The fate of 1D spin-charge separation away from Fermi points

Thomas L. Schmidt,1 Adilet Imambekov,2 and Leonid I. Glazman1

1Department of Physics, Yale University, 217 Prospect Street, New Haven, Connecticut 06520, USA
2Department of Physics and Astronomy, Rice University, Houston, Texas 77005, USA

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We consider the dynamic response functions of interacting one dimensional spin-1/2 fermions at arbitrary momenta. We build a nonperturbative zero-temperature theory of the threshold singularities using mobile impurity Hamiltonians. The interaction induced low-energy spin-charge separation and power-law threshold singularities survive away from Fermi points. We express the threshold exponents in terms of the spinon spectrum.

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The low-energy excitations of interacting spin-1/2 fermions confined to one dimension (1D) is well represented by two collective bosonic modes. These modes are the quantized waves of spin and charge densities. Their spectra are linear; the corresponding velocities, $v_s$ and $v_c$, differ from each other. A microscopic consideration of the repulsive interaction between spinful fermions leads to $v_c > v_s$.

The spectra of the collective modes can be probed in a momentum-resolved tunneling or in an ARPES experiment. In these methods, a spin-1/2 fermion with a given momentum tunnels into or out of the studied system. The tunneling inevitably perturbs each of the two collective modes. For example, in the case of a low-energy particle ($k_F \gg k - k_F > 0$), the small difference $k - k_F$ is shared between the excitations of the two modes. The Luttinger liquid (LL) theory predicts that at given $k$ the tunneling probability is singular at energies $v_s(k-k_F)$ and $v_c(k-k_F)$, corresponding to the entire momentum $k - k_F$ given to the “spinon” or “holon” belonging to the spin and charge mode, respectively. The exponents of the two power-law singularities depend on a single number, the LL parameter $K_c$ for the charge mode. The two sharp peaks in the momentum-resolved tunneling probability at energies associated with excitation of the two modes, $\omega = v_{s,c} \cdot (k - k_F)$, are the hallmark of the spin-charge separation in the LL.

The momentum-resolved tunneling rate is proportional to the fermionic spectral function $A(k, \omega) = \frac{1}{\pi} \text{Re} \int dt dx e^{i \omega t - i k x} \theta(t) \langle \psi(x, t), \psi^\dagger(0, 0) \rangle$. Recently, considerable progress was achieved in the analytical theory of dynamic responses of a 1D system away from the Fermi points for spinless fermions. The developed methods map the 1D dynamic response problem near the edge of support onto the “mobile quantum impurity” effective Hamiltonian. For spinless fermions in the weak-interaction limit, one may consider the generic spectrum of free fermions exactly, while treating their interaction perturbatively. For example, at $|k| < k_F$ the threshold coincides with the spectrum of a hole, which can be thought of as a mobile quantum impurity. Because of the interactions, it “shakes up” the fermions in the vicinity of Fermi points, leading to the orthogonality catastrophe and to the power-law behavior of $A(k, \omega)$ at the threshold $\omega_{s,c}$. The perturbation theory allows one to identify the quantum numbers of the impurity and to match the phenomenological theory of threshold exponents valid at any interaction strength with the weak-interaction limit.

In this Letter we build a nonperturbative zero-temperature theory of threshold singularities of dynamic responses of 1D spin-1/2 fermions at arbitrary $k$, shedding light on the fate of the spin-charge separation away from Fermi points. The obvious difficulty of the problem lies in the appearance of distinctly different spin and charge modes at any interaction strength. For weak interactions, these modes are degenerate, which renders perturbation theory inapplicable. We find that interaction induced spin-charge separation survives away from the Fermi points, and manifests itself in the choice of quantum numbers of the mobile impurity: for repulsive interactions, it only carries spin but no charge. We fix the parameters of the effective quantum impurity Hamiltonian using $SU(2)$ and Galilean invariance, and find the exponents of threshold singularities of various dynamic responses [density and spin structure factors as well as $A(k, \omega)$].

Away from $k = k_F$, the spectrum of the spinon mode $\epsilon_s(k)$ departs from the linear one, $v_s(k-k_F)$, and becomes a periodic function of $k$ with period $2k_F$. The threshold for $A(k, \omega)$ is located at $|\omega_k| = \epsilon_s(k - 2nk_F)$, with integer $n$ such that $k - 2nk_F \leq k_F$, see Fig. The spin-charge separation is preserved near the threshold at arbitrary $k$ in the following sense: if the energy $\omega$ of, say, the extracted fermion approaches the threshold $\omega_k$, then the momentum of a created spinon is approaching $k - k_F$. The rest of the energy, $\sim |\omega - \omega_k| \rightarrow 0$, is given to a holon; it inevitably resides near a Fermi point and may be described as a conventional linear LL. We express the exponents of various threshold singularities in terms of the derivatives of $\epsilon_s(k)$ with respect to $k$ and $\rho$, the density of the liquid [see Eq. and Table]. The obtained exponents are valid at arbitrary $k$, including the Fermi points.
Near the Fermi points, the exponents for $A(k, \omega)$ approach the universal values which depend only on $K_\sigma$. In the main region ($n = 0$), the exponent for $A(k, \omega < 0)$ coincides with the predictions of the linear LL (unlike in the spinless case [3]), while density and spin structure factor exponents approach 1/2.

Let us first introduce the refermionization of the linear LL Hamiltonian. Its charge ($c$) and spin ($s$) parts separate from each other, $H_0 = H_c + H_s$, and in terms of boson variables, these parts have the conventional form [1],

$$H_\nu = \frac{v_\nu}{2\pi} \int dx \left[ K_\nu (\partial_x \theta_\nu)^2 + \frac{1}{K_\nu} (\partial_x \phi_\nu)^2 \right]$$

for $\nu = c, s$. The canonically conjugate fields $\partial_x \theta_\nu$ and $\phi_\nu$ are the momentum and displacement operators of bosonic charge and spin density waves and satisfy $[\phi_\nu(x), \partial_y \theta_\mu(y)] = i\pi \delta_{\mu,\nu} \delta(x - y)$. The effects of the interaction are contained in the constants $K_\nu$, but most importantly interaction yields a difference between the velocities of charge and spin modes, thus removing the degeneracy characteristic of the free-fermion system. For repulsive interactions at the $SU(2)$ symmetric point, we can assume rather generally $K_\sigma = 1$ [1, 2]. Each of the Hamiltonians ($c, s$) may be refermionized in terms of free fermionic quasiparticles, the same way as it was suggested earlier [13, 14] for spinless particles. One obtains a free linear Hamiltonian in terms of left- and right-moving fermionic spin and charge quasiparticles $\tilde{\psi}_\alpha(x)$ ($\alpha = R, L = \pm$), spinons and holons, respectively [14]:

$$H_0 = -i \sum_{\nu=c,s} v_\nu \sum_{\alpha=R,L} \alpha \int dx : \tilde{\psi}_\alpha^\dagger(x) \nabla \tilde{\psi}_{\alpha\nu}(x) : .$$

The operators $\tilde{\psi}_{\alpha\nu}(x)$ have fermionic commutation relations, and their vacuum state has unit occupancy for negative (positive) momenta for $\alpha = R(L)$. The left- and right-moving components of the original fermions, defined by $\psi_\nu(x) = e^{-ikx} \psi_{\nu R} + e^{ikx} \psi_{\nu L}$, can be expressed in terms of refermionized quasiparticle operators as ($\sigma = \uparrow, \downarrow = \pm$)

$$\psi_{\alpha\sigma}(x) \propto \tilde{\psi}_{\alpha\uparrow}(x) F_{\alpha\uparrow}(x) \tilde{\psi}_{\alpha\downarrow}(x) F_{\alpha\downarrow}(x).$$

Here notations are such that $\tilde{\psi}_{\alpha\uparrow}(x)$ is the creation (annihilation) operator of a spinon, and the wave function of a spinon is $| \tilde{\psi}_{\alpha\uparrow}(x) \rangle$. Hence the interaction $A(k, \omega)$ is recovered.

The difference stems from the fact that $A(k, \omega)$ is the response functions at the spinon spectrum even for $|k - k_F| \ll k_F$. The difference stems from the fact that the leading-order band curvature of the spinons is cubic [15] as the $SU(2)$ symmetry enforces particle-hole symmetry for the spinon mode. If one attempts now to treat the leading interactions between spinons perturbatively in the spirit of Ref. [3], one finds that these interactions lead to $O(1)$ changes of the exponents in the response functions, since the difference in the velocities of spinons is proportional to $|k - k_F|^2$, unlike $|k - k_F|$ for holons.

Nevertheless, it is still possible to describe the spectral function at the spinon mode using mobile impurity models, as has been established in a number of articles [3, 12]. We shall first apply the approach to calculate $A(k, \omega)$ for $|k - k_F| \ll k_F, \omega < 0$ in the vicinity of the spinon mode $\omega \approx \epsilon_\sigma(k)$. According to Eq. (3), for energies close to $\epsilon_\sigma(k)$, the configurations of lowest energy will contain a spinon hole at momentum $k_d = k - k_F < 0$ and a holon at $k_p$. Therefore, we can project the Hamiltonian onto a band structure which consists of a “deep” right-moving spinon hole $\tilde{d}$ at momentum $k_d$ as well as spinon and holon states, $\tilde{s}$ and $\tilde{d}$, respectively, at the Fermi points. This leads to the Hamiltonian $H_0 + H_d + H_{\text{int}}$, where $H_0$ are given by

$$\frac{\delta_{\pm \nu}}{2\pi} = \left( \frac{1}{2} \pm \frac{1}{2} \right) \mp \frac{1}{8K_\nu} - \frac{K_\nu}{8}. \quad (4)$$

$K_\sigma = 1$ leads to $\delta_{-s} = 0$ and $\delta_{+s}/(2\pi) = 1 - 1/\sqrt{2}$. Within the linear spectrum approximation of the LL theory, the dynamics of the string operators is linear, and Eqs. (2)-(4) lead to the conventional LL results.

We first discuss the deviations from the LL results for the exponents of the spectral function at the charge mode in the vicinity of the Fermi point $+k_F$. Similar to the spinless case [9], one can describe the exponents by taking into account only the quadratic nonlinearity in the spectrum of charge quasiparticles. Interactions between charge quasiparticles lead only to small corrections to the exponent at the charge mode for $|k - k_F| \ll k_F$, and we obtain a singularity at the holon mass shell $A(k, \omega) \propto (\omega - \epsilon_\sigma(k))^{-\mu_c}$, where

$$\mu_c = 1 - \left( \frac{\delta_{-c}}{2\pi} \right)^2 - \left( \frac{\delta_{+s}}{2\pi} - 1 \right)^2 - \left( \frac{\delta_{+c}}{2\pi} \right)^2 . \quad (5)$$

The nonlinearity in $\epsilon_\sigma(k)$ changes the exponents compared to LL theory in an energy window of width $\sim (k - k_F)^2$ around $\epsilon_\sigma(k)$. Even at $K_\sigma \to 1$, the exponent $\mu_c \to \sqrt{2} - 1$, different from the LL prediction 1/2. Beyond this energy window, the usual LL behavior is recovered.

For the spin correlations, the density $\tilde{d}$ has to be raised to power $\delta_{+s}/(2\pi)$, which is $1 - 1/\sqrt{2}$.
is given by Eq. (2), while the other terms read

\[
H_d = \int dx \, \hat{d}(x)\left[\epsilon_s(k) - i\hbar v_d \nabla\right]\hat{d}(x),
\]

\[
H_{\text{int}} = \int dx \sum_{\alpha\nu} \hat{V}_{\alpha\nu}(k)\hat{\rho}_{\alpha\nu}(x)\hat{d}(x)\hat{\bar{d}}(x).
\]

The subbands were linearized around \(k_d\) and the Fermi points, and we used \(\hat{\psi}_{Rs} \propto \hat{\psi}_{Rs} + e^{ikd_x} \hat{d}\) and \(\hat{\psi}_{\alpha\nu} \propto \hat{\psi}_{s\nu}\) in all other cases. The velocity of the hole \(\hat{d}\) is given by \(v_d = \delta\epsilon_s(k)/\hbar k\) and \(\hat{V}_{\alpha\nu}(k)\) reflect the interaction of the hole with the modes near the Fermi points [16].

The interaction term \(H_{\text{int}}\) can be removed using a unitary transformation \([8–12]\). This will lead to additional phase shifts \(\Delta\delta_{\alpha\nu} = \hat{V}_{\alpha\nu}/(v_d - \alpha v_d)\) which determine the edge exponents. Except for \(\Delta\delta_{\epsilon_s}\), these additional phase shifts are small since the corresponding \(\hat{V}_{\alpha\nu}\) vanish for \(k \to k_F\), while the denominator remains finite. The remaining phase shift \(\Delta\delta_{\epsilon_s}\) can be fixed using the \(SU(2)\) symmetry [10]. In particular, this symmetry requires identical exponents of the spin correlation functions \(S^{-+}(k,\omega)\) and \(S^{++}(k,\omega)\), where e.g. \(S^{zz}(k,\omega) = \int dt dx e^{i\omega t - ikx}(\hat{S}^z(x,t)\hat{S}^z(0,0))\). We can calculate the exponents of these functions at the spinon mass shell for general phase shifts \(\Delta\delta_{\epsilon_s} = \delta_{\epsilon_s} + \Delta\delta_{\epsilon_s}\).

The leading exponents for \(S^{-+}(k,\omega)\) and \(S^{++}(k,\omega)\) read

\[
1 - \left[1 + \sqrt{2} \pm \delta_{\epsilon_s}(2\pi)\right]^2,
\]

respectively. Hence, the \(SU(2)\) symmetry leads to \(\delta_{\epsilon_s} = 0\) and rules out the interaction of the impurity with the low-energy spinons, and the LL result \(\mu_{LL}\) (see Fig. 1) for the exponent of the spectral function remains valid for \(|k - k_F| \ll k_F\).

Even beyond the universal regime \(|k - k_F| \ll k_F\), the exponents of the spectral function near its edge of support survive, and may be determined using mobile impurity Hamiltonians. Above analysis in the limit \(|k - k_F| \ll k_F\) indicates that in the vicinity of the Fermi points the impurity has the quantum numbers of a spinon, and the impurity Hamiltonian is given by Eq. (10). The continuous evolution of edge exponents implies that the same Hamiltonian should describe the edge exponents even significantly away from Fermi points. As a consequence of such a spin-charge separation beyond low energies, for Galilean-invariant systems we can express all exponents in terms of the edge position only, in the spinless case [10]. For this purpose, we rewrite Eq. (6) as

\[
H_{\text{int}} = \int dx \left[V_{R}\nabla\Theta_s - V_L\nabla\Theta_c + \frac{\phi_s}{2\pi} - \frac{\phi_c}{2\pi}\right]\hat{d}(x)\hat{\bar{d}}(x),
\]

since we saw previously that the coupling constant to the spin sector vanishes. The argument was based on \(SU(2)\)-symmetry and is valid beyond the low-energy regime.

The interaction term is removed [10] by a unitary transformation \(U^\dagger(H_c + H_d + H_{\text{int}})U\), where

\[
U = \exp\left[i \int dx \left[\frac{\Delta\delta_{\epsilon_s}}{2\pi}\left(\hat{\epsilon}_c\sqrt{K_c} - \hat{\epsilon}_s\sqrt{K_s}\right) - \frac{\Delta\delta_{\epsilon_s}}{2\pi}\left(\hat{\epsilon}_s\sqrt{K_c} - \hat{\epsilon}_c\sqrt{K_s}\right)\right]\right.
\]

\[
+ \left. \frac{\Delta\delta_{\epsilon_s}}{2\pi}\left(\hat{\epsilon}_s\sqrt{K_c} - \hat{\epsilon}_c\sqrt{K_s}\right) + \frac{\Delta\delta_{\epsilon_s}}{2\pi}\left(\hat{\epsilon}_c\sqrt{K_c} - \hat{\epsilon}_s\sqrt{K_s}\right)\right].
\]

FIG. 1: (Color online) Structure of the spectral function \(A(k,\omega)\) in the \((k,\omega)\)-plane (A) and along a cut for fixed \(0 < k < k_F\) (B,C). (A) Shaded areas indicate the regions where \(A(k,\omega)\) is nonzero. (B) Luttinger liquid (LL) results: in the vicinity of the spinon mass shell, \(A(k,\omega) \propto (\omega - v_0k)^{-\mu_{LL}}\), where \(\mu_{LL} = \frac{1}{2} - \frac{1}{4}(K_c + K_s - 2)\). (C) Schematic behavior of the spectral function beyond the LL approximation away from the Fermi points: the exponent \(\mu_{0,-}\) is different from \(\mu_{LL}\), the charge mode is smeared out, and for \(\omega > 0\) the region of finite support starts from \(-\epsilon_s(k)\) with an exponent \(\mu_{0,+}\) instead of \(-\epsilon_c(k)\).

\[
\theta_c\sqrt{K_c}\hat{d}(x)\hat{\bar{d}}(x)\right] \right. \left. \right\} \text{with phases } \Delta\delta_{\epsilon_s} \text{ defined by }
\]

\[
(V_L \mp V_R)K_c^{1/2} = -\Delta\delta_{\epsilon_s}(v_d + v_c) \pm \Delta\delta_{\epsilon_s}(v_d - v_c).
\]

\[
\text{The fermionic operator can be written as } \hat{\psi}_\uparrow = \hat{\bar{d}}e^{i(\theta_c - \phi_c)/\sqrt{2}} \text{ (with the second factor coming from a holon at } k \to k_F), \text{ and then its correlations can be evaluated using the operator } U \text{ similar to the spinless case.}
\]

\[
\text{The phases } \Delta\delta_{\epsilon_s} \text{ fix the exponents of the spectral function as well as of the dynamic density and spin structure factors} [3–12].
\]

In order to relate \(V_{L,R}\) to the spectrum \(\epsilon_s(k)\), we first consider the shift in energy due to a change in density. A uniform variation of the density by \(\delta\rho\) corresponding to a finite expectation value \(\langle \nabla\phi_c \rangle = -\hbar \delta\rho/\sqrt{2}\), and leads to the shift in the single-particle energy

\[
\delta\epsilon_s(k) = \left[\frac{\partial\epsilon_s(k)}{\partial\rho} + \frac{\partial\mu}{\partial\rho}\right] \delta\rho,
\]

where \(\mu\) denotes the chemical potential. We now need to calculate the same change in energy using Eq. (6). Unlike the spinless case [10], the spinon momentum changes

\[
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\]

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\]

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For the phase shifts, we thus obtain the result
\[ \varepsilon \] leads to a spinon hole at \( k \), as well as the shift in energy of the spinon \( \delta \varepsilon_s(k) \) calculated in Eq. (11). Moreover, \( m_\pm = (n+1/2 \pm 1/2) \) mod 2. The exponents for the spin structure factor \( S^{zz}(k, \omega) \) coincide with the exponents for \( S(k, \omega) \).

| \( A(k, \omega \geq 0) \) | \( (2n-1)k_F < k < (2n+1)k_F \) | \( A_{n, \pm} \) | \( 1 - \frac{1}{2} \left( \frac{(2n+1)K_c}{\sqrt{2}} + \frac{\delta^k + \delta^s}{2\pi} \right)^2 - \frac{1}{2} \left( \frac{1}{\sqrt{2K_c}} - \frac{\delta^k - \delta^s}{2\pi} \right)^2 - m_\pm^2 \) |
|---|---|---|---|
| \( S(k, \omega) \) | \( 2nk_F < k < (2n+1)k_F \) | \( \mu_{n, DFF}^\omega \) | \( 1 - \frac{1}{2} \left( \frac{2nK_c}{\sqrt{2}} + \frac{\delta^k + \delta^s}{2\pi} \right)^2 - \frac{1}{2} \left( \frac{1}{\sqrt{2K_c}} - \frac{\delta^k - \delta^s}{2\pi} \right)^2 \) |

**Table I:** Exponents for the spectral function \( A(k, \omega) \) (see Fig. 1 for notations) and the dynamic density structure factor \( S(k, \omega) \) at the edge of support. The exponents are determined in terms of the phase shifts \( \delta^k = \Delta \delta_{(k-2nk_F)} \) and \( \delta^s = \Delta \delta_{(2n+1)k_F-k} \) calculated in Eq. (11). Moreover, \( m_\pm = (n+1/2 \pm 1/2) \) mod 2. The exponents for the spin structure factor \( S^{zz}(k, \omega) \) coincide with the exponents for \( S(k, \omega) \).

under variation of the density, since the total momentum is fixed while the holon momentum changes following the shift of the Fermi point. Calculating the shift in the energy of the spinon \( d \) as well as the shift in energy of the holon at the Fermi point, and comparing with Eq. (17), one finds

\[ V_R + V_L = \frac{V_\varepsilon(k)}{2\sqrt{2}} = \frac{\partial \varepsilon_s(k)}{\partial \rho} + \frac{\pi}{2} \frac{\partial \varepsilon_s(k)}{\partial k} \] (8)

The second relation can be derived using Galilean invariance, which predicts \[ \frac{\rho}{14} \] that a uniform change in velocity \( u \) should lead to a change in energy of

\[ \delta \varepsilon_s(k) = mu \left[ k - \frac{\partial \varepsilon_s(k)}{\partial k} \right], \] (9)

where \( m \) is the bare mass. On the other hand, this change in velocity \( u \) will lead to a finite expectation value, \( \langle \nabla \theta \varepsilon \rangle = \sqrt{2mu} \). Because of the shift of the Fermi point, this leads to an energy shift of a holon at the right Fermi point \( K_c = v_F mu \), and a spinon energy shift \( -mu v_d \) due to the change of the spinon momentum. In addition, the interaction Hamiltonian yields a shift \( \sqrt{2mu}(V_L - V_R)/(2\pi) \). Combining these terms leads to

\[ V_L - V_R = \frac{k - k_F}{m} \] (10)

Equations (8) and (10) allow us to express \( V_{L,R} \) in terms of the derivatives of the single-particle spectrum \( \varepsilon_s(k) \).

For the phase shifts, we thus obtain the result

\[ \frac{\Delta \delta_{\pm, \varepsilon}(k)}{2\pi} = \frac{k - k_F}{mK_c} \pm \sqrt{K_c} \left( \frac{\partial \varepsilon_s(k)}{\partial \rho} + \frac{\partial \varepsilon_s(k)}{\partial k} \right) \] (11)

The predictions (11) and the absence of the coupling of the spinon impurity to the low-energy spinons can be explicitly checked for the case of the integrable Yang-Gaudin model based on its finite size spectrum [17]. Moreover, for \( k \to k_F \), Eqs. (8) and (10) yield \( V_L = V_R = 0 \) which reflects the absence of interactions between holons and spinons within the linear LL theory.

Finally, let us present the exponents of the spectral function in the regions \( (2n-1)k_F < k < (2n+1)k_F \) for integer \( n \). For \( \omega < 0 \), in the vicinity of \( \varepsilon_s(k) \), the symmetry of the edge position in \( k \) leads to a spinon hole at momentum \( k = (2n+1)k_F \) < 0, a holon at \( k_F \) as well as additional excitations which absorb the remaining momentum \( 2nk_F \). In terms of the original fermions, these “umklapp” excitations contain particles or holes at the Fermi points, and are thus characterized by four parameters. Fixing the total momentum at \( 2nk_F \) and requiring zero total charge and spin leaves one parameter \( m_- \), where the umklapp contribution to \( \psi_L \) can be represented as \( (\psi^\dagger_{R,t} \psi^\dagger_{L,t} - (n+m_-)(\psi^\dagger_{L,t} \psi^\dagger_{R,t})^{n-m_-}/2, \) and thus \( m_- \) has to satisfy the selection rule \( m_- = n \) (mod 2). The exponent of the spectral function is now determined by the phase shifts (11) taken at momentum \( k = 2nk_F \). For \( \omega > 0 \), one can also describe the exponents using the impurity Hamiltonian with the same parameters, but the states which determine the exponents are different. Following the same line of arguments as previously, one finds exponents which are formally identical to the \( \omega < 0 \) case, but where \( m_+ \) now has to satisfy a different selection rule, \( m_+ = n \) (mod 2). The leading exponents stem from \( m \) with smallest absolute value allowed by the selection rules, and the results are shown in Table I.

The exponents of the spin structure factor \( S^{zz}(k, \omega) \) and the dynamic structure factor \( S(k, \omega) = \int dt \rho_{L,R} e^{i(\omega t - kx)} \langle \rho(x, t) \rho(0, 0) \rangle \) can also be calculated similarly. It turns out that the exponents for \( S(k, \omega) \) coincide with those for \( S^{zz}(k, \omega) \), and they are shown in Table I.

In conclusion, we have calculated the exponents of dynamical correlation functions of 1D spinful interacting fermionic systems beyond the approximation of a linear spectrum, shedding light on the fate of the spin-charge separation away from Fermi points. In the low-energy sector near the Fermi points, we found universal phase shifts which control the exponents and depend only on the LL parameter \( K_c \). Beyond the low-energy regime, we were able to establish phenomenological relations between phase shifts and properties of the spinon spectrum.

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Note Added. – Recently, the preprint [18] appeared, where the exponents for the density and spin structure factors for \( L < k_F \) were derived.
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