Search-free direction-of-arrival estimation for transmit beamspace multiple-input multiple-output radar via tensor modelling and polynomial rooting

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Abstract

In order to improve the accuracy and resolution for transmit beamspace (TB) multiple-input multiple-output (MIMO) radar, a search-free direction-of-arrival (DOA) estimation method based on tensor modelling and polynomial rooting is proposed. In the proposed method, a 3-order tensor is designed to model the received signal of TB MIMO radar on the basis of the multi-linear property. The factor matrix with target DOA information is obtained by tensor decomposition via alternating least squares algorithm, and subsequently the DOA estimation is converted to independent minimisation of the virtual transmit beampattern for TB MIMO radar. By exploiting the Vandermonde structure of the transmit steering matrix, a polynomial function is constructed to solve the minimisation problem via polynomial rooting. The proposed method can fully take advantage of the TB technique and tensor modelling, and it requires no spectrum searching, which is different from the existing DOA estimation techniques for TB MIMO radar. The effectiveness of the proposed method is verified by simulations.

1 | INTRODUCTION

Multiple-input multiple-output (MIMO) radar, which emits different orthogonal waveforms into space and collects the received signal with multiple antennas, has been the focus of intensive research for over a decade [1–4]. Generally, MIMO radar can be categorised into collocated MIMO radar [1] and separated MIMO radar [2] according to the distance between the transmit and receive arrays. We mainly focus on the collocated MIMO radar.

In collocated MIMO radar, the received signal of each pair of transmit–receive channel can be separated by matched-filtering to achieve waveform diversity, which raises the system degree of freedom and improves the spatial resolution as well as parameter identifiability. These advantages have been exploited in many applications, especially in the area of direction-of-arrival (DOA) estimation [5–11]. Much of the literature has generalised classic DOA estimation algorithms from conventional-phased array radar to MIMO radar, such as multiple signal classification (MUSIC) [5, 6, 12], root-MUSIC [9, 11], and estimation of signal parameters via rotational invariance technique (ESPRIT) [8, 13–15]. Nevertheless, the orthogonal waveforms in MIMO radar generate an omnidirectional transmit beampattern, which deteriorates the DOA estimation performance as the coherent processing gain is lost. To tackle this problem, the transmit beamspace (TB) MIMO radar has been developed [6, 16]. In TB MIMO radar, with a number of waveforms less than the number of transmit elements, the emitted energy can be focussed on a given spatial region via the design of a TB matrix. This trade-off between the waveform diversity and spatial diversity mitigates the deterioration of target gain caused by the omnidirectional transmit beampattern and, hence, improves the DOA estimation performance.

Meanwhile, the multi-linear property of the received signal for MIMO radar has been demonstrated [17–20]. Methods such as parallel factors analysis [12] can be used to...
decompose the factor matrices of a tensor to conduct DOA estimation efficiently. It has been shown that the DOA estimation performance of tensor decomposition-based methods is better than that of covariance matrix-based methods [12, 19]. The covariance matrix-based methods ignore the multi-linear structure of the receive signal for each pair of transmit-receive channel and, therefore, perform poorly at low signal-to-noise ratio (SNR). Recently, the multi-linear property of the received signal for TB MIMO radar has also been discussed [21]. By exploiting the tensor model and TB technique, the DOA estimation performance is improved. However, the use of TB technique makes it difficult to estimate the target spatial information from the factor matrix directly, since the Vandermonde structure is destroyed. To conduct DOA estimation for TB MIMO radar, a special structured TB matrix has been designed in [16, 21, 22] to enforce the rotational invariance property (RIP) between two submatrices of the factor matrix. Then, an ESPRIT-like method is performed to compute the phase rotations that can be used as a look up table for finding target DOA, while the potential grating lobs are eliminated by spectrum searching. The aforementioned method can only utilise half of the TB matrix to synthesise the beamspace, and the coherent processing gain is degraded. The computational complexity is also increased. To fully exploit the advantages of TB technique and tensor modelling, it is necessary to develop an efficient DOA estimation method for TB MIMO radar.

Here, a search-free DOA estimation method based on tensor modelling and polynomial rooting is proposed to improve the accuracy and resolution performance of TB MIMO radar. Specifically, a 3-order tensor is firstly designed to model the received signal of MIMO radar on the basis of the multi-linear property. The factor matrix with target DOA information is obtained by the tensor decomposition via alternating least squares (ALSs) algorithm. Then, the DOA estimation is converted to the independent minimisation problem. By using the Vandermonde structure of the transmit steering matrix, a polynomial function is constructed to estimate the target DOA via polynomial rooting. The proposed method can fully take advantage of the TB technique and tensor modelling, and it requires no spectrum searching. Simulation results show that the proposed method can achieve better accuracy and higher resolution as compared with the existing DOA estimation techniques for TB MIMO radar.

The rest of this article is organised as follows: Section 2 describes the received signal model of conventional MIMO radar in a single pulse. In Section 3, a 3-order tensor that exploits the multi-linear property of the received signal in multiple pulses is designed for TB MIMO radar, and ALS algorithm is used to decompose the factor matrix that contains target spatial information. Conventional DOA estimation methods for TB MIMO radar are summarised at the beginning of Section 4. Then, a search-free DOA estimation method via polynomial rooting is proposed. Section 5 performs two simulation examples to evaluate the performance of the proposed DOA estimation method. Finally, Section 6 concludes the study.

1.1 Notation

Vectors, matrices and tensors are denoted by boldface lowercase, boldface uppercase and calligraphic letters, for example, \( \mathbf{z} \), \( \mathbf{Z} \) and \( \mathbf{Z} \), respectively. The transposition, Hermitian transposition, pseudo-inversion, Hadamard product, outer product, Kronecker product and Khatri-Rao product operations are denoted by \( (\cdot)^T \), \( (\cdot)^H \), \( (\cdot)^\dagger \), \( \circ \) and \( \otimes \), respectively, while \( \text{vec}(\cdot) \) stands for the operator which stacks the elements of a matrix/tensor one by one to a column vector. The notation \( \text{diag}(\cdot) \) represents a diagonal matrix using elements from \( \mathbf{z} \), while \( \| \mathbf{Z} \|_F \) and \( \| \mathbf{Z} \| \) are the Frobenius norm and Euclidean norm of \( \mathbf{Z} \), respectively. Moreover, \( \mathbf{I}_M \) and \( \mathbf{0}_{M \times N} \) stand for the identity matrix of size \( M \times M \) and an all-zero matrix of dimension \( M \times N \), respectively. The estimates of \( \mathbf{Z} \) and \( \mathbf{z} \) are denoted by \( \hat{\mathbf{Z}} \) and \( \hat{\mathbf{z}} \), respectively.

2 SIGNAL MODEL

Consider a collocated MIMO radar with \( M \) transmit elements organised in a uniform linear array (ULA) and \( N \) receive elements with arbitrary array configuration within a fixed aperture. The distance between transmit elements is \( d_t \), while the coordinates of the receive elements are denoted by \( \{\zeta_n|0 < \zeta_n \leq D, n = 1, \ldots, N\} \) with \( D \) being the aperture of the receive array. The \( M \times 1 \) transmit steering vector \( \mathbf{a}(\mathbf{\theta}) \) and the \( N \times 1 \) receive steering vector \( \mathbf{b}(\mathbf{\theta}) \) can be written as \( \mathbf{a}(\mathbf{\theta}) \triangleq [1, e^{-j2\pi d_t \sin \theta_1}, \ldots, e^{-j2\pi (M-1)d_t \sin \theta}]^T \) and \( \mathbf{b}(\mathbf{\theta}) \triangleq [e^{-j2\pi \zeta_1 \sin \theta}, e^{-j2\pi \zeta_2 \sin \theta}, \ldots, e^{-j2\pi \zeta_N \sin \theta}]^T \).

Let \( \mathbf{S} \triangleq [\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_M]^T \) be the \( M \times T_s \) matrix of the pre-designed waveforms, where \( T_s \) is the number of snapshots in a single pulse. For any two different transmit waveforms in \( \mathbf{S} \), the orthogonality property needs to be satisfied, that is, \( (1/T_s)\mathbf{S}\mathbf{S}^H = \mathbf{I}_M \). Assuming there are \( L \) targets at \( \mathbf{\theta}_l \), \( l = 1, 2, \ldots, L \), the received signal of all reflections from different targets can be formulated as \( \mathbf{X} = \sum_{l=1}^L \sigma_l \mathbf{b}(\mathbf{\theta}_l)\mathbf{a}^T(\mathbf{\theta}_l)\mathbf{S} + \mathbf{N} \), where \( \mathbf{N} \) is the matrix form of the Gaussian, zero-mean and white noise component, and \( \sigma_l^2 \) is the radar cross-section (RCS) fading coefficient (can be regarded as a function of target RCS). The compact form can be written as \( \mathbf{X} = \mathbf{B}\Sigma\mathbf{A}^T\mathbf{S} + \mathbf{N} \) [12], where \( \Sigma = \text{diag}(\mathbf{\sigma}) \) is a diagonal matrix standing for the target RCS information with \( \mathbf{\sigma} \triangleq [\sigma_1^2, \ldots, \sigma_L^2]^T \), the transmit and receive steering matrices for multiple targets can be given as:

\[
\mathbf{A} \triangleq [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \ldots, \mathbf{a}(\theta_L)]_{M \times L}, \quad \mathbf{B} \triangleq [\mathbf{b}(\theta_1), \mathbf{b}(\theta_2), \ldots, \mathbf{b}(\theta_L)]_{N \times L}. \tag{1}
\]

After matched-filtering, the received signal of MIMO radar in a single pulse can be denoted by:

\[
\mathbf{Y} = \mathbf{B}\Sigma\mathbf{A}^T + \mathbf{R} \tag{2}
\]
where $\mathbf{R} = (1/T_s) \mathbf{N} \mathbf{S}^{\mathcal{H}}$ is the noise residue. The DOA estimation is to determine $\{\theta_i\}_{i=1}^L$ from the observation of $\mathbf{Y}$.

3 | TB MIMO RADAR TENSOR MODEL

In a conventional MIMO radar, each transmit element emits a unique orthogonal waveform. This property improves the system’s degree of freedom and spatial resolution; however, the coherent signal processing gain is degraded as the transmit beampattern of MIMO radar is omnidirectional. Another problem regarding DOA estimation is that the use of Equation (2) is conducted on a per-pulse basis to update the result from pulse to pulse. If multiple pulses, the DOA estimation performance is deteriorated seriously at low SNR.

To tackle these problems, we introduce the TB technique [6] for MIMO radar to achieve a trade-off between the waveform diversity and spatial diversity. Then, a 3-order tensor model is designed to store the received signal of TB MIMO radar with multiple pulses to take advantage of the multi-linear property. The tensor modelling also improves the DOA estimation performance regarding both accuracy and resolution. The tensor decomposition method is performed to estimate the factor matrix that contains the target DOA information. The uniqueness decomposition condition of the constructed tensor is derived, which indicates the upper bound of the maximum number of targets that can be resolved.

3.1 | Tensor modelling

Consider the same array configuration. For TB MIMO radar, the emitted energy is focussed on region $\Theta$ via the design of TB matrix. The number of orthogonal waveforms $K$ is less than the number of transmit elements $M$. Hence, the waveform matrix can be given by randomly selecting $K$ different rows of $\mathbf{S}$. Without loss of generality, let us choose the first $K$ rows, that is, $\mathbf{S} \triangleq [\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K]^T$. The determination of the optimal value $K$ depends on the desired region $\Theta$ [6], which can be regarded as a prior information.

Let $\mathbf{W} \triangleq [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_K]_{M \times K}$ be the TB matrix. The design of TB matrix can be reduced to a convex optimisation problem [16, 22]. Note that every orthogonal waveform $\mathbf{s}_k, k = 1, 2, \ldots, K$, is emitted at various levels that determined by $K$ different transmit beamforming vectors $\mathbf{w}_k$, the received signal of TB MIMO radar in this case can be updated with $\mathbf{X} = \mathbf{B} \mathbf{S} (\mathbf{W}^H \mathbf{A})^T \mathbf{S} + \mathbf{N}$. After proper product of $\mathbf{X}$ by $(1/T_s) \mathbf{S}^{\mathcal{H}}$, the received signal can be denoted by $\mathbf{Y} = \mathbf{B} \mathbf{S} \mathbf{D}^{\mathcal{H}} + \mathbf{R}$, where $\mathbf{D} \triangleq \mathbf{W}^H \mathbf{A}$ and $\mathbf{R} = (1/T_s) \mathbf{N} \mathbf{S}^{\mathcal{H}}$. Like Equation (2), $\mathbf{Y}$ represents the received signal of TB MIMO radar in a single pulse. Let us consider $Q$ pulses in a single CPI and also consider the Doppler effect, the received signal for $q$-th pulse, $q = 1, 2, \ldots, Q$, can be generalised as

$$\mathbf{Y}^{(q)} = \mathbf{B} \mathbf{S} \mathbf{Z}^{(q)} \mathbf{D}^T + \mathbf{R}^{(q)}$$

where $\mathbf{R}^{(q)}$ is the noise residue, $\mathbf{Z}^{(q)} = \text{diag}(\xi^{(q)})$, the $L \times 1$ vector $\xi^{(q)} \triangleq [e^{j2\pi f_1 T_s}, e^{j2\pi f_2 T_s}, \ldots, e^{j2\pi f_q T_s}]^T$ stands for the Doppler shift at $q$-th pulse, $f_i$ is the Doppler frequency, and $T_p$ is the radar pulse width. Note that $\mathbf{S}$ and $\mathbf{Z}^{(q)}$ are both diagonal matrices, the product of them can be replaced by $\text{diag}(\eta_q)$ with $\eta_q = \sigma^* \xi^{(q)}$. Vectorising Equation (3), we can write:

$$\mathbf{z}_q = (\mathbf{D} \otimes \mathbf{B}) \eta_q + \mathbf{h}_q$$

where $\mathbf{h}_q$ is the vectorised result of $\mathbf{R}^{(q)}$. Concatenate these $KN \times 1$ vectors together, that is, $\mathbf{Z} \triangleq [\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_Q]$. The compact form can be given as:

$$\mathbf{Z} = (\mathbf{D} \otimes \mathbf{B}) \mathbf{C}^T + \mathbf{H}$$

where $\mathbf{C} \triangleq [\eta_1, \eta_2, \ldots, \eta_Q]^T$ and $\mathbf{H} \triangleq [\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_Q]$.

From Equation (5), it can be observed that $\mathbf{Z}$ is the matrix form of a 3-order tensor $\mathbf{Z} \in \mathbb{C}^{K \times N \times Q}$ unfolded across the third dimension. The three factor matrices of $\mathbf{Z}$ are $\mathbf{D}$, $\mathbf{B}$, and $\mathbf{C}$, respectively. Based on the concepts of canonical decomposition and parallel factors [18, 19], $\mathbf{Z}$ can be written as:

$$\mathbf{Z} \triangleq [\mathbf{D}, \mathbf{B}, \mathbf{C}] + \mathbf{H} = \sum_{l=1}^L \mathbf{d}_l \circ \mathbf{b}_l \circ \mathbf{c}_l + \mathbf{H}$$

where $\mathbf{d}_l$, $\mathbf{b}_l$, $\mathbf{c}_l$ are the $l$-th columns of $\mathbf{D}$, $\mathbf{B}$, $\mathbf{C}$, respectively, and $\mathbf{H}$ is the noise tensor of same size. To decompose all factor matrices simultaneously, the ALS algorithm [18] can be applied, that is, solving the following problems alternately among the three factor matrices $\mathbf{D}$, $\mathbf{B}$, and $\mathbf{C}$:

$$\min_{\mathbf{D}} \|\mathbf{z}_{(1)} - [(\mathbf{C} \otimes \mathbf{B}) \mathbf{D}^T]\|^2_F$$

$$\min_{\mathbf{B}} \|\mathbf{z}_{(2)} - [(\mathbf{C} \otimes \mathbf{D}) \mathbf{B}^T]\|^2_F$$

$$\min_{\mathbf{C}} \|\mathbf{z}_{(3)} - [(\mathbf{B} \otimes \mathbf{D}) \mathbf{C}^T]\|^2_F$$

where $\mathbf{z}_{(i)}$, $i = 1, 2, 3$, denotes the mould-i unfolding of the tensor $\mathbf{Z}$. During each alternating step, the objective function in Equation (7) is quadratic with respect to the optimised matrix parameter ($\mathbf{D}$, $\mathbf{B}$ or $\mathbf{C}$). Hence, the tensor decomposition can be conducted, and the factor matrix that contains the target DOA information can be estimated. The left singular matrices of $\mathbf{Z}_{(i)}$ after truncated SVD can be used to initialise the ALS algorithm.
Remark 1 It is assumed that the tensor rank, that is, the number of targets, is given as a priori information to conduct the ALS algorithm. If there is no information about the tensor rank, it is possible to estimate the value of L from Equation (5) based on criteria like minimum description length or Akaike information theoretic [23].

3.2 Parameter identifiability

Note that the ALS algorithm Equation (7) requires that the uniqueness condition of tensor decomposition holds. A sufficient condition is given by \( \sum_{i=1}^L k_i \geq 2L + (I-1) \) [17, 18], where \( k_i \) is the Kruskal-rank of the \( i \)-th factor matrix, \( I \) is the tensor order and \( L \) is the tensor rank. The upper bound of the tensor rank raises with the increase of tensor order at the level of Kruskal-rank of a matrix. Since the targets of interest are spatially distinct, a more general uniqueness condition can be obtained [24]:

\[
\min \{ (\eta_D \cdot \theta_D), (\eta_B \cdot \theta_B), (\theta_B \cdot \eta_C) \} \geq L \tag{8}
\]

where \( \eta_D, \theta_D, \eta_B, \theta_B, \) and \( \theta_C \) are the ranks of the corresponding factor matrices, respectively. Note that \( \eta_D \leq \min(K, L), \theta_B \leq \min(N, L) \) and \( \theta_C \leq \min(Q, L) \), and \( Q \geq \max \{ [M, N] \} \), the upper bound of \( L \) is therefore determined by \( \eta_D \cdot \theta_B \geq L \), which can be relaxed as \( KN \geq L \). This conclusion is identical to that of the signal subspace theory, that is, the maximum number of targets that can be resolved is determined by the dimension of the joint transmit and receive steering vector \((W^H \alpha(\theta)) \otimes \beta(\theta))\).

4 Proposed DOA estimation method for TB MIMO radar

4.1 Conventional DOA estimation method

It is worth noting that Equation (6) is identical to the tensor model of conventional MIMO radar if TB technique is not utilised, that is, \( W \) is replaced by the identity matrix. Under this circumstance, the first factor matrix of tensor \( Z \) after tensor decomposition should be \( \hat{A} \). The target DOAs can be estimated by directly using the Vandermonde structure of \( A \) that implies the signal subspace shift-invariance. Denoting \( U \) and \( \bar{U} \) as the submatrices of \( \hat{A} \) without the first and last rows, respectively, we can write

\[
U = \bar{U} \Gamma, \quad \Gamma = \text{diag}(\gamma) \tag{9}
\]

where \( \gamma \equiv [e^{-j2\pi d_1 \sin \theta_1}, e^{-j2\pi d_1 \sin \theta_2}, \ldots, e^{-j2\pi d_1 \sin \theta_L}]^T \). Since \( \Gamma \) is full rank, it can be estimated via the least square (LS) method, that is, \( \hat{\Gamma} = \bar{U}^H \bar{U} \). Hence, the target DOA can be computed by \( \hat{\theta}_l = \arcsin[\ln(\hat{u}_l)/2\pi d_1] \), where \( \hat{u}_l \) is the \( l \)-th eigenvalue of the matrix \( \bar{U}^H \bar{U} \).

However, in TB MIMO radar, the use of TB matrix destroys the Vandermonde structure of the first factor matrix. To obtain \( \hat{\theta}_l \) from \( \hat{D} \) after tensor decomposition, the RIP can be enforced at the transmit side by using a special structured TB matrix, given as \( W = \begin{bmatrix} W_1, \ W_2 \end{bmatrix} \), where \( W_2 \) is the flipped-conjugate version of \( W_1 \in \mathbb{C}^{M \times K/2} \), that is, each column of \( W_2 \) is generated by reversing the conjugate of the corresponding column of \( W_1 \). It can be shown that the first factor matrix of tensor \( Z \) in this case satisfies [21]:

\[
D_1 = D^\perp, \quad \Omega = \text{diag} \omega \tag{10}
\]

where \( D_1 = W_1^H A, D_2 = W_2^H A \) are the submatrices of \( D \) with the first and last \( K/2 \) rows, respectively, and \( \omega = \gamma^{M-1} \). Like Equation (9), we can use the LS method to estimate \( \omega \), which can also be regarded as the shift-invariance of a virtual ULA with \( (M-1)d_1 \) distance between two adjacent elements. The factor \( (M-1) \) causes the problem of grating lobes. A finite spectrum searching method can be used to eliminate those grating lobes for every target.

It can be seen that Equation (9) exploits the Vandermonde structure of the factor matrix; however, the omnidirectional transmit beampattern of MIMO radar degrades the DOA estimation performance. The use of Equation (10) enforces the RIP between two submatrices of the first factor matrix at the cost of declined number of transmit waveforms, which mitigates the advantage of TB technique. The DOA estimation performance can be degraded due to the loss of coherent processing gain. Note that an extra spectrum searching is also required, and the computational complexity is increased.

To make the most of the TB technique and tensor model and to reduce the computational complexity, it is necessary to recover target DOAs from the factor matrix \( D \) directly for any given TB matrix.

4.2 Search-free DOA estimation via polynomial rooting

Instead of using \( K/2 \) waveforms to enforce RIP at the transmit side, let us consider a pre-designed TB matrix \( W \in \mathbb{C}^{M \times K} \) with totally \( K \) orthogonal waveforms. By using the tensor model Equation (6), the first factor matrix after tensor decomposition of \( Z \), given as \( \hat{D} = W^H \hat{A} \), can be utilised to conduct DOA estimation. Particularly, let us consider the \( l \)-th column \( W^H \alpha(\hat{\theta}_l) = \hat{d}_l \), whose solution can be given by the LS method, \( \alpha(\hat{\theta}_l) = (W^H)^{\dagger} \hat{d}_l \). However, the number of constraints is less than the number of variables \( K \leq M \), LS method in this case has infinite solutions. To tackle this problem, let:

\[
\ell^{(l)} = \text{null}(W^H) \alpha(\hat{\theta}_l) = 0_{K \times 1}, \quad l = 1, 2, \ldots, L \tag{11}
\]

where \( W^H = [d_l, 0_{K \times (M-1)}] \). The DOA estimation in this case is to find the \( \hat{\theta}_l \) such that \( \alpha(\hat{\theta}_l) \subset \mathcal{N}(W^H) \).
denotes the null space of a matrix. For each target, Equation (11) describes $K$ different constraints, that is, $f_k^{(l)} = 0$ for any $k = 1, 2, \ldots, K$, where $f_k^{(l)}$ is the $k$-th element of the vector. These constraints can be further written as $\sum_{k=1}^{K}|f_k^{(l)}|^2 = 0$, since they hold if and only if $f_k^{(l)} = 0$ for any $k = 1, 2, \ldots, K$.

Define a polynomial $p(z_l) \triangleq [1, z_l, \ldots, z_l^{M-1}]^T$ with a complex generator $z_l$. When $z_l$ falls on the unit circle, for example, $z_l = e^{-j2\pi f_l \sin \theta}$, $p(z_l)$ can be regarded as a transmit steering vector at direction $\theta_l$. Then, we can write $f^{(l)} = V^H p(z_l)$ and further interpret it as

$$ F(z_l) \triangleq \sum_{k=1}^{K} |v_k^T p(z_l)|^2 = 0 \quad (12) $$

where $v_k^T$ is the $k$-th row of $V$. It can be shown that $F(z_l)$ can be regarded as the sum of $K$ different polynomial functions, which is still a polynomial function. Any $z_l$ satisfying Equation (12) belongs to the roots of the constructed polynomial function. Hence, the DOA estimation can be conducted by rooting Equation (12), and the target DOA can be computed by:

$$ \hat{\theta}_l = \arcsin[j\lambda \ln(\tilde{z}_l)/2\pi d_z] \quad (13) $$

where $\tilde{z}_l$ denotes the root of $F(z_l)$. Next, let us prove the uniqueness of $\tilde{z}_l$, that is, how to select the proper root. Define

$$ G(\theta) \triangleq f^{(l)H} f^{(l)} = \alpha^H(\theta) V^{(l)H} V^{(l)} \alpha(\theta) \quad (14) $$

which is a non-negative convex function. It can be observed that if $\hat{\theta}_l$ satisfies Equation (11), it is also the optimal point of the optimisation problem $\min_\theta G(\theta)$, that is, $\hat{\theta}_l \leftarrow \min_\theta G(\theta)$. Meanwhile, the polynomial function $F(z_l)$ can be rewritten as:

$$ F(z_l) = p^H(z_l) V^{(l)H} V^{(l)} p(z_l) \quad (15) $$

which has $M-1$ pairs of roots. The structure of $F(z_l)$ and $G(\theta)$ is similar. The minimisation of $G(\theta)$ equals to the minimisation of $F(z_l)$ with an extra constraint $|z_l| = 1$. In other words, the optimal set of $G(\theta)$ is the subset of the optimal set of $F(z_l)$.

Note that the solution of Equation (14) is unique, only one root of $F(z_l)$ will fall on the unit circle. Hence, the root that is nearest to the unit circle should be chosen as $\tilde{z}_l$ after polynomial rooting of $F(z_l)$. The procedures of the proposed DOA estimation method are summarised in Table 1.

**Remark 2** It is worth noting that $G(\theta)$ is always convex and the optimal value in Equation (14) is unique and achievable for steering vector $\alpha(\theta)$ with arbitrary structure. As a special case, the ULA structure enables the polynomial rooting method for optimising this objective function, which can be exploited by $F(z_l)$. Essentially, the proposed search-free DOA estimation method can be generalised to TB MIMO radar with arbitrary transmit array configuration by updating the structure of the steering vector.

### 4.3 Computational complexity

First, let us consider the computational complexity of the ALS algorithm. For each iteration, the number of flops that is required is about $O(4(I+1)LKNQ)$ [25]. In step 2 regarding polynomial rooting, it needs $O(2LM\log(2M) + KM^2)$ flops. Therefore, the proposed method requires approximately $O(4(I+1)LKNQP + 2LM\log(2M) + KM^2)$ flops, where $P$ is the number of iterations in the ALS algorithm. It can be found that the computational complexity is mostly determined by the tensor decomposition. For comparison, we analyse the computational complexity of the conventional DOA estimation methods described in Section 4.1 [12, 21], where the tensor modelling and the ALS algorithm are also used. Let us focus on the process of computing target DOA via the factor matrix in the second step. For conventional MIMO radar, the number of flops required by Equation (9) is about $O((M-1)^3 + (M-1)^2L + L^3)$. For TB MIMO radar with special structured TB matrix, the number of flops required by Equation (10) and the finite spectrum searching is roughly $O(K^3 + K^2L + L^3 + (KN)^3 + KMN L)$. An additional spectrum searching is required, which increases the computational complexity. However, our proposed method...
requires only \(O(KM^2)\) flops for building the matrix with polynomial coefficients plus \(O(2LM \log(2M))\) flops for polynomial rooting. The computational complexity of the proposed method surpasses the others.

5 | SIMULATION RESULTS

In this section, we present two simulation examples to evaluate the DOA estimation performance of the proposed method regarding root mean square error (RMSE) and probability of resolution. ESPRIT [13], TENSOR [12], TB-ESPRIT [16] and TB-TENSOR [21] are given for comparison. Throughout the simulations, a MIMO radar with \(M = 10\) and \(N = 10\) elements is assumed. Let \(d_i = \lambda/2\), and the receive elements are randomly spaced in a linear array with aperture of \(5\lambda\). The RCS fading coefficient \(\sigma^2\) is chosen from a standard Gaussian distribution as a complex value. The normalised Doppler shifts are \(f_1 = 0.1\) and \(f_2 = -0.25\). The number of pulses in a single CPI is \(Q = 64\). The number of Monte Carlo trials is \(J = 500\). The orthogonal waveforms used here are \(s_k(t) = \sqrt{\frac{1}{T_\tau}} e^{j2\pi f_k t}, k = 1, 2, …, K\).

When applying methods of ESPRIT and TENSOR, the TB matrix is the identity matrix. The methods of TB-ESPRIT and TB-TENSOR use identical TB matrix \(W = [W_1, W_2]\) with \(K/2\) acting waveforms to enforce RIP, while our proposed method applies \(W\) with totally \(K\) orthogonal waveforms as the TB matrix. The emitted energy in TB MIMO radar is focussed on \(\Theta:\ [15^\circ, 15^\circ]\).

5.1 | Example 1: RMSE versus SNR

In this example, two targets at \(\theta_i = [-10^\circ, 10^\circ]\) are assumed. The SNRs of two targets are both 5 dB. After tensor decomposition, the polynomial functions are constructed via Equation (15) and the target DOAs are estimated. The RMSE is computed by:

\[
RMSE = \sqrt{\frac{1}{2JL} \sum_{k=1}^{L} \sum_{j=1}^{J} (\hat{\theta}_i(j) - \theta_i(j))^2}.
\]

The RMSEs of methods tested are shown in Figure 1. The RMSE decreases stably with the rise of SNR. Results of ESPRIT and TENSOR are quite similar, where TENSOR method surpasses a little. By applying the TB technique, it can be observed that results of TB-ESPRIT and TB-TENSOR are improved as compared with their conventional MIMO counterparts, respectively. Nevertheless, the number of waveforms is half the optimal value to enforce RIP at transmit side, which mitigates the advantages of TB technique. In our proposed method, the design of polynomial function enables the search-free DOA estimation without additional requirement on TB matrix and avoids the spectrum searching. The well-known generalized sidelobe canceller structure can be used to demonstrate the relationship between root of polynomial nearest to the unit circle and target DOA information. Since our proposed method fully takes advantages of TB technique, the RMSE is substantially lower than those of the aforementioned methods.

5.2 | Example 2: probability of resolution versus SNR for two closely spaced targets

To evaluate the probability of resolution of two closely spaced targets, we assume that \(\theta_i = [10^\circ, 11^\circ]\). The other parameters are unchanged. These two targets are considered to be resolved if

\[
||\hat{\theta}_i - \theta_i|| \leq ||\theta_1 - \theta_2||, \quad l = 1, 2 \text{ holds for both targets.}
\]

In Figure 2, the probability of resolution of two closely spaced targets is shown. In particular, the closely spaced targets can be resolved with probability 1 when SNR is high. However, the resolution probability starts to decline with the decrease of SNR. It can be observed that our method achieves the lowest threshold. Hence, the DOA estimation performance of the proposed method surpasses other methods with better accuracy and higher resolution.

6 | CONCLUSION

A search-free DOA estimation method based on tensor modelling and polynomial rooting has been proposed to improve the accuracy and resolution for TB MIMO radar. In the proposed method, the DOA estimation has been converted to independent polynomial rooting problems by approaching the received signal via 3-order tensor modelling and tensor decomposition. The proposed DOA estimation method can fully take advantage of TB technique and tensor modelling. No spectrum searching is required, which reduces the computational complexity. The essence of the search-free DOA
estimation method is to minimise the virtual transmit beamspace pattern of TB MIMO radar. Simulation results have verified the performance improvement of the proposed method over the existed DOA estimation techniques for TB MIMO radar.

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