A note about rational surfaces as unions of affine planes

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Abstract
We prove that any smooth rational projective surface over the field of complex numbers has an open covering consisting of 3 subsets isomorphic to affine planes.

Keywords Smooth rational projective surface · Covering of surfaces · Surface blowup

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Since all smooth rational curves are isomorphic to $\mathbb{P}^1$, they can be seen as the union of two affine lines. In dimension two, as a consequence of the structure Theorem 1.3 below, all rational surfaces admit a covering of open subsets isomorphic to the affine plane. However, up to the authors’ knowledge, no general results are known on the minimal number of open subsets of such a covering, while some advances are known by computer algebrists in terms of surjectivity of parametrizations [1, 5, 6, 8]. In this short note we prove that all projective smooth rational surfaces behave like the projective plane in this aspect.

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1 Main result

Theorem 1.1 Let X be a projective smooth rational surface over the complex field. Then, there are three open subsets $U_0, U_1, U_2 \subset X$ such that:

1. $U_0 \cup U_1 \cup U_2 = X$.
2. For all $i = 0, 1, 2$, $U_i$ is isomorphic to the affine plane.

Remark 1.2 Note that the bound of three subsets in the covering is sharp. If the projective surface $X \subset \mathbb{P}^n$ is the union of two affine planes $U_0$ and $U_1$, then $Z = X - U_0$ is closed in $X$, so projective, and it is contained in $U_1 \cong \mathbb{A}^2$, so it must be finite. Since $Z$ is finite, there is a hyperplane $H \subset \mathbb{P}^n - Z$. Then the section $H \cap X$ is a projective curve contained in $X - Z = U_0 \cong \mathbb{A}^2$. Since $\mathbb{A}^2$ does not contain projective varieties of positive dimension, this is a contradiction.

To prove Theorem 1.1, we will use the following well-known result:

Theorem 1.3 (see e.g. [2, Theorem V.10]) Every non-singular rational surface can be obtained by repeatedly blowing up either $\mathbb{P}^2$ or the projective bundle $\mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-n))$ (the Hirzebruch surface $\Sigma_n$), for $n \neq 1$.

By Theorem 1.3, there exists a chain of morphisms $\pi = \pi_1 \circ \cdots \circ \pi_r : X \to M$ such that $M$ is either $\mathbb{P}^2$ or a Hirzebruch surface and $\pi_i : X_i \to X_{i-1}$ is the blowup of a smooth surface at a single point. Let $E$ be the exceptional divisor of $\pi$ and $E_i$ the exceptional divisor of $\pi_i$. Then, $\pi(E) \subset M$ is a finite set of closed points and $\pi_1(E_i)$ is one closed point. Moreover, $E_i \cong \mathbb{P}^1$ and $E$ is a finite union of smooth rational curves (in fact, $E_1$ and the proper transforms of all the $E_2, \ldots, E_r$). We begin by proving Theorem 1.1 for $X = M$ with care for the centers of the blowups:

Lemma 1.4 In the above conditions, there exist three open subsets $U_0^0, U_1^0, U_2^0$ such that:

1. $U_0^0 \cup U_1^0 \cup U_2^0 = M$.
2. For all $i = 0, 1, 2$, $U_i$ is isomorphic to the affine plane.
3. $\pi(E) \subset U_0^0 \cap U_1^0 \cap U_2^0$.

Proof The case $M = \mathbb{P}^2$ is well-known. Since $\pi(E)$ is finite and we work over an infinite field, one can choose three different projective lines $L_1, L_2$ and $L_3$ in $\mathbb{P}^2$ such that $\pi(E) \cap (L_1 \cup L_2 \cup L_3) = \emptyset = L_1 \cap L_2 \cap L_3$.

If $M$ is a Hirzebruch surface, then it is the projective bundle of a rank two vector bundle $O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-m)$ over $\mathbb{P}^1$. This means that there is a surjective morphism $p : M \to \mathbb{P}^1$ such that, for any point $P \in \mathbb{P}^1$, $p^{-1}(\mathbb{P}^1 - \{P\}) \cong (\mathbb{P}^1 - \{P\}) \times \mathbb{P}^1$. Then, since we work over an infinite field, one can choose a closed point $P_0 \in \mathbb{P}^1 - p(\pi(E))$ with its isomorphism $q_0 : p^{-1}(\mathbb{P}^1 - \{P_0\}) \to \mathbb{A}^1 \times \mathbb{P}^1$. Then, we choose a line $L_0 = \mathbb{A}^1 \times \{Q_0\}$, such that $q_0(\pi(E)) \cap L_0$ is empty. With this choice, $U_0^0 = q_0^{-1}(\mathbb{A}^1 \times \mathbb{P}^1 - L_0)$ is isomorphic to $\mathbb{A}^2$ and contains $\pi(E)$.

Then, $M - U_0^0$ is the union of two rational curves $C_1 := p^{-1}(P_0)$ and $C_2 := q_0^{-1}(L_0)$. Choosing $P_1 \in \mathbb{P}^1 - (p(\pi(E)) \cup \{P_0\})$ (again, the complement of a finite set), together with the isomorphism $q_1 : p^{-1}(\mathbb{P}^1 - \{P_1\}) \to \mathbb{A}^1 \times \mathbb{P}^1$, we have that $p^{-1}(\mathbb{P}^1 - \{P_1\})$ contains $C_1$ and $C_2$ with the exception of the point $R_1 := C_2 \cap p^{-1}(P_1)$ (the intersection of a section $\mathbb{A}^1 \to \mathbb{A}^1 \times \mathbb{P}^1$ with a fiber). We now choose a line $L_1 = \mathbb{A}^1 \times \{Q_1\}$ such that:

- $Q_1 \in \mathbb{P}^1$ is not in the second projection of $q_1(\pi(E)) \in \mathbb{A}^1 \times \mathbb{P}^1$; and
• $L_1 \neq q_1(C_2)$ (i.e. we are asking a constant section not to coincide with a given one, which is an open condition for $Q_1$), so the intersection of the two curves is finite.

Then, $U_1^0 = q_1^{-1}(A^1 \times \mathbb{P}^1 - L_1)$ is isomorphic to $\mathbb{A}^2$ and contains $\pi(E)$.

Now, $M - (U_0^0 \cup U_1^0) = (C_1 \cup C_2) - U_1^0$ is the finite set $A := \{R_1\} \cup q_1^{-1}(L_1 \cap q_1(C_2))$. Finally, we have again the complement of a finite set to choose $P_2 \in \pi^{-1}(P(A) \cup \{P_0, P_1\})$ with the isomorphism $q_2: p^{-1}(\mathbb{P}^1 - \{P_2\}) \rightarrow A^1 \times \mathbb{P}^1$, so we have $A \subset p^{-1}(\mathbb{P}^1 - \{P_2\})$. We now choose $L_2 = A^1 \times \{Q_2\}$ such that $Q_2 \in \mathbb{P}^1$ is not in the second projection of the finite set $q_2(A \cup \pi(E)) \subset A^1 \times \mathbb{P}^1$, and we define $U_2^0 = q_2^{-1}(A^1 \times \mathbb{P}^1 - L_2) \simeq \mathbb{A}^2$. Then $\pi(E) \subset U_2^0$ and, since $A \subset U_0^0$, we have that $U_0^0 \cup U_1^0 \cup U_2^0 = M$. □

**Remark 1.5** Let $\text{Bl}_P(\mathbb{A}^2)$ be the blowup of the affine plane at a point $P$. Consider a line $l$ passing through $P$ and define $U_l$ as the complement in $\text{Bl}_P(\mathbb{A}^2)$ of the proper transform of $l$. Just by changing coordinates, one has that all $U_l$ are isomorphic to each other. Since the case for $l$ being the vertical axis is well known to be isomorphic to $\mathbb{A}^2$, by restricting the defining projection $\pi: \text{Bl}_P(\mathbb{A}^2) \rightarrow \mathbb{A}^2$, we have morphisms $\pi_l: U_l \simeq \mathbb{A}^2 \rightarrow \mathbb{A}^2$. Moreover:

1. for $l_1 \neq l_2$, $U_{l_1} \cup U_{l_2} = \text{Bl}_P(\mathbb{A}^2)$.
2. if $E_{l_2}$ is the exceptional divisor of $\text{Bl}_P(\mathbb{A}^2)$, for any line $l$ passing through $P$, $E_{l_2} - U_l$ consists in one point, given by the isomorphism between $E_{l_2}$ and the $\mathbb{P}^1$ of all lines through $P$.
3. the restriction $\pi_{l_1}|_{U_{l_1} - E_{l_2}}$ is an isomorphism between $U_{l_1} - E_{l_2}$ and $\mathbb{A}^2 - l$.

**Lemma 1.6** Let $X$ be a smooth rational surface such that there exist three open subsets $U_0, U_1, U_2 \subset X$ with

1. $U_0 \cup U_1 \cup U_2 = X$.
2. For all $i = 0, 1, 2$, $U_i$ is isomorphic to the affine plane.

Consider a finite set $A_1 \subset U_0 \cap U_1 \cap U_2$. Let $P \in (U_0 \cap U_1 \cap U_2) - A_1$ be a point and consider $\pi: Y \rightarrow X$ to be the blowup of $X$ at $P$. Consider also a finite set $A_2$ in the exceptional divisor $E = \pi^{-1}(P) \subset Y$. Then, there are three open subsets $U_0', U_1', U_2' \subset Y$ such that

1. $U_0' \cup U_1' \cup U_2' = Y$.
2. For all $i = 0, 1, 2$, $U_i'$ is isomorphic to the affine plane.
3. Both $A_2$ and the proper transform of $A_1$ are contained in $U_0' \cap U_1' \cap U_2'$

**Remark 1.7** In the conditions of Lemma 1.6, note that for any $i, j = 0, 1, 2$, $i \neq j$, $X - (U_i \cup U_j)$ is a Zariski closed subset of a projective surface which is contained in $U_k \simeq \mathbb{A}^2$, with $i \neq k \neq j$. Since it is a projective scheme in an affine space, it must be finite.

**Proof** Taking into account Remark 1.5, consider a line $l_0 \subset U_0 \simeq \mathbb{A}^2$ through $P$ such that

• The intersection of $l_0$ with the finite set $X - (U_1 \cup U_2)$ (see Remark 1.7) is empty.
• $A_1 \cap l_0 = \emptyset$.
• The intersection point of the proper transform of $l_0$ with the exceptional divisor is not in $A_2$.

Then, we define $U_0'$ to be the open subset $U_{l_0}$ of the blowup of $U_0$ at $P$. $U_0$ is isomorphic to the affine plane, as said in Remark 1.5, and $Y - U_0'$ consists in the proper transform of $l_0 \cup (X - U_0)$. Therefore, it is one-dimensional.

Now, we choose a line $l_1 \subset U_1 \simeq \mathbb{A}^2$ such that the following open conditions are satisfied:

1. The intersection of $l_1$ with the finite set $X - (U_0 \cup U_2)$ (see Remark 1.7) is empty.
(2) the intersection multiplicity of \( l_1 \) and \( l_0 \) at \( P \) is 1 (note that \( l_1 \) is smooth at \( P \), so we are asking that \( l_1 \) is not the tangent line at \( P \) to \( l_0 \), when we see them in \( U_1 \)).

(3) \( l_1 \) does not contain any point in \( l_0 \cap (X - U_2) \) (note that \( l_0 \) is irreducible and \( P \subset l_0 \cap U_2 \), so such intersection is finite).

(4) \( A_1 \cap l_1 = \emptyset \).

(5) The intersection point of the proper transform of \( l_1 \) with the exceptional divisor is not in \( A_2 \).

Since we work over an infinite field, these conditions define a nonempty Zariski open subset to choose \( l_1 \) from. Now, we define \( U_i' \) to be \( U_i' \simeq \mathbb{A}^2 \). The whole exceptional divisor is in \( U_i' \cup U_1' \). Then, \( Y - (U_0' \cup U_1') \) is the proper transform of the finite sets \( B_1 = \{ l_0 \cap (X - U_0) \} \cap l_1 \) and \( B_2 = X - (U_0 \cup U_1) \).

Note that \( P \not\in B_1 \cup B_2 \subset U_2 \), so we choose a last line \( l_2 \subset U_2 \simeq \mathbb{A}^2 \) such that

- the intersection of \( l_2 \) with the finite set \( X - (U_0 \cup U_1) \) (see Remark 1.7) is empty,
- \( (A_1 \cup B_1 \cup B_2) \cap l_2 = \emptyset \), and
- the intersection point of the proper transform of \( l_2 \) with the exceptional divisor is not in \( A_2 \).

Defining \( U_2' = U_{l_2} \), one concludes the proof. \( \square \)

Remark 1.8 It is likely that a generalisation of Lemma 1.6 to higher dimension is possible. However, it is not yet known if all rational varieties of dimension greater than 2 are covered by open subsets isomorphic to open subsets of \( \mathbb{A}^n \) (see [7] for the original question). These varieties are known as plain [3] or uniformly rational [4] and it is possible that the main result can be extended to higher dimension for this type of varieties.

Proof of Theorem 1.1 By Lemma 1.4, we have that \( M = X_0 = U_0^0 \cup U_1^0 \cup U_2^0 \) with \( U_i^0 \simeq \mathbb{A}^2 \) and \( \pi(E) \subset U_0^0 \cap U_1^0 \cap U_2^0 \). Now we apply Lemma 1.6 to \( \pi_i : X_i \rightarrow X_{i-1} \), choosing

\[
A_1 = \left[ \pi_i \circ \pi_{i+1}(E_{i+1}) \cup \cdots \cup \pi_i \circ \cdots \circ \pi_r(E_r) \right] - \{ P_i \}
\]

(i.e. the points to be the center of future blowups outside \( \{ P_i \} \)) and

\[
A_2 = \left[ \pi_{i+1}(E_{i+1}) \cup \cdots \cup \pi_{i+1} \circ \cdots \circ \pi_r(E_r) \right] \cap E_i
\]

(i.e. the points to be center of future blowups in \( E_i \)). Note that any curve contracted by \( \pi_{i+1} \circ \cdots \circ \pi_i \) is contracted to a point in \( \pi_i^{-1}(A_1) \cup A_2 \). We then get \( U_0^i, U_1^i, U_2^i \) from \( U_0^{i-1}, U_1^{i-1}, U_2^{i-1} \) all isomorphic to \( \mathbb{A}^2 \) and covering \( X_i \), with all centers of future blowups in the intersection of the three open subsets. Then \( U_0^i, U_1^i \) and \( U_2^i \) are the three open subsets in the statement. \( \square \)

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