The Quicksort process

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Abstract

Quicksort on the fly returns the input of $n$ reals in increasing natural order during the sorting process. Correctly normalized the running time up to returning the $l$-th smallest out of $n$ seen as a process in $l$ converges weakly to a limiting process with path in the space of cadlag functions.

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1 Introduction

Quicksort was chosen as one of the 10 most important algorithms. Quicksort serves also as a challenging random divide-and-conquer algorithm for a mathematical analysis. Starting with the worst case, the best case and the expected running time [7] we know nowadays much finer results on the limiting distribution, the existence [13] via martingale methods and a characterization [14] as a stochastic fixed point. The running time for many versions, actually all versions I know of, can be analyzed [10, 9] by the tools (contraction method and Weighted Branching Process [18]) invented for Quicksort.

Around 2004 Conrado Martínez [8] came up with the algorithm Partial Quicksort, a mixture of Quickselect and standard Quicksort. Given as input a list of \( n \) numbers, find the numbers between the \( k \)-th and the \( l \)-th smallest in order. A special case is Quicksort on the fly, where the algorithm tries to finish always the left most list. The output is first the smallest then the second smallest element and so on as an incoming stream.

What can we say about the running time up to see the \( l \)-th smallest? Martínez gave an explicit formula for the expected time (4) and showed an asymptotic result. This was an indication for a possible distributional result. In 2010 Martínez and Roesler showed convergence of one dimensional marginals to a limit, which required already some effort. This paper is on the convergence of finite dimensional marginals and, moreover, the existence of a limiting process for Quicksort on the fly with cadlag paths in \( D \). Most of this work is in the dissertation of Ragab [12].

The existence of the Quicksort process, Theorem 1, uses a specific construction via a Weighted Branching Process (WBP), see section 2 for the WBP and section 3 for the existence. The construction uses a specific sequence of rvs \((R_n)\) with values in \( D \) and the convergence is a uniform convergence of paths in an \( L_2 \) sense. We could take the supremum norm in \( D \), since the approaching sequence has the same(!) jump points path wise as the limit.

For the convergence of the discrete processes to the limit we again use a WBP on the binary tree. Technically we embed the Quicksort process and the discrete version all at the same time into a WBP on the binary tree with very specific rvs. An additional parameter \( n \in \mathbb{N} \cup \{\infty\} \) stands for the input list length, \( n = \infty \) for the limit. The basic idea is to find appropriate rvs on the binary tree such that we have a limit (of \( R_n \)) as \( n \to \infty \). The main result of this paper is Theorem 2 on weak convergence of processes. This embedding
seems to be the right object in order to obtain stronger convergence results. Notice the interplay of rvs and distributions and that we have to take nice versions of rvs in order to obtain (only) distributional results. We face a similar interplay (and technique) in the analysis of Find \([3, 17, 6]\).

After this rough idea of the abstract embedding of this paper into the existing literature and of its contribution, we shall give some more details.

Quick sort on the fly sorts the input of \(n\) distinct numbers and returns first the smallest, than second smallest and so on at the time of identification. The output is a stream of data in time. We are interested in this flow of data and will analyze the difference to the average behavior.

Like in Quicksort \([4, 5]\), choose with a uniform distribution a pivot from the input set \(S\) of distinct numbers, split \(S\) into the set \(S_\prec\) of strictly smaller ones than the pivot, the pivot and the set \(S_\succ\) of strictly larger ones in this order. Then continue recalling Quicksort independently always for the left list \(S_\prec\). If \(S_\prec\) is empty continue with the next leftmost sublist recalling the algorithm. If \(S_\prec\) consists only of one element, output this number immediately and continue with the next leftmost list.

Let \(X(S, l), l = 1, 2, \ldots, |S|\) denote the number of comparisons made up to the event when the \(l\)-th smallest number in the set \(S\) is determined. The interpretation of \(X(S, l)\) is as proportional to the time of publishing the \(l\)-th smallest. The rv \(X(S, l) = X(S_\prec, l = |S| - l)\) satisfies the recursion

\[
(X(S, l))_{l=1}^{|S|} = (|S| - 1 + \mathbb{1}_{l \leq I} X(S_\prec, l) + \mathbb{1}_{l > I} (X(S_\prec, I - 1) + X(S_\succ, l - I)))_{l=1}^{|S|} \quad (1)
\]

for \(|S| \geq 2\). Here \(I = I(S) = |S_\prec| + 1\) denotes the rank of the pivot and has values in \(\{1, 2, \ldots, n\}\) with a uniform distribution. The rv \(I = I(S)\) is independent of all \(X\)-rvs on the right of equation (1). The rv \(X(S_\prec, \cdot)\) is 0 if \(S\) is the empty set or has only one element. Otherwise the rv satisfies a similar recursion, where \(I(S_\prec)\) is independent of \(I(S)\). (Mathematically correct: \(I(A), \emptyset \neq A \subset S\) are independent with \(I(A)\) a uniform distribution on \(A\) and the \(X\) are recursively defined.) Continuing this way we find the distribution of \(X(S, |S|)\) as the Quicksort distribution sorting \(S\) by standard Quicksort.

The equation (1) determines the distribution of \((X(S, l))_{l=1}^{|S|}\) via the distribution for smaller sets. The distribution depends on \(S\) only via the size \(n = |S|\) of \(S\). (Prove this by induction on \(n\) and notice the distribution of \(I(S)\) depends only on the size \(n\) of \(S\).) For that reason we write \(X(S, l) \overset{\text{D}}{=} X(|S|, l)\). For notational reasons as above, which corresponds nicely to the interpre-
tation, we use the boundary conditions \( X(n, 0) = 0 \) for all \( n \in \mathbb{N} \) and \( X(0, \cdot) = X(1, \cdot) \equiv 0 \). Then the above recursion writes for \( n \geq 2 \)

\[
(X(n, l))^n_{l=1} \overset{\mathcal{D}}{=} (n-1 + \sum_{l < I_n} X^1(I_n - 1, l) + \sum_{l \geq I_n} X^1(I_n - 1, I_n - 1) + X^2(n-I_{n}, l-I_{n})))
\]

(2)

and determines the distribution of \((X(n, l))^n_{l=0}, n \in \mathbb{N}\), by the previous ones. The rvs \( I_n, (X^i(j, k))_{k=0}^j, i = 1, 2, 1 \leq j < n \) are independent. The rv \( I_n \) has values in \( \{1, 2, \ldots, n\} \) with a uniform distribution, \( X^i(j, \cdot) \) has the same distribution as \( X(j, \cdot) \) given by recursion. Notice \( X(n, n) = X(n, n-1) \).

In our version of Quicksort we use internal randomness by picking the pivot with a uniform distribution. Like in standard Quicksort, we could instead of internal randomness also use external randomness. Choose as input an uniform distribution on all permutations \( \pi \) of order \( n \) and pick as pivot any, for example always the first in the list. Now \( X(\pi, \cdot) \) is a deterministic function depending on the input \( \pi \). Seen as a rv with random input \( \pi \) we face the same distribution as with internal randomness. The main advantage using internal randomness is that \( X(\pi, \cdot) \) has the same distribution for every input \( \pi \) of the same size. Alternatively we could start with \( n \) iid random variables uniformly on \([0, 1]\) as the input and choose as pivot always the first element of the list. The algorithm itself would be deterministic, the running time is a rv via the input of an iid sequence and has the same distribution as our \( X(n, \cdot) \).

From equation (2) we obtain a recursion for the expectation \( a(n, l) = E(X(n, l)), n \geq 2, 1 \leq l \leq n \)

\[
a(n, l) = n - 1 + \frac{1}{n} \sum_{j=1}^l (a(j - 1, j - 1) + a(n - j, l - j)) + \frac{1}{n} \sum_{j=l+1}^n a(j - 1, l) \quad (3)
\]

The term \( a(n, n) \) is the expectation of sorting \( n \) numbers by Quicksort. All \( a(n, l) \) are uniquely defined by the above equations and the starting conditions. Martínez [8] obtained the explicit formula

\[
a(n, l) = 2n + 2(n + 1)H_n - 2(n + 3 - l)H_{n+1-l} - 6l + 6 \quad (4)
\]

\( 1 \leq l \leq n \in \mathbb{N} \). \( H_j \) denotes the \( j \)-th harmonic number \( H_j = \sum_{i=1}^j \frac{1}{i} \). (Notice \( a(0, 0) = 0 = a(1, 0) = a(1, 1) \) but in general the formula (4) is not \( E X(n, l) \) for \( n \geq 2 \) and \( l = 0 \).) For Quicksort we obtain the well known formula \( a(n, n) = 2(n + 1)H_n - 4n \).
Martínez argued with Partial Quicksort $PQ(n, l)$, which for fixed $n, l$ sorts the $l$ smallest elements of a list. For more results and versions of it, optimality and one-dimensional distributions for Partial Quicksort see [10]. The Quicksort on the fly process is an extension of Partial Quicksort in the sense of taking $l$ as a time variable and considering processes. We find first up to the $l - 1$-smallest elements, then continue this search for the $l$-th smallest, then $l + 1$-smallest and so on.

Now we come to the distribution of the process $X(n, \cdot)$ in the limit, where $X$ is defined via (2). This includes the question, how much $X(n, l)$ differs from the average $a(n, l)$. Define the rvs

$$Y_n(l) = \frac{X(n, l) - E(X(n, l))}{n}$$

for $l = 0, 1, \ldots , n$. These rvs satisfy the recursion, $n \geq 2$

$$\left( Y_n(\frac{l}{n}) \right)_{1 \leq l \leq n} \overset{\mathcal{D}}{=} \left( C(n, l, I_n) + \mathbb{1}_{l < I_n} \frac{I_n - 1}{n} Y_{I_n - 1}(\frac{l}{I_n - 1}) \right) + \mathbb{1}_{l = I_n} \frac{I_n - 1}{n} Y_{I_n - 1}(1) + \mathbb{1}_{l > I_n} \left( \frac{n - I_n}{n} Y_{I_n - 1}(1) + \left( \frac{n - I_n}{n} Y_{I_n - 1}(\frac{l - I_n}{n}) \right) \right)_{1 \leq l \leq n}$$

$$C(n, l, i) = \frac{1}{n} (n - 1 - a(n, l)) + \mathbb{1}_{l < i} (a(i - 1, l + 1)) + \mathbb{1}_{l = i} (a(i - 1, i + n))$$

$$C(n, 0, i) = 0$$

for $l = 1, \ldots , n - 1$. Notice $Y_n$ is well defined, with the help of the indicator function, and there are no boundary conditions besides $Y_0 \equiv 0 \equiv Y_1$.

We extend the process $Y_n$ nicely to a process on the unit interval $[0, 1]$ with values in the space $D = D[0, 1]$ of cadlag functions (right continuous functions with existing left limits [1]) on the unit interval. This can be done by linear interpolation or a piece wise constant function. We shall use the extension

$$Y_n(t) := Y_n(\frac{nt}{n})$$

The process $Y_n$ with values in $D$ satisfies the recursion, we use $U_n = \frac{I_n}{n}$

$$Y_n \overset{\mathcal{D}}{=} \left( C(n, \lfloor nt \rfloor, I_n) + \mathbb{1}_{t < U_n} \frac{I_n - 1}{n} Y_{I_n - 1}(\frac{nt}{I_n - 1}) \right)$$
\[ + \mathbb{1}_{t \geq U_n}(\frac{I_n - 1}{n}Y^1_{I_n-1}(1) + (\frac{n - I_n}{n})Y^2_{n-I_n}(\frac{t - U_n}{1 - U_n}))_{t \in [0,1]} \]

for \( n \in \mathbb{N} \). In short notation

\[ Y_n \overset{\mathcal{D}}{=} \varphi_n(U_n, (Y^1_k)_{k<n}, (Y^2_k)_{k<n}) \quad (7) \]

for a suitable function \( \varphi_n \).

If \( n \to \infty \) then \( U_n \) converges in distribution to a rv \( U \) with a uniform distribution. We might expect that the process \( Y_n \) converges in some sense to a limiting process \( Y = (Y(t))_{t \in [0,1]} \) with values in \( D \) satisfying something like the stochastic fixed point equation

\[ Y \overset{\mathcal{D}}{=} (\mathbb{1}_{t < U}UY^1(\frac{t}{U}) + \mathbb{1}_{t \geq U}(UY^1(1) + (1 - U)Y^2(\frac{t - U}{1 - U}))) + C(t, U) \quad (8) \]

for a suitable function \( \varphi \). The rvs \( Y^1, Y^2, U \) are independent. \( Y^1 \) and \( Y^2 \) have the same distribution as \( Y \) and \( U \) is uniformly distributed on the unit interval \([0,1]\). The cost function \( C = C(\cdot, U) \) is given by

\[ C(t, x) := C(x) + 2\mathbb{1}_{t < x}(-1 + x + (1 - t) \ln(1 - t) - (1 - x) \ln(1 - x)) - (x - t) \ln(x - t) \quad (9) \]

\[ C(x) := 1 + 2x \ln x + 2(1 - x) \ln(1 - x) \quad (10) \]

and is the limit of \( C(n, l_n, i_n) \) as \( n \to \infty \) with \( \frac{t}{n} \to_t t, \frac{i}{n} \to_t x \), Proposition (10).

Our first major result, Theorem 5, states the existence of the \( Y \)-process with values in \( D \).

**Theorem 1** Let \( U^v, v \in V \), be iid rvs with a uniform distribution on \([0,1]\) and \( V \) be the binary tree. Then there exists a family \( Y^v, v \in V \), of rvs with values in \( D \) and all of the same distribution satisfying almost surely

\[ Y^v = \varphi(U^v, Y^v1, Y^v2) \quad (11) \]

\[ Y^v(t) = \mathbb{1}_{t < U^v}U^vY^v1(\frac{t}{U^v}) + \mathbb{1}_{t \geq U^v}(U^vY^v1(1) + (1 - U^v)Y^v2(\frac{t - U^v}{1 - U^v})) + C(t, U^v) \]

simultaneously for all \( t \in [0,1] \).
The above equation (11) for rvs implies the distributional equality and \( Y = Y^\emptyset \) satisfies the fixed point equation (8). A specific family \( Y^v \) is explicitly given in the paper. We call \( Y^\emptyset \) the Quicksort process and its distribution the Quicksort process distribution.

Our second major result, compare Theorem 8 and Corollary 9, states the weak convergence of \( Y_n \) to the Quicksort process constructed above.

**Theorem 2** All finite dimensional distributions of \( Y_n \) without the coordinate zero converge to those of the Quicksort process \( Y \) constructed above.

The first result is a probabilistic result, while the second is a measure theoretic one. We obtain both results via the Weighted Branching Process [16] and an explicitly given nice family of processes \( Y^v_n \) indexed by \( n \in \mathbb{N} \) and the binary tree. Basically we use the splitting \( U \)-rvs for the \( Y \)-process also for the \( Y_n \)-process.

Inspired by previous work on Find [6] one might choose the pure measure theoretic approach by first showing the convergence of finite dimensional distributions of \( Y_n \) by the contraction method obtaining a limiting measure on \( [0,1] \). Then show the tightness [1] of the sequence of measures in order to obtain a limiting measure with outer measure 1 of \( D \). The measure restricted to \( D \) provides the desired distribution. There exist rvs satisfying (8). (Statement (11) is slightly stronger.) Neininger and Sulzbach [11] study this approach in more generality using the Zolotarev metric.

We prefer here a probabilistic approach using rvs instead of measures. Actually we formulated a (seemingly) stronger statement, the existence of a family \( Y^v \) of rvs with values in \( D \) and indexed by the binary tree, which satisfies (8) almost everywhere as rvs. For the construction we used a.e. convergence of appropriate rvs in supremum metric. A similar probabilistic approach via the Skorodhod metric was used [3] for Find, the first process analysis of a stochastic algorithm.

Also for Quicksort we find the two approaches in the literature with different types of results. The probabilistic approach by Régnier [13] provided a limiting rv via an \( L_2 \)-martingale. The measure theoretic approach via the contraction method [15] and the backward view characterized the limiting distribution as unique solution of a stochastic fixed point equation. Via the approach of Weighted Branching Processes one can construct a family of tree indexed rvs satisfying the stochastic fixed point equation as rvs. (Another \( L_2 \)-martingale \( R_n = \sum_i L_v C^v \), different from Régnier’s, is the key to convergence.)
We stated the results for processes with values in $D$. This is because $D$ is preferred to the space $E$ of left continuous functions $f : [0,1] \to \mathbb{R}$ with existing right limits. Both spaces are isomorphic with respect to the topological, probabilistic and algebraic structure. (The easiest argument is via the map $[0,1] \ni x \mapsto 1-x$.) The space $D$ is preferred in probability theory ([1]) and this is our main argument for using the space of cadlag functions. This explains the motivation for our normalization in (5).

Using the same arguments, the path wise left continuous version of $Y$ is a solution in $E$ of the corresponding equation (11) in $E$. For the discrete setting and interpretation it seems on the first view more natural to use left continuous functions. If we know the $l$-th largest and are only interested in that one, we do not have to look to the future, we are done. The $D$ and $E$ versions are mathematically equivalent.

2 The Weighted Branching Process

We introduce Weighted Branching Processes (WBP) in general and will specialize to Quicksort examples on the binary tree.

Let $V$ be the Ulam-Harris tree $\mathcal{N}^* = \bigcup_{n=0}^{\infty} \mathcal{N}^n$ of all finite sequences of natural numbers including the empty sequence denoted by the empty set $\emptyset$. (By convention $\mathcal{N}^0 = \{\emptyset\}$.) $V$ is a rooted tree with root $\emptyset$ and the edges $(v, vi), v \in V, i \in \mathbb{N}$ in graph theoretical sense. We use the standard notation $v = (v_1, v_2, v_3, \ldots, v_n) = v_1 v_2 \ldots v_n$ for a vertex $v \in V$ of length $n = |v|$. We skip the empty set in the notation whenever possible. We use $V_n$ for $v \in V$ of length $n$ and $V_{\leq n}, V_{< n}$ appropriate.

A weighted branching process (WBP) is a tuple $(V, (G, *, H, \odot), (T, C))$. $(T, C)$ is a random variable with values in $G^{\mathcal{N}} \times H$ on some probability space. (Actually we need only this distribution for the description, but prefer here a probabilistic language.) $(G, *)$ is a measurable semi group $(* : G \times G \to G, (g, h) \mapsto g * h$ associative and measurable) with a neutral element $e (\forall g \in G : e * g = g = g * e)$ and a grave $\triangle (\forall g \in G : \triangle * g = \triangle = g * \triangle$, once in the grave forever in the grave). $(G, *)$ operates left on $H$ via $\odot : G \times H \to H$. $H$ is a measurable space and $G \times H$ endowed with the product $\sigma$-field.

Let $(T^v, C^v), v \in V$, be independent copies of $(T, C)$ on the same probability space $(\Omega, \mathcal{A}, P)$. The interpretation of $C^v$ is as a cost function on a vertex $v$ and $T^v$, where $T = (T_1, T_2, \ldots)$, is a transformation (weight) on the edge $(v, vi)$. The interpretation of $G$ is as a map from $H$ to $H$. If $H$ has an
additional structure then we might enlarge $G$ to have the induced structure. Examples are $H$ is a vector space or an ordered set and the extended $G$ will be a vector space or ordered set via the natural extension.

For a WBP define the path weights $L_w^v : \Omega \to G, \ v, w \in V$ on paths $(v,vw)$ recursively by $L_w^v = e$ and

$$L_w^{v}$ recursion$$

$$L_w^{v} = L_w^{v} * T_i^{vw}.$$ 

One of the basic assumptions of a WBP is the independence of families, but arbitrary dependence within a family. Let $A_n$ be the $\sigma$-field generated by all $T_i^{v}, C_v$ for $v \in V_{<n}, i \in N$. Then $L_w^v$ is measurable with respect to $A_{|vw|}$ and $C_v$ is independent of $A_{|v|}$. We will use this many times in the sequel.

For a WBP without costs we write also $(V, (G, *), T)$. We drop the vertex $\emptyset$ whenever possible, e.g. $L_v = L_{\emptyset}^v$ or later $R^\emptyset = R, S^\emptyset = S$. The interpretation of $L_w^v$ the grave is, we can not see the path from the root $v$ to $vw$. Mathematically, no value grave connected to the path $(v,vw)$ will ever contribute, like $L_w^v C^{vw}$ will be 0 in our examples if $L_w^v = 0$. By this construction we shall use freely other trees like $m$-ary trees $\{1,2,\ldots,m\}^* = \cup_{n\in\mathbb{N}_0} \{1,2,\ldots,m\}^n$ of all finite sequences over the finite alphabet $1,2,\ldots,m$ in an appropriate sense. For the $m$-ary tree we take $T = (T_1, T_2, \ldots, T_m, \Delta, \Delta, \ldots)$ instead of $T = (T_1, T_2, \ldots, T_m, \Delta, \Delta, \ldots)$ on the original tree $N^*$.

For the next sections we need the following two examples. Although they provide known results the novelty is the line of arguments, which can be generalized and which are the key for the Quicksort process.

**Example 1:** Quicksort distribution [14]: Here we show mainly the existence of the Quicksort distribution via an embedding into the WBP. Consider the WBP $(V = \{1,2\}^*, (\mathbb{R}, \cdot, \mathbb{R}, \cdot), ((U, 1-U), C(U)))$ with $U$ has a uniform distribution and

$$C(x) = 1 + 2x \ln x + 2(1-x) \ln(1-x)$$ as in (9).

$G$ is the multiplicative semi group $\mathbb{R}$ with the neutral element $e = 1$ and the grave $\Delta = 0$. $G$ operates left on $H = \mathbb{R}$ by multiplication. Let $U^v, \ v \in V$, be independent rvs with a uniform distribution on $[0,1]$. Put

$$T_1^v = U^v, \ T_2^v = 1 - U^v, \ C^v = C(U^v).$$

(For the general WBP with tree $N^*$ we could take $T_i^v = \Delta = 0$ for $i \geq 3$. The smaller binary tree is more suitable here.) Since $H$ is an ordered vector
space, we extend $G$ with the interpretation of maps to the ordered vector space generated by the maps.

The total weighted cost $R_m := \sum_{v \in V < m} L_v C^v$ up to the $m - 1$ generation is an $L_2$-martingale and converges in $L_2$ and a.e. to a rv $Q$ \[14\]. The distribution of $Q$ is called the Quicksort distribution. The distribution is uniquely characterized \[14, 2\] as the solution of the stochastic fixed point equation

$$Q \overset{D}{=} UQ_1 + (1 - U)Q_2 + C(U)$$  \hspace{1cm} (12)

with expectation 0. Here $\overset{D}{=}$ denotes equality in distribution. The random variables $U, Q_1, Q_2$ are independent, $U$ is uniformly distributed and $Q_1, Q_2$ have the same distribution as $Q$.

By the a.s. convergence of $R_m^v := \sum_{w \in V < m} L_w^v C^{vw}$ the rvs

$$Q^v := \sum_{w \in V} L_w^v C^{vw}$$  \hspace{1cm} (13)

exist and satisfy a.e.

$$Q^v = U^v Q^v_1 + (1 - U^v)Q^v_2 + C(U^v)$$  \hspace{1cm} (14)

for every $v \in V$. Of course the distribution of $Q^v$ is a solution of (12) and is the Quicksort distribution.

**Example 2:** Convergence of the discrete Quicksort distributions \[14\]:

The original problem concerns the number $X_n$ of comparisons to sort $n$ distinct reals. We use internal randomness. Then for $n \in \mathbb{N}$

$$X_n \overset{D}{=} n - 1 + X_{I_n-1}^1 + X_{n-I_n}^2$$

with $I_n, X^1, X^2$ are independent, $I_n$ has a uniform distribution on $1, \ldots, n$ and $X_i^1, X_i^2$ have the same distribution as $X_i$. The boundary conditions are $X_0$ and $X_1$ are identical 0. The expectation of $X_n$ is $a_n = a(n, n)$ as in \[4\]. The normalized rvs $Y_n = \frac{X_n - a_n}{n}$ satisfy the recursion

$$Y_n \overset{D}{=} \frac{I_n - 1}{n} Y_{I_n-1}^1 + \frac{n - I_n}{n} Y_{n-I_n}^2 + C_n(I_n)$$

where

$$C_n(i) := \frac{n - 1 - a_n + a_{i-1} + a_{n-i}}{n}$$
We come now to the abstract embedding of this example into a WBP with an additional parameter \( n \in \mathbb{N} \). Let \( H \) be the set of functions \( h : \mathbb{N}_0 \to \mathbb{R} \) and \( G \) the set \( H \times G_2 \) where \( G_2 \) are the functions \( g : \mathbb{N}_0 \to \mathbb{N}_0 \) satisfying \( g(0) = 0 \) and \( g(n) < n \) for all \( n \in \mathbb{N} \). The semi group structure is given by

\[
(f_1, g_1) * (f_2, g_2) = (f_1 f_2 \circ g_2, g_2 \circ g_1)
\]

and the operation on \( H \) via

\[
(f, g) \circ h = f h \circ g \quad \text{and} \quad ((f, g) \circ h)(n) = f(n) h(g(n))
\]

(\( \circ \) denotes composition and we use multiplication on \( \mathbb{R} \)).

The interpretation of \( (f, g) \in G \) is as a map on \( H \), where \( f \) is a multiplicative factor and \( g \) an index transformation. The operation * corresponds to the convolution of maps on \( H \). Since \( H \) is a vector space we may enlarge \( G \) naturally to a vector space.

Consider the binary tree \( V = \{1, 2\}^* \) and let \( U^v, v \in V \), be independent rvs with a uniform distribution. Let \( I^v_n := \lceil n U^v \rceil \) \((\text{upper Gauss bracket})\) and define the transformations on the edges \((v, v1), (v, v2)\) by

\[
J^v_1(n) := I^v_n - 1 \quad J^v_2(n) := n - I^v_n \\
T^v_1(n) := \left( \frac{J^v_1(n)}{n}, J^v_1(n) \right) \quad T^v_2(n) := \left( \frac{J^v_2(n)}{n}, J^v_2(n) \right)
\]

and the vertex weight

\[
C^v(n) := C_n(I^v_n)
\]

The rvs

\[
R^v_m := \sum_{w \in V_{<m}} I^w_v \otimes C^{vw}
\]

converge as \( m \to \infty \) a.e. and in \( L_2 \) to a limit \( R^v \) and satisfy

\[
R^v_m = \sum_i T^v_i \otimes R^v_{m-1} + C^v
\]

for \( m \in \mathbb{N} = \mathbb{N} \cup \{\infty\} \).

Notice \( R^v_m \) and \( R^v \) take values in \( H \) and \( R^v(n), n \in \mathbb{N} \), is a random variable with values in the reals. Notice the connection to the previous description of the Quicksort rv. \( Y_n \) from the introduction

\[
Y_n \overset{D}{=} R^\emptyset(n).
\]
$R^v(n)$ converge for every $v \in V$ in $L_2$ to the rv $Q^v$ from the Quicksort example [14]. We shall use $Q^v_n = R^v(n)$ in the sequel and drop the root whenever possible.

**Example 3:** Joint embedding: It is worth while to put the two examples together. Use $\overline{N}_0 = N_0 \cup \{\infty\}$ instead of $N_0$ in the second example and incorporate the first example via the value $\infty$.

### 3 The Quicksort Process

In this section we consider the Quicksort process. Let $D = D([0,1])$ be the vector space of cadlag functions $f : [0, 1] \to \mathbb{R}$ (right continuous with existing left limits). $D$ is endowed with the Skorohod $J_1$-metric

$$d(f, g) = \inf \{\epsilon > 0 \mid \exists \lambda \in \Lambda : \|f - g \circ \lambda\|_{\infty} < \epsilon, \|\lambda - \text{id}\|_{\infty} < \epsilon\}$$

(15)

where $\Lambda$ is the set of all bijective increasing functions $\lambda : [0, 1] \to [0, 1]$. We use the supremum norm $\|f\|_{\infty} = \sup_t |f(t)|$.

The space $(D, d)$ is a separable, non complete metric space, but a polish space [1]. The $\sigma$-field $\sigma(D)$ is the Borel-$\sigma$-field via the Skorodhod metric. The $\sigma$-field is isomorphic to the product $\sigma$-field $\mathbb{R}^A \cap D$ where $A$ is a dense subset of $[0, 1]$ containing the 1.

Let $\mathcal{F}(D)$ be the space of all measurable functions $X$ with values in $D$. For $1 \leq p < \infty$ let $\mathcal{F}_p(D)$ be the subspace such that

$$\|X\|_{\infty, p} := \|\|X\|_{\infty}\|_p < \infty$$

is finite. Here $\| \cdot \|_p, p < \infty$, denotes the usual $L_p$-norm for rvs. The map $\| \cdot \|_{\infty, p}$ is a pseudo metric on $\mathcal{F}_p(D)$. Let $\sim$ be the common equivalence relation

$$X \sim Y \iff P(X \neq Y) = 0$$

and $\mathcal{F}_p(D)$ be the set of equivalence classes $[X] = \{Y \in \mathcal{F}(D) \mid X \sim Y\}$ intersected with $\mathcal{F}_p(D)$. Then it is well known

**Proposition 3** $(\mathcal{F}_p(D), \| \cdot \|_{\infty, p})$ is a Banach space for $1 \leq p < \infty$ with the usual addition and multiplication

$$[f] + [g] = [f + g], \quad c[f] = [cf], \quad \|[f]\|_{p, \infty} = \|f\|_{\infty, p}.$$
In the following we will be careless and will not differ between functions and equivalence classes.

Let \((G, \ast)\) be the semi group \(G = D \times D^\uparrow\) where \(D^\uparrow\) consists of increasing functions \(D \ni g : [0,1] \to [0,1]\) and the semi group operation \(\ast\) is

\[
(f_1, g_1) \ast (f_2, g_2) := (f_1 f_2 \circ g_1, g_2 \circ g_1).
\]

\((G, \ast)\) has as neutral element \((1, \text{id})\), the function identically 1 and the identity, and the grave is \((0, \text{id})\). \(G\) operates left on \(H = D^\uparrow\) via

\[
(f, g) \circ h := fh \circ g.
\]

The tuple \((f, g) \in G\) has the interpretation of a map \(M_{f,g}\) from \(H\) to \(H\) acting as

\[
M_{f,g}(h)(t) = f(t)h(g(t)).
\]

The first coordinate \(f\) acts as a space transformation, the second coordinate \(g\) as a time transformation. The semi group structure \(\ast\) is the composition of the corresponding maps

\[
M_{(f_1, g_1) \ast (f_2, g_2)} = M_{f_1, g_1} \circ M_{f_2, g_2}.
\]

(Notice the order of the composition.) Since \(H\) is a vector space and \(\mathbb{R}\) is a lattice, we will embed \(G\) to maps \(H^H\) and use freely the induced structure \(+, \cdot, \lor\) on \(G\), \(\lor\) denotes the supremum, \(a \in \mathbb{R}\)

\[
(M_{f_1, g_1} + M_{f_2, g_2})(h) = M_{f_1, g_1}(h) + M_{f_2, g_2}(h)
\]

\[
a \cdot (M_{f,g})(h) = (a \cdot M_{f,g})(h)
\]

\[
(M_{f_1, g_1} \lor M_{f_2, g_2})(h) = (M_{f_1, g_1}(h)) \lor (M_{f_2, g_2}(h)).
\]

Let \(V = \{1, 2\}^*\) be the binary tree and let \((U^v, Q^v), \ v \in V\) be the rvs as in the Quicksort example in the WBP section. Define on the edges \((v, vi), \ v \in V, i \in \{1, 2\}\) the edge weights \(T^v = (T_1^v, T_2^v), T_i^v = (A_i^v, B_i^v)\) with values in \(G\) and the vertex weights \(C^v\) with values in \(H\) by

\[
A_i^v = \mathbb{1}_{[0, U^v]} U^v \quad B_i^v(t) = 1 \land \frac{t}{U^v}
\]

\[
A_2^v = \mathbb{1}_{[U^v, 1]} (1 - U^v) \quad B_2^v(t) = 0 \lor \frac{t - U^v}{1 - U^v}
\]
\[ C^v = C(\cdot, U^v) + 1_{[U^v,1]} Q^{v1} \]

where \( C(t, x) \) is given in (9). Define rv \( R^v_m \) with values in \( H \) by

\[
R^v_m := \sum_{w \in V < m} L^v_w \odot C^{vw} \tag{17}
\]

for \( m \in \mathbb{N}_0 \).

**Proposition 4** The rvs \( R^v_m \) satisfy

\[
R^v_m = 2 \sum_{i=1}^{T^v_i} R^v_{m-1} + C^v \tag{18}
\]

for all \( m \in \mathbb{N}, v \in V \).

The proof is straightforward and easy since \( \ast \) and \( + \) interchange. q.e.d.

**Theorem 5** Let \( (V = \{1, 2\}^*, (G, \ast, H, \odot), ((T_1, T_2), C)) \) be the weighted branching process as defined above. Then \( R^v_m \) converges in supremum metric on \( D \) as \( m \to \infty \) a.e. to a rv \( R^v \) for all \( v \in V \). The family \( R^v, v \in V \) satisfies

\[
R^v = 2 \sum_{i=1}^{T^v_i} R^v_{m-1} + C^v \tag{19}
\]

almost everywhere. For every \( p > 1 \) and \( v \in V \) holds

\[
\|\|\| R^v_m \|\|_p \leq \frac{8 + 2^{-1/p} k_p \|Q\|_p}{1 - k_p} \tag{20}
\]

where \( k_p = \left(\frac{2}{p+1}\right)^{1/p} \) and \( Q \) is a rv with the Quicksort distribution.

Proof: Let \( S^v_m = R^v_{m+1} - R^v_m = \sum_{w \in V_m} L^v_w \odot C^{vw}, S^v_0 = C^v \) and \( b_m := \|S^v_m\|_{2, \infty} \). Notice \( b_m = \|S^v_m\|_{\infty, 2} \) does not depend on the vertex \( v \) and

\[
S^v_m = \sum_{i=1}^{T^v_i} S^v_{m-1} \tag{18}
\]

by (18) for all \( m \in \mathbb{N}, v \in V \).

\[ b_m^2 \leq \frac{2}{3} b_{m-1}^2 \text{ for } m \in \mathbb{N} \]
\[
\begin{align*}
\quad b_m^2 &= E \sup_t (\mathbb{I}_{t < U} S_{m-1}^1(\frac{t}{U}) + \mathbb{I}_{t \geq U} (1 - U) S_{m-1}^2(\frac{t - U}{1 - U}))^2 \\
&= E \sup_t ((\mathbb{I}_{t < U} S_{m-1}^1(\frac{t}{U}))^2 + (\mathbb{I}_{t \geq U} (1 - U) S_{m-1}^2(\frac{t - U}{1 - U}))^2) \\
&\leq E((U\|S_{m-1}^1\|_\infty)^2) + E((1 - U)\|S_{m-1}^2\|_\infty)^2) \\
&= (E(U^2) + E((1 - U)^2))b_{m-1}^2 = \frac{2}{3} b_{m-1}^2
\end{align*}
\]

- \( b_m \leq \left(\frac{2}{3}\right)^{m/2} b_0 \) where \( b_0 = \|C\|_{\infty,2} < \infty \).
  
  Easy by recursion.

- \( R_m, \ m \in \mathbb{N}, \) is a Cauchy sequence in \((\mathcal{F}_2(H); \| \cdot \|_{\infty,2})\).
  
  For \( n_0 \leq m < n \) argue

\[
\|R_n - R_m\|_{\infty,2} \leq \| \sum_{l=m}^{n-1} S_l \|_{\infty,2}
\]

\[
\leq \sum_{l=m}^{n-1} \|S_l\|_{\infty,2} \leq \sum_{l \geq n_0} b_l
\]

\[
\leq b_0 \left(\frac{2}{3}\right)^{n_0/2} \frac{1}{1 - \sqrt{2/3}} \rightarrow_{n_0 \rightarrow \infty} 0
\]

- \( \sum_m \|S_m\|_{\infty} < \infty \) a.e.

\[
E(\sum_m \|S_m\|_{\infty}) = \sum_m \|S_m\|_{\infty} \leq \sum_m \|S_m\|_{\infty} \|_1 < \infty
\]

Let \( R \) be the limit of the Cauchy sequence \( R_m \). \( R(t) \) is the point wise limit of \( \sum_{m \in \mathbb{N}} S_m(t) \) for all \( t \in [0, 1] \).

- \( R \) is well defined a.e.

  Easy by the previous statements.

- \( R \in \mathcal{F}_2(D) \).

  Every \( R_m \) is in \( \mathcal{F}_2(D) \). Since we can write \( R = \sum_{m=0}^{\infty} S_m \) a.e. the triangle inequality provides the result.

  - Equation \([19]\) is true.

  The same is true for every \( R^v_m, \ v \in V, \) instead of \( R_m \) and call \( R^v_m \) the limit in a.s. sense. Then \( R^v_m \) satisfies \([18]\) and the a.e. convergence provides \([19]\).
For the last statement notice
\[ \| R_m \|_{\infty} \leq \| (T_1 R_{m-1}^1) \vee (T_2 R_{m-1}^2) \|_{\infty} + \| C \|_{\infty}. \]

Consider the WBP as above but with cost function the constant \( C^v := 8 + U^{|Q^v|} \), \( v \in V \), instead of \( C^v \). Define
\[ R^v_m := \sum_{j<m} \bigvee_{w \in V_j} L^v_{w} C^{vw}, \]
where the symbol \( \bigvee \) denotes the supremum. Then \( R^v_0 = 0 \) and \( R^v_m \) increases in point wise to \( R^v \) for all \( v \in V \) and
\[ R^v_m = T^v_1 R^v_{m-1} \vee T^v_2 R^v_{m-1} + C^v \]
for \( m \in \mathbb{N} \). By induction it is easy to show \( \| R^v_m \|_{\infty} \leq \overline{R}^v_m \). (Notice \( \| C \|_{\infty} \leq 8 \).) Then for \( p > 1 \)
\[ \| \overline{R}_m \|_p \leq \| T_1 \overline{R}_m^1 \vee T_2 \overline{R}_m^2 \|_p + \| C \|_p \]
\[ \leq \left( E|T_1 \overline{R}_m^1|^p + E|T_2 \overline{R}_m^2|^p \right)^{1/p} + 8 + \| UQ \|_p \]
\[ \leq \| \overline{R}_{m-1} \|_p \left( E|T_1|^p + E|T_2|^p \right)^{1/p} + 8 + \| UQ \|_p \]
\[ \leq k_p \| \overline{R}_{m-1} \|_p + 8 + \| UQ \|_p \]
\[ \leq \sum_{i=0}^{m} k^i_p (8 + \| U \|_p \| Q \|_p) \leq \frac{8 + \| U \|_p \| Q \|_p}{1 - k_p} \]
\[ \| \overline{R} \|_p = \lim_{m} \| \overline{R}_m \|_p \leq \frac{8 + (\frac{1}{p+1})^{1/p} \| Q \|_p}{1 - k_p} \]
q.e.d.

4 Convergence of the discrete Quicksort process.

In this section we prove the convergence of finite dimensional marginals of \( Y_n \) to \( Y \). We will define a nice version of \( Y_n \) such that in \( L_2 \)-norm \( Y_n(t) \) converges to \( Y(t) \) for every \( t \in [0, 1] \). This requires to define a nice family \( (Y^v_n)_n \) of
random processes with values in $D$, indexed by the tree $V$. We will include the
Quicksort process via the index $\infty$ and consider $(Y_n)_{n \in \mathbb{N}_0}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0, \infty\}$. Compare this construction to the examples 3 of the section on the WBP. We shall use the general notation of a WBP with binary tree, but now on a more
general function space.

Let $V = \{1, 2\}^*$ be the binary tree and

$$H = \{h : [0, 1] \times \mathbb{N}_0 \to \mathbb{R} \mid \forall n \in \mathbb{N}_0 : h(\cdot, n) \in D\}.$$  

Let $G_2$ be the set of all $g : [0, 1] \times \mathbb{N}_0 \to [0, 1] \times \mathbb{N}_0$ such that $g(\cdot, 0) \equiv 0$, $\forall n \in \mathbb{N}$ : $\Phi_2(g(\cdot, n)) < n$ and $\Phi_2(g(\cdot, \infty)) = \infty$ where $\Phi_2$ denotes the projection to the second coordinate.

Define $G = H \times G_2$ with the semi group operation $*$

$$(f_1, g_1) \ast (f_2, g_2) := (f_1 f_2 \circ g_1, g_2 \circ g_1).$$

$(G, \ast)$ has the neutral element $(1, \text{id})$, the function identically $1$ and the
identity, and the grave is $(0, \text{id})$. $G$ operates left on $H$ via

$$(f, g) \circ h := fh \circ g.$$  

The tuple $(f, g)$ has the interpretation of a map $M_{f,g}$ from $H$ to $H$ via

$$(M_{f,g}(h))(t, n) = f(t, n)h(g(t, n)).$$

The first coordinate $f$ acts as a space transformation, the second coordinate $g$ as a time and index transformation. Since $H$ is a vector space and ordered set we will embed $G$ into $H^H$ and use freely the operations $+, \cdot, \lor$

$$(M_{f,g} + M_{f_1,g_1})(h) = M_{f,g}(h) + M_{f_1,g_1}(h)$$

$$a \cdot (M_{f,g}(h)) = (a \cdot M_{f,g})(h)$$

$$(M_{f,g} \lor M_{f_1,g_1})(h) = ((M_{f,g}(h)) \lor ((M_{f_1,g_1})(h))$$

for $a \in \mathbb{R}$.

Let $U^v$, $v \in V$, be independent rvs with a uniform distribution. Let $Q^v, Q^v_i, v \in V$, be the rvs as in the Quicksort examples 1 and 2 in the WBP section. Define on the edges $(v, vi), v \in V, i \in \{1, 2\}$, the edge weights (transformations) $T^v = (T^v_1, T^v_2)$, $T^v_i = (A^v_i, (B^v_i, J^v_i))$ with values in $G$ and the vertex weights $C^v$ with values in $H$ by

$$I^v_n := \lfloor nU^v \rfloor \quad U^v_n := \frac{I^v_n}{n}$$

17
\[ J_1^v(t, n) := I_n^v - 1 \quad J_2^v(t, n) := n - I_n^v \]
\[ A_1^v(t, n) := \mathbb{I}_{t < U_n^v} \frac{I_n^v - 1}{n} \quad A_2^v(t, n) := \mathbb{I}_{t \geq U_n^v} (1 - \frac{I_n^v}{n}) \]
\[ B_1^v(t, n) := 1 \wedge \frac{|nt|}{I_n^v - 1} \quad B_2^v(t, n) := 0 \vee \frac{t - U_n^v}{1 - U_n^v} \]
\[ C^v(t, n) := C(n, |nt|, I_n^v) + \mathbb{I}_{t \geq U_n^v} \frac{I_n^v - 1}{n} Q_n^{v3} \]

for \( n \in \mathbb{N} \) and

\[ J_1^v(t, \infty) := \infty \quad J_2^v(t, \infty) := \infty \]
\[ A_1^v(t, \infty) := \mathbb{I}_{t < U^v} U^v \quad A_2^v(t, \infty) := \mathbb{I}_{t \geq U^v} (1 - U^v) \]
\[ B_1^v(t, \infty) := 1 \wedge \frac{t}{U^v} \quad B_2^v(t, \infty) := 0 \vee \frac{t - U^v}{1 - U^v} \]
\[ C^v(t, \infty) := C(t, U^v) + \mathbb{I}_{t \geq U^v} U^v Q_n^{v1} \]

\( t \in [0, 1] \). \( A_i^v(t, 0) \) is identically 0.

Define the rvs \( R_m^v \)

\[ R_m^v := \sum_{w \in V_{<m}} L_w^v \otimes C^{vw} \]

for \( m \in \mathbb{N}_0, v \in V \). Notice, \( R_m^v \) takes values in \( H \) and \( R_m^v(\cdot, \infty) \) is the same as the previous Quicksort \( R_m^v \) given in (17). The embedding analogous to the embedding in example 3 of the WBP section.

**Proposition 6** The rvs \( R_m^v \) satisfy

\[ R_m^v = \sum_{i=1}^{2} T_i^v \otimes R_{m-1}^{v_i} + C^v \quad (21) \]

for all \( m \in \mathbb{N}, v \in V \).

(By convention \( \infty - 1 = \infty \).) The proof is easy since * and + interchange. q.e.d.

For every \( n \in \mathbb{N} \) the function \( R_m^v(\cdot, n) \) converges a.e. as \( m \to \infty \) to \( R^v(\cdot, n) \) for every \( v \in V \). Notice the number of summands increase in \( m \) and \( R^v(\cdot, n) \) has only finitely many non zero summands. By induction it is easy to show
Proposition 7

\[ Y_n^v = R^v(\cdot, n) = \sum_{w \in V} L_w^v \odot C^{vw}(\cdot, n) \]

for \( n \in \mathbb{N}_0, v \in V \).

Proof: The cases \( n = 0 \) and \( n = 1 \) are easy, since both sides are 0. For the induction step use the representation after (6) for \( Y_n^v \) and (21) for \( R^v \). We show as an example the equality for the first term

\[ T_1^v \odot R^v(t, n) = A_1^v(t, n)R_{v1}(B_1^v(t, n), J_1^v(t, n)) \]

\[ = A_1^v(t, n)Y_{v1}^{v1}(B_1^v(t, n)) \]

\[ = \mathbb{1}_{t < U_w^v \frac{I_n^v - 1}{n}} Y_{I_{n-1}}^{v1}(1 \wedge \frac{nt}{I_n^v - 1}) \]

The rest follows the same line. q.e.d.

Theorem 8  In the above setting

\[ ||Y_n^v(t) - Y^v(t)||_2 \to_{n \to \infty} 0 \]

for all \( t \in (0, 1] \) and \( v \in V \).

Proof: Let \( S_m^v = R_{m+1}^v - R_m^v \) and \( a := \sup_{t \in [0, 1]} \sup_{n \in \mathbb{N}_0} E((C(t, n))^2) \). In the following let \( v, w, \overline{w} \in V, m, \overline{m} \in \mathbb{N}_0, t \in [0, 1] \).

Notice \( L_w^v = (A_w^v, B_w^v, J_w^v) \) acts as a map on \( H \) via \( (L_w^v \odot h)(t, n) =: A_w^v(t, n)h(B_w^v(t, n), J_w^v(t, n)) \). (We take this as the definition of \( (A_w^v, B_w^v, J_w^v) \).)

We will use \( E(L_w^v) \) also as an operator acting on \( H \) in the sense \( E(((L_w^v \odot h)(t, n) = ((E(L_w^v))(h))(t, n)) \). We use \( (L)^2 \) of an operator via \( ((L)^2)(h) = (L(h))^2 \).

Let \( A_m \) be the \( \sigma \)-field generated by all \( U_v^v, v \in V_{<m} \). The rv \( L_w^v \) is measurable with respect to \( A_{|vw|} \) and \( C^v \) is independent of \( A_{|v|} \).

\[ E(L_w^v \odot C^{vw}(t, n) \mid A_{|vw|}) = 0 \]

\[ E(L_w^v \odot C^{vw}(t, n) \mid A_{|vw|}) = E(A_w^v(t, n)E(C^{vw}(B_w^v, J_w^v)(t, n) \mid A_{|vw|})) = 0 \]

\[ E(L_w^v C^{vw}(t, n)L_{vw}^v C^{vw}(t, n)) = 0 \] for \( w \neq \overline{w} \).
For $|w| < |\overline{w}|$ (or vice versa) take the conditional expectation with respect to $A_{|w|}$ and use the previous statement. For $|w| = |\overline{w}|$ use the independence of $C^{vw}$ and $C^{\overline{w}w}$ given $A_{|w|}$ and argue the left hand side is

$$
E(A^v_w(t, n)A^{\overline{v}w}_w(t, n))E(C^{vw}((B^v_w, J^v_w)(t, n))C^{\overline{w}w}((B^{\overline{w}}_w, J^{\overline{w}}_w)(t, n)) | A_{|w|})
$$

$$
= E(A^v_w(t, n)A^{\overline{v}w}_w(t, n))(E(C^{vw}((B^v_w, J^v_w)(t, n)) | A_{|w|}))(E(C^{\overline{w}w}((B^{\overline{w}}_w, J^{\overline{w}}_w)(t, n)) | A_{|w|}))
$$

$$
= 0
$$

- $E(S_m^v S_m^\overline{w}) = 0$ for $m \neq \overline{m}$.
- Square out and use previous results.
- $E(S_m^v S_m^\overline{w}) = E \sum_{v \in V_m} (L^v_w \odot C^{vw})^2$
- Square out and use previous results.
- Let $b(n, i) = \frac{i+1}{n} \vee \frac{n-i}{n}$. Then $E(b(n, I_n))^2 \leq \frac{2}{3}$. 
- Notice $EI_n = \frac{n(n+1)}{2}$ and $E(I_n)^2 = \frac{n(n+1)(2n+1)}{6}$.

$$
E(b(n, I_n))^2 = \frac{1}{n^2}(n^2 + 2E(I_n)^2 - 2nEI_n - 2EI_n + 1)
$$

$$
= \ldots = \frac{2}{3} - \frac{1}{n} + \frac{1}{3n^2} \leq \frac{2}{3}
$$

- $\sup_n \sum_{w \in V_m} E(\sup_t A^v_w(t, n))^2 \leq \left(\frac{2}{3}\right)^m$

Let $A^v_w(\ast, n) = \sup_t A^v_w(t, n)$. Notice $\sup_t A^v_w(t, n) \leq b(n, I_n^v)$. The recursion for $A$ is

$$
A^v_{iw}(t, n) = A^v_i(t, n)A^{vi}_w((B^v_i, J^v_i)(t, n)).
$$

We obtain

$$
\sup_t \sum_i A^v_{iw}(t, n) \leq \sup_i A^v_{iw}(\ast, n) \leq b(n, I_n^v) \sup_i A^{vi}_w(\ast, J^v_i)
$$

This provides

$$
E((A_{iw}^v(\ast, n))^2) \leq E((b(n, I^v_n))^2 \sup_i E((A^{vi}_w(\ast, J^v_i(\ast, n))))^2 | A_{|v|}))
$$

$$
\leq \frac{2}{3} \sup_i E((A^{vi}_w(\ast, J^v_i(\ast, n)))^2)
$$

By an induction on the length of $w$ we obtain the claim.

- $\sup_{t, n} E((S_m^v(t, n))^2) \leq a \left(\frac{2}{3}\right)^m$
l.h.s. = \sum_{w \in V_m} E((A^v_w(t, n)C^v_w((B^v_w, J^v_w)(t, n)))^2) \\
= \sum_{w \in V_m} E((A^v_w(t, n))^2 E((C^v_w((B^v_w, J^v_w)(t, n)))^2 \mid A_{|w|}) \\
\leq a \sum_{w \in V_m} E((A^v_w(t, n))^2) \\
\leq a \sup_t \sum_{w \in V_m} E((A^v_w(t, n))^2) \\
\leq a \sum_{w \in V_m} \sup_t E((A^v_w(t, n))^2) \\
\leq a \left(\frac{2}{3}\right)^m \\

• R^v_m(t, n) is an L_2-martingale in m with respect to A_{|v|+m} for all (t, n) and v \in V. 

The martingale property follows by E(S^v_m(t, n) \mid A_{|v|+m}) = 0, the L_2-

statement by the previous claim.

• \sup_{t,n} E((R^v - R^v_m)(t, n))^2 \leq \frac{3a}{2} \left(\frac{2}{3}\right)^m \\

E((R^v - R^v_m)(t, n))^2 \leq \sum_{i \geq m} E(S^v_i(t, n))^2 \leq a \sum_{i \geq m} \left(\frac{2}{3}\right)^i = \frac{3a}{2} \left(\frac{2}{3}\right)^m \\

• E(R^v_m(t, n) - R^v_m(t, \infty))^2 \rightarrow_{n \rightarrow \infty} 0 for all t \in [0, 1] and m \in N. 

Without loss of generality let t be none of the finitely many splitting points
of the tree vV up to depths m. Estimate \|R^v_m(t, n) - R^v_m(t, \infty)\|_2 by the finite
sum over all w \in V_m of the terms \|L^v_w \odot C^w(t, n) - L^v_w \odot C^w(t, \infty)\|_2. We
shall show every such term converges to 0.

l.h.s. = \|A^v_w(t, n)C^w((B^v_w, J^v_w)(t, n)) - A^v_w(t, \infty)C^w((B^v_w, J^v_w)(t, \infty))\|_2 \\
\leq \|(A^v_w(t, n) - A^v_w(t, \infty))C^w((B^v_w, J^v_w)(t, n))\|_2 \\
+ \|A^v_w(t, \infty)(C^w((B^v_w, J^v_w)(t, n)) - C^w((B^v_w, J^v_w)(t, \infty))\|_2 \\
\leq \|A^v_w(t, n) - A^v_w(t, \infty)\|_2 \sup_{s \in [0, 1]} \|C^w(s, i)\|_2 \\
+ \|C^w((B^v_w, J^v_w)(t, n)) - C^w((B^v_w, J^v_w)(t, \infty))\|_2 \\

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The first term converges as $n \to \infty$ to 0 since the difference converges to 0 a.e. and is uniformly bounded by $1$.

Estimate the second term by the triangle inequality

$$\leq \|C(U_{jw}^{vw}(t,n)) - C(U_{vw}^{vw})\|_2 + \|\mathbb{I}_{B_{w}^{vw}(t,n) \geq U_{vw}^{vw}} J_{w1}^{v}(t,n) Q_{J_{w1}^{vw}(t,n)}^{vw1} - \mathbb{I}_{B_{w}^{vw}(t,\infty) < U_{vw}^{vw}} U_{vw}^{vw} Q_{J_{w1}^{vw}(t,n)}^{vw1}\|_2$$

The first term will converge to 0. Argue $U_{m}^{vw}$ converges a.e. to $U^{vw}$ for $m \to \infty$. Then dominated convergence provides the statement, since $J_{w1}^{v}(t,n) \to_n \infty$ a.e. and the function $C$ is bounded.

Estimate the second term by

$$\leq \|(\mathbb{I}_{B_{w}^{vw}(t,n) \geq U_{vw}^{vw}} J_{w1}^{v}(t,n) Q_{J_{w1}^{vw}(t,n)}^{vw1} - \mathbb{I}_{B_{w}^{vw}(t,\infty) < U_{vw}^{vw}} U_{vw}^{vw} Q_{J_{w1}^{vw}(t,n)}^{vw1})\|_2$$

The first term converges to 0 since the difference is bounded and converges a.e. to 0. The second term converges to 0 since the difference is bounded and converges a.e. to 0. For the third term notice $J_{w1}^{v}(t,n)$ converges a.e. to $\infty$ and $b_m := \|Q_m - Q\|_2 \to_m \to 0$. Argue

$$\|Q_{J_{w1}^{vw}(t,n)}^{vw1} - Q_{vw}^{vw1}\|_2 = E(E((Q_{J_{w1}^{vw}(t,n)}^{vw1} - Q_{vw}^{vw1})^2 | A_{J_{w1}^{vw}}))$$

$$= \|b_{J_{w1}^{vw}(t,n)}\|_2^2 \to_n 0$$

Combining the above results we obtain the Theorem. q.e.d.

We come now to the weak convergence of the processes. For a vector $\bar{t} = (t_1, t_2, \ldots, t_k) \in T^*$, $k \in \mathbb{N}$ and a real valued function $f : T \to \mathbb{R}$ let $f(\bar{t})$ be the vector $(f(t_1), f(t_2), \ldots, f(t_k))$. A finite dimensional distribution of a process $X = (X(t))_{t \in T}$ is the distribution of $X(\bar{t}), \bar{t} \in T^*$. A process $X_n$ converges weakly to a process $X$, if all finite dimensional distributions converge, i.e. $X_n(\bar{t})$ converges in distribution to $X(\bar{t})$ for all $\bar{t} \in T^*$.

**Theorem 9** The process $(Y_n(t))_{0 < t \leq 1}$ converges weakly to the process $(Y(t))_{0 < t \leq 1}$.

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This is an immediate consequence of Theorem 8. q.e.d.

The convergence is stated for the half open interval \((0, 1]\). We could also obtain convergence on \([0, 1]\) by redefining \(Y_n\) appropriate. However then we should be very careful about the recurrence relation. See the remark on left or right continuity (on \(D\) and \(E\)) at the end of the introduction.

**Proposition 10** If \(\frac{n}{n} \to t \in (0, 1]\), \(\frac{n}{n} \to x\) and \(t \neq x\) then

\[
C(n, l_n, i_n) \to C(t, x)
\]

where the \(C\)-functions are given in (6) and (9).

**Proof:** The recursion (2) for \(n \geq 2\) and \(1 \leq l \leq n\) implies a recursion for \(a(n, l) = EX(n, l)\) in \(n \geq 2\). The solution is given in (4) \([8]\). The solution provides the correct values \(EX(n, l)\) for the \((n, l)\)-tuples \((1, 1), (1, 0), (0, 0)\) but not for the tuples \((n, 0)\) for \(n \geq 2\). Therefore we have to be careful by plugging in.

The asymptotics of the harmonic numbers are

\[
H_n = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{10}{120n^4} + O(n^{-6})
\]

with \(\gamma = 0.577215...\) the Euler constant. We will use \(H_n = \ln n + \gamma + b_n\) with \(b_n = O(\frac{1}{n})\).

For \(n \geq 2\) and \(1 \leq l = l_n, i = i_n \leq n\) argue

\[
C(n, l, i) = I + II + III
\]

\[
I = \frac{1}{n} \left( n - 1 + a(i - 1, l) - a(n, l)\right)
\]

\[
II = \frac{1}{n} \left( n - 1 + a(i - 1, i - 1) - a(n, l)\right)
\]

\[
III = \frac{1}{n} \left( n - 1 + a(i - 1, i - 1) + a(n - i, l - i) - a(n, l)\right)
\]

\[
\lim_{n \to \infty} II = 0
\]

\[
\lim_{n \to \infty} I = 1 + x + \lim_{n \to \infty} \frac{2}{n}(2i(l - 1) - 6(l) - 6 - 2n + 6(l) - 6 + 2iH_{i-1} - 2(i - l + 2)H_{i-1} - 2(n + 1)H_n + 2(n - l + 3)H_{n-1})
\]

\[
= \frac{1}{n} \left( -1 + 2x + \lim_{n \to \infty} \frac{2}{n}(i(\ln(i - 1) + \gamma + b_{i-1}) - (i - l + 2)(\ln(i - l) + \gamma + b_{i-1})\right)
\]

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\[ -(n+1)(\ln n + \gamma + b_n) + (n-l+3)(\ln(n-l+1) + \gamma + b_{n-l+1}) \]
\[ = \mathbb{1}_{t< \theta}(-1 + 2x + \lim_{n \to \infty} \frac{2}{n} \left( i \ln \frac{i-1}{n} - (i-l+2) \ln \frac{i-l}{n} \right) \]
\[ -(n+1) \ln \frac{n}{n} + (n-l+3) \ln \frac{n-l+1}{n} \]
\[ = \mathbb{1}_{t< \theta}(-1 + 2x + 2x \ln x - 2(x-t) \ln(x-t) + 2(1-t) \ln(1-t)) \]

\[ \lim_{n \to \infty} III = \mathbb{1}_{t> \theta}(1 + \lim_{n} \frac{1}{n} \left( 2(i-1) - 6(i-2) + 2(n-i) - 6(l-i-1) - 2n + 6l \right) + 2(n-i+1)H_{n-i} - 2(n-l+3)H_{n-l+1} + 2iH_{i-1} - 6H_1 \]
\[ -2(n+1)H_n + 2(n-l+3)H_{n-l+1}) \]
\[ = \mathbb{1}_{t> \theta}(1 + 2(1-x) \ln(1-x) - 2(1-t) \ln(1-t) + 2x \ln x + 2(1-t) \ln(1-t)) \]

The statement follows easily.

q.e.d.

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