Entanglement Pre-thermalization in an Interaction Quench between Two Harmonic Oscillators

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Entanglement pre-thermalization (EP) is a quasi-stationary nonequilibrium state of a composite system in which each individual subsystem looks thermal but the entire system remains nonthermal due to quantum entanglement between subsystems. We theoretically study the dynamics of EP following a coherent split of a one-dimensional harmonic potential in which two interacting bosons are confined. This problem is equivalent to that of an interaction quench between two harmonic oscillators. We show that this simple model captures the bare essentials of EP; that is, each subsystem relaxes to an approximate thermal equilibrium, whereas the total system remains entangled. We find that a generalized Gibbs ensemble, which incorporates nonlocal conserved quantities, exactly describes the total system. In the presence of a symmetry-breaking perturbation, EP is quasi-stationary and eventually reaches thermal equilibrium. We analytically show that the lifetime of EP is inversely proportional to the magnitude of the perturbation.

I. INTRODUCTION

Over the last two decades or so, the foundation of quantum statistical mechanics has seen a resurgence of interest [1–4], motivated in large part by the advances in ultracold atom experiments [5–8]. One of the fundamental questions is how statistical mechanics emerges out of the unitary time evolution of quantum mechanics [9–14]. Recent studies have clarified that an isolated quantum system can effectively reach thermal equilibrium, where the memory of the initial state is not lost but only hidden in many-body correlations which cannot be captured exactly by any thermal ensemble. In the presence of a symmetry-breaking perturbation, EP is quasi-stationary and eventually reaches thermal equilibrium. We analytically show that the lifetime of EP is inversely proportional to the magnitude of the perturbation.
we post-select the state
\[ |\psi_0\rangle = \int dx_1 dx_2 \Psi_G(x_1, x_2) \psi_1^\dagger(x_1) \psi_2^\dagger(x_2) |\Omega\rangle \] (1)
as the one after the split. For \( t > 0 \), the system evolves from the initial state \( |\Omega\rangle \) as two independent harmonic oscillators since we assume that the two potentials are sufficiently displaced from each other so that the two bosons have no overlap and hence do not interact with each other.

The coherent splitting thus defined is equivalent to an interaction quench between two distinguishable particles with equal mass because quantum statistics plays no role after the split. (Note that the zero-temperature ground state of bosons is the same as that of distinguishable particles having the same mass.) We therefore consider the following Hamiltonian
\[ \hat{H}_\alpha = \frac{1}{2} (\hat{p}_1^2 + \hat{p}_2^2) + \frac{1}{2} (\hat{x}_1^2 + \hat{x}_2^2) + \alpha^2 (\hat{x}_1 - \hat{x}_2)^2, \] (2)
where the mass \( m \) of the particles and the frequency \( \omega \) of the harmonic potential are set to unity, and the dynamical variables \( \hat{x}_i \) and \( \hat{p}_i \) \((i = 1, 2)\) are assumed to satisfy the canonical commutation relations. Then the ground state \( |\psi_G; \alpha\rangle \) of Eq. (2) takes the form of Eq. (4). At time \( t = 0 \), the strength of interaction \( \alpha \) is suddenly changed to zero, and the system evolves in time according to \( \hat{H}_{\alpha=0} \).

In the following, we use the description of the interaction quench rather than that of the coherent splitting.

The Hamiltonian (2) is easily diagonalized for \( \alpha = 0 \) since it represents two noninteracting harmonic oscillators. We define the annihilation operators for each particle \( a_i \equiv \frac{1}{\sqrt{2}}(x_i + i p_i) \) \((i = 1, 2)\), and their linear combinations \( a_\pm \equiv \frac{1}{\sqrt{2}}(a_1 \pm a_2) \). Then we obtain the diagonalized Hamiltonian
\[ \hat{H}_{\alpha=0} = a_1^\dagger a_1 + a_2^\dagger a_2 = a_+^\dagger a_+ + a_-^\dagger a_-, \] (3)
where we have omitted the zero-point energy. We denote by \( |0\rangle \) the Fock vacuum defined by \( a_1 |0\rangle = a_2 |0\rangle = 0 \).

Then, the energy eigenstates are represented in two ways as
\[ |m, n\rangle = (a_1^\dagger)^m (a_2^\dagger)^n |0\rangle, \quad |m, n\rangle = (a_+^\dagger)^m (a_-^\dagger)^n |0\rangle. \] (4)

We note that the degenerate subspace with eigenenergy \( N \) (in units of \( \hbar \omega \)) is given by
\[ \mathcal{H}_N \equiv \text{span}\{|m, n\rangle | m + n = N\}, \] (5)
which can also be defined in terms of \( |m, n\rangle \).

For \( \alpha \neq 0 \), the Hamiltonian (2) is diagonalized through the changes of variables corresponding to the center-of-mass motion and the relative motion. Namely, we introduce \( a_{CM} = a_+ + a_{rel} \) and \( a_{rel} = \cosh r a_- + \sinh r a_-^\dagger \) with
\[ e^{4r} = 1 + a^2 \]. (6)
Then we obtain $\hat{H}_\alpha = a_{CM}^\dagger a_{CM} + \sqrt{1 + \alpha^2} a_{rel}^\dagger a_{rel}$. The ground state $|\text{GS}; \alpha \rangle$ of $\hat{H}_\alpha$ is defined by the following conditions: $a_{CM} |\text{GS}; \alpha \rangle = a_{rel} |\text{GS}; \alpha \rangle = 0$. We note that $|\text{GS}; \alpha \rangle$ is a squeezed vacuum:

$$|\text{GS}; \alpha \rangle = \frac{1}{\sqrt{\cosh r}} e^{-\frac{\tanh r}{2} (a^\dagger)^2} |0\rangle. \quad (7)$$

The time evolution is obtained as

$$|\Psi(t)\rangle = e^{-i\hat{H}_\alpha t} |\text{GS}; \alpha \rangle = \sum_{N=0}^{\infty} (-1)^N \sqrt{q_N e^{-2iNt}} |\Phi_N\rangle, \quad (8)$$

where $|\Phi_N\rangle = |0, 2N\rangle$ and

$$q_N = \frac{1}{\cosh r} \frac{1}{(N!)^2} \left(\frac{\tanh r}{2}\right)^{2N}. \quad (9)$$

In the following, we focus on the long-time average, which is described by the density matrix $\rho_\infty$ defined by

$$\rho_\infty = \langle \Psi(t) | \langle \Psi(t) \rangle = \sum_{N=0}^{\infty} q_N \langle \Phi_N | \Phi_N \rangle, \quad (10)$$

where $f(t) \equiv \lim_{r \to -\infty} T^{-1} \int_0^T \frac{df}{dt}$. We note that, since the energy spectrum has equal spacings, the temporal fluctuation around the long-time average is not negligible. In more realistic interacting models, however, this fluctuation is suppressed and the long-time average is known to describe an effective stationary state. Thus, we ignore the persistent temporal fluctuation in our model, and assume that the long-time average in our model gives an equivalent of the effective stationary state in realistic models.

### III. KEY FEATURES OF ENTANGLEMENT PRE-THERMALIZATION

In this section, we show that our simple model captures the key features of EP. Namely, in the long-time average, each subsystem approximately reaches thermal equilibrium, and the diagonal/off-diagonal decomposition holds true for the total system exactly as in Ref. [31], where all the information about subsystems is contained in the diagonal part and EP is characterized by the off-diagonal part.

#### A. Asymptotically thermal subsystem

To show the first key feature of EP that each subsystem can be described by a conventional statistical ensemble, let us consider the reduced density matrix of the subsystem $1$:

$$\rho_{\infty}^{(1)} = \text{tr}_2 \rho_\infty, \quad (11)$$

where $\text{tr}_2$ denotes the trace over the Hilbert space of the subsystem 2. Due to symmetry under the interchange of 1 and 2, the following argument also applies to $\rho_\infty^{(2)} = \text{tr}_1 \rho_\infty$. To take the partial trace, we expand $|\Phi_N\rangle$ in terms of $|m, n\rangle$ as $|\Phi_N\rangle = \sum_{m=0}^{2N} c_{m,2N-m} |m, 2N-m\rangle$ with $c_{m,n} \equiv \binom{m+n}{m} (-1)^m \sqrt{m! n!}$, which leads to

$$|\Phi_N\rangle \langle \Phi_N | = \sum_{m=0}^{2N} \sum_{m'=0}^{2N} c_{m,2N-m} c_{m',2N-m'} ^* \times |m, 2N-m\rangle \langle m', 2N-m' |. \quad (12)$$

From Eqs. (10) and (12), we obtain

$$\rho_{\infty}^{(1)} = \sum_{m=0}^{\infty} w_m |m\rangle \langle m|, \quad (13)$$

where $w_m = \sum_{N=|m|/2}^\infty q_N |c_{m,2N-m}|^2$ with $q_N$ defined in Eq. 4 represents the weight of each energy eigenstate and can be represented in terms of the hypergeometric function. By looking into the asymptotic behavior of $w_n$, we obtain $w_n \propto e^{-\beta n}$ for $n \gg 1$ with

$$\beta \equiv \ln (2 \coth r - 1), \quad (14)$$

which means that Eq. (13) is asymptotically thermal. The comparison between the actual weight $w_n$ and the asymptotic behavior is illustrated in Fig. 2. The approach to the asymptotic form is quite fast.

We note that $\beta$ can be regarded as the real temperature of the subsystem that is calculated from the total energy because the reduced density matrix $\rho_{\infty}^{(1)}$ itself is approximately thermal. In contrast, in Ref. [31], the reduced density matrix is not necessarily close to a thermal one and the temperature of the subsystem is determined by the least-squares fitting of correlation functions.

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**FIG. 2.** (Color Online) Weight $w_n$ of each energy eigenstate of the subsystem for $r = 0.1$ (square), 0.3 (circle), and 1.0 (triangle). The solid line shows the canonical weight $A e^{-\beta n}$ at the inverse temperature $\beta$, where $A$ is determined by the least-squares fitting.
We remark that our initial state is different from the so-called two-mode squeezed state (TMSS), which is given by \( |\text{TMSS}\rangle = \sum_{N=0}^{\infty} \frac{\lambda^N}{\sqrt{N!}} |N, N\rangle \) for \( |\lambda| < 1 \). The TMSS reduces to an exact thermal equilibrium state when either subsystem is traced out. However, this is not appropriate for addressing EP because, in each degenerate subspace \( \mathcal{H}_{2N} \), the state \( |N, N\rangle \) is a direct product of states in the subsystems, and therefore cannot support EP which requires entanglement, namely, a superposition state between degenerate subspaces.

### B. Diagonal/off-diagonal decomposition

Here we show the second key feature of EP that the total system in the long-time average \( \rho_\infty \) retains entanglement within degenerate subspaces. To this end, we show that the diagonal/off-diagonal decomposition

\[
\rho_\infty = \rho^d_\infty + \rho^{off-d}_\infty
\]

works exactly as in Ref. [31]. Here the diagonal \( \rho^d_\infty \) and off-diagonal \( \rho^{off-d}_\infty \) parts are given, respectively, as

\[
\rho^d_\infty = \sum_{N=0}^{\infty} q_N \sum_{m=0}^{2N} |c_{m,2N-m}|^2 |m, 2N-m\rangle \langle m, 2N-m|
\]

and

\[
\rho^{off-d}_\infty = \sum_{N=0}^{\infty} q_N \sum_{m=0}^{2N} \sum_{m'=0}^{2N} c_{m,2N-m} c^*_{m',2N-m'}
\]

\[
\times |m, 2N-m\rangle \langle m', 2N-m'|
\]

The off-diagonal part has no physical effect as long as we look at each subsystem alone. In fact, it vanishes when either subsystem is traced out, \( \text{tr}_i \rho^{off-d}_\infty = 0 \) \( (i = 1, 2) \). Therefore, the diagonal part contains all the information about individual subsystems in the sense that

\[
\text{tr}_i \rho^d_\infty = \text{tr}_i \rho_\infty \quad (i = 1, 2).
\]

Nonetheless, the off-diagonal part plays an essential role in the nonlocal correlation by which we mean the correlation between the different subsystems. To show this, we investigate the joint probability distribution in the coordinate basis

\[
P(x, y) = \langle x, y|\tilde{\rho}|x, y\rangle.
\]

The role of the off-diagonal part becomes evident when we compare \( P(x, x) \) and \( P(x, -x) \) calculated for both \( \rho_\infty \) and \( \rho^d_\infty \) as illustrated in Fig. 3. While they are different when evaluated for the actual state \( \rho_\infty \), they are equal when evaluated for \( \rho^d_\infty \), in which the off-diagonal part is absent. In other words, the joint probability distribution shows a richer variety of behavior due to \( \rho^{off-d}_\infty \) that cannot be captured by \( \rho^d_\infty \) alone.

We point out a close analogy of our argument to the so-called decoherence-free subspace (DFS) [32]. The simplest example of the DFS can be illustrated by a pair of qubits that are affected by random phase kicks [34]: \( (|0\rangle + e^{i\phi_1}|1\rangle)(|0\rangle + e^{i\phi_2}|1\rangle) \). If the random phases \( \phi_1 \) and \( \phi_2 \) are independent, averaging over them leads to the completely mixed state. However, if they are correlated, for example, as \( \phi_2 = -\phi_1 \), quantum coherence remains nonvanishing, even after averaging within a subspace spanned by \( |00\rangle \) and \( |11\rangle \). In our model, the long-time average corresponds to the phase averaging for each energy eigenstate. In this process, there are degenerate subspaces [3], where decoherence does not occur and quantum coherence is preserved.

### IV. Statistical-Mechanical Description of Entanglement Pre-Thermalization

So far, we have seen that our model captures the key features of EP. In this section, we discuss the statistical-mechanical description of EP. In Sec. IV A we consider the canonical(-like) ensemble by incorporating conserved quantities of each individual subsystem. In Sec. IV B we find an exact ensemble description by considering nonlocal conserved quantities that nontrivially act on the entire system.

### A. Local conserved quantities

Here we argue that conventional statistical ensembles cannot describe \( \rho_\infty \) well as long as we take into account only conserved quantities that involve each individual subsystem.

The canonical ensemble \( \rho_{\text{can}} \propto e^{-\beta H_{\alpha=0}} =
The entropy under the constraint of the total energy \( N_1 + N_2 \), is not sufficient because it does not reproduce the basic property of \( \rho_\infty \) that \( |m,n\rangle \) is populated only when \( m+n \) is even due to parity symmetry. This constraint can be incorporated if we consider an extra conserved quantity \((-1)^{N_1-N_2}\), which is unity for \( \rho_\infty \). The canonical ensemble with the constraint of \((-1)^{N_1-N_2}=1\) reads

\[
\rho_{\text{can}}' = \sum_{N=0}^{\infty} \frac{e^{-2\beta N}}{Z'} P_{2N},
\]

where \( P_{2N} = \sum_{m=0}^{2N} |m,2N-m\rangle \langle m,2N-m| \) denotes the projection operator on the subspace with the total energy \( 2N \) and \( Z' = \sum_{m=0}^{\infty} (2N+1)e^{-2\beta N} \). However, this still fails to approximate \( \rho_\infty \) because \( \rho_{\text{can}}' \) does not contain any off-diagonal elements in the eigenenergy basis.

This difficulty cannot be overcome even if we consider extra local conserved quantities. For instance, by considering any moment of the total energy \( \langle H_{\alpha=0}^m \rangle (m=2,3,\ldots) \), we obtain

\[
\rho_{\text{can}}'' = \sum_{N=0}^{\infty} q_N P_{2N},
\]

where \( q_N \) is the actual weight on each degenerate subspace. Although this ensemble contains many parameters, it cannot describe EP because there is no off-diagonal element. The ensemble \( \rho_{\text{can}}'' \) cannot approximate the long-time average \( \langle H_{\alpha=0}^\infty \rangle \) well because it leads to \( P(x,x) = P(x,-x) \) and does not hold in the actual long-time average. This difficulty persists even if we take into account each weight \( \langle m,n|\rho|m,n\rangle \) because it leads to \( \rho_{\infty}' \), where no off-diagonal contribution is present.

**B. Nonlocal conserved quantities**

In this subsection, we show that, by taking into account nonlocal conserved quantities in line with the rule of the generalized Gibbs ensemble, we can construct an ensemble describing the long-time average exactly.

To this end, we work in the \( a_\pm \) basis rather than the \( a_{1,2} \) basis. We note that nonlocal operators \( \hat{N}_\pm = a_\pm^\dagger a_\pm \) are conserved: \( \{H_{\alpha=0}, \hat{N}_\pm\} = 0 \) as seen from Eq. (17). By using these conserved quantities, we define a canonical-like ensemble

\[
\rho_{\text{NL}} = \frac{1}{Z_{\text{NL}}} e^{-\sum_{\sigma=\pm} \beta_\sigma \hat{N}_\sigma},
\]

where \( Z_{\text{NL}} = \text{tr} e^{-\sum_{\sigma=\pm} \beta_\sigma \hat{N}_\sigma} \), and the parameters \( \beta_\pm \) are determined by the conditions \( \text{tr}(\rho_{\text{NL}} \hat{N}_+^m) = \langle \psi(0)|\hat{N}_+^m|\psi(0)\rangle = 0 \) and \( \text{tr}(\rho_{\text{NL}} \hat{N}_-) = \langle \psi(0)|\hat{N}_-|\psi(0)\rangle = (\sinh r)^2 \) that can be solved, giving \( \beta_+ = 0 \) and \( \beta_- = 2\ln \coth r \). Then Eq. (22) reduces to

\[
\rho_{\text{NL}} = (1-e^{-\beta_-}) \sum_{N=0}^{\infty} e^{-\beta_- N} |0,N\rangle\langle 0,N|.
\]

We can improve the ensemble by adding yet another conserved quantity \( P_- \equiv (-1)^N \) as we have done in improving the ensemble from \( \rho_{\text{can}} \) to \( \rho_{\text{can}}' \). The improved ensemble is given by

\[
\rho_{\text{NL}}' = \frac{1}{Z_{\text{NL}}'} e^{-\sum_{\sigma=\pm} \beta'_\sigma \hat{N}_\sigma - \gamma P_-},
\]

where \( Z_{\text{NL}}' = \text{tr} e^{-\sum_{\sigma=\pm} \beta'_\sigma \hat{N}_\sigma - \gamma P_-} \) and the parameters \( \beta'_\pm \) and \( \gamma \) are determined by specifying the expectation values of \( \hat{N}_\pm \) and \( P_- \equiv (-1)^N \) in the initial state. Concretely speaking, our initial state \( |\psi(0)\rangle \) satisfies \( \langle \psi(0)|\hat{N}_+|\psi(0)\rangle = 0 \) and \( \langle \psi(0)|\hat{N}_-|\psi(0)\rangle = (\sinh r)^2 \) and \( \langle \psi(0)|P_-|\psi(0)\rangle = 1 \), which imply

\[
\rho_{\text{NL}}'' = (1-e^{-2\beta_-'}) \sum_{N=0}^{\infty} e^{-2\beta_-' N} |0,2N\rangle\langle 0,2N|,
\]

where \( \beta_-' = \frac{1}{2} \ln(1+\frac{2}{\sinh^2 r}) \) and the symbol \( \langle \cdot \cdot \cdot \rangle \) denotes the state in the \( a_\pm \) basis (see Eq. (11)). We note that \( \rho_{\text{NL}}'' \) is similar in form to \( \rho_{\infty} = \sum_{N=0}^{\infty} q_N |0,2N\rangle\langle 0,2N| \).

Furthermore, by taking more conserved quantities \( \{\hat{N}_m\}_{m=2}^{\infty} \) in line with the rule of the generalized Gibbs ensemble, we obtain the exact final-state distribution of the system:

\[
\rho_{\text{NL}}'' = \frac{1}{Z_{\text{NL}}''} e^{-\sum_{m=1}^{\infty} \beta''_m \hat{N}_m - \beta_+ \hat{N}_+ - \gamma P_-} = \rho_{\infty}.
\]

In deriving the last equality, we have used the solution \( \{\beta''_m\}_{m=1}^{\infty} \) to the set of equations \( \text{tr}(\rho_{\text{NL}} \hat{N}_m^m) = \langle \psi(0)|\hat{N}_m^m|\psi(0)\rangle \). This systematic improvement of \( \rho_{\text{NL}}'' \) to the exact result can be successfully made because \( \rho_{\infty} \) becomes diagonal in the nonlocal \( a_\pm \) basis (see Eq. (11)), while, in the local \( a_{1,2} \) basis, \( \rho_{\infty} \) involves the off-diagonal part \( \rho_{\infty}''_{\text{off}} \), which can be incorporated by neither the canonical ensemble nor its generalization.

Figure 4 shows how these ensembles give increasingly better results for the joint probability distribution. It is clearly seen that \( \rho_{\text{NL}}'' \) gives a better prediction than \( \rho_{\text{NL}}' \) even though \( \rho_{\text{NL}}'' \) has much less fitting parameters. This observation implies that an appropriate choice of basis is more important than the number of parameters. Figure 4 also shows that \( \rho_{\text{NL}}'' \) gives even better predictions than \( \rho_{\text{NL}}''_{\text{off}} \) as expected. As we have shown, by going further to \( \rho_{\text{NL}}'' \), we obtain the exact distribution.

Finally, we discuss how we could know, in more general situations, the best set of conserved quantities like \( \hat{N}_i \) (i = 1, 2). This is possible if we are given...
the expectation values of the lowest-order correlation in, say, the $a_i$ ($i = 1, 2$) basis:

$$A_{ij} = \langle \hat{\psi}(0)|a_j^\dagger a_i|\hat{\psi}(0)\rangle = \frac{(\sinh r)^2}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}_{ij}. \quad (27)$$

The natural candidate for the statistical ensemble describing $\rho_\infty$ is the canonical-like ensemble which is obtained by taking into account all the low-order conserved quantities. We note that there are four linearly independent conserved quantities since $Q = \sum_{i,j} a_i^\dagger Q_{ij} a_j$ for any $2 \times 2$ Hermitian matrix $Q_{ij}$ for any $2 \times 2$ Hermitian matrix $Q_{ij}$. Thus we can take $\sigma^\gamma = \sum_{i,j} a_i^\dagger \sigma_{ij} a_j$ ($\gamma = 0, 1, 2, 3$), where $\sigma_{ij}^\gamma$ is the $2 \times 2$ identity matrix and the other three are the Pauli matrices. Then, by introducing the Lagrange multipliers $\lambda_\gamma$ ($\gamma = 0, 1, 2, 3$), our lowest-order ensemble is given by

$$\rho_{LO} = \frac{1}{Z_{LO}} \exp \left(-\sum_{\gamma=0}^{3} \lambda_\gamma \sigma^\gamma \right), \quad (28)$$

where the Lagrange multipliers are determined by

$$\text{tr}(\rho_{LO} a_i^\dagger a_j) = A_{ij}. \quad (29)$$

By solving these equations, one can show $\lambda_2 = \lambda_3 = 0$, which implies that Eq. (25) reduces to Eq. (22) with $\beta_\pm = \lambda_0 \pm \lambda_1$. Thus, we obtain

$$\rho_{LO} = \rho_{NL} \quad (30)$$

without assuming a priori knowledge of the proper basis $a_\pm$. We note that this is also justified through the diagonalization of $A_{ij}$, which amounts to the change of basis from $a_{1,2}$ to $a_{\pm}$.

![Figure 4](image-url)  
**FIG. 4.** (Color Online) Distribution $P(x, y)$ with $x = y$ (upper two panels) and $x = -y$ (lower two panels) calculated for the long-time average (10) (thick solid), the canonical-like ensemble (21) (thin solid), and the canonical-like ensemble with nonlocal conserved quantities (25) (dashed) and (26) (dash-dotted) plotted for the quench magnitude of $r = 0.3$ (left two panels) and 1 (right two panels).

V. ENTANGLEMENT PRE-THERMALIZATION PLATEAU

EP originates from the degeneracy in the energy spectrum due to some symmetry of the system. However, the symmetry may only be approximate in reality. In this case, EP is observed as a transient phenomenon which eventually relaxes to a thermal equilibrium state. In this section, we demonstrate this by slightly breaking the symmetry between the two systems.

Suppose that the Hamiltonian after the coherent splitting is given, instead of Eq. (2), by

$$\hat{H}_{C} = (1 + \epsilon) a_1^\dagger a_1 + a_2^\dagger a_2 \quad (31)$$

and that the Hamiltonian before the splitting is given by Eq. (11) with $\epsilon = 0$. Here a small real number $\epsilon(\neq 0)$ quantifies the asymmetry between two potentials after the splitting. We assume that $\epsilon$ is an irrational number to ensure no degeneracy in the energy spectrum.

Under these conditions, only the diagonal part contributes to the long-time average. The time-evolved state is given, instead of Eq. (8), by

$$|\Psi_\epsilon(t)\rangle = \sum_{N=0}^{2N} \sum_{m=0}^{2N} (-1)^N \sqrt{qN c_m, 2N - m} \times e^{-i(2N + m \epsilon)t} |m, 2N - m\rangle, \quad (32)$$

and thus the long-time average is obtained as

$$\rho_\infty = |\langle \Psi_\epsilon(t) | \langle \Psi_\epsilon(t) \rangle = \rho_{LO}^1, \quad (33)$$

where $\rho_{LO}^1$ is given in Eq. (10). This implies that the system eventually loses the characteristics of EP and becomes describable by the standard statistical ensembles discussed above.

We now investigate the transient behavior by examining the time evolution of an observable

$$\hat{O} = (\hat{x}_1 - \hat{x}_2)^2. \quad (34)$$

The expectation value of this quantity for any state $\hat{\rho}$ is related to $P(x, y)$ defined in Eq. (19) as $\text{tr}(\hat{\rho}\hat{O}) = \int dx dy P(x, y)(x - y)^2$. In discussing the time evolution, we consider the following time-averaged quantity

$$O_T \equiv \int_0^T \frac{dt}{T} \langle \Psi_\epsilon(t) | \hat{O} | \Psi_\epsilon(t) \rangle \quad (35)$$

rather than the time-dependent expectation value $\langle \Psi_\epsilon(t) | \hat{O} | \Psi_\epsilon(t) \rangle$ since the latter keeps oscillating due to numerous equal energy-level spacings in the energy spectrum. In more realistic systems, this oscillation is known to be exponentially suppressed with increasing the system size (11, 22).

The time-averaged expectation value (35) is exactly calculated to give
FIG. 5. (Color Online) Time averages for the observable defined in Eq. (35) for the quench magnitude \( r = 0.3 \) (a) and 1 (b), where \( O_T \) is plotted for four different values of \( \epsilon = 0 \) (dash-dotted), \( 2\sqrt{2} \times 10^{-3} \) (dashed), \( \sqrt{2} \times 10^{-2} \) (dotted), and \( \sqrt{2} \times 10^{-1} \) (solid). The entanglement pre-thermalization plateau appears for sufficiently small \( \epsilon \).

One can easily confirm that, in the limit of \( \epsilon \to 0 \), Eq. (36) reduces to

\[
O_T = 1 + 2\sinh^2 r - \sinh(2r) \frac{\sin(2T)}{2T} \qquad (\epsilon = 0). \tag{37}
\]

Thus, in the absence of the symmetry-breaking perturbation \( \epsilon \), \( O_T \) converges to the EP value of \( 1 + 2\sinh^2 r \) which is given by \( \text{tr}(\hat{\rho}_\infty \hat{O}) \), whereas, in its presence, it approaches a different value \( 1 + \sinh^2 r \), which is given by \( \text{tr}(\hat{\rho}_\infty \hat{O}) \).

The EP plateau appears for sufficiently small \( \epsilon \) as shown in Fig. 5. For sufficiently small \( \epsilon \) such as \( 2\sqrt{2} \times 10^{-3} \) and \( \sqrt{2} \times 10^{-2} \), \( O_T \) first approaches the EP value, stay there for a while, and then converges to the value given by \( \text{tr}(\hat{\rho}_\infty \hat{O}) \). This is because the symmetry-breaking perturbation \( \epsilon \) introduces yet another energy scale and, hence, another time scale \( 1/\epsilon \). For \( T \ll 1/\epsilon \), the dynamics is well approximated by the one with \( \epsilon = 0 \), whereas it is modified after \( T = O(\epsilon^{-1}) \) as evident from Eq. (36). Thus, the duration of the EP plateau is of the order of \( 1/\epsilon \).

The EP plateau disappears if the symmetry-breaking perturbation \( \epsilon \) becomes comparable with the energy-level spacings of the unperturbed system. As illustrated in Fig. 5 for \( \epsilon = \sqrt{2} \times 10^{-1} \), the EP plateau disappears and \( O_T \) directly converges to \( \text{tr}(\hat{\rho}_\infty \hat{O}) \). This is because the time scale \( 1/\epsilon \) is comparable with the intrinsic scale, which has been set to unity in our discussion.

We expect that such a plateau-like behavior is commonly seen for generic systems that exhibit EP. EP derives from some symmetry that ensures the degeneracy in the energy spectrum and thus preserves quantum entanglement. On the other hand, in experimentally realistic systems, there may be symmetry-breaking perturbations, and EP appears only as a plateau. The duration of EP is expected to be \( O(\epsilon^{-1}) \), where \( \epsilon \) represents the magnitude of the perturbation, because the energy splitting of the degenerate eigenstates is \( O(\epsilon) \) and this does not affect the dynamics in that short-time scale.

VI. CONCLUSIONS

We have analyzed EP by using a simple model consisting of two harmonic oscillators and shown that this simple model involves the characteristics of EP proposed in Ref. [31]. Namely, while each individual subsystem is asymptotically thermal in the long-time limit, the total system is not due to quantum entanglement present in the initial state. We have taken a step further to identify the statistical ensemble that describes the nonthermal total system. We have shown that, if we know low-order correlations between the subsystems, we can construct a GGE-like ensemble, which exactly describes the total system. Finally, we have discussed the effect of small symmetry-breaking perturbations. If the perturbation is sufficiently small, EP is rather observed as a plateau in the time evolution of an observable.

Since the study of EP is still in its infancy, there are many open questions to be addressed. First of all, more examples are needed to gain a comprehensive understanding of EP. Second, it is important to study nonintegrable setups, since only integrable systems have been studied so far. The generalization to a mixed state in an open system is of interest to understand the range of applicability of EP. Our finding of a close relationship between EP and the DFS might give a useful insight into when quantum entanglement survives in the thermodynamic situation. Of course, experimental investigations are of great interest.

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