CLASSICAL AND $\mathbb{C}$-MOTIVIC ADAMS CHARTS

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Abstract. This document contains large-format Adams charts that compute 2-complete stable homotopy groups, both in the classical context and in the $\mathbb{C}$-motivic context. The charts are essentially complete through the 90-stem and contain partial results to the 95-stem.

This document contains large-format Adams charts that compute 2-complete stable homotopy groups, both in the classical context and in the $\mathbb{C}$-motivic context. The charts are essentially complete through the 90-stem and contain partial results to the 95-stem.

The charts are intended to be viewed electronically. The authors can supply versions that are suitable for printing.

Justifications for these computations appear in [3], [6], [7], [9], [14], [15], and [16]. Older references that justify many of the classical computations include [1], [2], [4], [10], [11], [12], and [13].

This document supersedes [5], which included Adams charts for the cofiber of $\tau$.

The charts associated to the cofiber of $\tau$ now appear in the separate manuscript [8].

1. Cohomology of the classical Steenrod algebra

This chart shows the cohomology of the classical Steenrod algebra, i.e., the $E_2$-page of the classical Adams spectral sequence, through the 110-stem.

- Black dots indicate copies of $\mathbb{F}_2$.
- Vertical lines indicate $h_0$ multiplications.
- Lines of slope 1 indicate $h_1$ multiplications.
- Lines of slope 1/3 indicate $h_2$ multiplications.

2. The classical Adams spectral sequence

This chart shows the classical Adams spectral sequence. The chart is complete to the 90-stem, with partial results through the 95-stem.

- Black dots indicate copies of $\mathbb{F}_2$.
- Vertical lines indicate $h_0$ multiplications.
- Lines of slope 1 indicate $h_1$ multiplications.
- Lines of slope 1/3 indicate $h_2$ multiplications.

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Key words and phrases. Adams spectral sequence, stable homotopy group, motivic stable homotopy group, cohomology of the Steenrod algebra.

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(5) Light blue lines of slope $-2$ indicate Adams $d_2$ differentials.
(6) Red lines of slope $-3$ indicate Adams $d_3$ differentials.
(7) Green lines of slope $-4$ indicate Adams $d_4$ differentials.
(8) Blue lines of slope $-5$ indicate Adams $d_5$ differentials.
(9) Orange lines of slope less than $-5$ indicate higher Adams differentials.
(10) Dashed lines indicate possible Adams differentials.

3. The $E_\infty$-page of the classical Adams spectral sequence

This chart indicates the $E_\infty$-page of the classical Adams spectral sequence. The chart is complete to the 90-stem, with partial results through the 95-stem. Beyond the 64-stem, not all hidden extensions have been resolved; see [7] for more details. See Section 2 for instructions on interpreting the chart. In addition:

(1) Unknown Adams differentials are indicated as dashed lines.
(2) Red lines indicate hidden 2 extensions. The dashed red lines in the 54-stem indicate that there is a hidden 2 extension, but its target is not known precisely.
(3) Blue lines indicate hidden $\eta$ extensions. The dashed blue lines in the 77-stem indicate that there is a hidden $\eta$ extension, but its source is not known precisely.
(4) Green lines indicate hidden $\nu$ extensions.

4. The $E_2$-page of the motivic Adams spectral sequence

This chart indicates the cohomology of the Steenrod algebra, i.e., the $\mathbb{C}$-motivic Adams $E_2$-page, through the 110-stem. Adams $d_2$ differentials are shown through the 95-stem. For legibility, the chart is divided into two pages with different scales.

(1) Black dots indicate copies of $M_2$.
(2) Red dots indicate copies of $M_2/\tau$.
(3) Blue dots indicate copies of $M_2/\tau^2$.
(4) Green dots indicate copies of $M_2/\tau^3$.
(5) Purple dots indicate copies of $M_2/\tau^4$.
(6) Vertical lines indicate $h_0$ multiplications. These lines might be black, red, blue, or green, depending on the $\tau$ torsion of the target.
(7) Lines of slope 1 indicate $h_1$ multiplications. These lines might be black, red, blue, or green, depending on the $\tau$ torsion of the target.
(8) Lines of slope 1/3 indicate $h_2$ multiplications. These lines might be black, red, blue, or green, depending on the $\tau$ torsion of the target.
(9) Red arrows indicate infinite towers of $h_1$ multiplications, all of which are annihilated by $\tau$.
(10) Magenta lines indicate that an extension hits $\tau$ times a generator. For example, $h_0 \cdot h_0 h_2 = \tau h_1^3$ in the 3-stem.
(11) Orange lines indicate that an extension hits $\tau^k$ times a generator, for some $k \geq 2$. For example, $h_0 \cdot h_0^3 x = \tau^3 h_0 c_0 g$ in the 37-stem.
(12) Blue lines of slope $-2$ indicate Adams $d_2$ differentials.
(13) Magenta lines of slope $-2$ indicate that an Adams $d_2$ differential hits $\tau$ times a generator. For example, $d_2(h_0 c_2) = \tau h_1^2 c_1$ in the 40-stem.
(14) Orange lines of slope $-2$ indicate that an Adams $d_2$ differential hits $\tau^2$ times a generator. For example, $d_2(h_0 y) = \tau^2 h_0 c_0 g$ in the 37-stem.
(15) Dashed lines indicate possible Adams $d_2$ differentials.
The use of color is well-illustrated by the element $h_2g^2$ in the 43-stem. The dot is green, indicating that $\tau^3h_2g^2$ is zero. The outgoing blue lines indicate that $h_0 \cdot h_2g^2$ and $h_2 \cdot h_2g^2$ are annihilated by $\tau^2$. The incoming magenta line indicates that $h_2 \cdot \tau g^2$ equals $\tau h_2g^2$, and the incoming orange line indicates that $h_1 \cdot Ph^3h_5$ equals $\tau^2h_2g^2$.

5. The $E_3$-page of the motivic Adams spectral sequence

This chart indicates the Adams $d_3$ differentials on the $E_3$-page of the motivic Adams spectral sequence. The chart is complete through the 95-stem, with indicated exceptions.

See Section 4 for instructions on interpreting the chart. In addition:

1. Blue lines of slope $-3$ indicate Adams $d_3$ differentials.
2. Magenta lines of slope $-3$ indicate that an Adams $d_3$ differential hits $\tau$ times a generator.
3. Orange lines of slope $-3$ indicate that an Adams $d_3$ differential hits $\tau^k$ times a generator for some $k \geq 2$.
4. Dashed lines indicate possible Adams $d_3$ differentials.

6. The $E_4$-page of the motivic Adams spectral sequence

This chart indicates the Adams $d_4$ differentials on the $E_4$-page of the motivic Adams spectral sequence. The chart is complete through the 95-stem.

See Section 4 for instructions on interpreting the chart. In addition:

1. Blue lines of slope $-4$ indicate Adams $d_4$ differentials.
2. Magenta lines of negative slope indicate that an Adams differential hits $\tau$ times a generator.
3. Orange lines of negative slope indicate that an Adams differential hits $\tau^k$ times a generator for some $k \geq 2$.
4. Dashed lines indicate possible Adams differentials.

7. The $E_5$-page of the motivic Adams spectral sequence

This chart indicates the Adams $d_5$ differentials on the $E_5$-page of the motivic Adams spectral sequence. The chart is complete through the 95-stem.

See Section 4 for instructions on interpreting the chart. In addition:

1. Blue lines of slope $-5$ indicate Adams $d_5$ differentials.
2. Magenta lines of negative slope indicate that an Adams differential hits $\tau$ times a generator.
3. Orange lines of negative slope indicate that an Adams differential hits $\tau^k$ times a generator for some $k \geq 2$.
4. Dashed lines indicate possible Adams differentials.

8. The $E_6$-page of the motivic Adams spectral sequence

This chart indicates the higher Adams differentials on the $E_6$-page of the motivic Adams spectral sequence. The chart is complete to the 90-stem, with partial results through the 95-stem.

See Section 4 for instructions on interpreting the chart. In addition:

1. Blue lines of negative slope indicate Adams $d_r$ differentials for some $r \geq 6$. 
(2) Magenta lines of negative slope indicate that an Adams differential hits $\tau$ times a generator.
(3) Orange lines of negative slope indicate that an Adams differential hits $\tau^k$ times a generator for some $k \geq 2$.
(4) Dashed lines indicate possible Adams differentials.

9. The $E_\infty$-page of the motivic Adams spectral sequence

This chart indicates the $E_\infty$-page of the motivic Adams spectral sequence. The chart is complete through the 90-stem, with partial results through the 95-stem. For clarity, hidden extensions by 2, $\eta$, and $\nu$ are not shown on this chart. See Section 4 for instructions on interpreting the chart. In addition:

(1) Green vertical lines indicate hidden $\tau$ extensions.
(2) Dashed lines of negative slope indicate unknown Adams differentials.

References

[1] M. G. Barratt, J. D. S. Jones, and M. E. Mahowald, Relations amongst Toda brackets and the Kervaire invariant in dimension 62, J. London Math. Soc. (2) 30 (1984), no. 3, 533–550, DOI 10.1112/jlms/s2-30.3.533. MR810962 (87g:55025)
[2] M. G. Barratt, M. E. Mahowald, and M. C. Tangora, Some differentials in the Adams spectral sequence, II, Topology 9 (1970), 309–316. MR0266215 (42 #1122)
[3] Robert Burklund, Daniel C. Isaksen, and Zhouli Xu, Synthetic stable stems, in preparation.
[4] Robert Bruner, A new differential in the Adams spectral sequence, Topology 23 (1984), no. 3, 271–276. DOI 10.1016/0040-9383(84)90010-7. MR770563 (86c:55016)
[5] Daniel C. Isaksen, Classical and motivic Adams charts (2014), available at arXiv:1401.4983.
[6] , Stable stems, Mem. Amer. Math. Soc. 262 (2019), no. 1269, viii+159, DOI 10.1090/memo/1269. MR4046815
[7] Daniel C. Isaksen, Guozhen Wang, and Zhouli Xu, More stable stems (2020), preprint, available at arXiv:2001.04511.
[8] , Classical algebraic Novikov charts and C-motivic Adams charts for the cofiber of $\tau$ (2019), preprint, available at s.wayne.edu/isaksen/adams-charts.
[9] Daniel C. Isaksen and Zhouli Xu, Motivic stable homotopy and the stable 51 and 52 stems, Topology Appl. 190 (2015), 31–34, DOI 10.1016/j.topol.2015.04.008. MR3349503
[10] Stanley O. Kochman, Stable homotopy groups of spheres, Lecture Notes in Mathematics, vol. 1423, Springer-Verlag, Berlin, 1990. A computer-assisted approach. MR1052407 (91j:55016)
[11] Stanley O. Kochman and Mark E. Mahowald, On the computation of stable stems, The Čech centennial (Boston, MA, 1993), Contemp. Math., vol. 181, Amer. Math. Soc., Providence, RI, 1995, pp. 299–316, DOI 10.1090/conm/181/02039. MR1320997 (96j:55018)
[12] Mark Mahowald and Martin Tangora, Some differentials in the Adams spectral sequence, Topology 6 (1967), 349–369. MR0214072 (35 #4924)
[13] J. Peter May, The cohomology of restricted Lie algebras and of Hopf algebras; application to the Steenrod algebra, Ph.D. dissertation, Princeton Univ., 1964.
[14] Guozhen Wang and Zhouli Xu, The triviality of the 61-stem in the stable homotopy groups of spheres, Ann. of Math. (2) 186 (2017), no. 2, 501–580, DOI 10.4007/annals.2017.186.2.3. MR3702672
[15] , Some extensions in the Adams spectral sequence and the 51-stem, Algebr. Geom. Topol. 18 (2018), no. 7, 3887–3906, DOI 10.2140/agt.2018.18.3887. MR3892234
[16] Zhouli Xu, The strong Kervaire invariant problem in dimension 62, Geom. Topol. 20 (2016), no. 3, 1611–1624, DOI 10.2140/gt.2016.20.1611. MR3523064
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