An action of the cactus group

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Let $\mathcal{M}_{0,n}(\mathbb{R})$ denote the Deligne-Mumford compactification of the moduli space of real curves of genus zero with $n$ marked points. Its points are the isomorphism classes of stable real curves of genus zero, that is, curves obtained by gluing $\mathbb{R}P^1$’s in a tree-like way, and such that each irreducible component has at least 3 special points. Let $[\mathcal{M}_{0,n+1}(\mathbb{R})/S_n]$ denote the quotient orbifold of $\mathcal{M}_{0,n+1}(\mathbb{R})$ by the action permuting the first $n$ marked points. In [3], J. Kamnitzer and the author showed that the cactus group $J_n := \pi_1([\mathcal{M}_{0,n+1}(\mathbb{R})/S_n])$ acts on tensor powers of Kashiwara crystals in a way similar to how the braid group acts on tensor powers of quantum group representations.

The big cactus group $J'_n$ is the fundamental group of $[\mathcal{M}_{0,n}(\mathbb{R})/S_n]$. It fits into a short exact sequence $0 \to \pi_1(\mathcal{M}_{0,n}(\mathbb{R})) \to J'_n \to S_n \to 0$, and its elements can be represented by movies, such as the following one:

\[ \infty \xrightarrow{\Delta} \circ \circ \xrightarrow{d_{ij}} \circ \circ \xrightarrow{d_{jk}} \circ \circ \xrightarrow{d_{ik}} \circ \circ \xrightarrow{\Delta} \infty \]

Let $\mathcal{F}_m := (\ast_n^*) \backslash SL_m$ be the variety of flags $0 \subset V_1 \subset \cdots \subset V_{m-1} \subset \mathbb{R}^m$, equipped with volume forms $\omega_i \in \Lambda^i V_i$. The goal of this note is to construct an action of $J'_n$ on the totally positive part $A(n)_{>0}$ of the variety $A(n) := (\mathcal{F}_m)^n/SL_m$. The space $A(n)_{>0}$ is a certain connected component of the locus $A(n)_{reg} \subset A(n)$, where the flags are in generic position. One gets similar actions on $((N\backslash G)^n/G)_{>0}$ for other reductive groups $G$. The space $A(n)_{>0}$ was introduced by Fock and Goncharov [1]. For $m = 2$, it agrees with the Teichmüller space of decorated ideal $n$-gons, that is, the space of isometry classes of hyperbolic $n$-gons with geodesic sides, vertices at infinity, and horocycles around each vertex. It is also an example of a cluster variety, i.e. it comes with special sets of coordinate systems, whose transition functions are given by cluster exchange relations [2]. For $m = 2$, the coordinates are due to Penner [4]. To each pair $i, j$ of vertices of the $n$-gon, he associates the quantity $\Delta_{ij} := \exp(d_{ij})$, where $d_{ij}$ denotes the hyperbolic length between the intersection points of the horocycles around $i$ and $j$, and the geodesic from $i$ to $j$. These coordinates are then subject to the following exchange relations [4]:

\[
\Delta_{ji} = \frac{\Delta_{ij} \Delta_{kl} + \Delta_{jk} \Delta_{il}}{\Delta_{ik}}.
\]

For general $m$, the coordinates on $A(n)$ are indexed by tuples $(i_1, \ldots, i_n) \in \mathbb{N}^n$ whose sum equals $m$, and such that at least two entries are non-zero. The coordinate $\Delta_{i_1 \ldots i_n}$ then assigns to $((V^1_1, \omega^1_1), \ldots, (V^n_1, \omega^n_1)) \in (\mathcal{F}_m)^n$ the ratio of
$\omega_1^1 \wedge \cdots \wedge \omega_n^n$ with the standard volume form on $\mathbb{R}^m$. These coordinates satisfy
\[
\Delta_{i_1 \ldots j \ldots k \ldots \ell} = \left( \Delta_{i_1+1 \ldots j \ldots k \ldots \ell} - \Delta_{i_1 \ldots j-1 \ldots k \ldots \ell} + \Delta_{i_1 \ldots j \ldots k+1 \ldots \ell} - \Delta_{i_1 \ldots j \ldots k \ldots \ell+1} \right) / \Delta_{i_1+1 \ldots j-1 \ldots k+1 \ldots \ell-1},
\]
which generalizes (1). Let $A(n) > 0$ be the locus where all the $\Delta$’s are $> 0$. It is a space isomorphic to $\mathbb{R}^{(n-2)(m+1)+n}$. Each triangulation provides such an isomorphism. More precisely, the isomorphism corresponding to a triangulation is given by the coordinates $\Delta_{i_0 \ldots 0 \ldots j_0 \ldots 0 \ldots k_0 \ldots 0}$, where $i, j, k$ are located at the vertices of the triangles. For example, for $n = 8$, $m = 4$, and the triangulation of the 8-gon, the corresponding coordinates on $A(n) > 0$ are in natural bijection with the bullets in the following figure:

We now explain a general machine for producing actions of $J'_n$ on various spaces. Suppose that we are given two manifolds $X_\triangle$ and $X_I$, equipped with maps

\[
\begin{align*}
\tau : & X_\triangle \to X_I \\
d_1 & \quad d_2 \\
d_3 & \quad d_4
\end{align*}
\]

subject to the relations $r^3 = 1$, $i^2 = 1$, and $d_i \circ r = r \circ d_{i-1}$. Such data can then be reinterpreted as a contravariant functor $X_\bullet : \mathcal{C} \to \{\text{manifolds}\}$ from the category $\mathcal{C} := \{\triangle \to I\}$, whose two objects are the oriented triangle “$\triangle$” and the unoriented interval “$I$”, and whose morphisms are the obvious embeddings and automorphisms. Let $\hat{\mathcal{C}}$ be the category whose objects are the 2-dimensional finite simplicial complexes with oriented 2-faces and connected links, and whose morphisms are the embeddings. There is an obvious inclusion $\mathcal{C} \hookrightarrow \hat{\mathcal{C}}$, and every object of $\hat{\mathcal{C}}$ can be written essentially uniquely as the colimit of a diagram in $\mathcal{C}$. Assuming $d_1 \times d_2 \times d_3 : X_\triangle \to X_I$ is a submersion, then there is a unique extension $X_\bullet$ to $\hat{\mathcal{C}}$ sending colimits to limits. For example, using that extension, we get $X_\bullet \cong X_\triangle \times X_I X_\triangle$.

**Theorem 1.** Let $X_\bullet$ be a functor as above, and denote by the same letter its canonical extension to $\hat{\mathcal{C}}$. Suppose that we are given isomorphisms

\[
\tau : X_\square \to X_\bullet \quad \text{and} \quad \theta : X_\triangle \to X_\square
\]

making the following diagrams commute:

\[
\begin{array}{c}
\begin{array}{ccc}
X_\square & \xrightarrow{\tau} & X_\bullet \\
\downarrow d_1' & & \downarrow d_4' \\
X_I & \xrightarrow{\theta} & X_\triangle
\end{array}
\end{array}
\]

where $d_i' : X_\square \to X_I$, $d_i'' : X_\square \to X_I$, $i = 1, 2, 3, 4$, are induced by the four face inclusions $I \hookrightarrow \square$. 


where $1/2 : X_{\Box} \to X_{\Box}$ and $1/2 : X_{\Box} \to X_{\Box}$ are induced by half turn rotation of the square.

where $1/4 : X_{\Box} \to X_{\Box}$ and $1/4 : X_{\Box} \to X_{\Box}$ are induced by rotation by a quarter turn.

note that “$1 \times \tau$” and “$\tau \times 1$” only become well defined once we have axioms 1) and 2).

then there is a natural action of $J'_n$ on the manifold that $X_\bullet$ associates to a triangulated $n$-gon. (For example, one gets an action of $J'_8$ on $X_\infty$).

We now use the above theorem to equip $\mathcal{A}(n)_{>0}$ with a $J'_n$ action. Indeed, the manifolds $X_\triangle := \mathcal{A}(3)_{>0}$ and $X_I := \mathcal{A}(2)_{>0}$ fit into a diagram $\mathfrak{D}$, and so provide a functor $\mathcal{C} \to \{\text{manifolds}\}$, and so provide a functor $\mathcal{C} \to \{\text{manifolds}\}$. The space associated to a triangulated $n$-gon is $\mathcal{A}(n)_{>0}$, as can be seen from the parameterization $[2]$. We let $\tau$ be the composite

$$\tau : X_{\Box} \xrightarrow{\sim} \mathcal{A}(4)_{>0} \xrightarrow{\sim} X_{\Box},$$

and $\theta$ be the map sending $(F_1, F_2, F_3) \in (\mathcal{F}_m)^3$ to $(F_3^+, F_2^+, F_1^+)$, where the orthogonal of a flag $F$ is given by $(V_1, \ldots, V_{m-1})^\perp := (V_{m-1}^-, \ldots, V_1^+)$, along with $\pm$ the obvious volume forms. The axioms 1) – 8) are then easy to check.

Both $\tau$ and $\theta$ are composites of cluster exchange relations. But the action of $J'_n$ on $\mathcal{A}(n)_{>0}$ is not cluster (it doesn’t satisfy the Laurent phenomenon; it doesn’t preserve the canonical presymplectic form). The reason is that $\theta$ is actually the composite of a cluster map with an automorphism that negates the cluster matrix. In particular, it negates the presymplectic form.

References

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