Berry phase for a spin 1/2 in a classical fluctuating field

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The effect of fluctuations in the classical control parameters on the Berry phase of a spin 1/2 interacting with an adiabatically cyclically varying magnetic field is analyzed. It is explicitly shown that in the adiabatic limit dephasing is due to fluctuations of the dynamical phase.

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Berry phase \cite{1} and related geometrical phases \cite{2,3,4,5,6,7,8,9,10,11,12,13,14} have received renewed interest in recent years due to several proposals for their use in the implementation of quantum computing gates\cite{15,16,17}. Such interest is motivated by the belief that geometric quantum gates should exhibit an intrinsic fault tolerance in their nature, i.e. proportional to the area spanned around the solid angle subtended by $\bar{\lambda}$ as a consequence the Berry phase depends only on $\vartheta, \phi$. A standard example is a slow precession of $A$ at an angle $\vartheta$ around the $z$ axis with angular velocity $\omega = 2\pi/T \ll B_0$. A straightforward calculation shows that

$$\gamma_{\psi} = \gamma_{\vartheta} = \oint_0^T A_{\vartheta}^\psi d\vartheta = \pi \cos \vartheta$$

where $\gamma_{\psi}$ and $\gamma_{\vartheta}$ are the Berry phases for a spin 1/2 in a classical fluctuating field.

The Hamiltonian, in appropriate units, takes the form

$$H(t) = \frac{1}{2} \mathbf{B}(t) \cdot \vec{\sigma}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, $\sigma_i$ are the Pauli operators and $\mathbf{B}(t) = B_0(t)\hat{n}(t)$ with the unit vector $\hat{n} = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$. The classical field $\mathbf{B}(t)$ acts as an external control parameter, as its direction and magnitude can be experimentally changed. When varied adiabatically the instantaneous energy eigenstates follow the direction of $\hat{n}$ and therefore can be expressed as

$$| \uparrow_n \rangle = e^{-i\varphi/2} \cos \frac{\vartheta}{2} | \uparrow \rangle + e^{i\varphi/2} \sin \frac{\vartheta}{2} | \downarrow \rangle$$

$$| \downarrow_n \rangle = e^{-i\varphi/2} \sin \frac{\vartheta}{2} | \uparrow \rangle - e^{i\varphi/2} \cos \frac{\vartheta}{2} | \downarrow \rangle$$

where $| \uparrow \rangle, | \downarrow \rangle$ are the eigenstates of the $\sigma_z$ operator.

When the time evolution is cyclic i.e. when after a time $T$ we have $\mathbf{B}(T) = \mathbf{B}(0)$ the energy eigenstates acquire a phase factor which contains a geometric correction to the dynamic phase:

$$| \uparrow_n(T) \rangle = e^{i\delta} e^{i\gamma_B} | \uparrow_n(0) \rangle$$

where the dynamic phase $\delta = \int_0^T B_0(t) dt$ and the Berry phase can be expressed in term of the so called Berry connection $A_{\vartheta}^\psi = i \langle \uparrow_n | \partial / \partial \varphi | \uparrow_n \rangle = \frac{1}{2} \cos \vartheta$ (4)

$$A_{\vartheta}^\phi = i \langle \uparrow_n | \partial / \partial \vartheta | \uparrow_n \rangle = 0$$

It is important to note that while the eigenenergies depend on $B_0(t)$ the eigenstates depend only on $\hat{n}(t)$. As a consequence the Berry phase depends only on $\vartheta, \varphi$. A standard example is a slow precession of $\mathbf{B}$ at an angle $\vartheta$ around the $z$ axis with angular velocity $\omega = 2\pi/T \ll B_0$. A straightforward calculation shows that

Note that the the Berry phase is proportional to the solid angle subtended by $\mathbf{B}$ with respect to the degeneracy $\mathbf{B} = 0$.

We are now in the position to extend our analysis to the case in which the magnetic field contains a fluctuating component. In this case Hamiltonian (1) is modified as follows

$$H(t) = \frac{1}{2} \mathbf{B}_F \cdot \vec{\sigma} = \frac{1}{2} (\mathbf{B}(t) + \mathbf{K}(t)) \cdot \vec{\sigma}$$

where we have divided the total magnetic field $\mathbf{B}_F$ into an average component $\mathbf{B}$ experimentally under our control and a fluctuating field $\mathbf{K}$. We will analyze the case in which $\mathbf{B}$ is a field of constant amplitude which undergoes a cyclic evolution while the components of $\mathbf{K}$ are random processes with zero average and small amplitude.
comparing to $\mathbf{B}$, in order to consider lowest order corrections. Finally we will assume that the fluctuations are characterized by timescales such that the adiabatic approximation holds. We will show that this is not an unphysical restriction.

We can now express the Berry phase in the presence of noise as

$$\gamma_B = \int_0^T (A_\varphi(\vartheta_0) + \delta A_\varphi)(\varphi_0 + \delta \varphi)dt$$

$$\cong \gamma_B^0 + \frac{2\pi}{T} \int_0^T \delta A_\varphi dt + A_\varphi(\vartheta_0) \int_0^T \delta \varphi dt$$

$$= \gamma_B^0 - \frac{\pi}{T} \int_0^T \sin \vartheta_0 \delta \varphi dt + A_\varphi(\vartheta_0) \delta \varphi(T)$$

where the average Berry phase $\gamma_B^0$ coincides with $\gamma_B$ in the absence of noise and it has been assumed $\delta \varphi(0) = 0$. The last term in (11) is a non-cyclic contribution which appears when, due to the presence of $K$, $B_T$ does not return to its original. In this case, instead of the geometrical phase definition given by Berry, which assumes that the Hamiltonian is periodic, we have to use the definition by Samuel and Bhandari [18] about non cyclic evolution. If this is done the third term does not appear and eq.(11) becomes:

$$\gamma_B = \gamma_B^0 - \frac{\pi}{T} \int_0^T \sin \vartheta_0 \delta \varphi dt$$

In order to proceed a physical model for the noise is needed, in other words a stochastic process for $K$ must be assigned. Given the probability distribution for the field it is straightforward to calculate the distribution for the Berry phase.

As a first step we express the trigonometric functions appearing in (12) in terms of the fluctuating field components $K_i$. This will be useful in calculating the probability distribution for $\gamma_B$. Let $\vartheta_0$, $\varphi_0$ be the polar angles of $\mathbf{B}$; $\vartheta$, $\varphi$ those of $B_T$; and $\delta \vartheta$, $\delta \varphi$ the first order differences between the polar angles of the two fields. Moreover let $B = |\mathbf{B}|$ be the modulus of $\mathbf{B}$. If we expand in Taylor series cos $\vartheta$ we obtain:

$$\cos(\vartheta_0 + \delta \vartheta) \cong \cos \vartheta_0 - \delta \vartheta \sin \vartheta_0 = \frac{B_3}{B} + \frac{K_3}{B^3} \mathbf{B} \cdot \mathbf{K}$$

and therefore

$$-\delta \vartheta \sin \vartheta_0 = \frac{K_3}{B} - \frac{B_3}{B^3} \mathbf{B} \cdot \mathbf{K}$$

Substituting equation (13) in (12) we find:

$$\gamma_B = \gamma_B^0 + \frac{\pi}{T} \int_0^T \left[ \frac{K_3}{B} - \frac{B_3}{B^3} \mathbf{B} \cdot \mathbf{K} \right] dt$$

From this expression it is possible to find the probability distribution for $\gamma_B$, once that for $K_i$ is known.

We will assume that the fluctuating field $K_i$ is a Ornstein-Uhlenbeck process, i.e it is gaussian, stationary and markovian with a lorentzial spectrum whose bandwidth $\Gamma_i$ we assume to be much less than the Bohr frequency of our energy eigenstates. However while we must have $\omega \ll B$ and $\Gamma_i \ll B$ we have no restriction on the
relative value of $\omega$ vs $\Gamma_i$. In order to allow for the possibility of anisotropic noise we assume that $K_3$ has variance $\sigma_3$ and bandwidth $\Gamma_3$ while $K_1$ and $K_2$ have $\sigma_{12}$ and $\Gamma_{12}$.

$$
\frac{\sigma^2}{\gamma} = 2\sigma^2_{12} \left( \frac{\pi \cos \theta_0 \sin \theta_0}{TB} \right)^2 \left[ \frac{(e^{-i\Gamma T} - 1)(\Gamma^2 - \omega^2)}{(\Gamma^2 + \omega^2)^2} \right] + \frac{\Gamma T}{\Gamma^2 + \omega^2} \left[ \frac{\Gamma T - 1 + e^{-i\Gamma T}}{\Gamma^2} \right] (15)
$$

This expression has an interesting limiting value when $\Gamma_i \ll \omega$ and $\Gamma_i \gg \omega$. To first order in $\Gamma T$:

$$
\frac{\sigma^2}{\gamma} = 4\sigma^2_{12} \left( \frac{\pi \cos \theta_0 \sin \theta_0}{B} \right)^2 \frac{\Gamma_{12} T}{(2\pi)^2} + 2\sigma^2_{12} \left( \frac{\pi \sin^2 \theta_0}{B} \right)^2 \left[ \frac{1}{12} \frac{\Gamma_{12} T}{6} \right] (16)
$$

We see that the leading term is $\sigma^2_{12} \left( \frac{\pi \sin^2 \theta_0}{B} \right)^2$ which tends to zero for little $\theta_0$.

When $T \gg \Gamma^{-1}$ the fluctuating field has time enough to make many uncorrelated oscillations during the cyclic evolution. In this case the effect of the fluctuations averages out and in the limit $(\Gamma T)^{-1} \to 0$ do not give contribution to the variance which tends to zero:

$$
\frac{\sigma^2}{\gamma} = 2\sigma^2_{12} \left( \frac{\pi \cos \theta_0 \sin \theta_0}{B} \right)^2 \frac{1}{\Gamma_{12} T} + 2\sigma^2_{12} \left( \frac{\pi \sin^2 \theta_0}{B} \right)^2 \left[ \frac{1}{\Gamma_{12} T} \right] (17)
$$

This is to be compared to the dynamical phase which grows linearly in $T$. This different behavior is due to the fact that while Berry phase corrections are proportional to $1/T \int K dt$, corrections to the dynamical phase are proportional to $\int K dt$. For a OU process the variance of the integral grows linearly for times long compared to the autocorrelation time of the field. This is analogous to the variance of the position of a brownian particle.

Until now we concentrated only in the geometrical phase. However during an adiabatic cyclic evolution the eigenstates acquire both the dynamical and geometrical phase. It is known that the dynamical phase $\delta$ is proportional to the modulus of the magnetic field. This means that the dynamical phase becomes a stochastic processes like the Berry phase. We can write $\delta$ in terms of the fields $\vec{B}$ and $\vec{K}$ as we did for the geometrical phase:

$$
\delta = \delta_0 + \int_0^T \frac{\vec{B} \cdot \vec{K}}{B} dt (18)
$$

where $\delta_0 = BT$. Note that expression $[13]$ is similar to $[14]$ for Berry phase. The difference is that while $\gamma_B$ comes from an integral in parameter space, $\delta$ from an integral in the time domain. For instance this means that if we double time $T$, $\gamma_B$ scales with $T^{-1}$ while the domain of integration of $\delta$ doubles. As we will see this is crucial for the different role of the two phases in dephasing.

Following the same steps as for the Berry phase it is possible to demonstrate that $\delta$ is a stochastic processes with a gaussian distribution. Now we analyze the effect of noise on the coherence of a system, in other words dephasing. Suppose we prepare the system in a state which is a superposition of the two eigenstate of the Hamiltonian:

$$
|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle (19)
$$

After a slowly cyclic evolution the eigenstates have acquired both the dynamical and geometrical phases and the final state is:

$$
|\psi\rangle = ae^{i\alpha}|\uparrow\rangle + be^{-i\alpha}|\downarrow\rangle (20)
$$

where $\alpha = \gamma_B + \delta$ is the total phase. In the presence of noise this phase is a random variable with a gaussian distribution $P(\alpha)$ then actually the system at the end of the evolution is in a mixed state.

It can be described by the density operator which is given by the expression:

$$
\rho = \int |\psi\rangle \langle \psi| P(\alpha) d\alpha (21)
$$

We want to stress that $P(\alpha) \neq P(\gamma_B) P(\delta)$, i.e. dynamical and geometrical phases are not independent processes because both depend on $K$.

If we insert eq. $[20]$ in the definition of $\rho$ we find that the population are unchanged while the coherence are shrunk by a factor $\exp(-2\sigma_3^2)$. In terms of the Bloch vector, this means that the $z$ component is unchanged while the component parallel to the $xy$ is reduced. This is what is called dephasing because the relative phase in a superposition is undefined.

In order to do not lose in generality and to compare the dynamical and geometrical phase we have studied the two case together. We have found the probability distribution $P(\alpha)$ for $\alpha$ which has mean value $\langle \alpha \rangle = \langle \gamma_B \rangle + \langle \delta \rangle$ and variance:

$$
\sigma^2_\alpha = 2\sigma_3^2 \left( \frac{\pi \cos \theta_0 \sin \theta_0}{T} + B \sin \theta_0 \right)^2 \times
$$

$$
\left[ \frac{(e^{-\Gamma_{13} T} - 1)(\Gamma_{13}^2 - \omega^2)}{(\Gamma_{13}^2 + \omega^2)^2} \right] + \frac{\Gamma_{12} T}{\Gamma_{12}^2 + \omega^2} + 2\sigma_3^2 \left( \frac{\pi \sin^2 \theta_0}{T} + B \cos \theta_0 \right)^2 \left[ \frac{\Gamma_{3} T - 1 + e^{-\Gamma_{3} T}}{\Gamma_{3}^2} \right] (22)
$$
In eq. (22) $\sigma^2_\alpha$ is the sum of two terms one coming from fluctuation in z direction and one in the xy plane. Each of these terms contains a factor in round brackets in which we recognize a geometrical term proportional to $1/T$, as we found in (15) and a dynamical one proportional to the Bohr frequency. Now because of the adiabaticity condition we have that the first is much less than the second. As a consequence the main contribution to dephasing has dynamical rather than geometrical origin. This does not mean of course that in a system in which Berry phase emerges dephasing is less than in a system in which it does not. What we have demonstrated is that fluctuations in Berry phase do not contribute considerably to dephasing.

Another relevant aspect to stress is that our calculation was performed under the assumptions of a constant circular precession, however our results are independent of the specific path executed by the magnetic field as long as the adiabatic approximation is valid.

It is worth noting that dephasing is not the only decoherence source in our system. The Bloch vector in fact does not return to its initial position since the magnetic field does not. To calculate the correct Bloch vector we have to average the final positions. However the contributions from this effect are proportional to $(K_i/B)^2$ and so we can neglect this effect at our level of approximation.

In this paper we have calculated the distribution of Berry phase in the presence of classical noise. Assuming a OU process for noise we found that under the assumption of small fluctuation Berry phase is a gaussian variable. We have calculated its mean value and the variance and we found that the variance diminishes as $1/T$. This is to be compared to the variance of the dynamical phase which grows linearly with $T$. This shows that adiabaticity and the geometrical aspects of Berry phase reduce fluctuations. This is what was expected but never demonstrated. Another aspect that makes Berry phase more robust than dynamical phase is that of dephasing. We showed that geometrical dephasing is much less than the dynamical one. This is due mainly to adiabaticity. This means that probably quantum gates based upon geometrical phase are more resistant. We would like to point out that after we submitted our manuscript a paper has appeared [19] in which the noise is treated fully quantum mechanically and similar conclusions have been drawn (but in a completely different setting) about the invariant of the geometric phase. These two results together complement each other.

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