Constant mean curvature hypersurfaces condensing along a submanifold

1 Introduction

Let $S$ be an oriented embedded (or possibly immersed) hypersurface in a Riemannian manifold $(M^{m+1}, g)$. The shape operator $A_S$ is the symmetric endomorphism of the tangent bundle of $S$ associated with the second fundamental form of $S$, $b_S$, by

$$b_S(X, Y) = g_S(A_S X, Y), \quad \forall X, Y \in TS; \quad \text{here} \quad g_S = g|_{TS}.$$ 

The eigenvalues $\kappa_i$ of the shape operator $A_S$ are the principal curvatures of the hypersurface $S$. The mean curvature of $S$ is defined to be the average of the principal curvatures of $S$, i.e.

$$H(S) := \frac{1}{m}(\kappa_1 + \ldots + \kappa_m).$$

Constant mean curvature hypersurfaces constitute a very important class of submanifolds in a compact Riemannian manifold $(M^{m+1}, g)$. In this paper we are interested in families of such submanifolds, with mean curvature varying from one member of the family to another, which ‘condense’ to a submanifold $K^k \subset M^{m+1}$ of codimension greater than 1. Under fairly reasonable geometric assumptions [9], the existence of such a family implies that $K$ is minimal. Two cases have been studied previously: Ye [11], [12] proved the existence of a local foliation by constant mean curvature hypersurfaces when $K$ is a point (which is required to be a nondegenerate critical point of the scalar curvature function); more recently, the second and third authors [9] proved existence of a partial foliation when $K$ is a nondegenerate geodesic. In this paper we extend the result and methods of [9] to handle the general case, when $K$ is an arbitrary nondegenerate minimal submanifold. No extra curvature hypotheses are required. In particular, this proves the existence of constant mean curvature hypersurfaces with nontrivial topology in any Riemannian manifold.

Let us describe our result in more detail. Let $K^k$ be a closed (possibly immersed) submanifold in $M^{m+1}$, $1 \leq k \leq m - 1$, and define the geodesic tube of radius $\rho$ about $K$ by

$$\bar{S}_\rho := \{ q \in M^{m+1} : \text{dist}_g(q, K) = \rho \}.$$ 

This is a smooth (immersed) hypersurface provided $\rho$ is smaller than the radius of curvature of $K$, and we henceforth always tacitly assume that this is the case. The mean curvature of this tube satisfies

$$H(\bar{S}_\rho) = \frac{n-1}{m} \rho^{-1} + O(1), \quad \text{as} \quad \rho \searrow 0,$$

with $n = m + 1 - k$ and hence it is plausible that we might be able to perturb this tube to a constant mean curvature hypersurface with $H = \frac{n-1}{m} \rho^{-1}$. This is not quite true since the mean

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curvature of $\tilde{S}_\rho$ is not sufficiently close to being constant, but when $K$ is minimal there is a better estimate

$$H(\tilde{S}_\rho) = \frac{n-1}{m} \rho^{-1} + O(\rho),$$

cf. §4. Even in this case, there are other more subtle obstructions to carrying out this procedure at certain radii $\rho$ related to eigenvalues of the linearized mean curvature operator on $\tilde{S}_\rho$, which in turn are related to a genuine bifurcation phenomenon, at least when $k = 1$. [9]. Thus we do not obtain existence of the constant mean curvature perturbation for every small radius.

**Theorem 1.1** Suppose that $K^k$ is a nondegenerate closed minimal submanifold $1 \leq k \leq m - 1$. Then there exists a sequence of disjoint nonempty intervals $I_\iota = (\rho^-_\iota, \rho^+_{\iota})$, $\rho^+_{\iota} \to 0$, such that for all $\rho \in I :\cup I_\iota$, the geodesic tube $\tilde{S}_\rho$ may be perturbed to a constant mean curvature hypersurface $S_\rho$ with $H = \frac{n-1}{m} \rho^{-1}$.

The nondegeneracy condition on $K$ is simply that the linearized mean curvature operator, also called the Jacobi operator, is invertible; this restriction is quite mild and holds generically [14]. As noted above, this result was already known when $k = 0, 1$, but the case $k > 1$ requires a more complicated analysis. This new approach is inspired by some recent work of Malchiodi and Montenegro in a somewhat different context [5], [9].

The hypersurface $S_\rho$ is a small perturbation of $\tilde{S}_\rho$ in the sense that it is the normal graph of some function (with $L^\infty$ norm bounded by a constant times $\rho^3$) over a submanifold obtained by ‘translating’ $K$ by a section of its normal bundle (with $L^\infty$ norm bounded by a constant times $\rho^2$); we refer to §3.1 for the precise formulation of the construction of $S_\rho$. When $K$ is embedded, then so are the hypersurfaces $S_\rho$ for $\rho$ sufficiently small. In addition, the hypersurfaces in each of the families $\{S_\rho\}_{\rho \in I_\iota}$ are leaves of a local foliation of some annular neighborhood of $K$.

That the construction fails for certain values of $\rho$ is related to a bifurcation phenomenon. When $k = 1$ the families of surfaces which bifurcate off are (perturbations of) Delaunay unduloids [5]; however, when $k \geq 2$, this bifurcation is only known to exist in special cases, and the geometry of the surfaces in the putative bifurcating branches is less clear. In any case, such bifurcations are inherent to the problem and occur also in [17] and in many other situations. Furthermore, the index of the hypersurfaces $S_\rho$, $\rho \in I_\iota$, tends to $+\infty$ as $i \to \infty$. On the other hand, we prove that the set $I = \cup_\iota I_\iota$ is quite dense near 0 in the sense that for any $q \geq 2$ there exists a $c_q > 0$ such that

$$|\mathcal{H}^1((0, \rho) \cap I) - \rho| \leq c_q \rho^q,$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure.

One way to describe the behavior of $S_\rho$ as $\rho$ tends to 0 is to consider the associated area and curvature densities of $S_\rho$ as $\rho$ tends to 0; these quantities, properly rescaled, are extremely close to the corresponding quantities for $\tilde{S}_\rho$, which in turn satisfy

$$\rho^{k-m} \mathcal{H}^m |\tilde{S}_\rho| \to (m-k)^{q/2} \omega_{m-k} K_k \cup K$$

and, for all $q \geq 1$,

$$\rho^{k-m+q} |A_{\tilde{S}_\rho}|^q \mathcal{H}^m |\tilde{S}_\rho| \to (m-k)^{q/2} \omega_{m-k} K_k \cup K$$

as $\rho \searrow 0$. Here $|A_S|^2 := \text{Tr}((A_S)\dagger A_S)$ is the norm squared of the shape operator. From the explicit estimates in the construction of $S_\rho$ one can deduce that (1.1) and (1.2) also hold when $\tilde{S}_\rho$ is replaced by $S_\rho$.

One can ask whether (1.1) and (1.2) hold for any family of constant mean curvature hypersurfaces which condense along $K$. It turns out that this is not the case: families of CMC hypersurfaces condensing along a nondegenerate geodesic which do not satisfy (1.1) are constructed in [5]. In another direction, it is plausible that one should be able to construct families of CMC hypersurfaces which condense along lower dimensional sets which are still minimal in an appropriate sense, but with singularities, for example a Steiner tree with geodesic edges. A simple example of this is when $\tilde{S}_\rho$ is obtained by homothetically rescaling a fixed Delaunay trinoid in
\[ R^3. \] The limit then is a union of three rays meeting at a common vertex, each ray having an associated density coming from the limiting Delaunay necksize on that end; each ray is minimal, of course, and the entire configuration is ‘balanced’ in the sense that the weighted sum of the vectors along the rays vanishes.

Keeping these various phenomena in mind, it is not clear whether our main result has a suitable converse, or whether it is possible to characterize the possible condensation sets of such families of CMC hypersurfaces. As a weak and tentative step in this direction we make the

**Conjecture:** Let \( S_j \) be a family of constant mean curvature hypersurfaces with mean curvature \( H_j \not\to \infty \); then for \( j \) sufficiently large, \( S_j \) is homologically trivial.

The intuition here is simply that if the \( S_j \) were indeed condensing on a lower dimensional (possibly singular) manifold \( K \), then \( S_j \) should bound a ‘tubular neighbourhood’ \( \bar{A}_j \) of \( K \). In any case, this circle of ideas merits further study.

In the next section we calculate the asymptotic expansion of the metric on \( M \) in Fermi coordinates around \( K \); this is applied in the (quite technical) \( \S 3 \) to derive the expansions of various geometric quantities for the tubes \( \bar{S}_\rho \) and their perturbations. This is used in \( \S 4 \) to obtain the expression for the mean curvature of the perturbed tubes, which gives us the equation which must be solved. An iteration scheme is introduced in \( \S 5 \) which allows us to find a preliminary perturbation for which the error term is much better, and estimates for the gaps in the spectrum of the linearization are obtained in \( \S 6 \); finally, the existence of the constant mean curvature hypersurfaces \( S_\rho \) is obtained in \( \S 7 \).

### 2 Expansion of the metric in Fermi coordinates near \( K \)

#### 2.1 Fermi coordinates

We now introduce Fermi coordinates in a neighborhood of \( K \). For a given \( p \in K \), there is a natural splitting

\[ T_pM = T_pK \oplus N_pK. \]

Choose orthonormal bases \( E_a, a = n + 1, \ldots, m + 1 \), for \( T_pK \), and \( E_i, i = 1, \ldots, n \), of \( N_pK \).

**Notation:** We shall always use the convention that indices \( a, b, c, \ldots \in \{ n + 1, \ldots, m + 1 \} \), indices \( i, j, k, \ldots \in \{ 1, \ldots, n \} \) and indices \( \alpha, \beta, \gamma, \ldots \in \{ 1, \ldots, m + 1 \} \).

Consider, in a neighborhood of \( p \) in \( K \), normal geodesic coordinates

\[ f(y) := \exp^K_p(y^a E_a), \quad y := (y^{n+1}, \ldots, y^{m+1}), \]

where \( \exp^K \) is the exponential map on \( K \) and summation over repeated indices is understood. This yields the coordinate vector fields \( X_a := f_*(\partial_{y^a}) \). For any \( E \in T_pK \), the curve

\[ s \mapsto \gamma_E(s) := \exp^K_p(sE), \]

is a geodesic in \( K \), so that

\[ \nabla_{X_a} X_b|_p \in N_pK. \]

We define the numbers \( \Gamma_{ab}^i \) by

\[ \nabla_{X_a} X_b|_p = \Gamma_{ab}^i E_i. \]

Now extend the \( E_i \) along each \( \gamma_E(s) \) so that they are parallel with respect to the induced connection on the normal bundle \( NK \). This yields an orthonormal frame field \( X_i \) for \( NK \) in a neighborhood of \( p \) in \( K \) which satisfies

\[ \nabla_{X_a} X_i|_p \in T_pK, \]

and hence defines coefficients \( \Gamma_{ai}^b \) by

\[ \nabla_{X_a} X_i|_p = \Gamma_{ai}^b E_b. \]
A coordinate system in a neighborhood of \( p \) in \( M \) is now defined by

\[
F(x, y) := \exp^M_f(y)(x^i X_i), \quad (x, y) := (x^1, \ldots, x^n, y^{n+1}, \ldots, y^{m+1}),
\]

with corresponding coordinate vector fields

\[
X_i := F_*(\partial_{x^i}) \quad \text{and} \quad X_\alpha := F_*(\partial_{y^\alpha}).
\]

By construction, \( X_\alpha \mid_p = E_\alpha \).

### 2.2 Taylor expansion of the metric

As usual, the Fermi coordinates above are defined so that the metric coefficients

\[
g_{\alpha\beta} = g(X_\alpha, X_\beta)
\]

equal \( \delta_{\alpha\beta} \) at \( p \); furthermore, \( g(X_\alpha, X_i) = 0 \) in some neighborhood of \( p \) in \( K \). This implies that

\[
X_b g(X_\alpha, X_i) = g(\nabla_{X_\alpha} X_i, X_i) + g(X_\alpha, \nabla_{X_b} X_i) = 0
\]
on \( K \), which yields the identity

\[
\Gamma^b_{ai} = -\Gamma^i_{ab}
\]
at \( p \).

Denote by \( \Gamma^b_a : N_p K \to \mathbb{R} \) the linear form

\[
\Gamma^b_a(E_i) := \Gamma^b_{ai}
\]

We now compute higher terms in the Taylor expansions of the functions \( g_{\alpha\beta} \). The metric coefficients at \( q := F(x, 0) \) are given in terms of geometric data at \( p := F(0, 0) \) and \( |x| = \text{dist}_g(p, q) \).

**Notation** The symbol \( \mathcal{O}(|x|^r) \) indicates a function such that it and its partial derivatives of any order, with respect to the vector fields \( X_\alpha \) and \( x^i X_j \), are bounded by \( c|x|^r \) in some fixed neighborhood of \( 0 \).

We begin with the expansion of the covariant derivative :

**Lemma 2.1** At the point of \( q = F(x, 0) \), the following expansions hold

\[
\begin{align*}
\nabla_{X_\alpha} X_j &= \mathcal{O}(|x|) X_\gamma, \\
\nabla_{X_a} X_b &= -\Gamma^b_a(E_i) X_i + \mathcal{O}(|x|) X_\gamma, \\
\nabla_{X_a} X_i &= \nabla_{X_a} X_\alpha = \Gamma^b_a(E_i) X_b + \mathcal{O}(|x|) X_\gamma,
\end{align*}
\]

**Proof:** We have by construction

\[
\nabla_{X_a} X_b = \Gamma^i_{ab} X_i + \mathcal{O}(|x|) X_\gamma
\]

and

\[
\nabla_{X_a} X_j = \nabla_{X_a} X_\alpha = \Gamma^b_{aj} X_b + \mathcal{O}(|x|) X_\gamma.
\]

Observe that, because we are using coordinate vector fields, \( \nabla_{X_a} X_\beta = \nabla_{X_\beta} X_a \) for any \( \alpha, \beta \). We also have \( \nabla_{X_i} X_j \mid_p = 0 \) since any \( X \in N_p K \) is tangent to the geodesic \( s \mapsto \exp^M_p(sX) \), and hence

\[
\nabla_{X_i + X_j} (X_i + X_j) \mid_p = 0
\]

Therefore

\[
\nabla_{X_i} X_j + \nabla_{X_j} X_i \mid_p = 0
\]

This completes the proof of the result.

We now give the expansion of the metric coefficients. The expansion of the \( g_{ij}, i, j = 1, \ldots, n \), agrees with the well known expansion for the metric in normal coordinates \[10], \[13], \[15\], but we briefly recall the proof here for completeness.
Proposition 2.1 At the point \( q = F(x,0) \), the following expansions hold

\[
\begin{align*}
g_{ij} &= \delta_{ij} + \frac{1}{3} g(R(E_k,E_i)E_i,E_k) x^k x^\ell + \mathcal{O}(|x|^3) \\
g_{ai} &= \mathcal{O}(|x|^2) \\
g_{ab} &= \delta_{ab} + 2 \Gamma^b_a(E_i) x^i + (g(R(E_k,E_a)E_i,E_b) + \Gamma^c_a(E_k) \Gamma^b_c(E_i)) x^k x^\ell + \mathcal{O}(|x|^3).
\end{align*}
\]

(2.5)

Proof: By construction, \( g_{\alpha\beta} = \delta_{\alpha\beta} \) at \( p \), and so

\[
g_{\alpha\beta} = \delta_{\alpha\beta} + \mathcal{O}(|x|).
\]

Now, from

\[
X_i g_{\alpha\beta} = g(\nabla X_i X_\alpha, X_\beta) + g(X_\alpha, \nabla X_i X_\beta),
\]

Lemma 2.1 and (2.3), we get

This yields the first order Taylor expansion

\[
g_{aj} = \mathcal{O}(|x|^2), \quad g_{ij} = \delta_{ij} + \mathcal{O}(|x|^2) \quad \text{and} \quad g_{ab} = \delta_{ab} + 2 \Gamma^b_a x^i + \mathcal{O}(|x|^2).
\]

To compute the second order terms, it suffices to compute \( X_k X_k g_{\alpha\beta} \) at \( p \) and polarize (i.e. replace \( X_k \) by \( X_i + X_j \), etc.). We compute

\[
X_k X_k g_{\alpha\beta} = g(\nabla^2 X_k X_\alpha, X_\beta) + g(X_\alpha, \nabla X_k X_\beta) + 2 g(\nabla X_k X_\alpha, \nabla X_k X_\beta)
\]

(2.6)

To proceed, first observe that

\[
\nabla X X |_{p'} = \nabla^2 X |_{p'} = 0
\]

at \( p' \in K \), for any \( X \in N_p K \). Indeed, for all \( p' \in K \), \( X \in N_p K \) is tangent to the geodesic \( s \rightarrow \exp_{p'}(sX) \), and so \( \nabla X X = \nabla^2 X = 0 \) at the point \( p' \).

In particular, taking \( X = X_k + \varepsilon X_j \), we obtain

\[
0 = \nabla_{X_k + \varepsilon X_j}^{X_k + \varepsilon X_j} (X_k + \varepsilon X_j) |_{p'}
\]

equating the coefficient of \( \varepsilon \) to 0 gives \( \nabla X_k \nabla X_k X_k |_{p'} = -2 \nabla X_k \nabla X_k X_j |_{p'} \), and hence

\[
3 \nabla^2 X_k X_j |_{p'} = R(E_k,E_j) E_k,
\]

So finally, using (2.6) together with the result of Lemma 2.1 we get

\[
X_k X_k g_{ij} |_{p'} = \frac{2}{3} g(R(E_k,E_i) E_k, E_j).
\]

The formula for the second order Taylor coefficient for \( g_{ij} \) now follows at once.

Recall that, since \( X_\gamma \) are coordinate vector fields, we have from (2.4)

\[
\nabla^2 X_\gamma X_\gamma = \nabla X_\gamma \nabla X_\gamma X_\gamma = \nabla X_\gamma \nabla X_k X_k + R(X_k, X_\gamma) X_k,
\]

Using (2.4), this yields

\[
X_k X_k g_{ab} = 2 g(R(X_k, X_a) X_k, X_b) + 2 g(\nabla X_k X_a, \nabla X_k X_b)
\]

\[
+ 2 g(\nabla X_a \nabla X_k X_k, X_b) + g(X_a, \nabla X_b \nabla X_k X_k)
\]

Using the result of Lemma 2.1 together with the fact that \( \nabla X X = 0 |_{p'} \) at \( p' \in K \) for any \( X \in N_p K \), we conclude that

\[
X_k X_k g_{ab} |_{p'} = 2 g(R(E_k,E_a) E_k, E_b) + 2 \Gamma^c_{a k} \Gamma^b_{c k}
\]

and this gives the formula for the second order Taylor expansion for \( g_{ab} \).

Later on, we will need an expansion of some covariant derivatives which is more accurate than the one given in Lemma 2.1. These are given in the :
Lemma 2.2 At the point $q = F(x,0)$, the following expansion holds

$$\nabla_{X_a} X_b = -\Gamma^b_{a}(E_j) X_j - g(R(E_i, E_a) E_j, E_b) x^i X_j$$

$$+ \frac{1}{2} (g(R(E_a, E_b) E_i, E_j) - \Gamma^b_{a}(E_i) \Gamma^c_{b}(E_j) - \Gamma^c_{a}(E_j) \Gamma^b_{c}(E_i)) x^i X_j$$

$$+ \mathcal{O}(|x|) X_c + \mathcal{O}(|x|^2) X_j. \quad (2.7)$$

Proof: We compute

$$X_i g(\nabla_{X_a} X_b, X_j) = g(\nabla_{X_i} \nabla_{X_a} X_b, X_j) + g(\nabla_{X_a} X_b, \nabla_{X_i} X_j)$$

$$= g(R(X_i, X_a) X_b, X_j) + g(\nabla_{X_a} \nabla_{X_i} X_b, X_j) + g(\nabla_{X_a} X_b, \nabla_{X_i} X_j)$$

Observe that, by construction, we have arranged in such a way that

$$\nabla_{X_a + \varepsilon X_b} X_i = (\Gamma^c_{ai} + \varepsilon \Gamma^c_{bi}) X_c$$

along the geodesic $s \rightarrow \exp^K_p(s(E_a + \varepsilon E_b))$. Hence

$$\nabla^2_{X_a + \varepsilon X_b} X_i = ((X_a + \varepsilon X_b)(\Gamma^c_{ai} + \varepsilon \Gamma^c_{bi})) X_c + (\Gamma^c_{ai} + \varepsilon \Gamma^c_{bi}) \nabla_{X_a + \varepsilon X_b} X_c \quad (2.8)$$

Evaluating this at the point $p$ and looking for the coefficient of $\varepsilon$, we obtain

$$(\nabla_{X_a} \nabla_{X_b} X_i + \nabla_{X_b} \nabla_{X_a} X_i) |_p - (\Gamma^c_{ai} \nabla_{X_b} X_c + \Gamma^c_{bi} \nabla_{X_a} X_c) |_p \in T_p K$$

Hence we get

$$g(\nabla_{X_a} \nabla_{X_b} X_i, X_j) |_p + g(\nabla_{X_b} \nabla_{X_a} X_i, X_j) |_p = \Gamma^c_{ai} g(\nabla_{X_b} X_c, X_j) |_p$$

$$+ \Gamma^c_{bi} g(\nabla_{X_a} X_c, X_j) |_p$$

$$= \Gamma^c_{ai} \Gamma^j_{bc} + \Gamma^c_{bi} \Gamma^j_{ac}$$

Finally, we use the fact that

$$g(\nabla_{X_a} \nabla_{X_b} X_i, X_j) = g(R(X_b, X_a) X_i, X_j) + g(\nabla_{X_a} \nabla_{X_b} X_i, X_j)$$

to conclude that, at the point $p$

$$2 g(\nabla_{E_a} \nabla_{E_b} E_i, E_j) |_p = g(R(E_a, E_b) E_i, E_j) + \Gamma^c_{ai} \Gamma^j_{bc} + \Gamma^c_{bi} \Gamma^j_{ac}$$

Collecting these estimates together with the fact that $\nabla_{E_i} E_j |_p = 0$ we conclude that

$$2 X_i g(\nabla_{X_a} X_b, X_j) |_p = -2 g(R(E_i, E_a) E_j, E_b) + g(R(E_a, E_b) E_i, E_j) + \Gamma^c_{ai} \Gamma^j_{bc} + \Gamma^c_{bi} \Gamma^j_{ac}$$

This easily implies (2.7). \qed

3 Geometry of tubes

We derive expansions as $\rho$ tends to 0 for the metric, second fundamental form and mean curvature of the tubes $\tilde{S}_\rho$ and their perturbations. This is an extension of the computation in [9].

3.1 Perturbed tubes

We now describe a suitable class of deformations of the geodesic tubes $\tilde{S}_\rho$, depending on a section $\Phi$ of $NK$ and a scalar function $w$ on the spherical normal bundle $SNK$. 

Fix $\rho > 0$. It will be convenient to introduce the scaled variable $\bar{y} = y/\rho$; we also use a local parametrization $z \to \Theta(z)$ of $S^{n-1}$. Now define the map

$$G(z, \bar{y}) := F(\rho (1 + w(z, \bar{y}))) \Theta(z) + \Phi(\rho \bar{y}, \rho \bar{y}),$$

and denote its image by $S_\rho(w, \Phi)$, so in particular

$$S_\rho(0, 0) = \bar{S}_\rho.$$

**Notation:** Because of the definition of these hypersurfaces using the exponential map, various vector fields we shall use may be regarded either as fields along $K$ or along $S_\rho(w, \Phi)$. To help allay this confusion, we write

$$\Phi := \Phi^j E_j, \quad \Psi := \Phi^j X_j, \quad \Psi_a := \partial_{y^a} \Phi^j X_j,$$

$$\Theta := \Theta^j E_j, \quad \Upsilon := \Theta^j X_j, \quad \Upsilon_i := \partial_{z^i} \Theta^j X_j.$$

These are all vectors in the tangent space $T_pM$ at the fixed point $p \in K$. On the other hand, the vectors

$$w_j := \partial_{z^j} w, \quad w_a := \partial_{y^a} w, \quad w_{ij} := \partial_{z^i, z^j} w, \quad w_{ab} := \partial_{y^a, y^b} w, \quad w_{aj} := \partial_{y^a, z^j} w.$$

In terms of all this notation, the tangent space to $S_\rho(w, \Phi)$ at any point is spanned by the vectors

$$Z_a = G_s(\partial_{y^a}) = \rho (X_a + w_a \Upsilon + \Psi_a), \quad a = n + 1, \ldots, m + 1$$

$$Z_j = G_s(\partial_{z^j}) = \rho ((1 + w) \Upsilon_j + w_j \Upsilon), \quad j = 1, \ldots, n - 1. \tag{3.9}$$

### 3.2 Notation for error terms

The formulas for the various geometric quantities of $S_\rho(\Phi, w)$ are potentially very complicated, and so it is important to condense notation as much as possible. Fortunately, we do not need to know the full structure of all of these quantities. Because it is so fundamental, we have isolated the notational conventions we shall use in this separate subsection.

Any expression of the form $L(w, \Phi)$ denotes a linear combination of the functions $w$ together with its derivatives with respect to the vector fields $\rho X_a$ and $X_i$ up to order 2, and $\Phi^j$ together with their derivatives with respect to the vector fields $X_a$ up to order 2. The coefficients are assumed to be smooth functions on $SNK$ which are bounded by a constant independent of $\rho$ in the $C^\infty$ topology (i.e. derivatives taken with respect to $X_a$ and $X_i$).

Similarly, an expression of the form $Q(w, \Phi)$ denotes a nonlinear operator in the functions $w$ together with its derivatives with respect to the vector fields $\rho X_a$ and $X_i$ up to order 2, and $\Phi^j$ together with their derivatives with respect to the vector fields $X_a$ up to order 2. Again, the coefficients of the Taylor expansion of the corresponding differential operator are smooth on $SNK$, and $Q$ which vanishes quadratically at $(w, \Phi) = (0, 0)$.

Finally, any term denoted $O(\rho^d)$ is a smooth function on $SNK$ which is bounded in $C^\infty(SNK)$ by a constant times $\rho^d$. 

7
3.3 The first fundamental form

The next step is the computation of the coefficients of the first fundamental form of \( S_\rho(w, \Phi) \).

We set
\[
q := G(z, 0) = F(\rho + w(z, 0)) \Theta(z) + \Phi(\rho z, 0)
\]
and \( p := G(0, 0) \). We obtain directly from (3.8) that
\[
g(X_a, X_b) = \delta_{ab} + 2 \rho \Gamma^b_a(\Theta) + O(\rho^2) + 2 \Gamma^b_a(\Phi) + \rho L(w, \Phi) + Q(w, \Phi)
\]
\[
g(X_i, X_j) = \delta_{ij} + \frac{\rho^2}{3} g(R(\Theta, E_i) \Theta, E_j) + O(\rho^3)
\]
\[
+ \frac{\rho}{3} (g(R(\Theta, E_i) \Phi, E_j) + g(R(\Phi, E_i) \Theta, E_j)) + \rho^2 L(w, \Phi) + Q(w, \Phi) \tag{3.10}
\]

3.3 The first fundamental form

Our next task is to understand the dependence on \( (w, \Phi) \) of the unit normal \( N \) to \( S_\rho(w, \Phi) \).

Proposition 3.1 We have
\[
\rho^{-2} g(Z_a, Z_b) = \delta_{ab} + 2 \rho \Gamma^b_a(\Theta) + O(\rho^2) + 2 \Gamma^b_a(\Phi) + \rho L(w, \Phi) + Q(w, \Phi)
\]
\[
\rho^{-2} g(Z_a, Z_j) = O(\rho^2) + L(w, \Phi) + Q(w, \Phi)
\]
\[
\rho^{-2} g(Z_i, Z_j) = g(\Theta_i, \Theta_j) + \frac{\rho^2}{3} g(R(\Theta, \Theta_i) \Theta, \Theta_j) + O(\rho^3) + 2 g(\Theta_i, \Theta_j) w
\]
\[
+ \frac{\rho}{3} (g(R(\Theta, \Theta_i) \Phi, \Theta_j) + g(R(\Theta, \Theta_j) \Phi, \Theta_i)) + \rho^2 L(w, \Phi) + Q(w, \Phi). \tag{3.13}
\]

3.4 The normal vector field

Our next task is to understand the dependence on \( (w, \Phi) \) of the unit normal \( N \) to \( S_\rho(w, \Phi) \).

Proposition 3.2 This unit normal vector field \( S_\rho(w, \Phi) \) has the expansion
\[
N := - \Upsilon + \alpha^j \Upsilon_j + \beta^a X_a + (\rho L(w, \Phi) + Q(w, \Phi)) X_a
\]
\[
+ (\rho^2 L(w, \Phi) + Q(w, \Phi)) X_j \tag{3.14}
\]

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where the coefficients $\alpha_j$ are solutions of the system

$$\alpha_j g(\Theta_j, \Theta_i) = w_i + \frac{\rho}{3} g(R(\Phi, \Theta) \Theta, \Theta_i), \quad i = 1, \ldots, n - 1,$$

and the coefficients $\beta^a$ are given by

$$\beta^a = w_a + g(\Phi_a, \Theta)$$

**Proof:** Define the vector field

$$\tilde{N} := -\Upsilon + A^j Z_j + B^a Z_a,$$

and choose the coefficients $A^j$ and $B^a$ so that $\tilde{N}$ is orthogonal to all of the $Z_b$ and $Z_i$. This leads to a linear system for $A^j$ and $B^a$.

We have the following expansions

$$g(\Upsilon, Z_a) = \rho w_a + \rho g(\Phi_a, \Theta) + \rho^2 L(w, \Phi) + \rho Q(w, \Phi)$$

(3.15)

$$g(\Upsilon, Z_j) = \rho w_j + \frac{\rho^2}{3} g(R(\Phi, \Theta) \Theta, \Theta_i) + \rho^3 L(w, \Phi) + \rho Q(w, \Phi)$$

These follow from (3.10) together with the fact that $g(\Upsilon, Z_a) = 0$ and $g(\Upsilon, Z_j) = 0$ when $w = 0$ and $\Phi = 0$.

Using Proposition 3.1, we get with little work

$$B^a = w_a + g(\Theta, \Phi_a) + \rho L(w, \Phi) + \frac{1}{\rho} Q(w, \Phi).$$

and

$$A^j g(\Theta_j, \Theta_i) = \frac{1}{\rho} w_i + \frac{1}{3} g(R(\Phi, \Theta) \Theta, \Theta_i) + \rho L(w, \Phi) + \frac{1}{\rho} Q(w, \Phi).$$

Recall also that $Z_j = \rho \Upsilon_j + \rho L(w, \Phi)$ and also that $Z_a = \rho X_a + \rho L(w, \Phi)$. Collecting these, together with the fact that $|\tilde{N}|_q = 1 + \rho^2 L(w, \Phi) + Q(w, \Phi)$.

we obtain

$$N := -\Upsilon + \frac{1}{\rho} (\alpha^j Z_j + \beta^a Z_a) + \left( L(w, \Phi) + \frac{1}{\rho} Q(w, \Phi) \right) Z_a$$

(3.16)

$$+ \left( \rho L(w, \Phi) + \frac{1}{\rho} Q(w, \Phi) \right) Z_j + (\rho^2 L(w, \Phi) + Q(w, \Phi)) \Upsilon$$

The result then follows at once. □

### 3.5 The second fundamental form

We now compute the second fundamental form. To simplify the computations below, we henceforth assume that, at the point $\Theta(z) \in S^{n-1}$,

$$g(\Theta_i, \Theta_j) = \delta_{ij} \quad \text{and} \quad \nabla_{\Theta_i} \Theta_j = 0, \quad i, j = 1, \ldots, n - 1$$

(3.17)

(where $\nabla$ is the connection on $TS^{n-1}$).
Proposition 3.3 The following expansions hold

\[ \rho^{-2} g(N, \nabla Z_a Z_a) = \Gamma_a^b(\Theta) + \rho g(R(\Theta, E_a) \Theta, E_a) + \rho \Gamma_a^b(\Theta) \Gamma^b_c(\Theta) + \mathcal{O}(\rho^2) \]

\[ - \frac{1}{\rho} w_{\bar{a}a} - g(\Phi_{aa} + R(\Phi, E_a) \Theta, \Theta) + \Gamma_a^b(\Theta) \Gamma^b_c(\Phi) w_j \Gamma_a^j(\Theta) \]

\[ + \rho L(w, \Phi) + \frac{1}{\rho} Q(w, \Phi), \]

\[ \rho^{-2} g(N, \nabla Z_a Z_j) = \frac{1}{\rho} + \frac{2}{\rho} \rho g(R(\Theta, \Theta_j) \Theta, \Theta_j) + \mathcal{O}(\rho^2) \]

\[ - \frac{1}{\rho} w_{jj} + \frac{1}{\rho} w + \frac{2}{\rho} g(R(\Phi, \Theta_j) \Theta, \Theta_j) \]

\[ + \rho L(w, \Phi) + \frac{1}{\rho} Q(w, \Phi) \]

\[ \rho^{-2} g(N, \nabla Z_a Z_b) = \Gamma_b^b(\Theta) - \frac{1}{\rho} w_{\bar{a}b} + \mathcal{O}(\rho) + L(w, \Phi) + \frac{1}{\rho} Q(w, \Phi) \quad a \neq b \]

\[ \rho^{-2} g(N, \nabla Z_a Z_j) = \mathcal{O}(\rho) + \frac{1}{\rho} L(w, \Phi) + \frac{1}{\rho} Q(w, \Phi) \quad i \neq j. \]

**Proof**: Some preliminary computations are needed. First note that by Lemma 2.1 we have

\[ \nabla_{X_a} X_b = -\Gamma_a^b(E_i) X_i + (\mathcal{O}(\rho) + L(w, \Phi) + Q(w, \Phi)) X_\gamma, \]

\[ \nabla_{X_a} X_j = (\mathcal{O}(\rho) + L(w, \Phi) + Q(w, \Phi)) X_\gamma, \]

\[ \nabla_{X_a} X_i = \Gamma_a^b(E_i) X_b + (\mathcal{O}(\rho) + L(w, \Phi) + Q(w, \Phi)) X_\gamma, \]

In particular, this, together with the expression of \( Z_a \) implies that

\[ \nabla_{Z_a} X_i = \rho \Gamma_a^b(E_i) X_b + (\mathcal{O}(\rho^2) + \rho L(w, \Phi) + \rho Q(w, \Phi)) X_\gamma, \]

\[ \nabla_{Z_a} X_b = -\rho \Gamma_a^b(E_i) X_i + (\mathcal{O}(\rho^2) + \rho L(w, \Phi) + \rho Q(w, \Phi)) X_\gamma, \]

We will also need the following expansion which follows from the result of Lemma 2.2

\[ \nabla_{X_a} X_b = -\Gamma_a^b(E_j) X_j - g(R(\rho \Theta + \Phi, E_a) E_j, E_b) X_j \]

\[ + \frac{1}{2} \left( g(R(E_a, E_b) \rho \Theta + \Phi, E_j) - \Gamma_a^b(\rho \Theta + \Phi) \Gamma^b_c(E_j) - \Gamma_a^b(\rho \Theta + \Phi) \Gamma^b_c(E_j) \right) X_j \]

\[ + (\mathcal{O}(\rho) + L(w, \Phi) + Q(w, \Phi)) X_c + (\mathcal{O}(\rho^2) + \rho L(w, \Phi) + Q(w, \Phi)) X_j. \]

Finally, we will need the expansions

\[ g(Y, X_a) = \rho L(w, \Phi) + Q(w, \Phi) \]

\[ g(Y, Y_j) = \rho L(w, \Phi) + Q(w, \Phi) \]

whose proof can be obtained as in §3.2, starting from the estimates (3.10).

**First estimate**: We estimate \( g(N, \nabla Z_a Z_b) \) when \( a = b \) since the corresponding estimate, when \( a \neq b \) is not as important and follows from the same proof. We must expand

\[ \rho^{-2} g(N, \nabla Z_a Z_a) = \rho^{-1} \left( g(N, \nabla Z_a X_a) + g(N, \nabla Z_a (w_{\bar{a}a} Y)) + g(N, \nabla Z_a \Psi_a) \right) \]

The estimate is broken into three steps:

**Step 1** From (3.14), we get

\[ g(N, Y) = -g(T, T) + \alpha^j g(T_j, Y) + \beta^b g(X_b, Y) + (\rho L^1(w, \Phi) + Q^1(w, \Phi)) g(X_c, Y) \]

\[ + \rho^2 L(w, \Phi) + Q(w, \Phi) g(X_j, Y) \]

\[ = -1 + \rho^2 L(w, \Phi) + Q(w, \Phi) \]
Substituting \( N = -\Upsilon + N + \Upsilon \) gives

\[
g(N, \nabla_{Z_a} \Upsilon) = -\frac{1}{2} \partial_{g^*} g(\Upsilon, \Upsilon) + g(N + \Upsilon, \nabla_{Z_a} \Upsilon)
\]

But it follows from (3.12) that

\[
\partial_{g^*} g(\Upsilon, \Upsilon) = \rho^2 L(w, \Phi) + Q(w, \Phi),
\]

and (3.20) together with the expression of \( N \) implies that

\[
g(N + \Upsilon, \nabla_{Z_a} \Upsilon) = \rho L(w, \Phi) + \rho Q(w, \Phi).
\]

Collecting these estimates we get

\[
g(N, \nabla_{Z_a} \Upsilon) = \rho L(w, \Phi) + Q(w, \Phi).
\]

Hence we conclude that

\[
g(N, \nabla_{Z_a} (w_a \Upsilon)) = w_{aa} g(N, \Upsilon) + w_a g(N, \nabla_{Z_a} \Upsilon) = -w_{aa} + Q(w, \Phi)
\]

**Step 2** Next,

\[g(N, \nabla_{Z_a} \Psi_a) = \rho g(N, \Psi_{aa}) + \Phi_a^j g(N, \nabla_{Z_a} X_j)\]

From (3.20), we have

\[
\Phi_a^j g(N, \nabla_{Z_a} X_j) = \rho L(w, \Phi) + Q(w, \Phi).
\]

Also, using the decomposition of \( N \) and (3.10), we have

\[
g(N, \Psi_{aa}) = -g(\Upsilon, \Psi_{aa}) + g(N + \Upsilon, \Psi_{aa}) = -g(\Theta, \Phi_{aa}) + \rho L(w, \Phi) + Q(w, \Phi)
\]

Collecting these gives

\[
g(N, \nabla_{Z_a} \Psi_a) = -\rho g(\Phi_{aa}, \Theta) + \rho^2 L(w, \Phi) + \rho Q(w, \Phi)).
\]

**Step 3** Expanding \( Z_a \) gives

\[
g(N, \nabla_{Z_a} X_a) = \rho g(N, \nabla_{X_a} X_a) + \rho w_a g(N, \nabla_{\Upsilon} X_a) + \rho \Phi_a^j g(N, \nabla_{X_j} X_b) = (3.23)
\]

With the help of (3.20), we evaluate

\[
g(N, \nabla_{\Upsilon} X_a) = O(\rho) + L(w, \Phi) + Q(w, \Phi)
\]

\[
g(N, \nabla_{X_a} X_a) = O(\rho) + L(w, \Phi) + Q(w, \Phi)
\]

\[
g(N + \Upsilon, \nabla_{X_a} X_a) = -\alpha^j \Gamma^a_a(\Theta_j) + \rho L(w, \Phi) + Q(w, \Phi),
\]

and plugging these into (3.23) already gives

\[
g(N, \nabla_{Z_a} X_a) = -\rho g(\Upsilon, \nabla_{X_a} X_a) + \rho^2 L(w, \Phi) + \rho Q(w, \Phi)
\]

Using (3.21) we get the expansion

\[
\nabla_{X_a} X_a = -\Gamma^a_a(E_j) X_j - g(R(\rho \Theta + \Phi, E_a) E_j, E_a) X_j + \Gamma^a_a(\rho \Theta + \Phi) \Gamma^a_a(E_j) X_j + (O(\rho) + L(w, \Phi) + Q(w, \Phi)) X_a + (O(\rho^2) + \rho L(w, \Phi) + Q(w, \Phi)) X_j,
\]

Finally, using (3.10) again, we conclude that

\[
g(N, \nabla_{Z_a} X_b) = \rho \Gamma^a_a(\Theta) + \rho^2 g(R(\Theta, E_a) \Theta, E_a) + \rho g(R(\Phi, E_a) \Theta, E_a) + \rho \Gamma^a_a(\rho \Theta + \Phi) \Gamma^a_a(\Theta) - \rho \alpha^j \Gamma^a_a(\Theta_j) + \rho^2 L(w, \Phi) + \rho Q(w, \Phi),
\]
which, together with the results of Step 1 and Step 2, completes the proof of the first estimate.

**Second estimate**: We estimate \( g(N, \nabla Z_i Z_j) \) when \( i = j \) since, just as before, the corresponding estimate, when \( i \neq j \) is not as important and follows similarly. This part is taken directly from [9]. Observe that, by Proposition 3.1, we can also write

\[
N = -\Upsilon + \frac{1}{\rho} \alpha^i Z_i + \hat{N},
\]

where

\[
\hat{N} = (L(w, \Phi) + Q(w, \Phi)) X_a + \left( \rho^2 L(w, \Phi) + Q(w, \Phi) \right) X_j.
\] (3.24)

Now write

\[
g(N, \nabla Z_j Z_j) = g(N, \nabla Z_j) = g(\nabla Z_j \Upsilon, Z_j) - g(\nabla Z_j (\alpha^i Z_i), Z_j)
\]

\[+ g(\hat{N}, \nabla Z_j) - \partial_{z_j} g(\hat{N}, Z_j)
\]

**Step 1**: By (3.19), we can estimate

\[
\nabla Z_j Z_j = \rho w_j Y_j + \rho w_{jj} Y_j + \rho (1 + w) \nabla Z_j Y_j + \rho w_j \nabla Z_j Y
\]

\[= (O(\rho^3) + \rho^2 L(w, \Phi) + \rho^2 L(w, \Phi) (L(w, \Phi) + Q(w, \Phi))) X_a
\]

\[+ (O(\rho^3) + \rho L(w, \Phi) + \rho^2 Q(w, \Phi)) X_k,
\]

Observe that the coefficient of \( X_a \) is slightly better than the coefficient of \( X_k \) since the first two terms only involve the \( X_k \). Using this together with (3.24) we conclude that

\[
g(\hat{N}, \nabla Z_j Z_j) = \rho^3 L(w, \Phi) + \rho Q(w, \Phi).
\]

**Step 2**: Next, using (3.24) together with (3.10), we find that

\[
\partial_{z_j} g(\hat{N}, Z_j) = \rho^3 L(w, \Phi) + \rho Q(w, \Phi).
\]

**Step 3**: We now estimate

\[
C := 2 g(\nabla Z_j \Upsilon, Z_j).
\]

It is convenient to define

\[
C' := \frac{2}{1 + w} g(\nabla Z_j (1 + w) \Upsilon, Z_j),
\]

It follows from (3.15) that

\[
C = C' + \rho Q(w, \Phi)
\]

hence it is enough to focus on the estimate of \( C' \). To analyze this term, let us revert for the moment and regard \( w \) and \( \Phi \) as functions of the coordinates \((z, \bar{y})\) and also consider \( \rho \) as a variable instead of just a parameter. Thus we consider

\[
\tilde{F}(\rho, z, \bar{y}) = F(\rho(1 + w(z, \bar{y})) \Upsilon(z) + \Phi(t \bar{y}), t \bar{y}).
\]

The coordinate vector fields \( Z_j \) are still equal to \( \tilde{F}_* (\partial_{z_j}) \), but now we also have \((1 + w) \Upsilon = \tilde{F}_* (\partial_{\rho})\), which is the identity we wish to use below. Now, from (3.18), we write

\[
C' = \frac{1}{1 + w} g(\nabla \partial_{\rho} Z_j, Z_j) = \frac{1}{1 + w} \partial_{\rho} g(Z_j, Z_j)
\]
Therefore, it follows from (3.13) in Proposition 3.1 that
\[
C = \frac{1}{1+w} \partial_r \left[ \rho^2 g(\Theta, \Theta) + \frac{4}{3} \rho g(R(\Theta, \Theta) \Theta, \Theta) + O(\rho^5) \right] + 2 \rho \rho L(w, \Phi) + \rho^2 Q(w, \Phi)
\]
\[
+ \rho^3 L(w, \Phi) + \rho Q(w, \Phi)
\]
\[
= \frac{1}{1+w} \left[ 2 \rho g(\Theta, \Theta) + \frac{4}{3} \rho^3 g(R(\Theta, \Theta) \Theta, \Theta) + O(\rho^4) \right] + 4 \rho \rho g(\Theta, \Theta) + \rho^2 (g(R(\Theta, \Theta) \Phi, \Theta) + g(R(\Theta, \Theta) \Phi, \Theta))
\]
\[
+ \rho^3 L(w, \Phi) + \rho Q(w, \Phi)
\]

**Step 4:** Finally, we must compute
\[
D := 2 g(\nabla Z_j (\alpha^i Z_i), Z_j)
\]
\[
= 2 g(Z_i, Z_j) \partial_j \alpha^i + 2 \alpha^i g(\nabla Z_j, Z_j)
\]
\[
= 2 g(Z_i, Z_j) \partial_j \alpha^i + \alpha^i \partial_j g(Z_j, Z_j)
\]
Observe that (3.17) implies
\[
\partial_j g(\Theta_i, \Theta_j) = 0
\]
at the point \(p\). Using this together with (3.13) and the expression for the \(\alpha^i\) given in Proposition 3.1, we get
\[
\alpha^i \partial_j g(Z_j, Z_j) = \rho^4 L(w, \Phi) + \rho^2 Q(w, \Phi)
\]
It follows from (3.13) and the definition of \(\alpha^i\) again that
\[
g(Z_i, Z_j) \partial_j \alpha^i = \rho^2 g(\Theta_i, \Theta_j) \partial_j \alpha^i + \rho^4 L(w, \Phi) + \rho^2 Q(w, \Phi)
\]
Therefore, it remains to estimate \(g(\Upsilon_i, \Upsilon_j) \partial_j \alpha^i\). By definition, we have
\[
\alpha^i g(\Theta_i, \Theta_j) = w_j + \frac{\rho}{3} g(R(\Phi, \Theta) \Theta, \Theta)
\]
Differentiating with respect to \(z^j\) we get
\[
(g(\Theta_i, \Theta_j) \partial_j \alpha^i + \alpha^i \partial_j g(\Theta_i, \Theta_j)) = w_j + \frac{\rho}{3} \partial_j g(R(\Phi, \Theta) \Theta, \Theta)
\]
Again, it follows from (3.17) that \(\partial_j g(\Theta_i, \Theta_j) = 0\). Moreover, using (3.20), we first estimate
\[
\nabla Z_j \Upsilon = \Upsilon_j + O(\rho^2) + \rho L(w, \Phi) + \rho Q(w, \Phi);
\]
and, using in addition (3.17), we also get
\[
\nabla Z_j \Upsilon_j = a \Upsilon_j + O(\rho^2) + \rho L(w, \Phi) + \rho Q(w, \Phi)
\]
for some \(a \in \mathbb{R}\). Reinserting this in (3.25) yields
\[
g(\Theta_i, \Theta_j) \partial_j \alpha^i = w_j + \frac{\rho}{3} g(R(\Phi, \Theta) \Theta, \Theta_j) + \frac{\rho}{3} g(R(\Phi, \Theta) \Theta_j, \Theta_j) + \rho^3 L(w, \Phi) + \rho^2 Q(w, \Phi),
\]
since \(R(\Theta, \Theta) = 0\).
Collecting these estimates, we conclude that

\[ D = \rho^2 w_{jj} + \frac{\rho^3}{3} g(R(\Phi, \Theta_j) \Theta, \Theta_j) + \rho^4 L(w, \Phi) + \rho^2 Q(w, \Phi) \]

since \( g(R(\Phi, \Theta_j) \Theta, \Theta_j) = 0 \). With the estimates of the previous steps, this finishes the proof of the estimate.

**Third estimate:** Decompose

\[ \frac{1}{\rho} g(N, \nabla Z \bar{a} Z_j) = g(N, \Upsilon_j) w_{\bar{a} j} + g(N, \Upsilon) w_{\bar{a} j} + (1 + w) g(N, \nabla Z \Upsilon_j) + w_j g(N, \nabla Z \Upsilon). \]

As above we use the expression of \( N \) given in (3.14) to estimate

\[ g(N, \Upsilon_j) = -g(\Upsilon, \Upsilon_j) + g(N + \Upsilon, \Upsilon_j) = L(w, \Phi) + Q(w, \Phi) \]

Similarly \( g(N, \Upsilon) = -1 + L(w, \Phi) + Q(w, \Phi) \)

But now, by (3.20), we have

\[ g(N, \nabla Z \Upsilon) = O(\rho^2) + \rho L(w, \Phi) + \rho Q(w, \Phi) \]

and, as already shown in Step 1

\[ g(N, \nabla Z \Upsilon) = \rho^2 L(w, \Phi) + Q(w, \Phi), \]

and the proof of the estimate follows directly. \( \square \)

## 4 The mean curvature of perturbed tubes

Collecting the estimates of the last subsection we obtain the expansion of the mean curvature of the hypersurface \( S_\rho(w, \Phi) \). In the coordinate system defined in the previous sections, we get

\[ \rho m H(w, \Phi) = n - 1 + \rho \Gamma^a_a(\Theta) + (g(R(\Theta, E_a) \Theta, E_a) + \frac{1}{3} g(R(\Theta, E_i) \Theta, E_i)) \rho^2 \]

\[ - \Gamma^c_a(\Theta) \Gamma^a_c(\Theta) \rho^2 + O(\rho^3) \]

\[ - (\rho^2 \Delta_K w + \Delta_{S^{n-1}} w + (n - 1) w) + 2 \rho \Gamma_{ab}(\Theta) w_{ab} \]

\[ - \rho g(\Delta_K \Phi + R(\Phi, E_a) E_a, \Theta) - \Gamma^c_a(\Phi) \Gamma^a_c(\Theta) \]

\[ + \rho^2 L(w, \Phi) + Q(w, \Phi). \]

We can simplify this rather complicated expression as follows. First, note that

\[ K \text{ minimal} \iff \Gamma^a_a = 0. \]

Next, define

\[ \mathcal{L}_\rho := - (\rho^2 \Delta_K + \Delta_{S^{n-1}} + (n - 1)), \quad (4.26) \]

as an operator on the spherical normal bundle \( S N K \) with the expression in any local coordinates. Also, the Jacobi (linearized mean curvature) operator, for \( K \) is defined by

\[ \mathcal{J} := \Delta^N - \mathcal{R}^N + \mathcal{B}^N, \quad (4.27) \]

cf. [3]. To explain the terms here, recall that the Levi-Civita connection for \( g \) induces not only the Levi-Civita connection on \( K \), but also a connection \( \nabla^N \) on the normal bundle \( NK \). The first term here is simply the rough Laplacian for this connection, i.e.

\[ \Delta^N = (\nabla^N)^* \nabla^N. \]
The second term is the contraction (in normal directions) of the curvature operator for this connection:

$$\mathcal{R}^N := (R(E_i, \cdot) E_i)^N,$$

where the $E_i$ are any orthonormal frame for $N_p K$. Finally, the second fundamental form

$$B : T_p K \times T_p K \to N_p K, \quad B(X, Y) := (\nabla_X Y)^N, \quad X, Y \in T_p K,$$

defines a symmetric operator

$$\mathcal{B}^N := B^* \circ B;$$

in terms of the coefficients $\Gamma^b_a := B(E_a, E_b), \quad g(\mathcal{B}^N X, Y) = \Gamma^b_a(X) \Gamma^a_b(Y)$. We also use the Ricci tensor

$$\text{Ric}(X, Y) = -g(R(X, E_r) Y, E_r), \quad X, Y \in T_p M.$$

In terms of all of this notation, we have the

**Proposition 4.1** Let $K$ be a minimal submanifold. Then the mean curvature of $T_p(w, \Phi)$ can be expanded as

$$\rho m H(w, \Phi) = (n - 1) + \left(\frac{2}{3} g(\mathcal{R}^N \Theta, \Theta) - \frac{1}{4} \text{Ric}(\Theta, \Theta) - g(\mathcal{B}^N \Theta, \Theta)\right) \rho^2 + O(\rho^3)$$

$$+ \, \mathcal{L}_w + \rho g(\mathcal{J} \Phi, \Theta) + O(\rho^3) \nabla^2_K w + \rho^2 L(w, \Phi) + Q(w, \Phi).$$

The equation $\rho m H = n - 1$ can now be written as

$$\mathcal{L}_w + \rho g(\mathcal{J} \Phi, \Theta) = -\left(\frac{2}{3} g(\mathcal{R}^N \Theta, \Theta) - \frac{1}{4} \text{Ric}(\Theta, \Theta) - g(\mathcal{B}^N \Theta, \Theta)\right) \rho^2 + O(\rho^3)$$

$$+ \, O(\rho^3) \nabla^2_K w + \rho^2 L(w, \Phi) + Q(w, \Phi).$$

(4.28)

**4.1 Decomposition of functions on $SNK$**

Before proceeding, we now state more clearly our notation for functions on $SNK$.

Let $(\varphi_j, \lambda_j)$ be the eigendata of $\Delta_{S^{n-1}}$, with eigenfunctions orthonormal and counted with multiplicity. These individual eigenfunctions do not make sense on all of $SNK$, but their span is a well-defined subspace $\mathcal{S} \subset L^2(SNK)$; thus $v \in \mathcal{S}$ if its restriction to each fibre of $SNK$ lies in the span of $\{\varphi_1, \ldots, \varphi_n\}$. We denote by $\Pi$ and $\Pi^\perp$ the $L^2$ orthogonal projections of $L^2(SNK)$ onto $\mathcal{S}$ and $\mathcal{S}^\perp$, respectively.

Now, given any function $v \in L^2(SNK)$, we write

$$\Pi v = g(\Phi, \Theta), \quad \Pi^\perp v = \rho w,$$

so $v = \rho w + g(\Phi, \Theta)$; here $\Phi$ is a section of the normal bundle $NK$, and the somewhat elaborate notation in the second summand here reflects the fact that any element of $\mathcal{S}$ can be written (locally) as the inner product of a section of $NK$ and the vector $\Theta$, whose components are the linear coordinate functions on each $S^{n-1}$. We shall often identify this summand with $\Phi$, and thus, in the following, $w$ and $\Phi$ will always represent the components of $v$ in $\mathcal{S}^\perp$ and $\mathcal{S}$, respectively. Thus

$$w = \frac{1}{\rho} \Pi^\perp v, \quad g(\Phi, \Theta) = \Pi v.$$

Later on we shall further decompose

$$w = w_0 + w_1$$

(4.29)

where $w_0$ is a function on $K$ and the integral of $w_1$ over each fibre of $SNK$ vanishes.

Note that $\mathcal{J}$ preserves $\mathcal{S}$ and is invertible since $K$ is a nondegerate minimal submanifold.
5 Improvement of the approximate solution

The first important step in solving (1.28) is to use an iteration scheme to find a sequence of approximate solutions \((w^{(i)}, \Phi^{(i)})\) for which the estimates for the error term are increasingly small:

\[
\rho m H(w^{(i)}, \Phi^{(i)}) = n - 1 + O(\rho^{i+3}).
\]

Letting \((w^{(0)}, \Phi^{(0)}) = (0, 0)\), we define the sequence \((w^{(i+1)}, \Phi^{(i+1)}) \in S^\perp \oplus S\) inductively as the unique solution to

\[
\mathcal{L}_0 w^{(i+1)} + \rho g(\mathfrak{J} \Phi^{(i+1)}, \Theta) = - \left( \frac{2}{3} g(R^N \Theta, \Theta) - \frac{1}{3} \text{Ric}(\Theta, \Theta) - g(B^N \Theta, \Theta) \right) \rho^2 + O(\rho^3)
- \rho^2 \Delta_K w^{(i)} + O(\rho^3) \nabla^2_K w^{(i)} + \rho^2 L(w^{(i)}, \Phi^{(i)}) + Q(w^{(i)}, \Phi^{(i)}).
\]

(5.30)

here

\[
\mathcal{L}_0 := - (\Delta_{S^{n-1}} + (n-1)).
\]

This equation becomes simpler when divided into its \(S^\perp\) and \(S\) components. Thus using that \(\mathcal{L}_0\) annihilates \(S\) and

\[
\frac{2}{3} g(R^N \Theta, \Theta) - \frac{1}{3} \text{Ric}(\Theta, \Theta) - g(B^N \Theta, \Theta) \in S^\perp
\]

since it is quadratic in \(\Theta\), (5.30) can be rewritten as the two separate equations:

\[
\mathcal{L}_0 w^{(i+1)} = - \left( \frac{2}{3} g(R^N \Theta, \Theta) - \frac{1}{3} \text{Ric}(\Theta, \Theta) - g(B^N \Theta, \Theta) \right) \rho^2 + O(\rho^3)
- \rho^2 \Delta_K w^{(i)} + O(\rho^3) \nabla^2_K w^{(i)} + \rho^2 L(w^{(i)}, \Phi^{(i)}) + Q(w^{(i)}, \Phi^{(i)}),
\]

and

\[
\mathfrak{J} \Phi^{(i+1)} = O(\rho^2) + O(\rho^3) \nabla^2_K w^{(i)} + \rho L(w^{(i)}, \Phi^{(i)}) + \rho^{-1} Q(w^{(i)}, \Phi^{(i)}).
\]

That there is a unique solution now follows directly from the invertibility of \(\mathfrak{J}\) on \(S\) and \(\mathcal{L}_0\) on \(S^\perp\), so the only issue is to obtain estimates.

**Lemma 5.1** For this sequence \((w^{(i)}, \Phi^{(i)})\), we have the estimates

\[
w^{(i)} = O(\rho^2) \quad \Phi^{(i)} = O(\rho^2),
\]

\[
w^{(i+1)} - w^{(i)} = O(\rho^{i+3}) \quad \Phi^{(i+1)} - \Phi^{(i)} = O(\rho^{i+2})
\]

for all \(i \geq 1\).

**Proof:** The estimates for \((w^{(i)}, \Phi^{(i)})\) are immediate, and the result for \(i > 1\) is proved by a standard induction using the general structure of the operators \(L\) and \(Q\). \(\square\)

Finally, replacing \((w, \Phi)\) by \((w^{(i)} + w, \Phi^{(i)} + \Phi)\) in (1.28), the equation we must solve becomes

\[
\frac{1}{\rho} \mathcal{L}_\rho w + g(\mathfrak{J} \Phi, \Theta) = O(\rho^{i+2}) + O(\rho^2) \nabla^2_K w + \rho \hat{L}(w, \Phi) + \frac{1}{\rho} \hat{Q}(w, \Phi).
\]

(5.31)

This is of course simply the expansion of the equation

\[
m H(w^{(i)} + w, \Phi^{(i)} + \Phi) = n - \frac{1}{\rho}.
\]

The linear and nonlinear operators appearing on the right are different from the ones before, but enjoy similar properties.
6 Estimating the spectrum of the linearized operators

We now examine the mapping properties of the linear operator

\[(w, \Phi) \mapsto \frac{1}{\rho} L_\rho w + g(\bar{\Phi}, \Theta) - O(\rho^2) \nabla_K^2 w - \rho \bar{L}^2(w, \Phi)\]  

(6.32)

which appears in (5.31). This is not precisely the usual Jacobi operator (applied to the function \(\rho w + g(\Phi, \Theta)\)), because we are parametrizing this hypersurface as a graph over \(S_\rho(w^{(i)}, \Phi^{(i)})\) using the vector field \(-\Upsilon\) rather than the unit normal.

To understand the difference between (6.32) and the Jacobi operator, recall that if \(N\) is the unit normal to a hypersurface \(\Sigma\) and \(\tilde{N}\) is any other transverse vector field, then hypersurfaces which are \(C^2\) close to \(\Sigma\) can be parameterized as either \(\Sigma \ni q \mapsto \exp_{\Sigma} M q (wN)\) or \(\Sigma \ni q \mapsto \exp_{\Sigma} M q (\tilde{w}\tilde{N})\).

The corresponding linearized mean curvature operators \(L_{\Sigma, N}\) and \(L_{\Sigma, \tilde{N}}\) are related by

\[L_{\Sigma, N}(g(N, \tilde{N}) w) + m(\tilde{N}^T H_{\Sigma}) w = L_{\Sigma, \tilde{N}} w,\]

here \(\tilde{N}^T\) is the orthogonal projection of \(\tilde{N}\) onto \(T\Sigma\). Since \(L_{\Sigma, N}\) is self-adjoint with respect to the usual inner product, we conclude that \(L_{\Sigma, \tilde{N}}\) is self-adjoint with respect to the inner product

\[\langle v, w \rangle := \int_{\Sigma} v w g(N, \tilde{N}) dA_{\Sigma}.\]

Now suppose that \(\Sigma = S_\rho(w^{(i)}, \Phi^{(i)})\) and \(\tilde{N} = \Upsilon\). From Lemma 5.1 and Proposition 3.2 we have

\[g(N, -\Upsilon) = 1 + O(\rho^2).\]

Furthermore, from Proposition 3.1 and Lemma 5.1, and the fact that \(K\) is minimal, the volume forms of the tubes \(S_\rho(w^{(i)}, \Phi^{(i)})\) and \(SNK\) are related by

\[\sqrt{\det(g_{S_\rho(w^{(i)}, \Phi^{(i)})})} = \rho^{k/2} (1 + O(\rho^2)) \sqrt{\det(g_{SNK})};\]

hence

\[A_\rho := g(N, -\Upsilon) \frac{\sqrt{\det(g_{S_\rho(w^{(i)}, \Phi^{(i)})})}}{\rho^{k/2} \sqrt{\det(g_{SNK})}} = 1 + O(\rho^2).\]

(6.33)

Now define

\[L_\rho v = L_\rho (\rho w + g(\Phi, \Theta)) := A_\rho \left(\frac{1}{\rho} L_\rho w + g(\bar{\Phi}, \Theta) + O(\rho^2) \nabla_K^2 w + \rho \bar{L}^2(w, \Phi)\right)\]

\[= \left(\frac{1}{\rho} L_\rho w + g(\bar{\Phi}, \Theta) + O(\rho^2) \nabla_K^2 w + \rho \bar{L}^2(w, \Phi)\right),\]

(6.34)

where the last equality follows from (6.33).

Finally, multiplying (5.31) by \(A_\rho\) gives one further equivalent form of this equation,

\[L_\rho v = O(\rho^{2+i}) + \frac{1}{\rho} \hat{Q} \left(\frac{1}{\rho} \Pi^i v, \Pi v\right),\]

(6.35)

where the nonlinear operator on the right has the same properties as before.

Associated to \(L_\rho\) is the quadratic form

\[Q_\rho(w, \Phi) := \int_{SNK} (\rho w + g(\Phi, \Theta)) L_\rho (\rho w + g(\Phi, \Theta)),\]

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and its corresponding polarization, the bilinear form $C_\rho$. We shall study these forms as perturbations of the model forms
\[
Q_0(w, \Phi) := \int_{SNK} (\rho^2 |\nabla_K w|^2 + |\nabla_{S^{n-1}} w|^2 - (n - 1) |w|^2) + \frac{\omega_{n-1}}{n} \int_K g(\Phi, \Phi)
\]
and associated polarization $C_0$.

To make precise the sense in which $Q_0$ and $Q_\rho$ are close, define the weighted norm
\[
\| (w, \Phi) \|_{H_\rho^1} := \int_{SNK} (\rho^2 |\nabla_K w|^2 + |\nabla_{S^{n-1}} w|^2 + |w|^2) + \omega_n \int_K (|\nabla_K \Phi|^2 + |\Phi|^2)
\]
and also
\[
\| (w, \Phi) \|_{L^2} := \int_{SNK} |w|^2 + \omega_n \int_K |\Phi|^2.
\]

Using (6.33) and the properties of $\bar{L}$, we have
\[
|C_\rho((w, \Phi), (w', \Phi')) - C_0((w, \Phi), (w', \Phi'))| \leq c \rho \|(w, \Phi)\|_{H_\rho^1} \|(w', \Phi')\|_{H_\rho^1}.
\]

### 6.1 Estimates for eigenfunctions with small eigenvalues

**Lemma 6.1** Let $\sigma$ be an eigenvalue of $L_\rho$ and $(w, \Phi)$ a corresponding eigenfunction. There exist constants $c, c_0 > 0$ such that if $|\sigma| \leq c_0$, then using the decomposition $w = w_0 + w_1$ from [4.29],
\[
\|(w - w_0, \Phi)\|_{H_\rho^1} \leq c \rho \|(w, \Phi)\|_{H_\rho^1}.
\]

**Proof:** For any $(w', \Phi')$,
\[
C_\rho((w, \Phi), (w', \Phi')) = \sigma \int_{SNK} (\rho^2 w w' + g(\Phi, \Theta)g(\Phi', \Theta))
\]
\[
= \sigma \int_{SNK} \rho^2 w w' + \sigma \frac{\omega_n}{n} \int_K g(\Phi, \Phi').
\]

In addition, (6.36) gives
\[
\left| \int_{SNK} (\rho^2 \nabla_K w \nabla_K w' + \nabla_{S^{n-1}} w \nabla_{S^{n-1}} w' - (n - 1 + \sigma) w w') + \frac{\omega_{n-1}}{n} \int_K (g(\Phi, \Phi') - \sigma g(\Phi, \Phi')) \right| \leq c \rho \|(w, \Phi)\|_{H_\rho^1} \|(w', \Phi')\|_{H_\rho^1}.
\]

**Step 1:** Take $w' = 0$ and $\Phi' = \Phi$ in (6.37); this yields
\[
\left| \int_K (g(\Phi, \Phi) + \sigma g(\Phi, \Phi)) \right| \leq c \rho \| (w, \Phi) \|_{H_\rho^1} \| (0, \Phi) \|_{H_\rho}
\]
Since $\Phi$ is invertible, there exists $c_1 > 0$ such that
\[
2 c_1 \| (0, \Phi) \|_{H_\rho}^2 \leq \left| \int_K g(\Phi, \Phi) \right|,
\]

hence
\[
(2 c_1 - |\sigma|) \| (0, \Phi) \|_{H_\rho}^2 \leq c \rho \| (w, \Phi) \|_{H_\rho}
\]
Assuming $c_1 \geq |\sigma|$, we conclude that
\[
\| (0, \Phi) \|_{H_\rho} \leq c \rho \| (w, \Phi) \|_{H_\rho}
\]
Step 2: Now use (6.38) with $\Phi' = 0$ and $w = w_1$ to get
\[
\left| \int_{\text{SKN}} (\rho^2 |\nabla_K w_1|^2 + |\nabla_{\text{SKN}} w_1|^2 - (n - 1 - \sigma) |w_1|^2) \right| \leq c \rho \|(w, \Phi)\|_{H^2} \|(w_1, 0)\|_{H^2}. \]
However, since $\Pi w_1 = 0$ and $\int_{\text{SKN}} w_1 = 0$, we have
\[
\int_{\text{SKN}} |\nabla_{\text{SKN}} w_1|^2 \geq 2n \int_{\text{SKN}} |w_1|^2,
\]
and hence
\[
\left| \int_{\text{SKN}} (\rho^2 |\nabla_K w_1|^2 + \frac{1}{2} |\nabla_{\text{SKN}} w_1|^2 + (1 - |\sigma|) |w_1|^2) \right| \leq c \rho \|(w, \Phi)\|_{H^2} \|(w_1, 0)\|_{H^2}.
\]
This implies that
\[
\|(w_1, 0)\|_{H^2} \leq c \rho \|(w, \Phi)\|_{H^2}
\]
provided $|\sigma| \leq 1/4$. This completes the proof if $c_0 = \min(c_1, 1/4)$.

### 6.2 Variation of small eigenvalues with respect to $\rho$

We shall need to obtain some information about the spectral gaps of $L_\rho$ when $\rho$ is small, and to do this, it is necessary to understand the rate of variation of the small eigenvalues of this operator.

**Lemma 6.2** There exist constants $c_0, c > 0$ such that, if $\sigma$ is an eigenvalue of $L_\rho$ with $|\sigma| < c_0$, then
\[
\rho \partial_\rho \sigma \geq 2(n - 1) - c \rho
\]
provided $\rho$ is small enough.

**Proof:** There is a well-known formula for the variation of a simple eigenvalue; complications arise in the presence of multiplicities, but a result of Kato [2] shows that if one considers the derivative of the eigenvalue as a multi-valued function, then an analogue of this same formula holds:
\[
\partial \sigma \in \left\{ \int_{\text{SKN}} v (\partial_\rho L_\rho) v : v = \rho w + g(\Phi, \Theta), \quad L_\rho v = \sigma v, \quad \|v\|_{L^2} = 1 \right\}.
\]
Hence we must provide bounds for the set on the right. We do this by comparing to the model case and using the bounds for eigenfunctions obtained in the last subsection.

Let $L_\rho v = \sigma v$, but rather than normalizing by $\|v\|_{L^2} = 1$, assume instead that $\|(w, \Phi)\|_{L^2} = 1$. In order to compute $\partial_\rho L_\rho$, recall that $w = \rho^{-1} \Pi^1 v$, so we can write
\[
L_\rho v = \frac{1}{\rho^2} L_\rho \Pi^1 v + g(\Phi, \Theta) + O(\rho) \nabla^2 K \Pi^1 v + \rho \hat{L}(\rho^{-1} \Pi^1 v, \Pi v).
\]

Since $\Pi$ and $\Pi^1$ are independent of $\rho$, we have
\[
\partial_\rho L_\rho v = \frac{2}{\rho^3} L_\rho (\Pi^1 v) + \frac{1}{\rho^2} (-\rho \Delta_K \Pi^1 v) + O(1) \nabla^2 K \Pi^1 v + \hat{L}(\rho^{-1} \Pi^1 v, \Pi v)
\]
\[
= -\frac{2}{\rho^2} L_\rho w + O(\rho) \nabla^2 K w + \hat{L}(w, \Phi).
\]

where the operator $\hat{L}$ varies from line to line but satisfies the usual assumptions. This now gives
\[
\left| \int_{\text{SKN}} v (\partial_\rho L_\rho) v + \frac{2}{\rho} \int_{\text{SKN}} (|\nabla_{\text{SKN}} w|^2 - (n - 1) |w|^2) \right| \leq c \|(w, \Phi)\|_{H^2}^2. \quad (6.38)
\]

Now, for this eigenfunction $v$, $Q_\rho(v, v) = \sigma \int \rho^2 |w|^2 + g(\Phi, \Phi)$, and hence by (6.38),
\[
\left| \int_{\text{SKN}} (\rho^2 |\nabla_K w|^2 + |\nabla_{\text{SKN}} w|^2 - (n - 1 + \sigma) |w|^2) + \frac{\omega_{n-1}}{n} \int_{\text{K}} (g(\Phi, \Phi) - \sigma g(\Phi, \Phi)) \right| \leq c \rho \|(w, \Phi)\|_{H^2}^2. \quad (6.39)
\]
By Lemma 6.1,
\[
\int_{SNK} |\nabla_{S^{n-1}} w|^2 + \int_K (|\nabla_K \Phi|^2 + |\Phi|^2) \leq c \rho \| (w, \Phi) \|^2_{H^1},
\]
and inserting this in (6.39) gives
\[
\int_{SNK} (\rho^2 |\nabla_K w|^2 - (n - 1 + \sigma) |w|^2) \leq c \rho \| (w, \Phi) \|^2_{H^1}. \tag{6.41}
\]
Adding these last two estimates now implies that
\[
\| (w, \Phi) \|^2_{H^1} \leq c \rho \| (w, \Phi) \|_{L^2} \leq c
\]
by our choice of normalization. From (6.40) again
\[
\int_{SNK} |\nabla_{S^{n-1}} w|^2 + \int_K (|\nabla_K \Phi|^2 + |\Phi|^2) \leq c \rho.
\]
Inserting this into (6.38), and using again that \( \| (w, \Phi) \|_{L^2} = 1 \), we get
\[
\left| \int_{SNK} v (\partial_{\rho} L_{\rho}) v - \frac{2}{\rho} (n - 1) \right| \leq c \tag{6.42}
\]
for all \( v \) such that \( L_{\rho} v = \sigma v \) and \( \| (w, \Phi) \|_{L^2} = 1 \).

This already implies that \( \partial_{\rho} \sigma > 0 \) for \( \rho \) small enough. But observing that we always have \( \| v \|_{L^2} \leq \| (w, \Phi) \|_{L^2} \), we conclude that
\[
\inf_{v : L_{\rho} v = \sigma \quad \| v \|_{L^2} = 1} \int_{SNK} v (\partial_{\rho} L_{\rho}) v \geq \inf_{v : \| (w, \Phi) \|_{L^2} = 1} \int_{SNK} v (\partial_{\rho} L_{\rho}) v
\]
and (6.42) implies that
\[
\partial_{\rho} \sigma \geq \frac{2}{\rho} (n - 1) - c.
\]
This completes the proof of the result. \( \square \)

### 6.3 The spectral gap at 0 of \( L_{\rho} \)

We can now prove a quantitative statement about the clustering of the spectrum at 0 of \( L_{\rho} \) as \( \rho \searrow 0 \). The ultimate goal is to estimate the norm of the inverse of this operator, but by self-adjointness, this is equivalent to an estimate on the size of the spectral gap at 0.

**Lemma 6.3** Fix any \( q \geq 2 \). Then there exists a sequence of disjoint nonempty intervals \( I_i = (\rho_i^-, \rho_i^+) \), \( \rho_i^\pm \to 0 \) and a constant \( c_q > 0 \) such that when \( \rho \in I := \cup_i I_i \), the operator \( L_{\rho} \) is invertible and
\[
L_{\rho}^{-1} : L^2(SNK) \to L^2(SNK)
\]
has norm bounded by \( c_q \rho^{-k-q+1} \), uniformly in \( \rho \in I \). Furthermore, \( I := \cup_i I_i \) satisfies
\[
|H^1((0, \rho) \cap I) - \rho| \leq c \rho^q, \quad \rho \searrow 0.
\]
**Proof:** An estimate for the size of the spectral gap at 0 is related to the spectral flow of \( L_\rho \), and so it suffices to find an asymptotic estimate for the number of negative eigenvalues of \( L_\rho \). Define the two quadratic forms

\[
Q^\pm(w, \Phi) := Q_0(w, \Phi) \pm \gamma \rho \| (w, \Phi) \|_{H^1_\rho}^2.
\]

From (6.36), if \( \gamma > 0 \), then \( Q^\pm \) is decreasing for \( \gamma \rho \geq 0 \), so it suffices to find an asymptotic estimate for the number of negative eigenvalues of \( D \) and hence the index of \( Q \) and this will give a two-sided bound for the index of \( W \).

Weyl’s asymptotic formula states that if \( 1 - \gamma \rho > 0 \), then the index of \( D^\pm \) equals the index of the minimal submanifold \( K \), and hence does not depend on \( \rho \). Next, if \( (1 - \gamma \rho) 2n - (n - 1 + \gamma \rho) > 0 \), then the index of \( D^\pm_1 \) equals 0. So it remains only to study the index of \( D^\pm_0 \). This is equal to the largest \( j \in \mathbb{N} \) such that

\[
(1 + \gamma \rho) \rho^2 \mu_j \leq (n - 1 + c \rho)
\]

If \( 1 - \gamma \rho > 0 \), then the index of \( D^\pm \) equals the index of the minimal submanifold \( K \), and hence does not depend on \( \rho \). Next, if \( (1 - \gamma \rho) 2n - (n - 1 + \gamma \rho) > 0 \), then the index of \( D^\pm_1 \) equals 0. So it remains only to study the index of \( D^\pm_0 \). This is equal to the largest \( j \in \mathbb{N} \) such that

\[
(1 + \gamma \rho) \rho^2 \mu_j \leq (n - 1 + c \rho)
\]

Weyl’s asymptotic formula states that

\[
\text{Ind } Q^\pm \sim c_K \rho^{-k},
\]

and hence the index of \( D^\pm_1 \), and finally \( \text{Ind } Q_\rho \) too, is asymptotic to \( c_K \rho^{-k} \).

Let \( \rho_i \rightarrow \gamma \rho \) be the decreasing sequence corresponding to the values at which the index of \( Q_\rho \) changes, counted according to the dimension of the nullspace of \( L_{\rho_i} \), i.e.

\[
\rho_i - \rho_i = \cdots = \rho_j < \rho_{j+1}
\]

if \( \dim \text{Ker } L_{\rho_i} = j + 1 - i \). This is well-defined since, by Lemma 6.2, the small eigenvalues of \( L_\rho \) are monotone increasing for \( \rho \) small enough and hence, the function \( \rho \rightarrow Q_\rho \) is monotone decreasing for \( \rho \) small.

The estimates for \( \text{Ind } Q_{2\rho} \) and \( \text{Ind } Q_\rho \) imply that

\[
r_\rho := \# \{ \rho_i < \rho, 2\rho \} \sim c \rho^{-k}.
\]

Letting \( l_\rho \) denote the sum of lengths of intervals \( (\rho_i, \rho_{i+1}) \) for which \( \rho_i \in (\rho, 2\rho) \) and \( (\rho_i - \rho) \leq \rho^{k+q} \), then we have \( l_\rho \leq c \rho^q \); from this we conclude that \( l_\rho \), the sum of lengths of all intervals \( (\rho_i, \rho_{i+1}) \) where \( \rho_i < \rho \) and \( \rho_i - \rho_{i+1} \leq \rho^{k+q} \) is also estimated by \( c \rho^q \).

Define

\[
\tilde{I} = \bigcup_{i \in J} (\rho_{i+1}, \rho_i), \quad \text{where} \quad i \in J \iff \rho_i - \rho_{i+1} \geq \rho_i^{k+q}.
\]

Then by the above, we have

\[
| \mathcal{H}_\rho^1((0, \rho) \cap I) - \rho | \leq c_q \rho^q.
\]

Finally, consider for any \( \rho \in (\rho_{i+1}, \rho_i), i \in J \), the eigenvalues of \( L_\rho \) which are closest to 0, say

\[
\sigma^-(\rho) < 0 < \sigma^+(\rho).
\]
(Thus for each $\rho \in (\rho_{i+1}, \rho_i)$, $\sigma^-(\rho) = \sigma_j$ where $j = \text{Ind } Q_{\rho_i}$.) By construction,
$$\lim_{\rho \searrow \rho_{i+1}} \sigma^+(\rho) = \lim_{\rho \searrow \rho_i} \sigma^-(\rho) = 0.$$ By Lemma 6.2
$$\sigma^-(\rho) \leq 2(n-1) \frac{\rho - \rho_i}{\rho_i} + c \rho_i^{k+q}, \quad \rho \in (\rho_{i+1}, \rho_i),$$ and
$$\sigma^+(\rho) \geq 2(n-1) \frac{\rho - \rho_{i+1}}{\rho_{i+1}} - c \rho_{i+1}^{k+q}, \quad \rho \in (\rho_{i+1}, \rho_i).$$ Hence by the monotonicity of small eigenvalues, if
$$\rho \in I := \bigcup_{i}(\rho_{i+1} + \frac{1}{3} \rho_i^{k+q}, \rho_i - \frac{1}{3} \rho_i^{k+q})$$ then the infimum of the absolute value of the eigenvalues of $\mathbb{L}_\rho$ is bounded from below by a constant (only depending on $n$) times $\rho_i^{k+q-1}$, provided $\rho$ is small enough. The result then follows at once. \hfill $\square$

7 Existence of constant mean curvature hypersurfaces

We now use the results of the previous sections in order to solve the equation (6.5) which reduces to find a fixed point
$$\rho w + g(\Phi, \Theta) = \mathbb{L}_\rho^{-1} \left(\mathcal{O}(\rho^{2+i}) + \frac{1}{\rho} \tilde{Q}(w, \Phi)\right).$$

Since any function $v$ defined on $\text{SNK}$ can be decomposed as $v = \rho w + g(\Phi, \Theta)$ where the function $w$ satisfies
$$\int_{S^{n-1}} w \varphi_j = 0$$ for all $j = 1, \ldots, n$, this equation can be re-written as
$$v = \mathbb{L}_\rho^{-1} \left(\mathcal{O}(\rho^{2+i}) + \frac{1}{\rho} \tilde{Q} \left(\frac{1}{\rho} \Pi^1 v, \Pi v\right)\right)$$

We start with the following elementary observation

Lemma 7.1 There exists a constant $c > 0$ such that
$$\rho^{2+\alpha} \|v\|_{C^{2,\alpha}} \leq c \rho^2 \|\mathbb{L}_\rho v\|_{C^{0,\alpha}} + c \rho^{-\frac{k}{2}} \|v\|_{L^2}$$

Proof: This is a simple application of (rescaled) standard elliptic estimates. We set $f := \mathbb{L}_\rho v$ and, as in §3.1, we use local normal coordinates $\bar{y} = y/\rho$ to parameterize a ball of radius $2\rho R$ in $K$, for some fixed small constant $R > 0$, and local coordinates $z$ to parameterize $S^{n-1}$. Define the functions
$$\bar{v}(z, \bar{y}) := v(z, \rho \bar{y}) \quad \text{and} \quad \tilde{f}(z, \bar{y}) := \rho^2 f(z, \rho \bar{y})$$
It is easy to check that $f := \mathbb{L}_\rho v$ translates into $\mathbb{L}_\rho \bar{v} = \tilde{f}$, where $\mathbb{L}_\rho$ is a second order elliptic operator whose coefficients are bounded uniformly in $\rho$ as $\rho$ tends to 0. Moreover, the principal part of $\mathbb{L}_\rho$ is the Laplace operator on $\text{SNK}$. Standard elliptic estimates yield
$$\|\bar{v}\|_{C^{2,\alpha}(B_{\rho R} \times S^{n-1})} \leq c \|\tilde{f}\|_{C^{0,\alpha}(B_{\rho R} \times S^{n-1})} + c \left(\int_{B_{\rho R} \times S^{n-1}} \int_{B_{2\rho R} \times S^{n-1}} |\bar{v}|^2 d\bar{y}\right)^{1/2}$$
where, to evaluate the Hölder norms in $C^{0,\alpha}$ one takes derivatives with respect to $\bar{y}$ and $z$. Going back to the functions $v$ and $f$ we have
$$\rho^{2+\alpha} \|v\|_{C^{2,\alpha}(B_{\rho R} \times S^{n-1})} \leq c \|\bar{v}\|_{C^{2,\alpha}(B_{\rho R} \times S^{n-1})}, \quad \|\tilde{f}\|_{C^{2,\alpha}(B_{\rho R} \times S^{n-1})} \leq c \rho^2 \|f\|_{C^{2,\alpha}(B_{\rho R} \times S^{n-1})}$$
and
\[
\left( \int_{S^{n-1}} \left( \int_{B_{2R}} |\tilde{v}|^2 \, d\bar{y} \right) \right)^{1/2} \leq c \rho^{-\frac{k}{2}} \left( \int_{S^{n-1}} \left( \int_{B_{2\rho R}} |v|^2 \, dy \right) \right)^{1/2}
\]
The result then follows at once.

We fix \( q \geq 2 \) and \( \alpha \in (0, 1) \). Collecting the result of Lemma 6.3 and the result of the previous Lemma, we conclude that, if \( \rho \in I \), then
\[
\|v\|_{C^{2,\alpha}} \leq c \rho^{-D} \|\mathbb{L}_\rho v\|_{C^{0,\alpha}}
\]
(7.43)
where the constant \( c > 0 \) does not depend on \( \rho \) and where \( D := 3\frac{k}{2} + q + 1 + \alpha \).

Given \( R > 0 \), set
\[
B(R) := \{ v \in C^{2,\alpha}(SNK) : \|v\|_{C^{2,\alpha}} \leq R \}.
\]
and define the mapping
\[
N_\rho(v) := \mathbb{L}_\rho^{-1} \left( \mathcal{O}(\rho^{2+i}) + \frac{1}{\rho} \tilde{Q} \left( \frac{1}{\rho} \Pi^1 v, \Pi v \right) \right)
\]
It follows from (7.43) that we have
\[
\|N_\rho(0)\|_{C^{2,\alpha}} \leq c_0 \rho^{2+i-D}
\]
for some constant \( c_0 > 0 \), independent of \( \rho \in I \).

We choose \( i \in \mathbb{N} \) such that \( i > 2D + 1 \). Using the properties of the operator \( \tilde{Q} \), it is easy to check that there exists \( \rho_0 > 0 \) such that, for all \( \rho \in (0, \rho_0) \cap I \),
\[
\|N_\rho(v)\|_{C^{2,\alpha}} \leq c_0 \rho^{2+i-D}
\]
and
\[
\|N_\rho(v) - N_\rho(v')\|_{C^{2,\alpha}} \leq c \rho^{i-1-2D} \|v - v'\|_{C^{2,\alpha}}
\]
for all \( v, v' \in B(c_0 \rho^{2+i-D}) \). Therefore the mapping \( N_\rho \) admits a (unique) fixed point \( v_\rho \) in \( B(c_0 \rho^{2+i-D}) \). This yields the existence of a constant mean curvature perturbation of the tube \( S_\rho(w^{(i)}, \Phi^{(i)}) \) for all \( \rho \in (0, \rho_0) \cap I \). The proof of the Theorem is complete.

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