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Abstract

From BFSS matrix theory considerations, it is expected that a single $D_{p+2}$-brane action can be obtained from $N D_p$-brane action in large $N$ limit. We examine and confirm this expectation by working out the details of DBI and Chern-Simons terms of $D_{p+2}$-brane action from $D_p$-brane action. We show that the same relation works for non-BPS, as well as BPS branes.
1 Introduction

D$p$-brane is a $(p+1)$-dimensional hypersurface in space-time defined by the property that open strings can end on it. They have been an active area of study and many remarkable properties of them have been discussed [1]. In particular, they have provided a useful tool for the study of black holes in string theory [2]; AdS/CFT correspondence is another offspring of D$p$-branes [3]. Moreover, it is well-known that IIA and IIB string theories have odd and even BPS D$p$-branes, respectively, and T-duality changes D$p$-branes in these theories to each other (or in other words it replaces a scalar field with a gauge field and vice versa). D$p$-branes are sources of RR $(P+1)$-form field in IIA and IIB string theories.

There is an interesting candidate for DLCQ of M-theory in terms of D$0$-branes, the BFSS matrix model [4]. This conjecture tells us that all D$p$-branes in IIA superstring theory can be described in terms of D$0$-branes and their bound states. This is a requirement for the BFSS matrix model as it is supposed that the theory of D$0$-branes describes the whole theory of DLCQ of M-theory. In particular, e.g. a flat D$2$-brane may be understand as a bound state of $N(N \rightarrow \infty)$ D$0$-branes. Therefore, understanding of connection among various D$p$-branes seems to be significant. It is well known that the low energy limit of the D-brane action reduce to the Yang-Mills(YM) action. In YM limit, this connection has already been addressed in the literature e.g. see [5] but here we use another way introduced in [6] and explain it more precisely. In fact in YM theory we consider fluctuations around two directions which are transverse to multiple D$p$-branes and it will be shown that these two directions play the role of components of gauge field in the world volume of D$p+2$-brane. We will then extend the prescription found at the level of YM theory to terms, for both BPS and non-BPS Dirac-Born-Infeld(DBI) action and their Chern-Simons actions as well. The D$p+2$-brane action is consistent with what is expected.

The multiple D$p$-branes theory has three parameters the string length, $l_s$, string coupling constant, $g_s$, and the number of D$p$-branes, $N$. Then our prescription can be explained as follow. Two transverse directions can be considered as a fuzzy torus³ (compact transverse direction on fuzzy torus) or on the other hand, for finite $N$, D$p$-branes are uniformly distributed on these two directions. Now two new parameters are added to our theory which are

³Although, other configurations can be considered.
radii of fuzzy torus, $R_1, R_2$. In order to take geometric interpretation, there
is a consistent limit which is when the number of multiple $D_p$-branes and
radii of fuzzy torus go to infinity like $\theta = \frac{R_1 R_2}{l_s N} \to 0$. As we will see in this
limit $D_p$-branes have been dissolved in the world volume of a $D_{p+2}$-brane.\footnote{The other limits are when $\theta$ is constant or goes to infinity. In both cases the final
theories are non-commutative.}

In comparison to two T-dualities replacing two transverse directions with
two components of gauge field (or vice versa) on $D_p$-branes, in our prescrip-
tion, we consider multiple $D_p$-branes and show that two transverse directions
replace with two components of gauge filed on $D_{p+2}$-brane. In opposite way,
one can argue that a $D_{p+2}$-brane becomes multiple $D_p$-branes if two compo-
nents of the gauge field are substituted by transverse scalar fields.

This paper is organized as following. In the two next sections the pre-
scription is introduced and applied to YM theory and DBI actions. We then
show that the non-BPS D-branes are consistent with above prescription and
in the section five Chern-Simons action is considered. The last section is
devoted to conclusion.

2 A large N limit of SYM theory

The bosonic part of a $p + 1$ dimensional $U(N)$ SYM theory is described by
the action

$$ S = -\frac{1}{2g_Y^2} \int d^{p+1}x \text{Tr} \left( (D_\mu X^I)^2 - \frac{1}{2} [X^I, X^J]^2 + \frac{1}{2} F_{\mu \nu}^2 \right), \quad (1) $$

where\footnote{In terms of D-brane and string theory parameters note that $g_Y^2 = (\lambda^2 T_p)^{-1} = (2\pi \lambda)^{-\frac{p}{2}} \lambda^{-1} g_s$, $\lambda = 2\pi l_s^2$ where $T_p$ is the brane tension.}

$$ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu], $$
$$ D_\mu \Phi = \partial_\mu \Phi + i[A_\mu, \Phi]. $$

(2)

$\mu (= 0, \ldots, p)$ and $I (= p+1, \ldots, 9)$. This action is also the action for N coincident
$D_p$-branes in the $\alpha' \to 0$ limit and in this picture $\mu$ and $I$ denote world
volume and transverse indices respectively. The above action has following
$U(N)$ gauge symmetry

$$ \delta_g X^I = i [X^I, \alpha], $$

$$ \delta_g A_\mu = \partial_\mu \alpha + i [A_\mu, \alpha]. $$

(3a)

(3b)
In order to describe a $D_{p+2}$-brane, our prescription two steps.

- The $9-p$ transverses directions are decomposed as
  \[ X^I \sim (y^{\mu=1,2}, X^{i=p+3...9}), \]  (4)

and the following replacement will then be done

\[ [\Phi(x), \Psi(x)] \rightarrow i\lambda \theta \{\Phi(x, y), \Psi(x, y)\}, \]  \((5a)\)
\[ \int d^{p+1}x \text{Tr}(\ast) \rightarrow \frac{1}{\lambda} \int d^{p+1}x d^2y(\ast), \]  \((5b)\)

where \(\{\Phi, \Psi\} = \epsilon^{\mu\nu} \partial_\mu \Phi \partial_\nu \Psi\) (it is fixed such that \(\epsilon_{\mu\nu} \{y^\mu, y^\nu\} = 2\)). \(\Phi\) and \(\Psi\) are arbitrary fields and \(\theta\) is a dimensionless constant. Note that, in new action, after above replacement all matrices in the YM theory have been changed with fields. Furthermore, these fields are functions of \(x, y\) and hence \(D_\mu X^i\) is no longer zero. Another point is that \((5a)\) is right replacement when the number of $D_p$-branes are large.

- Fluctuations around two transverse directions, \(X^{\bar{\mu}}\), is considered as
  \[ X^{\bar{\mu}} \equiv \frac{1}{\lambda \theta} y^{\bar{\mu}} - \epsilon^{\bar{\mu} \bar{\nu}} A_\nu. \]  \((6)\)

The form of above equation indicates that fluctuations play the role of extra components of the world volume $D_{p+2}$-brane gauge field which we will show it in detail.

Let's start with variation of scalar fields under gauge transformation \((3a)\) which leads to

\[ \delta_g X^I = -(\delta_g y^{\bar{\mu}}) \partial_{\bar{\mu}} X^I, \]  \((7)\)

where

\[ \delta_g y^{\bar{\mu}} = \lambda \theta \epsilon^{\bar{\mu} \bar{\nu}} \partial_{\bar{\nu}} \alpha. \]  \((8)\)

Variation of the new components of the gauge field can be found by using \((6)\) and \((7)\) which is

\[ \delta_g A_{\bar{\mu}} = \partial_{\bar{\mu}} \alpha + \lambda \theta \epsilon_{\bar{\mu} \bar{\nu}} \{X^{\bar{\nu}}, \alpha\}. \]  \((9)\)

The first term in the YM action can be simplified by using \((6)\) and we then have

\[ (D_{\mu} X^I)^2 = (D_{\mu} X^i)^2 + (F_{\mu\nu})^2, \]  \((10)\)

4
The second term gives \( \{X^i, X^j\}^2 \) and

\[
\{X^\mu, X^\nu\}^2 = \frac{2}{(\lambda \theta)^4} + \frac{1}{(\lambda \theta)^2}(F^\mu_\nu)^2 + \text{total derivative terms},
\]

\[
\{X^\mu, X^i\}^2 = \frac{1}{(\lambda \theta)^2}(D^\mu X^i)^2,
\]

where

\[
\{X^\mu, X^\nu\} = \frac{\epsilon^\mu_\nu}{(\lambda \theta)^2} - \frac{1}{\lambda \theta} \epsilon^\mu_\alpha F_{\alpha \beta} \epsilon^\beta_\nu,
\]

\[
\{X^\mu, X^i\} = \frac{\epsilon^\mu_\alpha}{\lambda \theta} D_\alpha X^i,
\]

and

\[
F^\mu_\nu = \partial^\mu A_\nu - \partial^\nu A^\mu - \lambda \theta \{A_\mu, A_\nu\},
\]

\[
F_{\mu \nu} = \partial_\mu A_\nu^\prime - \partial_\nu A_\mu^\prime - \lambda \theta \{A_\mu^\prime, A_\nu^\prime\},
\]

\[
D_\mu \Phi = \partial_\mu \Phi + \lambda \theta \{A_\mu, \Phi\}.
\]

By plugging the above expressions in (1), we arrive at a non-commutative \( D_{p+2} \)-brane action is appeared

\[
S = -\frac{1}{2g_Y M^2} \int d^{p+3}x \left( (D^\mu X^i)^2 + \frac{(\lambda \theta)^2}{4}\{X^i, X^j\}^2 + \frac{1}{2} F_{\mu \nu}^2 + \frac{1}{2(\lambda \theta)^2} \right),
\]

where

\[
F^\mu_\nu = \partial^\mu A_\nu - \partial^\nu A^\mu + \lambda \theta \{A_\mu, A_\nu\},
\]

\[
D_\mu \Phi = \partial_\mu \Phi + \lambda \theta \{A_\mu, \Phi\}.
\]

\( \mu = (\mu, \dot{\mu}) \) denotes the world volume index of \( D_{p+2} \)-brane and the above action is invariant under

\[
\delta_g A_\mu = \partial_\mu \alpha + \lambda \theta \{A_\mu, \alpha\},
\]

\[
\delta_g A^\mu_\dot{\mu} = \partial^\mu_\dot{\mu} \alpha + \lambda \theta \epsilon^\beta_\mu \{X^\beta, \alpha\}.
\]

As we will see in next section this action is a low energy limit of non-commutative \( D_{p+2} \)-brane action (22) where the B-field turns on as \( B^\mu_\nu = \frac{1}{\theta^2} \epsilon^\mu_\beta \dot{\epsilon}^\nu_\beta \). Hence the arbitrary constant, \( \theta \), plays the role of non-commutative parameter and it is acceptable to recover the commutative action when \( \theta \rightarrow 0 \). Obviously, the U(1) gauge theory action for \( D_{p+2} \)-brane is recovered although there is an extra term going to infinity in the action. This term can be considered as zero point energy of multiple \( D_p \)-branes dissolved in the \( D_{p+2} \)-brane world volume.
3 DBI theory

One of the remarkable properties of D\(_p\)-brane is that the U(1) gauge symmetry is enhanced to U(N) gauge symmetry for N coincident D\(_p\)-branes. The suitable action for multiple D\(_p\)-branes was introduced in [7]. The action is

\[
S = -T_p \int d^{p+1}x
\times \text{STr} \left( \sqrt{ - \det \left( P[E_{mn} + E_{mI}(Q^{-1} - \delta^I_J E_{Jn}] + \lambda F_{\mu\nu}) \det(Q^I_J) \right) } \right),
\]

where \( E_{mn} = G_{mn} + B_{mn} \), \( Q^I_J \equiv \delta^I_J + i\lambda[X^I, X^K]E_{KJ} \). The brane tension is \( T_p = \frac{2\pi}{g_s(2\pi l_s)^{p+1}} \) (\( l_s \) and \( g_s \) are string length and coupling) and "P" denotes pull-back of background metric and NSNS two-form(\( m, n = 0, ..., 9 \)). \( F_{\mu\nu} \) is field strength of gauge field living on the D\(_p\)-brane. \( \text{STr(...)} \) denotes that one takes a symmetrized average over all ordering of various fields appearing inside the trace.

Let us again consider that two longitudinal directions of D-branes fluctuate such that \( X^\mu = \frac{1}{\sqrt{\eta}} y^\mu - \epsilon^{\mu\alpha} A_\alpha \). Working, in static gauge i.e. \( \lambda X^\mu = x^\mu \), in flat space background and noting that the \( B \) is turned off in D\(_p\)-brane theory, the determinant in the action can be decomposed as

\[
\tilde{D} = \det \begin{pmatrix}
\eta_{\mu\nu} + \lambda F_{\mu\nu} & \lambda D_\mu X^\nu & \lambda D_\nu X^\mu \\
-\lambda D_\mu X^\nu & \delta^{\mu\nu} + i\lambda X^\mu \hat{\epsilon}^\nu & i\lambda X^\mu \hat{\epsilon}^\nu \\
-\lambda D_\nu X^\mu & i\lambda X^\mu \hat{\epsilon}^\nu & Q\delta^{k\ell}
\end{pmatrix}.
\]

Using (12) and

\[
D_\mu X^\nu = F_{\mu\nu},
\]

one can rewrite the determinant as

\[
\tilde{D} = \det \begin{pmatrix}
\eta_{\mu\nu} + \lambda F_{\mu\nu} & \lambda F_{\mu\alpha} \hat{\epsilon}^{\mu\alpha} & \lambda D_\mu X^i \\
-\lambda F_{\nu\alpha} \hat{\epsilon}^{\nu\alpha} & \delta^{\mu\nu} - \frac{1}{\epsilon} \epsilon^{\mu\nu} + \lambda \epsilon^{\mu\alpha} F_{\alpha\beta} \epsilon^{\beta\nu} & -\lambda \epsilon^{\mu\alpha} D_\alpha X^i \\
-\lambda D_{\nu} X^i & \lambda \epsilon^{\mu\alpha} D_\alpha X^i & Q\delta^{k\ell}
\end{pmatrix}.
\]

Using the standard trick for recombining the determinant[7]

\[
\tilde{D} = \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det (A - BD^{-1}C) \det(D),
\]
one can rearrange the action in a compact form similar to (17). As it can be seen, in the new action a constant $B$-field is turned on in new directions $\hat{\mu}$'s. Due to the facts that the covariant derivatives $D_{\hat{\mu}}$ and field strength $F_{\hat{\mu}\hat{\nu}}$ have now a non-vanishing first order $\theta$ term and the $B$ field is present, the new action is the $\theta \to 0$ limit of a non-commutative $U(1)$ $D_{p+2}$ brane action[8]. The general form of such an action is

$$\hat{S} = -T_p \int d^{p+1}x \left( e^{-\phi(\hat{A})} \times \sqrt{-\det \left( P_{\theta} [E_{mn}(\hat{A}) + E_{mI}(\hat{A})(Q^{-1} - \delta)IJ E_{Jn}(\hat{A})] + \lambda \hat{F}_{mn} \right) \det(Q_{IJ}^I)} \right)^{*N}$$

(22)

where $Q_{IJ}^I \equiv \delta_{IJ} + i\lambda[\hat{X}^I, \hat{X}^K]_M E_{KIJ}(\hat{A})$ and "*" denotes non-commutative fields. Here the $*N$ is the multiplication rule among various non-commutative fields [9]. One can show that in the limits where the non-commutativity parameter $\theta$ goes to zero this action reduces to DBI action.

4 Non-BPS DBI action

Besides of BPS branes, which are charges of RR fields, non-BPS branes may exist in the IIA(B) theory. The existence of these branes may cause the instability and breaks the supersymmetry. From the field theory of world volume of D-brane point of view, the existence of a tachyonic field $T$ is the reason of such phenomena.

In general, the action of non-BPS $D_p$-branes can be written as the product of the action of BPS $D_p$-branes and contribution which comes from the tachyon field such as

$$S_{Non-BPS} = T_p \int d^{p+1}x \ S_{BPS} F(T),$$

(23)

where $F(T)$ is the tachyonic contribution of the action of $D_p$-brane. The form of this action can be obtained by using the non-BPS $D_9$ (or $D_8$)-brane action and then performing the T-duality transformations[10]. Its most general form up to first order derivative of $T$ may be written as
\[ F(T) = V(T) - \Sigma_{n=1}^{\infty} f_n(T)\lambda^n((E^{\mu\nu} - E^{\mu I}E^{IJ}E_{IJ})D_\mu T D_\nu T + 2iE^{\mu K}E_{K J}[X^J, T]D_\mu T - E_{IJ}[X^I, T][X^J, T])^n, \]  

and \( V(T) \) is the potential for the tachyon. Functions \( f_n(T) \) are some even functions of the tachyon field [11], and although their explicit form are not known, there are some conjectures about it [12, 13].

Using the prescription (5a) in flat space background the second term vanishes and so \( F(T) \) reads as

\[ F(T) = V(T) - \Sigma_{n=1}^{\infty} f_n(T)\lambda^n((E^{\mu\nu} - E^{\mu I}E^{IJ}E_{IJ})D_\mu T D_\nu T + (\lambda\theta)^2E_{IJ}\{X^I, T\}\{X^J, T\})^n. \]

Applying \( X^{\mu} = \frac{1}{\alpha^2}y^\mu - \epsilon^{\mu\nu\lambda}A_\nu \) and noting that

\[ \epsilon_{\mu\nu}D^{\mu\nu}T = 0, \]

one easily finds that the tachyonic part of Non-BPS multiple D\(_p\)-branes action changes to tachyonic part of a Non-BPS D\(_{p+2}\)-brane action.

## 5 Chern-Simons term

In this section we examine that the Chern-Simons term of N D\(_p\)-brane action also reproduce the Chern-Simons term of a single D\(_{p+2}\)-brane in the large \( N \) limits discussed in previous sections. As it is well known, the world volume of D\(_p\) branes in IIA(B) string theory can couple to RR fields of rank lower than the dimension of brane and also can couple to RR fields with higher rank due to Myers terms[7]. In fact, the matrix representation of fields and then the non-commutativity of such fields in non-Abelian theories allows one to couple a combination of form fields of higher rank and commutators of scalar fields with world volume of D-brane in a covariant manner.

On the other hand, the Chern-Simons action is given by [7]

\[ S_{CS} = \mu_p \int STr \left( P[e^{i\lambda x i x}(\Sigma C^{(n)} e^B)] e^{\lambda F} \right), \]

where \( C^{(n)} \) denotes \((p+1)\)-forms and \( \mu_p \) is the RR charge of the brane and \( i_x \) is the exterior derivative in \( X \) direction which acts on RR form fields. This
special form of the action is necessary for compatibility with the T-duality transformations of various IIA and IIB fields. Note that all fields are in adjoint representation of the $U(N)$ gauge group.

The general couplings of these form fields with world volume of $D_p$-branes have many terms so, for simplicity, we study $D_1$-$D_3$ transition and only consider the couplings of all form fields up to the first power of fields.

Before we proceed, we mention that due to the non-commutativity of two longitudinal directions of $D_1$-branes, one may define [14, 15] a new two form field $Q^{(2)}$ which its components are

$$Q^{(2)} \equiv -i\lambda^3 \theta [X_{\dot{\mu}}, X_{\dot{\nu}}] dX^{\dot{\mu}} \wedge dX^{\dot{\nu}}.$$  

(27)

So, one should consider the couplings of this two form field with world volume of $D_1$-branes. We will see that this form fields has important role in this story. Finally, we will do our computations in the regime where $\theta \to 0$ and also consider the constant $C_{\dot{\mu}\dot{\nu}}$ and $F_{\dot{\mu}\dot{\nu}}$ fields such that

$$C_{\dot{\mu}\dot{\nu}} = \lambda^2 \epsilon_{\dot{\mu}\dot{\nu}}, \quad F_{\dot{\mu}\dot{\nu}} = \frac{\epsilon_{\dot{\mu}\dot{\nu}}}{\lambda}.$$  

(28)

By the above assumptions we have

$$\lambda^2 \{X^{\dot{\mu}}, X^{\dot{\nu}}\} \to \epsilon^{\dot{\mu}\dot{\nu}} \left( \frac{1}{\theta} - 1 \right), \quad Q_{\dot{\mu}\dot{\nu}} \to \lambda^2 \epsilon_{\dot{\mu}\dot{\nu}},$$

$$\lambda^2 \{X^{\dot{\mu}}, X^i\} \to \lambda \epsilon^{\dot{\mu}\dot{\nu}} D_\dot{\alpha} X^i, \quad \theta \{X^i, X^j\} \to 0,$$  

(29)

After all, recalling that in $D_1$-branes we have $B = 0$ then the Chern-Simons term is equal to

$$S_{CS} = \frac{1}{\lambda} \int \text{STr} \left( \mathcal{P} [e^{iA_{\dot{\mu}\dot{\nu}}(C^{(2)}(1 + Q^{(2)}))] e^{\lambda F}] \right)$$

$$\to \frac{1}{\lambda^2} \int d^4 x \left( \frac{1}{2} C_{mn} D_{\mu} X^n D_{\nu} X^m \right) \epsilon^{\mu\nu}$$

$$+ \frac{1}{\lambda^2} \int d^4 x \left( -\frac{\lambda^2 \theta}{2\lambda^2} \{X^m, X^n\} C_{mn} \right) \lambda F_{\mu\nu} \epsilon^{\mu\nu}$$

$$+ \frac{1}{\lambda^2} \int d^4 x \left( -\frac{\lambda^2 \theta}{2\lambda^2} \{X^m, X^n\} C_{mn} \right) Q_{\dot{\mu}\dot{\nu}} D_{\mu} X^{\dot{\mu}} D_{\nu} X^{\dot{\nu}} \epsilon^{\mu\nu}$$

$$+ \frac{1}{\lambda^2} \int d^4 x \left( -\frac{\lambda^2 \theta}{2\lambda^2} \{X^{\dot{\mu}}, X^{\dot{\nu}}\} Q_{\dot{\mu}\dot{\nu}} \right) C_{mn} D_{\mu} X^n D_{\nu} X^m \epsilon^{\mu\nu}$$

$$+ \frac{1}{\lambda^2} \int d^4 x \left( -\frac{\lambda^2 \theta}{2\lambda^2} \{X^{\dot{\mu}}, X^m\} C_{[m[n Q_{\dot{\mu}]\dot{\nu}]}} \right) D_{\mu} X^n D_{\nu} X^{\dot{\nu}} \epsilon^{\mu\nu},$$  

(30)
where the last term is antisymmetric for the pairs \((m, \dot{\mu})\) and \((n, \dot{\nu})\). Note also that due to dimensions of form fields any contraction with these fields leaves a factor \(\frac{1}{\lambda}\) in the action.

Using (12) and (19) one obtains

\[
C_{mn}D_\mu X^m D_\nu X^n = \frac{1}{\lambda^2} C_{\mu\nu} - \frac{1}{\lambda} C_{i[\mu, D_\nu]X^i} + C_{ij} D_\mu X^i D_\nu X^j
- \frac{1}{\lambda} C_{\dot{\mu}\dot{\nu}} F_{\dot{\nu}\dot{\lambda}} \epsilon^{\dot{\alpha}\dot{\lambda}} - C_{\dot{\mu}} D_{[\dot{\mu}} X^i F_{i\dot{\nu]}\dot{\lambda}} \epsilon^{\dot{\alpha}\dot{\lambda}} + \mathcal{O}(F^2),
\]

and one may rewrite the integrand of the above expression term by term as

\[
+ \frac{1}{2\lambda^2} (C_{mn}D_\mu X^m D_\nu X^n) \epsilon^{\mu\nu}
- \frac{1}{2\lambda^2} \left(2\left(\frac{1}{\theta} - 1\right) + 2C_{i[\dot{\mu}, D_\nu]X^i} \epsilon^{i\dot{\alpha}\dot{\lambda}} + \mathcal{O}(\theta)\right) F_{\mu\nu} \epsilon^{\mu\nu}
- \frac{1}{2\lambda^2} \left(\mathcal{O}(F^2)\right)
- \frac{1}{2\lambda^2} \left(2\left(\frac{1}{\theta} - 1\right)C_{mn} D_\mu X^m D_\nu X^n \right) \epsilon^{\mu\nu}
- \frac{1}{2\lambda^2} \left(2\left(\frac{1}{\theta} - 1\right)(C_{\mu\dot{\nu}} + C_{i\dot{\nu}} D_\mu X^i) F_{\nu\dot{\beta}} \epsilon^{\dot{\beta}\dot{\mu}} + \mathcal{O}(F^2)\right) \epsilon^{\mu\nu}
- \frac{1}{2\lambda^2} \left((C_{\mu} + C_{\dot{\mu}} D_\mu X^j) F_{\nu\dot{\beta}} D_\nu X^i \epsilon^{i\dot{\beta}\dot{\mu}} + \mathcal{O}(F^2)\right) \epsilon^{\mu\nu}
\]

where the two last terms come from the last term of (30).

Now, for D_3-brane, the Chern-Simons term up to first power of fields is equal to

\[
S_{CS} = \frac{1}{\lambda^2} \int \text{STr} \left( P[e^{i\lambda X^X (\Sigma C^{(n)}_e B)}] e^{\lambda F} \right)
\rightarrow \frac{1}{2\lambda^2} \left( \int d^2 x \left( C_{mn} B_{pq} D_\mu X^m D_\nu X^n D_\lambda X^p D_\theta X^q \right) \epsilon^{\mu\nu\lambda\dot{\theta}} + \text{antisym part} \right)
+ \frac{1}{2\lambda^2} \left( \int d^2 x \left( C_{mn} D_\mu X^m D_\nu X^n \lambda F_{\lambda\dot{\theta}} \right) \epsilon^{\mu\nu\lambda\dot{\theta}} + \text{antisym part} \right),
\]

where by antisym part we mean that, for example, the terms of \(C_{mn} B_{pq}\) are antisymmetric under the exchange of indices of \(C\) and \(B\) fields. Noting that in D_3-brane we have found \(P(B) = \frac{\epsilon^{\mu\nu}}{\theta}\), we rewrite the \(S_{CS}\) of D_3 brane as
\[
S_{CH} = \frac{1}{2\lambda^2} \int d^2 x \left( C_{\mu\nu} - 2C_{i\mu}D_{\nu}X^i + C_{ij}D_\mu X^i d_\nu X^j \right) \epsilon_{\mu\nu\tilde{\mu}\tilde{\nu}}
+ \frac{1}{2\lambda} \int d^2 x \left( C_{mn}D_\mu X^m D_\nu X^n F_{\lambda\bar{\theta}} \right) \epsilon^{\mu\nu\lambda\bar{\theta}}. \tag{34}
\]

It is not hard to show that the above D$_3$-brane Chern-Simons terms reproduce
the D$_1$-branes Chern-Simons terms. We see the role of the two form field \(Q^{(2)}\) in this D$_1$-D$_3$ transition in which some terms in D$_3$ such as \(C_{\mu i}F_{\nu\beta}D_\mu X^i \epsilon^\beta_\nu \epsilon^{\mu\nu}\)
come from the coupling of this form field with \(C^{(2)}\) and \(F^{(2)}\) in (30).

The general form of Chern-Simons action in non-commutative theories in
the presence of non-zero constant B field may be written as[14, 15, 16]

\[
S_{CS} = \mu_p \int \frac{\text{Pf} \hat{Q}}{\text{Pf} \theta} \left( P[e^{i x^i x^j} (\Sigma C^{(n)})] \epsilon^{B+\lambda F} \right), \tag{35}
\]

where \(\hat{Q}^{mn} = \theta^{mn} - \theta^m\alpha \hat{F}_{\alpha\beta} \theta^n\beta\) and \(\theta^{mn} = (B^{-1})^{mn}\), the Pf denotes the
Pfaffian of an antisymmetric matrix and all products are understood as * product. It can be seen that in the \(\theta \rightarrow 0\) limit this action and the action
(33) coincide with each other.

6 Conclusion

In this paper we have shown that in flat space multiple D$_p$-branes can be
considered as a D$_{p+2}$-brane for large number of D$_p$-branes. In fact matrix
valued \(p + 1\) dimension fields go to \(p + 3\) dimension fields. The point is that
two transverse directions to multiple D$_p$-branes appear as two components
of gauge field living on the D$_{p+2}$-brane. One can run this prescription in
opposite way and start with a D$_{p+2}$-brane and it finally leads to multiple D$_p$-
branes. In other words, one exchanges two scalar fields with two components
of gauge fields and at the end we have a D$_{p+2}$-brane or multiple D$_{p-2}$-branes.

Although, this setup has been done in flat space the above idea can be
generalized to curved background. In well-known curved background such as
pp-wave our half BPS D$_p$-branes have spherical symmetry and we may then
expect multiple D$_p$-branes lead to a spherical D$_{p+2}$-brane.

Moreover, by using the above prescription, we expect a relation between
\(mN\) D$_p$-branes and \(m\) D$_{p+2}$-branes in large N limit. Such idea is useful to
extract new understanding of the theory of multiple non-commutative D-branes.

Understanding of three dimensional conformal field theory (CFT) was an open issue for years [17]. Symmetries of this theory are consistent with symmetries of multiple M2-branes. Recently, a groundbreaking three dimensional CFT was presented in [18] known as BLG theory. In BLG theory, there are a lot of attempts to show that a M5-brane action leads to multiple M2-branes action or vice versa. We hope that uplifting the above results teach us more about M2-M5 relation. In the other hand in comparison to non-commutative D_p-brane, it seems that one should know about ”non-commutative” M5-brane to explain the relation correctly, although the geometry of M5-brane is not known by now [19].

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