigma(\theta) \geq -K|h'_\sigma(\theta)|$ with the same inequality holding for $h_\sigma$. Hence, via the maximum principle, $h_\sigma$ is less than or equal to its value at the end points. Thus

$$0 \geq h_\sigma(\theta) \geq \inf_{\theta \in [-1,1]} h_\sigma(\theta) =: -H. \quad (22)$$

The convexity estimate we intend to derive is $H \geq C > 0$ where $C$ depends only on $|x_0 - x_1|, \|u\|_{C^1}, g, c$. We use $C$ to indicate any positive constant depending only on these quantities.

Now, via semi-convexity, $g$-convex functions are locally Lipschitz. Thus for $\theta \in [-3/4, 3/4]$ and $\xi \in \mathbb{R}^n$ sufficiently small, (22) implies

$$h_\sigma(x_\theta + \xi) \leq h_\sigma(x_\theta) + C|\xi| \leq C|\xi|, \quad (23)$$

$$h_\sigma(x_\theta + \xi) \geq h_\sigma(x_\theta) - C|\xi| \geq -H - C|\xi|. \quad (24)$$

We let $\eta_\theta$ be a continuous unit normal vector field to the $g$-segment $\{x_\theta\}$. Fix $\delta > 0$ so that $x_{-1/2} + \delta \eta_{-1/2}$ and $x_{1/2} + \delta \eta_{1/2}$ lie in $\Omega$. For $\varepsilon \in [0, \delta]$ let $\{x_\theta^\varepsilon\}_{\theta \in [-1/2, 1/2]}$ be the $g$-segment with respect to $y_0, z_0$ joining $x_{-1/2} + \varepsilon \eta_{-1/2}$ to $x_{1/2} + \varepsilon \eta_{1/2}$. Using Lemma 2 for $\theta \in [-1/4, 1/4]$ combined with (23) and (24) implies

$$-C(\varepsilon + H) \leq \frac{d}{d\theta} h_\sigma(x_\theta^\varepsilon) \leq C(\varepsilon + H). \quad (25)$$

Here we have used that $|x_\theta - x_\theta^\varepsilon| < C\varepsilon$ for a Lipschitz constant independent of $\theta$.

This implies

$$\int_{-1/4}^{1/4} \frac{d^2}{d\theta^2} h_\sigma(x_\theta^\varepsilon) \, d\theta \leq C(\varepsilon + H), \quad (26)$$

and we come back to this in a moment. For now note that, since

$$\det [D^2u - A(-, u, Du)] \geq c \inf \det E > 0,$$

for any two orthogonal unit vectors $\xi, \eta$,

$$[D_{\xi \xi} u - g_{\xi \xi}(x, Y_u(x), Z_u(x))] \geq C.$$  

In particular for $\eta_\theta^\varepsilon$, a choice of unit normal vector field orthogonal to $\dot{x}_\theta^\varepsilon$ (continuous in $\theta, \varepsilon$) we have

$$C^{-1} \left[ D_{\xi_\theta^\varepsilon \xi_\theta^\varepsilon} u - g_{\xi_\theta^\varepsilon \xi_\theta^\varepsilon} (x, Y_u(x), Z_u(x)) \right] \geq |\dot{x}_\theta^\varepsilon|^2 \left[ D_{\eta_\theta^\varepsilon \eta_\theta^\varepsilon} u - g_{\eta_\theta^\varepsilon \eta_\theta^\varepsilon} (x, Y_u(x), Z_u(x)) \right]^{-1}. \quad \Box$$
Employing this and (25) in Lemma 1 gives
\[
\frac{d^2}{d\theta^2} h_\sigma (x_0^\varepsilon) \geq C |\dot{x}_0^\varepsilon|^2 \left( [D_{ij} u (x_0^\varepsilon) - g_{ij} (x_0^\varepsilon)] (\eta_0^\varepsilon)_i (\eta_0^\varepsilon)_j \right)^{-1} - C (\varepsilon + H),
\]
where initially this holds for \( h_0 \), and thus for \( h_\sigma \). Note that by (18) the \( D_p^2 A \) term in Lemma 1 is bounded below by \(-C/h'\) and subsequently controlled by (25).

Substituting (27) into (26) we obtain
\[
\int_{-1/4}^{1/4} |\dot{x}_0^\varepsilon|^2 \left( [D_{ij} u (x_0^\varepsilon) - g_{ij} (x_0^\varepsilon)] (\eta_0^\varepsilon)_i (\eta_0^\varepsilon)_j \right)^{-1} d\theta \leq C (\varepsilon + H),
\]
where we omit that \( g \) is evaluated at \((x_0^\varepsilon, Y_\sigma(x_0^\varepsilon), Z_\sigma(x_0^\varepsilon))\). An application of Jensen’s inequality implies
\[
\int_0^\delta \int_{-1/4}^{1/4} |\dot{x}_0^\varepsilon|^{-2} [D_{ij} u (x_0^\varepsilon) - g_{ij} (x_0^\varepsilon)] (\eta_0^\varepsilon)_i (\eta_0^\varepsilon)_j d\theta d\varepsilon \geq C \int_0^\delta \int_{-1/4}^{1/4} |\dot{x}_0^\varepsilon|^2 \left( [D_{ij} u (x_0^\varepsilon) - g_{ij} (x_0^\varepsilon)] (\eta_0^\varepsilon)_i (\eta_0^\varepsilon)_j \right)^{-1} d\theta \right)^{-1} d\varepsilon \geq \int_0^\delta \frac{C}{\varepsilon + H} d\varepsilon.
\]
This is the crux of the proof complete: the only way for the final integral to be bounded is if \( H \) is bounded away from 0. We’re left to show the integral (28) is bounded in terms of the allowed quantities, and approximate when \( u \) is not \( C^2 \).

To bound (28) use that \( \det E \neq 0 \) implies \(|\dot{x}_0^\varepsilon|\) is bounded below by a positive constant depending on \(|x_1 - x_0|\) and \( g \). This gives the estimate
\[
\int_0^\delta \int_{-1/4}^{1/4} |\dot{x}_0^\varepsilon|^{-2} [D_{ij} u (x_0^\varepsilon) - g_{ij} (x_0^\varepsilon)] (\eta_0^\varepsilon)_i (\eta_0^\varepsilon)_j d\theta d\varepsilon \leq C \int_0^\delta \int_{-1/4}^{1/4} [D_{ij} u (x_0^\varepsilon) - g_{ij} (x_0^\varepsilon)] (\eta_0^\varepsilon)_i (\eta_0^\varepsilon)_j d\theta d\varepsilon \leq C \int_0^\delta \int_{-1/4}^{1/4} \sum_i D_{ii} u (x_0^\varepsilon) - g_{ii} (x_0^\varepsilon) d\theta d\varepsilon.
\]

The final line is obtained using positivity of \( D_{ij} u - g_{ij} \) and is bounded in terms of \( \|g\|_{C^2} \) and sup \( |Du| \) (compute the integral in the Cartesian coordinates and note the Jacobian for this transformation is bounded). Thus returning to (28) and (29) we obtain \( H > C \) where \( C \) depends on the stated quantities.

**Step 2: Convexity estimates for Alexandrov solutions via a barrier argument**

We extend to Alexandrov solutions via a barrier argument. Suppose \( u \) is an Alexandrov solution of (14) that is not strictly convex. There is a support \( g (\cdot, y_0, z_0) \) touching at two points \( x_1, x_{-1} \). Using [18, Lemma 2.3] we also have \( u \equiv g (\cdot, y_0, z_0) \) along the \( g \)-segment joining these points (with respect to \( y_0, z_0 \)). Balls with sufficiently small radius are \( g \)-convex. This follows because, as noted in [14, §2.2], \( g \)-convexity requires the boundary curvatures minus a function depending only on \( \|g\|_{C^2} \) are positive. Thus we assume \( x_1, x_{-1} \) are sufficiently close to ensure that \( B \), the ball with radius \(|x_1 - x_{-1}|/2\) and centre \((x_1 + x_{-1})/2\) is \( g \)-convex with respect to \( Y_u(\Omega) \times Z_u(\Omega) \). Let \( \varepsilon > 0 \) be given, and let \( u_\varepsilon \) be the mollification of \( u \) with
h taken small enough to ensure \(|u - u_h| < \varepsilon/2 \) on \(\partial B\). The Dirichlet theory for GJE, [18, Lemma 4.6], yields a \(C^3\) solution of

\[
\text{det } D Y (\cdot, v_h, D v_h) = c/2 \text{ in } B, \\
v_h = u_h + \varepsilon \text{ on } \partial B,
\]

satisfying an estimate \(|D v_h| \leq K\) where \(K\) depends on the local Lipschitz constant of \(u\). Since \(v_h \geq u\) on \(\partial B\) the comparison principle, [18, Lemma 4.4] implies \(v_h \geq u\) in \(B\). Thus we can apply our previous argument and obtain strict \(g\)-convexity of \(v_h\) provided we note (20) and (21) are satisfied for \(\sigma = 2\varepsilon\). Hence at \(x_{\theta_{\varepsilon}}\) a point on the \(g\)-segment where the infimum defining \(H\) is obtained we have

\[
u \left( x_{\theta_{\varepsilon}} \right) - g \left( x_{\theta_{\varepsilon}}, y_0, z_0 \right) - 2\varepsilon \leq v_h \left( x_{\theta_{\varepsilon}} \right) - g \left( x_{\theta_{\varepsilon}}, y_0, z_0 \right) - 2\varepsilon \leq -H < 0.
\]

As \(\varepsilon \to 0\) we contradict that \(g(\cdot, y_0, z_0)\) supports \(u\). \(\Box\)

5 \(C^1\) regularity

For a \(g\)-convex function defined on \(\Omega\) with \(Y_u(\Omega) = \Omega^*\) its \(g^*\)-transform is the function defined on \(\Omega^*\) by

\[
v(y) := \sup_{x \in \Omega} g^*(x, y, u(x)).
\]

We list a few essential properties. Let us suppose \(y_0 \in Y_u(x_0)\). This means

\[
u(x) \geq g \left( x, y_0, g^* \left( x_0, y_0, u(x_0) \right) \right),
\]

taking \(g^* (x, y_0, \cdot)\) of both sides and using \(g_u^* < 0\) yields

\[
g^* (x, y_0, u(x)) \leq g^* \left( x_0, y_0, u(x_0) \right),
\]

so that \(v(y_0) = g^* (x_0, y_0, u(x_0))\). The definition of \(v\) implies for other \(y \in \Omega^*\) we have \(v(y) \geq g^* (x_0, y, u(x_0))\). Thus \(g^* (x_0, \cdot, u(x_0))\) is a \(g^*\)-support at \(y_0\). Which is to say \(x_0 \in X_v(Y_u(y_0))\).

We use this as follows. Suppose in addition \(u\) satisfies that for all \(E \subset \Omega\)

\[
|X_u(E)| \leq c^{-1}|E|.
\]

Take \(A \subset \Omega^*\) and let \(E_u\) denote the measure 0 set of points where \(u\) is not differentiable. Necessarily \(A \setminus Y_u(E_u) = Y_u(E)\) for some \(E \subset \Omega\). Our above reasoning implies \(E \subset X_v(Y_u(E))\). Hence

\[
|X_v(A)| \geq |X_v(A \setminus Y_u(E_u))| \geq |E| \geq c|A|.
\]  \(\text{(30)}\)

Corollary 1 follows: Let \(u\) be the function given in Corollary 1 and \(v\) its \(g^*\) transform defined on \(\Omega^*\). Theorem 1 holds in the dual form, that is, provided the relevant hypothesis are changed to their starred equivalents, Theorem 1 implies strict \(g^*\)-convexity. Thus the hypothesis of Corollary 1 along with (30) allow us to conclude \(v\) is strictly \(g^*\)-convex.

Suppose for a contradiction \(u\) is not \(C^1\). Then for some \(x_0\) the set \(Y_u(x_0)\) contains two distinct points, say \(y_0, y_1\). Our above working implies \(g^* (x_0, \cdot, u(x_0))\) is a support touching at \(y_0, y_1\). This contradicts strict \(g^*\)-convexity and proves the corollary.
Acknowledgements Thanks to Neil Trudinger who suggested extending the proof in [20] to generated Jacobian equations.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

A Proof of main lemma

In this appendix we provide the proof of Lemma 1.

Proof We first compute a differentiation formula for second derivatives along $g$-segments. We suppose

$$\frac{g_y}{g_z} (x_\theta, y_0, z_0) = \theta q_1 + (1 - \theta) q_0,$$  \hspace{1cm} (31)

and set $q = q_1 - q_0$. We begin with a formula for first derivatives. Since

$$\frac{d}{d\theta} = (x_\theta)_i D x_i$$  \hspace{1cm} (32)

we need to compute $(x_\theta)_i$. Differentiate (31) with respect to $\theta$ and obtain

$$\left[ \frac{g_{i,m}}{g_z} - \frac{g_{i,z} g_{z,m}}{g_z^2} \right] (x_\theta)_i = q_m,$$

from which it follows that

$$(x_\theta)_i = g_z E^{m,i} q_m,$$

where $E^{m,i}$ denotes the $m$, $i$th entry of $E^{-1}$. Thus (32) becomes

$$\frac{d}{d\theta} = g_z E^{m,i} q_m D x_i.$$  \hspace{1cm} (33)

Using this expression to compute second derivatives we have

$$\frac{d^2}{d\theta^2} = g_z E^{n,j} D x_j \left( g_z E^{m,i} D x_i \right) q_m q_n$$

$$= g_z^2 E^{n,j} E^{m,i} q_m q_n D x_i D x_j + g_z^2 q_m q_n E^{n,j} D x_j \left( E^{m,i} \right) D x_i$$

$$+ g_z g_{j,z} E^{n,j} E^{m,i} q_m q_n D x_i.$$  \hspace{1cm} (34)

The formula for differentiating an inverse yields

$$\frac{d^2}{d\theta^2} = (x_\theta)_i (x_\theta)_j D x_i D x_j - g_z^2 q_m q_n E^{n,j} E^{m,a} D x_j \left( E_{ab} \right) E^{b,i} D x_i$$

$$+ g_z g_{j,z} E^{n,j} E^{m,i} q_m q_n D x_i.$$  \hspace{1cm} (34)

1 We use the convention that subscripts before the comma denote differentiation with respect to $x$, and subscripts after the comma (which are not $z$) denote differentiation with respect to $y$. 

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Now compute

\[
D_{x_j} (E_{ab}) = D_{x_j} \left[ g_{a,b} - \frac{g_{a,z}g_{b,z}}{g_z} \right] = g_{aj,b} - \frac{g_{aj,z}g_{b,z}}{g_z} - \frac{g_{a,z}g_{j,b}}{g_z} + \frac{g_{j,z}g_{a,z}g_{b,z}}{g_z^2} = -\frac{g_{a,z}}{g_z} E_{jb} + E_{l,b} D_{pl} g_{aj}. \tag{35}
\]

Here we have used that

\[
E_{l,b} D_{pl} g_{aj}(\cdot, Y(\cdot, u, p), Z(\cdot, u, p)) = g_{aj,b} - \frac{g_{aj,z}g_{b,z}}{g_z},
\]

which follows by computing \(D_{pl} g_{aj}\), differentiating (4) with respect to \(p\) to express \(Z_p\) in terms of \(Y_p\), and employing \(E_{i,j} = D_{p,j} Y_{i}\) (which is obtained via calculations similar to those for (7)).

Substitute (35) into (34) to obtain

\[
\frac{d^2}{d\theta^2} = \langle \dot{x}_\theta \rangle_i \langle \dot{x}_\theta \rangle_j D_{x_i,x_j} - \frac{g_{i,j}^2 q_m q_n E_{m,j}^n E_{m,a}^n E_{l,b} D_{pl} g_{aj} E_{b,i}^n D_{x_i}}{q_m q_n} + \left[ g_{z} g_{a,z} E_{m,a}^n E_{i,j}^n E_{j,b}^n D_{x_i} + g_{z} g_{j,z} E_{m,a}^n E_{m,i}^n D_{x_i} \right] q_m q_n
\]

\[
= \langle \dot{x}_\theta \rangle_i \langle \dot{x}_\theta \rangle_j \left( D_{x_i,x_j} - D_{pl} g_{ij} D_{x_i} \right)
+ \left[ g_{z} g_{a,z} E_{m,a}^n E_{i,j}^n D_{x_i} + g_{z} g_{j,z} E_{m,a}^n E_{m,i}^n D_{x_i} \right] q_m q_n + g_{j,z} \left( E_{m,j}^n q_m \frac{d}{d\theta} + E_{m,j}^n q_n \frac{d}{d\theta} \right),
\]

where in the last equality we swapped the dummy indices \(i\) and \(a\) on the second term to allow us to collect like terms and also used (33).

Now let’s use this identity to compute \(h''(\theta)\). We have

\[
h''(\theta) = \left[ D_{ij} u (x_\theta) - g_{ij} (x_\theta, y_0, z_0) \right. - D_{pk} g_{ij} (x_\theta, y_0, z_0) \left( D_{k,j} u (x_\theta) - D_{k,j} g (x_\theta, y_0, z_0) \right) \right] \langle \dot{x}_\theta \rangle_i \langle \dot{x}_\theta \rangle_j
+ g_{j,z} \left( E_{m,j}^n q_m h' + E_{m,j}^n q_n h' \right).
\]

Terms on the final line are bounded below by \(-K |h'(\theta)|\). Now after adding and subtracting \(g_{ij}(x_\theta, y, z)\) for \(y = Y_u(x_\theta), z = Z_u(x_\theta)\) we have

\[
h''(\theta) \geq \left[ D_{ij} u (x_\theta) - g_{ij} (x_\theta, y_0, z_0) \right. - D_{pk} g_{ij} (x_\theta, y_0, z_0) \left( D_{k,j} u (x_\theta) - D_{k,j} g (x_\theta, y_0, z_0) \right) \right] \langle \dot{x}_\theta \rangle_i \langle \dot{x}_\theta \rangle_j
- K |h'(\theta)|.
\]
Set \( u_0 = g(x_0, y_0, z_0) \), \( u_1 = u(x_0) \) \( p_0 = g_x(x_0, y_0, z_0) \), and \( p_1 = Du(x_0) \). Then rewriting in terms of the matrix \( A \) we have

\[
\begin{align*}
  h'' &\geq [D_{ij}u(x_0) - g_{ij}(x_0, y, z)](x_0)_i(x_0)_j + [A_{ij}(x_0, u_1, p_1) - A_{ij}(x_0, u_0, p_0)](x_0)_i(x_0)_j - K|h'(\theta)| \\
  &= [D_{ij}u(x_0) - g_{ij}(x_0, y, z)](x_0)_i(x_0)_j + A_{ij, u}(x_0, u_1, p_1)(u_1 - u_0)(x_0)_i(x_0)_j \\
  &\quad - D_{jk}A_{ij}(x_0, u_0, p_0)(p_1 - p_0)](x_0)_i(x_0)_j - K|h'(\theta)|
\end{align*}
\]

Here \( u_\tau = \tau u + (1 - \tau)u_0 \) results from a Taylor series. Another Taylor series for \( f(t) := A_{ij}(x_0, u_0, tp_1 + (1 - t)p_0) \) and we obtain

\[
\begin{align*}
  h'' &\geq [D_{ij}u(x_0) - g_{ij}(x_0, y, z)](x_0)_i(x_0)_j + A_{ij, u}(x_0, u_1 - u_0)(x_0)_i(x_0)_j \\
  &\quad - K|h'(\theta)| + D_{jk}A_{ij}(x_0, u_0, p_1)(p_1 - p_0)(p_1 - p_0)
\end{align*}
\]

This is the desired formula. \( \square \)

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