Projective Ribbon Permutation Statistics: a Remnant of non-Abelian Braiding in Higher Dimensions

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In a recent paper, Teo and Kane proposed a 3D model in which the defects support Majorana fermion zero modes. They argued that exchanging and twisting these defects would implement a set $R$ of unitary transformations on the zero mode Hilbert space which is a ‘ghostly’ recollection of the action of the braid group on Ising anyons in 2D. In this paper, we find the group $T_{2n}$, which governs the statistics of these defects by analyzing the topology of the space $K_{2n}$ of configurations of $2n$ defects in a slowly spatially-varying gapped free fermion Hamiltonian: $T_{2n} \equiv \pi_1(K_{2n})$. We find that the group $T_{2n} = \mathbb{Z} \times \mathbb{T}_{2n}$, where the ‘ribbon permutation group’ $\mathbb{T}_{2n}$ is a mild enhancement of the permutation group $S_{2n}$; $T_{2n} \equiv \mathbb{Z}_2 \times E((\mathbb{Z}_2)^{2n} \times S_{2n})$. Here, $E((\mathbb{Z}_2)^{2n} \times S_{2n})$ is the ‘even part’ of $(\mathbb{Z}_2)^{2n} \times S_{2n}$, namely those elements for which the total parity of the element in $(\mathbb{Z}_2)^{2n}$ added to the parity of the permutation is even. Surprisingly, $R$ is only a projective representation of $T_{2n}$, a possibility proposed by Wilczek. Thus, Teo and Kane’s defects realize ‘Projective Ribbon Permutation Statistics’, which we show to be consistent with locality. We extend this phenomenon to other dimensions, co-dimensions, and symmetry classes. Since it is an essential input for our calculation, we review the topological classification of gapped free fermion systems and its relation to Bott periodicity.

PACS numbers:

I. INTRODUCTION

In two dimensions, the configuration space of $n$ point-like particles $C_n^{2D}$ is multiply-connected. Its first homotopy group, or fundamental group, is the $n$-particle braid group, $\pi_1(C_n^{2D}) = B_n$. The braid group $B_n$ is generated by counterclockwise exchanges $\sigma_i$ of the $i$th and $(i+1)$th particles satisfying the defining relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for} \quad |i - j| \geq 2$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for} \quad 1 \leq i \leq n - 2$$

(1)

This is an infinite group, even for only two particles, since $(\sigma_i)^m$ is a non-trivial element of the group for any $m > 0$. In fact, even if we consider distinguishable particles, the resulting group, called the ‘pure Braid group’ is non-trivial. (For two particles, the pure braid group consists of all even powers of $\sigma_1$.)

In quantum mechanics, the equation $\pi_1(C_n^{2D}) = B_n$ opens the door to the possibility of anyons\cite{1,2}. Higher-dimensional representations of the braid group give rise to non-Abelian anyons\cite{3,4,5}. There has recently been intense effort directed towards observing non-Abelian anyons due, in part, to their potential use for fault-tolerant quantum computation\cite{6,7,8}. One of the simplest models of non-Abelian anyons is called Ising anyons. They arise in theoretical models of the $\nu = 5/2$ fractional quantum Hall state\cite{9,10} (see also Ref. 12), chiral $p$-wave superconductors\cite{13,14}, a solvable model of spins on the honeycomb lattice\cite{15}, and interfaces between superconductors and either 3D topological insulators\cite{16} or spin-polarized semiconductors with strong spin-orbit coupling\cite{17}. A special feature of Ising anyons, which makes them relatively simple and connects them to BCS superconductivity, is that they can be understood in a free fermion picture.

A collection of $2n$ Ising anyons has a $2n-1$-dimensional Hilbert space (assuming fixed boundary condition). This can be understood in terms of $2n$ Majorana fermion operators $\gamma_i = \gamma_i^\dagger$, $i = 1, 2, \ldots, n$, one associated to each Ising anyon, satisfying the anticommutation rules

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}. \quad \text{(2)}$$

The Hilbert space of $2n$ Ising anyons with fixed boundary condition furnishes a representation of this Clifford algebra; by restricting to fixed boundary condition, we obtain a representation of products of an even number of $\gamma$ matrices, which has minimal dimension $2^{n-1}$.

$$\rho(\gamma_i) = e^{i\pi/8} \ e^{-\pi\gamma_i \gamma_{i+1}/4}. \quad \text{(3)}$$

(There is a variant of Ising anyons, associated with $SU(2)_2$ Chern-Simons theory, for which the phase factor $e^{i\pi/8}$ is replaced by $e^{-i\pi/8}$. In the fractional quantum Hall effect, Ising anyons are tensored with Abelian anyons to form more complicated models with more particle species; the phase factor depends on the model.) A key property, essential for applications to quantum computing, is that a pair of Ising anyons forms a two-state system. The two states correspond to the two eigenvalues $\pm 1$ of $\gamma_i \gamma_j$. No local degree of freedom can be associated with each anyon; if we insisted on doing so, we would have to say that each Ising anyon has $\sqrt{2}$ internal states. In superconducting contexts, the $\gamma_i$s are the Bogoliubov-de Gennes operators for zero-energy modes (or, simply, ‘zero modes’) in vortex cores; the vortices themselves are Ising anyons if they possess a single such zero mode $\gamma_i$. Although the Hilbert space is non-local in the sense that it cannot be decomposed into the tensor product of local Hilbert spaces associated with each anyon, the system is perfectly compatible
with locality and arises in local lattice models and quantum field theories.

In three or more dimensions, the configuration space of $n$ point-like particles is simply-connected if the particles are distinguishable. If the particles are indistinguishable, it is multiply-connected, $\pi_1(T^n) = S_n$. The generators of the permutation group satisfy the relations (1) and one more, $\sigma_1^2 = 1$. As a result of this last relation, the permutation group is finite. The one-dimensional representations of $S_n$ correspond to bosons and fermions. One might have hoped that higher-dimensional representations of $S_n$ would give rise to interesting 3D analogues of non-Abelian anyons. However, this is not the case, as shown in Ref. 18,19: any higher-dimensional representation of $S_n$ which is compatible with locality can be decomposed into the tensor product of local Hilbert spaces associated with each particle. For instance, suppose we had $2n$ spin-1/2 particles but ignored their spin values. Then we would have $2^{2n}$ states which would transform into each other under permutations. Clearly, if we discovered such a system, we would simply conclude that we were missing some quantum number and set about trying to measure it. This would simply lead us back to bosons and fermions with additional quantum numbers. (The color quantum number of quarks was conjectured by essentially this kind of reasoning.) The quantum information contained in these $2^{2n}$ states would not have any special protection.

The preceding considerations point to the following tension. The Clifford algebra (2) of Majorana fermion zero modes is not special to two dimensions. One could imagine a three (or higher) dimensional system with topological defects supporting such zero modes. But the Hilbert space of these topological defects would be $2^{n-1}$-dimensional, which manifestly cannot be decomposed into the tensor product of local Hilbert spaces associated with each particle, seemingly in contradiction with the results of Refs. 18,19 on higher-dimensional representations of the permutation group described above. However, as long as no one had a three or higher dimensional system in hand with topological defects supporting Majorana fermion zero modes, one could, perhaps, sweep this worry under the rug. Recently, however, Teo and Kane\textsuperscript{20} have shown that a 3D system which is simultaneously a superconductor and a topological insulator\textsuperscript{21-24} (which, in many but not all examples, is arranged by forming superconductor-topological insulator heterostructures) supports Majorana zero modes at point-like topological defects.

To make matters worse, Teo and Kane\textsuperscript{20} further showed that exchanging these defects enacts unitary operations on this $2^{n-1}$-dimensional Hilbert space which are essentially equal to (3). But we know that these unitary matrices form a representation of the braid group, which is not the relevant group in 3D. One would naively expect that the relevant group is the permutation group, but $S_n$ has no such representation (and even if it did, its use in this context would contradict locality, according to Ref. 18,19 and arguments in Ref. 25). So this begs the question: what is the group $T_{2n}$, for which Teo and Kane’s unitary transformations form a representation?

With the answer to this question in hand, we could address questions such as the following. We know that a 3D incarna-

tion of Ising anyons is one possible representation of $T_{2n}$, is a 3D version of other anyons another representation of $T_{2n}$?

Attempts to generalize the braiding of anyons to higher dimensions sometimes start with extended objects, whose configuration space may have fundamental group which is richer than the permutation group. Obviously, if one has line-like defects in 3D which are all oriented in the same direction, then one is essentially back to the 2D situation governed by the braid group. This is too trivial, but it is not clear what kind of extended objects in higher dimensions would be the best starting point. What is clear, however, is that Teo and Kane’s topological defects must really be some sort of extended objects. This is clear from the above-noted contradiction with the permutation group. It also follows from the ‘order parameter’ fields which must deform as the defects are moved, as we will discuss.

In this paper, we show that Teo and Kane’s defects are properly viewed as point-like defects connected pair-wise by ribbons. We call the resulting 2n-particle configuration space $K_{2n}$. We compute its fundamental group $\pi_1(K_{2n})$, which we denote by $T_{2n}$, and find that $T_{2n} \cong \mathbb{Z} \times T_{2n}$. Here, $T_{2n}$ is the ‘ribbon permutation group’, defined by $T_{2n} \equiv \mathbb{Z}_2 \times E((\mathbb{Z}_2)^{2n} \times S_{2n})$. The group $E((\mathbb{Z}_2)^{2n} \times S_{2n})$ is a non-split extension of the permutation group $S_{2n}$ by $\mathbb{Z}_2^{2n-1}$ which is defined as follows: it is the subgroup of $(\mathbb{Z}_2)^{2n} \times S_{2n}$, composed of those elements for which the total parity of the element in $(\mathbb{Z}_2)^{2n}$ added to the parity of the permutation is even. The ‘ribbon permutation group’ for $2n$ particles, by $T_{2n}$, is the fundamental group of the reduced space of $2n$-particle configurations.

Our analysis relies on the topological classification of gapped free fermion Hamiltonians\textsuperscript{26,27} – band insulators and superconductors – which is the setting in which Teo and Kane’s 3D defects and their motions are defined. The starting point for this classification is reducing the problem from classifying gapped Hamiltonians defined on a lattice to classifying Dirac equations with a spatially varying mass term.

One can motivate the reduction to a Dirac equation as Teo and Kane do: they start from a lattice Hamiltonian and assume that the parameters in the Hamiltonian vary smoothly in space, so that the Hamiltonian can be described as a function of both the momentum $k$ and the position $r$. Near the minimum of the band gap, the Hamiltonian can be expanded in a Dirac equation, with a position-dependent mass term. In fact, Kitaev\textsuperscript{27} has shown that the reduction to the Dirac equation with a spatially varying mass term can be derived much more generally: gapped lattice Hamiltonians, even if the parameters in the Hamiltonian do not vary smoothly in space, are stably equivalent to Dirac Hamiltonians with a spatially varying mass term. Here, equivalence of two Hamiltonians means that one can be smoothly deformed into the other while preserving locality of interactions and the spectral gap, while stable equivalence means that one can add additional “trivial” degrees of freedom (additional sites which have vanishing hopping matrix elements) to the original lattice Hamiltonian to obtain a Hamiltonian which is equivalent to a lattice discretization of the Dirac Hamiltonian.

Since this classification of Dirac Hamiltonians is essential
for the definition of $K_{2n}$, we give a self-contained review, following Kitaev’s analysis\textsuperscript{27}. Our exposition parallels the discussion of Bott periodicity in Milnor’s book\textsuperscript{28}. The basic idea is that each additional discrete symmetry which squares to $-1$ which we impose on the system is encapsulated by an anti-symmetric matrix which defines a complex structure on $\mathbb{R}^N$, where $N/2$ is the number of bands (or, equivalently, $N$ is the number of bands of Majorana fermions). For any given system, these are chosen and fixed. This leads to a progression of symmetric spaces $O(N) \to O(N)/U(N/2) \to U(N/2)/Sp(N/4) \to \ldots$ as the number of such symmetries is increased. Following Kitaev\textsuperscript{27}, we view the Hamiltonian as a final anti-symmetric matrix which must be chosen (and, thus, put almost on the same footing as the symmetries); it is defined by a choice of an arbitrary point in the next symmetric space in the progression. The space of such Hamiltonians is topologically-equivalent to that symmetric space. However, as the spatial dimension is increased, $\gamma$-matrices squaring to $+1$ must be chosen in order to expand the Hamiltonian in the form of the Dirac equation in the vicinity of a minimum of the band gap. These halve the dimension of subspaces of $\mathbb{R}^N$ by separating it into their $+1$ and $-1$ eigenspaces and thereby lead to the opposite progression of symmetric spaces. Thus, taking into account both the symmetries of the system and the spatial dimension, we conclude that the space of gapped Hamiltonians with no symmetries in $d = 3$ is topologically equivalent to $U(N)/O(N)$. (However, by the preceding considerations, the same symmetric space also, for instance, classifies systems with time-reversal symmetry in $d = 4$.) All such Hamiltonians can be continuously deformed into each other without closing the gap, $\pi_0(U(N)/O(N)) = 0$. However, there are topologically-stable point-like defects classified by $\pi_2(U(N)/O(N)) = \mathbb{Z}_2$. These are the defects whose multi-defect configuration space we study in order to see what happens when they are exchanged.

The second key ingredient in our analysis is 1950’s-vintage homotopy theory, which we use to compute $\pi_1(K_{2n})$. We apply the Pontryagin-Thom construction to show that $K_{2n}$, which includes not only the particle locations but also the full field configuration around the particles (i.e. the way in which the gapped free fermion Hamiltonian of the system explores $U(N)/O(N)$), is topologically-equivalent to a much simpler space, namely point-like defects connected pair-wise by ribbons. In order to then calculate $\pi_1(K_{2n})$, we rely on the long exact sequence of homotopy groups

\[ \ldots \to \pi_i(E) \to \pi_i(B) \to \pi_{i-1}(F) \to \pi_{i-1}(E) \to \ldots \] (4)

associated to a fibration defined by $F \to E \to B$. (In an exact sequence, the kernel of each map is equal to the image of the previous map.) This exact sequence may be familiar to some readers from Mermin’s review of the topological theory of defects\textsuperscript{30}, where a symmetry associated with the group $G$ is spontaneously broken to $H$, thereby leading to topological defects classified by homotopy groups $\pi_n(G/H)$. These can be computed by (4) with $E = G$, $F = H$, $B = G/H$, e.g. if $\pi_1(G) = \pi_0(G) = 0$, then $\pi_1(G/H) = \pi_0(H)$.

The ribbon permutation group is a rather weak enhancement of the permutation group and, indeed, we conclude that Teo and Kane’s unitary operations are not a representation of the ribbon permutation group. However, they are a projective representation of the ribbon permutation group. In a projective representation, the group multiplication rule is only respected up to a phase, a possibility allowed in quantum mechanics. A representation $\rho$ (sometimes called a linear representation) of some group $G$ is a map from the group to the group of linear transformations of some vector space such that the group multiplication law is reproduced:

\[ \rho(gh) = \rho(g) \cdot \rho(h) \] (5)

if $g, h \in G$. Particle statistics arising as a projective representation of some group realizes a proposal of Wilczek’s\textsuperscript{30}, albeit for the ribbon permutation group rather than the permutation group itself. This difference allows us to sidestep a criticism of Read\textsuperscript{25} based on locality, which Teo and Kane’s projective representation respects. The group $(\mathbb{Z}_2)^{2n-1}$ is generated by $2n - 1$ generators $x_1, x_2, \ldots, x_{2n-1}$ satisfying

\[ x_i^2 = 1 \]
\[ x_i x_j = x_j x_i \] for $|i - j| \geq 2$
\[ x_i x_{i+1} = z x_{i+1} x_i \]
\[ z^2 = 1 \] (6)

However, the projective representation of $(\mathbb{Z}_2)^{2n-1}$, which gives a subgroup of Teo and Kane’s transformations, is an ordinary linear representation of a $\mathbb{Z}_2$-central extension, called the extra special group $E_{2n-1}^1$:

\[ x_i^2 = 1 \]
\[ x_i x_j = x_j x_i \] for $|i - j| \geq 2$
\[ x_i x_{i+1} = z x_{i+1} x_i \]
\[ z^2 = 1 \] (7)

Here, $z$ generates the central extension, which we may take to be $z = -1$. The operations generated by the $x_i$s were dubbed ‘braidless operations’ by Teo and Kane\textsuperscript{30} because they could be enacted without moving the defects. While these operations form an Abelian subgroup of $\mathcal{T}_{2n}$, their representation on the Majorana zero mode Hilbert space is not Abelian – two such operations which twist the same defect anti-commute (e.g. $x_i$ and $x_{i+1}$).

The remaining sections of this paper will be as follows. In Section II, we rederive Teo and Kane’s zero modes and unitary transformations by simple pictorial and counting arguments in a ‘strong-coupling’ limit of their model. In Section III, we review the topological classification of free-fermion Hamiltonians, including topological insulators and superconductors. From this classification, we obtain the classifying space relevant to Teo and Kane’s model and, in turn, the topological classification of defects and their configuration space. In Section IV, we use a toy model to motivate a simple picture for the defects used by Teo and Kane and give a heuristic construction of the ribbon permutation group. In Section V, we give a full homotopy theory calculation. In Section VI, we compare the ribbon permutation group to Teo and Kane’s unitary transformations and conclude that the latter form a projective, rather than a linear, representation of the former. Finally, in Section VII, we review and discuss our results. Several appendices
contain technical details.

II. STRONG-COUPLING LIMIT OF THE TEO-KANE MODEL

In this section, we present a lattice model in $d$ dimensions which has, as its continuum limit in $d = 3$, the model discussed by Teo and Kane\textsuperscript{26}. In the limit that the mass terms in this model are large (which can be viewed as a ‘strong-coupling’ limit), a simple picture of topological defects (‘hedgehogs’) emerges. We show by a counting argument that hedgehogs possess Majorana zero modes which evolve as the hedgehogs are adiabatically moved. This adiabatic evolution is the 3D non-Abelian statistics which it is the main purpose of this paper to explain.

The strong coupling limit which we describe is the simplest way to derive the existence of Majorana zero modes and the unitary transformations of their Hilbert space which results from exchanging them. This section does not require the reader to be au courant with the topological classification of insulators and superconductors\textsuperscript{26,27}. (In the next section, we will review that classification in order to make our exposition self-contained.)

We use a hypercubic lattice in $d$-dimensions, with a single Majorana degree of freedom at each site. That is, for $d = 1$, we use a chain, in $d = 2$ we use a square lattice, in $d = 3$ we use a cubic lattice, and so on. We first construct a lattice model whose continuum limit is the Dirac equation with $2^d$-dimensional $\gamma$-matrices to reproduce the Dirac equation considered by Teo and Kane; we then show how to perturb this model to open a mass gap. We begin by considering only nearest neighbor couplings. The Hamiltonian $H$ is an antisymmetric Hermitian matrix. In $d = 1$, we can take the linear chain to give a lattice model with the Dirac equation as its continuum limit. That is, $H_{j,j+1} = i$ and $H_{j+1,j} = -i$. To describe this state in pictures, we draw these bonds as oriented lines, as shown in Fig. (1a), with the orientation indicating the sign of the bond. The continuum limit of this Hamiltonian is described by a Dirac equation with 2-dimensional $\gamma$ matrices. While this system can be described by a unit cell of a single site, we instead choose to describe it by a unit cell of two sites for convenience when adding mass terms later. In $d = 2$, we can take a $\pi$-flux state to obtain the Dirac equation in the continuum limit. A convenient gauge to take to describe the $\pi$-flux state is shown in Fig. (1b), with all the vertical bonds having the same orientation, and the orientation of the horizontal bonds alternating from row to row. The continuum limit here has 4-dimensional $\gamma$ matrices and we use a 4-site unit cell.

In general, in $d$ dimensions, we can obtain a Dirac equation with $2^d$-dimensional $\gamma$ matrices by the following iterative procedure. Let the “vertical” direction refer to the direction of the $d$-th basis vector. Having constructed the lattice Hamiltonian in $d - 1$ dimensions, we stack these Hamiltonians vertically on top of each other, with alternating signs in each layer. Then, we take all the vertical bonds to be oriented in the same direction. This Hamiltonian is invariant under translation in the vertical direction by distance 2. Thus, if $H_d - 1$ is the Hamiltonian in $d - 1$ dimensions, the Hamiltonian $H_d$ is given by

$$H_d = \left( \begin{array}{cc} H_{d-1} & 2 \sin(k/2)I \\ 2 \sin(k/2)I & -H_{d-1} \end{array} \right),$$

where $I$ is the identity matrix and $k$ is the momentum in the vertical direction. Near $k = 0$, this is

$$H_d \approx H_{d-1} \otimes \sigma_z + k \otimes \sigma_x.$$  

(9)

This iterative construction corresponds to an iterative construction of $\gamma$-matrices. Having constructed $d - 1$ different $2^{d-1}$-dimensional $\gamma$-matrices $\gamma_1, ..., \gamma_{d-1}$, we construct $d$ different $2^d$-dimensional $\gamma$-matrices, $\tilde{\gamma}_1, ..., \tilde{\gamma}_d$, by $\tilde{\gamma}_i = \gamma_i \otimes \sigma_z$ for $i = 1, ..., d - 1$, and $\tilde{\gamma}_d = I \otimes \sigma_x$.

In one dimension, dimerization of bonds corresponds to alternately strengthening and weakening the bonds as shown in Fig. (2). In two dimensions, we can dimerize in either the horizontal or vertical directions. In $d$-dimensions, we have $d$ different directions to dimerize. Dimerizing in the “vertical” direction gives, instead of (9), the result

$$H_d \approx H_{d-1} \otimes \sigma_z + k \otimes \sigma_x + m_d \otimes \sigma_y.$$

(10)

where $m_d$ is the dimerization strength. This corresponds to an iterative construction of mass matrices, $M_i$, as follows. In one dimension, we have $M_1 = i \sigma_y$. Given $d - 1$ different mass matrices in $d - 1$ dimensions, $M_i$, we construct $\tilde{M}_i$ in $d$-dimensions by $\tilde{M}_i = M_i \otimes \sigma_z$, for $i = 1, ..., d - 1$, and $\tilde{M}_d = iI \otimes \sigma_y$.

If the dimerization is non-zero, and constant, we can increase the dimerization strength without closing the gap until a strong coupling limit is reached. In one dimension, by increasing the dimerization strength, we eventually reach a fully dimerized configuration, in which each site has one non-vanishing bond connected to it. In two or more dimensions, the dimerization can be a combination of dimerization in different directions. However, if the dimerization is completely in one direction, for example the vertical direction, we increase the dimerization strength until the vertical bonds are fully dimerized. Simultaneously, we reduce the strength of the other bonds to zero without closing the gap. This is again
a fully dimerized state, the columnar state, with each site having one non-vanishing bond. Any configuration with uniform, small dimerization can be deformed into this pattern without closing the gap by rotating the direction of dimerization, increasing the strength of dimerization, and then setting the bonds in the other directions to zero.

It is important to understand that the ability to reach such a strong coupling limits depends on the perturbation of the Dirac equation that we consider; for dimerization, it is possible to reach a strong coupling limit, while if we had instead chosen to open a mass gap by adding, for example, diagonal bonds with imaginary coupling to the two-dimensional Dirac equations, we would open a mass gap by perturbing the Hamiltonian with the term $\gamma_1\gamma_2$, and such a perturbation cannot be continued to the strong coupling limit due to topological obstruction.

Further, if the dimerization is non-uniform then it may not be possible to reach a fully dimerized state without having defect sites. Consider the configurations in Fig. (3a) in $d = 1$ and in Fig. (3b) in $d = 2$. These are the strong coupling limits of the hedgehog configuration, and each contains a zero mode, a single unpaired site. This is one of the central results of the strong-coupling limit: topological defects have unpaired sites which, in turn, support Majorana zero modes.

Such strong-coupling hedgehog configurations can be constructed by the following iterative process in any dimension $d$. Let $x_d$ correspond to the coordinate in the vertical direction. For $x_d \geq 0$, stack $d - 1$-dimensional hedgehog configurations. Along the half-line given by $x_d > 0$ and $x_i = 0$ for $1 \leq i \leq d - 1$, arrange vertical bonds, oriented to connect the site with $x_d = 2k - 1$ to that with $x_d = 2k$, for $k \geq 1$. Along the lower half plane, given by $x_d < 0$, arrange vertical bonds oriented to connect a site with $x_d = -(2k - 1)$ to that with $-2k$, for $k \geq 1$. This procedure gives the $d = 2$ hedgehog in Fig. (3b) from the $d = 1$ hedgehog in Fig. (3a), and gives a strong coupling limit of the Teo-Kane hedgehog in $d = 3$. That is, the Teo-Kane hedgehog can be deformed into this configuration, without closing the gap.

So long as we consider only nearest-neighbor bonds, there is an integer index $\nu$ describing different dimerization patterns in the strong-coupling limit. This index, which is present in any dimension, arises from the sublattice symmetry of the system, and is closely-related to the U(1) symmetry of dimer models of spin systems. Label the two sublattices by A and B. Consider any set of sites, such that every site in that set has exactly one bond connected to it. (Recall that, in the strong coupling limit, every bond has strength 0 or 1 and every site has exactly one bond connected to it, except for defects.) Then, the number of bonds going from A sites in this set to B sites outside the set is exactly equal to the number of bonds going from B sites in this set to A sites outside the set. On the other hand, if there are defect sites in the set, then this rule is broken. Consider the region defined by the dashed line in Fig. (4a). We define the “flux” crossing the dashed line to be the number of bonds crossing that boundary which leave starting on an A site, minus the number which leave starting on a B site. The flux is non-zero in this case, but is unchanging as we increase the size of the region. This flux is the index $\nu$. By the argument given above for the existence of zero modes, $\nu$ computed for any region is equal to the number of Majorana zero modes contained within the region.

The index $\nu$ can be defined beyond the strong-coupling limit. Consider, for the sake of concreteness, $d = 3$. There are 3 possible dimerizations, one for each dimension, as we concluded in Eq. 10. In weak-coupling, the square of the gap is equal to the sum of the squares of the dimerizations. Thus, if we assume a fixed gap, we can model these dimerizations by a unit vector. The integer index discussed above is simply the total winding number of this unit vector on the boundary of any region.

However, once diagonal bonds are allowed, the integer index $\nu$ no longer counts zero modes. Instead, there is a $\mathbb{Z}_2$ index, equal to $\nu(\text{mod} 2)$ which counts zero modes modulo 2. To see this in the strong-coupling limit, consider the configuration in Fig. (4b). This is a configuration with $\nu = 2$ but no Majorana zero modes. However, a $\nu = 1$ configuration must still have a zero mode and, thus, any configuration with odd $\nu$ must have at least one zero mode.

In Fig. (4), we have chosen to orient the bonds from A to B sublattice to make it easier to compute $\nu$. However, the $\nu$ and its residue modulo 2, defined above are independent of the orientation of the bonds (which indicate the sign of terms in the Hamiltonian) and depend only on which sites are connected by bonds (which indicate which terms in the Hamiltonian are non-vanishing).

The $\nu(\text{mod} 2)$ with diagonal bonds is the same as Kitaev’s “Majorana number”\textsuperscript{15}. We can use this to show the existence of zero modes in the Teo-Kane hedgehog even outside the strong-coupling limit. Consider a hedgehog configuration. Outside some large distance $R$ from the center of the hedgehog, deform to the strong coupling limit without closing the gap. Then, outside a distance $R$, we can count $\nu(\text{mod} 2)$ by counting bonds leaving the region and we find a nonvanishing result relative to a reference configuration: if there are an even number of sites in the region then there are an odd number of

FIG. 3: (a) A one-dimensional hedgehog. (b) A two-dimensional hedgehog.
bonds leaving in a hedgehog configuration, and if there are an odd number of sites then there are an even number of bonds leaving. However, since this implies a nonvanishing Majorana number, there must be a zero mode inside the region, regardless of what the Hamiltonian inside is. We note that this is a highly non-trivial result in the weak-coupling limit, where the addition of weak diagonal bonds, all oriented the same direction, to the configuration of Fig. (1b) corresponds to adding the term $\gamma_1 \gamma_2$ to the Hamiltonian in $d = 2$. By the argument given above, even this Hamiltonian has a zero mode in the presence of a defect with non-zero $\nu$ (mod 2).

Given any two zero modes, corresponding to defect sites in the strong coupling limit, we can identify a string of sites connecting them. If we have a pair of defect sites on opposite sublattices, corresponding to opposite hedgehogs, then one particular string corresponds to the north pole of the order parameter, as in Fig. (5a). However, we can simply choose any arbitrary string. Let $\gamma_i, \gamma_j$ be the Majorana operators at the two defect sites. The operation $\gamma_i \rightarrow -\gamma_i, \gamma_j \rightarrow -\gamma_j$ can be implemented as follows. We begin with an adiabatic operation on one of the defect sites and the nearest 2 sites along the path $\theta = 0 \rightarrow \pi$. We then perform rotations on consecutive triples of sites along the defect line, which changes the orientation of pairs of neighboring bonds, arriving at the configuration in Fig. (5c). Finally, we rotate by $\pi$ in the plane containing the other defect site and the last site. This returns the system to the original configuration, having effected the desired operation.

Since we only consider adiabatic transformation, we can only perform orthogonal rotations with unit determinant. Thus, any transformation which swaps two defects and returns the bonds to their original configuration, must change the sign of one of the zero modes: $\gamma_i \rightarrow \gamma_j, \gamma_j \rightarrow -\gamma_i$. Indeed, any orthogonal transformation with determinant equal to minus one would change the sign of the fermion parity in the system, as the fermion parity operator is equal to the product of the $\gamma_i$ operators.

We used the ability to change the orientation of a pair of bonds in this construction. The fact that one can only change the orientation of bonds in pairs, and not the orientation of a single bond, is related to a global $Z_2$ invariant: the Hamiltonian is an anti-symmetric matrix and the sign of its Pfaffian cannot be changed without closing the gap. Changing the direction of a single bond changes the sign of this Pfaffian and so is not possible.

The above discussion left open the question of which zero changes its sign, i.e. is the effect of the exchange $\gamma_i \rightarrow \gamma_j, \gamma_j \rightarrow -\gamma_i$ or $\gamma_i \rightarrow -\gamma_j, \gamma_j \rightarrow \gamma_i$? The answer is that it depends on how the bonds are returned to their original configuration after the exchange is completed (which is a clue that the defects must be understood as extended objects, not point-like ones). For the bonds to be restored, one of the defects must be rotated by $2\pi$; the corresponding zero mode acquires a minus sign. We will discuss this in greater detail in a later section. The salient point here is that the effect of an exchange is a unitary transformation generated by the operator $e^{\pm \pi \gamma_i \gamma_j / 4}$. This is reminiscent of the representation of braid group generators for non-Abelian quasi-particles in the quantum Hall effect\(^9\) and vortices in chiral $p$-wave superconductors\(^14\), namely the braid group representation realized by Ising anyons\(^7\). But, of course, in 3D the braid group is not relevant, and the permutation group, which is associated with point-like particles in $d > 2$, does not have non-trivial higher-dimensional representations consistent with locality\(^18,19\). As noted in the introduction, this begs the question: what group are the unitary matrices $e^{\pm \pi \gamma_i \gamma_j / 4}$ representing?
III. TOPOLOGICAL CLASSIFICATION OF GAPPED FREE FERMION HAMILTONIANS

A. Setup of the Problem

In this section, we will briefly review the topological classification of translationally-invariant or slowly spatially-varying free-fermion Hamiltonians following Kitaev’s analysis in Ref. 27. (For a different perspective, see Schnyder et al.’s approach in Ref. 26.) The 3D Hamiltonian of the previous section is a specific example which fits within the general scheme and, by implication, the 3D non-Abelian statistics which we derived at the end of the previous section also holds for an entire class of models into which it can be deformed without closing the gap. Our discussion will follow the logic of Milnor’s treatment of Bott periodicity in Ref. 28.

Consider a system of $N$ flavors of electrons $c_j(k)$ in $d$ dimensions. The flavor index $j$ accounts for spin as well as the possibility of multiple bands. Since we will not be assuming charge conservation, it is convenient to express the complex fermion operators $c_j(k)$ in terms of real fermionic operators (Majorana fermions), $c_j(k) = (a_{2j-1}(k) + i a_{2j}(k))/2$ (the index $j$ now runs from 1 to $2N$). The momentum $k$ takes values in the Brillouin zone, which has the topology of the $d$-dimensional torus $T^d$. The Hamiltonian may be written in the form

$$H = \sum_{i,j,p} i A_{ij}(p) a_i(p) a_j(-p)$$

(12)

where, by Fermi statistics, $A_{ij}(p) = -A_{ji}(-p)$. Let us suppose that the Hamiltonian (12) has an energy gap $2\Delta$, by which we mean that its eigenvalues $E_\alpha(p)$ ($\alpha$ is an index labeling the eigenvalues of $H$) satisfy $|E_\alpha(p)| \geq \Delta$. The basic question which we address in this section is the following. What topological obstructions prevent us from continuously deforming one such gapped Hamiltonian into another?

Such an analysis can apply, as we will see, not only to free fermion Hamiltonians, but also to those interacting fermion Hamiltonians which, deep within ordered phases, are well-approximated by free-fermion Hamiltonians. (This can include rather non-trivial phases such as Ising anyons, but not Fibonacci anyons.) In such settings, the fermions may be emergent fermionic quasiparticles; if the interactions between these quasiparticles are irrelevant in the renormalization-group sense, then an analysis of free-fermion Hamiltonians can shed light on the phase diagrams of such systems. Thus, the analysis of free fermion Hamiltonians is equivalent to the analysis of interacting fermion ground states whose low-energy quasiparticle excitations are free fermions.

Let us begin by considering a few concrete examples, in order of increasing complexity.

B. Zero-Dimensional Systems

First, we analyze a zero-dimensional system which we will not assume to have any special symmetry. The Hamiltonian (12) takes the simpler form:

$$H = \sum_{i,j} i A_{ij} a_i a_j$$

(13)

where $A_{ij}$ is a $2N \times 2N$ antisymmetric matrix, $A_{ij} = -A_{ji}$. Any real antisymmetric matrix can be written in the form

$$A = O^T \begin{pmatrix} 0 & -\lambda_1 \\ \lambda_1 & 0 \\ \vdots & \vdots \end{pmatrix} O$$

(14)

where $O$ is an orthogonal matrix and the $\lambda_i$s are positive. The eigenvalues of $A$ come in pairs $\pm i \lambda_i$; thus, the absence of charge conservation can also be viewed as the presence of a particle-hole symmetry. By assumption, $\lambda_i \geq \Delta$ for all $i$. Clearly, we can continuously deform $A_{ij}$ without closing the gap so that $\lambda_i = \Delta$ for all $i$. (This is usually called ‘spectrum flattening’.) Then, we can write:

$$A = \Delta \cdot O^T J O$$

(15)

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ \vdots & \vdots \end{pmatrix}$$

(16)

The possible choices of $A_{ij}$ correspond to the possible choices of $O \in O(2N)$, modulo $O$ which commute with the matrix $J$. But the set of $O \in O(2N)$ satisfying $O^T J O = J$ is $U(N) \subset O(2N)$. Thus, the space of all possible zero-dimensional free fermionic Hamiltonians with $N$ single-particle energy levels is topologically-equivalent to the symmetric space $O(2N)/U(N)$.

This can be restated in more geometrical terms as follows. Let us here and henceforth take units in which $\Delta = 1$. Then the eigenvalues of $A$ are $\pm i$. If we view the $2N \times 2N$ matrix $A$ as a linear transformation on $\mathbb{R}^{2N}$, then it defines a complex structure. Consequently, we can view $\mathbb{R}^{2N}$ as $\mathbb{C}^N$ since multiplication of $\vec{v} \in \mathbb{R}^{2N}$ by a complex scalar can be defined as $(a + ib)\vec{v} = a\vec{v} + bA\vec{v}$. The set of complex structures on $\mathbb{R}^{2N}$ is given by performing an arbitrary $O(2N)$ rotation on a fixed complex structure, modulo the rotations of $\mathbb{C}^N$ which respect the complex structure, namely $U(N)$. Thus, once again, we conclude that the desired space of Hamiltonians is topologically-equivalent to $O(2N)/U(N)$.

What are the consequences of this equivalence? Consider the simplest case, $N = 1$. Then, the space of zero-dimensional Hamiltonians is topologically-equivalent to $O(2)/U(1) = \mathbb{Z}_2$: there are two classes of Hamiltonians, those in which the single fermionic level is unoccupied in the ground state, $c\bar{c} = (1 + i a_1 a_2)/2 = 0$, and those in which it is occupied. For larger $N$, $O(2N)/U(N)$ is a more com-
complicated space, but it still has two connected components, \( \pi_0(O(2N)/U(N)) = \mathbb{Z}_2 \), so that there are two classes of free fermion Hamiltonians, corresponding to even or odd numbers of occupied fermionic levels in the ground state.

Suppose, now, that we restrict ourselves to time-reversal invariant systems and, furthermore, to those time-reversal invariant systems which satisfy \( T^2 = -1 \), where \( T \) is the antiunitary operator generating time-reversal. Then, following Ref. 27, we write \( T a_i T^{-1} = (J_1)_{ij} a_j \). The matrix \( J_1 \) is antisymmetric and satisfies \( J_1^2 = -1 \). \( T \)-invariance of the Hamiltonian requires

\[
J_1 A = -AJ_1
\]

Since \( J_1 \) is antisymmetric and satisfies \( J_1^2 = -1 \), its eigenvalues are \( \pm i \). Therefore, \( J_1 \) defines a complex structure on \( \mathbb{R}^{2N} \) which may, consequently, be viewed as \( \mathbb{C}^N \). Now consider \( A \), which is also antisymmetric and satisfies \( A^2 = -1 \), in addition to anticommuting with \( J_1 \). It defines a quaternionic structure on \( \mathbb{C}^N \) which may, consequently, be viewed as \( \mathbb{H}^{N/2} \). Multiplication of \( \bar{v} \in \mathbb{R}^{2N} \) by a quaternion can be defined as:

\[
(a + bi + cj + dk)\bar{v} \equiv a\bar{v} + bJ_1\bar{v} + c\bar{A}\bar{v} + dJ_1A\bar{v}.
\]

The possible choices of \( A \) can be obtained from a fixed one by performing rotations of \( \mathbb{C}^N \), modulo those rotations which respect the quaternionic structure, namely \( \text{Sp}(N/2) \). Thus, the set of time-reversal-invariant zero-dimensional free fermionic Hamiltonians with \( T^2 = -1 \) is topologically-equivalent to \( \mathbb{U}(N)/\text{Sp}(N/2) \). Since \( \pi_0(\mathbb{U}(N)/\text{Sp}(N/2)) = 0 \), any such Hamiltonian can be continuously deformed into any other. This can be understood as a result of Kramers doubling: there must be an even number of fermions in the ground state so the division into two classes of the previous case does not exist here.

C. 2D Systems: \( T \)-breaking superconductors

Now, let us consider systems in more than zero dimensions. Once again, we will assume that charge is not conserved, and we will also assume that time-reversal symmetry is not preserved. For the sake of concreteness, let us consider a single band of spin-polarized electrons on a two-dimensional lattice. Let us suppose that the electrons condense into a (fully spin-polarized) \( p_x \)-wave superconductor. For fixed superconducting order parameter, the low-energy theory is a free fermion Hamiltonian for gapless fermionic excitations at the nodal points \( \pm \vec{k}_F \) \( \equiv (0, \pm p_F) \). We now ask the question, what other order parameters could develop which would fully gap the fermions? For fixed values of these order parameters, we have a free fermion Hamiltonian. Thus, these different possible order parameters correspond to different possible gapped free fermion Hamiltonians.

The low-energy Hamiltonian of a fully spin-polarized \( p_x \)-wave superconductor can be written in the form:

\[
H = \psi^\dagger (iv_\Delta \partial_x \tau_x + iv_F \partial_y \tau_z) \psi
\]

where \( v_F, v_\Delta \) are, respectively, the Fermi velocity and slope of the gap near the node. The Pauli matrices \( \tau \) act in the particle-hole space:

\[
\psi(k) \equiv \begin{pmatrix}
\psi_{kF+k}^\dagger \\
\psi_{-kF+k}^\dagger
\end{pmatrix}
\]

This Hamiltonian is invariant under the \( \mathbb{U}(1) \): \( \psi \rightarrow e^{i\theta} \psi \) which corresponds to conservation of momentum in the \( p_y \) direction (not to charge conservation). Since we will be considering perturbations which do not respect this symmetry, it is convenient to introduce Majorana fermions \( \chi_1, \chi_2 \) according to \( \psi = \chi_1 + i\chi_2 \). Then

\[
H = i\chi_a (v_\Delta \partial_x \tau_x + v_F \partial_y \tau_z) \chi_a + i\chi_a M_{ab} \chi_b
\]

with \( a = 1, 2 \). Note that we have suppressed the particle-hole index on which the Pauli matrices \( \tau \) act. Since \( \chi_1, \chi_2 \) are each a 2-component real spinor, this model has 4 real Majorana fields.

We now consider the possible mass terms which we could add to make this Hamiltonian fully gapped:

\[
H = i\chi_a (v_\Delta \partial_x \tau_x + v_F \partial_y \tau_z) \chi_a + i\chi_a M_{ab} \chi_b
\]

If we consider the possible order parameters which could develop in this system, it is clear that there are only two choices: an imaginary superconducting order parameter \( ip_y \) (which breaks time-reversal symmetry) and charge density-wave order (CDW). These take the form:

\[
M_{ab}^{ip_y} = \Delta_{ip_y} i\tau^y \delta_{ab}
\]

and

\[
M_{ab}^{CDW} = \rho_{2k_F} \tau^y \cos \theta \mu_{ab}^x + \sin \theta \mu_{ab}^y
\]

where \( \mu_{ab}^x \) are Pauli matrices and \( \theta \) is an arbitrary angle. For an analysis of the possible mass terms in the more complex situation of graphene-like systems, see, for instance, Ref. 32.

Let us consider the space of mass terms with a fixed energy gap \( \Delta \) which is the same for all 4 of the Majorana fermions in the model (i.e. a flattened mass spectrum). An arbitrary gapped Hamiltonian can be continuously deformed to one which satisfies this condition. Then we can have \( \Delta_{ip_y} = \pm \Delta \), \( \rho_{2k_F} = 0 \) or \( \rho_{2k_F} = \Delta \). \( \Delta_{ip_y} = 0 \) (in the latter case, arbitrary \( \theta \) is allowed). If both order parameters are present, then the energy gap is not the same for all fermions. It’s not that there’s anything wrong with such a Hamiltonian – indeed, one can imagine a system developing both kinds of order. Rather, it is that such a Hamiltonian can be continuously deformed to one with either \( \Delta_{ip_y} = 0 \) or \( \rho_{2k_F} = 0 \) without closing the gap. For instance, if \( \Delta_{ip_y} > \rho_{2k_F} \), then the Hamiltonian can be continuously deformed to one with \( \rho_{2k_F} = 0 \). (However if we try to deform it to a Hamiltonian with \( \Delta_{ip_y} = 0 \), the gap will close at \( \Delta_{ip_y} = \rho_{2k_F} \)) Hence, we conclude that the space of possible mass terms is topologically-equivalent to the disjoint union \( \mathbb{U}(1) \cup \mathbb{Z}_2 \): a single one-parameter family and two disjoint points.

Since \( \pi_0(\mathbb{U}(1) \cup \mathbb{Z}_2) = \mathbb{Z}_3 \), there are three distinct classes
of quadratic Hamiltonians for 4 flavors of Majorana fermions in 2D. The one-parameter family of CDW-ordered Hamiltonians counts as a single class since they can be continuously deformed into each other. The parameter $\theta$ is the phase of the CDW, which determines whether the density is maximum at the sites, the midpoints of the bonds, or somewhere in between. It is important to keep in mind, however, that, although there is no topological obstruction to continuously deforming one $\theta$ into another, there may be an energetic penalty which makes it costly. For instance, the coupling of the system to the lattice may prefer some particular value of $\theta$. The classification discussed here accounts only for topological obstructions; the possibility of energetic barriers must be analyzed by different methods.

We can restate the preceding analysis in the following, more abstract language. This formulation will make it clear that we haven’t overlooked a possible mass term and will generalize to more complicated free fermion models. Let us write $\tau_z \delta_{ab}$, $\gamma_2 = \tau_z \delta_{ab}$. Then

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}$$  \hspace{1cm} (24)

The Dirac Hamiltonian for $N = 4$ Majorana fermion fields takes the form

$$H = i\gamma_0(\gamma_i \partial_i + M)\chi$$  \hspace{1cm} (25)

The matrix $M$ plays the role that the matrix $A$ did in the zero-dimensional case. As in that case, we assume a flattened spectrum which here means that each Majorana fermion field has the same gap and that this gap is equal to 1. (It does not mean that the energy is independent of the momentum $k$.) In order to satisfy this, we must require that

$$\{\gamma_i, M\} = 0 \quad \text{and} \quad M^2 = -1$$  \hspace{1cm} (26)

Note that it is customary to write the Dirac Hamiltonian in a slightly different form,

$$H = \overline{\psi}(i\gamma_0 \gamma_i \partial_i + m)\psi$$  \hspace{1cm} (27)

which can be massaged into the form of (25) using $\overline{\psi} = \psi^\dag \gamma_0$:

$$H = \psi^\dag (i\gamma_0 \gamma_i \partial_i + m\gamma_0)\psi
= \psi^\dag (i\alpha_i \partial_i + m\beta)\psi
= i\psi^\dag (\alpha_i \partial_i - im\beta)\psi$$  \hspace{1cm} (28)

where $\alpha_i = \gamma_0 \gamma_i$ and $\beta = \gamma_0$. Thus, if we write $\gamma_i = \alpha_i$ and $M = -im\beta$ and consider Majorana fermions (or decompose Dirac fermions into Majoranas), we recover (25). We have used the form (25) so that it is analogous to (13), with $\{\gamma_i \partial_i + M\}$ replacing $A_{ij}$ and the $i$ pulled out front. Then, the matrix $M$ determines the gaps of the various modes in the same way as $A$ does in the zero-dimensional case. Similarly, assuming a ‘flattened’ spectrum leads to the condition $M^2 = -1$.

How many ways can we choose such an $M$? Since $\gamma_2 = 1$, its eigenvalues are $\pm 1$. Hence, viewed as a linear map from $\mathbb{R}^4$ to itself, this matrix divides $\mathbb{R}^4$ into two 2D subspaces $\mathbb{R}^4 = X_+ \oplus X_-$ with eigenvalue $\pm 1$ under $\gamma_2$, respectively.

For $\gamma_2 = \tau_z \delta_{ab}$, this is trivial:

$$X_+ = \text{span}\left\{\left(\begin{array}{c} 1 \\ 0 \end{array}\right) \otimes \left(\begin{array}{c} 1 \\ 0 \end{array}\right), \left(\begin{array}{c} 1 \\ 0 \end{array}\right) \otimes \left(\begin{array}{c} 0 \\ 1 \end{array}\right)\right\}$$  \hspace{1cm} (29)

where $\tau_z$ acts on the first spinor and the second spinor is indexed by $a = 1, 2$, i.e. is acted on by the Pauli matrices $\gamma^{x,z}$ in (23). This construction generalizes straightforwardly to arbitrary numbers $N$ of Majorana fermions, which is why we use it now.

Now $\gamma_i M$ commutes with $\gamma_2$ and satisfies $\{\gamma_i M\}^2 = 1$. Thus, it maps $X_+$ to itself and defines subspaces $X_+^k, X_+^\perp$ with eigenvalue $\pm 1$ under $\gamma_1 M$ (and equivalently for $X_-$). $X_+$ can be deformed into $X_+^1 \oplus X_+^\perp = X_+$. Choosing $M$ is thus equivalent to choosing a linear subspace $X_+^1$ of $X_+$.

This can be divided into three cases. If $\gamma_1 M$ has one positive eigenvalue and one negative one when acting on $X_+$ then the space of possible choices of $\gamma_1 M$ is equal to the space of 1D linear subspaces of $\mathbb{R}^2$, which is simply $U(1)$. If, on the other hand, $\gamma_1 M$ has two positive eigenvalues, then there is a unique choice, which is simply $M = \gamma_1 \gamma_2$. If $\gamma_1 M$ has two negative eigenvalues, then there is again a unique choice, $M = -\gamma_1 \gamma_2$. Therefore, the space of possible $M$s is topologically equivalent to $U(1) \cup \mathbb{Z}_2$.

Now, suppose that we have $2N$ Majorana fermions. Then $\gamma_2$ defines $N$-dimensional eigenspaces $X_+, X_-$ such that $\mathbb{R}^{2N} = X_+ \oplus X_-$ and $\gamma_1 M$ defines eigenspaces of $X_+$:

$$X_+^1 \oplus X_+^\perp = X_+.$$  \hspace{1cm} (30)

If $\gamma_1 M$ has $k$ positive eigenvalues and $N - k$ negative ones, then the space of possible choices of $\gamma_1 M$ is $O(N)/O(k) \times O(N-k)$, i.e. we can take the restriction of $\gamma_1 M$ to $X_+$ to be of the form

$$\gamma_1 M = O^T \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ -1 & \cdots & -1 \end{pmatrix} O$$

with $k$ diagonal entries equal to $+1$ and $N - k$ entries equal to $-1$. Thus, the space of Hamiltonians for $N$ flavors of free Majorana fermions is topologically equivalent to

$$\mathcal{M}_{2N} = \bigcup_{k=0}^{N} O(N)/(O(k) \times O(N-k))$$  \hspace{1cm} (31)

However, since $\pi_0(O(N)/(O(k) \times O(N-k))) = 0$, independent of $k$ (note that 0 is the group with a single element, not the empty set), $\pi_0(\mathcal{M}_{2N}) = \mathbb{Z}_{N+1}$.

In the model analyzed above, we had only a single spin-polarized band of electrons. By increasing the number of bands and allowing both spins, we can increase the number of flavors of Majorana fermions. In principle, the number of bands in a solid is infinity. Usually, we can introduce a cutoff and restrict attention to a few bands near the Fermi energy. However, for a purely topological classification, we can ignore energetics and consider all bands on equal footing. Then
we can take \( N \to \infty \), so that \( \pi_0(M_{\infty}) = \mathbb{Z} \). This classification permits us to deform Hamiltonians into each other so long as there is no topological obstruction, with no regard to how energetically costly the deformation may be. Thus, the \( 2N = 4 \) classification which we discussed above can perhaps be viewed as a ‘hybrid’ classification which looks for topological obstructions in a class of models with a fixed set of bands close to the Fermi energy.

But even this point of view is not really natural. The discussion above took as its starting point an expansion about a \( p_x \) superconductor; the \( p_x \) superconducting order parameter was assumed to be large and fixed while the \( i\phi_p \) and CDW order parameters were assumed to be small. In other words, we assumed that the system was at a point in parameter space at which the gap, though non-zero, was small at two points in the Brillouin zone (the intersection points of the nodal line in the \( p_x \) superconducting order parameter with the Fermi surface). This allowed us to expand the Hamiltonian about these points in the Brillouin zone and write it in Dirac form. And this may, indeed, be reasonable in a system in which \( p_x \) superconductivity is small and other orders are large. However, a topological classification should allow us to take the system into regimes in which \( p_x \) superconductivity is small and other orders are large. Suppose, for instance, that we took our model of spin-polarized electrons (which we assume, for simplicity, to be at half-filling) and made more precise and put on more solid mathematical footing using ideas which we discuss in Appendix B.

The topological classification described above holds not only for classes of translationally-invariant Hamiltonians such as (25), but also for topological defects within a class. Suppose, for instance, that we consider (25) with a mass which varies slowly as the origin is encircled at a great distance. We can ask whether such a Hamiltonian can be continuously deformed to a Hamil- 

\[
H = \psi_1 \left( i v_{\Delta} \partial_x \tau_x + i v_F \partial_y \tau_z \right) \psi_1 + \psi_2 \left( i v_{\Delta} \partial_y \tau_x + i v_F \partial_x \tau_z \right) \psi_2
\]

where \( v_F, v_{\Delta} \) are, respectively, the Fermi velocity and slope of the gap near the nodes; the subscripts 1,2 refer to the two sets of nodes \((\pm \pi/2, \pm \pi/2)\) and \((\mp \pi/2, \pm \pi/2)\); and \( \psi_A, A = 1, 2 \) are defined by:

\[
\psi_{1,2}(k) \equiv \begin{pmatrix} c_{(\pi/2, \pm \pi/2) + \vec{k}} \\ c_{(-\pi/2, \mp \pi/2) + \vec{k}} \end{pmatrix}
\]

If we introduce Majorana fermions \( \psi_A = \chi_{A1} + i \chi_{A2} \), then we can write this Hamiltonian with possible mass terms as:

\[
H = i \chi_{1a} \left( v_{\Delta} \partial_x \tau_x + v_F \partial_y \tau_z \right) \chi_{1a} + i \chi_{2a} \left( v_{\Delta} \partial_y \tau_x + v_F \partial_x \tau_z \right) \chi_{2a} + i \chi_A M_{Aa,Bb} \chi_{Bb}
\]

We have suppressed the spinor indices (e.g. \( \chi_{11} \) is a two-component spinor); with these indices included, \( M_{Aa,Bb} \) is an \( 8 \times 8 \) matrix. However, in order for the gap to be the same for all flavors, the mass matrix must anticommute with \( \tau_{x,z} \). Thus, \( M_{Aa,Bb} = \tau_y M_{Aa,Bb} \). The matrix \( M_{Aa,Bb} \) can have 0, 1, 2, 3, or 4 eigenvalues equal to +1 (with the rest being −1). The spaces of such mass terms are, respectively, \( O(4)/(O(1) \times O(3)), O(4)/(O(2) \times O(2)), O(4)/(O(3) \times O(1)) \), and 0. Mass terms with 0 or 4 eigenvalues equal to +1 correspond physically to \( \pm \sin \theta_x \) density wave order. \( \langle c_{k+Q}^\dagger c_k \rangle = \pm \sin k_x a \sin k_y a \). Mass terms with 2 eigenvalues equal to +1 and correspond physically to \( \pm \sin \theta_x \) density wave order at \((\pm \pi/2, \mp \pi/2)\). Clearly, this is the \( 2N = 8 \) case of the general classification discussed above. Thus, the same underlying physical degrees of freedom – a single band of spin-polarized electrons on a square lattice – correspond to either \( 2N = 4 \) or \( 2N = 8 \), depending on where the system is in parameter space. One can imagine regions of parameter space where the gap is small at an arbitrary number \( N \) of points. Thus, if we restrict ourselves to systems with a single band, then different regions of the parameter space (with different numbers of points at which the gap is small) will have very different topologies. Although such a classification may be a necessary evil in some contexts, it is far preferable, given the choice, to allow topology to work unfettered by energetics. Then, we can consider a large number \( n \) of bands. Suppose that the gap becomes small at \( r \) points in the Brillouin zone in each band. Then, the low-energy Hamiltonian takes the Dirac form for \( 2N = 2rn \) Majorana fermion fields. As we will see below, if \( N \) is sufficiently large, the topology of the space of possible mass terms will be independent of \( N \). Consequently, for \( n \) sufficiently large, the topology of the space of possible mass terms will be independent of \( r \). In other words, we are in the happy situation in which the topology of the space of Hamiltonians will be the same in the vicinity of any gap closing. But any gapped Hamiltonian can be continuously deformed so that the gap becomes small at some points in the Brillouin zone. Thus, the problem of classifying gapped free fermion Hamiltonians in \( d \)-dimensions is equivalent to the problem of classifying possible mass terms in a generic \( d \)-dimensional Dirac Hamiltonian so long as the number of bands is sufficiently large\(^{27}\). This statement can be made more precise and put on more solid mathematical footing using ideas which we discuss in Appendix B.

D. Classification of Topological Defects

The topological classification described above holds not only for classes of translationally-invariant Hamiltonians such as (25), but also for topological defects within a class. Suppose, for instance, that we consider (25) with a mass which varies slowly as the origin is encircled at a great distance. We can ask whether such a Hamiltonian can be continuously deformed into a uniform one. In a system in which the mass term is understood as arising as a result of some kind of underlying ordering such as superconductivity or CDW order, we are
simply talking about topological defects in an ordered media, but with the caveat that the order parameter is allowed to explore a very large space which may include many physically distinct or unnatural orders, subject only to the condition that the gap not close.

Let us, for the sake of concreteness, assume that we have a mass term with $N/2$ positive eigenvalues when restricted to the $+1$ eigenspace of $\gamma_2$. (For $N$ large, the answer obviously cannot depend on the number of positive eigenvalues $k$ so long as $k$ scales with $N$. Thus, we will denote the space $M_{2N}$ defined in Eq. 31 by $\mathbb{Z} \times O(N)/(O(N/2) \times O(N/2))$ where the integers in $\mathbb{Z}$ correspond to the number of positive eigenvalues of the mass term when restricted to the $+1$ eigenspace of $\gamma_2$. Then $M(r, \theta)$ defines a loop in $O(N)/(O(N/2) \times O(N/2))$ which cannot be continuously unwound if it corresponds to a nontrivial element of $\pi_1(O(N)/(O(N/2) \times O(N/2)))$.

To compute $\pi_1(O(N)/(O(N/2) \times O(N/2)))$, we parametrize $O(N)/(O(N/2) \times O(N/2))$ by symmetric matrices $K$ which satisfy $K^2 = 1$ and $\text{tr}(K) = 0$. (Such matrices decompose $\mathbb{R}^N$ into their $+1$ and $-1$ eigenspaces: $\mathbb{R}^N = V_+ \oplus V_-$. $K$ can be written in the form: $K = O^T K_0 O$, where $K_0$ has $N/2$ diagonal entries equal to $+1$ and $N/2$ equal to $-1$, i.e. $K_0 = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$. Note that any such $K$ itself is an orthogonal matrix, i.e. an element of $O(N)$; thus $O(N)/(O(N/2) \times O(N/2))$ can be viewed as a submanifold of $O(N)$ in a canonical way. Consider a curve $L(\lambda)$ in $O(N)/(O(N/2) \times O(N/2))$ with $L(0) = K$ and $L(\pi) = -K$. We will parametrize it as $L(\lambda) = K e^{A \lambda}$, where $A$ is in the Lie algebra of $O(N)$. In order for this loop to remain in $O(N)/(O(N/2) \times O(N/2))$, we need $(K e^{A \lambda})^2 = 1$. Since $(K e^{A \lambda})^2 = K e^{A K} e^{A} = e^{i\theta K} e^{A}$, this condition implies that $K A = -AK$. In order to have $L(\pi) = -K$, we need $A^2 = -1$. Such a curve is, in fact, a minimal geodesic from $K$ to $-K$. Each such geodesic can be represented by its midpoint $L(\pi/2) = K A$, so the space of such geodesics is equivalent to the space of matrices $A$ satisfying $A^2 = -1$ and $KA = -AK$. As discussed in Ref. 28, the space of minimal geodesics is a good enough approximation to the entire space of loops (essentially because an arbitrary loop can be deformed to a geodesic) that we can compute $\pi_1$ from the space of minimal geodesics just as well as from the space of loops. Thus, the loop space of $O(N/2)/(O(k) \times O(N/2 - k))$ is homotopically equivalent to the space of matrices $A$ satisfying $A^2 = -1$ and $KA = -AK$. Since it anticommutes with $K$, $KA$ maps the $+1$ eigenspace of $K$ to the $-1$ eigenspace. It is clearly a length-preserving map since $(KA)^2 = 1$ and, since the $\pm 1$ eigenspaces of $K$ are isomorphic to $\mathbb{R}^{N/2}$, $KA$ defines an element of $O(N/2)$. Thus a loop in $O(N)/(O(N/2) \times O(N/2))$ corresponds to an element of $O(N/2)$ or, in other words:

$$\pi_1(O(N)/(O(N/2) \times O(N/2))) = \pi_0(O(N/2)).$$

The latter group is simply $\mathbb{Z}_2$ since $O(N/2)$ has two connected components: (1) pure rotations and (2) rotations combined with a reflection.

It might come as a surprise that we find a $\mathbb{Z}_2$ classification for point-like defects in two dimensions. Indeed, if we require that the superconducting order parameter has fixed amplitude at infinity, then vortices of arbitrary winding number are stable and we have a $\mathbb{Z}$ classification. However, in the classification discussed here, we require a weaker condition be satisfied: that the fermionic gap remain constant. Consequently, a vortex configuration of even winding number can be unwound without closing the free fermion gap by, for instance, ‘rotating’ superconductivity into CDW order.

### E. 3D Systems with No Symmetry

With these examples under our belts, we now turn to the case which is of primary interest in this paper: free fermion systems in three dimensions without time-reversal or charge-conservation symmetry. We consider the Dirac Hamiltonian in 3D for a $2N$-component Majorana fermion field $\chi$:

$$H = i\chi \left( \partial_\tau \gamma_1 + \partial_\tau \gamma_2 + \partial_\tau \gamma_3 \right) \chi + i\chi M \chi.$$  

(36)

In the previous section, we discussed a lattice model which realizes (36) in its continuum limit with $2N = 8$. Different mass terms correspond to different quadratic perturbations of this model which open a gap (which can be viewed as order parameters which we are turning on at the mean-field level). We could classify such terms by considering, from a physical perspective, all such ways of opening a gap. However, we will instead determine the topology of the space of mass terms (and, thereby, the space of gapped free fermion Hamiltonians) by the same mathematical methods by which we analyzed the 2D case.

Since $\gamma_2^2 = 1$ and has vanishing trace, this matrix decomposes $\mathbb{R}^{2N}$ into its $\pm 1$ eigenspaces: $\mathbb{R}^{2N} = X_+ \oplus X_-$. Now $\left(\gamma_2 \gamma_3\right)^2 = -1$ and $\left[\gamma_1, \gamma_2 \gamma_3\right] = 0$. Therefore, $\gamma_2 \gamma_3$ is a complex structure on $X_+$ (and also on $X_-$), i.e. we can define multiplication of vectors $\vec{v} \in X_+$ by complex scalars according to $(a + bi)\vec{v} \equiv a\vec{v} + b\gamma_2 \gamma_3 \vec{v}$. (Consequently, we can view $X_+$ as $\mathbb{C}^{N/2}$.) Now, consider a possible mass term, with $M^2 = -1$. $(\gamma_2 M)^2 = 1$ and $[\gamma_1, \gamma_2 M] = 0$. Let $Y$ be the subspace of $X_+$ with eigenvalue $+1$ under $\gamma_2 M$. Since $\{\gamma_2 M, \gamma_2 \gamma_3\} = 0$, $\gamma_2 \gamma_3 Y$ is the subspace of $X_+$ with eigenvalue $-1$ under $\gamma_2 M$. In other words, $X_+ = Y \oplus \gamma_2 \gamma_3 Y$, i.e. $Y$ is a real subspace of $X_+$. Hence, the space of choices of $M$ is the space of real subspaces $Y \subset X_+$ (or, equivalently, of real subspaces $\mathbb{R}^{N/2} \subset \mathbb{C}^{N/2}$). Given any fixed real subspace $Y \subset X_+$, we can obtain all others by performing $U(N/2)$ rotations of $X_+$, but two such rotations give the same real subspace if they differ only by an $O(N/2)$ rotation of $Y$. Thus, the space of gapped Hamiltonians for $2N$ free Majorana fermion fields in 3D with no symmetry is topologically-equivalent to $U(N/2)/O(N/2)$. In the remaining sections of this paper, we will be discussing topological defects in such systems and their motions.
F. General Classification and Bott Periodicity

Before doing so, we pause for a minute to consider the classification in other dimensions and in the presence of symmetries such as time-reversal and charge conservation. We have seen that systems with no symmetry in $d = 0, 2, 3$ are classified by the spaces $O(2N)/U(N)$, $Z \times O(N)/O(N/2) \times O(N/2)$, and $U(N)/O(N/2)$, respectively. By similar methods, it can be shown that the $d = 1$ case is classified by $O(N)$. As we have seen, increasing the spatial dimension increases the number of $\gamma$ matrices by one. The problem of choosing $\gamma_1, \ldots, \gamma_d$ satisfying \( \{ \gamma_i, \gamma_j \} = 2\delta_{ij} \) and $M$ which anti-commutes with the $\gamma_i$s and squares to $-1$ leads us to subspaces of $\mathbb{R}^{2N}$ of smaller and smaller dimension, isometries between these spaces, or complex of quaternionic structures on these spaces. This leads the progression of spaces in the top row of Table I.

At the same time, we have seen that a time-reversal-invariant system in $d = 0$ is classified by $U(N)/Sp(N/2)$. Suppose that we add a discrete anti-unitary symmetry $SiS^{-1} = -i$ defined by

\[
S\alpha_i S^{-1} = (J)_{ij}\alpha_j
\]

which squares to $J^2 = -1$. It must anti-commute with the mass term

\[
\{J, M\} = 0
\]

in order to ensure invariance under the symmetry, so choosing $J$ amounts to adding a complex structure, which leads to the opposite progression of classifying spaces. Consider, as an example of the preceding statements, a time-reversal-invariant system in $d = 3$. Then time-reversal symmetry $T$ is an example of a symmetry generator $J$ discussed in the previous paragraph. We define a real subspace $X \subset X_+$, in a similar manner as above, but now as the subspace of $X_+$ with eigenvalue $+1$ under $\gamma_2T$, rather than under $\gamma_2M$. Once again, $X_+ = Y \oplus \gamma_3Y$. Now, $\{\gamma_3M, \gamma_2T\} = 0$, and $\{\gamma_3M, \gamma_2M\} = 1$, so the $+1$ eigenspace of $\gamma_3M$ is a linear subspace of $X$. The set of all such linear subspaces is $Z \times O(N/2)/O(N/2) \times O(N/2)$. But this is the same classifying space as for a system with no symmetry in $d = 2$ (apart from a reduction of $N$ by a factor of 2).

Thus, we are led to the list of classifying spaces for gapped free fermion Hamiltonians in Table I.

In this table, $Q$ refers to charge-conservation symmetry. Charge conservation is due to the invariance of the Hamiltonian of a system under the $U(1)$ symmetry $c_i \rightarrow e^{i\theta}c_i$. In terms of Majorana fermions $a_i$, defined according to $c_j = (a_{2j-1} + ia_{2j})/\sqrt{2}$, the symmetry takes the form $a_{2j-1} \rightarrow \cos \theta a_{2j-1} + \sin \theta a_{2j}, a_{2j} \rightarrow -\sin \theta a_{2j-1} + \cos \theta a_{2j}$. However, if a free fermion Hamiltonian is invariant under the discrete symmetry $c_i \rightarrow ic_i$, or, equivalently, $a_{2j-1} \rightarrow a_{2j}, a_{2j} \rightarrow -a_{2j-1}$, then it is automatically invariant under the full $U(1)$ as well, and conserves charge. Thus, we can treat charge conservation as a discrete symmetry $Q$ which is unitary, squares to $-1$, and commutes with the Hamiltonian (i.e. with the $\gamma$ matrices and $M$). Since $Q$ transforms $c_i \rightarrow ic_i$, it anti-commutes with $T$. Note further that if a system has time-reversal symmetry, then the product of time-reversal $T$ and charge conservation $Q$ is a discrete anti-unitary symmetry, $QT$ which anti-commutes with the Hamiltonian and with $T$ and squares to $-1$. Then $QT$ is defined by a choice of matrix $J$, analogous to $T$, as in Eq. 37. If the system is not time-reversal-invariant, then charge conservation is a unitary symmetry. It is easier then to work with complex fermions, and the classification of such systems falls into an entirely different sequence, as discussed in Appendix A.)

If a system is both time-reversal symmetric and charge-conserving, i.e. if it is a time-reversal invariant insulator, then it may have an additional symmetry which guarantees that the eigenvalues of the Hamiltonian come in $\pm E$ pairs, just as in a superconductor. An example of such a symmetry is the sublattice symmetry of Hamiltonians on a biparticle lattice in which fermions can hop directly from the $A$ sublattice to the $B$ sublattice but cannot hop directly between sites on the same sublattice. In such a case, the system is invariant under a unitary symmetry $\chi$ defined as follows. If we block diagonalize $\chi$ so that one block acts on sites in the $A$ sublattice and the other on sites in the $B$ sublattice, then we can write $\chi = \text{diag}(k, -k)$, i.e. $a_j(x) \rightarrow -k_ja_j(x)$ for $x \in A$ and $a_j(x) \rightarrow k_ja_j(x)$ for $x \in B$. This symmetry transforms the Hamiltonian to minus itself if $k^2 = 1$ or, in other words, if $\chi^2 = 1$. Then $\chi Q$ is a unitary symmetry which squares to $-1$ and anti-commutes with the Hamiltonian, $T$, and $QT$. Hence $\chi Q$, too, is defined by a choice of matrix $J$, as in Eq. 37. We will call such a symmetry a sublattice symmetry $\chi$ and a system satisfying this symmetry a ‘bipartite’ system which may have a different microscopic origin.

In an electron system, time-reversal ordinarily squares to $-1$, because the transformation law is $c_i \rightarrow c_i, c_i \rightarrow -c_i$, as we have thus far assumed in taking $J^2 = -1$. However, it is possible to have a system of fully spin-polarized electrons which has an anti-unitary symmetry $T$ which squares to $+1$. (One might object to calling this symmetry time-reversal because it doesn’t reverse the electron spins, but $T$ is a natural label because it is a symmetry which is just as good for the present purposes.) Then, since $J^2 = 1$, a choice of $J$ is similar to a choice of a $\gamma$ matrix. In general, symmetries (37) which square to $+1$ have the same effect on the topology of the space of free fermion Hamiltonians as adding dimensions since each such $J$ defines a subspace of half the dimension within the eigenspaces of the $\gamma$ matrices. This is true for systems with $T^2 = 1$.

SU(2) spin-rotation-invariant and time-reversal-invariant insulators (systems with $T$ and $Q$) effectively fall in this category. The Hamiltonian for such a system can be written in the form $H = h \otimes I_2$ where the second factor is the $2 \times 2$ identity matrix acting on the spin index. Then time-reversal can be written in the form $T = t \otimes io_y$, where $t^2 = 1$, and $Q$ can be written in the form $Q = q \otimes I_2$, so that $QT = qt \otimes io_y$, where $(qt)^2 = 1$. Thus, since the matrix $io_y$ squares to $-1$, the symmetries $T$ and $QT$ have effectively become symmetries which square to $+1$. They now move the system through the progression of classifying spaces in the same direction as increasing the dimension, i.e. in the opposite direction to symmetries which square to $-1$. Thus, SU(2) spin-rotation-invariant and
time-reversal-invariant insulators in $d$ dimensions are classified by the same space as systems with no symmetry in $d + 2$ dimensions. However, in a system which, in addition, has sublattice symmetry $\chi = x \otimes I_2$, we have $(qx)^{-1}$. Thus, sublattice symmetry is still a symmetry which squares to $-1$. Since the two symmetries which square to $+1$ ($T$ and $QT$) have the same effect as increasing the dimension while the symmetry which squares to $-1$ has the same effect as decreasing the dimension, SU(2) spin-rotation-invariant and time-reversal-invariant insulators with sublattice symmetry in $d$ dimensions are classified by the same space as systems with no symmetry in $d + 1$ dimensions (but with $N$ replaced by $N/4$). Similar considerations apply to superconductors with SU(2) spin-rotational symmetry.

In order to discuss topological defects in the systems discussed here, it is useful to return to the arguments which led to (35). By showing that the space of loops in $O(N/2) \times O(N/4)$ is well-approximated by $O(N/4)$, we not only showed that $\pi_1(O(N/2)/O(N/4)) = \pi_0(O(N/4))$ but, in fact, that $\pi_k(O(N/2)/O(N/4)) = \pi_{k-1}(O(N/4))$ (see Ref. 28). Continuing in the same way, we can approximate the loop space of $O(N/4)$ (i.e. the space of loops in $O(N/4)$) by minimal geodesics from $I$ to $-I$: $L^\ast(\lambda) = e^{\lambda A_1}$ where $A_1^2 = -1$. The midpoint of such a geodesic, $L^\ast(\pi/2) = A_1$ again defines a complex structure $A_1 = O^\ast J O$, where $J$ is given by (16) so that $\pi_1(O(N/4)) = \pi_{k-1}(O(N/4)/U(N/8))$. In a similar way, minimal geodesics in $O(N/4)/U(N/8)$ from $A_1$ to $-A_1$ can be parametrized by their mid-points $A_2$, which square to $-1$ and anti-commute with $A_1$, thereby defining a quaternionic structure, so that the loop space of $O(N/4)/U(N/8)$ is equivalent to $U(N/8)/Sp(N/8)$ and, hence $\pi_k(O(N/4)/U(N/8)) = \pi_{k-1}(U(N/8)/Sp(N/8))$. Thus, we see that the passage from one of the classifying spaces to its loop space is the same as the imposition of a symmetry such as time-reversal to a system classified by that space: both involve the choice of successive anticommuting complex structures. Continuing in this fashion (see Ref. 28), we recover Bott periodicity:

$$\pi_k(\text{O}(16N)) = \pi_{k-1}(\text{O}(16N)/\text{U}(8N)) = \pi_{k-2}(\text{U}(8N)/\text{Sp}(4N)) = \pi_{k-3}(\text{Z} \times \text{Sp}(4N)/(\text{Sp}(2N) \times \text{Sp}(2N))) = \pi_{k-4}(\text{Sp}(2N))$$

$$= \pi_{k-5}(\text{Sp}(2N)/\text{U}(2N)) = \pi_{k-6}(\text{U}(2N)/\text{O}(2N))$$

$$= \pi_{k-7}(\text{Z} \times \text{O}(2N)/\text{O}(N) \times \text{O}(N)) = \pi_{k-8}(\text{O}(N))$$

The approximations made at each step require that $N$ be in the stable limit, in which the desired homotopy groups are independent of $N$. For instance, $\pi_k(\text{O}(N))$ is independent of $N$ for $N > k/2$.

It is straightforward to compute $\pi_0$ for each of these groups:

$$\pi_0(\text{O}(N)) = \mathbb{Z}_2$$

$$\pi_0(\text{O}(2N)/\text{U}(2N)) = \mathbb{Z}_2$$

$$\pi_0(\text{O}(2N)/\text{Sp}(2N)) = 0$$

$$\pi_0(\text{Z} \times \text{Sp}(2N)/\text{Sp}(N) \times \text{Sp}(N)) = \mathbb{Z}$$

$$\pi_0(\text{Sp}(2N)/\text{U}(N)) = 0$$

$$\pi_0(\text{U}(2N)/\text{O}(N)) = 0$$

$$\pi_0(\text{Z} \times \text{O}(2N)/\text{O}(N) \times \text{O}(N)) = \mathbb{Z}$$

Combining (40) with (39), we can compute any of the stable homotopy groups of the above 8 classifying spaces. As discussed above, the space of gapped free fermion Hamiltonians in $d$-dimensions in a given symmetry class (determined by the number modulo 8 of symmetries squaring to $-1$ minus the number of those squaring to $+1$) is homotopically-equivalent to one of these classifying spaces. Thus, using (40) with (39) to compute the stable homotopy groups of these classi-

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**Table I:** The period-8 (in both dimension and number of symmetries) table of classifying spaces for free fermion Hamiltonians for $N$ complex = 2$N$ real (Majorana) fermion fields in dimensions $d = 0, 1, 2, 3, \ldots$ with no symmetries; time-reversal symmetry ($T$) only; time-reversal and charge conservation symmetries ($T$ and $Q$); time-reversal, charge conservation, and sublattice symmetries ($T$, $Q$, and $\chi$); and the latter two cases with SU(2) symmetry. As a result of the period-8 nature of the table, the top two rows could equally well be the bottom two rows of the table. Moving $p$ steps to the right and $p$ steps down leads to the same classifying space (but for $1/2^p$ as many fermion fields), which is a reflection of Bott periodicity, as explained in the text. The number of disconnected components of any such classifying space – i.e. the number of different phases in that in symmetry class and dimension – is given by the corresponding $\pi_0$, which may be found in Eq. 40. Higher homotopy groups, which classify defects, can be computed using Eq. 39. Table III, given in Appendix A, is the analogous table for charge-conserving Hamiltonians without time-reversal symmetry.

| dim.: | SU(2), $T$, $Q$ | SU(2), $T$, $Q$, $\chi$ | no symm. | $T$ only | $T$ and $Q$ | $T$, $Q$, $\chi$ |
|-------|----------------|--------------------------|----------|-----------|----------------|----------------|
| $Z \times \text{O}(N)$ | $\text{U}(N/2)/\text{O}(N/2)$ | $\text{Sp}(N/4)/\text{U}(N/4)$ | $\text{Sp}(N/8)$ | $Z \times \text{Sp}(N/8)$ | $\text{Sp}(N/16)$ | $\text{Sp}(N/2)$ |
| $\text{O}(N/4)$ | $Z \times \text{O}(N/4)$ | $\text{U}(N/8)/\text{O}(N/8)$ | $\text{Sp}(N/16)/\text{U}(N/16)$ | $\text{Sp}(N/32)$ | $\text{Sp}(N/4)/\text{U}(N/4)$ | $\text{Sp}(N/4)/\text{U}(N/4)$ |
fying spaces leads to a complete classification of topological states and topological defects in all dimensions and symmetry classes, as we now discuss.

Gapped Hamiltonians with a given symmetry and dimension are classified by \( \pi_0 \) of the corresponding classifying space in Table I. Due to Bott periodicity, the table is periodic along both directions of dimension and symmetry, so that there are 8 distinct symmetry classes. Ryu et al.\(^{26}\) denoted these classes using the Cartan classification of symmetric spaces, following the corresponding classification of disordered systems and random matrix theory\(^{34,35}\) which was applied to the (potentially-gapless) surface states of these systems. In this notation, systems with no symmetry are in class D, those with T only are in DIII, and those with T and Q are in AII. The other 5 symmetry classes, C, CI, CII, AI, and BDI arise, arise in systems which have spin-rotational symmetry or a sublattice symmetry. There are actually 2 more symmetry classes (denoted by A and AIII in the random matrix theory) which lie on a separate 2 \( \times \) 2 periodic table, which is less relevant to the present work and will be discussed in the Appendix A. In Table II we have listed examples of topologically-nontrivial states in physical dimensions 1,2,3 in all 8 symmetry classes. To help with the physical understanding of these symmetry classes, we have also listed the physical requirements for the realization of each symmetry class. In each dimension, there are two symmetry classes in which the topological states are classified by integer invariants and two symmetry classes in which the different states are distinguished by \( \mathbb{Z}_2 \) invariants. In all the cases in which a real material or a well-defined physical model is known with non-trivial \( \mathbb{Z}_2 \) or \( \mathbb{Z}_2 \) invariant, we have listed a typical example in the table. In some of the symmetry classes, non-trivial examples have not been realized yet, in which case we leave the topological classification \( \mathbb{Z}_2 \) or \( \mathbb{Z}_2 \) in the corresponding position in the table.

In one dimension, generic superconductors (class D) are classified by \( \mathbb{Z}_2 \), of which the nontrivial example is a \( p \)-wave superconductor with a single Majorana zero mode on the edge. The time-reversal invariant superconductors (class DIII) are also classified by \( \mathbb{Z}_2 \). The nontrivial example is a superconductor in which spin up electrons pair into a \( p \)-wave superconductor and spin down electron form another \( p \)-wave superconductor which is exactly the time-reversal of the spin-up one. Such a superconductor has two Majorana zero modes on the edge which form a Kramers pair and are topologically protected. The two integer classes are bipartite time-reversal invariant insulators with (CII) and without (BDI) spin-orbit coupling. An example of the BDI class is a graphene ribbon, or equivalently a carbon nanotube with a zigzag edge.\(^{36,37}\) The low-energy band structure of graphene and carbon nanotubes is well-described by a tight-binding model with nearest-neighbor hopping on a honeycomb lattice, which is bipartite. The integer-valued topological quantum number corresponds to the number of zero modes on the edge, which depends on the orientation of the nanotube. Because carbon has negligible spin-orbit coupling, to a good approximation it can be viewed as a system in the BDI class, but it can also be considered as a system in class CII when spin-orbit coupling is taken into account. In two dimensions, generic superconductors (class D) are classified by an integer, corresponding to the number of chiral Majorana edge states on the edge. The first nontrivial example was the \( p+iq \) wave superconductor, shown by Read and Green\(^{11}\) to have one chiral Majorana edge state. Non-trivial superconductors in symmetry class D are examples of topological superconductors. Some topological superconductors can be consistent with spin rotation symmetry; singlet superconductors (class C) are also classified by integer, with the simplest physical example a \( d+id \) wave superconductor. Similar to the 1D case, the time-reversal invariant superconductors (class DIII) are classified by \( \mathbb{Z}_2 \), of which the nontrivial example is a superconductor with \( p+iq \) pairing of spin-up electrons and \( p-ip \) pairing of spin-down electrons.\(^{26,38,39}\) The same symmetry class in 2D with a \( \mathbb{Z}_2 \) classification is composed of time-

| Symmetry classes | Physical realizations | \( d = 1 \) | \( d = 2 \) | \( d = 3 \) |
|------------------|----------------------|-------------|-------------|-------------|
| D                | SC                   | \( p \)-wave SC | \( (p+iq) \)-SC | 0           |
| DIII             | TRI SC               | \( \mathbb{Z}_2 \) | \( (p+iq)(p-ip) \)-SC | He\(^{3-}\)B |
| AII              | TRI ins.             | 0           | HgTe Quantum well | Bi\(_1-x\)Sb\(_x\), Bi\(_2\)Se\(_3\), etc. |
| C                | Singlet TRI ins.     | Carbon nanotube | 0           | \( \mathbb{Z}_2 \) |
| C                | Singlet TRI SC       | 0           | \( (d+id) \)-SC | 0           |
| AI               | TRI ins. w/o SOC     | 0           | 0           | 0           |
| BDI              | Bipartite TRI ins. w/o SOC | Carbon nanotube | 0 | 0 |
reversal invariant insulators (class AII), also known as quantum spin Hall insulators\textsuperscript{30-42}. The quantum spin Hall insulator phase has been theoretically predicted\textsuperscript{43} and experimentally realized\textsuperscript{44} in HgTe quantum wells. In three dimensions, time-reversal invariant insulators (class AII) are also classified by $Z_2$.\textsuperscript{21-23} The $Z_2$ topological invariant corresponds to a topological magneto-electric response with quantized coefficient $\theta = 0, \pi$.\textsuperscript{24} Several nontrivial topological insulators in this class have been theoretically predicted and experimentally realized, including Bi$_{1-x}$Sb$_x$ alloy\textsuperscript{45, 46} and the family of Bi$_2$Se$_3$, Bi$_2$Te$_3$, Sb$_2$Te$_3$.\textsuperscript{47-49} In 3D, time-reversal invariant superconductors (class DIII) are classified by an integer, corresponding to the number of massless Majorana cones on the surface.\textsuperscript{26} A nontrivial example with topological quantum number $N = 1$ turns out to be the B phase of He\textsuperscript{3}.\textsuperscript{26, 39, 50} The other classes with nontrivial topological classification in 3D are singlet time-reversal invariant superconductors (CI), classified by an integer; and bipartite time-reversal invariant insulators (CII), classified by $Z_2$. Some models have been proposed\textsuperscript{51} but no realistic material proposal or experimental realization has been found in these two cases. We would like to note that different physical systems can correspond to the same symmetry class. For example, bipartite superconductors are also classified by the BDI class.

The two remaining symmetry classes (unitary (A) and chiral unitary (AIII)) corresponds to systems with charge conservation symmetry but without time-reversal symmetry, which forms a separate $2 \times 2$ periodic table. For the sake of completeness, we carry out the preceding analysis for these two classes in Appendix A.

Topological defects in these states are classified by higher homotopy groups of the classifying spaces. Following the convention of Ref.\textsuperscript{27}, we name the classifying spaces by $R_{q,q} = 0, 1, 2, ..., 7$, with $R_1 = O(N)$, $R_2 = O(2N)/U(N)$, ..., $R_7 = U(N)/O(N)$, $R_0 = \mathbb{Z} \times O(2N)/O(N) \times O(N)$ in the order of Eq. (40). The symmetries in Table II can be labeled by $p = 0, 1, 2, ..., 7$, so that in $d$ dimensions and $p$-th symmetry class, the classifying space is $R_{2p-d}$. A topological defect with dimension $D$ ($D < d$) is classified by

$$\pi_{d-D-1}(R_{2p-d}) = \pi_0(R_{p-D+1})$$

which is determined by the zero-th homotopy groups in Eq. (40). The important conclusion we obtain from this formula is that the classification of topological defect is determined by the dimension of the defect $D$ and the symmetry class $p$, and is independent of the spatial dimension $d$. For $p$ and $D$, we obtain the same $8 \times 8$ periodic table for the classification of topological defects.\textsuperscript{52}

In the following we will focus on the 3D system with no symmetry, and discuss the generic cases in Sec. VII. Since a 3D system with no symmetry is classified by $R_2 = U(N)/O(N)$, point-like defects in such a system are classified by $\pi_2(U(N)/O(N))$. In the stable limit, $\pi_2(U(N)/O(N)) = \mathbb{Z}_2$. However, we note that, for smaller values of $N$ below the stable limit, the classification is a little different, e.g. $\pi_2(U(2)/O(2)) = \pi_2(U(1) \times S^2) = \mathbb{Z}$. The ‘8-band model’ in Ref. 20, which we discussed in the strong-coupling limit in the previous section, is an example of this particular ‘small $N$’ case, which is why the hedgehogs in that model are classified by a winding number $\in \mathbb{Z}$.

IV. EXCHANGING PARTICLES CONNECTED BY RIBBONS IN 3D

In this section, we will take an O(3) non-linear $\sigma$ model, i.e. one with target space $S^2$, as a toy model for our problem. It is essentially the ‘8-band’ model discussed in Ref. 20 and in the strong-coupling limit in Section II. As noted above, it is the $2N = 8$ limit of the classification reviewed above. It will be more familiar to most readers and easier to visualize than the full problem which we discuss in the next section. We will give a heuristic explanation of $\pi_1$ of the configurations of defects and will make a few comments about where our toy model goes wrong, compared to the full problem. In the next section, we will undertake a full and careful calculation of $\pi_1$ of the configurations of defects of a model with classifying space $U(N)/O(N)$.

A free fermion Hamiltonian with no symmetry, but limited to 8 bands, can be expanded about its minimum energy point in the Brillouin zone as

$$H = i\chi(\partial_i \gamma_i + n_i \Gamma_i)\chi$$

(42)

From the considerations in the previous section, we learned that the space of mass terms is $S^2$, which is why we have written the mass term in the above form with a unit vector $\vec{n} \in S^2$. (More precisely, the space of mass terms is $U(2)/O(2) = U(1) \times S^2$, and the mass term can be written $e^{i\theta_1 \gamma_1 \gamma_3 n_i \Gamma_i e^{-i\theta_1 \gamma_1 \gamma_3}}$. However, the extra $U(1)$ plays no role here and can be ignored.) In the model of Section II, the three components of $\vec{n}$ correspond to dimerization in each of the three directions on the cubic lattice. In Teo and Kane’s model they correspond to the real and imaginary parts of the superconducting order parameter and the sign of the Dirac mass at a band inversion.

FIG. 6: The pre-image of the North pole in the $S^2$ target space of $\vec{n}$ is a collection of arcs connecting hedgehogs. A fixed tangent vector at the North pole defines a vector field along the arcs, thereby making them framed arcs, which may be viewed as ribbons.
Now consider defects in the $\vec{n}$ field, which are classified by $\pi_2(S^2) = \mathbb{Z}$. Defects with winding number $\pm 1$ are positive and negative hedgehogs. For simplicity, we will focus on these; higher winding number hedgehogs can be built up from these. (In the real model, as opposed to the toy model, the ‘hedgehogs’ have a $\mathbb{Z}_2$ classification so there are no higher winding number hedgehogs and, in fact, they do not even have a sign.) As noted by Teo and Kane\textsuperscript{20}, the $\vec{n}$ field around a hedgehog can be visualized in a simplified way, following Wilszek and Zee’s discussion of the Hopf term in a $2 + 1$-D O(3) non-linear $\sigma$ model\textsuperscript{53}. The field $\vec{n}$ can be viewed as a map from the physical space where the electrons live, $\mathbb{R}^3$. If we assume that the total winding number is zero (equal numbers of + and - hedgehogs) and that $\vec{n}$ approaches a constant at $\infty$, we can compactify the physical space $\mathbb{R}^3$ so that it is $S^3$. The target space of the map is $S^2$. The pre-image of the north pole $N \in S^2$ is a set of arcs and loops. The choice of the north pole $N$ is arbitrary, and any other point on the sphere would be just as good for the following discussion. Let’s ignore the loops for the moment and focus on the arcs. Since $\vec{n}$ points in every direction at a hedgehog, the arcs terminate at hedgehogs. In fact, each arc connects a +1 hedgehog to a −1 hedgehog. We now pick an arbitrary unit vector in the tangent space of the sphere at $N$. This vector can be pulled back to $S^3$ to define a vector field along the arcs which is clearly normal to the arcs. This is a framing, which allows us to define, for instance, a self-winding number for an arc. Intuitively, we can think of a framing as a thickening of an arc into a ribbon. Thus, the field $\vec{n}$ allows us to define a set of framed arcs connecting the hedgehogs – in other words, a set of ribbons connecting the hedgehogs. As the normal vector twists around an arc, the ribbon twists, as shown in Fig. 6. (Although we will draw the ribbons as bands in the physical space, their width should not be taken seriously; they should really be viewed as arcs with a normal vector field.)

Although these ribbons are strongly reminiscent of particle trajectories, it is important to keep in mind that they are not. A collection of ribbons connecting hedgehogs defines a state of the system at an instant of time. Ribbons, unlike particle trajectories, can cross. They can break and reconnect as the system evolves in time. As hedgehogs are moved, the ribbons move with them.

A configuration of particles connected pairwise by ribbons is a seemingly crude approximation to the full texture defined by $\vec{n}$. However, according to the Pontryagin-Thom construction, as we describe in the next Section (and explain in Appendix F), it is just as good as the full texture for topological purposes. Thus, we focus on the space of particles connected pairwise by ribbons.

We now consider a collection of such particles and ribbons. For a topological discussion, all that we are interested in about the ribbons is how many times they twist, so we will not draw the framing vector but will, instead, be careful to put kinks into the arcs in order to keep track of twists in the ribbon, as depicted in Fig. 7. The fundamental group of their configuration space is the set of transformation which return the particles and ribbons to their initial configurations, with two such transformations identified if they can be continuously deformed into each other. Consider an exchange of two +1 hedgehogs, as depicted in Fig. 8. Although this brings the particles back to their initial positions (up to a permutation, which is equivalent to their initial configuration since the particles are identical), it does not bring the ribbons back to their initial configuration. Therefore, we need to do a further motion of the ribbons. By cutting and rejoining them as shown in Fig. 9a, a procedure which we call ‘recoupling’, we now have the ribbons connecting the same particles as in the initial configuration. But the ribbon on the left has a twist in it. So we rotate that particle by $-2\pi$ in order to undo the twist, as in Fig. 9b.

Let us use $t_i$ to denote such a transformation, defined by the sequence in Figs. 8, 9a, and 9b. The $t_i$s do not satisfy the multiplication rules of the permutation group. In particular, $t_i \neq t_i^{-1}$. The two transformations $t_i$ and $t_i^{-1}$ are not distinguished by whether the exchange is clockwise or counter-clockwise – this is immaterial since a clockwise exchange can be deformed into counter-clockwise one – but rather by which ribbon is left with a twist which must be undone by rotating one of the particles.

To see that the operations $t_i$, defined by the sequence in Figs. 8, 9a, and 9b, and $t_i^{-1}$, defined by the sequence in 10, are, in fact, inverses, it is useful to note that when they are performed sequentially, they involve two $2\pi$ twists of the same hedgehog. In 9b, it is the hedgehog on the left which is twisted; this hedgehog moves to the right in the first step of 10 and is twisted again in the fourth step. One should then note that a double twist in a ribbon can be undone continuously by using the ribbon to “lasso” the defect, a famous fact related to
can perform the similar transformations which exchange
permutes a pair of particles but also rotates one of them; any
FIG. 10: The sequence of moves which defines $t_i^{-1}$.
(Here, the $i^{\text{th}}$ particle is at the top left and the $(i + 1)^{\text{th}}$ is at the top right.) This may be contrasted with the sequence in Figs. 8, 9a, and 9b, which defines $\sigma_i^1$.

the existence of spin-1/2 and the fact that $\pi_1(SO(3)) = \mathbb{Z}_2$.
This is depicted in Fig. 16 in Appendix E. It will be helpful for our late discussion to keep in mind that $t_i$ not only permutes a pair of particles but also rotates one of them; any transformation built up by multiplying $t_i$s will enact as many $2\pi$ twists as pairwise permutations modulo two.

Thus far we have only discussed the $+1$ hedgehogs. We can perform the similar transformations which exchange $-1$ hedgehogs. We will not repeat the above discussion for $-1$ hedgehogs since the discussion would be so similar; furthermore, in the $N \to \infty$ model which is our main interest, defects do not carry a sign, so they can all be permuted with each other.

We have concluded that $t_i \neq t_i^{-1}$ and, therefore, the group of transformations which bring the hedgehogs and ribbons back to their initial configuration is not the permutation group. This leaves open the question: what is $t_i^2$? The answer is that $t_i^2$ can be continuously deformed into a transformation which doesn’t involve moving any of the particles – Teo and Kane’s ‘braidless operations’. Consider the transformation $x_i$ depicted in Fig. 11. Defect $i$ is rotated by $2\pi$, the twist is transferred from one ribbon to the other, and defect $i + 1$ is rotated by $-2\pi$. Since a $4\pi$ rotation can be unwound, as depicted in Fig. 16, $x_i^2 = 1$.

Intuitively, one expects that $x_i = t_i^2$ since neither $x_i$ nor $t_i^2$ permutes the particles and both of them involve $2\pi$ rotations of both particles $i$ and $i + 1$. To show that this is, in fact, the correct, we need to show that the history in Fig. 11 can be deformed into the sequence of Figs. 8, 9a, 9b repeated twice. If the history in Fig. 11 is viewed as a ‘movie’ and the sequence of Figs. 8, 9a, 9b repeated twice is viewed as another ‘movie’, then we need a one-parameter family of movies – or a ‘movie of movies’ – which connects the two movies. We will give an example of such a ‘movie of movies’ shortly. With this example in hand, the reader can verify that $x_i = t_i^2$ by drawing the corresponding pictures, but we will not do so here since this discussion is superseded, in any case, by the the next section, where a similar result is shown for the $N \to \infty$ problem by more general methods. We simply accept this identity for now.

We now consider the commutation relation for the $x_i$s. Clearly, for $|i - j| \geq 2$, $x_i x_j = x_j x_i$. It is also intuitive to conclude that

$$x_i x_{i+1} = x_{i+1} x_i$$

(43)
since the order in which twists are transferred is seemingly unimportant. However, since this is a crucial point, we verify it by showing in Figure 12 that the sequence of moves which defines $x_i x_{i+1}$ (a ‘movie’) can be continuously deformed into the sequence of moves which defines $x_{i+1} x_i$ (another ‘movie’). Such a deformation is a ‘movie of movies’; going from left-to-right in Fig. 12 corresponds to going forward in time while going from up to down corresponds to deforming from one movie to another.

Thus, we see that the equivalence class of motions of the defects (i.e. $\pi_1$ of their configuration space) has an Abelian subgroup generated by the $x_i$s. Since $x_i^2 = 1$ and they all commute with each other, this is simply $n - 1$ copies of $\mathbb{Z}_2$, or, simply, $(\mathbb{Z}_2)^{n-1}$.

In order to fully determine the group of transformations which bring the hedgehogs and ribbons back to their initial configuration, we need to check that the $t_i$s generate the full set of such transformations – i.e. that the transformations described above and those obtained by combining them exhaust the full set. In order to do this, we need the commutation relations of the $t_i$s with each other. Clearly, $t_i t_j = t_j t_i$ for
\[ |i - j| \geq 2 \] since distant operations which do not involve the same hedgehogs nor the same ribbons must commute. On the other hand operations involving the same hedgehogs or ribbons might not commute. For instance,

\[ t_i x_{i+1} = x_i x_{i+1} t_i \quad (44) \]

To see why this is true, note that if we perform \( x_{i+1} \) first, then defects \( i + 1 \) and \( i + 2 \) are twisted by \( 2\pi \). However, \( t_i \) then permutes \( i \) and \( i + 1 \) and twists \( i \) by \( 2\pi \). The \((i+1)\)th hedgehog was twisted by \( x_{i+1} \) and then permuted by \( t_i \) so that it ended up in the \( i \)th position, where it was twisted again in the last step in \( t_i \); two twists can be continuously deformed to zero, so this hedgehog is not twisted at all. The right-hand-side similarly permutes \( i \) and \( i + 1 \) and only twists \( i + 2 \) by \( 2\pi \). The reader may find it instructive to flesh out the above reasoning by constructing a movie of movies.

The multiplication rule which we have just described (but not fully justified) is that of a semi-direct product, which is completely natural in this context: when followed by a permutation, a transfer of twists ends up acting on the permuted defects. The twists \( x_i \) form the group \((\mathbb{Z}_2)^{n-1}\) which we can represent by \( n \)-component vectors all of whose entries are 0 or 1 which satisfy the constraint that the sum of the entries is even. The entries tell us whether a given hedgehog is twisted by \( 2\pi \) or not. In any product of \( x_i x_s \), an even number of hedgehogs is twisted by \( 2\pi \). Now consider, for \( n \) odd, the group elements given by

\[ \sigma_i = x_{n-1} x_{n-3} \ldots x_{i+3} x_{i+1} x_{i-2} x_{i-4} \ldots x_1 t_i \quad (45) \]

for \( i \) odd and

\[ \sigma_i = x_{n-1} x_{n-3} \ldots x_{i+2} x_{i+1} x_{i-1} x_{i-3} \ldots x_1 t_i \quad (46) \]

for \( i \) even. From (44), we see that \( \sigma_i^2 = 1 \). The group element \( \sigma_i \) permutes the \( i \)th and \((i + 1)\)th hedgehogs and twists all of the hedgehogs. Thus, the \( \sigma_i \)'s generate a copy of the permutation group \( S_n \). The \( \sigma_i \)'s do not commute with the \( x_i \)'s, however; instead they act according to the semi-direct product structure noted above. On the other hand, the situation is a bit different for \( n \) even. This may be a surprise since one might expect that \( n \) even is the same as \( n \) odd but with the last hedgehog held fixed far away. While this is true, exchanging the last hedgehog with the others brings in an additional layer of complexity which is not present for \( n \) odd. The construction above, Eqs. 45, 46, does not work. One of the hedgehogs will be left untwisted by such a construction; since subsequent \( \sigma_i \)'s will permute this untwisted hedgehog with others, we must keep track of the untwisted hedgehog and, therefore, the \( \sigma_i \)'s will not generate the permutation group. In the even hedgehog number case, the group of transformations has a \((\mathbb{Z}_2)^{n-1}\) subgroup, as in the odd case, but there isn’t an \( S_n \) subgroup, unlike in the odd case. To understand the even case, it is useful to note that in both cases, every transformation either (a) twists an even number of ribbons, which is the subgroup \((\mathbb{Z}_2)^{n-1}\); (b) performs an even permutation, which is the subgroup \( A_n \); or (c) twists an odd number of ribbons and performs an odd permutation. Another way of saying this is that the group of transformations is the ‘even part’ of \((\mathbb{Z}_2)^{n}\); the subgroup of \((\mathbb{Z}_2)^{n} \times S_n \) consisting of those elements whose \((\mathbb{Z}_2)^{n} \) parity added to their \( S_n \) parity is even. In the odd hedgehog number case, this is the semidirect product \((\mathbb{Z}_2)^{n-1} \times A_n \); in the even hedgehog number case, it is not. As we will see in Section VI, the difference between the even and odd hedgehog number cases is related to the fact that, for an even number of hedgehogs, the Hilbert space decomposes into even and odd total fermion number parity sectors. By contrast, the situation is simpler for an odd number of hedgehogs, where the parity of the total fermion number is not well-defined and the representation is reducible.

To summarize, we have given some plausible heuristic arguments that the ‘statistics’ of +1 hedgehogs in a model of \( 2N = 8 \) Majorana fermions is governed by a group \( E((\mathbb{Z}_2)^n \times S_n) \), the ‘even part’ of \((\mathbb{Z}_2)^n \times S_n \); those elements of \((\mathbb{Z}_2)^n \times S_n \) in which the parity of the sum of the entries of the element in \((\mathbb{Z}_2)^n \) added to the parity of the permutation in \( S_n \) is even. (The same group governs the −1 hedgehogs). Rather than devoting more time here to precisely determining the group for the toy model, we will move on to the prob-
lem which is our main concern here, a system of $2N \to \infty$ Majorana fermions. This problem is similar, with some important differences. (1) The target space is no longer $S^2$ but is, instead, $U(N)/O(N)$. (2) Consequently, the defects do not carry a sign. There is no preferred pairing into $\pm$ pairs; the defects are all on equal footing. All $2n$ of them can be exchanged. (3) The group obtained by computing $\pi_1$ of the space of configurations of $2n$ defects then becomes the direct product of the ‘ribbon permutation group’ $\mathcal{T}_{2n}$, where $\mathcal{T}_{2n} = \mathbb{Z} \times \mathcal{T}_{2n}^2$. The ribbon permutation group $\mathcal{T}_{2n}$ is given by $\mathcal{T}_{2n}^2 = \mathbb{Z} \times E((\mathbb{Z}_2)^{2n} \rtimes S_{2n})$, where $E((\mathbb{Z}_2)^{2n} \rtimes S_{2n})$ is the ‘even part’ of $(\mathbb{Z}_2)^{2n} \rtimes S_{2n}$.

V. FUNDAMENTAL GROUP OF THE MULTI-DEFECT CONFIGURATION SPACE

In Section III we concluded that the effective target space for the order parameter of a system of fermions in $3D$ with no symmetries is $U(N)/O(N)$ -- which, as is conventional, we will simply call $U/O$, dropping the $N$ in the large-$N$ limit. This enables us to rigorously define the space of topological configurations, $K_{2n}$, of $2n$ hedgehogs in a ball, and calculate its fundamental group $\pi_1(K_{2n})$, thereby elucidating Teo and Kane’s $S^3$ hedgehog motions and unitary transformations.

We now outline the steps involved in this calculation:

- We approximate the space $U/O$ by a cell complex (or CW complex), $\mathcal{C}$, a topological space constructed by taking the union of disks of different dimensions and specifying how the boundary of each higher-dimensional disk is identified with a subset of the lower-dimensional disks. This is a rather crude approximation in some respects, but it is sufficient for a homotopy computation.

- We divide the problem into (a) the motion of the hedgehogs and (b) the resulting deformation of the field configuration between the hedgehogs. This is accomplished by expressing the configuration space in the following way. Let us call the configuration space of $2n$ distinct points in three dimensions $X_{2n}$. (For the sake of mathematical convenience, we will take our physical system to be a ball $B^3$ and stipulate that the points must lie inside a ball $B^3$. Let’s denote the space of field configurations by $\mathcal{M}_{2n}$. This space is the space of maps to $U/O$ from $B^3$ with $2n$ points (at some standard locations) excised. The latter space is denoted by $B^3 \setminus 2n$ standard points. Since we will be approximating $U/O$ by $\mathcal{C}$, we can take $\mathcal{M}_{2n}$ to be the space of maps from $B^3 \setminus 2n$ standard points to $\mathcal{C}$ with boundary conditions at the $2n$ points specified below. Then, there is a fibration of spaces:

\[
\mathcal{M}_{2n} \to K_{2n} \quad \downarrow \\
X_{2n}
\]

- We introduce another two fibrations which further divide the problem into more manageable pieces:

\[
\begin{align*}
R_{2n} & \to K_{2n} \\
\downarrow & \\
Y_{2n} & \\
\end{align*}
\]

\[
\begin{align*}
N_{2n} & \to Y_{2n} \\
\downarrow & \\
X_{2n} & \\
\end{align*}
\]

The original fibration is kind of a “fiber-product” of the two new fibrations. Here, $R_{2n}$ is essentially the space of order parameter textures interpolating between the hedgehogs, and $Y_{2n}$ is the space of configurations of $2n$ points with infinitesimal spheres surrounding each point and maps from each of these spheres to $\mathcal{C}$. $N_{2n}$ is the space of maps from $2n$ infinitesimal spheres to $\mathcal{C}$, with each one of the spheres surrounding a different one of the $2n$ points (at some standard locations) excised from $B^3$. We call these order parameter maps from infinitesimal spheres to $\mathcal{C}$ “germs”.

- Having broken the problem down into smaller pieces by introducing these fibrations, we use the fact that a fibration $F \to E \to B$ induces a long exact sequence for homotopy groups

\[
\ldots \to \pi_i(E) \to \pi_i(B) \to \pi_{i-1}(F) \to \pi_{i-1}(E) \to \ldots
\]

For instance, applying this to the fibration $\mathcal{M}_{2n} \to K_{2n} \to X_{2n}$ leads to the exact sequence $\ldots \to \pi_1(\mathcal{M}_{2n}) \to \pi_1(K_{2n}) \to \pi_1(X_{2n}) \to 1$. It follows that $\pi_1(K_{2n})$ is an extension of the permutation group $S_{2n} = \pi_1(X_{2n})$. By itself, the above long exact sequence is not very helpful for computing any of the homotopy groups involved unless we can show by independent means that two of the homotopy groups are trivial. Then the homotopy groups which lie between the trivial ones in the sequence are tightly constrained.

- We directly compute that $\pi_1(N_{2n}) = (\mathbb{Z}_2)^{2n}$ and $\pi_1(X_{2n}) = S_{2n}$. We show that the homotopy exact sequence then implies that $\pi_1(Y_{2n}) = (\mathbb{Z}_2)^{2n} \rtimes S_{2n}$.

- We compute the homotopy groups of $R_{2n}$, defined by the fibration $R_{2n} \to K_{2n} \to Y_{2n}$. This computation involves a different way from the cell structure of thinking about the topology of a space, called the ‘Postnikov tower’, explained in detail Appendix E. The basic idea is to approximate a space with spaces with only a few non-trivial homotopy groups. (This is analogous to the cell structure, which has only a few non-trivial homology groups.) The simplest examples of such spaces are Eilenberg-Mac Lane spaces, which only have a single
non-trivial homotopy group. The Eilenberg-Mac Lane space $K(A, m)$ is defined for a group $A$ and integer $m$ as the space with homotopy group $\pi_m(K(A, m)) = A$ and $\pi_k(K(A, m)) = 0$ for all $k \neq m$. (The group $A$ must be Abelian for $m > 1$.) Such a space exists and is unique up to homotopy. A space $T$ with only two non-trivial homotopy groups can be constructed through the fibration $K(B, n) \to T \to K(A, m)$. The space $T$ has $\pi_m(T) = A$ and $\pi_n(T) = B$, as may be seen from the corresponding long exact sequence for homotopy groups. Continuing in this fashion, one can construct a sequence of such approximations $M_n$ to a space $M$. They are defined by $\pi_k(M_n) = \pi_k(M)$ for $k \leq n$ and $\pi_k(M_n) = 0$ for $k > n$. They can be constructed iteratively from the fibration $K(A, n) \to M_n \to M_{n-1}$, where $\pi_n(M) = A$.

- With $\pi_1(Y_{2n})$, $\pi_2(Y_{2n})$, $\pi_0(R_{2n})$ and $\pi_1(R_{2n})$ in hand, we compute the desired group $\pi_1(K_{2n})$ from the homotopy exact sequence.

We now go through these steps in detail.

**Approximating U/O by a cell complex.** Depending on microscopic details, gradients in the overall phase of the fermions may be so costly that we wish to consider only configurations in which this overall phase is fixed. We will refer to this as the scenario in which ‘phase symmetry is broken’. In this case, the effective target space is $SU/SO$, the non-phase factor of $U/O \cong U(1)/O(1) \times SU/SO$. In this case, we simplify matters by replacing $SU/SO$ by $\tilde{U}/O$, the universal cover of $U/O$. $\tilde{U}/O$ is homotopy equivalent to $SU/SO$, so this substitution is harmless. This substitution results in a reduced configuration space $\tilde{K}_{2n}$ and we will concentrate first on calculating $\pi_1(\tilde{K}_{2n})$. In an appendix, we show that this reduction essentially makes no difference: $\pi_1(K_{2n}) = \pi_1(\tilde{K}_{2n}) \times \mathbb{Z}$.

We now define a cell complex $C$ approximating $\tilde{U}/O$. In constructing this cell structure, we are not interested in the beautiful homogeneous nature of $\tilde{U}/O$ but rather only its homotopy type. The homotopy type of a space tells you everything you will need to know to study deformation classes of maps either into or out of that space. An important feature of any homotopy type is the list of homotopy groups (but these are by no means a complete characterization in general). For $\tilde{U}/O$, the homotopy groups are $\pi_i(\tilde{U}/O) = 0, \mathbb{Z}_2, \mathbb{Z}_2, 0, 0, 0, 0, 0, 0, 0, 0, 0$ for $i = 1, \ldots, 9$ and thereafter $\pi_i(\tilde{U}/O)$ cycles through the last eight groups. (For $U/O$, the first group would be $\mathbb{Z}$.)

Because $\tilde{U}/O$ is simply-connected, but has nontrivial $\pi_2$, it natural in building a cellular model for its homotopy type to begin with $S^2$. Since $\pi_2(S^2) = \mathbb{Z}$ and we only need a $\mathbb{Z}_2$ for $\pi_2(\tilde{U}/O)$, we should kill off the even elements by attaching a 3-cell $D^3$ using a degree-2 map of its boundary 2-sphere to the original $S^2$. For future reference, take this map to be $(\theta, \phi) \to (2\theta, \phi)$ in a polar coordinate system where the north pole $N = (\pi, 0)$. Similarly, a 4-cell is attached to achieve $\pi_3(U/O) \cong \mathbb{Z}_2$. The necessity of the 4-cell is proved (Fact 1) below.

The preceding logic leads us to the cell structure:

$C = S^2 \bigcup_{\text{degree}=2} D^3 \bigcup_{\text{2Hopf}} D^4 \bigcup_{\text{cells dimension} \geq 5}$ (47)

Since we are only trying to compute the fundamental group $\pi_1(K_{2n})$ from our various homotopy long exact sequences, we do not have to figure out the higher cells (dimension $\geq 5$) of $U/O$. We will, however, verify that $\pi_3(U/O)$ is generated by the Hopf map into the base $S^2 \subset \tilde{U}/O$.

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To summarize, we will henceforth assume that the order parameter takes values in the cell complex $C$. Although $C$ is a crude approximation for U/O, it is good enough for the topological calculations which follow.

Dividing the problem into the motion of the hedgehog centers and the deformation of the field configuration. Let us assume that our physical system is a ball of material $B^3$. Let $n \geq 0$ be the number of hedgehog pairs in the system. A configuration in $\tilde{K}_{2n}$ is a texture in the order parameter, $\Phi(x) : B^3 \to C$, which satisfies the following boundary conditions at the boundary of $B^3$ and at the $2n$ hedgehog locations (which are singularities in the order parameter). The order parameter has winding number 0 at the boundary of the ball, $\partial B^3$ and winding number 1 around each of the hedgehog centers. (Recall that $\pi_2(C) = \mathbb{Z}_2$, so the winding number can only be 0 or 1).

From its definition, $\tilde{K}_{2n}$ is the total space of a fibration:

$$M_{2n} \to \tilde{K}_{2n} \to X_{2n}$$

The above diagram suggests that we should think of the fibration $M_{2n} \to \tilde{K}_{2n} \to X_{2n}$ in the following way: above each point in $X_{2n}$ there is a fiber $M_{2n}$; the total space formed thereby is $\tilde{K}_{2n}$. (This is not quite a fiber bundle, since we do not require that there be local coordinate charts in which $\tilde{K}_{2n}$ is simply the direct product.) Here, $X_{2n}$ is the simply the configuration space of 2n distinct points in $B^3$. We write this formally as $X_{2n} = \prod_{i=1}^{2n} B^3 \setminus \text{big diagonal}$. (The big diagonal consists of $2n$-tuples of points in $B^3$ where at least two entries are identical.) The space $M_{2n}$ consists of maps from $B^3 \setminus 2n$ points in a fixed standard position to $C$ with the prescribed winding numbers given in the preceding paragraph.

**Germs of order parameter textures.** It is helpful to introduce an intermediate step in the fibration. Define a point in $Y_{2n}$, as a configuration in $X_{2n}$ together with a “germ” of $\Phi(x)$, which we call $\hat{\Phi}(x)$, defined only near $\partial B^3$ and the 2n points. The idea behind the germ $\hat{\Phi}(x)$ is to forget about the order parameter $\Phi(x)$ except for its behavior in an infinitesimal neighborhood around each hedgehog center and at the boundary of the system. $\hat{\Phi}(x)$ must satisfy the same boundary conditions
as $\Phi(x)$ itself. We take $\overline{\Phi}(x)$ to be constant on $\partial B^3$ and to have winding number 1 around each of the hedgehog centers. With this definition, we now have the fibration:

$$N_{2n} \rightarrow Y_{2n} \rightarrow X_{2n}$$

where $N_{2n}$ is the space of (germs of) order parameter textures $\Phi$ from the neighborhoods of the $2n$ fixed standard points and $\partial B^3$ to $C$. We will henceforth replace discussion of germs with the equivalent and simpler concept of maps on the boundary condition that the winding number is 1 on $\partial B^3$ with the equivalent and simpler concept of maps on $\partial B^3 \cup \bigcup_{i=1}^{2n} S_i^2$ where $S_i^2$ is a small sphere surrounding the $i$th standard point. Thus,

$$N_{2n} \subset \text{Maps}\left( \left( \partial B^3 \cup \bigcup_{i=1}^{2n} S_i^2 \right) \rightarrow C \right). \quad (48)$$

We now define $Q_{2n}$ as the ball $B^3$ with small balls (denoted below by interior($S_i^2$)) centered about the hedgehogs deleted:

$$Q_{2n} = \left( B^3 \setminus \bigcup_{i=1}^{2n} \text{interior}(S_i^2) \right) \quad (49)$$

for fixed standard positions $i = 1, \ldots, 2n$. Then $R_{2n}$ is the space of order parameter textures on $Q_{2n}$ which satisfy the boundary condition that the winding number is 0 on $\partial B^3$ and 1 on each of the small spheres. With this definition, we have the fibration:

$$R_{2n} \rightarrow K_{2n} \rightarrow Y_{2n}$$

Given the cell structure $C$, we can specify the boundary conditions for the order parameter precisely. On $\partial B^3$, the order parameter is equal to the North Pole in $S^2$. (Recall that $S^2 \subset C$ is the bottom cell of the structure $C$ which we are using to approximate $U/O$.) On each of the spheres $S_i^2$, the order parameter $\Phi(x)$ defines a map from $S_i^2 \rightarrow S^2$ which is the identity map (where, again $S^2$ is understood as a subset of the order parameter space $S^2 \subset C$). This ensures that the order parameter has the correct winding numbers at the boundaries of $Q_{2n}$. In essence, what we have done in writing Eq. V is to break up an order parameter texture containing hedgehogs into (a) the hedgehogs together with the order parameter on infinitesimal neighborhoods around them (i.e. ‘germs’) and (b) order parameter textures in the intervening regions between the hedgehogs. The space of configurations (a) is $Y_{2n}$; the space of configurations (b) is $R_{2n}$.

The name $R_{2n}$ is for “ribbons.” As we saw in Section IV, if the order parameter manifold were $S^2$, we could summarize an order parameter texture by looking at the inverse image of the North Pole $N \subset S^2$ and a fixed tangent vector at the North Pole. The inverse images form a collection of ribbons. Now, the order parameter manifold is actually (approximated by) $C$, but the bottom cell in $C$ is $S^2$. The effect of the 3-cell is that the hedgehogs lose their sign, so there is no well-defined “arrow” running lengthwise along the ribbons. The 4-cell allows the “twist” or framings of ribbons to be altered at will by $\pm 2$.

**Long exact sequence for homotopy groups.** It is very convenient to use fibrations to calculate homotopy groups. (For those interested in $K$-theory, the last two chapters of Milnor’s *Morse Theory*28 are a must read and exhibit these methods with clarity.) As noted above, fibrations have all the homotopy properties of fiber bundles but are (often) found arising between function spaces where it would be a lot of work – and probably a distraction from important business – to attempt to verify the existence of locally trivial coordinate charts. Operationally, fibrations share with fiber bundles the all-important “homotopy long exact sequence”:

$$F \rightarrow E \rightarrow B$$

we have:

$$\cdots \rightarrow \pi_{i+1}(B) \rightarrow \pi_i(F) \rightarrow \pi_i(E) \rightarrow \pi_i(B) \rightarrow \pi_{i-1}(F) \rightarrow \cdots$$

We now compute $\pi_1(Y_{2n})$ from the exact sequence:

$$\pi_2(X_{2n}) \xrightarrow{\partial} \pi_1(N_{2n}) \xrightarrow{\partial} \pi_1(Y_{2n}) \xrightarrow{\partial} \pi_1(X_{2n}) \xrightarrow{\partial} \pi_0(N_{2n})$$

We can compute two of the homotopy groups in this equation by inspection. $\pi_1(X_{2n})$ is clearly the symmetric group of point exchange:

$$\pi_1(X_{2n}) = S_{2n}. \quad (51)$$

Meanwhile, $\pi_1(N_{2n})$ amounts to (products of) loops of maps from the $S_i^2$ to $C$ and reduces to $2n$ copies of the third homotopy group of $C$ (and, therefore, to $\pi_3(\widetilde{U/O})$). Thus, $\pi_1(N_{2n}) = (\mathbb{Z}_2)^{2n}$:

$$\pi_1(N_{2n}) = \prod_{i=1}^{2n} \pi_1(\text{Maps}(S_i^2, \widetilde{U/O})) = \prod_{2n \text{ copies}} \pi_3(\widetilde{U/O})$$
Thus, we have a short exact sequence:

\[ \partial \]  

To proceed further, we need to evaluate boundary maps in the homotopy exact sequence. In Appendix D, we explain boundary maps through the example of the Hopf map. Consider Eq. 50. \( \pi_2(X_{2n}) \) is generated by the \( \binom{2}{n} \) different 2-parameter motions in which a pair of hedgehogs come close together and explore the 2-sphere of possible relative positions around their center of mass. This 2-parameter family of motions involves no “rotation” of the maps \( \bar{\Phi} \) which describe \( \pi_1(N_{2n}) \) (i.e. the order parameter configuration in the neighborhood of each hedgehog does not rotate as the hedgehogs are moved), so the leftmost \( \partial \) map in Eq. 50 is zero. Similarly, a simple exchange of hedgehogs produces no twist of the order parameter configuration in the neighborhood of either hedgehog, so the second \( \partial \) map of Eq. 50 is also zero. Thus, we have a short exact sequence:

\[ 1 \longrightarrow \mathbb{Z}_2^{2n} \overset{\alpha}{\longrightarrow} \pi_1(Y_{2n}) \overset{\beta}{\longrightarrow} S_{2n} \longrightarrow 1 \]

To derive this short exact sequence, we used the triviality of the boundary maps noted above and Eqs. 51, 52 to simplify Eq. 50.

There is a natural group homomorphism \( s \):

\[ s : S_{2n} \to \pi_1(Y_{2n}) \]

which associates to each permutation a motion of hedgehogs which permutes the hedgehogs in \( Y_{2n} \) but does not rotate the order parameter configuration \( \bar{\Phi} \) near the hedgehogs. Then \( \beta \circ s = id_{S_{2n}} \). In other words, the sequence is split:

\[ 1 \longrightarrow \mathbb{Z}_2^{2n} \overset{\alpha}{\longrightarrow} \pi_1(Y_{2n}) \overset{\beta}{\longrightarrow} s_{S_{2n}} \longrightarrow 1 \]

Thus, \( \pi_1(Y_{2n}) \) is a semi-direct product. To determine \( \pi_1(Y_{2n}) \) completely, it only remains to identify how \( s(S_{2n}) \) acts on the twist factors \( \mathbb{Z}_2^{2n} \) under conjugation. It is quite clear that this action is the only natural one available: \( s(p) \) acts on \( \mathbb{Z}_2^{2n} \) by applying the permutation \( p \) to the \( 2n \) coordinates of \( \mathbb{Z}_2^{2n} \). So, \( \pi_1(Y_{2n}) \cong \mathbb{Z}_2^{2n} \rtimes S_{2n} \) with group law:

\[ (v, p) \circ (v', p') = (v + p(v'), p \circ p') \]

where \( v \in \mathbb{Z}_2^{2n} \) is a \( \mathbb{Z}_2 \)-vector, \( p \in S_{2n} \), a permutation, and \( p(v') \) the natural action of \( S_{2n} \) on \( \mathbb{Z}_2^{2n} \) applied to \( v' \). Note that this is precisely the multiplication rule which we obtained pictorially in Section IV.

Computing the homotopy groups of \( R_{2n} \), the space of order parameter textures interpolating between the hedgehogs. Of course, computing \( \pi_1(Y_{2n}) \) only gets us part of the way home. Our ultimate goal is to compute \( \pi_1(K_{2n}) \). Thus, we now turn to the homotopy long exact sequence:

\[ \pi_2(Y_{2n}) \overset{\partial_2}{\longrightarrow} \pi_1(R_{2n}) \to \pi_1(K_{2n}) \to \pi_1(Y_{2n}) \overset{\partial_1}{\longrightarrow} \pi_0(R_{2n}) \]

First consider \( \partial_1 \). The kernel of \( \partial_1 \) is represented by loops in \( Y_{2n} \) which extend to loops in \( R_{2n} \). A loop \( \gamma \) in \( Y_{2n} \) is a motion of the hedgehogs together with rotations (about the spheres \( S_2^2 \)) of \( \bar{\Phi} \) which brings the system back to its initial configuration. If a loop \( \gamma \) is in kernel of \( \partial_1 \), then there is a corresponding loop in the configuration space of ribbons in \( B^3 \) (obtained by lifting \( \gamma \) to \( K_{2n} \)).

The next steps are to compute \( \pi_0(R_{2n}) \) and \( \pi_1(R_{2n}) \). These computations are detailed in Appendix E, where we see that using the cell structure \( C \) which we introduced for \( U/O \) which allows us to make all the calculations we need. We find that \( \pi_0(R_{2n}) = \mathbb{Z}_2 \) and \( \pi_1(R_{2n}) = (\mathbb{Z}_2)^{2n} \).

Thus, \( \pi_1(K_{2n}) \) sits in the following exact sequence:

\[ \pi_2(Y_{2n}) \overset{\partial_2}{\longrightarrow} \pi_1(R_{2n}) \to \pi_1(K_{2n}) \to \pi_1(Y_{2n}) \overset{\partial_1}{\longrightarrow} \pi_0(R_{2n}) \]

Recall that when we studied \( \pi_2(Y_{2n}) \), we found (exactly as in the case of \( \pi_2(X_{2n}) \)) that there are \( \binom{2}{n} \) generators corresponding to relative 2-parameter motions of any pair of hedgehogs around their center of mass. This can be used to understand the map \( \partial_2 : \pi_2(Y_{2n}) \to \pi_2(R_{2n}) \). The image of any center of mass 2-motion is the “bag” containing the corresponding pair of hedgehogs. Thus, \( \text{ker}(\partial_2) \cong \mathbb{Z}_2 \); it is \( \mathbb{Z}_2^{2n} \) modulo the even sublattice (vectors whose coordinate sum is zero in \( \mathbb{Z}_2 \)). Thus, we have the short exact sequence:

\[ 1 \longrightarrow \text{ker}(\partial_2) \to \pi_1(K_{2n}) \overset{\pi}{\longrightarrow} \text{ker}(\partial_1) \longrightarrow 1 \]

The kernel \( \text{ker}(\partial_1) \) consists of the part of \( \pi_1(Y) \) associated with even \((2\pi)\) twisting. As shown in Figure 9, a simple exchange is associated to a total twisting of ribbons by \( \pm 2\pi \).

Thus, \( \partial_1(v, p) = \sum_{i=1}^{2n} v_i + \text{parity}(p) \in \mathbb{Z}_2 \). We use the notation \( E(\mathbb{Z}_2^{m} \rtimes S_m) \) for \( \text{ker}(\partial_1) \).

Note: If \( m = 2 \), \( \mathbb{Z}_2^m \rtimes S_m \) is the dihedral group \( D_4 \) and its “even” subgroup \( \text{ker}(\partial_1) \cong \mathbb{Z}_4 \). This shows that for \( m \) even, the induced short exact sequence does not split, and the extension is more complicated:

\[ 1 \longrightarrow \mathbb{Z}_2^{2n-1} \longrightarrow E(\mathbb{Z}_2^{2n} \rtimes S_{2n}) \longrightarrow S_{2n} \longrightarrow 1 \]

There is a final step required to solve the extension problem 54 and finish the calculation of \( \pi_1(K_{2n}) \). We geometrically construct a homomorphism \( s : \text{ker}(\partial_1) = E(\mathbb{Z}_2^{2n} \rtimes S_{2n}) \to \pi_1(K_{2n}) \) which is a left inverse to the projection.

This will show that \( \pi_1(K_{2n}) \) is a semidirect product \( \mathbb{Z}_2 \rtimes \text{ker}(\partial_1) \), but since \( \mathbb{Z}_2 \) has no nontrivial automorphism, the semidirect product is actually direct:
\[ \pi_1(\overline{K}_{2n}) \cong \mathbb{Z}_2 \times E(\mathbb{Z}_2^{2n} \times S_{2n}) \]  

(55)

To construct \( s \), note that all elements of \( \ker(\partial_1) \) can be realized by a loop \( \gamma \) of maps into the bottom 2-cell of Eq. 47 \( S^2 \). Still confining the order parameter (map) to lie in \( S^2 \), such a loop lifts to an arc \( \tilde{\gamma} \) of ribbons representing an arc in \( \overline{K}_{2n} \). We may choose the lift so that as the ribbons move, they never "pass behind" the 2n hedgehogs. For example, we may place the hedgehogs on the sphere of radius \( \frac{1}{2} \) inside the 3-ball \( B^3 \) (assumed to have radius \( 1 \)) and then keep all ribbons inside \( B^3_1 \). These arcs may be surgured (still as preimages of \( N \subset \mathbb{S}^2 \)) so that they return to their original position except for a possible accumulation of normal twisting \( 2\pi \). Since \( \gamma \in \ker(\partial_1) \), \( t \) must be even. Now, allowing the order parameter (map) to leave \( S^2 \) and pass over the 4-cell (of Eq. 47), attached by \( 2\text{Hopf} : S^3 \rightarrow S^2 \), we may remove these even twists. (The 4-cell can introduce small closed ribbons with self-linking = 2 in a small ball. These small ribbons can be surgured into other ribbons.) This lifts a generating set of \( \ker(\partial_1) \) into \( \pi_1(\overline{K}_{2n}) \) as a set theoretic cross section (left inverse to \( \pi_1 \)). But what about relations? Because the entire loop is constant outside \( B^3_1 \), the corresponding homology class in \( H_2(Q \times I; \mathbb{Z}_2) \cong \pi_1(R) \) is trivial, so \( s \) is actually a group homomorphism.

### VI. REPRESENTATION THEORY OF THE RIBBON PERMUTATION GROUP

In this section, we discuss the mathematics of the group \( \mathcal{T}_m \) and its representation. The purpose of this section is to show that a direct factor of \( \mathcal{T}_m \), called the even ribbon permutation group, is a ghostly recollection of the braid group and the Teo-Kane representation of the even ribbon permutation group is a projectivized version of the Jones representation of the braid group at a \( 4^{th} \) root of unity, i.e. the representation relevant to Ising anyons.

#### A. Teo-Kane fundamental groups

In Section V, we consider only even number of hedgehogs for physical reasons. In this section, we will include the odd case for mathematical completeness.

The Teo-Kane fundamental group is the fundamental group of the Teo-Kane configuration space \( K_m \). As computed in Section V, \( \mathcal{T}_m = \pi_1(K_m) \cong \mathbb{Z} \times \mathbb{Z}_2 \times E(\mathbb{Z}_2^m \times S_m) \), where the subgroup \( \mathcal{G}_m = \mathbb{Z}_2 \times E(\mathbb{Z}_2^m \times S_m) \) is called the ribbon permutation group. Here, \( E(\mathbb{Z}_2^m \times S_m) \) is the subgroup of \( \mathbb{Z}_2^m \times S_m \) comprised of elements whose total parity in \( \mathbb{Z}_2^m \) added to their parity in \( S_m \) is even. In the following, we will call the group \( G_m = E(\mathbb{Z}_2^m \times S_m) \) the even ribbon permutation group because it consists of the part of \( \pi_1(Y_m) \) associated with even \( (2\pi) \) twisting. For the representations of the Teo-Kane fundamental groups, we will focus on the even ribbon permutation groups \( G_m \). No generality is lost if we consider only irreducible representations projectively because irreducibles of \( \mathbb{Z} \) and \( \mathbb{Z}_2 \) contribute only overall phases. But for reducible representations, the relative phases from representations of \( \mathbb{Z} \) and \( \mathbb{Z}_2 \) might have physical consequences in interferometer experiments.

The even ribbon permutation group \( G_m \) is an index-2 subgroup of \( \mathbb{Z}_2^m \rtimes S_m \). To have a better understanding of \( G_m \), we recall some facts about the important group \( \mathbb{Z}_2^m \rtimes S_m \). The group \( \mathbb{Z}_2^m \rtimes S_m \) is the symmetry group of the hypercube \( \mathbb{Z}_2^m \), therefore it is called the hyperoctahedral group, denoted as \( C_m \). \( C_m \) is also a Coxeter group of type \( B_m \) or \( C_m \), so in the mathematical literature it is also denoted as \( B_m \) or \( BC_m \). To avoid confusion with the braid group \( B_m \), we choose to use the hyperoctahedral group notation \( C_m \). The group \( C_m \) has a faithful representation as signed permutation matrices in the orthogonal group \( O(m) \): matrices with exactly one non-zero entry \( \pm 1 \) in each row and column. Therefore, it can also be realized as a subgroup of the permutation group \( S_{2m} \), called signed permutations: \( \sigma : \{ \pm 1, \pm 2, \ldots, \pm m \} \rightarrow \{ \pm 1, \pm 2, \ldots, \pm m \} \) such that \( \sigma(-i) = -\sigma(i) \).

We will denote elements in \( C_m \) by a pair \( (b, g) \), where \( b = (b_i) \in \mathbb{Z}_2^m \) and \( g \in S_m \). Recall the multiplication of two elements \( (b, g) \) and \( (c, h) \) is given by \( (b, g) \cdot (c, h) = (b + g.c, gh) \), where \( g.c \) is the action of \( g \) by permuting its coordinates. Let \( \{ e_i \} \) be the standard basis elements of \( \mathbb{R}^m \). To save notation, we will also use it for the basis element of \( \mathbb{Z}_2^m \). As a signed permutation matrix \( e_i \) introduces a \( -1 \) into the \( i^{th} \) coordinate \( x_i \). Let \( s_i \) be the transposition of \( S_m \) that interchanges \( i \) and \( i+1 \). As a signed permutation matrix, it interchanges the coordinates \( x_i, x_{i+1} \). There is a total parity map \( \text{det} : C_m \rightarrow \mathbb{Z}_2 \) defined as \( \text{det}(b, g) = \sum_{i=1}^m b_i + \text{parity}(g) \mod 2 \). We denote the total parity map as \( \text{det} \) because in the realization of \( C_m \) as signed permutation matrices in \( O(m) \), the total parity is just the determinant. Hence \( G_m \), as the kernel of \( \text{det} \), can be identified as a subgroup of \( SO(m) \). The set of elements \( t_i = (e_i, s_i), i = 1, \ldots, m-1 \) generates \( G_m \). As a signed permutation matrix, \( t_i(x_1, \ldots, x_i, x_{i+1}, \ldots, x_m) = (x_1, \ldots, -x_{i+1}, x_i, \ldots, x_m) \).

Given an element \( (b, g) \in C_m \), let \( \mathbb{Z}_2^{m-1} \) be identified as the subgroup of \( \mathbb{Z}_2^m \) such that \( \sum_{i=1}^m b_i = \text{even} \). Then we have:

**Proposition VI.1.**

1. For \( m \geq 2 \), the even ribbon permutation group \( G_m \) has a presentation as an abstract group

\[
< t_1, \ldots, t_{m-1} | t_i^4 = 1, i = 1, \ldots, m - 1, \\
(t_i^{-1} t_{i+1})^3 = (t_{i+1}^{-1} t_i)^3 = 1, i = 1, \ldots, m - 2 >.
\]

2. The exact sequence

\[ 1 \rightarrow \mathbb{Z}_2^{m-1} \rightarrow G_m \rightarrow S_m \rightarrow 1 \]

splits if and only if \( m \) is odd.

3. When \( m \) is even, a normalized 2-cocycle \( f(g, h) : S_m \times S_m \rightarrow \mathbb{Z}_2^{-m-1} \) for the extension of \( S_m \) by \( \mathbb{Z}_2^{-m} \) above
can be chosen as \( f(g, h) = 0 \) if \( g \) or \( h \) is even and \( f(g, h) = e_{g(1)} \) if \( g \) and \( h \) are both odd.

We briefly give the idea of the proof of Prop. VI.1. For (1), first we use a presentation of \( C_{m} \) as a Coxeter group of type \( B_{m} : \langle r_{1}, \ldots, r_{m} | r_{i}^{2} = 1, (r_{i} r_{j})^{3} = 1, i = 2, \ldots, m - 1 \rangle \). Then the Reidemeister-Schreier method allows us to deduce the presentation for \( G_{m} \). Above for (2), when \( m \) is odd, a section \( s \) for the splitting can be defined as \( s(g) = (0, g) \) if \( g \) is even and \( s(g) = ((11 \cdots 1), g) \) if \( g \) is odd. When \( m \) is even, that the sequence does not split follows from the argument in \(^{55} \). For (3), we choose a set map \( s(g) = (0, g) \) if \( g \) is even and \( s(g) = (e_{1}, g) \) if \( g \) is odd. Then a direct computation of the associated factor set as on page 91 of \(^{56} \) gives rise to our 2-cocycle.

As a remark, we note that there are another two obvious maps from the hyperoctahedral group \( C_{m} \) to \( \mathbb{Z}_{2} \). One of them is the sum of bits in \( b \) of \( (b, g) \). The kernel of this map is the Coxeter group of type \( D_{m} \), which is a semi-direct product of \( \mathbb{Z}_{2}^{m-1} \) with \( S_{m} \). The two groups \( D_{m} \) and \( G_{m} \) have the same order, and are isomorphic when \( m \) is odd (the two splittings induce the same action of \( S_{m} \) on \( \mathbb{Z}_{2}^{m-1} \)), but different when \( m \) is even. To see the difference, consider the order 2 automorphism \( \phi : C_{m} \to C_{m} \) given by \( \phi(x) = \det(x) x \). Its restriction is the identity on \( G_{m} \), but non-trivial on \( D_{m} \).

B. Teo-Kane unitary transformations

Suppose there are \( m \) hedgehogs in \( B^{3} \). A unitary transformation \( T_{ij} = e^{2 \pi i \gamma_{ij}} \) of the Majorana fermions is associated with the interchange of the \((i,j)\)-pair of the hedgehogs. Interchanging the same pair twice results in the "braided" operation \( T_{ij}^{2} = \gamma_{i} \gamma_{j} \). The Majorana fermions \( \gamma_{i}, i = 1, 2, \ldots, m \) form the Clifford algebra \( Cl_{m}(\mathbb{C}) \). Therefore, the Teo-Kane unitary transformations act as automorphisms of the Clifford algebra

\[
T_{ij} : \gamma^{i} \mapsto T_{ij} \gamma^{i} T_{ij}^{\dagger}.
\]

The projective nature of the Teo-Kane representation bears its head here already as an overall phase cannot be constrained by such actions. Do these unitary transformations afford a linear representation of the Teo-Kane fundamental group \( K_{m} \)? If so, what are their images? The surprising fact is that the Teo-Kane unitary transformations cannot give rise to a linear representation of the Teo-Kane fundamental group. The resulting representation is intrinsically projective. We consider only the even ribbon permutation group \( G_{m} \) from now on.

To define the Teo-Kane representation of \( G_{m} \), we use the presentation of \( G_{m} \) by \( t_{i}, i = 1, \ldots, m - 1 \) in Prop. VI.1. The associated unitary matrix for \( t_{i} \) comes from the Teo-Kane unitary transformation \( T_{i+1} \). As was alluded above, there is a phase ambiguity of the Teo-Kane unitary matrix. We will discuss this ambiguity more in the next subsection. For the discussion below, we choose any matrix realization of the Teo-Kane unitary transformation with respect to a basis of the Clifford algebra \( Cl_{m}(\mathbb{C}) \). A simple computation using the presentation of \( G_{m} \) in Prop. VI.1 verifies that the assignment of \( T_{i+1} \) to \( t_{i} \) indeed leads to a representation of \( G_{m} \). Another verification follows from a relation to the Jones representation in the next subsection. We can also check directly that this is indeed the right assignment for \( T_{i+1}^{2} : \gamma^{i} \rightarrow -\gamma^{i}, \gamma^{i+1} \rightarrow -\gamma^{i+1} \). In the Clifford algebra \( Cl_{m}(\mathbb{C}) \), \( \gamma_{i+1} \) correspond to the standard basis element \( e_{i}, e_{i+1} \). The element \( T_{i}^{2} \) of \( G_{m} \) is \((e_{i} + e_{i+1}, 1)\). As a signed permutation matrix, \( T_{i}^{2} \) sends \((x_{1}, \ldots, x_{i}, x_{i+1}, \ldots, x_{m}) \to (x_{1}, \ldots, -x_{i+1} \cdots, x_{m}) \), which agrees with the action of \( T_{i+1}^{2} \) on \( \gamma_{i}, \gamma_{i+1} \).

To see the projectivity, the interchange of the \((i,i+1)\)-pair hedgehogs corresponds to the element \( t_{i} = (e_{i}, s_{i}) \in G_{m} \). Performing the interchange twice gives rise to the element \( t_{i}^{2} = (e_{i} + e_{i+1}, 1) \), denoted as \( x_{m} \). Since \( x_{m} \) is a subgroup of \( G_{m} \) isomorphic \( \mathbb{Z}_{2}^{m-1} \), we obviously have \( x_{i} x_{i+1} = x_{i+1} x_{i} \). On the other hand, \( T_{i+1}^{2} T_{i+1, i+2} = \gamma^{i+1} \gamma^{i+2} \gamma^{i} \gamma^{i+2} = \gamma^{i} \gamma^{i+2} \), and \( T_{i+1}^{2} T_{i+1, i+2} T_{i+2, i+3} = \gamma^{i+1} \gamma^{i+2} \gamma^{i+3} \gamma^{i+2} \gamma^{i+3} = \gamma^{i} \gamma^{i+2} \gamma^{i+3} \). Since it is impossible to encode the \(-1\) in the \( x_{m} \)'s of \( G_{m} \), the representation has to be projective. Note that an overall phase in \( T_{ij} \) will not affect the conclusion. In the next subsection, we will see this projective representation comes from a linear representation of the braid group—the Jones representation at a \( 4^{th} \) root of unity and the \(-1\) is encoded in the Jones-Wenzl projector \( p_{3} = 0 \).

To understand the images of the Teo-Kane representation of \( G_{m} \), we observe that the Teo-Kane unitary transformations \( T_{ij} = e^{2 \pi i \gamma_{ij}} \) lie inside the even part \( Cl_{m}^{0}(\mathbb{C}) \) of \( Cl_{m}(\mathbb{C}) \). Therefore, the Teo-Kane representation of \( G_{m} \) is just the spinor representation projectivized to \( G_{m} \subset SO(\mathbb{C}) \). It follows that the projective image group of Teo-Kane representation of the even ribbon permutation group is \( G_{m} \) as an abstract group. Recall \( Cl_{m}^{0}(\mathbb{C}) \) is reducible into two irreducible sectors if \( m \) is even, and irreducible if \( m \) is odd. When \( m \) is even, it is important to know how the relative action of the center \( Z(G_{m}) \cong \mathbb{Z}_{2} \) of \( G_{m} \) on the two irreducible sectors. The center of \( G_{m} \) is generated by the element \( z = ((1 \cdots 1), 1) \), whose corresponding Teo-Kane unitary transformation is \( U = \gamma^{1} \gamma^{2} \cdots \gamma^{m} \) up to an overall phase. As we show in Appendix H, the relative phase of \( z \) on the two sectors is always \(-1\).

VII. DISCUSSION

We now review and discuss the main results derived in this paper. Using the topological classification of free fermion Hamiltonians \(^{26,27} \), we considered a system of fermions in 3D which is allowed to have arbitrary superconducting order parameter and arbitrary band structure; and is also allowed to develop any other possible symmetry-breaking order such as charge density-wave, etc. – i.e. we do not require that any symmetries are preserved. We argued in Section III that the space of possible gapped ground states of such a system is topologically equivalent to \( U(N)/O(N) \) for \( N \) large (for large \( N \), the topology of \( U(N)/O(N) \) becomes independent of \( N \)). By extension, if we can spatially vary the superconducting order parameter and band structure at will with no regard to the energy cost, then there will be topologically stable point-like
defects classified by $\pi_2(U(N)/O(N)) = \mathbb{Z}_2$.

This statement begs the question of whether one actually can vary the order parameter and band structure in order to create such defects. In a given system, the energy cost may simply be too high for the system to wind around $U(N)/O(N)$ in going around such a defect. (This energy cost, which would include the condensation energy of various order parameters, is not taken into account in the free fermion problem.) If we create such defects, they may be so costly that it is energetically favorable for them to simply unwind by closing the gap over large regions. (The energy cost associated with such an unwinding depends on the condensation energy of the order parameters involved, which is not included in the topological classification.) Thus, $U(N)/O(N)$ is not the target space of an order parameter in the usual sense because the different points in $U(N)/O(N)$ may not correspond to different ground states with the same energy. However, in Section II, we have given at least one concrete model of free fermions with no symmetries in 3D in which the topological defects predicted by the general classification are present and stable. Furthermore, Teo and Kane\textsuperscript{20} have proposed several devices in which these defects are simply superconducting vortices at the boundary of a topological insulator.

In order to understand the quantum mechanics of these defects, it is important to first understand their quantum statistics. To do this, we analyzed the multi-defect configuration space; its fundamental group governs defect statistics. The configuration space of $2n$ point-like defects of a system with ‘order parameter’ taking values in $U(N)/O(N)$ is the space which we call $K_{2n}$. It can be understood as a fibration. The base space is $X_{2n}$, the configuration space of $2n$ points (which we know has fundamental group $S_{2n}$ in dimension three and greater). Above each point in this base space there is a fiber $M_{2n}$ which is the space of maps from the ball $B^3$ to $2n$ fixed points to $U(N)/O(N)$ with winding number 1 about each of the $2n$ points. $K_{2n}$ is the total space of the fibration. In Section V, we found that its fundamental group is $\pi_1(K_{2n}) = \mathbb{Z} \times \mathbb{Z}_2 \times E(\mathbb{Z}_{2n}^2 \rtimes S_{2n})$, where $E(\mathbb{Z}_{2n}^2 \rtimes S_{2n})$ is the subgroup of $\mathbb{Z}_{2n}^2 \rtimes S_{2n}$ comprised of elements whose total parity in $\mathbb{Z}_{2n}^2$ added to their parity in $S_{2n}$ is even.

The fundamental group of the configuration space is the same as the group of equivalence classes of spacetime histories of a system with $2n$ point-like defects. Since these different equivalence classes cannot be continuously deformed into each other, quantum mechanics allows us to assign them different unitary matrices. These different unitaries form a representation of the fundamental group of the configuration space of the system. However, we found in Section VI that Teo and Kane’s unitary transformations are not a linear representation of $\pi_1(K_{2n})$, but a projective representation, which is to say that they represent $\pi_1(K_{2n})$ only up to a phase. Equivalently, Teo and Kane’s unitary transformations are an ordinary linear representation of a central extension of $\pi_1(K_{2n})$, as discussed in Section VI.

This surprise lurks in a seemingly innocuous set of defect motions: those in which defects $i$ and $j$ are rotated by $2\pi$ and the order parameter field surrounding them relaxes back to its initial configuration. This has the following effect on the Majorana fermion zero mode operators associated with the two defects:

$$\gamma_i \rightarrow -\gamma_i, \quad \gamma_j \rightarrow -\gamma_j \quad (56)$$

One might initially expect that two such motions, one affecting defects $i$ and $j$ and the other affecting $i$ and $k$, would commute since they simply multiply the operators involved by $-1$ signs. However, the unitary operator which generates (56) is\textsuperscript{59}:

$$U_{ij} = e^{i\theta} \gamma_i \gamma_j \quad (57)$$

Thus, the unitary operators $U_{ij}$ and $U_{ik}$ do not commute; they anti-commute:

$$U_{ij} U_{ik} = -U_{ik} U_{ij} \quad (58)$$

However, as shown in Figure 12, the corresponding classical motions can be continuously deformed into each other. Thus, a linear representation of the fundamental group of the classical configuration space would have these two operators commuting. Instead, the quantum mechanics of this system involves a projective representation.

Projective quasi-particle statistics were first proposed by Wilzcek\textsuperscript{30}, who suggested a projective representation of the permutation group in which generators $\sigma_j$ and $\sigma_k$ anti-commute for $|j - k| \geq 2$, rather than commuting. Read\textsuperscript{25} criticized this suggestion as being in conflict with locality. We can sharpen Read’s criticism as follows. Suppose that one can perform the operation $\sigma_1$ by acting on a region of space, called $A$, containing particles 1 and 2, and one can perform $\sigma_3$ by acting on a region called $B$, containing particles 3 and 4, and suppose that regions $A$ and $B$ are disjoint. Consider the following thought experiment: let Bob perform operation $\sigma_3$ at time $t = 0$ and let Bob then repeat operation $\sigma_3$ at time $t = 1$.

Let Alice prepare a spin in the state $|\uparrow\rangle\langle\uparrow|\otimes|\uparrow\rangle\langle\uparrow|$ at time $t = 0$ and then let Alice perform the following sequence of operations. At time $t = -\epsilon$, for some small $\epsilon$, she performs the unitary operation $|\uparrow\rangle\langle\uparrow| \otimes |\uparrow\rangle\langle\uparrow| \otimes |\uparrow\rangle\langle\uparrow| \otimes |I\rangle \otimes |I\rangle$, where $I$ is the identity operation, leaving the particles alone. At time $t = +\epsilon$, she performs the unitary operation $|\uparrow\rangle\langle\uparrow| \otimes |I\rangle \otimes |I\rangle \otimes |\uparrow\rangle\langle\uparrow| \otimes |\uparrow\rangle\langle\uparrow| \otimes |\uparrow\rangle\langle\uparrow| \otimes |I\rangle \otimes |I\rangle$. Thus, if the spin is up, she performs $\sigma_1$ at time $t = -\epsilon$, while if the spin is down, she does it at time $t = +\epsilon$. Finally, at time $t = 2$, Alice performs the operation $\sigma_1$ again. One may then show that, due to the anti-commutation of $\sigma_1$ and $\sigma_3$, that the spin ends in the state $(1/\sqrt{2})(|\uparrow\rangle + |\downarrow\rangle)$. However, if Bob had not performed any operations, the spin would have ended in the original state $(1/\sqrt{2})(|\uparrow\rangle + |\downarrow\rangle)$. Thus, by performing these interchange operations, Bob succeeds in transmitting information to a space-like separated region (if $A$ and $B$ are disjoint, and the time scale in the above thought experiment is sufficiently fast, then Alice and Bob are space-like separated throughout).

Having seen this criticism, we can also see how Teo and Kane’s construction evades it. The fundamental objects for Teo and Kane are not particles, but particles with ribbons attached. One may verify that, in every case where operations in Teo and Kane’s construction anti-commute, the two opera-
tions do not act on spatially disjoint regions due to the attached ribbons. That is, the interchange of particles also requires a rearrangement of the order parameter field.

While this argument explains why a projective representation does not violate causality, it does not really explain why a projective representation actually occurs in this system. Perhaps one clue is the fact that the hedgehogs have long-ranged interactions in any concrete model. Even in the “best-case scenario”, in which the underlying Hamiltonian of the system is $U(N)/O(N)$-invariant, there will be a linearly-diverging gradient energy for an isolated hedgehog configuration. Thus, there will be a linear long-ranged force between hedgehogs. Consequently, one might adopt the point of view that, as a result of these long-ranged interactions, the overall phase associated with a motion of the hedgehogs is not a purely topological quantity (but will, instead depend on details of the motion) and, therefore, need not faithfully represent the underlying fundamental group. As one motion is continuously deformed into another in Figure 12, the phase of the wavefunction varies continuously from $+1$ to $-1$ as the order parameter evolution is deformed. It is helpful to compare this to another example of a projective representation: a charged particle in a magnetic field $B$. Although the system is invariant under the Abelian group of translations, the quantum mechanics of the system is governed by a non-Abelian projective representation of this group (which may be viewed as a linear representation of the ‘magnetic translation group’).

A translation by $a$ in the $x$-direction, followed by a translation by $b$ in the $y$-direction differs in its action on the wavefunction from the same translations in the opposite order by a phase $abB/\Phi_0$ equal to the magnetic flux through the area $ab$ in units of the flux quantum $\Phi_0$. If we continuously deform these two sequences of translations into each other, the phase of the wavefunction varies continuously. For any trajectory along this one-parameter family of trajectories (or ‘movie of movies’), the resulting phase of the wavefunction is given by the magnetic flux enclosed by the composition of this trajectory and the inverse of the initial one. In our model of non-Abelian projective statistics, the phase changes continuously in the same way, but as a result of the evolution of the order parameter away from the defects, rather than as a result of a magnetic field.

An obvious question presents itself: is there a related theory in which the hedgehogs are no longer confined? Equivalently, could 3D objects with non-Abelian ribbon permutation statistics ever be the weakly-coupled low-energy quasiparticles of a system? The most straightforward route will not work: if we had a $U(N)$-invariant system and tried to gauge it to eliminate the linear confining force between hedgehogs, we would find that the theory is sick due to the chiral anomaly. If we doubled the number of fermions in order to eliminate the anomaly, there would be two Majorana modes in the core of each hedgehog, and their energies could be split away from zero by a local interaction. This is not surprising since ribbon permutation statistics would violate locality if the hedgehogs were truly decoupled (or had exponentially-decaying interactions). On the other hand, if the hedgehogs were to interact through a Coulomb interaction (or, perhaps, some other power-law), they would be neither decoupled nor confined, thereby satisfying the requirements that they satisfy locality and are low-energy particle-like excitations of the system. Elsewhere, we will describe a model which realizes this scenario.

The non-Abelian projective statistics studied in the 3D class with no symmetry can be generalized to arbitrary dimension. As shown in Eq. (41), the classification of topological defects is independent of the spatial dimension. Thus, in any dimension $d$ with no symmetry ($p = 0$), point-like ($D = 0$) topological defects are classified by $\pi_0(\mathcal{R}_{d-D+1}) = \pi_0(\mathcal{R}_1) = \mathbb{Z}_2$.

Moreover, it can be proved that analogous topological defects in different dimensions not only carry the same topological quantum number, but also have the same statistics. In Sec. V, we have defined the configuration space $\mathcal{M}_{2n}$, which is the space of maps from $B^3 \setminus \bigcup_{i=1}^{2n} B_i^3$ to $\mathcal{R}_7 = U/O$, with specific boundary conditions. Now if we consider point defects in the class with no symmetry in 4D, the configuration space $\mathcal{M}_{2n}$ is defined by maps from $B^4 \setminus \bigcup_{i=1}^{2n} B_i^4$ to the classifying space $\mathcal{R}_6 = Sp/U$. Noticing that $B^4 \setminus \bigcup_{i=1}^{2n} B_i^4$ is homotopy equivalent to the suspension of $B^3 \setminus \bigcup_{i=1}^{2n} B_i^3$, we obtain that $\mathcal{M}_{2n}$ is the space of maps from $B^3 \setminus \bigcup_{i=1}^{2n} B_i^3 \rightarrow \Omega (Sp/U)$, where $\Omega (Sp/U)$ is the loop space of $Sp/U$. Since $\Omega (Sp/U) \simeq U/O$, we obtain $\mathcal{M}_{2n} \simeq \mathcal{M}_{2n}$. Therefore, we have proved that the configuration space $\mathcal{M}_{2n}$ is independent of the spatial dimension $d$. On the other hand, the fundamental group of the configuration space $\mathcal{X}_{2n}$ of $2n$ distinct points in $\mathcal{B}^d$ is independent of $d$ as long as $d > 2$. Consequently, the space $\mathcal{K}_{2n}$ defined by the fibration $\mathcal{M}_{2n} \rightarrow \mathcal{K}_{2n} \rightarrow \mathcal{X}_{2n}$ is also topologically independent of spatial dimension $d$ for $d > 2$. Thus, the proof we did for $d = 3$ applies to generic dimension, and non-Abelian projective statistics exist in any spatial dimension $d > 3$ for point defects in the no-symmetry class. A similar analysis applies to extended defects with dimension $D > 0$. When the spatial dimension is increased by 1 and the symmetry class remains the same, the classifying space is always changed from $\mathcal{R}_{2p-d}$ to $\mathcal{R}_{2p-d}$. Consequently, at least for simple defects with the topology of $Sp$, the statistics is independent of spatial dimension $d$ as long as $d$ is large enough. For point defects, the “lower critical dimension” is $d = 3$, while for line defects, i.e. $D = 1$, the “lower critical dimension” is at least $d = 4$ since in $d = 3$ we can have braiding between loops.

Appendix A: Topological Classification of Hamiltonians with Charge Conservation but without Time-Reversal Symmetry

For the sake of completeness, in this appendix we give the topological classification of free fermion Hamiltonians with charge conservation but not time-reversal symmetry. We begin with the zero-dimensional case. The charge conservation condition is most transparent when the Hamiltonian is written in terms of complex fermion operators, $c_i, c_i^\dagger$, rather than
Majorana fermions:

\[ H = \sum_{i,j} h_{ij} c_i^\dagger c_j \]  
(A1)

The indices \( i, j \) range from 1 to \( N \), the number of bands. So long as there is one \( c \) and one \( c^\dagger \) in each term, the Hamiltonian will conserve charge. We make no further assumption about the Hamiltonian. As in our discussion in Section III, we are only interested in the topology of the space of such Hamiltonians, so two Hamiltonians are considered to be equivalent if they can be continuously deformed into each other without closing the gap. Thus, we can flatten the spectrum and assume that \( H \) and, therefore, \( h_{ij} \) only has eigenvalues \( \pm 1 \); any other gapped Hamiltonian can be continuously deformed so that it satisfies this condition. Then, \( h_{ij} \) can be written in the form:

\[
h = U^\dagger \begin{pmatrix} 1 & & & \cdots & \cdots & \cdots & \cdots \\ \vdots & 1 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \cdots & -1 & \cdots & \cdots \\ \vdots & \vdots & \cdots & \cdots & \ddots & \cdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & -1 \\ \end{pmatrix} U \]  
(A2)

with \( k \) diagonal entries equal to \( +1 \) and \( N-k \) diagonal entries equal to \( -1 \) for some \( k \). The space of such matrices \( h_{ij} \) is equal to the space of matrices \( U \), namely \( U(N) \), modulo those matrices \( U \) which commute with the diagonal matrix above, namely \( U(k) \times U(N-k) \). Thus, the space of matrices \( h_{ij} \) is topologically equivalent to

\[
\mathcal{M}_{2N} = \bigcup_{k=0}^{N} U(N)/(U(k) \times U(N-k)) \]  
(A3)

For \( N \) sufficiently large, the topology of these spaces cannot depend on \( k \), so we write this simply as \( \mathbb{Z} \times U(N)/(U(N/2) \times U(N/2)) \).

Now suppose that we have a 1D system. As in section III, we will consider the Dirac equation which approximates the Hamiltonian in the vicinity of the points in the Brillouin zone at which it becomes small. We write:

\[
H = \psi^\dagger (i \gamma_1 \partial_1 + M) \psi \]  
(A4)

Here, we have suppressed the band index on the fermion operators \( \psi^\dagger \) and \( \psi \). Both \( \gamma \) and \( M \) are \( N \times N \) matrices; \( \gamma_1 \) satisfies \( \text{tr}(\gamma_1) = 0 \) and \( \gamma_1^2 = 1 \). Then, the different fermion operators will have the same gap if

\[
\{\gamma_1, M\} = 0, \quad M^2 = 1 \]  
(A5)

Viewed as a linear operator on \( \mathbb{C}^N \), \( \gamma_1 \) defines two \( N/2 \)-dimensional subspaces, its eigenvalue \( +1 \) eigenspace and its eigenvalue \( -1 \) eigenspace. Let us call them \( X_+ \) and \( X_- \). Since \( \{\gamma_1, M\} = 0, M \) is a map from \( X_+ \) to \( X_- \). Thus, \( M \) is an isometry between two copies of \( \mathbb{C}^{N/2} \) and can be viewed as an element of \( U(N/2) \). This can be made more concrete by taking, without loss of generality,

\[
\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]  
(A6)

where \( I \) is the \( N/2 \times N/2 \) identity matrix. Then, \( M \) can be any matrix of the form

\[
M = \begin{pmatrix} 0 & U^\dagger \\ U & 0 \end{pmatrix} \]  
(A7)

for \( U \in U(N/2) \).

Now, consider a 2D system. The Dirac equation takes the form:

\[
H = \psi^\dagger (i \gamma_1 \partial_j + M) \psi \]  
(A8)

with \( j = 1, 2 \). As before, \( \gamma_1 \) defines two \( N/2 \)-dimensional subspaces, \( X_+ \) and \( X_- \). Then \( i \gamma_2 M \) commutes with \( \gamma_1 \) and squares to 1. Thus, it divides \( X_+ \) into two spaces, \( X_+^1 \) and \( X_+^2 \) which are the \( +1 \) and \( -1 \) eigenspaces for \( i \gamma_2 M \) (and it similarly divides for \( X_- \)). Thus, a choice of \( M \) is a choice of linear subspace \( X_+^1 \) of \( X_+ \) or, in other words,

\[
M \in \bigcup_{k=0}^{N/2} U(N/2)/(U(k) \times U(N/2-k)) = \mathbb{Z} \times U(N/2)/(U(N/4) \times U(N/4)) \]  
(A9)

Continuing in this way, we derive Table III analogous to Table I but for charge-conserving systems without time-reversal symmetry. Table III has period-2 as the dimension is increased whereas Table I had period-8.

The two rows in Table III correspond to the presence or absence of an additional unitary symmetry which requires the eigenvalues of the Hamiltonian to come in pairs with equal and opposite energy \( \pm E_i \), i.e. a sublattice symmetry, as discussed in Sec. IIIF. Consider a zero-dimensional system described by the Hamiltonian (A1) with such a symmetry. Then there is a unitary matrix \( U \) satisfying

\[
\{U, h\} = 0 \quad U^2 = 1 \]  
(A10)

We can now apply the same logic to \( U \) and \( M \) that we applied to \( \gamma_1 \) and \( M \) when considering one-dimensional systems. Thus, \( M \in U(N/2) \). More generally, it is clear that \( U \) plays the role of an extra \( \gamma \)-matrix, so the row of Table III for systems with sublattice symmetry is just shifted by one from the row of the table for systems without it.

In order to compute the homotopy groups of these classifying spaces, it is useful to follow the logic which we employed in Sections IIIID.F. We approximate the loop space of \( U(N) \) by minimal geodesics from \( I \) to \( -I \): \( L(\lambda) = e^{i\lambda A} \), where \( A \) is a Hermitian matrix satisfying \( A^2 = 1 \) so that \( L(0) = I \) and \( L(\pi) = -I \). Any such matrix \( A \) divides \( \mathbb{C}^N \) into \( +1 \) and \( -1 \)
TABLE III: The period-2 (in both dimension and number of symmetries) table of classifying spaces for charge-conserving free fermion Hamiltonians without time-reversal symmetry. The systems have $N$ complex fermion fields in dimensions $d = 0, 1, 2, 3, \ldots$ with no additional symmetries (apart from charge-conservation) or with an additional symmetry, such as a sublattice symmetry. Moving $p$ steps to the right and $q$ steps down leads to the same classifying space (but for $1/2^p$ as many fermion fields), which is a reflection of Bott periodicity, as explained in the text. The number of disconnected components of any such classifying space — i.e. the number of different phases in that symmetry class and dimension — is given by the corresponding $\pi_0$, which may be found in Eq. A13. Higher homotopy groups, which classify defects, can be computed using Eq. A12.

| Symmetry classes | Physical realizations | $d = 1$ | $d = 2$ | $d = 3$ |
|------------------|-----------------------|--------|--------|--------|
| A AII | Generic ins. | 0 | Quantum Hall (GaAs, etc.) | 0 | Z |
| AII | Bipartite ins. | | | | |

TABLE IV: Topological periodic table in physical dimensions 1, 2, 3 of the complex classes. A and AIII in the first column labels the symmetry classes in the Zirnbauer classification. The second column are the requirements to the physical systems which can realize the corresponding symmetry classes. The three columns $d = 1, 2, 3$ list the topological states in the spatial dimensions 1, 2, 3 respectively. 0 means the topological classification is trivial. The red label $Z$ stands for the topological state which have not been realized in realistic materials.

eigenspaces or, in other words, it can be written in the form

$$A = U^I \begin{pmatrix} \mathbb{I}_k & 0 \\ 0 & -\mathbb{I}_{N-k} \end{pmatrix} U$$  \hspace{1cm} (A11)$$

where $\mathbb{I}_m$ is the $m \times m$ identity matrix. Therefore $A \in \mathbb{Z} \times \mathbf{U}(N/2) \times \mathbf{U}(N/2)$. Thus, the loop space of $\mathbf{U}(N)$ can be approximated by $\mathbb{Z} \times \mathbf{U}(N/2) \times \mathbf{U}(N/2)$. Thus, $\pi_k(\mathbf{U}(N)) = \pi_{k-1}(\mathbb{Z} \times \mathbf{U}(N)/\mathbf{U}(N/2) \times \mathbf{U}(N/2))$.

To compute homotopy groups of $\mathbb{Z} \times \mathbf{U}(N)/\mathbf{U}(N/2) \times \mathbf{U}(N/2)$, consider the homotopy long exact sequence associated to the fibration

$$\mathbf{U}(N/2) \rightarrow \mathbf{U}(N) \rightarrow \mathbf{U}(N)/\mathbf{U}(N/2).$$

It tells us that

$$\cdots \pi_{i-1}(\mathbf{U}(N/2)) \rightarrow \pi_i(\mathbf{U}(N)) \rightarrow \pi_i(\mathbf{U}(N)/\mathbf{U}(N/2)) \rightarrow \cdots.$$  

If we are in the stable limit, i.e. for $N$ sufficiently large, then $\pi_{i-1}(\mathbf{U}(N/2)) = \pi_i(\mathbf{U}(N))$, so $\pi_i(\mathbf{U}(N)/\mathbf{U}(N/2)) = 0$. Now consider the fibration

$$\mathbf{U}(N/2) \rightarrow \mathbf{U}(N)/\mathbf{U}(N/2) \rightarrow \mathbf{U}(N)/(\mathbf{U}(N/2) \times \mathbf{U}(N/2)).$$

Then, since $\pi_i(\mathbf{U}(N)/\mathbf{U}(N/2)) = 0$, the associated homotopy exact sequence is simply

$$0 \rightarrow \pi_i(\mathbf{U}(N)/(\mathbf{U}(N/2) \times \mathbf{U}(N/2))) \rightarrow \pi_{i-1}(\mathbf{U}(N/2)) \rightarrow 0$$

so $\pi_i(\mathbf{U}(N)/(\mathbf{U}(N/2) \times \mathbf{U}(N/2))) = \pi_{i-1}(\mathbf{U}(N/2))$.

Combining these two relations, we have Bott periodicity for the unitary group:

$$\pi_k(\mathbf{U}(N)) = \pi_{k-1}(\mathbb{Z} \times \mathbf{U}(N)/(\mathbf{U}(N/2) \times \mathbf{U}(N/2))) = \pi_{k-2}(\mathbf{U}(N/2))$$  \hspace{1cm} (A12)$$

By inspection, we see that

$$\pi_0(\mathbb{Z} \times \mathbf{U}(N)/(\mathbf{U}(N/2) \times \mathbf{U}(N/2))) = \mathbb{Z}$$  \hspace{1cm} (A13)$$

Combining this with Eq. A12, we learn that the even homotopy groups of $\mathbf{U}(N)$ vanish and the odd ones are $\mathbb{Z}$; the reverse holds for $\mathbf{U}(N/2)/(\mathbf{U}(N/2) \times \mathbf{U}(N/2))$.

These homotopy groups can be used to classify the different possible phases and topological defects of systems in the symmetry classes/dimensions in Table III. Similar to Table II in Sec. III F, we list in Table IV the nontrivial topological physical systems in the two symmetry classes. In Ref. 26, these were called A and AIII, respectively, following the corresponding classification in random matrix theory. In A class (with charge conservation and without sublattice symmetry or time-reversal symmetry) a topological nontrivial state is classified by integer in 2d, which is the famous quantum Hall system. In the AIII class with sublattice symmetry, in 1d the carbon nanotube or graphene ribbon can be considered as an example with nontrivial edge states. Compared to the requirement of this symmetry class, carbon nanotube has too high symmetry because time-reversal symmetry is present. An orbital magnetic field can be coupled minimally to the carbon nanotube to get an example system with exactly the required symmetry of AIII class. In 3d there should be an integer classification of the AII class but no realistic topological nontrivial material is known yet.

Appendix B: Understanding the Role of Spatial Dimension in the Classification of Free Fermion Systems

In Sections III and A, we have seen that increasing the spatial dimension moves a system through the progression of classifying spaces oppositely to adding symmetries which
square to \(-1\) (but in the same direction as adding symmetries which square to 1). The classifying space for \(d\)-dimensional systems is the loop space of the classifying space for \(d+1\)-dimensional systems with the same symmetries. Symbolically,

\[
R_{2+p-d} = \Omega(R_{2+p-(d+1)})
\]

(B1)

where \(\Omega\) denotes the loop space and, as in Section III, \(p\) is the number of symmetries and \(d\) is the spatial dimension. Thus, in using Tables I and III, moving one step to the right, which increases the dimension and one step down, which increases the number of symmetries, leaves the classifying space the same.

We explained this in Sections III and A by expanding the Hamiltonian about the point(s) in the Brillouin zone where the number of symmetries changes. This leaves the classifying space the same.

Thus, for lattice-translational symmetry, we may employ

\[
KR^{n-d}(X \wedge \mathbb{T}^d) \simeq \tilde{KO}^n(X) \oplus \cdots \oplus \tilde{KO}^{-1}(X) \oplus \cdots
\]

(B3)

In this appendix, we want to sketch here a third way of understanding “spatial dimension = de-loop” that is more geometric than either of the alternatives above.

We warn the reader that this section is only a “sketch”, so there may be tricky analytic details regarding the precise definition of the controlled spaces which we have over-looked.

The math literature\(^7\) in controlled K-theory is constructed in a less naive context. We are not sure if there is something essential we are missing.

Let \(O_d\) be a limitingly large orthogonal group with integer \((\mathbb{Z})\) control and a topology (like compact/open) where sliding a disturbance off to \(\pm\infty\) converges to the identity. That is, \(O_d\) is the set of infinite matrices \(O_{I,J}\) with a finite-to-one map (decapitalizing) \(J \mapsto j \in \mathbb{Z}\) such that \(|O_{I,J}| \leq e^{-\text{const}(|i-j|)}\) and \(O_{I,J}(O_{I,K}) = \delta_{j,K}\) with the following identifications: Any matrix \(O_{I,J}\) with \(O_{I,0,J} = O_{0,0,J}\) is equivalent to the matrix with one less row and column obtained by removing the \(l^0\) row and \(l^0\) column.

By changing \(l\), the number of distinct \(J\) map that map to a given \(j\), we can choose any desired constant in the exponential decay \(\exp(-\text{const}(|i-j|))\). We could also have chosen to study the class of infinite matrices with \(O_{I,J} = 0\) for \(|i - j| > 1\); in this case, the matrix \(O_{I,J}\) is a banded matrix, which is nonzero only within distance \(l\) of the main diagonal. We refer to the case of exponential decay as “soft control”, while we refer to the case that \(O_{I,J} = 0\) for \(|i - j| > 1\) as “strict control”.

The key claim in this “third” approach is

Claim. \(O_d \simeq O/O \times O \times \mathbb{Z} = BO \times \mathbb{Z}\) (denotes weak homotopy type.)

We take for \(BO \times \mathbb{Z}\) the well known model of pairs, considered as formal differences, of transverse (not necessarily spanning) subspaces \((L_+, L_-)\) of a \(d\)-dimensional real vector space \(\mathbb{R}^d\) (with \(d\) finite and a limit taken as \(d \to \infty\)). The \(\mathbb{Z}\)-coordinate above is the index \(\dim L_+ - \dim L_-\).

Here are the maps: choose any cut in \(\mathbb{Z}\), say \(\Pi_+\) projects to \(J\) with \(j \geq 0\) and \(\Pi_-\) projects to \(J\) with \(j < 0\). Set

\[
f : O_d \to BO \times \mathbb{Z},
\]

\[
f(O) = (\ker(\Pi_+ \circ O|_{S_+}), \ker(\Pi_- \circ O|_{S_-})),
\]

where \(S_+(-)\) is the space of basis elements mapping to \([0, L]\) \(([-L, -1])\) where \(l \gg \text{the decay constant for } O_d\), and both kernels are regarded as subsets of the span \(\mathbb{R}^d\) of elements mapping to \([-L, L]\). Given \((L_+, L_-) \in BO \times \mathbb{Z}\), define \(g(L_+, L_-) \in O_d\) as follows. Think of \(L_+ \subset \mathbb{R}^d\) with \(P\) denoting the perpendicular subspace to \(\text{span}(L_+, L_-)\). The infinite vector space on which \(g(L_+, L_-)\) acts is spanned by a copy of \(\mathbb{R}^d\) for each \(i \in \mathbb{Z}\). \(O = g(L_+, L_-)\) acts as the identity on each \(P \times i\), translation by \(-1 \in \mathbb{Z}\) on each \(L_+ \times i\).
and translation by $+1 \in \mathbb{Z}$ on each $L_{\cdot \times i}$. Note: the use of the kernel above is correct in the case of strict control. In the case of soft control, we should replace the kernel by the projector onto right singular vectors of $\Pi_{+} \circ O|_{S_{+}}$ with singular value less than or equal to some quantity which is exponentially small in $L$.

It is immediate that $f \circ g = \text{Id}_{BO \times \mathbb{Z}}$. On the other hand, $g \circ f$ seems to turn a general $O \in O_{\mathbb{Z}}$ into a very special form (no rotation at all, just various subspaces sliding left and right.) However, we claim that there is a deformation retraction of $O_{\mathbb{Z}}$ to maps which take this simple form: $r : O_{\mathbb{Z}} \rightarrow \{\text{slides}\} \subseteq O_{\mathbb{Z}}$. First, think about a fiberwise $O$ (no sliding) which is $X$ over some $i \in \mathbb{Z}$ and $\text{Id}$ over other $j \neq i \in \mathbb{Z}$. Note that

$$
\begin{pmatrix}
X & 0 \\
0 & -X
\end{pmatrix} \sim 
\begin{pmatrix}
0 & \text{Id} \\
\text{Id} & 0
\end{pmatrix}
$$

via the canonical homotopy

$$
\cos \left( \frac{\pi t}{2} \right) \begin{pmatrix}
X & 0 \\
0 & -X
\end{pmatrix} + \sin \left( \frac{\pi t}{2} \right) \begin{pmatrix}
0 & \text{Id} \\
\text{Id} & 0
\end{pmatrix}.
$$

When $d$ is even (which we may assume) there are further canonical homotopies so that

$$
\begin{pmatrix}
0 & \text{Id} \\
\text{Id} & 0
\end{pmatrix} \sim 
\begin{pmatrix}
\sigma_{Z} & 0 \\
0 & \sigma_{Z}
\end{pmatrix} \sim 
\begin{pmatrix}
\text{Id} \\
0 & \sigma_{Z}
\end{pmatrix} \sim 
\begin{pmatrix}
\text{Id} \\
0 & 0
\end{pmatrix}
$$

Now a “Eilenberg swindle” runs $X$ (canonically) towards $+\infty$.

$$
\begin{pmatrix}
\text{Id} & X & \text{Id} \\
(i - 1) & (i + 1) & (i + 2) & (i + 3)
\end{pmatrix} \sim 
\begin{pmatrix}
\text{Id} & X & -X & X & -X & \cdots \\
(i - 1)(i + 1)(i + 2)(i + 3)(i + 4)
\end{pmatrix} \sim 
\begin{pmatrix}
\text{Id} & \text{Id} & \text{Id} & \text{Id} & \cdots \\
(i & (i + 1)(i + 2)(i + 3)
\end{pmatrix}
$$

Reusing this trick, any fiber-wise, though now it is better to slide from the center out to both $\pm \infty$. $X$ can be canonically connected to $\text{Id} \in O_{\mathbb{Z}}$. Similarly, one can (canonically) deform the general $O \in O_{\mathbb{Z}}$ to a sum of three pieces — fixed, left-sliding, right-sliding — and this is the deformation retraction $r$ written above.

We now promote the (weak) homotopy equivalence $O_{\mathbb{Z}} \simeq BO \times \mathbb{Z}$ to an entire table where the rows are all (weak) homotopy equivalences.

$$
\begin{align*}
O_{\mathbb{Z}} / U_{\mathbb{Z}} & \simeq O_{\mathbb{Z}} \\
O_{\mathbb{Z} + 1} & \simeq O_{\mathbb{Z}} \\
\cdots & \simeq \cdots
\end{align*}
$$

and see that the 1-parameter subgroup $\{e^{t \log O}\} \subseteq O_{\mathbb{Z}}$ retains $\mathbb{Z}$-control. This may be checked by writing

$$
\log J = \frac{\pi}{2} J,
$$

since $J^{2} = -I$. This yields $\mathbb{Z}$-controls on $\log J$ and hence on the entire 1-parameter subgroup.

One interesting feature of this approach is that we are able to classify different symmetry classes of controlled unitaries on a line. The controlled unitaries $U_{\mathbb{Z}}$ are classified by an integer. This integer is precisely the “flow” described by Kitaev.15. If the controlled unitary is assumed to be symmetric, it is in the orthogonal group, and again $O_{\mathbb{Z}}$ is classified by an integer. However, if the unitary is chosen to be self-dual, in this case $U_{\mathbb{Z}} / Sp_{\mathbb{Z}}$ has a $\mathbb{Z}$-classification. Ryu et. al.26 note (see Table I or Table IV of that paper) that each of the 10 different symmetric spaces can be obtained by considering the exponential of Hamiltonians lying in the 10 different symmetry classes; for example, given a Hermitian matrix $H$, the exponential $\exp(iHt)$ is a unitary matrix, while if $H$ is anti-symmetric and Hermitian, the exponential $\exp(iHt)$ is an orthogonal matrix. However, the classification of unitaries here implies that there can be an obstruction to writing a controlled unitary as an exponential $\exp(iHt)$ with $H$ of the correct symmetry class and

a codimension 1 hyperplane, in this case the line $y = -\frac{1}{2}$, and finds a pair of locally finite $\mathbb{Z}$-controlled kernels which define an element in $O_{\mathbb{Z}} / O_{\mathbb{Z}} \times O_{\mathbb{Z}} \times \mathbb{Z}$. However, there is one subtle point: the matrices $\Pi_{+}$ and $O$ are both controlled; however, this does not imply that the projector onto the kernel of $\Pi_{+} \circ O|_{S_{+}}$ is also controlled. If we knew that the singular values of $\Pi_{+} \circ O|_{S_{+}}$ had a spectral gap separating the zero singular values (or, in the case of soft control, the singular values which are exponentially small in $L$) from the rest of the spectrum, then we could show that the projector onto the kernel had soft control, with decay constant set by the gap, but in absence of knowledge of a gap, this seems to be a difficult technical step.

Moving down the array, say from $\simeq_{1}$ to $\simeq_{3}$, requires “looping” $\simeq_{1}$. We must check that $\Omega(O_{\mathbb{Z}}) \simeq O_{\mathbb{Z}} / U_{\mathbb{Z}}$ i.e. that the first step in Bott’s ladder of eight rungs holds with $\mathbb{Z}$-controls.

To do this, one must do Lie theory in $O_{\mathbb{Z}}$: find the shortest geodesics arcs from $\text{Id}$ to $-\text{Id}$ in $O_{\mathbb{Z}}$, understand which directions they pick out in the Lie algebra $O_{\mathbb{Z}}$ and that their midpoints are $\mathbb{Z}$-controlled complex structures, and finally make the analogs of Bott’s index calculations. The initial step is to consider a representative for a “$\mathbb{Z}$-controlled complex structure” (where coordinates are consecutive in $\mathbb{Z}$):
\( H \) also controlled.

Appendix C: Phase Symmetry Unbroken

If the phase symmetry of the order parameter is unbroken (i.e. if we allow gradients in the overall phase), then the effective sigma model target is \( U/O \), not \( U/O \). This means that at every stage of the calculation, an additional circle \( \mathbb{O}(1) \) or \( K(\mathbb{Z}, 1) \) product factor must be added to the target space. This leads to no change for \( \pi_2(Y) \) or \( \pi_1(R) \) (essentially because \( \pi_2(\mathbb{O}(1)) \cong 0 \), however \( \pi_1(Y) \) and \( \pi_0(R) \) pick up an additional integer which is the winding number around this circle. Thus, (C1) below

\[
\pi_2(Y) \xrightarrow{\partial_2} \pi_1(R) \xrightarrow{\pi_1(K_{2n})} \pi_1(Y) \xrightarrow{\partial_1} \pi_0(R) \\
\| \| \\
\mathbb{Z}^2n \times S_{2n} \quad \mathbb{Z}_2
\]

is replaced by (C2):

\[
\pi_2(Y_{\text{new}}) \xrightarrow{\partial_2} \pi_1(R_{\text{new}}) \xrightarrow{\pi_1(K_{2n}^{\text{new}})} \pi_1(Y_{\text{new}}) \xrightarrow{\partial_1} \pi_0(R_{\text{new}}) \\
\| \| \\
(\mathbb{Z}^2n \times \mathbb{Z}^2n) \times S_{2n} \quad \mathbb{Z}^2n \times \mathbb{Z}_2
\]

where \( S_{2n} \) acts on \( \mathbb{Z}^2n \) by permuting its standard generating set.

The new \( \mathbb{Z}^2n \) factor in \( \pi_0(R_{\text{new}}) \) represents rolling the order parameter around the phase circle as one moves radially into the hedgehog. Similarly, the \( \mathbb{Z}^2n \) in \( \pi_1(Y_{\text{new}}) \) comes from wrapping the whole hedgehog around the phase circle as the parameter is advanced. Clearly, the two \( \mathbb{Z}^2n \)'s cancel: \( \ker(\partial_1^{\text{new}}) = \ker(\partial_1) \). Thus,

\[
\pi_1(K_{2n}) \cong \pi_1(\bar{K}_{2n}) \times \mathbb{Z}_n
\]

where the extra factor \( \mathbb{Z}_n \) comes from a base point in \( B^3 \) wraps around the additional circle \( \mathbb{O}(1) \).

Appendix D: Hopf Map: an example of boundary maps in the homotopy exact sequence

The Hopf fibration is an example of a fibration (of a fiber bundle, in fact) which arises in several contexts in physics. It consists of a map (the Hopf map) from \( S^3 \) to \( S^2 \) such that the pre-image of each point in \( S^2 \) is an \( S^1 \). This is familiar from the representation of a vector (e.g. the Bloch sphere) by a spinor (e.g. a two-state system):

\[
n = z^1 \sigma z
\]

where \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) are Pauli matrices and the spinor

\[
z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}
\]

satisfies \( |z_1|^2 + |z_2|^2 = 1 \) so that \( n \) is a unit vector. Due to this normalization condition, the spinor \( z \) lives in \( S^3 \) while the vector \( n \) lives in \( S^2 \). Transforming \( z \rightarrow e^{i\theta} z \) leaves \( n \) invariant. Thus, the map (D1) maps circles in \( S^3 \) to points in \( S^2 \), as depicted below:

\[
U(1) \text{ phase} = S^1 \rightarrow S^3 = \text{normalized states in } \mathbb{C}^2 \\
\downarrow \\
S^2 = \text{Bloch sphere}
\]

One context, quite closely-related to the present discussion, in which the Hopf map arises is the \( O(3) \) non-linear \( \sigma \) model in \( 2 + 1 \)-dimensions. The order parameter field \( n(x) \) is a map from \( S^3 \) to \( S^2 \) (assuming that \( n(x) \) is constant at infinity, so that spacetime can be compactified into \( S^3 \)).

The topological term which can be added to such a model\(^{53} \) is:

\[
S_H = -\frac{1}{2\pi} \int d^2 x d\tau \epsilon_{\mu \nu \lambda} A_\mu F_{\nu \lambda}
\]

where \( F_{\nu \lambda} = \partial_\nu A_\lambda - \partial_\lambda A_\nu \) and

\[
\epsilon_{\mu \nu \lambda} \partial_\nu A_\lambda \equiv \frac{1}{8\pi} \epsilon^{\mu \nu \lambda} n \cdot \partial_\nu n \times \partial_\lambda n
\]

This term gives the linking number of skyrmion trajectories in the \( O(3) \) non-linear \( \sigma \) model. To see this, suppose that \( n(\infty) = (0, 0, 1) \). The pre-image of \( (0, 0, -1) \) (which we may view as the centers of skyrmions) is a set of closed circles, and the Hopf term gives their linking number.

We can use the Hopf map to explain boundary maps in the homotopy exact sequence: \( \pi_2(S^2) \xrightarrow{\partial} \pi_1(S^1) \):

\[
\pi_2(S^2) \xrightarrow{\partial} \pi_1(S^1) \xrightarrow{\pi_1(S^1)} \pi_1(S^3) \\
\| \| \\
\mathbb{Z} \quad \mathbb{Z} \quad 0
\]

which, we see, must be an isomorphism to be consistent with \( \pi_1(S^3) \cong 0 \).

But how is this boundary map defined? We think of a 2-sphere \( S^2 \) as a loop of circles \( S^1 \), as depicted in Fig. 14. Using the (defining) property of a fibration (see any book on algebraic topology), we may lift this loop of circles \( S^1 \) from the base \( S^2 \) to the total space \( S^3 \). The lifting is perfectly continuous, but it does not give us a sphere. Instead, it gives us a surface with boundary. This boundary is an \( S^1 \). What happens is that, at the very end of the loop of circles \( S^1 \), the lifting “explodes” the final circle \( S^1 \), whose map to the Bloch sphere is shrinking to a point, into a circle in \( S^3 \), as pictured in figure 15. In other words, an element of \( \pi_2(S^2) \) is lifted to a surface in \( S^3 \) whose boundary is a circle. This circle can wind around the fiber \( S^3 \), thus defining an element of \( \pi_1(S^1) \). This map
from an element of $\pi_2(S^2)$ to an element of $\pi_1(S^1)$ is called a boundary map since the element of $\pi_1(S^1)$ is the boundary of the lift to $S^3$ of the corresponding element of $\pi_2(S^2)$.

**FIG. 15:** The lifts of circles are depicted ‘upstairs’ with respect to the usual picture of Hopf circles filling $\mathbb{R}^3$. The lift at the end of the family is the biggest ellipse in this picture, not a single point, unlike the lift at the beginning of the family. Thus, the loop of circles no longer closes when lifted.

**Appendix E: Computation of $\pi_0(R_{2n}), \pi_1(R_{2n})$ using the “Postnikov tower” for $\widehat{U/O}$**

In this appendix, we compute the homotopy groups $\pi_0(R_{2n}), \pi_1(R_{2n})$.

Recall that the homotopy classes of maps from a 3-manifold $M^3$ to $S^2$ can be understood in terms of the ribbons which are the (framed) inverse image of the north pole $N \subset S^2$. Such ribbons serve as generators for the path components of Maps($M^3 \to S^2$) and the relations are “framed cobordisms,” that is, imbedded surfaces $(S; r_0, r_1) \subset (M^3 \times [0, 1]; M \times 0, M \times 1)$ where $S$ is a surface “cobording” between initial and final ribbons $r_0$ and $r_1$. $S$, like the ribbons themselves, has a normal framing which restricts to the oriented ribbon directions of both $r_0$ and $r_1$. This is a special case of what, in topology, is called the Pontryagin-Thom construction (PTC) (see Appendix F for more details).

Of course, there is more to $\widehat{U/O}$ than the bottom $S^2$. Let’s look next at the influence of the 3-cell $D^3$ on $R_{2n}$. Whereas $S^2 \setminus N$ is already contractible, to make $S^3 \cup_{\deg 2} D^3$ contractible, we must remove a diameter $\delta \subset D^3$ whose endpoints are glued to $N$. Making our maps $f$ transverse to $\delta \subset D^3$ (and not just $N \subset S^2$ as would be ordinarily done in the PTC) we see that the ribbons (and their framed cobordisms) are now subject to a “singularity” at the $f$-preimages of the origin $O \subset D^3$. The singularity is very mild: it is simply a point on the ribbon where the orientation (induced by comparing orientations on $M^3$ and $S^2$) longitudinally along the ribbon $r_0$ changes direction. The upshot is that the generators for $\pi_0(R_{2n})$ now consist of ribbons with no longitudinal arrows assigning a direction along the ribbon. (These generating ribbons are, of course, subject to the boundary condition that there are exactly $2n$ arc endpoints meeting each hedeghog exactly once at its marked point $f^{-1}(N)$.) Closed loops of ribbons may also arise. Because all maps from a 3-manifold deform into these lowest cells, $S^2 \cup_{\deg 2} D^3 \subset \widehat{U/O}$, all generators of $\pi_0(R_{2n})$ are of this form.

Finding the relations defining $\pi_0(R_{2n})$ requires looking at maps of a four manifold $(Q \times I, \partial (Q \times I))$ into $\widehat{U/O}$. To study these by a variant of the PTC, we would need to take into account all cells up to dimension 4, $sk^4(\widehat{U/O})$, the “4-skeleton.” Similarly, a full analysis of $\pi_1(R_{2n})$ would require looking at $sk^5(\widehat{U/O})$.

**Fact E.1.** $\pi_3(S^2 \cup_{\deg 2} D^3) \cong \mathbb{Z}_4$ and is generated by the Hopf map $h : S^3 \to S^2$.

**Proof.** The Whitehead exact sequence reads:

$$H_4(X) \longrightarrow \Gamma(\pi_2(X)) \longrightarrow \pi_3(X) \xrightarrow{Hurewicz} H_3(X)$$

where $\Gamma$ is Whitehead’s quadratic functor. It is known that $\Gamma(\mathbb{Z}_2) \cong \mathbb{Z}_4$. If $X = S^2 \cup_{\deg 2} D^3$, the outer integral homology groups vanish, so $\pi_3(X) \cong \mathbb{Z}_4$ as claimed.

To see that the generator $g \in \pi_3(X)$ deforms to $S^2$, make transverse to the origin $O \subset D^3$. Since $D^3$ is attached to $S^2$ with positive degree, the signed sum of inverse images $g^{-1}(O) = 0$. These may be paired and then removed by building a framed cobordism (see PTC appendix) from $g^{-1}(O)$ to $\emptyset$ inside $S^3 \times [0, 1]$. The map $g'$ associated to the $S^3 \times 1$ level avoids $O \subset D^3$ and so may be radially deformed into $S^2$.

**Consequence E.2.** $sk^4(\widehat{U/O}) = S^2 \cup_{\deg 2} D^3 \cup_{2 \text{Hopf}} D^4$

**Proof.** Consider the commutative diagram:
Applying the $\pi_3$ functor, we obtain:

\[ \begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} \\
\cong & & \\
\mathbb{Z} & \xrightarrow{\text{epimorphism}} & \mathbb{Z}_2
\end{array} \]

From the homotopy exact sequence of

\[ SO \to SU \]
\[ \frac{SU}{SO} \cong \widetilde{U/O} \]

we can see that

\[ \pi_3(SO) \xrightarrow{\times 2} \pi_3(SU) \to \pi_3\left(\frac{SU}{SO}\right) \to \pi_2(SO) \]
\[ \mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z} \quad \mathbb{Z}_2 \quad 1 \]

and we conclude that $\delta$ is an epimorphism. Thus, the Hopf map, which is order 4 in $X$, needs a 4-cell to be attached in order to make it order 2 in $\widetilde{U/O}$. 

We will not need this, but for the curious, $sk^5(\widetilde{U/O})$ requires one additional 5-cell which maps degree 2 over the 4-cell.

As remarked, studying maps into cell structures becomes laborious. Homotopy theory works best when studying maps out of cell structures and into fibrations. So, let us introduce the 2-stage Postnikov tower for $\widetilde{U/O}$.

The notation $K(\pi, n)$ is used for any connected space (generally infinite dimensional) that has only a single nontrivial homotopy group, $\pi_n(K(\pi, n)) = \pi$. It is easy to show that the homotopy type of $K(\pi, n)$ is unique. At the next stage of complexity come spaces $Z$ with two nontrivial homotopy groups, say $\pi_m(Z) = A$ and $\pi_n(Z) = B$, $m < n$. Such spaces (for simplicity we now assume $m \geq 2$) have a homotopy type determined by the total space $T$ of a fibration called a 2-stage Postnikov Tower:

\[ \begin{array}{ccc}
K(n, B) & \to & T \\
\downarrow & & \downarrow \\
K(m, A) & (\text{base}) & \\
\end{array} \]

To completely specify how the fibration twists and the homotopy type of $T$ we need to specify the “$k$” invariant $k \in H^{n+1}(K(m, A); B)$.

The space $K(m, A)$ classifies cohomology in the sense that for any space $S$, the homotopy classes of maps $[S, K(m, A)]$ naturally biject with $H^m(S; A)$. (There is a “fundamental” class $\iota \in H^m(K(m, A); A)$ so that for $f : S \to K(m, A)$ we associate $f^*\iota \in H^m(S; A)$.) Thus, the $k$ invariant is really a map:

\[ k : K(m, A) \to K(n + 1, B) \]

and $T$ is the pullback of the path loop fibration over $K(n + 1, B)$:

\[ \Omega K(n + 1, B) = K(n, B) \to \text{pt.} \]
\[ \downarrow \]
\[ K(n + 1, B) \]

\[ K(n, B) \]
\[ \downarrow \]
\[ T \to \text{pt.} \]
\[ \downarrow \]
\[ K(m, A) \xrightarrow{k} K(n + 1, B) \]

where we have denoted the contractible space of paths in $K(n + 1, B)$ starting at its base point by its homotopy model, a single point, pt.

For spaces with many homotopy groups, the Postnikov tower can be continued iteratively.

In our case, $\pi_i(\widetilde{U/O}) = 0, \mathbb{Z}_2, \mathbb{Z}_2, 0$ for $i = 1, 2, 3, 4$, so a 2-stage tower is adequate for computing $\pi_i(R)$ for $i = 0, 1$.

Thus, our model $T$ for $\widetilde{U/O}$ is now a fibration:

\[ \begin{array}{ccc}
K(\mathbb{Z}_2, 3) & \to & T \\
\downarrow & & \downarrow \\
K(\mathbb{Z}_2, 2) & & \\
\end{array} \]

Similar to the previously computed cell structure of $\widetilde{U/O}$,
the cells of \( K(\mathbb{Z}_2, 2) \) through dimension \( \leq 3 \) are the same. But for \( K(\mathbb{Z}_2, 2) \), the 4-cell kills the Hopf map and the 5-cell is now degree 4 over the 4-cell. The cell complex for \( K(\mathbb{Z}_2, 2) \) begins as follows:

\[
\begin{array}{cccc}
C_5 & \longrightarrow & C_4 & \longrightarrow & C_4 & \longrightarrow & C_2 & \longrightarrow & C_1 \\
\| & \times 4 & \| & \| & \| & \| & \| & \| & \|
Z & \longrightarrow & Z & \longrightarrow & Z & \longrightarrow & Z & \longrightarrow & 0
\end{array}
\]

The \( \mathbb{Z}_2 \)-cohomology is then the homology of the hom sequence of this chain complex into \( \mathbb{Z}_2 \):

\[
\mathbb{Z}_2 \leftarrow 0 \leftarrow \mathbb{Z}_2 \leftarrow 0 \leftarrow \mathbb{Z}_2 \leftarrow 0 \leftarrow 0
\]

Thus, one finds that \( H^4(K(\mathbb{Z}_2, 2); \mathbb{Z}_2) \cong \mathbb{Z}_2 \), so there are only two possible fibrations (up to homotopy type) as above, the product \( K(\mathbb{Z}_2, 2) \times K(\mathbb{Z}_2, 3) \) and a “twisted product.” Fact E.1 says that \( \pi_3(T) \) actually comes by composing the Hopf map into the (2-sphere inside the) base \( K(\mathbb{Z}_2, 2) \). This is clearly false for the product where that composition would factor through \( \pi_3(K(\mathbb{Z}_2, 2)) \cong 1 \). This shows that the model \( T \) for \( S^2 \) has nontrivial \( k \) invariant.

If \( X \) is a space and

\[
F \to E
\]

\[
\downarrow
\]

\[
B
\]

is a fibration, then there is a corresponding fibration of the space of maps:

\[
\mathcal{M}(X, F) \to \mathcal{M}(X, E)
\]

\[
\downarrow
\]

\[
\mathcal{M}(X, B)
\]

(We now use the usual comma instead of “\( \to \)” in the notation for map spaces.) This is our main tool; it enables us to tear apart maps into \( T \) into maps into \( K(\mathbb{Z}_2, 2) \) and \( K(\mathbb{Z}_2, 3) \) which are easy things to compute, namely cohomology groups. A small wrinkle is that because of the boundary conditions we will study relative maps and end up with relative cohomology groups, but these are also easily computed.

So, let us now return to the computation of \( \pi_0(R_{2n}) \). We study the fibration:

\[
R3 \to R
\]

\[
\downarrow
\]

\[
R2
\]

where \( R \) is our original \( R_{2n} \) (the subscript \( n \geq 0 \) is suppressed). \( R2 \) is the space \( \text{Maps}(Q \to K(\mathbb{Z}_2, 2)) \) with rigid boundary conditions (from \( \partial Q \) to \( S^2 \subset K(\mathbb{Z}_2, 2) \)) analogous to \( R \) but with maps to \( T \) replaced with maps to its base space \( K(\mathbb{Z}_2, 2) \). Similarly, \( R3 \) is Maps(\( Q \to K(\mathbb{Z}_2, 3) \)) with \( \partial Q \) mapping to the base point of \( K(\mathbb{Z}_2, 3) \). We have from the homotopy exact sequence:

\[
\pi_1(R2) \longrightarrow \pi_0(R3) \longrightarrow \pi_0(R) \longrightarrow \pi_0(R2)
\]

\[
\downarrow
\]

\[
H^3(Q, \partial Q; \mathbb{Z}_2)
\]

\[
\downarrow
\]

\[
H^2(Q, \partial Q; \mathbb{Z}_2) \cong H_1(Q; \mathbb{Z}_2)
\]

\[
\downarrow
\]

\[
\mathbb{Z}_2
\]

\[
\text{Poincare duality}
\]

\[
\cong 1
\]

The boundary map is always zero for product fibrations, but in this case, it is potentially affected by the \( k \) invariant (coming from the \( k \) invariant of \( T \)).

Following the usual procedure, \( \partial t \) may be understood as the “twist” (mod 2) in a ribbon induced by the “lasso move” shown in figure 16 below, which is well known to be 4\( \pi \).

![FIG. 16: By pulling a ribbon over a hedgehog and ‘lassoing’ it, a 4\( \pi \) twist is performed.](image)

Since 4\( \pi \) is an even multiple of 2\( \pi \), \( \partial t \) is in fact zero and \( \pi_0(R) \cong \mathbb{Z}_2 \). The two components can, on the level of framed ribbons, be identified with a total twisting of ribbons being an even or odd multiple of 2\( \pi \).

\( \pi_1(R) \) may be similarly computed:

\[
\cdots \longrightarrow \pi_1(R3) \longrightarrow \pi_1(R) \longrightarrow \pi_1(R2) \longrightarrow \cdots
\]

\[
\downarrow
\]

\[
H^3(Q \times I, \partial(Q \times I); \mathbb{Z}_2)
\]

\[
\downarrow
\]

\[
H^2(Q \times I, \partial(Q \times I); \mathbb{Z}_2)
\]

\[
\downarrow
\]

\[
H^1(Q \times I; \mathbb{Z}_2)
\]

\[
\downarrow
\]

\[
\mathbb{Z}_2
\]

\[
\text{Poincare duality}
\]

\[
\cong 1
\]

\[
\mathbb{Z}_2^{2n}
\]

Thus, \( \pi_1(R) \cong \mathbb{Z}_2^{2n} \). Intuitively, \( \pi_1(R) \) is represented by 2-dimensional “bags” or homology classes that a loop of ribbons, defining an element of \( \pi_1(R) \), sweeps out in \( Q \times I \).

Appendix F: Pontryagin-Thom Construction

Homotopy classes \( [M^d, S^k] \) may be studied geometrically. The idea is to make any map \( f : M^d \to S^k \) transverse to a base point \( N \in S^k \). Then \( f^{-1}(N) \) is a \( (d-k) \)-dimensional submanifold \( K \) of \( M^d \) equipped with a framing of its normal bundle obtained by pulling back a fixed normal \( k \)-frame to
N. This construction is reversible: Given a submanifold with normal framing $K^{d-k} \subset M$, there is a map $f : M^d \to S^k$ which wraps an $\eta$-neighborhood $\eta$ of $K$ over $S^k$ by using polar coordinates around the north pole $N$ and the framing to send each normal $D^k \subset \eta$ degree $= 1$ over $S^k$ and sends the rest of $M \setminus \eta$ to the south pole in $S^k$.

This discussion can be relativized. A homotopy $F : M \times [0, 1] \to S, F|_{M \times 0} = f, F|_{M \times 1} = g$ yields by inverse image $K$ of $N$ a "framed cobordism" from $K_0 \subset M \times 0$ to $K_1 \subset M \times 1$, that is, a manifold $K^{d-k+1} \subset M \times [0, 1]$ with a normal $k$-framing which restricts at 0 (1) to $K_0 (1)$ and its normal $k$-framing.

The maps to $S^k$ that are produced from a framed submanifold may seem extreme and unrepresentative since most points on the map to homotopy) exists for the part of the map which avoids $N$. Thus we have:

**Fact F.1.** The space of maps $\mathcal{M}(M^d, S^k)$ is (at least weakly) homotopy equivalent to the space of framed submanifolds $\{L^{d-k} \subset M^d\}$ (provided both are given reasonable topologies).

**Fact E.2.** **Fact F.1 holds in a relative setting, e.g. the \{space of homotopies\} \cong \{space of framed cobordisms\}.**

For example, merely on the level of $\pi_0$, this says that "homotopy classes of maps $(M^d, S^k)$ are in 1-1 correspondence with framed cobordism classes of framed submanifolds $K^{d-k} \subset M^d$.

This basic PTC may be generalized to maps $(N^d, M^k)$ whenever we can identify a “spine" $X \subset M^k$ so that $M \setminus X$ is contractible. One then studies $f^{-1}(X)$ (actually $f^{-1}(a germ of X)$ corresponding to the old framing data) instead of $f^{-1}(N)$. For this approach to be practical, $X$ and its neighborhood have to be fairly simple. In this extended setting, one must also keep track of the maps $f| : (f^{-1}(X)) \to X$ since this is no longer unique.

**Appendix G: Multiplication Table for $\pi_1(K_{2n})$**

We have shown $\mathcal{T}_{2n} = \pi_1(K_{2n}) \cong \mathbb{Z} \times \mathbb{Z}_2 \times G_{2n}$ and a short exact sequence (SES):

$$1 \longrightarrow G_{2n} \longrightarrow \mathbb{Z}_2^n \times S_{2n} \quad \text{total parity} \quad \mathbb{Z}_2 \longrightarrow 1$$

Thus, we have another SES:

$$1 \longrightarrow \mathbb{Z}_2^{2n-1} \longrightarrow G_{2n} \longrightarrow S_{2n} \longrightarrow 1$$

with a merely set theoretic section $s$. However, using $s$ (line 3.5 of $\theta$) we can write the multiplication table out for $G_{2n}$, hence $\pi_1(K_{2n})$ in terms of a twisted 2-cocycle $f$ on $S_{2n}$:

$$(a, g)(h, h) = (a + g.b + f(g, h), gh) \quad \text{(G1)}$$

where $a \in \mathbb{Z}_2^n, g, h \in S_{2n}$, $g$ acts on $\mathbb{Z}_2^{2n-1}$ by including $\mathbb{Z}_2^{2n-1} \subset \mathbb{Z}_2^n$ as the “even” vectors and letting $g$ permute the factors of $\mathbb{Z}_2^n$. It only remains to determine a twisted cocycle $f \in H^2(S_{2n}; \mathbb{Z}_2^{2n-1})$, where the action of $S_{2n}$ on $\mathbb{Z}_2^{2n-1}$ is as above. The cohomology class represented by $f$ is non-zero because the exact sequence does not split. As a check, we know (line 3.10 in $\theta$) that $f$ should obey:

$$g(f(h, k)) - f(g(h, k) + f(g, hk) - f(g, k) = 0 \quad \text{(G2)}$$

**Claim.** $f(g, h) = 0$ if $g$ or $h$ is even and $f(g, h) = e_{g(1)}$ if $g$ and $h$ are both odd, where $e_{g(1)}$ is the $g(1)^{\text{st}}$ standard generator of $\mathbb{Z}_2^n$.

**Appendix II: Ghostly recollection of the braid group**

To promote the Teo-Kane representation to a linear representation of the braid group $B_n$, we have the freedom to introduce an overall abelian phase to each $T_{i,j}$. There are several different choices related to different physical systems. For our purpose here, any choice suffices. The different choices lead to different linear images of the braid groups as the braid generator matrix has different orders.

The Ising TQFT can be realized using the Kauffman bracket by choosing $A = i e^{-\frac{2\pi}{5}}$. The braid group representation from a TQFT is not well-defined as a matrix representation because in general we do not have a canonical choice of the basis vectors. In the Kauffman bracket formulation of Ising TQFT, bases of representation spaces can be constructed using linear combinations of Temperley-Lieb (TL) diagrams or trivalent graphs. Then the representation of the braid groups is given by the Kauffman bracket interpreted as a map from $B_n$ to units of TL algebra.

The Temperley-Lieb-Jones algebra $TL_m(A)$ at the above chosen $A$ is isomorphic to the Clifford algebra. Therefore the $\gamma$-matrices $\{\gamma_i\}$ can be represented by TL generators $\{U_i\}$. In terms of the $\gamma$-matrices, the Jones representation in Kauffman bracket normalization is:

$$\rho_A(\sigma_i) = e^{-\frac{3\pi i}{4}} e^{-\frac{\pi}{2} \gamma_i \gamma_{i+1}}.$$  

Jones original representation from von Neumann algebra is:

$$\rho_A^J(\sigma_i) = e^{\frac{\pi i}{2}} e^{-\frac{\pi}{2} \gamma_i \gamma_{i+1}}.$$  

We will also refer to the $T_{i,i+1}$ representation of the braid group

$$\rho_A^\gamma(\sigma_i) = e^{-\frac{\pi}{2} \gamma_i \gamma_{i+1}}$$  

as the $\gamma$-matrix representation.

The ratio of the two distinct eigenvalues of the braid generator is $-i$, independent of the overall abelian phase. The
orders of the braid generator matrices in the three normalizations are 16, 4, 8, respectively. In the following, we will focus on the Jones representation.

Let $q = A^{-4} = 1$, then the Kauffman bracket is $\rho_q(\sigma_i) = \sigma_i = A + A^{-1}U_i$, where $U_i$'s are the TL generators. For convenience, we introduce the Jones generators of the TL algebras $e_i = U_i^d, d = \sqrt{2}, i = 1, \ldots, m - 1$. These $e_i$'s should not be confused with the basis element $e_i$'s of $\mathbb{R}^m$. The TL algebra in terms of $e_i$'s is:

$$e_i^j = e_i, e_i^0 = e_i, e_i e_{i+1} e_i = \frac{1}{d^2} e_i.$$  

The Jones representation is $\rho_A'(\sigma_i) = -A \rho_A(\sigma_i) = -1 + (1 + q)e_i$. Let $x_i = \langle \sigma_i^2 \rangle >$. Then we have: $x_i^2 = 1, x_i x_j = x_j x_i$ if $|i - j| > 1$, and $x_i x_{i+1} = -x_{i+1} x_i$. The $-1$ in the relation $x_i x_{i+1} = -x_{i+1} x_i$ is from the Jones-Wenzl projectors $p_i = 0$. By a simple calculation, the Jones-Wenzl projector $p_i$ in terms of $\{e_i, e_{i+1}\}$ is $p_i = 1 + 2(e_i e_{i+1} + e_{i+1} e_i - e_i e_{i+1})$.

Note that the sequence $1 \rightarrow 2 \rightarrow 1 \rightarrow 2 \rightarrow \cdots \rightarrow m - 1 \rightarrow 0$ generates $TL_m(A)$. Write $x_i$ in terms of $\gamma$-matrices, we have $x_i = \sqrt{-2 \gamma_3^{i+1}}$, which identifies $TL_m(A)$ with the even part $Cl_{2m}(C)$. On the other hand, if we set $v_i = (\sqrt{-1})^i x_i x_{i+1} x_i, i = 1, \ldots, m - 1$, a direct computation shows that $v_i^2 = v_i, v_i v_{j+1} + v_{j+1} v_i = 2\delta_{i,j}$. Therefore, $\{v_i, i = 1, \ldots, m - 1\}$ form the Clifford algebra $Cl_{m - 1}(C)$. Of course, the two different realizations $TL_m(A)$ as Clifford algebras are just the well known isomorphism $Cl_m(C) \cong Cl_{m-1}(C)$.

To identify the image of the Jones representation, we use the exact sequence $1 \rightarrow PB_m \rightarrow B_m \rightarrow S_m \rightarrow 1$. The image of the pure braid group $PB_m$ is generated by $\{x_i\}$. As an abstract group, the group can be presented with generators $x_i, i = 1 \cdots m - 1$ and relations $x_i^2 = 1, x_i x_j = x_j x_i$ if $|i - j| > 1$, and $x_i x_{i+1} = -x_{i+1} x_i$. This group, called the nearly-extra-special 2-group in $G = E_{m-1}$, is of order $2^m m!$. Consequently, the image $\tilde{G}_m$ of the Jones representation of $B_m$ fits into the exact sequence $1 \rightarrow E_{m-1} \rightarrow \tilde{G}_m \rightarrow S_m \rightarrow 1$. If we projectivize this sequence, we have $1 \rightarrow Z_{2^{m-1}} \rightarrow \tilde{G}_m \rightarrow S_m \rightarrow 1$. (This sequence splits if and only if $m$ is odd.) In this sense, the ribbon permutation group is a ghostly recollection of the braid group.

The center $Z(E_{m-1})$ of $E_{m-1}$ is $Z_2 = \{\pm 1\}$ if $m$ is even, and $Z_2 \times Z_2 = \{\pm 1, \pm 1, x_1 x_2 \cdots x_{m-1}\}$ if $m$ is odd. The representation of $E_{m-1}$ is faithful, but when $m$ is even, none of the irreducible sector is faithful because the central element $\pm x_1 x_2 \cdots x_{m-1}$ acts by $\pm 1$, too. The action of $x_1$ on the two irreducible sectors are the same for $i = 1, 2, \ldots, m - 2$ and differs only on $x_{m-1}$, which is $\pm 1$. When projectivized, note that the element $\alpha = (11 \cdots 1) \in Z_{2^m}$, as an element in $Z_2^m$, is invariant under the action of $S_m$, hence the projective image of each irreducible sector is isomorphic to $\mathbb{Z}_2^{m-2}$.

As comparison, we mention the related result in $^{25}$ The Jones representation there is in the $\gamma$-matrix normalization. The pure braid generators are of order 4. Hence the pure braid group image is a variant $E_{m-1}$ of $E_{m-1}$. The generators of the two groups are related by the change of $x_i$ to $-\sqrt{-1} x_i$. A presentation of $E_{m-1}$ with generators $x_i, i = 1 \cdots, m - 1$ has relations $x_i^2 = 1, x_i x_j = x_j x_i$ if $|i - j| > 1$, and $x_i x_{i+1} = -x_{i+1} x_i$. The center $Z(E_{m-1})$ is more complicated: $Z_2$ if $m$ is odd, and $Z_2 \times Z_2$ if $m = 4k + 1$ and $Z_4$ if $m = 4k + 3$. When $m$ is even, the image of each irreducible sector is identified as the lifting of $G_m$ to $Spin(m)$ in $^{25}$ It follows from the above discussion, the linear image of the Jones representation in $\gamma$ matrix-normalization fits into the exact sequence $1 \rightarrow Z_4 \times E_{m-1} \rightarrow G_m \rightarrow \tilde{G}_m \rightarrow Z_2 \times S_m \rightarrow 1$, where $\tilde{G}_m$ is the image of $B_m$ above.

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Upon finishing the paper we became aware of an independent work of J. Teo and C. L. Kane on the generic classification of topological defects.

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We note that neither Teo and Kane’s arguments nor ours determine the phase $\theta$. In a two-dimensional system of anyons, it is not an arbitrary phase. For instance, in a system of Ising anyons, $\theta$ must be equal to $\pi/8$. Any value of $\theta$ which is not a $2^{16}$th root of unity necessarily implies that the theory has additional structure and, therefore, additional particle types.

M. H. Freedman et al., in preparation.