Gravitational interactions between the stars during their encounters are quite essential in the evolution of star clusters. As a result of this kind of interactions, a cluster disrupts little by little by loosing stars (and the energy). The mass and the dimensions of a cluster decrease, the density increases.

The importance of the role of the encounters between the cluster stars was first emphasized by Ambartsumian (1938), who calculated the approximate disruption (dissipation) rate of a cluster. Similar results were derived a little later by Spitzer (1940), who also analysed the variations of cluster dimensions and density. Later, the encounter effect in star clusters was studied by Chandrasekhar (1942, 1943) and thereafter by a number of authors (Gurevich and Levin 1950, Skabitski 1950, Minin 1952, Woolley 1954b, von Hörner 1957). As a result of these studies (true enough, being not in all cases free of errors and insufficient arguments), the evolution of clusters outlined by Ambartsumian and Spitzer was specified and detailed at a certain lever.

Having in mind the need to analyse the cluster evolution in more detail, the purpose of the present paper is to derive the equations allowing to describe the evolution of clusters more precisely than it was done before. In this paper we analyse the equations enabling to follow the variations of the phase density and the gravitational potential of the cluster. We also study the formulas necessary to calculate the encounter function. We treat the clusters as isolated, spherical, almost steady systems of big number of gravitating particles of constant mass. This approximations are quite common in stellar dynamics when studying star clusters, and the properties of this kind of systems were studied in detail (Eddington 1914, 1916; Jeans 1916; ten Bruggencate 1927; Shiveshwarkar 1936; Kurth 1949, 1950, 1955; Camm 1952; Woolley 1954a, 1956). The cluster model that we use corresponds probably quite well to globular clusters. In case of open clusters the similarity is worse and in some case even completely absent (very sparse clusters can not be handled even approximately as isolated).

1. The phase density, the potential and the encounter function

When treating star clusters as systems consisting of big number of stars, we use the phase density \( \psi \) for the description of their structure. The phase density is the density of particles in six-dimensional phase space, where the coordinates are three rectangular coordinates of ordinary space and three corresponding velocity components. Due to the discrete nature of medium in phase space, the phase density is the smoothed density with the properties of probability density. The phase density can be defined as the number density of particles, the mass density or in some other way. It is important to distinguish between the total phase density and the partial phase density, latter corresponding to the unit interval of some particle’s characteristic, e.g. to unit interval of mass.

---

* Tartu Astron. Obs. Publ. 33, 75, 1957; Chapter 17 of the Thesis.
Integration of the total phase density as the phase mass density over the velocity space gives us the ordinary mass density in three-dimensional physical space \( \rho \) and enables to derive the gravitational potential of the system \( \Phi \). Potential \( \Phi \) calculated in this way is smoothed or regular potential, corresponding to continuous mass distribution in physical space. A randomly varying irregular potential is added to the regular potential, resulting from real discreteness of mass distribution. The existence of that potential is revealed in gravitational interactions during stellar encounters in physical space.

According to our assumption the number of stars is big, so the real gravitational potential of the system is highly smoothed and differs only slightly from regular potential. Hence the motion of stars is governed by regular potential, and the role of encounters is very small. Encounters may be revealed only after sufficiently long time, much longer than the “revolution time”. This conclusion is confirmed by the calculations of the relaxation time and the mean free path of particles (Chandrasekhar 1942).

To clarify the ways of system evolution due to encounters, we must follow the time evolution of the phase density \( \psi \). Evolution of the phase density is caused by the motion of stars in phase space. This motion consists of smooth regular motion, governed by the regular potential, and of more or less impulsive motions related to stellar encounters in physical space and governed by irregular potential. Correspondingly the evolution of the phase density is caused by non-stationarity of regular motion and by non-compensation of impulsive displacements. We designate the variation of the phase density per unit time resulting from the encounters as \( \chi \). This is called the encounter function.

At the initial stage of evolution of the system the variation of the phase density resulting from non-stationarity of the regular gravitational potential can be very large. But as it was mentioned already by Eddington (1916, 1921), after quite short time – of order of few revolution times – the system evolves due to the “mixing” process into a nearly-stationary state. Although the “mixing” process does not decrease the speed of phase density variation, it looses the correlation of phase density variations between different points of velocity space. As a result these variations will be like stochastic fluctuations, and the mean variation of the phase density in arbitrarily small volume element of the velocity space will tend to zero. The encounter effect is also supporting the establishment of stationarity, smoothing the inhomogeneities in particle distribution in velocity space.

Hence, if leaving aside the initial short stage of the system evolution, when non-stationarity may be high,† we can assume that the system is in nearly stationary state, and the deviations are described by slow action of the encounter effect, which changes step by step the structure of the system.

Having in mind approximately spherical form of most clusters, we may suppose that our system also has the spherical symmetry.

In case of the spherical symmetry, the system density \( \rho \) and thus the potential \( \Phi \) are functions of only the distance from system center \( r \) and time \( t \)

\[
\rho = \rho(r, t), \quad \Phi = \Phi(r, t).
\] (1.1)

* This stage of evolution in case of spherical systems was comprehensively studied by Kurth (1951). His results concern the very beginning of the evolution, when the results of the mixing process have not yet appeared. Kurth used his theory to analyse contemporary globular clusters. It is difficult to agree with this kind of application, because this initial stage of evolution is evidently finished for these clusters long ago.
We may assume the velocity distribution to be symmetrical about the radial direction. In this case $\psi$ and $\chi$ are functions of $v_t, v_r, r, t$ only, while $v_t$ and $v_r$ are the tangential and radial components of the particle velocity $\vec{v}$.

If the spherical system is precisely stationary, then the phase density is a function of two integrals of motion – the energy integral and the total angular momentum integral. Thus instead of arguments $v_t$ and $v_r$ it is suitable to use other arguments, supposing

$$\psi = \psi(p, q, r, t), \quad \chi = \chi(p, q, r, t),$$

where $p$ and $q$ are related to velocity in following way

$$2p = 2\Phi - v^2, \quad 2q = r^2v_t^2,$$

or

$$v^2 = 2(\Phi - p), \quad v_t^2 = 2q/r^2, \quad v_r^2 = 2(\Phi - p - q/r^2).$$

The arguments $p$ and $q$ are negative energy and half of the square of angular momentum per unit mass respectively. In spherically symmetric system $q$ remains constant during the regular motion and is thus an integral of motion. In a stationary system also $p$ would be an integral of motion. In this case $\psi$ would be a function of $p$ and $q$ only. In reality, however, $p$ is not constant, and the speed of its variation for a regularly moving point is described by the equation $d\Phi/dt - \vec{v} \nabla \Phi = \partial \Phi / \partial t$. But as the system is nearly stationary, $p$ varies very slowly and $\psi$ is a function of mainly $p$ and $q$.

The region of permitted values of $p$ and $q$ for given $r$ and $t$ is limited by the condition $v_r^2 \geq 0$. Two values of $v_r$ correspond to every possible set of $p$ and $q$ inside that region, being equal in size but with the opposite sign, therefore the functions $\psi$ and $\chi$ are two-valued. One branch of them corresponds to moving away from the center ($v_r > 0$), the other approaching to the center ($v_r < 0$). For $v_r = 0$ both branches coincide.

For the particles belonging to the system, the region of permitted values of $p$ and $q$ is limited also by the condition $p > 0$, because the values $p \leq 0$ corresponds to velocities equal or larger than the escape velocity (assuming $\Phi = 0$ for $r = \infty$). As a consequence of encounters, the particles may obtain all possible values of $p$ and $q$ and fill all the referred region. This means that the system radius is infinite (when the system is isolated). But in this case some part of the particles will have $p \leq 0$. These particles do not belong to the system any more and escape freely, as encounters act slowly. Hence their phase density is nearly zero. Therefore, we may suppose $\psi > 0$ for $p > 0$, and $\psi = 0$ for $p \leq 0$. $\psi$ must be smooth bounded function, as encounters smooth the irregularities of particle distribution. The encounter function may have for $p > 0$ both positive and negative values. For $p \leq 0$ it is positive up to some maximum value obtainable by a particle during encounter. This part of the encounter function describes the escape of particles from the system – the dissipation.

---

1 Designations $p$ and $q$ in here do not have any relation with canonical variables. [Later footnote.]
2. The equation for the phase density

In case of regular motion the volume of the phase space remains constant (Liouville’s theorem). Hence regular motion does not change the phase density at comoving point. At comoving point the phase density varies only because of stellar encounters. The variation of the phase density is thus described by the equation

\[ \frac{D\psi}{Dt} = \chi, \]  

(2.1)

where \( D/Dt \) is the Stokes derivative for regular motion.

Equation (2.1) is the well known kinetic Boltzmann equation. In stellar dynamics this equation is usually used by assuming \( \chi = 0 \), i.e. neglecting the encounter effect. In present case, it is essential to take it into account. In general form Eq. (2.1) was discussed in stellar dynamics by Charlier (1917), and later by Chandrasekhar (1943a–c) and Skabitski (1950). Mainly the right side of the equation was analysed, but not the application of the equation as a whole. Trying to simplify the equation, Skabitski as a first approximation substituted the left side of Eq. (2.1) with \( \partial\psi/\partial t \). However, this substitution is unjustified, because in this case the effect of encounters is reduced to the variation of the velocity distribution of particles only. In reality the variation of the spatial distribution is as important.* But despite of that Eq. (2.1) for the spherical systems will have quite simple form.

Due to the spherical symmetry we may put \( \psi \) and \( \chi \) according to Eq. (1.2) into Eq. (2.1). Taking into account that in case of regular motion \( p \) varies with the speed \( \partial\Phi/\partial t \), \( r \) with the speed \( v_r \) and \( q \) remains constant we have the equation

\[ \frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial p} \frac{\partial\Phi}{\partial t} + \frac{\partial\psi}{\partial r} v_r = \chi. \]  

(2.2)

Because at non-moving point of the phase space \( p \) varies also with the speed \( \partial\Phi/\partial t \) (q.v. Eq. (1.3)) and \( q \) and \( r \) remain constant, the sum of first two terms on left side of the equation is the speed of phase space variation at non-moving point of the phase space. Hence the equation may be written in form

\[ \left( \frac{\partial\psi}{\partial t} \right) + \frac{\partial\psi}{\partial r} v_r = \chi, \]  

(2.2a)

where \( (\partial/\partial t) \) is the time derivative at non-moving point of the phase space. Equation (2.2a) differs from Skabitski’s first approximation equation by including the second term on left side. This term takes into account the redistribution of particles in the physical space.

Let us take now into account the approximate stationarity of the system.*

* Assuming \( \partial\psi/\partial t = \chi \) Skabitski derived \( \partial\rho/\partial t = 0 \), expressing simply the conservation of the number and the mass of stars when dissipation is absent. He didn’t derive dissipation because of the used forms of \( \psi \) and \( \chi \). The similar error was made by Minin (1952). He in-explicitly assumed that the escape of particles from a given volume element of the system influences only the density in that volume element. In fact the influence spreads over all the space available for escaped particles.
Equation (2.2) concerns both branches of the phase density and the encounter function ($v_r > 0$ and $v_r < 0$). Designating the half-sums of both branches of $\psi$ and $\chi$ as $\Psi$ and $X$ and their half-differences as $\Delta \psi$ and $\Delta \chi$, we may replace Eq. (2.2a) with the pair of equations

$$
\frac{\partial \Psi}{\partial t} + \frac{\partial \Delta \psi}{\partial r} |v_r| = X, \quad \left( \frac{\partial \Delta \psi}{\partial t} \right) + \frac{\partial \psi}{\partial r} |v_r| = \Delta \chi.
$$

(2.3)

In collisionless system the stationarity would be strict, $\Psi$ would not depend explicitly on $r$ and $\Delta \psi$ would be zero. In reality, the system is not strictly stationary. But because it is close to stationarity and the encounter effect is very small, also $(\partial \psi/\partial t)$ and $\chi$ are very small and moreover $(\partial \Delta \psi/\partial t)$ and $\Delta \chi$ are very small. Therefore, on the basis of Eq. (2.3) we may suppose

$$
\Psi = \Psi(p, q, t).
$$

(2.4)

Now we may eliminate from the first equation of (2.3) the function $\Delta \psi$ and the argument $r$. To do this we first average the terms of the equation with the weight $dr/|v_r|$ over all possible values of $r$ for given $p, q, t$. After averaging the second term on left side of the equation will vanish, because at bounds of $r$ we have $\Delta \psi = 0$. We replace the quantities $\partial \Phi/\partial t$ and $X$ in the first term (in long form) and on the right side of the equation with their averaged values $\overline{\partial \Phi/\partial t}$ and $\overline{X}$, which are some functions of only $p, q, t$. Hence we have [q.v. Appendix 1]

$$
\frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial p} \frac{\partial \Phi}{\partial t} = \overline{X},
$$

(2.5)

or in other form (we use it further)

$$
\left( \frac{\partial \Psi}{\partial t} \right) - \frac{\partial \Psi}{\partial p} \left( \frac{\partial \Phi}{\partial t} - \overline{\partial \Phi/\partial t} \right) = \overline{X}.
$$

(2.5a)

In these equations

$$
\tau \frac{\partial \Phi}{\partial t} = \int_{r_1}^{r_2} \frac{\partial \Phi}{\partial t} \frac{dr}{|v_r|}, \quad \tau \overline{X} = \int_{r_1}^{r_2} X \frac{dr}{|v_r|},
$$

(2.6)

while

$$
\tau = \int_{r_1}^{r_2} \frac{dr}{|v_r|}.
$$

(2.7)

Here $v_r$ was substituted according to Eq. (1.4), $r_1$ and $r_2$ are the roots of the equation $v_r = 0$ for given $p, q, t$.

As $dr/v_r = dt$ and $p$ varies very slowly in case of regular motion ($q$ remains constant), we may handle the averaging used in deriving Eq. (2.5) as the averaging over time at regularly moving point. In this case the averaging spreads over the time interval when $r$ changes from $r_1$ to $r_2$. The referred time-interval (it is the half-period of particle oscillation along $r$) evidently equals to $\tau$. The averaging can be interpreted in another way, namely as the averaging over the phase space for given $p, q, t$. This results because the phase space volume element is $2\pi v_t \, d\nu \, dv_r \cdot 4\pi r^2 \, dr$ or according to (1.4) $8\pi dp dq \cdot dr/|v_r|$.

We can use Eq. (2.5) for real calculations to study the variation of the phase density due to encounters. According to this equation at a point moving in $p, q$ plane parallel to the axis $p$ with the velocity $dp/dt = \overline{\partial \Phi/\partial t}$, the variation speed of $\Psi$ equals to $\overline{X}$. If $\Psi$ is given at a certain initial moment of time and if we are able to calculate $\overline{\partial \Phi/\partial t}$ and $\overline{X}$ from known $\Psi$, then the equation enables to find $\Psi$ at any moment of time.
Equation (2.5) can be supplemented by the equation allowing to find $\Delta \psi$. Comparing Eq. (2.5a) with the first equation of (2.3) we have

$$\frac{\partial \Delta \psi}{\partial r} |_{v_r} = X - \bar{X} - \frac{\partial \Psi}{\partial p} \left( \frac{\partial \Phi}{\partial t} - \frac{\partial \bar{\Phi}}{\partial t} \right)$$

(2.8)

Because $X$ and $\partial \Phi/\partial t$ are very small, also $\Delta \psi$ is very small compared to $\Psi$. Hence $\Psi$ and $X$ nearly coincide with $\psi$ and $\chi$.

3. The equation for the potential. Initial conditions

The function $\Psi(p, q, t)$ describes the spatio-kinematic structure of our system of particles only when we know its potential $\Phi(r, t)$, because $p$ is related to the velocity $\vec{v}$ via potential. Therefore, to study the evolution of the system, we must find not only the evolution of $\Psi$ but also the evolution of $\Phi$. In addition, as we saw, the evolution of $\Psi$ can not be determined without knowing the evolution of $\Phi$. We must keep the potential in mind also when establishing the initial conditions for our problem. Giving the function $\Psi$ at some initial moment we must also give the corresponding function $\Phi$.

It is possible to find the relation between $\Psi$ and $\Phi$, and the equation for $\partial \Phi/\partial t$, starting with Poisson’s equation. Having in mind the spherical symmetry, the Poisson’s equation is

$$\frac{\partial^2}{\partial r^2} (r \Phi) + 4\pi G r \rho(r, t) = 0,$$

(3.1)

where $G$ is the gravitational constant. If under the phase density we assume the phase mass density, then the mass density in physical space $\rho$ is

$$\rho = \int_V \psi dV,$$

(3.2)

where $V$ is the volume element in velocity space and integration is over all that space. Because $dV = 2\pi v_t dv_t dv_r = 2\pi dp dq / r^2 |v_r|$ (q.v. Eq. (1.4)) and the sum of both branches of $\psi$ is $2\Psi$, Eq. (3.2) has now the form

$$\rho(r, t) = P(\Phi(r, t), r, t) = \frac{4\pi}{r^2} \int \int \Psi \frac{dp dq}{|v_r|},$$

(3.3)

where integration is over all $p$ and $q$ satisfying the condition $v_r^2 \geq 0$. The argument $t$ is contained in the function $P$ via $\Psi$, the arguments $\Phi$ and $r$ via $v_r$ (q.v. Eq. (1.4)). Substituting Eq. (3.3) into Eq. (3.1) we derive the equation

$$\frac{\partial^2}{\partial r^2} (r \Phi) + 4\pi G r P(\Phi, r, t) = 0.$$

(3.4)

Equations (3.3) and (3.4) just relate the functions $\Psi$ and $\Phi$. We can use these equations if the initial conditions for our problem are given.

As the argument $t$ in Eqs. (3.3) and (3.4) is not essential and $\Psi$ is independent of $r$, they coincide with equations describing the stationary spherical stellar system. To have a physically meaningful model, the function $\Psi$ must be non-negative and go to zero for
The potential $\Phi$ must satisfy the boundary conditions $r \Phi = 0$ for $r = 0$ and $r \partial (r \Phi) / \partial r = 0$ for $r = \infty$. The first of them results from the demand to have zero mass at the center of the system, the second – from the demand of the finite total mass of the system and from the condition $\Phi = 0$ for $r = \infty$. We may start the model construction by giving the function $\Psi(p, q)$ and taking into account the considerations above. Then after finding the function $P(\Phi, r)$ from Eq. (3.3), we derive $\Phi(r)$ from Eq. (3.4). But we may also give first the function $\Phi(r)$ and the function $\rho(r)$ related to it. We find the function $\Psi$ with the help of Eq. (3.3). The relation between $\Psi$ and $\Phi$ is not one-valued. Because the functions $\Phi$ and $\rho$ determine the function $P(\Phi, r)$ only along the curve $\Phi = \Phi(r)$, then $\Phi$ evidently does not determine $\Psi$ in a single way. But it may result that also $\Psi$ does not determine $\Phi$ in one-valued form. This occurs when the referred boundary conditions for $\Phi(r)$ happen to be insufficient for one-valued determination.

The models of stationary spherical stellar systems were studied by different authors. As an example, already Jeans (1916) and Eddington (1916) studied the models with spherical velocity distribution and with the density distribution similar to the density distribution of polytropic gaseous sphere. Afterwards these kind of models were studied by Camm (1952) who generalized them for the case of ellipsoidal velocity distribution. Also other models were analysed (q.v. for example Woolley 1954, 1956). Usually in the model construction the function $\Psi(p, q)$ was given and the potential $\Phi$ was found as the solution of Eq. (3.4). But already Eddington mentioned that it is possible to start from the function $\rho(r)$ (or $\Phi(r)$). He analysed the case when $\Psi$ is independent of $q$, i.e. when the velocity distribution is spherical. In this case $P$ is independent of $r$ and Eq. (3.3) turns into Abel integral equation (after integration over $q$).

By giving the initial conditions for our problem, it is suitable to use the models, where the encounters cause smooth variations without changing the model structure in a radical way. In such model $\Psi$ must be a smooth bounded function, vanishing only for $p \leq 0$. Hence the radius of the model must be infinite. The mass of the model must be surely finite. From the models studied up to now only one satisfies these conditions. This is the well known model with $\Psi \sim p^{7/2}$ and the Schuster density distribution thoroughly studied already by Eddington (1916) and recently by Skabitski (1950) and Minin (1952). But even this model is not completely suitable for us. If we use in calculations of an encounter function the approximate formulas given further in Section 5, we must suppose that $\partial \Psi / \partial p$ remains finite when $p$ decreases to zero. This condition is in contradiction with the law $\Psi \sim p^{7/2}$ and is moreover incompatible with the spherical velocity distribution. On the basis of Eq. (3.3), on can demonstrate that within this condition the system mass is finite ($\Phi \sim r^{-1}$, $\rho r^3 \to 0$ for $r \to \infty$) only when $\sqrt{q} \partial \Psi / \partial p \to 0$ for $p \to 0$ and $q \to \infty$.

Hence, to give the initial conditions we can not use the developed models but need to construct new ones.

Now we discuss the equation for $\partial \Phi / \partial t$. It was obtained from Poisson’s equation. Differentiating (3.1) with respect to $t$ we have

$$
\frac{\partial^2}{\partial r^2} \left( r \frac{\partial \Phi}{\partial t} \right) + 4\pi G r \frac{\partial \rho}{\partial t} = 0.
$$

(3.5)

It means that for small $p$ the function $\Psi$ decreases with $q$, i.e. in outer regions of spherical stellar systems the velocity distribution becomes radially elongated. [Later footnote.]
We must substitute into this equation

\[
\frac{\partial \rho}{\partial t} = \int_V \left( \frac{\partial \psi}{\partial t} \right) dV, \tag{3.6}
\]

where \((\partial/\partial t)\) again means differentiating with respect to \(t\) in non-moving point of the phase space. Going to variables \(p, q\), adding both branches of \(\Psi\) and substituting \((\partial \Psi/\partial t)\) according to (2.5a) we find

\[
\frac{\partial^2}{\partial r^2} \left( r \frac{\partial \Phi}{\partial t} \right) + 4\pi Gr \left[ F \left( \frac{\partial \Phi}{\partial t} - \overline{\frac{\partial \Phi}{\partial t}} \right) + H \right] = 0, \tag{3.7}
\]

where

\[
F = \frac{4\pi}{r^2} \int \int \frac{\partial \Psi}{\partial p} dp dq |v_r|, \\
F \overline{\frac{\partial \Phi}{\partial t}} = \frac{4\pi}{r^2} \int \int \frac{\partial \Psi}{\partial p} \overline{\frac{\partial \Phi}{\partial t}} dp dq |v_r|, \\
H = \frac{4\pi}{r^2} \int \int X dp dq |v_r|, \tag{3.8}
\]

The same result can be derived when differentiating Eq. (3.4) with respect to \(t\).

The function \(\overline{\frac{\partial \Phi}{\partial t}}\) in Eq. (3.7) is twice averaged \(\partial \Phi/\partial t\). But double averaging turns to single averaging. Substituting \(\overline{\frac{\partial \Phi}{\partial t}}\) according to Eq. (2.6) into the second formula of (3.8) and changing the order of integration we may write

\[
rF \overline{\frac{\partial \Phi}{\partial t}} = \int_0^\infty r' \frac{\partial \Phi'}{\partial t} f(r, r') dr', \tag{3.9}
\]

where

\[
f = \frac{4\pi}{r r'} \int \int \frac{\partial \Psi}{\partial p} r^{-1} \frac{dp dq}{|v_r v_{r'}|}, \tag{3.10}
\]

while apostrophe in \(\partial \Phi/\partial t\) and \(v_r\) means that the argument \(r\) is replaced with the argument \(r'\) and integration in Eq. (3.9) is over all \(p\) and \(q\) within the conditions \(v_r^2 \geq 0, v_{r'}^2 \geq 0\).

Equations (3.9) and (3.10) were written in form, where the first of them includes the term \(r' \partial \Phi'/\partial t\) in a similar way as (3.7). But surely the term \(\partial \Phi'/\partial t\) is averaged, not \(r' \partial \Phi'/\partial t\).

If the functions \(\Psi, \Phi,\) and \(X\) are known at a given moment of time, then we may find \(F\) and \(H\) as functions of \(r\), and \(f\) as function of \(r\) and \(r'\). Equation (3.7) can be solved as the equation for \(r \partial \Phi/\partial t\). To solve it we use the boundary conditions \(r \partial \Phi/\partial t = 0\) for \(r = 0\) and \(\partial (r \partial \Phi/\partial t)/\partial r = 0\) for \(r = \infty\).

Equation (3.7) together with (2.5) form a set of equations. Knowing the functions \(\Psi, \Phi,\) and \(X\) at initial moment of time we may find according to Eq. (3.7) \(\partial \Phi/\partial t\). Calculating thereafter \(\partial \Phi/\partial t\) we will find \(\partial \Psi/\partial t\) from Eq. (2.5). Thus the functions \(\Psi, \Phi\) are known for the subsequent moment of time. Calculating also the encounter function \(X\) we can find again \(\partial \Phi/\partial t\) and thereafter \(\partial \Psi/\partial t\). In that way, solving step by step Eqs. (2.5) and (3.7) we may determine \(\Psi\) and \(\Phi\) at any moment of time.
4. Calculation of the encounter function

Above we derived the equations allowing to follow the evolution of a spherical system of gravitating particles caused by the encounter effect. Now we begin to deal with calculations of the encounter function. From technical viewpoint this is perhaps the most difficult part of our problem.

As our system consists of big number of particles, the most important encounters are the ones having small impact parameter compared to the orbit’s dimensions. For the same reason the encounter time is short when compared with revolution time. Hence the encounter effect can be assumed to be similar to the elastic collision of particles (with the difference that during the encounters we have a completely different distribution of deflection angles). And besides, the encounters can be handled within sufficient precision as the encounters between two bodies. True, the encounters with impact parameter exceeding the mean distance between particles seem to resemble multiple collisions. But due to the weakness of interactions in case of distant encounters these interactions add together in a way which is similar to a series of two-body encounters. *

If treating the encounter of particles as two-body elastic collisions, the velocity vector of the center of mass and the values of particle velocities in respect to each other must remain unchanged during the encounter. If \( \vec{v}_1 \) and \( \vec{v}_2 \) are the velocities of particles with masses \( m_1 \) and \( m_2 \), then these velocities must change in a way that the center of mass in velocity space remains in place and the vector of relative velocity of particles \( \vec{w} \) keeps its value but rotates by some angle \( \varphi \) – the deflection of the motion direction. The value and the plane of the angle depend on the impact parameter \( D \) and on the impact azimuthal angle \( \vartheta \). The behavior of the particle velocities with respect to the center of mass \( \vec{u}_1 \) and \( \vec{u}_2 \) are similar to \( \vec{w} \). During the encounter the particles relocate their positions in velocity space remaining in diametrically opposite points on the spheres described around the center of mass by radii

\[
\begin{align*}
  u_1 &= \frac{m_2}{m_1 + m_2} w, & u_2 &= \frac{m_1}{m_1 + m_2} w.
\end{align*}
\]

If we take the positions of particles on the referred spheres before the encounter to be the poles of spherical coordinates, then after the encounter the spherical coordinates of their positions are \( \varphi, \vartheta \).

During the encounter the particle velocities \( \vec{v}_1 \) and \( \vec{v}_2 \) turn into some new velocities \( \vec{v}_1' \) and \( \vec{v}_2' \), which can be calculated from \( \varphi \) and \( \vartheta \). On the other side, during some particular encounter the velocities \( \vec{v}_1' \) and \( \vec{v}_2' \) may turn again into \( \vec{v}_1 \) and \( \vec{v}_2 \). Let us designate the phase number densities of particles with masses \( m_1 \) and \( m_2 \) as \( \psi_1 \) and \( \psi_2 \) for \( \vec{v}_1 \) and \( \vec{v}_2 \). For \( \vec{v}_1' \) and \( \vec{v}_2' \) the phase number densities are \( \psi_1' \) and \( \psi_2' \). Further, let us designate the function of relative deflection frequency per unit of solid angle as \( \Omega \) and the effective interaction radius of particles in physical space as \( D_0 \). If \( \vec{v}_1 \) and \( \vec{v}_2 \) lie within the volumes \( dV_1 \) and \( dV_2 \)

* Quite often it was assumed following Charlier (1917) and others, that encounters with impact parameter exceeding the mean distance between stars have little effect and they can be neglected. In fact the contribution of these encounters to the encounter function is quite significant. In statistical physics when discussing the interaction between charged particles, these encounters were taken into account by using two-body encounter model. This was done for example in a paper by Landau (1937). Recently the need to take into account distant encounters was mentioned again by Cohen et al. (1950).
of the velocity space respectively, and the deflection of the motion direction lies within the solid angle \( d\omega \) \((=\sin \varphi d\varphi d\theta)\), then the encounters remove from the volume \( dV_1 \) per unit time the number of particles with masses \( m_1 \) equal to \( \psi_1 dV_1 \cdot \psi_2 dV_2 \cdot \pi D_0^2 \omega \cdot \Omega \cdot d\omega \). On the other side, if \( \vec{v}_1' \) lies within the solid angle, taken with respect to the center of mass, and \( \vec{v}_2' \) within the volume \( dV_2' \), then the encounters add to the volume \( dV_1 \) per unit of time the number of particles equal to \( \psi_1 d\omega \cdot \psi_2 dV_2' \cdot \pi D_0^2 \omega \cdot \Omega \cdot dV_1 \). Besides, if we suppose that in both cases the center of mass lies within the same volume, then \( dV_2' = dV_2 \). Designating the encounter function of particles \( m_1 \) with respect to particles \( m_2 \) as \( \chi_{1,2} \) we derive for \( \chi_{1,2} \) after the integration a well known Boltzmann expression:

\[
\chi_{1,2} = \int \int (\psi_1' \psi_2' - \psi_1 \psi_2) \cdot \pi D_0^2 \omega \cdot \Omega d\omega dV_2. \tag{4.2}
\]

Integration is over all the velocity \( \vec{v}_2 \) space and over all deflections.

The encounter function \( \chi_{1,2} \) can be calculated also from the flux of the particles with mass \( m_1 \) per unit surface perpendicular to the flux. If we designate the vector of this flux \( \vec{i}_{1,2} \) then from the continuity equation in velocity space we have

\[
\chi_{1,2} = -\nabla \vec{i}_{1,2}, \tag{4.3}
\]

where \( \nabla \) must be taken in velocity space. Let us assume that particle displacement along the great circles of the spheres \( u = \text{const} \) and let the number of particles with deflection angles greater or equal to \( \varphi \) be \( 2\pi \sin \varphi \cdot J \). Taking again \( \vec{v}_1' \) in solid angle \( d\omega \) with respect to the center of mass and \( \vec{v}_2' \) in volume \( dV_2' \), we find that the flux of particles through the perpendicular surface element \( dS \) in vicinity of \( \vec{v}_1 \) equals to \( \psi_1' d\omega \cdot \psi_2 dV_2' \cdot \pi D_0^2 \omega \cdot u_1 J \cdot dS \), while again we can suppose \( dV_2' = dV_2 \). Now we find the expression for \( \vec{i}_{1,2} \)

\[
\vec{i}_{1,2} = \int \int \psi_1' \psi_2' \cdot \pi D_0^2 \omega \cdot u_1 \vec{J} d\omega dV_2, \tag{4.4}
\]

where the vector \( \vec{J} \) must be directed perpendicular to the vector \( \vec{w} = \vec{v}_2 - \vec{v}_1 \), with the azimuthal angle perpendicular to the azimuthal angle of \( \vec{v}_1 \). Besides, according to the definition of \( J \) we have

\[
J = \frac{1}{\sin \varphi} \int_0^\pi \Omega \sin \varphi d\varphi. \tag{4.5}
\]

Surely, it is easier to calculate \( \chi_{1,2} \) from Eq. (4.2), but in next Section where we derive approximate formula for \( \chi_{1,2} \), it is more convenient to use Eqs. (4.3) and (4.4).

As \( \psi_1 \) and \( \psi_2 \) can be handled as functions of \( p, q, r, t \) Eq. (4.2) or Eqs. (4.3) and (4.4) must give \( \chi_{1,2} \) as a function of the same arguments. Besides, as \( \psi_1 \) and \( \psi_2 \) were two-valued functions of \( p \) and \( q \), the function \( \chi_{1,2} \) is also two-valued. But the difference between two branches of \( \psi_1 \) and \( \psi_2 \) \((\nu_r > 0 \text{ and } \nu_r < 0)\) is not large. Hence, when we substitute the functions \( \psi_1 \) and \( \psi_2 \) with half-sums of two branches in both of them, we derive within sufficient precision the half-sum of both branches of \( \chi_{1,2} \). Just in this sense we need to know the encounter function when using the equations of previous Sections. But Eq. (4.2) and Eqs. (4.3) and (4.4) do not give directly the total encounter function \( \chi \). To have \( \chi \) as the variation of particle \( m_1 \) number phase density, we must integrate \( \chi_{1,2} \)

* In stellar dynamics the expression for the encounter function was derived first by Charlier (1917).
over all values of \( m_2 \). If we are interested in \( \chi \) as the variation of the total mass phase density (in this sense the encounter function was used in Sect. 3), we must multiply the previous encounter function by \( m_1 \) and integrate over \( m_1 \).

In order to use Eqs. (4.2) and (4.4), it is necessary to have some concrete expressions for \( \Omega \) and \( J \). They can be derived if the deflection angle \( \varphi \) as a function of impact parameter \( D \) is known. Functions \( \Omega \) and \( J \) are related to \( D \) and \( \varphi \) in following way

\[
\Omega = -\frac{1}{\pi \sin \varphi} \frac{D}{D_0} \frac{dD}{d\varphi}, \quad J = \frac{1}{2\pi \sin \varphi} \frac{D^2}{D_0^2}.
\]  

(4.6)

When \( D \) is not large compared to orbit dimension, the relation between \( D \) and \( \varphi \) can be taken as in two-body problem. For larger \( D \) our formulas are not valid. Although, as the role of very distant encounters is relatively small, we may approximately use our formulas for all encounters with \( D \) up to orbital dimensions. We take \( D_0 \) for the orbital dimension. The precise value of \( D_0 \) is not important, because the encounter effect depends only very weakly on \( D_0 \) (logarithmically). It is natural to assume that \( \varphi \to 0 \) for \( D \to D_0 \). In this case, \( D \) can be assumed as a function of \( \varphi \) in form of the following simple interpolation formula

\[
D^2 = D_0^2 \left(1 + \frac{\cos \varphi}{2} \frac{\lambda - 1}{\lambda - \cos \varphi} \right),
\]  

(4.7)

where

\[
\lambda - 1 = \frac{2G^2(m_1 + m_2)^2}{D_0^2 w^4}.
\]  

(4.8)

Equation (4.7) differs from the analogous formula for two-body problem only because instead of unity there is \( \lambda \) in the denominator of its second term.

By inserting (4.7) into (4.6) we derive the expressions for functions \( \Omega \) and \( J \). The expression for \( J \) is straightforward, the expression for \( \Omega \) is

\[
\Omega = \frac{1}{4\pi} \frac{\lambda^2 - 1}{(\lambda - \cos \varphi)^2}.
\]  

(4.9)

In case of great number of gravitating particles the right side of (4.8) is in general very small and thus \( \lambda \) is close to one. Hence Eq. (4.9) gives very large values of \( \Omega \) for small \( \varphi \), i.e. large relative frequency of small deflections. This is the well known property of gravitational encounters, being highly different from elastic collision of spherical particles for which, as it is known, the deflection distribution is uniform. Equation (4.9) gives uniform distribution when \( \lambda \) is large meaning that \( m_1 + m_2 \) must be large or \( w \) must be small. In this case the deflection angle distribution will be really close to uniform.

5. Approximate formula for the encounter function

It is quite complicated and time consuming to compute the encounter function on the basis of formulas above. The computations can be significantly simplified by taking into account that in most cases displacement of particles in velocity space due to encounters is very small. This allows us to calculate approximately \( \chi_{1,2} \) by expanding \( \psi_1 \) and \( \psi_2 \) in Taylor series and keeping only the first terms.
Let us use Eqs. (4.3) and (4.4) to calculate \( \chi_{1,2} \). Keeping in expansion of \( \psi'_1 \cdot \psi'_2 \) only linear terms and rejecting the terms which disappear in integration over \( \vartheta \) we derive Eq. (4.4) in following form

\[
\vec{t}_{1,2} = \int \int (u_2 \psi_1 \nabla J \psi_2 - u_1 \psi_2 \nabla J \psi_1) \sin \varphi \cdot \pi D_0^2 w \cdot u_1 J d\omega dV,
\]

where vector \( \nabla J \) is the component of \( \nabla \) along \( \vec{J} \). Equation (5.1) is similar to the equation derived by Landau (1937) and used by Skabitski (1950).

Let us integrate (5.1) over the deflection angles. With the help of Eqs. (4.1), (4.6)–(4.8) we find

\[
\vec{t}_{1,2} = \pi G^2 m_2 \int (m_1 \psi_1 \nabla_\perp \psi_2 - m_2 \psi_2 \nabla_\perp \psi_1) L w^{-1} dV,
\]

where

\[
L = \frac{1}{\lambda - 1} \int J \sin \varphi d\omega = \frac{1}{\lambda - 1} \int \Omega (1 - \cos \varphi) d\omega = \frac{\lambda + 1}{2} \ln \frac{\lambda + 1}{\lambda - 1} - 1
\]

and vector \( \nabla_\perp \) is the component of \( \nabla \) perpendicular to \( \vec{w} = \vec{v}_2 - \vec{v}_1 \). As \( w^2 \nabla_\perp = (w^2 - \vec{w} \vec{w}) \nabla \), from (5.2) results the following equation for \( \vec{t}_{1,2} \)

\[
\vec{t}_{1,2} = \pi G^2 m_2 (m_1 \vec{b} \psi_1 - m_2 \vec{B} \nabla \psi_1).
\]

Here

\[
\vec{B} = \int (w^2 - \vec{w} \vec{w}) w^{-3} L \psi_2 dV,
\]

\[
\vec{b} = \nabla \vec{B} = 2 \int \vec{w} w^{-3} L \psi_2 dV,
\]

while the expression for \( \vec{b} \) was derived under the assumption that \( L \) is independent of the direction of \( \vec{w} \).

As the velocity distribution is symmetrical about the radial direction, vectors \( \vec{b} \) and \( \nabla \psi_1 \) in Eq. (5.4) and also two main axes of tensor \( \vec{B} \) lie on the plane parallel to \( \vec{v} \) and \( \vec{v}_t \). Vector \( \vec{t}_{1,2} \) lies on the same plane. If in addition the velocity distribution is spherically symmetric, vectors \( \vec{b} \) and \( \nabla \psi_1 \) are parallel to \( \vec{v} \). In this case also one of the main axis of tensor \( \vec{B} \) lies in the same direction, while the other two main axes perpendicular to \( \vec{v} \) are equal to each other. Vector \( \vec{t}_{1,2} \) is also parallel to \( \vec{v} \).

Using the relation between \( \vec{B} \) and \( \vec{b} \) we may rewrite (5.4) in form

\[
\vec{t}_{1,2} = \pi G^2 m_2 [(m_1 + m_2) \vec{b} \psi_1 - m_2 \nabla \vec{B} \psi_1].
\]

Equation (5.4a) along with (4.3) can be seen as the diffusion equation of particles in velocity space [q.v. Appendix 2.1]. It is similar to Chandrasekhar’s (1943) diffusion equation. It is easy to see that the first term in (5.4a), which is similar to the corresponding term in Chandrasekhar’s equation, accounts for the systematic variation of \( \vec{v}_1 \) or the dynamical friction. Indeed, the encounters of particles \( m_1 \) and \( m_2 \) cause the systematic variation of \( \vec{v}_1 \) equal to \( -\vec{u}_1 (1 - \cos \varphi) \) per one encounter or \( -\psi_2 dV_2 \cdot \pi D_0^2 w \cdot \Omega d\omega \cdot \vec{u}_1 (1 - \cos \varphi) \) in volume \( dV_2 \), solid angle \( d\omega \) and unit time. We find after integration that systematic variation of \( \vec{v}_1 \) per unit time is equal to \( \pi G^2 m_2 (m_1 + m_2) \vec{b} \) (q.v. (4.1), (5.3) and (5.5)). Assuming the
spherical symmetry of $\psi_2$ and that $L$ is independent of $w$, our expression for dynamical friction coincides with Chandrasekhar’s expression, derived under the same assumptions. As it is seen from (5.5), the method used to calculate the $\vec{b}$ in case of $dL/dw = 0$ is similar to the method for calculating the gravitation acceleration, where instead of density we have $\psi_2$ and instead of the gravitational constant we have $-2L$. Hence in case of $dL/dw = 0$ and spherical symmetry of $\psi_2$, the dynamical friction depends only on $\psi_2$ for $v_2 \leq v_1$, just as it was derived by Chandrasekhar.

The second term in (5.4a) accounts (approximately) for the fluctuating part of $\vec{v}_1$ variation. It is not difficult to see that $2\pi G^2 m_2^2 \mathbf{B}$ is (within sufficient precision) the tensor of second moments of $\vec{v}_1$ variation per unit time. Meanwhile, we mention that the fluctuating term must be proportional to $\nabla \mathbf{B} \psi_1$ and not $\mathbf{B} \nabla \psi_1$, as it was taken by Chandrasekhar. Invalidity of Chandrasekhar’s expression is easy to show in case of $\psi_2$, as a function of the same arguments.

Inserting (5.4) (or (5.4a)) into (4.3) and accounting for the relation between $B$ and $\vec{b}$, we derive for the encounter function the following expression

$$\chi_{1,2} = \pi G^2 m_2 [m_1 a \psi_1 + (m_2 - m_1) b \nabla \psi_1 + m_2 B \nabla \nabla \psi_1], \quad (5.6)$$

where

$$a = -\nabla \vec{b} = 2 \int \frac{dL}{dw} w^{-2} \psi_2 dV_2 \quad (5.7)$$

(or $a = 8\pi L \psi_2$ in case of $dL/dw = 0$). Equation (5.6) together with (5.5) and (5.7) can be derived also from Eq. (4.2). But the calculations are more complicated, as in expansion of $\psi_2$, the second ordered terms must be used (as it was done by Landau (1937)).

Above we used $p$ and $q$ as arguments of the phase density and hence it is recommendable to move also in Eq. (5.6) to these variables (q.v. (1.4)). Considering that $\nabla = -\vec{v} \partial / \partial p + r^2 \vec{v}_t \partial / \partial q$ we find

$$\chi_{1,2} = \pi G^2 m_2 \left[ m_1 a \psi_1 + (m_2 - m_1) \left( a_1 \frac{\partial \psi_1}{\partial p} + a_2 \frac{\partial \psi_1}{\partial q} \right) + \right. \left. + m_2 \left( a_1' \frac{\partial \psi_1}{\partial p} + a_2' \frac{\partial \psi_1}{\partial q} + a_{11} \frac{\partial^2 \psi_1}{\partial p^2} + 2a_{12} \frac{\partial^2 \psi_1}{\partial p \partial q} + a_{22} \frac{\partial^2 \psi_1}{\partial q^2} \right) \right]. \quad (5.8)$$

Here, $a_1 = -\vec{b} \vec{v}$, $a_2 = r^2 \vec{b} \vec{v}_t$, $a_1' = -\mathbf{B} \nabla \vec{v} = -\mathbf{S} \mathbf{B}$, $a_2' = r^2 \mathbf{B} \nabla \vec{v}_t = r^2 \mathbf{S} \mathbf{B}$, $a_{11} = \mathbf{B} \vec{v} \vec{v}$, $a_{12} = -r^2 \vec{B} \vec{v} \vec{v}_t$, $a_{22} = r^2 \vec{B} \vec{v} \vec{v}_t$. The function $\psi_2$ is known, the coefficients $a$, $a_1$, $a_2$ etc. can be calculated as functions of $p$ and $q$. Then, knowing $\psi_1$ we derive from (5.8) the function $\chi_{1,2}$ as a function of the same arguments.

As was mentioned in Sect. 1, the encounter function for $p \leq 0$ describes the escape of particles from system. Yet, because $\psi_1 = 0$ for $p \leq 0$, Eq. (5.8) gives $\chi_{1,2} = 0$ for $p < 0$.

---

* The diffusion equation was treated correctly in paper by Cohen et al. (1950). However, we cannot agree with their opinion that equation similar to (4.2) can not be used in case of distant encounters. In reality, the diffusion equation results as a consequence of that formula (or of equivalent formulas (4.3) and (4.4)). Their remark that Landau’s equations (1937) do not allow to account the dynamical friction is also misleading.
This result is caused by the approximate nature of formulas in present Section. But they still enable, although approximately, to describe the escape of particles. Indeed, if the phase density gradient is finite on the inner surface of the escape velocity sphere, then we have a finite flux of particles, that have the escape velocity. We may identify this flux with the flux of particles escaping the system, similarly as it was done by Chandrasekhar (1943) in his calculations of cluster dissipation speed.∗

The same result for the escape speed of a particle can be derived from Eq. (5.6). According to that formula, the behavior of the encounter function on the surface of escape velocity sphere is like the delta-function. Supposing ψ₁ to be expressed as a function of p and q and using Eq. (5.8) we derive χ₁,2 for p = 0 in following form

\[
\chi^{(0)}_{1,2} = \pi G^2 m_2^2 \left( a_{11} \frac{\partial \psi_1}{\partial p} \right)_0 \delta(p),
\]

(5.9)

where index 0 under the parentheses designates the limit when p vanishes. The particles, that correspond to the distribution (5.9), can be handled just as escaping particles. True, in reality the behavior of the encounter function at p = 0 is not precisely like the delta-function, but has some, although not large, “width”, with about half of them having p > 0. But these particles form very rarefied media without further influence on system dynamics. We can suppose that they leave the system.

The condition \((\partial \psi / \partial p)_0 > 0\), that was necessary for dissipation according to equations of previous Section, may be not fulfilled for real systems, even if the system dissipates. However, if we use these formulas in description of the encounter effect, we shall find that the necessary condition for starting the dissipation is immediately present. Hence it is meaningful to include it among the initial conditions (it was mentioned in Sect. 3).

With this we finish the discussion about the equations and formulas for our problem. As we saw, the basic equations are Eq. (2.5) for the phase density and Eq. (3.7) for the potential. Simultaneous solution of these equations allows to follow the evolution of the system caused by the encounter effect. It is quite complicated to solve the equations, especially because of complications in calculations of the encounter function according to Eq. (4.2). But as we saw, the encounter function can be simplified somewhat by using the approximate formula (5.8).

In present paper we avoided drawing any conclusions about the possible evolution of star clusters. But in subsequent papers we hope to use the theory presented above for this

∗ As it was mentioned above we can not agree with the form of fluctuating term in Chandrasekhar’s equation. Thus his calculations need revision. Meanwhile, the calculations of the escape speed of stars with different masses from the cluster on the basis of Ambartsumian–Spitzer theory done in his book (Chandrasekhar 1942) are also disputable. In this calculations, he uniform energy distribution over all range of masses is assumed. This assumption is unjustified because the kinetic energy of stars is restricted by the escape velocity.
Later the theory developed here was generalized and applied in case of highly flattened Galactic subsystems (Chapter 22). Studies on the dynamical evolution of spherical stellar systems were not continued. We like to refer to the following papers on the evolution of spherical systems under the influence of irregular gravitational forces: M. Hénon, Ann. d’Ap. 24, 369, 1961; R.W. Michie, Ap.J. 133, 781, 1961 and M.N.R.A.S. 125, 124, 1963; T.A. Agekian, Astron. Zh. 40, 318, 1963 and 41, 523, 1964. [Later footnote.]
Appendices added in 1969

Appendix 1. Using the adiabatic invariants and integrals of motion in case of variable potential

For slowly varying gravitational potential we have adiabatic invariant

\[ P_r = \frac{1}{2\pi} \oint v_r dr. \quad (A1.1) \]

The invariant \( P_r \) is constant only in average. If a star moves, we have for the time derivative

\[ \dot{P}_r = \frac{\partial P_i}{\partial E} \dot{E} + \frac{\partial P_i}{\partial t} = -\frac{1}{\omega_r} \left( \frac{\partial \Phi}{\partial t} - \frac{\partial \Psi}{\partial t} \right), \quad (A1.2) \]

where \( E \) is the energy integral (in original paper and in Chapter 17 \( E = -p \)) and \( \omega_r \) is the frequency of \( r \) oscillations of the star.

If we use in \( \Psi \) the argument \( P_r \) instead of \( E \), i.e.

\[ \Psi = \Psi(P_r, I, t), \quad (A1.3) \]

where \( I \) is the integral of angular momentum (in original paper and in Chapter 17 it was designated as \( I^2 = 2q \)), we have instead of Eqs. (2.5) and (2.8)

\[ \frac{\partial \Psi}{\partial t} = \mathbf{X}, \quad \frac{\partial \Delta \psi}{\partial r} |v_r| = \mathbf{X} - \mathbf{X} - \frac{\partial \Psi}{\partial E} \frac{\partial \Phi}{\partial t}. \quad (A1.4) \]

Instead of \( P_r \) we can use also the integral of motion in slightly variable gravitational field

\[ K_r = P_r - \int \dot{P}_r dt = P_r - \int \left( \frac{\partial \Phi}{\partial t} - \frac{\partial \Psi}{\partial t} \right) \frac{dr}{v_r}. \quad (A1.5) \]

If

\[ \Psi = \Psi(K_r, I, t), \quad (A1.6) \]

we derive equations

\[ \frac{\partial \Psi}{\partial t} = \mathbf{X}, \quad \frac{\partial \Delta \psi}{\partial r} |v_r| = \mathbf{X} - \mathbf{X}. \quad (A1.7) \]

However, these simplifications are only formal and do not simplify the solution of equations because we must find the functions \( P_r(E, I, t) \) or \( K_r(E, I, r, t) \).

Appendix 2. The encounter function

1.° The formula for the encounter function

\[ X_{1,2} = -\nabla v \tilde{l}_{1,2} = -\nabla v \cdot (\tilde{a} \Psi_1) + \nabla v \cdot \nabla v (A \Psi_1), \quad (A2.1) \]

where

\[ A = \pi G^2 m_2^2 B, \quad \tilde{a} = \pi G^2 m_2 (m_1 + m_2) \tilde{b}, \quad (A2.2) \]

\[ B = \int (w^2 - \tilde{w} \tilde{w}) w^{-3} L \Psi_2 dV, \]

\[ \tilde{b} = \nabla_i B = 2 \int \tilde{w} w^{-3} L \Psi_2 dV \quad (A2.3) \]
is the Fokker-Planck equation for stellar encounters.

2.° In the case of equal stellar masses

\[ X_{1,2} = X, \quad \Psi_1 = \Psi_2 = \Psi, \quad m_1 = m_2 = m. \quad (A2.4) \]

Assuming the spherical velocity distribution

\[ \Psi = \Psi(p), \quad p = -E = \Phi - v^2/2 \quad (A2.5) \]

the encounter function has the spherical symmetry in velocity space.

Assuming for simplicity

\[ L = L_0 = \text{const}, \quad (A2.6) \]

the Fokker-Planck equation gives us in case of spherical velocity distribution

\[ X = 8\pi^2 G^2 m^2 L_0 \left( f_0 \Psi - f_1 \frac{d\Psi}{dp} + f_2 \frac{d^2\Psi}{dp^2} \right), \quad (A2.7) \]

where we find following expressions for functions \( f_1, f_2 \) and \( f_3 \)

\[ f_0 = \Psi \]

\[ f_1 = \int_0^p \Psi dp + (\Phi - p)^{-1/2} \int_0^\Phi (\Phi - p)^{1/2} \Psi dp \]

\[ f_2 = \frac{2}{3} (\Phi - p) \left[ \int_0^p \Psi dp + (\Phi - p)^{-3/2} \int_p^\Phi (\Phi - p)^{3/2} \Psi dp \right]. \quad (A2.8) \]

For the most simple polytropic model of a spherical star system, the phase density \( \Psi \) is proportional to \( p^s \), \( s = n - 3/2 \), where \( n \) is the polytrope index. In this case the integrals in expressions for \( f_1 \) and \( f_2 \) will be the elementary and incomplete B-functions, and if \( 2s \) is a natural number, the incomplete B-function reduces into elementary function.

In Fig. A1 the encounter function for the Schuster-Eddington model is given, i.e. for \( s = 7/2 \). Unit for \( X \) is \( \Psi_{p=\Phi}^2 \) (without a factor before parentheses in Eq. (A2.7)). The encounter function for Schuster-Eddington model was derived also by Skabitsky (1950).
3. Although in case of spherical velocity distribution the encounter function has the spherical symmetry, the averaged encounter function

\[
\overline{X} = \frac{\int X v_r^{-1} \, dr}{\int v_r^{-1} \, dr},
\]  

(A2.9)

occurring in our basic expression (2.5) doesn’t have spherical symmetry.

In order to illustrate it, we calculated \( \overline{X} \) for the Schuster-Eddington model. Figure A2 presents \( \overline{X} \) for different angular momentum integrals \( q \). The units for \( p, q \) and \( \overline{X} \) are \( \Phi_0, \Phi_0 r_0^2 \) and \( \Psi^2_{p=q} \) respectively (\( \Phi_0 \) is potential at the center of the model and \( r_0 \) is characteristic length in Schuster law). Line c corresponds to the circular velocity.

The lack of spherical symmetry in the averaged encounter function means that although the irregular forces do not explicitly destroy the spherical symmetry of the velocity distribution (just the opposite — they tend to remove non-sphericity if it will occur), the velocity distribution becomes non-spherical. This results due to the spatial redistribution of stars caused by simultaneous action of regular and irregular forces.

As can be seen from figure, the derivative \( \frac{\partial \Psi}{\partial q} \) for small \( p \) is negative, i.e. the velocity distribution in the periphery of the spherical stellar system becomes radially elongated (the term \( \frac{\partial \Phi}{\partial t} \frac{\partial \Psi}{\partial p} \) in Eq. (2.5) does not influence the result; it causes only slight shift in \( p \) of the curves of \( \Psi \)).

4. Derivation of the encounter function in case of spherical velocity distribution is quite simple. It is possible to use not only the Fokker-Planck equations but also more precise
formulas (Boltzmann formula in fact), which take into account large velocity variations in nearby encounters. This kind of calculations were made by R.v.d.R. Woolley (MNRAS 114, 1954) and T.A. Agekian (AZh 36, 41, 1959). They were done also by us (unpublished).

5.° In case of non-spherical velocity distribution, the calculations of the encounter function are highly complicated. Although the use of Fokker-Planck equation simplifies the problem, it remains complicated, as we need to calculate the vector \( \vec{b} \) and the tensor \( B \). For this reason it is recommended to simplify the expression for the encounter function. One of the possibilities is to use, following Chandrasekhar (1943a,b)* in Fokker-Planck equations (despite of the relation \( \vec{b} = \nabla B \))

\[
A = A, \quad \vec{a} = -\beta \vec{v}, \tag{A2.10}
\]

where \( A \) and \( \beta \) are scalars independent of velocity.

The expression for the encounter function will have form

\[
X = \beta \nabla_v (\vec{v} \Psi) + A \nabla_v^2 \Psi. \tag{A2.11}
\]

For the encounters between stars of equal masses we have the relation

\[
A = \beta \sigma^2, \tag{A2.12}
\]

where \( \sigma^2 \) is the mean of the components of velocity dispersion. This relation expresses that \( \sigma^2 \) remains constant in encounters (kinetic energy conservation).

While moving to the variables \( p, q \) we find

\[
X = \beta \left( 3 \Psi - [3 \sigma^2 + 2(\Phi - p)] \frac{\partial \Psi}{\partial p} + 2(r^2 \sigma^2 + q) \frac{\partial \Psi}{\partial q} + \right. \nonumber \\
\left. + 2 \sigma^2 (\Phi - p) \frac{\partial^2 \Psi}{\partial p^2} + 2r^2 \sigma^2 q \frac{\partial^2 \Psi}{\partial q^2} \right), \tag{A2.13}
\]

where \( \beta, \sigma, \Phi \) are functions of \( r \). In order to compare this formula to the more precise formula, we calculated \( X \) for the Schuster-Eddington model (spherical velocity distribution, \( \sigma^2 = \Phi/6 \)). The resulting curve is similar to the one represented in Fig. A1 with the exception that minimum and maximum of the curve are shifted to small \( p \).

---

* S. Chandrasekhar, Stochastic problems in physics and astrophysics, Rev. Mod. Phys. 15, 1, 1943.