ROBUST CONTROL OF A CAHN-HILLIARD-NAVIER-STOKES MODEL

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Abstract. We study in this article a class of robust control problems associated with a coupled Cahn-Hilliard-Navier-Stokes model in a two dimensional bounded domain. The model consists of the Navier-Stokes equations for the velocity, coupled with the Cahn-Hilliard model for the order (phase) parameter. We prove the existence and uniqueness of solutions and we derive a first-order necessary optimality condition for these robust control problems.

1. Introduction. It is well accepted that the incompressible Navier-Stokes equation governs the motions of single-phase fluids such as air or water. On the other hand, we are faced with the difficult problem of understanding the motion of binary fluid mixtures, that is fluids composed by either two phases of the same chemical species or phases of different composition. Diffuse interface models are well-known tools to describe the dynamics of complex (e.g., binary) fluids, [15]. For instance, this approach is used in [5] to describe cavitation phenomena in a flowing liquid. The model consists of the NS equation coupled with the phase-field system, cf., eg. [9, 15, 14, 16]. In the isothermal compressible case, the existence of a global weak solution is proved in [12]. In the incompressible isothermal case, neglecting chemical reactions and other forces, the model reduces to an evolution system which governs the fluid velocity \( v \) and the order parameter \( \phi \). This system can be written as a NS equation coupled with a convective Allen-Cahn equation, [15]. The associated initial and boundary value problem was studied in [15] in which the authors proved that the system generated a strongly continuous semigroup on a suitable phase space which possesses a global attractor. They also established the existence of an exponential attractor. This entails that the global attractor has a finite fractal dimension, which is estimated in [15] in terms of some model parameters. The dynamic of simple single-phase fluids has been widely investigated although some important issues remain unresolved, [28]. In the case of binary fluids, the analysis is even more complicate and the mathematical studied is still at its infancy as noted in [15]. As noted in [14], the mathematical analysis of binary fluid flows is far from being well understood. For instance, the spinodal decomposition under shear consists of a two-stage evolution of a homogeneous initial mixture: a phase separation stage in which some macroscopic patterns appear, then a shear stage in

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which these patterns organize themselves into parallel layers (see, e.g. [25] for experimental snapshots). This model has to take into account the chemical interactions between the two phases at the interface, achieved using a Cahn-Hilliard approach, as well as the hydrodynamic properties of the mixture (e.g., in the shear case), for which a Navier-Stokes equations with surface tension terms acting at the interface are needed. When the two fluids have the same constant density, the temperature differences are negligible and the diffuse interface between the two phases has a small but non-zero thickness, a well-known model is the so-called ”Model H” (cf. [19, 17]). This is a system of equations where an incompressible Navier-Stokes equation for the (mean) velocity $v$ is coupled with a convective Cahn-Hilliard equation for the order parameter $\phi$, which represents the relative concentration of one of the fluids.

The necessary conditions for optimal control problems governed by fluid mechanic models such as the NS systems have been studied by several authors (see for instance [31, 32, 30, 29, 21, 22, 23, 3]). In [29], the authors studied a Pontryagin’s maximum principle for optimal control problems (with a state constraint) governed by the 3D NS equations. In order to overcome the problem associated with the state constraint, the authors first defined a new penalty functional depending on a small parameter $\epsilon$ with which they approximated the original problem with a family of optimal control problems $P_\epsilon$ without state constraint. A Pontryagin’s maximum principle is derived for the approximate problem $P_\epsilon$ and the limit as $\epsilon$ goes to 0 yields an optimality condition for the original control problem with a state constraint. For control problems associated to the Cahn-Hilliard-Navier-Stokes (CH-NS) systems, not that much have been done. In [24], the author studied a Pontryagin’s maximum principle for optimal control problems (with a state constraint) governed by a 2D CH-NS model. Following the work of [29], he derived an optimality condition for the control problem. In [13], the authors studied some distributed optimal control problem associated with a CH-NS system. They proved the existence of a solution and derived a first-order optimality condition. Similar results are obtained in [18], where the authors studied an optimal boundary control problem associated with a time-discrete CH-NS system.

As described in [4], robust control theory, which generalizes optimal control theory, can be represented as a differential game between an engineer seeking the ”best” control which stabilizes the flow perturbation with limited control effort and, simultaneously, nature seeking the ”maximally malevolent” disturbance which destabilizes the flow perturbation with limited disturbance magnitude. In [4], the authors present a general framework for robust control problem in fluid mechanics. Given a fairly general cost functional $J = J(\psi, \phi)$, the authors in [4] proved the existence of a saddle point $(\bar{\psi}, \bar{\phi})$, which maximizes $J$ with respect to the disturbance $\psi$ and minimizes $J$ with respect to the control $f_2$, subject to the Navier-Stokes equations. In this article, we study some robust control problems associated with the CH-NS systems. We prove the existence and uniqueness of solutions using the framework given in [4]. Let us note that the coupling between the Navier-Stokes and the Cahn-Hilliard systems introduces in the system a highly nonlinear coupling term that makes the analysis of these control problems more involved.

The article is divided as follows. In the next section we present the CH-NS model and its mathematical setting. We recall from [8, 6, 7, 15, 14] some existence and uniqueness results as well as some a priori estimates on the solution. We also present some estimates for the linearized system. The third section presents the
robust control framework as well as the main results. We first study the linear problem and then, we consider the full nonlinear case. For each of these problems, we prove the existence and uniqueness of solution and we derive an associated first-order optimality condition. Finally, some a priori estimates on the adjoint systems are given in the Appendix.

2. The CH-NS model and its mathematical setting.

2.1. Governing equations. In this article, we consider a model of homogeneous incompressible two-phase flow. More precisely, we assume that the domain \( \mathcal{M} \) of the fluid is a bounded domain in \( \mathbb{R}^2 \). Then, we consider the system

\[
\begin{align*}
\frac{\partial v}{\partial t} - \nu_1 \Delta v + (v \cdot \nabla)v + \nabla p - K \mu \nabla \phi &= Q, \\
\text{div } v &= 0, \\
\frac{\partial \phi}{\partial t} + v \cdot \nabla \phi - \nu_3 \Delta \mu &= 0, \\
\mu &= -\nu_2 \Delta \phi + \alpha f(\phi),
\end{align*}
\]

in \( \mathcal{M} \times (0, +\infty) \).

In (1), the unknown functions are the velocity \( v = (v_1, v_2) \) of the fluid, its pressure \( p \) and the order (phase) parameter \( \phi \). The external volume force \( Q \) is given. The quantity \( \mu \) is the variational derivative of the following free energy functional

\[
\mathcal{F}(\phi) = \int_{\mathcal{M}} \left( \frac{\nu_2}{2} |\nabla \phi|^2 + \alpha F(\phi) \right) ds,
\]

where, e.g., \( F(r) = \int_0^r f(\zeta) d\zeta \). Here, the constants \( \nu_1 > 0, \nu_3 > 0 \) and \( K > 0 \) correspond to the kinematic viscosity of the fluid, the mobility constant and the capillarity (stress) coefficient respectively. Here \( \nu_2, \alpha > 0 \) are two physical parameters describing the interaction between the two phases. In particular, \( \nu_2 \) is related with the thickness of the interface separating the two fluids.

A typical example of potential \( F \) is that of logarithmic type. However, this potential is often replaced by a polynomial approximation of the type \( F(r) = \gamma_1 r^4 - \gamma_2 r^2, \gamma_1, \gamma_2 \) being positive constants. As noted in [14], (1) can be replaced by

\[
\frac{\partial v}{\partial t} - \nu_1 \Delta v + (v \cdot \nabla)v + \nabla \tilde{p} = -K \text{div} (\nabla \phi \otimes \nabla \phi) + Q,
\]

where \( \tilde{p} = p - K (\nu_2 |\nabla \phi|^2 + \alpha F(\phi)), \) since \( K \mu \nabla \phi = \nabla (K (\nu_2 |\nabla \phi|^2 + \alpha F(\phi))) - K \text{div} (\nabla \phi \otimes \nabla \phi) \). The stress tensor \( \nabla \phi \otimes \nabla \phi \) is considered the main contribution modeling capillary forces due to surface tension at the interface between the two phases of the fluid.

Regarding the boundary conditions for these models, we assume that the boundary conditions for \( \phi \) are the natural no-flux condition

\[
\partial_n \phi = \partial_n \Delta \phi = 0, \quad \text{on } \partial \mathcal{M} \times (0, \infty),
\]

where \( \partial \mathcal{M} \) is the boundary of \( \mathcal{M} \) and \( \eta \) is the outward normal to \( \partial \mathcal{M} \). These conditions ensure the mass conservation. Note that (4) implies that

\[
\partial_n \mu = 0, \quad \text{on } \partial \mathcal{M} \times (0, \infty).
\]

From (5), we deduce the conservation of the following quantity

\[
\langle \phi(t) \rangle = \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} \phi(x,t) dx,
\]
where $|\mathcal{M}|$ stands for the Lebesgue measure of $\mathcal{M}$. More precisely, we have
\begin{equation}
\langle \phi(t) \rangle = \langle \phi(0) \rangle, \quad \forall t \geq 0. \tag{7}
\end{equation}

Concerning the boundary condition for $v$, we assume the Dirichlet (no-slip) boundary condition
\begin{equation}
v = 0, \quad \text{on } \partial \mathcal{M} \times (0, \infty). \tag{8}
\end{equation}
Therefore we assume that there is no relative motion at the fluid-solid interface.

The initial condition is given by
\begin{equation}
(v, \phi)(0) = (v_0, \phi_0), \quad \text{in } \mathcal{M}. \tag{9}
\end{equation}

2.2. Mathematical setting. We first recall from [14] a weak formulation of (1), (4), (8)-(9). Hereafter, we assume that the domain $\mathcal{M}$ is bounded with a smooth boundary $\partial \mathcal{M}$ (e.g., of class $C^4$). We also assume that $f \in C^4(\mathbb{R})$ satisfies
\begin{equation}
\begin{cases}
\lim_{|r| \to +\infty} f'(r) > 0, \\
|f^k(r)| \leq c_f (1 + |r|^{m+1-k}), \quad \forall r \in \mathbb{R}, \quad k = 0, 1, 2, 3, 4,
\end{cases} \tag{10}
\end{equation}
where $c_f$ is some positive constant and $m \in [3, +\infty)$ is fixed.

Hereafter, if $X$ is a Banach space, we will denote by $X^*$ the dual space of $X$. To simplify the notations, the duality paring between $X$ and $X^*$ will be denoted $\langle \cdot, \cdot \rangle$ and the norm in $X^*$ will be denoted $\| \cdot \|_{X^*}$.

We set
\begin{equation}
\mathcal{V}_1 = \{ u \in C^\infty_c (\mathcal{M}) : \quad \text{div } u = 0 \text{ in } \mathcal{M} \}.
\end{equation}
We denote by $H_1$ and $\mathcal{V}_1$ the closure of $\mathcal{V}_1$ in $(L^2(\mathcal{M}))^2$ and $(H^1_0(\mathcal{M}))^2$ respectively. The scalar product in $H_1$ is denoted by $\langle \cdot, \cdot \rangle_{L^2}$ and the associated norm by $\| \cdot \|_{L^2}$. Moreover, the space $\mathcal{V}_1$ is endowed with the scalar product
\begin{equation}
((u, v)) = \sum_{i=1}^2 (\partial_{x_i} u, \partial_{x_i} v)_{L^2}, \quad \| u \| = ((u, u))^{1/2}.
\end{equation}

We now define the operator $A_0$ by
\begin{equation}
A_0 u = -\mathcal{P}_1 \Delta u, \quad \forall u \in D(A_0) = H^2(\mathcal{M}) \cap \mathcal{V}_1,
\end{equation}
where $\mathcal{P}_1$ is the Leray-Helmotz projector in $L^2(\mathcal{M})$ onto $H_1$. Then, $A_0$ is a self-adjoint positive unbounded operator in $H_1$ which is associated with the scalar product defined above. Furthermore, $A_0^{-1}$ is a compact linear operator on $H_1$ and $\| A_0 \cdot \|_{L^2}$ is a norm on $D(A_0)$ that is equivalent to the $H^2$-norm.

We introduce the linear nonnegative unbounded operator on $L^2(\mathcal{M})$
\begin{equation}
A_N \phi = -\Delta \phi, \quad \forall \phi \in D(A_N) = \{ \phi \in H^2(\mathcal{M}), \quad \partial_{\eta} \phi = 0, \quad \text{on } \partial \mathcal{M} \}, \tag{11}
\end{equation}
and we endow $D(A_N)$ with the norm $\| A_N \cdot \|_{L^2} + \| (\cdot) \|_{L^2}$, which is equivalent to the $H^2$-norm. We also define the linear positive unbounded operator on the Hilbert space $L^2_0(\mathcal{M})$ of the $L^2$-functions with null mean
\begin{equation}
B_N \phi = -\Delta \phi, \quad \forall \phi \in D(B_N) = D(A_N) \cap L^2_0(\mathcal{M}). \tag{12}
\end{equation}
Note that $B_N^{-1}$ is a compact linear operator on $L^2_0(\mathcal{M})$. More generally, we can define $B_N^s$ for any $s \in \mathbb{R}$, noting that $\| B_N^{s/2} \cdot \|_{L^2}$, $s > 0$, is an equivalent norm to the canonical $H^s$-norm on $D(B_N^{s/2}) \subset H^s(\mathcal{M}) \cap L^2_0(\mathcal{M})$. Also note that $A_N = B_N$ on $D(B_N)$. If $\phi$ is such that $\phi - \langle \phi \rangle \in D(B_N^{s/2})$, we have that $\| B_N^{s/2} (\phi - \langle \phi \rangle) \|_{L^2} + \| \langle \phi \rangle \|_{L^2}$ is equivalent to the $H^s$-norm. Moreover, we set $H^{-s}(\mathcal{M}) = (H^s(\mathcal{M}))^*$, whenever $s < 0$. 

\[ \text{Equation number} \]

We introduce the bilinear operators $B_0, B_1$ (and their associated trilinear forms $b_0, b_1$) as well as the coupling mapping $R_0$, which are defined from $D(A_0) \times D(A_0)$ into $H_1, D(A_0) \times D(A_N)$ into $L^2(M)$, and $L^2(M) \times (D(A_N) \cap H^3(M))$ into $H_1$, respectively. More precisely, we set

\[
\langle B_0(u, v), w \rangle = \int_M [(u \cdot \nabla)v] \cdot wdx = b_0(u, v, w), \ \forall u, v, w \in D(A_0),
\]

\[
\langle B_1(u, \phi), \rho \rangle = \int_M [(u \cdot \nabla)\phi]\rho dx = b_1(u, \phi, \rho), \ \forall u \in D(A_0), \ \phi, \rho \in D(A_N),
\]

\[
\langle R_0(\mu, \phi), w \rangle = \int_M \mu[\nabla \phi \cdot w]dx = b_1(w, \phi, \mu), \ \forall w \in D(A_0), \ \phi \in D(A_N) \cap H^3(M), \ \mu \in L^2(M).
\]

Note that

We recall that (due to the mass conservation) we have

\[
\langle \phi(t) \rangle = \langle \phi(0) \rangle = M_0, \ \forall t > 0.
\]

Thus, up to a shift of the order parameter field, we can always assume that the mean of $\phi$ is zero at the initial time and, therefore it will remain zero for all positive times. Hereafter, we assume that

\[
\langle \phi(t) \rangle = \langle \phi(0) \rangle = 0, \ \forall t > 0.
\]

We set

\[
\mathbb{Y} = H_1 \times D(B_N^{1/2}).
\]

The space $\mathbb{Y}$ is a complete metric space with respect to the norm

\[
||(u, \phi)||_{\mathbb{Y}}^2 = \mathcal{K}^{-1}||v||_{L^2}^2 + \nu_2||\nabla \phi||_{L^2}^2.
\]

We define the Hilbert space $\mathbb{V}$ by

\[
\mathbb{V} = V_1 \times D(B_N),
\]

endowed with the scalar products whose associated norm is

\[
||(v, \phi)||_{\mathbb{V}}^2 = ||v||^2 + |B_N \phi|_{L^2}^2.
\]

Finally we set

\[
H_2 = D(B_N^0), \ V_2 = D(B_N^{1/2}), \ H = H_1 \times H_2, \ V = V_1 \times V_2.
\]

Throughout this article, we shall denote by $c_i, K_i, K$ several positive constants that depend on the data $(v_0, \phi_0)$ and $Q$. We will also denote by $c$ a generic positive constant that depends on the domain $M$. To simplify the notations, we set (without loss of generality) $\nu_1 = \nu_2 = \nu_3 = \alpha = \mathcal{K} = 1$. 

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Using the notations above, we rewrite (1), (4), (8)-(6) as
\[
\begin{aligned}
&\frac{dv}{dt} + A_0v + B_0(v, v) - R_0(A_N\phi, \phi) = Q, \text{ in } V_t^*, \text{ a.e. in } (0, T), \\
&\mu = A_N\phi + f(\phi), \text{ a.e. in } M \times (0, T), \\
&\frac{d\phi}{dt} + A_N\mu + B_1(v, \phi) = 0, \text{ in } H^{-1}, \text{ a.e. in } (0, T),
\end{aligned}
\]
(24)
where hereafter \( Q = P_1Q \).

**Remark 1.** In the weak formulation (24), the term \( \mu \nabla \phi \) is replaced by \( A_N \nabla \phi \).

This is justified since \( \bar{f}'(\phi) \nabla \phi \) is the gradient \( F(\phi) \) and can be incorporated into the pressure gradient, see [14] for details. For the sake of convenience, as in [14] we will replace \( \mu \) in (24) by \( \bar{\mu} = \mu - \langle \mu \rangle \), that is \( \bar{\mu} = A_N\phi + f(\phi) - \langle f(\phi) \rangle \), a.e. in \( M \times (0, +\infty) \). Obviously we have \( \langle \bar{\mu}(t) \rangle = 0 \ \forall t > 0 \).

**Definition 2.1.** Suppose that \((v_0, \phi_0) \in \mathbb{V} \), \( Q \in L^2(0, T; V_1^*) \) and \( T > 0 \). A pair \((v, \phi)\) is called a weak solution to (24) and (9) on \([0, T]\) if it satisfies (24) and (9) in a weak sense on \([0, T]\) and

\[
(v, \phi) \in C([0, T]; \mathbb{V}) \cap L^2([0, T]; \mathbb{V}), \quad \frac{dv}{dt} \in L^2([0, T]; V_1^*), \quad \frac{d\phi}{dt} \in L^2([0, T]; V_2^*). \quad (25)
\]

If \((u_0, \phi_0) \in \mathbb{V} \), a weak solution \((v, \phi)\) is called a strong solution on the time interval \([0, T]\) if in addition to (25), it satisfies

\[
(v, \phi) \in C([0, T]; \mathbb{V}) \cap L^2(0, T; D(A_0) \times (D(B_N) \cap H^3(M))). \quad (26)
\]

The weak formulation of (24) was proposed and studied in [8, 6, 7, 15, 14] (see also [2, 1, 10]), where the existence and uniqueness results for weak and strong solutions were proved.

### 2.3. Some a priori estimates.

In this part, we first derive some a priori estimates on the solution to (24), (9).

Note that if \((v, \phi)\) is a smooth solution to (1), by taking the scalar product in \( H_1 \) of (1) with \( v \), then taking the scalar product in \( L^2(M) \) of (1) with \( \mu \), we derive that

\[
\frac{d}{dt} \left[ \frac{1}{2} \|v\|_{L^2}^2 + \mathcal{F}(\phi) \right] - <v, Q > + \|\nu\|_{L^2}^2 + |\nabla \mu|_{L^2}^2 = 0. \quad (27)
\]

**Proposition 1.** For \((v_0, \phi_0) \in \mathbb{V} \) and \( f \in C^1(\mathbb{R}) \). The system (24), (9) has a unique weak solution \((v, \phi)(t)\). Moreover, the following estimate holds:

\[
\begin{aligned}
&\|\langle v, \phi \rangle(t)\|_{H^1}^2 + \int_0^t (\|v(s)\|^2 + |\mu(s)|_{H^1}^2) \, ds \\
&\leq Q_0(\|\langle v, \phi \rangle(0)\|_{H^1}^2) + c \int_0^t \|Q\|_{V_1}^2 \, ds, \quad \forall t \geq 0,
\end{aligned}
\]
\[
\int_0^t (\|A_N\phi(s)\|_{L^2}^2 + |F(s)|_{H^3}^2) \, ds \leq Q_0(\|\langle v, \phi \rangle(0)\|_{H^1}^2) + c \int_0^t \|Q\|_{V_1}^2 \, ds, \quad \forall t \geq 0,
\]
(28)

\[
\int_0^t \frac{d}{dt} \langle v, \phi \rangle \|_{V_1}^2 \, ds \leq Q_0(\|\langle v, \phi \rangle(0)\|_{H^1}^2) + c \int_0^t \|Q\|_{V_1}^2 \, ds, \quad \forall t \geq 0,
\]

where hereafter, \( Q_0 \) will denote a monotone non-decreasing function independent of the time \( t \) and the initial conditions.
Proposition 2. The existence and uniqueness of weak solutions is proved in [8, 6, 7]. To derive (28), we proceed as in [14], (see Proposition 3.1 an Lemma 3.3 in [14]). To derive (28), we proceed follows. We take the scalar product in $L^2(M)$ of (24) with 2$\phi$. Adding the resulting equation to (27) gives

$$
\frac{dY_0}{dt} - 2(v, Q)_{L^2} + 2\|v\|^2 + 2|\nabla \mu|_{L^2}^2 = 0,
$$

(29)

where

$$
\mathcal{Y}_0(t) = |(v, \phi(t))_{V}|^2_0 + 2(F(\phi(t)), 1)_{L^2} + C_e,
$$

(30)

and $C_e = 2C_F|\mathcal{M}| > 0$, $|\mathcal{M}| > 0$ being the Lebesgue measure of $\mathcal{M}$, and $C_F > 0$ is a constant large enough to ensure that $E(t)$ is nonnegative.

Note that $F$ is bounded from below by a constant. With this choice of $C_e$, we can find $C_F > 0$ such that

$$
|(v(t), \phi(t))|^2_0 \leq \mathcal{Y}_0(t) \leq Q_0((v(t), \phi(t))|^2_0).
$$

(31)

It follows from (29) that

$$
\frac{dY_0}{dt} + \|v(t)\|^2 + 2|\nabla \mu(t)|_{L^2}^2 \leq c|Q|^2_{V},
$$

(32)

which gives

$$
\mathcal{Y}_0(t) + \int_0^t (\|v(s)\|^2 + 2|\nabla \mu(s)|_{L^2}^2)ds \leq \mathcal{Y}_0(0) + c \int_0^t |Q|^2_{V}ds,
$$

(33)

and (28) follows from (33) and (31).

Using (24), we can check (using well known regularity result) that

$$
\int_0^t (|A_N \phi(s)|_{L^2}^2 + |\phi(s)|_{L^2}^2)ds \leq Q_0((v, \phi)(0)|^2_0) + c \int_0^t |Q|^2_{V_1}ds,
$$

(34)

and (28) follows. Finally, it is clear that (28) follows from (28) and (14) - (16). \square

Proposition 2. Let $Q \in L^2(0, T; H_1)$, and $(v_0, \phi_0) \in V$, the system (24), (9) has a unique strong solution $(v, \phi)(t)$. Moreover the following estimate holds:

$$
\|(v, \phi)(t)\|^2_{V} + \int_0^T (|A_0 v(s)|_{L^2}^2 + |B_N^2 \phi(s)|_{L^2}^2 + |B_N \phi(s)|_{L^2}^2)ds
\leq c_1\|(v, \phi)(0)\|^2_{V} + c_1 \int_0^T |Q(s)|_{L^2}^2ds.
$$

(35)

$$
\int_0^T \left| \frac{d}{dt}(v, \phi) \right|^2_{L^2} \leq c_1\|(v, \phi)(0)\|^2_0 + c_1 \int_0^T |Q(s)|_{L^2}^2ds,
$$

(36)

$$
|B_N^3/2 \phi(t)|_{L^2}^2 \leq c_1\|(v, \phi)(0)\|^2_{V} + c_1 \int_0^T |Q(s)|_{L^2}^2ds.
$$

(37)

If $u_0^1 = (v_0^1, \phi_0^1), u_0^2 = (v_0^2, \phi_0^2) \in V$ and $Q_1, Q_2 \in L^2(0, T; H_1)$. If $u_i = (v_i, \phi_i), i = 1, 2$ is the solution to (24) corresponding to the initial condition $u_0^i$ and the forcing $Q_i$, then we have

$$
\|u_1(t) - u_2(t)\|^2_{V} + \int_0^T |A(u_1 - u_2)|_{L^2}^2ds \leq c_1 \left( \|u_0^1 - u_0^2\|^2_{V} + \int_0^T |Q_1 - Q_2|^2_{L^2}ds \right),
$$

(38)

where $c_1 = c_1(u_0^1, u_0^2, Q_1, Q_2)$. 

proof. The existence and uniqueness of strong solution is proved as in [8, 6, 7]. We only need to prove (35). We introduce the function $\tilde{\mu}(t) = \mu(t) - \langle \mu(t) \rangle$, and we observe that $\langle \tilde{\mu}(t) \rangle = 0$.

Taking the inner product in $H_1$ of (24) with $2A_0v$, the inner product in $L^2(M)$ of (24) with (24) and (24) with $2B_0^2\mu + 2\xi B_N^2\phi$ ($\xi > 0$ small enough) and $2B^2_N\phi$ respectively. Adding the resulting equalities gives

$$
\frac{d}{dt} \mathcal{Y}_1 + 2|A_0v|_{L^2} + 2B_N^2|\phi|_{L^2} + 2|B_N\tilde{\mu}|_{L^2} = 2(B_0(A_N\phi, \phi), A_0v)_{L^2}
$$

$$
-2(A_N(f(\phi) - (f(\phi)), B_N\tilde{\mu})_{L^2} - 2(B_1(v, \phi), B_N^2\phi)_{L^2}
$$

$$
-2(B_0(v, v), A_0v)_{L^2} + (Q, A_0v)_{L^2} + 2\xi(B_N\tilde{\mu}, B_N^2\phi)_{L^2}
$$

$$
+2\xi(B_N(f(\phi) - (f(\phi)), B_N^2\phi)_{L^2},
$$

where

$$\mathcal{Y}_1(t) = ||v(t)||^2 + |B_N\phi(t)|_{L^2}^2.$$

Note that

$$|2(R_0(A_N\phi, \phi), A_0v)_{L^2}| \leq 2B_N^2|\phi|_{L^2}||\nabla\phi||_{L^\infty} + |A_0v|_{L^2}
$$

$$\leq \frac{1}{8}|A_0v|_{L^2} + \frac{\xi}{8}|B_N\phi|_{L^2} + c|B_N\phi|_{L^2}||\nabla\phi||_{L^2},$$

$$2(B_0(v, v), A_0v)_{L^2} \leq \frac{1}{8}|A_0v|_{L^2} + c||v||_{L^2}^4$$

$$2(B_1(v, \phi), B_N^2\phi)_{L^2} \leq \frac{\xi}{8}|B_N^2\phi|_{L^2} + c||v||_{L^2}^2||v||_{L^2} + |\nabla\phi||_{L^2}^2|B_N\phi|_{L^2},$$

$$2\xi(B_N\tilde{\mu}, B_N^2\phi)_{L^2} \leq \frac{\xi}{4}|B_N\phi|_{L^2} + 4|B_N\tilde{\mu}|_{L^2}^2,$$

$$2(A_N(f(\phi) - (f(\phi)), B_N\tilde{\mu})_{L^2} \leq \xi|B_N\tilde{\mu}|_{L^2}^2 + c|B_N(f(\phi) - (f(\phi)))|_{L^2}^2$$

$$\leq \xi|B_N\tilde{\mu}|_{L^2}^2 + Q_0(|\phi|_{H^1})(1 + |B_N\phi|_{L^2}^2)|B_N\phi|_{L^2},$$

$$|2\xi(B_N(f(\phi) - (f(\phi)), B_N^2\phi)_{L^2} | \leq \frac{\xi}{4}|B_N\phi|_{L^2}^2 + Q_0(|\phi|_{H^1})(1 + |B_N\phi|_{L^2}^2)|B_N\phi|_{L^2}^2,$$

$$\frac{d}{dt} \mathcal{Y}_1 + |A_0v|_{L^2} + \frac{\xi}{4}|B_N\phi|_{L^2}^2 + (2 - 5\xi)|B_N\phi|_{L^2}^2 \leq \mathcal{G}(t) \mathcal{Y}_1(t) + c|Q|_{L^2}^2,$$

where

$$\mathcal{G}(t) = c(|\nabla\phi||_{L^2}^2|B_N|_{L^2}^2 + |v||_{L^2}^2 + |\nabla\phi||_{L^2}^2) + Q_0(|\phi|_{H^1})(1 + |B_N\phi|_{L^2}^2).$$

Note that from (28), we have

$$\int_0^t \mathcal{Y}_1(s) ds \leq Q_1(T, |v, \phi(0)|_{H^2}^2, \int_0^t |Q|_{L^2}^2 ds),$$

$$\int_0^t |\mathcal{G}(s)| ds \leq c\int_0^t (|\nabla\phi||_{L^2}^2|B_N\phi|_{L^2}^2 + |v||_{L^2}^2 + |\nabla\phi||_{L^2}^2) ds$$

$$+ \int_0^t Q_0(|\phi|_{H^1})(1 + |B_N\phi|_{L^2}) ds$$

$$\leq Q_1(T, |v, \phi(0)|_{H^2}^2, \int_0^t |Q|_{L^2}^2 ds).$$

Choosing $\xi = \frac{1}{5}$, it follows from (46)-(48) that

$$\mathcal{Y}_1(t) + \int_0^t (|A_0v|_{L^2}^2 + |B_N\phi|_{L^2}^2 + |B_N\tilde{\mu}|_{L^2}^2) ds$$

$$\leq c_1||v, \phi(0)||_{H^2}^2 + c_1 \int_0^t |Q|_{L^2}^2 ds, \forall t \in [0, T],$$
and (35) follows. We can easily check that (36) follows from (35) and (14)-(16).

To derive (37), we set \( w = \frac{dx}{dt}, \psi = \frac{d\phi}{dt} \). Then, differentiating (24) with respect to time gives,
\[
\frac{d\psi}{dt} + A_N^2 \psi + B_1(w, \phi) + B_1(v, \psi) + A_N f'(\phi) \psi = 0. 
\]
(50)

We multiply (50) by \( A_N \psi \) to derive that
\[
\frac{d}{dt} ||\psi||^2 + |B_N^{3/2} \psi|^2_{L^2} \leq |b_1(w, \phi, A_N \psi)| + |b_1(v, \psi, A_N \psi)| + |(A_N f'(\phi) \psi, A_N \psi)|_{L^2}. 
\]
(51)

Note that
\[
|b_1(w, \phi, A_N \psi)| \leq \frac{1}{4} |B_N^{3/2} \psi|^2_{L^2} + c|w|_{L^2}^2 ||A_N \phi||_{L^2}, 
\]
(52)
\[
|b_1(v, \psi, A_N \psi)| \leq \frac{1}{4} |B_N^{3/2} \psi|^2_{L^2} + c|v|_{L^2}^2 |A_N v||\psi||^2, 
\]
(53)
\[
|(A_N f'(\phi) \psi, A_N \psi)|_{L^2} \leq \frac{1}{4} |B_N^{3/2} \psi|^2_{L^2} + Q_0 ||\phi||^2. 
\]
(54)

It follows from (51)-(54) and (35)-(36) that
\[
||\psi(t)||^2 + \int_0^t |B_N^{3/2} \psi(s)|^2_{L^2} ds \leq c_1 ||(v, \phi)(0)||^2_{\psi} + c_1 \int_0^t |Q(s)|_{L^2}^2 ds. 
\]
(55)

Finally, (37) follows from (55) and (24). Note that
\[
|B_N^{3/2} \phi|^2_{L^2} \leq c \frac{d\phi}{dt} \frac{||\phi||^2_{L^2}}{||\phi||_{L^2}^2} + c |B_1(v, \phi)|_{L^2}^2 + c|A_N f'(\phi)||^2_{L^2} 
\]
(56)
\[
\leq c \frac{d\phi}{dt} \frac{||\phi||^2_{L^2}}{||\phi||_{L^2}^2} + c|v|_{L^2}||\phi||^2 ||A_N \phi||_{L^2} + Q_0 ||\phi||. 
\]

The proof of (38) is very similar to that of Proposition 3 given below.

2.4. Linearized system. Let \((v, \phi)\) be a strong solution to (24) given by Proposition 2. We consider the linearized system
\[
\begin{align*}
\frac{d}{dt} + A_0 w + B_0(w, v) + B_0(w, v) = R_0(A_N \phi, \psi) + R_0(A_N \psi, \phi) + g_1, \\
\mu = A_N \psi + f'(\phi) \psi, \\
\frac{d\dot{\psi}}{dt} + A_N \mu + B_1(v, \psi) + B_1(w, \phi) = g_2, \\
(w, \psi)(0) = (w_0, \psi_0),
\end{align*}
\]
(57)

or equivalently
\[
\begin{align*}
\frac{dw}{dt} + A_0 w + B_0(v, w) + B_0(w, v) = R_0(A_N \phi, \psi) + R_0(A_N \psi, \phi) + g_1, \\
\bar{\mu} = A_N \psi + f'(\phi) \psi - (f'(\phi) \psi), \\
\frac{d\dot{\psi}}{dt} + A_N \bar{\mu} + B_1(v, \psi) + B_1(w, \phi) = g_2, \\
(w, \psi)(0) = (w_0, \psi_0).
\end{align*}
\]
(58)

**Proposition 3.** For \((w_0, \psi_0) \in Y\) and \(g = (g_1, g_2) \in L^2(0, T; H)\), the system (58) has a unique solution \((w, \psi) \in L^2(0, T; V) \cap C(0, T; \mathbb{Y})\). Moreover, the following estimate holds true:
\[
||w(t)||^2_{\mathbb{Y}} + \int_0^T (||w(t)||^2_{\mathbb{Y}} + |\nabla \bar{\mu}(s)||^2_{L^2}) ds 
\leq c_1 ||(w_0, \psi_0)||^2_{\mathbb{Y}} + c_1 \int_0^T |g(s)|^2_{L^2} ds.
\]
(59)
Furthermore if \((w_0, \psi_0) \in \mathcal{V}\), then the solution \((w, \psi)\) to (58) satisfies the regularity
\[(w, \phi) \in L^2(0, T; D(A_0) \times D(B_N)) \cap C(0, T; \mathcal{V})\]
and the following estimates hold:
\[
\|w, \psi\|^2 + \int_0^T \|A_0 w(s)\|^2_{L^2} + |B_N^2 \psi(s)|^2_{L^2} + |B_N \bar{\mu}(s)|^2_{L^2} ds
\leq c_1 \|w_0, \psi_0\|^2 + c_1 \int_0^T |g(s)|_{L^2} ds,
\]
\[
\int_0^T \frac{d}{dt} \|w, \psi\|^2_{L^2} dt \leq c_1 \|w_0, \psi_0\|^2 + c_1 \int_0^T |g(s)|_{L^2} ds.
\]

**Proof.** Since \((v, \phi) \in L^\infty(0, T; \mathcal{V}) \cap L^2(0, T; D(A_0) \times D(B_N))\), using the known regularity results for parabolic equations such as the linearized Navier-Stokes system (see e.g., [28]), we can rigorously prove the existence and uniqueness of solution to (58) or (57). To derive (59), we multiplying (58) by \(w\). Then we take the duality of (58) and (58) with \(A_N \bar{\mu} - \xi A_N \psi\) and \(A_N \psi\) respectively, where \(\xi > 0\) is small enough and will be selected later. We derive that
\[
\frac{d}{dt} \|w\|^2_{L^2} + \|\psi\|^2_{L^2} + \|w\|^2 + \xi |A_N \psi|^2_{L^2} + |\nabla \bar{\mu}|^2_{L^2} = -b_0(w, v, w) + (R_0(A_N \phi, \psi), w)_{L^2} + (R_0(A_N \psi, \phi), w)_{L^2} - b_1(w, \phi, A_N \psi) - b_1(v, \psi, A_N \bar{\mu}) + \xi (f^\prime(\phi) \psi, A_N \psi)_{L^2} + (g_1, w)_{L^2} + (g_2, A_N \psi)_{L^2}. \tag{61}
\]

Note that
\[
|b_0(w, v, w)| \leq \frac{1}{8} \|w\|^2 + c \|\psi\|^2 \|w\|^2_{L^2}, \tag{62}
\]
\[
|(R_0(A_N \phi, \psi), w)_{L^2}| = |b_1(w, \phi, A_N \psi)| \leq \frac{1}{8} \|\psi\|^2 + \xi |A_N \psi|^2_{L^2} + \phi \|w\|^2_{L^2} \tag{63}
\]
\[
\|R_0(A_N \phi, \psi)\|_{L^2} = |b_1(w, \psi, A_N \phi)| \leq \frac{1}{8} \|\psi\|^2 + \xi |A_N \psi|^2_{L^2} + c \|w\|^2_{L^2} \|\psi\|^2 \|\phi\|^2_{H^2}, \tag{64}
\]
\[
\xi |f^\prime(\phi) \psi, A_N \psi\|_{L^2} \leq \frac{\xi}{8} |A_N \psi|^2_{L^2} + Q_0(\|\phi\|_{H^1}) \|\psi\|^2, \tag{65}
\]
\[
\xi |f^\prime(\phi) \psi, A_N \bar{\mu}\|_{L^2} \leq \xi |A_N \bar{\mu}|^2_{L^2} + \frac{1}{8} \|\nabla \bar{\mu}\|^2_{L^2} + Q_0(\|\phi\|_{H^1}) \|\psi\|^2, \tag{66}
\]
\[
|b_1(v, \psi, A_N \phi)| \leq \frac{\xi}{8} |A_N \psi|^2_{L^2} + c \|v\|^2_{L^2} \|\psi\|^2 \|\phi\|^2, \tag{67}
\]
\[
\xi |\bar{\mu}, A_N \psi\|_{L^2} \leq \frac{\xi}{8} |A_N \psi|^2_{L^2} + c \xi |\nabla \bar{\mu}|^2_{L^2}. \tag{68}
\]

Let
\[
\mathcal{Y}_2(t) = |w(t)|^2_{L^2} + |\nabla \psi|^2_{L^2}.
\]

It follows from (61)-(68) that
\[
\frac{d\mathcal{Y}_2}{dt} + c \|w\|^2 + (1 - \xi \epsilon) |\nabla \bar{\mu}|^2_{L^2} + c \xi |A_N \psi|^2_{L^2} \leq \mathcal{Y}_2(t) \mathcal{G}(t) + c \|g_1\|^2_{L^2} + c \|g_1\|^2_{L^2}, \tag{69}
\]
where
\[
\mathcal{G}(t) = c \|\psi\|^2 + \|\phi\|^2 |A_N \phi|^2_{L^2} + \|v\|^2_{L^2} \|\psi\|^2 + Q_0(\|\phi\|_{H^1}).
\]

Note that \(\mathcal{G}(t) \in L^1(0, T)\).
To prove (60)$_1$, we take the inner product of (58)$_1$ with $A_0w$ in $L^2(\mathcal{M})$. Then we take the inner product in $L^2(\mathcal{M})$ of (58)$_2$ and (58)$_3$ with $B_N^2\bar{\mu} - \xi B_N^2\psi$ and $B_N^2\psi$ respectively, where $\xi > 0$ will be selected later. We derive that

$$\frac{d\gamma_3}{dt} + |A_0w|_{L^2}^2 + \xi |B_N^2\psi|_{L^2}^2 + |B_N\bar{\mu}|_{L^2}^2 = \lambda_3(t), \quad (70)$$

where

$$\lambda_3(t) = -b_0(w, v, A_0w) - b_0(v, w, A_0w) + (R_0(B_N\phi, \psi) - A_0w)_{L^2} + (R_0(B_N\psi, \phi) - b_1(v, \psi, B_N^2\psi) + (f(\phi, \psi, B_N^2\bar{\mu})_{L^2} + \xi (f(\phi, \psi, B_N^2\psi)_{L^2} + (g_1, A_0w)_{L^2} + (g_2, B_N^2\psi)_{L^2},

\gamma_3 = \|w\|^2 + |B_N\psi|_{L^2}^2. \quad (71)$$

We note that

$$|b_0(w, v, A_0w)| \leq \frac{1}{8}|A_0w|_{L^2}^2 + \frac{1}{2}|v|_{L^2}^2 \|w\|^2, \quad (72)$$

$$|b_0(v, w, A_0w)| \leq \frac{1}{8}|A_0w|_{L^2}^2 + \frac{1}{2}|v|_{L^2}^2 |A_0w|_{L^2}^2 \|w\|^2, \quad (73)$$

$$|(R_0(A_N\phi, \psi), A_0w)_{L^2}| + |(R_0(B_N\psi, \phi), A_0w)_{L^2}| \leq \frac{1}{8}|A_0w|_{L^2}^2 + \xi |B_N^2\psi|_{L^2}^2 + c\|\psi\|_{L^2}^2 |B_N\phi|_{H^2}^2 + c |B_N\phi|_{H^2}^2 \|\psi\|_{H^3}^2 + c |B_N\phi|_{H^2}^2 |\phi|_{H^3}^2, \quad (74)$$

$$|b_1(w, \phi, B_N^2\psi)| \leq |(B_1(w, \phi), B_N^2\psi)_{L^2}| \leq \frac{1}{8}|(A_0w|_{L^2}^2 + \xi |B_N^2\psi|_{L^2}^2) + c\|w\|_{L^2}^2 |\phi|_{H^2}^2 + c \|w\|_{H^2}^2 \|\phi\|_{H^3}^2 + c |\phi|_{H^2}^2 \|\phi\|_{H^3}^2, \quad (75)$$

$$|(f(\phi, \psi, B_N^2\phi)| \leq c|A_N f(\phi, \psi, B_N\bar{\mu})_{L^2}| \leq \frac{\xi}{8}|B_N\bar{\mu}|_{L^2}^2 + Q_0(|\phi|_{H^3}^2) \|B_N\psi|_{L^2}^2, \quad (76)$$

$$\xi |(B_N\psi, B_N\bar{\mu})_{L^2}| \leq \frac{\xi}{8}|B_N\bar{\mu}|_{L^2}^2 + c |B_N\psi|_{L^2}^2. \quad (77)$$

It follows from (70)-(77) that (with $0 < \xi < 1$ small enough)

$$\frac{d\gamma_3}{dt} + |A_0w|_{L^2}^2 + \frac{\xi}{2}|B_N^2\psi|_{L^2}^2 + (1 - \xi)|B_N\bar{\mu}|_{L^2}^2 \leq \mathcal{G}(t)\gamma_3(t) + c |\phi|_{L^2}^2, \quad (78)$$

where

$$\mathcal{G}(t) = c|\phi|_{L^2}^2 + \|v\|_{V^*}^2 |A_0v|_{L^2}^2 + \|v\|_{V_i^*}^2 |\phi|_{H^2}^2 + |\phi|_{H^1}^2 |\phi|_{H^2} + |\phi|_{H^1}^2 |\phi|_{H^3} + |\phi|_{H^2}^2 + 1 + Q_0(|\phi|_{H^3}),$$

and (60)$_1$ follows from the Gronwall lemma. Note that $\mathcal{G}(t) \in L^1(0, T)$.

It is easy to check that (60)$_2$ follows from (60)$_1$ and the properties of the operators $B_0, B_1$ and $R_0$ given in (14)-(16).

3. **Robust control framework.** Let us first introduce some notations. We define the linear operator $A : V \to V^*$ by

$$\langle A_1, u_2 \rangle = \langle A_0v_1, v_2 \rangle + \langle A_N^2\phi_1, \phi_2 \rangle, \quad (79)$$

for $u_1 = (v_1, \phi_1), u_2 = (v_2, \phi_2) \in V$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $V$ and $V^*$ or between $V_i$ and $V_i^*$, $i = 1, 2$.

We also define the following trilinear functional and associated operator

$$\langle B(u_1, u_2, u_3) = b(u_1, u_2, u_3) = b_0(v_1, v_2, v_3) + b_1(v_1, \phi_2, \phi_3), \quad (80)$$

for $u_1 = (v_1, \phi_1), u_2 = (v_2, \phi_2), u_3 = (v_3, \phi_3) \in V$. 

\[\square\]
We introduce the bilinear operator \( R : V \times V \rightarrow V^* \) by
\[
R(u_1, u_2) = (R_0(A_N \phi_1, \phi_2), 0),
\]
for \( u_1 = (v_1, \phi_1), u_2 = (v_2, \phi_2) \in V. \)

Finally we set
\[
E(u_1) = (0, A_N f(\phi_1)), \quad (E'(u_1))u_2 = (0, A_N (f'(\phi_1)\phi_2)),
\]
for \( u_1 = (v_1, \phi_1), u_2 = (v_2, \phi_2) \in V. \)

To simplify the notations, we will also set
\[
B(u_1) = B(u_1, u_1), \quad R(u_1) = R(u_1, u_1).
\]

Then we can check that
\[
B'(u_1)u_2 = B(u_1, u_2) + B(u_2, u_1),
R'(u_1)u_2 = R(u_1, u_2) + R(u_2, u_1), \quad \forall u_1, u_2 \in V.
\]

With the above notations, if we set \( u = (v, \phi), u_0 = (v_0, \phi_0) \) and \( Q \equiv (Q, 0), \)
then we can rewrite (24) as:
\[
\frac{du}{dt} + Au + B(u, u) + Eu = R(u, u) + Q, \quad u(0) = u_0.
\]

Let \( u = (v, \phi) \) be the strong solution to (84) given by Proposition 2. If we set
\( \omega = (w, \psi), \omega_0 = (w_0, \psi_0) \) and \( g = (g_1, g_2) \), then the linearized system (58) can be rewritten as:
\[
\frac{d\omega}{dt} + A\omega + B'(u)\omega + E'(u)\omega = R'(u), \quad \omega + g, \quad \omega(0) = \omega_0.
\]

We will also consider the following adjoint system for which a result on the existence and uniqueness of solutions is given in Proposition 14 in the Appendix.
\[
- \frac{d\omega}{dt} + A\omega + (B'(u))^*\omega + (E'(u))^*\omega = (R'(u))^*\omega + g, \quad \omega(T) = \omega_0.
\]

For the control framework, in the spirit of the non-cooperative game discussed in [4], the interior forcing \( Q \) is decomposed into a disturbance \( f_1 \in L^2(0, T; H_1) \) and a control \( f_2 \in L^2(0, T; H_1) \). Thus we write
\[
Q = B_1 f_1 + B_2 f_2,
\]
where \( B_1 \) and \( B_2 \) are given bounded operators on \( L^2(M) \). The cost functional \( J \)
used in this section is defined by
\[
J(f_1, f_2) = \frac{1}{2} \int_0^T |C_1 u|^2 dy dt + \frac{1}{2} |C_2 u(T)|^2 y + \frac{1}{2} \int_0^T \left[ l^2 |f_2|^2 |L_2| - \gamma^2 |f_1|^2 |L_2|^2 \right] dt,
\]
where the flow \( u \) is related to the disturbance \( f_1 \) and the control \( f_2 \) through the CH-NS equations and the control parameters \( l \) and \( \gamma \) are given. The linear operators \( C_1 \) and \( C_2 \) are bounded or unbounded operators on \( Y \) satisfying
\[
|C_i u|^2 \leq \alpha_0 |u|^2 + \beta |u|^2 \quad \text{for} \quad i = 1, 2, \quad \forall u \in V,
\]
with \( \alpha_0 \geq 0, \beta \geq 0 \) and \( \alpha_0 + \beta > 0. \) Some cases of particular interest are:

- \( C_1 u = (d_1 v, 0), C_2 u = 0, \) for \( u = (v, \phi), \Rightarrow \) regulation of turbulent kinetic energy;

- \( C_1 u = (d_1 \nabla \times v, 0), C_2 u = 0, \) for \( u = (v, \phi), \Rightarrow \) regulation of the square of the vorticity.

The goal is to find a disturbance \( \tilde{f}_1 \) and a control \( \tilde{f}_2 \) such that \( (\tilde{f}_1, \tilde{f}_2) \) is a saddle point of the functional \( J. \)
3.1. The linear problem. In this subsection, the flow $u$ is related to the disturbance $f_1$ and the control $f_2$ through the linearized CH-NS
\[
\frac{du}{dt} + Au + E'(U)u + B'(U)u = R'(U)u + B_1 f_1 + B_2 f_2, \tag{90}
\]
which models small deviations of the flow perturbation $u = (v, \phi)$ from the desired target flow $U = (v, \Phi)$. The regularity required is given by
\[
f_1, f_2 \in L^2(0, T; L^2(M)), \quad u_0 \in V, \quad U \in L^\infty(0, T; V) \cap L^2(0, T; D(A)). \tag{91}
\]

**Proposition 4.** For $u_0 \in V$, the system (90) has a unique solution $u \in C((0, T; V) \cap L^2(0, T; D(A))$.

The mappings $(f_1, f_2) \mapsto u(f_1, f_2)$ from $(L^2(0, T; H_1))^2$ into $V$ and $(f_1, f_2) \mapsto u(f_1, f_2)|_T$ from $(L^2(0, T; H_1))^2$ into $V$ are affine and continuous.

For $u_0 \in V$ and $(f_1, f_2) \in (L^2(0, T; H_1))^2$, the mappings $(f_1, f_2) \mapsto u(f_1, f_2)$ and $(f_1, f_2) \mapsto u(f_1, f_2)|_T$ have Fréchet derivatives $u'(f_1^0, f_2^0)$, $u'(f_1^0, f_2^0)|_T$ in every direction $(f_1^1, f_2^1) \in (L^2(0, T; H_1))^2$. Finally the Fréchet derivative $u'(f_1^0, f_2^0)$ is the (unique) solution to the linear system
\[
\frac{du'}{dt} + Au' + B'(U)u' + E'(U)u' = R'(U)u' + B_1 f_1^0 + B_2 f_2^0, \tag{92}
\]
and it follows that $u' \in L^\infty(0, T; V) \cap L^2(0, T; D(A)).$

**Proof.** The proof of the existence and uniqueness of solution is given in Proposition 3. The rest of the proposition is proved as Proposition 9 given below for the full CH-NS model.

In this section, we consider the following robust control problem:

To find $(\bar{f}_1, \bar{f}_2) \in (L^2(0, T; H_1))^2$ such that
\[
\mathcal{J}(\bar{f}_1, \bar{f}_2) = \min_{f_1 \in X} \sup_{f_2 \in Y} \mathcal{J}(f_1, f_2) = \max_{f_2 \in Y} \inf_{f_1 \in X} \mathcal{J}(f_1, f_2), \tag{93}
\]
subject to the linearized CH-NS (90), where $X = L^2(0, T; H_1)$. Here the cost function is given by (88) where the flow $u$ is related to the disturbance $f_1$ and the control $f_2$ through the system (90).

The proof of the existence of a solution $(\bar{f}_1, \bar{f}_2)$ to the robust control problem for the linear case is based on the following result.

**Proposition 5.** Let $\mathcal{J}$ be a functional defined on $X \times Y$, where $X$ and $Y$ are non-empty, closed, unbounded, convex sets. If $\mathcal{J}$ satisfies:

1. $\forall f_1 \in X$, $f_2 \mapsto \mathcal{J}(f_1, f_2)$ is convex lower semi-continuous,
2. $\forall f_2 \in Y$, $f_1 \mapsto \mathcal{J}(f_1, f_2)$ is concave upper semi-continuous,
3. $\exists f_1^0 \in X$ such that $\lim_{\|f_2\|_Y \to +\infty} \mathcal{J}(f_1^0, f_2) = +\infty$,
4. $\exists f_2^0 \in Y$ such that $\lim_{\|f_1\|_X \to +\infty} \mathcal{J}(f_1, f_2^0) = -\infty$,

then the functional $\mathcal{J}$ has at least one saddle point $(\bar{f}_1, \bar{f}_2)$ and
\[
J(\bar{f}_1, \bar{f}_2) = \min_{f_2 \in Y} \sup_{f_1 \in X} \mathcal{J}(f_1, f_2) = \max_{f_2 \in Y} \inf_{f_1 \in X} \mathcal{J}(f_1, f_2).
\]

**Proof.** The proof is given in [11].
Proposition 6. There exists \( \gamma_0 = \gamma_0(\mathcal{M}, T, u_0, U) > 0 \) such that for \( \gamma > \gamma_0 \) and \( l > 0 \), the functional \( \mathcal{J} \) satisfies:

1) \( \forall f_1 \in X, \ f_2 \mapsto \mathcal{J}(f_1, f_2) \) is convex lower semi-continuous.
2) \( \forall f_2 \in X, \ f_1 \mapsto \mathcal{J}(f_1, f_2) \) is concave upper semi-continuous.
3) \( \mathcal{J}(0, f_2) \to +\infty \) as \( |f_2|_{L^2} \to +\infty \).
4) \( \mathcal{J}(f_1, 0) \to -\infty \) as \( |f_1|_{L^2} \to +\infty \).

Proof. We proceed by steps, checking these four conditions successively.

Condition 1: By Proposition 4, the mapping \( f_2 \mapsto \mathcal{J}(f_1, f_2) \) is lower semi-continuous. As the mapping \( f_2 \mapsto u(f_1, f_2) \) is affine, the convexity of the mapping \( f_2 \mapsto \mathcal{J}(f_1, f_2) \) follows.

Condition 2: By Proposition 4, the mapping \( f_1 \mapsto \mathcal{J}(f_1, f_2) \) is upper semi-continuous. To prove the concavity, it is enough to show that for every fixed \( f_1, f_2, h(r) = \mathcal{J}(f_1 + rf_2, f_2) \) is concave with respect to \( r \), i.e. \( h''(r) < 0 \).

Let \( u'(f_2, 0) = \left( \frac{Du}{Df_2} \right) \cdot f_1 \). Then

\[
h'(r) = \int_0^T \langle C_1 u, C_1 u' \rangle dt + \langle C_2 u(T), C_2 u'(T) \rangle - \gamma^2 \int_0^T \langle f_1 + rf_2, f_1'^2 \rangle dt, \tag{94}
\]

and

\[
h''(r) = \int_0^T |C_1 u'|^2 dt + |C_2 u'(T)|^2 - \gamma^2 \int_0^T |f_1|_{L^2}^2 dt. \tag{95}
\]

Note that \( u' \) satisfies

\[
\frac{d u'}{d t} + A u' + B'(U) u' + E'(U) u' = R'(U) u' + B_1 f_2, \quad u' = 0 \text{ at } t = 0, \ u' \in \mathcal{V}, \tag{96}
\]

and following the a priori estimates in the linear case we obtain

\[
|u'(t)|_{L^2}^2 \leq c_1 \int_0^t |f_1|^2_{L^2} ds, \quad \frac{1}{t} \int_0^t \| u' \|^2_{L^2} ds \leq c_1 \int_0^t |f_1|^2_{L^2} ds. \tag{97}
\]

Therefore

\[
\int_0^T |C_1 u'|^2 ds \leq \alpha \int_0^T |u'|_{L^2}^2 ds + \beta \int_0^T \| u' \|^2_{L^2} ds \leq c_1 \int_0^T |f_1|^2_{L^2} ds. \tag{98}
\]

Similarly we can check that

\[
|C_2 u'(T)|^2 \leq c_1 \int_0^T |f_1|^2_{L^2} ds. \tag{99}
\]

Using (95), (98)-(99), we conclude that under the assumption \( \gamma > \gamma_0 = \gamma_0(\mathcal{M}, T, u_0, U) \), we have \( h''(r) < 0 \) for all \( r \in \mathbb{R} \). Thus the concavity of the mapping \( f_1 \mapsto \mathcal{J}(f_1, f_2) \) follows.

Condition 3: Taking \( f_1 = 0 \), we have

\[
\mathcal{J}(0, f_2) \geq \frac{t^2}{2} \int_0^T |f_2|^2_{L^2} dt. \tag{100}
\]

Therefore \( \mathcal{J}(0, f_2) \to +\infty \) as \( |f_2|_{L^2} \to +\infty \).

Condition 4: taking \( f_2 = 0 \), we have

\[
\mathcal{J}(f_1, 0) = \frac{1}{2} \int_0^T |C_1 u|^2 dt + \frac{1}{2} |C_2 u(T)|^2 - \frac{\gamma^2}{2} \int_0^T |f_1|^2_{L^2} dt \\
\leq c_0 \int_0^T |f_1|^2_{L^2} dt + c_1 - \frac{\gamma^2}{2} \int_0^T |f_1|^2_{L^2} dt, \tag{101}
\]

\]
where \( c_0 = c_0(M, T, u_0, U) \) and \( c_1 = c_1(M, T, u_0, U) \) depend only on the data. We conclude that for \( \gamma \) large enough, \( \mathcal{J}(f_1, 0) \to -\infty \) as \( |f_1|_{L^2} \to +\infty \). 

**Theorem 3.1** (Existence of solution of the robust control problem in the linear case). For \( \gamma > \gamma_0 = \gamma_0(M, T, u_0, U) \) and \( l > 0 \), the robust control problem (93) has a unique solution \((f_1, f_2)\) that satisfies

\[
\mathcal{J}(f_1, f_2) \leq \mathcal{J}(\tilde{f}_1, f_2) \leq \mathcal{J}(f_1, \tilde{f}_2), \quad \forall (f_1, f_2) \in (L^2(0, T; L^2(\mathcal{M})))^2.
\]

**Proof.** The proof follows directly from Propositions 5 and 6. 

3.2. Characterization of the saddle point. In this subsection, we derive in a classical manner the adjoint equation associated with the robust control problem (93). We consider the following systems.

\[
\begin{align*}
\frac{du^\flat}{dt} + Au^\flat + B'(U)u^\flat + E'(U)u^\flat &= R'(U)u^\flat + B_1f_1^\flat + B_2f_2^\flat, \\
u^\flat &= 0 \text{ at } t = 0, \quad u^\flat \in \mathcal{V},
\end{align*}
\]

\(\tilde{u}(T) = C_2^*C_2u(T), \quad u^\flat \in \mathcal{V},\)

where \( u \) is given by (90) and hereafter, if \( M \) is a bounded operator defined from \( \mathcal{V} \) into \( \mathcal{V} \), \( M^* \) will denote the adjoint operator defined by

\[
\langle M^*u, v \rangle = \langle \tilde{u}, Mu \rangle, \quad \forall u, \tilde{u} \in \mathcal{V}.
\]

**Proposition 7.** Let \( u^\flat, \tilde{u} \) and \( u \) be the solutions of (103), (104) and (90) respectively. Then

\[
\int_0^T \langle C_1^*u, u^\flat \rangle dt + \langle C_2^*C_2u(T), u^\flat(T) \rangle = \int_0^T \langle B_1^*\tilde{u}, f_1^\flat \rangle dt + \int_0^T \langle B_2^*\tilde{u}, f_2^\flat \rangle dt.
\]

**Proof.**

\[
\int_0^T \langle C_1^*u, u^\flat \rangle dt + \langle C_2^*C_2u(T), u^\flat(T) \rangle = \int_0^T \left( \frac{du^\flat}{dt} + \nu Au^\flat + B'(U)u^\flat + E'(U)u^\flat - R'(U)u^\flat, u^\flat \right) dt \\
+ \langle C_2^*C_2u(T), u^\flat(T) \rangle = -\langle \tilde{u}(T), u^\flat(T) \rangle + \langle \tilde{u}(0), u^\flat(0) \rangle \\
+ \int_0^T \left( \frac{du^\flat}{dt} + \nu Au^\flat + B'(U)u^\flat + E'(U)u^\flat - R'(U)u^\flat, \tilde{u}^\flat \right) dt \\
+ \langle C_2^*C_2u(T), u^\flat(T) \rangle = \int_0^T \langle B_1^*f_1^\flat + B_2^*f_2^\flat, \tilde{u}^\flat \rangle dt = \int_0^T \langle B_1^*\tilde{u}, f_1^\flat \rangle dt + \int_0^T \langle B_2^*\tilde{u}, f_2^\flat \rangle dt.
\]

**Proposition 8.** Let \((f_1, f_2)\) be the solution to (93) given by Theorem 3.1. Then

\[
B_1^*\tilde{u} - \gamma^2 f_1 = 0, \quad B_2^*\tilde{u} + l^2 f_2 = 0,
\]

where \( \tilde{u} \) is given by (104).
Proof. We can easily check that
\[
\frac{D\mathcal{J}}{Df_1}(f_1, f_2) \cdot f_1^* = \int_0^T \left< c_1 u, \frac{Du}{Df_1} \cdot f_1^* \right> dt + \left< c_2 u, \frac{Du(T)}{Df_1} \cdot f_1^* \right> - \gamma^2 \int_0^T \langle f_1, f_1^* \rangle dt,
\]
and
\[
\frac{D\mathcal{J}}{Df_2}(f_1, f_2) \cdot f_2^* = \int_0^T \left< c_1 u, \frac{Du}{Df_2} \cdot f_2^* \right> dt + \left< c_2 u, \frac{Du(T)}{Df_2} \cdot f_2^* \right> + t^2 \int_0^T \langle f_2, f_2^* \rangle dt.
\]
Since \( \frac{D\mathcal{J}}{Df_1}(f_1, f_2) = \frac{D\mathcal{J}}{Df_2}(f_1, f_2) = 0 \), equality (108) follows from (109), (110) and (106).

3.3. The nonlinear problem. In this subsection, we consider the nonlinear case, that is the case where the flow \( u \) is related to the disturbance \( f_1 \) and the control \( f_2 \) through the following full CH-NS
\[
\frac{du}{dt} + Au + B(u, u) + E(u) = R(u, u) + B_1 f_1 + B_2 f_2, \quad u = u_0 \text{ at } t = 0, \quad u \in \mathbb{V}. \tag{111}
\]
Hereafter, for \( u_0 \in \mathbb{V} \), we will denote by \( u(f_1, f_2) = u \in L^\infty(0, T; \mathbb{V}) \cap L^2(0, T; D(A)) \) the unique solution to (111) corresponding to \( (f_1, f_2) \in (L^2(0, T; H_1))^2 \).

Proposition 9. For \( u_0 \in \mathbb{V} \) and \( (f_1, f_2) \in (L^2(0, T; H_1))^2 \), the mappings \((f_1, f_2) \mapsto u(f_1, f_2) \) and \((f_1, f_2) \mapsto u(f_1, f_2)|_T \) have Fréchet derivatives \( u'(f_1, f_2), u'(f_1, f_2)|_T \) in every direction \((f_1', f_2') \in (L^2(0, T; H_1))^2 \). Finally the Fréchet derivative \( u'(f_1', f_2') \) solves the linear system
\[
\frac{du'}{dt} + Au' + B'(u)u' + E'(u)u' = R'(u)u' + B_1 f_1' + B_2 f_2', \quad u' = 0 \text{ at } t = 0, \quad u' \in \mathbb{V}, \tag{112}
\]
and it follows that \( u' \in L^\infty(0, T; \mathbb{V}) \cap L^2(0, T; D(A)) \).

Proof. Let \( u = \mathcal{U}(f_1, f_2), u_h = \mathcal{U}(f_1 + h_1, f_2 + h_2) = u + w_h, h = (h_1, h_2) \), where \( f_1, f_2, h_1, h_2 \in L^2(\mathcal{M}) \). Let \( v_h \) be the solution to
\[
\frac{dv_h}{dt} + Av_h + E'(u)v_h + B'(u)v_h = R'(u)v_h + B_1 h_1 + B_2 h_2, \quad v_h(0) = 0. \tag{113}
\]
From Proposition 3, we have the following estimates on \( v_h \):
\[
\|v_h(t)\|_2^2 + \int_0^t |Av_h|_2^2 ds \leq c_1 \int_0^t (|B_1 h_1|_2^2 + |B_2 h_2|_2^2) ds. \tag{114}
\]
This shows that the map \((h_1, h_2) \mapsto v_h\) is continuous from \((L^2(0, T; L^2(\mathcal{M})))^2\) into \(C(0, T; \mathbb{V}) \cap L^2(0, T; D(A))\).
Now from (38), we derive that
\[
\|(u_h - u)(t)\|_V^2 + \int_0^t |A(u_h - u)|_{L^2}^2 ds \leq c_1 \int_0^t (|B_1 h_1|_{L^2}^2 + |B_2 h_2|_{L^2}^2) ds,
\]
i.e.,
\[
\|w_h(t)\|_V^2 + \int_0^t |Aw_h|_{L^2}^2 ds \leq c_1 \int_0^t (|B_1 h_1|_{L^2}^2 + |B_2 h_2|_{L^2}^2) ds.
\]
Let \(v = w_h - v_h\). Then \(v\) satisfies
\[
\frac{dv}{dt} + Av + E'(u)v + B'(u)v = R'(u)v + F_1, \quad v(0) = 0,
\]
where
\[
F_1 = -(B(u + w_h, u + w_h) - B(u, u)) + \bar{B}'(u)w_h + E'(u)w_h + (R(u + w_h, u + w_h) - R(u, u)) - R'(u)w_h - (E(u + w_h) - Eu)
\]
\[= -B(w_h, w_h) + R(w_h, w_h) + (E(u + w_h) - Eu) - E'(u)w_h.\]
Note that \(|E(u + w_h) - Eu) - E'(u)w_h|_{L^2} \leq c|E''(\sigma_1)(w_h)|_{L^2} \leq c_1\|w_h\|_V^2|Aw_h|_{L^2}^2,\]
where \(\sigma_1 = \theta(u + w_h) + (1 - \theta)u\), for some \(\theta \in (0, 1).\)
From (14)-(16), we also have
\[
|\|B(w_h, w_h) + R(w_h, w_h)\|_{L^2}^2 \leq c\|w_h\|_V^2|Aw_h|_{L^2}^2.
\]
It follows that
\[
|F_1|_{L^2}^2 \leq c_1\|w_h\|_V^2|Aw_h|_{L^2}^2.
\]
We derive from Proposition 3 that
\[
\|v(t)\|_V^2 + \int_0^t |Av|_{L^2}^2 ds \leq c_1 \int_0^t \|w_h\|_V^2|Aw_h|_{L^2}^2 ds
\]
\[\leq c_1 \sup \|w_h\|_V^2 \int_0^t |Aw_h|_{L^2}^2 ds \leq c_1 \left( \int_0^t |h|_{L^2}^2 ds \right)^2.
\]
Therefore \(U'\) defined by (12) is the Fréchet derivative of \(U\) at \((f_1, f_2)\).

Moreover, for \(i = 1, 2\), let \(\Phi^i = (f_1^i, f_2^i), w^i = U'(f_1^i, f_2^i)h\) solution to (12),
\(u^i = U(f_1^i, f_2^i), \ h = (h_1, h_2)\). Let \(w = u^1 - u^2\). Then
\[
\frac{dw}{dt} + Aw + E'(u^1)w + B'(u^1)w = R'(u^1)w - (R'(u^2) - R'(u^1))w^2 + (B'(u^2) - B'(u^1))w^2 + (E'(u^2) - E'(u^1))w^2, \quad w(0) = 0.
\]
Note that (see (14)-(16))
\[
|B'(u^2) - B'(u^1)|_{L^2} \leq c\|u^2 - u^1\|_V^{1/2}\|A(u^1 - u^2)\|_{L^2}^{1/2}\|w^2\|_V^{1/2}\|Aw^2\|_{L^2}^{1/2},
\]
\)[(R'(u^2) - R'(u^1))w^2|_{L^2} \leq c\|u^2 - u^1\|_V^{1/2}\|A(u^1 - u^2)\|_{L^2}^{1/2}\|w^2\|_V^{1/2}\|Aw^2\|_{L^2}^{1/2}.
\]

Since \(E(u) = (0, A_N f(\phi))\) for \(u = (v, \phi)\), we also have
\[
|(E'(u^2) - E'(u^1))w^2|_{L^2} \leq c_2\|u^2 - u^1\|_V\|w^2\|_V,
\]
where \(c_2 = c_2(\|u^1\|_V, \|u^2\|_V) > 0.\)
It follows that
\[
\|w(t)\|_V^2 + \int_0^t |Aw|^2 ds \\
\leq c_1 \int_0^t (\|u^2 - u^1\|_V |A(u^1 - u^2)|_{L^2} + c_2 \|u^2 - u^1\|_V^2) ds \\
\leq c_1 \sup (\|u^2 - u^1\|_V^2) \int_0^t |A(u^2 - u^1)|_{L^2} ds + c_2 \sup \|u^2 - u^1\|_V^2 \|w^2\|_V \\
\leq c_1 |\Phi^1 - \Phi^2|_{L^2(0,T;H_1)}.
\] (125)

Therefore
\[
\|\mathcal{U}'(f_1^1, f_1^2) - \mathcal{U}'(f_2^1, f_2^2)\|_{L^2(0,T;H_1, Y)} \leq c_1 |\Phi^1 - \Phi^2|_{L^2(0,T;H_1)}. \quad (126)
\]

Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two non-empty, closed, convex and bounded subsets of \( L^2(0,T;H_1) \). Let \( K > 0 \) such that
\[
\int_0^T (|f_1^1|^2_{L^2} + |f_2^1|^2_{L^2}) dt \leq K^2, \quad \forall (f_1, f_2) \in \mathcal{X} \times \mathcal{Y}. \quad (127)
\]

We consider the following robust control problem:
\[
\mathcal{J}(f_1, f_2) \leq \mathcal{J}(\bar{f}_1, f_2) \leq \mathcal{J}(\bar{f}_1, f_2), \quad \forall (f_1, f_2) \in \mathcal{X} \times \mathcal{Y}. \quad (128)
\]

Here the cost function is given by
\[
\mathcal{J}(f_1, f_2) = \frac{1}{2} \int_0^T |C_1 u|_V^2 dt + \frac{1}{2} |C_2 u(T)|_V^2 + \frac{1}{2} \int_0^T \left|f_2^1|^2_{L^2} - \gamma^2 |f_1|^2_{L^2}\right| dt, \quad (129)
\]

where the flow \( u \) is related to the disturbance \( f_1 \) and the control \( f_2 \) through the system (111). Hereafter \( c_0, c_1 \) denote numerical coefficients depending only on \( \mathcal{M}, T, u_0, \mathcal{X} \) and \( \mathcal{Y} \) and whose value may be different in each inequality.

The proof of the existence of a solution \((\bar{f}_1, \bar{f}_2)\) to the robust control problem for the nonlinear case if based on the following result.

**Proposition 10.** Let \( \mathcal{J} \) be a functional defined on \( \mathcal{X} \times \mathcal{Y} \), where \( \mathcal{X} \) and \( \mathcal{Y} \) are non-empty, closed, convex sets. If \( \mathcal{J} \) satisfies:

1. \( \forall f_2 \in \mathcal{Y}, \ f_1 \mapsto \mathcal{J}(f_1, f_2) \) is concave upper semi-continuous,
2. \( \forall f_1 \in \mathcal{X}, \ f_2 \mapsto \mathcal{J}(f_1, f_2) \) is convex lower semi-continuous,

then the functional \( \mathcal{J} \) has at least one saddle point \((\bar{f}_1, \bar{f}_2)\) on \( \mathcal{X} \times \mathcal{Y} \), which is defined by
\[
J(\bar{f}_1, f_2) = \min_{f_2 \in \mathcal{Y}} \max_{f_1 \in \mathcal{X}} \mathcal{J}(f_1, f_2) = \max_{f_1 \in \mathcal{X}} \min_{f_2 \in \mathcal{Y}} \mathcal{J}(f_1, f_2).
\]

**Proof.** The proof is given in [11]. \( \square \)

**Proposition 11.** 1) There exists \( \gamma_1 = \gamma_1(\mathcal{M}, T, \mathcal{X}, \mathcal{Y}, u_0) > 0 \) such that for \( \gamma > \gamma_1 \), we have: \( \forall f_2 \in \mathcal{Y}, \ f_1 \mapsto \mathcal{J}(f_1, f_2) \) is strictly concave upper semi-continuous.

2) There exists \( l_1 = l_1(\mathcal{M}, T, \mathcal{X}, \mathcal{Y}, u_0) > 0 \) such that for \( l > l_1 \), we have: \( \forall f_1 \in \mathcal{X}, \ f_2 \mapsto \mathcal{J}(f_1, f_2) \) is convex lower semi-continuous.

**Proof.** 1) Since the norm is lower semi-continuous, the mapping \( f_1 \mapsto \mathcal{J}(f_1, f_2) \) is upper semi-continuous. To prove the concavity, it is enough to show that for
every \( f_1, f'_1, f_2, h(r) = J(f_1 + rf'_2), f_2 \) is concave with respect to \( r \), near \( r = 0 \), i.e. \( h''(0) < 0 \). Let \( u'(f'_1, 0) = \left( \frac{D\nu}{\nu f'_1} \right) \cdot f'_1 \). Then

\[
h'(r) = \int_0^T \langle C_1 u, C_1 u' \rangle dt + \langle C_2 u(T), C_2 u'(T) \rangle - \gamma^2 \int_0^T \langle f_1 + rf'_2, f'_1 \rangle dt,
\]

(130)

where \( u' \) satisfies

\[
\frac{du'}{dt} + Au' + B'(u)u' + E'(u)u' = R'(u)u' + B_1 f'_1, \quad u' = 0 \text{ at } t = 0.
\]

(131)

From Proposition 3, we have

\[
\|u'(t)\|^2_{\bar{V}} + \int_0^t |Au'|^2_{L^2} ds \leq c_1 \int_0^t |f^*_1|^2_{L^2} ds.
\]

(132)

Similarly \( u'' = \left( \frac{D^2 u}{D^2 f_1} \right) \cdot f'_1 \cdot f'_1 \) satisfies

\[
\frac{du''}{dt} + Au'' + B''(u)u'' + E''(u)u'' = R''(u)u'' + F_1, \quad u'' = 0 \text{ at } t = 0, \quad u'' \in \mathcal{V},
\]

(133)

where

\[
F_1 = -B(\hat{u'}, u') - B(u', \hat{u'}) + R(\hat{u'}, u') + R(u', \hat{u'}) - E''(u)u'u',
\]

\[
u' = \left( \frac{Du}{Df_1} \right) \cdot f'_1, \quad \hat{u'} = \left( \frac{Du}{Df_1} \right) \cdot \hat{f'_1}.
\]

(134)

From the properties (14)-(16) of the operators \( B, R \) and \( E \), we have

\[
| - B(\hat{u'}, u') - B(u', \hat{u'}) + R(\hat{u'}, u') + R(u', \hat{u'}) |_{L^2} \leq c\|\hat{u'}\|_{\bar{V}}^{1/2} |A\hat{u'}|_{L^2}^{1/2} |u'|_{\bar{V}}^{1/2} |Au'|_{L^2}^{1/2},
\]

(135)

\[
|E''(u)u'u'|_{L^2} \leq c_2\|\hat{u'}\|_{\bar{V}} \|u'|_{\bar{V}}.
\]

(136)

where \( c_2 = c_2(\|u\|_{\bar{V}}) > 0 \).

It follows that

\[
|F_1|_{L^2} \leq c\|\hat{u'}\|_{\bar{V}}^{1/2} |A\hat{u'}|_{L^2}^{1/2} |u'|_{\bar{V}}^{1/2} |Au'|_{L^2}^{1/2} + c_2\|\hat{u'}\|_{\bar{V}} \|u'|_{\bar{V}}.
\]

(137)

For \( \hat{f'1} = f'_1 \) and \( \hat{u'} = u' \), we have

\[
\int_0^T |F_1|_{L^2} dt \leq c\int_0^T (\|u''\|_{\bar{V}}^2 |Au'|_{L^2}^{1/2} + c_2\|u'|_{\bar{V}}) dt
\]

\[
\leq c\sup_t\|u''\|_{\bar{V}}^2 \int_0^T |Au'|_{L^2}^{1/2} dt + c_2\sup_t\|u'|_{\bar{V}},
\]

(138)

which gives (see (132))

\[
\int_0^T |F_1|_{L^2} dt \leq c_0 \left( \int_0^T |f^*_1|_{L^2} dt \right)^2.
\]

(139)

Therefore, we have the following estimate

\[
|u''(t)|_{\bar{V}}^2 \leq c_0 \int_0^t |F_1|_{L^2}^2 ds \leq c_0 \left( \int_0^T |f^*_1|_{L^2}^2 dt \right)^2.
\]

(140)
Moreover,

\[ h''(r) = \int_0^T |C_1 u'|^2 \, dt + \int_0^T \langle C_1 u, C_1 u'' \rangle \, dt \]
\[ + |C_2 u'(T)|^2 + \langle C_2 u(T), C_2 u''(T) \rangle - \gamma^2 \int_0^T |f_1|^2 \, dt. \]  

(141)

We can check that

\[ \int_0^T |C_1 u'|^2 \, dt \leq D_1 \int_0^T |f_1|^2 \, dt, \]

(142)

\[ \int_0^T \langle C_1 u, C_1 u'' \rangle \, dt \leq \left( \int_0^T |C_1 u|^2 \, dt \right)^{\frac{1}{2}} \left( \int_0^T |C_1 u''|^2 \, dt \right)^{\frac{1}{2}} \]
\[ \leq c_0 \left( \int_0^T |C_1 u''|^2 \, dt \right)^{\frac{1}{2}} \]
\[ \leq c_0 \left( \int_0^T |\mathcal{F}_1|^2 \, dt \right)^{\frac{1}{2}} \leq D_2 \int_0^T |f_1|^2 \, dt, \]
\[ |C_2 u'(T)|^2 \leq D_3 \int_0^T |f_1|^2 \, dt, \]
\[ |\langle C_2 u(T), C_2 u''(T) \rangle| \leq |C_2 u(T)|_\gamma |C_2 u''(T)|_\gamma \leq D_4 \int_0^T |f_1|^2 \, dt, \]

(144)

where \( D_1 = D_1(\mathcal{M}, T, X, Y, u_0) \).

Finally, we have \( h''(0) < 0 \) for \( \gamma^2 > \gamma_1^2 \equiv D_1 + D_2 + D_3 + D_4 \), which proves that Condition 1 is satisfied.

To prove Condition 2, we first note that the map \( f_2 \mapsto \mathcal{J}(f_1, f_2) \) is upper semi-continuous since the norm is lower semi-continuous. For the convexity, it is enough to prove that \( g(r) = \mathcal{J}(f_1, f_2 + rf_2^r) \) is convex w.r.t. \( r \) near \( r = 0 \), i.e., \( g''(0) > 0 \).

Note that

\[ g''(r) = \int_0^T |C_1 u'|^2 \, dt + \int_0^T \langle C_1 u, C_1 u'' \rangle \, dt \]
\[ + |u'(T)|^2 + \langle C_2 u(T), C_2 u'' \rangle + t^2 \int_0^T |f_2|^2 \, dt, \]

(146)

where

\[ u' = \left( \frac{Du}{Df_2} \right) \cdot f_2^r, \quad u'' = \left( \frac{D^2 u}{Df_2^2} \right) \cdot f_2^r \cdot f_2^r. \]

(147)

Following similar estimates as in the proof of Condition 1, we arrive at

\[ g''(0) \geq (t^2 - t_1^2) \int_0^T |f_2|^2 \, dt > 0, \]

(148)

for \( t^2 > t_1^2 = D_2 + D_4 \). The strict convexity of the map \( f_2 \mapsto \mathcal{J}(f_1, f_2) \) follows.

From Propositions 10 and 11, the following results hold true.

**Proposition 12.** Assume that \( X \) and \( Y \) are non-empty, closed, bounded, convex subsets of \( L^2(0, T; H_1) \) and that \( l \) and \( \gamma \) are large enough. Then there exists a unique saddle point \( (f_1, f_2) \in X \times Y \) and an associated flow \( u(f_1, f_2) \), such that

\[ \mathcal{J}(f_1, f_2) \leq \mathcal{J}(\bar{f}_1, \bar{f}_2) \leq \mathcal{J}(\bar{f}_1, f_2), \quad \forall (f_1, f_2) \in X \times Y. \]

(149)
3.4. Identification of the gradients. We now state the main result of this section.

**Theorem 3.2.** Assume that \( \mathcal{X} \) and \( \mathcal{Y} \) are non-empty, closed, bounded, convex subset of \( L^2(0,T;L^2(M)) \) and that \( l \) and \( \gamma \) are large enough. The robust control problem (128) has a unique solution \((\bar{f}_1, \bar{f}_2) \in \mathcal{X} \times \mathcal{Y}\). Moreover, the gradients of the cost functional \( J \) are given by

\[
\frac{D J}{D f_1}(f_1, f_2) = B^*_1 \bar{u} - \gamma^2 f_1 \quad \text{and} \quad \frac{D J}{D f_2}(f_1, f_2) = B^*_2 \bar{u} + l^2 f_2,
\]

where \( \bar{u} \) is found from the solution \((u, \bar{u})\) of the following coupled systems

\[
\frac{du}{dt} + Au + B(u, u) + Eu = R(u, u) + B_1 f_1 + B_2 f_2, \quad u = u_0 \text{ at } t = 0, \quad u \in \mathbb{V}, \quad (150)
\]

\[
-\frac{d\bar{u}}{dt} + A\bar{u} + B'(u)^* \bar{u} + E'(u)^* \bar{u} = R'(u)^* \bar{u} + C_1^* C_1 u, \quad \bar{u}(T) = C_2^* C_2 u(T), \quad \bar{u} \in \mathbb{V}. \quad (151)
\]

**Proof.** The gradients \( \frac{D J}{D f_1} \) and \( \frac{D J}{D f_2} \) are determined as in the linear case by solving the system (150)-(151). The uniqueness of solutions is proved using the strict concavity and the strict convexity of the functional \( J \). □

3.5. Application to data assimilation. It is well known that chaotic problems, such as weather system, are highly susceptible to the small disturbance present in all physical systems. A classical control problem arising in meteorology and oceanography, in relation with data assimilation, is the adjustment of initial conditions in order to obtain a flow that agrees with a desired target flow (i.e., the observations).

Given a set of measurements of some actual flow \( \vartheta \) on \([0, T]\), the problem is to determine a "best" estimate as to the initial state of the model \( u \) that leads to the observed system behavior, while simultaneously forcing the model with a small component of the worst-case disturbance which perturbs \( u \) away from the observed system behavior \( \vartheta \), \([20, 26, 27]\).

Define \( \varphi = u - \vartheta \) as the difference between the estimated flow \( u \) and the observed flow \( \vartheta \). The cost function considered for this problem is given by

\[
J(f_1, f_2) = \frac{1}{2} \int_0^T |C_1 \varphi|^2_{\mathbb{V}} dt + \frac{1}{2} |C_2 \varphi(T)|^2_{\mathbb{V}} + \frac{l^2}{2} |f_2|^2_{L^2} - \frac{\gamma^2}{2} \int_0^T |f_1|^2_{L^2} dt. \quad (152)
\]

The measurements of the actual flow \( C_1 \varphi \) and \( C_2 \varphi(T) \) are assumed to be given. The linear operators \( C_1 \) and \( C_2 \) are bounded or unbounded operators on \( \mathbb{V} \) satisfying (89). For \( C_1 = 0 \) and \( C_2 = d_1 I \), where \( I \) is the identity operator, the goal is to match the potential vorticity of the estimated flow with that of the observed flow at the end time \( T \). More details on the functional \( J \) are given in [4].

The results given in this subsection are generalizations of those given in the previous subsection, therefore we will omit the details of the proofs. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two non-empty, closed, convex and bounded subsets of \( L^2(0,T;H_1) \) and \( \mathbb{V} \) respectively. Let \( K > 0 \) such that

\[
\int_0^T |f_1|^2_{L^2} dt + |f_2|^2_{L^2} \leq R^2, \quad \forall (f_1, f_2) \in \mathcal{X} \times \mathcal{Y}. \quad (153)
\]

Here we assume that \( B_2 \) is a bounded mapping from \( H_1 \) into \( \mathbb{V} \). We consider the following robust control problem:

To find \( (\bar{f}_1, \bar{f}_2) \in \mathcal{X} \times \mathcal{Y} \) such that

\[
J(f_1, f_2) \leq J(\bar{f}_1, f_2) \leq J(f_1, f_2), \quad \forall (f_1, f_2) \in \mathcal{X} \times \mathcal{Y}, \quad (154)
\]

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where \( \varpi = u - \partial \) is related to the control \( f_2 \) and the disturbance \( f_1 \) by the CH-NS
\[
\frac{du}{dt} + Au + B(u, u) + Eu = R(u, u) + B_1f_1, \quad u = B_2f_2 \text{ at } t = 0, \quad u \in \mathcal{V}, \quad (155)
\]
As for the body force control problem (128), we have the following results.

**Proposition 13.** Assume that \( X \) and \( \mathcal{V} \) are non-empty, closed, bounded, convex subset of \( L^2(0, T; L^2(M)) \) and \( \mathcal{V} \) respectively and that \( l \) and \( \gamma \) are large enough. The robust control problem (154) has a unique solution \( (f_1, f_2) \in X \times \mathcal{V} \). Moreover, the gradients of the cost functional \( J \) are given by
\[
\frac{DJ}{Df_1}(f_1, f_2) = B_1^* \tilde{u} - \gamma^2 f_1 \quad \text{and} \quad \frac{DJ}{Df_2}(f_1, f_2) = B_2^* \tilde{u}(0) + l^2 f_2,
\]
where \( \tilde{u} \) is found from the solution \( (u, \tilde{u}) \) of the following coupled systems
\[
\frac{du}{dt} + Au + B(u, u) + Eu = R(u, u) + B_1f_1, \quad u = B_2f_2 \text{ at } t = 0, \quad u \in \mathcal{V}, \quad (156)
\]
\[
-\frac{d\tilde{u}}{dt} + A\tilde{u} + B'(u)^*\tilde{u} + E'(u)^*\tilde{u} = R'(u)^*\tilde{u} + C_1^*C_1(u - \theta), \quad \tilde{u}(T) = C_2^*C_2(u - \theta)(T), \quad \tilde{u} \in \mathcal{V}. \quad (157)
\]

**Proof.** The proof is similar to that of Theorem 3.2. \( \square \)

4. **Appendix 1.** In this part, we study the existence and uniqueness of solutions to the adjoint systems.

Let \( u = (v, \phi) \) be the strong solution to (84) given by Proposition 2 with \( u_0 \in \mathcal{V} \).

We consider the following adjoint system:
\[
-\frac{d\omega}{dt} + A\omega + (B'(u))^*\omega + (E'(u))^*\omega = (R'(u))^*\omega + g, \quad \omega(T) = \omega_0, \quad (158)
\]
where the unknown is \( \omega = (w, \psi) \) and \( \omega_0 = (w_0, \psi_0) \) is given.

The existence and uniqueness of solution to (158) is given in [24].

**Proposition 14.** For \( \omega_0 = (w_0, \psi_0) \in \mathcal{V} \) and \( g \in L^2(0, T; \mathcal{V}^*) \), the system (158) has a unique solution \( \omega = (w, \psi) \in L^2(0, T; \mathcal{V}) \cap C(0, T; \mathcal{V}) \). Moreover, the following estimate holds true:
\[
|\langle w, \psi \rangle(t)\rangle^2 + \int_0^T \|\langle w, \psi \rangle\|^2 ds \leq c_1\|\langle w_0, \psi_0 \rangle\|^2 + c_1 \int_0^T |g|^4 ds. \quad (159)
\]
Furthermore, if \( (w_0, \psi_0) \in \mathcal{V} \) and \( g \in L^2(0, T; H) \), then the solution \( (w, \psi) \in C(0, T; \mathcal{V}) \cap L^2(0, T; D(A_0) \times D(B_N^2)) \) and the following estimates hold
\[
\|\langle w, \psi \rangle(t)\rangle^2 + \int_0^T (|A_0w(s)|^2 + |B_N^2 \psi(s)|^2 ds \leq c_1\|\langle w_0, \psi_0 \rangle\|^2 + c_1 \int_0^T |g(s)|^2 ds, \quad (160)
\]
\[
\int_0^T \frac{d}{dt} \|w, \psi\|^2 ds \leq c_1\|\langle w_0, \psi_0 \rangle\|^2 + c_1 \int_0^T |g(s)|^2 ds.
\]

**Proof.** The proof is given in [24]. For the reader’s convenience, we repeat it here. We recall that \( u = (v, \phi) \), \( \omega = (w, \psi) \). Since \( (v, \phi) \in L^\infty(0, T; \mathcal{V}) \cap L^2(0, T; D(A_0) \times D(B_N^2)) \), using the known regularity results for parabolic equations such as the linearized Navier-Stokes system (see e.g., [28]), we can rigorously prove the existence
and uniqueness of solution to (158). To derive (160), we first note that for \( \Theta = (\zeta, \varphi) \) regular enough, we have
\[
\langle (B^T(u))^\ast \omega, \Theta \rangle = \langle \omega, B^T(u) \Theta \rangle = \langle w, B_0(v, \zeta) + B_0(\zeta, v) + 2 (B_1(\zeta, \varphi) + B_1(\zeta, \phi)) = b_0(v, \zeta, w) + b_0(\zeta, v, w) + b_1(v, \varphi, \psi) + b_1(\zeta, \phi, \psi). \quad (161)
\]
We also have
\[
\langle (\hat{R}^T(u))^\ast \omega, \Theta \rangle = \langle \omega, \hat{R}^T(u) \Theta \rangle = \langle w, R_0(A_N \varphi, \varphi) + R_0(A_N \varphi, \phi) \rangle = b_1(w, \varphi, A_N \phi) + b_1(w, \phi, A_N \varphi). \quad (162)
\]
Multiplying (158) with \( \Theta = (w, \psi) \) in \( H \) and using (161)-(162) give
\[
- \frac{d}{dt} (|w|^2_{L^2} + |\psi|^2_{L^2}) + 2|w|^2 + 2 |B_N \psi|^2_{L^2} + 2b_0(w, v, w) + 2b_1(w, \varphi, \psi) + 2(2A_N f(\phi) \psi, A_N \psi)_{L^2} = 2b_1(w, \varphi, A_N \psi) + 2b_1(w, \phi, A_N \psi) + 2(g_1, w) + 2(g_2, A_N \psi). \quad (163)
\]
It follows that
\[
|(w, \psi)(t)|^2_{L^2} + \int_0^T (|w(s)|^2 + |B_N \psi(s)|^2_{L^2}) ds \leq c_1 |(w_0, \psi_0)|^2_{L^2} + c_1 \int_0^T |g|^2_{L^2}. \quad (164)
\]
Multiplying (158) with \( \Theta = (w, A_N \psi) \) in \( H \) and using (161)-(162) give
\[
- \frac{d}{dt} (|w|^2_{L^2} + |\psi|^2) + 2|w|^2 + 2 |B_N \psi|^2_{L^2} + 2b_0(w, v, w) + 2b_1(w, A_N \psi, \psi) + 2b_1(w, \phi, \psi) + 2(2A_N f(\phi) \psi, A_N \psi)_{L^2} = 2b_1(w, A_N \psi, A_N \phi) + 2b_1(w, \phi, A_N \psi) + 2(g_1, w) + 2(g_2, A_N \psi). \quad (165)
\]
We note that
\[
|b_0(w, v, w)| \leq \frac{1}{8} |w|^2 + c |w|^2_{L^2} |\psi|^2, \quad (166)
\]
\[
|b_1(w, \varphi, \psi)| = |b_1(w, \psi, A_N \psi)| \leq \frac{1}{8} |B_N^{3/2} \psi|^2_{L^2} + c |w|^2_{L^2} |\psi|^2 |\psi|^2, \quad (167)
\]
\[
|b_1(w, \phi, \psi)| \leq c |w|^2_{L^2} |\psi|^2_{L^2} |\psi|_{L^2} + |\phi|^4, \quad (168)
\]
\[
|b_1(w, A_N \phi, \psi)| = |b_1(w, \phi, A_N \psi)| \leq c |w|^2_{L^2} |\psi|^2_{L^2} |\psi|^2_{L^2} + c |\phi|^4, \quad (169)
\]
\[
|A_N f(\phi) \psi, A_N \psi)_{L^2} | \leq \frac{1}{8} |B_N^{3/2} \psi|^2_{L^2} + c |w|^2_{L^2} |\psi|^2_{L^2} |A_N \phi|^2_{L^2}, \quad (170)
\]
\[
|b_1(w, \phi, A_N \phi)| \leq |A_N^{1/2} B_1(w, \phi)|_{L^2} |A_N^{3/2} \phi|_{L^2} \leq c |\phi||\nabla \phi||_{\infty} + |w|^2_{L^2} |\phi|^2_{L^2} |A_N \phi|_{L^2} |A_N^{3/2} \phi|_{L^2} \leq \frac{1}{8} |A_N^{3/2} \phi|^2_{L^2} + c |w|^2_{L^2} |\phi|^2_{L^2} |A_N \phi|_{L^2} |A_N^{3/2} \phi|_{L^2} \leq c |\psi|^2_{L^2} |\phi|^2_{L^2} |A_N \phi|^2_{L^2} |A_N^{3/2} \phi|_{L^2}. \quad (171)
\]
It follows from (165)-(171) that
\[
- \frac{d}{dt} (|w|^2_{L^2} + |\psi|^2) + c(|w|^2 + |B_N^{3/2} \psi|^2_{L^2}) \leq G_1(t) (|w|^2_{L^2} + |\psi|^2) + c |g|^2_{L^2} + c |w|^2_{L^2} |\phi|^2_{L^2} |A_N^{3/2} \phi|_{L^2}, \quad (172)
\]
where
\[
G_1(t) = c |\psi|^2 + c |w|^2_{L^2} |\psi|^2 + c |g|^2_{L^2} + c |w|^2_{L^2} |\phi|^2_{L^2} |A_N^{3/2} \phi|_{L^2} + Q_0(|\phi|),
\]
which gives
\[ \frac{d}{dt}(|w(t)|^2 + ||\psi(t)||^2 + \int_0^T (||w||^2 + |B^N \psi|^2_2^2) dt) \leq c_1 (|w_0|^2 + ||\psi_0||^2) \]
(173)
and
\[ \frac{d}{dt}(|(w, \psi)(t)|^2 + \int_0^T \|(w, \psi)(s)||^2 ds) \leq c_1 (|w_0, \psi_0||^2 + \int_0^T |g(s)||^2 ds, \]
(174)
and (159) follows. Note that \( \mathcal{G}_2(t) \in L^1(0, T) \).

To prove (160), we multiply (158) by \((A_0 w, A_N \psi)\) and we use (161)-(162) to derive that
\[
-\frac{d}{dt}(||w||^2 + ||\psi||^2) + 2|A_0 w|^2_{L^2} + 2|B^N \psi|^2_2 + 2b_0(v, A_0 w, w)
+ 2b_0(A_0 w, v, w) + 2b_1(v, A_N \psi, \psi) + 2b_1(A_0 w, \phi, \psi)
= 2b_1(w, A_N \psi, \phi) + 2b_1(w, \phi, A^2_N \psi) + 2(g_1, A_0 w)_{L^2} + 2(g_2, A_N \psi)_{L^2}.
\]
(175)
We note that
\[ |b_0(v, A_0 w, w)| \leq c|v|^{1/2}_{L^2} ||v||^{1/2} ||w||^{1/2} |A_0 w|^2_{L^2}, \]
(176)
\[ |b_0(A_0 w, v, w)| \leq c|A_0 w|_{L^2} ||v||^{1/2} |A_0 v|^2_{L^2} ||w||^{1/2} |w|^2_{L^2}, \]
(177)
\[ |b_1(A_0 w, \phi, \psi)| \leq c|A_0 w|_{L^2} ||\phi||^{1/2} |A_N \phi|_{L^2} ||\psi||^{1/2} |\psi|^2_{L^2}, \]
(178)
\[ |b_1(v, A_N \psi, \psi)| = |b_1(v, \psi, A_N \psi)| \leq c|v|^{1/2}_{L^2} ||v||^{1/2} |B^N \psi|^2_{L^2} ||\psi||^{1/2} |\psi|^2_{L^2}, \]
(179)
\[ |b_1(w, A_N \psi, \phi)| \leq c|w|^{1/2}_{L^2} ||w||^{1/2} |B_N \phi|_{L^2} ||A_N \phi|^2_{L^2} ||\psi||^{1/2} |B^N \psi|^2_{L^2}, \]
(180)
\[ |b_1(w, \phi, A^2_N \psi)| \leq |A^2_N \psi|_{L^2} ||A_0 \phi||^{1/2} ||B^N \phi||^{1/2} |B^N \psi|^2_{L^2} + c||w||^{1/2}_{L^2} |B_N \phi||^{1/2} ||A_N \phi||^{1/2} |B^N \psi|^2_{L^2}, \]
(181)
\[ |(A_N f^\prime)(\phi, A_N \psi)| \leq \frac{1}{8} |B^N \psi|^2_{L^2} + Q_0(||\phi||^2 ||\psi||^2). \]
(182)
It follows from (175)-(182) that
\[ -\frac{d}{dt}(||w(t)||^2 + ||\psi(t)||^2) + |A_0 w|^2_{L^2} + |B^N \psi|^2_2 \]
\[ \leq \mathcal{G}_2(t)(||w||^2 + ||\psi||^2) + c|g(t)|^2_{L^2}, \]
(183)
where
\[ \mathcal{G}_2(t) = c(|v|^2_{L^2} ||v||^2 + ||v|| |A_0 v|^2_{L^2} + ||\phi||^2 |B_N \phi|^2_{L^2})
+ c(|B_N \phi||_{L^2} |B^N \phi|^2_{L^2} + ||\phi|| |B^N \phi||_{L^2}) + Q_0(||\phi||). \]
which gives
\[
\|w(t)\|^2 + \|\psi(t)\|^2 + \int_0^T (|A_0 w\|^2_{L^2} + |B_N^{3/2} \psi|^2_{L^2}) \, ds
\le c_1(\|w_0\|^2 + \|\psi_0\|^2) + c_1 \int_0^T |g(s)|^2_{L^2} \, ds,
\]  
(184)

and (160)_1 follows. Note that \(G_2(t) \in L^1(0, T)\).

We multiply (158) by \((A_0 w, A_N^2 \psi)\) and use (161)-(162) to derive that
\[
\begin{align*}
-\frac{d}{dt} (\|w\|^2 + |B_N \psi|_{L^2}^2) + 2|A_0 w|_{L^2}^2 + 2|B_N^{3/2} \psi|_{L^2}^2 + 2 b_0(v, A_0 w, w) \\
+ 2b_0(A_0 w, v, w) + 2b_1(v, A_N^2 \psi, \psi) + 2b_1(A_0 w, \phi, \phi) \\
+ 2(A_N f(\phi, \psi, A_N^3 \psi))_{L^2}
\end{align*}
= 2b_1(w, A_N^2 \psi, A_N \phi) + 2b_1(w, \phi, A_N^3 \psi) + 2(g_1, A_0 w)_{L^2} + 2(g_2, A_N \psi)_{L^2}.
\]  
(185)

We note that
\[
\begin{align*}
|b_1(v, A_N^2 \psi, \psi)| & \le |b_1(v, \psi, A_N^2 \psi)| \\
& \le c|v|^{1/2}_{L^2} \|v\|^{1/2}_{L^2} |A_N \psi|^{1/2}_{L^2} |A_N^2 \psi|_{L^2} \\
& \le \frac{1}{8} |A_N^2 \psi|_{L^2}^2 + c|v|_{L^2} \|v\| |A_N \psi|_{L^2}^2,
\end{align*}
\]  
(186)

\[
\begin{align*}
|b_1(w, A_N^2 \psi, A_N \phi)| & = |b_1(w, A_N \phi, A_N^2 \psi)| \\
& \le c|w|^{1/2}_{L^2} |A_0 w|^{1/2}_{L^2} |A_N^{3/2} \phi|_{L^2} |A_N^2 \psi|_{L^2} \\
& \le \frac{1}{8} (|A_0 w|_{L^2}^2 + |A_N^2 \psi|_{L^2}^2) + c|w|_{L^2}^2 |A_N^{3/2} \phi|_{L^2}^2,
\end{align*}
\]  
(187)

\[
|2(A_N f(\phi, \psi, A_N^3 \psi))_{L^2}| \le \frac{1}{8} |A_N^2 \psi|_{L^2}^2 + Q_0(\|B_N \phi|_{L^2}) |B_N \psi|_{L^2}^2,
\]  
(188)

\[
|b_1(w, \phi, A_N^3 \psi)| = |A_N B_1(w, \phi, A_N^2 \psi)| \\
\le c |A_0 w|_{L^2} \|\phi\|^{1/2}_{L^2} |A_N^{3/2} \phi|_{L^2} |A_N^2 \psi|_{L^2} + c |w|^{1/2}_{L^2} |A_0 w|^{1/2}_{L^2} |A_N^{3/2} \phi|_{L^2} |A_N^2 \psi|_{L^2} \\
\le \frac{1}{8} |A_N^2 \psi|_{L^2}^2 + c |A_0 w|_{L^2} \|\phi\| |A_N^2 \phi|_{L^2} + c |w|_{L^2}^2 |A_N^{3/2} \phi|_{L^2}^2.
\]  
(189)

It follows from (185)-(189) that
\[
-\frac{d}{dt} (\|w(t)\|^2 + |B_N \psi|_{L^2}^2) + |A_0 w|_{L^2}^2 + |B_N^2 \psi|_{L^2}^2
\le \mathcal{G}_3(t)(\|w\|^2 + |B_N \psi|_{L^2}^2) + \mathcal{H}_1(t),
\]  
(190)

where
\[
\begin{align*}
\mathcal{G}_3(t) &= \mathcal{G}_2(t) + c |A_N^{3/2} \phi|_{L^2}^2 Q_0(\|B_N \phi|_{L^2}), \\
\mathcal{H}_1(t) &= c |g|^2_{L^2} + c |A_0 w|_{L^2} \|\phi\| |A_N^{3/2} \phi|_{L^2},
\end{align*}
\]
and (160)_1 follows. Note that \(\mathcal{G}_3(t), \mathcal{H}(t) \in L^1(0, T)\). As in the proof of Proposition 3, we can easily prove that (160)_2 follows from (160)_1.

5. Appendix 2: An intuitive introduction to robust control theory. We recall from [4] some motivations for studying robust control problems such as the ones considered in this article. Consider the present problem as a differential game between an engineer seeking the best control \(f_2\) which stabilizes the flow perturbation with limited control effort and, simultaneously, nature seeking the maximally malevolent disturbance \(f_1\) which destabilizes the flow perturbation with limited disturbance magnitude. The parameter \(\gamma^2\) factors into such a competition as a weighting on the magnitude of the disturbance which nature can afford to offer, in a manner analogous to the parameter \(l^2\), which is a weighting on the magnitude of the control which the engineer can afford to offer. \(\square\)
The parameter $l^2$ may be interpreted as the price of the control to the engineer. The $l \to \infty$ limit corresponds to prohibitively expensive control, and results in $f_2 \to 0$ in the minimization with respect to $f_2$ for the present problem. Reduced values of $l$ increase the cost functional less upon the application of a control $f_2$. A non-zero control results whenever the control $f_2$ can affect the flow perturbation $(v, \phi)$ in such a way that the net cost functional $J$ is reduced.

The parameter $\gamma^2$ may be interpreted as the price of the disturbance to nature. The $\gamma \to \infty$ limit results in $f_1 \to 0$ in the maximization with respect to $f_1$, leading to the optimal control formulation of [3] for $f_2$ alone. Reduced values of $\gamma$ decrease the cost functional less upon the application of a disturbance. A non-zero disturbance results whenever the disturbance can affect the flow perturbation $u$ in such a way that the net cost functional $J$ is increased.

Solving for the control $f_2$ which is effective even in the presence of a disturbance $f_1$ which maximally spoils the control objective is a way of achieving system robustness. A control which works even in the presence of the malevolent disturbance $f_1$ will also be robust to a wide class of other possible disturbances. Put another way, the introduction of the worst-case disturbance in the robust approach is a means of detuning the optimal controls. It results in a set of controls which may have somewhat degraded performance when no disturbances are present. However, much greater system robustness (i.e., better performance) is attained in cases for which unknown disturbances are present in the system, and thus the approach is relevant for applications in physical systems, in which unpredictable disturbances are ubiquitous. In the present systems, for $\gamma < \gamma_0$ for some critical value $\gamma_0$ (an upper bound of which is established in this paper), the non-cooperative game is not known to have a finite solution; essentially, the malevolent disturbance wins. The control $f_2$ corresponding to $\gamma = \gamma_0$ results in a stable system even when nature is on the brink of making the system unstable. However, the control determined with $\gamma = \gamma_0$ is not always the most suitable, as it may result in a very large control magnitude and degraded performance in response to disturbances with structure more benign than the worst-case scenario. In the implementation, variation of $l$ and $\gamma$ provides the flexibility in the control design which is necessary to achieve the desired trade-offs between Gaussian and worst-case disturbance response and the control magnitude required.

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REFERENCES

[1] H. Abels, On a diffuse interface model for a two-phase flow of compressible viscous fluids, Indiana Univ. Math. J., 57 (2008), 659–698.
[2] H. Abels, On a diffuse interface model for two-phase flows of viscous, incompressible fluids with matched densities, Arch. Ration. Mech. Anal., 194 (2009), 463–506.
[3] F. Abergel and R. Temam, On some control problems in fluid mechanics, Theoret. Comput. Fluid Dynam., 1 (1990), 303–325.
[4] T. Bewley, R. Temam and M. Ziane, A general framework for robust control in fluid mechanics, Physica D, 138 (2000), 360–392.
[5] T. Blesgen, A generalization of the Navier-Stokes equation to two-phase flow, Physica D (Applied Physics), 32 (1999), 1119–1123.
[6] F. Boyer, Mathematical study of multi-phase flow under shear through order parameter formulation, Asymptotic Anal., 20 (1999), 175–212.
[7] F. Boyer, Nonhomogeneous cahn-hilliard fluids, Ann. Inst. H. Poincaré Anal. Non Linéaire, 18 (2001), 225–259.
[8] F. Boyer, A theoretical and numerical model for the study of incompressible mixture flows, *Computer and Fluids*, 31 (2002), 41–68.

[9] G. Caginalp, An analysis of a phase field model of a free boundary, *Arch. Rational Mech. Anal.*, 92 (1986), 205–245.

[10] C. Cao and C. G. Gal, Global solutions for the 2D NS-CH model for a two-phase flow of viscous, incompressible fluids with mixed partial viscosity and mobility, *Nonlinearity*, 25 (2012), 3211–3234.

[11] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, Series Classics in Applied Mathematics, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1999.

[12] E. Feireisl, H. Petzeltová, E. Rocca and G. Schimperna, Analysis of a phase-field model for two-phase compressible fluids, *Math. Methods Appl. Sci.*, 20 (2010), 1129–1160.

[13] S. Frigeri, E. Rocca and J. Sprekels, Optimal distributed control of a nonlocal Cahn-Hilliard/Navier-Stokes system in 2D, arXiv:1411.1627.

[14] C. G. Gal and M. Grasselli, Asymptotic behavior of a Cahn-Hilliard-Navier-Stokes system in 2D, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27 (2010), 401–436.

[15] C. G. Gal and M. Grasselli, Longtime behavior for a model of homogeneous incompressible two-phase flows, *Discrete Contin. Dyn. Syst.*, 28 (2010), 1–39.

[16] C. G. Gal and M. Grasselli, Trajectory attractors for binary fluid mixtures in 3D, *Chin. Ann. Math. Ser. B*, 31 (2010), 655–678.

[17] M. E. Gurtin, D. Polignone and J. Vinals, Two-phase binary fluid and immiscible fluids described by an order parameter, *Math. Models Methods Appl. Sci.*, 6 (1996), 8–15.

[18] M. Hintermüller and D. Wegner, Optimal control of a semidiscrete Cahn-Hilliard-Navier-Stokes system, *SIAM J. Control Optim.*, 52 (2014), 747–772.

[19] P. C. Hohenberg and B. I. Halperin, Theory of dynamical critical phenomena, *Rev. Modern Phys.*, 49 (1977), 435–479.

[20] F. X. LeDimet and V. Shutyaev, On data assimilation for quasilinear parabolic problems, *Russian J. Numer. Anal. Numer. Model.*, 16 (2001), 247–259.

[21] S. Li, Optimal controls of Boussinesq equations with state constraints, *Nonlinear Anal.*, 60 (2005), 1485–1508.

[22] X. Li and J. Yong, *Optimal Control Theory for Infinite Dimensional Systems*, Birkhäuser, Boston, 1995.

[23] J. L. Lions, Optimal Control of Systems governed by Partial Differential Equations, Springer-Verlag, New York, 1970.

[24] T. Tachim Medjo, Optimal control of a Cahn-Hilliard-Navier-Stokes model with state constraints, submitted, 2014.

[25] A. Onuki, Phase transition of fluids in shear flow, *J. Phys. Condens. Matter*, 9 (1997), 6119–6157.

[26] O. Talagrand, On the mathematics of data assimilation, *Tellus*, 33 (1981), 321–339.

[27] O. Talagrand and P. Courtier, Variational assimilation of meteorological observations with the adjoint vorticity equations i: Theory, *Q. J. R. Meteorol. Soc.*, 113 (1987), 1311–1328.

[28] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, volume 68, Appl. Math. Sci., Springer-Verlag, New York, second edition, 1997.

[29] G. Wang, Optimal controls of 3-dimensional Navier-Stokes equations with state constraints, *SIAM J. Control Optim.*, 41 (2002), 583–606.

[30] G. Wang, Pontryagin maximum principle of optimal control governed by fluid dynamic systems with two point boundary state constraint, *Nonlinear Anal.*, 51 (2002), 509–536.

[31] G. Wang, Pontryagin’s maximum principle for optimal control of the stationary Navier-Stokes equations, *Nonlinear Anal.*, 52 (2003), 1853–1866.

[32] G. Wang and L. Wang, Maximum principle of state-constrained optimal control governed by fluid dynamic systems, *Nonlinear Anal.*, 52 (2003), 1911–1931.

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