Ricci Flow on 3-dimensional Lie groups and 4-dimensional Ricci-flat manifolds

Kensuke Onda*

Graduate School of Mathematics, Nagoya University

Abstract

We study relation of the Ricci Flow on 3-dimensional Lie groups and 4-dimensional Ricci-flat manifolds. In particular, we construct Ricci-flat cohomogeneity one metrics for 3 dimensional Lie groups.

1 Introduction

In this paper, we will discuss some relationship between the Ricci flow of left-invariant Riemannian metrics on 3-dimensional unimodular simply-connected Lie group $G$ and the Ricci-flat metrics of cohomogeneity one on the space-time $\mathbb{R} \times G$.

First of all, we introduce the notion of cohomogeneity one metrics with respect to a Lie group $G$.

Definition 1.1. A pseudo-Riemannian manifold $(M, g)$ is a cohomogeneity one with respect to a Lie group $G$, if and only if $G$ is a subgroup of $\text{Isom}(M, g)$, and the codimension of principal orbits under the action of $G$ equals 1.

The simplest example of a cohomogeneity one metric arises from the standard action of $SO(n)$ to $\mathbb{R}^n$. The singular orbit is $\{0\}$, and the principal orbits are $S^{n-1}$ of various radii.

*kensuke.onda@math.nagoya-u.ac.jp
In this paper, we attempt to construct cohomogeneity one Einstein metrics (in particular Ricci-flat metrics) from the Ricci flow solutions of left-invariant metrics on 3-dimensional unimodular simply connected Lie groups. The Ricci-flat metrics obtained in this paper has the property that their sectional curvatures decay to 0 at one end toward which the metric is complete.

Such metrics was studied by Lorentz, Gibbons, Hawking, Pope and many people. Some of them is called ALF and the Taub-NUT metric on $\mathbb{R} \times SU(2)$ arising from the Ricci flow of the left $SU(2)$- and right $U(1)$-invariant metrics on $SU(2)$ is ALF.

Before proceeding to general 3-dimensional Lie groups, we examine the case of $SU(2)$, which is well known. There exists a left-invariant coframe $\{\theta^i\}_{i=1}^3$ on $SU(2)$ satisfying $d\theta^i = 2\theta^j \wedge \theta^k$, where $(i, j, k)$ are cyclic permutation of $\{1, 2, 3\}$. Then cohomogeneity one metrics with respect to $SU(2)$ is described as

$$g = dt^2 + a(t)^2(\theta^1)^2 + b(t)^2(\theta^2)^2 + c(t)^2(\theta^3)^2.$$  

(1.1)

Some special cases of (1.1) are listed in the following examples:

1. If $a = b = c$ are linear, then the metric $g$ has constant curvature $0$.

2. If $a$, $b$ and $c$ are constants, then the metric $g$ becomes a product metric on $\mathbb{R} \times SU(2)$.

3. If $a = b = c = \sin t$, then the metric $g$ has positive constant curvature.

4. If $a = b = c = \sinh t$, then the metric $g$ has negative constant curvature.

Thus, it is interesting to ask for which $\{a(t), b(t), c(t)\}$ the resulting cohomogeneity one metric is Einstein. In this paper, when the triple of functions $\{a(t), b(t), c(t)\}$ satisfies the Ricci flow equation, we examine whether the resulting cohomogeneity one metric is a Ricci-flat metric. This generalizes the above example 1.

First of all, we define the Ricci flow again. In this paper, we define that a 1-parameter family $g(t)$ of Riemannian metrics is the Ricci flow if and only if it solves

$$\frac{\partial}{\partial t} g(t)_{ij} = -\text{Ric}[g(t)]_{ij},$$

and a 1-parameter family of Riemannian metrics $g(t)$ is the backward Ricci flow if and only if it solves

$$\frac{\partial}{\partial t} g(t)_{ij} = \text{Ric}[g(t)]_{ij}.$$
In the Introduction, the Ricci flow was defined by

$$\frac{\partial}{\partial t} g(t)_{ij} = -2\text{Ric}[g]_{ij}.$$ 

The difference in 1 and 2 is just a choice of the scale and produces no trouble. Indeed, if we put $h := \frac{k}{2}g$, where $k$ is a positive constant, then

$$\frac{\partial}{\partial t} h(t)_{ij} = \left(\frac{k}{2}\right)(-2\text{Ric}[g]_{ij}) = -k\text{Ric}[h]_{ij},$$

because $\text{Ric}[h] = \text{Ric}[\frac{k}{2}g] = \text{Ric}[g]$. In this paper, if $k$ equals 1, we say that $h(t)$ is a solution to the Ricci flow. Also the backward Ricci flow (in this paper) is defined similarly. Note that if we change $t$ into $-t$, the Ricci flow equation changes into the backward Ricci flow equation.

Next, we review the Ricci flow on $SU(2)$. Let $\{F_i\}_{i=1}^3$ be an orthonormal frame of a left-invariant metric $g_3$ on $SU(2)$, satisfying $[F_i, F_j] = -2F_k$ $(i, j, k : \text{cyclic})$, and $\{\theta^i\}$ the dual coframe of $\{F_i\}_{i=1}^3$. The left-invariant metric $g_3$ is expressed as

$$g_3 = A(\theta^1)^2 + B(\theta^2)^2 + C(\theta^3)^2.$$ 

Then the Ricci flow is equivalent to the system of ODE’s:

$$\frac{d}{dt} A = \frac{(B - C)^2 - A^2}{BC},$$  
$$\frac{d}{dt} B = \frac{(C - A)^2 - B^2}{CA},$$  
$$\frac{d}{dt} C = \frac{(A - B)^2 - C^2}{AB},$$  

(1.2)

The behavior of solutions of (1.2) is known in [KM01, CK04].

**Proposition 1.2** ([KM01, CK04]). The solution of the Ricci flow equation exists on $(-\infty, T)$, where $T$ depends on the initial data, and if $t$ goes to $T$, then $g_3$ becomes asymptotically round and shrinks to a point.

Next we consider a cohomogeneity one metric with respect to $SU(2)$, given by

$$g = dt^2 + a(t)^2(\theta^1)^2 + b(t)^2(\theta^2)^2 + c(t)^2(\theta^3)^2.$$  

(1.3)
A typical example of cohomogeneity one metric with respect to $SU(2)$ is the Taub-NUT metric, given by
\[ g = \left( \frac{r + m}{r - m} \right) dr^2 + \left( r^2 - m^2 \right) \left( (\theta^1)^2 + (\theta^2)^2 \right) + 4m^2 \left( \frac{r - m}{r + m} \right) (\theta^3)^2. \]

As is well known, this metric has the following properties. First, this metric is a hyper-Kähler metric, and therefore this is a Ricci-flat metric. Secondly, we can put
\[ a = b = \left( r^2 - m^2 \right)^{1/2}, \quad c = 2m \left( \frac{r - m}{r + m} \right)^{1/2}, \]
and therefore the coefficient $a$ equals $b$, but $a$ is not equal to $c$ (i.e., the metric on $SU(2)$ part is left $SU(2)$- and right $U(1)$-invariant). Thirdly, the change of variables $h = \left( \frac{r + m}{r - m} \right)^{1/2}$ and $dt = h dr$ implies that this metric is regarded as a cohomogeneity one metric. Fourthly, it is easy to check that $a$, $b$ and $c$ satisfy
\[
\frac{da}{dt} = \frac{a^2 - (b - c)^2}{bc}, \quad \frac{db}{dt} = \frac{b^2 - (c - a)^2}{ca}, \quad \frac{dc}{dt} = \frac{c^2 - (a - b)^2}{ab}.
\]

This system is equivalent to the backward Ricci flow equation for left-invariant metrics on $SU(2)$. Thus we conclude that if coefficients $\{a(t), b(t), c(t)\}$ move along the Ricci flow or the backward Ricci flow on $SU(2)$, then the resulting cohomogeneity one metric \((1.3)\) is the Taub-NUT metric and therefore Ricci-flat.

**Theorem 1.3** ([Fri85,CGLP04]). If $a$, $b$ and $c$ satisfy the Ricci flow equations or the backward Ricci flow equations of a metric
\[ g_3 = a(\theta^1)^2 + b(\theta^2)^2 + c(\theta^3)^2 \] on $SU(2)$, then the cohomogeneity one metric
\[ g = dt^2 + a(t)^2(\theta^1)^2 + b(t)^2(\theta^2)^2 + c(t)^2(\theta^3)^2 \] with respect to $SU(2)$ on the space-time of the Ricci flow becomes a Ricci-flat metric.
It is important to pay attention to coefficients. Coefficients of metric (1.4) are $a$, $b$ and $c$. But, coefficients of metric (1.5) are $a^2$, $b^2$ and $c^2$. Therefore, $a$, $b$ and $c$ are positive (because we consider left-invariant Riemannian metrics on $SU(2)$). Even if $a$, $b$ and $c$ are positive, we have still freedom in introducing minus sign before $\{a^2, b^2, c^2\}$ in the attempt to discover cohomogeneity one metrics on the space-time of the Ricci flow (this freedom is really essential in the case of $E(1,1)$ and $SL(2,\mathbb{R})$).

Remark 1.4. Even if a cohomogeneity one metric

$$g = dt^2 + a(t)^2(\theta^1)^2 + b(t)^2(\theta^2)^2 + c(t)^2(\theta^3)^2$$

with respect to $SU(2)$ is Ricci-flat, it may not satisfy the Ricci flow equation nor the backward Ricci flow equation. We consider the Eguchi-Hanson metric as a typical example. The Eguchi-Hanson metric is given by

$$g = \frac{dr^2}{1 - (m/r)^4} + r^2\{(\theta^1)^2 + (\theta^2)^2\} + r^2(1 - (m/r)^4)(\theta^3)^2.$$  (1.6)

Put

$$a = b = r, \quad c = r\left(1 - \left(\frac{m}{r}\right)^4\right)^{\frac{1}{2}}.$$ 

By the coordinate transform,

$$h = \frac{1}{\left(1 - \left(\frac{m}{r}\right)^4\right)^{\frac{1}{2}}}, \quad dt = hdr,$$

the metric (1.6) is regarded as a cohomogeneity one metric. Also this metric (1.6) is a Ricci-flat metric, but coefficients $a$, $b$ and $c$ satisfy the following system of ODE’s:

$$\frac{da}{dt} = \frac{a^2 - (b - c)^2}{bc} + 2,$$
$$\frac{db}{dt} = \frac{b^2 - (c - a)^2}{ca} + 2,$$
$$\frac{dc}{dt} = \frac{c^2 - (a - b)^2}{ab} + 2.$$

These equations are neither the Ricci flow nor the backward Ricci flow.

We now study cohomogeneity one metrics constructed from the Ricci flow solution on other groups, and construct Ricci-flat metrics on their space-time. Ricci-flat metrics that we construct in this paper are listed in Table 1.
### Preparation

We present the definition of a Milnor frame.

**Definition 2.1.** Let \( \{F_i\}_{i=1}^3 \) be a left-invariant moving frame on \( G \). If \( \{F_i\}_{i=1}^3 \) satisfies

\[
[F_2, F_3] = n_1 F_1, \quad [F_3, F_1] = n_2 F_2, \quad [F_1, F_2] = n_3 F_3,
\]

where \( n_i \in \{\pm 1, 0\} \), then \( \{F_i\}_{i=1}^3 \) is called a Milnor frame.

As is well known, 3-dimensional unimodular simply-connected Lie groups were classified by Milnor [Mil76].

**Proposition 2.2 (Mil76).** Let \( \{F_i\} \) be a Milnor frame. For signatures of \( \{n_i\} \), 3-dimensional unimodular simply-connected Lie groups are determined as Table 2.

| Signature | Lie groups | description |
|-----------|------------|-------------|
| \((-1,-1,-1),(+1,+1,+1)\) | \( SU(2) \) | simple |
| \((-1,-1,+1),(-1,+1,+1)\) | \( SL(2,\mathbb{R}) \) | simple |
| \((-1,-1,0),(+1,+1,0)\) | \( E(2) \) | solvable |
| \((-1,0,+1)\) | \( E(1,1) \) | solvable |
| \((-1,0,0),(+1,0,0)\) | \( H_3 \) | nilpotent |
| \((0,0,0)\) | \( \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \) | commutative |

Table 2: 3-dimensional unimodular simply-connected Lie groups
Remark 2.3. We may change order of $n_i$.

Let $(M^4, g)$ be a cohomogeneity one 4-dimensional manifold with respect to 3-dimensional Lie group $G$. Let $\{F_i\}_{i=1}^3$ be a Milnor frame of $G$, and $\{\theta_i\}$ the dual coframe of $\{F_i\}$. Then the metric is expressed as follows:

$$g = dt^2 + a(t)^2(\theta^1)^2 + b(t)^2(\theta^2)^2 + c(t)^2(\theta^3)^2.$$  \hspace{1cm} (2.1)

From Theorem 1.1 (Koszul’s formula), the Levi-Civita connection is expressed as below.

**Proposition 2.4.** Let $g$ be the cohomogeneity one metric (2.1). Then the Levi-Civita connection is given by

$$\begin{pmatrix}
0 & \frac{a}{a} F_1 & \frac{1}{2} a \frac{b}{b} F_2 & \frac{1}{2} a \frac{c}{c} F_3 \\
\frac{a}{a} F_1 & -a \frac{\dot{a}}{\dot{a}} F_0 & -\frac{a}{a} \frac{b}{b} F_2 & -\frac{a}{a} \frac{c}{c} F_3 \\
\frac{b}{b} F_2 & \frac{1}{2} b \frac{c}{c} F_3 & -b \frac{\dot{b}}{\dot{b}} F_0 & -b \frac{c}{c} F_2 \\
\frac{c}{c} F_3 & \frac{1}{2} c \frac{b}{b} F_1 & -c \frac{\dot{c}}{\dot{c}} F_0 & -c \frac{a}{a} F_1
\end{pmatrix}$$  \hspace{1cm} (2.2)

Hence we obtain the Ricci tensor;

$$\text{Ric}(F_0, F_0) = R_{00} = -\dot{a} \frac{a}{a} - \frac{\dot{b}}{b} - \frac{\dot{c}}{c},$$  \hspace{1cm} (2.3a)

$$\text{Ric}(F_1, F_1) = R_{11} = -a \left(\frac{abc}{bc} \right) - \frac{(n_2 b^2 - n_3 c^2)^2 - n_1^2 a^4}{2 b^2 c^2},$$  \hspace{1cm} (2.3b)

$$\text{Ric}(F_2, F_2) = R_{22} = -b \left(\frac{abc}{ca} \right) - \frac{(n_3 c^2 - n_1 a^2)^2 - n_2^2 b^4}{2 c^2 a^2},$$  \hspace{1cm} (2.3c)

$$\text{Ric}(F_3, F_3) = R_{33} = -c \left(\frac{abc}{ab} \right) - \frac{(n_1 a^2 - n_2 b^2)^2 - n_3^2 c^4}{2 a^2 b^2},$$  \hspace{1cm} (2.3d)

and other components are 0. Remark that if $a$, $b$ and $c$ are constants, then the Ricci tensors $R_{11}$, $R_{22}$ and $R_{33}$ of the metric (2.1) are the Ricci tensors of a metric

$$g = a^2(\theta^1)^2 + b^2(\theta^2)^2 + c^2(\theta^3)^2.$$

### 3 The Heisenberg group

In this section, we consider the Heisenberg group $H_3$ and a cohomogeneity one with respect to $H_3$. Let $\{F_i\}$ be the Milnor frame of $H_3$, satisfying
\[ [F_1, F_2] = 0, [F_2, F_3] = F_1 \text{ and } [F_3, F_1] = 0; \text{ and } \{\theta_i\} \text{ the dual coframe of } \{F_i\}. \]

Then a left-invariant metric is expressed as

\[ g = a(\theta^1)^2 + b(\theta^2)^2 + c(\theta^3)^2. \] (3.1)

Therefore the Ricci flow equation is given by

\[ \frac{d}{dt}a = -\frac{a^2}{2bc}, \] (3.2a)
\[ \frac{d}{dt}b = \frac{a}{2c}, \] (3.2b)
\[ \frac{d}{dt}c = \frac{a}{2b}. \] (3.2c)

These equations were solved in [KM01] as follows:

\[
\begin{align*}
  a(t) &= a_0 b_0^\frac{1}{3} c_0^\frac{1}{3} (\frac{3}{2}t + b_0 c_0/a_0)^{-\frac{1}{3}}, \\
  b(t) &= a_0 b_0^\frac{2}{3} c_0^{-\frac{1}{3}} (\frac{3}{2}t + b_0 c_0/a_0)^{\frac{1}{3}}, \\
  c(t) &= a_0 b_0^{-\frac{1}{3}} c_0^\frac{2}{3} (\frac{3}{2}t + b_0 c_0/a_0)^{\frac{1}{3}},
\end{align*}
\] (3.3)

where \(a_0 = a(0), \ b_0 = b(0)\) and \(c_0 = c(0)\). In particular, the behavior of this solution is the following:

**Lemma 3.1 ([KM01])**. The metric (3.1) satisfying (3.2) has three properties.

1. The solution of the Ricci flow equation exists on \((-T, +\infty)\), where \(T = \frac{b_0 c_0}{a_0}\).

2. If \(t \to -T\), then \(a \to +\infty, b \to 0\) and \(c \to 0\).

3. If \(t \to +\infty\), then \(a \to 0, b \to +\infty\) and \(c \to +\infty\).

We use this lemma later to describe the resulting Ricci-flat metric on the space-time. Next we consider a cohomogeneity one metric with respect to the Heisenberg group \(H_3\). Let \((M^4, g)\) be a cohomogeneity one with respect to \(H_3\). Using the above frame, a cohomogeneity one metric is described as

\[ g = dt^2 + a(t)^2(\theta^1)^2 + b(t)^2(\theta^2)^2 + c(t)^2(\theta^3)^2. \] (3.4)
Put $F_0 = \frac{\partial}{\partial t}$, then $\{F_i\}$ becomes an orthogonal frame. The Ricci tensor of this metric is given by

\[
\begin{align*}
R_{00} &= -\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} - \frac{\ddot{c}}{c}, \\
R_{11} &= -a \left( \frac{(\dddot{abc})}{bc} \right) + \frac{a^4}{2b^2c^2}, \\
R_{22} &= -b \left( \frac{(\dddot{bca})}{ca} \right) - \frac{a^2}{2c^2}, \\
R_{33} &= -c \left( \frac{(\dddot{cab})}{ab} \right) - \frac{a^2}{2b^2}, \\
\end{align*}
\]

and other components are 0. Using (3.3), we construct a Ricci-flat metric in the following way:

**Theorem 3.2 ([GR00]).** Let $a$, $b$ and $c$ satisfy the Ricci flow equations on $H_3$:

\[
\begin{align*}
\frac{d}{dt}a &= -\frac{a^2}{2bc}, \\
\frac{d}{dt}b &= \frac{a}{2c}, \\
\frac{d}{dt}c &= \frac{a}{2b}. \\
\end{align*}
\]

Then $g$ becomes a Ricci-flat metric.

**Proof.** An easy computation shows that

\[
\begin{align*}
\frac{d^2}{dt^2}a &= \frac{a^3}{b^2c^2}, \\
\frac{d^2}{dt^2}b &= -\frac{a^2}{2bc^2}, \\
\frac{d^2}{dt^2}c &= -\frac{a^2}{2b^2c}, \\
\end{align*}
\]

and $ab$, $ac$ and $b/c$ are constants. Using this fact and the equation (2.3), we obtain the statement.

Sectional curvatures of this metric are

\[
\begin{align*}
K_{01} = K_{23} &= -\frac{a^2}{b^2c^2}, \\
K_{02} = K_{13} &= \frac{a^2}{2b^2c^2}, \\
K_{03} = K_{12} &= \frac{a^2}{2b^2c^2}. \\
\end{align*}
\]

From Proposition (3.1), we get the following proposition.
Proposition 3.3. Assume that $a_0b_0c_0$ is positive. Then

1. the parameter $t$ exists on $(-T, \infty)$, where $T = \frac{2b_0c_0}{3a_0}$.

2. If $t \to -T$, then $K_{ij} \to \pm \infty$.

3. If $t \to +\infty$, then $K_{ij} \to 0$.

Assume that $a_0b_0c_0$ is negative. Then

1. the parameter $t$ exists on $(-\infty, T')$, where $T' = -\frac{2b_0c_0}{3a_0}$.

2. If $t \to -\infty$, then $K_{ij} \to T'$.

3. If $t \to +\infty$, then $K_{ij} \to 0$.

Actually, this phenomenon is related with the hyper-Kähler structure. We confirm it from now on. Before we define almost complex structures, we put

\[ e_0 = dt, \quad e_1 = a\theta^1, \quad e_2 = b\theta^2, \quad e_3 = c\theta^3. \]

Then $\{e_i\}_{i=0}^3$ becomes an orthonormal coframe. Let $\{e_i\}$ be the orthonormal frame defined by $e^i(e_j) = \delta^i_j$. For $i = 1, 2$ and 3, we consider almost complex structures $J_i$ defined by

\[ J_i e_0 = e_i, \quad J_i e_j = -e_k, \quad (J_i)^2 = -id, \]

where $(i, j, k)$ are cyclic permutation of $\{1, 2, 3\}$. It is easy to check that $\{J_i\}$ satisfy $J_3 = -J_1J_2 = J_2J_1$. Moreover from direct calculation we have $N_{J_i}(F_j, F_k) = 0$ for all $i, j, k = 0, 1, 2, 3$ where $N_{J_i}$ is the Nijenhuis tensor of the almost complex structure $J_i$. Therefore the almost complex structures $\{J_i\}$ are all integrable. We consider a triple of 2-forms $(\omega^1, \omega^2, \omega^3)$, defined by

\[ \omega^i(X, Y) := g(J_iX, Y) \]

for $i = 1, 2, 3$. Then 2-forms $\omega_i$ are given by

\[ \omega^i = e^0 \wedge e^i - e^j \wedge e^k, \]

where $(i, j, k)$ are cyclic permutation of $\{1, 2, 3\}$. The above triple of 2-forms $(\omega^1, \omega^2, \omega^3)$ satisfies the following relations

\[ (\omega^i)^2 \neq 0, \quad \omega^i \wedge \omega^j = 0 \quad (i \neq j). \]
Since $de^0 = d(dt) = 0$, $de^1 = \dot{a}dt \wedge \theta^1 - a\theta^2 \wedge \theta^3$, $de^2 = \dot{b}dt \wedge \theta^2$, $de^3 = \dot{c}dt \wedge \theta^3$, we get
\[
d\omega^1 = (a - \dot{b}c - b\dot{c})dt \wedge \theta^2 \wedge \theta^3, \\
d\omega^2 = (\dot{a}c + a\dot{c})dt \wedge \theta^3 \wedge \theta^1, \\
d\omega^3 = (ab + a\dot{b})dt \wedge \theta^1 \wedge \theta^2.
\]

**Theorem 3.4.** Assume that the cohomogeneity one metric (3.4) has almost complex structures $J_i$. For any $i = 1, 2, 3$, $d\omega^i = 0$ if and only if $a$, $b$ and $c$ satisfy the Ricci flow equations on the Heisenberg group.

**Proof.** We see at once that $d\omega^i = 0$ if and only if
\[
a - \dot{b}c - b\dot{c} = 0, \quad (3.9a) \\
\dot{a}c + a\dot{c} = 0, \quad (3.9b) \\
\dot{a}b + a\dot{b} = 0. \quad (3.9c)
\]
Equations (3.9b) and (3.9c) imply that $ac$ and $ab$ are constants. Using equations (3.9a)-(3.9c), we compute that
\[
a = (bc)^{-1} \left( \frac{ab \cdot ac}{a^2} \right) = \left( \frac{1}{a^2} \right) \cdot a^2 bc = -2\dot{a} \frac{bc}{a}.
\]
Hence we get
\[
\dot{a} = -\frac{a^2}{2bc}.
\]
Since $\dot{a}b = -\dot{b}a = a^2/(2c)$ and $\dot{a}c = -\dot{c}a = a^2/(2b)$, we get
\[
\dot{b} = \frac{a}{2c}, \quad (3.11a) \\
\dot{c} = \frac{a}{2b}. \quad (3.11b)
\]
Therefore $a$, $b$ and $c$ satisfy the Ricci flow equations on $H_3$.

Conversely, $a$, $b$ and $c$ satisfy the Ricci flow equations on $H_3$, then $d\omega_i = 0$.\[\square\]

Therefore the cohomogeneity one metric with (3.2) becomes the hyper-Kähler metric.
The group of rigid motions of the Minkowski 2-space

In this section, we consider the group of rigid motions of the Minkowski 2-space. Let \( \{F_i\}_{i=1}^3 \) be the Milnor frame of \( E(1,1) \), satisfying \([F_1, F_2] = -F_3, [F_3, F_1] = 0, [F_2, F_3] = F_1\), and \( \{\theta_i\} \) the dual coframe of \( \{F_i\} \). Then a left-invariant metric is expressed as

\[
g = a(\theta^1)^2 + b(\theta^2)^2 + c(\theta^3)^2. \tag{4.1}\]

Therefore the Ricci flow equation is given by

\[
\begin{align*}
\frac{d}{dt}a &= \frac{c^2 - a^2}{2bc}, \\
\frac{d}{dt}b &= \frac{(a + c)^2}{2ac}, \\
\frac{d}{dt}c &= \frac{a^2 - c^2}{2ab}.
\end{align*} \tag{4.2}
\]

Lemma 4.1 ([KM01]). The Ricci flow on \( E(1,1) \) has the following properties.

1. The solution of the Ricci flow equation (4.2) exists on \((T, +\infty)\), where \( T > 0 \) is depending only on initial data.

2. The quantity \( ac \) and \( b(c - a) \) are conserved.

3. Put \( \rho := a/c \), then

\[
\begin{align*}
\frac{d}{dt}b &= \frac{(1 + \rho)^2}{2\rho}, \\
\frac{d}{dt}\rho &= \frac{1 - \rho^2}{b}.
\end{align*}
\]

4. \( \rho_0 < \rho_\infty \leq 1 \) or \( 1 \leq \rho_\infty < \rho_0 \).

5. If \( \rho \neq 1 \), then

\[
b = k_0 \sqrt{\rho} \frac{\sqrt{\rho}}{1 - \rho},
\]

where \( k_0 = b_0 \frac{|1 - \rho_0|}{\sqrt{\rho_0}} \).
6. If \( t \to +\infty \), then \( \rho \to 1 \) and \( b \to \infty \).

7. If \( t \to +\infty \), then \( a \) and \( c \) converge to \( a_\infty = c_\infty = \sqrt{a_0c_0} \).

We use this proposition later to describe the asymptotic behavior of the resulting Ricci-flat metric on the space-time.

**Remark 4.2.** If \( a = c \), then the metric (4.1) becomes a non-gradient expanding Ricci soliton. See Theorem ??.

Next, let \((M^4, g)\) be a cohomogeneity one with respect to \( E(1,1) \). Let \( \{F_i\}_{i=1}^3 \) and \( \{\theta_i\} \) be as before. Then a cohomogeneity one metric is expressed as

\[
g = dt^2 + a(t)^2(\theta^1)^2 + b(t)^2(\theta^2)^2 + c(t)^2(\theta^3)^2. \tag{4.3}
\]

Putting \( F_0 = \frac{\partial}{\partial t} \), we have an orthogonal frame \( \{F_i\}_{i=1}^4 \). The Ricci tensor is computed as

\[
R_{00} = -\frac{\ddot{a}}{a} - \frac{\ddot{b}}{b} - \frac{\ddot{c}}{c},
\]

\[
R_{11} = -a \frac{(\dot{a}bc)}{bc} + \frac{a^4 - c^4}{2b^2c^2},
\]

\[
R_{22} = -b \frac{(\dot{b}ca)}{ca} - \frac{(a^2 + c^2)^2}{2a^2c^2},
\]

\[
R_{33} = -c \frac{(\dot{c}ab)}{ab} + \frac{c^4 - a^4}{2a^2b^2},
\]

and other component are 0. The Ricci flow equation for (4.1) is equivalent to the following:

\[
\frac{d}{dt}a = \frac{c^2 - a^2}{2bc},
\]

\[
\frac{d}{dt}b = \frac{(c + a)^2}{2ac},
\]

\[
\frac{d}{dt}c = \frac{a^2 - c^2}{2ab}. \tag{4.5}
\]
Since \( \frac{d}{dt}(ac) = 0 \), it is easy to check that

\[
R_{22} = -b \frac{(bca)'}{ca} - \frac{(c^2 + a^2)^2}{2a^2c^2},
\]

\[
= -\ddot{b} - \frac{(c^2 + a^2)^2}{2a^2c^2},
\]

\[
= \frac{(c^2 - a^2)^2}{2a^2c^2} - \frac{(c^2 + a^2)^2}{2a^2c^2} = -2,
\]

and this metric is not a Ricci-flat metric.

**Remark 4.3.** The metric (4.3) with (4.5) is a Ricci soliton if and only if \( a = -c \), in other words, \( a, b \) and \( c \) are constants. Furthermore, this Ricci soliton is the product metric on \( \mathbb{R} \times E(1, 1) \) and a non-gradient expanding Ricci soliton. We can check in the following way. First, because \( E(1, 1) \approx \mathbb{R}^3 \), we can use standard coordinates \((x_1, x_2, x_3)\) on \( \mathbb{R}^3 \). Secondly, since left-invariant vector fields is given by

\[
F_1 = e^{x_2} \frac{\partial}{\partial x_3} + e^{-x_2} \frac{\partial}{\partial x_1},
\]

\[
F_2 = \frac{\partial}{\partial x_2},
\]

\[
F_3 = e^{x_2} \frac{\partial}{\partial x_3} - e^{-x_2} \frac{\partial}{\partial x_1},
\]

then we can write the Ricci soliton equation using standard coordinates \((x_1, x_2, x_3)\) on \( \mathbb{R}^3 \), and we will obtain conditions of the Ricci soliton, for example,

\[
X = \frac{\alpha t}{2} F_0 + e^{-x_2} \frac{\alpha x_3}{4} F_1 + \frac{\alpha}{4} F_2, \quad \alpha = \frac{4}{b^2}.
\]

Lastly, we check that \( \nabla_i X_j \) is not symmetric. Therefore this Ricci soliton is a non-gradient expanding Ricci soliton.

Next, we consider a cohomogeneity one metric

\[
g = dt^2 + a(t)^2(\theta^1)^2 - b(t)^2(\theta^2)^2 - c(t)^2(\theta^3)^2.
\]

(4.7)

This metric has signature \((2, 2)\). Put \( F_0 = \frac{\partial}{\partial t} \). Then \( \{ F_i \} \) is the orthogonal
frame. The Ricci tensor is given by

\[ R_{00} = -\frac{\ddot{a}}{a} - \frac{\dot{b}}{b} - \frac{\dot{c}}{c}, \]

\[ R_{11} = -a \frac{(\dot{abc})}{bc} + \frac{a^4 - c^4}{2b^2c^2}, \]

\[ R_{22} = b \frac{(\dot{bca})}{ca} + \frac{(c^2 - a^2)^2}{2a^2c^2}, \]

\[ R_{33} = c \frac{(\dot{cab})}{ab} + \frac{a^4 - c^4}{2a^2b^2}. \]

These formulas imply the following theorem:

**Theorem 4.4.** Let \( a, b \) and \( c \) satisfy the Ricci flow equations;

\[ \frac{d}{dt}a = \frac{c^2 - a^2}{2bc}, \]

\[ \frac{d}{dt}b = \frac{(c + a)^2}{2ca}, \]

\[ \frac{d}{dt}c = \frac{a^2 - c^2}{2ab}. \]

Then the metric (4.7) becomes a Ricci-flat metric.

**Proof.**

\[ \frac{d^2}{dt^2}a = \frac{(a^2 - c^2)(2a^2 + ac + c^2)}{ab^2c^2}, \]

\[ \frac{d^2}{dt^2}b = -\frac{(c^2 - a^2)^2}{2a^2bc^2}, \]

\[ \frac{d^2}{dt^2}c = \frac{(a^2 - c^2)(a^2 + ac + 2c^2)}{2a^2b^2c}, \]

and \( ac \) is a constant. Using this fact, we obtain the statement. \( \square \)

Sectional curvatures of the metric (4.7) satisfying (4.2) are computed as

\[ K_{01} = K_{23} = \frac{(c^2 - a^2)(2a^2 + ac + c^2)}{2a^2b^2c^2} = \frac{(1 - \rho^2)(2\rho^2 + \rho + 1)}{2\rho^2b^2}, \]

\[ K_{02} = K_{13} = \frac{(c^2 - a^2)^2}{2a^2b^2c^2} = \frac{(1 - \rho^2)^2}{2\rho^2b^2}, \]

\[ K_{03} = K_{12} = -\frac{(c^2 - a^2)(a^2 + ac + 2c^2)}{2a^2b^2c^2} = -\frac{(1 - \rho^2)(\rho^2 + \rho + 2)}{2\rho^2b^2}, \]

15
where \( \rho := a/c \). Notice that if \( a^2 \neq c^2 \), then signatures of sectional curvatures \( K_{ij} \) are decided. From Proposition (4.1), we get the following proposition.

**Proposition 4.5.** Let \( a, b \) and \( c \) satisfy the Ricci flow equations (4.2).

1. If \( a^2 = c^2 \), then the metric (4.7) becomes a zero curvature metric.
2. If \( a^2 \neq c^2 \), then the metric (4.7) is a Ricci-flat metric, but \( K_{ij} \) are not constant.

Assume that \( a_0c_0 \) and \( b_0 \) are positive.

1. The Ricci flow exists on \((-T, \infty)\), where \( T \) depends on the initial data.
2. If \( t \to +\infty \), then all sectional curvatures \( K_{ij} \to 0 \).
3. If \( t \to -T \), then \( K_{ij} \to \pm\infty \).

Assume that \( a_0c_0 \) is positive, and \( b_0 \) is negative.

1. The Ricci flow exists on \((-\infty, T)\), where \( T \) depends on initial data.
2. If \( t \to -T \), then sectional curvature \( K_{ij} \to 0 \).
3. If \( t \to \infty \), then \( K_{ij} \to \pm\infty \).

Assume that \( a_0c_0 \) is negative, and \( b_0 \) is positive. Then the behavior of \( a, b \) and \( c \) is the backward Ricci flow of \( E(2) \) (see (5.1)).

Assume that \( a_0c_0 \) and \( b_0 \) are negative. Then the behavior of \( a, b \) and \( c \) is the Ricci flow of \( E(2) \) (see (5.1)).

### 5 The group of rigid motions of the Euclidean 2-space

In this section, we consider the group \( E(2) \) of rigid motions of the Euclidean 2-space. Let \( \{F_i\}_{i=1}^3 \) be the Milnor frame of \( E(2) \), satisfying

\[
[F_2, F_3] = -F_1, \quad [F_3, F_1] = 0, \quad [F_1, F_2] = -F_3,
\]

and \( \{\theta_i\} \) dual coframe of \( \{F_i\} \). Then a left-invariant metric is expressed as

\[
g = a(\theta^1)^2 + b(\theta^2)^2 + c(\theta^3)^2.
\]
Therefore the Ricci flow equation is equivalent to

\[
\begin{cases}
\frac{da}{dt} = \frac{c^2 - a^2}{2bc}, \\
\frac{db}{dt} = \frac{(c - a)^2}{2ca}, \\
\frac{dc}{dt} = \frac{a^2 - c^2}{2ab}.
\end{cases}
\]  \tag{5.1}

Since \(E(1,1)\) and \(E(2)\) are solvable, so the Ricci flow equations (5.1) resemble as the Ricci flow equations (4.2). The behavior of the solution of (5.1) is known in [KM01,CK04].

**Lemma 5.1** ([KM01,CK04]). *The Ricci flow on \(E(2)\) has the following properties.*

1. The solution of the Ricci flow equation (5.1) exists on \((-T, \infty)\), where \(T > 0\) is depending only on initial data.

2. The quantities \(ac\) and \(b(c + a)\) are preserved.

3. Put \(k := a/c\). Then we get

\[
\frac{db}{dt} = \frac{(1 - k)^2}{2k},
\]
\[
\frac{dk}{dt} = \frac{1 - k^2}{b}.
\]

4. If \(t \to \infty\), then \(k \to 1\) and \(b \to b_\infty\).

5. The coefficient \(b\) satisfies

\[
b = l_0 \frac{\sqrt{k}}{1 + k},
\]

where \(l_0 = b_0 \frac{1 + k_0}{\sqrt{k_0}}\).

We use above proposition later to describe the asymptptotic property of the resulting Ricci-flat metric on the space-time.

**Remark 5.2.** Since \(E(2)\) is dynamically stable, the Ricci flow on \(E(2)\) converges to an Einstein metric. The details are written in [IJ92,Ses06,SSS08].
Next, we consider a cohomogeneity one metric with respect to $E(2)$. Let $\{F_i\}_{i=1}^3$ and $\{\theta_i\}$ be as before. Then the metric is expressed as

$$g = dt^2 + a(t)^2(\theta^1)^2 + b(t)^2(\theta^2)^2 + c(t)^2(\theta^3)^2.$$  \hfill (5.2)

The Ricci tensor of this metric is given by

\begin{align*}
R_{00} &= -\ddot{a}a - \ddot{b}b - \ddot{c}c, \\
R_{11} &= -a(\dddot{abc})bc + \frac{a^4 - c^4}{2b^2c^2}, \\
R_{22} &= -b(\dddot{bca})ca - \frac{(a^2 - c^2)^2}{2a^2c^2}, \\
R_{33} &= -c(\dddot{cab})ab + \frac{c^4 - a^4}{2a^2b^2},
\end{align*}

and other components are 0. Using the solution of (5.1), we construct a Ricci-flat metric on the space-time in the following way:

**Theorem 5.3.** If $a$, $b$ and $c$ satisfy the Ricci flow equations

$$\begin{cases}
\frac{da}{dt} = \frac{c^2 - a^2}{2bc}, \\
\frac{db}{dt} = \frac{(a - c)^2}{2ac}, \\
\frac{dc}{dt} = \frac{c^2 - a^2}{2ab},
\end{cases}$$ \hfill (5.4)

then this metric [(5.2)] becomes a Ricci-flat metric.

**Proof.** This is proved by routine calculations. \hfill $\Box$

Sectional curvatures of this metric are

\begin{align*}
K_{01} = K_{23} &= \frac{(c^2 - a^2)(2a^2 - ac + c^2)}{2a^2b^2c^2} = \frac{(1 - k^2)(2k^2 - k + 1)}{2b^2k^2}, \\
K_{02} = K_{13} &= \frac{(c^2 - a^2)^2}{2a^2b^2c^2} = \frac{(1 - k^2)^2}{2b^2k^2}, \\
K_{03} = K_{12} &= -\frac{(c^2 - a^2)(a^2 - ca + 2c^2)}{2a^2b^2c^2} = -\frac{(1 - k^2)(k^2 - k + 2)}{2b^2k^2},
\end{align*}

where $k := a/c$. We can observe similarities between the Ricci flow solutions on the space-times of the left-invariant Ricci flow solutions of $E(1,1)$ and $E(2)$. For instance, we get the following proposition.
Proposition 5.4. 1. If $a^2 = c^2$, then this metric is a flat curvature metric.

2. If $a^2 \neq c^2$, then this metric is a Ricci-flat metric but not a constant curvature metric.

Let $a_0 c_0$ and $b_0$ be positive.

1. The Ricci flow equation exists on $(-T, \infty)$, where $T$ depends on initial data.

2. If $t \to +\infty$, then $K_{ij} \to 0$.

Let $a_0 c_0$ be positive, and $b_0$ be negative.

1. The Ricci flow equation exists on $(-\infty, T)$, where $T$ depends on initial data.

2. If $t \to -T$, then sectional curvatures $K_{ij} \to 0$.

3. If $t \to \infty$, then $K_{ij} \to \pm \infty$.

Let $a_0 c_0$ be negative, and $b_0$ be positive. Then behavior of $a$, $b$ and $c$ is the backward Ricci flow of $E(1, 1)$ (see (4.2)). Let $a_0 c_0$ and $b_0$ be negative. Then behavior of $a$, $b$ and $c$ is the Ricci flow of $E(1, 1)$ (see (4.2)).

Next, we define almost complex structures. Using $E(2) \approx \mathbb{R}^3$, we can write

$$F_1 = \sin y \frac{\partial}{\partial x} + \cos y \frac{\partial}{\partial z},$$

$$F_2 = \frac{\partial}{\partial y},$$

$$F_3 = \cos y \frac{\partial}{\partial x} - \sin y \frac{\partial}{\partial z},$$

and

$$\theta^1 = \sin y \cdot dx + \cos y \cdot dz,$$

$$\theta^2 = dy,$$

$$\theta^3 = \cos y \cdot dx - \sin y \cdot dz.$$
Then the metric on $E(2)$ with $a = b = c = 1$ is

$$g_3 = (\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2 = dx^2 + dy^2 + dz^2.$$  

Let $\{e_i\}_{i=0}^3$ be an orthonormal coframe defined by

$$e^0 = dt, \quad e^1 = a\theta^1, \quad e^2 = b\theta^2, \quad e^3 = c\theta^3.$$  

Then $\{e^i\}_{i=0}^3$ becomes an orthonormal coframe of cohomogeneity one metric $g$. Let $\{e_i\}$ be the orthonormal frame, satisfying $e^i(e_j) = \delta^i_j$. Since $\mathbb{R}^4 \cong \mathbb{C}^2$ has natural almost complex structures $\{J_i\}$, we analogously define almost complex structures $J_1$ and $J_2$, satisfying

$$J_1e_0 = e_2, \quad J_1e_3 = e_1, \quad (J_1)^2 = -id,$$

$$J_2e_0 = \cos y \cdot e_1 - \sin y \cdot e_3,$$

$$J_2e_2 = \sin y \cdot e_1 + \cos y \cdot e_3,$$

$$(J_2)^2 = -id.$$  

If we put $J_3 := J_2J_1 = -J_1J_2$, then $J_3$ becomes an almost complex structure. From direct computation we have $N_{J_i}(F_j, F_k) = 0$ for all $i, j, k = 0, 1, 2, 3$ where $N_{J_i}$ is the Nijenhuis tensor of $J_i$. Therefore $J_i$‘s are all integrable. We consider a triple of 2-forms $(\omega^1, \omega^2, \omega^3)$, defined by

$$\omega^i(X, Y) := g(J_iX, Y)$$  

for $i = 1, 2, 3$. Then 2-forms $\omega_i$ are given by

$$\omega^1 = e^0 \wedge e^2 + e^3 \wedge e^1,$$

$$\omega^2 = \cos y \cdot (e^0 \wedge e^1 + e^2 \wedge e^3) - \sin y \cdot (e^0 \wedge e^3 + e^1 \wedge e^2),$$

$$\omega^3 = \sin y \cdot (e^0 \wedge e^1 + e^2 \wedge e^3) + \cos y \cdot (e^0 \wedge e^3 + e^1 \wedge e^2).$$  

The above triple of 2-forms $(\omega^1, \omega^2, \omega^3)$ satisfies the following relations:

$$(\omega^i)^2 \neq 0, \quad \omega^i \wedge \omega^j = 0 \ (i \neq j).$$  

Since $de^0 = d(dt) = 0, \ de^1 = \dot{a}dt \wedge \theta^1 + a\theta^2 \wedge \theta^3, \ de^2 = \dot{b}dt \wedge \theta^2, \ de^3 = \dot{c}dt \wedge \theta^3 + c\theta^1 \wedge \theta^2$, we get

$$d\omega^1 = (\dot{c}a + c\dot{a})dt \wedge \theta^3 \wedge \theta^1,$$

$$d\omega^2 = -(\dot{a}b + \dot{a}b + a - c) \sin y \cdot dt \wedge \theta^1 \wedge \theta^2 + (\dot{b}c + \dot{b}c + c - a) \cos y \cdot dt \wedge \theta^2 \wedge \theta^3,$$

$$d\omega^3 = -(\dot{a}b + \dot{a}b + a - c) \cos y \cdot dt \wedge \theta^1 \wedge \theta^2 + (\dot{b}c + \dot{b}c + c - a) \sin y \cdot dt \wedge \theta^2 \wedge \theta^3.$$  

20
Theorem 5.5. The cohomogeneity one metric $g$ expressed as (5.4), where the triple $\{a(t), b(t), c(t)\}$ satisfies the Ricci flow equation on $E(2)$, is a hyper-Kähler metric. Conversely the cohomogeneity one metric $g$ as in (5.2) satisfies $d\omega^i = 0$ for any $i = 1, 2, 3$, then the triple $\{a(t), b(t), c(t)\}$ satisfies the Ricci flow equations on $E(2)$.

Proof. It is straightforward to check that $d\omega^i = 0$ if and only if the following hold:

\begin{align}
\dot{c}a + c\dot{a} &= 0, \\
\dot{a}b + ab &= c - a, \\
\dot{b}c + bc &= a - c.
\end{align}

(5.11a) implies that $ca$ is constant. Solving the equations for $(\dot{a}, \dot{b}, \dot{c})$, we obtain the statement of the theorem. \qed

Therefore the cohomogeneity one metric (5.4) becomes a hyper-Kähler metric.

Remark 5.6. In this section, we obtain the cohomogeneity one metric with respect to $E(2)$ with the hyper-Kähler structure $\{J_i\}$. From the definition of $\{J_i\}$, the hyper-Kähler structure is non-trivial. The cohomogeneity one metric with respect to $E(2)$ with the trivial hyper-Kähler structure (i.e. $J_i e_0 = e_i, J_i e_j = e_k$, where $(i, j, k)$ are cyclic permutation of $\{1, 2, 3\}$,) is also a hyper-Kähler structure, however the triple of functions $\{a(t), b(t), c(t)\}$ does not satisfy the Ricci flow equation. So we do not consider it deeply.

6 \quad SL(2, \mathbb{R})

In this section, we consider $SL(2, \mathbb{R})$. Let $(M^4, g)$ be a cohomogeneity one with respect to $SL(2, \mathbb{R})$. Let $\{F_i\}_{i=1}^3$ be the Milnor frame of $SL(2, \mathbb{R})$, satisfying

$[F_1, F_2] = -F_3, \quad [F_2, F_3] = F_1, \quad [F_3, F_1] = F_2,$

and $\{\theta_i\}$ the dual coframe of $\{F_i\}$. The metric is expressed as

$$g = dt^2 + a(t)^2(\theta^1)^2 + b(t)^2(\theta^2)^2 + c(t)^2(\theta^3)^2.$$ (6.1)
The Ricci tensor of this metric is given by

\begin{align*}
R_{00} &= -\ddot{a} + \frac{\ddot{b}}{b} - \frac{\ddot{c}}{c}, \\
R_{11} &= -a \frac{(abc)}{bc} - \frac{(b^2 + c^2)^2 - a^4}{2b^2c^2}, \\
R_{22} &= -b \frac{(bca)}{ca} - \frac{(a^2 + c^2)^2 - b^4}{2a^2c^2}, \\
R_{33} &= -c \frac{(cab)}{ab} - \frac{(a^2 - b^2)^2 - c^4}{2a^2b^2},
\end{align*}

and other component are 0.

**Proposition 6.1.** If \(a, b\) and \(c\) satisfy the Ricci flow equations of \(SL(2, \mathbb{R})\)

\begin{align*}
\begin{aligned}
\frac{da}{dt} &= (b + c)^2 - a^2, \\
\frac{db}{dt} &= \frac{(c + a)^2}{2bc} - b^2, \\
\frac{dc}{dt} &= \frac{(a - b)^2}{2abc} - c^2,
\end{aligned}
\end{align*}

(6.3)

then this metric (6.1) satisfying (6.4) is not a Ricci-flat metric.

**Proof.** The proof is a straightforward calculation. \(\square\)

Next, we consider the metric of the form

\[ g = dt^2 - a(t)^2(\theta^1)^2 - b(t)^2(\theta^2)^2 + c(t)^2(\theta^3)^2. \]

(6.4)

This metric (6.4) has signature (2, 2).

The Ricci tensor of this metric is computed as

\begin{align*}
R_{00} &= -\ddot{a} + \frac{\ddot{b}}{b} - \frac{\ddot{c}}{c}, \\
R_{11} &= a \frac{(abc)}{bc} + \frac{(b^2 - c^2)^2 - a^4}{2b^2c^2}, \\
R_{22} &= b \frac{(bca)}{ca} + \frac{(a^2 - c^2)^2 - b^4}{2a^2c^2}, \\
R_{33} &= -c \frac{(cab)}{ab} - \frac{(a^2 - b^2)^2 - c^4}{2a^2b^2},
\end{align*}

22
and other components are 0. The Ricci flow equations of \( SL(2, \mathbb{R}) \) are given by

\[
\begin{align*}
\frac{d}{dt}a &= \frac{(b + c)^2 - a^2}{2bc}, \\
\frac{d}{dt}b &= \frac{(c + a)^2 - b^2}{2ca}, \\
\frac{d}{dt}c &= \frac{(a - b)^2 - c^2}{2ab}.
\end{align*}
\]

(6.6)

In this case, changing \( a \) into \( -a \) and \( b \) into \( -b \), therefore the Ricci flow equations of \( SL(2, \mathbb{R}) \) change into

\[
\begin{align*}
\frac{d}{dt}a &= \frac{(b - c)^2 - a^2}{2bc}, \\
\frac{d}{dt}b &= \frac{(c - a)^2 - b^2}{2ca}, \\
\frac{d}{dt}c &= \frac{(a - b)^2 - c^2}{2ab}.
\end{align*}
\]

(6.7)

This system of equations is nothing but the Ricci flow equations of left-invariant metrics on \( SU(2) \).

**Theorem 6.2.** If \( a, b \) and \( c \) satisfy the Ricci flow equations of \( SU(2) \), then the metric (6.4) satisfying (6.7) becomes a Ricci-flat metric of signature \((2, 2)\).

**Proof.** The proof method of this theorem is similar to Theorem 1.3. The proof is straightforward. \(\square\)

**References**

[Bes08] A. L. Besse, *Einstein manifolds*, Classics in Mathematics, Springer-Verlag, Berlin, 2008. Reprint of the 1987 edition.

[BD07] P. Baird and L. Danielo, *Three-dimensional Ricci solitons which project to surfaces*, J. Reine Angew. Math. **608** (2007), 65–91.

[CGLP04a] M. Cvetič, G. W. Gibbons, H. Lü, and C. N. Pope, *New cohomogeneity one metrics with Spin(7) holonomy*, J. Geom. Phys. **49** (2004), no. 3-4, 350–365.

[CGLP04b] ______, *Orientifolds and slumps in \( G_2 \) and Spin(7) metrics*, Ann. Physics **310** (2004), no. 2, 265–301.
[Nom81] K. Nomizu, Introduction to Modern Differential Geometry, Interscience Tracts in Pure and Applied Mathematics, No. 15 Vol. II, Shoka-bo, 1981 (Japanese).

[Nod08] T. Noda, A special Lagrangian fibration in the Taub-NUT space, J. Math. Soc. Japan 60 (2008), no. 3, 653–663.

[Per02] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, preprint: math. DG/0211159 (2002).

[Ses06] N. Sesum, Linear and dynamical stability of Ricci-flat metrics, Duke Math. J. 133 (2006), no. 1, 1–26.

[SSS08] O. C. Schnürer, F. Schulze, and M. Simon, Stability of Euclidean space under Ricci flow, Comm. Anal. Geom. 16 (2008), no. 1, 127–158.

[TY90] G. Tian and S.-T. Yau, Complete Kähler manifolds with zero Ricci curvature. I, J. Amer. Math. Soc. 3 (1990), no. 3, 579–609.

[Ura] H. Urakawa, Equivariant theory of Einstein metrics on Riemannian manifolds of cohomogeneity one, Unpublished.