Research Article

Hermite–Hadamard-Type Inequalities for Product of Functions by Using Convex Functions

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One of the many techniques to obtain a new convex function from the given functions is to calculate the product of these functions by imposing certain conditions on the functions. In general, the product of two or finite number of convex function needs not to be convex and, therefore, leads us to the study of product of these functions. In this paper, we reframe the idea of product of functions in the setting of generalized convex function to establish Hermite–Hadamard-type inequalities for these functions. We have analyzed different cases of double and triple integrals to derive some new results. The presented results can be viewed as the refinement and improvement of previously known results.

1. Introduction

Theory of convex functions has an essential role in different areas of mathematics, especially in optimization and modern analysis. Convex functions have many unique properties, for example, if a function is strictly convex, then it has a unique minimum on an open set. Even when the dimension of space is not finite, convex functions possess the similar properties and as a consequence, they are the examples of functionals in variation methods. A convex function via a random variable is bounded above by the expected value in the theory of probability. In fact, this is Jensen’s inequality, which can be used to reobtain many inequalities like the arithmetic-geometric mean inequality and Hölder’s inequality. One of the very fundamental results regarding convexity is the well-reputed Hermite–Hadamard inequality.

\[
p\left(\frac{\xi_1 + \xi_2}{2}\right) \leq \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} p(x)dx \leq \frac{p(\xi_1) + p(\xi_2)}{2} \tag{1}
\]

Convexity has been generalized in many aspects, and the classical Hermite–Hadamard inequality is viewed by these generalizations. In [1], Toader extended the idea of convexity by giving the definition of an \(m\)-convex function and constructed few results including Hermite–Hadamard-type inequalities. Further development on \(m\)-convex and \((a, m)\)-convex functions can be noticed in [1–9] and references therein. Further results on convex, \(m\)-convex and \((a, m, h)\)-convex functions can be seen in [10–21]. In [22], Pachpatte investigated the product of functions for developing Hermite–Hadamard-type inequalities by using the usual convexity. Recently, Noor et al. [23] introduced the idea of \((a, m, h)\)-convex function and established few fundamental inequalities for the class of twice-differentiable functions. Since the \((a, m, h)\)-convexity generalizes the concept of classical convexity, \(m\)-convexity, and \((a, m)\)-convex functions, the results therein are generalized. Also, the convex functions and their described generalizations are characterized by the product of functions in an elegant way. Motivated by these generalizations and [1, 3, 22, 23], we utilize the idea of product of functions in the setting of \((a, m, h)\)-convex function to establish Hermite–Hadamard-type inequalities for functions. We study different cases of double and triple integrals to derive some new results. This is the novel and innovative approach to
characterize the convex function with the product of $h$–convex, $m$–convex, $(s,m)$–convex, and $(a,s)$–convex functions. Our results unify several classes of functions like $(s,m)$–Godunova–Levin, $m$–Godunova–Levin, and $(a,s)$–Godunova–Levin functions with mild conditions in an elegant way.

2. Preliminaries

This section is devoted to few well-known definitions from the literature. In [1], Toader gave the definition of $m$–convex function in the following manner.

Definition 1. A function $p : I \rightarrow (0,\infty)$ is $m$–convex, if

$$p(\eta x_1 + m(1-\eta)x_2) \leq \eta p(x_1) + m(1-\eta)p(x_2),$$

(2)

for all $x_1, x_2 \in I$, $m \in [0,1]$, and $\eta \in [0,1]$.

In [24], Mihesan extended the idea of $m$–convex functions by introducing the idea of $(a,m)$–convex function as follows.

Definition 2. A function $p : I \rightarrow (0,\infty)$ is $(a,m)$–convex, if

$$p(\eta x_1 + m(1-\eta)x_2) \leq \eta^a p(x_1) + m(1-\eta^a)p(x_2),$$

(3)

for all $x_1, x_2 \in I$, $\eta \in [0,1]$, and $(a,m) \in [0,1]^2$.

In [23], Noor et al. generalized the idea of $m$–convexity and $(a,m)$–convexity in a more general way with the definition of $(a,m,h)$–convexity.

Definition 3. A function $p : I \rightarrow (0,\infty)$ is $(a,m,h)$–convex, if

$$p(\eta x_1 + m(1-\eta)x_2) \leq h(\eta^a) p(x_1) + mh(1-\eta^a)p(x_2),$$

(4)

for all $x_1, x_2 \in I$, $\eta \in [0,1]$, $(a,m) \in [0,1]^2$ and $h : [0,1] \rightarrow [0,1]$.

In [22], Pachpatte used the idea of product of functions for convex functions to establish the following result.

Theorem 1. Let $p$ and $q$ be nonnegative and convex functions on $[\xi_1, \xi_2]$ and further assume that they are real valued. Then, the following two inequalities hold:

$$\frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} p(x)q(x)dx \leq \frac{1}{3}M(\xi_1, \xi_2) + \frac{1}{6}N(\xi_1, \xi_2),$$

(5)

$$2p\left(\frac{\xi_1 + \xi_2}{2}\right)q\left(\frac{\xi_1 + \xi_2}{2}\right) \leq \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} p(x)q(x)dx + \frac{1}{6}M(\xi_1, \xi_2) + \frac{1}{3}N(\xi_1, \xi_2).$$

(6)

3. Main Results

This section contains the main results of our work involving product of $(a,m,h)$–convex functions to obtain Hermite–Hadamard-type integral inequalities for two functions $p$ and $q$. These inequalities have also been studied for double and triple integrals.

Theorem 2. Assume that $p$ and $q$ are nonnegative and real-valued functions with $pq \in L[\xi_1, \xi_2]$, where $\xi_1, \xi_2 \in I$ and $\xi_1 < \xi_2$. Furthermore, assume that $p$ and $q$ are $(a,m,h)$–convex on $[\xi_1, \xi_2]$. Then, we have the following inequality:

$$\frac{1}{m\xi_2 - \xi_1} \int_{\xi_1}^{m\xi_2} p(x)q(x)dx \leq h(\xi_1, \xi_2)L + h(\xi_2, \xi_1)M + h(\xi_1, \xi_2)N.$$  

(7)

Here,

$$H(\xi_1, \xi_2) = p(\xi_1)q(\xi_1),$$

$$H(\xi_2, \xi_1) = p(\xi_2)q(\xi_2),$$

$$H(\xi_1, \xi_2) = p(\xi_1)q(\xi_2) + p(\xi_2)q(\xi_1).$$

(8)

Also, $L = \int_{1}^{\xi_1} [h(\eta^a)]^2 d\eta$, $M = \int_{1}^{\xi_2} m^2 [h(1-\eta^a)]^2 d\eta$ and $N = \int_{0}^{1} mh(\eta^a)h(1-\eta^a)d\eta$.

Proof. Assume that $p$ and $q$ are $(a,m,h)$–convex on $[\xi_1, \xi_2]$, then,

$$p(\eta \xi_1 + m(1-\eta)\xi_2)q(\eta \xi_1 + m(1-\eta)\xi_2) \leq \eta h(\eta^a)p(\xi_3) + mh(1-\eta^a)p(\xi_3) \leq \eta^a h(\eta^a)p(\xi_3) + mh(1-\eta^a)p(\xi_3)$$

$$= \eta^a h(\eta^a)p(\xi_3) + mh(1-\eta^a)p(\xi_3) = mh(1-\eta^a)p(\xi_3)$$

(9)

Now,

$$\int_{0}^{1} p(\eta \xi_1 + m(1-\eta)\xi_2)q(\eta \xi_1 + m(1-\eta)\xi_2)d\eta \leq \left[ \int_{0}^{1} h(\eta^a)^2 d\eta \right] p(\xi_3)q(\xi_3) + \left[ \int_{0}^{1} m^2 [h(1-\eta^a)]^2 d\eta \right] p(\xi_2)q(\xi_2)$$

(10)

$$+ \left[ \int_{0}^{1} mh(\eta^a)h(1-\eta^a)[p(\xi_3)q(\xi_2) + p(\xi_2)q(\xi_3)]d\eta \right].$$
Thus,
\[
\int_0^1 \frac{1}{m\xi_2 - \xi_1} p(x)q(x)dx \leq \left[ \frac{1}{m\xi_2 - \xi_1} \int_{\xi_1}^{m\xi_2} p(x)q(x)dx \right] \left[ h(\xi_1, \xi_1) + h(\xi_2, \xi_2)M + h(\xi_1, \xi_2)N \right].
\]

(11)

Now, substituting \(\eta \xi_1 + m(1 - \eta)\xi_2 = x\), we obtain
\[
\int_0^1 \frac{1}{m\xi_2 - \xi_1} p(x)q(x)dx \leq \left[ \frac{1}{m\xi_2 - \xi_1} \int_{\xi_1}^{m\xi_2} p(x)q(x)dx \right] \left[ h(\xi_1, \xi_1) + h(\xi_2, \xi_2)M + h(\xi_1, \xi_2)N \right].
\]

(12)

Corollary 1. If \(h(\eta) = \eta\), then
\[
\lim_{\xi \to 0} \int_0^1 \frac{1}{m\xi_2 - \xi_1} p(x)q(x)dx \leq \left[ \frac{1}{m\xi_2 - \xi_1} \int_{\xi_1}^{m\xi_2} p(x)q(x)dx \right] \left[ h(\xi_1, \xi_1) + h(\xi_2, \xi_2)M + h(\xi_1, \xi_2)N \right].
\]

(13)

Theorem 3. For any two \((\alpha, m, h)\)-convex functions \(p\) and \(q\) with \(pq \in L[\xi_1, \xi_2]\), we have the following estimates:

\[
\int_0^1 \left[ \int_{\xi_1}^{\xi_2} p(x)q(x)dx \right] \leq \left[ \frac{1}{2\alpha + 1} H(\xi_1, \xi_1) + \frac{2m^2\alpha^2}{(2\alpha + 1)(\alpha + 1)} H(\xi_2, \xi_2) \right] + \frac{\max}{(2\alpha + 1)(\alpha + 1)} H(\xi_1, \xi_2).
\]

(14)

Remark 1. Corollary 1, along with \(m = 1\) and \(\alpha = 1\), gives inequality (5).

\[
2p\left(\frac{\xi_1 + m\xi_2}{2}\right)q\left(\frac{\xi_1 + m\xi_2}{2}\right) \leq \left[ \int_0^1 p(\eta \xi_1 + n(1 - \eta)\xi_2)q(\eta \xi_1 + m(1 - \eta)\xi_2)\eta d\eta \right] \left[ H(\xi_1, \xi_1) + n^2 H(\xi_2, \xi_2)S + \frac{1}{2} [H(\xi_1, \xi_2)]T \right].
\]

(15)

Proof. As \(p\) and \(q\) are \((\alpha, m, h)\)-convex, we have

\[
p\left(\frac{\xi_1 + m\xi_2}{2}\right)q\left(\frac{\xi_1 + m\xi_2}{2}\right) \leq \left[ \frac{1}{4} \left( p(\eta \xi_1 + m(1 - \eta)\xi_2) + p((1 - \eta)\xi_1 + m\eta\xi_2) \right) \right] \left[ q(\eta \xi_1 + m(1 - \eta)\xi_2) + q((1 - \eta)\xi_1 + m\eta\xi_2) \right].
\]

Also \(H(\xi_1, \xi_1), H(\xi_2, \xi_2),\) and \(H(\xi_1, \xi_2)\) are as defined in the above theorem.
\[
\begin{align*}
\frac{1}{4} [p(\eta \xi_1 + m(1 - \eta) \xi_2) q(\eta \xi_1 + m(1 - \eta) \xi_2) \\
+ p((1 - \eta) \xi_1 + m\eta \xi_2) q((1 - \eta) \xi_1 + m\eta \xi_2)] \\
+ \frac{1}{4} [p(\eta \xi_1 + m(1 - \eta) \xi_2) q((1 - \eta) \xi_1 + m\eta \xi_2)] \\
+ p((1 - \eta) \xi_1 + m\eta \xi_2) q(\eta \xi_1 + m(1 - \eta) \xi_2)] \\
\leq \frac{1}{4} [p(\eta \xi_1 + m(1 - \eta) \xi_2) q(\eta \xi_1 + m(1 - \eta) \xi_2) \\
+ p((1 - \eta) \xi_1 + m\eta \xi_2) q((1 - \eta) \xi_1 + m\eta \xi_2)] \\
+ \frac{1}{2} [h(\eta^a) p(\xi_1) + m^2 p(\xi_2) q(\xi_2)] \\
\leq \frac{1}{4} [h(1 - \eta^a) q(\xi_1) + m h(\eta^a) q(\xi_2)] \\
+ \frac{1}{4} [h(\eta^a)]^2 + [h(1 - \eta^a)]^2 [p(\xi_1) q(\xi_2) + p(\xi_2) q(\xi_1)] \\
\end{align*}
\]

On integrating,

\[
\begin{align*}
p\left(\frac{\xi_1 + m\xi_2}{2}\right) q\left(\frac{\xi_1 + m\xi_2}{2}\right) \leq \\
\frac{1}{4} \int_0^1 \left[ p(\eta \xi_1 + m(1 - \eta) \xi_2) q(\eta \xi_1 + m(1 - \eta) \xi_2) \\
+ p((1 - \eta) \xi_1 + m\eta \xi_2) q((1 - \eta) \xi_1 + m\eta \xi_2) \right] d\eta \\
+ \frac{1}{2} \int_0^1 h(\eta^a) h(1 - \eta^a) d\eta \left[ p(\xi_1) p(\xi_2) + m^2 p(\xi_2) q(\xi_2) \right] \\
+ \frac{1}{4} \int_0^1 \left[ m[h(\eta^a)]^2 + [h(1 - \eta^a)]^2 \right] d\eta \left[ p(\xi_1) q(\xi_2) + p(\xi_2) q(\xi_1) \right]
\end{align*}
\]

Thus, we obtain

\[
\begin{align*}
p\left(\frac{\xi_1 + m\xi_2}{2}\right) q\left(\frac{\xi_1 + m\xi_2}{2}\right) \leq \\
\frac{1}{2} \int_0^1 \left[ p(\eta \xi_1 + m(1 - \eta) \xi_2) q(\eta \xi_1 + m(1 - \eta) \xi_2) \right] d\eta \\
+ \frac{1}{2} \left[ H(\xi_1, \xi_1) + m H(\xi_2, \xi_2) \right] S + \frac{1}{4} \left[ H(\xi_1, \xi_2) T \right]
\end{align*}
\]
Now, substituting \( x = \eta \xi_1 + n(1 - \eta) \xi_2 \), we obtain
\[
2p\left(\frac{\xi_1 + m \xi_2}{2}\right)q\left(\frac{\xi_1 + m \xi_2}{2}\right) \leq \left[\frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} p(x)q(x)dx + \frac{\alpha}{(\alpha + 1)(2\alpha + 1)} [H(\xi_1, \xi_1) + mH(\xi_2, \xi_2)] + \frac{m(\alpha^2 + \alpha + 1)}{2(\alpha + 1)(2\alpha + 1)} H(\xi_1, \xi_2)\right].
\]

Corollary 2. If \( h(\eta) = \eta \), then
\[
2p\left(\frac{\xi_1 + m \xi_2}{2}\right)q\left(\frac{\xi_1 + m \xi_2}{2}\right) \leq \left[\frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} p(x)q(x)dx + \frac{\alpha}{(\alpha + 1)(2\alpha + 1)} [H(\xi_1, \xi_1) + mH(\xi_2, \xi_2)] + \frac{m(\alpha^2 + \alpha + 1)}{2(\alpha + 1)(2\alpha + 1)} H(\xi_1, \xi_2)\right].
\]

Remark 2. Corollary 2, along with \( m = 1 \) and \( \alpha = 1 \), gives inequality (6).

Theorem 4. Assume that \( p \) and \( q \) are \((\alpha, m, h)\)-convex functions satisfying all the conditions of the above theorem; then, the following inequality holds:
\[
\frac{1}{(\xi_2 - \xi_1)^2} \int_{\xi_1}^{\xi_2} \int_{0}^{1} p(\eta x + m(1 - \eta)y)q(\eta x + m(1 - \eta)y)d\eta dy dx \leq [L + M] \frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} p(x)q(x)dx + \frac{1}{2} \frac{1}{(\xi_2 - \xi_1)^2} N [H(\xi_1, \xi_1) + H(\xi_2, \xi_2) + H(\xi_1, \xi_2)],
\]

where
\[
L = \int_{0}^{1} [h(\eta^m)]^2 d\eta,
M = \int_{0}^{1} m^2 [h(1 - \eta^m)]^2 d\eta,
N = \int_{0}^{1} mh(\eta^m)h(1 - \eta^m)d\eta.
\]

Proof. Using the \((\alpha, m, h)\)-convexity,
\[
p(\eta x + m(1 - \eta)y)q(\eta x + m(1 - \eta)y) \leq \begin{bmatrix} [h(\eta^m)p(x) + mh(1 - \eta^m)p(y)] & [h(\eta^m)q(x) + mh(1 - \eta^m)q(y)] \\ [h(\eta^m)p(x) + mh(1 - \eta^m)p(y)] & [h(\eta^m)q(x) + mh(1 - \eta^m)q(y)] \end{bmatrix}
\]
\[
= [h(\eta^m)]^2 p(x)q(x) + m^2 [h(1 - \eta^m)] \times 2p(y)q(y) + [mh(\eta^m)h(1 - \eta^m)][p(x)q(y) + p(y)q(x)]
\]

Integrating on \([0, 1]\), we obtain
\[
\int_{0}^{1} p(\eta x + n(1 - \eta)y)q(\eta x + m(1 - \eta)y)d\eta \leq p(x)q(x) \int_{0}^{1} [h(\eta^m)]^2 d\eta + p(y)q(y) \int_{0}^{1} m^2 [h(1 - \eta^m)]^2 d\eta
\]
\[
+ [p(x)q(y) + p(y)q(x)] \int_{0}^{1} mh(\eta^m)h(1 - \eta^m)d\eta.
\]
Thus,
\[
\int_{\xi_1}^{\xi_2} p(\eta x + m(1 - \eta) y) q(\eta x + m(1 - \eta) y) d\eta \\
\leq [p(x)q(x)]L + [p(y)q(y)]M + [p(x)q(y) + p(y)q(x)]N.
\]
(25)

Now, integrating on the rectangle \([0, 1] \times [0, 1]\),
\[
\int_{\xi_1}^{\xi_2} \int_{\xi_1}^{\xi_2} p(\eta x + m(1 - \eta) y) q(\eta x + m(1 - \eta) y) d\eta dy dx
\leq L(\xi_2 - \xi_1) \int_{\xi_1}^{\xi_2} p(x) q(x) dx + M(\xi_2 - \xi_1) \int_{\xi_1}^{\xi_2} p(y) q(y) dy + N \left[ \int_{\xi_1}^{\xi_2} p(x) dx \times \int_{\xi_1}^{\xi_2} q(y) dy \right]
\]
(26)

Now, applying Hadamard’s inequality from right half to the above equation,
\[
\int_{\xi_1}^{\xi_2} \int_{\xi_1}^{\xi_2} p(\eta x + m(1 - \eta) y) q(\eta x + m(1 - \eta) y) d\eta dy dx
\leq [L + M](\xi_2 - \xi_1) \int_{\xi_1}^{\xi_2} p(x) q(x) dx + \frac{1}{2} N \left[ p(\xi_1) q(\xi_1) + p(\xi_2) q(\xi_2) + p(\xi_1) q(\xi_2) + p(\xi_2) q(\xi_1) \right]
\]
(27)
\[
= [L + M](\xi_2 - \xi_1) \int_{\xi_1}^{\xi_2} p(x) q(x) dx + \frac{1}{2} N \left[ H(\xi_1, \xi_1) + H(\xi_2, \xi_2) + H(\xi_1, \xi_2) \right]
\]

Thus, we obtain
\[
\frac{1}{(\xi_2 - \xi_1)^2} \int_{\xi_1}^{\xi_2} \int_{\xi_1}^{\xi_2} p(\eta x + m(1 - \eta) y) q(\eta x + m(1 - \eta) y) d\eta dy dx
\leq \frac{1}{\xi_2 - \xi_1} [L + M] \int_{\xi_1}^{\xi_2} p(x) q(x) dx + \frac{1}{2} \frac{1}{(\xi_2 - \xi_1)^2} N \left[ H(\xi_1, \xi_1) + H(\xi_2, \xi_2) + H(\xi_1, \xi_2) \right].
\]
(28)

Corollary 3. For \( h(\eta) = \eta \), we have
\[
\frac{1}{(\xi_2 - \xi_1)^2} \int_{\xi_1}^{\xi_2} \int_{\xi_1}^{\xi_2} p(\eta x + m(1 - \eta) y) q(\eta x + m(1 - \eta) y) d\eta dy dx
\leq \left[ \frac{2\alpha^2 m^2 + \alpha + 1}{(2\alpha + 1)(\alpha + 1)(\xi_2 - \xi_1)^2} \int_{\xi_1}^{\xi_2} p(x) q(x) dx \right]
\]
(29)
\[
\leq \left[ \frac{ma}{2(2\alpha + 1)(\alpha + 1)(\xi_2 - \xi_1)^2} \left[ H(\xi_1, \xi_1) + H(\xi_2, \xi_2) + H(\xi_1, \xi_2) \right] \right].
\]
Remark 3. For $\alpha = 1 = m$, we obtain equation (3) [22].

\[
\frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} \int_0^1 p\left( \eta x + m(1 - \eta) \frac{\xi_1 + \xi_2}{2} \right) q\left( \eta x + m(1 - \eta) \frac{\xi_1 + \xi_2}{2} \right) d\eta dx \\
\leq \frac{L}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} p(x) q(x) dx + \frac{1}{4} M \left[ H(\xi_1, \xi_1) + H(\xi_2, \xi_2) + H(\xi_1, \xi_2) \right] + \frac{N}{2(\xi_2 - \xi_1)} \left[ H(\xi_1, \xi_1) + H(\xi_2, \xi_2) + H(\xi_1, \xi_2) \right],
\]

where $L$, $M$, and $N$ are as in the above theorem.

Theorem 5. Assume that $p$ and $q$ are nonnegative real-valued functions such that they are $(\alpha, m, h)$-convex and $pq \in L[\xi_1, \xi_2]$; then,

\[
p\left( \eta x + m(1 - \eta) \frac{\xi_1 + \xi_2}{2} \right) q\left( \eta x + m(1 - \eta) \frac{\xi_1 + \xi_2}{2} \right) \\
\leq \left[ h(\eta^m) p(x) + mh(1 - \eta^m) p\left( \frac{\xi_1 + \xi_2}{2} \right) \right] \left[ h(\eta^m) q(x) + mh(1 - \eta^m) q\left( \frac{\xi_1 + \xi_2}{2} \right) \right] \\
= \left[ h(\eta^m) \right]^2 p(x) q(x) + m^2 [h(1 - \eta^m)]^2 p\left( \frac{\xi_1 + \xi_2}{2} \right) q\left( \frac{\xi_1 + \xi_2}{2} \right) \\
+ mh(\eta^m) h(1 - \eta^m) \left[ p(x) q\left( \frac{\xi_1 + \xi_2}{2} \right) + q(x) p\left( \frac{\xi_1 + \xi_2}{2} \right) \right].
\]

This implies that

\[
\int_0^1 p\left( \eta x + m(1 - \eta) \frac{\xi_1 + \xi_2}{2} \right) q\left( \eta x + m(1 - \eta) \frac{\xi_1 + \xi_2}{2} \right) d\eta \\
\leq p(x) q(x) \int_0^1 \left[ h(\eta^m) \right]^2 d\eta + p\left( \frac{\xi_1 + \xi_2}{2} \right) q\left( \frac{\xi_1 + \xi_2}{2} \right) \int_0^1 m^2 [h(1 - \eta^m)]^2 d\eta \\
+ \left[ p(x) q\left( \frac{\xi_1 + \xi_2}{2} \right) + q(x) p\left( \frac{\xi_1 + \xi_2}{2} \right) \right] \int_0^1 mh(\eta^m) h(1 - \eta^m) d\eta.
\]

Integrating on $[0, 1]$,

\[
\int_{\xi_1}^{\xi_2} \int_0^1 p\left( \eta x + m(1 - \eta) \frac{\xi_1 + \xi_2}{2} \right) q\left( \eta x + m(1 - \eta) \frac{\xi_1 + \xi_2}{2} \right) d\eta dx \\
\leq L \int_{\xi_1}^{\xi_2} p(x) q(x) dx + M (\xi_2 - \xi_1) \left[ p\left( \frac{\xi_1 + \xi_2}{2} \right) q\left( \frac{\xi_1 + \xi_2}{2} \right) + \left[ q\left( \frac{\xi_1 + \xi_2}{2} \right) \right] \int_{\xi_1}^{\xi_2} p(x) dx \\
+ \left( \frac{\xi_1 + \xi_2}{2} \right) \int_{\xi_1}^{\xi_2} q(x) dx \right] N.
\]
Now, using the right half of Hadamard’s inequality on the above equation,

\[
\int_{\xi_1}^{\xi_2} \int_{\xi_1}^{\xi_2} p \left( \eta x + n(1 - \eta) \left( \frac{\xi_1 + \xi_2}{2} \right) \right) q \left( \eta x + m(1 - \eta) \left( \frac{\xi_1 + \xi_2}{2} \right) \right) d\eta \, dx \\
\leq L \int_{\xi_1}^{\xi_2} p(x) q(x) \, dx + M (\xi_2 - \xi_1) \left[ \frac{p(\xi_1) + p(\xi_2)}{2} \right] + \left[ \frac{q(\xi_1) + q(\xi_2)}{2} \right] \right] N. 
\]

Hence, we obtain

\[
\frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} \int_{\xi_1}^{\xi_2} p \left( \eta x + n(1 - \eta) \left( \frac{\xi_1 + \xi_2}{2} \right) \right) q \left( \eta x + n(1 - \eta) \left( \frac{\xi_1 + \xi_2}{2} \right) \right) d\eta \, dx \\
\leq \frac{L}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} p(x) q(x) \, dx + \frac{M}{4} \left[ H(\xi_1, \xi_1) + H(\xi_2, \xi_2) + H(\xi_1, \xi_2) \right] + \frac{N}{2(\xi_2 - \xi_1)} \left[ h(\xi_1, \xi_1) + h(\xi_2, \xi_2) + h(\xi_1, \xi_2) \right].
\]

This completes the proof. \( \square \)

**Corollary 4.** If \( h(\eta) = \eta \), then

\[
\frac{1}{\xi_2 - \xi_1} \int_{\xi_1}^{\xi_2} \int_{\xi_1}^{\xi_2} p \left( \eta x + n(1 - \eta) \left( \frac{\xi_1 + \xi_2}{2} \right) \right) q \left( \eta x + n(1 - \eta) \left( \frac{\xi_1 + \xi_2}{2} \right) \right) d\eta \, dx \\
\leq \frac{1}{(\xi_2 - \xi_1)(2\alpha + 1)} \int_{\xi_1}^{\xi_2} p(x) q(x) \, dx + \frac{\alpha^2 m^2}{2(\alpha + 1)(2\alpha + 1)} \left[ h(\xi_1, \xi_1) + h(\xi_2, \xi_2) + h(\xi_1, \xi_2) \right]
\]

\[
+ \frac{m}{2(\xi_2 - \xi_1)(\alpha + 1)(2\alpha + 1)} \left[ h(\xi_1, \xi_1) + h(\xi_2, \xi_2) + h(\xi_1, \xi_2) \right].
\]

**Remark 4.** For \( \alpha = 1 = m \), we obtain equation (4) of [22].

**4. Conclusion**

In this paper, we utilize the product of functions to develop the class of generalized convex functions using two given functions. We have studied this product for \((\alpha, m)\)-convex functions. Afterwards, we applied this to investigate Hermite–Hadamard inequalities of various types. We have analyzed that for the specific value of \( h \), that is, to be identity function, these results coincide with the results for the product of \((\alpha, m)\)-convex functions. Moreover, the results are true for the said product in the sense of \( m \)-convexity with \( \alpha = 1 \). The comparison reflects that the obtained results improve and generalize the results for convex, \( m \)-convex, and \((\alpha, m)\)-convex functions in a peculiar way. For further interest of the readers in this direction, one may examine this product for invex, preinvex, \( m \)-preinvex, harmonically preinvex, and logarithmically preinvex functions. The idea is also interesting for fractional integrals and stochastic process for convex functions for new aspects in this regard [25].

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.
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