A dynamical evolution model on the black-hole horizon

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Abstract
This paper demonstrates a dynamical evolution model of the black-hole (BH) horizon. The result indicates that a kinetic area-cell model of the BH horizon can model the evolution of the BH due to the Hawking radiation, and this area-cell system can be considered as an interacting geometrical particle system. Thus, the evolution turns into a problem of statistical physics. In the present work, this problem is treated in the framework of non-equilibrium statistics. It is proposed that each area cell possesses energy like a microscopic black hole and has gravitational interaction with the other area cells. We consider both a non-interaction ideal system and a system with small nearest-neighbour interactions, and obtain an analytic expression of the expected value of the horizon area of a dynamical BH. We find that, after a long enough evolution, a dynamical BH with the Hawking radiation can be in equilibrium with a finite-temperature radiation field. However, we also find that the system has a critical point, and when the temperature of the radiation field surrounding the BH approaches the critical temperature of the BH, a critical slowing-down phenomenon occurs.

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The discovery of the thermal radiation of black holes (the Hawking radiation [1]) is a significant breakthrough in our understanding of black holes. However, after three decades, there is still disagreement on a number of important issues. For example, it is commonly accepted that black holes have finite temperature, which is proportional to the surface gravity of the event horizon and inversely proportional to the mass, and black holes can be in equilibrium with a finite-temperature radiation field. However, it is also known that the thermal radiation of black holes is a quantum effect. As the thermal radiation goes on, the black hole emits particles. As its mass decreases, the temperature rises and, as a result, the black hole emits particles
more quickly. It is thus foreseeable that the black hole cannot keep thermal equilibrium, and it will finally evaporate and reach zero mass. Now we turn to review the formation of a black hole from the classical point of view. A star in its late stage of evolution may evolve into a black hole, due to gravitational collapse. This black hole quickly reaches a stationary state, which can be characterized by three parameters: mass, charge and angular momentum. Even if we consider that the black holes with nonzero angular momentum and charge are in an active excited state, the Schwarzschild black holes with zero angular momentum and zero charge are in an inactive ground state, which is the end result of star evolution. In light of the above-mentioned points, questions naturally arise: Certainly a black hole with a certain temperature can be taken as a thermodynamic system, but can it be taken as a system in thermal equilibrium? Is equilibrium statistics applicable? Will thermal radiation really lead to the total evaporation of a black hole? What is the end result of star evolution? These are still open questions. Before the real theory of quantum gravity is completely established, can we find consistent answers?

In the present work, we aim at establishing a model of non-equilibrium dynamical evolution in the framework of statistical physics. The expected area of the event horizon is calculated and its long-time evolution is observed. These results are used to shed light on the above-mentioned questions.

The model. In the past few years, loop quantum gravity (LQG) has become a serious candidate for a non-perturbative quantum theory of gravity [2]. Its most notable prediction is the quantization of geometry [3]. As an analogy, in LQG, the fabric of space is like a weave of tiny threads, and each thread poking through a surface gives it a little bit of area. The surface area of a black hole (BH) then is generated by all the threads puncturing it. The event horizon is flat except at these punctures, where it can flex, and the microstates of the BH are defined by the different ways the event horizon can flex in or out [4].

According to the results of LQG, the horizon surface is split into a series of $N$ discrete cells (small surface elements at the Planck scale). Each cell has a discrete spectrum of area

$$A_i(j_i) = 8\pi l_p^2 \gamma \sqrt{j_i(j_i + 1)}, \quad i = 1, 2, 3, \ldots, N, \quad j_i = 0, 1/2, 1, 3/2, \ldots,$$

(1)

where $l_p$ is the Planck length, $\gamma$ is the Immirzi parameter and $j$ is the irreducible representation of the gauge group $SU(2)$. Due to the Hawking quantum evaporation, the geometrical area of these microscopic cells evolves with time.

Now, we can investigate the dynamical evolution of the event horizon of a BH from the quantum point of view. During the time interval of $t \sim t + dt$, the event horizon of a macroscopic BH can be considered as a set of area cells at the Planck scale. The area cells, with half-integer spins, $j = 0, 1/2, 1, 3/2, \ldots$ (irreducible representation of the $SU(2)$ gauge group), are characterized by quantized geometrical area and thus are referred to as quantum geometrical particles. For an individual area cell, it is a small quantum BH, and its energy has the same form as a microscopic black hole. However, adjacent area cells have gravitational

1 With respect to ‘time’, we must point out that, although it is true that there exists a notion of time in the 3 + 1 formulation, it is certainly untrue that the notion of time is unambiguous. However, in order to carry on the discussion of the dynamic evolution of black-hole horizon using non-equilibrium statistical mechanics, here we still give a clear definition of ‘time’ as an assumption. The extent to which this assumption is acceptable can be judged from the final result of the research.

2 As far as I know, the idea of ‘quantum geometrical particle’ was first given by Major and Setter in [5]. What I especially would like to emphasize here is that, except for the quantized geometric area (labelled by spin $j$), the other microscopic details of the geometric particles have been averaged out, in a way technically similar to the mean-field approximation used to treat ferromagnetic systems in condensed matter physics. This is an issue worthy of further discussion.
interaction and many area cells together form an interacting system of geometrical particles with spin. Since there can be various possible transitions among the area cells in different states, the collective transition of the system becomes the quantum evaporation of the macroscopic BH. We use a heat bath at temperature $T$ to simulate the radiation field surrounding the BH and suppose that the geometrical particle system is in contact with this heat bath, but it may be in a non-equilibrium state.

A system consisting of these geometrical particles with area (or spin) as a characteristic variable is very similar to a spin–lattice system in condensed matter physics. The contact of the system with a heat bath may reflect the situation of a BH in a radiation field. The Hawking evaporation leads the characteristic parameters to evolve with time. In this case, the evolution of the event horizon of a BH can be investigated by studying the dynamical behaviour of a geometrical particle system in non-equilibrium states. A generalized Glauber-type dynamics with single spin transition mechanism [6–8] can be very useful here. In this paper, we will reformulate the Glauber dynamics to make it suitable for the kinetic geometrical system and then apply it to a black hole with discrete area spectrum.

The formulation of the dynamical scheme. The area of each fixed cell on the event horizon of a BH is represented as a stochastic function of time $A_i(t)$ ($i = 1, \ldots, N$), which can be taken as discrete values. Transitions can occur among these values. The transition probability $W_i(A_i(t) \rightarrow \hat{A}_i(t))$ from the configuration $(A_1(t), A_2(t), \ldots, A_i(t), \ldots, A_N(t))$ to the configuration $(\hat{A}_1(t), A_2(t), \ldots, A_i(t), \ldots, A_N(t))$, in general, depends on the momentary values of the neighbouring cells as well as on the influence of the heat bath. For this reason, statistical correlations exist between different geometrical particles. Therefore, it is necessary to treat the entire $N$-particle system as a whole. The evolution of the particles’ area functions, which describes the evolution of the system, forms a Markov process of $N$ discrete random variables with a continuous time variable as argument.

We introduce a probability distribution function $P(\{A\}; t)$, which denotes the probability of the geometrical particle system being in the state of $\{A\} = (A_1, \ldots, A_i, \ldots, A_N)$ at time $t$. Let $W_i(A_i \rightarrow \hat{A}_i)$ be the transition probability per unit time that the $i$th particle transits from area $A_i$ to another possible area $\hat{A}_i$, while the others remain unchanged. Then, on the supposition of a single-particle transition, we may write the time derivative of the function $P(\{A\}; t)$ as

$$\frac{d}{dt}P(\{A\}; t) = \sum_i \sum_{\hat{A}_i} g(\hat{A}_i)[−W_i(A_i \rightarrow \hat{A}_i)P(\{A\}; t) + W_i(\hat{A}_i \rightarrow A_i)P(\ldots, \hat{A}_i, \ldots; t)],$$

where $g(\hat{A}_i)$ is the degeneracy of state $\hat{A}_i$. This is a probability equation, in which the first term on the right-hand side denotes the decrease of the probability distribution function $P(\{A\}; t)$ per unit time due to the transition of the particle state from the initial value $A_i$ ($i = 1, 2, \ldots, N$) to various possible final values $\hat{A}_i$ and the second term denotes the contrary situation. We shall refer to equation (2) as the master equation since its solution would contain the most complete description available of the system.

It is the most crucial step, obviously, to determine the transition probabilities, $W_i(A_i \rightarrow \hat{A}_i)$, $i = 1, 2, 3, \ldots, N$, before the master equation can possibly be solved. Then, how to determine the transition probabilities? We have both mathematical and physical considerations. On the one hand, mathematically, the transition probability must be positive definite and normalized, and physically, a thermodynamic system in a slowly varying process must have ergodicity and satisfy the detailed balance condition. On the other hand, the transition probabilities of the individual particles depend mainly on the momentary values of
the neighbouring particles as well as on the influence of the heat bath. Thus, the transition probability from \( A_i \) to \( \hat{A}_i \) must depend on the heat Boltzmann factor of the neighbouring particles. Based on these considerations, the following form of the transition probability

\[
W_i(A_i \rightarrow \hat{A}_i) = \frac{1}{Q_i} \exp \left[ -\beta H_{\text{eff}} \left( \hat{A}_i, \sum_w A_{i+w} \right) \right]
\]

(3)

is a natural choice, where \( Q_i \) is the normalization factor determined by the normalized condition \( \sum_{\hat{A}_i} g(\hat{A}_i) W_i(A_i \rightarrow \hat{A}_i) = 1 \) and the summation \( \sum_w \) is taken over the neighbouring particles of \( i \).

Usually, we are only interested in the expected value of a certain function of the area

\[
\langle f(\hat{A}_k) \rangle = \sum_{\{\hat{A}\}} g(\hat{A}_1) \cdots g(\hat{A}_N) f(\hat{A}_k) P(\{\hat{A}\}, t).
\]

(4)

According to definition (4) and the master equation (2), the time-evolution equation of \( \langle f(\hat{A}_k) \rangle \) can be derived (see the appendix)

\[
\frac{d}{dt} \langle f(\hat{A}_k) \rangle = -\langle f(\hat{A}_k) \rangle + \sum_{\{\hat{A}\}} g(\hat{A}_1) \cdots g(\hat{A}_N) \left( \sum_{\hat{A}_k} g(\hat{A}_k) f(\hat{A}_k) W_k(\hat{A}_k \rightarrow \hat{A}_k) \right) P(\{\hat{A}\}, t).
\]

(5)

### Application to the ideal geometrical particle system.

As a first step, we consider an ideal geometrical particle system. In this case, the total area of the horizon, \( A(t) = \sum_k A_k(t) \), can be taken as the parameter. Therefore, the time-evolution equation (5) can be simplified to the following form:

\[
\frac{d}{dt} \langle A(t) \rangle = -\langle A(t) \rangle + \sum_A g(A) \left( \sum_{\hat{A}} g(\hat{A}) \hat{A} W(\hat{A} \rightarrow A) \right) P(A; t).
\]

(6)

Because

\[
\sum_{\hat{A}} g(\hat{A}) \hat{A} W(A \rightarrow \hat{A}) = \frac{\sum_{\hat{A}} g(\hat{A}) \hat{A} \exp[-\beta H_{\text{eff}}(\hat{A})]}{\sum_{\hat{A}} g(\hat{A}) \exp[-\beta H_{\text{eff}}(\hat{A})]} = \langle A \rangle_{\text{eq}}
\]

(7)

and

\[
\sum_A g(A) P(A; t) = \text{Tr} P(A; t) \equiv 1,
\]

(8)

we can obtain

\[
\langle A(t) \rangle = \langle A \rangle_{\text{eq}} + \langle (A(0)) - \langle A \rangle_{\text{eq}} \rangle e^{-t}.
\]

(9)

Expression (9) means that the event horizon as a whole evolves in the form of an exponential decrease with time. Obviously, the result of a long-time evolution is \( \langle A(\infty) \rangle = \langle A \rangle_{\text{eq}} \). From equation (7), as long as the discrete spectrum expression of \( A \) is given, the expected value of the horizon’s area, \( \langle A \rangle_{\text{eq}} \), can be calculated.

### Application to interacting geometrical particle system.

In our consideration, an individual area cell (a quantum geometrical particle) is regarded as a microscopic quantum BH, and between the neighbouring geometrical particles there is gravitational interaction. We can adopt as a
hypothesis that the energy (mass)–area relation is a power law, and as a specific choice, we suppose that its energy (mass) is proportional to the square root of the area (Schwarzschild-type)\(^3\)

\[ \varepsilon_i = m_i \propto \sqrt{A_i(n_i)}, \]  
(10)

where \( A_i(n_i) = 4\pi l_p^2 \sqrt{n_i(n_i + 2)} = a_0 \sqrt{n_i(n_i + 2)}, \) \( a_0 = 4\pi l_p^2 \gamma \), \( n_i = 2j_i = 0, 1, 2, 3, \ldots \)

The gravitational interaction between the neighbouring geometrical particles, however, is still an unknown problem. But our understanding is that (1) the Planck scale \( l_p \) is the minimum length in the microscopic structure models of quantum gravity theory, and (2) the event horizon is flat except at those punctures generated by all the spin network’s edges puncturing it \([4]\).

So, we might as well adopt simply a classical gravitational potential as follows:

\[ u_{ij} = \begin{cases} \infty, & r_{ij} = 0, \\ -\frac{m_i m_j}{r_{ij}} \propto -\frac{1}{l_p} \sqrt{A_i(n_i)} \sqrt{A_j(n_j)}, & r_{ij} = \text{NN distance}, \\ 0, & \text{otherwise}. \end{cases} \]  
(11)

Thus, the effective Hamiltonian (with only nearest-neighbour (NN) interaction) can be written as

\[ \mathcal{H}_{\text{eff}} = \sum_i \varepsilon_i + \sum_{i<j} u_{ij} = m \sum_i [n_i(n_i + 2)]^{1/4} - u \sum_{(i,j)} [n_i(n_i + 2)n_j(n_j + 2)]^{1/4}, \]  
(12)

where \( m \) and \( u \) are the scaling factors \( (m > 0, u > 0) \) and each sum \( (i, j) \) runs over all NN pairs. Thus, the transition probability per unit time, that the characteristic quantity of the \( i \)th particle transits from one value \( n_i \) to another possible value \( \hat{n}_i \), can be written as

\[ W_i(A_i(n_i) \rightarrow \hat{A}_i(\tilde{n}_i)) = \frac{1}{Q_i} \exp \left\{ -\beta m[\hat{n}_i(\hat{n}_i + 2)]^{1/4} + \beta u \sum_w [\hat{n}_i(\hat{n}_i + 2)n_{i+w}(n_{i+w} + 2)]^{1/4} \right\}, \]  
(13)

where

\[ Q_i = \sum_{\hat{n}_i} (\hat{n}_i + 1) \exp \left\{ -\beta m[\hat{n}_i(\hat{n}_i + 2)]^{1/4} + \beta u \sum_w [\hat{n}_i(\hat{n}_i + 2)n_{i+w}(n_{i+w} + 2)]^{1/4} \right\}. \]

Here, the degeneracy of spin state \( j_i \), \( g(j_i) = 2j_i + 1 = \hat{n}_i + 1 \), is taken into account. At high temperature, the summation for \( \tilde{n}_i \) can be approximately replaced by an integral \( \sum_{\tilde{n}_i} \rightarrow \int d\tilde{n}_i \).

In order to get an analytical result, we consider the case of a weak gravitational interaction and a high environmental temperature. Then, one can obtain

\[ \sum_{\hat{n}_i} g(\hat{n}_i) \hat{A}_i(\tilde{n}_i) W_i(n_i \rightarrow \hat{n}_i) = \frac{20}{\beta^2 m^2 a_0} \left[ 1 + 2 \frac{u}{m} \frac{1}{\sqrt{a_0}} \sum_w \sqrt{A_{i+w}(n_{i+w})} + O \left( \left( \frac{u}{m} \right)^2 \right) \right], \]  
(14)

\[ \sum_{\hat{n}_i} g(\hat{n}_i) \hat{A}_i(\tilde{n}_i) W_i(n_i \rightarrow \hat{n}_i) = \frac{4}{\beta m \sqrt{a_0}} \left[ 1 + \frac{u}{m} \frac{1}{\sqrt{a_0}} \sum_w \sqrt{A_{i+w}(n_{i+w})} + O \left( \left( \frac{u}{m} \right)^2 \right) \right]. \]  
(15)

\(^3\) What is chosen here is the same as in \([3]\) (see formula (5)).
Therefore, we have the following evolution equations:

\[
\frac{d}{dt} \langle A_i(n_i) \rangle = 20 \frac{a_0}{\beta^2 m^2} - \langle A_i(n_i) \rangle + 40 \frac{\sqrt{a_0}}{\beta^2 m^2} \frac{u}{m} \sum_w \langle \sqrt{A_{i+w}(n_{i+w})} \rangle + O \left( \left( \frac{u}{m} \right)^2 \right),
\]

(16)

\[
\frac{d}{dt} \langle \sqrt{A_i(n_i)} \rangle = 4 \sqrt{a_0} - \langle \sqrt{A_i(n_i)} \rangle + 4 \frac{1}{\beta m} \frac{u}{m} \sum_w \langle \sqrt{A_{i+w}(n_{i+w})} \rangle + O \left( \left( \frac{u}{m} \right)^2 \right);
\]

(17)

equivalently,

\[
\frac{d}{dt} \left\{ \langle A(t) \rangle - \frac{10}{\beta m} \left[ \langle \sqrt{A_1(t)} \rangle + \langle \sqrt{A_2(t)} \rangle + \ldots \right] \right\}
\]

\[= - \frac{20 a_0}{\beta^2 m^2} N \left\{ \langle A(t) \rangle - \frac{10 \sqrt{a_0}}{\beta m} \left[ \langle \sqrt{A_1(t)} \rangle + \langle \sqrt{A_2(t)} \rangle + \ldots \right] \right\},
\]

(18)

\[
\frac{d}{dt} \left[ \langle \sqrt{A_1(t)} \rangle + \langle \sqrt{A_2(t)} \rangle + \ldots \right] = \frac{4 \sqrt{a_0}}{\beta m} N \left( 1 - \frac{16 u}{\beta m m} \right) \left[ \langle \sqrt{A_1(t)} \rangle + \langle \sqrt{A_2(t)} \rangle + \ldots \right].
\]

(19)

The exact solution is

\[
\langle A(t) \rangle = N \frac{20 a_0}{\beta^2 m^2} \left( \frac{2}{1 - \frac{16 u}{\beta m m}} - 1 \right) + \frac{10 \sqrt{a_0}}{\beta m} C_2 e^{-t/\tau} + C_1 e^{-t},
\]

(20)

where \( \langle A(t) \rangle = \sum_i \langle A_i(t) \rangle \) denotes the total area, \( C_1 \) and \( C_2 \) are the integral constants and \( \tau \) is the relaxation time of the system.

\[
\tau = \frac{1}{1 - \frac{16 u}{\beta m m}} = \left( 1 - \frac{\beta_c}{\beta} \right)^{-1} = \left( 1 - \frac{T}{T_c} \right)^{-1}.
\]

We note that the system has a critical point, \( \beta_c = \frac{16 u}{m} \). When the gravitational interaction is negligible, the system will evolve rapidly to the equilibrium state

\[
\langle A \rangle_0^\text{eq} = N \frac{20 a_0}{\beta^2 m^2} \propto l_s^6 N \left( \frac{k_B T}{m} \right)^2,
\]

which agrees with the result of [5]. However, a more complex dynamical behaviour will appear when the interaction cannot be ignored. If the temperature of the radiation field is much lower than the critical temperature \( T_c \), the system will still rapidly approach the equilibrium state

\[
\langle A \rangle_\text{eq} = N \frac{20 a_0}{\beta^2 m^2} \left( \frac{2}{1 - \frac{16 u}{\beta m m}} - 1 \right).
\]

However, if the temperature approaches the critical point, the system can hardly reach the equilibrium state, and this phenomenon is commonly known as the critical slowing down. It is surprising that the horizon area will increase continuously when the temperature is higher than the critical point. But this case should be excluded in the present discussion using this specific method. The reason is that, due to the very intensive heat exchange, we cannot regard the radiation field surrounding the BH as a heat bath with constant temperature.

This paper demonstrates a dynamical evolution model of the black-hole (BH) horizon with discrete area spectrum. The result indicates that the evolution of a BH due to the Hawking radiation can be modelled by a kinetic area-cell model of the BH horizon, and this area-cell system can be considered as an interacting geometrical particle system with spin. Thus, the evolution turns into a problem of statistical physics. In the present work, this problem is treated in the framework of non-equilibrium statistics, and the expected area of the event horizon is
obtained. We find that, after a long enough evolution, a dynamical BH with the Hawking radiation can be in equilibrium with a finite-temperature radiation field. However, we also find that the system has a critical point, and when the temperature of the radiation field surrounding the BH approaches the critical temperature of the BH, a critical slowing-down phenomenon occurs.

Of course, the present work is only a preliminary attempt on the evolution of the BH horizon in the framework of a non-equilibrium statistics. Any further study, such as to choose a well-defined Hamiltonian and to consider the possibility of creating or annihilating geometrical particles and so on, is very interesting.

I close this paper with some comments. Due to the discreteness of spacetime itself at the Planck scale, there is a minimum length, namely the Planck length, which is similar to the lattice constant in condensed matter. Thus, mature methods and viewpoints developed from condensed matter physics and statistical physics can be used for reference in the study of the discrete quantum spacetime. In this direction, some groups have made interesting and enlightening efforts (see, for example, [9–12]).

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Appendix. Proof of equation (5)

According to definition (4) of $\langle f(A_k) \rangle$ and the master equation (2), we have

$$
\frac{d}{dt} \langle f(A_k) \rangle = \sum_{\{A\}} \left( \prod_{\alpha=1}^{N} g(A_{\alpha}) \right) f(A_k) \sum_{i} \sum_{\tilde{A}_i} g(\tilde{A}_i) \left[ -W_i(A_i \rightarrow \tilde{A}_i) P([A], t) + W_i(\tilde{A}_i \rightarrow A_i) P([A_{\neq k}], \tilde{A}_k, t) \right] \\
+ \sum_{\{A\}} \left( \prod_{\alpha=1}^{N} g(A_{\alpha}) \right) f(A_k) \sum_{i(i \neq k)} \sum_{\tilde{A}_i} g(\tilde{A}_i) \left[ -W_i(A_i \rightarrow \tilde{A}_i) P([A], t) + W_i(\tilde{A}_i \rightarrow A_i) P([A_{\neq k}], \tilde{A}_k, t) \right] \\
+ \sum_{\{A\}} \left( \prod_{\alpha=1}^{N} g(A_{\alpha}) \right) f(A_k) \sum_{\tilde{A}_k} g(\tilde{A}_k) \left[ -W_k(A_k \rightarrow \tilde{A}_k) P([A], t) + W_k(\tilde{A}_k \rightarrow A_k) P([A_{\neq k}], \tilde{A}_k, t) \right].
$$

(A.1)

Looking at the $i \neq k$ term of (A.1),

$$
(i \neq k) \text{ term} = \sum_{\{A_{\alpha}\}} \left( \prod_{\alpha=1}^{N} g(A_{\alpha}) \right) f(A_k) \\
\times \sum_{i(i \neq k)} g(A_i) g(\tilde{A}_i) \left[ -W_i(A_i \rightarrow \tilde{A}_i) P([A], t) + W_i(\tilde{A}_i \rightarrow A_i) P([A_{\neq k}], \tilde{A}_k, t) \right].
$$

(A.2)
it is easy to see that this term equals zero, as long as \( \hat{A}_i \) exchange with \( A_i \) before doing the sum for \( \hat{A}_i \) and \( A_i \). So the surplus term of (A.1) is only the last term \((i = k)\)

\[
\begin{align*}
\frac{d}{dt} \langle f(A_k) \rangle & = \sum_{\{A\}} \left( \prod_{\alpha = 1}^{N} g(A_{\alpha}) \right) f(A_k) \left( \sum_{\hat{A}_k} g(\hat{A}_k) \right) \left[ -W_k(A_k \rightarrow \hat{A}_k) P([A], t) \right. \\
& \quad \left. + \sum_{A_1, \ldots, \hat{A}_k, \ldots, A_N} \sum_{\hat{A}_k} g(\hat{A}_k) f(\hat{A}_k) W_k(A_k \rightarrow \hat{A}_k) \right] P([A], t) \\
& \quad + \sum_{\{A\}} \left( \prod_{\alpha = 1}^{N} g(A_{\alpha}) \right) f(A_k) \left( \sum_{\hat{A}_k} g(\hat{A}_k) \right) \left[ W_k(\hat{A}_k \rightarrow A_k) P([A_j \neq k], \hat{A}_k, t) \right] \\
& \quad \left. - \langle f(A_k) \rangle + \sum_{\{A\}} \left( \prod_{\alpha = 1}^{N} g(A_{\alpha}) \right) \left( \sum_{\hat{A}_k} g(\hat{A}_k) f(\hat{A}_k) W_k(A_k \rightarrow \hat{A}_k) \right) P([A], t) \right),
\end{align*}
\]

in which the normalized condition \( \sum_{\hat{A}_k} g(\hat{A}_k) W_k(A_k \rightarrow \hat{A}_k) = 1 \) and the technique of exchange of \( \hat{A}_i \) for \( A_i \) were used. Hitherto, equation (5) has been proven exactly.

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