Correct Treatment of $\frac{1}{(\eta\cdot k)^p}$ Singularities in the Axial Gauge Propagator

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Abstract

The propagators in axial-type, light-cone and planar gauges contain $\frac{1}{(\eta\cdot k)^p}$-type singularities. These singularities have generally been treated by inventing prescriptions for them. In this work, we propose an alternative procedure for treating these singularities in the path integral formalism using the known way of treating the singularities in Lorentz gauges. To this end, we use a finite field-dependent BRS transformation that interpolates between Lorentz-type and the axial-type gauges. We arrive at the $\epsilon$-dependent tree propagator in the axial-type gauges. We examine the singularity structure of the propagator and find that the axial gauge propagator so constructed has no spurious poles (for real $k$). It however has a complicated structure in a small region near $\eta\cdot k = 0$. We show how this complicated structure can effectively be replaced by a much simpler propagator.

1 Introduction

As far as we know today, the known high energy physics is well explained by the Standard Model (SM). SM is an $SU(3) \times SU(2) \times U(1)$ nonabelian gauge

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theory \[1\]. Hence, the practical calculations in electroweak and strong interactions are calculations in a gauge theory requiring a choice of gauge. The two choices of gauges most frequently employed are the Lorentz-type and the axial-type gauges. (The latter include the light-cone gauges (LCG) and planar gauges, while the former include \(R_\xi\)-gauges in spontaneously broken gauge theories (SBGT).) The Lorentz-type gauges have been popular on account of their Lorentz covariance, simplicity of Feynman rules and availability of a gauge parameter to ensure gauge independence of physical results. The disadvantage of Lorentz-type gauges is however the presence of Faddeev-Popov ghosts and relatively large number of Feynman diagrams needed for evaluation of Green’s functions. The axial-type gauges, on the other hand, have the advantage of formal decoupling of ghosts \[2\]. This leads to a much smaller number of diagrams to be evaluated. These gauges are particularly useful in perturbative QCD calculations \[3\]. In fact the first QCD calculations were done in these gauges \[4\].

The main disadvantage of axial-type gauges arises from the lack of Lorentz covariance and especially from the appearance of \(1/(\eta \cdot k)^p\)-type spurious singularities in propagators. Much literature has been devoted to the question of how these singularities should be treated \[4, 5\]. Prescriptions have been proposed to deal with this issue: two important ones of these are the “principal-value prescription” (PVP) \[6\] and the Mandelstam-Leibbrandt (ML) prescriptions \[7\]. These, however, have lead in many cases to difficulties. The PVP procedure fails for LCG already at the on loop level and yields the wrong answer for the Wilson loop to order \(g^4\) \[2\].

Moreover, there are instances where the ghosts need to be taken into account \[4\]. In canonical quantization, the treatment of \(1/\eta \cdot k\)-type singularities has been given for axial gauges of the from \(A_1 + \lambda A_3 = 0\) \[8\] (This does not include LCG).

In this work, we advocate an ab-initio and fresh approach to the question of the axial gauge propagator based on earlier works \[9, 10, 11, 12, 13\]. The approach here utilizes a finite field-dependent BRS transformation established earlier between the Lorentz-type and the axial-type gauge \[9, 10, 11\]. This transformation has been used to write down a compact expression that interpolates between Green’s functions from the axial-type gauges to the Lorentz-type gauges \[12\]. We apply the results of \[12\] to the relation between the axial and the Lorentz gauge propagator. The procedure we adopt is detailed below.
We know how the $1/k^2$ singularities of the propagator are handled in the Lorentz gauges. These are in effect, treated by replacing $k^2$ by $k^2 + i\epsilon$. Where the poles have physical interpretation, this amounts to propagation of positive frequency waves into future and negative frequency waves into the past as the Feynman propagator for a physical field shows. This is taken into account in a Lorentz covariant manner by introduction of a term $\epsilon \int d^4x (A^2/2 - \bar{c}c)$ to the action. Introduction of such a term also has natural interpretation in the Minkowskian formulation of Lorentz gauge theories. This is elaborated in Section 3. We start from this well-established procedure in Lorentz gauges. We then perform a finite field-dependent BRS transformation \cite{9,10} (a non-local field transformation) that converts the Lorentz to the axial gauges. This procedure, following the work \cite{13} leads us to an expression for the tree $\epsilon$-dependent propagator for the gauge field in the axial gauges. We suggest that this expression, should, in principle be used for the axial gauge propagator. We analyze the singularity structure of the propagator and find that for real $\eta \cdot k$, there are no spurious poles. The propagator reduces to the usual propagator for $|\eta \cdot k| >> \epsilon$; however, it show a complex structure in a small region near $\eta \cdot k = 0$. We show that this propagator can equally well be replaced by an effective simpler expression.

Our prescription, by its very construction, has the desirable property that it preserves the value of the Wilson loop \cite{14}. This is so because the field transformation of \cite{9} and \cite{12} was explicitly constructed to preserve the expectation values of gauge-invariant observables as you go from gauge to gauge. This is unlike the other prescriptions where the property has to be imposed as a check on the prescription. For more comments, please see reference \cite{15}.

We now briefly state the plan of the paper. In Section 2, we review the results needed in this work. In particular we introduce the FFRBS transformations and the results of references \cite{12} and \cite{13}. In Section 3, we obtain the effect of $\epsilon \int d^4x (A^2/2 - \bar{c}c)$ term in the Lorentz gauge generating functional. In Section 4, we work out the $\epsilon$-dependent axial gauge propagator in detail. In Section 5, we examine the singularity structure of the propagator. In Section 6, we show how the propagator can effectively be replaced by a much simpler expression. Section 7 has the conclusion.
2 Summary of Results on FFBRS Transformation between Lorentz- and Axial-type Gauges

2.1 Notations and Conventions

We start with the Faddeev-Popov effective action (FPEA) in linear Lorentz-type gauges:

\[ S_{\text{eff}}^L[A, c, \bar{c}] = \int d^4x \left( -\frac{1}{4} F_{\mu\nu}^\alpha F^{\alpha,\mu\nu} \right) + S_{gf} + S_{gh}, \]  

(1)

where the gauge-fixing action \( S_{gf} \) is given by:

\[ S_{gf}^L = -\frac{1}{2\lambda} \int d^4x \sum_\alpha (\partial \cdot A^\alpha)^2 \equiv -\frac{1}{2\lambda} \int d^4x \sum_\alpha (f^L_\alpha[A])^2, \]  

(2)

and the ghost action \( S_{gh} \) is given by:

\[ S_{gh}^L = -\int d^4x \bar{c}^\alpha M^{\alpha\beta} c^\beta, \]  

(3)

where

\[ M^{\alpha\beta}[A(x)] \equiv \partial_\mu D_\mu^{\alpha\beta}(A, x). \]  

(4)

The covariant derivative is defined by:

\[ D_\mu^{\alpha\beta} \equiv \delta^{\alpha\beta} \partial_\mu + g f^{\alpha\beta\gamma} A_\gamma^\mu. \]  

(5)

In a similar manner, the FPEA in axial-type gauges, is given by:

\[ S_{gf}^A \equiv -\frac{1}{2\lambda} \int d^4x \sum_\alpha (\eta \cdot A^\alpha)^2 \equiv -\frac{1}{2\lambda} \sum_\alpha \int d^4x (f^A_\alpha[A])^2. \]  

(6)

We require \( \eta_\mu \) to be real, but otherwise unrestricted. and

\[ S_{gh}^A = -\int d^4x \bar{c}^\alpha \tilde{M}^{\alpha\beta} c^\beta, \]  

(7)

with

\[ \tilde{M}^{\alpha\beta} = \eta_\mu D_\mu^{\alpha\beta}. \]  

(8)
In the $\lambda \to 0$,
\[ e^{iS^A_{\text{eff}}} \sim \prod_{\alpha,x} \delta \left( \eta \cdot A^\alpha(x) \right). \] (9)

Thus, in the presence of the delta function, the $A$-dependent term in $\tilde{M}$ can be dropped leading to the formally ghost-free matrix. As is well known, $S^L_{\text{eff}}$ and $S^A_{\text{eff}}$ are invariant under the BRS transformations:
\[ \delta A^\alpha_\mu(x) = D^\alpha_\mu \beta(x) \delta \Lambda \]
\[ \delta c^\alpha(x) = -\frac{g}{2} \mathcal{F}^{\alpha\beta\gamma} c^\beta(x) c^\gamma(x) \delta \Lambda \]
\[ \delta \bar{c}^\alpha(x) = \frac{f^\alpha[A]}{\lambda} \delta \Lambda, \] (10)

where $f^\alpha[A] = \partial \cdot A^\alpha$ or $\eta \cdot A^\alpha$, depending on whether one has action in the Lorentz or the axial-type gauges. We also need the interpolating mixed gauge with
\[ S^M_{\text{g.f.}} = -\frac{1}{2\lambda} \int d^4x [\partial \cdot A^\alpha(1 - \kappa) + \kappa \eta \cdot A^\alpha]^2 \] (11)

and the associated ghost term
\[ S^M_{\text{gh}} = -\int d^4x \bar{c} \left[ (1 - \kappa) M + \kappa \tilde{M} \right] c. \] (12)

The net effective action $S^M_{\text{eff}}$ has a BRS symmetry under transformations (10) with $f^\alpha[A] \to (1 - \kappa) \partial \cdot A^\alpha + \kappa \eta \cdot A^\alpha$.

We write in general, the BRS transformations in this case as:
\[ \delta_{\text{BRS}} \phi_i = (\bar{\delta}_1 \phi_i + \kappa \bar{\delta}_2 \phi_i) \delta \Lambda. \] (13)

### 2.2 FFBRS Transformations

As observed by Joglekar and Mandal [9], in (10), $\delta \Lambda$ need not be infinitesimal nor need it be field-independent as long as it does not depend on $x$ explicitly for (10) to by a symmetry of FPEA In fact, the following finite field-dependent BRS (FFBRS) transformations were introduced:
\[ A'_{\mu}^\alpha = A^\alpha_\mu + D^\alpha_\mu \beta(x) \Theta[\phi] \]
\[ c' ^\alpha = c^\alpha - \frac{g}{2} \mathcal{F}^{\alpha\beta\gamma} c^\beta(x) c^\gamma(x) \Theta[\phi] \]
\[ \bar{c}' ^\alpha = \bar{c}^\alpha + \frac{f^\alpha[A]}{\lambda} \Theta[\phi], \] (14)
or generically

\[ \phi'_i(x) = \phi_i(x) + \delta_{\text{BRS}} \phi_i(x) \Theta[\phi], \] (15)

where \( \Theta[\phi] \) is an \( x \)-independent functional of \( A, c, \bar{c} \) (generically denoted by \( \phi_i \)) and these were also the symmetry of the FPEA. The transformations of the form (14) were used to connect actions of different kinds for Yang-Mills theory in \([1]\) and \([10]\). The FPEA is invariant under (14), but the functional measure is not invariant under the (nonlocal) transformations (14). The Jacobian for the FFBRS transformations can be expressed (in special cases dealt with in \([9, 10]\)) effectively as \( \exp(iS_1) \) and this \( S_1 \) explains the difference between the two effective actions. Such FFBRS transformations were constructed in \([9, 10]\) by integration of an infinitesimal field-dependent BRS (IFBRS) transformation:

\[ \frac{d \phi_i(x, \kappa)}{d \kappa} = \delta_{\text{BRS}} \phi_i(x, \kappa) \Theta'[\phi(x, \kappa)] \] (16)

The integration of (16) from \( \kappa = 0 \) to 1, leads to the FFBRS transformation of (15) with \( \phi(\kappa = 1) \equiv \phi' \) and \( \phi(\kappa = 0) = \phi \). Further \( \Theta \) in (13) was related to \( \Theta' \) by:

\[ \Theta[\phi] = \Theta'[\phi] \frac{\exp[f[\phi]] - 1}{f[\phi]}, \] (17)

where

\[ f[\phi] = \sum_i \int d^4x \frac{\delta \Theta'}{\delta \phi_i(x)} \delta_{\text{BRS}} \phi_i(x) \] (18)

FFBRS transformations of the type (15) were used to connect the FPEA in Lorentz-type gauges with gauge parameter \( \lambda \) to (i) the most general BRS/anti-BRS symmetric action in linear gauges, (ii) FPEA in quadratic gauges, (iii) the FPEA in Lorentz-type gauges with another gauge parameter \( \lambda' \) in \([3]\). It was also used to connect the former to FPEA in axial-type gauges in \([10]\). We shall now summarize the results of \([10]\) in 2.3.

### 2.3 FFBRS Transformation for Lorentz to Axial Gauge

We give the results for the FFBRS transformation that connects the Lorentz-type gauges (See \([1]\)) with gauge parameter \( \lambda \) to axial gauge (See \([3]\)) with
same gauge parameter \( \lambda \). [The same calculation can be used to connect it to axial gauges with another gauge parameter \( \lambda' \): one simply rescales \( \eta \) suitably.] They are obtained by integrating:

\[
\frac{d\phi_i(\kappa)}{d\kappa} = \delta_{\text{BRS}[\phi]} \Theta'[\phi],
\]

(19)

with

\[
\Theta' = i \int d^4x \bar{c}^\alpha (\partial \cdot A^\alpha - \eta \cdot A^\alpha).
\]

(20)

the consequent \( \Theta[\phi] \) is given by (17) with

\[
f[\phi] = i \int d^4x \left[ \frac{\partial \cdot A^\alpha}{\lambda} (\partial \cdot A^\alpha - \eta \cdot A^\alpha) + \bar{c} (\partial \cdot D - \eta \cdot D) c^\alpha \right].
\]

(21)

The meaning of these field transformations is as follows. Suppose we begin with vacuum expectation value of a gauge invariant functional \( G[\phi] \) in the Lorentz-type gauges:

\[
\langle\langle G[\phi] \rangle\rangle \equiv \int D\phi G[\phi] e^{iS_{\text{eff}}[\phi]}.
\]

(22)

Now, we perform the transformation \( \phi \to \phi' \) given by (15). Then we have [with \( G[\phi'] = G[\phi] \) by gauge invariance]

\[
\langle\langle G[\phi] \rangle\rangle \equiv \langle\langle G[\phi'] \rangle\rangle = \int D\phi' J[\phi'] G[\phi'] e^{iS_{\text{eff}}[\phi']}
\]

(23)

on account of the BRS invariance of \( S_{\text{eff}}^L \). Here \( J[\phi'] \) is the Jacobian

\[
D\phi = D\phi' J[\phi'].
\]

(24)

As was shown in [9], for the special case \( G[\phi] \equiv 1 \), the Jacobian \( J[\phi'] \) in (24), can be replaced by \( e^{iS_1[\phi']} \) where

\[
S_{\text{eff}}^L[\phi'] + S_1[\phi'] = S_{\text{eff}}^A[\phi'].
\]

(25)

As shown in Section III of [12], this replacement is valid for any gauge invariant \( G[\phi] \) functional of \( A \). If one were to live with vacuum expectation values of gauge invariant observables, the FFBRS in [9] would be sufficient. But as seen in [12], general Green’s functions need a modified treatment.
2.4 Relations between Green’s functions in Axial- and Lorentz-type gauges

The FFBRS in 2.3 was used to correlate arbitrary Green’s functions in the Lorentz-type and Axial-type gauges \[12\] Let \( O[\phi] \) represent any field operator (local or multi-local). Then the relation between the Green’s functions in the two gauges is given by:

\[
\langle \langle O[\phi] \rangle \rangle_A = \int D\phi' O[\phi'] e^{iS_{\text{eff}}[\phi']} \\
= \int D\phi \left( O[\phi] + \sum_i \delta_i \phi[\phi] \frac{\delta O}{\delta \phi_i} \right) e^{iS_{\text{eff}}[\phi]} \\
\equiv \langle \langle O[\phi] + \sum_i \delta_i \phi[\phi] \frac{\delta O}{\delta \phi_i} \rangle \rangle_L, \quad (26)
\]

where

\[
\phi' = \phi + \left( \tilde{\delta}_1[\phi] \Theta_1[\phi] + \tilde{\delta}_2[\phi] \Theta_2[\phi] \right) \Theta'[\phi] \\
\equiv \phi + \delta \phi[\phi] \quad (27)
\]

is an FFBRS \[4\] with

\[
\Theta_{1,2}[\phi] \equiv \int_0^1 d\kappa (1, \kappa) \exp \left( \kappa f_1[\phi] + \frac{\kappa^2}{2} f_2[\phi] \right); \quad (28)
\]

\[
f_1[\phi] \equiv i \int d^4x \left[ \frac{\partial \cdot A}{\lambda}(\partial \cdot A - \eta \cdot A) + \bar{c}(\partial \cdot D - \eta \cdot D) c \right] \\
f_2[\phi] \equiv -i \int d^4x (\partial \cdot A - \eta \cdot A)^2, \quad (29)
\]

and

\[
\Theta' \equiv i \int d^4x \bar{c}^\alpha (\partial \cdot A - \eta \cdot A). \quad (30)
\]

The relation \[26\] can be used to related the ordinary Green’s functions, operator Green’s functions, etc. in the two set of gauges depending on the choice of \( O[\phi] \).

A much more convenient and tractable result was also derived in \[13\]

\[
\langle O \rangle_A = \langle O \rangle_L + \int_0^1 d\kappa \int D\phi \sum_i \left( \tilde{\delta}_{1,i}[\phi] + \kappa \tilde{\delta}_{2,i}[\phi] \right) \Theta'[\phi] \frac{\delta O}{\delta \phi_i} e^{iS_{\text{eff}}}, \quad (31)
\]

8
where $S_{\text{eff}}^M$ is the FPEA for the mixed gauge function with the gauge fixing term defined in (11) and $\tilde{\delta}_{1,i}$ and $\tilde{\delta}_{2,i}$ have been defined in (13). Of course, $\tilde{\delta}_{2,i}$ is non-vanishing only for $\bar{c}$ field.

3 The General Procedure for Generating Prescription

In this section, we shall outline the general procedure for generating the correct treatment for $\frac{1}{\eta k}$ singularities in the axial-type gauges starting from the Lorentz-type gauges. In the Lorentz-type gauges, also there is a singularity in the propagator at $k^2 = 0$ in both the gauge and the ghost propagators. This, in analogy with the scalar particle, is dealt with by adding an $i\epsilon$ term, viz $k^2 \to k^2 + i\epsilon$ ($\epsilon$ is small positive) in the denominators. As is well known, this prescription allows the propagation of positive energy solutions into future and the negative energy solutions into the past. The role of this prescription in the Lorentz-type gauges can also be understood clearly in the Minkowskian formulation of quantum field theory. The above prescription is implemented by an addition of the term $-i\frac{\epsilon}{2}A_{\mu}A^\mu$ to the gauge field action. In the context of a scalar theory, the $i\epsilon\phi^2$ term provides a damping in the path integral:

$$W = \int D\phi e^{i(S + i\epsilon\phi^2)}$$

for large $\phi$. In the context of gauge theories, we expect the $\epsilon$-dependent term to be determined by similar damping provided in the transverse degrees of freedom. Then the form of the term in the context of a covariant formulation viz $-i\epsilon A_{\mu}A^\mu/2$ is determined by covariance. Thus, this treatment of the $\frac{1}{k^2}$-type singularity is well understood in the Lorentz-type gauges. Further, there are WT identities for Green’s functions that have terms that involve both the ghost and the gauge propagators. Their exact preservation requires that a similar modification be made in the ghost propagator poles $1/k^2 \to 1/k^2 + i\epsilon$.

Thus, the path integral for $\langle 0|0 \rangle$ in the Lorentz-type gauges we normally start with, is given in Minkowski space, by:

$$W^L = \int D\phi e^{i[S_{\text{eff}}^L - i\frac{\epsilon}{2}A_{\mu}A^\mu + i\epsilon\bar{c}c]}$$

$$\equiv \int D\phi e^{iS_{\text{eff}}^L[\phi] + iO_1[A,c,c,\bar{c},\epsilon]}.$$  

(33)
We now expect that if we start with this $W^L$ that has no pole prescription ambiguities and make suitable field transformation (as outlined in Section 2) to the axial gauges, we should obtain an ambiguity-free treatment of the axial gauge propagator. We thus imagine making the field transformation of (15) viz:

$$
\phi'_i(x) = \phi_i(x) + \delta_{\text{BRS}}[\phi_i] \Theta[\phi]
$$

with $\Theta[\phi]$ given explicitly by (17), (20) and (21).

As shown in [12], the effect of this field transformation can be evaluated via the formula (26), and is given by:

$$
W^L = \int D\phi e^{iS^L_{\text{eff}}[\phi] + i\epsilon O_1[\phi]}
$$

$$
= \int D\phi' e^{iS^A_{\text{eff}}[\phi'] + i\epsilon O'_1[\phi']}
$$

with

$$
O'_1 = O_1 + \sum_i \delta\phi_i[\phi] \frac{\delta O}{\delta\phi_i}.
$$

We regard the net exponent, including the new $O(\epsilon)$ terms, viz.

$$
S^A_{\text{eff}}' \equiv S^A_{\text{eff}} + \epsilon O'_1
$$

as given correct the treatment of the axial gauge poles. We can now, in principle, evaluate the effect of the $O'_1$ term by looking at the new effective quadratic form in (37). This turns out to be a more cumbersome procedure. We proceed along an alternate route as below.

Consider the effect of the net $\epsilon$-term in (37) on an axial gauge Green’s function:

$$
\langle O \rangle_A = \int D\phi' O[\phi'] e^{iS^A_{\text{eff}}[\phi']}
$$

This is given by the modification $S^A_{\text{eff}} \to S^A_{\text{eff}}'$ in (38), viz:

$$
\langle O \rangle_A = \int D\phi' O[\phi'] e^{iS^A_{\text{eff}} + i\epsilon O'_1}.
$$

We now proceed to relate (39) to the corresponding Green’s functions in Lorentz gauges as done in [12]. We reexpress $\langle O \rangle_A$ as:

$$
\langle O \rangle_A = \frac{1}{i} \frac{\delta}{\delta N} \int D\phi' e^{iS^A_{\text{eff}}[\phi'] + i\epsilon O'_1[\phi'] + iN O[\phi']} |_{N=0}
$$

$$
\equiv \frac{1}{i} \frac{\delta}{\delta N} W^A[N]|_{N=0}.
$$
Now, we consider the quantity:

\[ W^A[N] \equiv \int \mathcal{D}\phi' e^{i S_{\text{eff}}[\phi']} \{ e^{i O[\phi'] + i N O[\phi']} \} \]

\[ \equiv \int \mathcal{D}\phi' e^{i S_{\text{eff}}[\phi']} f[\phi'] \]  \hspace{1cm} (41)

We now apply the procedure of [12] (following equation (34) of that work) to the above expression where \( O[\phi'] \) there is replaced by the curly bracket above. Then, using identity (53) of [12], we obtain that

\[ W^A[N] = \int \mathcal{D}\phi e^{i S_{\text{eff}}[\phi]} \{ f[\phi] + \sum_i \delta \phi_i[\phi] \frac{\delta f}{\delta \phi_i} \}. \]  \hspace{1cm} (42)

Now,

\[ f[\phi] + \sum_i \delta \phi_i[\phi] \frac{\delta f}{\delta \phi_i} = \exp \left[ i \left( \epsilon (O'_1 + \sum_i \delta \phi_i[\phi] \frac{\delta O'_1}{\delta \phi_i}) + N(O + \sum_i \delta \phi_i[\phi] \frac{\delta O}{\delta \phi_i}) \right) \right]. \]  \hspace{1cm} (43)

In writing (43), we have used the nilpotency of \( \Theta' \) contained in each of \( \delta \phi_i[\phi] \) [See equation (27)]. Hence,

\[ W^A[N] = \int \mathcal{D}\phi e^{i S_{\text{eff}}[\phi]} e^{i \left( \epsilon (O'_1 + \sum_i \delta \phi_i[\phi] \frac{\delta O'_1}{\delta \phi_i}) + N(O + \sum_i \delta \phi_i[\phi] \frac{\delta O}{\delta \phi_i}) \right)} \]  \hspace{1cm} (44)

Now at \( N = 0 \), the above must coincide with \( W^L \) of [13]. Hence,

\[ \epsilon \left( O'_1 + \sum_i \delta \phi_i[\phi] \frac{\delta O'_1}{\delta \phi_i} \right) = \epsilon O_1[\phi] = -i \int d^4 x (A^2 / 2 - \bar{c} c) \]  \hspace{1cm} (45)

Now, following the transition from (53) to (54) of [12], we can make a transition in (44) above. This amounts to substitution of \( \delta \phi_i \) via (27) and (20). (Here we note that this is possible because the \( O(\epsilon) \) terms in (44) are independent of \( \kappa \)) We then have

\[ \langle O \rangle_A = \langle O \rangle_L + \int_0^1 d\kappa \int \mathcal{D}\phi e^{i S_{\text{eff}}[\phi, \kappa] - i\kappa \int d^4x (A^2 / 2 - \bar{c}c)} \sum_i \left( \delta_{1,i}[\phi] + \kappa \delta_{2,i}[\phi] \right) \Theta'[\phi] \frac{\delta O}{\delta \phi_i}. \]  \hspace{1cm} (46)
Thus, while the $O(\epsilon)$ terms needed to be calculated in the Green’s function calculation for (38) (viz $\epsilon O_1'$ are very complicated), when $\langle O \rangle_A$ is reexpressed as an integral over $\kappa$, the effect of $\epsilon$ terms in this integral is simply to modify $S_M[\phi, \kappa] \rightarrow S_M[\phi, \kappa] + i \epsilon \int d^4x (-A^2/2 + \bar{c}c)$. Thus, the form (16) facilitates the evaluation of the effect of $O(\epsilon)$ terms on the axial gauge Green’s functions, as the modification there is $\kappa$-independent.

We shall now use (16) to write down the expression for the axial gauge propagator. So we consider:

$$O[\phi] \equiv A^\alpha_\mu(x)A^\beta_\nu(y).$$

(47)

Then, with obvious notations

$$iG_{\mu\nu}^{A \alpha\beta}(x - y) = iG_{\mu\nu}^L \alpha\beta(x - y) + i \int_0^1 d\kappa \int D\phi e^{iS_{\text{eff}}^M[\phi, \kappa] - i\epsilon \int (A^2/2 - \bar{c}c)d^4x} \times \left[(D_\mu c)^\alpha(x)A^\beta_\nu(y) + A^{\alpha}_\mu(x)(D_\nu c)^\beta(y)\right] \int d^4z \bar{c}(z)(\partial \cdot A^\gamma - \eta \cdot A^\gamma)(z)$$

(48)

The above is an exact result for the relation between propagators valid to all orders. For obtaining the correct treatment for the $1/(\eta \cdot q)$-singularity, we are however interested in the tree propagator. For this, we collect the $O(g^0)$ terms on the right-hand side. Noting

$$\int D\phi e^{i\alpha^\alpha(x)\bar{c}^\beta(y)e^{i[S_{\text{eff}}^M - i\epsilon \int (A^2/2 - \bar{c}c)d^4x]} = i\delta^{\alpha\beta} \tilde{G}_{0M}(x - y)$$

$$= i\delta^{\alpha\beta} \int d^4q \frac{e^{-iq\cdot(x-y)}}{[(\kappa - 1)q^2 - i\kappa q \cdot \eta - i\epsilon]}. \quad \text{(49)}$$

We can write, for the tree propagator $G_{\mu\nu}^0$:

$$G_{\mu\nu}^{0A \alpha\beta}(x - y) = G_{\mu\nu}^{0L \alpha\beta}(x - y) + i \int_0^1 d\kappa \left[-i\partial_\mu \tilde{G}_{0M}^\alpha(x - y)(\partial_\nu^\gamma - \eta^\gamma)\tilde{G}_{\alpha\nu}^{0M \alpha\beta} + (\mu, x, \alpha) \leftrightarrow (\nu, y, \beta)\right].$$

(50)

In the next section, we shall use the result (50) to obtain the correct $\epsilon$-dependent propagator in the axial gauges.
4 Evaluation of the Axial Gauge Propagator

In this section, we shall evaluate the $\epsilon$-dependent axial gauge propagator using (50). We shall show that over most of the real $\eta \cdot k$ axial, we recover the naive axial gauge propagator. However, we find a non-trivial complex structure in a small region near $\eta \cdot k = 0$. In this section, we shall content ourselves with the algebraic study of $G_{\mu A}^{0 A}$. In the next section, we shall study, in detail, the analytic structure of $G_{\mu A}^{0 A}$ over the complex $\eta \cdot k$ plane.

We express (50), in momentum space as:

$$G_{\mu \nu}^{0 A}(k) = G_{\mu \nu}^{0 L}(k) + i \int_0^1 d\kappa \left[ k^\sigma \tilde{G}_{\mu \nu}^{0 M}(k, \kappa)(-i k^\sigma - \eta^\sigma)\tilde{G}_{\sigma \nu}^{0 M}(k, \kappa) + (\mu, k) \leftrightarrow (\nu, -k) \right].$$

(51)

Here

$$\tilde{G}_{\mu \rho}^{0 M}(k, \kappa) = \frac{1}{(\kappa - 1)k^2 - ik \eta \cdot k - i\epsilon}$$

(52)

and

$$\tilde{G}_{\mu \rho}^{0 M}(k, \kappa) = Z_{\mu \rho}^{-1}$$

(53)

with

$$Z_{\mu \nu} \equiv -(k^2 + i\epsilon) \left[ g_{\mu \nu} + \left( \frac{1}{\lambda} \right) \frac{k_\mu k_\nu}{k^2} + i\epsilon \frac{k(1 - \kappa)}{\lambda} \frac{k_\mu k_\rho}{k^2} + i\epsilon \frac{\eta_\mu k_\rho}{k^2} + i\epsilon \right].$$

(54)

is the quadratic form in momentum space arising from the $\epsilon$-dependent action $S_{\text{eff}}^{M} - i\epsilon \int (A^2/2 - \overline{c}c)$ in (48). [It should be emphasized $\tilde{G}_{\mu \nu}^{0 L}$ and $\tilde{G}_{\mu \nu}^{0 M}$ are only intermediate objects occurring in calculations and are not the actual ghost and gauge propagators in the mixed gauges. For example, while $\tilde{G}_{\mu \nu}^{0 M}(k, \kappa = 0)$ is equal to $\tilde{G}_{\mu \nu}^{0 L}$, $\tilde{G}_{\mu \nu}^{0 M}(k, \kappa = 1) \neq G_{\mu \nu}^{0 A}(k)$ because the latter has to be evaluated with the exact $\epsilon O_1'$ terms in the exponent and not $\epsilon (A^2/2 - \overline{c}c)$ as occurring in (48). The actual tree propagator in mixed gauges would similarly be evaluated with an appropriate term $\epsilon O_1[\phi, \kappa]$ and not from (48); this is not required in our evaluation.] We express $\tilde{G}_{\mu \rho}^{0 M}$ as:

$$\tilde{G}_{\mu \rho}^{0 M}(k) = -\frac{1}{k^2 + i\epsilon} \left[ g_{\mu \rho} + \right.$$
\[
\frac{\left[ (1 - \kappa)^2 - \lambda - \eta^2 - \frac{i \lambda (1 - \kappa)}{k^2 + i \epsilon} \eta \right] k_\mu k_\rho - i \kappa (1 - \kappa) k_\mu \eta \rho + \frac{\eta^2 \kappa}{k^2 + i \epsilon} k_\mu \eta \rho + i \frac{\kappa^2 \epsilon}{k^2 + i \epsilon} \eta_\mu \eta_\rho}{\left( - \frac{k^2}{k^2 + i \epsilon} \left[ (\eta \cdot k)^2 - \eta^2 k^2 + (k^2 + \eta^2) (k^2 + i \epsilon) \right] + 2k^2 \kappa - i \epsilon \lambda - k^2 \right).}
\]

(55)

\[\tilde{G}_{\mu\nu}^{0A} - \tilde{G}_{\mu\nu}^{0L} = \frac{-i}{(k^2 + i \epsilon)^2 (1 - i \xi_1 - i \xi_2)(1 - i \xi_2 + \xi_1^2 + i \xi_2 \xi_3)}\]

\[\times \int_0^1 d\kappa \left[ k_\mu k_\nu \left( \kappa + \left[ \frac{i \lambda - \xi_1 (1 - \lambda)}{\xi_1 + i \xi_3} \right] \right)(\xi_1 + i \xi_3) + \eta_\mu k_\nu \left( \kappa + \left[ \frac{1 - i \xi_2 (1 - \lambda)}{1 - i \xi_3 - i \xi_1} \right] \right)(1 - i \xi_1 + i \xi_2) \right]
\]

\[+ (k \to -k, \mu \leftrightarrow \nu) \]  

(56)

with

\[\xi_1 \equiv \frac{\eta \cdot k}{k^2 + i \epsilon};\]
\[\xi_2 \equiv \frac{\epsilon}{k^2 + i \epsilon};\]
\[\xi_3 \equiv \frac{\eta^2}{k^2 + i \epsilon};\]
\[a_1 \equiv \frac{1}{1 - i \xi_1 - i \xi_2};\]
\[\gamma \equiv \frac{(1 - i \xi_2)}{1 - i \xi_2 + \xi_1^2 + i \xi_2 \xi_3} = \frac{1 - i \xi_2}{D};\]
\[\beta \equiv \frac{1 + i \xi_2 (\lambda - 1)}{1 - i \xi_2 + \xi_1^2 + i \xi_2 \xi_3} = \gamma + \frac{i \xi_2 \lambda}{D}\]  

(57)

The quadratic in the denominator can be rewritten as \((\kappa - \kappa_1)(\kappa - \kappa_2)\) with

\[\kappa_{1,2} = \gamma \pm \sqrt{\gamma^2 - \beta} = \frac{1 - i \xi_2 \pm \sqrt{(1 - i \xi_2)^2 - [1 + i \xi_2 (\lambda - 1)](1 - i \xi_2 + \xi_1^2 + i \xi_2 \xi_3)}}{D}\]

\[\equiv \frac{1 - i \xi_2 \pm \sqrt{Y}}{D}\]  

(58)

We note that of the three zeros of the denominators, two are equal are \(\epsilon = 0\), since

\[\kappa_{1|\kappa=0} = \frac{1}{1 + \xi_1^2} + \sqrt{\left( \frac{1}{1 + \xi_1^2} \right)^2 - \frac{1}{1 + \xi_1^2}}\]

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We shall now state an important convention in defining the square roots in (58). The square root $\sqrt{Y}$ has branch points at $\pm \sqrt{-i\xi_2[(1-i\xi_2)^{-1}+\lambda(1-i\xi_2+i\xi_2\xi_3)]}$ and these lie a distance $O(\epsilon)$ away from the origin [For LCG, in the $k^2 = 0$ subspace, however, $\sqrt{Y} = i\sqrt{-\xi_1}$ has no branch cut in $\xi_1$-plane]. We choose the branch cut joining these. To obtain the value of $+\sqrt{Y}$ at any point $\xi'$ not on the branch cut, we consider $\sqrt{Y}$ for $\xi_1 = M\xi'$ as $M \to +\infty$. Then we can ignore $\epsilon$ terms in this case and $\sqrt{Y} = \sqrt{-\xi_1}$. This, we define to be $i\xi_1$. We then define $\sqrt{Y}$ for $\xi_1 = \xi'$ by requiring that the phase of $\sqrt{Y}$ is a continuous function of $M$ for $1 \leq M \leq \infty$. From this and from the fact that $Y \equiv Y(\xi_1^2)$, we learn that $\sqrt{Y}(-\xi_1) = -\sqrt{Y}(\xi_1)$. Hence, $\kappa_2(-\xi_1) = \kappa_1(\xi_1)$.

We further note that this prescription defines uniquely $\sqrt{Y}$ for real $\eta \cdot k \neq 0$ since the branch cut cuts the real $\eta \cdot k$ axis only at the origin $\eta \cdot k = 0$.

We further note that both the $k_\mu k_\nu$ and the $\eta_\mu k_\nu$ terms involve an integral of the same form

$$\int_0^1 \frac{d\kappa}{(\kappa-a_1)(\kappa-\kappa_1)(\kappa-\kappa_2)},$$

the constant $\alpha$ being different for the $k_\mu k_\nu$ and $\eta_\mu k_\nu$ terms. This can be evaluated and reorganized as:

$$\frac{(a_1 + \alpha)}{(a_1 - \kappa_1)(a_1 - \kappa_2)} \ln\left[\frac{1-a_1}{-a_1}\right] + \frac{(\kappa_1 + \alpha)}{(\kappa_1 - a_1)(\kappa_1 - \kappa_2)} \ln\left[\frac{1-\kappa_1}{-\kappa_1}\right] + \frac{(\kappa_2 + \alpha)}{(\kappa_2 - a_1)(\kappa_2 - \k_1)} \ln\left[\frac{1-\kappa_2}{-\kappa_2}\right]$$

$$\equiv \frac{1}{a_1 - \kappa_2 - \kappa_1} \left\{ \frac{(\alpha + a_1)}{a_1 - \kappa_1} \ln\left[\frac{\kappa_1 - \kappa_1 a_1}{a_1 - \kappa_1}\right] - \frac{(\kappa_2 + \alpha)}{\kappa_2 - \kappa_1} \ln\left[\frac{\kappa_2 - \kappa_1 \kappa_2}{\kappa_1 - \kappa_1 \kappa_2}\right] \right\}.$$

It is shown in Appendix A that the contribution of the second term vanishes in the limit $\epsilon \to 0$. Hence the propagator (56) is given in terms of the first term in (57):

$$\frac{a_1 + \alpha}{(a_1 - \kappa_1)(a_1 - \kappa_2)} \ln\left[\frac{\kappa_1 - \kappa_1 a_1}{a_1 - \kappa_1}\right]$$

substituted for (60) in (56). Hence, we shall study the structure of (62) in detail. The singularity structure of (62) is dependent on the denominators...
and the logarithm. The equation (62), in general reads:

\[
(a_1 + \alpha)D(1 - i\xi_1 - i\xi_2)^2 \ln \left[ \frac{-i(\xi_1 + \xi_2)}{1 - i\xi_2(1 - \lambda)} - \sqrt{\frac{1 - i\xi_2}{1 - i\xi_2(1 - \lambda)}} \right]
\]  

(63)

with

\[
P(\xi_1) \equiv \xi_1^2 + 2i\xi_1(1 - i\xi_2) + \frac{\lambda + i\xi_2(1 - 2\lambda) + \xi_2^2(1 - \lambda) + \xi_3}{1 - \lambda}.
\]  

(64)

The apparent complexity of (63) actually exists only in the small region of the \(\eta \cdot k\) complex plane near the origin. We note that for \(|a_1 - \kappa_1| < |a_1(1 - \kappa_1)|\), the expression (62) can be expressed as

\[
\frac{1}{a_1 - \kappa_1} \ln \left[ \frac{\kappa_1 - a_1\kappa_1}{a_1 - a_1\kappa_1} \right] = -\frac{1}{a_1 - a_1\kappa_1} + O(a_1 - \kappa_1).
\]  

(65)

The condition \(|a_1 - \kappa_1| < |a_1(1 - \kappa_1)|\) implies

\[
\text{Im} \left( \frac{\sqrt{- (\eta \cdot k)^2 - i\epsilon^2}}{\epsilon^2 + i\epsilon} \right) > \frac{1}{2}
\]  

(66)

and this covers all of real \(\eta \cdot k\) axis save the region \((-\epsilon, 0)\) for \(\eta^2 \neq 0\) and \((-\epsilon, \epsilon)\) for LCG. Thus, (65) reads neglecting \(O(\epsilon)\) terms

\[-\frac{1}{a_1(1 - \kappa_1)(a_1 - \kappa_2)}.
\]  

(67)

For \(|\eta \cdot k| > \epsilon\), this is easily seen to be

\[- \frac{(1 - i\xi_1)^2(1 + \xi_2^2)}{2\xi_1^2}
\]  

(68)

and leads to the usual behavior of the axial propagator when substituted into (66), which then reads (See Appendix C):

\[
\tilde{G}_{\mu\nu}^{0A} - \tilde{G}_{\mu\nu}^{0L} = -\frac{1}{k^2}k_\mu k_\nu \frac{(\lambda k^2 + \eta^2)}{(\eta \cdot k)^2 + (1 - \lambda)} + \frac{k_{[\mu}\eta_{\nu]}}{k^2 \eta \cdot k}.
\]  

(69)

We finally summarize our results. We find:

\[
\tilde{G}_{\mu\nu}^{0A} = \tilde{G}_{\mu\nu}^{0L} + \left( k_\mu k_\nu \Sigma_1 + \eta_\mu k_\nu \Sigma_2 \right) \ln \Sigma_3 + (k \to -k; \mu \leftrightarrow \nu)
\]  

(70)
where

\[
\Sigma_1 \equiv \frac{-(k^2 - i\eta \cdot k)}{\epsilon \Sigma} \left( \frac{\eta^2 + i\eta^2}{k^2 - i\eta k} + i\lambda - \frac{(1 - \lambda)\eta k}{k^2 + i\epsilon} \right)
\]
\[
\Sigma_2 \equiv \frac{-(k^2 - i\eta \cdot k)}{\epsilon \Sigma} \left( -\left[ \frac{k^2 + i\eta k}{k - i\eta k} \right] + 1 - \frac{i\epsilon(1 - \lambda)}{k^2 + i\epsilon} \right)
\]
\[
\Sigma_3 \equiv \frac{-i(\eta \cdot k + \epsilon)(k^2 + i\epsilon \lambda)}{(k^2 + i\epsilon)\left( -i\epsilon\lambda - \sqrt{k^4 - (k^2 + i\epsilon\lambda)\left[ k^2 + \frac{(\eta k)^2 + i\epsilon\eta^2}{k^2 + i\epsilon} \right]} \right)}.
\]

and

\[
\Sigma \equiv \left[ (1 - \lambda)[(\eta \cdot k)^2 + 2ik^2\eta \cdot k] + i\epsilon k^2(1 - 2\lambda) + \lambda(k^2 + i\epsilon)^2 + \eta^2(k^2 + i\epsilon) \right].
\]

5 Singularity Structure of the Propagator

In this section, we shall study the singularity structure of the propagator both on the real \( \eta \cdot k \) axis as well as the \( \eta \cdot k \) complex plane in general. The singularity structure on the real axis is important from the point of view of the well-defined nature of the propagator for real \( k_\mu \) while the singularity structure in the complex \( \eta \cdot k \) plane is relevant the question of Wick rotation.

As shown in Section 4, the quantity \((63)\) is relevant to both the propagator terms of \( k_\mu k_\nu \) and \( (k_\mu \eta_\nu + k_\nu \eta_\mu) \) kind. We shall first analyze its structure. The singularities of \((63)\) arise from those of \( P(\xi_1) \) and from those of \( \ln \) term. We shall first analyze the singularities of \( P(\xi_1) \).

\( P(\xi_1) \) is a quadratic polynomial in \( \xi_1 = \frac{\eta k}{k^2 + i\epsilon} \). It has two zeros; they are:

\[
\xi_1 = -i(1 - i\xi_2) \pm i\sqrt{\frac{1 - i\xi_2 + \xi_3}{1 - \lambda}},
\]

i.e. at

\[
\eta \cdot k = -ik^2 \pm i\sqrt{\frac{(k^2 + i\eta^2)(k^2 + i\epsilon)}{1 - \lambda}}.
\]

We note that the above roots vanish only when \( k^2 \) satisfies:

\[
\lambda k^4 + k^2(\eta^2 + i\epsilon) + i\epsilon\eta^2 = 0.
\]
The above equation for $k^2 = 0$ has physical (i.e. real $k^2$) root(s) only in $\eta^2 = 0$ (except for $\lambda = 1$). For $\eta^2 = 0$, it is easily seen that both the roots in (73) vanish at $k^2 = 0$. We thus note that in the light cone gauge ($\eta^2 = 0$), the point $k^2 = 0$ needs to be treated with care. We shall first discuss the case $\eta^2 \neq 0$.

Case I: $\eta^2 > 0$

(i) $k^2 > 0$: Hence, $(k^2 + \eta^2)k^2 > k^4 > 0$. We may set $\epsilon = 0$. The roots are:

$$\eta \cdot k = -ik^2 \left[ 1 + \sqrt{\frac{1}{1-\lambda} \left( 1 + \frac{\eta^2}{k^2} \right)} \right]$$

and lie on imaginary axis on either side of the real axis at a finite distance even as $\epsilon \to 0$.

(ii) $k^2 = 0$: The roots are at

$$\eta \cdot k = \pm i \sqrt{\frac{i\eta^2 \epsilon}{1-\lambda}} = \pm e^{\frac{3i\pi}{4}} \sqrt{\frac{\eta^2 \epsilon}{1-\lambda}}.$$  

The roots again lie on either side of the real axis at an infinitesimal distance on the line $\theta = \frac{3\pi}{4}$.

(iii) $0 > k^2 > -\eta^2$: Here $(k^2 + \eta^2)k^2 < 0$. We can therefore write the roots as:

$$\eta \cdot k = -ik^2 \pm \sqrt{\frac{|(k^2 + \eta^2)k^2| - i\epsilon|k^2 + \eta^2|}{1-\lambda}}.$$  

Again, we may set $\epsilon = 0$ here. Then, the roots

$$\eta \cdot k = -ik^2 \pm \sqrt{\frac{|(k^2 + \eta^2)k^2|}{1-\lambda}}$$

are on the same side of the real axis, i.e. in the the upper half plane (UHP) for $k^2 < 0$.

(iv) $k^2 < -\eta^2$: Here $0 < k^2(k^2 + \eta^2) < k^4$. Then (setting $\epsilon = 0$), the roots are at:

$$\eta \cdot k = i|k^2| \left[ 1 \pm \sqrt{\frac{1}{\lambda-1} \left( 1 - \frac{\eta^2}{|k^2|} \right)} \right],$$

and both lie in the UHP.

Case II: $\eta^2 < 0$
We express (73) as

\[ \eta \cdot k = i(-k^2) \pm i \sqrt{\left(\frac{-k^2 - \eta^2}{1 - \lambda}\right) \left(\frac{-k^2 - i\epsilon}{1 - \lambda}\right)}. \]  

(80)

We note that the above expression is analogous to the RHS of (73) with \(-\eta^2 > 0\) and \(k^2 \to -k^2, \ \epsilon \to -\epsilon\). Hence, a discussion parallel to that given above applies. We find:

(i) \(k^2 < 0\): The poles are at

\[ \eta \cdot k = i|k^2| \left[ 1 \pm \sqrt{\frac{1}{1 - \lambda} \left( 1 + \frac{\eta^2}{k^2} \right)} \right] \]  

(81)

and lie on imaginary axis on wither side of the real axis at a finite distance even as \(\epsilon \to 0\).

(ii) \(k^2 = 0\):

\[ \eta \cdot k = \sqrt{-i|\eta^2|\epsilon} = \pm \epsilon i \sqrt{\frac{|\eta^2|\epsilon}{1 - \lambda}}. \]  

(82)

The roots again lie on either side of the real axis at an infinitesimal distance on the line \(\theta = \frac{\pi}{4}\).

(iii) \(0 < k^2 < -\eta^2\): Here \((k^2 + \eta^2)k^2 < 0\). We can therefore write the roots as:

\[ \eta \cdot k = i k^2 \left[ 1 \pm \sqrt{\frac{1}{1 - \lambda} \left( 1 + |\eta^2| \right)} \right] \]  

(83)

and both lie in the UHP.

(iv) \(k^2 < -\eta^2\): Here \(0 < k^2(k^2 + \eta^2) < \eta^4\). Then (setting \(\epsilon = 0\)), the roots are at:

\[ \eta \cdot k = -i k^2 \left[ 1 \pm \sqrt{\frac{1}{\lambda - 1} \left( 1 - \frac{\eta^2}{k^2} \right)} \right], \]  

(84)

and both lie in the LHP.

Finally, we consider the important case of the light cone gauge (LCG). It is this case, where most of the difficulties associated with the prescriptions obtained by others are located.

Case III: \(\eta^2 = 0\) We note that the roots of (73) now read:

\[ \eta \cdot k = -i k^2 \pm i \sqrt{\frac{k^4 + i\epsilon k^2}{1 - \lambda}}. \]  

(85)
For $k \neq 0$, we can write the roots as:

$$
\eta \cdot k = -ik^2 \left[ 1 \mp \sqrt{\frac{1}{1 - \lambda} \left( 1 + i \frac{\epsilon}{k^2} \right)} \right].
$$

(86)

Without loss of generality, we may assume that $\epsilon < |k^2|$ and expand the square root. The we find:

$$
\eta \cdot k = -ik^2 \left[ 1 \mp \frac{1}{\sqrt{1 - \lambda}} \mp \frac{1}{8\sqrt{1 - \lambda}} k^2 + \frac{\epsilon}{2\sqrt{1 - \lambda}} \right].
$$

(87)

We note that for $0 < \lambda < 1$, the roots lie on either side of the real axis. In particular for small positive $\lambda$, we may set $\epsilon = 0$ and find:

$$
\eta \cdot k = -2ik^2, \quad \frac{i\lambda k^2}{2}.
$$

(88)

On the other hand, we may set $\lambda = 0$ in the beginning. Then we find it necessary to take into account the $O(\epsilon^2)$ term in (86). We then find:

$$
\eta \cdot k = -2ik^2, \quad \frac{i\epsilon}{8k^2}.
$$

(89)

Thus, in either case, the roots are at: (i) $\eta \cdot k = -2ik^2$, and (ii) on imaginary axis, in UHP for $k^2 > 0$ and LHP for $k^2 < 0$. We note that this discussion clearly fails for $k^2 = 0$ in the LCG. We then necessarily have to obtain the treatment for the LCG by a limiting procedure $\eta^2 \to 0$ for this subspace of momenta. From (73), for $k^2 = 0$, the roots $\eta \cdot k$ are:

$$
\eta \cdot k = \pm \sqrt{-i\epsilon \eta^2}.
$$

(90)

For $\eta \neq 0$, (irrespective of where $\eta^2 > 0$ or $< 0$) these lie on the opposite sides of the real axis at an infinitesimal distance away.

### 6 Effective Treatment for the Axial Propagator

We obtained an exact $\epsilon$-dependent expression for the axial gauge propagator from its connection to the Lorentz gauge Green’s functions. As remarked
earlier, the propagator is effectively the same as the usual one except in a small region near $\eta \cdot k = 0$. It is in this region that the many treatments of the propagator have been suggested, differ. The expression we have obtained ab initio, however, has complicated structure in this region. We wish to show that it can be replaced by a much simpler expression which yields the same (coordinate space) propagator and will facilitate the actual axial gauge calculations rather than using the expressions of (70) and (71).

We consider the tree axial gauge propagator in coordinate space:

$$\Delta_{\mu\nu}(x - y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot (x - y)} G_{\mu\nu}^0(k)$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} \int \frac{dk^0}{2\pi} e^{-ik^0(x^0 - y^0)} G_{\mu\nu}^0(k)$$  \hspace{1cm} (91)

We introduce the variable:

$$\zeta \equiv \eta^0 k^0 - \vec{\eta} \cdot \vec{k}.$$  \hspace{1cm} (92)

We shall deal with the case $\eta^0 \neq 0$. (This is always possible by a proper choice of a Lorentz frame.) We further assume for simplicity of treatment, $\eta^0 = 1$. (This can always be arranged by rescaling $\eta_\mu$ if necessary.) Thus, we use: $\zeta = k^0 - \vec{\eta} \cdot \vec{k}$ and $\vec{k}$ as integration variables. We express

$$\Delta_{\mu\nu}(x - y) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{x} - \vec{y}) - i\vec{\eta} \cdot \vec{k}(x^0 - y^0)} \int \frac{d\zeta}{(2\pi)} e^{-i\zeta(x^0 - y^0)} G_{\mu\nu}^0(\zeta, \vec{k}).$$  \hspace{1cm} (93)

We shall focus our attention on those terms in $G_{\mu\nu}^0(k) = G_{\mu\nu}^{0L} + [G_{\mu\nu}^{0A}(k) - G_{\mu\nu}^{0L}(k)]$ which have nontrivial structure near $\zeta = 0$. We shall in effect show that

$$\int_{-\infty}^{\infty} \frac{d\zeta}{(2\pi)} e^{-i\zeta(x^0 - y^0)} G_{\mu\nu}^{0A}(\zeta, \vec{k})$$

can, in the limit $\epsilon \to 0$, be effectively replaced by two terms: (i) one of which is

$$\int_{C} \frac{d\zeta}{(2\pi)} e^{-i\zeta(x^0 - y^0)} G_{\mu\nu,\text{eff}}^{0A}(\zeta, \vec{k})$$

where $G_{\mu\nu,\text{eff}}^{0A}$ has a simple structure near $\zeta = 0$ and the contour $C$ is a contour suitably distorted near $\zeta = 0$ as will be specified soon, (ii) and a contribution having a relatively simple form arising from the region near $\zeta = 0$. 21
To achieve this, we replace,

\[
\int_{-\infty}^{\infty} = \int_C d\zeta + \oint_{C_1} d\zeta
\]  

(96)

where \(C\) runs from \((-\infty, -\alpha \sqrt{\epsilon})\) and \((\alpha \sqrt{\epsilon}, \infty)\) and is completed by adding a semicircle of radius \(\alpha \sqrt{\epsilon}\) in the LHP. Contour \(C_1\) on the other hand is a closed contour to compensate \(C\) for the left hand side (See Fig 1). Here \(\alpha\) is a (large enough) arbitrary positive number. We then show that (i) on \(C\), \(G^{0A}_{\mu\nu}(\zeta, \vec{k})\), the \(\epsilon\)-dependence can, in fact, be ignored and be replaced by the naive axial propagator (with \(\zeta\) complex over the semicircle); (ii) The contour \(C_1\) can be shrunk so that the contribution over \(C_1\) can be replaced by that around the branch cut (from \(-i \sqrt{\epsilon \eta^2}\) to \(i \sqrt{\epsilon \eta^2}\)) which will then be evaluated (See Fig. 2). This contribution, in the limit \(\epsilon \to 0\), can be replaced by a simple expression as shown later.

[Here, we clarify the location of the contour on the left hand side of (96). We recall that in (70), the \(\ln\) factor:

\[
\ln \left( \frac{\zeta + \epsilon}{\sqrt{\zeta^2 + i\epsilon \eta^2}} \right).
\]  

(97)

is in principle, multi-valued. However, this factor, has arisen out of the expression (62) whose value is unambiguous in in the original complex integral \(\text{C} (60)\). In particular, we have already defined \(\sqrt{\zeta^2 + i\epsilon \eta^2}\) earlier for \(|\zeta| > \alpha \epsilon\), the factor boils down to

\[
\ln \left( \frac{\zeta}{i \zeta} \right) = -\frac{i \pi}{2}.
\]  

(98)

In the region \(|\zeta| \sim \epsilon\), we must define the phase of the logarithm recalling that it has arisen from the unambiguous integrals

\[
\int_0^1 \frac{d\kappa}{\kappa - a_1} = \ln \left( \frac{1 - a_1}{-a_1} \right)
\]  

(99)

\[
\int_0^1 \frac{d\kappa}{\kappa - \kappa_1} = \ln \left( \frac{1 - \kappa_1}{-\kappa_1} \right).
\]  

(100)

This defines the way the contour on the left hand side of (96) should be drawn near \(\zeta = 0, -\epsilon\).]
In this work, we shall only deal with the case $\eta^2 \neq 0$ and consider the propagator for $k^2 \neq 0$. We may then choose $\epsilon << |k^2|$, in which case we may replace $k^2/(k^2 + i\epsilon)$ by 1 wherever possible. We shall also set $\lambda = 0$ for simplicity. [In the case of LCG, we may need to keep $\lambda \neq 0$ till the end; we shall not deal with this here however.]

On the contour $C$, we have $|\zeta| >> \sqrt{\epsilon \eta^2}$ for sufficiently large $\alpha$. Here we can employ the treatment in Appendices B and C to conclude that the propagator over $C$ can be replaced by the naive axial propagator (with complex $\zeta$):

$$G^{0A}(k, \epsilon = 0) = -\frac{1}{k^2} \left( g_{\mu\nu} + \frac{(\lambda k^2 + \eta^2)}{(\eta \cdot k)^2} k_\mu k_\nu - \frac{k_\mu \eta_\nu}{\eta \cdot k} \right).$$

We recall from the discussion of Section 4, that for $k^2 \neq 0, \eta^2 \neq 0$, the singularities of $P(\xi_1)$ are at a finite distance away from $\zeta = 0$. Hence $C_1$ does not enclose these. We however, have the branch cut defined earlier (see discussion following (59)) and ones arising from the presence of the logarithm. We shrink the contour $C_1$ so that it in effect goes around the branch cut in the LHP, and receives a contribution proportional to the discontinuity in $ln(\sqrt{\xi^2 + i\epsilon \eta^2})$ across the branch cut.

We summarize the procedure followed for evaluating the effective replacement. We express

$$\int d^3k (2\pi)^3 e^{i\vec{k} \cdot (\vec{x} - \vec{y})} \int_{C_1} \frac{dk_0}{2\pi} e^{-ik_0(x^0 - y^0)} \left[ k_\mu k_\nu A_1(\zeta, \vec{k}) + \eta_\mu k_\nu A_2(\zeta, \vec{k}) + \eta_\nu k_\mu A_3(\zeta, \vec{k}) \right]$$

$$= -\partial^\mu_\alpha \partial^\nu \int d^3k (2\pi)^3 e^{i\vec{k} \cdot (\vec{x} - \vec{y}) - i\eta \vec{k} \cdot (x^0 - y^0)} \int_{C_1} \frac{d\zeta}{2\pi} e^{-i\zeta(x^0 - y^0)} A_1(\zeta, \vec{k})$$

$$+ i\eta_\mu \partial^\nu \int d^3k (2\pi)^3 e^{i\vec{k} \cdot (\vec{x} - \vec{y}) - i\vec{k} \cdot (x^0 - y^0)} \int_{C_1} \frac{d\zeta}{2\pi} e^{-i\zeta(x^0 - y^0)} A_2(\zeta, \vec{k})$$

$$+ i\eta_\nu \partial^\mu \int d^3k (2\pi)^3 e^{i\vec{k} \cdot (\vec{x} - \vec{y}) - i\vec{k} \cdot (x^0 - y^0)} \int_{C_1} \frac{d\zeta}{2\pi} e^{-i\zeta(x^0 - y^0)} A_3(\zeta, \vec{k})$$

We then evaluate

$$\int_{C_1} \frac{d\zeta}{2\pi} e^{-i\zeta(x^0 - y^0)} A_i(\zeta, \vec{k})$$

by replacing $C_1$ by a contour that goes around the branch cut as mentioned earlier. The contribution comes only from the discontinuity of $A_i$ across the
branch cut, and is equal to

\[ \int_{0}^{-i\sqrt{\epsilon \eta^2}} \frac{d\zeta}{(2\pi)} e^{-i\zeta(x^0-y^0)} \text{Disc}A_i(\zeta, \vec{k}) \]  

(104)

We expand \( \text{Disc}A_i(\zeta, \vec{k}) \) around \( \zeta = 0 \):

\[ \text{Disc}A_i(\zeta, \vec{k}) = \frac{1}{\epsilon} \sum_n a_{i(n)} \zeta^n \]  

(105)

where we have explicitly shown the \( \frac{1}{\epsilon} \) dependence of the discontinuity. We have:

\[
\begin{align*}
\frac{1}{\epsilon} \int_{0}^{-i\sqrt{\epsilon \eta^2}} & \frac{d\zeta}{(2\pi)} e^{-i\zeta(x^0-y^0)} \sum_{n=0}^{\infty} a_{i(n)} \zeta^n \\
&= \frac{1}{\epsilon} \int_{0}^{-i\sqrt{\epsilon \eta^2}} \frac{d\zeta}{(2\pi)} \left(a_{i(0)}[1 - i\zeta(x^0 - y^0)] + a_{i(1)} \zeta\right) + O(\sqrt{\epsilon}) \\
&= -a_{i(0)} \frac{1}{2\pi} i \sqrt{\frac{i\eta^2}{\epsilon}} e^{-\sqrt{i\eta^2(x^0-y^0)}} - \frac{i\eta^2}{2(2\pi)} a_{i(1)} + O(\sqrt{\epsilon}) \\
&= \frac{e^{-\sqrt{i\eta^2(x^0-y^0)}}}{2\pi} \left[-i \sqrt{\frac{i\eta^2}{\epsilon}} a_{i(0)} - \frac{i\eta^2}{2} a_{i(1)}\right] + O(\sqrt{\epsilon}) \\
&= \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} e^{-ik^0(x^0-y^0)} \delta \left(k^0 - \frac{1}{2} \frac{\epsilon \eta^2}{i} \right) \left[-i \sqrt{\frac{i\eta^2}{\epsilon}} a_{i(0)} - \frac{i\eta^2}{2} a_{i(1)}\right] + O(\sqrt{\epsilon}) \\
&= \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot (x-y)} \left\{ \delta \left(k^0 - \frac{1}{2} \frac{\epsilon \eta^2}{i} \right) \left[k_\mu k_\nu \left[-i \sqrt{\frac{i\eta^2}{\epsilon}} a_{1(0)} - \frac{i\eta^2}{2} a_{1(1)}\right]\right. \\
&+ \eta_\mu \eta_\nu \left[-i \sqrt{\frac{i\eta^2}{\epsilon}} a_{2(0)} - \frac{i\eta^2}{2} a_{2(1)}\right]\left. \\
&+ \eta_\nu \eta_\mu \left[-i \sqrt{\frac{i\eta^2}{\epsilon}} a_{3(0)} - \frac{i\eta^2}{2} a_{3(1)}\right]\right\} \]  

(106)

We can thus in effect replace the contribution in (102) by:

\[
\int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot (x-y)} \left\{ \delta \left(k^0 - \frac{1}{2} \frac{\epsilon \eta^2}{i} \right) \left[k_\mu k_\nu \left[-i \sqrt{\frac{i\eta^2}{\epsilon}} a_{1(0)} - \frac{i\eta^2}{2} a_{1(1)}\right]\right. \\
+ \eta_\mu \eta_\nu \left[-i \sqrt{\frac{i\eta^2}{\epsilon}} a_{2(0)} - \frac{i\eta^2}{2} a_{2(1)}\right]\left. \\
+ \eta_\nu \eta_\mu \left[-i \sqrt{\frac{i\eta^2}{\epsilon}} a_{3(0)} - \frac{i\eta^2}{2} a_{3(1)}\right]\right\} \]  

(107)

the curly bracket above gives the effective addition to the propagator.
We quote the values of $a_{i(n)}$, $n = 0, 1$; $i = 1, 2, 3$ for completeness:

\[
a_{1(0)} = -\frac{2i\eta^2}{\mathcal{K}_1} \ln(-1) = \frac{2\pi\eta^2}{\mathcal{K}_1},
\]

\[
a_{1(1)} = \frac{i\eta^2[\mathcal{K}_2 + k \to -k]}{\mathcal{K}_1^2} \ln(-1) = \frac{-4i\pi\eta^2[(\bar{\eta} \cdot \bar{k})^2 - \bar{k}^2]}{\mathcal{K}_1^2};
\]

\[
a_{2(0)} = 0,
\]

\[
a_{2(1)} = \frac{2i}{\mathcal{K}_1} \ln(-1) = -\frac{2\pi}{\mathcal{K}_1};
\]

\[
a_{3(0)} = 0,
\]

\[
a_{3(1)} = \frac{-2i}{\mathcal{K}_1} \ln(-1) = \frac{2\pi}{\mathcal{K}_1},
\]

where

\[
\mathcal{K}_1 \equiv \left((\bar{\eta} \cdot \bar{k})^2 - \bar{k}^2\right)(\eta^2 + i\epsilon);
\]

\[
\mathcal{K}_2 \equiv 2i\left((\bar{\eta} \cdot \bar{k})^2 - \bar{k}^2\right) + 2\bar{\eta} \cdot \bar{k}(\eta^2 + i\epsilon).
\]

In the treatment we have given in this section, we have found it necessary to keep $\eta^2$ nonzero in the intermediate stages. We can, however, define the LCG as a limit of the final result as $\eta^2 \to 0$. The procedure, here, is very analogous to the one we had to adopt in Appendix A, where we found it necessary to keep $\eta^2$ nonzero in the intermediate stages of calculations. We need, however, take this limit keeping $\epsilon$ nonzero. We find, by taking the limit $\eta^2 \to 0$ in \((107)-(109)\), that these extra terms vanish for LCG. This result can also be seen ab initio by looking at the contour integral over $C_1$. Here we note that the width of the branch cut shrinks as $\eta^2 \to 0$. This together with the fact that [for $\epsilon$ nonzero] the discontinuity is finite for $\eta^2 = 0$, leads to the above result.

7 Conclusion

In this work, we have dealt with the question of the treatment of $1/((\eta \cdot k)^p)$-type singularities in the axial/LCG gauges in an ab-initio manner. We have used the known treatment in the Lorentz gauges and connection between
Lorentz and axial gauges to achieve this. We have used the results established earlier on the FFBRS transformation that connect Green’s functions in these two gauges. We evaluated an \( \epsilon \)-dependent propagator in axial gauges via this procedure. We have suggested that this should give the correct way to deal with the \( 1/(\eta \cdot k)^p \)-type singularities in the axial propagator.

We find that this propagator, not surprisingly, coincides with the usual axial propagator except in a small region around \( \eta \cdot k = 0 \). The propagator does not show spurious poles at \( \eta \cdot k = 0 \). It is however complicated but predetermined in form in a small region near \( \eta \cdot k = 0 \). We have shown however that there is a way of effectively replacing this complicated structure of the propagator by a much simpler expression. We believe, this is a first ab-initio treatment of axial gauge poles in the path-integral formalism.

As mentioned in the introduction and elaborated in [15], the prescription obtained here also preserves the value of the Wilson loop.

A

In this appendix, we shall give the complete treatment for the second term in (61). In particular, we shall show that the contribution from this term vanishes as \( \epsilon \to 0 \).

We wish to consider the contribution of

\[
\frac{(\kappa_2 + \alpha)}{a_1 - \kappa_1} \left( \frac{1}{(\kappa_1 - \kappa_2)} \ln \left[ \frac{\kappa_2 - \kappa_1 \kappa_2}{\kappa_1 - \kappa_1 \kappa_2} \right] \right)
\]

(A1)

to the integral (60) occurring in the propagator expression (56). We, first, note that on account of \( \kappa_2(-\xi_1) = \kappa_1(\xi_1) \), we have that under \( \xi_1 \to -\xi_1 \),

\[
\frac{1}{(\kappa_1 - \kappa_2)} \ln \left[ \frac{\kappa_2 - \kappa_1 \kappa_2}{\kappa_1 - \kappa_1 \kappa_2} \right]
\]

(A2)

remains invariant.

Now, the contribution (A1) to the \( k_\mu k_\nu \) terms reads:

\[
k_\mu k_\nu \frac{(\xi_1 + i\xi_3)(\xi_1 + i\xi_3)}{(k^2 + i\epsilon)^2(1 - i\xi_1 - i\xi_2)(1 - i\xi_2 + \xi_1^2 + i\xi_2\xi_3)} \left[ \frac{i\lambda - \xi_1(1-\lambda)}{\xi_1 + i\xi_3} \right] \left[ \frac{1}{1 - i\xi_1 - i\xi_2 - \kappa_2} \right]
\]

26
\[
\begin{align*}
&\times \frac{1}{(\kappa_1 - \kappa_2)} \ln \left[ \frac{\kappa_2 - \kappa_1 \kappa_2}{\kappa_1 - \kappa_1 \kappa_2} \right] + (\xi_1 \to -\xi_1; \ \mu \leftrightarrow \nu) \\
&= -\frac{k_\mu k_\nu}{(k^2 + i\epsilon)^2(1 - i\xi_2 + \xi_2^2 + i\xi_2 \xi_3)} \frac{1}{(\kappa_1 - \kappa_2)} \ln \left[ \frac{\kappa_2 - \kappa_1 \kappa_2}{\kappa_1 - \kappa_1 \kappa_2} \right] \\
&\times \left[ \frac{(\xi_1 + i\xi_3)\kappa_2 + i\lambda - \xi_1(1 - \lambda) - (\xi_1 + i\xi_3)\kappa_2 + i\lambda + \xi_1(1 - \lambda)}{\kappa_1(1 + i\xi_1 - i\xi_2) - 1} \right].
\end{align*}
\tag{A3}
\]

The last square bracket in (A3) can be simplified as:
\[
\left[ \frac{(\xi_1 + i\xi_3)\kappa_2 + i\lambda - \xi_1(1 - \lambda)}{\kappa_1(1 + i\xi_1 - i\xi_2) - 1}[\kappa_2(1 - i\xi_1 - i\xi_2) - 1] + (\xi \to -\xi_1) \right].
\tag{A4}
\]

We note that the denominator is even under \(\xi_1 \to -\xi_1\). The numerator is now evaluated keeping in mind that \(\kappa_1 \kappa_2\) and \(\kappa_1 + \kappa_2\) are even under \(\xi_1 \to -\xi_1\) and \((\kappa_1 - \kappa_2)\) is odd. The net contribution of (A4) to the \(k_\mu k_\nu\) terms in the propagator reads:
\[
-\frac{k_\mu k_\nu}{(k^2 + i\epsilon)^2} f(\eta \cdot k, k^2, \eta^2, \lambda)
\tag{A5}
\]
with
\[
f \equiv \frac{1}{\sqrt{\gamma^2 - \beta}} \ln \left[ \frac{\kappa_2 - \kappa_1 \kappa_2}{\kappa_1 - \kappa_1 \kappa_2} \right] \frac{1}{1 - i\xi_2 + \xi_2^2 + i\xi_2 \xi_3} \\
\times i\xi_1^2(\beta - \gamma + \lambda\gamma) + i\xi_3[(\beta - \gamma) - i\xi_2 \beta] - \lambda[1 - \gamma(1 - i\xi_2)] + \sqrt{\gamma^2 - \beta^2} i\xi_1 \xi_2 (1 - \lambda).
\tag{A6}
\]

We note:
(i) for \(k^2 \neq 0\), \(\kappa_2 = 1/(1 + i\xi_2) = \kappa^*\) at \(\epsilon = 0\);
(ii) the denominator is well defined at \(\epsilon = 0\), for \(\eta \cdot k \neq 0\), \(k^2 \neq 0\);
(iii) \(\beta = \gamma\) at \(\lambda = 0\);
(iv) the numerator of (A6) vanishes at \(\lambda = 0, \epsilon = 0, k^2 \neq 0\).

Consequently,
\[
\lim_{\epsilon \to 0} f(\eta \cdot k \neq 0, k^2 \neq 0, \eta^2, \lambda = 0, \epsilon) = 0.
\tag{A7}
\]

Next, we treat the case \(k^2 = 0\). Here: (i) \(i\xi_2 = 0\); \(\xi_1 = -\frac{2k}{\epsilon}\.\)
(ii) \( \gamma = 0; \beta = \frac{\lambda^2}{(\eta \cdot k) + i \eta^2} \);

(iii) the denominator of (A6) simplifies to

\[
[-i\xi_1 \kappa_2 - 1][i\xi_1 \kappa_1 - 1] = \xi_1^2 \beta + 2i\xi_1 \sqrt{\gamma^2 - \beta} + 1 \quad (A8)
\]

Counting only the powers of \( \epsilon \):

\[
[-i\xi_1 \kappa_2 - 1][i\xi_1 \kappa_1 - 1] \equiv O(\epsilon^0);
\sqrt{\gamma^2 - \beta} \equiv O(\epsilon \sqrt{\lambda}), \lambda \neq 0;
\ln \left[ \frac{\kappa_2 - \kappa_1 \kappa_2}{\kappa_1 - \kappa_1 \kappa_2} \right] \equiv O(\epsilon^0);
\]

\[
\frac{1}{1 - i\xi_2 + \xi_1^2 + i\xi_2 \xi_3} \equiv O(\epsilon^2), \eta \cdot k \neq 0
\]

(numerator :) \( i\xi_1^2 \beta - i\lambda + \sqrt{\gamma^2 - \beta} i\xi_2 \xi_3 (1 - \lambda) \equiv O(\epsilon^0) \quad (A9)

Then:

\[
f(\eta \cdot k \neq 0, k^2 = 0, \eta^2, \lambda \neq 0, \epsilon) = 0(\epsilon) \quad (A10)
\]

and we find:

\[
\lim_{\epsilon \to 0} f(\eta \cdot k \neq 0, k^2 = 0, \eta^2, \lambda \neq 0, \epsilon) = 0. \quad (A11)
\]

We thereafter set \( \lambda = 0 \).

Finally, we shall deal with the subspace \( k^2 = \eta \cdot k = 0 \). Here, we note that \( \eta^2 \neq 0 \) is necessary to begin with for the procedure to be defined in the intermediate stages. Then:

\[
[-i\xi_1 \kappa_2 - 1][i\xi_1 \kappa_1 - 1] \equiv O(\epsilon^0);
\gamma = 0; \beta = \frac{\lambda \epsilon}{\eta^2}
\]

\[
\kappa_1 = -\kappa_2 = \sqrt{-\beta} = \sqrt{\gamma^2 - \beta} \equiv O\left(\frac{\lambda \epsilon}{\eta^2}\right);
\ln \left[ \frac{\kappa_2 - \kappa_1 \kappa_2}{\kappa_1 - \kappa_1 \kappa_2} \right] \equiv O(\epsilon^0);
\]

\[
\frac{1}{1 - i\xi_2 + \xi_1^2 + i\xi_2 \xi_3} \equiv O\left(\frac{\epsilon}{\eta^2}\right), \eta \cdot k \neq 0
\]

(numerator :) \( -i\lambda \equiv O(\epsilon^0 \lambda) \quad (A12)\)
Thus,
\[ f(\eta \cdot k \neq 0, k^2 = 0, \eta^2 \neq 0, \lambda \neq 0, \epsilon) = O(\sqrt{\frac{\lambda \epsilon}{\eta^2}}). \] (A13)
Thus we may set \( \lambda = 0. \) Then
\[ f(\eta \cdot k \neq 0, k^2 = 0, \eta^2 \neq 0, \lambda = 0, \epsilon) = 0. \] (A14)
We may then set \( \eta^2 = 0 \) at the end for LCG though \( \eta^2 \neq 0 \) is needed for the procedure to be well defined.

One can show that a similar analysis holds good for the contribution of (A1) to the \( \eta_\mu k_\nu \) terms in (56).

**B**

In this appendix, we shall consider the expression of the first term in (B1) involving
\[
\frac{1}{a_1 - \kappa_1} \ln \left[ \frac{\kappa_1 - a_1 \kappa_1}{a_1 - a_1 \kappa_1} \right] \quad (B1)
\]
which can be expanded in a Taylor series in powers of \( \frac{\kappa_1 - a_1}{a_1 (1 - \kappa_1)} \) in the domain defined by:
\[ |\kappa_1 - a_1| < |a_1 (1 - \kappa_1)|. \] (B2)
Further, the expression (B1) could be truncated as:
\[ - \frac{1}{a_1 (1 - \kappa_1)} + O(a_1 - \kappa_1) \] (B3)
and higher order terms neglected as \( \epsilon \to 0 \) if the expansion parameter
\[ \frac{\kappa_1 - a_1}{a_1 (1 - \kappa_1)} \to 0 \text{ as } \epsilon \to 0. \] (B4)
In this appendix, we shall seek the domains of validity of (B2) and (B4).

We divide out (B2) by \( |\kappa_1||a_1| \) (valid except where \( |\kappa_1| \) and \( |a_1| \) vanish). The equation (B2) then translates as:
\[ \left| \frac{1}{\kappa_1} - \frac{1}{a_1} \right| < \left| \frac{1}{\kappa_1} - 1 \right| \] (B5)
Defining $\frac{1}{\kappa_1} - 1 \equiv Y$ and $\frac{1}{a_1} - 1 \equiv -X = -i(\xi_1 + \xi_2)$, \(B2\) reads:

$$|Y + X| < |Y|.$$  \(\text{(B6)}\)

This simplifies to

$$\text{Re}\left(\frac{Y}{X}\right) < -\frac{1}{2}.$$  \(\text{(B7)}\)

We recall:

$$Y = \frac{1}{\kappa_1} - 1 = \frac{\kappa_2}{\beta} - 1 = \frac{\gamma}{\beta} - 1 - \sqrt{\left(\frac{\gamma}{\beta}\right)^2 - \frac{1}{\beta}}$$

$$= -i\xi_2\lambda - \sqrt{(1 - i\xi_2)(-\xi_2^2 - i\xi_2\xi_3) - i\xi_2\lambda(1 + \xi_2^2 - i\xi_2 + i\xi_2\xi_3)}$$

$$1 - i\xi_2(1 - \lambda)$$  \(\text{(B8)}\)

We analyze the condition \(\text{(B7)}\) at $\lambda = 0$, which is sufficient. The equation \(\text{(B7)}\) then comes:

$$\text{Im}\left[\sqrt{k^2 + i\epsilon}\frac{-i(\eta \cdot k)^2 - i\epsilon\eta^2}{\eta \cdot k + \epsilon}\right] > \frac{1}{2}.$$  \(\text{(B9)}\)

For $|\eta \cdot k| >> \epsilon$ and $|k^2| >> \epsilon$, it is evident that the left hand side is (using the convention for the square root as given below \(\text{(B3)}\))

$$\text{Im}\left[\frac{i\eta \cdot k}{\eta \cdot k + \epsilon}\right] \approx 1 > \frac{1}{2}.$$  \(\text{(B10)}\)

and thus \(\text{(B9)}\) is automatically satisfied In fact, for $\eta \cdot k \to 0^+$ also, (noting that the phase of $\sqrt{-(\eta \cdot k)^2 - i\epsilon\eta^2}$ varies continuously from $\frac{\pi}{2}$ as $\eta \cdot k \to 0^+$ from large values) \(\text{(B9)}\) reads, for $\eta^2 \neq 0$,

$$\text{Im}\left[\frac{\sqrt{-i\epsilon\eta^2}}{\epsilon}\right] = \frac{1}{\sqrt{2}} \frac{\sqrt{|\eta^2|}}{\sqrt{\epsilon}}.$$  \(\text{(B11)}\)

and is automatically $> \frac{1}{2}$ for $\epsilon$ sufficiently small. Also a shift of variables $\eta \cdot k + \epsilon = -\zeta$, will allow one to concluded in a similar manner that for $\zeta \to 0^+$ (i.e. $\eta \cdot k + \epsilon \to 0^-$) the condition \(\text{(B3)}\) is fulfilled for $\eta^2 \neq 0$. A careful analysis of \(\text{(B9)}\) in fact shows that \(\text{(B9)}\) is satisfied for $\eta^2 \neq 0$ and real $\eta \cdot k$ everywhere on the real $\eta \cdot k$ axis except the interval $(-\epsilon, 0)$. For $\eta^2 = 0,$
similarly, (B9) is valid over the real $\eta \cdot k$ axis except the interval $(-\epsilon, \epsilon)$. This is valid for any $k^2$.

Next, we analyze the condition (B4) required, in addition, for the truncation of the expansion (B3). The condition (B4), in notations below (B3) reads:

$$\left| 1 + \frac{X}{Y} \right| \rightarrow 0, \text{ as } \epsilon \rightarrow 0.$$  \hspace{1cm} (B12)

The condition (B12) can be analyzed at $\lambda = 0$. It then reads:

$$\left| 1 - \frac{i(\xi_1 + \xi_2)\sqrt{1 - i\xi_2}}{\sqrt{-\xi_1^2 - i\xi_2\xi_3}} \right| \rightarrow 0,$$  \hspace{1cm} (B13)

i.e.

$$\left| 1 - \frac{i(\eta \cdot k)\sqrt{\frac{k^2}{k^2 + i\epsilon}}}{\sqrt{-(\eta \cdot k)^2 - i\epsilon\eta^2}} \right| \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$  \hspace{1cm} (B14)

For $k^2 \neq 0$, and fixed $(\eta \cdot k)^2 \neq 0$, we can, in fact, by taking $\epsilon$ arbitrarily small, make the left-hand side of (B13) arbitrarily small. Thus, the conditions for the validity of the expansion of (B1) and its truncation of (B3) hold valid for any real $\eta \cdot k$ outside the interval $(-\epsilon, 0)$ for $\eta^2 \neq 0$ and $(-\epsilon, \epsilon)$ for $\eta^2 = 0$ provided $k^2 \neq 0$. The above analysis of (B13) however fails at $k^2 = 0$.

We find that for $k^2 = 0$, while the condition (B7) is satisfied in the domain mentioned, the condition (B12) is not satisfied. So, in this domain, we may not use the truncated expression for (B1).

C

In this appendix, we shall carry out the check that the propagator (56), via expression (62), in fact leads to the naive propagator at $\epsilon = 0$.

As noted in Appendix A, the contribution of the second term in (62), vanishes at $\epsilon = 0$. We shall now evaluate the contribution of the first term in (62). As noted in (68), this term reduces to:

$$\frac{1}{a_1 - \kappa_2} \frac{\alpha + a_1}{a_1} \ln \left[ \frac{\kappa_1 - a_1\kappa_2}{a_1 - a_1\kappa_1} \right] = -\left( \frac{a_1 + \alpha}{a_1} - i\xi_2 \right)^2 \frac{1}{2\xi_1^2}.$$  \hspace{1cm} (C1)
\( (\xi_1 = \frac{\eta \cdot k}{k^2} \text{ at } \epsilon = 0) \). Then the right-hand side of (56) at \( \epsilon = 0 \) reads:

\[
\frac{i}{(k^2)^2(1 - i\xi_1)(1 + \xi_1^2)} \left[ \frac{\xi_1 + i\xi_3}{1 + i\xi_1} + \frac{[i\lambda - \xi_1(1 - \lambda)]}{2\xi_1^2} \right] (1 - i\xi_1)^2(1 + \xi_1^2) k^2 \eta_{\mu} k_{\nu} \\
- (1 + i\xi_1) \left[ \frac{1}{1 + i\xi_1} - \frac{1}{1 - i\xi_1} \right] \eta_{\mu} k_{\nu} + (k \rightarrow -k; \mu \leftrightarrow \nu) 
\]

(C2)

A straightforward simplification then leads to the familiar result (69).

By using the identity

\[
\Sigma(\epsilon = 0) = 2i(\eta \cdot k)^2 k^2 \frac{d}{d\epsilon} \ln \Sigma_3|_{\epsilon=0} = \\
2i(\eta \cdot k)^2 k^2 \left[ -\frac{1}{\eta \cdot k} + i\frac{(\lambda - 1)}{2k^2} - \frac{i}{2(\eta \cdot k)^2} (\lambda k^2 + \eta^2) \right], 
\]

(C3)

one can show that the \( \epsilon = 0 \) limit of (70) also gives (69).

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Figure 1: Contours $C$ and $C_1$

Figure 2: Distortion of contour $C_1$; the dashed lines indicate the branch cuts