Random and Integrable Models in
Mathematics and Physics

Pierre van Moerbeke*

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Integrable Systems” at the “Centre de Recherche Mathématique”, Montréal, Canada (June
20-July 8, 2005). Department of Mathematics, Université Catholique de Louvain, 1348
Louvain-la-Neuve, Belgium and Brandeis University, Waltham, Mass 02454, USA. E-mail:
Pierre.Vanmoerbeke@uclouvain.be and vanmoerbeke@brandeis.edu. The support of a Na-
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During the last 15 years or so, and since the pioneering work of E. Wigner, F. Dyson and M. L. Mehta, random matrix theory, combinatorial and percolation questions have merged into a very lively area of research, producing an outburst of ideas, techniques and connections; in particular, this area contains a number of strikingly beautiful gems. The purpose of these five Montréal lectures is to present some of these gems in an elementary way, to develop some of the basic tools and to show the interplay between these topics. These lectures were written to be elementary, informal and reasonably self-contained and are aimed at researchers wishing to learn this vast and beautiful subject. My purpose was to explain these topics at an early stage,
rather than give the most general formulation. Throughout, my attitude has been to give what is strictly necessary to understand the subject. I have tried to provide the reader with plenty of references, although I may and probably will have forgotten some of them; if so, my apologies!

As we now know, random matrix theory has reached maturity and occupies a prominent place in mathematics, being at the crossroads of many subjects: number theory (zeroes of the Riemann zeta functions), integrable systems, asymptotics of orthogonal polynomials, infinite-dimensional diffusions, communication technology, financial mathematics, just to name a few. Almost 200 years ago A. Quetelet tried to establish universality of the normal distribution (mostly by empirical means). Here we are, trying to prove universality of the many beautiful statistical distributions which come up in random matrix theory and which slowly will find their way in everyday’s life.

This set of five lectures were given during the first week of a random matrix 2005-summer school at the “Centre de Recherches Mathématiques” in Montréal; about half of them are devoted to combinatorial models, whereas the remaining ones deal with related random matrix subjects. They have grown from another set of ten lectures I gave at Leeds (2002 London Mathematical Society Annual Lectures), and semester courses or lecture series at Brandeis University, at the University of California (Miller Institute, Berkeley), at the Universiteit Leuven (Francqui chair, KULeuven) and at the Université de Louvain (UCLouvain).

I would like to thank many friends, colleagues and graduate students in the audience(s), who have contributed to these lectures, through their comments, remarks, questions, etc..., especially Mark Adler, Ira Gessel, Alberto Grünbaum, Luc Haine, Arno Kuijlaars, Walter Van Assche, Pol Vanhaecke, and also Jonathan Delépine, Didier Vandercorstelen, Tom Claeys, Maurice Duits, Maarten Vanlessen, Aminul Huq, Dong Wang and many others.... Last, but not least, I would like to thank John Harnad for creating such a stimulating (and friendly) environment during the 2005-event on “random matrices” at Montréal. Finally, I would label it a success if this set of lectures motivated a few young people to enter this exciting subject.
1 Permutations, words, generalized permutations and percolation

1.1 Longest increasing subsequences in permutations, words and generalized permutations

(i) **Permutations** \( \pi := \pi_n \) of 1, \ldots, \( n \) are given by

\[
S_n \ni \pi_n = \begin{pmatrix} 1 & \ldots & n \\ j_1 & \ldots & j_n \end{pmatrix}, \quad 1 \leq j_1, \ldots, j_n \leq n \quad \text{all distinct integers}
\]

with \( \pi_n(k) = j_k \). Then

\[
\#S_n = n!.
\]

An *increasing subsequence* of \( \pi_n \in S_n \) is a sequence \( 1 \leq i_1 < \ldots < i_k \leq n \), such that \( \pi_n(i_1) < \ldots < \pi_n(i_k) \). Define

\[
L(\pi_n) = \text{length of a longest (strictly) increasing subsequence of } \pi_n.
\]

(1.1.1)

Notice that there may be many longest (strictly) increasing subsequences!

**Question:** Given uniform probability on \( S_n \), compute

\[
P^n(L(\pi_n) \leq k, \pi_n \in S_n) = \frac{\#\{L(\pi_n) \leq k, \pi_n \in S_n\}}{n!} = ?
\]

(Ulam’s problem 1961)

**Example:** for \( \pi_7 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 4 & 2 & 6 & 7 & 5 \end{pmatrix} \), we have \( L(\pi_7) = 4 \). A longest increasing sequence is underlined; it is not necessary unique.

(ii) **Words** \( \pi := \pi^q_n \) of length \( n \) from an alphabet 1, \ldots, \( q \) are given by integers

\[
S_q^n \ni \pi^q_n = \begin{pmatrix} 1 & \ldots & n \\ j_1 & \ldots & j_n \end{pmatrix}, \quad 1 \leq j_1, \ldots, j_n \leq q
\]

with \( \pi^q_n(k) = j_k \). Then

\[
\#S_q^n = q^n.
\]
An increasing subsequence of $\pi_n^q \in S_n^q$ is given by a sequence $1 \leq i_1 < \ldots < i_k \leq n$, such that $\pi_n^q(i_1) \leq \ldots \leq \pi_n^q(i_k)$. As before, define

$$L(\pi_n^q) = \text{length of the longest weakly increasing subsequence of } \pi_n^q.$$  

(1.1.2)

**Question:** Given uniform probability on $S_n^q$, compute

$$P_n^q (L(\pi_n^q) \leq k, \pi_n^q \in S_n^q) = \frac{\# \{L(\pi_n^q) \leq k, \pi_n^q \in S_n^q\}}{q^n} = ?$$

**Example:** for $\pi = \left(\begin{array}{cccc}1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 1 & 3 & 2 \end{array}\right) \in S_5^3$, we have $L(\pi) = 3$. A longest increasing sequence is underlined.

(iii) **Generalized permutations** $\pi := \pi_n^{p,q}$ are defined by an array of integers

$$GP_n^{p,q} \supseteq \pi_n^{p,q} = \left(\begin{array}{c}i_1 & \ldots & i_n \\ j_1 & \ldots & j_n \end{array}\right),$$

subjected to

$$1 \leq i_1 \leq i_2 \leq \ldots \leq i_n \leq p \quad \text{and} \quad 1 \leq j_1, \ldots, j_n \leq q$$

with $i_k = i_{k+1}$ implying $j_k \leq j_{k+1}$.

Then

$$\#GP_n^{p,q} = \binom{pq + n - 1}{n}.$$ 

An increasing subsequence of a generalized permutation $\pi$ is defined as

$$\left(\begin{array}{c}i_{r_1} \\ j_{r_1} \\ \ldots \\ i_{r_m} \\ j_{r_m} \end{array}\right) \subset \pi$$

with $r_1 \leq \ldots \leq r_m$ and $j_{r_1} \leq j_{r_2} \leq \ldots \leq j_{r_m}$. Define

$$L(\pi) := \text{length of the longest weakly increasing subsequence of } \pi.$$ 

**Example:** For $\left(\begin{array}{cccccc}1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 3 & 3 & 1 & 2 & 1 \end{array}\right) \in GP_{10}^{4,3}$, we have $L(\pi) = 5$.

For more information on these matters, see Stanley [80].
1.2 Young diagrams and Schur polynomials

Standard references to this subject are MacDonald [68], Sagan [77], Stanley [80], Stanton and White [81]. To set the notation, we remind the reader of a few basic facts.

- A partition $\lambda$ of $n$ (noted $\lambda \vdash n$) or a Young diagram $\lambda$ of weight $n$ is represented by a sequence of integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell \geq 0$, such that $n = |\lambda| := \lambda_1 + \ldots + \lambda_\ell$; $n = |\lambda|$ is called the weight. A dual Young diagram $\lambda^\top = ((\lambda^\top)_1 \geq (\lambda^\top)_2 \geq \ldots)$ is the diagram obtained by flipping the diagram $\lambda$ about its diagonal; set

$$\lambda^\top_i := (\lambda^\top)_i = \text{length of } i\text{th column of } \lambda.$$  \hfill (1.2.1)

Clearly $|\lambda| = |\lambda^\top|$. For future use, introduce the following notation:

$$Y := \{\text{all partitions } \lambda\}$$

$$Y_n := \{\text{all partitions } \lambda \vdash n\}$$

$$Y_p := \{\text{all partitions, with } \lambda^\top_1 \leq p\}$$

$$Y_p^n := \{\text{all partitions } \lambda \vdash n, \text{ with } \lambda^\top_1 \leq p\}$$  \hfill (1.2.2)

- A semi-standard Young tableau of shape $\lambda$ is an array of integers $a_{ij} > 0$ placed in box $(i, j)$ in the Young diagram $\lambda$, which are weakly increasing from left to right and strictly increasing from top to bottom.

- A standard Young tableau of shape $\lambda \vdash n$ is an array of integers $1, \ldots, n = |\lambda|$ placed in the Young diagram, which are strictly increasing from left to right and from top to bottom. For $\lambda \vdash n$, define

$$f^\lambda := \#\left\{\text{standard tableaux of shape } \lambda \right\}$$

- The Schur function $\tilde{s}_\lambda$ associated with a Young diagram $\lambda \vdash n$ is a symmetric function in the variables $x_1, x_2, \ldots$, (finite or infinite), defined as

$$\tilde{s}_\lambda(x_1, x_2, \ldots) = \sum_{\text{semi-standard tableaux } P \text{ of shape } \lambda} \prod_i x_i^{\# \text{times } i \text{ appears in } P}$$  \hfill (1.2.3)
It equals a polynomial \( s_\lambda(t_1, t_2, \ldots) \) (which will be denoted without the tilde) in the symmetric variables \( k t_k = \sum_{i \geq 1} x_i^k, \)

\[
\tilde{s}_\lambda(x_1, x_2, \ldots) = s_\lambda(t_1, t_2, \ldots) = \det (s_{\lambda_i-i+j}(t))_{1 \leq i, j \leq m}, \tag{1.2.4}
\]

for any \( m \geq n \). In this formula \( s_i(t) = 0 \) for \( i < 0 \) and \( s_i(t) \) for \( i \geq 0 \) is defined as

\[
e^{\sum_{i=0}^{\infty} t_i z^i} := \sum_{i \geq 0} s_i(t_1, t_2, \ldots) z^i.
\]

Note for \( \lambda \vdash n \),

\[
\tilde{s}_\lambda(x_1, x_2, \ldots) = f^\lambda x_1 \ldots x_n + \ldots = f^\lambda \frac{t_1^{\vert \lambda \vert}}{\vert \lambda \vert !} + \ldots
\]

- Given two partitions \( \lambda \supseteq \mu \), (i.e. \( \lambda_i \geq \mu_i \)), the diagram \( \lambda \setminus \mu \) denotes the diagram obtained by removing \( \mu \) form \( \lambda \). The skew-Schur polynomial \( s_{\lambda \setminus \mu} \) associated with a Young diagram \( \lambda \setminus \mu \vdash n \) is a symmetric function in the variables \( x_1, x_2, \ldots \), (finite or infinite), defined by

\[
\tilde{s}_{\lambda \setminus \mu}(x_1, x_2, \ldots) = \det (s_{\lambda_i-\mu_j-i+j}(t))_{1 \leq i, j \leq n}.
\]

Similarly, for \( \lambda \setminus \mu \vdash n \),

\[
\tilde{s}_{\lambda \setminus \mu}(x_1, x_2, \ldots) = f^{\lambda \setminus \mu} x_1 \ldots x_n + \ldots
\]

- The hook length of the \( i, j \)th box is defined by \( h^\lambda_{ij} := \lambda_i + \lambda_j^\top - i - j + 1 \). Also define

\[
h^\lambda := \prod_{(i,j) \in \lambda} h^\lambda_{ij} = \frac{\prod_{i=1}^{m} (m + \lambda_i - i)!}{\Delta_m(m + \lambda_1 - 1, \ldots, m + \lambda_m - m)}, \quad \text{for } m \geq \lambda_1^\top. \tag{1.2.5}
\]

\( h^\lambda_{ij} := \lambda_i + \lambda_j^\top - i - j + 1 \) is the hook length of the \( i, j \)th box in the Young diagram; i.e., the number of boxes covered by the hook formed by drawing a horizontal line emanating from the center of the box to the right and a vertical line emanating from the center of the box to the bottom of the diagram.
• The number of standard Young tableaux of a given shape $\lambda = (\lambda_1 \geq \ldots \geq \lambda_m)$ is given by

$$f^\lambda = \# \left\{ \text{standard tableaux of shape } \lambda \right\}$$

$$= \text{coefficient of } x_1 \ldots x_n \text{ in } \tilde{s}_\lambda(x)$$

$$= \frac{|\lambda|!}{u^{\lambda_1}} \tilde{s}_\lambda(x) \left|_{\sum_{i=1}^k x^k = \delta_{k1}u} \right. = \frac{|\lambda|!}{u^{\lambda_1}} s_\lambda(u, 0, 0, \ldots)$$

$$= \frac{|\lambda|!}{h^\lambda}$$

$$= |\lambda|! \det \left( \frac{1}{(\lambda_i - i + j)!} \right)_{1 \leq i, j \leq m}$$

$$= |\lambda|! \Delta_m(m + \lambda_1 - 1, \ldots, m + \lambda_m - m) \prod_{i=1}^m (m + \lambda_i - i)!$$

(1.2.6)

In particular, for any $m \geq \lambda_1^\top$ and arbitrary $u \in \mathbb{R}$,

$$s_\lambda(u, 0, 0, \ldots) = u^{\lambda_1} \frac{f^\lambda}{|\lambda|!} = u^{\lambda_1} \frac{\Delta_m(m + \lambda_1 - 1, \ldots, m + \lambda_m - m)}{\prod_{i=1}^m (m + \lambda_i - i)!}. \quad (1.2.7)$$

• The number of semi-standard Young tableaux of a given shape $\lambda \vdash n$, filled with numbers 1 to $q$ for $q \geq 1$:

$$\# \left\{ \text{semi-standard tableaux of shape } \lambda \right\}$$

$$\text{filled with numbers from 1 to } q$$

$$= \tilde{s}_\lambda(1, \ldots, 1, 0, 0, \ldots)$$

$$= s_\lambda(q, \frac{q}{2}, \frac{q}{3}, \ldots)$$

$$= \prod_{(i,j) \in \lambda} h^\lambda_{i,j}$$

---

2 using the Vandermonde determinant $\Delta_m(z_1, \ldots, z_m) = \prod_{1 \leq i < j \leq m} (z_i - z_j)$
\[
\Delta q(q + \lambda_1 - 1, \ldots, q + \lambda_q - q) = \left\{ \begin{array}{ll}
\prod_{i=1}^{q-1} i! & \text{when } q \geq \lambda_1^\top, \\
0 & \text{when } q < \lambda_1^\top,
\end{array} \right.
\]

(1.2.8)

using the fact that
\[
\prod_{(i,j) \in \lambda} (j - i + q) = \frac{\prod_{i=1}^{\lambda_1^\top}(q + \lambda_i - i)!}{\prod_{i=1}^{q-1} i!}.
\]

(1.2.9)

- Pieri’s formula: given an integer \( r \geq 0 \) and the Schur polynomial \( s_\mu \), the following holds
\[
s_\lambda s_r = \sum_{\mu \setminus \lambda = \text{horizontal strip} \quad \text{and} \quad |\mu \setminus \lambda| = r} s_\mu.
\]

(1.2.10)

Note \( \mu \setminus \lambda \) is an horizontal \( r \)-strip, when they interlace, \( \mu_1 \geq \lambda_1 \geq \mu_2 \geq \ldots \), and \( |\mu \setminus \lambda| = r \).

1.3 Robinson-Schensted-Knuth correspondence for generalized permutations

Define the set of \( p \times q \) integer matrices (see [80])
\[
\text{Mat}_{p,q} := \left\{ W = (w_{ij})_{1 \leq i \leq p, \ 1 \leq j \leq q} : w_{ij} \in \mathbb{Z}_{\geq 0} \text{ and } \sum_{i,j} w_{ij} = n \right\}
\]

Theorem 1.1 There is a 1-1 correspondence between the following three sets:
\[
GP_{n}^{p,q} \iff \left\{ \begin{array}{c}
two \text{ semi-standard Young tableaux } (P, Q), \\
of same, \ but \ arbitrary \ shape \ \lambda \vdash n, \ filled \\
resp. \ with \ integers \ (1, \ldots, p) \ and \ (1, \ldots, q)
\end{array} \right\} \iff \text{Mat}_{n}^{p,q}.
\]
\[
\pi \iff (P, Q) \iff W(\pi) = (w_{ij})_{1 \leq i \leq p, 1 \leq j \leq q}
\]
where

\[ w_{ij} = \# \left\{ \text{times that } \binom{i}{j} \in \pi \right\} . \]

Therefore, we have\(^3\)

\[
\binom{pq + n - 1}{n} = \# GP_{n}^{p,q} = \sum_{\lambda \vdash n} \tilde{s}_{\lambda}(1^p) \tilde{s}_{\lambda}(1^q) = \# Mat_{n}^{p,q}.
\] (1.3.1)

Also, we have equality between the length of the longest weakly increasing subsequence of the generalized permutation \(\pi\), the length of the first row of the associated Young diagram and the weight of the optimal path:

\[
L(\pi) = \lambda_1 = L(W) := \max_{\text{all such paths}} \left\{ \sum_{\text{paths starting from entry } (1,1) \text{ to } (p,q)} w_{ij}, \right. \sum_{\text{over right/down paths starting from entry } (1,1) \text{ to } (p,q)} \left. \right\}
\] (1.3.2)

**Sketch of Proof:**

Given a generalized permutation

\[
\pi = \begin{pmatrix}
i_1 & \ldots & i_n \\
j_1 & \ldots & j_n
\end{pmatrix},
\]

the correspondence constructs two semi-standard Young tableaux \(P, Q\) having the same shape \(\lambda\). This construction is inductive. Namely, having obtained two equally shaped Young diagrams \(P_k, Q_k\) from

\[
\begin{pmatrix}
i_1 & \ldots & i_k \\
j_1 & \ldots & j_k
\end{pmatrix}, \quad 1 \leq k \leq n
\]

with the numbers \((j_1, \ldots, j_k)\) in the boxes of \(P_k\) and the numbers \((i_1, \ldots, i_k)\) in the boxes of \(Q_k\), one forms a new diagram \(P_{k+1}\), by creating a new box in the first row of \(P\), containing the next number \(j_{k+1}\), according to the following rule:

(i) if \(j_{k+1} \geq \) all numbers appearing in the first row of \(P_k\), then one creates a new box with \(j_{k+1}\) in that box to the right of the first row,

\(^3\)Use the notation \(\tilde{s}_{\lambda}(1^p) = \tilde{s}_{\lambda}(1, \ldots, 1, 0, 0, \ldots)\).
(ii) if not, place $j_{k+1}$ in the box (of the first row) containing the smallest integer $> j_{k+1}$. The integer, which was in that box, then gets pushed down to the second row of $P_k$ according to the rule (i) or (ii), and the process starts afresh at the second row.

The diagram $Q$ is a bookkeeping device; namely, add a box (with the number $i_{k+1}$ in it) to $Q_k$ exactly at the place, where the new box has been added to $P_k$. This produces a new diagram $Q_{k+1}$ of same shape as $P_{k+1}$. The inverse of this map is constructed by reversing the steps above.

Formula (1.3.1) follows from (1.2.8).

**Example:** For $n = 10$, $p = 4$, $q = 3,$

$$GP_{n}^{p,q} \ni \pi = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 \\ 2 & 3 & 3 & 1 & 2 & 2 & 1 & 1 & 1 & 3 \end{pmatrix}, \text{ with } L(\pi) = 5$$

\[\downarrow\]

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 3 & 1 & 1 & 1 & 3 & 4 \\ 2 & 2 & 2 & 2 & 2 & 3 & 3 & 4 & \end{pmatrix}, \text{ with } L(\pi) = \lambda_1 = 5$$

\[\downarrow\]

$$W = \begin{pmatrix} \emptyset \rightarrow 1 & 2 \\ 0 \rightarrow 2 & 0 \\ 1 \rightarrow 0 & \emptyset \\ 1 \rightarrow 1 & \end{pmatrix}, \text{ with } L(\pi) = \sum_{(i,j) \in \text{path}} w_{ij} = 5$$

The RSK algorithm proceeds as follows:
adding 
\[
\begin{array}{cccc}
\begin{pmatrix} 2 \\ 1 \end{pmatrix} & \begin{pmatrix} 2 \\ 2 \end{pmatrix} & \begin{pmatrix} 2 \\ 2 \end{pmatrix} & \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\
\end{array}
\]

\[
P \quad 2 \quad 3 \quad 3 \\
2 \\
\Rightarrow 
\]

\[
Q \quad 1 \quad 1 \quad 1 \\
2 \\
\Rightarrow 
\]

\[
\begin{array}{cccc}
\begin{pmatrix} 3 \\ 2 \end{pmatrix} & \begin{pmatrix} 4 \\ 1 \end{pmatrix} & \begin{pmatrix} 4 \\ 3 \end{pmatrix} \\
\end{array}
\]

\[
\begin{array}{cccc}
1 \quad 1 \quad 1 \quad 2 \\
1 \quad 2 \quad 3 \\
3 \\
\Rightarrow 
\]

\[
\begin{array}{cccc}
1 \quad 1 \quad 1 \quad 3 \\
2 \quad 2 \quad 2 \\
3 \\
\Rightarrow 
\]

\[
\begin{array}{cccc}
1 \quad 1 \quad 1 \quad 3 \\
2 \quad 2 \quad 2 \\
3 \\
\Rightarrow 
\]

\[
= \begin{pmatrix} P \\ Q \end{pmatrix}
\]

yielding the set \((P, Q)\) of semi-standard Young tableaux above.

### 1.4 The Cauchy identity

**Theorem 1.2** Using the customary change of variables \(\sum_{k \geq 1} x_k^i = it_i\), \(\sum_{i \geq 1} y_i^j = is_i\), we have

\[
\prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} = \sum_{\lambda} \bar{s}_{\lambda}(x) \bar{s}_{\lambda}(y) = \sum_{\lambda} s_{\lambda}(t)s_{\lambda}(s) = e^{\sum_{i \geq 1} it_i s_i}.
\]
Proof: On the one hand, to every $\pi \in GP = \bigcup_{n,p,q} GP_{n}^{p,q}$, we associate a monomial, as follows

$$\pi \rightarrow \prod_{(i,j) \in \pi} x_i y_j \quad \text{(with multiplicities)} \quad (1.4.1)$$

Therefore, taking into account the multiplicity of $(i,j) \in \pi$,

$$\sum_{\pi \in GP} \prod_{(i,j) \in \pi} x_i y_j = \prod_{i,j \geq 1} \left(1 + x_i y_j + x_i^2 y_j^2 + x_i^3 y_j^3 + \ldots \right) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j}. \quad (1.4.2)$$

One must think of the product on the right hand side of (1.4.2) in a definite order, as follows:

$$\prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} = \frac{1}{1 - x_1 y_1} \frac{1}{1 - x_1 y_2} \frac{1}{1 - x_1 y_3} \ldots \times \frac{1}{1 - x_2 y_1} \frac{1}{1 - x_2 y_2} \frac{1}{1 - x_2 y_3} \ldots \times \ldots,$$

and similarly for the expanded version. Expanding out all the products,

$$\prod_{j \geq 1} \left(1 + x_1 y_j + x_1^2 y_j^2 + x_1^3 y_j^3 + \ldots \right) \prod_{j \geq 1} \left(1 + x_2 y_j + x_2^2 y_j^2 + x_2^3 y_j^3 + \ldots \right) \ldots, \quad (1.4.3)$$

leads to a sum of monomials, each of which can be interpreted as a generalized permutation, upon respecting the prescribed order. Vice-versa each generalized permutation can be found among those monomials. As an example illustrating identity (1.4.2), the monomial $x_1 y_2 x_1^2 y_3^2 x_3^2 y_2^2 x_3 y_3 x_4 y_1 x_4 y_2$, appearing in the expanded version of (1.4.3), maps into the generalized permutation

$$\begin{pmatrix} 1 & 1 & 1 & 3 & 3 & 4 & 4 \\ 2 & 3 & 3 & 2 & 2 & 3 & 1 & 2 \end{pmatrix}.$$
and vice-versa. On the other hand, to every couple of semi-standard Young tableaux \((P, Q)\), we associate

\[(P, Q) \rightarrow \prod_i x_i^{\# \text{times } i \text{ appears in } Q} \prod_j y_j^{\# \text{times } j \text{ appears in } P}.\]

Therefore, the Robinson-Schensted-Knuth construction, mapping the generalized permutation \(\pi\) into two semi-standard Young tableaux \((P, Q)\) of same shape \(\lambda\), implies

\[\prod_i x_i y_j = \prod_i x_i^{\# \text{times } i \text{ appears in } Q} \prod_j y_j^{\# \text{times } j \text{ appears in } P} \quad \forall \begin{pmatrix} i \\ j \end{pmatrix} \in \pi\]

Then, summing up over all \(\pi \in GP\), using the fact that RSK is a bijection, and using the definition of Schur polynomials, one computes

\[
\sum_{\pi \in GP} \prod_{\begin{pmatrix} i \\ j \end{pmatrix} \in \pi} x_i y_j = \\
\sum_{\text{all } (P, Q) \text{ with shape } P=\text{shape } Q} \prod_i x_i^{\# \text{times } i \text{ appears in } Q} \prod_j y_j^{\# \text{times } j \text{ appears in } P} = \\
\sum_{\lambda} \left( \sum_{\text{all } Q \text{ with shape } Q=\lambda} \prod_i x_i^{\# \text{times } i \text{ appears in } Q} \right) \left( \sum_{\text{all } P \text{ with shape } P=\lambda} \prod_j y_j^{\# \text{times } j \text{ appears in } P} \right) = \\
\sum_{\lambda} \tilde{s}_{\lambda}(x) \tilde{s}_{\lambda}(y),
\]
using the definition (1.2.3) of the Schur polynomial. The proof is finished by observing that

\[
\sum_{i \geq 1} it_is = \sum_{i \geq 1} \sum_{k \geq 1} \sum_{\ell \geq 1} \frac{x_k^i y_\ell^i}{i} = \sum_{k, \ell \geq 1} \sum_{i \geq 1} \frac{(x_k y_\ell)^i}{i} = \log \prod_{k, \ell \geq 1} (1 - x_k y_\ell)^{-1},
\]

ending the proof of Theorem 1.2.

\[\text{Proposition 1.3}
\]

For permutations, we have a 1-1 correspondence between

\[S_n \iff \left\{ \begin{array}{l}
\text{two standard Young tableaux } (P, Q), \\
\text{of same shape } \lambda \text{ and size } n, \\
\text{each filled with numbers } (1, \ldots, n)
\end{array} \right\} \iff \text{Mat}^{n,n}(0,1).
\]

\[\pi_n \iff (P, Q) \iff W(\pi) = (w_{ij})_{i,j \geq 1},
\]

\[1.5 \quad \text{Uniform Probability on Permutations, Plancherel Measure and Random Walks}
\]

\[1.5.1 \quad \text{Plancherel measure}
\]

In this section, one needs

\[\text{Mat}^{n,n}(0,1) := \left\{ W = (w_{ij})_{1 \leq i,j \leq n}, \begin{array}{l}
\text{with exactly one } 1 \\
in \text{each row and column} \\
\text{and otherwise all zeros}
\end{array} \right\}
\]

See [80, 25, 23, 26] and references within.
Uniform probability $P^n$ on $S_n$ induces a probability $\tilde{P}^n$ (Plancherel measure) on Young diagrams $Y_n$, given by ($m := \lambda^\top$)

$$\tilde{P}^n(\lambda) = \frac{1}{n!} \# \left\{ \text{permutations leading to shape } \lambda \right\} = \frac{(f^\lambda)^2}{n!} = n! \, s_\lambda(1,0,\ldots)^2$$

$$= n! \frac{\Delta_m(m + \lambda_1 - 1,\ldots,m + \lambda_m - m)^2}{\prod_{i=1}^{m} (m + \lambda_i - i)!}$$

and so

$$\#S_n = \sum_{\lambda\vdash n} (f^\lambda)^2 = n!.$$

Finally, the length of the longest increasing subsequence in permutation $\pi_n$, the length of the first row of the partition $\lambda$ and the weight of the optimal path $L(W)$ in the percolation matrix $W(\pi)$ are all equal:

$$L(\pi_n) = \lambda_1 = L(W) := \max_{\text{all such paths}} \left\{ \sum_{\text{over right/down paths starting from entry (1,1) to (n,n)}} w_{ij}, \text{ over right/down paths starting from entry (1,1) to (n,n)} \right\}.$$

Hence

$$P^n(L(\pi) \leq \ell) = \sum_{\lambda \in Y_n} \frac{(f^\lambda)^2}{n!} = n! \sum_{\lambda \in Y_n, \lambda_1 \leq \ell} s_\lambda(1,0,\ldots)^2.$$

Proof: A permutation is a generalized permutation, but with integers $i_1,\ldots,i_n$ and $j_1,\ldots,j_n$ all distinct and thus both tableaux $P$ and $Q$ are standard.

Consider now the uniform probability $P^n$ on permutations in $S_n$; from the RSK correspondence we have the one-to-one correspondence, given a fixed partition $\lambda$,

$$\left\{ \text{permutations leading to the shape } \lambda \right\} \iff \left\{ \text{standard tableaux of shape } \lambda, \text{ filled with integers } 1,\ldots,n \right\} \times \left\{ \text{standard tableaux of shape } \lambda, \text{ filled with integers } 1,\ldots,n \right\}$$
and thus, using formulae \((1.2.6)\) and \((1.2.8)\) and noticing that \(\tilde{s}_\lambda(1^q) = 0\) for \(\lambda^\top > q\),

\[
\tilde{P}^n(\lambda) = \frac{1}{n!} \# \{ \text{permutations leading to the shape } \lambda \} = \frac{(f^\lambda)^2}{n!}, \quad \lambda \in \mathbb{Y}_n,
\]

with

\[
\sum_{\lambda \in \mathbb{Y}_n} \tilde{P}^n(\lambda) = 1.
\]

Formula \((1.5.2)\) follows immediately from the explicit values \((1.2.6)\) and \((1.2.8)\) for \(f^\lambda\).

\[\textbf{Example}:\] For permutation \(\pi_5 = \left(\begin{array}{cccc}1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 3 & 2 \end{array}\right) \in S_5\), the RSK algorithm gives

\[
\begin{array}{cccccc}
\hline
P & \Rightarrow & 5 & 1 & 4 & 3 & 2 \\
& & 5 & 5 & 4 & 3 & 2 \\
& & & 5 & 4 & 3 & 1 \\
\hline
Q & \Rightarrow & 1 & 1 & 3 & 1 & 3 \\
& & 2 & 2 & 2 & 2 & 1 \\
& & & 4 & 4 & 3 & 0 \\
\hline
\end{array}
\]

Hence

\[
\pi_5 \downarrow
\]

\[
(P, Q) = \left(\begin{array}{c}
\begin{array}{c}
\text{standard}
\end{array}
\end{array}\right), \quad \left(\begin{array}{c}
\begin{array}{c}
\text{standard}
\end{array}
\end{array}\right) \quad \Rightarrow \quad \left(\begin{array}{c}
\begin{array}{c}
\text{standard}
\end{array}
\end{array}\right)
\]

\[
\left(\begin{array}{cccc}
(0) & 0 & 0 & 0 & 1 \\
\downarrow & 0 & 0 & 0 & 0 \\
\downarrow & 0 & 0 & 1 & 0 & 0 \\
\downarrow & 0 & 1 & 0 & 0 \\
\hline
\end{array}\right)
\]

\[\textbf{Remark}:\] The Robinson-Schensted-Knuth correspondence has the following properties

\[
\bullet \pi \mapsto (P, Q), \text{ then } \pi^{-1} \mapsto (Q, P)
\]
• length (longest increasing subsequence of $\pi$) = # (columns in $P$)
• length (longest decreasing subsequence of $\pi$) = # (rows in $P$)
• $\pi^2 = I$, then $\pi \mapsto (P, P)$
• $\pi^2 = I$, with $k$ fixed points, then $P$ has exactly $k$ columns of odd length.

(1.5.3)

1.5.2 Solitaire game

With Aldous and Diaconis [14], consider a deck of cards $1, \ldots, n$ thoroughly shuffled and put those cards one at a time into piles, as follows:

(1) a low card may be placed on a higher card, or can be put into a new pile to the right of the existing pile.

(2) only the top card of the pile is seen. If the card which turns up is higher than the card showing on the table, then start with that card a new pile to the right of the others.

**Question:** What is the optimal strategy which minimizes the number of piles?
**Answer:** put the next card always on the leftmost possible pile!

**Example:** Consider a deck of 7 cards, appearing in the order 3, 1, 4, 2, 6, 7, 5. The optimal strategy is as follows:

```
  3 3 3 4 3 4
  1 1 1 2 1 2
  3 4 6 3 4 6 7
  1 2 1 2 1 2 5
```

This optimal strategy leads to 4 piles! For a deck of 52 cards, you will find in the average between 10-13 piles and having 9 piles or less occurs approximately 5% of the times. It turns out that, for a given permutation, number of piles = length of longest increasing sequence.
1.5.3 Anticipating large $n$ asymptotics

Anticipating the results in section \[4.2\] and \[9.2\] given the permutations of $(1, \ldots, n)$, given a percolation matrix of size $n$ and given a card game of size $n$, the numbers fluctuate about $2\sqrt{n}$ like

$$L(\pi_n) = \text{length of the longest increasing subsequence}$$
$$= \text{weight of the optimal increasing path in the \((0, 1)\)-percolation matrix}$$
$$= \text{number of piles in the solitaire game}$$
$$\simeq 2\sqrt{n} + n^{1/6} \mathcal{F},$$

where $\mathcal{F}$ is a probability distribution, the Tracy-Widom distribution, with

$$E(\mathcal{F}) = -1.77109 \text{ and } \sigma(\mathcal{F}) = 0.9018.$$ 

In particular for $n = 52$ cards,

$$E(L(\pi_{52})) \simeq 2\sqrt{52} + (52)^{1/6}(-1.77109) = 11.0005.$$ 

The Tracy-Widom distribution will be discussed extensively in section \[9.2\].

1.5.4 A transition probability and Plancherel measure

**Proposition 1.4** \([91, 25]\) $P_n$ on $\mathbb{Y}_n$ can be constructed from $P_{n-1}$ on $\mathbb{Y}_{n-1}$, by means of a transition probability, as follows

$$P_n(\mu) = \sum_{\lambda \in \mathbb{Y}_{n-1}} P_{n-1}(\lambda)p(\lambda, \mu), \quad \mu \in \mathbb{Y}_n$$

where

$$p(\lambda, \mu) := \begin{cases} \frac{f^\mu}{f^n[\mu]} & \text{if } \lambda \in \mathbb{Y}_{n-1} \text{ and } \mu \in \mathbb{Y}_n \text{ are} \\ 0 & \text{otherwise} \end{cases}$$

is a transition probability, i.e.

$$\sum_{\mu \in \mathbb{Y}_n, \mu = \lambda + \square} p(\lambda, \mu) = 1, \quad \text{for fixed } \lambda.$$
Proof: Indeed, for fixed $\mu$, one computes

$$\sum_{\lambda \in \mathcal{Y}_{n-1}} P_{n-1}(\lambda)p(\lambda, \mu) = \sum_{\lambda \in \mathcal{Y}_{n-1}} \frac{(f^\lambda)^2}{(n-1)!} \left( \frac{f^\mu}{f^\lambda} \right) \sum_{\lambda \in \mathcal{Y}_{n-1}} \lambda, \mu \in \mathcal{Y}_n$$

$$= \frac{f^\mu}{n!} \sum_{\lambda \in \mathcal{Y}_{n-1}} f^\lambda \sum_{\lambda \in \mathcal{Y}_{n-1}} \lambda, \mu \in \mathcal{Y}_n$$

$$= \frac{(f^\mu)^2}{n!} = P_n(\mu).$$

Indeed, given a standard tableau of shape $\lambda$, filled with $1, \ldots, n-1$, adjoining a box to $\lambda$ such as to form a Young diagram $\mu$ and putting $n$ in that box yield a new standard tableau (of shape $\mu$).

That $p(\lambda, \mu)$ is a transition probability follows from Pieri’s formula (1.2.10), applied to $r = 1$, upon putting $t_i = \delta_{1i}$:

$$\sum_{\lambda \in \mathcal{Y}_n} p(\lambda, \mu) = \sum_{\lambda \in \mathcal{Y}_{n-1}} \frac{f^{\lambda+\square}}{|\lambda+\square|} \frac{1}{f^\lambda} = \sum_{\lambda \in \mathcal{Y}_{n-1}} \frac{f^{\lambda+\square}}{|\lambda+\square|!} \frac{|\lambda|!}{f^\lambda} = 1.$$

**Corollary 1.5** The following probability

$$P_n(\mu_1 \leq x_1, \ldots, \mu_k \leq x_k)$$

decreases, when $n$ increases.

Proof: Indeed

$$P_n(\mu_1 \leq x_1, \ldots, \mu_k \leq x_k)$$

$$= \sum_{\lambda \in \mathcal{Y}_{n-1}} \sum_{\mu = \lambda + \square} P_{n-1}(\lambda)p(\lambda, \mu)$$

$$= \sum_{\lambda \in \mathcal{Y}_{n-1}, \mu \in \mathcal{Y}_n} P_{n-1}(\lambda)p(\lambda, \mu)$$

$$\leq \sum_{\lambda \in \mathcal{Y}_{n-1}, \mu \in \mathcal{Y}_n} P_{n-1}(\lambda) = P_{n-1}(\lambda_1 \leq x_1, \ldots, \lambda_k \leq x_k),$$

proving Corollary 1.5.

---

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1.5.5 Random walks

Consider \( m \) random walkers in \( \mathbb{Z} \), starting from distinct points \( x := (x_1 < \ldots < x_m) \), such that, at each moment, only one walker moves either one step to the left or one step to the right. Notice that \( m \) walkers in \( \mathbb{Z} \), obeying this rule, is tantamount to a random walk in \( \mathbb{Z}^m \), where at each point the only moves are

\[ \pm e_1, \ldots, \pm e_m, \]

with all possible moves equally likely. That is to say the walk has at each point \( 2m \) possibilities and thus at time \( T \) there are \( (2m)^T \) different paths. Denote by \( P_x \) the probability for such a walk, where \( x \) refers to the initial condition. Requiring these walks not to intersect turns out to be closely related to the problem of longest increasing subsequences in random permutations, as is shown in the Proposition below. For skew-partitions, see section 1.2. For references, see [80, 10].

**Proposition 1.6**

\[
P_x \left( \begin{array}{c}
\text{that } m \text{ walkers in } \mathbb{Z}, \\
\text{reach } y_1 < \ldots < y_m \\
in T \text{ steps, without} \\
ever \text{ever intersecting}
\end{array} \right) = \frac{1}{(2m)^T} T \begin{pmatrix} T_L & T_R \end{pmatrix} \sum_{\lambda \text{ with } |\lambda|_L = T_L \atop |\lambda|_R = T_R} f^{\lambda \mu} f^{\lambda \nu}
\]

(1.5.4)

where \( \mu, \nu \) are fixed partitions defined by the points \( x_i \) and \( y_i \),

\[
\begin{align*}
\mu_k &= k - 1 - x_k, \quad \nu_k = k - 1 - y_k \\
T_L &= \frac{1}{2}(T + \sum_{i=1}^{m} (x_i - y_i)) = \frac{1}{2}(T - |\mu| + |\nu|) \\
T_R &= \frac{1}{2}(T - \sum_{i=1}^{m} (x_i - y_i)) = \frac{1}{2}(T + |\mu| - |\nu|) \\
T &= T_L + T_R, \quad \sum_{i=1}^{m} (x_i - y_i) = T_L - T_R.
\end{align*}
\]
In particular, close packing of the walkers at times 0 and T implies

\[
P_{1,\ldots,m}\left(\begin{array}{c}
\text{that } m \text{ walkers in } \mathbb{Z} \text{ reach} \\
1,\ldots,m \text{ in } 2n \text{ steps,}
\end{array}\right)
\]

without ever intersecting

\[
= \frac{1}{(2m)^{2n}} \binom{2n}{n} \sum_{\lambda^\top \leq m} (f^\lambda)^2
\]

\[
= (2n)! \frac{\#\{\pi_n \in S_n \text{ such that } L(\pi_n) \leq m\}}{n! (2m)^{2n}}
\]

(1.5.5)

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

Six non-intersecting walkers leaving from and returning to 1, \ldots, 6.

Proof: Step 1: Associate to a given walk a sequence of \(T_L\) \(L\)’s and \(T_R\) \(R\)’s:

\[
L \ R \ R \ R \ L \ R \ L \ L \ R \ \ldots \ R ,
\]

(1.5.6)
thus recording the nature of the move, left or right, at the first instant, at the second instant, etc...

If the $k^{th}$ walker is to go from $x_k$ to $y_k$, then

$$y_k - x_k = \# \left\{ \text{right moves for } k^{th} \text{ walker} \right\} - \# \left\{ \text{left moves for } k^{th} \text{ walker} \right\}$$

and so, if

$$T_L := \# \left\{ \text{left moves for all } m \text{ walkers} \right\} \quad \text{and} \quad T_R := \# \left\{ \text{right moves for all } m \text{ walkers} \right\},$$

we have, since at each instant exactly one walker moves,

$$T_R + T_L = T \quad \text{and} \quad T_R - T_L = \sum_{1}^{m} (y_k - x_k),$$

from which

$$T_{\frac{L}{L}} = \frac{1}{2} \left( T + \sum_{1}^{m} (x_k - y_k) \right).$$

Next, we show there is a canonical way to map a walk, corresponding to (1.5.6) into one with left moves only at instants $1,\ldots,T_L$ and then right moves at instants $T_L + 1,\ldots,T_L + T_R = T$, thus corresponding to a sequence

$$L \, L \, L \, \ldots \, L \quad R \, R \, R \, \ldots \, R.$$ (1.5.7)

This map, originally found by Forrester [42] takes on here a slightly different (but canonical) form. Indeed, in a typical sequence, as (1.5.6),

$$L \, R \, R \, \underline{R} \, L \, R \, L \, L \, \underline{R} \, \ldots \, R,$$ (1.5.8)

consider the first sequence $R \, L$ (underlined) you encounter, in reading from left to right. It corresponds to one of the following three configurations (in
the left column),

\[
\begin{align*}
L & \quad | \quad | \quad | \quad | \quad | \quad | \\
R & \quad | \quad | \quad | \quad | \quad | \\
\Rightarrow & \quad R \\
L & \quad | \quad | \quad | \quad | \quad | \quad | \\
R & \quad | \quad | \quad | \quad | \quad | \\
\Rightarrow & \quad R \\
L & \quad | \quad | \quad | \quad | \quad | \\
R & \quad | \quad | \quad | \quad | \quad |
\end{align*}
\]

which then can be transformed into a new configuration \( LR \), with same beginning and end, thus yielding a new sequence; in the third case the reflection occurs the first place it can. So, by the moves above, the original configuration (1.5.6) can be transformed in a new one. In the new sequence, pick again the first sequence \( RL \), reading from left to right, and use again one of the moves. So, this leads again to a new sequence, etc...

\[
\begin{align*}
L & \quad R \quad R \quad R \quad L \quad R \quad L \quad L \quad R \quad R \ldots R \\
L & \quad R \quad R \quad R \quad L \quad L \quad L \quad R \quad L \quad R \ldots R \\
L & \quad R \quad R \quad L \quad R \quad R \quad L \quad L \quad R \quad R \ldots R \\
L & \quad L \quad R \quad R \quad R \quad L \quad L \quad R \quad L \quad R \ldots R \\
\vdots
L & \quad L \quad L \quad \ldots \quad L \\
R & \quad R \quad R \quad \ldots \quad R \\
T_L & \quad T_R \\
\end{align*}
\]

Since this procedure is invertible, it gives a one-to-one map between all the left-right walks corresponding to a given sequence, with \( T_L \) L’s and \( T_R \) R’s

\[
L \quad R \quad R \quad R \quad L \quad R \quad L \quad L \quad R \ldots R ; \quad (1.5.10)
\]

and all the walks corresponding to

\[
\begin{align*}
L \quad L \quad L \quad \ldots \quad L \\
R & \quad R \quad R \quad \ldots \quad R \\
T_L & \quad T_R \\
\end{align*}
\]

(1.5.11)
Thus, a walk corresponding to (1.5.11) will map into \( T_{T_L}^T T_{T_R} \) different walks, corresponding to the \( T_{T_L}^T T_{T_R} \) number of permutations of \( T_L L \)'s and \( T_R R \)'s.

**Step 2:** To two standard tableaux \( P, Q \) of shape \( \lambda = (\lambda_1 \geq \ldots \geq \lambda_m > 0) \) we associate a random walk, going with (1.5.11), in the following way. Consider the situation where \( m \) walkers start at 0, 1, \ldots, \( m - 1 \).

\[ \begin{array}{cccccc}
& & & & & \\
1^{st} \text{ walker} & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\
2^{nd} \text{ walker} & c_{21} & c_{22} & c_{23} & & & \\
3^{rd} \text{ walker} & c_{31} & c_{32} & c_{33} & & & \\
& c_{41} & & & & & \\
\end{array} \]

- The 1\(^{st}\) walker starts at 0 and moves to the left, only at instants
  
  \[ c_{1i} = \text{content of box}(1, i) \in P \]

  and thus has made, in the end, \( \lambda_1 \) steps to the left.

\[ \vdots \]

- The \( k^{th} \) walker (1 \( \leq k \leq m \)) starts at \( k - 1 \) and moves to the left, only at instants
  
  \[ c_{ki} = \text{content of box}(k, i) \in P. \]

  and thus has made, in the end, \( \lambda_2 \) steps to the left.

\[ \vdots \]

- Finally, walker \( m = \lambda_1^\top \) walks according to the contents of the last row.

  Since the tableau is standard, filled with the numbers 1, \ldots, \( n \) the walkers never meet and at each moment exactly one walker moves, until instant \( n = |\lambda| \), during which they have moved from position

  \[ 0 < 1 < \ldots < k - 1 < \ldots < m - 1 \]
to position

\[-\lambda_1 + 0 < -\lambda_2 + 1 < \ldots < -\lambda_k + k - 1 < \ldots < -\lambda_m + m - 1\]

That is to say the final position is given by unfolding the right hand boundary of \(\lambda\), the horizontal (fat) segments refer to gaps between the final positions and vertical segments refer to contiguous final positions.

In the same fashion, one associates a similar walk to the other tableau \(Q\), with the walkers also moving left. These walkers will have reached the same position as in the first case, since the final position only depends on the shape of \(P\) or \(Q\). Therefore, reversing the time for the second set of walks, one puts the two walks together, thus yielding \(m\) non-intersecting walkers moving the first half of the time to the left and then the second half of the time to the right, as in the example below. Therefore the number of ways that \(m\) walkers start from and return to \(0, \ldots, m - 1\), without ever meeting each other, by first moving to the left and then to the right, is exactly \(\sum_{\lambda_1 \leq m} (f^\lambda)^2\).

\[
(P, Q) = \begin{pmatrix}
1 & 2 \\
3 & 4 \\
\text{standard} & \text{standard}
\end{pmatrix}, \quad \begin{pmatrix}
1 & 3 \\
2 & 4 \\
3 & 5
\end{pmatrix}
\]

More generally, an analogous argument shows that the number of ways that walkers leave from \(x_1 < \ldots < x_m\) and end up at \(y_1 < \ldots < y_m\) at time
\[ T, \text{ without intersection and by first moving to the left and then to the right, is given by} \]
\[ \sum_{\lambda^\top \leq m} f^{\lambda \mu} f^{\lambda \nu} \]  
\[ (1.5.12) \]

On the other hand, there are sequences of \( T_L \)'s and \( T_R \)'s, which combined with (1.5.12) yields formula (1.5.4).

In the close packing situation, one has \( \mu_k = \nu_k = 0 \) for all \( k \), and so \( \mu = \nu = \emptyset \) and \( T_L = T_R = T/2 \). With these data, (1.5.5) is an immediate consequence of (1.5.4).

\[ \text{1.6 Probability measure on words} \]

Remember from section 1.1 words \( \pi := \pi_n^q \) of length \( n \) from an alphabet \( 1, \ldots, q \) and from section 1.2 the set \( \mathbb{V}_n^q \) of partitions \( \lambda \vdash n \), with \( \lambda^\top \leq q \). Also define the set of \( n \times q \) matrices,
\[ \tilde{\text{Mat}}_{n,q}^n(0, 1) := \left\{ W = (w_{ij})_{1 \leq i,j \leq n}, \begin{array}{c} \text{with exactly one } 1 \\ \text{in each row} \\ \text{and otherwise all zeros} \end{array} \right\} \]

For references, see [80, 86].

**Proposition 1.7** In particular, for words, we have the 1-1 correspondence

\[ S_n^q \leftrightarrow \left\{ \begin{array}{l} \text{semi-standard and standard Young tableaux} \\ (P, Q) \text{ of same shape and of size } n, \text{ filled} \\ \text{resp., with integers } (1, \ldots, q) \text{ and } (1, \ldots, n), \end{array} \right\} \leftrightarrow \tilde{\text{Mat}}_{n,q}^n(0, 1). \]

\[ \pi \longleftrightarrow (P, Q) \longleftrightarrow W(\pi) = (w_{ij})_{i,j \geq 1}. \]  
\[ (1.6.1) \]

Uniform probability \( P_{n,q} \) on \( S_n^q \) induces a probability \( \tilde{P}_{n,q} \) on Young diagrams
$\lambda \in \mathcal{Y}_n^{(q)}$, given by

$$
\tilde{P}^{n,q}(\lambda) = \frac{1}{q^n} \tilde{\#} \left\{ \begin{array}{c}
\text{words in } S_n^q \\
\text{leading to shape } \lambda
\end{array} \right\} = \frac{f^\lambda \tilde{s}_\lambda(1^q)}{q^n} = \frac{n!}{q^n} s_\lambda(1,0,\ldots) s_\lambda(q, \frac{q}{2}, \frac{q}{3}, \ldots) = \frac{n!}{q^n} \Delta_q(q + \lambda_1 - 1, \ldots, q + \lambda_q - q)^2 \prod_{i=1}^{q-1} \frac{i!}{(q + \lambda_i - i)!}.
$$

(1.6.2)

Also,

$$
\# S_n^q = \sum_{\lambda \vdash n} f^\lambda \tilde{s}_\lambda(1^q) = q^n.
$$

(1.6.3)

Finally, given the correspondence (1.6.1), the length $L(\pi)$ of the longest weakly increasing subsequence of the word $\pi$ equals

$$
L(\pi) = \lambda_1 = L(W) := \max_{\text{all such paths}} \left\{ \sum_{\text{paths starting from entry (1,1) to (n,q)}} w_{ij} \right\},
$$

and thus

$$
P^{n,q}(L(\pi) \leq \ell) = \sum_{\substack{\lambda_1 \leq \ell \\
\lambda \in \mathcal{Y}_n^{(q)}}} \frac{n!}{q^n} s_\lambda(1,0,\ldots) s_\lambda(q, \frac{q}{2}, \frac{q}{3}, \ldots).
$$

Proof: A word is a special instance of generalized permutation, where the numbers $i_1, \ldots, i_n$ are all distinct. Therefore the RSK correspondence holds as before, except that $Q$ becomes a standard tableau; thus, a word maps to a pair of arbitrary Young tableaux $(P,Q)$, with $P$ semi-standard and $Q$ standard and converse. Also the correspondence with integer matrices is the same, with the extra requirement that the matrix contains 0 and 1’s, with each row containing exactly one 1.

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Consider now the uniform probability \( P_{n,q} \) on words in \( S_q^n \); from the RSK correspondence we have the one-to-one correspondence, given a fixed partition \( \lambda \),

\[
\left\{ \text{words in } S_q^n \quad \text{leading to shape } \lambda \right\} \leftrightarrow \left\{ \text{semi-standard tableaux of shape } \lambda, \text{ filled with integers } 1, \ldots, q \right\} \times \left\{ \text{standard tableaux of shape } \lambda, \text{ filled with integers } 1, \ldots, n \right\}
\]

and thus, using formulae (1.2.6) and (1.2.8) and noticing that \( \bar{s}_\lambda(1^q) = 0 \) for \( \lambda^\top \) > \( q \),

\[
\tilde{P}^{n,q}(\lambda) = \frac{1}{q^n} \# \left\{ \text{words leading to the shape } \lambda \right\} = \frac{\bar{s}_\lambda(1^q)f^\lambda}{q^n}, \quad \lambda \in \mathcal{Y}_n^q,
\]

with

\[
\sum_{\lambda \in \mathcal{Y}_n^q} \tilde{P}^{n,q}(\lambda) = 1.
\]

Formula (1.6.2) follows immediately from an explicit evaluation (1.2.6) and (1.2.8) for \( f^\lambda \) and \( s_\lambda(1^q) \).

\[\blacksquare\]

**Example:** For word

\[
\pi = \left( \begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
2 & 1 & 1 & 3 & 2 \\
1 & 3 & 4 & 1 & 3 \\
2 & 2 & 2 & 2 & 5
\end{array} \right) \in S_5^3
\]

the RSK algorithm gives

\[
\begin{array}{ccccc}
2 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
1 & 3 & 4 & 1 & 3 \\
2 & 2 & 2 & 2 & 5
\end{array}
\]

Hence

\[
\pi \mapsto \left( \begin{array}{cc}
\left( \begin{array}{cc}
1 & 1 \\
2 & 3
\end{array} \right) & \left( \begin{array}{cc}
1 & 3 \\
2 & 5
\end{array} \right)
\end{array} \right) \mapsto \left( \begin{array}{ccc}
0 & 1 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{array} \right).
\]
and $L(\pi) = \lambda_1 = L(W) = 3$.

## 1.7 Generalized Permutations, Percolation and growth models

The purpose of this section is to show that uniform probability, generalized permutations, percolation, queuing and growth models are intimately related. Their probabilities are ultimately given by the same formulae.

### 1.7.1 Probability on Young diagrams induced by uniform probability on generalized permutations

**Proposition 1.8** (Johansson [56]) Uniform probability $P_{p,q}^n$ on $\text{GP}_{p,q}^n$ induces a probability $\tilde{P}_{p,q}^n$ on Young diagrams $\gamma_{n}^{\text{min}(p,q)}$, given by

$$
\tilde{P}_{p,q}^n(\lambda) = \frac{1}{\# \text{GP}_{p,q}^n} \# \left\{ \begin{array}{l}
\text{generalized permutations} \\
\text{leading to shape } \lambda
\end{array} \right\}
\frac{\bar{s}_\lambda(1^p)\bar{s}_\lambda(1^q)}{\# \text{GP}_{p,q}^n}
= \frac{1}{\# \text{GP}_{p,q}^n} \prod_{j=0}^{q-1} \frac{1}{j!(p - q + j)!}
\Delta_{q}(q + \lambda_1 - 1, ..., q + \lambda_q - q)^2 \prod_{i=1}^{q} \frac{(p + \lambda_i - i)!}{(q + \lambda_i - i)!}, \text{ for } q \leq p,
$$

with

$$
\# \text{GP}_{p,q}^n = \sum_{\lambda \in \gamma_{n}^{\text{min}(p,q)}} \bar{s}_\lambda(1^p)\bar{s}_\lambda(1^q) = \binom{pq + n - 1}{n}.
$$

**Proof:** By the RSK correspondence, we have the one-to-one correspondence,

$$
\left\{ \begin{array}{l}
\pi \in \text{GP}_{p,q}^n \text{ leading} \\
to \text{the shape } \lambda
\end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l}
\text{semi-standard} \\
tableaux \text{ of shape} \\
\lambda, \text{ filled with} \\
\text{integers } 1, \ldots, q
\end{array} \right\} \times \left\{ \begin{array}{l}
\text{semi-standard} \\
tableaux \text{ of shape} \\
\lambda, \text{ filled with} \\
\text{integers } 1, \ldots, p
\end{array} \right\}
$$

There is no loss of generality in assuming $q \leq p$. 

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and thus, for $\lambda \in \mathcal{Y}_n^{\min(q,p)}$, we have

$$\tilde{P}_n^{p,q}(\lambda) = \frac{1}{\# \text{GP}_n^{p,q}} \{ \pi \in \text{GP}_n^{p,q} \text{ leading to the shape } \lambda \} = \frac{1}{\# \text{GP}_n^{p,q}} \tilde{s}_\lambda(1^q) \tilde{s}_\lambda(1^p);$$

when $q < \lambda^\top_1$, we have automatically $\tilde{s}_\lambda(1^q) = 0$, and thus

$$\sum_{\lambda \in \mathcal{Y}_n^{\min(q,p)}} \tilde{P}_n^{p,q}(\lambda) = 1.$$ 

Notice that for $m \geq \lambda^\top_1$,

$$\tilde{s}_\lambda(1^m) = \frac{\Delta_m (m + \lambda_1 - 1, \ldots, m + \lambda_m - m)}{\prod_{i=1}^{m-1} i!} = \prod_{1 \leq i < j \leq m} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

$$= \prod_{1 \leq i < j \leq q} \frac{(h_i - h_j)^2}{(j - i)^2} \prod_{i=1}^{q} \frac{h_i + j - q}{j - i}$$

$$= \frac{1}{\prod_{i=1}^{q} (j - i) \prod_{1 \leq i < j \leq q} (j - i)^2} \prod_{1 \leq i < j \leq q} \frac{(h_i - h_j)^2}{h_i^!} \prod_{i=1}^{q} \frac{(h_i + p - q)!}{h_i^!}.$$ 

(1.7.2)

Without loss of generality, assume $p \geq q$; then $\lambda_i = 0$ if $q < i \leq p$. Setting $h_j = \lambda_j + q - j$ for $j = 1, \ldots, q$, we have $h_1 > \ldots > h_q = \lambda_q \geq 0$. We now compute, using (1.2.8),

$$\tilde{s}_\lambda(1^p) \tilde{s}_\lambda(1^q)$$

$$= \prod_{1 \leq i < j \leq q} \frac{(h_i - h_j)^2}{(j - i)^2} \prod_{i=1}^{q} \frac{h_i + j - q}{j - i}$$

$$= \frac{1}{\prod_{i=1}^{q} (j - i) \prod_{1 \leq i < j \leq q} (j - i)^2} \prod_{1 \leq i < j \leq q} \frac{(h_i - h_j)^2}{h_i^!} \prod_{i=1}^{q} \frac{(h_i + p - q)!}{h_i^!}.$$ 

\[\begin{align*}
\text{\footnotesize[5]} & \text{Remember the notation from section 1.2: } \lambda \in \mathcal{Y}_n^{\min(q,p)} \text{ means the partition } \lambda \vdash n \text{ satisfies } \lambda^\top_1 \leq p, q. \\
\text{\footnotesize[6]} & \text{Note } \prod_{1 \leq i < j \leq q} (j - i) = \prod_{j=0}^{q-1} j!. \end{align*}\]
using
\[
\prod_{1 \leq i < j \leq q} (j - i) \prod_{i=1}^{q} \prod_{j=q+1}^{p} (j - i)
\]
\[
= \prod_{j=0}^{q-1} j! \prod_{i=1}^{q} (q + 1 - i)(q + 2 - i)(q + 3 - i)\ldots(p - i)
\]
\[
= q! \frac{(q + 1)!}{1!} \frac{(q + 2)!}{2!} \ldots \frac{(p - 1)!}{(p-q-1)!} \ldots(q - 1)!
\]
\[
= \prod_{j=0}^{q-1} (p - q + j)!
\]

This ends the proof of Proposition 1.8. \( \blacksquare \)

1.7.2 Percolation model with geometrically distributed entries

Consider the ensemble

\[ \text{Mat}^{(p,q)} = \{ p \times q \text{ matrices } M \text{ with entries } M_{ij} = 0, 1, 2, \ldots \} \]

with independent and geometrically distributed entries, for fixed \( 0 < \xi < 1 \),

\[ P(M_{ij} = k) = (1 - \xi)\xi^k, \quad k = 0, 1, 2, \ldots \]
Theorem 1.9 (Johansson [56]) Then

\[ L(M) := \max_{\text{all such paths}} \left\{ \sum_{i,j} M_{ij}, \ \text{over right/down paths starting from entry } (1,1) \text{ to } (p,q) \right\} \]

has the following distribution, assuming \( q \leq p \),

\[ P(L(M) \leq \ell) = \sum_{\lambda \in \mathbb{Y}_{\min(q,p)}} (1 - \xi)^{pq} \xi^{q|\lambda|} \mathcal{S}_\lambda(1^q) \mathcal{S}_\lambda(1^p) \]

\[ = Z_{p,q}^{-1} \sum_{\lambda \in \mathbb{Y}_{\min(q,p)}} \Delta_q(h_1, \ldots, h_q)^2 \prod_{i=1}^q \frac{(h_i + p - q)!}{h_i!} \xi^{h_i} \]

where

\[ Z_{p,q} = \xi^{q(a-1)/2} (1 - \xi)^{-pq} q! \prod_{j=0}^{q-1} j!(p - q + j)! \quad (1.7.3) \]

**Proof:** Then the probability that \( M \) be a given matrix \( A = (a_{ij})_{1 \leq i \leq p, 1 \leq j \leq q} \) equals

\[ P \left( M = (a_{ij})_{1 \leq i \leq p, 1 \leq j \leq q} \right) = \prod_{1 \leq i, j \leq p} P(M_{ij} = a_{ij}), \text{ using independence,} \]

\[ = \prod_{1 \leq i \leq p} (1 - \xi)^{a_{ij}} \]

\[ = (1 - \xi)^{pq} \xi^{\sum_{1 \leq i \leq p, 1 \leq j \leq q} a_{ij}} \]

\[ = (1 - \xi)^{pq} \xi^{\sum_{i,j} a_{ij}}. \]

This probability only depends on the total weight \(|A| = \sum_{i,j} a_{ij}\). Hence the matrices \( M \in \text{Mat}_{pq}^n \) have all equal probability and, in particular, due to the fact that, according to Theorem 1.1, the matrices in \( \text{Mat}_{pq}^n \) are in one-to-one correspondence with generalized permutations of size \( n \), with alphabets \( 1, \ldots, p \) and \( 1, \ldots, q \), one has

\[ P(|M| = n) = \# \text{GP}_{pq}^n (1 - \xi)^{pq} \xi^n. \]

\[ M \in \text{Mat}_{pq}^n \subset \text{Mat}^n_{pq}, \text{ means that } \sum M_{ij} = n. \]
We now compute

\[
P(L(M) \leq \ell \mid |M| = n)
\]

\[
= \frac{\#\{M \in \text{Mat}^{p,q}_n, L(M) \leq \ell\}}{\#\text{Mat}^{p,q}_n}
\]

\[
= \frac{\#\{\pi \in \text{GP}^{p,q}_n, L(\pi) \leq \ell\}}{\#\text{GP}^{p,q}_n}
\]

\[
= P^{p,q}_n(\lambda_1 \leq \ell)
\]

\[
= \frac{1}{\#\text{GP}^{p,q}_n} \sum_{\lambda_1 \leq \ell, |\lambda| = n} \tilde{s}_\lambda(1^q)\tilde{s}_\lambda(1^p).
\]

Hence,

\[
P(L(M) \leq \ell) = \sum_{n=0}^{\infty} P\left(L(M) \leq \ell \mid |M| = n\right) P(|M| = n)
\]

\[
= \sum_{n=0}^{\infty} \sum_{\lambda_1 \leq \ell, |\lambda| = n} \frac{1}{\#\text{GP}^{p,q}_n} \tilde{s}_\lambda(1^q)\tilde{s}_\lambda(1^p)(\#\text{GP}^{p,q}_n)(1 - \xi)^{pq}\xi^n
\]

\[
= \sum_{n=0}^{\infty} \sum_{\lambda_1 \leq \ell, |\lambda| = n} \tilde{s}_\lambda(1^q)\tilde{s}_\lambda(1^p)(1 - \xi)^{pq}\xi^{|\lambda|}
\]

\[
= \sum_{\lambda \in \mathbb{Y}} \tilde{s}_\lambda(1^q)\tilde{s}_\lambda(1^p)(1 - \xi)^{pq}\xi^{|\lambda|}.
\]

Now, using the expression for \(\tilde{s}_\lambda(1^p)\tilde{s}_\lambda(1^q)\) in Proposition [18], one computes, upon setting \(h_i = q + \lambda_i - i\), and noticing that \(\ell \geq \lambda_1 \geq \ldots \lambda_q \geq 0\) implies \(\ell + q - 1 \geq h_1 > \ldots > h_q \geq 0\),

\[
P(L(M) \leq \ell)
\]

\[
= \sum_{\lambda \in \mathbb{Y}, \lambda_1 \leq \ell} (1 - \xi)^{pq}\xi^{|\lambda|} \tilde{s}_\lambda(1^p)\tilde{s}_\lambda(1^q)
\]

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since the expression in the sum is symmetric in \(h_1, \ldots, h_q\). The normalization \(Z_{p,q}\) is as announced, ending the proof of \(\text{Theorem 1.9}\).\hfill \blacksquare

1.7.3 Percolation model with exponentially distributed entries

\textbf{Theorem 1.10} \cite{56} Consider the ensemble

\begin{equation}
\text{Mat}^{p,q} = \{p \times q \text{ matrices } M \text{ with } \mathbb{R}^+\text{-entries}\}
\end{equation}

with independent and exponentially distributed entries,

\begin{equation}
P(M_{ij} \leq t) = 1 - e^{-t}, \quad t \geq 0 \text{ and } \in \mathbb{R}.
\end{equation}

Then

\begin{equation}
L(M) = \max_{\text{all such paths}} \left\{ \sum_{\text{over right/lower paths starting from entry (1,1) to (p,q)}} M_{ij}, \right\}
\end{equation}

has the following distribution, (assuming \(q \leq p\), without loss of generality),

\begin{equation}
P(L(M) \leq t) = \frac{\int_{(0,t)^q} \Delta_q(x_1, \ldots, x_q)^2 \prod_{i=1}^q x_i^{p-q} e^{-x_i} dx_i}{\int_{(0,\infty)^q} \Delta_q(x_1, \ldots, x_q)^2 \prod_{i=1}^q x_i^{p-q} e^{-x_i} dx_i} = \frac{1}{Z_n} \int_{\mathcal{M} \in \mathcal{H}_q, \text{ Spectrum}(\mathcal{M}) \leq t} (\det \mathcal{M})^{p-q} e^{-\text{Tr} \mathcal{M}} d\mathcal{M},
\end{equation}

\textit{Remark:} It is remarkable that this percolation problem coincides with the probability that the spectrum of an Hermitian matrix does not exceed \(t\), where the matrix is taken from a (positive definite) random Hermitian ensemble with the Laguerre distribution, as appears in the second formula; this ensemble will be discussed much later in section \(8.2.2\).
Proof: For fixed $0 < \xi < 1$, let $X_\xi$ have a geometric distribution

$$P(X_\xi = k) = (1 - \xi)^k \quad 0 < \xi < 1, \quad k = 0, 1, 2, \ldots;$$

then in distribution

$$(1 - \xi)X_\xi \xrightarrow{\text{in distribution}} Y, \quad \text{for} \quad \xi \to 1$$

where $Y$ is an exponential distributed random variable. Indeed, setting $\varepsilon := 1 - \xi$, 

$$P((1 - \xi)X_\xi \leq t) = P(\varepsilon X_{1-\varepsilon} \leq t)$$

$$= \sum_{0 \leq k \leq t/\varepsilon} \varepsilon (1 - \varepsilon)^k$$

$$= \sum_{0 \leq k \leq t/\varepsilon} \varepsilon (1 - \varepsilon)^{\frac{t}{\varepsilon} k} \quad \text{(Riemann sum)}$$

$$\to \int_0^t ds e^{-s} = P(Y \leq t)$$

Then, setting $\xi - 1 - \varepsilon$, $t = \ell \varepsilon$, $\varepsilon h_i = x_i$ in formula (1.7.3) of Theorem 1.9 and letting $\varepsilon \to 0$, one computes

$$\lim_{\varepsilon \to 0} P_{\varepsilon}(L(M) \leq t)$$

$$= \lim_{\varepsilon \to 0} \frac{Z^{-1}}{\varepsilon q(q-1)+q(p-q)+q} \sum_{\max(h_i,\varepsilon) \leq \ell \varepsilon + (q-1)\varepsilon} \Delta_q(h_1\varepsilon, \ldots, h_q\varepsilon)^2 \prod_{i=1}^q \left( \prod_{k=1}^{p-q} \varepsilon (h_i + k) \right) (1 - \varepsilon)^{\sum h_i \varepsilon} \varepsilon^q$$

$$= \frac{1}{Z'} \int_{[0,\ell]^q} \Delta_q(x_1, \ldots, x_q)^2 \prod_{i=1}^q x_i^{p-q} e^{-x_i} dx_i$$

$$= \frac{1}{Z_n} \int_{M \in \mathcal{H}_q, \ \text{Spectrum}(M) \leq t} (\det M)^{p-q} e^{-\text{Tr} M} dM.$$ 

This last identity will be shown later in Proposition 7.2, ending the proof of Theorem 1.10.
1.7.4 Queuing problem

Consider servers 1, 2, ..., q waiting on customers 1, ..., p, with a “first-in first-out” service rule; see e.g., [21] and references within. At first, the system is empty and then p customers are placed in the first queue. When a customer $k$ is done with the $i$th server, he moves to the queue waiting to be served by the $i + 1$st server. Let

$$V(k, \ell) = \text{service time of customer } k \text{ by server } \ell$$

be all geometrically independently distributed

$$P(V(k, \ell) = t) = (1 - \xi)\xi^t, \quad t = 0, 1, 2, \ldots$$

**Theorem 1.11** The distribution of

$$D(p, q) = \left\{ \begin{array}{l}
\text{departure time for the last} \\
\text{customer} \begin{array}{c}
\text{at the last server}
\end{array}
\end{array} \right\}$$
is given by (assuming $q \leq p$)

$$P(D(p, q) \leq \ell) = \sum_{\lambda \in \gamma_{\text{min}}(q,p)} (1 - \xi)^{pq} \xi^{\lambda} s_{\lambda}(1^q)s_{\lambda}(1^p)$$

$$= Z_{q,p}^{-1} \sum_{\lambda \in \mathbb{N}} \Delta_q(h_1, \ldots, h_q) \prod_{i=1}^{q} \frac{(h_i + p - q)!}{h_i!} \xi^{h_i}$$

(1.7.4)

\textbf{Proof:} We show the problem is equivalent to the percolation problem discussed in Theorem 1.9. Indeed:

\underline{Step 1:} Setting $D(p, 0) = D(0, q) = 0$ for all $p, q$, we have for all $p, q \geq 1$,

$$D(p, q) = \max(D(p-1, q), D(p, q-1)) + V(p, q).$$

(1.7.5)

Indeed, if $D(p-1, q) \leq D(p, q-1)$, then customer $p-1$ has left server $q$ by the time customer $p$ reaches $q$, so that customer $p$ will not have to queue up and will be served immediately. Therefore

$$D(p, q) = D(p, q - 1) + V(p, q).$$

Now assume

$$D(p-1, q) \geq D(p, q-1);$$

then when customer $p$ reaches server $q$, then customer $p-1$ is still being served at queue $q$. Therefore the departure time of customer $p$ at queue $q$

$$D(p, q) = D(p-1, q) + V(p, q).$$

In particular

$$D(p, 1) = D(p-1, 1) + V(p, 1)$$

and

$$D(1, 2) = D(1, 1) + V(1, 2),$$

establishing (1.7.5).

\underline{Step 2:} we now prove

$$D(p, q) = \max_{\text{all such paths}} \left\{ \sum_{\text{over right/down paths}} V(i, j) \right\}$$

over right/down paths from entry $(1, 1)$ to $(p, q)$,

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where the paths are taken in the random matrix

\[ V = (V(i, j))_{1 \leq i \leq p, 1 \leq j \leq q}. \]

By a straightforward calculation,

\[
D(p, q) = \max(D(p - 1, q) + V(p, q), D(p, q - 1) + V(p, q)) \\
= \max \left( \max_{\text{all paths}} \sum_{\text{path}} V(i, j) + V(p, q), \right.
\]

\[
\left. \max_{\text{all paths}} \sum_{\text{path}} V(i, j) + V(p, q) \right) \\
= \max_{\text{all paths}} \left( \sum_{\text{path}} V(i, j) \right),
\]

ending the proof of Theorem 1.11.

1.7.5 Discrete polynuclear growth models

Consider geometric i.i.d. random variables \( \omega(x, t) \), with \( x \in \mathbb{Z}, t \in \mathbb{Z}_+ \),

\[
P(\omega(x, t) = k) = (1 - \xi)\xi^k, \quad k \in \mathbb{Z}_+,
\]

except

\[ \omega(x, t) = 0 \quad \text{if } t - x \text{ is even or } |x| > t. \]

Define inductively a growth process, with a height curve \( h(x, t) \), with \( x \in \mathbb{Z}, t \in \mathbb{Z}_+ \), given by

\[ h(x, 0) = 0, \]

\[ h(x, t + 1) = \max(h(x - 1, t), h(x, t), h(x + 1, t)) + \omega(x, t + 1). \]

For this model, see \[59, 79\] and \[66\].

**Theorem 1.12** The height curve at even sites \( 2x \) at times \( 2t - 1 \) is given by

\[ h(2x, 2t - 1) = \max_{\text{all such paths}} \left\{ \sum_{\text{over right/down}} V(i, j), \begin{array}{c} \text{paths starting from} \\ \text{entry } (1, 1) \text{ to } (t + x, t - x) \end{array} \right\} \]
where
\[ V(i, j) := \omega(i - j, i + j - 1). \]
Thus \( h(2x, 2t - 1) \) has again the same distribution as in \( (1.7.3) \).

Proof: It is based on the fact that, setting
\[ G(q, p) := h(q - p, q + p - 1), \]
one computes
\[ G(q, p) = \max(G(q - 1, p), G(q, p - 1)) + V(q, p) \]
\[ = \max \left\{ \sum_{ij} V_{ij}, \text{ over all right/down paths starting from entry (1, 1) to (q, p)} \right\}. \]

So
\[ h(2x, 2t - 1) = G(t + x, t - x), \]
establishing Theorem 1.12.

The figure below gives an example of such a growth process and also it shows that \( h(2x, 2t + 1) \) is given by the maximum of right/down paths starting at the upper-left corner and going to site \((t + x, t - x)\), where \( x \) is the running variable along the anti-diagonal.
2 Probability on partitions, Toeplitz and Fredholm determinants

Consider variables \( x = (x_1, x_2, \ldots) \) and \( y = (y_1, y_2, \ldots) \), and the corresponding symmetric functions

\[
t = (t_1, t_2, \ldots) \quad \text{and} \quad s = (s_1, s_2, \ldots)
\]

with

\[
kt_k = \sum_{i \geq 1} x_i^k \quad \text{and} \quad ks_k = \sum_{i \geq 1} y_i^k.
\]

Following Borodin-Okounkov-Olshanski (see \cite{23, 24, 25}), given arbitrary \( x, y \), define the (not necessarily positive) probability measure on \( \lambda \in \mathbb{Y} \),

\[
P_{x,y}(\lambda) := \frac{1}{Z} \hat{s}_\lambda(x) \hat{s}_\lambda(y) = \frac{1}{Z} s_\lambda(t) s_\lambda(s) \tag{2.0.1}
\]

with

\[
Z = \prod_{1 \leq i, j} (1 - x_i y_j)^{-1} = e^{\sum_{k \geq 1} k t_k s_k}.
\]

Indeed, by Cauchy’s identity,

\[
\sum_{\lambda \in \mathbb{Y}} P_{x,y}(\lambda) = \sum_{\lambda} s_\lambda(x) s_\lambda(y) \prod_{1 \leq i, j} (1 - x_i y_j) = \prod_{i, j \geq 1} (1 - x_i y_j)^{-1} \prod_{i, j \geq 1} (1 - x_i y_j) = 1.
\]

The main objective of this section is to compute

\[
P_{x,y}(\lambda_1 \leq n), \tag{2.0.2}
\]

which then will be specialized in the next section to specific \( x \)'s and \( y \)'s or, what is the same, to specific \( t \)'s and \( s \)'s. This probability \( \text{(2.0.2)} \) has three different expressions, one in terms of a determinant of a Toeplitz determinant, another in terms of an integral over the unitary group and still another in terms of a Fredholm determinant. The Toeplitz representation enables one to compute in an effective way the probability \( \text{(2.0.2)} \), whereas the Fredholm representation is useful, when taking limits for large permutations. In the
statement below, we need the Fredholm determinant of a kernel $K(i,j)$, with $i,j \in \mathbb{Z}$ and restricted to $[n, \infty)$; it is defined as
\[ \det \left( I - K(i,j) \right)_{[n,n+1,...]} := \sum_{m=0}^{\infty} (-1)^m \sum_{n \leq z_1 < ... < z_m \in \mathbb{Z}} \det(K(x_i,x_j))_{1 \leq i,j \leq m}. \] (2.0.3)

Now, one has the following statement [45, 23, 26, 85, 8]:

**Theorem 2.1** Given the “probability measure” (2.0.1), the following probability has three different expressions:

\[ P(\lambda \text{ with } \lambda_1 \leq n) = Z^{-1} \det \left( \int_{S^1} \frac{dz}{2\pi i z} e^{-\sum_{1}^{\infty} (t_{i}z^{i} + s_{i}z^{-i})} \right)_{1 \leq k,\ell \leq n} \]
\[ = Z^{-1} \int_{U(n)} e^{-\text{Tr} \sum_{i \geq 1} (t_{i}X^{i} + s_{i}\bar{X}^{i})} dX \]
\[ = \det \left( I - K(k,\ell) \right)_{[n,n+1,...]}, \] (2.0.4)

where $K(k,\ell)$ is a kernel

\[ K(k,\ell) = \left( \frac{1}{2\pi i} \right)^2 \int_{|w|=\rho<1} \int_{|z|=|w|^{-1}>1} \frac{dz \, dw}{z^{k+1}w^{-\ell}} e^{V(z)-V(w)} \]
\[ = \frac{1}{k-\ell} \left( \frac{1}{2\pi i} \right)^2 \int_{|w|=\rho<1} \int_{|z|=|w|^{-1}>1} \frac{dz \, dw}{z^{k+1}w^{-\ell}} \frac{d\, V(z) - w \, d\, V(w)}{z-w} e^{V(z)-V(w)} \]
for $k,\ell \in \mathbb{Z}$, with $k \neq \ell$, (2.0.5)

with
\[ V(z) := V_{t,s}(z) := -\sum_{i \geq 1} (t_{i}z^{-i} - s_{i}z^{i}). \]

**2.1 Probability on partitions expressed as Toeplitz determinants**

In this subsection, the first part of Theorem 2.1 will be reformulated as Proposition 2.2 and also demonstrated: (see e.g. Gessel [45], Tracy-Widom...
Proposition 2.2 Given the "probability measure"

\[ P(\lambda) = Z^{-1} s_\lambda(t)s_\lambda(s), \quad Z = e^{\sum_{i \geq 1} t_i s_i}, \]

the following holds

\[ P(\lambda \text{ with } \lambda_1 \leq n) = Z^{-1} \det \left( \int_{S^1} \frac{dz}{2\pi i} z^{k-\ell} e^{\sum_{i \geq 1} (t_i z^i + s_i z^{-i})} \right)_{1 \leq k, \ell \leq n} \]

and

\[ P(\lambda \text{ with } \lambda_1^\top \leq n) = Z^{-1} \det \left( \int_{S^1} \frac{dz}{2\pi i} z^{k-\ell} e^{\sum_{i \geq 1} (t_1 z^i - s_1 z^{-i})} \right)_{1 \leq k, \ell \leq n} \]

Proof: Consider the semi-infinite Toeplitz matrix

\[ m_\infty(t, s) = (\mu_{k\ell})_{k, \ell \geq 0}, \text{ with } \mu_{k\ell}(t, s) = \int_{S^1} z^{k-\ell} e^{\sum_{i \geq 1} (t_1 z^i - s_1 z^{-i})} \frac{dz}{2\pi i z}. \]

Note that

\[
\frac{\partial \mu_{k\ell}}{\partial t_m} = \int_{S^1} z^{k-\ell+m} e^{\sum_{i \geq 1} (t_1 z^i - s_1 z^{-i})} \frac{dz}{2\pi i z} = \mu_{k+m, \ell},
\]

\[
\frac{\partial \mu_{k\ell}}{\partial s_m} = -\int_{S^1} z^{k-\ell-m} e^{\sum_{i \geq 1} (t_1 z^i - s_1 z^{-i})} \frac{dz}{2\pi i z} = -\mu_{k, \ell+m},
\]

with initial condition \( \mu_{k\ell}(0, 0) = \delta_{k\ell} \). In matrix notation, this amounts to the system of differential equations\(^8\)

\[
\frac{\partial m_\infty}{\partial t_i} = \Lambda^i m_\infty \quad \text{and} \quad \frac{\partial m_\infty}{\partial s_i} = -m_\infty (\Lambda^\top)^i, \text{ with initial condition } m_\infty(0, 0) = I_\infty.
\]

The solution to this initial value problem is given by

\[
m_\infty(t, s) = (\mu_{k\ell}(t, s))_{k, \ell \geq 0} \tag{2.1.1}
\]

\(^8\)The operator \( \Lambda \) is the semi-infinite shift matrix, with zeroes everywhere, except for 1’s just above the diagonal, i.e., \((\Lambda v)_n = v_{n+1}\). \( I_\infty \) is the semi-infinite identity matrix.
\[ m_{\infty}(t, s) = e^{\sum_i t_i \Lambda^i} m_{\infty}(0, 0) e^{-\sum_i s_i \Lambda^\top_i}, \quad (2.1.2) \]

where

\[ e^{\sum_i t_i \Lambda^i} = \sum_{i=0}^{\infty} s_i(t) \Lambda^i = \begin{pmatrix} 1 & s_1(t) & s_2(t) & s_3(t) & \cdots \\ 0 & 1 & s_1(t) & s_2(t) & \cdots \\ 0 & 0 & 1 & s_1(t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & \cdots \end{pmatrix} = (s_{j-i}(t))_{1 \leq i < j < \infty}. \]

Then, by the uniqueness of solutions of ode’s, the two solutions coincide, and in particular the \( n \times n \) upper-left blocks of (2.1.1) and (2.1.2), namely

\[ m_n(t, s) = E_n(t)m_{\infty}(0, 0) E_n^\top (-s), \quad (2.1.3) \]

where

\[ E_n(t) = \begin{pmatrix} 1 & s_1(t) & s_2(t) & s_3(t) & \cdots & s_{n-1}(t) & \cdots \\ 0 & 1 & s_1(t) & s_2(t) & \cdots & s_{n-2}(t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots \end{pmatrix} = (s_{j-i}(t))_{1 \leq i < j < \infty}. \]

Therefore the determinants coincide:

\[ \det m_n(t, s) = \det(E_n(t)m_{\infty}(0, 0) E_n^\top (-s)). \quad (2.1.4) \]

We shall need to expand the right hand side of (2.1.4) in “Fourier series”, which is based on the following Lemma:

**Lemma 2.3** Given the semi-infinite initial condition \( m_{\infty}(0, 0) \), the expression below admits an expansion in Schur polynomials,

\[ \det(E_n(t)m_{\infty}(0, 0) E_n^\top (-s)) = \sum_{\lambda^\top \leq n, \nu^\top \leq n} \det(m^{\lambda, \nu}) s_{\lambda}(s_{\nu}(-s)), \quad \text{for } n > 0, \quad (2.1.5) \]

where the sum is taken over all Young diagrams \( \lambda \) and \( \nu \), with first columns \( \lambda^\top \leq n \) and \( \nu^\top \leq n \) and where

\[ m^{\lambda, \nu} := (\mu_{\lambda_i-i+n, \nu_j-j+n})_{1 \leq i, j \leq n}. \quad (2.1.6) \]
Proof: The proof of this Lemma is based on the Cauchy-Binet formula, which affirms that given two matrices $A_{(m,n)}$, $B_{(n,m)}$, for $n$ large $\geq m$

$$\det(AB) = \det \left( \sum_{i} a_{i\ell} b_{ik} \right)_{1\leq k, \ell \leq m}$$

$$= \sum_{1\leq i_{1} < \ldots < i_{m} \leq n} \det(a_{i_{k}, i_{\ell}})_{1\leq k, \ell \leq m} \det(b_{i_{k}, i_{\ell}})_{1\leq k, \ell \leq m}. \quad (2.1.7)$$

Note that every decreasing sequence $\infty > k_{n} > \ldots > k_{1} \geq 1$ can be mapped into a Young diagram $\lambda_{1} \geq \lambda_{2} \geq \ldots \geq \lambda_{n} \geq 0$, by setting $k_{j} = j + \lambda_{n+1-j}$. Relabeling the indices $i, j$ with $1 \leq i, j \leq n$, by setting $j':= n - j + 1$, $i':= n-i+1$, we have $1 \leq i', j' \leq n$ and $k_{j-i} = \lambda_{j'} - j' + i'$ and $k_{i-1} = \lambda_{i'} - i' + n$. In other terms, the sequence of integers $\infty > k_{n} > \ldots > k_{1} \geq 1$ leads to a partition $\lambda_{1} = k_{n} - n > \lambda_{2} = k_{n-1} - n + 1 > \ldots > \lambda_{n} = k_{1} - 1 \geq 0$. The same can be done for the sequence $1 \leq \ell_{1} < \ldots < \ell_{n} < \infty$, leading to the Young diagram $\nu$, using the same relabeling. Applying the Cauchy-Binet formula twice, expression (2.1.4) leads to:

$$\det \left( E_{n}(t) m_{\infty} (0, 0) E_{n}^{\top} (-s) \right)$$

$$= \sum_{1\leq k_{1} < \ldots < k_{n} < \infty} \det(s_{k_{j}-i}(t))_{1\leq i, j \leq n} \det \left( (m_{\infty} (0, 0) E_{n}^{\top} (-s))_{k_{i}, i} \right)_{1\leq i, \ell \leq n}$$

$$= \sum_{1\leq k_{1} < \ldots < k_{n} < \infty} \det(s_{k_{j}-i}(t))_{1\leq i, j \leq n} \det \left( (\mu_{k_{i}-1,j-1})_{1\leq i, j \leq n} \det(s_{i-j}(-s))_{1\leq i, j \leq n} \right)_{1\leq i, j \leq n}$$

$$= \sum_{1\leq k_{1} < \ldots < k_{n} < \infty} \det(s_{k_{j}-i}(t))_{1\leq i, j \leq n} \sum_{1\leq \ell_{1} < \ldots < \ell_{n} < \infty} \det(\mu_{k_{i}-1,\ell_{j}-1})_{1\leq i, j \leq n} \det(s_{\ell_{i}-j}(-s))_{1\leq i, j \leq n}$$

$$= \sum_{\lambda \in \mathbb{V} \lambda_{1} \leq n} \det(s_{\lambda, j'-i'}(t))_{1\leq i', j' \leq n}$$

$$\sum_{\nu \in \mathbb{V} \nu_{1} \leq n} \det(\mu_{\lambda', j'-i'+\nu, j'-i'}(-s))_{1\leq i', j' \leq n} \det(s_{\nu_{1}, i' + j'}(-s))_{1\leq i', j' \leq n}$$

$$= \sum_{\lambda, \nu \in \mathbb{V} \lambda_{1} \leq n} \det(m^{\lambda, \nu}) s_{\lambda}(t) s_{\nu}(-s),$$

which establishes Lemma 2.3. \hspace{1cm} \blacksquare
Continuing the proof of Proposition 2.2, apply now Lemma 2.3 to \( m_\infty(0, 0) = I_\infty \), leading to

\[
\det m^{\lambda, \nu} = \det (\mu_{\lambda_i - n, \nu_j - j + n})_{1 \leq i, j \leq n} \neq 0
\]

if and only if

\[
\lambda_i - i + n = \nu_i - i + n \quad \text{for all } 1 \leq i \leq n,
\]

i.e., \( \lambda = \nu \), in which case

\[
\det m^{\lambda, \lambda} = 1.
\]

Therefore, from (2.1.4), it follows

\[
\sum_{\lambda \in \mathcal{C}} s_\lambda(t) s_\lambda(-s) = \det \left( \int_{S^1} \frac{dz}{2\pi iz} z^{k-\ell} e^{\sum_1^\infty (t_i z^i + s_i z^{-i})} \right)_{1 \leq k, \ell \leq n}.
\]

So, we have, changing \( s \mapsto -s \),

\[
P(\lambda \text{ with } \lambda_1 \leq n) = Z^{-1} \sum_{\lambda \in \mathcal{C}} s_\lambda(t) s_\lambda(s)
\]

\[
= Z^{-1} \det \left( \int_{S^1} \frac{dz}{2\pi iz} z^{k-\ell} e^{\sum_1^\infty (t_i z^i + s_i z^{-i})} \right)_{1 \leq k, \ell \leq n},
\]

and, using \( s_\lambda(-t) = (-1)^{|\lambda|} s_{\lambda^T}(t) \), we also have

\[
P(\lambda \text{ with } \lambda_1 \leq n) = Z^{-1} \sum_{\lambda \in \mathcal{C}} s_\lambda(-t) s_\lambda(-s)
\]

\[
= Z^{-1} \sum_{\lambda \in \mathcal{C}} s_{\lambda^T}(-t) s_{\lambda^T}(-s)
\]

\[
= Z^{-1} \det \left( \int_{S^1} \frac{dz}{2\pi iz} z^{k-\ell} e^{\sum_1^\infty (t_i z^i + s_i z^{-i})} \right)_{1 \leq k, \ell \leq n},
\]

where

\[
Z = e^{\sum_1^\infty t_i s_i},
\]

ending the proof of Proposition 2.2. \( \blacksquare \)
2.2 The calculus of infinite wedge spaces

The material in this section can be found in Kac [61] and Kac-Raina [62], and many specific results are due to Borodin-Okounkov-Olshanski (see [23, 24, 25]). Given a vector space \( V = \bigoplus_{j \in \mathbb{Z}} \mathbb{C} v_j \) with inner-product \( \langle v_i, v_j \rangle = \delta_{ij} \), the infinite wedge space \( V^\infty = \Lambda^\infty V \) is defined as

\[
V^\infty = \text{span} \left\{ v_{s_1} \wedge v_{s_2} \wedge v_{s_3} \wedge \ldots, \quad s_1 > s_2 > \ldots, \quad s_k = -k \text{ for } k > 0 \right\}
\]

containing the “vacuum” \( f_\emptyset = v_{-1} \wedge v_{-2} \wedge \ldots \). The vector space \( V^\infty \) comes equipped with an inner-product \( \langle , \rangle \), making the basis vectors \( v_{s_1} \wedge v_{s_2} \wedge \ldots \) orthonormal. To each \( k \in \mathbb{Z} \), we associate two operations, a wedging with \( v_k \) and a contracting operator, removing a \( v_k \),

\[
\psi_k : V^\infty \to V^\infty : f \mapsto \psi_k(f) = v_k \wedge f
\]

\[
\psi_k^* : V^\infty \to V^\infty : v_{s_1} \wedge \ldots \wedge v_{s_i} \wedge \ldots \mapsto \sum_i (-1)^{i+1} \langle v_k, v_{s_i} \rangle v_{s_1} \wedge \ldots \wedge \hat{v}_{s_i} \wedge \ldots
\]

Note that

\[
\psi_k(f) = 0, \quad \text{if } v_k \text{ figures in } f
\]

\[
\psi_k^*(v_{s_1} \wedge \ldots) = 0, \quad \text{if } k \not\in (s_1, s_2, \ldots).
\]

Define the shift

\[
\Lambda^r := \sum_{k \in \mathbb{Z}} \psi_{k+r} \psi_k^*, \quad r \in \mathbb{Z}
\]

acting on \( V^\infty \) as follows

\[
\Lambda^r v_{s_1} \wedge v_{s_2} \wedge \ldots = v_{s_1+r} \wedge v_{s_2} \wedge v_{s_3} \wedge \ldots + v_{s_1} \wedge v_{s_2+r} \wedge v_{s_3} \wedge \ldots + v_{s_1} \wedge v_{s_2} \wedge v_{s_3+r} \wedge \ldots + \ldots
\]
One checks that

\[
[\Lambda^r, \psi_k] = \psi_{k+r}, \quad [\Lambda^r, \psi_k^*] = -\psi_{k-r}^*
\]  
(2.2.1)

\[
[\Lambda^k, \Lambda^\ell] = \ell \delta_{k,-\ell}
\]  
(2.2.2)

and hence

\[
\left[ \sum_{i \geq 1} t_i \Lambda^i, \sum_{j \geq 1} s_j \Lambda^{-j} \right] = - \sum_{i \geq 1} it_is_i.
\]  
(2.2.3)

**Lemma 2.4** *(Version of the Cauchy identity)*

\[
e^{\sum_{i \geq 1} t_i \Lambda^i} e^{-\sum_{j \geq 1} s_j \Lambda^{-j}} = e^{\sum_{i \geq 1} t_i \Lambda^i} e^{[\sum_{i \geq 1} t_i \Lambda^i, \sum_{j \geq 1} s_j \Lambda^{-j}]} e^{\sum_{i \geq 1} t_i \Lambda^i} (2.2.4)
\]

**Proof:** When two operators \(A\) and \(B\) commute with their commutator \([A, B]\), then according to Kac [61], p 308,

\[
e^A e^B = e^B e^A e^{[A, B]}.
\]

Setting \(A = \sum_{i \geq 1} t_i \Lambda^i\) and \(B = -\sum_{j \geq 1} s_j \Lambda^{-j}\), we find

\[
e^{\sum_{i \geq 1} t_i \Lambda^i} e^{-\sum_{j \geq 1} s_j \Lambda^{-j}} = e^{-\sum_{j \geq 1} s_j \Lambda^{-j}} e^{\sum_{i \geq 1} t_i \Lambda^i} e^{-\sum_{j \geq 1} s_j \Lambda^{-j}} e^{\sum_{i \geq 1} t_i \Lambda^i}
\]

It is useful to consider the generators of \(\psi_i\) and \(\psi_i^*\):

\[
\psi(z) = \sum_{i \in \mathbb{Z}} z^i \psi_i, \quad \psi^*(w) = \sum_{j \in \mathbb{Z}} w^{-j} \psi_j^*.
\]  
(2.2.5)

From (2.2.1), it follows that

\[
[\Lambda^r, \psi(z)] = \sum_{k} z^k [\Lambda^r, \psi_k] = \sum_{k} z^k \psi_{k+r} = \frac{1}{z^r} \psi(z),
\]

\[
[\Lambda^r, \psi^*(w)] = -\frac{1}{w^r} \psi^*(w).
\]

The two relations above lead to the following, by taking derivatives in \(t_i\) and \(s_i\) of the left hand side and setting all \(t_i = s_i = 0:\)
\[
e^{\pm \sum_{i=1}^{\infty} t_i \Lambda^i} \begin{cases} \psi(z) \\ \psi^*(w) \end{cases} e^{\pm \sum_{i}^{\infty} t_i / z^r \psi(z)} = \begin{cases} e^{\pm \sum_{i}^{\infty} t_i / w^r \psi^*(w)} \end{cases}
\]

\[
e^{\pm \sum_{i=1}^{\infty} s_i \Lambda^{-i}} \begin{cases} \psi(z) \\ \psi^*(w) \end{cases} e^{\mp \sum_{i}^{\infty} s_i / z^r \psi(z)} = \begin{cases} e^{\mp \sum_{i}^{\infty} s_i / w^r \psi^*(w)} \end{cases}
\]  \quad (2.2.6)

To each partition \( \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 0) \), associate a vector

\[
f_{\lambda} := v_{\lambda_{1-1}} \wedge v_{\lambda_{2-2}} \wedge \ldots \wedge v_{\lambda_{m-m}} \wedge v_{-(m+1)} \wedge v_{-(m+2)} \wedge \ldots \in V^\infty.
\]

The following lemma holds:

**Lemma 2.5**

\[
e^{\sum_{i \geq 1} t_i \Lambda^i} f_{\lambda} = \sum_{\mu \in Y} s_{\mu \lambda}(t) f_{\mu}, \quad e^{\sum_{i \geq 1} t_i \Lambda^{-i}} f_{\lambda} = \sum_{\mu \in Y} s_{\lambda \mu}(t) f_{\mu}. \quad (2.2.7)
\]

In particular,

\[
e^{\sum_{i \geq 1} t_i \Lambda^i} f_{\emptyset} = \sum_{\mu \in Y} s_{\mu}(t) f_{\mu} \quad \text{and} \quad e^{\sum_{i \geq 1} t_i \Lambda^{-i}} f_{\emptyset} = f_{\emptyset}. \quad (2.2.8)
\]

**Proof:** First notice that a matrix \( A \in GL_\infty \) acts on \( V^\infty \) as follows

\[
A(v_{s_1} \wedge v_{s_2} \wedge \ldots) = \sum_{s_1' > s_2' > \ldots} \det(A_{s_1', s_2', \ldots}^{s_1, s_2, \ldots}) v_{s_1'} \wedge v_{s_2'} \wedge \ldots,
\]

where

\[
A_{s_1', s_2', \ldots}^{s_1, s_2, \ldots} = \left\{ \text{matrix located at the intersection of the rows } s_1', s_2', \ldots \text{ and columns } s_1, s_2, \ldots \text{ of } A. \right\}.
\]

Here the rows and columns of the bi-infinite matrix are labeled by
Hence, for the bi-infinite matrix $e^{\sum t_i A^i}$,

$$
\det \left( \left( e^{\sum t_i A^i} \right)^{s_1, s_2, \ldots} \right) = \det (s_{s'_i - s_j})
$$

Setting $s_i = \lambda_i - i$ and defining $\mu_i$ in the formula below by $s'_i = \mu_i - i$, one checks

$$
e^{\sum t_i A^i} f_\lambda = e^{\sum t_i A^i} \left( v_{s_1} \land v_{s_2} \land \ldots \right)$$

$$
= \sum_{s'_1 > s'_2 > \ldots} \det \left( \left( e^{\sum t_i A^i} \right)^{s_1, s_2, \ldots} \right) v_{s'_1} \land v_{s'_2} \land \ldots
$$

$$
= \sum_{s'_1 > s'_2 > \ldots} \det (s_{s'_i - s_j}) \left( v_{s'_1} \land v_{s'_2} \land \ldots \right)
$$

$$
= \sum_{\mu_1 - 1 > \mu_2 - 2 > \ldots} \det (s_{(\mu_1 - i) - (\lambda_j - j)}(t)) \left( v_{\mu_1 - 1} \land v_{\mu_2 - 2} \land \ldots \right)
$$

$$
= \sum_{\mu \in Y} s_\mu \setminus \lambda(t) f_\mu
$$

The second identity in (2.2.7) is shown in the same way. Identities (2.2.8) follow immediately from (2.2.7), ending the proof of Lemma 2.5. 

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We also need:

Lemma 2.6 For

\[ \Psi^* b_k = \sum_{i \geq 1} b_{ki} \psi^*_i \quad \text{and} \quad \Psi a_k = \sum_{i \geq 1} a_{ki} \psi^*_i, \quad (2.2.9) \]

doing the following identity holds:

\[ \langle \Psi^*_a \ldots \Psi^*_a \Psi^*_b \ldots \Psi^*_b f_\emptyset, f_\emptyset \rangle = det \left( \langle \Psi^*_a \Psi^*_b f_\emptyset, f_\emptyset \rangle \right)_{1 \leq k, \ell \leq m} \quad (2.2.10) \]

**Proof:** First one computes for the \( \Psi \)'s as in (2.2.9),

\[ \Psi^*_b \ldots \Psi^*_b f_\emptyset = \sum_{i_1 > \ldots > i_m \geq 1} (-1)^{\sum_{i=1}^m (i_k+1)} \det (b_{ki})_{1 \leq k, \ell \leq m} \]

\[ v_1 \wedge \ldots \wedge \hat{v}_{-i_m} \wedge \ldots \wedge \hat{v}_{-i_{m-1}} \wedge \ldots \wedge \hat{v}_{-i_2} \wedge \ldots \wedge \hat{v}_{-i_1} \wedge \ldots \]

Then acting with the \( \Psi^*_a \) as in (2.2.9), it suffices to understand how it acts on the wedge products appearing in the expression above, namely:

\[ \Psi^*_a \ldots \Psi^*_a v_1 \wedge \ldots \wedge \hat{v}_{-i_m} \wedge \ldots \wedge \hat{v}_{-i_{m-1}} \wedge \ldots \wedge \hat{v}_{-i_2} \wedge \ldots \wedge \hat{v}_{-i_1} \wedge \ldots \]

\[ = (-1)^{\sum_{i=1}^m (i_k+1)} \det (a_{ki})_{1 \leq k, \ell \leq m} f_\emptyset \]

Thus, combining the two, one finds, using the Cauchy-Binet formula (2.1.7) in the last equality,

\[ \Psi^*_a \ldots \Psi^*_a \Psi^*_b \ldots \Psi^*_b f_\emptyset = \sum_{i_1 > \ldots > i_m \geq 1} \det (a_{ki})_{1 \leq k, \ell \leq m} \det (b_{ki})_{1 \leq k, \ell \leq m} f_\emptyset \]

\[ = \det \left( \sum_i a_{\ell i} b_{ki} \right)_{1 \leq k, \ell \leq m} f_\emptyset. \]

In particular for \( m = 1 \),

\[ \Psi^*_a \Psi^*_b f_\emptyset = \sum_i a_{\ell i} b_{ki} f_\emptyset. \]
Hence
\[
\langle \Psi_{a_1} \ldots \Psi_{a_m} \Psi_{b_n}^* \ldots \Psi_{b_1}^* f_\emptyset, f_\emptyset \rangle = \det \left( \sum_{i} a_{ki} b_{\ell i} \right)_{1 \leq k, \ell \leq m} = \det \left( \langle \Psi_{a_k} \Psi_{b_\ell}^* f_\emptyset, f_\emptyset \rangle \right)_{1 \leq k, \ell \leq m},
\]
ending the proof of Lemma 2.6. \hfill \blacksquare

2.3 Probability on partitions expressed as Fredholm determinants

Remember the definition (2.0.3) of the Fredholm determinant of a kernel \( K(i, j) \), with \( i, j \in \mathbb{Z} \) and restricted to \([n, \infty)\). This statement has appeared in the literature in some form (see Case-Geronimo [28]) and a more analytic formulation has appeared in Basor-Widom [22]. The proof given here is an “integrable” one, due to Borodin-Okounkov [26].

Proposition 2.7

\[
P(\lambda \text{ with } \lambda_1 \leq n) = \det \left( I - K(k, \ell) \right)_{[n, n+1, \ldots]},
\]

where \( K(k, \ell) \) is a kernel

\[
K(k, \ell) = \left( \frac{1}{2\pi i} \right)^2 \int_{|w|=\rho<1} \int_{|z|=\rho^{-1}>1} \frac{dz \, dw \, e^{V(z)-V(w)}}{z^{k+1} w^{-\ell}} \left( z - w \right)
\]

\[
= \frac{1}{k - \ell} \left( \frac{1}{2\pi i} \right)^2 \int_{|w|=\rho<1} \int_{|z|=\rho^{-1}>1} \frac{dz \, dw \, \frac{d}{dz} V(z) - w \frac{d}{dw} V(w)}{z^{k+1} w^{-\ell}} e^{V(z)-V(w)},
\]

for \( k \neq \ell \),

(2.3.1)

with

\[
V(z) = - \sum_{i \geq 1} (t_i z^{-i} - s_i z^i).
\]

The proof of Proposition 2.7 hinges on the following Lemma, due to Borodin-Okounkov [73, 74, 26]:

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Lemma 2.8 If $X = \{x_1, \ldots, x_m\} \subset \mathbb{Z}$ and $S(\lambda) := \{\lambda_1 - 1, \lambda_2 - 2, \lambda_3 - 3, \ldots\}$

$$P(\lambda \mid S(\lambda) \supset X) = \frac{1}{Z} \sum_{\lambda \text{such that } S(\lambda) \supset X} s_\lambda(t)s_\lambda(s) = \det(K(x_i, x_j))_{1 \leq i, j \leq m}.$$ 

Proof of Proposition 2.7: Setting $A_k := \{\lambda \mid k \in S(\lambda)\}$ with $S(\lambda) := \{\lambda_1 - 1, \lambda_2 - 2, \lambda_3 - 3, \ldots\}$, one computes (the $x_i$ below are integers)

$$\begin{align*}
P(\lambda \text{ with } \lambda_1 \leq n) \\
= P(\lambda \text{ with all } \lambda_i \leq n) \\
= P(\lambda \text{ with } S(\lambda) \cap \{n, n+1, n+2, \ldots\} = \emptyset) \\
= 1 - P(\lambda \text{ with } S(\lambda) \text{ contains some } k \text{ for } k \geq n) \\
= 1 - \sum_{n \leq i} P(A_i) + \sum_{n \leq i < j} P(A_i \cap A_j) - \sum_{n \leq i < j < k} P(A_i \cap A_j \cap A_k) + \ldots \\
\text{using Poincaré’s formula} \\
= \sum_{m=0}^{\infty} (-1)^m \sum_{n \leq x_1 < \ldots < x_m} P(\lambda \text{ with } \{x_1, \ldots, x_m\} \subset S(\lambda)) \\
= \sum_{m=0}^{\infty} (-1)^m \sum_{n \leq x_1 < \ldots < x_m} \det(K(x_i, x_j))_{1 \leq i, j \leq m} \\
= \det\left(I - K(i, j)\bigg|_{[n, n+1, \ldots]}\right),
\end{align*}$$

from which Proposition 2.7 follows.

Proof of Lemma 2.8: Remembering the probability measure introduced in (2.0.1), we have that

$$P(\lambda \mid S(\lambda) \supset X) = \frac{1}{Z} \sum_{\lambda \text{such that } S(\lambda) \supset X} s_\lambda(t)s_\lambda(s).$$

Next, from the wedging-contracting operation

$$\psi^*_x \psi_x f_\lambda = f_\lambda, \quad \text{if } x \in S_\lambda$$

$$= 0, \quad \text{if } x \notin S_\lambda,$$
and using both relations (2.2.7), one first computes:

\[
\langle e^{\sum_{i} s_i \Lambda^{-i}} \prod_{x \in X} \psi_x \psi_x^* e^{\sum_{i} t_i \Lambda^{i}} f_\emptyset, f_\emptyset \rangle = \langle e^{\sum_{i} s_i \Lambda^{-i}} \prod_{x \in X} \psi_x \psi_x^* \sum_{\lambda} s_\lambda(t) f_\lambda, f_\emptyset \rangle = \sum_{\lambda} s_\lambda(t) \langle e^{\sum_{i} s_i \Lambda^{-i}} f_\lambda, f_\emptyset \rangle = \sum_{\lambda} s_\lambda(t) \sum_{\mu \subseteq \lambda} \langle s_\lambda \mu(s) f_\mu, f_\emptyset \rangle = \sum_{\lambda} s_\lambda(t) \sum_{\mu \subseteq \lambda} s_\lambda \mu(s) (f_\mu, f_\emptyset) = \sum_{\lambda \text{ such that } S(\lambda) \supset X} s_\lambda(t) s_\lambda(s).
\]

Using this fact, one further computes (in the exponentials below the summation \(\Sigma\) stands for \(\sum_{1}^{\infty}\))

\[
\frac{1}{Z} \sum_{\lambda \text{ such that } S(\lambda) \supset X} s_\lambda(t) s_\lambda(s) = \frac{1}{Z} \langle e^{\sum_{i} s_i \Lambda^{-i}} \prod_{x \in X} \psi_x \psi_x^* e^{\sum_{i} t_i \Lambda^{i}} f_\emptyset, f_\emptyset \rangle = \frac{1}{Z} \left\langle e^{s_i \Lambda^{-i}} \psi_x \psi_x^* e^{t_i \Lambda^{i}} f_\emptyset, f_\emptyset \right\rangle \quad \text{using } \psi_x^i \psi_x^j = -\psi_x^j \psi_x^i \text{ for } i \neq j
\]

\[
= \frac{1}{Z} \left\langle e^{-\sum_{i} t_i \Lambda^{i}} e^{\sum_{i} s_i \Lambda^{-i}} \psi_x \psi_x^* \psi_{x_1} \psi_{x_1}^* \psi_{x_2} \psi_{x_2}^* \ldots e^{\sum_{i} t_i \Lambda^{i}} f_\emptyset, e^{-\sum_{i} t_i \Lambda^{i}} f_\emptyset, e^{-\sum_{i} t_i \Lambda^{i}} f_\emptyset \right\rangle \quad \text{using (2.2.8)}
\]

\[
= \frac{1}{Z} \left\langle e^{-\sum_{i} t_i \Lambda^{i}} e^{\sum_{i} s_i \Lambda^{-i}} \psi_x \psi_x^* \psi_{x_1} \psi_{x_1}^* \psi_{x_2} \psi_{x_2}^* \ldots e^{\sum_{i} t_i \Lambda^{i}} e^{-\sum_{i} s_i \Lambda^{-i}} f_\emptyset, f_\emptyset \right\rangle \quad \text{using \text{(2.2.8)}}
\]

\[
= \left\langle e^{-\sum_{i} t_i \Lambda^{i}} e^{\sum_{i} s_i \Lambda^{-i}} \psi_x \psi_x^* \psi_{x_1} \psi_{x_1}^* \psi_{x_2} \psi_{x_2}^* \ldots e^{\sum_{i} t_i \Lambda^{i}} e^{-\sum_{i} s_i \Lambda^{-i}} f_\emptyset, f_\emptyset \right\rangle \quad \text{using Cauchy's identity (2.2.4)}
\]

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\[
\langle \Psi_{x_m} \ldots \Psi_{x_1} \Psi^*_{x_1} \ldots \Psi^*_{x_m} f_0, f_0 \rangle,
\]

where
\[
\Psi_k = e^{-\sum t_i \Lambda^i} e^{\sum s_i \Lambda^{-i}} \psi_k e^{-\sum s_i \Lambda^{-i}} e^{\sum t_i \Lambda^i}
\]
\[
\Psi_k^* = e^{-\sum t_i \Lambda^i} e^{\sum s_i \Lambda^{-i}} \psi_k^* e^{-\sum s_i \Lambda^{-i}} e^{\sum t_i \Lambda^i}.
\]

Then, using Lemma 2.6, the expression above equals
\[
\langle \Psi_{x_m} \ldots \Psi_{x_1} \Psi^*_{x_1} \ldots \Psi^*_{x_m} f_0, f_0 \rangle = \det \left( \langle \Psi_{x_k} \Psi^*_{x_{\ell}} f_0, f_0 \rangle \right)_{1 \leq k, \ell \leq m}
\]
\[
= \det \left( K(x_k, x_{\ell}) \right)_{1 \leq k, \ell \leq m}
\]

where
\[
K(k, \ell) = \langle \Psi_k \Psi^*_{k} f_0, f_0 \rangle = \left( e^{-\sum t_i \Lambda^i} e^{\sum s_i \Lambda^{-i}} \psi_k \psi^*_k e^{-\sum s_i \Lambda^{-i}} e^{\sum t_i \Lambda^i} \right)
\]

Using
\[
\langle \psi_i \psi^*_j f_0, f_0 \rangle = 1 \text{ if } i = j < 0
\]
\[
= 0 \text{ otherwise}
\]

and setting
\[
V(z) = -\sum_{i \geq 1} (t_i z^{-i} - s_i z^i), \quad (2.3.2)
\]

the generating function of the \( K(k, \ell) \) takes on the following simple form:
\[
\sum_{k, \ell \in \mathbb{Z}} z^k w^{-\ell} K(k, \ell)
\]
\[
= e^{-\sum s_i \Lambda^{-i}} e^{-\sum t_i \Lambda^i} \psi(z) \psi^*(w) e^{-\sum s_i \Lambda^{-i}} e^{\sum t_i \Lambda^i} f_0, f_0 \rangle
\]
\[
= e^{-\sum s_i \Lambda^{-i}} e^{-\sum t_i \Lambda^i} \psi(z) e^{-\sum s_i \Lambda^{-i}} e^{\sum t_i \Lambda^i} \psi^*(w) e^{-\sum s_i \Lambda^{-i}} e^{\sum t_i \Lambda^i} f_0, f_0 \rangle
\]
\[
= e^{V(z)-V(w)} \langle \psi(z) \psi^*(w) f_0, f_0 \rangle \quad \text{using (2.2.6)}
\]
\[
= e^{V(z)-V(w)} \sum_{i,j \in \mathbb{Z}} z^i w^{-j} \langle \psi_i \psi^*_j f_0, f_0 \rangle
\]
\[
= e^{V(z)-V(w)} \sum_{i \geq 1} \left( \frac{w}{z} \right)^i
\]

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\[ = e^{V(z) - V(w)} \left( \frac{w}{z} \right) \left( 1 + \frac{w}{z} + \ldots \right) \]

\[ = e^{V(z) - V(w)} \left( \frac{w}{z} \right) \frac{1}{1 - \frac{w}{z}} \quad \text{for } |w| < |z| \]

\[ = e^{V(z) - V(w)} \frac{w}{z - w}, \]

and so

\[
K(k, \ell) = \left( \frac{1}{2\pi i} \right)^2 \oint_{|w| = \rho < 1} \oint_{|z| = \rho^{-1} > 1} dz \, dw \, \frac{e^{V(z) - V(w)}}{z^{k+1}w^{\ell+1}} \left( z/w - 1 \right),
\]

ending the proof of Proposition 2.8.

Formula (2.0.5) in the remark is obtained by setting \( z \mapsto tz, \ w \mapsto tw \) in (2.3.1),

\[
K(k, \ell) = \left( \frac{1}{2\pi i} \right)^2 \oint_{|w| = \rho < 1} \oint_{|z| = \rho^{-1} > 1} dz \, dw \, \frac{e^{V(tz) - V(tw)}}{t^{k-\ell}} \frac{1}{z^{k+1}w^{\ell+1}} \left( z/w - 1 \right),
\]

and taking \( \frac{\partial}{\partial t} \bigg|_{t=1} \) of both sides, using the \( t \)-independence of the left hand side, yielding (2.0.5) .

2.4 Probability on partitions expressed as \( U(n) \) integrals

This section deals with formula (2.0.4) in Theorem 2.1, stating that \( P(\lambda \mid \lambda_1 \leq n) \) can be expressed as a Unitary matrix integral. First we need a Lemma, whose proof can be found in [27]:

Lemma 2.9 If \( f \) is a symmetric function of the eigenvalues \( u_1, \ldots, u_n \) of the elements in \( U(n) \), then

\[
\int_{U(n)} f = \frac{1}{n!} \int_{(S^1)^n} |\Delta(u)|^2 f(u_1, \ldots, u_n) \prod_{j=1}^n \frac{du_j}{2\pi i u_j}.
\]

Proof: Set \( G := U(n) \) and \( T := \{ \text{diag}(u_1, \ldots, u_n) \}, \) with \( u_k = e^{i\theta_k} \in S^1 \), and let \( t \) and \( g \) denote the Lie algebras corresponding to \( T \) and \( G \). An element
in the quotient $G/T$ can be identified with $gT$, because $g' \in gT$ implies $g^{-1}g' \in T$, and thus there is a natural map
\[ T \times (G/T) \to G : (t, gT) \mapsto gtg^{-1}. \]

Note that the Jacobian $J$ of this map (with respect to the invariant measures on $T$, $G/T$ and $G$) only depends on $t \in t$, because of invariance of the measure under conjugation. For a function $f$ as above
\[ \int_G f = \frac{1}{n!} \int_{T \times (G/T)} f(gtg^{-1}) J(t) dtd(gT) = \frac{\text{vol}(G/T)}{n!} \int_T f(t) J(t) dt. \]

Denote the tangent space to $G/T$ at its base points by $t^\perp$, the orthogonal complement of $t$ in $g = t \oplus t^\perp$. Consider infinitesimal changes $t (1 + \varepsilon \xi)$ of $t$ with $\xi \in t$; also, an infinitesimal change of $1 \in G$, namely $1 \mapsto 1 + \varepsilon \eta$, with $\eta \in t^\perp$. Then
\[ gtg^{-1} - t = (1 + \varepsilon \eta)t(1 + \varepsilon \xi)(1 - \varepsilon \eta + O(\varepsilon^2)) - t \]
\[ = \varepsilon(t \xi + \eta t - t \eta) + O(\varepsilon^2) \]
\[ =: \varepsilon \rho + O(\varepsilon^2) \]
and so
\[ t^{-1} \rho = \xi + (t^{-1} \eta t - \eta) \in t \oplus t^\perp = g. \]
Thus the Jacobian of this map is given by
\[ J(t) = \det(A(t^{-1}) - I), \]
where $A(t)$ denotes the adjoint action by means of the diagonal matrix $t$, denoted by $t = \text{diag}(u_1, \ldots, u_n)$, with $|u_i| = 1$. Let $E_{jk}$ be the matrix with 1 at the $(j, k)$-th entry and 0 everywhere else. Then
\[ (A(t^{-1}) - I) E_{jk} = (u_j^{-1} u_k - 1) E_{jk} \]
and thus the matrix $A(t^{-1}) - I$ has $n(n - 1)$ eigenvalues $(u_j^{-1} u_k - 1)$, with $1 \leq j, k \leq n$ and $j \neq k$. Therefore
\[ \det(A(t^{-1}) - I) = \prod_{j \neq k} (u_j^{-1} u_k - 1) = \prod_{j < k} |u_j - u_k|^2, \]
using the fact that $u_j^{-1} = \bar{u}_j$. ■
Proposition 2.10 Given the “probability measure”

\[ P(\lambda) = Z^{-1}s_\lambda(t)s_\lambda(s), \quad Z = e^{\sum_{i \geq 1} t_i \lambda_i}, \]

the following holds

\[ P \left( \lambda \bigg| \lambda_1 \leq n \right) = Z^{-1} \int_{U(n)} e^{-Tr \sum_{i}^\infty (t_i X_i + s_i \bar{X}_i)} dX. \]

Proof: Using Lemma 2.9 and the following matrix identity,

\[ \sum_{\sigma \in S_n} \det \left( a_{i,\sigma(j)} b_{j,\sigma(j)} \right)_{1 \leq i, j \leq n} = \det (a_{ik})_{1 \leq i, k \leq n} \det (b_{ik})_{1 \leq i, k \leq n}, \]

one computes

\[ n! \int_{U(n)} e^{-Tr \sum_{i}^\infty (t_i X_i + s_i \bar{X}_i)} dX \]

\[ = \int_{(S^1)^n} \left| \Delta_n(z) \right|^2 \prod_{k=1}^n \left( e^{-\sum_{i}^\infty (t_i z_k^l + s_i \bar{z}_k^l)} \frac{dz_k}{2\pi iz_k} \right) \]

\[ = \sum_{\sigma \in S_n} \Delta_n(z) \Delta_n(\bar{z}) \prod_{k=1}^n \left( e^{-\sum_{i}^\infty (t_i z_k^l + s_i \bar{z}_k^l)} \frac{dz_k}{2\pi iz_k} \right) \]

\[ = \sum_{\sigma \in S_n} \prod_{1 \leq \ell, m \leq n} \left( \oint_{S^1} z^{\ell-1} z_k^{m-1} e^{-\sum_{i}^\infty (t_i z_k^l + s_i \bar{z}_k^l)} \frac{dz_k}{2\pi iz_k} \right) \]

ending the proof of Proposition 2.10.
3 Examples

3.1 Plancherel measure and Gessel’s theorem

The point of this section will be to restrict the probability

\[ P_{x,y}(\lambda) := \frac{1}{Z} \tilde{s}_\lambda(x) \tilde{s}_\lambda(y) = \frac{1}{Z} s_\lambda(t) s_\lambda(s) \]  

considered in section 2 (formula (2.0.1)) to the locus \( \mathcal{L}_1 \), defined for real \( \xi > 0 \) (expressed in \((x, y)\) and \((s, t)\) coordinates),

\[
\mathcal{L}_1 = \{(x, y) \text{ such that } \sum_{i \geq 1} x_i^k = \sum_{i \geq 1} y_i^k = \delta_{ki} \sqrt{\xi}\}
\]

\[
\mathcal{L}_1 = \{ \text{all } s_k = t_k = 0, \text{ except } t_1 = s_1 = \sqrt{\xi} \}. \tag{3.1.2}
\]

The reader is referred back to subsection 1.5.1 for a number of basic formulae.

**Theorem 3.1** [45, 85, 26, 8] For the permutation group \( S_k \), the generating function for distribution of the length of the longest increasing subsequence

\[ \tilde{P}^k(L(\pi_k) \leq n) \]

is given by

\[
e^{-\xi} \sum_{k=0}^{\infty} \frac{\xi^k}{k!} \tilde{P}^k(L(\pi_k) \leq n) = e^{-\xi} \det \left( \int_{S_1} \frac{dz}{2\pi i z} z^{j-\ell} e^{\sqrt{\xi} (z+z^{-1})} \right)_{1 \leq j, \ell \leq n}
\]

\[
= e^{-\xi} \det \left( J_{j-\ell}(2\sqrt{-\xi}) \right)_{1 \leq j, \ell \leq n}
\]

\[
= e^{-\xi} \int_{U(n)} e^{\sqrt{\xi} \text{Tr}(X+X^{-1})} dX
\]

\[
= \det \left( I - K(j, \ell) \right)_{[n,n+1,...]}^{(j, \ell)}
\]

where \( J_i(z) \) is the Bessel function and

\[
K(k, \ell) = \frac{\sqrt{\xi} (J_k(2\sqrt{\xi})J_{\ell+1}(2\sqrt{\xi}) - J_{k+1}(2\sqrt{\xi})J_\ell(2\sqrt{\xi}))}{k - \ell}, \quad (k \neq \ell)
\]

\[
= \sum_{n=1}^{\infty} J_{k+n}(2\sqrt{\xi}) J_{\ell+n}(2\sqrt{\xi}). \tag{3.1.3}
\]
Proof: For an arbitrary partition \( \lambda \in \mathbb{Y} \), and using formula (1.2.7), the restriction of \( P_{x,y} \) to the locus \( \mathcal{L}_1 \), as in (3.1.2), reads as follows:

\[
P^{\xi}(\lambda) := P_{x,y}(\lambda) \mid_{\mathcal{L}_1} = e^{-\sum_{k \geq 1} k \xi k \lambda \lambda}(t) s_{\lambda}(s) \bigg|_{t_i = \sqrt{\mathcal{S}_{\lambda i}^{\xi}}, s_i = \sqrt{\mathcal{S}_{\lambda i}^{\xi}}} = e^{-\xi|\lambda|/2} \frac{f^\lambda}{|\lambda|!} \frac{f^\lambda}{|\lambda|!} = e^{-\xi \frac{|\lambda|}{|\lambda|!}} \left( \frac{(f^\lambda)^2}{|\lambda|!} \right) = e^{-\xi \frac{|\lambda|}{|\lambda|!}} \tilde{P}^{(n)}(\lambda), \text{ for } n = |\lambda|,
\]

where \( \tilde{P}^{(n)}(\lambda) \) can be recognized as Plancherel measure on partitions in \( \mathbb{Y}_n \), as defined in section 1.5.1.

\[
\tilde{P}^{(n)}(\lambda) = \left( \frac{(f^\lambda)^2}{n!} \right), \quad \lambda \in \mathbb{Y}_n.
\]

It is clear that

\[
P^{\xi}(\lambda) = e^{-\xi \frac{|\lambda|}{|\lambda|!}} \left( \frac{(f^\lambda)^2}{|\lambda|!} \right), \quad \lambda \in \mathbb{Y},
\]

is a genuine probability \((\geq 0)\), called Poissonized Plancherel measure.

We now compute

\[
P^{\xi}(\lambda \text{ with } \lambda_1 \leq n) = e^{-\xi \sum_{\lambda \in \mathbb{Y}, \lambda_1 \leq n} \frac{|\lambda|}{|\lambda|!} \frac{(f^\lambda)^2}{|\lambda|!}} = e^{-\xi \sum_{k=0}^{\infty} \frac{\xi k}{k!} \tilde{P}^{(k)}(L(\pi_k) \leq n)},
\]
and thus using Theorem 2.1:

\[
P^\xi(\lambda \text{ with } \lambda_1 \leq n) = Z^{-1} \det \left( \int_{S^1} \frac{dz}{2\pi iz} z^{k-\ell} e^{-\sum_{1 \leq i \leq n} t_i z^i + s_i z^{-i}} \right)_{1 \leq k, \ell \leq n} |_{\mathcal{L}_1}
\]

\[
= e^{-\xi} \det \left( \int_{S^1} \frac{dz}{2\pi iz} z^{k-\ell} e^{-\sqrt{\xi}(z+z^{-1})} \right)_{1 \leq k, \ell \leq n}
\]

\[
= e^{-\xi} \int_{U(n)} e^{\sqrt{\xi} \text{Tr}(X+X^{-1})} dX
\]

\[
= e^{-\xi} \det \left( J_{k-\ell}(2\sqrt{-\xi}) \right)_{1 \leq k, \ell \leq n},
\]

where we used the fact that for a Toeplitz matrix

\[
\det(a^{k-\ell}c_{k-\ell})_{1 \leq k, \ell \leq n} = \det(c_{k-\ell})_{1 \leq k, \ell \leq n}, \quad a \neq 0.
\]

It also equals

\[
P^\xi(\lambda \text{ with } \lambda_1 \leq n) = \det \left( I - K(i, j) \right)_{[n, n+1, \ldots]}
\]

where \(K(i, j)\) is given by formula (2.0.5), where \(V(z) = \sqrt{\xi}(z - z^{-1})\). Since

\[
\frac{z \frac{d}{dz} V(z) - w \frac{d}{dw} V(w)}{z - w} = \sqrt{\xi} \left( 1 - \frac{1}{wz} \right),
\]

The Bessel function \(J_n(u)\) is defined by

\[
e^{u(t-1)} = \sum_{n=0}^{\infty} t^n J_n(2u)
\]

and thus

\[
e^{-\sqrt{\xi}(z+z^{-1})} = e^{\sqrt{\xi}(iz-(iz)^{-1})} = \sum (iz)^n J_n(2\sqrt{-\xi}).
\]
one checks

\[(k - \ell)K(k, \ell) = \frac{\sqrt{\xi}}{2(\pi i)^2} \oint \oint dz \, dw \left( \frac{e^{\sqrt{\xi}(z-z^{-1})} e^{\sqrt{\xi}(w^{-1}-w)}}{z^{k+1} w^{\ell}} \right) \]

\[= \sqrt{\xi} \left( J_k(2\sqrt{\xi}) J_{\ell+1}(2\sqrt{\xi}) - J_{k+1}(2\sqrt{\xi}) J_\ell(2\sqrt{\xi}) \right) \]

\[= (k - \ell) \sum_{n=1}^\infty J_{k+n}(2\sqrt{\xi}) J_{\ell+n}(2\sqrt{\xi}). \quad (3.1.4)\]

The last equality follows from the recurrence relation of Bessel functions

\[J_{k+1}(2z) = \frac{k}{z} J_k(2z) - J_{k-1}(2z).\]

Indeed, subtracting the two expressions

\[J_{\ell+1}(2z)J_k(2z) = \frac{\ell}{z} J_k(2z) J_\ell(2z) - J_{\ell-1}(2z) J_k(2z) \]

\[J_{k+1}(2z)J_\ell(2z) = \frac{k}{z} J_k(2z) J_\ell(2z) - J_{k-1}(2z) J_\ell(2z),\]

one finds

\[(\ell - k)J_k(2z) J_\ell(2z) = z(J_k(2z) J_{\ell+1}(2z) - J_{k+1}(2z) J_\ell(2z)) \]

\[-z(J_{k-1}(2z) J_\ell(2z) - J_{k}(2z) J_{\ell-1}(2z)) \]

implying

\[z(J_k(2z) J_{\ell+1}(2z) - J_{k+1}(2z) J_\ell(2z)) = (k - \ell) \sum_{n=1}^\infty J_{k+n}(2z) J_{\ell+n}(2z), \]

thus proving (3.1.4).
Remark: Incidentally, the fact that $P^\xi$ is a probability shows that Plancherel measure itself is a probability; indeed, for all $\xi$,

\[
    e^{-\xi} \sum_{n=0}^{\infty} \frac{\xi^n}{n!} = 1 = \sum_{\lambda \in Y} P^\xi(\lambda) = e^{-\xi} \sum_{n=0}^{\infty} \frac{\xi^n}{n!} \sum_{|\lambda|=n} \frac{(f^\lambda)^2}{|\lambda|!} = e^{-\xi} \sum_{n=0}^{\infty} \frac{\xi^n}{n!} \sum_{\lambda \in Y_n} \tilde{P}^n(\lambda);
\]

comparing the extremities leads to $\sum_{\lambda \in Y_n} \tilde{P}^n(\lambda) = 1$.

### 3.2 Probability on random words

Here also, restrict the probability (3.1.1) to the locus below, for $\xi > 0$ and $p \in \mathbb{Z}_{\geq 1}$,

\[
    \mathcal{L}_2 = \left\{ \sum_{i \geq 1} x_i^k = \delta_{k1}\xi, \ y_1 = \ldots = y_p = \beta, \ \text{all other } y_j = 0 \right\}
\]

\[
    = \{ t_k = \delta_{k1}\xi, \ ks_k = p\beta^k \}
\]

Recall from section 1.5.4 the probability $\tilde{P}^{k,p}$ on partitions, induced from uniform probability $P^{k,p}$ on words. This was studied by Tracy-Widom [86] and Borodin-Okounkov [26]; see also Johansson [58] and Adler-van Moerbeke [8].

**Theorem 3.2** For the set of words $S_n^p$, the generating function for the distribution of the length of the longest increasing subsequence, is given by (setting $\beta = 1$)

\[
    e^{-\xi} \sum_{k=0}^{\infty} \frac{(p\xi)^k}{k!} \tilde{P}^{k,p}(L(\pi_k) \leq n) = e^{-\xi} \det \left( \int_{S^p} \frac{dz}{2\pi i z} z^{k-\ell} e^{(1+z)^p} \right)_{1 \leq k, \ell \leq n}
\]

\[
    = e^{-\xi} \int_{U(n)} e^{\xi \text{Tr} \tilde{M}} \det(I + M)^p dM
\]

\[
    = \det \left( I - K(j, k) \right)_{(n, n+1, \ldots)}^{(n+1, n+2, \ldots)}
\]
with $K(j,k)$ a Christoffel-Darboux kernel of Charlier polynomials:

$$(j-k)K(j,k) = \frac{\xi}{(2\pi i)^2} \oint_{|z|=c_1} \oint_{|w|=c_2} dz \ dw \left( p \frac{(1-\xi z)^{-p-1}e^{-\xi z^{-1}}}{z^{j+1}} \frac{(1-\xi w)^{p-1}e^{\xi w^{-1}}}{w^{-k}} - \frac{(1-\xi z)^{-p}e^{-\xi z^{-1}}}{z^{j+2}} \frac{(1-\xi w)^{p}e^{\xi w^{-1}}}{w^{-k+1}} \right)$$

$$= (p)_{j+1}e^{-\xi^2} \left( \frac{1}{j!} F_1(-p, j+1; \xi^2) \frac{1}{(k+1)!} F_1(-p+1, k+2; \xi^2) \right.$$ 

$$- \frac{1}{(j+1)!} F_1(-q+1, j+2; \xi^2) \frac{1}{k!} F_1(-q, k+1; \xi^2) \left. \right) \quad (3.2.1)$$

where

$$(a)_j := a(a+1)...(a+j-1), \quad (a)_0 = 1$$

and $1 F_1(a, c; x)$ is the confluent hypergeometric function:

$$\frac{1}{2\pi i} \oint_{|z|=c_1} (1-\xi z)^{-p} e^{\xi z^{-1}} \frac{dz}{z^{m+1}} = \frac{(p)_m}{m!} e^{-\xi^2} \frac{1}{m!} F_1(1-p, m+1; \xi^2)$$

$$\frac{1}{2\pi i} \oint_{|w|=c_2} (1-\xi w)^q e^{\xi w^{-1}} \frac{dw}{w^{m-1}} = \frac{1}{m!} F_1(-q, m+1; \xi^2).$$

These functions are related to Charlier polynomials\(^{10}\).

\(^{10}\) Charlier polynomials $P(k; \alpha)$, with $k \in \mathbb{Z}_{\geq 0}$, are discrete orthonormal polynomials defined by the orthonormality condition

$$\sum_{k=0}^{\infty} P_n(k; \alpha) P_m(k; \alpha) w_\alpha(k) = \delta_{nm}, \quad \text{for} \quad w_\alpha(k) = e^{-\alpha} \frac{\alpha^k}{k!},$$

with generating function

$$\sum_{n=0}^{\infty} \alpha^{n/2} \frac{1}{\sqrt{n!}} P_n(k; \alpha) w^n = e^{-\alpha w}(1+w)^k.$$
Proof: This proof will be very sketchy, as more details will be given for the percolation case in the next section. One now computes

\[ P^{\xi,p}(\lambda \text{ with } \lambda_1 \leq n) = \sum_{\lambda_{\lambda_1 \leq n}} e^{-\sum_{i \geq 1} \xi s_i} \mathbf{s}(t) \mathbf{s}(s) \bigg|_{\mathcal{L}_2} \]

\[ = e^{-\xi \beta} \sum_{\lambda_{\lambda_1 \leq n}} \mathbf{s}(\xi, 0, \ldots) \mathbf{s}(p \beta, \frac{p \beta^2}{2}, \frac{p \beta^3}{3}, \ldots) \]

\[ = e^{-\xi \beta} \sum_{\lambda_{\lambda_1 \leq n}} (p \xi \beta)^{|\lambda|} \frac{\mathbf{s}(1, 0, \ldots) \mathbf{s}(p, p, p, \ldots)}{|\lambda|!} \]

\[ = e^{-\xi \beta} \sum_{k=0}^{\infty} \frac{(p \xi \beta)^k}{k!} \sum_{\lambda_{\lambda_1 \leq n} \lambda = k} \frac{\mathbf{s}(1^p)}{|\lambda|!} \]

\[ = e^{-\xi \beta} \sum_{k=0}^{\infty} \frac{(p \xi \beta)^k}{k!} \tilde{P}^{k,p}(L(\pi_k) \leq n). \]

In applying Theorem 2.1, one needs to compute

\[ \left. e^{-\sum_{i} (t_i z^i + s_i z^{-i})} \right|_{\mathcal{L}_2} = e^{-\xi z} e^{-\xi \beta} \sum_{i} \frac{1}{i!} \left( \frac{\beta}{z} \right)^i = e^{-\xi z} \left( 1 - \frac{\beta}{z} \right)^p \]

and

\[ \left. e^{V} \right|_{\mathcal{L}_2} = e^{-\sum_{i} (t_i z^{-i} - s_i z^i)} \bigg|_{\mathcal{L}_2} = e^{-\xi z^{-1} (1 - \beta z)^{-p}}. \]

Therefore

\[ P^{\xi,p}(\lambda_1 \leq n) = Z^{-1} \det \left( \int_{S^1} \frac{dz}{2\pi i z} z^{k-\ell} e^{-\sum_{i} (t_i z^i + s_i z^{-i})} \right)_{1 \leq k, \ell \leq n} \bigg|_{\mathcal{L}_2} \]

\[ = e^{-\xi \beta} \det \left( \int_{S^1} \frac{dz}{2\pi i z} z^{k-\ell} e^{-\xi z} (1 - \beta z^{-1})^p \right)_{1 \leq k, \ell \leq n} \]

\[ = e^{-\xi \beta} \det \left( \int_{S^1} \frac{dz}{2\pi i z} z^{k-\ell} e^{\xi z^{-1}} (1 + \beta z)^p \right)_{1 \leq k, \ell \leq n} \]

the change of variable \( z \mapsto -z^{-1} \)

\[ = e^{-\xi \beta} \int_{U(n)} e^{\xi TrM} \det(1 + \beta M)^p dM. \]
Then, one computes
\[
z \frac{d}{dz} V(z) - w \frac{d}{dw} V(w) = \frac{\beta p}{(1 - \beta z)(1 - \beta w)} - \frac{\xi}{zw},
\]
and
\[e^{V(z) - V(w)} = e^{-\xi z^{-1}}(1 - \beta z)^{-p}e^{\xi w^{-1}}(1 - \beta w)^p,
\]
leading after an appropriate rescaling, using \(\beta\) to formula (3.2.1). From footnote 10, the confluent hypergeometric functions turn out to be Charlier polynomials.

### 3.3 Percolation

Considering now the locus
\[\mathcal{L}_3 := \{kt_k = q\xi^{k/2}, \ ks_k = p\xi^{k/2}\},\]
one is led to the probability appearing in the generalized permutations and percolations (sections 1.5.4 and 1.5.5), namely
\[P(L(M) \leq \ell) = \sum_{\lambda} (1 - \xi)^{pq}\xi^{\lambda}[s_\lambda(q, \frac{q}{2}, \ldots)s_\lambda(p, \frac{p}{2}, \ldots)].\]

We now state: (see [26, 56])

**Theorem 3.3** Assuming \(q > p\), we have
\[
P(L(M) \leq \ell) = \frac{(1 - \xi)^{pq}}{\ell!} \int_{(S^1)^\ell} |\Delta_\ell(z)|^2 \prod_{j=1}^\ell (1 + \sqrt{\xi z_j})^q(1 + \sqrt{\xi z_j})^p \frac{dz_j}{2\pi iz_j}
\]
\[
= (1 - \xi)^{pq} \int_{U(\ell)} \det(1 + \sqrt{\xi}M)^q \det(1 + \sqrt{\xi}M)^p dM
\]
\[
= \det (I - K(i, j)) \bigg|_{[\ell, \ell+1, \ldots]}\]

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where \((C\) is a constant depending on \(p, q, \xi\))
\[
K(i, j) = C\frac{M_p(p + i)M_{p-1}(p + j) - M_{p-1}(p + i)M_p(p + i)}{i - j},
\]
where
\[
M_p(k) := M_p(k; q - p + 1, \xi) \quad \text{with} \quad k \in \mathbb{Z}_{\geq 0}.
\]
are Meixner polynomials\(^{11}\).

**Proof:** Using the restriction to locus \(L_3\), one finds
\[
e^{-\sum_{i=1}^{\infty}(t_iz^j + s_jz^{-j})}\bigg|_{L_3} = (1 - \sqrt{\xi} z)^q(1 - \sqrt{\xi} z^{-1})^p
\]
and
\[
e^V\bigg|_{L_3} = e^{-\sum_{i=1}^{\infty}(t_i z^{-i} + s_i z^i)}\bigg|_{L_3} = (1 - \sqrt{\xi} z^{-1})^q(1 - \sqrt{\xi} z)^{-p},
\]
and thus
\[
V(z) = q \log(1 - \sqrt{\xi} z^{-1}) - p \log(1 - \sqrt{\xi} z).
\]
Hence
\[
z \frac{d}{dz} V(z) - w \frac{d}{dw} V(w) = \frac{p\sqrt{\xi}}{(1 - \sqrt{\xi} z)(1 - \sqrt{\xi} w)} - \frac{q\sqrt{\xi}}{zw(1 - \sqrt{\xi} z^{-1})(1 - \sqrt{\xi} w^{-1})}
\]
and so
\[
\frac{1}{z^{k+1}w^{-\ell}} e^{V(z) - V(w)} \left( z \frac{d}{dz} V(z) - w \frac{d}{dw} V(w) \right) = \sqrt{\xi} \left( p \frac{(1 - \sqrt{\xi} z)^{-1}(1 - \sqrt{\xi} z^{-1})^q}{z^{k+1}} \frac{(1 - \sqrt{\xi} w)^{-1}(1 - \sqrt{\xi} w^{-1})^{-q}}{w^{-\ell}} - q \frac{(1 - \sqrt{\xi} z)^{-p}(1 - \sqrt{\xi} z^{-1})^{q-1}}{z^{k+2}} \frac{(1 - \sqrt{\xi} w)^{-1}(1 - \sqrt{\xi} w^{-1})^{-q-1}}{w^{-\ell+1}} \right)
\]

\(^{11}\)Meixner polynomials are discrete orthogonal polynomials defined on \(\mathbb{Z}_{\geq 0}\) by the orthogonality relations
\[
\sum_{k=0}^{\infty} \binom{\beta + k - 1}{k} \xi^k M_p(k; \beta, \xi) M_p'(k; \beta, \xi) = \frac{\xi^{-p}}{(1 - \xi)^p} \binom{\beta + p - 1}{p}^{-1} \delta_{pp'}, \quad \beta > 0,
\]
and with generating function
\[
\left(1 - \frac{z}{\xi}\right)^x (1 - z)^{-x - \beta} = \sum_{p=0}^{\infty} \frac{(\beta)_p z^p}{p!} M_p(x; \beta, \xi).
\]
Expanding in Laurent series, one finds
\[(1 - \eta z)^{-\beta}(1 - \eta z^{-1})^{\beta'}\]
\[
= \left( \sum_{i=0}^{\infty} \frac{(-\beta)(-\beta - 1) \ldots (-\beta - i + 1)}{i!} (-\eta)^i z^i \right)
\times \left( \sum_{j=0}^{\infty} \frac{\beta'(\beta' - 1) \ldots (\beta' - j + 1)}{j!} (-\eta)^j \frac{1}{z^j} \right)
\]
\[
= \sum_{m \in \mathbb{Z}} z^m \sum_{i,j \geq 0} \frac{\beta(\beta + 1) \ldots (\beta + i - 1)(-\beta')(-\beta' + 1) \ldots (-\beta' + j - 1)}{i! j!} \eta^{i+j}
\]
\[
= \sum_{m \in \mathbb{Z}} z^m \sum_{j=0}^{\infty} \frac{\beta(\beta + 1) \ldots (\beta + j + m - 1)(-\beta') \ldots (-\beta' + j - 1)}{(j + m)!} \eta^{2j+m}.
\]
The coefficient of \(z^m\) for \(m \geq 0\) reads
\[
\sum_{j=0}^{\infty} \frac{\beta(\beta + 1) \ldots (\beta + j + m - 1)(-\beta') \ldots (-\beta' + j - 1) \eta^{2j+m}}{(j + m)!} \frac{1}{j!} = \frac{(\beta)_m}{m!} \eta^m \]
\[\text{2F}_1(\beta + m, -\beta'; m + 1; \eta^2),\]
which is Gauss' Hypergeometric function\(^{12}\) and thus
\[
\oint \frac{(1 - \eta z)^{-\beta}(1 - \eta z^{-1})^{\beta'} \, dz}{2\pi iz^{k+1}} = \frac{(\beta)_k}{k!} \eta^k \text{2F}_1(-\beta', \beta + k; k + 1; \eta^2).
\]
One does a similar computation for the other piece in the kernel, where one computes the coefficient of \(z^{-m}\). Using this fact, using a standard linear transformation formula\(^{13}\) for the hypergeometric function, and using a polynomial property for hypergeometric functions\(^{14}\) and the formula for \(0 < p < q\)
\[
\frac{\Gamma(1 + p - q)}{\Gamma(1 - q)} = (-1)^{p+1} \frac{(q - 1)!}{(q - p - 1)!},
\]
\[\text{2F}_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}.\]
Notice that when \(a = -m < 0\) is an integer, then \(\text{2F}_1(a, b; c; z)\) is a polynomial of degree \(m\).

\(^{12}\) In general one has \(\text{2F}_1(a, b; c; z) = (1 - z)^{-b} \text{2F}_1(b, c - a; c; \frac{z}{1-z})\).

\(^{13}\) When \(p\) is a positive integer \((< q)\) and \(k\) an integer, then \(\text{2F}_1(-p, -q; k + 1; z)\) is a polynomial of degree \(p\) in \(z\), which satisfies \(\text{2F}_1(-p, -q; k + 1; z) = \frac{\Gamma(k+1)\Gamma(1-q+p)}{\Gamma(1-q)\Gamma(1-p)}(-z)^p \text{2F}_1(-p, -k - p; q - p + 1; \frac{1}{z})\).
one finds Meixner polynomials (assume \( q > p > 0, \ x \geq 0 \) and \( k \geq 0 \)) (see Nifikorov-Suslov-Uvarov [71] and Koekoek-Swartouw [65]):

\[
\frac{1}{2\pi i} \oint_{|\sqrt{\xi}|=\varepsilon_1} \frac{(1-\sqrt{\xi} \ z)^{-p-1}(1-\sqrt{\xi} \ z^{-1})^q}{z^{k+1}} \, dz
\]

\[
= \frac{(p+1)k}{k!} \xi^{k/2} \ _2F_1(p+1+k,-q;k+1;\xi)
\]

\[
= \frac{(p+1)k}{k!} \xi^{k/2}(1-\xi)^q \ _2F_1(-p,-q;k+1;\xi/\xi-1)
\]

\[
= \frac{(p+1)k\Gamma(1+p-q)}{(k+p)!\Gamma(1-q)} \frac{\xi^{p+k/2}}{(1-\xi)^{p-q}} \ _2F_1(-p,-k-p;q-p+1;\xi)
\]

\[
= \binom{q-1}{p} \frac{\xi^{p+k/2}}{(1-\xi)^{p-q}} M_p(p+k;q-p+1;\xi)
\]

where

\[ M_p(x; \beta, \xi) = \ _2F_1(-p,-x; \beta; \frac{\xi-1}{\xi}) = \left(\frac{1-\xi}{\xi}\right)^p \frac{1}{(\beta)_p} \frac{1}{x^p} + \ldots, \ x \in \mathbb{Z}_{\geq 0}
\]

\[ =: a_p x^p + \ldots, \ \text{with} \ \beta = q-p+1
\]

are Meixner polynomials in \( x \), satisfying the following orthogonality properties:

\[
\sum_{x=0}^{\infty} w(x) M_p(x; \beta, \xi) M_m(x; \beta, \xi) = \frac{p!}{\xi^p(1-\xi)^{\beta}(\beta)_p} \delta_{pm} =: h_p \delta_{pm},
\]

with weight

\[ w(x) := \frac{(\beta)_x}{x!} \xi^x = \frac{(1+q-p)x}{x!} \xi^x. \]
Using these facts, one computes

\[(k - \ell)K(k, \ell) = \frac{\sqrt{\xi}}{(2\pi i)^2} \int \int_{[\sqrt{\xi} |z| < c_1} dz \, dw \]

\[
\left( p (1 - \sqrt{\xi} z)^{-p-1} (1 - \sqrt{\xi} z^{-1})^q (1 - \sqrt{\xi} w)^{p-1} (1 - \sqrt{\xi} w^{-1})^{-q} \right)_{z^k+1}^{w^{-\ell}}
\]

\[
- q (1 - \sqrt{\xi} z)^{-p-1} (1 - \sqrt{\xi} z^{-1})^q (1 - \sqrt{\xi} w)^{p-1} (1 - \sqrt{\xi} w^{-1})^{-q-1} \right)_{z^k+2}^{w^{-\ell+1}}
\]

\[
= \frac{\alpha_{p-1}}{a_p h_{p-1}} \sqrt{w(p+k)w(p+\ell)} (M_p(p+k)M_{p-1}(p+\ell) - M_{p-1}(p+k)M_p(p+\ell)),
\]

where

\[M_p(p+k) := M_p(p+k; q - p + 1, \xi).\]

We also have, using Propositions 2.2 and 2.10 restricted to the locus \(L_3\),

\[P(L(M) \leq \ell)\]

\[= \sum_{\lambda_1 \leq \ell} (1 - \xi)^{p_{q|\lambda}}|s_{\lambda}(q, \frac{q}{2}, \ldots)s_{\lambda}(p, \frac{p}{2}, \ldots)\]

\[= e^{-\sum_{\lambda_1 \leq \ell} k \lambda_1 \eta_{\lambda}} \sum_{\lambda_1 \leq \ell} s_{\lambda}(t_1, t_2, \ldots)s_{\lambda}(s_1, s_2, \ldots) \bigg|_{kt_k = q \xi^k/2}^{ks_k = p \xi^k/2}
\]

\[= (1 - \xi)^{pq} \det \left( \int_{S^1} z^{\alpha - \alpha'} e^{-\sum_{j=1}^\infty (t_j z^j + s_j z^{-j})} \frac{dz}{2\pi iz} \right)_{1 \leq \alpha, \alpha' \leq \ell}
\]

\[= (1 - \xi)^{pq} \det \left( \int_{S^1} z^{\alpha - \alpha'} (1 - \sqrt{\xi}z)^q (1 - \sqrt{\xi}z^{-1})^p \frac{dz}{2\pi iz} \right)_{1 \leq \alpha, \alpha' \leq \ell}
\]

\[= \frac{(1 - \xi)^{pq}}{k!} \int_{(S^1)^k} |\Delta_t(z)|^2 \prod_{j=1}^k (1 + \sqrt{\xi}z_j)^q (1 + \sqrt{\xi}z_j)^p \frac{dz_j}{2\pi iz_j}
\]

\[= (1 - \xi)^{pq} \int_{U(\ell)} \det(1 + \sqrt{\xi}M)^q \det(1 + \sqrt{\xi}M)^p dM,
\]

thus ending the proof of Theorem 3.3.
4 Limit theorems

4.1 Limit for Plancherel measure

Stanislaw Ulam [88] raised in 1961 the question: how do you compute the probability $P^{(n)}(L(\pi_n) \leq k)$ that the length $L := L^{(n)}$ of the longest increasing sequence in a random permutation is smaller than $k$. What happens for very large permutations, i.e., when $n \to \infty$? By Monte-Carlo simulations, Ulam conjectured that

$$c := \lim_{n \to \infty} \frac{E^{(n)}(L)}{\sqrt{n}} \quad (4.1.1)$$

exists ($E^{(n)}$ denote the expectation with respect to $P^{(n)}$). A much older argument of Erdős and Szekeres [36] implied that $E^{(n)}(L) \geq \frac{1}{2}\sqrt{n - 1}$ and so $c \geq \frac{1}{2}$. Numerical computation by Baer and Brock [17] suggested $c = 2$. Hammersley [50] showed the existence of the limit (4.1.1); in 1977, Logan and Shepp [67] proved $c \geq 2$ and, at the same time, Vershik and Kerov [92] showed $c = 2$. More recently other proofs have appeared by Aldous and Diaconis [15], Seppäläinen [78] and Johansson [57]. Meanwhile, Gessel [45] found a generating function for the probability (with respect to $n$) and connected this problem with Toeplitz determinants. Monte-Carlo simulations by Odlyzko and Rains [72] suggested

$$c_0 \sim \lim_{n \to \infty} \frac{\text{Var}L^{(n)}}{n^{1/3}}, \quad c_1 = \lim_{n \to \infty} \frac{E^{(n)}(L^{(n)}) - 2\sqrt{n}}{n^{1/6}},$$

with $c_0 \sim 0.819$ and $c_1 \sim -1.758$.

In this section, we explain some of the ideas underlying this problem. It is convenient to write a partition $\lambda \in \mathcal{Y}$ in $(u, v)$-coordinates, as shown in the picture below.
Remember from (1.2.5) the hook length $h^\lambda$, defined as

$$h^\lambda = \prod_{(i,j) \in \lambda} h_{ij}^\lambda,$$

with $h_{ij}^\lambda = $ hook length $= \lambda_i + \lambda_j^\top - i - j - 1$,

where $\lambda_j^\top$ is the length of the $j$th column. Also remember from (1.2.6) the formula $f^\lambda$ expressed in terms of $h^\lambda$. Then one has the following Theorem, due to Vershik and Kerov [91]; see also Logan and Shepp [67]. A sketchy outline of the proof will be given here.

Consider Plancherel measure (see section 3.1), which using formula (1.2.6), can be written in terms of the hook length,

$$\bar{P}^{(n)}(\lambda) = \frac{(f^\lambda)^2}{n!} = \frac{n!}{(h^\lambda)^2} \text{ for } |\lambda| = n,$$

and define the function

$$\Omega(u) := \begin{cases} \frac{2}{n}(u \arcsin \frac{u}{2} + \sqrt{4 - u^2}), & \text{for } |u| \leq 2 \\ |u|, & \text{for } |u| \geq 2. \end{cases} \quad (4.1.2)$$
Theorem 4.1 (Vershik–Kerov-Logan-Shepp) Upon expressing \( \lambda \) in \((u,v)\)-coordinates, define the subset of partitions, for any \( \varepsilon > 0 \),
\[
\mathcal{Y}_n(\varepsilon) := \left\{ \lambda \in \mathcal{Y} \mid \sup_u \frac{1}{\sqrt{n}} \lambda(u\sqrt{n}) - \Omega(u) < \varepsilon \right\}.
\]
Then, for large \( n \), Plancherel measure concentrates on a Young diagram whose boundary has shape \( \Omega(u) \); i.e.,
\[
\lim_{n \to \infty} \tilde{P}(\mathcal{Y}_n(\varepsilon)) = 1.
\]
Moreover, for the length \( \lambda_1 \) of the first row, one has
\[
\lim_{n \to \infty} \tilde{P} \left( \left| \frac{\lambda_1}{2\sqrt{n}} - 1 \right| < \varepsilon \right) = 1,
\]
and thus for uniform measure on permutations (remembering \( L(\pi_n) = \text{the length of the longest increasing sequence in } \pi_n \) ), one has
\[
\lim_{n \to \infty} P \left( \left| \frac{L(\pi_n)}{2\sqrt{n}} - 1 \right| < \varepsilon \right) = 1.
\]

**Brief outline of the Proof:** In a first step, the following expression will be estimated, using Stirling’s formula\(^{15}\)
\[
-\frac{1}{n} \log P_n(\lambda)
\]
\[
= -\frac{1}{n} \log \frac{n!}{(h^\lambda)^2}
\]
\[
= 2n \log \prod_{(i,j) \in \lambda} h^\lambda_{ij} - \frac{1}{n} \log n!
\]
\[
= 2n \log \prod_{(i,j) \in \lambda} h^\lambda_{ij} - \log n - \frac{1}{n} \log \sqrt{2\pi n} + 1 + O(1), \text{ by Stirling}
\]
\[
= 1 + 2 \left( \log \prod_{(i,j) \in \lambda} h^\lambda_{ij} - n \log n^{1/2} \right) - \frac{1}{n} \log \sqrt{2\pi n} + O(1), \text{ using } |\lambda| = n,
\]
\[
= 1 + 2 \sum_{(i,j) \in \lambda} \frac{1}{n} \log \frac{h^\lambda_{ij}}{\sqrt{n}} + O \left( \frac{\log \sqrt{n}}{n} \right) \text{ (Riemann sum)}
\]

\(^{15}\) \( \log n! = n \log n + \log \sqrt{2\pi n} - n + \ldots \text{ for } n \not\to \infty. \)
\[ 
\longrightarrow 1 + 2 \int \int_{\{(x,y) : x,y \geq 0, y \leq F_\tilde{\lambda}(x)\}} dx \ dy \log(F_\tilde{\lambda}(x) - y + F_\tilde{\lambda}^{-1}(y) - x),
\]

assuming the partition \( \tilde{\lambda} = \frac{\lambda}{n} \) tends to a continuous curve \( y = F_\tilde{\lambda}(x) \) in \((x, y)\) coordinates. Then the Riemann sum above tends to the expression above for \( n \to \infty \); note \( 1/n \) is the area of a square in the Young diagram, after rescaling by \( 1/\sqrt{n} \) and thus turns into \( dx \ dy \) in the limit. Clearly, the expression \( F_\tilde{\lambda}(x) - y + F_\tilde{\lambda}^{-1}(y) - x \) is the hook length of the continuous curve, with respect to the point \((x, y)\). In the \((u, v)\)-coordinates, this hook length is particularly simple:

\[ F_\tilde{\lambda}(x) - y + F_\tilde{\lambda}^{-1}(y) - x = \sqrt{2}(u_2 - u_1), \quad \text{with } u_1 \leq u_2, \]

where \((u_1, \psi(u_1)) = (x, F_\tilde{\lambda}(x))\) and \((u_2, \psi(u_2)) = (F_\tilde{\lambda}^{-1}(y), y)\) and where \( v = \psi(u) \) denotes the curve \( y = F_\tilde{\lambda}(x) \) in \((u, v)\)-coordinates.

Keeping in mind Figure 4.1, consider two points \((u_1, \psi(u_1))\) and \((u_2, \psi(u_2))\) on the curve, with \( u_1 < u_2 \). The point \((x, y)\) such that the hook (emanating from \((x, y)\) parallel to the \((x, y)\)-coordinates) intersects the curve at the two points \((u_1, \psi(u_1))\) and \((u_2, \psi(u_2))\), is given by a point on the line emanating from \((u_1, \psi(u_1))\) in the \((1, -1)\)-direction; to be precise, the point

\[ (x, y) = (u_1, \psi(u_1)) + \frac{1}{2}(u_2 - \psi(u_2) - u_1 + \psi(u_1))(1, -1). \]

So the surface element \( dx \ dy \) is transformed into the surface element \( du_1 du_2 \) by means of the Jacobian, \( dx \ dy = \frac{1}{2}(1 + \psi'(u_1))(1 - \psi'(u_2)) du_1 du_2 \), and thus, further replacing \( u_1 \mapsto \sqrt{2} u_1 \) and \( u_2 \mapsto \sqrt{2} u_2 \), one is led to

\[ -\frac{1}{n} \log P_n(\lambda) \]

\[ \approx 1 + 2 \int \int_{\{(x,y) : x,y \geq 0, y \leq F(x)\}} dx \ dy \log((F(x) + F^+(y) - x - y) \]

\[ = 1 + \int_{u_1 < u_2} du_1 du_2 (1 + \psi'(u_1))(1 - \psi'(u_2)) \log(\sqrt{2}(u_2 - u_1)) \]

\[ = -\frac{1}{2} \int_{\mathbb{R}^2} \log(\sqrt{2}|u_2 - u_1|) f'(u_1) f'(u_2) du_1 du_2 + 2 \int_{|u| > 2} f(u) \arccosh \left| \frac{u}{2} \right| du \]

\[ = \frac{1}{2} \int_{\mathbb{R}^2} \left( \frac{f(u_1) - f(u_2)}{u_1 - u_2} \right)^2 du_1 du_2 + 2 \int_{|u| > 2} f(u) \arccosh \left| \frac{u}{2} \right| du \]

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upon setting $\psi(u) = \Omega(u) + f(u)$, where the function $\Omega(u)$ is defined in (4.1.2). The last identity is obtained by using Plancherel’s formula of Fourier analysis,
\[
\int_{\mathbb{R}} g_1(v)g_2(v)dv = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{g}_1(v)\widehat{g}_2(v)dv
\]
applied to the two functions
\[
g_1(v) := \int_{\mathbb{R}} du \log(\sqrt{2}|v-u_1|)f'(u_1) = \int_{\mathbb{R}} \frac{f(u_1)}{\sqrt{2}|v-u_1|}du_1
\]
and $g_2(v) := f'(v)$.

This shows the expression $-\frac{1}{n} \log P_n(\lambda)$ above is minimal (and = 0), when $f(u) = 0$; i.e. when the curve $\psi(u) = \Omega(u)$ and otherwise the integrals above $> 0$. So, when the integral equals $\epsilon > 0$, the expression $-\frac{1}{n} \log P_n(\lambda) \simeq \epsilon > 0$ and thus $P_n(\lambda) \simeq e^{-\epsilon n}$, which tends to 0 for $n \to \infty$. Only, when $\epsilon = 0$, is there a chance that $P_n(\lambda) = 1$; this happens only when $\psi = \Omega$.

4.2 Limit theorem for longest increasing sequences

In section 3.1, it was shown that a generating function for the probability of the length $L(\pi_k)$ of the longest increasing sequence in a random permutation is given in terms of a Bessel kernel:

\[
e^{-\xi} \sum_{k=0}^{\infty} \frac{\xi^k}{k!} P^k(L(\pi_k) \leq n) = \det(I - K(j, \ell))\bigg|_{[n,n+1,...]},
\]
where
\[
K(k, \ell) = \sum_{m=1}^{\infty} J_{k+m}(2\sqrt{\xi})J_{\ell+m}(2\sqrt{\xi}). \quad (4.2.1)
\]

In the statement below $A(x)$ is the classical Airy function
\[A(x) = \frac{1}{\pi} \int_{0}^{\infty} \cos \left(\frac{u^3}{3} + xu\right) du;\]
this function is well known to satisfy the ordinary differential equation $A''(x) = xA(x)$ and to behave asymptotically as
\[A(x) := \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi x^{1/4}}}(1 + O(x^{3/2})), \quad \text{as } x \to \infty.\]
Theorem 4.2 (J. Baik, P. Deift and K. Johansson [19]) The distribution of the length $L(\pi_n)$ of the longest increasing sequence in a random permutation behaves as

$$\lim_{n \to \infty} P^{(n)} \left( L(\pi_n) \leq 2n^{1/2} + xn^{1/6} \right) = \det \left( I - A\chi_{[x,\infty)} \right),$$

where

$$A(x,y) := \int_0^\infty du A(x + u) A(y + u) = \frac{A(x)A'(y) - A'(x)A(y)}{x - y}. \quad (4.2.2)$$

The proof of this Theorem, presented here, is due to A. Borodin, A. Okounkov and G. Olshanski [25]; see also [20]. Before giving this proof, the following estimates on Bessel functions are needed:

Lemma 4.3 [25] (i) The following holds for $r \to \infty$,

$$\left| r^{1/3} J_{2r +xr^{1/3}}(2r) - A(x) \right| = O(r^{-1/3}),$$

uniformly in $x$, when $x \in \text{compact } K \subset \mathbb{R}$.

(ii) For any $\delta > 0$, there exists $M > 0$ such that for $x, y > M$ and large enough $r$,

$$\left| \sum_{s=1}^\infty J_{2r +xr^{1/3} +s}(2r) J_{2r +yr^{1/3} +s}(2r) \right| < \delta r^{-1/3}.$$

Lemma 4.4 (de-Poissonization lemma) (Johansson [57]) Given $1 \geq F_0 \geq F_1 \geq F_2 \geq ... \geq 0$, and

$$F(\xi) := e^{-\xi} \sum_{k=0}^\infty \frac{\xi^k}{k!} F_k,$$

there exists $C > 0$ and $k_0$, such that

$$F(k + 4\sqrt{k\log k}) - \frac{C}{k^2} \leq F_k \leq F(k - 4\sqrt{k\log k}) + \frac{C}{k^2}, \quad \text{for all } k > k_0.$$
Sketch of proof of Theorem 4.2. Putting the following scaling in the Bessel kernel $K(k, \ell)$, as in (4.2.1), one obtains, setting $r := \sqrt{\xi}$,

$$
\xi^{1/6} K(2\xi^{1/2} + x\xi^{1/6}, 2\xi^{1/2} + y\xi^{1/6}) = \left( \sum_{k=1}^{N} + \sum_{k=N+1}^{\infty} \right) \xi^{-1/6} \left[ \xi^{1/6} J_{2\xi^{1/2}+(x+k\xi^{-1/6})\xi^{1/6}}(2\sqrt{\xi}) \right] \left[ \xi^{1/6} J_{2\xi^{1/2}+(y+k\xi^{-1/6})\xi^{1/6}}(2\sqrt{\xi}) \right]
$$

Fix $\delta > 0$ and pick $M$ as in Lemma 4.3(ii). Define $N := \lceil (M - m + 1) r^{1/3} \rceil = O(r^4)$, where $m$ is picked such that $x, y \geq m$ (which is possible, since $x$ and $y$ belong to a compact set). Then

$$
\left| \sum_{k=N+1}^{\infty} J_{2r+(x+kr^{-1/3})r^{1/3}}(2r) J_{2r+(y+kr^{-1/3})r^{1/3}}(2r) \right| = \left| \sum_{s=1}^{\infty} J_{2r+(x+M-m+1)+s}(2r) J_{2r+(y+M-m+1)+s}(2r) \right| < \delta r^{-1/3},
$$

the latter inequality holds, since

$$
x + (M - m + 1) = M + (x - m) + 1 > M.
$$

On the other hand,

$$
\left| r^{1/3} J_{2r+(x+kr^{-1/3})r^{1/3}}(2r) - A(x + kr^{-1/3}) \right| = O(r^{-1/3})
$$

uniformly for $x \in$ compact $K \subset \mathbb{R}$, and all $k$ such that $1 \leq k \leq N = \lceil (M - m + 1) r^{1/3} \rceil$. Indeed, for such $k$,

$$
m \leq x \leq x + kr^{-1/3} \leq x + M - m + 1 = M + (x - m) + 1
$$

and thus, for such $k$, the $x + kr^{-1/3}$'s belong to a compact set as well. Since the number of terms in the sum below is $N = \lceil (M - m + 1) r^{1/3} \rceil$, Lemma 4.3
(i) implies
\[
\sum_{k=1}^{N} \left( r^{1/3} J_{2r,x+1/3+k}(2r) r^{1/3} J_{2r,y+1/3+k}(2r) \right)
\]
\[
-A(x + kr^{-1/3})A(y + kr^{-1/3}) \right) = O(1). \quad (4.2.5)
\]

But the Riemann sum tends to an integral
\[
\sum_{k=1}^{(M-m+1)r^{-1/3}} A(x + kr^{-1/3})A(y + kr^{-1/3}) \rightarrow \int_{0}^{M-m+1} A(x+t)A(y+t)dt.
\]

Hence, combining estimates (4.2.4, 4.2.5, 4.2.6) and multiplying with \( r^{1/3} \) leads to
\[
\left| \sum_{k=1}^{\infty} J_{2r,x+1/3+k}(2r) J_{2r,y+1/3+k}(2r) - \int_{0}^{M-m+1} A(x+t)A(y+t)dt \right| \leq \delta + o(1).
\]

for \( r \to \infty \). Finally, letting \( \delta \to 0 \) and \( M \to \infty \) leads to the result. Thus the expression (4.2.3) tends to the Airy kernel
\[
A(x, y) := \int_{0}^{\infty} du A(x + u)A(y + u).
\]

for \( r = \sqrt{\xi} \to \infty \). Hence
\[
\lim_{\xi \to \infty} e^{-\xi} \sum_{n=0}^{\infty} \frac{\xi^n}{n!} P \left( L(\pi_n) \leq 2\xi^{1/2} + x\xi^{1/6} \right)
\]
\[
= \lim_{\xi \to \infty} \det \left( I - K(\ell, \ell') \right)_{[k,k+1,...]} = 2\xi^{1/2} + x\xi^{1/6}
\]
\[
= 1 + \sum_{k=1}^{\infty} (-1)^k \int_{x \leq z_1 \leq ... \leq z_k} \det \left( A(z_i, z_j) \right) \prod_{1 \leq i,j \leq k} d\bar{z}_i
\]
\[
= \det \left( I - \chi_{[x,\infty)} \right).
\]

Finally, one uses Johansson’s de-Poissonization Lemma 4.4. From Corollary 4.5 Plancherel measure \( P_n(\lambda_1 \leq x_1, \ldots, \lambda_k \leq x_k) \) decreases, when \( n \) increases, which is required by Lemma 4.4. It thus follows that
\[
\lim_{n \to \infty} P \left( L(\pi_n) \leq 2n^{1/2} + xn^{1/6} \right) = \det \left( I - \chi_{[x,\infty)} \right),
\]
ending the proof of Theorem 4.2.
4.3 Limit theorem for the geometrically distributed percolation model, when one side of the matrix tends to $\infty$

Consider Johansson’s percolation model in section 1.7.2 but with $p$ and $q$ interchanged. Consider the ensemble

$$\text{Mat}^{(q,p)} = \{ q \times p \text{ matrices } M \text{ with entries } M_{ij} = 0, 1, 2, \ldots \}$$

with independent and geometrically distributed entries, for fixed $0 < \xi < 1$,

$$P(M_{ij} = k) = (1 - \xi)^k \xi, \quad k = 0, 1, 2, \ldots$$

$$L(M) := \max_{\text{all such paths}} \left\{ \sum_{\text{over right/down paths starting from entry } (1,1) \text{ to } (q,p)} M_{ij}, \text{ entry } (1,1) \right\}$$

has the following distribution, assuming $q \leq p$,

$$P(L(M) \leq \ell) = Z_{p,q}^{-1} \sum_{\Delta_q(h_1, \ldots, h_q) \max(h_i) \leq \ell + q - 1} \Delta_q(h_1, \ldots, h_q)^2 \prod_{i=1}^q \frac{(h_i + p - q)!}{h_i!} \xi^{h_i}$$

where

$$Z_{p,q} = \xi^{\frac{q(q-1)}{2}} (1 - \xi)^{-pq} q! \prod_{j=0}^{q-1} j!(p - q + j)! \quad \text{(4.3.1)}$$

Assuming that the number of columns $p$ of the $q \times p$ random $M$ matrix above gets very large, as above, the maximal right/lower path starting from $(1,1)$ to $(q,p)$ consists, roughly speaking, of many horizontal stretches and $q$ small downward jumps. The $M_{ij}$ have the geometric distribution, with mean and standard deviation

$$E(M_{ij}) = \sum_{k=0}^{\infty} k P(M_{ij} = k) = (1 - \xi)\xi \sum_{k=1}^{\infty} k \xi^{k-1} = \frac{\xi}{1 - \xi}$$

$$\sigma^2(M_{ij}) = E(M_{ij}^2) - (E(M_{ij}))^2 = \sum_{k=0}^{\infty} k^2 P(M_{ij} = k) - \left( \frac{\xi}{1 - \xi} \right)^2$$

$$= \frac{\xi(\xi + 1)}{(1 - \xi)^2} - \left( \frac{\xi}{1 - \xi} \right)^2 = \frac{\xi}{(1 - \xi)^2}$$

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So, in the average,

$$L(M) \simeq pE(M_{ij}) = \frac{p\xi}{1 - \xi}, \text{ for } p \to \infty$$

with

$$\sigma^2 \left( L(M) - \frac{p\xi}{1 - \xi} \right) \simeq p\sigma^2(M_{ij}) = \frac{p\xi}{(1 - \xi)^2}.$$ 

Therefore, it seems natural to consider the variables

$$x_1 = \frac{L(M) - \frac{\xi}{1 - \xi}p}{\frac{\sqrt{p}}{1 - \xi}} = \frac{\lambda_1 - \frac{\xi}{1 - \xi}p}{\frac{\sqrt{p}}{1 - \xi}}$$

and, since $\lambda = (\lambda_1, \ldots, \lambda_q)$, with $q$ finite, all $\lambda_i$ should be on the same footing.

So, remembering from the proof of Theorem 1.9 that $h_i$ in (4.3.1) and $\lambda_i$ are related by

$$h_i = q + \lambda_i - i,$$

we set

$$x_i = \frac{\lambda_i - \frac{\xi}{1 - \xi}p}{\frac{\sqrt{p}}{1 - \xi}}.$$

Theorem 4.5 (Johansson [58]) The following limit holds:

$$\lim_{p \to \infty} P \left( \frac{L(M) - \frac{\xi}{1 - \xi}p}{\frac{\sqrt{p}}{1 - \xi}} \leq y \right) = \int_{(-\infty,y)^q} \Delta_q(x)^2 \prod_{1}^{q} e^{-x_i^2/2}dx_i,$$

which coincides with the probability that a $q \times q$ matrix from the Gaussian Hermitian ensemble (GUE) has its spectrum less than $y$; see section 8.2.7.
**Proof:** The main tool here is Stirling’s formula\(^{16}\). Taking into account \(h_i = q + \lambda_i - i\), we substitute

\[
\lambda_i = \frac{\xi p}{1 - \xi} + \frac{\sqrt{\xi p}}{1 - \xi} x_i
\]

in (4.3.1). The different pieces will now be computed for large \(p\) and fixed \(q\),

\[
\prod_{i=1}^{q} (h_i + p - q)!
\]

\[
= \prod_{i=1}^{q} (p + \lambda_i - i)!
\]

\[
= \prod_{i=1}^{q} \left( \frac{1}{1 - \xi} (p + x_i \sqrt{p\xi}) - i \right)!
\]

\[
= \left( \frac{2\pi p}{1 - \xi} \right)^{q/2} \prod_{i=1}^{q} \left( 1 + x_i \sqrt{\frac{\xi}{p}} - \frac{i(1 - \xi)}{p} \right)^{1/2} \left( 1 + O \left( \frac{1}{p} \right) \right)
\]

\[
\sum_{e=1}^{q} \left( \frac{p}{1 - \xi} + x_i \sqrt{\frac{\xi p}{1 - \xi}} - i \right) \left( \log \frac{p}{1 - \xi} - 1 + \log \left( 1 + x_i \sqrt{\frac{\xi}{p}} - \frac{i(1 - \xi)}{p} \right) \right)
\]

\[
= \left( \frac{2\pi p}{1 - \xi} \right)^{q/2} e^{-\frac{\xi}{2(1 - \xi)}} \sum_{i=1}^{q} x_i^2 - \frac{qp}{1 - \xi} + O \left( \frac{1}{\sqrt{p}} \right)
\]

\[
e \left( \frac{qp}{1 - \xi} + \frac{\sqrt{\xi p}}{1 - \xi} \sum_{i=1}^{q} x_i - \frac{q(q + 1)}{2} \right) \log \frac{p}{1 - \xi} \left( 1 + O \left( \frac{1}{p} \right) \right),
\]

\(^{16}\) Stirling’s formula:

\[
n! = \sqrt{2\pi n} \ e^{n \left( \log n - 1 \right)} \left( 1 + O \left( \frac{1}{n} \right) \right), \text{ for } n \to \infty,
\]
up upon expanding the logarithm in powers of $1/\sqrt{\ell}$. Similarly

$$
\prod_{i=1}^{q} h_i
= \prod_{i=1}^{q} (q + \lambda_i - i)!
= \prod_{i=1}^{q} \left( \frac{1}{1 - \xi} (p_\xi + x_i \sqrt{p_\xi}) + q - i \right)!
= \left( \frac{2\pi p_\xi}{1 - \xi} \right)^{q/2} \prod_{i=1}^{q} \left( 1 + \frac{x_i}{\sqrt{p_\xi}} + \frac{(q - i)(1 - \xi)}{p_\xi} \right)^{1/2} \left( 1 + O \left( \frac{1}{\sqrt{\ell}} \right) \right)
\sum_{i=1}^{q} \left( \frac{1}{1 - \xi} (p_\xi + x_i \sqrt{p_\xi}) + q - i \right) \left( \log \frac{p_\xi}{1 - \xi} - 1 \right)
\sum_{i=1}^{q} \left( \frac{1}{1 - \xi} (p_\xi + x_i \sqrt{p_\xi}) + q - i \right) \log \left( 1 + \frac{x_i}{\sqrt{p_\xi}} + \frac{(q - i)(1 - \xi)}{p_\xi} \right)
= \left( \frac{2\pi p_\xi}{1 - \xi} \right)^{q/2} \frac{1}{e} \frac{1}{2(1 - \xi)} \sum_{i=1}^{q} x_i^2 - \frac{q p_\xi}{1 - \xi} \left( 1 + O \left( \frac{1}{\sqrt{\ell}} \right) \right)
\left( \frac{q p_\xi}{1 - \xi} + \frac{\sqrt{p_\xi}}{1 - \xi} \sum_{i=1}^{q} x_i + \frac{q(q - 1)}{2} \right) \log \frac{p_\xi}{1 - \xi}
$$

Also

$$
\prod_{i=1}^{q} \xi^{b_i} = \prod_{i=1}^{q} \xi^{q + \lambda_i - i} = \sum_{i=1}^{q} \left( \frac{1}{1 - \xi} (p_\xi + x_i \sqrt{p_\xi}) + q - i \right) \log \xi
= e \left( \frac{q p_\xi}{1 - \xi} + \frac{\sqrt{p_\xi}}{1 - \xi} \sum_{i=1}^{q} x_i + \frac{q(q - 1)}{2} \right) \log \xi.
$$
Besides

\[
\prod_{j=0}^{q-1} (p - q + j)! = (2\pi p)^{q/2} \prod_{j=0}^{q-1} \left(1 - \frac{q - j}{p}\right)^{1/2} \left(1 + O\left(\frac{1}{p}\right)\right)
\]

\[
\sum_{e_j=0}^{q-1} (p - q + j) \left(\log p - 1 + \log \left(1 - \frac{q - j}{p}\right)\right)
\]

\[
= (2\pi p)^{q/2} e^{-p^2 + q p \log p - \frac{q(q + 1)}{2}} \log p \left(1 + O\left(\frac{1}{p}\right)\right)
\]

\[
= (2\pi)^{q/2} p^{q^2/2} e^{-q p} \left(1 + O\left(\frac{1}{p}\right)\right)
\]

and

\[
\Delta_q(h_1, ..., h_q) = \prod_{1 \leq i < j \leq q} (h_i - h_j)
\]

\[
= \prod_{1 \leq i < j \leq q} \left((x_i - x_j) \frac{\sqrt{x p}}{1 - \xi} - i + j\right)
\]

\[
= \left(\frac{\sqrt{x p}}{1 - \xi}\right)^{q(q-1)/2} \left(\prod_{1 \leq i < j \leq q} (x_i - x_j) + O\left(\frac{1}{\sqrt{p}}\right)\right)
\]

Remembering the relation \( h_i = q + \lambda_i - i \), and using the estimates before in equality \( \ast \) below, one computes

\[
Z_{p,q}^{-1} \sum_{h_i \in \mathbb{N}} \Delta_q(h_1, ..., h_q)^2 \prod_{1}^{q} \frac{(h_i + p - q)!}{h_i!} \xi^{h_i}
\]

\[
= \sum_{h_i \in \mathbb{N}^q \atop \max h_i \leq \ell + q - 1} \xi^{-\frac{q(q-1)}{2}} \prod_{j=0}^{q} (1 - \xi)^{\rho_j} \Delta_q(h_1, ..., h_q)^2 \prod_{i=1}^{q} \frac{(p + \lambda_i - i)!}{(q + \lambda_i - i)!} \xi^{h_i} \prod_{i=1}^{q} (p - q + i - 1)!
\]

\[
= \left(\frac{1}{(2\pi)^{q/2} \prod_i i!}\right) \sum_{x_i \leq y} \left(\frac{1 - \xi}{\sqrt{x p}}\right)^{q} \Delta(x)^{2} e^{-\frac{1}{2} \sum_{i=1}^{q} x_i} \left(1 + O\left(\frac{1}{\sqrt{p}}\right)\right)
\]

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\[
\int_{x_i \leq y} \frac{\Delta(x)^2 \prod_{1}^{q} e^{-x_i^2/2} \, dx_i}{\int_{\mathbb{R}^q} \Delta_q(x)^2 \prod_{1}^{q} e^{-x_i^2/2} \, dx_i},
\]

using Selberg’s formula

\[
\int_{\mathbb{R}^q} \Delta_q(x)^2 \prod_{1}^{q} e^{-x_i^2/2} \, dx_i = (2\pi)^{q/2} \prod_{1}^{q} i!
\]

and noticing that in

\[
h_i = q + \lambda_i - i = \frac{\xi p}{1 - \xi} + \frac{\sqrt{\xi p}}{1 - \xi} x_i + q - i
\]

an increment of one unit in \( \lambda_i \) implies an increment of \( x_i \) by

\[dx_i = \frac{1 - \xi}{\sqrt{p \xi}}.
\]

Therefore, one has

\[
\left(\frac{1 - \xi}{\sqrt{p \xi}}\right)^q \simeq \prod_{1}^{q} dx_i.
\]

Finally, for \( p \) large, one has

\[
h_i \leq \ell + q - 1 = \frac{\xi p}{1 - \xi} + \frac{\sqrt{\xi p}}{1 - \xi} y + q - 1 \iff x_i \leq y.
\]

The connection with the spectrum of Gaussian Hermitian matrices will be discussed in section 8.2.1. This ends the proof of Theorem 4.5.

4.4 Limit theorem for the geometrically distributed percolation model, when both sides of the matrix tend to \( \infty \)

The model considered here is the percolation model as in section 4.3 whose probability can, by section 3.3, be written as a Fredholm determinant of a Meixner kernel; see Theorem 3.3.
Theorem 4.6 (Johansson [56]) Given

\[ a = \frac{(1 + \sqrt{\xi \gamma})^2}{1 - \xi} - 1, \quad \rho = \left( \frac{\xi}{\gamma} \right)^{1/6} \frac{(\sqrt{\gamma} + \sqrt{\xi})^{2/3}(1 + \sqrt{\gamma \xi})^{2/3}}{1 - \xi}, \]

the following holds:

\[ \lim_{p, q \to \infty, q/p = \gamma \geq 1 \text{ fixed}} P \left( \frac{L(M_{q,p}) - ap}{\rho p^{1/3}} \leq y \right) = \mathcal{F}(y), \]

which is the Tracy-Widom distribution (see section 9.2). In other terms, the random variable \( L(M_{q,p}) \) behaves in distribution, like

\[ L(M_{q,p}) \sim ap + \rho p^{1/3} \mathcal{F}. \]
Sketch of Proof: Remember from Theorem 3.3, the probability $P(L(M_{q,p}) \leq z)$ is given by a Fredholm determinant of a Christoffel-Darboux kernel composed of Meixner polynomials. Proving the statement of Theorem 4.6 amounts to proving the limit of this Meixner kernel with the scaling mentioned in the Theorem tends to the Airy kernel, i.e.,

$$\lim_{p \to \infty} \rho p^{1/3} K_p(ap + \rho p^{1/3} \eta, ap + \rho p^{1/3} \eta') = A(\eta, \eta') := \frac{A(\eta)A'(\eta') - A' (\eta) A(\eta')}{\eta - \eta'}$$

where

$$K_p(x, y) = -\frac{\xi}{(1 - \xi)d_{p-1}^2} \sqrt{\left(\frac{x + \beta' - 1}{x}\right)\xi x \left(\frac{y + \beta' - 1}{y}\right)\xi y} \times \frac{m_p(x, \beta', \xi)m_{p-1}(y, \beta', \xi) - m_p(y, \beta', \xi)m_{p-1}(x, \beta', \xi)}{x - y}$$

for the Meixner polynomials, which have the following integral representation, a consequence of the the generating function (3.3.1), (slightly rescaled)

$$m_p(x, \beta', \xi) = p!\xi^{-x} \int_{|z| = r < 1} \frac{(\xi - z)^x}{z^p(1 - z)^{x+\beta'}2\pi iz}$$

Thus, it will suffice to prove the following limit

$$\lim_{p \to \infty} \rho p^{1/3} \left(\frac{y^p(y - 1)^{x+\gamma p - p + 1}}{(y - \xi)^x}\right)\bigg|_{y = -\sqrt{\xi}} \int_{|z| = r < 1} \frac{(z - \xi)^x}{z^p(z - 1)^{x+\gamma p - p + 1}2\pi iz} \bigg|_{z = \beta p + \rho p^{1/3} \eta} = CA(\eta), \quad (4.4.1)$$

where

$$\alpha := \frac{(\sqrt{\xi} + \sqrt{\gamma})^2}{1 - \xi}, \quad \beta = \alpha - \gamma + 1 = \frac{(\sqrt{\gamma} + 1)^2}{1 - \xi},$$

$$\rho = \left(\frac{\xi}{\gamma}\right)^{1/6} \frac{(\sqrt{\gamma} + \sqrt{\xi})^{2/3}(1 + \sqrt{\xi\gamma})^{2/3}}{1 - \xi}$$

$$C := \gamma^{-1/3} \xi^{-1/6}(\sqrt{\xi} + \sqrt{\gamma})^{1/3}(1 + \sqrt{\xi\gamma})^{1/3}.$$

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17 with $d_p = \frac{p!(p + \beta' - 1)!}{(1 - \xi)^p \xi^{p(\beta' - 1)}}$ and $\beta' = q - p + 1 = p\gamma - p + 1.$
In view of the saddle point method, define the function $F_p(z)$ such that

$$e^{pF_p(z)} := \frac{(z - \xi)^x}{z^p(z - 1)^x + |\gamma p| - p + 1} \bigg|_{x = \beta + \rho p^{1/3} \eta}. $$

Then one easily sees that

$$F_p(z) = F(z) + \rho p^{-2/3} \eta \log \frac{z - \xi}{z - 1} + (\gamma - \frac{|\gamma p|}{p}) \log(z - 1), \quad (4.4.2)$$

where the $p$-independent function $F(z)$ equals,

$$F(z) = \beta \log(z - \xi) - \alpha \log(z - 1) - \log(z). \quad (4.4.3)$$

For the specific values above of $\alpha$ and $\beta$, one checks the function $F(z)$ has a critical point at

$$z_c := -\sqrt{\xi/\gamma}, \quad \text{i.e.,}$$

$$F''(z_c) = F'''(z_c) = 0, \quad F'''(z_c) = \frac{2\gamma^{5/2}}{\xi(\sqrt{\gamma} + \sqrt{\xi})(1 + \sqrt{\xi\gamma})}$$

and thus

$$F(z) - F(z_c) = \frac{1}{6}(z - z_c)^3 F'''(z_c) + O((z - z_c)^4). \quad (4.4.4)$$

Setting

$$z = z_c(1 - p^{-1/3} sC)e^{itCp^{-1/3}} \quad (4.4.5)$$

in $F_p(z)$ as in $(4.4.2)$, one first computes this substitution in $F(z)$, taking into account $(4.4.4)$,

$$F(z) = F(z_c) + \frac{i(-z_cC)^3}{6p} (t + is)^3 F'''(z_c) + O(p^{-4/3})$$

$$= F(z_c) + \frac{i}{3p} (t + is)^3 + O(p^{-4/3}), \quad (4.4.6)$$

upon picking -in the last equality- the constant $C$ such that

$$(-z_cC)^3 F'''(z_c) = 2.$$

Also, substituting the $z$ of $(4.4.5)$ in the part of $F(z)$ (see $(4.4.3)$) containing $\eta.$
\[
\frac{\rho p^{2/3} \eta}{z - 1} \log \frac{z - \xi}{z_c - 1} = \frac{\rho \eta}{p^{2/3}} \log \frac{z_c - \xi}{z_c - 1} - \frac{\rho C z_c (\xi - 1)}{(z_c - 1) (z_c - \xi)} \frac{i \eta (t + is)}{p} + O(p^{-4/3}).
\]

Thus, adding the two contributions (4.4.6) and (4.4.7), one finds

\[
pF_p \left( z_c (1 - p^{-1/3} s C) e^{it C p^{-1/3}} \right) = pF_p(z_c) + \frac{i}{3} \left( (t + is)^3 + 3 \eta (t + is) \right) + O(p^{-1/3})
\]

One then considers two contributions of the contour integral about the circle \(|z| = r\) appearing in (4.4.1), a first one along the arc \((\pi - \delta_p, \pi + \delta_p)\), for \(\delta_p\) tending to 0 with \(p \to \infty\) and a second one about the complement of \((\pi - \delta_p, \pi + \delta_p)\). The latter tends to 0, whereas the former tends to the Airy function (keeping \(s\) fixed, and in particular \(= 1\))

\[
CA(\eta) = \frac{C}{2\pi} \int_\infty^\infty e^{i(t+is)^3 + \eta (t+is))} dt,
\]

upon noticing that \(dz/z = iC p^{-1/3} dt\) under the change of variable \(z \mapsto t\), given in (4.4.5), establishing limit (4.4.1) and finally the limit of the Meixner kernel and its Fredholm determinant. Further details of this proof can be found in Johansson [56]. The Fredholm determinant of the Airy kernel is precisely the Tracy-Widom distribution, as will be shown in section 9.2.

### 4.5 Limit theorem for the exponentially distributed percolation model, when both sides of the matrix tend to \(\infty\)

Referring to the exponentially distributed percolation model, discussed in section 1.10 we now state

**Theorem 4.7** (Johansson [56]) Given

\[
a = (1 + \sqrt{\gamma})^2, \quad \rho = \frac{(1 + \sqrt{\gamma})^{4/3}}{\gamma^{1/6}},
\]

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the following limit holds:

\[
\lim_{p, q \to \infty, \quad q/p = \gamma \geq 1 \text{ fixed}} P \left( \frac{L(M_{q,p}) - (p^{1/2} + q^{1/2})}{(p^{1/2} + q^{1/2})(p^{-1/2} + q^{-1/2})^{1/3}} \leq y \right)
\]

\[
= \lim_{p, q \to \infty, \quad q/p = \gamma \geq 1 \text{ fixed}} P \left( \frac{L(M_{q,p}) - ap}{\rho p^{1/3}} \leq y \right) = \mathcal{F}(u),
\]

which is again the Tracy-Widom distribution. Here again \( L(M_{q,p}) \) behaves, after some rescaling and in distribution, like the Tracy-Widom distribution, for large \( p \) and \( q \) such that \( q/p = \gamma \geq 1 \):

\[
L(M_{q,p}) \sim ap + \rho p^{1/3}\mathcal{F}
\]

Proof: In Theorem 1.10, one has shown that \( P(L(M) \leq t) \) equals the ratio of two integrals; this ratio will be shown in section 7 on random matrices (see Propositions 7.9 and 7.4) to equal a Fredholm determinant of a kernel corresponding to Laguerre polynomials:

\[
P(L(M) \leq t) = \frac{\int_{[0,t]} \Delta_p(x)^2 \prod_{i=1}^p x_i^{q-p} e^{-x_i} \, dx_i}{\int_{[0,\infty]} \Delta_p(x)^2 \prod_{i=1}^p x_i^{q-p} e^{-x_i} \, dx_i} = \det(I - K_p^{(\alpha)}(x,y)\chi_{[t,\infty]}),
\]

where

\[
K_p^{(\alpha)}(x,y) = \sqrt{\frac{\hbar_p}{\hbar_{p-1}}} (xy)^{\alpha/2} e^{-\frac{1}{2}(x+y)} \frac{\mathcal{L}_p^{(\alpha)}(x)\mathcal{L}_{p-1}^{(\alpha)}(y) - \mathcal{L}_p^{(\alpha)}\mathcal{L}_{p-1}^{(\alpha)}(x)}{x-y};
\]

in the formula above, the \( \mathcal{L}_p^{(\alpha)}(x) = \frac{1}{\sqrt{\hbar_p}} x^n + \ldots = (-1)^p \left( \frac{p!}{(p+\alpha)!} \right)^{1/2} L_p^{(\alpha)}(x) \) are the normalized Laguerre polynomials.\(^{18}\)

\[
= \int_0^\infty \mathcal{L}_n^{(\alpha)}(x) \mathcal{L}_m^{(\alpha)}(x) x^{\alpha} e^{-x} \, dx = \delta_{nm}.
\]

\(^{18}\) \( L_n^{(\alpha)}(y) = \sum_{m=0}^n (-1)^m \binom{n+\alpha}{n-m} \frac{m^m}{m!} \int_C \frac{z^{n-m+\alpha}}{(z-1)^{n-m}} \, dz \), where \( C \) is a circle about \( z = 1 \).
Therefore, using the precise values of $a$ and $\rho$ above,

$$P \left( \frac{L(M_{q,p}) - ap}{\rho p^{1/3}} \leq y \right) = \det (I - K(\xi, \eta) \chi_{[y, \infty)})$$

with

$$K(\xi, \eta) = bp^{1/3} K_p^{(\gamma-1)p} \left( ap + \rho p^{1/3} \xi, ap + \rho p^{1/3} \eta \right)$$

The result follows from an asymptotic formula for Laguerre polynomials

$$\lim_{p \to \infty} K_p^{(\gamma-1)p} \left( ap + \rho p^{1/3} \xi, ap + \rho p^{1/3} \eta \right) = A(\xi, \eta),$$

with $A(x, y)$ the Airy kernel, as in (4.2.2), namely

$$A(x, y) := \frac{A(x)A'(y) - A'(x)A(y)}{x - y}.$$  

The Fredholm determinant of the Airy kernel is the Tracy-Widom distribution. This ends the proof of Theorem [4.7].

5 Orthogonal polynomials for a time dependent weight and the KP equation

5.1 Orthogonal polynomials

The inner-product with regard to the weight $\rho(z)$ over $\mathbb{R}$, assuming $\rho(z)$ decays fast enough at the boundary of its support,$^{19}$

$$\langle f, g \rangle = \int_{\mathbb{R}} f(z)g(z)\rho(z)dz,$$  \hspace{1cm} (5.1.1)

leads to a moment matrix

$$m_n = (\mu_{ij})_{0 \leq i, j < n} = (\langle z^i, z^j \rangle)_{0 \leq i, j \leq n-1}.$$  \hspace{1cm} (5.1.2)

Since the $\mu_{ij}$ depends on $i + j$ only, this is a H"{a}nkel matrix, and thus symmetric. This is tantamount to the relation

$$\Lambda m_\infty = m_\infty \Lambda^\top,$$

---

$^{19}$In this section, the support of the weight $\rho(z)$ can be the whole of $\mathbb{R}$ or any other interval.
where $\Lambda$ denotes the semi-infinite shift matrix

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 1 & 0 & 0 & 0 & \ldots \\
0 & 0 & 1 & 0 & 0 & 0 & \ldots \\
\vdots & & & & & & \\
\end{pmatrix}
$$

Define $\tau_n := \det m_n$.

Consider the factorization of $m_\infty$ into a lower- times an upper-triangular matrix\footnote{This factorization is possible as long as $\tau_n := \det m_n \neq 0$ for all $n \geq 1$.}

$$m_\infty = S^{-1}S^\top - 1,$$

(5.1.3)

with $S = \text{lower triangular with non-zero diagonal elements}$. For any $z \in \mathbb{C}$, define the semi-infinite column

$$\chi(z) := (1, z, z^2, \ldots)^\top,$$

(5.1.4)

and functions $p_n(z)$ and $q_n(z)$,

$$p_n(z) := (S\chi(z))_n,$$

$$q_n(z) := (S^\top - 1\chi(z^{-1}))_n.$$

(5.1.5)

(1) The $p_n(z)$ are polynomials of degree $n$, orthonormal with regard to $\rho(z)$, and $q_n(z)$ is the Stieltjes transform of $p_n(z)$,

$$q_n(z) = z \int_{\mathbb{R}} \frac{p_n(u)\rho(u)}{z - u} du.$$

Indeed,

$$\left(\langle p_k, p_\ell \rangle\right)_{0 \leq k, \ell < \infty} = \int_{\mathbb{R}} S\chi(z)(S\chi(z))^\top \rho(z) dz = Sm_\infty S^\top = I.$$
Note that $S\chi(z)(S\chi(z))^\top$ is a semi-infinite matrix obtained by multiplying the semi-infinite column $S\chi(z)$ and row $(S\chi(z))^\top$. The definition of $q_n$, together with the decomposition $S^{-1} = S\chi\infty$, leads to

$$q_n(z) = (S^{-1}\chi(z^{-1})_n$$

$$= \sum_{j \geq 0} (Sm\infty)_n j z^{-j}$$

$$= \sum_{j \geq 0} z^{-j} \sum_{0 \leq \ell \leq n} S_n^\ell \mu_{\ell j}$$

$$= \sum_{j \geq 0} z^{-j} \sum_{0 \leq \ell \leq n} S_n^\ell \int_\mathbb{R} u^{\ell+j} \rho(u) du$$

$$= \int_\mathbb{R} S_n^\ell u^\ell \sum_{j \geq 0} \left( \frac{u}{z} \right)^j \rho(u) du$$

$$= z \int_\mathbb{R} \frac{p_n(u) \rho(u)}{z - u} du.$$

(2) The orthonormal polynomials $p_n$ have the following representation

$$p_n(z) = \frac{1}{\sqrt{\tau_n \tau_{n+1}}} \det \begin{pmatrix} 1 & \cdots & 1 \\ m_n & \cdots & m_n \\ \mu_{n,0} & \cdots & \mu_{n,n-1} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \\ z & \cdots & z^n \end{pmatrix}.$$

As a consequence, the monic orthogonal polynomials $\tilde{p}_n(z)$ are related to $p_n(z)$ as follows:

$$p_n(z) = \sqrt{\frac{\tau_n}{\tau_{n+1}}} \tilde{p}_n(z).$$

Defining $p'_n(z)$ to be the polynomial on the right hand side of (5.1.6), it suffices to show that for $k < n$,

$$\langle p'_n, p'_k \rangle = 0 \quad \text{and} \quad \langle p'_n, p'_n \rangle = 1,$$
thus leading to \( p_n = p'_n \). Indeed,

\[
\langle p'_n, z^k \rangle = \frac{1}{\sqrt{\tau_n \tau_{n+1}}} \det \begin{pmatrix} m_n & \langle 1, z^k \rangle \\ \langle z, z^k \rangle & \vdots \\ \langle z^n, z^k \rangle \end{pmatrix} \begin{cases} 0, & \text{for } k < n, \\ \sqrt{\frac{\tau_{n+1}}{\tau_n}} & \text{for } k = n, \end{cases}
\]

and thus for \( k = n \),

\[
\langle p'_n, p'_n \rangle = \sqrt{\frac{\tau_n}{\tau_{n+1}}} \langle p'_n, z^n + \ldots \rangle = \sqrt{\frac{\tau_n}{\tau_{n+1}}} \langle p'_n, z^n \rangle = 1,
\]

from which (5.1.6) follows. Formula (5.1.7) is a straightforward consequence.

(3) The monic orthogonal polynomials \( \tilde{p}_n \) and their Stieltjes transform have the following representation

\[
\tilde{p}_n(z) = z^n \det \left( \mu_{ij} - \frac{1}{z} \mu_{i,j+1} \right)_{0 \leq i,j \leq n-1} / \det(\mu_{ij})_{0 \leq i,j \leq n-1}
\]

\[
\int_{\mathbb{R}} \frac{\tilde{p}_n(u) \rho(u)}{z - u} du = z^{-n-1} \det \left( \mu_{ij} + \frac{1}{z} \mu_{i,j+1} + \frac{1}{z^2} \mu_{i,j+2} + \ldots \right)_{0 \leq i,j \leq n} / \det(\mu_{ij})_{0 \leq i,j \leq n-1}
\]

(5.1.8)

Proof: Setting

\[
\vec{\mu}_j = (\mu_{0,j}, \ldots, \mu_{n-1,j}) \in \mathbb{R}^n,
\]

one computes

\[
z^n \det \left( \mu_{ij} - z^{-1} \mu_{i,j+1} \right)_{0 \leq i,j \leq n-1} = \det(\vec{\mu}_j - \vec{\mu}_{i,j+1})_{0 \leq i,j \leq n-1}
\]

\[
= \det(\vec{\mu}_0 - \vec{\mu}_1, z\vec{\mu}_1 - \vec{\mu}_2, \ldots, z\vec{\mu}_{n-1} - \vec{\mu}_n)
\]

\[
= \det \left( \sum_{0}^{n-1} \frac{z\vec{\mu}_j - \vec{\mu}_{j+1}}{z^j}, \sum_{0}^{n-2} \frac{z\vec{\mu}_{j+1} - \vec{\mu}_{j+2}}{z^j}, \ldots, z\vec{\mu}_{n-1} - \vec{\mu}_n \right)
\]

by column operations

\[
= \det \left( \vec{\mu}_0 - \frac{\vec{\mu}_n}{z^{n-1}}, z\vec{\mu}_1 - \frac{\vec{\mu}_n}{z^{n-2}}, \ldots, z\vec{\mu}_{n-1} - \vec{\mu}_n \right)
\]

\[95\]
\[ \begin{vmatrix}
  z\bar{\mu}_0 - \frac{\bar{\mu}_n}{z^{n-1}} & 0 \\
  z\bar{\mu}_1 - \frac{\bar{\mu}_{n-2}}{z^{n-2}} & 0 \\
  \vdots & \vdots \\
  z\bar{\mu}_{n-1} - \bar{\mu}_n & 0 \\
  \bar{\mu}_n & z^n 
\end{vmatrix} \]

enlarging the matrix by one row and column

\[ \begin{vmatrix}
  z\bar{\mu}_0 & z \\
  z\bar{\mu}_1 & z^2 \\
  \vdots & \vdots \\
  z\bar{\mu}_{n-1} & z^n \\
  \bar{\mu}_n & z^n 
\end{vmatrix} \]

by adding a multiple of the last row to rows 1 to \( n \)

\[ = \frac{1}{z^n} \det \begin{vmatrix}
  \bar{\mu}_0 \\
  \bar{\mu}_1 \\
  \vdots \\
  \bar{\mu}_{n-1} \\
  \bar{\mu}_n 
\end{vmatrix} \]

\[ = \tau_n \tilde{p}_n(z). \]

Setting this time

\[ \bar{\mu}_j := (\mu_{0j}, ..., \mu_{nj}) \in \mathbb{R}^{n+1}, \]

one computes

\[ \det \left( \mu_{ij} + \frac{\mu_{i,j+1}}{z} + \frac{\mu_{i,j+2}}{z^2} + \ldots \right)_{0 \leq i, j \leq n} \]

\[ = \det \left( \sum_0^\infty \frac{\bar{\mu}_{j}^T}{z^j}, \sum_0^\infty \frac{\bar{\mu}_{j+1}^T}{z^{j+1}}, \ldots, \sum_0^\infty \frac{\bar{\mu}_{j+n}^T}{z^{j+n}} \right) \]

\[ = \det \left( \tilde{\mu}_0^T, \tilde{\mu}_1^T, \ldots, \tilde{\mu}_{n-1}^T, \sum_0^\infty \frac{\bar{\mu}_{j+n}^T}{z^j} \right) \]

\[ = z^n \det \left( \tilde{\mu}_0^T, \tilde{\mu}_1^T, \ldots, \tilde{\mu}_{n-1}^T, \sum_0^\infty \frac{\bar{\mu}_j^T}{z^j} \right) \]
\[
\int_{\mathbb{R}} \sum_{j=0}^{\infty} \left( \frac{u}{z} \right)^j \rho(u) \, du = \left. \frac{\tilde{p}_n(u) \rho(u)}{z - u} \right|_{u=0}^{u=\infty} \int_{\mathbb{R}} \tilde{p}_n(u) \rho(u) \, du
\]

Remark: Representation (5.1.8) for orthogonal polynomials \( p_n \) can also be deduced from Heine’s representation. However, representation (5.1.8) is much simpler.

(4) The vectors \( \tilde{p} \) and \( \tilde{q} \) are eigenvectors of the tridiagonal symmetric matrix
\[
L := S \Lambda S^{-1}.
\] (5.1.9)

Conjugating the shift matrix \( \Lambda \) by \( S \) yields a matrix
\[
L = S \Lambda S^{-1}
\]
\[
= S \Lambda m_\infty S^\top, \text{ using (5.1.3)},
\]
\[
= S \Lambda \Lambda^\top S^\top, \text{ using } \Lambda m_\infty = m_\infty \Lambda^\top,
\]
\[
= S(S^{-1}S^{\top-1})\Lambda^\top S^\top, \text{ using again (5.1.3)},
\]
\[
= (S \Lambda S^{-1})^\top = L^\top,
\]

which is symmetric and thus tridiagonal. Remembering \( \chi(z) = (1, z, z^2, ...)^\top \), and the shift \( (\Lambda v)_n = v_{n+1} \), we have
\[
\Lambda \chi(z) = z \chi(z) \text{ and } \Lambda^\top \chi(z^{-1}) = z \chi(z^{-1}) - z e_1, \text{ with } e_1 = (1, 0, 0, ...)^\top.
\]
Therefore, \( p(z) = S \chi(z) \) and \( q(z) = S^{T-1} \chi(z^{-1}) \) are eigenvectors, in the sense

\[
L_p = SAS^{-1}S\chi(z) = zS\chi(z) = zp
\]

\[
L^T q = S^{T-1} \Lambda^T S^T S^{T-1} \chi(z^{-1}) = zS^{T-1} \chi(z^{-1}) - zS^{T-1}e_1 = zq - zS^{T-1}e_1.
\]

Then, using \( L = L^T \), one is led to

\[
((L - zI)p)_n = 0, \quad \text{for } n \geq 0 \quad \text{and} \quad ((L - zI)q)_n = 0, \quad \text{for } n \geq 1.
\]

(5) The off-diagonal elements of the symmetric tridiagonal matrix \( L \) are given by

\[
L_{n-1,n} = \sqrt{\frac{h_n}{h_{n-1}}}. \tag{5.1.10}
\]

Since \( \langle \tilde{p}_n, \tilde{p}_n \rangle = h_n \), one has \( p_n(y) = \frac{1}{\sqrt{h_n}} \tilde{p}_n(y) \). From the three step relation \( Lp(y) = yp(y) \), it follows that

\[
\left( \frac{1}{\sqrt{h_{n-1}}} y^n + \ldots \right) = yp_{n-1}(y) = L_{n-1,n} p_n(y) + (\text{terms of degree } \leq n-1)
\]

\[
= L_{n-1,n} \left( \frac{1}{\sqrt{h_n}} y^n + \ldots \right),
\]

leading to statement (5.1.10).

5.2 Time dependent Orthogonal polynomials and the KP equation

Introduce now into the weight \( \rho(z) \) a dependence on parameters \( t = (t_1, t_2, \ldots) \), as follows

\[
\rho_t(z) := \rho(z)e^{\sum_{i=1}^{\infty} t_i z^i}. \tag{5.2.1}
\]

Consider the moment matrix \( m_n(t) \), as in (5.1.2), but now dependent on \( t \), and the factorization of \( m_\infty \) into lower- times upper-triangular \( t \)-dependent matrices, as in (5.1.3)

\[
m_\infty(t) = (\mu_{ij}(t))_{0 \leq i,j < \infty} = S^{-1}(t)S^{T-1}(t). \tag{5.2.2}
\]

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The Toda lattice mentioned in the Theorem below will require the following Lie algebra splitting
\[ \mathfrak{gl}(n) = \mathfrak{s} \oplus \mathfrak{b}, \tag{5.2.3} \]
into skew-symmetric matrices and (lower) Borel matrices.

Also, one needs in this section the Hirota symbol: given a polynomial \( p(t_1, t_2, \ldots) \) of a finite or infinite number of variables and functions \( f(t_1, t_2, \ldots) \) and \( g(t_1, t_2, \ldots) \), also depending on a finite or infinite number of variables \( t_i \), define the symbol
\[ p \left( \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \ldots \right) f \circ g := p \left( \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots \right) f(t + y)g(t - y) \bigg|_{y=0}. \tag{5.2.4} \]

The reader is reminded of the elementary Schur polynomials \( e \mathcal{P}^\infty \sum_{i \geq 0} s_i(t) z^i \) and for later use, set for \( \ell = 0, 1, 2, \ldots \),
\[ s_\ell(\partial) := s_\ell \left( \frac{\partial}{\partial t_1}, \frac{1}{2} \frac{\partial}{\partial t_2}, \ldots \right). \tag{5.2.5} \]

One also needs Taylor’s formula for a \( C^\infty \)-function \( f \):
\[ f(z + y) = e^{y \frac{\partial}{\partial z}} f(z), \tag{5.2.6} \]
which is seen by expanding the exponential. The following Lemma will also be used later in the proof of the bilinear relations:

**Lemma 5.1** \([4, 3]\) If \( f_\infty \) denotes the integral along a small circle about \( \infty \), the following identity holds (formal identity in terms of power series):
\[ \int_{\mathbb{R}} f(u)g(u)du = \frac{1}{2\pi i} \int_{f_\infty} dz \int_{\mathbb{R}} \frac{g(u)}{z - u} du, \tag{5.2.7} \]
for holomorphic \( f(z) = \sum_{i \geq 0} a_i z^i \) and \( g(z) \), the latter assumed to have all its moments.

**Proof:** For holomorphic functions \( f \) in \( \mathbb{C} \),
\[
\frac{1}{2\pi i} \int_{f_\infty} dz f(z) \int_{\mathbb{R}} \frac{g(u)}{z - u} du = \text{Res}_{z=\infty} \left( \sum_{i \geq 0} a_i z^i \right) \left( \frac{1}{z} \sum_{j \geq 0} z^{-j} \int_{\mathbb{R}} g(u) u^j du \right)
= \sum_{i \geq 0} a_i \int_{\mathbb{R}} g(u) u^i du
= \int_{\mathbb{R}} g(u) \sum_{i \geq 0} a_i u^i du
= \langle f, g \rangle, \tag{5.2.8}
\]
ending the proof of Lemma 5.1.

The next Theorem shows that the determinant of the \textit{time}-dependent moment matrices, satisfies the \textit{KP hierarchy}, a non-linear hierarchy, whereas in the next section, it will be shown that these same determinants satisfy \textit{Virasoro equations}. These two features will play an important role in random matrix theory. Notice that this result is very robust: it can be generalized from orthogonal polynomials to multiple orthogonal polynomials, from the KP hierarchy to multi-component KP hierarchies; see [12].

\textbf{Theorem 5.2} [6, 4] The determinants of the moment matrices, also representable as a multiple integral\footnote{1}$\Delta_n(z) = \prod_{1 \leq i < j \leq n}(z_i - z_j)$, satisfy the KP hierarchy (5.2.9)

\begin{equation}
\tau_n(t) := \det m_n(t) = \frac{1}{n!} \int_{E^n} \Delta_n^2(z) \prod_{k=1}^{n} \rho(z_k) e^{\sum_{i=1}^{\infty} t_i z_i} dz_k
\end{equation}

\begin{equation}
\text{(i) Eigenvectors of } L: \text{ The tridiagonal matrix } L(t) \text{ admits two independent eigenvectors:}
\end{equation}

\begin{equation}
(L(t)p(t; z))_n = zp_n(t; z), \quad n \geq 0
\end{equation}

\begin{equation}
(L(t)q(t; z))_n = zq_n(t; z), \quad n \geq 1.
\end{equation}

- $p_n(t; z)$ are $n$th degree polynomials in $z$, depending on $t \in \mathbb{C}^\infty$, orthonormal with respect to $\rho_t(z)$ (defined in (5.2.1)), and enjoying the following representations: (define $\chi(z) := (1, z, z^2, ...)^\top$)

\begin{equation}
p_n(t; z) := (S(t)\chi(z))_n = z^n h_n^{-1/2} \frac{\tau_n(t - [z^{-1}])}{\tau_n(t)}, \quad h_n := \frac{\tau_n+1(t)}{\tau_n(t)}
\end{equation}

- $q_n(t, z), \quad n \geq 0$, are Stieltjes transforms of the polynomials $p_n(t; z)$ and have the following \tau-function representations:

\begin{equation}
q_n(t; z) := z \int_{\mathbb{R}^n} \frac{p_n(t; u)}{z - u} \rho_t(u) du = (S^{\top-1}(t)\chi(z^{-1}))_n
\end{equation}

\begin{equation}
= z^{-n} h_n^{-1/2} \frac{\tau_{n+1}(t + [z^{-1}])}{\tau_n(t)}.
\end{equation}
(ii) The **standard Toda lattice**, i.e., the symmetric tridiagonal matrix

\[ L(t) := S(t) \Lambda S(t)^{-1} = \begin{pmatrix}
\frac{\partial}{\partial t_1} \log \frac{\tau_1}{\tau_0} & \left( \frac{\tau_0 \tau_2}{\tau_1} \right)^{1/2} & 0 \\
\left( \frac{\tau_0 \tau_2}{\tau_1} \right)^{1/2} & \frac{\partial}{\partial t_1} \log \frac{\tau_1}{\tau_2} & \left( \frac{\tau_1 \tau_3}{\tau_2} \right)^{1/2} \\
0 & \left( \frac{\tau_1 \tau_3}{\tau_2} \right)^{1/2} & \frac{\partial}{\partial t_1} \log \frac{\tau_2}{\tau_3} \\
\end{pmatrix} \ldots \]

satisfies the commuting equation\(^{22}\)

\[ \frac{\partial L}{\partial t_k} = \left[ \frac{1}{2} (L^k)_g, L \right] = - \left[ \frac{1}{2} (L^k)_b, L \right]. \quad (5.2.13) \]

(iii) The functions \( \tau_n(t) \) satisfy the following bilinear identity, for \( n \geq m+1 \), and all \( t, t' \in \mathbb{C}^\infty \), where one integrates along a small circle about \( \infty \),

\[ \oint_{z=\infty} \tau_n(t - [z^{-1}]) \tau_{m+1}(t' + [z^{-1}]) e^{\sum \tau_i (t_i - t_i') z^i} z^{n-m-1} dz = 0. \quad (5.2.14) \]

(iv) The **KP-hierarchy**\(^{23}\) for \( k = 0, 1, 2, \ldots \) and for all \( n = 1, 2 \ldots , \)

\[ \left( s_{k+4} \left( \frac{\partial}{\partial t_1}, \frac{1}{2} \frac{\partial}{\partial t_2}, \frac{1}{3} \frac{\partial}{\partial t_3}, \ldots \right) - \frac{1}{2} \frac{\partial^2}{\partial t_1 \partial t_{k+3}} \right) \tau_n \circ \tau_n = 0, \]

of which the first equation reads:

\[ \left( \left( \frac{\partial}{\partial t_1} \right)^4 + 3 \left( \frac{\partial}{\partial t_2} \right)^2 - 4 \frac{\partial^2}{\partial t_1 \partial t_3} \right) \log \tau_n + 6 \left( \frac{\partial^2}{\partial t_1^2} \log \tau_n \right)^2 = 0. \quad (5.2.15) \]

**Remark:** In order to connect with classical integrable theory, notice that, when \( \tau \) satisfies the equation above, the function

\[ q(t_1, t_2, \ldots) := 2 \frac{\partial^2 \log \tau_n}{\partial t_1^2} \]

\(^{22}\)in terms of the Lie algebra splitting \((5.2.3)\).

\(^{23}\)Remember the Hirota symbol \((5.2.4)\) and the Schur polynomial notation \((5.2.5)\).
satisfies the classical Kadomtsev-Petviashvili equation (KP equation):

\[ 3 \frac{\partial^2 q}{\partial t_2^2} - \frac{\partial}{\partial t_1} \left( 4 \frac{\partial q}{\partial t_3} - \frac{\partial^3 q}{\partial t_1^3} - 6q \frac{\partial q}{\partial t_1} \right) = 0. \] (5.2.16)

If \( q \) happens to be independent of \( t_2 \), then \( q \) satisfies the Korteweg-de Vries equation

\[ 4 \frac{\partial q}{\partial t_3} = \frac{\partial^3 q}{\partial t_1^3} + 6q \frac{\partial q}{\partial t_1}. \] (5.2.17)

\[ \text{Proof: Identity (5.2.9) follows from the general fact that the product of two matrices can be expressed as a symmetric sum of determinants, in particular the square of a Vandermonde can be expressed as a sum of determinants:} \]

\[ \Delta^2(u_1, \ldots, u_n) = \sum_{\sigma \in S_n} \det \left( u_{\sigma(k)}^{\ell+k-2} \right)_{1 \leq k, \ell \leq n}. \]

Indeed,

\[ n! \tau_n(t) = n! m_n(t) \]

\[ = \sum_{\sigma \in S_n} \det \left( \int_E z_{\sigma(k)}^{\ell+k-2} \rho_t(z_{\sigma(k)}) d z_{\sigma(k)} \right)_{1 \leq k, \ell \leq n} \]

\[ = \sum_{\sigma \in S_n} \int_{E^n} \det \left( z_{\sigma(k)}^{\ell+k-2} \right)_{1 \leq k, \ell \leq n} \rho_t(z_{\sigma(k)}) d z_{\sigma(k)} \]

\[ = \int_{E^n} \Delta_n^2(z) \prod_{k=1}^n \rho_t(z_k) d z_k. \]

(i) At first, note

\[ \mu_{ij}(t \mp [z^{-1}]) = \int_{\mathbb{R}} u^{i+j} e^{\sum_{i=1}^n (t_i \mp \frac{1}{z})} u^i \rho(u) du \]

\[ = \int_{\mathbb{R}} u^{i+j} \left( 1 - \frac{u}{z} \right)^{\pm 1} \rho(u) e^{\sum_{i=1}^n t_i u^i} du \]

\[ = \left\{ \begin{array}{l}
\mu_{i,j}(t) - \frac{1}{z} \mu_{i,j+1}(t) \\
\mu_{i,j}(t) + \frac{1}{z} \mu_{i,j+1}(t) + \frac{1}{z^2} \mu_{i,j+2}(t) + \ldots,
\end{array} \right. \]

\[ \sum_{\sigma \in S_n} \det \left( a, \sigma(j) \right)_{1 \leq i,j \leq n} = \det \left( a_{i,k} \right)_{1 \leq i,k \leq n} \]

\[ \sum_{\sigma \in S_n} \det \left( b, \sigma(j) \right)_{1 \leq i,j \leq n} = \det \left( b_{i,k} \right)_{1 \leq i,k \leq n}. \]

\[ \text{Indeed, } \sum_{\sigma \in S_n} \det \left( a_{i,j} \right)_{1 \leq i,j \leq n} = \det \left( a_{i,k} \right)_{1 \leq i,k \leq n} \det \left( b_{i,k} \right)_{1 \leq i,k \leq n} \].
which by formula (5.1.8) of section 5.1 leads at once to the following representation for the monic orthogonal polynomials \( \tilde{p}_n(t; z) \) and their Stieltjes transforms,

\[
\tilde{p}_n(t; z) = z^n \frac{\tau_n(t - [z^{-1}])}{\tau_n(t)},
\]

\[
z \int_{\mathbb{R}} \frac{\tilde{p}_n(t; u) \rho_t(u)}{z - u} du = z^{-n} \frac{\tau_{n+1}(t + [z^{-1}])}{\tau_n(t)}. \tag{5.2.18}
\]

(ii) The matrix \( L := S \Lambda S^{-1} \) satisfies the standard Toda lattice. One computes

\[
\frac{\partial \mu_{ij}}{\partial t_k} = \mu_{i+k,j} \text{ implying } \frac{\partial m_\infty}{\partial t_k} = \Lambda^k m_\infty.
\]

Then, using the factorization (5.2.2) and the definition (5.1.9) of \( L = S \Lambda S^{-1} \) of section 5.1, one computes

\[
0 = S \left( \Lambda^k m_\infty - \frac{\partial m_\infty}{\partial t_k} \right) S^T
\]

\[
= S \Lambda^k S^{-1} - S \frac{\partial}{\partial t_k} (S^{-1} S^T S^{-1}) S^T
\]

\[
= L^k + \frac{\partial S}{\partial t_k} S^{-1} + S^{-1} \frac{\partial S^T}{\partial t_k}.
\]

Upon taking the \((\cdot)\) and \((\cdot)_0\) parts of this equation (\(A_\cdot\) means the lower-triangular part of the matrix \(A\), including the diagonal and \(A_0\) the diagonal part) leads to

\[
(L^k)_- + \frac{\partial S}{\partial t_k} S^{-1} + \left( S^{-1} \frac{\partial S^T}{\partial t_k} \right)_0 = 0 \text{ and } \left( \frac{\partial S}{\partial t_k} S^{-1} \right)_0 = -\frac{1}{2} (L^k)_0.
\]

Upon observing that for any symmetric matrix the following holds,

\[
\begin{pmatrix} a & c \\ c & b \end{pmatrix}_b = \begin{pmatrix} a & 0 \\ 2c & b \end{pmatrix} = 2 \begin{pmatrix} a & c \\ c & b \end{pmatrix}_- - \begin{pmatrix} a & c \\ c & b \end{pmatrix}_0,
\]

it follows that the matrices \( L(t), S(t) \) and the vector \( p(t; z) = (p_n(t; z))_{n \geq 0} = S(t) \chi(z) \) satisfy the (commuting) differential equations and the eigenvalue problem

\[
\frac{\partial S}{\partial t_k} = -\frac{1}{2} (L^k)_b S, \quad L(t)p(t; z) = z p(t; z), \tag{5.2.19}
\]

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and thus the tridiagonal matrix $L$ satisfies the standard Toda lattice equations
\[
\frac{\partial L}{\partial t_k} = \frac{\partial}{\partial t_k} S \Lambda S^{-1} = \frac{\partial S}{\partial t_k} S^{-1} S \Lambda S^{-1} - S \Lambda S^{-1} \frac{\partial S}{\partial t_k} S^{-1} = -\left[ \frac{1}{2} (L^k)^b, L \right],
\]
with $p(t; z)$ satisfying
\[
\frac{\partial p}{\partial t_k} = \frac{\partial S}{\partial t_k} \chi(z) = -\frac{1}{2} (L^k)^b S \chi(z) = -\frac{1}{2} (L^k)^b p.
\]
The two leading terms of $p_n(t; z)$ look as follows, upon using (5.1.6) and (5.2.18):
\[
p_n(t; z) = \sqrt{\frac{\tau_n}{\tau_{n+1}}} \tilde{p}_n(t; z) = \frac{z^n \tau_n(t - [z^{-1}])}{\sqrt{\tau_n \tau_{n+1}}}
= \sqrt{\frac{\tau_n}{\tau_{n+1}}} z^n \left( 1 - z^{-1} \frac{\partial \tau_n}{\partial t_1} \frac{1}{\tau_n} + \ldots \right). \tag{5.2.21}
\]
Thus, $z^n$ admits the following representation in terms of the orthonormal polynomials $p_i$:
\[
z^n = \sqrt{\frac{\tau_n}{\tau_{n+1}}} \left( p_n + \frac{\partial \tau_n}{\partial t_1} \frac{z^{n-1}}{\sqrt{\tau_n \tau_{n+1}}} + O(z^{n-2}) \right)
= \sqrt{\frac{\tau_n}{\tau_{n+1}}} p_n + \frac{\partial \tau_n}{\partial t_1} \frac{z^{n-1}}{\sqrt{\tau_n \tau_{n+1}}} \tau_n + O(z^{n-2}). \tag{5.2.22}
\]
Then, using (5.2.21) in $zp_n$ and then using the representation (5.2.22) for $z^n$ and $z^{n+1}$, one checks that the diagonal entries $b_n$ and non-diagonal entries $a_n$ of $L$ are given by
\[
b_n = \langle zp_n, p_n \rangle = \sqrt{\frac{\tau_n}{\tau_{n+1}}} \left( \langle z^{n+1}, p_n \rangle - \langle z^n, p_n \rangle \frac{\partial \tau_n}{\partial t_1} \frac{1}{\tau_n} \right)
= \frac{\partial \tau_n}{\partial t_1} \frac{1}{\tau_n} - \frac{\partial \tau_n}{\partial t_1} \frac{1}{\tau_n}
= \frac{\partial}{\partial t_1} \log \frac{\tau_{n+1}}{\tau_n}.
\]
and
\[a_n = \langle zp_n, p_{n+1} \rangle = \sqrt{\frac{\tau_n}{\tau_{n+1}}} \langle z^{n+1} + \ldots, p_{n+1} \rangle = \sqrt{\frac{\tau_n \tau_{n+2}}{\tau_{n+1}^2}},\]
establishing (5.2.12).

(iii) The bilinear identity: The functions \(\tau_n(t)\) satisfy the following identity, for \(n \geq m + 1,\ t, t' \in \mathbb{C}^\infty\), where one integrates along a small circle about \(\infty\),
\[\oint_{\infty} \tau_n(t - [z^{-1}]) \tau_{m+1}(t' + [z^{-1}]) e^{\sum_t (t_i - t'_i) z^i} z^{n-m-1} dz = 0. \tag{5.2.23}\]
Indeed, using the \(\tau\)-function representation for the monic orthogonal polynomials and their Stieltjes transform (5.2.18), one checks:
\[\frac{1}{\tau_n(t) \tau_m(t')} \oint_{\infty} \tau_n(t - [z^{-1}]) \tau_{m+1}(t' + [z^{-1}]) e^{\sum_t (t_i - t'_i) z^i} z^{n-m-1} dz\]
\[= \oint_{\infty} \frac{\tau_n(t - [z^{-1}])}{\tau_n(t)} z^{-m} \frac{\tau_{m+1}(t' + [z^{-1}])}{\tau_m(t')} e^{\sum_t (t_i - t'_i) z^i} \frac{dz}{z} \]
\[= \oint_{\infty} dz e^{\sum_t (t_i - t'_i) z^i} \tilde{p}_n(t; z) \tilde{p}_m(t'; z) e^{\sum_t t_i z^i} \rho(z) dz, \text{ using Lemma 5.1} \]
\[= 2\pi i \oint_{\mathbb{R}} e^{\sum_t (t_i - t'_i) z^i} \tilde{p}_n(t; z) \tilde{p}_m(t'; z) e^{\sum_t t_i z^i} \rho(z) dz = 0, \text{ when } m \leq n - 1,\]
by orthogonality, establishing (5.2.23).

(iv) The KP hierarchy: Setting \(n = m + 1\) in (5.2.14), shifting \(t \mapsto t - y, t' \mapsto t + y\), evaluating the residue, Taylor expanding in \(y_k\) (see (5.2.6)) and using the notation
\[\tilde{\partial} = \left( \frac{\partial}{\partial t_1}, \frac{1}{2} \frac{\partial}{\partial t_2}, \frac{1}{3} \frac{\partial}{\partial t_3}, \ldots \right),\]
one computes the following residue about \(z = \infty\),
\[0 = \frac{1}{2\pi i} \oint_{\infty} dz e^{-\sum_t 2y_i z^i} \tau_n(t - y - [z^{-1}]) \tau_n(t + y + [z^{-1}])\]
\[= \frac{1}{2\pi i} \oint_{\mathbb{R}} dz e^{-\sum_t 2y_i z^i} e^{\sum_t \frac{\partial}{\partial y_i}} e^{\sum_t \frac{\partial}{\partial \tau_n(t - u)} \tau_n(t + u)} \bigg|_{u=0}\]
\[= \frac{1}{2\pi i} \oint_{\mathbb{R}} dz e^{-\sum_t 2y_i z^i} e^{\sum_t \frac{\partial}{\partial t} \tau_n(t)} \tau_n(t)\]
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\[
= \frac{1}{2\pi i} \oint dz \left( \sum_{0}^{\infty} z^i s_i(-2y) \right) \left( \sum_{0}^{\infty} z^{-j} s_j(\tilde{\partial}) \right) e^{\sum_{1}^{\infty} y_k \frac{\partial}{\partial k} \tau_n \circ \tau_n} \\
= e^{\sum_{1}^{\infty} y_k \frac{\partial}{\partial k}} \sum_{0}^{\infty} s_i(-2y)s_{i+1}(\tilde{\partial}) \tau_n \circ \tau_n \\
= \left( 1 + \sum_{1}^{\infty} y_j \frac{\partial}{\partial t_j} + O(y^2) \right) \left( \frac{\partial}{\partial t_1} + \sum_{1}^{\infty} s_{i+1}(\tilde{\partial})(-2y_i + O(y^2)) \right) \tau_n \circ \tau_n \\
= \left( \frac{\partial}{\partial t_1} + \sum_{1}^{\infty} y_k \left( \frac{\partial}{\partial t_k} \frac{\partial}{\partial t_1} - 2s_{k+1}(\tilde{\partial}) \right) \right) \tau_n \circ \tau_n + O(y^2),
\]

for arbitrary \( y_k \), implying

\[
\frac{\partial}{\partial t_1} \tau \circ \tau = 0 \quad \text{and} \quad \left( \frac{\partial^2}{\partial t_k \partial t_1} - 2s_{k+1}(\tilde{\partial}) \right) \tau_n \circ \tau_n = 0 \quad \text{for} \quad k = 1, 2, \ldots
\]

Taking into account the fact that trivially \( (\partial/\partial t_1) \tau \circ \tau = 0 \) and that the equation above is trivial for \( k = 1 \) and \( k = 2 \), one is led to the KP hierarchy:

\[
\left( s_{k+4} \left( \frac{\partial}{\partial t_1}, \frac{1}{2} \frac{\partial}{\partial t_2}, \frac{1}{3} \frac{\partial}{\partial t_3}, \ldots \right) - \frac{1}{2} \frac{\partial^2}{\partial t_1 \partial t_{k+3}} \right) \tau_n \circ \tau_n = 0, \quad \text{for} \quad k = 0, 1, 2, \ldots
\]

In particular, for \( k = 0 \), one computes

\[
s_4(t) := \frac{t_1^4}{4!} + \frac{1}{2} t_2 t_1^2 + t_3 t_1 + \frac{1}{2} t_2^2 + t_4,
\]

leading to the first equation in the hierarchy

\[
\left( \left( \frac{\partial}{\partial t_1} \right)^4 + 3 \left( \frac{\partial}{\partial t_2} \right)^2 - 4 \frac{\partial^2}{\partial t_1 \partial t_3} \right) \log \tau_n + 6 \left( \frac{\partial^2}{\partial t_1^2} \log \tau_n \right)^2 = 0. \quad (5.2.24)
\]

This ends the proof of Theorem 5.2. \[ \blacksquare \]

Remark: As mentioned earlier this method is very robust and can be generalized to other integrals, besides \( (5.2.9) \), upon using multiple orthogonal polynomials. Such integrals with appropriate multiple time-deformations lead to \( \tau \)-functions for multi-component KP hierarchies; see Adler-van Moerbeke-Vanhaecke [12].
6 Virasoro constraints

6.1 Virasoro constraints for $\beta$-integrals

Consider weights $\rho(z)dz = e^{-V(z)}dz$ with rational logarithmic derivative and $E$ a disjoint union of intervals:

$$-rac{\rho'}{\rho} = V'(z) = \frac{g}{f} = \sum_{i=0}^{\infty} b_i z^i \quad \text{and} \quad E = \bigcup_{1}^{r} [c_{2i-1}, c_{2i}] \subset F \subseteq \mathbb{R},$$

where $F = [A, B]$ is an interval such that

$$\lim_{z \to A, B} f(z)\rho(z)z^k = 0 \text{ for all } k \geq 0.$$

Consider an integral $I_n(t, c; \beta)$, generalizing (5.2.9), where $t := (t_1, t_2, ...) \text{ and } c = (c_1, c_2, ..., c_{2r});$ namely with a Vandermonde to the power $2\beta > 0$ instead of a square, and omitting the $n!$ appearing in (5.2.9),

$$I_n(t, c; \beta) := \int_{E^n} |\Delta_n(z)|^{2\beta} \prod_{k=1}^{n} \left( e^{\sum_{i=1}^{\infty} t_i z_i} \rho(z_k)dz_k \right) \text{ for } n > 0. \quad (6.1.1)$$

Then the following Theorem holds:

**Theorem 6.1** (Adler-van Moerbeke [8]) The multiple integrals $I_n(t) := I_n(t, c; \beta)$ with $I_0 = 1$, satisfy the following Virasoro constraints for all $k \geq -1$:

$$\left( -\sum_{1}^{2r} c_i^{k+1} f(c_i) \frac{\partial}{\partial c_i} + \sum_{i \geq 0} \left( a_i \mathbb{J}_{k+i,n}^{(2)}(t, n) - b_i \mathbb{J}_{k+i+1, n}^{(1)}(t, n) \right) \right) I_n(t) = 0, \quad (6.1.2)$$

where $\mathbb{J}_{k,n}^{(2)}(t, n)$ and $\mathbb{J}_{k,n}^{(1)}(t, n)$ are combined differential and multiplication (linear) operators. For all $n \in \mathbb{Z}$, the operators $\mathbb{J}_{k,n}^{(2)}(t, n)$ and $\mathbb{J}_{k,n}^{(1)}(t, n)$ form

$\Delta_n(z) = \prod_{1 \leq i < j \leq n} (z_i - z_j)$.

$\Delta_n(z) = \prod_{1 \leq i < j \leq n} (z_i - z_j)$.

$26$ When $E$ equals the whole range $F$, then the the first term, containing the partials with respect to the $c_i$’s, are absent in the formulae (6.1.3).
a Virasoro and a Heisenberg algebra respectively, interacting as follows

\[
\begin{align*}
\llbracket J^{(2)}_{k,n}, J^{(2)}_{\ell,n} \rrbracket &= (k - \ell) J^{(2)}_{k+\ell,n} + c \left( \frac{k^3 - k}{12} \right) \delta_{k,-\ell} \\
\llbracket J^{(2)}_{k,n}, J^{(1)}_{\ell,n} \rrbracket &= -\ell J^{(1)}_{k+\ell,n} + c' k(k + 1) \delta_{k,-\ell} \\
\llbracket J^{(1)}_{k,n}, J^{(1)}_{\ell,n} \rrbracket &= \frac{k}{2\beta} \delta_{k,-\ell},
\end{align*}
\]  

(6.1.3)

with “central charge”

\[
c = 1 - 6 \left( \beta^{1/2} - \beta^{-1/2} \right)^2 \quad \text{and} \quad c' = \frac{1}{2} \left( \frac{1}{\beta} - 1 \right).
\]  

(6.1.4)

**Remark 1:** The operators \( J^{(2)}_{k,n} = J^{(2)}_{k,n}(t,n) \)'s are defined as follows: (the normal ordering symbol “: :” means: always pull differentiation to the right, ignoring commutation rules)

\[
J^{(2)}_{k,n} = \beta \sum_{i+j=k} \llbracket J^{(1)}_{i,n}, J^{(1)}_{j,n} \rrbracket : + (1 - \beta) \left( (k + 1) J^{(1)}_{k,n} - k J^{(0)}_{k,n} \right),
\]  

(6.1.5)

in terms of the \( J^{(1)}_{k,n} = J^{(1)}_{k,n}(t,n) \)'s. Componentwise, we have

\[
J^{(1)}_{k,n}(t,n) = J^{(1)}_{k} + n J^{(0)}_{k} \quad \text{and} \quad J^{(0)}_{k,n} = n J^{(0)}_{k} = n \delta_{0k}
\]

and

\[
J^{(2)}_{k,n}(t,n) = \beta J^{(2)}_{k} + \left( 2n \beta + (k + 1)(1 - \beta) \right) J^{(1)}_{k} + n \left( n \beta + 1 - \beta \right) J^{(0)}_{k},
\]

where

\[
\begin{align*}
J^{(0)}_{k} &= \delta_{k0} \\
J^{(1)}_{k} &= \frac{\partial}{\partial t_k} + \frac{1}{2\beta} (-k)t_{-k} \\
J^{(2)}_{k} &= \sum_{i+j=k} \frac{\partial^2}{\partial t_i \partial t_j} + \frac{1}{\beta} \sum_{-i+j=k} it_i \frac{\partial}{\partial t_j} + \frac{1}{4\beta^2} \sum_{-i-j=k} it_i j t_j.
\end{align*}
\]  

(6.1.6)

The integer \( n \) appears explicitly in \( J^{(2)}_{k,n}(t,n) \) to indicate the explicit \( n \)-dependence of the \( n \)th component, besides \( t \).
Remark 2: In the case $\beta = 1$, the Virasoro generators (6.1.6) take on a particularly elegant form, namely for $n \geq 0$,
\[
J_{k,n}^{(2)}(t) = \sum_{i+j=k} J_{i,n}^{(1)}(t) J_{j,n}^{(1)}(t) = J_k^{(2)}(t) + 2nJ_k^{(1)}(t) + n^2\delta_{0,k}
\]
\[
J_{k,n}^{(1)}(t) = J_k^{(1)}(t) + n\delta_{0,k},
\]
with\(^{27}\)
\[
J_k^{(1)} = \frac{\partial}{\partial t_k} + \frac{1}{2}(-k)t_k, \quad J_k^{(2)} = \sum_{i+j=k} \frac{\partial^2}{\partial t_i \partial t_j} + \sum_{-i+j=k} it_i \frac{\partial}{\partial t_j} + \frac{1}{4} \sum_{-i-j=k} it_i j t_j.
\]

One now establishes the following Lemma:

Lemma 6.2 Setting
\[
dI_n(x) := |\Delta_n(x)|^{2\beta} \prod_{k=1}^{n} \left( e^{\sum_{i=0}^{\infty} t_i x_k^i} \rho(x_k) dx_k \right),
\]
the following variational formula holds:
\[
\left. \frac{d}{d\varepsilon} dI_n(x_i \mapsto x_i + \varepsilon f(x_i) x_i^{k+1}) \right|_{\varepsilon=0} = \sum_{\ell=0}^{\infty} \left( a_{k+\ell} J_{k+\ell,n}^{(2)} - b_{k+\ell} J_{k+\ell+1,n}^{(1)} \right) dI_n.
\]

Proof: Upon setting
\[
\mathcal{E}(x,t) := \prod_{k=1}^{n} e^{\sum_{i=0}^{\infty} t_i x_k^i} \rho(x_k),
\]
the following two relations hold:
\[
\left( \frac{1}{2} \sum_{i+j=k, i,j \geq 0} \frac{\partial^2}{\partial t_i \partial t_j} - \frac{n}{2} \delta_{k,0} \right) \mathcal{E} = \left( \sum_{1 \leq \alpha < \beta \leq n} x_{\alpha} x_{\beta} + \frac{k-1}{2} \sum_{1 \leq \alpha \leq n} x_{\alpha}^k \right) \mathcal{E},
\]
\[
\left( \frac{\partial}{\partial t_k} + n\delta_{k,0} \right) \mathcal{E} = \left( \sum_{1 \leq \alpha \leq n} x_{\alpha}^k \right) \mathcal{E}, \quad \text{all } k \geq 0.
\]

\(^{27}\)The expression $J_k^{(1)} = 0$ for $k = 0$. 

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So, the point now is to compute the $\varepsilon$-derivative
\[
\frac{d}{d\varepsilon} \left( |\Delta_n(x)|^{2\beta} e^{\sum_{k=1}^{n}(-V(x_k)+\sum_{i=1}^{\infty} t_i x_i)^{d}dx_1...dx_n} \right)_{x_i \to x_i + \varepsilon f(x_i)x_i^{k+1}} \bigg|_{\varepsilon=0},
\] (6.1.11)
which consists of three contributions:

**Contribution 1:**
\[
\frac{1}{2\beta |\Delta(x)|^{2\beta}} \frac{\partial}{\partial \varepsilon} \left| \Delta(x + \varepsilon f(x)x^{k+1}) \right|^{2\beta} \bigg|_{\varepsilon=0} = \sum_{1 \leq \alpha < \gamma \leq n} \frac{\partial}{\partial \varepsilon} \log \left( |x_\alpha - x_\gamma + \varepsilon(f(x_\alpha) x^{k+1}_\alpha - f(x_\gamma) x^{k+1}_\gamma)| \right) \bigg|_{\varepsilon=0} = \sum_{1 \leq \alpha < \gamma \leq n} \frac{f(x_\alpha) x^{k+1}_\alpha - f(x_\gamma) x^{k+1}_\gamma}{x_\alpha - x_\gamma} = \sum_{\ell=0}^{\infty} a_\ell \sum_{1 \leq \alpha < \gamma \leq n} \frac{x^{k+\ell+1}_\alpha - x^{k+\ell+1}_\gamma}{x_\alpha - x_\gamma}
\]
\[
= \sum_{\ell=0}^{\infty} a_\ell \left( \sum_{i+j=k+\ell+1, i,j \geq 0} x^i x^j + (n-1) \sum_{1 \leq \alpha \leq n} x^{\ell+k}_\alpha - \frac{n(n-1)}{2} \delta_{\ell+k,0} \right)
\]
\[
= \mathcal{E}^{-1} \sum_{\ell=0}^{\infty} a_\ell \left( \frac{1}{2} \sum_{i+j=k+\ell+1, i,j \geq 0} \frac{\partial^2}{\partial t_i \partial t_j} - \frac{n}{2} \delta_{k+\ell,0} \right) + \left( n - \frac{k+\ell+1}{2} \right) \left( \frac{\partial}{\partial t_{k+\ell}} + n \delta_{k+\ell,0} \right) - \frac{n(n-1)}{2} \delta_{k+\ell,0} \mathcal{E}, \text{ using (6.1.10)},
\]
\[
= \mathcal{E}^{-1} \sum_{\ell=0}^{\infty} a_\ell \left( \frac{1}{2} \sum_{i+j=k+\ell+1, i,j \geq 0} \frac{\partial^2}{\partial t_i \partial t_j} + \left( n - \frac{k+\ell+1}{2} \right) \frac{\partial}{\partial t_{k+\ell}} + \frac{n(n-1)}{2} \delta_{k+\ell,0} \right) \mathcal{E}.
\] (6.1.12)

**Contribution 2:** Using $f(x) = \sum_{0}^{\infty} a_i x^i$,}

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\[
\frac{\partial}{\partial \varepsilon} \prod_{1}^{n} d(x_{\alpha} + \varepsilon f(x_{\alpha})x_{\alpha}^{k+1}) \bigg|_{\varepsilon=0} = \sum_{1}^{n} \left( f'(x_{\alpha})x_{\alpha}^{k+1} + (k + 1)f(x_{\alpha})x_{\alpha}^{k} \right) \prod_{1}^{n} dx_i = \sum_{\ell=0}^{\infty} (\ell + k + 1)a_{\ell} \sum_{\alpha=1}^{n} x_{\alpha}^{k+\ell} \prod_{1}^{n} dx_i = \varepsilon^{-1} \sum_{\ell=0}^{\infty} (\ell + k + 1)a_{\ell} \left( \frac{\partial}{\partial t_{k+\ell}} + n\delta_{k+\ell,0} \right) \mathcal{E} \prod_{1}^{n} dx_i, \quad (6.1.13)
\]

Contribution 3: again using \( f(x) = \sum_{i=0}^{\infty} a_{i}x^{i} \),
\[
\frac{\partial}{\partial \varepsilon} \prod_{\alpha=1}^{n} \exp \left( -V \left( x_{\alpha} + \varepsilon f(x_{\alpha})x_{\alpha}^{k+1} \right) + \sum_{i=1}^{\infty} t_{i} \sum_{\alpha=1}^{n} \left( x_{\alpha} + \varepsilon f(x_{\alpha})x_{\alpha}^{k+1} \right)^{i} \right) \bigg|_{\varepsilon=0} = \left( -\sum_{\alpha=1}^{n} V'(x_{\alpha})f(x_{\alpha})x_{\alpha}^{k+1} + \sum_{i=1}^{\infty} t_{i} \sum_{\alpha=1}^{n} f(x_{\alpha})x_{\alpha}^{i+k} \right) \mathcal{E} = \left( -\sum_{\ell=0}^{\infty} b_{\ell} \sum_{\alpha=1}^{n} x_{\alpha}^{k+\ell+1} + \sum_{\ell \geq 0} a_{\ell} t_{i} \sum_{\alpha=1}^{n} x_{\alpha}^{i+k+\ell} \right) \mathcal{E} = \left( -\sum_{\ell=0}^{\infty} b_{\ell} \left( \frac{\partial}{\partial t_{k+\ell+1}} + n\delta_{k+\ell+1,0} \right) + \sum_{\ell=0}^{\infty} a_{\ell} \sum_{i=1}^{\infty} t_{i} \left( \frac{\partial}{\partial t_{i+k+\ell}} + n\delta_{i+k+\ell,0} \right) \right) \mathcal{E}. \quad (6.1.14)
\]

As mentioned, for knowing (6.1.8), we must add up the three contributions
1, 2 and 3, resulting in:
\[
\frac{\partial}{\partial \varepsilon} dI_{n}(x_{i} \mapsto x_{i} + \varepsilon f(x_{i})x_{i}^{k+1}) \bigg|_{\varepsilon=0} = \left( \sum_{\ell=0}^{\infty} a_{\ell} \left( \beta J_{k+\ell}^{(2)} + (2n\beta + (\ell + k + 1)(1 - \beta))J_{k+\ell}^{(1)} + n((n - 1)\beta + 1)\delta_{k+\ell,0} \right) \right.
\]
\[
- \left. \sum_{\ell=0}^{\infty} b_{\ell} \left( J_{k+\ell+1}^{(1)} + n\delta_{k+\ell+1,0} \right) \right) dI_{n}(x).
\]

Finally, the use of (6.1.6) ends the proof of Lemma 6.2. \[\blacksquare\]
Proof of Theorem 6.1: The change of integration variable \( x_i \mapsto x_i + \varepsilon f(x_i)x_i^{k+1} \) in the integral (6.1.1) leaves the integral invariant, but it induces a change of limits of integration, given by the inverse of the map above; namely the \( c_i \)'s in \( E = \bigcup_1^r [c_{2i-1}, c_{2i}] \), get mapped as follows

\[
c_i \mapsto c_i - \varepsilon f(c_i)c_i^{k+1} + O(\varepsilon^2).
\]

Therefore, setting \( E^\varepsilon = \bigcup_1^r [c_{2i-1} - \varepsilon f(c_{2i-1})c_{2i-1}^{k+1} + O(\varepsilon^2), c_{2i} - \varepsilon f(c_{2i})c_{2i}^{k+1} + O(\varepsilon^2)] \), we find, using Lemma 6.2 and the fundamental theorem of calculus,

\[
0 = \frac{\partial}{\partial \varepsilon} \int_{(E^\varepsilon)^n} |\Delta_{2n}(x + \varepsilon f(x)x^{k+1})| \prod_{i=1}^{2n} e^{-V(x_i + \varepsilon f(x_i)x_i^{k+1},t)} d(x_i + \varepsilon f(x_i)x_i^{k+1})
\]

\[
= - \sum_{i=1}^{2r} c_i^{k+1} f(c_i) \frac{\partial}{\partial c_i} + \sum_{\ell=0}^{\infty} \left( a_\ell J_{k+\ell,2n}^{(2)} - b_\ell J_{k+\ell+1,2n}^{(1)} \right) I_n(t, c, \beta),
\]

ending the proof of Theorem 6.1.

6.2 Examples

These examples are taken from \[1, 6, 7, 89\]; for the Laguerre ensemble, see also \[48\] and for the Jacobi ensemble, see \[49\].

Example 1 (GUE)

Here we pick

\[
\rho(z) = e^{-V(z)} = e^{-z^2}, \quad V' = g/f = 2z, \quad a_0 = 1, b_0 = 0, b_1 = 2, \text{ and all other } a_i, b_i = 0.
\]

Define the differential operators

\[
B_k := \sum_{i=1}^{2r} c_i^{k+1} \frac{\partial}{\partial c_i} \quad (6.2.1)
\]
in terms of the end points of the set \( E = \bigcup_{i=1}^{2r} [c_{2i-1}, c_{2i}] \subset \mathbb{R} \). From Theorem 6.1 the integrals

\[
I_n = \int_{E^n} \Delta_n(z)^2 \prod_{k=1}^{n} e^{-z_k^2 + \sum_{i=1}^{\infty} t_i z_k^i} d\mu_k
\]

satisfy the Virasoro constraints

\[
-B_k I_n = \left( -\tilde{g}^{(2)}_{k,n} + 2\tilde{g}^{(1)}_{k+2,n} \right) I_n, \quad k = -1, 0, 1, \ldots
\]

The first three constraints have the following form, upon setting \( F = \log I_n \) (this will turn out to be more convenient in the applications),

\[
-B_{-1} F = \left( 2 \frac{\partial}{\partial t_1} - \sum_{i \geq 2} i t_i \frac{\partial}{\partial t_{i-1}} \right) F - n t_1
\]

\[
-B_0 F = \left( 2 \frac{\partial}{\partial t_2} - \sum_{i \geq 1} i t_i \frac{\partial}{\partial t_i} \right) F - n^2
\]

\[
-B_1 F = \left( 2 \frac{\partial}{\partial t_3} - 2 n \frac{\partial}{\partial t_1} - \sum_{i \geq 1} i t_i \frac{\partial}{\partial t_{i+1}} \right) F
\]

For later use, take linear combinations such that each expression contains the pure differentiation term \( \partial F/\partial t_i \), yielding

\[
-\frac{1}{2} B_{-1} F =: D_{-1} F = \left( \frac{\partial}{\partial t_1} - \frac{1}{2} \sum_{i \geq 2} i t_i \frac{\partial}{\partial t_{i-1}} \right) F - \frac{n t_1}{2}
\]

\[
-\frac{1}{2} B_0 F =: D_0 F = \left( \frac{\partial}{\partial t_2} - \frac{1}{2} \sum_{i \geq 1} i t_i \frac{\partial}{\partial t_i} \right) F - \frac{n^2}{2}
\]

\[
-\frac{1}{2} (B_1 + n B_{-1}) F =: D_1 F = \left( \frac{\partial}{\partial t_3} - \frac{1}{2} \sum_{i \geq 1} i t_i \frac{\partial}{\partial t_{i+1}} - \frac{n}{2} \sum_{i \geq 2} i t_i \frac{\partial}{\partial t_{i-1}} \right) F - \frac{n^2 t_1}{2}.
\]

(6.2.4)

**Example 2 (Laguerre ensemble)**

Here, the weight is

\[ e^{-V} = z^a e^{-z}, \quad V' = \frac{g}{f} = \frac{z - a}{z}, \]

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Here define the (slightly different) differential operators

$$B_k := \sum_{i=1}^{2r} c_i^{k+2} \frac{\partial}{\partial c_i}.$$  \hspace{1cm} (6.2.5)

Thus from Theorem 6.1, the integrals

$$I_n = \int_{E^n} \Delta_n(z)^2 \prod_{k=1}^{n} z_k^2 e^{-z_k + \sum_{i=1}^{\infty} t_i z_i} \, dz_k$$  \hspace{1cm} (6.2.6)

satisfy the Virasoro constraints, for $k \geq -1$,

$$-B_k I_n = \left( -\mathbb{J}^{(2)}_{k+1,n} - a\mathbb{J}^{(1)}_{k+1,n} + \mathbb{J}^{(1)}_{k+2,n} \right) I_n.$$  \hspace{1cm} (6.2.7)

Written out, the first three have the form, again upon setting $F = \log I_n$,

$$-B_{-1} F = \left( \frac{\partial}{\partial t_1} - \sum_{i \geq 1} it_i \frac{\partial}{\partial t_i} \right) F - n(n + a)$$

$$-B_0 F = \left( \frac{\partial}{\partial t_2} - (2n + a) \frac{\partial}{\partial t_1} - \sum_{i \geq 1} it_i \frac{\partial}{\partial t_{i+1}} \right) F$$

$$-B_1 F = \left( \frac{\partial}{\partial t_3} - (2n + a) \frac{\partial}{\partial t_2} - \sum_{i \geq 1} it_i \frac{\partial}{\partial t_{i+2}} - \frac{\partial^2}{\partial t_1^2} \right) F - \left( \frac{\partial F}{\partial t_1} \right)^2 ;$$

Replacing the operators $B_i$ by linear combinations

$$D_1 = -B_{-1}$$

$$D_2 = -B_0 - (2n + a)B_{-1}$$

$$D_3 = -B_1 - (2n + a)B_0 - (2n + a)^2 B_{-1}$$  \hspace{1cm} (6.2.8)

yields expressions, each containing a pure derivative $\partial F/\partial t_i$.
\[ D_1 F = \frac{\partial F}{\partial t_1} - \sum_{i \geq 1} i t_i \frac{\partial F}{\partial t_i} - n(n + a) \]
\[ D_2 F = \frac{\partial F}{\partial t_2} + \sum_{i \geq 1} i t_i \left( -(2n + a) \frac{\partial}{\partial t_i} - \frac{\partial}{\partial t_{i+1}} \right) F - n(n + a)(2n + a) \]
\[ D_3 F = \frac{\partial F}{\partial t_3} - \sum_{i \geq 1} i t_i \left( (2n + a)^2 \frac{\partial}{\partial t_i} + (2n + a) \frac{\partial}{\partial t_{i+1}} + \frac{\partial}{\partial t_{i+2}} \right) F - \left( \frac{\partial^2 F}{\partial t_1^2} + \left( \frac{\partial F}{\partial t_1} \right)^2 \right) - n(n + a)(2n + a)^2. \]

Notice the non-linearity in this expression is due to the fact that one uses \( F = \log I_n \) rather than \( I_n \).

**Example 3 (Jacobi ensemble)**

The weight is given by
\[ \rho_{ab}(z) := e^{-V} = (1 - z)^a (1 + z)^b, \quad V' = \frac{g}{f} = \frac{a - b + (a + b)z}{1 - z^2} \]
\[ a_0 = 1, a_1 = 0, a_2 = -1, b_0 = a - b, b_1 = a + b, \text{ and all other } a_i, b_i = 0. \]

Here define
\[ B_k := \sum_{1}^{2r} c_i^{k+1} (1 - c_i^2) \frac{\partial}{\partial c_i}. \]

The integrals
\[ \int_{E^n} \Delta_n(z)^2 \prod_{k=1}^{n} (1 - z_k)^a (1 + z_k)^b e^{\sum_{i=1}^{\infty} t_i z_i} d\mu \quad (6.2.10) \]
satisfy the Virasoro constraints \((k \geq -1)\):
\[ - B_k I_n = \left( \mathfrak{g}^{(2)}_{k+2,n} - \mathfrak{g}^{(2)}_{k,n} + b_0 \mathfrak{g}^{(1)}_{k+1,n} + b_1 \mathfrak{g}^{(1)}_{k+2,n} \right) I_n. \quad (6.2.11) \]

Introducing the following notation
\[ \sigma = 2n + b_1 \]

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the first four having the following form, upon setting $F = \log I_n$,

\[- B_{-1} F = \left( \sigma \frac{\partial}{\partial t_1} + \sum_{i \geq 1} it_i \frac{\partial}{\partial t_{i+1}} - \sum_{i \geq 2} it_i \frac{\partial}{\partial t_{i-1}} \right) F + n(b_0 - t_1) \]

\[- B_0 F = \left( \sigma \frac{\partial}{\partial t_2} + b_0 \frac{\partial}{\partial t_1} + \sum_{i \geq 1} it_i \left( \frac{\partial}{\partial t_{i+2}} - \frac{\partial}{\partial t_i} \right) + \frac{\partial^2}{\partial t_1^2} \right) F + \left( \frac{\partial F}{\partial t_1} \right)^2 - \frac{n}{2}(\sigma - b_1) \]

\[- B_1 F = \left( \sigma \frac{\partial}{\partial t_3} + b_0 \frac{\partial}{\partial t_2} - (\sigma - b_1) \frac{\partial}{\partial t_1} + \sum_{i \geq 1} it_i \left( \frac{\partial}{\partial t_{i+3}} - \frac{\partial}{\partial t_{i+1}} \right) \right. \\
\left. \quad + 2 \frac{\partial^2}{\partial t_1 \partial t_2} \right) F + 2 \frac{\partial F}{\partial t_1} \frac{\partial F}{\partial t_2} \]

\[- B_2 F = \left( \sigma \frac{\partial}{\partial t_4} + b_0 \frac{\partial}{\partial t_3} - (\sigma - b_1) \frac{\partial}{\partial t_2} + \sum_{i \geq 1} it_i \left( \frac{\partial}{\partial t_{i+4}} - \frac{\partial}{\partial t_{i+2}} \right) \right. \\
\left. \quad + \frac{\partial^2}{\partial t_2^2} - \frac{\partial^2}{\partial t_1^2} + 2 \frac{\partial^2}{\partial t_1 \partial t_3} \right) F + \left( \frac{\partial F}{\partial t_2} \right)^2 - \left( \frac{\partial F}{\partial t_1} \right)^2 + 2 \frac{\partial F}{\partial t_1} \frac{\partial F}{\partial t_3} \right). \tag{6.2.12} \]

### 7 Random matrices

This whole section 7 is a very standard chapter of random matrix theory. Most of the results can be found e.g., in Mehta [69], Deift [32] and others.

#### 7.1 Haar measure on the space $\mathcal{H}_n$ of Hermitian matrices

Consider the most naïve measure (Haar measure)

\[dM := \prod_{i=1}^{n} dM_{ii} \prod_{1 \leq i < j \leq n} d\text{Re}M_{ij} \ d\text{Im}M_{ij} \] \tag{7.1.1}

on the space of Hermitian matrices

\[\mathcal{H}_n := \{ n \times n \text{ matrices such that } M^T = \bar{M} \}.\]
The parameters $M_{ij}$ in (7.1.1) are precisely the free ones in $M$. This measure turns out to be invariant under conjugation by unitary matrices: (see Mehta [69] and more recently Deift [32])

**Proposition 7.1** For a fixed $U \in SU(n)$, the map

$$\mathcal{H}_n \rightarrow \mathcal{H}_n : M \mapsto M' = U^{-1}MU$$

has the property

$$dM = dM', \quad \text{i.e.} \quad \left| \det \left( \frac{\partial M'}{\partial M} \right) \right| = 1.$$ 

**Proof:** Setting $M' = U^{-1}MU$, we have

$$\text{Tr} M^2 = \text{Tr} M'^2$$

and so

$$\sum_{i,j} M_{ij} M_{ji} = \sum_{i,j} M'_{ij} M'_{ji}.$$ 

Working out this identity, one finds

$$\sum_{i=1}^{n} M_{ii}^2 + 2 \sum_{1 \leq i < j \leq n} \left( (\Re M_{ij})^2 + (\Im M_{ij})^2 \right)$$

$$= \sum_{i=1}^{n} M'_{ii}^2 + 2 \sum_{1 \leq i < j \leq n} \left( (\Re M'_{ij})^2 + (\Im M'_{ij})^2 \right). \quad (7.1.2)$$

Setting

$$\vec{M} := (M_{11}, ..., M_{nn}; \Re M_{12}, ..., \Re M_{n-1,n}, \Im M_{12}, ..., \Im M_{n-1,n}),$$

identity (7.1.2) can be written, in terms of the usual inner product $\langle , \rangle$ in $\mathbb{R}^{n(n+1)/2}$,

$$\langle \vec{M}, D\vec{M} \rangle = \langle \vec{D'M'}, D\vec{M}' \rangle$$
for the $n^2 \times n^2$ diagonal matrix with $n$ 1’s and $n(n - 1)$ 2’s,

$$
D = \begin{pmatrix}
1 & & & \\
& \ddots & & \\
& & 1 & \\
& & & 2 \\
& & O & \\
& & & \ddots \\
O & & & \\
& & & 2
\end{pmatrix}.
$$

Let $V$ be the matrix transforming the vectors $M$ into $M'$

$$
\tilde{M}' = V \tilde{M}
$$

and so, (7.1.2) reads

$$
\langle \tilde{M}', D \tilde{M}' \rangle = \langle V \tilde{M}, DV \tilde{M} \rangle = \langle \tilde{M}, V^\top DV \tilde{M} \rangle,
$$

from which it follows that $D = V^\top DV$ and so

$$
0 \neq \det D = \det(V^\top DV) = (\det V)^2 \det D,
$$

implying

$$
|\det V| = 1.
$$

But for a linear transformation $\tilde{M}' = V \tilde{M}$ the Jacobian of the map is $V$ itself and so

$$
|\det \frac{\partial \tilde{M}'}{\partial M}| = |\det V| = 1,
$$

ending the proof of Proposition 7.1.

**Proposition 7.2** The diagonalization $M = U z U^{-1}$ leads to “polar” or “spectral” coordinates $M \mapsto (U, z)$, where $z = \text{diag}(z_1, \ldots, z_n)$, $z_i \in \mathbb{R}$. In these new coordinates

$$
dM = \Delta^2(z) dz_1 \ldots dz_n \ dU. \tag{7.1.3}
$$

**Proof:** Every matrix $M \in \mathcal{H}_n$ can be diagonalized,

$$
M = U z U^{-1},
$$

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with $U = e^A z e^{-A} \in SU(n)$ and

$$A = \sum_{1 \leq k < \ell \leq n} (a_{k\ell}(e_{k\ell} - e_{\ell k}) + i b_{k\ell}(e_{k\ell} + e_{\ell k})) \in su(n), \text{ with } a_{k\ell}, b_{k\ell} \in \mathbb{R}.$$ 

Then, using the definition of the measure $dM$, as in (7.1.1), and using the fact that $[A, z]$ is a Hermitian, one computes

$$dM\big|_{M=z} = d(e^A z e^{-A})\big|_{A=0} = d(z + [A, z] + O(A^2))\big|_{A=0}$$

$$= \prod_{1}^{n} dz_{i} \prod_{1 \leq j < k \leq n} d \Re[A, z]_{jk} d \Im[A, z]_{jk} \big|_{A=0}$$

$$= \prod_{1}^{n} dz_{i} \prod_{1 \leq j < k \leq n} (z_{j} - z_{k})^{2} \prod_{1 \leq j < k \leq n} da_{jk} db_{jk} \big|_{A=0}.$$ 

Since\(^{28}\)

$$[A, z] = \sum_{1 \leq k, \ell \leq n} (a_{k\ell}[e_{k\ell} - e_{\ell k}, z] + i b_{k\ell}[e_{k\ell} + e_{\ell k}, z])$$

$$= \sum_{1 \leq k, \ell \leq n} (a_{k\ell}(z_{\ell} - z_{k})(e_{k\ell} + e_{\ell k}) + i b_{k\ell}(z_{\ell} - z_{k})(e_{k\ell} - e_{\ell k})),$$

with

$$\Re[A, z]_{k, \ell} = a_{k\ell}(z_{\ell} - z_{k}), \quad \Im[A, z]_{k, \ell} = b_{k\ell}(z_{\ell} - z_{k}),$$

establishing (7.1.3) near $M = z$. By Lemma 7.1, $dM = d(U^{-1}MU')$ for any unitary matrix $U'$, implying the result (7.1.3) holds everywhere, establishing Lemma 7.2.

Remark: The set of Hermitian matrices is the tangent space to a symmetric space $G/K = SL(n, \mathbb{C})/SU(n)$. The argument in Theorem 7.2 can be generalized to many other symmetric spaces, as worked out in van Moerbeke [89] page 324-329.

\(^{28}\) $e_{k\ell}$ denotes the matrix having a 1 at the entry $(k, \ell)$ and 0 everywhere else.
7.2 Random Hermitian ensemble

Consider the probability distribution on the space \( \mathcal{H}_n \) of Hermitian matrices, in terms of Haar measure \( dM \), given by

\[
P(M \in dM) = \frac{1}{Z_n} e^{-\text{Tr}V(M)} dM,
\]

Let \( z_1 \leq z_2 \leq \ldots \leq z_n \) be the real eigenvalues of \( M \). Then

\[
P(z_1 \in dz_1, \ldots, z_n \in dz_n) = P(z_1, \ldots, z_n) dz_1 \ldots dz_n = \frac{1}{Z_n} \Delta^2(z) \prod_{i=1}^n e^{-V(z_i)} dz_i
\]

with

\[
Z_n = \int_{z_1 \leq \ldots \leq z_n} P(z_1, \ldots, z_n) \prod_{i=1}^n dz_i.
\]

Lemma 7.3

\[
P(M \in \mathcal{H}_n, \text{ spectrum } (M) \subseteq E) = \frac{\int_{E^n} \Delta^2(z) \prod_{i=1}^n e^{-V(z_i)} dz_i}{\int_{\mathbb{R}^n} \Delta^2(z) \prod_{i=1}^n e^{-V(z_i)} dz_i}.
\] (7.2.1)

Proof: Indeed

\[
P(M \in \mathcal{H}_n, \text{ spectrum } M \subseteq E)
\]

\[
= \frac{1}{Z_n} \int_{z_1 < \ldots < z_n} \Delta^2_n(z_1, \ldots, z_n) \prod_{i=1}^n \chi_E(z_i) e^{-V(z_i)} dz_i
\]

\[
= \frac{1}{Z_n n!} \sum_{\pi \in S_n} \int_{z_{\pi(1)} < \ldots < z_{\pi(n)}} \Delta^2_n(z_{\pi(1)}, \ldots, z_{\pi(n)}) \prod_{i=1}^n \chi_E(z_{\pi(i)}) e^{-V(z_{\pi(i)})} dz_{\pi(i)}
\]

\[
= \frac{1}{Z_n n!} \sum_{\pi \in S_n} \int_{z_{\pi(1)} < \ldots < z_{\pi(n)}} \Delta^2_n(z_1, \ldots, z_n) \prod_{i=1}^n \chi_E(z_i) e^{-V(z_i)} dz_i
\]
\[
E^n \frac{\Delta_n^2(z) \prod_{i=1}^{n} e^{-V(z_i)} dz_i}{\Delta_n^2(z) \prod_{i=1}^{n} e^{-V(z_i)} dz_i},
\]

showing Lemma 7.3.

Let \( p_0(z), p_1(z), p_2(z), \ldots \) be orthonormal polynomials with regard to the weight \( \rho(z) \) defined on \( \mathbb{R} \), as discussed in section 5.1,

\[
\int_{\mathbb{R}} p_i(z) p_j(z) \rho(z) dz = \delta_{ij},
\]

and \( \tilde{p}_n(z) \) be the monic orthogonal polynomials,

\[
\int_{\mathbb{R}} \tilde{p}_i(z) \tilde{p}_j(z) \rho(z) dz = h_i \delta_{ij}.
\]

Then,

\[
p_n(z) = \frac{1}{\sqrt{h_n}} (z^n + \ldots) = \frac{1}{\sqrt{h_n}} \tilde{p}_n(z).
\]

(7.2.2)

**Proposition 7.4** Setting

\[
Z_n = \int_{\mathbb{R}^n} \Delta_n^2(z) \prod_{k=1}^{n} \rho(z_k) dz_k,
\]

we have the identity

\[
Z_n^{-1} \Delta_n^2(z) \prod_{k=1}^{n} \rho(z_k) = \frac{1}{n!} \det(K_n(z_k, z_\ell))_{1 \leq k, \ell \leq n},
\]

(7.2.3)

where the symmetric kernel \( K_n \) is given by

\[
K_n(y, z) = \sqrt{\rho(y) \rho(z)} \sum_{j=0}^{n-1} p_j(y) p_j(z) \quad \text{(Christoffel-Darboux)}
\]

\[
eq \sqrt{\frac{h_n}{h_{n-1}}} \sqrt{\rho(y) \rho(z)} \frac{p_n(y) p_{n-1}(z) - p_{n-1}(y) p_n(z)}{y - z}.
\]

(7.2.4)

The kernel \( K_n(y, z) \) has the following “reproducing” property

\[
\int_{\mathbb{R}} K_n(y, u) K_n(u, z) du = K_n(y, z) \quad \text{and} \quad \int_{\mathbb{R}} K_n(z, z) dz = n.
\]
Proof: Notice that the Vandermonde $\Delta_n(z)$ can also be expressed as
$\det(\tilde{p}_{i-1}(z_j))_{1 \leq i, j \leq n}$ by row operations, where $\tilde{p}_i(z)$ can be chosen to be any
monic polynomial of degree $i$, and in particular the monic orthogonal poly-
nomial of degree $i$. Thus, one computes for the normalization $Z_n$,

$$Z_n = \int_{\mathbb{R}^n} \Delta^2(z) \prod_{i=1}^{n} \rho(z_i) dz_i$$

$$= \int_{\mathbb{R}^n} \det(\tilde{p}_{i-1}(z_j))_{1 \leq i, j \leq n} \prod_{i=1}^{n} \rho(z_i) dz_i$$

$$= \sum_{\pi, \pi' \in S_n} (-1)^{\pi + \pi'} \prod_{k=1}^{n} \int_{\mathbb{R}} \tilde{p}_{\pi(k)-1}(z_k) \tilde{p}_{\pi'(k)-1}(z_k) \rho(z_k) dz_k$$

$$= \sum_{\pi \in S_n} \prod_{k=1}^{n} \int_{\mathbb{R}} \tilde{p}_{\pi(k)-1}^2(z_k) \rho(z_k) dz_k$$, using the orthogonality of the $\tilde{p}_i$'s,

$$= n! \prod_{k=0}^{n-1} \int_{\mathbb{R}} \tilde{p}_k^2(z) \rho(z) dz = n! \prod_{k=0}^{n-1} h_k.$$

Then using the expression obtained for $Z_n$ and $(\det A)^2 = \det(AA^\top)$ in the
third equality, one further computes

$$Z_n^{-1} \Delta^2_n(z) \prod_{k=1}^{n} \rho(z_k)$$

$$= \frac{1}{n!} \prod_{k=0}^{n-1} h_k \det(\tilde{p}_{i-1}(z_j))_{1 \leq i, j \leq n} \prod_{k=1}^{n} \rho(z_k)$$

$$= \frac{1}{n!} \det \left( \frac{\tilde{p}_{i-1}(z_j)}{\sqrt{h_{i-1}}} \sqrt{\rho(z_j)} \right)_{1 \leq i, j \leq n} \det \left( \frac{\tilde{p}_{k-1}(z_\ell)}{\sqrt{h_{k-1}}} \sqrt{\rho(z_\ell)} \right)_{1 \leq k, \ell \leq n}$$

$$= \frac{1}{n!} \det \left( \sum_{j=1}^{n} \frac{\tilde{p}_{j-1}(z_k) \tilde{p}_{j-1}(z_\ell)}{\sqrt{h_{j-1}}} \sqrt{\rho(z_k)\rho(z_\ell)} \right)_{1 \leq k, \ell \leq n}$$

$$= \frac{1}{n!} \det(K_n(z_k, z_\ell))_{1 \leq k, \ell \leq n}.$$
One finally needs the classical Christoffel-Darboux identity: setting $p_j(y) = 0$ for $j < 0$, one checks

\[(y - z) \sum_{0}^{n-1} p_j(y)p_j(z) = \sum_{j=0}^{n-1} (a_{j,j-1}p_{j-1}(y) + a_{j,j}p_j(y) + a_{j,j+1}p_{j+1}(y)) p_j(z)\]

\[- \sum_{j=0}^{n-1} p_j(y) (a_{j,j-1}p_{j-1}(z) + a_{j,j}p_j(z) + a_{j,j+1}p_{j+1}(z))\]

\[= a_{n-1,n} (p_n(y)p_{n-1}(z) - p_{n-1}(y)p_n(z)),\]

and one uses (7.2.2). The reproducing property follows immediately from the Christoffel-Darboux representation of the kernel in terms of orthogonal polynomials. This proves Proposition 7.4.

### 7.3 Reproducing kernels

**Lemma 7.5** Let $K(x, y)$ be a symmetric kernel satisfying the reproducing property

\[\int_{\mathbb{R}} K(x, y)K(y, z)dy = K(x, z).\]

Then

\[\int \det (K(z_i, z_j))_{1 \leq i, j \leq n} dz \leq \left( \int_{\mathbb{R}} K(z, z)dz - n + 1 \right) \det(K(z_i, z_j))_{1 \leq i, j \leq n-1}, \quad (7.3.1)\]

where $dy$ stands for any measure on $\mathbb{R}$.

**Proof:** The proof of (7.3.1) is due to J. Verbaarschot [90], which proceeds in two steps:

**Step 1:** Let

\[M_n = \begin{pmatrix} M_{n-1} & m \\ \bar{m}^\top & \gamma \end{pmatrix}\]
be a $n \times n$ Hermitian matrix, with $M_{n-1}$ a $n - 1 \times n - 1$ Hermitian matrix. Then $m$ is a $(n - 1) \times 1$ column and $\gamma \in \mathbb{R}$. Then

$$\det M_n = \gamma \det M_{n-1} - \bar{m}^\top \tilde{M}_{n-1} m.$$ 

Indeed given the column $u \in \mathbb{C}^{n-1}$, one checks

$$\begin{pmatrix} I & 0 \\ \bar{u}^\top & 1 \end{pmatrix} \begin{pmatrix} M_{n-1} & m \\ \bar{m}^\top & \gamma \end{pmatrix} \begin{pmatrix} I \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} M_{n-1} & M_{n-1} u + m \\ \bar{u}^\top M_{n-1} + \bar{m}^\top & \bar{u}^\top M_{n-1} u + \bar{m}^\top u + \bar{u}^\top m + \gamma \end{pmatrix}$$

$$= \begin{pmatrix} M_{n-1} & 0 \\ 0 & \gamma - \bar{m}^\top M_{n-1} m \end{pmatrix}, \text{ upon setting } u = -M_{n-1}^{-1} m.$$

The latter assumes that $M_{n-1}$ is invertible. Furthermore, using $M_{n-1}^{-1} = (\det M_{n-1})^{-1} \tilde{M}_{n-1}$,

$$\det M_n = (\gamma - \bar{m}^\top M_{n-1}^{-1} m) \det M_{n-1}$$

$$= (\gamma - \bar{m}^\top \tilde{M}_{n-1} m (\det M_{n-1})^{-1}) \det M_{n-1}$$

$$= \gamma \det M_{n-1} - \bar{m}^\top \tilde{M}_{n-1} m.$$

**Step 2**: Proof of identity (7.3.1). Define the matrix

$$M_k := (K(z_i, z_j))_{1 \leq i, j \leq k} \quad \gamma := K(z_k, z_k)$$

and

$$m = \begin{pmatrix} K(z_1, z_k) \\ \vdots \\ K(z_{k-1}, z_k) \end{pmatrix};$$
$M_k$ is a symmetric matrix, since $K$ is a symmetric kernel. One finds, upon integration over $\mathbb{R}$,

$$
\int_{\mathbb{R}} \det(M_n) dz_n
= \det(M_{n-1}) \int_{\mathbb{R}} K(z,z) dz - \det(M_{n-1}) \int_{\mathbb{R}} \sum_{i,j=1}^{n-1} (\tilde{M}_{n-1})_{ij} K(z, z_n) dz
= \det(M_{n-1}) \int_{\mathbb{R}} K(z,z) dz - \det(M_{n-1}) \sum_{i,j=1}^{n-1} (\tilde{M}_{n-1})_{i,j} (M_{n-1})_{ji}
= \det(M_{n-1}) \left( \int_{\mathbb{R}} K(z,z) dz - (n-1) \right),
$$

establishing Lemma 7.5.

Lemma 7.6 If

\begin{align*}
(i) & \quad \int_{\mathbb{R}} K(x,y) K(y,z) dy = K(x,z) \\
(ii) & \quad \int_{\mathbb{R}} K(z,z) dz = n,
\end{align*}

then

$$
\int \cdots \int_{\mathbb{R}^{n-m}} \det(K(z_i, z_j))_{1 \leq i, j \leq n} dz_{m+1} \cdots dz_n = (n-m)! \det(K(z_i, z_j))_{1 \leq i, j \leq m}
$$

(7.3.2)

Proof: The proof proceeds by induction on $m$. On the one hand, assuming (7.3.2) to be true, integrate with regard to $z_m$ and use identity (7.3.1):

$$
\int \cdots \int_{\mathbb{R}^{n-m}} \det(K(z_i, z_j))_{1 \leq i, j \leq n} dz_m dz_{m+1} \cdots dz_n
$$
On the other hand for $m + 1 = n$, the statement follows at once from Lemma 7.5. This ends the proof of Lemma 7.6.

\[ n - m \int_R dz_m \det(K(z_i, z_j))_{1 \leq i, j \leq m} \]

\[ = (n - m)! \int_R K(z, z) dz - m + 1 \det(K(z_i, z_j))_{1 \leq i, j \leq m - 1} \]

\[ = (n - m)! (n - m + 1) \det(K(z_i, z_j))_{1 \leq i, j \leq m - 1} \]

\[ = (n - m + 1)! \det(K(z_i, z_j))_{1 \leq i, j \leq m - 1}. \]

### 7.4 Correlations and Fredholm determinants

For this section, see M. L. Mehta [69], P. Deift [32], Tracy-Widom [83] and others. Returning now to the probability distribution on the space $\mathcal{H}_n$ of Hermitian matrices (setting $\rho(z) := e^{-V(z)}$)

\[ P(M \in dM) = \frac{1}{Z_n} e^{-\text{Tr} V(M)} dM, \]

remember from Lemma 7.3 and Proposition 7.4,

\[ P(M \in \mathcal{H}_n, \text{spectra} M \subseteq E) = \int_{E^n} P_n(z) dz_1 \ldots dz_n \]

with

\[ P_n(z) = \frac{\Delta^2_n(z) \prod_1^n \rho(z_i) dz_i}{\int_{\mathbb{R}^n} \Delta^2_n(z) \prod_1^n \rho(z_i) dz_i} = \frac{1}{n!} \det(K_n(z_k, z_\ell))_{1 \leq k, \ell \leq n}, \quad (7.4.1) \]

with the kernel $K_n(y, z)$ defined in Proposition 7.4,

\[ K_n(x, y) = \sqrt{\rho(x) \rho(y)} \sum_{j=0}^{n-1} p_j(x)p_j(y). \quad (7.4.2) \]
Let $\mathbb{E}$ be the expectation associated with the probability $P$ above. Then one has the following “classical” Proposition for any subset $E \subset \mathbb{R}$ (for which a precise statement and proof was given by P. Deift [32]):

**Proposition 7.7** The 1- and 2-point correlations have the following meaning:

$$
\int_E K_n(z, z) \, dz = \mathbb{E}(\# \text{ of eigenvalues in } E)
$$

$$
\int_{E \times E} \det(K_n(z_i, z_j))_{1 \leq i, j \leq 2} \, dz_1 \, dz_2 = \mathbb{E}(\# \text{ of pairs of eigenvalues in } E),
$$

and thus

$$
K_n(z, z) = \frac{1}{d_z} \mathbb{E}(\# \text{ of eigenvalues in } d_z) \quad (7.4.3)
$$

**Proof:** Using (7.3.2) for $m = 1$ and (7.2.3), one computes

$$
\int_E K_n(z, z) \, dz
$$

$$
= \int_{\mathbb{R}} \chi_E(z_1) K_n(z_1, z_1) \, dz_1
$$

$$
= \frac{1}{(n-1)!} \int_{\mathbb{R}} dz_1 \chi_E(z_1) \int \cdots \int_{\mathbb{R}^{n-1}} \det(K_n(z_i, z_j))_{1 \leq i, j \leq n} \, dz_2 \cdots dz_n
$$

$$
= n \int_{\mathbb{R}} dz_1 \chi_E(z_1) \int \cdots \int_{\mathbb{R}^{n-1}} \frac{1}{Z_n} \Delta_n^2(z) \prod_{1}^{n} \rho(z_k) \, dz_2 \cdots dz_n
$$

$$
= \frac{n}{Z_n} \int_{\mathbb{R}^n} \chi_E(z_1) \Delta_n^2(z) \prod_{1}^{n} \rho(z_k) \, dz_k
$$

$$
= \frac{1}{Z_n} \int_{\mathbb{R}^n} \left( \sum_{i=1}^{n} \chi_E(z_i) \right) \Delta_n^2(z) \prod_{1}^{n} \rho(z_k) \, dz_k
$$

$$
= \frac{1}{Z_n} \int_{\mathbb{R}^n} \# \{ i \text{ such that } z_i \in E \} \Delta_n^2(z) \prod_{1}^{n} \rho(z_k) \, dz_k
$$

$$
= \mathbb{E}(\# \text{ of eigenvalues in } E).
$$

\[29\] If $x_1, x_2 \in E$, then it counts for 2 in the second formula.
A similar argument holds for the second identity of Proposition 7.7.

Consider disjoint intervals \( E_1, \ldots, E_m \) and integers \( 1 \leq n_1, \ldots, n_m \leq n \) and set \( n_{m+1} := n - \sum_{i=1}^{m} n_i \). Then for the \( n \times n \) Hermitian ensemble with \( P_n \) as in (7.4.1), one has:

\[
P(\text{exactly } n_i \text{ eigenvalues } \in E_i, 1 \leq i \leq m) = \left( \begin{array}{c} n \\ n_1, \ldots, n_m, n_{m+1} \end{array} \right) \int_{\mathbb{R}^n} \prod_{i=1}^{n_1} \chi_{E_1}(x_i) \prod_{i=n_1+1}^{n_1+n_2} \chi_{E_2}(x_i) \ldots \prod_{i=n_1+\ldots+n_{m}+1}^{n_{1+\ldots+n_{m+1}}} \chi_{E_{m}}(x_i) \prod_{i=\sum_{i=1}^{m} n_k+1}^{n} \chi_{(\cup_{i=1}^{m} E_i)^c}(x_i) P_n(x) dx_1 \ldots dx_n.
\]

(7.4.4)

This follows from the symmetry of \( P_n(x) \) under the permutation group; the multinomial coefficient takes into account the number of ways the event occurs.

**Lemma 7.8** The following identity holds

\[
\int_{\mathbb{R}^n} \prod_{k=1}^{n} \left( 1 + \sum_{i=1}^{m} \lambda_i \chi_{E_i}(x_k) \right) P_n(x) dx_1 \ldots dx_n = \det \left[ I + K_n(x, y) \sum_{i=1}^{m} \lambda_i \chi_{E_i}(y) \right].
\]

**Proof:** Upon setting

\[
\tilde{K}(x, y) := K_n(x, y) \sum_{i=1}^{m} \lambda_i \chi_{E_i}(y),
\]

and upon using the fact that \( \tilde{K}(x, y) \) has rank \( n \) in view of its special form (7.4.2), the Fredholm determinant can be computed as follows:

\[
\det(I + \tilde{K}(x, y)) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \int_{\mathbb{R}^\ell} \det \left[ \tilde{K}(x_i, x_j) \right]_{1 \leq i, j \leq \ell} dx_1 \ldots dx_\ell
\]

\[
= \sum_{\ell=0}^{n} \frac{1}{\ell!} \int_{\mathbb{R}^\ell} \det \left[ \tilde{K}(x_i, x_j) \right]_{1 \leq i, j \leq \ell} dx_1 \ldots dx_\ell
\]

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\[ \sum_{\ell=0}^{n} \frac{1}{\ell!} \int_{R^\ell} \det \left[ K_n(x_i, x_j) \sum_{k=1}^{m} \lambda_k \chi_{E_k}(x_j) \right] \, dx_1 \ldots dx_{\ell} \]

\[ = \sum_{\ell=0}^{n} \frac{1}{\ell!} \sum_{1 \leq s_1, \ldots, s_{\ell} \leq m} \lambda_{s_1} \ldots \lambda_{s_{\ell}} \int_{R^\ell} \, dx_1 \ldots dx_{\ell} \prod_{r=1}^{\ell} \chi_{E_{s_r}}(x_r) \det [K_n(x_i, x_j)]_{1 \leq i, j \leq \ell} \]

\[ = \int_{R^n} \sum_{\ell=0}^{n} \left( \begin{array}{c} n \\ \ell \end{array} \right) \sum_{1 \leq s_1, \ldots, s_{\ell} \leq m} \lambda_{s_1} \ldots \lambda_{s_{\ell}} \chi_{E_{s_1}}(x_1) \ldots \chi_{E_{s_{\ell}}}(x_{\ell}) \frac{1}{\ell!(n-\ell)!} \det [K_n(x_i, x_j)]_{1 \leq i, j \leq n} \, dx_1 \ldots dx_n, \text{ using Lemma } 7.6 \]

\[ = \int_{R^n} \sum_{\ell=0}^{n} \sum_{1 \leq s_1, \ldots, s_{\ell} \leq m} \lambda_{s_1} \ldots \lambda_{s_{\ell}} \chi_{E_{s_1}}(x_1) \ldots \chi_{E_{s_{\ell}}}(x_{\ell}) P_n(x) \, dx_1 \ldots dx_n \]

\[ = \int_{R^n} \prod_{k=1}^{m} \left( 1 + \sum_{i=1}^{m} \lambda_i \chi_{E_i}(x_k) \right) P_n(x) \, dx_1 \ldots dx_n, \]

establishing Lemma 7.8.

**Proposition 7.9** The Fredholm determinant is a generating function for the probabilities:

\[ P(\text{exactly } n_i \text{ eigenvalues } \in E_i, 1 \leq i \leq m) = \prod_{i=1}^{m} \frac{1}{n_i!} \left( \frac{\partial}{\partial \lambda_i} \right)^{n_i} \det \left[ I + \sum_{i=1}^{m} \lambda_i K_n(x, y) \chi_{E_i}(y) \right] \bigg|_{\lambda_i = -1}. \quad (7.4.5) \]

In particular

\[ P(\text{no eigenvalues } \in E_i, 1 \leq i \leq m) = \det \left[ I - K_n(x, y) \chi_{\cup_{i \neq \ell} E_i}(y) \right]. \quad (7.4.6) \]

**Proof:** The first equality below follows from Lemma 7.8. Concerning the second equality below, in order to carry out the differentiation \( \prod_{i=1}^{m} \frac{1}{n_i!} \left( \frac{\partial}{\partial \lambda_i} \right)^{n_i} \),
one chooses (keeping in mind the usual product rule of differentiation) a first
group of $n_1$ factors, a (distinct) second group of $n_2$ factors, ... , a $m$th group
of $n_m$ factors and finally the last group of $n - n_1 - \ldots - n_m$ remaining factors
among the product $\prod_{k=1}^{n} \left(1 + \sum_{i=1}^{m} \lambda_i \chi_{E_i}(x_k)\right)$. Then one differentiates the
first $m$ groups, leaving untouched the last group, where one sets $\lambda_i = -1$.
This explains the second equality below. Let $C_{n_1, \ldots, n_m}$ be the set of distinct
committees of size $n_1, n_2, \ldots, n_m, n_{m+1} := n - n_1 - \ldots - n_m$ formed with people
$1, 2, \ldots, n$:

\[
\prod_{i=1}^{m} \frac{1}{n_i!} \left(\frac{\partial}{\partial \lambda_i}\right)^{n_i} \det \left[ I + \sum_{i=1}^{m} \lambda_i K_n(x, y) \chi_{E_i}(y) \right] \bigg| \left. \text{all } \lambda_i = -1 \right|
\]

\[
= \prod_{i=1}^{m} \frac{1}{n_i!} \left(\frac{\partial}{\partial \lambda_i}\right)^{n_i} \int_{\mathbb{R}^n} \prod_{k=1}^{n} \left(1 + \sum_{i=1}^{m} \lambda_i \chi_{E_i}(x_k)\right) P_n(x) dx_1 \ldots dx_n \bigg| \left. \text{all } \lambda_i = -1 \right|
\]

\[
= \sum_{\sigma \in C_{n_1, \ldots, n_m}} \int_{\mathbb{R}^n} \prod_{i=1}^{n_1} \chi_{E_1}(x_{\sigma(i)}) \ldots \prod_{i=n_1+\ldots+n_{m-1}}^{n_1+\ldots+n_m} \chi_{E_m}(x_{\sigma(i)})
\]

\[
\prod_{i=n_1+\ldots+n_m+1}^{n} \left(1 - \sum_{\ell=1}^{m} \chi_{E_\ell}(x_{\sigma(i)})\right) P_n(x) dx_1 \ldots dx_n
\]

\[
= \left( \prod_{i=n_1+\ldots+n_m+1}^{n} \left(1 - \sum_{\ell=1}^{m} \chi_{E_\ell}(x_i)\right) \right) P_n(x) dx_1 \ldots dx_n
\]

as follows from (7.4.4), thus establishing identity (7.4.5), whereas (7.4.6)
follows from setting $n_1 = \ldots = n_m = 0$, completing the proof of Proposition
7.9.
8 The distribution of Hermitian matrix ensembles

8.1 Classical Hermitian matrix ensembles

8.1.1 The Gaussian Hermitian matrix ensemble (GUE)

Let $H_n$ be the Hermitian ensembles

$$H_n = \{ n \times n \text{ matrices } M \text{ satisfying } M^\top = \bar{M} \}.$$ 

The real and imaginary parts of the entries $M_{ij}$ of the $n \times n$ Hermitian matrix ($\bar{M} = M^\top$) are all independent and Gaussian; the variables $M_{ii}$, $1 \leq i \leq n$, $\Re M_{ij}$ and $\Im M_{ij}$, $1 \leq i < j \leq n$, which parametrize the full matrix have the following distribution (set $H_n \ni H = (u_{ij})$, with real $u_{ii}$ and off-diagonal elements $u_{ij} := v_{ij} + iw_{ij}$)

$$P(M_{ii} \in du_{ii}) = \frac{1}{\sqrt{\pi}} e^{-u_{ii}^2} du_{ii} \quad 1 \leq i \leq n$$

$$P(\Re M_{jk} \in dv_{jk}) = \frac{2}{\sqrt{\pi}} e^{-2v_{jk}^2} dv_{jk} \quad 1 \leq j < k \leq n$$

$$P(\Im M_{jk} \in dw_{jk}) = \frac{2}{\sqrt{\pi}} e^{-2w_{jk}^2} dw_{jk} \quad 1 \leq j < k \leq n.$$ 

Hence, using Haar measure (7.1.1),

$$P(M \in dH) = \prod_{1}^{n} P(M_{ii} \in du_{ii}) \prod_{1 \leq j < k \leq n} P(\Re M_{jk} \in dv_{jk})P(\Im M_{jk} \in dw_{jk})$$

$$= c_n \prod_{1}^{n} e^{-u_{ii}^2} \prod_{1 \leq j < k \leq n} e^{-2(v_{jk}^2 + w_{jk}^2)} \prod_{1}^{n} du_{ii} \prod_{1 \leq j < k \leq n} dv_{jk}dw_{jk}$$

$$= c_n e^{-\sum_{1 \leq i,j \leq n} \|u_{ij}\|^2} \prod_{1}^{n} du_{ii} \prod_{1 \leq j < k \leq n} d\Re u_{jk} d\Im u_{jk}$$

$$= c_n e^{-Tr H^2} dH = c_n \Delta_n^2(z) \prod_{1}^{n} e^{-z_i^2} dz_i, \quad H \in H_n, \quad (8.1.1)$$
using the representation of Haar measure in terms of spectral variables $z_i$ (see Proposition 7.2) and where

$$c_n = \left( \frac{2}{\pi} \right)^{n^2/2} \frac{1}{2^{n/2}}.$$ 

This constant can be computed by representing the integral of (8.1.1) over the full range $\mathbb{R}^n$ as a determinant of a moment matrix (as in (5.2.9)) of Gaussian integrals.

8.1.2 Estimating covariances of complex Gaussian populations and the Laguerre Hermitian ensemble

Consider the complex Gaussian population $\vec{x} = (x_1, \ldots, x_p)^\top$, with mean and covariance matrix given by

$$\vec{\mu} = \mathbb{E}(\vec{x}) = (\mu_1, \ldots, \mu_p)^\top, \quad \Sigma = (\mathbb{E}(x_i - \mu_i)(\bar{x}_j - \bar{\mu}_j))_{1 \leq i, j \leq p}$$

and density (for the complex inner-product $\langle \ , \ \rangle$),

$$\frac{1}{(2\pi)^{p/2}(\det \Sigma)^{1/2}} e^{-\frac{1}{4} \langle \vec{x} - \vec{\mu}, \Sigma^{-1}(\vec{x} - \vec{\mu}) \rangle}$$

Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p > 0$ be the eigenvalues of $\Sigma$. Taking $n$ samples of $(x_1, \ldots, x_p)^\top$, consider the normalized $p \times n$ sample matrix:

$$x = \begin{pmatrix} x_{11} - \frac{1}{n} \left( \sum_1^n x_{1i} \right) & x_{12} - \frac{1}{n} \left( \sum_1^n x_{1i} \right) & \cdots & x_{1n} - \frac{1}{n} \left( \sum_1^n x_{1i} \right) \\ x_{21} - \frac{1}{n} \left( \sum_1^n x_{2i} \right) & x_{22} - \frac{1}{n} \left( \sum_1^n x_{2i} \right) & \cdots & x_{2n} - \frac{1}{n} \left( \sum_1^n x_{2i} \right) \\ \vdots & \vdots & \ddots & \vdots \\ x_{p1} - \frac{1}{n} \left( \sum_1^n x_{pi} \right) & x_{p2} - \frac{1}{n} \left( \sum_1^n x_{pi} \right) & \cdots & x_{pn} - \frac{1}{n} \left( \sum_1^n x_{pi} \right) \end{pmatrix}$$

and the $p \times p$ sample covariance matrix,

$$S := \frac{1}{N-1} x \bar{x}^\top,$$

with eigenvalues $z_1, \ldots, z_p > 0$,

which is an unbiased estimator of $\Sigma$. It is a classical result that when $\Sigma = I$, the eigenvalues $z_1, \ldots, z_p > 0$ of $S$ have the Wishart distribution, a special case of the Laguerre Hermitian ensemble (see Hotelling [53] and also Muirhead [70]).

$$P_{n,p}(S \in dM) = c_{np} \Delta_p^2(z) \prod_{i=1}^p e^{-z_i} z_i^{-n-p-1} dz_i dU = e^{-\text{Tr} M (\det M)^{n-p-1}} dM.$$
8.1.3 Estimating the canonical correlations between two Gaussian populations and the Jacobi Hermitian ensemble

In testing the statistical independence of two complex Gaussian populations, one needs to know the distribution of canonical correlation coefficients. I present here the case of real Gaussian populations, not knowing whether the complex case has been worked out, although it should proceed in the same way. To set up the problem, consider $p+q$ normally distributed random variables $(X_1, \ldots, X_p)^\top$ and $(Y_1, \ldots, Y_q)^\top$ ($p \leq q$) with mean zero and covariance matrix

\[
\text{cov} \begin{pmatrix} X \\ Y \end{pmatrix} := \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^\top & \Sigma_{22} \end{pmatrix}.
\]

The method proposed by Hotelling [53] is to find linear transformations $U = L_1X$ and $V = L_2Y$ of $X$ and $Y$ having the property that the correlation between the first components $U_1$ and $V_1$ of the vectors $U$ and $V$ is maximal subject to the condition that $\text{Var} U_1 = \text{Var} V_1 = 1$; moreover, one requires the second components $U_2$ and $V_2$ to have maximal correlation subjected to

\[
\begin{aligned}
& (i) \quad \text{Var} U_2 = \text{Var} V_2 = 1 \\
& (ii) \quad U_2 \text{ and } V_2 \text{ are uncorrelated with both } U_1 \text{ and } V_1,
\end{aligned}
\]

etc. . .

Then there exist orthogonal matrices $O_p \in O(p)$, $O_q \in O(q)$ such that

\[
\Sigma_{11}^{-1/2} \Sigma_{12} \Sigma_{22}^{-1/2} = O_p^\top P O_q,
\]

where $P$ has the following form:
\[
P = \begin{pmatrix}
\rho_1 \\
\vdots \\
\rho_k \\
\rho_{k+1} \\
\vdots \\
\rho_p
\end{pmatrix}
\]

\[
P = \begin{pmatrix}
O \\
O \\
O \\
O \\
O
\end{pmatrix}
\]

\[
p, \quad k = \text{rank } \Sigma_{12},
\]

\[
1 \geq \rho_1 \geq \rho_2 \geq \ldots \geq \rho_k > 0, \quad \rho_{k+1} = \ldots = \rho_p = 0 \quad \text{(canonical correlation coefficients)},
\]

\[
\rho_i \text{ are solutions } (\geq 0) \text{ of } \det(\Sigma_{11}^{-1} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{12}^\top - \rho^2 I) = 0.
\]

Then the covariance matrix of the vectors

\[
U = L_1 X := O_p \Sigma_{11}^{-1/2} X \quad \text{and} \quad V = L_2 Y := O_q \Sigma_{22}^{-1/2} Y
\]

has the canonical form \((\det \Sigma_{\text{can}} = \prod_i (1 - \rho_i^2))\)

\[
\text{cov} \left( \begin{array}{c}
U \\
V
\end{array} \right) = \Sigma_{\text{can}} = \begin{pmatrix}
I_p & P \\
P^\top & I_q
\end{pmatrix},
\]

with

\[
\text{spectrum } \Sigma_{\text{can}} = 1, \ldots, 1, 1 - \rho_1, 1 + \rho_1, \ldots, 1 - \rho_p, 1 + \rho_p
\]

and inverse

\[
\Sigma_{\text{can}}^{-1} = \frac{1}{\prod_i (1 - \rho_i^2)^2} \begin{pmatrix}
I_p & -P \\
-P^\top & I_q
\end{pmatrix}.
\]

From here on, we may take \(\Sigma = \Sigma_{\text{can}}\). The \(n\) \((n \geq p + q)\) independent samples \((x_{11}, \ldots, x_{1p}, y_{11}, \ldots, y_{1q})^\top, \ldots, (x_{n1}, \ldots, x_{np}, y_{n1}, \ldots, y_{nq})^\top\), arising from observing \(\begin{pmatrix} X \\ Y \end{pmatrix}\) lead to a matrix \(\begin{pmatrix} x \\ y \end{pmatrix}\) of size \((p + q, n)\), having the normal distribution \[70\] (p. 79 and p. 539)

\[
(2\pi)^{-n(p+q)/2}(\det \Sigma)^{-n/2} \exp -\frac{1}{2} \text{Tr} \left( x^\top (\Sigma_{11}^{-1} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{12}^\top - \rho^2 I) \right)^{-1} \begin{pmatrix} x \\ y \end{pmatrix}
\]

\[
= (2\pi)^{-n(p+q)/2}(\det \Sigma)^{-n/2} e^{-\frac{1}{2} \text{Tr} \left( x^\top (\Sigma_{11}^{-1})_{11} x + y^\top (\Sigma_{11}^{-1})_{22} y + 2y^\top (\Sigma_{11}^{-1})_{12} x \right)}
\]

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The conditional distribution of $p \times n$ matrix $x$ given the $q \times n$ matrix $y$ is also normal:

$$
(det 2\pi \Omega)^{-n/2} e^{-\frac{1}{2} Tr \Omega^{-1}(x-Py)(x-Py)^T} \tag{8.1.2}
$$

with

$$
\Omega = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} = \text{diag}(1 - \rho_1^2, \ldots, 1 - \rho_p^2)
$$

$$
P = \Sigma_{12} \Sigma_{22}^{-1}.
$$

Then the maximum likelihood estimates $r_i$ of the $\rho_i$ satisfy the determinantal equation

$$
det(S_{11}^{-1} S_{12} S_{22}^{-1} S_{12}^\top - r^2 I) = 0, \tag{8.1.3}
$$

corresponding to

$$
S = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}^\top & S_{22} \end{pmatrix} := \begin{pmatrix} xx^\top & xy^\top \\ yx^\top & yy^\top \end{pmatrix},
$$

where $S_{ij}$ are the associated submatrices of the sample covariance matrix $S$.

**Remark:** The $r_i$ can also be viewed as $r_i = \cos \theta_i$, where the $\theta_1, \ldots, \theta_p$ are the critical angles between two planes in $\mathbb{R}^n$:

(i) a $p$-dimensional plane = span \{$(x_{11}, \ldots, x_{n1}), \ldots, (x_{1p}, \ldots, x_{np})$\}

(ii) a $q$-dimensional plane = span \{$(y_{11}, \ldots, y_{n1})^\top, \ldots, (y_{1q}, \ldots, y_{nq})$\}.

Since the $(q, n)$-matrix $y$ has rank$(y) = q$, there exists a matrix $H_n \in O(n)$ such that $yH_n = (y_1 \mid O)$; therefore acting on $x$ with $H_n$ leads to

$$
yH = (y_1 \mid O) \downarrow q, \quad xH_n = (u \mid v) \uparrow p. \tag{8.1.4}
$$

With this in mind,

$$
S_{12} S_{22}^{-1} S_{12}^\top - r^2 S_{11}
$$

$$
= xy^\top (yy^\top)^{-1} yx^\top - r^2 xx^\top
$$

$$
= xH(yH)^\top (yH(yH)^\top)^{-1} yH(xH)^\top - r^2 (xH)(xH)^\top
$$

$$
= (u \mid v) \begin{pmatrix} y_1^\top \\ O \end{pmatrix} \begin{pmatrix} y_1 \mid O \end{pmatrix}^{-1} \begin{pmatrix} y_1 \mid O \end{pmatrix} \begin{pmatrix} u^\top \\ v^\top \end{pmatrix} - r^2 (u \mid v) \begin{pmatrix} u^\top \\ v^\top \end{pmatrix}
$$

$$
= (u \mid v) \begin{pmatrix} I_q & O \\ O & 0_{n-q} \end{pmatrix} \begin{pmatrix} u^\top \\ v^\top \end{pmatrix} - r^2 (u \mid v) \begin{pmatrix} u^\top \\ v^\top \end{pmatrix}
$$

$$
= uu^\top - r^2 (uu^\top + vv^\top),
$$

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and so the equation (8.1.3) for the $r_i$ can be rewritten

$$\det(uu^\top - r^2(vv^\top + uu^\top)) = 0. \quad (8.1.5)$$

Then setting the forms (8.1.4) of $x$ and $y$ in the conditional distribution (8.1.2) of $x$ given $y$, one computes the following, setting $H := H_n$,

$$\text{Tr} \Omega^{-1}(x - Py)(x - Py)^\top$$

$$= \text{Tr} \Omega^{-1}(xH - PyH)(xH - PyH)^\top$$

$$= \text{Tr} \Omega^{-1}((u \mid v) - P(y_1 \mid O))(u \mid v) - P(y_1 \mid O))^\top$$

$$= \text{Tr} \Omega^{-1}(u - Py_1)(u - Py_1)^\top + \text{Tr} \Omega^{-1}vv^\top; \quad \Omega = \text{diag}(1 - \rho_2^2, \ldots, 1 - \rho_p^2);$$

this establishes the independence of the normal distributions $u$ and $v$, given the matrix $y$, with

$$u \equiv N(Py_1, \Omega), \quad v \equiv N(O, \Omega). \quad P = \text{diag}(\rho_1, \ldots, \rho_p).$$

Hence $uu^\top$ and $vv^\top$ are conditionally independent and both Wishart distributed; to be precise:

- The $p \times p$ matrices $vv^\top$ are Wishart distributed, given $y$, with $n - q$ degrees of freedom and covariance $\Omega$;

- The $p \times p$ matrices $uu^\top$ are non-centrally Wishart distributed, given $y$, with $q$ degrees of freedom, with covariance $\Omega$ and with non-centrality matrix

  $$\frac{1}{2}Py_1y_1^\top P^\top \Omega^{-1}.$$

- The marginal distribution of the $q \times q$ matrices $yy^\top$ are Wishart distributed, with $n$ degrees of freedom and covariance $I_q$, because the marginal distribution of $y$ is normal with covariance $I_q$.

To summarize, given the matrix $y$, the sample canonical correlation coefficients $r_1^2 > \ldots > r_p^2$ are the roots of

\[
(r_1^2 > \ldots > r_p^2) = \quad \text{roots of } \det(xy^\top(yy^\top)^{-1}yx^\top - r^2xx^\top) = 0 \\
= \quad \text{roots of } \det(uu^\top - r^2(uu^\top + vv^\top)) = 0 \\
= \quad \text{roots of } \det(uu^\top(uu^\top + vv^\top)^{-1} - r^2I) = 0.
\]
Then one shows that, knowing \( uu^\top \) and \( vv^\top \) are Wishart and conditionally independent, the conditional distribution of \( r_1^2 > \ldots > r_p^2 \), given the matrix \( y \), is given by

\[
\pi^{p^2/2}c_{n,p,q}e^{-\frac{1}{2}\text{Tr}\, Pyy^\top P^\top \Omega^{-1}} \Delta_p(r^2) \prod_{i=1}^{p} (r_i^2) \frac{1}{2}(q-p-1)(1-r_i^2) \frac{1}{2}(n-q-p-1),
\]

\[
\sum_{\lambda \in \mathcal{Y}} \frac{(n/2)_\lambda C_\lambda(1/2 Pyy^\top P^\top \Omega^{-1})}{(q/2)_\lambda C_\lambda(I_p) |\lambda|!} C_\lambda(R^2),
\]

where

\[
R^2 = \text{diag}(r_1^2, \ldots, r_p^2), \quad c_{n,p,q} = \frac{\Gamma_p(n/2)}{\Gamma_p(q/2)\Gamma_p((n-q)/2)\Gamma_p(p/2)},
\]

and where the \( C_\lambda \) are proportional to Jack polynomials corresponding to the partition \( \lambda \); for details see Muirhead [70] and Adler-van Moerbeke [11]. By taking the expectation with regard to \( y \) or, what is the same, by integrating over the matrix \( yy^\top \), which is Wishart distributed (see section 8.1.2), one obtains:

**Theorem 8.1** Let \( X_1, \ldots, X_p, Y_1, \ldots, Y_q \) \((p \leq q)\) be normally distributed random variables with zero means and covariance matrix \( \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \).

If \( \rho_1^2, \ldots, \rho_p^2 \) are the roots of \( \text{det}(\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^\top - \rho^2 I) = 0 \), then the maximum likelihood estimates \( r_1^2, \ldots, r_p^2 \) from a sample of size \( n \) \((n \geq p + q)\) are given by the roots of

\[
\text{det}(xy^\top(yy^\top)^{-1}y:xx^\top - r^2xx^\top) = 0.
\]

Then, assuming \( \rho_1^2 = \ldots = \rho_p^2 = 0 \), the joint density of the \( z_i = r_i^2 \) is given by the following density:

\[
\pi^{p^2/2}c_{n,p,q}\Delta_p(z) \prod_{i=1}^{p} z_i^{(q-p-1)/2}(1-z_i)^{(n-q-p-1)/2} dz_i.
\]  

(8.1.6)

---

\[^{30}\text{Using the standard notation, defined for a partition } \lambda,

\((a)_\lambda := \prod_i (a + (1 - i))_{\lambda_i}, \text{ with } (x)_n := x(x + 1)\ldots(x + n - 1), \ x_0 = 1.\]
Remark: Taking complex Gaussian populations should introduce in the formula above $\Delta^2_k(z)$ instead of $\Delta_k(z)$ and should remove the 1/2’s in the exponent.

8.2 The probability for the classical Hermitian random ensembles and PDE’s generalizing Painlevé

8.2.1 The Gaussian ensemble (GUE)

This section deals with the Gaussian Hermitian matrix ensemble, discussed in previous section. Given the disjoint union of intervals

$$E := \bigcup_{1}^{r} [c_{2i-1}, c_{2i}] \subseteq \mathbb{R},$$

define the algebra of differential operators

$$B_k = \sum_{i=1}^{2r} c_i^{k+1} \frac{\partial}{\partial c_i}. \quad (8.2.1)$$

The PDE (8.2.2) appearing below was obtained by Adler-Shiot-a-van Moerbeke [1, 5, 7], whereas the ODE (8.2.3) was first obtained by Tracy-Widom [84]. The method used here is different from the one of Tracy-Widom, who use the method proposed by Jimbo-Miwa-Mori-Sato [54]. John Harnad then shows in [51] the relationship between the PDE’s obtained by Tracy-Widom and by Adler-van Moerbeke.

Theorem 8.2 The log of the probability

$$\mathbb{P}_n := \mathbb{P}_n (\text{all } z_i \in E) = \frac{\int_{E^n} \Delta^2_n(z) \prod_{1}^{n} e^{-z_i^2} dz_i}{\int_{\mathbb{R}^n} \Delta^2_n(z) \prod_{1}^{n} e^{-z_i^2} dz_i}$$

satisfies the PDE

$$(B_{-1}^4 + 8nB_{-1}^2 + 12B_0^2 + 24B_0 - 16B_{-1}B_1) \log \mathbb{P}_n + 6(B_{-1}^2 \log \mathbb{P}_n)^2 = 0. \quad (8.2.2)$$

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In particular

\[ f(x) := \frac{d}{dx} \log \mathbb{P}_n(\max_i z_i \leq x) \]

satisfies the 3rd order ODE

\[ f''' + 6f^2 + 4(2n - x^2)f' + 4xf = 0, \quad (8.2.3) \]

which can be transformed into the Painlevé IV equation.

Proof: In Theorem 5.2, it was shown that integral (here we indicate the \( t \)-
and \( E \)-dependence)

\[ \tau_n(t; E) = \frac{1}{n!} \int_{E^n} \Delta_n^2(z) \frac{\prod}{1} e^{-z_i^2 + \sum_{i} t_i z_i} \, dz_i \quad (8.2.4) \]

satisfies the KP equation, regardless of \( E \),

\[ \left( \left( \frac{\partial}{\partial t_1} \right)^4 + 3 \left( \frac{\partial}{\partial t_2} \right)^2 - 4 \frac{\partial^2}{\partial t_1 \partial t_3} \right) \log \tau_n + 6 \left( \frac{\partial}{\partial t_1} \right)^2 \log \tau_n \right)^2 = 0, \quad (8.2.5) \]

and in Theorem 6.1 \( \tau_n(t; E) \) and \( \tau_n(t; \mathbb{R}) \) were shown to satisfy the same Virasoro constraints, and, in particular for the Gaussian case, the three equations (6.2.4), with the boundary term missing in the case \( \tau_n(t; \mathbb{R}) \).

Let \( T_i \) denote the pure \( t \)-differentiation appearing on the right hand side of (6.2.4), with “principal symbol” \( \frac{\partial}{\partial n+2} \):

\begin{align*}
T_{-1} & := \frac{\partial}{\partial t_1} - \frac{1}{2} \sum_{i \geq 2} it_i \frac{\partial}{\partial t_{i-1}} \\
T_0 & := \frac{\partial}{\partial t_2} - \frac{1}{2} \sum_{i \geq 1} it_i \frac{\partial}{\partial t_i} \\
T_1 & := \frac{\partial}{\partial t_3} - \frac{1}{2} \sum_{i \geq 1} it_i \frac{\partial}{\partial t_{i+1}} - \frac{n}{2} \sum_{i \geq 2} it_i \frac{\partial}{\partial t_{i-1}}. \quad (8.2.6)
\end{align*}
Recall the differential operators $D_i$ in terms of the boundary operators (8.2.1), appearing in the Virasoro constraints (6.2.4),

\[
\begin{align*}
D_{-1} &= -\frac{1}{2}B_{-1} \\
D_0 &= -\frac{1}{2}B_0 \\
D_1 &= -\frac{1}{2}(B_1 + \nu B_{-1}).
\end{align*}
\] (8.2.7)

With this notation, the Virasoro constraints (6.2.4) can be summarized as ($F := \log \tau_n(t; E)$)

\[
\begin{align*}
D_{-1}F &= T_{-1}F - \frac{nt_1}{2}, \quad D_0F = T_0F - \frac{n^2}{2}, \quad D_1F = T_1F - \frac{n^2t_1}{2}.
\end{align*}
\]

Expressing the action of $T_i$ on $t_1$, one finds

\[
\begin{align*}
T_{-1}t_1 &= 1 - t_2 \quad & T_{-1}^2t_1 &= T_{-1}(1 - t_2) = \frac{3}{2}t_3 \\
T_1t_1 &= -nt_2 \quad & T_{-1}^3t_1 &= T_{-1}T_{-1}^2t_1 = T_{-1}\left(\frac{3}{2}t_3\right) = -3t_4,
\end{align*}
\]

one computes consecutive powers of $D_i$ and their products, and one notices that $D_i$ involves differentiation with regard to the boundary terms only, implying in particular that $D_i$ and $T_i$ commute. In view of the form of the KP equation, containing only certain partials, and in view of the fact that the
Since one is actually interested in the integral (8.2.4) along the locus \( \{ t_i = 0 \} \), and since readily from (8.2.6) one has \( T_i \big| _{t_i} = \partial / \partial t_{i+2} \), one deduces from the equations above (8.2.8)\(^{31}\) that

\[
D_{-1}F = T_{-1}F - \frac{n t_1}{2}
\]

\[
D_{-1}^2 F = D_{-1} T_{-1} F = T_{-1} D_{-1} F = T_{-1} \left( T_{-1} F - \frac{n t_1}{2} \right) = T_{-1}^2 F - \frac{n}{2} (1 - t_2)
\]

\[
D_{-1}^3 F = D_{-1} T_{-1}^2 F = T_{-1}^2 D_{-1} F = T_{-1} \left( T_{-1} F - \frac{n t_1}{2} \right) = T_{-1}^3 F - \frac{3 n}{4} t_3
\]

\[
D_{-1}^4 F = D_{-1} T_{-1}^3 F = T_{-1}^3 D_{-1} F = T_{-1} \left( T_{-1} F - \frac{n t_1}{2} \right) = T_{-1}^4 F + \frac{3 n}{2} t_4.
\]

\[
D_1 F = T_1 F - \frac{n^2 t_1}{2}
\]

\[
D_{-1} D_1 F = D_{-1} T_1 F = T_1 D_{-1} F = T_1 \left( T_{-1} F - \frac{n^2 t_1}{2} \right) = T_1 T_{-1} F + \frac{n^3}{2} t_2
\]

\[
D_0 F = T_0 F - \frac{n^2}{2}
\]

\[
D_0^2 F = D_0 T_0 F = T_0 D_0 F = T_0 \left( T_0 F - \frac{n^2}{2} \right) = T_0^2 F.
\]  

Since one is actually interested in the integral (8.2.4) along the locus \( \mathcal{L} := \{ t_i = 0 \} \), and since readily from (8.2.6) one has \( T_i \big| _{t_i} = \partial / \partial t_{i+2} \), one deduces from the equations above (8.2.8)\(^{31}\)

\[
D_{-1}^2 F \bigg| _{\mathcal{L}} = T_{-1}^2 F \bigg| _{\mathcal{L}} - \frac{n}{2} = \frac{\partial^2 F}{\partial t_1^2} \bigg| _{\mathcal{L}} - \frac{n}{2}
\]

\[
D_{-1}^4 F \bigg| _{\mathcal{L}} = T_{-1}^4 F \bigg| _{\mathcal{L}} = \frac{\partial}{\partial t_1} T_{-1}^3 F \bigg| _{\mathcal{L}} = T_{-1}^3 \frac{\partial F}{\partial t_1} \bigg| _{\mathcal{L}} = \frac{\partial^4 F}{\partial t_1^4} \bigg| _{\mathcal{L}}
\]

\[
D_0^2 F \bigg| _{\mathcal{L}} = T_0^2 F \bigg| _{\mathcal{L}} = \frac{\partial}{\partial t_2} T_0 F \bigg| _{\mathcal{L}} = \left( \frac{\partial^2}{\partial t_2^2} - \frac{\partial}{\partial t_2} \right) F \bigg| _{\mathcal{L}}
\]

\[
D_{-1} D_1 F \bigg| _{\mathcal{L}} = T_1 T_{-1} F \bigg| _{\mathcal{L}} = \frac{\partial}{\partial t_3} T_{-1} F \bigg| _{\mathcal{L}} = \left( \frac{\partial^2}{\partial t_3^2} - \frac{3}{2} \frac{\partial}{\partial t_2} \right) F \bigg| _{\mathcal{L}}
\]

\[
D_0 F \bigg| _{\mathcal{L}} = T_0 F \bigg| _{\mathcal{L}} - \frac{n^2}{2} = \frac{\partial F}{\partial t_2} \bigg| _{\mathcal{L}} - \frac{n^2}{2}.
\]

\(^{31}\)Notice one also needs \( D_0 F \), because \( \partial F / \partial t_2 \) appears in the expressions \( D_0^2 F \) and \( D_{-1} D_1 F \) below.
By solving the five expressions above linearly in terms of the left hand side, one deduces

\[
\begin{align*}
\frac{\partial^2 F}{\partial t_1^2} \bigg|_L &= D^2_{-1} F \bigg|_L + \frac{n^2}{2}, & \frac{\partial^4 F}{\partial t_1^4} \bigg|_L &= D^4_{-1} F \bigg|_L \\
\frac{\partial F}{\partial t_2} \bigg|_L &= D_0 F \bigg|_L + \frac{n^2}{2}, & \frac{\partial^2 F}{\partial t_2^2} \bigg|_L &= D^2_{0} F \bigg|_L + D_0 F \bigg|_L + \frac{n^2}{2} \\
\frac{\partial^2 F}{\partial t_1 \partial t_3} \bigg|_L &= D_{-1} D_1 F \bigg|_L + \frac{3}{2} D_0 F \bigg|_L + \frac{3n^2}{4}.
\end{align*}
\]

So, substituting into the KP equation and expressing the \( D_i \) in terms of the \( B_i \) as in (8.2.7), one finds

\[
0 = \left( \left( \frac{\partial}{\partial t_1} \right)^4 + 3 \left( \frac{\partial}{\partial t_2} \right)^2 - 4 \frac{\partial^2}{\partial t_1 \partial t_3} \right) F + 6 \left( \left( \frac{\partial}{\partial t_1} \right)^2 F \right)^2 \bigg|_L
\]

\[
= D^4_{-1} + 6nD^2_{-1} F + 3D^2_0 F - 3D_0 F - 4D_{-1} D_1 F + 6(D^2_{-1} F)^2 \bigg|_L
\]

\[
= \frac{1}{16} \left( B^4_{-1} + 8nB^2_{-1} + 12B^2_0 + 24B_0 - 16B_{-1} B_1 \right) F + \frac{3}{8} (B^2_{-1} F)^2 \bigg|_L,
\]

which establishes (8.2.2) for (remember notation (8.2.4))

\[
F = \log \tau_n(0, E) = \log \mathbb{P}_n(\text{all } z_i \in E) + \log \tau_n(0, \mathbb{R}).
\]

Since the \( B_k \) are derivations with regard to the boundary points of the set \( E \) and since \( \log \tau_n(0, \mathbb{R}) \) is independent of those points, the equation (8.2.2) is also valid for \( \log \mathbb{P}_n \); it is an equation of order 4.

When \( E \) is a semi-infinite interval \((-\infty, x)\), then one has \( B_k = x^{k+1} \partial / \partial x \) and then, of course, PDE (8.2.2) turns into an ODE (8.2.3), of an order one less, involving \( f(x) := \frac{d}{dx} \log \mathbb{P}_n(\text{max}_i z_i \leq x) \). For the connection with Painlevé IV, see section 8.3, thus ending the proof of Theorem 8.2. \(\blacksquare\)
8.2.2 The Laguerre ensemble

Given $E \subset \mathbb{R}^+$ and the boundary operators

$$B_k := \sum_{i=1}^{2r} c_i^{k+2} \frac{\partial}{\partial c_i}, \text{ for } k = -1, 0, 1, \ldots,$$

the following statement holds: (see [1, 5, 7] for the PDE obtained below; the ODE was first obtained by Tracy-Widom [84])

**Theorem 8.3** The log of the probability

$$P_n := P_n(\text{all } z_i \in E) = \int_{E^n} \Delta_n^2(z) \prod_{i=1}^{n} z_i^a e^{-z_i} dz_i$$

$$\int_{(\mathbb{R}^+)^n} \Delta_n^2(z) \prod_{i=1}^{n} z_i^a e^{-z_i} dz_i$$

satisfies the PDE

$$\left( B_{-1}^4 - 2B_{-1}^3 + (1 - a^2)B_{-1}^2 - 4B_{-1}B_1 + 3B_0^2 \right) \log P_n$$

$$+ 6(B_{-1}^2 \log P_n)^2 - 4(B_{-1}^2 \log P_n)(B_{-1} \log P_n) = 0.$$

(8.2.9)

In particular, $f(x) := x \frac{d}{dx} \log P_n(\max_i z_i \leq x)$ satisfies

$$x^2 f'' + x f'' + 6x f^2 - 4 f f' - ((a - x)^2 - 4nx) f' - (2n + a - x) f = 0,$$

(8.2.10)

which can be transformed into the Painlevé V equation.

8.2.3 The Jacobi ensemble

The Jacobi ensemble is given by $(1 - z)^a(1 + z)^b$. For $E \subset [-1, +1]$, the boundary differential operators $B_k$ are now defined by

$$B_k = \sum_{i=1}^{2r} c_i^{k+1}(1 - c_i^2) \frac{\partial}{\partial c_i}.$$

Introduce the following parameters

$$r = 4(a^2 + b^2), \quad s = 2(a^2 - b^2), \quad q = 2(2n + a + b)^2.$$

$$r = a^2 + b^2, \quad s = a^2 - b^2, \quad q = (2n + a + b)^2.$$
Theorem 8.4 (Haine-Semengue [48] and Adler-van Moerbeke [7]) The following probability

$$\mathbb{P}_n := \mathbb{P}_n(\text{all } z_i \in E) = \frac{\int_{E^n} \Delta_n(z)^2 \prod_{k=1}^n (1 - z_i)^a (1 + z_i)^b \, dz_i}{\int_{[-1,1]^n} \Delta_n(z)^2 \prod_{k=1}^n (1 - z_i)^a (1 + z_i)^b \, dz_i} \quad (8.2.11)$$

satisfies the PDE:

$$\left( B_{-1}^4 + (q - 2r + 2)B_{-1}^2 + q(3B_0^2 - 2B_0 + 2B_2) + 4B_0B_{-1}^2 - 2(2q - 1)B_1B_{-1} + (2B_{-1}\log \mathbb{P}_n - s)(B_1 - B_{-1} + 2B_0B_{-1}) \right) \log \mathbb{P}_n$$

$$+ 2B_{-1}^2 \log \mathbb{P}_n \left( 2B_0 \log \mathbb{P}_n + 3B_{-1}^2 \log \mathbb{P}_n \right) = 0 \quad (8.2.12)$$

In particular, $f(x) := (1 - x^2) \frac{d}{dx} \log \mathbb{P}_n(\max_i \lambda_i \leq x)$ for $0 < x < 1$ satisfies:

$$(x^2 - 1)^2 f''' + 2(x^2 - 1) \left( x f'' - 3f'^2 \right) + \left( 8xf - q(x^2 - 1) - 2sx - 2r \right) f'$$

$$- f \left( 2f - qx - s \right) = 0, \quad (8.2.13)$$

which is a version of Painlevé VI.

*Proof of Theorems 8.3 and 8.4:* It goes along the same lines as Theorem 8.2 for GUE, namely using the Virasoro constraints (6.2.8) and (6.2.9), together with the KP equation (8.2.5). This then leads to the PDE’s (8.2.9) and (8.2.12). The ODE’s (8.2.10) and (8.2.13) are found by simple computation. For connections with the Painlevé equations see section 8.3. \(\blacksquare\)

### 8.3 Chazy and Painlevé equations

Each of these three equations (8.2.3), (8.2.10), (8.2.13) is of the Chazy form

$$f''' + \frac{P'}{P} f'' + \frac{6}{P} f'^2 - \frac{4P'}{P^2} f f' + \frac{P''}{P^2} f'^2 + \frac{4Q'}{P^2} f' - \frac{2Q'}{P^2} f + \frac{2R}{P^2} = 0, \quad (8.3.1)$$

with $P, Q, R$ having the form:

- **Gauss** $P(x) = 1$  \quad $4Q(x) = -4x^2 + 8n$  \quad $R = 0$
- **Laguerre** $P(x) = x$  \quad $4Q(x) = -(x - a)^2 + 4nx$  \quad $R = 0$
- **Jacobi** $P(x) = 1 - x^2$  \quad $4Q(x) = -(q(x^2 - 1) + 2sx + 2r)$  \quad $R = 0$
The differential equation (8.3.1) belongs to the general Chazy class
\[ f''' = F(z, f, f', f'') \]
where \( F \) is rational in \( f, f', f'' \) and locally analytic in \( z \), subjected to the requirement that the general solution be free of movable branch points; the latter is a branch point whose location depends on the integration constants. In his classification Chazy found thirteen cases, the first of which is given by equation (8.3.1), with arbitrary polynomials \( P(z), Q(z), R(z) \) of degree 3, 2, 1 respectively. Cosgrove and Scoufis [30, 29], show that this third order equation has a first integral, which is second order in \( f \) and quadratic in \( f'' \),

\[
f''' + \frac{4}{P^2} \left( (Pf'^2 + Qf' + R)f' - (P'f'^2 + Q'f' + R')f \right) \]
\[
+ \frac{1}{2} \left( P''f' + Q'' \right) f^2 - \frac{1}{6} P'''f^3 + c \right) = 0; \quad (8.3.2)
\]
c is the integration constant. Equations of the general form
\[ f''' = G(x, f, f') \]
are invariant under the map
\[ x \mapsto a_1 z + a_2 \quad \text{and} \quad f \mapsto \frac{a_5 f + a_6 z + a_7}{a_3 z + a_4}. \]

Using this map, the polynomial \( P(z) \) can be normalized to
\[ P(z) = z(z - 1), \quad z, \text{ or } 1. \]

Equation (8.3.2) is a master Painlevé equation, containing the 6 Painlevé equations, replacing \( f(z) \) by some new variable \( g(z) \). e.g.,

- \( g''^2 = -4g'^3 - 2g'(zg' - g) + A_1 \) (Painlevé II)
- \( g''^2 = -4g'^3 + 4(zg' - g)^2 + A_1g' + A_2 \) (Painlevé IV)
- \( (zg'')^2 = (zg' - g) \left( -4g'^2 + A_1(zg' - g) + A_2 \right) + A_3g' + A_4 \) (Painlevé V)
• \((z(z-1)g'')^2 = (zg' - g)
\left(4g'^2 - 4g'g' + A_2\right) + A_1g'^2 + A_3g' + A_4\)

(Painlevé VI)

Now, each of these Painlevé II, IV, V, VI equations can be transformed into
the standard Painlevé equations, which are all differential equations of the form
\[ f'' = F(z, f, f'), \text{rational in } f, f', \text{analytic in } z, \]
whose general solution has no movable critical points. Painlevé showed that
this requirement leads to 50 types of equations, six of which cannot be re-
duced to known equations.

9 Large Hermitian matrix ensembles

9.1 Equilibrium measure for GUE and Wigner’s semi-
circle

Remember according to (7.4.3), the average density of eigenvalues is given by
\[ K_n(z, z)dz. \]
Pastur and Marcenko \[76\] have proposed a method to compute
the average density of eigenvalues (equilibrium distribution), when \(n\) gets
very large. For a rigorous and very general approach, see Johansson \[55\],
who also studies the fluctuations of the linear statistics of the eigenvalues
about the equilibrium distribution.

Consider the case of a random Hermitian ensemble with probab-
ility defined by
\[ \frac{1}{Z_n} \int_{H_n(E)} dMe^{-\frac{1}{2n} \text{Tr}(M-A)^2} dM, \]
for a diagonal matrix \(A = (a_1, \ldots, a_n)\). Consider then the spectral function
of \(A\), namely \(d\sigma(\lambda) := \frac{1}{n} \sum \delta(\lambda - a_i)\). The Pastur-Marcenko method tells
us that the Stieltjes transform of the equilibrium measure of \(d\nu(\lambda)\), when
\(n \to \infty\), namely
\[ f(z) = \int_{-\infty}^{\infty} \frac{d\nu(\lambda)}{\lambda - z}, \quad \Im z \neq 0, \]
satisfies the integral equation
\[ f(z) = \int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{\lambda - z - v^2 f(z)}. \]
The density of the equilibrium distribution is then given by

\[ \frac{d\nu(z)}{dz} = \frac{1}{\pi} \Im m \, f(z). \]

When \( A = 0 \), the integral equation (9.1.1) becomes

\[ f(z) = \frac{1}{-z - v^2 f(z)} \]

with solution \( f(z) = \frac{1}{2\pi v^2} (-z \pm \sqrt{z^2 - 4v^2}) \), and thus one finds the classical semi-circle law,

\[ \frac{d\nu(z)}{dz} = \frac{1}{\pi} \Im m f(z) = \begin{cases} \frac{1}{2\pi v^2} \sqrt{4v^2 - z^2} & \text{for } -2v \leq z \leq 2v \\ 0 & \text{for } |z| \geq 2v, \end{cases} \]

concentrated on the interval \([-2v, 2v]\).

As an exercise, consider now the case, where \( v = 1 \) and where the diagonal matrix \( A \) has two distinct eigenvalues, namely

\[ A = \text{diag}(\alpha, \ldots, \alpha, \beta, \ldots, \beta). \]

See e.g., Adler-van Moerbeke [13]. The integral equation (9.1.1) becomes

\[ f - \frac{1 - p}{\beta - z - \bar{f}} - \frac{p}{\alpha - z - \bar{f}} = 0, \quad \text{for } 0 < p < 1, \]

which, upon clearing, leads to a cubic equation for \( g := f + z \),

\[ g^3 - (z + \alpha + \beta) g^2 + (z(\alpha + \beta) + \alpha\beta + 1) g - \alpha\beta z - (1 - p)\alpha - p\beta = 0, \]

having, as one checks, a quartic discriminant \( D_1(z) \) in \( z \). Since the roots of a cubic polynomial involve, in particular, the square root of the discriminant, the solution \( g(z) \) of the cubic will have a non-zero imaginary part, if and only if \( D_1(z) < 0 \). Thus one finds the following equilibrium density,

\[ \frac{d\nu(z)}{dz} = \frac{1}{\pi} \Im m f(z) = \begin{cases} \frac{1}{\pi} \Im m g(z) & \text{for } z \text{ such that } D_1(z) < 0 \\ 0 & \text{for } z \text{ such that } D_1(z) \geq 0 \end{cases} \]
Therefore the boundary of the support of the equilibrium measure will be
given by the real roots of $D_1(z) = 0$. Depending on the values of the
parameters $\alpha$, $\beta$ and $p$, there will be four real roots or two real roots, with a
critical situation where there are three real roots, i.e., when two of the four
real ones collide. The critical situation occurs exactly when the discriminant
$D_2(\alpha, \beta, p)$ (with regard to $z$) of $D_1(z)$ vanishes, namely when

$$D_2(\alpha, \beta, p) = 4p(1 - p)\gamma(\gamma^3 - 3\gamma^2 + 3\gamma(9p^2 - 9p + 1) - 1)^3 \bigg|_{\gamma = (\alpha - \beta)^2} = 0.$$ 

This polynomial has a positive root $\gamma$, which can be given explicitly, the
others being imaginary, and one checks that, when one has the relationship

$$\alpha - \beta = \frac{q + 1}{\sqrt{q^2 - q + 1}},$$

upon using the parametrization $p = \frac{1}{q^3 + 1}$, two of the four roots of $D_1(z)$ collide. This is to say, this is the precise
point at which the support of the equilibrium measure goes from two to one
interval. This then occurs exactly at value

$$z = \beta + \frac{2q - 1}{\sqrt{q^2 - q + 1}}$$
on the real line.

### 9.2 Soft edge scaling limit for GUE and the Tracy-Widom distribution

Consider the probability measure on eigenvalues $z_i$ of the $n \times n$ Gaussian
Hermitian ensemble (GUE)

$$\mathbb{P}_n(\text{all } z_i \in \tilde{E}) = \frac{\int_{\tilde{E}^n} \Delta_n^2(z) \prod_{1}^{n} e^{-z_i^2} dz_i}{\int_{\mathbb{R}^n} \Delta_n^2(z) \prod_{1}^{n} e^{-z_i^2} dz_i}.$$ 

Given the disjoint union $E := \bigcup [x_{2i-1}, x_{2i}] \subset \mathbb{R}$, define the gradient and the Euler operator with respect to the boundary points of the set $E$:

$$\nabla_x = \sum_{1}^{2r} \frac{\partial}{\partial x_i} \quad \text{and} \quad \mathcal{E}_x = \sum_{1}^{2r} x_i \frac{\partial}{\partial x_i}. \quad (9.2.1)$$
Remember the definition of the Fredholm determinant of a continuous kernel $K(x, y)$, the continuous analogue of the discrete kernel (2.0.3),

$$\det(I - K(x, y) \chi_E(y)) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_E dz_1 \ldots dz_n \det(K(z_i, z_j))_{1 \leq i, j \leq n}.$$

We now state: [84, 1, 5]

**Theorem 9.1** The gradient

$$f(x_1, \ldots, x_{2r}) := \nabla_x \log \mathbb{P}(E^c),$$

with

$$\mathbb{P}(E^c) := \lim_{n \to \infty} \mathbb{P}_n \left( \text{all } \sqrt{2n} \frac{1}{\sqrt{b}} \left( z_i - \sqrt{2n} \right) \in E^c \right),$$

satisfies the 3rd order non-linear PDE:

$$(\nabla^3_x - 4(E_x - \frac{1}{2})) f + 6(\nabla_x f)^2 = 0. \quad (9.2.2)$$

In particular, for $E = (x, \infty)$,

$$\mathcal{F}(x) := \lim_{n \to \infty} \mathbb{P}_n \left( \sqrt{2n} \frac{1}{\sqrt{b}} \left( z_{\max} - \sqrt{2n} \right) \leq x \right)$$

$$= \det(I - A \chi_{(x, \infty)})$$

$$= \exp \left(-\int_x^\infty (\alpha - x) g^2(\alpha) d\alpha \right), \quad (9.2.3)$$

is the Tracy-Widom distribution, with

$$A(x, y) := \frac{A(x)A'(y) - A'(x)A(y)}{x - y} = \int_y^\infty A(u + x) A(u + y) du$$

and $g(\alpha)$ the Hastings-McLeod (unique) solution of

$$\begin{cases} g'' = \alpha g + 2g^3 \\ g(\alpha) \sim e^{-\frac{2}{3} g^{3/2}} \text{ for } \alpha \nearrow \infty. \end{cases} \quad \text{(Painlevé II).}$$

\[32\]Remember the Airy function:

$$A(x) = \frac{1}{\pi} \int_0^\infty \cos \left( \frac{u^3}{3} + xu \right) du,$$

satisfying the ODE $A''(x) = xA(x)$.  

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Proof: Step 1: Applying Proposition 7.9, it follows that
\[ P_n(\text{all } z_i \in E^c) = \det (I - K_n(y, z)\chi_E(z)), \] (9.2.4)
where the kernel \( K_n(y, z) \) is given by Proposition 7.4,
\[ K_n(y, z) = \left( \frac{n}{2} \right)^{1/2} e^{-\frac{1}{2}(y^2 + z^2)} \frac{p_n(y)p_{n-1}(z) - p_{n-1}(y)p_n(z)}{y - z}. \] (9.2.5)
The \( p_n \)'s are orthonormal polynomials with respect to \( e^{-z^2} \) and thus proportional to the classical Hermite polynomials:
\[ p_n(y) := \frac{1}{2^{n/2}\sqrt{n!\pi^{1/4}}} H_n(y) = \frac{1}{\sqrt{h_n}} y^n + \ldots, \] (9.2.6)
with
\[ H_n(y) := e^{y^2} \left( -\frac{d}{dy} \right)^n e^{-y^2}, \quad h_n = \frac{\sqrt{n!}}{2^n}. \]

Step 2: The Plancherel-Rotach asymptotic formula (see Szegő [82]) says that
\[ e^{-x^2/2} \frac{n^{1/12}H_n(x)}{2^{n/2+1/4}\sqrt{n!\pi^{1/4}}} \bigg|_{x=\sqrt{2n+1}+\frac{t}{\sqrt{2(n-1)^{1/6}}}}^x = A(t) + O(n^{-2/3}), \]
uniformly for \( t \in \text{compact } K \subset \mathbb{C} \) and thus, in view of (9.2.6),
\[ e^{-\frac{x^2}{2}} p_n(x) \bigg|_{x=\sqrt{2n+1}+\frac{t}{\sqrt{2(n-1)^{1/6}}}} = 2^{1/4}n^{-\frac{1}{12}} \left( A(t) + O(n^{-\frac{2}{3}}) \right). \] (9.2.7)
Since the Hermite kernel (9.2.5) also involves \( p_{n-1}(x) \), one needs an estimate like (9.2.7), with the same scaling but for \( p_n \) replaced by \( p_{n-1} \). So, one needs the following:
\[
x = \sqrt{2n+1} + \frac{t}{\sqrt{2n^{1/6}}}
= \sqrt{(2n-1)(1 + \frac{2}{2n-1})} + \frac{t}{\sqrt{2(n-1)^{1/6}(1 + \frac{1}{n-1})^{1/6}}}
= \sqrt{2n-1} \left( 1 + \frac{1}{2n-1} + O\left( \frac{1}{n^2} \right) \right) + \frac{t}{\sqrt{2(n-1)^{1/6}}} \left( 1 + O\left( \frac{1}{n} \right) \right)
= \sqrt{2n-1} + \frac{t + \frac{1}{n^{1/6}}}{\sqrt{2(n-1)^{1/6}}} + O\left( \frac{1}{n^{7/6}} \right).
\]
Hence, from (9.2.7) it follows that

\[ e^{-\frac{x^2}{2}} p_{n-1}(x) \bigg|_{x=\sqrt{2n+1}+\frac{t}{\sqrt{2n^{1/6}}}+...} = e^{-\frac{x^2}{2}} p_{n-1}(x) \bigg|_{x=\sqrt{2n-1}+\frac{t}{\sqrt{2(n-1)^{1/6}}}+...} = 2^{1/4} n^{-1/3} \left( A(t + n^{-1/3}) + O(n^{-2/3}) \right). \]  

(9.2.8)

From the definition of the Fredholm determinant, one needs to find the limit of \( K_n(y;z)dz \). Therefore, in view of (9.2.5) and using the estimates (9.2.7) and (9.2.8),

\[ \lim_{n \to \infty} K_n(y;z)dz \bigg|_{y=(2n+1)^{1/2}+\frac{t}{\sqrt{n^{1/6}}}}^{y=(2n+1)^{1/2}+\frac{s}{\sqrt{n^{1/6}}}+...} = -\lim_{n \to \infty} \left( \frac{n}{2} \right)^{1/2} e^{-\frac{1}{2}(y^2+z^2)} \frac{p_n(y)(p_n(z) - p_{n-1}(z)) - p_n(z)(p_n(y) - p_{n-1}(y))}{(y-z)\sqrt{2} n^{1/6}} ds \bigg|_{z=(2n+1)^{1/2}+\frac{t}{\sqrt{n^{1/6}}}+...}^{y=(2n+1)^{1/2}+\frac{s}{\sqrt{n^{1/6}}}+...} \]

\[ = \lim_{n \to \infty} \left( \frac{n}{2} \right)^{1/2} (2^{1/4} n^{-1/3})^2 n^{-1/3} \frac{A(t)(A(s + n^{-1/3}) - A(s)) - A(s)(A(t + n^{-1/3}) - A(t))}{n^{-1/3}(t-s)} ds \]

\[ = \frac{A(t)A'(s) - A(s)A'(t)}{t-s} ds = A(t,s) ds \]

and thus for \( E := \bigcup_{1}^{2r}[x_{2i-1}, x_{2i}] \) and \( \tilde{E} := \bigcup_{1}^{2r}[c_{2i-1}, c_{2i}] \), related by

\[ \tilde{E} = \sqrt{2n} + \frac{E}{\sqrt{2n^{1/6}}}, \]  

(9.2.9)

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one has shown (upon taking the limit term by term in the sum defining the Fredholm determinant)

\[
\lim_{n \to \infty} \mathbb{P}_n(\text{all } z_i \in \tilde{E}^c) = \lim_{n \to \infty} \mathbb{P}_n(\text{all } z_i \in \sqrt{2n} + \frac{E^c}{\sqrt{2n}^{1/6}})
\]

\[
= \lim_{n \to \infty} \mathbb{P}_n(\text{all } z_i \in (2n + 1)^{1/2} + \frac{E^c}{\sqrt{2n}^{1/6}})
\]

\[
= \lim_{n \to \infty} \det(\mathcal{I} - K_n \chi_{\tilde{E}})_{\tilde{E} = (2n + 1)^{1/2} + \frac{E}{\sqrt{2n}^{1/6}}}
\]

\[
= \det(I - A \chi_{E}).
\]

**Step 3:** From Theorem 8.2, \(\mathbb{P}_n(\text{all } z_i \in \tilde{E}^c)\) satisfies, with regard to the boundary points \(c_i\) of \(\tilde{E}\), the PDE (8.2.2); thus setting that scaling into this PDE yields (remember \(\nabla_c\) and \(E_x\) are as in (9.2.1) and \(B_k = \sum c_i^{k+1} \frac{\partial}{\partial c_i}\))

\[
0 = (B_{-1}^4 + 8nB_{-2}^2 + 24B_0 - 16B_{-1}B_1) \log \mathbb{P}_n
\]

\[
+ 6(B_{-1}^2 \log \mathbb{P}_n)^2|_{c_i = \sqrt{2n + \sqrt{2n}^{1/6}}}
\]

\[
= 4n^{2/3} \left[ \left( \nabla_x^3 - 4(E_x - \frac{1}{2}) \right) \nabla_x \log \mathbb{P} + 6(\nabla_x^2 \log \mathbb{P})^2 \right] + o(n^{2/3}).
\]

Note that in this computation, the terms of order \(n^{4/3}\) cancel, because the leading term in

\[
12B_0^2 - 16B_{-1}B_1 = -4 \sum i^2 c_i^2 \left( \frac{\partial}{\partial c_i} \right)^2 + \ldots = -16n^{4/3} \nabla_x^2 + \ldots
\]

cancels versus the leading term in \(8nB_{-2}^2 = 16n^{4/3} \nabla_x^2 + \ldots\); thus only the terms of order \(n^{2/3}\) remain. Since in step 2 it was shown that the limit exists, the term in brackets vanishes, showing that \(\log \mathbb{P}(E^c)\) satisfies the PDE (9.2.2).

**Step 4:** In particular, upon picking \(E = (x, \infty)\), the PDE (9.2.2) for

\[
f(x) = \frac{\partial}{\partial x} \log \mathcal{F}(x) = \frac{\partial}{\partial x} \log \lim_{n \to \infty} \mathbb{P}_n \left( \sqrt{2n} \left( z_{\text{max}} - \sqrt{2n} \right) \leq x \right)
\]

becomes an ODE:

\[
f''' - 4xf' + 2f + 6f^2 = 0.
\]
Multiplying this equation with \( f'' \) and integrating from \(-\infty \) to \( x \) lead to the differential equation (the nature of the solution shows the integration constant must vanish)

\[
f''^2 + 4f'(f^2 - xf' + f) = 0. \tag{9.2.10}
\]

Then, setting

\[
\begin{align*}
f' &= -g^2 \\
f  &= g^2 - xg^2 - g^4
\end{align*} \tag{9.2.11}
\]

and, since then \( f'' = -2gg' \), an elementary computation shows that (9.2.11) is an obvious solution to equation (9.2.10). For (9.2.11) to be valid, the derivative of the right hand side of the second expression in (9.2.11) must equal the derivative of the right hand side of the first expression in (9.2.11), i.e. we must have:

\[
0 = (f)' - f' = (g^2 - xg^2 - g^4)' + g^2 = 2g'(g'' - 2g^3 - xg),
\]

and so \( g'' = 2g^3 + xg \). Hence

\[
\frac{\partial^2}{\partial x^2} \log \mathcal{F}(x) = f' = -g^2.
\]

Integrating once yields (assuming that \( g^2 \) decays fast enough at \( \infty \), which will be apparent later)

\[
\frac{\partial}{\partial x} \log \mathcal{F}(x) = \int_x^{\infty} g^2(u)du;
\]

integrating once more and further integrating by parts yield

\[
\log \mathcal{F}(x) = \int_x^{\infty} d\alpha \int_{\alpha}^{\infty} g^2(u)du = \int_x^{\infty} d\alpha \frac{d}{d\alpha} \alpha \int_{\alpha}^{\infty} g^2(u)du
\]

\[
= \alpha \int_{\alpha}^{\infty} g^2(u)du \bigg|_x^{\infty} + \int_x^{\infty} \alpha g^2(\alpha)d\alpha
\]

\[
= \int_x^{\infty} (\alpha - x)g^2(\alpha)d\alpha, \tag{9.2.12}
\]

confirming (9.2.3). For \( x \to \infty \), one checks that, on the one hand, from the definition of the Fredholm determinant of \( A \), the two leading terms are given by

\[
\mathcal{F}(x) = \det \left( I - A\chi_{[x,\infty)} \right) = 1 - \int_x^{\infty} A(z, z)dz + \ldots
\]

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and, on the other hand, from (9.2.12), the two leading terms are

\[ \mathcal{F}(x) = 1 - \int_x^\infty (\alpha - x)g^2(\alpha)d\alpha + \ldots \]

Therefore, comparing the two expressions above,

\[ \int_x^\infty (\alpha - x)g^2(\alpha)d\alpha = \int_x^\infty dz \, A(z, z) + \ldots \]

and upon taking two derivatives in \( x \),

\[ -g^2(x) = \frac{\partial}{\partial x}A(x, x) + \ldots \]

\[ = \frac{\partial}{\partial x} \int_x^\infty A(u + x)^2du + \ldots \]

\[ = \frac{\partial}{\partial x} \int_x^\infty A(u)^2du + \ldots \]

\[ = -A(x)^2 + \ldots, \]

showing that asymptotically \( g(x) \sim A(x) \) for \( x \to \infty \). It is classically known that asymptotically

\[ A(x) = \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}}(1 + \sum_{i=1}^{\infty} \frac{\alpha_i}{(x^{3/2})^i} + \ldots), \quad \text{as } x \to \infty. \]

The solution \( g(x) \) of \( g'' = 2g^3 + xg \) behaving like the Airy function at \( \infty \) is unique (Hastings-McLeod solution). It behaves like

\[ g(x) = \text{Ai}(x) + O\left(\frac{e^{-\frac{2}{3}x^{3/2}}}{x^{1/4}}\right) \quad \text{for } x \to \infty \]

\[ = \sqrt{-\frac{x}{2}} \left(1 + \frac{1}{8x^3} - \frac{73}{128x^6} + O(|x|^{-9})\right) \quad \text{for } x \to -\infty. \]

The Tracy-Widom distribution \( \mathcal{F} \) of mean and standard deviation (see Tracy-Widom [87])

\[ E(\mathcal{F}) = -1.77109 \quad \text{and} \quad \sigma(\mathcal{F}) = 0.9018 \]
has a density decaying for \( x \to \infty \) as (since \( F(x) \) tends to 1 for \( x \to \infty \))

\[
F'(x) = F(x) \int_x^\infty g^2(\alpha) d\alpha \sim \int_x^\infty A^2(\alpha) d\alpha \sim \frac{1}{8\pi x} e^{-\frac{4}{3}x^{3/2}} \quad \text{for} \quad x \to \infty.
\]

(9.2.13)

The last estimate is obtained by integration by parts:

\[
\int_x^\infty A^2(u) du = \int_x^\infty \frac{-1}{8\pi u} (1 + \sum_{i=1}^{\infty} \frac{c_i}{(u^{3/2})^i}) d(e^{-\frac{4}{3}u^{3/2}})
\]

\[
= \frac{e^{-\frac{4}{3}x^{3/2}}}{8\pi x} (1 + \sum_{i=1}^{\infty} \frac{c_i}{(x^{3/2})^i}) - \frac{1}{8\pi} \int_x^\infty \frac{e^{-\frac{4}{3}u^{3/2}}}{u^2} (1 + \sum_{i=1}^{\infty} \frac{c_i}{(u^{3/2})^i}).
\]

Then, following conjectures by Dyson [35] and Widom [93], Deift-Its-Krasovsky-Zhou [33] and Baik-Buckingham-DiFranco [18] give a representation of \( F(x) \) as an integral from \(-\infty\) to \( x \) and thus this provides an estimate for \( x \to -\infty \),

\[
F(x) = 2^{1/24} e^{\zeta(-1)} \frac{e^{-\frac{3}{12|x|^3}}}{|x|^{1/8}} \exp \left\{ \int_{-\infty}^x \left( R(y) - \frac{1}{4} y^2 + \frac{1}{8} y \right) dy \right\}
\]

\[
= 2^{1/24} e^{\zeta(-1)} \frac{e^{-\frac{3}{12|x|^3}}}{|x|^{1/8}} \left( 1 + \frac{3}{2^6 |x|^3} + O(|x|^{-6}) \right), \quad \text{for} \quad x \to -\infty,
\]

where

\[
R(y) = \int_{y}^\infty g(s)^2 ds = g'(y)^2 - yg(y)^2 - g(y)^4.
\]

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