A Robust Traffic Control Model Considering Uncertainties in Turning Ratios

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Abstract—The effects of uncertainties in model parameters on traffic flow control have recently drawn much research attention. Although certain parameters, such as capacity, initial densities, have been studied, the uncertainties in turning ratios have received few efforts. To fill this gap, this paper proposed a robust control model to deal with the uncertainties in the turning ratio by using distributionally robust chance constraints. The model offers an optimal solution over all possible distributions in accordance with given prior knowledge. Then, we apply this robust model on both a highway network and an urban network, and study the interactions between the uncertainties and the control inputs of the entire network.

Index Terms—Traffic control, Robust control, distributionally robust chance constraints, second order cone program

I. INTRODUCTION

TRAFFIC congestion has become a worldwide problem imposing a significant influence on both economy and environment. Traffic flow control modeling is one of the primary methods to improve the efficiency of transportation systems, and a myriad of control methods have been proposed in the past decades for both highway and urban networks. The general goal of a traffic control method is to improve the average performance of the system, such as decreasing delay and increasing throughput. For most control methods, traffic flow model parameters, such as fundamental diagram, and external inputs, such as traffic densities, are assumed to be known and deterministic. However, the uncertainties can potentially exist almost everywhere in a control problem, and such neglect of these uncertainties may lead to a poor control performance. The importance of considering the randomness in traffic control problems has been widely recognized, and many efforts have been made to handle the uncertainties.

For an urban network, it is commonly assumed all the vehicles travel at the same speed, which leads to parallel vehicle trajectories. As a result, the measure of effectiveness such as traffic delay is simplified. With traffic delay, considerations like the uncertainties in traffic arriving become an important parameter to take into account. A common way to study the effect of this is adding stochastic terms in the delay models. Heydecker \cite{1} summarized the progress of this method. Based on queueing theory, Newell \cite{2} developed a comprehensive approach to investigate the probabilistic arrivals. This method is based on steady-state analysis and may not be applicable for a short period study. To overcome this drawback, given the probability distribution functions (pdf) of arrival rates, Lo \cite{3} proposed a phase clearance reliability (PCR) framework to investigate the probability of overflow during consecutive cycles, and this method was implemented on an adaptive signal control method for an arterial street \cite{4}.

For a highway network, traffic flow is usually modeled by deterministic Partial Differential Equations (PDEs). The Lighthill-Whitham-Richards (LWR) \cite{5, 6} model might be the most famous macroscopic traffic flow model, in which a fundamental diagram representing the relationship between traffic density and speed is required to obtain the solution. Besides the travel demand, the fundamental diagram parameters such as capacity and congested speed can be random since they are affected by external factors such as weather condition and drive behavior. The stochastic nature of the capacity has been studied broadly \cite{7, 8, 9}. Furthermore, the initial density, as an input of optimization models, can be uncertain due to the sensor measurement errors. Based on the traffic flow control framework derived by Li et al. \cite{10, 11}, Liu et al. \cite{12, 13} proposed a model to investigate the effect of the uncertainties in the initial densities on the control performance through chance constraints.

In addition to the sources mentioned above, turning ratio is one of other main origins generating uncertainties. However, to the authors’ best knowledge, little effort has been put in developing robust models handling such uncertainties to improve control reliability. To fill this gap, a second order cone program (SOCP) is proposed to study the impact of the uncertainties of random turning ratios on the traffic flow control. Instead of assuming certain distributions for turning ratios, we only assume that historical data is available, and prior information such as estimation of mean and covariance can be achieved. The uncertainties are inserted into the optimization model through distributionally robust control chance constraints. Such constraints can be converted to SOC constraints \cite{14} and solved by commercial SOCP solvers such as MOSEK \cite{15}.

The rest of this paper is organized as follows. Section \cite{1} reviews the derivation of a traffic control model based on the analytical solution of the Hamilton-Jacobi (H-J) PDE. Based
on this linear program, section III proposes a SOCP to consider the uncertainties existing in turning ratios by distributionally robust chance constraints. The proposed model is robust over all the distributions compatible with the moments (mean and covariance matrices) estimated from historical data. Sections IV and V implement the proposed model on both a highway network and an urban network, and investigate the influence of the uncertainties on the controls (on-ramp inflows and boundary inflows). Section VI summarizes the contribution and provides some future research directions.

II. REVIEW OF THE LAX-HOPF SOLUTION

The proposed model is based on the Lax-Hopf solution [16], [17] of the H-J PDE. Li et al. [10], [11] has shown that a traffic flow model for a highway network can be modeled as an optimization program with linear constraints based on the Lax-Hopf solution. As the building block of our robust control model, this section reviews the derivation of the solution and corresponding constraints. Part II-A introduces the traffic flow model, initial and boundary conditions; part II-B presents the Lax-Hopf solutions; part II-C shows the constraints that the Lax-Hopf solutions need to be satisfied with due to the compatibility conditions. For details regarding the derivation and proof, the readers are referred to references [10], [18].

A. H-J PDE

LWR model [5], [6] is a widely used macroscopic model depicting relationships among traffic flow characteristics

$$\frac{\partial \rho(t, x)}{\partial t} + \frac{\partial \psi(\rho(t, x))}{\partial x} = 0,$$  (1)

where \(\rho(t, x)\) is the density of the point \(x\) away from a reference point at time \(t\), \(\psi\) denotes the experimental relationship, which is defined as fundamental diagram, between flow and density. A triangular fundamental diagram is utilized in this paper,

$$\psi(\rho) = \begin{cases} 
\frac{v_f \rho}{w(\rho - \rho_m)} & \rho \in [0, \rho_c], \\
\frac{w(\rho - \rho_m)}{\rho_c - \rho_m} & \rho \in [\rho_c, \rho_m],
\end{cases}$$  (2)

where \(v_f\) is the free flow speed, \(w\) is the congestion speed, \(\rho_c\) is the critical density where the flow reaches its capacity, \(\rho_m\) is the jam density, where the flow is zero due to the total congestion. Alternatively, by integrating the LWR PDE in space, another traffic flow model, H-J PDE, can be expressed as

$$\frac{\partial M(t, x)}{\partial t} - \psi(-\frac{\partial M(t, x)}{\partial x}) = 0,$$  (3)

where \(M(t, x)\), known as the Moskowitz function [19], denotes the index of the vehicles at point \((t, x)\). To solve this function, the spatial domain \([\xi, \chi]\) is divided evenly into \(k_{max}\) segments; the time domain \([0, t_{max}]\) is divided evenly into \(n_{max}\) segments. Let \(K = \{1, \ldots, k_{max}\}\) and \(N = \{1, \ldots, n_{max}\}\). Assuming the initial density in each spatial segment and the flow in each time step are constant, the piecewise affine initial condition \(M_k(t, x)\), upstream boundary condition \(\gamma_n(t, x)\), and downstream boundary condition \(\beta_n(t, x)\) are defined as

$$M_k(t, x) = \begin{cases} 
-\sum_{i=1}^{k-1} \rho(i) X \\
-\rho(k)(x - (k - 1)X), & \text{if } t = 0 \\
+\infty, & \text{and } x \in [(k - 1)X, kX] \\
& \text{otherwise}
\end{cases}$$  (4)

$$\gamma_n(t, x) = \begin{cases} 
\sum_{i=1}^{n-1} q_{in}(i)T \\
+q_{in}(n)(t - (n - 1)T), & \text{if } x = \xi \\
+\infty, & \text{and } t \in [(n - 1)T, nT] \\
& \text{otherwise}
\end{cases}$$  (5)

$$\beta_n(t, x) = \begin{cases} 
\sum_{i=1}^{n-1} q_{out}(i)T +q_{out}(n)(t - (n - 1)T) -\sum_{k=1}^{k_{max}} \rho(k)X, & \text{if } x = \chi \\
+\infty, & \text{and } t \in [(n - 1)T, nT] \\
& \text{otherwise}
\end{cases}$$  (6)

where \(X\) and \(T\) are the spatial segment length and time step size, respectively; \(\rho(i)\) is the initial density for the \(i\)th spatial segment, \(q_{in}(i)\) and \(q_{out}(i)\) are the inflow and outflow for the \(i\)th time step, respectively.

B. Lax-Hopf Solutions

The Barron-Jensen/Frankowska (B-J/F) solution [20], [21] fully characterized by the Lax-Hopf formula was incorporated in to solve the H-J equation.

**Definition 1:** (Value Condition): A value condition \(c(\cdot, \cdot)\) is a lower semicontinuous function defined on a subset of \([0, t_{max}] \times [\xi, \chi]\).

The initial conditions and boundary conditions are regarded as value conditions.

**Proposition 1:** (Lax-Hopf Formula): Let \(\psi(\cdot)\) be a concave and continuous Hamiltonian, and let \(c(\cdot, \cdot)\) be a value condition. The B-J/F solution \(M_c(\cdot, \cdot)\) to (3) associated with \(c(\cdot, \cdot)\) is defined [22], [23], [24] by

$$M_c(t, x) = \inf_{(u, T) \in (\varphi^*) \times R^+} (c(t - T, x + Tu) + T \varphi^*(u))$$  (7)

where \(\varphi^*(\cdot)\) is the Legendre-Fenchel transform of an upper semicontinuous Hamiltonian \(\psi(\cdot)\), which is given by,

$$\varphi^*(u) := \sup_{p \in \text{Dom}(\psi)} [p \cdot u + \psi(p)]$$  (8)

Based on this proposition, the Moskowitz solution from value conditions (4)-(6) can be expressed as [9]-[11]. For readers
interested in the derivation, see [16], [17] for more on the derivation.

\[
M_{M_k}(t, x) = \begin{cases} 
+ \infty, & \text{if } x \leq (k-1)X + tw \\
- \sum_{i=1}^{k-1} \rho(i)X + \rho(k), & \text{if } x \geq (k-1)X + v_j t \\
tw + (k-1)X - x, & \text{and } x \leq kX + v_j t \\
- \rho_m tw, & \text{and } \rho(k) \leq \rho_c
\end{cases}
\]

[C. Linear Constraints]

The Moskowitz solutions (9)–(11) show that each value (9a) = (10b) condition \(c(\cdot, \cdot)\) be minimum of a finite number of lower semi-continuous functions:

\[
\forall (t, x) \in [0, t_{\max}] \times [\xi, \chi], \quad c(t, x) := \min_{j \in J} c_j(t, x)
\]

The corresponding solution \(M_c(\cdot, \cdot)\) can be decomposed [22], [23] as

\[
\forall (t, x) \in [0, t_{\max}] \times [\xi, \chi], \quad M_c(t, x) := \min_{j \in J} M_{c_j}(t, x)
\]

Based on the Inf-morphism property, the Moskowitz solutions (9)–(11) have to satisfy the compatibility conditions [16].

**Proposition 3: (Compatibility Conditions):** Use the value condition \(c(t, x)\) and the corresponding solution in **Proposition 2.** The equality \(\forall (t, x) \in \text{Dom}(c), M_c(t, x) = c(t, x)\) is valid if and only if the inequalities below are satisfied,

\[
M_{c_j}(t, x) \geq c_i(t, x), \quad \forall (t, x) \in \text{Dom}(c_i), (i, j) \in J^2
\]

In detail, these constraints can be expanded as [10], [18].

\[
M_{M_k}(0, x_p) \geq M_p(0, x_p) \quad \forall (k, p) \in K^2
\]

\[
M_{M_k}(p T, \chi) \geq \beta_p(p T, \chi) \quad \forall k \in K, \quad \forall p \in N
\]

\[
M_{M_k}(\frac{\xi - x_j}{v_j} - (n-1) T, \xi) \geq \beta_p(\frac{\xi - x_j}{v_j}, \chi) \quad \forall k \in K, \quad \forall p \in N
\]

\[
\text{s.t.} \quad \frac{\xi - x_j}{v_j} \in [(p-1)T, p T]
\]

\[
M_{M_k}(p T, \xi) \geq \gamma_p(p T, \xi) \quad \forall k \in K, \quad \forall p \in N
\]

\[
M_{M_k}(n T + \frac{\xi - x_j}{v_j}, \xi) \geq \gamma_p(n T + \frac{\xi - x_j}{v_j}, \chi) \quad \forall k \in K, \quad \forall p \in N
\]

\[
\text{s.t.} \quad n T + \frac{\xi - x_j}{v_j} \in [(p-1)T, p T]
\]

These constraints are piecewise linear function of inflows and outflows. A traffic flow control model needs to satisfy these constraints to make the problem compatible. Unlike other traffic flow control methods [25], [26], [27] in which the PDEs are discretized to ODEs to employ available algorithm, such as gradient descent [28], to solve the optimization model, this framework does not require any discretization or approximation of the corresponding PDE.
III. ROBUST MODEL WITH DISTRIBUTIONALLY ROBUST CHANCE CONSTRAINTS

A. Deterministic Control Model

Founded on the Lax-Hopf solution, a traffic flow control model for a highway network can be expressed as,

\[
\min \ f(x) \\
\text{s.t. } \begin{bmatrix} q_i^z \\ q_{off}^z \end{bmatrix} = \begin{bmatrix} P_1^z \\ P_3^z \\ \mathcal{P} \end{bmatrix}, \ \forall z \in Z \tag{18}
\]

where \( l \) and \( z \) are the indexes of links and nodes, and \( L \) and \( Z \) are the sets of links and nodes. The decision variable \( x \) vector is

\[
x := \{q_{in}(i,j), q_{out}(i,j) : i \in N, j \in L\}, \tag{19}
\]

where \( N \) is the set of time steps. The first constraint means the inflows and outflows of all links need to satisfy the compatibility condition; the second constraint is the node model representing the flow transition. Note that a node has at least more than one incoming links (including on-ramps) or more than one outgoing links (including off-ramps).

Let \( N_i^z \) and \( N_i^r \) represent the number of outgoing links and incoming links at node \( z \). \( q_i^z \) and \( q_{out}^z \) are two column vectors denoting the incoming flows and outgoing flows at node \( z \), respectively; \( q_{in}^z \) and \( q_{off}^z \) are two scalars representing the on-ramp and off-ramp flows; \( P_1^z \) is a \( N_i^z \times N_i^z \) matrix of which each element \( P_1^z(i,j) \) means the proportion of the vehicles from the \( j \)th incoming link going into the \( i \)th outgoing link at node \( z \); \( P_2^z \) is a column vector with dimension of \( N_o \times 1 \) of which each element \( P_2^z(i) \) means the proportion of the vehicles from on-ramp going into the \( i \)th outgoing link; \( P_3^z \) is a row vector with dimension of \( 1 \times N_i \) of which each element \( P_3^z(j) \) means the proportion of the vehicles from the \( j \)th incoming link departing from the off-ramp. In addition, we assume no vehicles coming from an on-ramp would depart from the off-ramp at the same node.

B. Introduction of Distributionally Robust Chance Constraints

In reality, the turning ratio matrices are not always deterministic, and only prior information of their distributions such as moments can be extracted from historical data. Under this situation, the distributionally robust optimization model, in short DRO referred by Delage and Ye [29], is a proper method to study the effect of such uncertainties. This modeling framework has received considerable attention in research communities such as operations research and machine learning. A comprehensive review of DRO can be found in [30].

Let \( \xi \in \mathbb{R}^k \) denote the random parameters, its ambiguity set is defined as the set of distributions that consistent with the prior knowledge about the uncertainty. Assuming \( \mu \) and \( \Gamma \) is its expectation and covariance matrices, and they are the only information known. Then, its ambiguity set can be expressed as

\[
\mathcal{P} = \{ \mathcal{P} \in \mathbb{P}(\mathbb{R}^k) : \mathbb{E}_{\mathcal{P}}[\xi] = \mu, \ \mathbb{E}_{\mathcal{P}}[(\xi - \mu)(\xi - \mu)^T] = \Gamma \} \tag{20}
\]

The goal of a DRO is to optimize the worst-case objective value over the ambiguity set. For example, the objective function of a stochastic program in which the random parameters’ distributions are known can be expressed as

\[
\min_{x} \ R[f(x, \xi)] \tag{21}
\]

where \( x \) is the decision variable vector, \( R \) is the risk measure, such as expectation and Value at Risk (VaR). When an ambiguity set of \( \xi \) is given, this model can be transformed as a DRO

\[
\min_{x} \ \max_{P \in \mathcal{P}} \ R_P[f(x, \xi)] \tag{22}
\]

If the randomness is involved in constraints, a common way to develop robust model is to replace the deterministic constraints with chance constraints

\[
P[h(x, \xi) \geq 0] \geq 1 - \alpha \tag{23}
\]

which indicates that the constraint \( h(x, \xi) \geq 0 \) hold with confidence level of \( 1 - \alpha \). Similarly, if only ambiguity set is known, the distributionally robust chance constraint can be expressed as

\[
\min_{P \in \mathcal{P}} \ P[h(x, \xi) \geq 0] \geq 1 - \alpha \tag{24}
\]

which means the minimum of the probability, i.e. the worst case, that the constraint holds under all possible distributions is larger than the confidence level.

If \( h(x, \xi) \) is an affine function of \( x \), and the distributionally robust chance constrain can be expressed as

\[
\min_{P \in \mathcal{P}} \ P[\tilde{a}^Tx + \tilde{b} \leq 0] \geq 1 - \alpha \tag{25}
\]

Let \( d = [\tilde{a}^T, \tilde{b}]^T \). If \( \mathcal{P} \) is its ambiguity set with known expectation \( \tilde{d} \) and covariance matrix \( \tilde{\Gamma} \) and \( 1 - \alpha > 0.5 \), [25] can be converted to a convex second-order cone (SOC) constraint [14]

\[
\kappa_\alpha \sigma(\tilde{x}) + \phi(\tilde{x}) \leq 0, \quad \kappa_\alpha = \sqrt{(1 - \alpha)/\alpha}, \tag{26}
\]

where \( \tilde{x} = [x^T, 1]^T \), \( \phi(\tilde{x}) = \tilde{d}^T \tilde{x} \), \( \sigma(\tilde{x}) = \tilde{x}^T \tilde{\Gamma} \tilde{x} \) and \( \sigma(\tilde{x}) = \|\tilde{\Gamma}^{1/2} \tilde{x}\|^2 \). The proof can be found in [14].

C. Robust Control Model as a SOCP

The turning ratios are involved in the second constraint in [18]. However, we cannot convert these equality constraints to chance constraints directly due to the fact that for any feasible solution, the probability an equality constraint holds is always zero if the distribution is continuous. Therefore, we need to transform those constraints to inequality form before adding chance constraints. To this end, we make following declaration:

1. The on-ramps and off-ramps are regarded as links;
2. The links are divided into two groups: incoming boundary links and other links. Incoming boundary links are the links through which the vehicles flow into the network. For example, links 1 and 4 and all on-ramps in Figure [1] are incoming boundary links. Let \( L \) denote the set of links and \( L_{in} \) be the set of incoming boundary links.
3. All the nodes contain more than one incoming links or more than one outgoing links. Otherwise, the turning ratio is always 1. Let $Z$ be the set of nodes, and $L^+_Z$ and $L^-_Z$ be the incoming link set and outgoing link set of node $z$;

4. The turning ratios at different nodes are independent;

5. Let $P^z$ a $m_z \times n_z$ matrix where $m_z$ and $n_z$ are the number of outgoing links and incoming links at node $z$. $P^z(i, j)$ represents the ratio of vehicles from link $L^-_z(i)$ to link $L^+_z(j)$;

6. The inflows except for the incoming boundary links can be expressed as: $q_{in}(i, j) = \sum_{k \in L^-_z} P^z(j, k) q_{out}(i, k)$, $\forall j \in L^-_z/L_{in}$.

By this way, we can replace most of the $q_{in}(i, j)$’s with a function of $q_{out}(i, j)$’s, and the equality constraint in (18) can be removed. The new decision variable $x$ is defined as follows:

$$x := \{q_{in}(i,j) : i \in N, j \in L_{in}\} \cup \{q_{out}(i,j) : i \in N, j \in L\}.$$  

(27)

In addition, for the upstream boundary conditions [5] and Moskowitz solutions [10] related to $q_{in}(i, j)$’s, we need to reformat them as

$$\gamma_n(t, x, l) = \begin{cases} \sum_{i=1}^{n-1} (\sum_{k \in L^-_n} q_{out}(i, k) P^z(l, k)) T + (\sum_{k \in L^-_n} q_{out}(n, k) P^z(l, k)) (t - (n - 1) T), & \text{if } x = \xi \\
+\infty, & \text{otherwise} \end{cases}$$  

(28)

and

$$M_{\gamma_n}(t, x, l) = \begin{cases} \sum_{i=1}^{n-1} (\sum_{k \in L^-_n} q_{out}(i, k) P^z(l, k)) T + (\sum_{k \in L^-_n} q_{out}(n, k) P^z(l, k)) (t - \frac{\xi - \alpha}{\nu}) - (n - 1) T), & \text{if } t \leq (n - 1) T + \frac{\xi - \alpha}{\nu} \\
+\infty, & \text{if } t \geq (n - 1) T + \frac{\xi - \alpha}{\nu} \\
\sum_{i=1}^{n-1} (\sum_{k \in L^-_n} q_{out}(i, k) P^z(l, k)) T + \rho_v v_f (t - \frac{\xi - \alpha}{\nu} - nT), & \text{otherwise} \end{cases}$$  

(29)

where $l$ is the link index, $z$ is the node from which link $l$ starts, i.e. link $l$ is one of the outgoing links of node $z$. All other initial conditions and Moskowitz solutions will not be changed. Now, the $\gamma_n$’s and $M_{\gamma_n}$’s in the constraints (15)-(17) are replaced by (28) and (29). The new constraints are linear functions of $q_{out}$ with random coefficient $P$. Therefore, the chance constraint and distributionally robust chance constraint of each of them have the expression as (28) and (29).

Assume $P^z$ is the random turning ratio matrix at node $z$, let $P^z$ and $\Gamma$ be its expectation and covariance matrix. For the link $L^+_z(i)$, the distributionally robust chance constraints of (15)-(17) can be expressed as (30), (37) and (47). The derivation of the fourth constraint in (30) is shown in Appendix, and all others can be obtained in a similar way.

$$\begin{align*}
M_{Mb}(0, x_p) &\geq M_p(0, x_p) & \forall (k, p) \in K^2 \\
M_{Mb}(p T, \chi) &\geq \beta_p (p T, \chi) & \forall k \in K, \forall \chi \in N \\
M_{Mb}(\frac{\xi - \alpha}{\nu}, \chi) &\geq \beta_p (\frac{\xi - \alpha}{\nu}, \chi) & \forall k \in K, \forall \chi \in N \\
s.t. \quad \frac{\xi - \alpha}{\nu} \in [(p - 1) T, p T] \\
\kappa_{1n} || (\Gamma_p^z)^{-\frac{1}{2}} \hat{x}^p ||_2 + d^p_1(\hat{x}^p) &\leq 0, & \forall k \in K, \forall \chi \in N \\
\kappa_{1n} || (\Gamma_p^z)^{-\frac{1}{2}} \hat{x}^p ||_2 + d^p_2(\hat{x}^p) &\leq 0, & \forall k \in K, \forall \chi \in N \\
s.t. \quad \frac{\xi - \alpha}{\nu} - 1 \in [(p - 1) T, p T] \\
\end{align*}$$  

(30)

where

$$\begin{align*}
\hat{x}^p := \left[q_{out}(1, L^+_z(1)), q_{out}(2, L^+_z(1)), ..., q_{out}(p, L^+_z(1)), \right]^T \\
\hat{x}^p := \left[q_{out}(1, L^+_z(n_z)), q_{out}(2, L^+_z(n_z)), ..., q_{out}(p, L^+_z(n_z)), 1 \right]^T, \\
d^p_1 := \left[TP^z(L^+_z(1), L^-_z(1)), ..., TP^z(L^+_z(n_z), L^-_z(n_z)), -M_{Mb}(p T, \xi) \right]^T, \\
d^p_2 := \left[TP^z(L^+_z(1), L^-_z(1)), ..., TP^z(L^+_z(n_z), L^-_z(n_z)), -M_{Mb}(p T, \xi) \right]^T. \\
\end{align*}$$  

(31)

(32)
\[
\Gamma^i_1 := T^2 \begin{bmatrix}
\Gamma(1,1)_{p \times p} & \Gamma(1,2)_{p \times p} & \cdots & \Gamma(1,n_z)_{p \times p} & 0_{p \times 1} \\
\Gamma(2,1)_{p \times p} & \Gamma(2,2)_{p \times p} & \cdots & \Gamma(2,n_z)_{p \times p} & 0_{p \times 1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\Gamma(n_z,1)_{p \times p} & \Gamma(n_z,2)_{p \times p} & \cdots & \Gamma(n_z,n_z)_{p \times p} & 0_{p \times 1} \\
0_{1 \times p} & 0_{1 \times p} & \cdots & 0_{1 \times p} & 0_{1 \times 1}
\end{bmatrix},
\]

(33)

\[
\tilde{\mathbf{x}}^p_1 := \tilde{x}_1^p,
\]

(34)

\[
n_{p_1} := \left[ TP^z(L_{\text{out}}^z(i), L_{\text{in}}^z(1)), \ldots, P^z(L_{\text{out}}^z(i), L_{\text{in}}^z(1))t_1, \ldots \right],
\]

(35)

\[
\Gamma^p_2 := \left[ T^2 \Gamma(1,1)_{(p-1) \times (p-1)} \right] \left( \begin{array}{c}
(T_{t_1} \Gamma(1,1))_{1 \times (p-1)} \\
\vdots \\
(T_{t_1} \Gamma(n_z,1))_{1 \times (p-1)}
\end{array} \right] \left( \begin{array}{c}
(T_{t_1} \Gamma(1,1))_{1 \times (p-1)} \\
\vdots \\
(T_{t_1} \Gamma(n_z,1))_{1 \times (p-1)}
\end{array} \right) \left( \begin{array}{c}
0_{1 \times (p-1)} \\
0_{1 \times (p-1)} \\
0_{1 \times (p-1)}
\end{array} \right),
\]

(36)

where \(\Gamma(i,j)_{a \times b}\) indicates a \(a \times b\) matrix in which all the elements equal \(\Gamma(i, j)\) and \(t_1 = \left( \frac{z_{(p-1)} - n}{n} \right) \).
\[
\Gamma_0^p := T^2 \begin{bmatrix}
\Gamma(1, 1)_{n \times n} & \Gamma(1, 2)_{n \times n} & \cdots & \Gamma(1, n_z)_{n \times n} & 0_{n \times (p+1)} \\
\Gamma(2, 1)_{n \times n} & \Gamma(2, 2)_{n \times n} & \cdots & \Gamma(2, n_z)_{n \times n} & 0_{n \times (p+1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\Gamma(n_z, 1)_{n \times n} & \Gamma(n_z, 2)_{n \times n} & \cdots & \Gamma(n_z, n_z)_{n \times n} & 0_{n \times (p+1)} \\
0_{(p+1) \times n} & 0_{(p+1) \times n} & \cdots & 0_{(p+1) \times n} & 0_{(p+1) \times (p+1)}
\end{bmatrix},
\]

(43)

\[
\tilde{x}_0^p := \tilde{x}_4^p
\]

(44)

\[
\begin{aligned}
d_5^p := & -TP^z(L_{out}(i), L_{in}(1)), \ldots, -TP^z(L_{out}(i), L_{in}(1)), \\
& -TP^z(L_{out}(i), L_{in}(n_z)), \ldots, -TP^z(L_{out}(i), L_{in}(n_z)), \\
& -T_1, \ldots, -T_2, -\sum_{k=1}^{k_{\text{max}}} \rho(k)X,
\end{aligned}
\]

(45)

\[
\Gamma_0^p := \Gamma_4^p,
\]

(46)

where \(t_2 = nT + \frac{\chi - \xi}{\nu_f} - (p-1)T\).

\[
\begin{aligned}
\kappa_{\alpha}(\|\Gamma_0^p \tilde{x}_0^p\|_2^2 + d_5^p(\tilde{x}_0^p)) & \leq 0, \quad \forall(n, p) \in N^2, pT > nT + \frac{\chi - \xi}{\nu_f} \\
\kappa_{\alpha}(\|\Gamma_0^p \tilde{x}_0^p\|_2^2 + d_5^p(\tilde{x}_0^p)) & \leq 0, \quad \forall(n, p) \in N^2 \\
\text{s.t.} \quad nT + \frac{\chi - \xi}{\nu_f} & \in [(p-1)T, pT] \\
M_{\beta_{\text{n}}}(pT, \chi) & \geq \beta_p(pT, \chi) \quad \forall(n, p) \in N^2
\end{aligned}
\]

(47)

where

\[
\begin{aligned}
\tilde{x}_0^p := & (q_{out}(1, L_{out}(1)), q_{out}(2, L_{out}(1)), \ldots, q_{out}(n, L_{out}(1)), \ldots) \\
& q_{out}(1, L_{in}(n_z)), q_{out}(2, L_{in}(n_z)), \ldots, q_{out}(n, L_{in}(n_z)), \\
& q_{out}(1, L_{in}(1)), q_{out}(2, L_{in}(1)), \ldots, q_{out}(p, L_{out}(1)), 1]^T,
\end{aligned}
\]

(48)

\[
\begin{aligned}
d_0^p := & TP^z(L_{out}(i), L_{in}(1)), \ldots, TP^z(L_{out}(i), L_{in}(1)), \\
& TP^z(L_{out}(i), L_{in}(n_z)), \ldots, TP^z(L_{out}(i), L_{in}(n_z)), \\
& -T_1, \ldots, -T_2, -\sum_{k=1}^{k_{\text{max}}} \rho(k)X - \rho_{cvf}(p - n)T - \frac{\chi - \xi}{\nu_f}
\end{aligned}
\]

(49)

\[
\begin{aligned}
\Gamma_0^p := T^2 \begin{bmatrix}
\Gamma(1, 1)_{p \times p} & \Gamma(1, 2)_{p \times p} & \cdots & \Gamma(1, n_z)_{p \times p} & 0_{p \times (n+1)} \\
\Gamma(2, 1)_{p \times p} & \Gamma(2, 2)_{p \times p} & \cdots & \Gamma(2, n_z)_{p \times p} & 0_{p \times (n+1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\Gamma(n_z, 1)_{p \times p} & \Gamma(n_z, 2)_{p \times p} & \cdots & \Gamma(n_z, n_z)_{p \times p} & 0_{p \times (n+1)} \\
0_{(n+1) \times p} & 0_{(n+1) \times p} & \cdots & 0_{(n+1) \times p} & 0_{(n+1) \times (n+1)}
\end{bmatrix},
\end{aligned}
\]

(50)

\[
\tilde{x}_0^p := \tilde{x}_6^p
\]

(51)

\[
\begin{aligned}
d_7^p := & TP^z(L_{out}(i), L_{in}(1)), \ldots, P^z(L_{out}(i), L_{in}(1)), \ldots, \\
& TP^z(L_{out}(i), L_{in}(n_z)), \ldots, P^z(L_{out}(i), L_{in}(n_z)), \ldots \\
& -T_1, \ldots, -T_2, -\sum_{k=1}^{k_{\text{max}}} \rho(k)X - \rho_{cvf}(\xi - \chi) \left(\frac{1}{w} - \frac{1}{\nu_f}\right)
\end{aligned}
\]

(52)
Therefore, using this method to consider the uncertainties in turning ratios, we can convert the optimization model (15) to a program with SOC constraints. If the objective function is linear, the model becomes a SOCP.

IV. CASE STUDY ON A HIGHWAY NETWORK

This section implements the proposed framework on a network to test the impact of uncertainties in turning ratios on the control results.

A. Network and Problem Settings

The network employed is shown in Figure 1. This network consists of 6 links, 3 nodes (excluding the boundary nodes), 2 on-ramps and 2 off-ramps. Links 1 and 4 are incoming links. Each link has a length of 1.2km and is divided into 2 segments; the simulation time is 500s which is made into 25 equal time steps. Consider the free flow speed $v_f = 30$m/s, critical density $\rho_c = 0.0175$ veh/s/lane, capacity $C = 0.5250$ veh/s/lane, jam density $\rho_m = 0.2250$ vehs/s/lane, and the congestion speed $w = -5.5$m/s. Links 1-3 have 4 lanes, and links 4-6 have 3 lanes. The related sets, vectors and matrices are defined as follows:

1. Link sets: $L = \{i \in \mathbb{Z} : 1 \leq i \leq 6\}$, $L_{in} = \{1, 4\}$; 
2. Incoming and outgoing link sets: $L_{in}^2 = \{1, 4\}$, $L_{out}^2 = \{2, 5\}$; $L_{in}^3 = \{2, 7\}$, $L_{out}^3 = \{3, 9\}$; $L_{in}^5 = \{5, 8\}$, $L_{out}^5 = \{6, 10\}$; note that links 7, 8, 9 and 10 are on-ramp 1, on-ramp 2, off-ramp1 and off-ramp2, respectively;

Fig. 1. highway network layout.

3. The transition matrices at nodes connecting on-ramps and off-ramps are $P^1 = P^6 = \begin{bmatrix} 0.80 & 1.00 \\ 0.20 & 0.00 \end{bmatrix}$.

4. Covariance of turning ratio matrices: $\Gamma^2 = \begin{bmatrix} 0.005 & 0.001 \\ 0.001 & 0.005 \end{bmatrix}$, $\Gamma^3 = \begin{bmatrix} 0.005 & 0 \\ 0 & 0 \end{bmatrix}$, $\Gamma^6 = \begin{bmatrix} 0.005 & 0 \\ 0 & 0 \end{bmatrix}$.

5. The confidence level for each SOC constraint is equal. Since we assume that the vehicles from an on-ramp will not depart the highway from the off-ramp at the same node, the covariance matrices are sparse at such nodes. The optimization model is as follows

\[
\min - \sum_{i=1}^{n_{max}} \left( \sum_{j \in L_{in}} q_{out}(i, j) + \sum_{j \in L_{out}} q_{in}(i, j) \right) + m(q_{on}(i, 1) + q_{on}(i, 2))(n_{max} - i + 1) - hy(i) \\
\text{s.t.} \quad y(i) \geq n_{lance}(4)q_{out}(i, 1) - n_{lance}(1)q_{out}(i, 4), \quad \forall i \\
\quad y(i) \geq n_{lance}(1)q_{out}(i, 4) - n_{lance}(4)q_{out}(i, 1), \quad \forall i \\
\quad q_{on}(i, 1) \geq q_{out}(i, 2) / n_{lance}(2), \quad \forall i \in N \\
\quad q_{on}(i, 2) \geq q_{out}(i, 4)/n_{lance}(4), \quad \forall i \in N \\
\quad q_{out}(i, 3) \leq \psi'(\rho_3), \quad \forall i \in N \\
\quad q_{out}(i, 6) \leq \psi'(\rho_6), \quad \forall i \in N \\
\quad 15 - 17, \quad \forall j \in L_{in} \\
\quad 20 - 27, \quad \forall j \in L_{out} \\
\quad q_{out}(i, j) \geq 0, \quad q_{in}(i, j) \geq 0 \quad \forall i, \quad j
\]
a unit time, of nodes downstream which are nodes 4 and 7 in this case study, and

$$\psi'(\rho) = \begin{cases} C & \rho \in [0, \rho_c] \\ \psi(\rho) & \rho \in [\rho_c, \rho_m] \end{cases}$$

where $C$ is the capacity. The seventh and eighth constraint indicate that the incoming boundary links satisfy the deterministic compatibility conditions, and other links satisfy the distributionally robust chance constraints.

### B. Results

We studied three different scenarios in terms of the level of service of this network. The first scenario is that the network is under free flow condition; the second scenario is that the network is congested; the third case is that the network is partially congested.

1) **Free Flow Network**: Let the initial densities of every link are low than their critical densities, and $\rho = \rho_c$. $P^2 = \begin{bmatrix} 0.80 & 0.27 \\ 0.20 & 0.73 \end{bmatrix}$. This transition matrix makes the inflows of links 2 and 5 proportional to their capacity. Then, we investigate the influence of the uncertainty of turning ratio at one node on other control results. Figures 2 shows the control inputs considering the uncertainties of node 6. Every 25 points on the horizontal axis indicate the control inputs for an incoming link for 25 time steps. For example, the first 25 points represent the optimal inflow of link 1 during the simulation, and the range of 50-75 indicates the optimal inflow of on-ramp 1. The base case is the results without uncertainties. The change of the inflows from on-ramp 2 shows that, the distributionally robust chance constraints require the original constraints hold with a high probability for the worst distribution, and this will lead to a more conservative optimal solution. Due to the maximization objective function, the sum of the incoming flows from link 5 and the inflow of on-ramp 2 should be equal to the supply of link 6, if we are not certain about the turning ratio at node 6, we have to decrease the on-ramp inflows to ensure link 6 can accommodate all the incoming vehicles even for the worst case. This is the reason why the inflow of on-ramp 2 is lower than the deterministic model at the beginning of the simulation. For the same reason, the constraints (37) restrict the outflow of link 6 to make it less than the Moskowitz solution from the upstream conditions. Therefore, its outflow is less than its capacity although it is free flow condition downstream. This constraint adds a shockwave moving backward, and the supply of link 6 will decrease once the shockwave reaches its upstream end. This is the reason why there is a drop on the inflows of on-ramp 2 appearing in the middle of the simulation. In addition, under free flow condition, it is shown that there is little impact of an intersection (node 6) on upstream links (links 1 and 4) or other highways (on-ramp 1). Note that the drop of inflows of links 1 and 4, appearing at the end of the simulation, results from the deviation of the transition equation. In this case, $P^2C_1 + P^2C_4$ is a little larger than $C_2$, this will block a small part of vehicles and induces a shockwave on both links 1 and 4.

The control inputs considering the uncertainty at node 4 is shown in Figure 3. Due to the same reason mentioned above, the optimal inflows of links 1 and 4 drop at some point. Consequently, the inflows of two on-ramps increases since fewer vehicles upstream merge to the node. Therefore, the uncertainties decrease the inflow of the incoming links and increase the inflows of on-ramps downstream.

2) **Congested Network**: Let us assume a congested network with $\rho = 4\rho_c$. $P^2 = \begin{bmatrix} 0.80 & 0.27 \\ 0.20 & 0.73 \end{bmatrix}$. The corresponding control inputs are shown in Figures 4 and 5. Figure 4 shows a similar phenomenon as Figure 2. Unlike the free flow case,
Fig. 4. Optimal control inputs considering turning ratio uncertainty at node 3.

Fig. 5. Optimal control inputs considering turning ratio uncertainty at node 2.

Figure 5 shows that the on-ramps downstream are not impacted by the intersection upstream for the congested case. This is because although the inflows of links 1 and 4 are reduced, the whole network is still congested which means there are enough vehicles from the links merging with on-ramps. If the simulation horizon increases to some extent, the inflows of links 1 and 4 will further decrease and the inflows of on-ramps will be expected to increase.

3) Partially Congested Network: Let us consider another network with \( \rho_l = \begin{cases} 4\rho_l & l \in \{1, 2, 3, 4\} \\ 0.8\rho_l & l \in \{5, 6\} \end{cases} \). In this scenario, links 5 and 6 are under free flow condition, and all other links are congested since the demand of link 2 is higher than its supply, which is implied by the transition matrix at node 2. Figure 6 shows the control inputs with random turning ratios at node 3. This uncertainty reduces the on-ramp flows at the same node and increases the flows of on-ramp 2. Unlike the completely congested network, link 2 in this case blocks the vehicles on links 1 and 4, and since links 5 and 6 are under free flow conditions, on-ramp 2 can send more vehicles to maximize the total throughput. The two drops on links 1 and 4 are subjected different regimes or shockwaves. At the beginning of the simulation, shockwaves are generated at node 2 and 3. The shockwave at node 2 originates from the fact that, due to the transition matrix and the congestion condition, the number of vehicles passing through node 2 on links 1 and 4 is less than the number of vehicles desiring to. The shockwave at node 3 is generated by the robust constraints. Both shockwaves move backward and will arrive the upstream end of boundary links and restrict their inflows. Therefore, the inflows decrease by the same scale the first time while the second reduction increases with the confidence level \( 1 - \alpha \).

Figure 7 shows the control inputs with random turning ratios at node 2. Similarly, the on-ramp inflows downstream increases if it is under free flow condition and would not be impacted if it is congested.

Figure 8 shows the control inputs considering the uncertainties at node 6. Two phenomena need to be explained:

1. When the confidence level is high, e.g. \( 1 - \alpha = 0.95 \), the inflows of on-ramp 2 at the beginning is equal to the supply of link 6, and the outflow of link 5 is zero. This is because that, for a fixed outflow of link 5, due to the constraint (37), the outflow of link 6 is very small if \( 1 - \alpha \) is large. If we block the link 5 and proceed the vehicles from on-ramp 2, this will increase the outflows of link 6 in first several time steps substantially since we assume all the vehicle from on-ramp 2 will flow onto link 6. When this contribution to the objective value is more significant than the loss from blocking
Fig. 7. Optimal control inputs considering turning ratio uncertainty at node 2.

Fig. 8. Optimal control inputs considering turning ratio uncertainty at node 6.

2. When $1 - \alpha$ is small, there is a reduction on the inflow curve of on-ramp 2 followed by an increase. This is also caused by two distinct shockwaves. The first shockwave is generated when the vehicles on link 5 at the beginning of the simulation arrive at the downstream node of link 6. Let us call this shockwave $S_1$ and the equivalent density downstream $\mu_1$. In the following, since links 1 and 4 are blocked, there are not enough vehicles to come onto link 6 from link 5, so more vehicles from on-ramp 2 can flow in. Again, since there are no uncertainties in the turning ratios from on-ramp 2, the outflow of link 6 increases if more vehicles merge onto link 6. Therefore, a short time after the first shockwave is generated, a second shockwave $S_2$ is formed, and the corresponding density downstream $\mu_2 < \mu_1$. When these shockwaves touch the upstream node 6, the inflow of on-ramp 2 changes correspondingly. Since $\mu_2 < \mu_1$, the inflows should decrease and then increase.

V. CASE STUDY ON AN URBAN NETWORK

In addition to highway networks, this section applies the proposed framework on an urban network and the corresponding inflow controls are studied.

A. Network and Problem Settings

A sub-network of downtown Austin, TX, shown as the blue square in Figure 9, is employed. This network consists of 55 links and 20 nodes, and all the nodes are signalized. The link attributes, such as length and number of lanes, and signal timings, such as splits and offsets, are obtained from the database of VISTA [31] administered by the University of Texas at Austin. In this network, except for the Congress Avenue, all streets are one-way street. The model parameters are as follows: free flow speed $v_f = 13.5$ m/s, capacity $C = 0.3375$ veh/s/lane, critical density $\rho_c = 0.025$ veh/m/lane, jam density $\rho_m = 0.125$ veh/m/lane, and congestion speed $w = -3.86$ m/s. For simplicity, let the link lengths be unanimous and equal to 128 m. We divide each link into 2 even segments. Let the initial densities be 0 and $\rho_c$ in the upstream and downstream segments, respectively. Additionally, the simulation time is 300 s and is divided into 75 even time steps.

Similarly as the highway network, we assume the inflows of
incoming boundary links, shown as the blue arrows in Figure 9 are continuous and controllable. The optimization model is

\[
\min - \sum_{i=1}^{n_{\text{max}}} \left( \sum_{j \in L} q_{\text{out}}(i,j) + \sum_{j \in L_{\text{in}}} q_{\text{in}}(i,j) \right) (n_{\text{max}} - i + 1)
\]

\[
+ \omega \sum_{i=1}^{n_{\text{max}}} \sum_{j \in L_{\text{in}}} \left( q_{\text{in}}(i,j) - q_{\text{in}}(i+1,j) \right)^2
\]

s.t. \quad q_{\text{out}}(i,j) \leq e_j C_j, \quad \forall i \in N, j \in L_{\text{out}}

(15) - (17), \quad \forall j \in L_{\text{in}}

(30), (37), (47) \quad \forall j \in L/L_{\text{in}}

q_{\text{out}}(i,j) \leq C_j s(i,j), \quad \forall i \in N, j \in L

q_{\text{out}}(i,j) \geq 0, \quad q_{\text{in}}(i,j) \geq 0 \quad \forall i, j

(56)

The first term of the objective function is to maximize the sum of weighted outflows and inflows; the second term is a quadratic function which is used to reduce the inflow fluctuation, i.e. smooth the inflows. \( \omega \) is the corresponding weight. The first constraint adds an upper bound for the outflows of outgoing boundary links as a proportion of their capacities. In this example, \( e_j = 0.8 \) for all the outgoing boundary links. The fourth constraint adds the signal constraints where \( s(i,j) \)'s are binary parameters. \( s(i,j) = 1 \) when the phase serving \( j \) at time \( i \) is green, and \( s(i,j) = 0 \) otherwise. The rest ones are the same as the model for the highway network. Note that since most of the streets are one-way, all the vehicles from one link are served by the same phase. For a network consisting of two-way streets, we need to group movements that can be served by the same phase and treat the groups as separate links.

We assume all the vehicles go straight at an intersection with a probability of 0.8, and the rest turning ratios are equal to the turn ratios for the groups served by the same phase and treat the groups as separate links.

B. Results

Figures 10 and 11 show the optimal controls on those 10 incoming boundary links, shown in Figure 9 over the simulation time with different weights \( \omega \). The horizontal axis is the index of time steps while the vertical axis is the optimal inflows (veh/s). The differences between the ranges of vertical axis result from the different number of lanes. There is a trade-off between the smoothness of the control and the total throughput. The optimal control with \( \omega = 20 \) is smoother than \( \omega = 0.2 \) while the overall inflows are lower. Similarly as the highway network, Figure 10 shows that the optimal inflows considering uncertainties in turning ratios are lower than the base case, and a higher confidence level induces a larger reduction. It also shows that those optimal inflows are decreasing with time while the control of the base case does not present this trend. However, the optimal inflows of some links, such as links 4 and 5, of the base case in Figure 11 also decrease with time. The reason is that the weight of the total throughput in the objective function, \( (n_{\text{max}} - i + 1) \), decreases with time step \( i \), a large \( \omega \) may make the smooth term more significant the throughput term at some point, and the throughput is confined consequently.

VI. CONCLUSION

Based on the Lax-Hopf solution to the H-J PDE, this paper proposed a robust control model for highway networks to deal with the uncertainties in turning ratios. The uncertainties are inserted into the model by distributionally robust chance constraints and converted to SOC constraints. Then, multiple case studies for both highway and urban networks are conducted to investigate the influence of the uncertainties on the interactions between the control of incoming links. To the author’s best knowledge, there are few research studies focusing on the effect of uncertainties in turning ratios on traffic flow control. The proposed model in this paper can fit this gap well. One drawback of this model is that it does not consider the bounds of the random parameters, which indicates that the real ambiguity set is a subset of the one used in this paper. Therefore, this model may provide too conservative optimal solutions when the variance is large since the worst distribution may not belong to the real ambiguity set. How to overcome this drawback is a promising research direction. In addition, we do not have constraints for the demands at the boundaries of the networks, which denotes that we assume the demands are equal to capacities. Investigation of the uncertainties in demands is another interesting research topic.

VII. APPENDIX

A. Derivation of the SOC Constraint (30)

The deterministic version of (30) is (15) shown as

\[
M_k (0, x_p) \geq M_p (0, x_p) \forall (k, p) \in K^2
\]

\[
M_k (pT, \chi) \geq \beta_p (pT, \chi) \quad \forall k \in K, \quad \forall p \in N
\]

\[
M_k (\frac{x - x_f}{v_f}, \chi) \geq \beta_p (\frac{x - x_f}{v_f}, \chi) \quad \forall k \in K, \quad \forall p \in N
\]

s.t. \quad \frac{x - x_f}{v_f} \in [(p - 1)T, pT)

Since only \( q_{\text{in}} \)'s are expressed as a function of \( P^z \)

\[
q_{\text{in}}(i,j) = \sum_{k \in L_{\text{in}}^z} P^z(j,k) q_{\text{out}}(i,k), \quad \forall j \in L_{\text{in}}^z / L_{\text{in}},
\]

and \( M_k \)'s, \( M_p \)'s and \( \beta \)'s are independent of \( q_{\text{in}} \)'s, the first three constraints in (15) do not need to be changed. By substituting (25) into the fourth constraints, and let link \( l \) be the \( i \)th outgoing link at node \( z \), we obtain
In this constraint, the decision variables are $q_{out}(j, r)$’s, and $j = 1, 2, ..., p$ and $r = L_{in}^z(1), L_{in}^z(2), ..., L_{in}^z(|L_{in}^z|))$. Therefore, based on (26), the decision variable vector can be expressed as

$$\tilde{x}_p^p := \frac{P}{\sum_{j=1}^{p} (\sum_{r \in L_{in}^z} q_{out}(j, r)P^z(L_{out}^z(i), r)T - M_{M_k}(pT, \xi) \leq 0, \forall k \in K, \forall p \in N.} \tag{57}$$
Then, $d_k^n$ is the coefficient of each decision variable in (57), and it can be expressed as

$$d_k^n := \begin{bmatrix} TP^2(L_{\text{out}}(1), L_{\text{in}}(1)) & \cdots & TP^2(L_{\text{out}}(1), L_{\text{in}}(n_z)) \end{bmatrix}^T \Gamma \begin{bmatrix} (n_z-2) \times p \\ \vdots \\ TP^2(L_{\text{out}}(i), L_{\text{in}}(n_z)) \end{bmatrix},$$

where each block of $\tilde{a}_i^p$ and $\tilde{d}_i^p$ is the outflows for the same incoming link at $p$ time steps and the corresponding coefficients. Based on the dimension of the decision variable vector, the covariance matrix is a $p \times (n_z + 1)$ matrix. We assume the uncertainties of the turning ratios do not change over time steps, so

$$\text{covar}(\tilde{d}_i^p(i), \tilde{d}_j^p(j)) = \begin{cases} \Gamma(n, n) & \text{if } i, j \in \text{block } n \\ \Gamma(n, m) & \text{if } i \in \text{block } n, j \in \text{block } m \\ 0 & \text{if } i = n_p + 1 \text{ or } j = n_p + 1 \end{cases}$$

Therefore, the integral covariance matrix is

$$\Gamma_i^p := T^2 \begin{bmatrix} \Gamma(1,1)_{p \times p} & \Gamma(1,2)_{p \times p} & \cdots & \Gamma(1,n_z)_{p \times p} \\ \Gamma(2,1)_{p \times p} & \Gamma(2,2)_{p \times p} & \cdots & \Gamma(2,n_z)_{p \times p} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma(n_z,1)_{p \times p} & \Gamma(n_z,2)_{p \times p} & \cdots & \Gamma(n_z,n_z)_{p \times p} \end{bmatrix}$$

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