Point-splitting regularization of the stress tensor of a coupling scalar field in de Sitter space

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Abstract

We perform the point-splitting regularization on the vacuum stress tensor of a coupling scalar field in de Sitter space under the guidance from the adiabatically regularized Green’s function. For the massive scalar field with the minimal coupling $\xi = 0$, the 2nd order point-splitting regularization yields a finite vacuum stress tensor with a positive, constant energy density, which can be identified as the cosmological constant that drives de Sitter inflation. For the coupling $\xi \neq 0$, we find that, even if the regularized Green’s function is continuous, UV and IR convergent, the point-splitting regularization does not automatically lead to an appropriate stress tensor. The coupling $\xi R$ causes log divergent terms, as well as higher-order finite terms which depend upon the path of the coincidence limit. After removing these unwanted terms by extra treatments, the 2nd-order regularization for small couplings $\xi \in (0, 0.01)$, and respectively the 0th-order regularization for the conformal coupling $\xi = \frac{1}{6}$, yield a finite, constant vacuum stress tensor, in analogy to the case $\xi = 0$. For the massless field with $\xi = 0$ or $\xi = \frac{1}{6}$, the point-splitting regularization yields a vanishing vacuum stress tensor, and there is no conformal trace anomaly for $\xi = \frac{1}{6}$. If the 4th-order regularization were taken, the regularized energy density for general $\xi$ would be negative, which is inconsistent with the de Sitter inflation, and the regularized Green’s function would be singular at the zero mass, which is unphysical. In all these cases, the stress tensor from the point-splitting regularization is equal to that from the adiabatic one. We also discuss the issue of the adequate order of regularization.

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1 Introduction

The stress tensor and the Green’s function of a quantum field in vacuum state have ultra-violet (UV) divergences due to the zero-point fluctuations. Unlike in the Minkowski spacetime, these...
divergences may not simply be dropped [1, 2] since the finite part of fluctuations has gravitational
effects. For instance, the vacuum energies of the inflationary scalar field is a natural candidate for
the driving source of inflation expansion, and the vacuum fluctuations of the scalar field, together
with the perturbed metric field, can induce the CMB anisotropies and polarization [3–8]. To remove
the UV divergences, two classes of regularization methods have been proposed, the point-splitting
regularization [9–16] in the \(x\)-space, and the adiabatic regularization in the momentum \(k\)-space
[17–33]. The dimensional regularization [34–39], and the zeta function regularization [35, 36, 40]
work in the \(x\)-space and can be classified into the point-splitting.

The essence of a regularization program is to choose an appropriate subtraction term so that the
regularized vacuum stress tensor be UV and IR convergent, and respect the covariant conservation,
and, for a massive scalar field, the regularized vacuum energy density be positive. In de Sitter space,
the regularized vacuum stress tensor should also possess the maximum symmetry of de Sitter space.
The regularized spectral energy density and power spectrum should also be UV and IR convergent,
and positive. We refer to these as the desired properties of a regularized vacuum stress tensor. The
adiabatic regularization [17–33] deals with UV divergences in terms of the \(k\)-modes of a quantum
field. The subtraction term is systematically prescribed by the WKB approximate modes to certain
adiabatic order. There is no universal recipe of regularization for a general coupling. Related to
this, Ref. [18] assumed the minimal subtraction rule that only the minimum number of terms should
be subtracted to yield the convergent stress tensor. In the conventional prescription, for a scalar
field, the 4th-order subtraction is used for the stress tensor [18] and the 2nd-order for the power
spectrum [28]. However, in Ref. [41] we found that, for a massive scalar field in de Sitter space,
the 4th-order regularization leads to a negative spectral energy density, and that the conformal
trace anomaly for \(\xi = \frac{1}{6}\) is an artifact caused by the improper 4th-order subtraction term. We also
showed that, the 2nd-order adiabatic regularization for \(\xi = 0\) yields the positive, UV convergent
spectral energy density and power spectrum, and so does the 0th-order regularization for \(\xi = \frac{1}{6}\),
and there is no conformal trace anomaly [9–11,14,35–38,40]. In Ref. [42] we have studied the adiabatic
regularization of a massless scalar field, and the resulting stress tensor is zero for \(\xi = 0\) and \(\xi = \frac{1}{6}\,
agreeing with the massless limit of the massive field [41]. For these cases, the regularized spectral
stress tensor has been obtained analytically, from which follows the regularized stress tensor by the
numerical \(k\)-integratin.

The point-splitting regularization deals with the UV divergences in the \(x\)-space. In this method,
the stress tensor is constructed from the Green’s function via differentiations and the coincidence
limit. If the regularized Green’s function is available, one can use it to calculate the regularized stress
tensor. In literature, in lack of the full expression of regularized Green’s function, the unregularized
Green’s function is often expanded at small separation, and several UV divergent terms are removed.
But this kind of naive subtraction would cause new IR divergences. In Ref. [41], for the scalar field
with \(\xi = 0\) and \(\xi = \frac{1}{6}\) respectively, we have obtained the analytical expression of the regularized
Green’s function valid on the whole range of spacetime. This has been achieved by the Fourier
transformation of the adiabatically regularized power spectrum of a pertinent order.

In this paper, we shall perform the point-splitting regularization on the stress tensor, using the
2nd-order regularized Green’s functions for the minimal-coupling \(\xi = 0\) and the small coupling
\(\xi > 0\), and respectively the 0th-order regularized Green’s functions for the conformal coupling
\(\xi = \frac{1}{6}\) [41]. In these cases, we shall perform calculation in two equivalent schemes: One scheme is to
calculate the regularized stress tensor from the regularized Green’s function, another is to calculate
the unregularized, and subtraction stress tensors and then to take their difference. As we shall see,
given a well-defined, regularized Green’s function with \(\xi \neq 0\) may not automatically lead to an
appropriate stress tensor, and extra treatments are needed in both schemes. Besides, we shall also
perform the 4th-order regularization for a general \(\xi\) and point out the difficulties of its outcome.

In Sect. 2, we list the exact solution, the power spectrum, and the Green’s function of the
coupling massive scalar field in the vacuum state in de Sitter inflation.

In Sect. 3, we list the adiabatically regularized Green’s functions to be used in Sections 4, 5, 6.

In Sect. 4, for the minimally-coupling $\xi = 0$, we use the 2nd-order adiabatically regularized Green’s function to calculate the vacuum stress tensor by the point-splitting method. The regularized stress tensor is obtained with a positive energy density.

In Sect. 5, for a general coupling $\xi > 0$, we also adopt the 2nd-order point-splitting regularization. By extra treatments, we obtain the regularized stress tensor which has a positive energy density for small couplings $0 \leq \xi < \frac{1}{6}$ at a fixed parameter ratio $m^2 H^2 = 0.1$.

In Sect. 6, for the conformally-coupling $\xi = \frac{1}{6}$, we adopt the 0th-order point-splitting regularization. By the treatments analogous to Sect. 5, we obtain the stress tensor with a positive energy density.

In Sect. 7, as an examination, we perform the 4th-order point-splitting regularization for a general $\xi$. The regularized energy density is negative, and the regularized Green’s function is singular at the zero mass $m = 0$.

Sect. 8 gives the conclusions and discussions.

Appendix A lists some formulae of differentiation which are used in the context.

Appendix B shows certain terms which depend on the path of the coincidence limit.

## 2 The scalar field during de Sitter inflation

The metric of a flat Robertson-Walker spacetime is

$$ds^2 = a^2(\tau)[d\tau^2 - \delta_{ij}dx^i dx^j],$$

(1)

with the conformal time $\tau$. The Lagrangian density of a massive scalar field $\phi$ is

$$\mathcal{L} = \frac{1}{2} \sqrt{-g} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 - \xi R \phi^2),$$

(2)

and the field equation is

$$\Box \phi + m^2 + \xi R \phi = 0,$$

(3)

where $R = 6a''/a^3$ is the scalar curvature, $m$ is the mass, and $\xi$ is a coupling constant. For specific, we consider $0 \leq \xi \leq \frac{1}{6}$ in this paper. The energy momentum tensor of the scalar field is given by [25, 27]

$$T_{\mu\nu} = (1 - 2\xi)\partial_\mu \phi \partial_\nu \phi + (2\xi - \frac{1}{2})g_{\mu\nu} \partial^\sigma \phi \partial_\sigma \phi - 2\xi \phi_{,\mu\nu} \phi$$

$$+ \frac{1}{2} \xi g_{\mu\nu} \Box \phi - \xi (R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R + \frac{3}{2} \xi R g_{\mu\nu}) \phi^2 + \left(\frac{1}{2} - \frac{3}{2} \xi\right) g_{\mu\nu} m^2 \phi^2,$$

(4)

satisfying the conservation law $T^\mu_{\;\nu,\nu} = 0$. Using the field equation (3), it can be also written as

$$T_{\mu\nu} = (1 - 2\xi)\partial_\mu \phi \partial_\nu \phi + (2\xi - \frac{1}{2})g_{\mu\nu} \partial^\sigma \phi \partial_\sigma \phi - 2\xi \phi_{,\mu\nu} \phi$$

$$+ 2\xi g_{\mu\nu} \Box \phi - \xi G_{\mu\nu} \phi^2 + \frac{1}{2} g_{\mu\nu} m^2 \phi^2,$$

(5)

with $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. In the de Sitter space, $G_{\mu\nu} = -\frac{1}{2} g_{\mu\nu} R$ and $R = 12H^2$. The trace of (5) is

$$T^\mu_{\;\mu} = (6\xi - 1)\partial^\mu \phi \partial_\mu \phi + \xi (1 - 6\xi) R \phi^2 + 2(1 - 3\xi) m^2 \phi^2,$$

(6)
in particular,

\[ T_{\mu}^{\mu} = -\partial_{\mu} \phi \partial_{\mu} \phi + 2m^2 \phi^2 \quad \text{for } \xi = 0, \]

\[ T_{\mu}^{\mu} = m^2 \phi^2 \quad \text{for } \xi = \frac{1}{6}, \]

where the equation (3) has been used.

The field operator can be written in terms of its Fourier modes as the following

\[ \phi(r, \tau) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[ a_k \phi_k(\tau) e^{i k \cdot r} + a_k^* \phi_k^*(\tau) e^{-i k \cdot r} \right] \]

where \( a_k, a_k^* \) are the annihilation and creation operators and satisfy the canonical commutation relations. Since the field equation is linear, the \( k \)-modes are independent of each other. In this paper we consider de Sitter space with a scale factor \([43, 44]\)

\[ a(\tau) = \frac{1}{H|\tau|}, \quad -\infty < \tau \leq \tau_1, \]

where \( H \) is a constant, and \( \tau_1 \) is the ending time of inflation, and the scalar curvature \( R = 12H^2 \). Let \( \phi_k(\tau) = v_k(\tau)/a(\tau) \). Then the equation of \( k \)-mode \( v_k \) is

\[ v''_k + \left[ k^2 + \left( \frac{m^2}{H^2} + 12\xi - 2 \right) \tau^{-2} \right] v_k = 0. \]

The analytical solution is

\[ v_k(\tau) \equiv \sqrt{\frac{\pi}{2}} \sqrt{\frac{\nu + \frac{3}{2}}{2k}} H_{\nu + \frac{3}{2}}^{(1)}(k\tau), \]

with \( H_{\nu}^{(1)} \) being the Hankel function, and the conjugate \( v_k^* \) is another independent solution, where the variable \( x \equiv k|\tau| \), and

\[ \nu \equiv \left( \frac{9}{4} - \frac{m^2}{H^2} - 12\xi \right)^{1/2}. \]

In this paper we consider \( \nu \) being real. In the high \( k \) limit, the solution (11) approaches the positive-frequency mode \( v_k(\tau) \rightarrow \frac{1}{\sqrt{2k}} e^{-i k \cdot r} \). The Bunch-Davies (BD) vacuum state is defined as the state vector \( |0\rangle \) such that

\[ a_k |0\rangle = 0, \quad \text{for all } k. \]

The unregularized Green’s function in the BD vacuum state is given by the following \([13,34-36,41]\)

\[ G(x^\mu - x'^\mu) = \langle 0 | \phi(x^\mu) \phi(x'^\mu) | 0 \rangle = \frac{1}{(2\pi)^3} \int d^3k e^{i k \cdot (r - r')} \phi_k(\tau) \phi_k^*(\tau') \]

\[ = \frac{1}{8\pi a(\tau) a(\tau')} \frac{1}{|r - r'|} \int_0^\infty dk k \sin(k|r - r'|) H_{\nu}^{(1)}(k\tau) H_{\nu}^{(2)}(k\tau') \]

\[ = \frac{H^2}{16\pi^2} \Gamma\left( \frac{3}{2} - \nu \right) \Gamma\left( \nu + \frac{3}{2} \right) 2F_1 \left[ \frac{3}{2} + \nu, \frac{3}{2}, 2, 1 + \frac{\sigma}{2} \right], \]

where \( 2F_1 \) is the hypergeometric function \([45]\), and

\[ \sigma \equiv \frac{1}{(2\tau\tau')} \left[ (\tau - \tau')^2 - |r - r'|^2 \right]. \]
The Green’s function satisfies the equation
\[(\nabla_\mu \nabla^\mu + \xi R + m^2)G(x^\mu - x'^\mu) = 0.\] (17)
where \(\nabla_\mu\) is the covariant differentiation with respect to \(x\). It should be remarked that the analytic expression (15) is defined for \(\nu \neq \frac{3}{2}\) since the factor \(\Gamma\left(\frac{3}{2} - \nu\right)\) is divergent at \(\nu = \frac{3}{2}\) \((m = 0 = \xi)\), for which the Green’s function for \(\nu = \frac{3}{2}\) is given by (58) [41, 42]. The Green’s function at the equal time \(\tau = \tau'\) is
\[G(r - r') = \langle 0 | \phi(r, \tau) \phi(r', \tau) | 0 \rangle = \frac{1}{|r - r'|} \int_0^\infty \frac{\sin(k|r - r'|)}{k} \Delta_k^2(\tau) \, dk,\] (18)
and the auto-correlation function is
\[G(0) = \langle 0 | \phi(r, \tau) \phi(r, \tau) | 0 \rangle = \frac{1}{(2\pi)^3} \int d^3k |\phi_k(\tau)|^2 = \int_0^\infty \Delta_k^2(\tau) \frac{dk}{k},\] (19)
where the power spectrum is the following
\[\Delta_k^2(\tau) \equiv \frac{k^3}{2\pi^2 a^2} |\nu_k(\tau)|^2 = \frac{H^2}{8\pi} x^3 |H^{(1)}_{\nu}(x)|^2,\] (20)
which is nonnegative by definition. At low \(k\), \(\Delta_k^2 \propto k^{3-2\nu}\), giving an IR convergent auto-correlation (19) for \(\nu < \frac{3}{2}\). At high \(k\), \(\Delta_k^2 \propto k^3\left(\frac{1}{\pi^2} + \frac{4\nu^2 - 1}{16\pi^2 \nu^2}\right)\), leading to quadratic and logarithmic UV divergences of \(G(0)\). The corresponding Green’s function behaves as \(G(\sigma) \propto \sigma^{\nu - 3/2}\) at large \(\sigma\), which is IR convergent for \(\nu < \frac{3}{2}\). \(G(\sigma)\) is UV divergent at \(\sigma = 0\). In this paper we shall remove these UV divergences of the scalar field by regularization. (UV divergences also occur in the 2-point correlation function of relic gravitational wave [30, 31], and as well as in the 2-point correlation function of density perturbations [46–50] in Gaussian approximation, and we will not discuss these here.)

3 Adiabatic regularization of the Green’s function

There are two possible ways to remove the UV divergences of the Green’s function. One way is to adiabatically regularize the power spectrum and the perform the Fourier transformation of the regularized power spectrum, yielding the adiabatically regularized Green’s function which is both UV and IR convergent. This has been done in Refs. [41, 42]. Another way is to expand the Green’s function at small distance and to directly remove the UV divergences terms. However, as pointed in Refs. [41, 42], this kind of regularization in the position-space does not work easily. We give a brief summary as the following.

The expansion of \(G(\sigma)\) at small \(\sigma\) generally has the following form
\[G(\sigma) \simeq \frac{H^2}{16\pi^2} \left[-\frac{2}{\sigma} + \left(\frac{1}{4} - \nu^2\right) \ln \sigma + \text{const} \right] + O(\sigma),\]
where \(1/\sigma\) and \(\ln \sigma\) are UV divergent and should be removed. Often the exact subtraction terms for the Green’s function are not known, so that a Hadamard type of function is usually assumed as the following [9, 10, 13, 34–39]
\[G(\sigma)_{\text{sub}} = \frac{H^2}{16\pi^2} \left[-\frac{2}{\sigma} + \left(\frac{1}{4} - \nu^2\right) \ln \sigma \right],\] (21)
as an approximation to the exact subtraction terms at small distance. Some constant term can be added to (21) in order to ensure the covariant conservation. Although the \(\ln \sigma\) subtraction term of
(21) removes the log UV divergence at $\sigma = 0$, nevertheless, it will also cause a new IR divergence at $\sigma = \infty$ in the regularized Green’s function

$$G(\sigma)_{\text{reg}} = G(\sigma) - G(\sigma)_{\text{sub}}$$

for general $m$ and $\xi$. That is, the conventional Hadamard function (21) as a subtraction term is not valid at large $\sigma$, and one still lacks exact subtraction terms defined on the whole range of $\sigma$. In general, such an exact subtraction term is hard to find directly in position space. (However, for the special case $m = \xi = 0$, the exact subtraction term is easy to find, and the new $\ln \sigma$ IR divergence will not occur. See the paragraph around (58) later.) To find such an exact subtraction term for a general $\xi$, one can be assisted by the adiabatically regularized power spectrum defined in $k$-space. As shown in Refs. [41,42], for the cases $\xi = 0$ and $\xi = \frac{1}{6}$, subtracting off the UV divergent terms of the power spectrum in $k$-space to an appropriate adiabatic order gives a regularized, UV and IR convergent power spectrum. Then by the Fourier transformation of the regularized power spectrum, one obtains the adiabatically regularized, UV and IR convergent Green’s function. In this approach, the appropriate subtraction term for the Green’s function is the Fourier transformation of the subtraction term to the power spectrum, and is valid on the whole range of $\sigma$. As it turns out, its functional form is not simple, and quite different from the Hadamard function (21). In this paper, such kind of adiabatically regularized Green’s functions will be used in the point-splitting regularization, and we list the relevant formulae of the 2nd-order for $\xi = 0$ and the 0th-order for $\xi = \frac{1}{6}$ in the following.

The adequate regularization depends upon the coupling $\xi$. We assume the minimal subtraction rule that only the minimum number of terms should be subtracted to yield the convergent power spectrum. This was originally assumed for the stress tensor in Ref. [18]. For the minimally coupling $\xi = 0$, the 2nd-order regularization is adopted, and the regularized power spectrum is

$$\Delta_{k,\text{reg}}^2 = \frac{k^3}{2\pi^2 a^2} \left( |v_k|^2 - |v_k^{(2)}|^2 \right) = \frac{k^3}{2\pi^2 a^2} \left( |v_k|^2 - \frac{1}{2W(2)} \right),$$

where the 2nd-order effective inverse frequency

$$\frac{1}{W_k^{(2)}} = \frac{1}{\omega} - 3(\xi - \frac{1}{6}) \frac{a''/a}{\omega^3} + \frac{m^2(a''a + a')^2}{4\omega^5} - \frac{5m^4a''^2a^2}{8\omega^7}$$

with $\omega = \sqrt{k^2 + m^2a^2}$. (See (a29) in Ref. [41].) Replacing $\Delta_k^2$ by $\Delta_{k,\text{reg}}^2$ in eq.(18) yields the 2nd-order regularized Green’s function for $\xi = 0$ [41]

$$G(y)_{\text{reg}} = G(y) - G(y)_{\text{sub}}$$

$$= \frac{H^2}{8\pi^2} \frac{1}{y} \int_0^\infty dk k \sin(ky) \left( |H_\nu^{(1)}(k)|^2 - \frac{2}{\pi} \frac{1}{W(k)^2} \right),$$

where $y = |r - r'|$, $\nu = (\frac{9}{4} - \frac{m^2}{H^2})^{1/2}$, $G(y)$ is given by eq.(15), and the subtraction Green’s function is given by

$$G(y)_{\text{sub}} = \frac{H^2}{8\pi^2} \left[ \frac{w}{H} K_1 \left( \frac{w}{H} y \right) + (1 - 6\xi) K_0 \left( \frac{w}{H} y \right) + \frac{1}{2\pi} y K_1 \left( \frac{w}{H} y \right) - \frac{1}{2\pi} \frac{w^2}{2H} y^2 K_2 \left( \frac{w}{H} y \right) \right],$$

with $K_0$, $K_1$, and $K_2$ being the modified Hankel functions. The expression (26) differs from the naive expression (21). Fig.1 (a) shows that the 2nd-order regularized $\Delta_{k,\text{reg}}^2$ is positive, UV convergent and IR finite. Fig.1 (b) shows that the resulting $G(y)_{\text{reg}}$, is UV finite and IR convergent.

For the conformally coupling $\xi = \frac{1}{6}$, the 0th-order regularization is adopted

$$\Delta_{k,\text{reg}}^2 = \frac{k^3}{2\pi^2 a^2} \left( |v_k|^2 - |v_k^{(0)}|^2 \right) = \frac{k^3}{2\pi^2 a^2} \left( |v_k|^2 - \frac{1}{2W^{(0)}} \right),$$
Figure 1: (a): The 2nd-order regularized $\Delta_k^{\text{reg}}$ in (23) is IR and UV convergent. (b): The 2nd-order regularized $G(y)^{\text{reg}}$ in (25) is IR and UV convergent. The model for $\xi = 0$ and $\frac{m^2}{H^2} = 0.1$. Here $y = |r - r'|$ and $|\tau| = |\tau'| = 1$ for illustration.

where $1/W^{(0)} = 1/\omega$. The 0th-order regularized Green’s function for $\xi = 1/6$ is

\[
G(y)^{\text{reg}} = G(y) - G(y)^{\text{sub}} = \frac{H^2}{8\pi y} \int_0^\infty dk k \sin(ky) \left( |H^{(1)}(k)|^2 - \frac{2}{\pi} \right),
\]

(28)

where $\nu = (\frac{1}{4} - \frac{m^2}{H^2})^{1/2}$ for $\xi = 1/6$, and the subtraction Green’s function is given by

\[
G(y)^{\text{sub}} = \frac{H^2}{4\pi^2 y} m K_1\left(\frac{m H}{y}\right),
\]

(29)

also differing from the naive expression (21). Fig.2 (a) shows that the 0th-order regularized $\Delta_k^{\text{reg}}$ is positive, UV convergent and IR finite. Fig. 2 (b) shows that $G(y)^{\text{reg}}$ is UV finite and IR convergent.

For the nonequal time $\tau \neq \tau'$, by the maximal symmetry in de Sitter space, we just replace the variable $y \rightarrow \sqrt{-2\sigma}$ in the expressions (25) (28) and get the Green’s functions $G^{\text{reg}}(x - x')$. In the following sections we shall use the regularized Green’s functions to calculate the regularized stress tensor by the point-splitting method.

4 The 2nd-order regularized stress tensor with $\xi = 0$

The stress tensor of the scalar field contains UV divergences, which must be subtracted by regularization. Our goal is to achieve a regularized stress tensor with the desired properties mentioned in the introduction. In this section we study the $\xi = 0$ scalar field. For a clear comparison, we first summarize briefly the resulting stress tensor from adiabatic regularization [41, 42], and then present the point-splitting regularization in details, using the regularized Green’s function of the last section. The energy density and pressure of the $\xi = 0$ scalar field in the BD vacuum state are given by the expectation values

\[
\rho = \langle 0 | T_0^0 | 0 \rangle = \int_0^\infty \frac{dk}{k} \rho_k,
\]

(30)
The 0th-order regularized $\Delta^2_{k,\text{reg}}$ in (27) is IR and UV convergent. The 0th-order regularized $G(x)_{\text{reg}}$ in (28) is IR and UV convergent. The model $\xi = \frac{1}{6}$ and $\frac{m^2}{\Pi^2} = 0.1$.

\begin{align*}
p &= -\frac{1}{3} \langle 0 | T'_{ii} | 0 \rangle = \int_0^\infty p_k \frac{dk}{k}, \quad (31)
\end{align*}

where the spectral energy density and the spectral pressure are

\begin{align*}
\rho_k &= \frac{k^3}{4\pi^2 a^4} \left[ |v_k'|^2 + k^2 |v_k|^2 + m^2 a^2 |v_k|^2 + (6\xi - 1) \left( \frac{a'}{a} (v_k^* v_k + v_k v_k^*) - \frac{a'^2}{a^2} |v_k|^2 \right) \right], \quad (32)
\end{align*}

\begin{align*}
p_k &= \frac{k^3}{4\pi^2 a^4} \left[ \frac{1}{3} |v_k'|^2 + \frac{1}{3} k^2 |v_k|^2 - \frac{1}{3} m^2 a^2 |v_k|^2 + 2(\xi - \frac{1}{6}) \left( -2 |v_k|^2 + 3 \frac{a'}{a} (v_k^* v_k + v_k v_k^*) ight) ight. \\
&\quad \left. -3 \left( \frac{a'}{a} \right)^2 |v_k|^2 + 2(k^2 + m^2 a^2) |v_k|^2 + 12 \frac{a''}{a} |v_k|^2 \right]. \quad (33)
\end{align*}

$\rho_k$ is nonnegative, and $p_k$ can take both positive and negative values. For the minimal coupling $\xi = 0$ they reduce to

\begin{align*}
\rho_k &= \frac{k^3}{4\pi^2 a^4} \left( \left( \frac{v_k}{a} \right)'^2 + k^2 \frac{v_k}{a} |v_k|^2 + m^2 a^2 \frac{v_k}{a} |v_k|^2 \right), \quad (34)
\end{align*}

\begin{align*}
p_k &= \frac{k^3}{4\pi^2 a^4} \left( |v_k|^2 - \frac{1}{3} k^2 \frac{v_k}{a} |v_k|^2 - m^2 a^2 \frac{v_k}{a} |v_k|^2 \right). \quad (35)
\end{align*}

At low $k$, $\rho_k$ and $p_k$ are IR convergent and dominated by the mass term. At high $k$, $\rho_k$ and $p_k$ contain quartic, quadratic, and logarithmic UV divergences which are removed by adiabatic regularization. The 2nd-order adiabatic regularization is performed as the following [41]

\begin{align*}
\rho_{k,\text{reg}} &= \rho_k - \rho_k A_2 \\
&= \frac{k^3}{4\pi^2 a^4} \left( \left( \frac{v_k}{a} \right)'^2 + k^2 \frac{v_k}{a} |v_k|^2 + m^2 a^2 \frac{v_k}{a} |v_k|^2 \right) \\
&\quad - \frac{k^3}{4\pi^2 a^4} \left[ \omega + \frac{m^4 a^4}{8\omega^5} \frac{a''}{a^2} + (\xi - \frac{1}{6}) \left( -3 \frac{a'^2}{\omega^2 a^2} - \frac{3 m^2 a^2}{\omega^3} \right) \right], \quad (36)
\end{align*}

\begin{align*}
p_{k,\text{reg}} &= p_k - p_k A_2
\end{align*}
It should be remarked that the result (40) comes from the ordering: first the massless limit, then the 4th-order would remove too much, causing a spectral negative energy density. (As demonstrated in Ref. [41], for \( \xi = 0 \), the 0th-order regularization would not be able to remove all the UV divergences, and the 4th-order would remove too much, causing a spectral negative energy density.)

The 2nd-order adiabatically regularized stress tensor with \( \xi = 0 \) are obtained by the following \( k \)-integrations

\[
\rho_{\text{reg}} = \int_0^{\infty} (p_k - p_k a_2) \frac{dk}{k}, \quad p_{\text{reg}} = \int_0^{\infty} (p_k - p_k a_2) \frac{dk}{k},
\]

Interestingly, although the regularized spectra \( p_{k \text{reg}} \neq -p_{k \text{reg}}, \) nevertheless, \( \rho_{\text{reg}} = -p_{\text{reg}} \) after \( k \)-integration, That is, the regularized stress tensor in the vacuum satisfies the maximal symmetry in de Sitter space, \( \langle T_{\mu \nu} \rangle_{\text{reg}} = g_{\mu \nu} \rho_{\text{reg}}. \) For instance, \( p_{\text{reg}} = -p_{\text{reg}} \simeq 0.895913 \frac{H^4}{16 \pi} = 89.5913 \frac{m^4}{16 \pi} > 0 \) at \( \frac{m^2}{H^2} = 0.1, \) and \( \rho_{\text{reg}} = -p_{\text{reg}} \simeq 0.860342 \frac{H^4}{16 \pi} = 21.5086 \frac{m^4}{16 \pi} \) at \( \frac{m^2}{H^2} = 0.2. \) We plot \( \rho_{\text{reg}} \) as a function of \( m^2/H^2 \) in red dots in Fig.3 (b).

In the massless limit \( m = 0, \) the adiabatically regularized spectra (36) and (37) become zero

\[
\rho_{k \text{reg}} = 0 = p_{k \text{reg}} \quad \text{for} \quad m = \xi = 0.
\]

and the adiabatically regularized stress tensor becomes [41, 42]

\[
\langle T_{\mu \nu} \rangle_{\text{reg}} = 0, \quad \text{for} \quad m = \xi = 0.
\]

It should be remarked that the result (40) comes from the ordering: first the massless limit, then the \( k \)-integration.

In the point-splitting method [9, 10, 16, 35], the vacuum stress tensor is calculated by use of the Green’s function in \( x \)-space. In the first scheme one calculates

\[
\langle T_{\mu \nu} \rangle_{\text{reg}} = \lim_{x' \to x} \left[ \frac{1}{2} (1 - 2 \xi) (\nabla_\mu \nabla_\nu' + \nabla_\mu' \nabla_\nu) + (2 \xi - \frac{1}{2}) g_{\mu \nu} \nabla_\sigma \nabla_\sigma' - \xi (\nabla_\mu \nabla_\nu + \nabla_\mu' \nabla_\nu') + \xi g_{\mu \nu} (\nabla_\sigma \nabla^\sigma + \nabla_\sigma^\sigma) - \xi G_{\mu \nu} + \frac{1}{2} m^2 g_{\mu \nu} \right] G_{\text{reg}}(x - x'),
\]

where \( G_{\text{reg}}(x - x') \) is the adiabatically regularized Green’s function given by (25) for \( \xi = 0 \), and is a biscalar at \( x \) and at \( x' \), and \( \nabla_\mu \) and \( \nabla_\mu' \) are the covariant differentiation with respect to \( x \) and \( x' \) respectively.

Alternatively, in the second scheme, one calculates the unregularized stress tensor

\[
\langle T_{\mu \nu} \rangle = \lim_{x' \to x} \left[ \frac{1}{2} (1 - 2 \xi) (\nabla_\mu \nabla_\nu' + \nabla_\mu' \nabla_\nu) + (2 \xi - \frac{1}{2}) g_{\mu \nu} \nabla_\sigma \nabla_\sigma' - \xi (\nabla_\mu \nabla_\nu + \nabla_\mu' \nabla_\nu') + \xi g_{\mu \nu} (\nabla_\sigma \nabla^\sigma + \nabla_\sigma^\sigma) - \xi G_{\mu \nu} + \frac{1}{2} m^2 g_{\mu \nu} \right] G(x - x'),
\]

and the subtraction stress tensor

\[
\langle T_{\mu \nu} \rangle_{\text{sub}} = \lim_{x' \to x} \left[ \frac{1}{2} (1 - 2 \xi) (\nabla_\mu \nabla_\nu' + \nabla_\mu' \nabla_\nu) + (2 \xi - \frac{1}{2}) g_{\mu \nu} \nabla_\sigma \nabla_\sigma' - \xi (\nabla_\mu \nabla_\nu + \nabla_\mu' \nabla_\nu') + \xi g_{\mu \nu} (\nabla_\sigma \nabla^\sigma + \nabla_\sigma^\sigma) - \xi G_{\mu \nu} + \frac{1}{2} m^2 g_{\mu \nu} \right] G(x - x')_{\text{sub}},
\]
with $G(x - x')_{\text{sub}}$ being the subtraction Green’s function, and then takes the difference

$$
(T_{\mu\nu})_{\text{reg}} = (T_{\mu\nu}) - (T_{\mu\nu})_{\text{sub}}.
$$

(44)

The second scheme is often adopted in literature [9, 10, 13, 16, 35]. As we shall see, both schemes lead to the same result.

Now we calculate the stress tensor (41) for $\xi = 0$ by the first scheme of the point-splitting method. For this purpose, we only need the 2nd-order regularized $G(\sigma)_{\text{reg}}$ at small separation up to the order $\sigma^2$. By the maximal symmetry in de Sitter space, we replace $y \rightarrow \sqrt{-2\sigma}$ in (15) and (26), and expand them at small separation

$$
G(\sigma) = \frac{1}{16\pi^2} \left( -\frac{1}{\epsilon^2} + W \ln \epsilon^2 + X + Y \epsilon^2 \ln \epsilon^2 + Z \epsilon^2 \right) + O(\epsilon^3),
$$

(45)

$$
G(\sigma)_{\text{sub}} = \frac{1}{16\pi^2} \left( -\frac{1}{\epsilon^2} + W \ln \epsilon^2 + A + Y \epsilon^2 \ln \epsilon^2 + B \epsilon^2 \right) + O(\epsilon^3),
$$

(46)

where $\epsilon^2 \equiv \sigma/2H^2$ for simple notation, and the constants are

$$
W = m^2 + (\xi - \frac{1}{6})R,
$$

(47)

$$
X = (m^2 + (\xi - \frac{1}{6})R) \left( -1 + 2\gamma + \ln(-H^2) + \psi\left(\frac{3}{2} - \nu\right) + \psi\left(\frac{3}{2} + \nu\right) \right),
$$

(48)

$$
Y = \frac{1}{2}(m^2 + (\xi - \frac{1}{6})R)(m^2 + \xi R),
$$

(49)

$$
Z = \frac{1}{2}(m^2 + (\xi - \frac{1}{6})R)(m^2 + \xi R) \left( -\frac{5}{2} + 2\gamma + \ln(-H^2) + \psi\left(\frac{5}{2} - \nu\right) + \psi\left(\frac{5}{2} + \nu\right) \right),
$$

(50)

$$
A = (m^2 - \frac{1}{6}R) \left( -1 + 2\gamma + \ln(-m^2) \right) - \frac{R}{9},
$$

$$
B = -\frac{1}{2}m^2(m^2 - \frac{1}{6}R) \left( -\frac{5}{2} + 2\gamma + \ln(-m^2) \right) - \frac{5m^2R}{72},
$$

where the psi function $\psi(z) \equiv \Gamma'(z)/\Gamma(z)$ with $z \neq 0, -1, -2$ [51, 52]. Both expressions (45) (46) have a similar structure, and their difference is the 2nd-order adiabatically regularized Green’s function for $\xi = 0$ at small separation

$$
G(\sigma)_{\text{reg}} = \frac{1}{16\pi^2} \left[ (X - A) + (Z - B)\epsilon^2 \right] + O(\epsilon^3)
$$

(51)

with

$$
X - A = (m^2 - \frac{R}{6}) \left( \psi\left(\frac{3}{2} - \nu\right) + \psi\left(\frac{3}{2} + \nu\right) + \ln\left(\frac{R}{12m^2}\right) + \frac{R}{9},
$$

$$
Z - B = \frac{1}{2}m^2\left( \frac{R}{6} - m^2 \right) \left( \psi\left(\frac{3}{2} - \nu\right) + \psi\left(\frac{3}{2} + \nu\right) + \ln\left(\frac{R}{12m^2}\right) + \frac{R^2}{48} - \frac{m^2R}{18} \right),
$$

(52)

where a relation $\psi\left(\frac{3}{2} + \nu\right) + \psi\left(\frac{3}{2} - \nu\right) = \psi\left(\frac{3}{2} + \nu\right) + \psi\left(\frac{3}{2} - \nu\right) + \frac{R}{12m^2}$ has been used in (52). The expression (51) is valid only at $m \neq 0$, as it is derived from the Green’s function (15) which is undefined at $m = 0$ and $\xi = 0$. If we take massless limit of (51), using $\psi\left(\frac{3}{2} + \nu\right) + \psi\left(\frac{3}{2} - \nu\right) \simeq -\frac{3H^2}{m^2}$ at small $m$, we will get the auto-Green’s function

$$
G(0)_{\text{reg}} = \frac{1}{16\pi^2} (X - A) \simeq \frac{R^2}{384\pi^2 m^2}
$$

(53)

which is singular as $m \rightarrow 0$. The expression (53) and the conclusion of its invalidity at $m = 0$ and $\xi = 0$ agree with that of Ref. [53]. A singularity at $m = 0$ also occurs in the Green’s function of the
Proca massive vector field which does not reduce to the Maxwell field, and both fields are treated separately in the Minkowski spacetime [54]. Analogously, we shall give a separate treatment for the case $m = \xi = 0$ later around (58) (59). Ref. [53] adopted the 4th-order regularized Green’s function of Ref. [13], dropping the $R^2$ terms, and arrived at their (4.10), which corresponds to our 2nd-order auto Green’s function $G(0)_{\text{reg}}$ given by (53). But Ref. [53] did not calculate the regularized stress tensor though, then continued to explore possible quantum states other than the BD vacuum state. These are beyond the scope of our paper.

The expression (51) at small separation is a simple function of $\epsilon^2$, and contains neither $\ln \epsilon^2$ nor $\epsilon^2 \ln \epsilon^2$ terms. The stress tensor (41) for $\xi = 0$ becomes

$$\langle T_{\mu\nu}\rangle_{\text{reg}} = \lim_{x' \to x} \nabla_\mu \nabla_\nu G(x-x')_{\text{reg}} - \frac{1}{2} g_{\mu\nu} \nabla_\sigma \nabla_\sigma G(x-x')_{\text{reg}} + \frac{1}{2} g_{\mu\nu} m^2 G(x-x')_{\text{reg}} \right].$$

(54)

Substituting (51) into the above and using the formulae (119) and (120) in Appendix, we obtain the 2nd-order regularized vacuum stress tensor for $\xi = 0$

$$\langle T_{\mu\nu}\rangle_{\text{reg}} = \frac{1}{16\pi^2} \left[ \frac{1}{2} g_{\mu\nu} (Z - B) + \frac{1}{2} g_{\mu\nu} m^2 (X - A) \right]$$

$$= g_{\mu\nu} \Lambda,$$

(55)

where

$$\Lambda \equiv \frac{1}{64\pi^2} \left[ m^2 (m^2 - \frac{R}{6}) (\psi(\frac{3}{2} - \nu) + \psi(\frac{3}{2} + \nu) + \ln \frac{R}{12m^2}) + \frac{m^2 R}{9} + \frac{R^2}{24} \right].$$

(56)

As we have checked, the 2nd-order regularization of (7) also yields the trace of (55) consistently. The stress tensor (55) respects the covariant conservation of energy. As an important property, the vacuum stress tensor (55) is proportional to the metric, $\langle T_{\mu\nu}\rangle_{\text{reg}} \propto g_{\mu\nu}$, and satisfies the maximal symmetry in de Sitter space. The finite constant $\Lambda$ of (56) depends on the mass $m$ of the scalar field and the expansion rate $H$, and is naturally identified as, or part of, the cosmological constant that drives the de Sitter inflation [55]. From cosmological point of view, the cosmological constant is generally contributed by the vacuum stress tensors of more than one quantum field.

We compare the results from the point-splitting and from the adiabatic for the minimally coupling $\xi = 0$. Fig.3 (b) plots the 2nd-order point-splitting $\rho_{\text{reg}}$ from (55) in the blue line, and the 2nd-order adiabatic $\rho_{\text{reg}}$ from (38) in the red dots. The two results are equal over the whole range of $m^2/H^2$, positive and finite. Hence, both adiabatic and point-splitting regularization yield the same regularized stress tensor for $\xi = 0$.

Next we calculate the stress tensor by the second scheme of the point-splitting method. Substituting the unregularized Green’s function (45) into (42) for $\xi = 0$, using the formulae in Appendix, we get

$$\langle T_{\mu\nu}\rangle = \lim_{x' \to x} \frac{1}{16\pi^2} \left[ \frac{1}{2\epsilon^4} (\partial_\mu \partial_\nu + \partial_\mu \partial_\nu) \epsilon^2 - \frac{1}{\epsilon^4} (\partial_\nu \epsilon^2 \cdot \partial_\mu \epsilon^2 + \partial_\mu \epsilon^2 \cdot \partial_\nu \epsilon^2) \\
- \frac{W}{2\epsilon^4} (\partial_\mu \epsilon^2 \cdot \partial_\nu \epsilon^2 + \partial_\mu \epsilon^2 \cdot \partial_\nu \epsilon^2) + \frac{W}{2\epsilon^4} (\partial_\mu \partial_\nu \epsilon^2 \cdot \partial_\nu \epsilon^2) \\
+ \frac{1}{2} g_{\mu\nu} Y \ln \epsilon^2 + \frac{Y}{2\epsilon^4} (\partial_\mu \epsilon^2 \cdot \partial_\nu \epsilon^2 + \partial_\mu \epsilon^2 \cdot \partial_\nu \epsilon^2) \\
+ \frac{1}{2} g_{\mu\nu} \frac{R}{12\epsilon^2} + \frac{1}{2\epsilon^4} \frac{W}{2\epsilon^4} (Y + \frac{1}{2} Z) g_{\mu\nu} + \frac{1}{2} g_{\mu\nu} m^2 \left( - \frac{1}{\epsilon^2} + W \ln \epsilon^2 + X \right) \right].$$

(57)

The subtraction stress tensor $\langle T_{\mu\nu}\rangle_{\text{sub}}$ is obtained by substituting the subtraction Green’s function (46) into (43), and has an expression similar to (57) with the replacements $(X, Z) \to (A, B)$. Their difference $\langle T_{\mu\nu}\rangle_{\text{reg}} = \langle T_{\mu\nu}\rangle - \langle T_{\mu\nu}\rangle_{\text{sub}}$ is the same as the result (55) from the first scheme.
Now consider the case of $m = 0$ and $\xi = 0$, for which the formulae (15) (51) do not apply. We directly start with the unregularized Green function of the minimally-coupling massless scalar field [41, 42]

$$G(\sigma) = -\frac{H^2}{8\pi^2} \left[ \frac{1}{\sigma} + \ln\left( -\frac{2\pi^2}{\tau^2_0} \sigma \right) \right],$$

(58)

where $\tau_0$ is an arbitrary fixed constant. All the terms of (58) are UV divergent and should be subtracted off, and we take $G(\sigma)_{\text{sub}} = G(\sigma)$, so that the regularized vacuum Green function is zero,

$$G(\sigma)_{\text{reg}} = G(\sigma) - G(\sigma)_{\text{sub}} = 0,$$

(59)

and the regularized stress tensor is also zero, $\langle T_{\mu\nu} \rangle_{\text{reg}} = 0$, the same as (40) from adiabatic regularization. In the second scheme, the unregularized stress tensor is

$$\langle T_{\mu\nu} \rangle = \frac{1}{2} \left[ \nabla_\mu \nabla_\nu + \nabla_{\mu'} \nabla_\nu - g_{\mu\nu} \nabla_\sigma \nabla^{\sigma'} \right] G(x - x').$$

(60)

Using the formulae (119) through (128) in Appendix, we obtain

$$\langle T_{\mu\nu} \rangle = \lim_{x' \to x} -\frac{1}{32\pi^2} \left[ -\frac{1}{\epsilon^4} (\partial_\mu \partial_{\nu'} + \partial_{\mu'} \partial_\nu) \epsilon^2 + \frac{2}{\epsilon^6} (\partial_{\nu'} \epsilon^2 - \partial_{\mu'} \epsilon^2) \sigma + \frac{R}{6} \frac{1}{\epsilon^2} (\partial_\mu \partial_{\nu'} + \partial_{\mu'} \partial_\nu) \epsilon^2 + g_{\mu\nu} \frac{R}{12} \frac{1}{\epsilon^2} \right].$$

(61)

All the terms in (61) are UV divergent in the coincidence limit $x' \to x$ and should be subtracted off, so that $\langle T_{\mu\nu} \rangle_{\text{reg}} = 0$, also agreeing with (40).

As remarked earlier, the expressions (51) (53) (55) for $\xi = 0$ are valid only at $m \neq 0$. If we would take the massless limit of the stress tensor (55), by the expansion $\psi(\frac{3}{2} + \nu) + \psi(\frac{3}{2} - \nu) \simeq -\frac{3H^2}{m^2} + (\frac{11}{6} - 2\gamma)$ at small $m$, we would get

$$\lim_{m \to 0} \langle T_{\mu\nu} \rangle_{\text{reg}} = g_{\mu\nu} \frac{1}{64\pi^2} \left( \frac{R^2}{12} \right) \neq 0,$$

(62)

Figure 3: (a): The 2nd-order regularized spectral energy density $\rho_{k,\text{reg}}$ in (36) is positive, IR and UV convergent. The model $\xi = 0$ and $\frac{m^2}{H^2} = 0.1$. (b): For $\xi = 0$ the 2nd-order regularized energy density $\rho_{\text{reg}}$ is positive and finite for the whole range of $\frac{m^2}{H^2}$. Blue line: the point-splitting (55); Red dots: the adiabatic (38).
in contradiction to the result (40) at \( m = \xi = 0 \). So, our conclusion on the singularity at \( m = \xi = 0 \) is consistent with Ref. [53]. This issue is originated in the \( k \)-space. For \( \xi = 0 \), the spectral energy density \( \rho_{k_{\text{reg}}} \) in (36) has a massless limit \( \lim_{m \to 0} \rho_{k_{\text{reg}}} = 0 \) for any given \( k \), so its \( k \)-integration is \( \rho_{\text{reg}} = \int \lim_{m \to 0} \rho_{k_{\text{reg}}} \frac{dk}{k} = 0 \). If we would do the \( k \)-integration first and then take the massless limit, we would get the nonvanishing \( \rho_{\text{reg}} \neq 0 \) as (62), which is actually invalid at \( m = 0 \). Obviously, the ordering of the massless limit and the \( k \)-integration cannot be interchanged

\[
\lim_{m \to 0} \int \rho_{k_{\text{reg}}} \frac{1}{k} dk \neq \int \lim_{m \to 0} \rho_{k_{\text{reg}}} \frac{1}{k} dk.
\]  

(63)

This is because \( \frac{1}{k} \rho_{k_{\text{reg}}} \) for \( \xi = 0 \) is not dominantly convergent, i.e., there exists no non-negative integrable function \( g_k \) such that \( |\frac{1}{k} \rho_{k_{\text{reg}}}| \leq g_k \) for all \( k \) and \( m \). When \( k \) is sufficiently small, \( \frac{1}{k} \rho_{k_{\text{reg}}} \) is increasingly large, as shown in Fig. 4 (a).

![Figure 4: (a) \( \frac{1}{k} \rho_{k_{\text{reg}}} \) for \( \xi = 0 \). (b) \( \frac{1}{k} \rho_{k_{\text{reg}}} \) for \( \xi = \frac{1}{6} \). Red line: \( \frac{m^2}{\Pi^2} = 4 \times 10^{-4} \), Blue line: \( \frac{m^2}{\Pi^2} = 10^{-3} \).](image)

So far, for \( \xi = 0 \), in both schemes of the point-splitting, we have been guided by \( G(\sigma)_{\text{sub}} \) of (46) from the 2nd-order adiabatic regularization. Otherwise, it may not be easy to choose an appropriate subtraction stress tensor. The calculation of stress tensor is straightforward because the regularized Green’s function (51) is a linear function of \( \epsilon^2 \). Nevertheless, for a general \( \xi \neq 0 \), the regularized Green’s function may contain a term \( \epsilon^2 \ln \epsilon^2 \), and the calculation of stress tensor may not be so simple in the point-splitting method, as we shall see in the following sections.

## 5 The 2nd-order regularized stress tensor with small \( \xi > 0 \)

For a small \( \xi > 0 \), we also use (36) and (37) for the 2nd-order adiabatically regularized spectral stress tensor, and the \( k \)-integrations analogous to (38) give the regularized stress tensor. (Again, the 0th-order regularization would not remove all the UV divergences, and the 4th-order would lead to a negative energy density.)

We now calculate the stress tensor in the point-splitting method. The 2nd-order subtraction Green’s functions (26) for general \( \xi \) at small separation is

\[
G(\sigma)_{\text{sub}} = \frac{1}{16\pi^2} \left( -\frac{1}{\epsilon^2} + W \ln \epsilon^2 + C + D \epsilon^2 \ln \epsilon^2 + E \epsilon^2 \right),
\]  

(64)
where the constants are
\[
C = (m^2 + R(\xi - \frac{1}{6})(-1 + 2\gamma + \ln(-m^2)) + \xi R - \frac{R}{9},
\]
\[
D = -\frac{1}{2}m^2(m^2 + (\xi - \frac{1}{6})R + \xi R),
\]
\[
E = -\frac{1}{2}m^2(m^2 + R(\xi - \frac{1}{6}) + \xi R)(-\frac{5}{2} + 2\gamma + \ln(-m^2)) - \frac{5}{12}m^2R - \frac{1}{2}m^2\xi R.
\]  

(65)

The regularized Green’s function is the following difference
\[
G(\sigma)_{\text{reg}} = G(\sigma) - G(\sigma)_{\text{sub}} = \frac{1}{16\pi^2}(\langle X - C \rangle + (Y - D)\epsilon^2 \ln \epsilon^2 + (Z - E)\epsilon^2),
\]
where \(G(\sigma)\) is the un-regularized Green’s functions (45). Notice that (66) contains a term \(\sim \epsilon^2 \ln \epsilon^2\) with a coefficient \((Y - D) = -\frac{3}{4}\xi(\xi - \frac{1}{6})R^2\) which is of the 4th-order and arises from the coupling \(\xi R\). Although \(\epsilon^2 \ln \epsilon^2\) is continuous and UV convergent at \(\epsilon^2 = 0\), it will cause a UV divergent term \(\sim \ln \epsilon^2\) in the regularized stress tensor. Moreover, when \(G(\sigma)_{\text{reg}}\) is plugged into (41) to calculate the stress tensor, some unwanted 4th-order terms \(\sim R^2\) due to \(G(\sigma)_{\text{sub}}\) will come up. This is because \(G(\sigma)_{\text{sub}}\) of (64) satisfies an inhomogeneous equation as the following
\[
(\nabla_\mu \nabla^\mu + m^2 + \xi R)G(\sigma)_{\text{sub}}(\sigma) = \frac{1}{16\pi^2}(\langle X - C \rangle + (Y - D)\epsilon^2 \ln \epsilon^2 + (Z - E)\epsilon^2)R^2.
\]

(67)

So, instead of the first scheme (41), we shall work with the second scheme, using (42) and (43) in the following. By calculation, using the formulæ (119) — (132) in Appendix, the un-regularized stress tensor for general \(\xi\) is given by the following
\[
\langle T_{\mu\nu} \rangle = \lim_{\epsilon \to \epsilon^2} \frac{1}{16\pi^2} \left( \frac{1}{2}(1 - 2\xi) \left( \frac{1}{\epsilon^4} (\partial_\mu \partial_\nu + \partial_\mu' \partial_\nu') \epsilon^2 - \frac{2}{\epsilon^6} (\partial_\mu \epsilon^2 \cdot \partial_\nu \epsilon^2 + \partial_\mu' \epsilon^2 \cdot \partial_\nu' \epsilon^2) \right) 
+ \frac{1}{\epsilon^2} (\partial_\mu \partial_\nu + \partial_\mu' \partial_\nu') \epsilon^2 - \frac{1}{\epsilon^4} (\partial_\mu \epsilon^2 \cdot \partial_\nu \epsilon^2 + \partial_\mu' \epsilon^2 \cdot \partial_\nu' \epsilon^2) \right)
- \xi \left( \frac{1}{\epsilon^4} (\partial_\mu \partial_\nu + \partial_\mu' \partial_\nu') \epsilon^2 - \frac{2}{\epsilon^6} (\partial_\mu \epsilon^2 \cdot \partial_\nu \epsilon^2 + \partial_\mu' \epsilon^2 \cdot \partial_\nu' \epsilon^2) \right)
+ \frac{1}{\epsilon^2} (\partial_\mu \partial_\nu + \partial_\mu' \partial_\nu') \epsilon^2 - \frac{1}{\epsilon^4} (\partial_\mu \epsilon^2 \cdot \partial_\nu \epsilon^2 + \partial_\mu' \epsilon^2 \cdot \partial_\nu' \epsilon^2) \right)

+ \xi \left( \Gamma^\alpha_{\mu\nu} \partial_\alpha \epsilon^2 + \Gamma^\alpha'_{\mu'\nu'} \partial_\alpha' \epsilon^2 \right) \left( \frac{1}{\epsilon^4} + \frac{1}{\epsilon^2} \right)
+ g_{\mu\nu} \left( \frac{1}{6} (\xi + \frac{1}{4})R + \frac{W}{2} \right) \frac{1}{\epsilon^2}
+ \frac{1}{2} g_{\mu\nu} Y \ln \epsilon^2 + g_{\mu\nu} Y + \frac{1}{2} g_{\mu\nu} Z
+ \frac{1}{2} g_{\mu\nu} m^2 \left( -\frac{1}{\epsilon^2} + W \ln \epsilon^2 + X \right)
+ g_{\mu\nu} \frac{1}{2} \xi \frac{RW}{2} - \xi \left( \frac{1}{2} g_{\mu\nu} \frac{R}{4} \left( -\frac{1}{\epsilon^2} + W \ln \epsilon^2 + X \right) \right),
\]

which reduces to (57) when \(\xi = 0\). (Ref. [13] gave an expression of \(\langle T_{\mu\nu} \rangle\) in their eq.(3.17), which still contained some direction-dependent splitting vectors.) The subtraction stress tensor is
obtained by replacing \((X,Y,Z)\) by \((C,D,E)\) in the above

\[
\langle T_{\mu \nu} \rangle_{sub} = \lim_{x' \to x} \frac{1}{16\pi^2} \left[ \frac{1}{2}(1 - 2\xi) \left( \frac{1}{\epsilon^2} (\partial_\mu \partial_\nu + \partial_\mu' \partial_\nu') \epsilon^2 - \frac{2}{\epsilon^2} (\partial_\nu \epsilon^2 \cdot \partial_\mu' \epsilon^2 + \partial_\mu \epsilon^2 \cdot \partial_\nu' \epsilon^2) 
+ W \frac{1}{\epsilon^2} (\partial_\mu \partial_\nu + \partial_\mu' \partial_\nu') \epsilon^2 - W \frac{1}{\epsilon^2} (\partial_\mu' \epsilon^2 \cdot \partial_\nu \epsilon^2 + \partial_\nu' \epsilon^2 \cdot \partial_\mu \epsilon^2) 
+ D \frac{1}{\epsilon^2} (\partial_\mu \epsilon^2 \cdot \partial_\nu' \epsilon^2 + \partial_\mu' \epsilon^2 \cdot \partial_\nu \epsilon^2) \right) 
- \xi \left( \frac{1}{\epsilon^2} (\partial_\mu \partial_\nu + \partial_\mu' \partial_\nu') \epsilon^2 - \frac{2}{\epsilon^2} (\partial_\nu \epsilon^2 \cdot \partial_\mu' \epsilon^2 + \partial_\nu' \epsilon^2 \cdot \partial_\mu \epsilon^2) 
+ W \frac{1}{\epsilon^2} (\partial_\mu \partial_\nu + \partial_\mu' \partial_\nu') \epsilon^2 - W \frac{1}{\epsilon^2} (\partial_\mu' \epsilon^2 \cdot \partial_\nu \epsilon^2 + \partial_\nu' \epsilon^2 \cdot \partial_\mu \epsilon^2) 
+ D \frac{1}{\epsilon^2} (\partial_\mu \epsilon^2 \cdot \partial_\nu' \epsilon^2 + \partial_\mu' \epsilon^2 \cdot \partial_\nu \epsilon^2) \right) 
+ \xi (\Gamma_{\mu\nu}^\alpha \partial_\alpha \epsilon^2 + \Gamma_{\mu'\nu'}^\alpha \partial_\alpha \epsilon^2) \left( \frac{1}{\epsilon^2} + W \frac{1}{\epsilon^2} \right) 
+ g_{\mu\nu} \left( \frac{1}{6} (\xi + \frac{1}{4}) R + \frac{W}{2} \right) \frac{1}{\epsilon^2} 
+ \frac{1}{2} g_{\mu\nu} D \ln \epsilon^2 + g_{\mu\nu} D + \frac{1}{2} g_{\mu\nu} E 
+ \frac{1}{2} g_{\mu\nu} m^2 \left( \frac{1}{\epsilon^2} + W \ln \epsilon^2 + C \right) 
+ g_{\mu\nu} \frac{R W}{2} - \xi (-g_{\mu\nu} R \frac{1}{4})(\frac{1}{\epsilon^2} + W \ln \epsilon^2 + C) \right], \tag{69}
\]

The expressions (68) and (69) are lengthy. But, all \(\epsilon^{-4}\) and \(\epsilon^{-2}\) divergent terms will cancel between (68) and (69), and will be denoted as \((\epsilon^{-4}, \epsilon^{-2}\) terms\) for brevity. The four convergent terms occurring in (68) and (69) will be collectively denoted as

\[
P_{\mu\nu} \equiv \left( \frac{1}{2} - 2\xi \right) \frac{1}{\epsilon^2} (\partial_\mu \epsilon^2 \cdot \partial_\nu' \epsilon^2 + \partial_\mu' \epsilon^2 \cdot \partial_\nu \epsilon^2) - \frac{1}{\epsilon^2} (\partial_\mu \epsilon^2 \cdot \partial_\nu' \epsilon^2 + \partial_\mu' \epsilon^2 \cdot \partial_\nu \epsilon^2), \tag{70}
\]

which nevertheless depends in the path of the coincidence limit. See (133)–(137) in Appendix B. We write (68) and (69) briefly as the following

\[
\langle T_{\mu \nu} \rangle = \lim_{x' \to x} \frac{1}{16\pi^2} \left[ \left( \epsilon^{-4}, \epsilon^{-2}\) terms\right) + \frac{1}{2} g_{\mu\nu} Y \ln \epsilon^2 + g_{\mu\nu} Y + \frac{1}{2} g_{\mu\nu} Z + \frac{1}{2} g_{\mu\nu} m^2 (W \ln \epsilon^2 + X) 
+ g_{\mu\nu} \xi \frac{R W}{2} - \xi (-g_{\mu\nu} R \frac{1}{4}) W \ln \epsilon^2 - \xi (-g_{\mu\nu} R \frac{1}{4}) X \right], \tag{71}
\]

and

\[
\langle T_{\mu \nu} \rangle_{sub} = \lim_{x' \to x} \frac{1}{16\pi^2} \left[ \left( \epsilon^{-4}, \epsilon^{-2}\) terms\right) + \frac{1}{2} g_{\mu\nu} D \ln \epsilon^2 + g_{\mu\nu} D + \frac{1}{2} g_{\mu\nu} E + \frac{1}{2} g_{\mu\nu} m^2 (W \ln \epsilon^2 + C) 
+ g_{\mu\nu} \xi \frac{R W}{2} - \xi (-g_{\mu\nu} R \frac{1}{4}) W \ln \epsilon^2 - \xi (-g_{\mu\nu} R \frac{1}{4}) C \right]. \tag{72}
\]

Recall that, in the 2nd-order adiabatic regularization in \(k\)-space, only the 2nd adiabatic order terms \(\sim a'^2, a''\), are kept in the subtraction terms \(\rho_{kA2}\) and \(p_{kA2}\) in (36) and (37). To be consistent, in
where we keep up to the 2nd-order terms in the subtraction stress tensor (72). The 4th-order terms \( R^2 \) come only from the last line of (72):

\[
RW = R \left( m^2 + R(\xi - \frac{1}{6}) \right) = m^2 R + (\xi - \frac{1}{6}) R^2,
\]

\[
RC = R \left( m^2 + R(\xi - \frac{1}{6}) \right) (-1 + 2\gamma + \ln(-m^2)) + \xi R - \frac{R}{9},
\]

which can be dropped by the following replacements in (72),

\[
RW \to R[ W - (\xi - \frac{1}{6}) R],
\]

\[
RC \to R[ C - R(\xi - \frac{1}{6})(-1 + 2\gamma + \ln(-m^2)) - \xi R + \frac{R}{9}].
\]

With this replacement, the subtraction stress tensor (72) is modified to the following

\[
\langle T_{\mu\nu} \rangle_{sub} = \lim_{x \to x'} \frac{1}{16\pi^2} \left[ (\epsilon^{-4}, \epsilon^{-2} \text{ terms}) + DP_{\mu\nu}
\right.
\]

\[
+ \frac{1}{2} g_{\mu\nu} D \ln \epsilon^2 + g_{\mu\nu} D + \frac{1}{2} g_{\mu\nu} E + \frac{1}{2} g_{\mu\nu} m^2 (W \ln \epsilon^2 + C)
\]

\[
+ g_{\mu\nu} \xi \frac{R}{2} [ W - (\xi - \frac{1}{6}) R] - \xi (-g_{\mu\nu} \frac{R}{4})[ W - (\xi - \frac{1}{6}) R] \ln \epsilon^2
\]

\[
- \xi (-g_{\mu\nu} \frac{R}{4}) [ C - R(\xi - \frac{1}{6})(-1 + 2\gamma + \ln(-m^2)) - \xi R + \frac{R}{9}],
\]

which contains no terms \( \sim R^2 \). We take the difference between (71) and (77) and get

\[
\langle T_{\mu\nu} \rangle_{reg} = \frac{\langle Y - D \rangle}{16\pi^2} \lim_{x \to x} P_{\mu\nu} + g_{\mu\nu} \frac{1}{64\pi^2} \left[ -3(Y - D) + m^2 (X - C) - \frac{1}{4} (\xi - \frac{1}{6}) R^2 \right],
\]

where the terms \( \ln \epsilon^2 \) have been canceled. However, the term \( \sim \lim P_{\mu\nu} \) in (78) depends on the path of the coincidence limit, and does not possess the maximum symmetry in de Sitter space. See (133)- (137) in Appendix B. Dropping the \( P_{\mu\nu} \) term from (78), we obtain the 2nd-order regularized vacuum stress tensor with general \( \xi \)

\[
\langle T_{\mu\nu} \rangle_{reg} = g_{\mu\nu} \frac{1}{64\pi^2} \left[ -3(Y - D) + m^2 (X - C) - \frac{1}{4} (\xi - \frac{1}{6}) R^2 \right]
\]

\[
= g_{\mu\nu} \Lambda
\]

(79)

where

\[
\Lambda \equiv \frac{1}{64\pi^2} \left[ m^2 (m^2 + (\xi - \frac{1}{6}) R) \left( \frac{3}{2} \psi - \nu + \frac{3}{2} + \nu \right) - \ln(\frac{12m^2}{R}) \right]
\]

\[
- (\xi - \frac{1}{6}) m^2 R - \frac{m^2 R}{18} + \frac{3(\xi - \frac{1}{6})^2 R^2}{2}.
\]

The vacuum stress tensor (79) possesses the maximum symmetry in de Sitter space. The constant \( \Lambda \) of (80) is identified as the cosmological constant for \( \xi > 0 \). Setting \( \xi = 0 \), (80) will reduce to (56) consistently.

It is checked that (79) is equal to (38) for various \( \xi \) and \( m \). So the point-splitting and adiabatic regularization of 2nd-order yield the same result. Importantly, the 2nd-order regularized energy
density and spectral energy density are all positive for small couplings \( 0 \leq \xi < \frac{1}{7.04} \) at a fixed \( \frac{m^2}{H^2} = 0.1 \). As examples, we plot \( \rho_{k\,\text{reg}} \) and \( \rho_{\text{reg}} \) in Fig. 5 (a) and (b) for \( \xi = \frac{1}{10} \), and in Fig. 6 (a) and (b) for \( \xi = \frac{1}{7.04} \). Nevertheless, for large couplings \( \xi > \frac{1}{7.04} \), the 2nd-order regularized energy density and spectral energy density are negative. (Later we shall see that the 4th-order regularization also leads to negative energy density and spectral energy density for \( \xi > \frac{1}{7.04} \).)

![Graph](image1.png)

Figure 5: (a): The 2nd-order \( \rho_{k\,\text{reg}} \) is positive, IR and UV convergent. The model \( \xi = \frac{1}{10} \) and \( \frac{m^2}{H^2} = 0.1 \). (b): For \( \xi = \frac{1}{10} \), the 2nd-order \( \rho_{\text{reg}} \) is positive and finite for the whole range of \( \frac{m^2}{H^2} \).

![Graph](image2.png)

Figure 6: (a): The 2nd-order \( \rho_{k\,\text{reg}} \) is positive, IR and UV convergent. The model \( \xi = \frac{1}{7.04} \) and \( \frac{m^2}{H^2} = 0.1 \). (b): For \( \xi = \frac{1}{7.04} \), the 2nd-order \( \rho_{\text{reg}} \) is positive and finite for the whole range of \( \frac{m^2}{H^2} \).

The lesson from this section for \( \xi \neq 0 \) is that, even though \( G(\sigma)_{\text{reg}} \) is continuous, as well as UV and IR convergent, the point-splitting regularization does not automatically lead to an appropriate stress tensor. The coupling \( \xi R \) gives rise to \( \epsilon^2 \ln \epsilon^2 \) in \( G(\sigma)_{\text{reg}} \), and causes unwanted higher-order terms in the stress tensor, as well as some terms depending on the path of the coincidence limit. These need be treated in order to give an appropriate stress tensor which agrees with that from adiabatic regularization.
6 The 0th-order regularized stress tensor for $\xi = \frac{1}{6}$

We first list the main result from adiabatic regularization, and then give the point-splitting regularization. For a conformally-coupling $\xi = \frac{1}{6}$ massive field, the 0th-order adiabatic regularization is taken on the spectral stress tensor [41],

$$\rho_k^{\text{reg}} - \rho_k - \rho_{k0} = \frac{k^3}{4\pi^2a^4} \left[ |v_k'|^2 + k^2|v_k|^2 + m^2a^2|v_k|^2 \right] - \frac{k^3}{4\pi^2a^4}\omega,$$

(81)

$$p_k^{\text{reg}} - p_k - p_{k0} = \frac{k^3}{12\pi^2a^4} \left[ |v_k'|^2 + k^2|v_k|^2 - m^2a^2|v_k|^2 \right] - \frac{k^3}{12\pi^2a^4} \left( \omega - \frac{m^2a^2}{\omega} \right).$$

(82)

(The 2nd-, and 4th-order regularization would lead to a negative spectral energy density [41].) The 0th-order adiabatically regularized $\rho_k^{\text{reg}}$ and $p_k^{\text{reg}}$ are UV and IR convergent, and $\rho_k^{\text{reg}}$ is positive, as shown in Fig.7 (a). The 0th-order adiabatically regularized energy density and pressure are given by

$$\rho_{\text{reg}} = \int_0^\infty (\rho_k - \rho_{k0}) \frac{dk}{k}, \quad p_{\text{reg}} = \int_0^\infty (p_k - p_{k0}) \frac{dk}{k}.$$  

(83)

For examples, $\rho_{\text{reg}} = -p_{\text{reg}} \simeq 0.001786 \frac{H^4}{16\pi} = 0.1786\frac{m^4}{16\pi} > 0$ for $\frac{m^2}{H^2} = 0.1$, and $\rho_{\text{reg}} = -p_{\text{reg}} = 0.005221 \frac{H^4}{16\pi}$ for $\frac{m^2}{H^2} = 0.2$. We plot $\rho_{\text{reg}}$ in red dots in Fig.7 (b). The regularized vacuum stress tensor also satisfies the maximal symmetry in de Sitter space. In the massless limit $m = 0$ the regularized spectra and the stress tensor are vanishing

$$\rho_k^{\text{reg}} = 0 = p_k^{\text{reg}}, \quad \langle T_{\mu\nu}\rangle_{\text{reg}} = 0 \quad \text{for} \quad m = 0,$$

(84)

similar to (39) (40) of the case $\xi = 0$.

Now we calculate the stress tensor for $\xi = \frac{1}{6}$ by the point-splitting method. The simplest way is to take the vacuum expectation of eq.(8)

$$\langle T^\mu_{\phantom{\mu}\nu}\rangle_{\text{reg}} = m^2G(0)_{\text{reg}},$$

and, by the maximal symmetry, the 0th-order regularized vacuum stress tensor with $\xi = \frac{1}{6}$ is the following

$$\langle T_{\mu\nu}\rangle_{\text{reg}} = \frac{1}{4}g_{\mu\nu}\langle T^{\alpha}_{\phantom{\alpha}\alpha}\rangle_{\text{reg}} = \frac{1}{4}g_{\mu\nu}m^2G(0)_{\text{reg}}$$

$$= g_{\mu\nu}\Lambda,$$

(86)

where $G(0)_{\text{reg}}$ is the 0th-order regularized auto-correlation given by (92), and

$$\Lambda \equiv \frac{1}{4}m^2G(0)_{\text{reg}} = \frac{m^4}{64\pi^2} \left[ \psi\left(\frac{3}{2} - \nu\right) + \psi\left(\frac{3}{2} + \nu\right) + \ln \frac{R}{12m^2} \right]$$

(87)

with $\nu = (\frac{1}{4} - \frac{m^2}{H^2})^{1/2}$ for $\xi = \frac{1}{6}$. The merit of this simple derivation is that no differentiation is performed on the Green’s function. The finite constant of (87) also can be also identified as the cosmological constant for the case of conformally-coupling $\xi = \frac{1}{6}$.

We compare the results from the point-splitting and from the adiabatic for the conformally-coupling $\xi = \frac{1}{6}$. Fig.7 (b) plots $\rho_{\text{reg}}$ of (86) from the point-splitting in the blue line and $\rho_{\text{reg}}$ of (83) from the adiabatic in the red dots, the two are equal over the whole range $m^2/H^2$, positive and
finite. Consider the massless limit of (86). By the expansion
\[ \psi(\frac{3}{2} - \nu) + \psi(\frac{3}{2} + \nu) \simeq (1 - 2\gamma) + \frac{m^2}{H^2} \]
at small \( m \), we have \( \Lambda = 0 \) at \( m = 0 \), so that
\[ \langle T_{\mu\nu} \rangle_{\text{reg}} = 0 \quad \text{for} \quad m = 0, \quad (88) \]
also agreeing with (84). Thus, both the point-splitting and adiabatic regularization to the 0th-order yield a zero stress tensor for the conformally-coupling massless scalar field, and there is no trace anomaly.

\[ \rho_{k \text{reg}}(H^4/16\pi), \xi = \frac{1}{6} \]
\[ \rho_{\text{reg}}(H^4/16\pi), \xi = \frac{1}{6} \]

Figure 7: (a): The 0th-order \( \rho_{k \text{reg}} \) in (81) is positive, IR and UV convergent. The model \( \xi = \frac{1}{6} \) and \( \frac{m^2}{H^2} = 0.1 \). (b): For \( \xi = \frac{1}{6} \), the 0th-order \( \rho_{\text{reg}} \) is positive and finite for whole range of \( \frac{m^2}{H^2} \). Blue line: the point-splitting (87); Red dots: the adiabatic (83).

Here the ordering of the massless limit and the \( k \)-integration of \( \rho_{k \text{reg}} \) for \( \xi = \frac{1}{6} \) is interchangeable,
\[ \lim_{m \to 0} \int \rho_{k \text{reg}} \frac{1}{k} dk = \int \lim_{m \to 0} \rho_{k \text{reg}} \frac{1}{k} dk = 0, \quad (89) \]
in contrast to the case \( \xi = 0 \) of (63). This is because \( \frac{1}{6} \rho_{k \text{reg}} \) satisfies the requirement of the dominated convergence theorem. This property is also reflected by the fact that the Green’s function (15) is valid at \( m = 0 \) and \( \xi = \frac{1}{6} \). For the illustration, we plot \( \frac{1}{6} \rho_{k \text{reg}} \) with \( \frac{m^2}{H^2} = 10^{-3}, 4 \times 10^{-3} \) in Fig.4(b).

Alternatively, if we apply the formula (41),
\[ \langle T_{\mu\nu} \rangle_{\text{reg}} = \lim_{x \to x'} \left[ \frac{1}{3} (\nabla_\mu \nabla_\nu + \nabla_\mu \nabla_\nu) - \frac{1}{6} (\nabla_\mu \nabla_\nu + \nabla_\mu \nabla_\nu') - \frac{1}{6} g_{\mu\nu} \nabla_\sigma \nabla_\sigma' + \frac{1}{6} g_{\mu\nu} (\nabla_\sigma \nabla_\sigma + \nabla_\sigma' \nabla_\sigma') - \frac{1}{6} G_{\mu\nu} + \frac{1}{2} m^2 g_{\mu\nu} \right] G_{\text{reg}}(x - x'), \quad (90) \]
the calculation will be more involved than that of eq.(86), and we shall run into some problems caused by the coupling \( \frac{1}{6} R \), similar to the case \( \xi > 0 \) of Section 5. The unregularized Green’s function at small separation is (45) with \( \xi = \frac{1}{6} \) and \( W = m^2 \), and the 0th-order subtraction Green’s function (29) for \( \xi = \frac{1}{6} \) at small separation is
\[ G(\sigma)_{\text{sub}} = \frac{1}{16\pi^2} \left( -\frac{1}{\epsilon^2} + m^2 \ln \epsilon^2 + K + L \epsilon^2 \ln \epsilon^2 + M \epsilon^2 \right) + O(\epsilon^3), \quad (91) \]
with

\[ K = m^2 \left( -1 + 2\gamma + \ln(-m^2) \right), \]
\[ L = -\frac{m^4}{2}, \]
\[ M = -\frac{m^4}{4}(-5 + 4\gamma + 2\ln(-m^2)). \]

So, the difference between (45) and (91) gives the 0th-order regularized Green’s function at small distance

\[ G^{\text{reg}}(\sigma) \equiv \frac{1}{16\pi^2} \left( (X - K) + (Y - L)\epsilon^2 \ln \epsilon^2 + (Z - M)\epsilon^2 \right) \tag{92} \]

where \( X, Y, Z \) are given in (48) (49) (50), and

\[ X - K = m^2 \left( \ln \frac{R}{12m^2} + \psi \left( \frac{3}{2} - \nu \right) + \psi \left( \frac{3}{2} + \nu \right) \right), \]
\[ Y - L = -\frac{m^2 R}{12}, \]
\[ Z - M = -\frac{1}{2}m^4 \left( \ln \frac{R}{12m^2} + \psi \left( \frac{5}{2} - \nu \right) + \psi \left( \frac{5}{2} + \nu \right) \right), \]
\[ -\frac{1}{2}m^2 R \left( -5 + 4\gamma + 2\ln(-H^2) + 2\psi(-\frac{5}{2} - \nu) + 2\psi(-\frac{5}{2} + \nu) \right). \]

Note that the term \( \epsilon^2 \ln \epsilon^2 \) appears in (92), like (66) for general \( \xi \). A calculation shows that the regularized \( G^{\text{reg}}(\sigma) \) of (91) satisfies the inhomogeneous equation

\[ (\nabla^\nu \nabla_\mu + \frac{1}{6} R + m^2) G^{\text{reg}}(\sigma) = \frac{1}{16\pi^2} (Y - L) \left[ 1 + 4\gamma + 2\ln(-m^2\epsilon^2) \right], \tag{93} \]

which will cause unwanted higher order terms \( \sim R \) in the stress tensor. Thus, we shall work with the second scheme, using (42) and (43) in the following. The unregularized stress tensor is (71) with \( \xi = \frac{1}{6} \) and \( W = m^2 \),

\[
\langle T_{\mu\nu} \rangle = \lim_{x' \to x} \frac{1}{16\pi^2} \left[ \left( \epsilon^{-4}, \epsilon^{-2} \text{ terms} \right) + Y P_{\mu\nu} \right. \\
+ \frac{1}{2} g_{\mu\nu} Y \ln \epsilon^2 + g_{\mu\nu} Y + \frac{1}{2} g_{\mu\nu} Z + \frac{1}{2} g_{\mu\nu} m^2 (m^2 \ln \epsilon^2 + X) \\
+ g_{\mu\nu} \xi R \frac{R}{2} m^2 - \xi (-g_{\mu\nu} R \frac{R}{4}) m^2 \ln \epsilon^2 - \xi (-g_{\mu\nu} R \frac{R}{4}) X \bigg], \tag{94}
\]

and the substraction stress tensor is obtained by replacing \((X, Y, Z)\) by \((K, L, M)\) in (94),

\[
\langle T_{\mu\nu} \rangle_{\text{sub}} = \lim_{x' \to x} \frac{1}{16\pi^2} \left[ \left( \epsilon^{-4}, \epsilon^{-2} \text{ terms} \right) + L P_{\mu\nu} \right. \\
+ \frac{1}{2} g_{\mu\nu} L \ln \epsilon^2 + g_{\mu\nu} L + \frac{1}{2} g_{\mu\nu} M + \frac{1}{2} g_{\mu\nu} m^2 (m^2 \ln \epsilon^2 + K) \\
+ g_{\mu\nu} \xi R \frac{R}{2} m^2 - \xi (-g_{\mu\nu} R \frac{R}{4}) m^2 \ln \epsilon^2 - \xi (-g_{\mu\nu} R \frac{R}{4}) K \bigg], \tag{95}
\]

with \( \xi = \frac{1}{6} \). The last three terms in the above are of the 2nd-order \( \sim R \), and should be dropped, leading to

\[
\langle T_{\mu\nu} \rangle_{\text{sub}} = \lim_{x' \to x} \frac{1}{16\pi^2} \left[ \left( \epsilon^{-4}, \epsilon^{-2} \text{ terms} \right) + L P_{\mu\nu} \right. \\
+ \frac{1}{2} g_{\mu\nu} L \ln \epsilon^2 + g_{\mu\nu} L + \frac{1}{2} g_{\mu\nu} M + \frac{1}{2} g_{\mu\nu} m^2 (m^2 \ln \epsilon^2 + K) \bigg], \tag{96}
\]
where the coefficients \((K, L, M)\) are the 0th-order. Now the difference between \((94)\) and \((96)\) yields
\[
\langle T_{\mu\nu}\rangle_{\text{reg}} = (Y - L) \lim_{x' \to x} \frac{1}{16\pi^2} P_{\mu\nu} + g_{\mu\nu} \frac{1}{32\pi^2} \left[ (Z - M) + m^2(X - K) + \frac{R}{12}X \right],
\]
(97)
The term \(\sim P_{\mu\nu}\) is of the 2nd-order, depends on the path of the coincidence limit, and does not possess the maximum symmetry. Dropping it, we obtain
\[
\langle T_{\mu\nu}\rangle_{\text{reg}} = g_{\mu\nu} \frac{1}{32\pi^2} \left[ (Z - M) + m^2(X - K) + \frac{R}{12}X \right],
\]
(98)
which is equal to the result \((86)\) from the simple derivation.

For the case of \(m = 0\) and \(\xi = \frac{1}{6}\), the unregularized Green function \((15)\) is valid and reduces to the following simple form \([41, 42]\)
\[
G(\sigma) = -\frac{H^2}{8\pi^2} \frac{1}{\sigma},
\]
(99)
consisting of one divergent term only. After subtraction of this term, the regularized Green’s function is \(G(x, x')_{\text{reg}} = 0\). This result agrees with \((92)\) at \(m = 0\). So, \(\langle T_{\mu\nu}\rangle_{\text{reg}} = 0\), also agreeing with \((88)\) from the adiabatic regularization. In the second scheme, the unregularized stress tensor is
\[
\langle T_{\mu\nu}\rangle = \lim_{x' \to x} \left\{ \frac{1}{3} \left[ \nabla_\mu \nabla_\nu - \nabla_\mu' \nabla_\nu' \right] - \frac{1}{6} \left[ \nabla_\mu \nabla_\nu + \nabla_\mu' \nabla_\nu' \right] - \frac{1}{6} g_{\mu\nu} \nabla_\sigma \nabla_\sigma' \right\}
+ \frac{1}{6} g_{\mu\nu} \left( \nabla_\sigma \nabla_\sigma' - \frac{1}{6} G_{\mu\nu} \right) \left( x - x' \right)
= - \frac{1}{48\pi^2} \lim_{x' \to x} \left[ - \frac{1}{\epsilon^4} (\partial_\mu \partial_\nu + \partial_\nu' \partial_\mu') \epsilon_2^2 + \frac{1}{2\epsilon^4} (\partial_\mu \partial_\nu + \partial_\nu' \partial_\mu') \epsilon_2^2 \right.
+ \frac{2}{\epsilon^6} (\partial_\nu \epsilon_2^2 \cdot \partial_\mu \epsilon_2^2 + \partial_\nu' \epsilon_2^2 \cdot \partial_\mu' \epsilon_2^2) - \frac{1}{\epsilon^6} (\partial_\nu \epsilon_2^2 \cdot \partial_\mu \epsilon_2^2 + \partial_\nu' \epsilon_2^2 \cdot \partial_\mu' \epsilon_2^2)
\left. - \frac{1}{12} g_{\mu\nu} \right] \left. \frac{1}{\epsilon^2} - \frac{1}{2\epsilon^4} \Gamma^\alpha_{\mu\nu} \partial_\alpha \epsilon_2^2 - \frac{1}{2\epsilon^4} \Gamma^\alpha_{\mu\nu} \partial_\alpha \epsilon_2^2 \right].
\]
(100)
All the terms in \((100)\) are UV divergent and should be subtracted off, we also arrive at \(\langle T_{\mu\nu}\rangle_{\text{reg}} = 0\), the same as \((88)\).

### 7 The improper 4th-order regularization

We now examine the conventional 4th-order regularization for the scalar field with a general \(\xi\), and reveal its unphysical consequences. The 4th-order adiabatically regularized power spectrum with a general \(\xi\) is
\[
\Delta_k^2 = \frac{k^3}{2\pi^2 a^2} \left( |v_k|^2 - \frac{1}{2W(3)} \right),
\]
(101)
where the 4th-order effective inverse frequency is (see \((a38)\) in Ref. [41])
\[
(W_k^{(4)})^{-1} = \frac{1}{\omega} - 3(\xi - \frac{1}{6}) \frac{1}{\omega^3} \frac{a''}{a} + \frac{m^2(2a'' + a'^2)}{4\omega^5} - \frac{5m^4a'^2 a''}{8\omega^7}
- \frac{m^2(3a''^2 + 3a'' a')}{32\omega^9} + \frac{7m^4(3a''^2 + 3a'' a' + 18a'^2 a'' + 4a'^2a' a')}{96\omega^9}
- \frac{231m^6a^2(a'^4 + 2a''a'^2)}{32\omega^{11}} + \frac{1155m^8a'^4}{128\omega^{13}}
+ (\xi - \frac{1}{6}) \left[ \frac{3}{4\omega^5} \left( \frac{a''^2}{a^2} + \frac{a'' a'^2}{a^3} + 2\frac{a'^2 a''}{a^2} \right) - 2\frac{a'' a'^2}{a^2} \right],
\]
(102)
The 4th-order power spectrum $\Delta^2_{k,\text{reg}}$ is negative, as shown in Fig.8 (a) for $\xi = 0$, and in Fig.9 (a) for $\xi = \frac{1}{6}$. The negative power spectrum is unphysical. Obviously, the 4th-order regularization has subtracted off too much for the scalar field, and is discordant with the minimum subtraction rule [18]. Moreover, the 4th-order regularization will cause other difficulties, as we shall examine in the following.

![Plot of $\Delta^2_{k,\text{reg}}, \xi=0$](image1.png)

![Plot of $G_{\text{reg}}(y/(H^2/8\pi)), \xi=0$](image2.png)

Figure 8: (a): The 4th-order $\Delta^2_{k,\text{reg}}$ is negative. (b): the 4th-order $G_{\text{reg}}$. The model $\xi = 0$ and $m^2/H^2 = 0.1$.

![Plot of $\Delta^2_{k,\text{reg}}, \xi=\frac{1}{6}$](image3.png)

![Plot of $G_{\text{reg}}(y/(H^2/8\pi)), \xi=\frac{1}{6}$](image4.png)

Figure 9: (a): The 4th-order $\Delta^2_{k,\text{reg}}$ takes negative values. (b): the 4th-order $G_{\text{reg}}$. The model $\xi = \frac{1}{6}$ and $m^2/H^2 = 0.1$. 

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The 4th-order subtraction Green’s function is given by

\[ G(y)_{\text{sub}} = \frac{H^2}{8\pi} \int_0^\infty dk \frac{k \sin(ky)}{2} \left( \frac{1}{\pi W^{(4)}} \right) \]

\[ = \frac{H^2}{4\pi^2} \left[ \frac{m}{H} y K_1 \left( \frac{m}{H} y \right) \right. \]

\[ - 6(\xi - \frac{1}{6})K_0 \left( \frac{m}{H} y \right) \frac{m}{4H} y K_1 \left( \frac{m}{H} y \right) - \frac{1}{24} \frac{m^2}{H^2} y^2 K_2 \left( \frac{m}{H} y \right) \]

\[ + 9(\xi - \frac{1}{6}) + 54(\xi - \frac{1}{6})^2 \left( \frac{m}{H} y \right)^{-1} y K_1 \left( \frac{m}{H} y \right) - \frac{1}{4} + 10(\xi - \frac{1}{6}) y^2 K_2 \left( \frac{m}{H} y \right) \]

\[ + \frac{525 + 840(\xi - \frac{1}{6})}{32 \cdot 105} \left( \frac{m}{H} y \right)^3 y^3 K_3 \left( \frac{m}{H} y \right) - \frac{693}{32 \cdot 945} \left( \frac{m}{H} y \right)^2 y^4 K_4 \left( \frac{m}{H} y \right) \]

\[ + \frac{1155}{128 \cdot 10395} \left( \frac{m}{H} y \right)^3 y^5 K_5 \left( \frac{m}{H} y \right) \right]. \tag{103} \]

This 4th-order subtraction Green’s function has not been given before in literature. The first line in (103) is the 0th-order subtraction term, the first two lines belong to the 2nd-order subtraction term, and the remaining terms come from the 4th-order. Replacing \( y \rightarrow \sqrt{-2\sigma} \) in (103) gives \( G(\sigma)_{\text{sub}} \) for general spacetime separation \( \sigma \). The 4th-order regularized Green’s function is given by

\[ G(\sigma)_{\text{reg}} = G(\sigma) - G(\sigma)_{\text{sub}}, \tag{104} \]

where \( G(\sigma) \) is given by eq.(15). We plot \( G(\sigma)_{\text{reg}} \) in Fig.8 (b) and Fig.9 (b).

As has been found in Ref. [41], the 4th-order adiabatically regularized spectral energy density \( \rho_{k,\text{reg}} = \rho_k - \rho_{k,\text{A4}} \) takes negative values too, as illustrated in Fig.10 for \( \xi = 0 \) and \( \xi = \frac{1}{6} \).

![Figure 10](image-url)

Figure 10: The 4th-order \( \rho_{k,\text{reg}} \) takes negative values. (a): \( \xi = 0 \). (b): \( \xi = \frac{1}{6} \). The model \( \frac{m^2}{H^2} = 0.1 \).

We now calculate the stress tensor with general \( \xi \) by the 4th-order regularization in the point-splitting method. At small separation, the 4th-order subtraction Green’s function is

\[ G(\sigma)_{\text{sub}} = \frac{1}{16\pi^2} \left[ - \frac{1}{\epsilon^2} + W \ln \epsilon^2 + V + Y \epsilon^2 \ln \epsilon^2 + T \epsilon^2 \right] + O(\epsilon^3), \tag{105} \]
where
\[
V = \left( m^2 + R(\xi - \frac{1}{6}) \right) \left( 2\gamma - 1 + \ln(-m^2) \right) + \frac{R}{18} + (\xi - \frac{1}{6})R
\]
\[
- \frac{R^2}{2160m^2} + \frac{(\xi - \frac{1}{6})^2 R^2}{2m^2},
\]
\[
T = -\frac{1}{2} (m^2 + \xi R)(m^2 + (\xi - \frac{1}{6})R) \left( -\frac{5}{2} + 2\gamma + \ln(-m^2) + \frac{1}{4m^2} + \xi R \right)
\]
\[
- \left( \frac{m^2(\xi - \frac{1}{6})R}{2} + \frac{m^2 R}{36} + \frac{3(\xi - \frac{1}{6})^2 R^2}{4} + \frac{19R^2}{4320} + \frac{(\xi - \frac{1}{6})R^2}{9} \right).
\]
The difference between (45) and (105) is the 4th-order regularized Green’s function at small separation
\[
G(\sigma)_{\text{reg}} = G(0)_{\text{reg}} + \epsilon^2 G_{\epsilon},
\]
where
\[
G(0)_{\text{reg}} \equiv \frac{1}{16\pi^2}(X - V)
\]
\[
= \frac{1}{16\pi^2} \left[ (m^2 + (\xi - \frac{1}{6})R) \left( \psi\left(\frac{3}{2} - \nu\right) + \psi\left(\frac{3}{2} + \nu\right) - \ln\frac{12m^2}{R} \right) \right.
\]
\[
- (\xi - \frac{1}{6})R - \frac{1}{18} R - \frac{R^2 (\xi - \frac{1}{6})^2}{2m^2} + \frac{R^2}{2160m^2} \right],
\] (107)
\[
G_{\epsilon} \equiv \frac{1}{16\pi^2}(Z - T).
\] (108)
Our 4th-order (106) with (107) (108) at small separation is equal to (3.14) of Ref. [13], which did not give the subtraction Green’s function (103) valid for the whole range of \( \sigma \). Since (106) contains no \( \epsilon^2 \ln \epsilon^2 \) term, we do calculation in the first scheme. Plugging (106) into (41) leads to the following stress tensor,
\[
\langle T_{\mu\nu} \rangle_{\text{reg}} = g_{\mu\nu} \left[ \frac{1}{2} G_{\epsilon} + \frac{1}{2} m^2 G(0)_{\text{reg}} + \frac{1}{4} \xi RG(0)_{\text{reg}} \right],
\] (109)
which is independent of the path of coincidence limit, but still contains some unwanted 6th-order terms \( \sim R^3 \). This is because \( G_{\text{sub}}(\sigma) \) of (105) satisfies the inhomogeneous equation
\[
\lim_{x' \to x} (\nabla_{\sigma} \nabla^{\sigma} + m^2 + \xi R) G_{\text{sub}}(\sigma) = \frac{1}{16\pi^2} \left( -\frac{\xi}{2160m^2} + \frac{\xi(\xi - \frac{1}{6})^2}{2m^2} \right) R^3,
\] (110)
due to the coupling \( \xi R \). Requiring the 4th-order Green’s function to satisfy the homogeneous equation to the 4th-order,
\[
\lim_{x' \to x} \left[ \nabla^{\mu} \nabla_{\mu} + \xi R + m^2 \right] G_{\text{reg}}(\sigma) = 0,
\] (111)
\[
\xi RG_{\text{reg}}(0) = -\lim_{x' \to x} \left[ \nabla^{\mu} \nabla_{\mu} + m^2 \right] G_{\text{reg}}(\sigma) = -2G_{\epsilon} - m^2 G_{\text{reg}}(0).
\] (112)
By this relation, we can replace \( \xi RG_{\text{reg}}(0) \) in (109) by \( (-2G_{\epsilon} - m^2 G_{\text{reg}}(0)) \), and arrive at the 4th-order regularized stress tensor
\[
\langle T_{\mu\nu} \rangle_{\text{reg}} = g_{\mu\nu} \frac{1}{4} m^2 G_{\text{reg}}(0)
\]
\[
= g_{\mu\nu} \frac{1}{64\pi^2} \left[ m^2(m^2 + (\xi - \frac{1}{6})R) \left( \psi\left(\frac{3}{2} - \nu\right) + \psi\left(\frac{3}{2} + \nu\right) + \ln\frac{R}{12m^2} \right) \right.
\]
\[
- m^2(\xi - \frac{1}{6})R - \frac{m^2 R}{18} - \frac{(\xi - \frac{1}{6})^2 R^2}{2} + \frac{R^2}{2160} \right],
\] (113)

containing no $R^3$ terms.

(113) can be also derived by the second scheme. The subtraction stress tensor is obtained by replacing $(X, Z)$ by $(V, T)$,

$$\langle T_{\mu\nu}\rangle_{\text{sub}} = \lim_{x' \to x} \frac{1}{16\pi^2} \left[ \left( \epsilon^4, \epsilon^2 \text{ terms} \right) + Y P_{\mu\nu} + \frac{1}{2} g_{\mu\nu} Y \ln \epsilon^2 + g_{\mu\nu} Y + \frac{1}{2} g_{\mu\nu} T + \frac{1}{2} g_{\mu\nu} m^2 (W \ln \epsilon^2 + V) + g_{\mu\nu} \xi \frac{R}{2} W - \xi (-g_{\mu\nu} \frac{R}{4}) W \ln \epsilon^2 - \xi (-g_{\mu\nu} \frac{R}{4}) V \right].$$

The last term $RV$ of (114) contains $R^3$ which can be dropped by the replacement

$$RV \to R \left( V + \frac{R^2}{2160m^2} - \frac{(\xi - \frac{1}{6})^2 R^2}{2m^2} \right),$$

yielding

$$\langle T_{\mu\nu}\rangle_{\text{sub}} = \lim_{x' \to x} \frac{1}{16\pi^2} \left[ \left( \epsilon^4, \epsilon^2 \text{ terms} \right) + Y P_{\mu\nu} + \frac{1}{2} g_{\mu\nu} Y \ln \epsilon^2 + g_{\mu\nu} Y + \frac{1}{2} g_{\mu\nu} T + \frac{1}{2} g_{\mu\nu} m^2 (W \ln \epsilon^2 + V) + g_{\mu\nu} \xi \frac{R}{2} W - \xi (-g_{\mu\nu} \frac{R}{4}) W \ln \epsilon^2 - \xi (-g_{\mu\nu} \frac{R}{4}) \left( V + \frac{R^2}{2160m^2} - \frac{(\xi - \frac{1}{6})^2 R^2}{2m^2} \right) \right].$$

The difference between (71) and (115) yields the 4th-order regularized vacuum stress tensor

$$\langle T_{\mu\nu}\rangle_{\text{reg}} = g_{\mu\nu} \frac{1}{64\pi^2} \left[ 2(Z - T) + 2m^2 (X - V) + \xi R (X - V) - \frac{\xi R^3}{2160m^2} + \frac{\xi (\xi - \frac{1}{6})^2 R^2}{2m^2} \right],$$

which is equal to (113).

Now we examine several difficulties associated with the outcome of 4th-order regularization. Firstly the last two terms of $G(0)_{\text{reg}}$ in (107) are proportional to $m^{-2}$ and singular at $m = 0$, so that the 4th-order regularized Green’s function (106) is ill-defined in the massless limit. Associated with this is the so-called trace anomaly for $\xi = \frac{1}{6}$ in the massless limit [13],

$$\lim_{m \to 0} \langle T_{\mu\nu}\rangle_{\text{reg}} = m^2 G(0)_{\text{reg}} = \frac{R^2}{34560\pi^2},$$

which comes exactly from the last, singular term $\frac{R^2}{34560\pi^2}$ in (107). Obviously, the 4th-order result (117) is invalid since it is defined at the singular point $m = 0$ of the 4th-order regularized Green’s functions. The occurrence of the singular term and its associated trace anomaly are artifacts brought about by the 4th-order subtraction term. In contrast, the 2nd-order and 0th-order regularized Green’s functions, (51) and (92), contain no such kind of singular terms.

Next the resulting energy density of (113) is generally negative

$$\rho_{\text{reg}} < 0,$$

as shown in Fig.11 (a) for $\xi = \frac{1}{10}$, and in Fig.11 (b) for $\xi = \frac{1}{6}$. It is checked that the 4th-order adiabatic $\rho_{\text{reg}} = \int_0^\infty (\rho_k - \rho_k A_I) \frac{dk}{k}$ is also equal to the regularized energy density of (113). Such a negative vacuum energy is inconsistent with the de Sitter inflation that requires a positive vacuum energy. This is another vital difficulty of the 4th-order regularization.

From the above examinations it is clear that both the trace anomaly and the negative energy density are simultaneously caused by the over-subtraction of the 4th-order regularization which is discordant with the minimum subtraction rule. Hence, the 4th-order regularization, either the adiabatic or the point-splitting, [11,13,14,25,35,37,39,40] is an improper prescription for a massive scalar field in de Sitter space.
8 Conclusion and Discussions

We have carried out the point-splitting regularization of the stress tensor of the coupling massive scalar field in de Sitter inflation. The key of any regularization is to prescribe an appropriate subtraction term. In the point-splitting method, the stress tensor is constructed from the Green’s function in $x$-space, so the regularized Green’s function will be instrumental. In our previous work [41], the 2nd- and 0th-order adiabatically regularized Green’s functions with the coupling $\xi$ were obtained, and are used in this paper. For a given $\xi$, assuming the minimal subtraction rule [18], we have performed regularization on the stress tensor to the same adiabatic order as on the Green’s function, and in two alternative schemes: one is to calculate the regularized stress tensor from the regularized Green’s function, another is to calculate the unregularized, and subtraction stress tensors respectively and then to take their difference. In both schemes, we have found that, for $\xi \neq 0$, the point-splitting calculation may not automatically lead to an appropriate regularized stress tensor even when the regularized Green’s function is continuous and UV- and IR-convergent. After dropping unwanted higher-order terms, both schemes yield the same stress tensor, which is also equal to the outcome from adiabatic regularization. Comparatively, the second scheme involves more calculations of the divergent terms, and, nevertheless, is easier to pick out the unwanted higher-order terms.

For the minimal coupling $\xi = 0$ in Sections 4, we adopt the 2nd-order regularization which is sufficient to remove all the UV divergences and in accordance with the minimum subtraction rule. The 0-order regularization would not be able to remove all UV divergences, and the 4-order regularization would subtract off too much and lead to a negative spectral energy density. Using the 2nd-order regularized Green’s function, we have carried out differentiations and the coincidence limit, and obtained the 2nd-order regularized vacuum stress tensor (55), which is finite and constant, satisfies the maximal symmetry in de Sitter space, respects the covariant conservation, and its energy density is positive. Thus, it is identified as, or part of, the cosmological constant (55). The special case $m = \xi = 0$ needs a separate treatment in the point-splitting regularization, and the regularized vacuum Green’s function and stress tensor are zero, the same as the result from the adiabatic regularization.

The case of general $\xi > 0$ in Sections 5 is more involved than the case $\xi = 0$. The coupling $\xi R$ causes a term $\sim \epsilon^2 \ln \epsilon^2$ in the 2nd-order regularized Green’s function, and consequently brings a divergent term $\sim \ln \epsilon^2$ and other unwanted 4th-order terms in the regularized stress tensor. To avoid these, we remove the 4th-order terms from the subtraction stress tensor, just as we did.
in the adiabatic regularization. There is still a 4th-order term which depends upon the path of the coincidence limit and does not possess the maximum symmetry. After dropping this path-dependent term, the regularized stress tensor (79) becomes appropriate, and reduces to (55) when \( \xi = 0 \). In particular, we have found that, for small couplings, for instance \( \xi \in (0, \frac{1}{7.04}) \) at a fixed \( \frac{m^2}{H^2} = 0.1 \), the 2nd-order regularized energy density is positive, and also can be identified as the cosmological constant, like the \( \xi = 0 \) case. But, for large couplings, say \( \xi > \frac{1}{7.04} \) at a fixed \( \frac{m^2}{H^2} = 0.1 \), the regularized energy density and spectral energy density will still be negative.

For \( \xi = \frac{1}{6} \) in Sections 6, we adopt the 0th-order regularization which removes all the UV divergences and is in accordance with the minimum subtraction rule. If the 2nd-, or 4th-order regularization were adopted for \( \xi = \frac{1}{6} \), one would get a negative spectral energy density and a negative energy density. By the trace relation (85) and by the maximum symmetry, the 0th-order regularized stress tensor (86) follows straightforwardly without carrying out differentiations. Its energy density is positive, and also can be identified as, or part of, the cosmological constant. Alternatively, we have directly calculated the stress tensor, and, after extra treatments in analogy to the case \( \xi > 0 \), also arrived at (86). In the massless limit, the regularized vacuum stress tensor is zero, and there is no trace anomaly for the massless scalar field with \( \xi = \frac{1}{6} \).

The conventional 4th-order regularization is also examined in Section 7. We have calculated the 4th-order regularized Green’s function and stress tensor with general \( \xi \). We have demonstrated in Fig.11 that the 4th-order regularized vacuum energy density for a general \( \xi \) is negative, which is inconsistent with the de Sitter inflation that requires a positive vacuum energy. Moreover, the 4th-order regularized Green’s function (107) is singular at \( m = 0 \), and consequently its associated trace anomaly for \( \xi = \frac{1}{6} \) is ill-defined in the massless limit. These difficulties are caused by the over-subtraction under the conventional 4th-order regularization which is discordant with the minimum subtraction rule.

We now discuss the issue of the order of regularization. The outcome of our paper indicates that the order of regularization is very important in achieving an appropriate regularized stress tensor with the desired properties. However, there are little discussions on the issue of order of regularization in literature, even though there have been many of studies on regularization since 70’s, and almost all adopted the 4th-order for the stress tensor, by default, or implicitly. It seems to us that there is no unique recipe for the order of regularization except the desired properties of the stress tensor that we want to achieve. In this regard, the most closely related is the minimum subtraction rule suggested by Ref. [18] that only the minimum number of terms should be subtracted. The regularization order is actually implied by this rule, particularly, in the adiabatic regularization method, by which the subtraction terms are effectively grouped by the orders. In our paper, for \( \xi = \frac{1}{6} \), the 0th-order regularization yields an appropriate stress tensor and is in accordance with this rule. Similarly, for \( \xi = 0 \), the 2nd-order also works and is also in accordance with this rule. If the 4th-order were adopted for \( \xi = \frac{1}{6} \) and \( \xi = 0 \), it is discordant with the minimum subtraction rule, so as to yield a negative spectral energy density. Nevertheless, this does not rule out the 4th-order, which may be necessary in other cases. Our work has shown that an appropriate choice of the regularization order depends upon the coupling. In general, we speculate that this may depend also upon the type of quantum fields [56] and the symmetry of spacetime background, etc. All one can do is by trial and error, in each concrete case.

There occurs another issue of the regularization order for the Green’s function. As far as we know, Ref. [28] first performed the 2nd-order adiabatic regularization on the power spectrum for the \( \xi = 0 \) massive scalar field. In the point-splitting regularization, for the scalar field, \( \langle T_{\mu\nu} \rangle \) is actually constructed from \( G(x - x') \), and contains typical terms like \( \xi R G(x - x') \) and \( m^2 G(x - x') \), etc, and a regularization of Green’s function implies a regularization of stress tensor. Therefore, it is natural to conjecture that the order of regularization on the Green’s function should be equal to that on the stress tensor. Indeed, as our calculation shows, for \( \xi = \frac{1}{6} \) the 0th-order regularization
works for both $\langle T_{\mu\nu} \rangle$ and $G(x - x')$, and analogously for $\xi = 0$ the 2nd-order also works for both $\langle T_{\mu\nu} \rangle$ and $G(x - x')$. This is also true in the adiabatic regularization on the scalar field [41,42]. In these cases, both methods support the same order for the stress tensor and Green’s function. Nevertheless, this conjecture may not hold for other type of fields, such as vector fields and tensor fields, for which the Green’s functions possess multi components and the stress tensors is composed of several portions with different structure [56].

Comparing the two methods of regularization, the point-splitting in this paper and the adiabatic in Ref. [41,42], we see the following.

In the adiabatic regularization in $k$-space, one is able to get the subtraction terms to any desired order by the WKB approximation systematically, for the power spectrum and for the spectral stress tensor. On the other hand, in the point-splitting regularization in position space, the subtraction term for the Green’s function valid on the whole range is generally hard to find directly. The conventional Hadamard function as a subtraction term is only an approximation at small distance, and not valid on the whole range. With the help of the adiabatically regularized power spectrum, through the Fourier transformation, one will be able to get the adiabatically regularized Green’s function. However, even when the regularized Green’s function is given with the coupling $\xi R \neq 0$, one still needs extra treatments to drop certain higher-order terms from the subtraction stress tensor, and to drop the unwanted path-dependent terms from the regularized stress tensor.

In regard to the outcome, the two methods are complementary. The adiabatic regularization yields the regularized spectral stress tensor and the numerical, regularized stress tensor after $k$-integration. The point-splitting regularization yields the analytical, regularized stress tensor, but not the spectral stress tensor.

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A Some differentiation formulae

In this appendix, we list some formulae of the point-splitting method which are used in calculation of the stress tensor in the context. For simple notation, we introduce

$$
\epsilon^2 \equiv \frac{\sigma}{2H^2} = \frac{(\tau - \tau')^2 - (x - x')^2 - (y - y')^2 - (z - z')^2}{4H^2\tau\tau'},
$$

(118)

which is the one quarter of the square of the geodesic distance in the de Sitter space, and obeys the equation $\epsilon^2 = (\epsilon^2)^\mu(\epsilon^2)^\nu$ at small separation. Performing differentiations and then taking the coincidence limit, one obtains the basic formulae

$$
\lim_{x' \to x} \nabla_\mu \nabla_\nu \epsilon^2 = \lim_{x' \to x} \nabla_\mu \nabla_\nu' \epsilon^2 = - \lim_{x' \to x} \nabla_\mu \nabla_\nu' \epsilon^2 = \frac{1}{2} g_{\mu\nu},
$$

(119)

$$
\lim_{x' \to x} \nabla^\sigma \nabla_\sigma \epsilon^2 = \lim_{x' \to x} \nabla^\sigma' \nabla_\sigma' \epsilon^2 = - \lim_{x' \to x} \nabla^\sigma \nabla_\sigma' \epsilon^2 = 2.
$$

(120)

The following formulae are also involved in the context

$$
\lim_{x' \to x} \nabla_\mu \nabla_\nu \epsilon^2 = \lim_{x' \to x} \left( \frac{1}{\epsilon^4} \cdot \frac{1}{2} g_{\mu\nu} + \frac{2}{\epsilon^6} \cdot \partial_\nu \epsilon^2 \cdot \partial_\mu \epsilon^2 \right),
$$

(121)
\begin{align}
\lim_{x' \to x} \nabla_{\mu} \nabla_{\nu} \epsilon^{-2} &= \lim_{x' \to x} \left( -\frac{1}{\epsilon^4} \cdot \frac{1}{2} g_{\mu \nu} + \frac{2}{\epsilon^6} \cdot \partial_{\nu} \epsilon^2 \cdot \partial_{\mu} \epsilon^2 + \frac{1}{\epsilon^4} \Gamma_{\alpha}^{\mu} \partial_{\alpha} \epsilon^2 \right), \\
\lim_{x' \to x} \nabla_{\sigma} \nabla'_{\nu} \epsilon^{-2} &= \lim_{x' \to x} \left( \frac{1}{2\epsilon^2} \cdot (1 - \frac{\tau'}{\tau})^2 + \frac{R}{12 \epsilon^2} \right), \\
\lim_{x' \to x} \nabla_{\sigma} \nabla'_{\nu} \epsilon^{-2} &= \lim_{x' \to x} \nabla_{\sigma} \nabla'_{\nu} \epsilon^{-2} = \lim_{x' \to x} \left( -\frac{1}{6 \epsilon^2} \right), \\
\lim_{x' \to x} \nabla_{\mu} \nabla'_{\nu} \ln \epsilon^2 &= \lim_{x' \to x} \left( \frac{1}{2} g_{\mu \nu} \cdot \frac{1}{\epsilon^2} \cdot \partial_{\mu} \epsilon^2 \cdot \partial_{\nu} \epsilon^2 + \frac{1}{\epsilon^4} \right), \\
\lim_{x' \to x} \nabla_{\mu} \nabla'_{\nu} \ln \epsilon^2 &= \lim_{x' \to x} \left( \frac{1}{2} g_{\mu \nu} \cdot \frac{1}{\epsilon^2} \cdot \partial_{\mu} \epsilon^2 \cdot \partial_{\nu} \epsilon^2 + \frac{1}{\epsilon^4} \Gamma_{\nu \mu}^{\alpha} \partial_{\alpha} \epsilon^2 \right), \\
\lim_{x' \to x} \nabla_{\sigma} \nabla'_{\nu} \ln \epsilon^2 &= \lim_{x' \to x} \left( -\frac{1}{\epsilon^2} \right), \\
\lim_{x' \to x} \nabla_{\sigma} \nabla'_{\nu} \ln \epsilon^2 &= \lim_{x' \to x} \nabla_{\sigma} \nabla'_{\nu} \ln \epsilon^2 = \lim_{x' \to x} \left( \frac{1}{4} (\frac{4}{\epsilon^2} + R) \right), \\
\lim_{x' \to x} \nabla_{\mu} \nabla'_{\nu} \ln \epsilon^2 &= \lim_{x' \to x} \left( \frac{1}{\epsilon^2} \partial_{\mu} \epsilon^2 \cdot \partial_{\nu} \epsilon^2 - \frac{1}{2} g_{\mu \nu} (\ln \epsilon^2 + 1) \right), \\
\lim_{x' \to x} \nabla_{\mu} \nabla'_{\nu} \ln \epsilon^2 &= \lim_{x' \to x} \left( \frac{1}{\epsilon^2} \partial_{\mu} \epsilon^2 \cdot \partial_{\nu} \epsilon^2 + \frac{1}{2} g_{\mu \nu} (\ln \epsilon^2 + 1) \right), \\
\lim_{x' \to x} \nabla_{\sigma} \nabla'_{\nu} \ln \epsilon^2 &= \lim_{x' \to x} -(3 + 2 \ln \epsilon^2), \\
\lim_{x' \to x} \nabla_{\sigma} \nabla'_{\nu} \ln \epsilon^2 &= \lim_{x' \to x} \nabla_{\sigma} \nabla'_{\nu} \ln \epsilon^2 = \lim_{x' \to x} (3 + 2 \ln \epsilon^2). 
\end{align}

**B  The term depending on the path of coincidence limit**

In Sections 5 and 6, $P_{\mu \nu}$ defined by (70) shows up in the stress tensor (78) and (97) when the regularized Green’s function contains $\epsilon^2 \ln \epsilon^2$ for $\xi \neq 0$. Different paths of coincidence limit lead to different values of $\lim_{x' \to x} P_{\mu \nu}$. For instance, consider the 00'-component of the first term in (70),

$$\frac{1}{\epsilon^2} \partial_{x'} \epsilon^2 \cdot \partial_{y'} \epsilon^2 = \frac{1}{\epsilon^2} \left( -\frac{1}{\tau'} \epsilon^2 + \frac{\tau - \tau'}{2 H^2 \tau \tau'} \right) \cdot \left( -\frac{1}{\tau} \epsilon^2 - \frac{\tau - \tau'}{2 H^2 \tau \tau'} \right) = -\frac{1}{H^2 \tau \tau'} \frac{(\tau - \tau')^2}{(\tau - \tau')^2 - (x - x')^2 - (y - y')^2 - (z - z')^2}.$$

For the path $\tau' \to \tau$ followed by $\tilde{r}' \to \tilde{r}$, (133) gives

$$\lim_{\tilde{r}' \to \tilde{r}} \lim_{\tau' \to \tau} \frac{1}{\epsilon^2} \partial_{x'} \epsilon^2 \cdot \partial_{y'} \epsilon^2 = \lim_{\tilde{r}' \to \tilde{r}} -\frac{1}{H^2 \tau \tau'} \frac{0}{(\tau - \tau')^2 - (x - x')^2 - (y - y')^2 - (z - z')^2} = 0,$$

for the path $\tilde{r}' \to \tilde{r}$ followed by $\tau' \to \tau$, (133) gives

$$\lim_{\tau' \to \tau} \lim_{\tilde{r}' \to \tilde{r}} \frac{1}{\epsilon^2} \partial_{x'} \epsilon^2 \cdot \partial_{y'} \epsilon^2 = \lim_{\tau' \to \tau} -\frac{1}{H^2 \tau \tau'} \frac{(\tau - \tau')^2}{(\tau - \tau')^2} = -\frac{1}{H^2 \tau^2} = -a^2(\tau).$$

(134) and (135) are not equal. Similarly, other terms of $P_{\mu \nu}$ also depend on the path.

For $P_{\mu \nu}$ as a whole, detailed calculation shows that, for the path $\tilde{r}' \to \tilde{r}$ followed by $\tau' \to \tau$,

$$\lim_{\tau' \to \tau} \lim_{\tilde{r}' \to \tilde{r}} P_{\mu \nu} = a^2(\tau) \text{diag}(-1, 0, 0, 0).$$
For the path $\tau' \to \tau$ followed by $x' \to x$, and then irrespectively $y' \to y$, $z' \to z$,

$$\lim_{z' \to z} \lim_{y' \to y} \lim_{x' \to x} \lim_{\tau' \to \tau} P_{\mu\nu} = a^2(\tau) \text{diag}(0, 1, 0, 0).$$

(137)

which is not equal to (136). Other paths will give other values of $\lim P_{\mu\nu}$ which differ from (136) (137). Moreover, for any path, $\lim P_{\mu\nu}$ is not proportional to the metric $g_{\mu\nu}$, and does not respect the maximum symmetry in de Sitter space. Thus, $\lim P_{\mu\nu}$ is dropped from the stress tensor.