Learning-based Control of Unknown Linear Systems with Thompson Sampling

Yi Ouyang, Mukul Gagrani and Rahul Jain

Abstract—We propose a Thompson sampling-based learning algorithm for the Linear Quadratic (LQ) control problem with unknown system parameters. The algorithm is called Thompson sampling with dynamic episodes (TSDE) where two stopping criteria determine the lengths of the dynamic episodes in Thompson sampling. The first stopping criterion controls the growth rate of episode length. The second stopping criterion is triggered when the determinant of the sample covariance matrix is less than half of the previous value. We show under some conditions on the prior distribution that the expected (Bayesian) regret of TSDE accumulated up to time $T$ is bounded by $O(\sqrt{T})$. Here $O(\cdot)$ hides constants and logarithmic factors. This is the first $O(\sqrt{T})$ bound on expected regret of learning in LQ control. By introducing a re-initialization schedule, we also show that the algorithm is robust to time-varying drift in model parameters. Numerical simulations are provided to illustrate the performance of TSDE.

I. INTRODUCTION

That the model and its parameters are known precisely is a pervasive assumption. And yet, for real-world systems, this is hardly the case. Typically, a set is known in which the model parameters lie. Furthermore, for many problems, we do not have the luxury of first performing system identification, and then using that in designing controllers. Learning model parameters and the corresponding optimal controller must be performed simultaneously at the fastest possible non-asymptotic rate. Classical adaptive control [1–3] mostly provides asymptotic guarantees for non-stochastic systems. Results on Stochastic Adaptive Control are rather sparse. But recent advances in Online Learning [4] opens the possibility of using Online Learning-based methods for finding the optimal controllers to unknown stochastic systems.

In this paper, we consider a linear stochastic system with quadratic cost (an LQ system) with unknown parameters. If the true parameters are known, then the problem is the classic stochastic LQ control where optimal control is a linear function of the state. In the learning problem, however, the true system dynamics are unknown. This problem is also known as the adaptive control problem [5, 6].

The early works in the adaptive control literature made use of the certainty equivalence principle. The idea is to estimate the parameters from collected data and apply the optimal control by taking the estimates to be the true parameters. It was shown that the certainty equivalence principle may lead to the convergence of the estimated parameters to incorrect values [7] and thus results in suboptimal performance. This issue arises fundamentally from the lack of exploration. The controller must explore the environment to learn the system dynamics but at the same time it also needs to exploit the information available to minimize the accumulated cost. This leads to the well known exploitation-exploration trade-off in learning problems.

One approach to actively explore the environment is to add perturbations to the controls (see, for examples, [8], [9]). However, the persistence of perturbations lead to sub-optimal performance except in the asymptotic region. To overcome this issue, Campi and Kumar [10] proposed a cost-biased maximum likelihood algorithm and proved its asymptotic optimality. More recent works [11, 12] show a connection between the cost-biased maximum likelihood and the optimism in the face of uncertainty (OFU) principle [13] in online learning. The OFU principle handles the exploitation-exploration trade-off by making use of optimistic parameters. Based on the OFU principle, [11, 12] design algorithms that achieve $O(\sqrt{T})$ bounds on regret accumulated up to time $T$ with high probability. Here $O(\cdot)$ hides the constants and logarithmic factors. This regret scaling is believed to be optimal except for logarithmic factors because the similar linear bandit problem possesses a lower bound of $O(\sqrt{T})$ [14].

One drawback of the OFU-based algorithms is their computational requirements. Each step of an OFU-based algorithm requires optimistic parameters as the solution of an optimization problem. Solving the optimization is computationally expensive. In recent years, Thompson sampling (TS) has become a popular alternative to OFU due to its computational simplicity (see [15] for a recent tutorial). It has been successfully applied to multi-armed bandit problems [16–20] as well as to Markov Decision Processes (MDPs) [21, 22]. The idea dates back to 1933 due to Thompson [24]. TS-based algorithms generally proceed in episodes. At the beginning of each episode, parameters are randomly sampled from the posterior distribution maintained by the algorithm. Optimal control is applied according to the sampled parameters until the next episode begins. Without solving any optimization problem, TS-based algorithms are computationally more efficient than OFU-based algorithms.

The idea of TS has not been applied to learning in LQ control until very recently [23, 25]. One key challenge to adapt TS to LQ control is to appropriately design the length of the episodes. Abbasi-Yadkori and Szepesvri [22] designed a dynamic episode schedule for TS-based on their OFU-based
algorithm \cite{11}. Their TS-based algorithm was claimed to have a $\tilde{O}(\sqrt{T})$ growth, but a mistake in the proof of their regret bound was pointed out by \cite{26}. A modified dynamic episode schedule was proposed in \cite{25}, but it suffers a $\tilde{O}(T^{\frac{3}{4}})$ regret that is worse than the target $O(\sqrt{T})$ scaling. A related recent paper is \cite{27} which proposes a TS-based learning algorithm for finite state and finite action space stochastic control problems that is asymptotically optimal. Our focus is on non-asymptotic performance of learning-based control algorithms for stochastic linear systems that of course have both uncountable state and action spaces which is much more challenging.

In this paper, we consider the LQ control problem under two scenarios: with stationary parameters and with time-varying parameters. In the case of stationary parameters, we propose a Thompson sampling with dynamic episodes (TSDE) learning algorithm. In TSDE, there are two stopping criteria for an episode to end. The first stopping criterion controls the growth rate of episode length. The second stopping criterion is the doubling trick similar to the ones in \cite{11}, \cite{22}, \cite{25} that stops when the determinant of the sample covariance matrix becomes less than half of the previous value. Instead of a rate of episode length. The second stopping criterion is the TSDE-TV achieves sub-linear regret in some condition on the prior distribution, we show that the desired performance guarantee as the system cost may go some condition on the expected number of parameter jumps, we prove that TSDE-TV achieves sub-linear regret in $T$ which implies its asymptotical optimality under the average cost criterion. The performance of TSDE and TSDE-TV is also verified through numerical simulations.

II. PROBLEM FORMULATION

A. Preliminaries: Stochastic Linear Quadratic Control

Consider a linear system controlled by a controller. The system dynamics are given by

$$x_{t+1} = Ax_t + Bu_t + w_t,$$

where $x_t \in \mathbb{R}^n$ is the state of the system plant, $u_t \in \mathbb{R}^m$ is the control action by the controller, and $w_t$ is the system noise which has the standard Gaussian distribution $N(0, I)$. $A$ and $B$ are system matrices with proper dimensions. The initial state $x_1$ is assumed to be zero.

The control action $u_t = \pi_t(h_t)$ at time $t$ is a function $\pi_t$ of the history of observations $h_t = (x_{1:t}, u_{1:t-1})$ including states $x_{1:t} := (x_1, \ldots, x_t)$ and controls $u_{1:t-1} := (u_1, \ldots, u_{t-1})$. We call $\pi = (\pi_1, \pi_2, \ldots)$ a (adaptive) control policy. The control policy allows the possibility of randomization over control actions.

The cost incurred at time $t$ is a quadratic instantaneous function

$$c_t = x_t^T Q x_t + u_t^T R u_t$$

where $Q$ and $R$ are positive definite matrices.

Let $\theta^T = [A, B]$ be the system parameter including both the system matrices. Then $\theta \in \mathbb{R}^{d \times n}$ where $d = n + m$ with compact support $\Omega_\theta$. When $\theta$ is perfectly known to the controller, minimizing the infinite horizon average cost per stage is a standard stochastic Linear Quadratic (LQ) control problem. Let $J(\theta)$ be the optimal per stage cost under $\theta$. That is,

$$J(\theta) = \min_{\pi} \limsup_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[c_t | \theta]$$

It is well-known that the optimal cost is given by

$$J(\theta) = \text{tr}(S(\theta))$$

if the following Riccati equation has a unique positive definite solution $S(\theta)$.

$$S(\theta) = Q + A^T S(\theta) A$$

$$- A^T S(\theta) B (R + B^T S(\theta) B)^{-1} B^T S(\theta) A.$$  (5)

Furthermore, for any $\theta$ and any $x$, the optimal cost function $J(\theta)$ satisfies the Bellman equation

$$J(\theta) + x^T S(\theta) x = \min_u \left\{ x^T Q x + u^T R u + \mathbb{E} \left[ x_{t+1}^T(u) S(\theta) x_{t+1}(u) | x, \theta \right] \right\}$$

(6)

where $x_{t+1}(u) = \theta^T [x^T, u^T] + w_t$, and the optimal control that minimizes (6) is equal to

$$u = G(\theta) x$$

with the gain matrix $G(\theta)$ given by

$$G(\theta) = -(R + B^T S(\theta) B)^{-1} B^T S(\theta) A.$$  (8)

The problem we are interested in is the case when the system matrices $A, B$ are unknown. When $\theta^T = [A, B]$ is unknown, the problem becomes a reinforcement learning problem where the controller needs to learn the system parameter while minimizing the cost.

We first consider a learning problem with stationary parameters, and then with time-varying parameters.

B. Reinforcement Learning with Stationary Parameter

Consider the linear system

$$x_{t+1} = A_1 x_t + B_1 u_t + w_t,$$

where $A_1$ and $B_1$ are fixed but unknown system matrices. Let $\theta^T_1 = [A_1, B_1]$ be the model parameter. We adopt a Bayesian setup and assume that there is a prior distribution $\mu_1$ for $\theta_1$.

Since the actual parameter $\theta_1$ is unknown, we define the expected regret of a policy $\pi$ compared with the optimal cost
\[ J(\theta_1) \] to be
\[
R(T, \pi) = \mathbb{E} \left[ \sum_{t=1}^{T} \left( c_t - J(\theta_t) \right) \right].
\]

The above expectation is with respect to the randomness for \( W_t \), the prior distribution \( \mu_1 \) for \( \theta_1 \), and the randomized algorithm. The learning objective is to find a control algorithm that minimizes the expected regret.

### C. Reinforcement Learning with Time-Varying Parameter

Consider the time-varying system
\[
x_{t+1} = A_t x_t + B_t u_t + w_t,
\]
with system matrices \( A_t \) and \( B_t \). The model parameter \( \theta_t^T = [A_t, B_t] \) is time-varying and unknown to the controller.

We assume that the parameter \( (\theta_t, t = 1, 2, \ldots) \) is a jump process. When it jumps, the new parameter is generated from the prior distribution \( \mu_1 \). We use \( j_t \in \{0, 1\}, t = 1, 2, \ldots \) to indicate the jumps. Then \( \theta_t = \theta_{t-1} \) if \( j_t = 0 \), and \( \theta_t \) is generated (independently of the past) from \( \mu_1 \) if \( j_t = 1 \). The jump process \( (j_t, t = 1, 2, \ldots) \) is assumed to be independent of the system noise.

Since \( J(\theta_t) \) is the optimal cost under \( \theta_t \), we define the expected regret of a policy \( \pi \) to be
\[
R_{TV}(T, \pi) = \mathbb{E} \left[ \sum_{t=1}^{T} \left( c_t - J(\theta_t) \right) \right].
\]

The above expectation is with respect to the randomness for \( W_t \), the distribution for the jump process \( (\theta_t, t = 1, 2, \ldots) \), and the randomized algorithm. The learning objective is to find a control algorithm that minimizes the expected regret.

### III. THOMPSON SAMPLING BASED CONTROL POLICIES

In this section, we develop Thompson Sampling (TS)-based control policies for the problems with stationary and time-varying parameters.

#### A. Thompson Sampling for Stationary Parameter

For the reinforcement learning problem with stationary parameters, we make the following assumption on the prior distribution \( \mu_1 \).

**Assumption 1.** The prior distribution \( \mu_1 \) consists of independent Gaussian distributions projected on a compact support \( \Omega_1 \subset \mathbb{R}^{d \times n} \) such that for any \( \theta \in \Omega_1 \), the Riccati equation \([5]\) with \([A, B] = \theta^T \) has a unique positive definite solution. Specifically, there exist \( \hat{\theta}_i(1) \in \mathbb{R}^d \) for \( i = 1, \ldots, n \) and a positive definite matrix \( \Sigma_1 \in \mathbb{R}^{d \times d} \) such that for any \( \theta \in \mathbb{R}^{d \times n} \)
\[
\mu_1 = \tilde{\mu}_1 |_{\Omega_1}, \quad \tilde{\mu}_1(\theta) = \prod_{i=1}^{n} \tilde{\mu}_1(\theta(i))
\]
\[
\tilde{\mu}_1(\theta(i)) \equiv N(\hat{\theta}_i(i), \Sigma_1) \quad \text{for} \quad i = 1, \ldots, n.
\]

Here \( \theta(i) \) denotes \( \theta \)'s \( i \)th column \( (\theta = [\theta(1), \ldots, \theta(n)]) \).

Note that under the prior distribution, the mean \( \hat{\theta}_i(i) \) for each column of \( \theta_1 \) may be different, but they have the same covariance matrix \( \Sigma_1 \).

At each time \( t \), given the history of observations \( h_t = (x_{1:t}, u_{1:t-1}) \), we define \( \mu_t \) to be the posterior belief of \( \theta_1 \) given by
\[
\mu_t(\Theta) = P(\theta_1 \in \Theta | h_t).
\]

The posterior belief can be computed according to the following lemma.

**Lemma 1.** The posterior belief \( \mu_t \) on the parameter \( \theta_1 \) satisfies
\[
\mu_t = \tilde{\mu}_t|_{\Omega_1}, \quad \tilde{\mu}_t(\theta) = \prod_{i=1}^{n} \tilde{\mu}_t(\theta(i))
\]
\[
\tilde{\mu}_t(\theta(i)) \equiv N(\hat{\theta}_i(i), \Sigma_t)
\]
where \( \hat{\theta}_i(i), i = 1, \ldots, n, \) and \( \Sigma_t \) can be sequentially updated using observations as follows.
\[
\hat{\theta}_{t+1}(i) = \hat{\theta}_i(i) + \frac{\Sigma_t z_t (x_{t+1}(i) - \hat{\theta}_i(i)^T z_t)}{1 + z_t \Sigma_t z_t^T}
\]
\[
\Sigma_{t+1} = \Sigma_t - \frac{\Sigma_t z_t z_t^T \Sigma_t}{1 + z_t \Sigma_t z_t^T}
\]
where \( z_t = [x_t^T, u_t^T]^T \in \mathbb{R}^{n+m} \).

Lemma 1 can be proved using arguments for the least square estimator. For example, see [28] for a proof.

**Remark 1.** Instead of the Kalman filter-type equation \([19]\), \( \Sigma_t \) can also be computed by
\[
\Sigma_{t+1}^{-1} = \Sigma_t^{-1} + z_t z_t^T.
\]

Let’s introduce the Thompson Sampling with Dynamic Episodes (TSDE) learning algorithm.

**Algorithm 1 TSDE**

- **Input:** \( \Omega_1, \hat{\theta}_1, \Sigma_1 \)
- **Initialization:** \( t \leftarrow 1, t_k \leftarrow 0 \)
- **for** episodes \( k = 1, 2, \ldots \) **do**
  - \( T_{k-1} \leftarrow t - t_k \)
  - \( t_k \leftarrow t \)
  - Generate \( \tilde{\theta}_k \sim \mu_{t_k} \)
  - Compute \( G_k = G(\tilde{\theta}_k) \) from \([6;-7]\)
  - **while** \( t \leq t_k + T_{k-1} \) and \( \det(\Sigma_t) \geq 0.5 \det(\Sigma_{t_k}) \) **do**
    - Apply control \( u_t = G_k x_t \)
    - Observe new state \( x_{t+1} \)
    - Update \( \mu_{t+1} \) according to \([18;19]\)
    - \( t \leftarrow t + 1 \)

The TSDE algorithm operates in episodes. Let \( t_k \) be start time of the \( k \)th episode and \( T_k = t_{k+1} - t_k \) be the length of the episode with the convention \( T_0 = 1 \). From the description of the algorithm, \( t_1 = 1 \) and \( t_{k+1}, k \geq 1 \), is given by
\[
t_{k+1} = \min\{t > t_k : t > t_k + T_{k-1} \}
\]
or \( \det(\Sigma_t) < 0.5 \det(\Sigma_{t_k}) \).
At the beginning of episode \( k \), a parameter \( \theta_k \) is sampled from the posterior distribution \( \mu_{t_k} \). During each episode \( k \), controls are generated by the optimal gain \( G_k \) for the sampled parameter \( \theta_k \). One important feature of TSDE is that its episode lengths are not fixed. The length \( T_k \) of each episode is dynamically determined according to two stopping criteria: (i) \( t > t_k + T_k - 1 \), and (ii) \( \det(\Sigma_t) < 0.5 \det(\Sigma_{t_k}) \). The first stopping criterion provides that the episode length grows at a linear rate without triggering the second criterion. The second stopping criterion ensures that the determinant of sample covariance matrix during an episode should not be less than half of the determinant of sample covariance matrix at the beginning of this episode.

B. Thompson Sampling for Time-Varying Parameter

For the learning problem with stationary parameter, we assume that the prior distribution \( \mu \) which generates the parameter after each jump satisfies Assumption 1.

We now introduce the Time-Varying Thompson Sampling with Dynamic Episodes (TSDE-TV) learning algorithm.

Algorithm 2 TSDE-TV

Input: \( \Omega_1, \hat{\theta}_1, \Sigma_1 \) and a parameter \( q \)
Initialization: \( t \leftarrow 1, t_k \leftarrow 0, s_t \leftarrow 1, l \leftarrow 1 \)
for episodes \( k = 1, 2, \ldots \) do
\( t_{k-1} \leftarrow t - t_k \)
\( t_k \leftarrow t \)
Generate \( \hat{\theta}_k \sim \mu_{t_k} \)
Compute \( G_k = G(\hat{\theta}_k) \) from (9)–(14)
while \( t \leq t_k + T_{k-1} \) and \( \det(\Sigma_t) \geq 0.5 \det(\Sigma_{t_k}) \) do
if \( t \geq s_t + l^q \) then
Re-initialize: \( t_k \leftarrow t - 1, \hat{\theta}_t \leftarrow \hat{\theta}_1, \Sigma_t \leftarrow \Sigma_1 \)
\( s_t \leftarrow t, l \leftarrow l + 1 \)
break
else
Apply control \( u_t = G_k x_t \)
Observe new state \( x_{t+1} \)
Update \( \mu_{t+1} \) according to (13)–(15)
\( t \leftarrow t + 1 \)
end

In TSDE-TV, \( s_t \) is the time when the algorithm re-initializes. The idea of TSDE-TV is to re-initialize TSDE to adapt to the jumps of the model parameter. TSDE-TV re-initializes if the time difference between the current episode and the previous re-initialization is long enough. The time difference between two re-initializations is \( l^q \) which increases at a rate determined by the parameter \( q \).

IV. REGRET ANALYSIS FOR STATIONARY PARAMETER

In this section, we analyze the regret of TSDE in the stationary parameter case. In the regret analysis, we make the following assumption on the prior distribution.

Assumption 2. There exists a positive number \( \delta < 1 \) such that for any \( \theta \in \Omega_1 \), we have \( \rho(A_1 + B_1 G(\theta)) \leq \delta \). Here \( \rho(\cdot) \) is the spectral radius of a matrix, i.e. the largest absolute value of its eigenvalues.

This assumption ensures that the closed-loop system is stable under the learning algorithm. A weaker assumption in [12] can ensure that Assumption 2 is satisfied for \( \theta = \theta_k \) with high probability.

Since \( J(\cdot), S(\cdot) \), and \( G(\cdot) \) are well-defined functions on the compact set \( \Omega_1 \), there exists finite numbers \( M_f, M_b, M_S, \) and \( M_G \) such that \( J(\theta) \leq M_f, ||\theta|| \leq M_b, ||S(\theta)|| \leq M_S, \) and \( \|I, G(\theta)^\top\| \leq M_G \) for all \( \theta \in \Omega_1 \).

The main result of this section is the following bound on expected regret of TSDE in the stationary parameter case.

Theorem 1. Under Assumptions 7 and 2 the expected regret (10) of TSDE satisfies
\[
R(T, \text{TSDE}) \leq \tilde{O}\left(\sqrt{T}\right) \tag{22}
\]
where \( \tilde{O}(\cdot) \) hides all constants and logarithmic factors.

To prove Theorem 1 we first provide bounds on the system state and the number of episodes. Then, we give a decomposition for the expected regret and derive upper bounds for each term of the regret.

Let \( X_T = \max_{1 \leq t < T} ||x_t|| \) be the maximum value of the norm of the state and \( K_T \) be the number of episodes over the horizon \( T \). Then we have the following properties.

Lemma 2. For any \( j \geq 1 \) and any \( T \) we have
\[
\mathbb{E}\left[X_T^j\right] \leq O\left(\log(T)(1 - \delta)^{-j}\right) \tag{23}
\]

Lemma 3. The number of episodes is bounded by
\[
K_T \leq O\left(\sqrt{2dT \log(TX_T^2)}\right) \tag{24}
\]

The proofs of Lemmas 2 and 3 are in the appendix. Following the steps in [11] using the Bellman equation (6), for \( t_k \leq t < t_{k+1} \) during the \( k \)th episode, the cost of TSDE satisfies
\[
c_t = J(\hat{\theta}_k) + x_t^\top S(\hat{\theta}_k)x_t - \mathbb{E}\left[x_{t+1}^\top S(\hat{\theta}_k) x_{t+1} | x_t, \hat{\theta}_k\right] + (\theta_1^\top z_t)^\top S(\hat{\theta}_k) \theta_1^\top z_t - (\theta_k^\top z_t)^\top S(\hat{\theta}_k) \theta_k^\top z_t. \tag{25}
\]

Then from (25), the expected regret of TSDE can be decomposed into
\[
R(T, \text{TSDE}) = \mathbb{E}\left[\sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1}-1} c_t\right] - T \mathbb{E}\left[J(\theta_1)\right]
\]
where
\[
R_T = \mathbb{E}\left[\sum_{k=1}^{K_T} T_k J(\hat{\theta}_k)\right] - T \mathbb{E}\left[J(\theta_1)\right], \tag{27}
\]
\[
R_1 = \mathbb{E}\left[\sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1}-1} \left[x_t^\top S(\hat{\theta}_k)x_t - x_{t+1}^\top S(\hat{\theta}_k)x_{t+1}\right]\right], \tag{28}
\]
\[
R_2 = \mathbb{E}\left[\sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1}-1} \left[(\theta_1^\top z_t)^\top S(\hat{\theta}_k) \theta_1^\top z_t - (\theta_k^\top z_t)^\top S(\hat{\theta}_k) \theta_k^\top z_t\right]\right]. \tag{29}
\]
In the following, we proceed to derive bounds on $R_0$, $R_1$ and $R_2$.

As discussed in \cite{21, 26, 29}, one key property of Thompson/Posterior Sampling algorithms is that for any function $f$, $E[f(\theta_t)] = E[f(\theta_1)]$ if $\theta_t$ is sampled from the posterior distribution at time $t$. However, our TSDE algorithm has dynamic episodes that requires us to have the stopping-time version of the above property whose proof is in the appendix.

**Lemma 4.** Under TSDE, $t_k$ is a stopping time for any episode $k$. Then for any measurable function $f$ and any $\sigma(h_{t_k})$—measurable random variable $X$, we have

$$E[f(\hat{\theta}_k, X)] = E[f(\theta_1, X)].$$

(30)

Based on the key property of Lemma 4 we establish an upper bound on $R_0$.

**Lemma 5.** The first term $R_0$ is bounded as

$$R_0 \leq M_J E[K_T].$$

(31)

**Proof.** From monotone convergence theorem, we have

$$R_0 = E \left[\sum_{k=1}^{\infty} \mathbb{I}_{\{t_k \leq T\}} T_k J(\hat{\theta}_k) \right] - T E \left[ J(\theta_1) \right]$$

$$= \sum_{k=1}^{\infty} E \left[\mathbb{I}_{\{t_k \leq T\}} T_k J(\hat{\theta}_k) \right] - T E \left[ J(\theta_1) \right].$$

(32)

Note that the first stopping criterion of TSDE ensures that $T_k \leq T_{k-1} + 1$ for all $k$. Because $J(\hat{\theta}_k) \geq 0$, each term in the first summation satisfies

$$E \left[\mathbb{I}_{\{t_k \leq T\}} T_k J(\hat{\theta}_k) \right] \leq E \left[\mathbb{I}_{\{t_k \leq T\}} (T_{k-1} + 1) J(\hat{\theta}_k) \right].$$

(33)

Note that $\mathbb{I}_{\{t_k \leq T\}} (T_{k-1} + 1)$ is measurable with respect to $\sigma(h_{t_k})$. Then, Lemma 4 gives

$$E \left[\mathbb{I}_{\{t_k \leq T\}} (T_{k-1} + 1) J(\hat{\theta}_k) \right] = E \left[\mathbb{I}_{\{t_k \leq T\}} (T_{k-1} + 1) J(\theta_1) \right].$$

(34)

Combining the above equations, we get

$$R_0 \leq \sum_{k=1}^{\infty} E \left[\mathbb{I}_{\{t_k \leq T\}} (T_{k-1} + 1) J(\theta_1) \right] - T E \left[ J(\theta_1) \right]$$

$$= \sum_{k=1}^{K_T} (T_{k-1} + 1) E \left[ J(\theta_1) \right] - T E \left[ J(\theta_1) \right]$$

$$= E \left[ K_T J(\theta_1) \right] + E \left[ \sum_{k=1}^{K_T} (T_{k-1} - T) J(\theta_1) \right]$$

$$\leq M_J E \left[ K_T \right]$$

(35)

where the last equality holds because $J(\theta_1) \leq M_J$ and $\sum_{k=1}^{K_T} T_{k-1} \leq T$.

The term $R_1$ can be upper bounded using $K_T$ and $X_T$.

**Lemma 6.** The second term $R_1$ is bounded by

$$R_1 \leq M_S E \left[ K_T X_T^2 \right].$$

(36)

We now derive an upper bound for $R_2$.

**Lemma 7.** The third term $R_2$ is bounded by

$$R_2 \leq \tilde{O} \left( M_2 \sqrt{(T + E[K_T]) E[X_T^4 \log(T X_T^2)]} \right)$$

(39)

where $M_2 = M_S M_0 M_G \sqrt{\frac{2 \tilde{\lambda}_{\min}}{\lambda_{\min}^2}}$ and $\lambda_{\min}$ is the minimum eigenvalue of $\Sigma_1^{-1}$.

**Proof.** Each term inside the expectation of $R_2$ is equal to

$$\| S^{0.5}(\hat{\theta}_k) \theta_1^T z_t \|^2 - \| S^{0.5}(\hat{\theta}_k) \hat{\theta}_k^T z_t \|^2$$

$$= \left( \| S^{0.5}(\hat{\theta}_k) \theta_1^T z_t \| + \| S^{0.5}(\hat{\theta}_k) \hat{\theta}_k^T z_t \| \right)$$

$$\left( \| S^{0.5}(\hat{\theta}_k) \theta_1^T z_t \| - \| S^{0.5}(\hat{\theta}_k) \hat{\theta}_k^T z_t \| \right)$$

$$\leq \left( \| S^{0.5}(\hat{\theta}_k) \theta_1^T z_t \| + \| S^{0.5}(\hat{\theta}_k) \hat{\theta}_k^T z_t \| \right)$$

$$\| S^{0.5}(\hat{\theta}_k) (\theta_1 - \hat{\theta}_k)^T z_t \|. \quad (40)$$

Since $\| S^{0.5}(\hat{\theta}_k) \theta_1^T z_t \| \leq M_S^{0.5} M_0 M_G X_T$ for $\theta = \hat{\theta}_k$ or $\theta = \theta_1$, the above term can be further bounded by

$$2M_S^{0.5} M_0 M_G X_T \| S^{0.5}(\hat{\theta}_k) (\theta_1 - \hat{\theta}_k)^T z_t \|$$

$$\leq 2M_S^{0.5} M_0 M_G X_T \| (\theta_1 - \hat{\theta}_k)^T z_t \|. \quad (41)$$

Therefore,

$$R_2 \leq 2M_S M_0 M_G E \left[ X_T \sum_{k=1}^{K_T} t_{k+1} \right] \left( \| (\theta_1 - \hat{\theta}_k)^T z_t \| \right). \quad (42)$$
From Cauchy-Schwarz inequality, we have
\[
\mathbb{E} \left[ X_T \sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1} - 1} \| (\theta_1 - \tilde{\theta}_k)^\top z_t \| \right] \\
= \mathbb{E} \left[ X_T \sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1} - 1} \| (\Sigma_{t_k}^{-0.5} (\theta_1 - \tilde{\theta}_k)) \Sigma_{t_k}^{0.5} z_t \| \right] \\
\leq \mathbb{E} \left[ \sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1} - 1} \| \Sigma_{t_k}^{-0.5} (\theta_1 - \tilde{\theta}_k) \| \times X_T \| \Sigma_{t_k}^{0.5} z_t \| \right] \\
\leq \sqrt{\mathbb{E} \left[ \sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1} - 1} \| \Sigma_{t_k}^{-0.5} (\theta_1 - \tilde{\theta}_k) \|^2 \right] \sum_{k=1}^{K_T} X_T^2 \| \Sigma_{t_k}^{0.5} z_t \|^2} \tag{43}
\]

From Lemma 10 in the appendix, the first part of (43) is bounded by
\[
\sqrt{\mathbb{E} \left[ \sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1} - 1} \| \Sigma_{t_k}^{-0.5} (\theta_1 - \tilde{\theta}_k) \|^2 \right]} \leq \sqrt{4d\max(T + \mathbb{E}[K_T])}. \tag{44}
\]

For the second part of (43), note that
\[
\sum_{k=1}^{K_T} \sum_{t=t_k}^{t_{k+1} - 1} \| \Sigma_{t_k}^{0.5} z_t \|^2 = \sum_{t=1}^{T} z_t^\top \Sigma_t z_t \tag{45}
\]

Since \( \| z_t \| \leq M_G X_T \) for all \( t \leq T \), Lemma 8 of [22] implies
\[
\sum_{t=1}^{T} z_t^\top \Sigma_t z_t \\
\leq \sum_{t=1}^{T} \max(1, M_G^2 X_T^2 / \lambda_{\min}) \min(1, z_t^\top \Sigma_t z_t) \\
\leq 2d \max(1, M_G^2 X_T^2 / \lambda_{\min}) \log(\text{tr}(\Sigma_1^{-1}) + T M_G^2 X_T^2) \\
= O \left( \frac{2dM_G^2}{\lambda_{\min}} X_T^2 \log(T X_T^2) \right). \tag{46}
\]

Consequently, the second term of (43) is bounded by
\[
O \left( \frac{2dM_G^2}{\lambda_{\min}} \mathbb{E} \left[ X_T^4 \log(T X_T^2) \right] \right). \tag{47}
\]

Then, from (42), (43), (44) and (47), we obtain the result of the lemma. \( \square \)

Using the bounds on \( R_0, R_1 \) and \( R_2 \), we are now ready to prove Theorem 1.

**Proof of Theorem 1.** From the regret decomposition (26), Lemmas [5, 7] and the bound on \( K_T \) from Lemma 5 we obtain
\[
R(T, \text{TSDE}) \\
\leq O \left( M_2 \sqrt{T + \mathbb{E}[\sqrt{2dT \log(T X_T^2)}]} \right) \mathbb{E}[X_T^2 \log(T X_T^2)] \\
+ \mathbb{E} \left[ \sqrt{2dT \log(T X_T^2)}(M_J + M_S X_T^2) \right] \\
\leq \hat{O} \left( \sqrt{T + \mathbb{E}[\sqrt{T \log(T X_T^2)}]} \right) \mathbb{E}[X_T^2 \log(T X_T^2)] \\
+ \mathbb{E} \left[ \sqrt{T \log(T X_T^2)} X_T^2 \right]. \tag{48}
\]

From Lemma 11 in the appendix, we have \( \mathbb{E}[\sqrt{\log(T X_T^2)}] \leq O(1) \), \( \mathbb{E}[\sqrt{\log(T X_T^2)}] \leq \hat{O}(\delta^{-2}) \), and \( \mathbb{E}[X_T^2 \log(T X_T^2)] \leq \hat{O}(\delta^{-5}) \). Applying these bounds to (48) we get
\[
R(T, \text{TSDE}) \\
\leq \hat{O} \left( \sqrt{T + \sqrt{T}(1 - \rho)^{-5} + \sqrt{T}(1 - \delta)^{-2}} \right) \\
= \hat{O} \left( \sqrt{T(1 - \delta)^{-2.5}} \right). \tag{49}
\]

**V. REGRET ANALYSIS FOR TIME-VARYING PARAMETER**

We now present the regret analysis for the time-varying parameter case.

Since the true parameter varies over time, we make a stronger assumption on the prior distribution to ensure stability.

**Assumption 3.** There exists a positive number \( \delta < 1 \) such that for any \( \theta, \theta' \in \Omega_1 \), we have \( \rho(A + B G(\theta')) \leq \delta \) where \( \theta^\top = [A, B] \).

From the assumption, the closed-loop system is stable for any model parameter under the learning algorithm.

Let \( N_T = \mathbb{E} \left[ \sum_{t \leq T} j_t \right] \) be the expected number of jumps up to time \( T \). We now present the bound on expected regret of TSDE in the time-varying parameter case.

**Theorem 2.** Under Assumptions [7] and [3] the expected regret [12] of TSDE-TV satisfies
\[
R_{TV}(T, \text{TSDE-TV}) \leq \hat{O} \left( T^{4/5 + q} + T^{1/5 + \alpha} N_T \right) \tag{50}
\]

where \( \hat{O}() \) hides all constants and logarithmic factors.

From Theorem 2 we have the following corollary.

**Corollary 1.** If \( N_T \leq T^\alpha \) for some \( \alpha < 1 \), we can pick \( q = \frac{2(1-\alpha)}{1+2\alpha} \) such that
\[
R_{TV}(T, \text{TSDE-TV}) \leq \hat{O} \left( T^{4/5 + \alpha} \right). \tag{51}
\]

The corollary says that when the expected number of jumps is sub-linear in \( T \), with an appropriate choice of algorithm parameter \( q \), the TSDE-TV algorithm can achieve a sub-linear growth of expected regret.
Remark 2. Note that the sub-linear regret growth of TSDE-TV implies its asymptotically optimal performance under the average cost criterion.

We proceed to analyze the regret of TSDE-TV and prove Theorem 2.

Let $L_T$ be the number of re-initializations of TSDE-TV up to time $T$. Then, we can divide the time horizon $T$ into $L_T + 1$ phases. Let $s_1, s_2, \ldots, s_{L_T}$ be the times TSDE-TV re-initializes. Then the $l$th phase has length $D_l = s_l - s_{l-1}$ for $l = 1, 2, \ldots, (L_T + 1)$ (with the convention $s_0 = 1$ and $s_{L_T+1} = T+1$). From the specification of TSDE-TV we have $D_l = l^q$ for $l \leq L_T$.

We can now decompose the regret of TSDE-TV into two parts. The first part is the performance loss during the phases without parameter jumps, and the second part is the loss during the phases with at least one parameter jump. Specifically, we have

$$R_{TV}(T, \text{TSDE-TV}) = R_{TV,0} + R_{TV,1}$$

where

$$R_{TV,0} = \mathbb{E}\left[\sum_{l \leq L_T+1: j_l=0 \text{ for all } s_{l-1} \leq t < s_l} \left(c_t - J(\theta_t)\right)\right]$$

$$R_{TV,1} = \mathbb{E}\left[\sum_{l \leq L_T+1: j_l=1 \text{ for some } s_{l-1} \leq t < s_l} \left(c_t - J(\theta_t)\right)\right]$$

We use the stationary parameter result to bound the first part $R_{TV,0}$.

**Lemma 8.**

$$R_{TV,0} \leq \tilde{O}\left(L_T^{2+q}\right).$$

**Proof.** Since the model parameter remains the same for each phase $l$ in $R_{TV,0}$, the system during each of such phase is the same as the stationary parameter case. Therefore, the regret analysis for TSDE can be applied here because TSDE-TV is the same as TSDE during a phase. Note that the length of phase $l$ is $D_l$, so from Theorem 1 we get

$$R_{TV,0} = \mathbb{E}\left[\sum_{l \leq L_T+1: j_l=0 \text{ for all } s_{l-1} \leq t < s_l} \left(c_t - J(\theta_t)\right)\right]$$

$$\leq \mathbb{E}\left[\sum_{l \leq L_T+1: j_l=0 \text{ for all } s_{l-1} \leq t < s_l} \tilde{O}(\sqrt{D_l})\right]$$

$$\leq \tilde{O}\left(\sum_{l \leq L_T+1} \sqrt{D_l}\right)$$

$$\leq \tilde{O}\left(\sum_{l \leq L_T+1} l^{2}q\right) = \tilde{O}(L_T^{2+q}).$$

Using the expected number of parameter jumps $N_T$ we bound the second part of the regret in the lemma below.

**Lemma 9.**

$$R_{TV,1} \leq \tilde{O}\left(L_T^{q}N_T\right).$$

**Proof.** Note that for any $t \leq T$ we have

$$c_t - J(\theta_t) \leq c_t = x_t^T Q x_t + u_t^T Ru_t$$

$$\leq ||Q|| \cdot ||x_t||^2 + \max_{\theta \in \Theta_t} ||R|| \cdot ||G(\theta)|| \cdot ||x_t||^2$$

$$\leq \tilde{O}\left(X_T^2\right).$$

Therefore,

$$R_{TV,1} \leq \mathbb{E}\left[\sum_{l \leq L_T+1: j_l=1 \text{ for some } s_{l-1} \leq t < s_l} D_l \tilde{O}\left(X_T^2\right)\right]$$

$$\leq (L_T + 1)^q \mathbb{E}\left[\sum_{t \leq T} j_t \tilde{O}\left(X_T^2\right)\right]$$

where the last inequality holds because $D_l \leq (L_T + 1)^q$.

Since the jump process is independent of the system noise, the proof of Lemma 2 also hold when conditioned on $\sum_{t \leq T} j_t$. That is,

$$\mathbb{E}\left[X_T^2 \sum_{t \leq T} j_t\right] \leq \tilde{O}(1).$$

From (59) and (59) we get

$$R_{TV,1} \leq (L_T + 1)^q \mathbb{E}\left[\sum_{t \leq T} j_t \tilde{O}\left(X_T^2\right)\right]$$

$$= (L_T + 1)^q \mathbb{E}\left[\tilde{O}\left(X_T^2 \sum_{t \leq T} j_t\right)\right]$$

$$\leq (L_T + 1)^q \mathbb{E}\left[\sum_{t \leq T} j_t\right] = \tilde{O}(L_T^{q}N_T).$$

We now prove Theorem 2 using the above lemmas.

**Proof of Theorem 2.** From Lemmas 8 and 9 we have

$$R_{TV}(T, \text{TSDE-TV}) \leq \tilde{O}\left(L_T^{2+q} + L_T^{q}N_T\right).$$

Note that

$$T \geq \sum_{l \leq L_T} D_l = \sum_{l \leq L_T} l^q = \tilde{O}(L_T^{1+q}).$$

So $L_T \leq \tilde{O}(T^{1+q})$, and the proof of theorem is complete by applying this bound on $L_T$ in (62).

**VI. SIMULATIONS**

**A. Stationary Parameter**

In this section, we illustrate through numerical simulations the performance of the TSDE algorithm for different linear systems. The prior distribution used in TSDE are set according to (14) with $\hat{\theta}_1(1) = 1$, $\Sigma_1 = I$, and $\Omega_1 = \{\theta : \rho(A_1 + B_1 G(\theta)) \leq \delta\}$ where $\delta$ is a simulation parameter. The parameter $\delta$ can be seen as the level of accuracy of the prior distribution. The smaller $\delta$ is, the more accurate the prior distribution is for the true system parameters. Note that Assumption 2 holds when $\delta < 1$, but it does not hold when $\delta \geq 1$. 


In this paper, we have proposed a Thompson sampling with dynamic episodes (TSDE) learning algorithm for control of stochastic linear systems with quadratic costs. Under some dynamic episodes (TSDE) learning algorithm for control of stochastic linear systems with quadratic costs. Under some dynamic episodes (TSDE) learning algorithm for control of stochastic linear systems with quadratic costs. Under some
average regret per unit time goes to zero, thus the learning algorithm asymptotically learns the optimal control policy. We believe this is the first near-optimal guarantee on expected regret for a learning algorithm in LQ control. We have also shown that TSDE with a re-initialization schedule (i.e. the TSDE-TV algorithm) is robust to time-varying drift in model parameters. As long as the drift is not large, the algorithm can “track” the model parameters and find an approximately optimal control law. Numerical simulations confirm that TSDE indeed achieves sublinear regret which matches with the theoretical upper bounds. In addition to use of the Thompson sampling-based learning, the key novelty here is design of an exploration schedule that achieves sublinear regret.

REFERENCES

[1] K. J. Astrom and B. Wittenmark, Adaptive Control, Addison-Wesley Longman Publishing Co., Inc., 1994.
[2] S. Sastry and M. Bodson, Adaptive control: stability, convergence, and robustness. Prentice-Hall, Inc., 1989.
[3] K. Narendra and A. Annaswamy, Stable adaptive systems. Prentice-Hall, Inc., 1989.
[4] N. Cesa-Bianchi and G. Lugosi, Prediction, learning, and games. Cambridge university press, 2006.
[5] G. C. Goodwin and K. S. Sin, Adaptive filtering prediction and control. Courier Corporation, 2014.
[6] P. R. Kumar and P. Varaiya, Stochastic systems: Estimation, identification, and adaptive control. SIAM Classics, 2015.
[7] A. Becker, P. Kumar, and C.-Z. Wei, “Adaptive control with the stochastic approximation algorithm: Geometry and convergence,” IEEE Transactions on Automatic Control, vol. 30, no. 4, pp. 330–338, 1985.
[8] H.-F. Chen and L. Guo, “Convergence rate of least-squares identification and adaptive control for stochastic systems,” International Journal of Control, vol. 44, no. 5, pp. 1459–1476, 1986.
[9] L. Guo, “Self-convergence of weighted least-squares with applications to stochastic adaptive control,” IEEE Transactions on Automatic Control, vol. 41, no. 1, pp. 79–89, 1996.
[10] M. C. Campi and P. Kumar, “Adaptive linear quadratic gaussian control: the cost-biased approach revisited,” SIAM Journal on Control and Optimization, vol. 36, no. 6, pp. 1890–1907, 1998.
[11] Y. Abbasi-Yadkori and C. Szepesvári, “Regret bounds for the adaptive control of linear quadratic systems,” in Proceedings of the 24th Annual Conference on Learning Theory, pp. 1–21, 2011.
[12] M. Ibrahim, A. Javanmard, and B. V. Roy, “Efficient reinforcement learning for high dimensional linear quadratic systems,” in Advances in Neural Information Processing Systems (NIPS), pp. 2636–2644, 2012.
[13] T. L. Lai and H. Robbins, “Asymptotically efficient adaptive allocation rules,” Advances in applied mathematics, vol. 6, no. 1, pp. 4–22, 1985.
[14] V. Dani, T. P. Hayes, and S. M. Kakade, “Stochastic linear optimization under bandit feedback,” in COLT, pp. 355–366, 2008.
[15] D. Russo, B. Van Roy, A. Kazerouni, and I. Osband, “A tutorial on Thompson sampling,” arXiv preprint arXiv:1707.02038, 2017.
[16] S. L. Scott, “A modern bayesian look at the multi-armed bandit,” Applied Stochastic Models in Business and Industry, vol. 26, no. 6, pp. 639–658, 2010.
[17] O. Chapelle and L. Li, “An empirical evaluation of Thompson sampling,” in NIPS, 2011.
[18] S. Agrawal and N. Goyal, “Analysis of Thompson sampling for the multi-armed bandit problem,” in Conference on Learning Theory, pp. 39–1, 2012.
[19] E. Kaufmann, N. Korda, and R. Munos, “Thompson sampling: An asymptotically optimal finite-time analysis,” in International Conference on Algorithmic Learning Theory, pp. 199–213, Springer, 2012.
[20] S. Agrawal and N. Goyal, “Thompson sampling for contextual bandits with linear payoffs,” in ICML (3), pp. 127–135, 2013.
[21] I. Osband, D. Russo, and B. Van Roy, “(More) efficient reinforcement learning via posterior sampling,” in NIPS, 2013.
[22] Y. Abbasi-Yadkori and C. Szepesvári, “Bayesian optimal control of smoothly parameterized systems,” in UAI, 2015.
[23] A. Gopalan and S. Mannor, “Thompson sampling for learning parameterized markov decision processes,” in COLT, 2015.
[24] W. R. Thompson, “On the likelihood that one unknown probability exceeds another in view of the evidence of two samples,” Biometrika, vol. 25, no. 3/4, pp. 285–294, 1933.
Proof of Lemma 2. During the kth episode, we have $u_t = G(\theta_t)x_t$. Then,
\[
||x_{t+1}|| = ||(A_1 + B_1 G(\tilde{\theta}_t))x_t + w_t|| \\
\leq ||(A_1 + B_1 G(\tilde{\theta}_t))x_t|| + ||w_t|| \\
\leq \rho ||A_1 + B_1 G(\tilde{\theta}_t)|| ||x_t|| + ||w_t|| \\
\leq \delta ||x_t|| + ||w_t||
\]
(64)
where the second inequality is the property of spectral radius, and the last inequality follows from Assumption 2. Iteratively applying (64), we get
\[
||x_t|| \leq \sum_{\tau < t}^{\delta^{t-\tau-1} ||w_\tau||} \leq \sum_{\tau < t}^{\delta^{t-\tau-1} \max_{\tau \leq T} ||w_\tau||} \leq \frac{1}{1-\delta} \max_{\tau \leq T} ||w_\tau||.
\]
(65)
Therefore,
\[
X_T^2 \leq \left(\frac{1}{1-\delta} \max_{\tau \leq T} ||w_\tau||\right)^2 = (1-\delta)^{-j} \max_{\tau \leq T} ||w_\tau||^2.
\]
(66)
Then, it remains to bound $\mathbb{E}[[\max_{\tau \leq T} ||w_\tau||^2]]$. Following the steps of (50), we have
\[
\exp \left( \mathbb{E}[[\max_{\tau \leq T} ||w_\tau||^2]] \right) \leq \mathbb{E} \left[ \exp \left( \max_{\tau \leq T} ||w_\tau||^2 \right) \right] \\
= \mathbb{E} \left[ \max_{\tau \leq T} \left( ||w_\tau||^2 \right) \right] \\
\leq \mathbb{E} \left[ \sum_{\tau \leq T} \left( ||w_\tau||^2 \right) \right] \\
= T \mathbb{E} \left[ \exp \left( ||w_\tau||^2 \right) \right].
\]
(67)
Combining (66) and (67), we obtain
\[
\mathbb{E}[X_T^2] \leq (1-\delta)^{-j} \log \left( T \mathbb{E} \left[ \exp \left( ||w_\tau||^2 \right) \right] \right) = O((1-\delta)^{-j} \log(T)).
\]
(68)
Proof of Lemma 3. Define macro-episode with start times $t_{n_1}, i = 1, 2, \ldots$ where $t_{n_1} = t_1$ and $t_{n_{i+1}} = \min\{t_{n_i} > t_i : \det(\Sigma_{1:n}) < 0.5 \det(\Sigma_{1:n-1})\}$. The idea is that each macro-episode starts when the second stopping criterion happens. Let $M$ be the number of macro-episodes until time $T$ and define $n_{(M+1)} = K_T + 1$. Let $M$ be the set of episodes that is the first one in a macro-episode. Let $T_i = \sum_{k=1}^{n_i-1} T_k$ be the length of the $i$th macro-episode. By definition of macro-episodes, any episode except the last one in a macro-episode must be triggered by the first stopping criterion. Therefore, within the $i$th macro-episode, $T_k = T_{k-1} + 1$ for all $k = n_i, n_i + 1, \ldots, n_{i+1} - 2$. Hence,
\[
T_i = \sum_{k=n_i}^{n_{i+1}-n_i-1} T_k = \sum_{j=1}^{n_{i+1}-n_i-1} (T_{n_i-1} + j) + T_{n_{i+1}-1} - j \\
\geq \sum_{j=1}^{n_{i+1}-n_i} (j+1) + 1 = 0.5(n_{i+1} - n_i)(n_{i+1} - n_i - 1).
\]
Consequently, $n_{i+1} - n_i \leq \sqrt{2T_i}$ for all $i = 1, \ldots, M$. From this property, we obtain
\[
K_T = n_{M+1} - 1 = \sum_{i=1}^{M} (n_{i+1} - n_i) \leq \sum_{i=1}^{M} \sqrt{2T_i}.
\]
(70)
Using (70) and the fact that $\sum_{i=1}^{M} T_i = T$, we get
\[
K_T \leq \sum_{i=1}^{M} \sqrt{2T_i} \leq \sum_{i=1}^{M} \sqrt{2T_i} = \sqrt{2MT}.
\]
(71)
where the second inequality is by Cauchy-Schwarz.

Since the second stopping criterion is triggered whenever the determinant of sample covariance is half, we have
\[
\det(\Sigma_T^{-1}) \geq \det(\Sigma_{1:nM}^{-1}) \geq 2 \det(\Sigma_{1:nM}^{-1}) \\
> \cdots > 2^{M-1} \det(\Sigma_1^{-1})
\]
(72)
Since $(\text{tr}(\Sigma_T^{-1}))^d \geq \det(\Sigma_T^{-1})$, we have
\[
\text{tr}(\Sigma_T^{-1}) > (\text{det}(\Sigma_T^{-1}))^{1/d} \\
> 2^{(M-1)/d} \det(\Sigma_1^{-1})^{1/d} \geq 2^{(M-1)/d} \lambda_{\text{min}}
\]
(73)
where $\lambda_{\text{min}}$ is the minimum eigenvalue of $\Sigma_1^{-1}$. Note that
Hence, from (71) we obtain the claim of the lemma.

Then, 

\[ M \leq 1 + d \log \left( \frac{1}{\lambda_{\min}} \text{tr}(\Sigma^{-1}) + \sum_{t=1}^{T-1} z_t^T z_t \right) \]

\[ = O \left( d \log \left( \sum_{t=1}^{T-1} z_t^T z_t \right) \right). \]  

(76)

Note that, \( ||z_t|| = \| [I, G(\theta)^T] x_t \| \leq M_G \| x_t \| \). Consequently,

\[ M \leq O \left( d \log \left( M_G^2 \sum_{t=1}^{T-1} ||x_t||^2 \right) \right) = O \left( d \log \left( \sum_{t=1}^{T-1} ||x_t||^2 \right) \right) \]

\[ \leq O \left( d \log(TX^2_T) \right) \]  

(77)

Hence, from (71) we obtain the claim of the lemma. \( \square \)

**Lemma 10.** We have the following inequality:

\[ \mathbb{E} \left[ \sum_{k=1}^{K} t_{k+1}^{-1} ||\Sigma^{-0.5}(\theta_1 - \hat{\theta}_k)||^2 \right] \leq 4dn(T + \mathbb{E}[K_T]). \]  

(78)

**Proof.** From Lemma 9 of [22], we have

\[ ||\Sigma^{-0.5}(\theta_1 - \hat{\theta}_k)||^2 \]

\[ \leq ||\Sigma^{-0.5}(\theta_1 - \hat{\theta}_k)||^2 \frac{\det(\Sigma_{t_1})}{\det(\Sigma_{t})} \]

\[ \leq 2||\Sigma^{-0.5}(\theta_1 - \hat{\theta}_k)||^2 \]  

(79)

where the last inequality follows from the second stopping criterion of the algorithm. Therefore,

\[ \sum_{k=1}^{K_T} t_{k+1}^{-1} ||\Sigma^{-0.5}(\theta_1 - \hat{\theta}_k)||^2 \]

\[ \leq 2 \sum_{k=1}^{K_T} T_k ||\Sigma^{-0.5}(\theta_1 - \hat{\theta}_k)||^2. \]  

(80)

Using the idea of the proof of Lemma 5, we obtain

\[ \mathbb{E} \left[ \sum_{k=1}^{K_T} T_k ||\Sigma^{-0.5}(\theta_1 - \hat{\theta}_k)||^2 \right] \]

\[ = \sum_{k=1}^{K_T} \mathbb{E} \left[ I_{\{\tau_k \leq T\}} T_k ||\Sigma^{-0.5}(\theta_1 - \hat{\theta}_k)||^2 \right] \]

\[ \leq \sum_{k=1}^{K_T} \mathbb{E} \left[ I_{\{\tau_k \leq T\}} (T_{k-1} + 1) ||\Sigma^{-0.5}(\theta_1 - \hat{\theta}_k)||^2 \right]. \]  

(81)

Since \( I_{\{\tau_k \leq T\}} (T_{k-1} + 1) \) is measurable with respect to \( \sigma(h_{t_k}) \),

we get

\[ \mathbb{E} \left[ I_{\{\tau_k \leq T\}} (T_{k-1} + 1) ||\Sigma^{-0.5}(\theta_1 - \hat{\theta}_k)||^2 \right] \]

\[ = \mathbb{E} \left[ \mathbb{E} \left[ I_{\{\tau_k \leq T\}} (T_{k-1} + 1) ||\Sigma^{-0.5}(\theta_1 - \hat{\theta}_k)||^2 | h_{t_k} \right] \right] \]

\[ \leq \mathbb{E} \left[ I_{\{\tau_k \leq T\}} (T_{k-1} + 1) \mathbb{E} \left[ ||\Sigma^{-0.5}(\theta_1 - \hat{\theta}_k)||^2 | h_{t_k} \right] \right] \]

\[ \leq \mathbb{E} \left[ I_{\{\tau_k \leq T\}} (T_{k-1} + 1) 2dn \right] \]  

(82)

where the inequality holds because conditioned on \( h_{t_k} \), each column of \( \Sigma^{-0.5}(\theta_1 - \hat{\theta}_k) \) is the difference of two \( d \)-dimensional i.i.d. random vectors \( \sim N(0, I) \).

As a result,

\[ \mathbb{E} \left[ \sum_{k=1}^{K_T} t_{k+1}^{-1} ||\Sigma^{-0.5}(\theta_1 - \hat{\theta}_k)||^2 \right] \]

\[ \leq 4dn \mathbb{E} \left[ I_{\{\tau_k \leq T\}} (T_{k-1} + 1) \right] \]

\[ \leq 4dn \mathbb{E}[T + K_T]. \]  

(83)

\( \square \)

**Lemma 11.** The following bounds hold:

\[ \mathbb{E}[\sqrt{\log(X_T)}] \leq \tilde{O}(1) \]

\[ \mathbb{E}[\sqrt{\log(X_T)}X_T^2] \leq \tilde{O} \left( (1 - \delta)^{-2} \right), \]

\[ \mathbb{E}[X_T^4 \log(X_T)] \leq \tilde{O} \left( (1 - \delta)^{-5} \right). \]

**Proof.** Using Lemma 2 on \( X_T \), we get

\[ \mathbb{E}[\sqrt{\log(X_T)}X_T^2] \leq \sqrt{\mathbb{E}[\log(X_T)] \mathbb{E}[X_T^2]} \]

\[ \leq \sqrt{\mathbb{E}[\log(X_T)]} \mathbb{E}[X_T^2] \]

\[ \leq \tilde{O} \left( (1 - \delta)^{-2} \sqrt{\log(T)\log(T)(1 - \delta)^{-1}} \right) \]

\[ \leq \tilde{O} \left( (1 - \delta)^{-2} \right). \]  

(88)

Similarly,

\[ \mathbb{E}[\sqrt{\log(X_T)}X_T^2] \leq \sqrt{\mathbb{E}[\log(X_T)] \mathbb{E}[X_T^2]} \]

\[ \leq \tilde{O} \left( (1 - \delta)^{-2} \sqrt{\log(T)\log(T)(1 - \delta)^{-1}} \right) \]

\[ \leq \tilde{O} \left( (1 - \delta)^{-2} \right). \]  

(88)

Since \( \log(X_T) \leq X_T \), we have

\[ \mathbb{E}[X_T^4 \log(X_T)] \leq \mathbb{E}[X_T^5] \]

\[ \leq \tilde{O} \left( (1 - \delta)^{-5} \right) \]  

(89)

\( \square \)
University of California, Berkeley. His research interests include stochastic control, reinforcement learning, decentralized decision-making, and dynamic games with asymmetric information.

**Mukul Gagrani** received the B.tech and M.tech degree in Electrical Engineering from the Indian Institute of Technology, Kanpur, India in 2013. He is currently a PhD candidate in the department of electrical engineering at the University of Southern California, Los Angeles, CA. His research interests include decentralized stochastic control, stochastic scheduling and decision-making under uncertainty.

**Rahul Jain** is an associate professor and the K. C. Dahlberg Early Career Chair in the EE department at the University of Southern California. He received his PhD in EECS and an MA in Statistics from the University of California, Berkeley, his B.Tech from IIT Kanpur. He is winner of numerous awards including the NSF CAREER award, an IBM Faculty award and the ONR Young Investigator award. His research interests span stochastic systems, statistical learning, queueing systems and game theory with applications in communication networks, power systems, transportation and healthcare.