Quantization Analysis and Robust Design for Distributed Graph Filters

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Abstract—Distributed graph filters have recently found applications in wireless sensor networks (WSNs) to solve distributed tasks such as reaching consensus, signal denoising, and reconstruction. However, when employed over WSN, the graph filters should deal with the network limited energy, processing, and communication capabilities. Quantization plays a fundamental role to improve the latter but its effects on distributed graph filtering are little understood. WSNs are also prone to random link losses due to noise and interference. In this instance, the filter output is affected by both the quantization error and the topological randomness error, which, if it is not properly accounted in the filter design phase, may lead to an accumulated error through the filtering iterations and significantly degrade the performance. In this paper, we analyze how quantization affects distributed graph filtering over both time-invariant and time-varying graphs. We bring insights on the quantization effects for the two most common graph filters: the finite impulse response (FIR) and autoregressive moving average (ARMA) graph filter. Besides providing a comprehensive analysis, we devise theoretical performance guarantees on the filter performance when the quantization stepsize is fixed or changes dynamically over the filtering iterations. For FIR filters, we show that a dynamic quantization stepsize leads to more control on the quantization noise than the fixed-stepsize quantization. For ARMA graph filters, we show that decreasing the quantization stepsize over the iterations reduces the quantization noise to zero at the steady-state. In addition, we propose robust filter design strategies that minimize the quantization noise for both time-invariant and time-varying networks. Numerical experiments on synthetic and two real data sets corroborate our findings and show the different trade-offs between quantization bits, filter order, and robustness to topological randomness.

Index Terms—Graph signal processing; graph filters; quantization; subtractive dithering; time-varying graphs.

I. INTRODUCTION

Graph filters are enjoying an increasing popularity in graph signal processing (GSP) and graph convolutional neural networks [2], [3]. Their ability to be convolved with a graph signal renders graph filters versatile in a variety of applications ranging from recommender systems to spectral clustering [4, 9]. Graph filters find also application in wireless sensor networks (WSNs) [10, 13]. Here, the signal represents the sensor measurements and the WSN serves as a platform to perform distributed operations as well as a proxy to represent signal similarities in adjacent sensor nodes. Graph filters are useful for distributed signal representation [14], reconstruction [15, 16], denoising [17, 18], consensus [19, 20], and network coding [21]. Motivated by these applications, this paper focuses on distributed graph filtering.

In this work, we extend current art and evaluate quantization effects of both FIR and ARMA graph filters on time-invariant and time-varying topologies. Besides providing a broader analysis, we also devise theoretical performance guarantees on the filter performance when the quantization stepsize is fixed or changes dynamically over the filtering iterations. Further, we consider dithered quantization [37, 38] to make the assumption of quantization noise uncorrelated

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with input signals over the different graph filter iterations hold; an assumption commonly made in other works but unjustified. Our analysis sheds light on different tradeoffs in distributed filtering over WSN: FIR versus ARMA graph filter; fixed-stepszie quantization versus dynamically decreasing quantization stepsizes; and quantization rate versus link loss probability. The broad research question we are interested in is how quantization affects distributed graph filtering over both time-invariant and time-varying graphs. The specific contributions on how we answer this question are fourfold.

1) We study the quantization effects of distributed FIR graph filters. We analyze the impact of fixed and dynamic quantization stepsizze on the filtering performance and analyze their tradeoffs. We show that a dynamic quantization stepsizze allows more control on the quantization mean squared error (MSE) than fixed-size quantization. We devise also a robust filter design that minimizes the quantization noise.

2) We study the quantization effects of distributed ARMA graph filters. We analyze the impact of fixed and dynamic quantization stepsizze on the filtering performance and analyze their tradeoffs. We develop an ad-hoc dynamic quantization stepsizze framework that reduces the quantization MSE to zero at steady-state.

3) We conduct a statistical analysis to quantify the quantization effects on FIR and ARMA graph filters over random time-varying networks. We propose a novel filter design strategy that is robust to quantization and topological changes.

4) We characterize the different tradeoffs between the FIR and ARMA graph filters in terms of fixed-stepszie versus dynamically decreasing quantization stepsizes and between the quantization rate and the link loss probability.

The rest of this paper is organized as follows. Section II provides the background material. Sections III and IV analyze the quantization effects on FIR and ARMA graph filters, respectively. Section V contains the quantization analysis for random time-varying graphs. Section VI presents the numerical results. The paper conclusions are provided in Section VII.

II. BACKGROUND

Consider a graph $G = (\mathcal{V}, \mathcal{E})$ with node set $\mathcal{V} = \{1, \ldots, N\}$ and edge set $\mathcal{E}$ composed of tuples $(j, i)$ if there is a link between nodes $j$ and $i$. The set of all nodes connected to node $i$ is denoted by $N_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$. The graph is represented by its adjacency matrix $A$ whose $(j, i)$th entry is nonzero only if nodes $j$ and $i$ are connected. If the graph is undirected, it can also be represented by the graph Laplacian matrix $L$. To keep the discussion general for both directed and undirected graphs, we will use the so-called graph shift operator matrix $S$, which has plausible candidates $A$, $L$ or any of their normalized and translated forms. We shall only assume that the shift operator has an upper bounded spectral norm, i.e., $\|S\|_2 \leq \rho$.

On the vertices of $G$, a graph signal can be defined as a map from the vertex set (node set) to the set of real numbers, i.e., $x : \mathcal{V} \to \mathbb{R}$. We can denote the graph signal by a vector $x = [x_1, \ldots, x_N]^T$, whose $i$th entry $x_i$ denotes the signal at node $i$. WSNs match the above terminology: the nodes represent the sensors; the edges the communication links; and the signal the sensor data. By considering the eigendecomposition of the graph shift operator $S = U \Lambda U^{-1}$ with eigenvector matrix $U = [u_1, \ldots, u_N]$ and diagonal eigenvalue matrix $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N)$, we can alternatively analyze the graph signal $x$ by projecting it onto the shift operator eigenspace as $\hat{x} = U^{-1}x$. This projection is referred to as the graph Fourier transform (GFT) because the $i$th element $\hat{x}_i$ denotes how much eigenvector $u_i$ represents the variation of $x$ over $G$ and because the variation of the different eigenvectors can be ordered. The inverse GFT is $x = U\hat{x}$ and the eigenvalue $\lambda_i$ denotes the $i$th graph frequency $[2], [22]$.

A. Graph filter

A filtering operation on a graph combines locally the signal from node $i$ and the signals $\{x_j\}$ from all neighbors $j \in N_i$ of node $i$ to produce the output:

$$y_i = \sum_{j \in N_i \cup i} \phi_{ij} x_j$$  \hspace{1cm} (1)

for some scalar coefficients $\phi_{ij}$. By stacking all nodes’ output in vector $y = [y_1, \ldots, y_N]^T$, we can write (1) as $y = \mathbf{H}(S)x$, where the matrix $\mathbf{H}(S) : \mathbb{R}^N \to \mathbb{R}^N$ denotes the graph filter. The graph filter can be expressed as a function of the shift operator $S$ in different ways. Two widely used approaches are the FIR graph filter $[21], [22]$ and the ARMA graph filter $[11], [59]$.

FIR. An FIR graph filter is a polynomial of order $K$ in the shift operator $S$ with output:

$$y = \mathbf{H}(S)x = \sum_{k=0}^{K} \phi_k S^k x$$  \hspace{1cm} (2)

and scalar coefficients $\phi_0, \ldots, \phi_K$. The filtering behavior of $\mathbf{H}(S)$ can be viewed by means of the GFT:

$$h(\lambda) = \sum_{k=0}^{K} \phi_k \lambda^k \text{ for } \lambda \in [\lambda_{\text{min}}, \lambda_{\text{max}}]$$  \hspace{1cm} (3)

which is a polynomial in the generic graph frequency $\lambda$. This spectral representation allows to define a filtering operator by specifying the analytic function $h(\lambda) : \lambda \to \mathbb{R}$; hence, by approximating the latter with the polynomial in (3), we can implement it distributively over the nodes through the recursion $[2], [17]$. The distributed implementation is feasible because the shifted signal $x^{(1)} = Sx$ can be obtained through local exchanges between neighboring nodes in one communication round [cf. (1)]. The $k$th shifted signal can be obtained recursively as $x^{(k)} = Sx^{(k-1)}$, where nodes communicate to their neighbors the shifted signal $x^{(k-1)}$ obtained in the $(k - 1)$th communication round. The output $y$ of the FIR graph filter is obtained after $K$ iterations of exchanges between neighbors, implying that in total, each node $i$ exchanges $K \text{deg}(i)$ messages with its neighbors. This yields a communication complexity of order $O(MK)$.

ARMA. The ARMA graph filter extends (2) to a rational spectral response $[11]$:

1 Recent works consider also more general approaches such as the node-variant $[21]$ and the edge-variant graph filter $[23]$. To keep the exposition simple, we will discuss quantization of the two baseline approaches and leave the extension to the other methods for future research.
The quantization stepsize information to be transmitted [40].

Before proceeding with this analysis goal is to analyze the effects of dithered quantization to the implementation of distributed GSP operations over WSNs. Our shall analyze first quantization effects for static topologies and converges \((t \to \infty)\) to a steady-state only if the roots satisfy \(|\psi_k| \leq \rho\) for all \(k = 1, \ldots, K\), where \(\rho\) is the spectral radius of \(S\) [11].

The output of each branch \(w_{i}^{(k)}\) can be implemented distributively in a similar way as the FIR filters. The difference is that neighboring nodes exchange now the former output \(w_{i-1}^{(k)}\). Node \(i\) combines the shifted outputs \(w_{j}^{(k)}\) from all neighbors \(j \in N_i\) with its input signal \(x_i\) with coefficients (as given in (5)) to obtain the output \(w_{i}^{(k)}\). Finally, node \(i\) combines locally all branches’ outputs \(w_{i}^{(0)}, \ldots, w_{i}^{(K)}\) to obtain the overall ARMA\(K\) output \(y_i\) at iteration \(t\). This procedure accounts for \(K\) communication rounds between neighbors for each iteration \(t\); hence, the overall communication cost of the ARMA\(K\) filter for \(t = T\) iterations is of order \(O(K^3 T)\).

Equations (2) and (5) represent two fundamental algorithms to implement distributed GSP operations over WSNs. Our goal is to analyze the effects of dithered quantization to the filter outputs and account for it in the filter design phase. We shall analyze first quantization effects for static topologies in Sections III and IV and later for random time-varying topologies in Section V. Before proceeding with this analysis for the FIR graph filters, let us briefly introduce the conceptual terminology of dithered quantization.

### B. Dithered quantization

Quantizing consists of encoding the data prior to its transmission with a certain number of bits, reducing the amount of information to be transmitted [40].

Uniform quantizers map each input signal value to the nearest value of a finite set of quantization levels, where the quantization stepsize \(\Delta\) between two adjacent levels is constant [41]. We denote the quantized version of signal \(x\) as \(\tilde{x} = Q(x)\) and it is given by:

\[
\tilde{x} = x + n_q
\]

where \(n_q\) is the quantization noise. Although the quantization noise is deterministic, for a sufficiently small quantization stepsize \(\Delta\), it can be well modeled as a uniformly random variable with zero-mean and variance \(\Delta^2/12\), that is independent from the input [38], [42].

To control the quantization noise and ensure the uniform random variable assumption and independence from the input, we consider dithering quantization [37], [38], [42]. Dithering consists of adding a random additive signal \(n_d\), called dither, to the input signal \(x\) prior to quantization. Dithering is widely used in distributed signal processing [27], [30], [33], [43], which consists of iterative algorithms akin to distributed graph filtering. In subtractive dithered quantization, the dither signal is generated by a pseudo-random generator at the transmitter node and it is subtracted at the receiving node after transmission. The receiver node uses the same pseudo-random generator, which needs to be agreed prior to starting the communication. Let us denote \(x_d = x + n_d\) the dithered signal of \(x\). By applying quantization to the dithered signal \(x_d\), the transmitted signal becomes:

\[
x_d = Q(x_d) = Q(x + n_d) = x + n_q + n_d = \tilde{x} + n_d
\]

where signal \(\tilde{x}\) can be recovered by the receiver node by subtracting the dither \(n_d\) from the received signal \(x_d\).

The dither signal \(n_d\) also follows a uniform distribution with statistical properties:

\[
\mathbb{E}[n_d] = 0 \text{ and } \Sigma_q = \sigma_q^2 I = \frac{\Delta^2}{12} I.
\]

and with realizations independent of the input.

Two possible cases can be adopted when performing quantization with subtractive dithering, namely, a constant quantization stepsize for all iterations or a dynamically decreasing quantization stepsize over the iterations, which offers a benefit as compared to a fixed quantization stepsize. Decreasing the quantization stepsize implies transmitting more bits over the iterations but this increase of communication overhead improves the control over the quantization noise. In the sequel, we will analyze both cases.

### III. FIR Quantization Analysis

This section analyzes the quantization effects in FIR graph filters. We first discuss the fixed quantization stepsize and then the dynamically decreasing stepsize. Next, we formulate a filter design problem that is robust to quantization noise.

#### A. Fixed quantization stepsize

Consider the \(k\)th shifted signal \(x^{(k)} = S^k x\) exchanged with the neighbors. The quantized form of the latter is \(\tilde{x}^{(k)} = Q(x^{(k)}) = x^{(k)} + n_q^{(k)}\). At the filter initialization, we have \(x^{(0)} = x\), which quantized form is \(\tilde{x}^{(0)} = x^{(0)} + n_q^{(0)}\). This quantized signal is exchanged with neighbors leading to the quantized shifted signal \(\tilde{x}^{(1)} = S\tilde{x}^{(0)} = S(x^{(0)} + n_q^{(0)})\). Signal \(x^{(1)}\) is further quantized into \(\tilde{x}^{(1)}\) and subsequently transmitted to the neighboring nodes. The process is repeated \(K\) times. Based on the derivation in Appendix VIII-A the FIR filter output [cf. (2)] with quantization becomes:

\[
y^q = \sum_{k=0}^{K} \phi_k S^k x + \sum_{k=1}^{K} \sum_{\kappa=0}^{k-1} S^{k-\kappa} n_q^{(\kappa)} = \sum_{k=0}^{K} \phi_k S^k x + \sum_{\kappa=0}^{K-1} S^{K-\kappa} n_q^{(\kappa)}
\]
where the second term on the right-hand side of (10) accounts for the accumulated quantization error on the output:

\[
\epsilon = y^q - y = \sum_{k=1}^{K} \phi_k \sum_{\kappa=0}^{k-1} S^{k-\kappa} n_{q}^{\kappa}.
\] (11)

We analyze this quantization error in the spectral domain to ease the filter design. The following proposition provides a closed-form expression of the quantization noise mean squared error (MSE).

**Proposition 1.** Consider the FIR graph filter of order \( K \) in (2) with coefficients \( \phi_0, \ldots, \phi_K \) and quantization error \( \epsilon \) in (11) under fixed quantization stepsize. Consider also the graph Fourier transform \( \hat{\epsilon} = U^{-1} \epsilon \) of the error with respect to the shift operator \( S = U A U^{-1} \). The average quantization MSE per node \( \hat{\zeta_q} = \mathbb{E} \left[ \frac{1}{N} \text{tr}(\hat{\epsilon} \hat{\epsilon}^H) \right] \) is:

\[
\hat{\zeta_q} = \sum_{k=1}^{K} \phi_k \sum_{\kappa=0}^{k-1} \| A^{k-\kappa} \|_F^2.
\] (12)

where \( \| \cdot \|_F \) denotes the Frobenius norm and \( \sigma_q \) is the uniform quantizer standard deviation.

**Proof:** See Appendix VIII-B.

**Corollary 1.1.** Given the hypothesis of Proposition 1 and also a shift operator \( S \) with maximum eigenvalue \( \lambda_{\text{max}} \neq 1 \), the quantization MSE on the filter output \( \hat{\zeta_q} \) is bounded as:

\[
\frac{\sigma_q^2}{N} \sum_{k=1}^{K} \phi_k \eta_k \leq \hat{\zeta_q} \leq \frac{\sigma_q^2}{N} \sum_{k=1}^{K} \phi_k \eta_k
\] (13)

where \( \eta_k = (1 - \lambda_{\text{max}}^k)^{-1} (\lambda_{\text{max}}^2 - (\lambda_{\text{max}}^k + 1)) \).

**Proof:** See Appendix VIII-C.

The bounds in (13) suggest that by working with a fixed quantization stepsize, the MSE has always a Cramér-Rao lower-bound equivalence which cannot be overcome even by tuning the FIR coefficients in the design phase. In other words, even with a robust design strategy as the one in [34], we have an unavoidable error due to quantization that will affect the filter frequency response. To tackle this issue, we propose next an approach based on dynamically decreasing the quantization stepsize, which improves the control on the MSE. The caveat of this approach is that more bits are transmitted in the higher filter rounds (\( k \to K \)).

**B. Dynamically decreasing quantization stepsize**

Let us consider a quantization stepsize \( \Delta_k \) that decreases at each iteration \( k \). That is, less bits are transmitted for earlier values of \( k \) and more for \( k \to K \). The main result is given by the following proposition.

**Proposition 2.** Consider the FIR graph filter with shift operator \( S \) such that \( \lambda_{\text{max}} > 1 \). Consider also the input signal quantized with a uniform quantizer with decreasing quantization stepsize \( \Delta_k = (\lambda_{\text{max}})^{-k} \Delta_0 \). Then, the quantization MSE \( \hat{\zeta_q} \) of the FIR graph filter is upper bounded by:

\[
\hat{\zeta_q} \leq \frac{\Delta_0^2}{12(1 - \lambda_{\text{max}}^2)^{-1}} \phi_1
\] (14)

where \( \phi_1 = [\phi_1, \phi_2, \ldots, \phi_K]^T \) is the vector that contains the FIR coefficients, except the term for \( k = 0 \).

**Proof:** See Appendix VIII-D.

As opposed to Proposition 1, expression (14) shows that we have a clear control on the quantization MSE through \( \phi_1 \). Indeed, during the filter design phase, if we impose for the filter coefficients the condition that \( \mathbf{1}^\top \phi_1 \approx 0 \), we can reduce significantly the quantization MSE.

**C. Filter design**

Given a desired frequency response \( h^*(\lambda) \), we propose to design an FIR graph filter by solving the following convex optimization problem:

\[
\begin{aligned}
\text{minimize} & \quad \int \lambda \left| \sum_{k=0}^{K} \phi_k \lambda^k - h^*(\lambda) \right|^2 d\lambda \\
\text{subject to} & \quad \frac{1}{\Delta_0^2} \sum_{k=1}^{K} \phi_k \sum_{\kappa=0}^{k-1} \lambda_{\text{max}}^{2\kappa} (\lambda_{\text{max}}^2 - \lambda_{\text{max}}^{-k}) \leq \epsilon \\
& \quad \mathbf{1}^\top \phi_1 \leq \gamma \\
& \quad \Delta_k \leq \delta
\end{aligned}
\] (15)

For a finite small constant \( \epsilon \), the first constraint controls the upper bound of the quantization MSE in case of both fixed and decreasing quantization stepsize [cf. (5)]. For an infinite value of \( \gamma \), (15) leads to a similar optimization problem as [34] for the case of fixed quantization stepsize, while for the case of decreasing quantization stepsize, a finite small \( \gamma \) can be used. In the last constraint, \( \delta \) controls the maximum quantization stepsize, implying hence the control of the maximum number of bits used at each iteration, which we denote as \( \chi \).

By subquantizing the initial data of \( b_0 \) bits with an average of \( b \) bits at each iteration (i.e., \( b < b_0 \)), the communication cost of FIR graph filter in term of bits exchanged reduces to \( \mathcal{O}(MKb) \).

**IV. ARMA QUANTIZATION ANALYSIS**

This section analyzes the quantization effects on distributed ARMA graph filters. Since ARMA filters reach the designed frequency response at steady-state, the signal quantization will have also an effect on the filter convergence. We show in this section that the overall quantized MSE converges to zero if a dynamically decreasing quantization stepsize is considered, while this is not the case for the fixed stepsize-size quantizer.

**A. Fixed quantization stepsize**

Consider the parallel ARMA graph filter in (5) and let us indicate by \( w_t^{q(k)} = Q(w_t^{k}) = w_t^{k} + n_t^{q(k)} \)
the quantized signal of branch $k$ at iteration $t$, i.e., $w_t^{(k)}$. Here, $n_t^{(k)}$ denotes the respective quantization noise. Let also $w_t = [w_t^{(1)^T}, w_t^{(2)^T}, \ldots, w_t^{(K)^T}]^T$ be the $NK \times 1$ stacked vector containing all branches outputs and $n_t = [n_t^{(1)^T}, n_t^{(2)^T}, \ldots, n_t^{(K)^T}]^T$ the $NK \times 1$ stacked vector of quantization noise. Then, we can write the ARMA output $y_t$ due to quantization with the following compact notation:

$$w_t^q = (\Psi \otimes S)(w_{t-1}^q + n_{t-1}^q) + \varphi \otimes x$$  \hspace{1cm} (16)

where $\otimes$ indicates the Kronecker product, $\Psi = \text{diag}(\psi_1, \psi_2, \ldots, \psi_K)$ is the $K \times K$ diagonal matrix containing the former-output coefficients in the main diagonal and $\varphi = [\varphi_1, \varphi_2, \ldots, \varphi_K]^T$ is the $K \times 1$ coefficient vector associated to the input. By unfolding $w_t^q$ in (16) to all its terms, we have:

$$w_t^q = (\Psi \otimes S)^t w_0 + \sum_{r=0}^{t-1} (\Psi \otimes S)^r (\varphi \otimes x) + \sum_{r=0}^{t-1} (\Psi \otimes S)^{t-r} n_r$$  \hspace{1cm} (17)

where the first two terms on the right-hand side account for the ARMA output up to iteration $t$, while the third term $\epsilon_t^q = \sum_{r=0}^{t-1} (\Psi \otimes S)^{t-r} n_r$ accounts for the accumulated quantization noise.

To analyze the MSE for the ARMA filter, let us first denote by $w^* = \lim_{t \to \infty} w_t^q$ and by $y^* = \lim_{t \to \infty} y_t^q$ the steady-state values of $w_t^q$ and $y_t^q$ in (16), respectively. Let us also define the error:

$$\epsilon_t^q = (\Psi \otimes S)^t w_0 + \sum_{r=0}^{t-1} (\Psi \otimes S)^r (\varphi \otimes x) - w^*$$  \hspace{1cm} (18)

which indicates how close the output of all branches $w_t$ (without quantization) at iteration $t$ are w.r.t. the steady-state value $w^*$. We consider also the error $\epsilon_{yt}^q = y_t^q - y^*$ between the quantized ARMA output $y_t^q$ in (16) and the steady-state output $y^*$, which can be written as follows:

$$\epsilon_{yt}^q = (1^T \otimes I_K) e_t^q + (1^T \otimes I_K) e_t^q = \epsilon_t^q + \epsilon_{yt}^q$$  \hspace{1cm} (19)

where $e_t^q = (1^T \otimes I_K) e_t^q$ indicates how close the *unquantized* ARMA filter output $y_t$ at iteration $t$ is w.r.t. its steady-state $y^*$ and $\epsilon_{yt}^q = (1^T \otimes I_K) e_t^q$ accounts for the propagation of the quantization noise over the iterations. Then by simple algebra, the average MSE deviation per node of the error $\epsilon_{yt}^q$ in (19) can be similarly split as:

$$\zeta_{yt}^q = \frac{1}{N} \mathbb{E}[ \epsilon_t^q e_t^{H}] = \zeta_t^q + \zeta_{yt}^q$$  \hspace{1cm} (20a)

with:

$$\zeta_t^* = \frac{1}{N} \mathbb{E}[ \epsilon_t^q e_t^{H}(1^T \otimes I_K)^{H}]$$  \hspace{1cm} (20b)

$$\zeta_{yt}^q = \frac{1}{N} \mathbb{E}[ \epsilon_t^q e_t^{H}(1^T \otimes I_K)^{H}]$$  \hspace{1cm} (20c)

where we have used the linearity of the expectation w.r.t the trace and the independence of $x$, $w_0$ and $n_t^q$. $\zeta_t^*$ is the MSE for the case of unquantized filter from the steady-state output and $\zeta_{yt}^q$ is the quantization noise MSE at time $t$. The following proposition provides an upper bound on the quantization MSE.

**Proposition 3.** Consider the ARMA$_K$ graph filter of order $K$ in (16) with coefficients $\Psi$ and $\varphi$, and quantization error $\epsilon_{yt}^q$. Let $\psi_{\text{max}} = \max(|\psi_1|, |\psi_2|, \ldots, |\psi_K|)$ be the ARMA$_K$ coefficient with largest magnitude and let all $\text{ARMA}_K$ branches be stable i.e., $\psi_{\text{max}} \psi_{\text{max}} < 1$ for all $k = 1 \cdots K$. Consider also that the signal is quantized with a uniform quantizer with a fixed quantization stepsize $\Delta$. The quantization MSE $\zeta_{yt}^q$ of the filter output at iteration $t$ is upper bounded by:

$$\zeta_{yt}^q \leq \frac{K \Delta_0}{12} \tau^2 (\psi_{\text{max}} \lambda_{\text{max}})^2$$  \hspace{1cm} (21)

Further, the steady-state ($t \to \infty$) quantization MSE is:

$$\zeta_{yt}^q \to \infty \leq \frac{1}{1 - (\psi_{\text{max}} \lambda_{\text{max}})^2}$$  \hspace{1cm} (22)

**Proof:** See Appendix VIII-E

**Proposition 3** shows that the quantization MSE $\zeta_{yt}^q$ of ARMA graph filters is upper bounded by a term that depends on the shift operator maximum eigenvalue. At steady-state $t \to \infty$, the overall ARMA MSE in (20a) is governed by the quantization MSE $\zeta_{yt}^q$ since the deviation $\zeta_{yt}^q$ from the steady-state vanishes $\zeta_{yt}^q \to 0$ for convergent stable filters. Therefore, we conclude that a fixed quantization stepsize heavily affects the ARMA filter behavior, which even at the steady-state, although not divergent, might lead to a completely different filtering behavior.

The filtering behavior of the ARMA recursion will not be considerably affected by the quantization noise in the early regime (i.e., small value of $t$) as long as:

$$\zeta_{yt}^* \gg \zeta_{yt}^q.$$  \hspace{1cm} (23)

However, for larger $t$, this inequality will be violated and the overall ARMA MSE will be dominated by the quantization MSE $\zeta_{yt}^q$. While we might control $\zeta_{yt}^*$ in the design phase of FIR graph filters, we should consider the challenges encountered when designing convergent distributed ARMA filters [11], i.e., the difficulty to guarantee an accuracy-quantization robustness tradeoff. Rephrasing a non-convex design problem akin to (15) is possible, but because of non-convexity that may lead to suboptimal design solutions, in this work, we tackle this challenge by considering a decreasing quantization stepsize with $t$.

**B. Dynamically decreasing quantization stepsize**

Consider now a dynamic quantization stepsize size $\Delta_t$ that decreases with $t$ in a form that the quantization MSE $\zeta_{yt}^q$ decreases with $t$ at least with the rate of the unquantized ARMA error $\zeta_{yt}^q$ in (20a). The following proposition shows this can be achieved.

**Theorem 1.** Consider the ARMA$_K$ graph filter of order $K$ in (16) with coefficients $\Psi$ and $\varphi$, and quantization error $\epsilon_{yt}^q$. Let $\psi_{\text{max}} = \max(|\psi_1|, |\psi_2|, \ldots, |\psi_K|)$ be the ARMA$_K$ coefficient with largest magnitude and let all $\text{ARMA}_K$ branches be stable i.e., $\psi_{\text{max}} \psi_{\text{max}} < 1$ for all $k = 1 \cdots K$. Consider also that the signal is quantized with a uniform quantizer with a decreasing stepsize over the iterations $t$ as $\Delta_t = (\psi_{\text{max}} \lambda_{\text{max}})^2 \Delta_0$. The quantization MSE $\zeta_{yt}^q$ of the filter output at iteration $t$ is upper bounded by:

$$\zeta_{yt}^q \leq \frac{K \Delta_0}{12} \tau^2 (\psi_{\text{max}} \lambda_{\text{max}})^2$$  \hspace{1cm} (24)

which at the steady-state converges to zero ($\zeta_{yt}^q \to 0$) at a rate of $t(\psi_{\text{max}} \lambda_{\text{max}})^2$.
Theorem I shows that by adopting a decreasing quantization stepsize, the quantization MSE for the ARMA filters vanishes at the steady-state. This behavior is similar to the convergence error of the unquantized ARMA $\zeta_{qt}$ and suggests that at steady-state, we can reach the desired filter response. However, the quantization MSE converges with a rate $t(\psi_{\max}\lambda_{\max})^{2t}$ instead of $\psi_{\max}\lambda_{\max}$ at $t$. Faster convergence rates can be achieved by decreasing the quantization stepsize at a faster rate over time but this requires transmitting more bits for larger values of $t$.

Despite vanishing the quantization MSE at steady-state, the dynamic quantization stepsize comes together with a price. In particular, for large values of $t$, this implies that the quantization stepsize becomes infinitesimal; hence, the number of bits transmitted per round becomes that of the conventional ARMA graph filter [cf. (3)]. Nevertheless, this strategy reduces the communication efforts in the first iterations, i.e., we can start with a coarser $\Delta_0$. For $b_t$ being the number of bits transmitted at iteration $t$, the communication cost of the ARMA$_K$ graph filter per iteration is of order $O(MKb_t)$. If $b_t$ is the average number of bits transmitted over $T$ iterations, the ARMA$_K$ communication complexity amounts to $O(MKTb_t)$. The benefits of following this approach is that the ARMA design is readily available from the unquantized setting [11].

A related problem that can be of interest is to find the best sequence of quantization stepsize $\Delta_0, \Delta_1, \ldots, \Delta_t$ by taking into account the constraints of a given total bit budget $B$ available and a maximum number of iterations $t_{\max}$ and where $\Delta_t = (\psi_{\max}\lambda_{\max})^{t_0}$ $\Delta_0$. Note that the quantization stepsize $\Delta_t$ is defined as the ratio of the quantization range $r_t$ at iteration $t$ over the number of quantization intervals given by $\Delta_t = r_t^{-1/2^{(2t)}} = 2^{\parallel S^t \parallel_\infty/2^{(2t)}}$, where $\parallel S^t \parallel_\infty$ is the maximum size of the messages exchanged between nodes at iteration $t$ and that can be upper bounded by $\parallel S^t \parallel_\infty \leq \parallel S^t \parallel_2 \parallel x \parallel_2 \leq \rho' \parallel x \parallel_2 \leq \parallel x \parallel_2$ if the shift operator $S$ has a spectral radius bound i.e., $\rho \leq 1$. Thus, the best sequence of quantization stepsize can be obtained for $\rho \leq 1$, $\psi_{\max}\lambda_{\max} \neq 0$ and $\psi_{\max}\lambda_{\max} < 1$ by solving the problem $\sum_{t=0}^{t_{\max}} \log_2 \left( \frac{2^{ \parallel S^t \parallel_\infty/2^{(2t)}}}{\psi_{\max}\lambda_{\max}} \right) = B$, which implies $\Delta_0 = 2^{(1 - \frac{\rho'}{r_{t_{\max}}})} \parallel x \parallel_2 (\psi_{\max}\lambda_{\max})^{-t_{\max}}/2$.

V. QUANTIZATION ANALYSIS OVER TIME-VARYING GRAPHS

We now extend the quantization analysis to cases where the graph connectivity changes randomly over the filtering iterations. This scenario is expected to occur in applications of graph filtering over WSNs. For our analysis, we consider directly the more general dynamically decreasing quantization stepsize and the random edge sampling model from [25].

Definition 1 (Random edge sampling model [25]). Consider an underlying graph $G = (V, E)$. A random edge sampling (RES) graph realization $G_t = (V, E_t)$ of $G$ is composed of the same set of nodes $V$ and a random set of links $E_t \subseteq E$ that are activated (i.e., $(i, j) \in E_t$) with a probability $p_{ij} (0 < p_{ij} \leq 1)$. The links are activated independently over the graph and time and are mutually independent from the graph signal.

We consider the RES graph realization to model the link losses that occur at each filter iteration. As such, the RES model states that the realization $G_t = (V, E_t)$ at iteration $t$ is drawn from the underlying connectivity graph $G = (V, E)$, where the links $E_t \subseteq E$ are generated via an i.i.d. Bernoulli process with probability $p_{ij}$. Let then $P \in \mathbb{R}^{N \times N}$ denote the matrix that collects the link activation probabilities $p_{ij}$. Let also $S$, $S_t$, and $D$ denote, respectively, the shift operator of the underlying graph $G$, the graph realization $G_t$ at iteration $t$, and the expected graph $\bar{G}$. Since graph $\bar{G}$ has an upper bounded shift operator $\parallel \bar{G} \parallel_2 \leq \rho$, all its realizations $G_t$ have also an upper bounded shift operator $\parallel G_t \parallel_2 \leq \rho$ [46, 47].

Before, we proceed with the filter analysis, the following remark is in order. Under the RES model, if $S = A$ then the expected shift operator is $\bar{S} = E[A_t] = \bar{P} \circ A$. If $S = L_t$, then the expected shift operator is $\bar{S} = E[L_t] = D - (P \circ A)$, where $D = E[D_{ij}]$ is a diagonal matrix whose non zero entries are given by $D_{ii} = \sum_{j = 1}^{N} a_{ij} p_{ij}$.

A. FIR graph filters

When the FIR filter is run over RES graph realizations, the instantaneous shift operator $S_t$ is present in the filtering expression (22) and affects the output. To characterize this output, let us define the transition matrix of the RES graph realisations $G_t, \ldots, G_{t'}, \Theta(t', t) = \prod_{\tau = t}^{t'} S_{\tau}$ if $t' \geq t$ and $I$ if $t' < t$. The FIR filter output over a sequence of $K$ time-varying graphs is:

$$y_t = \sum_{k=0}^{K} \phi_k \Theta(t - 1, t - k) x$$

(25)

where the filter output is computed by considering all graph realizations from the iteration $t - K$ to $t$. From the independence of RES graph realizations, the expected FIR output is:

$$\bar{y}_t = E[y_t] = \sum_{k=0}^{K} \phi_k \overline{S^t} x$$

(26)

As shown in Appendix [VIII-G] the quantized FIR filter output over RES graph realizations can be written as $y_t = y_t + \epsilon_t$ where the quantization error $\epsilon_t$ has the expression:

$$\epsilon_t = \sum_{k=1}^{K} \sum_{\kappa=0}^{K-k-1} \phi_k \bar{\Theta}(t - \kappa - 1, t - k) n_{q}(\kappa).$$

(27)

The latter accounts for the percolation of the quantization noise $n_{q}(\kappa)$ over different random graph realizations. Since the quantization noise has a zero mean, the expected FIR output with quantization is $E[y_t^q] = \bar{y}_t$ [cf. (26)]. That is, in expectation, the FIR graph filter behaves as the filter in (25) operating on the expected graph with unquantized data.

To quantify the statistical impact of the quantization noise, we analyze the second order moment of the quantized output $y_t^q$ in the following proposition.

Proposition 4. Consider the FIR graph filter operating over the RES graph realizations $G_t$ [cf. Def. [7] with shift operators $S_t$ upper bounded as $\parallel S_t \parallel_2 \leq \rho$. Let also the filter input signal be quantized with a dynamic quantization stepsize size $\Delta_t$ at iteration $t$. The MSE of the filter output due to quantization

3Note that if $P$ has equal rows so that $p_{ij} = p_{i}$ for all $j \in V$ or has equal entries i.e. $p_{ij} = p$, we have $E[L_t] = P \circ L$.
and graph randomness $\xi_{yt}^{q} = E\left[ \frac{1}{N} \text{tr}(\mathbf{e}_t \mathbf{e}_t^H) \right]$ is upper bounded by:
\[
\xi_{yt}^{q} \leq \frac{1}{12} \sum_{k=1}^{K} \Delta_{k-1}^2 \left( \sum_{k=\infty}^{K} \rho^{k-\kappa+1} |\phi_k| \right)^2.
\] (28)

**Proof:** See Appendix VIII-H.

Note that Proposition 4 represents the worst-case bound for the graph randomness. This is similar to the unquantized graph filters over RES graphs [25] because the spectral radius $\rho$ accounts for all potential link losses (it is independent on the probabilities $p_{ij}$). On the other hand, this result serves as a proxy for the MSE to design a graph filter that is robust to both link losses and quantization error.

**Filter design.** Our goal is to design the filter coefficients $\phi_0, \ldots, \phi_K$ to reduce the quantization MSE in (28) while keeping the quantized graph filter output $y_t^q$ close in expectation to the unquantized output over the deterministic graph $G$; we denote the latter as $y^o = \sum_{k=0}^{K} \phi_k^O S^k x$. Then, let us consider the expected error due to quantization (bias):
\[
e = E[y_t^q - y^o] = E[y_t^q] - y^o.
\] (29)

While we can design the coefficients to minimize this bias, they will not account for the deviation around it. Therefore, we consider the more involved problem of finding the filter coefficients as a trade-off between the expected error of the filter output and the quantization MSE. For this, let us define the filtering matrix difference $E$:
\[
E = \sum_{k=0}^{K} \left( \phi_k S^k - \phi_k^O S^k \right)
\] (30)

that accounts for the response difference between the graph filtering over the expected graph $G$ and the graph filtering over the deterministic graph $G$. Then, we find the filter coefficients by solving the convex problem:
\[
\minimize_{\phi_k} \|E\|_F + \frac{\gamma}{12} \sum_{k=1}^{K} \Delta_{k-1}^2 \left( \sum_{k=\infty}^{K} \rho^{k-\kappa+1} |\phi_k| \right)^2
\] (31)

where $\|E\|_F$ is the Frobenius norm of (30) and $\gamma$ is a weighting factor trading-off the expected error and the quantization MSE.

**B. ARMA filters**

The parallel ARMA filter operating over random graphs has the branches outputs:
\[
w_t = (\Psi \otimes S_{t-1})w_{t-1} + \varphi \otimes x
\] (32)

which in the presence of quantization noise becomes:
\[
w_t^q = (\Psi \otimes S_{t-1})(w_{t-1}^q + n_{t-1}^q) + \varphi \otimes x
\] (33)

By expanding (33) to all the terms, we can write the overall ARMA filter output due to quantization as:
\[
w_t^q = \left( \prod_{\tau=0}^{t-1} \Psi \otimes S_{\tau} \right) w_0 + \varphi \otimes x + \sum_{\tau=1}^{t-1} \left( \prod_{\tau'=\tau-\tau}^{t-1} \Psi \otimes S_{\tau'} \right) (\varphi \otimes x) + \varepsilon_t^q
\]
\[
y_t^q = (1^T \otimes I_N)w_t^q
\] (34)

where in order to ease notation, we have denoted by $\varepsilon_t^q = \sum_{\tau=0}^{t-1} \left( \prod_{\tau'=\tau-\tau}^{t-1} \Psi \otimes S_{\tau'} \right) n_{\tau}^q$ the percolation of the quantization noise $n_{\tau}^q$ over the parallel ARMA branches up to time $t$. Then, let us consider the filter output error $\varepsilon_{yt} = y_t^q - y^o$ from the steady-state expected ARMA output $y^*$:
\[
\varepsilon_{yt} = \varepsilon_{yt}^q + \varepsilon_{yt}^q
\] (35)

where $\varepsilon_{yt}^q = (1^T \otimes I_N)\varepsilon_t^q$ is the quantization error on the output; $\varepsilon_{yt}^q = (1^T \otimes I_N)\varepsilon_t^q$ is the unquantized ARMA graph filter error at iteration $t$ w.r.t. to its steady-state $y^*$. Then, let us denote by $\varepsilon_t^q$ the unquantized ARMA error w.r.t. to the steady-state $w^*$, which is given by:
\[
\varepsilon_t^q = \left( \prod_{\tau=0}^{t-1} \Psi \otimes S_{\tau} \right) w_0 + \varphi \otimes x + \sum_{\tau=1}^{t-1} \left( \prod_{\tau'=\tau-\tau}^{t-1} \Psi \otimes S_{\tau'} \right) (\varphi \otimes x) - w^*.
\] (36)

Under the RES graph model and given the zero-mean quantization noise, it can be easily shown from (32) and (33) that $E[w_t^q] = E[w_t^q]$; i.e., in expectation both the quantized and unquantized ARMA filters give the same output. However, the quantization impacts on the second order moment of the filter output error $\varepsilon_{yt}^q$ in (35). We analyze next the MSE of the latter, which by simple algebra, can be split as:
\[
\xi_{yt} = \frac{1}{N} E[\text{tr}(E[y_t^q y_t^H])] = \xi_{yt}^q + \xi_{yt}^q.
\] (37a)

where:
\[
\xi_{yt}^q = \frac{1}{N} E[\text{tr}((1^T \otimes I_N)\varepsilon_t^q (\varepsilon_t^q)^H (1^T \otimes I_N)^H)]
\] (37b)
\[
\xi_{yt}^q = \frac{1}{N} E[\text{tr}((1^T \otimes I_N)\varepsilon_t^q (\varepsilon_t^q)^H (1^T \otimes I_N)^H)]
\] (37c)

and where we have used the linearity of the expectation w.r.t. the trace, the cyclic property of the trace, and the independence of $x, w_0$ and $n_t^q$. $\xi_{yt}$ is the MSE for the case of unquantized filter w.r.t. to its steady-state output. The next proposition provides an upper bound on the quantization MSE $\xi_{yt}^q$, when the quantization stepsize $\Delta_k$ decreases at each iteration $k$.

**Theorem 2.** Consider the ARMA$_K$ graph filter operating over RES graph realizations $G_t$ [cf. Def. II] with shift operators $S_t$ upper bounded as $\|S_t\|_2 \leq \rho$. Let $\psi_{\max} = \max(\|\psi_1\|,\|\psi_2\|,\ldots,\|\psi_K\|)$ be the ARMA$_K$ coefficient with largest magnitude and let all ARMA$_K$ branches be stable i.e., $\psi_{\max} \rho < 1$ for all $k = 1 \cdots K$. Let also the filter input signal be quantized with a uniform quantizer having a stepsize decreasing over the iterations $t$ as $\Delta_t = (\psi_{\max} \rho)^{t-\Delta_0}$. The MSE of the ARMA filter output at iteration $t$ due to quantization and graph randomness $\xi_t^q$ can be upper bounded by:
\[
\xi_t^q \leq \frac{K^2}{12} \Delta_0^2 t (\psi_{\max} \rho)^{2t}
\] (38)

making the quantization MSE converge to zero ($\xi_{yt}^q \rightarrow 0$) at a rate of $t(\psi_{\max} \rho)^{2t}$.

**Proof:** See Appendix VIII-I.

Theorem 2 highlights that the quantization MSE converges to zero when using a decreasing quantization stepsize, despite the random topological changes and the presence of quantization. This implies that there is no need to consider the quantization MSE in the design phase. However, contrarily to time-invariant graphs, the overall MSE of ARMA filters, which is affected by both the quantization $\xi_{yt}^q$ and the random variation part $\xi_{yt}^q$, cannot reach the desired filter response at steady-state ($t \rightarrow \infty$), because even if the quantization MSE $\xi_{yt}^q$ can be made to converge to zero, the unquantized MSE $\xi_{yt}$ does not converge to zero due to graph topological changes.
Similarly to time-invariant graphs in Section IV-B where we consider the constraints of a given total bit budget available and a maximum number of iterations \( t_{\text{max}} \), the best sequence of quantization stepizes is given by \( \Delta_0 = 2^{(1 - \frac{q}{t_{\text{max}}})} \| x \|_2 (\psi_{\text{max}} \rho)^{-\frac{q}{2}} \) and \( \Delta_t = (\psi_{\text{max}} \rho)^t \Delta_0 \) for \( \rho \leq 1 \) and \( 0 < \psi_{\text{max}} \rho < 1 \).

**Corollary 2.1.** Consider same settings as Theorem 2 with the input signal quantized with a uniform quantizer having a fixed quantization stepize \( \Delta \). The MSE of the filter output due to quantization and graph randomness \( \xi_{\text{qyt}} \) can be upper bounded by:

\[
\xi_{\text{qyt}} \leq K^2 \sigma_q^2 \left( \frac{(\psi_{\text{max}} \rho)^2}{(1 - (\psi_{\text{max}} \rho)^2)^{-1}} - (\psi_{\text{max}} \rho)^2 \right)^{t+1} \tag{39}
\]

which in the steady-state (\( t \to \infty \)) becomes:

\[
\xi_{\text{qyt} \to \infty} \leq K^2 \sigma_q^2 \left( \frac{(\psi_{\text{max}} \rho)^2}{1 - (\psi_{\text{max}} \rho)^2} \right)^2 \tag{40}
\]

**Proof.** By considering a fixed quantization stepize \( \Delta \), the upper bound of the MSE of ARMA filter due to quantization and graph randomness in (74) becomes:

\[
\xi_{\text{qyt}} \leq K^2 \sigma_q^2 \sum_{r=1}^t (\psi_{\text{max}} \rho)^{2r} \tag{41}
\]

By considering the upper bound in (41) is finite geometric series with argument smaller than 1, \( \xi_{\text{qyt}} \) can be upper bounded by (39).

VI. NUMERICAL EXPERIMENTS

This section corroborates our theoretical findings with numerical experiments on both synthetic and real data from the Molene and the Intel Berkeley sensor network.

A. Synthetic data

We consider WSNs with \( N = 100 \) sensor nodes, which are randomly and uniformly distributed over a square area of side 150 m. Each node can communicate with the neighbors within the transmission range \( R = 50 \) m. The latter forms a communication network that can be used to perform distributed graph filtering operations. In the sequel, we evaluate the quantized filters in the baseline ideal-low pass filter and signal denoising. To account for the graph randomness, we averaged the results over 1000 different realizations.

**Ideal low-pass filter.** We considered the FIR graph filter to approximate an ideal low-pass filter with frequency response \( h(\lambda) = 1 \) if \( \lambda \leq \lambda_c \) and zero otherwise. The shift operator is the normalized Laplacian \( L = D^{-1/2}LD^{-1/2} \) and the cut off frequency \( \lambda_c \) is half the spectrum. The input signal \( x \) has a white unitary spectrum w.r.t. the underlying graph \( G \).

**Tikhonov denoising.** We now evaluate the performance of the proposed solutions in distributed denoising. We assume a noisy graph signal \( x = z + n \), where \( z \) is the signal of interest and \( n \) is a zero mean additive noise. To recover signal \( z \), we solve the Tikhonov denoising problem:

\[
z^* = \arg\min_{z \in \mathbb{R}^N} \| x - z \|_2^2 + w \| z \|^2 \tag{42}
\]

for \( S = L \) or \( S = L_n \), and where the regularizer \( z^T Sz \) is based on the prior assumption the graph signal varies smoothly with respect to the underlying graph and \( w \) is the weighting factor trading smoothness and noise removal. The closed-form solution of (42) is an ARMA \( 1 \) filter \( z^* = (I + wS)^{-1}z \) with coefficients \( \psi = -w \) and \( \varphi = 1 \) (11). Hence, we can employ said filter to solve distributively the Tikhonov denoising problem.

In Fig. 2 we compare the MSE of the quantized and the unquantized outputs of FIR and ARMA graph filters over time-invariant graphs. The noise in this instance is zero-mean Gaussian with variance \( \sigma^2 = 0.2 \). We observe the ARMA graph filter with decreasing quantization stepsize significantly outperforms both the ARMA with fixed quantization stepsize and the FIR filter with optimized filter coefficients and decreasing quantization stepsize. The latter corroborates our finding in Theorem 2 that ARMA filters reach machine precision with a decreasing quantization stepsize.

We now evaluate the filters over time-varying graphs, by analyzing the average MSE between the quantized output seen in the steady-state.

Fig. 1. NSE of FIR graph filters over time-invariant graphs, when approximating an ideal low-pass filter. The filter coefficients are optimized by solving (15), where the maximum number of bits at each iteration is \( \chi = 32 \) bits. The results are compared to the Robust Filter Design (RobFD) solution [34].

Fig. 2. NSE between the quantized output and the unquantized output of FIR and ARMA filtering over time-invariant graphs for the Tikhonov denoising problem, where \( S = L_n \) and \( w = 0.3 \). The FIR filter coefficients are optimized by solving (15), with \( \chi = 25 \) bits. The x-axis is the filter order for the FIR filter and the number of iterations for the ARMA filter.
the time-varying graph $y^t$ and the unquantized output $y_t$ over the deterministic graph. As shown in Fig. 3(a), the ARMA graph filter presents significantly better performance than the FIR graph filter, when the link activation probability is $p = 0.95$ and the quantization stepsize is decreasing over the iterations. We also observe the average NSE for ARMA filters reduces considerably when the number of iterations increases. Notice also the NSE floor of ARMA filter is the value when the signal is quantized with all the available bits and where $\Delta_k$ is very small. The latter corresponds to the machine precision accuracy, corroborating our results in Theorem 2.

In Figs. 3(b), 3(c), we analyze the average NSE for different probabilities of link activation and different maximum numbers of bits used for quantization. ARMA filters with decreasing quantization stepsize achieves always the highest filtering accuracy with a significant margin compared to other filters. Fig. 3(b) shows that, as expected, better link connectivities (higher $p$) lead to lower errors as expected. It is also worth noticing that the graph filtering accuracy is less affected by topological changes due to link losses for lower filter orders $K$, as compared to higher filter orders. This is because the exchanges between nodes through problematic links reduce. This highlights the trade-off between the filter order and robustness to topological changes: a higher order should be preferred when the topology is highly stable. Fig. 3(c) shows that the average NSE decreases when the maximum number of bits used for quantization at each iteration is higher. This because increasing the quantization bits decreases the quantization stepsize at each iteration, which reduces as well the quantization errors accumulated among iterations. We can also observe that increasing the quantization bits does not lead necessarily to a noticeable improve of the filtering accuracy, especially for low probability of link activation, as compared to higher probability of link activation. We attribute this behavior to the large number of links that fall, therefore, the error due to link losses dominates that of quantization.

B. Real data

We now illustrate the performance of the proposed solutions for the graph signal interpolation task over time-invariant and time-varying topologies with two real data sets.

Molene weather data set. The Molene weather data contains hourly observations of temperature measurements of $N = 32$ weather stations collected in the region of Brest in France, for a total of 744 hours. We consider a geometric graph constructed from the node coordinates using the default nearest neighborhood approach, as in [13].

Let $x'$ be the observed graph signal $x$ with missing values. We aim at reconstructing the overall graph signal $x$ from the observations $x'$ by exploiting the smoothness of $x$ over the graph. This problem can be formulated as [48], [49]:
\[ \mathbf{x}^* = \arg\min_{\mathbf{x} \in \mathbb{R}^N} \| \mathbf{T} (\mathbf{x} - \mathbf{x}') \|_2^2 + w \mathbf{x}^\top \mathbf{S} \mathbf{x} \] (43)

where \( \mathbf{T} \) is a diagonal matrix with \( T_{ii} = 1 \) if \( x_i \) is known and \( T_{ii} = 0 \) otherwise and \( w \) is the weighting factor. The optimal solution of the convex optimization problem (43) is \( \mathbf{x}^* = (\mathbf{T} + w \mathbf{S})^{-1} \mathbf{x}' = (\mathbf{I} - \mathbf{S})^{-1} \mathbf{x}' \), which is an ARMA filter for the shift operator \( \mathbf{S} = \mathbf{T} + w \mathbf{S} - \mathbf{I} \) [11]. To generate missing values in the Molene weather data set, we randomly wipe off signal values up to certain percentage. Then, we analyze the MSE between the quantized output and the unquantized output of graph filters, for different percentages of missing values.

Fig. 4(a) shows the MSE decreases considerably at each iteration, particularly for ARMA filters. It is also worth noticing this decrease enhances when less data are missing.

**Intel Lab data set.** The Intel Berkeley Research Lab data set contains light data of \( N = 54 \) Mica2Dot sensor nodes distributed in an indoor environment over an area of \( 1200 \text{ m}^2 \) [50]. The communication between the sensor nodes is wireless and prone to channel noise and interference, leading to time-varying graph topological changes due to link losses. The probability of link activation of the nodes is about 0.13 with a standard deviation of 0.18. The underlying graph topology has high connectivity with an average node degree of 47, implying multiple communication paths exist between nodes, helping to make signal exchanged between nodes robust to link losses.

In Fig. 4(b) we analyze the average quantized MSE as a function of the missing values for the FIR and ARMA graph filters. Even though the graph filtering accuracy is affected by the accumulated quantization errors over iterations and the graph topological changes, ARMA filters provide a significant decrease in terms of MSE, when the number of iterations grows and the percentage of missing data is low. Fig. 4(c) depicts the MSE between the quantized graph signal output and the true signal for the two data sets as a function of the missing values. The results show that for both data sets a quite good performance in terms of signal reconstruction is achieved, especially with ARMA graph filters. This stresses our finding in Theorem 2, which means that with a decreasing quantization stepsize there is no need to perform a robust ARMA graph filter design since the proposed strategy achieves the optimal steady-state solution.

**VIII. APPENDIX**

**A. Quantized FIR graph filter output**

Considering \( \mathbf{x}(0) = \mathbf{x} \) and the quantized message at iteration \( k \), \( \hat{\mathbf{x}}(k) = \mathbf{x}(k) + \mathbf{n}_q(k) \), the output of the shifted signal with quantization is:

\[ \begin{align*}
\mathbf{x}(1) &= \mathbf{S} \mathbf{x}(0) + \mathbf{n}_q = \mathbf{S}(\mathbf{x}(0) + \mathbf{n}_q) = \mathbf{S} \mathbf{x} + \mathbf{S} \mathbf{n}_q \\
\mathbf{x}(2) &= \mathbf{S} \mathbf{x}(1) + \mathbf{S} \mathbf{n}_q = \mathbf{S}^2 \mathbf{x} + \mathbf{S}^2 \mathbf{n}_q \\
&\vdots \\
\mathbf{x}(k) &= \mathbf{S}^k \mathbf{x}(0) + \sum_{k=0}^{k-1} \mathbf{S}^k \mathbf{n}_q(k), \quad k \geq 1. \quad (44)
\end{align*} \]

From (44), the FIR graph filter output with quantization is:

\[ \begin{align*}
\mathbf{y}_q &= \phi_0 \mathbf{x} + \phi_1 (\mathbf{S} \mathbf{x} + \mathbf{n}_q) + \phi_2 (\mathbf{S}^2 \mathbf{x} + \mathbf{S} \mathbf{n}_q) + \mathbf{S} \mathbf{n}_q(1) + \cdots + \phi_k (\mathbf{S}^k \mathbf{x} + \mathbf{S}^{k-1} \mathbf{n}_q) + \mathbf{S} \mathbf{n}_q(1) + \cdots + \mathbf{S} \mathbf{n}_q(K-2) + \mathbf{n}_q(K-1) \\
&= \sum_{k=0}^{K} \phi_k \mathbf{S}^k \mathbf{x} + \sum_{k=1}^{K} \phi_k \mathbf{S}^{k-1} \mathbf{n}_q(k).
\end{align*} \]

**B. Proof of Proposition 1**

By applying the GFT on both sides of (11), the quantization error has the spectral response:

\[ \hat{\mathbf{e}} = \sum_{k=1}^{K} \phi_k \sum_{\kappa=0}^{k-1} \mathbf{A}^{k-\kappa} \mathbf{\hat{n}_q}(\kappa) \] (46)

where \( \mathbf{\hat{n}_q}(\kappa) \) is still i.i.d. with same statistics as \( \mathbf{n}_q(\kappa) \) iff \( \Sigma_{\mathbf{q}_\kappa} = \sigma_{\mathbf{q}_\kappa}^2 \mathbf{I} \). From the linearity of the expectation and from the matrix property \( (\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top \), the quantization noise covariance matrix becomes:

\[ \begin{align*}
\mathbb{E}[\mathbf{\hat{e}}^\top \mathbf{\hat{e}}] &= \sum_{k_1, k_2=1}^{K} \phi_{k_1} \phi_{k_2} \sum_{\kappa_1=0}^{k_1-1} \sum_{\kappa_2=0}^{k_2-1} \mathbf{A}^{\kappa_1, \kappa_2-\kappa_1} \mathbb{E}[\mathbf{n}_q(\kappa_1) (\mathbf{n}_q(\kappa_2))^\top] (\mathbf{A}^{k_2-\kappa_2})^\top. \\
(47)
\end{align*} \]

Given the quantization noise has independent realizations and a constant quantization stepsize \( \Delta \) for all iterations, we can rewrite (47) as:

\[ \begin{align*}
\mathbb{E}[\mathbf{\hat{e}}^\top \mathbf{\hat{e}}] &= \sum_{k=1}^{K} \phi_k \sum_{\kappa=0}^{k-1} \mathbf{A}^{k-\kappa} \Sigma_{\mathbf{q}_\kappa} (\mathbf{A}^{k-\kappa})^\top = \sigma_{\mathbf{q}_\kappa}^2 \sum_{k=1}^{K} \phi_k \sum_{\kappa=0}^{k-1} \mathbf{A}^{k-\kappa} (\mathbf{A}^{k-\kappa})^\top.
\end{align*} \]

Then, by substituting (48) into the MSE expression \( \hat{\mathbf{e}}_q = \frac{1}{\Delta} \text{tr}(\mathbb{E}[\mathbf{\hat{e}}^\top \mathbf{\hat{e}}]) \) and using the relation between the Frobenius norm and the trace \( \| \mathbf{A} \|_F = \sqrt{\text{tr}(\mathbf{A}^\top \mathbf{A})} \), result (12) yields.

**C. Proof of Corollary 1**

From (12) and the relation between the \( l_2 \)-norm and the Frobenius norm \( \| \mathbf{A} \|_F \leq \sqrt{n} \| \mathbf{A} \|_2 \) with \( r \) the rank of \( \mathbf{A} \) (at most \( N \)), \( \hat{\mathbf{e}}_q \) can be upper bounded as:

\[ \hat{\mathbf{e}}_q \leq N \sigma_{\max}^2 \sum_{k=1}^{K} \phi_k \sum_{\kappa=0}^{k-1} \| \mathbf{A}^{k-\kappa} \|_2 \leq N \sigma_{\max}^2 \sum_{k=1}^{K} \phi_k \sum_{\kappa=0}^{k-1} (\lambda_{\max})^{k-\kappa}. \]

(49)
Then, with the index change \( \sum_{k=1}^{K} \phi_k \sigma_{q}^2 \leq \sum_{k=1}^{K} \sigma_{q}^2 \), we obtain the finite geometric series whose argument is different from one by hypothesis; thus, \( \hat{\zeta}_q \) can be upper bounded as in (13). Similarly, by exploiting again the relationship between the \( l_2 \)-norm and Frobenius norm of matrices \( \| A \|_2 \leq \| A \|_F \) in (12), \( \hat{\zeta}_q \) can be likewise lower bounded to obtain (13).

D. Proof of Proposition 2

By equivalence to (12), the MSE on the filter output due to the quantization noise has the form:

\[
\hat{\zeta}_q = \frac{1}{N} \sum_{k=1}^{K} \phi_k \sigma_{q}^2 \| A + \kappa \|_F^2 \leq \frac{1}{N} \sum_{k=1}^{K} \phi_k \sigma_{q}^2 \| \lambda_{\max} \|_F^2
\]

By choosing the quantization stepsize \( \Delta_k = (\lambda_{\max})^{-k} \Delta_0 \) (50), we have:

\[
\hat{\zeta}_q \leq \frac{\Delta_0}{12} \sum_{k=1}^{K} \phi_k \sigma_{q}^2 \| \lambda_{\max} \|^2
\]

The bound (51) contains geometric series; thus, under the assumption \( \lambda_{\max} < 1 \), we have:

\[
\hat{\zeta}_q \leq \frac{\Delta_0}{12} \sum_{k=1}^{K} \phi_k \sigma_{q}^2 (1 - (\lambda_{\max})^k)
\]

where the final bound can be written as (14).

E. Proof of Proposition 3

By using (20b), the trace cyclic property \( \text{tr}(A B C) = \text{tr}(C A B) \), the inequality \( \text{tr}(A B) \leq \| A \|_2 \text{tr}(B) \)–which holds for any positive semi-definite matrix \( B \geq 0 \) and square matrix \( A \) of appropriate dimensions (51)–, and the linearity of the expectation w.r.t. the trace, we can write:

\[
\zeta_q^{\text{st}} = \frac{1}{N} E \left( \text{tr}\left( (I^T \otimes I_N) (I^T \otimes I_N) \right) \right) = \frac{1}{N} E \left( \text{tr}\left( \epsilon(\epsilon^T) \right) \right)
\]

Then, substituting \( \epsilon_q^2 = \sum_{k=0}^{t-1} \text{tr}(\Psi \otimes S)^k \geq \| S \|_F^2 \) in (53),

\[
E[\epsilon_q^2] = \sigma_q^2 \geq \sigma_q^2 \| S \|_F^2
\]

By using in (54) the index change \( \sum_{k=0}^{t-1} \text{tr}(A(\Phi(\Phi^T)) = \sum_{k=0}^{t-1} \text{tr}(A^T(\Lambda(\Lambda^T)) = \text{tr}(A^T(\Lambda^T)) \geq \text{tr}(A^T)) \), the Frobenius norm \( \| A \|_F = \sqrt{\text{tr}(A^T A)} \), and the triangle inequality of the norms \( \| A \|_2 \leq \| A \|_F \) (12), we have:

\[
\zeta_q^{\text{st}} \leq \frac{\Delta_0}{2} \sum_{k=1}^{K} \phi_k \sigma_{q}^2 \| \lambda_{\max} \|^2
\]

Then, from the Kronecker product identity \( \| A \otimes B \|_2 = \| A \|_2 \| B \|_2 \) and the \( l_2 \)-norm matrix norm expression \( \| A \|_2 = \sqrt{\max \text{eig}(A^T A)} \), we can further rewrite (55) as:

\[
\zeta_q^{\text{st}} \leq \frac{\Delta_0}{2} \sum_{k=1}^{K} \phi_k \sigma_{q}^2 \| \lambda_{\max} \|^2
\]

Finally, since (56) is a finite geometric series with an argument smaller than one, the quantization MSE \( \zeta_q^{\text{st}} \) can be upper bounded by (21).

F. Proof of Theorem 7

By equivalence to (54), but with a dynamic quantization stepsize, the MSE on the filter output due to the quantization noise is upper bounded by:

\[
\zeta_q^{\text{st}} \leq \frac{K}{N} \sum_{\tau=0}^{t-1} \sigma_{q}^2 \text{tr}( (\Psi \odot \Psi)^{t-\tau} ) (\|(\Psi \odot \Psi)^{t-\tau}\|^H)
\]

\[
\leq \frac{K}{12N} \sum_{\tau=1}^{t} \Delta_{\tau}^2 \| \Psi \odot \Psi \|^2 \leq \frac{1}{12N} \sum_{\tau=1}^{t} \Delta_{\tau}^2 (\psi_{\lambda_{\max}})^2
\]

(58)

where similarly to (55) and (56), we changed the summation index, used the expression of the Frobenius norm \( \| A \|_F = \sqrt{\text{tr}(A^T A)} \), and leveraged the norm properties. For the quantization stepsize \( \Delta_{\tau} = (\psi_{\lambda_{\max}})^{-\tau} \Delta_0 \), (58) can be further upper bounded as:

\[
\zeta_q^{\text{st}} \leq \frac{K}{12N} \sum_{\tau=1}^{t} (\psi_{\lambda_{\max}})^{2\tau} \Delta_0
\]

(59)

which can be easily rephrased as (24).

G. Quantized FIR graph filter over time-varying graphs

(59)

Considering \( x(0) = x \) and the quantized message at iteration \( k, x(k) = x(k) + n_q(k) \), the output of the shifted signal with quantization performed over \( G_t \) is:

\[
\begin{align*}
\psi(1) &= S_{t-1} x(0) = S_{t-1} (x(0) + n_q(0)), \\
\psi(2) &= S_{t-2} x(0) + S_{t-2} x(0) + n_q(0) + n_q(1), \\
& \vdots \\
\psi(k) &= \left( \prod_{t=1}^{k-1} S_t \right) x(0) + \sum_{n=0}^{k-1} \left( \prod_{t=1}^{k-n-1} S_t \right) n_q(n), \quad k \geq 1
\end{align*}
\]

(60)

The quantized output of FIR graph filter at iteration \( t \), performed over \( G_t \), with quantization effects, is given by:

\[
\begin{align*}
\psi(t) &= \psi(0) + \sum_{k=1}^{K} \sum_{k=1}^{K} \phi_k \Theta(t-1, t-k) x(0) + \sum_{n=0}^{k-1} \Theta(t-1-k, t-k) n_q(n), \\
& = \sum_{k=0}^{K} \phi_k \theta(t-1, t-k) x + \sum_{n=0}^{k-1} \theta(t-1-k, t-k) n_q(n), \quad k \geq 1
\end{align*}
\]

(61)

H. Proof of Proposition 4

By using \( \| x \|^2 = \text{tr}(x^H x) \) and rearranging the summation indices, we can write the MSE of the filter output due quantization and graph randomness as:

\[
\zeta_q^{\text{st}} = E\left[ \frac{1}{N} \text{tr}(e_t e_t^H) \right] = \frac{1}{N} E\left[ \| e_t \|^2 \right]
\]

(61)

Let then vector \( \omega(k, t) = \sum_{n=0}^{k-1} \phi_n \theta(t-1-k, t-k) n_q(n-1) \) account for the accumulated quantization noise over time-varying graphs. By using \( \| x \|^2 = x^H x \), we can write:

\[
E\left[ \left\| \sum_{k=1}^{K} \omega(k, t) \right\|^2 \right] = \sum_{k=1}^{K} E\left[ \omega(k, t)^H \omega(k, t) \right]
\]

(62)
Since the quantization errors are zero mean and independent from graph topology processes, we have:

\[ E\left[ \omega(k_1, t)\omega(k_2, t) \right] = \begin{cases} 0 & \text{if } k_1 \neq k_2 \\ \mathbb{E}[\|\omega(k_1, t)\|^2] & \text{if } k_1 = k_2 \end{cases} \tag{63} \]

Therefore, we can rewrite (63) as:

\[ \mathbb{E}\left[ \left\| \sum_{k=1}^{K} \omega(k, t) \right\|^2 \right] = \sum_{k=1}^{K} \mathbb{E}\left[ \|\omega(k, t)\|^2 \right]. \tag{64} \]

Using once again the norm property \( \|x\|^2 = \text{tr}(xx^H) \), the cyclic property of the trace \( \text{tr}(ABC) = \text{tr}(CAB) \), and the commutativity of the trace to respect to the expectation, we can write:

\[ \mathbb{E}\left[ \|e_t\|^2 \right] = \sum_{k=1}^{K} \mathbb{E}\left[ \left( \sum_{k=1}^{K} \phi_k \Theta(t-k, t-k) \right) n_k^{(k-1)} (n_k^{(k-1)})^H \right] \]

\[ \times \mathbb{E}\left[ \left( \sum_{k=1}^{K} \phi_k \Theta(t-k, t-k) \right) \right] \]

\[ \times \mathbb{E}\left[ \left( \sum_{k=1}^{K} \phi_k \Theta(t-k, t-k) \right) \right]^H \tag{65} \]

By using the inequality \( \text{tr}(AB) \leq \|A\|_2 \|B\|_2 \), we obtain:

\[ \mathbb{E}\left[ \|e_t\|^2 \right] \leq \sum_{k=1}^{K} \mathbb{E}\left[ \left( \sum_{k=1}^{K} \phi_k \Theta(t-k, t-k) \right) \right] \]

\[ \times \mathbb{E}\left[ \left( \sum_{k=1}^{K} \phi_k \Theta(t-k, t-k) \right) \right]^H \tag{66} \]

Since \( \text{tr}\left[ \mathbb{E}\left[ n_k^{(k)} (n_k^{(k)})^H \right] \right] = \text{tr}\left[ \mathbb{E}\left[ (n_k^{(k)})^H \right] \right] = N\sigma_x^2 \) and using the Jensen’s inequality of the spectral norm (\( \mathbb{E}[\|A\|_2] \leq \mathbb{E}[\|A\|_2] \)), we can further write:

\[ \mathbb{E}\left[ \|e_t\|^2 \right] \leq \sum_{k=1}^{K} \mathbb{E}\left[ \left( \sum_{k=1}^{K} \phi_k \Theta(t-k, t-k) \right) \right] \]

\[ \times \mathbb{E}\left[ \left( \sum_{k=1}^{K} \phi_k \Theta(t-k, t-k) \right) \right]^H \tag{67} \]

where \( \mathcal{T}(t, k) \) is:

\[ \mathcal{T}(t, k) = \left( \left\| \sum_{k=1}^{K} \phi_k \Theta(t-k, t-k) \right\|^2 \right) \]

\[ \left\| \sum_{k=1}^{K} \phi_k \Theta(t-k, t-k) \right\|^2 \tag{68} \]

By using the sub-multiplicativity property \( \|AB\|_2 \leq \|A\|_2 \|B\|_2 \) and subadditivity \( \|A + B\|_2 \leq \|A\|_2 + \|B\|_2 \) along with the upper bound of the shift operator \( \|S_i\|_2 \leq \rho \) for all \( t \), we upper bound (68) as:

\[ \mathcal{T}(t, k) \leq \left( \sum_{k=1}^{K} \rho^{k-1} |\phi_k| \right)^2 \tag{69} \]

Finally, by substituting (69) into (67) and computing the expectation, \( \xi_q^g \) can be upper bounded by (28).

I. Proof of Theorem 2

Similarly to (53), we can write the MSE of ARMA filter due to quantization and graph randomness (73) as:

\[ \xi_q^g \leq \frac{1}{N^2} \left( (1^T \otimes I_N)^H (1^T \otimes I_N) \right) \|\mathbb{E}[\epsilon_t^2 (\epsilon_t^2)^H]\| \leq \frac{K}{N} \|\mathbb{E}[\epsilon_t^2 (\epsilon_t^2)^H]\| \tag{70} \]

Then, by substituting \( \epsilon_t^2 \) with its expression, using the linearity of the expectation w.r.t the trace, the cyclic property of the trace \( \text{tr}(ABC) = \text{tr}(CAB) \), we can write:

\[ \text{tr}(\mathbb{E}[\epsilon_t^2 (\epsilon_t^2)^H]) = \sum_{\tau_1=0}^{t-1} \sum_{\tau_2=0}^{t-1} \text{tr} \left( \prod_{\zeta=\tau_1}^{\tau_2} \Psi \otimes S_i \right)^H \]

\[ \times \left( \prod_{\zeta=\tau_1}^{\tau_2} \Psi \otimes S_i \right) n_{\tau_2}^2 \tag{71} \]

By considering \( \mathbb{E}[n_{\tau_1}^2 (n_{\tau_2}^2)^H] = 0 \) if \( \tau_1 \neq \tau_2 \), using the inequality \( \text{tr}(AB) \leq \|A\|_2 \|B\|_2 \), assuming quantized representation with dynamic size i.e., \( \mathbb{E}[n_{\tau_1}^2 (n_{\tau_2}^2)^H] = KN\sigma_q^2 \), and using the Jensen’s inequality of the spectral norm (\( \mathbb{E}[\|A\|_2] \leq \mathbb{E}[\|A\|_2] \)), we can write:

\[ \text{tr}(\mathbb{E}[\epsilon_t^2 (\epsilon_t^2)^H]) \leq KN \sum_{\tau=0}^{t-1} \mathbb{E}\left[ \left( \prod_{\zeta=\tau}^{t-1} \Psi \otimes S_i \right)^H \left( \prod_{\zeta=\tau}^{t-1} \Psi \otimes S_i \right) \right] \]

\[ \times \left( \prod_{\zeta=\tau}^{t-1} \Psi \otimes S_i \right) n_{\tau_2}^2 \tag{72} \]

By using the sub-multiplicativity property of the spectral norm of a square matrix i.e., \( \|AB\|_2 \leq \|A\|_2 \|B\|_2 \), the property \( \|A \otimes B\|_2 \leq \|A\|_2 \|B\|_2 \) and assuming that the spectral norm of the shift operator used is upper bounded i.e., \( \|S_i\|_2 \leq \rho \) for all \( t \), we have:

\[ \mathbb{E}\left[ \left( \prod_{\zeta=\tau}^{t-1} \Psi \otimes S_i \right)^H \left( \prod_{\zeta=\tau}^{t-1} \Psi \otimes S_i \right) \right] \leq \left( \psi_{\text{max}} \right)^2 \tag{73} \]

By applying the expectation to (72) and combining it with (72) and (70), and making an index change using \( \sum_{\tau=0}^{t-1} c_{\tau} t^{-\tau} = \sum_{\tau=1}^{t} c_{\tau-\tau} \), we can write:

\[ \xi_q^g \leq K^2 \sum_{\tau=0}^{t-1} \mathbb{E}\left[ (\psi_{\text{max}})^2 \right] t^{-\tau} \leq K^2 \sum_{\tau=1}^{t} \mathbb{E}\left[ (\psi_{\text{max}})^2 \right] t^{-\tau} \]

\[ \leq K^2 \sum_{\tau=1}^{t} \mathbb{E}\left[ (\psi_{\text{max}})^2 \right] \tag{74} \]

With the choice of the quantization stepsize \( \Delta_t = (\psi_{\text{max}})^2 \Delta_0 \), the final bound in (73) becomes:

\[ \xi_q^g \leq K^2 \sum_{\tau=1}^{t} (\psi_{\text{max}})^2 \Delta_0 \tag{75} \]

Therefore, \( \xi_q^g \) can be upper bounded by (38).

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