Coding and Decoding for the Dynamic Decode and Forward Relay Protocol

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Abstract

We study the Dynamic Decode and Forward (DDF) protocol for a single half-duplex relay, single-antenna channel with quasi-static fading. The DDF protocol is well-known and has been analyzed in terms of the Diversity-Multiplexing Tradeoff (DMT) in the infinite block length limit. We characterize the finite block length DMT and give new explicit code constructions. The finite block length analysis illuminates a few key aspects that have been neglected in the previous literature: 1) we show that one dominating cause of degradation with respect to the infinite block length regime is the event of decoding error at the relay; 2) we explicitly take into account the fact that the destination does not generally know a priori the relay decision time at which the relay switches from listening to transmit mode. Both the above problems can be tackled by a careful design of the decoding algorithm. In particular, we introduce a decision rejection criterion at the relay based on Forney’s decision rule (a variant of the Neyman-Pearson rule), such that the relay triggers transmission only when its decision is reliable. Also, we show that a receiver based on the Generalized Likelihood Ratio Test rule that jointly decodes the relay decision time and the information message achieves the optimal DMT. Our results show that no cyclic redundancy check (CRC) for error detection or additional protocol overhead to communicate the decision time are needed for DDF. Finally, we investigate the use of minimum mean squared error generalized decision feedback equalizer (MMSE-GDFE) lattice decoding at both the relay and the destination, and show that it provides near optimal performance at moderate complexity.

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I. INTRODUCTION

Employing multiple antennas at the transmitter and the receiver of wireless communications is known to provide significant benefits in terms of both throughput (multiplexing gain) and reliability (diversity gain) (see [13] and references therein). When physical constraints limit the number of antennas that can be installed on a single wireless device (e.g., small sensors in sensor networks), the usage of cooperative wireless relay protocols is a promising alternative strategy. In these protocols, two or more terminals cooperate in order to mimic a super-user with multiple antennas.

The relay channel was introduced by van der Meulen [2] and was studied in detail by Cover and El Gamal [3], who characterized the capacity for the discrete memoryless as well as for the Gaussian degraded cases. The relay channel with fading was examined by Sendonaris et al., [4], where an achievable rate region was provided. In the case of slow fading, the outage behavior of half-duplex wireless relay channels was studied by Laneman et al., [5], and simple cooperative diversity protocols for signalling across these channels (such as amplify and forward and decode and forward) were introduced. In [15], Azarian et al. used the diversity-multiplexing tradeoff (DMT) formulation of [13] to study the outage behavior of slowly-fading relay channels in the high-SNR regime, and also introduced new classes of protocols such as the non-orthogonal amplify and forward (NAF) and the dynamic decode and forward (DDF). An improved DDF protocol based on code superposition was later proposed in [30]. The DDF protocol for the single relay case was subsequently studied in [16], where simplified variants of the protocol were introduced and some code design issues were addressed. Code design for the DDF protocol is also addressed in the recent contribution [29].

The present paper also focuses on the DDF protocol for the half-duplex, single relay single-antenna case. With respect to [16] and [29], we analyze explicitly the achievable DMT of practical codes with finite block length and propose a simple DMT optimal code construction that makes use of approximately universal codes for the parallel channel and of the Alamouti code. Approximately universal codes for the parallel channel may be obtained either from using a QAM base alphabet and a suitable unitary precoding matrix (lattice codes) or from permutation codes derived from universally decodable matrices (UDM) [21], [22]. We treat both cases and give construction examples and comparisons. Remarkably, our codes perform very close to the outage probability and have generally lower decoding complexity than those previously proposed.

Furthermore, we discuss two often neglected issues: 1) the effect of decoding errors at the relay, and how to mitigate it; 2) the fact that the destination does not generally know a priori the relay decision.
time. In order to tackle 1), we introduce a decision rejection criterion at the relay, such that the relay triggers transmission only when its decision is reliable. We show that the Forney’s decision rule (a variant of Neyman-Pearson rule) yields almost optimal performance with practical finite length codes, while previously proposed options suffer from significant degradation. In order to tackle 2), we treat the channel “seen at destination” as a compound channel, where each compound member corresponds to a different relay decision time. We prove that a receiver based on the Generalized Likelihood Ratio Test (GLRT) rule, that jointly decodes the relay decision time and the information message, achieves the optimal DMT. We also show that a simpler scheme that performs separate detection of the relay decision time, by ignoring the structure of the coded signal and treating it as random, is generally suboptimal and it becomes optimal only in the limit of infinite block length. As an aside, our results show that no side information channel or additional protocol overhead is needed in order to inform the destination about the relay decision time. This may yield to much simplified actual protocol design for the DDF scheme, at the cost of an augmented decoder at the destination.

With the lattice codes advocated in this paper, the decoder at the relay has to solve a closest lattice point problem with a rank deficient lattice matrix. It is well-known that standard sphere decoding [6], [7] yields exponential complexity in this case. In order to address this problem (again, often neglected in the current literature) we advocate the use of the minimum mean squared error generalized decision feedback equalizer (MMSE-GDFE) lattice decoder of [32], [34]. Via simulation of the performance of our explicitly constructed codes, we demonstrate that this lattice decoder is able to provide near optimal performance at moderate complexity.

In Section II, we introduce the system model we work with and review relevant previous results. Section III presents the main result of the present paper, a characterization of the DMT of the DDF protocol for finite block length. Explicit code constructions that achieve this DMT are provided in Section IV and methods to enable error detection at the relay and low complexity decoding of these codes are also dealt with.

II. PROBLEM DEFINITION AND BACKGROUND

A. System model

We consider the single relay channel shown in Fig. I where S, R and D denote the source, relay and destination, and $h$, $g_1$ and $g_2$ denote the fading coefficients between the source-relay, source-destination and relay-destination terminals, respectively.
The channel fading coefficients are i.i.d. \( CN(0,1) \) random variables, corresponding to i.i.d. Rayleigh fading. Following the standard outage setting [5], [15], [13], we assume that the channel coherence time is considerably larger than the allowed decoding delay. Invoking a time-scale decomposition argument (see for example [18]) this setting is modeled by the so-called quasi-static fading channel, where the channel coefficients are random but remain constant over the whole duration of a codeword, although the latter can be very large. We consider slotted transmission where a source codeword spans \( M \) slots of length \( T \) symbols each, resulting in a total block length of \( MT \).

The relay operates in half-duplex mode. In decode and forward protocols, the block of length \( MT \) symbols is split into two phases. In the first phase the relay is in listening mode and receives the signal from the source. At a certain instant, referred to as the decision time in the following, the relay tries to decode the source information message. In the second phase, from the decision time to the end of the block, the relay switches to transmit mode and sends symbols to help the destination decode the source message. The DDF protocol is characterized by the fact that the decision time is not fixed a priori. On the contrary, the relay decides when to decode and switch to transmit mode depending on the channel coefficient \( h \) and the received signal. Therefore, the decision time is a random variable \( \mathcal{M} \). Without loss of generality, we restrict the decision time to coincide with the end of a slot\(^1 \) i.e., \( \mathcal{M} \) takes on values in the set \( \{1, 2, \ldots, M\} \), where \( \mathcal{M} = M \) corresponds to the case where the relay does not help the destination. During phase 1 (listening phase) the signal received by the relay is

\[
y_{r,k} = h x_{s,k} + v_k, \quad k = 1, 2, \ldots, MT, \tag{1}
\]

\(^1\) Notice that \( T \) is a design parameter. Letting \( T = 1 \) provides an unrestricted decision time. In this way, there is no loss of generality in this assumption.

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Fig. 1. The single-antenna single relay fading channel.
and the signal received by the destination is
\[ y_k = g_1 x_{s,k} + w_k, \ k = 1, 2, \ldots, MT. \] (2)

During phase 2 (relay transmit phase), the signal received by the destination is
\[ y_k = g_1 x_{s,k} + g_2 x_{r,k} + w_k, \ k = MT + 1, MT + 2, \ldots, MT. \] (3)

Here, \( x_s = [x_{s,1} \cdots x_{s,MT}]^T \) denotes the source codeword, drawn from a code \( X_s \subset \mathbb{C}^{MT} \) of rate \( R \) bits per symbol. Without loss of generality, we may assume that the symbols \( x_{r,k} \) transmitted by the relay are from an auxiliary code \( X_r \subset \mathbb{C}^{MT} \) with rate \( R \) and block length \( MT \), but only the last \((M - M)T\) symbols of a codeword are effectively transmitted in phase 2, while in phase 1 the relay transmitter is idle because of the half-duplex constraint.

The noise at the relay and destination, denoted by \( v_k \sim \mathcal{CN}(0, \sigma_v^2) \) and \( w_k \sim \mathcal{CN}(0, \sigma_w^2) \), form two white mutually independent sequences. We impose the same per-symbol average power constraint for both the source and the relay, given by
\[ \mathbb{E} [ |x_{s,k}|^2], \ \mathbb{E} [ |x_{r,k}|^2] \leq E, \]
where \( E \) denotes the symbol energy, and define the SNRs of the S-D and the S-R links to be \( \rho = E/\sigma_w^2 \) and \( \rho' = E/\sigma_v^2 \), respectively.

For later use, we introduce the following notation: let \( y_{r,i}^j, y_{r,i}^{j,i}, x_{s,j}^i, \) and \( x_{r,j}^i \), each \( \in \mathbb{C}^{(j-i)T} \), denote respectively the received signals at the destination and at the relay from symbol time \( iT + 1 \) to \( jT \), the source transmit signal from time \( iT + 1 \) to \( jT \) and the relay transmit signal from time \( iT + 1 \) to \( jT \), where the latter is assumed to be zero for all times \( k \leq MT \). The quantities \( w_{r,i}^j \) and \( v_{r,i}^j \) are defined similarly.

B. Diversity-Multiplexing Tradeoff

A compact and convenient characterization of the tradeoff between rate and reliability of quasi-static fading channels in the high-SNR regime is provided by the DMT introduced in [13]. In this framework, rate and reliability are quantified in terms of the diversity gain \( d \) and spatial multiplexing gain \( r \). A family of coding systems, each of which operates at SNR \( \rho \) with rate \( R(\rho) \) and error probability \( P_e(\rho) \), achieves a point \((r, d)\) on the DMT plane if
\[ \lim_{\rho \to \infty} \frac{R(\rho)}{\log \rho} = r, \ \lim_{\rho \to \infty} \frac{\log P_e(\rho)}{\log \rho} = -d. \]

This latter relation is written as \( P_e(\rho) = \rho^{-d} \) in the exponential equality notation of [13].
We will use the DMT as our performance metric when we analyze cooperative diversity protocols. It is clear that the DMT of the MIMO channel with one receive and two transmit antennas provides an upper bound to the performance of any relay protocol for the channel of Fig. 1. This bound, known as the transmit diversity bound [5], is given by

\[ d_{\text{tx.div.bd}}(r) = 2(1 - r). \]

The DMT of the DDF protocol, proposed and analyzed in [15], is given by

\[
d^*(r) = \begin{cases} 
2(1 - r), & 0 \leq r \leq \frac{1}{2} \\
(1-r)/r, & \frac{1}{2} \leq r \leq 1 
\end{cases}
\]

This result is obtained by analyzing the information outage probability with Gaussian inputs, and it is achievable (e.g., by using a Gaussian random coding argument) in the limit of both \( M \to \infty \) and \( T \to \infty \). The relay decision time is given by

\[ M = \min \left\{ M, \frac{MR}{\log(1 + |h|^2 \rho')} \right\}, \]

i.e., \( M \) is set to the minimum \( m = 1, 2, \ldots, M - 1 \) such that the mutual information between \( X_{s,0}^m \) and \( Y_{r,0}^m \) for fixed and known \( h \), given by \( mT \log(1 + |h|^2 \rho') \), exceeds the number of information bits per message \( MT \). If such an \( m \) exists, the relay triggers the decoding of the whole information message and switches to the transmission mode. If no such \( m \) exists, then \( M = M \) and the relay remains silent.

Both the limit of large \( M \) and \( T \) are necessary to achieve the DDF DMT in (4). In fact, the normalized decision time \( M/M \) must converge to a continuous random variable distributed in \([0, 1]\) and, for every decision time \( M = m \), the number of symbols \( mT \) received by the relay must be arbitrarily large, such that the decoding error event coincides with the information outage error event. In this way, the corresponding probability of decoding error is arbitrarily close to the information outage probability

\[ P \left( \log(1 + |h|^2 \rho') \leq \frac{MR}{m} \right), \]

and the probability of undetected error (i.e., the relay accepts a wrong decision) is arbitrarily small. In brief, \( T \to \infty \) is necessary in order to fix the optimal decision time based only on the channel strength \( |h|^2 \) and be sure (with arbitrarily high probability) that the decoded message is the correct one.

We should also notice that, in the limit of \( T \to \infty \), the outage probability does not depend on the knowledge of \( h \) at the relay decoder and of \((g_1, g_2)\) at the destination decoder (see for example [8]). On the other hand, a common assumption made in previous works is that the destination knows exactly the relay decision time \( M \). In practice, this assumption requires some form of protocol to provide side
information to the destination. In the DMT analysis, one should pay great care to ensure that the error probability of such side information protocol does not dominate the decoding error probability, i.e., in designing any side information protocol we must ensure that its probability of error decreases not slower than $\rho^{-d^*(r)}$.

Practical code design for the DDF protocol considers finite, possibly very short, $M$ and $T$. In the following, we will make an explicit assumption of perfect receiver channel state information (CSIR), that is relatively easy to acquire using pilot symbols and is a common assumption in the DMT analysis of even finite-length codes (see [13] and [18]). On the contrary, we explicitly address the fact that the destination does not know a priori the relay decision time $M$ and tackle this problem by analyzing an augmented decoder based on the GLRT rule.

C. Existing DDF code designs

In [16], a variant of the DDF protocol is proposed where the relay code $X_r$ is such that the signal received at the destination reduces to an Alamouti constellation [10]. We will refer to this scheme as the “Alamouti-DDF” scheme, and review it briefly in the sequel since we make use of the same approach. With the Alamouti-DDF, assuming that the relay decodes correctly at the decision time $M = m$, the signal transmitted by the relay at time $k$ is given by [16]

$$x_{r,k} = \begin{cases} x_{s,k+1}^*, & k = mT + 1, mT + 3, \ldots \\ -x_{s,k-1}^*, & k = mT + 2, mT + 4, \ldots \end{cases}$$

which reduces the signal seen by the destination for $mT + 1 \leq k \leq MT$ to an Alamouti constellation. Through linear processing of the received signal $y_M^0$, the destination obtains the sufficient statistics for decoding, given by

$$\tilde{y}_k = \begin{cases} g_1 x_{s,k} + w_k, & k = 1, \ldots, mT \\ \sqrt{|g_1|^2 + |g_2|^2} x_{s,k} + \tilde{w}_k, & k = mT + 1, \ldots, MT \end{cases}$$

where the statistics of $\tilde{w}_k$ are identical to those of $w_k$. In this case, it is easy to see that the mutual information per symbol at the destination, for $M = m$ and i.i.d. Gaussian inputs, is given by

$$\frac{m}{M} \log \left( 1 + |g_1|^2 \rho \right) + \frac{M - m}{M} \log \left( 1 + (|g_1|^2 + |g_2|^2) \rho \right)$$

and coincides with that of the original DDF scheme defined by (2) and (3), when the codebooks $X_s$ and $X_r$ are also drawn independently from an i.i.d. Gaussian ensemble. Hence, the Alamouti-DDF modification entails no loss in DMT compared to the original DDF protocol [16].
III. DMT OF THE DDF PROTOCOL WITH FINITE LENGTH

In this section, we characterize the achievable DMT of the DDF protocol with finite $M$ and $T$. First, we find an upper bound on the DMT by letting $T \to \infty$, assuming that the destination has perfect knowledge of the relay decision time $M$, and using outage probability. Then, we shall analyze the performance of Gaussian random codes with finite length, with the assumption that the destination has no knowledge of $M$, and find a lower bound that matches the upper bound.

Since for i.i.d. Gaussian inputs the Alamouti-DDF yields the same mutual information as DDF, as far as outage probability is concerned we can refer to the channel defined in (7). This is a set of parallel channels for $m = 1, \ldots, M$, with dependent channel gains. In particular, there are two types of sub-channels: one representing the S-D link, and another set representing the composite (S,R)-D link (except for the case when $m = M$, which corresponds to when the relay remains inactive for the whole block; in this case, only the S-D link appears). The switching point between the two channels is controlled by the random variable $M$. We will refer to this channel as a random switch channel (RSC). Given a particular switching instant $M = m$, we will call the ensuing channel as a $m$-switch channel ($m$-SC). The RSC belongs to the class of “mixed channels” (see [9]), that is, a compound channel with an a priori probability distribution on the compound members. In this case, the probability distribution on the channel members (the $m$-SCs in (7)) is induced by the triple $(M, g_1, g_2)$.

A. Outage probability analysis

We compute the DMT of the RSC defined above for arbitrarily large $T$ under the assumption that the destination receiver has perfect knowledge of $M$, and hence find an upper bound on the DMT exponent $d_M^*(r)$ for the finite-length DDF protocol. This is established by the following theorem.

**Theorem 1:** The DMT of the single relay DDF scheme with decision times $m = 1, 2, \ldots, M$ and finite slot length $T \geq 1$ is upper bounded by

$$d_M^*(r) \leq d_{\text{out}}(r) = \min_{1 \leq m \leq M} \left\{ \tilde{d}_m(r) + d_m(r) \right\},$$

where

$$\tilde{d}_m(r) = \begin{cases} 
1 - \frac{Mr}{m-1}, & 0 \leq r \leq \frac{m-1}{M} \\
0, & \frac{m-1}{M} < r \leq \frac{m}{M} \\
\infty, & \frac{m}{M} < r \leq 1 
\end{cases} \quad (9)$$

$$d_m(r) = \begin{cases} 
2 - 2r, & m < \frac{M}{2} \\
\frac{M(1-r)}{m}, & m \geq \frac{M}{2} 
\end{cases} \quad (10)$$
for \( r \geq \frac{1}{2} \), and
\[
d_m(r) = \begin{cases} 
2 - 2r, & m < \frac{M}{2} \\
2 - \frac{rM}{M-m}, & \frac{M}{2} \leq m < M(1-r) \\
\frac{M(1-r)}{m}, & m \geq M(1-r)
\end{cases}
\] (11)
for \( r < \frac{1}{2} \).

**Proof:** Let \( M \) denote the random decision time as defined in (5) and \( P_{\text{out}}(r) \) denote the outage probability of the corresponding RSC. Also, let \( P^{m-\text{SC}}_{\text{out}}(r) \) denote the outage probability of the \( m \)-SC for given \( m \). Then, the law of total probability yields
\[
P_{\text{out}}(r) = \sum_{m=1}^{M} P(M = m) P^{m-\text{SC}}_{\text{out}}(r).
\] (12)

Since in the regime of very high SNR that characterizes the DMT, scaling SNR by a constant does not change the DMT, we allow both \( \rho, \rho' \to \infty \) and the DMT shall not depend on the (constant) ratio \( \rho'/\rho = \sigma_w^2/\sigma_v^2 \). Define
\[
P_{\text{out}}(r) \doteq \rho^{-d_{\text{out}}(r)},
\]
\[
P^{m-\text{SC}}_{\text{out}}(r) \doteq \rho^{-d_m(r)}, \quad 1 \leq m \leq M,
\]
\[
P(M = m) \doteq \rho^{-\overline{d}_m(r)}, \quad 1 \leq m \leq M.
\]
Then, it is clear from (12) that
\[
d_{\text{out}}(r) = \min_{1 \leq m \leq M} \{ \overline{d}_m(r) + d_m(r) \}.
\]
Furthermore, from standard arguments based on Fano inequality [13] and because here we are assuming that the destination receiver is enhanced by the side information on \( M \), it is also immediate to conclude that \( d^*_M(r) \leq d_{\text{out}}(r) \).

It remains to prove (9) and (10), (11). Notice that \( \overline{d}_m(r) \) is solely a function of the S-R link and \( d_m(r) \) is a function of the R-D and S-D links. We analyze these quantities separately as follows.

1) **Analysis of \( \rho^{-\overline{d}_m(r)} \):** Let’s consider first the case \( m < M \). Set \( R = r \log \rho \). The probability that the relay decodes after \( m \) sub-blocks \( P(M = m), 1 \leq m \leq M - 1 \), corresponds to the event
\[
\{ mT \log(1 + |h|^2 \rho') > MRT > (m-1)T \log(1 + |h|^2 \rho') \}
\]
\[
\Leftrightarrow \{ \frac{Mr}{m} \log \rho < \log(1 + |h|^2 \rho') < \frac{M(r-1)}{m-1} \log \rho \}
\]
\[
\Leftrightarrow \left\{ \frac{\rho^{\frac{Mr}{\rho}} - 1}{\rho} < |h|^2 < \frac{\rho^{\frac{M(r-1)}{\rho}} - 1}{\rho} \right\}.
\] (13)
Since $|h|^2$ is exponentially distributed and $\rho' \triangleq \rho$, we compute

$$P(M = m) \triangleq \int_{\rho \frac{Mr}{m-1}}^{\rho \frac{Mr}{m-1}} e^{-z} dz$$

$$= e^{-\rho \frac{Mr}{m-1}} - e^{-\rho \frac{Mr}{m-1}}.$$

According to the value of the multiplexing gain, we analyze the above quantity for each $1 \leq m < M$ as follows.

- $r > \frac{m}{M}$.
  
  This corresponds to $\frac{Mr}{m-1}, \frac{Mr}{m-1} - 1 > 0$. In this case\footnote{The notation $P \triangleq \rho^{-\infty}$ indicates that $P$ decreases faster than any polynomial function of $\rho$.}

$$P(M = m) \triangleq \rho^{-\infty}.$$

- $\frac{m-1}{M} < r \leq \frac{m}{M}$.
  
  This corresponds to $\frac{Mr}{m-1} - 1 \leq 0, \frac{Mr}{m-1} - 1 > 0$. In this case,

$$P(M = m) \triangleq \rho^0.$$

- $r \leq \frac{m-1}{M}$.
  
  This corresponds to $\frac{Mr}{m-1} - 1 \leq 0, \frac{Mr}{m-1} - 1 \leq 0$. In this case, using a power series expansion,

$$P(M = m) = \left[ 1 - \rho \frac{Mr}{m-1} + \rho^2 \frac{(\frac{Mr}{m-1})}{2!} + \cdots \right] - \left[ 1 - \rho \frac{Mr}{m-1} - 1 + \rho^2 \frac{(\frac{Mr}{m-1})}{2!} + \cdots \right]$$

$$= \rho^{\frac{Mr}{m-1}}.$$

A similar analysis for $P(M = M)$ results in

$$P\{M = M\} \triangleq \left\{ \begin{array}{ll}
\rho^{\frac{Mr}{m-1}}, & 0 \leq r \leq \frac{M-1}{M} \\
\rho^0, & \frac{M-1}{M} < r \leq 1
\end{array} \right.$$

Therefore, the result for all $1 \leq m \leq M$ can be compactly expressed by (9), shown in Fig. 2.

2) Analysis of $d_m(r)$: From (8), the outage probability of the $m$-SC is given by

$$P_{out}^{m-SC}(r) = P\{m-SC \leq MTR\},$$
A. DMT Analysis
Our aim is to compute the DMT of the RSC under the assumption of large $T$: The following theorem ... $r$. Notice

Define $\alpha$ and $\rho$ as described. Theorem 1: The outage probability $P_{out}^m$ is given by the infimum

$$d_m(r) = \inf_{\mathcal{B}} \{\alpha_1 + \alpha_2\},$$

where $\mathcal{B}$ is the two-dimensional region defined by the inequalities $(M - m) \max\{[1 - \alpha_1]_+, [1 - \alpha_2]_+\} + m[1 - \alpha_1]_+ \leq r M$ and $\alpha_i \geq 0 \ \forall \ i$.

Using Varadhan’s lemma [17], we obtain

$$d_m(r) = \inf_{\mathcal{B}} \{\alpha_1 + \alpha_2\}. \hspace{1cm} (14)$$

Define $\beta = \frac{m}{M}$. The region $\mathcal{B}$ is equivalently defined by

$$(1 - \beta) \max\{[1 - \alpha_1]_+, [1 - \alpha_2]_+\} + \beta[1 - \alpha_1]_+ \leq r,$$

$$\alpha_i \geq 0 \ \forall \ i.$$
concerned. We analyze $\mathcal{B}$ according to the following cases:

- $\alpha_1 \geq \alpha_2$:
  
  We have
  
  $$\beta(1 - \alpha_1) + (1 - \beta)(1 - \alpha_2) \leq r$$
  
  $$\Leftrightarrow \beta \alpha_1 + (1 - \beta)\alpha_2 \geq 1 - r.$$  

  This line has intercepts $\frac{1 - r}{\beta}$ and $\frac{1 - r}{1 - \beta}$ on the $\alpha_1$ and $\alpha_2$ axes respectively.

- $\alpha_1 < \alpha_2$:
  
  We have
  
  $$\beta(1 - \alpha_1) + (1 - \beta)(1 - \alpha_2) \leq r$$
  
  $$\Leftrightarrow \alpha_1 \geq 1 - r.$$  

The region $\mathcal{B}$ is depicted in Fig. 3. The solution to the problem in (3) corresponds to choosing the least non-negative $k$ such that the line $\alpha_1 + \alpha_2 = k$ touches $\mathcal{B}$. The analysis should be done according to whether $\frac{1 - r}{\beta} \geq 2 - 2r \Leftrightarrow \beta \leq \frac{1}{2}$ and whether $2 - 2r \geq 1 \Leftrightarrow r \leq \frac{1}{2}$. It is immediate from Fig. 3 that the solution to (14) when $r \geq \frac{1}{2}$ is at $(\alpha_1^*, \alpha_2^*) = \left(\frac{1 - r}{\beta}, 0\right)$ for $\beta \geq 0.5$, and $(\alpha_1^*, \alpha_2^*) = (1 - r, 1 - r)$ for $\beta < 0.5$. For the case when $r < \frac{1}{2}$, the solution to (14) is at $(\alpha_1^*, \alpha_2^*) = (1 - r, 1 - r)$ for $\beta < 0.5$,
at \((\alpha_1^*, \alpha_2^*) = \left(1, 1 - \frac{r}{\beta}\right)\) for \(\beta \geq 0.5\) and \(\frac{1-r}{\beta} > 1\), and at \((\alpha_1^*, \alpha_2^*) = \left(\frac{1-r}{\beta}, 0\right)\) for \(\beta \geq 0.5\) and \(\frac{1-r}{\beta} \leq 1\). The final solution is compactly expressed by (10), (11).

This concludes the proof of Theorem 1.

B. Achievability

We consider finite length \(T\) and no a priori knowledge of \(M\) at the destination decoder. We have the following result:

**Theorem 2:** The upper bound of Theorem 1 is achievable. Therefore, \(d_M^*(r) = d_{\text{out}}(r)\).

**Proof:** We consider the original DDF protocol (not the Alamouti variant) defined by (1), (2) and (3). For this channel we construct a particular coding scheme and analyze its performance.

**Codebook generation:** For given \(M, T\) and \(R\), we generate \(X_s \subset \mathbb{C}^{MT}\) and \(X_r \subset \mathbb{C}^{MT}\) of cardinality \(\rho^r MT\) independently, with i.i.d. components \(\sim \mathcal{CN}(0, E)\). We let \(x_s(\omega)\) and \(x_r(\omega)\) denote the codewords in \(X_s\) and in \(X_r\), respectively, corresponding to the information message \(\omega \in \{1, \ldots, \rho^r MT\}\).

**Relay decoding:** We define the relay outage event at slot \(m\) as

\[ O_m = \left\{ h \in \mathbb{C} : |h|^2 \leq \frac{\rho^r M}{\rho} - 1 \right\} \]  

(15)

Differently from the case of arbitrarily large \(T\), the relay may decode in error at time \(m\) even though \(h \notin O_m\). In the presence of such undetected error the relay would switch to transmit mode and send a codeword corresponding to an incorrect information message, thus jamming the destination receiver. In order to avoid this event we consider a bounded distance relay decoding decision function \(\psi_\delta\) defined as follows (see [12]): for \(m = 1, \ldots, M - 1\), define the regions \(S_m(\omega)\) of all points \(y \in \mathbb{C}^{mT}\) for which \(\omega\) is the unique message that is contained in a sphere of squared radius \(mT(1 + \delta)\sigma_v^2\) centered at \(y\), i.e., \(|y - h x_{s,0}^{m}(\omega)|^2 \leq mT(1 + \delta)\sigma_v^2\). Then, let \(\psi_\delta(y_{r,0}^m, h) = \hat{\omega} \in \{1, \ldots, \rho^r MT\}\) if both the following conditions are satisfied:

1) \(h \notin O_m\);
2) \(y_{r,0}^m \in S_m(\hat{\omega})\);

(the relay has perfect knowledge of its own channel coefficient \(h\), by the perfect CSIR assumption). If these conditions are satisfied, then \(M = m\) and the relay switches to transmit mode, sending the signal \(x_{r,m}^M(\hat{\omega})\) for the remaining part of the block. Otherwise, it refrains from making a decision and waits for the next slot.
It should be noticed that the condition 2) above is a test on the typicality of the estimated channel noise. In fact, if \( \omega \) is the transmitted message, we have that

\[
|y_{r,0}^m - h x_{s,0}^m(\omega)|^2 = |v_0^m|^2
\]

is a central chi-squared random variable with \( 2mT \) degrees of freedom and mean \( mT \sigma_n^2 \), that provides an empirical estimate of the noise variance.

**Destination decoding:** The destination is not aware of the relay decision time \( M \). Hence, it makes use of an augmented decoder that simultaneously detects the decision time and the information message according to the GLRT rule:

\[
\{ \hat{\omega}, \hat{m} \} = \arg\max_{\omega, m} p(y_0^M | \omega, m, g_1, g_2).
\]  (16)

where \( p(y_0^M | \omega, m, g_1, g_2) \) is the decoder likelihood function, i.e., the pdf of the signal received by the destination over the whole block length, under the hypothesis that the source transmitted the information message \( \omega \), that the relay decision time is \( m \), and given the channel coefficients \( g_1, g_2 \) (recall that we assume perfect CSIR).

**Error probability analysis:** Let \( E \) denote the decoding error event at the destination and \( E_r \) denote the decoding error event at the relay. We can write

\[
P(E) = \sum_{m=1}^{M} P(M = m)P(E|M = m)
\]

\[
= \sum_{m=1}^{M} P(M = m) (P(E, E_r|M = m) + P(E, \overline{E_r}|M = m))
\]

\[
\leq \sum_{m=1}^{M} P(M = m) (P(E_r|M = m) + P(E|E_r, M = m)P(E_r|M = m))
\]

\[
\leq \sum_{m=1}^{M} P(M = m) (P(E_r|M = m) + P(E|E_r, M = m)).
\]  (17)

First, we bound the effect of the undetected decision error at the relay. Our analysis follows closely the analysis of the MIMO-ARQ scheme in [12]. In fact, the relay applies a scheme very similar to ARQ: when it is sure about its decision it stops receiving and starts transmitting, while if it is not sure about

\( ^3 \)The complement of an event \( \mathcal{A} \) is denoted by \( \overline{\mathcal{A}} \).
its decision it waits for the next slot. We have
\[
P(E_r|M = m) = \rho^{-rMT} \sum_{\omega=1}^{\rho^M} P \left( \bigcup_{\tilde{\omega} \neq \omega} \left\{ y_{r,0}^m \in S_m(\tilde{\omega}) \right\} \mid \omega \right)
\leq P \left( |y_0^m|^2 > mT(1 + \delta)\sigma_v^2 \right)
\leq (1 + \delta)^{mT} e^{-mT\delta},
\]
where the last line follows from the Chernoff bound on the tail of the chi-squared distribution. Letting \( \delta = \mu \log \rho \), we find
\[
P(E_r|M = m) \leq \rho^{-mT\mu}.
\]
Notice that \( P(E_r|M = m) \geq \rho^{-d_m(r)} \) where \( d_m(r) \) is the exponent of the information outage probability of the \( m \)-SC channel given in (10), (11) and is not larger than 2. Hence, it is sufficient to choose \( \mu T > 2 \) in order to make the terms \( P(E_r|M = m) \) exponentially irrelevant in (17).

Next, let us examine the probabilities \( P(M = m) \). Let \( U_m = \bigcup_{\omega=1}^{\rho^M} S_m(\omega) \) denote the subset of the relay channel output space \( \mathbb{C}^{mT} \) such that if \( y_{r,0}^m \in U_m \) then there exists a unique codeword within the bounded distance decoder's decoding sphere centered at \( y_{r,0}^m \). For \( m = 1 \), we have
\[
P(M = 1) = P \left( \{ h \notin O_1 \} \cup \{ y_{r,0}^1 \in U_1 \} \right)
\leq P(h \notin O_1)
\leq \rho^{-\Pi_1(r)}.
\]
For brevity we let \( D_m = \{ h \notin O_m \} \cap \{ y_{r,0}^m \in U_m \} \). Then, for \( 1 < m < M \), we have
\[
P(M = m) = P \left( D_1, \ldots, D_{m-1}, D_m \right)
\leq P(D_{m-1}, D_m).
\]
For \( 1 < m < M \), from (20) we can write
\[
P(M = m) \leq P \left( \{ h \in O_{m-1} \} \cup \{ y_{r,0}^{m-1} \notin U_{m-1} \} , \{ h \notin O_{m-1} \} , \{ y_{r,0}^m \notin U_m \} \right)
\leq P \left( \{ h \in O_{m-1} \} , \{ h \notin O_m \} \right) + P \left( \{ h \notin O_{m-1} \} , \{ y_{r,0}^{m-1} \notin U_{m-1} \} \right),
\]
where the second inequality follows from the fact that for events \( A, B, C \) and \( D \), we have using the distributive law and the union bound that
\[
P \left( \{ A \cup B \} \cap \{ C \cap D \} \right) = P \left( \{ A \cup (B \cap \overline{A}) \} \cap \{ C \cap D \} \right)
\leq P(A \cap C) + P(B \cap \overline{A}).
\]
Finally, for \( m = M \), we have

\[
P(M = M) = P\left(\mathcal{D}_1, \ldots, \mathcal{D}_{M-1}\right) \leq P\left(h \notin \mathcal{O}_{M-1}\right) = P\left(\{h \notin \mathcal{O}_{M-1}\}, \{y_{r,0}^M \notin \mathcal{U}_{M-1}\}\right) + P\left(h \in \mathcal{O}_{M-1}\right).
\] (22)

We notice that the event \( \{h \in \mathcal{O}_{m-1}\} \cap \{h \notin \mathcal{O}_m\} \) coincides with (13) and therefore the first term in (21) decreases as \( \rho^{-\mathcal{D}_m(r)} \). It is also immediate to see that \( P\left(h \in \mathcal{O}_{M-1}\right) = \rho^{-\mathcal{D}_M(r)} \). Hence, we are left with the analysis of the probability

\[
P\left(\{h \notin \mathcal{O}_m\}, \{y_{r,0}^m \notin \mathcal{U}_m\}\right)
\] (23)

for all \( m = 1, \ldots, M - 1 \). Averaging with respect to the random coding ensemble, we may choose without loss of generality \( \omega = 1 \) as the reference transmitted message. We have

\[
\overline{\mathcal{U}}_m \subseteq \{|v_0^m|^2 > mT(1 + \delta)\sigma_v^2\} \cup \mathcal{R}_m(1),
\]

where \( \mathcal{R}_m(1) \) are the points \( y_{r,0}^m \) such that \( |y_{r,0}^m - hx_{s,0}^m(1)|^2 \leq mT(1 + \delta)\sigma_v^2 \), and there exists some \( \omega \neq 1 \) for which also \( |y_{r,0}^m - hx_{s,0}^m(\omega)|^2 \leq mT(1 + \delta)\sigma_v^2 \). Letting for brevity \( \Delta x(\omega) = x_{s,0}(\omega) - x_{s,0}^m(1) \), we can write

\[
\mathcal{R}_m(1) = \bigcup_{\omega \neq 1} \left\{|v_0^m - h\Delta x(\omega)|^2 \leq mT(1 + \delta)\sigma_v^2, |v_0^m|^2 \leq mT(1 + \delta)\sigma_v^2\right\}.
\]

Using the union bound and the Chernoff bound we have

\[
P\left(\{h \notin \mathcal{O}_m\}, \{y_{r,0}^m \notin \mathcal{U}_m\}\right) \leq P\left(|v_0^m|^2 \geq mT(1 + \delta)\sigma_v^2\right) + P\left(\{h \notin \mathcal{O}_m\}, \mathcal{R}_m(1)\right) \leq (1 + \delta)^m T e^{-mT\delta} + \sum_{\omega \neq 1} P\left(\{h \notin \mathcal{O}_m\}, \left\{|v_0^m|^2 \leq mT(1 + \delta)\sigma_v^2\right\}\right) \left\{|v_0^m - h\Delta x(\omega)|^2 \leq mT(1 + \delta)\sigma_v^2\right\}\right) \right)
\] (24)

Let us consider one term in the sum in the last line of (24) for a given message \( \omega \) and given channel \( h \), averaged over the random coding ensemble. Noticing that for vectors \( a \) and \( b \) and \( \Gamma > 0 \) we have

\[
\{|a + b|^2 \leq \Gamma, |b|^2 \leq \Gamma\} \subseteq \{|a|^2 \leq 4\Gamma\},
\]
we can bound this probability as
\[
P \left( \left\{ \left| y_0^m - h \Delta x(\omega) \right|^2 \leq mT(1 + \delta)\sigma_v^2 \right\}, \left\{ \left| y_0^m \right|^2 \leq mT(1 + \delta)\sigma_v^2 \right\} \middle| h \right) \]
\[
\leq P \left( \left| h \Delta x(\omega) \right|^2 \leq 4mT(1 + \delta)\sigma_v^2 \middle| h \right)
\]
\[
\overset{(a)}{=} P \left( \rho' \left| h \right|^2 \chi \leq 2mT(1 + \delta) \middle| h \right)
\]
\[
\leq \rho^{-mT[1 - \nu]_+},
\]
where (a) follows from the fact that for the randomly generated codewords, \( \chi = |\Delta x(\omega)|^2 / E \) is a central chi-squared random variable with mean \( 2mT \) and \( 2mT \) degrees of freedom, and the last line follows by letting \( |h|^2 = \rho^{-\nu} \) and from the fact that the chi-squared cdf satisfies \( P(\chi \leq u) = O(u^{mT}) \) for small \( u \) and \( P(\chi \leq u) = O(1) \) for large \( u \). Summing over the \( \rho^{rMT} - 1 \) messages \( \omega \neq 1 \) and integrating with respect to the pdf of \( |h|^2 \) over the set \( \overline{O}_m \), we obtain
\[
P \left( \left\{ h \notin \Theta_m \right\}, \left\{ y_{r,1}^m \notin \mathcal{U}_m \right\} \right) \leq \int \left\{ \nu \geq 0, [1 - \nu]_+ \geq \frac{\nu}{m} \right\} \rho^{-\nu} \rho^{-mT[1 - \nu]_+ + rMT} d\nu
\]
\[
= \rho^{-\tilde{d}_m(r)},
\]
where, from a standard application of Varadhan’s lemma, we have
\[
\tilde{d}_m(r) = \inf_{\nu \geq 0, [1 - \nu]_+ \geq \frac{\nu}{m}} \{ \nu + mT[1 - \nu]_+ - rMT \}.
\]
The domain of \( \nu \) over which the infimum is calculated is non-empty only for \( r \leq \frac{m}{M} \). This means that the set of channels for which the probability in (23) has a polynomial decrease is empty for \( r > \frac{m}{M} \) and therefore \( \tilde{d}_m(r) = \infty \) for \( r > \frac{m}{M} \). For \( r \leq \frac{m}{M} \) it is not hard to see that for all \( T \geq 1 \) we have \( \tilde{d}_m(r) = 1 - \frac{Mr}{m} \). Comparing \( \tilde{d}_m(r) \) with \( d_m(r) \) we see that the former dominates the latter for all \( r \in [0, 1] \). It follows that for our relay bounded distance decoder and the Gaussian random coding ensemble \( P(\mathcal{M} = m) \lesssim \rho^{-\tilde{d}_m(r)} \).

So far we have shown that in the upper bound (17) the terms \( P(E_r | \mathcal{M} = m) \) are asymptotically negligible and the terms \( P(\mathcal{M} = m) \) are upper bounded by the same exponent of the outage probability based, infinite \( T \), case. It remains to show that the terms \( P(E_r \mathcal{E}_r, \mathcal{M} = m) \) have exponent \( d_m(r) \) given in (10), (11), and the proof will be complete.

We consider the GLRT decoder at the destination. This decoder ignores the knowledge of the a priori distribution of \( \mathcal{M} \) and treats it as a deterministic unknown parameter. Hence, we are in the presence of a compound channel formed by the family of \( m \)-SC component channels, without any a priori knowledge of \( \mathcal{M} \).
Again, without loss of generality we assume message 1 is transmitted. While for the sake of notational simplicity, we omit the explicit conditioning with respect to $\mathcal{E}_r$, it is understood that the relay has perfect knowledge of the transmitted information message. We omit also the explicit conditioning with respect to CSIR and denote $y^M_{s,0}$ simply by $y$ since no ambiguity is possible at this point. Hence, the likelihood function $p(y^M_{s,0}|\omega, m, g_1, g_2)$ shall be denoted simply by $p(y|\omega, m)$. The pairwise error probability for some $\omega \neq 1$ can be upper bounded as follows:

$$P(1 \rightarrow \omega|M = m) = P \left( \max_{m'} p(y|1, m') \leq \max_{m'} p(y|\omega, m') \Bigg| M = m \right)$$

$$\leq P \left( p(y|1, m) \leq \max_{m'} p(y|\omega, m') \Bigg| M = m \right)$$

$$= P \left( \bigcup_{m' = 1}^{M} \{p(y|1, m) \leq p(y|\omega, m')\} \Bigg| M = m \right)$$

$$\leq \sum_{m' = 1}^{M} P \left( p(y|1, m) \leq p(y|\omega, m') \Bigg| M = m \right).$$

(29)

We shall analyze separately the terms inside the above sum, averaged over the random coding ensemble.

Define the event

$$\mathcal{E}_1 = \left\{ p(y|\omega, m') \geq 1 \right\}. \quad (30)$$

We first analyze the probability of the event $\mathcal{E}_1$, which we then use to compute $P(\mathcal{E})$. Assuming $M = m$, the actual received signal is

$$y^m_0 = g_1 x^m_{s,0}(1) + w^m_0$$

$$y^M_m = g_1 x^M_{s,m}(1) + g_2 x^M_{r,m}(1) + w^M_m.$$ 

We consider the case $m' \geq m$ and leave the case $m' \leq m$ to the reader, since it follows in an almost identical manner. Define the partial codeword differences $\Delta x^m_{s,0} = x^m_{s,0}(1) - x^m_{s,0}(\omega)$, $\Delta x^m_{s,m'} = x^m_{s,m'}(1) - x^m_{s,m}(\omega)$, $\Delta x^M_{s,m'} = x^M_{s,m'}(1) - x^M_{s,m}(\omega)$, and $\Delta x^M_{r,m'} = x^M_{s,m'}(1) - x^M_{s,m}(\omega)$. The error event $\mathcal{E}_1$ can be written as

$$\mathcal{E}_1 = \left\{ |g_1|^2 |\Delta x^m_{s,0}|^2 + 2\text{Re} \left\{ g_1 (w^m_0)^H \Delta x^m_{s,0} \right\} + |g_1 \Delta x^m_{s,m'} + g_2 x^m_{r,m'}(1)|^2 + 2\text{Re} \left\{ (w^m_{m'})^H g_1 \Delta x^m_{s,0} + g_2 x^m_{r,m'}(1) \right\} + |g_1 \Delta x^M_{s,m'} + g_2 \Delta x^M_{r,m'}|^2 + 2\text{Re} \left\{ (w^M_{m'})^H g_1 \Delta x^M_{s,m'} + g_2 \Delta x^M_{r,m'} \right\} \leq 0 \right\}. \quad (31)$$

After a little algebra, we obtain the compact expression

$$\mathcal{E}_1 = \left\{ 2\text{Re} \{z^H w \} \leq -|z|^2 \right\},$$

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where $z$ is defined as

$$z \triangleq \begin{bmatrix} g_1 \Delta x_{s,0}^m \\ g_1 \Delta x_{s,m}^{m'} + g_2 x_{r,m}^{m'} \\ g_1 \Delta_{s,m}^M + g_2 \Delta x_{r,m}^M \end{bmatrix}$$

For given codebooks $X_s, X_r$, the variance of $2\text{Re}\{z^H w\}$ is equal to $2|z|^2\sigma_v^2$, which leads to

$$P(E_1|X_s, X_r, M = m, g_1, g_2) \leq Q\left(\frac{|z|}{\sqrt{2\sigma_v^2}}\right) \leq e^{-|z|^2/(4\sigma_v^2)}.$$ 

Define the following notation,

$$\xi_i = [x_{s,i}(1) \ x_{s,i}(\omega) \ x_{r,i}(1) \ x_{r,i}(\omega)]^T, \ 1 \leq i \leq MT,$$

and

$$\xi \triangleq [\xi_1^T \ \xi_2^T \ \cdots \ \xi_{MT}^T]^T \in \mathbb{C}^{MT \times 1}.$$ 

It can be verified that $|z|^2 = \xi^H M \xi$, for a block diagonal $M$ of the form

$$M = \begin{bmatrix} M_1 \\ \vdots \\ M_{MT} \end{bmatrix},$$

where for $1 \leq k \leq mT$,

$$M_k = |g_1|^2 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

for $mT + 1 \leq k \leq m'T$,

$$M_k = \begin{bmatrix} |g_1|^2 & -|g_1|^2 & g_2 g_1^* & 0 \\ -|g_1|^2 & |g_1|^2 & -g_2 g_1^* & 0 \\ g_1 g_2^* & -g_1 g_2^* & |g_2|^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and for $m'T + 1 \leq k \leq MT$,

$$M_k = \begin{bmatrix} |g_1|^2 & -|g_1|^2 & g_2 g_1^* & -g_2 g_1^* \\ -|g_1|^2 & |g_1|^2 & -g_2 g_1^* & g_2 g_1^* \\ g_1 g_2^* & -g_1 g_2^* & |g_2|^2 & -|g_2|^2 \\ -g_1 g_2^* & g_1 g_2^* & -|g_2|^2 & |g_2|^2 \end{bmatrix}. $$
It turns out that the matrices $M_k$ have rank 1, for all $1 \leq k \leq MT$. It follows that the eigenvalues of each $M_k$ are $\text{tr}(M_k), 0, 0, 0$. We now average $P(\mathcal{E}_1|X_s, X_r, m, g_1, g_2)$ over the ensemble of random Gaussian codebooks. In order to do so, we use the following well-known result on the characteristic function of Hermitian quadratic form of complex Gaussian random variables (briefly, HQF-GRV).

**Lemma 3:** [19, Appendix 4] The characteristic function of the HQF-GRV $\Delta = z^H F z$, where $z \sim \mathbb{C}N(z, R)$ is given by

$$
\Phi_{\Delta}(s) = \mathbb{E} \left[ \exp(-s\Delta) \right] = \frac{\exp(-s z^H F (I + sRF)^{-1} z)}{\det(I + sRF)}.
$$

Therefore,

$$
P(\mathcal{E}_1| m, g_1, g_2) \leq \mathbb{E}_{X_s, X_r} \left[ e^{-|z|^2/(4\sigma_v^2)} \right] = \Phi_{|z|^2} \left( \frac{1}{4\sigma_v^2} \right) = \frac{1}{\det(I + \frac{\rho}{4} M)}.
$$

Explicitly, we have

$$
\frac{1}{\det(I + \frac{\rho}{4} M)} = \frac{1}{[1 + \rho|g_1|^2]^{MT}} \cdot \frac{1}{[1 + \rho(2|g_1|^2 + |g_2|^2)]^{m - m'}} \cdot \frac{1}{[1 + \rho(|g_1|^2 + |g_2|^2)]^{(M-m')R}}.
$$

(32)

We notice that (32) does not depend on $m'$, at least in the exponential equality sense. Summing over all $m' = 1, \ldots, M$ and over all messages $\omega \neq 1$, we eventually can bound the average probability of error of the GLRT decoder conditioned on $M = m$ and on $g_1, g_2$ as

$$
P(\mathcal{E}|\overline{\mathcal{E}}, M = m, g_1, g_2) \leq \sum_{\omega \neq 1} P(1 \rightarrow \omega | M = m, g_1, g_2)
$$

$$
\leq \sum_{\omega \neq 1} \sum_{m'=1}^M P(\mathcal{E}_1| m, g_1, g_2)
$$

$$
\leq \frac{M \rho^{MT}}{[1 + \rho|g_1|^2]^{MT}} \cdot \frac{1}{[1 + \rho(|g_1|^2 + |g_2|^2)]^{(M-m')R}}.
$$

(33)

Next, we shall evaluate the diversity exponent of $P(\mathcal{E}|\overline{\mathcal{E}}, M = m) = \mathbb{E}_{g_1, g_2}[P(\mathcal{E}|\overline{\mathcal{E}}, M = m, g_1, g_2)]$. In order to do so, we separate the outage event from the no-outage event. Define the outage event of the $m$-SC as

$$
\mathcal{A}_m = \{(M - m) \max\{1 - \alpha_1, 1 - \alpha_2\} + m[1 - \alpha_1] + rM \leq 0\}.
$$

(34)
Then,

\[ P(E|E_r, M = m) = P(E, A_m|E_r, M = m) + P(E, \overline{A}_m|E_r, M = m) \]

\[ \leq P(A_m) + P(E, \overline{A}_m|E_r, M = m). \]  

(35)

Recall that

\[ P(A_m) = P_{\text{out}}^{m-SC}(r) = \rho^{-d_m(r)}, \]

where \( d_m(r) \) is evaluated in (10), (11). In order to evaluate \( P(E, \overline{A}_m|E_r, M = m) \), we use (33) and write the exponential inequality

\[ P(E|E_r, M = m, g_1, g_2) \leq \rho^{-Tg_m(\alpha_1, \alpha_2, r)}, \]

where

\[ g_m(\alpha_1, \alpha_2, r) = (M - m) \max\{[1 - \alpha_1]_+, [1 - \alpha_2]_+\} + m[1 - \alpha_1]_+ - rM. \]  

(36)

Therefore, using again Varadhan’s lemma, we obtain

\[ P(E, \overline{A}_m|E_r, M = m) \approx \rho^{-d_{G,m}(r)}, \]

where

\[ d_{G,m}(r) = \inf_{\alpha_1, \alpha_2 \geq 0, g_m(\alpha_1, \alpha_2, r) > 0} \left\{ \alpha_1 + \alpha_2 + Tg_m(\alpha_1, \alpha_2, r) \right\}. \]  

(37)

The above infimum is achieved when \( g_m(\alpha_1, \alpha_2, r) \downarrow 0 \), yielding

\[ d_{G,m}(r) = d_m(r). \]

This concludes the proof of Theorem 2.

Remark. The proof of Theorem 2 is not only conceptually appealing, but also reveals a few very important and often neglected features that should be taken into account in the design of a DDF scheme. First, the proof sheds light on the fact that the relay must make its decision based not only on the outage condition, but also on the reliability of the decoding decision. Then, it shows also that despite the fact that the destination does not know the relay decision time, there is no need for an explicit protocol that provides this side information. In Appendix I, we analyze a simpler decoder, nicknamed relay activity detector, based on separated detection of the relay decision time by treating the codewords as random Gaussian signals (i.e., ignoring the structure of the code). We show that such a simple “energy detector”
is optimal if we let $T \to \infty$ first, and then consider the high SNR performance, but it is dramatically suboptimal if we do the limits in the reverse order. In fact, for any finite $T$, the relay activity detector yields a constant error probability, that does not vanish with SNR.

C. Computing the DMT and comparisons

Obtaining a closed-form solution to the DMT expression in Theorem 1 appears to be intractable. We plot in Fig. 4 values of $d^*_M(r)$ for $M = 2, 5, 10$ and $20$ in comparison with the optimal DMT of the DDF protocol (corresponding to $M = \infty$).

![Fig. 4. The DMT of the DDF channel with finitely many decoding decision times.](image)

With increasing $M$, $d^*_M(r)$ is seen to approach the optimal tradeoff very rapidly. For practical code design, even a relatively small value of $M$ is therefore expected to have close to optimal performance in terms of diversity.
Remark. The authors in [16] consider a related problem, where $T \to \infty$ and the relay is restricted to a finite number of decision times (say $N$). These time instants coincide with the end of blocks $\{M_j\}_{j=1}^{N}$, with $1 \leq M_1 < \cdots < M_N < M \ \forall \ j$ (notice: with this notation, in our case we would have $N = M$ and $M_j = j$). Further, define $M_0 \triangleq 0$, $M_{N+1} \triangleq M$, and a set of “waiting fractions” $\{f_j\}_{j=0}^{N+1}$ by $f_j \triangleq \frac{M_j}{M}$. Thus

$$f_0 = 0 < f_1 < \cdots < f_N < f_{N+1} = 1.$$  

In [16], it is proved that for any fixed $N$ no set of waiting fractions yields a DMT curve that dominates all others. Then, a particular set of waiting fractions are chosen that yield for any fixed $N$ a DMT curve that is not uniformly dominated by any other protocol with the same number of decision times $N$. The resulting DMT is derived and it is summarized by the following lemma from [16].

**Lemma 4:** [16] For the DDF protocol with a given number $N$ of decision times, let $f_1^p = \frac{1}{2}$ and

$$f_j^p = \frac{1 - f_{j-1}^p}{2 - \left(1 + \frac{1}{r}\right) f_{j-1}^p}, \ \text{for } 1 < j \leq N,$$

then no set of fractions uniformly dominates $\{f_j^p\}_{j=1}^{N}$. Further, the DMT corresponding to the set of fractions $\{f_j^p\}_{j=1}^{N}$ is given by

$$d^p(r) = 1 - r + \left[1 - \frac{r}{f_N^p}\right]. \ \tag{38}$$

A few interesting observations can be made about this result. As it is remarked in [16], the DMT obtained through $\{f_j^p\}$ is not asymptotically optimal, i.e., it does not converge to the optimal DMT of the DDF protocol as $N \to \infty$. Indeed, it is evident from (38) that $d^p(r)$ consists of two straight line segments, say $L_1$ for $0 \leq r \leq f_N^p$ and $L_2$ for $f_N^p \leq r \leq 1$. As $N \to \infty$, $L_1$ and $L_2$ can at best be tangential to the curved part of the DMT of the DDF protocol (i.e., the $0.5 \leq r \leq 1$ region) in (4). In particular, the optimal value of the DMT of the DDF protocol $d^*(0.5) = 1$ is never approached even in the limit by $d^p(r)$. In contrast, the DMT $d^*_M(r)$ derived in this paper is asymptotically optimal. As the number of decoding points increases, $d^*_M(r)$ dominates over $d^p(r)$ for almost all values of $r$, and is strictly less for only an exceedingly small range of values of $r$. Asymptotically, it is clear that the only set of points where $d^p(r)$ dominates $d^*_M(r)$ is a very small set of points around the point where $d^p(r)$ is tangent to the curved part of the DMT of the DDF protocol.

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4 According to the definition in [16], protocol A uniformly dominates protocol B if, for any multiplexing gain $r$, $d_A(r) \geq d_B(r)$. A protocol that is not uniformly dominated by any other protocol is said to be Pareto-optimal.
IV. DMT OPTIMAL CODES FOR THE SINGLE RELAY DDF CHANNEL

The authors in [15] used the ensemble of random Gaussian codes of asymptotically large block-lengths to show the achievability of the DMT of the DDF protocol. Subsequently, a construction of codes derived from cyclic division algebras (CDA) was shown to achieve the DMT of the DDF channel for arbitrary number of relays [28]; i.e., they achieve the corresponding tradeoff for a particular number of decoding instants. As we increase the block-length and the number of decoding instants, the DMT of these codes tends towards the optimal DMT of the DDF protocol given in (4). In a recent submission, [29], the authors present a division-algebraic construction based on the Alamouti code that is similar in flavor to the construction to be presented in this paper. However, for the codes in [29], the parameter $T$ is fixed to 2; on the contrary, we will see that our code construction is valid for arbitrary values of $T$ including the special case of $T = 1$, and is hence a minimum delay construction. Decoding these codes involves sphere or sequential decoding [31], [32] over a large dimensional lattice. It is hence of interest to construct codes that achieve the DMT of the DDF protocol and permit low complexity decoding. Since our construction is of minimum delay, the dimensionality of the lattice to be sphere decoded is half that of the corresponding case in [29].

In order to completely specify a signalling scheme $(X_s, \phi, X_r)$ for the DDF channel, we need to define the following:

1) A code $X_s$ that is used by the source.

2) A causal *decoding decision function* $\phi(\cdot, \cdot) : (\mathbb{C}, \mathbb{C}^{MT}) \rightarrow \{1, 2, \ldots, M\}$, that dictates when the relay attempts to decode the source’s transmission based on the S-R channel gain $h$ and the signal $y_r$ received at the relay. In particular, if $\phi(h, y_r) = M$, the relay will not attempt to decode the transmission of the source. If $\phi(h, y_r) = m, \ 1 \leq m < M$, then the relay attempts to decode the transmission of the source upon completion of the $m^{th}$ block. Because of the causality constraint, we assume that the output of $\phi$ at time $m$ depends only on $h$ (CSIR of the relay) and on $y_r^{m,0}$.

3) A code $X_r$ used by the relay. In the following, we will only consider the case that the relay implements the Alamouti-DDF scheme [16] given in (6); hence $X_r$ is the same as $X_s$ upto coordinate permutations, sign change and conjugation.

A. Design tradeoffs

Despite the importance of the decoder at the destination, as evidenced in the proof of Theorem 2, in this section we take a shortcut and we do not treat the the GLRT decoder explicitly. For the sake of simplicity, our simulations assume a genie-aided destination, ideally informed of the relay decision time.

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The main focus of this section is on the design of the codebook $X_s$ and on the efficient implementation of the relay decoding decision function, in order to trigger the relay transmission only when decisions are reliable.

Choosing a good relay decoding decision function $\phi$ is critical to ensure good performance: a conservative $\phi$ that makes the relay wait for too long before decoding results in low relay error probability $P(E_r)$, but increases the destination error probability $P(E, E_r)$ since the relay has less time to help the destination. Vice-versa, a $\phi$ that is too aggressive and makes the relay decode too early yields low $P(E, E_r)$ but results in a large $P(E_r)$, since the blocklength of the signal observed at the relay is too short to cope with atypical noise. We shall also see through simulations that undetected decoding errors at the relay have a huge impact on performance, since the relay ends up jamming the destination with high probability. We will present our choices of $X_s$ and $\phi$ in the following two subsections.

**B. Approximately universal $X_s$**

The equivalent channel resulting from the use of the Alamouti scheme for the relay code is a parallel channel with statistically dependent fading coefficients. We will choose $X_s$ to be a code of length $MT$ that is approximately universal over the parallel fading channel. A code that is approximately universal over the parallel channel (a notion introduced in [21]) meets the DMT over any parallel channel. Such a code has an error probability that decays exponentially with $\rho$ for all parallel channel gains such that the corresponding mutual information is larger than the coding rate, i.e., for all channel gains in the no-outage region. Therefore, such an approximately universal code $X_s$ meets the DMT of the relay DDF channel for any $M$. This means that, for any fixed rate $R$ and sufficiently large SNR, the decay of error probability with SNR of our code (with finite $T$) exhibits the same slope of outage probability. However, the "gap from outage" (i.e., the horizontal distance in dB between the outage probability and the actual probability of error) is not captured by the DMT optimality and in practice it may be very large, thus making a DMT-optimal scheme totally useless for practical purposes. We shall discuss ways to close this gap in the next section, by an appropriate choice of the relay decision function $\phi$.

We may obtain approximately universal $X_s$ from either suitable algebraic lattices [23], [24] or from permutation codes through UDMs [21], [22]. In the following we briefly review these constructions.

1) **Rotated QAM codes from algebraic lattices**: Let $\mathbb{L}$ be an $MT$-dimensional extension of $\mathbb{Q}(i)$ and let the Galois group $Gal(\mathbb{L} | \mathbb{Q}(i)) = \{ \sigma_1, \ldots, \sigma_{MT} \}$. Denote the ring of integers of $\mathbb{L}$ as $\mathbb{O}_L$ and let $\mathcal{I}$ be an ideal of $\mathbb{O}_L$. Let $N_{\mathbb{L} | \mathbb{Q}(i)}(\cdot)$ denote the algebraic norm from $\mathbb{L}$ to $\mathbb{Q}(i)$. We define the code $X_s$ as
follows:

\[
X_s = \left\{ \begin{bmatrix}
\sigma_1(\ell) \\
\sigma_2(\ell) \\
\vdots \\
\sigma_{MT}(\ell)
\end{bmatrix} : \ell \in S \right\},
\]

where \( S \) is some finite subset of \( J \). \( X_s \) has the desirable property of a “non-vanishing” product distance, since we have for each \( x \in X_s \) that

\[
\prod_{j=1}^{MT} |x_j| = \prod_{j=1}^{MT} |\sigma_j(\ell)| = |N_{L[i]}(\ell)| \geq 1,
\]

since the norm \( N_{L[i]}(\cdot) \) of an algebraic integer in \( L \) is an element of \( Z[i] \). This non-vanishing product distance property ensures that \( X_s \) is approximately universal over the parallel channel \([21], [29]\).

It can be verified that \( X_s \) can equivalently be rewritten as a lattice code, i.e.,

\[
X_s = \{ Gb : b \in B \},
\]

(39)

for suitable \( G \in C^{MT \times MT} \) and \( B \subset Z[i]^{MT} \). A particular choice of \( G \) and \( B \) that is good in terms of shaping consists of constructing \( G \) to be unitary and \( B \) to be a set of points in \( Z[i]^{MT} \) contained in a hypercube that is centered around the origin. For the algebraic details regarding the construction of such unitary \( G \), see [23], [24]. Notice also that choosing the information set \( B \) to be a bounded subset of \( Z[i]^{MT} \) corresponds, in practice, to choosing information symbols from a QAM alphabet, which is appealing for practical implementation. The rate of \( X_s \) in this case is

\[
R = \frac{\log |B|}{MT} \text{ bpcu.}
\]

Parameters for simulations: In the simulations to follow in Sec. IV-C we construct the matrix \( G \) using the cyclotomic construction given in [23]. For \( M = 4 \) and \( T = 1 \), \( G \) is a complex \( 4 \times 4 \) matrix, or equivalently a real \( 8 \times 8 \) matrix. We choose \( B \) to be a cartesian product of \( Q^2 \)-QAM alphabets,

\[
B = \{ a + ib : Q + 1 \leq a, b \leq Q - 1, \ a, b \ \text{odd} \}^{MT},
\]

for some even integer \( Q \). Thus \( |B| = Q^{2MT} \). For example, by choosing \( Q = 4 \) with \( M = 4 \) and \( T = 1 \) we obtain a rate of \( R = 4 \) bpcu.

\[5\text{Notice however that choosing } G \text{ unitary is optimal only when we are constrained to use a linear map to encode the information vector onto the code symbols. An alternate approach is to use a constellation carved out of a dense lattice in } \mathbb{R}^n \text{ and employ a non-linear sphere encoder and a mod-Λ MMSE-GDFE lattice decoder; this has been shown to yield significant performance improvements over unitary shaping [26], [27]. For simplicity of exposition, we will restrict our attention to the case of linear encoding in this paper.} \]
2) **Permutation codes from UDM**: Approximately universal code construction from UDM were introduced in [21] and a general algebraic construction valid for any number of sub-channels was provided in [22].

**Definition 1**: [22] Let \( n \) and \( L \) be some positive integers and let \( q \) be a prime power. The \( L \) matrices \( A_0, \ldots, A_{L-1} \) over \( \mathbb{F}_q \) of size \( n \times n \) are \((L,n,q)\)-UDMs if for every \((k_0, \ldots, k_{L-1})\) such that \( 0 \leq k_\ell \leq n \ \forall \ell, \sum_{\ell=0}^{L-1} k_\ell \geq n \), the \((\sum_{\ell=0}^{L-1} k_\ell) \times n\) matrix composed of the first \( k_0 \) rows of \( A_0 \), the first \( k_1 \) rows of \( A_1 \), ..., the first \( k_{L-1} \) rows of \( A_{L-1} \) has full rank.

The authors in [22] provide an algebraic construction of such \((L,n,q)\)-UDMs for any \( L \leq q + 1 \). It is shown in [21] that an approximately universal permutation code for the parallel channel with \( L \)-branches can be obtained from \((L,n,q)\)-UDMs, in the following manner. Assume that we have to transmit \( 2n \) information symbols from \( \mathbb{F}_q \). We encode independently \( n \)-symbols each onto the I and Q sub-channels. Let \( u \in \mathbb{F}_q^n \) denote the first \( n \) input information symbols. Map the sequence of \( \mathbb{F}_q^n \) symbols \( \{A_1 u, A_2 u, \ldots, A_L u\} \) componentwise onto a \( L \)-length vector of \( q^n \)-PAM symbols, and transmit the components on the I sub-channel. The next \( n \) information symbols are similarly encoded and transmitted on the Q sub-channel. The rate of such a permutation code is

\[
R = \frac{2n \log q}{L} \text{ bpcu.}
\]

In our case, we set \( L = MT \) to obtain codes for the DDF channel.

**Parameters for simulations**: The simulations involving permutation codes in Sec. IV-C for \( M = 4, T = 1 \) are derived from \((4,4,4)\)-UDMs, leading to \( R = 4 \) bpcu. In order to completely specify the code, we need to provide the mapping to PAM symbols that was used. We construct the Galois field \( \mathbb{F}_4 \) using the primitive polynomial \( X^2 + X + 1 \). Thus any element in \( \mathbb{F}_4 \) may be associated with a polynomial \( b_1 X + b_0 \), where the \( b_i \) are either 0 or 1, and \( X \) is a primitive element. Hence we may also associate each element in \( \mathbb{F}_4 \) with the binary string \( b_1 b_0 \). In order to map an \( \mathbb{F}_4^4 \) vector \( v \) (which is one of the \( A_j u \) considered previously) to the PAM alphabet, first concatenate the binary strings corresponding to \( v_i \in \mathbb{F}_4, i = 1, 2, 3, 4 \) to obtain an 8-length binary vector \( \mathbf{b} \). This binary vector is mapped to the centered 256-PAM alphabet by computing \( 2 \sum_{i=0}^{7} b_i 2^i - 255 \).
C. Decoding decision function $\phi$ and Forney’s decision rule

A first choice for $\phi$, which we shall denote $\phi_1$, would be to allow the relay to decode as soon as the mutual information between the source and the relay exceeds $MTR$, i.e.,

$$\phi_1(h) = \min \left\{ M, \left\lceil \frac{MR}{\log(1 + |h|^2\rho')} \right\rceil \right\},$$

where $\rho'$ is the SNR of the source-relay link. This rule is asymptotically optimal for large $T$, in fact it coincides with the rule in the original formulation of the DDF protocol [5]. For finite $T$, $\phi_1$ is suboptimal since it ignores the actual signal received by the relay, i.e., the atypical behavior of the noise may dominate the error probability for short block lengths. As an illustration of the inefficacy of this decision function at finite block-length, consider the simulation results in Fig. 5. In the simulations to follow, we choose $X_s$ to be a rotated QAM code. We will subsequently compare these results with those obtained by choosing $X_s$ to be a permutation code, and observe very similar trends. We consider ML decoding at both the relay and destination for all the simulations in this sub-section. Further, we assume that the source-relay link SNR $\rho'$ is 3 dB above the SNR $\rho$ of all other links in all our simulations (the X-axis on all our plots is the SNR $\rho$ in dBs).

The simulations in Fig. 5 are for the case when $X_s$ is a rotated QAM code, $T = 1$, $M = 4$ and $R = 4$ bits per channel use (bpcu). The seemingly strange non-monotonic behavior of the error probability can be understood by the following intuitive explanation. At low SNRs, the relay hardly ever triggers before $m = 4$, resulting in $P(\mathcal{E})$ being dominated by the error probability at the destination $P(\mathcal{E}, \mathcal{E}_r)$, and hence $P(\mathcal{E})$ is large and decreasing. Then, there is an intermediate region of SNR where the relay attempts to decode, but it decodes incorrectly with high probability and causes significant interference at the destination. Thus $P(\mathcal{E})$ is dominated by the relay error probability $P(\mathcal{E}_r)$, and increases in this region. For sufficiently large SNR, the relay decodes correctly with high probability and therefore helps the destination, thus providing the required cooperative diversity (slope of the overall error curve at high SNR). However, this happens at very large gap from the outage probability, that can be regarded as a de-facto optimal performance also for finite-length codes and not asymptotically high SNR. This simulation reveals a phenomenon that has been scantily treated in previous works: the effect of decoding errors at the relay clearly dominates the overall performance. This fact has often been neglected since it is neither captured by the $T \to \infty$ case, where the atypicality of the noise has no effect and triggering the relay based on the outage event is exact, nor by the DMT formulation, that does not capture the gap from outage, but just the asymptotic error probability curve slope.

One immediate remedy consists of adopting a conservative relay decoding decision function, which
Fig. 5. \( X_s \) is a rotated QAM code, \( T = 1 \), \( M = 4 \), \( R = 4 \) bpcu, relay implements \( \phi_1(\cdot) \).

we will denote as \( \phi_2 \), defined as

\[
\phi_2 = \min \left\{ M, \left[ \frac{MR}{\log(1 + |h|^2 \rho')} \right] + 1 \right\}.
\]

Simulation results using this strategy are shown in Fig. 6, once again for the case of \( X_s \) being a rotated QAM code, \( T = 1 \), \( M = 4 \) and \( R = 4 \) bpcu.

In this case, the relay error probability is so low that no errors were recorded in our Monte Carlo simulation (no such curve is shown in Fig. 6). The downside of this strategy however is that since the relay is over-conservative, it helps the transmitter too late, and the overall error probability \( P(\mathcal{E}, \mathcal{E}_r) \) suffers from significant degradation with respect to outage probability.

In [16], the authors prove that there is no loss in DMT for the DDF protocol using the Alamouti type relay by using the following relay decoder function \( \phi_3 \):

\[
\phi_3 = \min \left\{ M, \max \left\{ M/2, \left[ \frac{MR}{\log(1 + |h|^2 \rho')} \right] \right\} \right\}.
\]
i.e., the relay is allowed to decode and transmit only after the codeword from the source is at least half-way through. Simulation results of this protocol shown in Fig. 7 reveal that this scheme also suffers from a significant penalty at high SNRs due to the \( P(\mathcal{E}_r) \) term dominating the overall error probability.

In [16], the authors use an extra layer of cyclic-redundancy check (CRC) coding to enable the relay to perform error detection (and wait for another round if incorrect decoding is detected) - while this strategy is effective in reducing \( P(\mathcal{E}_r) \), there is an inherent loss of rate. In fact, it is shown in [12] for the MIMO-ARQ channel that CRC is suboptimal in terms of DMT since the undetected error probability must decrease with SNR at least with the same exponent of error probability itself, and this requires a number of CRC bits that grow linearly with \( \log \text{SNR} \). The same consideration applies here. Hence, we wish to avoid the use of CRC in order to detect if the relay decodes in error.

We present a novel strategy to enable error detection at the relay without further layers of coding at
the transmitter. We make use of a criterion introduced by Forney in [14] in the context of retransmission (ARQ) protocols to decide whether the decoder is in error or accept the decoding outcome. Here, we apply this criterion to the relay decoder, that we refer to as Forney’s decision rule. Interestingly, Forney’s decision rule is similar in essence to the bounded distance decoder that we have considered in the proof of Theorem 2. However, while the bounded distance decoder is easy to analyze but only asymptotically optimal, Forney’s decision rule has the remarkable property of striking an optimal balance between the probability of undetected error at the relay and the probability of rejecting the decision and waiting for the next slot (probability of decision “erasure”, in the language of [14]). To the best of our knowledge, this decoding decision rule was not proposed before in the context of relay cooperative communication.

We define the decoding decision function $\phi_F(h, y_r)$ using Forney’s decision rule as follows:

Fig. 7. $X_a$ is a rotated QAM code, $T = 1$, $M = 4$, $R = 4$ bpcu, relay implements $\phi_3(\cdot)$. 
1) If $\phi_1(h) = M$, don’t decode and set $\phi_F(h, y_r) = M$ (worthless trying to decode if we are in outage).

2) If $\phi_1(h) = m < M$, decode after the $m^{th}$ block and apply the following threshold test. Let $\tilde{\omega}$ denote the outcome of the relay decoder. Accept the decision and trigger the transmission mode if

$$
\frac{p(y^m_r | \tilde{\omega}, h)}{\sum_{\omega \neq \tilde{\omega}} p(y^m_r | \omega, h)} \geq \tau,
$$

(40)

where $\tau$ a suitable threshold set empirically for each SNR. If the threshold is not exceeded, wait for the next block and repeat this step until either the threshold is exceeded or $m = M$.

$\phi_F$ is found to be extremely effective in suppressing the error probability at the relay without being too conservative and refraining from helping the destination when possible. Simulation results for the case when $X_s$ is a rotated QAM code, $T = 1$, $M = 4$ and $R = 4$ bpcu are shown in Fig. 7.

---

Fig. 8. $X_s$ is a rotated QAM code, $T = 1$, $M = 4$, $R = 4$ bpcu, relay implements $\phi_F(\cdot)$. 

---
The error probability is within 1 dB from the corresponding outage probability.

Fig. 9 shows the results when we choose $X_s$ to be a permutation code, with $T = 1$, $M = 4$ and $R = 4$ bpcu. In this figure we considered two cases: the case of a genie aided relay, where a genie provides the relay with the source message as soon as the mutual information exceeds $MTR$, and the case where the relay performs minimum distance decoding in conjunction with Forney’s rule. The results are similar in flavour to the case where $X_s$ is a rotated QAM code, with the permutation code losing 1 dB with respect to the rotated QAM code. Notice that despite the DMT optimality, these codes may perform differently depending on their shaping and coding gain. In this case, it is apparent that the rotated QAM code outperforms the permutation code, although they achieve the same diversity.
D. Low complexity MMSE-GDFE Lattice Decoding

As we saw in (39), the choice of rotated QAM codes makes $X_s$ a lattice code. Let $\Lambda$ be the $2MT$-dimensional lattice corresponding to the generator matrix $G$ in (39). MMSE-GDFE lattice decoding has been shown to be DMT optimal for the class of lattice space-time (LaST) codes over MIMO channels [11], and has also been shown to perform well for deterministic structured LaST (S-LaST) codes [26], [27]. Let $V(\Lambda)$ denote the fundamental Voronoi cell of an $n$-dimensional lattice $\Lambda$ (See [20] for definitions relating to lattice theory). The lattice quantization function is defined by

$$ Q_\Lambda(y) \triangleq \arg\min_{\lambda \in \Lambda} |y - \lambda| $$

and the modulo-lattice function is given by

$$ y \mod \Lambda \triangleq y - Q_\Lambda(y). $$

In the sequel, we will work with the real channel model which is equivalent to (1), (2) and (3), obtained by writing signals explicitly in terms of their real and imaginary parts (see for example [1] for details regarding the equivalence between real and complex channel models). By slight abuse of notation, we will refer to the real equivalent of the complex vectors and matrices $x_s, y, y_r, \Lambda, G$ using the same notation. In order to use reduced complexity MMSE-GDFE lattice decoding [11], information needs to be encoded onto cosets of a sublattice $\Lambda_s$ of $\Lambda$, as follows. Choose $\Lambda_s = Q\Lambda$, where $Q \in \mathbb{Z}_+$. Thus $|\Lambda/\Lambda_s| = Q^{2MT}$. Let $\mathcal{C}$ denote the set of points $\{Gz \mid z \in \mathbb{Z}_Q^{2MT}\}$, where $\mathbb{Z}_Q \triangleq \{0, 1, \ldots, Q - 1\}$. The transmitter selects a codeword $c \in \mathcal{C}$, generates a pseudo-random dither signal $u$ with uniform distribution over $V(\Lambda_s)$, and obtains the transmitted codeword

$$ x_s = [c - u] \mod \Lambda_s. $$

Thus information is encoded onto the cosets of the partition $\Lambda/\Lambda_s$: $x_s$ is a coset representative of the coset onto which the information is encoded, and belongs to the fundamental Voronoi region of $\Lambda_s$. Let $\mathcal{C}_\omega$ denote the coset of $\Lambda_s$ in $\Lambda$ onto which the information corresponding to message $\omega$ is encoded. From (1), (2) and (3) the (real equivalent) received signals $y_r = y_{r,0}^m$ for $M = m$ at the relay and $y_s = y_{s,0}^M$ at the destination may be written as

$$ y_r = H_r x_s + v $$

and

$$ y_s = H x_s + w, $$

(41)

(42)
where \( \mathbf{H}_r \in \mathbb{C}^{2mT \times 2M_T} \) and \( \mathbf{H} \in \mathbb{C}^{2MT \times 2M_T} \) denote the (real) equivalent channels at the relay and destination, and \( \mathbf{v} \) and \( \mathbf{w} \) denote the (real equivalent) noise at the relay and destination respectively. Notice from (41) that decoding at the relay corresponds to solving an under-determined system of linear equations. We follow the approach of [34] in this case, where it was shown how MMSE-GDFE lattice decoding may be used to efficiently solve under-determined systems of linear equations. We focus on decoding at the relay in the sequel, the decoder at the destination is identical upon replacing the relevant signals and parameters at the relay with those at the destination. Let \( \mathbf{F} \) and \( \mathbf{B} \) denote the forward and backward filters of the MMSE-GDFE (see for example [11] for the definition of these matrices in terms of \( \mathbf{H}_r \) and the relay SNR). The relay produces the modified observation

\[
y'_r \triangleq \mathbf{Fy}_r + \mathbf{Bu},
\]

and computes

\[
\hat{\mathbf{z}} = \arg \min_{\mathbf{z} \in \mathbb{Z}^{2M_T}} |y'_r - \mathbf{Bz}|^2.
\]

The relay then decides in favor of the coset \( \mathcal{C}_\omega \) that contains the point

\[
[\mathbf{G}\hat{\mathbf{z}}] \mod \Lambda_s.
\]

In order to work with the lattice coding and decoding scheme, Forney’s decision rule (40) needs to be modified to take into account the fact that information is encoded onto cosets as against points in the lattice. Encoding information into cosets is equivalent to consider a modulo-\( \Lambda_s \) channel with output \( y'_r \mod \mathbf{B}\Lambda_s \). Hence, the relevant likelihood function is given by

\[
\bar{p}(y'_r, \omega, h) = \sum_{\lambda_s \in \Lambda_s} p_{\tilde{\omega}}(y'_r - \mathbf{B}(\mathbf{c}_\omega + \lambda_s)|h)
\]

with domain \( y'_r \in \mathcal{V}(\mathbf{B}\Lambda_s) \), where \( \mathbf{c}_\omega \) is a coset representative of \( \mathcal{C}_\omega \) and where \( p_{\tilde{\omega}}(w|h) \) denotes the pdf of the noise induced by the modulo-\( \Lambda_s \) channel with the dithering, that is, \( \tilde{w} = y'_r - \mathbf{Bx}_s \) where \( \mathbf{x}_s \) is the transmitted signal. Unfortunately, \( p_{\tilde{\omega}} \) is difficult if not impossible to determine in closed form. However, a good practical choice that works well for good shaping lattices is to let \( p_{\tilde{\omega}} \) be a Gaussian pdf with i.i.d. components \( \sim \mathcal{N}(0, \sigma_w^2/2) \) (see [11] for a theoretical asymptotic justification of Gaussianity in this context). Then, the proposed modification of Forney’s decision rule (40) for the lattice MMSE-GDFE decoder is given by: accept \( \tilde{\omega} \) at time \( m \) if

\[
\frac{\sum_{\lambda_s \in \Lambda_s} p_{\tilde{\omega}}(y'_r - \mathbf{B}(\mathbf{c}_\omega + \lambda_s)|h)}{\sum_{\tilde{\omega} \neq \omega \in \Lambda_s} \sum_{\lambda_s \in \Lambda_s} p_{\tilde{\omega}}(y'_r - \mathbf{B}(\mathbf{c}_\omega + \lambda_s)|h)} \geq \tau,
\]

where, again \( \tau \) is a suitable threshold set empirically for each SNR. The infinite sums at numeration and denominator can be safely truncated by restricting to a number of most likely lattice points, which may
be done as follows. Generate a list of $\mathcal{N} = \{B\lambda_i : \lambda_i \in \Lambda\}_{i=1}^{N}$ of lattice points of the lattice generated by $BG$ that are closest to $y_r'$. Such a list may be generated, for example, by using a standard lattice decoder with a sufficiently large search radius. For any given message $\omega$, check whether $\lambda_i$ belongs to the coset $c_{\omega} + \Lambda_s$. If yes, then this point makes a contribution towards the numerator of (43), else towards the denominator. If there exists $\hat{\omega}$ for which the corresponding ratio crosses the threshold $\tau$ then accept the decision, otherwise reject and wait for the next slot.

The modified Forney’s rule in (43) is seen to be quite effective for the case when MMSE-GDFE lattice decoding is performed at both the relay and the destination. The simulations in Fig. 10 compares the performance of rotated QAM codes with $T = 1$, $M = 4$, $R = 4$ bpcu that use Forney’s rule (40) with ML decoding and modified Forney’s rule (43) with MMSE-GDFE lattice decoding. The low complexity lattice decoder tracks the ML performance within 1 dB.

Fig. 10. $X_s$ is a rotated QAM code, $T = 1$, $M = 4$, $R = 4$ bpcu, relay implements Forney’s or modified Forney’s rule.
V. CONCLUSION

We presented a characterization of the achievable DMT of the single-relay DDF protocol for finite block length. Our achievability proof yields insight on the design of actual coding schemes. In particular, we stressed the importance of a relay decoding function that check the reliability of its decision, in order not to jam the destination. Also, we showed that the destination need not be aware of the relay decision time, since a GLRT-based decoder achieves optimal DMT performance. This may have some impact on the design of practical DDF protocols, since it essentially shows that no complicated side-information channel needs to be implemented in order to explicitly notify the destination about when the relay starts transmitting.

In our proofs, we considered a bounded distance decoder at the relay and an ensemble of random Gaussian codes. Then, we constructed practical and very simple codes based on lattices (rotated QAM constellations) and permutation codes. We demonstrated via simulation that the impact of undetected decoding errors at the relay may be huge. In order to tackle this problem, we have proposed the use of Forney’s decision rejection rule, that proves to be very effective.

Finally, we have investigated the use of a reduced complexity MMSE-GDFE lattice decoder and modulo-L lattice codes, that yields the well-known low-complexity decoding even at the relay. It should be remarked that the relay invariably has to decode an undetermined linear system, therefore standard sphere decoding algorithms fail.

A few comments relating to future work are in order here. We have not exploited the low-complexity quantization-based decoding approach for permutation codes [21], owing to the fact that it is not completely clear as to how we can apply a good decision rejection rule at the relay in this case. Another interesting problem relates to code design for finite but large $T$; in this case, neither the constructions presented in this paper, nor those in [28], [29] are fully controllable in terms of coding gain and both entail very high decoding complexity. Concatenation of short codes based on rotated QAM constellations or permutation codes with some form of outer coding (along the lines of [25]) may prove to be appropriate for this scenario.

APPENDIX I
SEPARETED RELAY ACTIVITY DETECTION

In this Appendix we treat a side problem. An intuitive low-complexity scheme for detecting the relay decision time consists of treating $M$ as a random parameter, and use ML detection by disregarding the structure of the channel codes. Intuitively, the destination should be able to detect a transition in the
received power, between the listening phase and the transmission phase of the relay. This approach is referred to as separated Relay Activity Detection (RAD), since the decision time and the source codeword are separately decoded, in contrast with the GLRT decoder analyzed in the proof of Theorem 2. We shall show that separated RAD yields no performance loss when we consider the limit of $T \to \infty$. On the contrary, it is suboptimal and actually may perform very poorly when limits are taken in the reverse order, that is, for each finite $T$ we consider the performance as SNR gets large.

We assume that the source uses an i.i.d. random Gaussian code and the relay implements the Alamouti-DDF scheme. As before, let $M$ denote the decision time. An ML decision time detector that is ignorant of the codebooks treats the channel input as a random Gaussian signal. The detection rule is given by

$$
\hat{M} = \arg \max_m p(y|M = m, g_1, g_2).
$$

where $p(y|M = m, g_1, g_2)$ shall be denoted in the following simply by $p(y|m)$ for simplicity, and it is given by

$$
p(y|m) = \frac{1}{\pi((|g_1|^2 \rho + 1))^T} \exp \left( -\frac{|y^m_0|^2}{|g_1|^2 \rho + 1} \right) \cdot \frac{1}{\pi (|g_1|^2 + |g_2|^2) \rho + 1} \exp \left( -\frac{|y^m_M|^2}{(|g_1|^2 + |g_2|^2) \rho + 1} \right). \tag{45}
$$

Suppose $M = m$, we define the pairwise error event

$$
\{m \to m'\} \triangleq \left\{ \frac{p(y|m')}{p(y|m)} \geq 1 \right\}.
$$

The detector error probability is lower bounded by

$$
P(M \neq \hat{M}) \geq \max_{m \neq m'} P(m \to m'),
$$

and is upper bounded by the union bound

$$
P(M \neq \hat{M}) \leq (M - 1) \max_{m \neq m'} P(m \to m').
$$

Hence, we shall study the diversity exponent of $P(m \to m')$ for general $m \neq m'$. If this does not depend on $m, m'$ we have determined the diversity exponent of the separated RAD.

A. Infinite block-length

If $M = m$ and $T \to \infty$, the law of large numbers yields the almost sure convergence of the limits:

$$
\frac{1}{T} |y^m_{n-1}|^2 \to |g_1|^2 \rho + 1, \quad 1 \leq n \leq m
$$
and
\[
\frac{1}{T} |y_{n-1}|^2 \to (|g_1|^2 + |g_2|^2)\rho + 1, \ m + 1 \leq n \leq M.
\]

Thus, for large $T$ we have
\[
p(y|m) \approx \exp \left\{ -MT - mT \log \left[ \pi(|g_1|^2\rho + 1) \right] - (M - m)T \log \left[ \pi((|g_1|^2 + |g_2|^2)\rho + 1) \right] \right\}.
\]

Consider the case $m' > m$ (the other case follows in the same way and it is omitted for brevity). We have
\[
p(y|m') \approx \exp \left\{ -mT - (M - m')T - (m' - m)T \frac{(|g_1|^2 + |g_2|^2)\rho + 1}{|g_1|^2\rho + 1} - m'T \log \left[ \pi(|g_1|^2\rho + 1) \right] - (M - m')T \log \left[ \pi((|g_1|^2 + |g_2|^2)\rho + 1) \right] \right\}.
\]

After some simplifications, the pairwise error probability for $T \to \infty$ is given by
\[
P(m \to m'|g_1, g_2) = P(1 - X_1 + \log X_1 \geq 0), \quad (46)
\]
where we let
\[
X_1 = \frac{|g_1|^2 + |g_2|^2)\rho + 1}{|g_1|^2\rho + 1}.
\]

Since $\log x \leq x - 1 \ \forall \ x \geq 0$, we see that $\{m \to m'\}$ can occur only if $|g_2|^2 = 0$, which is an event of measure 0. Therefore, we conclude that $P(m \to m') \downarrow 0$ for any fixed $\rho$, as $T \to \infty$. This shows that for the infinite $T$ case, even a very simple separated RAD scheme at the destination yields perfect knowledge of the relay decision time without any need of a side information channel that involves some protocol overhead.

**B. Finite block-length**

We now fix $T$ to be an arbitrary finite value and study the diversity exponent of $P(m \to m')$ as $\rho \to \infty$. Again, we consider only the case $m' > m$. The likelihood function for the hypothesis $m'$ when $M = m$ is given by
\[
p(y|m') = \exp \left\{ -m'T \log \left[ \pi(|g_1|^2\rho + 1) \right] - (M - m')T \log \pi \left[ (|g_1|^2 + |g_2|^2)\rho + 1 \right] \right. \\
- \frac{|y_0^m|^2 + |y_m^m|^2}{|g_1|^2\rho + 1} - \frac{|y_M^m|^2}{(|g_1|^2 + |g_2|^2)\rho + 1}\left. \right\}.
\]

After some algebra, we find that
\[
P(m \to m'|g_1, g_2) = P \left( \chi \leq \frac{(m' - m)T}{X_2} \log(1 + X_2) \right), \quad (47)
\]
where

\[ \chi = \frac{|y_m'|^2}{1 + (|g_1|^2 + |g_2|^2)\rho} \]

is a central chi-squared random variable with \(2T(m' - m)\) degrees of freedom and mean \(T(m' - m)\), and we define

\[ X_2 = \frac{|g_2|^2 \rho}{|g_1|^2 \rho + 1}. \]

As an aside, notice that \(\frac{1}{x} \log(1 + x)\) is a decreasing function of \(x\) that is less than \(1\) for all \(x > 0\), and approaches 1 for \(x \downarrow 0\). Therefore, the term \(\frac{(m' - m)T}{X_2} \log(1 + X_2)\) in (47) is always strictly less than \(\mathbb{E}[\chi] = (m' - m)T\) for all \(|g_2| > 0\). Therefore, as an application of the the large deviation theorem [17], we find that

\[ P(m \to m') \downarrow 0 \text{ exponentially with } T \text{ for all finite } \rho \text{ and } |g_2| > 0. \]

Thus, we recover in a more rigorous way the result obtained before by letting \(T \to \infty\) directly in the detector decision metric.

Returning to the case of finite \(T\), we have using well-known properties of the chi-squared distribution that

\[ P(\mathcal{X} \leq u) = \frac{1}{(m' - m)T!} u^{(m' - m)T} + O(u^{(m' - m)T+1}) \]

for small \(u\), and obviously

\[ P(\mathcal{X} \leq u) = O(1) \]

when \(u = \beta(m' - m)T\) for some constant \(\beta > 0\). Fix an arbitrary \(0 < \beta < 1\). From what was said before, there exists an \(x_2 > 0\) such that \(\frac{1}{x_2} \log(1 + x_2) = \beta\). Hence, consider the event

\[ \mathcal{E}(\rho, \beta) = \{X_2 \leq x_2\} = \{|g_2|^2 \rho \leq x_2(1 + |g_1|^2 \rho)\}. \] (48)

It is clear that for all \((g_1, g_2) \in \mathcal{E}(\rho, \beta)\), the pairwise error probability \(P(m \to m'|g_1, g_2)\) in (47) is exponentially equivalent to a constant as \(\rho \to \infty\), i.e.,

\[ P(m \to m'|g_1, g_2) \equiv \rho^0, \quad (g_1, g_2) \in \mathcal{E}(\rho, \beta). \]

Averaging with respect to \(g_1, g_2\), and using the standard variable substitution \(|g_1|^2 = \rho^{-\alpha_1}, |g_2|^2 = \rho^{-\alpha_2}\), we find

\[ P(m \to m') \geq \int_{\mathcal{E}} \rho^0 e^{-\rho^{-\alpha_1} - \rho^{-\alpha_2}} \rho^{-\alpha_1 - \alpha_2} d\alpha_1 d\alpha_2 \]

\[ = \int_{\mathcal{E}'} \rho^{-\alpha_1 - \alpha_2} d\alpha_1 d\alpha_2, \]

where, from \([48]\),

\[ \mathcal{E}' = \{(\alpha_1, \alpha_2) \in \mathbb{R}_+^2 : 1 - \alpha_2 \leq [1 - \alpha_1]_+\}. \]
Using Varadhan’s lemma, we find that the diversity exponent of the pairwise error probability is given by

\[
\Delta = \inf_{(\alpha_1, \alpha_2) \in \mathcal{E}'} \{\alpha_1 + \alpha_2\} = 0,
\]

since the point \(\alpha_1 = 0, \alpha_2 = 0\) belongs to the boundary of the region \(\mathcal{E}'\).

This shows that for any finite \(T\), a separated RAD scheme based on optimal (Maximum Likelihood) detection of the relay decision time \(M\) that ignores the codebook structure and treats the transmitted signals as random processes is very suboptimal. In fact, the probability of error of such a scheme is constant with SNR and eventually will dominate the performance of the whole destination decoder.

In some way, this result shows that the joint detection of the relay decision time and of the information message is necessary in order to achieve the optimal (infinite \(T\)) DDF DMT.

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