The theory of physical superselection sectors in terms of vertex operator algebra language

Haisheng Li
Department of Mathematics, University of California, Santa Cruz, CA 95064

Abstract We formulate an interpretation of the theory of physical superselection sectors in terms of vertex operator algebra language. Using this formulation we give a construction of simple current from a primary semisimple element of weight one. We then prove that if a rational vertex operator algebra $V$ has a simple current $M$ satisfying certain conditions, then $V \oplus M$ has a natural rational vertex operator (super)algebra structure. Applying our results to a vertex operator algebra associated to an affine Lie algebra, we construct its simple currents and the extension by a simple current. We also present two essentially equivalent constructions for twisted modules for an inner automorphism from the adjoint module or any untwisted module.

1 Introduction

This paper is motivated by some physical papers ([FRS], [MSc]) in the theory of superselection sectors ([DHR], [HK]). Let $A$ be the observable algebra for a quantum field theory and let $\pi_0$ be the vacuum representation on $H_0$ of $A$. In general, $A$ admits infinitely many inequivalent irreducible modules, so a criterion is needed to rule out the physical irrelevant modules. Let $\psi$ be an endomorphism of $A$. Then we obtain a representation $\pi_0\psi$ on $H_0$ of $A$. If $U$ is an element of $A$ with a left inverse $U^*$, then we have an endomorphism $\psi_U$ of $A$ defined by $\psi_U(a) = UaU^*$ for any $a \in A$. Consequently, we have a representation $\pi_0\psi_U$ on $H_0$ of $A$. In the algebraic theory of superselection sectors (see, e.g., [HK]), the physical superselection sectors consist of each equivalent class of irreducible representation $\pi$ which is equivalent to $\pi_0\psi$ for some endomorphism $\psi$ of $A$. If $W_i = (H_0, \pi_0\psi_i)$ ($i = 1, 2$) are superselection sectors, an intertwinner from $W_1$ to $W_2$ is defined to be a homomorphism of $A$-modules, and $W = (H_0, \pi_0\psi_2\psi_1)$ is defined to be the tensor product module of $W_1$ with $W_2$. Such a tensor product module is in general reducible, but it is assumed to be decomposable into irreducible ones. Furthermore, if $(H_0, \pi_0\psi_i)$ ($i = 1, 2, 3$) are three superselection sectors, then an intertwining operator of type $\left(\begin{array}{c} W_3 \\ W_1, W_2 \end{array}\right)$ is defined to be an intertwinner or a homomorphism $\phi$ of $A$-modules from $W = (H_0, \pi_0\psi_2\psi_1)$ to $W_3$ and fusion rules had also been defined accordingly.

On the other hand, vertex operator algebras were introduced by mathematician ([B], [FLM]), which are essentially chiral algebras formulated in [BPZ] in two-dimensional conformal field theory. Vertex operator algebras provide a powerful algebraic tool for studying the general structure of conformal field theory. For vertex operator algebra theory, the
notion of modules, intertwining operators and fusion rules have been defined in [FLM] and [FHL]. Furthermore, the notions of tensor product for modules have been also developed in [HL0-2] and [Li4]. The initial purpose of this paper is to interpret the physical superselection theory in terms of vertex operator algebra language ([B], [FLM], [FHL], [HL0-2], [Li4]).

Note that if $\sigma$ is an endomorphism of a vertex operator algebra $V$, then by definition ([B], [FLM], [FHL]), $\sigma$ preserves both the vacuum and the Virasoro element so that $\sigma$ preserves each homogeneous subspace of $V$. If $V$ is simple, i.e., $V$ is an irreducible $V$-module, it follows from Schur lemma that any nonzero endomorphism is a scalar. Then the twist of $V$ by $\sigma$ is isomorphic to $V$. Therefore, it is impossible to obtain all irreducible modules by twisting $V$ unless $V$ is holomorphic, i.e., any irreducible $V$-module is isomorphic to $V$. Having known the above fact, we turn to a certain associative algebra.

For any vertex operator algebra $V$, Frenkel and Zhu [FZ] constructed a topological $\mathbb{Z}$-graded associative algebra $U(V)$, which was called the universal enveloping algebra of $V$. Roughly speaking, $U(V)$ is the associative algebra with identity generated by all $a^\ell_n$, for $a \in V, n \in \mathbb{Z}$ with certain defining relations coming from the Jacobi identity and the Virasoro algebra relations. Then there is a natural 1-1 correspondence between the set of equivalence classes of lower truncated $\mathbb{Z}$-graded weak $V$-modules and the set of equivalence classes of continuous $\mathbb{Z}$-graded $U(V)$-modules. As an infinite-dimensional associative algebra, $U(V)$ has many nontrivial (continuous) endomorphisms. For instance, it was essentially proved in [Z] that $\text{Aut}U(V)$ contains $\text{PSL}(2, \mathbb{C})$ as a subgroup. It is reasonable to believe that $U(V)$ should play the role of the observable algebra in the algebraic quantum field theory.

Let $\Delta(z) \in U(V)\{z\}$ satisfy the conditions (2.7)-(2.10). Then it is proved that for any $V$-module $(M, Y_M(\cdot, z))$, $(\tilde{M}, Y_{\tilde{M}}(\cdot, z)) := (M, Y_M(\Delta(z)\cdot, z))$ is a $V$-module. Consequently, $\Delta(z)$ induces an endomorphism $\psi$ of $U(V)$ defined by

$$\psi(Y(a, z)) = Y(\Delta(z)a, z) \quad \text{for any } a \in V. \quad (1.1)$$

Our first theorem claims that if $\Delta(z)$ is invertible, $\tilde{M}$ is isomorphic to a tensor product module of $M$ with $\tilde{V}$ in the sense of [HL0-2] and [Li4]. This implies that $\tilde{V}$ is a simple current ([SY1-2], [FG]), i.e., the tensor functor associated to $\tilde{V}$ gives a permutation on the set of equivalence classes of irreducible $V$-modules. Next, we give a construction for such a $\Delta(z)$. Let $h$ be an semisimple element of a vertex operator algebra $V$ satisfying

$$L(n)h = \delta_{n,0}h, h(n)h = \delta_{n,1} \gamma 1 \quad \text{for any } n \in \mathbb{Z}_+, \quad (1.2)$$

where $\gamma$ is a rational number. Then we construct a $\Delta(h, z)$ (in Section 2) satisfying (2.7)-(2.10). Applying our results to a vertex operator algebra $L(\ell, 0)$ associated to an affine Lie

\footnote{It was pointed out by C. Dong that this condition is necessary}
algebra \( \tilde{g} \), we prove that if the fundamental (dominant integral) weight \( \lambda_i \) is cominimal [FG], then for any complex number \( \ell \neq -\Omega \) (the dual Coxeter number), \( L(\ell, \ell\lambda_i) \) is a simple current for \( L(\ell, 0) \). If \( \ell \) is a positive integer, this result has been proved in [FG] by calculating four-point functions.

Note that for all known rational vertex operator operator algebras (the definition is given in Section 2), there are only finitely many inequivalent irreducible modules and all irreducible modules are exactly those which are needed in conformal field theory. For instance, it was proved ([DL], [FZ], [Li2]) that for any positive integer \( \ell \), the set of equivalence classes of irreducible \( L(\ell, 0) \)-modules is exactly the set of equivalence classes of unitary highest weight \( \tilde{g} \)-modules of level \( \ell \). It is also known ([DMZ], [W]) that if \( c = 1 - \frac{6(p-q)^2}{pq} \), where \( p, q \in \{2, 3, \ldots\} \) and \( p \) and \( q \) are relatively prime, then the set of equivalence classes of irreducible \( L(c, 0) \)-modules is exactly the set of equivalence classes of lowest weight Virasoro modules in the minimal series given in [BPZ]. Therefore, at least for a rational vertex operator algebra \( V \), each irreducible \( V \)-module is a superselection sector so that we do not need a superselection rule to rule out irrelevant modules. Based this interpretation we conjecture that each irreducible \( V \)-module is isomorphic some twist of \( V \) by an endomorphism of \( U(V) \).

The famous moonshine module vertex operator algebra \( V^\natural \) [FLM] is the first mathematically rigorous construction of \( \mathbb{Z}_2 \)-orbifold theory, which is constructed by using a vertex operator algebra together with an irreducible \( \mathbb{Z}_2 \)-twisted module. From the construction of \( V^\natural \) it is an extension of a certain vertex operator algebra by a simple module. Recently Huang [Hua] has given a conceptual construction. To generalize the construction of \( V^\natural \), one is facing two problems, which are the existence of twisted modules and the extension of a vertex operator algebra by a simple module, respectively. In this paper, we consider a relatively simple case for the extension problem. Let \( h \in V \) satisfying the condition (1.2). Then we prove that under certain conditions, \( V \oplus \tilde{V} \) has a natural vertex operator (super)algebra structure. Furthermore, assuming that \( V \) is rational, we classify all irreducible modules for \( V \oplus \tilde{V} \) and prove that \( V \oplus \tilde{V} \) is rational. By restricting ourselves to a vertex operator algebra \( L(\ell, 0) \) associated to the affine Lie algebra \( s\ell_2 \), our results imply that \( L(\ell, 0) \oplus L(\ell, \frac{\ell}{2}) \) is a rational vertex operator algebra if \( \ell \) is a positive integral multiple of 4 and that \( L(\ell, 0) \oplus L(\ell, \frac{\ell}{2}) \) is a rational vertex operator superalgebra if \( \ell \) is a positive odd integral multiple of 2. In the notation of \( L(\ell, \frac{\ell}{2}) \), \( \frac{\ell}{2} \) is the spin number (half of the weight) in terms of physical language. These results are known to physicists (cf. [MSe]). All results have been greatly extended in [DLM].

In the last section, we consider the construction of twisted modules from \( V \). We present two essentially equivalent constructions for twisted modules (or sectors) for an inner automorphism from \( V \) or any (untwisted) module. This may shed some light for the construction of twisted modules (or sectors) in general.

**Acknowledgment** We thank Professors Dong, Lepowsky and Mason for many
useful discussions and Professor J. Fuchs for correcting a mistake about relation between
simple currents and cominimal weights as in [DLM]. We apologize for missing any related
references because of the lack of our knowledge.

2 An interpretation of superselection sectors in terms
of vertex operator algebra language

A vertex operator algebra ([B], [FLM]) is a quadruple \((V, Y, 1, \omega)\), where \(V\) is \(\mathbb{Z}\)-graded
vector space \(V = \oplus_{n \in \mathbb{Z}} V_n\) satisfying the conditions: for any \(n \in \mathbb{Z}\), \(\dim V_n < \infty\) and
\(V_n = 0\) for sufficiently small \(n\); where \(Y\) is a linear map
\[Y(\cdot, z) \rightarrow (\text{End} V)[[z, z^{-1}]],\]
\[Y(u, z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1}\]
such that for any \(u, v \in V\),
\[u_n v = 0 \quad \text{for } n \text{ sufficiently large}; \quad \text{(2.1)}\]
and that the Jacobi identity holds:
\[
z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y(u, z_1)Y(v, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y(v, z_2)Y(u, z_1)
= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y(Y(u, z_0)v, z_2); \quad \text{(2.2)}
\]
where 1 is a vector of \(V\), called the vacuum, satisfying the conditions:
\[Y(1, z) = \text{id}_V; \quad Y(u, z)1 \in V[[z]] \quad \text{and } \lim_{z \to 0} Y(u, z)1 = u; \quad \text{(2.3)}\]
and where \(\omega\) is a vector of \(V\), called the Virasoro element, such that if we set
\[L(z) = Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}, \quad \text{(2.4)}\]
the following Virasoro algebra relations hold:
\[[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12} (m^3 - m) \delta_{m+n,0} (\text{rank } V) \quad \text{(2.5)}\]
with \((\text{rank } V) \in \mathbb{C}, m, n \in \mathbb{Z}\); and the following relations hold:
\[
\frac{d}{dz} Y(v, z) = Y(L(-1)v, z), \quad L(0)u = (\text{wt} u)u = nu\quad \text{(2.6)}
\]
for \(u \in V_n\). This completes the definition.
A weak $V$-module is a vector space $M$ together with a linear map $Y_M(\cdot,z)$ from $V$ to $\text{End}M[[z,z^{-1}]]$ such that $Y_M(1,z) = id_M$, $Y_M(L(-1)a,z) = \frac{d}{dz} Y_M(a,z)$ for any $a \in V$ and a suitably adjusted Jacobi identity holds.

A $V$-module is a weak $V$-module $M$ which is a $\mathbb{C}$-graded vector space $M = \oplus_{h \in \mathbb{C}} M_h$, where $M_h$ is the eigenspace of $L(0)$ on $M$ with eigenvalue $h$, such that for any $h \in \mathbb{C}$, $\dim M_h < \infty$ and $M_{n+h} = 0$ for sufficiently large integer $n$.

A lower truncated $\mathbb{Z}$-graded weak $V$-module is a weak $V$-module $M$ together with a $\mathbb{Z}$-grading $M = \oplus_{n \in \mathbb{Z}} M(n)$ such that $M(n) = 0$ for sufficiently small integer $n$ and that

$$a_n M(m) \subseteq M(m + k - n - 1) \quad \text{for } a \in V_k, m, n, k \in \mathbb{Z}.$$ 

If any lower truncated $\mathbb{Z}$-graded weak $V$-module is completely reducible, we call $V$ rational $[\mathbb{Z}]$.

Let $W_i$ ($i = 1,2,3$) be $V$-modules. Then an intertwining operator of type $\begin{pmatrix} W_3 \\ W_1W_2 \end{pmatrix}$ is defined $[\text{FHL}]$ to be a linear map $I(\cdot,z)$ from $W_1$ to $\text{Hom}_\mathbb{C}(W_2,W_3\{z\})$ such that $I(L(-1)u,z) = \frac{d}{dz} I(u,z)$ for $u \in W_1$ and a suitably adjusted Jacobi identity holds. Denote by $I \begin{pmatrix} W_3 \\ W_1W_2 \end{pmatrix}$ the space of all intertwining operators of the indicated type. The dimension of this vector space is called the fusion rule of this type. Let $\text{Irr}(V)$ be the set of equivalence classes of irreducible $V$-modules and for any $V$-module $M$, denote the equivalence class of $M$ by $[M]$. For $[M_i], [M_j], [M_k] \in \text{Irr}(V)$, denote the fusion rule by $N_{ij}^k$. Suppose that all fusion rules among irreducible modules are finite. The fusion algebra or Verlinda algebra is defined to be the algebra linearly spanned by $\text{Irr}(V)$ with the multiplication:

$$[M_i] \cdot [M_j] = \sum_{k \in \text{Irr}(V)} N_{ij}^k [M_k].$$

Let $V$ be a vertex operator algebra and let $U(V)$ be the universal enveloping algebra of $V$ constructed by Frenkel and Zhu $[\text{FZ}]$. What we will use about $U(V)$ is an abstract fact that $U(V) = \oplus_{n \in \mathbb{Z}} U(V)_n$ is a $\mathbb{Z}$-graded topological associative algebra such that there is a natural 1-1 correspondence between the set of equivalence classes of lower truncated $\mathbb{Z}$-graded continuous $U(V)$-module and the set of equivalence classes of lower truncated $\mathbb{Z}$-graded weak $V$-module. Since we will not use any details of $U(V)$ we will not recall the construction and we refer the interested reader for $U(V)$ to $[\text{FZ}]$.

An endomorphism $\rho$ of $U(V)$ is said to be restricted if for any $a \in V$ and any integer $k$, there is an integer $m$ such that $\rho(a_n) \in \oplus_{j \leq k} U(V)_j$ for $n > m$. Then it is clear that for any $V$-module $M$, the $\rho$-twist of $M$ is a (weak) $V$-module. From the analysis made in introduction we conjecture that any irreducible $V$-module is isomorphic to a twist of the adjoint module $V$ by a restricted endomorphism of $U(V)$. 

5
Let $\Delta(z) \in U(V) \{z\}$ satisfy the following conditions:

\[
\begin{align*}
\Delta(z)a & \in V[z, z^{-1}]; \\
\Delta(z)1 &= 1; \\
[L(-1), \Delta(z)] &= -\frac{d}{dz}\Delta(z); \\
Y(\Delta(z_2 + z_0)a, z_0)\Delta(z_2) &= \Delta(z_2)Y(a, z_0) \quad \text{for any } a \in V.
\end{align*}
\] (2.7), (2.8), (2.9), (2.10)

Let $(M, Y_M(\cdot, z))$ be any $V$-module. Then for any $a, b \in V$ we have:

\[
\begin{align*}
z_0^{-1}\delta \left(\frac{z_1 - z_2}{z_0}\right) Y_M(\Delta(z_1)a, z_1)Y_M(\Delta(z_2)b, z_2) \\
- z_0^{-1}\delta \left(\frac{-z_2 + z_1}{z_0}\right) Y_M(\Delta(z_2)b, z_2)Y_M(\Delta(z_1)a, z_1) \\
= z_2^{-1}\delta \left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(\Delta(z_1)a, z_0)\Delta(z_2)b, z_2) \\
= z_2^{-1}\delta \left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(\Delta(z_2 + z_0)a, z_0)\Delta(z_2)b, z_2) \\
= z_2^{-1}\delta \left(\frac{z_1 - z_0}{z_2}\right) Y_M(\Delta(z_2)Y(a, z_0)b, z_2).
\end{align*}
\] (2.11)

The conditions (2.8) and (2.9) imply

\[
Y_M(\Delta(z)1, z) = \text{id}_M, \quad [L(-1), Y_M(\Delta(z)a, z)] = \frac{d}{dz}Y_M(\Delta(z)a, z) \quad \text{for } a \in V.
\] (2.12)

Therefore $(\tilde{M}, Y_M(\cdot, z)) = (M, Y_M(\Delta(z), z))$ is a weak $V$-module. Then each $\Delta(z)$ gives rise to a restricted endomorphism $\psi$ of $U(V)$ such that $\psi(Y(a, z)) = Y(\Delta(z)a, z)$ for any $a \in V$. Let $G(V)$ be the set of all $\Delta(z)$ satisfying the conditions (2.7)-(2.10).

**Lemma 2.1.** Let $\Delta(z) \in G(V)$, let $M^i (i = 1, 2, 3)$ be three $V$-modules and let $I(\cdot, z)$ be an intertwining operator of type $\left(\begin{array}{c} M^3 \\ M^1 M^2 \end{array}\right)$. Then $\tilde{I}(\cdot, z) = I(\Delta(z), z)$ is an intertwining operator of type $\left(\begin{array}{c} \tilde{M}^3 \\ M^1 M^2 \end{array}\right)$.

**Proof.** The $L(-1)$-derivative property for $\tilde{I}(\cdot, z)$ follows from the condition (2.9) immediately. For any $a \in V, u \in M^1$ we have:

\[
\begin{align*}
z_0^{-1}\delta \left(\frac{z_1 - z_2}{z_0}\right) Y_{\tilde{M}^3}(a, z_1)\tilde{I}(u, z_2) & - z_0^{-1}\delta \left(\frac{z_2 - z_1}{-z_0}\right) \tilde{I}(u, z_2)Y_{\tilde{M}^2}(a, z_1) \\
= z_0^{-1}\delta \left(\frac{z_1 - z_2}{z_0}\right) Y_{M^3}(\Delta(z_1)a, z_1)I(\Delta(z_2)u, z_2) \\
- z_0^{-1}\delta \left(\frac{z_2 - z_1}{-z_0}\right) I(\Delta(z_2)u, z_2)Y_{M^2}(\Delta(z_1)a, z_1) \\
= z_2^{-1}\delta \left(\frac{z_1 - z_0}{z_2}\right) I(Y_{M^1}(\Delta(z_1)a, z_0)\Delta(z_2)u, z_2)
\end{align*}
\]
\begin{align*}
= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) I(\Delta(z_2)Y_{M1}(a, z_0)u, z_2) \\
= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) \tilde{I}(Y_{M1}(a, z_0)u, z_2).
\end{align*}

Then the proof is complete. \qed

**Lemma 2.2.** Let \( \Delta(z) \in G(V) \) and let \( \psi \) be a \( V \)-homomorphism from a \( V \)-module \( W \) to another \( V \)-module \( M \). Then \( \psi \) is also a \( V \)-homomorphism from the \( V \)-module \( \tilde{W} \) to the \( V \)-module \( \tilde{M} \).

**Proof.** For any \( a \in V, u \in W \), we have:

\[
\psi(Y_{\tilde{W}}(a, z)u) = \psi(Y_{\tilde{W}}(\Delta(z)a, z)u) = Y_M(\Delta(z)a, z)\psi(u) = Y_{\tilde{M}}(a, z)\psi(u).
\]

Thus \( \psi \) is a \( V \)-homomorphism from \( \tilde{W} \) to \( \tilde{M} \). \qed

**Lemma 2.3.** Let \( \Delta_1(z), \Delta_2(z) \in G(V) \). Then \( \Delta_1(z)\Delta_2(z) \in G(V) \).

**Proof.** By assumption we have:

\[
\Delta_1(z)\Delta_2(z)1 = \Delta_1(z)1 = 1,
\]

\[
[L(-1), \Delta_1(z)\Delta_2(z)]
\]

\[
= [L(-1), \Delta_1(z)]\Delta_2(z) + \Delta_1(z)[L(-1), \Delta_2(z)]
\]

\[
= -\left( \frac{d}{dz}\Delta_1(z) \right) \Delta_2(z) + \Delta_1(z) \left( \frac{d}{dz}\Delta_2(z) \right)
\]

\[
= -\frac{d}{dz}(\Delta_1(z)\Delta_2(z)),
\]

and

\[
= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y(\Delta_1(z_1)\Delta_2(z_2)a, z_0)\Delta_1(z_2)\Delta_2(z_2)
\]

\[
= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) \Delta_1(z_2)Y(\Delta_2(z_1)a, z_0)\Delta_2(z_2)
\]

\[
= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) \Delta_1(z_2)\Delta_2(z_2)Y(a, z_0)
\]

for any \( a \in V \). Thus \( \Delta_1(z)\Delta_2(z) \in G(V) \). \qed

It is clear that \( id_V \in G(V) \), so that \( G(V) \) is a semigroup.

**Lemma 2.4.** Let \( \Delta(z) \in G(V) \) such that \( \Delta(z) \) has an inverse \( \Delta^{-1}(z) \in U(V)[[z, z^{-1}]] \). Then \( \Delta^{-1}(z) \in G(V) \).

**Proof.** First, we have: \( \Delta^{-1}(z)1 = 1 \). Since \( \Delta(z)\Delta^{-1}(z) = 1 \), we have:

\[
0 = \left( \frac{d}{dz}\Delta(z) \right) \Delta^{-1}(z) + \Delta(z)\frac{d}{dz}\Delta^{-1}(z).
\]

(2.18)
there exists a unique 

\[ V \]

\[ \text{twining operator} \]

holds: For any 

\[ (\psi \cdot, z) \]

extends canonically to a linear map from 

\[ (\Delta(\cdot, z), \cdot), z \]

Thus \[ \Delta^{-1}(z) \in G(V). \quad \square \]

Denote by \( G^0(V) \) the subgroup of all invertible elements of \( G(V) \). Let \( H(V) \) be the subgroup of \( G(V) \) consisting of each \( \Delta(z) \) such that \((V, Y(\Delta(z) \cdot, z))\) is isomorphic to \((V, Y(\cdot, z))\).

**Conjecture 2.5.** Let \( V \) be a vertex operator algebra. Then for any irreducible \( V \)-module \((M, Y_M(\cdot, z))\), there is a \( \Delta(z) \in G(V) \) such that \((V, Y(\Delta(z) \cdot, z))\) is isomorphic to \((M, Y_M(\cdot, z))\).

Recall the definition of tensor product for modules for a vertex operator algebra \( V \) from [Li4] (see [HL0-2] for a different version).

**Definition 2.6 [Li4].** Let \( M^1 \) and \( M^2 \) be two \( V \)-modules. A **tensor product** for the ordered pair \((M^1, M^2)\) is a pair \((M, F(\cdot, z))\) consisting of a \( V \)-module \( M \) and an intertwining operator \( F(\cdot, z) \) of type \((M, M^1, M^2)\) such that the following universal property holds: For any \( V \)-module \( W \) and any intertwining operator \( I(\cdot, z) \) of type \((W, M^1, M^2)\), there exists a unique \( V \)-homomorphism \( \psi \) from \( M \) to \( W \) such that \( I(\cdot, z) = \psi \circ F(\cdot, z) \).

(Here \( \psi \) extends canonically to a linear map from \( M\{z\} \) to \( W\{z\} \).)

**Proposition 2.7 [Li4].** If \((M^1, F(\cdot, z))\) is a tensor product for the ordered pair \((M^1, M^2)\) of \( V \)-modules, then for any \( V \)-module \( M^3 \), \( \text{Hom}_V(M, M^3) \) is linearly isomorphic to the space of intertwining operators of type \((M^3, M^1, M^2)\).

Let \( M \) be a module for \( V \) and let \( \Delta(z) \in G(V) \). Set \( W = V \), \( Y_W(\cdot, z) = Y_V(\Delta(z) \cdot, z) \).

In physical references, essentially \((M, Y_M(\Delta(z) \cdot, z))\) is defined to be the tensor product module of \( M \) with \( W \).

**Conjecture 2.8.** Let \( M \) be a module for \( V \) and let \( \Delta(z) \in G(V) \). Then \((\tilde{M}, \tilde{I}(\cdot, z)) := (M, Y_M(\Delta(z) \cdot, z))\) is a tensor product of \((M, \tilde{V})\) in the sense of [HL0-2] and [Li4], where \( \tilde{I}(\cdot, z) \) is defined by \( I(u, z)a = e^{zL(-1)}Y_M(a, -z)u \) for \( a \in V, u \in M \).
Proposition 2.9. Let \((W, F(\cdot, z))\) be a tensor product for a pair \((M^1, M^2)\) of \(V\)-modules and let \(\Delta(z) \in G^\circ(V)\). Then \((\tilde{W}, \tilde{F}(\cdot, z))\) is a tensor product of the pair \((\tilde{M}^1, \tilde{M}^2)\).

Proof. From Proposition 2.7 we have an intertwining operator \(\tilde{F}(\cdot, z) = F(\Delta(z)\cdot, z)\) of type \(\left( \begin{array}{c} \tilde{W} \\ M^1 \tilde{M}^2 \end{array} \right)\). Let \(M\) be any \(V\)-module and let \(I(\cdot, z)\) be any intertwining operator of type \(\left( \begin{array}{c} M \\ M^1 \tilde{M}^2 \end{array} \right)\). Then \(I(\Delta(z)^{-1}\cdot, z)\) is an intertwining operator of type \(\left( \begin{array}{c} \tilde{M} \\ M^1 \tilde{M}^2 \end{array} \right)\), where \((\tilde{M}, Y_M(\cdot, z)) = (M, Y_M(\Delta(z)^{-1}\cdot, z))\). By the universal property of \((W, F(\cdot, z))\), there is a unique \(V\)-homomorphism \(\psi\) from \(W\) to \(\tilde{M}\) such that \(\tilde{I}(\cdot, z) = \psi \circ \tilde{F}(\cdot, z)\). By Lemma 2.2, \(\psi\) is a \(V\)-homomorphism from \(W\) to \(M\). Since \(\Delta(z)u\) only involves finitely many terms, we have: \(I(\cdot, z) = \psi \circ F(\cdot, z)\). It is not difficult to check the uniqueness. Then the proof is complete. \(\blacksquare\)

Corollary 2.10. Let \(M\) be a \(V\)-module and let \(\Delta(z) \in G^\circ(V)\). Then \(\tilde{M}\) is isomorphic to the tensor product module of \(M\) with \(\tilde{V}\).

Proof. By Proposition 5.1.6 in [Li4], \((M, F(\cdot, z))\) is a tensor product for the pair \((M, V)\), where \(F(\cdot, z)\) is the transpose intertwining operator of \(Y_M(\cdot, z)\). By Proposition 2.9, \((\tilde{M}, \tilde{F}(\cdot, z))\) is a tensor product for \((M, \tilde{V})\). \(\square\)

The following definition is due to physicists (see for example [SY1-2], [FG]).

Definition 2.11. Let \(V\) be a vertex operator algebra. An irreducible \(V\)-module \(M\) is called a simple current if the tensor functor “\(M \times\)” is a permutation acting on the set of equivalence classes of irreducible \(V\)-modules.

By Corollary 2.10 we have:

Theorem 2.12. For any \(\Delta(z) \in G^\circ(V)\), \((V, Y(\Delta(z)\cdot, z))\) is a simple current \(V\)-module.

We will apply this result to vertex operator algebras associated to affine Lie algebras later. Combining Conjectures 2.5 and 2.8 we formulate the following conjecture.

Conjecture 2.13. The fusion algebra or the Verlinda algebra for vertex operator algebra \(V\) is isomorphic to the group algebra of \(G(V)/H(V)\) over the ground field \(\mathbb{C}\).

Next, we give a construction of \(\Delta(z)\) from a semisimple primary element of weight one. Let \(V\) be a vertex operator algebra and let \(h \in V\) satisfying the following conditions:

\[ L(n)h = \delta_{n,0}h, \quad h_n h = \delta_{n,1} \gamma 1 \quad \text{for any } n \in \mathbb{Z}_+, \]  

(2.21)

where \(\gamma\) is a fixed rational number. Furthermore, we assume that \(h_0\) semisimply acts on \(V\) with integral eigenvalues. From now on, we also freely use \(h(n)\) for \(h_n\). For any \(\alpha \in \mathbb{Q}\), set

\[ E^\pm(\alpha h, z) = \exp \left( \sum_{k=1}^{\infty} \frac{\alpha h(\pm k)}{k} z^\pm k \right). \]  

(2.22)

9
Then from [LW] we have:

$$E^+(\alpha h, z_1) E^-(\beta h, z_2) = \left(1 - \frac{z_2}{z_1}\right)^{-\gamma_0} E^-(\beta h, z_2) E^+(\alpha h, z_1). \quad (2.23)$$

Define

$$\Delta(h, z) = z^{h(0)} \exp \left( \sum_{k=1}^{\infty} \frac{h(k)}{-k} (-z)^{-k} \right) = z^{h(0)} E^+(-h, -z) \in U(V)\{z\}. \quad (2.24)$$

**Proposition 2.14.** Let $V$ be a vertex operator algebra and let $h$ be an element of $V$ satisfying (2.21). Then $\Delta(h, z) \in G^0(V)$.

Proposition 2.14 was essentially proved in [Li3], but for completeness, we present the proof. First we prove

**Lemma 2.15.** Let $h \in V$ satisfying (2.21). Then we have:

$$E^+(h, z_1) Y(a, z_2) E^+(-h, z_1) = Y(z_1^{h(0)} \Delta(-h, z_1 - z_2), z_2) \quad \text{for } a \in V. \quad (2.25)$$

**Proof.** For any $a \in V$, using the formula $[h(k), Y(a, z)] = \sum_{i=0}^{\infty} \left( \begin{array}{c} k \\ i \end{array} \right) z^{k-i} Y(h(i)a, z)$ we obtain

$$\left[ \sum_{k=1}^{\infty} \frac{h(k)}{k} z_1^{-k}, Y(a, z_2) \right]
= \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \frac{1}{k} \left( \begin{array}{c} k \\ i \end{array} \right) z_1^{-k} z_2^{-i} Y(h(i)a, z_2)
= \sum_{k=1}^{\infty} \frac{1}{k} z_1^{-k} z_2^{k} Y(h(0)a, z_2) + \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \frac{1}{k} \left( \begin{array}{c} k \\ i \end{array} \right) z_1^{-k} z_2^{k-i} Y(h(i)a, z_2)
= - \log \left(1 - \frac{z_2}{z_1}\right) Y(h(0)a, z_2)
+ \sum_{k=0}^{\infty} \sum_{i=1}^{\infty} \frac{1}{k+i} \left( \begin{array}{c} k+i \\ i \end{array} \right) z_1^{-k-i} z_2^{k} Y(h(i)a, z_2)
= - \log \left(1 - \frac{z_2}{z_1}\right) Y(h(0)a, z_2)
+ \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{i} (-1)^k \left( \begin{array}{c} -i \\ k \end{array} \right) z_1^{-k-i} z_2^{k} Y(h(i)a, z_2)
= - \log \left(1 - \frac{z_2}{z_1}\right) Y(h(0)a, z_2) + \sum_{i=1}^{\infty} \frac{1}{i} (z_1 - z_2)^{-i} Y(h(i)a, z_2). \quad (2.26)$$

Then

$$E^-(h, z_1) Y(a, z_2) E^-(-h, z_1)
= Y \left( \left(1 - \frac{z_2}{z_1}\right)^{-h(0)} E^+(h, z_1 - z_2)a, z_2 \right)
= Y \left(z_1^{h(0)} \Delta(-h, z_1 - z_2)a, z_2\right). \quad \square \quad (2.27)$$
Proof of Proposition 2.14. Since \([h(0), Y(u, z)] = Y(h(0)u, z)\) for any \(u \in V\), we have:

\[
zh^{h(0)}Y(u, z)z^{-h(0)} = Y(z^{h(0)}u, z) \quad \text{for any } u \in V.
\]

(2.28)

Then it follows from the construction of \(\Delta(h, z)\) and Lemma 2.15 that \(\Delta(h, z)\) satisfies (2.10). Since

\[
[L(-1), h(0)] = 0, \quad [L(-1), h(k)] = -kh(k - 1) \quad \text{for } k \in \mathbb{Z},
\]

we obtain

\[
[L(-1), \Delta(h, z)] = \sum_{k=1}^{\infty} h(k - 1)(-z)^{-k}\Delta(h, z) = \frac{d}{dz}\Delta(h, z).
\]

It is clear that \(\Delta(h, z)\) satisfies (2.7) and (2.8). Thus \(\Delta(h, z) \in G^0(V)\).

At the end of this section, we apply our results to some concrete examples. Let \(L\) be a positive-definite even lattice, let \(P\) be the dual lattice of \(L\) and let \(V_L\) be the vertex operator algebra constructed by Borcherds [B], Frenkel, Lepowsky and Meurman [FLM]. Then there is a 1-1 correspondence between the set of equivalence classes of irreducible modules for \(V_L\) and the set of cosets of \(P/L\) ([B], [FLM], [D1]). More specifically, \(V_P\) is a \(V_L\)-module with the following decomposition into irreducible modules:

\[
V_P = V_L + V_{L+\beta} + \cdots + V_{L+\beta_{k-1}}
\]

(2.29)

where \(k = |P/L|\).

Proposition 2.16. Let \(\beta \in P\). Then as a \(V_L\)-module, \((V_L, Y(\Delta(\beta, z)\cdot, z))\) is isomorphic to the \(V_L\)-module \(V_{L+\beta}\).

Proof. For any \(h' \in H = C \otimes_{\mathbb{Z}} L\), we have

\[
\Delta(\beta, z)h' = \Delta(\beta, z)h'(-1)1 = h' + z^{-1}\beta(h').
\]

(2.30)

Then \(\bar{Y}(h', z) = Y(h', z) + z^{-1}\beta(h')\). Thus \(\tilde{V}_L\) is a completely reducible module which is isomorphic to \(V_L\), for the Heisenberg algebra \(\bar{H}\) and the set of eigenfunctions of \(\bar{H}(0)\) on \(\tilde{V}_L\) is \(\beta + L\). Then it follows from [D1] that \((V_L, Y(\Delta(\beta, z)\cdot, z))\) is isomorphic to \(V_{L+\beta}\).

It follows from Proposition 2.16 that all irreducible \(V_L\)-modules can be obtained by using some \(\Delta(\beta, z)\) and that \((V_L, Y(\Delta(\beta, z)\cdot, z))\) is isomorphic to \((V_L, Y(\cdot, z))\) if and only if \(\beta \in L\). It is clear that \(\Delta(\beta, z)\) is invertible so that each irreducible module is a simple current. It is also clear that \(\Delta(\alpha, z)\Delta(\beta, z) = \Delta(\alpha + \beta, z)\) for \(\alpha, \beta \in P\). Let \(\beta_i \in P\) \((i = 1, 2)\). Then \(Y(\Delta(\beta_i, z)\cdot, z)\) is a nonzero intertwining operator of type

\[
\left( \frac{(V_L, Y(\Delta(\beta_1 + \beta_2, z)\cdot, z))}{(V_L, Y(\Delta(\beta_1, z)\cdot, z))(V_L, Y(\Delta(\beta_2, z)\cdot, z))} \right).
\]
Since each irreducible \( V_L \)-module is a simple current, all fusion rules are either zero or 1. This result on fusion rules has been obtained in [DL] using a different method. It is clear that Conjectures 3.5, 3.8 and 3.13 hold for \( V = V_L \).

Let \( g \) be a finite-dimensional simple Lie algebra with a fixed Cartan subalgebra \( H \), let \( \{\alpha_1, \ldots, \alpha_n\} \) be a set of positive roots and let \( \{e_i, f_i, h_i | i = 1, \ldots, n\} \) be the Chevalley generators. Let \( \theta = \sum_{i=1}^n a_i \alpha_i \) be the highest positive root and let \( \Omega \) be the dual Coxeter number of \( g \). Let \( \langle . , . \rangle \) be the normalized Killing form on \( g \) such that \( \langle \theta, \theta \rangle = 2 \). Let \( \lambda_i (i = 1, \ldots, n) \) be the fundamental weights of \( g \) and let \( P_+ \) be the set of dominant integral weights of \( g \). Recall from [Hum] that a dominant integral weight \( \lambda \) is said to be minimal if it is minimal in \( P_+ \). \( \lambda \) is said to be cominimal [FG] if \( \lambda' \) is minimal for the dual Lie algebra. From the table in [K], \( \lambda_i \) is cominimal if and only if \( a_i = 1 \). Let \( \tilde{g} \) be the affine Lie algebra [K]. For any complex number \( \ell \) and any weight \( \lambda \) of \( g \), let \( L(\ell, \lambda) \) be the irreducible highest weight \( \tilde{g} \)-module of level \( \ell \) with lowest weight \( \lambda \). It has been well known (cf. [FZ]) that \( L(\ell, 0) \) has a natural vertex operator algebra structure if \( \ell \neq -\Omega \).

**Proposition 2.17.** For any complex number \( \ell \neq -\Omega \), \( L(\ell, \ell \Lambda_i) \) is a simple current for \( L(\ell, 0) \) if \( \lambda_i \) is cominimal.

**Proof.** Choose \( h \in H \) such that \( \alpha_j(h) = \langle h, h_j \rangle = \delta_{i,j} \) for \( 1 \leq j \leq n \). Then we are going to show that \((V, Y(\Delta(h, z)\cdot, z))\) is isomorphic to \( L(\ell \Lambda_i) \). Since \( a_i = 1 \), \( \theta(h) = a_i = 1 \).

By definition we have:

\[
\begin{align*}
\Delta(h, z)h_j &= h_j + \ell \delta_{i,j} z^{-1}, \\
\Delta(h, z)e_i &= z e_i, \\
\Delta(h, z)f_i &= z^{-1} f_i, \\
\Delta(h, z) e_j &= e_j, \\
\Delta(h, z) f_j &= f_j, \\
\Delta(h, z) f_\theta &= z^{-1} f_\theta \quad \text{for } j \neq i.
\end{align*}
\]

In other words, the corresponding automorphism \( \psi \) of \( U(\tilde{g}) \) or \( U(L(\ell, 0)) \) satisfying the following conditions:

\[
\begin{align*}
\psi(h_i(n)) &= h_i(n) + \delta_{i,0} \ell, \\
\psi(e_i(n)) &= e_i(n + 1), \\
\psi(f_i(n)) &= f_i(n - 1); \\
\psi(h_j(n)) &= h_j(n), \\
\psi(e_j(n)) &= e_j(n), \\
\psi(f_j(n)) &= f_j(n) \quad \text{for } j \neq i, n \in \mathbb{Z},
\end{align*}
\]

and

\[
\psi(f_\theta(n)) = f_\theta(n - 1) \quad \text{for } n \in \mathbb{Z}.
\]

Then the vacuum vector \( 1 \) in \((V, Y(\Delta(h, z)\cdot, z))\) is a highest weight vector of weight \( \ell \Lambda_i \). Thus \((V, Y(\Delta(h, z)\cdot, z))\) is isomorphic to \( L(\ell, \ell \Lambda_i) \) as a \( \tilde{g} \)-module. By Theorem 2.12, \( L(\ell, \ell \Lambda_i) \) is a simple current.

**Remark 2.18:** Proposition 2.17 has been proved in [FG] by calculating the four point functions and it has also been proved in [F] that those are all simple currents except for \( E_8 \).
3 Extension of certain vertex operator algebras

The famous moonshine moonshine vertex operator algebra $V^\natural$ [FLM] was built as an extension of a vertex operator algebra by an irreducible module. (Another conceptual proof can be found in [Hua].) One important problem is to determine whether one can have an extension. The study of extension of a vertex operator algebra by a self-dual irreducible module with integral weights was initiated in [FHL] where they proved the duality or the Jacobi identity for two module elements and one algebra element. This section is devoted to the study of extension of certain vertex operator algebras by a simple current.

Let $W_i (i = 1, 2, 3)$ be three $V$-modules and let $I(\cdot, z)$ be an intertwining operator of type $\left( \begin{array}{c} W_3 \\ W_1 W_2 \end{array} \right)$. The contragredient intertwining operator $I'(\cdot, z)$ of $I(\cdot, z)$ is an intertwining operator of type $\left( \begin{array}{c} W_2' \\ W_1 W_3' \end{array} \right)$ defined ([FHL], [HL0-2]) as follows:

$$\langle u_2, I'(u_1, z)u_3' \rangle_2 = \langle I(e^{zL(1)}e^{\pi i L(0)}z^{-2L(0)}u_1, z^{-1})u_2, u_3' \rangle_3$$ (3.1)

for $u_i \in W_i, u_i' \in W_i'$. The transpose intertwining operator $I'(\cdot, z)$ of $I(\cdot, z)$ is an intertwining operator of type $\left( \begin{array}{c} W_3 \\ W_2 W_1 \end{array} \right)$ defined ([FHL], [HL0-2]) as follows:

$$I'(u_2, z)u_1 = e^{zL(-1)}I(u_1, e^{\pi i z})u_2 \quad \text{for } u_i \in W_i.$$ (3.2)

It was proved ([HL0-2], [Li4]) that

$$I \left( \begin{array}{c} W_3 \\ W_1 W_2 \end{array} \right) \simeq I \left( \begin{array}{c} W_3 \\ W_2 W_1 \end{array} \right) \simeq I \left( \begin{array}{c} W_2' \\ W_1 W_3' \end{array} \right).$$ (3.3)

Let $V$ be a self-dual vertex operator algebra, i.e., $V \simeq V'$ as a $V$-module and let $M$ be an irreducible $V$-module. From [Li1] we have:

$$I \left( \begin{array}{c} M' \\ VM \end{array} \right) \simeq \text{Hom}_V(M, M'), \quad \dim \text{Hom}_V(M, M') \leq 1.$$ (3.4)

Then $\dim I \left( \begin{array}{c} V \\ MM \end{array} \right) \leq 1$. Thus $\dim I \left( \begin{array}{c} V \\ MM \end{array} \right) = 1$ is equivalent to that $M$ is self-dual. From [Li1], any invariant bilinear form on $M$ is either symmetric or skew-symmetric. Let $Y(\cdot, z)$ be a nonzero intertwining operator of type $\left( \begin{array}{c} V \\ MM \end{array} \right)$ and let $(\cdot, \cdot)$ be the corresponding bilinear form on $M$. Then we have the following minor generalization of Proposition 5.6.1 in [FHL]:

13
Proposition 3.1. The bilinear form $(\cdot, \cdot)_M$ is symmetric (resp. skew-symmetric) if and only if
\[ Y(u, z)v = \pm e^{zL(-1)}Y(e^{2\pi i L(0)}v, e^{\pi iz})u \quad \text{for } u, v \in M, \] (3.5)
where “+” (resp. “−”) corresponds to a symmetric (resp. skew-symmetric) bilinear form.

Proposition 3.2 [FHL]. Let $V$ be a self-dual vertex operator algebra and let $M$ be a self-dual $V$-module with integral weights. Then the duality or the Jacobi identity for three elements in $V \cup M$ consisting of at most two module elements holds.

Let $V$ be a vertex operator algebra and let $M$ be a simple current with integral weights. The question is whether $V \oplus M$ always has a natural vertex operator algebra structure which extends $V$. In this section, our main goal is to prove that if $h \in V$ satisfies (2.21) such that $\gamma$ is an integer and $\tilde{V}$ is isomorphic to the adjoint module $V$, then $V \oplus \tilde{V}$ is a vertex operator (super)algebra.

From now on, we assume that $h \in V$ satisfies (2.21) such that $\gamma$ is an integer and $\tilde{V}$ is isomorphic to the adjoint module $V$. From Section 2, for any $V$-module $(M, Y_M(\cdot, z))$, we have a $V$-module $(\tilde{M}, Y_{\tilde{M}}(\cdot, z)) = (M, Y_M(\Delta(h, z) \cdot, z))$. This yields a linear isomorphism (the identity map) $\psi_M$ from $\tilde{M}$ onto $M$ such that
\[ \psi_M(Y(a, z)u) = Y(\Delta(h, z)a, z)\psi_M(u) \quad \text{for } a \in V, u \in \tilde{M}. \] (3.6)
Since $\Delta(h, z)^{-1} = \Delta(-h, z)$, (3.6) is equivalent to
\[ \psi_M(Y(\Delta(-h, z)a, z)u) = Y(a, z)\psi_M(u). \] (3.7)
Let $\pi_0$ be an isomorphism from $V$ onto $\tilde{V}$ and set $\phi_V = \psi_{\tilde{V}} \pi_0$. Since $\psi_{\tilde{V}}$ is a linear isomorphism from $\tilde{V}$ onto $\tilde{V}$, $\phi_V$ is a linear isomorphism from $V$ onto $\tilde{V}$ such that
\[ \phi_V(Y(a, z)b) = Y(\Delta(h, z)a, z)\phi_V(b), \] (3.8)
or equivalently
\[ \phi_V(Y(\Delta(-h, z)a, z)b) = Y(a, z)\phi_V(b) \quad \text{for } a, b \in V. \] (3.9)

Set $\tilde{V} = V \oplus \tilde{V}$. For any $u \in \tilde{V}, a \in V$ we define
\[ \tilde{Y}(a, z) := e^{zL(-1)}Y(a, -z)u, \] (3.10)
and for any $u, v \in \tilde{V}$, we define
\[ \tilde{Y}(u, z)v := \pi_0^{-1}\psi_{\tilde{V}}^{-1}\tilde{Y}(\Delta(h, z)u, z)\psi_V(v) = \phi_{\tilde{V}}^{-1}\tilde{Y}(\Delta(h, z)u, z)\psi_V(v). \] (3.11)

By Lemma 2.1 $\tilde{Y}(\cdot, z)$ are intertwining operators of types $\left( \begin{array}{c} \tilde{V} \\ VV \end{array} \right)$ and $\left( \begin{array}{c} V \\ \tilde{V}V \end{array} \right)$, respectively.
Lemma 3.3. The following identities hold
\[ e^{z(L(1)-h(1))}e^{-zL(1)} = \exp \left( \sum_{k=1}^{\infty} \frac{h(k)}{k} (-z)^k \right) = E^+(h, -z), \] (3.12)
\[ e^{z(L(-1)+h(-1))}e^{-zL(-1)} = \exp \left( \sum_{k=1}^{\infty} \frac{h(-k)}{k} z^k \right) = E^-(h, z). \] (3.13)

Proof. Set
\[ X(z) = e^{z(L(1)-h(1))}e^{-zL(1)} \exp \left( \sum_{k=1}^{\infty} \frac{h(k)}{-k} (-z)^k \right). \] (3.14)

Since \( X(z) = 1 \) when \( z = 0 \), it is sufficient to prove \( \frac{d}{dz} X(z) = 0 \). It follows from the Jacobi identity and (2.21) that \([L(1), h(k)] = -kh(k+1)\) for \( k \in \mathbb{Z} \). Then using induction on \( n \) we get
\[ h(1)L(1)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!} L(1)^{n-k} h(k+1). \] (3.15)

Thus
\[ h(1)e^{zL(1)} = e^{zL(1)} \sum_{k=1}^{\infty} h(k)z^{k-1}. \] (3.16)

Using (3.14) we obtain
\[
\begin{align*}
\frac{d}{dz} X(z) &= e^{z(L(1)-h(1))}(L(1) - h(1))e^{-zL(1)} \exp \left( \sum_{k=1}^{\infty} \frac{h(k)}{-k} (-z)^k \right) \\
&\quad - e^{z(L(1)-h(1))}L(1)e^{-zL(1)} \exp \left( \sum_{k=1}^{\infty} \frac{h(k)}{-k} (-z)^k \right) + X(z) \sum_{k=1}^{\infty} h(k)(-z)^{k-1} \\
&\quad - e^{z(L(1)-h(1))}h(1)e^{-zL(1)} \exp \left( \sum_{k=1}^{\infty} \frac{h(k)}{-k} (-z)^k \right) + X(z) \sum_{k=1}^{\infty} h(k)(-z)^{k-1} \\
&\quad - e^{z(L(1)-h(1))}e^{-zL(1)} \left( \sum_{k=1}^{\infty} h(k)(-z)^{k-1} \right) \exp \left( \sum_{k=1}^{\infty} \frac{h(k)}{-k} (-z)^k \right) \\
&\quad + X(z) \sum_{k=1}^{\infty} h(k)(-z)^{k-1}
\end{align*}
\]
\[ = 0. \] (3.17)

This proves the first identity. Using the symmetry: \( L(1) \mapsto -L(-1), h(k) \mapsto h(-k) \) for \( k = 1, 2, \cdots \), one can obtain the second identity. \( \square \)
Since $\Delta(h, z)h = h + z^{-1}\gamma$, we get
\[
\psi_W h(n) = (h(n) + \delta_{n,0}\gamma)\psi_W, \quad \phi_W h(n) = (h(n) + \delta_{n,0}\gamma)\phi_W \quad \text{for } n \in \mathbb{Z}. \quad (3.18)
\]

Then from the definition of $E^\pm(h, z)$ and $\Delta(h, z)$ we obtain
\[
\begin{align*}
\phi_W E^\pm(h, z) &= E^\pm(h, z)\phi_W, \quad \psi_W E^\pm(h, z) = E^\pm(h, z)\psi_W; \quad (3.19) \\
\psi_W \Delta(h, z) &= z^\gamma \Delta(h, z)\psi_W, \quad \phi_W \Delta(h, z) = z^\gamma \Delta(h, z)\phi_W. \quad (3.20)
\end{align*}
\]

**Lemma 3.4.** Let $W$ be any $V$-module. Then the following identities hold
\[
\begin{align*}
\psi_W e^{zL(-1)}\psi_W^{-1} e^{-zL(-1)} &= E^-(h, z), \quad e^{zL(-1)}\psi_W e^{-zL(-1)}\psi_W^{-1} = E^-(h, z), \quad (3.21) \\
\phi_W e^{zL(-1)}\phi_W^{-1} e^{-zL(-1)} &= E^-(h, z), \quad e^{zL(-1)}\phi_W e^{-zL(-1)}\phi_W^{-1} = E^-(h, z). \quad (3.22)
\end{align*}
\]

**Proof.** By a simple calculation we get
\[
\Delta(h, z)\omega = \omega + z^{-1}h + \frac{1}{2}\gamma z^{-2}, \quad \Delta(-h, z)\omega = \omega - z^{-1}h + \frac{1}{2}\gamma z^{-2}. \quad (3.23)
\]

Then from (3.6) and (3.7) we obtain
\[
\psi_W L(-1) = (L(-1) + h(-1))\psi_W, \quad L(-1)\psi_W = \psi_W (L(-1) - h(-1)). \quad (3.24)
\]

Thus
\[
\psi_W e^{zL(-1)}\psi_W^{-1} = e^{z(L(-1)+h(-1))}, \quad \psi_W^{-1} e^{zL(-1)}\psi_W = e^{z(L(-1)-h(-1))}. \quad (3.25)
\]

By Lemma 3.3 we obtain
\[
\psi_W e^{zL(-1)}\psi_W^{-1} e^{-zL(-1)} = E^-(h, z), \quad \phi_W^{-1} e^{zL(-1)}\phi_W e^{-zL(-1)} = E^-(h, z).
\]

Similarly, one can prove the identities for $\phi_W$. □

**Lemma 3.5.** (a) For any $a, b \in V$, we have
\[
Y(\psi_V \phi_V a, z)b = E^-(h, z)^2\psi_V \phi_V Y(a, z)\Delta(-h, -z)^2 b. \quad (3.26)
\]

(b) For any $u \in \tilde{V}, a \in V$, we have
\[
\tilde{Y}(u, z)a = E^-(h, z)Y(\Delta(h, z)\phi_V^{-1}(u), z)\phi_V \Delta(-h, -z)a; \quad (3.27)
\]

(c) For any $u, v \in \tilde{V}$, we have
\[
\tilde{Y}(u, z)v = z^{-\gamma}E^-(h, z)Y(\Delta(h, z)\phi_V^{-1}(u), z)\Delta(-h, -z)\psi_V(v). \quad (3.28)
\]
Lemma 3.4 we obtain

\[ u, v \]

following condition:

\[ \text{The defined intertwining operator} \]

\[ \text{Proof.} \] Using the skew-symmetry ([B], [FHL]) \( Y(u, z)v = e^{zL(-1)}Y(v, -z)u \) and Lemma 3.4 we obtain

\[
\begin{align*}
\bar{Y}(\psi_Y \phi_Y a, z)b &= e^{zL(-1)}Y(b, -z)\psi_Y \phi_Y a \\
&= e^{zL(-1)}\psi_Y \phi_Y (\Delta(-h, -z)^2 b, -z)a \\
&= e^{zL(-1)}\psi_Y \phi_Y e^{-zL(-1)}Y(a, z)\Delta(-h, -z)^2 b \\
&= E^{-}(-h, z)^2 \psi_Y \phi_Y Y(a, z)\Delta(-h, -z)^2 b. \quad (3.29)
\end{align*}
\]

Similarly we obtain

\[
\begin{align*}
\bar{Y}(u, z)a &= e^{zL(-1)}Y(a, -z)u \\
&= e^{zL(-1)}\phi_Y Y(\Delta(-h, -z)a, -z)\phi_Y^{-1}(u) \\
&= e^{zL(-1)}\phi_Y e^{-zL(-1)}Y(\phi_Y^{-1}(u), z)\Delta(-h, -z)a \\
&= e^{zL(-1)}\phi_Y e^{-zL(-1)}\phi_Y^{-1}Y(\Delta(h, z)\phi_Y^{-1}(u), z)\phi_Y \Delta(-h, -z)a \\
&= E^{-}(-h, z)Y(\Delta(h, z)\phi_Y^{-1}(u), z)\phi_Y \Delta(-h, -z)a. \quad (3.30)
\end{align*}
\]

For any \( u, v \in \tilde{V} \), we get

\[
\begin{align*}
\bar{Y}(u, z)v &= \phi_Y^{-1}\bar{Y}(\Delta(h, z)u, z)\psi_Y (v) \\
&= \phi_Y^{-1}e^{zL(-1)}Y(\psi_Y (v), -z)\Delta(h, z)u \\
&= \phi_Y^{-1}e^{zL(-1)}\phi_Y Y(\Delta(-h, -z)\psi_Y (v), -z)\phi_Y^{-1}\Delta(h, z)u \\
&= \phi_Y^{-1}e^{zL(-1)}\phi_Y e^{-zL(-1)}Y(\phi_Y^{-1}\Delta(h, z)u, z)\Delta(-h, -z)\psi_Y (v) \\
&= E^{-}(-h, z)Y(\phi_Y^{-1}\Delta(h, z)u, z)\Delta(-h, -z)\psi_Y (v) \\
&= z^{-\gamma}E^{-}(-h, z)Y(\Delta(h, z)\phi_Y^{-1}(u), z)\Delta(-h, -z)\psi_Y (v). \quad (3.31)
\end{align*}
\]

This proves the lemma. \( \square \)

Lemma 3.6. The defined intertwining operator \( \bar{Y}(\cdot, z) \) of type \( \left( \begin{array}{c} V \\ \tilde{V} \end{array} \right) \) satisfies the following condition:

\[ \bar{Y}(u, z)v = (-1)^\gamma e^{zL(-1)}\bar{Y}(v, -z)u \quad \text{for any} \ u, v \in \tilde{V}. \quad (3.32) \]

Proof. For any \( u, v \in \tilde{V} \), using Lemma 3.5 we obtain

\[
\begin{align*}
e^{zL(-1)}\bar{Y}(v, -z)u \\
&= e^{zL(-1)}(-z)^{-\gamma}E^{-}(-h, -z)Y(\Delta(h, -z)\phi_Y^{-1}(v), -z)\Delta(-h, -z)\psi_Y (v) \\
&= e^{zL(-1)}(-z)^{-\gamma}E^{-}(-h, -z)Y(\Delta(h, -z)\phi_Y^{-1}(v), -z)\Delta(-h, -z)\phi_Y \phi_Y^{-1}(u) \\
&= (-z)^{-\gamma}z^{2\gamma}e^{zL(-1)}E^{-}(-h, -z)Y(\Delta(h, -z)\phi_Y^{-1}(v), -z)\psi_Y \phi_Y \Delta(-h, -z)\phi_Y^{-1}(u) \\
&= (-z)^{\gamma}e^{zL(-1)}E^{-}(-h, -z)\psi_Y \phi_Y Y(\Delta(-h, -z)\phi_Y^{-1}(v), -z)\Delta(-h, -z)\phi_Y^{-1}(u)
\end{align*}
\]
\[ (-z)\gamma e^{zL(-1)}E^-(h, -z)\psi_V \phi_V \]
\[ \cdot e^{-zL(-1)}Y(\Delta(h, z)\phi_V^{-1}(u), z)\Delta(-h, -z)\phi_V^{-1}(v) \]
\[ = (-z)\gamma (-z)^{-2}e^{zL(-1)}E^-(h, -z)\psi_V \phi_V e^{-zL(-1)}. \]
\[ \cdot Y(\Delta(h, z)\phi_V^{-1}(u), z)\phi_V^{-1}\psi_V^{-1} \Delta(-h, -z)\psi_V(v) \]
\[ = (-z)^{-\gamma}e^{zL(-1)}E^-(h, -z)\psi_V \phi_V e^{-zL(-1)}\phi_V^{-1}\psi_V^{-1} \]
\[ \cdot Y(\Delta(h, z)\phi_V^{-1}(u), z)\Delta(-h, -z)\psi_V(v) \quad (3.33) \]

and
\[ (-1)^{\gamma}Y(u, z)v = (-z)^{-\gamma}E^-(h, z)Y(\Delta(h, z)\phi_V^{-1}(u), z)\Delta(-h, -z)\psi_V(v). \quad (3.34) \]

Using Lemma 3.4 we get
\[ e^{zL(-1)}E^-(h, -z)\psi_V \phi_V e^{-zL(-1)}\phi_V^{-1}\psi_V^{-1} \]
\[ = e^{zL(-1)}e^{-zL(-1)}\psi_V e^{zL(-1)}\phi_V e^{-zL(-1)}\phi_V^{-1}\psi_V^{-1} \]
\[ = \psi_V e^{zL(-1)}\phi_V e^{-zL(-1)}\phi_V^{-1}\psi_V^{-1} \]
\[ = \psi_V E^-(h, z)\psi_V^{-1} \]
\[ = E^-(h, z). \quad (3.35) \]

Then (3.32) follows immediately. \( \square \)

**Lemma 3.7.** For any \( a \in V \), the following identity hold

\[ Y(E^-(h, z_1)a, z_2) \]
\[ = E^-(h, z_1 + z_2)E^-(h, z_2)Y(a, z_2)z_2^{-h(0)}E^+(h, z_2)(z_2 + z_1)^h(0)E^+(h, z_2 + z_1). \quad (3.36) \]

**Proof.** It is equivalent to prove that

\[ E^-(h, z_1 + z_2)Y(E^-(h, z_1)a, z_2)(z_2 + z_1)^{-h(0)}E^+(h, z_2 + z_1) \]
\[ = E^-(h, z_2)Y(a, z_2)z_2^{-h(0)}E^+(h, z_2). \quad (3.37) \]

Since it is true when \( z_1 = 0 \), it is enough for us to prove that partial derivatives for both sides with respect to the variable \( z_1 \) are equal. For any positive integer \( k \) and for any \( u \in V \), from the Jacobi identity we get

\[ Y(h(-k)u, z_2) \]
\[ = \sum_{i=0}^{\infty} \left( \begin{array}{c} -k \\ i \end{array} \right) ((-z_2)^i h(-k - i)Y(u, z_2) - (-z_2)^{-k-i}Y(u, z_2)h(i)). \quad (3.38) \]
Using the binomial coefficient formulas: \( \binom{i-k}{i} = (-1)^i \binom{k-1}{i} \) and \( \binom{-i}{k} = (-1)^{k+i-1} \binom{-k-1}{i-1} \), then using (3.38) for \( u = E^-(h, z_1) \) we obtain

\[
\frac{\partial}{\partial z_1} Y(E^-(h, z_1)a, z_2)
\]

\[
= Y \left( \sum_{k=1}^{\infty} z_1^{k-1} h(-k) E^-(h, z_1)a, z_2 \right)
\]

\[
= \sum_{k=1}^{\infty} z_1^{k-1} Y \left( h(-k) E^-(h, z_1)a, z_2 \right)
\]

\[
= \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \binom{-k}{i} z_1^{k-1} (-z_2)^i h(-k-i) Y(u, z_2)
\]

\[- \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \binom{-k}{i} z_1^{k-1} (-z_2)^{-k-i} Y(u, z_2) h(i)
\]

\[
= \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \binom{i-k}{i} z_1^{k-i-1} (-z_2)^i h(-k) Y(u, z_2)
\]

\[- \sum_{k=1}^{\infty} z_1^{k-1} (-z_2)^{-k} Y(u, z_2) h(0)
\]

\[- \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \binom{-k}{i} z_1^{k-1} (-z_2)^{-k-i} Y(u, z_2) h(i)
\]

\[
= \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} (1)^i \binom{k-1}{i} z_1^{k-i-1} (-z_2)^i h(-k) Y(u, z_2)
\]

\[+ (z_2 + z_1)^{-1} Y(u, z_2) h(0) - \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \binom{-i}{k} z_1^{-1} (-z_2)^{-i-k} Y(u, z_2) h(k)
\]

\[
= \sum_{k=1}^{\infty} (z_1 + z_2)^{-1} h(-k) Y(u, z_2) + (z_2 + z_1)^{-1} Y(u, z_2) h(0)
\]

\[- \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} (-1)^{k+i-1} \binom{-k-1}{i-1} z_1^{-1} (-z_2)^{-i-k} Y(u, z_2) h(k)
\]

\[
= \sum_{k=1}^{\infty} (z_1 + z_2)^{-1} h(-k) Y(u, z_2) + (z_2 + z_1)^{-1} Y(u, z_2) h(0)
\]

\[+ \sum_{k=1}^{\infty} (z_2 + z_1)^{-k-1} Y(u, z_2) h(k).
\]

Therefore

\[
\frac{\partial}{\partial z_1} E^-(h, z_1 + z_2) Y(E^-(h, z_1)a, z_2)(z_2 + z_1)^{-h(0)} E^+(h, z_2 + z_1) = 0.
\]

Then the proof is complete. □
Lemma 3.8. For \( a \in V \), we have

\[
E^-(h, z_1)Y(a, z_2)E^-(h, z_1) = Y(\Delta(-h, z_2 - z_1)\Delta(h, z_2)a, z_2).
\]

(3.41)

Proof. For any \( a \in V \), we have:

\[
[h(-k), Y(a, z)] = \sum_{i=0}^{\infty} \binom{-k}{i} z^{-k-i}Y(h(i)a, z).
\]

(3.42)

Noticing that \( \frac{1}{k} \binom{-k}{i} = (-1)^{i+k} \binom{-i}{k} \) for any nonnegative integers \( k \) and \( i \), we have:

\[
\left[ \sum_{k=1}^{\infty} \frac{h(-k)}{k} z_1^k, Y(a, z_2) \right] = \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \frac{1}{k} \binom{-k}{i} z_1^k z_2^{-k-i}Y(h(i)a, z_2)
\]

\[
= \sum_{k=1}^{\infty} \frac{1}{k} z_1^k z_2^{-k}Y(h(0)a, z_2) + \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \frac{1}{k} \binom{-k}{i} z_1^k z_2^{-k-i}Y(h(i)a, z_2)
\]

\[
= - \log \left( 1 - \frac{z_1}{z_2} \right) Y(h(0)a, z_2)
\]

\[
+ \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} (-1)^{i+k} \frac{1}{i} \binom{-i}{k} z_1^k z_2^{-k-i}Y(h(i)a, z_2)
\]

\[
= - \log \left( 1 - \frac{z_1}{z_2} \right) Y(h(0)a, z_2)
\]

\[
+ \sum_{i=1}^{\infty} \frac{1}{i} \left( (-z_2 + z_1)^{-i} - (-z_2)^{-i} \right) Y(h(i)a, z_2).
\]

(3.43)

Then

\[
E^-(h, z_1)Y(a, z_2)E^-(h, z_1) = Y \left( \left( 1 - \frac{z_1}{z_2} \right)^{-h(0)} E^+(h, -z_2 + z_1)E^+(h, -z_2)a, z_2 \right)
\]

\[
= Y \left( \Delta(-h, z_2 - z_1)\Delta(h, z_2)a, z_2 \right). \quad \Box
\]

(3.44)

Now we are ready to prove our main theorem.

Theorem 3.9. Let \( V \) be a vertex operator algebra and let \( h \in V \) satisfying the conditions (2.21) with an integer \( \gamma \). Then a) if \( \gamma \) is even, \( V \bigoplus \tilde{V} \) is a vertex operator algebra. b) if \( \gamma \) is odd, \( V \bigoplus \tilde{V} \) is a vertex operator superalgebra.

Proof. From [FHL], we only need to prove the Jacobi identity for three module elements. For any \( u, v, w \in \tilde{V} \), since \( Y(v, z)w \in V((z)) \), using Lemmas 3.5 and 3.8 we
obtain

\[ \bar{Y}(u, z_1)\bar{Y}(v, z_2) \]
\[ = E^-(\Delta(h, z_1), \phi_V^{-1}(u), z_1)\phi_V \Delta(-h, -z_1)\bar{Y}(v, z_2)w \]
\[ = E^-(\Delta(h, z_1), \phi_V^{-1}(u), z_1)\phi_V \Delta(-h, -z_1) \cdot \]
\[ \cdot z_2^{-\gamma} E^-(\Delta(h, z_2), \phi_V^{-1}v, z_2)\Delta(-h, -z_2)\psi_V(w) \]
\[ = E^-(\Delta(h, z_1), \phi_V^{-1}(u), z_1)E^-(\Delta(h, z_2), \phi_V^{-1}v, z_2) \left( 1 - \frac{z_2}{z_1} \right)^\gamma \cdot \]
\[ \cdot \phi_V \Delta(-h, -z_1)\phi_V^{-1}v, z_2)\Delta(-h, -z_2)\psi_V(w) \]
\[ = E^-(\Delta(h, z_1), \phi_V^{-1}(u), z_1)E^-(\Delta(h, z_2), \phi_V^{-1}v, z_2) \left( 1 - \frac{z_2}{z_1} \right)^\gamma \cdot \]
\[ \cdot \phi_V \Delta(-h, -z_1 + z_2)\Delta(h, z_2)\phi_V^{-1}v, z_2)\Delta(-h, -z_2)\psi_V(w) \]
\[ = E^-(\Delta(h, z_1), \phi_V^{-1}(u), z_1)E^-(\Delta(h, z_1 - z_2), \phi_V^{-1}v, z_2) \left( 1 - \frac{z_2}{z_1} \right)^\gamma \cdot \]
\[ \cdot \phi_V \Delta(-h, -z_1 + z_2)\Delta(h, z_2)\phi_V^{-1}v, z_2)\Delta(-h, -z_2)\psi_V(w) \]
\[ = \phi_V \psi_V E^-(\Delta(h, z_1), \phi_V^{-1}(u), z_1)Y(\Delta(-h, z_1 - z_2), \phi_V^{-1}v, z_2) \cdot \]
\[ \cdot Y(\Delta(-h, -z_1 + z_2), \phi_V^{-1}v, z_2)\Delta(-h, -z_1)\Delta(-h, -z_2)w. \]  

(3.45)

Symmetrically, we have:

\[ \bar{Y}(v, z_2)\bar{Y}(u, z_1)w \]
\[ = \phi_V \psi_V E^-(\Delta(h, z_1), \phi_V^{-1}(u), z_1)Y(\Delta(-h, z_2 - z_1), \phi_V^{-1}v, z_2) \cdot \]
\[ \cdot Y(\Delta(-h, -z_2 + z_1), \phi_V^{-1}u, z_1)\Delta(-h, -z_1)\Delta(-h, -z_2)w. \]  

(3.46)

Then

\[ z_0^{-1}\delta \left( \frac{z_1 - z_2}{z_0} \right) \bar{Y}(u, z_1)\bar{Y}(v, z_2)w - (-1)^\gamma z_0^{-1}\delta \left( \frac{z_2 - z_1}{z_0} \right) \bar{Y}(v, z_2)\bar{Y}(u, z_1)w \]
\[ = z_0^{-1}\delta \left( \frac{z_1 - z_2}{z_0} \right) \phi_V \psi_V E^-(\Delta(-h, z_1), \phi_V^{-1}v, z_2) \cdot \]
\[ \cdot Y(\Delta(-h, z_2), \phi_V^{-1}u, z_1)\Delta(-h, -z_2)w \cdot \]
\[ \cdot Y(\Delta(-h, -z_0), \phi_V^{-1}v, z_2)\Delta(-h, -z_2)w \cdot \]
\[ \cdot Y(\Delta(-h, -z_0), \phi_V^{-1}v, z_2)\Delta(-h, -z_2)w \cdot \]
\[ = z_2^{-1}\delta \left( \frac{z_1 - z_0}{z_2} \right) \phi_V \psi_V E^-(\Delta(-h, z_1), \phi_V^{-1}v, z_2) \cdot \]
\[ \cdot Y(\Delta(-h, -z_2), \phi_V^{-1}u, z_1)\Delta(-h, -z_2)w \cdot \]
\[ \cdot Y(\Delta(-h, -z_0), \phi_V^{-1}v, z_2)\Delta(-h, -z_2)w \cdot \]
\[ \cdot Y(\Delta(-h, -z_0), \phi_V^{-1}v, z_2)\Delta(-h, -z_2)w \cdot \]  

(3.47)

where

\[ A = \Delta(-h, z_0), B = \Delta(-h, -z_0), C = \Delta(-h, -z_1)\Delta(-h, -z_2)w. \]  

(3.48)
On the other hand, using Lemmas 3.5 and 3.7 we obtain

\[
\begin{align*}
&z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y(\bar{Y}(u, z_0) v, z_2) w \\
&= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y \left( E^- (-h, z_0) Y(\phi_V^{-1} \Delta(h, z_0) u, z_0) \Delta(-h, -z_0) \psi_V(v), z_2 \right) w \\
&= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) E^- (-h, z_0 + z_2) E^-(h, z_2) \\
&\cdot Y \left( \phi_V^{-1} \Delta(h, z_0) u, z_0 \right) \Delta(-h, -z_0) \psi_V(v), z_2 \right) \cdot z \cdot h^0 E^+(h, z_2 + z_0) w \\
&= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) E^- (-h, z_0 + z_2) E^-(h, z_2) \\
&\cdot Y \left( \phi_V^{-1} \Delta(h, z_0) u, z_0 \right) \Delta(-h, -z_0) \psi_V(v), z_2 \right) \Delta(h, -z_2) \Delta(-h, -z_2 - z_0) w \\
&= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) E^- (-h, z_0) E^-(h, z_2) z_0^{-\gamma} \\
&\cdot Y \left( \phi_V^{-1} \Delta(h, z_0) \psi_V^{-1}(u), z_0 \right) \Delta(-h, -z_0) \psi_V(v), z_2 \right) \Delta(h, -z_2) \Delta(-h, -z_1) w \\
&= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) E^- (-h, z_0) E^-(h, z_2) z_0^{-\gamma} \cdot z_0^{2\gamma} \\
&\cdot Y \left( \phi_V^{-1}(u), z_0 \right) \psi_V \phi_V \Delta(-h, -z_0) \phi_V^{-1}(v), z_2 \right) \Delta(h, -z_2) \Delta(-h, -z_1) w \\
&= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) \psi_V \phi_V E^- (-h, z_0) E^-(h, z_2) z_0^{\gamma} \\
&\cdot Y \left( \phi_V^{-1}(u), z_0 \right) \Delta(-h, -z_0) \phi_V^{-1}(v), z_2 \right) \Delta(-h, -z_2) \Delta(-h, -z_1) w \\(3.49)
\end{align*}
\]

Therefore, the Jacobi identity holds. Then the proof is complete. \[\square\]

**Remark 3.10.** Under the assumption of Theorem 3.9, suppose that $V$ is a simple vertex operator algebra. Then $\tilde{V} = V \oplus \tilde{V}$ is a completely reducible $V$-module. Let $P_0$ and $P_1$ be the projection maps from $\tilde{V}$ onto $V$ and $\tilde{V}$, respectively. Then $P_0$ and $P_1$ are $V$-homomorphisms. Let $W$ be any irreducible $V$-submodule of $\tilde{V}$. Then $P_0$ and $P_1$ are $V$-homomorphisms from $W$ to $V$ and $\tilde{V}$, respectively. Thus, if $\tilde{V}$ as a $V$-module is not isomorphic to $V$, either $P_0(W) = 0$ or $P_1(W) = 0$, so that $W = V$ or $W = \tilde{V}$. Therefore, if $\tilde{V}$ as a $V$-module is not isomorphic to $V$, $\tilde{V}$ is a simple vertex operator superalgebra.

Next, continuing with Theorem 3.9 we study modules for the vertex operator (super)algebra $\tilde{V}$. We define a linear endomorphism $\sigma$ of $\tilde{V}$ by: $\sigma|_V = 1$, $\sigma|_{\tilde{V}} = -1$. Then it is clear that $\sigma$ is an order-two automorphism of the vertex operator algebra $\tilde{V}$. Recall that $\pi_0$ is a $V$-isomorphism from $V$ onto $\tilde{V}$. Let $W$ be any $V$-module. Then we shall show that $\tilde{W}$ is isomorphic to $W$ as a $V$-module. For simplicity we shall prove this only for an irreducible $V$-module $W$. 

22
Proposition 3.11. Let $W$ be an irreducible $V$-module. Then the linear map

$$I(\cdot, z) : \ V \to \text{Hom}(W, \tilde{W})\{z\};$$

$$I(a, z)w = e^{zL(-1)}\psi^{-1}_W\psi^{-1}_W e^{-zL(-1)}Y_W(\psi_V\psi_{\tilde{V}}\pi_0(a), z)\Delta(2h, -z)w. \quad (3.50)$$

is a nonzero intertwining operator of type \( \begin{pmatrix} W \\ VW \end{pmatrix} \). Consequently, $\pi_W := I(1, z)$ is a $V$-isomorphism from $W$ onto $\tilde{W}$.

**Proof.** We shall construct $I(\cdot, z)$ from the intertwining operator $Y_W(\cdot, z)$ through the following sequence:

$$\begin{pmatrix} W \\ VW \end{pmatrix} \to \begin{pmatrix} W \\ WV \end{pmatrix} \to \begin{pmatrix} \tilde{W} \\ W\tilde{V} \end{pmatrix} \to \begin{pmatrix} \tilde{W} \\ WV \end{pmatrix} \to \begin{pmatrix} \tilde{W} \\ V\tilde{W} \end{pmatrix}. \quad (3.51)$$

Starting from the intertwining operator $Y_W(\cdot, z)$ of type \( \begin{pmatrix} W \\ VW \end{pmatrix} \) we have an intertwining operator $I_1(\cdot, z)$ defined as follows [FHL]:

$$I_1(w, z)a = e^{zL(-1)}Y_W(a, -z)w \quad \text{for any } a \in V, w \in W. \quad (3.52)$$

Since $Y_W(\cdot, z)$ is not zero, $I_1(\cdot, z)$ is not zero either. By Lemma 2.1 we have an intertwining operator $I_2(\cdot, z)$ of type \( \begin{pmatrix} \tilde{W} \\ W\tilde{V} \end{pmatrix} \) defined by

$$I_2(w, z)u = \psi^{-1}_W\psi^{-1}_W I_1(\Delta(2h, z)w, z)\psi_V\psi_{\tilde{V}}u \quad \text{for any } u \in \tilde{V}, w \in W. \quad (3.53)$$

Using the isomorphism $\pi_0$ from $\tilde{V}$ onto $V$ we obtain an intertwining operator $I_3(\cdot, z)$ of type \( \begin{pmatrix} \tilde{W} \\ V\tilde{W} \end{pmatrix} \) defined as follows:

$$I_3(w, z)a = I_2(w, z)\pi_0a \quad \text{for any } w \in W, a \in V. \quad (3.54)$$

Finally we obtain an intertwining operator $I(\cdot, z)$ of type \( \begin{pmatrix} \tilde{W} \\ V\tilde{W} \end{pmatrix} \) defined by

$$I(a, z)w = e^{zL(-1)}I_3(w, -z)a \quad \text{for any } a \in V, w \in W. \quad (3.55)$$

Composing all intertwining operators together we obtain

$$I(a, z)w = e^{zL(-1)}I_3(w, -z)a$$

$$= e^{zL(-1)}I_2(w, -z)\pi_0(a)$$

$$= e^{zL(-1)}\psi^{-1}_W\psi^{-1}_W I_1(\Delta(2h, -z)w, -z)\psi_V\psi_{\tilde{V}}\pi_0(a)$$

$$= e^{zL(-1)}\psi^{-1}_W\psi^{-1}_W e^{-zL(-1)}Y_W(\psi_V\psi_{\tilde{V}}\pi_0(a), z)\Delta(2h, -z)w. \quad (3.56)$$
\[
\frac{d}{dz} I(1, z) = I(L(-1)1, z) = 0, \tag{3.57}
\]

\(I(1, z)\) is a constant linear map from \(W\) to \(\tilde{W}\). Since \([Y(a, z_1), I(1, z)] = 0\) for any \(a \in V\), \(I(1, z)\) is a \(V\)-homomorphism from \(W\) to \(\tilde{W}\). Since \(W\) and \(\tilde{W}\) are irreducible \(V\)-modules, \(I(1, z)\) is a \(V\)-isomorphism. Then the proof is complete. \(\square\)

Let \(W\) be an irreducible \(V\)-module (with finite-dimensional homogeneous subspaces). Since \(h(0)\) preserves each homogeneous subspace of \(M\) and \(h(0)\) is semisimple on \(V\), \(h(0)\) acts semisimply on \(W\). Suppose that \(h(0)\) has only rational eigenvalues on \(W\). Set \(\phi_W = \psi_W \pi_W\). That is, \(\phi_W\) is a linear isomorphism from \(W\) onto \(\tilde{W}\) such that

\[
\phi_W(Y(a, z)w) = Y(\Delta(h, z)a, z)\phi_W(w) \quad \text{for } a \in V, w \in W. \tag{3.58}
\]

Next we shall make \(\tilde{W} := W \oplus \tilde{W}\) a \(\tilde{V}\)-module or \(\sigma\)-twisted \(\tilde{V}\)-module. In order to do this we need intertwining operators of types \(\begin{pmatrix} \tilde{W} \\ \tilde{V} \end{pmatrix}\) and \(\begin{pmatrix} W \\ \tilde{W} \end{pmatrix}\).

Following [DL], for any \(a \in Q\) we define \((-z)^a = e^{a \pi i}\). For any \(a \in V, w \in W\), define \(F(u, z)a = e^{zL(-1)}Y_W(a, -z)u\). Then \(F(\cdot, z)\) is an intertwining operator of type \(\begin{pmatrix} W \\ W \end{pmatrix}\).

For any \(u \in \tilde{V}, w \in W\), define \(F_1(w, z)u = \psi_W^{-1}F(\Delta(h, z)w, z)\psi_V(u)\). Then from Lemma 2.1, \(F_1(\cdot, z)\) is an intertwining operator of type \(\begin{pmatrix} \tilde{W} \\ \tilde{W} \end{pmatrix}\). For any \(u \in \tilde{V}, w \in W\) we define \(Y(u, z)w = e^{zL(-1)}F_1(w, -z)u\). Then \(Y(\cdot, z)\) is an intertwining operator of type \(\begin{pmatrix} \tilde{W} \\ \tilde{W} \end{pmatrix}\).

That is,

\[
Y(u, z)w = e^{zL(-1)}F_1(w, -z)u
= e^{zL(-1)}\psi_W^{-1}F(\Delta(h, -z)w, -z)\psi_V(u)
= e^{zL(-1)}\psi_W^{-1}e^{-zL(-1)}Y_W(\psi_V(u), z)\Delta(h, -z)w
= E^-(h, z)\psi_W^{-1}Y(\psi_V(u), z)\Delta(h, -z)w. \tag{3.59}
\]

Then we obtain an intertwining operator of type \(\begin{pmatrix} \tilde{W} \\ \tilde{V} \end{pmatrix}\).

From Lemma 2.1 we obtain an intertwining operator \(\tilde{Y}(\cdot, z)\) of type \(\begin{pmatrix} \tilde{W} \\ \tilde{V} \end{pmatrix}\). Then \(Y(\cdot, z) := \pi_W^{-1}\tilde{Y}(\cdot, z)\) is an intertwining operator of type \(\begin{pmatrix} W \\ \tilde{V} \end{pmatrix}\). For any \(u \in \tilde{V}, \tilde{w} \in \tilde{W}\), we have:

\[
Y(u, z)\tilde{w} = \pi_W^{-1}\tilde{Y}(u, z)\tilde{w}
= \pi_W^{-1}\psi_W^{-1}Y(\Delta(h, z)u, z)\psi_W\tilde{w}
= E^-(h, z)\pi_W^{-1}\psi_W^{-1}\psi_WY(\psi_V\Delta(h, z)(u), z)\Delta(h, -z)\psi_W\tilde{w}. \tag{3.60}
\]
Then we obtain an intertwining operator of type \( \left( \begin{array}{c} W \\ \hat{VW} \end{array} \right) \).

Similar to Lemma 3.6 we have:

**Lemma 3.12.** For \( a \in V, u \in \hat{V}, w \in W, \tilde{w} \in \hat{W}, \) we have

\[
Y(\psi_V \phi_V a, z)u = E^-(2h, z)\psi_W \psi_{\hat{W}} \pi_W Y(a, z) \Delta(-2h, -z)u, \\
Y(u, z)w = E^-(h, z)\phi_W Y(\phi_V^{-1} u, z) \Delta(-h, -z)w, \\
Y(u, z)\tilde{w} = (-1)^{\gamma} E^-(h, z)\psi_W Y(\phi_V^{-1} u, z) \Delta(-h, -z)\tilde{w}.
\]

**Proof.** Recall from Proposition 3.11 that \( I(\cdot, z) \) is an intertwining operator of type \( \left( \begin{array}{c} \hat{W} \\ VW \end{array} \right) \). Then \( \pi_W^{-1} I(\cdot, z) \) is an intertwining operator of type \( \left( \begin{array}{c} W \\ VW \end{array} \right) \). Since \( I \left( \begin{array}{c} W \\ VW \end{array} \right) \) is one-dimensional and \( \pi_W^{-1} I(1, z) = id_W = Y_W(1, z) \), we get \( \pi_W^{-1} I(\cdot, z) = Y_W(\cdot, z) \). Thus

\[
Y_W(a, z)w = \pi_W^{-1} e^{z_L(-1)} \psi_W^{-1} \psi_{\hat{W}}^{-1} e^{-z_L(-1)} Y_W(\psi_V \phi_V \pi_0(a), z) \Delta(2h, -z)w.
\]

Therefore (by Lemma 3.4)

\[
Y_W(\psi_V \phi_V \pi_0(a), z)w = e^{z_L(-1)} \psi_W \psi_{\hat{W}}^{-1} \pi_W Y_W(a, z) \Delta(-2h, -z)w = E^-(2h, z)\psi_W \psi_{\hat{W}}^{-1} \pi_W Y_W(a, z) \Delta(-2h, -z)w.
\]

For any \( u \in \hat{V}, w \in W, \) using the first identity we obtain

\[
Y(u, z)w = E^-(h, z)\psi_W \psi_{\hat{W}}^{-1} \psi_W^{-1} Y(\psi_V \phi_V \phi_V^{-1} u, z) \Delta(h, -z)w = E^-(h, z)\psi_W \psi_{\hat{W}} Y(\phi_V^{-1} u, z) \Delta(-h, -z)w.
\]

Similarly, for any \( u \in \hat{V}, \tilde{w} \in \hat{W}, \) we obtain

\[
Y(u, z)\tilde{w} = E^-(h, z)\psi_W \psi_{\hat{W}} Y(\phi_V^{-1} \Delta(h, z)u, z) \Delta(-h, -z)\psi_W \tilde{w} = z^{-\gamma} E^-(h, z)\psi_W Y(\Delta(h, z)\phi_V^{-1} u, z) \Delta(-h, -z)\psi_W \tilde{w} = (-1)^{\gamma} E^-(h, z)\psi_W Y(\phi_V^{-1} u, z) \Delta(-h, -z)\tilde{w}.
\]

Then the proof is complete. \( \square \)

Recall ([D2], [FFR], [FLM]) that a \( \sigma \)-twisted \( \hat{V} \)-module is a \( \mathbb{C} \)-graded vector space \( M = \bigoplus_{h \in \mathbb{C}} M_h \) satisfying all conditions for a module except that \( Y_M(a, z) \in \text{End} M[[z^{\frac{1}{\gamma}}, z^{-\frac{1}{\gamma}}]] \) for \( a \in \hat{V} \) and that the Jacobi identity is replaced by the following twisted Jacobi identity:

\[
z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_M(u, z_1)Y_M(v, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{-z_0} \right) Y_M(v, z_2)Y_M(u, z_1) = z_2^{-1} \left( \frac{z_1 - z_0}{z_2} \right)^{-r/2} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_M(Y(u, z_0)v, z_2)
\]

(3.68)
for $u \in V^r$, where $r = 0, 1$ and $V^0 = V, V^1 = \tilde{V}$.

**Theorem 3.13.** Let $W$ be an irreducible $V$-module such that $h(0)$ has only half-integral eigenvalues on $W$. Then $W \oplus \tilde{W}$ is either a $\tilde{V}$-module or a $\sigma$-twisted $\tilde{V}$-module.

**Proof.** It is similar to the proof of Theorem 3.9. We only need to prove the Jacobi identity for $(u, v, w)$, where $u, v \in \tilde{V}, w \in W \cup \tilde{W}$.

**Case 1:** $w \in W$. By Lemma 3.12 we obtain

$$Y(u, z_1)Y(v, z_2)w = (-1)^7 E^{-}(h, z_1)\psi_W Y(\phi_V^{-1}u, z_1) \Delta(h, -z_1).$$

$$\cdot E^{-}(h, z_2)\phi_W Y(\phi_V^{-1}v, z_2) \Delta(h, -z_2)w$$

$$= (-1)^7 \left(1 - \frac{z_1}{z_2}\right)^7 E^{-}(h, z_1)\psi_W Y(\phi_V^{-1}u, z_1) \cdot$$

$$\cdot E^{-}(h, z_2)\Delta(h, -z_2) \phi_W Y(\phi_V^{-1}v, z_2) \Delta(h, -z_2)w$$

$$= (-1)^7 \left(1 - \frac{z_1}{z_2}\right)^7 E^{-}(h, z_1)\psi_W Y(\Delta(h, z_1 - z_2) \Delta(h, z_1)\phi_V^{-1}u, z_1) \cdot$$

$$\cdot \Delta(h, -z_2) \phi_W Y(\phi_V^{-1}v, z_2) \Delta(h, -z_2)w$$

$$= (z_1 - z_2)^7 E^{-}(h, z_1)\psi_W \phi_W Y(\Delta(h, z_1 - z_2) \phi_V^{-1}u, z_1) \cdot$$

$$\cdot Y(\Delta(h, -z_1) \phi_V^{-1}v, z_2) \Delta(h, -z_1) \Delta(h, -z_2)w.$$  \hfill (3.69)

Then

$$z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1)Y(v, z_2)w$$

$$= z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0}\right) z_0^7 E^{-}(h, z_1)\psi_W \phi_W Y(\Delta(h, z_0) \phi_V^{-1}(u), z_1) \cdot$$

$$\cdot Y(\Delta(h, -z_0) \phi_V^{-1}(v), z_2) \Delta(h, -z_1) \Delta(h, -z_2)w.$$  \hfill (3.70)

Symmetrically, we have:

$$z_0^{-1} \delta \left(\frac{z_2 - z_1}{z_0}\right) Y(v, z_2)Y(u, z_1)w$$

$$= z_0^{-1} \delta \left(\frac{z_2 - z_1}{z_0}\right) (-z_0)^7 E^{-}(h, z_1)\psi_W \phi_W Y(\Delta(h, -z_0) \phi_V^{-1}(v), z_2) \cdot$$

$$\cdot Y(\Delta(h, z_0) \phi_V^{-1}(u), z_1) \Delta(h, -z_1) \Delta(h, -z_2)w.$$  \hfill (3.71)

Therefore

$$z_0^{-1} \delta \left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1)Y(v, z_2)w - (-1)^7 z_0^{-1} \delta \left(\frac{z_2 - z_1}{z_0}\right) Y(v, z_2)Y(u, z_1)w$$

$$= z_2^{-1} \delta \left(\frac{z_1 - z_0}{z_2}\right) z_2^7 E^{-}(h, z_1)\psi_W \phi_W \cdot$$

$$\cdot Y(\Delta(h, -z_0) \phi_V^{-1}(u), z_0) \Delta(h, -z_0) \phi_V^{-1}(v), z_2)C.$$  \hfill (3.72)
where \( C = \Delta(-h, -z_1)\Delta(-h, -z_2)w \). On the other hand, we have:

\[
\begin{align*}
&z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0)v, z_2)w \\
&= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) z_0^\gamma Y\left(E^\gamma(-h, z_0)Y(\Delta(h, z_0)\phi^{-1}_V u, z_0)\Delta(-h, -z_0)\psi_V(v), z_2\right)w \\
&= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) E^\gamma(-h, z_1)E^\gamma(-h, z_2)z_0^\gamma \\
&\quad \cdot Y\left(Y(\Delta(h, z_0)\phi^{-1}_V u, z_0)\Delta(-h, -z_0)\psi_V(v), z_2\right) \\
&\quad \cdot \Delta(h, -z_2)\Delta(-h, -(z_2 + z_0))w \\
&= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) E^\gamma(-h, z_1)E^\gamma(-h, z_2)z_0^\gamma \\
&\quad \cdot Y\left(\psi_V\phi_V Y(\Delta(-h, z_0)\phi^{-1}_V u, z_0)\Delta(-h, -z_0)\phi^{-1}_V v, z_2\right) \\
&\quad \cdot \Delta(h, -z_2)\Delta(-h, -(z_2 + z_0))w \\
&= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) E^\gamma(-h, z_1)E^\gamma(-h, z_2)z_0^\gamma \\
&\quad \cdot \psi_W\phi_W Y(\Delta(-h, z_0)\phi^{-1}_V u, z_0)\Delta(-h, -z_0)\phi^{-1}_V v, z_2) \\
&\quad \cdot \Delta(-h, z_2)\Delta(-h, -z_2 - z_0)w.
\end{align*}
\]

(3.73)

Suppose \( h(0)w = \alpha w \). Then

\[
\begin{align*}
&z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) \Delta(-h, -z_2)\Delta(-h, -z_2 - z_0)w \\
&= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) (-z_2 - z_0)^\alpha \Delta(-h, -z_2)E^+(h, z_2 + z_0)w \\
&= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) (-z_2 - z_0)^\alpha \Delta(-h, -z_2)E^+(h, z_1)w \\
&= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) (-z_2 - z_0)^\alpha \Delta(-h, -z_2)\Delta(-h, -z_1)w \\
&= z_1^{-1}\delta\left(\frac{z_2 + z_0}{z_1}\right) \left(\frac{z_2 + z_0}{z_1}\right)^\alpha \Delta(-h, -z_2)\Delta(-h, -z_1)w \\
&= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) \left(\frac{z_1 - z_0}{z_2}\right)^\alpha \Delta(-h, -z_2)\Delta(-h, -z_1)w.
\end{align*}
\]

(3.74)

Thus

\[
\begin{align*}
&z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0)v, z_2)w \\
&= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) E^\gamma(-h, z_0 + z_2)E^\gamma(-h, z_2)z_0^\gamma \\
&\quad \cdot \psi_W\phi_W Y(\Delta(-h, z_0)\phi^{-1}_V u, z_0)\Delta(-h, -z_0)\phi^{-1}_V v, z_2) \left(\frac{z_1 - z_0}{z_2}\right)^\alpha C. \ (3.75)
\end{align*}
\]

27
Thus
\[ z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y(u, z_1)Y(v, z_2)w - (-1)^{\gamma} z_0^{-1} \delta \left( \frac{z_2 - z_1}{-z_0} \right) Y(v, z_2)Y(u, z_1)w \]
\[ = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) \left( \frac{z_1 - z_0}{z_2} \right)^\alpha Y(Y(u, z_0)v, z_2)w \]  \hspace{1cm} (3.76)
if \( h(0)w = \alpha w \).

Case 2: \( w \in \tilde{W} \). Similarly, for \( u, v \in \tilde{V}, \tilde{w} \in \tilde{W} \), we have:
\[ Y(u, z_1)Y(v, z_2)\tilde{w} \]
\[ = E^-(-h, z_1)\phi_W Y(\phi_V^{-1}u, z_1)\Delta(-h, -z_1) \cdot \cdot (-1)^\gamma E^-(-h, z_2)\psi_W Y(\phi_V^{-1}v, z_2)\Delta(-h, -z_2)\tilde{w} \]
\[ = (z_1 - z_2)\gamma \phi_W \psi_W E^-(-h, z_1)E^-(-h, z_2)Y(\Delta(-h, z_1 - z_2)\phi_V^{-1}(u, z_1) \cdot \cdot Y(\Delta(-h, -z_1)\Delta(-h, -z_2)\tilde{w}. \]  \hspace{1cm} (3.77)

Then
\[ z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y(u, z_1)Y(v, z_2)\tilde{w} \]
\[ = z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) z_0^\gamma \phi_W \psi_W E^-(-h, z_1)E^-(-h, z_2)Y(\Delta(-h, z_0)\phi_V^{-1}u, z_1) \cdot \cdot Y(\Delta(-h, -z_1)\Delta(-h, -z_2)\tilde{w}, \]  \hspace{1cm} (3.78)
and
\[ z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y(v, z_2)Y(u, z_1)\tilde{w} \]
\[ = z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) (-z_0)^\gamma \phi_W \psi_W E^-(-h, z_1)E^-(-h, z_2)Y(\Delta(-h, -z_0)\phi_V^{-1}v, z_2) \cdot \cdot Y(\Delta(-h, -z_1)\Delta(-h, -z_2)\tilde{w}. \]  \hspace{1cm} (3.79)

Thus
\[ z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y(u, z_1)Y(v, z_2)\tilde{w} \]
\[ -(-1)^\gamma z_0^{-1} \delta \left( \frac{z_2 - z_1}{-z_0} \right) Y(v, z_2)Y(u, z_1)\tilde{w} \]
\[ = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) \phi_W \psi_W E^-(-h, z_1)E^-(-h, z_2) \cdot \cdot Y \left( Y(\Delta(-h, z_0)\phi_V^{-1}u, z_0)\Delta(-h, -z_0)\phi_V^{-1}v, z_2) \Delta(-h, -z_1)\Delta(-h, -z_2)\tilde{w}. \]  \hspace{1cm} (3.80)

On the other hand, using the calculation in case 1, we obtain
\[ z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y(Y(u, z_0)v, z_2)\tilde{w} \]
\[ z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) E^{-}(-h, z_1) E^{-}(h, z_2) \delta_0 \cdot Y \left( \psi_V \phi_V Y(\Delta(-h, z_0) \phi_V^{-1} u, z_0) \Delta(-h, -z_0) \phi_V^{-1} v, z_2 \right) \cdot \Delta(h, -z_2) \Delta(-h, -(z_2 + z_0)) \tilde{w} \]
\[ = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) E^{-}(-h, z_1) E^{-}(h, z_2) \delta_0 \psi_W^{-1} \cdot Y \left( \psi_V \phi_V Y(\Delta(-h, z_0) \phi_V^{-1} u, z_0) \Delta(-h, -z_0) \phi_V^{-1} v, z_2 \right) \cdot \psi_W \Delta(h, -z_2) \Delta(-h, -(z_2 + z_0)) \tilde{w} \]
\[ = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) E^{-}(-h, z_1) E^{-}(h, z_2) \delta_0 2 \gamma \psi_W^{-1} \cdot Y \left( \psi_V \phi_V Y(\Delta(-h, z_0) \phi_V^{-1} u, z_0) \Delta(-h, -z_0) \phi_V^{-1} v, z_2 \right) \cdot \psi_W \Delta(h, -z_2) \Delta(-h, -(z_2 + z_0)) \tilde{w} \]
\[ = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) E^{-}(-h, z_0 + z_2) E^{-}(-h, z_2) \delta_0 2 \gamma \cdot \psi_W \phi_W Y(\Delta(h, z_2) \Delta(-h, -z_0) \phi_V^{-1} u, z_0) \Delta(-h, -z_0) \phi_V^{-1} v, z_2) \Delta(-2h, -z_2) \psi_W G \]
\[ = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) E^{-}(-h, z_0 + z_2) E^{-}(-h, z_2) \delta_0 \cdot \phi_W Y(\Delta(h, z_2) \Delta(-h, -z_0) \phi_V^{-1} u, z_0) \Delta(-h, -z_0) \phi_V^{-1} v, z_2) \left( \frac{z_2 + z_0}{z_1} \right)^{h(0)} \psi_W G \]
\[ = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) E^{-}(-h, z_0 + z_2) E^{-}(-h, z_2) \delta_0 \cdot \phi_W \psi_W Y(\Delta(-h, -z_0) \phi_V^{-1} u, z_0) \Delta(-h, -z_0) \phi_V^{-1} v, z_2) \left( \frac{z_2 + z_0}{z_1} \right)^{h(0)} G, \tag{3.81} \]

where \( G = \Delta(-h, -z_1) \Delta(-h, -z_2) \tilde{w} \). Then the proof is complete. \( \square \)

**Proposition 3.14.** Under the assumption of Theorem 3.13, if \( \tilde{W} \) as a \( V \)-module is not isomorphic to \( W \), then \( W \oplus \tilde{W} \) is an irreducible \( V \)-module, and if \( \bar{W} \) as a \( V \)-module is isomorphic to \( W \), then \( W \oplus \bar{W} \) is a direct sum of two irreducible \( V \)-modules, each of which is isomorphic to \( W \) as a \( V \)-module.

**Proof.** If \( \tilde{W} \) as a \( V \)-module is not isomorphic to \( W \), the irreducibility of \( W \oplus \tilde{W} \) follows from the argument in Remark 3.10. For the rest of proof, we assume that \( \bar{W} \) as a \( V \)-module is isomorphic to \( W \). Let \( f \) be a \( V \)-isomorphism from \( W \) onto \( \bar{W} \). Then \( \phi_W^{-1} f \psi_W f \) is a \( V \)-automorphism of \( W \). Since \( W \) is an irreducible \( V \)-module, from Schur lemma by multiplying a constant we may choose \( f \) such that

\[ \phi_W^{-1} f \psi_W f = \text{id}_W. \tag{3.82} \]

By definition, for any \( u \in \tilde{V}, w \in W, \tilde{w} \in \bar{W} \), we have:

\[ Y(u, z)w = E^{-}(-h, z) \psi_W \pi_W Y(\phi_V^{-1} u, z) \Delta(-h, -z)w, \tag{3.83} \]
\[ Y(u, z)\tilde{w} = E^{-}(-h, z) \psi_W Y(\phi_V^{-1} u, z) \Delta(-h, -z)\tilde{w}. \tag{3.84} \]
Since $f(w) \in \tilde{W}$ by Lemma 3.12 we get
\[
\begin{align*}
fY(u, z)f(w) &= E^{-}(-h, z)f \psi_W Y(\phi_{\tilde{V}}^{-1}u, z)\Delta(-h, -z)f(w) \\
&= E^{-}(-h, z)f \psi_W Y(\phi_{\tilde{V}}^{-1}u, z)\Delta(-h, -z)f(w) \\
&= E^{-}(-h, z)f \psi_W fY(\phi_{\tilde{V}}^{-1}u, z)(w) \\
&= E^{-}(-h, z)\phi_{\tilde{V}}Y(\phi_{\tilde{V}}^{-1}u, z)\Delta(-h, -z)(w) \\
&= Y(u, z)w. \quad (3.85)
\end{align*}
\]
Therefore $W_0 = \{(w, f(w))|w \in W\}$ and $W_1 = \{(w, -f(w))|w \in W\}$ are submodules of $\tilde{W} = W \oplus \tilde{W}$ and $\tilde{W}$ is the direct sum of $W_0$ and $W_1$. Since $W_0$ and $W_1$ are irreducible $V$-modules, they are irreducible $\tilde{V}$-modules. Then proof is complete. \hfill \Box

**Remark 3.15:** Let $V$ be a vertex operator algebra, let $M$ be a $V$-module and let $U$ be any subspace of $M$. Then it follows from the associativity ([DL], [FHL]) that the linear span $V \cdot U$ of all $a_nU$ for $a \in V, n \in \mathbf{Z}$ is a submodule of $M$ (cf. Lemma 6.1.1 [Li4]). Furthermore, if $V$ is a simple vertex operator algebra, then $Y(a, z)u \neq 0$ for $0 \neq a \in V, 0 \neq u \in M$ ([DL]). This is due to the fact that the annihilating space $\text{Ann}_V(u) = \{a \in V|Y(a, z)u = 0\}$ is an ideal of $V$.

Next we classify all irreducible $\tilde{V}$-modules. Recall that $\sigma$ is an involution of $\tilde{V}$. For any $\tilde{V}$-module $(\tilde{M}, Y_M(\cdot, z))$, we have a $\tilde{V}$-module $(\tilde{M}, Y_M(\sigma \cdot, z))$. For convenience, we denote this module by $M^\sigma$.

**Lemma 3.16.** If $M$ is a $\tilde{V}$-module such that $M$ is an irreducible $V$-module, then $M^\sigma$ is not isomorphic to $M$ as a $\tilde{V}$-module.

**Proof.** Suppose $M^\sigma$ as a $\tilde{V}$-module is isomorphic to $M$, i.e., there is a linear automorphism $\psi$ of $M$ satisfying
\[
\begin{align*}
\psi(Y_M(a, z)u) &= Y_M(a, z)\psi(u) \quad \text{for } a \in V, u \in M, \quad (3.86) \\
\psi(Y_M(a, z)u) &= -Y_M(a, z)\psi(u) \quad \text{for } a \in \tilde{V}, u \in M. \quad (3.87)
\end{align*}
\]
Since $\psi$ is an automorphism of $M$ as a $V$-module, by Schur lemma $\psi = \alpha \text{id}_M$ for some nonzero number $\alpha$. Then we have:
\[
\alpha Y_M(a, z)u = -\alpha Y_M(a, z)u \quad \text{for } a \in \tilde{V}, u \in M. \quad (3.88)
\]
Thus $Y_M(a, z)u = 0$ for any $a \in \tilde{V}, u \in M$. It is a contradiction. \hfill \Box

**Proposition 3.17**\footnote{It was pointed out by Professor Dong that it also follows from Theorem 5.4 in [DM].} Let $M_1$ and $M_2$ be $\tilde{V}$-modules such that $M_1$ and $M_2$ are isomorphic irreducible $V$-modules. Then $M_1$ is isomorphic to either $M_2$ or $M_2'$.\footnote{It was pointed out by Professor Dong that it also follows from Theorem 5.4 in [DM].}
**Proof.** Let \( f \) be a \( V \)-isomorphism from \( M_1 \) onto \( M_2 \). Then \( f^{-1}Y_2(\cdot, z)f \) is an intertwining operator of type \( \left( \begin{array}{c} M_1 \\ \tilde{V}M_1 \end{array} \right) \). Since \( \tilde{V} \) is a simple current \( V \)-module, the corresponding fusion rule is one, so that there is a nonzero number \( \alpha \) such that

\[
Y_1(a, z)u = \alpha f^{-1}Y_2(a, z)f(u) \quad \text{for any } a \in \tilde{V}, u \in M_1.
\]

Then

\[
Y_1(a, z_1)Y_1(b, z_2)u = \alpha^2 f^{-1}Y_2(a, z_1)Y_2(b, z_2)f(u),
\]

\[
Y_1(b, z_2)Y_1(a, z_1)u = \alpha^2 f^{-1}Y_2(b, z_2)Y_2(a, z_1)f(u),
\]

\[
Y_1(Y(a, z_0)b, z_2)f(f(a, z_0)b, z_2)u = Y_2(Y(a, z_0)b, z_2)f(u)
\]

for any \( a, b \in \tilde{V}, u \in M_1 \). From the Jacobi identity we obtain \( \alpha^2 = 1 \). Thus \( \alpha = 1 \) or \( \alpha = -1 \). Therefore \( M_1 \) is isomorphic to either \( M_2 \) or \( M_2^\sigma \). \( \square \)

**Proposition 3.18.** Let \( W_1 = W_{11} \oplus W_{12} \) and \( W_2 = W_{21} \oplus W_{22} \) be two \( \tilde{V} \)-modules satisfying the following conditions: (1) Each \( W_{ij} \) is an irreducible \( V \)-module. (2) \( W_{11} \) as a \( V \)-module is isomorphic to \( W_{21} \). (3) \( a_nW_{11} \subseteq W_{i2}, a_nW_{12} \subseteq W_{i1} \) for \( a \in \tilde{V}, n \in \mathbb{Z}, i = 1, 2 \). Then \( W_1 \) and \( W_2 \) are isomorphic \( \tilde{V} \)-modules.

**Proof.** Let \( f_1 \) be a \( V \)-isomorphism from \( W_{11} \) onto \( W_{21} \). Since \( Y_2(\cdot, z)f_1 \) is an intertwining operator of type \( \left( \begin{array}{c} W_{22} \\ \tilde{V}W_{11} \end{array} \right) \) and \( (W_{12}, Y_1(\cdot, z)) \) is a tensor product of \( (\tilde{V}, W_{11}) \), there is a \( V \)-homomorphism \( f_2 \) from \( W_{21} \) to \( W_{22} \) such that

\[
Y_2(a, z)f_1(u) = f_2Y_1(a, z)u \quad \text{for any } a \in \tilde{V}, u \in W_{11}.
\]

Let \( f = f_1 \oplus f_2 \) be the \( V \)-homomorphism from \( W_1 \) to \( W_2 \). Then

\[
f(Y_1(a, z)u) = Y_2(a, z)f(u) \quad \text{for any } a \in V_0, u \in W_1,
\]

\[
f(Y_1(a, z)v) = Y_2(a, z)f(u) \quad \text{for any } a \in V_1, v \in W_{11}.
\]

For any \( a, b \in \tilde{V}, u \in W_{11} \), from the associativity (cf. [DL], [FHL]) we have:

\[
f(Y_1(a, z_1)Y_1(b, z_2)f(u) = Y_1(a, z_1)Y_1(b, z_2)f(u).
\]

Since \( Y_1(a, z_1)Y_1(b, z_2)f(u) = Y_1(a, z_1)f(Y_1(b, z_2)f(u) \), we obtain

\[
f(Y_1(a, z_1)Y_1(b, z_2)u) = Y_1(a, z_1)f(Y_1(b, z_2)u).
\]

Since \( W_{12} \) is linearly spanned by all the coefficients of \( Y_1(b, z)u \) for \( b \in V_1, u \in W_{11} \), we have:

\[
f(Y_1(a, z)v) = Y_2(a, z)f(v) \quad \text{for any } a \in \tilde{V}, v \in W_{12}.
\]
Then $f$ is a $V$-homomorphism from $W_1$ to $W_2$. It is clear that $f$ is an isomorphism because each $W_{ij}$ is an irreducible $V$-module.

**Theorem 3.19.** If the vertex operator algebra $V$ is rational, then $\tilde{V}$ is rational and any irreducible $\tilde{V}$-module is isomorphic to one of those obtained in Proposition 3.14.

**Proof.** Since $V$ is rational, any $\tilde{V}$-module $M$ is a completely reducible $V$-module. Let $W$ be any irreducible $\tilde{V}$-submodule of $M$. From Remark 3.10, $\tilde{V} \cdot W$ is a $\tilde{V}$-submodule of $M$ and $\tilde{V} \cdot W = W + V \cdot W$. It follows from Corollary 2.10 that $(\tilde{W}, \tilde{Y}(\cdot, z))$ is a tensor product of $(W, \tilde{V})$. Let $Y_t(\cdot, z)$ be the transpose intertwining operator of $Y(\cdot, z)$. Then $(\tilde{W}, \tilde{Y}_t(\cdot, z))$ is a tensor product of $(\tilde{V}, W)$ (Lemma 5.1.6 [Li4]). Since $\tilde{V}$ and $W$ are $V$-modules, $Y(\cdot, z)$ is an intertwining operator, so that there is a unique $V$-homomorphism $f_1$ from $W$ to $\tilde{V} \cdot W$ satisfying

$$f_1 \tilde{Y}_t(\tilde{w}, z) a = Y(a, z) u \in \tilde{V} \cdot W \quad \text{for } \tilde{w} \in \tilde{W}, a \in \tilde{V}. \quad (3.99)$$

Consequently, $\tilde{V} \cdot W$ (isomorphic to $\tilde{W}$) is an irreducible $V$-module. If $\tilde{V} \cdot W = W$, then $W$ is an irreducible $\tilde{V}$-submodule of $M$. If $\tilde{W} \neq W$, then $\tilde{V} \cdot W = W \oplus V \cdot W$. By Lemma 3.16, $\tilde{V} \cdot W$ is a completely reducible $V$-module. Then $M$ is a direct sum of irreducible $\tilde{V}$-modules.

Let us return to the case for an affine Lie algebra $\hat{g}$.

**Theorem 3.20.** Let $\ell$ be a rational number and let $\lambda_i$ be a cominimal weight such that $L(\ell, \ell \lambda_i)$ is self-dual and that $\gamma = \ell(\lambda_i, \lambda_i)$ is an integer. Then $L(\ell, 0) \oplus L(\ell, \ell \lambda_i)$ is a vertex operator algebra if $\gamma$ is even and $L(\ell, 0) \oplus L(\ell, \ell \lambda_i)$ is a vertex operator superalgebra if $\gamma$ is odd.

**Proof.** It is clear that $L(\ell, 0)$ is self-dual for any $\ell$. Since $L(\ell, \ell \lambda_i)$ is self-dual, there is a nonzero intertwining operator of type $L(\ell, \ell \lambda_i)/L(\ell, \ell \lambda_i)$. Because $L(\ell, \ell \lambda_i)$ is a simple current, by Corollary 2.10 $L(\ell, \ell \lambda_i)$ is isomorphic to $L(\ell, 0)$. Then it follows from Theorem 3.9 immediately.

Next we go back to a specific example. Let $\{e, f, h\}$ be a basis for $sl_2$ such that $[h, e] = e$, $[h, f] = -f$, $[e, f] = h$. Let $\tilde{sl}_2$ be the affine Lie algebra with respect to the normalized Killing form such that the square length of $h$ is $\frac{1}{2}$. For any complex numbers $\ell$ and $j$, denote by $L(\ell, j)$ the irreducible highest weight module of level $\ell$ with spin $j$ for $\tilde{sl}_2$. It is well known ([DL], [FZ], [Li2]) that if $\ell$ is a positive integer, $L(\ell, 0)$ is rational and the set $\{L(\ell, j)|2j \in \mathbb{N}, 2j \leq \ell\}$ of all standard $\tilde{sl}_2$-modules of level $\ell$ is the set of equivalence classes of irreducible $L(\ell, 0)$-modules.

**Corollary 3.21.** (a) Let $\ell$ be any complex number such that $\ell \neq -2$. Then $L(\ell, \frac{\ell}{2})$ has a natural $L(\ell, 0)$-module structure.

(b) Let $k$ be a positive integer and let $\ell = 4k$. Then $L(\ell, 0) \oplus L(\ell, 2k)$ has a natural vertex operator algebra structure with an order-two automorphism $\sigma$ such that $\sigma|_{L(\ell, 0)} = \text{id}$, $\sigma|_{L(\ell, 2k)} = -\text{id}$. 

32
Proof. Since $\langle h, h \rangle = \frac{1}{2}$, we get $\gamma = \ell \langle h, h \rangle = \frac{1}{2} \ell$. Then it immediately follows from Theorem 3.9. □

Corollary 3.22. Let $k$ be a positive integer and let $\ell = 4k$. Then (a) the spaces $L(\ell, j) \oplus L(\ell, 2k - j)$ for $j \in \mathbb{Z}, 0 \leq j < k$, $L(\ell, k)$ and $L(\ell, k)^\sigma$ have natural irreducible module structures for the vertex operator algebra $L(\ell, 0) \oplus L(\ell, 2k)$.

(b) $L(\ell, j) \oplus L(\ell, 2k - j)$ for $j \in \frac{1}{2} + \mathbb{Z}, 0 \leq j < k$ have natural irreducible $\sigma$-twisted module structures for the vertex operator algebra $L(\ell, 0) \oplus L(\ell, 2k)$.

(c) The vertex operator algebra $L(\ell, 0) \oplus L(\ell, 2k)$ is rational and any irreducible module for $L(\ell, 0) \oplus L(\ell, 2k)$ is isomorphic to one of the modules in (a).

Remark 3.23: In the physical references (cf. [MSe]), the existence of twisted modules in Corollary 3.22 (b) was neglected.

4 Twisted modules for inner automorphisms

In this section, we present two constructions of twisted modules from any untwisted modules for an inner automorphism of a vertex operator algebra and we prove that they are essentially equivalent.

Let $h \in V$ satisfying (2.21). Furthermore, we assume that $h(0)$ semisimply acts on $V$ with rational eigenvalues. It is clear that $e^{2\pi i h(0)}$ is an automorphism of $V$. If the denominators of all eigenvalues of $h(0)$ are bounded, then $e^{2\pi i h(0)}$ is of finite order.

Proposition 4.1 [Li3]. Let $(M, Y_M(\cdot, z))$ be any $V$-module. Then $(\tilde{M}, \tilde{Y}(\cdot, z)) = (M, Y(\Delta(h, z)\cdot, z))$ is a $\sigma$-twisted $V$-module, where $\sigma = e^{2\pi i h(0)}$.

This is the first construction for twisted modules from any untwisted modules for an inner automorphism. This construction is also closely related to shifted vertex operators in [Le].

Let $V = \oplus_{n \in \mathbb{Q}} V(n)$ be a $\mathbb{Q}$-graded vertex operator algebra $V$, i.e., $V$ satisfies all axioms of a vertex operator algebra except that it may have nonintegral weights. Let $\sigma = e^{2\pi i L(0)}$.

For any $a \in V(n), n \in \mathbb{Z}$, we have: $\sigma(a) = a$. In particular, $\sigma(1) = 1$ and $\sigma(\omega) = \omega$. For any $a \in V(\alpha), b \in V(\beta), \alpha, \beta \in \mathbb{Q}$, we have:

$$e^{2\pi i L(0)}(a_n b) = e^{2(\text{wt} a + \text{wt} b - n - 1)} (a_n b) = e^{2(\text{wt} a + \text{wt} b)\pi i} (a_n b) = (e^{2\pi i L(0)} a)_n (e^{2\pi i L(0)} b). \quad (4.1)$$

Therefore $\sigma = e^{2\pi i L(0)}$ is an automorphism of $V$ as a vertex operator algebra.

It is clear that $\sigma_T$ is of finite order if and only if there is a positive integer $T$ such that all weights are contained in $\frac{1}{T} \mathbb{Z}$. In general, $\sigma$ may be of infinite order. Let $M = \oplus_{a \in \mathbb{C}} M(a)$ be any $V$-module and let $M' = \oplus_{a \in \mathbb{C}} M'_(a)$ be the restricted dual [FHL] of $M$. Define

$$\langle Y(a, z) u', v \rangle = \langle u', Y(e^{2\pi i L(0)} z^{-2\pi i L(0)} a, z^{-1}) v \rangle \quad (4.2)$$

33
for any \( a \in V, u' \in M', v \in M \).

**Proposition 4.2.** Let \( V \) be a \( \mathbb{Q} \)-graded vertex operator algebra and let \( M \) be a \( V \)-module. Then \((M', Y(\cdot, z))\) defined above is a \( \sigma^2 \)-twisted \( V \)-module.

**Proof.** It is a slight generalization of contragredient module theory in [FHL]. But for completeness, we present the details. First we recall the following formulas from [FHL].

\[
\begin{align*}
z^{L(0)} L(-1) &= z^{-1} L(-1) z^{L(0)}, \\
e^{zL(1)} L(-1) &= L(-1) e^{zL(1)} + 2z e^{zL(1)} L(0) - z^2 e^{zL(1)} L(1), \\
z^{L(0)} Y(a, z_0) z^{-L(0)} &= Y(z^{L(0)} a, z_0 z), \\
e^{zL(1)} Y(a, z_0) e^{-zL(1)} &= Y \left( e^{z(1-zz_0)} (1 - z_0 z)^{-2L(0)} a, \frac{z_0}{1 - z z_0} \right).
\end{align*}
\]

Let \( a \in V, u' \in M', v \in M \). Then

\[
\frac{d}{dz} \langle Y(a, z) u', v \rangle = \frac{d}{dz} \langle u', Y(e^{zL(1)} e^{\pi iL(0)} z^{-2L(0)} a, z^{-1}) v \rangle
= -z^{-2} \langle u', Y(L(-1) e^{zL(1)} e^{\pi iL(0)} z^{-2L(0)} a, z^{-1}) v \rangle
+ \langle u', Y(L(1) e^{zL(1)} e^{\pi iL(0)} z^{-2L(0)} a, z^{-1}) v \rangle
- 2z^{-1} \langle u', Y(e^{zL(1)} L(0) e^{\pi iL(0)} z^{-2L(0)} a, z^{-1}) v \rangle.
\]

On the other hand, we have:

\[
\begin{align*}
\langle Y(L(-1) a, z) u', v \rangle &= \langle u', Y(e^{zL(1)} e^{\pi iL(0)} z^{-2L(0)} a, z^{-1}) v \rangle \\
&= -z^{-2} \langle u', Y(L(-1) e^{zL(1)} e^{\pi iL(0)} z^{-2L(0)} a, z^{-1}) v \rangle \\
&= -z^{-2} \langle u', Y(L(-1) e^{zL(1)} e^{\pi iL(0)} z^{-2L(0)} a, z^{-1}) v \rangle \\
&- 2z^{-1} \langle u', Y(e^{zL(1)} L(0) e^{\pi iL(0)} z^{-2L(0)} a, z^{-1}) v \rangle.
\end{align*}
\]

Thus

\[
\frac{d}{dz} \langle Y(a, z) u', v \rangle = \langle Y(L(-1) a, z) u', v \rangle \quad \text{for } a \in V, u' \in M', v \in M.
\]

For \( a, b \in V \), similar to [FHL] we obtain

\[
\begin{align*}
z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y(v, z_2^{-1}) Y(u, z_1^{-1}) \\
- z_0^{-1} \delta \left( \frac{z_2 - z_1}{-z_0} \right) Y(u, z_1^{-1}) Y(v, z_2^{-1})
\end{align*}
\]

34
where

\[ u = e^{z_1 L(1)} e^{\pi iL(0)} z_1^{-2L(0)} a, \quad v = e^{z_2 L(1)} e^{\pi iL(0)} z_2^{-2L(0)} b. \]  

Since the right-hand side of the desired twisted Jacobi identity should be

\[ z_1^{-1} \delta \left( \frac{z_2 + z_0}{z_1} \right) \left( \frac{z_1 - z_0}{z_2} \right)^{-2r} Y \left( e^{z_1 L(1)} e^{\pi iL(0)} z_1^{-2L(0)} a, -\frac{z_0}{z_1 z_2} \right) e^{z_2 L(1)} e^{\pi iL(0)} z_2^{-2L(0)}, \]  

where \( r \) is the weight of \( a \), i.e., \( L(0)a = ra \), it is sufficient to prove

\[ z_1^{-1} \delta \left( \frac{z_2 + z_0}{z_1} \right) \left( \frac{z_1 - z_0}{z_2} \right)^{-2r} e^{z_2 L(1)} e^{\pi iL(0)} z_2^{-2L(0)} Y(a, z_0), \]  

or, equivalently to prove

\[ z_1^{-1} \delta \left( \frac{z_2 + z_0}{z_1} \right) \left( \frac{z_1 - z_0}{z_2} \right)^{-2r} Y \left( e^{z_1 L(1)} e^{\pi iL(0)} z_1^{-2L(0)} a, -\frac{z_0}{z_1 z_2} \right) e^{z_2 L(1)} e^{\pi iL(0)} z_2^{-2L(0)}, \]  

By the conjugation formulas (4.3)-(4.6), we have:

\[ e^{z_2 L(1)} e^{\pi iL(0)} z_2^{-2L(0)} Y(a, z_0) z_2^{2L(0)} e^{-\pi iL(0)} e^{-z_2 L(1)} = e^{z_2 L(1)} e^{\pi iL(0)} Y \left( z_2^{-2L(0)} a, -z_2^{-2} z_0 \right) e^{-\pi iL(0)} e^{-z_2 L(1)} \]

\[ = e^{z_2 L(1)} Y \left( e^{\pi iL(0)} z_2^{-2L(0)} a, -z_2^{-2} z_0 \right) e^{-z_2 L(1)} \]

\[ = Y \left( e^{z_2(1+z_0 z_2^{-1})L(1)} (1 + z_0 z_2^{-1})^{-2L(0)} e^{\pi iL(0)} z_2^{-2L(0)} a, -\frac{z_2^{-2} z_0}{1 + z_0 z_2^{-1}} \right) \]

\[ = Y \left( e^{(z_2+z_0)L(1)} e^{\pi iL(0)} (z_2 + z_0)^{-2L(0)} a, -\frac{z_0}{z_2(z_2 + z_0)} \right). \]
Thus

$$z_1^{-1} \delta \left( \frac{z_2 + z_0}{z_1} \right) \left( \frac{z_1 - z_0}{z_2} \right)^{-2r} \cdot e^{z_2 L(1)} e^{\pi i L(0)} z_2^{-2L(0)} Y(a, z_0) z_2^{2L(0)} e^{-\pi i L(0)} e^{-z_2 L(1)}$$

$$= z_1^{-1} \delta \left( \frac{z_2 + z_0}{z_1} \right) \left( \frac{z_2 + z_0}{z_1} \right)^{2r} \cdot Y \left( e^{(z_2 + z_0)L(1)} e^{\pi i L(0)} (z_2 + z_0)^{-2L(0)} a, -\frac{z_0}{z_2^2 (z_2 + z_0)} \right)$$

$$= z_1^{-1} \delta \left( \frac{z_2 + z_0}{z_1} \right) Y \left( e^{z_1 L(1)} e^{\pi i L(0)} z_1^{-2L(0)} a, -\frac{z_0}{z_2 z_1} \right). \quad (4.16)$$

Then the proof is complete. \( \square \)

**Remark 4.3.** Let \((V, \omega)\) be a vertex operator algebra and let \(e\) be a Virasoro element of \(V\) such that \(e(-1) = L(-1)\) and that \(e(0)\) is semisimple on \(V\) with rational eigenvalues. Then \((V, e)\) is a \(Q\)-graded vertex (operator) algebra. By Proposition 4.3, we obtain a \(\sigma^2\)-twisted weak module for \((V, \omega)\), where \(\sigma = e^{2\pi i e(0)}\).

**Remark 4.4.** For Frenkel, Lepowsky and Meurman’s Moonshine module vertex operator algebra \(V^\natural\) [FLM], many involutions have been constructed in [M] as formal exponentials of the weight-zero component \(e(0)\) of the vertex operator associated to a Virasoro element \(e\). Unfortunately, \(e(-1) \neq L(-1)\). It is very interesting to know how we can modify this procedure to obtain (irreducible) twisted modules for those involutions.

**Remark 4.5.** Let \(V\) be a vertex operator algebra, let \(\hat{h} \in V\) satisfying (2.21) and set \(e = \omega + L(-1) h\). Then \(Y(e, z) = \sum_{n \in \mathbb{Z}} e(n) z^{-n-2}\) gives a representation of the Virasoro algebra on \(V\) such that \(e(-1) = L(-1)\) (cf. [DLinM]). In general, \((V, Y(\cdot, z), e)\) is a \(Q\)-graded vertex operator algebra. Combining this new Virasoro element with Proposition 4.2, we get the second construction for twisted modules from any untwisted modules for an inner automorphism.

As a matter of fact, the above two constructions are essentially the same. The following proposition gives the connection between the two constructions of twisted modules from untwisted modules for an inner automorphism.

**Proposition 4.6.** Let \(V\) be a vertex operator algebra, let \(\hat{h} \in V\) satisfy the conditions (2.21), let \((M, Y_M(\cdot, z))\) be a \(V\)-module and let \((M', Y_M'(\cdot, z), \omega)\) be the contragredient module of \(M\) with respect to the Virasoro element \(\omega\). Suppose that the restricted dual spaces of \((M, Y_M(\cdot, z), \omega)\) and \((M, Y_M(\cdot, z), \omega + L(-1) h)\) are the same. Then \((M, Y(\Delta(-2h, z) e^{-\pi i h(0)}, \cdot, z))\) as a \(e^{4\pi i h(0)}\)-twisted \(V\)-module is isomorphic to the contragredient module of \((M', Y_M'(\cdot, z), \omega)\) with respect to the Virasoro element \(\omega + h(-2) 1\).

**Proof.** First from [FHL] we have the following formula:

$$z L(0) e^{z_0 L(1)} = e^{z_0 z L(1)} z L(0). \quad (4.17)$$

36
Set $e = \omega + h(-2)1$. Then

$$e(m) = L(m) - (m + 1)h(m) \quad \text{for } m \in \mathbb{Z}. \quad (4.18)$$

In particular, we have:

$$e(-1) = L(-1), \; e(0) = L(0) - h(0), \; e(1) = L(1) - h(1). \quad (4.19)$$

For any $a \in V, u \in M, v' \in M'$, we have:

$$\langle Y(a, z)u, v' \rangle = \langle u, Y(e^{zL(1)}(-z^{-2})L(0)a, z^{-1})v' \rangle = \langle Y(e^{z^{-1}e(1)}(-z^2)e^{zL(1)}(-z^{-2})L(0)a, z)u, v' \rangle. \quad (4.20)$$

Then

$$e^{z^{-1}e(1)}(-z^2)e^{zL(1)}(-z^{-2})L(0) = e^{z^{-1}(L(1)-2h(1))}(-z^2)^L(0)-h(0)e^{zL(1)}(-z^{-2})L(0) = z^{-2h(0)}e^{-z^{-1}(L(1)-2h(1))}(-z^2)^L(0)e^{zL(1)}(-z^{-2})L(0) = z^{-2h(0)}e^{z^{-1}(L(1)-2h(1))}e^{-z^{-1}L(1)}e^{-\pi ih(0)}. \quad (4.21)$$

For $a \in V, u \in M, v' \in M'$, using Lemma 3.3 we obtain

$$\langle Y(a, z)u, v' \rangle = \langle Y(z^{-2h(0)}e^{z^{-1}(L(1)-2h(1))}e^{-z^{-1}L(1)}e^{-\pi ih(0)}a, z)u, v' \rangle = \langle Y(\Delta(-2h, z)e^{-\pi ih(0)}a, z)u, v' \rangle. \quad (4.22)$$

Then the proof is complete. \(\square\)

Combining Proposition 2.16 with Proposition 4.6 we get

**Corollary 4.7.** Let $L$ be a positive-definite even lattice. Then every irreducible $V_L$-module can be considered as a contragredient module of the adjoint module with respect to some Virasoro element.

**References**

[BPZ] A. Belavin, A. M. Polyakov, A. B. Zamolodchikov, Infinite conformal symmetries in two-dimensional quantum field theory, *Nucl. Phys. B* 241 (1984), 333-380.

[B] R. E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Natl. Acad. Sci. USA* 83 (1986), 3068-3071.
[D1] C. Dong, Vertex algebras associated with even lattices, *J. of Alg.* 160 (1993), 245-265.

[D2] C. Dong, Twisted modules for vertex operator algebras associated with even lattices, *J. of Alg.* 165 (1994), 90-112.

[D3] C. Dong, Representations of the moonshine module vertex operator algebra, *Contemporary Math.* 175 (1994), 27-36.

[DHR] Doplicher, S., Haag, R., Roberts, J. E., Local observables and particle statistics I, II, *Comm. Math. Phys.* 23 (1971) 199-230, 35 (1974), 49-85.

[DL] C. Dong and J. Lepowsky, Generalized Vertex Algebras and Relative Vertex Operators, Progress in Math. *Vol. 112*, Birkhäuser, Boston, 1993.

[DLiM] C. Dong, H. Li and G. Mason, Simple currents and extensions of vertex operator algebras, preprint, q-alg/9504008.

[DLinM] C. Dong, Z. Lin and G. Mason, On vertex operator algebras as $sl_2$-modules, in *Proc. on groups and related topics*, Columbus, May, 1993, ed. by K. Harada, S. Sehgal and R. Solomon, Walter de Gruyter, Berlin-New York.

[DM] C. Dong and G. Mason, On the operator content of nilpotent orbifold models, preprint, UCSC 1994, hep-th/9412109.

[DMZ] C. Dong, G. Mason and Y. Zhu, Discrete series of the Virasoro algebras and the Moonshine module, *Proc. Symp. Pure Math., American Math. Soc.* 56 II (1994), 295-316.

[F] J. Fuchs, Simple WZW currents, *Comm. Math. Phys.* 136 (1991) 345-356.

[FFR] Alex J. Feingold, Igor B. Frenkel and John F. X. Ries, *Spinor Construction of Vertex Operator Algebras, Triality, and $E_8^{(1)}$*, Contemporary Math. 121 (1991).

[FG] J. Fuchs and D. Gepner, On the connection between WZW and free field theories, *Nucl. Phys. B* 294 (1988) 30-42.

[FGV1] J. Fuchs, A. Ganchev and P. Vecsernyés, Level one WZW superselection sectors, *Comm. Math. Phys.* 146 (1992) 553-583.

[FGV2] J. Fuchs, A. Ganchev and P. Vecsernyés, Simple WZW superselection sectors, *Lett. Math. Phys.* 28 (1993) 31-41.
[FHL] I. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, preprint, 1989; Memoirs Amer. Math. Soc. 104, 1993.

[FLM] I. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, Pure and Appl. Math. Vol. 134, Academic Press, Boston, 1988.

[FRS] K. Fredenhagen, K.-H. Rehren and B. Schroer, Superselection sectors with Braid group statistics and exchange algebras I, Comm. Math. Phys. 125 (1989), 201-226.

[FZ] I. Frenkel and Y.-C. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, Duke Math. J. 66 (1992), 123-168.

[HK] Hagg, R., Kastler, D., An algebraic approach to field theory, J. Math. Phys. 5 (1964), 848-861.

[HL0] Y.-Z. Huang and J. Lepowsky, Toward a theory of tensor product for representations for a vertex operator algebra, in Proc. 20th International Conference on Differential Geometric Methods in Theoretical Physics, New York, 1991, ed. S. Catto and A. Rocha, World Scientific, Singapore, 1992, Vol. 1, 344-354.

[HL1] Y.-Z. Huang and J. Lepowsky, A theory of tensor product for module category of a vertex operator algebra, I, preprint (1993).

[HL2] Y.-Z. Huang and J. Lepowsky, A theory of tensor product for module category of a vertex operator algebra, II, preprint (1993).

[Hua] Y.-Z. Huang, A non-meromorphic extension of the Moonshine module vertex operator algebra, Contemporary Math., to appear.

[Hum] J. E. Humphreys, Introduction to Lie Algebras and Representation Theory, Graduate Texts in Mathematics 9, Springer-Verlag, New York, 1972.

[K] V. G. Kac, Infinite-dimensional Lie algebras, 3rd ed., Cambridge Univ. Press, Cambridge, 1990.

[Le] J. Lepowsky, Calculus of twisted vertex operators, Proc. Natl. Acad. Sci. USA 82 (1985), 8295-8299.

[Li1] H.-S. Li, Symmetric invariant bilinear forms on vertex operator algebras, Journal of Pure and Applied Algebra, 96 (1994), 279-297.

[Li2] H.-S. Li, Local systems of vertex operators, vertex superalgebras and modules, hep-th/9406187, Journal of Pure and Applied Algebra, to appear.
[Li3] H.-S. Li, Local systems of twisted vertex operators, vertex superalgebras and twisted modules, q-alg/9504022, *Contemporary Math.*, to appear.

[Li4] H.-S. Li, Representation theory and tensor product theory of vertex operator algebras, hep-th/9406211, Ph.D. thesis, Rutgers University, 1994.

[LW] J. Lepowsky and R. L. Wilson, A new family algebras underlying the Rogers-Ramanujan identities and generalization, *Proc. Natl. Acad. Sci. USA*, 78, 7254-7258 (1981).

[M] M. Miyamoto, Griess algebras and Virasoro elements in vertex operator algebras, preprint (1994).

[MSc] G. Mack and V. Schomerus, Conformal field algebras with quantum symmetry from the theory of superselection sectors, *Comm. Math. Phys.* 134 (1990), 139-196.

[MSe] G. Moore and N. Seiberg, Classical and quantum conformal field theory, *Comm. Math. Phys.* 123 (1989), 177-254.

[SY1] Schellekens, A. N. and Yankielowick, S., Extended chiral algebras and modular invariant partition functions, *Nucl. Phys.* B327 (1989), 673-703.

[SY2] Schellekens, A. N. and Yankielowick, S., Modular invariants from simple currents. An explicit proof. *Phys. Lett.* B227 (1989), 387-391.

[W] W.-Q. Wang, Rationality of Virasoro vertex operator algebras, *Duke Math. J. IMRN*, Vol. 71, No. 1 (1993), 197-211.

[Z] Y.-C. Zhu, Vertex operator algebras, elliptic functions and modular forms, Ph.D. thesis, Yale University, 1990.