Self-Similarity between 3-dimensional Magnetostatics and 4-dimensional Electrodynamics

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Abstract

It is shown that Euclidean electrodynamics is the exact 4-dimensional analogue of 3-dimensional magnetostatics. This concept is related to a 4-dimensional generalization of the cross product between two vectors where the only essential modification is made to the tensor rank of the involved arguments. Parallels to the Schläfi-Coxeter theory of Platonic polytopes are pointed out.
1 Introduction

In a recent paper by Silagadze [6], an alternative derivation of the Maxwell equations is given which mixes classical and quantum concepts. Furthermore, that author presents a 7-dimensional generalization of the cross product between two vectors where the structure constants are by part admitted to be zero, even in case of all three indices being different from each other.

It is well-known that velocity and acceleration are the first and the second derivative of the coordinate vector relative to time and that these relations are remaining true at the level of the Euclidean representation of special relativity if the coordinate vector is replaced by the event vector and time is replaced by proper time. In this paper, we will show that exactly this receipt also transforms magnetostatics into electrodynamics, without further assumptions to be performed. This procedure is associated with a 4-dimensional (formula language) translation of the 3-dimensional cross product between two vectors which makes a compromise on the tensor rank of the linked variables, but not on the Levi-Civita character of its structure constants.

This paper is organized as follows. In Sec. 2, we are preparing the mathematical foundations for the reinterpretation of special relativity in the context of self-similarity, to be analysed in Sec. 3. The problematic around the energy-momentum tensor is discussed in Sec. 4. The final section is devoted to interdisciplinary perspectives.

2 Exterior Derivatives and the Cross Product

Let $\mathbf{b} = \mathbf{b}(\mathbf{x})$ denote an arbitrary 3-dimensional real coordinate-dependent vector field. The tensorial translation (in the sense of formula language) of its exterior derivative

$$d(\mathbf{b} \cdot d\mathbf{x}) = \text{curl} \mathbf{b} \cdot d\mathbf{A}$$

(with $\text{curl} \mathbf{b} = \varepsilon^{ijk} \hat{e}_i \partial_j b_k$ (using sum convention in three dimensions, the Euclidean Levi-Civita tensor $\varepsilon^{ijk} = \varepsilon^{ijk}$, and canonical base vectors $\hat{e}_i$ with $\hat{e}_i \cdot \mathbf{b} = b_i \forall i$) and the vectored areal element $d\mathbf{A} = \varepsilon^{mnr} \hat{e}_m dx_n \wedge dx_r$ (turning to the normal of the differential of area—and with calligraphic $\mathbf{A}$, in order to distinguish it from alphabetically similar figurative symbols which will be worked with below, like "$\mathbf{A}$" and "$A$") is known as $d(\mathbf{b} \cdot dx^i) = \frac{1}{2} (\text{CURL} b)_{ij} dx^j \wedge dx^k$, with the components of the tensorial CURL (we are using the ad-hoc notation "CURL" rather than "Curl" in order to facilitate the optical distinction between "curl" and "Curl") given by $(\text{CURL} b)_{ij} = \partial_i b_j - \partial_j b_i$. Indeed the CURL is not only feasible in three dimensions, but can also be comfortably generalized to four dimensions (cf. def. (26) below), belonging to the most genuine second differential form there, as the curl is in three dimensions.

Therefore the general understanding up to now is that the 4-dimensional CURL is the 4-dimensional analogue of the 3-dimensional curl. The intention
of this paper is to demonstrate that this interpretation guides into an impasse in that way that it conceals the aspects presented below.

The counter-suggestion is to assign the label "curl" to the third differential form in four dimensions. Let \( C_{\alpha\beta} \) with \( C_{\alpha\beta} = -C_{\beta\alpha} \) be a totally antisymmetric tensor of rank 2 in real 4-space. Then it has the same tensor transformation properties (just the given antisymmetry) as any result of a CURL in real 4-space. The third differential form in four dimensions can now be written as

\[
d(C_{\alpha\beta} \, dx^\alpha \wedge dx^\beta) = (\text{curl} \, C)_\gamma \, dA^\gamma
\]

with \( \text{curl} \, C := \varepsilon^{\alpha\beta\gamma\delta} \, \partial_\alpha \, C_{\beta\gamma} \) and the element of 3-dimensional hypersurface "area" in 4-space \( dA := \varepsilon^{\mu\nu\rho\sigma} \, dx_\mu \wedge dx_\nu \wedge dx_\rho \wedge dx_\sigma \), where again \( \varepsilon_{\alpha\beta\gamma\delta} = \varepsilon^{\alpha\beta\gamma\delta} \) is the Euclidean representation of the Levi-Civita tensor in four dimensions (it is rather trivial to hint at the existence of a corresponding integral version according to the general integral theorem of Stokes

\[
\int_M \, d\omega = \oint_{\partial M} \, \omega
\]

where \( \omega \) is any \((p-1)\) form being continuously differentiable in an open superset of any compact orientated smooth \( p \)-dimensional submanifold \( M \) in \( D \)-dimensional flat space, with boundary \( \partial M \) in induced orientation–just identify \( \omega = C_{\alpha\beta} \, dx^\alpha \wedge dx^\beta \).

If we demand that

\[
\text{curl} \, C = \partial \times C
\]

in four dimensions, analogically to

\[
\text{curl} \, b = \partial \times b
\]

in three dimensions, then the 4-dimensional cross product acts on a 4-dimensional vector \( b \) and an antisymmetric 4-dimensional dyad \( C \) according to

\[
b \times C := \varepsilon^{\alpha\beta\gamma\delta} \, \partial_\alpha \, b_\beta \, C_{\gamma\delta} ,
\]

also adhering to the 4-dimensional Euclidean version of the Levi-Civita tensor \( \varepsilon_{\alpha\beta\gamma\delta} = \varepsilon^{\alpha\beta\gamma\delta} \). This definition is self-consistent in every respect and will be motivated physically in the next section (including the geometric concept of orthogonality, which remains applicable in 4-space–it will be enlightened in the discussion of the 4-dimensional reinterpretation of chain of eq. (18)).

For this purpose, we finally need further auxiliary definitions for the ensuing differential operators, acting on vectors \( b \) and antisymmetric dyads \( C \) in real 4-space: \( d(b_\alpha \, dA^\alpha) = \text{div} \, b \, dV \) with

\[
\text{div} \, b := \partial_\beta \, b^\beta
\]

and Euclidean 4-dimensional \( dV = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \),

\[
\text{div} \, C := \partial_\alpha \, C^{\alpha\beta} \, \hat{e}_\beta ,
\]

as well as (the following definition is motivated by the insertion of a unit dyad w.r.t. (3) and (5), supplying one index for the argument of the 4-dimensional
cross product and the other index as supplement to the result of that cross product, where $1/2$ is a compensating normalization factor)

$$ (\text{curl } b)_{\alpha\beta} := \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} \partial^\gamma b^\delta $$  

(8)

so that the 4-dimensional curl converts an antisymmetric dyad into a vector, and vice versa. Please realize that the 4-dimensional Levi-Civita tensor behaves anti-cyclic relative to an exchange of its indices, unlike its 3-dimensional counterpart. All these relations can and will be taken over for the Euclidean representation of special relativistic spacetime, where both the metrics and its inverse are equivalent to a Kronecker Delta tensor, but the fourth component of any physical vector exhibits to be purely imaginary.

### 3 A Novel Access to Special Relativity

In order to facilitate the comparison between 3-dimensional space and 4-dimensional Euclidean spacetime, we will reserve ourselves to drop the tensor specification—or the vector bold type, respectively—when regarding 3-dimensional objects, from now on as already done in case of 4-space objects, where a corresponding convention exists anyway. This means that we want by intention to leave any tensor of any rank ambiguous with respect to its relevant dimensionality ($D = 3$ for space and $D = 4$ for Euclidean spacetime) if and only if its contingent indices are omitted. This concept will always leave undecided whether a scalar has to be treated as an invariant of classical mechanics (or other nonrelativistic physics, else) or as an invariant of special relativity, which can indeed differ by a velocity-dependent factor.

We start (here as well as later on, the anticipated superscript $^t$ designates a transposition, which turns a row vector into a column vector, e.g.) with the coordinate vector, that then changes from

$$ x = {}^t(x_1, x_2, x_3) $$  

(9)

to

$$ x = {}^t(x_1, x_2, x_3, x_4), \quad x_4 = i c t $$  

(10)

if space ($D = 3$) is replaced by Euclidean spacetime ($D = 4$). In both cases, we want to define a generalized length $L(x)$ by the main value of the square root

$$ L(x) := \sqrt{\bar{x} \cdot x} \, |\text{main value}| $$  

(11)

which is selected by fixing the complex phase angle $\arg(L(x))$ within the interval $]-\pi, \pi]$. It is obvious that $L(x)$ is the real geometric length in the event of $D = 3$ (referring to space), and something like a pseudo-length in the event of $D = 4$ (referring to Euclidean spacetime: the imaginary unit $i$ in (10) shall be taken seriously in the framework of this topic), which...
is nevertheless conserved strictly w.r.t. all transformations of the system of reference there. In both cases ( \( D = 3 \) and \( D = 4 \) ), the associated (actual or generalized, respectively) scalar product is bilinear (in formal excess of sesquilinearity for \( D = 4 \)) and equivalent to the corresponding clear-cut matrix product: \( x \cdot y = t^i x_i y_i = t^i y_i x_i \). Note that \( L(x) \) can become purely imaginary for \( D = 4 \).

Now we are able to specify time in both scenarios ( \( D = 3 \) and \( D = 4 \)). The time \( T \) alters from normal time
\[
T = t
\] (12)
for \( D = 3 \) to proper time
\[
T = -i \int L(dx) \frac{\text{sgn}(dt)}{c}
\] (13)
for \( D = 4 \), with \( L(dx) \) relating to \( D = 4 \) of course (but \( dt \) still refers to normal time \( t \)).

The formulae (9) up to (13) are supporting the specifications of velocity
\[
V = \frac{dx}{dT}
\] (14)
and acceleration
\[
A = \frac{d^2x}{dT^2}
\] (15)
(italic non-bold typewriter face for distinction from "\( A \)", which will be defined later) simultaneously for both scenarios of reference ( \( D = 3 \) and \( D = 4 \)).

We designate this behaviour as non-Mandelbrot self-similarity and we will use the underlying principle to rediscover special relativity by the reinterpretation of the most fundamental laws of classical mechanics and magnetostatics, employing (9) up to (13).

In a first tentative step, we try to extrapolate the definition of momentum
\[
P = m \frac{dx}{dT}
\] (16)
and the (simplest mechanical one-body description for the) equations of motion for force
\[
F = m \frac{d^2x}{dT^2}
\] (17)
(italic non-bold typewriter face for distinction from "\( F_{\mu \nu} \)", which will be defined later) to special relativity, taking advantage of the pre-results in (14) and (15). If we interpret \( m \) as normal mass for \( D = 3 \) and as the relativistically (i.e.: w.r.t. special relativity) invariant rest mass for \( D = 4 \) and if we demand that \( P \) and \( F \) have to be 4-momentum and 4-force in Euclidean special relativity, then in fact (16) and (17) remain true for \( D = 4 \)!
In electrostatics, the electric field is constant relative to time (in the scope of the probe time interval of reference) and no (macroscopically relevant) magnetic field does appear. We formally define magnetostatics by the complementary situation that the magnetic field is static and that there is effectively no non-zero electric field present in a given system. Imagine the situation inside a Faraday cage surrounding circuits of constant direct current (coils are admitted—we are thinking of the idealized classical picture of a co-moving probe charge $q$ with mass $m$), or discard the entity of the Lorentz force (skipping (18) and (19), v.i.) and just refer to the most superficial description of magnetism between permanent magnets, for example.

Regarding this magnetostatic type of situation, we can argue that

$$\frac{d}{dT} L^2(V) = \frac{d}{dT} V^2 = 2 \cdot V \cdot A = \frac{2}{m} V \cdot F = 2 \frac{q}{m} V \cdot (V \times B) = 0 \quad ,$$

where (11), (14), (15), and (17) are applied for $D = 3$, specifying $F$ to be the magnetostatic (static relative to the magnetic field $B$, not relative to $x$ : $V$ does not vanish in general) Lorentz force (in système international d’unités, SI, or with $c = 1$)

$$F = q V \times B .$$

(18) and (19) (within the scope of (18)) are telling us that

$$L(V) = \text{const}$$

so that $V$ can only change its direction, which can be handled as a process of pure revolving of $V$ itself, with $\frac{q}{m} B$ being the angular velocity pseudotensor of the associated rotation of $V$. The rank of that angular velocity pseudotensor $\frac{q}{m} B$ corresponds to the number of simultaneously possible linearly independent momentary planes of rotation. It is clear that this rank is one for $D = 3$, confirming that the magnetic field is a pseudovector. On the other hand, the $D = 4$ analogue for $B$ has to be an angular velocity pseudotensor of the second rank for the same reason, being realized by an antisymmetric dyad.

From (11) and (14), we can deduce that $L(V) = i c$ for $D = 4$, with $i$ being the imaginary unit and $c$ being the velocity of light (which will be set equal to unity subsequently) because (10) and (13) are supplying contributions to (14) which are not totally independent of each other. This implies that (20) is always fulfilled for $D = 4$, referring to full electrodynamics of special relativity (indeed, this aspect is relevant for the whole domain of special relativity, including relativistic mechanics as well—for the ensuing illustration, we will, however, focus upon the picture of electrodynamics)!

We would hence expect that there is an angular velocity representation for acceleration or (the comparison is rendered possible by (17)) force like (19) for $D = 4$, concerning the 4-dimensional interpretation of the cross product given in (5). This is in fact possible if we choose $F$ to be the 4-vector...
formulation of the complete electrodynamic Lorentz force, \( q \) to be the relativistically (i.e.: w.r.t. special relativity) invariant rest charge, \( V \) to be given by (14), and \( B \) to be the antisymmetric dyad with the components

\[
B_{\mu\nu} = \frac{1}{2} \tilde{F}_{\mu\nu} = \frac{1}{4} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma},
\]

(21)

where \( F_{\mu\nu} \) and \( \tilde{F}_{\mu\nu} \) are the actual and the dual field strength tensor in Euclidean special relativity, respectively. Because of (5), all steps taken in (18) are valid for \( D = 4 \) as well, telling us that \( V \) is perpendicular to \((V \times B)\), independent of the dimensionality of reference (\( D = 3 \) or \( D = 4 \)).

In order to check the self-consistency of the preceding considerations, we can investigate what the 4-dimensional (formula language) translation of the magnetostatic differential relations

\[
\text{div} \, B = 0 \quad (22)
\]

and (the field equations for a field representation relative to the classical vacuum, using CGS units with \( c = 1 \) and that redefinition of the current density which gets rid of the insignificant conversion factor (4\(\pi\)) )

\[
\text{curl} \, B = j \quad (23)
\]

are resulting in if we take \( j \) to be the special relativistic 4-current density (in corresponding normalization) and if we evaluate the auxiliary relationships (2), (3), (5), (7), (10), and (21). This means that we continue applying the principle of self-similarity. The resulting \( D = 4 \) interpretations of the equations (22) and (23) can be retranslated into relationships between the common physical quantities (tensors of rank 0 and 1 since objects like a stress tensor are not relevant here) of the nonrelativistic description of electrodynamics, referring to time and 3-dimensional position space. In this manner, we obtain the full homogeneous Maxwell equations from (22) and the full inhomogeneous Maxwell equations from (23)! According to the fact that the repeated application of an exterior derivative is zero in flat space, we can automatically derive the continuity equation

\[
\text{div} \, j = \text{div} \, \text{curl} \, B \equiv 0 , \quad (24)
\]

both for nonrelativistic magnetostatics (\( D = 3 \)) and for relativistic electrodynamics (\( D = 4 \)).

The integral versions \( \oint B \cdot dA = 0 \) and \( \oint B \cdot ds = \int j \cdot dA \) of (22) and (23) are valid for \( D = 3 \) and \( D = 4 \) if we apply the general version of Stokes’ theorem as mentioned in Sec. 2 (using the 4-definition of \( dA \) given there, \( B \cdot ds := B_{a\beta} dx^a \wedge dx^\beta \), and \( B \cdot dA := B_{a\beta} \hat{e}^a dA^\beta \) for \( D = 4 \)). The adaptation of the general integral theorem of Stokes to Euclidean special relativity with (10) will not provide anything useful, but is exempt from problems. Don’t confound it with its notorious special formulation for 2-dimensional
submanifolds of 3-dimensional position space in nonrelativistic physics, to be applied for (23) in the event of \( D = 3 \).

The lemma of Poincaré predicts that (22) is implying a potential representation

\[
B = \text{curl} \ A
\]

for \( D = 3 \) and sufficiently mathematically benign \( B \). Although we can immediately convince ourselves that (25) remains valid if \( A \) is reinterpreted as the standard 4-potential in Euclidean special relativity and \( B \) and \( \text{curl} \) are viewed in the context of (21) and (8), the way of arguing has to be modified since the genuine \( \text{curl} \) in four dimensions is given by (2), and not by (8).

We can improve the situation by explicitly introducing the second differential form

\[
d(b_\alpha \, dx^\alpha) = (\text{chi} \, b)_{\mu\nu} \, dx^\mu \wedge dx^\nu \tag{26}
\]

with \( (\text{chi} \, b)_{\mu\nu} := \frac{1}{2} (\partial_\mu \, b_\nu - \partial_\nu \, b_\mu) \) in 4-space (assigning the less misapprehensive expression “chiasm” to half the tensorial CURL in four dimensions) and the 4-dimensional extension

\[
E_{\mu\nu} = \frac{i}{2} \, F_{\mu\nu} \tag{27}
\]

of the electric field \( E_m \), where \( i \) is the imaginary unit and \( F_{\mu\nu} \) is again the field strength tensor in Euclidean special relativity. Then (2), (21), (22), and (27) are yielding \( \text{curl} \, E = 0 \), which is implying \( E = \text{chi} \, (i \, A) \) according to the lemma of Poincaré (again, \( i \) is the imaginary unit) and can be converted into (25), using (8), (21), (26), and (27). We can conclude that (22) is implying (25) for \( D = 4 \) if both the lemma of Poincaré and the duality of \( E \) and \( B \) are utilized. For \( D = 3 \) on the other hand, \( E \) and \( B \) are not dual relative to each other, and thus merely the lemma of Poincaré is required for deriving (25).

It is remarkable that the demonstrated analogy between magnetostatics in three dimensions and Euclidean electrodynamics in four dimensions does fit up to the tiniest detail. Hence this non-Mandelbrot type of self-similarity concept is a powerful and quick tool for checking the normalization of conversion factors in Minkowski spacetime since a remodeling of metrics is in most cases less sophisticated than infinitesimal calculations in relativistic mechanics, as well as in electrodynamics.

4 Visiting General Relativity

The rôle of the symmetric energy-momentum tensor in electrodynamics is far less clear. Let us denote it by \( \Theta^{(2)}_{\mu\nu} \), for reasons explained below. If we use the 4-dimensional Euclidean electric and magnetic fields defined in (21) and (27) we can write

\[
\Theta^{(2)}_{\mu\nu} = - (E_{\mu\rho} \, E_{\sigma\nu} + B_{\mu\rho} \, B_{\sigma\nu}) \, g^{\rho\sigma} \sqrt{g}, \tag{28}
\]
where $g$ is the determinant of the metric tensor $g_{\mu\nu}$, $\sqrt{g}$ is used for sakes of completeness (in the framework of a density-like definition for any energy-momentum tensor, like here) only since it is equal to unity in Euclidean special relativistic electrodynamics.

Electrodynamics is based on the unitary group $\text{U}(1)$, whose Lie algebra has one generator only, given by the scalar factor $2^{-0.5}$ if the same normalization standard is applied as in any unitary group or subgroup (i.e. any actual or special orthogonal or unitary group). By generalizing (28) to any Yang-Mills theory, we obtain an equation which can be formally expressed in terms of

$$G^{(k)}_{\mu\nu} = \left( \delta^\alpha_\mu g^{\beta\gamma} \delta^\delta_\nu + g^{\alpha\gamma} g^{\beta\delta} \right) \frac{g}{2k} \Omega^{(k)}_{\alpha\beta\gamma\delta}$$  \hspace{1cm} (29)$$

with

$$\Omega^{(k)}_{\alpha\beta\gamma\delta} = - \Omega^{(k)}_{\beta\alpha\gamma\delta} = - \Omega^{(k)}_{\alpha\beta\delta\gamma} = \Omega^{(k)}_{\gamma\delta\alpha\beta}$$ \hspace{1cm} (30)$$

using $k = 2$ and

$$\Omega^{(2)}_{\alpha\beta\gamma\delta} = \frac{1}{2} \text{trace} \left( F_{\alpha\beta} F_{\gamma\delta} \right).$$ \hspace{1cm} (31)$$

The formulae (29) and (30) are structurally reminiscent of the Einstein field equations in general relativity. These are given by (29) for $k = 1$ if

$$G^{(1)}_{\alpha\beta\gamma\delta} = \frac{1}{\kappa} R_{\alpha\beta\gamma\delta},$$ \hspace{1cm} (32)$$

using the Einstein gravitational constant $\kappa$, the Riemann tensor $R_{\alpha\beta\gamma\delta}$, and identifying $G^{(1)}_{\mu\nu}$ with the standard energy-momentum tensor of general relativity in $(i\,c\,t)$-Euclidean formulation (spacetime is curved, but the fourth component of any physical rank one tensor is chosen to be purely imaginary). Furthermore, (30) is fulfilled for $k = 1$ w.r.t. (32), too.

It is possible to introduce an SO(4) field strength tensor in classical $(i\,c\,t)$-Euclidean general relativity via

$$F_{\mu\nu} = \frac{1}{i g} \left[ D_\mu, D_\nu \right]_-, \text{ with } D_\mu = \partial_\mu + i g A_\mu,$$ \hspace{1cm} (33)$$

like in $\text{U}(1)$ or $\text{SU}(N)$ if the true physical coupling $g$ (this is not the same "$g" as in (28) or (29)–the charge $q$ of an electron is equal to $g$ in case of $\text{U}(1)$ electrodynamics, corresponding to the square root of the fine structure constant there) is replaced by the artificially fixed pseudo-coupling

$$g \equiv \text{const} = -2$$ \hspace{1cm} (34)$$

and

$$A_\mu = \frac{i}{4} V^{a\alpha} V^{b\beta}_{\alpha;\mu} \left( \hat{e}_a \cdot \hat{e}_b - \hat{e}_b \cdot \hat{e}_a \right),$$ \hspace{1cm} (35)$$

where $i$ is the imaginary unit, $\hat{e}_a$ are the basis vectors of the locally associated Frenetian vierbein [2–5], $V^a$ (having no reference to "$V^a", as it is given
by (14) is an indexed representation of the vierbein transformation matrices w.r.t. Cartan base vectors $\hat{e}_\alpha$, and the semicolon marks the covariant derivative referring to the following adjacent index in the sense of classical general relativity. By sticking to (33) up to (35), we can extend the definition (31) to be relevant for general relativity, with (30) remaining true for $k = 2$ in this situation. Specially (no pun, allusion, or reference to "special relativity") for general relativity then

$$\Omega^{(2)}_{\alpha\beta\gamma\delta} = \frac{\kappa^2}{2} \Omega^{(1)}_{\alpha\beta\mu\nu} g^{\mu\rho} g^{\nu\sigma} \Omega^{(1)}_{\rho\sigma\gamma\delta}, \quad (36)$$

demonstrating a quasi-quadratic relationship.

(29), (30), and (36) are illuminating the closest structural coherencies between the self-similarity scheme of Sec. 3 and general relativity at the level of classical (macroscopic) physics. One should be aware of the circumstance that (29) is specifying the physical ingredients of the regarded energy-momentum tensor for $k = 2$, while it is rendering equations of motion when $k = 1$.

5 Concluding Remarks

The discussion of non-Mandelbrot self-similarity in Sec. 3 is engendering the impression that several of the most fundamental laws of classical mechanics and magnetostatics, regarding our normal 3-dimensional space, are repeated once more at the higher level of Euclidean special relativity if the product of the imaginary unit, the velocity of light, and our normal time is added as a fourth dimension. Therefore the laws of nonrelativistic mechanics and magnetostatics can be considered as 3-dimensional copies of perfectly analogous hyper-laws in 4-dimensional space. Apparently, nature is allocating resembling rôles to 3-dimensional and 4-dimensional space.

It is striking to diagnose a similar phenomenon in aesthetics. The $D$-dimensional generalization of a polyhedron is called a polytope. A $D$-dimensional Platonic (introducing the required generalization for this conception just hereby) polytope is homogeneously bounded by facets in the same style and size, being $(D-1)$-dimensional Platonic polytopes, if $D \geq 1$. We have to distinguish between regular Platonic polytopes which are convex and related constructions, being classified as Platonic star polytopes (by defining a regular star polytope to be irregular in the sense of a normal convex polytope, for reasons of clarity). For instance, both pentagon and pentagram are homogeneously surrounded by five lines of the same size each, with the line being the unique Platonic polytope in one dimension. It is intuitively evident that the pentagon is a 2-dimensional regular Platonic polytope and that the pentagram is a 2-dimensional Platonic star polytope.

The entirety of all Platonic polytopes has been predicted theoretically by Schläfli in the 19th century and visualized up to dimensionality five by Coxeter in the 20th century [1]. Let $N_s(D)$ be the number of all different
Platonic polytopes in $D$ dimensions including star polytopes, $N_p(D)$ be the corresponding number for regular Platonic polytopes exclusively, and let $D$ be a natural number in this context. According to the Schl"afli-Coxeter theory, we obtain $N_s(D) = D^2$ and $N_p(D) = D + 2$ both for $D = 3$ and for $D = 4$, while $N_s(D) = N_p(D)$ for all other dimensionalities $D$, with $N_p(0) = N_p(1) = 1$, $N_p(2) = +\infty$, and $N_p(D) = 3$ for all $D > 4$. Again, we observe a distinct similarity between 3-dimensional and 4-dimensional space.

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