The third post-Newtonian gravitational waveforms for compact binary systems in general orbits: instantaneous terms

Chandra Kant Mishra,1,2,3, K. G. Arun,4 and Bala R. Iyer1

1Raman Research Institute, Bangalore 560 080, India
2Indian Institute of Science, Bangalore 560 012, India
3International Centre for Theoretical Sciences, Tata Institute of Fundamental Research, Bangalore 560012, India
4Chennai Mathematical Institute, Siruseri 603103, India

(Dated: January 29, 2015)

We compute the instantaneous contributions to the spherical harmonic modes of gravitational waveforms from compact binary systems in general orbits up to the third post-Newtonian order. We further extend these results for compact binaries in quasi-elliptical orbits using the 3PN quasi-Keplerian representation of the conserved dynamics of compact binaries in eccentric orbits. Using the multipolar post-Minkowskian formalism, starting from the different mass and current type multipole moments, we compute the spin weighted spherical harmonic decomposition of the instantaneous part of the gravitational waveform. These are terms which are functions of the retarded time and do not depend on the history of the binary evolution. Together with the hereditary part, which depends on the binary’s dynamical history, these waveforms form the basis for construction of accurate templates for the detection of gravitational wave signals from binaries moving in quasi-elliptical orbits.

PACS numbers: 04.25.Nx, 04.30.-w, 97.60.Jd, 97.60.Lf

I. INTRODUCTION

Compact binary systems composed of neutron stars (NSs) and/or black holes (BHs) are one of the most promising sources for the second generation of earth bound gravitational-wave (GW) detectors such as Advanced LIGO 1 and Advanced Virgo 2 as well as for the proposed space based detector eLISA 3. Detection of such systems in GW detectors relies on a data-analysis technique known as matched filtering which in turn requires very accurate modelling of GW signals from these sources 4. The compact binaries are known to have significant eccentricities when they are formed. However, since the GW radiation reaction effects tend to circularize the binary’s orbit 5, 6, for most long-lived binary systems one can expect that their orbits would have circularized by the time they enter the sensitivity band of ground based detectors. This has motivated the GW community to perform searches of GW signals from coalescing compact binary (CCBs) systems using circular orbit templates.

Many astrophysical scenarios have been proposed which suggest the possible existence of close eccentric binary systems. One such scenario may exist in the cores of dense globular clusters due to a mechanism known as Kozai mechanism 7. This mechanism can also come into the effect in scenarios involving formation of hierarchical triples of supermassive black holes due to subsequent mergers of galaxies 8. Another scenario might involve formation of close eccentric compact binary systems in dense stellar systems like globular clusters 9. Compact binaries involving intermediate mass BHs in globular clusters might be seen in eLISA band with residual eccentricities of 0.1 \( \lesssim e \lesssim 0.2 \) 10. Other scenarios involve formation of close eccentric compact binary systems at centers of galaxies 11 and NS-BH binary systems which can become eccentric as a consequence of multi-stage mass transfer from the NS to the BH 12. In the light of these possibilities it becomes necessary to compute accurate waveforms accounting for eccentricity of the binary’s orbit.

A number of investigations concerning the sensitivity of searches using circular orbit templates to detect eccentric binary systems have been performed in the past. The first such investigation was presented in Ref. 13 where the authors studied the loss of SNR in detecting signals from binaries with residual eccentricities using circular orbit templates with leading order effects (both conservative and secular). They argued that even if the system has a residual eccentricity \( (e_0 \lesssim 0.13 \) for binary system with two 1.4\( M_\odot \) neutron stars or \( e_0 \lesssim 0.3 \) for a binary with two...
6M⊙ black holes), use of circular orbit templates will be sufficient to detect signals from such systems. However, this result has been subsequently weakened due to two independent investigations [14, 18]. Both of the investigations suggest that if the eccentricity of the binary when it enters the sensitivity band of detector is greater than 0.1, then it will not be possible to detect such systems using circular orbit templates. These investigations only dealt with sources that will be seen in ground based detectors. However, the capabilities of circular waveforms to detect signals from the coalescence of supermassive black holes (visible in the eLISA frequency band) have been investigated in [16]. The results presented in Ref. [16] suggest that even to search signals from sources with initial eccentricities of the order $10^{-4}$ one would need waveforms which accurately account for the effects of eccentricities. In addition, in a recent work, Huerta and Brown [17] showed that searches for CCBs with eccentricity $\geq 0.05$ would require eccentric template banks to avoid significant loss in the sensitivity of the search. Lastly, systematic errors due to the orbital eccentricity in measuring the source parameters of double NS systems was investigated recently by Favata [18] which again indicated the necessity to incorporate the effects of eccentricity to measure the parameters of a double NS system if it has non-negligible eccentricity when detected.

Evolution of a compact binary system can be divided into three stages: the early inspiral, late inspiral and merger and the final ringdown. The early inspiral phase can be very well modelled using the approximation schemes such as multipolar post-Minkowskian (MPM) approximation matched to PN [10] whereas the late inspiral, merger and ringdown phases can be modelled using Numerical Relativity (NR) [20] or effective one body approach [21]. In fact, it is now possible to perform numerical simulations to track the evolution of the BH binary systems over many inspiral orbits and the subsequent merger and ringdown phases. However, high computational cost of generating numerical waveforms covering the entire parameter space of coalescing binary black holes (BBHs) has led to the construction of hybrid waveforms (by combining PN and NR waveforms), which further are used to phenomenologically construct a waveform model which has sufficient overlap with the hybrid waveform [22, 24]. In addition to this, one needs to check the consistency between these two waveforms (PN & NR) in a regime where both of them are valid. This would not only tell us about the compatibility of the two waveforms but also would indicate the limits up to which PN waveforms are reliable. There have been many such investigations involving nonspinning BBH in quasi-circular orbits [25, 38] and quasi-eccentric orbits [39, 40]. The need for such comparisons and matching of the two waveforms (PN and NR) has led to the high accuracy computations of spherical harmonic modes of the PN waveforms in case of CCBs moving in quasi-circular orbits [40, 42]. Evidently, in order to perform similar comparisons for eccentric binaries, one would need high accuracy eccentric PN waveforms for such systems.

The leading order (or Newtonian) expressions for the GW polarizations ($h_+$ and $h_{\times}$) were obtained in the context of spacecraft Doppler detection of GWs from an isolated compact binary in eccentric orbit [43]. This work was then extended to 1PN and the next 1.5PN order in [44, 48]. At the 2PN order, the transverse-traceless radiation field ($h_{ij}^{TT}$) due to an isolated binary composed of two compact stars moving in eccentric orbits was computed in [49, 50]. Although the two works, [49] and [50], followed two different approaches, their final findings were in perfect agreement with each other. Under the adiabatic approximation, associated 2PN GW polarizations ($h_+, h_{\times}$) were obtained in [51] for the inspiral phase of binaries in quasi-eccentric orbits. Later in Refs. [52, 53], the method of variation of constants was used to compute post-adiabatic corrections (varying on the orbital time scale and $1/c^5$ times smaller) to the secular variation due to radiation reaction. Using the 3PN generalized quasi-Keplerian representation of the conservative dynamics of compact binary systems with arbitrary mass ratios moving in eccentric orbits presented in [54], Ref. [55] provides the evolution of the orbital phase with relative 1PN accuracy (absolute 3.5PN). The energy and angular momentum fluxes as well as evolution of orbital elements up to 3PN order was calculated in Refs. [56, 57]. Recently, computations of the frequency domain waveforms and the orbital dynamics (both at the 2PN order) were presented for eccentric binaries in harmonic coordinates [58]. On the NR front, the first simulations involving nonspinning equal mass BBHs in bound eccentric orbits were performed in [59, 60] and the effects of eccentricity on the final mass and spin were studied. Another recent work [61] presents numerical simulations for a nonspinning equal mass binary system with an initial eccentricity of $e \sim 0.1$ and compares the NR waveforms with those of the PN models.

In this paper we present the computation of instantaneous part\(^2\) of various modes of the waveform ($h^{\ell m}$) using the basis of spherical harmonics of spin weight -2 for general orbits. In addition we also specialize to the case of compact binaries in quasi-elliptical orbits and provide 3PN instantaneous expressions for various modes using 3PN quasi-Keplerian representation of the conserved dynamics of compact binaries in eccentric orbits [54, 56]. Note

1 They chose a lower cut-off for the fitting factor (FF$\text{min} = 90\%$) corresponding to a loss in event rates of about less than 27%.
2 The part of the gravitational radiation which depends on the state of its source at a given retarded time.
again that investigations presented here involve only the contributions from the instantaneous terms which must be complemented by computations accounting for the hereditary effects.\(^3\) Computations of hereditary parts to various modes of the waveform will form the basis for a companion paper \([61]\).

This paper is organized in the following manner. In Sec. II we first introduce general formulas for spherical harmonic modes of the gravitational waveform, \(h^{\ell m}\), in terms of the radiative mass and current multipole moments, \(U^{\ell m}\) and \(V^{\ell m}\). Section IIIA recalls some of the important aspects of the MPM-PN formalism and lists various inputs that are needed for computing 3PN expressions for various modes. These inputs involve relations connecting radiative moments to source moments, expressions for various source multipole moments for an isolated compact binary system and equations of motion. In Sec. IV we provide our results related to the instantaneous part of the spherical harmonic modes of the waveform for a nonspinning compact binary system in terms of variables that describe the radiation from a generic compact binary. We find that these expressions are quite large and run over several pages. Keeping this in mind we choose to list only the 3PN expression for the dominant mode \((h_{22})\) in the main text of the paper and list all the relevant modes contributing to the 3PN waveform in a separate file readable in MATHEMATICA (Hlm-GenOrb.m) that will be made available on the journal web-page as supplementary material along with the paper. In Sec. V we specialize to the case of CCBs moving in quasi-elliptical orbits and provide the 3PN expression for the dominant mode, \(h_{22}\), in terms of the time-eccentricity \(e_t\), a PN parameter related to the orbital frequency \(x\) and the eccentric anomaly \(u\). Similar to the general orbit case, in the case of CCBs in quasi-elliptical orbits, expressions for all the relevant modes contributing to the 3PN waveform will be listed in a separate file (Hlm-EllOrb.m). Finally in Sec. VI we conclude the paper by providing a brief summary of our results and the future plans.

II. SPHERICAL HARMONIC MODES OF THE GRAVITATIONAL WAVEFORM

For an isolated source of GWs, the spherical harmonic modes of the waveform \((h^{\ell m})\), in terms of the radiative mass-type \((U^{\ell m})\) and current-type multipole moments \((V^{\ell m})\) \([40–42, 62]\), are given as

\[
h^{\ell m} = -\frac{G}{\sqrt{2} R^\ell+2} \left[ U^{\ell m} - \frac{i}{c} V^{\ell m} \right],
\]

where, \(R\) is the distance of the source in radiative coordinates, \(G\) is Newton’s gravitational constant and \(c\) is the speed of the light. The radiative multipole moments, \(U^{\ell m}\) and \(V^{\ell m}\), appearing above are related to the symmetric trace-free (STF) radiative moments \(U_L\) and \(V_L\) as

\[
U^{\ell m} = \frac{4}{\ell !} \sqrt{\frac{(\ell + 1)(\ell + 2)}{2(\ell + 1)(\ell + 2)}} \alpha_L^{\ell m} U_L,
\]

\[
V^{\ell m} = -\frac{8}{\ell !} \sqrt{\frac{\ell(\ell + 2)}{2(\ell + 1)(\ell + 2)}} \alpha_L^{\ell m} V_L.
\]

Here \(\alpha_L^{\ell m}\) denote STF tensors which connect the usual basis of spherical harmonics \(Y^{\ell m}(\Theta, \Phi)\) to the set of STF tensors \(N^{(L)}(\Theta, \Phi)\) as\(^4\)

\[
N^{(L)}(\Theta, \Phi) = \sum_{m=-\ell}^{\ell} \alpha_L^{\ell m} Y^{\ell m}(\Theta, \Phi),
\]

\[
Y^{\ell m}(\Theta, \Phi) = \frac{(2\ell + 1)!!}{4\pi \ell !} \alpha_L^{\ell m} N^{(L)}(\Theta, \Phi).
\]

In the above, \(\mathbf{N} = \mathbf{X}/R\) is a unit vector pointing towards the detector along the line joining the source to the detector. For instance, if the binary’s plane is the \(x-y\) plane then \(\mathbf{N}\), in terms of angles \((\Theta, \Phi)\) giving the location of the binary, can be given as

\[
\mathbf{N} = \sin \Theta \cos \Phi \mathbf{\hat{e}}_x + \sin \Theta \sin \Phi \mathbf{\hat{e}}_y + \cos \Theta \mathbf{\hat{e}}_z.
\]

\(^3\) The part of the gravitational radiation which depends on the entire dynamical history of the source and is complementary to the instantaneous part of the radiation.

\(^4\) Here \(L = i_1 i_2 \cdots i_\ell\) represents a multi-index composed of \(\ell\) spatial indices and the angular brackets \((\cdot)\) surrounding indices denote symmetric trace-free projections.
The STF tensorial coefficients $\alpha_L^{\ell m}$ in terms of $N_{(i_1, \cdots, i_r)}$ and $Y^{\ell m}(\Theta, \Phi)$ can be written as \(^5\)

$$\alpha_L^{\ell m} = \int d\Omega N_{(L)} Y^{\ell m}. \quad (2.5)$$

where the usual basis of spherical harmonics is given as

$$Y^{\ell m}(\Theta, \Phi) = (-)^m \frac{1}{2\ell!} \left[ \frac{2\ell + 1}{4\pi} (\ell + m)! \right]^{1/2} c^m e^m \Phi (\sin \Theta)^m \frac{d^{\ell+m}}{d(cos \Theta)^{\ell+m}} (\cos^2 \Theta - 1)^{\ell}. \quad (2.6)$$

It is important to note that for nonspinning binaries, there exists a mode separation as pointed out in Ref. \(^{41}\) and explicitly shown in Ref. \(^{42}\). The mode $h^{\ell m}$ is completely determined by mass-type radiative multipole moment ($U^{\ell m}$) when $\ell + m$ is even, and by current-type radiative multipole moment ($V^{\ell m}$) when $\ell + m$ is odd. This allows us to write for various modes

$$h^{\ell m} = -\frac{G}{\sqrt{2} R \epsilon_{\ell+2}} U^{\ell m} \quad \text{if } \ell + m \text{ is even}, \quad (2.7a)$$

$$h^{\ell m} = -\frac{i G}{\sqrt{2} R \epsilon_{\ell+3}} V^{\ell m} \quad \text{if } \ell + m \text{ is odd}. \quad (2.7b)$$

### III. INPUTS FOR COMPUTING THE 3PN WAVEFORM

#### A. Relations connecting the radiative moments to the source moments

In the MPM-PN formalism \(^{13, 63, 67}\), the radiative multipole moments ($U_L$, $V_L$) are first written in terms of two sets of canonical moments ($M_L$, $S_L$), which in turn are expressed in terms of six sets of source moments ($I_L$, $J_L$, $W_L$, $Y_L$, $X_L$, $Z_L$). Relations connecting radiative moments to the canonical moments and those connecting the canonical moments to the source moments, with the PN accuracy desired for the waveform computations at the 3PN order, have been established and have been listed in Ref. \(^{40}\) (see Eqs. (5.4-5.8) and Eqs. (5.9-5.11) there). Using these inputs we can parametrize the set of radiative moments (and hence the modes) in terms of source multipole moments. Below we list all the relevant radiative multipole moments in terms of the source multipole moments with PN accuracy desired for the present work. Further, these expressions can be decomposed into two parts namely the instantaneous contribution and the hereditary contribution.

The only radiative moment required at the 3PN order is the one related to the mass quadrupole ($U_{ij}$) and is given by

$$U_{ij} = U_{ij}^{\text{inst}} + U_{ij}^{\text{hered}}, \quad (3.1)$$

where the instantaneous and hereditary parts in terms of the source multipole moments read

$$U_{ij}^{\text{inst}}(U) = I_{ij}^{(2)}(U) + \frac{G}{c^3} \left\{ \frac{1}{7} I_{a(i)}^{(5)} I_{j)a} - \frac{5}{7} I_{a(i)}^{(4)} I_{j)a} - \frac{2}{7} I_{a(i)}^{(3)} I_{j)a} J_b + \frac{1}{3} \varepsilon_{ab(i} I_{j)a} J_b + 4 \left[ W^{(4)} I_{ij} + W^{(3)} I_{ij}^{(1)} - W^{(2)} I_{ij}^{(2)} - W^{(1)} I_{ij}^{(3)} \right] + O \left( \frac{1}{c^7} \right), \quad (3.2a)$$

$$U_{ij}^{\text{hered}}(U) = \frac{2GM}{c^3} \int_{-\infty}^{U} d\tau \left[ \ln \left( \frac{U - \tau}{2\tau_0} \right) + \frac{11}{12} I_{ij}^{(4)}(\tau) + \frac{G}{c^3} \left\{ -\frac{2}{7} \int_{-\infty}^{U} d\tau I_{a(i)}^{(3)}(\tau) I_{j)a}^{(3)}(\tau) \right\} + \frac{57}{70} \ln \left( \frac{U - \tau}{2\tau_0} \right) + \frac{124627}{44100} \right] I_{ij}^{(5)}(\tau) + O \left( \frac{1}{c^7} \right). \quad (3.2b)$$

---

\(^5\) The notation used in \(^{40, 12}\) (which we follow here) to the one in \(^{41, 62}\) is related by $Y_L^{\ell m} = \frac{(2\ell + 1)^{\ell m}}{4\pi} \alpha_L^{\ell m}$. 

In the above, the quantity $M$ represents the mass multipole moment or the Arnowitt, Deser and Misner (ADM) mass of the source. The constant $r_0$ appearing in the above integrals is related to an arbitrary length scale $r_0$ by $\tau_0 = r_0/c$ and was originally introduced in the MPM formalism. Note that, numbers in the parenthesis (appearing as superscripts of the source moments) denote the $p^{th}$ time derivatives. The Levi-Civita tensor is denoted by $\varepsilon_{ijk}$, such that $\varepsilon_{123} = +1$ and $\mathcal{O}(1/c^7)$ indicates that we ignore contributions of the order 3.5PN and higher.

As may be seen from the above, computing the instantaneous part requires source multipole moments given at a retarded time $U$. On the other hand, the hereditary part involves integrals over time and would require the knowledge of the source multipole moments at any instant of time before $U$ in the past dynamical history of the source. Further, the hereditary terms are of two kinds: those with and without the logarithmic factors (see Eq. (3.2b) above). The first integral appearing in Eq. (3.2b) (the one with the logarithmic kernel inside) is called the tail-integral and the one in the last line is called tail-of-tail integral whereas the integral without the logarithmic factor (in the first line) is known as the memory integral. This paper only focuses on computing the instantaneous contribution to various modes of gravitational waveforms and the computation of hereditary contributions shall be discussed elsewhere [61].

Moments required with 2.5PN accuracy are mass octupole moment $U_{ijk}$ and current quadrupole moment $V_{ij}$. The mass octupole moment $U_{ijk}$ is given as

$$U_{ijk} = U_{ijk}^{\text{inst}} + U_{ijk}^{\text{hered}},$$

where $U_{ij}^{\text{inst}}$ and $U_{ijk}^{\text{hered}}$ in terms of source multipole moments read

$$U_{ijk}^{\text{inst}}(U) = \frac{G}{c^3} \left\{ \frac{4}{3} I_{i(k}^{(3)} (U) - 9 \frac{I_{i(k}^{(4)} (U)}{4} + 3 \frac{I_{i(k}^{(5)} (U)}{4} - \frac{5}{4} I_{a(i}^{(1)} (U) - \frac{I_{a(i}^{(5)} (U)}{4} + \frac{I_{a(i}^{(6)} (U)}{12} \right\},$$

$$U_{ijk}^{\text{hered}}(U) = \frac{2GM}{c^3} \int_{-\infty}^{U} d\tau \left[ \ln \left( \frac{U - \tau}{2\tau_0} \right) + \frac{97}{60} I_{i(k}^{(5)} (\tau) \right] + \frac{G}{c^3} \left\{ \int_{-\infty}^{U} d\tau \left[ -\frac{3}{3} I_{a(i}^{(3)} (\tau) I_{j(k}^{(4)} (\tau) \right] \} + \mathcal{O} \left( \frac{1}{c^6} \right).$$

The current quadrupole moment $V_{ij}$ is given as

$$V_{ij} = V_{ij}^{\text{inst}} + V_{ij}^{\text{hered}},$$

where $V_{ij}^{\text{inst}}$ and $V_{ij}^{\text{hered}}$ in terms of source multipole moments read

$$V_{ij}^{\text{inst}}(U) = \frac{J_{ij}^{(2)} (U)}{\tau_0} + \frac{G}{c^3} \left\{ \frac{4}{3} J_{ij}^{(3)} (U) + 8I_{a(i}^{(2)} (U) + 27J_{a(i}^{(4)} (U) - 9I_{a(i}^{(5)} (U) - \frac{9}{8} I_{a(i}^{(6)} (U) + \frac{1}{2} \varepsilon_{abc} \left[ 3J_{a(k}^{(3)} (U) + 353 I_{a(k}^{(4)} (U) + 113 I_{a(k}^{(5)} (U) \right] + \frac{15}{8} I_{a(k}^{(6)} (U) + \frac{3}{8} I_{a(k}^{(6)} (U) + 14 \left[ \varepsilon_{abc} \left( -\frac{J_{j(k}^{(3)} W_a - 2J_{j(k}^{(3)} W_a + 12 J_{j(k}^{(3)} W_a \right) + \mathcal{O} \left( \frac{1}{c^6} \right).$$

At the 2PN order the required moments are $U_{ijkl}$ and $V_{ij}$. The moment $U_{ijkl}$ is given by
\[ U_{ijkl} = U_{ijkl}^{\text{inst}} + U_{ijkl}^{\text{hered}} , \]

where \( U_{ijkl}^{\text{inst}} \) and \( U_{ijkl}^{\text{hered}} \) in terms of the source multipole moments read

\[
U_{ijkl}^{\text{inst}}(U) = \frac{G}{c^3} \left\{ \frac{21}{5} I_{ijkl}^{(5)} - \frac{63}{5} I_{ijkl}^{(4)} + \frac{102}{5} I_{ijkl}^{(3)} \right\} + O \left( \frac{1}{c^5} \right),
\]

\[
U_{ijkl}^{\text{hered}}(U) = \frac{G}{c^3} \left\{ 2M \int_{-\infty}^{U} d\tau \left[ \ln \left( \frac{U - \tau}{2\tau_0} \right) + \frac{59}{30} I_{ijkl}^{(6)}(\tau) + \frac{1}{5} \int_{-\infty}^{U} d\tau I_{ijkl}^{(3)}(\tau) \right] \right\} + O \left( \frac{1}{c^5} \right).
\]

The moment \( V_{ijk} \) is given by

\[ V_{ijk} = V_{ijk}^{\text{inst}} + V_{ijk}^{\text{hered}} , \]

where \( V_{ijk}^{\text{inst}} \) and \( V_{ijk}^{\text{hered}} \) in terms of the source multipole moments read

\[
V_{ijk}^{\text{inst}}(U) = J_{ijk}^{(3)}(U) + \frac{G}{c^3} \left\{ \frac{1}{10} I_{ijkl}^{(5)} - \frac{1}{2} I_{lijk}^{(6)} - 2J_{ijkl}^{(4)} \right\} + O \left( \frac{1}{c^5} \right),
\]

\[
V_{ijk}^{\text{hered}}(U) = \frac{2GM}{c^3} \int_{-\infty}^{U} d\tau \left[ \ln \left( \frac{U - \tau}{2\tau_0} \right) + \frac{5}{3} J_{ijk}^{(5)}(\tau) \right] + O \left( \frac{1}{c^5} \right).
\]

The moments required at the 1.5PN order are \( U_{ijklm} \) and \( V_{ijkl} \). The mass-type moment \( U_{ijklm} \) is given as

\[ U_{ijklm} = U_{ijklm}^{\text{inst}} + U_{ijklm}^{\text{hered}} , \]

where \( U_{ijklm}^{\text{inst}} \) and \( U_{ijklm}^{\text{hered}} \) in terms of the source multipole moments read

\[
U_{ijklm}^{\text{inst}}(U) = J_{ijklm}^{(5)}(U) + \frac{G}{c^3} \left\{ -\frac{710}{21} I_{ijklm}^{(3)} - \frac{265}{7} I_{ijklm}^{(2)} - \frac{120}{7} I_{ijklm}^{(4)} - \frac{155}{7} I_{ijklm}^{(5)} \right\} + O \left( \frac{1}{c^5} \right),
\]

\[
U_{ijklm}^{\text{hered}}(U) = \frac{G}{c^3} \left\{ 2M \int_{-\infty}^{U} d\tau \left[ \ln \left( \frac{U - \tau}{2\tau_0} \right) + \frac{232}{105} I_{ijklm}^{(7)}(\tau) + \frac{20}{21} \int_{-\infty}^{U} d\tau I_{ijklm}^{(3)}(\tau) \right] \right\} + O \left( \frac{1}{c^5} \right).
\]

The current-type moment \( V_{ijkl} \) is given by

\[ V_{ijkl} = V_{ijkl}^{\text{inst}} + V_{ijkl}^{\text{hered}} , \]

where \( V_{ijkl}^{\text{inst}} \) and \( V_{ijkl}^{\text{hered}} \) in terms of the source multipole moments read

\[
V_{ijkl}^{\text{inst}}(U) = J_{ijkl}^{(4)}(U) + \frac{G}{c^3} \left\{ -\frac{35}{3} S_{ijkl}^{(2)} + \frac{25}{3} I_{ijkl}^{(2)} - \frac{65}{6} J_{ijkl}^{(3)} - \frac{25}{6} f_{ijkl}^{(4)} - \frac{25}{6} I_{ijkl}^{(4)} - \frac{19}{6} I_{ijkl}^{(5)} \right\} + O \left( \frac{1}{c^5} \right),
\]

\[
V_{ijkl}^{\text{hered}}(U) = \frac{2GM}{c^3} \int_{-\infty}^{U} d\tau \left[ \ln \left( \frac{U - \tau}{2\tau_0} \right) + \frac{119}{60} J_{ijkl}^{(6)}(\tau) \right] + O \left( \frac{1}{c^5} \right).
\]
Other mass-type moments $U_L$ contributing to 3PN waveform are given as

$$U_L = U_L^{\text{inst}} + U_L^{\text{hered}},$$

where $U_L^{\text{inst}}$ and $U_L^{\text{hered}}$ in terms of source multipole moments read

$$U_L^{\text{inst}}(U) = I_L^{(i)}(U) + \mathcal{O}\left( \frac{1}{c^3} \right),$$

$$U_L^{\text{hered}}(U) = \mathcal{O}\left( \frac{1}{c^3} \right).$$

Other current-type moments $V_L$ contributing to 3PN waveform are given as

$$V_L = V_L^{\text{inst}} + V_L^{\text{hered}},$$

where $V_L^{\text{inst}}$ and $V_L^{\text{hered}}$ in terms of source multipole moments read

$$V_L^{\text{inst}}(U) = J_L^{(i)}(U) + \mathcal{O}\left( \frac{1}{c^3} \right),$$

$$V_L^{\text{hered}}(U) = \mathcal{O}\left( \frac{1}{c^3} \right).$$

\[ B. \text{ Source multipole moments in general dynamical variables} \]

What we need next are expressions for various source multipole moments with the PN accuracy sufficient for the present computation. Expressions for various multipole moments presented here are generalizations of related circular orbit expressions presented in Refs. [40, 68] to the case of general orbits and have been computed using the methods presented in Refs. [64, 69]. We skip all the details of the computation and list the final expressions for the source multipole moments related to a source composed of two nonspinning compact objects moving in general orbits.

The only moment required here with 3PN accuracy is the mass quadrupole, $I_{ij}$, which for CCBs in general orbit was computed in Ref. [69] and listed in Ref. [56] in standard harmonic (SH) coordinates.\footnote{Note that Ref. [56] lists explicit expressions for all the source multipole moments for binaries in general orbits needed for computing 3PN energy flux.} As was argued in Ref. [56], though the use of SH coordinate is useful in performing algebraic checks on PN computations, quantities when expressed in these coordinates involve some\emph{gauge-dependent} logarithmic terms and are not suitable for numerical calculations. It was suggested in Ref. [66] that such logarithms can be transformed away by using some coordinate transformations. They showed how the use of a modified harmonic (MH) coordinate system (or alternatively an ADM coordinate system) removes these logarithms. We skip the details related to those transformations and directly write expression for the mass quadrupole moment in MH coordinates. In MH coordinate, 3PN $I_{ij}$ reads

$$I_{ij} = \nu m \left\{ A_1 \left[ \frac{24}{7^c} \frac{\nu}{r^2} \frac{G m^2}{2^c} \right] x_{(i)(j)} + A_2 \left[ \frac{r}{7^c} \frac{\nu}{r^2} + \frac{48}{7^c} \frac{G m^2}{2^c} \right] \right. x_{(i)(v)(j)} + A_3 \left[ \frac{r^2}{7^c} \frac{\nu}{r^2} v_{(i)(v)(j)} \right] \right\} + \mathcal{O}\left( \frac{1}{c^7} \right), \quad (3.19)$$

where,

$$A_1 = 1 + \frac{1}{c^2} \left[ v^2 \left( \frac{29}{42} - \frac{29}{14} \nu \right) + \frac{G m}{r} \left( \frac{5}{7} + \frac{8}{7} \nu \right) \right] + \frac{1}{c^4} \left[ \frac{G m}{r} v^2 \left( \frac{2021}{756} - \frac{5947}{756} \nu \right) + \left( \frac{35}{756} - \frac{953}{756} \right) + \frac{337}{252} \nu^2 \right] + v^4 \left( \frac{253}{504} - \frac{1835}{504} \nu^2 \right)$$

$$+ \frac{3545}{504} \nu^2 \right) \right\} + \frac{1}{c^6} \left[ v^6 \left( \frac{4561}{11088} - \frac{7993}{11088} \nu + \frac{117067}{5544} \nu^2 - \frac{328663}{11088} \nu^3 \right) + \frac{G m v^2}{r^2} \left( \frac{8539}{20790} \right) \right]$$

$$+ \frac{1}{c^8} \left[ v^8 \left( \frac{4561}{11088} - \frac{7993}{11088} \nu + \frac{117067}{5544} \nu^2 - \frac{328663}{11088} \nu^3 \right) + \frac{G m v^2}{r^2} \left( \frac{8539}{20790} \right) \right]$$
The expression for mass octupole, $I_{ijk}$, at 2.5PN order reads

$$I_{ijk} = -\nu m \Delta \left\{ B_1 - \frac{56}{9} \nu \frac{G^2 m^2}{c^2} \frac{\dot{r}}{r^3} x_{(ij)} + \left[ B_2 \frac{r \dot{r}}{c^2} + \frac{\nu r}{c^2} \left( \frac{232}{15} \frac{G^2 m^2}{r^2} - \frac{12}{5} \frac{G m}{r} \nu^2 \right) \right] x_{(i)j} x_{k} \right\} + B_3 \frac{r^2}{c^2} x_{(ij)k} + B_4 \frac{r^2}{c^2} \frac{\dot{r}}{r} v_{(ijk)} + \mathcal{O} \left( \frac{1}{c^6} \right),$$

(3.21)

where,

$$B_1 = 1 + \frac{1}{c^2} \left[ \frac{\nu^2}{6} \left( \frac{5}{6} - \frac{19 \nu}{6} \right) + \frac{G m}{r} \left( \frac{5}{6} + \frac{13 \nu}{6} \right) \right] + \frac{1}{c^4} \left[ \frac{G m^2}{r^2} \right] \left( \frac{3853}{1320} - \frac{14257 \nu}{1320} - \frac{17371 \nu^2}{1320} \right) - \frac{1347 \nu^2}{440} \right],$$

(3.22a)

$$B_2 = -(1 - 2 \nu) \frac{1}{c^2} \left[ \nu^2 \left( \frac{13}{22} - \frac{107 \nu}{22} - \frac{102 \nu^2}{11} \right) + \frac{G m}{r} \left( \frac{2461}{660} + \frac{8689 \nu}{660} + \frac{1389 \nu^2}{220} \right) \right].$$

(3.22b)
\[
B_3 = 1 - 2\nu + \frac{1}{c^2} \left[ \frac{Gm}{r} \left( \frac{1949}{330} + 62\nu - \frac{483\nu^2}{55} \right) + v^2 \left( \frac{61}{110} - \frac{519\nu}{110} + \frac{504\nu^2}{55} \right) \right] + \mathcal{O} \left( \frac{1}{c^6} \right),
\]

\[
B_4 = \left( \frac{13}{55} - \frac{52\nu}{55} + \frac{39\nu^2}{55} \right). \tag{3.22d}
\]

The remaining mass-type source multipole moments with PN accuracy required in the present work read

\[
I_{ijkl} = \nu m \left\{ x_{ijkl} \left[ 1 - 3\nu + \frac{1}{c^2} \left[ \frac{Gm}{r} \left( -\frac{10}{11} + \frac{61\nu}{11} - \frac{105\nu^2}{11} \right) + v^2 \left( \frac{103}{110} - \frac{147\nu}{22} + \frac{279\nu^2}{22} \right) \right] \right] + \frac{1}{c^4} \left[ v^4 \left( \frac{3649}{5720} - \frac{5019\nu}{5720} + \frac{112357\nu^2}{2860} - \frac{325687\nu^3}{5720} \right) + \frac{G^2 m^2}{r^2} \left( -\frac{15549}{1010} + \frac{9457\nu}{715} + \frac{7961\nu^2}{143} \right) \right]
\]

\[
- \frac{5829\nu^3}{286} + G m \left( \frac{11049\nu}{3575} + \frac{152489\nu^2}{7150} + \frac{15124\nu^2}{715} + \frac{46934\nu^3}{715} \right) + \frac{G^2 m^2}{r^2} \left( -\frac{15549}{1010} + \frac{9457\nu}{715} + \frac{7961\nu^2}{143} \right) \right]
\]

\[
+ \frac{12619\nu}{7150} - \frac{10557\nu^2}{1430} + \frac{9671\nu^3}{715} \right] \right] + x_{ijklb} \frac{r^b}{c^2} \left[ \frac{-72}{55} + \frac{72\nu}{11} - \frac{72\nu^2}{11} + \frac{1}{c^2} \left[ \frac{Gm}{r} \left( -\frac{103}{110} + \frac{147\nu}{22} - \frac{279\nu^2}{22} \right) \right] \right]
\]

\[
\times \left( \frac{-15463}{3575} + \frac{98374\nu}{3575} - \frac{25606\nu^2}{715} - \frac{18839\nu^3}{715} \right) + v^2 \left( -\frac{476}{715} + \frac{1228\nu}{143} - \frac{23512\nu^2}{715} \right)
\]

\[
+ \frac{25796\nu^3}{715} \right] \right] + x_{ijklb} \frac{r^b}{c^2} \left[ \frac{78}{55} - \frac{78\nu}{11} + \frac{78\nu^2}{11} + \frac{1}{c^2} \left[ v^2 \left( \frac{553}{715} - \frac{6913\nu}{715} + \frac{25994\nu^2}{715} \right) \right]
\]

\[
- \frac{28207\nu^3}{715} \right] + G m \left( \frac{27818}{3575} - \frac{72474\nu}{3575} + \frac{17202\nu^2}{715} - \frac{27568\nu^3}{715} \right) \right] + r^2 \left( -\frac{4}{13} + \frac{28\nu}{13} \right)
\]

\[
- \frac{56\nu^2}{13} + \frac{28\nu^3}{13} \right] \right] + x_{ijklb} \frac{r^b}{c^2} \left[ \frac{304}{715} + \frac{2128\nu}{715} + \frac{2128\nu^3}{715} \right] + v_{ijklb} \frac{r^b}{c^2} \left[ \frac{1}{c^2} \right]
\]

\[
\times \left( \frac{71}{715} - \frac{497\nu}{715} + \frac{994\nu^2}{715} - \frac{497\nu^3}{715} \right) \right] + O \left( \frac{1}{c^6} \right). \tag{3.23a}
\]

\[
I_{ijklm} = -\nu m \Delta \left\{ x_{ijklm} \left[ 1 - 2\nu + \frac{1}{c^2} \left[ \frac{Gm}{r} \left( -\frac{25}{26} + \frac{139\nu}{26} - \frac{109\nu^2}{13} \right) + v^2 \left( \frac{79}{78} - \frac{511\nu}{78} \right) \right] \right] + \frac{137\nu^2}{13} \right] \right) + x_{ijklm} \frac{r^l}{c^2} \left[ \frac{20}{13} + \frac{80\nu}{13} - \frac{60\nu^2}{13} \right] + x_{ijklm} \frac{r^l}{c^2} \left[ \frac{39}{39} - \frac{280\nu}{39} + \frac{70\nu^2}{13} \right]
\]

\[
+ O \left( \frac{1}{c^6} \right). \tag{3.23b}
\]

\[
I_{ijklmn} = \nu m \left\{ x_{ijklmn} \left[ 1 - 5\nu + 5\nu^2 + \frac{1}{c^2} \left[ v^2 \left( \frac{15}{14} - \frac{21\nu}{2} + 33\nu^2 - \frac{63\nu^3}{2} \right) - \frac{Gm}{r} \left( 1 - 9\nu 
\right) + 14\nu^2 - 7\nu^3 \right) \right] \right) \right] + O \left( \frac{1}{c^6} \right), \tag{3.23c}
\]

\[
I_{ijklmno} = -\nu m \Delta \left( 1 - 4\nu + 3\nu^2 \right) x_{ijklmno} + O \left( \frac{1}{c^6} \right), \tag{3.23d}
\]

\[
I_{ijklmnop} = \nu m \left( 1 - 7\nu + 14\nu^2 - 7\nu^3 \right) x_{ijklmnop} + O \left( \frac{1}{c^6} \right). \tag{3.23e}
\]

The current quadrupole moment is needed at 2.5PN order and given as

\[
J_{ij} = -\nu m \Delta \left\{ C_1 - \frac{62\nu G^2 m^2}{7c^5} \right\} \epsilon_{ab(i)x_a)y_b} + \frac{C_2}{c^5} + \nu \frac{Gm}{r} \left( \frac{216Gm}{35r} - \frac{4}{5} v^2 \right) \epsilon_{ab(i)x_a} + O \left( \frac{1}{c^6} \right) \tag{3.24}
\]
where,

\[
C_1 = 1 + \frac{1}{c^2} \left[ \nu^2 \left( \frac{13}{28} - \frac{17\nu}{7} \right) + \frac{Gm}{r} \left( \frac{27}{14} + \frac{15\nu}{7} \right) \right] + \frac{1}{c^2} \left[ \frac{Gm}{r} \nu^2 \left( \frac{671}{252} - \frac{1297\nu}{126} - \frac{121\nu^2}{12} \right) \right] \\
+ \frac{Gm}{r} \nu^2 \left( \frac{5}{252} - \frac{241\nu}{252} - \frac{335\nu^2}{84} \right) + \frac{G^2m^2}{r^2} \left( \frac{43}{252} - \frac{1543\nu}{126} + \frac{293\nu^2}{84} \right) + \nu^4 \left( \frac{29}{84} - \frac{11\nu}{3} \right) + \frac{505\nu^2}{56} \right],
\]

\[
C_2 = \frac{5}{28} (1 - 2\nu) + \frac{1}{504} c^2 \left[ \frac{Gm}{r} \left( 824 + 1348\nu - 1038\nu^2 \right) + 75\nu^2 (1 - 7\nu + 12\nu^2) \right].
\]

Other current-type source multipole moments with PN accuracies sufficient for present calculations read

\[
J_{ijk} = \nu m \epsilon_{abi} \left\{ x_{ikj} v_b \left[ 1 - 3\nu + \frac{1}{c^2} \left[ \frac{Gm}{r} \left( \frac{14}{9} - \frac{16\nu}{9} - \frac{86\nu^2}{9} \right) + \nu^2 \left( \frac{41}{90} - \frac{77\nu}{18} + \frac{185\nu^2}{18} \right) \right] \right] + \frac{1}{c^2} \left[ \nu^2 \left( \frac{1349}{3960} - \frac{4159\nu}{792} + \frac{52409\nu^2}{1980} - \frac{171539\nu^3}{3960} \right) + \frac{G^2m^2}{r^2} \left( \frac{45}{44} - \frac{988\nu}{99} + \frac{9925\nu^2}{198} \right) \right] \right. \\
\left. - \frac{8093\nu^3}{396} + \frac{Gm}{r} \nu^2 \left( \frac{23}{396} - \frac{637\nu}{990} - \frac{1861\nu^2}{990} + \frac{3222\nu^3}{1980} \right) \right] + \frac{Gm}{r} \nu^2 \left( \frac{1597}{660} - \frac{19381\nu}{990} \right) \right] \\
+ \frac{6307\nu^3}{198} + \frac{21127\nu^3}{396} \right] \right] + x_j v_{kj} v_a r \dot{r} \left[ \frac{2}{9} - \frac{10\nu}{9} + \frac{10\nu^2}{9} + \frac{1}{c^2} \left[ \nu^2 \left( \frac{73}{495} - \frac{814\nu}{495} + \frac{3002\nu^2}{495} \right) \right] \right. \\
\left. - \frac{3151\nu^3}{495} + \frac{Gm}{r} \left( \frac{133}{66} - \frac{81\nu}{55} - \frac{3914\nu^2}{330} + \frac{3089\nu^3}{330} \right) \right] + v_{kj} v_a \frac{r^2}{c^2} \left( \frac{7}{45} (1 - 5\nu + 5\nu^2) \right) \right] \\
+ \frac{1}{c^2} \left[ \nu^2 \left( \frac{119}{990} - \frac{259\nu}{198} + \frac{2219\nu^2}{495} - \frac{4529\nu^3}{990} \right) + \dot{r}^2 \left( \frac{14}{165} - \frac{98\nu}{165} + \frac{196\nu^2}{165} - \frac{98\nu^3}{165} \right) \right] \\
\left. + \frac{Gm}{r} \left( \frac{751}{495} - \frac{1792\nu}{198} + \frac{227\nu^2}{99} + \frac{427\nu^3}{99} \right) \right] \right] \right) + \mathcal{O} \left( \frac{1}{c^5} \right),
\]

\[
J_{ijkl} = - \nu m \Delta \epsilon_{abi} \left\{ x_{ikl} v_a \left[ 1 - 2\nu + \frac{1}{c^2} \left[ \frac{Gm}{r} \left( \frac{15}{11} + \frac{35\nu}{44} - \frac{185\nu^2}{22} \right) + \nu^2 \left( \frac{5}{11} - \frac{95\nu}{22} \right) \right] \right] \\
+ \frac{195\nu^2}{22} \right] \right] \right) + \frac{5}{22} x_j v_{kj} v_a \frac{r^2}{c^2} (1 - 4\nu + 3\nu^2) + \frac{4}{11} x_j v_{kj} v_a \frac{r^2}{c^2} (1 - 4\nu + 3\nu^2) \right] \\
+ \mathcal{O} \left( \frac{1}{c^5} \right),
\]

\[
J_{ijklm} = \nu m \epsilon_{abi} \left\{ x_{iklm} v_a \left[ 1 - 5\nu + 5\nu^2 + \frac{1}{c^2} \left[ \nu^2 \left( \frac{83}{182} - \frac{161\nu}{26} + \frac{317\nu^2}{13} - \frac{707\nu^3}{26} \right) \right] \right] \\
+ \frac{Gm}{r} \left( \frac{81}{65} - \frac{138\nu}{65} - \frac{210\nu^2}{13} + \frac{339\nu^3}{13} \right) \right] \right) \right] + \frac{20}{91} x_{iklm} v_a \frac{r^2 \dot{r}}{c^2} (1 - 7\nu + 14\nu^2 - 7\nu^3) \right] + \mathcal{O} \left( \frac{1}{c^5} \right),
\]

\[
J_{ijklmn} = - \nu m \Delta \epsilon_{ab(i} x_{jklmn)} v_a \left( 1 - 4\nu + 3\nu^2 \right) + \mathcal{O} \left( \frac{1}{c^5} \right),
\]

\[
J_{ijklmno} = \nu m \epsilon_{abi} x_{ijklmno} v_a \left( 1 - 7\nu + 14\nu^2 - 7\nu^3 \right) + \mathcal{O} \left( \frac{1}{c^5} \right).
\]

The required gauge moments, the monopolar moment \( W \) and two dipolar moments \( W_i \) and \( Y_i \) are finally given by,

\[
W = \frac{1}{3} \nu m r \dot{r} + \mathcal{O} \left( \frac{1}{c^5} \right),
\]

\[
W_i = \frac{1}{10} \nu m \Delta \nu^2 \left[ \nu^3 - \frac{3}{r} \dot{x}_i \right] + \mathcal{O} \left( \frac{1}{c^5} \right),
\]

\[
Y_i = \mathcal{O} \left( \frac{1}{c^5} \right).
\]
\[ Y_i = \frac{1}{5} \nu m \Delta \left[ \frac{1}{2} \frac{G m}{r} x^i - \frac{1}{2} v^2 x^i - \frac{3}{2} r \dot{v} x^i \right] + \mathcal{O} \left( \frac{1}{c^2} \right). \]  

(3.27c)

### C. The post-Newtonian compact binary dynamics

Since relations connecting the radiative multipole moment to the source multipole moment involve time derivatives of the source multipole moments, computations of various modes will require the knowledge of the equations of motion (EOM) with the PN accuracy with which one wants to compute various modes. Before we write expression for the EOM, with the PN accuracy required for the present work, let us recall the definitions of various dynamical variables as (EOM) with the PN accuracy with which one wants to compute various modes. Before we write expression for the EOM reduced to CM frame associated with MH coordinate which is given in [75] and takes the following form as discussed in the previous section (about using MH or ADM coordinates instead of SH coordinates) we wish to use accurate expression for relative acceleration, reduced to CM frame, in SH coordinates, were obtained in [74]. However, multipole moments in the center-of-mass frame of the system, we need 3PN EOM reduced to CM frame. The 3PN the position and the velocity of individual constituents of the binary. Since we are using expressions for the source multipole moments, in the center-of-mass frame of the system, we need 3PN EOM reduced to CM frame. The 3PN accurate expression for relative acceleration, reduced to CM frame, in SH coordinates, were obtained in [74]. However, as discussed in the previous section (about using MH or ADM coordinates instead of SH coordinates) we wish to use EOM reduced to CM frame associated with MH coordinate which is given in [75] and takes the following form

\[ a^i = -\frac{G m}{r^2} \left[ P_1 - \frac{\nu}{c^3} \left( 136 \frac{G^2 m^2}{r^2} \dot{v} + \frac{24 G m}{r^2} \dot{v} v^2 \right) \right] n^i + \left[ P_2 \frac{\dot{r}}{c^2} + \frac{\nu}{c^3} \left( \frac{24 G^2 m^2}{r^2} + \frac{8 G m}{r} v^2 \right) \right] v^i, \]

(3.34)

where,

\[
P_1 = 1 + \frac{1}{c^2} \left[ \frac{G m}{r} (-4 - 2\nu) - \frac{3\dot{v}^2 \nu}{2} + v^2 (1 + 3\nu) \right] + \frac{1}{c^4} \left[ \frac{G^2 m^2}{r^2} \left( 9 + \frac{87\nu}{4} \right) + \dot{v}^4 \left( \frac{15\nu}{8} - \frac{45\nu^2}{8} \right) \right] + v^4 \left( 3\nu - 4\nu^2 \right) + \frac{G m}{r} \dot{r}^2 (-2 - 25\nu - 2\nu^2) + v^2 \left( \frac{G m}{r} \left( -13\nu \frac{2}{2} + 2\nu^2 \right) + \dot{v}^2 \left( -\frac{9\nu}{2} + 6\nu^2 \right) \right) \]

\[
P_2 = \frac{G m}{r} \left( 2 - 3\nu - 2\nu^2 \right) + \frac{G m}{r} \dot{v} \left( -1 - 5\nu - 2\nu^2 \right) + v^2 \left( \frac{G m}{r} \left( 1 + 8\nu \frac{2}{2} + 2\nu^2 \right) + \dot{v}^2 \left( -\frac{9\nu}{2} + 6\nu^2 \right) \right).
\]
\[ P_2 = -4 + 2 \nu + \frac{1}{c^2} \left[ v^2 \left( -\frac{15 \nu}{2} - 2 \nu^2 \right) + v^2 \left( \frac{9 \nu}{2} + 3 \nu^2 \right) + \frac{G m}{r} \left( \frac{2}{2} + \frac{41 \nu}{2} + 4 \nu^2 \right) \right] \]
\[ + \frac{1}{c^2} \left[ v^4 \left( -\frac{45 \nu}{8} + 15 \nu^2 + 15 \nu^3 \right) + v^4 \left( -\frac{65 \nu}{8} + 19 \nu^2 + 6 \nu^3 \right) + \frac{G^2 m^2}{r^2} \left( -4 - \frac{5849 \nu}{840} - \frac{123 \nu^2 \nu}{32} \right) \right] \]
\[ + 25 \nu^2 + 8 \nu \]
\[ \left( \frac{329 \nu}{6} + \frac{59 \nu^2}{2} + 18 \nu^3 \right) + v^2 \left( \frac{12 \nu}{4} - \frac{111 \nu^2}{4} - 12 \nu^3 \right) + \frac{G m}{r} \left( -15 \nu \right) \]
\[ -27 \nu^2 - 10 \nu^3 \right) \].

Now we have all the inputs which are needed to compute the instantaneous expressions for various spherical harmonic modes \((h^{\ell m})\) associated with 3PN gravitational waveforms of GW signals from CCBs moving in general orbits. With this motivation we shall proceed towards the next section where we shall present our results.

IV. INSTANTANEOUS TERMS IN THE 3PN GRAVITATIONAL WAVEFORM FOR CCBs IN GENERAL ORBITS

Combining Eq. (4.1) and Eq. (4.2) we can write the instantaneous part of various modes as

\[ h^{\ell m}_{\text{inst}} = -\frac{G}{\sqrt{2} R c^{\ell+2} \ell!} \sqrt{\frac{(\ell + 1)(\ell + 2)}{2\ell(\ell - 1)}} \alpha^{\ell m}_{\ell L} U^{\ell m}_{L} \]
\[ \quad \text{if } \ell + m \text{ is even,} \]
\[ h^{\ell m}_{\text{inst}} = -\frac{i G}{\sqrt{2} R c^{\ell+3} \ell!} \sqrt{\frac{\ell(\ell + 2)}{2(\ell + 1)(\ell - 1)}} \alpha^{\ell m}_{\ell L} V^{\ell m}_{L} \]
\[ \quad \text{if } \ell + m \text{ is odd.} \]

Relations connecting instantaneous part of STF radiative moments \((U^{\ell m}_{L} \text{ and } V^{\ell m}_{L})\) to the source multipole moments have been listed in the previous section. This allows one to write the instantaneous part of various modes \((h^{\ell m}_{\text{inst}})\) in terms of the source multipole moments. With expressions for the source multipole moments for CCBs moving in general orbits and their relevant time derivatives, one can write expressions for various modes in terms of dynamical variables related to the position and velocity \((r, \dot{r}, \phi, v)\).  

Again, since \(v^2 = \dot{r}^2 + r^2 \dot{\phi}^2\), we can write various modes of the waveform in terms of the dynamical variables, namely, the radial separation \((r)\), radial velocity \((\dot{r})\), orbital phase \((\phi)\) and the angular velocity \((\dot{\phi})\). The structure of \(h^{\ell m}\) reads

\[ h^{\ell m}_{\text{inst}} = \frac{4 G m \nu}{c^2 R} \sqrt{\frac{\pi}{5}} e^{-i m \phi} \dot{H}^{\ell m}_{\text{inst}}. \]

For the dominant mode \((\ell = 2, m = 2)\), with 3PN accuracy, various PN pieces of the coefficient \(\dot{H}^{\ell m}_{\text{inst}}\) read

---

7 Alternatively, one can also compute various modes associated with the gravitational waveform using polarization waveforms (see Sec. II and IX of Ref. [10] for the details).
\[ \hat{H}_{\text{inst}}^{22} \text{Newt} = \frac{Gm}{r} + r^2 \dot{\phi}^2 + 2i r \dot{\phi} - \dot{r}^2, \quad (4.4a) \]

\[ \hat{H}_{\text{inst}}^{22} \text{1PN} = \frac{1}{c^2} \left[ \frac{Gm^2}{r^2} \left( -5 + \frac{\nu}{2} - 9i \frac{\nu}{7} \right) \phi^2 + \frac{Gm^2}{r} \left( -\frac{15}{14} - \frac{16 \nu}{7} \right) r^4 + \frac{9i}{7} \right] + r^3 \left( \frac{9i}{7} - 27i \frac{\nu}{7} \right) \phi^3, \quad (4.4b) \]

\[ \hat{H}_{\text{inst}}^{22} \text{2PN} = \frac{1}{c^4} \left[ \frac{G^2 m^3}{r^3} \left( \frac{757}{63} + \frac{181 \nu}{36} + 79 \nu^2 - \frac{126}{126} + \left( -\frac{83}{168} + \frac{589 \nu}{168} - \frac{1111 \nu^2}{168} \right) \phi^2 + \frac{G^2 m^2}{r} \left( \frac{11891}{1512} + \frac{5225 \nu}{216} + \frac{13133 \nu^2}{1512} \right) \phi^2 + \frac{Gm^3}{r} \left( \frac{835}{252} - \frac{2995 \nu^2}{252} + \frac{19 \nu^3}{252} \right) \phi^3 + \frac{r^6}{r^2} \left( \frac{Gm}{r} \left( \frac{863i}{126} + \frac{73i \nu}{63} \right) \phi \right) + \frac{r}{r^2} \left( \frac{3pi}{84} + \frac{1111 \nu^2}{84} \right) \phi^3 \right], \quad (4.4c) \]

\[ \hat{H}_{\text{inst}}^{22} \text{2.5PN} = \frac{1}{c^5} \left[ -\frac{122 G^2 m^2 \nu^3}{35r^2} + \frac{468i G^2 m^3 \nu \phi}{35r^2} + \frac{184i G^2 m^2 \nu \phi}{35r^2} - \frac{316}{35} G^2 m^2 r \nu \phi^3 \right] + \frac{Gm^5}{r} \left( \frac{507i}{1232} - \frac{12125i \nu^2}{1232} \right) \phi^2 + \frac{G^3 m^3 \nu \dot{\phi}}{r} + \frac{184i G^2 m^2 \nu \phi}{35r^2}, \quad (4.4d) \]
Finally, circular-orbit limit of the instantaneous $h_{\ell m}$ can be obtained by replacing related expressions for $\phi(=\omega)$, $\dot{r}$ and $r$ given in Sec. IV of [40].

Note that $h_{\ell m}$ can directly be used to write the polarization waveforms $(h_+, h_\times)$ using the standard decomposition of $h_+$ and $h_\times$ in terms of spherical harmonic modes of spin weight -2 given in Ref. [40, 41],

$$h_+ - ih_\times = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} h_{\ell m}^\ell \ Y_{-2}^\ell (\Theta, \Phi),$$

(4.5)

where $Y_{-2}^\ell$'s (the spin-weighted spherical harmonics of weight $-2$) are functions of the spherical angles $(\Theta, \Phi)$ defining the binary's location, and given as

$$Y_{-2}^\ell = \sqrt{\frac{2\ell + 1}{4\pi}} d_{-2}^\ell (\Theta) e^{im\Phi},$$

(4.6a)

$$d_{-2}^\ell = \sum_{k=k_1}^{k_2} \frac{(-1)^k}{k!} \frac{\sqrt{(\ell + m)! (\ell - m)! (\ell + 2)! (\ell - 2)!}}{(k + m - 2)! (k + m - k)!} \left( \cos \frac{\Theta}{2} \right)^{2\ell + m - 2k - 2} \left( \sin \frac{\Theta}{2} \right)^{2k - m + 2},$$

(4.6b)

with $k_1 = \max(0, m - 2)$ and $k_2 = \min(\ell + m, \ell - 2)$.

Further, the polarization waveform can be used to write the transverse-traceless part of the radiation field $(h_{ij})$ by using the following relation [40, 41]

$$h_{ij}^{TT} = 2 \left( h_+ e_+^{ij} + h_\times e_\times^{ij} \right),$$

(4.7)

where

$$e_+^{ij} = \frac{1}{2} (P_i P_j - Q_i Q_j) \quad \text{and} \quad e_\times^{ij} = \frac{1}{2} (P_i Q_j + P_j Q_i)$$

(4.8a)

(4.8b)

Here $P$ and $Q$ are the two unit polarization vectors and they have been chosen following the convention used in [40].
V. QUASI-KEPLERIAN REPRESENTATION

In the previous section we listed the 3PN accurate expression for the dominant mode \((h^{22})\) in terms of the variables \(r, \dot{r}, \phi\) and \(\dot{\phi}\). Although, this kind of representation is the most generic one, while specializing to the case of elliptical orbits it is usually convenient to express the relevant quantities as functions of parameters associated with elliptical orbits. For instance, expressions for various \(h_{\ell m}\) suitable for describing the radiation from binaries in elliptical orbits can be obtained by replacing the dynamical variable \(r, \dot{r}, \phi\) and \(\dot{\phi}\) in terms of the parameters associated with elliptical orbits in related general orbit \(h_{\ell m}\) expressions obtained in the previous section. In order to be consistent with the PN accuracy of the results in the present work we need to know the 3PN generalized quasi-Keplerian (QK) representation of the conservative dynamics of the binary moving in eccentric orbits, which indeed is available to us due to the work of Memmesheimer, Gopakumar and Schäfer (hereafter MGS) \([54]\).

The QK representation was first introduced by Damour and Deruelle \([76]\) and dealt with the binary dynamics at 1PN order. The generalized QK representation at 2PN order in ADM type coordinates was given in Ref. \([77–79]\). MGS provides the 3PN generalized QK representation in both the ADM and MH coordinates, which involve expressions of the orbital elements associated with the orbit of the binary in terms of the conserved energy and orbital angular momentum of the binary. Before we get into the details of the parametrization, we first summarize equations describing the radial and angular motion of the binary in terms of various orbital elements associated with elliptical orbits (see Refs. \([53, 56]\) for details). In the parametric form, the radial separation, \(r\), is given by

\[
r = a_r(1 - e_r \cos u),
\]

where \(a_r\) is the semi-major axis of the orbit and \(e_r\) is the eccentricity of the orbit (both labeled after the radial coordinate, \(r\)). The quantity \(u\) is called eccentric anomaly and at the 3PN order it is related to the mean anomaly \((l)\) by the relation

\[
l = u - e_t \sin u + f_t \sin V + g_t [V - u] + i_t \sin 2V + h_t \sin 3V.
\]

The orbital phase, \(\phi\), at the 3PN order reads

\[
\phi = \phi_P + K [V + f_\phi \sin 2V + g_\phi \sin 3V + i_\phi \sin 4V \\
+ h_\phi \sin 5V],
\]

where \(\phi_P\) is the initial phase at the first passage of the periastron and \(V\) is the true anomaly that takes the form

\[
V \equiv V(u, e_\phi) = 2 \arctan \left( \sqrt{\frac{1 + e_\phi}{1 - e_\phi}} \tan \left( \frac{u}{2} \right) \right).
\]

Also, the mean anomaly, \(l\), is related to the time as

\[
l = n(t - t_P),
\]

where \(t_P\) is the instant of the first passage at the periastron and \(n = 2\pi/P\) is the mean motion with \(P\) being the orbital period.

In addition to this, expressions for the radial and angular velocity can be given as

\[
\dot{r} = a_r e_r \sin u \left( \frac{\partial l}{\partial u} \right)^{-1} \frac{\partial l}{\partial t},
\]

\[
\dot{\phi} = K (1 + 2f_\phi \cos 2V + 3g_\phi \cos 3V + 4i_\phi \cos 4V \\
+ 5h_\phi \cos 5V) \frac{\partial V}{\partial u} \left( \frac{\partial l}{\partial u} \right)^{-1} \frac{\partial l}{\partial t},
\]

It may be seen from Eqs. \((5.2), (5.3)\) and \((5.5)\) that

\[
\frac{\partial l}{\partial t} = n,
\]

\[
\frac{\partial l}{\partial u} = 1 - e_t \cos u + f_t \cos V \frac{\partial V}{\partial u} + g_t (\frac{\partial V}{\partial u} - 1)
\]


\[ + 2 i t \cos 2V \frac{\partial V}{\partial u} + 3 h t \cos 3V \frac{\partial V}{\partial u}. \]  

(5.7b)

\[ \frac{\partial V}{\partial u} = \frac{(1 - e^2_\phi)^{1/2}}{(1 - e_\phi \cos u)}, \]  

(5.7c)

In the above, \( e_\phi \) and \( e_\ell \) denote eccentricities related to the coordinates \( \phi \) and \( t \), respectively. \( K \) is related to the advance of the periastron per orbit, and is given by \( K = \Phi/(2 \pi) \), where \( \Phi \) is the angle of return to the periastron. In this parametrization, \( f_j, f_\phi, g_t \) and \( g_\phi \) contribute both at 2PN and 3PN order whereas \( i_t, i_\phi, h_t \) and \( h_\phi \) contribute only at the 3PN order (see Ref. [56] for related details).

Once we have written equations connecting the generic dynamical variables \((r, \dot{r}, \phi, \dot{\phi})\) to the orbital elements of the elliptical orbit, we can use inputs from MGS to express them in terms of a suitable set of parameters of our choice. The main result of MGS is that, it provides 3PN accurate expressions for various orbital elements \((a_r, e_t, e_r, e_\phi, \ldots)\) associated with the elliptical orbits in terms of the 3PN accurate conserved energy per unit reduced mass \((E)\) and the parameter \( h \), related to the reduced angular momentum \((J)\), by \( h = J/Gm. \) Using these relations one can express the dynamical variables \( r, \dot{r}, \phi \) and \( \dot{\phi} \) in terms of \( E, h \) and \( u \). Here one should note that this is not the only way in which the orbital dynamics can be parametrized. In fact, one can re-express \( E \) and \( h \) in terms of any of the two orbital elements to write equations describing the orbital motion of the binary; however a parametrization involving gauge invariant parameters is sometimes preferred as such parametrization is suitable for making comparisons with related numerical results. This led MGS to use a parametrization involving \( n \) and \( K = \Phi/(2 \pi) \) (both are independent of the coordinate system used when expressed in terms of \( E \) and \( h \) in order to describe the orbital motion of the binary in elliptical orbits).\(^9\) Here, \( n \) is the mean motion and \( K = \Phi/(2 \pi) \) denotes the angle of the advance of the periastron per orbital revolution. In a work related to the phasing of the GWs from inspiralling compact binary in elliptical orbit due to Damour, Gopakumar and Iyer \([52]\), the orbital dynamics has been described using \( n \) and \( e_t \) as parametrizing variables. Following the conventions of \([52]\), Königsdöfer and Gopakumar \([53]\) provided the 3PN accurate expressions for \( r, \dot{r}, \phi \) and \( \dot{\phi} \) in terms of \( n, e_t \) and \( u \).\(^9\) Reference \([56]\) makes an alternative choice of parametrization in terms of variables \( x \) and \( e_t \), where \( x \) is related to the orbital frequency \( \omega \) by, \( x = (Gm \omega/e^3)^{2/3} \), and is independent of the choice of the coordinate system used.\(^10\) The choice \( \{x, e_t\} \) as a parameterizing variable leads to expressions which can be reduced to those related to the circular orbit case \((e_t \to 0)\) which uses \( x \) as the expansion parameter. In addition to this, in another related work Hinder et al. \([59]\) compared the two parameterizations and concluded that the choice of \( x \) as compared to \( n \) provides better agreement with NR results.\(^12\)

Following the arguments presented above we choose to parametrize the dynamical variables \( \{r, \dot{r}, \phi, \dot{\phi}\} \) in terms of the QK variables \( \{x, e_t, u\} \) which can further be used to write expressions for various \( h_{m,n} \) in terms of the QK variables. Note that Ref. \([33]\) already lists the 3PN expressions for \( r, \dot{r}, \phi, \dot{\phi} \) in terms of \( \{x, e_t, u\} \), however, they numerically differentiate \( r \) to obtain \( \dot{r} \) and perform numerical integration of \( \phi \) to obtain \( \dot{\phi} \) in order to avoid the use of long and complicated expressions for \( \dot{r} \) and \( \dot{\phi} \) in their numerical code. As mentioned above, Ref. \([53]\) lists 3PN accurate expressions for \( \{r, \dot{r}, \phi, \dot{\phi}\} \) in terms of the QK variables \( \{n, e_t, u\} \) in MH coordinates (see Eqs. (23)-(28) there). In fact, they use a variable \( \xi \) related to \( n \) by \( \xi = (Gmn/e^4)^{1/3} \). Hence to obtain related expressions parametrized in terms of \( \{x, e_t, u\} \) all we need to know is how \( \xi \) is related to \( x \) and \( e_t \). The relation between \( \xi \) and our QK variables \( x \) and \( e_t \) with 3PN accuracy in MH coordinates is given as

\[ \xi = \frac{x^{3/2}}{(1 - e^2_t)^3} \left( 1 - 3e_t^2 + 3e^4_t - e_t^6 + x \left( -3 + 6e_t^2 - 3e^4_t \right) \right) + x^3 \left[ \frac{3}{2} + \nu \left( \frac{457}{4} - \frac{123\pi^2}{32} \right) - 7\nu^2 + \left( \frac{267}{4} + \nu \left( \frac{279}{2} - \frac{123\pi^2}{128} \right) - 40\nu^2 \right) e_t^2 + \left( \frac{39}{2} + \frac{55\nu}{4} - \frac{65\nu^2}{8} \right) e_t^4 \right], \]

\[ 8 \text{ in Ref. [56], which uses results obtained in MGS, orbital elements are expressed in terms of the parameters } \{\epsilon, j\} \text{ instead of } \{E, h\}, \text{ where } \epsilon \text{ and } j \text{ are defined as } \epsilon = -2 E/c^2 \text{ and } j = -2 E h^2. \]

\[ 9 \text{ In fact, MGS uses } x_{\text{MGS}} = (Gmn/e^4)^{2/3} \text{ and the parameter } k' = (K - 1)/3. \]

\[ 10 \text{ In fact, } [53] \text{ is the extension of } [52] \text{ and discusses the 3PN conservative dynamics of the binaries in elliptical orbits.} \]

\[ 11 \text{ Parameters } n \text{ and } \omega \text{ are related by } n = K \omega, \text{ and } K \text{ has been defined above.} \]

\[ 12 \text{ They separately discuss these two PN models and call them x-model and n-model.} \]
\[
+h_{\text{inst}}^\ell m = \frac{4 G m \nu x}{c^2 R} \sqrt{\frac{\pi}{5}} e^{-i m \phi} H_{\text{inst}}^\ell m. \tag{5.9}
\]

The instantaneous part of dominant mode \((h_{\text{inst}}^{22})\) reads
\[
h_{\text{inst}}^{22} = \frac{4 G m \nu x}{c^2 R} \sqrt{\frac{\pi}{5}} e^{-2i \phi} H_{\text{inst}}^{22}. \tag{5.10a}
\]

where various PN pieces appearing in Eq. (5.11a) are listed in appendix A. Before we move to the concluding section, we would like to make a few remarks about the results presented here.

We observe logarithmic dependences on the arbitrary length scale \(r_0\) in Eq. (A1c) through the quantity \(x_0\) which is related to \(r_0\) by \(x_0 = (Gm/c^2 r_0)\). This dependence is due to the presence of terms involving logarithms of \(r_0\) in the expression for the mass quadrupole moment \((I_{ij})\) given by Eq. (3.19)-(3.20). As was discussed in Sec. III B we would expect such dependences to disappear from final expression for various modes (for instance see Ref. 40 which lists \(h_{\text{inst}}^\ell m\) for binaries in quasi-circular orbits). As has been observed in Refs. 10, 41, 58, 70, it turns out the hereditary contribution has equal and opposite dependences on the arbitrary length scale \(r_0\) and cancels out from the final expression. Since we do not provide hereditary contributions in this paper such cancellation can not be shown here. However, we explicitly show this cancellation in [61] which deals with computation of hereditary effects in various modes at 3PN order.

In this paper we used 3PN accurate QK representation which describes 3PN conserved dynamics of CCBs in eccentric orbits to obtain various modes in terms of QK variables \((x, e_t, u)\). However, it should be noted that the parameters \(x, e_t\) evolve with time over radiation reaction time scales and these secular effects starts to show at 2.5PN order [52, 80]. Hence, in order to correctly account for the reactive dynamics of the binary at 3PN order our results should always be coupled with equations describing secular time-evolution of \(x\) and \(e_t\). Equations describing the secular evolution of orbital elements with relative 3PN accuracies were presented in Ref. 80 (see Sec. VI there). Evolution equations (due to instantaneous terms in energy and angular momentum loss) for orbital frequency \((d\omega/dt)\) (related to \(x\) by \(\omega = (c^3 x^{3/2}/G m)\)) and for time-eccentricity parameter \((de_t/dt)\) in terms of \(x\) and \(e_t\) have been listed as Eqs. (6.14)-(6.15) and Eqs. (6.18)-(6.19), respectively, in Ref. 80.

\[\text{VI. SUMMARY AND CONCLUDING REMARKS}\]

In this paper we presented the computations of the instantaneous contributions to all the relevant modes of the 3PN accurate gravitational waveform of the GW signal from non-spinning coalescing compact binaries in general orbits.
The expression for the instantaneous part of the dominant mode \( h_{\text{inst}}^{\ell m} \), in terms of the variables \( r, \dot{r}, \phi \text{ and } \dot{\phi} \), has been given by Eqs. (A3)\textsuperscript{a}, (A3)\textsuperscript{b} above, whereas expressions for other sub-dominant modes (along with the dominant mode) have been listed in a separate file \( \text{Hlm-GenOrb.m} \) that is being made available along with the paper. Next, we specialized to the case of CCBs in quasi-elliptical orbits using the 3PN quasi-Keplerian representation of the conserved dynamics of compact binaries in eccentric orbits in Sec. [V]. Related 3PN accurate expressions for the instantaneous part of the dominant mode \( h_{\text{inst}}^{\ell m} \), in terms of the variables, namely, the time-eccentricity \( e_t \), a PN parameter \( x \) and eccentric anomaly \( u \), is given by Eqs. (5.10a)-(5.11a) and Eq. (A1). The expressions for other sub-dominant modes (along with the dominant mode) have been listed in a separate file \( \text{Hlm-EllOrb.m} \) that is being made available along with the paper.

We once again remind the readers that results presented here only account for the contributions from instantaneous terms in the waveform which must be complemented by computations accounting for hereditary effects. Our investigations suggest that it is not possible to provide closed form analytical expressions for the hereditary terms for binaries moving in general orbits. Moreover, even for the special case of CCBs in quasi-elliptical orbits it may not possible to have closed form analytical expressions for hereditary terms valid for systems with arbitrary eccentricities. However, we find that such computations can be performed assuming an expansion in the eccentricity parameter \( e_t \) \[61\]. Unlike the results presented in this paper which can be applied to a binary with arbitrary eccentricity, results of \[61\] can only be applied to systems with small eccentricities. However, the positive side of the work is that, we shall have complete 3PN analytical expression for the waveform for binaries in quasi-elliptical orbits that can be used for comparison with related Numerical Relativity results, which is one of motivations for high PN order computations of gravitational waveforms. In addition, with complete waveforms at hand one would able to write complete polarization waveforms (as discussed in Sec. [V]) which would be useful for the data-analysis purposes.

Acknowledgments

We thank L Blanchet and G Faye for useful discussions. CKM thanks Chennai Mathematical Institute for hospitality. KGA thanks Raman Research Institute for hospitality. KGA’s research was partly funded by a grant from Infosys Foundation. All the calculations reported in this paper are carried out with the aid of the algebraic computing software MATHEMATICA.

Appendix A: Various PN pieces of the coefficient \( \hat{H}^{22} \) associated with the dominant mode \( h^{22} \)

\[
(H_{\text{inst}}^{22})_{\text{Newt}} = \frac{2}{(1 - e_t \cos u)^2} \left\{ 1 - e_t^2 - \frac{1}{2} (e_t \cos u) + \frac{1}{2} (e_t \cos u)^2 + i (e_t \sin u) \sqrt{1 - e_t^2} \right\}, \tag{A1a}
\]

\[
(H_{\text{inst}}^{22})_{1\text{PN}} = \frac{x}{42 (1 - e_t^2)(1 - e_t \cos u)^3} \left\{ 214 - 110\nu + e_t^2 (64 + 46\nu) + e_t^4 (-278 + 64\nu) + (e_t \cos u) (-405 + 123\nu + e_t^2 (207 - 89\nu) + e_t^4 (114 - 34\nu)) + (e_t \cos u)^2 (54 + 34\nu + e_t^2 (114 - 34\nu)) + (e_t \cos u)^3 (-27 - 17\nu + e_t^2 (-57 + 17\nu)) + i (e_t \sin u) \sqrt{1 - e_t^2} \left( -20 - 38\nu + e_t^2 (272 - 46\nu) + (e_t \cos u) (-138 + 50\nu + e_t^2 (-114 + 34\nu)) \right) \right\}, \tag{A1b}
\]

\[
(H_{\text{inst}}^{22})_{2\text{PN}} = \frac{x^2}{3024 \left( 1 - e_t^2 \right)^2 (1 - e_t \cos u)^5} \left\{ -38932 - 17836\nu + 8188\nu^2 + e_t^2 (182850 - 92982\nu - 3966\nu^2) + e_t^4 (-196098 + 212448\nu - 30360\nu^2) + e_t^6 (53374 - 133790\nu + 39866\nu^2) + e_t^8 (-1194 + 32160\nu - 13728\nu^2) + (e_t \cos u) (-4628 + 84514\nu - 30202\nu^2 + e_t^2 (-121926 - 34230\nu + 65946\nu^2) + e_t^4 (199848 - 80970\nu - 41286\nu^2) + e_t^6 (-35494 + 3470\nu + 5542\nu^2) + (e_t \cos u)^2 (14904 + 7158\nu + 18558\nu^2 + e_t^2 (-151740 + 4032\nu - 27288\nu^2) + e_t^4 (4320 + 86670\nu - 1098\nu^2) + e_t^6 (-18684 + 1100\nu + 9828\nu^2)) + (e_t \cos u)^3 (104628 - 84714\nu - 9786\nu^2 + e_t^2 (140196 - 58056\nu + 19248\nu^2) \right\}. \tag{A1c}
\]
\[ H^{22}_{\text{inst}} = \frac{i}{105} \left( \frac{x}{x_0} \right)^{\nu^5/2} \left\{ \sqrt{1 - \nu^2} \left( -2352 + 1500e_t^2 + 1404 (e_t \cos u) - 552 (e_t \cos u)^2 \right) + i (e_t \sin u) (2539 - 2175e_t^2 + 2 (e_t \cos u) - 366 (e_t \cos u)^2) \right\}, \]

\[ \langle H^{22}_{\text{inst}} \rangle_{3\text{PN}} = \frac{x^3}{1277337600} \left( \frac{x}{x_0} \right)^3 \left( 1 - e_t \cos u \right)^8 \left\{ 186870371328 - 20826685440 \log (1 - e_t \cos u) + 20826685440 \log \left( \frac{x}{x_0} \right) + 15657907200 \nu - 13806489600 \nu^2 + 2934656000 \nu^4 + e_t^2 \left( -319951363584 + 75496734720 \log (1 - e_t \cos u) - 75496734720 \log \left( \frac{x}{x_0} \right) + 572456762880 + 18002476800 \pi^2 \right) + 128784537600 \nu^2 - 2970944000 \nu^3 \right\} \]

\[ + e_t^4 \left( 7924875264 - 101530091520 \log (1 - e_t \cos u) + 101530091520 \log \left( \frac{x}{x_0} \right) + 1455622894080 - 43369603200 \pi^2 \right) - 240132902400 \nu^2 + 5117260800 \nu^3 \]

\[ + e_t^6 \left( 292424159232 + 59876720640 \log (1 - e_t \cos u) - 59876720640 \log \left( \frac{x}{x_0} \right) + 157942629760 + 34368634800 \pi^2 \right) + 359519040000 \nu^2 - 32956160000 \nu^3 \]

\[ + e_t^8 \left( -171895234560 - 13016678400 \log (1 - e_t \cos u) + 13016678400 \log \left( \frac{x}{x_0} \right) + 873551592960 - 9001238400 \pi^2 - 365015424000 \nu^2 + 53854515200 \nu^3 \right) \]

\[ + e_t^{10} \left( 4469829120 - 197763148800 \nu + 146424576000 \nu^2 - 32956160000 \nu^3 \right) \]

\[ + e_t^{12} \left( 157363200 + 4813747200 \nu - 12775219200 \nu^2 + 6083481600 \nu^3 \right) \]
\[
+ (e_t \cos u) \left( -815811686400 + 6508332000 \log (1 - e_t \cos u) - 65083392000 \times \log \left( \frac{x}{x_0} \right) + \nu \left( -10777029760 - 2454883200 \pi^2 \right) + 70743936000 \nu^2 \right) - 211858048000 \nu^3 + e_t^2 (1636480788480 - 234300211200 \log (1 - e_t \cos u)) + 234300211200 \log \left( \frac{x}{x_0} \right) + \nu \left( 1487299445760 - 49915958400 \pi^2 \right) - 420029798400 \nu^2 + 30425203200 \nu^3 \right) + e_t^4 (-691922419200 + 312400281600 \times \log (1 - e_t \cos u) - 3124002816000 \log \left( \frac{x}{x_0} \right) + \nu \left( -346330315008 \right) + 1358368704000 \pi^2 + 230527872000 \nu^2 + 22174886400 \nu^3 \right) + e_t^6 (-529844974080 - 182233497600 \log (1 - e_t \cos u) + 182233497600 \log \left( \frac{x}{x_0} \right) + \nu \left( 3240401410560 \right) - 106378272000 \pi^2 - 90015513600 \nu^3 - 31134338800 \nu^3 \right) + e_t^8 (500039447040 - 39050035200 \log (1 - e_t \cos u) - 39050035200 \log \left( \frac{x}{x_0} \right) + \nu \left( -1788108142080 \right) - 260655360000 \nu^3 - 530788864000 \nu^3 \right) + e_t^2 (-280211731120 + 234300211200 \log (1 - e_t \cos u) - 234300211200 \log \left( \frac{x}{x_0} \right) + \nu \left( -2357003520000 \right) + 662818464000 \pi^2 + 972179174400 \nu^2 - 130893849600 \nu^3 \right) + e_t^4 (1596715430400 - 312400281600 \log (1 - e_t \cos u) + 312400281600 \log \left( \frac{x}{x_0} \right) + \nu \left( 3961710581760 \right) - 1710235296000 \pi^2 - 332844249600 \nu^2 + 78350246400 \nu^3 \right) + e_t^6 (-86515975680 + 182233497600 \log (1 - e_t \cos u) - 182233497600 \log \left( \frac{x}{x_0} \right) + \nu \left( -1923370721280 \right) + 117834339600 \pi^2 - 784967323200 \nu^2 + 19523494400 \nu^3 \right) + e_t^8 (495126120960 - 39050035200 \log (1 - e_t \cos u) + 39050035200 \log \left( \frac{x}{x_0} \right) + \nu \left( 102831694840 \right) - 270037152000 \pi^2 - 47218214400 \nu^2 - 15916761600 \nu^3 \right) + e_t^{10} (6089771520 - 229606041600 \nu + 124910784000 \nu^2 - 4142016000 \nu^3) + (e_t \cos u)^3 (-334437120000 + 13016678400 \log (1 - e_t \cos u) - 13016678400 \log \left( \frac{x}{x_0} \right) + \nu \left( -3817315046400 \right) + 842843232000 \pi^2 + 4574204928000 \nu^2 - 489369472000 \nu^3 + e_t^2 (2153965992960 - 52066713600 \log (1 - e_t \cos u) + 52066713600 \log \left( \frac{x}{x_0} \right) + \nu \left( 2622913935360 \right) - 875575008000 \pi^2 - 9691491840000 \nu^2 + 1467561088000 \nu^3 \right) + e_t^4 (-713296212480 + 78100070400 \log (1 - e_t \cos u) - 78100070400 \log \left( \frac{x}{x_0} \right) + \nu \left( -3996933972480 \right) + 139110408000 \pi^2 + 791266867200 \nu^2 - 1509366912000 \nu^3 \right) + e_t^6 (564367726080 - 52066713600 \log (1 - e_t \cos u) + 52066713600 \log \left( \frac{x}{x_0} \right) + \nu (239607559680)
\[ -58917196800\pi^2 + 810237081600\nu^2 + 61642892800\nu^3 + e_1^6 (189541040640) + 13016678400 \log(1 - e_1 \cos u) - 13016678400 \log \left( \frac{x}{x_0} \right) + \nu (-103681282560) + 9001238400\nu^2 - 76422144000\nu^2 - 12815411200\nu^3 + e_1^{10} (-337881600) + 3036268800\nu - 27567820800\nu^2 + 4290048000\nu^3) + (e_1 \cos u)^4 (-68785478080) + 13016678400 \log(1 - e_1 \cos u) - 13016678400 \log \left( \frac{x}{x_0} \right) + \nu (5807370984960) - 153021052800\pi^2 - 455413900800\nu^2 + 17977395200\nu^3 + e_1^2 (1490127552000) - 39050035200 \log(1 - e_1 \cos u) + 39050035200 \log \left( \frac{x}{x_0} \right) + \nu (-384485191680) + 90830678400\pi^2 - 203492160000\nu^2 - 53794060800\nu^3 + e_1^4 (-664152491520) + 39050035200 \log(1 - e_1 \cos u) - 39050035200 \log \left( \frac{x}{x_0} \right) + \nu (2723950126080) - 99013622400\pi^2 - 465812467200\nu^2 + 53517811200\nu^3 + e_1^6 (-224123312640) - 13016678400 \log(1 - e_1 \cos u) + 13016678400 \log \left( \frac{x}{x_0} \right) + \nu (255670333440) + 18002476800\pi^2 - 514787366400\nu^2 - 175630280800\nu^3 + e_1^8 (-33410741760) - 27429388800\nu - 3469593600\nu^2 - 138124800\nu^3) + (e_1 \cos u)^5 (787721912832) - 5206671360 \log(1 - e_1 \cos u) + 5206671360 \log \left( \frac{x}{x_0} \right) + \nu (-5142719623680) + 140746636800\pi^2 - 348355737600\nu^2 - 7195404800\nu^3 + e_1^2 (1833256917504) + 15620014080 \log(1 - e_1 \cos u) - 15620014080 \log \left( \frac{x}{x_0} \right) + \nu (-2284122493440) - 38459836800\pi^2 + 100566796800\nu^2 + 20515392000\nu^3) + e_1^4 (477180919296) - 15620014080 \log(1 - e_1 \cos u) + 15620014080 \log \left( \frac{x}{x_0} \right) + \nu (-597097059840) + 45824486400\pi^2 - 7296998400\nu^2 - 18373747200\nu^3) + e_1^6 (-10724639232) + 5206671360 \log(1 - e_1 \cos u) - 5206671360 \log \left( \frac{x}{x_0} \right) + \nu (-375209879040) + 4909766400\pi^2 + 28825536000\nu^2 + 3982937600\nu^3) + e_1^8 (12237465600) + 24072192000\nu + 7993420800\nu^2 + 1070822400\nu^3) + (e_1 \cos u)^6 (-411239646720 + \nu (2829259175040 - 74464790400\pi^2) - 2076331392000\nu^2 + 3495795200\nu^3) + e_1^2 (-1455591651840 + \nu (2185925860600 - 4091472000\pi^2) - 72499219200\nu^2 - 10543577600\nu^3) + e_1^4 (9839024640 + \nu (-146180229120 + 7364649600\pi^2) + 15925593600\nu^2 + 10655961600\nu^3) + e_1^6 (-1425415680 + 162493286400\nu - 690507648000\nu^2 - 3664371200\nu^3) + e_1^8 (-1385565000 - 6721920000\nu + 4220928000\nu^2 + 56192000\nu^3) + (e_1 \cos u)^7 (132699700800 + \nu (-909674841600 + 22991243200\pi^2) + 71153510400\nu^2 - 196672000\nu^3 + e_1^2 (518098291200 + \nu (-814416422400 + 5728060800\pi^2) + 246203596800\nu^2 + 590016000\nu^3) + e_1^4 (-20603980800 + 7260261200\nu + 9760665600\nu^2 - 590016000\nu^3) + e_1^6 (-4850496000 - 2352672000\nu + 1477324800\nu^2 + 196672000\nu^3) + (e_1 \cos u)^8 (-18955814400 + \nu (129953548800 - 3273177600\pi^2) - 10164787200\nu^2 + 28096000\nu^3 + e_1^2 (-74014041600 + \nu (116345203200 - 818294400\pi^2) - 351719424000\nu^2 - 84288000\nu^3) + e_1^4 (3714854400 - 10371801600\nu)\]
\[-13943808000\nu^2 + 84288000\nu^3 + e_t^5 (692928000 + 336096000\nu - 211046400\nu^2 \\
-28096000\nu^3) + i (e_c \sin u) (7451136000 + \nu (-22530816000 + 2727648000\nu^2) \\
+13191552000\nu^2 + e_t^5 (-3877620000 + \nu (693492940800 - 7091884800\nu^2) \\
-37407744000\nu^2) + e_t^4 (417871872000 + \nu (-487101542400 + 4364236800\nu^2) \\
+7299072000\nu^2) + e_t^6 (-1046200000 + 18916761600\nu + 23478681600\nu^2) \\
+ (e_c \cos u) (-24117350400 + \nu (1018828800000 - 13092710400\nu^2 - 56932761600\nu^2 \\
+ e_t^5 (1711176192000 + \nu (-342405550800 + 36004953600\nu^2 + 194885222400\nu^2) \\
+ e_t^4 (-189410984000 + \nu (2447865446400 - 22912243200\pi^2) - 12043468800\nu^2) \\
+ e_t^6 (421406496000 - 42638745600 - 125908992000\nu^2) + (e_c \cos u)^2 (19631462400 \\
+ \nu (-176598057600 + 2454883200\nu^2) + 89961062400\nu^2 + e_t^5 (-3063785472000 \\
+ \nu (6872258764800 - 73646469600\nu^2 - 423163699200\nu^2) + e_t^6 (3526280325200 \\
+ \nu (-5087179468800 + 4990766400\nu^2) + 59122483200\nu^2) + e_t^6 (-65934950400 \\
-19099238400\nu + 274080153600\nu^2) + (e_c \cos u)^3 (136249344000 + \nu (1381795430400 \\
-2182118400\nu^2 - 54499737600\nu^2 + e_t^2 (289864396800 + \nu (-7118338867200 \\
+76374144000\nu^2 + 491957542800\nu^2) + e_t^4 (-352362700800 + \nu (5612904259200 \\
-5455296000\nu^2) + 130653388800\nu^2) + e_t^6 (488733696000 + 12444917760\nu \\
-306084326400\nu^2) + (e_c \cos u)^4 (-292571136000 + \nu (-358171545600 \\
+8182944000\nu^2 - 4926873600\nu^2 + e_t^2 (-1608532992000 + \nu (3959624908800 \\
-4091472000\nu^2 - 324626227200\nu^2) + e_t^4 (208297267200 + \nu (-348537292800 \\
+3273177600\nu^2 + 146711347200\nu^2) + e_t^6 (-181868544000 - 116116070400\nu \\
+182841753600\nu^2) + (e_c \cos u)^5 (149935104000 - 107418009600\nu + 18977587200\nu^2 \\
+ e_t^2 (54697420800 + \nu (-109394841600 + 9819532800\nu^2) + 116055244800\nu^2) \\
+ e_t^4 (-73903104000 + \nu (1166391705600 - 9819532800\nu^2 - 82114560000\nu^2) \\
+ e_t^6 (42121728000 + 3497472000\nu - 52918272000\nu^2) + (e_c \cos u)^6 (-2326579200 \\
+ \nu (56253542400 - 545529600\nu^2) - 5778432000\nu^2 + e_t^6 (-9671270400 \\
+ \nu (110966169600 - 545529600\nu^2) - 17700249600\nu^2) + e_t^4 (12910233600 \\
+ \nu (-166733107200 + 1091059200\nu^2) + 1824768000\nu^2) + e_t^6 (-912384000 \\
-4866048000\nu + 5231001600\nu^2)) + \sqrt{1 - e_t^2} (-39536640000 + \nu (-37022515200 \\
+1091059200\nu^2 - 4866048000\nu^2 + e_t^2 (-18338918400 + \nu (472987756800 \\
-545529600\nu^2 - 25911705600\nu^2) + e_t^4 (33210777600 + \nu (-469492531200 \\
+4364236800\nu^2) + 1277337600\nu^2) + e_t^6 (-10918195200 + 34427289600\nu \\
+18004377600\nu^2 + (e_c \cos u) (285956352000 + \nu (49893004800 - 4637001600\nu^2) \\
+26793676800\nu^2 + e_t^2 (80974080000 + \nu (-2359414886400 + 27549244800\nu^2) \\
+14248396800\nu^2) + e_t^4 (-150018379200 + \nu (2355425740800 - 22912123400\nu^2) \\
-65813292000\nu^2) + e_t^6 (442810368000 - 106231910400\nu - 103464345600\nu^2) \\
+ (e_c \cos u)^2 (-70238361600 + \nu (-158623027200 + 8728473600\nu^2) - 32176742400\nu^2 \\
+ e_t^2 (-1736266752000 + \nu (510681600000 - 57826137600\nu^2) - 35826278400\nu^2) \\
+ e_t^4 (285895526400 + \nu (-4921064755200 + 49097664000\nu^2) + 151399184000\nu^2) \\
+ e_t^6 (-68854579200 + 80168140800\nu + 23904460800\nu^2)) + (e_c \cos u)^3 (714016512000 \\
+ \nu (650230732800 - 12819945600\nu^2) - 68824166400\nu^2 + e_t^2 (266461747200 \\
+ \nu (-653489971200 + 67372905600\nu^2) + 540040089600\nu^2) + e_t^4 (-308362982400
\begin{align*}
& -459194112000\nu + 253455436800\nu^2 - 6922828800\nu^3 + e^4_t (-294220800 \\
& +9305395200\nu - 23887411200\nu^2 + 1261209600\nu^3)) + (e_t \cos u)^2 (-901804861440 \\
& +78100070400 \log (1 - e_t \cos u) - 78100070400 \log \left(\frac{x}{x_0}\right) + \nu (1567758735360 \\
& -34368364800\pi^2) - 4901222400\nu^2 + 16293504000\nu^3 + e^6_t (1178660367360 \\
& -234300211200 \log (1 - e_t \cos u) + 234300211200 \log \left(\frac{x}{x_0}\right) + \nu (638531481600 \\
& +26185420800\pi^2) + 85797964800\nu^2 - 23253580800\nu^3) + e^4_t (-2558005079040 \\
& +234300211200 \log (1 - e_t \cos u) - 234300211200 \log \left(\frac{x}{x_0}\right) + \nu (-627219118080 \\
& -65463552000\pi^2) + 636385766400\nu^2 + 18039628800\nu^3) + e^6_t (1322237813760 \\
& -78100070400 \log (1 - e_t \cos u) + 78100070400 \log \left(\frac{x}{x_0}\right) + \nu (-12701306880 \\
& -9819532800\nu^2) - 123584793600\nu^2 + 10020948000\nu^3) + e^6_t (-31024880640 \\
& +275186073600\nu - 105116083200\nu^2 - 19404288000\nu^3)) + (e_t \cos u)^3 (452697154560 \\
& -26033356800 \log (1 - e_t \cos u) + 26033356800 \log \left(\frac{x}{x_0}\right) + \nu (-2649367956480 \\
& +58917196800\pi^2) + 75336499200\nu^2 - 10847257600\nu^3 + e^2_t (1093220843520 \\
& +78100070400 \log (1 - e_t \cos u) - 78100070400 \log \left(\frac{x}{x_0}\right) + \nu (5716638720 \\
& -49097664000\pi^2) - 361501670400\nu^2 + 204555264000\nu^3) + e^4_t (211920660480 \\
& -78100070400 \log (1 - e_t \cos u) + 78100070400 \log \left(\frac{x}{x_0}\right) + \nu (602037273600 \\
& +19639065600\pi^2) - 554518809600\nu^2 - 36066355200\nu^3) + e^6_t (-442958069760 \\
& +26033356800 \log (1 - e_t \cos u) - 26033356800 \log \left(\frac{x}{x_0}\right) + \nu (-367892106240 \\
& +32731776000\nu^2) + 163837209600\nu^2 - 5878784000\nu^3) + e^6_t (5034931200 \\
& -45901670400\nu + 25404595200\nu^2 + 1499212800\nu^3)) + (e_t \cos u)^4 (-287513978880 \\
& +\nu (2242649057280 - 52370841600\pi^2) - 56454220800\nu^2 - 9712896000\nu^3 \\
& +e^2_t (-115839312960 + \nu (-333825269760 + 39278131200\pi^2) + 302949196800\nu^2 \\
& +1949107200\nu^3) + e^4_t (427268290560 + \nu (-205133137920 - 11456121600\pi^2) \\
& +280446028800\nu^2 + 20456678400\nu^3) + e^6_t (28699361280 + 13786521600\nu \\
& -38359372800\nu^2 - 2274432000\nu^3)) + (e_t \cos u)^5 (169541542800 \\
& +\nu (-933238149120 + 24548832000\pi^2) + 9044275200\nu^2 + 2341017600\nu^3 \\
& +e^2_t (328893465600 + \nu (69430671360 - 16365888000\pi^2) - 74749132800\nu^2 \\
& -58116096000\nu^3) + e^4_t (-9019353600 + \nu (163341803520 + 1636588800\pi^2) \\
& -128144793600\nu^2 - 3793766400\nu^3) + e^6_t (-122646726400 - 36156672000\nu \\
& -15830016000\nu^2 - 3996672000\nu^3) + (e_t \cos u)^6 (-29502489600 + \nu (159620889600 \\
& -49097664000\nu^2) - 1152000000\nu^2 + 8153600000\nu^3 + e^2_t (-48882355200 + \nu (6869283840 \\
& +32731776000\pi^2) + 2940825600\nu^2 - 1280256000\nu^3) + e^4_t (11003212800 - 50441702400\nu \\
& +30168576000\nu^2 + 13800192000\nu^3) + e^6_t (1385856000 + 6721920000\nu - 4220928000\nu^2 \\
& -56192000\nu^3))))}.
\end{align*}
[59] U. Sperhake et al., Phys. Rev. D 78, 064069 (2008), arXiv:0710.3823.
[60] I. Hinder, B. Vaishnav, F. Herrmann, D. M. Shoemaker, and P. Laguna, Phys. Rev. D 77, 081502 (2008), arXiv:0710.5167.
[61] C. K. Mishra et al., in preparation (2015).
[62] K. Thorne, Rev. Mod. Phys. 52, 299 (1980).
[63] L. Blanchet and G. Faye, Phys. Rev. D 63, 062005 (2001), gr-qc/0007051.
[64] L. Blanchet, B. R. Iyer, and B. Joguet, Phys. Rev. D 65, 064005 (2002), Erratum-ibid 71, 129903(E) (2005), gr-qc/0105098.
[65] L. Blanchet and B. R. Iyer, Phys. Rev. D 71, 024004 (2004), gr-qc/0409094.
[66] L. Blanchet, T. Damour, and G. Esposito-Farèse, Phys. Rev. D 69, 124007 (2004), gr-qc/0311052.
[67] L. Blanchet, T. Damour, G. Esposito-Farèse, and B. R. Iyer, Phys. Rev. D 71, 124004 (2005), gr-qc/0503044.
[68] K. G. Arun, L. Blanchet, B. R. Iyer, and M. S. S. Qusailah, Class. Quantum Grav. 21, 3771 (2004), erratum-ibid. 22, 3115 (2005), gr-qc/0404185.
[69] L. Blanchet and B. R. Iyer, Phys. Rev. D 71, 024004 (2005), gr-qc/0409094.
[70] L. Blanchet, Class. Quantum Grav. 15, 113 (1998), gr-qc/9710038.
[71] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D 63, 044021 (2001), erratum-ibid 66, 029901(E) (2002).
[72] V. de Andrade, L. Blanchet, and G. Faye, Class. Quantum Grav. 18, 753 (2001).
[73] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Lett. B 513, 147 (2001).
[74] L. Blanchet and B. R. Iyer, Class. Quantum Grav. 20, 755 (2003), gr-qc/0209089.
[75] T. Mora and C. M. Will, Phys. Rev. D 69, 104021 (2004), gr-qc/0312082.
[76] T. Damour and N. Deruelle, Annales Inst. H. Poincaré Phys. Théor. 43, 107 (1985).
[77] T. Damour and G. Schäfer, Nuovo Cim. B101, 127 (1988).
[78] G. Schäfer and N. Wex, Phys. Lett. A 174, 196 (1993), Erratum, ibid 177, 461(E) (1993).
[79] N. Wex, Class. Quant. Grav. 12, 983 (1995).
[80] K. G. Arun, L. Blanchet, B. R. Iyer, and S. Sinha, Phys. Rev. D 80, 124018 (2009), arXiv:0908.3854.