List-coloring embedded graphs

Zdeněk Dvořák* Ken-ichi Kawarabayashi†

Abstract
For any fixed surface Σ of genus g, we give an algorithm to decide whether a graph G of girth at least five embedded in Σ is colorable from an assignment of lists of size three in time \( O(|V(G)|) \). Furthermore, we can allow a subgraph (of any size) with at most s components to be precolored, at the expense of increasing the time complexity of the algorithm to \( O(|V(G)|^{K(g+s)+1}) \) for some absolute constant K; in both cases, the multiplicative constant hidden in the \( O- \)notation depends on g and s. This also enables us to find such a coloring when it exists. The idea of the algorithm can be applied to other similar problems, e.g., 5-list-coloring of graphs on surfaces.

1 Introduction
In general, deciding 3-colorability of a planar graph is NP-complete [2]. On the other hand, Grötzsch [8] proved that any triangle-free planar graph is 3-colorable. A quadratic algorithm to find such a 3-coloring follows from his proof; the time complexity was later improved to \( O(n \log n) \) by Kowalik [12] and finally to linear by Dvořák et al. [4]. The situation is significantly more complicated for graphs embedded in surfaces. Nevertheless, Dvořák et al. [5] gave a linear-time algorithm to decide whether a triangle-free graph embedded in a fixed surface is 3-colorable.

Motivated by subproblems appearing in many coloring proofs, Vizing [18] and Erdős et al. [7] introduced the notion of list coloring. A list assignment for a graph G is a function \( L \) that assigns to each vertex \( v \in V(G) \) a list \( L(v) \) of colors. An L-coloring is a function \( \varphi : V(G) \rightarrow \bigcup_i L(v) \) such that \( \varphi(v) \in L(v) \) for every \( v \in V(G) \) and \( \varphi(u) \neq \varphi(v) \) whenever \( u, v \) are adjacent vertices of G. If G admits an L-coloring, then it is L-colorable. A graph G is k-choosable if it is L-colorable for every list assignment L such that \( |L(v)| \geq k \) for all \( v \in V(G) \).

A natural question is whether triangle-free planar graphs are also 3-choosable. This is not the case, as demonstrated by Voigt [19]. Let us remark that this implies that deciding whether a triangle-free planar graph is colorable from a given assignment of lists of size three is NP-complete, by a straightforward reduction from 3-colorability of planar graphs. Therefore, we need to restrict the graphs

*Computer Science Institute, Charles University, Prague, Czech Republic. E-mail: rado@iuuk.mff.cuni.cz. The work leading to this invention has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013)/ERC grant agreement no. 259385.
†National Institute of Informatics, Tokyo, Japan and JST ERATO Kawarabayashi Project. E-mail: keniti@nii.ac.jp.
in this setting even more. Thomassen \[15\] proved that every planar graph of girth at least five is 3-choosable. His proof also gives a quadratic algorithm to find such a coloring.

In this paper, we consider the problem of 3-list-coloring a graph of girth at least five embedded in a fixed surface. Our main result is a linear-time algorithm to decide whether a graph of girth at least five embedded in a fixed surface is colorable from given lists of size three. Furthermore, the algorithm can be modified to decide in linear time whether an embedded graph of girth at least five is 3-choosable. This problem is \(\Pi_2\)-complete for triangle-free embedded graphs \[9\]. Also, let us remark that it is not even known whether all projective planar graphs of girth at least five are 3-choosable.

Another interesting area of study is the precoloring extension problem, where we are given a coloring \(\psi\) of a subgraph \(F\) of the considered graph \(G\) and we want to decide whether there exists a (list-)coloring of \(G\) that extends \(\psi\). Unless some restriction is placed on \(F\), it is easy to see that the problem is NP-complete for planar graphs of arbitrarily large genus, even for ordinary 3-coloring (e.g., by a reduction from 3-colorability of planar graphs, where we replace each edge by two paths of odd length whose vertices are adjacent to vertices colored by 1 and 2, respectively). The known algorithms for (list-)coloring graphs on surfaces \[16, 17, 4\] allow some vertices to be precolored, but the number of precolored vertices needs to be bounded by a constant in order to achieve a polynomial time complexity. Our algorithm allows an arbitrary number of precolored vertices, assuming that the subgraph induced by them has a bounded number of components (at most \(s\)), at the expense of increasing its time complexity to \(O(|V(G)|K(g+s)+1)\) for some absolute constant \(K\). Since we can handle arbitrarily large precolored subgraphs, the algorithm can be used to find the coloring when it exists, by coloring the vertices one by one.

The algorithm is based on a bound on the size of critical planar graphs with one precolored face. Consider a graph \(G\), a subgraph (not necessarily induced) \(S \subseteq G\) and an assignment \(L\) of lists to vertices of \(G\). A graph \(G\) is \(S\)-critical (with respect to \(L\)) if for every proper subgraph \(G' \subset G\) such that \(S \subseteq G'\), there exists an \(L\)-coloring of \(S\) that does not extend to an \(L\)-coloring of \(G\), but extends to an \(L\)-coloring of \(G'\); i.e., removal of any edge of \(E(G) \setminus E(S)\) affects which colorings of \(S\) extend to the rest of the graph. Dvořák and Kawarabayashi \[3\] proved the following.

**Theorem 1.** Let \(G\) be a plane graph of girth at least 5 with the outer face \(F\) bounded by a cycle, and let \(L\) be an assignment of lists of size three to vertices of \(G\). If \(G\) is \(F\)-critical with respect to \(L\), then \(|V(G)| \leq 37\ell(F)/3\).

Our algorithm can be applied in any other setting where the result analogous to Theorem 1 holds (the bound on the size of the critical graph must be linear in the length of the precolored face). For example, Postle \[14\] gave a similar bound for 5-list-coloring.

**Theorem 2 (Postle).** There exists a constant \(c\) with the following property. Let \(G\) be a plane graph with the outer face \(F\) bounded by a cycle, and let \(L\) be an assignment of lists of size five to vertices of \(G\). If \(G\) is \(F\)-critical with respect to \(L\), then \(|V(G)| \leq c\ell(F)\).

Using this theorem instead of Theorem 1, we obtain a polynomial-time algorithm for extending a precoloring of a subgraph with bounded number of
components in graphs embedded in a fixed surface and with lists of size five. Let us remark that a linear-time algorithm for testing colorability of an embedded graph from lists of size five (but with only a constant number of precolored vertices) was previously given by Kawarabayashi and Mohar [11].

The basic idea of our algorithm is to use Theorem 1 to show that all vertices of critical graphs embedded in a fixed surface with sufficiently large edge-width are in logarithmic distance from the precolored subgraph (for planar graphs with one precolored cycle, this was first observed by Postle [14]). This enables us to restrict the problem to graphs of logarithmic tree-width.

In Section 2, we prove the result on the distance from the precolored subgraph in critical graphs and use it to design a list-coloring algorithm for graphs with large edge-width. The algorithm for the general case (which follows by a standard dynamic programming idea, used before e.g. in [5]) is described in Section 3. The 3-choosability case is discussed in Section 4.

2 Distances in critical graphs

We use the following consequence of Theorem 1.

Corollary 3. Let \( G \) be a plane graph of girth at least 5 and let \( S \) be a set of vertices such that each vertex in \( S \) is incident with the outer face of \( G \). Let \( L \) be an assignment of lists of size three to vertices of \( G \). If \( G \) is \( S \)-critical with respect to \( L \), then \(|V(G)| \leq 50|S|\).

Proof. Let \( k = |S| \). Thomassen [15] proved that a precoloring of any vertex of a planar graph \( G \) of girth at least five extends to a coloring of \( G \) from arbitrary lists of size three. Consequently, we can assume that \( k \geq 2 \). Let \( s_1, s_2, \ldots, s_k \) be the vertices of \( S \), and let \( G' \) be the graph obtained from \( G \) by joining \( s_i \) with \( s_{i+1} \) by a new path of length four (where \( s_{k+1} = s_1 \)) for \( 1 \leq i \leq k \). Let \( C \) be the cycle in \( G' \) formed by the newly added paths. Note that we can choose the ordering of the vertices of \( S \) so that \( G' \) is a plane graph with a face bounded by \( C \). Give the new vertices arbitrary lists of size three, and note that \( G' \) is \( C \)-critical with respect to the resulting list assignment. Furthermore, \( G' \) has girth at least five, and by Theorem 1 we have \(|V(G)| < |V(G')| \leq 37\ell(C)/3 = 37 \cdot 4|S|/3 < 50|S|\). □

A surface is a compact 2-dimensional manifold (possibly disconnected or with boundary). A useful tool for dealing with surfaces is the operation of cutting along prescribed curves. To avoid technical complications, we restrict ourselves to cutting along subgraphs of some graph embedded in the surface, as follows.

Let \( Q \) be a graph embedded in a surface \( \Sigma \) without boundary. Let \( \Sigma - Q \) be the open space obtained from \( \Sigma \) by removing the edges and vertices of the embedding of \( Q \). Let \( \Sigma_Q \) be the surface (possibly disconnected) with boundary such that the interior of \( \Sigma_Q \) is homeomorphic to \( \Sigma - Q \). Let \( \theta: \Sigma_Q \rightarrow \Sigma \) be the continuous extension of this homeomorphism. Let \( \Sigma_Q \) stand for the surface without boundary obtained from \( \Sigma_Q \) by capping each component of its boundary by a disk. Suppose that \( G \) is embedded in \( \Sigma \) so that the intersection of the embeddings of \( G \) and \( Q \) is a subgraph of both \( G \) and \( Q \) (i.e., a point of this intersection is a vertex in \( G \) iff it is a vertex in \( Q \), and if a point of this intersection belongs to an edge \( e \) of \( G \) or \( Q \), then \( e \) is an edge drawn in the
same way both in $G$ and $Q$). Then, we say that the embeddings of $G$ and $Q$ are compatible and we let $G_Q = \theta^{-1}(G)$. Note that each edge of $G \cap Q$ corresponds to two edges of $G_Q$. Similarly, each vertex $v \in V(G) \cap V(Q)$ corresponds to $\deg_Q(v)$ vertices of $G_Q$. The basic property of criticality is that it is preserved by this cutting operation. If $S$ is a subgraph of $G$, then let $S^Q$ denote the graph $\theta^{-1}(S \cup (G \cap Q))$. If $L$ is a list assignment for $G$, then let $L_Q$ denote the list assignment for $G_Q$ such that $L_Q(v) = L(\theta(v))$ for $v \in V(G_Q)$.

**Lemma 4.** Let $G$ and $Q$ be graphs with compatible embeddings in a surface $\Sigma$ without boundary, and let $L$ be a list assignment for $G$. Let $S$ be a subgraph of $G$. If $G$ is $S$-critical with respect to $L$, then $G_Q$ is $S^Q$-critical with respect to $L_Q$.

**Proof.** Let $\theta: \Sigma_Q \to \Sigma$ be the mapping from the definition of cutting along $Q$. Consider an arbitrary proper subgraph $G'$ of $G_Q$ such that $S^Q \subseteq G'$, and let $G'' = \theta(G')$. Note that $G''$ is a proper subgraph of $G$ containing $S \cup (G \cap Q)$, and thus there exists an $L$-coloring of $S$ that extends to an $L$-coloring $\varphi''$ of $G''$, but does not extend to an $L$-coloring of $G$. Let $\varphi'$ be the $L_Q$-coloring of $G'$ defined by $\varphi'(v) = \varphi''(\theta(v))$ for $v \in V(G')$. Observe that the restriction of $\varphi'$ to $S^Q$ does not extend to an $L_Q$-coloring of $G_Q$, as otherwise the image of this $L_Q$-coloring under $\theta$ would be an $L$-coloring of $G$ extending $\varphi$. Since the choice of $G'$ was arbitrary, this implies that $G_Q$ is $S^Q$-critical.

Similarly, we can prove the following.

**Lemma 5.** Let $S$ be a subgraph of a graph $G$ and let $L$ be a list assignment for $G$. If $G$ is $S$-critical with respect to $L$ and $G = G_1 \cup G_2$, then $G_1$ is $((G_2 \cup S) \cap G_1)$-critical with respect to $L$.

In particular, if $G_1$ is a connected component of $G$, then $G$ is $(S \cap G_1)$-critical.

Consider a surface $\Sigma$ without boundary and a graph $G$ with a 2-cell embedding in $\Sigma$. The radial graph $H$ of $G$ is the bipartite graph with vertex set consisting of the vertices and faces of $G$, and edge set corresponding to the incidence relation between vertices and faces of $G$ (with multiplicities—i.e., if a vertex $v$ appears $k$ times in the facial walk of a face $f$ of $G$, then $v$ and $f$ are joined by $k$ edges in the radial graph of $G$). The radial graph has a natural embedding in $\Sigma$, where each vertex $v \in V(H) \cap V(G)$ is drawn at the same position as in the embedding of $G$, each vertex $f \in V(H) \setminus V(G)$ is drawn inside the corresponding face $f$ of $G$, and each face of $H$ has length four and contains exactly one edge of $G$.

Postle [14] observed that Corollary 3 and Lemma 4 imply a logarithmic bound on the distances in a critical plane graph with all precolored vertices incident with the same face. We include a proof of this claim giving a slightly better multiplicative constant in the bound.

**Theorem 6** (Postle). Let $G$ be a connected plane graph of girth at least 5 and let $S$ be a set of vertices such that each vertex in $S$ is incident with the outer face $f$ of $G$. Let $L$ be an assignment of lists of size three to vertices of $G$. Let $H$ be the radial graph of $G$. If $G$ is $S$-critical with respect to $L$, then every vertex of $H$ is at distance at most $398 + 100\log |S|$ from the vertex corresponding to $f$. 

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Proof. Let $F$ be the set of all faces incident with vertices of $S$ (including $f$). Let $S_0 = S$. Let $S_i$ be the set of vertices of $V(G) \setminus S$ adjacent to the elements of $F$ in $H$. For an integer $i \geq 2$, let $S_i$ denote the set of vertices of $H$ at distance exactly $2i - 1$ from $F$ in $H$ (all these vertices belong to $V(G)$). Let $G_i$ denote the subgraph of $G$ induced by $\bigcup_{j \geq i} S_j$. Let $n_i = |V(G_i)| = \sum_{j \geq i} |S_j|$. Note that $G_i$ is $S_i$-critical by Lemma 5 and thus $n_i \leq 50|S_i|$ by Corollary 8. It follows that $\frac{1}{n_i} = 1 - \frac{n_i - n_{i+1}}{n_i} = 1 - \frac{|S_i|}{n_i} \leq \frac{49}{50}$ for every $i \geq 0$, and consequently $n_i \leq 50|S|0.98^i$.

Therefore, for $k > \frac{\log 50|S|}{\log 0.98}$, we have $n_k < 1$, and since $n_k$ is a nonnegative integer, it is zero. Consequently, every vertex of $H$ is at distance at most $2k - 2$ from $F$, and thus at distance at most $2k$ from $f$. The claim of the theorem follows.

We now aim to prove a similar claim for graphs on surfaces and possibly with precolored vertices incident with several faces. We need an auxiliary result on surface cutting first. We use the following standard properties of graphs on surfaces [13]. Given a spanning tree $T$ of a graph, a set $S$ of edges not belonging to $T$ and a vertex $v$, let $T_{S,v}$ denote the graph consisting of $S$ and of the paths in $T$ joining $v$ with the vertices incident with edges of $S$.

Lemma 7. Let $H$ be a graph with a 2-cell embedding in a connected surface $\Sigma$ of Euler genus $g > 0$ and without boundary. Let $T$ be a spanning tree of $H$ and $v$ a vertex of $H$. Then, there exists a set $S \subseteq E(H) \setminus E(T)$ of size $g$ such that $\Sigma_{T_{S,v}}$ is a disk.

Lemma 8. Let $G$ be a 2-cell embedding in a connected surface $\Sigma$ without boundary and let $H$ be a natural embedding of the radial graph of $G$. Let $G'$ be an induced subgraph of $G$. Vertices $u, v \in V(H) \setminus V(G')$ lie in the same face of $G'$ if and only if there exists a path in $H - V(G')$ joining $u$ with $v$.

Let $G$ be a graph with 2-cell embedding in a surface $\Sigma$ and let $F$ be a set of faces of $G$. A cycle $C \subseteq G$ is $F$-contractible if there exists a disk $\Delta \subseteq \Sigma$ bounded by $C$ disjoint with all the faces of $F$.

Lemma 9. Let $G$ be a 2-cell embedding in a connected surface $\Sigma$ without boundary and let $F$ be a set of faces of $G$ and $r$ an integer, such that every cycle in $G$ of length less than $r$ is $F$-contractible. Let $H$ be a natural embedding of the radial graph of $G$. There exists a spanning tree $T$ of $H$, a vertex $v \in V(G)$, closed disks $\Delta_1, \ldots, \Delta_k \subset \Sigma$ (for some $k \geq 0$) and either a point or a closed disk $\Delta_0$, such that the following holds:

1. $\Delta_0, \ldots, \Delta_k$ are disjoint with the faces in $F$ and have pairwise disjoint interiors.
2. If $\Delta_0$ is a point, then it is equal to $v$. If $\Delta_0$ is a closed disk, then the boundary of $\Delta_0$ is a cycle of length at most $r - 1$ in $G$, the vertex $v$ lies in $\Delta_0$ and the distance in $H$ from $v$ to any vertex in the boundary of $\Delta_0$ is the same.
3. For $1 \leq i \leq k$, the boundary of $\Delta_i$ is a cycle of length at most $r - 1$ in $G$.
4. $T \cap \Delta_0$ is connected.
5. For $1 \leq i \leq k$, the graph obtained from $T$ by removing the vertices contained in the interior of $\Delta_i$ is connected.

6. Every path $P \subseteq T$ starting with a vertex in the boundary of $\Delta_0$ such that no internal vertex of $P$ is contained in $\Delta_0 \cup \ldots \Delta_k$ has length at most $2|V(G)|/r + 2$.

Proof. Choose an arbitrary vertex $v \in V(G)$, and let $T$ be a breadth-first search tree of $H$ rooted in $v$. Let $G_i$ denote the subgraph of $G$ induced by the vertices whose distance from $v$ in $H$ is $2i$. Let $i_0$ be the largest index such that $G_{i_0}$ contains a cycle of length at most $r - 1$ bounding a disk containing $v$ and disjoint with the faces in $F$, and let $\Delta_0$ be the disk; if no such index exists, then let $i_0 = 0$ and let $\Delta_0 = v$. Let $i_1 > i_0$ be the smallest index such that $|V(G_{i_1})| < r$ ($G_{i_1}$ can be empty). Since every cycle in $G_{i_1}$ has length at most $r - 1$, each such cycle is $F$-contractible. Let $\Delta_1, \ldots, \Delta_k$ be the maximal disks whose boundaries are cycles in $G_{i_1}$ and that are disjoint with the faces of $F$. Note that the interiors of the disks are pairwise disjoint. Since $i_0 < i_1$, the boundary of $\Delta_0$ is disjoint with the boundaries of $\Delta_1, \ldots, \Delta_k$, and the choice of $i_0$ implies that $\Delta_0$ is actually disjoint with $\Delta_1, \ldots, \Delta_k$.

Let us show that $T$ and the disks $\Delta_0, \ldots, \Delta_k$ satisfy the conclusions of the lemma. The first three properties follow by the construction. For the next two properties, the following observation is useful: for every $i > 0$, letting $f_i$ be the face of $G_i$ that contains $v$, a vertex $u \in V(H)$ is at distance at most $2i - 1$ from $v$ in $H$ if and only if $u$ lies in $f_i$—this follows from Lemma \[\text{ }\] as $u$ is at distance at most $2i - 1$ from $v$ in $H$ if and only if there exists a path from $u$ to $v$ in $H - V(G_i)$. Furthermore, the same argument implies that all vertices of $G_i$ are incident with this face $f_i$.

Therefore, if the boundary of $\Delta_0$ is a cycle in $G_{i_0}$, then $f_{i_0}$ is contained in $\Delta_0$, and consequently all vertices of $T$ at distance at most $2i_0$ from $v$ are contained inside $\Delta_0$. This implies that for every vertex $u$ in the boundary of $\Delta_0$, the path from $u$ to $v$ in $T$ is contained in $\Delta_0$. Note that every path in $T$ that contains $v$ is either contained in $\Delta_0$ or intersects the boundary of $\Delta_0$, and thus there exists a path from every vertex of $T \cap \Delta_0$ to $v$ contained in $T \cap \Delta_0$.

The fourth property follows.

For the fifth property, note that $v \notin \Delta_1 \cup \ldots \cup \Delta_k$, and that $\Sigma - G_i \subseteq f_i \cup \Delta_1 \cup \ldots \cup \Delta_k$. Therefore, exactly the vertices of $H$ at distance at least $2i_1 + 1$ are contained in the interiors of $\Delta_1, \ldots, \Delta_k$. Since all vertices in the boundaries of these disks have distance $2i_1$ from $v$ and $T$ is a breadth-first search tree, there exists no path in $T$ with endvertices outside of $\Delta_1$ and with an internal vertex in the interior of $\Delta_i$ for any $1 \leq i \leq k$. The fifth property follows.

By the choice of $i_1$, we have $|V(G_i)| \geq r$ when $i_0 + 1 \leq i \leq i_1 - 1$. Since the graphs $G_{i_0+1}, \ldots, G_{i_1-1}$ are pairwise vertex disjoint, we have $|V(G)| \geq (i_1 - i_0 - 1)r$. Every path $P \subseteq T$ with the properties described in the fifth claim contains at most one vertex of $G_i$ for $i_0 \leq i \leq i_1$, and thus it has length at most $2(i_1 - i_0)$. The last property follows.

Combined with Theorem \[\text{ }\] and Lemma \[\text{ }\] we obtain the result on surface cutting we mentioned before.
Lemma 10. Let $G$ be a graph of girth at least five with a 2-cell embedding in a connected surface $\Sigma$ of Euler genus $g$ and without boundary, and let $F$ be a set of faces of $G$ and $r$ an integer such that every cycle in $G$ of length less than $r$ is $F$-contractible. Let $S$ be a set of vertices of $G$ such that every vertex in $S$ is incident with a face in $F$. Let $L$ be an assignment of lists of size three to $V(G)$. Let $H$ be a natural embedding of the radial graph of $G$. If $G$ is $S$-critical with respect to $L$, then there exists a subgraph $Q \subseteq H$ with at most $(2|V(G)|/r + 800 + 200 \log r)(2g + |F|)$ edges such that $\Sigma_Q$ is a disk.

Proof. Note that $2g + |F| \geq 1$, as planar graphs of girth at least five are 3-choosable. Let $T$, $v$, $\Delta_0$, $\ldots$, $\Delta_k$ be obtained by Lemma 8. By Lemma 10 the subgraph of $G$ embedded in $\Delta_i$ is $F_i$-critical for $0 \leq i \leq k$, where $F_i$ is the intersection of the boundary of $\Delta_i$ with $G$. By Theorem 6 we conclude that every vertex of $T$ is at distance at most $2|V(G)|/r + 798 + 200 \log r$ from $v$ in $T$. Let $S \subseteq E(H) \setminus E(T)$ be the set of edges given by Lemma 7 where we set $S = \emptyset$ when $g = 0$. We let $Q$ consist of $T_{S,v}$ and of the paths in $T$ joining $v$ with the vertices corresponding to the faces in $F$. \hfill \Box

Together with Corollary 8 this bounds the size of critical graphs.

Lemma 11. Let $G$ be a graph of girth at least five with an embedding in a surface $\Sigma$ of Euler genus $g$ and without boundary, and let $F$ be a set of faces of $G$. Let $c$ be the number of components of $\Sigma$. For a component $\Sigma'$ of $\Sigma$, let $q(\Sigma') = 2g' + n'$, where $g'$ is the Euler genus of $\Sigma'$ and $n'$ is the number of faces of $F$ in $\Sigma'$. Let $q$ be the maximum $q(\Sigma')$ over all components $\Sigma'$ of $\Sigma$.

Suppose that every cycle in $G$ of length less than $200q$ is $F$-contractible. Let $S$ be a set of vertices of $G$ such that every vertex in $S$ is incident with a face in $F$. Let $L$ be an assignment of lists of size three to $V(G)$. If $G$ is $S$-critical with respect to $L$, then $|V(G)| \leq 100|S| + 40000(2g + |F| - c)(10 + \log q)$.

Proof. We can assume that $\Sigma$ is connected, as otherwise we can consider each component separately. Similarly, we can assume that the embedding of $G$ is 2-cell, as otherwise we can cut $\Sigma$ along a non-$F$-contractible curve contained inside a face of $G$ and cap the resulting hole(s) with disk(s)—this simplifies the embedding, does not increase $q$ and when it increases $|F|$, it either decreases $g$ or increases $c$. Note that $2g + |F| \geq 1$, as planar graphs of girth at least five are 3-choosable. If $2g + |F| = 1$, then the claim follows from Corollary 9, hence assume that $2g + |F| \geq 2$.

Let $H$ be a natural embedding of the radial graph of $G$. Let $Q$ be the subgraph of $H$ obtained by Lemma 10. By Lemma 4 $G_Q$ is $S^Q$-critical with respect to $L$. Note that $|S^Q| \leq |S| + \sum_{v \in V(Q)} \deg_Q(v) \leq |S| + |E(Q)| \leq |S| + |V(G)|/100 + (2g + |F|)(2000 + 200 \log q)$. By Corollary 8, we have $|V(G)| \leq |V(G_Q)| \leq 50|S^Q| \leq 50|S| + |V(G)|/2 + 50(2g + |F|)(2000 + 200 \log q)).$ The claim of the lemma follows, since $2g + |F| \leq 2(2g + |F| - 1)$. \hfill \Box

Using this lemma (instead of Corollary 8), we prove the main result of this section—a bound on distances for critical graphs on surfaces analogous to Theorem 6.

Theorem 12. Let $G$ be a graph of girth at least five with an embedding in a surface $\Sigma$ of Euler genus $g$ and without boundary. Let $S$ be a set of vertices of $G$ and $F$ a set of faces of $G$ such that every vertex in $S$ is incident with
a face in $F$. Suppose that every cycle in $G$ of length less than $200g$ is $F$-contractible, where $q$ is defined as in Lemma 17. Let $L$ be an assignment of lists of size three to $V(G)$. If $G$ is $S$-critical with respect to $L$, then every vertex of $G$ either is at distance less than $200(C + 5 + \log(1 + |S|/(C + 1)))$ from $S$ or belongs to a connected component of $G$ with at most $100C$ vertices, where $C = 400(2g + |F| - 1)(10 + \log q)$.

**Proof.** Each component of $G$ that contains no vertex of $S$ has at most $100C$ vertices by Lemma 11. Therefore, we can assume that $S$ intersects all components of $G$.

For $i \geq 0$, let $S_i$ be the set of vertices of $G$ at distance exactly $i$ from $S$. Let $G_i$ denote the subgraph of $G$ induced by $\bigcup_{j \geq i} S_j$. Let $n_i = |V(G_i)| = \sum_{j \geq i} |S_j|$. Note that $G_i$ is $S_i$-critical by Lemma 5 and all vertices of $S_i$ are incident with one of at most $|F|$ faces of $G_i$, and thus $n_i \leq 100|S_i| + 100C$ by Lemma 11.

If $|S_i| > C$, this implies that $n_i < 200|S_i|$, and thus $n_{i+1} = 1 - \frac{n_i}{n_{i+1}} < \frac{199}{200}$ for every $i \geq 0$. Let $k$ be the smallest index such that $|S_k| < C + 1$. For $0 \leq i \leq k$, we have $|S_i| \leq n_i \leq 100(|S| + C)0.995^i$, and thus $k \leq 200(5 + \log(1 + |S|/(C + 1)))$. Since $n_k \leq 100|S_k| + 100C \leq 200C$, we have $n_{k+200C} = 0$. Consequently, every vertex of $G$ is at distance less than $k + 200C$ from $S$. The claim of the theorem follows.

This theorem enables us to bound the tree-width of critical graphs, using the following result of Eppstein [6].

**Theorem 13** (Eppstein). There exists a constant $c_k$ such that every graph $G$ of Euler genus $g$ and radius $r$ has tree-width at most $c_k(g + 1)r$. Furthermore, the tree decomposition of this width can be found in time $O((g + 1)r|V(G)|)$.

**Corollary 14.** Let $g, s \geq 0$ be fixed integers. Let $C = 0$ if $g = s = 0$ and $C = 400(2g + s - 1)(10 + \log(2g + s))$ otherwise. There exists an algorithm with the following specification. The input of the algorithm is a graph $G$ of girth at least five embedded in a surface $\Sigma$ of Euler genus at most $g$ and without boundary, a set $F$ of at most $s$ faces of $G$ such that every cycle in $G$ of length at most $100C$ is $F$-contractible, a set $S$ of vertices incident with the faces of $F$, and a list assignment $L$ such that $|L(v)| = 3$ for $v \in V(G) \setminus S$ and $|L(v)| = 1$ for $v \in S$. The algorithm correctly decides whether $G$ is $L$-colorable. The time complexity of the algorithm is $O((|S| + 1)^{K(g+s)}|V(G)|)$ for some absolute constant $K$.

**Proof.** If $g = s = 0$, then $G$ is $L$-colorable by Thomassen [15]; hence, assume that $g + s \geq 1$. Let $G_1$ be the subgraph of $G$ induced by vertices at distance less than $200(C + 5 + \log(1 + |S|/(C + 1)))$ from $S$. Note that if $G$ is not $L$-colorable, then it contains a subgraph $G_0$ with $S \subseteq V(G_0)$ that is $S$-critical and $L$-colorable. Suppose that $G_0$ has a component $Q$ that does not contain a vertex of $S$. By Theorem 12, $Q$ has at most $100C$ vertices. Since planar graphs of girth at least five are 3-choosable, $Q$ contains a non-contractible cycle of length at most $|V(Q)| \leq 100C$, contrary to the assumptions. Therefore, every component of $G_0$ contains a vertex of $S$, and by Theorem 12, we have $G_0 \subseteq G_1$. It follows that $G$ is $L$-colorable if and only if $G_1$ is $L$-colorable.

Let $G_2$ be the graph obtained from $G_1$ by adding a new vertex adjacent to all vertices of $S$. Note that $G_2$ has radius at most $200(C + 5 + \log(1 + |S|/(C + 1))) = O(\log |S|)$ by Theorem 12 and that $G_2$ can be embedded in a surface of Euler genus at most $995$ by adding a new vertex adjacent to all vertices of $S$.
genus at most \( g + 2s - 2 \). By Theorem 13, \( G_2 \) (and thus also \( G_1 \)) has tree-width at most \( K'(g + s) \log(|S| + 1) \) for some absolute constant \( K' \). We apply the standard dynamic programming algorithm [10] for list-coloring graphs from lists of bounded size, which (for lists of size at most three) has time complexity \( e^{O(tw(G_1))|V(G_1)|} \). Since \( |V(G_1)| \leq |V(G)| \), this gives the desired bound on the time complexity of the algorithm.

Let us remark that in the situation of Corollary 14, it is easy to find an \( L \)-coloring of \( G \) when it exists in time \( O(|V(G)|^{K(2 + \log(2)) + 2}) \) as follows. We can assume that \( S \) is an independent set, as edges joining vertices of the same color would prevent the existence of an \( L \)-coloring, and edges joining vertices of different colors are irrelevant. If \( V(G) = S \), then the list assignment \( L \) gives an \( L \)-coloring of \( G \). Otherwise, there exists a vertex \( v \in V(G) \setminus S \) incident with a face of \( F \). Let \( S' = S \cup \{v\} \), and using the algorithm of Corollary 14 try all three possible colors for \( v \) and test whether the corresponding coloring of \( S' \) extends to an \( L \)-coloring of \( G \). If this succeeds for at least one color \( c \), set \( L(v) := \{c\}, S := S' \) and repeat the process until the whole graph is colored.

3 The algorithm

A standard dynamic programming approach enables us to deal with short non-\( F \)-contractible cycles. Let us first consider the case of a graph embedded in the sphere with precolored vertices incident with at most two faces.

**Lemma 15.** Let \( d \) be an integer such that \( 4 \leq d \leq [100C_0] \), where \( C_0 = 400(10 + \log 2) \). There exists an algorithm \( A_d \) with the following specification. The input of the algorithm is a graph \( G \) of girth at least five embedded in the sphere, a set \( F \) of at most 2 faces of \( G \) such that every cycle in \( G \) of length at most \( d \) is \( F \)-contractible, a set \( S \) of vertices incident with the faces of \( F \), and a list assignment \( L \) such that \( |L(v)| = 3 \) for \( v \in V(G) \setminus S \) and \( |L(v)| = 1 \) for \( v \in S \). The algorithm correctly decides whether \( G \) is \( L \)-colorable. The time complexity of the algorithm is \( O((|S| + 1)^{K_0}|V(G)|) \) for some absolute constant \( K_0 \).

**Proof.** Let \( K_0 = 2K \) for the constant \( K \) from Corollary 14. We proceed by induction on \( d \), starting from the largest value. If \( d = [100C_0] \), then we can use the algorithm of Corollary 14 as \( A_d \). Hence, suppose that \( d < [100C_0] \) and as the induction hypothesis assume that the algorithm \( A_{d+1} \) exists.

We now describe the algorithm \( A_d \). If \( |F| \leq 1 \), then we use the algorithm of Corollary 14 since then every cycle is \( F \)-contractible; thus, we can assume that \( F \) consists of two faces \( f_1 \) and \( f_2 \). If an edge joins two vertices \( u, v \in S \) such that \( L(u) = L(v) \), then \( G \) is not \( L \)-colorable. Otherwise, \( G \) is \( L \)-colorable if and only if \( G - uv \) is \( L \)-colorable. Consequently, we can also assume that \( S \) forms an independent set.

Note that a cycle in \( G \) is not \( F \)-contractible if and only if it separates \( f_1 \) from \( f_2 \). Consider the dual \( G' \) of \( G \) and a maximum flow from \( f_1 \) to \( f_2 \) in \( G' \) (where all edges have capacity 1). The size of the flow is equal to the size of a minimum cut between \( f_1 \) and \( f_2 \) in \( G' \), which corresponds to the shortest cycle separating \( f_1 \) from \( f_2 \) in \( G \). If the size of the flow is at least \( d + 2 \) (which can be verified in \( O(|V(G)|) \) using the Ford-Fulkerson algorithm by stopping after we improve the flow \( d + 2 \) times), then every cycle in \( G \) of length at most \( d + 1 \) is
$F$-contractible and the lemma follows by induction. Therefore, assume that the maximum flow has size exactly $d + 1$. Then, there exists a unique $(d + 1)$-cycle $Q_1$ in $G$ separating $f_1$ from $f_2$ which is nearest to $f_1$, corresponding to the cut of size $d + 1$ in $G'$ bounding the set of vertices that can be reached from $f_1$ by augmenting paths. Let $G_1$ be the graph obtained from $G$ by removing all vertices between $f_1$ and $Q_1$, including $V(f_1)$ but not $V(Q_1)$, as well as all edges of $Q_1$. In $G_1$, we find the $(d + 1)$-cycle $Q_2$ separating the face corresponding to $Q_1$ from $f_2$ that is nearest to $Q_1$, and by removing the part between $Q_1$ and $Q_2$ as well as the edges of $Q_2$, we obtain a graph $G_2$. We repeat this as long as the current graph contains such a cycle. Let $Q_1, Q_2, \ldots, Q_n$ be the sequence of $(d + 1)$-cycles obtained in this way, let $Q_0 = S \cap V(f_1)$ and let $Q_{n+1} = S \cap V(f_2)$.

Note that we can inherit the flow from the dual of $G$ to the dual of $G_{i+1}$ and that in order to find the cycle $Q_{i+1}$, the algorithm visits only the vertices in $V(G_{i+1}) \setminus V(G_i)$, for $1 \leq i \leq n - 1$. Consequently we can find this sequence of cycles in time $O(|V(G)|)$.

For $0 \leq i \leq n$, let $H_i$ be the subgraph of $G$ between $Q_i$ and $Q_{i+1}$, not including the edges of $Q_i$ and $Q_{i+1}$. Note that all cycles of length at most $d + 1$ in $H_i$ are \{$Q_i, Q_{i+1}$\}-contractible and that for $j \in \{i, i + 1\}$, we either have $|V(Q_j)| = d + 1$ or $L$ assigns a unique coloring to $Q_j$. Furthermore, $n = O(|V(G)|)$ and only the vertices of $Q_1 \cup Q_2 \cup \ldots \cup Q_n$ can belong to several of the graphs $H_i$; thus, $\sum_{i=0}^n |V(H_i)| = O(|V(G)|)$. For $0 \leq i \leq n$, we determine the set $\Psi_i$ of all $L$-colorings of $V(Q_i) \cup V(Q_{i+1})$ that extend to an $L$-coloring of $H_i$, excluding those incompatible with the edges of $Q_i \cup Q_{i+1}$. This can be done by applying the algorithm $\mathcal{A}_{d,1}$ to $H_i$ repeatedly, precoloring the vertices of $Q_i \cup Q_{i+1}$ in all possible ways (the number of $L$-colorings of $Q_i \cup Q_{i+1}$ is bounded by the constant $3^{2d+2}$). The total time spend by this is $O((|S| + 1)^{K_0}|V(G)|)$.

Finally, for $1 \leq i \leq n + 1$, we determine the set $\Phi_i$ of all $L$-colorings of $V(Q_0) \cup V(Q_i)$ that extend to an $L$-coloring of the subgraph of $G$ between $Q_0$ and $Q_i$ (inclusive)—we already know the set $\Phi_1 = \Psi_0$, and $\Phi_{i+1}$ can be obtained by combining $\Phi_i$ with $\Psi_i$ in constant time. The time to determine $\Phi_{n+1}$ is thus $O(|V(G)|)$. The graph $G$ is $L$-colorable if and only if $\Phi_{n+1}$ is nonempty. \hfill \Box

Let us note that if $d = 4$, then the assumption on $F$-contractibility of all cycles of length at most $d$ is void, since the graph has girth at least five; hence, we can drop the assumption entirely.

A similar argument is used to give the algorithm in the general case. Given a graph $G$ embedded in a surface $\Sigma$ with boundary, a non-contractible cycle $C$ in $G$ is almost contractible if there exists a component $b$ of the boundary of $\Sigma$ such that $C$ is contractible in the surface obtained from $\Sigma$ by capping $b$ with a disk; in this case, we say that $C$ surrounds $b$.

**Theorem 16.** Let $g, s \geq 0$ be fixed integers. There exists an algorithm with the following specification. The input of the algorithm is a graph $G$ of girth at least five embedded in a surface $\Sigma$ of Euler genus at most $g$ and without boundary, a set $F$ of at most $s$ faces of $G$, a set $S$ of vertices incident with the faces of $F$, and a list assignment $L$ such that $|L(v)| = 3$ for $v \in V(G) \setminus S$ and $|L(v)| = 1$ for $v \in S$. The algorithm correctly decides whether $G$ is $L$-colorable. The time complexity of the algorithm is $O((|S| + 1)^{K(g+s)}|V(G)|)$ for some absolute constant $K$. 

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Proof. We proceed by induction, assuming that the theorem holds for all surfaces of Euler genus less than \( g \), and for all graphs embedded in a surface of Euler genus \( g \) with precolored vertices incident with fewer than \( s \) faces. If \( g = 0 \) and \( s \leq 2 \), then we can use the algorithm \( A_1 \) of Lemma 15. Therefore, assume that \( g > 0 \) or \( s > 2 \). Let \( C = 400(2g + s - 1)(10 + \log(2g + s)) \). As usual, we can assume that \( S \) forms an independent set and that \( G \) is connected and its embedding is 2-cell.

Let us drill a hole in each face of \( F \), obtaining a surface \( \Sigma' \). We use the algorithm (inspired by the result of Cabello and Mohar [11]) described in Dvořák et al. [5] before the proof of Theorem 8.3 to test whether the embedding of \( G \) in \( \Sigma' \) contains a non-contractible cycle of length less than 100\( C \) that is not almost contractible, in time \( O(|V(G)|) \). Suppose that \( Q \) is such a cycle. For each \( L \)-coloring \( \psi \) of \( Q \), let \( L^\psi \) be the list assignment such that \( L^\psi(v) = \{\psi(v)\} \) for \( v \in V(Q) \) and \( L^\psi(v) = L(v) \) for \( v \in V(G) \setminus V(Q) \). There are only constantly many choices for \( \psi \), and \( G \) is \( L \)-colorable if and only if it is \( L^\psi \)-colorable for some \( L \)-coloring \( \psi \) of \( Q \). Note that \( G \) is \( L^\psi \)-colorable if and only if \( G_Q \) is \( L^\psi_Q \)-colorable, and each component of the surface \( \Sigma_Q \) gives rise to a simpler instance of the problem (either the component has smaller Euler genus than \( \Sigma \), or it is homeomorphic to \( \Sigma \), but the precolored vertices in \( G_Q \) are incident with fewer than \( |F| \) faces in this component). The claim of the theorem follows by induction applied to each of the components (and all the possible colorings \( \psi \)).

Therefore, we can assume that all non-contractible cycles of length at most 100\( C \) in the embedding of \( G \) on \( \Sigma' \) are almost contractible. Since \( g > 0 \) or \( s > 2 \), the component of the boundary of \( \Sigma' \) surrounded by such a cycle is unique. The closer inspection of the algorithm of Dvořák et al. [5] shows that it actually implements a data structure representing the graph \( G \) that can be initialized in time \( O(|V(G)|) \) and supports the following operations in a constant time:

- Remove an edge or an isolated vertex.
- Decide whether a vertex belongs to a non-contractible cycle of length at most 100\( C \), and if that is the case, return such a cycle; furthermore, if the cycle is almost contractible, it decides which component of the boundary it surrounds.

Using this data structure, we process the vertices of \( G \) one by one, testing whether they belong to an almost contractible cycle \( Q \) of length at most 100\( C \). If that is the case, we remove all vertices and edges of \( G \) between \( Q \) and the surrounded component \( b \) of the boundary (including the edges, but not the vertices, of \( Q \)). The part of graph to be removed can be found by a depth-first search from \( b \) in time proportional to the size of the removed part. We then continue the process with the remaining vertices of \( G \). We end up with a subgraph \( G' \) of \( G \). For each component \( b \) of the boundary of \( \Sigma' \), let \( S_b \) denote the subset of \( S \) incident with the face of \( G \) corresponding to \( b \). Let \( G_b \) denote the removed part of \( G \) between \( b \) and \( G' \); if nonempty, \( V(G_b) \cap V(G') \) is a vertex set of a cycle \( Q_b \) in \( G_b \) of length at most 100\( C \) that surrounds \( b \) in the embedding of \( G \) in \( \Sigma' \). For each \( b \), we determine the set of all \( L \)-colorings of \( S_b \cup V(Q_b) \) that extend to an \( L \)-coloring of \( G_b \) using the algorithm \( A_1 \) of Lemma 15. Furthermore, note that the embedding of \( G' \) in \( \Sigma' \) contains no non-contractible cycle of length at most 100\( C \), thus we can determine the set of all \( L \)-colorings of its intersection with \( S \) and with the cycles \( Q_b \) using the
algorithm of Corollary 14. Combining these sets (whose size is bounded by a constant depending on $g$ and $s$), we can decide whether $G$ is $L$-colorable. Observe that this algorithm has the required time complexity.

Also, a straightforward cutting argument enables us to deal with precolored vertices being contained in connected subgraphs instead of incident with a common face.

**Corollary 17.** Let $g, s \geq 0$ be fixed integers. There exists an algorithm with the following specification. The input of the algorithm is a graph $G$ of girth at least five embedded in a surface $\Sigma$ of Euler genus at most $g$ and without boundary, a subgraph $Q$ of $G$ with at most $s$ components, and a list assignment $L$ such that $|L(v)| = 3$ for $v \in V(G) \setminus V(Q)$ and $|L(v)| = 1$ for $v \in V(Q)$. The algorithm correctly decides whether $G$ is $L$-colorable. The time complexity of the algorithm is $O\big(|V(Q)| + 1\big)^K(g+s)|V(G)|)$ for some absolute constant $K$.

**Proof.** We can assume that $Q$ is a forest and each component of $Q$ has at least three edges (otherwise, remove or add edges to $Q$ to make these conditions hold). Consequently, $\Sigma_Q$ is homeomorphic to $\Sigma$, $G_Q$ has girth at least five and each component of $Q$ corresponds to a cycle in $G_Q$ bounding a face. Let $S = V(Q_Q)$ and note that $|S| = 2|E(Q)| < 2|V(Q)|$. Note also that $G$ is $L$-colorable if and only if $G_Q$ is $L_Q$-colorable, since all vertices of $Q$ have lists of size one. The claim then follows by Theorem 16.

Again, by coloring the vertices of $G$ one by one, we can actually find an $L$-coloring of $G$ when it exists, in time $O\big(|V(G)|^{K(g+\max(s,1)) + 2}\big)$.

## 4 Choosability

Here, we extend the algorithm of Theorem 16 to the case of 3-choosability. The basic ingredients are the following generalizations of Corollary 14 and Lemma 15.

Let $S$ be a set of vertices. We say that two list assignments for $S$ are are equivalent when they differ only by renaming the colors. Let $\mathcal{C}(S)$ denote a maximal set of pairwise non-equivalent assignments of lists of size three to $S$, and note that $\mathcal{C}(S)$ is finite. If $G$ is a graph with $S \subseteq V(G)$, then let $\mathcal{C}(S,G)$ denote the set consisting of all pairs $(L_0, \Psi)$ such that $L_0 \in \mathcal{C}(S)$, $\Psi$ is a set of $L_0$-colorings of $S$ and there exists a list assignment $L$ for $G$ such that $|L(v)| = 3$ for $v \in V(G) \setminus S$, $L(v) = L_0(v)$ for $v \in S$ and an $L_0$-coloring $\psi$ of $S$ extends to an $L$-coloring of $G$ if and only if $\psi \in \Psi$.

**Corollary 18.** Let $g, s \geq 0$ be fixed integers. Let $C = 0$ if $g = s = 0$ and $C = 400(2g + s - 1)(10 + \log(2g + s))$ otherwise. There exists a function $f_{g,s}$ and an algorithm with the following specification. The input of the algorithm is a graph $G$ of girth at least five embedded in a surface $\Sigma$ of Euler genus at most $g$ and without boundary, a set $F$ of at most $s$ faces of $G$ such that every cycle in $G$ of length at most $100C$ is $F$-contractible and a set $S$ of vertices incident with the faces of $F$. The algorithm outputs the set $\mathcal{C}(S,G)$ in time $O(f_{g,s}(|S|)|V(G)|)$.

**Proof.** If $g = s = 0$, then $G$ is 3-choosable by Thomassen [15], and $\mathcal{C}(S,G)$ consists of the pair $\psi_0$, where $L_0$ is the null list assignment and $\psi_0$ is the null coloring. Hence, assume that $g + s \geq 1$. Let $G_1$ be the subgraph of...
$G$ induced by vertices at distance less than $200(C + 5 + \log(1 + |S|/(C + 1)))$ from $S$. Consider any list assignment $L$ such that $|L(v)| = 3$ for $v \in V(G) \setminus S$ and $|L(v)| = 1$ for $v \in S$. As in Corollary 13 we have that $G$ is $L$-colorable if and only if $G_1$ is $L$-colorable. Consequently, $C(S, G) = C(S, G_1)$, and since $G_1$ has bounded treewidth, we can determine $C(S, G_1)$ in linear time by a standard dynamic programming algorithm.

Lemma 19. Let $d$ be an integer such that $4 \leq d \leq [100C_0]$, where $C_0 = 400(10 + \log 2)$. There exists a function $f_d$ and an algorithm $A'_d$ with the following specification. The input of the algorithm is a graph $G$ of girth at least five embedded in the sphere, a set $F$ of at most 2 faces of $G$ such that every cycle in $G$ of length at most $d$ is $F$-contractible and a set $S$ of vertices incident with the faces of $F$. The algorithm outputs a set $C(S, G)$ in time $O(f_d(|S|)|V(G)|)$.

Proof. As in Lemma 18 we proceed by induction on $d$, starting from the largest value (we use Corollary 13 as the basic case). Hence, suppose that $d < [100C_0]$ and as the induction hypothesis assume that the algorithm $A'_{d+1}$ exists. We find a maximal sequence $Q_1, Q_2, \ldots, Q_n$ of non-crossing $(d + 1)$-cycles separating the faces $f_1$ and $f_2$ of $F$ in the same way as in Lemma 18 and let $Q_0 = S \cap V(f_1)$ and $Q_{n+1} = S \cap V(f_2)$. For $0 \leq i \leq n$, we then apply $A'_{d+1}$ to the subgraph $H_i$ of $G$ between $Q_i$ and $Q_{i+1}$, obtaining $C(V(Q_i) \cup (Q_{i+1}), H_i)$. Let $G_k = \bigcup_{0 \leq i \leq k} H_i$. We now determine $C(V(Q_0) \cup V(Q_{k+1}), G_k)$ for $0 \leq k \leq n$ by induction on $k$. Since $G_0 = H_0$, we can assume that $k \geq 1$ and that $C_1 = C(V(Q_0) \cup V(Q_k), G_{k-1})$ is already known. Let $C_2 = C(V(Q_k) \cup V(Q_{k+1}), H_k)$. By combining $C_1$ with $C_2$, we determine $C_3 = C(V(Q_0) \cup V(Q_k) \cup V(Q_{k+1}), G_k)$—a pair $(L_0, \Psi)$ belongs to $C_3$ if and only if there exist $(L'_0, \Psi') \in C_1$ and $(L''_0, \Psi'') \in C_2$ such that $L'_0$ and $L''_0$ are restrictions of $L_0$ to the respective sets and $\Psi$ consists of colorings $\psi$ such that $\Psi'$ and $\Psi''$ contain the restriction of $\psi$ to the respective sets. The set $C(V(Q_0) \cup V(Q_{k+1}), G_k)$ is obtained by including all pairs $(L_0, \Psi)$ such that there exists $(L'_0, \Psi') \in C_3$ such that $L'_0$ extends $L_0$ and $\Psi$ consists of $L_0$-colorings $\psi$ such that an extension of $\psi$ is contained in $\Psi'$. In the end, we have $C(S, G) = C(V(Q_0) \cup V(Q_{n+1}), G_n)$.

Now, to decide whether a graph $G$ of girth at least five embedded in a fixed surface of genus $g$ is 3-choosable, we cut $G$ along short cycles as in the proof of Theorem 10. This way, we express $G$ as an edge-disjoint union of graphs $G_1, \ldots, G_k$, where $k$ is bounded by a function of $g$ and the set $S$ of vertices contained in at least two of these graphs has size bounded by a function of $g$. Furthermore, for $1 \leq i \leq k$, the graph $G_i$ with the set $S \cap V(G_i)$ satisfies either assumptions of Corollary 13 or of Lemma 19 with $d = 4$. Consequently, we can determine $C(S, G)$ in linear time. The graph $G$ is 3-choosable if and only if $\Psi \neq \emptyset$ for all $(L_0, \Psi) \in C(S, G)$.

5 Concluding remarks

The degree of the polynomial bounding the time complexity of our algorithm depends mostly on the bound on the distance given by Theorem 12. The algorithm is easy to implement, especially in the planar case. While the bound in Theorem 12 is rather high, it seems likely that it is possible to reduce it
significantly. This could make the algorithm of Corollary 17 practical, at least for the special case of a planar graph with a connected precolored subgraph.

Nevertheless, an interesting open question is whether we can eliminate the dependence of the exponent on the genus entirely, and thus obtain an FPT algorithm for the problem of extension of a precoloring of a connected subgraph of a graph embedded in a fixed surface. The same remark holds for the case of finding a coloring of a graph without precolored subgraph, where we currently have to introduce a precolored subgraph whose size may be up to $\Omega(\sqrt{|V(G)|})$.

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