Autonomous Ticking Clocks from Axiomatic Principles

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There are many different types of time keeping devices. We coin the phrase ticking clock to describe those which — simply put — “tick” at approximately regular intervals. Various important results have been derived for ticking clocks, and more are in the pipeline. It is thus important to understand the underlying models on which these results are founded. The aim of this manuscript is to introduce a new ticking clock model from axiomatic principles that overcomes concerns in the community about the physicality of the assumptions made in previous models. The model we introduce is autonomous and requires only finite memory per unit of time for which it is in operation. It can also achieve the same accuracy as those reported in [1] while retaining full autonomy such as the model in [2]. What is more, [2] is revealed to be a special case of the new ticking clock model.

1 Introduction and basics

Clocks form part of our everyday lives. Understanding their fundamental limitations is an interesting and rich theoretical problem of study which may yield important design principles for improved future clocks. However, results pertinent to the performance of clocks are only of relevance if the theoretical clock models underpinning them capture the relevant properties. Therefore, understanding the clock models which underlie the results about clock performance, is as important as the results themselves.

Before discussing our findings and their motivation, let us first describe two different types of time keeping devices generically referred to as “clocks” in order to set the scene for this work. We coin the phrase ticking clocks to refer to one type and call the other stopwatches. In the literature, both devices are often simply referred to as “clocks” e.g. [1–9] yet it is worth introducing distinct names due to their different character. A stopwatch is a device which measures the elapsed time between two external events. There will be a starting time (e.g. the beginning of a race) and a stopping time (e.g. when the winner crosses the finish line). The stopwatch will attempt to measure the elapsed time. The earliest types of quantum clocks considered in the literature were of this form [10, 11]. These are also the types of clocks one often considers in a metrology setting, since it is equivalent to measuring a phase. However, the action of measuring the stopwatch disturbs its internal dynamics, thus changing the outcome statistics of later time measurements.

On the other hand, one can consider a ticking clock which, roughly speaking, is a device which emits ticks at approximately regular intervals. A typical wall clock is a good classical example. Here one can listen or watch the clock face and will know in real-time when it ticks. Analogously to the above example, we will want our mathematical formulation of the ticking clock to allow continuous observations of whether it has ticked or not, without affecting its internal dynamics. Since there is no such requirement imposed on stopwatches; ticking clocks and stopwatches require very different mathematical formulations.

Ticking clocks and stopwatches are also physically very distinct objects. The following two examples illustrate this point quite nicely. Firstly, consider a race and measuring the elapsed time between the winner leaving the starting line and crossing the finishing line with a stopwatch. This task can also be carried out by a ticking clock, at least to an accuracy to within plus or minus the time between two consecutive ticks. However, what about if you arranged to meet a friend at a given location at, say, 13:00h tomorrow? If you were only equipped with a stopwatch with no other time reference, you would hopelessly fail to be on time. The reason for this negative predicament, is that you would have no external signal (like the winner crossing the finish line in the previous example) to know when to stop your stopwatch — when you eventually press the “stop button”, it may indicate that only 1 second has passed or maybe one week. One may hope to remedy this predicament by resetting their stopwatch immediately after it was stopped; and trying again while keeping a record of the previous outcome. However, what about if you arranged to meet a friend at a given location at, say, 13:00h tomorrow? If you were only equipped with a stopwatch with no other time reference, you would hopelessly fail to be on time. The reason for this negative predicament, is that you would have no external signal (like the winner crossing the finish line in the previous example) to know when to stop your stopwatch — when you eventually press the “stop button”, it may indicate that only 1 second has passed or maybe one week. One may hope to remedy this predicament by resetting their stopwatch immediately after it was stopped; and trying again while keeping a record of the previous outcome. However, this would only lead to a finite number of completely irregular instances when you would know what the time was. Consequently, you would almost surely be very late for the meeting with your friend.

The above hypothetical example involving the stopwatch, while conveying an important point, is a bit far fetched from our everyday experience since we do, in fact, always have access to ticking clocks — albeit bad ones — such as the visual difference between day and night. To study such scenarios, one could investigate a different type of time keeping device formed by
combining a stopwatch and a ticking clock to take advantage of the best properties of both. Atomic clocks are a good example of such devices. We will leave their study to future work.

If either the stopwatch or ticking clock is quantum mechanical in nature, then from a mathematical perspective, both these devices output information from the clock on Hilbert space $\mathcal{H}_C$ to the “outside”. This information transfer can be made explicit within the model by including a register. In the case of a stopwatch, it will record the outcome of a measurement. But this information retrieval from the clock via a POVM is passive i.e. its retrieval is triggered by an external signal, and the clock reacts passively to the measurement [5]. This is in contrast to the information transfer to the register of a ticking clock, in which the information transfer is triggered internally by the clock mechanism itself, with no help from external triggers. Both stopwatches and ticking clocks can be modelled by multipartite Hilbert spaces. The most two common elements we will discuss concern the bipartition $\mathcal{H}_C \otimes \mathcal{H}_R$. In keeping with the terminology described in section 3.1.

Demanding eq. (1) has some immediate consequences, the most important of which is that the clockwork is fully determined at all times by the “smallest coordinate time step”. In the case of continuous coordinate time, under appropriate continuity assumptions, this reads:

$$M_{C \rightarrow C}^{t_1+t_2} = M_{C \rightarrow C}^t \circ M_{C \rightarrow C}^\delta,$$

for all $t_1, t_2 \in \mathcal{S}_C$. This condition has been justified by considering the opposite scenario: suppose that $M_{C \rightarrow C}^C$ were not divisible, i.e. eq. (1) does not hold for some $t_1, t_2 \in \mathcal{S}_C$. Then, the channel being applied per unit of coordinate time (discrete or continuous) should have to depend on knowledge of the value of coordinate time itself. In other words, the device may need an additional time reference external to the setup. However, by definition the clockwork is supposed to contain all sources of timing necessary for the ticking clock to function. Requirement eq. (1) is discussed further in section 3.1.

The authors of [6] do not use the terminology “ticking clocks” and instead refer to their devices as “clocks”. In keeping with the terminology introduced in this manuscript, we will use the former denomination.

2 The ticking clock model of [6]

The authors start by describing their model for a discrete coordinate time ticking clock. The continuous coordinate time ticking clock, is then realised by allowing the discrete time step parameter $t \rightarrow 0^+$. One will find that $t/\delta$ times. We will use this notation thought this manuscript.

The authors of [6] construct a ticking clock model by describing how the clockwork they introduce interacts with a register. We review their model in the next section.
to which the temporal information coming from the clockwork is recorded. The register is formed by a tensor product space, \( \mathcal{H}_{R_1} \times \mathcal{H}_{R_2} \times \mathcal{H}_{R_3} \times \ldots \) over local registers \( \mathcal{H}_{R_i} \) which are all isomorphic to a fixed register \( \mathcal{H}_{R_t} \).

The authors define a channel \( \mathcal{M}^\delta_{C \rightarrow CR_1} : L(\mathcal{H}_C) \to L(\mathcal{H} \oplus \mathcal{H}_{R_t}) \) for some fixed \( \delta > 0 \). This gives rise to the state of the register after \( N \) applications of the channel; denoted \( \rho_{R_t}(N) = \rho_{R_t}^{R_1 R_2 R_3 R_4 \ldots} \). After \( l \in \mathbb{N}_{\geq 0} \) discrete coordinate time steps, it has local states give by

\[
\rho_{R_l} := \left\{ \begin{array}{ll}
\text{tr}_C \left[ \mathcal{M}^\delta_{C \rightarrow CR_1} \left( \rho_0^C \right) \right], & \text{for } l = 1, 2, \ldots, N, \\
\rho_0^R, & \text{for } l = N + 1, \ldots
\end{array} \right.
\]

where \( \rho_0^R \) is the \( l^{th} \) initial local state, and \( \mathcal{M}^\delta_{C \rightarrow CR_1} \) is defined recursively by applying the channel of the clockwork \( l \) times: \( \mathcal{M}^\delta_{C \rightarrow CR_1} := \mathcal{M}^\delta_{C \rightarrow CR_1} \left( \text{tr}_R \left[ \mathcal{M}^{(m-1)\delta}_{C \rightarrow CR_1} \left( \rho_0^C \right) \right] \right), m \in \mathbb{N}_{\geq 0}. \)

As discussed in section 1, every application of the channel \( \mathcal{M}^\delta_{C \rightarrow CR_1} \) needed in the construction of eq. (3) corresponds to one time step of discrete coordinate time. The register is thus like a conveyor belt, in the sense that for the \( 1^{st} \) application of the channel \( \mathcal{M}^\delta_{C \rightarrow CR_1} \), the clockwork interacts with the 1st register \( \mathcal{H}_{R_1} \). The registers are then instantaneously moved along by one to the left so that the clockwork now interacts with the 2nd register \( \mathcal{H}_{R_2} \) for the second application of the channel \( \mathcal{M}^\delta_{C \rightarrow CR_1} \). This process is repeated indefinitely. The local states of the register \( \text{tr}_C \left[ \mathcal{M}^\delta_{C \rightarrow CR_1} \left( \rho_0^C \right) \right] \), generated via the \( l^{th} \) application of the channel, determine whether a tick occurred or not. The intuition is that the clockwork is releasing some temporal information at every application of the channel, so that after a sufficiently large number of applications of the channel it will contain too little temporal information to be useful and the register states \( \rho_{R_l} \) for sufficiently large \( l \), contain very little temporal information. The registers could contain tick/no-tick information by having the channel write “0” to the \( l^{th} \) register, \( \text{tr}_C \left[ \mathcal{M}^\delta_{C \rightarrow CR_1} \left( \rho_0^C \right) \right] = |0\rangle|0\rangle_{R_t}, \) in the case of no-tick, or a “1” in the case of a tick, \( \text{tr}_C \left[ \mathcal{M}^\delta_{C \rightarrow CR_1} \left( \rho_0^C \right) \right] = |1\rangle|1\rangle_{R_t}. \) Here \( |0\rangle \) and \( |1\rangle \) are two orthogonal states.

The authors then introduce a continuous coordinate ticking clock by demanding that the channel \( \text{tr}_R \left[ \mathcal{M}^\delta_{C \rightarrow CR_1} \right] \) satisfies an \( \epsilon \)-continuity condition,

\[
\| \text{tr}_R \left[ \mathcal{M}^\delta_{C \rightarrow CR_1} \right] - I_C \|_o \leq \epsilon(\delta),
\]

where \( \epsilon(\delta) \to 0^+ \) as \( \delta \to 0^+ \) and \( I \) is the identity channel, \( \| \cdot \|_o \) is the diamond norm. The authors then specify that one applies the channel \( \mathcal{M}^\delta_{C \rightarrow CR_1} \) a number of times which is proportional to \( \delta \) with \( \delta \) of order \( 1/\epsilon \), so as to achieve non-trivial ticking clock dynamics. The continuum limit case was further studied in [1] with a few additional physically motivated constraints introduced.

The system which moves the register one site to the left after every application of the clockwork channel \( \mathcal{M}^\delta_{C \rightarrow CR_1} \), is referred to as a gear system. The gear system is an integral part of the ticking clock. It may be a mechanical system such as a rack and pinion, or non mechanical such as a kinetic degree of freedom associated with the register, turning it into flying qubits on a line in rectilinear motion.

The gear system is thus represented by a channel \( G_{R_T \rightarrow R_T} \) when applied to a product register state

\[
\rho_{R_T} = \sigma_{R_1} \otimes \sigma_{R_2} \otimes \sigma_{R_3} \otimes \ldots
\]

\( m \) times achieves \( G_{R_T \rightarrow R_T}^m (\rho_{R_T}) = \sigma_{R_{1+m}} \otimes \sigma_{R_{2+m}} \otimes \sigma_{R_{3+m}} \otimes \ldots \). This gear system is only considered implicitly, without an explicit channel for it. See fig. 2 a).

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**Figure 1:**

1. a) Register from [6]: In the continuous coordinate time limit (\( \delta \to 0 \)), during any finite time interval (e.g. between two consecutive ticks), the clockwork needs to interact with an infinite number of qubit registers \( R_t \) — one at a time and sequentially. The blue triangle indicates the location of the clockwork relative to the register site it is writing to. The grey bar represents the infinite number of register sites.

2. b) Depiction of a digital wrist watch. Here between seconds (or “ticks”) the digital display does not change, and during any finite coordinate time interval, the display exhibits a finite number of distinguishable states. This is analogous to the register in the new model presented here: Between ticks, the register does not change.

However, if such a gear system channel existed,
rather than applying it to the clockwork channel $\mathcal{M}_{\text{CR}}^{\mathcal{C}}$, one could arguably apply it to a much simpler channel — bypassing the clockwork altogether — and achieve an idealised ticking clock. In this instance, this means a ticking clock for which $\text{tr}[^{T}p_{\mathcal{R}}(t)p_{\mathcal{R}}(t')] = \delta(t - t')$ for all $t, t' \in S_{\mathcal{R}}$, where $\delta(\cdot)$ is the Dirac-delta function in the continuous coordinate time limit [see fig. 2 b)], and a Kronecker-delta in the discrete coordinate time case. This highlights the 1st drawback with such a model: the gear system channel — while it is not supposed to contain temporal information — actually can function like an idealised ticking clock. While this argument is rather indirect, the following argument is direct in the sense that it applies to all ticking clocks of the authors using the gear system in conjunction with the clockwork as intended.

If one has a ticking clock, and no other resource, then they should arguably not be able to determine the precise time between ticks — to do that, one would need an additional time keeping device, such as a very precise stopwatch. However, by simply counting the number of zeros between the ones in the register, one can determine precisely the time between ticks. This is a direct consequence of the gear system moving the register along by one qubit sequentially in perfect tandem with the passing of coordinate time. Observe also that this holds independently of how regular the ticks are — it could be a ticking clock which is very accurate and the ticks occur at highly regular intervals, or very imprecise with ticks occurring randomly with respect to coordinate time.

One might hope to remedy this by simply removing the gear system all together and allowing the clockwork to always write to the same initial qubit register at all times. While this indeed means that the number of zeros (“no-ticks”) emitted by the clockwork between the ones (“ticks”) is now not recorded in the register as one would like, the state of the register in the instances when ticks occur will now be the same regardless of how may times the clock has ticked. Likewise, the register will also be in the same state (|00⟩|0⟩$_{\mathcal{R}_{1}}$) between any two consecutive ticks. As such, in the continuous time limit, the register will be in the state |1⟩|1⟩$_{\mathcal{R}_{1}}$, a measure zero amount of time, and in the state |00⟩|0⟩$_{\mathcal{R}_{1}}$ almost always. This behaviour is clearly problematic and at odds with that of familiar ticking clocks, such as a wall clock. Ideally, one satisfactory option would be to assign one qubit of memory to which the zeros (“no tick”) information is written to, and a new register qubit to be allocated to the output of the clockwork every time the ticking clock ticks. Therefore, at any instance, one would be able to determine the time by simply reading the number of ones in the register, but since the no tick information is always overwritten, one would not be able to determine how much coordinate time has passed between ticks. This hypothetical solution is unfortunately not possible, since it would require the gear system channel $G_{\mathcal{R}_{1}}$ to know when the clockwork is going to tick, yet since it acts solely on the register, it cannot do so. This highlights the difficulty of removing the gear system or altering its behaviour in a beneficial way. In section 4, we will show, via explicit construction, a satisfactory solution.

Note that the authors do suggest that a high precision gear system is not needed. Their argument is based on assuming that the gear system can fail with some probability $p$, where fail means that the gear channel $G_{\mathcal{R}_{1}}$ is replaced with the channel $\tilde{G}_{\mathcal{R}_{1}} = p\mathcal{I}_{\mathcal{R}_{1}} + (1 - p)G_{\mathcal{R}_{1}}$, with $\mathcal{I}_{\mathcal{R}_{1}}$ the identity channel on $\mathcal{H}_{\mathcal{R}_{1}}$. However, their reasoning is based on the fact that replacing the channel $G_{\mathcal{R}_{1}}$ with this one, incurs (at most) an irrelevant change in the state of the clockwork at arbitrary coordinate

Figure 2: a) Illustration of the ticking clock model in [6]: All the local register site qubits $\sigma_{R_{1}}, \sigma_{R_{2}} \ldots$ are initially set to |0⟩|0⟩. The channel $\mathcal{M}_{\text{CR}}^{\mathcal{C}} := G_{\mathcal{R}_{1}} \otimes \mathcal{M}_{\text{CR}}^{\mathcal{C}} |\sigma_{R_{1}}⟩⟨\sigma_{R_{1}}| \otimes \mathcal{I}_{\mathcal{R}_{1}} (\otimes \mathcal{I}_{\mathcal{R}_{2}} (\otimes \cdots))$, where $\mathcal{I}_{\mathcal{R}_{1}}$ is the identity channel, is then applied repeatedly at times $t = \delta, 2\delta, 3\delta, \ldots$. The state of the register at some fixed time $t > 0$ is obtained by setting $N = t/\delta$ and taking limit $\delta \to 0^{+}$. The number of 1’s corresponds to the number of ticks which have occurred in time interval [0, t].

b) Same scenario as in a) but now swapping the clockwork channel $\mathcal{M}_{\text{CR}}^{\mathcal{C}}$ with the Pauli X channel $\sigma_{X}$ which maps |0⟩|0⟩ to |1⟩|1⟩. The register now records the time with zero error, even though the Pauli X channel produces no temporal information, unlike the channel it replaced. We thus see that all the temporal information comes solely from the gear system $G_{\mathcal{R}_{1}}$. Alternatively, the same observation holds when using $|1⟩|1⟩ \otimes \mathcal{I}_{\mathcal{R}_{1}} (\otimes \mathcal{I}_{\mathcal{R}_{2}} (\otimes \cdots))$, rather than $\sigma_{X}$. Even in scenario a), the gear system is functioning as a perfect stopwatch: by counting the number of zeros between ticks, one can determine precisely the coordinate time interval between ticks.
times \( t \in S_\epsilon = (0, \delta, 2\delta, \ldots) \). However, there are several issues with this approach. On the one hand, no study of the induced change in the register is produced; yet the accuracy of the ticking clocks according to their measure (the Alternative Ticks Game), is solely a function of the register states on \( \mathcal{H}_{R_T} \) in the large coordinate time limit. Second, in the continuous ticking clock limit (\( \delta \to 0^+ \)), the gear system moves the register continuously, and any physically motivated gear system may produce errors which are irreconcilable with the error model described above. Furthermore, errors in the gear system are accumulative, and if the gear system writes the tick to an incorrect location in the register, it is possible to change the outcome of the alternative ticks game, thus changing its accuracy according to this measure.

3 Two basic principles for ticking clocks models: Finite running memory and Self-timing

We now present two basic principles to physically motivate descriptions of ticking clocks. The first is conceptually desirable, but arguably not necessary, while the second is more essential.

3.1 Self-timing

Understanding the underlying timing resources of a ticking clock is an important task. Otherwise, any physical implementation of it may require unaccounted for timing resources. Therefore identifying and quantifying such resources is important. A simple counter-example where the timing resources are unaccounted for, is a clock model with unitary dynamics governed by a time-dependent Hamiltonian over the clockwork and register.

One should distinguish the concept of self-timing from that of autonomy. An autonomous ticking clock can be thought of as one in which all resources for the clock to run can be explicitly accounted for. An example of an autonomous clock is [2]. Such ticking clocks are clearly also self-timing but the contrary is not necessarily true. The extent to which the ticking clock model presented in this manuscript is autonomous, will be discussed in sections 5 and 8.1.

We say that the (continuous coordinate time) ticking clock is self-timing if given the clocks initial state \( \rho_{CR_T} \) and a register space \( \mathcal{H}_{R_T} \), its one-parameter channel on the clockwork and register

\[
\mathcal{M}_{CR_T \to CR_T}^t(\rho_{CR_T}),
\]

is divisible: \( \mathcal{M}_{CR_T \to CR_T}^{t_1+t_2} = \mathcal{M}_{CR_T \to CR_T}^{t_1} \circ \mathcal{M}_{CR_T \to CR_T}^{t_2} \) for all \( t_1, t_2 \geq 0 \). The reasoning is that if this where not the case, then one could use systems alien to the register and clockwork to provide timing. In fact, after the identification of the register space \( \mathcal{H}_{R_T} \), the smallest additional space ones needs to include so that eq. (6) is satisfied (if such a space exists), is a means with which to identify a clockwork space.

One may wonder why we do not demand this divisibility requirement directly for the gear system. Yet, the point of the clockwork is to provide all the timing — the register should be a passive element. However, we will see in proposition 3 that while the clockwork will indeed be divisible is most circumstances, there will be others in which it may not, yet the register will not be providing a source of timing in these cases.

This definition of self-timing differs from that of self-containment from [6] in that the output in eq. (6) is on the entire register \( R_T \) rather than an individual register subsite which is isometric to \( R_1 \). Therefore the self-contained ticking clocks from [6] are not necessarily self-timing according to the above definition. One can of course make them self-timing by including explicitly the channel for the gear system together with that of the clockwork; however, the estimates of the ticking clock’s precision in [6] are solely based on Hilbert space dimension of clockwork alone, and thus do not take into account the dimensionality of the gear system, yet the gear system itself can be used as an idealised clock (as discussed in section 2).
3.2 Finite running memory

A requirement for any realistic model of a ticking clock is that it only utilises finite resources per unit of coordinate time. In this section we introduce a definition which captures this notion for the clock’s register by demanding that the clockwork can only invoke a finite change on it per unit of coordinate time.

We say that a ticking clock requires finite running memory if for every tuple \((\epsilon > 0, t > 0, \rho_{RT}(0) \in \mathcal{S}(\mathcal{H}_{RT}))\) there exists a projector \(\hat{P}\) onto a finite dimensional subspace \(\mathcal{H}_P \subseteq \mathcal{H}_{RT}\), such that

\[
\left\| \rho_{RT}(t) - \hat{P}\rho_{RT}(t)\hat{P} - (\hat{P}_\perp\rho_{RT}(t)\hat{P} + \text{h.c.}) \right\|_1 \leq \epsilon, \tag{7}
\]

where \(\rho_{RT}(t) := \text{tr}_{C}[\mathcal{M}_{\mathcal{C}\to\mathcal{C}}(\rho_{\mathcal{C}T})]\) is the state of the register at coordinate time \(t\), and \(\hat{P}_\perp := 1_{\mathcal{H}_T} - \hat{P}\). See fig. 3 for a graphical illustration. Note that it is important that the condition holds for all initial register states on \(\mathcal{H}_{RT}\), even if some initial register states are not relevant for the functioning of the ticking clock. This is because such states are physical and if the ticking clock is a physically realistic model, it should satisfy the finite running memory requirement even in these scenarios — regardless of whether the register is correctly encoding the information from the clockwork in such cases. This reasoning is analogous to why quantum information theorists demand quantum channels be completely positive rather than just positive — even if they do not intend to apply their channels on entangled states.

Observe that the register in the ticking clock model of [6] does not satisfy the finite running memory requirement since the output on the register is independent of the initial register state, and the clockwork has to interact with infinitely many copies of the register subspace \(\mathcal{H}_T\) in any finite interval of coordinate time; see fig. 1 a).

One may feel that infinite dimensional registers are physical since, indeed, spaces with continuous spectrum are physical. Consider for example the case in which the register is “a particle in a box” \(\mathcal{H}_{RT} = L^2[0,1]\). One could in principle store an infinite amount of information in the box by partitioning it into infinitesimally small orthogonal compartments. However, due to technological constraints, such information would not be retrievable nor writable, and a more realistic setup would be to store only a finite amount of information in finitely many partitions — each one, containing an infinite number of orthogonal states. Any resolvable reader would then consist of a projective measure \(\{\hat{P}_n\}_{n \in \mathbb{N}}\), where each \(\hat{P}_n\) projects onto one compartment of the register. This way, while each \(\hat{P}_n\) may project onto an infinite dimensional subspace, one can never discern between different orthogonal states on the subspace. Under such a condition, the finite running memory condition given by eq. (7) should still hold when \(\hat{P}\) is replaced with any linear combination of a finite number of projectors \(\hat{P}_n\). What is more, we would also require that the ticking clock cannot write an infinite amount of information to every register subspace. One way to ensure this, would be to require that the ticking clock channel written in Kraus form, \(\mathcal{M}_{\mathcal{C}T\to\mathcal{C}T}(\cdot) = \sum_{n} K_n(t)(\cdot)K_n^*(t)\), has Kraus operators which admit an expression \(K_n(t) = \sum_{l \in \mathbb{N}} \gamma_{l,n}(t)\hat{P}_l\) for some operators \(\gamma_{l,n}(t) \in \mathcal{B}(\mathcal{H}_{RT})\).

For simplicity, the model we introduce in the following section will satisfy the former finite running memory condition; eq. (7). This is to say, the projectors \(\hat{P}_l\) will project onto finite dimensional spaces. It could however be generalised to contain a register satisfying the latter condition also.

4 New ticking clock model

We now propose a ticking clock model through a set of physically motivated axioms which will be self-timing and of finite running memory. Unlike the model discussed in section 2, it will be a continuous coordinate time model from the outset. We discuss its accuracy in section 7.

The following describes the extension of a clockwork channel \(\mathcal{M}_{\mathcal{C}\to\mathcal{C}}\) to include the interaction with the register \(\mathcal{H}_{RT}\). All the conditions on how the ticking clock functions will be laid-out in this section. While some of these will be similar to those of section 2, the new model will not assume any of the conditions nor setup from said section. For example, it will not require a gear system.

The tick register here is also different to that described in section 2. It consists of \(N_T + 1\) orthonormal states \([0]_{RT}, [1]_{RT}, [2]_{RT}, \ldots, [N_T]_{RT}\) representing no tick, 1 tick, 2 ticks, \ldots, \(N_T \in \mathbb{N}_{>0}\) ticks respectively. While it is clear that any ticking clock with a finite dimensional tick register satisfies the finite running memory condition of section 3.2; in this case, the fulfilment of this condition is not inherently related to its finite dimensionality. Indeed, if one takes the infinite dimensional limit \(N_T \to \infty\) in the ticking clock model in section 5 which results from the axioms of the current section, the resulting ticking clock also satisfies the finite running memory condition of section 3.2. To start with, we describe a periodic register which resembles the familiar clock which repeats itself whenever the memory is full, e.g. every 12 or 24 hours. It naturally satisfies \(|n\rangle_{RT} = |n \text{ mod } N_T + 1\rangle_{RT}\); for \(n \in \mathbb{Z}\). Later in this section we will consider a variant of this.

In order for a device to be considered a ticking clock, it should satisfy some conditions on its clockwork and tick registers. After introducing the follow-
ing shorthand notation, we discuss 5 such conditions.

\[ M_{C \rightarrow CR_{T}}^{t,k}(\cdot) := M_{CR_{T} \rightarrow CR_{T}}^{t}(\cdot) \otimes |k|k_{R_{T}} \]  \quad \text{(8)}

\[ B(H_C) \rightarrow B(H_C \otimes H_{R_{T}}), \]

\[ k = 0, 1, \ldots, N_{T} \] to denote the ticking clock channel with a finite dimensional clockwork when acting on the \( k \)th register state.

1) **Time invariance symmetry condition:**

\[ \text{tr}_{R_{T}}\left[ M_{C \rightarrow CR_{T}}^{t,k}(\rho_{C}) |k + l|k + l_{R_{T}} \right] \]

is \( k \) independent for all \( t \geq 0, \rho_{C} \in \mathcal{S}(H_{C}) \) and \( l \in \mathbb{Z}; k = 0, 1, \ldots, N_{T} \) s.t. \( k + l = 0, 1, \ldots, N_{T} \). Physically, this condition means that the dynamics of the clockwork is invariant under translation of the input and output states of the register by the same amount. This is what one expects from a ticking clock: e.g. the probability of a ticking clock ticking 2 hours in the future according to coordinate time (and the state of the clockwork at this time), given that the clock’s register was initiated to 3pm, should be the same as if it were initiated to 6pm.

The instances of eq. (9) for which \( l \) is negative correspond to the state of the clockwork when the register is found to have “un-ticked”, i.e. that one finds the register to be in a state corresponding to an earlier time than it was initiated to, while coordinate time has increased.

2) **The time isolation condition:** For all \( t \geq 0 \), the ticking clock channel is Markovian:

\[ M_{CR_{T} \rightarrow CR_{T}}^{t_{1}+t_{2}} = M_{CR_{T} \rightarrow CR_{T}}^{t_{2}} \circ M_{CR_{T} \rightarrow CR_{T}}^{t_{1}} \]  \quad \text{(10)}

for all \( t_{1}, t_{2} \geq 0 \). This condition implies that no temporal information can come from systems alien to the clockwork and register. Together with the next condition, this guarantees the ticking clock is self-timing.

3) **The zeroth order condition:**

\[ M_{CR_{T} \rightarrow CR_{T}}^{0} = I_{CR_{T}} \]  \quad \text{(11a)}

\[ \lim_{t \rightarrow 0^{+}} \| M_{CR_{T} \rightarrow CR_{T}}^{t} - I_{CR_{T}} \| = 0. \]  \quad \text{(11b)}

This condition simply states that if no time has passed, then no change is permitted in the ticking clock.

The next condition concerns the probabilities of ticks. We denote the probability that the \( l \)th tick has occurred but not the \((l+1)\)th tick, at coordinate time \( t \), given the \(|k|k_{R_{T}}\) register state as input, by

\[ \tilde{p}_{l}^{(k)}(t) := \text{tr}\left[ M_{C \rightarrow CR_{T}}^{t,k}(\rho_{C}) |l|l_{R_{T}} \right]. \]  \quad \text{(12)}

4) **The leading order condition:**

\[ \lim_{t \rightarrow 0^{+}} \frac{\sum_{l=0}^{N_{T}} \tilde{p}_{l}^{(k)}(t)}{\tilde{p}_{0}^{(k)}(t)} = 0, \]

for all \( \rho_{C} \in \mathcal{S}(H_{C}) \) where \( f(k) = k + 1 \) mod. \( N_{T} + 1 \). This condition imposes the constraint that the clock cannot “skip a tick”. More precisely, between a coordinate time at which \( k \) ticks have occurred, and a later coordinate time at which \( l > k \) ticks have occurred, the probability that ticks \( k + 1, k + 2, \ldots, l - 1 \) have occurred is one.

Conditions 1) to 4) provide the necessary ingredients to define a ticking clock with a periodic register, but before doing so, it is advantageous to consider a distinct scenario which we call cut-off register.

In this scenario, the register will stop changing when it is full. For \( N_{T} = 11 \), an analogous wall clock would be one which you start at 12:00 midnight and it stops ticking at 11am the next day. Both cut-off and periodic register models have clear advantages and disadvantages: While the periodic ticking clock will never stop ticking, one can only determine the time up to multiples of its period (although this can be circumvented by counting the ticks in real time); but while this issue does not arise in the cut-off case, it is only useful for keeping track of time until it runs out of memory. Both types of register exhibit some common characteristics, see fig. 1 b).

While conditions similar to 2) and 4) can be defined for the cut-off register, due to the asymmetry in its boundary conditions, it is complicated to do so. A more direct and intuitive requirement is the following:

5) **The cut-off register condition:** for every ticking clock channel with a cut-off register \( M_{CR_{T} \rightarrow CR_{T}}^{T} \), there exists a ticking clock channel with a periodic register denoted \( M_{CR_{T} \rightarrow CR_{T}}^{T} \) and satisfying conditions 1) to 4), such that in the \( t \rightarrow 0^{+} \) limit:

\[ M_{C \rightarrow CR_{T}}^{T,k}(\rho_{C}) = M_{C \rightarrow CR_{T}}^{T,k}(\rho_{C}) + o(t) \]  \quad \text{(14a)}

for \( k = 0, 1, \ldots, N_{T} - 1 \) and

\[ \text{tr}_{C}\left[ M_{C \rightarrow CR_{T}}^{T,N_{T}}(\rho_{C}) \right] = |N_{T}|N_{T}_{R_{T}} + o(t), \]  \quad \text{(14b)}

for all \( \rho_{C} \in \mathcal{S}(H_{C}) \) and where \( o(\cdot) \) is little-o notation. This requirement captures some of the behaviour of the ticking clock with a periodic register, while enforcing the condition that the last register state \(|N_{T}|N_{T}_{R_{T}}\) as input, it can no longer invoke a change in the register — i.e. it “stops ticking”.

After these general remarks, we are now ready to state the technical definition of a ticking clock:
Definition 1 (Ticking clock). A ticking clock is a pair \((\rho_{\text{CrT}}^0, (M^t_{\text{C(CrT)}})_{t \geq 0}), \) with \(\rho_{\text{CrT}}^0 \in \mathcal{S}(\mathcal{H}_C \otimes \mathcal{H}_{R_T})\) the state of the clockwork at coordinate time \(t = 0,\) and where the interaction between the clockwork and register, governed by the channel \(M^t_{\text{C(CrT)}}\), satisfies conditions 1) to 4) in the case of a periodic register, and conditions 2) and 5) in the case of a cut-off register.

Observe that for a ticking clock with a cut-off register, the state of the clockwork given the final state of the register \(|N_T\rangle\langle N_T|\) as input, is of no relevance, since it does not affect the state of the register (and hence the tick statistics) anymore. We emphasize this point with the following definition.

Definition 2 (Clockwork equivalence). Two ticking clocks with a cut-off register are said to be clockwork equivalent if their underlying ticking clock channel with a periodic register in eq. (14a), namely \(M^t_{\text{CrT} \rightarrow \text{CrT}}\), is the same in both cases but the states of their clockwork when inputting the state register \(|N_T\rangle\langle N_T|\), namely

\[ t_{\text{CrT}} \left[ M^{t,N_T}_{\text{C \rightarrow CrT}}(\rho_C) \right], \tag{15} \]

differ for some \(t \geq 0\) and for all \(\rho_C \in \mathcal{S}(\mathcal{H}_C)\).

At present the registers are a collection of \(N_T + 1\) orthonormal states without more structure. One may furthermore demand that if local structure is given to the register states via some distance measure \(\text{Dis}(|l\rangle_{R_T}, |m\rangle_{R_T}) \geq 0\) that it satisfies \(\text{Dis}(|l\rangle_{R_T}, |m\rangle_{R_T}) < \infty\) for all \(l, m = 0, 1, \ldots, N_T\). This requirement is physically motivated by noting that when it is imposed, and the register has a local structure, condition 4) (eq. (13)) implies that the clockwork does not have to "travel" an infinite distance in finite time to write the next tick to the register — which would by unphysical. Furthermore, one can minimise the speed of sound in the register by arranging the local sites on the register so that \(\text{Dis}(|l\rangle_{R_T}, |m\rangle_{R_T}) = g(|l - m|)\) for some monotonically increasing function \(g\) in the case of a cut-off register.\(^4\) The simplest example of such a set-up is when the register is embedded into \(H^\otimes N_T+1\), where \(H\) is the space of a qubit spanned by \(|0\rangle, |1\rangle\) and we identify \(|n\rangle_{R_T} = |l\rangle^\otimes n \otimes |0\rangle^\otimes N_T+1-n\), and \(\text{Dis}(|l\rangle_{R_T}, |m\rangle_{R_T}) = |l - m|; l, n = 0, 1, \ldots, N_T\). In this case, at the instance when the \((k+1)\)th tick occurs, the clockwork "flips" the qubit "next to" the \(k\)th qubit.

In this manuscript we will work with clocks with classical registers \(R_T\). These are registers which are only permitted to be measured in the fixed basis \(|0\rangle_{R_T}, |1\rangle_{R_T}, \ldots, |N_T\rangle_{R_T}\). This condition, together with the condition that the clock's accuracy can only depend on what is measured on the register — rather than the specific state of the clockwork, or any joint measurements on both systems — implies without loss of generality, that the channel \(M^t_{\text{C(CrT)}}\) can be written in the following form (since the off diagonal elements in the register can never affect the outcome of the register measurements):

\[ M^t_{\text{C(CrT)}}(\rho_C^0) = \sum_{n=0}^{N_T} \rho_C^{(n)}(t) \otimes |n\rangle\langle n|_{R_T}, \tag{16} \]

for all \(k = 0, 1, \ldots, N_T; t \geq 0; \rho_C \in \mathcal{S}(\mathcal{H}_C)\), where \(\rho_C^{(n)}(t)\) are arbitrary subnormalised states on the clockwork. We will therefore take eq. (16) as the defining property of a classical register:

Definition 3 (Classical register). A ticking clock \((\rho^0_{\text{CrT}}, (M^t_{\text{C(CrT)}})_{t \geq 0})\) has a classical register if the channels \(M^t_{\text{C(CrT)}}\) are of the form eq. (16) for all \(t \geq 0, k = 0, 1, \ldots, N_T\).

This definition is not to be confused with demanding that the register is itself classical, but moreover should be interpreted as requiring that we are only allowed to extract classical information from it. Moreover, it allows for the register to be “continuously observed” without changing the properties of the ticking clock — analogously to how one can continuously look at a wall clock or listen for its ticks without disturbing the dynamics of its clockwork (and hence accuracy). This may come as a surprise for two reasons: For one, clearly the state of the ticking clock in eq. (16), before and after measuring the register in the basis \(|n\rangle_{R_T}\) is different. Secondly, the Zeno effect [13–15] dictates that if a quantum system is continuously measured, then it will stop evolving all together — which is clearly at odds with the desired properties of a ticking clock.

The solution to the 1st apparent problem is to recall that the state of the register is a probabilistic mixture and thus the change in the state due to the register’s measurement is due to a change in our knowledge about which state the register is in. This is analogous to the description of any purely classical ticking clock which is not perfectly accurate: while in every run of the ticking clock in which it is continuously observed, the state of the register will always be known exactly; in order to calculate the statistics associated with its accuracy, one will need the ensemble of all possible ticking clock runs weighted by the probability that each trajectory occurs. The 2nd apparent problem is resolved by showing that the Zeno effect does not apply to continuous measurements of the register when it is of the form eq. (16).

The following proposition formalises the previous two remarks by showing that one can continuously measure the register without affecting the statistics...
associated with the probabilistic distribution of ticks. Before stating it, we need to introduce some notation and definitions:

Let $P_t(\cdot) := |\langle l|\rho_T\rangle| |\langle l|\rho_T\rangle|/\text{tr}(\cdot) |\langle l|\rho_T\rangle|$. $B(H_C \otimes H_{R_T}) \rightarrow B(H_C \otimes H_{R_T})$ denote the channel which takes any ticking clock state and outputs the state of a ticking clock after measuring the register in the register basis $\{|0\rangle_{R_T}, |1\rangle_{R_T}, \ldots, |N_T\rangle_{R_T}\}$ and finding the register to be in the state $|l\rangle|l\rangle_{R_T}$.

**Definition 4** (Measured channels). Given a ticking clock ($\rho_{CR}^0, (M^t_{CR_T \rightarrow CR_T})_{t \geq 0}$), we call the following channel $B(H_C \otimes H_{R_T}) \rightarrow B(H_C \otimes H_{R_T})$ a measured channel:

$$C M^t_{CR_T \rightarrow CR_T}[s_N](\cdot) := \sum_{n=1}^{N} (P_n \circ M^t_{CR_T \rightarrow CR_T}) (\cdot),$$

where $s_N := (l_n, t_n)^{N}_{n=1}$ is the sequence of measurement outcome indices $l_n = 0, 1, \ldots, N_T$ and times $t_n \geq 0$; $\sum_{n=1}^{N} t_n = t$. In the case that $M^t_{CR_T \rightarrow CR_T}$ has a classical register we call the channel a classical register measured channel.

The channel eq. (17) corresponds to the state of the ticking clock at coordinate time $t$ when the free evolution of the ticking clock was interrupted at times $t_n$ by register measurements with outcomes $|l_n\rangle|l_n\rangle_{R_T}$. Let Prob$[s_N]$ be the probability that the sequence of outcomes with indices $l_1, l_2, \ldots, l_N$ at times $t_1, t_2, \ldots, t_N$ occurs. We denote the set of all sequences of outcomes at times $(t_1, t_2, \ldots, t_N) =: t$ by

$$S_N(t):= \{ (l_n, t_n)^{N}_{n=1} : l_n \in \{0, 1, \ldots, N_T\} \}. \quad (18)$$

**Proposition 1** (Measured register equivalence). For all coordinate times $t_n \geq 0$ s.t. $\sum_{n=1}^{N} t_n = t$ and for all $N \in \mathbb{N}_{>0}$, the dynamics of any ticking clock with a classical register is equal to that of the ensemble of classical register measured channels, where the ensemble is weighted by the probability of the classical register measured channel occurring:

$$M^t_{CR_T \rightarrow CR_T}(\rho_C \otimes |k\rangle|k\rangle_{R_T}) = \sum_{s_N \in S_N(t)} \text{Prob}[s_N] C M^t_{CR_T \rightarrow CR_T}[s_N](\rho_C \otimes |k\rangle|k\rangle_{R_T})$$

for all $(l_n)^{N}_{n=1}, N \in \mathbb{N}_{>0}, t \geq 0, k = 0, 1, \ldots, N_T,$ and $\rho_C \in S(H_C)$.

See section B.1 for proof. A direct consequence of proposition 1 is that if we choose $t_n = t/N$ followed by taking the limit $N \rightarrow \infty$ for fixed $t$ on the r.h.s. of eq. (19), we are in the regime of continuous measurements proposed in the Zeno effect. However, in this continuous measurement case, proposition 1 certifies that the ticking clock channel $M^t_{CR_T \rightarrow CR_T}(\rho_C \otimes |k\rangle|k\rangle_{R_T})$ still adequately describes the statistics. Indeed, if the register starts in the state $|k\rangle|k\rangle_{R_T}$, the probability $\text{Prob}[s_N]$ specialised to the case of finding the register in the state $|k\rangle|k\rangle_{R_T}$ for all time $t \in [0, T]$, for some $\tau > 0$ would have to be one if Zeno’s mechanism were to hold. We will later see that it is however only true for some irrelevant trivial clocks.

For later purposes, it is useful to introduce a notion of a “classical ticking clock”. This notion of classicality is effectively the same as the one introduced in [1] but stated for the ticking clock introduced in this manuscript (definition 1).

**Definition 5** (Classical ticking clock). We call a ticking clock ($\rho_{CR}^0, (M^t_{CR_T \rightarrow CR_T})_{t \geq 0}$) classical, if there exists a basis $\{|l\rangle_1\}$ spanning the clockwork Hilbert space $H_C$, for which the clockwork remains incoherent in this basis at all coordinate times:

$$\text{tr}_{R_T}[M^t_{CR_T \rightarrow CR_T}(\rho_{CR}^0)] = \sum_{t} p_t(t) |l\rangle|l\rangle_C, \quad \forall t \geq 0. \quad (20)$$

Likewise, we call a ticking clock a quantum ticking clock if it does not satisfy the classical ticking clock criterion. Thus unless otherwise specified, a ticking clock may be quantum or classical.

5 Autonomous dynamics

In this section we formulate a representation of the ticking clock channel $M^t_{CR_T \rightarrow CR_T}$ which holds if and only if the ticking clock satisfies the axiomatic definition 1, up to some stated equivalence. An alternative — more technical in nature — representation is left to appendix section A. Intrepid explorers may want to detour into section A before continuing here. The more cursory reader should leave section A alone.

**Proposition 2** (Explicit ticking clock representation). The pair $(\rho_{CR}^0, (M^t_{CR_T \rightarrow CR_T})_{t \geq 0})$ form a ticking clock (definition 1) with a classical tick register, up to clockwork equivalence (definition 2), if and only if there exists a Hermitian operator $H$ as well as two finite sequences of operators $(L_j)^{N_L}_{j=1}$ and $(J_j)^{N_J}_{j=1}$ on $B(H_C)$; which are all $k$ and $t$ independent, such that for all $t \geq 0$ and $N_T \in \mathbb{N}_{>0}$,

$$M^t_{CR_T \rightarrow CR_T}(\cdot) = e^{t\mathcal{L}_{CR_T}}(\cdot), \quad (21a)$$

$$\mathcal{L}_{CR_T}(\cdot) = -i[H, (\cdot)] + \sum_{j=1}^{N_L} \hat{L}_j(\cdot)\hat{L}^+_j - \frac{1}{2}\{\hat{L}^+_j\hat{L}_j, (\cdot)\} + \sum_{j=1}^{N_J} \hat{J}^{(1)}_j(\cdot)\hat{J}^{(1)*}_j - \frac{1}{2}\{\hat{J}^{(1)*}_j\hat{J}^{(1)}_j, (\cdot)\}. \quad (21b)$$
where the operators are $\hat{H} = H \otimes \mathbb{1}_{R_T}$, $\hat{L}_j = L_j \otimes \mathbb{1}_{R_T}$, $j^{(i)}_j = J_j \otimes O^{(i)}_{R_T}$, with

$$O^{(i)}_{R_T} := |1\rangle\langle 1|_{R_T} + |2\rangle\langle 2|_{R_T} + \cdots + |N_T\rangle\langle N_T|_{R_T}.$$  \hspace{1cm} (21c)

In the cut-off register case $l = 0$, while $l = 1$ for the periodic register case.

The proof follows straightforwardly from a more technical representation (lemma 1) discussed in appendix section A. Specifically, if one expands to leading order in $t$ the channel $\mathcal{M}^{k}_{C \rightarrow CR_T}$ for $k = 0, 1, \ldots, N_T$ using eq. (21a), one finds eq. (30b). Furthermore, since eq. (21a) is manifestly Markovian, the expression eq. (30a) also holds. Since it was established in lemma 1, that this form of the channel is both necessary and sufficient for the channel to be a ticking clock (definition 1), we conclude the proof of the proposition.

Observe that the Lindbladian in eq. (21b) only requires local coupling between the orthogonal register states, according to the distance measure $\text{Dist}(\cdot, \cdot)$ introduced in section 4 after definition 1. The dynamics of the register also clearly satisfies the finite running memory condition in section 3.2.

While the ticking clock model in [6], in the case of continuous coordinate time $t$, does have a dynamical semigroup representation from the clockwork and individual tick registers to itself, $L(\mathcal{H}_C \otimes \mathcal{H}_{R_j}) \to L(\mathcal{H}_C \otimes \mathcal{H}_{R_j})$ for all $j \in \mathbb{N}$, (see [1]) a dynamical semigroup representation on the clockwork and the total tick register has not been shown to exist. As explained previously in section 2, since the dynamics of the register is dependent on the details of the gear system in their model, such a formulation would inevitably need to include a description of the gear system used which, for any realistic gear system, would arguably lead to unaccounted for sources of error.

The following proposition shows that an effective Markovian dynamical semigroup for the clockwork can always be found.

**Proposition 3** (Clockwork representation). Consider a ticking clock with a classical periodic register (definitions 1 and 3) written in the representation of proposition 2. Its clockwork channel, defined via

$$\mathcal{M}^{k}_{C \rightarrow C}(\cdot) := \text{tr}_{R_T}[\mathcal{M}^{k}_{CR_T \rightarrow CR_T}(\cdot) \otimes |k\rangle\langle k|_{R_T}]$$  \hspace{1cm} (22)

is $k$-independent for all $t \geq 0$ and of the form

$$\mathcal{M}^{k}_{C \rightarrow C}(\cdot) = e^{t\mathcal{L}_C}(\cdot),$$  \hspace{1cm} (23)

with $\mathcal{L}_C$ equal to the r.h.s. of eq. (21b) under the replacements $\hat{H} \mapsto H$, $\hat{L}_j \mapsto L_j$ and $J_j^{(i)} \mapsto J_j$. What is more, for every ticking clock with a classical cut-off register written in the representation of proposition 2, there exists a ticking clock which is clockwork equivalent (definition 2), such that its clockwork is $k$-independent and given by eq. (23).

The proof is constructive and can be found in section B.3. In the case of the cut-off register, the representation used in the proof for which eq. (23) holds, has a ticking clock channel whose clockwork still produces ticks when the input register state is $|N_T\rangle\langle N_T|_{R_T}$, but does not write them to the register. This can be contrasted with the clockwork equivalent ticking clock with cut-off register representation in proposition 2. In this case, the clockwork stops producing ticks when the input register state is $|N_T\rangle\langle N_T|_{R_T}$.

Dynamical semigroups of the form eq. (21a) have been shown to have a microscopic description in which the system (which in the present case would constitute the clockwork and total register) interacts with an infinite dimensional environment on $\mathcal{H}_E$ via a time independent Hamiltonian under the appropriate limits. In particular, the Hamiltonian $\mathcal{H}_{tot}$ which leads to dynamics eqs. (21a) and (21b) is of the form

$$\mathcal{H}_{tot} = H \otimes \mathbb{1}_{R_T E} - \mathcal{H}_{CR_T} \otimes \mathbb{1}_E \otimes \mathcal{H}_{R_T} \otimes \mathcal{H}_E + V, \hspace{1cm} (24)$$

where $\mathcal{H}_{CR_T} \in B(\mathcal{H}_C \otimes \mathcal{H}_{R_T})$ is tuned to counteract a shift in energy on the clockwork and register due to interactions with the environment with local Hamiltonian $\mathcal{H}_E$, while $V$ mediates the interaction between the register, clockwork, and environment. It takes on the form $V = \sum_n \mathbb{1}_{R_T} \otimes A_n^{(L)} \otimes B_n^{(R)} + O_{R_T}^{(i)} \otimes A_n^{(J)} \otimes B_n^{(J)}$ with $O_{R_T}^{(i)}$ given by eq. (21c). The $(A_n^{(L)})_n$ and $(A_n^{(J)})_n$ terms give rise to the operators $(L_j)$, and $(J_j)$ respectively while terms $(B_n^{(L)})_n,(B_n^{(J)})_n$ are suitably chosen local terms acting on the environment. There are two known types of limiting procedures which one can apply to eq. (24) to achieve dynamics of the form eqs. (21a) and (21b). One in which the time-scales of the environment are much shorter than those of the system — the so-called “weak coupling limit” — and the other where the time scales are reversed — called the “singular coupling limit”. See [16] for physical insight into these limits and [17] for how to re-scale the interactions to interchange between them. In [18] it is proven that all Lindblad operators in eq. (21b) are achievable via appropriate choice of the terms eq. (24) in the singular coupling limit. Finally, observe that the interaction term $V$ in eq. (24) only requires local coupling to the register.

## 6 Ticking Clock Examples

In this section we will see how clocks from the literature are either a special case of the ticking clocks introduced here (definition 1), or that they can be easily adapted to be of this form. In all three examples,
yields $\rho_{H}$ the local Hamiltonian for the qubits and ladder while $\Sigma$ takes on the form.

Here we consider the same clockwork but when its coupling to the register results from the axioms introduced in section 4, rather then the model [6]. It was introduced in the context of the model from [6]. As with the example of section 6.2, it can easily be adapted to the ticking clock model definition 1. We find $N_L = d$, with

$$L_j = 0, \quad J_j = \sqrt{2V_j} |\psi_C)(t_j|, \quad (27)$$

where $\{|t_j\rangle\}_{j=1}^d$ is an orthonormal basis for $\mathcal{H}_C$. The state $\rho_{\Sigma} = |\psi_C\rangle\langle\psi_C|$ is both the initial state of the clockwork and the state which it is reset to after each tick. It is called the Quasi-ideal clock state and follows a complex Gaussian distribution in the $\{|t_j\rangle\}_{j=1}^d$ basis. The coefficients $\{V_j > 0\}_{j=1}^d$ follow a peaked distribution; see [1] for details. The Hamiltonian $H$ is a ladder Hamiltonian with equally spaced energy gaps and diagonal in the Fourier transform basis generated from $\{|t_j\rangle\}_{j=1}^d$.

The free dynamics of the clockwork according to $H$ allows the complex Gaussian amplitude distribution to “move coherently” in the $\{|t_j\rangle\}_{j=1}^d$ basis until the peak of the distribution overlaps with the peak of the distribution $\{|V_j > 0\}_{j=1}^d$, at which point a tick occurs and the clockwork is reset and starts again. Note that the statistics of the 1st tick are invariant under the exchange of the Lindblad operators in eq. (27) with the simpler form $L_j = J_j = 0$ for $j = 1, 2, \ldots, d - 1$ and $L_d = 0$, $J_d = \sum_{j=1}^d \sqrt{2V_j} |j\rangle\langle t_j|$. Here $\{|j\rangle\in \mathcal{H}_C\}_{j}$ is any orthonormal basis since in this case the clockwork does not need to be reset after it ticks — as can be seen formally from eq. (38).

6.3 Ladder Ticking Clock

This clock is a classical ticking clock (definition 5) which was defined in [19] and proven to be the most accurate classical continuous coordinate time clock in [1] in the context of the model [6]. As with the example of section 6.2, it can easily be adapted to the ticking clock model definition 1. We find $N_L = d$, with

$$L_j = |e_{j+1}\rangle\langle e_j|, \quad J_j = 0, \quad (28a)$$
$$L_d = 0, \quad J_d = |c_1\rangle\langle c_d|, \quad (28b)$$

for some orthonormal basis $\{|e_j\rangle\}_{j=1}^d$. The clocks initial state is $\rho_{\Sigma} = |c_1\rangle\langle c_1|$ and since it is a classical clock, the Hamiltonian term vanishes, $H = 0$.

This clock can also be approached thermodynamically as in [2], in the limit of semi-classical dynamics and of infinite entropy cost.

7 Measures of accuracy

We will call an accuracy measure of a ticking clock, to any quantity which can be written solely as a function of measurement outcomes of the register on $\mathcal{H}_{R_T}$, at different coordinate times. It can only depend on the state of the clockwork indirectly, via its coupling

$$\rho_{\Sigma} = |\psi_C\rangle\langle\psi_C|$$

where $\{|t_j\rangle\}_{j=1}^d$ is an orthonormal basis for $\mathcal{H}_C$. The state $\rho_{\Sigma} = |\psi_C\rangle\langle\psi_C|$ is the initial state of the clockwork and the state which it is reset to after each tick. It is called the Quasi-ideal clock state and follows a complex Gaussian distribution in the $\{|t_j\rangle\}_{j=1}^d$ basis. The coefficients $\{V_j > 0\}_{j=1}^d$ follow a peaked distribution; see [1] for details. The Hamiltonian $H$ is a ladder Hamiltonian with equally spaced energy gaps and diagonal in the Fourier transform basis generated from $\{|t_j\rangle\}_{j=1}^d$.

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to the register, but not on the state of the clockwork directly. This is important since with the accuracy measure one wants to capture how good the clockwork is at emitting temporal information to the “outside”. The quantities $P_{\text{tick}}, P_{\text{no tick}}, P^{(k)}_{\text{tick}}$ in eqs. (32), (33) and (36) are manifestly of this form. It is worth noting that while in the description of the theoretical ticking clock model one has knowledge of the state of the register and the value of coordinate time, in an actual physical implementation of any ticking clock, one does not. As such, any measure of accuracy may be hard to calculate experimentally and may require many repeated experiments involving multiple copies of any given ticking clock in order to garner enough statistics. This constraint of the model, is a virtue not a weakness however, since by definition the only information we should have access to (i.e. which should be stored in a register), is the information about ticks — not coordinate time directly. The storage of coordinate time in the register was an inherent drawback of earlier ticking clock models.

We will now consider the cut-off register model, since in this case the quantities $P_{\text{tick}}, P_{\text{no tick}}, P^{(k)}_{\text{tick}}$ take on exactly the same expression in terms of how they are related to dynamics on the clockwork as presented in [1] for the model [6]. The clock accuracy of the $k$th tick $R_k$, introduced in [1, 2], is the ratio of the mean and standard deviation of $P^{(k)}_{\text{tick}}(t)$ with respect to coordinate time $t$. As such, there is a one-to-one relation between the ticking clocks in [1] and the ones introduced in this manuscript for $k = 1, 2, \ldots, N_T$. Subsequently, all the theorems about the accuracy of ticking clocks in [1] also apply to those introduced here with a cut-off register. For instance, the most accurate classical ticking clock (see section 6.3) satisfies $R_k = kd$ for $k = 1, 2, \ldots, N_T$; $d \in \mathbb{N}_0$; where $d$ is the Hilbert space dimension of the clockwork. The thermodynamic clock in section 6.1 can also achieve a similar accuracy, see [2]. On the other hand, there exists a quantum ticking clock (see section 6.2) whose accuracy is lower bounded by

$$R_k = kR_1, \quad R_1 \geq d^{2-\varepsilon} + o(d^{2-\varepsilon}),$$

for all $k = 1, 2, \ldots, N_T$ and for all fixed $\varepsilon > 0$ in the large $d$ limit [1]. Recently it has been shown that this bound is essentially tight for ticking clocks which only tick once [20]. It remains an open question whether a quantum ticking clock which ticks more than once can have ticks which have a higher accuracy.

The alternative ticks game measure of accuracy [6, 19], is applicable to the ticking clock models developed here, with the difference that the referee will need to play the game in real-time, rather than comparing register states at the end since the registers in the new model do not record the coordinate time corresponding to when the tick occurred.

8 Conclusions and Outlook

8.1 Conclusions

We started by discussing the ticking clock model of [6] which was one of the first theoretical models of a ticking clock. This revolutionary work inspired follow-up papers yet also some legitimate concerns from the community regarding the physicality of its foundations which we formalise and discuss. We then introduce an axiomatic definition of a new ticking clock based on physical principles and derive explicit solutions to its equations of motion. It is self-evident that the aforementioned drawbacks do not apply to the new ticking clock model introduced here. We furthermore show that the new equations of motion admit a fully autonomous realisation.

With every ticking clock, one can associate a set of delay functions which determine the accuracy measures of the ticking clock. We show that there is a one-to-one relation between the set of delay functions produced by the new ticking clock model introduced here and that of [6]. Consequently, bounds on the accuracy of the clocks in [6], derived in [1], apply also to the new ticking clock model presented here. Therefore, the main conclusion of [1], namely that quantum ticking clocks are more accurate than classical ones, applies also to the new ticking clock model presented here.

Another ticking clock model, based on thermodynamic principles was introduced in [2]. This model has many positive points, such as being fully autonomous and physically well motivated. It does however have some drawbacks, such as not being derived from first principles and having a reported accuracy which is substantially lower than that of the quantum ticking clock in [1]. Consequently, the results of this manuscript imply that both of the desirable properties of the ticking clock models [2, 6] are achievable in one physically transparent model: full autonomy and high accuracy. This is considered the most important conclusion of this manuscript. What’s more we have seen that the ticking clock in [2] is in fact a special case of the ticking clock model introduced here.

Here by autonomous it is understood that for every ticking clock according to the new definition 1, there exists a large macroscopic environment, a time independent and local Hamiltonian over the clockwork, register, and environment, which represents said ticking clock. Note that this environment need not necessarily be thermal. Other possibilities such as a vacuum state, may turn out to be necessary in some cases. Whether indeed a (or several) thermal baths at various temperatures(s) are sufficient for the most

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6By “community” we refer to comments from anonymous referees and senior scientists during conference talks.
accurate clocks is an open question.

One would like to be able to continuously monitor the tick register of a ticking clock in real-time — analogous to how one can listen for the ticks of a wall clock in real-time. At first sight, this may seem impossible since the quantum Zeno effect dictates that continuous observation of a quantum system causes it to freeze its motion; which is clearly not a desirable property of a clock. We show that one can continuously observe the tick register without affecting its statistics and hence its accuracy.

8.2 Outlook

It is worth discussing some of the multiple future directions this work opens up. On the one hand there are questions such as what is the most accurate quantum clock, how much entropy is produced every time the ticking clock ticks, or how would one build such a ticking clock in practice. On the other hand, one can consider extensions to the formalism itself. One very practical such extension would be to include noise from the environment and the possibility of leveraging tools from quantum error correction to protect ticks against such adversarial noise. Such studies could also be applied to other types of extensions to the model. We provide 4 such examples:

1) Clocks in a network: One can readily extend the model introduced in this manuscript to take into account a source of external timing. One way to consider external timing in the case of a ticking clock was introduced in [9] and formulated in terms of the continuous time limit of the ticking clock of [6]. This extension could also be formulated for the ticking clocks introduced in this manuscript. We now discuss two other types of extensions which to date have not been considered in the literature thus far.

2) Relativistic ticking clock model: The ticking clock models in the literature (including this one), are not relativistic. Making them so, would be an interesting endeavour. Since we do not have a fully consistent theory of gravity yet, such work would be highly speculative. One approach is to attempt to make relativistic versions of the axioms for ticking clocks presented in section 4, by stating how the observable statistics and invariant quantities in these axioms transform relativistically. Another method would be to employ the same approach used in [8] to construct a relativistic quantum stopwatch. In such a semi-classical approach, one would include in the ticking clock set-up an additional kinematic degree of freedom associated with the ticking clock’s momentum and position. One would then expand to leading order in relativistic corrections the general relativistic equations for time dilation, following a similar procedure as to as in [21].

3) Relaxation of the axioms: One could consider variants of the ticking clock models presented here by changing or disregarding some of the conditions 1) to 5) in section 4. An obvious choice would be to consider “Time variant asymmetric ticking clocks”, namely those which do not satisfy condition 1) [eq. (9)]. An example of such a ticking clock channel with a classical register would be eqs. (21a) and (21b) in proposition 2 under the replacement $j^{(l)} \rightarrow \sum_{k=0}^{N_T} J_{j,k} \otimes |k + 1 \bmod N_T + 1\rangle\langle k|_{\mathbb{R}^T} \left(1 - \delta_{k,N_T+1}\right)$, where $J_{j,k}$ are arbitrary operators on the clockwork.

4) Unitary ticking clock model: The model proposed in this manuscript takes on the form of a one-parameter dynamical semigroup over the clockwork and tick register. The clockwork provides the necessary timing while the register stores the tick information. The potential accuracy of the ticking clock depends on the properties of the clockwork, such as its dimensionality or energy. We have discussed how this can be dilated via the aid of an infinite-dimensional environment to unitary dynamics with a time independent Hamiltonian, using standard limiting procedures.

One could consider a ticking clock whose dynamics are unitarily driven by a time independent Hamiltonian with a finite environment instead. It would however have no classical counterpart (definition 5) nor allow for a classical register (definition 3). The lack of a classical register would mean that it could suffer from the Zeno effect, and thus there would be a critical observation frequency which if surpassed, the observations would start to change the accuracy of the clock significantly and if frequent enough, might even stop the clock ticking altogether. Understanding and quantifying the properties of such models could be an interesting future line of research.

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Appendices

A Implicit Ticking Clock Representation

In this appendix we will formulate a representation of the ticking clock channel $\mathcal{M}^t_{\text{CR} \rightarrow \text{CR}_t}$ which holds if and only if the ticking clock satisfies the axiomatic definition 1, up to some stated equivalence. Unlike the representation of the ticking clock channel from section 5, this representation is technical in nature. Its presentation is followed by some technical implications. The cursory reader is advised to skip this section.

We start with the following lemma which asserts that the ticking clock from section 4 can equivalently be specified in terms of generators acting on the clockwork space $\mathcal{H}_C$.

**Lemma 1** (Implicit ticking clock representation). The pair $(\rho^0_{\text{CR}_t}, (\mathcal{M}^t_{\text{CR} \rightarrow \text{CR}_t})_{t \geq 0})$ form a ticking clock (definition 1) with a classical tick register, up to clockwork equivalence (definition 2), if and only if there exists a Hermitian operator $H$ as well as two finite sequences of operators $(L_j)_{j=1}^{N_L}$ and $(J_j)_{j=1}^{N_L}$ on $\mathcal{B}(\mathcal{H}_C)$; which are all $k$ and $t$ independent, such that for all $t \geq 0$ and $k = 0, 1, \ldots, N_T$:

$$
\mathcal{M}^{t,k}_{\text{CR} \rightarrow \text{CR}_t}(\rho^0_C) = \lim_{N \rightarrow +\infty} L^{(N-1)}_{\text{CR}_t \rightarrow \text{CR}_t} \circ \mathcal{M}^{t/N,k}_{\text{CR} \rightarrow \text{CR}_t}(\rho^0_C),
$$

where

$$
\mathcal{M}^{t/N,k}_{\text{CR} \rightarrow \text{CR}_t}(\cdot) = (\cdot) \otimes |k\rangle \langle k|_{\text{CR}_t} + \left( \frac{t}{N} \right) C(1,k)(\cdot) \otimes |k+1\rangle \langle k+1|_{\text{CR}_t} + F^{t/N,k}_{\text{CR} \rightarrow \text{CR}_t}(\cdot),
$$

with

$$
C(1,k)(\cdot) := -i[H,\cdot] - \sum_{j=1}^{N_L} \frac{1}{2} \{L_j^\dagger L_j + \theta(k)J_j^\dagger J_j,\cdot\} + L_j(\cdot)L_j^\dagger,
$$

$$
C(2,k)(\cdot) := \theta(k) \sum_{j=1}^{N_L} J_j(\cdot)J_j^\dagger,
$$

and $F^{t,k}_{\text{CR} \rightarrow \text{CR}_t}(\rho^0_C) = o(\delta)$ entry-wise. $\theta(k) = 1$ for all $k$ in the periodic register case and $\theta(k) = 1 - \delta_{k,N_T}$ in the cut-off register case, where $\delta_\cdot$ is the Kronecker delta.

The proof of this lemma, which uses some elements of the proof of Lindblad’s representation theorem [22, 23], is provided in Appendix section B.2.

As regards to the dynamics on $\mathcal{H}_C \otimes \mathcal{H}_{\text{RT}_t}$, eq. (30a) is completely determined by eq. (30b) in terms of the initial state and operators $H$, $(L_j)$, $(J_j)$. To see this, 1st note that applying the channel $\mathcal{M}^{t/N}_{\text{CR} \rightarrow \text{CR}_t}$ to both sides of eq. (30b) one obtains the composition law

$$
\mathcal{M}^{t/N}_{\text{CR} \rightarrow \text{CR}_t} \circ \mathcal{M}^{t/N}_{\text{CR} \rightarrow \text{CR}_t}(\cdot) = \mathcal{M}^{t/N}_{\text{CR} \rightarrow \text{CR}_t}(\cdot) + (t/N) \mathcal{M}^{t/N}_{\text{CR} \rightarrow \text{CR}_t}(C(1,0)(\cdot)) + (t/N) \mathcal{M}^{t/N+1}_{\text{CR} \rightarrow \text{CR}_t}(C(2,0)(\cdot)) + F^{t,k}_{\text{CR} \rightarrow \text{CR}_t}(\cdot),
$$

with $l = 0, 1, \ldots, N_T$. This establishes eq. (30a) inductively. Every one of the $N$ applications of the channel $\mathcal{M}^{t/N}_{\text{CR} \rightarrow \text{CR}_t}$ in eq. (30a), has a direct physical meaning. Up to order $o(t/N)$ contributions, the $i$th application of $\mathcal{M}^{t/N}_{\text{CR} \rightarrow \text{CR}_t}$ takes the state from the previous time step $(i-1)$ applications of $\mathcal{M}^{t/N}_{\text{CR} \rightarrow \text{CR}_t}$, and updates the state and probability so that if $t$ ticks had occurred in the previous time step, then either no tick occurs or the ticking clock ticks once, in the $i$th time step. Other processes, such as the clock “loosing a tick” or ticking more than once in the time step $t/N$ can only occur with probability $o(t/N)$. However, when taking the $N \rightarrow +\infty$ limit in eq. (30a) the order $o(t/N)$ terms vanish. As such they are irrelevant and can be set to zero if one wishes. Furthermore, the requirement that $F^{t,k}_{\text{CR} \rightarrow \text{CR}_t}(\rho_C) = o(\delta)$ entry-wise, holds if and only if $\|F^{t,k}_{\text{CR} \rightarrow \text{CR}_t}(\rho_C)\|_p = o(\delta)$ for any $p > 0$ where $\|\cdot\|_p$ is the operator norm induced by the vector $p$-norm. This is shown in lemma 2 in the appendix.

Observe that we have not placed any restrictions on $N_L \in \mathbb{N}$. It turns out that without loss of generality, one can set $N_L = d^2 - 1$, where $d$ is the Hilbert space dimension of the clockwork. This is because for any two finite sequences $(L_j)_{j=1}^{N_L}, (J_j)_{j=1}^{N_L}$ giving rise to eq. (30a), there exists a new set $(L_j^1)_{j=1}^{N_L}, (J_j^1)_{j=1}^{N_L}$ which gives
rise to the same dynamics in eq. (30a) upon their substitution. This follows from simple variants of well known proofs in quantum channel representation theory, as shown in lemma 3 in the appendix.

If in the definition of a ticking clock (definition 1), we remove the condition that the channel from $\mathcal{B}(\mathcal{H}_C \otimes \mathcal{H}_{R_T})$ to $\mathcal{B}(\mathcal{H}_C \otimes \mathcal{H}_{R_T})$ is Markovian [eq. (10)], then lemma 1 still holds if one traces out the register in both sides of eq. (30a) to produce the channel $\mathcal{M}_{C \rightarrow C}$. In such cases, it appears that the full channel $\mathcal{M}_{CR_T \rightarrow CR_T}$ producing the dynamics on the register is undetermined other than for an infinitesimal time step. While such channels do not allow one to determine all properties of the clock, it does allow one to determine the probability of the ticking clock “ticking” during infinitesimal time step $[t, t + dt]$. Denoting this probability $P_{\text{tick}}(\rho_C(t)) dt$, one has that $P_{\text{tick}}(\rho_C(t)) = \sum_{k=0}^{N_T} p_k P_{\text{tick}}^{(k)}(\rho_C(t))$ where $P_{\text{tick}}^{(k)}(\rho_C(t))$ is the probability density corresponding to ticking during coordinate time interval $[t, t + dt]$, given that the probability of the clockwork and register being in state $\rho_C(t) \otimes |k\rangle|k\rangle_{R_T}$ at time $t$, was $p_k$. For all probability distributions $(p_k)_k$ it takes the value

$$P_{\text{tick}}(\rho_C(t)) = \sum_{k=0}^{N_T} p_k \lim_{\delta \to 0^+} \frac{\text{tr} \left[ |k+1\rangle\langle k+1|_{R_T} \left( \mathcal{M}_{C \rightarrow CR_T}^{(k)}(\rho_C(t)) - \rho_C(t) \otimes |k\rangle|k\rangle_{R_T} \right) \right]}{\delta},$$

where $\rho_C(t) = \mathcal{M}_{C \rightarrow C}(\rho_C)$ is the clockwork state at coordinate time $t$. Likewise, the probability of not ticking in said interval, $P_{\text{no tick}}(\rho_C(t)) dt$, can also be easily calculated without knowledge of the channel $\mathcal{M}_{CR_T \rightarrow CR_T}$ either:

$$P_{\text{no tick}}(\rho_C(t)) = 1 - P_{\text{tick}}(\rho_C(t)) = \sum_{k=0}^{N_T} \frac{1}{2^{N_T}} \left( \sum_{j=1}^{N} \text{tr} \left[ -\{L_j, J_j\} + J_j \rho_C(t) J_j^\dagger \right] + L_j \rho_C(t) L_j^\dagger \right).$$

However, such probabilities are not so useful for determining measures of ticking clock accuracy, as discussed in section 7.

For example, a more useful quantity is the probability of producing the $k^{th}$ tick during coordinate time interval $[t, t + dt]$. Or in other words, the probability that during time interval $[0, t]$, the ticking clock ticked $k - 1$ times and then produces a tick during time interval $[t, t + dt]$. Denoting this probability $P_{\text{ticks}}^{(k)}(t) dt$, one finds

$$P_{\text{ticks}}^{(k)}(t) = \lim_{\delta \to 0^+} \frac{\text{tr} \left[ |k\rangle|k\rangle_{R_T} \left( \mathcal{M}_{CR_T \rightarrow CR_T}^{(k)}(\rho_{CR_T}(t)) - \rho_{CR_T}(t) \right) \right]}{\delta},$$

where $\rho_{CR_T}(t)$ is the un-normalised outcome of a measurement when the register is found to be in the $|k-1\rangle|k-1\rangle_{R_T}$ state,

$$\rho_{CR_T}(t) = |k-1\rangle|k-1\rangle_{R_T} \left( \mathcal{M}_{C \rightarrow CR_T}^{(k-1)}(\rho_{CR_T}(t)) - \rho_{CR_T}(t) \right),$$

and we have assumed that the clockwork is in state $\rho_C$ and the register in the “no-tick state” $|0\rangle|0\rangle_{R_T}$ at coordinate time $t = 0$. Since the outcome of the measurement is a product state and the probability of ticking when the register is in a particular state, is independent of that state, $P_{\text{ticks}}^{(k)}(t)$ can be written as

$$P_{\text{ticks}}^{(k)}(t) = P_{\text{tick}}(\rho_{C}^{(k-1)}(t)), \quad P_{\text{ticks}}^{(k-1)}(t) = \text{tr}_{R_T} \left[ |k-1\rangle|k-1\rangle_{R_T} \left( \mathcal{M}_{C \rightarrow CR_T}^{(k-1)}(\rho_{C}) \right) \right].$$

For example, consider the case of the 1st tick in the case of the cut-off register model. A simple calculation finds that $\rho_{C}^{(0)}(t)$ is generated via the clockwork’s dynamics with the tick generating channel removed:

$$\rho_{C}^{(0)}(t) = \text{tr}_{R_T} \left[ |0\rangle|0\rangle_{R_T} \left( \mathcal{M}_{C \rightarrow CR_T}^{(0)}(\rho_{C}) \right) \right]$$

$$= \lim_{N \to +\infty} \text{tr}_{R_T} \left[ |0\rangle|0\rangle_{R_T} \mathcal{M}_{C \rightarrow CR_T}^{(0)}(\rho_{C}) |0\rangle|0\rangle_{R_T}^{\otimes N} \right],$$

$$= e^{t\mathcal{L}_{C}^{(0)}}(\rho_{C}), \quad \mathcal{L}_{C}^{(0)}(\cdot) = \mathcal{L}_{(1,0)}(\cdot).$$
In the case of the periodic register model, the above expression for \( \rho_C^{(0)}(t) \) does not hold, since the equality in line (38) is false. Physically speaking, this is because in the periodic register case, when the register runs out of memory, a tick is produced in the initial memory state \( |0\rangle |0\rangle_{R_T} \) and thus the dynamics of the clock at times after the 1st tick has occurred are still relevant for the 1st tick’s statistics. This is not the case in the infinite register limit \( N_T \to +\infty \) nor for the cut-off register case.

B. Proofs

B.1 Proof of proposition 1

**Proposition 1** (Measured register equivalence). *For all coordinate times \( t_n \geq 0 \) s.t. \( \sum_{n=1}^{N} t_n = t \) and for all \( N \in \mathbb{N}_{>0} \), the dynamics of any ticking clock with a classical register is equal to that of the ensemble of classical register measured channels, where the ensemble is weighted by the probability of the classical register measured channel occurring:

\[
\mathcal{M}^{t,k}_{CRT \rightarrow CRT}(\rho_C \otimes |k\rangle\langle k|_{R_T}) = \sum_{s_N \in S_N(t)} \text{Prob}[s_N] \mathcal{C} \mathcal{M}^{t}_{CRT \rightarrow CRT}[s_N](\rho_C \otimes |k\rangle\langle k|_{R_T})
\]

for all \( (t_n)^N \), \( N \in \mathbb{N}_{>0} \), \( t \geq 0 \), \( k = 0, 1, \ldots, N_T \), and \( pc \in S(\mathcal{H}_C) \).

**Proof.** To start with, observe that for a ticking clock \( (\rho_C, \mathcal{M}^{t}_{CRT \rightarrow CRT}) \) with a classical register, one has for all \( t \geq 0; k = 0, 1, \ldots, N_T; \rho_C^0 \in S(\mathcal{H}_C) \),

\[
\mathcal{M}^{t,k}_{CRT \rightarrow CRT}(\rho_C^0) = \sum_{l=0}^{N_T} \text{tr}_{R_T} \left[ |l\rangle\langle l|_{R_T} \mathcal{M}^{t,k}_{CRT \rightarrow CRT}(\rho_C^0) \right] \otimes |l\rangle\langle l|_{R_T}
\]

(41)

where \( \text{Prob}[l, t] := \text{tr} \left[ |l\rangle\langle l|_{R_T} \mathcal{M}^{t,k}_{CRT \rightarrow CRT}(\rho_C^0) \right] \). Thus, using the Markovian property of channel \( \mathcal{M}^{t}_{CRT \rightarrow CRT} \) [condition 2]), followed by iteratively substituting the above equation,

\[
\mathcal{M}^{t,k}_{CRT \rightarrow CRT}(\rho_C^0) = \bigcap_{n=1}^{N} \mathcal{M}^{t,k}_{CRT \rightarrow CRT}(\rho_C^0) \otimes |k\rangle\langle k|_{R_T}
\]

(43)

\[
= \bigcap_{n=2}^{N} \mathcal{M}^{t,k}_{CRT \rightarrow CRT} \odot \sum_{l=0}^{N_T} \text{Prob}[l, t_1] \mathcal{P}_{l_1} \odot \mathcal{M}^{t}_{CRT \rightarrow CRT}(\rho_C^0) \otimes |k\rangle\langle k|_{R_T}
\]

(44)

\[
= \bigcap_{n=3}^{N} \mathcal{M}^{t,k}_{CRT \rightarrow CRT} \odot \sum_{l_2=0}^{N_T} \text{Prob}[l_2, t_2] \mathcal{P}_{l_2} \odot \mathcal{M}^{t,k}_{CRT \rightarrow CRT} \odot \sum_{l_1=0}^{N_T} \text{Prob}[l_1, t_1] \mathcal{P}_{l_1} \odot \mathcal{M}^{t}_{CRT \rightarrow CRT}(\rho_C^0) \otimes |k\rangle\langle k|_{R_T}
\]

(45)

\[
= \sum_{l_1=0}^{N_T} \sum_{l_2=0}^{N_T} \ldots \sum_{l_N=0}^{N_T} \text{Prob}[l_1, t_1] \cdots \text{Prob}[l_N, t_N] \bigcap_{n=1}^{N} \mathcal{P}_{l_n} \odot \mathcal{M}^{t,k}_{CRT \rightarrow CRT}(\rho_C^0) \otimes |k\rangle\langle k|_{R_T}
\]

(46)

Observe that \( \text{Prob}[l_1, t_1] \cdots \text{Prob}[l_N, t_N] = \text{Prob}[s_N] \). Taking into account definition 4 we complete the proof. \( \blacksquare \)

B.2 Proof of lemma 1

**Lemma 1** (Implicit ticking clock representation). *The pair \( (\rho_C^{(0)}; (\mathcal{M}^{t}_{CRT \rightarrow CRT})_{t \geq 0}) \) form a ticking clock (definition 1) with a classical tick register, up to clockwork equivalence (definition 2), if and only if there exists a Hermitian operator \( H \) as well as two finite sequences of operators \( (L_j)_{j=1}^{J} \) and \( (J_j)_{j=1}^{J} \) on \( \mathcal{B}(\mathcal{H}_C) \); which are all \( k \) and \( t \) independent, such that for all \( t \geq 0 \) and \( k = 0, 1, \ldots, N_T \);

\[
\mathcal{M}^{t}_{C \rightarrow CRT}(\rho_C^0) = \lim_{N \to +\infty} \left( \mathcal{M}^{t,N}_{C \rightarrow CRT} \right)^{(N-1)} \odot \mathcal{M}^{t/N}_{C \rightarrow CRT}(\rho_C^0),
\]

(30a)
Consider an expansion of the form
\[ M_{C\to CR_T}^{\ell,N,k}(\cdot) = (\cdot) \otimes |k\rangle_{R_T} + \left( \frac{\ell}{N} \right) C_{(1,k)}(\cdot) \otimes |k\rangle_{R_T} + \left( \frac{\ell}{N} \right) C_{(2,k)}(\cdot) \otimes |k+1\rangle_{R_T} + F_{C\to CR_T}^{\ell,N,k}(\cdot), \quad (30b) \]
with
\[ C_{(1,k)}(\cdot) := -i[H,\cdot] - \sum_{j=1}^{N_t} \frac{1}{2} \{ L_j^+ L_j + \theta(k) J_j^+ J_j, \cdot \} + L_j(\cdot)L_j^+, \quad (30c) \]
\[ C_{(2,k)}(\cdot) := \theta(k) \sum_{j=1}^{N_t} J_j(\cdot)J_j^+, \quad (30d) \]
and \( F_{C\to CR_T}^{\delta,k}(\rho_C) = \alpha(\delta) \) entry-wise. \( \theta(k) = 1 \) for all \( k \) in the periodic register case and \( \theta(k) = 1 - \delta_k,N_T \) in the cut-off register case, where \( \delta_k \) is the Kronecker delta.

**Proof.** First we will prove the proposition for the case of a ticking clock with a periodic register. The case of a cut-off register will then be straightforward. To start with, we consider the most general representation of a channel in the Kraus form, namely
\[ M_{C\to CR_T}^{\ell,k}(\rho_C) = \sum_{j=0}^{N_t} Q_j^{(k)}(t) \rho_C Q_j^{(k)^\dagger}(t), \quad (48) \]
where
\[ Q_j^{(k)}(t) : B(H_C) \to B(H_C \otimes H_{R_T}) \quad (49) \]
and \( \sum_j Q_j^{(k)}(t)^\dagger Q_j^{(k)}(t) = 1_C. \) We now expand the \( Q_j^{(k)}(t) \) operators in the register basis. By making the identification \( N_j^{(k)}(l,t) := |l\rangle_{R_T} Q_j^{(k)}(t) : B(H_C) \to B(H_C) \) this yields
\[ M_{C\to CR_T}^{\ell,k}(\rho_C) = \sum_{l,l'=0}^{N_T} \sum_{j=0}^{N_t} N_j^{(k)}(l,t) \rho_C N_j^{(k)^\dagger}(l',t) \otimes |l\rangle\langle l|_{R_T}, \quad (50) \]
since the register is classical, the off diagonal terms must vanish due to compatibility with eq. (16). We therefore have
\[ M_{C\to CR_T}^{\ell,k}(\rho_C) = \sum_{l=0}^{N_T} \sum_{j=0}^{N_t} N_j^{(k)}(l,t) \rho_C N_j^{(k)^\dagger}(l,t) \otimes |l\rangle\langle l|_{R_T}, \quad (51) \]
for all \( k = 0,1,\ldots,N_T \); \( t \geq 0. \) Observe that
\[ \text{tr}_{R_T} \left[ M_{C\to CR_T}^{\ell,k}(\rho_C) |l\rangle\langle l|_{R_T} \right] = \sum_{j=0}^{N_t} N_j^{(k)}(l,t) \rho_C N_j^{(k)^\dagger}(l,t). \quad (52) \]
Moreover, in the periodic register case, \( |l\rangle_{R_T} = |l \mod. N_T + 1 \rangle_{R_T}. \) For convenience, we therefore extend the definition of the operators \( N_j^{(k)}(l,t) \) in the periodic case from \( l = 0,1,\ldots,N_T \) to \( l \in \mathbb{Z} \) by defining
\[ N_j^{(k)}(l,t) = N_j^{(k)}(l \mod. N_T + 1, t). \quad (53) \]
Consider an expansion of the form \( N_j^{(k)}(l,t) = \sum_{n,m} a_{n,m}^{(j,k)}(l,t) |n\rangle|m\rangle_C \) and states \( \rho_C(p) \) which are pure and diagonal in this basis, \( \rho_C(p) = |p\rangle\langle p|_C. \) Therefore
\[ \text{tr}_{R_T} \left[ M_{C\to CR_T}^{\ell,k}(\rho_C(p)) |l\rangle\langle l|_{R_T} \right] \tilde{p}_l^{(k)}(t) = \sum_{j=0}^{N_t} \sum_{n,m \neq n} a_{n,m}^{(j,k)}(l,t) a_{m,n}^{(j,k)^*}(l,t) |n\rangle\langle m| + \sum_{n} a_{n,p}^{(j,k)}(l,t) a_{n,p}^{(j,k)^*}(l,t) |n\rangle\langle n|. \quad (54) \]
Now noting the definition of $\tilde{p}_j^{(k)}(t)$ (eq. (12)) and taking the trace on both sides, we find

$$1 = \frac{\sum_{n,j} |a_{n,p}^{(j,k)}(l,t)|^2}{\tilde{p}_j^{(k)}(t)}. \quad (55)$$

Therefore,

$$\left| a_{n,p}^{(j,k)}(l,t) \right| \leq \sqrt{\tilde{p}_j^{(k)}(t)} \quad \forall j = 0, 1, \ldots, N_L; \; n, p = 0, 1, \ldots, d - 1; \; l, k = 0, 1, \ldots, N_T; \; t \geq 0, \quad (56)$$

where $d \in \mathbb{N}_{>0}$ is the dimension of the Hilbert space of the clockwork. We will now use inequality eq. (56) together with condition 4) [eq. (13)] to show an important limit. To start with, denote the entries of a matrix $M$ by $[M]_{ab}$ and observe that eq. (56) implies

$$\lim_{t \to 0^+} \left| \frac{\sum_{j=0}^{N_L} \sum_{l, \delta t \in \{k,f(k)\}} N_j^{(k)}(l,t) \rho_C N_j^{(k)\dagger}(l,t)}{\tilde{p}_j^{(k)}(t)} \right|_{[ab]} = \lim_{t \to 0^+} \left| \frac{\sum_{j=0}^{N_L} \sum_{l, \delta t \in \{k,f(k)\}} a_{n,m}^{(j,k)}(l,t) [\rho_C]_{mn} a_{b,n}^{(j,k)*}(l,t)}{\tilde{p}_j^{(k)}(t)} \right| \leq (N_L + 1) \left( \sum_{m,n} |[\rho_C]_{mn}| \right) \lim_{t \to 0^+} \frac{\tilde{p}_j^{(k)}(t)}{\tilde{p}_j^{(k)}(t)} = 0, \quad (57)$$

for all $k = 0, 1, \ldots, N_T; \; a, b = 0, 1, \ldots, d - 1; \; \rho_C \in \mathcal{S}(\mathcal{H}_C)$. In the last line in eq. (57) we have used eq. (13). Therefore,

$$\sum_{l=0}^{N_T} \sum_{i \in \{k,f(k)\}} N_j^{(k)}(l,t) \rho_C N_j^{(k)\dagger}(l,t) = \alpha \left( \tilde{p}_j^{(k)}(l,t) \right) \quad (58)$$

entry-wise in the $t \to 0^+$ limit for all $k = 0, 1, \ldots, N_T$. Since every term $N_j^{(k)}(l,t) \rho_C N_j^{(k)\dagger}(l,t)$ in the above summation is positive semi-definite, we thus have

$$\mathcal{M}_{C\to CR_T}^{t,k}(\rho_C) = \sum_{l \in \{k,f(k)\}} \sum_{j=0}^{N_L} N_j^{(k)}(l,\delta t) \rho_C N_j^{(k)\dagger}(l,\delta t) \otimes |l+l|_{\mathbb{R}_T} + \alpha \left( \tilde{p}_j^{(k)}(\delta t) \right), \quad (59)$$

$$= \sum_{l \in \{0,f(k)\} \setminus \{k\}} \sum_{j=0}^{N_L} N_j^{(k)}(l+k,\delta t) \rho_C N_j^{(k)\dagger}(l+k,\delta t) \otimes |l+k|_{\mathbb{R}_T} + \alpha \left( \tilde{p}_j^{(k)}(\delta t) \right), \quad (60)$$

for all $k = 0, 1, \ldots, N_T; \; t \geq 0; \; \rho_C \in \mathcal{S}(\mathcal{H}_C)$. On the other hand, from eq. (51) it follows that the state of the clockwork, given the register is measured to be in the state $|l+l|_{\mathbb{R}_T}$ for $l \in \mathbb{Z}$, is

$$\text{tr}_{\mathbb{R}_T} \left[ \mathcal{M}_{C\to CR_T}^{t,k}(\rho_C) |l+l|_{\mathbb{R}_T} \right] = \sum_{j=0}^{N_L} N_j^{(k)}(l+k,t) \rho_C N_j^{(k)\dagger}(l+k,t), \quad (61)$$

By virtue of condition 1) (eq. (9)), we have that every matrix component of the r.h.s. of eq. (61) is $k$ independent for all $t \geq 0; \; \rho_C \in \mathcal{S}(\mathcal{H}_C)$ and $l \in \mathbb{Z}; \; k = 0, 1, \ldots, N_T$ s.t. $k+l = 0, 1, \ldots, N_T$ in the periodic register case. Therefore, in particular, we have

$$\sum_{j=0}^{N_L} N_j^{(k)}(l+k,t) \rho_C N_j^{(k)\dagger}(l+k,t) = \sum_{j=0}^{N_L} N_j^{(0)}(l,t) \rho_C N_j^{(0)\dagger}(l,t) \quad (62)$$

for all $t \geq 0; \; \rho_C \in \mathcal{S}(\mathcal{H}_C)$ and $l \in \mathbb{Z}; \; k = 0, 1, \ldots, N_T$ s.t. $k+l = 0, 1, \ldots, N_T$ in the periodic case. Plugging eq. (62) into eq. (60), yields

$$\mathcal{M}_{C\to CR_T}^{t,k}(\rho_C) = \sum_{l \in \{0,f(k)\} \setminus \{k\}} \sum_{j=0}^{N_L} N_j^{(0)}(l,\delta t) \rho_C N_j^{(0)\dagger}(l,\delta t) \otimes |l+k|_{\mathbb{R}_T} + \alpha \left( \tilde{p}_j^{(k)}(\delta t) \right), \quad (63)$$
for all $k = 0, 1, \ldots, N_T - 1$; $\rho_C \in \mathcal{S}(\mathcal{H}_C)$ in the periodic register case. For the case $k = N_T$ in the above equation, recall that due to eq. (53) we have that

$$\sum_{j=0}^{N_L} N_j^{(0)} (-N_T, t) \rho_C N_j^{(0)\dagger} (-N_T, t) = \sum_{j=0}^{N_L} N_j^{(0)} (1, t) \rho_C N_j^{(0)\dagger} (1, t).$$

(64)

For the periodic register case, taking into account eqs. (62) and (64) we have

$$\tilde{p}_{j(k)}(t) = \left[ \sum_{j=0}^{N_L} N_j^{(k)} (k + 1, t) \rho_C N_j^{(k)\dagger} (k + 1, t) \right] = \left[ \sum_{j=0}^{N_L} N_j^{(0)} (1, t) \rho_C N_j^{(0)\dagger} (1, t) \right] = \tilde{p}_{1}^{(0)}(t),$$

(65)

for all $k = 0, 1, \ldots, N_T$; $t \geq 0$; $\rho_C \in \mathcal{S}(\mathcal{H}_C)$. Therefore, for all $k = 0, 1, \ldots, N_T$; $t \geq 0$; $\rho_C \in \mathcal{S}(\mathcal{H}_C)$; eq. (63) reduces to

$$\mathcal{M}_{C\rightarrow CR_T}^{\delta t} (\rho_C) = \sum_{l \in \{0, 1\}} \sum_{j=0}^{N_L} N_j^{(0)} (l, \delta t) \rho_C N_j^{(0)\dagger} (l, \delta t) \otimes |l+k|/|l+k|_{R_T} + o \left( \sum_{j=0}^{N_L} N_j^{(0)} (l, \delta t) \right).$$

(66)

It follows from eqs. (10), (11a) and (11b) of condition 3), that the clockwork channel admits a power-law expansion in $t$ (see uniformly continuous semigroup in [12]). Specifically, there exits an operator $A_{CR_T}$ on $\mathcal{B}(\mathcal{H}_C \otimes \mathcal{H}_{R_T})$ such that

$$\mathcal{M}_{CR_T \rightarrow CR_T}^{\delta t} = e^{t A_{CR_T}} := \sum_{n=0}^{\infty} \frac{t^n A_{CR_T}^n}{n!}.$$ 

(67)

Therefore, w.l.o.g. we can use the following ansatz:

Let $n_1 \leq N_L$ of the elements of the sequence $(N_j^{(0)}(0, \delta t))_j$ be of linear order in $\delta t$ while the others are of order $\sqrt{\delta t}$. Specifically, let

$$\left( N_j^{(0)} (0, \delta t) = I_j + (-iH_j + K_j) \delta t \right)_{j=0}^{n_1}, \quad \left( N_j^{(0)} (0, \delta t) = L_j \sqrt{\delta t} \right)_{j=n_1+1}^{N_L},$$

(68)

where $H_j, K_j$ are Hermitian and the operators $L_j, H_j, K_j, I_j$ are all $t$ independent. Similarly, we can employ the same form of the expansion for the sequence $(N_j^{(1)}(1, \delta t))_j$ associated with the register state $|k+1)(k+1|_{R_T}$:

$$\left( N_j^{(1)} (1, \delta t) = I_j + (-i\tilde{H}_j + \tilde{K}_j) \delta t \right)_{j=0}^{n_1}, \quad \left( N_j^{(1)} (1, \delta t) = J_j \sqrt{\delta t} \right)_{j=n_1+1}^{N_L},$$

(69)

where $\tilde{H}_j, \tilde{K}_j$ are Hermitian and the operators $J_j, \tilde{H}_j, \tilde{K}_j, I_j'$, are all $t$ independent operators.

We first fix the zeroth order terms by noting that $\mathcal{M}_{C \rightarrow C}^{0} := \text{tr}_{R_T} \left[ \mathcal{M}_{C \rightarrow CR_T}^{0} \right]$ has to be the identity channel due to condition 3) [eq. (11a)] and eq. (66). It hence follows:

$$\mathcal{M}_{C \rightarrow C}^{0} (\rho_C) = \sum_{j=0}^{n_1} I_j \rho_C I_j^\dagger + I_j' \rho_C I_j'^\dagger = \rho_C$$

(70)

for all $\rho_C \in \mathcal{S}(\mathcal{H}_C)$. Two sets of Kraus operators $(\hat{K}_l)_l=0^{n_1}$, $(\hat{K}_l')_l=0^{n_1}$ give rise to the same quantum channel (i.e. $\sum_l \hat{K}_l \hat{K}_l^\dagger = \sum_l \hat{K}_l' \hat{K}_l'^\dagger$) iff there exists an $n_1$ by $n_1$ unitary $V$ with entries $V_{l[m]} \in \mathbb{C}$ such that $\hat{K}_l' = \sum_m V_{l[m]} \hat{K}_m$ for all $l = 0, 1, \ldots, n_1$. See e.g. [24, 25] for a proof. Note that this even covers the case in which one or both of the channels have less than $n_1 + 1$ Kraus operator elements, since we can always choose to define additional Kraus operators which are equal to zero. Therefore, since $(I_j = c_j I, I_j' = c_j' I)$ with $\sum_{j=0}^{n_1} |c_j|^2 + |c_j'|^2 = 1$ are solutions to eq. (70), this unitary equivalence theorem implies that it is the only family of solutions. On the other hand, one also finds

$$\lim_{t \rightarrow 0^+} \tilde{p}_{0}^{(0)}(t) = \frac{n_1}{2} \text{tr} \left[ I_j \rho_C I_j^\dagger \right] = 1,$$

(71)

$$\lim_{t \rightarrow 0^+} \tilde{p}_{1}^{(0)}(t) = \frac{n_1}{2} \text{tr} \left[ I_j' \rho_C I_j'^\dagger \right] = 0,$$

(72)
for all \( \rho_C \in \mathcal{S}(\mathcal{H}_C) \). The last equality in eq. (71) follows from invoking condition 3) [eq. (11b)] and the definition of \( \tilde{p}^{(0)}_\tau \), while the last equality in eq. (72) follows from conservation of probability. Since \( I_j^\dagger \rho_C I_j \) is positive semi-definite, it follows that \( I_j^\dagger = 0 \) for all \( j = 0, 1, \ldots, n_1 \) and thus we find \( I_j = c_j I \) with \( \sum_{j=0}^{n_1} |c_j|^2 = 1 \) for all \( j = 0, 1, \ldots, n_1 \). It thus follows from plugging in the above ansatz for \( N_j^{(0)}(0, t) \) and \( N_j^{(0)}(1, t) \) into the definition of \( \tilde{p}_j^{(0)} \) that

\[
\alpha(\tilde{p}_j^{(0)}(\delta t)) = \alpha(\delta t).
\]

(73)

Expanding eq. (66) we thus find

\[
\mathcal{M}^{\delta t, k}_{C \to \mathcal{C}}(\rho_C) = \rho_C \otimes |k\rangle\langle k|_{\mathcal{R}_T} + \left[ \rho_C \sum_{j=0}^{n_1} (c_j^R H_j + c_j^I K_j) \right] \otimes |k\rangle\langle k|_{\mathcal{R}_T} \delta t + \left\{ \rho_C, \sum_{j=0}^{n_1} (c_j^R H_j + c_j^I K_j) \right\} \otimes |k\rangle\langle k|_{\mathcal{R}_T} \delta t
\]

\[
+ \sum_{j=n_1+1}^{N_L} \left( L_j \rho_C L_j^\dagger \otimes |k\rangle\langle k|_{\mathcal{R}_T} + J_j \rho_C J_j^\dagger \otimes |k+1\rangle\langle k+1|_{\mathcal{R}_T} \right) \delta t + o(\delta t),
\]

(74)

where \( c_j^R, c_j^I \) are the real and imaginary parts of \( c_j \) respectively. Now observe that by defining Kraus operators \( H := \sum_{j=0}^{n_1} (c_j^R H_j + c_j^I K_j) \), \( K := \sum_{j=0}^{n_1} (c_j^R H_j + c_j^I K_j) \) one can exchange eqs. (68) and (69) with

\[
N_0^{(0)}(0, \delta t) = I + (-iH + K)\delta t, \quad \left( N_j^{(0)}(0, \delta t) = L_j \sqrt{\delta t} \right)_{j=1}^{N_L} = 1,
\]

\[
N_0^{(0)}(1, \delta t) = 0, \quad \left( N_j^{(0)}(1, \delta t) = J_j \sqrt{\delta t} \right)_{j=1}^{N_L} = 1,
\]

(75)

and obtain the same solution as eq. (74) to order \( o(\delta t) \) up to a relabelling of the summation indices obtaining

\[
\mathcal{M}^{\delta t, k}_{C \to \mathcal{C} R_T}(\rho_C) = \rho_C \otimes |k\rangle\langle k|_{\mathcal{R}_T} - \delta t \left( i[H, \rho_C] - \{K, \rho_C\} - \sum_{j=1}^{N_L} L_j \rho_C L_j^\dagger \right) \otimes |k\rangle\langle k|_{\mathcal{R}_T}
\]

\[
+ \delta t \sum_{j=1}^{N_L} J_j \rho_C J_j^\dagger \otimes |k+1\rangle\langle k+1|_{\mathcal{R}_T} + o(\delta t)
\]

(76)

for all \( k = 0, 1, \ldots, N_T; \ t \geq 0; \ \rho_C \in \mathcal{S}(\mathcal{H}_C) \). Furthermore, taking into account the normalisation of the Kraus operators [eq. (49)], from eq. (75) we obtain a solution for \( K \), namely

\[
K = -\frac{1}{2} \sum_{j=1}^{N_L} \left( L_j L_j^\dagger + J_j J_j^\dagger \right).
\]

(77)

In the case of a periodic register, eq. (30b) in the lemma statement follows by pugging in eq. (77) into eq. (76). Equation (31) in the lemma statement follows by recalling eq. (8) and using eq. (76). By applying the divisibility of the channel [condition 2), eq. (10)] recursively \( N \in \mathbb{N}_{>0} \) times we find

\[
\left( \mathcal{M}^{t/N}_{C \to \mathcal{C} R_T} \right)^o = \mathcal{M}^{t/N}_{C \to \mathcal{C} R_T},
\]

(78)

for all \( t \geq 0 \). Equation (30a) then follows by recalling the notation eq. (8) and taking the limit \( N \to +\infty \). This concludes the “only if” part of the lemma for a periodic register.

Finally, to verify the converse part of the lemma in the case of a periodic register, one simply has to check that for all Hermitian operators \( H \) and families of operators \( (L_j)_{j=1}^{n_1}, (J_j)_{j=1}^{n_1} \) acting on \( \mathcal{B}(\mathcal{H}_C) \), the equations in the lemma statement satisfy the conditions 1) to 4) in section 4. We do this in the following. For conciseness, we will refer to the sequence of such operators on \( \mathcal{B}(\mathcal{H}_C) \) by \( \mathcal{D} = (H, (L_j)_j, (J_j)_j) \).

To verify that 1) [eq. (9)] holds for all \( \mathcal{D} \), first note that tr\(_{\mathcal{R}_T} \left[ (\mathcal{M}^{\tau}_{\mathcal{C} R_T} \to \mathcal{C} R_T)^o \mathcal{M}^{k}_{\mathcal{C} \to \mathcal{C} R_T}(\rho_C) |k+m\rangle\langle k+m| \right] \) is \( k \) independent (up to an order \( o(\tau) \) term) for all \( l \in \mathbb{N}_{>0}, m \in \mathbb{Z}, \ k = 0, 1, \ldots, N_T \) s.t. \( k+m = 0, 1, \ldots, N_T \), for all \( \tau \geq 0; \ \rho_C \in \mathcal{S}(\mathcal{H}_C) \); and \( \mathcal{D} \). Hence condition 1) [eq. (9)] follows by choosing \( \tau = t/N, \ l = N - 1 \) and taking the \( N \to \infty \) limit such that the \( o(\tau) \) terms vanish.

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21
To verify that eq. (10) in condition 2) holds for all $\mathcal{D}$, one needs to verify that
\[
\lim_{N \rightarrow \infty} \left( \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{(t_1+2t_2)/N} \right)^{\circ N} (\rho_C \otimes |k\rangle\langle k|) \tag{79}
\]
and
\[
\lim_{N_1 \rightarrow \infty} \lim_{N_2 \rightarrow \infty} \left( \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_1/N_1} \circ \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_2/N_2} \right)^{\circ N_1} \circ \left( \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_2/N_2} \right)^{\circ N_2} (\rho_C \otimes |k\rangle\langle k|) \tag{80}
\]
are equal for all $t_1, t_2 \geq 0$; $k = 0, 1, \ldots, N_T$; $\rho_C \in \mathcal{S}(\mathcal{H}_C)$; and $\mathcal{D}$. To do so, we first note by explicit calculation using eq. (30b) that
\[
\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_1/N} \circ \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_2/N} (\rho_C \otimes |k\rangle\langle k|) = \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{(t_1+t_2)/N} (\rho_C \otimes |k\rangle\langle k|) + o(1/N). \tag{81}
\]

Therefore
\[
\lim_{N \rightarrow \infty} \left( \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{(t_1+t_2)/N} \right)^{\circ N} (\rho_C \otimes |k\rangle\langle k|) = \lim_{N \rightarrow \infty} \left( \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_1/N} \circ \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t_2/N} \right)^{\circ N} (\rho_C \otimes |k\rangle\langle k|) \tag{82}
\]

which is equal to eq. (80) due to continuity. The confirmation that eqs. (11a) and (11b) in condition 3) hold for all $\mathcal{D}$, follows straightforwardly from eq. (30b):
\[
\lim_{t \rightarrow 0^+} \lim_{N \rightarrow \infty} \left( \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}^{t/N} \right)^{\circ N} (\rho_C \otimes |k\rangle\langle k|) = \lim_{t \rightarrow 0^+} \mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T} (\rho_C \otimes |k\rangle\langle k|) = \rho_C \otimes |k\rangle\langle k|, \tag{83}
\]

for all $\rho_C \in \mathcal{S}(\mathcal{H}_C)$, $\mathcal{D}$ and where in the second equality we used the Markovianity of the channel (which we have just proven) and the penultimate line uses eq. (30b).

Finally, the verification that condition 4) holds for all $\mathcal{D}$ is straightforward. Plugging in eq. (30b) into definition eq. (12) and proceeding similarly to as in the above equation, one finds
\[
\frac{\sum_{t=0}^{N_T} \tilde{p}_j^{(k)}(t)}{\sum_{j \in \mathcal{E}(k)} \tilde{p}_j^{(k)}(t)} = \lim_{t \rightarrow 0^+} \frac{o(t)}{ct} = 0, \tag{90}
\]
for all $k = 0, 1, \ldots, N_T$; $\rho_C \in \mathcal{S}(\mathcal{H}_C)$; and $\mathcal{D}$, where $c > 0$ is a constant. This concludes the proof of the converse part of the lemma for the case of a periodic register. We now proceed to the case of a cut-off register.

We have just proven that $\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}$ is the channel of a ticking clock with a classical register of the periodic type, iff it has the form stated in the lemma. Therefore a ticking clock with a classical register of the cut-off type can only satisfy the 1st part of condition 5) (eq. (14a)) iff the ticking clock $\mathcal{M}_{\text{CR}_T \rightarrow \text{CR}_T}$ in eq. (14a) is of the form of that in the lemma statement for $k = 0, 1, \ldots, N_T - 1$. Since eq. (30b) is the same for both cut-off and periodic register types, for $k = 0, 1, \ldots, N_T - 1$, this holds true. Furthermore, by direct calculation of eq. (30b) in the case of a cut-off register and $k = N_T$, we see that it satisfies eq. (14b) for $k = N_T$. While eq. (30b) in the case of a cut-off register and $k = N_T$ is clearly not necessary for it to satisfy eq. (14b) for $k = N_T$, it is necessary to satisfy eq. (14b) for $k = N_T$ up to clockwork equivalence (definition 2). This can be verified by noting that eq. (14b) for $k = N_T$ implies that the state of the register and clockwork must be a product state up to order $o(\delta t)$.

We have thus far verified that condition 5) holds, up to clockwork equivalence, for a ticking clock with a classical register of the cut-off type iff eq. (30b) in the lemma statement holds. By definition of a ticking clock (definition 1), we only need to verify that condition 2) [eq. (10)] holds for a ticking clock with a cut-off register. The case eq. (10) is verified analogously to the periodic register case above.
**B.3 Proof of proposition 3**

**Proposition 3 (Clockwork representation).** Consider a ticking clock with a classical periodic register (definitions 1 and 3) written in the representation of proposition 2. Its clockwork channel, defined via

\[ \mathcal{M}_{C \to C}^t(\cdot) := \tr_{R_T} [\mathcal{M}_{CR_T \to CR_T}(\cdot) \otimes |k\rangle\langle k|_{R_T})] \]  

(22)

is \( k \)-independent for all \( t \geq 0 \) and of the form

\[ \mathcal{M}_{C \to C}^t(\cdot) = e^{t\mathcal{L}_C(\cdot)} \]

(23)

with \( \mathcal{L}_C \) equal to the r.h.s. of eq. (21b) under the replacements \( \tilde{H} \mapsto H \), \( \tilde{L}_j \mapsto L_j \) and \( \tilde{J}_j(\delta) \mapsto J_j \). What is more, for every ticking clock with a classical cut-off register written in the representation of proposition 2, there exists a ticking clock which is clockwork equivalent (definition 2), such that its clockwork is \( k \)-independent and given by eq. (23).

**Proof.** Consider a ticking clock \((\rho_C^0, \{M_{CR_T \to CR_T}^t\}_{t \geq 0})\) with a classical register given by the following expression for all \( t \geq 0 \) and \( k = 0, 1, \ldots, N_T\):

\[ \mathcal{M}_{C \to CR_T}^t(\rho_C^0) = \lim_{N \to +\infty} \left( \mathcal{M}_{CR_T \to CR_T}^{t/N}(\cdot) \otimes \mathcal{M}_{C \to CR_T}^{t/N,k}(\rho_C^0) \right). \]

(91a)

where

\[ \mathcal{M}_{C \to CR_T}^{t/N,k}(\cdot) = (\cdot) \otimes |k\rangle\langle k|_{R_T} + \left( \frac{t}{N} \right) \mathcal{C}_{(1)}(\cdot) \otimes |k\rangle\langle k|_{R_T} + \left( \frac{t}{N} \right) \mathcal{C}_{(2)}(\cdot) \otimes |k+1\rangle\langle k+1|_{R_T}, \]

(91b)

with

\[ \mathcal{C}_{(1)}(\cdot) := -i[H, \cdot] - \sum_{j=1}^{N_L} \frac{1}{2} \left( L_j^\dagger L_j + J_j^\dagger J_j \right), \]

(91c)

\[ \mathcal{C}_{(2)}(\cdot) := \sum_{j=1}^{N_L} J_j(\cdot) J_j^\dagger, \]

(91d)

and where \( H \in \mathcal{B}(\mathcal{H}_C) \) is Hermitian and \((L_j), (J_j)\) are arbitrary operators in \( \mathcal{B}(\mathcal{H}_C) \) and

\[ |l\rangle_{R_T} = \begin{cases} \lfloor l \mod N_T \rfloor_{R_T} & \text{for } l \in \mathbb{N} \text{ in the periodic register case,} \\ \lfloor N_T \rfloor_{R_T} & \text{for } l = N_T, N_T + 1, N_T + 2, \ldots \text{ in the cut-off register case.} \end{cases} \]

(92)

By comparison with the ticking clock representation in lemma 1 and taking into account the definition of clockwork equivalence (definition 2), we see that all ticking clocks with a classical register can be written in this form, up to clockwork equivalence. Observe that

\[ \mathcal{M}_{C \to C}^t(\cdot) = \lim_{N \to +\infty} \tr_{R_T} \left[ \left( \mathcal{M}_{CR_T \to CR_T}^{t/N}(\cdot) \otimes \mathcal{M}_{C \to CR_T}^{t/N,k}(\cdot) \right)^{(N-1)} \circ \mathcal{M}_{C \to CR_T}^{t/N,k}(\cdot) \right]. \]

(93)

Furthermore, observe that one has the following expansion

\[ \left( \mathcal{M}_{CR_T \to CR_T}^{t/N}(\cdot) \right)^{(m-1)} \circ \mathcal{M}_{C \to CR_T}^{t/N,k}(\cdot) = \sum_{l=0}^{m \cdot \delta t} M_{m \cdot \delta t}^l(\cdot) \otimes |l\rangle\langle l + k|_{R_T}, \]

(94)

for some \( M_{m \cdot \delta t}^l \in \mathcal{B}(\mathcal{H}_C) \) which may be \( k \)-dependent. To see the a solution of the form eq. (94) exists, note that every application of the channel eq. (91b) only contains terms which either keep the support of the register the same, i.e. has support on \( |k\rangle\langle k|_{R_T} \), or increases by one, i.e. has support on \( |k+1\rangle\langle k+1|_{R_T} \). Furthermore, the summation ranges from 0 to \( m \) after \( m \) applications of the channel, which follows easily inductively.

Hence by comparing eqs. (93) and (94) one sees that to prove that \( \mathcal{M}_{C \to C}^t \) is \( k \)-independent, it suffices to show that the channels \( \left( M_{m \cdot \delta t}^l(\cdot) \right)^m_{l=0} \) are \( k \)-independent for all \( m \in \mathbb{N}_0, \delta t \geq 0 \).
This is most easily shown by induction. We start by showing that \( M_1^{(l, \delta t)}(\cdot) \) are k-independent by equating

\[
M_{C \rightarrow CR_T}^{k, l} (\cdot) = \sum_{l=0}^{1} M_1^{(l, \delta t)} (\cdot) \otimes |l + k\rangle |l + k\rangle_{RT} \tag{95}
\]

with eqs. (91c) and (91d) to find \( M_1^{(0, \delta t)} = T_C + \delta t C_1(\cdot), M_1^{(1, \delta t)} = \delta t C_2(\cdot) \). Therefore, \( (M_1^{(l, \delta t)}(\cdot))_l^{m+1} \) are k-independent since \( C_1(\cdot) \) and \( C_2(\cdot) \) are. Now assume \( (M_m^{(l, \delta t)}(\cdot))_l^{m+1} \) are k-independent. We show that it follows that \( (M_{m+1}^{(l, \delta t)}(\cdot))_l^{m+1} \) are k-independent:

\[
\sum_{l=0}^{m+1} M_{m+1}^{(l, \delta t)} (\cdot) \otimes |l + k\rangle |l + k\rangle_{RT} = (M_{CR_T \rightarrow CR_T}^{k, l})^m \circ M_{C \rightarrow CR_T}^{k, l} (\cdot) \tag{96}
\]

\[
= M_{CR_T \rightarrow CR_T}^{k, l} \circ \left( \sum_{l=0}^{m} M_m^{(l, \delta t)} (\cdot) \otimes |l + k\rangle |l + k\rangle_{RT} + M_1^{(1, \delta t)} \circ M_1^{(l, \delta t)} (\cdot) \otimes |l + k + 1\rangle |l + k + 1\rangle_{RT} \right) \tag{97}
\]

\[
\sum_{l=0}^{m} M_1^{(l, \delta t)} \circ M_m^{(l, \delta t)} (\cdot) \otimes |l + k\rangle |l + k\rangle_{RT} + M_1^{(1, \delta t)} \circ M_m^{(l, \delta t)} (\cdot) \otimes |l + k + 1\rangle |l + k + 1\rangle_{RT} \tag{98}
\]

Therefore, since \( M_1^{(0, \delta t)} \), \( M_1^{(1, \delta t)} \) are manifestly k-independent and \( M_m^{(l, \delta t)} \) are k-independent by assumption, it follows by equating terms in lines (96) and (99), that \( (M_{m+1}^{(l, \delta t)}(\cdot))_l^{m+1} \) are k-independent. Hence by induction, we conclude that \( (M_m^{(l, \delta t)}(\cdot))_l^{m+1} \) are k-independent for all \( m \in \mathbb{N}_0 \) and thus that eq. (93) is k-independent also.

Observe that the only difference between the periodic register case and the cut-off register case, are the kets \( |l\rangle_{RT} \), which are defined in eq. (92). Since \( \text{tr}[|l\rangle \langle l|_{RT}] = 1 \) for all \( l \in \mathbb{N}_{\geq 0} \), we conclude that eq. (93) is the same in both cases.

Finally, proceeding similarly to the above inductive proof, we observe that

\[
M_{C \rightarrow C}^{l, \delta t} (\cdot) = \lim_{N \rightarrow +\infty} \left( M_{C \rightarrow C}^{l, \delta t/N} \right)^{\otimes N}, \tag{100}
\]

where

\[
\left( M_{C \rightarrow C}^{l, \delta t/N} \right)^{\otimes N} (\cdot) = \sum_{l=0}^{N} M_{C \rightarrow C}^{l, \delta t/N} (\cdot). \tag{101}
\]

It is now straightforward to verify that the above channel constitutes a dynamical semigroup thus admitting a generator representation of the form \( M_{C \rightarrow C}^{l, \delta t} (\cdot) = e^{\delta t G_C (\cdot)} \) with

\[
G_C (\cdot) = \lim_{t \rightarrow 0^+} \frac{M_{C \rightarrow C}^{l, \delta t} - T_{CR_T}}{t} = \lim_{t \rightarrow 0^+} \frac{M_1^{(0, \delta t)} + M_1^{(1, \delta t)} - T_{CR_T}}{t} = C_1(\cdot) + C_2(\cdot). \tag{102}
\]

Thus \( G_C (\cdot) \) is equal to the r.h.s. of eq. (21b) under the replacements \( \tilde{H} \mapsto H, \tilde{L}_j \mapsto L_j \) and \( \tilde{J}_j^{(l)} \mapsto J_j \). □

B.4 Proofs of lemmas 2 and 3

**Lemma 2** (Entry-wise and p-norm equivalence). Let the complex finite dimensional matrix \( A \in \mathbb{C}^l \times \mathbb{C}^m \) have entries denoted by \( A_{qr} \in \mathbb{C} \). Let \( \| \cdot \|_p \) denote the operator norm on \( \mathbb{C}^l \times \mathbb{C}^m \) induced by the vector p-norm on vector in \( \mathbb{C}^m \). Let \( o(\delta) \) denote “little o” notation for some limit \( \delta \rightarrow a \). It follows that

\[
A_{qr} = o(\delta) \tag{103}
\]

for all \( q = 1, 2, 3, \ldots l; r = 1, 2, 3, \ldots m \) if and only if

\[
\| A \|_p = o(\delta). \tag{104}
\]

The statement holds for any \( p > 0 \).
Proof. Given the expression for the operator norm, namely

\[ \|A\|_p = \sup_{v \in \mathbb{R}^n; \|v\|_2 \leq 1} \left( \sum_{q=1}^{l} \sum_{r=1}^{m} |A_{q,r,v_r}|^p \right)^{1/p}, \tag{105} \]

the direction \( A_{q,r} = o(\delta) \) \( \forall q,r \implies \|A\|_p = o(\delta) \) follows easily. To prove the converse, we will use proof by contradiction. Suppose \( \|A\|_p = o(\delta) \), and by contradiction, assume that there exists matrix entry \( A_{st} \) s.t. \( A_{st} \neq o(\delta) \). Therefore,

\[ \lim_{\delta \to a} \frac{|A_{st}|}{\delta} > 0. \tag{106} \]

We can now use the definition of the operator norm to achieve the lower bound

\[ \|A\|_p \geq \left( \sum_{q=1}^{l} \sum_{r=1}^{m} |A_{q,r,d_{r,t}}|^p \right)^{1/p} = \left( \sum_{q=1}^{l} |A_{qt}|^p \right)^{1/p} \geq |A_{st}|. \tag{107} \]

Therefore, dividing both sides by \( \delta \) followed by taking the limit \( \delta \to a \) we achieve using eq. (106) that \( \lim_{\delta \to a} \frac{\|A\|_p}{\delta} > 0 \). This contradicts the assertion that \( \|A\|_p = o(\delta) \).

**Lemma 3** (Maximum number of Lindblad operators needed). Consider a clockwork of Hilbert space dimension \( d \in \mathbb{N} \). For every Hermitian operator \( H \) and two finite sequences of operators \( (L_j)_{j=1}^{N_L}, (J_j)_{j=1}^{N_L} \) on \( B(H_\mathbb{C}) \) giving rise to the channel \( \mathcal{M}_{C^{\infty} CR_T}(\cdot) \) via eq. (30a); there exists \( 2(d^2 - 1) \) new operators \( (L'_j)_{j=1}^{d^2 - 1}, (J'_j)_{j=1}^{d^2 - 1} \) on \( B(H_\mathbb{C}) \) such that the channel \( \mathcal{M}^{t,k}_{C^{\infty} CR_T}(\cdot) \) is invariant under the mappings

\[ \sum_{j=1}^{N_L} \frac{1}{2} \{ L_j^i L_j + \theta(k) J_j^i J_j, \cdot \} + L_j(\cdot) L_j^i \mapsto \sum_{j=1}^{d^2 - 1} \frac{1}{2} \{ L_j^i L_j + \theta(k) J'_j^i J'_j, \cdot \} + L_j(\cdot) L_j^i \tag{108} \]

\[ \sum_{j=1}^{N_L} J_j(\cdot) J_j^i \mapsto \sum_{j=1}^{d^2 - 1} J'_j(\cdot) J'_j^i \tag{109} \]

in eqs. (30b) and (30d) respectively.

Proof. In the proof of lemma 1, \( N_L \) is simply a non negative integer arising from writing an arbitrary implementation of the channel \( \mathcal{M}^{t,k}_{C^{\infty} CR_T}(\cdot) \) is Kraus form. To prove lemma 3, it will suffice to prove that without loss of generality, \( N_L \) in the proof of lemma 1 can be chosen to be equal to \( d^2 - 1 \). To do so, we start by recalling eq. (51) in the proof of lemma 1:

\[ \mathcal{M}_{C^{\infty} CR_T}(\rho_C) = \sum_{l=0}^{N_T} \sum_{j=0}^{N_T} N_j^{(k)}(l,t) \rho_C N_j^{(k)}(l,t) \otimes |l\rangle \langle l|_{R_T}, \tag{110} \]

for all \( k = 0, 1, \ldots, N_T; t \geq 0 \). Recall also that \( N_j^{(k)}(l,t) := |l\rangle \langle l|_{R_T} Q_j^{(k)}(t) : B(H_\mathbb{C}) \to B(H_\mathbb{C}) \) where \( Q_j^{(k)}(t) : B(H_\mathbb{C}) \to B(H_\mathbb{C} \otimes H_{R_T}) \) and

\[ \sum_{j=0}^{N_L} Q_j^{(k)}(t) \dagger Q_j^{(k)}(t) = \mathbb{I}_C. \tag{111} \]

Using the resolution of the identity, eq. (111) implies

\[ \sum_{l=0}^{N_T} \sum_{j=0}^{N_T} N_j^{(k)}(l,t) \otimes N_j^{(k)}(l,t) = \mathbb{I}_C. \tag{112} \]

Thus since the basis \( \{|l\rangle_{R_T}\}_{l=0}^{N_T} \) is orthogonal, the operators \( \{N_j^{(k)}(l,t)\}_{j=0}^{N_L} \) are completely arbitrary and independent from \( \{N_j^{(k)}(l',t)\}_{j=0}^{N_L} \) for all \( l \neq l', l = 0, 1, 2, \ldots, N_T; \) up to the normalisation imposed by eq. (112). The lemma now follows directly from Choi’s theorem [26]. To see this, note that the channel

\[ \sum_{j=0}^{N_L} N_j^{(k)}(l,t) \otimes N_j^{(k)}(l,t) : B(H_\mathbb{C}) \to B(H_\mathbb{C}), \tag{113} \]
is completely positive for all $N_L \in \mathbb{N}_{>0}$. Therefore, via Choi’s theorem there exists operators $(N_j^{(k)\dagger})_{j=1}^{d^2-1}$ such that

$$\sum_{j=0}^{N_L} N_j^{(k)}(l,t) \cdot N_j^{(k)\dagger}(l,t) = \sum_{j=0}^{d^2-1} N_j^{(k)\dagger}(l,t) \cdot N_j^{(k)\dagger}(l,t), \quad (114)$$

where

$$\sum_{l=0}^{N_T} \sum_{j=0}^{d^2-1} N_j^{(k)\dagger}(l,t) \cdot N_j^{(k)\dagger}(l,t) = \mathbb{1}_C. \quad (115)$$

However, since the operators $(N_j^{(k)})_{j=1}^{N_L}$ were arbitrary to begin with (up to the aforementioned normalisation which the operators $(N_j^{(k)\dagger}(l,t))_{j=1}^{d^2-1}$ also satisfy), we can always choose $N_L = d^2 - 1$ from the outset. \[\blacksquare\]