FROM LARGE N QUANTUM MECHANICS
TO PLANAR QUANTUM FIELD THEORY*

J. Wosiek

M. Smoluchowski Institute of Physics, Jagellonian University
Reymonta 4, 30-059 Krakow, Poland
E-mail: wosiek@th.if.uj.edu.pl

We review a performance of Fock space methods in calculating spectra of a range of supersymmetric models with gauge symmetry. Examples include: a) SU(2) Supersymmetric Yang Mills Quantum Mechanics in four euclidean dimensions, b) Quantum Mechanics of one fermion and one boson with infinite number of colours, and c) planar 1+1 dimensional Yang Mills theories with adjoint matter. Infrared divergencies of the latter theories with scalars are briefly discussed and a possible dynamical solution of the problem is suggested.

1 Quantum mechanics and a space reduced field theory

It is usually believed that simple quantum mechanics of the finite number of degrees of freedom can hardly teach us something about the quantum field theory with all its subtleties like renormalization, spontaneous symmetry breaking, etc. This is in general correct, however there exists a class of quantum mechanics which are being intensively studied precisely because of their connection to the field theoretical systems [1] [2] [3]. Consider the following hamiltonian

\[ H = \frac{1}{2} \sum_{i=1}^{D-1} p_i^2 + \frac{g^2}{4} \sum_{a=1}^{N^2-1} \sum_{b=1}^{N^2-1} f_{abc} f_{ade} x_d \frac{\partial}{\partial x_c} x_e \frac{\partial}{\partial x_e} + \frac{i g}{2} \sum_{a=1}^{N^2-1} \psi^a \Gamma^k \psi^a x_k, \]

It describes a quantum mechanical system with the finite number of degrees of freedom (e.g. 15 for D=4 and N=2) and results from the dimensional reduction of the D dimensional supersymmetric Yang-Mills theory to one point in the D-1 dimensional space. In
spite of the reduction, it still has many nontrivial properties inherited from the parent, space extended, theory \[4, 5, 6\] and can teach us quite a lot about the latter. Even the SU(2) model is not soluble for D=4. We have studied it numerically by constructing a gauge invariant basis of bosonic and fermionic Fock states, cutting off the total number of bosonic quanta and subsequently increasing the cutoff \[7\]. The algorithm worked very satisfactorily and we were able to uncover a rich, manifestly supersymmetric spectrum with coexisting localized and non-localized states, SUSY vacua and a fractional bulk value of the Witten index \[8, 9, 10\]. All these results agree with \[11\], and in some cases extend \[3\], theoretical predictions.

2 Large N quantum mechanics

There is much interest in the large N limit of quantum systems and above approach turns out to be very useful in studying such models as well. Direct calculations allow to obtain spectra for the first few lowest values of N. An extrapolation to \(N = \infty\) does not seem feasible, however. Nevertheless, it appears that one can calculate analytically matrix elements of typical hamiltonians directly at \(N = \infty\). \[12\]. We have studied with this method a close cousin of the space reduced \(SYM_{2}\) in the planar limit. Its hamiltonian reads

\[
H = \{Q, Q^\dagger\}, Q = \sqrt{2} Tr[f a^\dagger (1 + g a)], Q^\dagger = \sqrt{2} Tr[f^\dagger (1 + ga^\dagger) a],
\]

or explicitly

\[
H = Tr[a^\dagger a + g(a^{12} a + a^{\dagger 2} a^2) + g^2 a^{\dagger 2} a^2]
+ Tr[ f^\dagger f + g(f^\dagger f a^\dagger a + f^\dagger (a^\dagger + a) f]
+ g^2(f^\dagger a f a^\dagger f + f^\dagger a a^\dagger f + f^\dagger f a^\dagger a + f^\dagger a^\dagger f a)].
\]

It was found that, in spite of its simplicity, the system exhibits many interesting phenomena: unbroken supersymmetry, the phase transition in the ’t Hooft coupling, at \(\lambda = 1\), exact duality between the strong and weak coupling phases, rearrangement of supermultiplets across the transition point and emergence if the new vacua in the strong coupling phase \[12, 13, 14\]. All this was first discovered numerically and subsequently confirmed by the analytic solution. Moreover, at strong coupling the model is exactly equivalent to the XXZ Heisenberg spin chain and, at the same time, to the lattice gas of q-bosons. This proves existence of a hidden supersymmetry in these well explored statistical models \[15, 16\].
3 Planar field theories in 1+1 dimensions

Extension to the field theoretical systems is in principle straightforward. One has to diagonalize the Hamiltonian matrix calculated in the physical basis of harmonic oscillators. The obviously crucial difference is that now we deal with the infinite number of (e.g. momentum) degrees of freedom. Nevertheless one can define cutoff schemes which allow to extract meaningful continuum physics. There exist two popular ways to introduce a cutoff.

Light Cone Discretization (LCD) \cite{17} replaces the total momentum $P$ of a proton, say, by an integer $K$. This momentum can then be split between various numbers of partons, each carrying, an integer momentum $r_i > 0$. Therefore all partitions of $K$ into sets of integers $\{r\}$ define a finite Fock space bound by only one cutoff, $K$, which discretizes momenta and, at the same time, cuts the multiplicities of partons \cite{17,18,19}. The Hamiltonian matrix $\langle \{r\}|H|\{s\}\rangle$ is then calculated and diagonalized, similarly as in the case of quantum mechanics. The whole art consists of the meaningful extrapolation with $K \to \infty$.

The second approach consists of solving the Integral Equations (IE) \cite{20} which are equivalent to the eigenequation $H|\Phi\rangle = M^2|\Phi\rangle$ (in the (LC) formulation a Hamiltonian is proportional to a mass$^2$ operator). Decomposing a bound state into its Fock components $|\Phi\rangle = \sum_{n=2}^{\infty} \int [dx] \delta(1-x_1-x_2-...x_n) \Phi_n(x_1,x_2,...x_n)|x_1,...,x_n\rangle$, turns the eigenequation into an infinite hierarchy of integral equations

$$M^2\Phi_n(x_1...x_n) = A \otimes \Phi_n + B \otimes \Phi_{n-2} + C \otimes \Phi_{n+2},$$

(2)

where each term is a convolution of the wave function with the amplitude for scattering, emission and fusion of partons respectively. Appropriate amplitudes can be readily read from the LC form of the hamiltonian. In practice one has to cut the hierarchy limiting the multiplicity of partons. At fixed maximal multiplicity $n_{\text{max}}$ the LCD is indeed the momentum-discretized version of IE cut to $n \leq n_{\text{max}}$. In Fig.1 we compare the LCD simulations restricted to $n_{\text{max}} = 2$ with the IE solution at the same multiplicity. The agreement is satisfactory, however very fine discretization is required to see the convergence ($K \sim 2000$\footnote{In contrast solving integral equations with two partons required only few basis functions}). This is due to the singular nature of the scattering amplitude ($A \sim P_1^1$). If one allows for arbitrary number of partons in the LCD approach the rapid growth of the number of states with $K$ excludes $K > 25$ which makes extrapolation to $K = \infty$ rather delicate \cite{18,19,20}.
4 Infrared divergencies for theories with scalars

The previous discussion applies to a generic LC Hamiltonians. The calculations reported in Fig. 3 were done for the $YM_2$ with adjoined fermions. There is also some interest in $YM_2$ with adjoined scalar matter (which can be thought of as the dimensional reduction of $YM_3$), and finally one may combine both into the two-dimensional supersymmetric Yang-Mills theory, $SYM_2$ [21]. However introducing scalars brings one more complication: the integral equations have infrared logarithmic divergence. It appears explicitly in the mass term which is part of the diagonal (in multiplicity) transition in [2]. One way to dispose of it is the ”mass renormalization” introduced in [20]. This might be possible for the $YM_2$ with a scalar matter, however for the supersymmetric model such a procedure would break supersymmetry. The attractive possibility to maintain SUSY would occur if the above divergence was canceled dynamically by other contributions [22].

Some support for this idea is provided by Fig. 2 where we compare the lowest mass obtained by the LCD simulations with (open circles) and without (filled circles) the multiplicity cutoff. Clearly the dependence on the cutoff is weaker (indicating possible
Second piece of evidence comes from careful inspection of the integral equations \( \text{(2)} \) which shows that, when the wave functions have a suitable divergence at \( x \sim 0 \), additional divergences occur which in fact cancel the original IR divergence of the mass term \( \text{[22]} \). For this mechanism to work one has to include all Fock components in eqns \( \text{(2)} \). This is consistent with the message learnt from Fig.2. In fact the whole mechanism is analogous to the classical Bloch-Nordsieck treatment of the IR singularities in QED as discussed in the following Section.

Figure 2: The LCD results without (filled) and with the multiplicity cut (open, \( n_{\max} = 2 \)).
5 The Bloch-Nordsieck inspired toy model

A good insight into the mechanism of the above dynamical cancelations is provided by the following Light Cone model hamiltonian

\[
H = \int_0^P dk \left[ a_k a_k^\dagger - j_k (a_k a_k^\dagger + a_k^\dagger a_k) + j_k^2 \right] = \sum_k H_k, \quad j_k = \frac{g}{\sqrt{k}}
\]

(3)

\[
H_k = A_k^\dagger A_k, \quad A_k = e^{-i P_k j_k a_k} e^{i P_k j_k}, \quad P_k = \frac{1}{i \sqrt{2}} (a_k - a_k^\dagger).
\]

(4)

The model is of course soluble, the eigenstates are the BN coherent states

\[
|n_k\rangle_{\text{new}} = \frac{A_{k}^n}{\sqrt{n!}} |0\rangle_{\text{new}} = e^{-i P_k j_k} |n_k\rangle_{\text{old}}
\]

(5)

and the eigenvalues are integer \( M^2 = \sum_k n_k \).

The integral equations (6) can be easily derived

\[
M^2 f_n(x_1, \ldots, x_n) = \left(n + \int_0^1 j(x)^2 dx\right) f_n(x_1, \ldots, x_n)
\]

\[
- \sum_{i=1,n} j(x_i) f_{n-1}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) - \int_0^1 j(x) f_{n+1}(x, x, x, x, x) dx
\]

(6)

and are divergent for the sources (3), but in fact they must give (and they do!) the above finite spectrum.

We expect that the similar cancelations occur in the case of \( Y M_2 \) with adjoint scalars and also in \( SYM_2 \). Figure 3 lends some support for this analogy. On the left hand we show numerical solutions of (6) for the first three eigenvalues as the function of the IR cutoff \( \epsilon \). Three curves for each eigenvalue correspond to increasing multiplicities (top to bottom: \( n_{\text{max}} = 2, 3, 4 \)). At finite \( n_{\text{max}} \) eigenvalues are divergent at small \( \epsilon \) as expected. However for increasing \( n_{\text{max}} \) the divergence is shifted towards smaller and smaller \( \epsilon \) and the integer values of \( M^2 \) are better and better approximated. On the right hand we compare in the same way the masses obtained from the LCD and IE for \( Y M_2 \) with adjoint scalars [22]. Both calculations were done only in the two-parton sector. Results are not far from each other showing the onset of the divergence at low \( \epsilon \). It is expected that as we include higher Fock sectors the two curves would converge and the IR divergence will shift to yet smaller \( \epsilon \), similarly to the toy model example.
In the summary, reliable extrapolation of LCD data requires very large cutoffs, while the IE do not have this problem. On the other hand, LCD samples better multiparton Fock states which is difficult to achieve with the IE. In that sense the two methods are complementary and only comparison of both can provide unbiased information about many-parton phenomena in the continuum limit of quantum field theories including QCD.

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