DO WE UNDERSTAND BLACK HOLE ENTROPY? ∗

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ABSTRACT

I review various proposals for the nature of black hole entropy and for the mechanism behind the operation of the generalized second law. I stress the merits of entanglement entropy qua black hole entropy, and point out that, from an operational viewpoint, entanglement entropy is perfectly finite. Problems with this identification such as the multispecies problem and the trivialization of the information puzzle are mentioned. This last leads me to associate black hole entropy rather with the multiplicity of density operators which describe a black hole according to exterior observers. I relate this identification to Sorkin’s proof of the generalized second law. I discuss in some depth Frolov and Page’s proof of the same law, finding it relevant only for scattering of microsystems by a black hole. Assuming that the law is generally valid I make evident the existence of the universal bound on entropy regardless of issues of acceleration buoyancy, and discuss the question of why macroscopic objects cannot emerge in the Hawking radiance.

1. Introduction

Three intricately related issues have characterized black hole thermodynamics for the better part of two decades: the meaning of black hole entropy, the mechanism behind the operation of the generalized second law, and the information loss puzzle. Black hole entropy and the generalized second law were introduced in 1972.1−3 A lot of activity in black hole thermodynamics followed Hawking’s 1974–75 papers describing the Hawking radiance.4,5 The information puzzle dates from Hawking’s 1976 paper.6 Interest in these matters mellowed at the end of that decade. From the early 1990’s there has been a intense resurgence of interest in all three issues leading to much debate, illumination and confusion. Today, well into its third decade of development, black hole thermodynamics remains intellectually stimulating and puzzling at once. What follows is not so much a full review of the first two issues, as my impression of some promising directions which are likely to influence resolution of the information puzzle and lead to insights outside the immediate subject.

Black hole entropy had some predecessors: Christodoulou’s irreducible mass,7 Wheeler’s suggestion of a demon who violates the second law with help of a black hole,8 Penrose and Floyd’s observation that the event horizon area tends to grow9 and Hawking’s area theorem.10 Carter11 and Bardeen, Carter and Hawking12 were aware of the analogy between horizon area and entropy as reflected in their first and second laws of black hole mechanics, but did not take the analogy seriously. The view that horizon area divided by Planck’s length square is really an entropy, not just an analog of entropy,1−3 met initially with opposition12,8,13 but was embraced

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widely after Hawking’s demonstration\textsuperscript{5} that black holes radiate thermally. By the end of the 1970’s it was generally accepted that a black hole, at least a quasistatically and semiclassically evolving one, is endowed with an entropy (throughout I use units with $G = c = k$, but display $\hbar$)

$$S_{BH} = A/(4\hbar) \quad (1)$$

Today it is clear that if one sticks to general relativity or to dilaton type gravity theories in $3 + 1$ dimensions, and matter has normal properties, Eq. (1) is widely valid.\textsuperscript{14}

As a geometric property, black hole entropy could be granted thermodynamic status only because of two points. First, one can derive from it a temperature by the thermodynamic relation $T = (\partial M/\partial S_{BH})$ with $M$ the black hole mass–energy\textsuperscript{5} which happens to have the same form as Hawking’s radiance temperature [$T_{BH} = \hbar/(8\pi M)$ for Schwarzschild]; in fact this is the way the proportionality constant in Eq. (1) was first calibrated.\textsuperscript{4,5} However, the ulterior meaning of black hole entropy has remained a mystery. Second, black hole entropy enters into the generalized second of thermodynamics (GSL) on the same footing as ordinary matter–radiation entropy $S_{rad+mat}$: for a transformation of a closed system including black holes

$$\Delta S_{BH} + \Delta S_{rad+mat} \geq 0 \quad (2)$$

This law has proved quite successful. Suffice it to recall that when it was originally formulated,\textsuperscript{1,2} Hawking’s radiance was still a thing of the future, yet the GSL was found to be satisfied by the Hawking process (in its semiclassical form).\textsuperscript{15,16} Since then a number of succesful tests of the GSL have been carried out. Two questions arose: what mechanism insures that the generalized entropy grows in any situation, and are there any exceptions to the law? These are not trivial questions: understanding why the ordinary second law (with no black holes) works in the quantum world is just beginning to crystallize a century and a half after Carnot, Clausius and Kelvin (see Ref. 17 for a nice recap).

The Hawking “evaporation” of a black hole brings with it the information puzzle.\textsuperscript{6} Recall the essentials. Hawking’s original derivation and subsequent work show the radiance to have a thermal character (quasi–Planckian spectrum and thermal statistics mode–by–mode).\textsuperscript{18,15} This is usually traced to the picture of pair formation out of the vacuum for modes that skim the event horizon on their way out to future null–infinity $J^+$. One of each pair goes out to contribute to the Hawking radiance; its companion is lost down the black hole. The quantum state of the Hawking radiation by itself lacks the quantum correlations with the “lost” quanta which are part and parcel of the original pure vacuum state at past null–infinity $J^-$. Hence the Hawking radiation all by itself is in a mixed state. It happens to be a nearly maximally mixed state, and so is thermal. If the black hole truly disappears by evaporation, one is left with a thermal (mixed) state of radiation with nothing to correlate with in order to reconstitute the pure state. Hawking concluded from this that black hole evaporation catalyzes unitarity violation, that quantum mechanics is not fully correct in the presence of black hole horizons, and that contrary to the venerable rules, a pure state can become mixed.\textsuperscript{6} This strong claim forms the basis of the information puzzle or paradox in black hole physics.

The three issues are actually one in the sense that when people find out how to fundamentally resolve one of them, they will have resolved all three. In the last
few years it has been fashionable to explore these issues in the framework of exactly solvable field-theoretic toy models in $1+1$ dimensions.\textsuperscript{19} I wish my comments to be interpreted in unfashionable $3+1$ dimensions. What is lost in exactness of treatment this way is balanced by the realism of the conclusions.

2. The Meaning of Black Hole Entropy

“Entropy” must be one of the most abused terms in physics. We all agree that Boltzmann’s entropy derived from the one-particle distribution function of a gas, and Gibbs’ canonical ensemble entropy are closely related to Clausius’s thermodynamic entropy. Somewhat more removed, but still clearly related to phenomenological entropy, is Shannon entropy\textsuperscript{20} – the measure of unavailable information,

$$S = - \sum_A p_A \ln p_A$$  \hspace{1cm} (3)

Most likely unrelated to it are Kolmogorov entropy in the theory of chaotic flows, or Chaitin’s algorithmic entropy in the theory of computation.

Although there can be little doubt that black hole entropy corresponds closely to a phenomenological entropy, its deeper meaning has remained mysterious. Is it similar to that of ordinary entropy, \textit{i.e.} the logarithm of a count of internal black hole states associated with a single black hole exterior?\textsuperscript{2,15,16} Is it the logarithm of the number of ways in which the black hole might be formed?\textsuperscript{2,16} Is it the logarithm of the number of horizon quantum states?\textsuperscript{21−23} Does it stand for information lost in the transcendence of the hallowed principle of unitary evolution?\textsuperscript{6,24} I would claim that at this stage the usefulness of any proposed interpretation of black hole entropy turns on how well it relates to the original “statistical” aspect of entropy as a measure of disorder, missing information, multiplicity of microstates compatible with a given macrostate, \textit{etc.}

In Hawking’s field theoretic approach, which served as model for nearly all work in the 1970’s and 1980’s, and in the venerable surface gravity method,\textsuperscript{12,25} black hole temperature is the primary quantity, and the black hole entropy is recovered from Clausius’s rule $S = \int dM/T$. The statistical aspect is not exposed. Wald’s Noether charge method,\textsuperscript{26,27} the method of deficit angle,\textsuperscript{28} and the method of field redefinition\textsuperscript{27} are likewise good for calculating black hole entropy in unfamiliar situations, but leave one mostly in the dark as to its statistical meaning. In the Gibbons–Hawking Euclidean method\textsuperscript{29} the black hole entropy is basically classical: the $A/(4\hbar)$ contribution appears at tree level, \textit{i.e.}, to lowest order in $\hbar$ in the functional integral. Yet in statistical mechanics of fields, statistical entropy first appears at one-loop level. Thus although the Gibbons–Hawking approach has proved fruitful for calculating the value of the black hole entropy in novel situations,\textsuperscript{14} it is not in itself a statistical interpretation of black hole entropy (my early reaction to the Gibbons–Hawking approach is recorded in the discussion to Ref. 30). One might expect that going beyond tree level might bring in truly statistical features of entropy. Thus enters entanglement entropy.

2.1. Why Entanglement Entropy?

Entanglement entropy was used very early in relativity to understand the Unruh effect as resulting from ignoring the states beyond the Rindler horizon.\textsuperscript{31,32} The last year witnessed a renaissance of the interpretation of black hole entropy in terms
of quantum entanglement entropy proposed by Bombelli, Koul, Lee and Sorkin\textsuperscript{33} (henceforth BKLS) in a classic paper from the quiet period of the subjects’s history. The idea was rediscovered by Srednicki\textsuperscript{34} who pointed out that the global vacuum state of a scalar field in flat spacetime, when restricted to the exterior of an imaginary sphere, is in a mixed state there. The density matrix of this mixed state arises from tracing out those parts of the global state that reside inside the sphere; its entropy is evidently related to the unknown information about the sphere’s interior. This entropy is nonvanishing only because the exterior state is correlated with the interior one (or entangled in the sense that the parts of the singlet state of two electrons $|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle$ are entangled). In the sphere’s case the quantum entanglement entropy comes out to be proportional to the sphere’s surface area, albeit with a coefficient which diverges quadratically in the high frequency cutoff.\textsuperscript{34}

The main points had been made earlier by BKLS. They also gave reasons for relating at least part of the black hole entropy to entanglement entropy of the state outside black hole. In particular, they pointed out that whereas for an ordinary “black box” situation the emergence of entanglement entropy out of a pure state is to a large extent a matter of choice for the observer, for the black hole case the horizon’s presence makes its emergence mandatory. They noted that because the black hole exterior evolves autonomously – no information is fed into it from inside the horizon – one can expect a second law to apply to an entropy defined exclusively in it. BKLS were aware that the divergence of the entanglement entropy is due to high frequency modes near the horizon, and suggested that the physical entropy is finite due to quantum fluctuations of the geometry at the horizon.

Entanglement entropy has lately been explored further by Susskind, Thorlacius and Uglum\textsuperscript{23} with an eye on the relation between entanglement and radiation entropy. Holzhey\textsuperscript{35} and Callan and Wilczek\textsuperscript{36} have made use of clever techniques for computing it, concluding with BKLS and Srednicki that a plane boundary in Minkowski spacetime, when the quantum state beyond it is ignored, gets ascribed entanglement entropy proportional to the area of the boundary with an ultraviolet quadratically divergent coefficient. Kabat and Strassler\textsuperscript{37} further show that the density operator in question is thermal irrespective of the nature of the field. Holzhey, Larsen and Wilczek\textsuperscript{38} explore a method to regularize the divergence in conformal field theories.

2.2. Entanglement Entropy is Operationally Finite

The divergence of entanglement entropy, common to flat and black hole spacetimes, has puzzled people. But, at least in flat spacetime, the problem is a red herring: when the operational procedure behind the formal “tracing” is a physical one, there cannot be a divergence. To see why this is so, let me first state the problem as usually conceived. The global vacuum state will be denoted by $|0\rangle$. The space is divided by a boundary into an interior and exterior region. Let a complete basis of states for the interior region be denoted by $\{|a\rangle\}$ and one for the exterior one by $\{|A\rangle\}$. Now suppose that $\{|a\rangle\} \otimes \{|A\rangle\}$ is a basis for the global states. Then it is possible to represent the vacuum state as

$$|0\rangle = \sum_{aA} C_{aA} |a\rangle \otimes |A\rangle$$

where the $C_{aA}$ are complex numbers. If nothing is known about the interior side of the boundary, then one obtains the exterior state by assigning each interior state
an equal weight, i.e., by tracing $|0\rangle\langle 0|$ over the basis $\{|a\rangle\}$ and then normalizing:

$$\hat{r}_{\text{ext}} = \frac{\text{Tr}_a |0\rangle\langle 0|}{\text{Tr}_a |0\rangle\langle 0|} = \frac{\sum_{ab} C_{aA} C^*_{bA} |A\rangle\langle B|}{\sum_a \sum_a |C_{aA}|^2}$$

(5)

It is the von Neumann entropy $S = -\text{Tr}_A \hat{r}_{\text{ext}} \ln \hat{r}_{\text{ext}}$ which is the entanglement entropy. It diverges because there are many high frequency modes in the sum in Eq. (4), and thus an infinity of states are traced over.

However, the kind of trace in Eq. (5) does not correspond to any operational prescription. It is untrue, in general, that one knows nothing about the interior state. For example, if the region selected is spherical of radius $R$, then just from the fact that the spacetime is flat to some accuracy, one knows that the energy $E$ associated with the interior region has to be small. Of course, the global vacuum has zero energy, but one is discussing the energy of the state left after tracing – a different one. In fact if the boundary delineating the region being traced out were absolutely sharp, the uncertainty principle might suggest a very large energy for it. Thus we think of that boundary as slightly fuzzy.

Anyway, we can write $E/R = \xi \ll 1$ where $\xi$ is the relativistic quality parameter $(GM/c^2 R$ in dimensional notation). Thus in forming the density operator for the exterior region, one should assign equal nonvanishing weights only to the interior states with energy below $E$. Equivalently, one should trace $|0\rangle\langle 0| \Theta(E-H_{\text{int}})$ instead of just $|0\rangle\langle 0|$ over $\{|a\rangle\}$; here $H_{\text{int}}$ is the Hamiltonian for the interior degrees of freedom. In the expression for the new density operator, $\hat{r}_{\text{ext}}(E)$, all sums over $a$ are to be confined to states $|a\rangle$ with energy below $E$. The claim is that the physical entropy $S_{\text{ext}}(E) = -\text{Tr}_A \hat{r}_{\text{ext}}(E) \ln \hat{r}_{\text{ext}}(E)$ is finite for bounded $E$.

To see this most easily suppose the basis $\{|A\rangle\}$ diagonalizes $\hat{r}_{\text{ext}}(E)$. Then the eigenvalues of $\hat{r}_{\text{ext}}(E)$,

$$p_A(E) = \frac{\sum_a |C_{aA}|^2}{\sum_a \sum_a |C_{aA}|^2}$$

(6)

are the probabilities for the exterior states $\{|A\rangle\}$; here a prime on a sum means it is restricted to states with energy below $E$. Then the von Neumann entropy of $\hat{r}_{\text{ext}}(E)$ is just the Shannon entropy:

$$S_{\text{ext}}(E) = -\sum_A p_A(E) \ln p_A(E)$$

(7)

I now argue that this entropy is bounded from above by $\ln N(E)$, where $N(E)$ is the number of interior quantum states $|a\rangle$ with energy below $E$, itself a number easy to bound.

The first step is the well known symmetry theorem whose proof in the present context goes as follows (see Ref. 34). Define $\hat{r}_{\text{int}}$ by the analog of Eq. (5), but with the trace taken over $\{|A\rangle\}$. This is the state of the interior region when one ignores the information about the exterior. However, as before, interior states $|a\rangle$ with energies above $E$ are not allowed. Thus the rows of the matrix $C_{aA}$ corresponding to such states are to be amputated in a physical discussion. In effect, given the information that the interior region has little energy, the full $C_{aA}$ does not give the correct global quantum state compatible with that information. Call the amputated matrix $\tilde{C}$. Then the manifestly positive definite matrix $R_{\text{ext}} = \tilde{C}^\dagger C^\dagger / \text{Tr} C^\dagger C$ represents $\hat{r}_{\text{ext}}$ (see Eq. (5)) while $R_{\text{int}} = C^* \tilde{C}^\dagger / \text{Tr} C^* \tilde{C}^\dagger$ represents $\hat{r}_{\text{int}}$ ($T$ denotes “transpose”).
Because the sets \{\langle A \rangle\} and \{\langle a \rangle\} are inequivalent, \(R_{\text{int}} \neq R_{\text{ext}}\). However, \(\text{Tr} C^* C^T = \text{Tr} C^\dagger C\) (transposing does not affect traces). By the cyclic invariance of the trace of a product, it is easy to extend this to \(\text{Tr} (C^* C^T)^n = \text{Tr} (C^\dagger C)^n\) for \(n = 2, 3, \ldots\). Equivalently, \(\sum_A p_A^n = \sum_a p_a^n\) where \(p_a\) is an eigenvalue of \(\hat{\rho}_{\text{int}}\) defined in analogy with Eq. (6). This last relation is true for all \(n\) only if \(R_{\text{int}}\) and \(R_{\text{ext}}\) have the same list of nonvanishing eigenvalues (the number of zero eigenvalues may be different\(^{40}\)). Now because the von Neumann entropy of \(\hat{\rho}_{\text{int}}\), \(S_{\text{int}}\), can be expressed in terms of \(p_a\) in analogy with Eq. (7),

\[
S_{\text{ext}}(E) = S_{\text{int}}(E) = -\sum_a p_a \ln p_a
\]

Of course this key result would likewise be valid formally had one not excluded the high energy \(\langle a \rangle\) states. However, since \(S = S_{\text{ext}}(\infty) = \infty\), that result would not be interesting.

The maximum possible value of \(S_{\text{int}}(E)\) is obtained when all \(p_a\) are equal. If there are \(N(E)\) interior \(\langle a \rangle\) states below energy \(E\), then the sum in Eq. (8) equals \(\ln N(E)\), the microcanonical entropy of the interior as a function of energy. Thus one finds for the entanglement entropy according to the exterior observer

\[
S_{\text{ext}}(E) < \ln N(E)
\]

The terms of the problem require that the states \(\langle a \rangle\) counted by \(N(E)\) be confined to the interior region. One way to enforce this is to subject the field to a boundary condition at the surface between the regions, which amounts to putting the system represented by \(\hat{H}_{\text{int}}\) in a box and ignoring the exterior. Suppose the field is free, and thus described by some one–particle Hamiltonian \(\hat{h}_1\). Then a semianalytical argument\(^{41}\) assures one that for any box shape

\[
\max [\ln N(E)/E] \approx [\zeta(\hat{h}_1, 4)]^{1/4}
\]

where

\[
\zeta(\hat{h}_1, 4) \equiv \text{Tr} \ \hat{h}_1^{-4} = g_1 \epsilon_1^{-4} + g_2 \epsilon_2^{-4} + \ldots
\]

is the analog of Riemann’s zeta function \(\zeta(4)\), but with the one–particle eigenenergies \(\epsilon_j\) (each with multiplicity \(g_j\)) in the box replacing the positive integers. This result has been checked\(^{41}\) by counting all many–particle states in a box up to energy \(E\) for electromagnetic, scalar and neutrino fields. The boxes were either spherical, or rectangular with various aspect ratios. The boundary conditions were Neumann or Dirichlet for the scalar, conducting boundary for the electromagnetic, or zero energy outflow for the neutrino field. The results confirm Eq. (10) to about 5% accuracy. It is already plain from the comparison of inequality (9) with the approximation (10) that the entanglement entropy arising from ignoring the interior region is bounded so long as it is recognized that the interior region has limited energy \(E\). The entanglement entropy grows at most as fast as \(E\).

The approximation (10) can be traded for the rigorous bound\(^ {42,43}\)

\[
\ln N(E) < [4! \zeta(\hat{h}_1, 4)]^{1/4} E,
\]

where a bar indicates that the eigenenergies and degeneracies used to calculate the zeta function are to be those appropriate for a sphere with radius \(R\) equal to the
circumscribing radius of the box (which can be of any shape and topology). Now the terms in Eq. (11) typically drop off rapidly. And since only \( [\tilde{\zeta}(\hat{h}_1, 4)]^{1/4} \) is of concern, and \( g_i \) should not be large compared to unity, a passable approximation to \( [\tilde{\zeta}(\hat{h}_1, 4)]^{1/4} = 1/\varepsilon \). On dimensional grounds one expects, for a massless field, that \( \varepsilon \sim h/R \). If the field is massive, \( \varepsilon \) should be larger. Thus for a massless field one expects \( [\tilde{\zeta}(\hat{h}_1, 4)]^{1/4} \approx R/h \), with a smaller value for a massive field. Explicit calculation of \( \tilde{\zeta}(\hat{h}_1, 4) \) for electromagnetic, scalar and neutrino fields\(^{44,42}\) confirm this. One can cover every type of known field by replacing (12) by the (rather generous) uniform bound

\[
\ln N(E) < 2\pi RE/h \tag{13}
\]

which I like to call the universal entropy bound.\(^{44}\)

Put all this together. From the definition of relativistic quality parameter one has \( E = \xi R \). Substitution in bound (13) and that in inequality (9) gives

\[
S_{\text{ext}} < 2\pi \xi R^2/h \tag{14}
\]

which is the desired formula. This bound on entanglement entropy scales up with area of the circumscribing sphere, but the coefficient is not infinite (for a nearly flat spacetime system, \( \xi \ll 1 \)). Let me now cavalierly push the formula beyond its intent to \( \xi \rightarrow 1 \) (the black hole regime). Obviously the entanglement entropy could very well approach \( \pi R^2/h \) which is of the order of the black hole entropy, Eq. (1) The identification of the two\(^{34}\) seems reasonable on this grounds.

An obvious caveat about the above argument is that it pushes bound (13), which is well established in flat spacetime, to a strong gravity situation. The strong gravitational redshift in the black hole vicinity may well allow many states based on arbitrarily high (local) frequency modes to be included in the count of states for a system with finite global energy \( E \) (isn’t this what the Hawking process is about ?). Thus, although it is clear than in flat spacetime the entanglement entropy is finite in physically well defined situations, the analogous claim about curved spacetime awaits proof of the analog of bound (13) for strong gravity. A quite independent argument that black hole entropy calculated as entanglement entropy will come out finite as a result of renormalization of the gravitational constant is put forth by Susskind and Uglum.\(^{45}\)

2.3. The Multiplicity of Species Problem

Another thorny problem with equating entanglement and black hole entropy is that since each field in nature must make its contribution to the entanglement entropy, black hole entropy should scale up with the number of field species in nature. Yet Eq. (1) says nothing about number of species ! An interesting resolution suggested by Sorkin\(^{46}\) and ’t Hooft\(^{47}\) is that indeed different species contribute, but that the contributions of the actual species in nature exactly add up to \( A/(4\hbar) \). The point of view here is that the list of elementary particle species is prearranged to chime with gravitational physics. The results of Sec. 2.2 can be used to show that this is not out of the question. One adds up the specific values of \( \tilde{\zeta}(\hat{h}_1, 4) \) for the species found in nature to form a grand zeta function for “matter”. Taking into account three species of single–helicity neutrinos, six species of quarks, three of leptons and eight gluons together with all their antiparticles, as well as the photon, the \( W^\pm \) and \( Z \) bosons, and a Higgs doublet of complex scalars (for simplicity I think of all species as massless), one gets\(^{41,42} \) \( \tilde{\zeta}(\text{grand, 4}) = 9.45R^4/h^4 \). Repeating the
argument based on inequality (12), one is led to replace inequality (14) by $S_{\text{ext}} < 3.88\xi R^2/h$. Thus it is not inconceivable that due to the gravitational and other interactions, $S_{\text{ext}}$ ends up being $\pi R^2/h$ in the strong gravity limit, as appropriate for black hole entropy.

A very different resolution is offered by Jacobson. The argument is, roughly, that the effective action of every field quantized in curved spacetime carries a piece that looks like the Hilbert action, so that every such field makes a correction to the value of $G^{-1}$. For $n$ fields the correction to $G^{-1}$ is proportional to $n$. But the entanglement entropy contributed by $n$ fields is also proportional to $n$. Thus, if all of $G^{-1}$ comes from effective actions (Sakharov’s vision of effective gravity) the entanglement entropy will scale up as $G^{-1}$. It thus makes sense to identify the entanglement entropy and the black hole entropy; the latter, $c^3 A/(4G\hbar)$ in dimensional form, also scales like $G^{-1}$. A similar argument is given by Susskind and Ughum. This resolution of the multiplicity problem depends on black hole entropy being all entanglement entropy. As I argue below, this identification seems to resolve the information loss puzzle in a somewhat too trivial way.

This section would be incomplete without reference to the resolution due to Frolov. Its background is Frolov and Novikov’s identification of black hole entropy with the entanglement entropy of the mixed state obtained by tracing over the exterior states in the global vacuum of a field. Note that it is a logical consequence of the nature of quantum entanglement that one cannot have the black hole entropy residing in one region and arising from ignorance of the state of the degrees of freedom in that same region. Accordingly, Frolov and Novikov’s black hole entropy “resides” inside the black hole. The BKLS and Frolov–Novikov viewpoints do not necessarily clash because the symmetry theorem certifies that, because the global state is pure, the entanglement entropy comes out the same either way.

Frolov worried about the dependence of this entanglement entropy on the number of matter fields, and reconsidered the identification. He recalls that the free energy $F$ of a system depends on an external parameter $\lambda$, say, the dependence of mode frequencies on it. Thus $dF = -SDT + \Pi d\lambda$. Here $S$ is the usual statistical entropy, and the extra term is usually interpreted as work. Frolov regards black hole temperature as an external parameter, at least within York’s picture of the black hole enclosed in a cavity whose wall is kept a fixed temperature. Since mode frequencies scale inversely with black hole mass, Frolov considers them as proportional to $T_{BH}$; then $d\lambda \propto dT_{BH}$ and Frolov interprets the entire coefficient of $-dT_{BH}$ in $dF$, not just $S$, as $S_{BH}$. He calculates that the new terms mostly cancel out the entanglement entropy’s contribution to $S$. Since entanglement entropy is a one–loop contribution, Frolov finds $S_{BH}$ to be close to the Gibbons–Hawking tree–level entropy. If little of the Hilbert action is induced by quantum corrections, this last entropy is independent of the number of species, and so the species problem is resolved.

I find Frolov’s view of temperature somewhat confusing. In addition, and on a more practical level, I note that because of the negative specific heat of the Schwarzschild black hole, it is possible for such a hole to be in stable canonical ensemble with temperature as a parameter only in a very small container. What to do about black hole entropy for a black hole in empty space or one in a large cavity? Mode frequencies of radiation of a free black hole are not functions of its temperature. Thus at best, Frolov’s reinterpretation of the Frolov–Novikov paper is limited in scope. However, as I discuss now, some reidentification of what
is meant by black hole entropy is indeed needed for another reason.

2.4. A Proposal for Black Hole Entropy

Consider a black hole formed from collapse of a classical object. A (quantum) scalar field, originally in the vacuum state, propagates on this background. On a Cauchy hypersurface like $\sigma \cup v_1$ in the Penrose diagram of Fig. 1, the entanglement entropy of the state $\hat{\rho}_{v_1}(\sigma)$ arising from tracing $|0\rangle\langle 0|$ over the interior quantum states on $v_1$ is, according to the BKLS viewpoint, just $S_{BH}$. It follows from the symmetry theorem that the interior entanglement entropy of the state $\hat{\rho}_\sigma(v_1)$ which arises from tracing over the exterior states on $\sigma$ must also be equal to $S_{BH}$ since the

Fig.1: Penrose diagram for an evaporating Schwarzschild black hole showing the semihypersurfaces $v_j$ inside the horizon $\mathcal{H}$ and $\sigma$ outside it.
global state of the scalar field is pure. But this “interior” entropy can be identified with the fine-grained entropy of the Hawking radiation since it arises precisely from ignoring information about states in the black hole exterior. If the semiclassical picture is any guide, the horizon area will shrink and thus $S_{BH}$ must decrease as time goes on. But then the radiation entropy, which is at all times equal to $S_{BH}$ by the symmetry theorem, must decrease and end up by vanishing as the black hole fizzes out. If this conclusion is correct, it means that the radiation’s state becomes fully pure in the limit. This eventuality would obviously remove the information puzzle.

Yet despite venerable arguments in favor of such an outcome, no sign of this returning of the radiation to purity is seen either in the semiclassical or beyond semiclassical calculations. This has engendered the thought that the puzzle cannot even be properly stated without detailed understanding of trans-Planckian physics. There is thus an obvious problem with the argument in the preceding paragraph. I infer from this that one should not rigidly equate entanglement entropy with black hole entropy. To be sure, this is no new insight. BKLS stated that entanglement entropy is only a part of $S_{BH}$, and Callan and Wilczek claimed that it is only a correction to the tree level part of $S_{BH}$.

Anyway, tracing over the interior states leaves open the question of which semihypersurface $v$ this is being done on. The exterior spacelike semihypersurface $\sigma$ of interest (see Fig. 1) can be continued in any number of ways $v_1, v_2, v_3, \ldots$ — inside the horizon to the central point. Tracing over the states on the typical semihypersurface $v$ gives a density operator $\hat{\rho}_v(\sigma)$ for the black hole exterior. How does $\hat{\rho}_v(\sigma)$ depend on the choice of $v$? Classically it does not. The global vacuum density operator $|0\rangle\langle0|$ is unevolving in the Heisenberg picture. The interior observables do evolve in that picture and so must their eigenstates. However, their evolution from $v_1$ to $v_2$, say, is unitary: no information is fed from the exterior since the two spacelike semihypersurfaces meet at the horizon. Thus if one performs the trace of $|0\rangle\langle0|$ over interior states in a representation based on eigenstates of observables, one can expect the resulting density operator $\hat{\rho}_v(\sigma)$ to be the same for all choices of $v$ because the change from $v_1$ to $v_2$ is equivalent to a change of basis, and traces are unaffected by a change of basis. Thus, classically, $\hat{\rho}_v(\sigma)$ is unique for given $\sigma$.

But quantum fluctuations of the geometry are bound to smear the meeting point of the various $v_j$ at the horizon (this smearing is related to that invoked by BKLS to regularize the entanglement entropy). The unitary relation between interior eigenstates states on the various semihypersurfaces $v$ cannot be relied upon because in the face of the fluctuations the very meaning of the statement “spacelike semihypersurfaces meet at the horizon” becomes fuzzy. My guess is that because of this $\hat{\rho}_v(\sigma)$ depends slightly on $v$.

However, the entropies of the various $\hat{\rho}_v(\sigma)$, namely $S_{v_1}(\sigma), S_{v_2}(\sigma), \ldots$ are identical. By the symmetry theorem $S_{v_1}(\sigma)$ equals the entropy $S_{\sigma}(v_1)$ of the state induced on $v_1$ by tracing $|0\rangle\langle0|$ defined on $\sigma \cup v_1$ over the states in $\sigma$, and analogously for $v_2, v_3, \ldots$. Since the global state $|0\rangle\langle0|$ and the semihypersurface $\sigma$ are both fixed, the trace, $\hat{\rho}_\sigma(v)$, and the corresponding entropy $S_{\sigma}(v)$ have to be the same on all $v$. That and the symmetry theorem gives $S_{v_1}(\sigma) = S_{v_2}(\sigma) = \ldots$.

Since there are many possible, albeit quite similar, density operators on $\sigma$, $S_{\sigma}(v)$ is not the full expression of the statistical uncertainty on $\sigma$ about the black hole interior. According to information theory, if the states of a system can be classified into several classes $\{\kappa\}$, one gets the total uncertainty (entropy) as the sum of the
expression $- \sum_{\kappa} p_{\kappa} \ln p_{\kappa}$, where $p_{\kappa}$ is the probability of class $\kappa$, and the weighted average of the intrinsic entropies of the various classes (the weighing factors again being $p_{\kappa}$). Obviously in the present case the $\kappa$ stand for the $\hat{\rho}_v(\sigma)$. In the spirit of Laplace’s principle of ignorance, I shall assume that there are effectively $N$ equally likely $\hat{\rho}_v(\sigma)$, where $N$ is a finite number set by the amplitude of the quantum fuzziness alluded to above. Then

$$\text{Uncertainty on } \sigma = \ln N + N^{-1} \sum_j S_{v_j}(\sigma) = \ln N + S_v(\sigma). \quad (15)$$

Now $S_v(\sigma) = S_\sigma(v)$ in this expression is equal to the Hawking radiation fine–grained entropy on $\sigma$ (c.f. argument at the beginning of this section). I now interpret $\ln N$ as $S_{BH}$ because it is the extra uncertainty about the state in the black hole interior that is independent of the type of quantum state or field being considered. By construction this black hole entropy is independent of the number of matter fields. And in this interpretation the information puzzle is not trivially removed: the eventual disappearance of $\ln N$ as the horizon contracts does not force $S_\sigma(v)$ to vanish, though the eventual “purification” of the Hawking radiation is certainly not forbidden. It remains to be seen whether, because of quantum fluctuations, $N$ is indeed finite, and whether its logarithm indeed scales as horizon area. Since within the interpretation just offered $\ln N + S_v(\sigma)$ is evidently the generalized entropy of Eq. (2), a more immediate question is why does this quantity tend to rise? In other words, what makes the GSL work?

3. The Generalized Second Law at Work

3.1. Early Arguments for the Validity of the GSL

Early “proofs” of the GSL used gedankenexperiments to show that a loss of material entropy into a black hole is typically compensated by growing $S_{BH}$. With the advent of Hawking’s radiance, thermodynamic arguments were given that any decrease in $S_{BH}$ is more than offset by the growth of the radiance’s entropy. Hawking’s argument is that since the radiance is dumped into a low temperature environment, the increase in radiation entropy more than compensates for the reduction of entropy of the hotter black hole. Sewell argued that the work done by a system whose intensive parameters (temperature, electric potential) are set by a black hole should not, as in ordinary thermodynamics, exceed the reduction in its Gibbs free energy. By the conservation laws this is equivalent to requiring a growth in generalized entropy. These arguments make it seem that black hole thermodynamics is within the province of ordinary thermodynamics; however, they leave one in the dark about the statistical reasons for the GSL.

I demonstrated early that the statistics of the outgoing Hawking radiance (also found in Ref. 18) make it as entropic as allowed by the spectrum that filters through the potential barrier around the black hole; this remains true even when thermal radiation of any temperature $T$ impinges on the hole. The GSL is satisfied in both processes mode–by–mode. The processes of emission or reemission of incident radiation are irreversible except when $T = T_{BH}$, making it plain that the radiation entropy studied is a coarse-grained one. Page has calculated that the Hawking radiance of a hole in free space carries 1.619 times more entropy than would be required to break even according to the GSL. All the approaches mentioned so far
assume that $T_{BH}$ derives from the horizon area, and do not explain why/whether the GSL always works and how it fits in with quantum mechanics (which does not require an increase in entropy).

3.2. Modern Proofs of the GSL

This last issue was first studied by Sorkin\textsuperscript{17} in a seminal paper, a jumping off point for the BKLS paper. Sorkin ignores the black hole interior, and assumes the exterior and horizon can be described by a density operator $\hat{\rho}_{\text{ext}}(\sigma)$ (here $\sigma$ is still an exterior spacelike semihypersurface). He notes that $\hat{\rho}_{\text{ext}}(\sigma)$ must evolve autonomously (not influenced by the goings on beyond the horizon), at least in a classical picture of geometry. Its properties of positive definiteness, hermiticity and unit trace are expected to be preserved by this evolution. Sorkin further assumes, on the ground of conservation of energy for the whole system, that the maximum possible value of $S_{\text{ext}}(\sigma)$, the von Neumann entropy of $\hat{\rho}_{\text{ext}}(\sigma)$, is unaffected by evolution. He then proves that all this leads to the growth of $S_{\text{ext}}(\sigma)$ as $\sigma$ is pushed forward in time.

Sorkin regarded this an embryonic proof of the GSL, valid for dynamical black holes as well as quasistatic ones. He did not discuss how to split $S_{\text{ext}}(\sigma)$ into black hole and radiation parts. It is clear from Sorkin’s characterization of $\hat{\rho}_{\text{ext}}(\sigma)$ with my “uncertainty on $\sigma$”. (But I see no direct way to construct Sorkin’s $\hat{\rho}_{\text{ext}}(\sigma)$ from my $\hat{\rho}_{V}(\sigma)$). One thus gets a natural split for $S_{\text{ext}}(\sigma)$, Eq. (15). Thus Sorkin’s is a proof that $\ln \mathcal{N} + S_{V}(\sigma)$ must increase as $\sigma$ advances.

A very different proof of the GSL for quasistatic changes of a black hole has been formulated by Frolov and Page,\textsuperscript{58} who were influenced by Zurek and Thorne.\textsuperscript{59} For an eternal black hole, Frolov and Page consider the exterior mixed initial state $\hat{\rho}_{\text{initial}}$ to have a factorable form $\hat{\rho}_{\text{up}} \otimes \hat{\rho}_{\text{in}}$, where “up” denotes radiation modes coming up from the past horizon (in the picture of an eternal black hole – equivalent to Hawking radiation modes for an evaporating one), and “in” denotes modes ingoing from $\mathcal{J}^{-}$. The von Neumann entropies are thus related by $S_{\text{initial}} = S_{\text{up}} + S_{\text{in}}$. The state $\hat{\rho}_{\text{initial}}$ is assumed to evolve unitarily to a final state $\hat{\rho}_{\text{final}}$ so that $S_{\text{final}} = S_{\text{initial}}$. The natural modes for this last are “out” modes escaping to $\mathcal{J}^{+}$ and “down” modes falling into the future horizon. By tracing $\hat{\rho}_{\text{final}}$ over states formed out of “out” modes they obtain $\hat{\rho}_{\text{down}}$ and by tracing out “down” type states they obtain $\hat{\rho}_{\text{out}}$. Because of correlations between “out” and “down” quanta, $\hat{\rho}_{\text{final}}$ is not factorable as $\hat{\rho}_{\text{out}} \otimes \hat{\rho}_{\text{down}}$. In fact the correlations imply that $S_{\text{final}} < S_{\text{out}} + S_{\text{down}}$. Frolov and Page thus obtain

$$\Delta S_{\text{rad+mat}} = S_{\text{out}} - S_{\text{in}} > S_{\text{up}} - S_{\text{down}}$$

(16)

In this approach no attempt is made to follow the entropy changes moment by moment; only the overall change in ordinary entropy $\Delta S_{\text{rad+mat}}$ is of import.

In terms of the energies measured at infinity, the change in black hole entropy is evidently $\Delta S_{BH} = (E_{\text{in}}(\infty) - E_{\text{out}}(\infty))/T_{BH}$. By conservation of energy $E_{\text{in}}(\infty) - E_{\text{out}}(\infty) = E_{\text{down}}(\infty) - E_{\text{up}}(\infty)$. Converting the energies to the frame of a local observer corotating near the horizon (or at rest near it in the Schwarzschild case), and using inequality (16), Frolov and Page are led to

$$\Delta S_{BH} + \Delta S_{\text{rad+mat}} > [S_{\text{up}} - E_{\text{up}}(\text{local})/T_{0}] - [S_{\text{down}} - E_{\text{down}}(\text{local})/T_{0}]$$

(17)

where $T_{0}$ is $T_{BH}$ blueshifted to the local observer’s frame: $T_{0}/T_{BH} = E(\text{local})/E(\infty)$. Frolov and Page regard the “up” and “down” systems as strictly equivalent by time
reversal invariance. The “up” states coming out of the past horizon are supposed to be in equilibrium at global temperature $T_{BH}$ and thus at $T_0$ in the local observer’s frame. The “down” modes form the same system, but in some other state. Frolov and Page recall that when $S$ and $E$ are properties of a thermodynamic system in any state, and $T_0$ is some fixed temperature, $S - E/T_0$ attains its maximum when the system is in equilibrium at temperature $T_0$. Thus, conclude Frolov and Page, the r.h.s. of inequality (17) must be positive, and the GSL (2) follows.

How general is the Frolov–Page proof of the GSL? It is, of course, limited by its reliance on the semiclassical approximation (classical geometry driven by averages of quantum stress tensor). This weakness is remediable. Fiola, Preskill, Strominger and Trivedi have recently devised a proof of the GSL in 1+1 dimension dilaton gravity which goes beyond semiclassical considerations. However, that proof is restricted to very special situations, and works only if a new type of entropy is ascribed to coherent radiation states. Frolov and Page’s proof certainly has a wider applicability. But it does have a loophole: the assumed equivalence of “up” and “down” systems by time reversal invariance. The eternal black hole (Kruskal spacetime) is time–reversal invariant as assumed; the realistic radiating black hole is not (a time reverted black hole is not a black hole). Can one project this equivalence of systems from the former to the later?

What is involved in the statement that $S - E/T_0$ is maximum at equilibrium at temperature $T_0$? The state of the matter and radiation is encoded in some density operator $\hat{\rho}$. In terms of the hamiltonian $\hat{H}, E = \text{Tr}(\hat{\rho}\hat{H})$ while $S = -\text{Tr}(\hat{\rho}\ln\hat{\rho})$. Thus the variation $\delta(S - T_0E)$ under a small variation $\delta\hat{\rho}$ is

$$\delta(S - T_0E) = \text{Tr}[\delta\hat{\rho}(\hat{H} + T_0\ln\hat{\rho} + T_0)]$$

so that $S - T_0E$ has an extremum under variations that preserve $\text{Tr}\hat{\rho} = 1$ where $\hat{\rho}$ satisfies $\hat{H} + T_0\ln\hat{\rho} + T_0 - \lambda = 0$ with $\lambda$ a Lagrange multiplier. Obviously there is a unique solution $\hat{\rho} \propto \exp(-\hat{H}/T_0)$, i.e., there is one extremum of $S - T_0E$, a thermal (equilibrium) state with temperature $T_0$. This extremum is a maximum since for fixed $E$, $S$ attains a maximum in equilibrium. Thus the r.h.s. of Eq. (17) is indeed nonnegative provided the “up” and “down” states are described by the selfsame hamiltonian.

For the eternal black hole time reversal invariance does indeed guarantee equivalence of “up” and “down” hamiltonians. Compare now an evaporating black hole made by collapse with an eternal black hole of like parameters. Assuming a complete set of states, each of the relevant hamiltonians can be expanded in the usual form $\hat{H} = \sum |j\rangle\langle j| \epsilon_j$. If the time variation of the evaporating black hole’s parameters may be ignored, the “down” states and eigenenergies for the two black holes are in detailed correspondence, so that the “down” hamiltonians are equivalent. Thus the “down” hamiltonian for the evaporating black hole is equivalent to the “up” hamiltonian of the eternal black hole. But the equivalence cannot be carried one step further. The Hawking “up” states from an evaporating black hole emerge through the time dependent geometry of the collapsing object. Thus they cannot be put into exact correspondance with “up” states for the eternal black hole which emerge right into the stationary geometry. This is particularly true of early emerging “up” states. Thus the exact equivalence of “up” and “down” hamiltonians for the realistic evaporating black hole is in question since the comparison must be over a complete set of states.

The above mathematical nicety may well prove irrelevant for the Frolov–Page
proof when it is the scattering of microscopic systems off the black hole which is under consideration. However, for events involving an evaporating black hole and macroscopic objects, the sets of “up” and “down” modes are distinctly different. Macroscopic objects are bound states of many quanta of elementary fields. As discussed below, over the black hole evaporation lifetime such an object occurs in the Hawking radiance only with exponentially small probability. Thus even if emitted, the object is emitted by a black hole whose parameters cannot be regarded as stationary even in rough approximation. The comparison of the realistic and eternal black holes is thus murky since the former evolves drastically over the relevant time span. The equivalence of the “up” and “down” hamiltonians is thus unclear, and inequality (17) cannot be exploited.

3.3. The Universal Entropy Bound from the GSL

As just mentioned, the Frolov–Page proof is unconvincing for a situation where macroscopic matter falls into a black hole. Such a situation occurs frequently, e.g., astrophysical accretion onto a black hole. The Sorkin proof does seem to apply. Thus I assume that the GSL is also valid in such a situation. There are then interesting consequences.

First consider dropping a spherical macroscopic system of mass $E$, radius $R$ and entropy $s$ into a Schwarzschild black hole of mass $M \gg E$ from a large distance $D \gg M$ away. The black hole gains mass $E$, which it then proceeds to radiate over time $\tau$. At the end of this process the black hole is back at mass $M$. Were the emission reversible, the radiated entropy would be $E/T_{BH}$. As mentioned, the emission is actually irreversible, and the entropy emitted is a factor $\mu > 1$ larger. Thus the overall change in generalized entropy is

$$\Delta S_{BH} + \Delta S_{rad} = \mu E/T_{BH} - s$$

From numerical work Page\textsuperscript{56} estimates $\mu = 1.35 - 1.64$ depending on the species radiated. One can certainly choose $M$ larger than $R$ by an order of magnitude, say, so that the system will fall into the hole without being torn up: $M = \gamma R$ with $\gamma = a few$. Thus, if the GSL is obeyed, the restriction

$$s < 8\mu\gamma\pi RE/h$$

must be valid. It is clear from the argument that there is no need for $\mu\gamma$ to be arbitrarily large. Thus from the GSL one infers a bound on the entropy of a rather arbitrary – but not strongly gravitating – system in terms of its total gravitating energy and size. Note that this bound is compatible with bound (13) which comes from statistical mechanics in flat spacetime.

One objection that could be raised to the above line of argument is that Hawking radiation pressure might keep the system from being absorbed by the hole, thus obviating the conclusion. This is not so. Approximate the Hawking radiance as black body radiance of temperature $\hbar/(8\pi M)$ from a sphere of radius $2M$, the energy flux at distance $r$ from the hole is

$$F(r) = \frac{\hbar}{61,440(\pi Mr)^2}$$

resulting in a radiation force $f_{rad}(r) = \pi R^2 F(r)$ on the infalling sphere. Writing the Newtonian gravitational force as $f_{grav}(r) = ME/r^2$ one sees that

$$\frac{f_{rad}(r)}{f_{grav}(r)} = \frac{\hbar R^2}{61,440\pi^2 M^3 E}$$
The size of a macroscopic system always exceeds its Compton length. Thus for any macroscopic sphere able to fall whole into the hole $h/E < R < M$. Therefore, $f_{\text{rad}}(r)/g_{\text{rad}}(r) \ll 1$ throughout the fall until very close to the hole where the Newtonian approximations used must fail. By then the game is up, and the system must surely be swallowed up. It is also clear that the system falls essentially geodesically (more on this below).

The objection might be refurbished by relying on the radiation pressure of a large number of massless species to overpower gravity and drive the system away. So let me pretend the number of species in nature is large. However, the relevant number, $n$, is the number of species actually represented in the radiation flowing out during the time that the sphere is falling in. I shall take $D$ to be such that the infall time equals the time $\tau$ to radiate energy $E$. Then the number of radiation species into which $E$ is converted is also $n$. Thus from Eq. (21) one sees that the hole radiates the energy $E$ in time $\tau \approx 5 \times 10^4 E M^2 h^{-1} n^{-1}$. Since $D \approx (3\tau/\sqrt{2})^{2/3} M^{1/3}$, one checks that $D \approx 2.2 \times 10^3 (ME/n\hbar)^{2/3} M$. Now, the typical Hawking quantum bears an energy of order $T_{\text{BH}}$, so the number of quanta radiated is $\approx 8\pi ME/\hbar$. Since a species will be effective at braking the fall only if represented by at least one quantum, one has $n < 8\pi ME/\hbar$. As a result, $D \gg M$ as required by the discussion. Multiplying the ratio (22) by $n$ and recalling that $h/E < R < M$, one sees that

$$f_{\text{rad}}(r) < \frac{R^2}{7680\pi M^2} \ll 1$$

Radiation pressure thus fails to modify appreciably the geodesic fall of the sphere, and bound (20) follows.

I conclude that the GSL requires for its functioning a property of ordinary macroscopic matter encapsulated in bound (20). This is consistent with the tighter and more definite bound (13) established from statistical arguments in flat space-time. This last granted, the GSL is seen to be safe from the invasion of a black hole’s airspace by macroscopic entropy–bearing objects. It is interesting that this profoundly gravitational law “knows” about prosaic physics. This last statement has been at the heart of a protracted controversy in which Unruh and Wald have argued that the GSL can take care of itself with no help from the entropy bound by means of the buoyancy of objects in the Unruh acceleration radiation. Yet in the gedankenexperiment above buoyancy is irrelevant: the sphere falls freely, radiation pressure makes a small perturbation to its unaccelerated worldline, and so there is no Unruh–Wald buoyancy. Evidently, the GSL’s functioning does depend on properties of ordinary matter. (For a recent demonstration that the entropy bound (13) follows from the GSL even in circumstances where buoyancy is present see Ref. 62 and references cited therein.)

3.4. Do Black Holes Emit TV Sets?

Nothing illustrated so well to my generation the force of the “no hair” principle than Wheeler’s proverbial TV set falling into a black hole. But if a black hole can radiate, are TVs emitted in the Hawking radiance? The thermodynamic notion that anything can be found in a thermal radiation bath would suggest the answer is yes. This principle, however, must be applied cautiously. First, a system of energy $E$ appears spontaneously in a thermal bath only when the temperature is at least of order $E$. A black hole cannot be hotter than the Planck–Wheeler temperature.
Thus the only TVs that could be expected to appear are those lighter than the Planck–Wheeler mass $\sim 10^{-5}$ gm.

Further, the TV should be recalcitrant to dissociation. In the primordial plasma at redshift $z = 10^5$ there were no hydrogen atoms, not because it was not in equilibrium, but because the corresponding temperature of $3 \times 10^5$ K is way above the ionization temperature of hydrogen. There were $^4$He nuclei then because their dissociation temperature is way above $3 \times 10^5$ K. According to all this logic, Wheeler TVs weighing much less than the Planck–Wheeler mass, and having a very high dissociation temperature, should show up in Hawking radian ce whose temperature is of order of the TV’s rest energy. Yet, as I show now, TV sets or other macroscopic systems do not occur measurably in any Hawking radiance.

Suppose a macroscopic object (a TV for short) of size $R$ has rest energy $E$ and a degeneracy factor $g$. The last reflects the complexity of the composite system, so that $g$ could be very large. The object will get emitted in an available Hawking mode with probability $g \exp(-E/T_{\text{BH}})$. Actually, if the TV is measurably excited at temperature $T_{\text{BH}}$ one should replace $g$ by an appropriate partition function; I ignore such complications. Over the Hawking evaporation lifetime $\sim M^3/\hbar$ there emerge of order $M^2/\hbar$ “up” modes of each species. Thus the probability that the hole emits a TV over its lifetime amounts to $p \sim (M^2/\hbar)g \exp(-8\pi ME/\hbar)$.

Obviously $\ln g$ plays the role of internal entropy of the object. From the bound (13) one may infer that $\ln g < 2\pi RE/\hbar$. Thus $p < (M^2/\hbar)\exp[2\pi(R - 4M)E/\hbar]$. However, in order for the TV to be emitted whole it must be smaller than the hole: $R < 2M$. Hence $p < (M^2/\hbar)\exp(-4\pi ME/\hbar)$. But obviously the particles composing the TV (masses $\ll E$) must have Compton lengths smaller than $R < 2M$ so that $EM/\hbar \gg 1$. It follows that the argument of the exponent is very large, so that $p$ is exponentially small. Thus in practice an evaporating black hole does not emit TVs or any macroscopic objects. This “selection rule” depends on the bound on entropy.

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