Local–global principle for reduced norms over function fields of $p$-adic curves

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Local–global principle for reduced norms over function fields of \( p \)-adic curves

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Abstract

Let \( K \) be a (non-archimedean) local field and let \( F \) be the function field of a curve over \( K \). Let \( D \) be a central simple algebra over \( F \) of period \( n \) and \( \lambda \in F^* \). We show that if \( n \) is coprime to the characteristic of the residue field of \( K \) and \( D \cdot (\lambda) = 0 \) in \( H^3(F, \mu_n^{\otimes 2}) \), then \( \lambda \) is a reduced norm from \( D \). This leads to a Hasse principle for the group \( \text{SL}_1(D) \), namely, an element \( \lambda \in F^* \) is a reduced norm from \( D \) if and only if it is a reduced norm locally at all discrete valuations of \( F \).

1. Introduction

Let \( K \) be a \( p \)-adic field and \( F \) a function field in one variable over \( K \). Let \( \Omega_F \) be the set of all discrete valuations of \( F \). Let \( G \) be a semi-simple simply connected linear algebraic group defined over \( F \). It was conjectured in [CPS12] that the Hasse principle holds for principal homogeneous spaces under \( G \) over \( F \) with respect to \( \Omega_F \); i.e. if \( X \) is a principal homogeneous space under \( G \) over \( F \) with \( X(F_\nu) \neq \emptyset \) for all \( \nu \in \Omega_F \), then \( X(F) \neq \emptyset \). If \( G = \text{SL}_1(D) \), where \( D \) is a central simple algebra over \( F \) of square-free index \( n \), it follows from the injectivity of the Rost invariant [MS90] and a Hasse principle for \( H^3(F, \mu_n^{\otimes 2}) \) due to Kato [Kat86] that this conjecture holds. This conjecture has been settled for classical groups of type \( B_n, C_n \) and \( D_n \) [Hu14, Pre13]. It is also settled for groups of type \( 2A_n \) with the assumption that \( n + 1 \) is square-free [Hu14, Pre13].

The main aim of this paper is to prove that the conjecture holds for \( \text{SL}_1(D) \) for any central simple algebra \( D \) over \( F \) with period coprime to \( p \). In fact we prove the following theorem (cf. Theorem 11.1).

Theorem 1.1. Let \( K \) be a local field and \( F \) a function field in one variable over \( K \). Let \( D \) be a central simple algebra over \( F \) of period coprime to the characteristic of the residue field of \( K \) and \( \lambda \in F^* \). If \( D \cdot (\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2}) \), then \( \lambda \) is a reduced norm from \( D \).

This, together with Kato’s result on the Hasse principle for \( H^3(F, \mu_n^{\otimes 2}) \), gives the following theorem (cf. Corollary 11.2).

Theorem 1.2. Let \( K \) be a local field and \( F \) a function field in one variable over \( K \). Let \( \Omega_F \) be the set of discrete valuations of \( F \). Let \( D \) be a central simple algebra over \( F \) of period \( n \) coprime to the characteristic of the residue field of \( K \) and \( \lambda \in F^* \). If \( \lambda \) is a reduced norm from \( D \otimes F_\nu \) for all \( \nu \in \Omega_F \), then \( \lambda \) is a reduced norm from \( D \).
In fact we may restrict the set of discrete valuations to the set of divisorial discrete valuations of $F$; namely, those discrete valuations of $F$ centered on a regular proper model of $F$ over the ring of integers in $K$.

Here are the main steps in the proof. We reduce to the case where $D$ is a division algebra of period $\ell^n$ with $\ell$ a prime not equal to $p$. Given a central division algebra $D$ over $F$ of period $n = \ell^n$ with $\ell \neq p$ and $\lambda \in F^*$ with $D \cdot (\lambda) = 0 \in H^3(F, \mu_n^\otimes)$, we construct an extension $L$ of $F$ of degree $\ell$, and $\mu \in L^*$ such that $N_{L/F}(\mu) = \lambda$, $(D \otimes L) \cdot (\mu) = 0$ and the index of $D \otimes L$ is strictly smaller than the index of $D$. Then, by induction on the index of $D$, $\mu$ is a reduced norm from $D \otimes L$ and hence $N_{L/F}(\mu) = \lambda$ is a reduced norm from $D$.

Let $X$ be a regular proper two-dimensional scheme over the ring of integers in $K$ with function field $F$ and $X_0$ the reduced special fiber of $X$. By the patching techniques of Harbater, Hartmann and Krashen [HH10, HHK09], construction of such a pair $(L, \mu)$ is reduced to the construction of compatible pairs $(L_x, \mu_x)$ over $F_x$ for all $x \in X_0$ (7.5), where for any $x \in X_0$, $F_x$ is the field of fractions of the completion of the regular local ring at $x$ on $X$. We use local and global class field theory to construct such local pairs $(L_x, \mu_x)$. Our proof does not immediately extend to the more general situation where $F$ is a function field in one variable over a complete discretely valued field with arbitrary residue field.

Here is a brief description of the organization of the paper. In §3 we prove a few technical results concerning central simple algebras and reduced norms over global fields. These results are key to the later patching construction of the fields $L_x$ and $\mu_x \in L_x$ with required properties.

In §4 we prove the following local variant of Theorem 1.1.

**Theorem 1.3.** Let $F$ be a complete discrete valued field with residue field $\kappa$. Suppose that $\kappa$ is a local field or a global field. Suppose further that if $\kappa$ is a global field, then either $n$ is odd or $\kappa$ has no real places. Let $D$ be a central simple algebra over $F$ of period $n$. Suppose that $n$ is coprime to char($\kappa$). Let $\alpha \in H^2(F, \mu_n)$ be the class of $D$ and $\lambda \in F^*$. If $\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_n^\otimes)$, then $\lambda$ is a reduced norm from $D$.

Let $A$ be a complete regular local ring of dimension 2 with residue field $\kappa$ finite, field of fractions $F$ and maximal ideal $m = (\pi, \delta)$. Let $\ell$ be a prime not equal to char($\kappa$). Let $D$ be a central simple algebra over $F$ of period $\ell^n$ with $n \geq 1$ and $\alpha$ the class of $D$ in $H^2(F, \mu_n)$. Suppose that $D$ is unramified on $A$, except possibly at $\pi$ and $\delta$. In §5 we analyze the structure of $D$. We prove that the index of $D$ is equal to the period of $D$. A similar analysis is done by Saltman [Sal97] with the additional assumption that $F$ contains all the primitive $\ell^n$th roots of unity, where $\ell^n$ is the period of $D$. Let $\lambda \in F^*$. Suppose that $\lambda = u \pi^r \delta^s$ for some unit $u \in A$ and $r, s \in \mathbb{Z}$ and $\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_n^\otimes)$. In §6 we construct possible pairs $(L, \mu)$ with $L/F$ of degree $\ell$, $\mu \in L$ such that $N_{L/F}(\mu) = \lambda$, ind$(D \otimes L) < \text{ind}(D)$ and $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^\otimes)$.

Let $K$ be a local field and $F$ a function field of a curve over $K$. Let $\ell$ be a prime not equal to the characteristic of the residue field of $K$. $D$ a central division algebra over $F$ of period $\ell^n$ and $\alpha$ the class of $D$ in $H^2(F, \mu_n)$. Let $\lambda \in F^*$ with $\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_n^\otimes)$. Let $X$ be a normal proper model of $F$ over the ring of integers in $K$ and $X_0$ its reduced special fiber. In §7 we reduce the construction of $(L, \mu)$ to the construction of local $(L_x, \mu_x)$ for all $x \in X_0$ with some compatible conditions along the ‘branches’.

Further, assume that $X$ is regular and ram$_X(\alpha) \cup \text{supp}_X(\lambda) \cup X_0$ is a union of regular curves with normal crossings. In §8, we group the components of $X_0$ into eight types depending on the valuation of $\lambda$, the index of $D$ and the ramification type of $D$ along those components. We call some nodal points of $X_0$ as special points depending on the type of components passing through

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the point. We also say that two components of $X_0$ are type 2 connected if there is a sequence of curves of type 2 connecting these two components. We prove that there is a regular proper model of $F$ with no special points and no type 2 connection between certain types of components (Proposition 8.6).

Starting with a model constructed in Proposition 8.6, in §9 we construct $(L_P, \mu_P)$ for all nodal points of $X_0$ (Proposition 9.8) with the required properties. In §10, using the class field results of §3, we construct $(L_\eta, \mu_\eta)$ for each of the components $\eta$ of $X_0$ which are compatible with $(L_P, \mu_P)$ when $P$ is in the component $\eta$.

Finally, in §11, we prove the main results by piecing together all the constructions of §§7, 9 and 10.

2. Preliminaries

In this section we recall a few definitions and facts about Brauer groups, Galois cohomology groups, residue homomorphisms and unramified Galois cohomology groups. We refer the reader to [Col95] and [GS06].

Let $K$ be a field and $n \geq 1$. Let $\text{Br}(K)$ be the $n$-torsion subgroup of the Brauer group $\text{Br}(K)$. Assume that $n$ is coprime to the characteristic of $K$. Let $\mu_n$ be the group of $n$th roots of unity. For $d \geq 1$ and $m \geq 0$, let $H^d(K, \mu_n^{\otimes m})$ denote the $d$th Galois cohomology group of $K$ with values in $\mu_n^{\otimes m}$. We have $H^1(K, \mu_n) \cong K^*/K^{*n}$ and $H^2(K, \mu_n) \cong \text{Br}(K)$. For $a \in K^*$, let $(a)_n \in H^1(K, \mu_n)$ denote the image of the class of $a$ in $K^*/K^{*n}$. When there is no ambiguity of $n$, we drop $n$ and denote $(a)_n$ by $(a)$. If $K$ is a product of finitely many fields $K_i$, we denote $\prod H^d(K_i, \mu_n^{\otimes m})$ by $H^d(K, \mu_n^{\otimes m})$.

Let $K_s$ be a separable closure of $K$. Then $H^1(K, \mathbb{Z}/n\mathbb{Z}) = \text{Hom}_{\text{cont}}(\text{Gal}(K_s/K), \mathbb{Z}/n\mathbb{Z})$. Let $\chi : \text{Gal}(K_s/K) \to \mathbb{Z}/n\mathbb{Z}$ be a continuous homomorphism and $E$ the fixed field of $\ker(\chi)$. Then $E/K$ is a cyclic extension of degree $d$ equal to the order of the image of $\chi$. Hence the degree of $E$ divides $n$. Let $\sigma \in \text{Gal}(K_s/K)$ with $\chi(\sigma) = n/[E : K]$ modulo $n\mathbb{Z}$. Then $\chi$ is uniquely determined by the pair $(E, \sigma)$. Thus every element of $H^1(K, \mathbb{Z}/n\mathbb{Z})$ is uniquely represented by a pair $(E, \sigma)$, where $E/K$ is a cyclic extension of degree $t$ dividing $n$ and $\sigma$ a generator of $\text{Gal}(E/K)$. Let $r \geq 1$. Then $(E, \sigma)^r \in H^1(K, \mathbb{Z}/n\mathbb{Z})$ is represented by the pair $(E', \sigma')$, where $E'$ is the fixed field of the subgroup of $\text{Gal}(E/K)$ generated by $\sigma^{t/d}$, where $d = \gcd(t, r)$, and $\sigma' = \sigma^r$, where $rr' + tr' = d$. In particular, if $r$ is coprime to $n$, then $(E, \sigma)^r = (E, \sigma^r)$ with $rr' \equiv 1$ modulo $t$. Let $(E, \sigma) \in H^1(K, \mathbb{Z}/n\mathbb{Z})$ and $\chi : \text{Gal}(K_s/K) \to \mathbb{Z}/n\mathbb{Z}$ be the associated homomorphism. Let $L/K$ be a field extension. Then we have the restriction homomorphism $\text{Gal}(L_s/L) \to \text{Gal}(K_s/K)$. Let $\chi_L$ be the composition of $\chi$ with this restriction. Let $E_L/L$ be the fixed field of $\ker(\chi_L)$ and $\sigma_L$ be the corresponding generator of $\text{Gal}(E_L/L)$. Then $(E_L, \sigma_L)$ is the image of $(E, \sigma)$ under the restriction map $H^1(K, \mathbb{Z}/n\mathbb{Z}) \to H^1(L, \mathbb{Z}/n\mathbb{Z})$. Further, $E \otimes_K L \cong \prod E_L$.

Let $(E, \sigma) \in H^1(K, \mathbb{Z}/n\mathbb{Z})$ and $\lambda \in K^*$. Let $(E, \sigma, \lambda) = (E/K, \sigma, \lambda)$ denote the cyclic algebra over $K$,

$$(E, \sigma, \lambda) = E \oplus Ey \oplus \cdots \oplus Ey^{n-1},$$

with $y^n = \lambda$ and $ya = \sigma(a)y$. The cyclic algebra $(E, \sigma, \lambda)$ is a central simple algebra and its index is the order of $\lambda$ in $K^*/N_{E/K}(E^*)$ [Alb61, Theorem 18, p. 98]. The pair $(E, \sigma)$ represents an element in $H^1(K, \mathbb{Z}/n\mathbb{Z})$ and the element $(E, \sigma) \cdot (\lambda) \in H^2(K, \mu_n)$ is represented by the central simple algebra $(E, \sigma, \lambda)$. In particular, $(E, \sigma, \lambda) \otimes E$ is a matrix algebra and hence $\text{ind}(E, \sigma, \lambda) \leq [E : K]$.

For $\lambda, \mu \in K^*$ we have [Alb61, p. 97]

$$(E, \sigma, \lambda) + (E, \sigma, \mu) = (E, \sigma, \lambda\mu) \in H^2(K, \mu_n).$$

In particular, $(E, \sigma, \lambda^{-1}) = -(E, \sigma, \lambda)$.
Let \((E, \sigma, \lambda)\) be a cyclic algebra over a field \(K\) and \(L/K\) be a field extension. Then \((E, \sigma, \lambda) \otimes L\) is Brauer equivalent to \((E_L, \sigma_L, \lambda)\). In particular, if \(L\) is a finite extension of \(K\) and \(EL\) is the composite of \(E\) and \(L\) in an algebraic closure of \(K\), then \(EL/L\) is cyclic with Galois group isomorphic to a subgroup of the Galois group of \(E/K\) and \((E, \sigma, \lambda) \otimes L\) is Brauer equivalent to \((EL, \sigma_L, \lambda)\).

By an abuse of notation, when the role of \(\sigma\) is not important or is clear from the context, we denote \((E, \sigma, \lambda)\) by \((E, \lambda)\).

**Lemma 2.1.** Let \(E/K\) be a cyclic extension of degree \(n\), \(\sigma\) a generator of \(\text{Gal}(E/K)\) and \(\lambda \in K^\ast\). Let \(m\) be a factor of \(n\) and \(d = n/m\). Let \(M/K\) be the subextension of \(E/K\) with \([M : K] = m\). Then \((E, \lambda) \otimes K(\sqrt[n]{\lambda}) = (M(\sqrt[n]{\lambda}), \sqrt[n]{\lambda})\).

**Proof.** We have \((E, \lambda)^d = (M, \sigma) \in H^1(K, \mathbb{Z}/n\mathbb{Z})\) and hence
\[
(E, \lambda) \otimes K(\sqrt[n]{\lambda}) = (E(\sqrt[n]{\lambda}), \lambda)
= (E(\sqrt[n]{\lambda}), (\sqrt[n]{\lambda})^d)
= ((E(\sqrt[n]{\lambda}))^d, (\sqrt[n]{\lambda}))
= (M(\sqrt[n]{\lambda}), \sqrt[n]{\lambda}).
\]
\[
\square
\]

**Lemma 2.2.** Let \(K\) be a complete discretely valued field and \(\ell\) a prime. Let \(L/K\) be a cyclic field extension or the split extension of degree \(\ell\) and \(\mu \in L^\ast\). Then there exists \(\theta \in L\) with \(N_{L/K}(\theta) = 1\) such that \(L = K(\mu\theta)\) and \(\theta\) is sufficiently close to 1.

**Proof.** Since \([L : K]\) is a prime, if \(\mu \notin K\), then \(L = K(\mu)\). In this case \(\theta = 1\) has the required properties.

Suppose that \(\mu \in K\). If \(L = \prod K\), let \(\theta_0 \in K^\ast \setminus \{\pm 1\}\) be sufficiently close to 1 and \(\theta = (\theta_0, \theta_0^{-1}, 1, \ldots, 1)\). Suppose that \(L\) is a field. Let \(\sigma\) be a generator of \(\text{Gal}(L/K)\). Suppose that \(\text{char}(K) \neq \ell\) contains a primitive \(\ell\)th root of unity. Since \(L/K\) is cyclic, we have \(L = K(\sqrt[\ell]{a})\) for some \(a \in K^\ast\). For any sufficiently large \(n\), \(\theta = (1 + \pi^n\sqrt[\ell]{a})^{-1}\sigma(1 + \pi^n\sqrt[\ell]{a}) \in L\) has the required properties.

Suppose that \(\text{char}(K) = \ell\) or \(K\) contains no primitive \(\ell\)th root of unity. Since \(L/K\) is separable, we have \(L = K(\alpha)\) for some \(\alpha \in L^\ast\). Let \(\theta = (1 + \sigma(\pi^n\alpha))/(1 + \pi^n\alpha)\). Then \(\theta \neq 1\) and \(N_{L/K}(\theta) = 1\). Suppose that \(\theta \in K\). Then \(\theta^d = N_{L/K}(\theta) = 1\) and hence \(\theta = 1\), leading to a contradiction. Hence \(\theta \notin K\). Therefore for sufficiently large \(n\), \(\theta\) has the required properties. \(\square\)

**Lemma 2.3.** Let \(K\) be a field and \(E/K\) be a finite extension of degree coprime to \(\text{char}(K)\). Let \(L/K\) be a subextension of \(E/K\) and \(e = [E : L]\). Suppose \(L/K\) is Galois and \(E = L(\sqrt[n]{\pi})\) for some \(\pi \in L^\ast\). Then \(E/K\) is Galois if and only if \(E\) contains a primitive \(e\)th root of unity and, for every \(\tau \in \text{Gal}(L/K)\), \(\tau(\pi) \in E^e\).

**Proof.** Suppose that \(E/K\) is Galois. Let \(f(X) = X^e - \pi \in L[X]\). Since \([E : L] = e\) and \(E = L(\sqrt[n]{\pi})\), \(f(X)\) is irreducible in \(L[X]\). Since \(f(X)\) has one root in \(E\) and \(E/L\) is Galois, \(f(X)\) has all the roots in \(E\). Hence \(E\) contains a primitive \(e\)th root of unity. Let \(\tau \in \text{Gal}(L/K)\). Then \(\tau\) can be extended to an automorphism \(\tilde{\tau}\) of \(E\). We have \(\tau(\pi) = \tilde{\tau}(\pi) = (\tilde{\tau}(\sqrt[n]{\pi}))^e \in E^e\).

Conversely, suppose that \(E\) contains a primitive \(e\)th root of unity and \(\tau(\pi) \in E^e\) for every \(\tau \in \text{Gal}(L/K)\). Let
\[
g(X) = \prod_{\tau \in \text{Gal}(L/K)} (X^e - \tau(\pi)).
\]
Then $g(X) \in K[X]$ and $g(X)$ splits completely in $E$. Since $e$ is coprime to $\text{char}(K)$, the splitting field $E_0$ of $g(X)$ over $K$ is Galois. Since $L/K$ is Galois and $E$ is the composite of $L$ and $E_0$, $E/K$ is Galois.

The following lemma is well known.

**Lemma 2.4.** Let $K$ be a complete discretely valued field with residue field $\kappa$ and $\pi \in K$ a parameter. Let $e$ be a natural number coprime to the characteristic of $\kappa$. If $L/K$ is a totally ramified extension of degree $e$, then $L = K(\sqrt[e]{\nu \pi})$ for some $\nu \in K$ which is a unit in the valuation ring of $K$. Further, if $e$ is a parameter at the valuation $L$.

**Proof.** Since $K$ is a complete discretely valued field, there is a unique extension of the valuation $\nu$ on $K$ to a valuation $\nu_L$ on $L$. Since $L/K$ is totally ramified extension of degree $e$ and $e$ is coprime to char($\kappa$), the residue field of $L$ is $\kappa$ and $\nu_L(\pi) = e$. Let $\pi_L \in L$ with $\nu_L(\pi_L) = 1$. Then $\pi = w\pi_L^e$ for some $w \in L$ with $\nu_L(w) = 0$. Since the residue field of $L$ is same as the residue field of $K$, there exists $v \in K$ with $\nu(v) = 0$ and the image of $v^{-1}$ is the same as the image of $w$ in the residue field $\kappa$. Since $L$ is complete and $e$ is coprime to char($\kappa$), by Hensel’s lemma, there exists $u \in L$ such that $w = v^{-1}u^e$. Thus $\pi = w\pi_L^e = v^{-1}u^e\pi_L^e = v^{-1}(u\pi_L)^e$. In particular, $v\pi \in L^e$ and hence $L = K(\sqrt[e]{\nu \pi})$.

Suppose that $\theta \in K^\times$, $\theta \not\in \pm K^e$ and $-\theta$ is a norm from $L$. Let $\mu \in L$ with $N_{L/K}(\mu) = -\theta$. Since $L = K(\sqrt[e]{\nu \pi})$ with $v \in K$ a unit in the valuation ring of $K$ and $\pi \in K$ a parameter, $\sqrt[e]{\nu \pi} \in L$ is a parameter at the valuation of $L$. Write $\mu = w_0(\sqrt[e]{\nu \pi})^s$ for some $w_0 \in L$ a unit at the valuation of $L$ and $s \in \mathbb{Z}$. As above, we have $w_0 = v_1 u_1^e$ for some $v_1 \in K$ and $u_1 \in L$. Since $v_1 \in K$, we have

$$-\theta = N_{L/K}(\mu) = N_{L/K}(w_0(\sqrt[e]{\nu \pi})^s) = N_{L/K}(v_1 u_1^e(\sqrt[e]{\nu \pi})^s) = v_1^e N_{L/K}(u_1)^e(-1)^{(e+1)s}(v_1)^s = a^e(-1)^s(v_1)^s.$$ 

where $a = v_1 N_{L/K}(u_1)(-1)^e$. Hence $\theta = (-1)^{s+1}(v_1)^s \in K^*/K^e$. Since $\theta \not\in \pm K^e$ and $e$ is a power of $\ell$, $s$ is coprime to $\ell$. In particular, $(-1)^{s+1} \in K$ and hence $K(\sqrt[e]{\nu \pi}) = K(\sqrt[e]{v_1}) = K(\sqrt[e]{\nu \pi}) = L$.

Throughout this paper by a local field we mean a non-archimedean local field.

**Lemma 2.5.** Let $k$ be a local field and $\ell$ a prime not equal to the characteristic of the residue field of $k$. Let $L_0/k$ be an extension of degree $\ell$ and $\theta_0 \in k^\times$. If $\theta_0 \not\in \pm k^e$ and $-\theta_0$ is a norm from $L_0$, then $L_0 = k(\sqrt[e]{\theta_0})$.

**Proof.** Suppose that $L_0/k$ is ramified. Since $\theta_0 \not\in \pm k^e$, by Lemma 2.4, $L_0 = k(\sqrt[e]{\theta_0})$.

Suppose that $L_0/k$ is unramified. Let $\pi$ be a parameter in $k$ and write $\theta_0 = w\pi^r$ with $w$ a unit in the valuation ring of $k$. Since $\theta_0$ is a norm from $L_0$, $\ell$ divides $r$ and $k(\sqrt[e]{\theta_0}) = k(\sqrt[e]{\pi})$ is an unramified extension of $k$ of degree $\ell$. Since $k$ is a local field, there is only one unramified field extension of $k$ of degree $\ell$ and hence $L_0 = k(\sqrt[e]{\theta_0})$.

**Lemma 2.6.** Suppose $K$ is a complete discretely valued field with residue field $\kappa$ a local field. Let $\ell$ be a prime not equal to the characteristic of the residue field of $\kappa$. Let $L/K$ be a field extension of degree $\ell$ and $\theta \in K^\times$. If $\theta \not\in \pm K^e$ and $-\theta$ is a norm from $L$, then $L \simeq K(\sqrt[e]{\theta})$. 414
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Proof. If \( L/K \) is a ramified extension, then by Lemma 2.4, \( L \simeq K(\sqrt[\ell]{\theta}) \). Suppose that \( L/K \) is an unramified extension. Since \(-\theta\) is a norm from \( L/K \), the valuation of \( \theta \) is divisible by \( \ell \). Thus, without loss of generality, we assume that \( \theta \) has valuation zero. Let \( L_0 \) be the residue field of \( L \) and \( \bar{\theta} \) be the image of \( \theta \) in \( \kappa \). Then \( L_0/\kappa \) is a field extension of degree \( \ell \) and \(-\bar{\theta}\) is a norm from \( L_0 \). Since \( \theta \not\in \pm K^{\times\ell} \), \( \bar{\theta} \not\in \pm \kappa^{\ell} \). Since \( \kappa \) is a local field, \( L_0 \simeq \kappa(\sqrt[\ell]{\bar{\theta}}) \) (Lemma 2.5) and hence \( L \simeq K(\sqrt[\ell]{\theta}) \).

For \( L = \prod_{i=1}^r L_i \), let \( \sigma \) be the automorphism of \( L \) given by \( \sigma(a_1, \ldots, a_\ell) = (a_2, \ldots, a_\ell, a_1) \). Set Gal\((L/K) = \{ \sigma^i \mid 0 \leq i \leq \ell - 1 \} \). Then any \( \sigma^i, 1 \leq i \leq \ell - 1 \), is called a generator of Gal\((L/K)\).

Lemma 2.7. Let \( K \) be a field and \( \ell \) a prime not equal to the characteristic of \( K \). Let \( L \) be a cyclic extension of \( K \) or the split extension of degree \( \ell \) and \( \sigma \) a generator of the Galois group of \( L/K \). Suppose that there exists an integer \( t \geq 1 \) such that \( K \) does not contain a primitive \( \ell \)th root of unity. Let \( \mu \in L \) with \( N_{L/K}(\mu) = 1 \) and \( m \geq t \). If \( \mu \in L^{\ell^m} \), then there exists \( b \in L^* \) such that \( \mu = b^{-\ell^m} \sigma(b^{\ell^m}) \).

Proof. Suppose \( L = \prod_{i=1}^r K_i \) and \( \mu \in L^{\ell^s} \) for some \( s \geq 1 \) with \( N_{L/K}(\mu) = 1 \). Then \( \mu = (\theta_1^{\ell^s}, \ldots, \theta_\ell^{\ell^s}) \in L \) with \( \theta_1^{\ell^s} \cdots \theta_\ell^{\ell^s} = 1 \). Without loss of generality we assume that \( \sigma \) is given by \( \sigma(a_1, \ldots, a_\ell) = (a_2, \ldots, a_\ell, a_1) \). Let \( b = (1, b_1, \ldots, b_{\ell-1}) \in L^* \), where \( b_i = \theta_1 \cdots \theta_i \). Then \( \mu = b^{-\ell^s} \sigma(b^{\ell^s}) \).

Suppose \( L/K \) is a cyclic field extension. Write \( \mu = \mu_0^{\ell^m} \) for some \( \mu_0 \in L \). Let \( \mu_1 = \mu_0^{\ell^m} \). Then \( \mu_1 = \mu_1^{\ell^m} \) \( \Leftrightarrow \) \( \mu_1 = \mu_1^{\ell^m} \). Let \( \theta_0 = N_{L/K}(\mu_0) \) and \( \theta_1 = N_{L/K}(\mu_1) \). Then \( \theta_1 = \theta_0^{\ell^m} \). Since \( N_{L/K}(\mu_1) = 1 \), we have \( \theta_1 = \theta_1^{\ell^m} \). If \( \ell \neq 1 \), then \( K \) contains a primitive \( \ell^m \)th root of unity. Since \( m \geq t \) and \( K \) has no primitive \( \ell^t \)th root of unity, \( \theta_1 = 1 \). Hence \( N_{L/K}(\mu_1) = 1 \) and by Hilbert’s Theorem 90, \( \mu_1 = b^{-\ell^s} \sigma(b^{\ell^s}) \) for some \( b \in L \). Thus \( \mu = \mu_1^{\ell^m} = b^{-\ell^s} \sigma(b^{\ell^s}) \).

We end this section with the following well-known fact.

Lemma 2.8. Let \( k \) be a local field and \( \ell \) a prime not equal to \( \text{char}(k) \). If \( \theta \in k^* \), then there exist a field extension \( L/k \) of degree \( \ell \) and \( \mu \in L^* \) such that \( N_{L/k}(\mu) = \theta \).

Proof. Let \( \nu \) be the discrete valuation on \( k \) and \( \theta \in k^* \). Without loss of generality we assume that \( 0 \leq \nu(\theta) < \ell \). Suppose \( \nu(\theta) > 0 \). Let \( L = k(\sqrt[\ell]{\theta}) \) and \( \mu = -\sqrt[\ell]{\theta} \in L \). Then \( N_{L/k}(\mu) = \theta \).

Suppose \( \nu(\theta) = 0 \). Then let \( L/k \) be the unramified extension of degree \( \ell \). Then \( \theta \) is a norm from \( L \) (cf. [Ser79, p. 82, Proposition 3 and Remark 1]).

3. Global fields

In this section we prove a few technical results concerning Brauer groups of global fields and reduced norms. We begin with the following lemma.

Lemma 3.1. Let \( k \) be a global field, \( \ell \) a prime not equal to \( \text{char}(k) \), \( n, d \geq 2 \) and \( r \geq 1 \) be integers. Let \( E_0 \) be a cyclic extension of \( k \), \( \sigma_0 \) a generator of the Galois group of \( E_0/k \) and \( \theta_0 \in k^* \). Let \( \beta \in H^2(k, \mu_r) \) be such that \( r\ell \beta = (E_0, \sigma_0, \theta_0) \in H^2(k, \mu_r) \). Let \( S \) be a finite set of places of \( k \) containing all the places of \( k \) with \( \beta \otimes k_p \neq 0 \). For each \( \nu \in S \), let \( L_\nu/k_\nu \) be a cyclic field extension of degree \( \ell \) or \( L_\nu \) be the split extension of \( k_\nu \) of degree \( \ell \) and \( \mu_\nu \in L_\nu^* \). Suppose that:

1. \( N_{L_\nu/k_\nu}(\mu_\nu) = \theta_0 ;\)
2. \( r\beta \otimes L_\nu = (E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu) ;\)
(3) \( \text{ind}(\beta \otimes E_0 \otimes L_\nu) < d; \)
(4) \( k \) contains a primitive \( \ell \)th root of unity.

Then there exist a field extension \( L_0/k \) of degree \( \ell \) and \( \mu_0 \in L_0 \) such that:

(1) \( N_{L_0/k}(\mu_0) = \theta_0; \)
(2) \( r\beta \otimes L_0 = (E_0 \otimes L_0, \sigma_0 \otimes 1, \mu_0); \)
(3) \( \text{ind}(\beta \otimes E_0 \otimes L_0) < d; \)
(4) \( L_0 \otimes k_\nu \simeq L_\nu \) for all \( \nu \in S; \)
(5) \( \mu_0 \) is close to \( \mu_\nu \) for all \( \nu \in S \).

**Proof.** Let \( \Omega_k \) be the set of all places of \( k \) and

\[ S' = S \cup \{ \nu \in \Omega_k \mid \theta_0 \text{ is not a unit at } \nu \text{ or } E_0/k \text{ is ramified at } \nu \}. \]

Let \( \nu \in S' \setminus S. \) Then \( \beta \otimes k_\nu = 0. \) Since \( k \) contains a primitive \( \ell \)th root of unity, there exists a cyclic field extension \( L_\nu \) of \( k_\nu \) of degree \( \ell \) such that \( \theta_0 \in N(L_\nu^*) \) (cf. the proof of Lemma 2.8). Let \( \mu_\nu \in L_\nu \) with \( N_{L_\nu/k_\nu}(\mu_\nu) = \theta_0. \) Since \( \beta \otimes k_\nu = 0, \text{ind}(\beta \otimes E_0 \otimes L_\nu) = 1 < d. \) Since the corestriction map \( \text{cor} : H^2(L_\nu, \mu_\nu) \to H^2(k_\nu, \mu_\nu) \) is injective (cf. [Lor08, Theorem 10, p. 237]) and \( \text{cor}(E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu) = (E_0 \otimes k_\nu, \sigma_0 \otimes 1, \theta_0) = r\beta \otimes k_\nu = 0, \) \( (E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu) = 0 = r\beta \otimes L_\nu. \) Thus, if necessary, by enlarging \( S, \) we assume that \( S \) contains all those places \( \nu \) of \( k \) with either \( \theta_0 \) not a unit at \( \nu \) or \( E_0/k \) ramified at \( \nu \) and that there is at least one \( \nu \in S \) such that \( L_\nu \) is a field extension of \( k_\nu \) of degree \( \ell. \)

Let \( \nu \in S. \) By Lemma 2.2, there exists \( \theta_\nu \in L_\nu \) such that \( N_{L_\nu/k_\nu}(\theta_\nu) = 1, L_\nu = k_\nu(\theta_\nu \mu_\nu) \) and \( \theta_\nu \) is sufficiently close to 1. In particular, \( \theta_\nu \in L_\nu^* \) and hence \( r\beta \otimes L_\nu = (E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu) = (E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu \theta_\nu). \) Thus, replacing \( \mu_\nu \) by \( \mu_\nu \theta_\nu, \) we assume that \( L_\nu = k_\nu(\mu_\nu). \) Let \( f_\nu(X) = X^\ell + b_{-1,\nu}X^{\ell-1} + \cdots + b_{1,\nu}X + (-1)\ell \theta_0 \in k_\nu[X] \) be the minimal polynomial of \( \mu_\nu \) over \( k_\nu. \)

By Chebotarev’s density theorem [FJ08, Theorem 6.3.1], there exists \( v_0 \in \Omega_k \setminus S \) such that \( E_0 \otimes k_{v_0} \) is the split extension of \( k_{v_0}. \) By the strong approximation theorem [CF67, p. 67], choose \( b_j \in k, 1 \leq j \leq \ell - 1, \) such that each \( b_j \) is sufficiently close to \( b_j, \nu \) for all \( \nu \in S \) and each \( b_j \) is an integer at all \( \nu \not\in S \setminus \{ v_0 \}. \) Let \( L_0 = k[X]/(X^\ell + b_{-1,0}X^{\ell-1} + \cdots + b_{1,0}X + (-1)\ell \theta_0) \) and \( \mu_0 \in L_0 \) be the image of \( X. \) We now show that \( L_0 \) and \( \mu_0 \) have the required properties.

Since each \( b_j \) is sufficiently close to \( b_{j,\nu} \) at each \( \nu \in S, \) it follows from Krasner’s lemma that there exists an isomorphism \( L_0 \otimes k_\nu \simeq L_\nu \) with the image of \( \mu_0 \otimes 1 \) in \( L_\nu \) close to \( \mu_\nu \) for all \( \nu \in S \) (cf. [Ser79, ch. II, §2]). Since \( L_\nu \) is a field extension of \( k_\nu \) of degree \( \ell \) for at least one \( \nu \in S, \) \( L_0 \) is a field extension of degree \( \ell \) over \( k. \) Since \( X^\ell + b_{-1,\nu}X^{\ell-1} + \cdots + (-1)\ell \theta_0 \) is the minimal polynomial of \( \mu_0, \) we have \( N(\mu_0) = \theta_0. \)

To show that \( \text{ind}(\beta \otimes E_0 \otimes L_\nu) < d \) and \( r\beta = (E_0, \sigma_0, \mu_0) \in H^2(L_0, \mu_\nu), \) by the Hasse–Brauer–Noether theorem (cf. [CF67, p. 187]), it is enough to show that for every place \( w \) of \( L_0, \) \( \text{ind}(\beta \otimes E_0 \otimes L_w) < d \) and \( r\beta \otimes L_w = (E_0, \sigma_0, \mu_0) \otimes L_w \in H^2(L_w, \mu_\nu). \)

Let \( w \) be a place of \( L_0 \) and \( \nu \) a place of \( k \) lying below \( w. \) Suppose that \( \nu \in S. \) Then \( L_0 \otimes k_\nu \simeq L_\nu. \)

By the assumption on \( L_\nu, \) we have \( \text{ind}(\beta \otimes E_0 \otimes k_\nu) < d. \) Since \( \mu_\nu \) is close to \( \mu_0, \) we have \( r\beta \otimes L_\nu = (E_0 \otimes L_\nu, \sigma_0, \mu_\nu) = (E_0 \otimes L \otimes k_\nu, \sigma_0, \mu_0). \)

Suppose that \( \nu \not\in S \) and \( \nu \neq v_0. \) Then \( \theta_0 \) is a unit at \( \nu, E_0/k \) is unramified at \( \nu \) and \( \beta \otimes k_\nu = 0. \) Since each \( b_j \) is an integer at \( \nu \) and \( \mu_0 \) is a root of the polynomial \( X^\ell + b_{-1,\nu}X^{\ell-1} + \cdots + b_{1,\nu}X + (-1)\ell \theta_0, \mu_0 \) is an integer at \( w. \) Since \( \theta_0 \) is a unit at \( \nu, \mu_0 \) is a unit at \( w. \) In particular,
(E_0 \otimes L_w, \sigma_0, \mu_0) = 0 = r \beta \otimes L_w. If \nu = \nu_0, then by the choice of \nu_0, \beta \otimes k_\nu = 0, E_0 \otimes k_\nu is the split extension of k_\nu and hence (E_0, \sigma_0, \mu_0) \otimes L_w = 0 = r \beta \otimes L_w. 

\begin{proof}
Let k be a global field, \ell a prime not equal to \text{char}(k), n and r \geq 1 integers. Suppose that either \ell \neq 2 or k has no real place. Let \theta_0 \in k^* and \beta \in H^2(k, \mu_r). Suppose that r \ell \beta = 0 \in H^2(k, \mu_r) and \beta \neq 0. Then there exist a field extension L_0/k of degree \ell and \mu_0 \in L_0 such that N_{L_0/k}(\mu_0) = \theta_0, r \beta \otimes L_0 = 0 and \text{ind}(\beta \otimes L_0) < \text{ind}(\beta).

We use the following notation for the rest of this section:
- k a global field with no real places and \theta_0 \in k^*;
- \ell a prime not equal to \text{char}(k);
- k contains a primitive \ell\text{th} root of unity;
- E_0/k a cyclic extension of degree a power of \ell and \sigma_0 a generator of \text{Gal}(E_0/k);
- n \geq 1;
- \beta \in H^2(k, \mu_n) with r \ell \beta = (E_0, \sigma_0, \theta_0) for some r \geq 1.

**Lemma 3.3.** Suppose that r \beta \otimes E_0 \neq 0. If \nu is a place of k and L_\nu/k_\nu a field extension of degree \ell such that \theta_0 \in N_{L_\nu/k_\nu}(L_\nu^*), then \text{ind}(\beta \otimes E_0 \otimes L_\nu) < \text{ind}(\beta \otimes E_0).

\begin{proof}
Write r \ell = m \ell^d with m coprime to \ell. Then d \geq 1. Since m \ell^d \beta = r \ell \beta = (E_0, \sigma_0, \theta_0), we have m \ell^d \beta \otimes E_0 = 0. Since m is coprime to \ell and the period of \beta is a power of \ell, it follows that \ell^d \beta \otimes E_0 = 0. Since r \beta \otimes E_0 \neq 0, \ell^d-1 \beta \otimes E_0 \neq 0 and \text{per}(\beta \otimes E_0) = \ell^d.

Let \nu be a place of k and L_\nu/k_\nu a field extension of degree \ell such that \theta_0 \in N_{L_\nu/k_\nu}(L_\nu^*). Suppose that L_\nu is not contained in E_0 \otimes k_\nu. Then [E_0 \otimes L_\nu : E_0 \otimes k_\nu] = \ell and hence \text{ind}(\beta \otimes E_0 \otimes L_\nu) < \text{ind}(\beta \otimes E_0) [\text{CF67}, p. 131]. Suppose that L_\nu is contained in E_0 \otimes k_\nu. Then E_0 \otimes L_\nu = \prod E_i with each E_i a cyclic extension of k_\nu. Since E_0/k is a Galois extension, E_i \simeq E_j for all i and j and m \ell^d \beta \otimes k_\nu = (E_0, \sigma_0, \theta_0) \otimes k_\nu = (E_i, \sigma_i, \theta_0) for all i, for suitable generators \sigma_i of \text{Gal}(E_i/k_\nu). Since L_\nu is a field and contained in E_0 \otimes k_\nu, L_\nu is contained in E_i for all i. Since \theta_0 is a norm from L_\nu, \theta_0\vert_{E_i/k_\nu} \in N_{E_i/k_\nu}(E_i^*) \subseteq \mathcal{N}_{E_i/k_\nu}(E_i^*) [\text{Alb61}, p. 75], \text{per}(E_i, \sigma_i, \theta_0) \leq [E_i : k_\nu]/\ell < [E_i : k_\nu].

Suppose that \text{per}(\beta \otimes k_\nu) \leq [E_i : k_\nu]. Since k_\nu is a local field, \text{per}(\beta \otimes E_i) = 1. Thus \text{per}(\beta \otimes E_0 \otimes k_\nu) = \text{per}(\beta \otimes E_i) = 1 = \ell^d = \text{per}(\beta \otimes E_0).

Suppose that \text{per}(\beta \otimes k_\nu) > [E_i : k_\nu]. Since m \ell^d \beta \otimes k_\nu = (E_i, \sigma_i, \theta_0) and m is coprime to \ell, we have \text{per}(\beta \otimes k_\nu) \leq \ell^d \text{per}(E_i, \sigma_i, \theta_0). Since k_\nu is a local field,

\[\text{per}(\beta \otimes E_0 \otimes k_\nu) = \text{per}(\beta \otimes E_i) = \frac{\text{per}(\beta \otimes k_\nu)}{[E_i : k_\nu]} \leq \frac{\ell^d \text{per}(E_i, \sigma_i, \theta_0)}{[E_i : k_\nu]} < \ell^d = \text{per}(\beta \otimes E_0).\]

Since k_\nu is a local field, period equals index and hence the lemma follows. \qed

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Proposition 3.4. Suppose that \( r\beta \otimes E_0 \neq 0 \). Then there exist a field extension \( L_0/k \) of degree \( \ell \) and \( \mu_0 \in L_0 \) such that:

1. \( N_{L_0/k}(\mu_0) = \theta_0 \);
2. \( \text{ind}(\beta \otimes E_0 \otimes L_0) < \text{ind}(\beta \otimes E_0) \);
3. \( r\beta \otimes L_0 = (E_0 \otimes L_0, \sigma_0 \otimes 1, \mu_0) \).

Proof. Let \( S \) be the finite set of places of \( k \) consisting of all those places \( \nu \) with \( \beta \otimes k_\nu \neq 0 \). Let \( \nu \in S \). By Lemma 2.8, we have a field extension \( L_\nu/k_\nu \) of degree \( \ell \) and \( \mu_\nu \in L_\nu \) such that \( N_{L_\nu/k_\nu}(\mu_\nu) = \theta_0 \) and, by Lemma 3.3, \( \text{ind}(\beta \otimes E_0 \otimes L_\nu) < \text{ind}(\beta \otimes E_0) \). Since \( \text{cor}_{L_\nu/k_\nu}(r\beta \otimes L_\nu) = r\ell \beta = (E_0 \otimes k_\nu, \sigma_\nu, \theta_0) = \text{cor}_{L_\nu/k_\nu}(E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu) \) and the corestriction map here is injective (cf. [Lor08, Theorem 10, p. 237]), we have \( r\beta \otimes L_\nu = (E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu) \).

By Lemma 3.1, we have the required \( L_0 \) and \( \mu_0 \). \qed

Proposition 3.5. Suppose that \( r\beta \otimes E_0 = 0 \) and \( E_0 \neq k \). Let \( L_0 \) be the unique subfield of \( E_0 \) of degree \( \ell \) over \( k \). Then there exists \( \mu_0 \in L_0 \) such that:

1. \( N_{L_0/k}(\mu_0) = \theta_0 \);
2. \( r\beta \otimes L_0 = (E_0 \otimes L_0, \sigma_0 \otimes 1, \mu_0) \).

Proof. Since \( r\beta \otimes E_0 = 0 \) and \( E_0/k \) is a cyclic extension, we have \( r\beta = (E_0, \sigma_0, \mu') \) for some \( \mu' \in k \). We have \( (E_0, \sigma_0, \mu') \ell = (E_0, \sigma_0, \theta_0) \). Thus \( \theta_0 = N_{E_0/k}(y)\mu'. \ell \). Let \( \mu_0 = N_{E_0/L_0}(y)\mu' \in L_0 \).

Since \( L_0 \subset E_0 \), we have \( r\beta \otimes L_0 = (E_0/L_0, \sigma_0, \mu') = (E_0/L_0, \sigma_0, N_{E_0/L_0}(y)\mu') = (E_0/L_0, \sigma_0, \mu_0) \) (cf. §2) and \( N_{L_0/k}(\mu_0) = N_{L_0/k}(N_{E_0/L_0}(y))\mu' = \theta_0. \) \qed

Corollary 3.6. Suppose that \( r\beta \otimes E_0 = 0 \) and \( E_0 \neq k \). Let \( L_0 \) be the unique subfield of \( E_0 \) of degree \( \ell \) over \( k \). Let \( S \) be a finite set of places of \( k \). Suppose that for each \( \nu \in S \) there exists \( \mu_\nu \in L_0 \otimes k_\nu \) such that:

\begin{itemize}
  \item \( N_{L_0/k}(\mu_\nu) = \theta_0 \);
  \item \( r\beta \otimes L_0 \otimes k_\nu = (E_0 \otimes L_0 \otimes k_\nu, \sigma_0 \otimes 1, \mu_\nu) \).
\end{itemize}

Then there exists \( \mu \in L_0 \) such that:

1. \( N_{L_0/k}(\mu) = \theta_0 \);
2. \( r\beta \otimes L_0 = (E_0 \otimes L_0, \sigma_0 \otimes 1, \mu) \);
3. \( \mu \) is close to \( \mu_\nu \) for all \( \nu \in S \).

Proof. By Proposition 3.5, there exists \( \mu_0 \in L_0 \) such that:

\begin{itemize}
  \item \( N_{L_0/k}(\mu_0) = \theta_0 \);
  \item \( r\beta \otimes L_0 = (E_0 \otimes L_0, \sigma_0 \otimes 1, \mu_0) \).
\end{itemize}

Let \( \nu \in S \). Then we have:

\begin{itemize}
  \item \( N_{L_0/k}(\mu_0) = \theta_0 = N_{L_0 \otimes k_\nu/k_\nu}(\mu_\nu) \);
  \item \( (E_0 \otimes L_0 \otimes k_\nu, \sigma_0 \otimes 1, \mu_0) = (E_0 \otimes L_0 \otimes k_\nu, \sigma_0 \otimes 1, \mu_\nu) \).
\end{itemize}

Let \( b_\nu = \mu_\nu \mu_0^{-1} \in L_0 \otimes k_\nu \). Then \( N_{L_0 \otimes k_\nu}(b_\nu) = 1 \) and \( (E_0 \otimes L_0 \otimes k_\nu, \sigma_0 \otimes 1, b_\nu) = 0 \). Thus, there exists \( a_\nu \in E_0 \otimes L_0 \otimes k_\nu \) with \( N_{E_0 \otimes L_0 \otimes k_\nu/L_\nu}(a_\nu) = b_\nu \). We have \( N_{E_0 \otimes L_0 \otimes k_\nu/L_\nu}(a_\nu) = N_{L_0 \otimes k_\nu}(b_\nu) = 1 \). Since \( E_0/k \) is a cyclic extension with \( \sigma_0 \) a generator of \( \text{Gal}(E_0/k) \), for each \( \nu \in S \), there exists \( c_\nu \in E_0 \otimes L_0 \otimes k_\nu \) such that \( a_\nu = c_\nu^{-1}(\sigma_0 \otimes 1)(c_\nu) \). By weak approximation, there exists \( c \in E_0 \otimes L_0 \) such that \( c \) is close to \( c_\nu \) for all \( \nu \in S \). Let \( a = c^{-1}(\sigma \otimes 1)(c) \in E_0 \otimes L_0 \) and \( \mu = \mu_0 N_{E_0 \otimes L_0/L_\nu}(a) \in L_0 \). Then \( \mu \) has all the required properties. \qed
4. Complete discretely valued fields

Let $F$ be a field with a discrete valuation $\nu$, valuation ring $R$ and residue field $\kappa$. Suppose that $n$ is coprime to the characteristic of $\kappa$. For any $d \geq 1$, we have the residue map $\partial_F : H^d(F, \mu_n) \rightarrow H^{d-1}(\kappa, \mu_{n^{d-1}})$. We also denote $\partial_F$ by $\partial$. An element $\alpha$ in $H^d(F, \mu_n)$ is called unramified at $\nu$ or $R$ if $\partial(\alpha) = 0$. The subgroup of all unramified elements is denoted by $H^d_{nr}(F/R, \mu_n)$ or simply $H^d_{nr}(F, \mu_n)$. Suppose that $F$ is complete with respect to $\nu$. Then we have an isomorphism $H^d(\kappa, \mu_n) \sim H^d_{nr}(F, \mu_n)$ and the composition $H^d(\kappa, \mu_n) \sim H^d_{nr}(F, \mu_n) \hookrightarrow H^d(F, \mu_n)$ is denoted by $\iota_n$ or simply $\iota$.

Let $F$ be a complete discretely valued field with residue field $\kappa$, $\nu$ the discrete valuation on $F$ and $\pi \in F^*$ a parameter. Suppose that $n$ is coprime to the characteristic of $\kappa$. Let $\partial : H^2(F, \mu_n) \rightarrow H^1(\kappa, \mathbb{Z}/n\mathbb{Z})$ be the residue homomorphism. Let $E/F$ be a cyclic unramified extension of degree $n$ with residue field $E_0$ and $\sigma$ a generator of $\text{Gal}(E/F)$ with $\sigma_0 \in \text{Gal}(E_0/\kappa)$ induced by $\sigma$. Then $(E, \sigma, \pi)$ is a division algebra over $F$ of degree $n$. For any $\lambda \in F^*$, we have

$$\partial(E, \sigma, \lambda) = (E_0, \sigma_0)^{\nu(\lambda)}.$$ 

For $\lambda, \mu \in F^*$, we have

$$\partial((E, \sigma, \lambda) \cdot (\mu)) = (E_0, \sigma_0) \cdot ((-1)^{\nu(\lambda)\nu(\mu)}\theta),$$

where $\theta$ is the image of $\lambda^{\nu(\mu)}/\mu^{\nu(\lambda)}$ in the residue field.

Suppose $E_0$ is a cyclic extension of $\kappa$ of degree $n$. Then there is a unique unramified cyclic extension $E/F$ of degree $n$ with residue field $E_0$. Let $\sigma_0$ be a generator of $\text{Gal}(E_0/\kappa)$ and $\sigma \in \text{Gal}(E/F)$ be the lift of $\sigma_0$. Then $\sigma$ is a generator of $\text{Gal}(E/F)$. We call the pair $(E, \sigma)$ the lift of $(E_0, \sigma_0)$.

We use the following notation throughout this section:

- $(F, \nu)$ a complete discretely valued field;
- $\kappa$ the residue field of $F$;
- $\pi \in F^*$ a parameter at $\nu$;
- $n \geq 2$ an integer coprime to $\text{char}(\kappa)$;
- $D$ a central simple algebra over $F$ of period $n$;
- $\alpha \in H^2(F, \mu_n)$ the class representing $\iota_n$.

Let $\lambda \in F^*$. In this section we analyze the condition $\alpha \cdot (\lambda) = 0$ and we use this analysis in the proof of our main result (§10). We also prove that if $\kappa$ is either a local field or a global field and $\alpha \cdot (\lambda) = 0$ in $H^2(F, \mu_n)$, then $\lambda$ is a reduced norm from $D$.

Let $E_0$ be the cyclic extension of $\kappa$ and $\sigma_0 \in \text{Gal}(E_0/\kappa)$ be such that $\partial(\alpha) = (E_0, \sigma_0)$. Let $(E, \sigma)$ be the lift of $(E_0, \sigma_0)$. The pair $(E, \sigma)$ or $E$ is called the lift of the residue of $\alpha$. The following lemma is well known.

**Lemma 4.1.** Let $\alpha \in H^2(F, \mu_n)$, $(E, \sigma)$ the lift of the residue of $\alpha$. Then $\alpha = \alpha' + (E, \sigma, \pi)$ for some $\alpha' \in H^2_{nr}(F, \mu_n)$. Further, $\alpha' \otimes E = \alpha \otimes E$ is independent of the choice of $\pi$.

**Proof.** Since $\partial(E, \sigma, \pi) = \partial(\alpha)$, $\alpha' = \alpha - (E, \sigma, \pi) \in H^2_{nr}(F, \mu_n)$ and $\alpha = \alpha' + (E, \sigma, \pi)$. $\square$

**Lemma 4.2.** Let $\alpha \in H^2(F, \mu_n)$. If $\alpha = \alpha' + (E, \sigma, \pi)$ as in Lemma 4.1, then $\text{ind}(\alpha) = \text{ind}(\alpha' \otimes E)[E : F] = \text{ind}(\alpha \otimes E)[E : F]$.

**Proof.** Cf. [FS39, Proposition 1(3)] and [JW90, 5.15]. $\square$

**Lemma 4.3.** Let $E$ be the lift of the residue of $\alpha$. Suppose there exists a totally ramified extension $M/F$ which splits $\alpha$. Then $\alpha \otimes E = 0$. 419
Proof. Write \( \alpha = \alpha' + (E, \sigma, \pi) \) as in Lemma 4.1. Since \( \alpha' \otimes E = \alpha \otimes E \), we have \( \alpha' \otimes E \otimes M = 0 \). Since \( E \otimes M/E \) is totally ramified, the residue field of \( E \otimes M \) is the same as the residue field of \( E \). Since \( \alpha' \otimes E \otimes M = 0 \) and \( \alpha' \otimes E \) is unramified, it follows from [Ser03, 7.9 and 8.4] that \( \alpha' \otimes E = 0 \) and hence \( \alpha \otimes E = 0 \). \( \square \)

For an element \( \zeta \in H^m(F, A) \) for any abelian group \( A \), let \( \per(\zeta) \) denote the order of \( \zeta \) in the group \( H^m(F, A) \).

Lemma 4.4. Let \( \alpha \in H^2(F, \mu_n) \) and \( (E, \sigma) \) be the lift of the residue of \( \alpha \). If \( \alpha \otimes E = 0 \), then \( \alpha = (E, \sigma, u\pi) \) for some \( u \in F^* \) which is a unit at the discrete valuation, and \( \per(\alpha) = \ind(\alpha) \).

Proof. We have \( \alpha = \alpha' + (E, \sigma, \pi) \) as in Lemma 4.1. Since \( \alpha' \otimes E = \alpha \otimes E = 0 \), we have \( \alpha' = (E, \sigma, u) \) for \( u \in F^* \). Since \( E/F \) and \( \alpha' \) are unramified at the discrete valuation of \( F \), \( u \) is a unit at the discrete valuation of \( F \). We have \( \alpha = (E, \sigma, u) + (E, \sigma, \pi) = (E, \sigma, u\pi) \). Since \( E/F \) is an unramified extension and \( u\pi \) is a parameter, \( (E, \sigma, u\pi) \) is a division algebra and its period is \( [E : F] \). In particular, \( \ind(\alpha) = \per(\alpha) \). \( \square \)

Theorem 4.5. Let \( F \) be a complete discretely valued field with residue field \( \kappa \). Suppose that \( \kappa \) is a local field. Let \( \ell \) be a prime not equal to the characteristic of \( \kappa \), \( n = \ell^d \) and \( \alpha \in H^2(F, \mu_n) \). Then \( \per(\alpha) = \ind(\alpha) \).

Proof. Write \( \alpha = \alpha' + (E, \sigma, \pi) \) as in Lemma 4.1. Then \( E \) is an unramified cyclic extension of \( F \) with \( \partial(\alpha) = (E_0, \sigma_0) \) and \( \alpha' \) is unramified at the discrete valuation of \( F \). Let \( \overline{\alpha}' \) be the image of \( \alpha' \) in \( H^2(\kappa, \mu_n) \).

Suppose that \( \per(\partial(\alpha)) = \per(\alpha) \). Then \( \per(\partial(\alpha)) = [E_0 : \kappa] \). Since \( F \) is complete and \( E/F \) is an unramified extension, we have \( [E_0 : \kappa] = [E : F] \). Thus,

\[
0 = \per(\alpha)\alpha = \per(\alpha)\alpha' + (E, \sigma, \pi) = \per(\alpha)\alpha' + \per(\alpha)(E, \sigma, \pi) = \per(\alpha)\alpha' + [E : F](E, \sigma, \pi) = \per(\alpha)\alpha'.
\]

In particular, \( \per(\alpha') \) divides \( \per(\alpha) = [E_0, \kappa] = [E : F] \). Since \( \kappa \) is a local field, \( \overline{\alpha}' \otimes E_0 \) is zero [CF67, p. 131] and hence \( \alpha' \otimes E \) is zero. By Lemma 4.4, we have \( \alpha = (E, \sigma, \theta \pi) \) for some \( \theta \in F \) which is a unit in the valuation ring. In particular, \( \alpha \) is cyclic and \( \ind(\alpha) = \per(\alpha) = [E : F] \).

Suppose that \( \per(\partial(\alpha)) \neq \per(\alpha) \). Then \( \per(\partial(\alpha)) < \per(\alpha) \). Since \( \per(\partial(\alpha)) = \per(\theta, \sigma, \pi) \), we have \( \per(\alpha) = \per(\alpha') \). Since \( \kappa \) is a local field, \( \per(\overline{\alpha}') = \ind(\overline{\alpha}') \). Since \( \per(\overline{\alpha}') = \per(\alpha') \) and \( \per(\partial(\alpha)) = [E_0 : \kappa] \), we have \( [E_0 : \kappa] < \per(\overline{\alpha}') \). Since \( \kappa \) is a local field,

\[
\ind(\alpha' \otimes E_0) = \frac{\per(\overline{\alpha}')}{[E_0 : \kappa]}.
\]

Since \( E \) is a complete discretely valued field with residue field \( E_0 \) and \( \alpha' \) is unramified at the discrete valuation of \( E \), we have \( \ind(\alpha' \otimes E) = \ind(\overline{\alpha}' \otimes E_0) \). Thus, we have

\[
\ind(\alpha) = \ind(\alpha' \otimes E)[E : F] \quad \text{ (by Lemma 4.2)}
\]
\[
= \ind(\overline{\alpha}' \otimes E_0)[E_0 : \kappa]
\]
\[
= \frac{\per(\overline{\alpha}')}{[E_0 : \kappa]}[E_0 : \kappa]
\]
\[
= \per(\overline{\alpha}') = \per(\alpha). \quad \square
\]
Proposition 4.6 [Kat79, Corollary 2, p. 331]. Suppose that $\kappa$ is a local field. If $L/F$ is a finite field extension, then the corestriction homomorphism $H^3(L, \mu_n^{\otimes 2}) \to H^3(F, \mu_n^{\otimes 2})$ is bijective.

Proof. Let $\kappa'$ be the residue field of $L$. Since $\kappa$ and $\kappa'$ are local fields, $H^3(\kappa, \mu_n^{\otimes 2}) = H^3(\kappa', \mu_n^{\otimes 2}) = 0$ [Ser97, p. 86]. Since $F$ and $L$ are complete discretely valued fields, the residue homomorphisms $H^3(F, \mu_n^{\otimes 2}) \xrightarrow{\partial_F} H^2(\kappa, \mu_n)$ and $H^3(L, \mu_n^{\otimes 2}) \xrightarrow{\partial_L} H^2(\kappa', \mu_n)$ are isomorphisms (cf. [Ser03, 7.9]). The proposition follows from the commutative diagram

\[
\begin{array}{ccc}
H^3(L, \mu_n^{\otimes 2}) & \xrightarrow{\partial_L} & H^2(\kappa, \mu_n) \\
\downarrow & & \downarrow \\
H^3(F, \mu_n^{\otimes 2}) & \xrightarrow{\partial_F} & H^2(\kappa', \mu_n)
\end{array}
\]

where the vertical arrows are the corestriction maps [Ser03, 8.6]. \hfill $\square$

Lemma 4.7. Let $\ell$ be a prime not equal to char($\kappa$) and $n = \ell^d$ for some $d \geq 1$. Let $\alpha \in H^2(F, \mu_n)$ and $\lambda \in F^\times$. Write $\lambda = \theta \pi^r$ for some $\theta, \pi \in F$ with $\nu(\theta) = 0$ and $\nu(\pi) = 1$. Let $(E, \sigma)$ be the lift of the residue of $\alpha$ and $\alpha = \alpha' + (E, \sigma, \pi)$ as in Lemma 4.1. Then

$$\partial(\alpha \cdot (-\lambda)) = 0 \iff r\alpha' = (E, \sigma, (\theta^{-1})^{r+1}\lambda) \iff r\alpha = (E, \sigma, (-1)^{r+1}\lambda).$$

In particular, if $\partial(\alpha \cdot (-\lambda)) = 0$ and $r = \nu(\lambda)$ is coprime to $\ell$, then $\text{ind}(\alpha \otimes F(\sqrt{\lambda})) = \text{ind}(\alpha)$ and $\partial_{F(\sqrt{\lambda})}(\alpha \cdot (-\sqrt{\lambda})) = 0$.

Proof. Since $r\alpha = r\alpha' + (E, \sigma, \pi^r)$ and $\lambda = \theta \pi^r$, $r\alpha = (E, \sigma, (-1)^{r+1}\lambda)$ if and only if $r\alpha' = (E, \sigma, (-1)^{r+1}\theta)$. We have

$$\partial(\alpha \cdot (-\lambda)) = \partial((\alpha' + (E, \sigma, \pi)) \cdot (-\theta\pi^r)) = r\alpha' + (E_0, \sigma_0, (-1)^{r+1}\theta^{-1}),$$

where $\partial(\alpha) = (E_0, \sigma_0)$. Thus $\partial(\alpha \cdot (-\lambda)) = 0$ if and only if $r\alpha' + (E_0, \sigma_0, (-1)^{r+1}\theta^{-1}) = 0$ if and only if $r\alpha' = (E_0, \sigma_0, (-1)^{r+1}\theta)$ if and only if $r\alpha' = (E, \sigma, (-1)^{r+1}\theta)$ ($F$ being complete).

Suppose $r = \nu(\lambda)$ is coprime to $\ell$ and $\partial(\alpha \cdot (-\lambda)) = 0$. Clearly $(-1)^{r+1}$ is an $\ell^{th}$ power in $F$. Thus, we have $r\alpha = (E, \sigma, (-1)^{r+1}\lambda) = (E, \sigma, \lambda)$. Since $r$ is coprime to $\ell$, we have

$$\text{ind}(\alpha) = \text{ind}(r\alpha) = \text{ind}(E, \sigma, \lambda) = [E : F]$$

and

$$\text{ind}(\alpha \otimes F(\sqrt{\lambda})) = \text{ind}(r\alpha \otimes F(\sqrt{\lambda})) = \text{ind}(E(\sqrt{\lambda}), \sigma, \lambda)$$

$$= [E(\sqrt{\lambda}) : F(\sqrt{\lambda})]/\ell < [E : F] = \text{ind}(\alpha).$$

Further, $\partial_{F(\sqrt{\lambda})}(r\alpha \cdot (-\sqrt{\lambda})) = \partial_{F(\sqrt{\lambda})}((E, \sigma, \lambda) \cdot (-\sqrt{\lambda})) = (E_0, \sigma_0) \cdot ((-1)^{r+1}\theta).$ If $\ell$ is even, then $(-1)^{r+1} = 1$. If $\ell$ is odd, then $n$ is odd and $(-1)^{r+1}$ is an $n$th power. Thus, in either case,

$(E_0, \sigma_0) \cdot ((-1)^{r+1}\theta) = 0 \in H^2(\kappa, \mu_n).$ Since $r$ is coprime to $\ell$, $\partial_{F(\sqrt{\lambda})}(\alpha \cdot (-\sqrt{\lambda})) = 0$. \hfill $\square$

Lemma 4.8. Let $n \geq 2$ be coprime to $\text{char}(\kappa)$ and $\ell$ a prime which divides $n$. Let $\alpha \in H^2(F, \mu_n)$, $\lambda = \theta \pi^r \in F^\times$ with $\theta$ a unit in the valuation ring of $F$, $\pi$ a parameter and $\alpha = \alpha' + (E, \sigma, \pi)$ be as in Lemma 4.1. Let $L_0/\kappa$ be an extension of degree $\ell$ and $\mu_0 \in L_0$. Suppose that:
\begin{itemize}
  \item $N_{L_0/\kappa}(\mu_0) = -\overline{\theta}$;
  \item $\mu r^\alpha \otimes L_0 = (E_0 \otimes L_0, \sigma_0 \otimes 1, (-1)^r \mu_0)$.
\end{itemize}

Let $L/F$ be the unramified extension of degree $\ell$ with residue field $L_0$. Then, there exists $\mu \in L$ such that:

\begin{itemize}
  \item $\mu$ a unit in the valuation ring of $L$;
  \item $\overline{\mu} = \mu_0$;
  \item $N_{L/F}(\mu) = -\theta$;
  \item $\alpha \cdot (\mu \pi^\ast) \in H^3(L, \mu_1^0)$ is unramified.
\end{itemize}

\textbf{Proof.} Since $L$ is a prime and $[L_0 : \kappa] = \ell$, $L_0 = \kappa(\mu_0)$ for any $\mu_0 \in L_0 \setminus \kappa$. Let $g(X) = X^\ell + b_{\ell-1}X^{\ell-1} + \cdots + b_1 X + b_0 \in \kappa[X]$ be the minimal polynomial of $\mu_0$ over $\kappa$. Let $a_i$ be in the valuation ring of $F$ mapping to $b_i$ and $f(X) = X^\ell + a_{\ell-1}X^{\ell-1} + \cdots + a_1 X + a_0 \in F[X]$. Suppose $\mu_0 \notin \kappa$. Then we take $\mu_0 = 0$. Since $N_{L_0/\kappa}(\mu_0) = -\overline{\theta}$, we have $b_0 = (-1)^r \overline{\theta}$. Let $a_0 = (-1)^r \theta$. Since $g(X)$ is irreducible in $\kappa[X]$, $f(X) \in F[X]$ is irreducible. Then $L = F[X]/(f)$. Let $\mu \in L$ be the class of $X$. Then the image of $\mu$ is $\mu_0$ and $N_{L/F}(\mu) = -\theta$. Suppose $\mu_0 \in \kappa$. Then $-\overline{\theta} = N_{L_0/\kappa}(\mu_0) = \mu_0^\ast$. Since $F$ is a complete discretely valued field and $\ell$ is coprime to $\text{char}(\kappa)$, there exists $\mu \in F$ which is a unit in the valuation ring of $F$ which maps to $\mu_0$ and $\mu^\ell = -\theta$.

Since $L/F$, $E/F$ and $\alpha'$ are unramified at the discrete valuation of $F$, we have $\partial L(\alpha' \cdot (\mu \pi^\ast)) = r \alpha^\ell \otimes L_0$ and $\partial L((E, \sigma, \pi) \cdot (\mu \pi^\ast)) = (E_0 \otimes L_0, \sigma_0 \otimes 1, (-1)^r \mu_0^{-1})$. Since $\alpha = \alpha' + (E, \sigma, \pi)$, we have

$$
\partial L(\alpha \cdot (\mu \pi^\ast)) = \partial L((\alpha' \cdot L) \cdot (\mu \pi^\ast)) + \partial L((E, \sigma, \pi) \cdot (\mu \pi^\ast))
$$

$$
= r \alpha^\ell \otimes L_0 + (E_0 \otimes L_0, \sigma_0 \otimes 1, (-1)^r \mu_0^{-1})
$$

$$
= 0.
$$

\textbf{Lemma 4.9.} Suppose that $\kappa$ is a local field. Let $\ell$ be a prime not equal to $\text{char}(k)$ and $n$ a power of $\ell$. Let $\alpha \in H^2(F, \mu_n)$ and $\lambda \in F^\ast$. Suppose $\lambda \not\in \pm F^\ast$, $\alpha \cdot (-\lambda) = 0$. Then $\text{ind}(\alpha \otimes F(\sqrt[k]{\lambda})) < \text{ind}(\alpha)$ and $\alpha \cdot (-\sqrt[k]{\lambda}) = 0 \in H^3(F(\sqrt[k]{\lambda}), \mu_n^0)$.

\textbf{Proof.} Since $\lambda \notin F^\ast$, and $N_{F(\sqrt[k]{\lambda})/F}(\sqrt[k]{\lambda}) = -\lambda$, we have $\text{cor}_{F(\sqrt[k]{\lambda})/F}(\alpha \cdot (-\sqrt[k]{\lambda})) = \alpha \cdot (-\lambda) = 0$. Hence, by Proposition 4.6, $\alpha \cdot (-\sqrt[k]{\lambda}) = 0 \in H^3(F(\sqrt[k]{\lambda}), \mu_n^0)$.

Suppose $r = \nu(\lambda)$ is coprime to $\ell$. Then, by Lemma 4.7, we have $\text{ind}(\alpha \otimes F(\sqrt[k]{\lambda})) < \text{ind}(\alpha)$.

Suppose that $\nu(\lambda)$ is divisible by $\ell$. Write $\lambda = \theta \pi^{\ell d}$, with $\theta \in F$ a unit in the valuation ring of $F$. Since $\lambda \not\in \pm F^\ast$, $\theta \notin F^\ast$.

Write $\alpha = \alpha' + (E, \sigma, \pi)$ as in Lemma 4.1. Then $\text{ind}(\alpha) = \text{ind}(\alpha' \otimes E)[E : F]$ (cf. Lemma 4.2) and $\text{ind}(\alpha \otimes F(\sqrt[k]{\lambda})) < \text{ind}(\alpha' \otimes E)[E : F(\sqrt[k]{\lambda})]$. (cf. Lemma 4.2), $\text{ind}(\alpha \otimes F(\sqrt[k]{\lambda})) = \text{ind}(\alpha' \otimes E)[E : F(\sqrt[k]{\lambda})] = \text{ind}(\alpha' \otimes E)[E : F] / \ell < \text{ind}(\alpha)$.

Suppose $\alpha' \otimes E = 0$. Then, by Lemma 4.4, $\alpha = (E, \sigma, u \pi)$ for some unit $u$ in the valuation ring of $F$. Since $\alpha \cdot (-\lambda) = 0$, $\alpha = (E, \sigma, u \pi)$. Since $E/F$ is unramified with residue field $E_0$, $u, \theta$ are units in the valuation ring of $F$ and $\pi$ is a parameter, by taking the residue of $\alpha \cdot (-\lambda) = 0$, we see that $(E_0, \sigma_0, -(-1)^{\ell d} \overline{\theta}^{-\ell d} \pi^d) = 0 \in H^2(\kappa, \mu_n)$ (cf. Lemma 4.7). In particular, $-(-1)^{\ell d} \overline{\theta}^{-\ell d} \pi^d$ is a norm from $E_0$. Since $[E_0 : \kappa]$ is a power of $\ell$ and $E_0/\kappa$ is cyclic, there exists a subextension $L_0$ of $E_0$ such that $[L_0 : \kappa] = \ell$. Then $-(-1)^{\ell d} \overline{\theta}^{-\ell d}$ is a norm from $L_0$ and hence $-\overline{\theta}$ is a norm from $L_0$. Since $\pm \overline{\theta}$ is not in $K^{\ell d}$, by Lemma 2.5, $L_0 = \kappa(\sqrt[k]{\theta})$. In particular, $\sqrt[k]{\theta} \in E_0$ and hence $\sqrt[k]{\theta} \in E$. Also $\text{ind}(\alpha \otimes F(\sqrt[k]{\lambda})) < \text{ind}(\alpha)$.

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Suppose that $\sqrt{\theta} \not\in E$. Then, as above, $\alpha' \otimes E \neq 0$. Since $E$ is an unramified extension of $F$ and $\theta$ is a unit in the valuation ring of $E$, $E(\sqrt{\theta})$ is an unramified extension of $F$ with residue field $E_0(\sqrt{\theta})$, where $E_0$ is the residue field of $E$ and $\overline{\theta}$ is the image of $\theta$ in the residue field. Since $F$ is a complete discretely valued field and $\theta$ is not an $\ell$th power in $E$, $\overline{\theta}$ is not an $\ell$th power in $E_0$ and $[E_0(\sqrt{\theta}) : E_0] = \ell$. Since $\alpha' \otimes E \neq 0$, $\overline{\alpha'} \otimes E_0 \neq 0$. Since $E_0$ is a local field and $\text{ind}(\overline{\alpha'})$ is a power of $\ell$, $\text{ind}(\overline{\alpha'} \otimes E_0(\sqrt{\theta})) < \text{ind}(\overline{\alpha'} \otimes E_0)$ [CF67, p. 131]. Hence $\text{ind}(\alpha' \otimes E(\sqrt{\theta})) < \text{ind}(\alpha' \otimes E)$ and $\text{ind}(\alpha \otimes F(\sqrt{\theta})) < \text{ind}(\alpha)$ (cf. Lemma 4.2). □

**Lemma 4.10.** Suppose $\kappa$ is a local field. Let $\ell$ be a prime not equal to $\text{char}(\kappa)$ and $n = \ell^d$. Let $\alpha \in H^2(F, \mu_n)$ and $\lambda \in F^*$. Suppose that $\kappa$ contains a primitive $\ell$th root of unity. If $\alpha \neq 0$ and $\alpha \cdot (-\lambda) = 0 \in H^3(F, \mu_{n^2}^{2\ell})$, then there exist a cyclic field extension $L/F$ of degree $\ell$ and $\mu \in L^*$ such that $N_{L/F}(\mu) = -\lambda$, $\text{ind}(\alpha \otimes L) < \text{ind}(\alpha)$ and $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{2\ell})$. Further, if $\nu(\lambda)$ is divisible by $\ell$, then one can choose $L/F$ unramified.

**Proof.** Suppose $\lambda \not\in \pm F^{\ell \times}$. Let $L = F(\sqrt{\lambda})$ and $\mu = -\sqrt{\lambda}$. Then, by Lemma 4.9, $\text{ind}(\alpha \otimes L) < \text{ind}(\alpha)$ and $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{2\ell})$. Clearly $N_{L/F}(\mu) = -\lambda$, and if $\nu(\lambda)$ is a multiple of $\ell$, then $L/F$ is unramified.

Suppose $\lambda \in F^{\ell \times}$ or $-\lambda \in F^{\ell \times}$. Write $\alpha = \alpha' + (E, \sigma, \pi)$ as in Lemma 4.1.

Suppose that $\alpha' \otimes E = 0$. Then, by Lemma 4.4, $\alpha = (E, \sigma, u\pi)$ for some $u \in F^*$ which is a unit in the valuation ring of $F$. Since $\alpha \neq 0$, $E \neq F$. Let $L$ be the unique subfield of $E$ with $L/F$ of degree $\ell$. Then $\text{ind}(\alpha \otimes L) < \text{ind}(\alpha)$.

Suppose $-\lambda \in F^{\ell \times}$. Then $-\lambda = \mu^\ell$ for some $\mu \in F^*$ and $N_{L/F}(\mu) = \mu^\ell = -\lambda$. Since $\text{cor}_{L/F}(\alpha \cdot (\mu)) = \alpha \cdot (\mu^\ell) = \alpha \cdot (-\lambda) = 0$, by Proposition 4.6, we have $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{2\ell})$.

Suppose $-\lambda \not\in F^{\ell \times}$, $\ell = 2$ and $-1 \not\in F^{\ell \times}$ with $\nu(-1) = 0$. Since $\alpha \cdot (-\lambda) = 0$ and $\alpha = (E, \sigma, u\pi)$, by taking the residue of $\alpha \cdot (-\lambda)$, we see that $(E_0, \sigma_0) \cdot (-\pi^2\overline{\theta}^{-2}) = 0$. In particular, $-u^2\theta^{-2}$ is a norm from $E$. Thus $-1$ is a norm from $L$. Let $v \in L$ such that $N_{L/F}(v) = -1$ and $\mu = v\theta \pi^r$. Then $N_{L/F}(\mu) = N_{L/F}(v)(\theta \pi^r)^2 = -\lambda$. Since $\text{cor}(\alpha \cdot (\mu)) = \alpha \cdot (-\lambda) = 0 \in H^3(F, \mu_n^{2\ell})$, $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{2\ell})$ (cf. Proposition 4.6).

Suppose that $\alpha' \otimes E \neq 0$. Let $E_0$ be the residue field of $E$. Then $E_0/\kappa$ is a cyclic field extension of $\kappa$ of degree equal to the degree of $E/\ell$. Let $\overline{\alpha'}$ be the image of $\alpha'$ in $H^2(\kappa, \mu_n)$. Since $\lambda \in F^{\ell \times}$ or $-\lambda \in F^{\ell \times}$, $-\lambda = \epsilon \theta^r\pi^s$ with $\epsilon = \pm 1$ and $\theta \in F^*$ a unit at $\nu$. Since $E$ is a complete discretely valued field, $\overline{\alpha'} \otimes E_0 \neq 0$. Since $\kappa$ is a local field and contains a primitive $\ell$th root of unity, there exist a cyclic extension $L_0/\kappa$ of degree $\ell$ and $\mu_0 \in L_0$ such that $N_{L_0/\kappa}(\mu_0) = \epsilon \theta^r$ (cf. the proof of Lemma 2.8). Let $L/F$ be the unramified extension of degree $\ell$ with residue field $L_0$. Since $F$ is complete, $\theta^r \in N_{L/F}(L^*)$. Let $\mu' \in L^*$ such that $N_{L/F}(\mu') = \theta^r$ and $\mu = \mu' \pi^r$. Then $N_{L/F}(\mu) = -\lambda$. Suppose that $L_0 \not\subset E_0$. Since $\kappa$ is a local field, $\text{ind}(\overline{\alpha'} \otimes E_0 \otimes L_0) < \text{ind}(\overline{\alpha} \otimes E_0)$. Since $E$ is a complete discretely valued field with residue field $E_0$, $\text{ind}(\alpha \otimes E \otimes L) < \text{ind}(\alpha \otimes E)$.

Suppose that $L_0 \subset E_0$. Then $L \subset E$. Since $L/F$ is unramified, $\partial(\alpha \otimes L) = \partial(\alpha) \otimes L_0$ (cf. [Col95, Proposition 3.3.1]) and hence the decomposition $\alpha = \alpha' \otimes L + (E \otimes L, \sigma \otimes 1, \pi)$ is as in Lemma 4.1. Thus, by Lemma 4.2, $\text{ind}(\alpha \otimes L) < \text{ind}(\alpha)$. Since $-\lambda = N_{L/F}(\mu)$, as above, we have $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{2\ell})$.

**Lemma 4.11.** Suppose that $\kappa$ is a global field. Let $\ell$ be a prime not equal to $\text{char}(\kappa)$ and $n = \ell^d$. Suppose that either $n$ is odd or $\kappa$ has no real places. Let $\alpha \in H^2(F, \mu_n)$ and $\lambda \in F^*$. If $\alpha \neq 0$ and $\alpha \cdot (-\lambda) = 0 \in H^3(F, \mu_n^{2\ell})$, then there exist a field extension $L/F$ of degree $\ell$ and $\mu \in L^*$ such that $N_{L/F}(\mu) = -\lambda$, $\text{ind}(\alpha \otimes L) < \text{ind}(\alpha)$ and $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{2\ell})$. □
Proof. Suppose that \( \nu(\lambda) \) is coprime to \( \ell \). Then, by Lemma 4.7, \( L = F(\sqrt[n]{\lambda}) \) and \( \mu = -\sqrt[n]{\lambda} \) has the required properties.

Suppose that \( \nu(\lambda) \) is divisible by \( \ell \). Let \( \pi \) be a parameter in \( F \). Then \( \lambda = \theta \pi^{\ell} \) with \( \nu(\theta) = 0 \).

Write \( \alpha = \alpha' + (E, \sigma, \pi) \) as in Lemma 4.1. Let \( \overline{\alpha}' \) be the image of \( \alpha' \) in \( H^2(\kappa, \mu_n) \) and \( \theta_0 \) the image of \( \theta \) in \( \kappa \). Since \( \alpha' \cdot (\lambda) = 0 \), by Lemma 4.7, we have \( r\ell \overline{\alpha}' = (E_0, \sigma_0, (-1)^{\ell+1} \theta_0) \), where \( E_0 \) is the residue field of \( E \) and \( \sigma_0 \) induced by \( \sigma \).

Suppose that \( r\ell \overline{\alpha}' \otimes E_0 \neq 0 \). Then, by Proposition 3.4, there exist an extension \( L_0/\kappa \) of degree \( \ell \) and \( \mu_0 \in \mathcal{L}_0 \) such that \( N_{L_0/\kappa}(\mu_0) = (\ell+1)^{\ell+1} \theta_0 \) and \( \nu(\alpha') \otimes \mathcal{L}_0 = \mathcal{L}_0 \). Thus, replacing \( \mathcal{L}_0 \) by \( \mathcal{L}_0 \), we assume that \( \mathcal{L}_0 = (E_0, \sigma_0, (-1)^{\ell} \mu_0) \).

Let \( L \) be the unramified extension of \( F \) of degree \( \ell \) with residue field \( \mathcal{L}_0 \). Then, as in the last paragraph of the proof of Lemma 4.10, \( \nu(\alpha) \otimes L < \nu(\alpha) \). By Lemma 4.8, there exists \( \mu \in \mathcal{L} \) with the required properties.

\[ \square \]

**Theorem 4.12.** Let \( F \) be a complete discretely valued field with residue field \( \kappa \). Suppose that \( \kappa \) is a local field or a global field. Suppose that either \( n \) is odd or \( \kappa \) has no real places. Let \( D \) be a central simple algebra over \( F \) of period \( n \). Suppose that \( n \) is coprime to \( \text{char}(\kappa) \). Let \( \alpha \in H^3(\mathbf{F}_n) \) be the class of \( D \) and \( \lambda \in \mathbf{F}^* \). If \( \alpha \cdot (\lambda) = 0 \in H^3(\mathbf{F}_n^{\mathbf{Q}(2)}) \), then \( \lambda \) is a reduced norm from \( D \).

**Proof.** Write \( n = \ell_1^{d_1} \cdots \ell_r^{d_r} \), \( \ell_i \) distinct primes, \( d_i > 0 \), \( D = D_1 \otimes \cdots \otimes D_r \) with each \( D_i \) a central simple algebra over \( F \) of period power of \( \ell_i \) \cite[ch. V, Theorem 18]{Alb61}. Let \( \alpha_i \) be the corresponding cohomology class of \( D_i \). Since the \( \ell_i \) are distinct primes, \( \alpha_i \cdot (\lambda) = 0 \) if and only if \( \alpha \cdot (\lambda) = 0 \) and \( \lambda \) is a reduced norm from \( D \) if and only if \( \lambda \) is a reduced norm from each \( D_i \). Thus without loss of generality we assume that \( \text{per}(D) = \ell^d \) for some prime \( \ell \).

We prove the theorem by induction on the index of \( D \). Suppose that \( \text{ind}(D) = 1 \). Then every element of \( \mathbf{F}^* \) is a reduced norm from \( D \). We assume that \( \text{ind}(D) = n = \ell^d \geq 2 \).

Let \( \lambda \in \mathbf{F}^* \) with \( \alpha \cdot (\lambda) = 0 \in H^3(\mathbf{F}_n^{\mathbf{Q}(2)}) \). Let \( \rho \) be a primitive \( \ell \)-th root of unity. Since \( [F(\rho) : F] \) is coprime to \( n \), \( \lambda \) is a reduced norm from \( F \) if and only if \( \lambda \) is a reduced norm from \( D \otimes F(\rho) \). Thus, replacing \( F \) by \( F(\rho) \), we assume that \( \rho \in F \).

Since \( \kappa \) is either a local field or a global field, by Lemmas 4.10 and 4.11, there exist an extension \( L/F \) of degree \( \ell \) and \( \mu \in L^* \) such that \( N_{L/F}(\mu) = \lambda \), \( \alpha \cdot (\mu) = 0 \) and \( \text{ind}(\alpha \otimes L) < \text{ind}(\alpha) \). Thus, by induction, \( \mu \) is a reduced norm from \( D \otimes L \). Since \( N_{L/F}(\mu) = \lambda \), \( \lambda \) is a reduced norm from \( D \).

\[ \square \]

The following technical lemma is used in § 6.

**Lemma 4.13.** Let \( \kappa \) be a finite field and \( K \) a function field of a curve over \( \kappa \). Let \( u, v, w \in \kappa^* \) and \( \lambda \in K^* \). Let \( \ell \) be a prime not equal to \( \text{char}(\kappa) \) and \( \theta = wu\lambda \). If \( \kappa \) contains a primitive \( \ell \)-th root of unity and \( w \notin \kappa^{\pm \ell} \), then for \( r \geq 1 \), the element \((v, \sqrt[r]{\theta})_{\ell} \in H^2(K(\sqrt[r]{\theta}), \mu_{\ell}) \) is trivial over \( K(\sqrt[r]{\theta}, \sqrt[r]{v} + u\lambda) \).

**Proof.** Let \( L = K(\sqrt[r]{\theta}, \sqrt[r]{v} + u\lambda) \) and \( \beta = (v, \sqrt[r]{\theta})_{\ell} \). Since \( L \) is a global field, to show that \( \beta \otimes L \) is trivial, it is enough to show that \( \beta \otimes L_\nu \) is trivial for every discrete valuation \( \nu \) of \( L \). Let \( \nu \) be a
discrete valuation of $L$. Since $v \in \kappa^*$, $v$ is a unit at $\nu$. If $\theta$ is a unit at $\nu$, then $\beta \otimes L$ is unramified at $\nu$ and hence $\beta \otimes L_\nu$ is trivial. Suppose that $\theta$ is not a unit at $\nu$. Since $u$ and $w$ are units at $\nu$, $\lambda$ is not a unit. Suppose that $\nu(\lambda) > 0$. Then $v \in L_\nu^e$ and hence $\beta \otimes L_\nu$ is trivial. Suppose that $\nu(\lambda) < 0$. Then $\sqrt{u\lambda} \in L_\nu$. Since $r \geq 1$, $\theta = uv\lambda$ and $\sqrt{\theta} \in L_\nu$, we have $\sqrt{\theta} = \sqrt{uw\lambda} \in L_\nu$. Hence $\sqrt{w} \in L_\nu$. Since $w \in \kappa^* \setminus \kappa^e$, $v \in \kappa^*$ and $\kappa$ is a finite field, $\sqrt{v} \in \kappa(\sqrt{w})$. Since $\kappa(\sqrt{w}) \subseteq L_\nu$, $\beta \otimes L_\nu$ is trivial.

We end this section with the following well-known fact.

**Lemma 4.14.** Let $L/F$ be a cyclic extension of degree $n$, $\tau$ a generator of $\text{Gal}(L/F)$ and $\theta \in F^*$. If $\nu(\theta)$ is coprime to $n$ and $\text{ind}(L/F, \tau, \theta) = [L : F]$, then $[L : F] = \text{per}(\partial(L/F, \tau, \theta))$.

**Proof.** Let $\beta = (L/F, \tau, \theta)$ and $m = \text{per}(\partial(\beta))$. Since $n = [L : F] = \text{ind}(\beta)$, $m$ divides $n$. Since $\nu(\theta)$ is coprime to $n$, $F(\sqrt[n]{\theta})/F$ is a totally ramified extension of degree $m$ with residue field equal to the residue field $\kappa$ of $F$. Since $\partial(\beta \otimes F(\sqrt[n]{\theta})) = m\partial(\beta)$, $\beta \otimes F(\sqrt[n]{\theta})$ is unramified. Since $F(\sqrt[n]{\theta})/F(\sqrt[n]{\theta})$ is totally ramified and $\beta \otimes F(\sqrt[n]{\theta})$ is trivial, $\beta \otimes F(\sqrt[n]{\theta})$ is trivial (cf. Lemma 4.3). Hence $n = m$.

**5. Brauer group: complete two-dimensional regular local rings**

Let $X$ be an integral regular scheme with function field $F$. For every point $x$ of $X$, let $\cO_{X,x}$ be the regular local ring at $x$ and $\kappa(x)$ the residue field at $x$. Let $\cO_{X,x}$ be the completion of $\cO_{X,x}$ at its maximal ideal $m_x$ and $F_x$ the field of fractions of $\cO_{X,x}$. Then every codimension one point $x$ of $X$ gives a discrete valuation $\nu_x$ on $F$. Let $n \geq 1$ be an integer which is a unit on $X$. For any $d \geq 1$, the residue homomorphism $H^d(F, \mu_n^{\otimes j}) \to H^d-1(\kappa(x), \mu_n^{\otimes (j-1)})$ at the discrete valuation $\nu_x$ is denoted by $\partial_x$. An element $\alpha \in H^d(F, \mu_n^{\otimes m})$ is said to be ramified at $x$ if $\partial_x(\alpha) \neq 0$ and unramified at $x$ if $\partial_x(\alpha) = 0$. If $X = \text{Spec}(A)$ and $x$ is a point of $X$ given by $(\pi)$, $\pi$ a prime element, we also denote $F_x$ by $F_\pi$ and $\kappa(x)$ by $\kappa(\pi)$.

Throughout this section $A$ denotes a complete regular local ring of dimension 2 with residue field $\kappa$ and $F$ its field of fractions. Let $\ell$ be a prime not equal to the characteristic of $\kappa$ and $n = \ell^d$ for some $d \geq 1$. Let $m = (\pi, \delta)$ be the maximal ideal of $A$. For any prime $p \in A$, let $F_p$ be the completion of the field of fractions of the completion of the local ring $A_{(p)}$ at $p$ and $\kappa(p)$ the residue field at $p$.

**Lemma 5.1.** Let $E_\pi$ be an unramified Galois extension of $F_\pi$ of degree coprime to $\text{char}(\kappa)$. Then there exists a Galois extension $E$ of $F$ of degree $[E_\pi : F_\pi]$ which is unramified on $A$, except possibly at $\delta$ and $\text{Gal}(E/F) \simeq \text{Gal}(E_\pi/F_\pi)$. Further, if the residue field of $E_\pi$ is unramified over $\kappa(\pi)$, then $E/F$ can be chosen to be unramified on $A$.

**Proof.** Since $A$ is complete and $m = (\pi, \delta)$, $\kappa(\pi)$ is a complete discretely valued field with residue field $\kappa$ and the image $\tilde{\delta}$ of $\delta$ as a parameter. Let $E_0$ be the residue field of $E_\pi$. Then $E_0/\kappa(\pi)$ is a Galois extension with $\text{Gal}(E_0/\kappa(\pi)) \simeq \text{Gal}(E_\pi/F_\pi)$. Let $L_0$ be the maximal unramified extension of $\kappa(\pi)$ contained in $E_0$. Then $L_0$ is also a complete discretely valued field with $\tilde{\delta}$ as a parameter and $L_0/\kappa(\pi)$ is Galois. Since $E_0/L_0$ is a totally ramified extension of degree coprime to $\text{char}(\kappa)$, we have $E_0 = L_0(\sqrt[n]{v\delta})$ for some $v \in L_0$ which is a unit at the discrete valuation of $L_0$ (cf. Lemma 2.4).

Since $E_0/\kappa(\pi)$ is a Galois extension, $E_0/L_0$ is a Galois extension. Let $\kappa_0$ be the residue field of $E_0$. Then the residue field of $L_0$ is also $\kappa_0$. Since $\kappa_0$ is a Galois extension of $\kappa$ and $A$ is complete,
there exists a Galois extension $L$ of $F$ which is unramified on $A$ with residue field $\kappa_0$. Let $B$ be the integral closure of $A$ in $L$. Then $B$ is a regular local ring with residue field $\kappa_0$ (cf. [PS14, Lemma 3.1]). Let $u \in B$ be a lift of $\overline{\tau}$ in $\kappa_0$.

Let $E = L(\sqrt[n]{\tau})$. Since $L/F$ is unramified on $A$, $E/F$ is unramified on $A$, except possibly at $\delta$. In particular, $E/F$ is unramified at $\pi$ with residue field $E_0$. By construction, $[E:F] = [E_0: \kappa(\pi)]$. Hence $E \otimes F_\pi \simeq E_\pi$.

Since $L/F$ is a Galois extension which is unramified at $\pi$, we have $\text{Gal}(L/F) \simeq \text{Gal}(L_0/\kappa(\pi))$. Let $\tau \in \text{Gal}(L/F)$ and $\overline{\tau} \in \text{Gal}(L_0/\kappa(\pi))$ be the image of $\tau$. Since $E_0/\kappa(\pi)$ is Galois and $E_0 = L_0(\sqrt[n]{\tau})$, by Lemma 2.3, $E_0$ contains a primitive $n$th root of unity $\rho$ and $\tau(\sqrt[n]{\overline{\tau}}) \in E_0^\vee$. In particular, $\rho \in \kappa_0$. Since $B$ is complete with residue field $\kappa_0$, $\rho \in B$ and hence $\rho \in L \subseteq E$. Since $\tau(\sqrt[n]{\overline{\tau}}) = \tau(\sqrt[n]{\overline{\tau}})\overline{\tau}$ and $\overline{\tau}(\sqrt[n]{\overline{\tau}}) \in E_0^\vee, \overline{\tau}(\sqrt[n]{\overline{\tau}}) \in E_0^\vee$. Since $\overline{\tau}(\sqrt[n]{\overline{\tau}})$ and $\overline{\tau}(\sqrt[n]{\overline{\tau}})$ are units at the discrete valuation of $L_0$ and $E_0/L_0$ is totally ramified, $\tau(\sqrt[n]{\overline{\tau}})/\tau(\sqrt[n]{\overline{\tau}}) \in L^\circ$. Thus, by Lemma 2.3, $E/F$ is Galois. Since $E \otimes F_\pi \simeq E_\pi$, $\text{Gal}(E/F) \simeq \text{Gal}(E_\pi/F_\pi)$.

Further, if the residue field $E_0$ of $E_\pi$ is unramified, then $E_0 = L_0$ and hence $E = L$ is unramified on $A$.

Since $A$ is complete and $(\pi, \delta)$ is the maximal ideal of $A$, $A/(\pi)$ is a complete discrete valuation ring with $\delta$ as a parameter and $A/(\delta)$ is a complete discrete valuation ring with $\pi$ as a parameter. The next lemma follows from [Kat86, Proposition 1.7].

**Lemma 5.2 [Kat86, Proposition 1.7].** Let $m \geq 1$ and $\alpha \in H^m(F, \mu_n^{\otimes(m-1)})$. Suppose that $\alpha$ is unramified on $A$, except possibly at $\pi$ and $\delta$. Then

$$\partial_\delta(\partial_\pi(\alpha)) = -\partial_\pi(\partial_\delta(\alpha)).$$

Let $H^m_{nr}(F, \mu_n^{\otimes(m-1)})$ be the intersection of the kernels of the residue homomorphisms $\partial_\theta : H^m(F, \mu_n^{\otimes(m-1)}) \to H^m(\kappa(\theta), \mu_n^{\otimes(m-2)})$ for all primes $\theta \in A$. The next lemma follows from the purity theorem of Gabber.

**Lemma 5.3.** For $m = 1, 2$, we have $H^m_{nr}(F, \mu_n^{\otimes(m-1)}) \simeq H^m(F, \mu_n^{\otimes(m-1)})$. For $m \geq 3$, we have a surjection $H^m(\kappa, \mu_n^{\otimes(m-1)}) \to H^m_{nr}(F, \mu_n^{\otimes(m-1)})$. In particular, if $\kappa$ is a finite field and $m \geq 2$, then $H^m_{nr}(F, \mu_n^{\otimes(m-1)}) = 0$.

**Proof.** For $m \geq 1$, by the purity theorem of Gabber (cf. [Rio14, ch. XVI]), we have a surjection $H^m_{et}(A, \mu_n^{\otimes(m-1)}) \to H^m_{nr}(F, \mu_n^{\otimes(m-1)})$. Since $A$ is complete, we have $H^m_{et}(A, \mu_n^{\otimes(m-1)}) \simeq H^m(\kappa, \mu_n^{\otimes(m-1)})$ (cf. [Mil80, Corollary 2.7, p. 224]). Thus we have a surjection $H^m(\kappa, \mu_n^{\otimes(m-1)}) \to H^m_{nr}(F, \mu_n^{\otimes(m-1)})$. For $m = 1$ and 2, since the map $H^m_{et}(A, \mu_n^{\otimes(m-1)}) \to H^m_{nr}(F, \mu_n^{\otimes(m-1)})$ is injective (cf. [MO60, Theorem 7.2]), we have $H^m_{nr}(F, \mu_n^{\otimes(m-1)}) \simeq H^m(\kappa, \mu_n^{\otimes(m-1)})$.

Suppose $\kappa$ is a finite field and $m \geq 2$. Since $H^m(\kappa, \mu_n^{\otimes(m-1)}) = 0$ (cf. [Ser79, § 3.3 p. 80]), we have $H^m_{nr}(F, \mu_n^{\otimes(m-1)}) = 0$. 

**Lemma 5.4.** Let $1 \leq m \leq 3$ and $\alpha \in H^m(F, \mu_n^{\otimes(m-1)})$. Suppose that $\alpha$ is unramified, except possibly at $\pi$. Then there exist $\alpha_0 \in H^m(F, \mu_n^{\otimes(m-1)})$ and $\beta \in H^{m-1}(F, \mu_n^{\otimes(m-2)})$ which are unramified on $A$ such that

$$\alpha = \alpha_0 + \beta \cdot (\pi).$$

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Let $\beta_0 = \partial_n(\alpha)$. By Lemma 5.2, $\beta_0 \in H^{m-1}(\kappa(\pi), \mu_n^\otimes(m-2))$ is unramified on $A/(\pi)$. Since $A/(\pi)$ is a complete discrete valuation ring with residue field $\kappa$, we have $H^{m-1}_{nr}(\kappa(\pi), \mu_n^\otimes(m-2)) \simeq H^{m-1}(\kappa, \mu_n^\otimes(m-2))$ (cf. Lemma 5.3). Since $A$ is complete, we have $H^{m-1}_{nr}(F, \mu_n^\otimes(m-1)) \simeq H^{m-1}(\kappa, \mu_n^\otimes(m-1))$ (cf. Lemma 5.3). Thus, there exists $\beta \in H^{m-1}_{nr}(F, \mu_n^\otimes(m-1))$ which is the lift of $\beta_0$. Then $\alpha_0 = \alpha - \beta(\pi)$ is unramified on $A$. Hence $\alpha = \alpha_0 + \beta(\pi)$. \hfill \Box

**Corollary 5.5.** Let $1 \leq m \leq 3$ and $\alpha \in H^m(F, \mu_n^\otimes(m-1))$ is unramified on $A$, except possibly at $\pi$ and $\delta$. If $\alpha \otimes F_\delta = 0$, then $\alpha = 0$. In particular, if $\alpha_1, \alpha_2 \in H^m(F, \mu_n^\otimes(m-1))$ unramified on $A$, except possibly at $\pi$ and $\delta$ and $\alpha_1 \otimes F_\delta = \alpha_2 \otimes F_\delta$, then $\alpha_1 = \alpha_2$.

**Proof.** Since $\alpha \otimes F_\delta = 0$, $\alpha$ is unramified at $\delta$. Thus $\alpha$ is unramified on $A$, except possibly at $\pi$. By Lemma 5.4, we have $\alpha = \alpha_0 + \beta(\pi)$ for some $\alpha_0 \in H^m(F, \mu_n^\otimes(m-2))$ and $\beta \in H^{m-1}(F, \mu_n^\otimes(m-2))$ which are unramified on $A$. Since $\alpha \otimes F_\delta = 0$, we have $(\beta \cdot (\pi)) \otimes F_\delta = -\alpha_0 \otimes F_\delta$. Since $\beta \cdot (\pi)$ and $\alpha_0$ are unramified at $\delta$, we have $\beta \cdot (\pi) = -\alpha_0$, where the bar denotes the image over $\kappa(\delta)$. Since $\kappa(\delta)$ is a complete discrete valued field with $\pi$ as a parameter, by taking the residues, we see that the image of $\beta$ is zero in $H^{m-1}(\kappa, \mu_n^\otimes(m-2))$. Since $A$ is a complete regular local ring, $\beta = 0$ (cf. Lemma 5.3). Hence $\alpha = \alpha_0$ is unramified on $A$. Let $\alpha' \in H^m(\kappa, \mu_n^\otimes(m-1))$ which maps to $\alpha$ (cf. Lemma 5.3). Let $\mathcal{A}_\delta$ be the completion of the localization of $A$ at $\delta$. Since $\mathcal{A}_\delta$ is a complete discrete valuation ring, the natural map $H^n_{et}(\mathcal{A}_\delta, \mu_n^\otimes(m-1)) \rightarrow H^m(F_\delta, \mu_n^\otimes(m-1))$ is injective [Col95, §3.6]. Thus, since $\alpha \otimes F_\delta = 0$, $\alpha' \otimes \mathcal{A}_\delta = 0 \in H^n_{et}(\mathcal{A}_\delta, \mu_n^\otimes(m-1))$. In particular, $\alpha' \otimes A/(\delta) = 0 \in H^m_{et}(A/(\delta), \mu_n^\otimes(m-1))$ and hence $\alpha' \otimes \kappa = 0 \in H^m(\kappa, \mu_n^\otimes(m-1))$. Since $A$ is a complete regular local ring, $\alpha' = 0$ (cf. [Mil80, Corollary 2.7, p. 224]) and hence $\alpha = 0$. \hfill \Box

If $\text{char}(F) = \text{char}(\kappa)$, the above corollary follows from [Hu17, Lemma 2.2].

**Corollary 5.6.** Let $1 \leq m \leq 3$ and $\alpha \in H^m(F, \mu_n^\otimes(m-1))$. If $\alpha$ is unramified on $A$, except possibly at $\pi$ and $\delta$, then $\text{per}(\alpha) = \text{per}(\alpha \otimes F_\pi) = \text{per}(\alpha \otimes F_\delta)$.

**Proof.** Suppose $t = \text{per}(\alpha \otimes F_\delta)$. Then $t \alpha \otimes F_\delta = 0$ and hence, by Corollary 5.5, $t \alpha = 0$. Since $\text{per}(\alpha \otimes F_\delta) \leq \text{per}(\alpha)$, it follows that $\text{per}(\alpha) = \text{per}(\alpha \otimes F_\delta)$. Similarly, $\text{per}(\alpha) = \text{per}(\alpha \otimes F_\pi)$. \hfill \Box

**Corollary 5.7.** Suppose that $\kappa$ is a finite field. Let $\alpha \in H^2(F, \mu_n)$. If $\alpha$ is unramified, except at $\pi$ and $\delta$, then there exist a cyclic extension $E/F$ and $\sigma \in \text{Gal}(E/F)$ a generator, $u \in A$ a unit, and $0 \leq i, j < n$ such that $\alpha = (E, \sigma, u^{\pi^i}\delta^j)$ with $E/F$ unramified on $A$, except at $\pi$ and $\delta$ and $i = 1$, or $E/F$ unramified on $A$, except at $\pi$ and $j = 1$.

**Proof.** Since $n$ is a power of the prime $\ell$ and $\eta = 0$, $\text{per}(\partial_\pi(\alpha))$ and $\text{per}(\partial_\delta(\alpha))$ are powers of $\ell$. Let $d'$ be the maximum of $\text{per}(\partial_\pi(\alpha))$ and $\text{per}(\partial_\delta(\alpha))$. Then $\partial_\pi(d' \alpha) = d' \partial_\pi(\alpha) = 0$ and $\partial_\delta(d' \alpha) = d' \partial_\delta(\alpha) = 0$. In particular, $d' \alpha$ is unramified on $A$. Since $\kappa$ is a finite field, $d' \alpha = 0$. Hence $\text{per}(\alpha)$ divides $d'$ and $d' = \text{per}(\alpha)$. Thus $\text{per}(\alpha) = \text{per}(\partial_\pi(\alpha))$ or $\text{per}(\partial_\delta(\alpha))$.

Suppose that $\text{per}(\alpha) = \text{per}(\partial_\pi(\alpha))$. Since $\partial_\pi(\alpha \otimes F_\pi) = \partial_\pi(\alpha)$, we have $\text{per}(\partial_\pi(\alpha)) \leq \text{per}(\alpha \otimes F_\pi) \leq \text{per}(\alpha)$. Thus $\text{per}(\alpha \otimes F_\pi) = \text{per}(\partial_\pi(\alpha \otimes F_\pi))$. Let $(E_0, \sigma_0) = \partial_\pi(\alpha \otimes F_\pi)$ and $(E/\pi, \sigma)$ be the lift of $(E_0, \sigma_0)$. Then $\kappa(\pi) = [E_0 : \kappa(\pi)] = \text{per}(\partial_\pi(\alpha \otimes F_\pi)) = \text{per}(\alpha \otimes F_\pi)$. Write $\alpha \otimes F_\pi = \alpha' + (E_\pi, \sigma, \pi)$ as in Lemma 4.1. Let $\pi$ be the image of $\alpha'$ over $\kappa(\pi)$. Since $\kappa(\pi)$ is a local field and $\text{per}(\pi)$ divides $\text{per}(\alpha \otimes F_\pi) = [E_0 : \kappa(\pi)]$, we have $\pi \otimes E_0 = 0$ and hence $\alpha' \otimes E_\pi = 0$. Since $\alpha \otimes E_\pi = \alpha' \otimes E_\pi = 0$, by Lemma 4.4, we have $\alpha \otimes F_\pi = (E/\pi, \sigma, \theta \pi)$ for some cyclic unramified extension $E/\pi$ and $\theta \in F_\pi$ a unit in the valuation ring of $F_\pi$. 427
By Lemma 5.1, there exists a Galois extension $E/F$ which is unramified on $A$, except possibly at $(\delta)$, such that $E \otimes F_\pi \simeq E_\pi$. Since $E_\pi/F_\pi$ is cyclic, $E/F$ is cyclic. Since $\theta \in F_\pi$ is a unit in the valuation ring of $F_\pi$ and the residue field of $F_\pi$ is a complete discretely valued field with $\delta$ as parameter, we can write $\theta = u\delta^j \theta_0$ for some unit $u \in A$, $\theta_1 \in F_\pi$ and $0 \leq j \leq n - 1$. Then $\alpha \otimes F_\pi \simeq (E, \sigma, u\delta^j \pi) \otimes F_\pi$. Thus, by Corollary 5.5, we have $\alpha = (E, \sigma, u\delta^j \pi)$.

If $\text{per}(\alpha) = \text{per}(\partial_3(\alpha))$, then, as above, we get $\alpha = (E, \sigma, \omega^i \delta)$ for some cyclic extension $E/F$ which is unramified on $A$, except possibly at $\pi$. \hfill \Box

The following proposition is proved in [RS13, 2.4] under the assumption that $F$ contains a primitive $n$th root of unity.

**Proposition 5.8.** Suppose that $\kappa$ is a finite field. Let $\alpha \in H^2(F, \mu_n)$. If $\alpha$ is unramified on $A$, except possibly at $(\pi)$ and $(\delta)$, then $\text{ind}(\alpha) = \text{ind}(\alpha \otimes F_\pi) = \text{ind}(\alpha \otimes F_\delta)$.

**Proof.** Suppose that $\alpha$ is unramified on $A$, except possibly at $(\pi)$ and $(\delta)$. Then, by Corollary 5.7, we assume without loss of generality that $\alpha = (E/F, \sigma, \pi \delta^j)$ with $E/F$ unramified on $A$, except possibly at $\delta$. Then $\text{ind}(\alpha) \leq [E : F]$. Since $E/F$ is unramified on $A$ except possibly at $\delta$, we have $[E : F] = [E_\pi : F_\pi]$ and $\text{ind}(\alpha \otimes F_\pi) = [E_\pi : F_\pi]$. Thus $[E : F] = [E_\pi : F_\pi] = \text{ind}(\alpha \otimes F_\pi) \leq \text{ind}(\alpha) \leq [E : F]$ and hence $[E : F] = \text{ind}(\alpha \otimes F_\pi) = \text{ind}(\alpha)$. \hfill \Box

**Corollary 5.9.** Suppose that $\kappa$ is a finite field. Let $\alpha \in H^2(F, \mu_n)$. If $\alpha$ is unramified on $A$, except possibly at $(\pi)$ and $(\delta)$, then $\text{per}(\alpha) = \text{per}(\alpha \otimes F_\pi)$.

**Proof.** By Corollary 5.6, $\text{per}(\alpha) = \text{per}(\alpha \otimes F_\pi)$, and by Theorem 4.5, $\text{ind}(\alpha \otimes F_\pi) = \text{ind}(\alpha \otimes F_\delta)$. Thus $\text{per}(\alpha) = \text{ind}(\alpha \otimes F_\pi)$. By Proposition 5.8, we have $\text{ind}(\alpha) = \text{per}(\alpha)$. \hfill \Box

Let $\mathcal{X}$ be an integral regular two-dimensional scheme with field of fractions $F$. For each $x \in \mathcal{X}$, let $F_x$ denote the field of fractions of the completion of the local ring at $x$. The following proposition follows from [HHK15b].

**Proposition 5.10.** Let $\alpha \in H^2(F, \mu_n)$. Let $\phi : \mathcal{X} \rightarrow \text{Spec}(A)$ be a sequence of blow-ups and $V = \phi^{-1}(m)$. Then $\text{ind}(\alpha) = \text{l.c.m.}\{\text{ind}(\alpha \otimes F_x) \mid x \in V\}$.

**Proof.** Let $\eta$ be the generic point of an irreducible component of an exceptional curves in $\mathcal{X}$. Then, arguing as in [HHK15a, Theorems 9.2 and 9.12], we get that $\text{ind}(\alpha \otimes F_\eta) = \text{ind}(\alpha \otimes F_U)$ for some nonempty open set $U$ of the closure of $\eta$. Since $A$ is a complete regular local ring of dimension 2, the proposition follows by [HHK15b, Lemma 4.6 and Example 4.16]. \hfill \Box

We end this section with the following well-known results.

**Lemma 5.11.** Let $E/F$ be a cyclic extension of degree $\ell^d$ for some $d \geq 1$. If $E/F$ is unramified on $A$, except possibly at $\delta$, then there exist a subextension $E_{nr}$ of $E/F$ and $w \in E_{nr}$ which is a unit in the integral closure of $A$ in $E_{nr}$ such that $E_{nr}/F$ is unramified on $A$ and $E = E_{nr}(\sqrt[\ell^s w\delta})$ for some $s \geq 0$. Further, if $\kappa$ is a finite field containing a primitive $\ell$th root of unity and $0 < e < d$, then $N_{E/F}(\sqrt[\ell^s w\delta}) = w_1 \ell^{s-e}$ with $w_1 \in A$ a unit and not an $\ell$th power in $A$.

**Proof.** Let $E(\pi)$ be the residue field of $E$ at $\pi$. Since $E/F$ is unramified at $A$, except possibly at $\delta$, by Corollary 5.6 (with $m = 1$), $[E(\pi) : \kappa(\pi)] = [E : F]$. Since $E/F$ is cyclic, $E(\pi)/\kappa(\pi)$ is cyclic. As in the proof of Lemma 5.1, there exist a cyclic extension $E_0/F$ unramified on $A$ and a
unit \( w \) in the integral closure of \( A \) in \( E_0 \) such that the residue field of \( E_0(\sqrt[w]{w}) \) at \( \pi \) is \( E(\pi) \). By Corollary 5.5 (with \( m = 1 \), we have \( E \simeq E_0(\sqrt[w]{w}) \). Let \( E_{nr} = E_0 \). Then \( E_{nr} \) has the required properties. Since \( [E : F] = t^d \) and \( [E : E_{nr}] = t^f \), we have \( [E_{nr} : F] = t^f \), where \( f = d - e \).

Suppose that \( \kappa \) is a finite field and contains a primitive \( \ell \)-th root of unity. Let \( B \) be the integral closure of \( A \) in \( E_{nr} \). Then \( B \) is a complete regular local ring with residue field \( \kappa' \) a finite extension of \( \kappa \).

Let \( w_0 = N_{E_{nr}/F}(w) \in A^* \) and \( \overline{w}_0 \in \kappa^* \). Suppose that \( w_0 \in A^\ell \). Then \( \overline{w}_0 \in \kappa^\ell \). Since \( \kappa \) contains a primitive \( \ell \)-th root of unity, we have \( |\kappa^\ell/\kappa^\ell| = |\kappa^*/\kappa^\ell| = \ell \). Since it is surjective from \( \kappa' \) to \( \kappa \), the norm map induces an isomorphism from \( \kappa^\ell/\kappa^\ell \) to \( \kappa^*/\kappa^\ell \). Thus the image of \( w \) in \( \kappa' \) is an \( \ell \)-th power. Since \( B \) is a complete regular local ring, \( w \in B^\ell \). Suppose \( 0 < e < d \). Then \( \sqrt[\ell]{\delta} \in E \). Since \( E_{nr}/F \) is a nontrivial unramified extension and \( F(\sqrt[\ell]{\delta})/F \) is a nontrivial extension of \( F \) which is totally ramified at \( \delta \), we have two distinct subextensions of \( E/F \) of degree \( \ell \), in contradiction to the fact that \( E/F \) is cyclic. Hence \( w_0 \not\in A^\ell \). Further, we have \( N_{E/F}(\sqrt[w]{w}) = N_{E_{nr}/F}(w_{0}^{\ell+1}) \delta_{E} = (\ell+1)^{\ell} w_{0} \delta_{E} \). Since \( f > 0 \), \( w_1 = (\ell+1)^{\ell} w_0 \) is not an \( \ell \)-th power in \( A \).

\[ \square \]

**Lemma 5.12.** Suppose \( \kappa \) is a perfect field. Let \( L_{\pi}/F_{\pi} \) be an unramified field extension of degree \( N \). Then there exists a field extension \( L/F \) of degree \( N \) such that \( L \otimes F_{\pi} \simeq L_{\pi} \) and the integral closure of \( A \) in \( L \) is regular.

**Proof.** Let \( L(\pi) \) be the residue field of \( L_{\pi} \). Suppose that \( L(\pi)/\kappa(\pi) \) is unramified at the discrete valuation of \( A/\kappa(\pi) \). Let \( \kappa' \) be the residue field of \( L(\pi) \). Then \( \kappa'/\kappa \) is an extension of degree \( N \). Write \( \kappa' = \kappa[T]/(f(T)) \) for some monic polynomial. Let \( g(T) \in A[T] \) be a monic polynomial which is a lift of \( f(T) \). Then clearly \( L = F[T]/(g(T)) \) has the required properties.

Suppose \( L(\pi)/\kappa(\pi) \) is ramified. Let \( L(\pi)_{nr} \) be the maximal unramified extension of \( \kappa(\pi) \) contained in \( L(\pi) \). Let \( \tilde{L}_{\pi} \) be the subextension of \( L_{\pi} \) with residue field \( L(\pi)_{nr} \). Then, as above, there exists a field extension \( \tilde{L}/F_{\pi} \) such that \( \tilde{L} \otimes F_{\pi} \simeq \tilde{L}_{\pi} \). Let \( \tilde{A} \) be the integral closure of \( A \) in \( \tilde{L} \). Then \( \tilde{A} \) is a regular local ring with \( (\pi, \delta) \) as the maximal ideal. Thus, replacing \( F \) by \( \tilde{L}_{\pi} \), we assume that \( L(\pi)/\kappa(\pi) \) is totally ramified. Hence \( L(\pi) = \kappa(\pi)[T]/(f(T)) \) with \( f(T) = T^N + a_{N-1}T^{N-1} + \cdots + a_1 T + a_0 \in A[T] \) and a unit \( v \in A \), where the bar denotes the image in \( A/\kappa(\pi) \). Let \( g(T) = T^N + a_{N-1}T^{N-1} + \cdots + a_1 T + a_0 \in A[T] \). Let \( L = F[T]/(g(T)) \) and \( B = A[T]/(g(T)) \). Let \( \tilde{m} \) be a maximal ideal of \( B \). Let \( t \) be the image of \( T \) in \( B \). We have \( t(t^{N-1} + a_{N-1}T^{N-2} + \cdots + a_1 T + a_0) = -v \delta \). Since \( \delta \in m \subset \tilde{m} \), it follows that \( t \in \tilde{m} \). Since \( B/(\pi, t) \simeq \kappa \), \( \tilde{m} = (\pi, t) \) is the unique maximal ideal of \( B \) and hence \( B \) is a regular local ring. In particular, \( B \) is integrally closed and hence \( B \) is the integral closure of \( A \) in \( L \).

\[ \square \]

**Remark 5.13.** Let \( L_{\pi}/F_{\pi} \) be an unramified extension of degree \( N \) and \( L/F \) be the extension of degree \( N \) as in the proof of Lemma 5.12. Let \( B \) be the integral closure of \( A \) in \( L \). Then, by the construction of \( L, (\pi, \delta) \) is the maximal ideal of \( B \) for some \( \delta' \in B \) such that \( \delta' \) is the only prime in \( B \) lying over \( \delta \) and \( N_{L/F}(\delta') = v \delta^f \) for some unit \( v \in A \) and \( f \geq 1 \).

### 6. Reduced norms: complete two-dimensional regular local rings

Throughout this section we fix the following notation:

- \( A \) a complete two-dimensional regular local ring;
- \( F \) the field of fractions of \( A \);
- \( m = (\pi, \delta) \) the maximal ideal of \( A \);
\( \bullet \ \kappa = A/m \) a finite field;
\( \bullet \ \ell \) a prime not equal to \( \text{char}(\kappa); \)
\( \bullet \ n = \ell^d. \)
\( \bullet \ \alpha \in H^2(F, \mu_n) \) is unramified on \( A, \) except possibly at \( (\pi) \) and \( (\delta); \)
\( \bullet \ \lambda = w\pi^s\delta^t, \ w \in A \) a unit and \( s, t \in \mathbb{Z} \) with \( 1 \leq s, t < n. \)

The aim of this section is to prove that if \( \alpha \neq 0 \) and \( \alpha \cdot (\lambda) = 0, \) then there exist an extension \( L/F \) of degree \( \ell \) and \( \mu \in L \) such that \( \text{ind}(\alpha \otimes L) < \text{ind}(\alpha) \) and \( N_{L/F}(\mu) = \lambda. \) We assume that:

\( \bullet \ \ F \) contains a primitive \( \ell \)th root of unity.

We begin with the following lemma.

**Lemma 6.1.** If \( \alpha \cdot (\lambda) = 0, \) then \( s\alpha = (E, \sigma, (-1)^{s+1}\lambda) \) for some cyclic extension \( E \) of \( F \) which is unramified on \( A, \) except possibly at \( \delta. \) In particular, if \( s \) is coprime to \( \ell, \) then \( \alpha = (E', \sigma', (-1)^{s+1}\lambda) \) for some cyclic extension \( E' \) of \( F \) which is unramified on \( A, \) except possibly at \( \delta. \)

**Proof.** By Lemma 4.7, there exists an unramified cyclic extension \( E_\pi \) of \( F \) such that \( s\alpha \otimes F_\pi = (E_\pi, \sigma, (-1)^{s+1}\lambda). \) By Lemma 5.1, there exists a cyclic extension \( E \) of \( F \) which is unramified on \( A, \) except possibly at \( \delta. \) Since \( E/F \) is unramified on \( A, \) except possibly at \( \delta \) and \( \lambda = w\pi^s\delta^t \) with \( w \) a unit in \( A, \) \( (E, \sigma, (-1)^{s+1}\lambda) \) is unramified on \( A, \) except possibly at \( (\pi) \) and \( (\delta). \) Since \( \alpha \) is unramified on \( A, \) except possibly at \( (\pi) \) and \( (\delta), \) \( s\alpha - (E, \sigma, (-1)^{s+1}\lambda) \) is unramified on \( A, \) except possibly at \( (\pi) \) and \( (\delta). \) Since \( s\alpha \otimes F_\pi = (E_\pi, \sigma, (-1)^{s+1}\lambda) \) is unramified on \( A, \) except possibly at \( (\pi) \) and \( (\delta), \) \( \alpha \) is unramified on \( A, \) except possibly at \( (\pi) \) and \( (\delta). \) In particular, \( \text{ind}(\alpha \otimes L) < \text{ind}(\alpha) \) and \( \alpha \cdot (\sqrt[\ell]{\lambda}) = 0. \) □

**Lemma 6.2.** Suppose that \( \alpha \cdot (\lambda) = 0 \) and \( \lambda \notin \pm F^{\ell^d}. \) If \( \alpha \neq 0, \) then \( \text{ind}(\alpha \otimes F(\sqrt[\ell]{\lambda})) < \text{ind}(\alpha) \) and \( \alpha \cdot (\sqrt[\ell]{\lambda}) = 0. \)

**Proof.** Suppose that \( s \) is coprime to \( \ell. \) Then, by Lemma 6.1, \( \alpha = (E', \sigma', (-1)^{s+1}\lambda) \) for some cyclic extension \( E' \) of \( F \) which is unramified on \( A, \) except possibly at \( \delta. \) Since \( \nu_\pi(\lambda) = s \) is coprime to \( \ell \) and \( E'/F \) is unramified at \( \pi, \) it follows that \( \text{ind}(\alpha) = [E' : F]. \) In particular, \( \text{ind}(\alpha \otimes F(\sqrt[\ell]{(-1)^{s+1}\lambda})) \) is an unramified cyclic extension \( A, \) except possibly at \( (\pi) \) and \( (\delta). \) Since \( s \) is coprime to \( \ell, \) we have \( (\ell)^s = -(\epsilon)^s \) for some \( \epsilon \in \pm 1 \) and hence \( F(\sqrt[\ell]{(-1)^{s+1}\lambda}) = F(\sqrt[\ell]{\lambda}). \) Similarly, if \( t \) is coprime to \( \ell, \) then \( \text{ind}(\alpha \otimes F(\sqrt[\ell]{\lambda})) < \text{ind}(\alpha) \) and \( \alpha \cdot (\sqrt[\ell]{\lambda}) = (E', \sigma', (-1)^{s+1}\lambda) \cdot (\sqrt[\ell]{\lambda}) = 0. \)

Suppose that \( s \) and \( t \) are divisible by \( \ell. \) Since \( \lambda = w\pi^s\delta^t \), we have \( F(\sqrt[\ell]{\lambda}) = F(\sqrt[\ell]{\lambda}). \) Let \( L = F(\sqrt[\ell]{\lambda}) \) and \( B \) be the integral closure of \( A \) in \( L. \) Since \( w \) is a unit in \( A, \) by [PS14, Lemma 3.1], \( B \) is a complete regular local ring with maximal ideal generated by \( \pi \) and \( \delta \). Since \( \lambda \notin \pm F^{\ell^d} \) and \( A \) is a complete regular local ring, the images of \( \pm w \) in \( A/m \) are not \( \ell \)th powers. Since \( A/(\pi) \) is also a complete regular local ring with residue field \( A/m, \) the images of \( \pm w \) in \( A/(\pi) \) are not \( \ell \)th powers. Since \( F_\pi \) is a complete discretely valued field with residue field the field of fractions of \( A/\pi \), the images of \( \pm w \) in \( F_\pi \) are not \( \ell \)th powers. Since \( \text{cores}(\alpha \cdot (\sqrt[\ell]{\lambda})) = \text{cores}(\alpha \cdot (\lambda)) = 0, \) by Proposition 4.6, \( \text{cores}(\alpha \cdot (\sqrt[\ell]{\lambda})) \otimes L_\pi = 0 \) and \( \text{cores}(\alpha \cdot (\sqrt[\ell]{\lambda})) \otimes L_\delta = 0. \) Hence, by Corollary 5.5, \( \alpha \cdot (\sqrt[\ell]{\lambda}) = 0. \)

**Lemma 6.3.** Suppose \( \alpha = (E/F, \sigma, w\pi^d\delta^m) \) for some \( m \geq 0, \) \( u \) a unit in \( A, \) \( E/F \) a cyclic extension of degree \( \ell^d \) which is unramified on \( A, \) except possibly at \( \delta, \) and \( \sigma \) a generator of \( \text{Gal}(E/F). \) Let \( \ell' \) be the ramification index of \( E/F \) at \( \delta \) and \( f = d - e. \) Let \( i \geq 1 \) be such that \( \ell' + \ell di > \ell m. \) Let \( \nu \in A \) be a unit which is not in \( F^{\ast \ell} \) and \( L = F^{\ell'}(\nu\ell^{f}/\ell^{m} - \ell^{m} + u\pi). \) If \( f > 0, \) then \( \text{ind}(\alpha \otimes L) < \text{ind}(\alpha). \)
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Proof. Let \( B \) be the integral closure of \( A \) in \( L \) and \( r = \ell^f + \ell^{di} - \ell m \). Since \( \ell^f + \ell^{di} > \ell m \), \( L = F(\sqrt[\ell^f]{\nu} + u\pi) \) and \( v\nu^r + u\pi \) is a regular prime in \( A \). Thus \( B \) is a complete regular local ring (cf. [PS14, Lemma 3.2]) and \( \pi \), \( \delta \) remain primes in \( B \). Note that \( \pi \) and \( \delta \) may not generate the maximal ideal of \( B \). Let \( L_\pi \) and \( L_\delta \) be the completions of \( L \) at the discrete valuations given by \( \pi \) and \( \delta \), respectively. Since \( v \not\in F^{\ell^f} \), \( F(\sqrt[\ell^f]{\nu}) \) is the unique extension of \( F \) of degree \( \ell \), which is unramified on \( A \). Since \( f > 0 \), there is a subextension of \( E \) of degree \( \ell \) over \( F \) which is unramified on \( A \) and hence \( F(\sqrt[\ell^f]{\nu}) \subset E \).

Since \( E/F \) is unramified on \( A \), except possibly at \( \delta \), \( [E : F] = [E_\pi : F_\pi] \) and hence \( \text{ind}(\alpha) = \text{per}(\alpha) = [E : F] \) (Proposition 5.8).

Since \( r \) is divisible by \( \ell \), \( L_\pi \cong F_\pi(\sqrt[\ell]{\nu}) \) and hence \( L_\pi \subset E_\pi \). Thus \( \text{ind}(\alpha \otimes L_\pi) < \text{ind}(\alpha) \). Since \( r > 0 \), \( L_\delta \cong F_\delta(\sqrt[\ell]{\nu}) \). Since \( \alpha = (E/F, \sigma, u\pi\delta^{\ell m}) \), \( \text{ind}(\alpha \otimes L_\delta) < [E \otimes L_\delta] \subset [E : F] \). In particular, \( \text{per}(\alpha \otimes L_\pi) < \text{ind}(\alpha) \) and \( \text{per}(\alpha \otimes L_\delta) < \text{ind}(\alpha) \). Since \( \alpha \otimes L \) is unramified on \( B \), except possibly at \( \pi \) and \( \delta \), and \( H^2(B, \mu_\ell) = 0 \), \( \text{per}(\alpha \otimes L) < \text{ind}(\alpha) \). If \( f = 1 \), then \( \text{per}(\alpha \otimes L) < \text{ind}(\alpha) = \ell \) and hence \( \text{per}(\alpha \otimes L) = \text{ind}(\alpha \otimes L) = 1 < \text{ind}(\alpha) \). Suppose that \( d \geq 2 \).

Let \( \phi : \mathcal{X} \to \text{Spec}(B) \) be a sequence of blow-ups such that the ramification locus of \( \alpha \otimes \Delta \) is a union of regular curves with normal crossings. Let \( V = \phi^{-1}(P) \). To show that \( \text{ind}(\alpha \otimes L) < \text{ind}(\alpha) \), by Proposition 5.10, it is enough to show that for every point \( x \) of \( V \), \( \text{ind}(\alpha \otimes L_x) < \text{ind}(\alpha) \).

Let \( x \in V \) be a closed point. Then, by Corollary 5.9, \( \text{ind}(\alpha \otimes L_x) = \text{per}(\alpha \otimes L_x) \). Since \( \text{per}(\alpha \otimes L_x) < \text{ind}(\alpha) \), \( \text{ind}(\alpha \otimes L_x) < \text{ind}(\alpha) \).

Let \( x \in V \) be a codimension zero point. Then \( \phi(x) \) is the closed point of \( \text{Spec}(B) \). Let \( \nu \) be the discrete valuation of \( L \) given by \( x \). Then \( \kappa(\nu) \cong \kappa(\ell t) \) for some finite extension \( \kappa' \) over \( \kappa \) and a variable \( t \) over \( \kappa \). Let \( \nu \) be the restriction of \( \nu \) to \( F \).

Suppose that \( \nu(\delta^r) < \nu(\pi) \). Then \( L \otimes F_\nu = F_\nu(\sqrt[\ell^f]{nu}) \). Since \( \ell \) divides \( r \), \( L \otimes F_\nu = F_\nu(\sqrt[\ell]{nu}) \). Since \( F(\sqrt[\ell]{nu}) \subset E \), \( \text{ind}(\alpha \otimes L \otimes F_\nu) < \text{ind}(\alpha) \). Suppose that \( \nu(\delta^r) > \nu(\pi) \). Then \( L \otimes F_\nu = F_\nu(\sqrt[\ell]{\nu}) \) and, as above, \( \text{ind}(\alpha \otimes L \otimes F_\nu) < \text{ind}(\alpha) \). Suppose that \( \nu(\delta^r) = \nu(\pi) \). Let \( g = \pi/\delta^r \). Then \( g \) is a unit at \( \nu \) and \( L_\nu = F_\nu(\sqrt[\ell]{nu} + ug) \). We have \( u\pi\delta^{\ell m} = u\pi\delta^{\ell m} = u\pi\delta^{\ell f} \) and

\[
\alpha \otimes F_\nu = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, u\pi\delta^{\ell m}) = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, u\pi\delta^{\ell f} + \ell^{di}).
\]

Since \( [E : F] = \ell^d \), \( \alpha \otimes F_\nu = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, u\pi\delta^{\ell f}) \). Suppose that \( f = d \). Then \( E/F \) is unramified and hence every element of \( A^* \) is a norm from \( E \). Thus \( \alpha \otimes F_\nu = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, w_0 u\pi) \) for any \( w_0 \in A^* \). Suppose that \( f < d \). Then \( \epsilon = \ell^e - f > 0 \) and hence, by Lemma 5.11, we have \( E = E_{\ell^e}(\sqrt[\ell^e]{w\delta}) \), for some unit \( w \) in the integral closure of \( A \) in \( E_{\ell^e} \), with \( N_{E/F}(\sqrt[\ell^e]{w\delta}) = w_1^{\ell^e} \) with \( w_1 \in A^* \setminus A^{\ell^e} \). Thus

\[
\alpha \otimes F_\nu = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, u\pi\delta^{\ell f}) = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, w_0 u\pi),
\]

with \( w_0 = w_1^{-1} \). Hence, in either case, we have \( \alpha \otimes F_\nu = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, w_0 u\pi) \) with \( w_0 \not\in A^{\ell^e} \).

If \( E \otimes F_\nu \) is not a field, then \( \text{ind}(\alpha \otimes F_\nu) < [E : F] \). Suppose \( E \otimes F_\nu \) is a field. Let \( \theta = w_0 u\pi \).

Since \( \alpha \otimes F_\nu = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, \theta) \), \( \text{ind}(\alpha \otimes L \otimes F_\nu) \leq \text{ind}(\alpha \otimes L \otimes F_\nu(\ell^{\ell^e-1}\theta)) \cdot [L \otimes F_\nu(\ell^{\ell^e-1}\theta) : L \otimes F_\nu] \). Since \( [L \otimes F_\nu(\ell^{\ell^e-1}\theta) : L \otimes F_\nu] \leq \ell^{\ell^e-1} < [E : F] \), it is enough to show that \( \alpha \otimes L \otimes F_\nu(\ell^{\ell^e-1}\theta) \) is trivial.

Since \( F(\sqrt[\ell^f]{nu})/F \) is the unique subextension of \( E/F \) of degree \( \ell \) and \( [E : F] = \ell^d \), we have \( \alpha \otimes F_\nu(\ell^{\ell^e-1}\theta) = (F_\nu(\ell^{\ell^e-1}\theta, \sqrt[\ell^f]{nu})/F_\nu(\ell^{\ell^e-1}\theta), \sigma, \ell^{\ell^e-1}\theta) \) (cf. Lemma 2.1). Let \( M = F_\nu(\ell^{\ell^e-1}\theta) \). Since \( \kappa \) contains a primitive \( \ell^e \)th root of unity, we have \( \alpha \otimes M = (M, \ell^{\ell^e-1}\theta) \). Then \( M \) is a complete discretely valued field. Since \( g \) is a unit at \( \nu \), \( \theta \) is a unit at \( \nu \). Hence the residue field of \( M \)
is $\kappa(\nu)(\sqrt[d-1]{\delta})$. Since $\theta$ and $v$ are units at $\nu$, $\alpha \otimes M = (v, \sqrt[d-1]{\delta})$ is unramified at the discrete valuation of $M$. Hence it is enough to show that the specialization $\beta$ of $\alpha \otimes M$ is trivial over $\kappa(\nu)(\sqrt[d-1]{\delta}) \otimes L_0$, where $L_0$ is the residue field of $L \otimes F_\nu$ at $\nu$.

Suppose that $L_\delta/F_\nu$ is ramified. Since $L_\delta = F_\nu(\sqrt[d]{v + ug})$, $v + ug$ is not a unit at $\nu$. Thus $v = -ug$ modulo $F_\nu^{*d}$ and $\theta = w_0ug = -w_0v$ modulo $F_\nu^{*d}$. In particular, $\sqrt[d]{\delta} = \sqrt[d]{-w_0v}$ modulo $M^{*d}$. Since $\pi, \pi_0 \in \kappa$ and $\kappa$ a finite field, $\beta = (\pi, \sqrt[d-1]{\delta}) = (\pi, \sqrt[d-1]{-w_0\pi})$ is trivial.

Suppose that $L_\delta/F_\nu$ is unramified. Then $L_0 = \kappa(\nu)(\sqrt[d]{\pi + w_0\pi})$. Since $\kappa(\nu)$ is a global field of positive characteristic and $d - 1 \geq 1$, by Lemma 4.13, $\beta \otimes L_0(\sqrt[d-1]{\delta}) = 0$. □

Lemma 6.4. Suppose $L_\pi/F_\pi$ and $L_\delta/F_\delta$ are unramified cyclic field extensions of degree $l$ and $\mu_\pi \in L_\pi$, $\mu_\delta \in L_\delta$ such that:

- $-\lambda = N_{L_\pi/F_\pi}(\mu_\pi)$ and $-\lambda = N_{L_\delta/F_\delta}(\mu_\delta)$;
- $\alpha \cdot (\mu_\pi) = 0 \in H^3(L_\pi, \mu_\pi^{\otimes 2})$, $\alpha \cdot (\mu_\delta) = 0 \in H^3(L_\delta, \mu_\delta^{\otimes 2})$;
- $\alpha = 0$ or $\alpha \neq 0$, $\text{ind}(\alpha \otimes L_\pi) < \text{ind}(\alpha)$ and $\text{ind}(\alpha \otimes L_\delta) < \text{ind}(\alpha)$.

Then there exist a cyclic extension $L/F$ of degree $l$ and $\mu \in L$ such that:

- $-\lambda = N_{L/F}(\mu)$;
- $\alpha \cdot (\mu) = 0 \in H^3(L, \mu^{\otimes 2})$;
- $L \otimes F_\pi \cong L_\pi$ and $L \otimes F_\delta \cong L_\delta$;
- if $\alpha \neq 0$, then $\text{ind}(\alpha \otimes L) < \text{ind}(\alpha)$.

Proof. Since $\alpha \cdot (\mu_\pi) = 0 \in H^3(L_\pi, \mu_\pi^{\otimes 2})$ and $-\lambda = N_{L_\pi/F_\pi}(\mu_\pi)$, by taking the corestriction, we see that $\alpha \cdot (-\lambda) = 0 \in H^3(F_\pi, \mu_\pi^{\otimes 2})$. Since $\alpha \cdot (-\lambda)$ is unramified on $A$, except possibly at $\pi$ and $\delta$, by Corollary 5.5, $\alpha \cdot (-\lambda) = 0$.

Suppose that $\lambda \not\equiv \pm F^{*l}$. Then, by Lemmas 2.6 and 6.2, $L = F(\sqrt[l]{\lambda})$ and $\mu = -\sqrt[l]{\lambda}$ have the required properties.

Suppose that $\lambda \in F^{*l}$ or $-\lambda \in F^{*l}$. Let $L(\pi)$ and $L(\delta)$ be the residue fields of $L_\pi$ and $L_\delta$, respectively. Since $L_\pi/F_\pi$ and $L_\delta/F_\delta$ are unramified cyclic extensions of degree $l$, $L(\pi)/\kappa(\pi)$ and $L(\delta)/\kappa(\delta)$ are cyclic extensions of degree $l$. Since $F$ contains a primitive $l^{th}$ root of unity, we have $L(\pi) = \kappa(\pi)[X]/(X^l - a)$ and $L(\delta) = \kappa(\delta)[X]/(X^l - b)$ for some $a \in \kappa(\pi)$ and $b \in \kappa(\delta)$. Since $\kappa(\pi)$ is a complete discretely valued field with $\delta$ a parameter, without loss of generality we assume that $a = \overline{w_1\pi^{l'}}$ for some unit $w_1 \in A$ and $\epsilon = 0$ or $1$. Similarly, we have $b = \overline{w_2\pi^{l'}}$ for some unit $w_2 \in A$ and $\epsilon' = 0$ or $1$.

Suppose $\alpha = 0$. If $-\lambda \in F^{*l}$, then $L = F(\sqrt[l]{u_1\delta^{l+e} + u_2\pi^{l'+e}})$ and $\mu = \sqrt[l]{-\lambda} \in F \subset L$ have the required properties. Suppose $-\lambda \not\in F^{*l}$. Then $\lambda \in F^{*l}$ and hence $\ell = 2$ and $-1 \not\in F^{*2}$. In particular, $-1 \not\in \kappa(\pi)^{*2}$ and $-1 \not\in \kappa(\delta)^{*2}$. Since $-\lambda$ is a norm from $L_\pi$ and $L_\delta$, $-1$ is a norm from $L_\pi$ and $L_\delta$. Thus $-1$ is a norm from the extensions $L(\pi)/\kappa(\pi)$ and $L(\delta)/\kappa(\delta)$. Hence $L(\pi)/\kappa(\pi)$ and $L(\delta)/\kappa(\delta)$ are unramified and hence $\epsilon = \epsilon' = 0$. Let $L$ be the degree two extension of $F$ which is unramified on $A$. Then $-1$ is a norm from $L$. Hence there exists $\mu \in L$ such that $N_{L/F}(\mu) = -\lambda$ and $L, \mu$ have the required properties.

Suppose that $\alpha \neq 0$. Then $\text{ind}(\alpha \otimes L_\pi) < \text{ind}(\alpha)$ and $\text{ind}(\alpha \otimes L_\delta) < \text{ind}(\alpha)$.

By Corollary 5.7, we assume that $\alpha = (E/F, \sigma, \varpi \delta^j)$ for some cyclic extension $E/F$ which is unramified on $A$, except possibly at $\delta$, $\varpi$ a unit in $A$ and $j \geq 0$. Then $\text{ind}(\alpha) = [E : F]$. Let $E_0$ be the residue field of $E$ at $\pi$. Then $[E : F] = [E_0 : \kappa(\pi)]$. Since $\partial_\pi(\alpha) = (E_0/\kappa(\pi), \sigma)$, per$(\partial_\delta(\alpha)) = [E : F] = \text{ind}(\alpha)$. Since $L_\pi/F_\pi$ is an unramified extension of degree $l$, $\pi$ is a parameter in $L_\pi$ and hence $\text{ind}(\alpha \otimes L_\pi) = [E_{L_\pi} : L_\pi]$. Since $\text{ind}(\alpha \otimes L_\pi) < \text{ind}(\alpha) = [E_{L_\pi} : F_\pi]$, $[E_{L_\pi} : L_\pi] < [E_{L_\pi} : F_\pi]$ and hence $L_\pi \subseteq E_{L_\pi}$. Thus the residue field $L(\pi)$ of $L_\pi$ is the unique subextension of $E_0/\kappa(\pi)$ of degree $l$.  

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Suppose that $\epsilon = \epsilon' = 0$. Since $L_\pi$ and $L_\delta$ are fields, $u_1$ and $u_2$ are not $\ell$th powers. Let $L/F$ be the unique cyclic field extension of degree $\ell$ which is unramified on $A$. Then $L \otimes F_{\pi} \simeq L_\pi$ and $L \otimes F_{\delta} \simeq L_\delta$. Let $B$ be the integral closure of $A$ in $L$. Then $B$ is a regular local ring with maximal ideal $(\pi, \delta)$ and hence, by Proposition 5.8, $\text{ind}(\alpha \otimes L) < \text{ind}(\alpha)$.

Suppose $\epsilon = 1$. Then $L_{\pi} = F_{\pi}(\sqrt[\ell]{u_1 \delta})$ and $L(\pi) = \kappa(\pi)(\sqrt[\ell]{u_1 \delta})$. Since $E_0/\kappa(\pi)$ is a cyclic extension containing a totally ramified extension, $E_0/\kappa(\pi)$ is a totally ramified cyclic extension. Thus $\kappa(\pi)$ contains a primitive $\ell^d$th root of unity and $E_0 = \kappa(\pi)(\sqrt[\ell]{u_1 \delta})$ (cf. Lemmas 2.3 and 2.4). In particular, $F$ contains a primitive $\ell^d$th root of unity and $\alpha = (u_1 \delta, u \pi \delta^j) = (u_1 \delta, u' \pi)$. Then $\delta_0(\alpha) = \kappa(\delta)(\sqrt[\ell]{u' \pi})$. Since $L_{\delta}/F_{\delta}$ is an unramified extension of degree $\ell$ with $\text{ind}(\alpha \otimes L_{\delta}) < \text{ind}(\alpha)$, the residue field $L(\delta)$ of $L_{\delta}$ is the unique subfield of $\kappa(\delta)(\sqrt[\ell]{u' \pi})$ of degree $\ell$ over $\kappa(\delta)$. Hence $L(\delta) = \kappa(\delta)(\sqrt[\ell]{u' \pi})$. Since $L(\delta) = \kappa(\delta)(\sqrt[\ell]{u \pi \delta^j})$, we have $\epsilon' = 1$ and $u' = u_2$ modulo $F^{\ell \delta}$. Hence $\alpha = (u_1 \delta, u_2 \pi)$. Let $L = F(\sqrt[\ell]{u_1 \delta} + u_2 \pi)$. Then $L \otimes F_{\pi} \simeq L_\pi$ and $L \otimes F_{\delta} \simeq L_\delta$. Since for any $a, b \in F^*$, $(a, b) = (a+b, -a^{-1}b)$, we have $\alpha = (u_1 \delta + u_2 \pi, -u_1^{-1} \delta - u_2 \pi)$. In particular, $\text{ind}(\alpha \otimes L) < \text{ind}(\alpha)$.

Suppose that $\epsilon = 0$ and $\epsilon' = 1$. Suppose $j$ is coprime to $\ell$. Then, by Lemma 4.14, $\text{ind}(\alpha) = \text{per}(\delta_0(\alpha))$, and, as in the proof of Corollary 5.7, we have $\alpha = (E'/F, \sigma', v \delta \pi^j)$ for some cyclic extension $E'/F$ which is unramified on $A$, except possibly at $\pi$. Thus, we have the required extension as in the case $\epsilon = 1$.

Suppose $j$ is divisible by $\ell$. Since $\epsilon = 0$, $L_\pi = F_{\pi}(\sqrt[\ell]{u_1})$. Since the residue field $L(\pi)$ of $L_\pi$ is contained in the residue field $E_0$ of $E$ at $\pi$, $F(\sqrt[\ell]{u_1}) \subset E$ and hence $E/F$ is not totally ramified at $\delta$. Since $E/F$ is unramified on $A$, except possibly at $\delta$, by Lemma 5.11, $E = E_{nr}(\sqrt[\ell]{u \pi \delta^j})$ for some unit $w$ in the integral closure of $A$ in $E_{nr}$. Suppose $e = 0$. Then $E = E_{nr}/F$ is unramified on $A$. Since $\kappa$ is a finite field and $A$ is complete, every unit in $A$ is a norm from $E/F$. Thus, multiplying $u \pi \delta^j$ by a norm from $E/F$, we assume that $\alpha = (E/F, \sigma, u_2 \pi \delta^j)$. Suppose that $e > 0$. Then, by Lemma 5.11, $N_{E/F}(\sqrt[\ell]{u \pi \delta^j}) = w_1^{j \ell \delta f}$ with $w_1 \in A^* \setminus A^{\ell \delta}$. Since $A^*/A^{\ell \delta}$ is a cyclic group of order dividing $\ell^d$, we have $u^{-1}u_2 = w_1^{j'}$ modulo $A^{\ell \delta}$. In particular, $N_{E/F}(\sqrt[\ell]{u \pi \delta^j})^j = w_1^{j \ell \delta f}j' = u^{-1}u_2^{\ell \delta f}j'$ modulo $A^{\ell \delta}$. Hence, we have $\alpha = (E/F, \sigma, u_2 \pi \delta^{j+j' \ell})$ for some $j'$. Since $j$ is divisible by $\ell$ and $f \geq 1$, $j+j' \ell$ is divisible by $\ell$. Hence, we assume that $\alpha = (E/F, \sigma, u_2 \pi \delta^m)$ for some $m$. Thus, by Lemma 6.3, there exists $i \geq 0$ such that $\text{ind}(\alpha \otimes L) < \text{ind}(\alpha)$ for $L = F(\sqrt[\ell]{u \pi \delta^l + u_2 \pi \delta^m})$.

By choice, we have that $L/F$ is the unique unramified extension or $L = F(\sqrt[\ell]{u \pi \delta^l + u_2 \pi \delta^m})$ with $\ell f = m$. Let $B$ be the integral closure of $A$ in $L$. Then $B$ is a complete regular local ring with $\pi$ and $\delta$ remain prime in $B$.

Suppose $\lambda \in F^{*\ell}$. Since $\lambda = -w_0 \pi^s \delta^t$, we have $\lambda = w_0 \pi^s \delta^t$ for some unit $w_0 \in A$. Let $\mu = w_0 \pi^s \delta^t \in F$. Then $N_{L/F}(\mu) = \mu \ell = -\lambda$. Since $\alpha \cdot (-\lambda) = 0$, by Proposition 4.6, $\alpha \cdot (\mu) = 0$ in $H^3(L_{\pi}, \mu_{\ell \delta}^{\otimes 2})$ and $H^3(L_{\delta}, \mu_{\ell \delta}^{\otimes 2})$. Hence $\alpha \cdot (\mu)$ is unramified at all height one prime ideals of $B$. Since $B$ is a complete regular local ring with residue field finite, $\alpha \cdot (\mu) = 0$ (Lemma 5.3).

Suppose that $\lambda \notin F^{*\ell}$. Then $\lambda \in F^{*\ell}$, $\ell = 2$ and $-1 \notin F^{*\ell}$. Hence $-1 \notin F^{*\ell}$ and $-1 \notin F^{*\ell}$. In particular, $-1 \notin \kappa(\pi)^{*2}$, $-1 \notin \kappa(\delta)^{*2}$. Since $\lambda \in F^{*\ell}$ and $-\lambda$ is a norm from $L_\pi$ and $L_{\delta}$, $-1$ is a norm from $L_{\pi}$ and $L_{\delta}$. Hence $-1$ is a norm from $L(\pi)$ and $L(\delta)$. Since $\kappa(\pi)$ and $\kappa(\delta)$ are local fields with residue fields of characteristic not equal to 2, we have $\text{L}(\pi) \simeq \kappa(\pi)(\sqrt{-1})$ and $\text{L}(\delta) \simeq \kappa(\delta)(\sqrt{-1})$. Let $L = F(\sqrt{-1})$. Since $\kappa$ is a finite field of characteristic not equal to 2, $-1$ is a norm from $L$. Since $\lambda \in F^{*\ell}$, there exists $\mu \in L$ such that $N_{L/F}(\mu) = -\lambda$. Further, $L$ and $\mu$ have the required properties.

□
Lemma 6.5. Suppose that $\nu_\pi(\lambda)$ is divisible by $\ell$, $\alpha$ is unramified on $A$, except possibly at $\pi$ and $\delta$, and $\alpha \cdot (-\lambda) = 0$. Let $L_\pi$ be a finite product of unramified finite field extensions of $F_\pi$ with $\dim_{F_\pi}(L_\pi) = \ell$, $\mu_\pi \in L_\pi$ and $d_0 \geq 2$ such that:

- $N_{L_\pi/F_\pi}(\mu_\pi) = -\lambda$;
- $\text{ind}(\alpha \otimes L_\pi) < d_0$;
- $\alpha \cdot (\mu_\pi) = 0$ in $H^3(L_\pi, \mu_{n}^{\otimes 2})$.

Then there exist an étale algebra $L$ over $F$ of degree $\ell$ and $\mu \in L$ such that:

- $N_{L/F}(\mu) = -\lambda$;
- $\text{ind}(\alpha \otimes L) < d_0$;
- $\alpha \cdot (\mu) = 0$ in $H^3(L, \mu_{n}^{\otimes 2})$; and
- there is an isomorphism $\phi : L_\pi \to L \otimes F_\pi$ with

$$\phi(\mu_\pi)(\mu \otimes 1)^{-1} \in (L \otimes F_\pi)^{\ell^m},$$

for all $m \geq 1$.

Further, if $L_\pi/F_\pi$ is a field extension with the residue field of $L_\pi$ unramified over $\kappa(\pi)$, then $L$ can be chosen to be a field extension with $L/F$ unramified on $A$.

Proof. Since $\nu_\pi(\lambda)$ is divisible by $\ell$, $\lambda = w\pi^{s_1}\delta^t$ for some $w \in A$ a unit. Write $L_\pi = \prod_1^q L_{\pi,i}$ with $L_{\pi,i}/F_\pi$ a finite unramified extension and $\mu_\pi = (\mu_1, \ldots, \mu_q)$ with $\mu_i \in L_{\pi,i}$. Since $L_{\pi,i}/F_\pi$ is unramified, $\pi$ is a parameter in $L_{\pi,i}$ for all $i$. Write $\mu_i = \theta_i \pi^{t_i}$ for some $\theta_i \in L_{\pi,i}$ a unit at $\pi$.

Let $\theta = (\theta_1, \ldots, \theta_q) \in L_\pi$. Since $N_{L_\pi/F_\pi}(\mu_\pi) = \lambda = w\pi^{s_1}\delta^t$, we have $N_{L_{\pi,i}/F_\pi}(\theta_i) = w^\delta_i$.

For each $i$, let $L_i/F$ be a field extension with $L_i \otimes F_\pi \simeq L_{\pi,i}$ as in Lemma 5.12. Let $B_i$ be the integral closure of $A$ in $L_i$. Then each $B_i$ is regular local ring with maximal ideal $(\pi, \delta_i)$ for some prime $\delta_i$ with $N_{L_i/F}(\delta_i) = v_i \delta_i^{f_i}$ for some unit $v_i \in A$ and $f_i \geq 1$ (Remark 5.13). Then the residue field $L_i(\pi)$ of $L_i$ at the discrete valuation given by $\pi$ is the field of fractions of $B_i/(\pi)$. In particular, $L_i(\pi)$ is a complete discrete valued field with $\delta_i \in B_i/(\pi)$ as a parameter. We identify $L_{\pi,i}$ with $L_i \otimes F_\pi$ and assume that $\mu_i \in L_i \otimes F_\pi$.

For $1 \leq i \leq q$, let $\theta_i$ be the image of $\theta_i$ in $L_i(\pi)$. Then $\theta_i = \overline{v_i} \delta_i^{f_i}$ for some unit $v_i \in B_i$ and $t_i \in \mathbb{Z}$. Since $N_{L_{\pi,i}/F_\pi}(\theta_i) = w^\delta_i$ and $N_{L_{\pi,i}/F_\pi}(\delta_i) = v_i \delta_i^{f_i}$, we have $\prod_1^q N_{L_i(\pi)/\kappa(\pi)}(\theta_i) = \prod_1^q N_{L_i(\pi)/\kappa(\pi)}(\overline{v_i}) \prod_1^q (\overline{v_i} \delta_i^{f_i} \overline{\delta_i}) = \overline{w} \delta_i^{f_i}$. Hence

$$\sum f_i t_i = t \quad \text{and} \quad N_{L_1(\pi)/\kappa(\pi)}(w_1) = \overline{w} \prod_1^q N_{L_i(\pi)/\kappa(\pi)}(\overline{w_i})^{-1} \prod_1^q \overline{v_i}^{-t_i}.$$
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Further, if $L/F_π$ is a field extension such that the residue field $L_π(π)$ of $L_π$ is an unramified extension of $κ(π)$, then by the choice of $L$, $L/F$ is a field extension with $L/F$ unramified on $A$ (see the proof of Lemma 5.12).

Lemma 6.6. Suppose that $α = (E/F, σ, wπδ^m)$ for some cyclic extension $E/F$ which is unramified on $A$, except possibly at $δ$. Let $E_δ$ be the lift of the residue of $α$ at $δ$. If $t_1α ⊗ E_δ = 0$ for some $t_1$, then there exists an integer $r_1 ≥ 0$ such that $w_1δ^{mr_1−t_1}$ is a norm from the extension $E/F$ for some unit $w_1 ∈ A$.

Proof. Write $α ⊗ F_δ = α' + (E_δ/F_δ, σ_δ, δ)$ as in Lemma 4.1. Suppose that $t_1α ⊗ E_δ = 0$. Since $α ⊗ E_δ = α' ⊗ E_δ$, $t_1α' ⊗ E_δ = 0$. Hence $t_1α' = (E_δ, σ_δ, δ)$ for some $θ ∈ F_δ$. Since $α'$ and $E_δ/F_δ$ are unramified at $δ$, we assume that $θ ∈ F_δ$ is a unit at $δ$. Since the residue field $κ(δ)$ of $F_δ$ is a complete discretely valued field with the image of $π$ as a parameter, without loss of generality we assume that $θ = w_0π^{r_1}$ for unit $w_0 ∈ A$ and $r_1 ≥ 0$. Let $λ_1 = w_0π^{r_1}δ^{t_1}$. Since $t_1α' = (E_δ, σ_δ, δ)$, by Lemma 4.7, $δ_1(α · (λ_1)) = 0$. Since $κ(δ)$ is a local field, $α · (λ_1) = 0 ∈ H^3(F_δ, μ_n^{⊗2})$ (cf. the proof of Proposition 4.6). Since $α$ is unramified on $A$, except possibly at $π$ and $δ$, by Corollary 5.5, $α · (λ_1) = 0 ∈ H^3(F, μ_n^{⊗2})$. We have

$$0 = ∂_π(α · (λ_1)) = ∂_π((E/F, σ, wπδ^m) · (w_0π^{r_1}δ^{t_1})) = (E(π)/κ(π), σ, (-1)^{r_1}π^{t_1}w_0^{−1}δ^{mr_1−t_1}).$$

Since $(E/F, σ, (-1)^{r_1}w_0^{−1}δ^{mr_1−t_1})$ is unramified on $A$, except possibly at $π$ and $δ$, by Corollary 5.5, $(E/F, σ, (-1)^{r_1}w_0^{−1}δ^{mr_1−t_1}) = 0$. In particular, $(-1)^{r_1}w_0^{−1}δ^{mr_1−t_1}$ is a norm from the extension $E/F$.

Lemma 6.7. Suppose that $α · (−λ) = 0$ and $λ = wπ^sδ^t_1$ for some unit $w ∈ A$ and $s$ coprime to $ℓ$. Let $E_δ$ be the lift of the residue of $α$ at $δ$. If $t_1α ⊗ E_δ = 0$, then there exists $θ ∈ A$ such that:

- $α · (θ) = 0$
- $ν_α(θ) = 0$
- $ν_δ(θ) = t_1$

Proof. Since $s$ is coprime to $ℓ$, by Lemma 6.1, $α = (E/F, σ, (-1)^{s+1}λ)$ for some cyclic extension $E/F$ which is unramified on $A$, except possibly at $δ$. Let $r = [F: E]$. Since $r$ is a power of $ℓ$ and $s$ is coprime to $ℓ$, there exists an integer $s' ≥ 1$ such that $ss' ≡ 1$ modulo $r$. We have

$$α = α^{ss'} = (E/F, σ, (-1)^{s+1}wπ^{sδ^t_1ℓ^{s'}}),$$

$$= (E/F, σ)^s · ((-1)^{s+1}wπ^{sδ^t_1ℓ^{s'}})^{s'},$$

$$= (E/F, σ)^s · ((-1)^s w^{s'} π^{sδ^t_1ℓ^{s'}}).$$

Since $s$ is coprime to $ℓ$, we also have $((E/F, σ)^s = (E/F, σ^{s'})$ (cf. §2) and hence $α = (E/F, σ^{s'})$, $((-1)^{s'} w^{s'} π^{sδ^t_1ℓ^{s'}})$. Thus, by Lemma 6.6, there exist a unit $w_1 ∈ A$ and $r_1 ≥ 0$ such that $w_1δ^{s't_1ℓ^{r_1−t_1}}$ is a norm from $E/F$. Since $s'r_1−1$ is coprime to $ℓ$, $s'r_1-1$ is coprime to $r$ and hence there exists an integer $r_2 ≥ 0$ such that $(s'r_1−1)r_2 ≡ 1$ modulo $r$. In particular, $w_1^{r_2δ^t_1} ≡ (w_1δ^{s't_1ℓ^{r_1−t_1}})^{r_2}$ modulo $F^{s'}$ and hence $w_1^{r_2δ^t_1}$ is a norm from $E/F$. Thus $θ = w_1^{r_2δ^t_1}$ has the required properties.

Lemma 6.8. Let $E_π$ and $E_δ$ be the lift of the residues of $α$ at $π$ and $δ$, respectively. Suppose that $λ = wπ^{s'fδ^t_1ℓ}$ for some unit $w ∈ A$. If $α · (−λ) = 0$, $s_1α ⊗ E_π = 0$ and $t_1α ⊗ E_δ = 0$, then there exists $θ ∈ A$ such that:

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• \( \alpha \cdot (\theta) = 0; \)
• \( \nu_\pi(\theta) = s_1; \)
• \( \nu_\delta(\theta) = t_1. \)

Proof. By Corollary 5.7, we assume that \( \alpha = (E/F, \sigma, u_\pi \delta^m) \) for some extension \( E/F \) which is unramified on \( A \), except possibly at \( \delta \) and \( m \geq 0 \). Without loss of generality, we assume that \( 0 \leq m < [E : F] \). By Lemma 6.6, there exist an integer \( r_1 \geq 0 \) and a unit \( w_1 \in A \) such that \( w_1 \delta^{mr_1-t_1} \) is a norm from \( E/F \). Let \( r = [E : F] \) and \( \theta = (-u_\pi + \delta^t)^{r_1-s_1}w_1^{-1}(-u)^{s_1}w \). Since \( r - m > 0 \), we have \( \nu_\pi(\theta) = s_1 \) and \( \nu_\delta(\theta) = t_1. \)

Now we show that \( \alpha \cdot (\theta) = 0. \) Let \( \gamma \) be a prime in \( A \) with \( (\gamma) \neq (\pi) \) and \( (\gamma) \neq (\delta) \). Since \( \alpha \) is unramified on \( A \), except possibly at \( \pi \) and \( \delta \), if \( \gamma \) does not divide \( \theta \), then \( \alpha \cdot (\theta) \) is unramified at \( \gamma \). Suppose \( \gamma \) divides \( \theta \). Then \( \gamma = -u_\pi + \delta^t \). Thus \( u_\pi \delta^m \equiv \delta^t \) modulo \( \gamma \). Since \( \delta \cdot (\alpha \cdot (\theta)) = (E(\gamma), \sigma, \bar{u_\pi \delta^m})^{r_1-s_1} \), where \( E(\gamma) \) is the residue field of \( E \) at \( \gamma \) and bar denotes the image modulo \( \gamma \), we have \( \delta \cdot (\alpha \cdot (\theta)) = (E(\gamma), \sigma, \bar{u_\pi \delta^m})^{r_1-s_1} = (E(\gamma), \sigma, \bar{\delta^t})^{r_1-s_1} = 0. \) Hence \( \alpha \cdot (\theta) \) is unramified on \( A \), except possibly at \( \pi \) and \( \delta \).

We have \( (-u_\pi + \delta^t)^{r_1-s_1} \equiv \delta^{r_1-s_1}w_1^{-1}(-u)^{s_1}w_1^{-1}(w_1 \delta^{mr_1-t_1})^{-1} \) modulo \( F_\pi \).

Since \( w_1 \delta^{mr_1-t_1} \) is a norm from \( E/F \) and \( r = [E : F] \), we have
\[
(\alpha \cdot (\theta)) \otimes F_\pi = (E/F, \sigma, u_\pi \delta^m) \cdot ((-u_\pi \delta^m)^{s_1}(w_1 \delta^{mr_1-t_1})^{-1}) \otimes F_\pi = (E/F, \sigma, u_\pi \delta^m) \cdot ((-u_\pi \delta^m)^{s_1}) \otimes F_\pi = 0.
\]

Thus, by Corollary 5.5, we have \( \alpha \cdot (\theta) = 0. \) \( \square \)

7. Patching

We fix the following data:
• \( R \) a complete discrete valuation ring;
• \( K \) the field of fractions of \( R \);
• \( \kappa \) the residue field of \( R \);
• \( \ell \) a prime not equal to \( \text{char}(\kappa) \) and \( n = \ell^d \) for some \( d \geq 1 \);
• \( X \) a smooth projective geometrically integral curve over \( K \);
• \( F \) the function field of \( X \);
• \( \alpha \in H^2(F, \mu_n), \alpha \neq 0; \)
• \( \lambda \in F^* \) with \( \alpha \cdot (-\lambda) = 0; \)
• \( \mathcal{X} \) a normal proper model of \( X \) over \( R \) and \( X_0 \) the reduced special fiber of \( \mathcal{X} \);
• \( \mathcal{P}_0 \) the finite set of closed points of \( X_0 \) consisting of all the points of intersection of irreducible components of \( X_0 \).

We recall the following notation from [HH10, §6] and [HHK09, §3.3]. For \( x \in \mathcal{X} \), let \( \hat{A}_x \) be the completion of the local ring \( A_x \) at \( x \) on \( \mathcal{X} \), \( F_x \) the field of fractions of \( \hat{A}_x \) and \( \kappa(x) \) the residue field at \( x \). Let \( \eta \) be a codimension zero point of \( X_0 \) and \( U \subset \eta \) be a nonempty open subset. Let \( A_U \) be the ring of all those functions in \( F \) which are regular at every closed point of \( U \). Let \( t \) be a parameter in \( R \). Then \( t \in A_U \). Let \( \hat{A}_U \) be the \( (t) \)-adic completion of \( A_U \) and \( F_U \) be the field of fractions of \( \hat{A}_U \). Then \( F \subseteq F_U \subseteq F_\eta \).

Let \( \eta \in X_0 \) be a codimension zero point and \( P \in X_0 \) be a closed point such that \( P \) is in the closure of \( \eta \). By an abuse of notation, we denote the closure of \( \eta \) by \( \eta \) and say that \( P \) is a point of \( \eta \). A branch is a height one prime ideal \( \varphi \) of \( \hat{A}_P \) containing \( t \). Let \( \varphi \) be a branch. Let \( A_\varphi \) be
the completion of the localization of $\hat{A}_P$ at $\varphi$ and $F_\varphi$ the field of fractions of $\hat{A}_\varphi$. The contraction $\varphi \cap A_P$ of $\varphi$ to $A_P$ is a height one prime ideal and hence a branch $\varphi$ uniquely determines an irreducible component $\eta$ of $X_0$ containing $P$.

Suppose further that $X$ is a regular proper model of $X$ over $R$ and $X_0$ is a union of regular curves with normal crossings. Then $A_x$, $\hat{A}_x$ are regular local rings. Every branch $\varphi$ is uniquely determined by a pair $(P, \eta)$ where $\eta$ is a codimension zero point of $X_0$ and $P \in \eta$ is a closed point. In this case, $F_\varphi$ is the completion of $F_P$ at the discrete valuation of $F_P$ given by $\eta$. We also denote $F_\varphi$ by $F_{P, \eta}$. Note that the residue field $\kappa(\eta)_P$ of $\hat{A}_\varphi$ is the completion of the residue field $\kappa(\eta)$ at the discrete valuation given by $P$.

We begin with the following result, which follows from [HHK15a, Theorem 9.11] (cf. the proof of [PS15, Theorem 2.4]).

**Proposition 7.1.** For each irreducible component $X_\eta$ of $X_0$, let $U_\eta$ be a nonempty proper open subset of $X_\eta$ and $\mathcal{P} = X_0 \setminus \bigcup_{\eta} U_\eta$, where $\eta$ runs over the codimension zero points of $X_0$. Suppose that $\mathcal{P}_0 \subseteq \mathcal{P}$. Let $L$ be a finite extension of $F$. Suppose that there exists $N \geq 1$ such that for each codimension zero point $\eta$ of $X_0$, $\text{ind}(\alpha \otimes L \otimes F_{U_\eta}) \leq N$, and for every closed point $P \in \mathcal{P}$, $\text{ind}(\alpha \otimes L \otimes F_P) \leq N$. Then $\text{ind}(\alpha \otimes L) \leq N$.

**Proof.** Let $\mathcal{Y}$ be the integral closure of $X$ in $L$ and $\phi : \mathcal{Y} \to X$ be the induced map. Let $\mathcal{P}'$ be a finite set of closed points of $\mathcal{Y}$ containing the inverse image of $\mathcal{P}$ under $\phi$. Let $U$ be an irreducible component of $Y \setminus \mathcal{P}'$. Then $\phi(U) \subset U_\eta$ for some $U_\eta$ and there is a homomorphism of algebras from $L \otimes F_{U_\eta}$ to $L_U$. (Note that $L \otimes F_{U_\eta}$ may be a product of fields.) Since $\text{ind}(\alpha \otimes L \otimes F_{U_\eta}) \leq N$, we have $\text{ind}(\alpha \otimes L_U) \leq N$. Let $Q \in \mathcal{P}'$. Suppose $\phi(Q) = P \in \mathcal{P}$. Then there is a homomorphism of algebras from $L \otimes F_P$ to $L_Q$. (Once again note that $L \otimes F_P$ may be a product of fields.) Since $\text{ind}(\alpha \otimes L \otimes F_P) \leq N$, $\text{ind}(\alpha \otimes L_Q) \leq N$. Suppose that $\phi(Q) \in U_\eta$ for some $U_\eta$. Then there is a homomorphism of algebras from $L \otimes F_{U_\eta}$ to $L_Q$. Thus $\text{ind}(\alpha \otimes L_Q) \leq N$. Therefore, by [HHK15a, Theorem 9.11], $\text{ind}(\alpha \otimes L) \leq N$. \hfill \qed

**Lemma 7.2.** Let $\eta$ be a codimension zero point of $X_0$. Suppose there exist a field extension or split extension $L_\eta/F_\eta$ of degree $\ell$ and $\mu_\eta \in L_\eta$ such that:

1. $N_{L_\eta/F_\eta}(\mu_\eta) = -\lambda$;
2. $\text{ind}(\alpha \otimes L_\eta) < \text{ind}(\alpha)$;
3. $\alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_\eta^{\otimes 2})$.

Then there exist a nonempty open subset $U_\eta$ of $\eta$, a split or field extension $L_{U_\eta}/F_{U_\eta}$ of degree $\ell$ and $\mu_{U_\eta} \in L_{U_\eta}$ such that:

1. $N_{L_{U_\eta}/F_{U_\eta}}(\mu_{U_\eta}) = -\lambda$;
2. $\text{ind}(\alpha \otimes L_{U_\eta}) < \text{ind}(\alpha)$;
3. $\alpha \cdot (\mu_{U_\eta}) = 0 \in H^3(L_{U_\eta}, \mu_{U_\eta}^{\otimes 2})$;
4. there is an isomorphism $\phi_{U_\eta} : L_{U_\eta} \otimes F_\eta \to L_\eta$ with $\phi_{U_\eta}(\mu_{U_\eta} \otimes 1)\mu_{U_\eta}^{-1} \in L_\eta^{\ell m}$ for all $m \geq 1$.

Further, if $L_\eta/F_\eta$ is cyclic, then $L_{U_\eta}/F_{U_\eta}$ is cyclic.

**Proof.** Suppose $L_\eta = \prod F_\eta$ is the split extension of degree $\ell$. Write $\mu_\eta = (\mu_1, \ldots, \mu_\ell)$ with $\mu_i \in F_\eta$. Then $-\lambda = N_{L_\eta/F_\eta}(\mu_\eta) = \sum_{i=1}^\ell \mu_i$. Since $\text{ind}(\alpha \otimes L_\eta) = \text{ind}(\alpha \otimes F_\eta) < \text{ind}(\alpha)$, by [HHK15a, Proposition 5.8], [KMRT98, Proposition 1.17], there exists a nonempty open subset $U_\eta$ of $\eta$ such that $\text{ind}(\alpha \otimes F_{U_\eta}) < \text{ind}(\alpha)$. Since $F_\eta$ is the completion of $F$ at the discrete valuation given
by $\eta$, there exist $\theta_i \in F^*$, $1 \leq i \leq \ell$, such that $\theta_i \mu_i^{-1} \equiv 1$ modulo the maximal ideal of $\hat{R}_\eta$. Let $L_{U_\eta} = \prod F_{U_\eta}$ and $\mu_{U_\eta} = (\lambda(\theta_2 \cdots \theta_{i-1})^{-1}, \theta_2, \ldots, \theta_{\ell}) \in L_{U_\eta}$. Then $N_{L_{U_\eta}/F_{U_\eta}}(\mu_{U_\eta}) = -\lambda$. Since $\alpha \cdot (\theta_i) \in H^3(F_{U_\eta}, \mu_{U_\eta}^{\otimes 2})$ and $\alpha \cdot (\theta_i) = 0 \in H^3(F_{U_\eta}, \mu_{U_\eta}^{\otimes 2})$, by [HIK14, Proposition 3.2.2], there exists a nonempty open subset $V_\eta \subseteq U_\eta$ such that $\alpha \cdot (\theta_i) = 0 \in H^3(F_{U_\eta}, \mu_{U_\eta}^{\otimes 2})$. By replacing $U_\eta$ by $V_\eta$, we have the required $L_{U_\eta}$ and $\mu_{U_\eta} \in L_{U_\eta}$.

Suppose that $L_\eta/F_\eta$ is a field extension of degree $\ell$. Let $F^h_\eta$ be the henselization of $F$ at the discrete valuation $\eta$. Then there exists a field extension $L^h_\eta/F^h_\eta$ of degree $\ell$ with an isomorphism $\phi^h_\eta : L^h_\eta \otimes_{F^h_\eta} F_\eta \rightarrow L_\eta$. We identify $L^h_\eta$ with a subfield of $L_\eta$ through $\phi^h_\eta$. Further, if $L_\eta/F_\eta$ is a cyclic extension, then $L^h_\eta/F^h_\eta$ is also a cyclic extension. Let $\pi_\eta, \mu_\eta \in L^h_\eta$ be a parameter. Then $\pi_\eta$ is also a parameter in $L_\eta$. Write $\mu_\eta = u_\eta \pi_\eta^r$ for some $u_\eta \in L_\eta$ a unit at $\eta$. Since $N_{L_{U_\eta}/F_{U_\eta}}(\mu_\eta) = -\lambda$, we have $-\lambda = N_{L_\eta/F_\eta}(u_\eta)N_{L_\eta/F_\eta}(\pi_\eta)$. Since $u_\eta \in L_\eta$ is a unit at $\eta$, $N_{L_\eta/F_\eta}(u_\eta) \in F_\eta$ is a unit at $\eta$. By [Art69, Theorem 1.10], there exists $u^h_\eta \in L^h_\eta$ such that $N_{L_{U_\eta}/F_{U_\eta}}(u^h_\eta) = N_{L_\eta/F_\eta}(u_\eta)$ and $u^h_\eta \equiv u_\eta$ modulo the maximal ideal of the valuation ring of $L^h_\eta$. Let $\mu^h_\eta = u^h_\eta \pi^r_\eta \in L^h_\eta$. Then $\alpha \cdot (\mu^h_\eta) = \alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_\eta^{\otimes 2})$ and hence $\alpha \cdot (\mu^h_\eta) = 0 \in H^3(L^h_\eta, \mu_\eta^{\otimes 2})$ (cf. the proof of [HIK14, Proposition 3.2.2]). Since $F^h_\eta$ is the filtered direct limit of the fields $F_\eta$, where $\eta$ ranges over the nonempty open subset of $\eta$ [HIK14, Lemma 3.2.1], there exist a nonempty open subset $U_{\eta} \subseteq L_{\eta}$, a field extension $L_{U_{\eta}}/F_{U_{\eta}}$ of degree $\ell$ and $\mu_{U_{\eta}} \in L_{U_{\eta}}$ such that $N_{L_{U_{\eta}}/F_{U_{\eta}}}(\mu_{U_{\eta}}) = -\lambda$ and there is an isomorphism $\phi^h_{U_{\eta}} : L_{U_{\eta}} \otimes_{F^h_{U_{\eta}}} F_{U_{\eta}} \cong L^h_{U_{\eta}}$ with $\phi^h_{U_{\eta}}(\mu_{U_{\eta}}) = \mu^h_{U_{\eta}}$. Since $u^h_{U_{\eta}} \equiv u_\eta$ modulo the maximal ideal of the valuation ring of $L_{U_{\eta}}$, $\mu_{U_{\eta}} = u_{\eta} \pi^r_{U_{\eta}}$ and $\mu^h_{U_{\eta}} = u_{\eta}^h \pi^r_{U_{\eta}}$, it follows that $\phi^h_{U_{\eta}}(\mu_{U_{\eta}} \cdot 1) \mu_{U_{\eta}}^{-1} \in L_{U_{\eta}}^{\otimes m}$ for all $m \geq 1$. By shrinking $U_{\eta}$, we assume that $\alpha \cdot (\mu_{U_{\eta}}) = 0 \in H^3(L_{U_{\eta}}, \mu_{U_{\eta}}^{\otimes 2})$ [HIK14, Proposition 3.2.2]. Further, if $L_\eta/F_\eta$ is cyclic, by shrinking $U_{\eta}$, we can assume that $L_{U_{\eta}}/F_{U_{\eta}}$ is cyclic. 

For the rest of this section we assume that for each point $x$ of $X_0$, there exist an étale algebra $L_x/F_x$ of degree $\ell$ and $\mu_x \in L_x$ such that:

1. $N_{L_x/F_x}(\mu_x) = -\lambda$;
2. $\alpha \cdot (\mu_x) = 0 \in H^3(L_x, \mu_x^{\otimes 2})$;
3. $\ind(\alpha \otimes L_x) < \ind(\alpha)$;
4. for any branch $(P, \eta)$ there is an isomorphism $\phi_{P, \eta} : L_{\eta} \otimes_{F_{P, \eta}} F_{P, \eta} \rightarrow L_P \otimes F_{P, \eta}$ such that $\phi_{P, \eta}(\mu_{P, \eta}) \mu_{P, \eta}^{-1} \in (L_P \otimes F_{P, \eta})^{\otimes m}$ for all $m \geq 1$;
5. if $x = \eta$ is a codimension zero point of $X_0$, then $L_\eta/F_\eta$ is either a field or the split extension.

**Lemma 7.3.** There exist:

- a field extension $L/F$ of degree $\ell$;
- a nonempty open proper subset $U_\eta$ of $\eta$ for every codimension zero point $\eta$ of $X_0$ and $\mu'_{U_\eta} \in L \otimes F_{U_\eta}$;
- for every $P \in \mathcal{P} = X_0 \setminus U_\eta$, $\mu'_{P} \in L \otimes F_{P}$, such that:
  1. $\ind(\alpha \otimes L) < \ind(\alpha)$;
  2. $N_{L \otimes F_{U_\eta}/F_{U_\eta}}(\mu'_{U_\eta}) = -\lambda$ and $\alpha \cdot (\mu'_{U_\eta}) = 0 \in H^3(L \otimes F_{U_\eta}, \mu_{U_\eta}^{\otimes 2})$ for all codimension zero points $\eta$ of $X_0$;
  3. $N_{L \otimes F_{P}/F_{P}}(\mu'_{P}) = -\lambda$ and $\alpha \cdot (\mu'_{P}) = 0 \in H^3(L \otimes F_{P}, \mu_{P}^{\otimes 2})$ for all $P \in \mathcal{P}$;
  4. for any branch $(P, \eta)$, $\mu'_{U_\eta} \mu'_{P}^{-1} \in (L_P \otimes F_{P, \eta})^{\otimes m}$ for all $m \geq 1$.

Further, if for each $x \in X_0$, $L_x/F_x$ is cyclic or split, then $L/F$ is cyclic.
Local–global principle

Proof. Let $\eta$ be a codimension zero point of $X_0$. By assumption, there exists a field or split extension $L_0/F_0$ and $\mu_0 \in L_0$ such that $N_{L_0/F_0}(\mu_0) = -\lambda$, $\alpha \cdot (\mu_0) = 0 \in H^3(L_0, \mu_n^{\otimes 2})$ and $\text{ind}(\alpha \otimes L_0) < \text{ind}(\alpha)$. By Lemma 7.2, there exist a nonempty open set $U_\eta$ of $\eta$, a field or split extension $L_{U_\eta}/F_{U_\eta}$ of degree $\ell$ and $\mu_{U_\eta} \in L_{U_\eta}$ such that $N_{L_{U_\eta}/F_{U_\eta}}(\mu_{U_\eta}) = -\lambda$, $\alpha \cdot (\mu_{U_\eta}) = 0 \in H^3(L_{U_\eta}, \mu_n^{\otimes 2})$, $\text{ind}(\alpha \otimes L_{U_\eta}) < \text{ind}(\alpha)$, $\phi_{\eta}: L_{U_\eta} \otimes F_{\eta} \to L_{\eta}$ an isomorphism $\phi_{U_\eta}(\mu_{U_\eta} \otimes 1)\mu_1 \in L_{\eta}$ for all $m \geq 1$. By shrinking $U_\eta$, if necessary, we assume that $\mathcal{P}_0 \cap U_\eta = \emptyset$.

Let $\mathcal{P} = X_0 \setminus \bigcup_n U_\eta$ and $P \in \mathcal{P}$. Then, by assumption, we have an étale algebra $L_P/F_P$ of degree $\ell$ and for every branch $(P, \eta)$ there is an isomorphism $\phi_{P, \eta} : L_\eta \otimes F_{P, \eta} \to L_P \otimes F_{P, \eta}$. Thus $\phi_{P, U_\eta} = \phi_{P, \eta}(\phi_\eta \otimes 1) : L_{U_\eta} \otimes F_{\eta} \otimes F_{P, \eta} \to L_P \otimes F_{P, \eta}$ is an isomorphism. Thus, by [HH10, Theorem 7.1], there exists an extension $L/F$ of degree $\ell$ with isomorphisms $\phi_{U_\eta} : L \otimes F_{U_\eta} \to L_{U_\eta}$ for all codimension zero points $\eta$ of $X_0$ and $\phi_P : L \otimes F_P \to L_P$ for all $P \in \mathcal{P}$ with the following commutative diagram:

$$
\begin{array}{ccc}
L \otimes F_{U_\eta} \otimes F_{P, \eta} & \xrightarrow{\phi_{U_\eta} \otimes 1} & L_{U_\eta} \otimes F_{\eta} \otimes F_{P, \eta} \\
\downarrow & & \downarrow \\
L \otimes F_P \otimes F_{P, \eta} & \xrightarrow{\phi_P \otimes 1} & L_P \otimes F_{P, \eta}
\end{array}
$$

where the vertical arrow on the left is the natural map. Further, if each $L_x/F_x$ is cyclic or split for all $x \in X_0$, then $L/F$ is cyclic [HH10, Theorem 7.1].

Since $\text{ind}(\alpha \otimes L \otimes F_{U_\eta}) < \text{ind}(\alpha)$ for all codimension zero points of $X_0$ and $\text{ind}(\alpha \otimes L \otimes F_\eta) < \text{ind}(\alpha)$ for all $P \in \mathcal{P}$, by Proposition 7.1, $\text{ind}(\alpha \otimes L) < \text{ind}(\alpha)$. In particular, $L$ is a field.

For every codimension zero point $\eta$ of $X_0$, let $\mu_{U_\eta} = (\phi_{U_\eta})^{-1}(\mu_{U_\eta}) \in L \otimes F_{U_\eta}$, and for every $P \in \mathcal{P}$, let $\mu_P = (\phi_P)^{-1}(\mu_P) \in L \otimes F_P$. Since $\phi_{U_\eta}$ and $\phi_P$ are isomorphisms, we have the required properties. 

\begin{proposition}
Suppose that for every branch $\varphi = (P, \eta)$, there exists $t_\varphi \geq 0$ such that $F_{P, \eta}$ has no primitive $\ell^t$-th root of unity. Let $L/F$ be a cyclic field extension of degree $\ell$. Suppose that:

- for every codimension zero point $\eta$ of $X_0$, there exist a nonempty open proper subset $U_\eta$ of $\eta$ and $\mu_{U_\eta} \in L \otimes F_{U_\eta}$;
- for every $P \in \mathcal{P} = X_0 \setminus \bigcup_n U_\eta$, $\mu_P \in L \otimes F_P$,

such that:

1. $N_{L_\otimes F_{P, \eta}/F_{P, \eta}}(\mu_{U_\eta}) = -\lambda$ and $\alpha \cdot (\mu_{U_\eta}) = 0 \in H^3(L \otimes F_{U_\eta}, \mu_n^{\otimes 2})$ for all codimension zero points $\eta$ of $X_0$;
2. $N_{L_\otimes F_{P, \eta}/F_{P, \eta}}(\mu_P) = -\lambda$ and $\alpha \cdot (\mu_P) = 0 \in H^3(L \otimes F_P, \mu_n^{\otimes 2})$ for all $P \in \mathcal{P}$;
3. for any branch $(P, \eta)$, $\mu_{U_\eta} \mu_P^{-1} \in (L \otimes F_{P, \eta})^\ell_m$ for all $m \geq 1$.

Then there exists $\mu \in L^*$ such that:

- $N_{L/F}(\mu) = -\lambda$; and
- $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{\otimes 2})$.

\end{proposition}

\begin{proof}
Let $\sigma$ be a generator of $\text{Gal}(L/F)$. Let $\varphi = (P, \eta)$ be a branch. Since $N_{L_\otimes F_{P, \eta}/F_{P, \eta}}(\mu_{U_\eta}) = N_{L_\otimes F_{P, \eta}/F_{P, \eta}}(\mu_P)$, by Lemma 2.7, there exists $\theta_{P, \eta} \in L \otimes F_{P, \eta}$ such that $\mu_{U_\eta} \mu_P^{-1} = \theta_{P, \eta}^\ell \sigma(\theta_{P, \eta})$. Applying [HHK09, Theorem 3.6] for the rational group $R_{L/F}G_m$, there exist $\theta_{U_\eta} \in L \otimes F_{U_\eta}$ and $\theta_P \in L \otimes F_P$ for every codimension zero point $\eta$ of $X_0$ and $P \in \mathcal{P}$ such that for every branch $(P, \eta)$, $\theta_{P, \eta} = \theta_{U_\eta} \theta_P$.

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Let $μ^{α}_{U_η} = μ^{α}_{U_η}θ^{ed}_Uθ^{ed}_U ∈ L ⊗ F_U$ and $μ^{α}_{P} = μ^{α}_{P}θ^{ed}_Pθ^{ed}_P ∈ L ⊗ F_P$. If $(P,η)$ is a branch, then we have

$$μ^{α}_{U_η}θ^{ed}_Uθ^{ed}_U = μ^{α}_{P}θ^{ed}_Pθ^{ed}_P = μ^{α}_{P}θ^{ed}_Pθ^{ed}_Pθ^{ed}_Uθ^{ed}_U.$$ 

Hence, by [HH10, Proposition 6.3], there exists $μ ∈ L$ such that $μ = μ^{α}_{U_η}$ and $μ = μ^{α}_{P}$ for every codimension zero point $η$ of $X_0$ and $P ∈ Ψ$. Clearly, $N_{L/F}(μ) = -λ$ over $F$. Let $P ∈ Ψ$. Since $α · (μ_P) = 0$ and $α · (θ_P^{ed}) = 0$, $α · (μ) = 0 ∈ H^3(L ⊗ F_P, μ^{ed}_n)$ for every codimension zero point $η$ of $X_0$. Let $Ψ$ be the normal closure of $Ψ$ in $L$ and $Y_0$ the reduced special fiber of $Ψ$. Let $η'$ be a codimension zero point of $Y_0$. Then the image $η$ of $η'$ in $Χ$ is a codimension zero point. Since $F_η ⊂ L_{η'}$, we have a map $L ⊗ F_{U_{η'}} → L_{η'}$ and hence $α · (μ) = 0 ∈ H^3(L_{η'}$, $μ^{ed}_{n})$. Let $Q$ be a closed point of $Y_0$ and $P$ its image in $X_0$. Suppose $P ∈ U_{η'}$ for some $η$. Since $F_{U_{η'}} ⊂ F_P ⊂ L_Q$, it follows that $α · (μ) = 0 ∈ H^3(L_Q, μ^{ed}_{n})$. Suppose $P ∈ Ψ$. Since $F_P ⊂ L_Q$, we have $α · (μ) = 0 ∈ H^3(L_Q, μ^{ed}_{n})$. Hence, by [HHK14, Theorem 3.2.3], $α · (μ) = 0$ in $H^3(L, μ^{ed}_{n})$.

**Proposition 7.5.** Suppose that for every branch $ϕ = (P,η)$, there exists $t_φ ≥ 0$ such that $F_{P,η}$ has no primitive $ℓ^t_φ$th root of unity. Let $L/F$ be an extension of degree $ℓ$ as in Lemma 7.3. Then there exist a field extension $N/F$ of degree coprime to $ℓ$ and $μ ∈ (L ⊗ N)^*$ such that:

- $N_{L⊗N/F}(μ) = -λ$; and
- $α · (μ) = 0 ∈ H^3(L ⊗ N, μ^{ed}_n)$.

**Proof.** Let $L/F$, $U_{η'}$, $Ψ$, $μ^{α}_{U_η}$ and $μ^{α}_{P}$ be as in Lemma 7.3. Since $L/F$ is an extension of degree $ℓ$, there exists a field extension $N/F$ of degree coprime to $ℓ$ such that $L ⊗ N$ is a cyclic extension field extension $N$ of degree $ℓ$.

Let $Ψ$ be the integral closure of $Ψ$ in $N$ and $Y_0$ the reduced special fiber of $Ψ$. Let $ϕ : Y_0 → X_0$ be the induced morphism.

Let $η' ∈ Y_0$ be a codimension zero point. Then $η = ϕ(η') ∈ X_0$ is a codimension zero point. Let $U_{η'} = ϕ^{-1}(U_η) ∩ Y_0$. Then $U_{η'}$ is a proper open subset of $Y_0$ and we have an inclusion $F_{U_{η'}} ⊂ N_{U_{η'}}$. Let $μ^{α}_{U_{η'}} ∈ (L ⊗ F N) ⊗ N N_{U_{η'}}$ be the image of $μ^{α}_{U_η}$ under the natural map $L ⊗ F → (L ⊗ F N) ⊗ N N_{U_{η'}}$. Then we have $N_{L⊗F,N}(μ^{α}_{U_{η'}}) = -λ$ and $α · (μ^{α}_{U_{η'}}) = 0 ∈ H^3((L ⊗ F N) ⊗ N N_{U_{η'}}, μ^{ed}_n)$.

Let $Ψ' = Y_0 \cup U_{η'}$. Let $Q ∈ Ψ'$ and $P = ϕ(Q) ∈ X_0$. Then $P ∈ Ψ$ and $P ⊂ N_Q$ and $Q = ϕ(Q) ⊂ X_0$. Then $P ∈ Ψ$ and $F_P ⊂ N_Q$. Let $μ^{α}_{Q} ∈ (L ⊗ F N) ⊗ N N_Q$ be the image of $μ^{α}_{P}$ under the natural map $L ⊗ F → (L ⊗ F N) ⊗ N N_Q$. Then we have $N_{L⊗F,N}⊙N N_Q/N_Q(μ^{α}_{Q}) = -λ$ and $α · (μ^{α}_{Q}) = 0 ∈ H^3((L ⊗ F N) ⊗ N N_Q, μ^{ed}_n)$.

Let $ϕ' = (Q, η')$ be a branch in $Y_0$ and $P = ϕ(Q)$, $η = ϕ(η')$. Then $(P, η)$ is a branch in $X_0$. Since $μ^{α}_{U_{η'}} μ^{α}_{P} ∈ (L ⊗ F_{P,η})^{ed}_{n}$ for all $m ≥ 0$, it follows that $μ^{α}_{U_{η'}} μ^{α}_{P} ∈ (L ⊗ F N) ⊗ N N_{Q,η'}^{ed}_{n}$ for all $m ≥ 0$. Since there exists $t_φ ≥ 0$, such that $F_{P,η}$ has no primitive $ℓ^t_φ$th root of unity and $N_{Q,η'}/F_{P,η}$ is a finite extension, there exists $t_{ϕ'} ≥ 0$ such that $N_{Q,η'}$ contains no primitive $ℓ^{t_{ϕ'}}$th root of unity.
Since $L \otimes_F N$ is a cyclic extension of degree $\ell$, by Proposition 7.4, there exist $\mu' \in L \otimes_F N$ such that $N_{L \otimes_F N/N}(\mu') = -\lambda$ and $\alpha \cdot (\mu') = 0 \in H^3(L \otimes_F N, \mu_n^{\otimes 2})$.

8. Types of points, special points and type 2 connections

Let $F, \alpha \in H^2(F, \mu_n)$, $\lambda \in F^*$ with $\alpha \cdot (-\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$, $\mathcal{X}$ and $X_0$ be as in §7. Further, assume that:

- $\kappa$ is a finite field;
- $\mathcal{X}$ is regular such that $\text{ram}_{\mathcal{X}}(\alpha) \cup \text{supp}_{\mathcal{X}}(\lambda) \cup X_0$ is a union of regular curves with normal crossings;
- the intersection of any two distinct irreducible curves in $X_0$ is at most one closed point.

We fix the following notation:

- $\mathcal{P}$ is the set of points of intersection of distinct irreducible curves in $X_0$;
- $\mathcal{O}_{\mathcal{X}}, \mathcal{P}$ is the semi-local ring at the points of $\mathcal{P}$ on $\mathcal{X}$;
- if a codimension zero point $\eta$ of $X_0$ contains a closed point $P \in \mathcal{P}$, then $\pi_{\eta} \in \mathcal{O}_{\mathcal{X}, \mathcal{P}}$ is a prime defining $\eta$ on $\mathcal{O}_{\mathcal{X}, \mathcal{P}}$.

Let $\eta$ be a codimension zero point of $X_0$. For the rest of this paper, let $(E_\eta, \sigma_\eta)$ denote the lift of the residue of $\alpha$ at $\eta$. Since $\alpha \in H^2(F, \mu_n)$ with $n$ a power of $\ell$, $[E_\eta : F_\eta]$ is a power of $\ell$. If $\alpha$ is unramified at $\eta$, then $E_\eta = F_\eta$ and let $M_\eta = F_\eta$. If $\alpha$ is ramified at $\eta$, then $E_\eta \neq F_\eta$ and there is a unique subextension of $E_\eta$ of degree $\ell$ and we denote it by $M_\eta$.

Remark 8.1. Let $\eta$ be a codimension zero point of $X_0$. Suppose $\alpha$ is ramified at $\eta$. Since $\text{ind}(\alpha \otimes F_\eta) = \text{ind}(\alpha \otimes E_\eta)[E_\eta : F_\eta]$ (cf. Lemma 4.2) and $M_\eta \subset E_\eta$, it follows that $\text{ind}(\alpha \otimes M_\eta) < \text{ind}(\alpha)$.

We divide the codimension zero points $\eta$ of $X_0$ as follows.

Type 1: $\nu_\eta(\lambda)$ is coprime to $\ell$ and $\text{ind}(\alpha \otimes F_\eta) = \text{ind}(\alpha)$.

Type 2: $\nu_\eta(\lambda)$ is coprime to $\ell$ and $\text{ind}(\alpha \otimes F_\eta) < \text{ind}(\alpha)$.

Type 3: $\nu_\eta(\lambda) = r\ell, r\alpha \otimes E_\eta \neq 0$ and $\text{ind}(\alpha \otimes F_\eta) = \text{ind}(\alpha)$.

Type 4: $\nu_\eta(\lambda) = r\ell, r\alpha \otimes E_\eta \neq 0$ and $\text{ind}(\alpha \otimes F_\eta) < \text{ind}(\alpha)$.

Type 5: $\nu_\eta(\lambda) = r\ell, r\alpha \otimes E_\eta = 0$ and $\text{ind}(\alpha \otimes F_\eta) = \text{ind}(\alpha)$.

Type 6: $\nu_\eta(\lambda) = r\ell, r\alpha \otimes E_\eta = 0$ and $\text{ind}(\alpha \otimes F_\eta) < \text{ind}(\alpha)$.

Let $P$ be a closed point of $\mathcal{X}$. Suppose $P$ is the point of intersection of two distinct codimension zero points $\eta_1$ and $\eta_2$ of $X_0$. We say that the point $P$ is a:

1. special point of type $I$ if $\eta_1$ is of type 1 and $\eta_2$ is of type 2;
2. special point of type $II$ if $\eta_1$ is of type 1 and $\eta_2$ is of type 4;
3. special point of type $III$ if $\eta_1$ is of type 3 or 5 and $\eta_2$ is of type 4;
4. special point of type $IV$ if $\eta_1$ is of type 1, 3 or 5 and $\eta_2$ is of type 5 with $M_{\eta_2} \otimes F_{P, \eta_2}$ not a field.

Lemma 8.2. Suppose that $\eta$ is a codimension zero point of $X_0$ and $P$ a point of $\eta$. Suppose that $\alpha$ is ramified at $\eta$. Let $(E_\eta, \sigma_\eta)$ be the lift of residue of $\alpha$ at $\eta$. If $E_\eta \otimes F_{P, \eta}$ is not a field, then $\text{ind}(\alpha \otimes F_P) < \text{ind}(\alpha)$.

Proof. Suppose that $E_\eta \otimes F_{P, \eta}$ is not a field. Since $E_\eta/F_\eta$ is a cyclic extension, $E_\eta \otimes F_{P, \eta} \simeq \prod E_{\eta, P}$ with $[E_{\eta, P} : F_{P, \eta}] < [E_\eta : F_\eta]$. We have $(E_\eta, \sigma_\eta, \pi_\eta) \otimes F_{P, \eta} = (E_{\eta, P}, \sigma_\eta, \pi_\eta)$ (cf. §2).
Write \( \alpha \otimes F_\eta = \alpha_1 + (E_\eta, \sigma_\eta, \pi_\eta) \) as in Lemma 4.1. Then \( \alpha \otimes F_{P,\eta} = \alpha_1 \otimes F_{P,\eta} + (E_{\eta,P}, \sigma_\eta, \pi_\eta) \). By Lemma 4.2, we have \( \text{ind}(\alpha \otimes F_{\eta}) = \text{ind}(\alpha_1 \otimes E_\eta)[E_\eta : F_\eta] \). We have

\[
\text{ind}(\alpha \otimes F_{P,\eta}) = \text{ind}(\alpha_1 \otimes E_{\eta,P})[E_{\eta,P} : F_{P,\eta}]
\leq \text{ind}(\alpha_1 \otimes E_\eta)[E_\eta : F_\eta]
< \text{ind}(\alpha \otimes E_\eta)[E_\eta : F_\eta]
= \text{ind}(\alpha \otimes F_\eta).
\]

Thus, by Proposition 5.8, \( \text{ind}(\alpha \otimes F_P) < \text{ind}(\alpha) \).

\[\Box\]

Lemma 8.3. Let \( \eta \in X_0 \) be a point of codimension zero and \( P \) a closed point on \( \eta \). Let \( X_P \to X \) be the blow-up at \( P \) and \( \gamma \) the exceptional curve in \( X_P \). If \( E_\eta \otimes F_{P,\eta} \) is not a field or \( \eta \) is of type 2, 4 or 6, then \( \gamma \) is of type 2, 4 or 6.

Proof. If \( E_\eta \otimes F_{P,\eta} \) is not a field, then by Lemma 8.2, \( \text{ind}(\alpha \otimes F_P) < \text{ind}(\alpha) \). If \( \eta \) is of type 2, 4 or 6, then \( \text{ind}(\alpha \otimes F_\eta) < \text{ind}(\alpha) \) and hence, by Proposition 5.8, \( \text{ind}(\alpha \otimes F_P) < \text{ind}(\alpha) \). Since \( F_P \subset F_\gamma \), we have \( \text{ind}(\alpha \otimes F_\gamma) \leq \text{ind}(\alpha \otimes F_P) < \text{ind}(\alpha) \). Hence \( \gamma \) is of type 2, 4 or 6.

\[\Box\]

Lemma 8.4. Let \( \eta_1 \) and \( \eta_2 \) be two distinct codimension zero points of \( X_0 \) intersecting at a closed point \( \psi \). Suppose that \( \eta_1 \) is of type 1 or 2 and \( \eta_2 \) is of type 2. Then there exists a sequence of blow-ups \( \psi : X' \to X \) such that if \( \eta_i \) are the strict transforms of \( \eta_i \), then:

1. \( \psi : X' \to X \) is an isomorphism;
2. \( \psi^{-1}(P) \) is the union of irreducible regular curves \( \gamma_1, \ldots, \gamma_m \);
3. \( \eta_1 \cap \gamma_1 = \{P_0\}, \eta_2 \cap \gamma_{i+1} = \{P_i\}, \gamma_1 \cap \gamma_2 = \emptyset \) for all \( i > 1 \), \( \eta_2 \cap \gamma_i = \emptyset \) for all \( i < m \), \( \eta_1 \cap \eta_2 = \emptyset \), \( \gamma_i \cap \gamma_j = \emptyset \) for all \( i < j \neq i + 1 \);
4. \( \gamma_1 \) and \( \gamma_m \) are of type 6 and \( \gamma_i, 1 < i < m \), are of type 2, 4 or 6;
5. \( \psi^{-1}(P) \) has no special points.

Proof. Let \( X_P \to X \) be the blow-up of \( X \) at \( P \) and \( \gamma \) the exceptional curve in \( X_P \). Let \( \eta_i \) be the strict transform of \( \eta_i \). Then \( \eta_1 \) intersects \( \gamma \) only at one point \( P_0 \) and \( \eta_2 \) intersects \( \gamma \) at only one point \( P_1 \). Since \( \eta_2 \) is of type 2, by Lemma 8.3, \( \gamma \) is of type 2, 4 or 6 and hence \( P_1 \) is not a special point.

Let \( s_1 = \nu_{\eta_1}(\lambda), s_2 = \nu_{\eta_2}(\lambda) \). Then \( \nu_\gamma(\lambda) = s_1 + s_2 \). Suppose \( s_1 + s_2 = \ell d + r_0 \) for some integer \( r_0 \), where \( \ell d = \text{ind}(\alpha) \). Since \( \ell d \alpha = 0, \ell d r_0 \alpha = 0 \). Thus, \( \gamma \) is of type 6. Hence \( P_0 \) is not a special point and \( X_P \) has all the required properties.

Suppose \( s_1 + s_2 = \ell d r_0 \) with \( t \leq d \) and \( r_0 \) coprime to \( \ell \). Then label the points \( P_0 \) and \( P_1 \) and let \( \gamma_1 \) and \( \gamma_2 \) be the exceptional curves in this blow-up. Then we have \( \nu_{\gamma_1}(\lambda) = 2s_1 + s_2 \) and \( \nu_{\gamma_2}(\lambda) = s_1 + 2s_2 \). If \( 2s_1 + s_2 \) is not of the form \( \ell d + r_1 \) for some \( r_1 \geq 1 \), then blow up the point of intersection of the strict transforms of \( \gamma_1 \) and \( \gamma_2 \). If \( s_1 + 2s_2 \) is not of the form \( \ell d + r_2 \) for some \( r_2 \geq 1 \), then blow up the point of intersection of the strict transforms of \( \eta_2 \) and \( \gamma_2 \). Since \( s_1 \) and \( s_2 \) are coprime to \( \ell \), there exist \( i \) and \( j \) such that \( is_1 + s_2 = \ell d + r_1 \) and \( s_1 + js_2 = \ell d + r' \) for some \( r, r' \geq 1 \). Thus, we get the required finite sequence of blow-ups.

\[\Box\]

Proposition 8.5. There exists a regular proper model of \( F \) with no special points.

Proof. Let \( P \in \mathcal{P} \). Then there exist two codimension zero points \( \eta_1 \) and \( \eta_2 \) of \( X_0 \) intersecting at \( P \).

Suppose that \( P \) is a special point of type I. Let \( \psi : X' \to X \) be a sequence of blow-ups as in Lemma 8.4. Then there are no special points in \( \psi^{-1}(P) \). Since there are only finitely many
special points in $\mathcal{X}$, replacing $\mathcal{X}$ by a finite sequence of blow-ups at all special points of type I, we assume that $\mathcal{X}$ has no special points of type I.

Suppose $P$ is a special point of type II. Without loss of generality we assume that $\eta_1$ is of type 1 and $\eta_2$ is of type 4. Let $\mathcal{X}_P \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ at $P$ and $\gamma$ the exceptional curve in $\mathcal{X}_P$. Since $\eta_2$ is of type 4, by Lemma 8.3, $\gamma$ is of type 2, 4 or 6. Since $\eta_1$ is of type 1 and $\eta_2$ is of type 4, $\nu_{\eta_1}(\lambda)$ is coprime to $\ell$ and $\nu_{\eta_2}(\lambda)$ is divisible by $\ell$. Since $\nu_\gamma(\lambda) = \nu_{\eta_1}(\lambda) + \nu_{\eta_2}(\lambda)$, $\nu_\gamma(\lambda)$ is coprime to $\ell$ and hence $\gamma$ is of type 2. Let $\tilde{\eta}_i$ be the strict transform of $\eta_i$ in $\mathcal{X}_P$. Then $\tilde{\eta}_1$ and $\gamma$ intersect at only one point $Q_1$. Since $\gamma$ is of type 2, $Q_1$ is a special point of type I and $Q_2$ is not a special point. Thus, as above, by replacing $\mathcal{X}$ by a sequence of blow-ups of $\mathcal{X}$, we assume that $\mathcal{X}$ has no special points of type I or II.

Suppose $P$ is a special point of type III. Without loss of generality assume that $\eta_1$ is of type 3 or 5 and $\eta_2$ is of type 4. Let $\mathcal{X}_P \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ at $P$, $\gamma$, $\tilde{\eta}_i$, and $Q_1$ be as above. Since $\eta_2$ is of type 4, by Lemma 8.3, $\gamma$ is of type 2, 4 or 6. Since $\nu_{\eta_1}(\lambda)$ and $\nu_{\eta_2}(\lambda)$ are divisible by $\ell$, $\nu_\gamma(\lambda) = \nu_{\eta_1}(\lambda) + \nu_{\eta_2}(\lambda)$ is divisible by $\ell$. Thus $\gamma$ is of type 4 or 6. Hence $Q_2$ is not a special point. By Corollary 5.7, $\alpha \otimes F_P = (E_P, \sigma, u\pi^{d_1}_{\eta_1} \pi^{d_2}_{\eta_2})$ for some cyclic extension $E_P/F_P$, $u \in \mathcal{A}_P$ a unit, and at least one of the $d_i$ is coprime to $\ell$ (in fact equal to 1). In particular, $\alpha \otimes F_P$ is split by the extension $F_P(\sqrt{u\pi^{d_1}_{\eta_1} \pi^{d_2}_{\eta_2}})$, where $m$ is the degree of $E_P/F_P$ which is a power of $\ell$. Suppose $d_1 + d_2$ is coprime to $\ell$. Since $\nu_\gamma(u\pi^{d_1}_{\eta_1} \pi^{d_2}_{\eta_2}) = d_1 + d_2$, $F_P(\sqrt{u\pi^{d_1}_{\eta_1} \pi^{d_2}_{\eta_2}})$ is totally ramified at $\gamma$. Thus, by Lemma 4.3, $\gamma$ is of type 6. Hence $Q_1$ is not a special point. Suppose that $d_1 + d_2$ is divisible by $\ell$. Let $\pi_\gamma$ be a prime defining $\gamma$ at $Q_1$. Then we have $w\pi^{d_1}_{\eta_1} \pi^{d_2}_{\eta_2} = w_1 \pi^{d_1}_{\eta_1} \pi^{d_2}_{\eta_2}$ for some unit $w_1$ at $Q_1$. Since one of $d_i$ is coprime to $\ell$ and $d_1 + d_2$ is divisible by $\ell$, the $d_i$ are not divisible by $\ell$. In particular, $2d_1 + d_2$ is coprime to $\ell$. Let $\mathcal{X}_{Q_1}$ be the blow-up of $\mathcal{X}_P$ at $Q_1$ and $\gamma'$ be the generic point of the exceptional curve in $\mathcal{X}_{Q_1}$. Then $\nu_{\gamma'}(u\pi^{d_1}_{\eta_1} \pi^{d_2}_{\eta_2}) = \nu_{\gamma'}(w_1 \pi^{d_1}_{\eta_1} \pi^{d_2}_{\eta_2}) = 2d_1 + d_2$. Since $2d_1 + d_2$ is coprime to $\ell$, once again by Lemma 4.3, $\gamma'$ is of type 6. In particular, no point on the exceptional curve in $\mathcal{X}_{Q_1}$ is a special point. Thus, replacing $\mathcal{X}$ by a sequence of blow-ups, we assume that $\mathcal{X}$ has no special points of type I, II or III.

Suppose $P$ is a special point of type IV. Without loss of generality assume that $\eta_1$ is of type 1, 3 or 5 and $\eta_2$ is of type 5, with $M_{\eta_2} \otimes F_{P,\eta_2}$ not a field. Let $\mathcal{X}_P \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ at $P$ and $\gamma$, $\tilde{\eta}_i$, $Q_1$ be as above. Since $M_{\eta_2} \otimes F_{P,\eta_2}$ is not a field, by Lemma 8.3, $\gamma$ is of type 2, 4 or 6. If $\gamma$ is of type 6, then $Q_1$ and $Q_2$ are not special points. Suppose $\gamma$ is of type 2 or 4. Then $Q_1$ and $Q_2$ are special points of type I, II or III. Thus, as above, by replacing $\mathcal{X}$ by a sequence of blow-ups of $\mathcal{X}$, we assume that $\mathcal{X}$ has no special points.

Let $\eta$ and $\eta'$ be two codimension zero points of $X_0$ (need not be distinct). A type 2 connection from $\eta$ to $\eta'$ is a sequence of distinct codimension zero points $\eta_1, \ldots, \eta_n$ of $X_0$ of type 2 such that $\eta$ intersects $\eta_1$, $\eta'$ intersects $\eta_n$, $\eta_i$ intersects $\eta_{i+1}$ for all $1 \leq i \leq n - 1$, $\eta$ does not intersect $\eta_i$ for $i > 1$, $\eta'$ does not intersect $\eta_i$ for $i < n$, $\eta_i$ does not intersect $\eta_j$ for $i < j \neq i + 1$ and if $\eta = \eta'$, then $n \geq 2$.

We note that if $\eta$ is a codimension zero point of $X_0$ of type 2 and $\eta'$ is any other codimension zero point of $X_0$ intersecting $\eta$ at a closed point, then there is a type 2 connection from $\eta$ to $\eta'$. This can be seen by taking $n = 1$ and $\eta_1 = \eta$.

Proposition 8.6. There exists a regular proper model $\mathcal{X}$ of $F$ such that:

1. $\mathcal{X}$ has no special points;  
2. if $\eta_1$ and $\eta_2$ are two (not necessarily distinct) codimension zero points of $X_0$ with $\eta_1$ of type 3 or 5 and $\eta_2$ of type 3, 4 or 5, then there is no type 2 connection between $\eta_1$ and $\eta_2$.
Proof. Let $\mathcal{X}$ be a regular proper model with no special points (Proposition 8.5). Let $m(\mathcal{X})$ be the number of type 2 connections between a point of type 3 or 5 and a point of type 3, 4 or 5. We prove the proposition by induction on $m(\mathcal{X})$. Suppose $m(\mathcal{X}) \geq 1$. We show that there is a sequence of blow-ups $\mathcal{X}'$ of $\mathcal{X}$ with no special points and $m(\mathcal{X}') < m(\mathcal{X})$.

Let $\eta$ be a codimension zero point of $X_0$ of type 3 or 5 and $\eta'$ a codimension zero point of $X_0$ of types 3, 4 or 5. Suppose there is a type 2 connection from $\eta$ to $\eta'$. Then there exist distinct codimension zero points $\eta_1, \ldots, \eta_n$ of $X_0$ of type 2 with $\eta$ intersecting $\eta_1$, $\eta'$ intersecting $\eta_n$ and $\eta_i$ intersecting $\eta_{i+1}$ for $i = 1, \ldots, n - 1$.

Suppose $n = 1$. Let $Q$ be the point of the intersection of $\eta$ and $\eta_1$. Let $\mathcal{X}_Q \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ at $Q$ and $\gamma$ the exceptional curve in $\mathcal{X}_Q$. Since $\eta_1$ is of type 2, by Lemma 8.3, $\gamma$ is of type 2, 4 or 6. Since $\eta$ is of type 3 or 5 and $\eta_1$ is of type 2, $\ell$ divides $\nu_\eta(\lambda)$ and $\ell$ does not divide $\nu_{\eta_1}(\lambda)$. Since $\nu_{\gamma}(\lambda) = \nu_\eta(\lambda) + \nu_{\eta_1}(\lambda)$, $\nu_{\gamma}(\lambda)$ is not divisible by $\ell$ and hence $\gamma$ is of type 2. Let $\tilde{\eta}$ and $\tilde{\eta}_1$ be the strict transform of $\eta$ and $\eta_1$ in $\mathcal{X}_Q$. Since $\gamma$ is a point of type 2, the points of intersection of $\tilde{\eta}$ and $\tilde{\eta}_1$ with $\gamma$ are not special points. Hence $\mathcal{X}_Q$ has no special points. Replacing $\mathcal{X}$ by $\mathcal{X}_Q$, we assume that $n \geq 2$ and $\mathcal{X}$ has no special points.

Let $P$ be the point of intersection of $\eta_1$ and $\eta_2$. Let $\mathcal{X}'$ be as in Lemma 8.4. Then $\mathcal{X}'$ has no special points and all the exceptional curves in $\mathcal{X}'$ are of type 2, 4 or 6 and the exceptional curves which intersect the strict transforms of $\eta_1$ and $\eta_2$ are of type 6. In particular, the number of type 2 connections between the strict transforms of $\eta$ and $\eta'$ is one less than the number of type 2 connections between $\eta$ and $\eta'$. Since all the exceptional curves in $\mathcal{X}'$ are of type 2, 4 or 6, $m(\mathcal{X}') = m(\mathcal{X}) - 1$. Thus, by induction, we have a regular proper model with the required properties.

Lemma 8.7. Let $\mathcal{X}$ be as in Proposition 8.6 and $X_0$ the special fiber of $\mathcal{X}$. Let $\eta$ be a codimension zero point of $X_0$ of type 2 and $\eta'$ a codimension zero point of $X_0$ of type 3 or 5. Suppose there is a type 2 connection from $\eta$ to $\eta'$. If there is a type 2 connection from $\eta$ to a type 3 or 5 point $\eta''$, then $\eta'' = \eta'$. Further, if $\eta_1, \ldots, \eta_n$ are codimension zero points of $X_0$ of type 2 giving a type 2 connection from $\eta$ to $\eta'$ and $\gamma_1, \ldots, \gamma_m$ codimension zero points of $X_0$ of type 2 giving another type 2 connection from $\eta$ to $\eta'$, then $n = m$ and $\eta_i = \gamma_i$ for all $i$.

Proof. Suppose $\eta''$ is a codimension zero point of $X_0$ of type 3 or 5 with type 2 connection to $\eta$. Since $\eta$ is of type 2, there is a type 2 connection from $\eta'$ to $\eta''$. Since no two points of type 3 or 5 have a type 2 connection (cf. Proposition 8.6), $\eta'' = \eta'$. Suppose $\gamma_1, \ldots, \gamma_m$ is of type 2 connection from $\eta$ to $\eta'$. If $m \neq n$ or $\eta_i \neq \gamma_i$ for some $i$, then we will have a type 2 connection from $\eta'$ to $\eta'$ and hence a contradiction to the choice of $\mathcal{X}$ (cf. Proposition 8.6). Thus $n = m$ and $\eta_i = \gamma_i$ for all $i$.

Let $\eta$ be a codimension zero point of $X_0$ of type 2 and $\eta'$ be a codimension zero point of $X_0$ of type 3 or 5. Suppose there is a type 2 connection $\eta_1, \ldots, \eta_n$ from $\eta$ to $\eta'$. Then, by Lemma 8.7, $\eta'$ and $\eta_n$ are uniquely defined by $\eta$. We call this point of intersection of $\eta_n$ with $\eta'$ the point of type 2 intersection of $\eta$ and $\eta'$. Once again note that such a closed point is uniquely defined by $\eta$.

9. Choice of $L_P$ and $\mu_P$ at closed points

Let $F, \alpha \in H^2(F, \mu_n)$, $\lambda \in F^*$ with $\alpha \cdot (-\lambda) = 0$ in $H^3(F, \mu_n^{\otimes 2})$, $\mathcal{X}$ and $X_0$ be as in (§§ 7 and 8). Throughout this section we assume that $\mathcal{X}$ has no special points and if $\eta_1$ and $\eta_2$ are two (not
necessarily distinct) codimension zero points of \( X_0 \) with \( \eta_1 \) is of type 3 or 5 and \( \eta_2 \) is of type 3, 4 or 5, then there is no type 2 connection between \( \eta_1 \) and \( \eta_2 \). Further, assume that \( F \) contains a primitive \( \ell \)th root of unity.

Let \( \eta \) be a codimension zero point of \( X_0 \) of type 5. Then we call \( \eta \) of \textit{type 5a} if \( \alpha \) is unramified at \( \eta \) and of \textit{type 5b} if \( \alpha \) is ramified at \( \eta \). Suppose \( \eta \) is of type 5b. Then \( \alpha \) is ramified and hence \( M_\eta \) is the unique subextension of \( E_\eta \) of degree \( \ell \), where \((E_\eta, \sigma_\eta)\) is the lift of the residue of \( \alpha \).

**Lemma 9.1.** Let \( \eta \) be a codimension zero point of \( X_0 \) of type 5b. Then \( \text{ind}(\alpha \otimes M_\eta) < \text{ind}(\alpha) \) and there exists \( \mu_\eta \in M_\eta \) such that \( N_{M_\eta/F_\eta}(\mu_\eta) = -\lambda \) and \( \alpha \cdot (\mu_\eta) = 0 \) in \( H^3(M_\eta, \mu_\eta^{\otimes 2}) \).

**Proof.** Since \( \eta \) is of type 5b, \( \alpha \) is ramified at \( \eta \), \( \nu_\eta(\lambda) = r\ell \), \( r\alpha \otimes E_\eta = 0 \) and \( E_\eta \neq F_\eta \). Thus, as in the proof of Lemma 4.11, there exists \( \mu_\eta \in M_\eta \) such that \( N_{M_\eta/F_\eta}(\mu_\eta) = -\lambda \) and \( \alpha \cdot (\mu_\eta) = 0 \).

**Lemma 9.2.** Let \( P \in \mathcal{A} \), and \( \eta_1 \) and \( \eta_2 \) be codimension zero points of \( X_0 \) containing \( P \). Suppose that \( \eta_1 \) and \( \eta_2 \) are of type 5. Then there exist a cyclic field extension \( L_P/F_P \) of degree \( \ell \) and \( \mu_P \in L_P \) such that:

1. \( N_{L_P/F_P}(\mu_P) = -\lambda \);
2. \( \text{ind}(\alpha \otimes L_P) < \text{ind}(\alpha) \);
3. \( \alpha \cdot (\mu_P) = 0 \) in \( H^3(L_P, \mu_\eta^{\otimes 2}) \);
4. \( \text{if } \eta_i \text{ is of type 5a, then } L_P \otimes F_{P,\eta_i}/F_{P,\eta_i} \text{ is an unramified field extension;} \)
5. \( \text{if } \eta_i \text{ is of type 5b, then } L_P \otimes F_{P,\eta_i} \simeq M_{\eta_i} \otimes F_{P,\eta_i} \).

**Proof.** Since \( \mathcal{A} \) has no special points, \( P \) is not a special point of type IV. Since \( \eta_1 \) and \( \eta_2 \) are of type 5 intersecting at \( P \), \( M_{\eta_1} \otimes F_{P,\eta_1} \) and \( M_{\eta_2} \otimes F_{P,\eta_2} \) are fields. Suppose \( \eta_i \) is of type 5a. If \( \alpha \otimes F_{P,\eta_i} = 0 \), then let \( L_{P,\eta_i} \) be any cyclic unramified field extension with \( -\lambda \) a norm and \( \mu_{\eta_i} \in L_{P,\eta_i} \) with \( N_{L_{P,\eta_i}/F_{P,\eta_i}}(\mu_{\eta_i}) = -\lambda \). If \( \alpha \otimes F_{P,\eta_i} \neq 0 \), then let \( L_{P,\eta_i} \) be a cyclic unramified field extension of degree \( \ell \) and \( \mu_{\eta_i} \) be as in Lemma 4.10. Suppose \( \eta_i \) is of type 5b. Let \( L_{P,\eta_i} = M_{\eta_i} \otimes F_{P,\eta_i} \) and \( \mu_{\eta_i} \in M_{\eta_i} \) be as in Lemma 9.1. Then, by choice \( L_{P,\eta_i}/F_{P,\eta_i} \) are unramified field extensions. By applying Lemma 6.4 to \( L_{P,\eta_i} \) and \( \mu_{\eta_i} \), there exist a cyclic field extension \( L_P/F_P \) and \( \mu_P \in L_P \) with the required properties.

**Lemma 9.3.** Let \( \eta \) be a codimension zero point of \( X_0 \) with \( \nu_\eta(\lambda) \) a multiple of \( \ell \) and \( P \) a closed point on \( \eta \). Then there exists a cyclic unramified field extension \( L_{P,\eta}/F_{P,\eta} \) of degree \( \ell \) and \( \mu_{P,\eta} \in L_{P,\eta} \) such that \( N_{L_{P,\eta}/F_{P,\eta}}(\mu_{P,\eta}) = -\lambda \) and \( \alpha \cdot (\mu_{P,\eta}) = 0 \). Further, if \( \eta \) is of type 3 or 4, then \( \text{ind}(\alpha \otimes E_\eta \otimes L_{P,\eta}) < \text{ind}(\alpha \otimes E_\eta) \).

**Proof.** Since \( \nu_\eta(\lambda) \) is divisible by \( \ell \), write \( \lambda = \theta\pi_\eta^{\ell} \) for some \( \theta \in F_\eta \) a unit at \( \eta \) and integer \( r \). Write \( \alpha \otimes F_\eta = \alpha' + (E_\eta, \sigma_\eta, \pi_\eta) \) as in Lemma 4.1. Let \( \overline{\alpha'} \) be the image of \( \alpha' \) in \( H^2(\kappa(\eta), \mu_\eta) \) and \( \theta_0 \) be the image of \( \theta \) in \( \kappa(\eta) \). Since \( \kappa(\eta)_P \) is a local field containing a primitive \( \ell \)th root of unity, there exists a cyclic field extension \( L(\eta)_P/\kappa(\eta)_P \) of degree \( \ell \) such that \( \theta_0 \) is a norm from \( L(\eta)_P \) (cf. the proof of Lemma 2.8). Let \( L_{P,\eta}/F_{P,\eta} \) be the unramified extension of degree \( \ell \) with residue field \( L(\eta)_P \). Since \( \overline{\theta} \) is a norm from \( L(\eta)_P \), \( -\theta \) is a norm from \( L_{P,\eta} \) and hence \( -\lambda = -\theta\pi_\eta^{\ell} \) is a norm from \( L_{P,\eta} \). Since \( N_{L_{P,\eta}/F_{P,\eta}}(\mu_{P,\eta}) = -\lambda \), \( L_{P,\eta}/F_{P,\eta} \) is a field extension and \( \alpha \cdot (\lambda) = 0 \), by Proposition 4.6, we have \( \alpha \cdot (\mu_{P,\eta}) = 0 \).

Suppose \( \eta \) is of type 3 or 4. Then \( r\alpha' \otimes E_\eta = r\alpha \otimes E_\eta \neq 0 \) and hence \( r\alpha' \otimes E(\eta) \neq 0 \). Thus, by Lemma 3.3, \( \text{ind}(\overline{\alpha'} \otimes E(\eta) \otimes L(\eta)_P) < \text{ind}(\overline{\alpha'} \otimes E(\eta)) \). Suppose \( \alpha \otimes E_\eta \otimes L_{P,\eta} \neq 0 \). Since \( \alpha \otimes E_\eta = \alpha' \otimes E_\eta \), \( \alpha' \otimes E_\eta = 0 \) and hence \( \overline{\alpha'} \otimes E(\eta) \neq 0 \) by Proposition 4.6, we have \( \alpha \cdot (\mu_{P,\eta}) = 0 \).

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Lemma 9.4. Let $P \in \mathcal{P}$, and $\eta_1$ and $\eta_2$ be codimension zero points of $X_0$ containing $P$. Suppose that $\eta_1$ is of type 2 and $\eta_2$ is of type 5 or 6. Then there exist $\mu_1 \in F_P$, $1 \leq i \leq \ell$, such that:

1. $\mu_1 \cdots \mu_{\ell} = -\lambda$;
2. $\nu_{\eta_1}(\mu_1) = \nu_{\eta_1}(\lambda)$, $\nu_{\eta_1}(\mu_i) = 0$ for $i \geq 2$;
3. $\nu_{\eta_2}(\mu_i) = \nu_{\eta_2}(\lambda)/\ell$ for all $i \geq 1$;
4. $\alpha \cdot (\mu_i) = 0 \in H^3(F_P, \mu_i^{\otimes 2})$.

Proof. Since $\eta_1$ is of type 2 and $\eta_2$ is of type 5 or 6, we have $\lambda = w \pi_{\eta_1} r_1 \pi_{\eta_2}^2$ with $r_1$ coprime to $\ell$ and $r_2 \otimes E_{\eta_2} = 0$. Hence, by Lemma 6.7, there exists $\theta \in F_P$ such that $\alpha \cdot (\theta) = 0$, $\nu_{\eta_1}(\theta) = 0$ and $\nu_{\eta_2} = \nu_{\eta_2}(\theta) = r_2$. For $i \geq 2$, let $\alpha = \theta$ and $\mu_1 = -\lambda \theta^{1-\ell}$. Then the $\mu_i$ have the required properties.

Lemma 9.5. Let $P \in \mathcal{P}$, and $\eta_1$ and $\eta_2$ be codimension zero points of $X_0$ containing $P$. Suppose that $\eta_1$ and $\eta_2$ are of type 5 or 6. Then there exist $\mu_1 \in F_P$, $1 \leq i \leq \ell$, such that:

1. $\mu_1 \cdots \mu_{\ell} = -\lambda$;
2. $\nu_{\eta_1}(\mu_i) = \nu_{\eta_1}(\lambda)/\ell$ for all $i \geq 0$ and $j = 1, 2$;
3. $\alpha \cdot (\mu_i) = 0 \in H^3(F_P, \mu_i^{\otimes 2})$.

Proof. Since $\eta_1$ and $\eta_2$ are of type 5 or 6, by Lemma 6.8, there exists $\theta \in F_P$ such that $\alpha \cdot (\theta) = 0$ and $\nu_{\eta_1}(\theta) = \nu_{\eta_1}(\lambda)/\ell$ for $i = 1, 2$. For $i \geq 2$, let $\mu_1 = \theta$ and $\mu_1 = -\lambda \theta^{1-\ell} \in F_P$. Then the $\mu_i$ have the required properties.

Lemma 9.6. Let $P \in \mathcal{P}$, $\eta_1$ be a codimension zero point of $X_0$ of type 3 and $\eta_2$ a codimension zero point of $X_0$ of type 5. Suppose $\eta_1$ and $\eta_2$ intersect at $P$. Then there exist a cyclic field extension $L_P/F_P$ of degree $\ell$ and $\mu_P \in L_P$ such that:

1. $N_{L_P/F_P}(\mu_P) = -\lambda$;
2. $\text{ind}(\alpha \otimes L_P) < \text{ind}(\alpha)$;
3. $\alpha \cdot (\mu_P) = 0 \in H^3(L_P, \mu_P^{\otimes 2})$;
4. $L_P \otimes F_{P, \eta_1} = F_{P, \eta_2}$ is an unramified field extension for $i = 1, 2$;
5. $\text{ind}(\alpha \otimes (E_{\eta_1} \otimes F_{P, \eta_1}) \otimes (L_P \otimes F_{P, \eta_1})) < \text{ind}(\alpha \otimes E_{\eta_1})$;
6. If $\ell_2$ is of type 5b, then $L_P \otimes F_{P, \eta_2} \simeq M_{\eta_2} \otimes F_{P, \eta_2}$.

Proof. Suppose $\lambda \notin \pm F_{\eta}^{\pm}$. Let $L_P = F_P(\sqrt{\lambda})$ and $\mu_P = -\sqrt{\lambda}$. Then $N_{L_P/F_P}(\mu_P) = -\lambda$ and, by Lemma 6.2, (2) and (3) are satisfied. Since $\eta_1$ is of type 3 or 5, $\nu_{\eta_1}(\lambda)$ is divisible by $\ell$ and hence (4) is satisfied. Since $\lambda \notin F_{\eta}^{\pm}$, case (5) does not arise. Suppose that $\eta_2$ is of type 5b. Since $\mathcal{X}$ has no special points, $M_{\eta_2} \otimes F_{P, \eta_2}$ is a field. Since $-\lambda$ is a norm from $M_{\eta_2}$ (Lemma 9.1), by Lemma 2.6, we have $L_P \otimes F_{P, \eta_2} \simeq M_{\eta_2} \otimes F_{P, \eta_2}$.

Suppose that $\lambda \in F_{\eta}^+$ or $-\lambda \in F_{\eta}^-$. Let $L_{P, \eta_1}$ and $\mu_{P, \eta_1} \in L_{P, \eta_1}$ be as in Lemma 9.3. Write $\alpha \otimes E_{\eta_1} = \alpha_1 + (E_{\eta_1}, \sigma_1, \pi_{\eta_1})$ as in Lemma 4.1. Then, by Lemma 4.2, we have $\text{ind}(\alpha \otimes E_{\eta_1}) = \text{ind}(\alpha \otimes E_{\eta_1})[(E_{\eta_1}, \sigma_1, \pi_{\eta_1})].$ Since $\eta_1$ is of type 3, by the choice of $L_{P, \eta_1}$ (cf. Lemma 9.3), $\text{ind}(\alpha \otimes E_{\eta_1} \otimes L_{P, \eta_1}) < \text{ind}(\alpha \otimes E_{\eta_1}).$ We have $\text{ind}(\alpha \otimes L_{P, \eta_1}) < \text{ind}(\alpha \otimes E_{\eta_1} \otimes L_{P, \eta_1})[(E_{\eta_1}, \sigma_1, \pi_{\eta_1})] < \text{ind}(\alpha \otimes E_{\eta_1})[(E_{\eta_1}, \sigma_1, \pi_{\eta_1})] = \text{ind}(\alpha)$.

Suppose that $\eta_2$ is of type 5a. Let $L_{P, \eta_2}$ and $\mu_{P, \eta_2} \in L_{P, \eta_2}$ be as in Lemma 9.3. Since $\eta_2$ is of type 5a, $\alpha$ is unramified at $\eta_2$. Since $L_{P, \eta_2} / F_{P, \eta_2}$ is an unramified field extension, $\text{ind}(\alpha \otimes L_{P, \eta_2}) < \text{ind}(\alpha)$.
Suppose \( \eta_2 \) is of type 5b. Since \( \mathcal{R} \) has no special points, \( M_{\eta_2} \otimes F_{P,\eta_2} \) is a field. Let \( L_{P,\eta_2} = M_{\eta_2} \otimes F_{P,\eta_2} \). Then, by Lemma 9.1, there exists \( \mu_{P,\eta_2} \in L_{P,\eta_2} \) such that \( N_{L_{P,\eta_2}/F_{P,\eta_2}}(\mu_{P,\eta_2}) = -\lambda \), \( \text{ind}(\alpha \otimes L_{P,\eta_2}) < \text{ind}(\alpha) \) and \( \alpha \cdot (\mu_{P,\eta_2}) = 0 \).

Then, by Lemma 6.4, there exist \( L_P \) and \( \mu_P \) with the required properties.

**Lemma 9.7.** Let \( P \in \mathcal{P}, \) and \( \eta_1 \) and \( \eta_2 \) be codimension zero points of \( X_0 \) of type 3, 4 or 6. Suppose \( \eta_1 \) and \( \eta_2 \) intersect at \( P \). Then there exist a cyclic field extension \( L_P/F_P \) of degree \( \ell \) and \( \mu_P \in L_P \) such that:

1. \( N_{L_P/F_P}(\mu_P) = -\lambda; \)
2. \( \text{ind}(\alpha \otimes L_P) < \text{ind}(\alpha); \)
3. \( \alpha \cdot (\mu_P) = 0 \in H^3(L_P, \mu_n \otimes 2); \)
4. \( L_P \otimes F_{P,\eta_1}/F_{P,\eta_1} \) is an unramified field extension;
5. if \( \eta_i \) is of type 3, \( \lambda \in F_{P,\eta}^* \) or \( -\lambda \in F_{P,\eta}^* \), then \( \text{ind}(\alpha \otimes (E_{\eta_i} \otimes F_{P,\eta_i}) \otimes (L_P \otimes F_{P,\eta})) < \text{ind}(\alpha \otimes E_{\eta_i}). \)

**Proof.** Suppose \( \lambda \notin \pm F_{P,\eta}^* \). Then, as in the proof of Lemma 9.6, \( L_P = F_P(\sqrt{\lambda}) \) and \( \mu_P = -\sqrt{\lambda} \) have the required properties.

Suppose that \( \lambda \in F_P^* \) or \( -\lambda \in F_P^* \). For \( i = 1, 2 \), let \( L_{P,\eta_i} \) and \( \mu_{P,\eta_i} \in L_{P,\eta_i} \) be as in Lemma 9.3. If \( \eta_i \) is of type 3, then as in the proof of Lemma 9.6, \( \text{ind}(\alpha \otimes L_{P,\eta_i}) < \text{ind}(\alpha) \). Suppose \( \eta_i \) is of type 4 or 6. Then \( \text{ind}(\alpha \otimes E_{\eta_i}) < \text{ind}(\alpha) \) and hence \( \text{ind}(\alpha \otimes L_{P,\eta_i}) < \text{ind}(\alpha) \).

Then, by Lemma 6.4, there exist \( L_P \) and \( \mu_P \) with the required properties.

**Proposition 9.8.** Let \( P \in \mathcal{P} \). Then there exist a cyclic field extension or split extension \( L_P/F_P \) of degree \( \ell \) and \( \mu_P \in L_P \) such that:

1. \( N_{L_P/F_P}(\mu_P) = -\lambda; \)
2. \( \text{ind}(\alpha \otimes L_P) < \text{ind}(\alpha); \)
3. \( \alpha \cdot (\mu_P) = 0 \in H^3(L_P, \mu_n \otimes 2). \)

Further, suppose \( \eta \) is a codimension zero point of \( X_0 \) containing \( P \).

4. If \( \eta \) is of type 1, then \( L_P = F_P(\sqrt{\lambda}) \) and \( \mu_P = -\sqrt{\lambda} \).
5. Suppose \( \eta \) is of type 2 with a type 2 connection to a type 5 point \( \eta' \). Let \( Q \) be the type 2 intersection point of \( \eta \) and \( \eta' \). If \( M_{\eta'} \otimes F_{Q,\eta'} \) is not a field, then \( L_P = \prod F_P \) and \( \mu_P = (\theta_1, \ldots, \theta_\ell) \) with \( \theta_i \in F_P, \nu_\eta(\theta_i) = \nu_\eta(\lambda) \) and \( \nu_\eta(\theta_i) = 0 \) for \( i \geq 2 \).
6. Suppose \( \eta \) is of type 2 with a type 2 connection to a type 5 point \( \eta' \). Let \( Q \) be the type 2 intersection point of \( \eta \) and \( \eta' \). If \( M_{\eta'} \otimes F_{Q,\eta'} \) is a field, then \( L_P = F_P(\sqrt{\lambda}) \) and \( \mu_P = -\sqrt{\lambda} \).
7. Suppose \( \eta \) is of type 2 and there is no type 2 connection from \( \eta \) to any type 5 point. Then \( L_P = F_P(\sqrt{\lambda}) \) and \( \mu_P = -\sqrt{\lambda} \).
8. If \( \eta \) is of type 3, then \( L_P \otimes F_{P,\eta}/F_{P,\eta} \) is an unramified field extension. Further, if \( \lambda \in F_P^* \) or \( -\lambda \in F_P^* \), then \( \text{ind}(\alpha \otimes (E_{\eta} \otimes F_{P,\eta}) \otimes (L_P \otimes F_{P,\eta})) < \text{ind}(\alpha \otimes E_{\eta}). \)
9. If \( \eta \) is of type 4, then \( L_P \otimes F_{P,\eta}/F_{P,\eta} \) is an unramified field extension.
10. If \( \eta \) is of type 5a, then \( L_P \otimes F_{P,\eta}/F_{P,\eta} \) is an unramified field extension.
11. If \( \eta \) is of type 5b, then \( L_P \otimes F_{P,\eta} \simeq M_\eta \otimes F_{P,\eta}, \) and if \( L_P = \prod F_P, \) then \( \mu_P = (\theta_1, \ldots, \theta_\ell) \) with \( \nu_\eta(\theta_i) = \nu_\eta(\lambda)/\ell \).
(12) If $\eta$ is of type 6, then either $L_P \otimes F_{P,\eta}/F_{P,\eta}$ is an unramified field extension or $L_P = \prod F_P$, with $\mu_P = (\theta_1, \ldots, \theta_l)$ and $\nu(P) = \nu(\lambda)/\ell$.

**Proof.** Let $\eta_1$ and $\eta_2$ be two codimension zero points of $X_0$ intersecting at $P$. By the choice of $\mathcal{X}$, $X_0$ is a union of regular curves with normal crossings and hence there are no other codimension zero points of $X_0$ passing through $P$.

**Case I.** Suppose that either $\eta_1$ or $\eta_2$, say $\eta_1$, is of type 1. Then $\nu_{\eta_1}(\lambda)$ is coprime to $\ell$ and hence $\lambda \notin \pm F_{P,\eta}$, $L_P = F_P(\sqrt{\lambda})$ and $\mu_P = -\sqrt{\lambda}$. Then, by Lemma 6.2, $L_P$ and $\mu_P$ satisfy (1), (2) and (3). By choice (4) is satisfied. Since $\mathcal{X}$ has no special points, $\eta_2$ is not of type 2 or 4. Thus (5), (6), (7) and (9) do not arise. Suppose $\eta_2$ is of type 3, 5 or 6. Then $\nu_{\eta_2}(\lambda)$ is divisible by $\ell$ and hence $L_P \otimes F_{P,\eta_2}/F_{P,\eta_2}$ is an unramified field extension. Thus (8), (10) and (12) are satisfied. Suppose $\eta_2$ is of type 5b. Since $\mathcal{X}$ has no special points and $\eta_1$ is of type 1, $M_{\eta_2} \otimes F_{P,\eta_2}$ is a field. Since $-\lambda$ is a norm from the extension $M_{\eta_2}/F_{\eta_2}$ (Lemma 9.1) and $\lambda \notin \pm F_{P,\eta_2}$ (Corollary 5.6), by (Lemma 2.6), $M_{\eta_2} \otimes F_{P,\eta_2} \simeq F_{P,\eta_2}(\sqrt{\lambda})$ and hence (11) is satisfied.

**Case II.** Suppose neither $\eta_1$ nor $\eta_2$ is of type 1. Suppose either $\eta_1$ or $\eta_2$ is of type 2, say $\eta_1$ is of type 2. Then $\nu_{\eta_1}(\lambda)$ is coprime to $\ell$ and hence $\lambda \notin \pm F_{P,\eta}$. Suppose that $\eta_1$ has type 2 connection to a codimension zero point $\eta'$ of $X_0$ of type 5. Let $Q$ be the closed point on $\eta'$ which is the type 2 intersection point of $\eta_1$ and $\eta'$. By the choice of $\mathcal{X}$ (cf. Proposition 8.6), $\eta_2$ is of type 2, 5 or 6. Note that if $\eta_2$ is also of type 2, then $Q$ is also the point of type 2 intersection of $\eta_2$ and $\eta'$. Thus if both $\eta_1$ and $\eta_2$ are of type 2, $\eta'$ and $Q$ do not depend on whether we start with $\eta_1$ or $\eta_2$.

Suppose that $M_{\eta_1} \otimes F_{P,\eta_1}$ is not a field. Let $L_P = \prod F_P$. Suppose $\eta_2$ is of type 2. Then let $\mu_P = (\lambda_1, \ldots, 1) \in L_P = \prod F_P$. Suppose $\eta_2$ is of type 5. Then by the assumption on $\mathcal{X}$, $\eta_2 = \eta'$, $Q = P$. Thus $M_{\eta_2} \otimes F_{P,\eta_2} = M_{\eta'} \otimes F_{Q,\eta'}$ is not a field and hence $\eta_2$ is of type 5b. Let $\mu_i \in F_P$ be as in Lemma 9.4, and $\mu_P = (\mu_1, \ldots, \mu_l)$. Suppose $\eta_2$ is of type 6. Let $\mu_i \in F_P$ be as in Lemma 9.4, and $\mu_P = (\mu_1, \ldots, \mu_l) \in L_P$. Then $L_P$ and $\mu_P$ satisfy (1) and (3). Since $\eta_1$ is of type 2, $\text{ind}(\alpha \otimes F_{\eta_1}) < \text{ind}(\alpha)$ and hence, by Proposition 5.8, $\text{ind}(\alpha \otimes F_P) < \text{ind}(\alpha)$ and (2) is satisfied. Since neither $\eta_1$ nor $\eta_2$ is of type 1, case (4) does not arise. By choice $L_P$ satisfies (5).

Since there is only one type 5 point with a type 2 connection to $\eta_1$ or $\eta_2$, case (6) does not arise. Clearly case (7) does not arise. Since $\eta_2$ is not of type 3, 4 or 5a, cases (8), (9) and (10) do not arise. By the choice of $L_P$ and $\mu_P$, (11) and (12) are satisfied.

Suppose $M_{\eta_2} \otimes F_{Q,\eta_2}$ is a field. Let $L_P = F_P(\sqrt{\lambda})$ and $\mu_P = -\sqrt{\lambda}$. Since $\lambda \notin \pm F_{P,\eta}$, by Lemma 6.2, $L_P$ and $\mu_P$ satisfy (1), (2) and (3). As above, cases (4), (5), (7), (8) and (9) do not arise. By choice (6) is satisfied. Suppose $\eta_2$ is of type 5. Then $\eta_1 = \eta'$, $Q = P$ and $\nu_{\eta_2}(\lambda)$ is divisible by $\ell$ and hence (10) is satisfied. Suppose $\eta_2$ is of type 5b. Since $M_{\eta_2} \otimes F_{P,\eta_2}$ is a field, as in case I, $M_{\eta_2} \otimes F_{P,\eta_2} \simeq L_P \otimes F_{P,\eta_2}$ and hence (11) is satisfied. If $\eta_2$ is of type 6, then $\nu_{\eta_2}(\lambda)$ is divisible by $\ell$ and $L_P \otimes F_{P,\eta_2}/F_{P,\eta_2}$ is an unramified field extension and hence (12) is satisfied.

Suppose that $\eta_1$ has no type 2 connection to a point of type 5. In particular, $\eta_2$ is not of type 5. Then, let $L_P = F_P(\sqrt{\lambda})$ and $\mu_P = -\sqrt{\lambda}$. Then, by Lemma 6.2, $L_P$ and $\mu_P$ satisfy (1), (2) and (3). Since neither $\eta_1$ nor $\eta_2$ has type 2 connection to a point of type 5, (5) and (6) do not arise. By the choice of $L_P$ and $\mu_P$, (7) is satisfied. If $\eta_2$ is of type 3, 4 or 6, then $\nu_{\eta_2}(\lambda)$ is divisible by $\ell$ and (8), (9) and (12) are satisfied. Since neither $\eta_1$ nor $\eta_2$ is of type 5, (10) and (11) do not arise.

**Case III.** Suppose neither of $\eta_i$ is of type 1 or 2. Suppose that one of the $\eta_i$, say $\eta_1$, is of type 3. Since $\mathcal{X}$ has no special points, $\eta_2$ is not of type 4 and hence $\eta_2$ is of type 3, 5 or 6. If $\eta_2$ is of type 5, let $L_P$ and $\mu_P$ be as in Lemma 9.6. If $\eta_2$ is of type 3 or 6, let $L_P$ and $\mu_P$ be as
in Lemma 9.7. Then, (1), (2), (3), (8), (9), (10), (11) and (12) are satisfied and the other cases do not arise.

Case IV. Suppose neither of \( \eta_i \) is of type 1, 2 or 3. Suppose that one of the \( \eta_i \), say \( \eta_1 \), is of type 4. Since \( \mathcal{X} \) has no special points, \( \eta_2 \) is not of type 5. Hence \( \eta_2 \) is of type 4 or 6. Let \( L_P \) and \( \mu_P \) be as in Lemma 9.7. Then \( L_P \) and \( \mu_P \) have the required properties.

Case V. Suppose neither of \( \eta_i \) is of type 1, 2, 3 or 4. Suppose that one of the \( \eta_i \) is of type 5, say \( \eta_1 \) is of type 5. Then \( \eta_2 \) is of type 5 or 6. Suppose that \( \eta_2 \) is of type 5. Since \( \mathcal{X} \) has no special points, \( M_{\eta_1} \otimes F_{P,\eta_1} \) are fields for \( i = 1, 2 \). Let \( L_P \) and \( \mu_P \) be as in Lemma 9.2. Then \( L_P \) and \( \mu_P \) have the required properties.

Suppose that \( \eta_2 \) is of type 6. Suppose that \( \eta_1 \) is of type 5a. Let \( L_{P,\eta_1} \) and \( \mu_{P,\eta_1} \) be as in Lemma 4.10. Since \( \nu_i(\lambda) \) is divisible by \( \ell \), by the construction of \( L_{P,\eta_1} \), \( L_{P,\eta_1}/F_{P,\eta_1} \) are unramified. Let \( L_P, \mu_P \in L_P \) be as in Lemma 6.4. Then \( L_P, \mu_P \) have the required properties. Suppose that \( \eta_1 \) is of type 5b. Suppose \( M_{\eta_1} \otimes F_{P,\eta_1} \) is a field with the residue field \( M(\eta_1)P \) of \( M_{\eta_1} \otimes F_{P,\eta_1} \) unramified over \( \kappa(\eta_1)P \). Let \( L_{P,\eta_1} = M_{\eta_1} \otimes F_{P,\eta_1} \) and \( \mu_{\eta_1} \in M_{\eta_1} \) with \( N_{M_{\eta_1}/F_{\eta_i}}(\mu_{\eta_1}) = -\lambda \) (cf. Lemma 9.1). Let \( L_P \) and \( \mu_P \) be as in Lemma 6.5 with \( L_P \otimes F_{P,\eta_1} \cong L_{P,\eta_1} \). Then \( L_P \) is a field with \( L_P/F_P \) unramified on \( A_P \) (cf. Lemma 6.5) and hence \( L_P \) and \( \mu_P \) have the required properties.

Suppose that \( M_{\eta_1} \otimes F_{P,\eta_1} \) is a field extension and the residue field \( M(\eta_1)P \) of \( M_{\eta_1} \otimes F_{P,\eta_1} \) is ramified over \( \kappa(\eta_1)P \). Then \( M_{\eta_1} \otimes F_{P,\eta_1} = F_{P,\eta_1}(\sqrt{v_P\pi_{\eta_1}}) \) for some unit \( v_P \) at \( P \) (cf. proof of Lemma 6.4). Since \( \lambda = w_P\pi_{\eta_1}^\ell \pi_{\eta_2}^2 \) for some unit \( w_P \) at \( P \) and \( -\lambda \) is a norm from \( M_{\eta_1} \otimes F_{P,\eta_1} \), it follows that the image \( -\overline{w_P} \) of \( w_P \) in \( \kappa(\eta_1)P \) is a norm from \( M(\eta_1)P \). Since \( w_P \) is a unit and \( M(\eta_1)P/\kappa(\eta_1)P \) is a ramified extension, it follows that \( -w_P \in F_{P,\eta_1}^\ell \) and hence \( -w_P \in F_{P,\eta_1}^\ell \). Let \( L_P = F_P(\sqrt{v_P\pi_{\eta_2}^2 + \pi_{\eta_1}}) \) and \( \mu_P = \sqrt{-\lambda} \in F_P \). Then \( N_{L_P/F_P}(\mu_P) = -\lambda \). Since \( \eta_2 \) is of type 6, ind\((\alpha \otimes \pi_{\eta_1}) < \text{ind}(\alpha) \) and hence, by Proposition 5.8, \( \text{ind}(\alpha \otimes \pi_{\eta_1}) < \text{ind}(\alpha) \). In particular, ind\((\alpha \otimes L_P) < \text{ind}(\alpha) \). Let \( B_P \) be the integral closure of the local ring \( A_P \) at \( P \) in \( L_P \). Since the maximal ideal \( m_P \) at \( P \) is equal to \( (\pi_{\eta_1}, \pi_{\eta_2}) \), \( v_P\pi_{\eta_2} + \pi_{\eta_1} \) is a regular prime and hence \( B_P \) is a regular local ring. Since \( \text{cor}_{L_P/F_{P,\eta_1}}(\alpha \cdot (\mu_P)) = \alpha \cdot (-\lambda) = 0 \) and \( L_{P,\eta_1}/F_{P,\eta_1} \) is a field extension, by Proposition 4.6, \( \alpha \cdot (\mu_P) = 0 \) in \( H^3(L_P \otimes F_{P,\eta_1}, \mu_{\eta_1}^\otimes 2) \) for \( i = 1, 2 \). In particular, \( \alpha \cdot (\mu_P) \) is unramified on \( B_P \) and hence \( \alpha \cdot (\mu_P) = 0 \) (cf. Lemma 5.3). Thus \( L_P \) and \( \mu_P \) satisfy the required properties.

Suppose that \( M_{\eta_1} \otimes F_{P,\eta_1} \) is not a field. Let \( L_P = \prod F_P \) and \( \mu_i \in F_P \) be as in Lemma 9.5, and \( \mu_P = (\mu_1, \ldots, \mu_i) \in L_P \). Then \( L_P \) and \( \mu_P \) have the required properties.

Case VI. Suppose neither of \( \eta_i \) is of type 1, 2, 3, 4 or 5. Then, \( \eta_1 \) and \( \eta_2 \) are of type 6. Let \( L_P \) and \( \mu_P \) be as in Lemma 9.7. Then \( L_P \) and \( \mu_P \) have the required properties.

\subsection{10. Choice of \( L_\eta \) and \( \mu_\eta \) at codimension zero points}

Let \( F, n = \ell^4, \alpha \in H^2(F, \mu_n), \lambda \in F^* \) with \( \alpha \neq 0, \alpha \cdot (-\lambda) = 0 \in H^3(F, \mu_n^\otimes 2), \mathcal{X}, X_0 \) and \( \mathcal{P} \) be as in §§ 7–9. Assume that \( \mathcal{X} \) has no special points and that there is no type 2 connection between a codimension zero point of \( X_0 \) of type 3 or 5 and a codimension zero point of \( X_0 \) of type 3, 4 or 5.

For a codimension zero point \( \eta \in X_0 \), let \( \mathcal{P}_\eta = \eta \cap \mathcal{P} \).

\begin{proposition}
Let \( \eta \) be a codimension zero point of \( X_0 \) of type 1. For each \( P \in \mathcal{P}_\eta \), let \((L_P, \mu_P)\) be chosen as in Proposition 9.8, and \( L_\eta = F_\eta(\sqrt{\lambda}) \) and \( \mu_\eta = -\sqrt{\lambda} \in L_\eta \). Then:
\begin{enumerate}
\item \( N_{L_\eta/F_\eta}(\mu_\eta) = -\lambda \);
\item \( \alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_\eta^\otimes 2) \);
\end{enumerate}
\end{proposition}
(3) \(\text{ind}(\alpha \otimes L_\eta) < \text{ind}(\alpha)\);
(4) for \(P \in \mathcal{P}_\eta\), there is an isomorphism \(\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}\) and
\[\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} = 1.\]

**Proof.** By choice, we have \(N_{L_\eta/F_\eta}(\mu_\eta) = -\lambda\). Since \(\eta\) is of type 1, \(\nu_\eta(\lambda)\) is coprime to \(\ell\) and hence by Lemma 4.7, \(L_\eta\) and \(\mu_\eta\) satisfy (2) and (3). Let \(P \in \mathcal{P}_\eta\). Since \(\eta\) is of type 1, by the choice of \(L_P\) and \(\mu_P\) (cf. Proposition 9.8(4)), we have \(L_P = F_P(\sqrt{\lambda})\) and \(\mu_P = -\sqrt{\lambda}\). Hence \(L_\eta\) and \(\mu_\eta\) satisfy (4).

**Lemma 10.2.** Let \(\eta\) be a codimension zero point of \(X_0\). For each \(P \in \mathcal{P}_\eta\), let \(\theta_P \in F_P\) with \(\alpha \cdot (\theta_P) = 0 \in H^3(F_{P,\eta}, \mu_\eta^{\otimes 2})\). Suppose \(\nu_\eta(\theta_P) = 0\) for all \(P \in \mathcal{P}_\eta\). Then there exists \(\theta_\eta \in F_\eta\) such that:

1. \(\alpha \cdot (\theta_\eta) = 0 \in H^3(F_\eta, \mu_\eta^{\otimes 2})\);
2. for \(P \in \mathcal{P}_\eta\), \(\theta_P^{-1}\theta_\eta \in F_{P,\eta}^{\text{em}}\) for all \(m \geq 1\).

**Proof.** Let \(\pi_\eta \in F_\eta\) be a parameter. Write \(\alpha \otimes F_\eta = \alpha' + (E_\eta, \sigma_\eta, \pi_\eta)\) as in Lemma 4.1. Let \(E(\eta)\) be the residue field of \(E_\eta\). Since \(\alpha \cdot (\theta_P) = 0 \in H^3(F_{P,\eta}, \mu_\eta^{\otimes 2})\) and \(\nu_\eta(\theta_P) = 0\), by Lemma 4.7, we have \((E(\eta) \otimes \kappa(\eta)_P, \sigma_0, \theta_P) = 0 \in H^2(\kappa(\eta)_P, \mu_\eta)\), where \(\theta_P\) is the image of \(\theta_P \in \kappa(\eta)_P\). Hence \(\theta_P\) is a norm from \(E(\eta) \otimes \kappa(\eta)_P\) for all \(P \in \mathcal{P}_\eta\). For \(P \in \mathcal{P}_\eta\), let \(\theta_P \in E(\eta) \otimes \kappa(\eta)_P\) with \(N_{E(\eta) \otimes \kappa(\eta)_P/\kappa(\eta)}(\theta_P) = \theta_P\). By weak approximation, there exists \(\theta \in E(\eta) \otimes \kappa(\eta)\) which is sufficiently close to \(\theta_P\) for all \(P \in \mathcal{P}_\eta\). Let \(\theta_0 = N_{E(\eta)/\kappa(\eta)}(\theta) \in \kappa(\eta)\). Then \(\theta_0\) is sufficiently close to \(\theta_P\) for all \(P \in \mathcal{P}_\eta\). In particular, \(\theta_0^{-1}\theta_P \in \kappa(\eta)^{\text{em}}_P\) for all \(m \geq 1\). Let \(\theta_\eta \in F_\eta\) have image \(\theta_0\) in \(\kappa(\eta)\). Then \((E_\eta, \sigma_\eta, \theta_\eta) = 0\) and hence, by Lemma 4.7, \(\alpha \cdot (\theta_\eta) = 0\). Since \(\theta_0^{-1}\theta_P \in \kappa(\eta)^{\text{em}}_P\) for all \(m \geq 1\) and \(F_{P,\eta}\) is a complete discretely valued field with residue field \(\kappa(\eta)_P\), it follows that \(\theta_P^{-1}\theta_\eta \in F_{P,\eta}^{\text{em}}\) for all \(m \geq 1\).

**Proposition 10.3.** Let \(\eta\) be a codimension zero point of \(X_0\) of type 2. Suppose there is a type 2 connection between \(\eta\) and a codimension zero point \(\eta'\) of \(X_0\) of type 5. Let \(Q\) be the point of type 2 intersection of \(\eta\) and \(\eta'\). Suppose that \(M_{\eta'} \otimes F_{Q,\eta'}\) is not a field. For each \(P \in \mathcal{P}_\eta\), let \(\mu_P = (\theta_1^P, \ldots, \theta_\ell^P) \in L_P = \prod F_P\) be as in Proposition 9.8(5). Let \(L_\eta = \prod F_\eta\). Then there exists \(\mu_\eta = (\theta_1^\eta, \ldots, \theta_\ell^\eta) \in L_\eta\) such that:

1. \(N_{L_\eta/F_\eta}(\mu_\eta) = -\lambda\);
2. \(\alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_\eta^{\otimes 2})\);
3. \(\text{ind}(\alpha \otimes L_\eta) < \text{ind}(\alpha)\);
4. \(\mu_P^{-1}\mu_\eta \in (L_\eta \otimes F_{P,\eta})^{\text{em}}\) for all \(P \in \mathcal{P}_\eta\) and \(m \geq 1\).

**Proof.** Let \(i \geq 2\). By choice (cf. Proposition 9.8(5)), we have \(\nu_\eta(\theta^P_\ell) = 0\) and \(\alpha \cdot (\theta^P_\ell) = 0 \in H^3(F_{P,\eta}, \mu_\eta^{\otimes 2})\) for all \(P \in \mathcal{P}_\eta\). By Lemma 10.2, there exists \(\theta_\ell^\eta \in F_\eta\) such that \(\alpha \cdot (\theta^\ell_1) = 0 \in H^3(F_\eta, \mu_\eta^{\otimes 2})\) and \(\theta^\ell_1^{-1}\theta^\ell_\ell \in F_{P,\eta}^{\text{em}}\) for all \(P \in \mathcal{P}_\eta\) and \(m \geq 1\). Let \(\theta^\ell_1 = -\lambda(\theta^\ell_2 \cdots \theta^\ell_\ell)^{-1}\). Then \(\theta_1^\ell \cdots \theta_\ell^\ell = -\lambda\) and \((\theta^\ell_1)^{-1}\theta^\ell_1 \in F_{P,\eta}^{\text{em}}\) for all \(m \geq 1\). Since \(\alpha \cdot (\theta^\ell_1) = 0 \in H^3(F_\eta, \mu_\eta^{\otimes 2})\) for \(i \geq 2\), we have \(\alpha \cdot (\theta^\ell_1) = 0 \in H^3(F_\eta, \mu_\eta^{\otimes 2})\). Let \(L_\eta = \prod F_\eta\) and \(\mu_\eta = (\theta^\ell_1, \ldots, \theta^\ell_\ell) \in L_\eta\). Since \(\eta\) is of type 2, \(\text{ind}(\alpha \otimes F_\eta) < \text{ind}(\alpha)\) and hence \(L_\eta, \mu_\eta\) have the required properties.

**Proposition 10.4.** Let \(\eta\) be a codimension zero point of \(X_0\) of type 2. For each \(P \in \mathcal{P}_\eta\), let \((L_P, \mu_P)\) be chosen as in Proposition 9.8. Suppose one of the following holds:
there is a type 2 connection between \( \eta \) and codimension zero point \( \eta' \) of \( X_0 \) of type 5 with 

\( Q \) the point of type 2 intersection of \( \eta \) and \( \eta' \), and \( M_{\eta'} \otimes F_{Q, \eta'} \) is a field;

there is no type 2 connection between \( \eta \) and any codimension zero point of \( X_0 \) of type 5.

Let \( L_\eta = F_\eta(\sqrt[\ell]{\lambda}) \) and \( \mu_\eta = -\sqrt[\ell]{\lambda} \). Then:

1. \( N_{L_\eta/F_\eta}(\mu_\eta) = -\lambda; \)
2. \( \alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_\eta^{\otimes 2}); \)
3. \( \text{ind}(\alpha \otimes L_\eta) < \text{ind}(\alpha); \)
4. for \( P \in \mathcal{P}_\eta \), there is an isomorphism \( \phi_{P, \eta} : L_\eta \otimes F_{P, \eta} \to L_P \otimes F_{P, \eta} \) and

\[
\phi_{P, \eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} = 1.
\]

Proof. Since \( \nu_\eta(\lambda) \) is coprime to \( \ell \), by Lemma 4.7, \( \alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_\eta^{\otimes 2}) \) and \( \text{ind}(\alpha \otimes L_\eta) < \text{ind}(\alpha) \). Clearly, \( N_{L_\eta/F_\eta}(\mu_\eta) = \lambda \). By the choice of \( (L_P, \mu_P) \) (cf. Proposition 9.8), for \( P \in \mathcal{P}_\eta \), we have \( L_P = F_P(\sqrt[\ell]{\lambda}) \) and \( \mu_P = -\sqrt[\ell]{\lambda} \). Thus \( L_\eta \) and \( \mu_\eta \) have the required properties.

Lemma 10.5. Let \( \eta \) be a codimension zero point of \( X_0 \) of type 3, 4 or 5a. Let \( P \in \eta \). Suppose there exists \( L_{P, \eta} = \frac{F_{P, \eta}}{E_{P, \eta}} \) an unramified field extension of degree \( \ell \) and \( \mu_{P, \eta} \in L_{P, \eta} \) such that:

1. \( N_{L_{P, \eta}/F_{P, \eta}}(\mu_{P, \eta}) = -\lambda; \)
2. \( \text{ind}(\alpha \otimes L_{P, \eta}) < \text{ind}(\alpha); \)
3. \( \alpha \cdot (\mu_{P, \eta}) = 0 \in H^3(L_{P, \eta}, \mu_{P, \eta}^{\otimes 2}); \)
4. if \( \eta \) is of type 3, \( \lambda \in F_{P, \eta}^\ell \) or \( -\lambda \in F_{P, \eta}^\ell \), then \( \text{ind}(\alpha \otimes (E_{P, \eta} \otimes F_{P, \eta}) \otimes (L_{P, \eta})) < \text{ind}(\alpha \otimes E_{P, \eta}) \).

Then \( \text{ind}(\alpha \otimes (E_{P, \eta} \otimes F_{P, \eta}) \otimes (L_{P, \eta})) < \text{ind}(\alpha)/[E_{P, \eta} : E_\eta] \).

Proof. Write \( \alpha \otimes F_\eta = \alpha' + (E_\eta, \sigma_\eta, \pi_\eta) \) as in Lemma 4.1. Then, by Lemma 4.2, \( \text{ind}(\alpha \otimes F_\eta) = \text{ind}(\alpha \otimes E_\eta)[E_\eta : F_\eta] = \text{ind}(\alpha \otimes E_\eta)[E_\eta : F_{P, \eta}] \). Let \( t = [E_\eta : F_\eta] \) and \( \beta \) be the image of \( \alpha' \) in \( H^2(\kappa(\eta), \mu_\eta) \).

Suppose \( \eta \) is of type 4. Then \( \text{ind}(\alpha \otimes F_\eta) < \text{ind}(\alpha) \) and hence \( \text{ind}(\alpha \otimes E_\eta) = \text{ind}(\alpha \otimes F_\eta)/t < \text{ind}(\alpha)/t \). Thus \( \text{ind}(\alpha \otimes (E_\eta \otimes F_{P, \eta}) \otimes (L_{P, \eta})) < \text{ind}(\alpha \otimes E_\eta) < \text{ind}(\alpha)/t \).

Suppose that \( \eta \) is of type 5a. Then \( \alpha \) is unramified at \( \eta \) and hence \( E_\eta = F_\eta \) and \( t = 1 \). The lemma is clear if \( \alpha \otimes F_{P, \eta} = 0 \). Suppose \( \alpha \otimes F_{P, \eta} \neq 0 \). Then \( \beta \otimes \kappa(\eta)_P \neq 0 \). Since \( L_{P, \eta} \) is an unramified field extension, the residue field \( L_P(\eta) \) of \( L_{P, \eta} \) is a field extension of \( \kappa(\eta)_P \) of degree \( \ell \). Since \( \kappa(\eta)_P \) is a local field and \( \beta \) is divisible by \( \ell \), \( \text{ind}(\beta \otimes L_P(\eta)) < \text{ind}(\beta)[CF67, p. 131] \).

In particular, \( \text{ind}(\alpha \otimes L_{P, \eta}) < \text{ind}(\alpha) \).

Suppose that \( \eta \) is of type 3. Then \( r\alpha \otimes E_\eta \neq 0 \) and hence \( r\alpha' \otimes E_\eta = r\alpha \otimes E_\eta \neq 0 \). In particular, \( r\beta \otimes E(\eta) \neq 0 \) and \( \text{ind}(\alpha \otimes F_\eta) > t \). Suppose \( \lambda \in F_{P, \eta}^\ell \) or \( -\lambda \in F_{P, \eta}^\ell \). Then, by the choice of \( L_{P, \eta} \), \( \text{ind}(\alpha \otimes (E_\eta \otimes F_{P, \eta}) \otimes (L_{P, \eta})) < \text{ind}(\alpha \otimes E_\eta) < \text{ind}(\alpha)/t \). Suppose \( \lambda \notin \pm F_{P, \eta}^\ell \). Then \( \lambda \) is not a norm from \( L_{P, \eta} \), by Lemma 2.6, \( L_{P, \eta} \simeq F_{P, \eta}(\sqrt[\ell]{\lambda}) \). Since \( \eta \) is of type 3, \( \nu_\eta(\lambda) = r\ell \) and \( \lambda = \theta_\eta \pi_\eta^\ell \) with \( \theta_\eta \in F_\eta \) a unit at \( \eta \). Let \( \overline{\theta}_\eta \) be the image of \( \theta_\eta \) in \( \kappa(\eta) \). Then \( \overline{\theta}_\eta \not\in \kappa(\eta)^\ell \) and \( L_{P, \eta} = \kappa(\eta)_P(\sqrt[\ell]{\overline{\theta}_\eta}) \). Since \( \alpha \cdot (\lambda) = 0 \), by Lemma 4.7, \( r\ell\alpha' = (E_\eta, \sigma_\eta, (1)^{\ell+1}\theta_\eta) \) and hence \( r\ell\beta = (E(\eta), \sigma_0, (1)^{\ell+1}\theta_\eta) \). Since \( \overline{\theta}_\eta \) is a norm from \( L_P(\eta) \) and \( L_P(\eta)/\kappa(\eta)_P \) is an extension of degree \( \ell \), \( (1)^{\ell+1}\overline{\theta}_\eta \) is a norm from \( L_P(\eta) \). Thus, by Lemma 3.3, \( \text{ind}(\beta \otimes E(\eta)_P \otimes L_P(\eta)) < \text{ind}(\beta \otimes E(\eta)) \). Thus

\[
\text{ind}(\alpha \otimes (E_\eta \otimes F_{P, \eta}) \otimes (L_{P, \eta})) = \text{ind}(\alpha' \otimes (E_\eta \otimes F_{P, \eta}) \otimes (L_{P, \eta}))
\]

\[
= \text{ind}(\beta \otimes E(\eta)_P \otimes L_P(\eta))
\]

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< ind(β ⊗ E(η)) = ind(α' ⊗ E_0) \\
= ind(α ⊗ E_0) = ind(α)/t. \hfill \square

**Proposition 10.6.** Let η be a codimension zero point of $X_0$ of type 3, 4 or 5α. For each $P \in \mathcal{P}_\eta$, let $(L_P, \mu_P)$ be chosen as in Proposition 9.8. Then there exist an unramified field extension $L_\eta/F_\eta$ of degree $\ell$ and $\mu_\eta \in L_\eta$ such that:

1. $N_{L_\eta/F_\eta}(\mu_\eta) = -\lambda$;
2. $\alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_\eta^2)$;
3. $\text{ind}(\alpha \otimes E_0) < \text{ind}(\alpha)$;
4. for $P \in \mathcal{P}_\eta$, there is an isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \rightarrow L_P \otimes F_{P,\eta}$ and

$$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^\ell\mu,$$

for all $m \geq 1$.

**Proof.** Since $\eta$ is of type 3, 4 or 5α, we have $\nu_\eta(\lambda) = r\ell$ for some integer $r$ and $\lambda = \theta_\eta\pi_\eta^p$ for some parameter $\pi_\eta$ at $\eta$ and $\theta_\eta \in F_\eta$ a unit at $\eta$. Write $\alpha \otimes E_\eta = \alpha' + (\nu_\eta, \sigma_\eta, \pi_\eta)$ as in Lemma 4.1. By Lemma 4.7, $r\ell\alpha' = (\nu_\eta, \sigma_\eta, (-1)^{r\ell+1}\theta_\eta)$. Let $\beta$ be the image of $\alpha'$ in $H^2(\kappa(\eta), \mu_\eta)$ and $E(\eta)$ the residue field of $E_\eta$. Then $r\ell\beta = (\nu(\eta), \sigma_0, (-1)^{r\ell+1}\theta_0) \in H^2(\kappa(\eta), \mu_\eta)$, where $\sigma_0$ is the automorphism of $E(\eta)$ induced by $\sigma_\eta$ and $\theta_0$ is the image of $\theta_\eta$ in $\kappa(\eta)$.

Let $S$ be a finite set of places of $\kappa(\eta)$ containing the places given by closed points of $\mathcal{P}_\eta$ and places $\nu$ of $\kappa(\eta)$ with $\beta \otimes \kappa(\eta)_\nu \neq 0$. Let $t = [E_\eta : F_\eta]$. For each $\nu \in S$, we now give a field extension $L_\nu/\kappa(\eta)_\nu$ of degree $\ell$ and $\mu_\nu \in L_\nu$ satisfying the conditions of Lemma 3.1 with $E_0 = E(\eta)$ and $d = \text{ind}(\alpha)/t$.

Let $\nu \in S$. Then $\nu$ is given by a closed point $P$ of $\eta$. If $P \in \mathcal{P}$, let $L_{P,\eta} = L_P \otimes F_{P,\eta}$ and $\mu_{P,\eta} = \mu_P \otimes 1 \in L_{P,\eta}$. Suppose that $P \notin \mathcal{P}$. Suppose that $\lambda \notin \pm F_P^\ell$. Then $\lambda \notin \pm F_P^\ell$. Let $L_{P,\eta} = F_{P,\eta}(\sqrt{\lambda})$ and $\mu_{P,\eta} = -\sqrt{\lambda}$. Suppose that $\lambda \in F_P^\ell$ or $-\lambda \in F_P^\ell$. Let $L_{P,\eta}/F_{P,\eta}$ be a cyclic unramified field extension of degree $\ell$ and $\mu_{P,\eta} \in L_{P,\eta}$ as in Lemma 9.3. Since $L_{P,\eta}/F_{P,\eta}$ is an unramified field extension of degree $\ell$, $\pi_\eta$ is a parameter in $L_{P,\eta}$ and the residue field $L_P(\eta)$ is a field extension of $\kappa(\eta)P$ of degree $\ell$. Let $L_\nu = L_P(\eta)$. Since $N_{L_{P,\eta}/F_{P,\eta}}(\mu_{P,\eta}) = -\lambda$, $\mu_{P,\eta} = \theta_{P,\eta}\pi_\eta^p$ for some $\theta_{P,\eta} \in L_{P,\eta}$ which is a unit at $\eta$. Let $\mu_\nu$ be the image of $\theta_{P,\eta}$ in $L_\nu = L_P(\eta)$. Then $N_{L_\nu/\kappa(\eta)}(\mu_\nu) = -\theta_0$. Since the corestriction map $H^2(L_\nu, \mu_\eta) \rightarrow H^2(\kappa(\eta)_\nu, \mu_\eta)$ is injective, $r\beta \otimes L_\nu = (E_0 \otimes L_\nu, \sigma_0 \otimes 1, (-1)^r\mu_0)$. By Lemma 10.5, we have $\text{ind}(\alpha \otimes (E_\eta \otimes F_{P,\eta}) \otimes L_{P,\eta}) < \text{ind}(\alpha)/t$. Since $\alpha \otimes E_\eta = \alpha' \otimes E_0$, we have $\text{ind}(\alpha' \otimes (E_\eta \otimes F_{P,\eta}) \otimes L_{P,\eta}) < \text{ind}(\alpha)/t$. Since $\text{ind}(\beta \otimes E_\eta \otimes L_\nu) = \text{ind}(\alpha' \otimes (E_\eta \otimes F_{P,\eta}) \otimes (L_{P,\eta}))$, $\text{ind}(\beta \otimes E_0 \otimes L_\nu) < \text{ind}(\alpha)/t$.

Since $\kappa(\eta)$ is a global field, by Lemma 3.1, there exist a field extension $L_0/\kappa(\eta)$ of degree $\ell$ and $\mu_0 \in L_0$ such that:

1. $N_{L_0/k}(\mu_0) = -\theta_0$;
2. $r\beta \otimes L_0 = (E(\eta) \otimes L_0, \sigma_0 \otimes 1, (-1)^r\mu_0)$;
3. $\text{ind}(\beta \otimes E(\eta) \otimes L_0) < \text{ind}(\alpha)/t$;
4. $L_0 \otimes \kappa(\eta)_P \simeq L_P(\eta)$ for all $P \in \mathcal{P}_\eta$;
5. $\mu_0$ is close to $\delta_{P,\eta}$ for all $P \in \mathcal{P}_\eta$.

Then, by Lemma 4.8, there exist a field extension $L_\eta/F_\eta$ of degree $\ell$ and $\mu \in L_\eta$ such that:

- the residue field of $L_\eta$ is $L_\eta$;
- $\mu$ is a unit in the valuation ring of $L_\eta$;
Proof. Let $E(\eta)$ and $M(\eta)$ be the residue fields of $E_\eta$ and $M_\eta$ at $\eta$. Since $\eta$ is of type 5b, $M(\eta)$ is the unique subfield of $E(\eta)$ with $M(\eta)/\kappa(\eta)$ a cyclic field extension of degree $\ell$. Let $\pi_\eta$ be a parameter at $\eta$. Since $\eta$ is of type 5, $\nu_\eta(\lambda) = r\ell$ and $\lambda = \theta_\eta \pi_\eta^{r\ell}$ for some $\theta_\eta \in F$ a unit at $\eta$. Let $\overline{\theta}_\eta$ be the image of $\theta_\eta$ in $\kappa(\eta)$. Let $P \in \mathcal{P}_\eta$. Suppose $M_\eta \otimes F_{P,\eta}$ is not a field. Then, by the choice of $\mu_P$ (cf. Proposition 9.8(11)), we have $\mu_P = \mu_\eta \pi_\eta^{r\ell}$, where $\mu_\eta = (\theta_1', \ldots, \theta_\ell') \in M_\eta \otimes F_{P,\eta}$ a unit at $\eta$. Let $\overline{\mu}_P$ be the image of $\mu_\eta$ in $\kappa(\eta)$ of type 5b. For each $P \in \mathcal{P}_\eta$, there is an isomorphism $\phi_{P,\eta} : M_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and

$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_\eta \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^\ell$,

for all $m \geq 1$.

PROPOSITION 10.7. Let $\eta$ be a codimension zero point of $X_0$ of type 5b. Let $(E_\eta, \sigma_\eta)$ be the lift of the residue of $\alpha$ at $\eta$ and $M_\eta$ be the unique subfield of $E_\eta$ with $M_\eta/F_\eta$ a cyclic extension of degree $\ell$. For each $P \in \mathcal{P}_\eta$, let $L_P$ and $\mu_P$ be as in Proposition 9.8. Then there exists $\mu_\eta \in M_\eta$ such that:

1. $N_{M_\eta/F_\eta}(\mu_\eta) = -\lambda$;
2. $\alpha \cdot (\mu_\eta) = 0 \in H^3(M_\eta, \mu_\eta^\otimes 2)$;
3. $\text{ind}(\alpha \otimes M_\eta) < \text{ind}(\alpha \otimes F_\eta)$;
4. for $P \in \mathcal{P}_\eta$, there is an isomorphism $\phi_{P,\eta} : M_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and

$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_\eta \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^\ell$,

for all $m \geq 1$.

Proof. Let $E(\eta)$ and $M(\eta)$ be the residue fields of $E_\eta$ and $M_\eta$ at $\eta$. Since $\eta$ is of type 5b, $M(\eta)$ is the unique subfield of $E(\eta)$ with $M(\eta)/\kappa(\eta)$ a cyclic field extension of degree $\ell$. Let $\pi_\eta$ be a parameter at $\eta$. Since $\eta$ is of type 5, $\nu_\eta(\lambda) = r\ell$ and $\lambda = \theta_\eta \pi_\eta^{r\ell}$ for some $\theta_\eta \in F$ a unit at $\eta$. Let $\overline{\theta}_\eta$ be the image of $\theta_\eta$ in $\kappa(\eta)$. Let $P \in \mathcal{P}_\eta$. Suppose $M_\eta \otimes F_{P,\eta}$ is not a field. Then, by the choice of $\mu_P$ (cf. Proposition 9.8(11)), we have $\mu_P = \mu_\eta \pi_\eta^{r\ell}$, where $\mu_\eta = (\theta_1', \ldots, \theta_\ell') \in M_\eta \otimes F_{P,\eta}$ a unit at $\eta$. Let $\overline{\mu}_P$ be the image of $\mu_\eta$ in $\kappa(\eta)$ of type 5b. For each $P \in \mathcal{P}_\eta$, there is an isomorphism $\phi_{P,\eta} : M_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and

$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_\eta \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^\ell$,

for all $m \geq 1$.

PROPOSITION 10.8. Let $\eta$ be a codimension zero point of $X_0$ of type 6. For each $P \in \mathcal{P}_\eta$, let $L_P$ and $\mu_P$ be as in Proposition 9.8. Then there exist an unramified field extension $L_\eta/F_\eta$ of degree $\ell$ and $\mu_\eta \in L_\eta$ such that:

1. $N_{L_\eta/F_\eta}(\mu_\eta) = -\lambda$;
2. $\alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_\eta^\otimes 2)$;

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Proof. Let $P \in \mathcal{P}_η$. Suppose $L_η \otimes F_{P_η}$ is a field. Let $L_η(η), \overline{η}_{P,η} \in L_η(η), θ_0 \in κ(η)$ and $β$ be as in the proof of Proposition 10.6. Then, as in the same proof, we have $η_{L_η(η)/κ(η)}(\overline{η}_{P}) = −θ_0$ and $\text{ind}(β \otimes E_0 \otimes L_η(η)) < \text{ind}(α)/[E_0 : F_η]$. As in the proof of Proposition 10.7, we have $rβ \otimes L_η(η) = (E_0 \otimes L_η(η), σ_0 \otimes 1, (−1)^{i} \overline{η}_{P})$.

If $L_η/F_η$ is not a field, by choice (cf. Proposition 9.8(12)), we have $μ_η = (θ_1 π_η^{r}, ..., θ_τ π_η^{r})$. Since $α \cdot (μ_η) = 0$ in $H^3(L_η, μ_η^{⊗2}) = \prod H^3(F_η, μ_η^{⊗2})$, we have $α \cdot (θ_1 π_η^{r}) = 0 \in H^3(F_η, μ_η^{⊗2})$. Thus, by Lemma 4.7, we have $rβ \otimes κ(η)_P = (E_0, σ_0 \otimes 1, (−1)^{i} \overline{η})$ for all $i$. Since $L_η(η) = \prod κ(η)_P$ and $\overline{η}_{P} = (\overline{η}_1, ..., \overline{η}_κ)$, we have $rβ \otimes L_η(η) = (E_0 \otimes L_η(η), σ_0 \otimes 1, (−1)^{i} \overline{η}_{P})$.

As in the proof of Proposition 10.6, we construct $L_η$ and $μ_η$ with the required properties. □

Lemma 10.9. Let $η$ be a codimension zero point of $X_0$ and $P$ a closed point on $η$. Suppose there exists $θ_η \in F_η$ such that $α \cdot (θ_η) = 0 \in H^3(F_η, μ_η^{⊗2})$. Then there exists $θ_P \in F_P$ such that $α \cdot (θ_P) = 0 \in H^3(F_P, μ_η^{⊗2})$, $ν_η(θ_P) = ν_η(θ_η)$ and $θ_P^{-1}θ_η \in F_{P_η}^{\text{fr}}$, for all $m \geq 1$.

Proof. Let $π$ be a prime representing $η$ at $P$. Since $X_0 \cup \text{ram} (α)$ has normal crossings, there exists a prime $δ$ at $P$ such that the maximal ideal at $P$ is generated by $π$ and $δ$, and $α$ is unramified at $P$, except possibly at $π$ and $δ$. Since $F_{P_η}$ is a complete discretely valued field with $π$ as a parameter, $θ_η = wπ^{s}$ for some $w \in F_η$ unit at $η$. Since the residue field $κ(η)_P$ of $F_{P_η}$ is a complete discretely valued field with $δ$ as a parameter, we have $\overline{w} = \overline{π}^{δ}$ for some $u \in F_P$ unit at $P$. Let $θ_P = uδ^{s}π^{s}$. Then clearly $ν_η(θ_η) = ν_η(θ_P)$ and $θ_P^{-1}θ_η \in F_{P_η}^{\text{fr}}$, for all $m \geq 1$. Since $α \cdot (θ_P)$ is unramified at $P$, except possibly at $π$ and $δ$, and $α \cdot (θ_P) = α \cdot (θ_η) = 0 \in H^3(F_P, μ_η^{⊗2})$, by Corollary 5.5, $α \cdot (θ_P) = 0 \in H^3(F_P, μ_η^{⊗2})$. □

11. The main theorem

Theorem 11.1. Let $K$ be a local field with residue field $κ$ and $F$ the function field of a curve over $K$. Let $D$ be a central simple algebra over $F$ of period $n$, $α$ its class in $H^2(F, μ_n)$, and $λ ∈ F^*$. If $α \cdot (−λ) = 0$ and $n$ is coprime to $\text{char}(κ)$, then $−λ$ is a reduced norm from $D^*$.

Proof. As in the proof of Theorem 4.12, we assume that $n = ℓ^d$ for prime $ℓ$ with $ℓ \neq \text{char}(κ)$ and $F$ contains a primitive $ℓth$ root of unity. We prove the theorem by induction on $\text{ind}(D)$.

The case $\text{ind}(D) = 1$ is clear. Assume that $\text{ind}(D) > 1$.

Without loss of generality we assume that $K$ is algebraically closed in $F$. Let $X$ be a regular projective geometrically irreducible curve over $K$ with $K(X) = F$. Let $R$ be the ring of integers in $K$ and $κ$ its residue field. Let $X'$ be a regular proper model of $F$ over $R$ such that the union of $\text{ram}(α)$, $\supp_{X}(λ)$ and the special fiber $X_0$ of $X'$ is a union of regular curves with normal crossings. By Proposition 8.6, we assume that $X'$ has no special points, and there is no type 2 connection between codimension zero points of $X_0$ of type 3 or 5, and codimension zero points of $X_0$ of type 3, 4 or 5.

Let $\mathcal{P}$ be the set of nodal points of $X_0$. For each $P \in \mathcal{P}$, let $L_η$ and $μ_η$ be as in Proposition 9.8. Let $η$ be a codimension zero point of $X_0$ and $\mathcal{P}_η = \mathcal{P} \cap η$. Let $L_η$ and $μ_η$ be as in Propositions 10.1, 10.3, 10.4, 10.6, 10.7 or 10.8 depending on the type of $η$. Then $L_η/F_{P_η}$ is a field or the split extension of degree $ℓ$ and $μ_η \in L_η$ such that:
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(1) $N_{L_{\eta}/F_{\eta}}(\mu_\eta) = -\lambda$;
(2) $\alpha \cdot (\mu_\eta) = 0 \in H^3(L_{\eta}, \mu_\eta \otimes \mathbb{Q}^2)$;
(3) $\text{ind}(\alpha \otimes L_{\eta}) < \text{ind}(\alpha)$;
(4) for $P \in \mathcal{P}_\eta$, there is an isomorphism $\phi_{P,\eta} : L_{\eta} \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and

$$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^m,$$

for all $m \geq 1$.

Let $P \in \mathcal{P}$ be a closed point with $P \not\in \mathcal{P}$. Then there is a unique codimension zero point $\eta$ of $X_0$ with $P \in \eta$. We give a choice of an étale algebra $L_P/F_P$ of degree $\ell$ and $\mu_P \in L_P^*$ such that:

(1) $N_{L_P/F_P}(\mu_P) = -\lambda$;
(2) $\text{ind}(\alpha \otimes L_P) < \text{ind}(\alpha)$;
(3) $\alpha \cdot (\mu_P) = 0 \in H^3(L_P, \mu_P \otimes \mathbb{Q}^2)$;
(4) there is an isomorphism $\phi_{P,\eta} : L_{\eta} \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and

$$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^m,$$

for all $m \geq 1$.

Suppose that $\eta$ is of type 1. Let $L_P = F_P(\sqrt{\lambda})$ and $\mu_P = -\sqrt{\lambda}$. Then, by Lemma 6.2 and Proposition 10.1, $L_P$ and $\mu_P$ have the required properties.

Suppose that $\eta$ is of type 2. Suppose that there is a type 2 connection to a codimension zero point $\eta'$ of $X_0$ of type 5. Let $Q$ be the point of type 2 intersection $\eta$ and $\eta'$. Suppose that $M_{\eta'} \otimes F_{Q,\eta'}$ not a field. Then, by choice (cf. Proposition 10.3), we have $L_{\eta} = \prod F_{\eta_i}$ and $\mu_{\eta} = (\theta_1, \ldots, \theta_\ell)$. Since $\alpha \cdot (\mu_{\eta}) = 0$, we have $\alpha \cdot (\theta_i) = 0$. For each $i, 2 \leq i \leq \ell$, by Lemma 10.9, there exists $\theta_{i,P} \in F_P$ such that $\alpha \cdot (\theta_{i,P}) = 0 \in H^3(F_P, \mu_P \otimes \mathbb{Q}^2)$ and $\theta_i^{-1}\theta_{i,P} \in F_{P,\eta}$, for all $m \geq 1$. Let $\theta_i = -\lambda(\theta_{i,2} \cdots \theta_{i,P})^{-1}$. Then $L_P = \prod F_P$ and $\mu_P = (\theta_{1,P}, \ldots, \theta_{\ell,P})$ have the required properties. Suppose that $M_{\eta'} \otimes F_{Q,\eta'}$ is a field or there is no type 2 connection from $\eta$ to any point of type 5. Then, by choice (Proposition 10.4), we have $L_{\eta} = F_\eta(\sqrt{\lambda})$ and $\mu_{\eta} = -\sqrt{\lambda}$. Hence $L_P = F_P(\sqrt{\lambda})$ and $\mu_P = -\sqrt{\lambda} \in L_P$ have the required properties (cf. Lemma 6.2).

Suppose that $\eta$ is not of type 1 or 2. Then, by choice, $L_{\eta}/F_{\eta}$ is an unramified field extension of degree $\ell$ or the split extension of degree $\ell$. Let $\hat{A}_P$ be the completion of the local ring at $P$ and $\pi$ a prime in $\hat{A}_P$ defining $\eta$ at $P$. Since $P \not\in \mathcal{P}$ and ram $\pi$ (\alpha) is union of regular curves with normal crossings, there exists a prime $\delta \in \hat{A}_P$ such that $\alpha$ is unramified on $\hat{A}_P$, except possibly at $\pi$ and $\delta$. Further, $\lambda = w\pi^r\delta^s$ for some unit $w \in \hat{A}_P$. Since $\eta$ is not of type 1 or 2, $\nu_\eta(\lambda) = r$ is divisible by $\ell$. Thus, by Lemma 6.5, there exist an étale algebra $L_P/F_P$ and $\mu_P \in L_P$ such that:

(1) $L_P \otimes F_{P,\eta} \simeq L_{\eta} \otimes F_{P,\eta}$;
(2) $\text{ind}(\alpha \otimes L_P) < \text{ind}(\alpha)$;
(3) $\alpha \cdot (\mu_P) = 0 \in H^3(L_P, \mu_P \otimes \mathbb{Q}^2)$;
(4) there is an isomorphism $\phi_{P,\eta} : L_{\eta} \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and

$$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^m,$$

for all $m \geq 1$.

Thus for every $x \in X_0$, we have chosen an étale algebra $L_x/F_x$ of degree $\ell$ and $\mu_x \in L_x$ such that:

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(1) $N_{L_x/F_x}(\mu_x) = -\lambda$;
(2) $\alpha\cdot(\mu_x) = 0 \in H^3(L_x, \mu_n^{\otimes 2})$;
(3) $\text{ind}(\alpha \otimes L_x) \leq \text{ind}(\alpha)$;
(4) for any branch $(P, \eta)$, there is an isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and
$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^m$, for all $m \geq 1$. Further, if $\eta$ is a codimension zero point of $X_0$, then $L_\eta/F_\eta$ is field or the split extension.

Let $(P, \eta)$ be a branch. Since $\kappa(P)$ is a finite field, there exists $t_P$ such that $\kappa(P)$ has no $\ell^{t_P}$th primitive root of unity. Since $\kappa(\eta)_P$ is a complete discretely valued field with residue field $\kappa(P)$, $\kappa(\eta)_P$ has no $\ell^{t_P}$th primitive root of unity. Since $F_{P,\eta}$ is a complete discretely valued field with residue field $\kappa(\eta)_P$, $F_{P,\eta}$ has no $\ell^{t_P}$th primitive root of unity.

Let $L/F$ be a degree $\ell$ extension as in Lemma 7.3. Then $\text{ind}(\alpha \otimes L) \leq \text{ind}(\alpha)$. Note that for every closed point $P$ of $X_0$, the residue field $\kappa(P)$ at $P$ is a finite field. Thus, for every closed point $P$ of $X_0$, there exists $t_P \geq d$ such that there is no primitive $\ell^{t_P}$th root of unity in $\kappa(P)$. Thus, by Proposition 7.5), there exist a field extension $N/F$ of degree coprime to $\ell$ and $\mu \in L \otimes N$ such that:

- $N_{\ell \otimes N/F}(\mu) = -\lambda$; and
- $\alpha \cdot (\mu) = 0 \in H^3(L \otimes N, \mu_n^{\otimes 2})$.

Since $L \otimes N$ is also a function field of a curve over a local field, by induction hypotheses, $\mu$ is a reduced norm from $D \otimes L \otimes N$ and hence $-\lambda = N_{\ell \otimes N/F}(\mu)$ is a reduced norm from $D$.

Since $N_{\ell \otimes N/F}(\lambda) = (-\lambda)^{[N:F]}$, $(-\lambda)^{[N:F]}$ is a norm from $D$. Since $[N:F]$ is coprime to $\ell$, $-\lambda$ is a reduced norm from $D$.

**Corollary 11.2.** Let $K$ be a local field with residue field $\kappa$ and $F$ the function field of a curve over $K$. Let $\Omega$ be the set of divisorial discrete valuations of $F$. Let $D$ be a central simple algebra over $F$ of period coprime to $\text{char}(\kappa)$ and $\lambda \in F$. If $\lambda$ is a reduced norm from $D \otimes F_\omega$ for all $\omega \in \Omega$, then $\lambda$ is a reduced norm from $D$.

**Proof.** Let $n$ be the period of $D$ and $\alpha \in H^2(F, \mu_n)$ be the class of $D$. Since $\lambda$ is a reduced norm from $F_\omega$ for all $\omega \in \Omega_F$, $\alpha \cdot (\lambda) = 0 \in H^3(F_\omega, \mu_n^{\otimes 2})$ for all $\omega \in \Omega$. Thus, by [Kat86, Proposition 5.2], $\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$ and by Theorem 11.1, $\lambda$ is a reduced norm from $D$. $\Box$

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