On the Newton polyhedron of a Jacobian pair

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Abstract. We introduce and describe the Newton polyhedron related to a “minimal” counterexample to the Jacobian conjecture. This description allows us to obtain a sharper estimate for the geometric degree of the polynomial mapping given by a Jacobian pair and to give a new proof in the case of the Abhyankar’s two characteristic pairs.

Keywords: Jacobian conjecture, Newton polytopes.

To the memory of Anatoliy Georgievich Vitushkin, one of the champions of the Jacobian Conjecture

§ 1. Introduction

Assume that \( f, g \in \mathbb{C}[x, y] \) (where \( \mathbb{C} \) is the field of complex numbers) satisfy

\[
J(f, g) = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} = 1
\]

and provide a counterexample to the JC (the Jacobian conjecture claiming that \( \mathbb{C}[f, g] = \mathbb{C}[x, y] \); see [1]). It has been known for many years that then there exists an automorphism \( \xi \) of \( \mathbb{C}[x, y] \) such that the Newton polygon \( N(\xi(f)) \) of \( \xi(f) \) contains a vertex \( v = (m, n) \), where \( n > m > 0 \), and is included in a trapezium with one vertex \( v \), two edges parallel to the \( y \)-axis and to the bisectrix of the first quadrant adjacent to \( v \), and two edges belonging to the coordinate axes (see [2]–[15]). This was improved quite recently by Pierrette Cassou-Noguès, who showed that \( N(f) \) has no edge parallel to the bisectrix (see [16] and [17]).

So we assume below that \( N(f) \) is included in such a trapezium with leading vertex \( (m, n) \). We also may assume that \( N(f) \) and \( N(g) \) contain the origin as a vertex and are similar (an easy consequence of the relation \( J(f, g) = 1 \)), that the coefficients of the leading vertices of \( f \) and \( g \) are equal to 1 (this can be achieved by an appropriate re-scaling of \( x, y \) and \( f, g \), \( \deg_y(g) > \deg_y(f) \) and \( \deg_y(f) \) does not divide \( \deg_y(g) \) (otherwise we can replace the pair \( f, g \) by a “smaller” pair \( f, g - cf^k \)).

Here are the restrictions on \( N(f) \) known at present and it is not clear how to tighten them further by working with \( N(f) \) only. To proceed with this line of

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research I will consider an irreducible algebraic dependence of \( x, f \) and \( g \) and obtain information about the Newton polyhedron of this dependence.

§ 2. Algebraic dependence of \( x, f \) and \( g \)

We may regard \( f \) and \( g \) as polynomials in one variable \( y \) over \( \mathbb{C}(x) \). It is well known that two polynomials in one variable over a field \( K \) are algebraically dependent over \( K \) (see [18]). Therefore \( f \) and \( g \) are algebraically dependent over \( \mathbb{C}(x) \).

We may choose a dependence \( P(F, G) = P(x, F, G) \in \mathbb{C}(x)[F, G] \) (that is, \( P(x, f, g) = 0 \)) such that \( \deg_G(P) \) is least possible and hence \( P \) is irreducible as an element of \( \mathbb{C}(x)[F, G] \) with coefficients in \( \mathbb{C}[x] \) (since we can multiply a dependence by the least common denominator of the coefficients) and assume that these polynomial coefficients have no common divisor.

§ 3. A connection between \( G \) and \( y \)

Let \( G \) be an algebraic function of \( x \) and \( F \) given by \( P(x, F, G) = 0 \) and \( y \) an algebraic function of \( x \) and \( F \) given by \( F - f(x, y) = 0 \).

Lemma 1. \( y \in \mathbb{C}(x, f, g) \) and \( y \in \mathbb{C}(f(c, y), g(c, y)) \) for any \( c \in \mathbb{C} \).

Proof. By Lióroth’s Theorem, we have \( \mathbb{C}(f(c, y), g(c, y)) = \mathbb{C}(r(y)) \), where \( r \) is a rational function (see [18]). We can replace \( r \) by its linear fractional transformation and assume that \( r = p_1(y)/p_2(y) \), where \( p_1, p_2 \in \mathbb{C}[y] \) and \( \deg(p_1) > \deg(p_2) \).

We can assume without loss of generality that \( p_1 \) and \( p_2 \) are relatively prime polynomials. Then \( f(c, y) = F_1(r)/F_2(r) \) for some polynomials \( F_1 \) and \( F_2 \), where \( d_1 = \deg(F_1) > d_2 = \deg(F_2) \), and

\[
f(c, y) = \frac{F_{1,0}p_1^{d_1} + \cdots + F_{1,d_1}p_2^{d_1}}{(F_{2,0}p_1^{d_2} + \cdots + F_{2,d_2}p_2^{d_2})p_2^{d_1-d_2}}.
\]

Hence \( p_2 = 1 \) and \( r \) is a polynomial. Since \( 1 = J(f, g)|_{x=c} \in r'(y)\mathbb{C}[y] \), we have \( r'(y) \in \mathbb{C} \). Therefore \( y \in \mathbb{C}(f(c, y), g(c, y)) \) and \( y \in \mathbb{C}(x, f, g) \). Since \( x, f \) and \( g \) are algebraically dependent, we can represent \( y \) as a polynomial in \( g \) (with coefficients in \( \mathbb{C}(x, f) \)). \( \Box \)

Remark. It is easy to prove that \( y \in \mathbb{C}(x, f, g) \) using only the Jacobian condition. Indeed, \( \partial f/\partial y = P_9/P_x \) and \( \partial g/\partial y = -P_f/P_x \) since \( P(x, f, g) = 0 \), and hence \( \partial/\partial y \) acts on \( \mathbb{C}(x, f, g) \). But this does not imply that \( y \in \mathbb{C}(f(c, y), g(c, y)) \) for all \( c \in \mathbb{C} \).

There is a one-to-one correspondence between the roots \( y_i \) of \( f(x, y) - F \) and \( G_i \) of \( P(x, F, G) \) in any extension of \( \mathbb{C}(x, F) \) which contain these roots. Indeed, \( G_i = g(x, y_i) \) and \( y_i = R(G_i) \), where \( y = R(G) \in \mathbb{C}(x, F)[G] \).

§ 4. The Newton polytope of a polynomial

Let \( p \in \mathbb{C}[x_1, \ldots, x_n] \) be a polynomial in \( n \) variables. Represent each monomial of \( p \) by the lattice point in \( n \)-dimensional space with coordinate vector equal to the degree vector of this monomial. The convex hull \( \mathcal{N}(p) \) of the points so obtained is called the Newton polytope of \( p \). In the two- and three-dimensional cases, \( \mathcal{N}(p) \) is referred to as the Newton polygon and Newton polyhedron, respectively.
§ 5. The weight degree function

We define a function \( \deg_w(p) \) on \( \mathbb{C}[x_1, \ldots, x_n] \) as follows. First, we take weights given by \( w(x_i) = \alpha_i \), where \( \alpha_i \in \mathbb{R} \) (real numbers), and put \( w(x_1^{j_1} \cdots x_n^{j_n}) = \sum \alpha_i j_i \). When \( p \in \mathbb{C}[x_1, \ldots, x_n] \) we define the support \( \text{supp}(p) \) as the tuple of all the monomials appearing in \( p \) with non-zero coefficients. Then \( \deg_w(p) = \max(w(\mu)\mid \mu \in \text{supp}(p)) \). The polynomial \( p \) can be written as \( p = \sum p_1 \), where the \( p_i \) are forms homogeneous relative to \( \deg_w \). The leading form \( p_w \) of \( p \) with respect to \( \deg_w \) is the form of maximal weight in this representation.

For a non-zero weight degree function, the monomials appearing in the support of the leading form of \( p \) correspond to the points of a face \( \Phi \) of \( \mathcal{N}(p) \). If the codimension of \( \Phi \) equals \( i \), then there is a cone of dimension \( i \) of weight degree functions corresponding to \( \Phi \). The leading forms corresponding to these weights coincide and we denote them by \( p(\Phi) \).

The correspondence between faces and weight degree functions is one-to-one on faces of codimension 1 when \( \alpha_1, \ldots, \alpha_n \) are coprime integers. We sometimes refer to such a weight degree function as the function corresponding to the face.

§ 6. The roots of \( F = f(x, y) \)

Newton introduced a polygon, which we call the Newton polygon, to find a solution \( y \) of \( p(x, y) = 0 \) in terms of \( x \) (see [19]). Here is a process of obtaining such a solution. Consider an edge \( e \) of \( \mathcal{N}(p) \) that is not parallel to the \( x \)-axis and take the weight that corresponds to \( e \). Then the leading form \( p(e) \) allows us to determine the first summand of the solution as follows. Consider the equation \( p(e) = 0 \). Since \( p(e) \) is a homogeneous form and \( \alpha = w(x) \neq 0 \), the solutions of this equation are \( y = c_i x^{\beta/\alpha} \), where \( \beta = w(y) \) and \( c_i \in \mathbb{C} \). Choose any solution \( c_i x^{\beta/\alpha} \) and replace \( p(x, y) \) by \( p_1(x, y) = p(x, c_i x^{\beta/\alpha} + y) \). Though \( p_1 \) is not necessarily a polynomial in \( x \), we can define the Newton polygon of \( p_1 \) in the same way as for polynomials. The only difference is that \( \text{supp}(p_1) \) may contain monomials \( x^\mu y^\nu \), where \( \mu \) is in \( \mathbb{Q} \) rather than in \( \mathbb{Z} \). In what follows we use these Newton polygons and Newton polyhedra. The polygon \( \mathcal{N}(p_1) \) contains the degree vertex \( v \) of \( e \), that is, the vertex with the coordinate \( y \) equal to \( \deg_y(p_w) \) of the polynomial \( p_w \) in \( y \) and an edge \( e' \) that is a modification of \( e \) (\( e' \) may collapse to \( v \)). Take the order vertex \( v_1 \) of \( e' \), that is, the vertex with the coordinate \( y \) equal to the order \( \text{ord}_y(p_w) \) of \( p_w \) as a polynomial in \( y \) (if \( v_1 = v \) take \( e' = v \)). Use the edge \( e_1 \) for which \( v_1 \) is the degree vertex to determine the next summand, and so on.

After a possibly countable number of steps we obtain a vertex \( v_\mu \) and an edge \( e_\mu \) for which \( v_\mu \) is not the degree vertex, that is, either \( e_\mu \) is horizontal or the degree vertex of \( e_\mu \) has \( y \)-coordinate larger than that of \( v_\mu \). This is possible only if \( \mathcal{N}(p_\mu) \) has no vertex on the \( x \)-axis. Therefore \( p_\mu(x, 0) = 0 \) and a solution is obtained.

When the characteristic is zero the process of constructing a solution is more straightforward than might seem from this description. The denominators of fractional powers of \( x \) (when the denominators and numerators of these rational numbers are relatively prime) do not exceed \( \deg_y(p) \). Indeed, for any initial weight there are at most \( \deg_y(p) \) solutions while the summand \( c x^{M/N} \) can be replaced by \( c \varepsilon^M x^{M/N} \), where \( \varepsilon^N = 1 \), which gives at least \( N \) different solutions.
If \( \text{deg}_y(p) = n \) and we want to obtain all \( n \) solutions, we must choose the first edge \( e \) appropriately. Consider \( p_w \) with \( w(x) = 0 \) and \( w(y) = 1 \). This leading form corresponds to a horizontal edge with “left” and “right” vertices \( v_l \) and \( v_r \), or a vertex \( v \) in the case when \( v_l = v_r \). If we choose \( e \) with the degree vertex \( v_r \), we obtain \( n \) solutions with decreasing powers of \( x \) and if we choose \( e \) with the degree vertex \( v_l \), we obtain \( n \) solutions with increasing powers of \( x \). When \( v_l = v_r = v \), take the “right” edge containing \( v \) to obtain \( n \) solutions with decreasing powers of \( x \) and the “left” edge containing \( v \) to obtain \( n \) solutions with increasing powers of \( x \).

We can apply Newton’s approach to finding solutions \( F - f(x, y) = 0 \) in an appropriate extension of \( \mathbb{C}(x, F) \). To do this we have to take the weights \( w(x), w(F) \) and \( w(y) \) to be such that the corresponding face (possibly the edge) of \( \mathcal{N}(F - f(x, y)) \) contains the leading vertex \((m, n)\) of \( \mathcal{N}(f(x, y)) \) and proceed as above. Of course, the process would be much harder to visualize but it can be made two-dimensional if the weights \( \alpha = w(x) \) are \( \rho = w(F) \) are commensurable. If \( w(x) = 0 \), say, we can replace \( \mathbb{C} \) by the algebraic closure \( K \) of \( \mathbb{C}(x) \) and make computations over \( K \). If \( w(x) \neq 0 \), we can take \( K \) to be the algebraic closure of \( \mathbb{C}(z) \), where \( z = x^{-\rho/\alpha}F \), put \( x = t^{d_1} \) and \( F = z t^{-d_2} \), where \( d_1, d_2 \in \mathbb{Z} \) are such that \(-d_1/d_2 = \alpha/\rho \), and regard \( F - f(x, y) = z t^{-d_2} - f(t^{d_1}, y) \) as a polynomial in \( y, t \) and \( t^{-1} \) over \( K \).

\section{The Newton polyhedron \( \mathcal{N}(P) \)}

In this section we will find some restrictions on \( \mathcal{N}(P) \).

Note that \( \text{deg}_y(g^{\text{deg}_y(f)} - f^{\text{deg}_y(g)}) < \text{deg}_y(f) \text{deg}_y(g) \) because of the shape of \( \mathcal{N}(f) \) and \( \mathcal{N}(g) \). It is known that the leading form of \( P(x, F, G) \) relative to the weight given by \( w(x) = 0, w(F) = \text{deg}_y(f) \) and \( w(G) = \text{deg}_y(g) \) is \( p_0(x)(G^{a_0} - F^{b_0})^\nu \), where \( a_0/b_0 = \text{deg}_y(f)/\text{deg}_y(g) \), \((a_0, b_0) = 1 \) and \( b_0\nu = \text{deg}_F(P), a_0\nu = \text{deg}_G(P) \) (see [20–22]).

It follows from Lemma 1 that \( \text{deg}_G(P) = [\mathbb{C}(x, f, g) : \mathbb{C}(x, f)] = [\mathbb{C}(x, f, g) : \mathbb{C}(x, f)] = \text{deg}_y(f) \) and \( \text{deg}_G(P_\lambda) = \text{deg}_y(f(\lambda, y)) \) when \( \lambda \in \mathbb{C} \), where \( P_\lambda \) is an irreducible dependence between \( f(\lambda, y) \) and \( g(\lambda, y) \) (recall that \( y \in \mathbb{C}(x, f, g) \) and \( y \in \mathbb{C}(f(\lambda, y), g(\lambda, y)) \)).

Furthermore, \( \text{deg}_G(P) = \text{deg}_G(P_\lambda) \) for all \( \lambda \in \mathbb{C}^* \) since \( \text{deg}_y(f(\lambda, y)) = \text{deg}_y(f) \) for all \( \lambda \in \mathbb{C}^* \). Hence \( P_\lambda(F, G) \) is proportional to \( P(\lambda, F, G) \) for all \( \lambda \in \mathbb{C}^* \) and \( p_0(\lambda) = 0 \) is possible only if \( \lambda = 0 \). Therefore \( p_0(x) = c_0 x^d \), and \( (c_0 x^d)^{-1} P \) is a polynomial monic in \( G \) (with coefficients in \( \mathbb{C}[x, x^{-1}] \)). From now on \( P \) is this monic polynomial.

If \( p(x, F, G) \) is a Laurent polynomial in \( x \). The face \( \Phi_b \) is below the plane \( FOG \) when \( P(x, F, G) \) is a Laurent polynomial in \( x \).

Since the leading form of \( P \) relative to the weight given by \( w(x) = 0, w(F) = \text{deg}_y(f) \) and \( w(G) = \text{deg}_y(g) \) is \( (G^{a_0} - F^{b_0})^\nu \), the \( x \)-axis cannot be parallel to \( \Phi_a \) or \( \Phi_b \).

One can use \( \mathcal{N}(P) \) to find a representation of \( G \) as a fractional power series in \( x \) and \( F \) using the approach discussed in §6.
7.1. The face $\Phi_b$. Assume that the face $\Phi_b$ (the lower face containing $E$) is below the plane $FOG$. Since the $x$-axis is not parallel to $\Phi_b$, we can choose the corresponding weight by taking $w(x) = 1$, $w(F) = \rho < 0$ and $w(G) = \sigma < 0$. Of course, $\rho, \sigma \in \mathbb{Q}$. Expansions of $G$ as well as the corresponding expansions of $y$ relative to this weight are by the components with the increasing weight.

Consider the leading form $P(\Phi_b)$ and its factorization into irreducible factors. If all these factors depend on only two variables, then $P(\Phi_b) = \phi_1(x, F)\phi_2(x, G) \times \phi_3(F, G)$ and $\Phi_b$ is either an interval, a parallelogram or a hexagon with parallel opposite sides. Since $\Phi_b$ is not any of these ($\Phi_b$ is not $E$ and it cannot contain an edge parallel to $E$ of the same length), $P(\Phi_b)$ has an irreducible factor $Q(x, F, G)$ that depends on $x, F$ and $G$. Denote by $\tilde{G}$ a root of $Q(x, F, G) = 0$ and by $G$ the root of $P(x, F, G) = 0$ for which $\tilde{G}$ is the leading form. Then $f(x, y) = F$ and $g(x, y) = G$ when $y = R(x, F)[\tilde{G}]$. (The reader should regard the roots $G, \tilde{G}$ and $y$ as fractional power series in $x$ over the algebraic closure of the field $\mathbb{C}(z)$, $z = x^{-\rho}F$.)

We can write $\tilde{y} = \sum_{j=0}^{k} y_j$, where the $y_j$ are the homogeneous components of $\tilde{y}$. Since $f(x, y) = F$, there exists a $k$ for which $y_j = c_j x^{\nu_j}$, $c_j \in \mathbb{C}$, $\mu_j \in \mathbb{Q}$, if $j \leq k$ and $y_{k+1} \notin \mathbb{C}(x)$.

We can also get $\tilde{y}$ from the Newton polyhedron of $F - f(x, y)$. The terms $y_j$ when $j \leq k$ are obtained by a resolution process applied to $\mathbb{N}(f)$ and the term $y_{k+1}$ is defined by a face $\Psi$ of this polyhedron which contains $0, 0, 1$, that is, the vertex corresponding to $F$ (otherwise $y_{k+1} \notin \mathbb{C}(x)$). The face $\Psi$ corresponds to the weight given by $w(x) = 1$, $w(F) = \rho$ and $w(y) = \alpha = w(y_{k+1})$, and $\Psi$ contains an edge $e \in xOy$ of $\mathbb{N}(f, \sum_{j=0}^{k} y_j + y)$.

Put

$$f_k(x, y) = f\left(x, \sum_{j=0}^{k} y_j + y\right) \quad \text{and} \quad g_k(x, y) = g\left(x, \sum_{j=0}^{k} y_j + y\right)$$

(then $\mathbb{N}(f_k)$ contains the edge $e$ and $w(f_k) = \rho$) and denote by $f_k(e)$ and $g_k(e)$ the leading forms of $f_k$ and $g_k$ for the weight $w$. Thus $f_k(e)(x, y_{k+1}) = F$ by the definition of $y_{k+1}$. Also, $g_k(e)(x, y_{k+1}) \neq 0$ (recall that $y_{k+1} \notin \mathbb{C}(x)$). Since $g_k(x, \sum_{j=k+1}^{\infty} y_j) = \tilde{G}$, we must have $g_k(e)(x, y_{k+1}) = \tilde{G}$.

If $J(f_k(e), g_k(e)) = 0$, then $g_k(e)(x, y_{k+1}) = cF^\lambda$, $c \in \mathbb{C}^*$ (since $f_k(e)$ is a homogeneous form of non-zero weight, any homogeneous form algebraically dependent with $f_k(e)$ is proportional to a rational power of $f_k(e)$). But $\tilde{G}$ depends on $x$, and so $J(f_k(e), g_k(e)) \neq 0$. Since $J(f_k, g_k) = 1$, this implies that $J(f_k(e), g_k(e)) = 1$.

Since the expansion $\tilde{y}$ is by components with increasing weight given by $w(x) > 0$ and $w(f_k) < 0$, the leading vertex $(m, n)$ must be below the line containing $e$. The following consideration shows that this is impossible. We have $w(g_k) = w(G) = \sigma < 0$ and $\rho + \sigma = w(x) + w(y)$ to make $J(f_k(e), g_k(e)) = 1$ possible. Therefore $\rho = w(x) + w(y) - \sigma = 1 + \alpha - \sigma$ and the points $(\rho, 0)$ and $(1 - \sigma, 1)$ have the same weight $\rho$. (Recall that $w(x) = 1$, $w(y) = \alpha$, $w(F) = \rho$ and $w(G) = \sigma$). Thus they both belong to the line containing the edge $e$. But this line intersects the bisectrix of the first quadrant in a point with coordinates smaller than $1$ since $\rho < 0$ and $\sigma < 0$, and the vertex $(m, n)$ is above this line.
Hence $\Phi_b$ cannot be below $FOG$ and $P(x, F, G) \in \mathbb{C}[x, F, G]$. On the other hand, $P(0, f(x, 0), g(x, 0)) = 0$ and the Newton polygon of this dependence is not an edge.\textsuperscript{1} Therefore $\Phi_b$ is not an edge and belongs to $FOG$.

7.2. The face $\Phi_a$. For the other face $\Phi_a$ containing $\mathcal{E}$, choose the weight in such a way that $w(x) = 1$, $w(F) = \rho > 0$ and $w(G) = \sigma > 0$. An expansion of $G$ relative to this weight is by components with decreasing weight.

By repeating considerations in the previous subsection verbatim, we obtain an edge $e$ of the corresponding polygon $\mathcal{N}(f_1)$ that belongs to the line containing the points $(\rho, 0)$ and $(1 - \sigma, 1)$ and running below the leading vertex $(m, n)$.

Therefore $\rho + n[1 - \sigma - \rho] \geq m$, that is, $n - m \geq n(\rho + \sigma) - \rho$. Also $\sigma = (b_0/a_0)\rho$ because $\Phi_a$ contains $\mathcal{E}$ and

$$n - m \geq \left[ n\left(1 + \frac{b_0}{a_0}\right) - 1\right] \rho, \quad \rho \leq \frac{(n - m)a_0}{n(a_0 + b_0) - a_0}, \quad \sigma \leq \frac{(n - m)b_0}{n(a_0 + b_0) - a_0}.$$ 

Hence,

$$\deg_x(P) \leq n\sigma \leq (n - m)\frac{nb_0}{n(a_0 + b_0) - a_0}.$$

If these inequalities are not strict, then the edge $e$ contains $(m, n)$, that is, $e$ is the (right) leading edge. Since $\rho < 1$ and $\sigma < 1$, this would imply that $f(x, 0)$ and $g(x, 0)$ are constants and then $J(f, g) = 1$ is impossible. Therefore $(m, n)$ does not belong to $e$ and the inequalities are strict.

From Lemma 1 we have $\mathbb{C}(x, f, g) = \mathbb{C}(x, y)$. Therefore the degree $[\mathbb{C}(x, y) : \mathbb{C}(f, g)]$ of the field extension is equal to $\deg_x(P)$ and

$$[\mathbb{C}(x, y) : \mathbb{C}(f, g)] < (n - m)\frac{nb_0}{n(a_0 + b_0) - a_0}.$$ 

This estimate is sharper than the estimate $m + n$ obtained by Yitang Zhang (see [23]).

It is known that $[\mathbb{C}(x, y) : \mathbb{C}(f, g)]$ is at least 6 if $J(f, g) = 1$ (see [24]–[29]). Hence the difference $n - m > 6$.

7.3. Edges of $\mathcal{N}(P)$. An edge of $\mathcal{N}(P)$ can be parallel to one of the coordinate planes $GOx$ or $FOG$ and then the leading form of $G$ which corresponds to this edge is $cx^r$ or $cF^r$, where $c \in \mathbb{C}^*$ and $r \in \mathbb{Q}$. An edge parallel to $GOx$ does not correspond to the leading form of $G$.

If $E$ is a slanting edge, that is, an edge which is not parallel to either coordinate plane, then at least one of the corresponding leading forms is $\overline{G} = cx^{r_1}F^{r_2}$, where $c \in \mathbb{C}^*$ and $r_i \in \mathbb{Q}^*$. In this case we have more freedom in choosing a weight which corresponds to $E$ and, with an appropriate choice, the edge $e \in \mathcal{N}(f_k)$ (see §7.1) collapses to a vertex and both $f_k(e)$ and $g_k(e)$ are monomials. Since $J(f_k(e), g_k(e)) = 1$ and $\deg_x(f_k(e))$, $\deg_y(g_k(e))$ are non-negative integers, either $\deg_y(g_k(e)) = 0$ or $\deg_y(f_k(e)) = 0$. If $\deg_y(g_k(e)) = 0$, then

$$\overline{G} = g_k(e)(x, y_{k+1}) = cx^{r_1}$$

\textsuperscript{1}If $g(x, 0)^b = cf(x, 0)^a$, then $(f, g)$ cannot be a counterexample to JC because $\mathbb{C}(f(x, 0), g(x, 0)) = \mathbb{C}(x)$.\textsuperscript{1}
and the edge $E$ is parallel to $GOx$. If $\deg_y(f_k(e)) = 0$, then $f_k(e) = cx^s$, while $f_k(e)(x, y_{k+1}) = F$.

Hence $N(P)$ has no slanting edges.

7.4. Non-vertical and non-horizontal faces. Consider again the face $\Phi_a$. This face belongs to a slanting plane containing $E$ which intersects the first octant in a triangle $\Delta$. Since all the edges of $\Phi_a$ are parallel to the coordinate planes and $\Phi_a$ contains $E$, the face $\Phi_a$ is either $\Delta$ or a trapezium obtained from $\Delta$ by cutting it with an edge $E_1$ parallel to $E$.

If $\Phi_a$ is a trapezium, then the same consideration applied to $E_1$ shows that the next face is also a triangle or a trapezium, and so on until we reach the face parallel to $FOG$.

7.5. Horizontal faces. The polyhedron $N(P)$ has a non-degenerate horizontal face $\Phi_b \subset FOG$ (the “floor”). It also has a “ceiling”, which may degenerate into a vertex. Let us replace $f$ and $g$ by $f - c_1$ and $g - c_2$, where $c_i \in \mathbb{C}$ and $(c_1, c_2)$ is a “general pair”. Then the corresponding Newton polyhedron has a triangular floor (with a vertex at the origin) and a triangular ceiling (with a vertex on the $x$-axis).

7.6. The shape of $N(P)$. Collecting the information about $N(P)$ obtained above, we conclude that all its vertices are in the coordinate planes $FOx$ and $GOx$; it contains two horizontal faces which are right triangles with the right angles at the origin and on the $x$-axis, a face $\Phi_G$ in $FOx$ and a face $\Phi_F$ in $GOx$, which are polygons with the same number of vertices, and all the remaining faces are trapezia obtained by connecting the corresponding vertices of $\Phi_F$ and $\Phi_G$ by edges parallel to $E$.

To give a new proof that in the case of two characteristic pairs a counterexample is impossible (see [3]), we will estimate $\rho$ from below.

§ 8. An estimate of $\rho$ from below

To get an estimate for $\rho$ of the face $\Phi_a$ from below we need to know more about $P(x, F, G)$.

Consider $f, g \in \mathbb{C}(x)[y]$. The first necessary ingredient is the expansion of $g$ as a power series of $f$ in an appropriate algebra relative to the weight given by $w(y) = 1$ and $w(x) = 0$.

8.1. Expansion of $g$. Consider the ring $L = \mathbb{C}[x^{-1}, x]$ of Laurent polynomials in $x$. Define $A$ to be the algebra of asymptotic power series in $y$ with coefficients in $L$, that is, the elements of $A$ are $\sum_{i=k}^{i=k} y_i y^i$, where $y_i \in L$ and $y_k \neq 0$. When $a = \sum_{i=k}^{i=k} y_i y^i$ put $|a| = y_k y^k$.

Lemma 2 (on the radical). If $r \in \mathbb{Q}$ is a rational number, $|a| = cx^l y^k$, $c \in \mathbb{C}$, and $|a|^r \in A$, then $a^r \in A$.

Proof. By the binomial theorem,

$$a^r = |a|^r \sum_{j=0}^{\infty} \binom{r}{j} \left( \sum_{i=-\infty}^{i=-k} \frac{y_i}{y_k} y^{i-k} \right)^j$$
because
\[ a = |a| \left( 1 + \sum_{i=k}^{i=k-1} \frac{y_i}{y_k} y^{i-k} \right). \]

Since all the \( y_i/y_k \in L \), the element \( a^r \in A \). \( \square \)

Regard \( f(x,y) \) and \( g(x,y) \) as elements of \( A \). Then \([f] = x^m y^n \) and \([g] = c_0 [f]^{\lambda_0} \), where \( \lambda_0 = b_0/a_0 \) (see §§1 and 7). By lemma 2, \( f^{\lambda_0} \in A \) and hence \( g_1 = g - c_0 [f]^{\lambda_0} \in A \) (here \( c_0 = 1 \)). Since \( J(f, g_1) = 1 \), either \( J([f], [g_1]) = 0 \) or \( J([f], [g_1]) = 1 \). If \( J([f], [g_1]) = 0 \), then \([g_1] = c_1 [f]^{\lambda_1} \), \( c_1 \in \mathbb{C} \), \( \lambda_1 \in \mathbb{Q} \), and we can put \( g_2 = g - c_0 f^{\lambda_0} - c_1 f^{\lambda_1} \), which is in \( A \) for the same reasons as \( g_1 \). We can proceed until we obtain \( g_\kappa = g - \sum_{i=0}^{\kappa-1} c_i f^{\lambda_i} \in A \) for which \( J([f], [g_\kappa]) = 1 \), that is, \( J(x^m y^n, [g_\kappa]) = 1 \). Therefore,
\[ |g_\kappa| = \left( c_\kappa (x^m y^n)^{(1-n)/n} - \frac{1}{n-m} x^{1-m} y^{1-n} \right), \]

where \( c_\kappa \in \mathbb{C} \). If \( c_\kappa \neq 0 \), then \((x^m y^n)^{(1-n)/n} \in A \) and \( m/n \in \mathbb{Z} \), which is impossible since \( 0 < m < n \). Thus \(|g_\kappa| = (m-n)^{-1} x^{1-m} y^{1-n} \) and

\[ g = \sum_{i=0}^{\kappa-1} c_i f^{\lambda_i} + g_\kappa, \quad c_i \in \mathbb{C}, \quad \text{(1)} \]

where \( \deg_y([f]^{\lambda_i}) > 1 - n \) and \( \deg_y([g_\kappa]) = 1 - n \), and \(|g_\kappa| = (m-n)^{-1} x^{1-m} y^{1-n} = (m-n)^{-1} x^{(n-m)/n} [f]^{\lambda_\kappa} \), where \( \lambda_\kappa = (1-n)/n \).

To obtain a “complete” expansion
\[ g = \sum_{i=0}^{\infty} c_i f^{\lambda_i} \quad \text{(2)} \]

through \( x \) and \( f \) we must extend \( A \) to a larger algebra \( B \) with elements \( \sum_{i=-\infty}^{i=k} y_i y^i \), where \( y_i \in L_n = \mathbb{C}[x^{-m/n}, x^{m/n}] \) in which \( f^{1/n} \) is defined. Indeed, \(|x^{-m/n} f^{1/n}| = y \) and we can obtain an expansion for \( g \) with \( c_i \in L_n \).

It is clear that \( \lambda_i = n_i/n, \ n_i \in \mathbb{Z} \). Since \( \deg_g(P) = n \) and \( \lambda_\kappa = (1-n)/n \), all \( n \) roots \( G_j \) of \( P(x, F, G) = 0 \) in \( B \) can be obtained from \( G = \sum_{i=0}^{\infty} c_i F^{n_i/n} \) by the substitutions \( F^{1/n} \to \varepsilon^j F^{1/n}, \ j = 0, 1, \ldots, n-1 \), where \( \varepsilon \) is a primitive root of unity of degree \( n \).

8.2. A monomial of \( P(x, F, G) \) containing a power of \( x \). The polyhedron \( \mathcal{N}(P) \) contains the edge \( E \) with vertices \((n_0, 0, 0) \) and \((0, n, 0) \), where \( n_0 = \lambda_0 n \) (in the coordinate system \( F G x \)). Hence, if \( \mathcal{N}(P) \) contains a vertex \((i, j, k) \), then \( \lambda_0 n \rho \geq i \rho + j \sigma + k = (i + \lambda_0 j) \rho + k \) and \( \rho \geq k/((\lambda_0 (n-j) - i) \), which gives a meaningful estimate for \( \rho \) when \( k > 0 \).

The following algorithm provides an irreducible relation for polynomials \( f, g \in \mathbb{C}(x)[y] \).

Put \( \tilde{g}_0 = g \). Assume that after \( s \) steps we have obtained \( \tilde{g}_0, \ldots, \tilde{g}_s \in \mathbb{C}(x,y) \). Denote \( \deg_y(\tilde{g}_i) \) by \( m_i \) and the greatest common divisor of \( n, m_0, \ldots, m_i \) by \( d_i \). Put \( d_{-1} = n \) and \( a_i = d_{i-1}/d_i \), for \( 0 \leq i \leq s \). (Clearly, \( a_s m_s \) is divisible by \( d_{s-1} \) and \( a_s \) is the smallest integer with this property.)
We say that a monomial \( m = f_1^{1/0} \cdots f_s^{j_s} \) is \( s \)-standard if \( 0 \leq j_k < a_k \), \( k = 0, \ldots, s \). Take an \((s-1)\)-standard monomial \( m_{s,0} \) with \( \text{deg}_y(m_{s,0}) = a_s m_s \) and \( k_0 \) in \( K = \mathbb{C}(x) \) for which \( m_{s,1} = \text{deg}_y(g_s^{a_s} - k_0 m_{s,0}) < a_s m_s \). If \( m_{s,1} \) is divisible by \( d_s \), take an \( s \)-standard monomial \( m_{s,1} \) with \( \text{deg}_y(m_{s,1}) = m_{s,1} \) and \( k_1 \) in \( K \) for which \( \text{deg}_y(g_s^{a_s} - k_0 m_{s,0} - k_1 m_{s,1}) < m_{s,1} \), and so on.

If after a finite number of reductions, an \( m_{s,i} \) which is not divisible by \( d_s \) is obtained, denote the corresponding expression by \( \widetilde{g}_{s+1} \) and take the next step. After a finite number of steps we obtain an irreducible relation.

This algorithm was suggested in [22] with a proof that it works. In the zero characteristic case it is also shown there that all the \( \widetilde{g}_i \) are polynomials in \( f \) and \( g \) (that is, there are no negative powers of \( f \) in the standard monomials).

We can rewrite (1) as

\[
|g| = \sum_{i=0}^{\kappa-1} c_i f^{n_i/n} + g_\kappa, \quad c_i \in \mathbb{C}, \tag{3}
\]

where \( |g_\kappa| = (m-n)^{-1}|xy/|f|\). Applying the algorithm to this expansion, after several steps we will get the “the last” \( \widetilde{g}_\kappa \) with

\[
|\widetilde{g}_\kappa| = c \left| \frac{xy}{f} g_0^{-a_0 - 1} g_1^{-a_1 - 1} \cdots g_{\kappa-1}^{-a_{\kappa-1} - 1} \right|.
\]

In the case of two characteristic pairs, we have \( \kappa = 1 \) and \( |\widetilde{g}_1| = c |(xy/f) \tilde{g}_0^{a_0 - 1}| \).

If we put \( |f| = (x^a y^b)^{a_0} \) and \( |g| = (x^a y^b)^{b_0} \), then \( P = \tilde{g}_1^{a_0} = c x^{b-a} f^i \tilde{g}_0^{a_0} \cdots \), where \( |x^{b-a} f^i \tilde{g}_0^{a_0 - 1}| = |(xy/f) \tilde{g}_0^{a_0 - 1}|^{a_0 - 1} \). Therefore,

\[
\rho \geq \frac{b-a}{\lambda_0(n-j) - i} = \frac{b-a}{\lambda_0 (ba_0 - j) - i}.
\]

Since \( |x^{b-a} f^i \tilde{g}_0^{a_0 - 1}| = |(xy/f) \tilde{g}_0^{a_0 - 1}|^{a_0 - 1} \), we have \( a_0 i + b_0 j = 1 - a_0 b + b_0 (a_0 - 1) b \) and \( i + \lambda_0 j = (bb_0 a_0 - ba_0 - bb_0 + b_0)/a_0 \) (recall that \( \lambda_0 = b_0/a_0 \)). Hence,

\[
\rho \geq \frac{b-a}{\lambda_0 (bb_0 - ba_0 - bb_0 + b_0) - (b-a) a_0} = \frac{(b-a) a_0}{ba_0 b_0 - (bb_0 a_0 - ba_0 - bb_0 + b_0) + (b-a) a_0}.
\]

On the other hand,

\[
\rho \leq \frac{(n-m) a_0}{n (a_0 + b_0) - a_0} = \frac{(b-a) a_0^2}{ba_0 (a_0 + b_0) - a_0} = \frac{(b-a) a_0}{b (a_0 + b_0) - 1}.
\]

and we have a contradiction.

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