Are there maximally correlated states?

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A measure of total correlations cannot increase under deterministic local operations. We show that, for any number of systems, this condition alone does not guarantee the existence of maximally correlated states. Namely, there is no state that simultaneously maximizes all the measures satisfying it. If, in addition, the measures do not increase with probability unity under local measurements, then such states exist for two systems. They are the maximally entangled states. For a larger number of systems, it depends on their Hilbert space dimensions.

In order to apprehend quantum entanglement, many measures have been introduced [1]. Some of them have clear operational meanings, such as distillable entanglement or entanglement cost [2, 4], others are readily computable, such as negativity [3, 6]. They all have an essential property that makes them proper entanglement quantifiers, which is the monotonicity under local operations and classical communication [1, 3, 7, 8]. More precisely, when a multipartite state is deterministically changed into another using such means, the measure does not have a higher value for the obtained state than for the original one. A measure satisfying this requirement is called entanglement monotone. Similar monotones have been defined for other quantum resources, such as nonuniformity, coherence, or asymmetry [9, 10].

A multipartite state is said to be not more entangled than another when the former can be deterministically obtained from the latter by local operations and classical communication, or, equivalently, when any entanglement monotone is not larger for the former than for the latter. This defines a partial order. Two states can be incomparable, which means that, in going from one to the other, some entanglement monotones increase whereas others decrease. However, there are states that simultaneously minimize all the entanglement monotones. They are the separable states which are the mixtures of product states [17]. Moreover, for bipartite systems, there are so-called maximally entangled states which are not less entangled than any other state on the same Hilbert space [4, 13, 19].

The entanglement ordering of multipartite states is based on deterministic operations. Performing a local measurement and selecting a specific outcome can result in a state more or less entangled than the initial one, with or without classical communication. So, the values of an entanglement monotone for the state before the measurement and for the states after it are not necessarily related in a particular way. However, many familiar entanglement monotones are nonincreasing on average under local measurements [1, 8]. This means that the average of the postmeasurement amounts of entanglement, each weighted with the corresponding measurement probability, is not larger than the premeasurement amount of entanglement. Other quantum resource monotones are nonincreasing on average under appropriate measurements [10, 19].

In this paper, we address the issue of ordering multipartite states according to total correlations. We are, in particular, interested in the possible existence of maximally correlated states. The correlations between systems that do not influence each other cannot increase. The time evolution of independent systems is given by deterministic local operations. Thus, a measure of total correlations cannot increase under such operations [21, 21].

This condition suffices to ensure that any product state is not more correlated than any other state. We name a measure with this property a correlation monotone. As we will see, if no other requirement is imposed, there is no maximally correlated state on any Hilbert space, in stark contrast with the case of entanglement. When only correlation monotones which are nonincreasing on average under local measurements are considered, there are maximally correlated states for bipartite systems. In fact, the same result follows from a weaker condition, namely, that the measures do not increase with probability unity under local measurements. For more than two systems, the existence or not of maximally correlated states for these correlation orderings depends on the Hilbert space dimensions of the systems.

For $N$ systems with Hilbert spaces $\mathcal{H}_n$, a local operation is characterized by Kraus operators of the form

$$K_q = K_q \otimes I_q^\alpha, \quad I_q^\alpha \otimes K_q, \quad \text{or} \quad I_q^\alpha \otimes K_q \otimes I_q^\alpha,$$

(1)

where $I_q^\alpha = \otimes_{m<n} I_m$ and $I_q^\alpha = \otimes_{m>n} I_m$ with $I_n$ the identity operator on $\mathcal{H}_n$. The local operators $K_q$ map $\mathcal{H}_n$ into an Hilbert space $\mathcal{H}_n'$ possibly different from $\mathcal{H}_n$. They form a complete set of Kraus operators, namely, they satisfy $\sum_q K_q^\dagger K_q = I_n$. If $\mathcal{H}_n'$ is the same for every $q$, a deterministic operation $\Lambda$ can be defined. It changes the state $\rho$ of the $N$ systems into $\Lambda(\rho) = \sum_q K_q \rho K_q^\dagger$. As discussed in the introduction, we consider measures $C$, termed correlation monotones, that obey the following Condition.

**Condition 1.** The measure $C$ is nonincreasing under deterministic local operations, i.e., $C(\Lambda(\rho)) \leq C(\rho)$ for any state $\rho$ and deterministic local operation $\Lambda$.

A measure of total correlations can, for example, be a minimal distance to the set of product states, $C(\rho) = \inf_{(\rho_n, \lambda)} D(\rho, \otimes_n \rho_n)$, where the infimum is taken over all the density operators of the considered systems. Provided $D$ satisfies $D(\Lambda(\omega), \Lambda(\omega')) \leq D(\omega, \omega')$ for any quantum operation $\Lambda$, $C$ is a correlation monotone.
Some possible choices for $D$ are the relative entropy, the Hellinger distance, or the Bures distance \cite{22,24}. The above definition gives the total mutual information

$$I(\rho) = \sum_{n=1}^{N} S(\rho^{(n)}) - S(\rho),$$

for the relative entropy \cite{22}. In this expression, $S$ is the von Neumann entropy and $\rho^{(n)} = tr_{n \neq n} \mathcal{H}_n \rho$ is the state of system $n$, where $tr_{\mathcal{H}}$ denotes the partial trace over the Hilbert space $\mathcal{H}$.

Any two product states can be transformed into one another by local operations and so a correlation monotone $C$ assumes the same value for all the product states. Moreover, $C(\rho)$ cannot be smaller than this value since any state $\rho$ can be changed into a product state by local operations. It is usually set to zero as there is no correlation between systems in product states. The above mentioned measures of total correlations vanish for product states. Another characteristic of correlation monotones is that they do not depend explicitly on the Hilbert spaces $\mathcal{H}_n$, in particular on their dimensions. This follows from the Proposition below.

**Proposition 1.** Consider any $r \times r$ Hermitian matrix $M$ with trace unity, $r$ integer vectors $i_s = (i_{s,n})_{n=1}^N$, and correlation monotone $C$.

The value $C(\rho)$ is the same for all the states $\rho = \sum_{s,t=1}^{N} M_{s,t} |i_s\rangle \langle i_t|$, where $M_{s,t}$ denotes the elements of $M$ and $|i_s\rangle = \otimes_{n=1}^N |i_{s,n}\rangle_n$ with $|i\rangle_n$ any orthonormal states of any Hilbert space $\mathcal{H}_n$.

The proofs of the Propositions are given in the Appendix. For pure bipartite states, Proposition \[1\] yields the following. Any such state $|\psi\rangle$ can be written as $|\psi\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i} |i\rangle_1 \otimes |i\rangle_2$, where $r$ is its Schmidt rank, $\lambda_i$ denotes its Schmidt coefficients, and $|i\rangle_n$ are orthonormal states of $\mathcal{H}_n$. The corresponding $r \times r$ matrix $M$ is given by $M_{s,t} = \sqrt{\lambda_s \lambda_t}$ and the $r$ vectors $i_s$ by $i_s = (s,s)$. These vectors are the only Schmidt rank and coefficients of $|\psi\rangle$.

We say that $\rho$ is not more correlated than $\rho'$, according to Condition \[1\] if and only if $C(\rho) \geq C(\rho')$ for any correlation monotone $C$. We denote this relation by $\prec_1 \rho'$. It is a preorder, i.e., $\rho \prec_1 \rho$ and if $\rho \prec_1 \rho'$ and $\rho' \prec_1 \rho''$ then $\rho \prec_1 \rho''$. States $\rho$ and $\rho'$ such that $\rho \prec_1 \rho'$ and $\rho' \prec_1 \rho$, which means that $C(\rho) = C(\rho')$ for any correlation monotone $C$, are said to be equally correlated for $\prec_1$. This is, for instance, the case of two pure bipartite states with identical Schmidt coefficients. States $\rho$ and $\rho'$ such that $C(\rho') - C(\rho)$ is positive for some correlation monotones $C$ and negative for others are said to be incomparable for $\prec_1$. The ordering $\prec_1$ can also be characterized as follows.

**Proposition 2.** Let $\rho$ and $\rho'$ be two states on any $N$-partite Hilbert spaces. The three following statements are equivalent.

i) some deterministic local operations change $\rho'$ into $\rho$, ii) $\rho \sim_1 \rho'$, iii) $C(\rho) \leq C(\rho')$ for any correlation monotone $C$ vanishing only for product states.

For measures of total correlations, the class of correlations monotones which vanish only for product states is of interest. An example is the total mutual information \[2\]. The above Proposition shows that adding this restriction does not change the ordering of the multipartite states.

As seen above, any product state is not more correlated, according to Condition \[1\] than any other state. One can wonder whether, similarly, there are states that simultaneously maximize all the correlation monotones. This may be meaningful only for a given Hilbert space $\mathcal{H} = \otimes_{n=1}^N \mathcal{H}_n$. For instance, for spaces $\mathcal{H}_n$ of identical dimension $d$, the maximum value of the total mutual information \[2\] on $\mathcal{H}$, $N \ln d$, is strictly increasing with $d$. The following general result can be shown.

**Proposition 3.** For any $N$ Hilbert spaces $\mathcal{H}_n$, there is no maximally correlated state on $\mathcal{H} = \otimes_{n=1}^N \mathcal{H}_n$ for $\prec_1$, i.e., no state $\rho'$ on $\mathcal{H}$ such that $\rho \prec_1 \rho'$ for any $\rho$ on $\mathcal{H}$.

To prove the non-existence of maximally correlated states for $\prec_1$ in the particular case $N = 2$, the Proposition below is used.

**Proposition 4.** Two pure bipartite states with the same Schmidt rank have identical Schmidt coefficients or are incomparable for $\prec_1$.

As an example, consider the pure states $|\psi_\eta\rangle$ with Schmidt rank 2 and coefficients $\epsilon$ and $1-\epsilon$. For any $\eta > 0$, which can be chosen as small as wished, there is a correlation monotone $C$ such that $C(|\psi_\eta\rangle) > C(|\psi_\epsilon\rangle)$ for any $\epsilon \neq \eta$.

We now examine specific classes of correlation monotones $C$ for which $C(|\psi_\eta\rangle) > C(|\psi_\epsilon\rangle)$ is a nondecreasing function of $\epsilon$ on $[0,1/2]$. They are defined by the Conditions below. A local measurement is characterized by a complete set of Kraus operators $\tilde{K}_{q,s}$ of the form of eq. \[1\]. When the measurement is performed on systems in the state $\rho$, the probability $p_q$ of outcome $q$ and the corresponding postmeasurement state $\rho_q$ are given by

$$p_q = \sum_{s=1}^{t_q} tr(\rho \tilde{K}_{q,s} \tilde{K}_{q,s}^\dagger), \quad \rho_q = \sum_{s=1}^{t_q} \tilde{K}_{q,s} \rho \tilde{K}_{q,s}^\dagger / p_q.$$

The number $t_q$ of terms in the above sums can depend on $q$. If $t_q = 1$, the measurement is said to be efficient \[25\]. Note that here the codomains of the local operators $\tilde{K}_{q,s}$ and $\tilde{K}_{q',s}$ can be different from one another provided $q' \neq q$. A very common requirement for measures of entanglement or of any quantum resource \[1\] is \[12\] that reads here as follows.

**Condition 2.** The measure $C$ is nonincreasing on average under local measurements, i.e., $\sum_q p_q C(\rho_q) \leq C(\rho)$ for any state $\rho$ and complete set of Kraus operators \[1\], where $p_q$ and $\rho_q$ are given by eq. \[3\].

We also consider the following weaker requirement.
Condition 3. The measure $C$ does not increase with probability unity under local measurements, i.e., $\min_{\rho} C(\rho_q) \leq C(\rho)$ for any state $\rho$ and complete set of Kraus operators $\{(1)\}$, where $\rho_q$ is given by eq. (3).

Clearly, if $C$ satisfies Condition 2 then it also satisfies Condition 3. For correlation monotones, it is enough to impose Condition 2 or Condition 3 for efficient projective measurements, as shown by the Proposition below.

Proposition 5. A correlation monotone $C$ fulfills Condition 2 (Condition 3) if and only if $C$ fulfills it for efficient local measurements whose Kraus operators $\{(1)\}$ are projectors.

Using this Proposition, it can be shown that the total mutual information $\{(2)\}$ satisfies Condition 2 see the Appendix.

It is usual to derive the analogue of Condition 4 for measures of entanglement or of other quantum resources assuming they are convex and obey the analogue of Condition 2 $\{(5)\}$. These assumptions lead here to very particular correlation monotones, as shown by the Proposition below.

Proposition 6. For any $\rho$, if a function of the $\rho$-partite states is convex and nonincreasing on average under local measurements, then it is an entanglement monotone.

We remark that, as a consequence of this Proposition, for any measure $C$ fulfilling Condition 2 there is an entanglement monotone that does not exceed $C$, which is the convex hull of $C$. Condition 3 together with Condition 4 leads to the following result.

Proposition 7. If $C$ is a correlation monotone obeying Condition 3 then, for any pure bipartite state $|\psi\rangle$ with Schmidt rank $m$, $C(|\psi\rangle \langle \psi|) = s((\lambda_s)_{s=1}^{1})$ where $s$ is an entropy, i.e., $s(\lambda) \leq s(\mu)$ when the vector $\lambda$ majorizes the vector $\mu$.

As is well known, a pure bipartite state $|\psi\rangle$ is not more entangled than another one $|\psi'\rangle$ if and only if the vector majorized of $|\psi\rangle$ does not exceed $|\psi'\rangle$. Consequently, due to Proposition 4 a correlation monotone fulfilling Condition 3 necessarily coincides with an entanglement monotone for pure bipartite states.

Similarly to the ordering $\leq$, we define the preorders $\leq_2$, $\leq_3$ with correlation monotones fulfilling Condition 2 and $\leq_3$ with correlation monotones fulfilling Condition 3. These three orderings are related by $\rho \leq_1 \rho' \Rightarrow \rho \leq_3 \rho' \Rightarrow \rho \leq_2 \rho'$. Contrary to $\leq_1$, for $\leq_2$ and $\leq_3$, there are maximally correlated states on bipartite Hilbert spaces. This results from the following Proposition.

Proposition 8. Let $\rho$ be a state on any bipartite Hilbert space $\mathcal{H}$. The following three statements are equivalent. 

i) $\rho$ is maximally entangled on $\mathcal{H}$,

ii) $\rho$ is maximally correlated on $\mathcal{H}$ for $\leq_3$,

iii) $\rho$ is maximally correlated on $\mathcal{H}$ for $\leq_2$.

A state $\rho'$ is maximally entangled on $\mathcal{H}$ if and only if $E(\rho) \leq E(\rho')$ for any entanglement monotone $E$ and state $\rho$ on $\mathcal{H}$. Such states exist for bipartite Hilbert spaces. When the dimension $d_1$ of $\mathcal{H}_1$ is not larger than the dimension $d_2$ of $\mathcal{H}_2$, which can always be assumed without loss of generality, they are necessarily of the form

$$\rho = \sum_{q=1}^{Q} p_q|\bar{q}\rangle \langle \bar{q}| \text{ with } |\bar{q}\rangle = \sum_{i=1}^{d_2} |i\rangle \otimes |q, i\rangle_2 / \sqrt{d_1}, \quad (4)$$

where $|i\rangle_1$ and $|q, i\rangle_2$ are orthonormal states of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and the probabilities $p_q$ sum to unity. This follows from the fact that the only states that maximise the entanglement of formation on $\mathcal{H}$ are given by eq. (4). If $d_2 \geq 2d_1$, $\rho$ can be mixed. Reciprocally, any state $\{(4)\}$ is maximally entangled on $\mathcal{H}$ as it can be transformed into the pure maximally entangled state $|1\rangle$ by the deterministic local operation with Kraus operators $I_1 \otimes \sum_{q=1}^{d_2} |1, i\rangle_2 \langle q, i|$ where $q = 1, \ldots, Q$ and, if $d_2 \neq Q$, $I_1 \otimes (I_2 - \sum_{q=1}^{Q} |q, i\rangle_2 \langle q, i|)$. Note that this operation changes a state given by eq. (4) with any Schmidt coefficients $\lambda_i$, in place of $1/d_1$, into a pure state. As a result of Proposition 8 the maximally correlated states on $\mathcal{H}$ for $\leq_2$ and $\leq_3$ are also the density operators $\{(4)\}$.

If a bipartite correlation monotone $C$ is not maximum, on some Hilbert space, for the maximally entangled states, then, due to Proposition 8, it does not satisfy Condition 3. This means that, for at least one state $\rho$, there is a local measurement such that $C(\rho_q) > C(\rho)$ for every outcome $q$, where $\rho_q$ denotes the postmeasurement states given by eq. 3. This is actually the case for all the maximally entangled states which do not maximise $C$, as we show now. Such a state $\rho$ on $\mathcal{H}$ is given by eq. (4). Assume there are states on $\mathcal{H}$ for which $C$ is larger than $C(\rho)$. Any state is not more correlated, according to Condition 4, than a pure state on the same Hilbert space, see the proof of Proposition 8. So, there are pure states $|\psi\rangle$ of $\mathcal{H}$ with Schmidt coefficients $\lambda_i = 1/d_1$ such that $C(|\psi\rangle \langle \psi|) = c > C(\rho)$. The local measurement with Kraus operators $\tilde{K}_q = \sum_{i=1}^{d_1} \sqrt{\lambda_i} |i\rangle_1 (i + q - 1) \otimes I_2$ where $q = 1, \ldots, d_1$, performed on $\rho$, gives $C(\rho_q) = c$ for every outcome $q$.

Interesting examples of bipartite correlation monotones that do not obey Condition 3 can be defined from Bell inequalities. The relation between Bell nonlocality and entanglement is not straightforward $[17, 29, 30]$. A Bell inequality reads $\sum_{s, t, x, y} \beta_{s, t, x, y}^e p(s, t|x, y) \leq b$ where $\beta_{s, t, x, y}^e$ are real coefficients and $p(s, t|x, y)$ is the probability of the outcomes $s$ and $t$ of the local measurements $x$ and $y$, respectively. When the set of probabilities is Bell local, the left side can approach but not exceed $b$. The largest value of this left side, for a given state $\rho$, is

$$B(\rho) = \sup_{(F(s)_{|#s}|n, r, x s, t, x, y)} \sum_{s, t, x, y} \beta_{s, t, x, y}^{e} \text{ tr} \left( \rho F_{s|x}^{(1)} \otimes F_{t|y}^{(2)} \right),$$

where the supremum is taken over the positive operators $F_{s|x}$ such that $\sum_{s} F_{s|x}^{(n)} = I_n$. The measure $B$ satisfies
Condition since, for any complete set of Kraus operators \( K_q : \mathcal{H}_n \rightarrow \mathcal{H}_n' \) and positive operators \( F_i' \) on \( \mathcal{H}_n' \), summing to the corresponding identity operator, the operators \( F_i = \sum K_q F_i' K_q^\dagger \) are positive and sum to \( I_n \). If \( B(\rho) > b \), the Bell inequality can be violated by \( \rho \). For some Bell inequalities, \( B \) is maximum for partially entangled states \([22]\) and so does not fulfill Condition. It follows from the above discussion that a maximal violation of the Bell inequality can nevertheless be achieved with a maximally entangled state \( \rho \) by first performing a local measurement on it, and for every outcome of the measurement. Note that this does not even suppose that \( B(\rho) > b \). In the context of Bell nonlocality, performing a local measurement and selecting an outcome is termed filtering.

For more than two systems, the existence or not of maximally correlated states for \( <_2 \) or \( <_3 \) depends on the dimensions of the Hilbert spaces \( \mathcal{H}_n \). The two following Propositions can be proved.

**Proposition 9.** Consider any \( N \) Hilbert spaces \( \mathcal{H}_n \) of respective dimensions \( d_n \) and denote by \( \mathcal{F} \) the set of the nonempty subsets \( \mathcal{E} \) of \( \{1, \ldots, N\} \) such that \( \Pi_{n \in \mathcal{E}} d_n \leq \sqrt{d} \) where \( d = \prod_{n=1}^N d_n \).

- If \( \rho \) is a maximally correlated state on \( \otimes_{n=1}^N \mathcal{H}_n \) for \( <_2 \) or \( <_3 \), then, for any \( \mathcal{E} \in \mathcal{F} \), \( \text{tr}_{\otimes_{n \in \mathcal{E}} \mathcal{H}_n} \rho \) is the maximally mixed state on \( \otimes_{n \in \mathcal{E}} \mathcal{H}_n \).
- If, moreover, \( \max_{\mathcal{E} \in \mathcal{F}} \Pi_{n \in \mathcal{E}} d_n > \sqrt{d}/2 \), then \( \rho \) is pure.

For identical dimensions \( d_n \), the pure states with the property stated in the above Proposition are known as absolutely maximally entangled states \([31, 33]\). Furthermore, in this case, the above condition on the dimensions is always satisfied for even \( N \) and never for odd \( N \). For \( d_n = 2 \), absolutely maximally entangled states do not exist when \( N = 4 \) \([31]\) and \( N \geq 8 \) \([32]\). So, due to Proposition \([3]\) for \( N \) two-level systems, there is no maximally correlated state neither for \( <_2 \) nor for \( <_3 \) when \( N \) is even and different from 6. Using the results of Ref.\([34]\), such states can be shown to exist on the Hilbert spaces considered in the Proposition below.

**Proposition 10.** Let \( \mathcal{H}_n \) be any \( N \) Hilbert spaces of respective dimensions \( d_n \). If \( d_N \geq d \) where \( d = \prod_{n=1}^N d_n \), then there are maximally correlated states on \( \mathcal{H} = \otimes_{n=1}^N \mathcal{H}_n \) for \( <_2 \) and \( <_3 \), which are also maximally entangled on \( \mathcal{H} \) and are of the form

\[
\sum_{i_1 \ldots, i_N=1}^{d_1 \ldots, d_N} \otimes_{n=1}^{N-1} |i_n\rangle_n \otimes |i_1, \ldots, i_{N-1}\rangle_N / \sqrt{d},
\]

where \( |i\rangle_n \) and \( |i, j, \ldots\rangle_N \) are orthonormal states of \( \mathcal{H}_n \) with \( n < N \) and \( \mathcal{H}_N \), respectively.

In summary, we have shown that there is no maximally correlated state when the measures of total correlations are only required to be correlation monotones. In contrast, there exist such states if, in addition, they do not increase with probability unity under local measurements. In this case, there are always maximally correlated states for two systems, which are the corresponding maximally entangled states. For a larger number of systems, two opposite examples have been considered. We have furthermore seen that a correlation monotone fulfilling the above mentioned condition necessarily coincides with an entanglement monotone for pure bipartite states. Bell inequalities maximally violated by partially entangled states have also been discussed. Correlation monotones that can increase with probability unity under local measurements have been defined from them. It would be of interest to determine whether the existence of maximally correlated states can follow from a condition weaker than that considered here.

**Appendix: Proofs**

In this Appendix, we prove the results discussed in the main text.

**Proof of Proposition 1**

Consider the states \( \rho_k = \sum_{s,t=1}^r M_{s,t} |i_{s}\rangle_{kk} |i_{t}\rangle \) where \( k = 1 \) or 2 and \( |i_{s}\rangle = \otimes_{n=1}^N |i_{s,n}\rangle_{k,n} \) with \( |i\rangle_{k,n} \) any orthonormal states of any Hilbert space \( \mathcal{H}_{k,n} \). Let us define the Hilbert spaces \( \mathcal{H}_n = \mathcal{H}_{1,n} \otimes \mathcal{H}_{2,n} \). The states \( \rho_1 \) and \( \rho_1' = \rho_1 \otimes |i_2\rangle \otimes |i_2\rangle \) can be changed into each other by the local transformations \( \rho \rightarrow \rho \otimes |i_1, n\rangle_2, n_2, \langle i_1, n| \) and \( \rho \rightarrow \text{tr}_{\mathcal{H}_{2,n}} \rho \), and so \( C(\rho_1) = C(\rho_1') \). The state \( \rho_1 \) is changed into \( \rho_2 = |i_1\rangle_1 \langle i_1| \otimes _\rho_2 \) by applying the \( N \) operations with Kraus operators \( K^{(n)} = \otimes_{m \neq n} I_m \otimes U_n \) where \( U_n \) is the identity operator on \( \mathcal{H}_n \) and \( U_n \) is a unitary operator on \( \mathcal{H}_n \) such that \( |i_1, 1, n\rangle_2 = |i_1, 1, n\rangle_2, |j\rangle_1 \langle j|_{2,n} \). The state \( \rho_2 \) is transformed back into \( \rho_1 \) by applying the \( N \) operations with Kraus operators \( (K^{(n)})^\dagger \), and hence \( C(\rho_2) = C(\rho_1) \), which leads to \( C(\rho_2) = C(\rho_1) \) and finishes the proof.

**Proof of Proposition 2**

We first show that (i) implies (ii). If \( \rho' \) can be changed into \( \rho \) by deterministic local operations, there are \( N \) such operations \( \Lambda^{(n)} \), each corresponding to a system \( n \), such that \( \rho = \otimes_{m=1}^N \Lambda^{(m)}(\rho') \). For any \( \Lambda \) obeying Condition 1, \( C[\Lambda^{(N)}(\rho)] \leq C(\rho) \) and \( C[\otimes_{m=1}^N \Lambda^{(m)}(\rho') \leq C[\otimes_{m=1}^N \Lambda^{(m)}(\rho')] \) for \( n = 2, \ldots, N \), and so \( C(\rho) \leq C(\rho') \).

The implication (ii)\(\Rightarrow\) (iii) follows directly from the definition of the preorder \( < \).

We finally show that not (i) implies not (iii). Let \( \rho \) and \( \rho' \) be two states such that \( \rho' \) cannot be changed into \( \rho \) by deterministic local operations. Consequently, \( \rho \) is not a product state. Assume first that \( \rho' \) is a product state, and define a function \( C \) as follows. For any state \( \rho, C(\rho) = 0 \) if \( \rho \) is a product state and \( C(\rho) = c > 0 \) otherwise. In particular, \( C(\rho') = 0 \) and \( C(\rho) = c \). Consider
any state $\tilde{\rho}$ and any deterministic local operation $\Lambda$. If $C(\tilde{\rho}) = c$ then $C[\Lambda(\tilde{\rho})] \leq C(\tilde{\rho})$ is trivially fulfilled. If $C(\tilde{\rho}) = c > 0$ then $\tilde{\rho}$ can be changed into $\tilde{\rho}$ by local operations and $\tilde{\rho}$ is not a product state, and $\tilde{\rho}$ vanishes only for product states. Consider any state $\tilde{\rho}$ and any deterministic local operation $\Lambda$. If $C(\tilde{\rho}) = c$ then $C[\Lambda(\tilde{\rho})] \leq C(\tilde{\rho})$ is trivially fulfilled. If $C(\tilde{\rho}) = c > 0$ then $\tilde{\rho}$ can be changed into $\tilde{\rho}$ by local operations, and hence also into $\Lambda(\tilde{\rho})$, which gives $C[\Lambda(\tilde{\rho})] \leq c$. If $C(\tilde{\rho}) = 0$, then $\tilde{\rho}$ is a product state, and so is $\Lambda(\tilde{\rho})$, and hence $C[\Lambda(\tilde{\rho})] = 0$.

Assume now that $\rho'$ is not a product state. Since any state can be transformed into any product state by local operations, a function $C$ can be defined as follows. For any state $\tilde{\rho}$, $C(\tilde{\rho}) = 0$ if $\tilde{\rho}$ is a product state, $C(\tilde{\rho}) = c' > 0$ if $\rho'$ can be changed into $\tilde{\rho}$ by local operations and $\tilde{\rho}$ is a product state, and $C(\tilde{\rho}) = c > c'$ otherwise. In particular, $C(\rho') = c'$, $C(\rho) = c$, and $\rho$ vanishes only for product states. Consider any state $\tilde{\rho}$ and any deterministic local operation $\Lambda$. If $C(\tilde{\rho}) = c$ then $C[\Lambda(\tilde{\rho})] \leq C(\tilde{\rho})$ is trivially fulfilled. If $C(\tilde{\rho}) = c'$, then $\tilde{\rho}$ can be changed into $\tilde{\rho}$ by local operations, and hence also into $\Lambda(\tilde{\rho})$, which gives $C[\Lambda(\tilde{\rho})] \leq c$. If $C(\tilde{\rho}) = 0$, then $\tilde{\rho}$ is a product state, and so is $\Lambda(\tilde{\rho})$, and hence $C[\Lambda(\tilde{\rho})] = 0$.

**Proof of Proposition 3**

Assume there is $\rho'$ on $H$ such that $\rho \prec \rho'$ for any $\rho$ on $H$. We first treat the case $N > 2$. Define the measures $C_{\{x,y\}}(\rho)$, where $\{x,y\} \subset \{1, \ldots, N\}$, as follows. Consider any $N$-partite state $\rho = \sum_{i} M_{s,i} |i_s,i_n\rangle\langle i_s,i_n|$, where $|i_s\rangle = \otimes_{n=1}^{N} |i_{s,n}\rangle$ with $|i_{s,n}\rangle$ any orthonormal states of any Hilbert space $H_n$, and the corresponding reduced density operator for any two systems $x$ and $y$,

$$\rho_{\{x,y\}} = \sum_{i} M_{s,i} \prod_{n \neq x,y} \delta_{i_{s,n},i_{s,n}'} |i_{s,x},i_{s,y}\rangle\langle i_{s,x},i_{s,y}|,$$

where $|i_{s,x},i_{s,y}\rangle = |i_{s,x}\rangle \otimes |i_{s,y}\rangle$, and let $C_{\{x,y\}}(\rho) = E_f(\rho_{\{x,y\}})$, where $E_f$ is the entanglement of formation, which is a correlation monotone. The above expression for $\rho_{\{x,y\}}$ makes apparent that, due to Proposition 1, $C_{\{x,y\}}(\rho)$ depends only on the matrix elements $M_{s,t}$ and the integer vectors $i_s$. For deterministic local operations $\Lambda$ acting on system $x$ or system $y$, with local Kraus operators $K_q$, $C_{\{x,y\}}[\Lambda(\rho)] = E_f[\Lambda(\rho_{\{x,y\}})]$ where $\Lambda$ is the operation with Kraus operators $K_q \otimes I_y$ or $I_x \otimes K_q$, respectively. For all the other deterministic local operations, $C_{\{x,y\}}[\Lambda(\rho)] = E_f[\rho_{\{x,y\}}]$. Consequently, $C_{\{x,y\}}(\rho)$ is a correlation monotone, and hence $C_{\{x,y\}}(\rho) \leq C_{\{x,y\}}(\rho')$ for any $\rho$ on $H$.

For any $\rho$ on $H_x \otimes H_y$, the above inequality with $\rho = \otimes_{n \neq x,y} \rho_n \otimes \rho /\Pi_{n \neq x,y} d_n$, where $d_n$ is the dimension of the Hilbert space, shows that $E_f(\rho) \leq E_f(\rho')$. In other words, $\rho_{\{x,y\}}'$ maximizes $E_f$ on $H_x \otimes H_y$. Thus, assuming, without loss of generality, that $d_x \leq d_y$, it reads

$$\rho'_{\{x,y\}} = \sum_{q} \rho_{q} |q\rangle\langle q|$$

where $|q\rangle_x$ and $|q\rangle_y$ are orthonormal states of $H_x$ and $H_y$, respectively, and so $\rho' = \sum_{q,q'} \langle q\rangle |q\rangle |q'\rangle \otimes \Omega_{q,q'}$, where $\Omega_{q,q'}$ are operators on $\otimes_{n \neq x,y} H_n$ such that $tr \Omega_{q,q'} = p_q \delta_{q,q'}$. Consider now $\rho'_{\{x,z\}}$ where $z \neq y$.

From the above expression for $\rho'$, it follows that $\rho'_{\{x,z\}} = d_x^{-1} I_x \otimes tr_{\otimes x,y,z} \sum \Omega_{q,q'}$ and thus $E_f(\rho'_{\{x,z\}}) = 0$, which contradicts that $E_f$ is maximum on $H_x \otimes H_z$ for $\rho'_{\{x,z\}}$.

In the particular case $N = 2$, since $\rho'$ maximizes the correlation monotone $E_f$ on $H$, it is of the form of eq. (5) with $x = 1$ and $y = 2$, assuming, without loss of generality, that $d_1 \leq d_2$ [18]. The operation with Kraus operators $\sqrt{\mu_1} K_1 \otimes U_2$ where $U_2$ is a unitary operator on $H_2$ such that $|q, i\rangle_2 = U_2 |1, i\rangle_2$ transforms $|1\rangle|1\rangle$ into $\rho'$, and so $\rho' \prec_1 |1\rangle|1\rangle$. This implies that $|1\rangle|1\rangle$ is a pure maximally correlated state on $H$ for $\prec_1$, which is not possible, due to Proposition 4.

**Proof of Proposition 4**

Consider two pure bipartite states $|\psi\rangle$ and $|\psi'\rangle$ with the same Schmidt rank $d$ such that $|\langle \psi' | \psi\rangle| = |\langle \psi' | \psi\rangle|$. Denote $\lambda$ the Schmidt coefficients of $|\psi\rangle$. Define $\phi = \sum_{i=1}^{d} \sqrt{\lambda_i} |i\rangle \otimes |i\rangle$ where $\{|i\rangle\}$ is an orthonormal basis of an Hilbert space $H_0$ of dimension $d$, $\phi' = \phi \otimes H_0$ whose Schmidt coefficients are those of $|\phi'\rangle$, $\rho' = |\phi'\rangle \langle \phi'|$. From Proposition 1, it follows that $\rho \prec \rho'$. Thus, due to Proposition 2, there are two deterministic local operators $\Lambda_n$, each corresponding to a system $n$, such that $\rho = \Lambda_1 \otimes \Lambda_2(\rho')$. We denote the eigenstates of $\Lambda_2(\rho')$ by $|\tilde{\rho}\rangle$. As $\Lambda_1$ is linear and $\rho$ is pure, the above equality gives $\rho = \Lambda_1(|\tilde{\rho}\rangle \langle \tilde{\rho}|)$ for any $\rho$. Kraus operators of $\Lambda_1$ are of the form $K_q \otimes I_0$ where $I_0$ is the identity operator on $H_0$ and the linear maps $K_q : H_0 \to H_0$ are such that $\sum_q K_q^\dagger K_q = I_0$. The state $|\tilde{\rho}\rangle$ can be written as $|\tilde{\rho}\rangle = \sum_{i=1}^{d} \sqrt{\mu_i} |i\rangle \otimes |i\rangle$ where $\{|i\rangle\}$ and $\{|i\rangle\}$ are orthonormal bases of $H_0$ and some coefficients $\mu_i$ may vanish. Since $\Lambda_1$ changes $|\tilde{\rho}\rangle$ into the pure state $\rho$, one has, for any $q$ and $i$,

$$\sqrt{\mu_i} K_q |i\rangle = \sqrt{\mu_q} \sum_{j=1}^{d} \sqrt{\lambda_j} \langle i|j|\rangle |j\rangle,$$

where $p_q = \langle \tilde{\rho} | K_q \otimes I_0 | \tilde{\rho}\rangle$. Since $\lambda_j \neq 0$ for any $j$ and $\{|i\rangle\}$ is a basis of $H_0$, the above sum is nonzero for any $i$. Moreover, at least one $p_q$ is nonvanishing. So, $\mu_i \neq 0$ for any $i$, and hence $K_q = \sqrt{\mu_q} K$ where $K : H_0 \to H_0$ is independent of $q$. It is necessarily a unitary operator, and so $\Lambda_1$ is a unitary operation, and $\Lambda_2(\rho')$ a pure state with Schmidt coefficients $\lambda_l$. By following the same steps as above, with $\Lambda_2(\rho')$, $\Lambda_1$, and $|\tilde{\rho}\rangle$ in lieu of, respectively, $\rho$, $\Lambda_1$, and $|\tilde{\rho}\rangle$, one can shown that $\Lambda_2$ is also a unitary operation and the Schmidt coefficients of $|\psi'\rangle$ are $\lambda_l$. 


Proof of Proposition 5

Let $\rho$ be any state on any multipartite Hilbert space $H_1 \otimes \cdots \otimes H_n$. It is enough to consider a measurement characterized by a complete set of Kraus operators of the form $K_{q,s} = K_{q,s} \otimes I_r^s$ where $K_{q,s} : H_1 \to H_1(q,s)$, $q = 1, \ldots, d$, and $s = 1, \ldots, r$. Let $H'_1$ be an Hilbert space of dimension larger than those of the spaces $H_1(q,s)$, and define $K_{q,s}' : H_1 \to H'_1(q,s)$ by $(j'|K_{q,s}'|i) = (j|K_{q,s}|i)$ for $j$ not larger than the total mutual information fulfills Condition 2

The total mutual information can be written as $I(\rho) = \min_\delta S(\rho||\delta)$ where $S(\rho||\delta)$ is the quantum relative entropy and the minimum is taken over all the product states on the same Hilbert space as $\rho$ [22]. For any states $\rho$ and $\delta$ on any Hilbert space $H$ and Kraus operators $K_q$ such that $\sum_q K_q^\dagger K_q$ is the identity operator on $H$, it can be shown that $S(\rho||\delta) \geq \sum_q p_q S(\rho_q||\delta_q)$ where

$$p_q = \text{tr}(K_q^\dagger K_q), \quad \rho_q = K_q \rho K_q^\dagger / p_q,$$

and $\delta_q$ is given by a similar expression with $\delta$ in place of $\rho$ [10]. If $H = \otimes_{n=1}^N H_n$, $\delta$ is a product state on $H$, and the operators $K_q$ are of the form

$$K_q = K_q \otimes I_r^q, \quad I_r^{q,s} \otimes K_q, \quad \text{or} \quad I_r^q \otimes K_q \otimes I_r^s,$$

then the density operators $\delta_q$ are also product states. Thus, the above expression for the total mutual information gives $S(\rho||\delta) \geq \sum_q p_q I(\rho_q)$, which leads to the result.

Proof of Proposition 6

Consider a function $C$ of the $N$-partite states convex and nonincreasing on average under local measurements. Let us first show that $C$ is a correlation monotone. Let $\rho$ be any $N$-partite state and $\Lambda$ any deterministic local operation with Kraus operators $\tilde{K}_q$. The convexity of $C$ yields $C[\Lambda(\rho)] \leq \sum_q p_q C(\rho_q)$ where $p_q$ and $\rho_q$ are given by eq. (6). So, since $C$ satisfies Condition 2, it also satisfies Condition 1. An entanglement monotone is nonincreasing under state transformations involving only local operations and classical communication. Such a transformation is a sequence of maps involving only local operations and one-way classical communication. Thus, it is enough to show that $C[\Lambda(\rho)] \leq C(\rho)$ for any map $\Lambda$ given by $\Lambda(\rho) = \sum_q \Lambda(q)(\tilde{K}_q \rho \tilde{K}_q^\dagger)$ with a complete set of Kraus operators $K_q = K_q \otimes I_r^q$ and deterministic local operations $\Lambda(q)$ whose Kraus operators are of the form $I_r \otimes K_q \otimes I_r^q$. Since $C$ is convex and obeys Condition 1, and $C[\Lambda(\rho)] \leq \sum_q p_q C(\rho_q)$ where $p_q$ and $\rho_q$ are given by eq. (6). The wished inequality follows immediately from Condition 2.

Proof of Proposition 8

Due to Proposition 1, for any pure bipartite state $\rho = |\psi\rangle\langle\psi|$, $C(\rho) = s(\lambda)$ where $\lambda = (\lambda_i)_{i=1}^r$ with the Schmidt rank $r$ and coefficients $\lambda_i$ of $|\psi\rangle$. Consider any vector $\lambda'$ majorized by $\lambda$ [27]. Any pure state $\rho'$ with Schmidt coefficients $\lambda'_i$ can be changed into $\rho$ by local operations and one-way classical communication, i.e., $\rho = \sum_q \Lambda(q)(\tilde{K}_q \rho \tilde{K}_q^\dagger)$ with a complete set of Kraus operators $K_q = K_q \otimes I_r^q$ and deterministic local operations $\Lambda(q)$ whose Kraus operators are of the form $I_r \otimes K_q \otimes I_r^q$. Since $\rho$ is pure and $\Lambda(q)$ linear, $\rho = \Lambda(q)(\rho'_q)$ for any $q$, where $\rho'_q$ is given by eq. (10) with $\rho'$ in place of $\rho$. Thus, $C(\rho) \leq C(\rho'_q)$ for any $q$, and so $C(\rho) \leq C(\rho')$, which can be rewritten as $s(\lambda) \leq s(\lambda')$.

Proof of Proposition 7

Assume, without loss of generality, that the dimension $d_1$ of $H_1$ is not larger than the dimension $d_2$ of $H_2$. We first show that (i) implies (ii). Let $\rho'$ be a pure maximally entangled state on $H = H_1 \otimes H_2$, i.e., with Schmidt coefficients $1/d_1$. Consider any state $\rho$ on $H$ and denote its eigenvalues by $\mu_q$ and the corresponding eigenstates by $|\psi_q\rangle$. Let us introduce the Hilbert spaces $H'_1$ and $H'_2 = H_2 \otimes H_2$. Provided the dimension of $H'_2$ is not smaller than $d_1 d_2$, $\rho$ can be expressed as $\rho = \text{tr}_{H'_2} |\psi\rangle\langle\psi|$ with orthonormal eigenstates $|\psi_q\rangle$ of $H'_2$. The state $|\psi\rangle$ can also be written as $|\psi\rangle = \sum_{i=1}^{d_1 d_2} \sqrt{\lambda_i} |i_1\rangle \otimes |i_2\rangle$ where $|i_1\rangle$ and $|i_2\rangle$ are orthonormal states of $H_1$ and $H_2$, respectively, and $r \leq d_1$. The vector $(\lambda_i)_{i=1}^{d_1 d_2}$ majorizes $(1/d_1, \ldots, 1/d_1)$ [27]. Thus, $\rho'$ can be changed into the pure state $\tilde{\rho} = |\psi\rangle\langle\psi|$ by local operations and one-way classical communication [28]. Consequently, for any measure $C$ fulfilling Conditions 1 and 3, $C(\tilde{\rho}) \leq C(\rho')$, see the proof of Proposition 7, which
means $\rho \prec_3 \rho'$. From the relation between the orderings $\prec_1$ and $\prec_3$ and the fact that $\text{tr}_{H_2}'$ is a local operation on $H_2$, it follows that $\rho \prec_3 \tilde{\rho}$, and so $\rho \prec_3 \rho'$. If there are mixed maximally entangled states on $H$, they can be transformed into pure maximally entangled states on $H$ by local operations, as shown in the main text, and are hence maximally correlated on $H$ for $\prec_3$.

The implication (ii)$\Rightarrow$(iii) directly follows from the fact that any two states $\rho$ and $\rho'$ such that $\rho \prec_3 \rho'$ necessarily satisfy $\rho \prec_2 \rho'$.

Consider a maximally correlated state $\rho'$ on $H$ for $\prec_2$ and a maximally entangled state $\rho$ on $H$. For any correlation monotone $C$ obeying Condition 2, $C(\rho) \leq C(\rho')$.

The entanglement of formation $E_f$ is such a measure, and so $E_f(\rho') = E_f(\rho)$. Consequently, $\rho'$ is given by eq. (18), and is thus maximally entangled on $H$, as shown in the main text.

Proof of Proposition 9

Consider any $N$ Hilbert spaces $H_n$ and any nonempty set $E \subseteq \{1, \ldots, N\}$, and define the measure $C_E$ as the entanglement of formation $E_f$ for the bipartition $H = H^e \otimes H^d$ where $H = \otimes_{n \in E} H_n$, $H^e = \otimes_{n \notin E} H_n$ and $H^d = \otimes_{n \notin E} H_n$. It is a correlation monotone fulfilling Condition 2, since $E_f$ is, and operators $\{\hat{K}_q\}$ can be rewritten as $\hat{K}_q = I^e \otimes \hat{K}_q$ or $\hat{K}_q = \hat{K}_q \otimes I^d$ where $\hat{K}_q$ acts on $H^d$ or $H^e$, respectively, and $I^e$ and $I^d$ are the identity operators on $H^e$ and $H^d$, respectively. Assume there is a maximally correlated state $\rho$ on $H$ for $\prec_2$ or $\prec_3$. It maximizes all the monotones $C_E$ on $H$. If $E \subseteq F$, the dimension $d^e$ of $H^e$ is not larger than the dimension $d^d = d/\sqrt{d}$ of $H^d$. Thus, $\rho$ is given by eq. (5) with $d_x$ replaced by $d^e$, the states $\{\lambda_i\}_n$ by orthonormal states of $H^e$, and the states $\{|i\}_n$ by orthonormal states of $H^d$. Consequently, $\text{tr}_{H^e}' \rho$ is the maximally mixed state on $H^e$. Moreover, $\rho$ is necessarily pure if $d^e < 2d^d$.

Proof of Proposition 10

Consider any state $\rho$ on $H = \otimes_{n=1}^N H_n$. Let us introduce the Hilbert spaces $H_N$ and $H'_N = H_N \otimes H'_N$. Provided the dimension of $H_N'$ is not smaller than that of $H$, $\rho$ can be expressed as $\rho = \text{tr}_{H_N'} \psi(\psi|\psi)$, where $|\psi\rangle \in \otimes_{n=1}^{N-1} H_n \otimes H_N$, see the proof of Proposition 8. The state $|\psi\rangle$ can always be written as

$$|\psi\rangle = \sum_{i_1,\ldots,i_{N-1}=1}^{d_1,\ldots,d_{N-1}} \lambda_{i_1,\ldots,i_{N-1}} \otimes_{n=1}^{N-1} |i_n\rangle \otimes |i_1,\ldots,i_{N-1}\rangle_N,$$

where $|i_n\rangle$ are orthonormal states of $H_n$, and $|i_1,\ldots,i_N\rangle_N$ are normalized states of $H_N$. We denote by $\mathcal{H}_N$ a subspace of $H_N$ of dimension $d = \Pi_{n=1}^{N-1} d_n$ that contains the states $|i_1,\ldots,i_N\rangle_N$. From the relation between the orderings $\prec_1$ and $\prec_3$ and the fact that $\text{tr}_{H_N^e}'$ is a local operation on $H_N$, it follows that $\rho \prec_3 \tilde{\rho}$ where $\tilde{\rho} = |\psi\rangle \langle \psi|\psi\rangle$.

Consider $\rho' = |\psi\rangle \langle \psi|\psi\rangle$ and $\tilde{\rho}' = |\tilde{\psi}\rangle \langle \tilde{\psi}|\tilde{\psi}\rangle$ where $|\psi\rangle$ and $|\tilde{\psi}\rangle$ are given by the above expression with $\lambda_{i_1,\ldots,i_{N-1}} = 1/\sqrt{d}a$ and orthonormal states $|i_1,\ldots,i_N\rangle_N$ of $H_N$ and $H'_N$, respectively. The states $\rho'$ and $\tilde{\rho}'$ are maximally correlated for $\prec_1$, and so also for $\prec_3$, see Proposition 1. There is a set of Kraus operators $M_q = \otimes_{n=1}^{N-1} U_q(n) \otimes K_q$ where $U_q(n)$ denotes a unitary operator on $H_n$ and the linear maps $K_q : H_N^e \rightarrow H_N$ are such that $\sum_q K_q K_q^\dagger = I_N$ with $I_N$ the identity operator on $H_N$ for which $|\psi\rangle = \sqrt{d}M_q |\tilde{\psi}\rangle$. This equality can be rewritten as $\tilde{\rho} = U_q \otimes I_N \tilde{\rho}' U_q^\dagger \otimes I_N$ where $U_q = \otimes_{n=1}^{N-1} U_q(n)$ and $\tilde{\rho}'$ is given by eq. (5) with $K_q = I_N^e \otimes K_q$ and $\tilde{\rho}'$ in place of $\rho$. Thus, for any correlation monotone fulfilling Condition 3, $C(\tilde{\rho}) = C(\tilde{\rho}')$ for any $q$, and so $C(\tilde{\rho}) \leq C(\tilde{\rho}')$. This means $\tilde{\rho} \prec_3 \tilde{\rho}'$, and hence $\rho \prec_3 \rho'$ and, due to the relation between the orderings $\prec_3$ and $\prec_2$, $\rho \prec_2 \rho'$.
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