ASYMPTOTIC FALSE DISCOVERY CONTROL OF THE
BENJAMINI-HOCHELBG PROCEDURE FOR PAIRWISE
COMPARISONS

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Abstract. In a one-way analysis-of-variance (ANOVA) model, the number of all pairwise comparisons can be large even when there are only a moderate number of groups. Motivated by this, we consider a regime with a growing number of groups, and prove that for testing pairwise comparisons the BH procedure (Benjamini and Hochberg, 1995) can offer asymptotic control on false discoveries, despite that the t-statistics involved do not exhibit the well-known positive dependence structure called the PRDS to guarantee exact false discovery rate (FDR) control (Benjamini and Yekutieli, 2001). Sharing Tukey’s viewpoint that the difference in the means of any two groups cannot be exactly zero, our main result is stated in terms of the control on the directional false discovery rate and directional false discovery proportion (Benjamini and Yekutieli, 2005). A key technical contribution is that we have shown the dependence among the t-statistics to be weak enough to induce a convergence result typically needed for establishing asymptotic FDR control. Our analysis does not adhere to stylized assumptions such as normality, variance homogeneity and a balanced design, and thus provides a theoretical grounding for applications in more general situations.

1. Introduction

Suppose we have $m$ independent groups of observations $X_i = \{X_{ki}, 1 \leq k \leq n_i\}$, $1 \leq i \leq m$, where for each $i$, $X_{ki}$’s are independent and identically distributed random variables with mean $\mu_i$ and variance $\sigma_i^2$. The pairwise comparison problem

\begin{align}
H_{ij} : \mu_i &= \mu_j \quad \text{against}\quad K_{ij} : \mu_i \neq \mu_j, \quad 1 \leq i < j \leq m
\end{align}

has been widely studied since Tukey (1953)’s early work on multiple comparisons. In the early days, Tukey (1953) and Kramer (1956) independently proposed their famous procedure for testing (1.1), based on the studentized range distribution, with the goal of controlling the family-wise error rate (FWER). Such developments culminated in the work of Hayter (1984) who established the conservativeness of the Tukey-Kramer procedure in the affirmative. In modern applications when $m$ can be large, the number of hypotheses $q := \binom{m}{2} = m(m-1)/2$ to consider is even larger, making any testing procedure aiming to control FWER too conservative to be useful. Hence there is a strong case for using the false discovery rate (FDR) proposed by Benjamini and Hochberg (1995), defined as the expectation of the false
discovery proportion (FDP),
\[
\frac{\text{#hypotheses incorrectly rejected}}{\text{#hypotheses rejected}},
\]
as a more appropriate type 1 error measure for pairwise comparisons, given its scalability to the number of rejections (“discoveries”) made.

The original step-up procedure proposed in Benjamini and Hochberg (1995) (widely known as the BH procedure, or BH for short) is proven to control the FDR at a pre-specified level \(0 < \alpha < 1\) when the test statistics are independent. However, the test statistics for the pairwise comparison problem (1.1), namely, the two-sample t-statistics,

\[
T_{ij} = \frac{\bar{X}_i - \bar{X}_j}{\sqrt{\hat{\sigma}^2_i/n_i + \hat{\sigma}^2_j/n_j}}, \quad 1 \leq i \neq j \leq m,
\]

where \(\bar{X}_i = \frac{1}{n_i} \sum_{k=1}^{n_i} X_{ki}\) is the sample mean and \(\hat{\sigma}^2_i = \frac{1}{n_i-1} \sum_{k=1}^{n_i} (X_{ki} - \bar{X}_i)^2\) is the sample variance for group \(i\), are apparently dependent in a specific pattern. If for each unique index pair \(i < j\), we let

\[
p_{ij,U} = 1 - F_{ij}(T_{ij}) \quad \text{and} \quad p_{ij,L} = F_{ij}(T_{ij}),
\]

be the one-sided upper and lower tailed p-values of \(T_{ij}\), where \(F_{ij}\) is the (approximate) cumulative distribution function of \(T_{ij}\) under \(H_{ij}\), the BH procedure would stipulate that we first sort all the two-sided p-values, defined as

\[
p_{ij} = 2 \min(p_{ij,L}, p_{ij,U}) \text{ for each pair } i < j,
\]

into their order statistics

\[
p(1) \leq p(2) \leq \cdots \leq p(q),
\]

and reject the null hypothesis \(H_{ij}\) whenever \(p_{ij} \leq p(\hat{k}) \leq i \alpha / q\) (and if \(\hat{k}\) is the maximum over an empty set then no rejection is made). Despite the dependence among the statistics in (1.2), many studies suggested that the BH procedure can still provide valid FDR control based on extensive simulations (Blair and Hochberg, 1995, Keselman, Cribbie, and Holland, 1999, Williams, Jones, and Tukey, 1999). The subsequent well-cited work of Benjamini and Yekutieli (2001) has proven the validity of BH under “positive dependence” of the test statistics. Unfortunately, this particular dependence condition, widely known as PRDS in the literature, is not satisfied by the two-sample t-statistics in (1.2); see Yekutieli (2008a, Example 2.6). Indeed, even under the seemingly innocuous setup with a balanced design, as well as normality and homogeneity of error variances, establishing exact FDR control of the BH procedure for pairwise comparisons remains a hard open problem, as Benjamini and Yekutieli (2001, p.1882) also pointed out: “Another important open question is whether the same procedure controls the FDR when testing pairwise comparisons of normal means, either Studentized or not.”

In real applications, the number of tests \(q\) can become large for only a moderate number of groups \(m\); for example, \((m, q) = (10, 45)\) in Oishi et al (1998), \((m, q) = (41, 820)\) in Williams et al (1999) and \((m, q) = (72, 2556)\) in Pawluk-Kolc
et al (2006). Motivated by this, we here pursue another avenue and show, from an empirical-process viewpoint, that the BH procedure is asymptotically valid for the pairwise comparison problem when \( m \) tends to \( \infty \) at a rate controlled by the available sample size. This approach is perhaps best represented by the FDR works of Storey et al (2004) and Genovese and Wasserman (2004), and a key ingredient in proving BH’s asymptotic validity is the convergence of an underlying empirical process, which is \textit{assumed} to hold under some general weak dependence among the test statistics (Storey et al, 2004, p.193). In this respect, one main offering of our work is that we have shown the dependence among the two-sample t-statistics in (1.2) to be “weak enough” to warrant such convergence in the form of a uniform weak law of large numbers (Section 3), which is not immediately obvious and also naturally leads to results on the asymptotic control of the FDP; compare Storey et al (2004, Theorem 6). Moreover, an asymptotic treatment allows us the flexibility to do away with standard assumptions such as normality, variance homogeneity and a balanced design, which can be easily violated in applications.

Originally, Tukey (1962, 1991) argued that, for pairwise comparisons, the difference \( \mu_i - \mu_j \) for any pair \((i, j)\) can never be exactly zero in reality and at best be close to zero to many decimal points. As such a type 1 error of rejecting a true null hypothesis \( H_{ij} \) can never occur. Nevertheless, one can still make sense of the BH procedure by making a slight modification: Following the rejection of a hypothesis \( H_{ij} \), a declaration of the sign of the difference \( \mu_i - \mu_j \) must be made by the practitioner. If \( T_{ij} > 0 \), then \( \text{sgn}(\mu_i - \mu_j) \) will be declared as positive, and vice versa. Making a wrong sign declaration will constitute what is known as a \textit{directional error}. Following more recent works such as Benjamini and Hochberg (2000) and Benjamini and Yekutieli (2005), we acknowledge the possibility that some null hypotheses in (1.1) could in fact be true, and define a directional error more generally as follows:

\textbf{Definition 1.1.} A directional error for a hypothesis \( H_{ij} \) is made if, either \( \text{sgn}(\mu_i - \mu_j) \) is incorrectly declared after a rejection of \( H_{ij} \) when \( \mu_i - \mu_j \neq 0 \), or \( H_{ij} \) is rejected when \( \mu_i - \mu_j = 0 \).

Hence, a directional error is equivalent to a usual type 1 error for \( H_{ij} \) when \( \mu_i - \mu_j = 0 \). Under Definition 1.1, we formally define the \textit{directional false discovery proportion} (dFDP) as

\begin{equation}
\text{dFDP} = \frac{\#\{(i, j) : \text{a directional error is made for } (i, j)\}}{\#\{(i, j) : H_{ij} \text{ is rejected}\}}
\end{equation}

and the directional false discovery rate (dFDR) as the expectation thereof. To accommodate Tukey’s viewpoint, our main result (Theorem 2.1) is thus stated in terms of dFDR and dFDP control. To the best of our knowledge, asymptotic results on dFDR/dFDP control have not appeared elsewhere in the literature. While our techniques can certainly be employed to prove a similar theorem in terms of the original FDR, it is instructive to demonstrate how they can be adopted to prove a result for dFDR, using the pairwise comparison problem as a showcase.
1.1. Organization and notation. Section 2 states and discusses our main result, which is immediately followed by its proof that relies on a key uniform weak law of large numbers proved later in Section 3. Sections 4 conducts a simple numerical study to shed light on our main result, while 5 concludes with a discussion of open issues. To fix notation, we let \( n := \max_i n_i \). Throughout, \( C, c \) are positive constants whose values, unless otherwise specified, are understood to not depend on \((m, n)\), and may vary from place to place. The dependence of \( m = m_n \) on \( n \) is implicit.

The “big-O, little-o” notation is as usual: For two sequences of real numbers \( \{a_n\} \) and \( \{b_n\} \), \( a_n = O(b_n) \) means that \( |a_n/b_n| \) is bounded, and \( a_n = o(b_n) \) means that \( a_n/b_n \rightarrow 0 \) as \( m, n \rightarrow \infty \). The notation \( \text{Card}(A) \) denotes the cardinality of a set \( A \). \( \Phi(\cdot) \) is the cumulative distribution function of the standard normal and \( \bar{\Phi}(\cdot) := 1 - \Phi(\cdot) \); \( \phi(\cdot) \) is the standard normal density. \( \log_d \) means taking log for \( d \) times and for two functions \( f \) and \( g \), \( f(x) \sim g(x), x \rightarrow \infty \) means \( f(x)/g(x) \rightarrow 1 \) as \( x \) goes to infinity. For real numbers \( a, b, \in \mathbb{R} \), \( a \lor b \) and \( a \land b \) are shorthand for \( \max(a, b) \) and \( \min(a, b) \); if \( a \) and \( b \) are non-negative, \( a \lesssim b \) means there exists a constant \( C \) not depending on \((m, n)\) such that \( a \leq Cb \) holds. The Euclidean norm of a vector \( x \in \mathbb{R}^p \) is denoted as \( \|x\| \).

2. Main results

We first formally define the version of the BH procedure accompanied by sign declaration upon the rejection of a null hypotheses in (1.1).

**Definition 2.1** (Level-\( \alpha \) BH with sign declaration for pairwise comparisons).

(i) Obtain the two-sided p-values \( p_{|ij|} \) according to (1.2), (1.3) and (1.4), and sort them as the order statistics \( p(1) \leq p(2) \leq \ldots \leq p(q) \), where \( q = m(m - 1)/2 \).

(ii) Let \( \hat{k} = \max\{k : p(k) \leq k\alpha/q\} \), and reject \( H_{ij} \) whenever \( p_{|ij|} \leq p(\hat{k}) \). No rejection will be made if \( \hat{k} \) cannot be properly defined.

(iii) For each rejected \( H_{ij} \), declare \( \text{sgn}(\mu_i - \mu_j) > 0 \) if \( T_{ij} > 0 \), and declare \( \text{sgn}(\mu_i - \mu_j) < 0 \) if \( T_{ij} < 0 \).

Note that the \( F_{ij} \)'s for forming the p-values do not have to be the exact null distribution functions of the \( T_{ij} \)'s. Our main result, Theorem 2.1 below, establishes the validity of the above BH procedure under the following assumptions:

**Assumption 1** (Approximate balance and homogeneity). There exist \( 0 < c_L \leq 1 \leq c_U \) such that for all \( 1 \leq i \neq j \leq m \),

\[ c_L \leq \sigma_i^2/\sigma_j^2, \quad n_i/n_j \leq c_U. \]

**Assumption 2** (Asymptotic regime and moments). Suppose that for some constants \( r, C, \nu, K > 0 \),

\[ m \leq Cn^r \quad \text{and} \quad \max_{1 \leq i \leq m} \mathbb{E}|(X_{1i} - \mu_i)/\sigma_i|^{2r+4r+\nu} \leq K. \]
Assumption 3 (Uniform Cramér-type moderate deviations of the reference null distributions). For all \( i < j \), \( F_{ij}'s \) are monotone increasing distribution functions, with the symmetric properties that 
\[
\max_{1 \leq i < j \leq m} \sup_{0 \leq t \leq 2\sqrt{\log m}} \left| \frac{1 - F_{ij}(t)}{\Phi(t)} - 1 \right| = O(n^{-c}) \text{ as } m, n \to \infty
\]
for some \( c > 0 \), where the possible dependence of the distribution functions \( F_{ij} \) on 
\((m, n_1, \ldots, n_m)\) is suppressed for notational simplicity.

These are very modest assumptions. Assumption 1 allows for an imbalanced design and variance non-homogeneity in a controlled manner, while Assumption 2 allows the growth of \( m \) in \( n \) to be of a polynomial order that depends on the variables’ moments. In particular, \( 4r + \nu \) can in fact be less than 1, despite the commensurate limited growth rate of \( m \) in that case. In Assumption 3, the symmetry property of a reference null distribution functions \( F_{ij} \) covers the usual normal and Student’s \( t \) distributions. The same assumption also requires them to be approximately normal in a uniform way. This later condition trivially holds when all \( F_{ij}'s \) are taken to be \( \Phi(\cdot) \). By the Cramér-type moderate deviation for \( t \)-statistics (Jing et al, 2003), it will also hold when the \( F_{ij}'s \) are taken as \( t \)-distribution functions with, say, \( \min(n_i, n_j) - 1 \) degrees of freedom (Scheffé, 1943), under the approximately balanced design and asymptotic regime specified in Assumptions 1 and 2.

**Theorem 2.1** (Asymptotic FDR and FDP control for pairwise comparisons). Let 
\( \mathcal{H}_0 := \{(i, j) : i < j, \mu_i - \mu_j = 0\} \) be the set of null hypothesis indices and define 
\( q_0 := |\mathcal{H}_0| \). Under Assumptions 1 to 3 and the condition
\[
\text{Card} \left\{ (i, j) : 1 \leq i < j \leq m, \frac{|\mu_i - \mu_j|}{\sqrt{\sigma_i^2/n_i + \sigma_j^2/n_j}} \geq 8c_U \sqrt{\log m} \right\} \geq 1,
\]
the testing procedure in Definition 2.1 has the properties that
\[
\limsup_{m,n} \frac{d\text{FDR}}{\alpha/2} \leq 1
\]
and for any \( \epsilon > 0 \),
\[
P \left( d\text{FDP} < \alpha/2 \left( 1 + \frac{q_0}{q} \right) + \epsilon \right) \longrightarrow 1
\]
as \( m, n \to \infty \), where the \( d\text{FDR} \) and \( d\text{FDP} \) are defined as in Section 7.

The \( d\text{FDR} \) bound of the form \( \frac{\alpha}{2}(1 + \frac{q_0}{q}) \) has appeared in the exact results established by Benjamini and Yekutieli (2005, Corollary 3 and Corollary 6) for test statistics that are independent or positively dependent. (More precisely, their bound takes the form \( \frac{\alpha}{2}(1 + \frac{N_0}{N}) \) where \( N \) and \( N_0 \) respectively denote the number of hypotheses and the number of true nulls in a given multiple testing problem.) The term \( \frac{\alpha}{2}(1 + \frac{q_0}{q}) \) becomes \( \alpha/2 \) when no hypotheses are true \((q_0 = 0)\). The latter is intuitive because in an extremely error-prone situation with all differences \( \mu_i - \mu_j \)
being very close to but not exactly zero, every \( T_{ij} \) stochastically behaves almost as
if $\mu_i - \mu_j = 0$, in which case upon rejecting $H_{ij}$, there is an approximate 1/2 chance of making false declaration about $\text{sgn}(\mu_i - \mu_j)$. In numerical studies where none of the $\mu_i$’s are set to be equal, Williams et al. (1999) showed that the dFDR can be controlled at $\alpha/2$ by the BH procedure in Definition 2.1 at level $\alpha$.

Lastly, (2.1) is imposed for the probabilistic control of the dFDP in (2.3). In particular only one $(i, j)$ pair is required to give a more prominent signal (which will, however, necessarily imply that $O(m)$ many other pairs will also give similarly prominent signals; see (2.5)). Ultimately, this is to ensure that with probability tending to 1, our BH procedure in Definition 2.1 will find a p-value cutoff that is not too close to zero; see displays (2.5) and (2.7). The existence of such a non-zero cutoff is actually assumed in the main asymptotic theorem of Storey et al. (2004, Theorem 4), which represents the line of approach our current work is based on. In fact, Liu and Shao (2014, Proposition 2.1) has shown that even in the most ideal multiple testing setting with independent and exact p-values, the BH procedure cannot control the FDP with an overwhelming probability as the number of tested hypotheses increases if the number of non-null hypotheses does not grow in tandem; hence, (2.1) is a kind of near necessary condition to guarantee asymptotic dFDP/FDP-type control. In a sense, our result complements a result of Yekutieli (2008a, Corollary 3.5) which says that, under some conditions, the BH procedure is valid for testing pairwise comparisons when the complete null hypothesis is true, i.e. $\mu_i = \mu_j$ for all pairs $i < j$.

We now give the main part of the proof for Theorem 2.1; the weak convergence of a key empirical process, which underpins this proof, will be treated separately in Section 3.

2.1. Proof of Theorem 2.1. It suffices to show (2.3), since it implies that

$$\limsup_{m,n} \frac{\mathbb{E}[dFDP]}{\frac{q}{2}(1 + \frac{q_0}{q})} \leq 1 + \epsilon$$

for any $\epsilon > 0$, and the arbitrariness of $\epsilon$ will give (2.2). The following notation will be used throughout this section and the next: Let $\mathcal{H}_+ := \{(i, j) : i < j, \mu_i - \mu_j > 0\}$ and $\mathcal{H}_- := \{(i, j) : i < j, \mu_i - \mu_j < 0\}$. Define $q_+ := |\mathcal{H}_+|$ and $q_- = |\mathcal{H}_-|$, and hence $q = q_+ + q_- + q_0$. For notational brevity, we will also use $\sum_{\mathcal{H}_+}$ to denote a summation over all pairs $(i, j)$ in the set $\mathcal{H}_+$, and use $\sum_{\mathcal{H}_-}, \sum_{\mathcal{H}_0}, \sum_{\mathcal{H}_+ \cup \mathcal{H}_0}$ and $\sum_{\mathcal{H}_- \cup \mathcal{H}_0}$ similarly. Finally, for each pair $i \neq j$, we define the centered two-sample t-statistic

$$T_{ij} = \frac{(\bar{X}_i - \bar{X}_j) - (\mu_i - \mu_j)}{\sqrt{\hat{\sigma}^2/n_i + \hat{\sigma}^2/n_j}}$$

Note that the rejection rule in Definition 2.1 (ii) is equivalent to the classical BH procedure, so by Theorem 2 in Benjamini and Hochberg (1995) it is equivalent to a procedure that rejects $H_{ij}$ whenever $|p_{ij}| < \hat{\alpha}$, where

$$\hat{\alpha} = \sup \left\{ 0 \leq \hat{\alpha} \leq 1 : \hat{\alpha} \leq \frac{\alpha}{q} \sum_{i<j} 1_{(p_{ij} \leq \hat{\alpha})} \right\}.$$
Since, with probability one, \( \frac{2}{q} \sum_{i<j} 1_{(p_{ij} \leq \hat{\alpha})} \) is right continuous as a function in \( \hat{\alpha} \) on the interval \([0, 1]\), by elementary arguments it can be shown that
\[
(2.5) \quad \hat{\alpha} = \frac{\alpha}{q} \sum_{i<j} 1_{(p_{ij} \leq \hat{\alpha})}.
\]

If, for every \( \epsilon > 0 \), we can show that, as \( m, n \to \infty \),
\[
(2.6) \quad P\left( \sup_{2\Phi(\sqrt{2\log m}) \leq \hat{\alpha} \leq 1} \frac{\sum_{H_+} 1_{(p_{ij} \leq \hat{\alpha}/2)} + \sum_{H_-} 1_{(p_{ij} \leq \hat{\alpha}/2)} + \sum_{H_0} 1_{(p_{ij} \leq \hat{\alpha})}}{\hat{\alpha}(q + q_0)/2} \leq 1 + \epsilon \right) \to_p 1
\]
and
\[
(2.7) \quad P(2\Phi(\sqrt{2\log m}) \leq \hat{\alpha} \leq 1) \to_p 1,
\]
then (2.3) is proven since, by (2.5), the expression
\[
\frac{2q}{\alpha(q + q_0)}dFDP = \frac{2q|\sum_{H_+} 1_{(p_{ij} \leq \hat{\alpha}/2)} + \sum_{H_-} 1_{(p_{ij} \leq \hat{\alpha}/2)} + \sum_{H_0} 1_{(p_{ij} \leq \hat{\alpha})}|}{\alpha(q + q_0)} \leq 1 + \epsilon
\]
is less than \( 1 + \epsilon \) with probability tending to one. (2.6) essentially amounts to proving a uniform law of large number, which is deferred to Section 3. We will focus on showing the probabilistic bound for the cutoff point \( \hat{\alpha} \) in (2.7) for the rest of this section.

We claim that, under Assumption [1] there exists a subset \( H_s \subset \{(i,j): 1 \leq i < j \leq m\} \) such that,
\[
(2.8) \quad \text{Card}\{H_s\} = \lfloor m/2 \rfloor \quad \text{and} \quad \frac{|\mu_i - \mu_j|}{\sqrt{\sigma_i^2/n_i + \sigma_j^2/n_j}} \geq 4\sqrt{\log m} \quad \text{for each} \quad (i,j) \in H_s.
\]

By (2.1), without loss of generality we can assume
\[
(2.9) \quad \frac{|\mu_1 - \mu_2|}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}} \geq 8c_U\sqrt{\log m}.
\]

There can only be two cases: If Card\{\( j: 3 \leq j \leq m, \frac{|\mu_1 - \mu_2|}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}} \geq 4\sqrt{\log m} \} \geq m/2 \), then (2.8) is satisfied; otherwise, we have
\[
(2.10) \quad \text{Card}\left\{ j: 3 \leq j \leq m, \frac{|\mu_1 - \mu_2|}{\sqrt{\sigma_1^2/n_1 + \sigma_j^2/n_j}} < 4\sqrt{\log m} \right\} \geq m/2.
\]

Since \( |\mu_2 - \mu_j| \geq |\mu_2 - \mu_1| - |\mu_1 - \mu_j| \), by Assumption [1]
\[
\frac{|\mu_2 - \mu_j|}{\sqrt{\sigma_2^2/n_2 + \sigma_j^2/n_j}} \geq \frac{|\mu_2 - \mu_1|}{\sqrt{\sigma_2^2/n_2 + \sigma_1^2/n_1}} - c_U\frac{|\mu_j - \mu_1|}{\sqrt{\sigma_2^2/n_2 + \sigma_1^2/n_1}}.
\]

Recalling that \( c_U \geq 1 \), together with (2.9) and (2.10) we have
\[
\text{Card}\left\{ j: 3 \leq j \leq m, \frac{|\mu_2 - \mu_j|}{\sqrt{\sigma_2^2/n_2 + \sigma_j^2/n_j}} \geq 4\sqrt{\log m} \right\} \geq m/2
\]
and hence (2.8) also holds. The maximal inequality in the appendix (Lemma A.1)
also states that

\[
P \left( \max_{1 \leq i \leq m} \left| \frac{\hat{\sigma}^2_i}{\sigma^2_i} - 1 \right| > (\log m)^{-2} \right) = O(n^{-r-c}),
\]

for some \( c > 0 \), which, together with (2.8), gives (2.11)

\[
P \left( \min_{(i,j) \in H^s} \left( \frac{|\mu_i - \mu_j|}{\sqrt{\hat{\sigma}^2_i/n_i + \hat{\sigma}^2_j/n_j}} \geq 3.9\sqrt{\log m} \right) \geq \beta \right) = O(\log m)^{-1/2} \rightarrow 0.
\]

On the other hand, by the Cramér-type moderate deviation for two-sample t-
statistics (Lemma C.1), under Assumptions 1 and 2 we have

\[
\frac{P(T_{ij} \geq s) - \Phi(s)}{\Phi(s)} = 1 + O \left( \left( \frac{1 + s}{n^{1/2-1/(4r+2+\nu)}} \right)^{4r+2+\nu} \right)
\]

uniformly in \( s \in [0, o(n^{1/2-1/(4r+2+\nu)})] \) and \( 1 \leq i < j \leq m \). A union bound then
implies (2.13)

\[
P \left( \max_{(i,j) \in H^s} \left| \tilde{T}_{ij} \right| \geq \sqrt{2 \log m} \right) \leq \frac{m}{2} \Phi(\sqrt{2 \log m}) = O((\log m)^{-1/2}) \rightarrow 0
\]

using the fact that \( \Phi(x) \leq (2\pi)^{-1/2}x^{-1/2} \exp(-x^2/2) \) for all \( x > 0 \).

In view of (2.11) and (2.13), as well as \( 3.9 - \sqrt{2} > 2 \), we have that

\[
P \left( \sum_{1 \leq i < j \leq m} 1(\{T_{ij}\} \geq 2\sqrt{\log m}) \geq \frac{m}{2} \right) \rightarrow 1.
\]

By Assumption 3 \( |T_{ij}| \geq 2\sqrt{\log m} \) implies \( p_{i,j} = o(m^{-2}) \), so recognizing \( q = O(m^2) \)
we have

\[
P \left( \sum_{1 \leq i < j \leq m} 1(p_{i,j} \leq 1/q) \geq \frac{m}{2} \right) \rightarrow 1,
\]

which in turn gives (2.14)

\[
P \left( 2\Phi(\sqrt{2 \log m}) \leq \frac{\alpha}{q} \sum_{1 \leq i < j \leq m} 1(p_{i,j} \leq 2\Phi(\sqrt{2 \log m})) \right) \rightarrow 1
\]

since \( 2\Phi(\sqrt{2 \log m}) = O((\log m)^{-1/2}m^{-1}). \) By the definition of \( \hat{\alpha} \) this implies (2.7).

3. PROOF OF THE UNIFORM LAW OF LARGE NUMBERS

In this section, we will establish the following weak convergence of a key empirical
process, which underpins our proof in the prior section:
Lemma 3.1 (Uniform weak law of large numbers). Suppose Assumptions 7 to 10 hold. Then

\[ \sup_{0 \leq t \leq \sqrt{2 \log m}} \left| \sum_{(i,j) \in H_+ \cup H_0} \mathbf{1}_{\Phi^{-1}(\bar{p}_{ij,L}) \geq t} + \sum_{(i,j) \in H_- \cup H_0} \mathbf{1}_{\Phi^{-1}(\bar{p}_{ij,U}) \geq t} \right| \Phi(t)(q + q_0) \rightarrow p 0, \]

where \( \bar{p}_{ij,L} := F_{ij}(T_{ij}) \) and \( \bar{p}_{ij,U} := 1 - F_{ij}(T_{ij}) \) are respectively one-sided lower and upper tailed p-values computed from the centered t-statistics in (2.4).

Note that this lemma automatically leads to (2.6). This can be easily seen to be true by observing that for each \((i,j)\), the events \( \{p_{ij,L} \leq \hat{\alpha}/2\} \), \( \{p_{ij,U} \leq \hat{\alpha}/2\} \) and \( \{p_{|ij|} \leq \hat{\alpha}\} \) are identical to \( \{\Phi^{-1}(p_{ij,L}) \geq \Phi^{-1}(\hat{\alpha}/2)\} \), \( \{\Phi^{-1}(p_{ij,U}) \geq \Phi^{-1}(\hat{\alpha}/2)\} \) and \( \{\Phi^{-1}(p_{|ij|}) \geq \Phi^{-1}(\hat{\alpha})\} \) respectively, and that \( \bar{p}_{ij,L} \leq p_{ij,L} \) for \((i,j) \in H_+ \cup H_0\), \( \bar{p}_{ij,U} \leq p_{ij,U} \) for \((i,j) \in H_- \cup H_0\).

Our proof of Lemma 3.1 essentially shows that for every threshold \( t \), the empirical process in question can be bounded by terms that converge to zero using Chebyshev’s inequality. This strategy has also been adopted previously by Liu and Shao (2014) to prove the FDP controlling property of the BH procedure under weak dependence among one-sample t-statistics. In the current context of pairwise comparisons with two-sample t-statistics, the key observation making this strategy possible is that, on rewriting the difference bounded in display (3.1) as

\[ \frac{1}{(q + q_0)^2} \sum_{(i,j) \in H_0 \cup H_+ \cup H_- \{\{i', j'\} \in \{i,j\} \cap \{i', j'\} \}} \left( \frac{P(\Phi^{-1}(\bar{p}_{ij,L}) \geq t)}{\Phi(t)} - 1 \right) \left( \frac{P(\Phi^{-1}(\bar{p}_{ij,U}) \geq t)}{\Phi(t)} - 1 \right), \]

upon expansion, the highest-order terms of its second moment are seen to involve pairs of index duples that do not overlap. For example, one such term is

\[ \frac{1}{(q + q_0)^2} \sum_{(i,j) \in H_0 \cup H_+ \cup H_- \{\{i', j'\} \in \{i,j\} \cap \{i', j'\} \}} \left( \frac{P(\Phi^{-1}(\bar{p}_{ij,L}) \geq t)}{\Phi(t)} - 1 \right) \left( \frac{P(\Phi^{-1}(\bar{p}_{ij,U}) \geq t)}{\Phi(t)} - 1 \right). \]

Here, each summand takes the product form \( \left( \frac{P(\Phi^{-1}(\bar{p}_{ij,L}) \geq t)}{\Phi(t)} - 1 \right) \left( \frac{P(\Phi^{-1}(\bar{p}_{ij,U}) \geq t)}{\Phi(t)} - 1 \right) \) due to the independence between \( \bar{p}_{ij,L} \) and \( \bar{p}_{ij,U} \) for \(|\{i, j\} \cap \{i', j'\}| = 0\). Since the number of summands, which equals \( \binom{m}{2} \left( \frac{m - 2}{2} \right) \), and the divisor \((q + q_0)^2\) are both of order \( O(m^4) \), the whole term should approach zero, since \( \frac{P(\Phi^{-1}(\bar{p}_{ij,L}) \geq t)}{\Phi(t)} - 1 \) can be expected to vanish in a uniform manner. The ensuing proof formalizes this.

3.1. Proof of Lemma 3.1 Let 0 = t_0 < t_1 < ... < t_g = \sqrt{2 \log m} be such that \( t_l - t_{l-1} = v_m \) for \( 1 \leq l \leq g - 1 \) and \( t_g - t_{g-1} \leq v_m \), where

\[ v_m = (\sqrt{2 \log m \log_4 m})^{-1} = (t_g \log_4 m)^{-1}. \]
By the mean value theorem, for some $c_t \in (t_{l-1}, t_l)$,

\begin{equation}
\frac{\Phi(t_l)}{\Phi(t_{l-1})} = \frac{\Phi(t_{l-1}) - v_m \phi(c_t)}{\Phi(t_{l-1})} = 1 - \frac{v_m \phi(c_t)}{\Phi(t_{l-1})} = 1 + o(1).
\end{equation}

(3.2)

Since

\[
0 < \frac{v_m \phi(c_t)}{\Phi(t_{l-1})} \leq \frac{v_m \phi(t_{l-1})}{\Phi(t_{l-1})} = (\log m)^{-1} \frac{\phi(t_{l-1})}{t_g \Phi(t_{l-1})}
\]

and that, whenever, $m \geq 2$

\[
\left| \frac{\phi(t)}{t_g \Phi(t)} \right| \leq C \text{ for some universal constant } C \text{ and all } t \in [0, t_g]
\]

(the latter fact comes from the well-known fact that $\frac{\phi(t)}{t \Phi(t)} \sim 1$ as $t \to \infty$; see Victor et al (2009, p.113) for instance), we have that

\begin{equation}
\max_{1 \leq i \leq g} \left| \frac{\Phi(t_l)}{\Phi(t_{l-1})} - 1 \right| \to 0 \text{ as } m, n \to \infty.
\end{equation}

(3.3)

For any $t \in [t_{l-1}, t_l]$, $l = 1, \ldots, g$, note that

\begin{equation}
\frac{1}{\Phi(t_l)} \leq \frac{1}{\Phi(t_{l-1})} \leq \frac{1}{\Phi(t)} \quad \text{and} \quad \frac{1}{\Phi(t_{l-1})} \leq \frac{1}{\Phi(t)} \leq \frac{1}{\Phi(t_l)}.
\end{equation}

(3.4)

Combining (3.3) and (3.4) implies that, for proving Lemma 3.1, it suffices to show

\begin{equation}
\max_{0 \leq i \leq g} \left| \frac{\sum_{(i,j) \in H_+ \cup H_0} 1(\phi^{-1}(p_{ij,L}) \geq t_i) + \sum_{(i,j) \in H_- \cup H_0} 1(\phi^{-1}(p_{ij,U}) \geq t_i)}{\Phi(t_l)(q + q_0)} - 1 \right| \to_p 0;
\end{equation}

(3.5)

to this end, for a given $\epsilon > 0$, we will bound the probability

\[
P \left( \max_{0 \leq i \leq g} \left| \frac{\sum_{(i,j) \in H_+ \cup H_0} 1(\phi^{-1}(p_{ij,L}) \geq t_i) + \sum_{(i,j) \in H_- \cup H_0} 1(\phi^{-1}(p_{ij,U}) \geq t_i)}{\Phi(t_l)(q + q_0)} - 1 \right| > \epsilon \right)
\]

for the rest of this section.

First, note that

\begin{equation}
P \left( \max_{0 \leq i \leq g} \left| \frac{\sum_{H_+ \cup H_0} 1(\phi^{-1}(p_{ij,L}) \geq t_i) + \sum_{H_- \cup H_0} 1(\phi^{-1}(p_{ij,U}) \geq t_i)}{\Phi(t_l)(q + q_0)} - 1 \right| > \epsilon \right) \leq \\
\sum_{0 \leq i \leq g} P \left( \frac{\sum_{H_+ \cup H_0} 1(\phi^{-1}(p_{ij,L}) \geq t_i) + \sum_{H_- \cup H_0} 1(\phi^{-1}(p_{ij,U}) \geq t_i)}{\Phi(t_l)(q + q_0)} > 1 + \epsilon \right) + \\
\sum_{0 \leq i \leq g} P \left( \frac{\sum_{H_+ \cup H_0} 1(\phi^{-1}(p_{ij,L}) \geq t_i) + \sum_{H_- \cup H_0} 1(\phi^{-1}(p_{ij,U}) \geq t_i)}{\Phi(t_l)(q + q_0)} < 1 - \epsilon \right).
\end{equation}

(3.6)
For any \( l = 1, \ldots, g \), from (3.4) we know that
\[
\frac{1(\bar{\Phi}(\bar{p}_{ij,l}) \geq t_l)}{\Phi(t_l)} \leq \frac{1(\bar{\Phi}(\bar{p}_{ij,l}) \geq t_l)}{\Phi(t_l)} \Phi(t_l) - \Phi(t_l-1) \Phi(t_l) + t_l \Phi(t_l-1), \text{ for any } t \in [t_l, t_l-1].
\]

From this and (3.3), we can deduce that, for a large enough \( n \) (and \( m \)) and all \( l = 1, \ldots, g \),
\[
\frac{1(\bar{\Phi}(\bar{p}_{ij,l}) \geq t_l)}{\Phi(t_l)} \leq \frac{1(\bar{\Phi}(\bar{p}_{ij,l}) \geq t_l)}{\Phi(t_l)} \left(1 + \frac{\epsilon}{2 + \epsilon}\right) \text{ and }\]
\[
\frac{1(\bar{\Phi}(\bar{p}_{ij,l}) \geq t_l)}{\Phi(t_l)} \leq \frac{1(\bar{\Phi}(\bar{p}_{ij,l}) \geq t_l)}{\Phi(t_l)} \left(1 + \frac{\epsilon}{2 + \epsilon}\right) \text{ for any } t \in [t_l-1, t_l],
\]

which implies that
\[
(3.7) \quad \sum_{1 \leq l \leq g} P \left( \frac{\sum_{H \cup H_0} 1(\bar{\Phi}^{-1}(\bar{p}_{ij,l}) \geq t_l) + \sum_{H \cup H_0} 1(\bar{\Phi}^{-1}(\bar{p}_{ij,l}) \geq t_l)}{\Phi(t_l)(t + q_0)} > 1 + \epsilon \right) \leq v_m^{-1} \int_0^{t_g} \left( \frac{\sum_{H \cup H_0} 1(\bar{\Phi}^{-1}(\bar{p}_{ij,l}) \geq t_l) + \sum_{H \cup H_0} 1(\bar{\Phi}^{-1}(\bar{p}_{ij,l}) \geq t_l)}{\Phi(t_l)(t + q_0)} > 1 + \epsilon/2 \right) dt,
\]
for large enough \( n \) (and \( m \)).

With a completely analogous argument, one can then deduce that
\[
(3.8) \quad \sum_{0 \leq l \leq g-1} P \left( \frac{\sum_{H \cup H_0} 1(\bar{\Phi}^{-1}(\bar{p}_{ij,l}) \geq t_l) + \sum_{H \cup H_0} 1(\bar{\Phi}^{-1}(\bar{p}_{ij,l}) \geq t_l)}{\Phi(t_l)(t + q_0)} < 1 - \epsilon \right) \leq v_m^{-1} \int_0^{t_g} \left( \frac{\sum_{H \cup H_0} 1(\bar{\Phi}^{-1}(\bar{p}_{ij,l}) \geq t_l) + \sum_{H \cup H_0} 1(\bar{\Phi}^{-1}(\bar{p}_{ij,l}) \geq t_l)}{\Phi(t_l)(t + q_0)} < 1 - \epsilon/2 \right) dt,
\]
for large enough \( n \) (and \( m \)).

Combining (3.6), (3.7) and (3.8),
\[
(3.9) \quad P \left( \max_{0 \leq l \leq g} \left| \frac{\sum_{H \cup H_0} 1(\bar{\Phi}^{-1}(\bar{p}_{ij,l}) \geq t_l) + \sum_{H \cup H_0} 1(\bar{\Phi}^{-1}(\bar{p}_{ij,l}) \geq t_l)}{\Phi(t_l)(t + q_0)} - 1 \right| > \epsilon \right) \leq v_m^{-1} \int_0^{t_g} \left( \frac{\sum_{H \cup H_0} 1(\bar{\Phi}^{-1}(\bar{p}_{ij,l}) \geq t_l) + \sum_{H \cup H_0} 1(\bar{\Phi}^{-1}(\bar{p}_{ij,l}) \geq t_l)}{\Phi(t_l)(t + q_0)} - 1 \right| > \epsilon/2 \right) dt + \sum_{l=0,g} P \left( \frac{\sum_{H \cup H_0} 1(\bar{\Phi}^{-1}(\bar{p}_{ij,l}) \geq t_l) + \sum_{H \cup H_0} 1(\bar{\Phi}^{-1}(\bar{p}_{ij,l}) \geq t_l)}{\Phi(t_l)(t + q_0)} - 1 \right| > \epsilon \right),
\]
for large enough \( n \) (and \( m \)).
In fact for any \( t \in [0, \sqrt{2 \log m}] \), for large enough \( n \) the probabilities of the type in (3.9) above satisfy

\[
P \left( \left\| \frac{\sum_{H_+ \cup H_0} 1(\Phi^{-1}(\Phi^{-1}(\bar{p}_{ij}, L) \geq t) + \sum_{H_- \cup H_0} 1(\Phi^{-1}(\Phi^{-1}(\bar{p}_{ij}, U) \geq t))}{\Phi(t)(q + q_0)} - 1 \right\| > \epsilon \right) =
\]

\[
P \left( \left\| \sum_{H_+ \cup H_0} \left( 1(-\bar{T}_{ij} \geq t_{ij}) - \tilde{\Phi}(t) \right) + \sum_{H_- \cup H_0} \left( 1(\bar{T}_{ij} \geq t_{ij}) - \tilde{\Phi}(t) \right) \right\| > \tilde{\Phi}(t)(q + q_0) \epsilon \right)
\]

where

\[ t_{ij} = \Phi^{-1}(\Phi(t)(1 + \epsilon|\bar{T}_{ij}|)), \]

for a number \( \epsilon_{|\bar{T}_{ij}|} \) that depends on the absolute value \(|\bar{T}_{ij}|\) but has the property that

\[
|\epsilon_{|\bar{T}_{ij}|}| \leq \epsilon_n \text{ for all } (i, j) \text{ a.s., for some deterministic sequence } \epsilon_n = O(n^{-c}).
\]

The argument leading to (3.10) is a bit delicate and is deferred to Appendix B.1.

Continuing from (3.10), with Chebyshev’s inequality, we get that for \( t \in [0, \sqrt{2 \log m}] \) and sufficiently large \( n \),

\[
P \left( \left\| \frac{\sum_{H_+ \cup H_0} 1(\Phi^{-1}(\Phi^{-1}(\bar{p}_{ij}, L) \geq t) + \sum_{H_- \cup H_0} 1(\Phi^{-1}(\Phi^{-1}(\bar{p}_{ij}, U) \geq t))}{\Phi(t)(q + q_0)} - 1 \right\| > \epsilon \right) \leq \left( e\Phi(t)(q + q_0) \right)^{-2}
\]

\[
\times \sum_{(i, j) \in H_0 \cup H_0} \left\{ P(-\bar{T}_{ij} \geq t_{ij}, -\bar{T}_{i'j'} \leq t_{i'j'}) + \tilde{\Phi}(t)^2 - \tilde{\Phi}(t) P(-\bar{T}_{ij} \geq t_{ij}) + \tilde{\Phi}(t) P(-\bar{T}_{i'j'} \geq t_{i'j'}) \right\}
\]

\[
2 \sum_{\{(i, j) \in H_0 \cup H_0, (i', j') \in H_0 \cup H_0, |\{(i, j) \}| \cap |\{(i', j')\}| = \ell \}} \left\{ P(-\bar{T}_{ij} \geq t_{ij}, T_{i'j'} \geq t_{i'j'}) + \tilde{\Phi}(t)^2 - \tilde{\Phi}(t) P(-\bar{T}_{ij} \geq t_{ij}) + \tilde{\Phi}(t) P(T_{i'j'} \geq t_{i'j'}) \right\}
\]

\[
\sum_{\{(i, j) \in H_0 \cup H_0, (i', j') \in H_0 \cup H_0, |\{(i, j) \}| \cap |\{(i', j')\}| = \ell \}} \left\{ P(T_{ij} \geq t_{ij}, T_{i'j'} \geq t_{i'j'}) + \tilde{\Phi}(t)^2 - \tilde{\Phi}(t) P(T_{ij} \geq t_{ij}) + \tilde{\Phi}(t) P(T_{i'j'} \geq t_{i'j'}) \right\}
\]

Note that, for each of the three sums appearing on the right hand side, the combinations over pairs of index duples \( \{i, j\} \) and \( \{i', j'\} \) overlapping for 0, 1 and 2 elements respectively give rise to \( O(m^4), O(m^3) \) and \( O(m^2) \) many summands. To finish the proof we need the following lemma whose proof is given in Appendix B.2 where we define

\[
T_{ij}^* = \frac{(\bar{X}_i - \mu_i) - (\bar{X}_j - \mu_j)}{\sqrt{\sigma_i^2/n_i + \sigma_j^2/n_j}}.
\]

**Lemma 3.2.** Under our Assumptions [4] to [5] we have, for large enough \( n \),
By applying the lemma to the right hand side of (3.12), we continue to get that for $t \in [0, \sqrt{2}\log m]$ and sufficiently large $n$,

\[
P\left(\frac{\sum_{H_i H_0} 1_{(\Phi^{-1}(\hat{p}_{ij,k} t) > t)} + \sum_{H_i H_0} 1_{(\Phi^{-1}(\hat{p}_{ij,k} t) < t)} - 1}{\Phi(t)(q + q_0)} > \epsilon\right)
\]

\[
\leq \sum_{(i,j) \in H_0 \cup H_+} P(-T_{ij} \geq t_{ij}, \bar{T}_{ij} \geq t_{ij}) + \sum_{(i,j) \in H_0 \cup H_+} P(-T_{ij} \geq t_{ij}, -\bar{T}_{ij} \geq t_{ij}) + \sum_{(i,j) \in H_0 \cup H_+} P(T_{ij} \geq t_{ij}, \bar{T}_{ij} \geq t_{ij}) + \sum_{(i,j) \in H_0 \cup H_+} P(T_{ij} \geq t_{ij}, -\bar{T}_{ij} \geq t_{ij})
\]

\[
\leq C(1 + t)^{-2} \exp(-t^2/(1 + \delta)) + O(n^{-r-c}),
\]

uniformly both in $0 \leq t \leq \sqrt{2}\log m$ and in all pairs of dupes $(i_1, j_1), (i_2, j_2)$ such that $|\{i_1, j_1\} \cap \{i_2, j_2\}| = 1$, $i_1 \neq j_1$ and $i_2 \neq j_2$.

We now explain each of the annotated (in)equalities above:

(a) It results from the applications of Lemma 3.2(i) to the right hand side of (3.12), and recognizing that $q + q_0$ is of the same order as $m^2$, and...
(b) From the maximal inequality for the sample variances (Lemma \textup{A.1}), we have that with probability of at least $1 - O(n^{-r-c})$,  
\[
\sqrt{\frac{\sigma^2_i/n_i + \sigma^2_j/n_j}{\sigma^2_i/n_i + \sigma^2_j/n_j}} = \sqrt{1 - \frac{(\sigma^2_i - \sigma^2_j)/n_i + (\sigma^2_j - \sigma^2_i)/n_j}{\sigma^2_i/n_i + \sigma^2_j/n_j}} \geq \sqrt{1 - (\log m)^{-2}}
\]
for all $(i, j)$, which also implies that  
\[
P(-\bar{T}_{ij} \geq t_{ij}, \bar{T}_{ij} \geq t_{ij}^*) \leq P(-T_{ij}^* \geq (1 - (\log m)^{-2})t_{ij}, T_{ij}^* \geq (1 - (\log m)^{-2})t_{ij}^*) + O(n^{-r-c}).
\]
To save space we have only shown the summands satisfying $(i, j) \in \mathcal{H}_0 \cup \mathcal{H}_+$ and $(i', j') \in \mathcal{H}_0 \cup \mathcal{H}_-$, where the other summands are absorbed into $\lesssim$.

(c) This results from the application of Lemma \textup{3.2}(ii).

(d) We used the fact that $(1 + t)^{-1} \phi(t) \lesssim \Phi(t)$.

(e) Substituting $\sqrt{2 \log m}$ for $t$.

Collecting (3.9), (3.10) and (3.13), we have proved (3.5) under the asymptotic regime in Assumption \textup{2}.

4. Numerical studies

We conducted a numeric study to shed light on our main theoretical result (Theorem 2.1). For simplicity, we only consider balanced and homogenous setups where $n_1 = \cdots = n_m = n$ and $\sigma_1 = \cdots = \sigma_m$. We generate $m$ population means $\mu_1, \ldots, \mu_m$ such that the first $\mu_1, \ldots, \mu_{m_0}$ are set to be zero, with $m_0 \leq m$ picked in such a way that  
\[
m_0 = \max \left\{ m' : 1 \leq m' \leq m \text{ and } {m' \choose 2} \leq \beta {m \choose 2} \right\},
\]
for some $\beta \in [0, 1]$, and the other $\mu_{m_0+1}, \ldots, \mu_m$ are i.i.d. realizations of a mean-zero normal distribution  
\[
N(\text{mean} = 0, \text{sd} = \text{effect size}),
\]
where \textit{effect size} is a chosen value for the standard deviation, named as such since the larger \textit{effect size} is, the larger are the magnitudes of the pairwise differences $\mu_i - \mu_j$, $m_0+1 \leq i < j \leq m$; note that $\beta$ is roughly the same as $q_0/q$, the proportion of true nulls. Then for each $1 \leq i \leq m$, we generate $n$ i.i.d. data  
\[
X_{ki} = \mu_i + \epsilon_{ki}, \quad 1 \leq k \leq n,
\]
where $\epsilon_{ki}$ are independent $t$-distributed error terms with 12 degrees of freedom, and apply the BH procedure in Definition 2.1 at level $\alpha$ using $\Phi$ as the reference distribution function to calibrate the p-values. For a given set of $\mu_1, \ldots, \mu_m$, the experiment is repeated 500 times to empirically estimate the performance of the BH procedure. At $\alpha = 0.2$, the empirical probabilities of the dFDP meeting (i.e. being less than) the desired target $\frac{q}{n}(1 + \frac{q}{n})$ and the empirical dFDR’s are reported in Tables \textup{1} to \textup{6} for $\beta = 0, 0.25, 0.5$, as well as for different combinations of $(m, n, \textit{effect size})$; additional results for $\alpha = 0.1$ and $\alpha = 0.3$ are shown in Appendix \textup{D}.
Note that from the proof of Theorem 2.1 in Section 2, it is seen that the condition (2.1) is a kind of minimalist assumption to ensure that, asymptotically, the small two-sided p-values are prevalent enough so that the event

$2\Phi(\sqrt{2 \log m}) \leq \frac{\alpha \sum_{1 \leq i < j \leq m} 1_{(p_{ij} \leq 2\Phi(\sqrt{2 \log m}))}}{q}$

happens with an overwhelming probability so that the p-value cutoff $\hat{\alpha}$ is not too small; revisit (2.7) and (2.14). For finite $m$ and $n$, the event in (4.1) is only true when there are enough small p-values, so one may generally expect better dFDP control in a signal rich environment. This is borne out by Tables 1, 3 and 5, where it is seen that as $\beta$ increases, i.e. more true nulls are present in the system, the empirical probabilities $P(dFDP \leq \frac{\alpha}{2} (1 + q_0/q))$ become progressively lower than 1. Moreover, for each of Tables 1, 3 and 5, the larger the effect size, the larger are these probabilities generally tending to become, which also provide further evidence for the near necessity of a condition like (2.1). The empirical probabilities in Table 5 for effect size $= 0.05$ violate the latter trend somewhat, as they tend to be larger than the same numbers for effect size $= 0.25$ when $m$ or $n$ are smaller. However, since (2.3) in Theorem 2.1 is an asymptotic statement for large $m, n$, numerical results for finite $m, n$ should be interpreted with caution.

Lastly, except when $n$ is too small compared to $m$, the dFDR targets are met as seen in Tables 2, 4 and 6.

5. Discussion

In summary, our work has established the validity of the BH procedure for pairwise comparisons in a flexible asymptotic framework suitable for a relatively large number of groups which does not require a strictly balanced design and variance homogeneity (Assumptions 1), by demonstrating that the dependence among all the two-sample t-statistics is weak enough to induce a requisite uniform weak law of large numbers. Using the Cramér-type moderate deviation as a tool, our result is present under minimal moment assumptions, and hence the growth rate of the number of groups is polynomial in the sample size (Assumption 2), as opposed to exponentially in the sample size had sub-Gaussian tails been assumed like in the related work of Liu and Shao (2014) focusing on one-sample t-statistics. On a related note, while we have assumed that the p-values are calibrated with deterministic reference distributions such as the standard normal (Assumption 3), it is also possible to establish our results for calibration with bootstrap distributions, which has the potential to better the approximation accuracy (Delaigle et al., 2011; Fan et al., 2007). However, as demonstrated in the numerical studies of Liu and Shao (2014), for heavy-tailed situations, bootstrap calibration has limited advantage in practice; we decided not to pursue this embellishment for simplicity.

While the BH procedure serves as a benchmarking procedure, in recent years, other multiple-testing methods for controlling FDR-related quantities are in active development. It was brought to our attention by the Associate Editor that, to account for the dependence among the statistics for FDR testing, a recent series of
works by Fan et al. (2019), Zhou et al. (2018) and other references therein impose an *approximate* factor model on the covariance structure between the test statistics, i.e. the covariance matrix is assumed to be the sum of a low rank and an approximately diagonal *uniqueness* matrix. In a nutshell, their proposal is to subtract away the common factor structure among the statistics from the data prior to forming the p-values for the BH, with the goal of offering better false discovery control and improving the testing efficiency. The pair comparison (1.1) does have a natural factor structure: Let $X = (X_1, \ldots, X_m) = d (X_{k1}, \ldots, X_{km})^T$, i.e. a random vector having the same distribution as $(X_{k1}, \ldots, X_{km})^T$ whose corresponding mean is $\mu_X = (\mu_1, \ldots, \mu_m)^T$, and define the $(\binom{m}{2})$-vector

\begin{equation}
Y = (\underbrace{X_1 - X_2, \ldots, X_1 - X_m}_{m-1 \text{ times}}, \underbrace{X_2 - X_3, \ldots, X_2 - X_m}_{m-2 \text{ times}}, \ldots, \underbrace{X_{m-1} - X_m}_{1 \text{ times}})^T.
\end{equation}

One can then write the factor representation

\begin{equation}
Y = \mu_Y + Lf,
\end{equation}

where

\begin{equation*}
\mu_Y := (\underbrace{\mu_1 - \mu_2, \ldots, \mu_1 - \mu_m}_{m-1 \text{ times}}, \underbrace{\mu_2 - \mu_3, \ldots, \mu_2 - \mu_m}_{m-2 \text{ times}}, \ldots, \underbrace{\mu_{m-1} - \mu_m}_{1 \text{ times}})^T.
\end{equation*}

captures the mean differences in (1.1). $f := X - \mu_X$ serves as an *unobserved* mean-zero hidden factor with $m$ elements, and $L = (L_{ij,l})$ is a $(\binom{m}{2}) \times m$ loading matrix such that

\begin{equation*}
L_{ij,l} = \begin{cases} 
1 & \text{if } l = i \\
-1 & \text{if } l = j \\
0 & \text{if otherwise}
\end{cases}
\end{equation*}

for $1 \leq i < j \leq m$ and $1 \leq l \leq m$. However, in the oracle case where $L$ and $f$ are hypothetically assumed to be observed, there is conceptual difficulty of applying Fan et al. (2019), Zhou et al. (2018)’s approach to test the mean differences in $\mu_Y$, because upon subtracting away the factor structure, the residual $Y - Lf$ is a degenerate vector with no randomness to form the p-values, due to the lack of the usual idiosyncratic errors accounting for the uniqueness variances in the factor representation (5.2). In unreported numerical studies, we have also experimented with the practical version of their approach where the underlying factors and loadings are estimated from the data $Y$ (and hence not necessarily coinciding with $Lf$), but the resulting false discovery control is very unstable; this is most possibly due to the fact that the underlying theory of Fan et al. (2019, Assumption 1(iv)) still relies upon a non-degenerate uniqueness component in the covariance structure.

In this paper, we treated the classical setup with only a single one-way ANOVA model. By comparison, one typical modern application in genomics involves an experimental design where the measured expression levels of thousands, or even tens of thousands, of genes under several treatment groups are described by a multitude of one-way ANOVA models, each of which corresponds to a gene and has its own set of pairwise comparisons (Reiner-Benaim, 2007, Yekutieli, 2008a, Yekutieli et al. 2006). A natural question is whether our techniques can be used to
prove, in an asymptotic regime where the number of genes tends to infinity, that the BH procedure is still valid when applied to the pairwise comparisons across all the genes en masse. This will certainly come down to how dependent the expression measurements between different genes are (Reiner et al., 2003). One can also ask whether there are other more preferable procedures than the BH; in particular, hierarchical testing procedures have been proposed in the recent literature (Hassall and Mead, 2018; Yekutieli, 2008b). These are beyond the scope here and we will leave them for future research.
Table 1. Estimates of $P(dFDP \leq \frac{a}{2} (1 + q_0/q))$ for the BH procedure in Definition 2.1 at level $\alpha = 0.2$, based on 500 repetitions of data generated from the model in Section 4, for combinations of $(m, n)$ and $q_0/q = \beta = 0$.

| $m \setminus n$ | 20   | 40   | 100  | 200  | 400  | 600  | 20   | 40   | 100  | 200  | 400  | 600  |
|-----------------|------|------|------|------|------|------|------|------|------|------|------|------|
|                 | effect size = 0.05 | effect size = 0.25 | effect size = 0.05 | effect size = 0.25 | effect size = 0.05 | effect size = 0.25 |
| 15              | 0.86 | 0.88 | 0.90 | 0.91 | 0.92 | 0.91 | 0.88 | 0.94 | 0.99 | 1.00 | 1.00 | 1.00 |
| 30              | 0.85 | 0.87 | 0.91 | 0.91 | 0.93 | 0.98 | 0.96 | 0.99 | 1.00 | 1.00 | 1.00 | 1.00 |
| 50              | 0.78 | 0.87 | 0.90 | 0.91 | 0.95 | 0.99 | 0.96 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 70              | 0.75 | 0.85 | 0.88 | 0.93 | 0.97 | 1.00 | 0.97 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 90              | 0.71 | 0.83 | 0.87 | 0.90 | 0.98 | 0.99 | 0.97 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 120             | 0.71 | 0.83 | 0.89 | 0.92 | 0.98 | 1.00 | 0.99 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |

Table 2. Estimates of $dFDR$ for the BH procedure in Definition 2.1 at level $\alpha = 0.2$, based on 500 repetitions of data generated from the model in Section 4, for combinations of $(m, n)$ and $q_0/q = \beta = 0$; the dFDR target is $\frac{a}{2} (1 + q_0/q) = 0.1$.

| $m \setminus n$ | 20   | 40   | 100  | 200  | 400  | 600  | 20   | 40   | 100  | 200  | 400  | 600  |
|-----------------|------|------|------|------|------|------|------|------|------|------|------|------|
|                 | effect size = 0.05 | effect size = 0.25 | effect size = 0.05 | effect size = 0.25 | effect size = 0.05 | effect size = 0.25 |
| 15              | 0.97 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 0.99 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 30              | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 50              | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 70              | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 90              | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 120             | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
Table 3. Estimates of $P(dFDP \leq \frac{2}{d} (1 + q_0/q))$ for the BH procedure in Definition 2.1 at level $\alpha = 0.2$, based on 500 repetitions of data generated from the model in Section 4, for combinations of $(m, n)$ and $q_0/q \approx \beta = 0.25$.

| effect size = 0.05 | effect size = 0.25 |
|-------------------|-------------------|
| 15                | 0.84 0.85 0.88 0.89 0.86 0.86 | 0.81 0.83 0.84 0.90 0.95 0.95 |
| 30                | 0.82 0.84 0.87 0.85 0.81 0.82 | 0.77 0.83 0.95 0.98 0.98 0.99 |
| 50                | 0.76 0.85 0.88 0.84 0.82 0.82 | 0.72 0.81 0.93 0.97 1.00 1.00 |
| 70                | 0.73 0.83 0.85 0.86 0.81 0.79 | 0.66 0.80 0.92 0.97 0.99 1.00 |
| 90                | 0.69 0.81 0.84 0.85 0.83 0.82 | 0.63 0.78 0.94 0.99 1.00 1.00 |
| 120               | 0.67 0.81 0.86 0.86 0.80 0.85 | 0.67 0.85 0.99 1.00 1.00 1.00 |

| effect size = 0.45 | effect size = 0.65 |
|-------------------|-------------------|
| 15                | 0.76 0.84 0.93 0.96 0.97 0.98 | 0.83 0.88 0.94 0.97 0.99 0.99 |
| 30                | 0.88 0.94 0.98 0.99 0.99 1.00 | 0.94 0.97 0.99 0.99 1.00 1.00 |
| 50                | 0.84 0.94 0.99 1.00 1.00 1.00 | 0.95 0.97 0.99 1.00 1.00 1.00 |
| 70                | 0.83 0.95 0.99 0.99 1.00 1.00 | 0.93 0.99 1.00 0.99 1.00 1.00 |
| 90                | 0.87 0.95 0.99 1.00 1.00 1.00 | 0.95 0.99 1.00 1.00 1.00 1.00 |
| 120               | 0.92 0.99 1.00 1.00 1.00 1.00 | 0.98 1.00 1.00 1.00 1.00 1.00 |

Table 4. Estimates of $dFDR$ for the BH procedure in Definition 2.1 at level $\alpha = 0.2$, based on 500 repetitions of data generated from the model in Section 4, for combinations of $(m, n)$ and $q_0/q \approx \beta = 0.25$; the dFDR target is $\frac{2}{d} (1 + q_0/q) \approx 0.125$.

| effect size = 0.05 | effect size = 0.25 |
|-------------------|-------------------|
| 15                | 0.10 0.08 0.07 0.05 0.06 0.05 | 0.07 0.05 0.05 0.05 0.05 0.05 |
| 30                | 0.09 0.08 0.06 0.05 0.06 0.06 | 0.07 0.06 0.06 0.05 0.05 0.05 |
| 50                | 0.15 0.08 0.06 0.07 0.06 0.06 | 0.08 0.07 0.06 0.06 0.06 0.06 |
| 70                | 0.16 0.10 0.07 0.05 0.06 0.06 | 0.10 0.08 0.06 0.06 0.06 0.06 |
| 90                | 0.18 0.11 0.08 0.06 0.05 0.06 | 0.11 0.08 0.06 0.06 0.06 0.06 |
| 120               | 0.18 0.12 0.06 0.06 0.06 0.06 | 0.10 0.08 0.06 0.06 0.06 0.06 |

| effect size = 0.45 | effect size = 0.65 |
|-------------------|-------------------|
| 15                | 0.06 0.05 0.05 0.04 0.04 0.04 | 0.05 0.05 0.04 0.04 0.04 0.04 |
| 30                | 0.06 0.06 0.05 0.05 0.06 0.05 | 0.06 0.06 0.05 0.05 0.05 0.05 |
| 50                | 0.07 0.06 0.06 0.06 0.05 0.05 | 0.06 0.06 0.05 0.05 0.05 0.05 |
| 70                | 0.08 0.07 0.06 0.06 0.06 0.06 | 0.07 0.06 0.06 0.05 0.05 0.05 |
| 90                | 0.08 0.07 0.06 0.06 0.06 0.06 | 0.07 0.06 0.06 0.05 0.05 0.05 |
| 120               | 0.08 0.07 0.06 0.06 0.05 0.05 | 0.07 0.06 0.05 0.05 0.05 0.05 |
Table 5. Estimates of $P(dFDP \leq \frac{\alpha}{2} (1 + q_0/q))$ for the BH procedure in Definition 2.1 at level $\alpha = 0.2$, based on 500 repetitions of data are generated from the model in Section 4, for combinations of $(m, n)$ and $q_0/q \approx \beta = 0.5$.

| $m \setminus n$ | 20  | 40  | 100 | 200 | 400 | 600 | 20  | 40  | 100 | 200 | 400 | 600 |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\text{effect size} = 0.05$ | 0.83 | 0.84 | 0.88 | 0.87 | 0.84 | 0.85 | 0.82 | 0.80 | 0.76 | 0.72 | 0.76 | 0.73 |
| $\text{effect size} = 0.25$ | 0.81 | 0.85 | 0.86 | 0.85 | 0.80 | 0.77 | 0.75 | 0.71 | 0.73 | 0.78 | 0.78 | 0.80 |
| $\text{effect size} = 0.45$ | 0.73 | 0.84 | 0.86 | 0.84 | 0.80 | 0.79 | 0.67 | 0.66 | 0.72 | 0.77 | 0.78 | 0.82 |
| $\text{effect size} = 0.65$ | 0.72 | 0.81 | 0.84 | 0.86 | 0.78 | 0.76 | 0.59 | 0.65 | 0.71 | 0.79 | 0.80 | 0.83 |
| $\text{effect size} = 0.05$ | 0.67 | 0.80 | 0.80 | 0.84 | 0.81 | 0.75 | 0.55 | 0.64 | 0.73 | 0.84 | 0.89 | 0.91 |
| $\text{effect size} = 0.25$ | 0.65 | 0.80 | 0.86 | 0.83 | 0.81 | 0.78 | 0.54 | 0.63 | 0.80 | 0.88 | 0.92 | 0.95 |

Table 6. Estimates of $dFDR$ for the BH procedure in Definition 2.1 at level $\alpha = 0.2$, based on 500 repetitions of data generated from the model in Section 4, for combinations of $(m, n)$ and $q_0/q \approx \beta = 0.5$; the dFDR target is $\frac{\alpha}{2} (1 + q_0/q) \approx \beta = 0.15$.

| $m \setminus n$ | 20  | 40  | 100 | 200 | 400 | 600 | 20  | 40  | 100 | 200 | 400 | 600 |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\text{effect size} = 0.05$ | 0.13 | 0.12 | 0.08 | 0.09 | 0.10 | 0.08 | 0.10 | 0.10 | 0.08 | 0.09 | 0.08 | 0.09 |
| $\text{effect size} = 0.25$ | 0.14 | 0.11 | 0.09 | 0.09 | 0.09 | 0.10 | 0.11 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 |
| $\text{effect size} = 0.45$ | 0.19 | 0.12 | 0.10 | 0.10 | 0.10 | 0.09 | 0.15 | 0.12 | 0.11 | 0.11 | 0.10 | 0.10 |
| $\text{effect size} = 0.65$ | 0.21 | 0.15 | 0.11 | 0.08 | 0.10 | 0.09 | 0.17 | 0.12 | 0.11 | 0.11 | 0.11 | 0.11 |
| $\text{effect size} = 0.05$ | 0.24 | 0.15 | 0.13 | 0.08 | 0.08 | 0.08 | 0.16 | 0.12 | 0.11 | 0.10 | 0.10 | 0.10 |
| $\text{effect size} = 0.25$ | 0.24 | 0.14 | 0.08 | 0.09 | 0.07 | 0.08 | 0.17 | 0.13 | 0.11 | 0.10 | 0.10 | 0.10 |
| $\text{effect size} = 0.45$ | 0.14 | 0.11 | 0.11 | 0.10 | 0.10 | 0.10 | 0.12 | 0.11 | 0.10 | 0.10 | 0.10 | 0.10 |
| $\text{effect size} = 0.65$ | 0.13 | 0.11 | 0.10 | 0.10 | 0.10 | 0.10 | 0.12 | 0.11 | 0.10 | 0.10 | 0.10 | 0.10 |
APPENDIX A. MAXIMAL INEQUALITY OF THE SAMPLE VARIANCES

The following inequality has been used in the proofs of Sections 2 and 3.

Lemma A.1 (Maximal inequality of the sample variances). Under Assumption 2, we have

\[ P\left( \max_{1 \leq i \leq m} \left| \frac{\hat{\sigma}_i^2}{\sigma_i^2} - 1 \right| > (\log m)^{-2} \right) = O(n^{-r-c}) \text{ as } m, n \to \infty. \]

Proof. Note that for all \( 1 \leq i \leq m \),

\[ \frac{\hat{\sigma}_i^2}{\sigma_i^2} = \sum_{k=1}^{n} \frac{(X_{ki} - \bar{X}_i)^2}{(n-1)\sigma_i^2} = \sum_{k=1}^{n} \frac{(X_{ki} - \mu_i)^2}{(n-1)\sigma_i^2} - \frac{n(\bar{X}_i - \mu_i)^2}{(n-1)\sigma_i^2}, \]

which in turn gives

\[ \left| \frac{\hat{\sigma}_i^2}{\sigma_i^2} - 1 \right| \leq \frac{n(\bar{X}_i - \mu_i)^2}{(n-1)\sigma_i^2} + \frac{\sum_{k=1}^{n} (X_{ki} - \mu_i)^2 - \sigma_i^2}{(n-1)\sigma_i^2} + \frac{n}{n-1} - 1 = \frac{\sum_{k=1}^{n} (X_{ki} - \mu_i)^2}{\sigma_i \sqrt{n(n-1)}} \cdot \frac{1}{n-1} \left\{ \frac{n}{n-1} \sum_{k=1}^{n} \left( \frac{(X_{ki} - \mu_i)^2 - 1}{\sigma_i^2} \right) + 1 \right\}. \]

Hence, by a union bound, we get that

\[ P\left( \max_{1 \leq i \leq m} \left| \frac{\hat{\sigma}_i^2}{\sigma_i^2} - 1 \right| > (\log m)^{-2} \right) \leq P\left( \frac{1}{n-1} > \frac{(\log m)^{-2}}{3} \right) \]

\[ + P\left( \max_{1 \leq i \leq m} \left[ \frac{\sum_{k=1}^{n} (X_{ki} - \mu_i)^2}{\sigma_i} \right] > \sqrt{\frac{n(n-1)}{3}} \left( \log m \right)^{-1} \right) \]

\[ + P\left( \max_{1 \leq i \leq m} \left[ \sum_{k=1}^{n} \left( \frac{(X_{ki} - \mu_i)^2}{\sigma_i^2} - 1 \right) \right] > \frac{(\log m)^{-2}(n-1)}{3} \right). \]

Since \( m \leq Cn^r \) by Assumption 2, the first term on the right hand side in (A.1) is zero for large enough \( n \) (and \( m \)), and it suffices to show the last two terms are of order \( O(n^{-r-c}) \). The first term can be bounded as

\[ P\left( \max_{1 \leq i \leq m} \left[ \frac{\sum_{k=1}^{n} (X_{ki} - \mu_i)^2}{\sigma_i} \right] > \sqrt{\frac{n(n-1)}{3}} \left( \log m \right)^{-1} \right) \]

\[ \leq \sum_{i=1}^{m} P\left( \frac{\sum_{k=1}^{n} (X_{ki} - \mu_i)^2}{\sigma_i} > \sqrt{\frac{n(n-1)}{3}} \left( \log m \right)^{-1} \right) \]

\[ \leq m \frac{n^{1+2r+\nu/2}(\log m)^{2+4r+\nu}}{n^{2+4r+\nu}} = m \frac{(\log m)^{2+4r+\nu}}{n^{1+2r+\nu/2}} \leq n^{-r-c}, \]

for some \( c > 0 \), where the inequalities \( \leq \) follow from [Rosenthal 1970, Theorem 3)'s inequality and Assumption 2. For the second term, we have

\[ P\left( \max_{1 \leq i \leq m} \left[ \sum_{k=1}^{n} \left( \frac{(X_{ki} - \mu_i)^2}{\sigma_i^2} - 1 \right) \right] > \frac{(\log m)^{-2}(n-1)}{3} \right) \leq \]

\[ \sum_{i=1}^{m} P\left( \frac{\sum_{k=1}^{n} (X_{ki} - \mu_i)^2}{\sigma_i^2} > \frac{(\log m)^{-2}(n-1)}{3} \right). \]
For $1 + 2r + \nu/2 \leq 2$, by von Bahr and Esseen [1965, Theorem 2’s inequality, Exp 6.1] and Assumption 2 to give, for large enough $n$ and $r$

$$1 + 2r + \nu/2 \leq 2$$

Since

$$s = \text{log } a$$

can be bounded by a term of the order $O(m(\log m)^{4r+2+\nu} n^{-2r-\nu/2}) = O(n^{-r-c})$. If $1 + 2r + \nu/2 > 2$, then we can apply a Fuk-Nagaev type inequality \cite{Liu et al, 2008} [Lemma 6.1] and Assumption 2 to give, for large enough $n$,

$$\sum_{i=1}^{m} P \left( \sum_{k=1}^{n} \left( \frac{(X_{ki} - \mu_k)^2}{\sigma_k^2} - 1 \right) > \frac{n-1}{3(\log m)^2} \right)$$

$$\leq \sum_{i=1}^{m} \left\{ \sum_{k=1}^{n} P \left( \left| \frac{(X_{ki} - \mu_k)^2}{\sigma_k^2} - 1 \right| > \frac{c(n-1)}{3(\log m)^2} \right) + \exp \left( - \frac{n}{C(\log m)^4} \right) + Cn^{-\max(2r+\nu,2)} \right\}$$

$$\lesssim m \left( \frac{(\log m)^{2+4r+\nu}}{n^{2r+\nu/2}} + n^{-\max(2r+\nu,2)} \right) = O(n^{-r-c})$$

for some small enough $c > 0$ and large enough $C > 0$.

□

Appendix B. Further proofs for Section 3

We first introduce a lemma that will be used to finish the proofs.

**Lemma B.1.** Consider an asymptotic regime where $m, n \to \infty$. For any constant $c' > \sqrt{2}$ and a sequence $b_n = o(1)$, it must be true that

$$\Phi^{-1}(\Phi(t)(1 + b_n)) \leq c' \sqrt{\log m}$$

for all $t \in [0, \sqrt{2\log m}]$ and a sufficiently large $n$ (and $m$).

**Proof.** Let $t_{b_n} = \Phi^{-1}(\Phi(t)(1 + b_n))$. Since $t \leq \sqrt{2\log m}$, it must be always true that $t_{b_n} \leq a_m$ for a sufficiently large $n$, where $a_m = \Phi^{-1}(\Phi(\sqrt{2\log m})(1 + b_n))$. Since $s\Phi(s)/\phi(s) \to 1$ as $s \to \infty$, it must be that

$$\frac{\phi(a_m)\sqrt{2\log m}}{a_m\phi(\sqrt{2\log m})} = \frac{\Phi(a_m)\phi(a_m)}{\Phi(a_m)a_m} \times \frac{\sqrt{2\log m}}{\Phi(a_m)a_m} \times \frac{\phi(a_m)}{\Phi(a_m)a_m} \times (1 + b_n)$$

$$= \frac{\Phi(\sqrt{2\log m})\sqrt{2\log m}}{\phi(\sqrt{2\log m})} \times \frac{\phi(a_m)}{\Phi(a_m)a_m} \times (1 + b_n)$$

By taking log on both side of the preceding display, one immediately gets that

$$\frac{a_m^2}{2} = \log \left( \sqrt{2\log m} \right) - \log a_m + \frac{2\log m}{2} - \log \left( 1 + o(1) \right),$$

which, in light of the fact that $\log a_m \to \infty$ as $n, m \to \infty$, implies that

$$t_{b_n} \leq a_m \leq \sqrt{2\log m + 2\log(\sqrt{2\log m}) + o(1)} \leq c' \sqrt{\log m}$$

for a large enough $n$ (and hence $m$) and $c' > \sqrt{2}$. □
B.1. Proof of (3.10). We will show that for a sufficiently large $n$, the event equivalences

$$\bar{p}_{ij,L} \leq \Phi(t) \iff \bar{T}_{ij} \geq \Phi^{-1}(\Phi(t)(1 + \epsilon_{|T_{ij}|}))$$

and

$$\bar{p}_{ij,U} \leq \Phi(t) \iff \bar{T}_{ij} \geq \Phi^{-1}(\Phi(t)(1 + \epsilon_{|T_{ij}|})),$$

with $\epsilon_{|T_{ij}|}$ having the property in (3.11). To show this we first note the following two events are identical, by their definitions:

\[(B.1) \quad \{\bar{p}_{ij,L} \leq \Phi(t)\} = \{1 - F_{ij}(-\bar{T}_{ij}) \leq \Phi(t)\}.\]

Fix a constant $c' \in (\sqrt{2}, 2)$. For a sufficiently large $n$ the event in (B.1) can be realized for all $0 \leq t \leq \sqrt{2 \log m}$ if $-\bar{T}_{ij} \geq c' \sqrt{\log m}$ since, with Assumption 3\footnote{Assumption 3},

\[
1 - F_{ij}(-\bar{T}_{ij}) \leq 1 - F_{ij}(c' \sqrt{\log m}) = \Phi(c' \sqrt{\log m})(1 + o(1)) \leq \Phi(\sqrt{2 \log m}) \leq \Phi(t)
\]

when $m$ is large enough. But since the event in (B.1) may also be true for some $-\bar{T}_{ij} < c' \sqrt{\log m}$, using Assumption 3\footnote{Assumption 3} and the identity (B.1) again, we can conclude, conditioning on $-\bar{T}_{ij} < c' \sqrt{\log m}$, the equivalence of events

\[
\bar{p}_{ij,L} \leq \Phi(t) \iff \frac{1 - F_{ij}(-\bar{T}_{ij})}{\Phi(-\bar{T}_{ij})} \leq \Phi(t) \iff -\bar{T}_{ij} \geq \Phi^{-1}(\Phi(t)(1 + \epsilon_{|T_{ij}|})).
\]

for large enough $n$, where $\epsilon_{|T_{ij}|}$ has the property in (3.11). However, in light of Lemma 3.1\footnote{Lemma 3.1} for large enough $n$, $-\bar{T}_{ij} \geq c' \sqrt{\log m}$ will necessarily imply $-\bar{T}_{ij} \geq \Phi^{-1}(\Phi(t)(1 + \epsilon_{|T_{ij}|}))$, and hence the train of equivalence in the preceding display is also true without conditioning on $-\bar{T}_{ij} < c' \sqrt{\log m}$. By a completely analogous argument we also have the equivalence of events

\[
\bar{p}_{ij,U} \leq \Phi(t) \iff \bar{T}_{ij} \geq \Phi^{-1}(\Phi(t)(1 + \epsilon_{|T_{ij}|}))
\]

for large enough $n$.

B.2. Proof of Lemma 3.2. For $\epsilon_n$ is as in (3.11), we let

\[
t_U = \Phi^{-1}(\Phi(t)(1 - \epsilon_n)) \quad \text{and} \quad t_L = \Phi^{-1}(\Phi(t)(1 + \epsilon_n)),
\]

where the subscripts are suggestive of the fact that $t_L$ is always less than $t_U$. We also note that

\[
(B.2) \quad t_L \leq t_{ij} \leq t_U \quad \text{for all} \quad (i,j) \quad \text{and} \quad t \in [0, \sqrt{2 \log m}].
\]

Proof of Lemma 3.2(i). In fact, it suffices to show that for sufficiently large $m$,

\[
(B.3) \quad \frac{P(\bar{T}_{ij} \geq t_U)}{\Phi(t)} = 1 + O(n^{-\epsilon}),
\]

\[
(B.4) \quad \frac{P(\bar{T}_{ij} \geq t_L)}{\Phi(t)} = 1 + O(n^{-\epsilon}),
\]

both uniformly in $t \in [0, \sqrt{2 \log m}]$ and all $(i,j)$, in light of (B.2).
where \(\tilde{L}(\text{Lemma C.1})\), we have

\[
P(\bar{T}_{ij} \geq t_U) = (1 - \epsilon_n) \left(1 + O(1) \left(\frac{1 + t_U}{n^{1/2 - \nu/(2\nu + \nu)}}\right)^{4r + 2 + \nu}\right)
\]

uniformly in all \((i, j)\) and \(0 \leq t \leq \sqrt{2 \log m}\), for sufficiently large \(n\).

(B.3): Since \(t_L\) can be a negative number, Cramér-type moderate deviation results (Lemma C.1) is not directly applicable. However, we can do a separation argument. For any \(t \in [1, \sqrt{2 \log m}]\), for large enough \(n\) it must be that \(t_L \geq 0\), hence with the same argument for proving (B.3), we have

\[
P(\bar{T}_{ij} \geq t_L) = 1 + O(n^{-c}) \quad \text{uniformly in } t \in [1, \sqrt{2 \log m}] \text{ and all } (i, j).
\]

For any \(t \in [0, 1)\), we have

\[
P(\bar{T}_{ij} \geq t_L) = \frac{P(\bar{T}_{ij} \geq t) - P(\bar{T}_{ij} \geq t_L)}{\Phi(t)}
\]

(B.5)

\[
= 1 + O(n^{-c}) + \frac{P(\bar{T}_{ij} \geq t_L) - P(\bar{T}_{ij} \geq t)}{\Phi(t)}
\]

uniformly in \((i, j)\) and \(t \in [0, 1)\), where the last equality is based on Assumption \(2\) and the Cramér-type moderate deviation (Lemma C.1). But we also note that

\[
\left|\frac{P(\bar{T}_{ij} \geq t_L) - P(\bar{T}_{ij} \geq t)}{\Phi(t)}\right| \leq \frac{|P(\bar{T}_{ij} \geq t_L) - \Phi(t_L)|}{\Phi(t)} + \frac{|P(\bar{T}_{ij} \geq t) - \Phi(t)|}{\Phi(t)} + \frac{\left|\Phi(t) - \Phi(t_L)\right|}{\Phi(t)}.
\]

Given that \(\Phi(t) > C > 0\) for some \(C\) on interval \([0, 1)\), on the right-hand side of (B.6) the first two terms are of order \(O(n^{-c})\) by Lemma C.1 (compare (2.12)), and the last term is so due to a bounded first derivative of \(\Phi^{-1}\) on the interval \([\Phi(1), \Phi(-1)]\), all uniformly in \((i, j)\) and \(t \in [0, 1)\).

\(\square\)

**Proof of Lemma 3.2 (ii).** Given (B.2), it suffices to show, for sufficiently large \(n\),

\[
P(\bar{T}_{i,j_1}^* \geq \tilde{t}, \bar{T}_{i,j_2}^* \geq \tilde{t}) \leq C(1 + t)^{-2} \exp(-\tilde{t}^2/(1 + \delta))
\]

uniformly in \(0 \leq t \leq \sqrt{2 \log m}\) and \(|\{i, j_1\} \cap \{i_2, j_2\}| = 1\) for some constant \(\delta > 0\), where \(\tilde{t} := (1 - (\log m)^{-2})t_L\).

We first recognize that for any fixed pair \(i \neq j\), \(T_{ij}^*\) can be rewritten as the standardized sum

\[
T_{ij}^* = \frac{\sum_{k=1}^{n_i} (X_{ki} - \mu_i) - \sum_{k=1}^{n_j} n_j^{-1} (X_{kj} - \mu_j)}{\sqrt{n_i \sigma_i^2 + n_j^2 \sigma_j^2/n_j}} = \frac{\sum_{k=1}^{n_i \vee n_j} \eta_{k,ij}}{\sqrt{n_{ij}}},
\]

where...
are independent random variables with mean 0, and

$$\eta_{k,ij} := \begin{cases} \frac{(X_{k(i)} - \mu_i) - n_i^{-1/2}(X_{k(j)} - \mu_j)}{\sigma_i} & \text{if } k \leq n_i \wedge n_j, 1 \leq k \leq n_i \vee n_j, \\ \frac{(X_{k(i)} - \mu_i)1_{(n_i > n_j)} - n_i^{-1/2}(X_{k(j)} - \mu_j)1_{(n_i < n_j)}}{\sigma_i} & \text{if } k > n_i \wedge n_j \end{cases}$$

are independent random variables with mean 0, and

$$a_{ij} := n_i + \frac{n_i^2 \sigma_i^2}{n_j \sigma_j^2}.$$ 

Without loss of generality, we will prove the lemma by assuming that \(i_1 = i_2\), and make the identification \(i_1 = i_2 = i, j_1 = j, j_2 = l\) with three distinct indices \(i, j, l\) from now on. For each \(k = 1, \ldots, n_i \vee n_j \vee n_l\), let

$$\eta_k = (\eta_{k,ij}1_{(k \leq n_i \vee n_j)}/\sqrt{a_{ij}}, \eta_{k,il}1_{(k \leq n_i \vee n_l)}/\sqrt{a_{il}})^T$$

and

$$\Sigma = \text{Cov} \left( \sum_{k=1}^{n_i \vee n_j \vee n_l} \eta_k \eta_k^T \right),$$

where the dependence of \(\eta_k\) on the particular choice of the triple \((i, j, l)\) is suppressed for brevity; this gives rise to the alternative expression

$$(T^*_ij, T^*_il)^T = \sum_{k=1}^{n_i \vee n_j \vee n_l} \eta_k.$$ 

Later we will use the properties that

$$(B.7) \quad \mathbb{E} [\eta_k] = (0, 0)^T,$$

$$(B.8) \quad \Sigma = \begin{pmatrix} \Sigma_{1,1} & \Sigma_{1,2} \\ \Sigma_{2,1} & \Sigma_{2,2} \end{pmatrix}$$

is a 2 \times 2 matrix with 1’s on the diagonal and

$$(B.9) \quad \left(1 + c_{ij}^2 \right)^{-1} \leq \Sigma_{1,2} = \text{Cov}(T^*_ij, T^*_il) = \sqrt{\left(1 + \frac{\sigma_i^2 n_i}{\sigma_j^2 n_j} \right)^{-1} \left(1 + \frac{\sigma_j^2 n_j}{\sigma_i^2 n_i} \right)^{-1} \left(1 + c_{ij}^2 \right)^{-1}},$$

where \((B.9)\) follows from Assumption \[4\]. For \(1 \leq k \leq n_i \vee n_j \vee n_l\), by defining the the truncations

$$\hat{\eta}_k = (\tilde{\eta}_{k,ij}, \tilde{\eta}_{k,il})^T := \eta_k 1_{(\|\eta_k\| \leq (\log m)^{-1})} - \mathbb{E} [\eta_k 1_{(\|\eta_k\| \leq (\log m)^{-1})}]$$

and

$$\tilde{\eta}_k = \eta_k - \hat{\eta}_k = \eta_k 1_{(\|\eta_k\| > (\log m)^{-1})} - \mathbb{E} [\eta_k 1_{(\|\eta_k\| > (\log m)^{-1})}],$$

whose dependence on \((i, j, l)\) is again suppressed, we get that

$$(B.10) \quad P \left( T^*_ij \geq \tilde{i}, T^*_il \geq \tilde{l} \right) \leq P \left( \left\| \sum_{k=1}^{n_i \vee n_j} \tilde{\eta}_k \right\| \geq (\log m)^{-2} \right) +$$

$$P \left( \sum_{k=1}^{n_i \vee n_j} \tilde{\eta}_{k,ij} \geq \tilde{i} - (\log m)^{-2}, \sum_{k=1}^{n_i \vee n_l} \tilde{\eta}_{k,il} \geq \tilde{l} - (\log m)^{-2} \right).$$
By Assumptions 1 and 2 as well as Jensen’s inequality, there exists some constant $\kappa > 0$, such that

\begin{equation}
(B.11) \quad \max_{i \neq j \neq l} \max_{1 \leq k \leq n_i \land n_j \land n_l} \mathbb{E}[\|\eta_k\|^{4r+2+\nu}] \leq \kappa/n^{2r+1+\nu/2},
\end{equation}

which also implies

\begin{align}
\max_{i \neq j \neq l} \left\| \sum_{k=1}^{n_i \land n_j \land n_l} \mathbb{E}\left[\eta_k 1(\|\eta_k\|>(\log m)^{-t})\right] \right\| \\
\leq \max_{i \neq j \neq l} \sum_{k=1}^{n_i \land n_j \land n_l} \mathbb{E}[\|\eta_k\|1(\|\eta_k\|>(\log m)^{-t})] \\
\leq \max_{i \neq j \neq l} \sum_{k=1}^{n_i \land n_j \land n_l} (\log m)^{-4t} \mathbb{E}\left[\left\| \frac{\eta_k}{(\log m)^{-4t}} \right\|^{4r+2+\nu}\right] \\
\leq \kappa (\log m)^{16r+4+4\nu} n^{-2r+\nu/2} = o((\log m)^{-2})
\end{align}

Hence for a sufficiently large $n$, in order for $\| \sum_{k=1}^{n_i \land n_j \land n_l} \tilde{\eta}_k \|$ to be greater than $(\log m)^{-2}$, from the definition of $\tilde{\eta}_k$ it can be seen that at least one of $\tilde{\eta}_k 1(\|\eta_k\|>(\log m)^{-t})$, $k = 1, \ldots, n_i \lor n_j \lor n_l$, must be non-zero, which gives

\begin{align}
\max_{i \neq j \neq l} P\left(\left\| \sum_{k=1}^{n_i \land n_j \land n_l} \tilde{\eta}_k \right\| > (\log m)^{-2}\right) \\
\leq \max_{i \neq j \neq l} P\left(\left\| \sum_{k=1}^{n_i \land n_j \land n_l} \eta_k 1(\|\eta_k\|>(\log m)^{-t}) \right\| > \frac{(\log m)^{-2}}{2} - \frac{\kappa (\log m)^{16r+4+4\nu}}{n^{2r+\nu/2}}\right) \\
\leq \max_{i \neq j \neq l} n P\left(\|\eta_k\| > \frac{(\log m)^{-2}}{2} - \frac{\kappa (\log m)^{16r+4+4\nu}}{n^{2r+\nu/2}}\right) \\
\leq \kappa n^{-2r-\nu/2} \left(\frac{(\log m)^{-2}}{2} - \frac{\kappa (\log m)^{16r+4+4\nu}}{n^{2r+\nu/2}}\right)^{4r+2+\nu} \\
= \kappa n^{-2r-\nu/2} \left(\frac{(\log m)^{-2}}{2} - o((\log m)^{-2})\right)^{4r+2+\nu} = O(n^{-\nu}).
\end{align}

From (B.10) it remains to show, for some $\delta > 0$,

\begin{align}
\max_{i \neq j \neq l} \max_{0 \leq t \leq 2 \log m} P\left(\sum_{k=1}^{n_i \land n_j} \tilde{\eta}_{k,ij} \geq \tilde{t} - (\log m)^{-2}, \sum_{k=1}^{n_i \land n_l} \tilde{\eta}_{k,il} \geq \tilde{t} - (\log m)^{-2}\right) \\
\leq C (1+t)^2 \exp\left(-\frac{t^2}{1+\delta}\right).
\end{align}
Since $\|\hat{\eta}_k\|$ are bounded by $2(\log m)^{-4}$, by taking $\tau = 2(\log m)^{-4}$ for $\tau$ in Lemma C.3 [Zaitsev, 1987], we have

$$P\left(\sum_k \hat{\eta}_{k,ij} \geq \hat{t} - (\log m)^{-2}, \sum_k \hat{\eta}_{k,il} \geq \hat{t} - (\log m)^{-2}\right)$$

$$\leq P\left(W_1 > \hat{t} - 2(\log m)^{-2}, W_2 > \hat{t} - 2(\log m)^{-2}\right) + c_1 \exp\left(-\left(\frac{(\log m)^2}{c_2}\right)\right)$$

for some absolute constants $c_1, c_2 > 0$, where $(W_1, W_2)^T$ is a bivariate Gaussian vector with mean zero and covariance structure $\hat{\Sigma} := E[\sum_{k=1}^{n} \hat{\eta}_k \hat{\eta}_k^T]$. Note that the term $c_1 \exp\left(-\left(\frac{(\log m)^2}{c_2}\right)\right)$ is less than $C(1 + t)^{-2} \exp(-t^2/(1 + \delta))$ for some $C, \delta > 0$ on the range $t \in [0, \sqrt{2\log m}]$, and we are left with uniformly bounding the probability $P(\hat{W}_1 > \hat{t} - 2(\log m)^{-2}, \hat{W}_2 > \hat{t} - 2(\log m)^{-2})$ in the preceding display.

By (B.7) and (B.12),

$$\max_{i \neq j \neq l} \left\| \mathbb{E}\left[ \sum_{k=1}^{n} \eta_k \mathbf{1}(\|\eta_k\| \leq (\log m)^{-4}) \right] \right\| = o((\log m)^{-2}),$$

hence, by letting $\| \cdot \|_\infty$ denote the matrix max norm,

$$\max_{i \neq j \neq l} \|\Sigma - \hat{\Sigma}\|_\infty \leq \max_{i \neq j \neq l} \left\| \mathbb{E}\left[ \sum_{k=1}^{n} \eta_k \eta_k^T \mathbf{1}(\|\eta_k\| \leq (\log m)^{-4}) \right] \right\| + o((\log m)^{-4})$$

$$\leq \max_{i \neq j \neq l} \sum_{k=1}^{n} \mathbb{E}[\|\eta_k\|^2 \mathbf{1}(\|\eta_k\| > (\log m)^{-4})] + o((\log m)^{-4})$$

$$= (\log m)^{-8} \max_{i \neq j \neq l} \sum_{k=1}^{n} \mathbb{E}\left[ \left\| \frac{\eta_k}{(\log m)^{-4}} \right\|^2 \mathbf{1}(\|\eta_k\| > (\log m)^{-4}) \right] + o((\log m)^{-4})$$

(B.13)

$$\leq (\log m)^{16r + 4\nu} \frac{k}{n^{2r + \nu/2}} + o((\log m)^{-4}) = o((\log m)^{-2}),$$

where the last inequality follows from (B.11). By Lemma C.3 [Berman, 1962], for $0 \leq t \leq \sqrt{2\log m}$ and large enough $n$,

(B.14) \hspace{1cm} P(\hat{W}_1 > \hat{t} - 2(\log m)^{-2}, \hat{W}_2 > \hat{t} - 2(\log m)^{-2}) \leq \frac{C}{(1 + \min\left(\hat{t} - 2(\log m)^{-2}, \frac{\hat{t} - 2(\log m)^{-2}}{\sqrt{\Sigma_{11}}}, \frac{\hat{t} - 2(\log m)^{-2}}{\sqrt{\Sigma_{22}}}\right))^2} \exp\left(-\left(\frac{\hat{t} - 2(\log m)^{-2}}{\sqrt{\Sigma_{11}}}, \frac{\hat{t} - 2(\log m)^{-2}}{\sqrt{\Sigma_{22}}}\right)^2\right).$$

By the bound in (B.8), (B.9), (B.13) and elementary calculations, the right hand side can further be bounded by

$$\frac{C}{(1 + t)^2} \exp(-t^2/(1 + \delta))$$

for some other constants $C > 0$ and $1 > \delta > 0$, uniformly in $(i, j, l)$ and $0 \leq t \leq \sqrt{2\log m}$. These elementary calculations are left to the readers, but the fact that
$t_L = \Phi^{-1}(\Phi(t)(1 + \epsilon_n))$ converges to $t$ uniformly in $t \in [0, \sqrt{2 \log m}]$ is helpful: It is obvious how $t_L \to t$ uniformly for $t \in [0, 2]$, since $\Phi^{-1}$ has bounded derivative on $[\Phi(2), \Phi(0)]$. On the interval $t \in [2, \sqrt{2 \log m}]$, by inverse function theorem and with mean value theorem,

$$|t_L - t| \leq |\phi(t)|^{-1} |\Phi(t)| \epsilon_n \leq \frac{\epsilon_n}{t} \to 0$$

uniformly in $t$ since $\epsilon_n = O(n^{-c})$. \qed
APPENDIX C. TECHNICAL LEMMAS

Lemma C.1. (Chang et al., 2016, Theorem 2.3) Let \( \{Y_1, \ldots, Y_n\} \) be two independent i.i.d samples such that \( \mathbb{E}[Y_1] = \mathbb{E}[Z_1] = 0 \) and \( \mathbb{E}[|Y_1|^{2+\delta}], \mathbb{E}[|Z_1|^{2+\delta}] < \infty \) for some positive number \( \delta \in (0, 1) \), and let \( \mathbb{E}[Y_2] = \sigma^2_y \) and \( \mathbb{E}[Z_1^2] = \sigma_z^2 \). Consider the two sample t-statistic

\[
T = \frac{\bar{Y} - \bar{Z}}{\sqrt{\frac{\sigma^2_y}{n_y} + \frac{\sigma^2_z}{n_z}}},
\]

where \( \bar{Y} = n_y^{-1} \sum_{1 \leq k \leq n_y} Y_k \), \( \bar{Z} = n_z^{-1} \sum_{1 \leq k \leq n_z} Z_k \),

\[
\sigma^2_y = \frac{\sum_{1 \leq k \leq n_y} (Y_k - \bar{Y})^2}{n_y - 1}, \quad \sigma^2_z = \frac{\sum_{1 \leq k \leq n_z} (Z_k - \bar{Z})^2}{n_z - 1}.
\]

There exist absolute constants \( A, a > 0 \) such that

\[
P(T \geq s) = \frac{\Phi(s)}{\Phi(0)} = 1 + O(1)(1 + s)^{2+\delta} \left( \frac{\mathbb{E}[|Y_1|^{2+\delta}]}{\sigma^2_y} + \frac{\mathbb{E}[|Z_1|^{2+\delta}]}{\sigma^2_z} \right),
\]

holds for \( 0 \leq s \leq a \min \left( \frac{\sigma^2_y}{\mathbb{E}[|Y_1|^{2+\delta}]}, \frac{\sigma^2_z}{\mathbb{E}[|Z_1|^{2+\delta}]}, \frac{\sigma^2_y^{1/2} - 1/2 + \delta}{\mathbb{E}[|Y_1|^{2+\delta}]} \right) \), where \( \theta(1) = A \).

Lemma C.2. (Berman, 1962, Lemma 2) If \( (Y, Z) \) have a bivariate normal distribution with \( \mathbb{E}[Y] = \mathbb{E}[Z] = 0, \mathbb{E}[Y^2] = \mathbb{E}[Z^2] = 1 \) and \( \mathbb{E}[YZ] = r \), then

\[
\lim_{r \to 0} \frac{P(Y > c, Z > c)}{2 \pi (1 - r)^{1/2} c^{-1} \exp(-c^2/2 \pi)} = 1
\]

uniformly for all \( r \) such that \( |r| \leq \delta \), for any \( \delta \in (0, 1) \).

Lemma C.3. (Zaitsev, 1984, Theorem 1.1) Let \( \tau > 0 \) and \( \xi_1, \ldots, \xi_n \in \mathbb{R}^p \) be independent mean-zero random vectors such that for all \( t, u \in \mathbb{R}^p \),

\[
\mathbb{E}[((\xi_i^T t)^2 (\xi_i^T u)^{\omega-2})] \leq \frac{\omega!}{2} r^{\omega-2} \|u\|^{\omega-2} \mathbb{E}[((\xi_i^T t)^2)]
\]

for every \( \omega = 3, 4, \ldots \), and define \( S = \sum_{i=1}^n \xi_i \). Let \( B \) be a Borel set in \( \mathbb{R}^p \) and, for \( \lambda > 0 \), \( B^\lambda \) be its \( \lambda \)-neighborhood defined by

\[
B^\lambda = \left\{ y \in \mathbb{R}^p : \inf_{x \in B} \|x - y\| < \lambda \right\}.
\]

If \( \mu_S \) and \( \mu_{N_{0,\text{Con}(S)}} \) are respectively the probability measures of \( S \) and of a \( p \)-variate normal distribution with mean \( 0 \) and the same covariance structure as \( S \), then for all \( \lambda \geq 0 \),

\[
\sup_B \left\{ \left( \mu_S(B) - \mu_{N_{0,\text{Con}(S)}}(B^\lambda) \right) \vee \left( \mu_{N_{0,\text{Con}(S)}}(B) - \mu_S(B^\lambda) \right) \right\} \leq c_1(p) \exp \left( -\frac{\lambda}{c_2(p) r} \right),
\]

where \( c_1(p), c_2(p) > 0 \) are constants depending on \( p \) only, and the supremum is taken over all Borel sets.

APPENDIX D. ADDITIONAL NUMERICAL STUDIES

We provide extra simulation results for \( \alpha = 0.3, 0.1 \), for the setups in Section 4.
Table 7. Estimates of $P(dFDP \leq \frac{\alpha}{2} (1 + q_0/q))$ for the BH procedure in Definition 2.1 at level $\alpha = 0.3$, based on 500 repetitions of data generated from the one-way ANOVA model in Section 4, for combinations of $(m, n)$ and $q_0/q = 0$.

| $m \backslash n$ | 20   | 40   | 100  | 200  | 400  | 600  | 20   | 40   | 100  | 200  | 400  | 600  |
|-----------------|------|------|------|------|------|------|------|------|------|------|------|------|
|                 | effect size = 0.05 | effect size = 0.25 | effect size = 0.45 | effect size = 0.65 |
| 15              | 0.79 | 0.83 | 0.87 | 0.88 | 0.91 | 0.93 | 0.89 | 0.97 | 1.00 | 1.00 | 1.00 | 1.00 |
| 30              | 0.79 | 0.82 | 0.90 | 0.92 | 0.96 | 0.99 | 0.99 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 50              | 0.73 | 0.82 | 0.89 | 0.95 | 0.98 | 1.00 | 0.99 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 70              | 0.69 | 0.82 | 0.87 | 0.93 | 0.99 | 1.00 | 0.99 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 90              | 0.67 | 0.79 | 0.85 | 0.94 | 0.99 | 1.00 | 0.99 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 120             | 0.64 | 0.77 | 0.88 | 0.95 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |

Table 8. Estimates of $dFDR$ for the BH procedure in Definition 2.1 at level $\alpha = 0.3$, based on 500 repetitions of data generated from the one-way ANOVA model in Section 4, for combinations of $(m, n)$ and $q_0/q = 0$; the $dFDR$ target is $\frac{\alpha}{2} (1 + q_0/q) = 0.15$.

| $m \backslash n$ | 20   | 40   | 100  | 200  | 400  | 600  | 20   | 40   | 100  | 200  | 400  | 600  |
|-----------------|------|------|------|------|------|------|------|------|------|------|------|------|
|                 | effect size = 0.05 | effect size = 0.25 | effect size = 0.45 | effect size = 0.65 |
| 15              | 0.11 | 0.08 | 0.05 | 0.04 | 0.04 | 0.04 | 0.05 | 0.04 | 0.03 | 0.02 | 0.01 | 0.01 |
| 30              | 0.09 | 0.07 | 0.04 | 0.04 | 0.04 | 0.04 | 0.04 | 0.03 | 0.02 | 0.02 | 0.01 | 0.01 |
| 50              | 0.12 | 0.07 | 0.05 | 0.04 | 0.04 | 0.04 | 0.05 | 0.04 | 0.02 | 0.02 | 0.01 | 0.01 |
| 70              | 0.14 | 0.07 | 0.05 | 0.04 | 0.04 | 0.04 | 0.05 | 0.04 | 0.02 | 0.02 | 0.01 | 0.01 |
| 90              | 0.14 | 0.08 | 0.06 | 0.04 | 0.04 | 0.04 | 0.05 | 0.04 | 0.03 | 0.02 | 0.01 | 0.01 |
| 120             | 0.14 | 0.08 | 0.05 | 0.04 | 0.04 | 0.04 | 0.05 | 0.04 | 0.03 | 0.02 | 0.01 | 0.01 |
Table 9. Estimates of $P(dFDP \leq \frac{a}{2} (1 + q_0/q))$ for the BH procedure in Definition 2.1 at level $\alpha = 0.3$, based on 500 repetitions of data generated from the one-way ANOVA model in Section 4, for combinations of $(m, n)$ and $q_0/q \approx 0.25$.

| $m \backslash n$ | 20 | 40 | 100 | 200 | 400 | 600 | effect size = 0.05 | effect size = 0.25 |
|-----------------|----|----|-----|-----|-----|-----|-------------------|-------------------|
| 15              | 0.77| 0.82| 0.83| 0.85| 0.81| 0.81| 0.76 0.81 0.90| 0.94 0.98 0.99 |
| 30              | 0.77| 0.79| 0.83| 0.81| 0.79| 0.83| 0.80 0.88 0.98| 1.00 1.00 1.00 |
| 50              | 0.68| 0.77| 0.80| 0.81| 0.79| 0.82| 0.74 0.86 0.98| 1.00 1.00 1.00 |
| 70              | 0.66| 0.76| 0.80| 0.82| 0.79| 0.81| 0.70 0.85 0.98| 0.99 1.00 1.00 |
| 90              | 0.62| 0.75| 0.78| 0.79| 0.80| 0.84| 0.70 0.85 0.98| 0.99 1.00 1.00 |
| 120             | 0.59| 0.73| 0.79| 0.80| 0.80| 0.89| 0.77 0.93 1.00| 1.00 1.00 1.00 |

**Table 10.** Estimates of dFDR for the BH procedure in Definition 2.1 at level $\alpha = 0.3$, based on 500 repetitions of data generated from the one-way ANOVA model in Section 4, for combinations of $(m, n)$ and $q_0/q \approx 0.25$; the dFDR target is $\frac{a}{2} (1 + q_0/q) \approx 0.1875$.

| $m \backslash n$ | 20 | 40 | 100 | 200 | 400 | 600 | effect size = 0.45 | effect size = 0.65 |
|-----------------|----|----|-----|-----|-----|-----|-------------------|-------------------|
| 15              | 0.79| 0.88| 0.97| 0.99| 1.00| 1.00| 0.90 0.95 0.99| 1.00 1.00 1.00 |
| 30              | 0.95| 0.99| 1.00| 1.00| 1.00| 1.00| 0.99 1.00 1.00| 1.00 1.00 1.00 |
| 50              | 0.93| 0.98| 1.00| 1.00| 1.00| 1.00| 0.99 1.00 1.00| 1.00 1.00 1.00 |
| 70              | 0.91| 0.99| 1.00| 1.00| 1.00| 1.00| 0.98 1.00 1.00| 1.00 1.00 1.00 |
| 90              | 0.93| 0.99| 1.00| 1.00| 1.00| 1.00| 0.99 1.00 1.00| 1.00 1.00 1.00 |
| 120             | 0.97| 1.00| 1.00| 1.00| 1.00| 1.00| 0.99 1.00 1.00| 1.00 1.00 1.00 |

| $m \backslash n$ | 20 | 40 | 100 | 200 | 400 | 600 | effect size = 0.05 | effect size = 0.25 |
|-----------------|----|----|-----|-----|-----|-----|-------------------|-------------------|
| 15              | 0.15| 0.11| 0.09| 0.08| 0.08| 0.08| 0.11 0.08 0.07| 0.07 0.07 0.07 |
| 30              | 0.13| 0.11| 0.08| 0.08| 0.09| 0.09| 0.10 0.09 0.09| 0.08 0.08 0.08 |
| 50              | 0.19| 0.13| 0.10| 0.09| 0.10| 0.09| 0.12 0.10 0.09| 0.09 0.08 0.08 |
| 70              | 0.20| 0.13| 0.10| 0.09| 0.10| 0.10| 0.13 0.12 0.10| 0.09 0.09 0.09 |
| 90              | 0.22| 0.15| 0.11| 0.09| 0.09| 0.09| 0.13 0.11 0.10| 0.09 0.09 0.09 |
| 120             | 0.23| 0.15| 0.10| 0.08| 0.10| 0.10| 0.14 0.11 0.10| 0.09 0.09 0.08 |

| $m \backslash n$ | 20 | 40 | 100 | 200 | 400 | 600 | effect size = 0.45 | effect size = 0.65 |
|-----------------|----|----|-----|-----|-----|-----|-------------------|-------------------|
| 15              | 0.09| 0.08| 0.07| 0.07| 0.06| 0.06| 0.08 0.07 0.06| 0.06 0.06 0.06 |
| 30              | 0.09| 0.09| 0.08| 0.08| 0.08| 0.07| 0.09 0.08 0.07| 0.07 0.07 0.07 |
| 50              | 0.10| 0.09| 0.08| 0.08| 0.08| 0.08| 0.10 0.09 0.08| 0.08 0.08 0.08 |
| 70              | 0.11| 0.10| 0.09| 0.09| 0.08| 0.08| 0.11 0.10 0.08| 0.08 0.08 0.08 |
| 90              | 0.12| 0.10| 0.09| 0.09| 0.08| 0.08| 0.10 0.09 0.08| 0.08 0.08 0.08 |
| 120             | 0.11| 0.10| 0.09| 0.08| 0.08| 0.08| 0.10 0.09 0.08| 0.08 0.08 0.08 |
Table 11. Estimates of $P(dFDP \leq \frac{\alpha}{2} (1 + q_0/q))$ for the BH procedure in Definition 2.1 at level $\alpha = 0.3$, based on 500 repetitions of data generated from the one-way ANOVA model in Section 4 for combinations of $(m, n)$ and $q_0/q \approx 0.5$.

| $m \backslash n$ | 20   | 40   | 100  | 200  | 400  | 600  | 20   | 40   | 100  | 200  | 400  | 600  |
|-----------------|------|------|------|------|------|------|------|------|------|------|------|------|
|                 | effect size = 0.05 |      |      |      |      |      |      |      |      |      |      |      |
| 15              | 0.74 | 0.78 | 0.82 | 0.79 | 0.78 | 0.80 | 0.72 | 0.74 | 0.73 | 0.73 | 0.79 | 0.77 |
| 30              | 0.75 | 0.78 | 0.82 | 0.81 | 0.73 | 0.73 | 0.71 | 0.69 | 0.78 | 0.83 | 0.86 | 0.85 |
| 50              | 0.67 | 0.75 | 0.80 | 0.77 | 0.73 | 0.74 | 0.62 | 0.67 | 0.76 | 0.82 | 0.86 | 0.88 |
| 70              | 0.64 | 0.75 | 0.80 | 0.81 | 0.74 | 0.72 | 0.55 | 0.64 | 0.74 | 0.83 | 0.87 | 0.89 |
| 90              | 0.59 | 0.74 | 0.74 | 0.77 | 0.77 | 0.72 | 0.53 | 0.67 | 0.80 | 0.89 | 0.95 | 0.95 |
| 120             | 0.58 | 0.73 | 0.78 | 0.80 | 0.74 | 0.75 | 0.52 | 0.67 | 0.87 | 0.92 | 0.97 | 0.97 |

|                 | effect size = 0.25 |      |      |      |      |      |      |      |      |      |      |      |
| 15              | 0.67 | 0.69 | 0.75 | 0.78 | 0.85 | 0.84 | 0.68 | 0.71 | 0.78 | 0.83 | 0.86 | 0.87 |
| 30              | 0.70 | 0.77 | 0.85 | 0.89 | 0.90 | 0.88 | 0.76 | 0.81 | 0.88 | 0.91 | 0.90 | 0.89 |
| 50              | 0.65 | 0.73 | 0.85 | 0.87 | 0.92 | 0.91 | 0.76 | 0.80 | 0.90 | 0.90 | 0.93 | 0.94 |
| 70              | 0.64 | 0.76 | 0.83 | 0.90 | 0.92 | 0.93 | 0.69 | 0.83 | 0.87 | 0.93 | 0.94 | 0.94 |
| 90              | 0.67 | 0.80 | 0.91 | 0.96 | 0.97 | 0.98 | 0.78 | 0.88 | 0.95 | 0.97 | 0.98 | 0.99 |
| 120             | 0.72 | 0.88 | 0.97 | 0.98 | 0.99 | 0.99 | 0.84 | 0.94 | 0.98 | 0.99 | 1.00 | 0.99 |

Table 12. Estimates of $dFDR$ for the BH procedure in Definition 2.1 at level $\alpha = 0.3$, based on 500 repetitions of data generated from the one-way ANOVA model in Section 4 for combinations of $(m, n)$ and $q_0/q \approx 0.5$; the dFDR target is $\frac{\alpha}{2} (1 + q_0/q) \approx 0.225$.

| $m \backslash n$ | 20 | 40 | 100 | 200 | 400 | 600 | 20 | 40 | 100 | 200 | 400 | 600 |
|-----------------|----|----|-----|-----|-----|-----|----|----|-----|-----|-----|-----|
|                 | effect size = 0.05 |      |      |      |      |      |      |      |      |      |      |      |
| 15              | 0.19 | 0.16 | 0.13 | 0.14 | 0.13 | 0.12 | 0.16 | 0.13 | 0.13 | 0.13 | 0.13 | 0.13 |
| 30              | 0.18 | 0.16 | 0.12 | 0.12 | 0.15 | 0.14 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 | 0.15 |
| 50              | 0.24 | 0.18 | 0.14 | 0.14 | 0.14 | 0.13 | 0.19 | 0.17 | 0.16 | 0.16 | 0.15 | 0.15 |
| 70              | 0.27 | 0.18 | 0.14 | 0.11 | 0.13 | 0.14 | 0.22 | 0.18 | 0.16 | 0.16 | 0.16 | 0.16 |
| 90              | 0.29 | 0.19 | 0.17 | 0.12 | 0.12 | 0.13 | 0.21 | 0.18 | 0.16 | 0.16 | 0.15 | 0.15 |
| 120             | 0.29 | 0.19 | 0.14 | 0.11 | 0.13 | 0.13 | 0.21 | 0.18 | 0.16 | 0.16 | 0.15 | 0.15 |

|                 | effect size = 0.25 |      |      |      |      |      |      |      |      |      |      |      |
| 15              | 0.15 | 0.14 | 0.13 | 0.13 | 0.13 | 0.13 | 0.15 | 0.14 | 0.13 | 0.13 | 0.12 | 0.12 |
| 30              | 0.16 | 0.16 | 0.15 | 0.14 | 0.15 | 0.14 | 0.16 | 0.15 | 0.15 | 0.14 | 0.14 | 0.14 |
| 50              | 0.18 | 0.17 | 0.15 | 0.15 | 0.15 | 0.15 | 0.17 | 0.16 | 0.15 | 0.15 | 0.15 | 0.15 |
| 70              | 0.19 | 0.17 | 0.16 | 0.16 | 0.16 | 0.15 | 0.18 | 0.17 | 0.15 | 0.15 | 0.16 | 0.15 |
| 90              | 0.19 | 0.17 | 0.16 | 0.15 | 0.15 | 0.15 | 0.18 | 0.16 | 0.15 | 0.15 | 0.15 | 0.15 |
| 120             | 0.19 | 0.16 | 0.15 | 0.15 | 0.15 | 0.15 | 0.17 | 0.16 | 0.15 | 0.15 | 0.15 | 0.15 |
Table 13. Estimates of $P(dFDP \leq \frac{\alpha}{2} (1 + q_0/q))$ for the BH procedure in Definition 2.1 at level $\alpha = 0.1$, based on 500 repetitions of data generated from the one-way ANOVA model in Section 4 for combinations of $(m, n)$ and $q_0/q = 0$.

| $m \setminus n$ | 20 | 40 | 100 | 200 | 400 | 600 | effect size = 0.05 | effect size = 0.25 |
|-----------------|----|----|-----|-----|-----|-----|-------------------|-------------------|
| 15              | 0.92 | 0.94 | 0.96 | 0.94 | 0.92 | 0.92 | 0.91 | 0.90 | 0.96 | 0.99 | 1.00 | 1.00 |
| 30              | 0.91 | 0.93 | 0.95 | 0.94 | 0.92 | 0.95 | 0.91 | 0.98 | 1.00 | 1.00 | 1.00 | 1.00 |
| 50              | 0.87 | 0.94 | 0.93 | 0.92 | 0.94 | 0.95 | 0.88 | 0.98 | 1.00 | 1.00 | 1.00 | 1.00 |
| 70              | 0.83 | 0.92 | 0.93 | 0.94 | 0.94 | 0.98 | 0.90 | 0.99 | 1.00 | 1.00 | 1.00 | 1.00 |
| 90              | 0.83 | 0.91 | 0.92 | 0.91 | 0.96 | 0.97 | 0.89 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 120             | 0.79 | 0.90 | 0.92 | 0.91 | 0.95 | 0.98 | 0.93 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |

| effect size = 0.45 | effect size = 0.65 |
|-------------------|-------------------|
| 15                | 0.92 | 0.98 | 1.00 | 1.00 | 1.00 | 1.00 | 0.98 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 30                | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 50                | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 70                | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 90                | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 120               | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |

Table 14. Estimates of dFDR for the BH procedure in Definition 2.1 at level $\alpha = 0.1$, based on 500 repetitions of data generated from the one-way ANOVA model in Section 4 for combinations of $(m, n)$ and $q_0/q = 0$; the dFDR target is $\frac{\alpha}{2} (1 + q_0/q) = 0.05$.

| $m \setminus n$ | 20 | 40 | 100 | 200 | 400 | 600 | effect size = 0.05 | effect size = 0.25 |
|-----------------|----|----|-----|-----|-----|-----|-------------------|-------------------|
| 15              | 0.04 | 0.03 | 0.02 | 0.01 | 0.02 | 0.01 | 0.02 | 0.01 | 0.01 | 0.01 | 0.00 | 0.00 |
| 30              | 0.04 | 0.03 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.01 | 0.00 | 0.00 |
| 50              | 0.06 | 0.02 | 0.01 | 0.01 | 0.01 | 0.01 | 0.02 | 0.01 | 0.01 | 0.01 | 0.00 | 0.00 |
| 70              | 0.08 | 0.03 | 0.02 | 0.01 | 0.01 | 0.01 | 0.02 | 0.01 | 0.01 | 0.01 | 0.00 | 0.00 |
| 90              | 0.07 | 0.03 | 0.02 | 0.01 | 0.01 | 0.01 | 0.02 | 0.01 | 0.01 | 0.01 | 0.01 | 0.00 |
| 120             | 0.08 | 0.04 | 0.02 | 0.01 | 0.01 | 0.01 | 0.02 | 0.01 | 0.01 | 0.01 | 0.01 | 0.00 |

| effect size = 0.45 | effect size = 0.65 |
|-------------------|-------------------|
| 15                | 0.01 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 30                | 0.01 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 50                | 0.01 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 70                | 0.01 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 90                | 0.01 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 120               | 0.01 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
Table 15. Estimates of $P(dFDP \leq \frac{\alpha}{2}(1 + q_0/q))$ for the BH procedure in Definition 2.1 at level $\alpha = 0.1$, based on 500 repetitions of data generated from the one-way ANOVA model in Section 4 for combinations of $(m, n)$ and $q_0/q \approx 0.25$.

| $m \setminus n$ | 20   | 40   | 100  | 200  | 400  | 600  | 20   | 40   | 100  | 200  | 400  | 600  |
|-----------------|------|------|------|------|------|------|------|------|------|------|------|------|
|                 |      |      |      |      |      |      |      |      |      |      |      |      |
| effect size = 0.05 | 0.92 | 0.93 | 0.94 | 0.96 | 0.92 | 0.93 | 0.89 | 0.88 | 0.82 | 0.85 | 0.86 | 0.89 |
| effect size = 0.25 | 0.89 | 0.91 | 0.94 | 0.93 | 0.86 | 0.85 | 0.82 | 0.79 | 0.89 | 0.92 | 0.92 | 0.93 |
| effect size = 0.45 | 0.86 | 0.91 | 0.94 | 0.91 | 0.90 | 0.87 | 0.78 | 0.77 | 0.87 | 0.92 | 0.94 | 0.95 |
| effect size = 0.65 | 0.81 | 0.89 | 0.91 | 0.92 | 0.86 | 0.83 | 0.69 | 0.73 | 0.85 | 0.93 | 0.96 | 0.96 |
| effect size = 0.05 | 0.80 | 0.89 | 0.91 | 0.92 | 0.89 | 0.86 | 0.69 | 0.75 | 0.87 | 0.94 | 0.97 | 0.99 |
| effect size = 0.25 | 0.78 | 0.89 | 0.93 | 0.92 | 0.85 | 0.85 | 0.65 | 0.77 | 0.92 | 0.98 | 0.99 | 0.99 |

Table 16. Estimates of $dFDR$ for the BH procedure in Definition 2.1 at level $\alpha = 0.1$, based on 500 repetitions of data generated from the one-way ANOVA model in Section 4 for combinations of $(m, n)$ and $q_0/q \approx 0.25$; the dFDR target is $\frac{\alpha}{2}(1 + q_0/q) \approx 0.0625$.

| $m \setminus n$ | 20   | 40   | 100  | 200  | 400  | 600  | 20   | 40   | 100  | 200  | 400  | 600  |
|-----------------|------|------|------|------|------|------|------|------|------|------|------|------|
|                 |      |      |      |      |      |      |      |      |      |      |      |      |
| effect size = 0.05 | 0.06 | 0.04 | 0.04 | 0.02 | 0.03 | 0.02 | 0.04 | 0.02 | 0.03 | 0.02 | 0.02 | 0.02 |
| effect size = 0.25 | 0.06 | 0.05 | 0.03 | 0.02 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 | 0.03 |
| effect size = 0.45 | 0.09 | 0.05 | 0.03 | 0.03 | 0.02 | 0.03 | 0.05 | 0.04 | 0.03 | 0.03 | 0.03 | 0.03 |
| effect size = 0.65 | 0.12 | 0.06 | 0.04 | 0.03 | 0.03 | 0.03 | 0.06 | 0.04 | 0.03 | 0.03 | 0.03 | 0.03 |
| effect size = 0.05 | 0.12 | 0.07 | 0.04 | 0.03 | 0.02 | 0.03 | 0.06 | 0.04 | 0.03 | 0.03 | 0.03 | 0.03 |
| effect size = 0.25 | 0.12 | 0.06 | 0.03 | 0.03 | 0.03 | 0.03 | 0.06 | 0.04 | 0.03 | 0.03 | 0.03 | 0.03 |
| effect size = 0.45 | 0.12 | 0.06 | 0.03 | 0.03 | 0.03 | 0.03 | 0.06 | 0.04 | 0.03 | 0.03 | 0.03 | 0.03 |
| effect size = 0.65 | 0.12 | 0.06 | 0.03 | 0.03 | 0.03 | 0.03 | 0.06 | 0.04 | 0.03 | 0.03 | 0.03 | 0.03 |
Table 17. Estimates of $P(dFDP \leq \frac{\alpha}{2} (1 + q_0/q))$ for the BH procedure in Definition 2.1 at level $\alpha = 0.1$, based on 500 repetitions of data generated from the one-way ANOVA model in Section 4, for combinations of $(m, n)$ and $q_0/q \approx 0.5$.

| $m \setminus n$ | 20 | 40 | 100 | 200 | 400 | 600 | 20 | 40 | 100 | 200 | 400 | 600 |
|----------------|----|----|-----|-----|-----|-----|----|----|-----|-----|-----|-----|
|                 | effect size = 0.05 | effect size = 0.25 | effect size = 0.45 | effect size = 0.65 |
| 15              | 0.90 | 0.91 | 0.94 | 0.94 | 0.93 | 0.93 | 0.89 | 0.89 | 0.85 | 0.77 | 0.77 | 0.74 |
| 30              | 0.88 | 0.91 | 0.94 | 0.92 | 0.89 | 0.86 | 0.83 | 0.78 | 0.74 | 0.77 | 0.72 | 0.75 |
| 50              | 0.85 | 0.90 | 0.93 | 0.90 | 0.90 | 0.87 | 0.76 | 0.75 | 0.72 | 0.72 | 0.72 | 0.76 |
| 70              | 0.79 | 0.89 | 0.91 | 0.93 | 0.86 | 0.84 | 0.68 | 0.69 | 0.71 | 0.74 | 0.74 | 0.77 |
| 90              | 0.78 | 0.88 | 0.89 | 0.93 | 0.88 | 0.83 | 0.64 | 0.66 | 0.68 | 0.79 | 0.82 | 0.85 |
| 120             | 0.76 | 0.88 | 0.93 | 0.91 | 0.88 | 0.86 | 0.62 | 0.66 | 0.73 | 0.81 | 0.85 | 0.86 |

Table 18. Estimates of $dFDR$ for the BH procedure in Definition 2.1 at level $\alpha = 0.1$, based on 500 repetitions of data generated from the one-way ANOVA model in Section 4, for combinations of $(m, n)$ and $q_0/q \approx 0.5$; the dFDR target is $\frac{\alpha}{2} (1 + q_0/q) \approx 0.075$.

| $m \setminus n$ | 20 | 40 | 100 | 200 | 400 | 600 | 20 | 40 | 100 | 200 | 400 | 600 |
|----------------|----|----|-----|-----|-----|-----|----|----|-----|-----|-----|-----|
|                 | effect size = 0.05 | effect size = 0.25 | effect size = 0.45 | effect size = 0.65 |
| 15              | 0.08 | 0.07 | 0.04 | 0.04 | 0.05 | 0.04 | 0.06 | 0.05 | 0.04 | 0.04 | 0.04 | 0.04 |
| 30              | 0.08 | 0.07 | 0.04 | 0.05 | 0.05 | 0.05 | 0.07 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| 50              | 0.11 | 0.08 | 0.05 | 0.06 | 0.04 | 0.04 | 0.08 | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 |
| 70              | 0.16 | 0.09 | 0.06 | 0.04 | 0.05 | 0.05 | 0.11 | 0.07 | 0.05 | 0.05 | 0.06 | 0.05 |
| 90              | 0.17 | 0.09 | 0.07 | 0.04 | 0.03 | 0.04 | 0.11 | 0.07 | 0.06 | 0.05 | 0.05 | 0.05 |
| 120             | 0.16 | 0.08 | 0.05 | 0.04 | 0.04 | 0.04 | 0.10 | 0.07 | 0.06 | 0.05 | 0.05 | 0.05 |

|         | effect size = 0.45 | effect size = 0.65 |
|----------------|-----------------|-----------------|
| 15              | 0.06 | 0.05 | 0.04 | 0.04 | 0.04 | 0.04 | 0.05 | 0.05 | 0.04 | 0.04 | 0.04 | 0.04 |
| 30              | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| 50              | 0.08 | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 | 0.07 | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 |
| 70              | 0.08 | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 | 0.07 | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 |
| 90              | 0.08 | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 | 0.07 | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 |
| 120             | 0.08 | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 | 0.07 | 0.06 | 0.05 | 0.05 | 0.05 | 0.05 |
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