Degenerations of elliptic curves and cusp singularities

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Abstract. This paper gives more or less explicit equations for all two-dimensional cusp singularities of embedding dimension at least 4. They are closely related to Felix Klein’s equations for universal curves with level $n$ structure. The main technical result is a description of the versal deformation of an $n$-gon in $\mathbb{P}^{n-1}$. The final section contains the equations for smoothings of simple elliptic singularities (of multiplicity $n \leq 9$).

Equations of cusp singularities are known for multiplicity at most 5 [Ka]. Cusps are quite well understood in terms of generators of the local ring, described as Fourier series; the cusp is constructed as compactification of the quotient of the product of upper half spaces $H^2$ by the semi-direct product of a lattice in a real quadratic field and an infinite cyclic group. The precise relations between these generators however are not known even when equations for the cusp are available. Still the combinatorics of the associated continued fractions provide the pattern for the equations.

The relation between cusp equations and families of elliptic curves can already be seen in the hypersurface case. The cusps of multiplicity 3 are the $T_{pqr}$-singularities with $3 \leq p \leq q \leq r$ and at least one strict inequality. With a slightly different notation I write

$$X_0^{a_0+1} + X_1^{a_1+1} + X_2^{a_2+1} - X_0X_1X_2 = 0;$$

the periodic continued fraction $[[a_0,a_1,a_2]]$ determines the cusp. The equation resembles the Hesse normal form

$$X_0^3 + X_1^3 + X_2^3 - 3\mu X_0X_1X_2 = 0,$$

which describes the universal elliptic curve with level 3 structure. For four values of the parameter ($\mu^3 = 1$ and $\mu = \infty$) the curve degenerates into a triangle. The projectivised tangent cone of the cusp is the triangle corresponding to $\mu = \infty$ in the Hesse normal form, if all $a_i$ are at least 3. The other possibilities, a line and a conic ($X_0^3 - X_0X_1X_2$) or a nodal cubic ($X_0^3 + X_1^3 - X_0X_1X_2$), do not occur as degenerate curve with level 3 structure. Therefore it is better to forget about extra structures and to directly consider the versal deformation of the triangle:

$$X_0X_1X_2 - t_0X_0^3 - t_1X_1^3 - t_2X_2^3 = 0,$$

in which the vertices can be smoothed separately. The regular sequence $(t_0 - X_0^{a_0-2}, t_1 - X_1^{a_1-2}, t_2 - X_2^{a_2-2})$ cuts out the cusp singularity from the total space of this deformation.

For higher multiplicities I also start from equations for elliptic curves with a level structure. For odd $n$ they were written down by Felix Klein (see [K–F]). The number of equations becomes rather large, but due to the symmetries of the elliptic curve and the total family one needs only a few different types. In fact, they can all be combined in the formula

$$F_{\alpha\beta\gamma\delta} = s_{\alpha-\beta}s_{\gamma-\delta}X_{\alpha+\beta}X_{\gamma+\delta} + s_{\alpha-\gamma}s_{\delta-\beta}X_{\alpha+\gamma}X_{\delta+\beta} + s_{\alpha-\delta}s_{\beta-\gamma}X_{\alpha+\delta}X_{\beta+\gamma},$$

with the $s_\alpha$ depending on the modulus of the curve. For special values of the modular parameter (at the ‘cusps’) the elliptic curve degenerates to an $n$-gon. I adapt the equations to give the versal deformation of the standard $n$-gon (with the as vertices coordinate points in their natural order). Equations of cusps are again obtained by the specialisation $t_i = X_i^{n_i-2}$. The change in the formulas for the modular curve involve the deformation parameters $t_i$ making the coefficients $s_\alpha$ into power series in the $t_i$. Therefore the resulting cusp equations...
are not longer polynomial if $n > 5$. For even $n$ similar results can be obtained, basically with the same formulas.

For the special case of degenerate cusps Ruud Pellikaan found the equations using that the normalisation of a degenerate cusp consists of cyclic quotient singularities [Pe]. A degenerate cusp is a non-normal surface singularity which can also be described by numbers $[[a_1, \ldots, a_n]]$, but now $a_i = \infty$ for at least one $i$; one formally puts $x_i^{\infty} = 0$. To describe the equations I use the abbreviation $X_n^{a_i-2} = t_i$ with the convention that $t_i = 0$ if $a_i = \infty$.

**Proposition** (Pellikaan). Let $X(a_1, \ldots, a_n)$ be a degenerate cusp with $n > 3$. The $n(n-3)/2$ equations

$$
F_{ij} : X_iX_j = X_{i+1}\left(\prod_{k=i+1}^{j-1} t_k\right) X_{j-1} + X_{j+1}\left(\prod_{k=j+1}^{i-1} t_k\right) X_{i-1}
$$

with $i - j \neq -1, 0, 1$ generate the ideal of the cusp.

Pellikaan’s formulas applied to isolated cusps with multiplicity 4 and 5 give the usual form of the equations, but for higher multiplicities one needs correction terms, which were already computed by Pellikaan in the case $n = 6$. My cusp equations have a similar structure; they specialise to both cases mentioned, of degenerate cusps and elliptic curves.

The infinitesimal deformations of cusp singularities were determined by Kurt Behnke using the Fourier series approach [Be]. In terms of my equations I can only give them for multiplicity at most 6; in those cases I know also the versal deformation. I gave the formulas for $n = 6$. Even for simple elliptic singularities I do not have a general formula for infinitesimal deformations, but I did compute them and also the versal deformation in all cases where the singularity is smoothable (i.e., up to $n = 9$). A geometric description of the versal deformation of simple elliptic singularities can be found in [Mé].

In the first Section I review the equations for elliptic curves and their symmetries and indicate the modifications for the case of even degree. The examples introduce the equations which are used for the computations of the versal deformation in the last Section. The second Section contains the main technical result, the versal deformation of an $n$-gon. The third Section applies it to the cusp equations.

### 1. Elliptic curves with level $n$ structure

Equations for elliptic curves of degree $n$ were first written down by Bianchi [Bi] for small odd $n$ and for general odd $n$ by Felix Klein [Kl]. Their formulas describe the universal family of elliptic curves with level $n$ structure (in fact as scheme over $\text{Spec} \mathbb{Z}[1/n]$ [Ve]). Hurwitz [Hu1] realised that in the case of even $n$ basically the same embedding works, although it does not give the universal family. The basic reference for these results is [K–F, Fünfber Abschnitt]; see also [Hul].

(1.1) Let $E = \mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be an elliptic curve and let $n \geq 5$ be an odd integer. Klein defined functions $X_\alpha(u), \alpha \in \mathbb{Z}/n$, whose zeroes in the period parallelogram are the points $\frac{\omega_1}{n}, \frac{\omega_2}{n}, i = 0, \ldots, n - 1$. The values of these functions at the origin define constants $s_\alpha := X_\alpha(0)$.

**Proposition** (Klein) [K–F, V.1 §9]. The functions $X_\alpha$ embed $E$ in $\mathbb{P}^{n-1}$. The ideal of $E$ is generated by the quadratic equations:

$$
F_{\alpha\beta\gamma\delta} = s_{\alpha-\beta}s_{\gamma-\delta}X_{\alpha+\beta}X_{\gamma+\delta} + s_{\alpha-\gamma}s_{\delta-\beta}X_{\alpha+\gamma}X_{\delta+\beta} + s_{\alpha-\delta}s_{\beta-\gamma}X_{\alpha+\delta}X_{\beta+\gamma},
$$

with the indices taking values in $\mathbb{Z}/n$.

These equations are a consequence of the $\sigma$-relation:

$$
\sigma(t + u)\sigma(t - u)\sigma(v + w)\sigma(v - w) + \sigma(t + v)\sigma(t - v)\sigma(w + u)\sigma(w - u)
$$

$$
+ \sigma(t + w)\sigma(t - w)\sigma(u + v)\sigma(u - v) = 0 \quad (1.)
$$
about which Weierstraß wrote: ‘Mann kann, ohne von der Function \( \sigma(u) \) irgend etwas zu wissen, direct nachweisen, dass es eine vier willkürliche Constanten enthaltende (transcendente) ganze Function der Veränderlichen \( u \) gibt, welche für \( \sigma(u) \) in die Gleichung (1.) eingesetzt, dieselbe befriedigt’ [We] (for a discussion of this functional equation see [Hu 2] or [Ha, I p. 187]).

Each function \( X_\alpha \) is defined as translate of the function \( \sigma(u; \omega_1, \omega_2/n) \) multiplied by a suitable exponential factor, which I describe below for completeness; the equation \( F_{\alpha\beta\gamma\delta} \) is obtained from (1.) by the specialisation \( t = \frac{1}{n}u + \frac{\omega_1}{n} \omega_1, \ldots, v = \frac{1}{n}u + \frac{\omega_1}{n} \omega_1 \). Each term requires the same exponential factor. To define the \( \tilde{X}_\alpha \) one starts with the functions:

\[
\sigma_{\lambda, \mu}(u; \omega_1, \omega_2) := e^{(\lambda n_1 + \mu n_2)(u - \frac{\omega_1 + \omega_2}{2})} \sigma(u - \lambda \omega_1 - \mu \omega_2; \omega_1, \omega_2),
\]

where the constants \( n_i \) are the periods of the integral of the second kind. They satisfy the Legendre relation \( \omega_1 \eta_2 - \omega_2 \eta_1 = 2\pi i \) (I follow Klein and Fricke and take the imaginary part of \( \omega_1/\omega_2 \) positive). If \( (\omega_1, \omega_2) = (\omega_1, \omega_2/n) \), and \( \eta_1, \eta_2 \) are the corresponding periods, then

\[
\frac{\eta_1}{\omega_1} - \frac{n \eta_1}{\omega_1} = \frac{\eta_2}{\omega_2} - \frac{n \eta_2}{\omega_2} =: G_1
\]

and \( X_\alpha \) is now defined as

\[
X_\alpha(u) = \rho (-1)^\alpha e^{-G_1 u^2} \sigma_{n, 0}(u; \omega_1, \frac{\omega_2}{n})
\]

with \( \rho \) a final term independent of \( \alpha \) to make the \( X_\alpha \) themselves modular forms with simple transformation behaviour. Note that the functions \( X_\alpha \) are defined for all \( \alpha \in \mathbb{Z} \), and satisfy \( X_{\alpha+n}(u) = X_\alpha(u) \).

(1.2) For even \( n \) similar results are obtained with functions \( X_\alpha(u) \) whose zeroes are not \( n \)-torsion points themselves, but shifted by \( u_0 = \frac{1}{2} \omega_1 + \frac{1}{2n} \omega_2 \). More precisely,

\[
X_\alpha(u) = \rho e^{-\frac{\pi i}{2}} \frac{\pi i}{2} e^{-G_1 u^2} \sigma_{\frac{\pi}{2}}(u; \omega_1, \frac{\omega_2}{n})
\]

The same \( \sigma \)-relation (1.) now gives equations involving constants \( \sigma_{n, 0}(0; \omega_1, \frac{\omega_2}{n}) \) [K–F, p. 268]; a small computation shows that they are proportional to \( e^{\alpha \pi i/n} X_\alpha(u_0) \). Therefore I define

\[
s_\alpha := e^{\alpha \pi i/n} X_\alpha(u_0).
\]

**Proposition.** For even \( n \) the ideal of \( E \) in \( \mathbb{P}^{n-1} \) is generated by the functions \( F_{\alpha\beta\gamma\delta} \), defined by the same formula as for odd \( n \), but with indices all in \( \mathbb{Z}/n \) or in \( \frac{1}{2} + \mathbb{Z}/n \).

(1.3) Different quartuples \( (\alpha, \beta, \gamma, \delta) \) lead to the same equation. To analyse the situation I use different indices and write

\[
F_{ijk}^h = s_{k+j}s_{k-j}X_{h+i}X_{h-i} - s_{k+i}s_{k-i}X_{h+j}X_{h-j} + s_{j+i}s_{j-i}X_{h+k}X_{h-k}.
\]

The choice \( h + i = \alpha + \beta \), etc., leads to the following equations (modulo \( n \)) determining the transition from one set of indices to another:

\[
\begin{align*}
\alpha + \beta + \gamma + \delta &= 2h & h + i + j + k &= 2\alpha \\
\alpha + \beta - \gamma - \delta &= 2i & h + i - j - k &= 2\beta \\
\alpha - \beta + \gamma - \delta &= 2j & h - i + j - k &= 2\gamma \\
\alpha - \beta - \gamma + \delta &= 2k & h - i - j + k &= 2\delta.
\end{align*}
\]
Putting \( h - i = \alpha + \beta \), etc., gives the solution \( (h - \alpha, h - \beta, h - \gamma, h - \delta) \) instead of \( (\alpha, \beta, \gamma, \delta) \). Other choices just lead to a permutation of the indices.

For odd \( n \) division by two with integer result is always possible, so it suffices to take \( h \in \mathbb{Z}/n \) and \( 0 \leq i < j < k < \frac{n}{2} \). For even \( n \) there are two types of equation, with \( 2h \) even or odd. In the latter case all indices \( h, i, j, k \) and \((h + \frac{1}{2}, \frac{3}{2} - i, \frac{1}{2} - j, \frac{3}{2} - k)\) give the same equation I can take \( 0 \leq h < \frac{n}{2} \) and \( 0 \leq i < j < k \leq \frac{n}{2} \).

(1.4) The system of equations \( F_{\alpha\beta\gamma\delta} \) admits a large symmetry group. The action of the involution of the elliptic curve and translation by points of order \( n \) is given by the following formulas:

i) \( X_\alpha(-u) = (-1)^n X_{-\alpha}(u) \)

ii) \( X_\alpha(u + \frac{\omega}{n}) = (-1)^n e^{n(u + \frac{\omega}{n})} X_{\alpha-1}(u) \)

iii) \( X_\alpha(u + \frac{\omega}{n}) = (-1)^n e^{n(u + \frac{\omega}{n})} \varepsilon^{\alpha} X_\alpha(u) \),

where \( \varepsilon = e^{2\pi i/n} \). These formulas describe the action of the Heisenberg group, see [Hul, I.2.4]. The exponential factor is the same for all \( X_\alpha \), so in the action on \( \mathbb{P}^{n-1} \) it drops out.

(1.5) To investigate the symmetries coming from the action of the modular group the dependence of the \( X_\alpha \) on \( \tau = \omega_1/\omega_2 \) has to be considered. For odd \( n \) the functions \( s_\alpha(\tau) := X_\alpha(0; \omega_1, \omega_2) \) are modular forms for the principal congruence subgroup \( \Gamma(n) \) [K-F, p. 280] and they embed the modular curve \( X(n) \) into \( \mathbb{P}^{\frac{n-1}{2}} \) (there are \( \frac{n-1}{2} \) essentially different non-zero \( s_\alpha \), as \( s_\alpha = -s_{-\alpha} = -s_{n-\alpha} \) by formula i) in (1.5)) and the equations \( F_{\alpha\beta\gamma\delta} \) describe the universal elliptic curve with level \( n \) structure [Ba]. For even \( n \) one has \( s_{\alpha+n} = -s_{\alpha} \), so in this case \( s_{n-\alpha} = s_\alpha \). The number of essentially different constants is \( n/2 \).

In either case relations between the \( s_\alpha \) are obtained by further specialising the equations \( F_{\alpha\beta\gamma\delta} \) or \( F_{ij}^h \), by putting \( u = 0 \) for \( n \) odd and \( u = u_0 \) for even \( n \); in the latter case one has to introduce an extra factor \((-1)\) if the index sum is bigger than \( n \) : \( X_i X_j \) specialises to \( s_i s_j \) if \( 0 \leq i + j < n \), but to \(-s_i s_j \) for \( n \leq i + j < 2n \).

**Lemma.** For \( n \geq 6 \) the following equations hold:

\[
s_{j-i} s_{i-k} s_{i+j} s_{k+l} - s_{k-i} s_{i-j} s_{i+k} s_{l+j} + s_{l-i} s_{k-j} s_{i+l} s_{j+k} = 0 ,
\]

with \( 0 \leq i < j < k < l \leq n/2 \).

If \( n = p \) is prime these equations define the modular curve \( X(p) \subset \mathbb{P}^{\frac{n-1}{2}} \) [Vé], but they do not necessarily generate the homogeneous ideal of the curve (see the example \( n = 11 \) below). For general odd \( n \) the curve \( X(n) \) is only an irreducible component of the zero locus.

The action of the modular substitutions \( S = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \) and \( T = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \) is [K-F, V.2 §7]:

\[
S : \quad X'_\alpha = \varepsilon^{-\alpha(n-\alpha)/2} X_\alpha
\]

\[
T : \quad X'_\alpha = \frac{i^{n-3}}{\sqrt{n}} \sum_{\beta=0}^{n-1} \varepsilon^{-\alpha\beta} X_\beta .
\]

For even \( n \) one has [K-F, V.2 §8]:

\[
S : \quad X'_\alpha = \varepsilon^{\frac{n}{4}} \frac{\omega}{n^2} X_\alpha
\]

\[
T : \quad X'_\alpha = \frac{1}{\sqrt{n}} \sum_{\beta=0}^{n-1} \varepsilon^{-\alpha\beta} X_\beta .
\]
The $X_\alpha$ are now modular forms of level $2n$ or $4n$; if $n$ is divisible by 4, then the appropriate congruence subgroup is
\[
\Gamma(n, 2n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid a \equiv d \equiv 1 \pmod{n}, \quad b \equiv c \equiv 0 \pmod{2n} \right\}.
\]
For $n$ the double of an odd integer one has the additional conditions
\[
a \equiv d \equiv 1 + kn, \quad b + c \equiv nk(k + 1) \pmod{4n}
\]
for the $X_\alpha$ to remain unchanged [K–F, p. 289], but they only exclude substitutions which multiply all $X_\alpha$ by $-1$.

**Proposition.** The elliptic curves $\mathbb{C}/\mathbb{Z} \omega_1 + \mathbb{Z} \omega_2$ and $\mathbb{C}/\mathbb{Z} \omega'_1 + \mathbb{Z} \omega'_2$ have the same image in $\mathbb{P}^{n-1}$ if and only if $\omega_1/\omega_2$ and $\omega'_1/\omega'_2$ are equivalent under $\Gamma(n)$.

**Proof.** The case $n$ odd is contained in the results of Bartsch [Ba] and Vélu [Vé], so it suffices to look at the action of representatives of $\Gamma(n)/\Gamma(n, 2n)$ for even $n$. To a common factor the substitution $S^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ transforms the $X_\alpha$ into $(-1)^{\alpha}X_\alpha$, whereas $\omega'_0 = \omega_0 + \frac{1}{2}\omega_2$, so also $s_\alpha$ is transformed into $(-1)^{\alpha}s_\alpha$. The equations $F_{i,j,k}^h$ are invariant: if all indices are integers, then $h - i \equiv h + i \pmod{2}$, whereas $h - i \not\equiv h + i \pmod{2}$ for elements of $1/2 + \mathbb{Z}$.

One computes that $\begin{pmatrix} 1 \\ 0^n \end{pmatrix}$ acts as $\mathbb{I}S^{-n}T^{-1}$ sends (up to a common factor) $X_\alpha$ to $X_\alpha - \frac{1}{2}$, whereas $\omega'_0 = \omega_0 + \frac{1}{2}\omega_1$. Therefore the ideal is again invariant and $s'_\alpha = s_\alpha$ as $X'_\alpha(u'_0) = X_\alpha - \frac{1}{2}(u_0 + \frac{1}{2}\omega_1) = X_\alpha(u_0)$ (up to a common factor).

The double cover of the modular curve $X(n)$ given by the involution $s_\alpha \mapsto (-1)^{\alpha}s_\alpha$ satisfies the equations of the Lemma. So the equations $F_{\alpha\beta\gamma\delta}$ describe a family of curves of genus one over $X(n)$.

(1.6) The equations are the $4 \times 4$ Pfaffians of the anti-symmetric matrix $P$ with entries $P_{\alpha\beta} = s_{\beta-\alpha}X_{\beta+\alpha}$. For odd $n$ one obtains all equations in this way, while for even $n$ a second matrix is necessary, say with $P'_{\alpha\beta} = s_{\beta-\alpha}X_{\beta+\alpha+1}$, to yield the equations with half-integer coefficients. This way of writing the equations entails relations with linear coefficients.

As the equations are not linearly independent (if $n > 5$) one has also relations whose coefficients depend only on the parameters $s_i$; they involve the equations between them. It is not difficult to write down a minimal set of generators of the ideal of the elliptic curve, but then the linear relations are much more complicated.

**Proposition.** The syzygies between the equations $F_{\alpha\beta\gamma\delta}$ of the elliptic curve $E$ are generated by the relations:
\[
R_{ij,k}^h : \quad -s_{k+i}s_{k-i}F_{ijl}^h + s_{l+i}s_{l-i}F_{ijk}^h + s_{j+i}s_{j-i}F_{ikl}^h = 0.
\]
and the linear relations:
\[
R_{\alpha\beta\gamma\delta\varepsilon} : \quad s_{\alpha-\beta}X_{\alpha+\beta}F_{\alpha\gamma\delta\varepsilon} + s_{\alpha-\gamma}X_{\alpha+\gamma}F_{\beta\alpha\delta\varepsilon} + s_{\alpha-\delta}X_{\alpha+\delta}F_{\beta\gamma\alpha\varepsilon} + s_{\alpha-\varepsilon}X_{\alpha+\varepsilon}F_{\beta\gamma\delta\alpha} = 0.
\]

The proposition follows from the analogous statement for the more general equations of the degeneration which is the subject of the next section.

(1.7) The curve $\mathcal{H}/\Gamma(n)$ can be compactified with $\frac{1}{2}n^2 \prod_{p|n}(1 - \frac{1}{p^2})$ cusps. Over a cusp the universal elliptic curve (for odd $n$) degenerates to an $n$-gon. Each subgroup $\mathbb{Z}/n \subset \mathbb{Z}/n \times \mathbb{Z}/n$ has $n$ invariant hyperplanes forming an $n$-simplex whose vertices are the fixed points of the action on $\mathbb{P}^{n-1}$. For the subgroup generated by $(1, 0)$ it is the simplex of reference, with vertices $e_\alpha$. By joining the vertices $e_\alpha$ and $e_{\alpha+m}$ by lines I obtain an $n$-gon if $(m, p) = 1$. This construction yields $\phi(n)/2$ different $n$-gons. Each of them is a generalised elliptic curve
whose origin is the intersection with the \((n-3)/2\)-dimensional subspace given by the equations \(X_\alpha + X_{n-\alpha} = 0\); the intersection point lies on the line joining \(e_{n-m}\) and \(e_{n+m}\) (consider the indices as elements of \(\mathbb{Z}/n\) or take \(m\) odd). Defining equations for the \(n\)-gon are

\[
F_{0j_{n-m}}^h = s_{n-m}^2 X_{h+j} X_{h-j}.
\]

In the same way \(n\)-simplices can be constructed for the other of the \(n!\prod_{\overline{p}} (1 + \frac{1}{p})\) cyclic subgroups of order \(n\) in \(\mathbb{Z}/n \times \mathbb{Z}/n\); they are permuted by the modular group.

If \((n, m) = d\) then the lines joining \(e_\alpha\) to \(e_{\alpha+m}\) in the standard simplex form \(d\) \(n/d\)-gons. The equations still reduce to \(F_{0j_{n-m}}^h = s_{n-m}^2 X_{h+3} X_{h+3j}\); if \(n/d = 3\) they define \(d\) planes (a plane triangle does not admit a quadratic equation), otherwise the zero locus consists \(d\) \(n/d\)-gons. It follows that the equations between the \(s_\alpha\) have solutions other than the curve \(X(n)\), if \(n\) is not a prime.

For an example of a degeneration with \(n\) even see the case \(n = 8\) below.

(1.8) Example: \(n = 7\). I write the equations with the \(\sigma\)-constants \(s_1, s_2\) and \(s_4\); for every prime \(p = 4k - 1\) it is convenient to use the quadratic residues as indices. The modular curve is the Klein quartic

\[
s_1s_4^3 + s_2s_1^3 + s_4s_2^3.
\]

The elliptic curves are defined by 14 linearly independent equations, out of 28 quadrics \(F_{ijk}^h\), of which I write only the ones with \(h = 0\); the others can be obtained by cyclic permutation of the \(X_\alpha\).

\[
s_1s_4X_0^2 + s_2^2X_1X_6 - s_1^2X_2X_5
\]
\[
s_1s_2X_0^2 + s_4^2X_2X_5 - s_2^2X_4X_4
\]
\[
s_2s_4X_0^2 + s_1^2X_3X_4 - s_4^2X_1X_6
\]
\[
s_1s_4X_3X_4 + s_2s_4X_2X_5 + s_1s_2X_1X_6
\]

The modular substitution \(U\) with \(\omega_1' = 4\omega_1, \omega_2' = 2\omega_2/4 \pmod{7}\) operates by \(X'_\alpha = X_{4\alpha}\) (see [K–F, p. 302]), so it acts by cyclic permutation of the indices \((1, 2, 4)\) and \((6, 5, 3)\); it permutes the first three equations, whereas the fourth is invariant. This reduces the number of essentially different equations to two. Although all equations are specified by only one formula \(F_{ijkl}^h\), there is no notable action of the available symmetry groups which permutes the remaining two. However under the action of \(T\) one single equation is transformed into a linear combination of all 28 equations.

(1.9) Example: \(n = 9\). Again I write only equations of the form \(F_{ijk}^0\). An appropriate modular substitution \(U\) now permutes the indices \((1, 4, 7)\) and their complements \((8, 5, 2)\) leaving \((0, 3, 9)\) invariant. The ten equations split into four types:

\[
s_1^2X_2X_7 - s_2^2X_1X_8 + s_1s_3X_0^2
\]
\[
s_2^2X_3X_6 - s_3^2X_1X_8 - s_4s_7X_0^2
\]
\[
s_1s_3X_3X_6 + s_4s_7X_2X_7 - s_1s_4X_1X_8
\]
\[
s_1s_3X_4X_5 + s_3s_4X_2X_7 + s_3s_7X_1X_8.
\]

Notice that the last equation \((F_{124}^0)\) can be divided by \(s_3\), to give \(s_1X_4X_5 + s_4X_2X_7 + s_7X_1X_8\). As equations between the coefficients I get:

\[
s_3s_1 + s_7s_4 - s_1s_4
\]
\[
s_3s_4 + s_1s_7 - s_4s_7
\]
\[
s_3s_7 + s_1s_1 - s_7s_1
\]
\[
(s_1s_7 + s_4s_1 + s_7s_4)s_3.
\]
In this case the equations do not generate the ideal of the modular curve, which is a curve of degree nine given by: \( s_1^2 s_7 + s_4^2 s_1 + s_2 s_7^2 + s_4 s_7^2 + s_7 s_3^2 - s_3^3 = 0 \). The four equations above have also four isolated points as solution: \((0 : 1 : 0 : 0)\) and three points \((1 : 0 : \omega : \omega^2)\) with \( \omega^3 = 1 \). For these points the equations in the \( X_\alpha \) do not define a curve.

There are 36 cusps. The relevant subgroups of \( \mathbb{Z}/n \times \mathbb{Z}/n \) are the ones generated by \((1, l)\), \(0 \leq l < 9\), \((0, 1)\), \((3, 1)\) and \((6, 1)\). The nine cusps of the last three subgroups satisfy \( s_3 = 0 \), so they are the intersection points of the plane cubics \( s_1^2 s_7 + s_4^2 s_1 + s_2 s_7^2 + s_4 s_7^2 + s_7 s_3^2 \). The other 27 cusps lie above the intersection of \( s_1^2 s_7 + s_4^2 s_1 + s_2 s_7^2 + s_4 s_7^2 + s_7 s_3^2 \) and its Hessian \( s_1^2 + s_4^2 + s_2^2 - 3 s_1 s_4 s_7 \).

**Example**: \( n = 11 \). I write only the equations for the modular curve \( X(11) \) using the variables \( s_1, s_3, s_9, s_5 \) and \( s_4 \) (in this order) [K–F, V.5 §2]. The automorphism group of the curve is a group \( G_{660} \) of order 660; I single out the transformations \( U \) which is congruent to \((3 \ 0 \ 4)\) acting by cyclic permutation of the coordinates (namely by \( s_\alpha \mapsto s_{3\alpha} \)), and \( S \) which multiplies \( s_\alpha \) by \( \varepsilon = e^{2\pi i/11} \).

Specialising the equations \( F_{\alpha\beta\gamma\delta} \) yields ten equations in different eigenspaces for the induced action of \( S \). They split into two groups of five, which are permuted among each other by the action of \( U \). Therefore it suffices to write down the following two:

\[
\begin{align*}
& s_1 s_9^3 + s_9 s_5^3 + s_3 s_4^3 \\
& s_1^2 s_9 s_5 - s_3 s_9 s_5 + s_1 s_3 s_4^2 .
\end{align*}
\]

The equations do not generate the homogeneous ideal of the curve, as there is a zero dimensional embedded component. By subtracting the second equation multiplied with \( s_4 \) from \( s_1 \) times the first one obtains an expression which is divisible by \( s_9 \). In total one gets five extra equations of the type:

\[ s_3^2 s_5^2 + s_3 s_4^3 - s_1 s_3^2 s_4 + s_1 s_9 s_5 s_4 . \]

Felix Klein’s original way to obtain the equations starts from the invariant

\[ \Phi = s_1^2 s_3 + s_3^2 s_9 + s_5^2 s_5 + s_5^2 s_4 + s_4^2 s_1 \]

of the group \( G_{660} \). Its Hessian is a quintic threefold with the modular curve as double locus, and the fifteen \( 4 \times 4 \) minors give the equations of this curve.

**Example**: \( n = 4 \). There are two equations:

\[
\begin{align*}
& F_{012}^0 : \quad s_1^2 X_0^2 - s_2^2 X_1 X_3 + s_1^2 X_2^2 \\
& F_{012}^1 : \quad s_1^2 X_1^2 - s_2^2 X_0 X_2 + s_1^2 X_3^2 .
\end{align*}
\]

The second equation is \( F_{0123} \); to form the first one needs non-integral indices. Note that \( X_0(u_0) = 0 \), \( X_1^2(u_0) = - s_1^2 \), \( X_2^2(u_0) = - s_2^2 \) and \( X_3^2(u_0) = s_1^2 \).

**Example**: \( n = 6 \). The equations \( F_{ijk}^0 \) are

\[
\begin{align*}
& s_1 s_3 X_0^2 - s_2^2 X_1 X_5 + s_1^2 X_2 X_4 \\
& s_2 X_0^2 - s_3^2 X_1 X_5 + s_1^2 X_3^2 \\
& s_3 X_0^2 - s_3^2 X_2 X_4 + s_2^2 X_3^2 \\
& s_1^2 X_1 X_5 - s_2^2 X_2 X_4 + s_1 s_3 X_3^2 .
\end{align*}
\]

For \( 2h \) odd there is only one equation. For \( h = \frac{1}{2} \) one has

\[ s_1 s_2 X_0 X_1 - s_2 s_3 X_2 X_5 + s_1 s_2 X_3 X_4 , \]
which can be divided by \( s_2 \) to give:

\[
s_1X_0X_1 - s_3X_2X_5 + s_1X_3X_4.
\]

The relation between the coefficients \( s_\alpha \) is:

\[
s_1^4 - s_2^4 + s_1s_3^3.
\]

Remark that only even powers of \( s_2 \) occur (after the division).

**Example**: \( n = 8 \). Even as polynomials in the \( X_\alpha \) and \( s_\alpha \) the ten equations \( F_{ijk}^0 \) are not linearly independent. The substitution \((h; i, j, k) \mapsto (h + 4; i, j, k)\) increases all indices \( \alpha \) of the \( X_\alpha \) by 4; it sends e.g. \( F_{012}^0 \) to \( F_{234}^0 \). The automorphism \((s_1, s_2, s_3, s_4) \mapsto (-s_3, -s_2, s_1, s_4)\) of order four of the base curve can be extended to the total space by \((X_0, X_1, X_2, X_3, X_4, X_5, X_6, X_7) \mapsto (X_0, X_5, X_2, X_7, X_4, X_1, X_6, X_3)\). The different types of equations are:

\[
\begin{align*}
& s_1s_3X_0^2 - s_2^2X_1X_7 + s_1^2X_2X_6 \\
& s_2s_4X_0^2 - s_3^2X_1X_7 + s_1^2X_3X_5 \\
& s_3^2X_0^2 - s_2^2X_1X_7 + s_1^2X_4^2 \\
& s_2^2X_0^2 - s_3^2X_2X_6 + s_1^2X_4^2 \\
& s_1s_3X_1X_7 - s_2s_4X_2X_6 + s_1s_3X_3X_5.
\end{align*}
\]

With \( 2h = 1 \) one obtains (out of four):

\[
\begin{align*}
& s_1s_4X_0X_1 - s_2s_3X_2X_7 + s_1s_2X_3X_6 \\
& s_2s_3X_0X_1 - s_3s_4X_2X_7 + s_1s_2X_4X_5.
\end{align*}
\]

The curve defined by the relations between the coefficients \( s_\alpha \) is determinantal, given by the minors of the matrix:

\[
\begin{pmatrix}
  s_3^2 - s_1^2 & s_2s_4 & s_1s_3 \\
  s_2^2 & s_1^2 + s_3^2 & s_2^2
\end{pmatrix}.
\]

It has four ordinary double points, at the coordinate vertices \( e_2 \) and \( e_4 \), and at \( s_2 = s_4 = s_1^2 + s_3^2 = 0 \). In particular, if \( s_4 = 1 \) then \( s_2 = s_3^2 - s_1^4 \) and \( s_1s_3 - (s_3^2 - s_1^4)^2(s_2^2 - s_1^2) = 0 \). The curve over the branch with tangent \( s_1 = 0 \) specialises for \( s_3 \rightarrow 0 \) to the 8-gon joining the vertices in order, while the other branch gives the one joining \( e_i \) to \( e_{i+3} \).

**2. The degeneration**

(2.1) I concentrate on the case \( n \) odd because the relation to the universal curve is then more direct, but the final formulas will also hold for even \( n \).

The \( n \)-gon with equations \( X_\alpha X_\beta = 0 \) for \(|\alpha - \beta| > 1\) is the fibre of the universal family over the cusp \( P \) with coordinates \( s_2^{-1} = 1 \), \( s_\alpha = 0 \) for \( 1 \leq \alpha < (n - 1)/2 \); it corresponds to \( \tau = i\infty \). In [D-R, VII.4] a Tate curve with \( n \) sides is constructed; it is a generalised elliptic curve with level \( n \) structure, defined over the unit disc \( D \subset \mathbb{C} \) with coordinate \( q^{1/2} \), and it provides an isomorphism between the germs \((D, 0)\) and \((X(n), P)\). A modification of this construction leads to the versal deformation of the \( n \)-gon as abstract curve of genus one, i.e., disregarding the group structure. As to infinitesimal deformations, for an \( n \)-gon \( Y \) one computes that \( \dim T^1_Y = n \).
Construction. Let S be a smooth analytic germ and consider n sections $t_0, \ldots, t_{n-1} \in \Gamma(S, \mathcal{O}_S)$. Define $t_i$ for all $i \in \mathbb{Z}$ by $t_{i+n} = t_i$. Set $T := \prod_{i=0}^{n-1} t_i$. Construct a space $\overline{Y}$ from charts $(U_i)_{i \in \mathbb{Z}}$ with $U_i \subset S \times \mathbb{C}^2$ given by $x_i y_i - t_i = 0$; here $\mathbb{C}^2$ is a copy of $\mathbb{C}$ with coordinates $(x_i, y_i)$. The gluing from $U_i$ to $U_{i+1}$ is determined by $x_i y_i = 1$. This makes $V_i := U_i \cap U_{i-1}$ into $S \times \mathbb{C}^*$ with coordinate ring $\mathcal{O}_S\{x_i, x_i^{-1}\} = \mathcal{O}_S\{y_i, y_i^{-1}\}$, as one has $y_i = t_i x_i^{-1}$ and $x_{i-1} = t_{i-1} y_{i-1}^{-1}$. Outside $\prod t_i = 0$ all $U_i$ are identified and one has $x_1 = t_0^{-1} x_0$, $x_2 = (t_0 t_1)^{-1} x_0$, etc.

Let $g$ be the section of $V_n$ with $x_n(g) = 1$, so $x_0(g) = T$. Multiplication by $g$ is a well-defined operation on the union $Y$ of all $V_i$, and extends to $\overline{Y}$: it is given by $g: V_i \to V_{i+n}$, $x_i + n(a) = x_i(a)$ for every section of $Y \to S$.

The quotient $\overline{Y}/g^\infty$ is a family of curves of genus one.

(2.2) If all $t_i$ are equal to the same section $t$, then a group structure can be introduced on $Y$ by putting $V_i \times V_j \to V_{i+j}$; $x_i y_j(ab) = x_i(a) x_j(b)$. It extends to an action of $Y$ on $\overline{Y}$. Let $g$ be a section of $Y$ contained in a $V_n$ with $n \neq 0$. Multiplication by any section $g$ of $Y$, contained in $V_n$ with $n \neq 0$, defines a $\mathbb{Z}$-action on $\overline{Y}$, the quotient $\overline{Y}/g^\infty$ is a Tate curve. In particular, if $t = 0$ and $g$ is the section $x_n = 1$ of $V_n$, then $\overline{Y}/g^\infty$ is the standard $n$-gon over $S$.

For $S$ the unit disc $D \subset \mathbb{C}$ with coordinate $q^{1/2}$, section $t = q^{1/2}$ and section $g$ of $V_n$ with $x_n(g) = 1$ (so $x_0(g) = g$), denoted by $g$ again, the quotient $\overline{Y}/g^\infty$ is the Tate curve with $n$ sides. By the modular interpretation the $s_n$ can be written as functions of the variable $q^{1/2}$. An explicit $q$-expansion can be derived from formula (10.6) in [Vé], or from formula (3) p. 281 in [K–F]. Here I only note that $s_n$ is in first approximation proportional to $(-1)^{-\alpha} (q^{1/2})^{-\frac{n(\alpha+1)}{2}}$.

(2.3) Theorem. The versal deformation of the $n$-gon has a smooth base space with deformation parameters $t_i$; it is given by the equations

$$F_{ijk}^h = s_{k+j}s_{k-j} \left( \prod_{m=i}^{j-1} T_{h-m}^{i-j} \right) X_{h+i} X_{h-i} - s_{k+i}s_{k-i} X_{h+j} X_{h-j}$$

$$+ s_{j+i}s_{j-i} \left( \prod_{m=j+1}^{k} T_{h+m}^{i-j} \right) X_{h+k} X_{h-k}.$$

Here $T_i = \prod_{k=i}^{n} t_k$, and $T := \prod_{k=0}^{n-1} t_k$. Furthermore $h \in \mathbb{Z}/n\mathbb{Z}$ for odd $n$, and $h \in \mathbb{Z}/2\mathbb{Z}/n\mathbb{Z}$ for even $n$, and $0 \leq i < j < k \leq n/2$; for even $n$ the numbers $i$, $j$ and $k$ are integral if and only if $h$ is integral. The equations $F_{ijk}^h$ form a $\mathbb{A}$ minimal set of generators of the ideal. The coefficients $s_n$ in the equations are holomorphic functions of $T$ with $s_n(0) = (-1)^{n(\alpha+1)}$ and they satisfy:

$$-s_{k+i}s_{k-i} X_{k-j} + s_{j+i}s_{j-i} X_{j+k} X_{i-k} T^{k-j} = 0,$$

for $0 \leq i < j < k < l \leq n/2$.

Proof. I will not express the functions $X_{\alpha}$ in terms of the coordinates on the space $Y$, so the proof will be indirect: I shall establish that the given equations define a deformation of the $n$-gon by studying the syzygies.

As an explanation of the equations I offer the circumstance that for fixed $T \neq 0$ they are really the same as Klein and Fricke’s equations: I describe an explicit coordinate transformation. For odd $n$ it involves the $n$th root of each $t_i$, whereas for even $n$ one needs the $2n$th root. I write $t_i = \tau_i^n$ for all $n$, so for even $n$ there is the additional indeterminacy of a square root. Consider the transformation which replaces $X_i$ by $\prod_{j} \tau_{i}^{a_{ij}} X_i$, and $s_i$ by $\prod_{j} \tau_{j}^{b_{ij}} s_i$ with
\[ a_{ij} = (j - i)(n - j + i)/2 \] for \( i \leq j \leq i + n \); the exponents \( a_{ij} \) are solutions to the equations:

\[
2a_{ij} = a_{i,j-1} + a_{i,j+1} + 1, \quad j \neq i,
\]
\[
n = a_{i,i-1} + a_{i,i+1} + 1,
\]
\[
a_{i,i-j} = a_{i,i+j},
\]
\[
a_{ii} = 0.
\]

The \( b_i \) satisfy similar equations: \( b_i = (i-1)(n-i-1)/2 \) for \( 0 < i < n \). Note that the exponent of \( q^{h} \) in the \( q \)-expansion of \( s_i \) given above is \(-i(n-i)/2\), which differs from \(-b_i\) by \((n-1)/2\), independently of \( i \).

This coordinate change transforms the equations to multiples of the elliptic curve equations. I check here only the equations for the parameters. The exponent of \( \prod \tau_i \) as coefficient of the monomial \( s_{k+l} s_{k-l} s_t s_{l-j} \) is:

\[
\frac{(k+i-1)(n-i-k-1)}{2} + \frac{(k-i-1)(n-k+i-1)}{2} + \frac{(l+i-1)(n-i-l-1)}{2} + \frac{(l-i-1)(n-l+i-1)}{2} = (k + l - 2)n - k^2 - i^2 - l^2 - j^2 + 2,
\]

which by symmetry is the same as for \( s_{l+i} s_{l-i} s_{k+j} s_{k-j} \); due to the extra term \( T^{k-j} \) the third summand gives:

\[
((j + l - 2)n - j^2 - l^2 - k^2 + 2) + n(k-j) = (k + l - 2)n - k^2 - i^2 - l^2 - j^2 + 2.
\]

Note that for even \( n \) this is an integral power of \( \prod \tau_i \).

The \( q \)-expansion mentioned above shows that solutions to the \( s_i \) equations exist, which are power series in \( T \).

**Lemma 1.** Let \( 0 \leq i < j < k < l \leq n/2 \). The following relations hold:

\[
R^{h;i}_{jkl} : s_{l+i} s_{l-i} F^{h}_{ijk} = s_{k+i} s_{k-i} F^{h}_{ijkl} + s_{j+i} s_{j-i} \left( \prod_{m=j+1}^{k} T_{h+m}^{h-m} \right) F^{h}_{ikl} = 0
\]

\[
R^{h;j}_{ikl} : s_{l+j} s_{l-j} F^{h}_{ijkl} = s_{k+j} s_{k-j} F^{h}_{ijkl} + s_{j+i} s_{j-i} \left( \prod_{m=j+1}^{k} T_{h+m}^{h-m} \right) F^{h}_{jkl} = 0
\]

\[
R^{h;k}_{ijl} : s_{l+k} s_{l-k} \left( \prod_{m=j}^{k-1} T_{h+m}^{h-m} \right) F^{h}_{ijkl} = s_{k+j} s_{k-j} F^{h}_{ikl} + s_{k+i} s_{k-i} F^{h}_{jkl} = 0
\]

\[
R^{h;l}_{ijk} : s_{l+k} s_{l-k} \left( \prod_{m=j}^{k-1} T_{h+m}^{h-m} \right) F^{h}_{ijkl} = s_{l+j} s_{l-j} F^{h}_{ikl} + s_{l+i} s_{l-i} F^{h}_{jkl} = 0.
\]

The proof consists of a simple verification. The assumption \( 0 \leq i < j < k < l \leq n/2 \) and the division into four cases is needed to be able to specify the terms involving \( \tau_i \).

The four-term relations are best formulated in terms of equations of the form \( F_{\alpha \beta \gamma \delta} \). The precise form of the equation \( F_{\alpha \beta \gamma \delta} \) depends on the order of the six points \( \alpha + \beta, \gamma + \delta, \alpha + \gamma, \delta + \beta, \alpha + \delta, \beta + \gamma \). If \( \alpha < \beta < \gamma < \delta \), then \( X_{\alpha+\gamma} X_{\beta+\delta} \) is the 'middle monomial', the one occurring without \( t_i \)-term, as \( \alpha + \beta \) and \( \gamma + \delta \) lie between \( \beta + \delta \) and \( \alpha + \gamma \), whereas \( \alpha + \delta \) and \( \beta + \gamma \) lie on the other side. As I really work modulo \( n \) I can always arrange for a given index to be the smallest.
Lemma 4. Suppose $0 \leq \alpha < \beta < \gamma < \delta < \varepsilon$. Let $D_2$ be the product of factors $T_i^j$, occurring in $F_{\beta \gamma \alpha}$ as coefficient of $X_{\beta + \alpha}X_{\gamma + \varepsilon}$, and $C_2$ that of $X_{\alpha + \varepsilon}X_{\beta + \delta}$ in $F_{\beta \alpha \delta \varepsilon}$. Then

$$D_2 s_{\alpha - \beta} X_{\alpha + \beta} F_{\alpha \gamma \delta \varepsilon} + s_{\alpha - \gamma} X_{\alpha + \gamma} F_{\beta \alpha \delta \varepsilon} + s_{\alpha - \delta} X_{\alpha + \delta} F_{\beta \gamma \alpha \varepsilon} + C_2 s_{\alpha - \varepsilon} X_{\alpha + \varepsilon} F_{\beta \gamma \delta \alpha} = 0.$$  

Proof. Write $B_i, \ldots, E_2$ for the product of the factors $T_i^j$ occurring in the equations involved. The same substitution as before shows the existence of a relation of the form

$$A_1 s_{\gamma - \delta} X_{\delta + \varepsilon} + A_2 s_{\alpha - \gamma} X_{\alpha + \gamma} F_{\beta \alpha \delta \varepsilon} + A_3 s_{\alpha - \delta} X_{\alpha + \delta} F_{\beta \gamma \alpha \varepsilon} + A_4 s_{\alpha - \varepsilon} X_{\alpha + \varepsilon} F_{\beta \gamma \delta \alpha} = 0.$$  

Lemma 2. Suppose $0 \leq \alpha < \beta < \gamma < \delta < \varepsilon$. Let $D_2$ be the product of factors $T_i^j$, occurring in $F_{\beta \gamma \alpha}$ as coefficient of $X_{\beta + \alpha}X_{\gamma + \varepsilon}$, and $C_2$ that of $X_{\alpha + \varepsilon}X_{\beta + \delta}$ in $F_{\beta \alpha \delta \varepsilon}$. Then

$$D_2 s_{\alpha - \beta} X_{\alpha + \beta} F_{\alpha \gamma \delta \varepsilon} + s_{\alpha - \gamma} X_{\alpha + \gamma} F_{\beta \alpha \delta \varepsilon} + s_{\alpha - \delta} X_{\alpha + \delta} F_{\beta \gamma \alpha \varepsilon} + C_2 s_{\alpha - \varepsilon} X_{\alpha + \varepsilon} F_{\beta \gamma \delta \alpha} = 0.$$  

Proof. Write $B_i, \ldots, E_2$ for the product of the factors $T_i^j$ occurring in the equations involved. The same substitution as before shows the existence of a relation of the form

$$A_1 s_{\gamma - \delta} X_{\delta + \varepsilon} + A_2 s_{\alpha - \gamma} X_{\alpha + \gamma} F_{\beta \alpha \delta \varepsilon} + A_3 s_{\alpha - \delta} X_{\alpha + \delta} F_{\beta \gamma \alpha \varepsilon} + A_4 s_{\alpha - \varepsilon} X_{\alpha + \varepsilon} F_{\beta \gamma \delta \alpha} = 0.$$  

Lemma 3. The equations $F_{ijk}^h$ generate the ideal of the total space of the degeneration, while the relations of Lemma 1 and Lemma 2 generate the syzygies between them.

Proof. The relations $F_{ijk}^{h_{ij}}$ and $F_{ijl}^{h_{ij}}$ show that all equations $F_{ijk}^h$ can be expressed terms of the equations $F_{ijk}^{h_{ij}}$ with nonzero coefficient, whereas for even $n$ one has $(n - 2)/2$ equations with $2h$ even and $(n - 4)/2$ with $2h$ odd. If $t_i = 0$ for all $i$, these equations generate the ideal of the $n$-gon.

The equations $F_{j-1,j,j+1}^h$ are linearly independent; for fixed $h$ one can use induction on $j$: the monomial $X_{h-j+1}X_{h-j}$ in the equation $F_{j-1,j,j+1}^h$ with nonzero coefficient, as $T \neq 0$; note that if $T = 0$, then from some $j_0$ onwards the smallest monomial occurring in $F_{j-1,j,j+1}^h$ is $X_{h-j}X_{h-j}$. As to the relations is suffices to lift the generators of the syzygy module for the $n$-gon, which are of the form $(X_{i}X_{j})X_{k} - (X_{i}X_{k})X_{j} = 0$. I write the formula for $(i,j,k) = (2\gamma, \gamma + \beta, \gamma + \alpha)$. To be specific I suppose that $\alpha < \beta < \gamma$. The two equations to consider are

$$F_{\alpha,\gamma,\gamma - 1,\gamma + 1} = s_{\alpha - \gamma + 1} s_{\gamma + 2} T_{\gamma + 1}^{\alpha + \gamma - 1} X_{\alpha + \gamma - 1} X_{\gamma + 1} - s_{\alpha - \gamma} s_{\gamma + 1} X_{\alpha + \gamma} X_{\gamma} + s_{\alpha - \gamma - 1} s_{\gamma + 2} T_{\gamma + 1}^{\alpha + \gamma - 1} X_{\alpha + \gamma + 1} X_{\gamma + 1}$$

and

$$F_{\gamma,\beta,\gamma - 1,\gamma + 1} = s_{\gamma - \beta + 1} s_{\beta + 2} T_{\beta + 1}^{\gamma - 1} X_{\beta + \gamma - 1} X_{\gamma + 1} - s_{\gamma - \beta} s_{\gamma + 1} X_{\beta + \gamma} X_{\gamma} + s_{\gamma - \beta - 1} s_{\beta + 2} T_{\gamma + 1}^{\gamma - 1} X_{\beta + \gamma + 1} X_{\gamma + 1} .$$

They occur in the relation

$$s_{\gamma - \alpha} X_{\gamma + \alpha} F_{\gamma,\beta,\gamma - 1,\gamma + 1} + s_{\gamma - \beta} X_{\gamma + \beta} F_{\alpha,\gamma,\gamma - 1,\gamma + 1} + s_{\gamma - \beta} X_{\gamma + \beta} F_{\alpha,\gamma,\gamma - 1,\gamma + 1} + s_{\gamma + 1} X_{\gamma + 1} T_{\beta + 1}^{\gamma - 1} F_{\alpha,\beta,\gamma,\gamma + 1} + s_{\gamma + 1} X_{\gamma + 1} T_{\gamma + 1}^{\gamma - 1} F_{\alpha,\gamma,\gamma - 1,\gamma + 1} = 0 .$$

Lemma 4. The deformation in the $t_{\alpha}$-direction smooths the double point of the $n$-gon at the vertex $e_\alpha$ of the coordinate simplex, whereas the curve has a node at $e_\alpha$ if $t_\alpha = 0$.

Proof. Consider the standard affine charts in $\mathbb{P}^{n-1}$. It suffices to look at $X_0 = 1$. Let $x_\alpha = X_\alpha/X_0$. All coordinates $x_\alpha$ except $x_1$ and $x_{n-1}$ can be eliminated successively. I start with the equation

$$F_{012}^{-1} = s_2^2 x_{n-2} = s_3 s_1 t_{n-1} x_{n-1}^2 + s_1^2 T_{1}^{n-3} x_1 x_{n-3} .$$

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Using it the equation $F_{\frac{3}{4}}^{\frac{2}{2}}$ gives

$$(s_3 s_2 - s_4 s_1^2 s_2^{-2} t_1^{n-1} x_1) x_{n-3} = s_4 s_3^2 s_2^{-2} t_{n-2} t_{n-1}^2 x_{n-1}^3 + s_2 s_1 T_1^{n-4} x_1 x_{n-4}.$$ 

In the neighbourhood of the origin the coefficient of $x_{n-3}$ is a unit so the equation can be used to eliminate $x_{n-3}$. By induction

$$x_{n-i} = (\text{unit}) \cdot t_{n-i+1} t_{n-i+2} \cdots t_{n-1} x_{n-1} + (\text{unit}) \cdot T_1^{n-i-1} x_1 x_{n-i-1}.$$ 

In particular,

$$x_2 = (\text{unit}) \cdot t_3 t_4 \cdots t_{n-2} x_{n-1} + (\text{unit}) \cdot t_1 x_1^2.$$ 

Going back one finds

$$x_i = (\text{unit}) \cdot t_{i+1} t_{i+2} \cdots t_{n-1} x_{n-1} + (\text{unit}) \cdot t_{i-1} t_{i-2} \cdots t_{1} x_{i}.$$ 

Finally the equation $F_{012}^0$ gives

$$(\text{unit}) \cdot x_1 x_{n-1} - (\text{unit}) \cdot t_{n-2} t_{n-3} \cdots t_1 x_{n-1}^2 x_{n-1}^n - (\text{unit}) \cdot t_2 t_3 \cdots t_{n-1} x_{n-1} = t_0,$$

which proves the claims. \(\square\)

**Example:** \(n = 6\). I write two of the equations $F_{ijk}^0$ (the other two are obtained by replacing $X_\alpha$, $t_\alpha$ by $X_{\alpha+3}$ and $t_{\alpha+3}$):

\[
s_1 s_2 t_0 X_0^2 - s_2^2 X_1 X_5 + s_1^2 t_2 t_3 t_4 X_2 X_4
\]

\[
s_2^2 t_0 X_0^2 - s_2^2 X_1 X_5 + s_1^2 t_2 t_3 t_4 X_2 X_4
\]

Furthermore $F_{\frac{4}{2}}^{\frac{2}{2}} = s_2 (s_1 t_0 t_1 X_0 X_1 - s_3 X_2 X_5 + s_1 t_3 t_4 X_3 X_4)$. The relation between the coefficients is $s_1 t_1^2 - s_3^2 + s_1 s_3^3$. A solution is $(s_1, s_2, s_3) = (1, 1 + T, 1)$. Ruud Pellikkaan obtained this formula (in the cusp case, with $t_i = X_i^{-2}$) by direct computations with syzygies.

**Example:** \(n = 9\). The equations on the coefficients become:

\[
s_3^3 s_4 T - s_1 s_3^3 - s_7^3 s_4
\]

\[
s_3^3 s_7 T^2 + s_3^3 s_4 - s_7^3 s_7
\]

\[
s_3^3 s_7 T - s_3^3 s_7 - s_3^3 s_1
\]

\[
s_3^3 s_7 T + s_3^3 s_4 + s_3^3 s_3
\]

As the $s_\alpha$ are nonzero for small $T$ the equations reduce to: $s_3^3 s_7 T + s_3^3 s_1 + s_3^3 s_4$ and $s_1 s_3^2 T + s_4 s_7^2 + s_3^3 s_3 - s_3^3$. As solution in closed form one can take $s_1 = s_7 = 1, s_4 = -\frac{1}{2}(1 + \sqrt{1 - 4T}),$ and $s_3 = (\frac{1}{2}(1 - T)(1 + \sqrt{1 - 4T}))^{1/3}$.

3. Cusp singularities

**Example:** \(n = 9\). A cusp singularity is a normal isolated two-dimensional singularity, whose minimal resolution has a cycle of rational curves as exceptional divisor. The analytical type of the singularity is completely determined by the self-intersections $-b_i$ of these curves. If the exceptional divisor is irreducible, it is a nodal curve.

The name cusp comes from the connection with Hilbert modular forms; the standard reference is [Hi]; see also [Be, Ge]. Let $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field. Two lattices
$M_1$, $M_2$ in $K$ are strictly equivalent if $M_2 = \lambda M_1$ for some totally positive $\lambda \in K$ (i.e., both $\lambda$ and its conjugate $\lambda'$ are positive). A lattice $M$ is always strictly equivalent to one of the form $\mathbb{Z} \oplus \mathbb{Z} w$ with $0 < w' < 1 < w$. Let $U^+_M$ be the group of totally positive elements $\varepsilon$ of $K$, satisfying $\varepsilon M = M$, and let $V$ be a subgroup of finite index $s$ in $U^+_M$. The semidirect product $G(M, V) = M \rtimes V$ acts freely and properly discontinuously on the product of upper half spaces $\mathfrak{H}^2$. The quotient $G(M, V) \setminus \mathfrak{H}^2$ can be compactified by one cusp $\infty$ to a normal complex space $X(M, V)$. The number $w$ has a periodic continued fraction expansion $w = [b_1, \ldots, b_i]$ with primitive period of length $r/s$. The numbers $b_i$ are the same as before, minus the self-intersection of the exceptional curves on the minimal resolution $X$, except in the case $r = 1$; then $-b_1$ is the degree of the normal bundle of the composed map $\mathbb{P}^1 \to E_1 \subset X$.

The resolution can be constructed by toroidal methods. To this end one considers the embedding of $M_+$, the totally positive elements of $M$, in $\mathbb{R}^2_+$, given by the two embeddings of $K$ into $\mathbb{R}$. The lattice points on the boundary of the convex hull of $M_+$ define a fan and with it one constructs the manifold $Y$ containing an infinite chain of rational curves, which comes with an embedding of $Y := M \setminus \mathfrak{H}^2$, and an action of $V$. The orbit space $Y/V$ is the minimal resolution of $X(M, V)$. This construction is similar to that of the Tate curve in the previous section.

The dual lattice $M^* = \{ \mu \in K \mid \text{Tr}(\alpha \mu) \in \mathbb{Z} \text{ for all } \alpha \in M \}$ is strictly equivalent to $M = \mathbb{Z} \oplus \mathbb{Z} w^*$ with $w^* = (2 - w' / (1 - w'))$; here $\text{Tr}(\alpha) = \alpha + \alpha'$. The continued fraction $w = [a_1, \ldots, a_n]$ (again with the primitive period repeated $s$ times) is the dual of $[b_1, \ldots, b_i]$ in reverse order: the sequence $(b_1, \ldots, b_i)$ starts with $a_n - 3$ twos.

The local ring of the cusp consists of certain Fourier series, of the form

$$f(z) = \sum_{\mu \in M^* \cup \{0\}} a_{\mu} \exp 2\pi i (\mu z_1 + \mu' z_2),$$

with $a_{\mu} = a_{\mu'}$ for all $\varepsilon \in V$. Abbreviate $e(\mu z) = \exp 2\pi i (\mu z_1 + \mu' z_2)$. Consider the embedding of $M^*$ in $\mathbb{R}^2$ defined by $i(\mu) = (\mu, \mu')$. Let $i(A_i)$ be the lattice points on the boundary of the convex hull of $i(M^*_+)$.

Put $A_0 = 1$, $A_{-1} = w$ and define $w_k := A_{k-1}/A_k$. One has $A_{k-1} + A_{k+1} = a_k A_k$, or equivalently $w_k = a_k - 1/w_{k+1}$, so $w_k = [a_k, a_{k+1}, \ldots]$. Then $A_n$ is the unique generator $\varepsilon_1$ of $V$ with $0 < \varepsilon_1 < 1$. For $\mu \in M^*$ define ‘Poincaré series’ $F_\mu(z) := \sum_{\varepsilon \in V} e(\varepsilon z)$. If $n \geq 3$, the $n$ functions $F_{A_1}, \ldots, F_{A_n}$ generate the local ring of the cusp, and define an embedding in $(\mathbb{C}^n, 0)$; for $n = 1, 2$ one has to take some extra generators (see [Co]). The exact relations between these generators are not known. Even in the hypersurface case ($n \leq 3$) the equation is not polynomial, but its Newton diagram has the standard form (for an explicit example see [Co]).

(3.2) Proposition. The specialisation $t_i = X_i^{a_i-2}$ in the equations for the deformation of the $n$-gon in Theorem (2.3) gives rise to defining equations of a cusp singularity with $w = [a_1, \ldots, a_n]$.

Proof. The equations of (2.3) describe a deformation of the cone over the $n$-gon over a smooth base space. The specialisation equations $t_i - X_i^{a_i-2}$ form a regular sequence, so the resulting ideal is the ideal of a surface singularity. Equations of the tangent cone are obtained by putting $t_i = 0$ if $a_i > 2$ (and $t_i = 1$ if $a_i - 2 = 0$). As at least one $a_i > 2$, the product $T = \prod t_i$ vanishes. The projectivised tangent cone consists of a cycle of rational curves, with as many double points as the number of $i$ with $a_i > 2$.

Singular points on the first blow-up can only occur in double points of the exceptional locus. Suppose $a_0 > 2$. Let $X_0 = x_0$, $X_i = x_0 x_i$ be the description of the blow-up in one affine chart. The strict transform of the surface is given by the same equations as in the proof of Lemma 4, with $t_i = x_0^{-a_i-2} x_i^{-a_i-2}$ for $i \neq 0$ and $t_0 = x_0^{-a_0-2}$. It is smooth at the origin if $a_0 = 3$, and otherwise there is a singularity of type $A_{a_0-3}$. As cusp singularities are characterised by their exceptional divisors, this computation shows that the surface singularity is indeed a cusp associated with $w = [a_1, \ldots, a_n]$. \qed
Weakly minimal improvement has a cycle of rational curves as exceptional divisor. To construct the improvement one takes the minimal embedded resolution $Y$ of $X_i$. Define $i = 1, \ldots, a_i - 2$, and put

$$t_i = X_i^{a_i - 2} + v_i^{(1)} X_i^{a_i - 3} + \ldots + v_i^{(a_i - 3)} X_i + v_i^{(a_i - 2)}.$$ 

This family has codimension $n$ in $T^1$, as $\dim T^1 = \sum (a_i - 1)$ [Be]. For $n > 9$ it forms an irreducible component of the deformation space: simple elliptic singularities of multiplicity $n > 9$ have only equisingular unobstructed deformations, of codimension $n$.

This deformation is similar to the Artin component of cyclic quotient singularities. There the additional deformations are obtained by adding an extra summand $v_i^{(a_i - 1)}/X_i$ to each $t_i$ [Ar]. In the case at hand it is difficult to make sense of such a formula, as the $s_i$ depend on the $t_i$; for $n = 6$ this idea works, as the formulas in the next section show.

(3.3) Degenerate cusp are a class of weakly normal, non-isolated surface singularities, first introduced and described by Shepherd-Barron [SB]. For such singularities the role of the resolution is taken over by an improvement. By definition an improvement $\pi: Y \to X$ is a point modification, such that $Y$ is weakly normal, the singular locus $S$ of $Y$ is the strict transform of $\text{Sing}(X)$, while $S$, the normalisation $\tilde{Y}$ of $Y$ and the inverse image $\tilde{S}$ of $S$ on $\tilde{Y}$ are all smooth [St]. Define $Y$ to be weakly minimal, if $\tilde{Y}$ contains only $(-1)$-curves that intersect $\tilde{S}$.

**Definition.** A degenerate cusp singularity is a weakly normal surface singularity, such that a weakly minimal improvement has a cycle of rational curves as exceptional divisor.

The normalisation $\tilde{X}$ of $X$ consists of a disjoint union of cyclic quotient singularities $X_i$ (including the case of $A_0$, a smooth germ), and the inverse image of $\text{Sing}(X)$ consists of two transversal lines. To construct the improvement one takes the minimal embedded resolution $Y_i \to X_i$ of these lines, and glues the $Y_i$ along the strict transform of the lines to form $Y$.

One can still associate a cycle of numbers $[[a_1, \ldots, a_n]]$ to $X$. In the non-degenerate case the canonical model contains an $A_{i-3}$-singularity for each $i$ with $a_i \geq 3$; on the ‘canonical improvement’ of a degenerate cusp (which for $n \geq 3$ is just the first blow-up) an $A_{\infty}$-singularity then corresponds to $a_i = \infty$.

(3.4) The same procedure as for non-degenerate cusps yields equations: put $t_i = X_i^{a_i - 2}$. The equations simplify because $t_i = 0$, if $a_i = \infty$, and therefore $T = \prod t_i = 0$. This allows to take $s_\alpha = 1$ (if $\alpha$ is odd). The resulting equations were originally obtained by Ruud Pellikaan from the equations for cyclic quotients.

**Proposition (Pellikaan).** Let $X(a_1, \ldots, a_n)$ be a degenerate cusp with $n > 3$. The $n(n-3)/2$ equations

$$F_{ij} := F_{i,j}^{(1)} : X_i X_j = X_{i+1} \left( \prod_{k=i+1}^{j-1} X_k^{a_k-2} \right) X_{j-1} + X_{j+1} \left( \prod_{k=j+1}^{i-1} X_k^{a_k-2} \right) X_{i-1}$$

with $i - j \neq -1, 0, 1$ generate the ideal of the cusp.

For degenerate cusps the equations can be ‘blown down’ if the cycle $[[a_1, \ldots, a_n]]$ contains a $1$. This means that I formally write $t_i = 1/X_i$, if $a_i = 1$. Suppose $a_n = \infty$. If $i \neq 1, n - 1$, then the equation $F_{i-1,i+1}: X_{i-1} X_{i+1} - X_i$ allows to eliminate $X_i$, and substituting $1/X_{i-1} X_{i+1}$ for $t_i$ shows that the new equations correspond to $[[a_1, \ldots, a_{i-1}, 1, a_{i+1} - 1, \ldots, a_{n-1}, \infty]]$. The case $a_{n-1} = 1$ gives the equation $X_{n-1} = X_n X_{n-2} - X_1 T_1^{a-3} X_{n-3}$. As $t_n = 0$ the only equations in which $t_{n-1}$ appears are the $F_{kn}$, where the combination
contrary to the
t
These deformation parameters also enter the equations through the formula

\[ X_1X_3 = X_0^{a_0} + X_2^{a_2} \]
\[ X_0X_2 = X_1^{a_1} + X_3. \]

Elimination of \( X_3 \) gives the equation of \( T_{a_0,a_1+1,a_2} \), the cusp with cycle \([a_0 - 1,a_1,a_2 - 1]\):

\[ X_0X_1X_2 - X_1^{a_1+1} + X_0^{a_0} + X_2^{a_2}. \]

In particular, equations for the cycle \([2,\ldots,2,1,\infty]\) blow down to \( T_{23\infty} \), the cusp with cycle \([\infty]\).

**Proposition.** Every degenerate cusp deforms to \( T_{23\infty} \) and is therefore smoothable.

**Proof.** I can suppose that \( a_0 = \infty \). As noted above \( t_{n-1} \) occurs only in the combination \( t_{n-1}X_{n-1} \) in the equations \( F_{ij} \); but note that this does not hold for general \( F_{i,j,k}^h \). The relations between the \( F_{ij} \) can easily be computed directly to be:

\[
\begin{align*}
X_iF_{ij} - X_jF_{ik} + X_{i+1}T_{i,j+1}^{j-1}F_{j-1,k} &= 0 \\
X_iF_{ik} - X_jF_{ij} + X_{k-1}T_{j+1}^{i-1}F_{i,j+1} &= 0 \\
X_iF_{0j} - X_jF_{0i} - X_iT_1^{i-1}F_{i-1,j} + X_{n-1}T_j^{n-1}F_{i,j+1} &= 0 \\
X_iF_{0j} - X_0F_{ij} + X_{n-1}T_j^{n-1}F_{i,j+1} + X_{n-1}T_{i+1}^{n-1}F_{i+1,j} &= 0.
\end{align*}
\]

Again \( t_{n-1} \) only occurs together with \( X_{n-1} \). The required deformation can now be defined by putting \( t_0 = 0, t_i = X_i^{n_i-2} + \varepsilon \) for \( 1 \leq i \leq n - 2 \) and \( t_{n-1} = X_{n-1}^{a_{n-1}-2} + \varepsilon/X_{n-1} \).

**4. Smoothings of simple elliptic singularities**

In this section I use the symmetric equations to describe deformations of simple elliptic singularities. The dimension of \( T^1 \), the vector space of infinitesimal deformations, is \( n + 1 \) for a singularity of multiplicity \( n \); the graded parts satisfy \( \dim T^1(-1) = n \) and \( \dim T^1(0) = 1 \). I have not been able to find the general formula for the perturbation of the equations \( F_{i,j,k}^h \). Therefore I restrict myself to \( n \leq 9 \), the cases in which cones over elliptic curves of degree \( n \) are smoothable. I only give the results of my computations.

**(4.1) Example: \( n = 6 \).** For this multiplicity the versal deformation of cusps can be given. It is known that the base is up to a smooth factor independent of the specific cusp; this circumstance enables me to give a general formula.

I introduce deformation parameters \( v_i(= v_i^{(a_i+1)}) \), while the others are implicit in the formula

\[ t_i = X_{i}^{a_i-2} + v_i^{(1)}X_{i}^{a_i-3} + \ldots + v_i^{(a_i-3)}X_i + v_i^{(a_i-2)}. \]

These deformation parameters also enter the equations through the formula \( s_i^4T - s_i^3 + s_1s_3^2 \), contrary to the \( v_i \). The deformation of the simple elliptic singularity is obtained by putting \( t_1 = 1 \).

I describe the equations \( F_{012}^0, F_{013}^0 \) and \( F_{1/2,3/2,5/2}^1 = F_{0124} \):

\[
\begin{align*}
s_2^2t_5t_0t_1X_1X_5 - s_2^2X_2X_4 + s_1s_3t_3X_3^2 \\
+ s_3s_2^2v_3X_3 + s_1s_2t_0(v_5t_1X_1 + v_1t_5X_5) + s_3^2s_3t_1t_5v_0X_0 - s_1^2t_0t_3t_5(v_2t_4X_4 + v_4t_2X_2) \\
+ s_2^4t_0v_1v_5 + s_1s_3s_2t_1t_5v_0^2 - s_1^2s_2t_0t_3t_5v_2v_4
\end{align*}
\]
\[ s_2^2t_0X_0^2 - s_2^3X_1X_5 + s_1^2t_2^2t_4X_4^2 \\
+ (s_3^3 + 2s_1^3T)v_0X_0 + 2s_1s_2^3X_3t_2t_3t_4v_3 + s_1s_3^2t_3(v_2t_4X_4 + v_4t_2X_2) \\
+ s_2^2s_3^2t_2t_4v_4 + s_1^2t_2t_4v_5 + s_1^2s_2t_1t_2t_3t_5v_6 \\
\]

\[ s_1t_0t_1X_0X_1 - s_3X_2X_5 + s_1t_3t_4X_3X_4 \\
+ s_2^3(v_3t_4X_4 + v_1t_0X_0 + v_0t_1X_1 + v_4t_3X_4) + s_2^2s_3t_0t_1t_3t_4v_5v_0 + (1/2s_3^3 + s_3^2T)(v_0v_1 + v_3v_4) \]

The base space is (up to a smooth factor) the cone over \( \mathbb{P}^1 \times \mathbb{P}^2 \):

\[
\begin{pmatrix}
  v_0 \\
  v_2 \\
  v_4 \\
  v_3 \\
  v_5 \\
  v_1
\end{pmatrix}
\]

The infinitesimal deformation in the \( v_0 \) direction can be obtained by formally putting \( t_0 = X_0^{a_0^{-2}} + v_0s_2^2/s_1X_0 \), using the other equations and making coordinate transformations involving the 'unit' \( 1 + s_1^2t_2t_3t_4v_0/4s_2^2X_0 \).

**Example:** \( n = 7 \). The equations of the total space are not very enlightening, but I give them for completeness. The equation \( F_{012}^0 \) is changed into:

\[
s_1s_4X_0^2 + s_2^2X_1X_6 - s_3^2X_2X_5 + (s_2^3s_4 - s_3^3)s_1s_2s_4v_0X_0 \\
+ 2(s_2s_4 - s_3^2)s_1s_2^2(v_0X_1 + v_1X_6) - (s_2^3 - s_3^2s_4)s_1^3(v_5X_2 + v_2X_5) - 3s_3^3s_2s_4(v_4X_3 + v_3X_4) \\
+ 2(s_2s_4^2 - s_2s_4^2 - s_1^2s_2^2s_4v_0v_6 + (s_2^3 + 3s_2^3s_2^2s_4 - 7s_2s_4^4)s_4s_1^3v_2v_5 \\
+ (4s_1s_4^2 - 3s_1s_2s_4^2 + 2s_5^2)s_1s_2^2v_3v_4
\]

and \( F_{123}^0 \) into:

\[
s_1s_2X_1X_6 + s_2s_4X_2X_5 + s_1s_4X_3X_4 + 3s_2^2s_2^2s_4v_0X_0 \\
+ (s_1^2 - s_2^2s_4)s_1s_2^2(v_0X_1 + v_1X_6) - (s_1^2 - s_2^2s_4)s_2s_4^2(v_5X_2 + v_2X_5) + (s_1^2 - s_1^2s_2)s_4s_1^3(v_4X_3 + v_3X_4) \\
- (4s_1s_4^2 - 3s_1s_2s_4^2 + 2s_5^2)s_1s_2^2s_4v_1v_6 - (4s_2s_4^2 - 3s_1s_2s_4^2 + 2s_5^2)s_1s_2^2s_4v_2v_5 \\
- (4s_1s_4 - 3s_1s_2s_4^2 + 2s_5^2)s_1s_2^2s_4v_3v_4 .
\]

The equations can be slightly simplified by replacing the deformation parameters by \( v_i/s_4 \) (for \( s_4 = 0 \) the elliptic curve degenerates); using the equation of the modular curve the resulting formulas are still polynomial. However such an operation breaks the symmetry.

The base space of the versal deformation in negative degree is the cone over an elliptic scroll of degree 7 in \( \mathbb{P}^6 \). The seven equations of the scroll are of the following type:

\[ s_1s_2s_4v_0^2 + s_1s_4^2v_1v_6 + s_1s_2v_2v_5 + s_4s_2^2v_3v_4 . \]

Actually these equations do not describe the family of scrolls over the modular curve; from each equation one can obtain three other ones by multiplying with \( s_i \), using the modular equation and dividing by \( s_4 \).

**Example:** \( n = 8 \). To describe the deformation it is enough to give the equations \( F_{012}^0, F_{123}^0, F_{234}^3, F_{345}^4, F_{456}^5, F_{567}^6, \) and \( F_{678}^7, \) which are all \( F_{j-1,j+1}^h \) with \( h = 0 \) or \( h = 1/2 \). Due to the modular symmetries the following three equations suffice.

\[
s_1^3X_2X_6 - s_2^3X_1X_7 + s_1s_3X_0^2 \\
- v_0X_0s_2^3 + v_4X_4 + s_2s_4v_0s_2^4s_4^2 + (v_3X_5 + v_5X_3 + s_2s_4v_0s_2^4v_3v_5)s_4s_3s_4 \\
+ (v_7X_1 + v_1X_7)s_1s_3s_4 + (v_6X_2 + v_2X_6 + s_2s_4v_0s_2^4v_6)(s_4s_2^4s_4 - s_1s_2s_3) - s_2(s_4^4 + s_3^4)s_4s_4v_7
\]

\[
16
\]
\[ s_1 s_3 X_3 X_5 + s_1 s_3 X_1 X_7 - s_2 s_4 X_2 X_6 \]
\[ + (v_0 X_0 + v_4 X_4)s_1 s_3 s_2 (s_2^2 - s_1^2) + (v_0 X_2 + v_2 X_6)s_2 (s_3^4 + s_1^4) \]
\[ - s_2^2 s_1^2 s_3 (s_1^2 v_4^2 + s_3^2 v_0^2) - s_1 s_2^2 s_3 s_4 v_1 v_7 - s_2^2 s_3^5 v_2 v_6 \]

\[ s_1 s_2 X_3 X_6 - s_2 s_3 X_2 X_7 + s_1 s_4 X_0 X_1 \]
\[ + (v_4 X_5 + v_5 X_4)s_1 s_2 s_4 + (v_7 X_2 + v_2 X_7)s_2^3 s_3 s_1^2 + (v_0 X_1 + v_1 X_0)(s_1 s_2^3 s_4 - s_2^3 s_3 s_4) \]
\[ - (2s_1 s_2^5 v_4^2 - s_3^2 s_4 s_1^2 + s_1 s_2 s_3^2 v_2 v_7 - s_2^2 s_3 s_4 v_4 v_5 \]

The base space (for fixed \( s_\alpha \)) is given by twelve linearly independent quadrics, eight with odd index sum, two of which come from the following equations with \( 2h = 1 \):

\[
\begin{align*}
s_1 s_2 v_2 v_7 + s_3 s_2 v_3 v_6 - s_3 s_4 v_0 v_1 \\
s_1 s_2 v_3 v_6 + s_3 s_2 v_2 v_7 - s_3 s_4 v_4 v_5 \\
s_1 s_2 v_0 v_1 - s_3 s_2 v_4 v_5 + s_1 s_4 v_2 v_7 \\
s_1 s_2 v_4 v_5 - s_3 s_2 v_0 v_1 + s_1 s_4 v_3 v_6
\end{align*}
\]

and four even ones, which can be written as extension of the matrix

\[
\begin{pmatrix}
 s_3^2 - s_1^2 & s_2 s_4 & s_1 s_3 & v_1 v_7 - v_3 v_5 \\
 s_2^2 & s_1^2 + s_3^2 & s_2^2 & v_2^2 - v_0^2
\end{pmatrix}
\]

describing the base curve; I have only written the \( h = 0 \) terms. The base space consists of five components, four planes given by the ideals:

\[
\begin{align*}
(v_7, v_5, v_3, v_1, v_2 - v_6, v_0 - v_4) \\
(v_7, v_5, v_3, v_1, v_2 + v_0, v_0 + v_4) \\
(v_6, v_4, v_2, v_0, v_3 - v_7, v_1 - v_5) \\
(v_6, v_4, v_2, v_0, v_3 + v_7, v_1 + v_5)
\end{align*}
\]

and one component of degree 8, which is isomorphic to the cone over the elliptic curve. In fact, the ideal of the component can be obtained from the ideal of the curve by the substitution \( X_i \mapsto v_i \) for even \( i \) and \( X_i \mapsto v_{i+4} \) for odd \( i \).

**Example:** \( n = 9 \). The equations \( F_{ijh}^0 \) contain (for fixed value of the modulus) three linear independent ones; I choose \( F_{013}^0, F_{043}^0 \) and \( F_{073}^0 \). As they are permuted by the action of the modular substitution \( U \) it suffices to give the deformation of \( F_{013}^0 \):

\[
\begin{align*}
-s_4 s_7 X_0^2 + s_1^2 X_3 X_6 - s_3^2 (X_1 + s_1 (s_1^2 - s_4 s_7) v_1) (X_8 + s_1 (s_1^2 - s_4 s_7) v_8) \\
-s_7 s_4 (s_4^3 + 3 s_3^3 + 2 s_2^3) v_0 X_0 + s_1^2 (s_1^2 + 2 s_3^2) (v_6 X_3 + v_3 X_6) \\
+s_2^3 s_1 (s_1 s_7 - s_2^2) (v_7 X_2 + v_2 X_7) + s_3^2 s_1 (s_4 s_1 - s_2^2) (v_5 X_4 + v_4 X_5) \\
-s_4 s_7 (s_1^4 + s_1^3 + s_2^3) (2 s_3^3 + s_2^3) v_0^2 + s_3^2 s_1 (s_3^2 + s_2^3) v_3 v_6 \\
+s_3^2 s_1 (s_1 s_7 - s_2^2) (s_1^2 + 2 s_1^2 - s_4 s_7 + s_3^2) v_2 v_7 + s_3^2 s_7 (s_2^2 - 3 s_4^2 + 8 s_4^2 s_4^2 + 4 s_1^2 s_7 - s_1 s_7^2) v_4 v_5
\end{align*}
\]

The base space is given by nine times two equations:

\[
\begin{align*}
s_3 (v_0^2 - v_3 v_6) & - s_4 v_1 v_8 - s_1 v_7 v_2 - s_7 v_4 v_5 \\
s_1 v_1 v_8 + s_7 v_7 v_2 + s_4 v_4 v_5
\end{align*}
\]

In the actual computation these equations come out multiplied with some factors in the modular parameters \( s_i \), which vanish at the cusps. As the elliptic curve is supposed to be
smooth there is no harm in dividing by those factors. One of them is the Hessian $H = s_1^3 + s_4^3 + s_7^3 - 3s_1s_4s_7$. The base space has nine irreducible components, three of which are given by the ideal

$$(v_1, v_4, v_7, v_2, v_5, v_8, v_0^2 - v_3v_6, v_3^3 - v_0v_6, v_6^2 - v_0v_3)$$

and the other six by cyclic permutation. Furthermore there is an embedded component. The equations for the reduction of the base space include $v_i v_j$ if $i \not\equiv j \pmod{3}$, which simplifies the equation $F_{013}$ considerably. The group of points of order nine of the elliptic curve acts on the set of components with exactly satisfies the equations above. The projection 

$Y_0 = s_3(s_7 - s_4)(X_1 + X_8) + s_3(s_4 - s_1)(X_2 + X_7) + s_3(s_1 - s_7)(X_4 + X_5)$

$- (s_1s_4 + s_4s_7 + s_1s_7)(X_3 + X_6) - (s_1^2 + s_2^2 + s_7^2)X_0$

$Y_3 = s_3(s_4 - s_1)(X_1 + X_5) + s_3(s_1 - s_7)(X_8 + X_7) + s_3(s_7 - s_4)(X_4 + X_2)$

$- (s_1s_4 + s_4s_7 + s_1s_7)(X_0 + X_6) - (s_1^2 + s_2^2 + s_7^2)X_2$

$Y_6 = s_3(s_4 - s_1)(X_1 + X_2) + s_3(s_4 - s_1)(X_4 + X_8) + s_3(s_7 - s_4)(X_5 + X_7)$

$- (s_1s_4 + s_4s_7 + s_1s_7)(X_3 + X_0) - (s_1^2 + s_2^2 + s_7^2)X_6$,

from the space spanned by six suitable chosen points maps the elliptic curve $E$ onto the plane cubic

$f = s_3^2(Y_0^3 + Y_3^3 + Y_6^3) + (s_1^3 + s_4^3 + s_7^3 + 6s_1s_4s_7)Y_0Y_3Y_6$.

The functions $Y_i$ can be obtained from the sum $\sum X_i X_j X_k$, where the sum ranges over $i, j, k$ in different residue classes modulo 3 with $i + j + k \equiv l \pmod{9}$, by specialising the first two factors of each product. The equation $f$ together with the following nine polynomials

$X_0 = -(s_3^2Y_0^2 + 3s_1s_4s_7Y_3Y_6)Y_0$

$X_3 = -(s_3^2Y_3^2 + 3s_1s_4s_7Y_0Y_6)Y_3$

$X_6 = -(s_3^2Y_6^2 + 3s_1s_4s_7Y_3Y_0)Y_6$

$X_1 = s_3^2(s_4Y_0^2Y_3 + s_1Y_3^2Y_6 + s_7Y_6^2Y_0)$

$X_4 = s_3^2(s_7Y_0^2Y_3 + s_4Y_3^2Y_6 + s_1Y_6^2Y_0)$

$X_7 = s_3^2(s_1Y_0^2Y_3 + s_7Y_3^2Y_6 + s_4Y_6^2Y_0)$

$X_2 = s_3^2(s_1Y_0^2Y_6 + s_4Y_3^2Y_0 + s_7Y_6^2Y_3)$

$X_5 = s_3^2(s_7Y_0^2Y_6 + s_1Y_3^2Y_0 + s_4Y_6^2Y_3)$

$X_8 = s_3^2(s_4Y_0^2Y_6 + s_7Y_3^2Y_0 + s_1Y_6^2Y_3)$

forms a basis of the linear system of plane cubics, except when $H = s_1^3 + s_4^3 + s_7^3 - 3s_1s_4s_7 = 0$ or $s_3 = 0$, which are precisely the equations of the cusps of the modular curve. Upon setting $v_0 = f/H$ the image of $\mathbb{P}^2$ exactly satisfies the equations above.

The group of points of order nine of the elliptic curve acts on the set of components with a points of order three giving rise to an automorphism of each of the components of the total space.

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