An approach toward a finite-dimensional definition of twisted $K$-theory

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Abstract

This is an expository account of the following result: we can construct a group by means of twisted $\mathbb{Z}_2$-graded vectorial bundles which is isomorphic to $K$-theory twisted by any degree three integral cohomology class.

1 Introduction

Topological $K$-theory admits a twisting by a degree three integral cohomology class. The resulting $K$-theory, known as twisted $K$-theory [2], has its origin in the works of Donovan-Karoubi [8] and Rosenberg [15], and has applications to D-brane charges [5, 12, 17], Verlinde algebras [9] and so on.

As is well-known, ordinary $K$-theory has definitions by means of:

1. vector bundles;
2. the $C^*$-algebra of continuous functions; and
3. the space of Fredholm operators.

Twisted $K$-theory is usually defined by twisting the definitions (2) or (3). For a definition parallel to (1), there are the notions of twisted vector bundles [13, 14, 16], see also [5, 8, 12], and of bundle gerbe $K$-modules [4]. However, the definitions by means of these geometric objects are only valid for twisted $K$-theory whose “twisting”, a third integral cohomology class, is of finite order: otherwise, there is no non-trivial such geometric object in finite-dimensions.

Toward a finite-dimensional definition of twisted $K$-theory valid for degree three integral cohomology classes of infinite order, we explain in this article Furuta’s notion of generalized vector bundles [10], which we call vectorial bundles. We also explain a notion of finite dimensional approximation of Fredholm operators, which provides a linear version of the finite dimensional approximation of the monopole equations [11]. We can use these notions to construct a group and an isomorphism from $K$-theory twisted by any degree three integral cohomology class. The proof of the result is only outlined. The detailed treatment will be provided elsewhere.

A possible application of the result above is to generalize the notions of 2-vector bundles [3, 4]. A 2-vector bundle of rank 1 due to Brylinski [6] is a stack which reproduces the category of twisted vector bundles. Replacing
the category of vector bundles by that of vectorial bundles, we can directly
genralize the 2-vector bundles in [6]. Similarly, we can also generalize the 2-
vector bundles due to Baas, Dundas and Rognes [3], which they studied in an
approach to geometric realization of elliptic cohomology. It seems interesting to
apply their study of 2-vector bundles to the generalization of 2-vector bundles
made of vectorial bundles.

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2 Twisted $K$-theory and twisted vector bundles

In this section, we review the definition of twisted $K$-theory by means of the
space of Fredholm operators, following [2]. We also review the notion of twisted
vector bundles [4, 5, 8, 12, 13, 14, 16].

Unless otherwise mentioned, $X$ is a compact manifold through this article.

2.1 Twisted $K$-theory

The twisted $K$-theory we consider in this article is associated to a degree three
integral cohomology class. To give the precise definition, we represent the class
by a projective unitary bundle. Let $\mathcal{H}$ be a separable Hilbert space of infinite
dimension, and $PU(\mathcal{H})$ the projective unitary group $PU(\mathcal{H}) = U(\mathcal{H})/U(1)$.

**Definition 2.1.** For a principal $PU(\mathcal{H})$-bundle over $X$, we define the twisted $K$-

group $K_P(X)$ to be the fiberwise homotopy classes of sections of the associate-
d bundle $P \times_{Ad} \mathcal{F}(\mathcal{H})$ over $X$, where $PU(\mathcal{H})$ acts on the space $\mathcal{F}(\mathcal{H})$ of Fredholm
operators on $\mathcal{H}$ by adjoint.

In the above definition, the topologies on $PU(\mathcal{H})$ and $\mathcal{F}(\mathcal{H})$ are understood
to come from the operator norm. Notice that we can also use the compact-open
topology in the sense of [2].

If $P$ is a trivial bundle, then $K_P(X)$ is exactly the homotopy classes of
continuous functions $X \to \mathcal{F}(\mathcal{H})$. Thus, in this case, we recover the $K$-group of
$X$ by the well-known fact that $\mathcal{F}(\mathcal{H})$ is a classifying space of $K$-theory [1].

$PU(\mathcal{H})$-bundles over $X$ are classified by $H^3(X, \mathbb{Z})$: since $U(\mathcal{H})$ is contractible
by Kuiper’s theorem, $PU(\mathcal{H})$ is homotopy equivalent to the Eilenberg-MacLane
space $K(\mathbb{Z}, 2)$, so that the classifying space $BPU(\mathcal{H})$ is homotopy equivalent to
$K(\mathbb{Z}, 3)$. If $P$ and $P'$ are isomorphic $PU(\mathcal{H})$-bundles, then the twisted $K$-groups
$K_P(X)$ and $K_{P'}(X)$ are also (non-canonically) isomorphic. So we often speak
of “twisted $K$-theory twisted by a class in $H^3(X, \mathbb{Z})$”.

We will call the cohomology class corresponding to $P$ the Dixmier-Douady
class, and denote it by $\delta(P) \in H^3(X, \mathbb{Z})$. For later convenience, we recall here
the construction of $\delta(P)$: take an open cover $\mathcal{U} = \{U_\alpha\}$ of $X$ so that:
• there are local sections $s_\alpha : U_\alpha \to P_{|U_\alpha}$;
• there are lifts $g_{\alpha\beta} : U_{\alpha\beta} \to U(H)$ of the transition functions $\mathcal{F}_{\alpha\beta} : U_{\alpha\beta} \to PU(H)$.

Here we write $U_{\alpha\beta}$ for the overlap $U_\alpha \cap U_\beta$, and the transition function is defined by the relation $s_\alpha \mathcal{F}_{\alpha\beta} = s_\beta$. Because of the cocycle condition for $\{\mathcal{F}_{\alpha\beta}\}$, we can find a map $c_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to U(1)$ such that $g_{\alpha\beta} g_{\beta\gamma} = c_{\alpha\beta\gamma} g_{\alpha\gamma}$. These maps comprise a Čech 2-cocycle $(c_{\alpha\beta\gamma}) \in \check{Z}^2(\mathcal{U}, U(1))$ with its coefficients in the sheaf of germs of $U(1)$-valued functions, which represents $\delta(P)$ through the isomorphism $\check{H}^2(X, U(1)) \cong H^3(X, \mathbb{Z})$.

### 2.2 Twisted vector bundles

For a $PU(H)$-bundle $P$ over $X$, a twisted vector bundle consists essentially of the data $(\mathcal{U}, E_\alpha, \phi_{\alpha\beta})$:

- an open cover $\mathcal{U} = \{U_\alpha\}$ of $X$;
- vector bundles $E_\alpha$ over $U_\alpha$;
- isomorphisms of vector bundles $\phi_{\alpha\beta} : E_\beta|_{U_{\alpha\beta}} \to E_\alpha|_{U_{\alpha\beta}}$ over $U_{\alpha\beta}$ satisfying the “twisted cocycle condition” on $U_{\alpha\beta\gamma}$:

$$\phi_{\alpha\beta} \phi_{\beta\gamma} = c_{\alpha\beta\gamma} \phi_{\alpha\gamma},$$

where $c_{\alpha\beta\gamma}$ is as in the previous subsection.

In the rigorous definition of twisted vector bundles, we have to include the choices of the local sections $s_\alpha$ and the lifts $g_{\alpha\beta}$. Though it is crucial to specify these choices in considering isomorphism classes of twisted vector bundles, we omit them for simplicity.

The isomorphism classes of twisted vector bundles $\text{Vect}_P(X)$ gives rise to a semi-group by the direct sum of local vector bundles. Let $K(\text{Vect}_P(X))$ denote the group given by applying the Grothendieck construction to $\text{Vect}_P(X)$. Then the following fact is known. (See [5, 8, 12, 13, 16].)

**Proposition 2.2.** For a $PU(H)$-bundle whose Dixmier-Douady class $\delta(P)$ is of finite order, there exists an isomorphism:

$$K_P(X) \to K(\text{Vect}_P(X)).$$

Instead of twisted vector bundles, we can use bundle gerbe $K$-modules to obtain an equivalent result [4, 7].

The rank of a twisted vector bundle is a multiple of the order of $\delta(P)$. This can be seen readily as follows. Suppose that a twisted vector bundle has a finite rank $r$. Taking the determinant of the twisted cocycle condition, we have:

$$\det \phi_{\alpha\beta} \det \phi_{\beta\gamma} = c_{\alpha\beta\gamma}^r \det \phi_{\alpha\gamma}.$$  

Hence $(c_{\alpha\beta\gamma}^r) \in \check{Z}^2(\mathcal{U}, U(1))$ is a coboundary and $r\delta(P) = 0$. 

3
Because of the above remark, there are no non-trivial twisted vector bundles in the case where $\delta(P)$ is infinite order. So we cannot use twisted vector bundles of finite dimensions to realize $K_P(X)$ generally. In spite of this fact, collections of locally defined vector bundles seem to have the potential in defining $K_P(X)$ by means of finite-dimensional objects. An approach is to use the usual technique proving the isomorphism $K(X) \cong \{X, F(H)\}$. In this approach, however, some complications prevent us from transparent management, in particular, in giving equivalence relation. The usage of Furuta’s generalized vector bundle provides a more efficient approach, which we explain in the next section.

3 Furuta’s generalized vector bundle

In this section, we explain a generalization of the notion of vector bundles introduced by M. Furuta [10]. We call the generalized vector bundles vectorial bundles for short. This notion is closely related to a finite-dimensional approximation of Fredholm operators. Applying these notions, we approach to our problem of defining twisted $K$-theory finite-dimensionally.

3.1 Approximation of a Fredholm operator

We begin with the simplest situation. A $\mathbb{Z}_2$-graded vectorial bundle over a single point is a pair $(E, h)$ consisting of:

- a $\mathbb{Z}_2$-graded Hermitian vector space $E = E^0 \oplus E^1$ of finite rank; and
- a Hermitian map $h : E \to E$ of degree 1.

By using a $\mathbb{Z}_2$-graded vectorial bundle over a point, we can approximate a single Fredholm operator as follows. Let $A : H \to H$ be a Fredholm operator. For simplicity, we assume the kernel or cokernel of $A$ is non-trivial. We define the $\mathbb{Z}_2$-graded Hilbert space $\hat{H}$ by $\hat{H} = H \oplus H$, and the self-adjoint Fredholm operator $\hat{A} : \hat{H} \to \hat{H}$ of degree 1 by $\hat{A} = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$. By the assumption, the spectrum $\sigma(\hat{A}^2)$ of the non-negative operator $\hat{A}^2$ contains $0$. Since $\hat{A}$ is also Fredholm, $0 \in \sigma(\hat{A}^2)$ is a discrete spectrum. Hence there is a positive number $\mu$ such that:

- $\mu \notin \sigma(\hat{A}^2)$;
- the subset $\sigma(\hat{A}^2) \cap [0, \mu)$ consists of a finite number of eigenvalues;
- the eigenspace $\text{Ker}(\hat{A}^2 - \lambda)$ is finite dimensional for $\lambda \in \sigma(\hat{A}^2) \cap [0, \mu)$.

Let $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n < \mu$ be the distinct eigenvalues in $\sigma(\hat{A}^2) \cap [0, \mu)$. Then we have the following orthogonal decomposition of $\hat{H}$:

$$\hat{H} = (\hat{H}, \hat{A})_{\lambda_1} \oplus (\hat{H}, \hat{A})_{\lambda_2} \oplus \cdots \oplus (\hat{H}, \hat{A})_{\lambda_n} \oplus \hat{H}',$$

where $(\hat{H}, \hat{A})_{\lambda} = \text{Ker}(\hat{A}^2 - \lambda)$ is the eigenspace of $\hat{A}^2$ with its eigenvalue $\lambda$, and $\hat{H}'$ is the orthogonal complement. Notice that $\hat{A}$ preserves each eigenspace as well as the orthogonal complement. More precisely, $\hat{A}$ restricts to the trivial
map on $(\hat{\mathcal{H}}, \hat{A})_{\lambda_1} \cong \text{Ker} A \oplus \text{Ker} A^*$, while $\hat{A}$ induces isomorphisms on $(\mathcal{H}, \hat{A})_{\lambda_i}$ for $i > 1$ and $\mathcal{H}'$.

Now, cutting off the infinite-dimensional part $\hat{\mathcal{H}}'$, we define $E$ and $h$ by $E = \bigoplus_i (\hat{\mathcal{H}}, \hat{A})_{\lambda_i}$ and $h = \hat{A}|_E$. The pair $(E, h)$ is nothing but a $\mathbb{Z}_2$-graded vectorial bundle over a point.

As a finite-dimensional approximation of a single Fredholm operator, we obtained a $\mathbb{Z}_2$-graded vectorial bundle over a single point. As will be explained in Subsection 3.3 a similar approximation is possible for a family of Fredholm operators parameterized by $X$. The resulting object is a $\mathbb{Z}_2$-graded vectorial bundle over $X$.

### 3.2 Definition of vectorial bundle

We now introduce $\mathbb{Z}_2$-graded vectorial bundles:

**Definition 3.1.** Let $(U, (E_\alpha, h_\alpha), \phi_{\alpha \beta})$ be the following data:

- an open cover $U = \{U_\alpha\}$ of $X$;
- $\mathbb{Z}_2$-graded Hermitian vector bundles $E_\alpha$ over $U_\alpha$;
- Hermitian maps $h_\alpha : E_\alpha \rightarrow E_\alpha$ of degree 1;
- vector bundle maps $\phi_{\alpha \beta} : E_{\beta}|_{U_{\alpha \beta}} \rightarrow E_{\alpha}|_{U_{\alpha \beta}}$ of degree 0 over $U_{\alpha \beta}$ such that $h_\alpha \phi_{\alpha \beta} = \phi_{\alpha \beta} h_\beta$.

A $\mathbb{Z}_2$-graded vectorial bundle over $X$ is defined to be data $(U, (E_\alpha, h_\alpha), \phi_{\alpha \beta})$ satisfying the following conditions:

$$\phi_{\alpha \alpha} \cong 1 \text{ on } U_\alpha,$$

$$\phi_{\alpha \beta} \phi_{\beta \gamma} \cong \phi_{\alpha \gamma} \text{ on } U_{\alpha \beta \gamma}.$$

In the above definition, the symbol $\cong$ stands for an equivalence relation.

For any point $x \in U_\alpha$, there are a neighborhood $V \subset U_\alpha$ of $x$ and a positive number $\mu$ such that: for all $y \in V$ and $v \in (E_\alpha, h_\alpha)_y, < \mu$ we have $\phi_{\alpha \alpha}(v) = v$.

Here $(E_\alpha, h_\alpha)_y, < \mu$ is the subspace in the fiber of $E_\alpha$ at $y$ given by the direct sum of eigenspaces of $(h_\alpha)_y^2$ whose eigenvalues are less than $\mu$:

$$(E_\alpha, h_\alpha)_y, < \mu = \bigoplus_{\lambda < \mu} \text{Ker} ((h_\alpha)_y^2 - \lambda) = \bigoplus_{\lambda < \mu} \{v \in (E_\alpha)_y | (h_\alpha)_y^2 v = \lambda v\}.$$

The meaning of the second condition $\phi_{\alpha \beta} \phi_{\beta \gamma} \cong \phi_{\alpha \gamma}$ is now obvious.

**Definition 3.2.** Let $E = (U, (E_\alpha, h_\alpha), \phi_{\alpha \beta})$ and $E' = (U, (E'_\alpha, h'_\alpha), \phi'_{\alpha \beta})$ be $\mathbb{Z}_2$-graded vectorial bundles over $X$.

(a) A set $(f_\alpha)$ of vector bundle maps $f_\alpha : E_\alpha \rightarrow E'_\alpha$ of degree 0 such that $f_\alpha h_\alpha = h'_\alpha f_\alpha$ on $U_\alpha$ is said to be a homomorphism from $E$ to $E'$, if we have $f_\alpha \phi_{\alpha \beta} \cong \phi'_{\alpha \beta} f_\beta$ on $U_{\alpha \beta}$.
(b) A homomorphism $(f_\alpha) : E \to E'$ is said to be an isomorphism, if there exists a homomorphism $(f'_\alpha) : E' \to E$ such that $f_\alpha f'_\alpha = 1$ and $f'_\alpha f_\alpha = 1$ on $U_\alpha$.

In the above definition of homomorphism, $E$ and $E'$ share the same open cover. In the case where they have different open covers $\mathcal{U}$ and $\mathcal{U}'$ respectively, it suffices to take a common refinement of $\mathcal{U}$ and $\mathcal{U}'$.

**Definition 3.3.** A homotopy between $\mathbb{Z}_2$-graded vectorial bundles $E$ and $E'$ over $X$ is defined to be a $\mathbb{Z}_2$-graded vectorial bundle $\hat{E}$ over $X \times [0,1]$ such that $E$ and $E'$ are isomorphic to $\hat{E}|_{X \times \{0\}}$ and $\hat{E}|_{X \times \{1\}}$, respectively.

We write $KF(X)$ for the set of homotopy classes of isomorphism classes of $\mathbb{Z}_2$-graded vectorial bundles. The set $KF(X)$ gives rise to a group by means of the direct sum of vector bundles given locally.

A $\mathbb{Z}_2$-graded (ordinary) vector bundle $E$ gives an example of a $\mathbb{Z}_2$-graded vectorial bundle by setting $\mathcal{U} = \{X\}$ and $h = 0$. This construction induces a well-defined homomorphism $K(X) \to KF(X)$. In [10], Furuta proved:

**Proposition 3.4.** The homomorphism $K(X) \to KF(X)$ is an isomorphism.

### 3.3 Approximation of a family of Fredholm operators

As a family version of the construction in Subsection 3.1, we can show:

**Lemma 3.5.** Let $A = \{A_x\} : X \to \mathcal{F}(\mathcal{H})$ be a continuous map. For any point $p \in X$, there are a neighborhood $U_p$ of $p$ and a positive number $\mu_p$ such that the following family of vector spaces gives rise to a vector bundle over $U_p$:

$$
\bigcup_{x \in U_p} (\mathcal{H}, \hat{A}_x)_{<\mu_p} = \bigcup_{x \in U_p} \bigoplus_{\lambda < \mu_p} \text{Ker}(\hat{A}_x^2 - \lambda).
$$

A key to this lemma is that eigenvalues of $A_x$ is continuous in $x$.

By means of the lemma, the family of Fredholm operators $A : X \to \mathcal{F}(\mathcal{H})$ yields a $\mathbb{Z}_2$-graded vectorial bundle $\{(U_p)_{p \in X}, (E_{U_p}, h_{U_p}), (\phi_{U_p, U_q})\}$, where the $\mathbb{Z}_2$-graded vector bundle $E_{U_p}$ is that in Lemma 3.5 the Hermitian map $h_{U_p}$ is given by restricting the Fredholm operator: $h_{U_p}|_{x} = \hat{A}_x|_{E_{U_p}}$, and the map of vector bundles $\phi_{U_p, U_q} : E_{U_q} \to E_{U_p}$ is the following composition of the natural inclusion and the orthogonal projection:

$$
\bigcup_{x \in U_p \cap U_q} (\mathcal{H}, \hat{A}_x)_{<\mu_p} \to (U_p \cap U_q) \times \mathcal{H} \to \bigcup_{x \in U_p \cap U_q} (\mathcal{H}, \hat{A}_x)_{<\mu_p}.
$$

The construction above induces a well-defined homomorphism

$$
\alpha : [X, \mathcal{F}(\mathcal{H})] \to KF(X).
$$
This homomorphism is compatible with the isomorphism \( \text{ind} : [X, F(H)] \rightarrow K(X) \) in \([\text{II}]\). Namely, the following diagram is commutative:

\[
\begin{array}{ccc}
[X, F(H)] & \xrightarrow{\text{ind}} & [X, F(H)] \\
\downarrow & & \downarrow \alpha \\
K(X) & \longrightarrow & KF(X).
\end{array}
\]

The compatibility follows from the fact that one can realize any vector bundle \( E \rightarrow X \) as \( E = \bigcup_{x \in X} \text{Ker}A_x \) by taking \( A : X \rightarrow F(H) \) such that \( \sigma(A^2_x) = \{0, 1\} \). (See the proof of the surjectivity of \( \text{ind} \) in \([\text{II}]\).)

### 3.4 Twisted vectorial bundle and twisted \( K \)-theory

We now apply vectorial bundles and finite dimensional approximations explained so far to twisted \( K \)-theory.

Recall that twisted vector bundles are defined by “twisting” the ordinary cocycle condition for vector bundles. In a similar way, for a \( PU(H) \)-bundle \( P \), we define a twisted \( \mathbb{Z}_2 \)-graded vectorial bundle by replacing the “cocycle condition” \( \phi_{\alpha \beta} \phi_{\beta \gamma} = \phi_{\alpha \gamma} \) in Definition 3.1 by the “twisted cocycle condition”:

\[ \phi_{\alpha \beta} \phi_{\beta \gamma} = \epsilon_{\alpha \beta \gamma} \phi_{\alpha \gamma}. \]

A twisted \( \mathbb{Z}_2 \)-graded vectorial bundle can be constructed from a section \( \tilde{A} : X \rightarrow P \times_{Ad} F(H) \). The section gives a set of maps \( \{ A_p : W_p \rightarrow F(H) \}_{p \in X} \) such that \( A_p = g_{pq}^{-1} A_q g_{pq} \), where \( W_p \) is an open set containing \( p \) and \( g_{pq} : W_p \cap W_q \rightarrow U(H) \) is a lift of transition function of \( P \). Now, we use Lemma 3.5 to define a Hermitian vector bundle over \( U_p \subset W_p \) by \( E_{U_p} = \bigcup_{x \in U_p} (\tilde{H}, (\tilde{A}_p)_x)_{<\mu_p}. \) The map \( A_p \) also defines a Hermitian map \( h_{U_p} \) on \( E_{U_p} \) by restriction. If we define \( \tilde{h}_{U_p, U_q} : E_{U_q} \rightarrow E_{U_p} \) by the following composition:

\[
\bigcup_{x \in U_{pq}} (\tilde{H}, (\tilde{A}_p)_x)_{<\mu_p} \rightarrow U_{pq} \times \tilde{H} \xrightarrow{id \times g_{pq}^{-1}} U_{pq} \times \tilde{H} \rightarrow \bigcup_{x \in U_{pq}} (\tilde{H}, (\tilde{A}_p)_x)_{<\mu_p},
\]

then \( \{ U_p \}, (E_{U_p}, h_{U_p}, \tilde{h}_{U_p, U_q}) \) is a twisted \( \mathbb{Z}_2 \)-graded vectorial bundle.

Introducing isomorphisms and homotopies in a similar way, we obtain the group \( KF_P(X) \) of homotopy classes of isomorphism classes of twisted \( \mathbb{Z}_2 \)-graded vectorial bundles. The above construction of twisted vectorial bundles induces the well-defined homomorphism

\[ \alpha : K_P(X) \longrightarrow KF_P(X). \]

Since this map generalizes \( \alpha : [X, F(H)] \rightarrow KF(X) \), it is reasonable to expect that \( \alpha \) gives rise to an isomorphism. In fact, we have:

**Theorem 3.6.** For any \( PU(H) \)-bundle \( P \) over a compact manifold \( X \), the homomorphism \( \alpha : K_P(X) \longrightarrow KF_P(X) \) is bijective.

We sketch the proof of this result in the next subsection.
3.5 Sketch of the proof of Theorem 3.6

The fundamental idea to prove Theorem 3.6 is to construct a kind of generalized cohomology theory on CW complexes by means of $KF_P(X)$.

As is known [1, 2, 7], the twisted $K$-group $K_P(X)$ fits into a certain generalized cohomology theory $\{K^*_P(X, Y)\}_{n \in \mathbb{Z}}$. In particular, for a CW pair $(X, Y)$ equipped with a $PU(\mathcal{H})$-bundle $P \to X$, we have the long exact sequence:

$$\cdots \to K^*_{P|Y}(X) \to K^*_{P}(X, Y) \to K^*_{P}(X) \to K^*_{P|Y}(Y) \to \cdots.$$ 

Note that we can identify $K^*_{P}(X, Y)$ with $K^*_{P\times I}(X \times I, Y \times I \cup X \times \partial I)$, and $\delta_0 : K^0_{P|Y}(Y) \to K^1_{P}(X, Y)$ with the composition of the following maps:

$$K^0_{P|Y}(Y) \xrightarrow{\beta} K^0_{P|Y \times D^2}(Y \times D^2, Y \times S^1) = K^2_{P|Y}(Y) \xrightarrow{\delta_1} K^1_{P}(X, Y).$$

Here $\beta$ induces the Bott periodicity, and is given by “multiplying” a map $T : D^2 \to F(\mathcal{H})$ representing the generator of $K(D^2, S^1)$.

To construct a similar cohomology theory, we define $KF^*_P(X, Y)$ by using twisted $\mathbb{Z}_2$-graded vectorial bundles on $X$ whose support do not intersect $Y$. (The support of a twisted $\mathbb{Z}_2$-graded vectorial bundle $E = (U, (E_\alpha, h_\alpha), \phi_{\alpha\beta})$ on $X$ is the closure of the points $x \in X$ such that $(h_\alpha)_x$ is not invertible for an $\alpha$.) For $n \geq 0$, we put:

$$K^*_{F_P}(X, Y) = K^*_P(X \times I^n, Y \times I^n \cup X \times \partial I^n).$$

Then $K^*_{F_P}(X, Y)$ satisfies the (suitably modified) homotopy axiom and the excision axiom in the Eilenberg-Steenrod axioms. In a way parallel to the method in [1], we can also introduce a natural map $\delta_{-n} : K^*_{P|Y}(Y) \to K^*_{F_P}(X, Y)$, and obtain the long exact sequence for a pair:

$$\cdots \to K^*_{F_P}(X) \to K^*_{P|Y}(Y) \to K^*_{P}(X, Y) \to K^*_{P|Y}(Y).$$

To extend this sequence, we put $K^*_{F_P}(X, Y) = K^*_{F_P}(X, Y)$ and define $\delta_0 : K^0_{P|Y}(Y) \to K^0_{F_P}(X, Y)$ to be the composition of:

$$K^0_{P|Y}(Y) \xrightarrow{\beta} K^0_{P|Y \times D^2}(Y \times D^2, Y \times S^1) = K^2_{P|Y}(Y) \xrightarrow{\delta_1} K^1_{P}(X, Y),$$

where $\beta$ is given by tensoring a vector bundle representing the generator of $K(D^2, S^1)$. Then the composition of $K^0_{P}(X) \to K^0_{P|Y}(Y) \to K^0_{F_P}(X, Y)$ is trivial. (This sequence is not yet shown to be exact at this stage.)

The spaces $X \times I^n$ and $Y \times I^n \cup X \times \partial I^n$ in the definition of $K^*_{F_P}(X, Y)$ are used in that of $K^*_{P}(X, Y)$. Hence the finite-dimensional approximation induces
the natural homomorphism \( \alpha_{-n} : K^{-n}_P(X, Y) \to KF^{-n}_p(X, Y) \) for \( n \geq -1 \). We can readily see that \( \delta_{-n} \) \((n \geq 1)\) commutes with \( \alpha_{-n} \), since \( \delta_{-n} \) is essentially defined by an inclusion map of spaces. If \( X \) is compact, then \( \beta \) commutes with \( \alpha_{-n} \), so that \( \delta_0 \) does. The key to this fact is that the compactness allows us to choose a map \( T : D^2 \to F(H) \) realizing the generator of \( K(D^2, S^1) \) in a way appropriate for the finite-dimensional approximation.

Now, for a finite CW complex \( X \) and a \( PU(H) \)-bundle \( P \to X \), we can prove the bijectivity of \( \alpha_{-n} : K^{-n}_P(X) \to KF^{-n}_P(X) \) \((n \geq 0)\) by the induction on the number of cells in \( X \). Notice that, if \( P \) is trivial, then an argument by using Proposition 3.4 implies the bijectivity of \( \alpha_{-n} \), hence \( \delta_{-n} \) is essentially defined by an inclusion map of spaces. If \( X \) is compact, then \( \beta \) commutes with \( \alpha_{-n} \), so that \( \delta_0 \) does. The key to this fact is that the compactness allows us to choose a map \( T : D^2 \to F(H) \) realizing the generator of \( K(D^2, S^1) \) in a way appropriate for the finite-dimensional approximation.

We can assume that the first and the forth columns are bijective in the induction. The excision axiom implies that \( K^{-n}_P(X, Y) \cong K^{-n}_P(D^q, S^{q-1}) \) and \( KF^{-n}_P(X, Y) \cong KF^{-n}_P(D^q, S^{q-1}) \). Since any \( PU(H) \)-bundle over \( D^q \) is trivial, the second and fifth columns are also bijective. Thus, so is the third column. (The exactness at \( KF^{-n}_P(X, Y) \) is not necessary in the five-term lemma.)

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