INFINITE DIMENSIONAL ORTHOGONAL PRESERVING QUADRATIC 
STOCHASTIC OPERATORS

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Abstract. In the present paper, we study infinite dimensional orthogonal preserving quadratic stochastic operators (OP QSO). A full description of OP QSOs in terms of their canonical form and heredity coefficient’s values is provided. Furthermore, some properties of OP QSOs and their fixed points are studied.

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1. Introduction

The history of quadratic stochastic operators (QSOs) is traced back to Bernstein’s work [4] where such kind of operators appeared from the problems of population genetics (see also [14]). These kind of operators describe time evolution of variety species in biology and are represented by so-called Lotka-Volterra (LV) systems [23], but currently in the present, there are many papers devoted to these operators owing to the fact that they have plentiful applications especially in modelings in many different fields such as biology [11, 19] (population and disease dynamics), physics [20, 22] (non-equilibrium statistical mechanics), economics, and mathematics [14, 19, 22] (replicator dynamics and games).

A quadratic stochastic operator is usually used to present the time evolution of species in biology, which arises as follows. By considering an evolution of species in biology as given in the situation where \( I = \{1, 2, \ldots, n\} \) is the \( n \) type of species (or traits) in a population, the probability distribution of the species in an early state of that population is \( x^{(0)} = (x^{(0)}_1, \ldots, x^{(0)}_n) \). On a side note, we define \( P_{ij,k} \) as the probability of an individual in the \( i \)th species and \( j \)th species to cross-fertilize and produce an individual from \( k \)th species (trait). Given \( x^{(0)} = (x^{(0)}_1, \ldots, x^{(0)}_n) \), we can find the probability distribution of the first generation, \( x^{(1)} = (x^{(1)}_1, \ldots, x^{(1)}_n) \) by using a total probability, i.e.,

\[
x^{(1)}_k = \sum_{i,j=1}^{n} P_{ij,k} x^{(0)}_i x^{(0)}_j, \quad k \in \{1, \ldots, n\}.
\]

This relation defines an operator which is denoted by \( V \) and it is called quadratic stochastic operator (QSO). Each QSO maps the simplex \( S^{n-1} = \{x \in \mathbb{R}^n \mid x_i \geq 0, \sum_{i=1}^{n} x_i = 1\} \) into itself. Moreover, the operator \( V \) can be interpreted as an evolutionary operator that describes the sequence of generations in terms of probability distributions if the values of \( P_{ij,k} \) and the distribution of the current generation are given. The most well-known class in the theory QSO is a Volterra one, namely whose heredity coefficients satisfy

\[
P_{ij,k} = 0 \text{ if } k \notin \{i,j\}.
\]
The condition (1.1), biologically, means that each individual can inherit only the species of the parents. The dynamics of Volterra QSO was studied in [9, 8]. Nevertheless, not all QSOs are of Volterra-type, therefore, the understanding of the dynamics of non-Volterra QSO still remains open. We refer the reader to [10, 17] as the exposition of the recent achievements and open problems in the theory of the QSO can be further researched.

One of the main problems in the theory of nonlinear operator is to study the limiting behavior of nonlinear operators. To this day, there are a handful of studies dedicated to the exploration of the dynamics of higher dimensional systems despite the fact that it is a very exquisite and important topic. Although, most research has been focused on the simplex $S^{n-1}$, but there are models where the probability distribution is given on a countable set, which means that the corresponding QSO is defined on an infinite-dimensional space.

The simplest case of the infinite-dimensional space is the Banach space $\ell_1$ of absolutely summable sequences. It is worth mentioning that some infinite dimensional QSOs were studied in [13, 15, 16].

On the other hand, from [21] with the results of [18] we conclude that a QSO (acting on finite dimensional simplex) is surjective, if and only if, it is orthogonal preserving (OP) QSO. Here by the orthogonality of distributions we mean their disjointness. We cannot afford to ignore the surjectivity of a quadratic operator is strongly tied up with nonlinear optimization problems [3]. Furthermore, any orthogonal preserving QSO is a permutation of Volterra QSO in [1, 18]. Yet, if we look at the same problem in the infinite dimensional setting, the last statement becomes incorrect. Also in [1], we have considered a special class of orthogonal preserving operators for which an analogous result was obtained replicated in the finite dimensional setting. Unfortunately, this type of result is wrong in a general setting. Therefore, in this paper, we go on a voyage of discovery in an attempt to describe the orthogonality preserving infinite dimensional quadratic stochastic operators in a general case. We notice that every linear stochastic operators can be considered as a particular case of QSO. In the later case, there are many papers that are devoted to the orthogonal preserving linear operators defined on various Banach spaces (see for example [2, 5, 6, 7, 12, 24]), once the nonlinearity appears in operators, then all existing methods (for linear operators) are no longer applicable. The simplest nonlinearity is quadratic which for these kinds of we fully describe, OP QSOs in terms of the their heredity coefficients, and provide their canonical forms. Last but not least, we provide certain examples of such kind of operators along with the properties of OP QSOs and their fixed points.

2. Orthogonal Preserving QSO

Let $E$ be a subset of $\mathbb{N}$. Denote

$$S^E = \left\{ x = (x_i)_{i \in E} \in \mathbb{R}^E : x_i \geq 0, \sum_{i \in E} x_i = 1 \right\}. $$

In what follows, by $e_i$ we denote the standard basis in $S^E$, i.e. $e_i = (\delta_{ik})_{k \in E}$ ($i \in E$), where $\delta_{ij}$ is the Kroneker delta.

Let $V$ be a mapping defined by

$$(2.1) \quad V(x)_k = \sum_{i,j \in E} P_{ij,k} x_i x_j, \quad k \in E$$
here, \( \{P_{ij,k}\} \) are hereditary coefficients which satisfy
\[(2.2) \quad P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{k \in E} P_{ij,k} = 1, \quad i, j, k \in E \]

One can see that \( V \) maps \( S^E \) into itself and \( V \) is called Quadratic Stochastic Operator (QSO) [15].

By support of \( x = (x_i)_{i \in E} \in S^E \) we mean a set \( \text{Supp}(x) = \{i \in E : x_i \neq 0\} \). A sequence \( \{A_k\} \) of sets is called cover of a set \( B \) if \( \bigcup_{k=1}^{\infty} A_k = B \) and \( A_i \cap A_j = \emptyset \) for \( i, j \in \mathbb{N} \) (\( i \neq j \)).

Recall that two vectors \( x = (x_k), y = (y_k) \) belonging to \( S^E \) are called orthogonal (denoted by \( x \perp y \)) if \( \text{supp}(x) \cap \text{supp}(y) = \emptyset \). If \( x, y \in S^E \), then one can see that \( x \perp y \) if and only if \( x \circ y = 0 \) (or \( x_k \cdot y_k = 0 \) for all \( k \in E \)). Here, \( \circ \) stands for the standard dot product.

**Definition 2.1.** A QSO \( V \) given by \((2.1)\) is called orthogonal preserving QSO (OP QSO) if for any \( x, y \in S \) with \( x \perp y \) one has \( V(x) \perp V(y) \).

Let \( T \) be a stochastic matrix given by \((t_{ij})_{i,j \in E} \), where \( t_{ij} \geq 0, \sum_{j \in E} t_{ij} = 1 \) \((i \in \mathbb{N})\). Then one can define a linear operator (which is called linear stochastic operator (LSO))
\[(2.3) \quad (Tx)_k = (x^T)_k = \sum_{i \in E} t_{ik} x_i, \quad x \in S^E, \quad k \in E. \]

Due to stochasticity of \( T \), the operator \( T \) maps \( S^E \) into itself. Note that each LSO can be considered as a particular case of q.s.o. Indeed, let us define
\[(2.4) \quad P^{(T)}_{ij,k} = \frac{t_{ik} + t_{jk}}{2}. \]

Then one can see that \( \{P^{(T)}_{ij,k}\} \) satisfies \((2.2)\), and for the corresponding q.s.o. \( V_T \) we have
\[
(V_T(x))_k = \sum_{i, j \in E} P^{(T)}_{ij,k} x_i x_j \\
= \sum_{i, j \in E} \left( \frac{t_{ik} + t_{jk}}{2} \right) x_i x_j \\
= \sum_{i \in E} t_{ik} x_i \\
= (Tx)_k, \quad \forall k \in \mathbb{N},
\]
i.e. \( V(x) = Tx \) for all \( x \in S^E \). This implies that all results holding for QSO are valid for LSO.

**Remark 2.2.** Let \( T \) be a LSO, then for any \( x = (x_k) \in S^E \), \( T \) can be written as follows
\[
T(x) = \sum_{k \in E} x_k T(e_k).
\]

Therefore, a LSO \( T \) defined on \( S^E \) is orthogonal preserving if and only if \( T(e_i) \perp T(e_j) \) for all \( i \neq j \). Indeed, it is enough to show that the last statement implies OP of \( T \). Let us take \( x, y \in S^E \) such that \( x \perp y \) (i.e. \( x \circ y = 0 \)). Then from
\[
T(x) = \sum_{t \in E} x_t T(e_t), \quad T(y) = \sum_{m \in E} y_m T(e_m)
\]
with $T(e_l) \circ T(e_m) = \delta_{lm}$ we find

$$T(x) \circ T(y) = \left( \sum_{\ell \in E} x_\ell T(e_\ell) \right) \circ \left( \sum_{m \in E} y_m T(e_m) \right)$$

$$= \sum_{\ell, m \in E} x_\ell y_m T(e_\ell) \circ T(e_m)$$

$$= \sum_{\ell \in E} x_\ell y_\ell$$

$$= 0$$

Now using (2.3) we conclude that $T$ is OP if and only if for the stochastic matrix $(t_{ij})$ one has $t_i \perp t_j$ for all $i, j \in \mathbb{N}$ with $i \neq j$. Here $t_k = (t_{ki})_{i \in E}$, $k \in E$ for all $i \neq j$.

When we consider the QSO, then similar kind of result is not valid, but we use some ideas from the mentioned remark.

Remark 2.3. We first note that if $V$ is an OP QSO, then the system $\{V(e_k)\}$ is also orthogonal. Therefore, to describe OP QSO it is enough for us just to fix this (i.e. $\{V(e_k)\}$) system. Indeed, let us denote by $V$ the set of all OP QSO $V$ such that $V(e_k) = F_k$ for some orthogonal system $F_k$ in $S$. Now, let us assume that an OP QSO $\tilde{V}$ such that $\tilde{V}(\tilde{F}_k) = F'_k$, where $\{\tilde{F}_k\}$ and $\{F'_k\}$ are orthogonal systems in $S$. On the other hand, if one considers $\{\tilde{V}(e_k)\}$ then the system is also has to be orthogonal in $S$ i.e., $\tilde{V}(e_k) = I_k$, where $\{I_k\}$ is an orthogonal system in $S$. Hence $\tilde{V}$ is an element of $\mathcal{V}$.

Recall [16] that a QSO $V : S^E \rightarrow S^E$ is called Volterra if one has

(2.5) \[ P_{ij,k} = 0 \text{ if } k \notin \{i, j\}, \quad i, j, k \in E. \]

Remark 2.4. In [16] it was given an alternative definition Volterra operator in terms of extremal elements of $S^E$.

One can check [17] that a QSO $V$ is Volterra if and only if one has

$$ (V(x))_k = x_k \left( 1 + \sum_{i \in E} a_{ki} x_i \right), \quad k \in E, $$

where $a_{ki} = 2P_{ik,k} - 1$ $(i, k \in E$. One can see that $a_{ki} = -a_{ik}$. This representation leads us to the following definitions.

Definition 2.5. A QSO $V : S^E \rightarrow S^E$ is called $\pi$-Volterra if there is a permutation $\pi$ of $E$ such that $V$ has the following form

$$ V(x)_k = x_{\pi(k)} \left( 1 + \sum_{i \in E} a_{\pi(k)i} x_i \right) $$

where $a_{\pi(k)i} = 2P_{\pi(k),i} - 1$, $a_{\pi(k)i} = -a_{i\pi(k)}$ for any $i, k \in E$.

In [1, 18] it has been proved the following result.

Theorem 2.6 ([1, 18]). Let $E = \{1, 2, \ldots, n\}$ and $V$ be a QSO on $S^E$. Then the following statements are equivalent:
(i) $V$ is orthogonal preserving;
(ii) $V$ is $\pi$-Volterra QSO.

In what follows, for the sake of convenience we denote $S$ instead of $S^n$.

**Remark 2.7.** We notice that the vertices of the finite simplex $S^E$ ($E = \{1, 2, \ldots, n\}$) are described by the elements $e_k = (\delta_{ik})_{i \in E}$. Therefore, any OP QSO on $S^{n-1}$ is a permuted Volterra QSO (see Theorem 2.6). However, if we consider $S$, then one can see that there are many orthogonal systems in $S$, which differ from the system $\{e_k\}$. For example

$$F^{(1/2)}_1 = \left(\frac{1}{2}, \frac{1}{2}, 0, \ldots\right), F^{(1/2)}_2 = \left(0, 0, \frac{1}{2}, \frac{1}{2}, 0, \ldots\right),$$

(2.6) $$\ldots, F^{(1/2)}_k = \left(0, 0, \ldots, \frac{1}{2}, \frac{1}{2}, 0, \ldots\right), \ldots$$

Another crucial moment is that for a given orthogonal system $\{F_k\}$ in $S$, the set

$$\bigcup_{k}(\text{supp}(F_k))$$

may not equal to $\mathbb{N}$. For example, we have $\bigcup_{k=2}^{\infty}(\text{supp}(e_k)) = \mathbb{N} \setminus \{1\}$. All these make the description of OP QSOs is more challenging than the finite dimensional setting.

In [1] a special class of infinite dimensional OP QSOs have been studied for which an analogous of Theorem 2.6 holds.

**Theorem 2.8 ([1]).** Let $V$ be a QSO on $S$ such that $V(e_i) = e_{\pi(i)}$ for some permutation $\pi : \mathbb{N} \to \mathbb{N}$. Then $V$ is an OP QSO if and only if $V$ is $\pi$-Volterra QSO.

Recall that an orthogonal basis $\{F_k\}_{k=1}^{\infty}$ in $S$ is called total if for any $x \in S$ one finds $\{\lambda_i\}_{i=1}^{\infty}, \lambda \geq 0, \sum_{i=1}^{\infty} \lambda_i = 1$ such that

$$x = \sum_{i=1}^{\infty} \lambda_i F_i$$

**Theorem 2.9.** Let $\{F_k\}_{k=1}^{\infty}$ be an orthogonal basis in $S$. The following conditions are equivalent

(i) $\{F_i\}_{i=1}^{\infty}$ is total;
(ii) For every $k \in \mathbb{N}$ one has $|\text{supp}(F_k)| = 1$ and $\bigcup_{k=1}^{\infty} \text{supp}(F_k) = \mathbb{N}$.

**Proof.** (i) $\Rightarrow$ (ii). Assume contrary i.e., there exists some $k_0 \in \mathbb{N}$ such that $|\text{supp}(F_{k_0})| \geq 2$. Now, take $m \in \text{supp}(F_{k_0})$. If one considers $e_m \in S$, then due to the totality of $\{F_k\}$ we have

$$e_m = \sum_{i=1}^{\infty} \lambda_i F_i$$

This means that $\lambda_i = 0$ for $i \neq k_0$, so $e_m = F_{k_0}$, which contradicts to $|\text{supp}(F_{k_0})| \geq 2$. Now, if $\bigcup_{k=1}^{\infty} \text{supp}(F_k) \subset \mathbb{N}$, then for $\ell \in \mathbb{N} \setminus \bigcup_{k=1}^{\infty} \text{supp}(F_k)$, the vector $e_\ell$ can not be represent as a convex combination of $\{F_k\}$. Hence, we infer the statement (ii).

(ii) $\Rightarrow$ (i). If (ii) holds, then the system $\{F_k\}_{k=1}^{\infty}$ is a permutation of the standard basis $\{e_k\}_{k=1}^{\infty}$, which is clearly total. $\Box$
Corollary 2.10. Let \( \{F_k\}_{k=1}^\infty \) be an orthogonal system in \( S \) and \( V \) is an OP QSO on \( S \) such that \( V(e_k) = F_k \), for all \( k \in \mathbb{N} \). Then the following statements are equivalent

(i) \( V \) is an \( \pi - \text{Volterra QSO} \);
(ii) \( F_k \) is total.

3. Description of OP QSOs

In this section, we are going to describe infinite dimensional OP QSOs.

Let \( V \) be a QSO on \( S \) whose heredity coefficients are \( \{P_{ij,k}\} \). Let us introduce the following vectors

\[ \mathbb{P}_{ij} = (P_{ij,1}, \ldots, P_{ij,n}, \ldots) \quad \text{for any } i, j \in \mathbb{N} \]

One can see that for every \( i, j \in \mathbb{N} \) the vector \( \mathbb{P}_{ij} \) belongs to \( S \). Next result describes OP QSOs in terms of the vectors \( \{\mathbb{P}_{ij}\} \).

Theorem 3.1. Let \( V \) be a QSO. Then the following conditions are equivalent:

(i) \( V \) is an OP QSO;
(ii) For any \( A, B \subset \mathbb{N} \) with \( A \cap B = \emptyset \) one has \( \mathbb{P}_{ij} \perp \mathbb{P}_{uv} \) for all \( i, j \in A \) and \( u, v \in B \).

Proof. (i) \( \Rightarrow \) (ii). Take any \( A, B \subset \mathbb{N} \) with \( A \cap B = \emptyset \). Then chose two elements \( x, y \in S \) such that \( \text{supp}(x) = A \) and \( \text{supp}(y) = B \). From the condition \( A \cap B = \emptyset \) one concludes that \( x \perp y \).

From the definition of QSO, we have

\[ V(x) = \left( \sum_{i,j \in \text{supp}(x)} P_{ij,k}x_i x_j \right)_{k=1}^\infty, \quad V(y) = \left( \sum_{u,v \in \text{supp}(y)} P_{uv,k}y_u y_v \right)_{k=1}^\infty. \]

Due to the orthogonal preserving property of \( V \) one has \( V(x) \circ V(y) = 0 \), therefore one gets

\[
V(x) \circ V(y) = \sum_{k=1}^\infty \left( \sum_{i,j \in \text{supp}(x)} P_{ij,k}x_i x_j \right) \left( \sum_{u,v \in \text{supp}(y)} P_{uv,k}y_u y_v \right)
= \sum_{i,j \in \text{supp}(x)} \sum_{u,v \in \text{supp}(y)} \left( \sum_{k=1}^\infty P_{ij,k}P_{uv,k} \right) x_i x_j y_u y_v
= 0.
\]

According to \( i, j \in \text{supp}(x) \) and \( u, v \in \text{supp}(y) \) (i.e., \( x_i > 0, y_u > 0 \) for any \( i \in \text{supp}(x) \) and \( u \in \text{supp}(y) \)) from the last equalities, we conclude that

\[
\sum_{k=1}^\infty P_{ij,k}P_{uv,k} = 0
\]

which means \( \mathbb{P}_{ij} \circ \mathbb{P}_{uv} = 0 \) for all \( i, j \in A \) and \( u, v \in B \).

Now let us prove (ii) \( \Rightarrow \) (i). Now, take \( x, y \in S \) such that \( x \perp y \), then from (3.2) one finds

\[
V(x) \circ V(x) = \sum_{i,j \in \text{supp}(x)} \sum_{u,v \in \text{supp}(y)} \left( \mathbb{P}_{ij} \circ \mathbb{P}_{uv} \right) x_i x_j y_u y_v
\]
Due the fact \( \text{supp}(x) \cap \text{supp}(y) = \emptyset \) and the assumption (ii) we immediately obtain \( V(x) \circ V(x) = 0 \), i.e. \( V(x) \perp V(x) \). This completes the proof. \( \square \)

From this theorem we immediately get the following corollary.

**Corollary 3.2.** Let \( V \) be an OP QSO, then for any \( i \neq j \) (\( i, j \in \mathbb{N} \)) one has \( \mathbb{P}_{ii} \perp \mathbb{P}_{jj} \).

**Remark 3.3.** If a QSO \( V \) is given by a stochastic matrix (see (2.4)) then from Corollary 3.2 we infer that \( V \) is OP if and only if \( t_i \perp t_j \) for all \( i, j \in \mathbb{N} \) (\( i \neq j \)). This recovers the result of Remark 2.2.

One can infer that from Theorem 3.1 it is difficult to write representation of OP QSO. Therefore, for a given OP QSO \( V \) we denote \( \mathbb{F}_k = V(e_k), k \in \mathbb{N} \). The system \( \mathcal{F} = \{ \mathbb{F}_k \} \) is orthogonal. In what follows, we denote \( \mathbb{F}_k = (f_{k,i})_{i \in \mathbb{N}} \). One can see that \( f_{k,i} = 0 \) if \( i \notin \text{supp}(\mathbb{F}_k) \).

Henceforth, \( |A| \) is referred to the cardinality of a set \( A \) and denote

\[
\mathcal{C}_\mathcal{F} = \mathbb{N} \setminus \left( \bigcup_{k \in \mathbb{N}} \text{supp}(\mathbb{F}_k) \right)
\]

**Theorem 3.4.** Let \( \mathcal{F} = \{ \mathbb{F}_k \} \) be an orthogonal system and \( V \) be a QSO on \( S \) such that \( V(e_k) = \mathbb{F}_k, k \in \mathbb{N} \). Then, \( V(x) \) is an OP QSO if and only if it has the following form: for any \( x \in S \)

| (a) | for any \( m \in \text{supp}(\mathbb{F}_k) \) |
|-----|----------------|
| (b) | for any \( c \in \mathcal{C}_\mathcal{F} \), \( V(x)_c \) takes one of the following form |
| (I) | if there is no \( P_{ij,c} > 0 \) for every \( i, j \in \mathbb{N} \) then \( V(x)_c = 0 \) or |
| (II) | if there exists at least one \( P_{i,cj,c} > 0 \), then \( V(x)_c \) has one of the following form: |
| (i) | if there is no \( P_{ij,c} > 0 \) for \( j \in \{i_c, j_c\} \) where \( i \in \mathbb{N} \setminus \{j\} \), then |

\[
V(x)_c = 2P_{ij,c}x_ix_j
\]

| (ii) | if there exists \( P_{i_cj_c,c} > 0 \) for either \( j = i_c \) or \( j = j_c \) (here let \( j = i_c \)), then \( V(x)_c \) has one of the following form: |
| (1) | if \( P_{i_cj_c,c} > 0 \) then |

\[
V(x)_c = 2P_{ij,c}x_j + P_{ij,c}x_j + P_{i_cj_c,c}x_{i_c}x_{j_c} + P_{i_cj_c,c}x_{i_c}x_{j_c}
\]

| (2) | if \( P_{i_cj_c,c} = 0 \) then |

\[
V(x)_c = 2x_{i_c} \left( P_{ij,c}x_j + \sum_{i \neq i_c, j_c} P_{i_c,c}x_i \right)
\]

**Proof.** Let us start with ”if” part, i.e. we assume that \( V \) is an OP QSOs. From the assumption \( V(e_k) = \mathbb{F}_k \) and the definition of QSO we have

\[
V(e_k) = (P_{kk,1}, \ldots, P_{kk,m}, \ldots) = \mathbb{F}_k
\]
This implies that
\[(3.4)\]
\[P_{k,k,m} = \begin{cases} 0 & \text{if } m \not\in \text{supp}(F_k), \\ f_{k,m} & \text{if } m \in \text{supp}(F_k), \end{cases} \]

By choosing
\[(3.5)\]
\[x_k = (x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots) \text{ such that } x_i > 0, \text{ for } i \in \mathbb{N}\setminus\{k\}\]
and \(e_k\) one has \(x_k \perp e_k\). Due to the assumption, we infer that \(V(x_k) \perp V(e_k)\). It is clear that
\[V(x_k) = \left( \sum_{i,j \neq k} P_{ij,1} x_i x_j, \ldots, \sum_{i,j \neq k} P_{ij,m} x_i x_j, \ldots \right).\]

Thus, from the fact \(V(x_k) \circ V(e_k) = 0\) and (3.5), we immediately find
\[\sum_{i,j \neq k} P_{ij,m} x_i x_j = 0 \Rightarrow P_{ij,m} = 0 \text{ for any } i, j \neq k \text{ and } m \in \text{supp}(F_k).\]

Hence, for any \(m \in \text{supp}(F_{\pi(k)})\) and for any \(x \in S\)
\[V(x)_m = \sum_{i,j = 1}^{\infty} P_{ij,m} x_i x_j \]
\[= P_{k,k,m} x_k^2 + \sum_{i \neq k} P_{ik,m} x_i x_k + \sum_{j \neq k} P_{kj,m} x_j x_k \]

Keeping in mind \(P_{ik,m} = P_{ki,m}, x_k = 1 - \sum_{i \neq k} x_i\) and (3.4), \((V(x))_m\) reduces to
\[V(x)_m = x_k \left( f_{k,m} + \sum_{i \neq k} (2P_{ik,m} - f_{k,m}) x_i \right) = x_k \left( f_{k,m} + \sum_{i=1}^{\infty} a^{(m)}_{ik} x_i \right) \]
which shows (a).

Next, let us consider \(c \in C_x\). Then
\[(V(x))_c = \sum_{i,j = 1}^{\infty} P_{ij,c} x_i x_j \]
\[= \sum_{i=1}^{\infty} P_{ii,c} x_i^2 + \sum_{i \neq 1}^{\infty} P_{i1,c} x_1 x_i + \cdots + \sum_{i \neq n}^{\infty} P_{in,c} x_n x_i + \cdots \]
\[= \sum_{i=1}^{\infty} P_{ii,c} x_i^2 + 2 \sum_{i=2}^{\infty} P_{i1,c} x_1 x_i + \cdots + 2 \sum_{i=n}^{\infty} P_{in,c} x_n x_i + \cdots \]
\[= \sum_{i=1}^{\infty} P_{ii,c} x_i^2 + 2 \sum_{j=1}^{\infty} \sum_{i=j+1}^{\infty} P_{ij,c} x_i x_j \]

Taking into account (3.4), one gets \(P_{kk,c} = 0\) for any \(k \in \mathbb{N}\) and \(c \in C_x\). Therefore
\[V(x)_c = 2 \sum_{j=1}^{\infty} \sum_{i=j+1}^{\infty} P_{ij,c} x_i x_j \]
\[(3.6)\]
First, we assume that there exist \( i_c, j_c \in \mathbb{N} \) such that \( P_{i_c, j_c} > 0 \) (if it is not the case, then we get (I) i.e., \( V(x)_c = 0 \)). Next, let us choose two vectors from the simplex \( S \) as follows

\[
x^{(i_c, j_c)} = \left( 0, \ldots, 0, \frac{1}{2}, 0, \ldots, 0, \frac{1}{2}, 0 \right)
\]

\[
y^{[i_c, j_c]} = (y_1, \ldots, y_{i_c-1}, 0, y_{i_c}+1, \ldots, y_{j_c} - 1, 0, y_{j_c} + 1, \ldots)
\]

where \( y_i > 0 \) for any \( i \in \mathbb{N} \setminus \{i_c, j_c\} \). Clearly \( x^{(i_c, j_c)} \) is orthogonal to \( y^{[i_c, j_c]} \), hence by assumption on \( V \)

\[
V(x^{(i_c, j_c)}) \perp V(y^{[i_c, j_c]})
\]

From the part (a), one gets

\[
V(x^{(i_c, j_c)})_k \cdot V(y^{[i_c, j_c]})_k = 0 \quad \forall \ k \in \bigcup_{i \in \mathbb{N}} supp(F_i)
\]

Using (3.6), one has

\[
V(x^{(i_c, j_c)})_c = \frac{1}{2} P_{i_c, j_c, c} \quad \text{and} \quad V(y^{[i_c, j_c]})_c = 2 \sum_{j=1}^{\infty} \sum_{i=j+1}^{\infty} P_{i_c, j_c} y_i y_j
\]

Due to (3.7) and the assumption \( P_{i_c, j_c} > 0 \) one infers that \( V(x^{(i_c, j_c)})_c \cdot V(y^{[i_c, j_c]})_c = 0 \) whence

\[
P_{i_c, j_c} = 0, \quad \forall \ i, j \in \mathbb{N} \setminus \{i_c, j_c\}
\]

Moreover, we are interested to find the following coefficients

\[
P_{i_c, i_c}, \quad P_{i_c, j_c} \quad \text{for all} \quad i \in \mathbb{N} \setminus \{i_c, j_c\}
\]

Furthermore, we assume, there exists \( i_{c_0} \) such that \( P_{i_{c_0}, j_c} > 0 \) for either \( j = i_c \) or \( j = j_c \) (here let \( j = i_c \)) (if it is not the case, then \( V(x)_c = P_{i_c, j_c} x_{i_c} y_{j_c} \) which gives (i)). Without the loss of generality, we may consider \( i_{c_0} < i_c \). Next, let us choose

\[
x^{(i_{c_0}, i_c)} = \left( 0, \ldots, 0, \frac{1}{2}, 0, \ldots, 0, \frac{1}{2}, 0 \right)
\]

\[
y^{[i_{c_0}, i_c]} = (y_1, \ldots, y_{i_{c_0}-1}, 0, y_{i_{c_0}+1}, \ldots, y_{i_c} - 1, 0, y_{i_c} + 1, \ldots)
\]

Using the facts from (3.6) and (3.8), one finds

\[
V(x^{(i_{c_0}, i_c)})_c = \frac{1}{2} P_{i_{c_0}, i_c, c} \quad \text{and} \quad V(y^{[i_{c_0}, i_c]})_c = 2 \sum_{i=1}^{\infty} \sum_{i \neq i_{c_0}, i_{c_0}} P_{i_c, j_c} y_i y_{j_c}
\]

Hence, by the same argument as before \( P_{i_c, j_c} = 0 \) for any \( i \in \mathbb{N} \setminus \{i_{c_0}, i_c\} \). Here, we consider two subcases:
Case 1. Let $P_{i_0,j_c,c} > 0$. By the same argument as before and choosing

$$x^{(i_0,j_c)} = \left(0, \ldots, 0, \frac{1}{2}, 0, \ldots, 0, \frac{1}{2}, 0\right)$$

we obtain $P_{i_k,c} = 0$ for any $i \in \mathbb{N}\setminus\{i_0, j_c\}$. Therefore, in this case, we can write $V(x)_c$ in the form given by (1).

Case 2. In this case, we suppose that $P_{i_0,j_c,c} = 0$. Then, it is clear that we find (2).

Now let us turn to "only if" part. This part comes directly from the fact $x \perp y$, i.e. $x_k \cdot y_k = 0$ for all $k \in \mathbb{N}$. The orthogonality of $x$ and $y$ implies that, for any fixed $k \in \mathbb{N}$, either $x_k = 0$ or $y_k = 0$. Therefore, if $m \in \text{supp}(\mathbb{F}_{\pi(k)})$, $k \in \mathbb{N}$, then from (a) one finds $V(x)_m \cdot V(y)_m = 0$.

Using (b) one can check that we have

$$V(x)_c \cdot V(y)_c = 0 \quad \text{for all } c \in C_F$$

This completes the proof. $\square$

We point out that if $F = \{\mathbb{F}_k\}$ is an orthogonal system, then for any injective mapping $\pi : \mathbb{N} \to \mathbb{N}$, the system $F_{\pi} = \{\mathbb{F}_{\pi(k)}\}$ is also orthogonal. Hence, the previous theorem will still remain valid for $\{\mathbb{F}_{\pi(k)}\}$.

Corollary 3.5. Let $F = \{\mathbb{F}_k\}$ be an orthogonal system and $V$ be a QSO such that $V(e_k) = \mathbb{F}_{\pi(k)}$, $k \in \mathbb{N}$, for some injective mapping $\pi : \mathbb{N} \to \mathbb{N}$. Then, $V(x)$ is an OP QSO if and only if it has the following form, for any $x \in S$:

(a) For any $m \in \text{supp}(\mathbb{F}_{\pi(k)})$

$$V(x)_m = x_k \left(f_{\pi(k),m} + \sum_{i=1}^{\infty} a^{(m)}_{ik} x_i\right)$$

where $a^{(m)}_{ik} = 2P_{ik,m} - f_{\pi(k),m}$ and set $a^{(m)}_{kk} = 0$.

(b) For any $c \in C_F$, $V(x)_c$ takes one of the following form

(I) If there is no $P_{ij,c} > 0$ for every $i, j \in \mathbb{N}$, then $V(x)_c = 0$ or

(II) If there exist at least one $P_{ij,c,c} > 0$, then $V(x)_c$ has one of the following form:

(i) If there is no $P_{ij,c} > 0$ for $j \in \{i, j_c\}$ where $i \in \mathbb{N}\setminus\{j\}$, then

$$V(x)_c = 2P_{ij,c,c} x_i x_{jc}$$

(ii) If there exist $P_{ij,c,c} > 0$ for either $j = i_c$ or $j = j_c$ (here let $j = i_c$), then $V(x)_c$ has one of the following form:

(1) If $P_{i_0,j_c,c} > 0$ then

$$V(x)_c = 2 \left(P_{i_0,j_c,c} x_{i_0} x_{jc} + P_{i_0,j_c,c} x_{i_0} x_{jc} + P_{i_0,j_c,c} x_{i_0} x_{i_c}\right)$$
(2) If $P_{i_0,j,c} = 0$ then

$$V(x)_c = 2x_c \left( P_{i_0,j,c}x_{j,c} + \sum_{i \neq i_0,j} P_{i,c}x_i \right)$$

An immediate consequence of the theorem is the following result.

**Corollary 3.6.** Let $F = \{\mathbb{F}_k\}$ be an orthogonal system and $V$ be a QSO such that $V(e_k) = \mathbb{F}_{\pi(k)}$, $k \in \mathbb{N}$, for some injective mapping $\pi : \mathbb{N} \rightarrow \mathbb{N}$. Then $V(x)$ is an OP QSO if and only if the heredity coefficients $P_{ij,c}$ satisfy the following conditions:

(a) $P_{i,k} = f_{\pi(i),k}$ for $k \in \text{supp}(\mathbb{F}_{\pi(i)})$, and $P_{ij,k} = 0$ for $k \notin \{\text{supp}(\mathbb{F}_{\pi(i)}) \cup \text{supp}(\mathbb{F}_{\pi(j)})\}$

(b) The coefficients $P_{ij,c}$, where $c \in C_{F_k}$, satisfy one of the following conditions:

(I) $P_{ij,c} = 0$ for all $i, j \in \mathbb{N}$ or

(II) If there exist $P_{i,j,c} > 0$, then $P_{ij,c} = 0$ for any $i, j \in \mathbb{N}\{i, j, c\}$. Further, the other coefficients must satisfy one of the following:

(i) $P_{ij,c} = 0$ for $j \in \{i, j, c\}$ for all $i \in \mathbb{N}\{j\}$ or

(ii) If there exist $P_{i_0,j,c} > 0$ for either $j = i_0$ or $j = j_0$ (here we let $j = i_0$), then $P_{ij,c} = 0$ for any $i \in \mathbb{N}\{i, j, i_0\}$. Moreover one of the following must be satisfied:

(1) $P_{i_0,j,c} > 0$, then $P_{i,c} = 0$ for any $i \in \mathbb{N}\{i, j, i_0\}$ or

(2) $P_{i_0,j,c} = 0$

**Corollary 3.7.** Let $F = \{\mathbb{F}_k\}$ be an orthogonal system and $T$ is a LSO on $S$ such that $T(e_k) = \mathbb{F}_{\pi(k)}$ for any $k \in \mathbb{N}$ and an injective mapping $\pi : \mathbb{N} \rightarrow \mathbb{N}$, then $T$ is an OP linear stochastic operator if and only if $T$ takes the following form:

(i) For any $m \in \text{supp}(\mathbb{F}_{\pi(k)})$

$$T(x)_m = f_{k,m}x_k$$

(ii) For any $c \in C_{F_k}$

$$T(x)_c = 0$$

**Remark 3.8.** Let $F = \{\mathbb{F}_k\}$ be an orthogonal system. One of the important class of infinite dimensional OP QSO is when the union of the supports of $\{\mathbb{F}_k\}$ cover $\mathbb{N}$. So, let $V$ be a QSO such that $V(e_k) = \mathbb{F}_{\pi(k)}$ for some injective mapping $\pi : \mathbb{N} \rightarrow \mathbb{N}$, and $C_{F_k} = \emptyset$. Then $V$ is OP if and only if one has

(i) $V$ has the form given by

$$V(x)_m = x_k \left( f_{\pi(k),m} + \sum_{i=1}^{\infty} a_{ik}^{(m)} x_i \right)$$

for any $m \in \text{supp}(\mathbb{F}_{\pi(k)})$.

(ii) The heredity coefficients $P_{ij,k}$ satisfy

$$P_{i,k} = f_{i,k} \quad \forall k \in \text{supp}(\mathbb{F}_{\pi(i)}) \quad \text{and} \quad P_{ij,k} = 0 \quad \forall k \notin \{\text{supp}(\mathbb{F}_{\pi(i)}) \cup \text{supp}(\mathbb{F}_{\pi(j)})\}$$

Now, it is natural to consider an orthogonal system $F = \{\mathbb{F}_k\}$ of $S$ such that the support of each (or some) $\mathbb{F}_k$ is countable. Let us provide an example of such kind of orthogonal system.
Take $A_j = \{j^{2n} : n \geq 0\}$, $j \in 2\mathbb{N} - 1$. It is clear that $\{A_j\}$ is a cover for $\mathbb{N}$. Now, for each $j \in 2\mathbb{N} - 1$ we define $F_j = (f_m^{(j)})_{m=1}^{\infty}$ as follows: for each $j \in 2\mathbb{N} - 1$ define
\[
 f_m^{(j)} = \begin{cases} 
 \frac{j-1}{j} \left(\frac{1}{2}\right)^n, & m = j^{2n}, \ n \geq 0, \\
 0, & m \notin A_j.
\end{cases}
\]

One can see that the system $\{F_j\}_{j \in 2\mathbb{N} - 1}$ is orthogonal and $\text{supp}(F_j) = A_j$, $j \in 2\mathbb{N} - 1$.

Let us consider some examples of OP QSO defined on $S$.

**Example 3.9.** Now we are going to produce an example of quadratic shift operator. Assume that a QSO $V$ such that $V(e_i) = e_{i+1}$ for every $i \in \mathbb{N}$. From Corollary 3.6 one gets $P_{i,i+1} = 1$ for any $i \in \mathbb{N}$. Choose $P_{1,2} = 0$ for any $i \geq 2$. Next, we take for any $k \geq 2$
\[
 P_{ik,k+1} = 1 \text{ for } i \in \{1, 2, \ldots, k - 1\} \text{ and } P_{ik,k+1} = 0 \text{ for } i \geq k + 1
\]
From the selected heredity coefficients, we have $P_{ij,1} = 0$ for any $i, j \in \mathbb{N}$ and it is clear that they satisfy (2.2) hence $V$ is well-defined. Thus, using Theorem 3.3 one gets
\[
 (V(x))_k = \begin{cases}
 0, & k = 1,
 x_1^2, & k = 2,
 x_k \left(1 + \sum_{i=1, i \neq k}^{\infty} (2P_{ik,k+1} - 1)x_i \right), & k \geq 3
\end{cases}
\]
\[
 = \begin{cases}
 0, & k = 1,
 x_1^2, & k = 2,
 x_k \left(\sum_{i=1}^{k-1} 2x_i + x_k \quad \forall \right) & k \geq 3
\end{cases}
\]

Note that $V$ is a concrete example of nonlinear shift operator.

4. **Properties of OP QSO**

In this section we are going to investigate some properties of infinite dimensional OP QSO.

In what follows, we consider proper subsets of $\mathbb{N}$, i.e. $\alpha \subset \mathbb{N}$ with $\alpha \neq \mathbb{N}$. For a given $\alpha \subset \mathbb{N}$, we denote
\[
 \Gamma_\alpha = \{x \in S : x_i = 0, \ \forall i \notin \alpha\}, \quad \text{ri} \Gamma_\alpha = \{x \in \Gamma_\alpha : x_i > 0, \ \forall i \in \alpha\}
\]
By $Fix(V)$ we denote the set of all fixed points of $V$, i.e. $Fix(V) = \{x \in S : V(x) = x\}$. Let $\mathcal{F} = \{F_k\}$ be an orthogonal system of $S$. By $\mathcal{V}_\mathcal{F}$ we denote the set of all OP QSO which are generated by the orthogonal system $\mathcal{F}$, i.e. $V \in \mathcal{V}_\mathcal{F}$ means $V(e_k) = F_k$ for any $k \in \mathbb{N}$.

Denote
\[
 \text{supp}(\mathcal{F}) = \bigcup_{k=1}^{\infty} \text{supp}(F_k)
\]

**Lemma 4.1.** Let $\mathcal{F} = \{F_k\}$ be an orthogonal system such that $\text{supp}(\mathcal{F}) = \mathbb{N}$ and $V \in \mathcal{V}_\mathcal{F}$. Then for any $\alpha \subset \mathbb{N}$ one has
(i) $V(\Gamma_\alpha) \subset \Gamma_{\alpha'}$;
(ii) $V(\text{ri} \Gamma_\alpha) \subset \text{ri} \Gamma_{\alpha'}$. 

where
\[ \alpha' = \bigcup_{\ell \in \alpha} \text{supp}(F_{\ell}). \]

**Proof.** (i) Let \( V \in \mathcal{V}_F \), then due to Remark 3.8 \( V \) takes the following form

\[ (4.1) \quad V(x)_k = x_\ell \left( f_{\ell,k} + \sum_{i=1}^{\infty} a^{(k)}_{i\ell} x_i \right) \]

for any \( k \in \text{supp}(F_{\ell}), \ell \in \mathbb{N}. \)

Now let \( x = (x_1, x_2, \ldots) \in \Gamma_\alpha \), then \( x_\ell = 0 \) for any \( \ell \notin \alpha \), hence from (4.1) one finds

\[ (4.2) \quad V(x)_k = 0, \quad \text{for all } k \in \text{supp}(F_{\ell}), \ell \notin \alpha, \]

this is the assertion (i).

Now take \( x = (x_1, x_2, \ldots) \in ri \Gamma_\alpha \), then \( x_\ell > 0 \) for all \( \ell \in \alpha \). From (4.1) one gets

\[
V(x)_k = x_\ell \left( f_{\ell,k} + \sum_{i=1}^{\infty} a^{(k)}_{i\ell} x_i \right)
\[
= x_\ell \left( f_{\ell,k} + \sum_{i=1 \atop i \neq \ell}^{\infty} (2P_{i\ell,k} - f_{i,k}) x_i \right)
\[
= x_\ell \left( f_{\ell,k} + \sum_{i=1 \atop i \neq \ell}^{\infty} 2P_{i\ell,k} x_i - f_{\ell,k} \sum_{i=1 \atop i \neq \ell}^{\infty} x_i \right)
\[
= x_\ell \left( f_{\ell,k} + \sum_{i=1 \atop i \neq \ell}^{\infty} 2P_{i\ell,k} x_i - f_{\ell,k} (1 - x_\ell) \right)
\[
= x_\ell \left( \sum_{i \in \alpha \atop i \neq \ell} 2P_{i\ell,k} x_i + f_{i,k} x_\ell \right)
\[
\geq f_{\ell,k} x_\ell^2 > 0
\]

for \( k \in \text{supp}(F_{\ell}) \). This means \( V(ri \Gamma_\alpha) \subset ri \Gamma'_{\alpha'} \). Moreover, using (4.3) we have

\[
\text{supp}(V(x)) = \bigcup_{\ell \in \alpha} \text{supp}(F_{\ell}).
\]

This completes the proof. \( \square \)
Now it is natural to consider the case $\text{supp}(\mathcal{F}) \subseteq \mathbb{N}$. According to Theorem 3.3 for any $c \in \mathcal{C}_\mathcal{F}$ (here as before, $\mathcal{C} = \mathbb{N} \setminus \text{supp}(\mathcal{F})$), $V(x)_c$ takes one of the following form

\[
\begin{align*}
(i) & \quad V(x)_c = 0 \\
(ii) & \quad V(x)_c = 2P_{i \in \mathcal{C}, c}x_i x_c \\
(iii) & \quad V(x)_c = 2\left(P_{i \neq i_c, c}x_i x_j + P_{i \neq i_c, c}x_i x_{i_0} + P_{i \neq i_c, c}x_i x_{i_0} x_i \right) \\
(iv) & \quad V(x)_c = 2x_i \left(P_{i \neq i_c, c}x_j + \sum_{i = 1}^\infty P_{i \neq i_c, c}x_i \right)
\end{align*}
\]

From now on, let us keep the notation that we have used in Theorem 3.3 (i.e., $i_c, j_c, i_0$). To get an analogous result like in Lemma 4.1, it is enough for us to study the coordinates belonging to $\mathcal{C}_\mathcal{F}$ while $V(x)_c$ takes one of the forms given by (ii), (iii) and (iv), since the case $m \in \text{supp}(\mathcal{F}_k)$ is already described by Lemma 4.1.

Let us take $\alpha \subseteq \mathbb{N}$. Now we consider the mentioned cases one by one.

**CASE (ii).** In this case, we have the following possibilities:

(I) $i_c, j_c \in \alpha$; (II) $i_c \in \alpha, j_c \notin \alpha$; (III) $j_c \in \alpha, i_c \notin \alpha$; (IV) $i_c, j_c \notin \alpha$.

**CASE (iii).** In this case, we have the following ones:

(I) $i_c, j_c, i_0 \in \alpha$; (II) $i_c \in \alpha, j_c, i_0 \notin \alpha$; (III) $i_c, j_c \in \alpha, i_0 \notin \alpha$; (IV) $i_c, i_0 \in \alpha, j_c \notin \alpha$.

(V) $j_c \in \alpha, i_c, i_0 \notin \alpha$; (VI) $j_c, i_0 \in \alpha, i_c \notin \alpha$; (VII) $i_c, j_c, i_0 \notin \alpha$; (VIII) $i_0 \in \alpha, j_c, i_c \notin \alpha$.

**CASE (iv).** This case is the same like CASE (ii).

**Remark 4.2.** Let $V \in \mathcal{V}_\mathcal{F}$ such that $\text{supp}(\mathcal{F}) \subseteq \mathbb{N}$. For any $\alpha \subseteq \mathbb{N}$ we have the following statements:

(a) Let $c \in \mathcal{C}_\mathcal{F}$, then $V(x)_c$ takes the form as given by (ii). If (I) is satisfied then $V(x)_c > 0$ and in the other cases $V(x)_c = 0$.

(b) Let $c \in \mathcal{C}_\mathcal{F}$ then $V(x)_c$ takes the form as given by (iii). If (I), (III), (IV) and (VI) are satisfied then $V(x)_c > 0$ and in the other cases $V(x)_c = 0$.

(c) Let $c \in \mathcal{C}_\mathcal{F}$ then $V(x)_c$ takes the form as given by (iv). If

- (I) is satisfied then $V(x)_c > 0$
- (II) is satisfied and there exist $i_0 \in \alpha$ such that $P_{i_0, c} > 0$ (if not, then $V(x)_c = 0$),
then $V(x)_c > 0$

In the other cases $V(x)_c = 0$.

Let $V$ be a OP QSO generated by an orthogonal system $\mathcal{F} = \{\mathcal{F}_k\}$, i.e. $V(e_k) = \mathcal{F}_k, k \in \mathbb{N}$. Now want to distinguish a set where some of elements of the system $\mathcal{F}$ coincides with certain elements of the standard basis. Namely, let us denote

$$\beta = \{k \in \mathbb{N} : \mathcal{F}_k = e_i \text{ for some } i \in \mathbb{N}\}$$

**Theorem 4.3.** Let $V \in \mathcal{V}_\mathcal{F}$. If $\beta = \emptyset$, then for any $\alpha \subseteq \mathbb{N}$, one has

$$\text{Fix}(V) \notin \Gamma_\alpha.$$  

Moreover, if the fixed point exists, then Fix$(V) \in \text{riS}$.
Proof. Assume that for a fixed point \( \mathbf{x}_0 \in S \) one has \( \mathbf{x}_0 \in \Gamma_\alpha \) for some \( \alpha \subset \mathbb{N} \). This means
\[
V(\mathbf{x}_0)_k = 0 \quad \text{if} \quad k \notin \alpha, \\
V(\mathbf{x}_0)_k > 0 \quad \text{if} \quad k \in \alpha.
\]

Now we consider two separate cases: \((\text{supp}(\mathcal{F}) = \mathbb{N})\) and \((\text{supp}(\mathcal{F}) \subset \mathbb{N})\).

Case 1. Let us suppose \((\text{supp}(\mathcal{F}) = \mathbb{N})\). Since \( \mathbf{x}_0 \) is a fixed point, then one has
\[
\text{supp}(V(\mathbf{x}_0)) = \alpha
\]
(4.5)

On the other hand, due to the assumption \( \beta = \emptyset \) and from Lemma 4.1, we get
\[
|\text{supp}(\mathcal{F}_k)| \geq 2 \quad \text{for} \quad k \in \alpha
\]
and
\[
\text{supp}(V(\mathbf{x}_0)) = \bigcup_{\ell \in \alpha} \text{supp}(\mathcal{F}_\ell)
\]
Therefore
\[
|\text{supp}(V(\mathbf{x}_0))| > |\alpha|
\]
which contradicts to (4.5). Therefore, the fixed point cannot be in the face \( \Gamma_\alpha \) for any \( \alpha \subset \mathbb{N} \).

Part 2 \((\text{supp}(\mathcal{F}) \subset \mathbb{N})\). Take any \( \alpha \subset \mathbb{N} \). Now we are going to consider the following three possible cases: \( \mathcal{C}_\mathcal{F} \cap \alpha = \emptyset \), \( \mathcal{C}_\mathcal{F} \cap \alpha \neq \emptyset \), \( \alpha \not\subset \mathcal{C}_\mathcal{F} \), and \( \alpha \subset \mathcal{C}_\mathcal{F} \).

In the first case, we obtain the desired result by the same argument as in Part 1.

Now we consider the case: \( \alpha \not\subset \mathcal{C}_\mathcal{F} \neq \emptyset \). Let \( \mathbf{x}_0 \in ri\Gamma_\alpha \) which implies (4.5). On the other hand, we have \( |\text{supp}(\mathcal{F}_{\pi(k)})| \geq 2 \) for all \( k \in \alpha \not\subset \mathcal{C}_\mathcal{F} \), therefore using Lemma 4.1 one concludes that
\[
|\text{supp}(V(\mathbf{x}_0))| = |\{\alpha \cap \mathcal{C}_\mathcal{F}\} \cup \bigcup_{k \in \alpha \not\subset \mathcal{C}_\mathcal{F}} \text{supp}(\mathcal{F}_{\pi(k)})| > |\alpha|
\]
which contradicts to (4.5).

Let us turn to the last case, i.e. \( \alpha \subset \mathcal{C}_\mathcal{F} \). Due to \( \mathbf{x}_0 \in ri\Gamma_\alpha \) we get (4.3) and
\[
\sum_{k \in \alpha} V(\mathbf{x}_0)_k = 1
\]
(4.7)

On the other hand, by taking into account that \( \mathbf{x}_0 \in ri\Gamma_\alpha \) and \( P_{i,c} = 0 \) for any \( i \in \mathbb{N} \), \( c \in \mathcal{C}_\mathcal{F} \), then one finds
\[
\sum_{k \in \alpha} V(\mathbf{x}_0)_k = \sum_{k \in \alpha} \sum_{i,j \in \alpha} P_{i,j,k} x_i x_j = \sum_{i,j \in \alpha} x_i x_j \left( \sum_{k \in \alpha} P_{i,j,k} \right)
\]
(4.8)

Since \( \sum_{k \in \mathbb{N}} P_{i,j,k} = 1 \), we then obtain
\[
\sum_{k \in \alpha} V(\mathbf{x}_0)_k \leq \sum_{i,j \in \alpha} x_i x_j = \sum_{i \in \alpha} x_i \left( \sum_{j \in \alpha \setminus \{i\}} x_j \right)
\]
(4.9)

Again \( \mathbf{x}_0 \in ri\Gamma_\alpha \) implies
\[
\sum_{j \in \alpha \setminus \{i\}} x_j < \sum_{j \in \alpha} x_j = 1 \quad \text{for any} \quad i \in \alpha
\]
Therefore,
\[
\sum_{k \in \alpha} V(x_0)_k \leq \sum_{i,j \in \alpha, i \neq j} x_i x_j < \sum_{i \in \alpha} x_i = 1
\]
which contradicts to (4.7).

Furthermore, according to the arbitrariness of \(\alpha \subset \mathbb{N}\), we infer that if a fixed point \(x_0\) exists, then \(x_0 \in riS\). This completes the proof. \(\square\)

**Remark 4.4.** Let \(\mathcal{F} = \{F_k\}\) be an orthogonal system in \(S\) and \(\alpha \subset \mathbb{N}\). Then we have
\[
\text{supp}(\{F_k\}_{k \in \alpha}) = \text{supp}(\{e_k\}_{k \in \alpha})
\]
if and only if there is a permutation \(\pi_\alpha\) of \(\alpha\) such that \(\{F_k\}_{k \in \alpha} = \{e_{\pi_\alpha(k)}\}_{k \in \alpha}\).

**Theorem 4.5.** Let \(V\) be an OP QSO generated by \(V(e_k) = F_k\) for any \(k \in \mathbb{N}\) and let set \(\beta \neq \emptyset\). Assume that for any \(\alpha \subset \beta\) one has
\[
\{F_k\}_{k \in \alpha} \neq \{e_{\pi_\alpha(k)}\}_{k \in \alpha}
\]
for any permutation \(\pi_\alpha\) of \(\alpha\). Then for any \(\alpha \subset \mathbb{N}\) one has
\[
\text{Fix}(V) \notin \Gamma_\alpha
\]
Moreover, if a fixed point exists, then \(\text{Fix}(V) \in riS\).

**Proof.** Assume that for a fixed point \(x_0 \in S\) one has \(x_0 \in \Gamma_\alpha\) for some \(\alpha \subset \mathbb{N}\). Without loss of generality we may assume that \(x_0 \in ri\Gamma_\alpha\). Now we consider two possibilities \(\text{supp}(\mathcal{F}) = \mathbb{N}\) and \(\text{supp}(\mathcal{F}) \subset \mathbb{N}\).

**Part 1 (\(\text{supp}(\mathcal{F}) = \mathbb{N}\)).** There are several possibilities:

(a) \(\alpha \cap \beta = \emptyset\)
(b) \(\alpha \cap \beta \neq \emptyset, \alpha \not\subset \beta\)
(c) \(\alpha \subset \beta\)

Cases (a) and (b) follow from the same argument as in the proof of Theorem 4.3, since there exists some \(k_0 \in \alpha \setminus \beta\) such that \(\text{supp}(F_{k_0}) \geq 2\).

Let us consider the case (c), i.e. \(\alpha \subset \beta\). Due to our assumption, we have
\[
\text{supp}(x_0) = \text{supp}(V(x_0)) = \text{supp}(\{e_k\}_{k \in \alpha}) = \alpha
\]
From Lemma 4.1 one gets that
\[
\text{supp}(V(x_0)) = \text{supp}(\{F_k\}_{k \in \alpha})
\]
From (4.11), (4.12) and Remark 4.4 we conclude that there is a permutation \(\pi_\alpha\) of \(\alpha\) such that
\[
\{F_k\}_{k \in \alpha} = \{e_{\pi_\alpha(k)}\}_{k \in \alpha}
\]
which contradicts to the assumption of the theorem.

**Part 2 (\(\text{supp}(\mathcal{F}) \subset \mathbb{N}\)).** Since we have already considered all possible situations of \(\alpha\) and \(\beta\), therefore, then it is enough for us to consider the following cases: \(\alpha \cap C_\mathcal{F} = \emptyset\), \(\alpha \cap C_\mathcal{F} \neq \emptyset\), \(\alpha \not\subset C_\mathcal{F}\) and \(\alpha \subset C_\mathcal{F}\). These cases can be proceeded by the same argument as in the proof of Theorem 4.3. This completes the proof. \(\square\)

Now we want provide certain examples which satisfy the conditions of the last theorem.
Example 4.6. Let us consider the following orthogonal system:

\[
F_1 = \left( \frac{1}{2}, \frac{1}{2}, 0, \ldots \right), \quad F_2 = e_3, \quad F_3 = e_4, \quad F_4 = e_5,
\]

\[
F_n = \begin{pmatrix}
0, \ldots, 0, \frac{1}{2}, \frac{1}{2}, 0, \ldots \n\end{pmatrix} \quad \text{for } n \geq 5
\]

Let \( V \) be generated as follows \( V(e_k) = F_k, \ k \in \mathbb{N} \). One can see that the set \( \beta = \{2, 3, 4\} \) and for any subset \( A \subset \beta \) we have

\[
\{F_k\}_{k \in A} \neq \{e_k\}_{k \in A}
\]

Then, due to Theorem 4.5 for any \( \alpha \subset \mathbb{N} \), we have \( \operatorname{Fix}(V) \notin \Gamma_\alpha \).

Example 4.7. Let us consider the following orthogonal system:

\[
F_1 = \left( \frac{1}{2}, \frac{1}{2}, 0, \ldots \right), \quad F_2 = e_6, \quad F_3 = e_7, \quad F_4 = e_8,
\]

\[
F_n = \begin{pmatrix}
0, \ldots, 0, \frac{1}{2}, \frac{1}{2}, 0, \ldots \n\end{pmatrix} \quad \text{for } n \geq 5
\]

Let \( V \) be generated as follows \( V(e_k) = F_k, \ k \in \mathbb{N} \). One can see that \( \beta = \{2, 3, 4\} \) and \( C_F = \{3, 4, 5\} \). Moreover, one has for any subset \( A \subset \beta \)

\[
\{F_k\}_{k \in A} \neq \{e_k\}_{k \in A}
\]

Then, due to Theorem 4.5 for any \( \alpha \subset \mathbb{N} \), we have \( \operatorname{Fix}(V) \notin \Gamma_\alpha \).

It is well-known that an infinite-dimensional simplex \( S \) is not compact either in \( \ell_1 \) topology, nor in a weak topology, therefore, the existence of a fixed point of any QSO \( V \) defined on \( S \) is not always true.

Example 4.8. Let us consider an OP QSO \( V \) defined by

\[
V(x_1, x_2, \ldots, x_n, \ldots) = (0, x_1, x_2, \ldots, x_n, \ldots)
\]

where \( (x_n) \in S \). It is easy to see that this operator has no fixed points belonging to \( S \).

Next result provides a sufficient condition for the existence of a fixed point of OP QSO.

Proposition 4.9. Let \( V \in \mathcal{V}_F \) with \( \operatorname{supp}(F) = \mathbb{N} \). If \( \beta \neq \emptyset \) and there exists a subset \( \alpha \subset \beta \) with \( |\alpha| < \infty \) such that

\[
\{e_{\pi(k)}\}_{k \in \alpha} = \{F_k\}_{k \in \alpha}
\]

for some permutation \( \pi \) of \( \alpha \). Then there exists a fixed point \( x_0 \in \Gamma_\alpha \).

Proof. Let \( \alpha = \{i_1, \ldots, i_n\} \subset \beta \). By the definition of \( V \) we infer that

\[
(4.13) \quad V(e_{i_k}) = e_{\pi(i_k)} \quad \text{for all } k \in \{1, \ldots, n\}
\]

and

\[
V(e_m) = F_m \quad \text{for all } m \in \mathbb{N} \setminus \alpha
\]
Due to Corollary 3.8, the operator $V$ can be written in the following form, for any $x \in \Gamma_\alpha$

$$
\begin{align*}
V(x)_{\pi(i)} &= x_i \left( 1 + \sum_{\ell \neq i, \ell \in \alpha} (2P_{i,j} - 1) x_\ell \right), \quad i \in \alpha \\
V(x)_k &= 0 \text{ if } k \notin \alpha
\end{align*}
$$

This implies that $V(\Gamma_\alpha) \subset \Gamma_\alpha$. The compactness of $\Gamma_\alpha$ with the Brouwer fixed-point Theorem yields the existence of a fixed point $x_0 \in \Gamma_\alpha$ of $V$. This finishes the proof. □

Immediately from the last proposition, one concludes the following corollary.

**Corollary 4.10.** Let $V \in \mathcal{V}_F$ with $\text{supp}(F) \subset \mathbb{N}$ and $\beta \neq \emptyset$. If $V(x)_c = 0$ for any $c \in \mathcal{C}_F$ and there exists a subset $\alpha \subseteq \beta$ with $|\alpha| < \infty$ such that

$${\{ e_{\pi(k)} \}_{k \in \alpha}} = {\{ F_k \}_{k \in \alpha}}$$

for some permutation $\pi$ of $\alpha$. Then there exists a fixed point $x_0 \in \Gamma_\alpha$.

We provide an example of OP QSO that has fixed point.

**Example 4.11.** Let us consider the following orthogonal system:

$$
F_1 = e_1, \quad F_2 = e_2, \quad F_n = \begin{pmatrix} 0, \ldots, 0, \frac{1}{2}, \frac{1}{2}, 0, \ldots \end{pmatrix} \text{ for } n \geq 3
$$

Now let $V$ be an OP QSO such that

$$
V(e_1) = F_2, \quad V(e_2) = F_1, \quad V(e_k) = F_{k-1} \text{ for } k \geq 3
$$

One can see that $\beta = \{1, 2\}$, and for a permutation $\pi(1) = 2$, $\pi(1) = 2$, we have $\{ e_{\pi(k)} \}_{k \in \beta} = \{ F_k \}_{k \in \beta}$. For any $x \in \text{ri} \Gamma_\beta$, using Corollary 3.8, one gets

$$
\begin{align*}
V(x)_1 &= x_2 \left( 1 + (2P_{1,2} - 1)x_1 \right) \\
V(x)_2 &= x_1 \left( 1 + (2P_{2,1} - 1)x_2 \right)
\end{align*}
$$

In particular, assume that $P_{1,2} = P_{2,1} = \frac{1}{2}$, then clearly we have $\left( \frac{1}{2}, \frac{1}{2} \right)$ as a fixed point for the system (4.15). Clearly, $\left( \frac{1}{2}, \frac{1}{2}, 0, \ldots \right) \in \Gamma_{\{1,2\}}$ is a fixed point for $V$.

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