ANALYTIC $m$-ISOMETRIES HAVING THE WANDERING SUBSPACE PROPERTY

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Abstract. In this article, generalizing a result due to Shimorin, we show that all $(m + 1)$-concave operators satisfying a certain set of operator inequalities must have Wold-type decomposition. As an application, generalizing a result of Richter, we show that an analytic $(m + 1)$-isometry with one dimensional co-kernel, which satisfies this set of operator inequalities can be represented as an operator of multiplication by the coordinate function on a Dirichlet-type space induced from an $m$-tuple of positive Borel measures on the unit circle.

1. Introduction

In what follows, $\mathcal{H}$ will denote a complex separable Hilbert space and $\mathcal{B}(\mathcal{H})$ will stand for the set of all bounded linear operators on $\mathcal{H}$. A closed subspace $W$ of $\mathcal{H}$ is called $T$-invariant if $TW \subseteq W$. Further, the hyper-range of $T$ is denoted by $\mathcal{H}_\infty(T)$ and is defined to be

$$\mathcal{H}_\infty(T) = \bigcap_{n=0}^{\infty} T^n(\mathcal{H}).$$

For an isometry $T$ in $\mathcal{B}(\mathcal{H})$, the classical Wold-Kolmogorov decomposition theorem says that $\mathcal{H}_\infty(T)$ is a reducing subspace for $T$ and $\mathcal{H}$ can be decomposed in the following way:

$$\mathcal{H} = \mathcal{H}_\infty(T) \oplus \mathcal{H}_a(T),$$

where $\mathcal{H}_a(T) = \bigoplus_{n=0}^{\infty} T^n(\ker T^*)$. Furthermore, $T|_{\mathcal{H}_\infty(T)}$ is a unitary operator and $T|_{\mathcal{H}_a(T)}$ is a unilateral forward shift operator [22]. In 2001, Shimorin [31] introduced Wold-type decomposition to study a much bigger class of operators than that of isometries. An operator $T$ in $\mathcal{B}(\mathcal{H})$ is said to admit Wold-type decomposition if the following statements hold:

(i) $\mathcal{H}_\infty(T)$ is a reducing subspace for $T$ and $T|_{\mathcal{H}_\infty(T)}$ is a unitary operator.

(ii) $\mathcal{H} = \mathcal{H}_\infty(T) \oplus [\ker T^*]_T$,

where for a non-empty subset $E$ of $\mathcal{H}$, $[E]_T$ denotes the smallest closed $T$-invariant subspace of $\mathcal{H}$ containing $E$ i.e. $[E]_T = \bigvee_{n \geq 0} T^n(E)$.

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An operator $T$ in $B(\mathcal{H})$ is called analytic (pure) if $\mathcal{H}_\infty(T) = \{0\}$. An analytic operator $T$ in $B(\mathcal{H})$ admits Wold-type decomposition if and only if

$$[\ker T^*]^\perp = \mathcal{H}. \quad (1)$$

Following [31, Definition 2.4], we say that an operator $T$ in $B(\mathcal{H})$ has wandering subspace property if it satisfies (1). The term “wandering subspace” is originally attributed to Halmos [15]. There is a prominent role of wandering subspaces in the structure theory of $z$-invariant subspaces of a Hilbert space of analytic functions. Now onwards we will use the notation $M_z$ to denote the multiplication operator by the coordinate function $z$ on a Hilbert space of analytic functions. A well known result of Beurling [7] states that every non-zero $M_z$-invariant subspace $W$ of the classical Hardy space $H^2$ is of the form $W = \theta H^2$ for some inner function $\theta \in H^2$. Moreover, the operator $M_z |_W$ has the wandering subspace property with wandering subspace equal to span$\{\theta\}$.

In 1988, Richter [26, Theorem 1] shows that if an analytic operator $T$ satisfies

$$T^* T - 2 | T^* T | + I \leq 0 \quad (2)$$

then $T$ has wandering subspace property. As a consequence, he described the structure of $M_z$-invariant subspaces of the classical Dirichlet space.

In [5], Aleman, Richter and Sundberg show that for every non-zero $M_z$-invariant subspace $W$ of the classical Bergman space $A^2(\mathbb{D})$, the operator $M_z |_W$ also has the wandering subspace property. Wandering subspace for $M_z |_W$ plays a crucial role in the study of the factorization theory of functions in the Bergman space, see [16]. In literature, wandering subspace property has been studied extensively (see for instances [24, 18, 19, 20].

In 2001, Shimorin shows that if an operator $T$ in $B(\mathcal{H})$ satisfies one of the following conditions:

(i) $T^* T - 2 | T^* T | + I \leq 0$,

(ii) $T T^* + (T^* T)^{-1} \leq 2 | I$,

then $T$ admits Wold-type decomposition [31, Theorem 3.6]. This gives an alternative proof of the result of Richter [26, Theorem 1] and generalizes the result of Hedenmalm, Jakobsson and Shimorin on wandering subspace property for multiplication operators on a class of weighted Bergman space [17]. An operator $T$ in $B(\mathcal{H})$ is called concave if it satisfies (2). In literature, it is also known as $2$-hyperexpansive operator, see [8, 21]. For a positive integer $m$, an operator $T$ in $B(\mathcal{H})$ is said to be $m$-concave (resp. $m$-isometry) if $\beta_m(T) \leq 0$ (resp. $\beta_m(T) = 0$), where the $m$-th defect operator $\beta_m(T)$ is defined by

$$\beta_m(T) := \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} T^{*j} T^j.$$

We also set that $\beta_0(T) = I$. An operator $T$ in $B(\mathcal{H})$ is said to be expansive (norm-increasing) if $\beta_1(T) \geq 0$. In [31, Theorem 3.6], Shimorin shows that any $2$-concave operator (necessarily norm-increasing) admits Wold-type decomposition. In the same paper, he asked the following question:

**Question 1.1.** [31, p. 185] If an operator $T$ in $B(\mathcal{H})$ is expansive and $m$-concave for some $m \geq 3$, then does $T$ admit Wold-type decomposition?
The answer is not yet known even in the class of $m$-isometries. Recently, it has been shown that there are plenty of non-expansive analytic cyclic 3-isometries which fail to have wandering subspace property (see [3]). To answer the above question, in case of $m$-concave operators, the best known result to our knowledge is the following theorem due to Shimorin (see [31, Theorem 3.8]).

**Theorem 1.2.** Let $T \in \mathcal{B}(\mathcal{H})$ be expansive and satisfy the operator inequality

$$T^* 2T^2 - 3T^*T + 3I - T^*T' - P_{\ker T^*} \leq 0,$$

where $P_{\ker T^*}$ is the orthogonal projection of $\mathcal{H}$ onto $\ker T^*$ and $T' = (T^*T)^{-1}$, is the Cauchy dual of $T$. Then $T$ is a 3-concave operator and admits Wold-type decomposition.

In this article, we establish the Wold-type decomposition for a bigger class of norm-increasing $m$-concave operators than the class in Theorem 1.2. For a left invertible operator $T$ in $\mathcal{B}(\mathcal{H})$, we use the notation $L_T$ to denote the left inverse $(T^*T)^{-1}$ of $T$. One of the main results in this article is the following theorem.

**Theorem 1.3.** Let $T$ be an $(m + 1)$-concave operator in $\mathcal{B}(\mathcal{H})$ with some $m \geq 1$. If $T$ satisfies the following inequalities:

$$\beta_r(T) \geq \sum_{k=1}^{\infty} L_T^k \beta_{r+1}(T)L_T^k$$

for $r = 1, \ldots, m - 1$, then $T$ admits the Wold-type decomposition.

In case $m = 1$, the conditions in (3) are vacuously satisfied. For $m > 1$, we will see in Theorem 1.4 that the conditions (3) play an important role among the class of cyclic analytic $(m + 1)$-isometries. Agler and Stankus studied the class of $m$-isometries extensively, see [1, 2, 3, 4]. In 1991, Richter [27] obtains a model for cyclic analytic 2-isometries. He shows that every cyclic analytic 2-isometry is unitarily equivalent to the multiplication operator $M_z$ on a Dirichlet-type space $D(\mu)$ for some positive Borel measure $\mu$ on the unit circle $\mathbb{T}$. Olofsson [28] generalized this result of Richter to the case of arbitrary 2-isometries. Recently, Rydhe [30] obtained a model for the class of cyclic $m$-isometries in terms of the multiplication operator $M_z$ on Dirichlet-type spaces. Motivated by the Rydhe’s model, in this article, we further concentrate our study towards establishing a Richter type model for analytic $m$-isometries satisfying (3).

We will use the notation $\mathcal{M}_+(\mathbb{T})$ to denote the set of all finite positive Borel measures on the unit circle $\mathbb{T}$. Let $\mathcal{O}(\mathbb{D})$ denote the vector space of all holomorphic functions on the unit disc $\mathbb{D}$. Following [30], for a measure $\mu$ in $\mathcal{M}_+(\mathbb{T})$ and for a positive integer $n$, we define the semi-norm $D_{\mu,n}(f)$, for every $f \in \mathcal{O}(\mathbb{D})$, by

$$D_{\mu,n}(f) := \frac{1}{n!(n-1)!} \int_{\mathbb{D}} |f^{(n)}(z)|^2 P_n(z)(1 - |z|^2)^{n-1} dA(z),$$

where $dA$ is the normalized Lebesgue area measure on $\mathbb{D}$ and $P_n(z)$ is the Poisson integral of the measure $\mu$. Although we consider the same semi-norms $D_{\mu,n}(\cdot)$ as described in [30], our approach differs in considering the associated Hilbert space $\mathcal{H}_\mu$ defined below. As a consequence of Theorem 3.9 we observe that for any $m$-tuple $\mu = (\mu_1, \ldots, \mu_m)$ of finite positive Borel measures on the unit circle $\mathbb{T}$, the space $\mathcal{H}_\mu$ coincides with that in [30].
Let $\mu = (\mu_1, \ldots, \mu_m)$ be an $m$-tuple of finite positive Borel measures on $\mathbb{T}$ and $H_\mu$ be the linear subspace of $O(D)$ given by

$$H_\mu := \{ f \in O(D) : D_{\mu_{j,j}}(f) < \infty \text{ for each } j = 1, \ldots, m \}.$$  

For $f \in H_\mu$ we associate the norm $\| f \|_\mu$ given by

$$\| f \|_\mu^2 = \| f \|_{H_\mu}^2 + \sum_{j=1}^m D_{\mu_{j,j}}(f).$$

The space $H_\mu$ with respect to the norm $\| \cdot \|_\mu$ turns out to be a reproducing kernel Hilbert space and the operator $M_z$ is an analytic $(m+1)$-isometry on $H_\mu$. Furthermore we show that the operator $M_z$ on $H_\mu$ satisfies (3).

Theorem. Let $m \in \mathbb{N}$ and $\mu = (\mu_1, \ldots, \mu_m)$ be a tuple of finite positive Borel measures on the unit circle $\mathbb{T}$ and consider the associated Hilbert space $H_\mu$. Then the multiplication operator $M_z$ on the Hilbert space $H_\mu$ is a $(m+1)$-isometry such that for each $r = 1, \ldots, m-1$,

$$\beta_r(M_z) \geq \sum_{n=1}^\infty L_{M_z}^n \beta_{r+1}(M_z) L_{M_z}^n.$$

This allows us to conclude that the operator $M_z$ on $H_\mu$ has wandering subspace property. Moreover the set of all polynomials are dense in $H_\mu$. Lastly, we establish the following model theorem for analytic $(m+1)$-isometry satisfying (3).

**Theorem 1.4.** Suppose that $T$ is an analytic $(m+1)$-isometry satisfying

(i) $\beta_r(T) \geq \sum_{k=1}^\infty L_T^k \beta_{r+1}(T) L_T^k$ for $r = 1, \ldots, m-1$

(ii) $\dim(\ker T^*) = 1$,

for some positive integer $m$. Then there exists a unique $m$-tuple of finite positive Borel measures $\mu = (\mu_1, \ldots, \mu_m)$ on the unit circle $\mathbb{T}$ such that the operator $T$ is unitarily equivalent to the multiplication operator $M_z$ on $H_\mu$.

2. Wandering subspace property for a class of $m$-concave operators

The following two combinatorial identities in Lemma 2.1 will be be essential for the proof of Lemmas 2.2 and 2.4. The first one is known as the Hockey-stick identity and second one is called Chu-Vandermonde’s identity which can be found in [12, p. 46] and [9, p. 44] respectively.

**Lemma 2.1.** (i) If $n$ and $p$ are non-negative integers with $n \geq p$, then

$$\sum_{j=p}^n \binom{j}{p} = \binom{n+1}{p+1}.$$

(ii) If $z, w$ are two complex numbers and $n$ is a non-negative integer, then

$$\binom{z + w}{n} = \sum_{j=0}^n \binom{z}{j} \binom{w}{n-j}.$$

**Lemma 2.2.** If $n, p, m$ are three positive integers with $n \geq p + m$, then

$$\sum_{j=p+m}^n (-1)^{j-p-m} \binom{n}{j} \binom{j-m}{p} = \binom{n-p-1}{n-m-p}.$$
Proof. Note that
\[
\sum_{j=p+m}^{n} (-1)^{j-p-m} \binom{n}{j} \left( \frac{j-m}{p} \right) = \sum_{j=0}^{n-p-m} (-1)^j \binom{n}{j+p+m} \left( \frac{j+p}{p} \right)
\]
\[
= \sum_{j=0}^{n-p-m} (-1)^j \binom{j+p}{j} \left( \frac{n}{n-p-m-j} \right)
\]
\[
= \sum_{j=0}^{n-p-m} (-p+1) \binom{n}{j} \left( \frac{n}{n-p-m-j} \right)
\]
\[
= \left( \frac{n-p-1}{n-p-m} \right).
\]
Here the third equality holds since \((-1)^i \binom{k}{i} = (-k-i+1)^i, k \in \mathbb{C}, i \in \mathbb{Z}_+, \) and the fourth equality follows from (ii) of Lemma 2.1. \(\square\)

The following lemma can be found in [14, Corollary 2.4 and Theorem 2.5]. We are presenting a slightly different proof.

**Lemma 2.3.** Let \(T\) be an \(m\)-concave operator on a Hilbert space \(\mathcal{H}\) for some \(m \in \mathbb{N}\). Then for all positive integers \(n \geq m\),
\[
T^{*-n}T^n \leq \sum_{j=0}^{m-1} \binom{n}{j} \beta_j(T).
\]

Moreover, \(\beta_{m-1}(T) \geq 0\).

**Proof.** Let \(T\) be a bounded linear operator on \(\mathcal{H}\). From [10, Proposition 2.1] (see also [13, Theorem 8.2]), one obtains
\[
T^{*-n}T^n = \sum_{j=0}^{n} \binom{n}{j} \beta_j(T),
\]
and for \(j \geq m\),
\[
\beta_j(T) = \sum_{k=0}^{j-m} (-1)^{j-k-m} \binom{j-m}{k} T^{*k} \beta_{m}(T) T^k.
\]

Therefore
\[
\sum_{j=m}^{n} \binom{n}{j} \beta_j(T) = \sum_{j=m}^{n} \sum_{k=0}^{j-m} (-1)^{j-k-m} \binom{n}{j} \binom{j-m}{k} T^{*k} \beta_{m}(T) T^k
\]
\[
= \sum_{p=0}^{n-m} \left( \sum_{j=p+m}^{n} (-1)^{j-p-m} \binom{n}{j} \binom{j-m}{p} \right) T^{*p} \beta_{m}(T) T^p
\]
\[
= \sum_{p=0}^{n-m} \left( \frac{n-p-1}{n-p-m} \right) T^{*p} \beta_{m}(T) T^p.
\]
Here the last equality follows from Lemma 2.2. Since \(\beta_m(T) \leq 0\), it follows that \(\sum_{j=m}^{n} \binom{n}{j} \beta_j(T) \leq 0\). This put together with (5) gives the inequality (4). To see
the moreover part, we observe that for \( n \geq m \),
\[
\frac{1}{\binom{n}{m-1}} \left( \sum_{j=0}^{m-1} \binom{n}{j} \beta_j(T) \right) \geq \frac{1}{\binom{n}{m-1}} T^*n T^n \geq 0.
\]
Since \( \binom{n}{j} \) is polynomial in \( n \) of degree \( j \), taking limit as \( n \to \infty \) in (6) completes the proof. \( \square \)

The condition (ii) in the following lemma, when specialized to \( m = 1 \), is vacuously true and in this case the proof presented here is essentially present in [27, p. 209].

**Lemma 2.4.** Let \( m \) be a positive integer. Let \( T \) be a bounded linear operator on a Hilbert space \( H \) satisfying
\[
(i) \quad \beta_m(T) \geq 0 \\
(ii) \quad \beta_r(T) \geq \sum_{n=1}^{\infty} L^*_T n \beta_{r+1}(T) L^n_T; \quad r = 1, \ldots, m-1.
\]
Then the following inequalities hold
\[
\sum_{n=r+1}^{\infty} \binom{n}{r} L^*_T n \beta_{r+1}(T) L^n_T \leq I \quad r = 1, \ldots, m-1.
\]

**Proof.** It is evident from the hypothesis that \( \beta_r(T) \geq 0 \) for \( r = 1, \ldots, m-1 \). Furthermore, for any \( r \in \{1, \ldots, m-1\} \), we have
\[
\sum_{i=r}^{\infty} \binom{i-1}{r-1} L^*_T i \beta_r(T) L^i_T \geq \sum_{i=r}^{\infty} \binom{i-1}{r-1} \sum_{n=1}^{\infty} L^*_T n \beta_{r+1}(T) L^n_T
\]
\[
= \sum_{n=r+1}^{\infty} \binom{n-1}{i-r-1} L^*_T n \beta_{r+1}(T) L^n_T
\]
\[
= \sum_{n=r+1}^{\infty} \binom{n}{r} L^*_T n \beta_{r+1}(T) L^n_T,
\]
where the last equality follows from Lemma 2.1(i). Also the above inequality shows that to complete the proof it suffices to show \( \sum_{i=1}^{\infty} L^*_T i \beta_1(T) L^i_T \leq I \). Now, it follows from [27, p. 209] that,
\[
\|x\|^2 = \sum_{i=0}^{n-1} \|P L^i_T x\|^2 + \|L^*_T n x\|^2 + \sum_{i=0}^{n} \|D L^i_T x\|^2,
\]
where \( D \) is the positive square root of \( \beta_1(T) \) and \( P = I - TL_T \). Thus we have,
\[
\left\langle \sum_{i=1}^{\infty} L^*_T i \beta_1(T) L^i_T x, x \right\rangle = \sum_{i=0}^{\infty} \|D L^i_T x\|^2 \leq \|x\|^2.
\]
Consequently, \( \sum_{i=1}^{\infty} L^*_T i \beta_1(T) L^i_T \leq I \). This completes the proof. \( \square \)

Now, we present the proof of the main theorem (Theorem 1.3) of this section. This theorem establishes Wold-type decomposition for a large class of \((m + 1)\)-concave operators. The techniques involved here are motivated from those in [27, Theorem 1].
Proof of Theorem 1.3. Since $T$ is $(m + 1)$-concave, by Lemma 2.3, we have $\beta_m(T) \geq 0$. This put together with (3) implies that $\beta_1(T) \geq 0$, i.e. $T$ is expansive. Since $T$ is $(m + 1)$-concave and expansive, by [31] Proposition 3.4, $\mathcal{H}_\infty(T)$ is a reducing subspace for $T$ and $T|_{\mathcal{H}_\infty(T)}$ is a unitary operator. Define $S := T|_{\mathcal{H}_\infty(T)}$ and note that $L_S = L_T|_{\mathcal{H}_\infty(T)}$. The operator $S$ is analytic, $(m + 1)$-concave and satisfies the analogous inequalities (3). We shall show that

$$\bigvee \{ S^n(\ker S^*) : n \in \mathbb{Z}_+ \} = \mathcal{H}_\infty(T)^\perp.$$  

Using Lemmas 2.3 and 2.4 note that for $x \in \mathcal{H}$ and for any $k, l \in \mathbb{N}$ with $l \geq k \geq m + 1$, we have

$$\inf_{k \leq n \leq l} \left( \|S^n L_S^n x\|^2 - \|L_S^n x\|^2 \right) \sum_{n=k}^{l} \frac{1}{n} \leq \sum_{n=k}^{l} \frac{\|S^n L_S^n x\|^2 - \|L_S^n x\|^2}{n} \leq \sum_{n=k}^{l} \sum_{j=1}^{m} \frac{1}{j} \left( \frac{n-1}{j-1} \right) \langle L_S^n \beta_j(S) L_S^n x, x \rangle = \sum_{j=1}^{m} \frac{1}{j} \sum_{n=k}^{l} \left( \frac{n-1}{j-1} \right) \langle L_S^n \beta_j(S) L_S^n x, x \rangle \leq \left( \sum_{j=1}^{m} \frac{1}{j} \right) \|x\|^2.$$  

The following argument is essentially same as in [27]. For the sake of completeness we give the details below. Since $\left( \|L_S^n x\| \right)$ is a decreasing sequence and $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series, we have $\lim \inf \|S^n L_S^n x\| = \lim \|L_S^n x\|$. Thus $\left( S^n L_S^n x \right)$ is a bounded sequence in $\mathcal{H}$ and therefore there exists a convergent subsequence $\left( S^n L_S^n x \right)$, converging to $y$ (say) in weak topology of $\mathcal{H}$. As $S^n$ is expansive for each $n \in \mathbb{N}$, the range $R(S^n)$ is a closed subspace of $\mathcal{H}_\infty(T)^\perp$ and $y \in R(S^n)$. As $S$ is analytic, we note that $y = 0$. Thus $(I - S^n L^n)x \rightarrow x$ weakly. As $(I - S^n L^n)x \in \bigvee \{ S^n(\ker S^*) : n \in \mathbb{Z}_+ \}, x \in \bigvee \{ S^n(\ker S^*) : n \in \mathbb{Z}_+ \}$. Thus we conclude that $S$ has wandering subspace property.

We obtain the following result of Shimorin [31] Theorem 3.4 as a corollary of Theorem 1.3.

Corollary 2.5 (Shimorin). Let $T$ be an expansive operator in $\mathcal{B}(\mathcal{H})$ satisfying the inequality

$$T^* T^2 - 3T^* T + 3I - L_T^* L_T - P \leq 0,$$

where $P$ is the orthogonal projection onto $\ker T^*$. Then $T$ is $3$-concave and has Wold-type decomposition.

Proof. Since $TL_T$ is an orthogonal projection onto the range of $T$, we have $I - P = TL_T = L_T^* T^* TL_T$. Note that (7) is equivalent to

$$\beta_2(T) \leq \beta_1(T) - L_T^* \beta_1(T) L_T.$$

From (3), we see that $T^* \beta_2(T) T \leq \beta_2(T)$ which in turn implies that $\beta_3(T) \leq 0$. Hence $T$ is $3$-concave. Again, from (8), for all $n \in \mathbb{N}$, we get

$$\sum_{k=0}^{n} L_T^k \beta_2(T) L_T^k \leq \beta_1(T) - L_T^{n+1} \beta_1(T) L_T^{n+1} \leq \beta_1(T).$$
Since $T$ is 3-concave, $\beta_2(T) \geq 0$. Hence, (9) in particular implies that
\[ \sum_{k=1}^{\infty} L_T^k \beta_2(T) L_T^{k} \leq \beta_1(T). \]

Now applying Theorem 1.3, we see that $T$ has Wold-type decomposition. \hfill \Box

In Remark 3.11 we shall see a class of expansive linear operators where (7) does not get satisfied whereas it lies in the realm of (3).

3. DIRICHLET-TYPE SPACES

For $\alpha \in \mathbb{R}$ and an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the unit disc $\mathbb{D}$, consider the norm $\| f \|_{D_{\alpha}}$ defined by
\begin{equation}
\| f \|_{D_{\alpha}}^2 = \sum_{n=0}^{\infty} (n+1)^{\alpha} |a_n|^2.
\end{equation}

Let $D_{\alpha} := \{ f \in \mathcal{O}(\mathbb{D}) : \| f \|_{D_{\alpha}}^2 < \infty \}$. It is well known that the space $D_{\alpha}$ equipped with the norm $\| f \|_{D_{\alpha}}$ is a Hilbert space (see [32]). Note that for $\alpha \leq \beta$, we have $\| f \|_{D_{\alpha}} \leq \| f \|_{D_{\beta}}$ and consequently we have $D_{\beta} \subseteq D_{\alpha}$. Here we remark that many classical functional Hilbert spaces are given by $D_{\alpha}$, for example, $D_{-1}$, $D_0$ and $D_1$ are the Bergman space, the Hardy space and the Dirichlet space on the unit disc $\mathbb{D}$ respectively. From (10), it can easily be seen that
\begin{equation}
f \in D_{\alpha} \text{ if and only if } f' \in D_{\alpha-2}.
\end{equation}

For $\alpha < 0$, the norms $\| \cdot \|_{D_{\alpha}}$ and $\| \cdot \|_{\alpha}$ on $D_{\alpha}$ are equivalent [32 Lemma 2], where for each $f \in D_{\alpha}$, $\| f \|_{\alpha}$ is given by
\begin{equation}
\| f \|_{\alpha}^2 := \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^{-\alpha-1} dA(z).
\end{equation}

Suppose $\mu$ is a finite positive Borel measure on the unit circle $\mathbb{T}$. The Poisson integral of the measure $\mu$ is given by
\[ P_\mu(z) = \int_{\mathbb{T}} P(z, \zeta) d\mu(\zeta), \quad z \in \mathbb{D}, \]
where $P(z, \zeta) = \frac{1-|z|^2}{|z-\zeta|^2}$, $z \in \mathbb{D}, \zeta \in \mathbb{T}$, is the Poisson kernel for the unit disc $\mathbb{D}$. The Poisson integral $P_\mu(z)$ is a positive harmonic function on $\mathbb{D}$ and conversely every positive harmonic function on $\mathbb{D}$ is the Poisson integral of some finite positive Borel measure on the unit circle $\mathbb{T}$ [28 Ch. 11]. For every positive integer $n$ and an analytic function $f$ on the unit disc $\mathbb{D}$, define the semi-norm $D_{\mu,n}(f)$, by
\begin{equation}
D_{\mu,n}(f) := \frac{1}{n!(n-1)!} \int_{\mathbb{D}} |f^{(n)}(z)|^2 P_\mu(z) (1-|z|^2)^{n-1} dA(z),
\end{equation}
where $dA$ is the normalized Lebesgue area measure on the unit disc $\mathbb{D}$. Let $d\sigma$ be the normalized arclength measure on the unit circle $\mathbb{T}$. For $f \in \mathcal{O}(\mathbb{D})$, we use the notation $D_{\mu,0}(f)$ to denote the integral
\[ D_{\mu,0}(f) := \lim_{R \to 1} \int_{\mathbb{T}} |f(R\zeta)|^2 P_\mu(R\zeta) d\sigma(\zeta). \]
It is well known that for a measure $\mu \in \mathcal{M}_+(\mathbb{T})$, the measure $P_\mu(R\zeta)\,d\sigma(\zeta)$ converges to $\mu$ in weak*-topology as $R \to 1$, (see [29] Theorem 3.3.4)). Consequently it follows that for a function $f \in \mathcal{O}(\mathbb{D})$, which extends continuously upto the closed disc $\overline{\mathbb{D}}$, 

$$D_{\mu,0}(f) = \int_{\mathbb{T}} |f(\zeta)|^2 \,d\mu(\zeta).$$

For a positive integer $n$ and a finite positive Borel measure $\mu$ in $\mathcal{M}_+(\mathbb{T})$, let $\mathcal{H}_{\mu,n}$ be the linear subspace of $\mathcal{O}(\mathbb{D})$ given by

$$\mathcal{H}_{\mu,n} := \{ f \in \mathcal{O}(\mathbb{D}) : D_{\mu,n}(f) < \infty \}.$$

For $\mu = 0$, we set $\mathcal{H}_{\mu,n} = H^2$, the Hardy space on unit disc $\mathbb{D}$. It is straightforward to see that the set of all polynomials $\mathbb{C}[z]$ is contained in $\mathcal{H}_{\mu,n}$ and the linear space $\mathcal{H}_{\mu,n}$ equipped with semi-norm $\sqrt{D_{\mu,n}(\cdot)}$ is a semi-inner product space. Following proposition gives the relationship between $\mathcal{H}_{\mu,n}$ and the space $D_\alpha$.

**Proposition 3.1.** Let $\mu$ be a finite positive Borel measure on $\mathbb{T}$. Then,

$$\mathcal{H}_{\mu,n} \subseteq D_{n-1}, \quad n \geq 1,$$

$$D_{n+1} \subseteq \mathcal{H}_{\mu,n}, \quad n \geq 2.$$ 

In particular, for every positive integer $n$, $\mathcal{H}_{\mu,n}$ is contained in the Hardy space $H^2$.

**Proof.** From [28], p. 236], it is easy to see that the Poisson kernel satisfies the following estimates:

$$\frac{1 - |z|^2}{4} \leq P(z, \zeta) \leq \frac{4}{(1 - |z|^2)^2}, \quad z \in \mathbb{D}, \quad \zeta \in \mathbb{T}.$$ 

This in turn implies that for $n \in \mathbb{N}$ and $z \in \mathbb{D}$, we have

$$\frac{\mu(T)}{4}(1 - |z|^2)^n \leq P_\mu(z)(1 - |z|^2)^{n-1} \leq 4\mu(T)(1 - |z|^2)^{n-2}. \quad (14)$$

Using (12) along with the above estimates in (14), it follows that for $f \in \mathcal{O}(\mathbb{D})$,

$$D_{\mu,n}(f) \geq \frac{\mu(T)}{4n!(n-1)!} \|f^{(n)}\|_{-(n+1)}, \quad n \geq 1, \quad (15)$$

$$D_{\mu,n}(f) \leq \frac{4\mu(T)}{n!(n-1)!} \|f^{(n)}\|_{-(n-1)}, \quad n \geq 2. \quad (16)$$

Note that for $\alpha < 0$, the norm $\|f\|_\alpha$ is equivalent to the norm $\|f\|_{D_\alpha}$. Using this fact together with (14), (15), and (16), we obtain the desired conclusion. \hfill \Box

The following proposition shows that the function $z$ is a multiplier for $\mathcal{H}_{\mu,n}$, that is, $f \in \mathcal{H}_{\mu,n}$ implies $zf \in \mathcal{H}_{\mu,n}$. We will see in Lemma 3.5(ii) that converse also holds.

**Proposition 3.2.** Let $\mu \in \mathcal{M}_+(\mathbb{T})$ and $n \in \mathbb{N}$. The function $z$ is a multiplier for the semi-inner product space $\mathcal{H}_{\mu,n}$.

**Proof.** Let $\mu$ be a finite positive Borel measure on $\mathbb{T}$ and $n$ be a positive integer. Then $d\nu := (1 - |z|^2)^{n-1}P_\mu(z)\,dA(z)$ is a weighted area measure on $\mathbb{D}$. Note that for a function $f \in \mathcal{O}(\mathbb{D})$, we have $(zf)^{(n)} = zf^{(n)} + nf^{(n-1)}$. Suppose $f$ is in $\mathcal{H}_{\mu,n}$ then we easily see that $zf^{(n)} \in L^2(\mathbb{D}, d\nu)$. Thus to show that $zf \in \mathcal{H}_{\mu,n}$, it is sufficient to show that $f^{(n-1)} \in L^2(\mathbb{D}, d\nu)$. 

\hfill \Box
Case \( n = 1 \): In this case, we note that \( zf' \in L^2(\mathbb{D}, dv) \). Also from [30], there exists \( C > 0 \) such that \( \|f\|_{L^2(\mathbb{D}, dv)} \leq C\|f\|_{H^2} \). Using Proposition 3.1 we get \( zf \in \mathcal{H}_{\mu,1} \).

Case \( n \geq 2 \): By Proposition 3.1 we have \( \mathcal{H}_{\mu,n} \subseteq D_{n-1} \). Therefore \( f \in D_{n-1} \). By repeated use of (11), one obtains \( \|f^{(n-1)}\|_{-(n-1)} \leq \infty \). Using (12) and (14), we get
\[
4\mu(T)\|f^{(n-1)}\|^2_{-(n-1)} = 4\mu(T) \int_{\mathbb{D}} |f^{(n-1)}(z)|^2 (1 - |z|^2)^{n-2} dA(z)
\geq \int_{\mathbb{D}} |f^{(n-1)}(z)|^2 (1 - |z|^2)^{n-1} P_{\mu}(z) dA(z)
= \|f^{(n-1)}\|^2_{L^2(\mathbb{D}, dv)}.
\]
Thus \( zf \in \mathcal{H}_{\mu,n} \).

Let \( \mathcal{O}_s(\mathbb{D}) \) denote the set of all holomorphic functions on \( \mathbb{D} \) which extend smoothly up to its closure \( \overline{\mathbb{D}} \). For \( f \in \mathcal{O}_s(\mathbb{D}) \) and for each \( j \in \mathbb{Z}_+ \), Rydhe (see [30, Proposition 3.4]) has established the following relationship for \( D_{\mu,j}(f) \),
\[
D_{\mu,j+1}(zf) - D_{\mu,j+1}(f) = D_{\mu,j}(f).
\]

Now in the following proposition we will establish the same formula for a larger class of functions, namely for the space \( \mathcal{H}_{\mu,n} \).

**Proposition 3.3.** Let \( \mu \) be a finite positive Borel measure and \( n \) be a positive integer. Then for \( f \in \mathcal{H}_{\mu,n} \) and \( 0 \leq k \leq n-1 \),
\[
D_{\mu,k+1}(zf) - D_{\mu,k+1}(f) = D_{\mu,k}(f).
\]

**Proof.** Let \( \mu \in \mathcal{M}_+(\mathbb{T}) \) and \( n \in \mathbb{N} \). Fix \( 0 \leq k \leq n-1 \) and \( 0 < R < 1 \). For any \( f \in \mathcal{O}(\mathbb{D}) \), define
\[
D(\mu, k, R, f) := \begin{cases} \frac{1}{\pi(k-1)!} \int_{RD} |f^{(k)}(z)|^2 (R^2 - |z|^2)^{k-1} P_{\mu}(z) dA(z), & k \geq 1, \\ \int_{\mathbb{T}} |f(R\zeta)|^2 P_{\mu}(R\zeta) d\sigma(\zeta), & k = 0. \end{cases}
\]

Following identity will be essential in proving this proposition.
\[
D(\mu, k + 1, R, zf) - R^2 D(\mu, k + 1, R, f) = R^2 D(\mu, k, R, f), \quad f \in \mathcal{O}(\mathbb{D}).
\]

Consider the positive measure \( d\mu_n(\zeta) = P_{\mu}(R\zeta) d\sigma(\zeta) \). Since \( z \mapsto P_{\mu}(Rz) \) is a harmonic function in a neighborhood of the closed disc \( \overline{\mathbb{D}} \) and \( P_{\mu_n}(\zeta) = P_{\mu}(R\zeta) \) for every \( \zeta \in \mathbb{T} \), it follows that
\[
P_{\mu_n}(w) = P_{\mu}(Rw), \quad w \in \overline{\mathbb{D}}.
\]
Note that $R^k(zf)(k+1)(Rw) = (zf)^{(k+1)}(w)$ and $R^{k+1}f^{(k+1)}(Rw) = f^{(k+1)}(w)$, for every $0 < R < 1$, $w \in \mathbb{D}$, where $f^k(w) := f(Rw)$ for $w \in \mathbb{D}$. Thus it follows that

$$D(\mu, k + 1, R, zf) - R^2 D(\mu, k + 1, R, f) = R^2 (D_{\mu, k+1}(zf) - D_{\mu, k+1}(f))$$

$$= R^2 \int_{\mathbb{D}} |f_{\mu, k}^{(k)}(w)|^2 (1 - |w|^2)^{k-1} P_{\mu, k} (w) dA(w)$$

$$= R^2 \int_{\mathbb{D}} |f^{(k)}(z)|^2 (R^2 - |z|^2)^{k-1} P_{\mu, k} (z) dA(z)$$

$$= R^2 D(\mu, k, R, f).$$

This completes the proof of (17). Now observe that the function $R \mapsto D(\mu, k, R, f)$ is non-decreasing on the interval $(0, 1)$. Using monotone convergence theorem, we obtain that

$$\lim_{R \to 1} D(\mu, k, R, f) = D_{\mu, k}(f), \quad f \in \mathcal{O}(\mathbb{D}), \quad k \in \mathbb{Z}_+.$$

Since $zf \in \mathcal{H}_{\mu, n}$ for every $f \in \mathcal{H}_{\mu, n}$, we obtain our desired result from (17) and by taking the limit as $R \to 1$. 

As an immediate corollary of the above proposition along with Proposition 3.2, we get the following result.

**Corollary 3.4.** For every finite positive Borel measure $\mu$ and every positive integer $k$, $\mathcal{H}_{\mu, k+1} \subseteq \mathcal{H}_{\mu, k}$.

For any analytic map $f$ on the unit disc $\mathbb{D}$, let $Lf$ be defined by

$$Lf(z) = \frac{f(z) - f(0)}{z}, \quad z \in \mathbb{D}.$$\n
Note that $Lf$ is also analytic and $D_{\mu, j}(Lf) = D_{\mu, j}(f)$ for $j \in \mathbb{N}$ and $\mu \in \mathcal{M}_+(\mathbb{T})$.

**Lemma 3.5.** Let $\mu \in \mathcal{M}_+(\mathbb{T})$ and $n \in \mathbb{N}$. Then for every $f \in \mathcal{O}(\mathbb{D})$, we have

(i) $D_{\mu, n}(Lf) \leq D_{\mu, n}(f)$.

(ii) $f \in \mathcal{H}_{\mu, n}$ if and only if $zf \in \mathcal{H}_{\mu, n}$.

**Proof.** Let $\mu \in \mathcal{M}_+(\mathbb{T})$ and $n \in \mathbb{N}$. Suppose $f$ is an analytic map on the unit disc $\mathbb{D}$. As an application of (17), for any $0 < R < 1$, we obtain

$$R^2 D(\mu, n, R, f) \leq D(\mu, n, R, zf).$$

Now taking limit as $R \to 1$, we get $D_{\mu, n}(f) \leq D_{\mu, n}(zf)$. Since $D_{\mu, n}(Lf) = D_{\mu, n}(f)$, it follows that $D_{\mu, n}(Lf) \leq D_{\mu, n}(f)$. This also gives us that $f \in \mathcal{H}_{\mu, n}$ whenever $zf \in \mathcal{H}_{\mu, n}$. The remaining part follows from Proposition 3.2. 

The following lemma can be thought as a generalization of [27, Lemma 3.3]. This will be an essential ingredient in proving Theorem 3.9.

**Lemma 3.6.** Let $n$ be a positive integer and let $\mu$ be a finite positive Borel measure on the unit circle $\mathbb{T}$. Then for any function $f$ in $\mathcal{H}_{\mu, n}$, we have

$$\sum_{k=1}^{\infty} D_{\mu, j}(L^k f) = D_{\mu, j+1}(f), \quad 0 \leq j \leq n - 1.$$
Proof. Let \( j \in \{0, \ldots, n - 1\} \) and \( 0 < R < 1 \). By equation (17), we obtain
\[
D(\mu, j + 1, R, f) = R^2 D(\mu, j + 1, R, Lf) + R^2 D(\mu, j, R, Lf).
\]
Applying this inductively, for every positive integer \( k \), one obtains
\[
(18) \quad D(\mu, j + 1, R, f) = R^{2k} D(\mu, j + 1, R, L^k f) + \sum_{i=1}^{k} R^{2i} D(\mu, j, R, L^i f).
\]
By Corollary 3.3, we have \( f \in \mathcal{H}_{\mu,j+1} \) and by repeated applications of Lemma 3.5 we get
\[
D(\mu, j + 1, R, L^k f) \leq D_{\mu,j+1}(L^k f) \leq D_{\mu,j+1}(f).
\]
Thus, it follows that \( \lim_{k \to \infty} R^{2k} D(\mu, j + 1, R, L^k f) = 0 \). Hence, by (18), we see that the series \( \sum_{i=1}^{\infty} R^{2i} D(\mu, j, R, L^i f) \) is convergent and
\[
(19) \quad \sum_{i=1}^{\infty} R^{2i} D(\mu, j + 1, R, L^i f) = D(\mu, j + 1, R, f).
\]
Now, note that, for any \( k \in \mathbb{N} \),
\[
\sum_{i=1}^{k} D_{\mu,j}(L^i f) = \lim_{R \to 1} \sum_{i=1}^{k} R^{2i} D_{\mu,j}(L^i f) \leq \lim_{R \to 1} D(\mu, j + 1, R, f) = D_{\mu,j+1}(f).
\]
Hence \( \sum_{i=1}^{\infty} D_{\mu,j}(L^i f) \leq D_{\mu,j+1}(f) \). Also, it follows from equation (19) that,
\[
D_{\mu,j+1}(f) = \lim_{R \to 1} D(\mu, j + 1, R, f) \leq \sum_{i=1}^{\infty} D_{\mu,j}(L^i f).
\]
This completes the proof of lemma. \( \square \)

Let \( \mu \) be a finite positive Borel measure, \( f \) be an arbitrary but fixed function in \( \mathcal{H}_{\mu,1} \) and \( j \in \mathbb{Z}_+ \). Let \( \Delta D_{\mu,j}(f) \) be the forward difference of \( D_{\mu,j}(f) \) given by
\[
\Delta D_{\mu,j}(f) := D_{\mu,j}(zf) - D_{\mu,j}(f).
\]
Inductively, we define \( \Delta^{k+1} D_{\mu,j}(f) = \Delta(\Delta^k D_{\mu,j}(f)) \) for every \( k \in \mathbb{N} \). We shall adopt the convention that \( \Delta^0 \) is the identity operator. From Proposition 3.3 it follows that
\[
D_{\mu,0}(f) = \lim_{R \to 1} \int_{\mathbb{T}} |f(R\zeta)|^2 \mu(R\zeta) d\sigma(\zeta) < \infty.
\]
Now it is straightforward to verify that \( \Delta D_{\mu,0}(f) = 0 \). An induction argument will give us that for each \( n \in \mathbb{Z}_+ \),
\[
(20) \quad \Delta^l D_{\mu,n}(f) = \begin{cases} D_{\mu,n-l}(f), & 0 \leq l \leq n, \\ 0, & l > n + 1. \end{cases}
\]

Lemma 3.7. For \( j, r \in \mathbb{Z}_+ \) and a finite positive Borel measure \( \mu \) on \( \mathbb{T} \),
\[
\Delta^r D_{\mu,j}(f) = \sum_{n=1}^{\infty} \Delta^{r+1} D_{\mu,j}(L^n f) = \begin{cases} D_{\mu,0}(f), & r = j, \\ 0, & r \neq j. \end{cases}
\]

Proof. In view of Proposition 3.3, we get that \( \Delta^{r+1} D_{\mu,j}(L^n f) = D_{\mu,j-r-1}(L^n f) \), with the convention that \( D_{\mu,l}(f) = 0 \) for any negative integer \( l \). And now using Lemma 3.6 we get the desired identity. \( \square \)
Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)$ be an $m$-tuple of positive Borel measures on the unit circle $\mathbb{T}$. Let $\mathcal{H}_\mu$ denote the linear space given by

$$\mathcal{H}_\mu := \bigcap_{j=1}^m \mathcal{H}_{\mu_j} = \left\{ f \in \mathcal{O}(\mathbb{D}) : D_{\mu_j}(f) < \infty \text{ for } j = 1, \ldots, m \right\}.$$  

From Proposition 3.8, it follows that $\mathcal{H}_\mu \subseteq H^2$. With the help of this fact, we associate a norm $\| \cdot \|_\mu$ to the linear space $\mathcal{H}_\mu$ given by

$$\| f \|^2_\mu := \| f \|_{H^2}^2 + \sum_{j=1}^m D_{\mu_j}(f),$$

where $\| f \|_{H^2}$ denotes the Hardy norm of $f$ for any $f \in H^2$. It is straightforward to verify that the linear space $\mathcal{H}_\mu$ is a Hilbert space with respect to the norm $\| \cdot \|_\mu$. Since $\| f \|_{H^2} \leq \| f \|_\mu$ for every $f \in \mathcal{H}_\mu$, it follows that $\mathcal{H}_\mu$ is a reproducing kernel Hilbert space.

**Proposition 3.8.** Suppose $m \in \mathbb{N}$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)$ is an $m$-tuple of finite positive Borel measures on the unit circle $\mathbb{T}$. Then the multiplication operator $M_z$ on $\mathcal{H}_\mu$ is a bounded, analytic $(m+1)$-isometry.

**Proof.** In view of Corollary 3.5, it follows that $zf \in \mathcal{H}_\mu$ whenever $f \in \mathcal{H}_\mu$. Since $\mathcal{H}_\mu$ is a reproducing kernel Hilbert space, by closed graph theorem, it follows that $M_z$ on $\mathcal{H}_\mu$ is bounded. As $\mathcal{H}_\mu \subseteq \mathcal{O}(\mathbb{D})$, the operator $M_z$ on $\mathcal{H}_\mu$ is analytic. Note that

$$\sum_{j=0}^{m+1} \binom{m+1}{j} (-1)^{m+1-j} z^j f^2_\mu = \sum_{k=1}^m \Delta^{m+1} D_{\mu_k}(f), \quad f \in \mathcal{H}_\mu.$$ 

In view of (20), it follows that the operator $M_z$ on $\mathcal{H}_\mu$ is a $(m+1)$ isometry. $\square$

Following Lemma 3.3(ii), we see that every function $f$ in $\mathcal{H}_\mu$ has the following decomposition.

$$f(z) = f(0) + zg(z), \quad g \in \mathcal{H}_\mu.$$ 

Since $(1, \mu g) = 0$, for every $g \in \mathcal{H}_\mu$, it follows that $g = \ker M_z^* = \text{span}\{1\}.$

In view of Lemma 3.3(i), we observe that $L$ is a bounded operator on $\mathcal{H}_\mu$. Thus the operator $L$ is a left inverse of $M_z$ with $\ker L = \ker M_z^*$. Hence it follows that the operator $L$ on $\mathcal{H}_\mu$ coincides with the operator $L_{M_z} = (M_z^* M_z)^{-1} M_z^*$. In the following theorem we show that the operator $M_z$ on $\mathcal{H}_\mu$ satisfies (3).

**Theorem 3.9.** Let $m \in \mathbb{N}$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)$ be an $m$-tuple of finite positive Borel measures on the unit circle $\mathbb{T}$. Then for any $r = 1, \ldots, m$, the operator $M_z$ on $\mathcal{H}_\mu$ satisfies

$$(22) \quad \langle \beta_r(M_z)f, f \rangle = D_{\mu_r,0}(f) + \sum_{n=1}^\infty \langle \beta_{r+1}(M_z)L_{M_z}^n f, L_{M_z}^n f \rangle, \quad f \in \mathcal{H}_\mu.$$ 

In particular,

$$\beta_r(M_z) = \sum_{n=1}^\infty L_{M_z}^n \beta_{r+1}(M_z)L_{M_z}^n.$$
Proof. Since \( \|zf\|_{H^2} = \|f\|_{H^2} \) for every \( f \in \mathcal{H}_\mu \), it follows that for any \( r = 1, \ldots , m \),
\[
\langle \beta_r(M_z)f, f \rangle = \sum_{j=1}^{m} \Delta^r D_{\mu_j,j}(f), \quad f \in \mathcal{H}_\mu .
\]
Using Lemma 3.7, we see that for any \( f \in \mathcal{H}_\mu \),
\[
\langle \beta_r(M_z)f, f \rangle \leq \sum_{n=1}^{\infty} \langle \beta_{r+1}(M_z)L^n_{M_z}f, L^n_{M_z}f \rangle = \sum_{j=1}^{m} \Delta^r D_{\mu_j,j}(f) - \sum_{j=1}^{m} \sum_{n=1}^{\infty} \Delta^{r+1} D_{\mu_j,j}(L^n_{M_z}f) = D_{\mu_r,0}(f).
\]
This establishes the equation (22) and completes the proof of the theorem. \( \square \)

Corollary 3.10. The multiplication operator \( M_z \) on \( \mathcal{H}_\mu \) has wandering subspace property. In particular the set of polynomials are dense in \( \mathcal{H}_\mu \).

Proof. By Proposition 3.8, \( M_z \) on \( \mathcal{H}_\mu \) is a bounded \((m + 1)\)-isometry. Using Theorem 1.3, together with (21), the corollary is proved. \( \square \)

We conclude this section with the remark that the class of 3-concave operators satisfying (8) is larger than that considered in (1).

Remark 3.11. We point out that by Proposition 3.8 and Theorem 3.9 for any \( m \)-tuple \( \mu = (\mu_1, \ldots , \mu_m) \) of positive Borel measures on the unit circle \( \mathbb{T} \), the multiplication operator \( M_z \) on \( \mathcal{H}_\mu \) is an \((m + 1)\)-isometry and satisfies (8). Now let \( m = 2 \) and suppose that \( M_z \) on \( \mathcal{H}_\mu \) satisfies (7). Then using the equivalence of (7) and (8) we have
\[
\langle \beta_2(M_z)1, 1 \rangle \leq \langle \beta_1(M_z) - L_{M_z}^1 \beta_1(M_z)L_{M_z}^1, 1 \rangle = \langle \beta_2(M_z)1, 1 \rangle.
\]
Thus, by (22), we obtain \( D_{\mu_2,0}(1) \leq D_{\mu_1,0}(1) \), that is, \( \mu_2(\mathbb{T}) \leq \mu_1(\mathbb{T}) \). Hence we conclude that the class of 3-concave operators which satisfy (8) is strictly larger than the class of operators which satisfy (7).

4. Model for class of \( m \)-isometries

In this section, we obtain a model for a class of analytic \((m + 1)\)-isometries. We start with the following lemma which is necessary for the proof of main theorem in this section.

Lemma 4.1. Let \( A \) and \( T \) be two operators in \( B(\mathcal{H}) \) and \( e \) be an arbitrary but fixed vector in \( \mathcal{H} \). If \( A \) is positive and \( T^*AT = A \), then there exists \( \mu \in \mathcal{M}_+(\mathbb{T}) \) such that
\[
\langle AT^j(e), T^j(e) \rangle = \int_{\mathbb{T}} \zeta^{l-j}d\mu(\zeta), \quad j, l \in \mathbb{Z}_+ .
\]

Proof. Set \( \phi(j,l) = \langle AT^j(e), T^j(e) \rangle, \quad j, l \in \mathbb{Z}_+ \). It is straightforward to verify that the matrix \( (\phi(j,l))_{j,l=0}^{n} \) is a Toeplitz matrix, that is, \( \phi(j+1,l+1) = \phi(j,l) \) for all \( j, l \in \mathbb{Z}_+ \). Furthermore, for any \( n \in \mathbb{N} \), and \( c_1, \ldots , c_n \in \mathbb{C} \), we have
\[
\sum_{j,l=1}^{n} \phi(j,l)c_j \bar{c}_l = \langle Av, v \rangle \geq 0 ,
\]
where \( v = \sum_{i=1}^{n} c_i T^i(e) \). Thus the Toeplitz matrix \( ((\phi(j,l)))_{j,l=0}^{\infty} \) is formally positive. Hence, from the characterization of trigonometric moment sequence \[25, \text{Theorem 2.14}], it follows that there exists a positive Borel measure \( \mu \) on \( T \) such that
\[
\phi(j,l) = \int \zeta^{l-j} d\mu(\zeta), \quad j, l \in \mathbb{Z}_+.
\]
This completes the proof. \( \square \)

Before we state the main result of this section, we prove the following lemma which characterizes the unitary equivalence class of the multiplication operator \( M_z \) on \( \mathcal{H}(\mu) \) spaces.

**Lemma 4.2.** Let \( \mu = (\mu_1, \ldots, \mu_m) \) and \( \nu = (\nu_1, \ldots, \nu_m) \) be two \( m \)-tuples of positive Borel measures on the unit circle \( T \). Then the operators \( M_z \) on \( \mathcal{H}_\mu \) and \( M_z \) on \( \mathcal{H}_\nu \) are unitarily equivalent if and only if \( \mu_r = \nu_r \) for all \( r = 1, \ldots, m \).

**Proof.** Let \( M_z(\mu) \) and \( M_z(\nu) \) denote the operators \( M_z \) on \( \mathcal{H}_\mu \) and \( \mathcal{H}_\nu \) respectively. Suppose that \( M_z(\mu) \) and \( M_z(\nu) \) are unitarily equivalent. Then there exists a unitary operator \( U : \mathcal{H}_\mu \to \mathcal{H}_\nu \) such that \( UM_z(\mu) = M_z(\nu)U \). Since
\[
\text{ker } M_z(\mu)^* = \text{ker } M_z(\nu)^* = \text{span}\{1\},
\]
it follows that \( U^*(1) = \zeta \) for some \( \zeta \in T \). By replacing \( U \) by \( \zeta U \), without loss of generality, we can assume that \( U^*(1) = 1 \). Hence \( U(1) = 1 \), and therefore, for any polynomial \( p \), \( U(p(M_z(\mu))(1)) = p(M_z(\nu))(U(1) = p) \). Also, for \( r = 1, \ldots, m \), by a routine verification, we see that
\[
U \left( \beta_r(M_z(\mu)) - \sum_{k=1}^{\infty} L_{M_z(\mu)}^k \beta_{r+1}(M_z(\mu)) L_{M_z(\mu)}^k \right)
= \left( \beta_r(M_z(\nu)) - \sum_{k=1}^{\infty} L_{M_z(\nu)}^k \beta_{r+1}(M_z(\nu)) L_{M_z(\nu)}^k \right)U.
\]
Hence by Theorem \[3, \text{Theorem 5.1}\] it follows that \( D_{\nu, 0}(Uf) = D_{\nu, 0}(f) \) for all \( f \) in \( \mathcal{H}_\mu \), and consequently, \( D_{\nu, 0}(p) = D_{\nu, 0}(p) \) for all polynomial \( p \). Hence, by \[13\] and polarization identity, we conclude that \( \mu_r = \nu_r \) for \( r = 1, \ldots, m \). \( \square \)

The following theorem, providing a canonical model for the class of analytic \((m+1)\)-isometries having one dimensional co-kernel satisfying \[3\], generalizes \[25, \text{Theorem 5.1}\].

**Theorem 4.3.** Suppose that \( T \) is an analytic \((m+1)\)-isometry satisfying
\[
(\text{i}) \quad \beta_r(T) \geq \sum_{k=1}^{\infty} L_T^k \beta_{r+1}(T) L_T^k \text{ for } r = 1, \ldots, m - 1,
(\text{ii}) \quad \text{dim}(\ker T^*) = 1.
\]
Then there exists a unique \( m \)-tuple of positive Borel measures \( \mu = (\mu_1, \ldots, \mu_m) \) on the unit circle \( T \) such that the operator \( T \) is unitarily equivalent to the multiplication operator \( M_z \) on \( \mathcal{H}_\mu \).

**Proof.** Let \( T \) be an analytic \((m+1)\)-isometry satisfying (i) and (ii). From Theorem \[14, \text{Theorem 1.3}\] \( T \) has wandering subspace property. Let \( e \) be a non-zero unit vector in \( \ker T^* \). Thus the linear span of \( \{T^n e : n \in \mathbb{Z}_+\} \) is dense in \( \mathcal{H} \). Note that as \( T \) is left invertible, it follows that the set \( \{T^n e : n \in \mathbb{Z}_+\} \) is linearly independent.
Since $T$ is an $(m+1)$-isometry, we have $T^* \beta_m(T)T = \beta_m(T)$. From Theorem 2.3, it follows that $\beta_m(T) \geq 0$. Moreover, since $L_T T = I$ and $T^* \beta_r(T)T - \beta_{r+1}(T) = \beta_r(T)$, it follows that for each $r = 1, \ldots, m$,

$$T^* \left( \beta_r(T) - \sum_{k=1}^{\infty} L_T^k \beta_{r+1}(T) L_T^k \right) T = \beta_r(T) + \beta_{r+1}(T) - \sum_{k=1}^{\infty} T^* L_T^k \beta_{r+1}(T) L_T^k T
$$

Thus following Lemma 4.1, there exists an $m$-tuple of positive Borel measures $\mu = (\mu_1, \ldots, \mu_m)$ on the unit circle $\mathbb{T}$ such that

$$\langle \beta_m(M_z \zeta^j, \zeta^j) = \int_{\mathbb{T}} \zeta^{j-l} d\mu_m(\zeta), \quad j, l \in \mathbb{Z}_+,$$

for each $r = 1, \ldots, m$. We will show that $T$ is unitarily equivalent to the multiplication operator $M_z$ by coordinate function on $H_\mu$.

In view of Theorem 3.9 together with the polarization identity and (23), we have

$$\langle \beta_m(M_z, z^j) = \int_{\mathbb{T}} \sum_{k=1}^{\infty} L_T^k \beta_{r+1}(T) L_T^k T^j(e), T^j(e) \rangle, \quad j, l \in \mathbb{Z}_+.$$

Since $L_T T = I$ and $L_T(e) = 0$, we get that

$$L_T^n T^j(e) = \begin{cases} T^{j-n} e & \text{if } n \leq j, \\ 0 & \text{if } n > j. \end{cases}$$

In a similar manner, we also have

$$L_{M_z}^n z^j = \begin{cases} z^{j-n} & \text{if } n \leq j, \\ 0 & \text{if } n > j. \end{cases}$$

This combined together with (24) gives

$$\langle \beta_m(M_z) L_{M_z}^j, L_{M_z}^j \rangle = \langle \beta_m(T) L_T^j(e), L_T^j(e) \rangle, \quad k, l, j \in \mathbb{Z}_+.$$

Again using Theorem 3.9 along with the polarization identity and (23), one obtains

$$\langle \beta_r(M_z) - \sum_{k=1}^{\infty} L_{M_z}^k \beta_{r+1}(M_z) L_{M_z}^k \rangle \zeta^j, \zeta^j \rangle = \langle \beta_r(T) - \sum_{k=1}^{\infty} L_T^k \beta_{r+1}(T) L_T^k \rangle T^j(e), T^j(e) \rangle, \quad j, l \in \mathbb{Z}_+,.$$

for every $r = 1, \ldots, m - 1$. Using this together with (25), inductively, we have

$$\langle \beta_r(M_z) z^j, z^j \rangle = \langle \beta_r(T) T^j(e), T^j(e) \rangle, \quad j, l \in \mathbb{Z}_+,.$$

for every $r = m, \ldots, 1$. From the case of $r = 1$, it follows that

$$\langle z^{j+1}, z^{j+1} \rangle - \langle z^j, z^j \rangle = \langle T^{j+1} e, T^{j+1} e \rangle - \langle T^j e, T^j e \rangle, \quad j, l \in \mathbb{Z}_+.$$

We also have $\|1\| = \|e\| = 1$ and $\langle z^j, 1 \rangle = \langle T^j e, e \rangle = 0$ for every $l > 1$. Hence inductively we obtain that

$$\langle z^j, z^j \rangle = \langle T^j e, T^j e \rangle, \quad j, l \in \mathbb{Z}_+. $$
Now consider the map $U$ defined on the linear span of \( \{ T^n e : n \in \mathbb{Z}_+ \} \) given by
\[
U \left( \sum_{j=0}^{k} c_j T^j e \right) := \sum_{j=0}^{k} c_j z^j, \quad e_1, \ldots, e_n \in \mathbb{C}.
\]
From (26) it follows that $U$ is an isometry from the linear span of \( \{ T^n e : n \in \mathbb{Z}_+ \} \) onto the set of all polynomials in $H_\mu$ and the equality $UT = M_U \lambda I$ holds on the linear span of \( \{ T^n e : n \in \mathbb{Z}_+ \} \). Since polynomials are dense in $H_\mu$ (by Corollary 3.10) the map $U$ extend as a unitary map from $H$ onto $H_\mu$ satisfying $UT = M_U \lambda I$. The uniqueness of the tuple $\mu$ follows from Lemma 12. This completes the proof.

We conclude this section by showing that every analytic $m$-isometry with non trivial finite dimensional co-kernel lies in the class of $B_n(D)$, Cowen-Douglas class operator associated to unit disc $D$, see [11] for the definition and other properties.

**Proposition 4.4.** Let $T$ be an analytic $m$-isometry in $B(H)$ with $\dim(\ker T^*) = n$, for some positive integers $m$ and $n$. Then it follows that
(i) $\sigma(T) = \overline{D}$ and $\sigma_{ap}(T) = \mathbb{T},$
(ii) $T^* \in B_n(D),$

where $\sigma(T)$ and $\sigma_{ap}(T)$ denote the spectrum and the approximate point spectrum of the operator $T$ respectively.

**Proof.** Since $T$ is an $m$-isometry, it follows that $\sigma_{ap}(T) \subseteq \mathbb{T}$ (see [2] Lemma 1.21). Thus $T - \lambda I$ is bounded below for every $\lambda \in \mathbb{D}$. This gives us that $T - \lambda I$ is semi-Fredholm for every $\lambda \in \mathbb{D}$. Since $\ker T^*$ is $n$ dimensional, using continuity of the semi-Fredholm index, we conclude that $\dim(\ker(T^* - \lambda I)) = n$, for every $\lambda \in \mathbb{D}$. Consequently, it follows that $\sigma(T) = \overline{D}$ and $\sigma_{ap}(T) = \mathbb{T}$.

In order to show that $T^* \in B_n(D)$, it is sufficient to show that
\[
\bigvee \left\{ \ker(T^* - \lambda I) : \lambda \in \mathbb{D} \right\} = H.
\]

Note that $(T^* - \lambda I)$ is Fredholm for every $\lambda \in \mathbb{D}$ with Fredholm index equal to $n$. Now following [11] Proposition 1.11], we obtain that there exist holomorphic $H$-valued functions $\{e_i(z) : i = 1, \ldots, n\}$ defined on some neighborhood $\Omega$ of 0 such that $\{e_1(z), \ldots, e_n(z)\}$ forms a basis for $\ker(T^* - zI)$ for every $z \in \Omega$. As $T$ is bounded below, it follows that $T^k$ is also bounded below for every $k \in \mathbb{N}$. Now from general properties of Fredholm index, see [10] Ch.11, 3.7], we obtain that $\dim(\ker T^{*k}) = kn$, for every $k \in \mathbb{N}$. Furthermore from the proof of [11] Lemma 1.22], one may infer that
\[
\text{span}\{e_1(0), \ldots, e_n(0), \ldots, e_1^{(k-1)}(0), \ldots, e_n^{(k-1)}(0)\} = \ker T^{*k}, \quad k \in \mathbb{N}.
\]
It follows that $\ker T^{*k}$ is contained in $\bigvee \left\{ \ker(T^* - \lambda I) : \lambda \in \mathbb{D} \right\}$ for every $k \in \mathbb{N}$. Note that the hyper-range $H_{\infty}(T)$ of $T$ is given by
\[
\{ \ker T^{*k} : k \in \mathbb{N} \} \perp = H_{\infty}(T).
\]
As $T$ is analytic by assumption, we obtain that $\bigvee \{ \ker(T^* - \lambda I) : \lambda \in \mathbb{D} \} = H$. □

In view of Proposition 3.8 together with (21) the following corollary is now immediate. The result in [27] Corollary 3.8(b,c)] can be seen as a special case of the following corollary.

**Corollary 4.5.** The operator $M_z$ on $H_\mu$ has the following properties:
(i) $\sigma(M_z) = \mathbb{T}$ and $\sigma_{ap}(M_z) = T$.
(ii) $M^*_z \in B_1(\mathbb{D})$.

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