Moments of the ruin time in a perturbed Cramér-Lundberg model

Philipp Lukas Strietzel* and Anita Behme*

August 12, 2021

Abstract

We present formulae for the moments of the ruin time in a Lévy risk model. From these we derive the asymptotic behaviour of the moments of the ruin time, as the initial capital tends to infinity. In the perturbed Cramér-Lundberg model with phase-type distributed claims, we explicitly compute the first two moments of the ruin time in terms of roots and derivatives of the corresponding Laplace exponent. In the special case of exponential claims we provide explicit formulae for the first two moments of the ruin time in terms of the model parameters. All our considerations distinguish between the profitable and the unprofitable setting.

2020 Mathematics subject classification. 60G51, 60G40, 91G05

Keywords: Cramér-Lundberg risk process; exponential claims; Laplace transforms; moments; phase-type distributions; ruin theory; ruin time; time to ruin

1 Introduction

Let \( X = (X_t)_{t \geq 0} \) be a perturbed Cramér-Lundberg risk model, i.e. set

\[
X_t := x + pt - \sum_{i=1}^{N_t} S_i + \sigma B_t, \quad t \geq 0,
\]

where \( x \geq 0 \) is interpreted as initial capital, \( p > 0 \) denotes a constant premium rate, the Poisson process \((N_t)_{t \geq 0}\) represents the claim counting process, and the i.i.d. positive random variables \( \{S_i, i \in \mathbb{N}\} \) are the claim size variables and independent of \((N_t)_{t \geq 0}\). Moreover, the perturbation is scaled by \( \sigma \geq 0 \), and \((B_t)_{t \geq 0}\) denotes a standard Brownian motion that is independent of all other sources of randomness.

In this article we compute moments of the time to ruin or ruin time

\[
\tau^-_0 := \inf \{t \geq 0 : X_t < 0\},
\]

*Technische Universität Dresden, Institut für Mathematische Stochastik, Fakultät Mathematik, 01062 Dresden, Germany, anita.behme@tu-dresden.de and philipp.strietzel@tu-dresden.de phone: +49-351-463-32425, fax: +49-351-463-37251.
or, more precisely, the ruin time, given that ruin happens

$$(\tau_0 - 0 < \infty),$$

of $(X_t)_{t \geq 0}$ as in (1.1), where in the applications considered in this paper we will assume the claims $\{S_i, i \in \mathbb{N}\}$ to have a phase-type distribution.

Note that in the non-profitable case, i.e. whenever $\mathbb{E}[X_1] \in [-\infty, 0]$, the risk process $X$ enters the negative half-line almost surely and hence $\tau_0^- = (\tau_0^- | \tau_0^- < \infty)$. However, in the profitable setting, where the net-profit condition $\mathbb{E}[X_1] > 0$ holds, the term ruin time will be typically used for the conditioned quantity (1.3).

The ruin time $\tau_0^-$ is an extensively studied quantity in the field of actuarial mathematics. In particular, for the classical Cramér-Lundberg risk model without perturbation, i.e. for

$$X_t := x + pt - \sum_{i=1}^{N_t} S_i, \quad t \geq 0,$$  

(1.4)

various authors have studied the time to ruin or the joint distribution of the time to ruin, the surplus immediately before ruin, and the deficit at ruin, using different techniques, see e.g. [15, 21, 22] to mention just a few. In particular, in [22] and [10], recursion formulae for the moments of the ruin time are provided. In [8] the approach of [22] is taken up. This leads to closed form expressions for the $k$-th moment of the ruin time in the case of exponential claims, see [8], and a Mathematica program that deals with the heavy algebra involved in calculating explicit moments of the ruin time of a general Cramér-Lundberg process provided in [7]. For discrete claim size distributions, moments of the ruin time have also been computed in [23].

Furthermore, e.g. in [5, 24], approximations of the moments of the ruin time are considered. In [12] upper bounds of the expected ruin time are derived using the duality of the Cramér-Lundberg model with a single server queueing system. The latter work has been extended in [13] to a renewal risk model with phase-type distributed claims.

In the context of the perturbed Cramér-Lundberg model or of an even more general Lévy risk model, where $(X_t)_{t \geq 0}$ is chosen to be any spectrally negative Lévy process, $\tau_0^-$ is typically referred to as exit time or first passage time. Most results on $\tau_0^-$ and related quantities in this setting are, however, stated in terms of Laplace transforms, see e.g. [6, Sec. 9.5] for an overview. Recently, in [3], we proved necessary and sufficient conditions for finiteness of general moments of the ruin time of spectrally negative Lévy processes, and provided general formulae for integer moments of the ruin time of spectrally negative Lévy processes. In the present article, these results shall first be used to derive asymptotic expressions for integer moments of the ruin time. Following, they will be applied in the setting of the perturbed Cramér-Lundberg model with phase-type distributed claims.

The outline of this article is thus as follows. After a brief explanation of some preliminaries in Section 2, in Section 3 we apply the results from [3] to the ruin time of the perturbed Cramér-Lundberg model and derive the asymptotics of integer moments of the ruin time. Afterwards, we provide formulas for the first two moments of the ruin time of the perturbed Cramér-Lundberg model with phase-type distributed claims in Section 4, while in the subsequent Section 5 we restrict to exponentially distributed claims which allows for even more explicit results. The final Section 6 contains the proofs of our main results.
2 Spectrally negative Lévy processes and scale functions

The process \((X_t)_{t\geq 0}\) defined in (1.1) is a special case of a spectrally negative Lévy process, i.e. of a càdlàg stochastic process with independent and stationary increments that does not admit positive jumps. Spectrally negative Lévy processes are typically characterized by their so-called Laplace exponent \(\psi(\theta) := \frac{1}{t} \log \mathbb{E} \left[ e^{\theta X_t} \right] \), which takes the form

\[
\psi(\theta) = c\theta + \frac{1}{2} \sigma^2 \theta^2 + \int_{(-\infty,0)} \left( e^{\theta x} - 1 - \theta x \mathbb{1}_{\{|x|<1\}} \right) \Pi(dx),
\]

with constants \(c \in \mathbb{R}, \sigma^2 \geq 0\) and a Lévy measure \(\Pi\) satisfying

\[
\int_{(-\infty,0)} (1 \wedge x^2) \Pi(dx) < \infty.
\]

In the special case of the perturbed Cramér-Lundberg model (1.1) this reduces to

\[
\psi(\theta) = p\theta + \frac{1}{2} \sigma^2 \theta^2 + \lambda \int_{(0,\infty)} (e^{-\theta x} - 1) F(dx),
\]

where \(F\) denotes the cdf of the claim sizes \(\{S_i, i \in \mathbb{N}\}\), \(\lambda \geq 0\) is the intensity of the claim arrival process \((N_t)_{t\geq 0}\), and \(p = c - \int_{|x|<1} x \Pi(dx)\).

Our formulas for the moments of \(\tau_0^+\) or \(\tau_0^-|\tau_0^- < \infty\) rely on \(q\)-scale functions of the spectrally negative Lévy process \(X\). Recall that for any \(q \geq 0\) the \(q\)-scale function \(W^q(x) : \mathbb{R} \to [0, \infty)\) of \(X\) is the unique function satisfying

\[
\int_0^\infty e^{-\beta x} W^q(x) \, dx = \frac{1}{\psi'(\beta) - q},
\]

for all \(\beta > \Phi(q) := \sup\{\theta \geq 0 : \psi(\theta) = q\}, \ q \geq 0\),

and such that \(W^q(x) = 0\) for \(x < 0\).

Note that \(q \mapsto W^q(x)\) may be extended analytically to \(\mathbb{C}\), which means especially that it is infinitely often differentiable with bounded derivatives on \([0, \infty)\).

The Laplace exponent’s right inverse \(q \mapsto \Phi(q)\) is strictly monotone increasing on \([0, \infty)\), infinitely often differentiable on \((0, \infty)\) and such that

\[
\Phi(0) = 0 \iff \psi'(0+) \geq 0 \quad \text{while} \quad \Phi(0) > 0 \iff \psi'(0+) < 0.
\]

Moreover, \(q \mapsto \Phi(q)\) is the well-defined inverse of \(\psi(\theta)\) on the interval \([\Phi(0), \infty)\), i.e.

\[
\Phi(\psi(\theta)) = \theta \quad \text{and} \quad \psi(\Phi(q)) = q, \quad \forall \theta \in [\Phi(0), \infty), \ q \geq 0.
\]

We refer to [19] for proofs of the given properties and a more thorough discussion of (spectrally negative) Lévy processes and scale functions. More detailed accounts on scale functions and their numerous applications can be found in [2] and [18].

Throughout this article \(\partial^k_q f(q, x)\) denotes the \(k\)-th derivative of a function \(f\) with respect to \(q\), while \(\partial_q := \partial^1_q\). In case of only one parameter, we will usually omit the subscript.
3 Moments of the ruin time

Throughout this section we consider a spectrally negative Lévy process \( X = (X_t)_{t \geq 0} \), with Laplace exponent \( \psi \) given in (2.1). We assume the process \( (X_t)_{t \geq 0} \) to be defined on a filtered probability space \((\Omega, \mathcal{F}, (\mathbb{F}_t), \mathbb{P})\), and as usual we write \( \mathbb{P}_x \), \( \mathbb{E}_x \) for the law and expectation of \( X \) given \( X_0 = x \), respectively.

To avoid trivialities we exclude the case that \( X \) is a pure drift, thus we always have \( \mathbb{P}_x(\tau^-_0 < \infty) > 0 \). If \( X \) is of infinite variation, i.e. if \( \sigma^2 > 0 \) or if \( \int_{|x| < 1} x \Pi(dx) = \infty \), we have \( \mathbb{P}_0(\tau^-_0 = 0) = 1 \), and hence in this case we exclude the initial capital \( x = 0 \) from our considerations.

As shown in [3], for any \( x \geq 0 \) and any \( k \in \mathbb{N} \) the \( k \)-th moment of the ruin time is given by

\[
\mathbb{E}_x[(\tau^-_0)^k | \tau^-_0 < \infty] = \frac{(-1)^k}{\mathbb{P}_x(\tau^-_0 < \infty)} \lim_{q \downarrow 0} \left( k \int_0^x \partial^k_q W(q)(y) \, dy - \sum_{\ell=0}^{k-1} \binom{k}{\ell} \left( \frac{q}{\Phi(q)} \right)^\ell \left( \frac{\partial^\ell_q W(q)(x)}{\Phi(q)} \right) \right),
\]

where the left-hand side is finite if and only if the right-hand side is finite.

Moreover, it follows from the results in [3], that the \( k \)-th moment of the ruin time is finite if and only if one of the following two assumptions holds:

(i) \( \mathbb{E}_0[X_1] = \psi'(0+) < 0 \),

(ii) \( \mathbb{E}_0[X_1] = \psi'(0+) > 0 \) and \( \mathbb{E}[[X_1^{k+1}] < \infty \).

We will thus exclude the case \( \mathbb{E}_0[X_1] = \psi'(0+) = 0 \) from now on. We also note that in the case of a perturbed Cramér-Lundberg model (1.1) the assumption \( \mathbb{E}_0[[X_1]^{k+1}] < \infty \) is equivalent to \( \mathbb{E}[S_1^{k+1}] < \infty \), cf. [19], Thm. 3.8.

The above formula can be used to further study the \( k \)-th moment of the ruin time via its Laplace transform. In the unprofitable case, this is done in the next theorem whose proof is given in Section 6.

**Theorem 3.1.** Let \( X = (X_t)_{t \geq 0} \) be a spectrally negative Lévy process with Laplace exponent \( \psi \) as in (2.1) such that \( \psi'(0+) = \mathbb{E}_0[X_1] \in [-\infty, 0) \) and fix \( x \geq 0 \) (\( x > 0 \) if \( X \) is of infinite variation). Then \( \mathbb{P}_x(\tau^-_0 < \infty) = 1 \) and for all \( k \in \mathbb{N} \) the Laplace transform of \( \mathbb{E}_x[(\tau^-_0)^k] \) is given by

\[
\int_0^\infty e^{-\beta x} \mathbb{E}_x[(\tau^-_0)^k] \, dx = (-1)^k \cdot k! \cdot \left( \frac{1}{\beta} \cdot \frac{1}{\psi(\beta)^k} - \sum_{\ell=1}^{k} \frac{1}{\ell!} \lim_{q \downarrow 0} \partial_q^\ell \left( \frac{q}{\Phi(q)} \right) \cdot \frac{1}{\psi(\beta)^{k-\ell+1}} \right),
\]

for all \( \beta > 0 \), where for \( \beta = \Phi(0) > 0 \) the right hand side of (3.2) has to be understood in the limiting sense yielding

\[
\int_0^\infty e^{-\Phi(0) x} \mathbb{E}_x[(\tau^-_0)^k] \, dx = (-1)^k \lim_{q \downarrow 0} \partial_q^k \left( \frac{1}{\Phi(q)} \right).
\]

Moreover, for all \( k \in \mathbb{N} \),

\[
\lim_{x \to \infty} \frac{\mathbb{E}_x[(\tau^-_0)^k]}{x^k} = \frac{1}{\mathbb{E}_0[X_1]^k}.
\]
An analogue result on the asymptotic behaviour of the moments of the ruin time in the profitable case does not hold, as we will see from the examples in the following sections. Actually, whenever \( \psi'(0+) > 0 \) we have \( \mathbb{P}_x(\tau_0^- < \infty) = 1 - \psi'(0+)W(0)(x) \), cf. [19, Thm. 8.1], and hence the pre-factor in (3.1) depends on \( x \). Thus we can only consider the Laplace transform of \( \mathbb{E}_x[(\tau_0^-)^k \cdot 1_{\{\tau_0^- < \infty\}}] \) in this case as done in the following Proposition. Again, the proof is postponed to Section 6.

**Proposition 3.2.** Let \( X = (X_t)_{t \geq 0} \) be a spectrally negative Lévy process with Laplace exponent \( \psi \) as in (2.1) such that \( \psi'(0+) = \mathbb{E}_0[X_1] \in (0, \infty) \) and fix \( x \geq 0 \) (\( x > 0 \) if \( X \) is of infinite variation). Choose \( k \in \mathbb{N} \) such that \( \mathbb{E}[X_1]^{k+1} < \infty \). Then for any \( \beta > 0 \)

\[
\int_0^\infty e^{-\beta x} \mathbb{E}_x[(\tau_0^-)^k \cdot 1_{\{\tau_0^- < \infty\}}] \, dx = (-1)^k \cdot k! \left( \frac{1}{\beta \psi'(\beta)} - \frac{\psi'(0+)}{\psi(\beta)^{k+1}} \right) - \sum_{\ell=1}^k \frac{1}{\ell! \cdot \psi(\beta)^{k-\ell+1}} \sum_{j=1}^\ell \frac{\psi^{(j+1)}(0+)}{(j+1)} \cdot B_{\ell,j} \left( \Phi'(0+), \ldots, \Phi^{(\ell-j+1)}(0+) \right),
\]

where \( B_{\ell,j} \) denote the partial Bell polynomials, and moreover

\[
\lim_{x \to \infty} \mathbb{E}_x[(\tau_0^-)^k \cdot 1_{\{\tau_0^- < \infty\}}] = \lim_{x \to \infty} \mathbb{E}_x[(\tau_0^-)^k | \tau_0^- < \infty] : \mathbb{P}_x(\tau_0^- < \infty) \to 0.
\]

Lastly, let us mention that - under the assumptions (i) or (ii) on page 4 - formulas for the first two moments of the ruin time have been derived in [3], and for the reader’s convenience we collect these here in the following proposition.

**Proposition 3.3.** Let \( X = (X_t)_{t \geq 0} \) be a spectrally negative Lévy process with Laplace exponent \( \psi \) as in (2.1) and fix \( x \geq 0 \) (\( x > 0 \) if \( X \) is of infinite variation).

(i) Assume \( \mathbb{E}_0[X_1] = \psi'(0+) < 0 \), then all moments of \( \tau_0^- \) are finite and in particular

\[
\mathbb{E}_x[\tau_0^-] = \frac{1}{\Phi(0)} W^{(0)}(x) - \int_0^x W^{(0)}(y) \, dy,
\]

\[
\mathbb{E}_x[(\tau_0^-)^2] = 2 \int_0^x \lim_{q \downarrow 0} \partial_q W^{(q)}(y) \, dy - \frac{2}{\Phi(0)} \lim_{q \downarrow 0} \partial_q W^{(q)}(x) + \frac{2W^{(0)}(x)}{\Phi(0)^2 \psi'(\Phi(0))}.
\]

(ii) Assume that \( \mathbb{E}_0[X_1] = \psi'(0+) > 0 \). If \( \mathbb{E}_0[X_1^2] < \infty \), then

\[
\mathbb{E}_x[\tau_0^- | \tau_0^- < \infty] = \frac{\psi'(0+) \cdot \lim_{q \downarrow 0} \partial_q W^{(q)}(x)) + \frac{\psi^\prime(0+)}{2\psi'(0+)^2} \cdot W^{(0)}(x) - \int_0^x W^{(0)}(y) \, dy}{1 - \psi'(0+)W^{(0)}(x)} < \infty.
\]

Moreover, if \( \mathbb{E}_0[|X_1|^3] < \infty \), then

\[
\mathbb{E}_x[(\tau_0^-)^2 | \tau_0^- < \infty] = \frac{1}{1 - \psi'(0+)W^{(0)}(x)} \left( 2 \lim_{q \downarrow 0} \int_0^x \partial_q W^{(q)}(y) \, dy - \psi'(0+) \lim_{q \downarrow 0} \partial_q^2 W^{(q)}(x) \right) + \frac{\psi'(0+)^2}{3\psi'(0+)^3} \cdot W^{(0)}(x) < \infty.
\]


4 Phase-type distributed claims

From here onwards, we consider the perturbed Crámer-Lundberg risk model $X = (X_t)_{t \geq 0}$ as in (1.1), where we exclude the trivial case that $X$ is a pure drift, as well as the case $\sigma^2 > 0$ and $x = 0$.

In this section, we assume that $\{S_i, i \in \mathbb{N}\}$ is a sequence of i.i.d. random variables with phase-type distribution, i.e. $S_i \sim \text{PH}_d(\alpha, T)$, with the cdf and density of $S_i$ being given by

$$F(z) = 1 - e^{z^T \alpha \mathbf{1}}, \quad \text{and} \quad f(z) = e^{z^T \alpha \mathbf{t}},$$

respectively. Here, $d \in \mathbb{N}, \alpha \in \mathbb{R}_d^d$ with $\|\alpha\|_1 = 1$, $T \in \mathbb{R}^{d \times d}$ is an invertible subintensity matrix, $\mathbf{1}$ is the $d$-dimensional column vector of $1$'s, and $\mathbf{t} := -T \mathbf{1}$ is a vector in $\mathbb{R}^d$. It is well known, see e.g. [4] Thm. 3.1.16 and Cor. 3.1.18, that in this case

$$E[S_1] = \alpha (-T)^{-1} \mathbf{1}, \quad \text{and} \quad E[S_1^2] = 2\alpha (-T)^{-2} \mathbf{1},$$

and that all moments of $S_i$ exist. Further, the Laplace exponent of the process $(X_t)_{t \geq 0}$ in the current setting easily follows from (2.2) and [4] Thm. 3.1.19 to be

$$\psi(\theta) = p\theta + \frac{1}{2}\theta^2 \sigma^2 + \lambda \left( \theta \mathbf{I} - T \right)^{-1} t - 1, \quad \theta \geq 0. \quad (4.11)$$

Moreover, whenever $\psi'(0+) \neq 0$, the $q$-scale functions of $(X_t)_{t \geq 0}$ in the current setting are known explicitly. Namely, let

$$\mathcal{R}_q := \{ z \in \mathbb{C} : \psi(z) = q \text{ and } z \neq \Phi(q) \}, \quad q \geq 0, \quad (4.12)$$

be the set of (possibly complex) $q$-roots of $\psi$. Then all $z \in \mathcal{R}_q$ have non-positive real part $\Re(z) \leq 0$. Further, we write $n = |\mathcal{R}_q|$ for the number of distinct roots in $\mathcal{R}_q$, denote $\mathcal{R}_q = \{ \phi_i(q), q = 1, \ldots, n \}$ and assume w.l.o.g. that $\Re(\phi_n(q)) \leq \ldots \leq \Re(\phi_1(q)) \leq 0$. Then, assuming that the multiplicity of each $z \in \mathcal{R}_q$ is one, the $q$-scale function of the process (1.1) with $S_i \sim \text{PH}_d(\alpha, T)$ is given by

$$W^{(q)}(x) = \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))} + \sum_{i=1}^{n} \frac{e^{\phi_i(q)x}}{\psi'('q_i(q))}, \quad x \geq 0, q \geq 0, \quad (4.13)$$

as stated in [16] Eq. (5)], which relies on [13] Sec. 5.4. For $\psi'(0+) < 0$ the above form of the $q$-scale function has also been given in [9], while a special case of $\psi'(0+) > 0$ can also be found in [20] Eq. (9)]. Note that several of these sources also consider the case of multiple roots, but due to the more lengthy form of the resulting scale functions we shall exclude this case in our exposures.

The explicit form of the scale functions (4.13) allows for an evaluation of the formulas given in Proposition 3.3 which results in semi-explicit formulae for the first two moments of the ruin time for the considered processes as they will be presented in the following two theorems. The technical and lengthy proofs of both theorems are postponed to Section 6.

We start with our result in the unprofitable setting.

**Theorem 4.1.** Let $(X_t)_{t \geq 0}$ have the Laplace exponent (4.11) and assume that $\psi'(0+) = E_0[X_1] < 0$. Then for all $x \geq 0$ ($x > 0$ if $\sigma^2 > 0$)

$$E_x[\tau^{-}] = -\frac{x}{\psi'(0+)} + C_1^1 + \epsilon_1(x), \quad (4.14)$$
\[
\mathbb{E}_x[(\tau_0^-)^2] = \frac{x^2}{\psi'(0+)^2} - \frac{2}{\psi'(0+)^2} \left( \psi''(0+) + \frac{1}{\Phi(0)} \right) \cdot x + C_2^\uparrow + \epsilon_2(x), \tag{4.15}\]
with
\[
\epsilon_1(x) := \sum_{i=2}^n e^{\phi_i(0)x} \cdot \frac{1}{\psi'(\phi_i(0))} \left( \frac{1}{\Phi(0)} - \frac{1}{\phi_i(0)} \right),
\]
\[
\epsilon_2(x) := 2 \sum_{i=2}^n \left( e^{\phi_i(0)x} \cdot \frac{1}{\psi'(\phi_i(0))^2} \left( \frac{1}{\phi_i(0)} - \frac{1}{\Phi(0)} \right) + e^{\phi_i(0)x} \cdot \frac{\psi''(\phi_i(0))}{\psi'(\phi_i(0))^3} \left( \frac{1}{\Phi(0)} - \frac{1}{\phi_i(0)} \right) \right)
\]
\[
+ e^{\phi_i(0)x} \left( \frac{1}{\Phi(0)^2 \psi'(\Phi(0)) \psi'(\phi_i(0))} - \frac{1}{\psi'(\phi_i(0))^2 \phi_i(0)^2} \right),
\]
and
\[
C_1^\uparrow := \frac{1}{\Phi(0)p} \mathbb{1}_{\{\sigma^2=0\}} - \epsilon_1(0) = \frac{1}{\Phi(0) \psi'(0+)^2} + \frac{\psi''(0+)}{2 \psi'(0+)^2},
\]
\[
C_2^\uparrow := \frac{2}{\Phi(0)^2 \psi'(\Phi(0)) p} \mathbb{1}_{\{\sigma^2=0\}} - \epsilon_2(0)
\]
\[
= \frac{2 \psi''(0+)}{\Phi(0) \phi_i(0)^4} + \frac{2}{\Phi(0)^2 \psi'(\Phi(0)) \psi'(0+)^3} + \frac{3 \psi'''(0+)^2}{2 \psi'(0+)^4} - \frac{2 \psi''''(0+)}{3 \psi'(0+)^5}.
\]
In particular
\[
\mathbb{E}_0[\tau_0^-] = \frac{1}{\Phi(0)p} \mathbb{1}_{\{\sigma^2=0\}}, \quad \text{and} \quad \mathbb{E}_0[(\tau_0^-)^2] = \frac{2}{\Phi(0)^2 \psi'(\Phi(0))} \mathbb{1}_{\{\sigma^2=0\}},
\]
while for \( x \to \infty \) we have \( \epsilon_j(x) \to 0, \ j = 1, 2, \) exponentially fast, since \( \Re \phi_i(0) < 0 \) for all \( i = 2, \ldots, n. \)

In the profitable setting, the first two moments of the ruin time can be expressed as follows.

**Theorem 4.2.** Let \((X_t)_{t \geq 0}\) have the Laplace exponent \((4.11)\) and assume that \(\psi'(0+) = \mathbb{E}_0[X_1] > 0.\) Then for all \( x \geq 0 \) (\( x > 0 \) if \( \sigma^2 > 0 \))

\[
\mathbb{E}_x[\tau_0^- | \tau_0^- < \infty] = -\left( \sum_{i=1}^n e^{\phi_i(0+)x} \right)^{-1} \left( \sum_{i=1}^n \frac{e^{\phi_i(0+)x} \cdot x}{\psi'(\phi_i(0+))^2} \right)
\]
\[
- \sum_{i=1}^n e^{\phi_i(0+)x} \cdot \left( \frac{\psi''(\phi_i(0+))}{\psi'(\phi_i(0+))^3} - \frac{\psi''(0+)}{2 \psi'(0+)^2 \phi_i(0+)^2} + \frac{1}{\psi'(0+)^2 \phi_i(0+)^2} \right), \tag{4.16}
\]
and

\[
\mathbb{E}_x[(\tau_0^-)^2 | \tau_0^- < \infty] = \left( \sum_{i=1}^n \frac{e^{\phi_i(0+)x}}{\psi'(\phi_i(0+))} \right)^{-1} \left( \sum_{i=1}^n \frac{e^{\phi_i(0+)x}}{\psi'(\phi_i(0+))^3} \cdot \frac{x^2}{\psi'(\phi_i(0+))^3} + B_i^\uparrow x + C_i^\uparrow \right), \tag{4.17}
\]
with
\[
B_i^\uparrow := \frac{\psi''(0+)}{\psi'(0+)^2 \psi'(\phi_i(0+))^2} - \frac{2}{\psi'(\phi_i(0+))^2 \phi_i(0+)^2 \psi'(0+)} - \frac{3 \psi'''(\phi_i(0+))}{\psi'(\phi_i(0+))^4}.
\]
and  \( C_i^\uparrow := \frac{2}{\psi''(\phi_i(0+))\phi_i(0+)^2\psi''(0+)} - \frac{\psi'''(\phi_i(0+))}{\psi''(\phi_i(0+))^3} + \frac{\psi''''(0+)}{3\psi''(\phi_i(0+))\psi''(0+)^3} - \frac{\psi''(0+)^2}{2\psi''(\phi_i(0+))\psi''(0+)} - \frac{\psi'''(\phi_i(0+))}{\psi''(\phi_i(0+))} \cdot B_i^\uparrow. \)

In particular

\[
\mathbb{E}_0[\tau_0^- | \tau_0^- < \infty] = \left( \frac{1}{p} \mathbb{I}_{\{\sigma^2=0\}} - \frac{1}{\psi'(0+)} \right)^{-1} \cdot \frac{\psi''(0+)}{2\psi'(0+)^2} p \mathbb{I}_{\{\sigma^2=0\}},
\]

and  \( \mathbb{E}_0[\tau_0^- | \tau_0^- < \infty] \) is

\[
\frac{1}{p} \mathbb{I}_{\{\sigma^2=0\}} - \frac{1}{\psi'(0+)} \cdot \frac{\psi'''(0+)}{3\psi'(0+)^3} - \frac{\psi''(0+)^2}{2\psi'(0+)^4} \cdot \frac{1}{p} \mathbb{I}_{\{\sigma^2=0\}}.
\]

At this point, let us emphasize that the obtained formulas in Theorems 4.1 and 4.2 may seem complicated at first sight, but they can be evaluated rather easily in concrete cases as long as \( n \) is not too big: As the Laplace exponent \( \psi \) is given explicitly in (4.11), its derivatives and roots can be determined by standard procedures. In Figure 1 we provide evaluations of the obtained formulas for Cramér-Lundberg processes with phase-type distributions.

Remark 4.3. Note that, by the method explained in [18, Sec. 5.4], it is possible to show that a representation for the \( q \)-scale function as in (4.13) exists for any Lévy process that has a rational transform, i.e., any Lévy process whose Laplace exponent is a rational function. In the case \( \psi'(0^+) < 0 \) this has been done in [9]. This also allows to extend the results stated above in Theorems 4.1 and 4.2 in the context of phase-type distributions to this wider class of processes, since our proofs only rely on the specific representation of the scale function.
Figure 1: Expected ruin time (given it is finite) on top, and standard deviation of the ruin time (given it is finite) on bottom, in the perturbed Cramér-Lundberg model with different phase-type distributed claims and different choices of $\sigma^2 \geq 0$. The parameters chosen are $p = 1 = \lambda$, such that $\psi'(0) \approx -0.5 < 0$ (left), and $p = 2$, $\lambda = 1$, such that $\psi'(0) \approx 0.5 > 0$ (right).

5 Exponentially distributed claims

As before, we consider the perturbed Cramér-Lundberg model $(X_t)_{t \geq 0}$ of the form (1.1), where we exclude the case that $X$ is a pure drift, as well as the case $\sigma^2 > 0$ and $x = 0$.

In this section the claim sizes $\{S_i, i \in \mathbb{N}\}$ are supposed to be i.i.d. exponential random variables with parameter $\gamma > 0$.

Whenever $\sigma^2 = 0$, we recover the classical Cramér-Lundberg process with exponential claims, a model that has shown to be extremely well-treatable, as, e.g., ruin probabilities or excess of loss distributions at the time of ruin can be determined in closed form, cf. [1]. Also the ruin time in this model has been studied before, e.g. in [14, Chapter 9.3], where the first moment of the ruin time is computed, or more recently in [8], where explicit formulae for arbitrary integer moments and a pdf of the ruin time are derived. However, all those results are only valid if $\sigma^2 = 0$ and the safety loading condition $p - \lambda \gamma > 0$ is fulfilled, i.e. if the model is profitable with $\mathbb{E}[X_1] = \psi'(0+) > 0$.

With the methods derived above, we are now able to present explicit formulae for the first and second moment of the ruin time, whenever $\psi'(0+) \neq 0$ and for all choices of $\sigma^2 \geq 0$, i.e., we will extend the existing results in two directions by considering a perturbed CL model both in a profitable and in an unprofitable scenario.

As the exponential distribution is the simplest representative of a phase-type distribution, we can apply the results from the last section. First of all note that the Laplace exponent
ψ of \( X_t \) in the current setting is

\[
\psi(\theta) = p\theta + \frac{1}{2}\theta^2\sigma^2 + \lambda\left(\frac{\gamma}{\theta + \gamma} - 1\right) = p\theta + \frac{1}{2}\theta^2\sigma^2 - \frac{\theta\lambda}{\theta + \gamma}, \quad \theta \neq -\gamma,
\] (5.18)

with derivatives

\[
\psi'(\theta) = p + \theta\sigma^2 - \frac{\lambda\gamma}{(\theta + \gamma)^2}, \quad \psi''(\theta) = \sigma^2 + \frac{2\lambda\gamma}{(\theta + \gamma)^3}, \quad \text{and} \quad \psi'''(\theta) = -\frac{6\lambda\gamma}{(\theta + \gamma)^4}.
\] (5.19)

Furthermore, the \( q \)-scale function of \( X \) is of the form (4.13), where one can see from (5.18) that for every \( q > 0 \) the equation \( \psi'(\theta) = q \) has exactly three real solutions \( \phi_2(q) < -\gamma < \phi_1(q) < 0 < \Phi(q) \), cf. [18, Ex. 1.3]. Moreover, as \( \sigma \) tends to zero, \( \phi_2(q) \to -\infty \), and in the limiting case \( \sigma^2 = 0 \) only two \( q \)-roots of \( \psi \) exist, which are then easily computed to be, cf. [18],

\[
\zeta_{1,2} = \frac{1}{2p} \left( \lambda + q - \gamma p \pm \sqrt{(\lambda + q - \gamma p)^2 + 4q\gamma p} \right),
\]

such that in particular

\[
\Phi(0) = \begin{cases} \frac{\lambda p - \gamma}{p}, & \text{if } \psi'(0+) < 0, \\ 0, & \text{if } \psi'(0+) > 0, \\ \frac{-\psi(0+) - \lambda}{p}, & \text{if } \psi'(0+) > 0. \end{cases}
\]

(5.20)

For \( \sigma^2 > 0 \) no explicit representation of \( \Phi(q), \phi_1(q) \) and \( \phi_2(q), q > 0 \), is known. However, for \( q = 0 \), it is easily checked that the three roots are given by

\[
\zeta_1 = 0, \quad \text{and} \quad \zeta_{2,3} = \frac{1}{2\sigma^2} \left( -\gamma\sigma^2 - 2p \pm \sqrt{(\gamma\sigma^2 - 2p)^2 + 8\lambda\sigma^2} \right),
\]

where in the case \( \psi'(0+) = p - \frac{\lambda}{\gamma} < 0 \) it holds \( \phi_2(0) = \zeta_3 < \phi_1(0) = 0 < \Phi(0) = \zeta_2 \), while for \( \psi'(0+) > 0 \) we have to reorder and obtain \( \phi_2(0) = \zeta_3 < \phi_1(0) = \zeta_2 < \Phi(0) = 0 \).

The following two corollaries are now immediate consequences of Theorems 4.1 and 4.2, respectively, and can be shown by standard algebra.

Again we start with the unprofitable setting.

**Corollary 5.1.** Let \( (X_t)_{t\geq 0} \) have the Laplace exponent (5.18) with \( \sigma^2 > 0 \) and assume that \( E_0[X_1] = \psi'(0+) = p - \frac{\lambda}{\gamma} < 0 \). Set \( r := \sqrt{(\gamma\sigma^2 - 2p)^2 + 8\lambda\sigma^2} \). Then for all \( x > 0 \)

\[
E_x[\tau_0^-] = \gamma \cdot \frac{x}{\lambda - p\gamma} \cdot x + \left( 1 - e^{-\frac{\sigma^2 + 2p+rx}{2\sigma^2}} \right) \cdot C^\perp_1,
\]

\[
E_x[(\tau_0^-)^2] = \frac{\gamma^2}{(\lambda - p\gamma)^2} \cdot x^2 + \left( B^\perp - e^{-\frac{\sigma^2 + 2p+rx}{2\sigma^2}} \left( \frac{4\lambda\gamma\sigma^4}{(\gamma\sigma^2 - 2p - r)^2} + \frac{\gamma\sigma^2 + r}{2r} \right) \right) \cdot x
\]

\[
+ \left( 1 - e^{-\frac{\sigma^2 + 2p+rx}{2\sigma^2}} \right) \cdot C^\perp_2,
\]

with

\[
B^\perp = \frac{2\gamma^2}{(\lambda - p\gamma)^2} \left( \frac{2\lambda}{\gamma(\lambda - p\gamma)} + \frac{\gamma\sigma^2}{(\lambda - p\gamma)^2} + \frac{2\sigma^2}{\gamma\sigma^2 + 2p - r} \right),
\]

\[
C^\perp_1 = \frac{\lambda}{(\lambda - p\gamma)^2} + \frac{\gamma^2\sigma^2}{2(\lambda - p\gamma)^2} + \frac{2\gamma^2}{(\lambda - p\gamma)(\gamma\sigma^2 + 2p - r)},
\]

\[
C^\perp_2 = -\frac{4\lambda}{(\lambda - p\gamma)^3} + \frac{3\sigma^2 + 2\lambda^2}{2(\lambda - p\gamma)^3} + \frac{4\gamma\lambda\sigma^4 + 8\lambda\gamma\sigma^2}{(\lambda - p\gamma)^3(\gamma\sigma^2 - 2p - r)} + \frac{2\gamma}{(\lambda - p\gamma)(\gamma\sigma^2 - 2p - r)^2} \left( \frac{\lambda\gamma}{(\gamma\sigma^2 - 2p - r)^2} + \frac{\gamma\sigma^2 + r}{8r} \right)^{-1}.
\]
In the profitable setting, the representations of the first two moments are as follows.

**Corollary 5.2.** Let \((X_t)_{t \geq 0}\) have the Laplace exponent \((5.18)\) with \(\sigma^2 > 0\) and assume that \(E_0[X_1] = \psi'(0+) = p - \frac{\lambda}{\gamma} > 0\). Set \(r := \sqrt{(\gamma\sigma^2 - 2p)^2 + 8\lambda\sigma^2}\). Then for all \(x > 0\)

\[
E_x[\tau_0^-|\tau_0^- < \infty] = \frac{x \cdot \left(\frac{4\lambda\gamma^4}{(\gamma\sigma^2 - 2p - r)^2} + \frac{\gamma^2 - r}{2}\right)^{-1} + A^\uparrow_x}{1 + e^{\text{Exp}(x)}} + \frac{x \cdot \left(\frac{4\lambda\gamma^4}{(\gamma\sigma^2 - 2p - r)^2} + \frac{\gamma^2 + r}{2}\right)^{-1} + A^\downarrow_x}{1 + e^{\text{Exp}(x)^{-1}}},
\]

\[
E_x[\tau_0^-|\tau_0^- < \infty] = \frac{x^2 \cdot \left(\frac{4\lambda\gamma^4}{(\gamma\sigma^2 - 2p - r)^2} + \frac{\gamma^2 - r}{2}\right)^{-2} + x \cdot B^\uparrow_x + C^\uparrow}{1 + e^{\text{Exp}(x)}};
\]

\[
E_x[\tau_0^-|\tau_0^- < \infty] = \frac{x^2 \cdot \left(\frac{4\lambda\gamma^4}{(\gamma\sigma^2 - 2p - r)^2} + \frac{\gamma^2 + r}{2}\right)^{-2} + x \cdot B^\downarrow_x + C^\downarrow}{1 + e^{\text{Exp}(x)^{-1}}},
\]

with

\[
A^\uparrow_x := -\frac{\lambda}{(\gamma - \lambda)^2} - \frac{2\gamma^2}{(\gamma - \lambda)^2} - \frac{2\gamma^2}{(\gamma - \lambda)(\gamma\sigma^2 + 2p + r)} + \left(\sigma^2 + \frac{16\lambda\gamma^6}{(\gamma\sigma^2 - 2p - r)^2}\right) \left(\frac{4\lambda\gamma^4}{(\gamma\sigma^2 - 2p - r)^2} + \frac{\gamma^2 + r}{2}\right)^{-2},
\]

\[
B^\uparrow_x := -\left(\frac{4\lambda\gamma^4}{(\gamma\sigma^2 - 2p - r)^2} + \frac{\gamma^2 + r}{2}\right)^{-1} \left[\frac{2\lambda + \gamma^2}{(\gamma - \lambda)^2} + \frac{4\gamma^2}{(\gamma - \lambda)(\gamma\sigma^2 + 2p + r)}\right]
+ \left(\frac{4\lambda\gamma^4}{(\gamma\sigma^2 - 2p - r)^2} + \frac{\gamma^2 + r}{2}\right)^{-3} \left[3\sigma^2 + \frac{48\lambda\gamma^6}{(\gamma\sigma^2 - 2p - r)^4}\right],
\]

\[
C^\uparrow_x := -\frac{2\lambda}{(\gamma - \lambda)^3} - \frac{(2\lambda + \gamma^2)^2}{(\gamma - \lambda)^4} - \left(\frac{4\lambda\gamma^4}{(\gamma\sigma^2 + 2p + r)^2} + \frac{\gamma^2}{2}\right)^{-3} \left[\frac{8\gamma^4}{(\gamma - \lambda)^4} + \frac{16\lambda\gamma^6}{(\gamma\sigma^2 - 2p - r)^2}\right] B^\uparrow_x,
\]

and

\[
e^{\text{Exp}(x)} := \left(\frac{\gamma^2 - 2p - r}{\gamma^2 - 2p + r}\right)^2 \cdot \frac{8\lambda\gamma^4 + (\gamma^2 - 2p - r)^2(\gamma^2 - r)}{8\lambda\gamma^4 + (\gamma^2 - 2p - r)^2(\gamma^2 + r)} \cdot e^{-\frac{\lambda x}{\gamma}}.
\]

**Remark 5.3.** Observe that due to \(-\frac{\lambda}{\sigma^2} = -\sqrt{(\gamma - 2p)^2 + 8\lambda\sigma^2} < 0\), the function \(e^{\text{Exp}(x)}\) in Corollary 5.2 above decreases exponentially as \(x\) grows. Thus for large \(x\) the behaviour of the first and second moment of the ruin time is dominated by the nominator of the first summand in \((5.21)\) and \((5.22)\), respectively. In particular, the first moment shows an approximately affine linear behaviour, while the second moment shows an approximately quadratic behaviour, whenever \(x\) is large.

Inserting the roots \((5.20)\) in Theorems 4.1 and 4.2 we can also easily derive the first two moments of the ruin time in the classical Cramér-Lundberg model. Note that in the profitable case \(\psi'(0+) > 0\) these formulas have already been obtained in [8].

**Corollary 5.4.** Let \((X_t)_{t \geq 0}\) have the Laplace exponent \((5.18)\) with \(\sigma^2 = 0\). Then in the unprofitable case \(E_0[X_1] = \psi'(0+) = p - \frac{\lambda}{\gamma} < 0\) for all \(x \geq 0\)

\[
E_x[\tau_0^-] = \frac{\gamma x + 1}{\lambda - p\gamma},
\]

and

\[
E_x[\tau_0^-] = \frac{\gamma^2 x^2}{(\lambda - p\gamma)^2} + \frac{2\gamma x(2\lambda - p\gamma)}{(\lambda - p\gamma)^3} + \frac{2\lambda}{(\lambda - p\gamma)^3}.
\]
In the profitable case $E_0[X_1] = \psi'(0+) = p - \frac{\lambda}{\gamma} > 0$ we get for all $x \geq 0$

$$E_x[\tau_0^- | \tau_0^- < \infty] = \frac{\lambda x + 1}{p\gamma - \lambda},$$

and

$$E_x[(\tau_0^-)^2 | \tau_0^- < \infty] = \frac{(\lambda^2 x^2)}{(p\gamma - \lambda)^2} + \frac{2\lambda x (2p\gamma - \lambda)}{(p\gamma - \lambda)^3} + \frac{2p\gamma}{(p\gamma - \lambda)^3}.$$

Moreover, for $E_0[X_1] = \psi'(0+) \neq 0$ we may summarize to get

$$\text{Var}_x(\tau_0^- | \tau_0^- < \infty) = \frac{2\lambda\gamma x}{|p\gamma - \lambda|^3} + \frac{p\gamma + \lambda}{|p\gamma - \lambda|^3}, \quad x \geq 0.$$

Comparing the obtained formulas in Corollaries 5.1, 5.2 for $\sigma^2 > 0$ with the results in Corollary 5.4, we immediately see that the additional Brownian motion in the perturbed Cramér-Lundberg model has a big impact on the ruin time. For small $x$ this is intuitively clear. For large $x$, we note that for $\psi'(0+) < 0$ the $\epsilon_{1,2}$-terms in Corollary 5.1 vanish and the influence of $\sigma^2$ can only be seen in the appearing constants, while the ascent in $x$ of the expected ruin time is untouched by $\sigma^2$. This behaviour coincides with the asymptotics shown in Theorem 3.1. In the case $\psi'(0+) > 0$ treated in Corollary 5.2 however also the ascent in $x$ of the expected ruin time does depend on $\sigma^2$ as we already saw in Figure 1.

6 Proofs

6.1 Proofs for the results in Section 3

Proof of Theorem 3.1

We will first prove Theorem 3.1 under an additional condition as formulated in the upcoming proposition. Thereafter we will argue that this additional condition is always fulfilled and hence can be discarded.

Note that it follows from [19, Eq (8.10)] that under the assumption $\psi'(0+) = E[X_1] < 0$ we have $P_x(\tau_0^- < \infty) = 1$. We may thus abbreviate throughout this section

$$u_k(x) := E_x[(\tau_0^-)^k] = E_x[(\tau_0^-)^k | \tau_0^- < \infty], \quad k \in \mathbb{N}.$$ 

Further, we set $W := W^{(0)}$, and define the function $\eta : [0, \infty) \to [0, \infty)$ by setting $\eta(q) := q/\Phi(q)$.

Proposition 6.1. Consider the setting of Theorem 3.1 and assume additionally that for some $k \in \mathbb{N}$, there exist $A_k, B_k > 0$ such that

$$u_k(x) \leq A_k + B_k x^k \quad \text{for all } x \geq 0. \quad (6.1)$$

Then the Laplace transform of $u_k$ exists for all $\beta > 0$ and it is given by (3.2) and (3.3). Moreover (3.4) holds for the chosen $k$.

Proof. To compute the Laplace transform of $u_k$ observe first that it has been shown in [3], that Equation (3.1) can be rewritten as

$$E_x[(\tau_0^-)^k | \tau_0^- < \infty] = \frac{(-1)^k \cdot k!}{P_x(\tau_0^- < \infty)} \left( \int_0^x W^{*k}(y) dy - \sum_{\ell=0}^{k} \frac{\eta^{(\ell)}(0+)}{\ell!} W^{*(k-\ell+1)}(x) \right), \quad (6.2)$$
where $W^*k$ denotes the $k$-fold convolution of the 0-scale function $W$ with itself. 

Hereby, as mentioned, the assumption $\psi'((0+) = E[X_1] < 0$ implies $\mathbb{P}_x(\tau_0 < \infty) = 1$. Moreover, in [3] it has been shown that in this case $\eta^{(\ell)}(0+)) < \infty$ for any $\ell \in \mathbb{N}$. Additionally, $\eta^{(0)}(0-) = \lim_{q \to 0} q/\Phi(q) = 0$ in this case, cf. [19 Thm. 8.1(i)], such that (6.2) reduces to

$$u_k(x) = (-1)^k \cdot k! \left( \int_0^x W^*k(y) \, dy - \sum_{\ell=1}^k \frac{\eta^{(\ell)}(0+)}{\ell!} W^*(k-\ell+1)(x) \right), \quad k \in \mathbb{N}. \quad (6.3)$$

Now recall the definition of $W$ in (2.3) as the inverse Laplace transform of $1/\psi(\beta)$. Using standard calculation rules for Laplace transforms, cf. [11 Chapter XIII] or [25], we thus obtain for any $\beta > \Phi(0)$

$$\int_0^\infty e^{-\beta x} u_k(x) \, dx = (-1)^k \cdot k! \int_0^\infty e^{-\beta x} \left( \int_0^x W^*k(y) \, dy - \sum_{\ell=1}^k \frac{\eta^{(\ell)}(0+)}{\ell!} W^*(k-\ell+1)(x) \right) \, dx$$

$$= (-1)^k \cdot k! \cdot \left( \frac{1}{\beta} \cdot \frac{1}{\psi(\beta)^k} - \sum_{\ell=1}^k \frac{\eta^{(\ell)}(0+)}{\ell!} \cdot \frac{1}{\psi(\beta)^{k-\ell+1}} \right) =: \mathcal{L}_k(\beta),$$

which agrees with the formula in (3.2). Furthermore, assumption (6.1) implies that the Laplace transform of $u_k$ exists also for $0 < \beta \leq \Phi(0)$, cf. [25 Thm. II.2.1]. Hence (3.2) holds for all $\beta > 0$ as claimed, cf. [25 Thm. II.6.3].

Further, to compute $\mathcal{L}_k(\Phi(0))$ and thus prove (3.3), we consider the limit

$$\lim_{\beta \to \Phi(0)} \mathcal{L}_k(\beta) = (-1)^k \cdot k! \cdot \lim_{\beta \to \Phi(0)} \left( \frac{1}{\beta} \cdot \frac{1}{\psi(\beta)^k} - \sum_{\ell=1}^k \frac{\eta^{(\ell)}(0+)}{\ell!} \cdot \frac{1}{\psi(\beta)^{k-\ell+1}} \right). \quad (6.4)$$

Recall that for $q \geq 0$ we have $\psi(\Phi(q)) = q$ and that $\Phi$ is a continuous function on $[0, \infty)$. Thus,

$$\lim_{\beta \to \Phi(0)} \frac{1}{\beta} \cdot \frac{1}{\psi(\beta)^k} - \sum_{\ell=1}^k \frac{\eta^{(\ell)}(0+)}{\ell!} \cdot \frac{1}{\psi(\beta)^{k-\ell+1}} = \lim_{\gamma \downarrow 0} \frac{1}{\Phi(\gamma)} - \sum_{\ell=1}^k \frac{\eta^{(\ell)}(0+)}{\ell!} \cdot \frac{1}{\gamma^{k-\ell+1}}$$

$$= \lim_{\gamma \downarrow 0} \frac{1}{\Phi(\gamma)} - \sum_{\ell=1}^k \frac{\eta^{(\ell)}(0+)}{\ell!} \cdot \frac{\gamma^{\ell-1}}{\gamma^{k}}$$

$$=: \lim_{\gamma \downarrow 0} g_k(\gamma).$$

Hereby, as $\eta(q) = \frac{q}{\Phi(q)}$, the general Leibniz rule implies that

$$\eta^{(k)}(q) = \sum_{\ell=0}^k \binom{k}{\ell} \cdot (\partial_q q)^{\ell} \cdot (\partial_q^{k-\ell} (\Phi(q)-1)) = q \cdot \partial_q^k (\Phi(q)-1) + k \cdot \partial_q^{k-1} (\Phi(q)-1).$$

for any $k \geq 1$, and hence, since $\psi'(0+) < 0$, we have

$$\eta^{(k)}(0+) = k \cdot \lim_{q \downarrow 0} \partial_q^{k-1} (\Phi(q)-1), \quad k \in \mathbb{N}. \quad (6.5)$$

Note that the limit on the right-hand side is finite by the same reasoning as in [3 Proof of Thm. 3.1(i)]. We now compute

$$\lim_{\gamma \downarrow 0} g_k(\gamma) = \lim_{\gamma \downarrow 0} \frac{1}{\Phi(\gamma)} - \sum_{\ell=2}^k \frac{\eta^{(\ell)}(0+)}{\ell!} \cdot \frac{\gamma^{\ell-1}}{\gamma^k}, \quad (6.6)$$
where for \( k = 1 \) with (6.5)

\[
\lim_{\gamma \downarrow 0} g_1(\gamma) = \lim_{\gamma \downarrow 0} \frac{1}{\Phi(\gamma)} - \frac{1}{\Phi(0)} = \lim \frac{\partial}{\gamma} \left( \frac{1}{\Phi(\gamma)} \right) = \frac{\eta''(0+)}{2}.
\]

For \( k \geq 2 \) we proceed from (6.6) using l’Hôpital’s rule to obtain

\[
\lim_{\gamma \downarrow 0} g_k(\gamma) = \lim_{\gamma \downarrow 0} \frac{\partial}{\gamma} \left( \frac{1}{\Phi(\gamma)} \right) - \sum_{l=2}^{k-1} \frac{\eta^{(l)}(0+)}{l!} (l-1) \cdot \gamma^{l-2} \cdot k^{-1} - \sum_{l=3}^{k} \frac{\eta^{(l)}(0+)}{l!} (l-1) \cdot \gamma^{l-2} \cdot k^{-1},
\]

which, again in the light of (6.5), implies for \( k = 2 \)

\[
\lim_{\gamma \downarrow 0} g_2(\gamma) = \frac{1}{2} \lim_{\gamma \downarrow 0} \frac{\partial}{\gamma} \left( \frac{1}{\Phi(\gamma)} \right) - \lim_{\gamma \downarrow 0} \frac{\partial}{\gamma} \left( \frac{1}{\Phi(\gamma)} \right) = \frac{\eta''(0+)}{3!}.
\]

Further iterating this argument yields

\[
\lim_{\gamma \downarrow 0} g_k(\gamma) = \frac{1}{k!} \lim_{\gamma \downarrow 0} \frac{\partial}{\gamma} \left( \frac{1}{\Phi(\gamma)} \right) = \frac{\eta^{(k+1)}(0+)}{(k+1)!}, \quad k \in \mathbb{N},
\]

and inserting this in (6.4) proves

\[
\lim_{\beta \to \Phi(0)} \mathcal{L}_k(\beta) = (-1)^k \cdot k! \cdot \frac{\eta^{(k+1)}(0+)}{(k+1)!} = (-1)^k \cdot \frac{k!}{k+1} \cdot \eta^{(k+1)}(0+) = (-1)^k \cdot \lim_{q \to 0} \frac{\partial^k}{\partial \gamma^k} \left( \frac{1}{\Phi(q)} \right)
\]

which is (3.3).

To prove the asymptotics as stated in (3.4) we aim to apply a Tauberian theorem, as presented in [11, Ch. XIII.5, Thm. 4]. Note that this theorem is applicable, since \( u_k : [0, \infty) \to [0, \infty) \) is finite (see [3] or Section 3), and monotonely increasing in \( x \). Hence, \( U_k(x) := \int_0^x u_k(x) \, dx \) defines an improper distribution function. Moreover, we have

\[
\lim_{\beta \downarrow 0} \mathcal{L}_k(\beta) = (-1)^k \cdot k! \cdot \lim_{\beta \downarrow 0} \frac{\beta^k}{\psi(\beta)^k} \cdot \frac{\beta^{k+1}}{\psi(\beta)^{k+1}} = (-1)^k \cdot k! \cdot \frac{1}{\psi'(0+)^k},
\]

since \( \beta/\psi(\beta) \to \psi'(0+)^{-1} \) by l’Hôpital’s rule. Hence,

\[
\mathcal{L}_k(\beta) \sim \frac{1}{\beta^{k+1}} \cdot \left( (-1)^k \cdot k! \cdot \frac{1}{\psi'(0+)^k} \right), \quad \text{as } \beta \to 0.
\]

Now [11, Ch. XIII.5, Thm. 4] yields the claim. \( \square \)

It remains to prove that in the setting of Theorem 3.1 the assumption (6.1) is always valid for all \( k \in \mathbb{N} \). We will show this via induction. The next lemma provides the initial case \( k = 1 \) while the proposition thereafter covers the induction step.

**Lemma 6.2.** Under the assumptions of Theorem 3.1 there exist \( A, B > 0 \) such that

\[
u_1(x) \leq A + B \cdot x \quad \text{for any } x \geq 0.
\]
Proof. We use a semi-explicit representation of the scale function as given in \([2, \text{Eq. (29)}]\), which reads
\[
W^{(q)}(x) = \Phi'(q) \cdot \left( e^{\Phi(q) x} - \mathbb{P}_x(T_{[0]} < e_q) \right) = \frac{1}{\psi'(\Phi(q))} \cdot \left( e^{\Phi(q) x} - \mathbb{P}_x(T_{[0]} < e_q) \right),
\]
where \(T_{[0]} := \inf\{t \geq 0 : X_t = 0\}\) denotes the first hitting time of 0 and \(e_q\) is an exponentially distributed, independent, random time with parameter \(q\). Note that \(\Phi'(q) = \frac{1}{\psi'(\Phi(q))}\) can be shown by a simple application of the chain rule; see also (6.17). For \(q = 0\) we thus have
\[
W(x) = \frac{1}{\psi'(\Phi(0))} \cdot e^{\Phi(0) x} - \mathbb{P}_x(T_{[0]} < \infty) = \frac{1}{\psi'(\Phi(0))} \cdot e^{\Phi(0) x} - h(x),
\]
where \(h(x) = [0, \psi'(\Phi(0))^{-1}]\) for all \(x \geq 0\). Using the explicit form of \(u_1\) shown in [3] (see also (3.7)) we now derive
\[
u_1(x) = \frac{1}{\Phi(0)} W^{(0)}(x) - \int_0^x W^{(0)}(y) \, dy \leq \frac{1}{\Phi(0)} \frac{1}{\psi'(\Phi(0))} e^{\Phi(0) x} - \int_0^x \left( \frac{1}{\psi'(\Phi(0))} \cdot e^{\Phi(0) y} - \frac{1}{\psi'(\Phi(0))} \right) \, dy
\]
\[
= \frac{1}{\Phi(0) \cdot \psi'(\Phi(0))} + \frac{1}{\psi'(\Phi(0))} \cdot x,
\]
which yields the claim. \(\square\)

**Proposition 6.3.** Consider the setting of Theorem [3.1] and assume (6.1) holds for some \(k \in \mathbb{N}, A_k, B_k > 0\). Then there also exist \(A_{k+1}, B_{k+1} > 0\) such that
\[
u_{k+1}(x) \leq A_{k+1} + B_{k+1} x^{k+1} \quad \text{for all } x \geq 0.
\]

**Proof.** Fix \(k \in \mathbb{N}\) such that (6.1) holds for some \(A_k, B_k > 0\) and recall the representation (6.3) of \(\nu_k\) used in the proof of Proposition 6.1 which yields
\[
u_{k+1}(x) = (-1)^{k+1} (k+1)! \left( \int_0^x W^{*(k+1)}(y) \, dy - \sum_{\ell=1}^k \frac{\eta^{(\ell)}(0+)}{\ell!} W^{*(k+\ell+1)}(x) \right)
\]
\[
= (-1)^{k+1} \eta^{(k+1)}(0+) \cdot W(x)
\]
\[
= -(k+1) \cdot (\nu_k \ast W)(x) + (-1)^k \cdot \eta^{(k+1)}(0+) \cdot W(x),
\]

since
\[
\int_0^x W^{*(k+1)}(y) \, dy = \int_0^x \int_0^y W^{*(k)}(y-z) W(z) \, dz \, dy = \int_0^x \int_0^{x-z} W^{*(k)}(y) \, dy \, W(z) \, dz = \left( \int_0^x W^{*(k)}(y) \, dy \ast W \right)(x).
\]

Using the representation (6.7) of the scale function \(W\) we thus obtain
\[
u_{k+1}(x) = -(k+1) \int_0^x u_k(x-y) \left( \frac{e^{\Phi(0)y}}{\psi'(\Phi(0))} - h(y) \right) \, dy + (-1)^k \eta^{(k+1)}(0+) \left( \frac{e^{\Phi(0)x}}{\psi'(\Phi(0))} - h(x) \right)
\]
\[
= -(k+1) \int_0^x u_k(x-y) \frac{e^{\Phi(0)y}}{\psi'(\Phi(0))} \, dy + (k+1) \int_0^x u_k(x-y) h(y) \, dy
\]
\[
+ (-1)^k \cdot \eta^{(k+1)}(0+) \left( \frac{1}{\psi'(\Phi(0))} \left( \Phi(0) \int_0^x e^{\Phi(0)y} dy + 1 \right) - h(x) \right)
\]

Moreover, by assumption and boundedness of $h$ results in [3], we note that the third summand $s_3(q)$ as our induction hypothesis implies (3.4) by Proposition 6.1. Hence, by Proposition 6.1, Equation (3.3) holds, which in sight of (6.5) implies that strict convexity of $\psi$ for some constants $A, B > 0$ and any $x \geq 0$, since $\psi(\Phi(0)) > 0$ which in turn follows from the strict convexity of $\psi$ and $\psi(0) = \psi(\Phi(0)) = 0$. It remains to consider

\[
0 \leq s_2(x) \leq \frac{k + 1}{\psi'(\Phi(0))} \int_0^x u_k(y) \, dy \leq \frac{k + 1}{\psi'(\Phi(0))} \int_0^x (A + By^k) \, dy \leq A^* + B^* x^{k+1},
\]

for some constants $A^*, B^* > 0$ and any $x \geq 0$, since $\psi(\Phi(0)) > 0$ which in turn follows from the strict convexity of $\psi$ and $\psi(0) = \psi(\Phi(0)) = 0$. It remains to consider

\[
s_1(x) = \frac{e^{\Phi(0)x}}{\psi'(\Phi(0))} \int_0^x e^{-\Phi(0)z} \left( (-1)^k \eta^{(k+1)}(0+) \phi(0) - (k + 1) \cdot u_k(z) \right) \, dz
\]

=:

\[
\int_0^x e^{-\Phi(0)z} f(z) \, dz.
\]

By Proposition 6.1, Equation (3.3) holds, which in sight of (6.5) implies that

\[
\lim_{x \to \infty} \int_0^x e^{-\Phi(0)z} f(z) \, dz = \lim_{x \to \infty} \int_0^x \left( (-1)^k \eta^{(k+1)}(0+) \phi(0) - (k + 1) \int_0^x e^{-\Phi(0)z} u_k(z) \, dz \right) \, dz
\]

=:

\[
(-1)^k \eta^{(k+1)}(0+) - (k + 1) \int_0^\infty e^{-\Phi(0)z} u_k(z) \, dz
\]

= 0,

and hence we may use l’Hôpital’s rule to obtain

\[
\psi'(\Phi(0)) \cdot \lim_{x \to \infty} \frac{s_1(x)}{x^k} = \lim_{x \to \infty} \frac{\int_0^x e^{-\Phi(0)z} f(z) \, dz}{x^k \cdot e^{-\Phi(0)x}} = \lim_{x \to \infty} \frac{e^{-\Phi(0)x} f(x)}{x \cdot x^{k-1} \cdot e^{-\Phi(0)x} - x^k \cdot \phi(0) \cdot e^{-\Phi(0)x}}
\]

= \lim_{x \to \infty} \frac{f(x)}{x \cdot x^{k-1} \cdot \phi(0) - x^k \cdot \phi(0)} = \lim_{x \to \infty} \frac{- (k + 1) \cdot u_k(x)}{x \cdot x^{k-1} \cdot \phi(0) - x^k \cdot \phi(0)}
\]

= \frac{k + 1}{\phi(0)} \cdot \frac{(-1)^k}{\psi'(0+)^{k}},

as our induction hypothesis implies (3.4) by Proposition 6.1. Hence, $s_1, s_2$ and $s_3$ in (6.8) are bounded polynomially with degree at most $k + 1$ which completes our proof.

**Proof of Proposition 3.2**

Throughout this section we abbreviate

\[ u_k(x) := \mathbb{E}_x[(\tau_0^{-})^k 1_{\{\tau_0^- < \infty\}}] = \mathbb{P}_x(\tau_0^- < \infty) \mathbb{E}_x[(\tau_0^-)^k | \tau_0^- < \infty], \quad k \in \mathbb{N}. \]

Further, as above, we set $W := W(0)$, define the function $\eta : [0, \infty) \to [0, \infty)$ by setting $\eta(q) := q/\Phi(q)$, and additionally set

\[ \varphi(q) := \frac{\psi(q)}{q}, \quad q > 0. \]

The proof of Proposition 3.2 will be split into two parts. First, however, we prove the following auxiliary result.

16
Lemma 6.4. Assume $k \in \mathbb{N}$ is such that $\psi^{(k+1)}(0+) \text{ is finite. Then}$

$$\lim_{\beta \downarrow 0} \varphi^{(k)}(\beta) = \lim_{\beta \downarrow 0} \partial_\beta^k \left( \frac{\psi(\beta)}{\beta} \right) = \frac{\psi^{(k+1)}(0+)}{k+1}.$$  

Proof. By the general Leibniz rule

$$\varphi^{(k)}(\beta) = \sum_{\ell=0}^{k} \binom{k}{\ell} \cdot \partial^{k-\ell}(\beta^{-1}) \cdot \partial^\ell(\psi(\beta)) = \sum_{\ell=0}^{k} \frac{k!}{\ell!} \cdot (-1)^{k-\ell} \cdot \beta^{-k+\ell-1} \cdot \psi^{(\ell)}(\beta) = k! \cdot (-1)^k \cdot \beta^{-k+1} \cdot \sum_{\ell=0}^{k} (-1)^{\ell} \cdot \frac{\beta^\ell}{\ell!} \cdot \psi^{(\ell)}(\beta),$$

where, by assumption, $\psi^{(\ell)}(0+) \text{ is finite for all } \ell = 1, \ldots, k+1.$

Inspired by the Taylor expansion of $\psi$ we define the function $\psi_- : [0, \infty) \rightarrow \mathbb{R}$ via

$$\psi_-(\beta) := \sum_{\ell=0}^{k+1} \frac{\psi^{(0+)}(0+)}{\ell!} (-\beta)^\ell = \sum_{\ell=0}^{k+1} (-1)^{\ell} \frac{\beta^\ell}{\ell!} \psi^{(\ell)}(0+), \quad \beta \geq 0.$$  

Clearly, $\psi_-$ is infinitely often differentiable on $(0, \infty)$ with

$$\psi_-(\beta) := \sum_{\ell=0}^{k+1} (-1)^{\ell} \frac{\beta^{\ell-n}}{\ell!} \psi^{(0+)} \xrightarrow{\beta \downarrow 0} (-1)^n \cdot \psi^{(n)}(0+), \quad \text{for } n \leq k+1. \quad (6.9)$$

Moreover,

$$\beta^{k+1} \varphi^{(k)}(\beta) - \psi_-(\beta) = \sum_{\ell=0}^{k} (-1)^{\ell} \frac{\beta^\ell}{\ell!} \left( \psi^{(\ell)}(\beta) - \psi^{(\ell)}(0+) \right) - (-1)^{k+1} \frac{\beta^{k+1}}{(k+1)!} \psi^{(k+1)}(0+),$$

which, after rearrangement, implies

$$\lim_{\beta \downarrow 0} \frac{\varphi^{(k)}(\beta)}{\beta^{k+1}} = \lim_{\beta \downarrow 0} \sum_{\ell=0}^{k} (-1)^{\ell} \frac{\beta^\ell}{\ell!} \left( \psi^{(\ell)}(\beta) - \psi^{(\ell)}(0+) \right) + \psi_-(\beta) + (-1)^{k+1} \frac{\psi^{(k+1)}(0+)}{(k+1)!}, \quad (6.10)$$

such that it only remains to prove that the limit on the right hand side vanishes. Hereby, the nominator $N_0(\beta)$ in the limit on the right hand side tends to 0 as $\beta \downarrow 0.$ Moreover, the general Leibniz rule yields for any $n \geq 0$

$$N_n(\beta) := \partial_\beta^n \left( \sum_{\ell=0}^{k} (-1)^{\ell} \frac{\beta^\ell}{\ell!} \left( \psi^{(\ell)}(\beta) - \psi^{(\ell)}(0+) \right) + \psi_-(\beta) \right) = \sum_{\ell=0}^{k} (-1)^{\ell} \frac{n!}{\ell!} \cdot \sum_{j=0}^{n} \frac{n^j}{j!(\ell-j)!} \beta^{\ell-j} \cdot \partial_\beta^{n-j} \left( \psi^{(\ell)}(\beta) - \psi^{(\ell)}(0+) \right) + \psi^{(n)}_-(\beta), \quad (6.11)$$

and thus for $n \leq k+1,$ in the light of $(6.9),$ we obtain

$$\lim_{\beta \downarrow 0} N_n(\beta) = \sum_{\ell=0}^{n-1} (-1)^{\ell} \frac{n!}{\ell!} \cdot \left( \frac{n}{\ell} \cdot \lim_{\beta \downarrow 0} \partial_\beta^{n-\ell} \left( \psi^{(\ell)}(\beta) - \psi^{(\ell)}(0+) \right) \right) + \lim_{\beta \downarrow 0} \psi^{(n)}_-(\beta)$$

$$= \sum_{\ell=0}^{n-1} (-1)^{\ell} \frac{n!}{\ell!} \psi^{(n)}(0+) + (-1)^n \cdot \psi^{(n)}(0+)$$
\[ \psi^{(n)}(0+) \cdot \sum_{\ell=0}^{n} (-1)^{\ell} \binom{n}{\ell} = 0. \]

which ensures that we may \((k+1)\)-fold apply l’Hospital’s rule to the limit in \((6.10)\). This gives

\[ \lim_{\beta \downarrow 0} \varphi^{(k)}(\beta) = \frac{(-1)^{k}}{k+1} \cdot \lim_{\beta \downarrow 0} N_{k+1}(\beta) + \frac{\psi^{(k+1)}(0+)}{k+1} = \frac{\psi^{(k+1)}(0+)}{k+1}, \]

as stated.

Now we can compute the Laplace transform of \(u_k\).

**Proof of Proposition 3.2, Equation (3.5).** Observe that in complete analogy to the proof of Proposition 6.1, we get

\[ u_k(x) = (-1)^k \cdot k! \left( \int_0^x W^k(y) \, dy - \sum_{\ell=0}^{k} \frac{\eta^{(\ell)}(0+)}{\ell!} \cdot W^{k-\ell+1}(x) \right), \quad k \in \mathbb{N}, \]

with \(\eta(0+) > 0\), hence \(\psi'(0+) > 0\), while by assumption and the results in \([3]\) we know that \(\eta^{(\ell)}(0+)\) is finite for all \(\ell = 0, \ldots, k\). Hence, by the same standard computations as in the proof of Proposition 6.1, we obtain

\[ \int_0^\infty e^{-\beta x} u_k(x) \, dx = (-1)^k \cdot k! \left( \frac{1}{\beta \psi(\beta)} - \sum_{\ell=0}^{k} \frac{1}{\ell! \cdot \psi(\beta)k-\ell+1} \cdot \eta^{(\ell)}(0+) \right) \]

\[ = (-1)^k \cdot k! \left( \frac{1}{\beta \psi(\beta)} - \frac{\psi'(0+)}{\psi(\beta)^{k+1}} - \sum_{\ell=1}^{k} \frac{1}{\ell! \cdot \psi(\beta)k-\ell+1} \cdot \eta^{(\ell)}(0+) \right), \quad (6.12) \]

since \(\eta(0+) = \psi'(0+)\). Note that as \(\psi'(0+) > 0\) implies \(\Phi(0) = 0\), Equation \((2.3)\) holds for all \(\beta > 0\) and this carries over to \((6.12)\).

As \(\varphi(\beta) = \frac{\psi(\beta)}{\beta}\) immediately implies \(\varphi(\Phi(q)) = \frac{q}{\Phi(q)} = \eta(q)\), we may now apply Faà di Bruno’s formula, cf. \([17]\, \text{Eq. (2.2)}\), which yields for \(\ell = 1, \ldots, k\)

\[ \eta^{(\ell)}(q) = \partial^{\ell} \varphi(\Phi(q)) = \sum_{j=1}^{\ell} \varphi^{(j)}(\Phi(q)) \cdot B_{\ell,j} \left( \Phi'(q), \ldots, \Phi^{(\ell-j+1)}(q) \right), \]

where \(B_{\ell,j}\) denote the partial Bell polynomials. Thus, as \(\Phi(q) \downarrow 0\) for \(q \downarrow 0\), Lemma 6.4 implies

\[ \eta^{(\ell)}(0+) = \sum_{j=1}^{\ell} \frac{\psi^{(j+1)}(0+)}{(j+1)} \cdot B_{\ell,j} \left( \Phi'(0+), \ldots, \Phi^{(\ell-j+1)}(0+) \right), \quad \ell = 1, \ldots, k, \quad (6.13) \]

and inserting this in \((6.12)\) gives \((3.5)\). \(\square\)

**Remark 6.5.** In \([3]\), it was shown that in the profitable case for any \(\kappa > 0\)

\[ \mathbb{E}_x[(\tau_0^-)^\kappa | \tau_0^- < \infty] \iff \lim_{q \downarrow 0} |D_q^\kappa \eta(q)| < \infty \iff \mathbb{E}_0[|X_1|^{\kappa+1}] < \infty, \]

with \(D_q^\kappa\) denoting the \(\kappa\)-th fractional derivative with respect to \(q\). While the proof of the first equivalence turned out to be simple, the given proof of the second equivalence involved various arguments from the toolbox of Lévy processes and subordinators. For all special cases where \(\kappa = k \in \mathbb{N}\) however, the reasoning used above to obtain \((6.13)\) provides an alternative proof for this second equivalence in a purely analytical way.
In order to prove the remainder of Proposition 3.2, we need a preliminary result on the relations between derivatives of $\eta$ and $\psi$ as given in the next Lemma.

**Lemma 6.6.** Let $\psi'(0+) \in (0, \infty)$. Choose $k \in \mathbb{N}$ such that $\psi^{(k+1)}(0+)$ is finite, then

\[
\lim_{q \downarrow 0} \partial_q^n \left( \sum_{\ell=0}^k \frac{\eta^{(\ell)}(0+)}{\ell!} \cdot \psi(q)^\ell \right) = \frac{\psi^{(n+1)}(0+)}{(n+1)}, \quad n = 1, \ldots, k. \tag{6.14}
\]

**Proof.** First note that $\eta(\psi(q)) = \frac{\psi(q)}{\Phi(\psi(q))} = \frac{\psi(q)}{q} = \varphi(q)$ and thus, using Faà di Bruno’s formula, cf. [17, Eq. (2.2)], we obtain

\[
\partial_q^n \varphi(q) = \sum_{\ell=1}^k \eta^{(\ell)}(q) \cdot B_{k,\ell} \left( \psi(q), \ldots, \psi^{(k-\ell+1)}(q) \right),
\]

where $B_{k,\ell}$ denote the partial Bell polynomials. Since $\psi(0) = 0$ and due to Lemma 6.4, taking the limit $q \downarrow 0$ on both sides of this formula implies

\[
\sum_{\ell=1}^k \eta^{(\ell)}(0+) \cdot B_{k,\ell} \left( \psi'(0+), \ldots, \psi^{(k-\ell+1)}(0+) \right) = \frac{\psi^{(k+1)}(0+)}{k+1}. \tag{6.15}
\]

Further, Faà di Bruno’s formula yields for any $\ell \geq 1$ and $n = 1, \ldots, k$

\[
\partial_q^n \left( \psi(q)^\ell \right) = \sum_{j=0}^{\ell} \frac{\ell!}{(\ell-j)!} \cdot \psi(q)^{\ell-j} \cdot B_{n,j} \left( \psi'(q), \ldots, \psi^{(n-j+1)}(q) \right)
\]

for any $n \geq \ell$, as $q \downarrow 0$.

Now, as clearly $\partial_q^n \left( \psi(0)^0 \right) = \partial_q^n 1 = 0$ for any $n \geq 1$, we observe that for any $n = 1, \ldots, k$

\[
\lim_{q \downarrow 0} \partial_q^n \left( \sum_{\ell=0}^k \frac{\eta^{(\ell)}(0+)}{\ell!} \cdot \psi(q)^\ell \right) = \sum_{\ell=1}^k \eta^{(\ell)}(0+) \cdot B_{n,\ell} \left( \psi'(0+), \ldots, \psi^{(n-\ell+1)}(0+) \right). \tag{6.16}
\]

and (6.14) follows immediately from (6.15). \hfill \square

We are now ready to provide the remainder of the proof of Proposition 3.2.

**Proof of Proposition 3.2.** Again, we aim to use a standard Tauberian theorem to prove the stated asymptotics. First, observe that by [15, Lemma 2.3] the scale function $x \mapsto W(x)$ is almost everywhere differentiable and that left and right derivatives of $W(x)$ exist on $(0, \infty)$. Thus, in view of (6.2), it follows that $u'_{k}(x)$ also exists almost everywhere and the expression is well-defined. Further, from (6.12) it follows via integration by parts that

\[
\beta \cdot (-1)^k k! \left( \frac{1}{\beta \psi(\beta)^k} - \sum_{\ell=0}^k \frac{1}{\ell! \beta(\beta)^{k-\ell+1}} \eta^{(\ell)}(0+) \right) = \beta \int_0^{\infty} e^{-\beta x} u_k(x) \, dx
\]

\[
= \int_0^{\infty} e^{-\beta x} u'_{k}(x) \, dx + u_k(0+).
\]

Hereby, a $k$-fold application of l’Hôpital’s rule using Lemma 6.4 and (6.14) yields

\[
\lim_{\beta \downarrow 0} \beta \cdot (-1)^k k! \left( \frac{1}{\beta \psi(\beta)^k} - \sum_{\ell=0}^k \frac{1}{\ell! \beta(\beta)^{k-\ell+1}} \eta^{(\ell)}(0+) \right)
\]
which implies the statement.

Thus \( \int_{0}^{\infty} e^{-\beta x} u_k'(x) \, dx \to -u_k(0+) \) as \( \beta \to 0 \) and [229] Ch. V, Thm. 4.3] yields that

\[
\lim_{x \to \infty} (u_k(x) - u_k(0+)) = \lim_{x \to \infty} \int_{0}^{x} u_k'(y) \, dy = -u_k(0+),
\]

which implies the statement.

\[
\square
\]

### 6.2 Proofs for the results in Section 4

Before we start to present the proofs of Theorems 4.1 and 4.2, recall that \( q \mapsto \Phi(q) \) is the well-defined inverse of \( \psi(\theta) \) on the interval \( \Phi(0), \infty \), and hence applying the chain rule on \( q \mapsto q = \psi(\Phi(q)) \) we observe that

\[
\Phi'(q) := \partial_q \Phi(q) = \frac{1}{\psi'(\Phi(q))}, \quad q \geq 0,
\]

(6.17)

where the case \( q = 0 \) is interpreted in the limiting sense \( q \downarrow 0 \). Moreover, further differentiation of (6.17) yields

\[
\Phi''(q) = -\Phi''(\Phi(q))\Phi'(q)^3 = -\frac{\psi''(\Phi(q))}{\psi'(\Phi(q))^3},
\]

and

\[
\Phi'''(q) = 3 \cdot \frac{\psi''(\Phi(q))^2}{\psi'(\Phi(q))^5} - \frac{\psi'''(\Phi(q))}{\psi'(\Phi(q))^4}.
\]

(6.18)

(6.19)

The next two Lemmas provide several useful equalities that are valid in the setting considered in Section 4. Note that to shorten notation in this section we will omit all obvious limits and abbreviate \( \psi'(0) := \psi'(0+) \) and alike.

**Lemma 6.7.** If \( \psi'(0) > 0 \), then

\[
\frac{\psi''(0)}{2\psi'(0)^2} = \sum_{i=1}^{n} \frac{1}{\psi'(\phi_i(0)) \cdot \phi_i(0)},
\]

(6.20)

and

\[
\frac{3\psi''(0)^2}{2\psi'(0)^4} - \frac{2\psi'''(0)}{3\psi'(0)^3} = \sum_{i=1}^{n} \left( \frac{2}{\psi'(\phi_i(0))^2 \phi_i(0)^2} + \frac{2\psi''(\phi_i(0))}{\psi'(\Phi(0))^3 \phi_i(0)} \right).
\]

(6.21)

If \( \psi'(0) < 0 \), then

\[
\frac{\psi''(0)}{2\psi'(0)^2} = \sum_{i=2}^{n} \frac{1}{\psi'(\phi_i(0)) \phi_i(0)} + \frac{1}{\psi'(\Phi(0)) \Phi(0)},
\]

(6.22)

and

\[
\frac{3\psi''(0)^2}{2\psi'(0)^4} - \frac{2\psi'''(0)}{3\psi'(0)^3} = \sum_{i=2}^{n} \left( \frac{2}{\psi'(\phi_i(0))^2 \phi_i(0)^2} + \frac{2}{\psi'(\Phi(0))^2 \Phi(0)^2} \right).
\]

(6.23)
Proof. We use an approach similar to the one used in [2, Section 10.2]. Recall that \( \phi_i(q), \) \( i = 0, \ldots, n \) are the solutions of \( \psi(\theta) - q = 0, \) where \( \phi_0(q) = \Phi(q) \) and \( \phi_i(q) \in \mathcal{R}_q, \) \( i = 1, \ldots, n \) as in (4.12). Then, using partial fraction decomposition, we may conclude that on \( \{(\theta, q) \in [0, \infty)^2 : \psi(\theta) \neq q\} \) it holds that

\[
\frac{1}{\psi(\theta) - q} = \sum_{i=0}^{n} \frac{1}{\psi'(\phi_i(q)) \cdot (\theta - \phi_i(q))}.
\]

Setting \( \theta = 0 \) this implies

\[
\frac{1}{q} = \sum_{i=0}^{n} \frac{1}{\psi'(\phi_i(q)) \cdot \phi_i(q)}, \quad q > 0.
\]  \(\text{(6.24)}\)

Assume first that \( \psi'(0) > 0 \), then \( \Phi(0) = 0 \) and \( \Re \phi_i(0) < 0, \) \( i = 1, \ldots, n, \) and hence \( \text{(6.24)} \) implies

\[
\frac{1}{q} \cdot \frac{\Phi'(q) \cdot \Phi(q)}{\Phi'(q)} = \sum_{i=0}^{n} \frac{1}{\psi'(\phi_i(q)) \cdot \phi_i(q)} \cdot \frac{q^{i\theta} \cdot 1}{q^{i\theta} \cdot \phi_i(q)},
\]  \(\text{(6.25)}\)

where the limit of the left hand side can be computed using \( \text{(6.17)} \) and by a double application of l’Hospital’s rule as

\[
\lim_{q \downarrow 0} \left( \frac{1}{q} - \frac{\Phi'(q)}{\Phi(q)} \right) = \lim_{q \downarrow 0} \frac{\Phi(q) - q \Phi'(q)}{\Phi(q)} = \lim_{q \downarrow 0} \frac{q \Phi''(q) - q \Phi''(q)}{q \Phi''(q) - q \Phi''(q)} = \lim_{q \downarrow 0} \frac{-q \Phi''(q)}{\Phi'(q) + q \Phi'(q)}
\]

Via \( \text{(6.18)} \) this proves \( \text{(6.20)} \).

In the case \( \psi'(0+) < 0 \), we have \( \Phi(0) > 0, \phi_1(0) = 0 \) and \( \Re \phi_i(0) < 0, \) \( i = 2, \ldots, n, \) cf. [18 Prop. 5.4 (i)]. Hence we conclude from \( \text{(6.24)} \) that

\[
\frac{1}{q} \cdot \frac{\Phi'(q) \cdot \Phi(q)}{\Phi'(q)} = \sum_{i=0}^{n} \frac{1}{\psi'(\phi_i(q)) \cdot \phi_i(q)} \cdot \frac{q^{i\theta} \cdot 1}{q^{i\theta} \cdot \phi_i(q)},
\]

and \( \text{(6.22)} \) follows in analogy to the above, since analogously to \( \text{(6.17)} \) it holds that

\[
\phi_i'(q) = \psi'(\phi_i(q))^{-1}, \quad i = 1, \ldots, n.
\]  \(\text{(6.26)}\)

To prove \( \text{(6.21)} \) and \( \text{(6.23)} \) we differentiate \( \text{(6.24)} \) with respect to \( q \) which then yields via \( \text{(6.26)} \)

\[
\frac{1}{q^2} = \sum_{i=0}^{n} \frac{\psi''(\phi_i(q))}{\psi'(\phi_i(q))^3 \phi_i(q)} + \frac{1}{\psi'(\phi_i(q))^2 \phi_i(q)^2}, \quad q > 0.
\]

Again assume first that \( \psi'(0+) > 0 \) and recall that \( \Phi(0) = 0 \) and \( \Re \phi_i(0) < 0, \) \( i = 1, \ldots, n, \) such that we may conclude that

\[
\frac{1}{q^2} \cdot \frac{\psi''(\Phi(q))}{\psi'(\Phi(q))^3 \Phi(q)} - \frac{1}{\psi'(\Phi(q))^2 \Phi(q)^2} = \sum_{i=1}^{n} \frac{\psi''(\phi_i(q))}{\psi'(\phi_i(q))^3 \phi_i(q)} + \frac{1}{\psi'(\phi_i(q))^2 \phi_i(q)^2}
\]

\[
\lim_{q \downarrow 0} \sum_{i=1}^{n} \frac{1}{\psi'(\phi_i(0))^2 \phi_i(0)^2} + \frac{\psi''(\phi_i(0))}{\psi'(\phi_i(0))^3 \phi_i(0)^3}.
\]
To take the limit on the left hand side we reformulate via \((6.17)\) and \((6.18)\) and then apply l’Hospital’s rule four times. To shorten notation we will omit the argument of \(\Phi\) and its derivatives in the following computation.

\[
\lim_{q \downarrow 0} \left( \frac{1}{q^2} - \frac{\psi''(\Phi(q))}{\psi'(\Phi(q))^3 \Phi(q)} - \frac{1}{\psi'(\Phi(q))^2 \Phi(q)^2} \right) = \lim_{q \downarrow 0} \left( \frac{\Phi(q)^2 + 4q^2 \Phi''(q) \Phi(q) - q^2 \Phi''(q)^2}{q^2 \Phi(q)^2} \right) \\
= \lim_{q \downarrow 0} \left( \frac{q^2 \Phi''(q) - q^2 \Phi''(q)^2 + 2q \Phi''(q) - 2q \Phi''(q)^2}{2q^2 \Phi'(q)^2} \right) \\
= \lim_{q \downarrow 0} \left( \frac{q^2 \Phi''(q) + 4q \Phi'''(q) + 4q \Phi''(q) - q^2 (\Phi''(q))^2 - 4q \Phi''(q)'^2}{2q^2 \Phi'(q)^2 + 8q \Phi'(q)^2 + 2q \Phi''(q)^2} \right) \\
= \lim_{q \downarrow 0} \left( \frac{q^2 \Phi''(q) + 2q \Phi''(q)^2 + 8q \Phi''(q) + 6q \Phi'(q) - 2q \Phi''(q)^2}{2q^2 \Phi'(q)^2 + 12q \Phi'(q)^2 + 12q \Phi''(q)^2 + 12q \Phi''(q)^2} \right) \\
= \lim_{q \downarrow 0} \left( \frac{q^2 \Phi''(q) + 2q \Phi''(q)^2 + 16q \Phi''(q) + 14q \Phi'(q) - 2q \Phi''(q)^2}{2q^2 \Phi'(q)^2 + 16q \Phi'(q)^2 + 6q \Phi''(q)^2 + 2q \Phi''(q)^2 + 24q \Phi''(q) + 24(q \Phi''(q))^2} \right) \\
= \frac{8 \Phi''(0) \Phi'(0)^2 - 6 \Phi'(0)^2}{24 \Phi'(0)^2} = \frac{\Phi''(0)}{3 \Phi'(0)} - \frac{\Phi'(0)^2}{4 \Phi'(0)^2}.
\]

Inserting \((6.19)\) we thus conclude

\[
\lim_{q \downarrow 0} \left( \frac{1}{q^2} - \frac{\psi''(\Phi(q))}{\psi'(\Phi(q))^3 \Phi(q)} - \frac{1}{\psi'(\Phi(q))^2 \Phi(q)^2} \right) = \frac{\psi''(\Phi(0))^2}{\psi'(\Phi(0))^4} - \frac{\psi'''(\Phi(0))}{3 \psi'(\Phi(0))^5} - \frac{\psi''(\Phi(0))^2}{4 \psi'(\Phi(0))^4} \\
= \frac{3 \psi''(\Phi(0))^2}{4 \psi'(\Phi(0))^4} - \frac{\psi'''(\Phi(0))}{3 \psi'(\Phi(0))^5},
\]

which proves \((6.21)\). Again, equation \((6.23)\) follows in complete analogy.

**Lemma 6.8.** If \(\psi'(0) > 0\), then

\[
\frac{1}{p} \mathbb{1}_{\{\sigma^2=0\}} - \frac{1}{\psi'(0)} = \sum_{i=1}^{n} \frac{1}{\psi'(\phi_i(0))}, \quad \text{(6.27)}
\]

\[
\frac{\psi''(0)}{\psi'(0)^3} = \sum_{i=1}^{n} \frac{\psi''(\phi_i(0))}{\psi'(\phi_i(0))^3}, \quad \text{(6.28)}
\]

and

\[
\frac{3 \psi''(0)^2}{\psi'(0)^5} - \frac{\psi'''(0)}{\psi'(0)^4} = \sum_{i=1}^{n} \left( \frac{3 \psi''(\phi_i(0))^2}{\psi'(\phi_i(0))^5} - \frac{\psi'''(\phi_i(0))}{\psi'(\phi_i(0))^4} \right). \quad \text{(6.29)}
\]

If \(\psi'(0) < 0\), then

\[
\frac{1}{p} \mathbb{1}_{\{\sigma^2=0\}} - \frac{1}{\psi'(0)} = \sum_{i=1}^{n} \frac{1}{\psi'(\phi_i(0))}, \quad \text{(6.30)}
\]

and

\[
\frac{\psi''(0)}{\psi'(0)^3} = \sum_{i=1}^{n} \frac{\psi''(\phi_i(0))}{\psi'(\phi_i(0))^3}, \quad \text{(6.31)}
\]

**Proof.** Note that in the present setting of the perturbed CL model, cf. [18, Lemma 3.1],

\[
W'(\psi)(0) = \frac{1}{p} \cdot \mathbb{1}_{\{\sigma^2=0\}}.
\]

22
Inserting this in (4.13) thus yields
\[ \frac{1}{p} \cdot 1_{\{\sigma^2 = 0\}} = \frac{1}{\psi'(\Phi(q))} + \sum_{i=1}^{n} \frac{1}{\psi'(\phi_i(q))}, \quad q \geq 0. \] (6.32)

Letting \( q \downarrow 0 \) in the cases \( \psi'(0) > 0 \) and \( \psi'(0) < 0 \) we obtain (6.27) and (6.30), respectively. Furthermore, differentiating (6.32) with respect to \( q \) proves
\[ 0 = -\frac{\psi''(\Phi(q))}{\psi'(\Phi(q))^3} - \sum_{i=1}^{n} \frac{\psi''(\phi_i(q))}{\psi'(\phi_i(q))^3}, \quad q > 0, \] (6.33)

and again letting \( q \downarrow 0 \) this gives (6.28) and (6.31), respectively. Lastly, another differentiation of (6.33) proves
\[ 0 = 3\frac{\psi''(\Phi(q))}{\psi'(\Phi(q))^5} - \frac{\psi''(\Phi(q))}{\psi'(\Phi(q))^4} + \sum_{i=1}^{n} \left( \frac{\psi''(\phi_i(q))}{\psi'(\phi_i(q))^5} - \frac{\psi''(\phi_i(q))}{\psi'(\phi_i(q))^4} \right), \quad q > 0, \] (6.34)

and letting \( q \downarrow 0 \) in the case \( \psi'(0) > 0 \) we get (6.29).

We are now in the position to provide the proofs of Theorems 4.1 and 4.2.

Proof of Theorem 4.1. From (4.13) we obtain
\[ \int_{0}^{x} W^*(0)(y) \, dy = \int_{0}^{x} \left( \frac{e^{\Phi(0)y}}{\psi'(0)} + \sum_{i=1}^{n} e^{\phi_i(0)y/\psi'(\phi_i(0))} \right) \, dy, \]
where \( \Phi(0) > 0 \) and \( \Re \phi_i(0) < 0 \) for all \( i = 2, \ldots, n \), while \( \phi_1(0) = 0 \), cf. [16, Sec. 2.5]. Thus for all \( i = 2, \ldots, n \) and likewise for the term involving \( \Phi(0) \)
\[ \int_{0}^{x} \frac{e^{\phi_i(0)y}}{\psi'(\phi_i(0))} \, dy = \int_{0}^{x} \frac{1}{\psi'(0)} \, dy = \frac{x}{\psi'(0)}, \quad x \geq 0, \]

Hence it follows from (3.7) and (4.13) that
\[ E_x[\tau_0] = \frac{1}{\Phi(0)} W^*(0)(x) - \int_{0}^{x} W^*(0)(y) \, dy \]
\[ = \frac{1}{\psi'(\Phi(0)) \cdot \Phi(0)} + \frac{1}{\Phi(0) \cdot \psi'(0)} - \frac{x}{\psi'(0)} + \sum_{i=2}^{n} \left( \frac{e^{\phi_i(0)x}}{\Phi(0) \cdot \psi'(\phi_i(0))} - \frac{e^{\phi_i(0)x}}{\psi'(\phi_i(0)) \cdot \phi_i(0)} \right) \]
\[ = -x \frac{1}{\psi'(0)} + \frac{1}{\Phi(0)} \left( \frac{1}{\psi'(\Phi(0))} + \frac{1}{\psi'(0)} \right) \]
\[ + \sum_{i=2}^{n} \frac{e^{\phi_i(0)x}}{\psi'(\phi_i(0))} \left( \frac{1}{\Phi(0)} - \frac{1}{\phi_i(0)} \right) + \frac{1}{\psi'(\phi_i(0)) \cdot \phi_i(0)}, \]
which, by (6.22), proves (4.14) with
\[ C_1 = \frac{1}{\Phi(0) \psi'(0)} + \frac{\psi''(0)}{2 \psi'(0)^2}. \]
Moreover, via \(6.30\) and \(6.22\) we observe that
\[
\epsilon_1(0) = \sum_{i=2}^{n} \frac{1}{\psi'(\phi_i(0))} \left( \frac{1}{\Phi(0)} - \frac{1}{\phi_i(0)} \right)
\]
\[
= \frac{1}{\Phi(0)} \left( \frac{1}{p} \mathbb{1}_{\{\sigma^2=0\}} - \frac{1}{\psi'(\Phi(0))} - \frac{1}{\psi'(0)} \right) - \left( \frac{\psi''(0)}{2\psi'(0)^2} - \frac{1}{\psi'(\Phi(0))\Phi(0)} \right)
\]
\[
= \frac{1}{\Phi(0)} \frac{1}{p} \mathbb{1}_{\{\sigma^2=0\}} - C_1',
\]
which proves the stated form of \(C_1'\) as well as the formula for \(\mathbb{E}[\tau_0]\).

For the proof of \(4.15\) we need to evaluate \(3.8\) and hence additionally have to consider the derivative of \(W^{(q)}(x)\) w.r.t. \(q\), which is by \(4.13\)
\[
\partial_q W^{(q)}(x) = \partial_q \left( \frac{e^{\Phi(x)}}{\psi'(\Phi(x))} \right) + \sum_{i=1}^{n} \partial_q \left( \frac{e^{\psi_i(x)}}{\psi'(\phi_i(x))} \right).
\]
Applying the quotient rule we obtain for \(i = 1, \ldots, n\) (and likewise for the term involving \(\Phi(q)\))
\[
\partial_q \left( \frac{e^{\psi_i(x)}}{\psi'(\phi_i(x))} \right) = \frac{\psi'(\phi_i(x))e^{\psi_i(x)} - e^{\psi_i(x)}\psi''(\phi_i(x))\phi_i'(x)}{\psi'(\phi_i(x))^2} = \frac{e^{\psi_i(x)}}{\psi'(\phi_i(x))^2} \left( x - \psi''(\phi_i(x)) \right),
\]
where we have used \(6.26\) in the second equality. Hence,
\[
\partial_q W^{(q)}(x) = \frac{e^{\Phi(x)}}{\psi'(\Phi(x))^2} \left( x - \psi''(\Phi(x)) \right) + \sum_{i=1}^{n} \frac{e^{\psi_i(x)}}{\psi'(\phi_i(x))^2} \left( x - \psi''(\phi_i(x)) \right),
\]
and as \(q \downarrow 0\) we obtain with \(\Phi(0) > 0, \phi_1(0) = 0\) and \(\Re \phi_i(0) < 0, i = 2, \ldots, n\),
\[
\lim_{q \downarrow 0} \partial_q W^{(q)}(x) = \frac{e^{\Phi(x)}}{\psi'(\Phi(x))^2} \left( x - \psi''(\Phi(x)) \right) + \frac{1}{\psi'(\phi_1(0))^2} \left( x - \psi''(\phi_1(0)) \right)
\]
\[
+ \sum_{i=2}^{n} \frac{e^{\psi_i(x)}}{\psi'(\phi_i(0))^2} \left( x - \psi''(\phi_i(0)) \right).
\]
Lastly we need to calculate \(\int_0^x \lim_{q \downarrow 0} \partial_q W^{(q)}(y) \, dy\) where a straightforward integration of the single summands in \(6.36\) yields
\[
\int_0^x \frac{1}{\psi'(y)^2} \left( y - \frac{\psi''(y)}{\psi'(y)} \right) \, dy = \frac{1}{2\psi'(0)^2} \cdot x^2 - \frac{\psi''(0)}{\psi'(0)^3} \cdot x
\]
while for \(i = 2, \ldots, n\) (and likewise for the term involving \(\Phi(0)\))
\[
\int_0^x \frac{e^{\psi_i(y)}}{\psi'(\phi_i(0))^2} \left( y - \psi''(\phi_i(0)) \right) \, dy
\]
\[
= \frac{1}{\psi'(\phi_i(0))^2} \cdot \left( e^{\psi_i(x)} \left( \frac{x}{\phi_i(0)} - \frac{1}{\phi_i(0)} \right) + \frac{1}{\phi_i(0)} \right)
\]
\[
- \frac{\psi''(\phi_i(0))}{\psi'(\phi_i(0))^3} \left( \frac{1}{\phi_i(0)} \cdot e^{\phi_i(x)} - \frac{1}{\phi_i(0)} \right)
\]
\[
= e^{\psi_i(x)} \cdot \frac{x}{\psi'(\phi_i(0))^2} - e^{\phi_i(x)} \cdot \frac{1}{\phi_i(0)} \cdot \frac{\psi''(\phi_i(0))}{\psi'(\phi_i(0))} + \frac{\psi''(\phi_i(0))}{\psi'(\phi_i(0))^2} \frac{1}{\phi_i(0)}.
\]
Putting all terms together and inserting into \(3.8\) now yields after some rearrangement
\[
\mathbb{E}[\tau_0^-]^2 = 2 \cdot \int_0^x \lim_{q \downarrow 0} \partial_q W^{(q)}(y) \, dy - \frac{2}{\Phi(0)} \cdot \lim_{q \downarrow 0} \partial_q W^{(q)}(x) + \frac{2W^{(0)}(x)}{\Phi(0)^2\psi'(\Phi(0))}.
\]
\[
\frac{x^2}{\psi'(0)^2} - \frac{2}{\psi'(0)^2} \left( \frac{\psi''(0)}{\psi'(0)} + \frac{1}{\Phi(0)} \right) \cdot x + \frac{2}{\psi'(0)^2 \Phi(0)} + \sum_{i=2}^{n} \frac{2}{\psi'(0)^2 \Phi(0)} \frac{\psi''(\phi_i(0))}{\psi'(\phi_i(0))} e_{\phi_i(0)} x + 2 \frac{\psi''(0)}{\Phi(0) \psi'(0)^3} + \frac{2}{\Phi(0)^2 \psi'(0) \psi''(0)} e_{\phi_i(0)} x - e_{\phi_i(0)} x \left( \frac{1}{\psi'(\phi_i(0))^2 \phi_i(0)^2} - \frac{1}{\Phi(0)^2 \psi'(\phi_i(0)) \psi''(\phi_i(0))} \right).
\]

This proves (4.15) where from (6.23) it follows that

\[
C_2^+ = \frac{2}{\psi'(0)^2 \psi''(0)} + \frac{2}{\Phi(0)^2 \psi'(0) \psi''(0)} + \frac{3}{2 \psi'(0)^4} - \frac{2}{3 \psi'(0)^3}.
\]

To finish the proof we note that via (6.23), (6.30), and (6.31)

\[
e_2(0) = 2 \sum_{i=2}^{n} \left( \frac{\psi''(\phi_i(0))}{\psi'(\phi_i(0))^3} \left( \frac{1}{\Phi(0)} - \frac{1}{\phi_i(0)} \right) \right.
\] + \left( \frac{1}{\psi'(\Phi(0))} \psi'(\phi_i(0)) - \frac{1}{\psi'(\phi_i(0))^2 \phi_i(0)^2} \right) 
\] - \frac{2}{\Phi(0)^2} \left( \frac{\psi''(\Phi(0))}{\psi'(\Phi(0))^3} + \frac{1}{\psi'(\Phi(0))^3} \right) + \frac{2}{\Phi(0)^2 \psi'(\Phi(0))} \left( \frac{1}{\Phi(0)} - \frac{1}{\psi'(0)} \right) - \frac{3}{2 \psi'(0)^4} - \frac{2}{3 \psi'(0)^3} - \frac{2}{\psi'(\Phi(0))^2 \Phi(0)^2} - \frac{2}{\psi'(\Phi(0))^2 \Phi(0)}
\] = \frac{2}{\Phi(0)^2 \psi'(\Phi(0))} \frac{1}{\Phi(0)} - 1 \cdot e_{\phi_i(0)} - C_2^+.
\]

which yields the desired representation of \(C_2^+\) and the formula for \(E_\mathcal{E}[\tau_0^-]^2\). □

**Proof of Theorem 4.3.** First of all note that all appearing moments are finite due to finiteness of all moments of phase-type distributions.

To prove (4.16) we evaluate (3.9) and note that in comparison to the case handled in the proof of Theorem 4.1 where \(\psi'(0^+) < 0\), essentially \(\Phi(q)\) and \(\phi_i(q)\) change their roles, since in the present case of \(\psi'(0^+) > 0\) we have \(\Phi(0) = 0\), while \(\Re \phi_i(0) < 0\), \(i = 1, \ldots, n\).

With this observation we immediately obtain in analogy to the computations in the proof of Theorem 4.1

\[
W^{(0)}(x) = \frac{1}{\psi'(0)^2} + \sum_{i=1}^{n} \frac{e_{\phi_i(0)} x}{\psi'(\phi_i(0)) \cdot \phi_i(0)}.
\]

\[
\int_0^x W^{(0)}(y) \, dy = \frac{x}{\psi'(0)^2} + \sum_{i=1}^{n} \frac{e_{\phi_i(0)} x}{\psi'(\phi_i(0)) \cdot \phi_i(0)} - 1,
\]

and

\[
\lim_{q \to 0} \frac{\partial_q}{\partial_q} W^{(q)}(x) = \frac{1}{\psi'(0)^2} \left( x - \psi''(0) \psi'(0) \right) + \sum_{i=1}^{n} \frac{e_{\phi_i(0)} x}{\psi'(\phi_i(0))^2} \left( x - \psi''(\phi_i(0)) \right).
\]

Inserting these terms in (3.9) then yields by simple rearrangement

\[
E_\mathcal{E}[\tau_0^- | \tau_0^- < \infty] = \frac{\psi'(0^+) \cdot \lim_{q \to 0} \frac{\partial_q}{\partial_q} W^{(q)}(x) + \frac{\psi''(0^+) \cdot W^{(0)}(x) - \int_0^x W^{(0)}(y) \, dy}{1 - \psi'(0^+) \cdot W^{(0)}(x)}}{1 - \psi'(0^+) \cdot W^{(0)}(x)}.
\]
\[-\left(\frac{\psi'(\phi_i(0))}{\phi_i(0)}\right)^n \cdot \left(-\frac{\psi''(0)}{2\psi'(0)^3} + \frac{1}{\psi'(0)} \sum_{i=1}^{n} \frac{1}{\psi'(\phi_i(0)) \cdot \phi_i(0)} \right) + \sum_{i=1}^{n} \left( \frac{\psi'(\phi_i(0))}{\psi'(\phi_i(0))^2} - \frac{\psi'(0)}{2\psi'(0)} \left( \frac{\psi''(0)}{\psi'(0)^2} \phi_i(0) \right) \right) \]\n
and (4.16) follows via (6.20).

To prove (4.17), in addition to (6.37), (6.38), and (6.39), we need to differentiate to get the terms \( \lim_{q \downarrow 0} \int_0^x \partial_q W(q)(y) \ dy \) and \( \lim_{q \downarrow 0} \partial_q^2 W(q)(x) \). Hereby, from (6.35),

\[
\int_0^x \partial_q W(q)(y) \ dy = \int_0^x \left( \frac{e^{\Phi(q)y}}{\psi'(\Phi(q))^2} \left( y - \frac{\psi''(\Phi(q))}{\psi'(\Phi(q))} \right) \right) dy + \sum_{i=1}^{n} \int_0^x \left( \frac{e^{\phi_i(q)y}}{\psi'(\phi_i(q))^2} \left( y - \frac{\psi''(\phi_i(q))}{\psi'(\phi_i(q))} \right) \right) dy,
\]

with

\[
\int_0^x \left( \frac{e^{\phi_i(q)y}}{\psi'(\phi_i(q))^2} \left( y - \frac{\psi''(\phi_i(q))}{\psi'(\phi_i(q))} \right) \right) dy = \frac{e^{\phi_i(q)x}}{\psi'(\phi_i(q))^2} \left( x - \frac{1}{\phi_i(q)^2} \right) + \frac{1}{\psi'(\phi_i(q))^2} - \frac{e^{\phi_i(q)x}}{\psi'(\phi_i(q))^3} \phi_i(q) - \frac{\psi''(\phi_i(q))}{\psi'(\phi_i(q))^3} \phi_i(q)
\]

and likewise for the term involving \( \Phi(q) \). Moreover, as \( \Phi(0) = 0 \) we derive for \( q \downarrow 0 \) via double application of l’Hôpital’s rule and using (6.17) and (6.18).

\[
\lim_{q \downarrow 0} \left( \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))^2} \left( x - \frac{1}{\Phi(q)^2} \right) + \frac{1}{\psi'(\Phi(q))^2} \right)
\]

\[
\lim_{q \downarrow 0} \left( \frac{e^{\Phi(q)x} \Phi^2 - \Phi^2}{\Phi^2} + e^{\Phi(q)x} \Phi_\Phi - \Phi'' \Phi \right)
\]

\[
\lim_{q \downarrow 0} \left( \frac{e^{\Phi(q)x} \Phi^2 \Phi + 3e^{\Phi(q)x} \Phi \Phi_\Phi - e^{\Phi(q)x} \Phi_\Phi \Phi + \Phi'' \Phi + e^{\Phi(q)x} \Phi \Phi_\Phi - \Phi'' \Phi \Phi} {2\Phi'} \right)
\]

\[
\lim_{q \downarrow 0} \left( \frac{e^{\Phi(q)x} \Phi^2 \Phi + 3e^{\Phi(q)x} \Phi \Phi_\Phi - e^{\Phi(q)x} \Phi_\Phi \Phi + \Phi'' \Phi + e^{\Phi(q)x} \Phi \Phi_\Phi - \Phi'' \Phi \Phi} {2\Phi'} \right)
\]

\[
\lim_{q \downarrow 0} \left( \frac{\Phi^2 \Phi + 2\Phi'} {2\Phi'} \right) = \frac{\Phi^2(0)} {2\Phi'^{(0)2}} = \frac{\Phi^2(0)} {2\Phi'(0)^2} + \Phi''(0)x = \frac{x^2}{2\Phi'(0)^2} - \frac{\psi''(0)} {\psi'(0)^2} x.
\]

As \( \Re \phi_i(0) < 0, i = 1, \ldots, n \), we conclude that altogether

\[
\lim_{q \downarrow 0} \int_0^x \partial_q W(q)(y) \ dy = \frac{x^2} {2\Phi'(0)^2} - \frac{\psi''(0)} {\psi'(0)^2} x
\]

\[
+ \sum_{i=1}^{n} \left( \frac{e^{\phi_i(0)x}}{\psi'(\phi_i(0))^2} \left( x - \frac{1}{\phi_i(0)^2} \right) + \frac{1}{\psi'(\phi_i(0))^2} \phi_i(0) - (e^{\phi_i(0)x} - 1) \frac{\psi''(\phi_i(0))}{\psi'(\phi_i(0))^2} \phi_i(0) \right).
\]
Moreover, again from (6.35)
\[
\partial_q^2 W(q)(x) = \partial_q \left( \frac{e^{\phi(q)x}}{\psi'(\Phi(q))^2} \left( x - \frac{\psi''(\Phi(q))}{\psi'(\Phi(q))} \right) \right) + \sum_{i=1}^{n} \partial_q \left( \frac{e^{\phi_i(q)x}}{\psi'(\phi_i(q))^2} \left( x - \frac{\psi''(\phi_i(q))}{\psi'(\phi_i(q))} \right) \right),
\]
with
\[
\partial_q \left( \frac{e^{\phi(q)x}}{\psi'(\phi(q))^2} \left( x - \frac{\psi''(\phi(q))}{\psi'(\phi(q))} \right) \right) = \frac{e^{\phi(q)x}x^2}{\psi'(\phi(q))^3} - 3e^{\phi(q)x}x \frac{\psi''(\phi(q))}{\psi'(\phi(q))^4} + 3e^{\phi(q)x}x \frac{\psi''(\phi(q))^2}{\psi'(\phi(q))^5} - e^{\phi(q)x} \frac{\psi''(\phi(q))}{\psi'(\phi(q))^4},
\]
and likewise for the term involving \( \Phi(q) \), such that for \( q \downarrow 0 \)
\[
\lim_{q \downarrow 0} \partial_q^2 W(q)(x) = \frac{\partial_q W(q)(x)}{\psi'(0)^2} \left( 2 \cdot \lim_{q \downarrow 0} \int_{0}^{x} \partial_q W(q)(y) \, dy - \psi'(0) \lim_{q \downarrow 0} \partial_q W(q)(x) \right)
\]
\[
\lim_{q \downarrow 0} \partial_q W(q)(x) = \frac{\partial_q W(q)(x)}{\psi'(0)^2} \left( \frac{2}{1 - \psi'(0) \cdot W(0)(x)} \cdot \frac{\partial_q W(q)(y) \, dy}{\psi'(0)^2} \right)
\]
\[
\lim_{q \downarrow 0} \partial_q W(q)(x) = \left( \sum_{i=1}^{n} \frac{e^{\phi_i(0)x}}{\psi'(\phi_i(0))^3} \right) \cdot \left( \frac{2}{\psi'(0)^2} - \frac{3 \psi''(\phi_i(0))}{\psi'(\phi_i(0))^3} \right)
\]
\[
+ \sum_{i=1}^{n} \frac{e^{\phi_i(0)x}}{\psi'(\phi_i(0))^3} \left( \frac{2}{\psi'(\phi_i(0))^3} \psi''(\phi_i(0)) - \frac{3 \psi''(\phi_i(0))^2}{\psi'(\phi_i(0))^4} + \psi''(\phi_i(0)) \psi'(\phi_i(0))^2 \psi'(\phi_i(0))^2 \right)
\]
\[
+ \sum_{i=1}^{n} \frac{e^{\phi_i(0)x}}{\psi'(\phi_i(0))^3} \left( \frac{2}{\psi'(\phi_i(0))^3} \psi''(\phi_i(0))^2 \psi'(\phi_i(0))^2 \psi'(\phi_i(0))^2 - \psi''(\phi_i(0))^2 \psi'(\phi_i(0))^2 \psi'(\phi_i(0))^2 \right)
\]
\[
- \sum_{i=1}^{n} \left( \frac{2}{\psi'(\phi_i(0))^3} \psi''(\phi_i(0))^2 \psi'(\phi_i(0))^2 \psi'(\phi_i(0))^2 + \frac{2}{\psi'(\phi_i(0))^3} \psi''(\phi_i(0))^2 \psi'(\phi_i(0))^2 \psi'(\phi_i(0))^2 \right)
\].

In sight of (6.21) the proof of (4.17) is finished. Finally, setting \( x = 0 \) in (4.16), we get
\[
\mathbb{E}_0[\tau_0^- | \tau_0^- < \infty] = \left( \sum_{i=1}^{n} \frac{1}{\psi'(\phi_i(0)))} \right)^{-1}
\]
\[
\cdot \left( - \sum_{i=1}^{n} \left( \frac{\psi''(\phi_i(0)))}{\psi'(\phi_i(0))^3} - \frac{\psi''(0+)}{2\psi'(0+)^2}\psi'(\phi_i(0))) + \frac{1}{\psi'(0+)^3}\phi_i(0+) \right) \right),
\]
and via (6.29), (6.27), and (6.28) this yields the stated result. Similarly, setting \( x = 0 \) in (4.17) gives
\[
\mathbb{E}_0[(\tau_0^-)^2 | \tau_0^- < \infty] = \left( \sum_{i=1}^{n} \frac{1}{\psi'(\phi_i(0)))} \right)^{-1} \cdot \sum_{i=1}^{n} C_i \].
where, via (6.21), (6.27), (6.28), and (6.29), it can easily be shown that

\[ \sum_{i=1}^{n} C_i^3 = \left( \frac{\psi'''(0)}{3\psi'(0)^3} - \frac{\psi''(0)^2}{2\psi'(0)^4} \right) \frac{1}{p} \mathbb{1}_{\{\sigma^2=0\}}. \]

Inserting this, the stated formula for \( \mathbb{E}_0[(\tau_0^-)^2 | \tau_0^- < \infty] \) follows via (6.27).

References

[1] S. Asmussen and H. Albrecher. *Ruin probabilities*. World Scientific, 2nd edition, 2010.

[2] F. Avram, D. Grahovac, and C. Vardar-Acar. The W, Z scale functions kit for first passage problems of spectrally negative Lévy processes, and applications to control problems. *ESAIM: Probability and Statistics*, 24:454–525, 2020.

[3] A. Behme and P.L. Strietzel. On moments of downwards passage times for spectrally negative Lévy processes. 2021. Preprint. Available on arXiv:2106.00401.

[4] M. Bladt and B.F. Nielsen. *Matrix-Exponential Distributions in Applied Probability*. Springer, 2017.

[5] D.C.M. Dickson and H.R. Waters. The distribution of the time to ruin in the classical risk model. *ASTIN Bulletin*, 32(2):299–313, 2002.

[6] R.A. Doney. Fluctuation theory for Lévy processes. In Jean Picard, editor, *Lecture Notes in Mathematics*, volume 1897. Springer, Berlin, 2007.

[7] S. Drekic, J.E. Stafford, and G.E. Willmot. Symbolic calculation of the moments of the time of ruin. *Insurance: Mathematics and Economics*, 34(1):109–120, 2004.

[8] S. Drekic and G.E. Willmot. On the density and moments of the time of ruin with exponential claims. *ASTIN Bulletin*, 33(1):11–21, 2003.

[9] M. Egami and K. Yamazaki. Phase-type fitting of scale functions for spectrally negative Lévy processes. *Journal of Computational and Applied Mathematics*, 264:1–22, 2014.

[10] A.D. Egidio dos Reis. On the moments of ruin and recovery times. *Insurance: Mathematics and Economics*, 27(3):331 – 343, 2000.

[11] W. Feller. *An Introduction to Probability Theory and its Applications. Part 2*. Wiley, New York, NY, 2, edition, 1971.

[12] E. Frostig. Upper bounds on the expected time to ruin and on the expected recovery time. *Advances in Applied Probability*, 36(2):377–397, 2004.

[13] E. Frostig, S.M. Pitts, and K. Politis. The time to ruin and the number of claims until ruin for phase-type claims. *Insurance: Mathematics and Economics*, 51(1):19–25, 2012.

[14] H.U. Gerber. *An Introduction to Mathematical Risk Theory*. S. S. Huebner Foundation for Insurance Education, Wharton School, University of Pennsylvania, 1979.

[15] H.U. Gerber and E.S.W. Shiu. On the time value of ruin. *North American Actuarial Journal*, 2(1):48–72, 1998.

[16] J. Ivanovs. On scale functions for Lévy processes with negative phase-type jumps. *Queueing Systems*, 98:3–19, 2021.
[17] W.P. Johnson. The curious history of Faà di Bruno’s formula. The American Mathematical Monthly, 109(3):217–234, 2002.

[18] A. Kuznetsov, A.E. Kyprianou, and V. Rivero. The theory of scale functions for spectrally negative Lévy processes. In Lévy Matters II, Springer Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2013.

[19] A.E. Kyprianou. Fluctuations of Lévy processes with Applications. Springer, 2nd edition, 2014.

[20] A.E. Kyprianou and Z. Palmowski. Distributional study of De Finetti’s dividend problem for a general Lévy insurance risk process. Journal of Applied Probability, 44(2):428–443, 2007.

[21] X.S. Lin and G.E. Willmot. Analysis of a defective renewal equation arising in ruin theory. Insurance: Mathematics and Economics, 25(1):63–84, 1999.

[22] X.S. Lin and G.E. Willmot. The moments of the time of ruin, the surplus before ruin, and the deficit at ruin. Insurance: Mathematics and Economics, 27(1):19 – 44, 2000.

[23] P. Picard and C. Lefèvre. The moments of ruin time in the classical risk model with discrete claim size distribution. Insurance: Mathematics and Economics, 23(2):157–172, 1998.

[24] S.M. Pitts and K. Politis. Approximations for the moments of ruin time in the compound Poisson model. Insurance: Mathematics and Economics, 42(2):668–679, 2008.

[25] D.V. Widder. The Laplace Transform. Princeton University Press, 1946.