\textbf{S-adic characterization of minimal ternary dendric subshifts}

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Abstract

Dendric subshifts are defined by combinatorial restrictions of the extensions of the words in its language. This family generalizes well-known families of subshifts such as Sturmian subshifts, Arnoux-Rauzy subshifts and codings of interval exchange transformations. It is known that any minimal dendric subshifts has a primitive $S$-adic representation where the morphisms in $S$ are positive tame automorphisms of the free group generated by the alphabet. In this paper we investigate those $S$-adic representations, heading towards an $S$-adic characterization of this family. We obtain such a characterization in the ternary case, involving a directed graph with 9 vertices.

1 Introduction

Dendric subshifts are defined in terms of extension graphs that describe the left and right extensions of their factors. Extension graphs are bipartite graphs that can be roughly be described as follows: if $u$ is a word in the language $\mathcal{L}(X)$ of the subshift $X$, one puts an edge between the left and right copies of letters $a$ and $b$ such that $aub$ is in $\mathcal{L}(X)$. A subshift is then said to be dendric if the extension graph of every word of its language is a tree. These subshifts were initially defined through their languages under the name of tree sets [BDFD +15a] and were studied in a series of papers. They generalize classical families of subshifts such as Sturmian subshifts [MH40], Arnoux-Rauzy subshifts [AR91], codings of regular interval exchange transformations [Ose66, Arn63] (IET) or else subshifts arising from the application of the Cassaigne multidimensional continued fraction algorithm [CLL17] (MCF).

Minimal dendric subshifts exhibit striking combinatorial [BDFD +15d, BDD +18], algebraic [BDFD +15a, BDFD +15c] and ergodic properties [BBD +ar]. They for instance have factor complexity $\#(\mathcal{L}(X) \cap A^n) = (\#A-1)n+1$ [BDFD +15a] and topological rank $\#A$ [BBD +ar], where $A$ is the alphabet of the subshift. They also fall into the class of subshifts satisfying the regular bispecial condition [DF20], which implies that the number of their ergodic measures is at most $\#A/2$. An important property for our work is that the derivated subshift of a minimal dendric subshift is again a minimal dendric subshift on the same alphabet, where derivation is here understood as derivation by return words (see Section 3 for definitions). This allows to give $S$-adic representations of such subshifts [Per96], i.e., to define a set $S$ of endomorphisms

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of the free monoid $A^*$ and a sequence $\sigma = (\sigma_n)_{n \geq 1} \in S^\mathbb{N}$ such that
\[ X = \{ x \in A^Z \mid u \in L(x) \Rightarrow \exists n \in \mathbb{N}, a \in A : u \in L(\sigma_1\sigma_2 \cdots \sigma_n(a)) \}. \]
The sequence $\sigma$ is called an $S$-adic representation of the subshift.

$S$-adic representations are a classical tool that allows to study several properties of subshifts such as the factor complexity [DLR13, DDMP20], the number of ergodic measures [BD14, BHL20b, BHL20a], the dimension group and topological rank $[\text{BBD}^+\text{ar}]$ or yet the automorphism group [EM20]. In the case of minimal dendric subshifts, the involved endomorphisms are particular tame automorphisms of the free group generated by the alphabet [BDFD$^+$15d, BDD$^+18$]. This in particular allows to prove that minimal dendric subshifts have topological rank equal to the cardinality of the alphabet and that ergodic measures are completely determined by the measures of the letter cylinders $[\text{BBD}^+\text{ar}, \text{BHL20a}]$.

An important open problem concerning $S$-adic representations is the $S$-adic conjecture whose goal is to give an $S$-adic representation of subshifts with at most linear complexity [Ler12], i.e., to find a stronger notion of $S$-adicity such that a subshift has at most linear factor complexity if and only if it is “strongly $S$-adic”. In this article we attack this conjecture by studying $S$-adic representations of minimal dendric subshifts. Our main result is the following that gives an $S_3$-adic characterization of minimal dendric subshifts over a ternary alphabet, where $S_3$ is defined in Section 5.1. It involves a labeled directed graph $\mathcal{G}$ with 9 vertices and which is non-deterministic, i.e., a given morphism may label several edges leaving a given vertex.

**Theorem 1.** A subshift $(X, S)$ is a minimal dendric subshift over $A_3 = \{1, 2, 3\}$ if and only if it has a primitive $S_3$-adic representation $\sigma \in S_3^\mathbb{N}$ that labels a path in the graph $\mathcal{G}$ represented in Figure 1.

We then characterize, within this graph or its deterministic version (with 16 vertices), the well-known families of Arnoux-Rauzy subshifts (Proposition 33) and of coding of regular IET (Theorem 51). We also show that subshifts arising from the Cassaigne MCF are never Arnoux-Rauzy subshifts, nor codings of regular IET (Proposition 53).

Observe that we do not focus only on the ternary case. We investigate the $S$-adic representations of minimal dendric subshifts over any alphabet obtained when considering derivations by return words to letters. We for instance show that when taking the image $Y$ of subshift $X$ under a morphism in $S$, the extension graphs of long enough factors of $Y$ are the image of the extension graph of factors of $X$ under some graph homomorphism (Proposition 13). This allows us to introduce the notion of dendric preserving morphism for $X$ which is the fundamental notion for the construction of the graph $\mathcal{G}$. We also characterize the morphisms $\sigma$ of $S$ that are dendric preserving for all $X$ using Arnoux-Rauzy morphisms (Proposition 17).

The paper is organized as follows. We start by giving, in Section 2, the basic definitions for the study of subshifts. We introduce the notion of extension graph of a word, of dendric subshift and of $S$-adic representation of a subshift.

In Section 3, we recall the existence of an $S$-adic representation using return words for minimal subshifts (Theorem 4) and the link between return words and Rauzy graph.

In Section 4, we then study the relation between words in a subshift and in its image by a strongly left proper morphism (Proposition 7). We deduce from it a link between the extension graphs (Proposition 13) using graph morphisms and we prove that the injective and strongly left proper morphisms that preserve dendricity can be characterized using Arnoux-Rauzy morphisms (Proposition 17).
In Section 5 we study the notions and results of Section 4 in the case of a ternary alphabet. We then prove the main result of this paper (Theorem 1) which gives an S-adic characterization of ternary dendric subshifts. For interval exchanges, we first recall Theorem 51 in the ternary case using a subgraph of the graph obtained in the dendric case. We also prove that the families of Cassaigne subshifts, of Arnoux-Rauzy subshifts and of regular interval exchanges are disjoint.

2 Preliminaries

2.1 Words, languages and subshifts

Let \( A \) be a finite alphabet of cardinality \( d \geq 2 \). Let us denote by \( \varepsilon \) the empty word of the free monoid \( A^* \) (endowed with concatenation), and by \( A^\mathbb{Z} \) the set of bi-infinite words over \( A \). For a bi-infinite word \( x \in A^\mathbb{Z} \), and for \( i, j \in \mathbb{Z} \) with \( i \leq j \), the notation \( x_{[i,j]} \) (resp., \( x_{[i,j)} \)) stands for \( x_i \cdots x_{j-1} \) (resp., \( x_i \cdots x_j \)) with the convention \( x_{[i,i)} = \varepsilon \). For a word \( w = w_1 \cdots w_\ell \in A^\ell \), its length is denoted \( |w| \) and equals \( \ell \). We say that a word \( u \) is a factor of a word \( w \) if there exist words \( p, s \) such that \( w = pus \). If \( p = \varepsilon \) (resp., \( s = \varepsilon \)) we say that \( u \) is a prefix (resp., suffix) of \( w \). For a word \( u \in A^* \), an index \( 1 \leq j \leq \ell \) such that \( w_j \cdots w_{j+|w|-1} = u \) is called an occurrence of \( u \) in \( w \) and we use the same term for bi-infinite word in \( A^\mathbb{Z} \). The number of occurrences of a word \( u \in A^* \) in a finite word \( w \) is denoted as \( |w|_u \).

The set \( A^\mathbb{Z} \) endowed with the product topology of the discrete topology on each copy of \( A \) is topologically a Cantor set. The shift map \( S \) defined by \( S((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}} \) is a homeomorphism of \( A^\mathbb{Z} \). A subshift is a pair \( (X, S) \) where \( X \) is a closed shift-invariant subset of some \( A^\mathbb{Z} \). It is thus a topological dynamical system. It is minimal if the only closed shift-invariant subset \( Y \subset X \) are \( \emptyset \) and \( X \). Equivalently, \((X, S)\) is minimal if and only if the orbit of every \( x \in X \) is dense in \( X \). Usually we say that the set \( X \) is itself a subshift.

The language of a sequence \( x \in A^\mathbb{Z} \) is its set of factors and is denoted \( \mathcal{L}(x) \). For a subshift \( X \), its language \( \mathcal{L}(X) = \bigcup_{x \in X} \mathcal{L}(x) \) and we set \( \mathcal{L}_n(X) = \mathcal{L}(X) \cap A^n, n \in \mathbb{N} \). Its factor complexity is the function \( p_X : \mathbb{N} \to \mathbb{N} \) defined by \( p_X(n) = \# \mathcal{L}_n(X) \). We say that a subshift \( X \) is over \( A \) if \( \mathcal{L}_1(X) = A \).

2.2 Extension graphs and dendric subshifts

Dendric subshifts are defined with respect to combinatorial properties of their language expressed in terms of extension graphs. Let \( F \) be a factorial set of finite words on the alphabet \( A \). For \( w \in F \), we define the sets of left, right and bi-extensions of \( w \) by

\[
E_F^-(w) = \{ a \in A \mid aw \in F \}; \\
E_F^+(w) = \{ b \in A \mid wb \in F \}; \\
E_F(w) = \{(a, b) \in A \times A \mid awb \in F \}.
\]

The elements of \( E_F^-(w), E_F^+(w) \) and \( E_F(w) \) are respectively called the left extensions, the right extensions and the bi-extensions of \( w \) in \( F \). If \( X \) is a subshift on \( A \), we will use the terminology extensions in \( X \) instead of extensions in \( \mathcal{L}(X) \) and the index \( \mathcal{L}(X) \) will be replaced by \( X \) or
even omitted if the context is clear. Observe that as \( X \subset A^Z \), the set \( E_X(w) \) completely determines \( E_X^-(w) \) and \( E_X^+(w) \). A word \( w \) is said to be right special (resp., left special) if \( #(E^+(w)) \geq 2 \) (resp., \( #(E^-(w)) \geq 2 \)). It is bispecial if it is both left and right special. The factor complexity of a subshift is completely governed by the extensions of its special factors. In particular, we have the following result.

**Proposition 2** (Cassaigne and Nicolas [CN10]). Let \( X \) be a subshift. For all \( n \), we have

\[
p_X(n + 1) - p_X(n) = \sum_{w \in \mathcal{L}_n(X)} (\#E^+(w) - 1)
= \sum_{w \in \mathcal{L}_n(X)} (\#E^-(w) - 1)
\]

In addition, if for every bispecial factor \( w \in \mathcal{L}(X) \), one has

\[
\#E(w) - \#E^-(w) - \#E^+(w) + 1 = 0,
\]

then \( p_X(n) = (p_X(1) - 1)n + 1 \) for every \( n \).

For a word \( w \in F \), we consider the undirected bipartite graph \( E_F(w) \) called its extension graph with respect to \( F \) and defined as follows: its set of vertices is the disjoint union of \( E_F^-(w) \) and \( E_F^+(w) \) and its edges are the pairs \((a, b) \in E_F^-(w) \times E_F^+(w) \) such that \( awb \in F \). For an illustration, see Example 3 below. We say that \( w \) is dendric if \( E(w) \) is a tree. We then say that a subshift \( X \) is a dendric subshift if all its factors are dendric in \( \mathcal{L}(X) \). Note that every non-bispecial word is trivially dendric. By Proposition 2 we immediately deduce that any dendric subshift has factor complexity \( p_X(n) = (p_X(1) - 1)n + 1 \) for every \( n \).

**Example 3.** Let \( \sigma \) be the Fibonacci substitution defined over the alphabet \( \{a, b\} \) by \( \sigma : a \mapsto ab, b \mapsto a \) and consider the subshift generated by \( \sigma \) (i.e., the set of bi-infinite words over \( \mathcal{A} \) whose factors belong to some \( \sigma^*(a) \)). The extension graphs of the empty word and of the two letters \( a \) and \( b \) are represented in Figure 1.

\[
\begin{align*}
E(\varepsilon) & \\
E(a) & \\
\end{align*}
\]

![Extension graphs](image)

Figure 1: The extension graphs of \( \varepsilon \) (on the left), \( a \) (in the center) and \( b \) (on the right) are trees.

### 2.3 S-adicity

Let \( \mathcal{A}, \mathcal{B} \) be finite alphabets with cardinality at least 2. By a morphism \( \sigma : \mathcal{A}^* \to \mathcal{B}^* \), we mean a non-erasing monoid homomorphism (also called a substitution when \( \mathcal{A} = \mathcal{B} \)). By non-erasing, we mean that the image of any letter is a non-empty word. We stress the fact that all morphisms are assumed to be non-erasing in the following. Using concatenation, we
extend $\sigma$ to $A^\mathbb{N}$ and $A^\mathbb{Z}$. In particular, if $X$ is a subshift over $A$, the image of $X$ under $\sigma$ is the subshift

$$Y = \{S^k\sigma(x) \mid x \in X, 0 \leq k < |\sigma(x_0)|\}.$$  

The morphism $\sigma$ is said to be left proper (resp., right proper) when there exists a letter $\ell \in B$ such that for all $a \in A$, $\sigma(a)$ starts with $\ell$ (resp., ends with $\ell$). It is strongly left proper (resp., strongly right proper) if it is left proper (resp., right proper) and the starting letter (resp., ending letter) $\ell$ only occurs once in each image $\sigma(a)$, $a \in A$. It is said to be proper if it is both left and right proper. With a left proper morphism $\sigma : A^* \to B^*$ with first letter $\ell$, we associate a right proper morphism $\bar{\sigma} : A^* \to B^*$ by

$$\sigma(a)\ell = \ell \bar{\sigma}(a), \ \forall a \in A.$$  

Let $\sigma = (\sigma_n : A_{n+1}^* \to A_n^*)_{n \geq 1}$ be a sequence of morphisms such that $\max_{a \in A_n} |\sigma_1 \circ \cdots \circ \sigma_{n-1}(a)|$ goes to infinity when $n$ increases. We assume that all the alphabets $A_n$ are minimal, in the sense that for all $n \in \mathbb{N}$ and $b \in A_n$, there exists $a \in A_{n+1}$ such that $b$ is a factor of $\sigma_n(a)$. For $1 \leq n < N$, we define the morphisms $\sigma_{[n,N]} = \sigma_{n} \circ \sigma_{n+1} \circ \cdots \circ \sigma_{N-1}$ and $\sigma_{[n,N]} = \sigma_{n} \circ \sigma_{n+1} \circ \cdots \circ \sigma_{N}$. For $n \geq 1$, the language $L^{(n)}(\sigma)$ of level $n$ associated with $\sigma$ is defined by

$$L^{(n)}(\sigma) = \{w \in A_n^* \mid w \text{ occurs in } \sigma_{[n,N]}(a) \text{ for some } a \in A_N \text{ and } N > n\}.$$  

As $\max_{a \in A_n} |\sigma_{[1,n]}(a)|$ goes to infinity when $n$ increases, $L^{(n)}(\sigma)$ defines a non-empty subshift $X^{(n)}_{\sigma}$ that we call the subshift generated by $L^{(n)}(\sigma)$. More precisely, $X^{(n)}_{\sigma}$ is the set of points $x \in A_n^\mathbb{Z}$ such that $L(x) \subseteq L^{(n)}(\sigma)$. Note that it may happen that $L(X^{(n)}_{\sigma})$ is strictly contained in $L^{(n)}(\sigma)$. Also observe that for all $n$, $X^{(n)}_{\sigma}$ is the image of $X^{(n+1)}_{\sigma}$ under $\sigma_n$.  

We set $L(\sigma) = L^{(1)}(\sigma)$, $X_{\sigma} = X^{(1)}_{\sigma}$ and call $X_{\sigma}$ the $S$-adic subshift generated by the directive sequence $\sigma$. We also say that the directive sequence $\sigma$ is an $S$-adic representation of $X_{\sigma}$.

We say that $\sigma$ is primitive if, for any $n \geq 1$, there exists $N > n$ such that for all $(a,b) \in A_n \times A_N$, $a$ occurs in $\sigma_{[n,N]}(b)$. Observe that if $\sigma$ is primitive, then $\min_{a \in A_n} |\sigma_{[1,n]}(a)|$ goes to infinity when $n$ increases, $L(\sigma_{[1,n]}) = L^{(n)}(\sigma)$, and $X^{(n)}_{\sigma}$ is a minimal subshift (see for instance [Dur00] Lemma 7]).

We say that $\sigma$ is ((strongly) left, (strongly) right) proper whenever each morphism $\sigma_n$ is ((strongly) left, (strongly) right) proper. We also say that $\sigma$ is injective if each morphism $\sigma_n$ is injective (seen as an application from $A_{n+1}^*$ to $A_n^*$). By abuse of language, we say that a subshift is a (strongly left or right proper, primitive, injective) $S$-adic subshift if there exists a (strongly left or right proper, primitive, injective) sequence of morphisms $\sigma$ such that $X = X_{\sigma}$.

### 3 $S$-adicity using return words and shapes of Rauzy graphs

#### 3.1 $S$-adicity using return words and derived subshifts

Let $X$ be a minimal subshift over the alphabet $A$ and let $w \in L(X)$ be a non-empty word. A return word to $w$ in $X$ is a non-empty word $r$ such that $w$ is a prefix of $rw$ and, $rw \in L(X)$ and $rw$ contains exactly two occurrences of $w$ (one as a prefix and one as a suffix). We let
\(R_X(w)\) denote the set of return words to \(w\) in \(X\) and we omit the subscript \(X\) whenever it is clear from the context. The subshift \(X\) being minimal, \(R(w)\) is always finite.

Let \(w \in L(X)\) be a non-empty word and write \(R_X(w) = \{1, \ldots, \#(R_X(w))\}\). A morphism \(\sigma : R(w)^* \to R(w)^*\) is a coding morphism associated with \(w\) if \(\sigma(R(w)) = R(w)\). It is trivially injective. Let us consider the set \(D_w(X) = \{x \in R(w)^Z \mid \sigma(x) \in X\}\). It is a minimal subshift, called the derived subshift of \(X\) (with respect to \(w\)). We know show that derivation of minimal subshifts allows to build left proper and primitive \(S\)-adic representations of a minimal subshift. We inductively define the sequences \((a_n)_{n \geq 1}\), \((R_n)_{n \geq 1}\), \((X_n)_{n \geq 1}\) and \((\sigma_n)_{n \geq 1}\) by

- \(X_1 = X\), \(R_1 = A\) and \(a_1 \in A\);
- for all \(n\), \(R_{n+1} = R_X(a_n)\), \(\sigma_n : R_{n+1}^* \to R_n^*\) is a coding morphism associated with \(a_n\), \(X_{n+1} = D_{a_n}(X_n)\) and \(a_{n+1} \in R_{n+1}\).

Observe that the sequence \((a_n)_{n \geq 1}\) is not uniquely defined as well as the morphism \(\sigma_n\) (even if \(a_n\) is fixed). However, to avoid heavy considerations when we deal with sequences of morphisms obtained in this way, we will speak about “the” sequence \((\sigma_n)_{n \geq 1}\) and it is understood that we may consider any such sequence. Also observe that as we consider derived subshifts with respect to letters, each coding morphism \(\sigma_n\) is strongly left proper.

**Theorem 4** (Durand [Dur98]). Let \(X\) be a minimal subshift. Using the notation defined above, the sequence of morphisms \(\sigma = (\sigma_n : R_{n+1}^* \to R_n^*)_{n \geq 1}\) is a strongly left proper, primitive and injective \(S\)-adic representation of \(X\). In particular, for all \(n\), we have \(X_n = X_{\sigma}^{(n)}\).

In the case of minimal dendric subshifts, the \(S\)-adic representation \(\sigma\) can be made stronger. This is summarized by the following result. Recall that if \(F_A\) is the free group generated by \(A\), an automorphism \(\alpha\) of \(F_A\) is tame if it belongs to the monoid generated by the permutations of \(A\) and by the elementary automorphisms

\[
\lambda_{a,b} : \begin{cases} 
a \mapsto ab, 
\end{cases}
\quad\text{and}\quad
\tilde{\lambda}_{a,b} : \begin{cases} 
a \mapsto ba, 
\end{cases}
\end{equation}

\begin{equation}
\quad\text{for } c \neq a,
\quad\text{for } c \neq a.
\end{equation}

**Theorem 5** (Berthé et al. [BDFD+15]). Let \(X\) be a minimal dendric subshift over the alphabet \(A = \{1, \ldots, d\}\). For any \(w \in L(X)\), \(D_w(X)\) is a minimal dendric subshift over \(A\) and the coding morphism associated with \(w\) is a tame automorphism of \(F_A\). As a consequence, if \(\sigma = (\sigma_n)_{n \geq 1}\) is the primitive directive sequence of Theorem 4, then all morphisms \(\sigma_n\) are strongly left proper tame automorphisms of \(F_A\).

### 3.2 Rauzy graphs

Let \(X\) be a subshift over an alphabet \(A\). The **Rauzy graph of order** \(n\) of \(X\), is the directed graph \(G_n(X)\) whose set of vertices is \(L_n(X)\) and there is an edge from \(u\) to \(v\) if there are letters \(a, b\) such that \(ab = av \in L(X)\); this edge is labeled by \(a\). If \(X\) is minimal, then any Rauzy graph \(G_n(X)\) is strongly connected.

If \(w \in L(X)\) is non-empty, then any return word \(r\) to \(w\) in \(X\) labels a path from \(w\) to \(w\) in \(G_n(X)\). Indeed, the word \(u = rw = u_1u_2 \cdots u_N\) being in \(L(X)\), \(u_i \in A\) for all \(i\), we can associate with \(r = u_1u_2 \cdots u_{N-n}\) the path

\[
w = u_1 \quad \mapsto \quad u_2 \quad \mapsto \quad \cdots \quad \mapsto \quad u_N \quad = \quad w.
\]
Observe that \( r \) being a return word, all vertices \( u_{[i,i+n-1]} ; i \notin \{1,N-n+1\} \), are different from \( w \). Observe also that the converse does not hold, i.e., there might exist paths from \( w \) to \( w \) whose label is not a return word (see Example 6 below). However, the shape of the Rauzy graph \( G_1(X) \) provides restrictions on the possible return words to a letter \( a \) in \( X \). For instance, in the minimal case, for all letters \( a,b,c \in \mathcal{L}(X) \), \( b \neq c \), there is an edge from \( a \) to \( b \) in \( G_1(X) \) if and only if there is a return word \( r \) to \( c \) in which \( ab \) occurs. Furthermore, the extension graph of the empty word having for edges the pairs \((a,b) \in E_X(\varepsilon)\), it completely determines \( \mathcal{L}_2(X) \), hence the Rauzy graph \( G_1(X) \). Therefore, the extension graph \( E_X(\varepsilon) \) provides restrictions on the possible return words to letters in \( X \).

Example 6. The Rauzy graph of order 2 of the subshift \( X \) generated by the Fibonacci substitution is given in Figure 2. For any \( n \geq 1 \), the word \( a(ab)^n \) labels a path from \( aa \) to \( aa \). However, as \( aaa \) is not an element of \( \mathcal{L}(X) \), neither is \( (ab)^3 \) thus \( a(ab)^n \) is not a return word to \( aa \) if \( n \geq 3 \).

4 Bispecial factors in S-adic subshifts

4.1 Description of bispecial factors in injective strongly left proper S-adic subshifts

Our aim is to describe bispecial factors and their bi-extensions in an \( S \)-adic subshift \( X_\sigma \). A classical way to do this is to “desubstitute” a bispecial factor \( u \), i.e., to find the set of “minimal” factors \( v_i \) in \( \mathcal{L}(X_\sigma(k)) \) such that \( u \) is a factor of the words \( \sigma_{[1,k]}(v_i) \) and then to deduce the extensions of \( u \) from those of the \( v_i \)’s. The easiest case is when the set of \( v_i \)’s is singleton. The next result states that this is the case for injective and strongly left proper directive sequences.

If \( X \) is a subshift over \( A \) and \( \sigma : A^* \rightarrow B^* \) is a morphism, then for any words \( v \in \mathcal{L}(X) \) and \( x,y \in B^* \), we define the sets

\[
E_{X,x}(v) = \{ a \in E_X^-(v) \mid \sigma(a) \in B^*x \};
\]

\[
E_{X,y}^+(v) = \{ b \in E_X^+(v) \mid \sigma(b) \in yB^* \};
\]

\[
E_{X,x,y}(v) = E_X(v) \cap (E_{X,x}(v) \times E_{X,y}^+(v)).
\]

Proposition 7. Let \( X \) be a subshift over \( A \) and \( \sigma : A^* \rightarrow B^* \) be an injective and strongly left proper morphism (with first letter \( \ell \)), \( Y \) the image of \( X \) under \( \sigma \) and \( u \) a non-empty word in \( \mathcal{L}(Y) \). If \( \ell \) does not occur in \( u \), then there is \( b \in A \) such that \( u \) is a non-prefix factor of \( \sigma(b) \). Otherwise, there is a unique triplet \((s,v,p) \in B^* \times \mathcal{L}(X) \times B^* \) for which there exists a pair \((a,b) \in E_X(v) \) such that \( u = s\sigma(v)p \) with

1. \( s \) a strict suffix of \( \sigma(a) \);

Figure 2: Rauzy graph of order 2 for the Fibonacci subshift
2. $p$ a strict prefix of $\sigma(b)$ and if $p = \varepsilon$, then $v = v'c$ for some letter $c$ such that $E_{X,s,\sigma(c)}(v') = E_{X,s}(v) \times \{c\}$.

In the latter case, the left, right and bi-extensions of $u$ are governed by those of $v$ through the relation

$$E_Y(u) = \{(a', b') \in \mathcal{B} \times \mathcal{B} : \exists (a, b) \in E_{X,s,p}(v) : \sigma(a) \in \mathcal{B}^*a's \land \sigma(b)\ell \in pb'\mathcal{B}^*\}. \quad (1)$$

**Proof.** Since $u$ is a word in $\mathcal{L}(Y)$, it is a factor of $\sigma(w)$ for some $w \in \mathcal{L}(X)$. The word $u$ being non-empty, any such $w$ is non-empty as well. We say that $w$ is covering $u$ if $u$ is a factor of $\sigma(w)$ and for any proper factor $w'$ of $w$, $u$ is not a factor of $\sigma(w')$. Recall that $\ell$ is the first letter of $\sigma(a)$ for all $a \in A$. Thus, if $|u|_{\ell} = 0$, then any word $w$ covering $u$ is a letter and $u$ is a non-prefix factor of $\sigma(w)$.

Now assume that $|u|_{\ell} \geq 1$ and let $u = s'u'p'$ with $|s'|_{\ell} = 0$, $p' \in \ell(A \setminus \{\ell\})^*$ and $u' \in \{\varepsilon\} \cup \ell A^*$. As the letter $\ell$ occurs only as a prefix in any image $\sigma(a)$, $a \in A$, any word $w$ covering $u$ is of the form $xv'y$ with $x \in \mathcal{A} \cup \{\varepsilon\}$ and $y \in \mathcal{A}$, where one has $\sigma(v') = u'$, $s'$ is a suffix of $\sigma(x)$ (which is proper if $x \neq \varepsilon$) and $p'$ is a prefix of $\sigma(y)$. As $\sigma$ is injective, the word $v'$ is uniquely defined, i.e., $v'$ does not depend on $w$.

If there is such a covering word $w$ for which $p'$ is a proper prefix of $\sigma(y)$, then the triplet $(s', v', p')$ satisfies the requirements. Indeed, either the pair $(x, y)$ is in $E_X(v')$ or, if $x = \varepsilon$, then we may take any pair $(a, y) \in E_X(v')$. It will thus remain to show that it is the unique triplet.

If there is no such covering word $w$, then for any covering word $w' = x'v'y'$ ($v'$ is common to all covering words), one has $p' = \sigma(y')$. As $\sigma$ is injective, there is a letter $c$ such that all covering words $w$ are of the form $w = x'v'c$ with $u = s'\sigma(v')c$ and $s'$ suffix of $\sigma(x')$. This shows that with $v = v'c$, the triplet $(s, v, p) = (s', v, \varepsilon)$ satisfies the requirement. Indeed, one has $E_{X,s',\sigma(c)}(v') = E_{X,s'}(v) \times \{c\}$ and either we consider any pair $(a, b) \in E_X(v)$ (if $x \neq \varepsilon$), or we may consider any pair $(a, b) \in E_X(v)$ (if $x = \varepsilon$).

Let us now show that $(s, v, p)$ is unique. Assuming that $(\tilde{s}, \tilde{v}, \tilde{p})$ satisfies the conditions, then $|\tilde{s}|_{\ell} = 0$ and $\sigma(\tilde{v})\tilde{p}$ starts with $\ell$. Thus, $\tilde{s} = s'$ and $\sigma(\tilde{v})\tilde{p} = \sigma(v')p'$. Since $\tilde{p}$ and $p'$ are both prefixes of words in $\sigma(A)$ (and $\sigma$ is strongly left proper), there are only two possibilities: either $\tilde{p} = p'$ (hence $\tilde{v} = v'$) or $\tilde{p} = \varepsilon$. In the latter situation, since $p' \neq \varepsilon$, there exists a unique $c \in A$ such that $\tilde{v} = v'c$ and $p' = \sigma(c)$. To prove the uniqueness of $(s, v, p)$, it remains to show that $(s', v', p')$ and $(s', v'c, \varepsilon)$ cannot both verify the hypotheses. If the second decomposition does, then $E_{X,s',\sigma(c)}(v') = E_{X,s'}(v'c) \times \{c\}$. Thus there is no $(a, b) \in E_X(v')$ for which $s'$ is a proper suffix of $\sigma(a)$ and $p'$ is a proper prefix of $\sigma(b)$. Otherwise stated, the triplet $(s', v', p')$ does not satisfy the conditions, hence the conclusion.

Let us now prove Equation (1). The inclusion

$$E_Y(u) \supset \{(a', b') \in \mathcal{B} \times \mathcal{B} : \exists (a, b) \in E_{X,s,p}(v) : \sigma(a) \in \mathcal{B}^*a's \land \sigma(b)\ell \in pb'\mathcal{B}^*\}$$

is trivial. For the other one, assume that $(a', b')$ is in $E_Y(u)$. Thus we have $a'ub' \in \mathcal{L}(Y)$. Let $w \in \mathcal{L}(X)$ be a covering word for $a'ub'$. By definition of $v$, $v$ is a factor of $w$, thus one has $w = xvy$ for some words $x, y$. We then have $a'ub' = a'\sigma(v)p'b'$, with a's a suffix of $\sigma(x)$ and $p'b'$ a prefix of $\sigma(y)$. In particular, $x$ and $y$ are non-empty. Let $a$ be the last letter of $x$ and $b$ be the first letter of $y$. We have $(a, b) \in E_X(v)$ and, as $|s|_{\ell} = 0$, $s$ is a proper suffix of $\sigma(a)$, from which we have $\sigma(a) \in \mathcal{B}^*a's$. We also have that $p$ is a prefix of $\sigma(b)$. If it is strict, then $p'b'$ is a prefix of $\sigma(b)$ so that $\sigma(b)\ell \in pb'\mathcal{B}^*$. Otherwise, $p = \sigma(b)$, $y$ has length at least 2 and
b' is the first letter of σ(c), where c is such that bc is prefix of y. Otherwise stated, b' = ℓ and we indeed have σ(b)ℓ ∈ pβB*

Remark 8. Observe that, as we have seen in the previous proof (and using the same notation), as s is a proper suffix of σ(a), the letter ℓ does not occur in it. As a consequence, for any a' ∈ E_{X,s}(v), s is a proper suffix of σ(a').

Whenever u and v are as in the previous lemma with |u|ℓ ≥ 1, the word v is called the antecedent of u under σ and u is said to be an extended image of v. Thus, the antecedent is defined only for words containing an occurrence of the letter ℓ and an extended image always contains an occurrence of ℓ. Whenever u is a bispecial factor, the next result gives additional information about s and p. We first need to define the following notation. If σ: A^* → B^* is a morphism and a_1, a_2 ∈ A, let us denote by s(a_1, a_2) (resp., p(a_1, a_2)) the longest common suffix (resp., prefix) between σ(a_1) and σ(a_2).

Corollary 9. Let X be a subshift over A, σ: A^* → B^* be an injective and strongly left proper morphism (with first letter ℓ), Y the image of X under σ and v a word in L(X). A word u is a bispecial extended image of v if and only if there exist (a_1, b_1), (a_2, b_2) ∈ E_X(v) with a_1 ≠ a_2, b_1 ≠ b_2 and u = s(a_1, a_2)σ(v)p(b_1, b_2). In particular, the antecedent v of a bispecial word u is bispecial.

Proof. First assume that there exist (a_1, b_1), (a_2, b_2) ∈ E_X(v) with a_1 ≠ a_2, b_1 ≠ b_2 and u = s(a_1, a_2)σ(v)p(b_1, b_2). Let us fix s = s(a_1, a_2) and p = p(b_1, b_2). Since σ is injective, σ(a_1) ≠ σ(a_2), hence s is a proper suffix of one of them. Thus, as σ is strongly left proper, s does not contain any occurrence of the letter ℓ. As a consequence, s is a proper suffix of both σ(a_1) and σ(a_2). In particular, u is left special. The same reasoning shows that p is a proper prefix of σ(b_i) for some i ∈ {1, 2} and that u is right special, hence bispecial. Furthermore, p is non-empty since it admits ℓ as a prefix. The pair (a_i, b_i) thus satisfies Proposition 7.

Now assume that u is a bispecial extended image of v with u = σ(v)p. Since u is bispecial, there exist (a'_1, b'_1), (a'_2, b'_2) ∈ E_Y(u) with a'_1 ≠ a'_2 and b'_1 ≠ b'_2. From Equation 9 there exist (a_1, b_1), (a_2, b_2) ∈ E_{X,s,p}(v) such that

σ(a_i) ∈ B^*a'_is and σ(b_i)ℓ ∈ pβB^*

for all i ∈ {1, 2}. We deduce that s = s(a_1, a_2) and a_1 ≠ a_2. As b'_1 and b'_2 cannot be simultaneously equal to ℓ, we deduce that b_1 ≠ b_2 and that p = p(b_1, b_2).

Example 10. Consider the morphism

\[
\begin{align*}
\sigma : \quad & a \mapsto ℓab \\
& b \mapsto ℓa \\
& c \mapsto ℓc \\
& d \mapsto ℓcd \\
& e \mapsto ℓce 
\end{align*}
\]

and assume that v is a bispecial factor of X ⊂ \{a, b, c, d, e\}^\mathbb{Z} whose extension graph is
By Corollary 9, the word \( v \) admits \( \sigma(v)\ell, \sigma(v)\ell_a \) and \( \sigma(v)\ell_c \) as bispecial extended images. Using Proposition 7, their extension graphs are given in Figure 3.

Assume that \( X_\sigma \) is an \( S \)-adic subshift where the directive sequence \( \sigma = (\sigma_n : A_{n+1}^* \to A_n^*)_{n \geq 1} \) is primitive and contains only strongly left proper injective morphisms. The directive sequence \( \sigma \) being primitive, the sequence \( (\min_{a \in A_n} |\sigma_{[1,n]}(a)|)_{n \geq 1} \) goes to infinity. Hence, iterating Proposition 7 with any word \( u \in \mathcal{L}(X_\sigma) \) one can associate a unique finite sequence \( (u_1, u_2, \ldots, u_k) \) such that \( u_1 = u \) and, for \( i < k \), \( u_{i+1} \in \mathcal{L}(X_{\sigma_{k}}^{(i+1)}) \) is the antecedent of \( u_i \) under \( \sigma_i \). We say that \( u \) is a descendant of each \( u_i \), \( 1 \leq i \leq k \), and, reciprocally, that each \( u_i \), \( 1 \leq i \leq k \), is an ancestor of \( u \). The word \( u_k \in \mathcal{L}(X_{\sigma_k}^{(k)}) \) is its oldest ancestor and it is either empty or a non-prefix factor of \( \sigma_k(b) \) for some letter \( b \in A_{k+1} \). Observe that with our definition, \( u \) is an ancestor and a descendant of itself.

Let \( X \) be an \( S \)-adic subshift with a strongly left proper and injective \( S \)-adic representation \( \sigma \). Let \( u \) be a bispecial factor of \( X \). From Corollary 9 all ancestors of \( u \) are bispecial factors of some \( X_{\sigma_k}^{(k)} \). From Proposition 7, the extensions of \( u \) are completely governed by those of its oldest ancestor. More precisely, we have the following direct corollary.

**Corollary 11.** For all \( k \geq 1 \), there is a finite number of bispecial factors of \( X_{\sigma_k}^{(k)} \) that do not have an antecedent under \( \sigma_k \). They are called initial bispecial factors of order \( k \). Furthermore, for any bispecial factor \( u \) of \( X \), there is a unique \( k \geq 1 \) and a unique initial bispecial factor \( v \in \mathcal{L}(X_{\sigma_k}^{(k)}) \) such that \( u \) is a descendant of \( v \). Finally, \( \mathcal{E}_X(u) \) depends only on \( \mathcal{E}_{X_{\sigma_k}^{(k)}}(v) \), i.e., if \( Y \) is a subshift such that \( \mathcal{E}_Y(v) = \mathcal{E}_{X_{\sigma_k}^{(k)}}(v) \) and if \( Z \) is the image of \( Y \) under \( \sigma_{[1,k]} \), then \( \mathcal{E}_Z(u) = \mathcal{E}_X(u) \).

**4.2 Action of morphisms on extension graphs**

Proposition 7 shows that whenever \( v \) is the antecedent of \( u \) under \( \sigma \), the extension graph \( \mathcal{E}_Y(u) \) is the image under a graph morphism of a subgraph of \( \mathcal{E}_X(v) \) (where by subgraph we mean
the subgraph generated by a subset of edges). In particular, if \( \#(E_X^{-}(u)) = \#(E_X^{-}(v)) \) and \( \#(E_X^{+}(u)) = \#(E_X^{+}(v)) \), then \( E_X(u) \) and \( E_X(v) \) are isomorphic. In this section, we formalize this observation and study the behavior of a tree structure when we consider the extension graphs of bispecial extended images.

In this section, \( X \) is a subshift over \( A, \sigma : A^{\ast} \to B^{\ast} \) an injective and strongly left proper morphism (with first letter \( \ell \)), \( Y \) the image of \( X \) under \( \sigma \) and \( v \) a bispecial word in \( L(X) \). By Corollary 9, the bispecial extended images of \( v \) under \( \sigma \) are the words \( u \) of the form \( s\sigma(v)p \) where \( s = s(a_1, a_2) \) and \( p = p(b_1, b_2) \) for some \( (a_1, b_1), (a_2, b_2) \in E_X(v) \) such that \( a_1 \neq a_2 \) and \( b_1 \neq b_2 \). For any such \( \sigma \) and \( v \), we introduce the following notations:

\[
S(\sigma) = \{ s(a_1, a_2) \mid a_1, a_2 \in A, a_1 \neq a_2 \}; \\
P(\sigma) = \{ p(b_1, b_2) \mid b_1, b_2 \in A, b_1 \neq b_2 \}; \\
S_v(\sigma) = \{ s(a_1, a_2) \mid a_1, a_2 \in E_X^{-}(v), a_1 \neq a_2 \}; \\
P_v(\sigma) = \{ p(b_1, b_2) \mid b_1, b_2 \in E_X^{+}(v), b_1 \neq b_2 \}; \\
\overline{S}_v(\sigma) = S_v(\sigma) \cup \sigma(E_X^{-}(v)); \\
\overline{P}_v(\sigma) = P_v(\sigma) \cup \sigma(E_X^{+}(v))\ell.
\]

The prefix strict order (resp., suffix strict order) defines a tree structure on \( \overline{P}_v(\sigma) \) (resp., on \( \overline{S}_v(\sigma) \)), where the root \( p_0 \) (resp., \( s_0 \)) is the smallest word of the set. In particular, \( E_{X,s_0}^{-}(v) = E_{X,s_0}^{+}(v) \). Furthermore, the leaves of \( \overline{P}_v(\sigma) \) (resp., \( \overline{S}_v(\sigma) \)) are exactly the elements of \( \sigma(E_X^{+}(v))\ell \) (resp., \( \sigma(E_X^{-}(v)) \)) and every internal node (i.e., every element of \( P_v(\sigma) \) or \( S_v(\sigma) \)) has at least two children.

For \( (s, p) \in S_v(\sigma) \times P_v(\sigma) \), we define the subgraph \( E_{X,s,p}(v) \) of \( E_X(v) \) whose vertices are those involved by the edges in \( E_{X,s,p}(v) \). Observe that the sets of left and right vertices of \( E_{X,s,p}(v) \) are respectively included in \( E_{X,s}^{-}(v) \) and \( E_{X,p}^{+}(v) \). Furthermore, if \( s' \in S_\sigma(v) \) (resp., \( p' \in P_\sigma(v) \)) is a child of \( s \) (resp., of \( p \)), then \( E_{X,s',p}(v) \) (resp., \( E_{X,s,p'}(v) \)) is a subgraph of \( E_{X,s,p}(v) \).

**Example 12.** Using the notations from Example 10, we have the following tree structures on \( \overline{S}_v(\sigma) \) and \( \overline{P}_v(\sigma) \):

\[
\begin{array}{c}
\overline{S}_v(\sigma) \\
\ell_a \\
\ell_b \\
\ell_c \\
\ell_d
\end{array}
\quad
\begin{array}{c}
\overline{P}_v(\sigma) \\
\ell \\
\ell_a \\
\ell_c
\end{array}
\]

These structures help us understand the construction of the extension graphs of Figure 3. Let \( s \in S_v(\sigma) \) and \( p \in P_v(\sigma) \) be such that \( u = s\sigma(v)p \). The extension graph of \( u \) can be obtained from the extension graph of \( v \) as follows:

1. Start by selecting the elements of \( E_{X,s}^{-}(v) \) and \( E_{X,p}^{+}(v) \) (see Figure 4). These elements are the letters such that the corresponding leaf in \( \overline{S}_v(\sigma) \) (resp., \( \overline{P}_v(\sigma) \)) is in the subtree with root \( s \) (resp., \( p \)).
2. Take the subgraph of $E_X(v)$ with only the vertices which are in these two sets and remove the isolated vertices that were created. This gives the graph $E_{X,s,p}(v)$ (see Figure 5).

3. For any letter $a \in A$, merge the vertices $b$ on the left side such that $\sigma(b) \in A^*$ as into a new left vertex labeled by $a$. In other words, for any left vertex $b$, map it to the left vertex labeled by the letter $a$ such that the leaf corresponding to $b$ is in the subtree whose root is the only child of $s$ ending by $as$. Do the same on the right side with the vertices $b$ such that $\sigma(b) \ell \in paA^*$ (see Figure 6).

| $\sigma(v)\ell$             | $\sigma(v)\ell a$          | $\sigma(v)\ell c$          |
|-----------------------------|-----------------------------|-----------------------------|
| $E^-_{X,\varepsilon}(v) = \{a, b, c, d\}$ | $E^-_{X,\varepsilon}(v) = \{a, b, c, d\}$ | $E^-_{X,\varepsilon}(v) = \{a, b, c, d\}$ |
| $E^+_{X,\ell}(v) = \{a, b, c, d, e\}$     | $E^+_{X,\ell a}(v) = \{a, b\}$          | $E^+_{X,\ell c}(v) = \{c, d, e\}$          |

Figure 4: Step 1

![Figure 4: Step 1](image)

$E_{X,\varepsilon,\ell}(v)$ $E_{X,\varepsilon,\ell a}(v)$ $E_{X,\varepsilon,\ell c}(v)$

![Figure 5: Step 2](image)

$\sigma(v)\ell$ | Left side | Right side |
|----------------|-----------|------------|
| $a \mapsto b$  | $a \mapsto a$ |
| $b \mapsto a$  | $b \mapsto a$ |
| $c \mapsto c$  | $c \mapsto c$ |
| $d \mapsto d$  | $d \mapsto c$ |

$\sigma(v)\ell a$ | Left side | Right side |
|----------------|-----------|------------|
| $a \mapsto b$  | $a \mapsto b$ |
| $b \mapsto a$  | $b \mapsto b$ |
| $c \mapsto c$  | $c \mapsto c$ |

$\sigma(v)\ell c$ | Left side | Right side |
|----------------|-----------|------------|
| $c \mapsto c$  | $c \mapsto \ell$ |
| $d \mapsto d$  | $d \mapsto d$ |

Figure 6: Step 3

The next result gives a more formal description of this construction and directly follows from Equation (1). Observe that for right extensions, we have to split the cases because if $p' \in \sigma(E^+_X(v))\ell$, then the set $E^+_{X,p'}(v)$ is empty.
Proposition 13. If \((s, p) \in S_v(\sigma) \times P_v(\sigma)\) is such that \(u = s\sigma(v)p\) is an extended image of \(v\), the extension graph of \(u\) is the image of \(E_{X,s,p}(v)\) under the graph morphism \(\varphi_{v,s,p} : E_{X,s,p}(v) \to E_Y(u)\) that

- for every child \(s'\) of \(s\) in \(S_v(\sigma)\), maps all vertices of \(E_{X,s'}(v)\) to the left vertex with label \(a \in B\) such that \(s' \in B^*as\);
- for every child \(p' \in P_v(\sigma)\) of \(p\), maps all vertices of \(E_{X,p'}(v)\) to the right vertex with label \(b \in B\) such that \(p' \in pbB^*\);
- for every child \(p' \in \sigma(E_X^+(v))\) of \(p\), if \(p' = \sigma(a)\) maps to \(a\), maps to the right vertex with label \(b \in B\) such that \(p' \in pbB^*\).

In particular, if \(S_v(\sigma) = \{s_0\}\) and \(P_v(\sigma) = \{p_0\}\), then \(v\) has a unique bispecial extended image \(u\) and the associated morphism \(\varphi_{v,s,p}\) is an isomorphism.

Observe that the morphism \(\varphi_{v,s,p}\) of the previous result acts independently on the left and right vertices of \(E_{X,s,p}(v)\), i.e.

\[
\varphi_{v,s,\ell}(L) = \varphi_{v,s,\ell}(L) \quad \text{and} \quad \varphi_{v,s,\ell}(R) = \varphi_{v,s,\ell}(R),
\]

where \(\varphi_{v,s,\ell}\) acts only on the left vertices of \(E_{X,s,p}(v)\) and \(\varphi_{v,s,\ell}\) acts only on the right vertices of \(E_{X,s,p}(v)\). Furthermore, if a letter \(a\) belongs to \(E_{X,s}(v) \cap E_{X,s}(v')\) (resp., to \(E_{X,s}(v) \cap E_{X,s}(v')\)) then \(\varphi_{v,s}(a) = \varphi_{v,s}(a)\) (resp., \(\varphi_{v,s}(a) = \varphi_{v,s}(a)\)). Thus we can define partial maps \(\varphi_{L,s} : A \to B\) by \(\varphi_{L,s}(a) = \varphi_{L,s}(a)\) whenever \(a \in E_{X,s}(e)\) and by \(\varphi_{L,s}(a) = \varphi_{L,s}(a)\) whenever \(a \in E_{X,s}(e)\).

4.3 Stability of dendricity

In this section, we use the results and notations of the previous section to understand under which conditions a dendric bispecial factor only has dendric bispecial extended images under some morphism. We then characterize the morphisms for which every dendric bispecial factor only has dendric bispecial extended images.

Proposition 14. Let \(X\) be a subshift over \(A\) and \(\sigma : A^* \to B^*\) be an injective and strongly left proper morphism. If \(v \in \mathcal{L}(X)\) is a dendric bispecial factor, then all the bispecial extended images of \(v\) under \(\sigma\) are dendric if and only if the following conditions are satisfied

1. for every \(s \in S_v(\sigma) \setminus \{s_0\}\), \(E_{X,s,p}(v)\) is a tree;
2. for every \(p \in P_v(\sigma) \setminus \{p_0\}\), \(E_{X,s,p}(v)\) is a tree.

Proof. Let us first assume that every bispecial extended image of \(v\) is dendric. We show item 2.

Consider a maximal word \(s'\) (for the suffix order) in \(\{s(a_1, a_2) \mid a_1 \in A_1, a_2 \in A_2\}\). Let also \(B_i, i \in \{1, 2\}\), denote the set of right vertices \(E_{X,s,p}\) that are connected to vertices of \(A_i\) and consider a maximal word \(p'\) (for the prefix order) in \(\{p(b_1', b_2') \mid b_1' \in B_1, b_2' \in B_2\}\). We claim that the extension graph of the bispecial extended image \(u = s'\sigma(v)p'\) is not connected.
Let \( Y \) be the image of \( X \) under \( \sigma \) and let \( \varphi \) be the morphism from \( \mathcal{E}_{X,s',p'}(v) \) to \( \mathcal{E}_Y(u) \) given by Proposition 13. By maximality of \( s' \) and \( p' \), the morphism \( \varphi \) identifies two left vertices \( a, a' \) (resp., right vertices \( b, b' \)) only if they belong to the same \( A_i \) (resp., \( B_i \)). This implies that \( \mathcal{E}_Y(u) \) is not connected, which is a contradiction.

Let us now show that, under the hypothesis 1 and 2, any bispecial extended image of \( v \) is dendric. By Corollary 9, the bispecial extended images of \( v \) are of the form \( s\sigma(v)p \in \mathcal{L}(Y) \) where \( s \) is in \( S_\sigma(\sigma) \) and \( p \) is in \( P_\nu(\sigma) \).

For any such pair \( (s, p) \), we first show that the graph \( \mathcal{E}_{X,s,p}(v) \) is a tree. It is trivially acyclic as it is a subgraph of \( \mathcal{E}_{X,s,p_0}(v) \), which is assumed to be a tree. Let us show that it is connected. Let \( b_1, b_2 \) be right vertices of \( \mathcal{E}_{X,s,p}(v) \). As \( \mathcal{E}_{X,s,p}(v) \) is a subgraph of both \( \mathcal{E}_{X,s,p_0}(v) \) and \( \mathcal{E}_{X,s,p}(v) \) which are trees, there exist a path \( q \) in \( \mathcal{E}_{X,s,p_0}(v) \) and a path \( q' \) in \( \mathcal{E}_{X,s,p}(v) \) connecting \( b_1 \) and \( b_2 \). In particular, the right vertices occurring in \( q \) belong to \( E_{X,s}^+(v) \) and the left vertices occurring in \( q' \) belong to \( E_{X,s}^-(v) \). As \( \mathcal{E}_{X,s,p_0}(v) \) and \( \mathcal{E}_{X,s,p}(v) \) are both subgraphs of \( \mathcal{E}_{X,s,p_0}(v) \), which is a tree, the paths \( q \) and \( q' \) coincide. It means that this path only goes through left vertices belonging to \( E_{X,s}^-(v) \) and through right vertices belonging to \( E_{X,s}^+(v) \). This implies that it is a path of \( \mathcal{E}_{X,s,p}(v) \), hence that \( b_1 \) and \( b_2 \) are connected in \( \mathcal{E}_{X,s,p}(v) \).

We now show that if \( u = s\sigma(v)p \) is an extended image of \( v \), then it is dendric. By Proposition 13, \( \mathcal{E}_Y(u) \) is the image of \( \mathcal{E}_{X,s,p}(v) \) under some graph morphism \( \varphi \), hence it is connected. We proceed by contradiction to show that it is acyclic. Assume that \( c = (a_1, b_1, a_2, b_2, \ldots, a_n, b_n, a_1), n \geq 2 \), is a non-trivial cycle in \( \mathcal{E}_Y(u) \), where \( a_1, \ldots, a_n \) are left vertices and \( b_1, \ldots, b_n \) are right vertices. Again by Proposition 13 for every \( i \leq n \), there exist

- a child \( s_i \) of \( s \) in \( S_\sigma(\sigma) \);
- a child \( p_i \) of \( p \) in \( P_\nu(\sigma) \);
- left vertices \( a'_i, a''_i \) of \( \mathcal{E}_{X,s,p}(v) \), belonging to \( E_{X,s_i}^-(v) \);
- right vertices \( b'_i, b''_i \) of \( \mathcal{E}_{X,s,p}(v) \), belonging to \( E_{X,p_i}^+(v) \);

such that \( \varphi(a'_i) = \varphi(a''_i) = a_i, \varphi(b'_i) = \varphi(b''_i) = b_i \) and such that the pairs

\[
(a_j', b_j'), (a_{j+1}', b_{j+1}'), \quad j < n,
\]

\[
(a_n', b_n'), (a'_1, b'_1),
\]

are edges of \( \mathcal{E}_{X,s,p}(v) \).

Observe that as \( \mathcal{E}_{X,s,p}(v) \) is a tree, for each \( i \leq n \) there is a unique simple path \( q_i \) from \( a''_i \) to \( a'_i \) and a unique simple path \( q'_i \) from \( b'_i \) to \( b''_i \). As \( \mathcal{E}_{X,s_i,p}(v) \) and \( \mathcal{E}_{X,p_i}(v) \) are sub-trees of \( \mathcal{E}_{X,s,p}(v) \), the path \( q_i \) is a path of \( \mathcal{E}_{X,s_i,p}(v) \) and the path \( q'_i \) is a path of \( \mathcal{E}_{X,p_i}(v) \) (otherwise this would contradict the fact that \( \mathcal{E}_{X,s,p}(v) \) is acyclic). In particular, each path \( \varphi(q_i) \) (resp., \( \varphi(q'_i) \)) reduces to the length-0 path \( a_i \) (resp., \( b_i \)). In \( \mathcal{E}_{X,s,p}(v) \), we thus have the cycle

\[
c' = (a''_{q_1}, a'_{q_1}, b'_{q_1}, a''_{q_2}, a'_{q_2}, b'_{q_2}, \ldots, a''_{q_n}, a'_{q_n}, b''_{q_n}, a''_{q_1}).
\]

It is not trivial as its image by \( \varphi \) reduces to the non-trivial cycle \( c \) of \( \mathcal{E}_Y(u) \). This contradicts the fact that \( \mathcal{E}_{X,s,p}(v) \) is a tree, which ends the proof.
We say that a strongly left proper morphism $\sigma$ is *dendric preserving* for $v \in \mathcal{L}(X)$ if all bispecial extended images of $v$ under $\sigma$ are dendric. We let $\text{DP}_X(v)$ denote the set of such morphisms, or simply $\text{DP}(v)$ when the context is clear.

**Remark 15.** Due to the previous result, we have the following properties.

1. If $S_v(\sigma) = \{s_0\}$, then $\sigma$ is dendric preserving for $v$ if and only if for every $p \in \mathcal{P}_v(\sigma)$, $E_{X,s_0,p}(v)$ is a tree. This is in particular the case if $\#(E_{X}(v)) = 2$.

2. If $\mathcal{P}_v(\sigma) = \{p_0\}$, then $\sigma$ is dendric preserving for $v$ if and only if for every $s \in S_v(\sigma)$, $E_{X,s,p_0}(v)$ is a tree. This is in particular the case if $\#(E_{X}^{-}(v)) = 2$.

The previous result is illustrated in the preceding examples. Indeed, for $\sigma$ and $v$ as in Example 10, we have $S_v(\sigma) = \{s_0\} = \{\varepsilon\}$ and $\mathcal{P}_v(\sigma) = \{\ell, \ell a, \ell c\}$. The extension graphs $E_{X,\varepsilon,\ell a}(v)$ and $E_{X,\varepsilon,\ell c}(v)$ in Figure 5 being trees, $v$ has only dendric bispecial extended images, as already observed in Figure 3.

Another consequence of Proposition 14 is given by the following corollary.

**Corollary 16.** Let $\sigma$ be an injective and strongly left proper morphism on the alphabet $A$. The following properties are equivalent.

1. The sets $S(\sigma)$ and $\mathcal{P}(\sigma)$ both only contain one element.

2. For any subshift $X$ on the alphabet $A$ and any dendric bispecial factor $v \in \mathcal{L}(X)$, $\sigma$ is dendric preserving for $v$.

**Proof.** If $S(\sigma) = \{s_0\}$ and $\mathcal{P}(\sigma) = \{p_0\}$, the fact that $\sigma$ is dendric preserving for any dendric bispecial word $v$ is a direct consequence of Proposition 14 since $S_v(\sigma) \subset S(\sigma)$ and $\mathcal{P}_v(\sigma) \subset \mathcal{P}(\sigma)$.

To prove the other implication, let us assume that $S(\sigma)$ contains at least two elements $s_1$ and $s_2$ and show that there exist a subshift $X$ and a dendric bispecial factor $v \in \mathcal{L}(X)$ such that $\sigma$ is not dendric preserving for $v$. We can assume that $s_1$ is not a suffix of $s_2$. By definition of $S(\sigma)$, there exist three different letters $a, b, c \in A$ such that

$$s_1 = s(a, b) \quad \text{and} \quad s_2 = s(a, c).$$

Let $a_1, \ldots, a_n$ be the elements of $A \setminus \{a, b, c\}$. Let us denote by $X$ the subshift coding the regular interval exchange transformation represented below (for precise definitions and more details about interval exchanges, see Subsection 6.2).

The extension graph of the word $\varepsilon$ is given by
It is a tree thus ε is bispecial dendric. However, the graph $\mathcal{E}_{X,s_{1},p_{0}}(\varepsilon)$ is not connected as it does not contain the vertex c on the left but both a and b are left vertices. By Proposition 14, $\sigma$ is not planar preserving for $\varepsilon$. This proves that $\mathcal{S}(\sigma)$ must contain exactly one element. Similarly, $\mathcal{P}(\sigma)$ also contains one element.

The previous result characterizes injective and strongly left proper morphisms for which every dendric bispecial factor has only dendric bispecial extended images. However, the condition does not imply that the image of a dendric subshift by such a morphism $\sigma$ is again dendric. Indeed, the result gives information only on the bispecial factors that are extended images under $\sigma$, i.e., that have an antecedent. The next result characterizes those morphisms $\sigma$ for which even the new initial bispecial factors are dendric. For any letter $a \in \mathcal{A}$, let $\alpha_{a}$ and $\bar{\alpha}_{a}$ denote the so-called Arnoux-Rauzy morphisms

$$
\alpha_{a}(b) = \begin{cases} 
a & \text{if } b = a, \\
ab & \text{otherwise}, 
\end{cases}
\quad \bar{\alpha}_{a}(b) = \begin{cases} 
a & \text{if } b = a, \\
ba & \text{otherwise}.
\end{cases}
$$

**Proposition 17.** The injective and strongly left proper morphisms preserving dendricity, i.e. such that the image of any dendric subshift is a dendric subshift, are exactly the morphisms

$$
\bar{\alpha}_{a_{1}} \circ \cdots \circ \bar{\alpha}_{a_{n}} \circ \alpha_{\ell} \circ \pi
$$

for any $n \geq 0$, any $a_{1}, \ldots, a_{n} \in \mathcal{A} \setminus \{\ell\}$ and any permutation $\pi$ of $\mathcal{A}$.

**Proof.** Using Corollary 16 an injective and strongly left proper morphism $\sigma$ for the letter $\ell$ preserves dendricity if and only if the following conditions are satisfied:

1. the sets $\mathcal{S}(\sigma)$ and $\mathcal{P}(\sigma)$ both only contain one element;
2. if $F_{\sigma}$ is the set

$$
F_{\sigma} = \text{Fac}(\{\sigma(a)\ell : a \in \mathcal{A}\}),
$$

then any word $w \in F_{\sigma}$ such that $|w|_{\ell} = 0$ is dendric in $F_{\sigma}$.

Let $\sigma = \bar{\alpha}_{a_{1}} \circ \cdots \circ \bar{\alpha}_{a_{n}} \circ \alpha_{\ell} \circ \pi$ for $a_{1}, \ldots, a_{n} \in \mathcal{A} \setminus \{\ell\}$. It is easily verified that $\sigma$ is injective and strongly left proper for the letter $\ell$. As conditions 1 and 2 only depend on the set $\sigma(\mathcal{A})$, one can assume that $\pi = id$. For any letter $c$, $\ell c$ is a prefix of $\alpha_{\ell}(c)\ell$ thus

$$(\bar{\alpha}_{a_{1}} \circ \cdots \circ \bar{\alpha}_{a_{n}}(\ell)) c$$

is a prefix of $\sigma(c)\ell$. This shows that

$$
\mathcal{P}(\sigma) = \{\bar{\alpha}_{a_{1}} \circ \cdots \circ \bar{\alpha}_{a_{n}}(\ell)\}.
$$

Similarly, $c\ell$ is a suffix of $\ell\bar{\alpha}_{\ell}(c)$ thus

$$
c(\alpha_{a_{1}} \circ \cdots \circ \alpha_{a_{n}}(\ell))$$

is a suffix of

$$
\ell(\alpha_{a_{1}} \circ \cdots \circ \alpha_{a_{n}} \circ \bar{\alpha}_{\ell}(c)) = (\bar{\alpha}_{a_{1}} \circ \cdots \circ \bar{\alpha}_{a_{n}} \circ \alpha_{\ell}(c)) \ell = \sigma(c)\ell
$$

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thus \( S(\sigma) \) only contains one element and \( \sigma \) satisfies the condition 1. To prove condition 2 let us proceed by induction on \( n \). If \( n = 0 \), then the only bispecial word is \( \varepsilon \) and it is easy to verify that it is dendric. If \( \sigma' = \bar{\alpha}_{a_2} \circ \cdots \circ \bar{\alpha}_{a_n} \circ \alpha_\ell \) satisfies condition 2, then simple adaptations of Proposition 7 and of Proposition 14 tell us that, as \( \bar{\alpha}_{a_1} \) is strongly right proper for the letter \( a_1 \) and the images of letters by \( \bar{\alpha}_{a_1} \) have a unique longest common suffix and a unique longest prefix, it suffices to prove that the words \( w \in F_\sigma \) such that \( |w|_{a_1} = 0 \) are dendric. These words are the elements of \( \{ \varepsilon \} \cup A \setminus \{ a_1 \} \), of which only \( \varepsilon \) is bispecial. The conclusion follows.

Let us now assume that \( \sigma' \) is a strongly left proper morphism for the letter \( \ell \) which satisfies conditions 1 and 2. As \( S(\sigma') = \{ s_0 \} \), for any letter \( a \in A \), \( a s_0 \) is suffix of some \( \sigma'(b) \). In particular, for \( a = \ell \), this implies that there exists \( b \in A \) such that \( \ell s_0 = \sigma'(b) \) and that, for any letter \( c \neq b \), \( \sigma'(c) \) is strictly longer than \( \sigma'(b) \). Similarly, \( p_0 \ell \) is prefix of some \( \sigma(b') \ell \) thus \( \sigma'(b') = p_0 \) and, as \( \sigma'(b') \) must be strictly shorter than any other \( \sigma'(a) \), we obtain \( b = b' \) and \( p_0 = \ell s_0 \). We can thus assume that \( \sigma' = \sigma \circ \pi \) where \( \pi \) is a permutation of \( A \) such that \( \ell s_0 a \) is a prefix of \( \sigma(a) \ell \) for all \( a \in A \). In particular, \( \sigma(\ell) = \ell s_0 \). We have

\[
F_\sigma = F_{\sigma'}.
\]

By construction, for any prefix \( u \) of \( s_0 \ell \) and any suffix \( v \) of \( \ell s_0 \),

\[
E^-_{F_\sigma}(u) = A \quad \text{and} \quad E^+_{F_\sigma} = A.
\]

In particular, if \( s_0 \ell \in a A^* \) and \( \ell s_0 \in A^* b \), then

\[
E_{F_\sigma}(\varepsilon) = (A \times \{ a \}) \cup (\{ b \} \times A)
\]

because \( \varepsilon \) is dendric in \( E_{F_\sigma} \). Thus, any occurrence of \( c \neq b \) in \( F_\sigma \) can only be followed by an occurrence of \( a \). Let us use this observation to prove that \( \sigma = \bar{\alpha}_{a_1} \circ \cdots \circ \bar{\alpha}_{a_n} \circ \alpha_\ell \) for some letters \( a_1, \ldots, a_n \in A \setminus \{ \ell \} \). If \( s_0 = \varepsilon \), then \( a = b = \ell \) thus \( \sigma(c) = \ell c \) for all \( c \in A \setminus \{ \ell \} \) and \( \sigma = \alpha_\ell \). If \( s_0 \) is not empty, then \( a \) is the first letter of \( s_0 \) and \( b \) the last one. In particular, \( a \) and \( b \) cannot be the letter \( \ell \) and, as \( s_0 \ell \) is an element of \( F_\sigma \), \( a \) cannot only be followed by occurrences of \( a \) thus \( a \) must be equal to \( b \). For any letter \( c \in A \), we know that \( \sigma(c) \) begins with \( \ell \) and that any letter \( d \neq a \) in \( \sigma(c) \) is followed by an \( a \) thus

\[
\sigma(c) \in \ell a (\{ a \} \cup (A \setminus \{ \ell \}) a)^*.
\]

Let us define the morphism \( \tau \) such that

\[
\sigma(c) = \bar{\alpha}_a \circ \tau(c).
\]

This morphism is unique and \( \tau(c) \) is obtained by removing an occurrence of \( a \) after each letter \( d \neq a \) in \( \sigma(c) \). By construction, \( \tau \) is injective and strongly left proper for the letter \( \ell \). In addition, \( s_0 \) begins and ends with the letter \( a \) thus there exists \( s'_0 \in A^* \) such that

\[
a\bar{\alpha}_a(s'_0) = s_0.
\]

It is easy to check that, as \( \ell s_0 c \) is a prefix of \( \sigma(c) \ell \), \( \ell s'_0 c \) is a prefix of \( \tau(c) \ell \) and that, as \( c s_0 \) is a suffix of \( \sigma(c) \), \( c s'_0 \) is a suffix of \( \tau(c) \) for all \( c \in A \). Thus, \( S(\tau) \) and \( P(\tau) \) both contain only one element and \( \tau \) satisfies the condition 1. In addition, for all \( w \in F_\tau \) and all \( c, d \in A \)

\[
cwd \in F_\tau \iff ca\bar{\alpha}_a(w)d \in F_\sigma.
\]
Indeed, this equivalence is direct if \( c \neq a \) and, for \( c = a \), it derives from the fact that \( a \neq \ell \) thus, if \( awd \in F_\tau \), then there exists \( c' \) such that \( c'awd \in F_\tau \). As a consequence, the extension graph of \( w \) in \( F_\tau \) is the same as the extension graph of \( a\hat{\alpha}_a(w) \) in \( F_\sigma \) and \( \tau \) satisfies the condition 2. By construction, we have \( |s_0'| < |s_0| \) thus we can conclude by iterating the proof on \( \tau \).

\[ \square \]

5 The case of ternary minimal dendric subshifts

In Section 3, we showed that any minimal dendric subshift \( X \) over the alphabet \( \mathcal{A} \) is \( S \)-adic with \( S \) a set of tame automorphisms of \( F_\mathcal{A} \), a directive sequence of \( X \) being given by Theorem 3. In this section, we give an \( S \)-adic characterization of minimal dendric subshifts over the alphabet \( \mathcal{A}_3 = \{1, 2, 3\} \). More precisely, we strengthen Theorem 3 by exhibiting a set \( S_3 \) (several choices are possible) and by characterizing which are the sequences \( (\sigma_n)_{n \geq 1} \in S_3^\mathbb{N} \) that are a directive sequence of a minimal dendric subshift.

5.1 Return morphisms in the ternary case

Assume that \( a, b, c \) are such that \( \mathcal{A}_3 = \{a, b, c\} \). Let us start with an example. Assume that \( X \) is a minimal dendric subshift over \( \mathcal{A}_3 \) and that the extension graph of \( \varepsilon \) in \( X \) is

\[ a \\
\longleftarrow b \\
c \longrightarrow b \\
\longleftarrow c \]

The associated Rauzy graph \( G_1(X) \) is

\[ a \\
\longleftarrow b \\
c \longrightarrow c \]

From it, we deduce that

- the return words to \( a \) are \( a \), \( ab \) and \( acb \);
- the return words to \( b \) are of the form \( ba^k \) or \( ba^k c \) with \( k \geq 1 \);
- the return words to \( c \) belong to \( c(ba^+)^+ \).

An additional restriction concerning the powers of \( a \) occurring in the return words to \( b \) can be deduced from the fact that \( X \) is dendric. We claim that if \( ba^k \) or \( ba^k c \) is a return word to \( b \), then the other return words can not be of the form \( ba^{k+\ell} \) or \( ba^{k+\ell} c \) for some \( \ell \) such that \( |k-\ell| \geq 2 \). Indeed, if for instance both \( ba^k \), \( k \geq 1 \), and \( ba^{k+\ell} \), \( \ell \geq k+2 \), are return words, then by definition of return words, the words \( ba^k b \) and \( ba^{k+\ell} b \) belong to \( \mathcal{L}(X) \). This implies that there is a cycle in the extension graph of \( a^k \), contradicting the fact that \( X \) is dendric. Therefore, the set of return words to \( b \) is one of the following for some \( k \geq 1 \):

\[ \{ba^k, ba^{k+1}, ba^k c\}, \quad \{ba^k, ba^{k+1}, ba^{k+1} c\}, \quad \{ba^k c, ba^{k+1} c, ba^k\}, \quad \{ba^k c, ba^{k+1} c, ba^{k+1}\}. \]
Return words to $c$ are less easily described. Since $\mathcal{R}(c) \subset c(ba^+)$ and $\#(\mathcal{R}(c)) = 3$, the set $\mathcal{R}(c)$ is determined by three sequences $(k_i^{(j)})_{1 \leq i \leq n_j}, j \in \{1, 2, 3\}$ such that

$$\mathcal{R}(c) = \{cba^{k_i^{(j)}}ba^{k_2^{(j)}} \cdots ba^{k_{n_j}^{(j)}} | j \in \{1, 2, 3\}\}.$$

Similar arguments show that there must exist $k \geq 1$ such that \{(k_i^{(j)})_{1 \leq i \leq n_j}, 1 \leq i \leq n_j\} = \{k, k + 1\}$, but precisely describing the three sequences $(k_i^{(j)})_{1 \leq i \leq n_j}, j \in \{1, 2, 3\}$, is much more tricky. The main reason for this difference is that the letter $c$ is not left special in $X$. If $u$ is the smallest left special factor having $c$ as a suffix, then writing $u = vc$, we have $v\mathcal{R}(c) = \mathcal{R}(u)v$. To better understand the possible sequences $(k_i^{(j)})_{1 \leq i \leq n_j}$, we thus need the Rauzy graph of order $|u|$ of $X$ and not just $G_1(X)$.

With the notation of Theorem 4, any choice of sequence of letters $(a_n)_{n \geq 1}$ leads to a directive sequence of $X$. Consequently, in the sequel we will only consider return words to left special letters with the “simplest” return words. In other words, if the extension graph of the empty word in $X$ is as in the previous example, we will only consider the coding morphisms associated with the left special letter $a$.

The possible extension graphs of the empty word for minimal dendric subshifts on $A_3$ are given in Figures 7 and 8. They must satisfy two conditions: $E(\varepsilon)$ must be a tree and the associated Rauzy graph $G_1(X)$ must be strongly connected (by minimality of $X$). We always assume that $a$ is a left special letter (recall that $a$ represent any letter in $A_3 = \{1, 2, 3\}$) and we present these associated Rauzy graph of order 1 as well as coding morphisms associated with $\mathcal{R}(a)$. Whenever some power appear in an image, we always have $k \geq 1$. The reason why we only have $k$ and $k + 1$ as exponent is the same as in the previous example: a bigger difference would contradict dendricity by inducing a cycle in some extension graph. From this list, we consider the set

$$\mathcal{S}_3 = \{\alpha_a, \beta_{abc}, \gamma_{abc}, \delta_{abc}, \zeta_{abc}, \eta_{abc} | \{a, b, c\} = A_3, k \geq 1\}$$

and obtain the following result.

![Figure 7: The case with a unique left special letter and/or a unique right special letter](image-url)
Proposition 18. Any ternary minimal dendric subshift $X$ has a primitive $S_3$-adic representation $\sigma$ and, for each such representation, $X^{(n)}_{\sigma}$ is a ternary minimal dendric subshift for each $n$.

Proof. For the existence, we consider the construction of a directive sequence following Theorem 4 where at each step, we choose the left special letter $a_n$ for which $\sigma_n$ is in $S_3$. Using Theorem 4 and Theorem 5, we obtain that for each $S_3$-adic representation $\sigma$ and each $n$, $X^{(n)}_{\sigma}$ is a ternary minimal dendric subshift.

5.2 Conditions for having only dendric bispecial extended images

Assume that $X$ is a minimal subshift over $A$ and that $v \in \mathcal{L}(X)$ is a bispecial factor and let $Y$ be the image of $X$ under some injective and strongly left proper morphism $\sigma : A^* \rightarrow B^*$. Recall from Section 4.2 (and, in particular, Proposition 13) that the extension graph of any bispecial extended image of $v$ under $\sigma$ is the image under two consecutive graph morphisms $\varphi_s^{(L)}$ and $\varphi_p^{(R)}$ of a subgraph of $\mathcal{E}_X(v)$. In this section, we determine those graph morphisms when $\sigma$ is a morphism in $S_3$ and we give necessary and sufficient conditions on the extension graph of $v \in \mathcal{L}(X)$ so that $v$ only has dendric bispecial extended images. In this particular case, since the alphabet has cardinality 3, $P(\sigma)$ and $S(\sigma)$ have cardinality at most 2. Tables 1 and 2 define the (possibly partial) maps $\varphi_s^{(L)}$, $\varphi_p^{(R)} : A_3 \rightarrow A_3$ associated with each morphism $\sigma \in S_3$.

A direct application of Proposition 14 shows that whenever $v$ is dendric, then $v$ has only dendric bispecial extended images if and only if the following conditions are satisfied:

1. either $S_v(\sigma) = \{s_0\}$, or both $S_v(\sigma) = \{s_0, s\}$ and $\mathcal{E}_{X,s,s_0,p_0}(v)$ is a tree;

2. either $P_v(\sigma) = \{p_0\}$, or both $P_v(\sigma) = \{p_0, p\}$ and $\mathcal{E}_{X,s_0,p}(v)$ is a tree.

We first give a handier interpretation of these conditions. Observe that for convenience, we actually characterize the dendric bispecial factors $v \in \mathcal{L}(X)$ that have a non-dendric bispecial extended images. When considering a letter $a \in A_3$ as a vertex of $\mathcal{E}(v)$, we respectively write $a_L$ or $a_R$ to emphasize that $a$ is considered as a left or right vertex. For $a \in A_3$ and $v \in \mathcal{L}(X)$, we define the following conditions on $v$:

Figure 8: The cases with two left special letters and two right special letters
there exists a permutation $C_1$ above is.

Proof. The negation of item 2 of Proposition 14 is equivalent to “there exists
a non-dendric bispecial extended image under $\sigma$ if and only if there is a letter $a \in A_3$ for which one of the following conditions is satisfied:

1. $v$ satisfies $C_L(a)$ and $\sigma \in \{\beta_{abc}, \delta_{abc}, \gamma_{abc}, \eta_{abc} \mid k \geq 1\}$;

2. $v$ satisfies $C_R(a)$ and $\sigma \in \{\gamma_{abc}, \delta_{abc}, \gamma_{abc}, \eta_{bca} \mid k \geq 1\}$.

Remark 19. If $v \in L(X)$ is a dendric factor, then in the condition $C_i(a)$, the considered
subgraph is not connected if and only if $a_i$ has at least two neighbors that are not leaves, i.e.,
that have degree at least 2. In particular, $v$ is bispecial. Observe also that as $E(v)$ is a tree,
this implies that the $i$-side of $E(v)$ contains three vertices. Hence, another equivalent condition
of $C_i(a)$ when $E_X(v)$ is a tree is that, writing $A_3 = \{a, b, c\}$, the path from $b_i$ to $c_i$ has length 4.

Proposition 20. Let $X$ be a subshift over $A_3$, $\sigma$ be a morphism in $S_3$ and $Y$ be the image of
$X$ under $\sigma$.

If $v \in L(X)$ is a dendric bispecial factor, then $v$ has a non-dendric bispecial extended image
under $\sigma$ if and only if there is a letter $a \in A_3$ for which one of the following conditions is satisfied:

1. $v$ satisfies $C_L(a)$ and $\sigma \in \{\beta_{abc}, \delta_{abc}, \gamma_{abc}, \eta_{abc} \mid k \geq 1\}$;

2. $v$ satisfies $C_R(a)$ and $\sigma \in \{\gamma_{abc}, \delta_{abc}, \gamma_{abc}, \eta_{bca} \mid k \geq 1\}$.

Example 21. Assume that $v$ is a dendric bispecial factor in some minimal ternary subshift
$X$ with extension graph.

| $\sigma$ | $S(\sigma)$ | $\varphi_{\sigma}^{(L)}$ | $\varphi_{\sigma}^{(R)}$ |
|----------|-------------|--------------------------|--------------------------|
| $\alpha_a$ | $\{\varepsilon\}$ | id | $\{a\}$ | id |
| $\beta_{abc}$ | $\{\varepsilon, b\}$ | $\varphi_{\varepsilon}^{(L)}$ | $\{a\}$ | id |
| $\varphi_{\beta}^{(L)}$ | $a \mapsto a$ | $b, c \mapsto b$ | |
| $\varphi_{\beta}^{(R)}$ | $a \mapsto a$ | $b, c \mapsto b$ | |
| $\gamma_{abc}$ | $\{\varepsilon\}$ | id | $\{a, ab\}$ | $\varphi_{\gamma}^{(R)}$ |
| $\varphi_{\gamma}^{(R)}$ | $b \mapsto a$ | $c \mapsto c$ | $b \mapsto a$ | $c \mapsto c$ |
| $\varphi_{\gamma}^{ab}$ | $b \mapsto a$ | $c \mapsto c$ | $b \mapsto a$ | $c \mapsto c$ |

Table 1: Definition of the graph morphisms $\varphi_{\sigma}^{(L)}$ and $\varphi_{\sigma}^{(R)}$ associated with the morphisms $\alpha_a$, $\beta_{abc}$ and $\gamma_{abc}$.
Table 2: Definition of the graph morphisms $\varphi_s^{(L)}$ and $\varphi_p^{(R)}$ associated with the morphisms $\delta^{(k)}_{abc}$, $\zeta^{(k)}_{abc}$ and $\eta_{abc}$

Thus $v$ satisfies $C_L(c)$ and $C_R(a)$. If $Y_1$ is the image of $X$ under $\beta_{abc}$, the bispecial extended images of $v$ in $Y_1$ are $u_1 = \beta_{abc}(v)a$ and $u_2 = b\beta_{abc}(v)a$ and they have the following extension graphs:

Similarly, if $Y_2$ is the image of $X$ under $\beta_{cab}$, the bispecial extended images of $v$ in $Y_2$ are $w_1 = \beta_{cab}(v)c$ and $w_2 = a\beta_{cab}(v)c$ and they have the following extension graphs:

5.3 Ternary dendric-preserving morphisms

Assuming that $v \in \mathcal{L}(X)$ is a dendric bispecial factor, Proposition 20 characterizes under which conditions $v$ has a non-dendric bispecial extended image under $\sigma \in S_3$ or, in other words, under which conditions $\sigma \in S_3$ is not dendric preserving for $v$. As we consider the ternary case, the set $DP(v)$ of dendric preserving morphisms for $v$ will be implicitly restricted
Table 3: Conditions for the empty word

| $\sigma$ | $\alpha_a$ | $\beta_{abc}$ | $\gamma_{abc}$ | $\delta_{abc}^{(k)}$ | $\varphi_{abc}^{(k)}$ | $\eta_{abc}$ |
|----------|------------|---------------|----------------|----------------------|----------------------|-------------|
| $A_L(\varepsilon)$ | $\emptyset$ | $\{a\}$ | $\emptyset$ | $\{c\}$ | $\{\bar{c}\}$ | $\{b\}$ |
| $A_R(\varepsilon)$ | $\emptyset$ | $\emptyset$ | $\{a\}$ | $\{a\}$ | $\{c\}$ | $\{c\}$ |

to morphisms from $S_3$. We set

$$
A_L(v) = \{ a \in A_3 \mid v \text{ satisfies } C_L(a) \},
A_R(v) = \{ a \in A_3 \mid v \text{ satisfies } C_R(a) \}
$$

and, if $X$ is a ternary dendric subshift,

$$
A_L(X) = \bigcup_{v \in \mathcal{L}(X)} A_L(v);
A_R(X) = \bigcup_{v \in \mathcal{L}(X)} A_R(v);
\text{DP}(X) = \bigcap_{v \in \mathcal{L}(X)} \text{DP}(v).
$$

Using Proposition 20, the sets $A_L(X)$ and $A_R(X)$ completely determine the set $\text{DP}(X)$ of all morphisms in $S_3$ that are dendric-preserving for $X$, i.e. that are dendric-preserving for all $v \in \mathcal{L}(X)$. In this section, we in particular show that $A_L(X)$ and $A_R(X)$ contain at most one letter and we show that, when $Y$ is the image of $X$ under $\sigma \in \text{DP}(X)$, $A_L(Y)$ (resp., $A_R(Y)$) is completely determined by $A_L(X)$ (resp., $A_R(X)$) and $\sigma$. The next lemma is a trivial consequence of Remark 19.

**Lemma 22.** Let $X$ be a subshift over $A_3$. For every dendric bispecial factor $v \in \mathcal{L}(X)$, $A_L(v)$ (resp., $A_R(v)$) contains at most one letter.

**Lemma 23.** Let $X$ be a subshift over $A_3$ which is the image under $\sigma \in S_3$ of another subshift $Z$ over $A_3$. The sets $A_L(\varepsilon)$ and $A_R(\varepsilon)$ are given in Table 3 where $\varepsilon$ is considered as a factor of $X$.

**Proof.** Indeed, the morphism $\sigma$ completely determines the extension graph $E_X(\varepsilon)$. The result thus directly follows from the definition of the conditions $C_L(a)$ and $C_R(a)$. \qed

**Lemma 24.** If $X$ is a ternary dendric subshift and if $Y$ is the image of $X$ under some morphism $\sigma \in S_3$, then $Y$ is dendric if and only if $\sigma \in \text{DP}(X)$.

**Proof.** Indeed, if $\sigma \in \text{DP}(X)$, every bispecial extended image of a bispecial factor of $X$ is dendric by definition of $\text{DP}(X)$. Any other bispecial factor of $Y$ is the empty word or a non-prefix factor of an image $\sigma(a)$, $a \in A_3$ (by Proposition 7). It suffices to check that any such bispecial factor is dendric when $\sigma$ belongs to $S_3$ to prove that $Y$ is dendric. Assume now that $Y$ is dendric. If $\sigma$ is not in $\text{DP}(X)$, then there exists $v \in \mathcal{L}(X)$ such that $\sigma \notin \text{DP}(v)$ thus $v$ has an extended image in $Y$ which is not dendric. \qed

We say that a morphism $\sigma \in S_3$ is left-invariant (resp., right-invariant) if $\mathcal{S}(\sigma)$ (resp., $\mathcal{P}(\sigma)$) is a singleton, i.e. $\mathcal{S}(\sigma) = \{s_0\}$ (resp., $\mathcal{P}(\sigma) = \{p_0\}$). The next lemma directly follows from the definition of the morphisms $\alpha_a$, $\beta_{abc}$, $\gamma_{abc}$, $\delta_{abc}^{(k)}$, $\varphi_{abc}^{(k)}$ and $\eta_{abc}$.
Lemma 25.  
1. $\alpha_a$ is both left-invariant and right-invariant;
2. $\beta_{abc}$ is right-invariant, but not left-invariant;
3. $\gamma_{abc}$ is left-invariant, but not right-invariant;
4. $\varepsilon_{abc}, \zeta_{abc}$ and $\eta_{abc}$ neither are left-invariant, nor right-invariant.

We let $S_{LI}$ and $S_{RI}$ respectively denote the left-invariant and right-invariant morphisms, i.e.,

$$S_{LI} = \{\alpha_a, \gamma_{abc} | A_3 = \{a, b, c\}\} \quad \text{and} \quad S_{RI} = \{\alpha_a, \beta_{abc} | A_3 = \{a, b, c\}\}.$$ 

Observe that if $\sigma \in S_{LI}$ (resp., $\sigma \in S_{RI}$), then the associated graph morphism $\varphi_{s_0}(L)$ (resp., $\varphi_{s_0}(R)$) is the identity. Moreover, from Table 3 $\sigma \in S_{LI}$ (resp., $\sigma \in S_{RI}$) if and only if $A_L(\varepsilon) = \emptyset$ (resp., $A_R(\varepsilon) = \emptyset$).

Lemma 26. Let $X$ be a subshift over $A_3$ and $v \in L(X)$ be a dendric bispecial factor. If $\sigma \in S_3$ is not left-invariant (resp., not right-invariant), then any dendric extended image $u$ of $v$ is such that $A_L(u) = \emptyset$ (resp., $A_R(u) = \emptyset$).

In particular, if $X$ is dendric and $\sigma \in DP(X)$ is not left-invariant (resp., not right-invariant) and if $Y$ is the image of $X$ under $\sigma$, then $A_L(Y) = A_L(\varepsilon) \neq \emptyset$ (resp., $A_R(Y) = A_R(\varepsilon) \neq \emptyset$), where $\varepsilon$ is considered as a factor of $Y$.

Proof. Let us show the result for a non-left-invariant morphism, the other case being symmetric. By definition of left-invariance, $S(\sigma)$ contains two elements $s_0$ and $s_1$. It suffices to check in Tables 1 and 2 that for each of them, the range of the associated graph morphism $\varphi_{s_i}(L)$ has cardinality 2. As the left vertices of $E_Y(u)$ are images of the left vertices of $E_X(v)$, $E_Y(u)$ contains at most two left vertices. From Remark 19 $u$ thus cannot satisfies $C_L(a)$ for any letter $a \in A_3$.

Now assume that $\sigma \in DP(X)$. By Lemma 24 $Y$ is a dendric subshift. Let $u$ be a non-empty factor of $Y$. By Proposition 7 either $u$ is a non-prefix factor of $\sigma(a)$ for some letter $a \in A_3$, or $u$ is an extended image of a factor $v \in L(X)$. In the first case, it suffices to check that $\#(E_Y(u)) \leq 2$, which implies that $A_L(u) = \emptyset$. In the second case, as $X$ is dendric, $v$ is also dendric so by the first part of the lemma, $A_L(u) = \emptyset$. Thus $A_L(Y) = A_L(\varepsilon)$ and, by Lemma 23 it is non-empty.

Lemma 27. Let $X$ be a subshift over $A_3$ and $v \in L(X)$ be a dendric bispecial factor. If $\sigma \in DP(v)$ is left-invariant (resp., right-invariant), then

$$A_L(v) = \bigcup_{u \text{ bispecial extended image of } v} A_L(u);$$

$$A_R(v) = \bigcup_{u \text{ bispecial extended image of } v} A_R(u).$$

In particular, if $X$ is dendric and $\sigma \in DP(X)$ is left-invariant (resp., right-invariant) and $Y$ is the image of $X$ under $\sigma$, then $A_L(Y) = A_L(X)$ (resp., $A_R(Y) = A_R(X)$).

Proof. Let us assume that $\sigma$ is left-invariant, the other case is symmetric. We first show that if $u$ is a bispecial extended image of $v$, then $A_L(u) \subset A_L(v)$. As $S(\sigma) = \{s_0\}$, there exists $p \in P(\sigma)$ such that $u = s_0\sigma(v)p$. Assume that $a \in A_L(u)$. Writing $A_3 = \{a, b, c\}$, Remark 19
states that the path \( q \) from \( b_L \) to \( c_L \) in \( E_Y(u) \) has length 4. This path \( q \) is the image under \( \varphi_{s_0,p} \) of a path \( q' \) of length at least 4 in \( E_X(v) \). As \( E_X(v) \) is a tree with at most 6 vertices and the extremities of \( q' \) are left vertices, the path has length exactly 4. As \( \varphi_{s_0,0} \) is the identity, we conclude that \( q' \) is a path of length 4 from \( b_L \) to \( c_L \) in \( E_X(v) \), hence that \( a \in A_L(v) \).

If \( A_L(v) = \emptyset \), then Equality (2) is direct. If \( A_L(v) \neq \emptyset \), we show that there exists exactly one bispecial extended image \( u \) of \( v \) such that \( A_L(u) \neq \emptyset \) and for this one, \( A_L(u) = A_L(v) \).

As \( \alpha \) is left-invariant, we have by Lemma 25 that \( \sigma = \alpha_i \) or \( \sigma = \gamma_{ijk} \) for some \( i, j, k \) such that \( A_3 = \{i, j, k\} \).

If \( \sigma = \alpha_i \), then as \( \alpha_i \) is also right-invariant (see Lemma 25), Proposition 13 implies that \( u \) is the unique bispecial extended image of \( v \) and that \( E_Y(u) = E_X(v) \) (the graph morphism is the identity), hence that \( A_L(u) = A_L(v) \).

If \( \sigma = \gamma_{ijk} \), then as \( P(\sigma) = \{i, ij\} \), (see Table 1), Corollary 9 implies that \( v \) has at most two bispecial extended images \( u_1 = \sigma(v)i \) and \( u_2 = \sigma(v)ij \). Let \( \varphi_1 = \varphi_{\epsilon,0} i \) and \( \varphi_2 = \varphi_{\epsilon,ij} \). The morphism \( \varphi_{\epsilon}^{(L)} \) is the identity thus we have \( \varphi_1 = \varphi_i^{(R)} \) and \( \varphi_2 = \varphi_{ij}^{(R)} \).

As \( A_L(v) \neq \emptyset \), by Lemma 22 there is a letter \( a \in A_3 \) such that \( A_L(v) = \{a\} \). Using Remark 19 the path \( q \) from \( b_L \) to \( c_L \) has length 4 in \( E_X(v) \). Let us write \( q = (b_L, x_R, a_L, y_R, c_L) \), with \( x, y \in A_3 \).

If \( i \in \{x, y\} \), we assume without loss of generality that \( i = x \). Then we have \( \varphi_1(q) = (b_L, i, a_L, j, c_L) \) which is a path of length 4 from \( b_L \) to \( c_L \) in \( E_Y(u_1) \). By Remark 19 and Lemma 22 one has \( A_L(u_1) = \{a\} \). On the other hand, \( i \) is not a right vertex of \( E_X,\epsilon,ij(v) \). Thus \( q \) is not a path in \( E_X,\epsilon,ij(v) \). As \( \varphi_2 \) is a graph isomorphism that is invariant on the left vertices, there is no path from \( b_L \) to \( c_L \) in \( E_Y(u_2) \). Using again Remark 19 \( a \notin A_L(u_2) \). But as \( A_L(u_2) \subset A_L(v) \) by the first part of the proof, we get \( A_L(u_2) = \emptyset \).

If \( \{x, y\} = \{j, k\} \), we assume without loss of generality that \( (x, y) = (j, k) \). Then we have \( \varphi_2(q) = (b_L, i, a_L, k, c_L) \) which is a path of length 4 from \( b_L \) to \( c_L \) in \( E_Y(u_2) \). By Remark 19 and Lemma 22 one has \( A_L(u_2) = \{a\} \). On the other hand, we have \( \varphi_1(j) = \varphi_1(k) = j \), which shows that \( (b_L, j, c_L) \) is a path of length 2 from \( b_L \) to \( c_L \) in \( E_Y(u_1) \). Using again Remark 19 \( a \notin A_L(u_1) \). But as \( A_L(u_1) \subset A_L(v) \) by the first part of the proof, we get \( A_L(u_1) = \emptyset \).

Let us finally show that \( A_L(Y) = A_L(X) \). With \( \sigma \in \{\alpha_i, \gamma_{ijk}\} \), the non-prefix factor of \( \sigma(a), a \in A_3 \), are not bispecial. Hence, using Proposition 7 \( b \) a bispecial factor \( u \in L(Y) \) is either empty, or a bispecial extended image of some bispecial factor \( v \in L(X) \). As \( \alpha \) is left-invariant, we have \( A_L(\epsilon) = \emptyset \) by Lemma 23. Using Equation (2), we get

\[
A_L(Y) = \bigcup_{v \in L(Y), \text{bispecial \( u \) bispecial extended image of} v} A_L(u) = \bigcup_{v \in L(X), \text{bispecial}} A_L(v) = A_L(X),
\]

which ends the proof.

The following corollary is a direct consequence of Lemmas 26 and 27.

**Corollary 28.** Let \( X \) be a dendric subshift over \( A_3 \), \( \sigma \in DP(X) \) and \( Y \) the image of \( X \) under \( \sigma \). If \( A_L(Y) = \emptyset \), then \( A_L(X) = \emptyset \) and \( \sigma \) is left-invariant. Respectively, if \( A_R(Y) = \emptyset \), then \( A_R(X) = \emptyset \) and \( \sigma \) is right-invariant.
Proposition 29. Let $X$ be a ternary minimal dendric subshift with $S_3$-adic representation $\sigma = (\sigma_n)_{n \geq 1}$. Then $A_L(X)$ and $A_R(X)$ contain at most one letter. Moreover,

1. $A_L(X) = \emptyset$ if and only if $\sigma$ belongs to $S_{LI}^N$, if and only if $X$ has a unique left special factor of each length;

2. $A_R(X) = \emptyset$ if and only if $\sigma$ belongs to $S_{RI}^N$, if and only if $X$ has a unique right special factor of each length.

Proof. Using the notation of Section 2.3, we have $X = X_{\sigma}$ and for each $n \geq 1$, $X_{\sigma}^{(n+1)}$ is dendric by Proposition 18, thus $\sigma_n \in DP(X_{\sigma}^{(n+1)})$ by Lemma 24. We first show item 1.

Assuming that $A_L(X) = \emptyset$, we have $\sigma \in S_{LI}^N$ by Corollary 28.

Now assuming that $\sigma$ belongs to $S_{LI}^N$, we deduce that any bispecial factor of $X$ is a descendant of the empty word in some $X_{\sigma}^{(n)}$. Using Figure 7, the bispecial factor $v = \varepsilon \in L(X_{\sigma}^{(n)})$ has a unique right extension $va$, $a \in A_3$, which is left special and it satisfies $E^-_{X_{\sigma}^{(n)}}(va) = A_3$.

It then suffices to observe, using Proposition 7 that this property is preserved by taking bispecial extended images under some morphism $\sigma \in S_{LI}^N$. This shows that any bispecial factor $u$, and hence any left special factor, of $X$ satisfies $E^-_{X}(u) = A_3$. Proposition 2 then implies that $X$ has a unique left special factor of each length.

Finally assume that $X$ has a unique left special factor $u_n$ of each length $n$. By Proposition 2, we have $E^-_X(u_n) = A_3$. Using Remark 19, the set $A_L(u_n)$ is non-empty if and only if there are two letters $x, y \in A_3$ such that both $u_n x$ and $u_n y$ are left special factors of $X$. This implies that $A_L(u_n) = \emptyset$. Any non-left-special factor $u$ is such that $A_L(u) = \emptyset$ by Remark 19, hence $A_L(X) = \emptyset$.

The proof of item 2 is symmetric. We finish the proof by showing that $A_L(X)$ and $A_R(X)$ contain at most one letter. Assume by contrary that $a, b \in A_L(X)$ for some different letters $a$ and $b$. By Lemmas 22, 26 and 27, all morphisms $\sigma_n$ are left-invariant. But then item 1 implies that $A_L(X) = \emptyset$. □

5.4 $S_3$-adic characterization of minimal ternary dendric subshifts

By Proposition 29, any ternary minimal dendric subshift $X$ satisfies $\#A_L(X), \#A_R(X) \leq 1$. To alleviate notations in what follows, we consider the alphabet $A_0 = A_3 \cup \{0\}$ and we write $A_L(X) = a$ instead of $A_L(X) = \{a\}$ and $A_L(X) = 0$ instead of $A_L(X) = \emptyset$ (and similarly for $A_R(X)$). We then define the equivalence relation $\sim$ on the set of minimal ternary dendric subshifts by

$$X \sim Y \iff (A_L(X), A_R(X)) = (A_L(Y), A_R(Y)).$$

For all $l, r \in A_0$, we let $[l, r]$ denote the equivalence class of all minimal ternary dendric subshifts satisfying $(A_L(X), A_R(X)) = (l, r)$.

Lemma 30. Let $X$ and $Y$ be minimal ternary dendric subshifts. We have $X \sim Y$ if and only if $\text{DP}(X) = \text{DP}(Y)$. Furthermore, if $X \sim Y$ and $\sigma \in \text{DP}(X)$ and $X'$ and $Y'$ are the respective images of $X$ and $Y$ under $\sigma$, then $X' \sim Y'$.

Proof. The equivalence between $X \sim Y$ and $\text{DP}(X) = \text{DP}(Y)$ follows from Proposition 20. The second part of the statement follows from Lemma 26 and Lemma 27. □
Using the previous lemma, we can define, for each equivalence class $C = [l, r]$, the set $\text{DP}(C) = \text{DP}(X)$, where $X \in C$. Furthermore, for any $\sigma \in \text{DP}(C)$, there is a unique equivalence class $C'$ such that when $X \in C$ and $Y$ is the image of $X$ under $\sigma$, then $Y \in C'$. We call $C'$ the image of $C$ under $\sigma$.

**Lemma 31.** Let $C = [l, r]$ for some $l, r \in A_0$ and let $\sigma \in \text{DP}(C)$.

1. If $\sigma$ is left-invariant, then for all $l' \in A_0$, we have $\sigma \in \text{DP}([l', r])$.

2. If $\sigma$ is not left-invariant, then there is a unique $l' \in A_3$ such that $\sigma \notin \text{DP}([l', r])$.

3. If $\sigma$ is right-invariant, then for all $r' \in A_0$, we have $\sigma \in \text{DP}([l, r'])$.

4. If $\sigma$ is not right-invariant, then there is a unique $r' \in A_3$ such that $\sigma \notin \text{DP}([l, r'])$.

**Proof.** It follows from Proposition 20 and Lemma 25.

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** Let us first define the following directed graph $G'$. The set of vertices is the set of equivalent classes of the relation $\sim$. For every vertices $C, C' \in V$, and every morphism $\sigma \in S_3$, there is an edge from $C$ to $C'$ with label $\sigma$ if $\sigma \in \text{DP}(C')$ and $C$ is the image of $C'$ under $\sigma$.

Assume that $X$ is a minimal ternary dendric subshift. By Proposition 18, $X$ has a primitive $S_3$-adic representation $\sigma$ where for each $n$, $X^{(n)}{\sigma}$ is a ternary minimal dendric subshift. For every $n$, $X^{(n)}{\sigma}$ is the image of $X^{(n+1)}{\sigma}$ under $\sigma_n$ and we thus have $\sigma_n \in \text{DP}(X^{(n+1)}{\sigma})$. By definition of the graph, $P = ([A_L(X^{(n)}{\sigma}), A_R(X^{(n)}{\sigma}))]_{n \geq 1}$ is a path in $G'$ with label $\sigma$.

Now consider a primitive sequence $\sigma$ labeling a path in $G'$ and let us show that the subshift $X_\sigma$ is minimal and dendric. It is minimal by primitiveness of $\sigma$. If it is not dendric, there exists a bispecial factor $u \in \mathcal{L}(X_\sigma)$ which is not dendric. Using Corollary 11, there is a unique $k \geq 1$ and a unique initial bispecial factor $v$ in $\mathcal{L}(X^{(k)}_\sigma)$ such that $u$ is a descendant of $v$ and $E_X(u)$ depends only on $E_{X^{(k)}_\sigma}(v)$. By definition of initial bispecial factors, $v$ is either the empty word or a non-prefix factor of $\sigma_k(a)$ for some letter $a$. In both cases, $E_{X^{(k)}_\sigma}(v)$ is completely determined by $\sigma_k$ and is a tree. By definition of the edges of $G'$, the morphism $\sigma_1 \cdots \sigma_{k-1}$ is dendric-preserving for $v$. Thus $E_X(u)$ is a tree, which is a contradiction.

We finally show that we can reduce $G'$ to the subgraph $G$ obtained by deleting the vertices of the form $[0, r]$ and of the form $[l, 0]$. This subgraph contains 9 vertices and is represented in Figure 9. We show that for every path $P$ in $G'$, there is a path $P'$ in $G$ with the same label.

We thus consider a path $P = ([l_n, r_n])_{n \geq 1} \in G'$ with label $\sigma = (\sigma_n)_{n \geq 1}$ (hence $\sigma_n$ labels the edge from $[l_n, r_n]$ to $[l_{n+1}, r_{n+1}]$). If $P$ is not a path in $G$, then there exists $N$ such that $l_N$ or $r_N$ is 0. Assume that $N$ is the smallest integer such that $l_N = 0$. By Corollary 28 we deduce that for all $n \geq N$, $\sigma_n$ is left-invariant and $l_n = 0$. Using Lemmas 26 and 27 and 31 we have that for all $n \geq N$ and every $l' \in A_3$, $\sigma_n$ is in $\text{DP}([l', r_{n+1}])$ and labels the edge from $[l', r_n]$ to $[l', r_{n+1}]$. Thus for every $l' \in A_3$, the sequence $P_{l'} = ([l', r_n])_{n \geq N}$ is a path in $G'$ with label $(\sigma_n)_{n \geq N}$.

If $N = 1$, we have found a path $P'$ with label $\sigma$ and that does not go through vertices of the form $[0, r]$. If $N > 1$, then by Lemma 27, $\sigma_{N-1}$ is not left-invariant. By Lemma 31 we can choose $l' \in A_3$ such that $\sigma_{N-1} \in \text{DP}([l', r_N])$. Then by Lemmas 26 and 27, $\sigma_{N-1}$ labels the edge from $[l_{N-1}, r_{N-1}]$ to $[l', r_N]$. We finally consider the path $P'$ consisting in the prefix
of \( \mathcal{P} \) from \([l_1, r_1] \) to \([l_{N-1}, r_{N-1}] \) followed by the edge from \([l_{N-1}, r_{N-1}] \) to \([l', r_N] \), followed by the path \( \mathcal{P}_i \). This path has label \( \sigma \) and does not go through vertices of the form \([0, r] \). Observe that if \( \mathcal{P}' = ([l'_n, r'_n])_{n \geq 1} \), then for all \( n \), we have \( r'_n = r_n \).

Starting from \( \mathcal{P}' \) we similarly find another path \( \mathcal{P}'' = ([l''_n, r''_n])_{n \geq 1} \) with label \( \sigma \) and such that for all \( n \), \( l''_n = l'_n \neq 0 \) and \( r''_n \neq 0 \). This concludes the proof.

The graph \( \mathcal{G} \) represented in Figure 9 is a co-deterministic automaton, i.e., for every vertex \( C \) and every morphism \( \sigma \in \mathcal{S}_3 \), there is at most one edge with label \( \sigma \) reaching \( C \). This follows from Lemmas 23, 26 and 27. The edges are computed using Table 3, Lemmas 26 and 27 (to determine the starting vertex of a given morphism) and Proposition 20 (to determine the allowed target vertices).

This is summarized in Table 4, where for each \( x \in \mathcal{A}_0 \), \( \bar{x} \) denotes the set \( \mathcal{A}_3 \setminus \{x\} \).

| Morphism | Starting vertex | Target vertex | Conditions |
|-----------|----------------|--------------|------------|
| \( \alpha_a \) | \([l, r]\) | \([l, r]\) | \(l, r \in \emptyset\) |
| \( \beta_{abc} \) | \([a, r]\) | \([l, r]\) | \(l \in \bar{a}, r \in \emptyset\) |
| \( \gamma_{abc} \) | \([l, \bar{a}]\) | \([l, r]\) | \(r \in \bar{a}\) |
| \( \delta_{abc}^{(k)} \) | \([c, a]\) | \([l, r]\) | \(l \in \bar{a}\) |
| \( \zeta_{abc}^{(k)} \) | \([c, c]\) | \([l, r]\) | \(l \in \bar{a}\) |
| \( \eta_{abc} \) | \([\bar{b}, \bar{c}]\) | \([l, r]\) | \(l \in \bar{a}, r \in \bar{c}\) |

Table 4: List of edges of the graph \( \mathcal{G} \) represented in Figure 9

We now build a deterministic automaton \( \mathcal{G}_d \) which is equivalent to \( \mathcal{G} \) (where all vertices of \( \mathcal{G} \) are initial and final). The states of the automaton are the sets

\[ l \times \bar{r} = \{ [x, y] \mid x \in l, y \in \bar{r} \}, \ l, r \in \mathcal{A}_0. \]

The initial state is \( \emptyset \times \emptyset \) and all states are final. For each morphism \( \sigma \in \mathcal{S}_3 \), there is a transition from \( l \times \bar{r} \) to \( l' \times \bar{r}' \) whenever \( l' \times \bar{r}' \) is the set of vertices reachable from \( l \times \bar{r} \) in \( \mathcal{G} \) by reading \( \sigma \). Thus the edges are as given in Table 5.

| Morphism | Starting vertex | Target vertex | Conditions |
|-----------|----------------|--------------|------------|
| \( \alpha_a \) | \(l \times \bar{r}\) | \(l \times \bar{r}\) | \(l, r \in \emptyset\) |
| \( \beta_{abc} \) | \(l \times \bar{r}\) | \(\bar{a} \times \bar{r}\) | \(l \in \bar{a}, \bar{r} \in \mathcal{A}_0\) |
| \( \gamma_{abc} \) | \(l \times \bar{r}\) | \(l \times \bar{a}\) | \(l \in \mathcal{A}_0, \bar{r} \in \bar{a} \cup \{0\}\) |
| \( \delta_{abc}^{(k)} \) | \(l \times \bar{r}\) | \(\bar{a} \times \bar{a}\) | \(l \in \bar{c} \cup \{0\}, \bar{r} \in \bar{a} \cup \{0\}\) |
| \( \zeta_{abc}^{(k)} \) | \(l \times \bar{r}\) | \(\bar{a} \times \bar{a}\) | \(l, r \in \bar{c} \cup \{0\}\) |
| \( \eta_{abc} \) | \(l \times \bar{r}\) | \(\bar{a} \times \bar{c}\) | \(l \in \bar{b} \cup \{0\}, \bar{r} \in \bar{c} \cup \{0\}\) |

Table 5: List of transitions in the deterministic automaton \( \mathcal{G}_d \)

The automaton \( \mathcal{G}_d \) has 16 vertices and except for the fact that it is not complete (i.e., the sink state has been removed), it is the minimal automaton. It has four strongly connected
Every \( \alpha_a \) labels a loop on every vertex; \( \beta_{abc} \) and \( \beta_{acb} \) label the vertical edges leaving any vertex of the form \([a, x]\); \( \gamma_{abc} \) and \( \gamma_{acb} \) label the horizontal edges leaving any vertex of the form \([x, a]\).

The morphism \( \delta^{(k)} \) labels the edges from \([c, a]\) to \([x, y]\) for all \(x, y \in \{b, c\}\). Only the edges with label \( \delta^{(k)}_{123} \) (solid), \( \delta^{(k)}_{132} \) (dashed) and \( \delta^{(k)}_{213} \) (dotted) are represented.

The morphism \( \zeta^{(k)} \) labels the edges from \([c, c]\) to \([x, y]\) for all \(x, y \in \{b, c\}\). Only the edges with label \( \zeta^{(k)}_{132} \) (solid) and \( \zeta^{(k)}_{312} \) are represented.

The morphism \( \eta^{(k)} \) labels the edges from \([b, c]\) to \([x, y]\) for all \((x, y) \in \{b, c\} \times \{a, b\}\). Only the edges with label \( \eta_{231} \) (solid), \( \eta_{321} \) (dashed) and \( \eta_{132} \) (dotted) are represented.

Figure 9: A subshift on \( A_3 \) is minimal and dendric if and only if it has a primitive \( S_3 \)-adic representation labeling an infinite path in this graph denoted \( G \). For the sake of clarity, we represent 4 copies of the graph and show only a part of the edges in each copy, assuming that \( A_3 = \{a, b, c\} \).
The components that are

\[ C_1 = \{0 \times 0\} \]
\[ C_2 = \{0 \times r \mid r \in A_3\} \]
\[ C_3 = \{l \times 0 \mid l \in A_3\} \]
\[ C_4 = \{l \times r \mid l, r \in A_3\} \]

The component \( C_1 \) thus consists in one vertex with three loops, one for each morphism \( \alpha_a \). By Proposition \( \ref{prop:arnoux-rauzy} \) any minimal dendric subshift whose \( S_3 \)-adic representation labels a path that stays in \( C_1 \) has exactly one left and one right special factor of each length. This class of minimal dendric subshifts exactly corresponds to the class of Arnoux-Rauzy subshifts \( \text{[AR91]} \).

The component \( C_2 \) consists in three vertices and any edge in this component is labeled by a left-invariant morphism. This component is accessible only from \( C_1 \) and any edge from \( C_1 \) to \( C_2 \) is labeled by a morphism \( \gamma_{abc} \). By Proposition \( \ref{prop:arnoux-rauzy} \) any minimal dendric subshift whose \( S_3 \)-adic representation labels a path that ends in \( C_2 \) has exactly one left and two right special factors of each length.

The component \( C_3 \) is symmetric to \( C_2 \) where the involved morphisms are the right-invariant morphisms. By Proposition \( \ref{prop:arnoux-rauzy} \) any minimal dendric subshift whose \( S_3 \)-adic representation labels a path that ends in \( C_3 \) has exactly two left and one right special factors of each length.

The component \( C_4 \) consists in 9 vertices. It is reachable from \( C_1 \) only by morphisms that are neither right- nor left-invariant, from \( C_2 \) only by morphisms that are not left-invariant and from \( C_3 \) only by morphisms that are not right-invariant. By Proposition \( \ref{prop:arnoux-rauzy} \) any minimal dendric subshift whose \( S_3 \)-adic representation labels a path that ends in \( C_4 \) has exactly two left and two right special factors of each length. Observe that if we restrict \( G_d \) to the component \( C_4 \) in which we consider that all states are initial, then this automaton is again equivalent to the initial one. Thus, except for the fact that all states are initial, we have a deterministic automaton with 9 states.

**Remark 32.** A very similar \( S \)-adic characterization of minimal subshifts with factor complexity \( 1 \leq p(n+1) - p(n) \leq 2 \) was given by the third author \( \text{[Ler14]} \). This \( S \)-adic characterization also involves a graph with four strongly connected components, each one corresponding to having exactly one or two left or right special factors of each length.

### 6 Descriptions in \( G \) of well-known families of minimal ternary dendric subshifts

The class of minimal dendric ternary subshifts contains several classes of well-known families of subshifts, namely Arnoux-Rauzy subshifts, codings of regular 3-interval exchange transformations and Cassaigne subshifts. In this section, we study the \( S_3 \)-adic representations of these particular families in the light of \( G \) and \( G_d \).

#### 6.1 Arnoux-Rauzy subshifts

A subshift \( X \subset A_3^\mathbb{Z} \) is an Arnoux-Rauzy subshift if it is a minimal subshift with factor complexity \( p(n) = 2n + 1 \) that has exactly one left and one right special factor of each length. Another equivalent definition is that \( X \) admits a primitive \( S_3 \)-adic representation \( \sigma \in \{\alpha_1, \alpha_2, \alpha_3\}^\mathbb{N} \) \( \text{[AR91]} \). As already mentioned, we get the following result.
Proposition 33. A subshift $X \subset \mathcal{A}_3^\mathbb{Z}$ is an Arnoux-Rauzy subshift if and only if it has a primitive $S_3$-adic characterization that labels a path in the component $C_1$ of $\mathcal{G}_d$.

6.2 Interval exchange subshifts

Let us recall the definition of an interval exchange transformation. A semi-interval is a non-empty subset of the real line of the form $[\alpha, \beta]=\{z \in \mathbb{R} \mid \alpha \leq z < \beta\}$. Thus it is a left-closed and right-open interval. For two semi-intervals $\Delta, \Gamma$, we denote $\Delta < \Gamma$ if $x < y$ for any $x \in \Delta$ and $y \in \Gamma$.

Let $(\mathcal{A}, \leq)$ be an ordered set. A partition $(I_a)_{a \in \mathcal{A}}$ of $[0,1]$ in semi-intervals is ordered if $a < b$ implies $I_a < I_b$.

Let $\mathcal{A}$ be a finite set ordered by two total orders $\leq_1$ and $\leq_2$. Let $(I_a)_{a \in \mathcal{A}}$ be a partition of $[0,1]$ in semi-intervals ordered for $\leq_1$. Let $\lambda_a$ be the length of $I_a$. Let $\mu_a = \sum_{b \leq_1 a} \lambda_b$ and $\nu_a = \sum_{b \leq_2 a} \lambda_b$. Set $\xi_a = \nu_a - \mu_a$. The interval exchange transformation relative to $(I_a)_{a \in \mathcal{A}}$ is the map $T : [0,1] \rightarrow [0,1]$ defined by

$$T(z) = z + \xi_a \quad \text{if} \quad z \in I_a.$$ 

Observe that the restriction of $T$ to $I_a$ is a translation onto $J_a = T(I_a)$, that $\mu_a$ is the right boundary of $I_a$ and that $\nu_a$ is the right boundary of $J_a$. We additionally denote by $\iota_a$ the left boundary of $I_a$ and by $\kappa_a$ the left boundary of $J_a$. Thus

$$I_a = [\iota_a, \mu_a], \quad J_a = [\kappa_a, \nu_a].$$

Note that $a <_2 b$ implies $\nu_a < \nu_b$ and thus $J_a < J_b$. This shows that the family $(J_a)_{a \in \mathcal{A}}$ is a partition of $[0,1]$ ordered for $<_2$. In particular, the transformation $T$ defines a bijection from $[0,1]$ onto itself.

An interval exchange transformation relative to $(I_a)_{a \in \mathcal{A}}$ is also said to be on the alphabet $\mathcal{A}$. The values $(\xi_a)_{a \in \mathcal{A}}$ are called the translation values of the transformation $T$.

Example 34. Let $R$ be the interval exchange transformation corresponding to $\mathcal{A} = \{a, b\}$, $a <_1 b$, $b <_2 a$, $I_a = [0, 1 - \alpha]$, $I_b = [1 - \alpha, 1]$. The transformation $R$ is the rotation of angle $\alpha$ on the semi-interval $[0,1]$ defined by $R(z) = z + \alpha \mod 1$.

\begin{center}
\begin{tabular}{c}
0 & b & a & 1 - \alpha & b & 1 \\
\hline
0 & b & a & 1 & a & 1
\end{tabular}
\end{center}

Since $\leq_1$ and $\leq_2$ are total orders, there exists a unique permutation $\pi$ of $\mathcal{A}$ such that $a <_1 b$ if and only if $\pi(a) <_2 \pi(b)$. Conversely, $\leq_2$ is determined by $\leq_1$ and $\pi$ and $\leq_1$ is determined by $\leq_2$ and $\pi$. The permutation $\pi$ is said to be associated with $T$.

If we set $\mathcal{A} = \{a_1, a_2, \ldots, a_s\}$ with $a_1 <_1 a_2 <_1 \cdots <_1 a_s$, the pair $(\lambda, \pi)$ formed by the family $\lambda = (\lambda_a)_{a \in \mathcal{A}}$ and the permutation $\pi$ determines the map $T$. We will also denote $T$ as $T_{\lambda, \pi}$. The transformation $T$ is also said to be an $s$-interval exchange transformation.

Example 35. A 3-interval exchange transformation where the associated permutation is the cycle $\pi = (abc)$.

\begin{center}
\begin{tabular}{c}
0 & a & b & c & 1 \\
\hline
0 & b & c & a & 1
\end{tabular}
\end{center}
6.2.1 Regular interval exchange transformations

The orbit of a point \( z \in [0, 1] \) is the set \( \{T^n(z) \mid n \in \mathbb{Z} \} \). The transformation \( T \) is said to be minimal if, for any \( z \in [0, 1] \), the orbit of \( z \) is dense in \( [0, 1] \).

Set \( \mathcal{A} = \{a_1, a_2, \ldots, a_s\} \) with \( a_1 < a_2 < \cdots < a_s \), \( \mu_i = \mu_{a_i} \) and \( \kappa_i = \kappa_{a_i} \). The points \( 0, \mu_1, \ldots, \mu_{s-1} \) form the set of separation points of \( T \), denoted \( \text{Sep}(T) \). Note that the singular points of the transformation \( T \) (that is the points \( z \in [0, 1] \) at which \( T \) is not continuous) are among the separation points but that the converse is not true in general (see Example 35).

An interval exchange transformation \( T_{\lambda, \pi} \) is called regular if the orbits of the nonzero separation points \( \mu_1, \ldots, \mu_{s-1} \) are infinite and disjoint. Note that the orbit of 0 cannot be disjoint of the others since one has \( T(\mu_i) = 0 \) for some \( i \) with \( 1 \leq i \leq s-1 \). A regular interval exchange transformation is also said to be without connections or to satisfy the idoc condition (where idoc stands for infinite disjoint orbit condition). Note that since \( \kappa_2 = T(\mu_1), \ldots, \kappa_s = T(\mu_{s-1}) \), \( T \) is regular if and only if the orbits of \( \kappa_2, \ldots, \kappa_s \) are infinite and disjoint. As an example, the 2-interval exchange transformation of Example 34 which is the rotation of angle \( \alpha \) is regular if and only if \( \alpha \) is irrational. The following result is due to Keane.

**Theorem 36** (Keane [Kea75]). A regular interval exchange transformation is minimal.

The converse is not true. Indeed, consider the rotation of angle \( \alpha \) with \( \alpha \) irrational, as a 3-interval exchange transformation with \( \lambda = (1 - 2\alpha, \alpha, \alpha) \) and \( \pi = (132) \). The transformation is minimal as any rotation of irrational angle but it is not regular since \( \mu_1 = 1 - 2\alpha, \mu_2 = 1 - \alpha \) and thus \( \mu_2 = T(\mu_1) \).

The following necessary condition for minimality of an interval exchange transformation is useful. A permutation \( \pi \) of an ordered set \( \mathcal{A} \) is called decomposable if there exists an element \( b \in \mathcal{A} \) such that the set \( \mathcal{B} \) of elements strictly less than \( b \) is non-empty and such that \( \pi(\mathcal{B}) = \mathcal{B} \). Otherwise it is called indecomposable. If an interval exchange transformation \( T = T_{\lambda, \pi} \) is minimal, the permutation \( \pi \) is indecomposable. Indeed, if \( \mathcal{B} \) is a set as above, the set \( S = \cup_{a \in \mathcal{B}} I_a \) is closed under \( T \) and strictly included in \([0, 1] \). The following example shows that the indecomposability of \( \pi \) is not sufficient for \( T \) to be minimal.

**Example 37.** Let \( \mathcal{A} = \{a, b, c\} \) and \( \lambda \) be such that \( \lambda_a = \lambda_b \). Let \( \pi \) be the transposition \((ac)\). Then \( \pi \) is indecomposable but \( T_{\lambda, \pi} \) is not minimal since it is the identity on \( I_b \).

6.2.2 Natural coding and planar dendric subshifts

Let \( T \) be an interval exchange transformation relative to \( (I_a)_{a \in \mathcal{A}} \). We say that a word \( w = b_0b_1 \cdots b_{m-1} \in \mathcal{A}^* \) is admissible for \( T \) if the set
\[
I_w = I_{b_0} \cap T^{-1}(I_{b_1}) \cap \cdots \cap T^{-m+1}(I_{b_{m-1}})
\]
is non-empty. The language of \( T \) is the set \( \mathcal{L}_T \) of admissible words for \( T \). It uniquely defines the subshift
\[
X_T = \{x \in \mathcal{A}^\mathbb{Z} \mid \mathcal{L}(x) \subset \mathcal{L}_T\}
\]
that we call as the natural coding of \( T \). If \( T \) is regular, then \( X_T \) is minimal, hence \( X_T = \{x \in \mathcal{A}^\mathbb{Z} \mid \mathcal{L}(x) = \mathcal{L}_T\} \). Ferenczi and Zamboni gave the following combinatorial characterization of natural codings of regular interval exchange transformations.
Theorem 38 (Ferenczi and Zamboni [FZ08]). A subshift $X$ over $A$ is the natural coding of a regular interval exchange transformation with orders $\leq_1$ and $\leq_2$ if and only if it is minimal, $A \subset L(X)$ and it satisfies the following conditions:

1. for every $w \in L(X)$, $E^X_X(w)$ is an interval for $\leq_2$ and $E^X_X(w)$ is an interval for $\leq_1$;
2. for every $w \in L(X)$ and all $(a_1, b_1), (a_2, b_2) \in E_X(w)$, if $a_1 \leq a_2$, then $b_1 \leq b_2$;
3. for every $w \in L(X)$ and all $a_1, a_2 \in E^*_X(w)$, if $a_1$ and $a_2$ are consecutive for $\leq_2$, then $E^*_X(a_1w) \cap E^*_X(a_2w)$ is a singleton.

This result can be reformulated as follows. Let $\leq_L$ and $\leq_R$ be two total orders on $A$. A factor $w$ of a subshift $X \subset A^\mathbb{Z}$ is said to be planar for $(\leq_L, \leq_R)$ if for all $(a_1, b_1), (a_2, b_2) \in E_X(w)$, if $a_1 \leq_L a_2$, then $b_1 \leq_R b_2$. This implies that, placing the left and right vertices of $E_X(w)$ on parallel lines and ordering them respectively by $\leq_L$ and $\leq_R$, the edges of $E_X(w)$ may be drawn as straight noncrossing segments, resulting in a planar graph. A minimal dendric subshift over $A$ is said to be planar if there exists two total orders $\leq_L$ and $\leq_R$ on $A$ such that every $w \in L(X)$ is planar for $(\leq_L, \leq_R)$. Theorem 38 states that a subshift $X$ over $A$ is the natural coding of a regular interval exchange transformation if and only if $X$ is a minimal planar dendric subshift such that $A \subset L(X)$ [BDFD+15b]. The following example shows that the Tribonacci dendric subshift is not planar.

Example 39. Let $X$ be the Tribonacci subshift obtained with the substitution $\sigma$ defined by $\sigma(1) = 12$, $\sigma(2) = 13$, $\sigma(3) = 1$. The extension graphs of $\varepsilon$, 1 and 121 are given below and it is easily seen that it is not possible to find two orders on $A_3$ making the three graphs planar.

\[
\begin{align*}
E_X(\varepsilon) & \quad E_X(1) & \quad E_X(121) \\
1 & 3 & \quad 2 & 1 & \quad 3 & 2 \\
2 & \quad 3 & 2 & 1 & \quad 3 & 1 \\
3 & \quad 1 & \quad 1 & 2 & \quad 2 & 3
\end{align*}
\]

Observe that a regular interval exchange provides orders for which the natural coding is planar dendric: $(\leq_L, \leq_R) = (\leq_2, \leq_1)$. It is also clear that if a minimal dendric subshift $X$ is planar for the orders $(\leq_L, \leq_R)$, then it is also planar for the dual orders $(\leq_L^*, \leq_R^*)$, where the dual order $\leq^*$ of $\leq$ is defined by $x \leq^* y \iff y \leq x$. We now show that the pairs $(\leq_L, \leq_R)$ and $(\leq_L^*, \leq_R^*)$ are the only possibilities.

Proposition 40. Let $X$ be a minimal dendric subshift which is planar with respect to the orders $(\leq_L, \leq_R)$.

1. Any long enough left special factor $w$ is such that $E^*_X(w)$ is equal to $\{a, b\}$, where $a, b$ are consecutive for $\leq_L$. Furthermore, for every two $\leq_L$-consecutive letters $a, b$ and for all $n \in \mathbb{N}$, there is a (unique) left special factor $w$ of length $n$ such that $\{a, b\} \subset E^*_X(w)$.
2. The same holds on the right, i.e., any long enough right special factor $w$ is such that $E^*_X(w)$ is equal to $\{a, b\}$, where $a, b$ are consecutive for $\leq_R$. Furthermore, for every two $\leq_R$-consecutive letters $a, b$ and for all $n \in \mathbb{N}$, there is a (unique) right special factor $w$ of length $n$ such that $\{a, b\} \subset E^*_X(w)$.
Proof. By Theorem 38, $X$ is the natural coding of a regular interval exchange transformation $T$ relative to $(I_w)_{w \in A}$. For every $n \in \mathbb{N}$, the transformation $T^n$ is a regular interval exchange transformation relative to $(I_w)_{w \in L_n(X)}$ [BDFD+15b]. For each $w$, we set $J_w = T^{|w|} I_w$ and $\kappa_w$ the left boundary of $J_w$. By definition, for every $w \in L(X)$, the word $w$ is admissible and for all $a, b \in A$, we have

\[ a \in E_X^-(w) \Leftrightarrow J_a \cap I_w \neq \emptyset; \]

\[ b \in E_X^+(w) \Leftrightarrow J_b \cap I_w \neq \emptyset. \]

Therefore, $w$ is left special if and only if $I_w$ contains a point $\kappa_a$, $a \in A$, as an interior point. In this case, $a$ and its $\leq_L$-predecessor belong to $E_X^-(w)$. Similarly, it is right special if and only if $J_w$ contains a point $\kappa_b$, $b \in A$, as an interior point. In this case, $b$ and its $\leq_R$-predecessor belong to $E_X^+(w)$. By minimality, we have $\min_{w \in L_n(X)} |I_w| = \min_{w \in L_n(X)} |J_w| \to 0$ whenever $n$ goes to infinity. Therefore, for all large enough $n$, every interval $I_w$ or $J_w$, $w \in L_n(X)$, contains at most one point $\kappa_a$ and at most one point $\kappa_b$, $a, b \in A$, as interior points, showing that $E_X^-(w), E_X^+(w) \leq 2$ and that the (left or right) extensions of $w$ are consecutive. Since both $(I_w)_{w \in L_n(X)}$ and $(J_w)_{w \in L_n(X)}$ are partitions of $[0,1]$, every $\kappa_a$ and every $\kappa_b$ belong to some intervals $I_w$ and $J_w'$ and they are interior points because $T$ is regular. This ends the proof. \[ \square \]

Corollary 41. Let $X$ be a minimal dendric subshift. If $X$ is planar with respect to the orders $(\leq_L, \leq_R)$ and $(\geq_L, \geq_R)$, then either $(\leq_L, \leq_R) = (\geq_L, \geq_R)$ or $(\leq_L, \leq_R) = (\lessdot_L, \lessdot_R)$.

Proof. This follows from Proposition 40. The left extensions of the long enough left special factors determine the letters that are consecutive for $\leq_L$. Then, the dual order $\leq_L^*$ is the unique order that has exactly the same set of consecutive letters. The same holds for $\leq_R$-consecutive letters. It then suffices to observe that if $X$ is planar with respect to $(\leq_L, \leq_R)$, then it is not planar with respect to $(\leq_L^*, \leq_R)$, nor to $(\leq_L, \leq_R^*)$. Indeed, let $x_m, y_m$ (resp., $x_M, y_M$) denote the minimal (resp., maximal) elements for $\leq_L$ and $\leq_R$. As $E_X(\varepsilon)$ is planar for $(\leq_L, \leq_R)$, both $(x_m, y_m)$ and $(x_M, y_M)$ are edges of $E_X(\varepsilon)$ thus it is not planar for $(\leq_L^*, \leq_R^*)$, nor for $(\leq_L, \leq_R^*)$ as $x_m <_L x_M$ and $y_m <_R y_M$.

Notice that, on the ternary alphabet $A_3$, if the pair of orders $(\leq_L, \leq_R)$ can correspond to a regular interval exchange, then the set $\{(\leq_L, \leq_R), (\leq_L^*, \leq_R^*)\}$ is completely determined by the middle letters of $\leq_L$ and $\leq_R$, i.e. the letters $l, r \in A_3$ for which there exist $a, b, c, d \in A_3$ such that

\[ a <_L l <_L b \quad \text{and} \quad c <_R r <_R d. \]

Indeed, if $l, r$ are the middle letters of two orders $(\leq_L, \leq_R)$ that can correspond to a regular interval exchange $T_{\lambda, \pi}$, then, for any $x \in A_3 \setminus \{l, r\}$, one cannot have

\[ (x <_L l \quad \text{and} \quad x <_R r) \quad \text{or} \quad (l <_L x \quad \text{and} \quad r <_R x), \]

as otherwise the permutation $\pi$ would be decomposable. Thus, if $\{y_L\} = A_3 \setminus \{x, l\}$ and $\{y_R\} = A_3 \setminus \{x, r\}$, the two possible pairs of orders are

\[ (x <_L l <_L y_L, y_R <_R r <_R x) \quad \text{and} \quad (y_L <_L l <_L x, x <_R r <_R y_R) \]

and the set $\{(\leq_L, \leq_R), (\leq_L^*, \leq_R^*)\}$ is unique. Let us denote it $o(l, r)$. Remark that, for every pair $(l, r)$, the set $o(l, r)$ is non empty as we can always find a regular interval exchange with the middle intervals labeled by these letters.
6.2.3 $S_3$-adic representations of regular 3-interval exchange transformations

From Theorem 38, we know that the natural coding of a regular 3-interval exchange transformation is a minimal ternary dendric subshift. By Theorem 11 it thus admits a primitive $S_3$-adic representation which labels a path in $\mathcal{G}$. In this section, we characterize those labeled paths in $\mathcal{G}$.

**Lemma 42.** If $X$ is a minimal planar dendric subshift over $\mathcal{A}_3$, then its $S_3$-adic representation does not belong to $S^*_3(S^N_{LI} \cup S^N_{RI})$. In particular, $X$ belongs to $[l, r]$ for some $l, r \in \mathcal{A}_3$.

*Proof.* Let $\sigma$ be an $S_3$-adic representation of $X$. By Propositions 29 and 10 $\sigma$ does not belong to $S^N_{LI} \cup S^N_{RI}$. To prove that $\sigma$ does not belong to $S^*_3(S^N_{LI} \cup S^N_{RI})$, we observe that for all $N \geq 0$, $X^{(N+1)}$ is a minimal planar dendric subshift over $\mathcal{A}_3$. Indeed, $X^{(N+1)}$ is a derived subshift of $X^{(N)}$ and, by Theorem 4.13 and Theorem 38, every derived subshift of a minimal planar dendric subshift over $\mathcal{A}_3$ is a minimal planar dendric subshift over $\mathcal{A}_3$. The second part of the statement directly follows from Proposition 29.

The next lemma relates the orders of a minimal planar dendric subshift $X$ to its equivalence class $[\mathcal{A}_l(X), \mathcal{A}_R(X)]$.

**Lemma 43.** Let $X$ be a subshift over $\mathcal{A}_3$ and $v \in \mathcal{L}(X)$ a dendric bispecial factor which is planar for the orders $(\leq_L, \leq_R)$.

1. If $\mathcal{A}_l(v) = l$ for $l \in \mathcal{A}_3$, then $(\leq_L, \leq_R) \in o(l, r)$ for some $r \in \mathcal{A}_3$;

2. If $\mathcal{A}_R(v) = r$ for $r \in \mathcal{A}_3$, then $(\leq_L, \leq_R) \in o(l, r)$ for some $l \in \mathcal{A}_3$.

In particular, if $X$ is a planar dendric subshift of $[l, r]$, $l, r \in \mathcal{A}_3$, then $X$ is only planar for the orders of $o(l, r)$.

*Proof.* Let us first treat the case where $\mathcal{A}_L(v) = l$, the other case is symmetric. Consider $a, b$ such that $\mathcal{A}_3 = \{a, b, l\}$. Since $\mathcal{A}_L(v) = l$, by Remark 19 there is a simple path of length 4 from $a_L$ to $b_L$ in $\mathcal{E}_X(v)$; we denote it by $(a_L, x, r, a_L, y, r, b_L)$. Thus we have $(a, x), (l, x), (l, y), (b, y) \in E_X(v)$. Since $v$ is planar for $(\leq_L, \leq_R)$, we have

$$(a <_L l \text{ and } x <_R y) \text{ or } (l <_L a \text{ and } y <_R x)$$

and

$$(l <_L b \text{ and } x <_R y) \text{ or } (b <_L l \text{ and } y <_R x).$$

Thus, we have the two following possibilities

$$(a <_L l <_L b \text{ and } x <_R y) \text{ or } (b <_L l <_L a \text{ and } y <_R x).$$

In both cases, $l$ is the middle letter for $\leq_L$, which implies that $(\leq_L, \leq_R) \in o(l, r)$ for some $r \in \mathcal{A}_3$.

If $X$ is in the class $[l, r]$ then there exist $v, v' \in \mathcal{L}(X)$ such that $\mathcal{A}_L(v) = l$ and $\mathcal{A}_R(v') = r$. Using the first part of the result, any pair of orders for which $X$ is planar is in $o(l, r)$.

**Corollary 44.** Let $Y$ be any subshift over $\mathcal{A}_3$ such that $\mathcal{L}_1(X) = \mathcal{A}_3$ and let $X$ be the image of $Y$ under some morphism $\sigma \in S_3$. 

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1. If $\sigma \in \{ \alpha_a \mid a \in A_3 \}$, then up to taking the dual orders, there are four pairs of orders for which $\varepsilon$ is planar (in $X$); they are represented in Table 6.

2. If $\sigma \notin \{ \alpha_a \mid a \in A_3 \}$, then up to taking the dual orders, there is a unique pair of orders for which $\varepsilon$ is planar (in $X$); they are represented in Table 7. In addition, the other initial bispecial factors of $X$ are also planar for these orders.

Table 6: Planar extension graphs of $\varepsilon$ in images under $\alpha_a$

Table 7: Planar extension graphs of $\varepsilon$ in images under $\sigma$ for $\sigma \notin \{ \alpha_a \mid a \in A_3 \}$

Proof. It is clear that the pairs of orders given in Tables 6 and 7 make $E_X(\varepsilon)$ planar.

Whenever $\sigma = \alpha_a$, it is easily seen that for $\varepsilon$ to be planar, the letter $a$ must be minimal for $\leq_L$ and maximal for $\leq_R$ or vice-versa. It then suffices to check that Table 6 gives all such pairs of orders (when also considering the dual orders).

Whenever $\sigma = \beta_{abc}$, $\{a, b, c\} = A_3$, Table 5 shows that $A_L(\varepsilon) = a$ thus, by Lemma 13, $a$ must be the middle letter of the $\leq_L$. One can easily check that $\leq_R$ is then completely determined. The case $\sigma = \gamma_{abc}$ is symmetric.

Whenever $\sigma \notin \{ \alpha_a, \beta_{abc}, \gamma_{abc} \mid \{a, b, c\} = A_3 \}$, we have $A_L(\varepsilon), A_R(\varepsilon) \neq 0$. This case thus directly follows from Lemma 13.

Recall that the non-empty initial bispecial factors of $X$ are non prefix factors of a word in $\sigma(A_3)$. The only such factors that are bispecial are

- $c^l, l \leq k$ when $\sigma = \delta_{abc}^{(k)}$;
- $c^l, l \leq k$ when $\sigma = \zeta_{abc}^{(k)}$.

Their extension graphs are represented in Table 8 and one can see that they are indeed planar for the orders that make $E_X(\varepsilon)$ planar.
For every $\sigma \in S_3$, we let $o(\sigma)$ denote the set of pairs of orders for which $\varepsilon$ is planar in images under $\sigma$. The following is a direct consequence of Lemmas 42 and 43 and Corollary 44.

**Proposition 45.** If $X$ is a minimal dendric planar subshift over $A_3$, then it has an $S_3$-adic representation $(\sigma_n)_{n \geq 1}$ labeling a path $([l_n, r_n])_{n \geq 1}$ in $G$ such that, for each $n \geq 1$, $X^{(n)}$ is planar for the orders of $o(l_n, r_n)$ and $o(l_n, r_n) \subset o(\sigma_n)$.

It is clear that not every path in $G$ has a label $(\sigma_n)_{n \geq 1}$ which is the $S_3$-adic representation of a minimal ternary planar dendric subshift $X$. One reason is given by Proposition 45: the morphism $\sigma_1 \in S_3$ may label edges whose starting vertex $[l, r]$ does not satisfy $o(l, r) \subset o(\sigma_1)$. For instance, the morphism $\beta_{123}$ labels two edges leaving the vertex $[1, 1]$, two edges leaving the vertex $[1, 2]$ and two edges leaving the vertex $[1, 3]$ but only $[1, 3]$ satisfies $o(l, r) \subset o(\beta_{123})$.

We will prove that another reason is that, even though the morphism $\sigma_1$ may label several edges leaving the vertex $[A_L(X), A_R(X)]$, the class of the subshift $X^{(2)}$ is uniquely defined by $\sigma$ and by $[A_L(X), A_R(X)]$.

This motivates the following definitions. Observe that they are valid on an alphabet of any size.

If $\leq L$ and $\leq R$ are two total orders on $A$, a partial map $\varphi : A \to A$ is order preserving from $\leq L$ to $\leq R$ if

1. for all $s \in S(\sigma)$, $\varphi_s^{(L)}$ is order preserving from $\leq L$ to $\leq L$;
2. for all $p \in P(\sigma)$, $\varphi_p^{(R)}$ is order preserving from $\leq R$ to $\leq R$.

In this definition, the graph morphisms $\varphi_s^{(L)}$ and $\varphi_p^{(R)}$ are seen as applications on the letters.

From the graph point of view, saying that $\varphi_s^{(L)}$ is order preserving from $\leq L$ to $\leq R$ means that, starting from a bipartite graph ordered by $\leq L$ on the left and applying $\varphi_s^{(L)}$, the vertices in the image graph will be ordered by $\leq L$ on the left.

**Example 46.** Let $\sigma = \eta_{abc}$. By Table 2, the associated morphisms $\varphi_s^{(L)}$ and $\varphi_p^{(R)}$ are given by

$$\varphi_\varepsilon^{(L)} : \begin{cases} a \mapsto b \\ b, c \mapsto c \end{cases}, \quad \varphi_c^{(L)} : \begin{cases} b \mapsto b \\ c \mapsto a \end{cases}, \quad \varphi_a^{(R)} : \begin{cases} a, b \mapsto b \\ c \mapsto c \end{cases}, \quad \varphi_{ab}^{(R)} : \begin{cases} a \mapsto a \\ b \mapsto c \end{cases}.$$ 

Let us consider the following orders on $A_3$:

$$b <_L c <_L a, \quad c <_R a <_R b, \quad c <_L b <_L a \quad \text{and} \quad b <_R a <_R c.$$ 

By definition, both $\varphi_\varepsilon^{(L)}$ and $\varphi_c^{(L)}$ are order preserving from $\leq L$ to $\leq L$ and $\varphi_a^{(R)}$ is order preserving from $\leq R$ to $\leq R$ but $\varphi_{ab}^{(R)}$ is not. Thus $\sigma$ is not planar preserving from $(\leq L, \leq R)$ to $(\leq L, \leq R)$.

If $v$ is a word whose extension graph is given by

```
  b --------- c
 /             \
 c ----- a     \
 \
 a --------- b
```
then the extension graphs of its extended images $\sigma(v)a$, $\sigma(v)ab$, $\sigma\sigma(v)a$ and $\sigma\sigma(v)ab$ are given respectively by

\[
\begin{array}{ccc}
  c & \rightarrow & b \\
  b & \rightarrow & c
\end{array}
\quad
\begin{array}{ccc}
  a & \rightarrow & c \\
  b & \rightarrow & a
\end{array}
\quad
\begin{array}{ccc}
  b & \rightarrow & c \\
  a & \rightarrow & c
\end{array}
\quad
\begin{array}{ccc}
  c & \rightarrow & b \\
  a & \rightarrow & c
\end{array}
\]

It is easy to see that $v$ is planar for $(\leq_L, \leq_R)$ and that only two of its extended images are planar for $(\leq_L, \leq_R)$.

The terminology planar preserving comes from the fact that the situation of the previous example does not occur if $\sigma$ is planar preserving from $(\leq_L, \leq_R)$ to $(\leq_L, \leq_R)$. This is formalized in the following result.

**Proposition 47.** Let $X$ be a dendric subshift, $v \in \mathcal{L}(X)$ and $\sigma \in \text{DP}(v)$ which is planar preserving from $(\leq_L, \leq_R)$ to $(\leq_L, \leq_R)$. Then $v$ is planar for $(\leq_L, \leq_R)$ if and only if every bispecial extended image of $v$ under $\sigma$ is planar for $(\leq_L, \leq_R)$.

**Proof.** Let $Y$ denote the image of $X$ by $\sigma$. Let us first assume that $\mathcal{E}_X(v)$ is a tree but not planar for $(\leq_L, \leq_R)$. There exist $x'_1, x'_2, y'_1, y'_2 \in \mathcal{A}$ such that $(x'_1, y'_1), (x'_2, y'_2)$ belong to $E_X(v)$ and are crossing edges in $\mathcal{E}_X(v)$, i.e., they satisfy

\[ x'_1 \prec_L x'_2 \quad \text{and} \quad y'_2 \prec_R y'_1. \]

If $s = s(x'_1, x'_2)$ and $p = p(y'_1, y'_2)$, then the extension graph of the bispecial extended image $u = s\sigma(v)p$ of $v$ is the image of a subgraph of $\mathcal{E}_X(v)$ by the morphisms $\varphi^{(L)}_s$ and $\varphi^{(R)}_p$. If we denote

\[
\begin{align*}
  x_1 &= \varphi^{(L)}_s(x'_1), \\
  x_2 &= \varphi^{(L)}_s(x'_2), \\
  y_1 &= \varphi^{(R)}_p(y'_1) \\
  y_2 &= \varphi^{(R)}_p(y'_2),
\end{align*}
\]

then $(x_1, y_1)$ and $(x_2, y_2)$ belong to $E_Y(u)$ and, since $\sigma$ is planar preserving from $(\leq_L, \leq_R)$ to $(\leq_L, \leq_R)$, they are crossing edges of $\mathcal{E}_Y(u)$, i.e.,

\[ x_1 \prec_L x_2 \quad \text{and} \quad y_2 \prec_R y_1. \]

Thus $v$ has a bispecial extended image whose extension graph is not planar for $(\leq_L, \leq_R)$.

Assume now that there exists a bispecial extended image $u$ of $v$ whose extension graph is not planar for $(\leq_L, \leq_R)$. As $\sigma$ is in $\text{DP}(v)$, the graph $\mathcal{E}_Y(u)$ is a tree. If $\mathcal{E}_Y(u)$ is not planar for $(\leq_L, \leq_R)$, there exist $x_1, x_2, y_1, y_2 \in \mathcal{A}$ such that $(x_1, y_1), (x_2, y_2) \in E_Y(u)$ are crossing edges in $\mathcal{E}(u)$, i.e.,

\[ x_1 \prec_L x_2 \quad \text{and} \quad y_2 \prec_R y_1. \]

By Proposition 13, the graph $\mathcal{E}_Y(u)$ is the image of a subgraph of $\mathcal{E}_X(v)$ by some morphisms $\varphi^{(L)}_s$ and $\varphi^{(R)}_p$. Thus there exist $x'_1, x'_2, y'_1, y'_2 \in \mathcal{A}$ such that $(x'_1, y'_1), (x'_2, y'_2)$ belong to $E_X(v)$ and satisfy

\[
\begin{align*}
  \varphi^{(L)}_s(x'_1) &= x_1, \\
  \varphi^{(L)}_s(x'_2) &= x_2, \\
  \varphi^{(R)}_p(y'_1) &= y_1 \\
  \varphi^{(R)}_p(y'_2) &= y_2.
\end{align*}
\]

Since $\sigma$ is planar preserving from $(\leq_L, \leq_R)$ to $(\leq_L, \leq_R)$, we must have, $x'_1 \prec_L x'_2$ and $y'_2 \prec_R y'_2$, which contradicts the fact that $\mathcal{E}_X(v)$ is planar for $(\leq_L, \leq_R)$. \qed
Observe that $\sigma$ is planar preserving from $(\preceq_L, \preceq_R)$ to $(\leq_L, \leq_R)$ if and only if it is planar preserving from $(\leq_L^*, \leq_R^*)$ to $(\leq_L^*, \leq_R^*)$.

Thus, in the ternary case we say that $\sigma$ is planar preserving from $o(l, r)$ to $o(l', r')$ if it is planar preserving from $(\preceq_L, \preceq_R)$ to $(\leq_L, \leq_R)$ for some $(\preceq_L, \preceq_R) \in o(l, r)$ and some $(\leq_L, \leq_R) \in o(l', r')$. The next result will imply that the subgraph of $G$ representing the $S_3$-adic representations of minimal ternary planar dendric subshifts is deterministic.

**Lemma 48.** For every $\sigma \in S_3$ and every $l, r \in A_3$, there exists a unique pair $(l', r') \in A_3^2$ such that $\sigma$ is planar preserving from $o(l', r')$ to $o(l, r)$.

**Proof.** We actually prove the following stronger result: for every $\sigma \in S_3$ and every total order $\leq_L$ on $A_3$, there is a unique total order $\preceq_L$ on $A_3$ such that for all $s \in S(\sigma)$, $\varphi_s^{(L)}$ is order preserving from $\preceq_L$ to $\leq_L$. Similarly, for every total order $\leq_R$ on $A_3$, there is a unique total order $\preceq_R$ on $A_3$ such that for all $p \in P(\sigma)$, $\varphi_p^{(R)}$ is order preserving from $\preceq_R$ to $\leq_R$.

This follows from the fact that either $S(\sigma) = \{s_0\}$, in which case $\varphi_s^{(L)}$ is the identity, or $S(\sigma) = \{s_0, s\}$ and $\varphi_{s_0}^{(L)}(A_3)$ and $\varphi_s^{(L)}(A_3)$ both contain two letters (and not the same). The same holds on the right. \qed

The next example shows how we can build $(\preceq_L, \preceq_R)$ from $(\leq_L, \leq_R)$ and $\sigma \in S_3$ such that $\sigma$ is planar preserving from $(\leq_L, \leq_R)$ to $(\leq_L, \leq_R)$.

**Example 49.** The morphism $\delta^{(k)}_{abc}$ labels edges from $[c, a]$ to $[x, y]$ for all $x, y \in \{b, c\}$ (see Figure 9). Up to considering the dual orders, the pair of orders in $o(c, a)$ is

$$a <_L c <_L b \quad \text{and} \quad b <_R a <_R c.$$  

From Table 3, we have $S(\delta^{(k)}_{abc}) = \{\varepsilon, c^k\}$, $P(\delta^{(k)}_{abc}) = \{a, abc^k\}$ and

$$\varphi_{\varepsilon}^{(L)} : \begin{cases} a \mapsto a \\ b, c \mapsto c \end{cases}, \quad \varphi_{c^k}^{(L)} : \begin{cases} b \mapsto b \\ c \mapsto c \end{cases}, \quad \varphi_{a}^{(R)} : \begin{cases} a \mapsto a \\ b, c \mapsto b \end{cases}, \quad \varphi_{abc}^{(R)} : \begin{cases} b \mapsto a \\ c \mapsto c \end{cases}.$$  

We have $\varphi_{\varepsilon}^{(L)}(b) = \varphi_{\varepsilon}^{(L)}(c) = c$, $\varphi_{c^k}^{(L)}(a) = a$ and $a <_L c$. Thus for $\varphi_{\varepsilon}^{(L)}$ to be order preserving from $\preceq_L$ to $\leq_L$, we must have $a <_L b, c$. Then, from $\varphi_{c^k}^{(L)}(c) = c <_L b = \varphi_{c^k}^{(L)}(b)$, we deduce that we must have $c <_L b$, so that

$$a <_L c <_L b.$$  

We similarly deduce that for $\delta^{(k)}_{abc}$ to be planar preserving from $(\preceq_L, \preceq_R)$ to $(\leq_L, \leq_R)$, we must have

$$b <_R c <_R a.$$  

Therefore, $\delta^{(k)}_{abc}$ is planar preserving from $o(c, a)$ to $o(x, y)$ if and only if $(x, y) = (c, c)$.

The list of all edges $([l, r], \sigma, [l', r'])$ of $G$ for which $o(l, r) \subset o(\sigma)$ and $\sigma$ is planar preserving from $o(l', r')$ to $o(l, r)$ is given in Table 3.

**Lemma 50.** Let $X \in [l, r]$ and $Y \in [l', r']$ be two ternary dendric planar subshifts such that $X$ is the image of $Y$ under $\sigma$ for $\sigma \in S_3$. Then $\sigma$ is planar preserving from $o(l', r')$ to $o(l, r)$.  

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Table 9: List of edges ([l, r], [l', r']) of G for which o(l, r) ⊂ o(σ) and σ is planar preserving from o(l', r') to o(l, r)

| σ | α_1 | β_abc | γ_abc | δ_abc | ζ_abc | η_abc |
|---|---|---|---|---|---|---|
| [l, r] | [b, b] | [b, c] | [c, b] | [c, c] | [a, c] | [c, a] |
| [l', r'] | [b, b] | [b, c] | [c, b] | [c, c] | [c, c] | [c, a] |

Proof. As X is planar, it is for the orders of o(l, r) thus o(l, r) ⊂ o(σ). By the previous lemma, there exists a unique pair of letters ([l''', r''']) such that σ is planar preserving from o(l''', r''') to o(l, r). For all v ∈ L(Y), its bispecial extended images are in L(X) thus are planar for the orders of o(l, r). Using Proposition 44, v is planar for the orders of o(l'', r''). As it is true for any v ∈ L(Y), Y is planar for the orders of o(l'', r'') and, by Lemma 43, l'' = l' and r'' = r', thus σ is planar preserving from o(l', r') to o(l, r).

We can now prove a result similar to Theorem 1 but in the case of interval exchanges. The graph that we obtain is the subgraph of the graph G in Figure 9 (where we only kept the edges ([l, r], [l', r']) such that o(l, r) ⊂ o(σ) and σ is planar preserving from o(l', r') to o(l, r). These edges are given by Table 9.

Theorem 51. A subshift X over A_3 is a coding of a regular interval exchange if and only if it has a primitive S_3-adic representation σ ∈ S_3^n that labels a path in the graph represented in Figure 11.

Proof. Let us assume that X is the coding of a regular interval exchange. From Proposition 45, X has an S_3-adic representation σ = (σ_n)_{n≥1} labeling a path ([l_n, r_n])_{n≥1} in the graph G such that, for each n ≥ 1, X^{(n)} is planar exactly for the orders of o(l_n, r_n) and o(l_n, r_n) ⊂ o(σ_n). By Lemma 50, σ_n is planar preserving from o(l_{n+1}, r_{n+1}) to o(l_n, r_n). Thus, all the edges of the path are preserved in the subgraph represented in Figure 10.

Assume now that X has a primitive S_3-adic representation σ ∈ S_3^n labeling a path ([l_n, r_n])_{n≥1} in the graph of Figure 10. As this graph is a subgraph of the graph G, X is minimal dendric. Let us show that X is planar for the orders of o(l_1, r_1). Let (≤L, ≤R) be an element of o(l_1, r_1). The initial bispecial factor of X are planar for (≤L, ≤R) by Corollary 44 as (≤L, ≤R) ∈ o(l_1, r_1) ⊂ o(σ_1). Let u ∈ L(X) be a non-initial bispecial factor of X. Using Corollary 11, there is a unique n > 1 and a unique initial bispecial factor v in L(X^{(n)}_{σ}) such that u is a descendant of v. From Corollary 44, v is planar for any order of o(l_n, r_n) ⊂ o(σ_n). By construction of the subgraph and by Proposition 47, the extension graph of u is planar for (≤L, ≤R).

6.3 Cassaigne subshifts

A subshift X over A_3 is a Cassaigne subshift if it has a primitive C-adic representation, where C = {c_1, c_2} and

\[
\begin{align*}
   c_1 : & \begin{cases} 
       1 \mapsto 1 \\
       2 \mapsto 13 \\
       3 \mapsto 2
   \end{cases} \\
   c_2 : & \begin{cases} 
       1 \mapsto 2 \\
       2 \mapsto 13 \\
       3 \mapsto 3
   \end{cases}
\end{align*}
\]
Figure 10: A subshift over $A_3$ is the coding of an interval exchange if and only if it has a primitive $S_3$-adic representation labeling an infinite path in this graph. We stress the fact that, compared with Figure 9, the vertices $[1,3]$ and $[3,1]$ are exchanged.
Cassaigne subshifts are minimal ternary dendric subshifts and a directive sequence \((\sigma_n)_{n \geq 1} \in \mathcal{C}_N\) is primitive if and only if it cannot be eventually factorized over \(\{c_1^2, c_2^2\}\), i.e., there is no \(N \in \mathbb{N}\) such that for all \(n \geq N\), \(c_{N+2n} = c_{N+2n+1}\) \cite{LLL17}.

By considering products of morphisms, we obtain that a subshift is a Cassaigne subshift if and only if it has a primitive \(\mathcal{C}'\)-adic representation where \(\mathcal{C}' = \{c_{11}, c_{12}, c_{122}, c_{211}, c_{121}, c_{212}\}\) and

\[
\begin{align*}
c_{11} = c_1^2 : & \quad \begin{cases} 1 \mapsto 1, \\ 2 \mapsto 12, \\ 3 \mapsto 13 \end{cases} \\
c_{122} = c_1 c_2^2 : & \quad \begin{cases} 1 \mapsto 12, \\ 2 \mapsto 132, \\ 3 \mapsto 2 \end{cases} \\
c_{121} = c_1 c_2 c_1 : & \quad \begin{cases} 1 \mapsto 13, \\ 2 \mapsto 132, \\ 3 \mapsto 12 \end{cases} \\
c_{22} = c_2^2 : & \quad \begin{cases} 1 \mapsto 13, \\ 2 \mapsto 23, \\ 3 \mapsto 3 \end{cases} \\
c_{211} = c_2 c_1^2 : & \quad \begin{cases} 1 \mapsto 2, \\ 2 \mapsto 213, \\ 3 \mapsto 23 \end{cases} \\
c_{212} = c_2 c_1 c_2 : & \quad \begin{cases} 1 \mapsto 23, \\ 2 \mapsto 213, \\ 3 \mapsto 13 \end{cases}
\end{align*}
\]

Thus a subshift is a Cassaigne subshift if and only if it has a primitive \(\mathcal{S}\)-adic representation using morphisms from the set

\[
\{\alpha_1, \alpha_3, \gamma_{213} \pi_{132}, \beta_{231} \pi_{213}, \eta_{132} \pi_{132}, \eta_{321} \pi_{321}\}\]

or, equivalently, if and only if it has a primitive \(\mathcal{S}_C\)-adic representation where

\[
\mathcal{S}_C = \{\alpha_1, \alpha_3, \gamma_{213} \pi_{132}, \beta_{231} \pi_{213}, \eta_{132} \pi_{132}, \eta_{321} \pi_{321}\}.
\]

**Lemma 52.** Let \(X\) and \(Y\) be two dendric subshifts over \(\mathcal{A}_3\) such that \(Y = \pi(X)\) where \(\pi\) is a permutation on \(\mathcal{A}_3\). For any \(l, r \in \mathcal{A}_0\), if \(X \in [l, r]\), then \(Y \in [\pi(l), \pi(r)]\) where we take the convention \(\pi(0) = 0\).

**Proof.** It directly follows from Remark \[19\] \(\square\)

**Proposition 53.** There is no Cassaigne subshift which is an Arnoux-Rauzy or the coding of a regular interval exchange.

**Proof.** Let \(X\) be a Cassaigne subshift and \(\sigma = (\sigma_n)_{n \geq 1}\) be a primitive \(\mathcal{S}_C\)-adic representation of \(X\). Firstly, it is clear that \(X\) is not an Arnoux-Rauzy subshift. Using Proposition \[20\], it would indeed require \(\sigma\) to contain only left-invariant and right-invariant morphisms (extending these notions to morphisms in \(\mathcal{S}_C\)), hence to be in \(\{\alpha_1, \alpha_3\}^N\). It then suffice to observe that there is no primitive sequence in \(\{\alpha_1, \alpha_3\}^N\).

For each \(n\), let us denote

\[
\sigma_n = \mu_n \pi_n
\]

where \(\mu_n \in \mathcal{S}_3\) and \(\pi_n\) is a permutation on \(\mathcal{A}_3\). The sequence \(\sigma' = (\sigma_n')_{n \geq 1}\) where

\[
\sigma_{2n-1}' = \mu_n \quad \text{and} \quad \sigma_{2n}' = \pi_n
\]

is also a primitive \(\mathcal{S}\)-adic representation of \(X\). Let us assume that \(X\) is the coding of a regular interval exchange and prove that we reach a contradiction. By \cite{BDF+15} Theorem 4.13, as \(X_{\sigma_n^{(n+1)}}\) is either a derived subshift of \(X_{\sigma_n^{(n)}}\) if \(n\) is odd or a permutation of \(X_{\sigma_n^{(n)}}\) if \(n\) is even, it is the coding of a regular interval exchange for all \(n \geq 1\). Using Lemma \[42\], let us denote

\[
X_{\sigma_n^{(n)}} \in [l_n, r_n], \quad l_n, r_n \in \mathcal{A}_3.
\]
By Lemma 50, \( \mu_n \) is planar preserving from \( o(l_{2n-1}, r_{2n-1}) \) to \( o(l_{2n}, r_{2n}) \) and, by Lemma 52:

\[
l_{2n+1} = \pi_n(l_{2n}) \quad \text{and} \quad r_{2n+1} = \pi_n(r_{2n}).
\]

Thus, \( (l_{2n+1}, r_{2n+1}) \) can be fully determined by \( (l_{2n-1}, r_{2n-1}) \) and \( \sigma_n \) using Table 9. In conclusion, the sequence \( \sigma \) labels an infinite path in the following graph:

\[
\begin{array}{ccc}
[1,1] & \circlearrowright \alpha_3 & [1,2] \\
& \cup & [1,3] \\
\alpha_3 & \downarrow & \alpha_3 \\
[2,1] & \beta_{231}\pi_{213} & [2,2] \\
& \downarrow & \alpha_1, \alpha_3 \\
\eta_{321}\pi_{321} & \gamma_{213}\pi_{132} & [2,3] \\
[3,1] & \eta_{132}\pi_{132} & [3,2] \\
& \alpha_1 & \alpha_1 \\
& \eta_{132}\pi_{132} & [3,3]
\end{array}
\]

This is absurd as the only infinite paths in this graph belong to the set \( S^*_C \{ \alpha_1, \alpha_3 \}^\mathbb{N} \) and do not correspond to primitive sequences. \( \Box \)

7 Further work

The problem of finding an \( \mathcal{S} \)-adic characterization of minimal dendric subshift over larger alphabets is still open. For any fixed alphabet \( \mathcal{A}_k = \{1, 2, \ldots, k\} \), one could define a set \( \mathcal{S}_k \) of injective and strongly left proper morphisms over \( \mathcal{A}_k \) such that any minimal dendric subshift has a primitive \( \mathcal{S}_k \)-adic representation, thus extending Proposition 18. It is likely that there exists a graph \( G_k \) that allows to extend Theorem 1 but its definition is much more tricky. Indeed, a key point in the ternary case is that we were able to interpret Proposition 14 with the conditions \( C_L(a) \) and \( C_R(a) \) (see Section 5.2), which then allows to define the vertices of \( G \). What could be an equivalent interpretation on \( \mathcal{A}_k \) is not so clear.

Another interesting question would be to study the \( \mathcal{S} \)-adic representations obtained by factorizing the morphisms of \( \mathcal{S}_k \) into elementary automorphisms. Such factorizations always exist (see Theorem 5) and therefore yield to another graph \( G'_k \) where the edges are labeled by elementary automorphisms of \( F_{\mathcal{A}_k} \). Numerical computations show that the minimal graph obtained from the graph \( G \) of Theorem 1 has more than 200 vertices. Of course, this graph depends on the choices that we made for the set \( \mathcal{S}_3 \) of morphisms. Finding a handier graph in the ternary case is an open question.

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