On the Kirchheim-Magnani counterexample to metric differentiability

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In Kirchheim-Magnani [7] the authors construct a left invariant distance $\rho$ on the Heisenberg group such that the identity map $id$ is 1-Lipschitz but it is not metrically differentiable anywhere.

In this short note we give an interpretation of the Kirchheim-Magnani counterexample to metric differentiability. In fact we show that they construct something which fails shortly from being a dilatation structure.

Dilatation structures have been introduced in [2]. These structures are related to conical group [3], which form a particular class of contractible groups and are a slight generalization of Carnot groups.

Carnot groups, in particular the Heisenberg group, appear as infinitesimal models of sub-riemannian manifolds [1], [6]. In [5] we explain how the formalism of dilatation structures applies to sub-riemannian geometry.

Further on we shall use the notations, definitions and results concerning dilatation structures, as found in [2], [3] or [5].

We shall construct a structure $(H(1), \rho, \bar{\delta})$ on $H(1)$ which satisfies all axioms of a dilatation structure, excepting A3 and A4. We prove that for $(H(1), \rho, \bar{\delta})$ the axiom A4 implies A3. Finally we prove that A4 for $(H(1), \rho, \bar{\delta})$ is equivalent with $id$ metrically differentiable from $(H(1), d)$ to $(H(1), \rho)$, where $d$ is a left invariant CC distance.

For other relations between dilatation structures and differentiability in metric spaces see [4].

1 Metric differentiability for conical groups

The general definition of metric differentiability for conical groups is formulated exactly as the same notion for Carnot groups.
Definition 1.1 Let \((N,d,\delta)\) be a conical group. A continuous function \(\eta : N \to [0, +\infty)\) is a seminorm if:

(a) \(\eta(\delta \varepsilon x) = \varepsilon \eta(x)\) for any \(x \in N\) and \(\varepsilon > 0\),

(b) \(\eta(xy) \leq \eta(x) + \eta(y)\) for any \(x, y \in N\).

Let \((N, \delta, d)\) be a conical group, \((X, \rho)\) a metric space, \(A \subset N\) an open set and \(x \in A\). A function \(f : A \to X\) is metrically differentiable in \(x\) if there is a seminorm \(\eta_x : N \to [0, +\infty)\) such that

\[
\frac{1}{\varepsilon} \rho(f(x \delta \varepsilon v), f(x)) - \eta_x(v) \to 0
\]
as \(\varepsilon \to 0\), uniformly with respect to \(v\) in compact neighbourhood of the neutral element \(e \in N\).

2 Kirchheim-Magnani counterexample to metric differentiability

For the elements of the Heisenberg group \(H(1) = \mathbb{R}^2 \times \mathbb{R}\) we use the notation \(\tilde{x} = (x, \bar{x})\), with \(\tilde{x} \in H(1), x \in \mathbb{R}^2, \bar{x} \in \mathbb{R}\). In this subsection we shall use the following operation on \(H(1)\):

\[
\tilde{x} \tilde{y} = (x, \bar{x})(y, \bar{y}) = (x + y, \bar{x} + \bar{y} + 2\omega(x, y)),
\]
where \(\omega\) is the canonical symplectic form on \(\mathbb{R}^2\). On \(H(1)\) we consider the left invariant distance \(d\) uniquely determined by the formula:

\[
d((0, 0), (x, \bar{x})) = \max \left\{ \|x\|, \sqrt{|\bar{x}|} \right\}.
\]

The construction by Kirchheim and Magnani is described further. Take an invertible, non decreasing function \(g : [0, +\infty) \to [0, +\infty)\), continuous at 0, such that \(g(0) = 0\).

For a function \(g\) which is well chosen, the function \(\rho : H(1) \to [0, +\infty)\),

\[
\rho(\tilde{x}) = \max \{\|x\|, g(\|\bar{x}\|)\}
\]
induces a left invariant invariant distance on \(H(1)\) (we use the same symbol)

\[
\rho(\tilde{x}, \tilde{y}) = \rho(\tilde{x}^{-1} \tilde{y}).
\]

In order to check this it is sufficient to prove that for any \(\tilde{x}, \tilde{y} \in H(1)\) we have

\[
\rho(\tilde{x} \tilde{y}) \leq \rho(\tilde{x}) + \rho(\tilde{y}),
\]
and that \(\rho(\tilde{x}) = 0\) if and only if \(\tilde{x} = (0, 0)\). The following result is theorem 2.1 [7].
Theorem 2.1 (Kirchheim-Magnani) If the function $g$ has the expression

$$g^{-1}(t) = k(t) + t^2$$

for any $t > 0$, where $k : [0, +\infty) \to [0, +\infty)$ is a convex function, strictly increasing, continuous at 0, and such that $k(0) = 0$, then the function $\rho$ induces a left invariant distance (denoted also by $\rho$). Moreover, the identity function $id$ is 1-Lipschitz from $(H(1), d)$ to $(H(1), \rho)$.

3 Interpretation in terms of dilatation structures

Further we shall work with a function $g$ satisfying the hypothesis of theorem 2.1, and with the associated function $\rho$ described in the previous subsection.

Definition 3.1 Define for any $\varepsilon > 0$, the function

$$\bar{\delta}_\varepsilon(x, \bar{x}) = (\varepsilon x, \text{sgn}(\bar{x})g^{-1}(\varepsilon g(|\bar{x}|)))$$

for any $\tilde{x} = (x, \bar{x}) \in H(1)$.

We define the following field of dilatations $\bar{\delta}$ by: for any $\varepsilon > 0$ and $\tilde{x}, \tilde{y} \in H(1)$ let

$$\bar{\delta}^\varepsilon \tilde{y} = \tilde{x} \bar{\delta}(\tilde{x}^{-1} \tilde{y})$$

For any $\varepsilon > 0$ and $\tilde{x}, \tilde{y} \in H(1)$ we define

$$\bar{\beta}_\varepsilon(\tilde{x}, \tilde{y}) = \bar{\delta}_\varepsilon^{-1}(\bar{\delta}_\varepsilon(\tilde{x})\bar{\delta}_\varepsilon(\tilde{y}))$$

We want to know when $(H(1), \rho, \bar{\delta})$ is a dilatation structure.

Proposition 3.2 The structure $(H(1), \rho, \bar{\delta})$ satisfies the axioms A0, A1, A2. Moreover, A4 implies A3.

Proof. It is easy to check that for any $\varepsilon, \mu \in (0, +\infty)$ we have

$$\bar{\delta}_\varepsilon \bar{\delta}_\mu = \bar{\delta}_{\varepsilon \mu}$$

and that $id = \delta_1$.

Moreover, from $g$ non decreasing and continuous at 0 we deduce that

$$\lim_{\varepsilon \to 0} \bar{\delta}_\varepsilon \tilde{x} = (0, 0),$$

uniformly with respect to $\tilde{x}$ in compact sets.

Another computation shows that

$$\rho(\bar{\delta}_\varepsilon \tilde{x}) = \varepsilon \rho(\tilde{x})$$
for any \( \tilde{x} \in H(1) \) and \( \varepsilon > 0 \). Otherwise stated, the function \( \rho \) is homogeneous with respect to \( \tilde{\delta} \).

All that is left to prove is that A4 implies A3. Remark that \( \tilde{\delta} \) is left invariant (in the sense of transport by left translations in \( H(1) \)) and the distance \( \rho \) is also left invariant. Then axiom A4 takes the form: there exists the limit

\[
\lim_{\varepsilon \to 0} \beta_{\varepsilon}(\tilde{x}, \tilde{y}) = \beta(\tilde{x}, \tilde{y}) \in H(1)
\]  

(3.0.1)
uniform with respect to \( \tilde{x}, \tilde{y} \in K, K \) compact set.

From the homogeneity of the function \( \rho \) with respect to \( \tilde{\delta} \) we deduce that for any \( \tilde{x}, \tilde{y} \in H(1) \) we have:

\[
\frac{1}{\varepsilon} \rho(\tilde{\delta}(\tilde{x}), \tilde{\delta}(\tilde{y})) = \rho(\beta_{\varepsilon}(\tilde{x}^{-1}, \tilde{y})).
\]

From the left invariance of \( \tilde{\delta} \) and \( \rho \) it follows that A4 implies A3. □

**Theorem 3.3** If the triple \((H(1), \rho, \tilde{\delta})\) is a dilatation structure then \( id \) is metrically differentiable from \((H(1), d)\) to \((H(1), \rho)\).

**Proof.** We know that the triple \((H(1), \rho, \tilde{\delta})\) is a dilatation structure if and only if (3.0.1) is true. Taking (3.0.1) as hypothesis we deduce that the identity function is derivable from \((H(1), d, \delta)\) to \((H(1), \rho, \tilde{\delta})\). Indeed, computation shows that \( id \) derivable is equivalent to the existence of the limit

\[
\lim_{\varepsilon \to 0} \tilde{\delta}^{-1}_{\varepsilon} \tilde{\delta}_{\varepsilon} \tilde{u} = (u, \text{sgn}(\tilde{u}))g^{-1}(\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} g(\varepsilon^2 | \tilde{u} |))
\]
uniform with respect to \( \tilde{u} \) in compact set. Therefore the function \( id \) is derivable everywhere if and only if the uniform limit, with respect to \( \tilde{u} \) in compact set:

\[
A(\tilde{u}) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} g(\varepsilon^2 | \tilde{u} |)
\]  

(3.0.2)
exists. We want to show that (3.0.1) implies the existence of this limit.

For this we shall use an equivalent (isomorphic) description of \((H(1), \rho, \tilde{\delta})\). Consider the function \( F : H(1) \to H(1) \), defined by

\[
F(x, \tilde{x}) = (x, \text{sgn}(\tilde{x})g(| \tilde{x} |)).
\]
The function \( F \) is invertible because \( g \) is invertible. For any \( \varepsilon > 0 \) let \( \tilde{\delta}_{\varepsilon} \) be the usual dilatations:

\[
\tilde{\delta}_{\varepsilon}(x, \tilde{x}) = (\varepsilon x, \varepsilon \tilde{x}).
\]
It is then straightforward that

\[
\tilde{\delta}_{\varepsilon} = F^{-1} \tilde{\delta}_{\varepsilon} F,
\]
for any \( \varepsilon > 0 \).
The function $F$ can be made into a group isomorphism by re-defining the group operation on $H(1)$

$$
\tilde{x} \cdot \tilde{y} = F((x, h(\tilde{x}))(y, h(\tilde{y})),
$$
where $h$ is the function

$$
h(t) = \text{sgn}(t)(t^2 + k(|t|)).
$$

Let $\mu$ be the transported left invariant distance on $H(1)$, defined by

$$
\mu(F(\tilde{x}), F(\tilde{y})) = \rho(\tilde{x}, \tilde{y}).
$$

Remark that $\mu$ has the simple expression

$$
\mu((0, 0), (x, \bar{x})) = \max\{|x|, |ar{x}|\}.
$$

Exactly as before we can construct the structure $\hat{\delta}$ by

$$
\hat{\delta}_\varepsilon \tilde{x} \cdot \hat{\delta}_\varepsilon (\tilde{x}^{-1} \cdot \tilde{y}).
$$

We get a dilatation structure $(H(1), \mu, \hat{\delta})$ isomorphic with $(H(1), \rho, \bar{\delta})$.

The identity function $id$ is derivable from $(H(1), d, \delta)$ to $(H(1), \rho, \bar{\delta})$ if and only if the function $F$ is derivable from $(H(1), d, \delta)$ to $(H(1), \mu, \hat{\delta})$.

The axiom A4 for the dilatation structure $(H(1), \mu, \hat{\delta})$ implies that for any $\tilde{x}, \tilde{y} \in H(1)$ the limit exists

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mu(\frac{1}{2} \omega(x, y) + |\tilde{x}||\bar{\varepsilon}| + |\tilde{x}|^2 + sgn(\tilde{x})k(\varepsilon |\bar{\varepsilon}|)) + sgn(\tilde{y})k(\varepsilon |\bar{\varepsilon}|) + \sgn(\tilde{y})k(|\tilde{y}|),
$$

uniform with respect to $\tilde{y}$ in compact set. Take in the previous limit $\tilde{x} = \tilde{y} = 0$ and denote $\bar{u} = \frac{1}{2} \omega(x, y)$. We get (3.0.2), therefore we proved that $id$ is derivable from $(H(1), d, \delta)$ to $(H(1), \rho, \bar{\delta})$.

Finally, the derivability of $id$ implies the metric differentiability. Indeed, we use (3.0.2) to compute $\nu$, the metric differential of $id$. We obtain that

$$
\nu_\varepsilon = \mu((x, A(\tilde{x}))) = \max\{|x|, |A(\tilde{u})|\}.
$$

The proof is done. □

In the counterexample of Kirchheim and Magnani the identity function $id$ is not metric differentiable, therefore the corresponding triple $(H(1), \rho, \bar{\delta})$ is not a dilatation structure.

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