Barrier crossing induced by very slow external noise

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Abstract

We consider the motion of a particle in a force field subjected to adiabatic, fluctuations of external origin. We do not put the restriction on the type of stochastic process that the noise is Gaussian. Based on a method developed earlier by us [ J. Phys. A 31 (1998) 3937, 7301] we have derived the equation of motion for probability distribution function for the particle on a coarse-grained timescale $\Delta t$ assuming that it satisfies the separation of timescales; $|\mu|^{-1} \ll \Delta t \ll \tau_c$, where $\tau_c$ is the correlation time of fluctuations. $|\mu|^{-1}$ refers to the inverse of the damping rate (or, the largest of the eigenvalues of the unperturbed system) and sets the shortest timescale in the dynamics in contrast to the conventional theory of fast fluctuations. The equation includes a third order noise term. We solve the equation for a Kramers’ type potential and show that although the system is thermodynamically open, appropriate boundary conditions allow the distinct steady states. Based on the exact solution of the third order equation for the linearized potential and the condition for attainment of the steady states we calculate the adiabatic noise-induced rate of escape of a particle confined in a well. A typical variation of the escape rate as a function of dissipation which is reminiscent of Kramers’ turn-over problem, has been demonstrated.

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I. INTRODUCTION

About a century ago Einstein formulated the problem of Brownian motion as what is known today as stochastic processes. To explain the motion of a Brownian particle on an observable macroscopic scale he introduced the coarse-graining of space and time in the dynamics. This has served as the basis for generation of successive levels of description from the microscopic to macroscopic realm in subsequent many other formulations. Although there exists no general program of coarse-graining it has been possible to describe the dynamics fairly realistically in terms of some systematic separation of timescales consistent with experiments.

In dealing with the stochastic processes one essentially examines the average motion of a system subjected to fluctuations which may be fast or slow depending on the correlation time $\tau_c$ of the fluctuations compared to coarse-grained time scale, $\Delta t$ over which one observes the average motion of the particle. While the stochastic processes with short correlation time are well understood, significant progress has been made in recent times in theories dealing with arbitrary long correlation time. However, one important constraint in this context needs to be emphasized. This is that the separation of timescales is just not enough and one has to specify further the nature of stochastic process by assuming it to be either Markovian or Gaussian or both. In fact these lie at the heart of the overwhelming majority of the traditional theories so far. The Ornstein-Uhlenbeck process is an age old standard paradigm in this respect.

Very recently we have proposed a method for analyzing the stochastic dynamics of a particle subjected to external, adiabatically slow fluctuations where we do not limit ourselves to Markovian or Gaussian processes. It is apparent that to go beyond Markovian and Gaussian approximations the standard procedure of expansion of Master equation (based on Markov approximation) leading to Kramers-Moyal [KM] equation or characteristic function method (based on the calculation of moments and, in general, used for Gaussian processes) are not suitable for the purpose. Based on adiabatic following approximation we have derived the appropriate equations of motion for the probability distribution function on a coarse-grained timescale $\Delta t$ assuming that $\Delta t$ satisfies

$$\frac{1}{|\mu|} \ll \Delta t \ll \tau_c$$

where $\frac{1}{|\mu|}$ refers to the inverse of the damping rate (or the inverse of the largest eigenvalue of the ‘unperturbed’ system. This is in contrast to fast fluctuations characterized by the inequality $\tau_c \ll \Delta t \ll \frac{1}{|\mu|}$). $\frac{1}{|\mu|}$ therefore refers to the shortest timescale in the dynamics in the present investigation compared to short correlation time $\tau_c$ of the fast stochastic
processes. The separation of the timescales (1.1) has been critically analyzed by Masoliver et al. in their treatment of generalized Langevin equation for studying diffusion with external noise.

In this paper we extend our earlier treatment to analyze and present a solution for the problem of motion of a particle in a force field subjected to external, adiabatic noise pertaining to the separation of timescales (1.1). The potential we consider here is of Kramers’ type as shown in Fig.(1). The physical motivation behind the solution is two fold :

First, since it is subjected to fluctuations of external origin, the system is thermodynamically open (the close system on the other hand is characterized by internal noise which satisfies the fluctuation-dissipation relation). We search for the condition of an appropriate steady state for this open system. Similar studies for Gaussian processes have been carried out by Masoliver and others. The study of open systems are specifically relevant for describing the effect of pump fluctuations on the emission of a dye laser, effect of fluctuating rate constants on a chemical reaction, and effect of noise on parametric oscillator, etc. Second, once the condition for attainment of the steady state is realized, it becomes possible to consider the situation such that the particle in a force field, i.e., originally confined in a potential well may escape under the influence of external adiabatic noise by maintaining a steady state probability current over the barrier. It is therefore pertinent to calculate the rate of escape induced by this nonthermal activation in the spirit of Kramers and Smoluchowski and to elucidate the aspects of dependence of escape rate on dissipation. The counterpart of the latter issue in the theory of fast fluctuations is the well known turn-over problem.

We thus intend to touch the three issues in the theory of adiabatic fluctuations in the subsequent sections : First, based on the method of ‘adiabatic following approximations’ together with a systematic separation of timescales (1.1) to carry out an expansion in $|\mu|^{-1}$ as developed by us in two earlier papers, we obtain an equation of motion for probability distribution function for a particle in a force field simultaneously subjected to external adiabatic noise. The correlations of fluctuating forces give rise to second and third order diffusion coefficients. We discuss this issue in Sec. II. Second, although the system is thermodynamically open we look for the physically allowed steady states and show that application of the appropriate boundary conditions leads to distinct steady states. Sec. III is devoted to this aspect. Third, based on the exact solution of third order equation for the linearized potential and the condition for attainment of the steady states, we address the problem of escape of a particle confined in a well in the spirit of Kramers-Smoluchowski theory. This forms the subject matter of Sec. IV. We conclude this paper by summarizing the main results and their experimental relevance in Conclusion.
II. MOTION OF A PARTICLE IN A FORCE FIELD IN PRESENCE OF ADIABATIC FLUCTUATIONS

The equation of motion of a particle of unit mass in an one-dimensional extension where it is acted upon by an external field of force corresponding to a potential $V(x)$ and an external, adiabatic stochastic force $\xi(t)$, can be written as follows

$$\dot{x} = -\frac{1}{\beta} V'(x) + \frac{1}{\beta} \alpha \xi(t) .$$

(2.1)

Here we have considered the overdamped limit. $\beta$ is the dissipation constant and $\alpha$ which is necessary to keep track of the order of perturbation, is a parameter determining the size of fluctuations of the external noise $\xi(t)$. We emphasize here the two points: (i) we do not put restriction on the type of stochastic process $\xi(t)$ that the noise is Gaussian. This has attracted so much attention in the recent literature that it is necessary to point out that no such assumption has been made. The only restriction we make on the nature of the stochastic process $\xi(t)$ is that its correlation time $\tau_c$ is very long, i.e., it corresponds to the separation of the timescales implied in the inequality (1.1). Also note that the inequality implies the overdamped limit. (ii) We assume for convenience, without any loss of generality, that $\langle \xi(t) \rangle = 0$

In a preceding paper we have derived the equation of motion for probability density distribution function $P(x, t)$ in phase space corresponding to the Langevin description (2) where the associated timescale satisfies the inequality (1.1). We have shown that $P(x, t)$ obeys the differential equation of motion which contains third order terms (beyond the usual Fokker-Planck terms) leading to non-Gaussian noise. The appearance of these terms is generic for the stochastic process we consider here. The general expression for time evolution of probability density function is given by

$$\frac{\partial P(x, t)}{\partial t} = \left\{ -\nabla \cdot F_0 + \alpha^2 \nabla \cdot \int_0^\infty \langle F_1 \nabla_{-\tau} \cdot F_1(x^{-\tau}) \rangle \left| \frac{dx^{-\tau}}{dx} \right| d\tau \\
-\alpha^2 \nabla \cdot \int_0^\infty \tau \langle F_1 \nabla_{-\tau} \cdot \dot{F}_1(x^{-\tau}) \rangle \left| \frac{dx^{-\tau}}{dx} \right| d\tau \\
+\alpha^2 \nabla \cdot \int_0^\infty \tau \langle F_1 \nabla_{-\tau} \cdot F_1(x^{-\tau}) \nabla_{-\tau} \cdot F_0(x^{-\tau}) \rangle \left| \frac{dx^{-\tau}}{dx} \right| d\tau \right\} P(x, t) ,$$

(2.2)

where $F_0$ and $F_1$ refer to the unperturbed and the fluctuating terms, respectively, corresponding to Eq.(2) as given by

$$F_0 = -\frac{1}{\beta} V'(x) \quad \text{and} \quad F_1 = \frac{1}{\beta} \alpha \xi(t) .$$

(2.3)
The symbol $\nabla$ is used for the operator that differentiates everything that comes after it with respect to $x$. $\nabla_{-\tau}$ denotes the differentiation with respect to $x_{-\tau}$. The Jacobian $\left| \frac{dx_{-\tau}}{dx} \right|$ defines the mapping $x \to x_{-\tau}$ for the unperturbed motion and is given by

$$\left| \frac{dx_{-\tau}}{dx} \right| = 1 + \frac{1}{\beta} V''(x) \tau + \mathcal{O}(\tau^2) ,$$

where it has been assumed that $x$ varies very little in $\tau$ (in the scale of $\frac{1}{\beta}$). Also explicitly we have

$$\nabla_{-\tau} = \left[ 1 - \frac{1}{\beta} V''(x_{-\tau}) \tau \right] \frac{\partial}{\partial x} ,$$

$$F_0(x_{-\tau}) = -\frac{1}{\beta} V'(x_{-\tau}) = -\frac{1}{\beta} [V'(x) - \tau V''(x)]$$

and

$$F_1(x_{-\tau}) = \frac{1}{\beta} \xi(t - \tau) , \quad \dot{F}_1(x_{-\tau}) = \frac{1}{\beta} \frac{d\xi(t)}{dt} \bigg|_{(t - \tau)} .$$

Making use of the relations (4-8) and after collecting the terms of the order of $\alpha^2$, all the four terms in Eq.(3) can be simplified further to obtain [ some details are outlined in the Appendix-A ]:

$$\frac{\partial P(x,t)}{\partial t} = \frac{1}{\beta} \frac{\partial}{\partial x} \left[ V'(x) P(x,t) \right] + \alpha^2 c_{01} \frac{1}{\beta^2} \frac{\partial^2 P(x,t)}{\partial x^2} - \alpha^2 c_2 \frac{1}{\beta^3} \frac{\partial^3}{\partial x^3} \left[ V'(x) P(x,t) \right].$$

$c_0, c_1$ and $c_2$ in Eq.(9) are given by

$$c_{01} = c_0 - c_1$$
$$c_0 = \int_0^\infty \langle \xi(t) \xi(t - \tau) \rangle \, d\tau$$
$$c_1 = \tau c_0 \langle \xi^2(t) \rangle - c_0$$
$$c_2 = \int_0^\infty \tau \langle \xi(t) \xi(t - \tau) \rangle \, d\tau .$$

We now put $\alpha = 1$ for the rest of the treatment.

The above equation describes the time evolution of an overdamped particle in a force field (derivable from a potential $V(x)$) simultaneously subjected to an external adiabatic stochastic force. $c_0, c_1$ and $c_2$ measure the strength of the noise term. While the first term in Eq.(9) can be identified as the usual deterministic dynamical term, the second and the third terms refer to second and third order diffusion coefficients due to stochasticity $\xi(t)$. The remarkable departure from the standard form of Fokker-Planck equation (Smoluchowski equation) is due to the presence of the third order noise.
Let us now digress a little bit about Eq. (2.8) which forms the basis of this paper. The very appearance of third order term apparently suggests an immediate bearing with the familiar KM expansion. We emphasize that Eq. (2.8) is not a truncated KM expansion. The KM expansion is based on an expansion of Master equation in a series in $\tau$ and the terms of $O(\tau^2)$ are neglected. Or in other words the coefficients in a KM expansion originate from the corresponding moments (of the distribution function) which are assumed to be linear in $\tau$. This severely restricts the use of higher order moments for finite $\tau$. This is mathematically formalized in terms of Pawula Theorem (whose proof concerns the relation between the moments pertaining to the KM expansion only. Note that the familiar Wigner equation for probability distribution for a cubic potential is a third order equation which has nothing to do with a KM expansion or Pawula theorem) which precludes the occurrence of nonzero terms beyond Fokker-Planck in a KM expansion. Thus if there is any finite third order term it has to be treated as a perturbation. In many situations one encounters serious interpretive difficulties since the probability distribution functions often turn out to be negative.

The expansion leading to Eq. (2.8) on the other hand is based on an expansion [Eq. (9) of] in $\alpha\beta^{-1}$ where $\alpha$ is the strength of noise and $\beta$ is the damping constant. Note that $\beta^{-1}$ defines the shortest time scale in the dynamics. That the expansion is in $\alpha\beta^{-1}$ is evident from the appearance of $\frac{1}{\beta}, \frac{1}{\beta^2}$ and $\frac{1}{\beta^3}$ factors in the successive terms in Eq. (2.8). Although the convergence of formal KM expansion, in general, is not guaranteed (it is built in from outside as done by van Kampen in terms of $\Omega^{-1}$ expansion) it has been proved that our expansion scheme is convergent in the adiabatic following limit. The appearance of third derivative term is generic since it appears in the same order ($\alpha^2$-order) as the second derivative term in the present expansion scheme and is characteristic of non-Gaussian features. Thus the third order term cannot be treated as a perturbation.

Eq. (2.8) is derived on the basis of two approximations: (i) adiabatic following approximation which results in a convergent expansion in $\alpha\beta^{-1}$ (valid for very slow noise processes corresponding to the timescales (1.1) which is complementary to the cummulant expansion based on an expansion in $\alpha\tau_c$ (valid for very fast noise processes). (ii) decoupling approximation. We have calculated the error caused by decoupling and shown that it is of the order of $\alpha^2\beta^{-1}$. The corresponding error in decoupling in the case of fast fluctuations is $\alpha^2\tau_c$. We explicitly point out that no other approximation is required to develop the theory further as done in this paper and given the appropriate boundary conditions the derived solutions are well behaved positive definite probability distribution functions.
III. THE SOLUTION OF THIRD ORDER EQUATION FOR A KRAMERS’ TYPE POTENTIAL: STEADY STATE PROBABILITY DENSITY

A. The general solution

To start with we now recast the third order equation (9) in the form of the familiar continuity equation and identify the current $S(x, t)$ as follows;

$$\frac{\partial P(x, t)}{\partial t} = -\frac{\partial S(x, t)}{\partial x} \quad (3.1)$$

where

$$S(x, t) = -\frac{1}{\beta} V'(x) P - \frac{c_{01}}{\beta^2} \frac{\partial P}{\partial x} + \frac{c_2}{\beta^3} \frac{\partial^2}{\partial x^2} \{V'(x)P\} \quad (3.2)$$

In the steady state one puts

$$\frac{\partial P(x, t)}{\partial t} = 0 \quad (3.3)$$

to obtain the following steady state current $J$ as given by

$$J = -\frac{1}{\beta} V'(x) P_s - \frac{c_{01}}{\beta^2} \frac{\partial P_s}{\partial x} + \frac{c_2}{\beta^3} \frac{\partial^2}{\partial x^2} \{V'(x)P_s\} \quad (3.4)$$

where $P_s$ is the steady state probability distribution. Henceforth for the sake of brevity we omit the subscript $s$ from $P_s$ for all the subsequent calculations to follow and denote $P(x)$ as the steady state probability distribution function.

Multiplying both sides of Eq.(14) by $\beta^3/c_2$ we obtain

$$\frac{d^2}{dx^2} \{V'(x)P\} - a \frac{dP}{dx} - b \{V'(x)P\} = \frac{\beta^3 J}{c_2} \quad (3.5)$$

where $a$ and $b$ are given by

$$a = \frac{\beta c_{01}}{c_2} \quad \text{and} \quad b = \frac{\beta^2}{c_2} \quad (3.6)$$

We now explicitly make use of the Kramers’ type potential $V(x)$ as shown in Fig.(1) for the problem. We consider by linearizing it at $x = 0$,

$$V(x) = E_0 - \frac{1}{2} \omega_0^2 x^2 \quad (3.7)$$

where $\omega_0$ refers to frequency of the inverted well. $E_0$ defines the potential at the barrier top.

The probability distribution function $P(x)$ therefore satisfies the following equation
\[
x \frac{d^2 P}{dx^2} + \left( \frac{2 \omega_0^2 + a}{\omega_0^2} \right) \frac{dP}{dx} - b x P = -\frac{\beta^3 J}{c_2 \omega_0^2} .
\]

(3.8)

Defining
\[
\gamma = \frac{2 \omega_0^2 + a}{\omega_0^2} \quad \text{and} \quad D = -\frac{\beta^3 J}{c_2 \omega_0^2} ,
\]

(3.9)

the Eq.(18) reads as
\[
x \frac{d^2 P}{dx^2} + \gamma \frac{dP}{dx} - b x P = D .
\]

(3.10)

It is convenient to make the following substitution
\[
P(x) = x^{\frac{\nu}{2}}(1 - \gamma) W(x)
\]

(3.11)
in Eq.(20) to obtain
\[
x^2 \frac{d^2 W}{dx^2} + x \frac{dW}{dx} - \left[ \frac{1}{4} \left( \gamma - 1 \right)^2 + b x^2 \right] W = D x^{\frac{1}{2}(1 + \nu)} .
\]

(3.12)

Let
\[
\nu = \frac{1}{2} (\gamma - 1) .
\]

(3.13)

We then rewrite Eq.(22) as follows,
\[
x^2 \frac{d^2 W}{dx^2} + x \frac{dW}{dx} - (\nu^2 + b x^2) W = D x^{1 + \nu} .
\]

(3.14)

From the definitions of \( \gamma \) and \( a \) (Eqs.(19) and (23)) we have
\[
\nu = \frac{1}{2} \left( 1 + \frac{a}{\omega_0^2} \right) = \frac{1}{2} \left( 1 + \frac{\beta c_{01}}{\omega_0^2 c_2} \right) .
\]

(3.15)

The structure of \( c_{01} = c_0 - c_1 \) suggests (see the definition (10)) that by virtue of adiabatic stochasticity, \( c_1 \) is much smaller compared to \( c_0 \) and \( c_{01} \) is always positive. This ensures that \( \nu \) as defined in Eq.(25) is always positive.

It is convenient to make a further substitution for independent variable \( x \) as
\[
\zeta = \sqrt{b} x .
\]

(3.16)

This reduces Eq.(24) to
\[
\zeta^2 \frac{d^2 W}{d\zeta^2} + \zeta \frac{dW}{d\zeta} - (\nu^2 + \zeta^2) W = \frac{D}{b^{\frac{1}{2}(1+\nu)}} \zeta^{1+\nu} .
\]

(3.17)
The homogenous counterpart corresponding to the above Eq.(3.17) is the standard modified Bessel equation of order $\nu$. The general solution of Eq.(3.17) can be written as

$$W(\zeta) = A I_\nu(\zeta) + B K_\nu(\zeta) + \frac{D}{\frac{1}{\sqrt{b}}(1+\nu)} I_\nu(\zeta) \int^\zeta \zeta'^\nu K_\nu(\zeta') d\zeta' - \frac{D}{\frac{1}{\sqrt{b}}(1+\nu)} K_\nu(\zeta) \int^\zeta \zeta'^\nu I_\nu(\zeta') d\zeta',$$

(3.18)

where $I_\nu(\zeta)$ and $K_\nu(\zeta)$ are modified Bessel functions of order $\nu$; $A$ and $B$ are the two arbitrary constants of integration corresponding to the homogenous part of Eq.(3.17). The $D$ containing term results from the particular integral corresponding to the inhomogenous contribution of Eq.(3.17) obtained by the method of variation of parameters. Making use of the relations (26) and (21) we revert back to the original variables $x$ and $P(x)$ to obtain the general solution of Eq.(20) as

$$P(x) = A x^{-\nu} I_\nu(\sqrt{bx}) + B x^{-\nu} K_\nu(\sqrt{bx})$$

$$+ D x^{-\nu} \left[ I_\nu(\sqrt{bx}) \int^{\sqrt{bx}} x'^\nu K_\nu(\sqrt{bx'}) dx' - K_\nu(\sqrt{bx}) \int^{\sqrt{bx}} x'^\nu I_\nu(\sqrt{bx'}) dx' \right].$$

(3.19)

**B. The boundary conditions and the normalized probability distribution**

At this juncture it is necessary to specify the boundary conditions. We impose the following natural boundary conditions on the solution (29)

(i) $P(x)$ vanishes for $|x| \to \infty$

(ii) $P(x)$ remains finite at $x = 0$.

To this end we proceed as follows:

(i) First we note that a modified Bessel equation of the form

$$x^2 y'' + xy' - (x^2 + \nu^2) y = 0$$

(3.20)

has an irregular singular point as $x \to \infty$. The leading behavior of the solutions are

$$y(x) = I_\nu(x) \sim C_1 x^{-1/2} e^x, \quad x \to \infty$$

(3.21a)

and

$$y(x) = K_\nu(x) \sim C_2 x^{-1/2} e^{-x}, \quad x \to \infty.$$  

(3.21b)

We thus observe that $I_\nu(x)$ diverges exponentially for large $x$. By applying the boundary condition (i), i.e., $P(x)$ vanishes for large $x$ we see that the constant $A$ in the general solution (29) must be zero. Therefore we have
\[ P(x) = B \, x^{-\nu} \, K_\nu(\sqrt{bx}) + D \, x^{-\nu} \, I_\nu(\sqrt{bx}) \int_{\sqrt{bx}}^{\infty} x'^{\nu} \, K_\nu(\sqrt{bx'}) \, dx' \\
- D \, x^{-\nu} \, K_\nu(\sqrt{bx}) \int_{\sqrt{bx}}^{\infty} x'^{\nu} \, I_\nu(\sqrt{bx'}) \, dx' . \] (3.22)

\( B \) and \( D \) are the two remaining constants to be determined.

(ii) We now turn to the second boundary condition, i.e., the finiteness of \( P(x) \) at \( x = 0 \). For a fixed \( \nu (\nu > 0) \) we know that

\[ K_\nu(x) \sim \frac{1}{2} \Gamma(\nu) \left( \frac{1}{2} x \right)^{-\nu} , \quad \text{for} \quad x \sim 0 \] (3.23)

which implies that \( K_\nu(x) \) has a singularity at \( x = 0 \). This reveals that \( P(x) \) is singular at \( x = 0 \). Our strategy here is to remove this singularity by having an appropriate relation between the two remaining constants \( B \) and \( D \) in the solution (36). To derive this relation we proceed as follows:

We first derive the asymptotic expansion for the solution of the modified Bessel equation (30) whose leading behavior is (31a). To do this we peel off the leading behavior by substituting

\[ y(x) = C_1 \, x^{-1/2} \, e^x \, w(x) \] (3.24)

into the modified Bessel equation (30). The equation satisfied by \( w(x) \) is

\[ x^2 \, w''(x) + 2x^2 \, w'(x) + \left( \frac{1}{4} - \nu^2 \right) w(x) = 0 . \] (3.25)

We seek a solution of this equation of the form \( w(x) = 1 + \varepsilon(x) \) with \( \varepsilon(x) \ll 1 \) \((x \to \infty) \). \( \varepsilon(x) \) satisfies the equation

\[ x^2 \, \varepsilon''(x) + 2x^2 \, \varepsilon'(x) + \left( \frac{1}{4} - \nu^2 \right) \varepsilon(x) + \left( \frac{1}{4} - \nu^2 \right) = 0 , \] (3.26)

which may be simplified by the approximations

\[ \left( \frac{1}{4} - \nu^2 \right) \varepsilon \ll \frac{1}{4} - \nu^2 , \quad x^2 \varepsilon'' \ll x^2 \varepsilon' ; \quad x \to \infty . \] (3.27)

We make the second of these approximations because we anticipate that \( \varepsilon \) decays like a power of \( x \) as \( x \to \infty \). The resulting asymptotic differential equation is

\[ 2x^2 \, \varepsilon' \sim \left( \nu^2 - \frac{1}{4} \right) , \quad x \to \infty . \]

Ordinarily the solution to this equation would be \( \varepsilon(x) \sim \tilde{c} \) as \( x \to \infty \) where \( \tilde{c} \) is an integration constant. However since \( \varepsilon(x) \ll 1 \) as \( x \to \infty \) we must set \( \tilde{c} = 0 \). The leading behavior of \( \varepsilon(x) \) is then given by
\[ \varepsilon(x) \sim \left( \frac{1}{8} - \frac{1}{2} \nu^2 \right) x^{-1}, \ x \to \infty. \]

This kind of analysis can be repeated to obtain all the terms in the asymptotic expansion of \( w(x) \) as \( x \to \infty \). However the leading behavior of \( \varepsilon(x) \) suggests that \( w(x) \) has a series expansion in inverse power of \( x \). Thus to simplify the analysis we assume at the outset that

\[ w(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}, \ (x \to \infty, \ a_0 = 1). \tag{3.28} \]

Substituting this expression into the differential equation for \( w(x) \) gives

\[ \sum_{n=0}^{\infty} n(n+1) a_n x^{-n} - 2 \sum_{n=0}^{\infty} n a_n x^{1-n} + \left( \frac{1}{4} - \nu^2 \right) \sum_{n=0}^{\infty} a_n x^{-n} \sim 0. \tag{3.29} \]

Since the coefficients of any asymptotic power series are unique we equate to zero the coefficients of all powers of \( \frac{1}{x} \) in the above relation

\[ x^{-n} : \left[ \left( n + \frac{1}{2} \right)^2 - \nu^2 \right] a_n - 2 (n+1) a_{n+1} = 0, \ n = 0, 1, 2 \ldots. \tag{3.30} \]

Solving this recursion relation and using \( a_0 = 1 \) we obtain

\[ w(x) \sim 1 - \frac{(4\nu^2 - 1^2)}{1! 8x} + \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)}{2! (8x)^2} \ldots, \ x \to \infty. \tag{3.31} \]

From the ratio test we see that the radius of convergence \( R \) of (41) is

\[ R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \frac{2(n+1)}{(n+1)^2 - \nu^2} = 0 \tag{3.32} \]

unless the series (41) terminates, which it does when

\[ \nu = \pm \frac{1}{2}, \ \pm \frac{3}{2}, \ \pm \frac{5}{2}, \ldots. \]

When this happens, the finite series (41) when multiplied by \( e^{-x}/\sqrt{x} \) gives an exact solution to the modified Bessel equation.

Similarly, the complete asymptotic series for the function whose leading behavior is given by (31b) is

\[ y(x) \sim C_2 x^{-1/2} e^{-x} w(x) \]

where

\[ w(x) \sim 1 + \frac{(4\nu^2 - 1^2)}{1! 8x} + \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)}{2! (8x)^2} \ldots, \ x \to \infty. \tag{3.33} \]
By global analysis the two constants $C_1$ and $C_2$ can be derived as
\begin{align}
C_1 &= (2\pi)^{-1/2} \\
C_2 &= \sqrt{\pi/2},
\end{align}
(3.34)
to get the modified Bessel functions $I_\nu(x)$ and $K_\nu(x)$ respectively.

Thus, in general, $I_\nu(x)$ and $K_\nu(x)$ are represented by infinite power series. The above analysis reveals that for $\nu = \pm 1/2, \pm 3/2, \ldots$, the series terminates and from the asymptotic expansion we get an exact solution. In such a case, i.e., when the power series terminates it is possible to find a relation between the two constants $B$ and $D$ of the general solution (3.22), such that the singularity of $P(x)$ at $x = 0$ is removed. Thus both the boundary conditions (i) and (ii) [i.e., $P(x) = 0$ for $|x| \to \infty$ and $P(x)$ is finite at $x = 0$] are satisfied by the solution (3.22) provided $\nu$ is an odd half integers, i.e.,

$$\nu = n + \frac{1}{2}, \quad n = 0, 1, 2, \ldots$$
(3.35)

By Eq.(25) we have
\[
\frac{1}{2} \left[ 1 + \frac{\beta c_{01}}{\omega_0^2 c_2} \right] = n + \frac{1}{2}, \quad n = 0, 1, 2, \ldots .
\]
But $n = 0$ implies $c_{01} = 0$ or $c_0 = c_1$ which is not allowed physically. Therefore we have
\[
\frac{\beta c_{01}}{2\omega_0^2 c_2} = n \quad ; \quad n = 1, 2, 3, \ldots ,
\]
(3.36)
In passing, we mention here that the integers $n$ characterize the distinct physically allowed steady states of the thermodynamically open systems.

With the above mentioned restricted values of $\nu$ ($\nu = n + \frac{1}{2}, \quad n = 1, 2, 3, \ldots$) we now explicitly calculate the several quantities [ details are given in Appendix-B] which appeared in the general solution $P(x)$ in Eq.(3.22). We finally obtain:
\[
P_{n+\frac{1}{2}}(x) = B \sqrt{\frac{\pi}{2 b^{1/2}}} \sum_{k=0}^{n} f_k^n \frac{e^{-\sqrt{b}x}}{x^{k+n+1}}
\]
\[- \frac{D}{2\sqrt{b}} \sum_{i=0}^{n} \sum_{k=0}^{n} (-1)^{i+k} \sum_{j=0}^{n-k} \left\{ (-1)^i + (-1)^{i-n} \right\} f_i^n f_k^n \frac{(n-k)!}{j!} \frac{x^{j-i-n-1}}{(\sqrt{b})^{n-k-j+1}}
\]
where we have defined
\[
f_k^n = \frac{(n+k)!}{2^k b^{k/2} k! (n-k)!}.
\]
(3.37)

We are now in a position to remove the singularity of $P(x)$ at $x = 0$. To achieve this we seek a relation between the constants $B$ and $D$. For this we now expand the exponential
in Eq.(47). It is convenient to write a few nontrivial steps explicitly after this expansion. We thus have

\[
P_{n+\frac{1}{2}}(x) = B \sqrt{\frac{\pi}{2 b^{1/2}}} \sum_{k=0}^{n} \sum_{\alpha=0}^{\infty} (-1)\alpha \left(\frac{\sqrt{b}}{\alpha!}\right)^{\alpha} \frac{f_k^n}{x^{k+n+1-\alpha}}
\]

\[
- \frac{D}{2\sqrt{b}} \sum_{i=0}^{n} \sum_{k=0}^{n-k} (-1)^i f_i^n f_k^n \frac{(n-k)!}{j!} \frac{1}{(\sqrt{b})^{n-k-j+1}} \frac{1}{x^{i+n+1-j}}
\]

\[
- \frac{D}{2\sqrt{b}} \sum_{i=0}^{n} \sum_{k=0}^{n-k} (-1)^{j-n} f_i^n f_k^n \frac{(n-k)!}{j!} \frac{1}{(\sqrt{b})^{n-k-j+1}} \frac{1}{x^{i+n+1-j}}
\]

\[
= B \sqrt{\frac{\pi}{2 b^{1/2}}} \sum_{k=0}^{n} \sum_{\alpha=0}^{\infty} (-1)\alpha \left(\frac{\sqrt{b}}{\alpha!}\right)^{\alpha} \frac{f_k^n}{x^{k+n+1-\alpha}}
\]

\[
- \frac{D}{2\sqrt{b}} \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^i f_i^n f_0^n \frac{n!}{j!} \frac{1}{(\sqrt{b})^{n-j+1}} \frac{1}{x^{i+n+1-j}}
\]

\[
- \frac{D}{2\sqrt{b}} \sum_{i=0}^{n} \sum_{j=0}^{n-2} (-1)^i f_i^n f_1^n \frac{(n-1)!}{j!} \frac{1}{(\sqrt{b})^{n-1-j+1}} \frac{1}{x^{i+n+1-j}}
\]

\[
- \frac{D}{2\sqrt{b}} \sum_{i=0}^{n} \sum_{j=0}^{n-3} (-1)^i f_i^n f_2^n \frac{(n-2)!}{j!} \frac{1}{(\sqrt{b})^{n-2-j+1}} \frac{1}{x^{i+n+1-j}}
\]

\[
- \ldots
\]

\[
= B \sqrt{\frac{\pi}{2 b^{1/2}}} \sum_{k=0}^{n} \sum_{\alpha=0}^{\infty} (-1)\alpha \left(\frac{\sqrt{b}}{\alpha!}\right)^{\alpha} \frac{f_k^n}{x^{k+n+1-\alpha}}
\]

\[
- \frac{D}{2\sqrt{b}} \sum_{k=0}^{n} \sum_{j=0}^{n} (-1)^k f_k^n f_0^n \frac{n!}{j!} \frac{1}{(\sqrt{b})^{n-j+1}} \frac{1}{x^{k+n+1-j}}
\]

\[
+ \sum_{j=0}^{n-1} (-1)^k f_k^n f_1^n \frac{(n-1)!}{j!} \frac{1}{(\sqrt{b})^{n-j}} \frac{1}{x^{k+n+1-j}} + \ldots
\]

\[
- \frac{D}{2\sqrt{b}} \sum_{k=0}^{n} \sum_{j=0}^{n-2} (-1)^{j-n} f_k^n f_1^n \frac{n!}{j!} \frac{1}{(\sqrt{b})^{n-j+1}} \frac{1}{x^{k+n+1-j}}
\]

\[
+ \sum_{j=0}^{n-1} (-1)^{j-n} f_k^n f_1^n \frac{(n-1)!}{j!} \frac{1}{(\sqrt{b})^{n-j}} \frac{1}{x^{k+n+1-j}} + \ldots
\]

(3.39)

(where we have replaced the dummy index \(i\) by \(k\) in the \(D\)-containing sums)

To remove the singularity at \(x = 0\) from the above expression for \(P_{n+\frac{1}{2}}(x)\), \(B\) must be
related to $D$ in such a way that all powers of $\frac{1}{x^{n+1}}$ from the summation are equated to zero. This gives

$$B \sqrt{\frac{\pi}{2b^{1/2}}} \left( -1 \right)^n \left( \frac{\sqrt{b}}{n!} \right)^n f_k^n - \frac{D}{2\sqrt{b}} f_k^n f_0^n \frac{1}{\sqrt{b}} - \frac{D}{2\sqrt{b}} f_k^n f_0^n \frac{1}{\sqrt{b}} = 0 \quad (3.40)$$

Note that the relevant coefficients contribute from the $B$-containing term for $\alpha = 1$ and from $D$-containing term for $j = n$. Putting the value of $f_0^n$ from Eq.(48) in Eq.(50) we obtain the desired relation between $B$ and $D$.

$$B = (-1)^n D \frac{n!}{\sqrt{\pi}} 2^{(2n+1)/2} \frac{1}{b^{(2n+3)/4}} \quad (3.41)$$

From Eq.(51) it reveals that the relation between $B$ and $D$ is independent of any dummy index $i, j$ or $k$ and only depends on $n$. Thus for any $n$ the relation is unique. With this choice the probability $P_{n+\frac{1}{2}}(x)$ (Eq.(47)) becomes finite for all $x$ and is given by

$$P_{n+\frac{1}{2}}(x) = (-1)^n D 2^n n! \frac{1}{b^{(n+2)/2}} \sum_{k=0}^{n} f_k^n e^{-\sqrt{bx}} \frac{1}{x^{k+n+1}}$$

$$- \frac{D}{2\sqrt{b}} \sum_{i=0}^{n} \sum_{k=0}^{n} \sum_{j=0}^{n-k} \left( (-1)^i + (-1)^j - n \right) f_i^n f_k^n \frac{(n-k)!}{j!} \frac{x^{i-n-1}}{(\sqrt{b})^{n-k-j+1}} \quad (3.42)$$

The above probability distribution function which remains undetermined upto the constant $D$ (which we are to determine shortly) vanishes as $x \to \infty$ and remains finite at the barrier top $x = 0$. For illustration, we explicitly write some of the probability distribution functions. For $n = 1$ case we have,

$$P_{3/2}(x) = -2D b^{-3/2} e^{-\sqrt{bx}} \left( \frac{1}{x^2} + \frac{1}{\sqrt{bx^3}} \right) - \frac{D}{b} \left( \frac{1}{x} - \frac{2}{bx^3} \right) \quad ; \quad x > 0 \quad (3.43a)$$

$$= -2D b^{-3/2} e^{\sqrt{bx}} \left( \frac{1}{x^2} - \frac{1}{\sqrt{bx^3}} \right) + \frac{D}{b} \left( \frac{1}{x} - \frac{2}{bx^3} \right) \quad ; \quad x < 0 \quad (3.43b)$$

$$= -\frac{2D}{3\sqrt{b}} \quad ; \quad x = 0 \quad (3.43c)$$

where the probability distributions for $x < 0$ are obtained by noting the symmetry of the differential equation.

To normalize the probability and thereby to calculate the current over the top of the barrier located at $x = 0$, we consider the condition for normalization:

$$\int_{-\infty}^{+\infty} P_{n+\frac{1}{2}}(x) \ dx = 1 \quad (3.44)$$

Since the probability decreases rapidly (due to the presence of the modified Bessel function $K_\nu(x)$) we approximate the integral (54) by
\[ \int_{-\Delta}^{\Delta} P_{n+\frac{1}{2}}(x) \, dx = 1 \quad ; \quad \Delta \text{ large but finite} \quad (3.45) \]

where \( \pm \Delta \) approximately indicates the two zeroes of the inverted potential \( V(x) \). By Eq.(17), \( \Delta \) may be expressed in terms of \( E_0 \) the height of the barrier as,

\[ \Delta = \left( \frac{2E_0}{\omega_0^2} \right)^{1/2}. \]

By symmetry we rewrite

\[ \int_{0}^{\Delta} P_{n+\frac{1}{2}}(x) \, dx = 1/2 \quad ; \quad \Delta \text{ large but finite} \quad (3.46) \]

Due to the presence of the powers of \( \frac{1}{x} \) in the expression for \( P_{n+\frac{1}{2}}(x) \), it is not possible to carry out the integration in Eq.(56) in a straightforward manner. We therefore employ a limiting procedure and make use of the expression (52) for \( P_{n+\frac{1}{2}}(x) \) in (56) to obtain an expression for the following normalization constant \( D \)

\[ D = \frac{b \text{Ein}(\sqrt{b}\Delta) + 2^{n+1} n! \sum_{k=0}^{n} \frac{\sum_{j=0}^{k+n-1} (-1)^k k! k!(n-k)!}{2^k (n+k-j)!}}{2 \text{Ein}(\sqrt{b}\Delta) + 2^{n+1} n! \sum_{k=0}^{n} \frac{\sum_{j=0}^{k+n-1} (-1)^k k! k!(n-k)!}{2^k (n+k-j)!}}, \quad (3.47) \]

where \( \Delta \) is large but finite and the function \( \text{Ein}(x) \) is defined as

\[ \text{Ein}(x) = \sum_{k=1}^{\infty} (-1)^k \frac{x^k}{k! k}. \quad (3.48) \]

The expression (52) together with the normalization constant \( D \) as given by (57) yields the complete and exact analytical expression for the probability distribution function \( P_{n+\frac{1}{2}}(x) \). In Fig.(2) we draw two typical normalized probabilities for \( n = 1 \) and \( n = 2 \) for the parametric values \( \beta = 1.0, \Delta = 2.5 \) and \( c_2 = 0.9 \). The effect of third order noise is illustrated in Fig.(3), where we exhibit the probability distribution functions for several values of the third order noise strength \( c_2 \). One observes that the distribution gets flattened as \( c_2 \) increases as expected.

**IV. DYNAMICS OF BARRIER CROSSING INDUCED BY ADIABATIC NOISE**

We now extend the above analysis to elucidate the following problem of dynamics of barrier crossing.

An overdamped particle moves in an external field of force and in addition to this is subject to an adiabatically fluctuating force of external origin. The conditions are such that the particle is originally caught in the potential well in \( V(x) \), but may escape in course of time over the barrier. Our object here is to calculate the rate of escape from this well. The
analogous problem for the case of fast fluctuations is the celebrated Kramers’ problem of Brownian motion in phase space. Our calculation rests on the third order equation of motion (9) obeyed by probability distribution function derived in Sec. II.

To proceed further we employ the popular flux-over-population method originated by Farkas many years ago. The calculation rests on the evaluation of two quantities; (i) the steady state current $J$ over the barrier top, (located at $x = 0$) that results if the particles are continuously fed into the domain of attraction (say, in the region of left well) and are subsequently and continuously removed in the neighboring domain of attraction. (ii) Steady state population $n_a$ in the initial domain of attraction, i.e., the left well. The rate is defined by

$$\mathcal{K} = J/n_a \ ,$$

The method has been used by Kramers in his seminal work on barrier crossing dynamics and many others over the several decades.

A. Calculation of $n_a$

For calculation of $n_a$ it is necessary to evaluate the stationary probability density $P_b(x)$ near the bottom of the well corresponding to a zero current ($J = 0$) situation along the $x$ co-ordinate.

We first linearize the potential $V(x)$ as shown in Fig.(1) near the bottom of the left well, at $x = -\Delta$, so that we approximate

$$V(x) \simeq \frac{1}{2} \omega_b^2 (x + \Delta)^2 \ ,$$

where $\omega_b$ refers to the frequency at the bottom of the left well. Eq.(60) and $J = 0$ condition reduce the third order equation of motion (Eq.(9)) for probability density $P_b(x)$ inside the well to the following form:

$$(x + \Delta) \frac{d^2 P_b}{dx^2} + \gamma' \frac{dP_b}{dx} - b (x + \Delta) P_b = 0 \ .$$

The above equation is valid near the bottom of the left well ($x \simeq -\Delta$). Here $\gamma'$ is defined as

$$\gamma' = \frac{2\omega_b^2 - a}{\omega_b^2} \ ,$$

where $a$ is as given by Eq.(16). Putting $z = x + \Delta$ and $y = zP_b$, Eq.(61) can be transformed as follows :

$$z^2 \frac{d^2 y}{dz^2} - (2 - \gamma') z \frac{dy}{dz} + [(2 - \gamma') - b z^2] y = 0 \ .$$

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From Eqs. (62) and (16) we have
\[ 2 - \gamma' = \frac{\beta c_{01}}{c_2 \omega_0^2}. \] (4.6)

Note that \(2 - \gamma'\) is a positive quantity. We write
\[ 2 - \gamma' = \sigma. \] (4.7)

Therefore Eq. (63) is given by
\[ z^2 \frac{d^2 y}{dz^2} - \sigma z \frac{dy}{dz} + [\sigma - b z^2] y = 0. \] (4.8)

Substitution of \(y(z) = z^{\frac{1}{2}(\sigma+1)} W(z)\) in Eq. (66) yields
\[ z^2 \frac{d^2 W}{dz^2} + z \frac{dW}{dz} - [\nu^2 + b z^2] W = 0, \] (4.9)

where
\[ \nu' = \frac{\sigma - 1}{2}. \] (4.10)

The solutions of Eq. (67) are again the modified Bessel functions, \(I_{\nu'}(\sqrt{b}z)\) and \(K_{\nu'}(\sqrt{b}z)\). Reverting back to original variables, the general solution for the steady state probability distribution near the bottom of the left well is given by
\[ P_b(x) = A' (x + \Delta)^{\nu'} I_{\nu'}[\sqrt{b} (x + \Delta)] + B' (x + \Delta)^{\nu'} K_{\nu'}[\sqrt{b} (x + \Delta)]. \] (4.11)

\(A'\) and \(B'\) are the two arbitrary constants of integration.

By demanding that \(P_b(x)\) must vanish at infinity, we require
\[ A' = 0 \] (4.12)

and therefore
\[ P_b(x) = B' (x + \Delta)^{\nu'} K_{\nu'}[\sqrt{b} (x + \Delta)]. \] (4.13)

Furthermore, we note that although \(K_{\nu'}\) itself is singular at \(x = -\Delta\) the presence of \((x+\Delta)^{\nu'}\) in Eq. (71) assures that the probability (71) remains finite at \(x = -\Delta\). To verify this assertion, we use the property of modified Bessel function \(K_{\nu'}(z)\) for fixed \(\nu'\) and \(z \to 0\).
\[ K_{\nu'}(z) \sim \frac{1}{2} \Gamma(\nu') \left(\frac{1}{2} z\right)^{-\nu'}, \text{ Re } \nu' > 0. \] (4.14)
Hence

\[ P_b(-\Delta) = B' \lim_{x \to -\Delta} \frac{(x + \Delta)^{\mu'}}{K_{\mu'}[\sqrt{b}(x + \Delta)]} \]

\[ = B' \frac{\Gamma(\mu') \, 2^{\mu'-1}}{b^{\mu'/2}}. \]  

(4.15)

We thus see that the steady state probability \( P_b \) (given by Eq.(71)) is finite at the bottom of the left well.

The above solution \( P_b(x) \) must now be subject to the following boundary condition:

\[ P_b(-\Delta) = \frac{P_n + 1}{2} \]  

(4.16)

where the stationary probability \( P_n + \frac{1}{2} \) corresponds to the vanishing current \( J = 0 \) along \( x \) pertaining to the homogenous version of Eq.(20). As usual, \( P_n + \frac{1}{2} \) must also satisfy the boundary condition that for \( |x| \to \infty \), \( P_n + \frac{1}{2} \) vanishes. Such a solution is immediately apparent from our earlier analysis of Sec. III. Thus

\[ P_n + \frac{1}{2}(x) = \frac{B}{\sqrt{\pi}} \frac{2^{\mu'/2}}{\Gamma(\mu')} \sum_{k=0}^{n} \frac{(-1)^{k+n+1} f_k e^{\sqrt{\Delta} b(x + \Delta)}}{x^{k+n+1}} ; \quad x > 0 \]  

(4.17a)

\[ = B \frac{2^{\mu'/2}}{b^{\mu'/2}} \sum_{k=0}^{n} \frac{(-1)^{k+n+1} f_k e^{\sqrt{\Delta} b(x + \Delta)}}{x^{k+n+1}} ; \quad x < 0 \]  

(4.17b)

where \( f_k \) is as defined in Eq.(48).

Making use of Eqs.(73) and (75) in Eq.(74) we obtain a relation between \( B \) and \( B' \).

\[ B' = B \frac{\sqrt{\pi}}{2 \, b^{1/2}} \frac{b^{\mu'/2}}{\Gamma(\mu')} \frac{1}{2^{\mu'-1}} \sum_{k=0}^{n} (-1)^{k+n+1} f_k \frac{e^{\sqrt{\Delta} b}}{x^{k+n+1}} . \]  

(4.18)

Therefore \( P_b(x) \) in Eq.(71) may be expressed as

\[ P_b(x) = B \frac{\sqrt{\pi}}{2 \, b^{1/2}} \frac{b^{\mu'/2}}{\Gamma(\mu')} \frac{1}{2^{\mu'-1}} \sum_{k=0}^{n} (-1)^{k+n+1} f_k \frac{e^{\sqrt{\Delta} b}}{x^{k+n+1}} \]  

(4.19)

The above distribution which is valid near the bottom of the left well may be used to calculate the population inside the left well as,

\[ n_a = 2 \int_{-\Delta}^{0} P_b(x) \, dx . \]  

(4.20)

Due to the presence of \( K_{\mu'}(x) \) the probability \( P_b(x) \) is a rapidly decreasing function. We may extend the above integration limit to infinity. This yields ( using Eq.(71) )

\[ n_a = B' \frac{\sqrt{\pi} \, 2^{\mu'} \, \Gamma(\mu' + \frac{1}{2})}{\Gamma(\mu')} \]  

(4.21)

Using the relations (76) and (79) we finally have

\[ n_a = \sqrt{2} \, \pi \, B \, \frac{\Gamma(\mu' + \frac{1}{2})}{\Gamma(\mu')} \sum_{k=0}^{n} (-1)^{k+n+1} f_k \frac{e^{\sqrt{\Delta} b}}{x^{k+n+1}} b^{-3/4} . \]  

(4.22)
B. Calculation of escape rate

Having determined the population \( n_a \) of the left well and the steady state current \( J \) from Eqs.(80) and (19), respectively, we are now in a position to calculate the escape rate in terms of the Eq.(59). We thus obtain (for calculation of \( J \) the linearization of the potential \( V(x) \) at \( x = 0 \) which results in Eq.(17) has been carried out)

\[
K_{n+\frac{1}{2}} = J/n_a = \frac{c_2 \omega_0^2}{\sqrt{2} \pi \beta^3} \left( \frac{D}{B} \right) \frac{\Gamma(\nu')}{\Gamma(\nu' + \frac{1}{2})} \left\{ \frac{b^{3/4}}{\sum_{k=0}^{n} (-1)^{k+n+1} f_k^n \frac{e^{\sqrt{\Delta} \Delta}}{\Delta^{k+n+1}}} \right\} .
\]  

(4.23)

Making use of the relation (51) between the two constants \( B \) and \( D \) in Eq.(81) one finds

\[
K_{n+\frac{1}{2}} = (-1)^{n+1} \frac{c_2 \omega_0^2}{\sqrt{2} \pi \beta^3} \frac{\Gamma(\nu')}{\Gamma(\nu' + \frac{1}{2})} \frac{1}{n!} \frac{1}{2^{n+\frac{3}{2}}} \left\{ \frac{b^{(n+3)/2}}{\sum_{k=0}^{n} (-1)^{k+n+1} f_k^n \frac{e^{\sqrt{\Delta} \Delta}}{\Delta^{k+n+1}}} \right\} .
\]  

(4.24)

The expression (82) can be simplified further by noting the following relations. First, we have from Eqs.(19) and (23)

\[
\nu = \frac{1}{2} (\gamma - 1) \quad \text{where} \quad \gamma = 2 + \frac{a}{\omega_0^2} ,
\]  

(4.25a)

and from Eqs.(62), (65) and (68)

\[
\nu' = \frac{1}{2} (1 - \gamma') \quad \text{where} \quad \gamma' = 2 - \frac{a}{\omega_b^2} .
\]  

(4.25b)

Thus we have

\[
\nu + \nu' = \frac{1}{2} a \left( \frac{\omega_0^2 + \omega_b^2}{\omega_0^2 \omega_b^2} \right) .
\]  

(4.26)

Furthermore Eqs.(16) and Eq.(3.36) suggest that

\[
a = 2 n \omega_0^2 ; \quad b = \frac{\beta^2}{c_2} ; \quad n = 1, 2, \ldots
\]  

(4.27)

Use of Eq.(85) in Eq.(83b) yields

\[
\nu' = n \frac{\omega_0^2}{\omega_b^2} - \frac{1}{2} ; \quad n = 1, 2, \ldots
\]  

(4.28)

The above relation together with the expression for \( f_k^n \) (Eq.(48)) leads to the formula for the transition rate \( K_{n+\frac{1}{2}} \) as follows:

\[
K_{n+\frac{1}{2}} = \frac{c_2 \omega_0^2}{\sqrt{2} \pi \beta^3} \frac{1}{n!} \frac{\Gamma(n \omega_0^2 \omega_b^{-2} - \frac{1}{2})}{\Gamma(n \omega_0^2 \omega_b^{-2})} \left\{ \frac{e^{-\sqrt{\Delta} \Delta}}{\sum_{k=0}^{n} (-1)^{k} \frac{k!}{(n-k)!} \frac{\Delta^{k+n+1} 1}{b^{n+k+3/2}}} \right\} .
\]  

(4.29)
Δ-s refer to the zero’s of the potential $V(x)$ as shown in Fig.(1) and by virtue of linearization of $V(x)$ at $x = 0$ (Eq.(17)) $\Delta$ is approximately given by

$$\Delta = \left(\frac{2E_0}{\omega_0^2}\right)^{1/2}.$$  \hspace{1cm} (4.30)

The above expression can be made more transparent by demonstrating a representative transition rate, say, for $n = 1$ as follows:

$$K_{3/2} = \frac{1}{8\sqrt{\pi}} \frac{c_{01}}{c_2^2} \frac{\Gamma\left(2c_2\omega_2^2, \frac{1}{2}\right)}{\Gamma\left(2c_2\omega_2^2\beta c_{01}ight)} \Delta^2 \beta^2 \exp\left(-\frac{\beta \Delta}{\sqrt{c_2}}\right).$$ \hspace{1cm} (4.31)

The above expressions are analogous to Kramers’ formula for the rate of escape from a potential well over a finite barrier of height $E_0$ under the influence of an external nonthermal adiabatic noise. What we have shown here is that within a linearized description of the potential, the corresponding diffusion process can be described exactly for dissipation $\beta$ pertaining to the timescale $\frac{1}{\beta} \ll \Delta t \ll \tau_c$.

The escape rate expressions derived above suggest that the rate approaches zero both for $\beta \to \infty$ and $\beta \to 0$. This behavior is somewhat reminiscent of Kramers’ theory, where it was noted earlier that these two limiting behaviors imply a maximal rate at some damping value $\beta$. The rate therefore undergoes a turnover in a form of a bell-shaped curve. In Fig.(4) we plot a representative variation of the escape rate versus dissipation $\beta$ for different third order noise strength. With increasing friction, the rate undergoes a turnover from an increasing behavior at low friction to an inverse behavior in the high friction limit.

Since the driving noise is of nonthermal origin, the escape rate expression is devoid of any temperature. Temperature is characteristic of a closed thermodynamic system at equilibrium. What we have here instead is the ratio of strength of nonthermal noise $\sqrt{c_2}$ to dissipation $\beta$ in the exponential factor of the rate expressions. As Ma pointed out that in the steady state such a parameter might play the role of temperature in the open system which characterizes the steady state.

As emphasized earlier that since the noise is external and the noise and dissipation have no common mechanistic origin (in contrast to what one observes in theory of Brownian motion) the steady state is not allowed arbitrarily. The system can attain the steady states depending on the specific integers $n \ (n = 1, 2, \ldots)$ which uniquely connect the parameters $c_0, c_1, c_2$ according to the relation derived in Sec. III.

V. CONCLUSIONS

In this paper we have presented a solution for the problem of motion of a particle in a force field simultaneously subject to an external adiabatic noise characterized by long
correlation time without keeping any restriction on the type of noise that it is Gaussian. Specifically we have calculated the rate of escape of the particle over the barrier initially confined in a well induced by nonthermal fluctuations. The theory rests on a perturbative expansion in $|\mu|^{-1}$ (where $|\mu|$ is the damping constant or the largest eigenvalue of the unperturbed system) pertaining to the separation of timescales (1.1) as carried out in our earlier papers. The main conclusions of our study are as follows:

(i) An analogue of Smoluchowski equation in the case of adiabatic non-Gaussian noise processes (Eq.(9)) has been proposed.

(ii) The undergoing stochastic process is characterized by third order noise which is responsible for non-Gaussian features.

(iii) Given the appropriate boundary conditions the third order equation admits of exact solution for the linearized potential.

(iv) The interplay of the characteristic linear dissipation of the system and the external noise leads to physically allowed distinct steady states subject to appropriate boundary conditions.

(v) In the spirit of Kramers theory we have solved the problem of escape of the particle confined in a well and have shown that the escape rate exhibits a turnover as one passes from the relatively low dissipative to the strong dissipative regime (Eq.(87)).

In conclusion, we have thus discussed a number of basic issues in the classical theory of motion of a particle in a force field in presence of external, adiabatic fluctuations. In view of several experimental investigations on external noise-induced processes in the past, the study of thermodynamically open systems has been specially relevant in various areas of physical and chemical sciences. We particularly mention the following examples:

A dye laser with fluctuating pump parameter shows interesting qualitative changes in the stationary distributions. The shape of the distribution also changes as a function of increasing fluctuation strength in the case of a two-species chemical reaction with a fluctuating rate coefficient. The early work on electronic parametric oscillator driven by external noise which exhibits the transition from non-oscillatory to oscillatory behavior is also worth-mentioning. Although the driving noise processes in the above mentioned cases are fast, suitable extension to adiabatic noise limit (such a typical case had been discussed by us earlier in detail in connection with population inversion in a two-level atom by adiabatically varying the field strength) might lead to experimental situations which are relevant to the present theoretical context. We believe that studies in this direction are worth-pursuing.
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APPENDIX A: SIMPLIFICATIONS OF THE TERMS IN EQ.(3)

Making use of the relations (4-8) we simplify below the four terms as appeared in Eq.(3).

First term :

\[- \nabla \cdot F_0 P(x,t) = - \frac{\partial}{\partial x} \left[ - \frac{1}{\beta} V'(x) P(x,t) \right] \]

\[= \frac{1}{\beta} \frac{\partial}{\partial x} \left[ V'(x) P(x,t) \right]. \quad (A1)\]

Second term :

\[\begin{align*}
\alpha^2 \nabla \cdot \int_0^\infty \left\{ \langle F_1 \nabla \tau \cdot F_1(x-\tau) \rangle \left| \frac{dx-\tau}{dx} \right| d\tau \right\} P(x,t) \\
= \alpha^2 \frac{\partial}{\partial x} \int_0^\infty \langle \frac{1}{\beta} \xi(t) \left[ 1 - \frac{1}{\beta} V''(x-\tau) \right] \frac{\partial}{\partial x} \xi(t-\tau) \rangle \left[ 1 + \frac{1}{\beta} V''(x) \right] d\tau P(x,t) \\
= \alpha^2 \frac{1}{\beta^2} \frac{\partial^2}{\partial x^2} \int_0^\infty \langle \xi(t) \xi(t-\tau) \rangle \left[ 1 - O(\tau^2) \right] d\tau P(x,t) \\
= \alpha^2 \frac{1}{\beta^2} \frac{\partial^2}{\partial x^2} \int_0^\infty \langle \xi(t) \xi(t-\tau) \rangle d\tau P(x,t) \\
= \alpha^2 c_0 \frac{1}{\beta^2} \frac{\partial^2}{\partial x^2} P. \quad (A2)\end{align*}\]

Third term :

\[\begin{align*}
- \alpha^2 \nabla \cdot \int_0^\infty \left\{ \tau \langle F_1 \nabla \tau \cdot \tilde{F}_1(x-\tau) \rangle \left| \frac{dx-\tau}{dx} \right| d\tau \right\} P(x,t) \\
= - \alpha^2 \frac{\partial}{\partial x} \int_0^\infty \tau \langle \frac{1}{\beta} \xi(t) \left[ 1 - \frac{1}{\beta} V''(x-\tau) \right] \frac{\partial}{\partial x} \xi(t) \rangle \left[ 1 + \frac{1}{\beta} V''(x) \right] d\tau P(x,t) \\
\simeq - \alpha^2 c_1 \frac{1}{\beta^2} \frac{\partial^2}{\partial x^2} P. \quad (A3)\end{align*}\]

Fourth term :
\[ \alpha^2 \nabla \cdot \int_0^\infty \left\{ \tau \langle F_1 \nabla - \tau \rangle \nabla \cdot F_0(x^{-\tau}) \right\} \left[ \frac{dx^{-\tau}}{dx} \right] \, d\tau \, P(x, t) \]

\[ = \alpha^2 \frac{\partial}{\partial x} \int_0^\infty \tau \left( \frac{1}{\beta} \xi^2 \right) \left[ 1 - \frac{1}{\beta} V''(x^{-\tau}) \right] \frac{\partial}{\partial x} \frac{1}{\beta} \xi(t - \tau) \left[ 1 - \frac{1}{\beta} V''(x^{-\tau}) \right] \frac{\partial}{\partial x} \times \left( \frac{1}{\beta} \right) [V'(x) - \tau V''(x)] \right] d\tau \, P(x, t) \]

\[ = -\alpha^2 c_2 \frac{1}{\beta^3} \frac{\partial^3}{\partial x^3} \int_0^\infty \tau \langle \xi(t) \xi(t - \tau) \rangle [V'(x) - \tau V''(x)] \left[ 1 + \frac{1}{\beta} V''(x) \right] d\tau \, P(x, t) \]

\[ = -\alpha^2 c_2 \frac{1}{\beta^3} \frac{\partial^3}{\partial x^3} [V'(x) \, P(x, t)] . \] (A4)

**APPENDIX B: CALCULATION OF VARIOUS QUANTITIES FOR SIMPLIFICATION OF EQ.(32)**

We explicitly calculate the several quantities which appeared in the expression for \( P(x) \) in Eq.(3.22). These are necessary for the derivation of Eq.(47). To this end we first note that

\[ K_{n+\frac{1}{2}}(\sqrt{bx}) = \sqrt{\frac{\pi}{2 \, b^{1/2}}} \frac{e^{-\sqrt{bx}}}{\sqrt{x}} \sum_{k=0}^{n} \frac{(n + k)!}{2^k \, b^{k/2} \, k! \, (n - k)!} \left( \frac{1}{x^k} \right) , \] (B1)

\[ I_{n+\frac{1}{2}}(\sqrt{bx}) = \frac{1}{\sqrt{2 \, \pi \, b^{1/2}}} \frac{e^{\sqrt{bx}}}{\sqrt{x}} \sum_{k=0}^{n} (-1)^k \frac{(n + k)!}{2^k \, b^{k/2} \, k! \, (n - k)!} \left( \frac{1}{x^k} \right) . \] (B2)

Using the expression for \( f^n_k \) (Eq.(48))

\[ f^n_k = \frac{(n + k)!}{2^k \, b^{k/2} \, k! \, (n - k)!} \]

we rewrite

\[ K_{n+\frac{1}{2}}(\sqrt{bx}) = \sqrt{\frac{\pi}{2 \, b^{1/2}}} \sum_{k=0}^{n} f^n_k \frac{e^{-\sqrt{bx}}}{x^{k+\frac{1}{2}}} , \] (B3)

\[ I_{n+\frac{1}{2}}(\sqrt{bx}) = \frac{1}{\sqrt{2 \, \pi \, b^{1/2}}} \sum_{k=0}^{n} (-1)^k f^n_k \frac{e^{\sqrt{bx}}}{x^{k+\frac{1}{2}}} . \] (B4)

Therefore we have

\[ x^n K_\nu(\sqrt{bx}) = \sqrt{\frac{\pi}{2 \, b^{1/2}}} \sum_{k=0}^{n} f^n_k \frac{e^{-\sqrt{bx}}}{x^{n-k}} \]

and

23
\[ \int \sqrt{bx} x^\nu K_\nu(\sqrt{bx}) \, dx = \sqrt{\frac{\pi}{2 \ b^{1/2}}} \sum_{k=0}^{n} f_k^n \int \sqrt{bx} \ e^{-\sqrt{bx}} x^{n-k} \, dx . \]  
\hspace{1cm} \text{(B6)}

Integrating successively by parts \( n-k \) times we obtain

\[ \int \sqrt{bx} x^\nu K_\nu(\sqrt{bx}) \, dx = -\sqrt{\frac{\pi}{2 \ b^{1/2}}} \sum_{k=0}^{n-k} f_k^n \frac{(n-k)!}{j!} \frac{x^j}{(\sqrt{b})^{n-k-j+1}} . \]  
\hspace{1cm} \text{(B7)}

Similarly we have

\[ \int \sqrt{bx} x^\nu I_\nu(\sqrt{bx}) \, dx = \frac{1}{\sqrt{2 \pi \ b^{1/2}}} \sum_{k=0}^{n-k} f_k^n \frac{(n-k)!}{j!} \frac{x^j}{(\sqrt{b})^{n-k-j+1}} . \]  
\hspace{1cm} \text{(B8)}

Hence from (B7) and (B4) we obtain

\[ x^{-\nu} I_\nu(\sqrt{bx}) \int \sqrt{bx} x^\nu K_\nu(\sqrt{bx}) \, dx \]
\[ = x^{-(n+\frac{1}{2})} \left( -\sqrt{\frac{\pi}{2 \ b^{1/2}}} \sum_{k=0}^{n-k} f_k^n \frac{(n-k)!}{j!} \frac{x^j}{(\sqrt{b})^{n-k-j+1}} \right) \]
\[ \times \frac{1}{\sqrt{2 \pi \ b^{1/2}}} \sum_{i=0}^{n} (-1)^i f_i^n e^{\sqrt{bx}} \frac{1}{x^{i+\frac{1}{2}}} \]
\[ = -\frac{1}{2 \sqrt{b}} \sum_{i=0}^{n} \sum_{k=0}^{n-k} \sum_{j=0}^{n-k} (-1)^j f_i^n f_k^n \frac{(n-k)!}{j!} \frac{x^{j-i-n-1}}{(\sqrt{b})^{n-k-j+1}} . \]  
\hspace{1cm} \text{(B9)}

Similarly,

\[ x^{-\nu} K_\nu(\sqrt{bx}) \int \sqrt{bx} x^\nu I_\nu(\sqrt{bx}) \, dx \]
\[ = \frac{1}{2 \sqrt{b}} \sum_{i=0}^{n} \sum_{k=0}^{n-k} \sum_{j=0}^{n-k} (-1)^{j-n} f_i^n f_k^n \frac{(n-k)!}{j!} \frac{x^{j-i-n-1}}{(\sqrt{b})^{n-k-j+1}} . \]  
\hspace{1cm} \text{(B10)}

The expressions (B3,B4) and (B9,B10) can now be utilized in Eq. (3.22) to obtain the expression for probability distribution function \( P(x) \) (Eq. (47)).
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FIGURES

FIG. 1. A schematic plot of the Kramers’ type potential $V(x)$.

FIG. 2. The normalized probability distribution function $P_{n+\frac{1}{2}}(x)$ is plotted as a function of $x$ for $n = 1$ and $n = 2$ ( $\beta = 1.0$, $\Delta = 2.5$ and $c_2 = 0.9$).

FIG. 3. The normalized probability distribution function $P_{3/2}(x)$ is plotted as a function of $x$ for various values of the third order noise strength $c_2$ ( $\beta = 1.0$ and $\Delta = 2.5$).

FIG. 4. Escape rate $K_{3/2}$ is plotted as a function of the characteristic dissipation $\beta$ of the system for various values of $c_2$ ( $c_{01} = 7.0$, $\omega_b = 0.80$ and $\Delta = 2.5$).
Fig. (2)

\[ P_{n+1/2}(x) \]

For \( n = 1 \) and \( n = 2 \).
Fig. (3)

\[ P_{3/2}(x) \]

\[ c_2 = 0.05 \]
\[ c_2 = 0.20 \]
\[ c_2 = 0.90 \]
Fig.(4)

$c_2 = 0.2$
$c_2 = 0.5$
$c_2 = 0.9$

$K_{3/2}$ vs. $\beta$