ON UNIFORMIZABLE REPRESENTATION FOR ABELIAN INTEGRALS

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Abstract. We show how the uniformizable representation for Abelian integrals of nontrivial genera arise. The technique makes use of the famous Chudnovsky’s Fuchsian linear differential equations and their relation to the sixth Painlevé transcendent.

1. Introduction

In this note we exhibit first examples of uniformizable $\tau$-representation for functions having an additive automorphic property, i.e., Abelian integrals on 1-dimensional orbifolds of a negative constant curvature. Orbifolds are generalizations of a notion Riemann surface for the case when fundamental group of this ‘manifold’ has elements of finite or formally infinite order. These are called usually conic singularities or punctures. Matrix representations for fundamental groups of 1-dimensional orbifolds are described by Fuchsian linear ordinary differential equations (ODEs) of 2nd order. Their singularities are precisely the conic points on these orbifolds.

In work [1] we showed how Fuchsian equations

$$p \Psi'' + p' \Psi' + (x + A) \Psi = 0, \quad p := x(x - \alpha)(x - \beta)$$

arise in the theory of Painlevé equations and in note [2] we announced a first explicit example of a solvable allied Schwarz ODE

$$[u, \tau] = -2 \varphi(2u), \quad u(\tau) = \frac{\varphi_3(\tau)}{\varphi_2(\tau)} \cdot 2F_1\left(\frac{1}{2}, \frac{1}{4}, \frac{5}{4}, \varphi_2(\tau) \cdot \varphi_4(\tau)\right),$$

where $u$ is an elliptic holomorphic integral. Below are some details on these examples and methods of getting explicit formulae.

2. Schwarz equation and equations on tori

As is well known Schwarz’s equations are the 3rd order ODEs coming from linear ODEs of the form $\Psi_{xx} = \frac{1}{2} Q(x) \Psi$. The ratio of its two linearly independent solutions

$$\tau = \frac{\Psi_1(x)}{\Psi_2(x)}$$

defines $\tau$ as function of $x$ and conversely. We may write down an autonomic ODE defining $x$ as function of $\tau$. If, as usual,

$$\{f, z\} := \frac{f_{zz}}{f_z} - \frac{3}{2} \frac{f_{xx}}{f_x^2},$$

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defines the standard Schwarz derivative then the inverted function \( x = \chi(\tau) \) satisfies the ODE

\[
[x, \tau] = Q(x),
\]

wherein the following notation has been adopted \([x, \tau] := -\{\tau, x\}\), that is

\[
[x, \tau] := \frac{x_{\tau\tau\tau}}{x_\tau^3} - \frac{3}{2} \frac{x_\tau^2}{x_\tau^3}.
\]

We shall deal with the two examples of Fuchsian equations of the class (1):

\[
\begin{align*}
x(x-1)(x+1)\Psi'' + (3x^2 - 1)\Psi' + (x + 0)\Psi &= 0, \\
x(x^2 + 3x + 3)\Psi'' + (3x^2 + 6x + 3)\Psi' + (x + 1)\Psi &= 0,
\end{align*}
\]

They belong to the set of four equations known as Chudnovsky’s ones [3]. With the help of well-known linear transformation \( \Psi \mapsto \psi \) we can transform these equations into the normal form

\[
\begin{align*}
\psi'' &= \frac{1}{4} \frac{(x^2 + 1)^2}{x(x - 1)(x + 1)^2} \psi, \\
\psi'' &= \frac{1}{4} \frac{(x + 1)(x + 3)(x^2 + 3)}{x^2(x^2 + 3x + 3)^2} \psi.
\end{align*}
\]

It is known that each of these equations has the four parabolic singularities on Riemann sphere and, thereby, define two punctured spheres; the simplest orbifolds of genus zero. Halphen used original trick to relate these equations with elliptic functions corresponding to algebraic curves of the form \( y^2 = x(x-1)(x+1) \) and \( y^2 = x(x^2 + 3x + 3) \). They have the standard Weierstrassian models

\[
\begin{align*}
y^2 &= 4x^3 - 4x \quad \text{and} \quad y^2 &= 4x^3 - 4
\end{align*}
\]

respectively, i.e., Gauss’ lemniscate and the equi-anharmonic elliptic curve.

Let us make the substitutions \( x = \wp(u; g_2, g_3) \) in equations (3) and (4), where invariants \( g_2, g_3 \) are chosen according to the Weierstrassian models above. We shall obtain Fuchsian equations in variable \( u \) and, then, Schwarz’s equation of the form (2). An easy computation gives

\[
[u, \tau] = -2\wp(2u; g_2, g_3).
\]

and this equation is equivalent to equations on tori considered for the first time in classical work [5]. Thus, if we find solutions \( u = u(\tau) \) to this equation we obtain nontrivial examples of uniformizable representation for the object \( u \) which is an Abelian integral, because

\[
u = \wp^{-1}(x; g_2, g_3).
\]

On the other hand, above mentioned Fuchsian and Schwarz’s equations correspond to orbifolds with punctures and therefore their curvature is not equal to zero but is a negative constant. The theory of such orbifolds is very nontrivial. Automorphic functions on them (Klein’s Hauptmoduln) are known, a few as they are, but examples of additively automorphic objects are absent hitherto.
3. Holomorphic elliptic integrals and hypergeometric functions

3.1. Llemniscate. We have in this case

\[ \pm u = \int_{\infty}^{x} \frac{du}{\sqrt{4u^3 - 4u}} = \ldots \]

It follows that

\[ \ldots = \frac{1}{2} \int_{\infty}^{x} u^{-\frac{1}{2}} (u^2 - 1)^{-\frac{1}{2}} du \]

which is a particular case of a consequence of the integral definition to the hypergeometric \( \text{}_2F_1 \)-function. More precisely, the standard integral definition of the \( \text{}_2F_1 \)-functions is through the definite integral \[ \text{Sect. 2.1.3} \]

\[ \text{}_2F_1(\alpha, \beta; \gamma | z) := \frac{\Gamma(\gamma)}{\Gamma(\beta) \Gamma(\gamma - \beta)} \int_{0}^{1} u^{\beta-1} (1-u)^{\gamma-\beta-1} (1-zu)^{-\alpha} du \]

but in some cases this definition may be rewritten in terms of non-definite integrals. Changing the integration variable \( u \mapsto zu \), the upper limit \( u = 1 \) gets mapped to \( u = z \). Putting further \( \gamma - \beta - 1 = 0 \), we arrive at a definition of this particular case for the \( \text{}_2F_1 \)-function through the indefinite integral. Clearly, all the other cases are in fact variations of this scheme\(^1\). We obtain (see also tables in \[17\])

\[ \int_{0}^{z} u^{\alpha-1} (u-1)^{-\beta} du = \frac{e^{\pi i \beta}}{\alpha^\beta} (z^\alpha - 2F_1(\beta, \alpha; \alpha + 1 | z), \quad \Re(\alpha) > 0, \quad (6) \]

and, therefore,

\[ \int_{\infty}^{z} u^{\alpha-1} (u-1)^{-\beta} du = \frac{z^{\alpha-\beta}}{\alpha - \beta} (z^\alpha - 2F_1(\beta - \alpha; \beta - \alpha + 1 | z^{-1}), \quad \Re(\beta - \alpha) > 0. \quad (7) \]

The lemniscate integral under question is thus transformed into

\[ \pm u = \frac{1}{\sqrt{x}} \cdot 2F_1\left( \frac{1}{2}, \frac{1}{4}; \frac{5}{4}, \frac{1}{x^2} \right). \quad (8) \]

We also know that \( \tau \)-representation for all the Chudnovsky equations are expressed through the standard elliptic modular functions, namely, Jacobi’s \( \vartheta \)-constants. By this means, correlating these two points, we obtain an explicit solution to the Schwarz equation (5).

Let us use the Hauptmodul \( x = \chi(\tau) \) for the 1st Chudnovsky equation (3):

\[ \chi(\tau) = \frac{\vartheta_2^2(\tau)}{\vartheta_3^2(\tau)}, \]

where

\[ \vartheta_2(\tau) := e^{\frac{1}{8} \pi i \tau} \sum_{-\infty}^{\infty} e^{(k^2 + k) \pi i \tau}, \quad \vartheta_3(\tau) := \sum_{-\infty}^{\infty} e^{k^2 \pi i \tau}, \quad \vartheta_4(\tau) := \sum_{-\infty}^{\infty} (-1)^k e^{k^2 \pi i \tau}. \]

\(^1\)To all appearances, the variable upper limit integral formulae for definition to the \( \text{}_2F_1 \)-functions was observed for the first time in the 1876 dissertation by Tikhomandritskii [8, p. 78] before the known 1881 Goursat dissertation.
To put it differently this function solves equation

\[ [x, \tau] = -\frac{1}{2} \frac{(x^2 + 1)^2}{(x^3 - x)^2}. \]

Substituting this \( \chi(\tau) \) into (8), we get that function

\[ u(\tau) = \frac{\vartheta_3(\tau)}{\vartheta_2(\tau)} \cdot 2F_1 \left( \frac{1}{2}, \frac{1}{4}; \frac{5}{4} \left| \frac{\vartheta_3^2(\tau)}{\vartheta_2^2(\tau)} \right| \right) \]

solves Eq. (5) under \((g_2, g_3) = (4, 0)\).

3.2. Equi-anharmonic curve. First we shift \( x \)-variable \( x = z - 1 \) in Eq. (4) to obtain the canonical form \( y^2 = 4z^3 - 4 \) with \((g_2, g_3) = (0, 4)\). Halphen’s transformation and Hauptmodul \( x = \chi(\tau) \) [6] in this case have the form

\[ z = \wp(u; 0, 4), \quad z = 9\frac{\eta^3(9\tau)}{\eta^3(\tau)} + 1, \]

where \( \eta(\tau) := e^{\frac{2\pi i}{3}} \prod_k (1 - e^{2\pi i k\tau}) \) is the Dedekind eta-function. This Hauptmodul \( z(\tau) \) satisfies the equation

\[ [z, \tau] = -\frac{1}{2} \frac{z(z^3 + 8)}{(z^3 - 1)^2} \]

and one can show that the change \( z \mapsto u \) above transforms this Schwarz equation into Eq. (5). We have, according to (7),

\[ \pm u = \int_{\infty}^{z} \frac{du}{\sqrt{4u^3 - 4}} = \frac{1}{\sqrt{z}} \cdot 2F_1 \left( \frac{1}{2}, \frac{1}{6}; \frac{7}{6} \left| \frac{1}{z^3} \right| \right) \]

and, hence,

\[ u = \left( 9\frac{\eta^3(9\tau)}{\eta^3(\tau)} + 1 \right)^{-\frac{1}{2}} \cdot 2F_1 \left( \frac{1}{2}, \frac{1}{6}; \frac{7}{6}, \left\{ 9\frac{\eta^3(9\tau)}{\eta^3(\tau)} + 1 \right\}^{-3} \right). \]

This is a very nontrivial exercise to check directly that this function solves Eq. (5).

Remark. All the Chudnovsky Hauptmoduln are single-valued functions and, hence, in spite of a square root in the last formula, this expression provides a single-valued object in the neighborhood of point \( z = \infty \). Indeed, making the change \( z^2 = z^{-1} \) in Schwarz’s equation for \( z \) above, we get

\[ [z, \tau] = -\frac{1}{2} \frac{1}{z^2} + \cdots \]

and, hence, \( z = z(\tau) \) has an exponentially single-valued behavior in \( \tau \) about point \( z = \infty \) and \( 2F_1 \)-function is, by definition, a single valued Taylor series at the origin \( z = 0 \).

We can, however, get an explicit root-free solution to this problem with the help of the following trick. Use formula (7) and the fact that holomorphic integral is defined up to an additive constant:

\[ u = \int_{\infty}^{z} \frac{du}{\sqrt{4u^3 - 4}} = \int_{0}^{\infty} \frac{du}{\sqrt{4u^3 - 4}} + \int_{0}^{\infty} \frac{du}{\sqrt{4u^3 - 4}}. \]
The first of these integrals is a transcendental constant \( u_0 \) such that \( \wp(u_0) = 0 \), i.e., zero of the Weierstrass \( \wp(z; 0, 4) \)-function. It is computed into elliptic integrals

\[
u_0 = \frac{i}{2\sqrt[3]{3}} F\left(\sqrt[3]{3}(\sqrt[3]{3} - 1); \frac{1}{4}(\sqrt{6} + \sqrt{2})\right) - \frac{1}{\sqrt[3]{3}} K\left(\frac{\sqrt[3]{3} - 1}{2\sqrt[2]{2}}\right) = \frac{i}{6} B\left(\frac{1}{6}, \frac{1}{3}\right) = i \cdot 1.402182105325 \ldots,
\]

where \( F \) and \( K \) are the standard non-complete and complete elliptic integrals \([4]\) and \( B \) is the Euler beta-function. The second integral, upon application of (6), becomes

\[
\int_0^x \frac{du}{\sqrt{4u^3 - 4}} = \frac{i}{2} x \cdot 2 F_1\left(\frac{1}{2}, \frac{1}{3}; \frac{4}{3} x^3\right).
\]

Turning to the Hauptmodul (variable) \( z(\tau) \), we get finally

\[
\pm u(\tau) = u_0 + \frac{i}{2} \left(9 \eta^3(9\tau) + 1\right) \cdot 2 F_1\left(\frac{1}{2}, \frac{1}{3}; \frac{4}{3} \left\{9 \eta^3(9\tau) + 1\right\}^3\right).
\]

Single-valuedness of this expression is now obvious; it solves Eq. (5).

4. Abelian integrals for genus \( g > 1 \)

If \( \alpha, \beta \) in formula (6) are the rational numbers then one has an Abelian integral belonging to rationality that may have genus greater than unity. The integral may be holomorphic, meromorphic, or logarithmic one. If, on the other hand, we have a single-valued \( \tau \)-representation for \( z^{\alpha} \), then we get automatically an explicit \( \tau \)-representation for this integral. This is a nontrivial consequence because, in general, Abelian integrals are not expressible in terms of any known functions and, thereby, we get analogs of Weierstrass’ uniformization theory for elliptic curves.

Recall that Weierstrass’ theory, in its complete description, includes closed and self-contained collection of the base functions and differentials

\[
w^2 = 4z^3 - az - b, \quad \{z = \wp(\tau), \ w = \wp'(\tau)\}, \quad dz = \wp'(\tau) d\tau,
\]

and the two principal integrals

\[
\Pi := \int z \frac{dz}{w} = -\zeta(u), \quad \Xi := \frac{1}{2} \int \frac{w + \wp'(\alpha)}{z - \wp(\alpha)} \frac{dz}{w} = \ln \frac{\sigma(u - \alpha)}{\sigma(u)} + \zeta(\alpha)u,
\]

as functions of the fundamental holomorphic object

\[
u := \int \frac{dz}{w}, \quad \nu(\tau) = \tau.
\]

By this means, having the base holomorphic and meromorphic integrals we can manipulate with this integrals and functions in exactly the same manner as we do with Weierstrassian objects \( \sigma, \zeta, \wp, \) and \( \wp' \). Insomuch as no one explicit formula for such a theory was known hitherto, it is, perhaps, not without interest to exhibit examples.
4.1. Higher genera. Examples. Consider the set of integrals

\[ \int_0^z \frac{u^m}{\sqrt[4]{u(u^4-1)}} du. \]

Looking them at the integrals for the algebraic irrationality \( w^k = z(z^4-1) \), we find that they can be either holomorphic or meromorphic ones. For example, in case \( k = 2 \) (hyperelliptic curves) we have two base holomorphic integrals when \( m = 0, 1 \) and two base meromorphic ones when \( m = 2 \) or 3. The only thing we need now is the formula for Hauptmodul \( z = z(\tau) \). It is known explicitly in cases when \( z \), as a solution to corresponding Fuchs–Schwarz’s equation, has only parabolic singularities; that is Fuchsian equation has punctures at all the points \( z = \{0, \pm 1, \pm i, \infty\} \):

\[ z = \frac{\vartheta_2(\tau)}{\vartheta_3(\tau)}. \]

We derive from (6)

\[ \int_0^z u^{\alpha-1}(u^n-1)^{-\beta} du = \frac{\alpha^\beta}{\alpha} z^{\alpha} \cdot _2F_1 \left( \beta, \frac{\alpha}{n}; \frac{\alpha}{n} + 1 \mid z^n \right), \]

\[ \int_0^\infty u^{\alpha-1}(u^n-1)^{-\beta} du = \frac{z^{\alpha-n\beta}}{\alpha-n\beta} \cdot _2F_1 \left( \beta, \frac{\alpha-n\beta}{n}; \frac{\alpha}{n} + 1 \mid \frac{1}{z^n} \right) \]

and \( \alpha = m + 1 - \frac{1}{k} \). The value \( n \) does not affect on single-valuedness of the \( \tau \)-representation and under \( k = 2 \) we have to do the square root of \( z(\tau) \). One can show that ratio of any two Jacobi’s \( \vartheta \)-constant is a complete square. In particular

\[ \sqrt[2]{\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)}} = \sqrt{2} \frac{\vartheta_2(\tau)}{\vartheta_2(\tau/2)}. \]

It follows that for the famous algebraic curve \( w^2 = z^5 - z \) we can obtain a complete set of base Abelian integrals.

**Theorem.** Every Abelian (homolomorphic, meromorphic, or logarithmic) integral belonging to the algebraic irrationality \( w^2 = z^5 - z \) has a uniformizing \( \tau \)-representation through the Jacobi’s \( \vartheta \)-functions of the two base holomorphic integrals.

**Proof.** This curve admits a representation in form of two isomorphic elliptic curves. Indeed, it is suffice to use the well-known formulae by Jacobi–Legendre for covering of the curve

\[ y^2 = x(x-1)(x-A)(x-B)(x-AB). \]

We have in this case the following cover of torus \( (u) \):

\[ \begin{cases} 
\varphi(u) = -\frac{(\sqrt{A} \pm \sqrt{B})^2}{(x-A)(x-B)} x - \frac{1}{3} (k^z + 1) \\
\varphi'(u) = \frac{2(\sqrt{A} \pm \sqrt{B})^2}{\sqrt{(1-A)(1-B)} (x-A)^2(x-B)^2} y 
\end{cases} \]
where

\[ k^\pm = -\frac{(\sqrt{A} \pm \sqrt{B})^2}{(1 - A)(1 - B)}. \]

Then expression \( \wp'(u)^2 = 4\wp^3(u) - g_2\wp(u) - g_3 \) is equivalent to the hyperelliptic curve above. Reduction of holomorphic differentials has the form

\[ du = \frac{1}{2 \sqrt{(1 - A)(1 - B)(x \mp \sqrt{A\sqrt{B}})}} dx. \]

and the curve \( w^2 = z^5 - z \) corresponds to the following parameters

\[ A = -1, \quad B = i, \quad k^\pm = \frac{1}{2}(1 \pm \sqrt{2}), \]

\[ \wp' = 4\wp^3 - \frac{5}{3}\wp \pm \frac{7}{27}\sqrt{2} = \]

\[ = 4\left\{ \wp + \frac{i}{3}(3 \pm \sqrt{2}) \right\} \left\{ \wp - \left( \pm \frac{i}{3}\sqrt{2} \right) \right\} \left\{ \wp - \frac{i}{3}(3 \mp \sqrt{2}) \right\}. \]

With use of (12) we derive

\[ \mathfrak{U} = \int \frac{u^m du}{\sqrt{u(u^4 - 1)}} = \frac{2\sqrt{2}i}{2m + 1} \frac{\wp_2^{m+1}(\tau)}{\wp_2^{m}(\tau) \wp_2(\frac{\tau}{2})} \cdot {}_2F_1\left( \frac{1}{2} m + \frac{1}{4} m - \frac{1}{8}, \frac{1}{4} m + \frac{9}{8}, \frac{\wp_2^{m}(\tau)}{\wp_2^{m}(\tau)} \right) \]

and these expressions provide not only two holomorphic integrals \( \mathfrak{U}_1, \mathfrak{U}_2 \) \( (m = 0, 1) \) but the meromorphic ones as well. In some cases we have holomorphic integrals as functions of \( \tau \) we can compute any other Abelian integral in terms of 2-dimensional \( \Theta \)-functions. On the other hand, by virtue of the cover above, all the \( \Theta \)-functions are reducible to Jacobi’s \( \theta \). It follows that all the Abelian integrals are expressed through such \( \theta \)-functions having their arguments the two holomorphic integrals above. These \( \theta \)-formulae can be transformed into the Weierstrass \( \wp, \wp', \zeta, \sigma \) because all the integrals are reducible to the objects (9)–(11). \( \blacksquare \)

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