The orbit spaces $G_{n,2}/T^n$ and the Chow quotients $G_{n,2}/((\mathbb{C}^*)^n$ of the Grassmann manifolds $G_{n,2}$

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Abstract

The focus of our paper is on the complex Grassmann manifolds $G_{n,2}$ which appear as one of the fundamental objects in developing the interaction between algebraic geometry and algebraic topology. In his well-known paper Kapranov has proved that the Deligne-Mumford compactification $\overline{M}(0, n)$ of $n$-pointed curves of genus zero can be realized as the Chow quotient $G_{n,2}/((\mathbb{C}^*)^n$. In our recent papers, the constructive description of the orbit space $G_{n,2}/T^n$ has been obtained. In getting this result our notions of the CW-complex of the admissible polytopes and the universal space of parameters $F_n$ for $T^n$-action on $G_{n,2}$ were of essential use. Using technique of the wonderful compactification, in this paper it is given an explicit construction of the space $F_n$. Together with Keel’s description of $\overline{M}(0, n)$, this construction enabled us to obtain an explicit diffeomorphism between $F_n$ and $\overline{M}(0, n)$. Thus, we showed that the space $G_{n,2}/((\mathbb{C}^*)^n$ can be realized as our universal space of parameters $F_n$. In this way, we give description of the structure in $G_{n,2}/((\mathbb{C}^*)^n$, that is $\overline{M}(0, n)$ in terms of the CW-complex of the admissible polytopes for $G_{n,2}$ and their spaces of parameters. 

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1 Introduction

The questions about the action of the maximal algebraic torus as well as the induced action of the compact torus on the complex Grassmann manifolds naturally arise in many areas of mathematics, [17], [14], [15], [16], [18]. It is well known that the Grassmann manifolds are a fundamental object for the problems which
develop an interaction between algebraic geometry and algebraic topology. The subject of toric geometry are algebraic manifolds which consist of the closure of one orbit of algebraic torus. The equivariant structure of these manifolds can be effectively described in terms of combinatorial structure of a moment polytope. The issue which naturally arises in this context is to study algebraic manifolds with algebraic torus action whose family of algebraic torus orbits gives, in a way, good stratification of a manifold. The first such nontrivial examples are the Grassmann manifolds $G_{n,2}$ which can be as well interpreted as the space of all projective lines in $\mathbb{C}P^{n-1}$. In addition, from the results of Gel’fand-Serganova [17] it follows that the structure of the algebraic torus orbit stratification, that is the gluing of the strata, in the case of $G_{n,2}$ can be explicitly described. This is no more true for the Grassmannians $G_{n,k}$, $n \geq 7$, $k \geq 3$, as it is demonstrated on the example $G_{7,3}$ in [17], see also [4]. Further on, we generalized this property of $G_{n,2}$ and introduced in [4] the notion of the CW complex of the strata.

In this paper we establish the connection between the well-known constructions from algebraic geometry based on the notion of wonderful compactification and the results on equivariant algebraic topology of the Grassmann manifolds.

The motivation to tackle this issue has come to us from the problem of describing the universal space of parameters $\mathcal{F}_n$ for the canonical $T^m$-action on a Grassmannian $G_{n,2}$. The universal spaces of parameters have been introduced in our theory of $(2m, k)$-manifolds, see [4]. This theory focuses on smooth $2m$-dimensional manifolds with a smooth action of $k$-dimensional compact torus, which satisfy given set of axioms. The important class of such manifolds is consisted of the manifolds for which an action of a compact torus is induced by the action of an algebraic torus. Among these manifolds a beautiful role have the manifolds $G_{n,2}$, which are $(2m, k)$-manifold for $m = 2(n-2)$ and $k = n-1$. The universal space of parameters $\mathcal{F}$ for a $(2m, k)$-manifold $M^{2m}$ with an effective action of the compact torus $T^k$, $k \leq m$, is a compactification of the space of parameters $F$ of the main stratum. Its general definition is given in [4]. The universal space of parameters for the Grassmannian $G_{5,2}$ in regard with the canonical action of $T^5$ is defined and explicitly described in [3]. More precisely, it is proved that in this case the universal space of parameters is the blow up of the cubic surface $c_1' c_2' c_3 = c_1 c_2 c_3$ in $(\mathbb{C}P^1)^3$ at one point, being the same as the blow up of $\mathbb{C}P^2$ at four points in general position. This space is in algebraic geometry known as the del Pezzo surface of degree $5$, see also [25]. Further on, in [20] it is proved that the Grothendieck-Knudsen compactification of the moduli space $\mathcal{M}(0, n)$ of genus zero curves with $n$ marked distinct points provides universal space of parameters for $G_{n,2}$ in regard
to the canonical $T^n$-action. From the one side, the method in [20] appeals on the Gel’fand MacPherson correspondence [15] between the space $F_n$ and configurations of all pairwise distinct points in $(\mathbb{C}P^1)^n$ up to automorphism of $\mathbb{C}P^1$ and, from the other side, on the description of the Grothendieck-Knudsen compactification $\overline{M}(0, n)$ of the moduli space $M(0, n)$ of genus zero curves with $n$ marked distinct points via cross ratios as described by McDuff-Salamon in [24]. There is also the notion of the Chow quotient $G_{n,2}/(\mathbb{C}^*)^n$ defined by Kapranov in [18] for which he proved to coincide with the space $\overline{M}(0, n)$. It is well known fact that $\overline{M}(0, 5)$ is the del Pezzo surface of degree 5, while algebro-geometric characterization of the space $\overline{M}(0, n)$ for $n > 5$ is an open well known problem. In the paper [6] it has been shown that the problem of algebro-topological characterization of these spaces is related to well known problems of complex cobordisms theory.

In this paper we provide new description of the universal space of parameters $F_n$ for $G_{n,2}$ which comes purely from equivariant topology of the Grassmann manifolds $G_{n,2}$. We obtain a smooth, compact manifold for which we prove to be diffeomorphic to the moduli space $\overline{M}(0, n)$ of genus zero stable curves with $n$ marked distinct points, that is to the Chow quotient $G_{n,2}/(\mathbb{C}^*)^n$. In this context, the main task to be considered can be formulated as follows: given an algebraic variety $X$ in $(\mathbb{C}P^1)^N$ which is an open subset of its closure $\overline{X}$ in $(\mathbb{C}P^1)^N$, and a family of its automorphism $A(X)$, to find a compactification $\mathcal{X}$ of $X$ for which there is the projection $p : \mathcal{X} \to \overline{X}$ which restriction on $X$ is an identity, and any automorphism $f \in A(X)$ extends to the automorphism of $\mathcal{X}$.

Here we take $X$ to be the coordinate record of the space of parameters $F_n$ of the main stratum in a standard chart for $G_{n,2}$ which is defined by the Plücker coordinates, while for $A(F_n)$ we take all automorphism of $F_n$ induced by the coordinate changes between a fixed chart and all other charts. In order to find $\mathcal{F}_n$ we resolve the singularities of the closure $\overline{F}_n \subset (\mathbb{C}P^1)^N$, which arise in the context of extension of the automorphisms from $A(F_n)$, using the constructions of algebraic geometry known as wonderful compactification. We prove that in such a way the smooth manifold $\mathcal{F}_n$ is obtained, which is diffeomorphic to the Chow quotient $G_{n,2}/(\mathbb{C}^*)^n$ that is, to the Grothendieck-Knudsen compactification of the space $\overline{M}(0, n)$.

The wonderful compactification of a complex manifold $M$ is a compact manifold $\overline{X}$, such that $D = \overline{X} \setminus M$ is a divisor with normal crossings in $\overline{X}$ whose irreducible components are smooth, and any number of connected components of $D$ intersect
transversally. Such strong conditions on compactification turn out to be of essential importance for many algebraic and geometric problems, such as the problems of enumerative algebraic geometry and Schubert enumerative problems, description of the rational homotopy geometry and Schubert enumerative problems, description of the rational homotopy type of $M$, its mixed Hodge structure, the Chow ring, etc. The notion of wonderful compactification firstly appeared in the paper [11] of De Concini-Procesi in the context of an equivariant compactification of the symmetric spaces $G/H$, see also [23] and [26] for a comprehensive overview of the subject. This idea has been further developed and applied in many directions, such as Fulton-MacPherson compactification in [13], De Concini-Procesi wonderful models [11], [12], the wonderful compactification of Li [22] and more recently the projective wonderful models of toric arrangements by De Concini-Gaiffi and others [7], [8], [9].

Furthermore, this paper finds the advantage of wonderful compactification for the description of the equivariant topology of the Grassmannians $G_{n,2}$ for the canonical $T^n$-action. More explicitly, it turns out that the wonderful compactification of arrangements of subvarieties from [22] can be successfully applied for the compactification of the space of parameters $F_n$ of the main stratum $W_n \subset G_{n,2}$. In this way, we obtain the smooth manifold $F_n$ which enables us to construct the model $U_n = F_n \times \Delta_{n,2}$ for the orbit space $G_{n,2}/T^n$ meaning that there exists a continuous surjection $p_n : U_n \to G_{n,2}/T^n$, see [5].

This Chow quotient is the compactification of the orbit space $W_n/(\mathbb{C}^*)^n$ for the main stratum $W_n \subset G_{n,2}$ given in terms of the Chow variety for $G_{n,2}$, see [18], [14]. The build up components for this compactification are described in [18] in terms of the maximal algebraic torus orbits in $G_{n,2}$. In this paper, we push this further and show that the build up components of $G_{n,2}/(\mathbb{C}^*)^n$ can be described in terms of the CW complex of admissible polytopes and their spaces of parameters. In this description, following the ideas of the Chow quotient from [18], the key role have the cortèges of $(n - 1)$-dimensional admissible polytopes which give polyhedral decompositions for $\Delta_{n,2}$. We relate this to our description of the space $F_n$ and provide an explicit expression for the build up components of $G_{5,2}/(\mathbb{C}^*)^5$. 

5
2 Space of parameters $F_n$ of the main stratum for $G_{n,2}$

2.1 The embedding of $F_n$ in $(\mathbb{C}P^1)^N$

First we recall from [3] and [4] the notions of the main stratum $W_n$ in $G_{n,2}$ as well as its space of parameters $F_n$ as a background introduction into the objects we are going to consider.

The main stratum $W_n \subset G_{n,2}$ is characterized by the condition that its points have all non-zero Plücker coordinates, which implies that $W_n$ belongs to any standard chart $M_{ij} = \{ L \in G_{n,2} | P_{ij}(L) \neq 0 \}$, $1 \leq i < j \leq n$ of a Grassmann $G_{n,2}$. The main stratum $W_n$ is invariant under the canonical action of the algebraic torus $(\mathbb{C}^\ast)^n$ and the orbit space $F_n = W_n / (\mathbb{C}^\ast)^n$ is said to be the space of parameters for $W_n$. In a fixed chart, the main stratum is given by the following system of equations:

$$c'_{ij}z_iw_j = c_{ij}z_jw_i, \quad 3 \leq i < j \leq n,$$

where the parameters $(c_{ij} : c'_{ij}) \in \mathbb{C}P^1$ and $c_{ij}, c'_{ij} \neq 0$ and $c_{ij} \neq c'_{ij}$ for all $3 \leq i < j \leq n$.

The number of parameters $(c_{ij} : c'_{ij})$ is $N = \binom{n-2}{2}$ and from (1) it follows that these parameters satisfy the following equations:

$$c'_{ij}c_{ik}c'_{jk} = c_{ij}c_{ik}c_{jk}, \quad 3 \leq i < j < k \leq n.$$  \hspace{1cm} (2)

The number of these equations is $\binom{n-3}{2}$.

We see from (2) that the parameters $(c_{ij} : c'_{ij})$ satisfy the following relations:

$$(c_{ij} : c'_{ij}) = (c'_{3i}c_{3j} : c_{3i}c'_{3j}), \quad 4 \leq i < j \leq n.$$  \hspace{1cm} (3)

$$(c_{3i} : c'_{3i}) \neq (c_{3j} : c'_{3j}), \quad 4 \leq i < j \leq n.$$  \hspace{1cm} (4)

The number of these relations is $M = \binom{n-3}{2}$. Therefore, we obtain that the space $F_n$ is homeomorphic to the space

$$F_n \cong (\mathbb{C}P_A^1)^{n-3} \setminus \Delta,$$  \hspace{1cm} (5)

where $A = \{(0 : 1), (1 : 0), (1 : 1)\}$ and $\Delta = \bigcup_{3 \leq i < j \leq n} \Delta_{ij}$ for the diagonals $\Delta_{ij} = \{(c_{34} : c'_{34}), \ldots, (c_{n-1,n} : c'_{n-1,n}) \in (\mathbb{C}P_A^1)^{n-3} | (c_{3i} : c'_{3i}) = (c_{3j} : c'_{3j})\}$. 


The relations (2) give an embedding of the space $F_n$ into $(\mathbb{C}P^1)^N$, $N = \binom{n-2}{2}$, that is:

**Lemma 1.** The space of parameters $F_n$ of the main stratum $W_n$ is given by (2) as a subspace in $(\mathbb{C}P^1)^N$, $N = \binom{n-2}{2}$.

Moreover, we see that $F_n$ is an open algebraic manifold in $\bar{F}_n \subset (\mathbb{C}P^1)^N$, $N = \binom{n-2}{2}$.

**Proposition 1.** The compactification $\bar{F}_n$ of $F_n$ in $(\mathbb{C}P^1)^N$ is a smooth algebraic variety given by

$$\{(c_{ij} : c'_{ij})_{3 \leq i < j \leq n} \in (\mathbb{C}P^1)^N \mid c'_{ij}c_{ik}c'_{jk} = c_{ij}c_{ik}c_{jk}, 3 \leq i < j < k \leq n\}.$$

**Proof.** We consider the gradients of the functions $f_{ijk} = c_{ij}c'_{ik}c'_{jk} - c'_{ij}c_{ik}c_{jk}$ which define $\bar{F}_n$. It can be directly verified that there are $M = \binom{n-3}{2}$ linearly independent vectors among these gradients at any point of $\bar{F}_n$. This implies that $\bar{F}_n$ is a smooth algebraic variety of real dimension $2(N - M)$ for $N = \binom{n-2}{2}$ and $M = \binom{n-3}{2}$.

### 2.2 Requirements on the compactification of $F_n$

The motivation for our approach in finding the compactification for the space of parameters $F_n$ of the main stratum $W_n$ in $G_{n,2}$ comes from our work on description of topology of the orbit space $G_{n,2}/T^n$. Using Plücker coordinates one can define the stratification of a Grassmannian $G_{n,2}$, that is $G_{n,2} = \cup_{\sigma} W_\sigma$, where $\sigma \subset \{\{i, j\} \subset \{1, \ldots, n\}, i \neq j\}$. The strata $W_\sigma = \{L \in G_{n,2} | P_{ij}(L) \neq 0, \text{ for } \{i, j\} \in \sigma\}$ are pairwise disjoint and $T^n$-invariant, even $(\mathbb{C}^*)^n$-invariant. This induces the stratification of the orbit space

$$G_{n,2}/T^n = \cup_\sigma W_\sigma/T^n.$$

The main stratum $W_n$ is a dense set in $G_{n,2}$ and the torus $T^{n-1} = T^n/S^1$ acts freely on it. In [3] we proved that

$$W_n/T^n \cong \Delta_{n,2} \times F_n,$$
where $\Delta_{n,2}$ is the hypersimplex. Moreover, for any stratum $W_\sigma$ we proved that

$$W_\sigma/T^n \cong P_\sigma \times F_\sigma,$$

where $P_\sigma$ is a subpolytope in $\Delta_{n,2}$ given by $P_\sigma = \text{convhull}\{\Lambda_{ij} = e_i + e_j \in \mathbb{R}^n, \{i, j\} \in \sigma\}$ and $F_\sigma = W_\sigma/(\mathbb{C}^*)^n$. Note that all this implies that to any stratum $W_\sigma$ it can be assigned the subspace $\tilde{F}_\sigma \subset F_n$ for any compactification $F_n$ for $F_n$. In order to describe the orbit space $G_{n,2}/T^n$ we look for such compactification $F_n$ for $F_n$ for which $\tilde{F}_n \subset (\mathbb{C}P^1)^N$ and all possible $\tilde{F}_\sigma \subset \tilde{F}_n$, and make corrections along those $\tilde{F}_\sigma$ which are not independent of a chart $M_{ij}$.

### 2.3 Automorphisms of $F_n$ induced by the changes of coordinates

The main stratum $W_n$ belongs to any chart defined by the Plücker coordinates. It follows from (1) and (2) that the transition maps between the charts produce the automorphisms of the space of parameters $F_n$ of the main stratum.

We deduce explicitly the automorphisms of $F_n$ induced by the transition maps between the charts $M_{12}$ and $M_{ij}$, $i < j$, $\{1, 2\} \neq \{i, j\}$. Denote the local coordinates in the chart $M_{12}$ by

$$z_3, \ldots, z_n, w_3, \ldots, w_n$$

and let

$$z'_1, \ldots, z'_{i-1}, z'_{i+1}, \ldots, z'_{j-1}, z'_j, \ldots, z'_n,$$

$$w'_1, \ldots, w'_{i-1}, w'_{i+1}, \ldots, w'_{j-1}, w'_j, \ldots, w'_n$$

be the local coordinates in the chart $M_{ij}$.

Let further $(c_{pq} : c'_{pq})$, $3 \leq p < q \leq n$ be the coordinate record of the space $F_n$ in the chart $M_{12}$ and $(d_{kl} : d'_{kl})$, $1 \leq k < l \leq n$, $k, l \neq i, j$ be the coordinate record of $F_n$ in a chart $M_{ij}$.

We differentiate the following cases.
1) $i = 1, \, 3 \leq j \leq n$, that is we consider the chart $M_{1j}$. In this case we have that

$$
\begin{align*}
   z_2' &= -\frac{z_j}{w_j}, & z_k' &= \frac{z_kw_j - z_jw_k}{w_j}, & k \geq 3, \\
   w_2' &= \frac{1}{w_j}, & w_k' &= \frac{w_k}{w_j}, & k \geq 3.
\end{align*}
$$

(6)

Lemma 2. The coordinates $(c_{pq} : c_{pq}')$ and $(d_{kl} : d_{kl}')$ of the space of parameters $F_n$ in the charts $M_{12}$ and $M_{1j}, \, 3 \leq j \leq n$ are related by

$$
\begin{align*}
   (d_{2l} : d_{2l}') &= (c_{jl} : c_{jl} - c_{jl}'), & 3 \leq l \leq n, & l \neq j, \\
   (d_{kl} : d_{kl}') &= (c_{jk}(c_{jl} - c_{jl}') : c_{jk}(c_{jl} - c_{jl}')), & 3 \leq k < l \leq n, & k, l \neq j.
\end{align*}
$$

The second expression can be written as

$$
\begin{align*}
   (d_{kl} : d_{kl}') &= (c_{kl}c_{jl}(c_{jk} - c_{jk}'): c_{kl}c_{jk}(c_{jl} - c_{jl}')).
\end{align*}
$$

Proof. In the chart $M_{1j}$ the main stratum writes as

$$
\begin{align*}
   d_{kl}'z_kw_l' &= d_{kl}z_kw_k, & 2 \leq k < l \leq n, & k, l \neq j.
\end{align*}
$$

Substituting the relations (6) into these equations of the main stratum, we obtain

$$
\begin{align*}
   -d_{2l}' \cdot \frac{z_j}{w_j} \cdot \frac{w_l}{w_j} &= d_{2l} \cdot \frac{z_kw_j - z_jw_k}{w_j} \cdot \frac{1}{w_j},
\end{align*}
$$

which implies

$$
\begin{align*}
   d_{2l}' &= -d_{2l}(\frac{z_kw_j}{z_kw_l} - 1) = d_{2l}(1 - \frac{c_{jl}'}{c_{jl}}),
\end{align*}
$$

that is

$$
\begin{align*}
   (d_{2l} : d_{2l}') &= (c_{jl} : c_{jl} - c_{jl}').
\end{align*}
$$

For $k \geq 3$ in an analogous way we obtain

$$
\begin{align*}
   d_{kl}'w_l(z_kw_j - z_jw_k) &= d_{kl}w_k(z_kw_j - z_jw_l),
\end{align*}
$$

which implies

$$
\begin{align*}
   d_{kl}' &= d_{kl} \cdot \frac{w_k}{w_l} \cdot \frac{z_kw_l(c_{jl}' - 1)}{z_kw_l(c_{jk}' - 1)},
\end{align*}
$$

that is

$$
\begin{align*}
   (d_{kl} : d_{kl}') &= (c_{jl}(c_{jk}' - c_{jk}) : c_{jk}(c_{jl}' - c_{jl}')).
\end{align*}
$$

□
2) \( i = 2, \ 3 \leq j \leq n \), that is we consider the chart \( M_{2j} \). We have that

\[
\begin{align*}
  z'_1 &= -\frac{w_j}{z_j}, \quad z'_k = \frac{z_j w_k - z_k w_j}{z_j}, \quad k \geq 3, \\
  w'_1 &= \frac{1}{z_j}, \quad w'_k = \frac{z_k}{z_j}, \quad k \geq 3.
\end{align*}
\]

Substituting this into equations of the main stratum written in the chart \( M_{2j} \) we obtain:

**Lemma 3.** The coordinates \((c_{pq} : c'_{pq})\) and \((d_{kl} : d'_{kl})\) of the space of parameters \( F_n \) in the charts \( M_{12} \) and \( M_{2j} \), \( 3 \leq j \leq n \) are related by

\[
\begin{align*}
  (d_{kl} : d'_{kl}) &= (c'_{jl} : c_{jl} - c_{jl}), \quad 3 \leq l \leq n, \ l \neq j, \\
  (d_{kl} : d'_{kl}) &= (c_{jl}(c_{jk} - c'_{jk}) : c_{jk}(c_{jl} - c'_{jl})), \quad 3 \leq k < l \leq n, \ k, l \neq j.
\end{align*}
\]

3) \( 3 \leq i < j \leq n \), we are in the chart \( M_{ij} \) and

\[
\begin{align*}
  z'_1 &= -\frac{w_j}{z_j} - z_i w_j - z_i w_i, \quad z'_2 = -\frac{z_j}{z_i w_j - z_i w_i}, \quad z'_k = \frac{z_i w_k - z_j w_j}{z_i w_j - z_i w_i}, \quad k \geq 3. \\
  w'_1 &= \frac{w_i}{z_j w_i - z_i w_j}, \quad w'_2 = -\frac{z_i}{z_j w_i - z_i w_j}, \quad w'_k = \frac{z_i w_k - z_j w_j}{z_j w_i - z_i w_j}, \quad k \geq 3.
\end{align*}
\]

For the coordinates of the space of parameters \( F_n \) we obtain:

**Lemma 4.** The coordinates \((c_{ik} : c'_{ik})\) and \((d_{kl} : d'_{kl})\) of the space of parameters \( F_n \) in the charts \( M_{12} \) and \( M_{ij} \), \( 3 \leq i < j \leq n \) are related by

\[
\begin{align*}
  (d_{12} : d'_{12}) &= (c_{ij} : c'_{ij}), \\
  (d_{1l} : d'_{1l}) &= (c'_{jl}(c_{il} - c'_{il}) : c_{il}(c_{jl} - c'_{jl})), \quad 3 \leq l \leq n, \ l \neq i, j, \\
  (d_{2l} : d'_{2l}) &= (c_{jl}(c_{il} - c'_{il}) : c_{il}(c_{jl} - c'_{jl})), \quad 3 \leq l \leq n, \ l \neq i, j, \\
  (d_{kl} : d'_{kl}) &= ((c'_{jk} - c_{jk})(c_{il} - c'_{il})c'_{jk}c'_{kl} : (c_{ik} - c'_{ik})(c_{jl} - c'_{jl})c_{jk}c_{kl}), \quad 3 \leq k < l \leq n, \ k, l \neq i, j.
\end{align*}
\]

**Remark 1.** In this way we obtain the family \( \{f_{12,ij}\} \ i = 1, \ j \geq 3 \) or \( 2 \leq i < j \leq n \) of homeomorphisms of the space \( F_n \). Note that any homeomorphism \( f_{kl,pq} \) of \( F_n \) induced by the transition map between the charts \( M_{kl} \) and \( M_{pq} \) can be represented as \( f_{kl,pq} = f_{12,kl}^{-1} \circ f_{12,pq} \).
3 Universal space of parameters $F_n$ and wonderful compactification

We want to find the compactification $F_n$ for $F_n$ such that the homeomorphisms \{\(f_{12,ij}\)\} of $F_n$ induced by the transition maps between the chart $M_{12}$ and a chart $M_{ij}$, extend to the homeomorphisms of $F_n$. According to Remark 1 it is enough to consider just the family \{\(f_{12,ij}\)\}, since then any homeomorphism \{\(f_{kl,pq}\)\} of $F_n$ induced by the transition map between any two charts $M_{kl}$ and $M_{pq}$ can be canonically extended to the homeomorphism of $F_n$.

3.1 The compactification $\bar{F}_n$ of $F_n$ in $(\mathbb{C}P^1)^N$

We start with $\bar{F}_n \subset (\mathbb{C}P^1)^N$, $N = \binom{n-2}{2}$ given by

$$\bar{F}_n = \{(c_{ij} : c'_{ij}) \mid 3 \leq i < j \leq n, c_{ik}c_{il} = c'_{ik}c'_{il}, 3 \leq i < k < l \leq n\},$$

which is, by Proposition 1, a smooth algebraic variety obtained as a compactification of $F_n$ in $(\mathbb{C}P^1)^N$.

The compactification $\bar{F}_n$ is not the one we are looking for as it is obvious that the homeomorphisms of $F_n$ defined in previous lemmata for $n > 4$ can not be continuously extended to $\bar{F}_n$. Note that for $n = 4$ these homeomorphisms extend to the homeomorphisms of $\bar{F}_4 = \mathbb{C}P^1$.

In more detail, the boundary of $F_n$ in $\bar{F}_n$ is $\bar{F}_n \setminus F_n$ and it consists of the points $(c_{ij} : c'_{ij}) \in \bar{F}_n$ such that $c_{ij} = 0$ or $c'_{ij} = 0$ or $c_{ij} = c'_{ij}$ for some $3 \leq i < j \leq n$. The homeomorphisms given by previous lemmata do not continuously extend to the points from the boundary of $\bar{F}_n$ which satisfy $c_{jk} = c'_{jk}$ and $c_{jl} = c'_{jl}$ for some $3 \leq j < k < l \leq n$. Moreover, even if they do continuously extend to some subset from $\bar{F}_n \setminus F_n$ these extensions do not have to be homeomorphisms. For example, the subvariety in $\bar{F}_n \setminus F_n$ given by $c'_{34} = c'_{35} = 0$ maps by the continuous extension of the homeomorphism $f_{12,13}$, which is defined by Lemma 2 for $j = 3$, to the subvariety given by $d_{24} = d'_{24}$, $d_{25} = d'_{25}$ and $d_{45} = d'_{45}$. In particular for $n = 5$ this means that the subvariety $(1 : 0), (1 : 0), (c_{45} : c'_{45})$ maps to the point $((1 : 1), (1 : 1), (1 : 1))$, see also [3]. Thus, this extension can not be a homeomorphism.

To overcome these problems the idea is to blow up the smooth, compact variety $\bar{F}_n$ along the singular subvarieties for \{\(f_{12,ij}\)\} which consist of all singular points
for these homeomorphisms in $\bar{F}_n$. In order to do that we use the technique from algebraic geometry known as the wonderful compactification of an arrangement of subvarieties.

### 3.2 Basic facts on wonderful compactification

A wonderful compactification is a kind of compactification of a variety aimed to resolve singularities of a variety which appear in some context. The smooth compactification $\bar{X}$ of a complex manifold $M$ is wonderful in the sense that $D = \bar{X} \setminus M$ is a divisor with normal crossings in $\bar{X}$ whose irreducible components are smooth and any number of connected components of $D$ intersect transversally. There are several compactifications in the literature which provide examples of wonderful compactification. We first mention the compactification of the symmetric spaces given by De Concini and Procesi [10] in which $M$ is a symmetric space $G/H$ of an adjoint semisimple Lie group and $\bar{X}$ is a smooth, compact variety with an $G$-action such that $\bar{X}$ has an open orbit isomorphic to $G/H$ with finitely many $G$-orbits, the orbit closures are all smooth and any number of orbit closures intersect transversally. The other example is the compactification of configuration spaces given by Fulton and MacPherson [13] in which $M$ is an open subset of the Cartesian product $X^n$ of a given nonsingular variety $X$, that is $M$ is defined as the complement of all diagonals, while $\bar{X}$ is defined by a sequence of blowups of $X^n$ along the nonsingular subvarieties corresponding to all diagonals. An example of wonderful compactification is as well the compactification of arrangements of complements of linear subspaces given also by De Concini and Procesi [11] in which $M$ is a finite-dimensional vector space and $\bar{X}$ is obtained by replacing any given family of its subspaces by a divisor with normal crossings. In the recent paper of Li [22] the wonderful compactification of arrangement of subvarieties is described and in this case $M$ is a nonsingular variety and $\bar{X}$ is obtained by replacing any given arrangement of subvarieties by a divisor with normal crossings.

It has been proved that any of these compactifications can be constructed by a sequences of blow ups along appropriate subvarieties and their transforms.

We follow here the paper of Li [22] to recall the basic facts on wonderful compactification for the setting we are going to use.

**Definition 1.** Let $Y$ be a nonsingular variety over an algebraically closed field of arbitrary characteristic. Let $\mathcal{G}$ be a nonempty building set and let $Y^\circ = Y \setminus$
The closure of the image of the naturally closed embedding

\[ Y^0 \hookrightarrow \prod_{G \in \mathcal{G}} \text{Bl}_G Y \]

is called the wonderful compactification of \( \mathcal{G} \) and it is denoted by \( Y_\mathcal{G} \).

We formulate two crucial theorems from [22], the first one states that the wonderful compactification \( Y_\mathcal{G} \) is a nonsingular variety and the second one describes \( Y_\mathcal{G} \) as the series of blow-ups determined by the subvarieties from \( \mathcal{G} \).

**Theorem 1.** Let \( Y \) be a nonsingular variety and let \( \mathcal{G} \) be a nonempty building set of subvarieties of \( Y \). Then the wonderful compactification \( Y_\mathcal{G} \) is a nonsingular variety. Moreover, for any \( G \in \mathcal{G} \) there is a nonsingular divisor \( D_G \subset Y_\mathcal{G} \) such that

1. the union of these divisors is \( Y_\mathcal{G} \setminus Y^0 \);
2. any set of these divisors meet transversally. An intersection of divisors \( D_{T_1} \cap \ldots \cap D_{T_r} \) is nonempty exactly when \( \{T_1, \ldots, T_r\} \) form a \( \mathcal{G} \)-nest.

**Theorem 2.** Let \( Y \) be a nonsingular variety and let \( \mathcal{G} \) be a nonempty building set of subvarieties of \( Y \). Let the building set \( \mathcal{G} = \{G_1, \ldots, G_Q\} \) is ordered such that the sets of subvarieties \( \{G_1, \ldots, G_i\} \) form a building set for any \( 1 \leq i \leq Q \). Then the iterated blow-ups give the smooth variety

\[ X_\mathcal{G} = \text{Bl}_{\bar{G}_Q} \cdots \text{Bl}_{\bar{G}_2} \text{Bl}_{G_1} Y, \]

where \( \bar{G}_i \) is a nonsingular variety obtained as the iterated dominant transform of \( G_i \) in \( \text{Bl}_{\bar{G}_{i-1}} \cdots \text{Bl}_{\bar{G}_2} \text{Bl}_{G_1} Y \), \( 2 \leq i \leq Q \) and the smooth manifold \( X_\mathcal{G} \) coincides with the wonderful compactification \( Y_\mathcal{G} \).

We explain shortly the notions which are used in these theorems.

**Definition 2.** A simple arrangement of subvarieties of a nonsingular variety \( Y \) is a finite set \( \mathcal{S} = \{S_i\} \) of nonsingular closed subvarieties \( S_i \), properly contained in \( Y \), that satisfy the following conditions:

1. \( S_i \) and \( S_j \) intersect cleanly;
2. \( S_i \cap S_j \) either is equal to some \( S_k \) or it is empty.
Here two closed non-singular subvarieties $S_1$ and $S_2$ in $Y$ are said to intersect cleanly if their intersection is nonsingular and their tangent bundles satisfy $T(S_1 \cap S_2) = T(S_1) |_{(S_1 \cap S_2) \cap T(S_2)} |_{(S_1 \cap S_2)}$.

**Definition 3.** Let $S$ be an arrangement of subvarieties of $Y$. A subset $G \subseteq S$ is called a building set of $S$ if, for all $S \in S$, the minimal elements in $\{ G \in G : S \subseteq G \}$ intersect transversally and their intersection is $S$.

Here the subvarieties $S_1, \ldots, S_k$ intersect transversally if $k = 1$ or

$$\text{codim} \left( \bigcap_{i=1}^{k} T(S_i), T(Y) \right) = \sum_{i=1}^{k} \text{codim}(S_i, Y).$$

A finite set $G$ of nonsingular subvarieties of $Y$ is called a building set if the set of all possible intersections of collections of subvarieties from $G$ form an arrangement $S$ and if $G$ is a building set of $S$. Then $S$ is called the arrangement induced by $G$.

We point to the following observation which we will find useful.

**Lemma 5.** Let a finite set of nonsingular subvarieties $G$ of nonsingular variety $Y$ satisfies the following:

- $G$ contains all intersections of its elements;
- any two elements of $G$ intersect cleanly.

Then $G$ is a building set.

*Proof.* In this case the set $G$ is a simple arrangement whose building set is $G$ since for any $S \in G$ we have that $S$ is the smallest element of the set $\{ G \in G : S \subseteq G \}$.

We recall also the meaning of blow-ups in Theorem 2.

**Definition 4.** Let $Z$ be a nonsingular subvariety of a nonsingular variety $Y$, then let $Bl_Z Y$ be the blow-up of $Y$ along $Z$ and let $\pi : Bl_Z Y \to Y$ be the canonical projection. For any irreducible subvariety $V$ of $Y$ the dominant transform $\tilde{V}$ is defined to be...
• the strict transform of \( V \) if \( V \not\subseteq Z \), which is the closure of \( \pi^{-1}(V \setminus (V \cap Z)) \) in \( Bl_Z Y \),

• the scheme-theoretic inverse \( \pi^{-1}(V) \) if \( V \subseteq Z \).

The reason for introducing the notion of the dominant transform is to correct the fact that the strict transform of a subvariety contained in the center of blow-up is empty. In our applications it will always be satisfied that \( V \not\subseteq Z \), so \( \tilde{V} \) will always be a strict transform.

We also shortly comment on the proof of Theorem 2, see [22]. The proof essentially relies on the result proved in the same paper: Let \( Y \) be a nonsingular and \( F \) is a minimal element in the building set \( \mathcal{G} = \{G_1, \ldots, G_N\} \) with the induced arrangement \( \mathcal{S} \) and \( E \) is an exceptional divisor in the blow up \( Bl_F Y \). Then the collection of subvarieties \( \tilde{S} \) in \( Bl_F Y \) defined by

\[
\tilde{S} = \{\tilde{S}\}_{S \in \mathcal{S}} \cup \{\tilde{S} \cap E\}_{\emptyset \neq S \cap F \subseteq S}
\]

is an arrangement of subvarieties in \( Bl_F Y \) and \( \tilde{\mathcal{G}} = \{\tilde{G}\}_{G \in \mathcal{G}} \) is a building set in \( \tilde{S} \).

Then the idea is to order (partially) the set \( \mathcal{G} \) according to the inclusion relation and to start with the blow up of \( Y \) along the minimal element \( F \), and, using just mentioned result, to iteratively repeat the procedure with the dominant transform \( \tilde{\mathcal{G}} \) of \( \mathcal{G} \) in \( Bl_F Y \).

### 3.3 The space \( \mathcal{F}_n \) as the wonderful compactification on \( \tilde{F}_n \)

Let \( \tilde{F}_n \subset (\mathbb{C}P^1)^N \) be as given in Proposition 11 and let

\[
\hat{F}_I = \tilde{F}_n \cap \{(c_{ik} : c_{ik}') = (c_{kl} : c_{kl}') = (c_{kl} : c_{kl}') = (1 : 1)\}, \tag{7}
\]

for \( I = \{i, k, l\} \in \{I \subset \{1, \ldots, n\}, |I| = 3\} \) and \( n \geq 5 \).

We take \( Y = \tilde{F}_n \) and take the building set \( \mathcal{G}_n \) to be

• \( \mathcal{G}_n = \emptyset \) for \( n = 4 \),

• \( \mathcal{G}_n = \{G = \bigcap_I \hat{F}_I \subset \tilde{F}_n\} \), that is all possible nonempty intersection of \( \hat{F}_I \)'s.

An element \( G \in \mathcal{G}_n \) of the form \( G = \hat{F}_{I_1} \cap \cdots \cap \hat{F}_{I_k} \) we denote by \( \hat{F}_{I_1, \ldots, I_k} \).
Lemma 6. $G_n$ is a building set.

Proof. Any intersection of elements from $G_n$ belongs to $G_n$. The set $G_n$ is a simple arrangement since obviously intersection of two elements from $G_n$ is either empty either it belongs to $G_n$ and any two elements intersect cleanly. We see this from the description of the subvarieties $F_{i_1,...,i_k} \subset (\mathbb{C}P^1)^N$ given by (2) and from the observation that for $S_1 = \tilde{F}_{i_1,...,i_k}, S_2 = \tilde{F}_{j_1,...,j_l} \in S$ it holds

$$S_1 \cap S_2 = \tilde{F}_{i_1,...,i_k,j_1,...,j_l}.$$  

Lemma 3.3 then implies that $G_n$ is a building set. □

Next we prove that the building set $G_n$ satisfies the condition of Theorem 2

Lemma 7. The defined building set $G_n$ can be ordered as $G_n = \{G_1, \ldots, G_Q\}$ such that the sets of subvarieties $\{G_1, \ldots, G_i\}$ form a building sets for any $1 \leq i \leq Q$.

Proof. We assign to an element $G = \tilde{F}_{i_1,...,i_k} \in G_n$ the number $o(G)$ which is equal to the number of coordinates of the points $F \subset (\mathbb{C}P^1)^N$ which are determined by the set $I_1 \cup \cdots \cup I_k$. In other words $o(G)$ is the number of coordinates of the form $(1 : 1)$ common for all points from $G$. For example, if $k = 2$ and $I_1 = 345, I_2 = 346$ then $o(G) = 6$, then for $I_1 = 345, I_2 = 367$ we have that $o(G) = 10$, while in the case $I_1 \cap I_2 = \emptyset$ in general we have $o(G) = 6$.

We define an equivalence relation on $G$ by: $G_1, G_2$ are in relation if and only if $o(G_1) = o(G_2)$. Denote by $\tilde{G}_1, \ldots, \tilde{G}_p$ the corresponding equivalence classes.

We assume that we order these equivalence classes by an order which is oppositely compatible with the corresponding numbers $o(\tilde{G}_i)$ that is $i < j$ if and only if $o(\tilde{G}_i) > o(\tilde{G}_j)$. It implies that $\tilde{G}_1$ contains only the point $S = (1 : 1)^N$, while $\tilde{G}_p$ is consisted consists of all elements $\tilde{F}_j$.

We further order the elements of $G_n$ as follows: $G_1 = (1 : 1)^N$, then we put the elements from $\tilde{G}_1$ in an arbitrary order, after that we put the elements from $\tilde{G}_2$ in an arbitrary order and so on, at the end we put the elements of $\tilde{G}_p$, that is $\tilde{F}_j$ in an arbitrary order. We denote this order by $G_n = \{G_1, \ldots, G_Q\}$. Since the elements of $G$ intersect cleanly it follows that the set $\{G_1, \ldots, G_i\}$ is a building set for any $1 \leq i \leq Q$. □

By $F_n$ we denote the smooth, compact manifold $Y_G$ which is the wonderful compactification with the building set $G = G_n$ and $Y = \tilde{F}_n$. 

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Theorem 3. The homeomorphisms of $F_n$ given by the set $\mathcal{A}$ extend to the homeomorphisms of $F_n$.

Remark 2. Note that for $n = 5$ the building set $\mathcal{G}_5$ consists of one point $P = ((1 : 1), (1 : 1), (1 : 1))$ and, therefore, $\mathcal{F}_5 = Bl_p \bar{F}_5$, compare to [3]. For $n = 6$, the description of $\mathcal{F}_6$ is no more trivial and allows us to demonstrate the general approach.

We have the situation that we are given a smooth manifolds $F_n \subset (\mathbb{C}P^1)^N$, which is an open subset in $\bar{F}_n \subset (\mathbb{C}P^1)^N$ and the group of automorphisms $\mathcal{A} = \{f_{ij,kl}\}$ for $F_n$ which are induced by the transition maps between the charts $M_{ij}$ and $M_{kl}$ for $G_{n,2}$. The manifold $\mathcal{F}_n$ which is a compactification for $F_n$ satisfies the desired property stated in the introduction:

Theorem 3. The homeomorphisms of $F_n$ given by the set $\mathcal{A}$ extend to the homeomorphisms of $F_n$.

Proof. Since it holds $f_{ij,kl} = f_{12,ij}^{-1} \circ f_{12,kl}$, it is enough to prove the statement for the homeomorphisms $f_{12,ij}$. We demonstrate the proof for the homeomorphisms $f_{12,ij}$ given by Lemma 2, the other cases go in an analogous way. We discuss first the homeomorphic extension of the homeomorphisms $f_{12,ij}$ to the boundary $\bar{F}_n \setminus F_n$. This boundary is given by the conditions $(c_{kl} : c'_{kl}) = (1 : 0)$ or $(0 : 1)$ or $(1 : 1)$ for some $3 \leq k < l \leq n$. We analyze each of these cases using the equations $c_{kl}c_{kp}c_{lp} = c_{kl}'c_{kp}'c_{lp}'$ which define $\bar{F}_n$ and the expressions for $f_{12,ij}$ given by Lemma 2.

- We first assume that $k \neq j$. For $c'_{kl} = 0$ we have that $c'_{kp} = 0$ or $c_{lp} = 0$, which implies that $d'_{kl} = 0$ and, $d''_{kp} = 0$ or $d'_{lp} = 0$. For $c_{kl} = 0$ we have that $c_{kp} = 0$ or $c_{lp}' = 0$, which gives $d_{kl} = 0$ and, $d_{kp} = 0$ or $d_{lp} = 0$. For $c_{kl} = c'_{kl}$ we have $(c_{lp} : c'_{lp}) = (c_{kp} : c'_{kp})$, which implies $d_{kl} = d'_{kl}$ and $(d_{lp} : d'_{lp}) = (d_{kp} : d'_{kp})$.

- We assume now that $k = j$. If $c_{jl} = c_{jp} = 0$ we have that $d_{2l} = d_{2p} = 0$, while for $c_{jl} = c_{ip} = 0$ we have that $d_{2l} = 0$ and $d'_{lp} = 0$. If $c'_{jl} = c_{jp} = 0$, then $(c_{dp} : c'_{dp})$ can be an arbitrary element form $\mathbb{C}P^1$, while we have that $(d_{2l} : d'_{2l}) = (d_{2p} : d'_{2p}) = (d_{lp} : d'_{lp}) = (1 : 1)$. Note that in this case $f_{12,ij}$ extends to such elements of the boundary but it can not be a homeomorphism. If $c'_{jl} = c_{dp} = 0$ we obtain that $d_{2l} = d'_{2p}$ and $d_{lp} = 0$. For $c_{jl} = c'_{jl}$ we have that $(c_{dp} : c'_{dp}) = (c_{dp} : c'_{dp})$ which implies $d_{2l} = 0$ and $(d_{lp} : d'_{lp}) = (d_{jp} : d'_{jp})$.  

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Altogether we conclude that $f_{12,1j}$ can not be continuously extended to the subvarieties $\hat{F}_I \subset \bar{F}_n \setminus F_n$, $I = \{j, l, p\}$ given by $(c_{jl} : c_{jl}') = (c_{jp} : c_{jp}') = (c_{lp} : c_{lp}') = (1 : 1)$ and that it can be continuously, but not homeomorphically, extended to the subvarieties $\bar{F}_I \subset \bar{F}_n \setminus F_n$, $I = \{j, l, p\}$ given by $(c_{jl} : c_{jl}') = (c_{jp} : c_{jp}') = (1 : 0)$.

We denote by $G(j)$ the family of subvarieties consisting of all possible non-empty intersections of the subvarieties $\hat{F}_I$ and by $H(j)$ the family of all possible non-empty intersections of the subvarieties $\bar{F}_I$. From the previous discussion we see that the map $f_{12,1j}$ extends homeomorphically to the complement in $\bar{F}_n$ of the union of the subvarieties from $G(j)$ and $H(j)$, that is to $\bar{F}_n \setminus (G(j) \cup H(j))$.

Moreover, note that the preimages of the subvarieties $\hat{F}_I$ by these extensions of $f_{12,1j}$ are the subvarieties $\bar{F}_I$.

We extend a homeomorphism $f_{12,1j}$ to the homeomorphism $\tilde{f}_{12,1j} : F_n \to F_n$ as follows:

- On the complement of the union of subvarieties from $G(j)$ and $H(j)$ the map $\tilde{f}_{12,1j}$ is given by the natural homeomorphic extension of $f_{12,1j}$.

- Let $S \in H(j)$, then $S = \hat{F}_{I_1} \cap \cdots \cap \hat{F}_{I_k}$ for some $I_1, \ldots, I_k \subset \{1, \ldots, n\}$, $|I| = 3$, $j \in I$ and let $\hat{S} \in G(j)$ is given by $\hat{S} = \tilde{F}_{I_1} \cap \cdots \cap \tilde{F}_{I_k}$. Then we define $\tilde{f}_{12,1j}$ to map homeomorphically $S$ to an exceptional divisor $E(S)$ for $\hat{S}$ in $\mathcal{F}_n$. This can be naturally done because of the previous observation on behavior of an extension of the map $f_{12,1j}$ on the subvarieties $\hat{F}_{(j,l,p)}$.

- Let $E(\hat{S}) \subset F_n$ be an exceptional divisor for $\hat{S} \in G(j)$ where $\hat{S} = \hat{F}_{I_1} \cap \cdots \cap \hat{F}_{I_k}$, $I_1, \ldots, I_k \in \{1, \ldots, n\}$, $|I| = 3$, $j \in I$. We define $\tilde{f}_{12,1j}$ to map homeomorphically $E(\hat{S})$ to $\tilde{S} = \tilde{F}_{I_1} \cap \cdots \cap \tilde{F}_{I_k}$, as the inverse of the previously defined extension $\tilde{f}_{12,1j} : \tilde{S} \to E(\hat{S})$.

\[ \square \]
4 $\mathcal{F}_n$ and moduli space $\overline{M}_{0,n}$

4.1 The main result

We denote by $\mathcal{M}_{0,n}$, as usual, the moduli space of curves of genus 0 with $n$ marked distinct points. The space $\mathcal{M}_{0,n}$ parametrizes $n$-tuples of distinct points on the Riemann sphere $\mathbb{C}P^1$ up to biholomorphisms, that is

$$\mathcal{M}_{0,n} = \left((\mathbb{C}P^1)^n \setminus \Delta\right)/PGL_2(\mathbb{C}),$$

where $\Delta = \bigcup_{i \neq j} \{(x_1, \ldots, x_n) \in (\mathbb{C}P^1)^n | x_i = x_j\}$. It follows that $\mathcal{M}_{0,n}$ can be identified with

$$\mathcal{M}_{0,n} = \{(x_1, \ldots, x_{n-3}) \in \mathbb{C}P^{n-3} | x_i \neq 0, 1, \infty, x_i \neq x_j\}.$$

For example $\mathcal{M}_{0,3}$ is a points, while $\mathcal{M}_{0,4} = \mathbb{C}P^1 \setminus \{0, 1, \infty\}$. Note that the moduli space $\mathcal{M}_{0,n}$ coincides with our space of parameters of the main stratum $F_n$, compare to (5).

The moduli space $\overline{\mathcal{M}}_{0,n}$ is the space of biholomorphism classes of stable curves of genus 0 with $n$ marked distinct points. It is a compact, complex manifold of dimension $n-3$ in which $\mathcal{M}_{0,n}$ is a Zariski-open subset. The moduli space $\overline{\mathcal{M}}_{0,n}$ is a compactification of $\mathcal{M}_{0,n}$ known as the Grothendieck-Deligne-Knudsen-Mumford compactification.

In [19] Keel has given an alternative, to that of Grothendieck-Knudson, construction of the smooth complete variety $\overline{\mathcal{M}}_{0,n}$. It has been been noted in [22] that Theorem 2 when applied to his construction immediately leads that $\overline{\mathcal{M}}_{0,n}$ is a wonderful compactification $Y_G$ where $Y = (\mathbb{C}P^1)^{n-3}$ and the building set $G$ consists of the set of all diagonals and augmented diagonals. More precisely, $G$ consists of

$$\Delta_I = \{(c_4, \ldots, c_n) \in (\mathbb{C}P^1)^{n-3} | c_i = c_j \text{ for all } i, j \in I\},$$

$$\Delta_I,a = \{(c_4, \ldots, c_n) \in (\mathbb{C}P^1)^{n-3} | c_i = a \text{ for all } i \in I\},$$

where $I \subseteq \{4, \ldots, n\}$, $|I| \geq 2$ and $a \in \{0, 1, \infty\}$. The corresponding arrangement is the set of all intersections of elements in $G$.

Using this, we prove that our compactification $\mathcal{F}_n$ of $F_n$, which is obtained as the wonderful compactification of the building set $G_n$ consisting of the nonsingular subvarieties in $\overline{F}_n$ given in Section 3.3 coincides with the Grothendieck-Deligne-Knudsen-Mumford compactification of $\mathcal{M}_{0,n} = F_n$. 

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Theorem 4. The manifold $\mathcal{F}_n$ is diffeomorphic to the manifold $\overline{\mathcal{M}}_{0,n}$, $n \geq 4$.

Remark 3. Recall that for $n = 4$ it is already known that $\mathcal{F}_4 = \mathbb{C}P^1 = \overline{\mathcal{M}}_{0,4}$. Also, for $n = 5$ it has already been noted in [3], Remark 7.13 that $\mathcal{F}_5$ coincides with $\overline{\mathcal{M}}_{(0,5)}$.

4.2 $\mathcal{F}_6$ and $\overline{\mathcal{M}}_{0,6}$

For the sake of clearness we first elaborate Theorem 4 and prove Theorem 5 in the case of Grassmannian $G_{6,2}$. In addition, $G_{6,2}$ is of particular importance in algebraic geometry being one of the six Severi varieties, [21]. For $n = 6$ we start with the manifold

\[ \tilde{F}_6 = \{ ((c_{34} : c'_{34}), (c_{35} : c'_{35}), (c_{36} : c'_{36}), (c_{45} : c'_{45}), (c_{46} : c'_{46}), (c_{56} : c'_{56})) \in (\mathbb{C}P^1)^6, \]

\[ c_{34}c_{35}c'_{45} = c_{34}c'_{35}c_{45}, \quad c_{34}c_{36}c'_{46} = c_{34}c'_{36}c_{46}, \]

\[ c_{35}c_{36}c'_{56} = c_{35}c'_{36}c_{56}, \quad c_{45}c_{46}c'_{56} = c_{45}c'_{46}c_{56}, \}

The building set is given by the following subvarieties in $\tilde{F}_6$:

\[ \hat{F}_{345} = \{ ((1 : 1), (1 : 1), (c_{36} : c'_{36}), (1 : 1), (c_{46} : c'_{46}), (c_{56} : c'_{56})) \}, \]

\[ c_{36}c'_{46} = c'_{36}c_{46}, \quad c_{36}c'_{56} = c'_{36}c_{56}, \quad c_{46}c'_{56} = c'_{46}c_{56} \}

\[ \hat{F}_{346} = \{ ((1 : 1), (c_{35} : c'_{35}), (1 : 1), (c_{45} : c'_{45}), (1 : 1), (c_{56} : c'_{56})) \}, \]

\[ c_{35}c'_{45} = c'_{35}c_{45}, \quad c_{35}c'_{56} = c'_{35}c_{56}, \quad c_{45}c'_{56} = c'_{45}c_{56} \}

\[ \hat{F}_{356} = \{ ((c_{34} : c'_{34}), (1 : 1), (1 : 1), (c_{45} : c'_{45}), (c_{46} : c'_{46}), (1 : 1)) \}, \]

\[ c_{34}c'_{45} = c'_{34}c_{45}, \quad c_{34}c'_{46} = c'_{34}c_{46}, \quad c_{45}c'_{46} = c'_{45}c_{46} \}

\[ \hat{F}_{456} = \{ ((c_{34} : c'_{34}), (c_{35} : c'_{35}), (c_{36} : c'_{36}), (1 : 1), (1 : 1), (1 : 1)) \}, \]

\[ c_{34}c'_{35} = c'_{34}c_{35}, \quad c_{34}c'_{36} = c'_{34}c_{36}, \quad c_{35}c'_{36} = c_{35}c'_{36} \}

together with the point $S = (1 : 1)^6$. At this point any of these two subvarieties intersect.

The smooth, compact manifold $\mathcal{F}_6$ is a wonderful compactification with the building set $\mathcal{G}_6 = \{ S, \hat{F}_{345}, \hat{F}_{346}, \hat{F}_{356}, \hat{F}_{456} \}$ that is

\[ \mathcal{F}_6 = Bl_{\hat{F}_{345}} Bl_{\hat{F}_{346}} Bl_{\hat{F}_{356}} Bl_{\hat{F}_{456}} Bl_S \tilde{F}_6. \]
Note that the dominant transform \( \tilde{F}_{ijk} \) in \( Bl_s \tilde{F}_6 \) of any of the submanifolds \( \tilde{F}_{ijk} \), \( 3 \leq i < j < k \leq 6 \) intersect an exceptional divisor \( CP^2 \) at an one point. In addition, the four points obtained in such a way are different. It implies that the wonderful compactification given by (8) does not depend on an order of the blow-ups along the subvarieties \( F_{ijk} \).

We show that the manifold \( F_6 \) coincide with the moduli space \( \overline{M}_{0,6} \). As we already mentioned the construction from [19] describes \( \overline{M}_{0,6} \) as a sequence of blow-ups, which is then used in [22] to note that \( \overline{M}_{0,6} \) is a wonderful compactification \( Y_G \) where \( Y = (CP^1)^3 \) and the building set \( G \) consists of the set of all diagonals
\[
\Delta_I = \{(p_1, p_2, p_3) \in (CP^1)^3 | p_i = p_j \text{ for } i, j \in I \}
\]
and augmented diagonals
\[
\Delta_{I,a} = \{(p_1, p_2, p_3) \in (CP^1)^3 | p_i = a \forall i \in I \}
\]
where \( I \subset \{1, 2, 3\} \), \( |I| \geq 2 \) and \( a \in A = \{0, 1, \infty\} \).

Note that \( \Delta_I \) is a complex two-dimensional submanifold in \( (CP^1)^3 \) for \( |I| = 2 \), so the blowing up \( (CP^1)^3 \) along the diagonals \( \Delta_I \), \( |I| = 2 \) leaves \( (CP^1)^3 \) unchanged. Thus, in the wonderful compactification which describes \( \overline{M}_{0,6} \) it is enough to consider, as a building set, the complete diagonal \( \Delta_{123} \) and augmented diagonals \( \Delta_{I,a} \).

**Theorem 5.** The manifold \( F_6 \) is diffeomorphic to the space \( \overline{M}_{0,6} \).

**Proof.** Let us consider the smooth map \( f : (CP^1)^3 \to (CP^1)^6 \) given by
\[
f((c_{34} : c'_{34}), (c_{35} : c'_{35}), (c_{36} : c'_{36})) =
((c_{34} : c'_{34}), (c_{35} : c'_{35}), (c_{36} : c'_{36}), (c'_{34}c_{35} : c'_{34}c'_{35}), (c'_{34}c_{36} : c'_{35}c'_{36}), (c'_{34}c_{36} : c'_{35}c'_{36})).
\]
This map is not defined at the points of the following submanifolds in \( (CP^1)^3 \):
\[
\Delta_{\{1,2\},\infty} = \{(1 : 0), (1 : 0), (c_{36} : c'_{36})\}, \quad \Delta_{\{1,2\},0} = \{(0 : 1), (0 : 1), (c_{36} : c'_{36})\},
\]
\[
\Delta_{\{1,3\},\infty} = \{(1 : 0), (c_{35} : c'_{35}), (1 : 0)\}, \quad \Delta_{\{1,3\},0} = \{(0 : 1), (c_{35} : c'_{35}), (0 : 1)\},
\]
\[
\Delta_{\{2,3\},\infty} = \{(c_{34} : c'_{34}), (1 : 0), (1 : 0)\}, \quad \Delta_{\{2,3\},0} = \{(c_{34} : c'_{34}), (0 : 1), (0 : 1)\}.
\]
Then for \( P = ((1 : 0), (1 : 0), (1 : 0)) \) and \( Q = ((0 : 1), (0 : 1), (0 : 1)) \)
\[
G' = \{\Delta_{\{i,j\},a}, a = 0, \infty\} \cup \{P, Q\}
\]
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is a building set in \((\mathbb{C}P^1)^3\) and we consider the wonderful compactification \(Z = (\mathbb{C}P^1)_g^0\). The map \(f\) extends to the diffeomorphism \(\bar{f}\) between \(Z\) and \(\bar{F}_6\). For example the points of the divisor \(\mathbb{C}P^1\) along the submanifold \(\Delta_{1,2,0}\), the map \(\bar{f}\) maps by

\[
\bar{f}((1 : 0), (1 : 0), (c_{36} : c'_{36}), (x_1 : x_2)) = ((1 : 0), (1 : 0), (c_{36} : c'_{36}), (x_1 : x_2), (0 : 1), (0 : 1)),
\]

while the points \((x_1 : x_2 : x_3)\) of the divisor \(\mathbb{C}P^2\) at the point \(P\), the map \(\bar{f}\) maps by

\[
\bar{f}(P, (x_1 : x_2 : x_3)) = ((1 : 0), (1 : 0), (1 : 0), (x_1 : x_2), (x_1 : x_3), (x_2 : x_3)).
\]

Since in the neighborhood \(((1 : c'_{34}), (1 : c'_{35}), (1 : c'_{36}))\) of \(P = ((1 : 0), (1 : 0), (1 : 0))\) it holds

\[
c'_{34}x_2 = c'_{35}x_1, \quad c'_{34}x_3 = c'_{35}x_1, \quad c'_{35}x_3 = c'_{36}x_2,
\]

it follows that the points \(((1 : 0), (1 : 0), (1 : 0), (x_1 : x_2), (x_1 : x_3), (x_2 : x_3))\) belong to \(\bar{F}_6\).

In order to finish the proof it is left to note that the wonderful compactification for \(F_6\) with the building set consisting of \(\bar{F}_{ijk}\) and \(S\) corresponds to the wonderful compactification for \(Z\) with the building set consisting of \(\Delta_{1,2,1}, \Delta_{1,3,1}, \Delta_{2,3,1}, R = ((1 : 1), (1 : 1), (1 : 1))\) and the diagonal \(\Delta_{123}\). It follows that \(F_6\) and \(\bar{M}_{0,6}\) are diffeomorphic.

\[\square\]

### 4.3 The proof of the main result

**Proof.** (of Theorem 4). The proof proceeds in an analogous way as for \(n = 6\). According to \(\mathbf{3}\), \(\mathbf{4}\) and \(\mathbf{5}\) one can start with the smooth map \(f : (\mathbb{C}P^1)^{n-3} \to (\mathbb{C}P^1)^N, N = \binom{n-2}{2}\) given by

\[
f((c_{34} : c'_{34}), \ldots, (c_{3n} : c'_{3n})) =
\]

\[
((c_{34} : c'_{34}), \ldots, (c_{3n} : c'_{3n}), (c'_{34}c_{35} : c'_{34}c_{35}), \ldots, (c'_{3n-1}c_{3n} : c_{3n-1}c'_{3n})).
\]

The map \(f\) is not defined at the points of the submanifolds \(G'_{pq}, G'_{pq} \subset (\mathbb{C}P^1)^{n-3}, 4 \leq p < q \leq n\) given by

\[
G_{pq} = \{(c_{3i} : c_{3j}) \in (\mathbb{C}P^1)^{n-3} | (c_{3p} : c'_{3p}) = (c_{3q} : c'_{3q}) = (1 : 0)\}
\]

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\[ G'_{pq} = \{ ((c_3i : c_3j)) \in (\mathbb{C}P^1)^{n-3} | (c_3p : c_3'p) = (c_3q : c_3'q) = (0 : 1) \}. \]

From (3) and (4) it obviously follows that the map \( f \) gives a diffeomorphism between \((\mathbb{C}P^1)^{n-3} \setminus \Delta\) and the space of parameters of the main stratum \( F_n \).

It is easy to verify that the set \( G' \) of all possible intersections of the subvarieties \( G_{pq}, G'_{pq} \) form a building set. Let \( Z \) be a smooth manifold obtained as the wonderful compactification of \((\mathbb{C}P^1)^{n-3}\) with the building set \( G' \) that is \( Z = (\mathbb{C}P^1)^{n-3} \).

Then, as in the case \( n = 6 \), we see that the map \( f \) extends to the diffeomorphism \( \bar{f} \) between \( Z \) and \( \bar{F}_n \).

Let further
\[ H_{pq} = \{ ((c_3i : c_3j)) \in (\mathbb{C}P^1)^{n-3} | (c_3p : c_3'p) = (c_3q : c_3'q) = (1 : 1) \} \]
and let \( \bar{H}_{pq} \) be a proper transform of \( H_{pq} \) in \( Z \). The set \( G'' \) of all possible intersections of subvarieties \( \bar{H}_{pq} \) is a building. It is explained in [22], subsection 4.4, that the manifold \( \overline{\mathcal{M}}(0, n) \) coincides with the wonderful compactification \( Z_{G''} \). It is left to note that diffeomorphism \( \bar{f} \) extends to the diffeomorphism between the wonderful compactification \( Z_{G''} \) and the wonderful compactification \( (\bar{F}_n)_G \), where the building set \( G \) is given by all possible intersections of the subvarieties (7). Thus, the smooth manifolds \( \overline{\mathcal{M}}(0, n) \) and \( \mathcal{F}_n \) coincides.

5.1 Basic facts on Chow varieties and Chow quotient \( G_{n,2} // (\mathbb{C}^*)^n \)

We follow the monograph [14], Chapter 4 to recall the basic facts on Chow varieties, while for the notion of Chow quotient \( G_{n,k} // (\mathbb{C}^*)^n \) we follow the paper of Kapranov [18]. The idea behind the definition of the Chow quotient is the construction from algebraic geometry known as the Chow varieties, that is compact varieties whose points parametrize algebraic cycles in a given variety of the same dimension and degree. The Chow variety for \( G_{n,k} \) which is needed in the definition of the Chow quotient can be defined as follows. Let \( \delta \in H_{2(n-1)}(G_{n,k}, \mathbb{Z}) \) be the homology class of the closure of a generic \((\mathbb{C}^*)^n\) - orbit in \( G_{n,k} \) and let \( C_{2(n-1)}(G_{n,k}, \delta) \) denotes the set of all algebraic cycles in \( G_{n,k} \) of dimension \( 2(n-1) \) whose homology class is \( \delta \). The Grassmann manifolds \( G_{n,k} \) embeds into \( \mathbb{C}P^N, N = \binom{n}{k} - 1 \) via Plücker embedding, so let \( d \in H_{2(n-1)}(\mathbb{C}P^N, \mathbb{Z}) \approx \)
\(Z\) be the image of the class \(\delta\) under this embedding. Now, consider the set 
\(G(N, d, 2(n - 1))\) of algebraic cycles in \(\mathbb{C}P^N\) of dimension \(2(n - 1)\) and degree \(d\). In other words, one considers algebraic cycles whose multiplicity in 
\(H_{2(n-1)}(\mathbb{C}P^N, \mathbb{Z})\), regarded to the canonical generator, is \(d\). Denote by \(\mathcal{B}\) the coordinate ring of \(G_{n,k}\) via the Plücker embedding, that is the quotient of the 
polynomial ring \(\mathbb{C}[z_1, \ldots z_{N+1}]\) by the Plücker relations and denote by \(\mathcal{B}_d\) a complex linear subspace which is given by the homogeneous part of \(\mathcal{B}\) of degree \(d\).

By the theorem of Chow and van der Waerden, the set 
\(G(N, d, 2(n - 1))\) becomes 
a closed projective algebraic variety, in particular compact, via Chow embedding \(G(N, d, 2(n - 1)) \to P(\mathcal{B}_d)\). The set \(C_{2(n-1)}(G_{n,k}, \delta)\) endowed with the resulting structure of the algebraic variety via \(C_{2(n-1)}(G_{n,k}, \delta) \subset G(N, d, 2(n - 1))\) is the needed Chow variety for \(G_{n,k}\).

In more detail, the mentioned Chow embedding is defined as follows, see [14]. For any irreducible algebraic cycle \(X \in G(N, d, 2(n - 1))\) one can consider the 
set \(\mathcal{Z}(X)\) of all \((N - 2(n - 1) - 1)\)-dimensional projective subspaces \(L\) in \(\mathbb{C}P^N\) which intersect \(X\). The set \(\mathcal{Z}(X)\) is a subvariety in the Grassmannian \(G(N, N - 2(n - 1) + 1)\). It can be proved that \(\mathcal{Z}(X)\) is defined by some element \(R_X \in \mathcal{B}_d\) which is unique up to constant factor and \(R_X\) is called the Chow form of \(X\). If \(X\) is not irreducible cycle then \(X = \sum a_i X_i\), where \(X_i\) are \(2(n - 1)\)-dimensional closed irreducible varieties and \(a_i\) are non-negative integer coefficients, and the 
Chow form for \(X\) is defined by \(R_X = \prod R_{X_i}^{a_i} \in \mathcal{B}_d\). The map \(X \to R_X\) defines an embedding of \(G(N, d, 2(n - 1))\) into the projective space \(P(\mathcal{B}_d)\) which is called the Chow embedding.

In order to define the Chow quotient one considers the natural map 
\[ W/((\mathbb{C}^*)^n \to C_{2(n-1)}(G_{n,k}, \delta), \ x \to (\mathbb{C}^*)^n \cdot x, \]
where \(W\) is the main stratum in \(G_{n,k}\) which consists of the points whose all 
Plücker coordinates are non-zero. By definition, the Chow quotient \(G_{n,k}/((\mathbb{C}^*)^n)\) 
is the closure of the image of this map.

We recall the following results from [18], Propositions (1.2.11), (1.2.15) which give the description of the build up components of \(W/((\mathbb{C}^*)^n)\) in \(G_{n,k}/((\mathbb{C}^*)^n)\).

**Proposition 2.** The algebraic cycles in the Chow quotient \(G_{n,k}/((\mathbb{C}^*)^n)\) are of the 
form \(Z = \sum Z_i\), where \(Z_i\) are the closures of \((\mathbb{C}^*)^n\)-orbits in \(G_{n,k}\) such that the 
matroid polytopes \(\mu(Z_i)\) give a polyhedral decomposition of \(\Delta_{n,k}\).

We point out that the matroid polytopes defined in [18] in the case \(G_{n,2}\) coincide 
with our admissible polytopes [5]. Note that, in our terminology, the Chow quo-
tient gives actually a compactification of the space of parameters $F_n$ of the main stratum.

We want to emphasize that the Chow quotient $X//H$ can be defined for any complex projective variety $X \subset \mathbb{CP}^N$ with an action of an algebraic group $H$, see [18]. Namely, the orbit closure $\overline{H \cdot x}$ is a compact subvariety in $X$ for any point $x \in X$ and for a small Zariski open $H$-invariant subset $U \subset X$ which consists of generic points all varieties $\overline{H \cdot x}$ for $x \in U$ have the same dimension $m$ and represent the same homology class $\delta \in H_{2m}(X, \mathbb{Z})$. One can consider the Chow variety $C_{2m}(X, \delta) \subset G(N, d, 2m)$, where $d$ is an image of $\delta$ by an embedding $X \to \mathbb{CP}^N$, and the Chow quotient $X//H$ is the closure of the image of the map $U/H \to C_{2m}(X, \delta)$ defined by $x \to x \cdot \overline{H}$.

Using Gel’fand-MacPherson construction Kapranov in [18] proved that for $k = 2$ the Chow quotient $G_{n,2}/((\mathbb{C}^*)^n$ is isomorphic to the Chow quotient $(\mathbb{CP}^k)^n/\text{GL}(k)$. Appealing on this, he constructed an isomorphism between the Chow quotient $G_{n,2}/((\mathbb{C}^*)^n$ and the Grothendieck-Knudsen compactification $\overline{M}_{0,n}$. In addition using the construction of such isomorphism Kapranov in [18] provided the description of the quotient $G_{n,2}/((\mathbb{C}^*)^n$ as the sequence of blow ups along some specified subvarieties in $\mathbb{CP}^{n-3}$.

In conclusion, we want to point that Theorem 4 implies that our construction of the space $F_n$ provides the new, purely topological approach for the description of the Chow quotient $G_{n,2}/((\mathbb{C}^*)^n$.

5.2 The structures of $G_{n,2}/((\mathbb{C}^*)^n$ and $G_{n,2}/T^n$

In [5] we described the orbit space $G_{n,2}/T^n$ in terms of CW complex of admissible polytopes and the universal space of parameters $F_n$. In this description, fundamental role has the chamber decomposition of $\Delta_{n,2}$ induced by the admissible polytopes. Using the results of [18], first we describe the Chow quotient $G_{n,2}/((\mathbb{C}^*)^n$ in terms of cortèges of admissible polytopes which give the polyhedral decomposition for $\Delta_{n,2}$ and the spaces of parameters of these polytopes. In addition, we describe $G_{n,2}/((\mathbb{C}^*)^n$ in terms of the virtual spaces of parameters for the admissible polytopes which form a $(n - 1)$-dimensional chamber in $\Delta_{n,2}$.

Let $P$ denotes the family of admissible polytopes of dimension $n - 1$ for the standard $T^n$ - action on $G_{n,2}$. Let the set $\{P_1, \ldots, P_l\}$ consists of all subfamilies $P_i = \{P_{i_1}, \ldots, P_{i_s}\} \subset P$ such that the polytopes $P_{i_1}, \ldots, P_{i_s}$ give a polyhedral
Proof. We define the bijection \( \sigma \) parameter \( Z_c \). We assign to a family \( \mathcal{P}_i = \{P_{i_1}, \ldots, P_{i_s}\} \) the set \( \mathcal{W}_i = \{W_{i_1}, \ldots, W_{i_s}\} \), where \( \mathcal{W}_{ij} \) are the strata in \( G_{n,2} \) which correspond to the admissible polytopes \( P_{ij} \), that is \( \mu(W_{ij}) = \tilde{P}_{ij} \). By factorizing these strata by \((\mathbb{C}^*)^n\)-action, we can assign to any \( \mathcal{P}_i \) the set \( \{F_{i_1}, \ldots, F_{i_s}\} \).

Further, we introduce the space of parameters \( \mathcal{F}_i \) of a family \( \mathcal{P}_i \) as the multisubset product of the set \( \{F_{i_1}, \ldots, F_{i_s}\} \), meaning that it is a topological space homeomorphic to the direct product \( \mathcal{F}_i = F_{i_1} \times \cdots \times F_{i_s} \).

It follows from Proposition 2 that the Chow quotient \( G_{n,2}/(\mathbb{C}^*)^n \) is the disjoint union of the connected components \( \mathcal{C}_i \), each consisting of algebraic cycles determined by a family \( \mathcal{P}_i \), \( 1 \leq i \leq l \). In particular, the complement of \( F_i \) in \( G_{n,2}/(\mathbb{C}^*)^n \) is the disjoint union of the components \( \mathcal{C}_i \) for which \( \mathcal{P}_i \neq \{\Delta_{n,2}\} \).

Proposition 3. There is a bijection between \( \mathcal{F}_i \) and \( \mathcal{C}_i \) for any \( 1 \leq i \leq l \).

Proof. We define the bijection \( g_i : \mathcal{F}_i \rightarrow \mathcal{C}_i \) as follows:

\[
g_i(c_{i_1}, \ldots, c_{i_s}) = Z_{i_1}(c_{i_1}) + \cdots + Z_{i_s}(c_{i_s}),
\]

where \( Z_{ij}(c_{ij}) \) is the \((\mathbb{C}^*)^n\)-orbit from the stratum \( W_{ij} \) which is determined by the parameter \( c_{ij} \).

The Chow quotient \( G_{n,2}/(\mathbb{C}^*)^n \) and the complement of \( F_i \) in \( G_{n,2}/(\mathbb{C}^*)^n \) can be interpreted in the following way as well.

Let \( P_\sigma \) is an admissible polytope and consider the set \( \mathcal{P}_\sigma = \{\mathcal{P}_{\sigma,1}, \ldots, \mathcal{P}_{\sigma,s}\} \subset \mathcal{P} \) which consists of all decompositions \( \mathcal{P}_{\sigma,i} \) of \( \Delta_{n,2} \) which contain \( P_\sigma \), that is \( \mathcal{P}_{\sigma,i} \in \mathcal{P} \) if and only if \( P_\sigma \in \mathcal{P}_{\sigma,i} \).

Let \( \tilde{Z}_{\sigma,i} \subset G_{n,2}/(\mathbb{C}^*)^n \) be the family of algebraic cycles determined by the decompositions \( \mathcal{P}_{\sigma,i} = \{P_{\sigma_{i_1}}, \ldots, P_{\sigma_{i_q}}\} \). These cycles are, by Proposition 3, of the form

\[
Z_{\sigma_{i_1}}(c_{\sigma_{i_1}}) + \cdots + Z_{\sigma_{i_q}}(c_{\sigma_{i_q}}),
\]

for \( (c_{\sigma_{i_1}}, \ldots, c_{\sigma_{i_q}}) \in F_{\sigma_{i_1}} \times \cdots \times F_{\sigma_{i_q}} \), where \( Z_{\sigma_{i_j}}(c_{\sigma_{i_j}}) \) is the closure of the algebraic torus orbit in \( W_{\sigma_{i_j}} \) and this orbit is determined by \( c_{\sigma_{i_j}} \in F_{\sigma_{i_j}} = W_{\sigma_{i_j}}/(\mathbb{C}^*)^n \). Let further

\[
\tilde{Z}_\sigma = \bigcup_{i=1}^{s} \tilde{Z}_{\sigma,i}.
\]
Proposition 4. There exists the projection \( p_\sigma : \tilde{Z}_\sigma \to F_\sigma \) for any admissible set \( \sigma \).

Proof. For an algebraic cycle \( Z \in \tilde{Z}_\sigma \) there exists a \((\mathbb{C}^*)^n\)-orbit \( Z_\sigma \) from the stratum \( W_\sigma \) whose admissible polytopes is \( P_\sigma \) and which is an irreducible summand of \( Z \). Since \( F_\sigma = W_\sigma/(\mathbb{C}^*)^n \) there is the canonical \((\mathbb{C}^*)^n\)-invariant projection \( q_\sigma : W_\sigma \to F_\sigma \). We define \( p_\sigma(Z) = q_\sigma(Z_\sigma) \).

Recall [16], [5] that the admissible polytopes define the chamber decomposition for \( \Delta_{n,2} \): for some subset \( \omega \) of all admissible sets, \( C_\omega \) is a chamber if

\[ C_\omega = \bigcap_{\sigma \in \omega} \overset{\circ}{P}_\sigma, \quad C_\omega \cap \overset{\circ}{P}_\sigma = \emptyset. \]

Theorem 6. Let \( C_\omega \subset \Delta_{n,2} \) be a chamber such that \( \dim C_\omega = n - 1 \). Then \( C_\omega \) defines the decomposition of \( G_{n,2}/(\mathbb{C}^*)^n \) into disjoint union, that is:

\[ \bigcup_{\sigma \in \omega} \tilde{Z}_\sigma = G_{n,2}/(\mathbb{C}^*)^n. \quad (9) \]

Proof. Let \( Z \in G_{n,2}/(\mathbb{C}^*)^n \). Then \( Z \) is determined by some decomposition \( \mathcal{P}_i = \{ P_{\sigma_1}, \ldots, P_{\sigma_l} \} \). Note that for any admissible polytope \( P_\sigma \) it holds \( C_\omega \subset \overset{\circ}{P}_\sigma \) either \( C_\omega \cap \overset{\circ}{P}_\sigma = \emptyset \). Thus, there exists \( P_{\sigma_{ij}} \in \mathcal{P}_i \) such that \( C_\omega \subset \overset{\circ}{P}_{\sigma_{ij}} \) meaning that \( \sigma_{ij} \in \omega \). Therefore, \( Z \in \tilde{Z}_{\sigma_{ij}} \), which implies that \( Z \in \bigcup_{\sigma \in \omega} \tilde{Z}_\sigma \).

In order to prove that the union (9) is disjoint, we note that for \( \sigma_1, \sigma_2 \in \omega \) we have that \( C_\omega \subset \overset{\circ}{P}_{\sigma_1}, \overset{\circ}{P}_{\sigma_2} \), that is \( \overset{\circ}{P}_{\sigma_1} \cap \overset{\circ}{P}_{\sigma_2} \neq \emptyset \). It implies that there is no decomposition for \( \Delta_{n,2} \) which contains both \( P_{\sigma_1} \) and \( P_{\sigma_2} \). Thus, there is no algebraic cycle \( Z \in G_{n,2}/(\mathbb{C}^*)^n \) such that \( Z \in \tilde{Z}_{\sigma_1} \) and \( Z \in \tilde{Z}_{\sigma_2} \).

Note that \( \tilde{Z}_\sigma = F_n \) for \( P_\sigma = \Delta_{n,2} \), which implies that (9) describes as well the build up components to \( F_n \) in \( G_{n,2}/(\mathbb{C}^*)^n \), which are in general no more disjoint.

Remark 4. In our papers [3], [4], for the purpose of description of an orbit space \( G_{n,2}/T^n \), we introduced the notion of virtual space of parameters \( \bar{F}_\sigma \) for a stratum \( W_\sigma \). Note that the properties of the spaces \( \tilde{Z}_\sigma \) formulated by Proposition 4 and Theorem 6 confirm that the spaces \( \tilde{Z}_\sigma \) correspond to the spaces \( \bar{F}_\sigma \) for which \( \dim P_\sigma = n - 1 \). In particular, Theorem 7 from [5] is an analogue of Theorem 6.
Theorem 4 and the result of [18] that the manifolds $G_{n,2}/((\mathbb{C}^*)^n)$ and $\mathcal{M}(0,n)$ are isomorphic imply that the universal space of parameters $\mathcal{F}_n$ describe the topology of the gluing of the build up components in $G_{n,2}/((\mathbb{C}^*)^n)$. Having the description of $\mathcal{F}_n$ as the wonderful compactification of $\bar{\mathcal{F}}_n$ we give an explicit demonstration of this correspondence in the cases $n = 4$ and $n = 5$.

5.3 $G_{4,2}/((\mathbb{C}^*)^4)$ and $G_{4,2}/T^4$

For $n = 4$ the build up components in $G_{4,2}/((\mathbb{C}^*)^4)$ consist of three points and they glue together with $F_1 \cong \mathbb{C}P^1_A$ in $G_{4,2}/((\mathbb{C}^*)^4)$ to give $\mathbb{C}P^1_A$. This is observed in [18], but also independently follows from [2] and Theorem 4. Using our notation, we describe this in the following way. There are exactly three decompositions of an octahedron $\Delta_{4,2}$ that is $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$ and they are given by the three pairs of four-sided complementary pyramids. The space of parameters for a stratum for any of these pyramids is a point. Then Proposition 3 implies that $(G_{4,2}/((\mathbb{C}^*)^4)) \setminus F_1$ consists of three points, that is $G_{4,2}/((\mathbb{C}^*)^4) \cong \mathbb{C}P^1_A$. More precisely, these three points correspond to the algebraic cycles formed of the $(\mathbb{C}^*)^4$-orbits whose admissible polytopes are complementary pyramids in the octahedron $\Delta_{4,2}$. In addition, for any pyramid $P_\sigma$ we have that $\tilde{Z}_\sigma$ is given by the one algebraic cycle, which implies that $F_\sigma = \tilde{Z}_\sigma$.

5.4 $G_{5,2}/((\mathbb{C}^*)^5)$ and $G_{5,2}/T^5$

In the case $n = 5$, we use the results from [3] to formulate the correspondence between $\mathcal{F}_5$ and the Chow quotient $G_{5,2}/((\mathbb{C}^*)^5)$. First, we immediately have the following:

**Lemma 8.** There are 25 decompositions of the hypersimplex $\Delta_{5,2}$ given by the admissible polytopes for $T^5$-action on $G_{5,2}$. There are given by the pairs $\{K_{ij}, P_{ij}\}$, $1 \leq i < j \leq 5$ and the triples $\{P_{ij}, K_{ij,kl}, P_{kl}\}$, $1 \leq i < j \leq 5$, $1 \leq k < l \leq 5$, $\{i, j\} \cap \{k, l\} = \emptyset$. Here $K_{ij}$ is a polytope with 9 vertices which does not contain the vertex $\Lambda_{ij}$, then $P_{ij}$ is seven-sided pyramid with the apex $\Lambda_{ij}$, while $K_{ij,kl}$ is a polytope with 8 vertices which does not contain the vertices $\Lambda_{ij}$ and $\Lambda_{kl}$.

The space of parameters for a admissible polytope $K_{ij}$ is $\mathbb{C}P^1_A$, while for the polytopes $P_{ij}$ and $K_{ij,kl}$ it is a point. Then Proposition 3 gives:
**Corollary 1.** The disjoint build up components for $F_5$ in $G_{5,2}/(\mathbb{C}^*)^5$ are $C_{ij} \cong \mathbb{C}P^1_A$ and the points $C_{ij,kl}$ for $1 \leq i < j \leq 5$, $1 \leq k < l \leq 5$. A component $C_{ij}$ consists of the cycles of the form $Z_{ij,9}(c) + Z_{ij,7}$ while a component $C_{ij,kl}$ consists of the cycle $Z_{ij,7} + Z_{ij,kl} + Z_{kl,7}$, where $c \in \mathbb{C}P^1_A$. The irreducible algebraic cycles here are as follows:

- $Z_{ij,9}(c)$ is the closure of a orbit from the stratum whose admissible polytope is $K_{ij}$;
- $Z_{ij,kl}$ is the closure of the orbit whose admissible polytope $K_{ij,kl}$;
- $Z_{ij,7}$ is the closure of the orbit whose admissible polytope $P_{ij}$.

We proved in [3] that the universal space of parameters $\mathcal{F}_5$ is the blow up of the surface $\bar{F}_5 = \{(c_1 : c_1), (c_2 : c_2), (c_3 : c_3) \in (\mathbb{C}P^1)^3, c_1c_2c_3 = c'_1c'_2c'_3\}$ at the point $((1 : 1), (1 : 1), (1 : 1))$. The identification of $\mathcal{F}_5$ with $G_{5,2}/(\mathbb{C}^*)^5$ translates to the gluing of the build up components in $G_{5,2}/(\mathbb{C}^*)^5$ as follows:

**Corollary 2.** The gluing of the build up components in $G_{5,2}/(\mathbb{C}^*)^5$ corresponds to the compactification of $F_5 \subset \mathcal{F}_5$ by the following pattern:

- the cycles $Z_{ij}(c) = Z_{ij,9}(c) + Z_{ij,7} \in C_{ij}$ correspond to the subvarieties in $\mathcal{F}_5$ as follows:
  - $Z_{23}(c) = ((0 : 1), (0 : 1), (c : c'))$,
  - $Z_{24}(c) = ((1 : 0), (c : c'), (0 : 1))$,
  - $Z_{25}(c) = ((c : c'), (1 : 0), (1 : 0))$,
  - $Z_{13}(c) = ((1 : 0), (1 : 0), (c : c'))$,
  - $Z_{14}(c) = ((0 : 1), (c : c'), (1 : 0))$,
  - $Z_{15}(c) = ((c : c'), (0 : 1), (0 : 1))$,
  - $Z_{34}(c) = ((1 : 1), (c : c'), (c : c'))$,
  - $Z_{35}(c) = ((c : c'), (1 : 1), (c' : c))$,
  - $Z_{45}(c) = ((c : c'), (c : c'), (1 : 1))$,
  - $Z_{12}(c)$ to the points from $\mathbb{C}P^1_A$ of the divisor $\mathbb{C}P^1$.
- the cycles $Z_{ij,kl} = Z_{ij,7} + Z_{ij,kl} + Z_{kl,7} \in C_{ij,kl}$ correspond to the points:
\[-Z_{14,23} = ((0 : 1), (0 : 1), (1 : 0)), \ Z_{13,24} = ((1 : 0), (1 : 0), (0 : 1)) ,\]
\[-Z_{15,24} = ((1 : 0), (0 : 1), (0 : 1)), \ Z_{23,45} = ((0 : 1), (0 : 1), (1 : 1)),\]
\[-Z_{24,35} = ((1 : 0), (1 : 1), (0 : 1)), \ Z_{25,34} = ((1 : 1), (1 : 0), (1 : 0)).\]

We also directly establish the correspondence of the subspaces \(\tilde{Z}_{ij,9}, \tilde{Z}_{ij,7}\) and \(\tilde{Z}_{ij,kl}\) with the subspaces in \(F_5\).

**Corollary 3.** The spaces \(\tilde{Z}_{ij,9}, \tilde{Z}_{ij,7}, \tilde{Z}_{ij,kl} \subset G_{5,2}/((\mathbb{C}^*)^5)\) are homeomorphic to \(\mathbb{C}P^1\), \(\mathbb{C}P^1\) and a point respectively. The corresponding spaces in \(F_5\) are

\[
\tilde{Z}_{ij,9} = C_{ij}, \quad \tilde{Z}_{ij,kl} = \mathcal{C}_{ij,kl},
\]
\[
\tilde{Z}_{ij,7} = C_{ij} \cup (\bigcup_{k,l \in \{1, \ldots, 5\}\setminus\{i,j\}} \mathcal{C}_{ij,kl}).
\]

Note that the spaces \(\tilde{Z}_{ij,9}, \tilde{Z}_{ij,7}\) and \(\tilde{Z}_{ij,k}\) coincide with the virtual spaces of parameters \(\tilde{F}_{ij,9}, \tilde{F}_{ij,7}\) and \(\tilde{F}_{ij,k}\) for the corresponding strata in \(G_{5,2}\) from [5].

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