Relativistic non-instantaneous action-at-a-distance interactions

Domingo J. Louis-Martinez
Science One Program and
Department of Physics and Astronomy,
University of British Columbia
Vancouver, Canada

Abstract

Relativistic action-at-a-distance theories with interactions that propagate at the speed of light in vacuum are investigated. We consider the most general action depending on the velocities and relative positions of the particles. The Poincaré invariant parameters that label successive events along the world lines can be identified with the proper times of the particles provided that certain conditions are imposed on the interaction terms in the action. Further conditions on the interaction terms arise from the requirement that mass be a scalar. A generic class of theories with interactions that satisfy these conditions is found. The relativistic equations of motion for these theories are presented. We obtain exact circular orbits solutions of the relativistic one-body problem. The exact relativistic one-body Hamiltonian is also derived. The theory has three components: a linearly rising potential, a Coulomb-like interaction and a dynamical component to the Poincaré invariant mass. At the quantum level we obtain the generalized Klein-Gordon-Fock equation and the Dirac equation.
In the past thirty years or so, a great deal of work has been focused on the problem of relativistic bound states [1] - [14], particularly on the relativistic equations for quark-antiquark bound states and the problem of deriving the meson spectrum[15], [16]. Within the relativistic action-at-a-distance formulation of Wheeler and Feynman [17] - [22] (for electrodynamics) solutions of the two-body Dirac equations [23], [24] have been found for positronium [23]. The spectrum obtained by this method agrees with the result of quantum field theory [25], [26] at least up to the $\alpha^4$ order. The approach has also been applied to mesons [27]. There is strong experimental evidence that for large separations the interaction between quarks can be effectively described by a linearly rising potential [16]. Several relativistic generalizations of a linearly rising potential have been studied [28] - [32]. From quantum chromodynamics, using the Wilson loops approach [33], it has been established that the quark-antiquark bound states are effectively described by a static potential, which is a sum of a linearly rising potential and a Coulomb-like interaction: $V = \sigma r - \frac{k}{r}$ [34].

In this Letter we extend the approach of Wheeler and Feynman to explore what types of interparticle interactions are allowed in special relativity. We assume that the interactions travel at the speed of light in vacuum and that the theory can be described by an action principle for which the interaction terms in the action do not depend on the four-vector accelerations or on higher derivatives.

We find explicitly the most general theory that satisfies these conditions. In the static limit we find the theory has three components: a linearly rising potential, a Coulomb-like interaction and a dynamical component to the Poincaré invariant mass.

We obtain the relativistic equations of motion for $N$ particles and apply these results to the relativistic one-body problem, for which we obtain explicitly the Hamiltonian. Quantum mechanical equations for spinless particles and for spin $\frac{1}{2}$ particles are presented at the end of the Letter. The possibility of considering the effect of a dynamical component to the quark masses in current phenomenological models is naturally suggested by the results obtained here.

Let us consider a system of $N$ interacting relativistic particles. Let $m_i$ ($i = 1, 2, ..., N$) be the mass of particle $i$, $c$ is the speed of light and $\lambda_i$ a Poincaré invariant parameter labelling
the events along the world line $z_i^\mu(\lambda_i)$ of particle $i$ in Minkowski spacetime. We denote $\dot{z}_i^\mu = \frac{dz_i^\mu}{d\lambda_i}$.

The metric tensor: $\eta_{\mu\nu} = diag(+1, -1, -1, -1)$. We can write the Poincaré invariants [3], [4]:

$$\zeta_i = \dot{z}_i^2$$ (1)

$$\xi_{ij} = (\dot{z}_i \dot{z}_j)$$ (2)

$$\gamma_{ij} = (\dot{z}_i (z_j - z_i))$$ (3)

$$\rho_{ij} = (z_i - z_j)^2$$ (4)

Let us consider the action:

$$S = \sum_i m_i c \int d\lambda_i \zeta_i + \sum_i \sum_{j \neq i} g_i g_j c \int \int d\lambda_i d\lambda_j F(\xi_{ij}, \gamma_{ij}, \gamma_{ji}, \zeta_i, \zeta_j) \delta(\rho_{ij})$$ (5)

The Dirac delta function in (5) accounts for the interactions propagating at the speed of light forward and backward in time.

Without loss of generality, we assume the function $F$ to be symmetric:

$$F(\xi_{ij}, \gamma_{ij}, \gamma_{ji}, \zeta_i, \zeta_j) = F(\xi_{ij}, \gamma_{ji}, \gamma_{ij}, \zeta_j, \zeta_i)$$ (6)

The Minkowski equations of motion for $N$ interacting relativistic particles can be derived from the action (5) using the variational principle. We find $^1$:

$$m_i \ddot{z}_i^\mu = K_i^\mu$$ (7)

$^1$Equations (7 - 11) are derived from (5) by varying $z_{i\mu}$ in the action and integrating by parts, taking into account that:

$$\frac{d}{d\lambda_i}(\delta(\rho_{ij})) = \left(\frac{\partial}{\partial z_{ij}}\right)\frac{d}{d\lambda_j}(\delta(\rho_{ij})) = \frac{\gamma_{ij}}{\gamma_{ji}} \frac{d}{d\lambda_j}(\delta(\rho_{ij}))$$
where,

\[ K_i^\mu = \frac{g_i}{c^2} \sum_j g_j \int d\lambda_j \delta(\rho_{ij}) (A_{ij}^\mu + B_{ij}^{\mu\nu} \ddot{z}_{ij}^\nu + C_{ij}^{\mu\nu} \dot{z}_{ij}^\nu) \]  \hspace{1cm} (8)

\[ A_{ij}^\mu = \frac{\partial F}{\partial z_{ij}} - \frac{\partial^2 F}{\partial z_{ij} \partial \dot{z}_{ij}} \dot{z}_{ij}^\eta + \frac{\zeta_{ij}}{\gamma_{ji}} \left( (z_{i}^\mu - z_{j}^\mu) F + \gamma_{ij} \frac{\partial F}{\partial \dot{z}_{ij}} \right) \]

\[ + \frac{1}{\gamma_{ji}} \left( -\dot{z}_{i}^\mu F + (z_{i}^\mu - z_{j}^\mu) \frac{\partial F}{\partial z_{ij}} \dot{z}_{j}^\eta + \xi_{ij} \frac{\partial F}{\partial \dot{z}_{ij}} + \gamma_{ij} \frac{\partial^2 F}{\partial \dot{z}_{ij} \partial \dot{z}_{jij}} \dot{z}_{j}^\eta \right) \]  \hspace{1cm} (9)

\[ B_{ij}^{\mu\nu} = -\frac{\partial^2 F}{\partial \dot{z}_{ij} \partial \dot{z}_{jij}} \]  \hspace{1cm} (10)

\[ C_{ij}^{\mu\nu} = \frac{(z_{i}^\mu - z_{j}^\mu)}{\gamma_{ji}} \left( \frac{\partial F}{\partial z_{ij}} - \frac{(z_{i}^\mu - z_{j}^\mu)}{\gamma_{ji}} F \right) + \gamma_{ij} \left( \frac{\partial^2 F}{\partial \dot{z}_{ij} \partial \dot{z}_{jij}} - \frac{(z_{i}^\mu - z_{j}^\mu)}{\gamma_{ji}} \frac{\partial F}{\partial \dot{z}_{ij}} \right) \]  \hspace{1cm} (11)

In order to identify \( \lambda_i \) with \( s_i = c\tau_i \), where \( \tau_i \) is the particle’s proper time in flat spacetime, one needs to impose the well known conditions:

\[ (K_i \dot{z}_i) = 0 \]  \hspace{1cm} (12)

The conditions (12) guarantee that, for all solutions of the equations of motion, \( \zeta_i (i = 1, 2, ..., N) \) are constants (which by simple scaling can be made equal to 1):

\[ d\lambda_i^2 = \eta_{\mu\nu} dz_{i}^{\mu} dz_{i}^{\nu} \]  \hspace{1cm} (13)

Taking into account Eqs (8) and (9, 10, 11) we can see that Eqs (12) lead to the following conditions on \( F \):

\[ \gamma_{ji} \dot{z}_{i}^{\eta} \frac{\partial \tilde{F}}{\partial \dot{z}_{i}^{\eta}} - \gamma_{ij} \dot{z}_{j}^{\eta} \frac{\partial \tilde{F}}{\partial \dot{z}_{j}^{\eta}} - \left( \xi_{ij} + \zeta_{j} \gamma_{ji} \right) \tilde{F} = 0 \]  \hspace{1cm} (14)

\[ \frac{\partial \tilde{F}}{\partial \dot{z}_{ij}^{\nu}} = 0 \]  \hspace{1cm} (15)
\[(z'^{\nu}_i - z'^{\nu}_j) \hat{F} - \gamma_{ji} \frac{\partial \hat{F}}{\partial \dot{z}^{\nu}_j} = 0 \]  
(16)

where,

\[\hat{F} = F - \dot{z}_{i\mu} \frac{\partial F}{\partial \dot{z}_{i\mu}}\]  
(17)

\(F\) is a function of the invariants \(\xi_{ij}, \gamma_{ij}, \gamma_{ji}, \zeta_i\) and \(\zeta_j\). From (15) it follows that \(\hat{F}\) does not depend on \(\xi_{ij}, \gamma_{ij}\) and \(\zeta_i\). Therefore, we can rewrite the conditions (14) and (16) on \(\hat{F}(\gamma_{ji}, \zeta_j)\) as follows:

\[\left(\xi_{ij} + \zeta_j \gamma_{ji} \right) \left( \gamma_{ji} \frac{\partial \hat{F}}{\partial \gamma_{ji}} - \hat{F} \right) = 0 \]  
(18)

\[\left( z'^{\nu}_i - z'^{\nu}_j \right) \left( \gamma_{ji} \frac{\partial \hat{F}}{\partial \gamma_{ji}} - \hat{F} \right) + 2 \dot{z}^{\nu}_j \gamma_{ji} \frac{\partial \hat{F}}{\partial \zeta_j} = 0 \]  
(19)

From (18) and (19) it follows that \(\hat{F}\) must satisfy the partial differential equations:

\[\gamma_{ji} \frac{\partial \hat{F}}{\partial \gamma_{ji}} - \hat{F} = 0 \]  
(20)

\[\frac{\partial \hat{F}}{\partial \zeta_j} = 0 \]  
(21)

From (20) and (21) it follows that \(\hat{F}\) must be of the form: \(\hat{F} = \kappa \gamma_{ji}\), where \(\kappa\) is a constant. Therefore, from this result and (17), it follows that \(F\) must obey the following two conditions:

\[F - \xi_{ij} \frac{\partial F}{\partial \xi_{ij}} - \gamma_{ij} \frac{\partial F}{\partial \gamma_{ij}} - 2 \zeta_i \frac{\partial F}{\partial \zeta_i} = \kappa \gamma_{ji} \]  
(22)

\[F - \xi_{ij} \frac{\partial F}{\partial \xi_{ij}} - \gamma_{ji} \frac{\partial F}{\partial \gamma_{ji}} - 2 \zeta_j \frac{\partial F}{\partial \zeta_j} = \kappa \gamma_{ij} \]  
(23)

The most general solution to the system of partial differential equations (22) and (23) can be written in the form:
\[ F = \kappa (\gamma_{ij} + \gamma_{ji}) + f (\xi_{ij}, \zeta_{ij}, \eta_{ij}, \eta_{ji}) \]  

(24)

where,

\[ \zeta_{ij} = \sqrt{\zeta_i \zeta_j} \]  

(25)

\[ \eta_{ij} = \frac{\gamma_{ij}}{\sqrt{\zeta_i}} \]  

(26)

\[ \eta_{ji} = \frac{\gamma_{ji}}{\sqrt{\zeta_j}} \]  

(27)

and the function \( f \) is homogeneous of degree one in the first two arguments and symmetric:

\[ f (t \xi_{ij}, t \zeta_{ij}, \eta_{ij}, \eta_{ji}) = tf (\xi_{ij}, \zeta_{ij}, \eta_{ij}, \eta_{ji}) \]  

(28)

\[ f (\xi_{ij}, \zeta_{ij}, \eta_{ji}, \eta_{ij}) = f (\xi_{ij}, \zeta_{ij}, \eta_{ij}, \eta_{ji}) \]  

(29)

We have considered the most general action depending on the velocities and relative positions for \( N \) relativistic interacting particles (5). We have found that, in order to identify the Poincaré invariant parameters \( \lambda_i \) with \( s_i = c\tau_i \), the interaction terms in the action must be of the form (24).

The equations of motion (7) can be rewritten as follows:

\[ \ddot{z}_{ij} + \ddot{z}_{ji} = K^\mu_i \]  

(30)

where,

\[ \ddot{M}_i^{\mu\nu} = \ddot{z}_{ij}^{\mu\nu} \]  

(31)

\[ \ddot{K}_i^\mu = \frac{g_i}{c^2} \sum_{j \neq i} g_j \int ds_j \delta (\rho_{ij}) (A_{ij}^{\mu\nu} \ddot{z}_{j\nu} + C_{ij}^{\mu\nu} \ddot{z}_{j\nu}) \]  

(32)
Notice that the four-vector force $\vec{K}_i$ does not depend on $\dot{z}_i$.

In (30) $\vec{M}_i$ can be interpreted as the particle’s mass, a tensor under Poincaré’s transformations.

If we assume mass to be a scalar quantity, which we denote as $\bar{m}_i$ ($i = 1, 2, ..., N$), then the tensor $\vec{M}_i$ should be of the form:

$$\vec{M}_i^{\mu\nu} = \bar{m}_i \eta^{\mu\nu} + \bar{n}_i^{\mu} \dot{z}_i^{\nu}$$ (33)

From the expressions (31), (10) and (24) we find that the requirement of a Poincaré invariant mass (33) translates into the following new conditions on the interaction function $f$:

$$\frac{\partial^2 f}{\partial \eta_{ij}^2} = 0$$ (34)

$$\frac{\partial^2 f}{\partial \eta_{ji}^2} = 0$$ (35)

$$\frac{\partial^2 f}{\partial \xi_{ij}^2} = 0$$ (36)

$$\frac{\partial^2 f}{\partial \xi_{ij} \partial \eta_{ij}} = 0$$ (37)

$$\frac{\partial^2 f}{\partial \xi_{ij} \partial \eta_{ji}} = 0$$ (38)

$$\frac{\partial}{\partial \xi_{ij}} \left( \zeta_{ij} \frac{\partial f}{\partial \zeta_{ij}} - \eta_{ij} \frac{\partial f}{\partial \eta_{ij}} \right) = 0$$ (39)

$$\frac{\partial}{\partial \xi_{ij}} \left( \zeta_{ij} \frac{\partial f}{\partial \zeta_{ij}} - \eta_{ji} \frac{\partial f}{\partial \eta_{ji}} \right) = 0$$ (40)

$$\frac{\partial}{\partial \eta_{ij}} \left( \zeta_{ij} \frac{\partial f}{\partial \zeta_{ij}} - \eta_{ij} \frac{\partial f}{\partial \eta_{ij}} \right) = 0$$ (41)
\[
\frac{\partial}{\partial \eta_{ji}} \left( \zeta_{ij} \frac{\partial f}{\partial \zeta_{ij}} - \eta_{ji} \frac{\partial f}{\partial \eta_{ji}} \right) = 0 \tag{42}
\]

The most general solution to the system of differential equations (34 - 42) that also satisfies the conditions (28) and (29) can be written in the following form:

\[
f = \alpha \zeta_{ij} \eta_{ji} + \beta \xi_{ij} + \gamma \zeta_{ij} + \delta \zeta_{ij} (\eta_{ij} + \eta_{ji}) \tag{43}
\]

where \(\alpha, \beta, \gamma\) and \(\delta\) are constants.

Substituting (24) and (43) into the expression (5) for the action functional we obtain:

\[
S = \sum_i m_i c \int d\lambda_i \zeta_i + \sum_i \sum_{j \neq i} \frac{g_i g_j}{c} \int \int d\lambda_i d\lambda_j \delta (\rho_{ij}) \left( \left( \kappa + \delta \zeta_j \right)^{1/2} \gamma_{ij} + \left( \kappa + \delta \zeta_i \right)^{1/2} \gamma_{ji} \right) \\
+ \sum_i \sum_{j \neq i} \frac{g_i g_j}{c} \int \int d\lambda_i d\lambda_j \delta (\rho_{ij}) \left( \alpha \gamma_{ij} \gamma_{ji} + \beta \xi_{ij} + \gamma \zeta_{ij} \gamma_{ji} + \delta \zeta_{ij} \gamma_{ji} \right) \tag{44}
\]

Notice that the second term in (44) vanishes identically, since:

\[
\int d\lambda_j \delta (\rho_{ij}) \gamma_{ji} = \frac{1}{2} \sum_{s = -1}^1 \int d\lambda_j \delta (\lambda_j - \lambda_j^{(i,s)}) \frac{\gamma_{ji}}{|\gamma_{ji}|} = 0 \tag{45}
\]

In (45), \(\lambda_j^{(i,s)}\) are the two roots of the equation:

\[
\left( z_i (\lambda_i) - z_j (\lambda_j^{(i,s)}) \right)^2 = 0 \tag{46}
\]

Therefore, we have found that the most general theory with an action functional of the form (5) for which:

1. the Poincaré invariant parameters \(\lambda_i = s_i = c \tau_i\),

2. all the masses are scalars under Poincaré transformations,

is given by the action:

\[
S = \sum_i m_i c \int d\lambda_i \zeta_i + \sum_i \sum_{j \neq i} \frac{g_i g_j}{c} \int \int d\lambda_i d\lambda_j \delta (\rho_{ij}) \left( \alpha \gamma_{ij} \gamma_{ji} + \beta \xi_{ij} + \gamma \zeta_{ij} \zeta_{ji} \right) \tag{47}
\]
From the action (47), we obtain the equations of motion for a system of $N$ interacting relativistic particles:

$$
\bar{m}_i \ddot{z}_i^\mu = \bar{K}_i^\mu
$$

(48)

The Poincare invariant masses are equal to:

$$
\bar{m}_i = m_i + \gamma \frac{g_i}{c^2} \sum_{j \neq i} g_j \int ds_j \delta ((z_i - z_j)^2)
$$

(49)

The four-vector forces $\bar{K}_i^\mu$, ($i = 1, 2, ..., N$) can be expressed in the compact form:

$$
\bar{K}_i^\mu = g_i \left( F_i^{\mu \nu} \dot{z}_i^\nu + \Gamma_i^\mu_{\alpha \beta} \dot{z}_i^\alpha \dot{z}_i^\beta \right)
$$

(50)

where $F_i^{\mu \nu}$ is an antisymmetric tensor ($F_i^{\nu \mu} = -F_i^{\mu \nu}$) given by the following formula:

$$
F_i^{\mu \nu} = \sum_{j \neq i} \frac{g_j}{c^2} \int ds_j \frac{\delta (\rho_{ij})}{\gamma_{ji}^2} \left[ \alpha \gamma_{ji}^2 \left[ (z_i^\mu - z_j^\mu) \dot{z}_j^\nu - \dot{z}_j^\mu (z_i^\nu - z_j^\nu) \right] 
+ \beta \left[ ((z_i^\mu - z_j^\mu) \dot{z}_j^\nu - \dot{z}_j^\mu (z_i^\nu - z_j^\nu)) (1 - \sigma_{ji}) + ((z_i^\mu - z_j^\mu) \dot{z}_j^\nu - \dot{z}_j^\mu (z_i^\nu - z_j^\nu)) \gamma_{ji} \right] \right]
$$

(51)

and $\Gamma_i^\mu_{\alpha \beta}$ is a symmetric tensor in flat space-time ($\Gamma_i^\mu_{\alpha \beta} = \Gamma_i^\mu_{\beta \alpha}$) given by the expression:

$$
\Gamma_i^\mu_{\alpha \beta} = \gamma \sum_{j \neq i} \frac{g_j}{c^2} \int ds_j \frac{\delta (\rho_{ij})}{\gamma_{ji}^2}
\left[ \left( (z_i^\mu - z_j^\mu) \eta_{\alpha \beta} - \frac{1}{2} \left( \delta_\alpha^\mu (z_i^\beta - z_j^\beta) + \delta_\beta^\mu (z_i^\alpha - z_j^\alpha) \right) \right) (1 - \sigma_{ji}) 
- \left( \dot{z}_j^\mu \eta_{\alpha \beta} - \frac{1}{2} (\delta_\alpha^\mu \dot{z}_j^\beta + \delta_\beta^\mu \dot{z}_j^\alpha) \right) \gamma_{ji} \right]
$$

(52)

In (51, 52):

$$
\sigma_{ji} = (\dot{z}_j (z_i - z_j))
$$

(53)
The relativistic equations of motion (48 - 52) admit exact solutions for any number of particles. The details of the calculation will be given in a future publication.

In the remaining part of this Letter we will consider the relativistic one-body problem.

Consider a particle of mass $m_1 = m$ and charge $g_1 = g$ interacting with a heavy particle (of mass $m_2 \gg m_1$ and charge $g_2 = \tilde{g}$) at rest ($\vec{v}_2 = 0$). We obtain the equations of motion:

$$\left( m + \frac{\gamma g_1 \tilde{g} c^2 r}{c^2} \right) \left( \vec{a} + \frac{(\vec{v} \vec{a})}{c^2 (1 - \frac{v^2}{c^2})} \right) = \frac{g \tilde{g} q}{r} \left[ \left( 1 - \frac{v^2}{c^2} \right)^{\frac{1}{2}} \left( \alpha + \frac{\beta}{r^2} \right) \vec{r} + \frac{\gamma}{r^2} \left( 1 - \frac{v^2}{c^2} \right) \vec{r} + \left( \frac{\vec{r} \vec{v}}{c^2} \right) \vec{v} \right]$$

These equations follow from (48 - 52).

The one-body equations of motion (54) can also be derived from the Lagrangian:

$$L = -mc^2 \left( 1 + \frac{\gamma g_1 \tilde{g}}{mc^2 r} \right) \left( 1 - \frac{v^2}{c^2} \right)^{\frac{1}{2}} + g \tilde{g} \left( \alpha r - \frac{\beta}{r} \right)$$

The exact relativistic one-body Hamiltonian is given by the expression:

$$H = mc^2 \left( \left( 1 + \frac{\gamma g_1 \tilde{g}}{mc^2 r} \right)^2 + \frac{p^2}{m^2 c^2} \right)^{\frac{1}{2}} - g \tilde{g} \left( \alpha r - \frac{\beta}{r} \right)$$

The conserved energy $E$ and angular momentum $\vec{L}$ can be found to be:

$$E = \left( \frac{mc^2 + \frac{\gamma g_1 \tilde{g}}{r}}{1 - \frac{v^2}{c^2}} \right)^{\frac{1}{2}} - g \tilde{g} \left( \alpha r - \frac{\beta}{r} \right)$$

$$\vec{L} = \left( \frac{mc^2 + \frac{\gamma g_1 \tilde{g}}{r}}{1 - \frac{v^2}{c^2}} \right)^{\frac{1}{2}} \vec{r} \times \vec{v}$$

The motion of the relativistic particle is planar, with polar coordinates $(r, \phi)$ obeying the equations:

$$\frac{d\phi}{dt} = \frac{c^2 L}{r^2 \left( E + g \tilde{g} \left( \alpha r - \frac{\beta}{r} \right) \right)}$$
\[
\frac{1}{c^2} \left( \frac{dr}{dt} \right)^2 = 1 - \frac{\left( m + \frac{\gamma g^2}{c^2 r} \right)^2 c^4 + \frac{L^2 c^2}{r^2}}{(E + g \tilde{g} \left( \alpha r - \frac{\beta}{r} \right))^2} \quad (60)
\]

The equations of motion admit circular orbits solutions with the angular frequency \( \omega \) and the radius \( r \) obeying the generalized Kepler relation:

\[
\omega^2 = -\frac{g \tilde{g}}{mr^3} \left[ \gamma + (\alpha r^2 + \beta) \left( 1 + \frac{\gamma g \tilde{g}}{mc^2 r} + \frac{g^2 \tilde{g}^2 (\alpha r^2 + \beta)^2}{4mc^4 r^2} \right)^{\frac{1}{2}} + \frac{g \tilde{g} (\alpha r^2 + \beta)}{2mc^2 r} \right] \quad (61)
\]

For circular orbits the energy is given by the expression:

\[
E = mc^2 \left( 1 + \frac{\gamma g \tilde{g}}{mc^2 r} + \frac{g^2 \tilde{g}^2 (\alpha r^2 + \beta)^2}{4mc^4 r^2} \right)^{\frac{1}{2}} + \frac{g \tilde{g}}{2} \left( -3ar + \frac{\beta}{r} \right) \quad (62)
\]

and the angular momentum is:

\[
L = \left( 1 + \frac{\gamma g \tilde{g}}{mc^2 r} \right) \left[ -g \tilde{g} mr \left[ \gamma + (\alpha r^2 + \beta) \left( 1 + \frac{\gamma g \tilde{g}}{mc^2 r} + \frac{g^2 \tilde{g}^2 (\alpha r^2 + \beta)^2}{4mc^4 r^2} \right)^{\frac{1}{2}} + \frac{g \tilde{g} (\alpha r^2 + \beta)}{2mc^2 r} \right] \right]^\frac{1}{2}
\]

\[
\left( i\hbar \frac{\partial}{\partial t} + g \tilde{g} \left( \alpha r - \frac{\beta}{r} \right) \right)^2 \Psi (t, \vec{r}) = m^2 c^4 \left( 1 + \frac{\gamma g \tilde{g}}{mc^2 r} \right)^2 \Psi (t, \vec{r}) - \hbar^2 c^2 \nabla^2 \Psi (t, \vec{r}) \quad (64)
\]

For stationary states, \( \Psi (t, \vec{r}) = e^{-iEt/\hbar} \psi (\vec{r}) \), in spherical coordinates we can separate the radial and angular parts of the wave function in the standard way:

\[
\psi (\vec{r}) = R(r) Y_{lm} (\theta, \phi) \quad (65)
\]

where \( Y_{lm} (\theta, \phi) \) are the spherical harmonics. For the radial part we obtain the equation:
\[
\left( -\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) + \frac{l(l+1)}{r^2} \right) R = \frac{1}{\hbar^2 c^2} \left[ \left( E + g \tilde{g} \left( \alpha r - \beta \frac{1}{r} \right) \right)^2 - m^2 c^4 \left( 1 + \frac{\gamma g \tilde{g}}{mc^2 r} \right) \right]^2 R
\]

For spin- \( \frac{1}{2} \) particles we can write the Dirac equation as follows:

\[
\frac{i\hbar}{\partial t} \Psi = \hat{H} \Psi
\]

where,

\[
\hat{H} = i\hbar c \vec{\alpha} \vec{\nabla} - \beta_o m \left( 1 + \frac{\gamma g \tilde{g}}{mc^2 r} \right) - g \tilde{g} \left( \alpha r - \beta \frac{1}{r} \right)
\]

One can easily check that the total angular momentum \( \vec{L} + \frac{1}{2} \hbar \vec{\sigma} \) commutes with the Hamiltonian operator (68).

The matrices \( \vec{\alpha}, \beta_o \) and \( \vec{\sigma}_o \) may be written in the standard way in terms of the Pauli matrices \( \vec{\sigma} \) and the unit matrix \( I \).

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