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Methods of Functional Analysis and Topology

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Abstract. Let $f : M \to \mathbb{R}$ be a Morse function on a smooth closed surface, $V$ be a connected component of some critical level of $f$, and $\mathcal{E}_V$ be its atom. Let also $S(f)$ be a stabilizer of the function $f$ under the right action of the group of diffeomorphisms $\text{Diff}(M)$ on the space of smooth functions on $M$, and $S_V(f) = \{h \in S(f) | h(V) = V\}$. The group $S_V(f)$ acts on the set $\pi_0 \partial \mathcal{E}_V$ of connected components of the boundary of $\mathcal{E}_V$. Therefore we have a homomorphism $\phi : S(f) \to \text{Aut}(\pi_0 \partial \mathcal{E}_V)$. Let also $G = \phi(S(f))$ be the image of $S(f)$ in $\text{Aut}(\pi_0 \partial \mathcal{E}_V)$.

Suppose that the inclusion $\partial \mathcal{E}_V \subset M \setminus V$ induces a bijection $\pi_0 \partial \mathcal{E}_V \to \partial \mathcal{E}_V | \partial M$.

Let $H$ be a subgroup of $G$. We present a sufficient condition for existence of a section $s : H \to S_V(f)$ of the homomorphism $\phi$, so, the action of $H$ on $\partial \mathcal{E}_V$ lifts to the $H$-action on $M$ by $f$-preserving diffeomorphisms of $M$.

This result holds for a larger class of smooth functions $f : M \to \mathbb{R}$ having the following property: for each critical point $z$ of $f$ the germ of $f$ at $z$ is smoothly equivalent to a homogeneous polynomial $\mathbb{R}^2 \to \mathbb{R}$ without multiple linear factors.

1. Introduction

Let $M$ be a smooth compact surface. The group of diffeomorphisms $\mathcal{D}(M)$ of $M$ acts on the space of smooth functions $C^\infty(M)$ by the rule

\[ C^\infty(M) \times \mathcal{D}(M) \to C^\infty(M), \quad (f, h) \mapsto f \circ h. \]

The set $S(f) = \{h \in \mathcal{D}(M) | f \circ h = f\}$ is called the stabilizer of the function $f$ under action (1.1). Endow $C^\infty(M)$ and $\mathcal{D}(M)$ with the corresponding Whitney topologies. The topology on $\mathcal{D}(M)$ induces a certain topology on the stabilizer $S(f)$.

Let $\mathcal{F}(M) \subset C^\infty(M, \mathbb{R})$ be the set of smooth functions satisfying the following two conditions:

- (B) the function $f$ takes a constant value at each connected component of $\partial M$, and all critical points of $f$ belong to the interior of $M$;
- (P) for each critical point $x$ of $f$ the germ $(f, x)$ of $f$ at $x$ is smoothly equivalent to some homogeneous polynomial $f_x : \mathbb{R}^2 \to \mathbb{R}$ without multiple linear factors.

It is well-known that each homogeneous polynomial $f_x : \mathbb{R}^2 \to \mathbb{R}$ splits into a product of linear and irreducible over $\mathbb{R}$ quadratic factors. Condition (P) means that

\[ f_x = \prod_{i=1}^n L_i \cdot \prod_{j=1}^m Q_j^{q_j}, \]

where $L_i(x, y) = a_i x + b_i y$ is a linear form, $Q_j = c_j x^2 + d_j x y + e_j y^2$ is an irreducible quadratic form such that $L_i/L_{i'} \neq \text{const}$ for $i \neq i'$, and $Q_j/Q_{j'} \neq \text{const}$ for $j \neq j'$. So, if $\deg f_x \geq 2$, then $0$ is an isolated critical point of $f_x$.

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Recall that if \( f : (\mathbb{C}, 0) \rightarrow (\mathbb{R}, 0) \) is a germ of \( C^\infty \)-function such that \( 0 \in \mathbb{R}^2 \) is an isolated critical point of \( f \), then there is a germ of a homeomorphism \( h : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0) \) such that

\[
    f_x \circ h(z) = \begin{cases} 
        \pm |z|^2, & \text{if } 0 \text{ is a local extremum}, \[4]. \\
        \text{Re}(z^n), & \text{for some } n \in \mathbb{N} \text{ otherwise}, \[13].
    \end{cases}
\]

If \( 0 \) is not a local extreme, then the number \( n \) does not depend of a particular choice of \( h \). In this case the point \( 0 \) will be called a generalized \( n \)-saddle, or simply an \( n \)-saddle. The number \( n \) corresponds to the number of linear factors in \( \pm z \). Examples of level sets foliations near isolated critical points are given in Fig. [12].

![Figure 1.1. Level set foliations in neighborhoods of isolated points](image)

(a) local extreme, (b) 1-saddle, (c) 3-saddle

Let \( \text{Morse}_\partial(M) \) be the space of Morse functions on \( M \), which satisfy condition (B), \( f \in \text{Morse}_\partial(M) \), and \( x \) be a critical point of \( f \). Then, by Morse Lemma, there exists a coordinate system \((t, s)\) near \( x \) such that the function \( f \) has one of the following forms

\[
    f(s, t) = \pm s^2 \pm t^2,
\]

which are, obviously, homogeneous polynomials without multiple factors. This implies that \( \text{Morse}_\partial(M) \) is a subspace of \( \mathcal{F}(M) \).

Let \( f \in \mathcal{F}(M) \) be a smooth function and \( c \in \mathbb{R} \) be a real number. A connected component \( C \) of the level set \( f^{-1}(c) \) is called critical if it contains at least one critical point, otherwise, \( C \) is called regular. Let \( \Delta \) be a foliation of \( M \) into connected components of level sets of \( f \). It is well-known that the quotient-space \( M/\Delta \) has a structure of 1-dimensional CW complex. The space \( M/\Delta \) is called the Kronrod-Reeb graph, or simply, KR-graph of \( f \). We will denote it by \( \Gamma_f \). Let \( p_f : M \rightarrow \Gamma_f \) be a projection of \( M \) onto \( \Gamma_f \). Then vertices of \( \Gamma_f \) correspond to connected components of critical level sets of the function \( f \).

It should be noted that the function \( f \in \mathcal{F}(M) \) can be represented as the composition

\[
    f = \phi \circ p_f : M \xrightarrow{p_f} \Gamma_f \xrightarrow{\phi'} \mathbb{R},
\]

where \( \phi' \) is the map induced by \( f \). Let \( h \in \mathcal{S}(f) \). Then \( f \circ h = f \), and we have \( h(f^{-1}(t)) = f^{-1}(t) \) for all \( t \in \mathbb{R} \). Hence \( h \) interchanges connected components of level sets of the function \( f \) and therefore it induces an automorphism \( \rho(h) \) of KR-graph \( \Gamma_f \) such that the following diagram is commutative:

In other words, we have a homomorphism \( \rho : \mathcal{S}(f) \rightarrow \text{Aut}(\Gamma_f) \). Let \( G = \rho(\mathcal{S}(f)) \) be the image of \( \mathcal{S}(f) \) in \( \text{Aut}(\Gamma_f) \). It is easy to show that the group \( G \) is finite.

Let \( v \) be a vertex of \( \Gamma_f \) and \( G_v = \{ g \in G \mid g(v) = v \} \) be the stabilizer of \( v \) under the action of \( G \) on \( \Gamma_f \). An arbitrary connected closed \( G_v \)-invariant neighborhood of \( v \) in \( \Gamma_f \) containing no other vertices of \( \Gamma_f \) will be called a star of \( v \). We denote it by \( \text{st}(v) \).

The set \( G_v^{oc} = \{ g|_{\text{st}(v)} \mid g \in G \} \) which consists of restrictions of elements of \( G_v \) onto the star \( \text{st}(v) \) is a subgroup of \( \text{Aut}(\text{st}(v)) \). This group will be called a local stabilizer of \( v \). Let also \( r : G_v \rightarrow G_v^{oc} \) be the map defined by \( r(g) = g|_{\text{st}(v)} \) for \( g \in G_v \), i.e., \( r \) is the restriction map.

Let \( v \) be a vertex of \( \Gamma_f \), and \( V = p_f^{-1}(v) \) be the corresponding connected component of the critical level set \( f^{-1}(p_f(v)) \).
Definition 1.1. A vertex $v$ of the graph $\Gamma_f$ will be called special if there is a bijection between connected components of $\text{st}(v) \setminus v$ and $M \setminus V$. The corresponding connected component $V = \overline{p_f^{-1}}(v)$ will be called special.

It follows from definition of KR-graph $\Gamma_f$ that for a special vertex $v$ there is a 1–1 correspondence between connected components of complement to $v$ in $\text{st}(v)$ and connected components of $\Gamma_f \setminus v$.

Note that a special component $V$ gives a partition $\Xi$ of the surface $M$ whose 0-dimensional elements are vertices of $V$, 1-dimensional elements are edges of $V$, and 2-dimensional elements are connected components of complement of $V$ in $M$. Since $M$ is compact, it follows that $\Xi$ has a finite number of elements in each dimension.

2. Main result

Let $f \in \mathcal{F}(M)$. Suppose that its Kronrod-Reeb graph $\Gamma_f$ contains a special vertex $v$, and $V$ be the special component of level set of $f$ which corresponds to $v$.

Let $S_v(f) = \{ h \in S(f) \mid h(V) = V \}$ be a subgroup of $S(f)$ leaving $V$ invariant. It is easy to see that $\rho(S_v(f)) \subset G_v$. We denote by $\phi$ the map

$$\phi = r \circ \rho : S_v(f) \xrightarrow{\rho} G_v \xrightarrow{r} G^{\text{loc}}_v.$$  

Let $H$ be a subgroup of $G^{\text{loc}}_v$ and $\mathcal{H} = \phi^{-1}(H)$ be a subgroup of $S_v(f)$. We will say that the group $H$ has property (C) if the following conditions hold.

(C) Let $h \in \mathcal{H}$, and $E$ be a 2-dimensional element of $\Xi$. Suppose that $h(E) = E$.

Then $h(e) = e$ for all other $e \in \Xi$, and the map $h$ preserves orientation of each element of $\Xi$.

Lemma 2.1. If $\mathcal{H} = \phi^{-1}(H)$ has property (C), then $H$ acts on the set of all elements of the partition $\Xi$. Moreover this action is free on the set of 2-dimensional elements of $\Xi$.

Proof. Let $g \in H$, and $h \in \mathcal{H}$ be a diffeomorphism such that $\phi(h) = g$. Define the map $\tau : H \times \Xi \to \Xi$ by the following rule

$$\tau(g, e) = h(e), \quad e \in \Xi.$$  

We claim that this definition does not depend of a particular choice of such $h$. Let $h_1, h_2 \in \mathcal{H}$ be diffeomorphisms such that $\phi(h_1) = \phi(h_2)$. Then $\phi(h_1 \circ h_2^{-1}) = 1_H$, where $1_H$ be the unit of $H$. By definition of the unit $1_H$, we have $(h_1 \circ h_2^{-1})(E) = E$ for each 2-dimensional component $E$ of $\Xi$. Then, by condition (C), $(h_1 \circ h_2^{-1})(e) = e$ for other $e \in \Xi$. Hence $h_1(e) = h_2(e)$. So, the map $\tau$ is well-defined. It is easy to see that $\tau(1_H, e) = e$, where $1_H$ is the unit of $H$, and $\tau(g_1, \tau(g_2, e)) = \tau(g_1 \circ g_2, e)$ for each $g_1, g_2 \in H$, and $e \in \Xi$. Thus $\tau$ is an $H$-action on $\Xi$.

Suppose $h \in \mathcal{H}$ is such that $h(E) = E$ for some 2-dimensional component $E$ of $\Xi$. Then, by condition (C), $h(E') = E'$ for each 2-dimensional component $E'$ of $\Xi$. Hence, $h = \text{id}$, so the $H$-action on the set of 2-dimensional components of $\Xi$ is free. $\square$

Thus condition (C) implies that $H << \text{combinatorially} >>$ acts of $M$ i.e., it ensures invariance of the partition $\Xi$ under the action of $H$ on $M$. Our aim is to prove that in fact this «combinatorial» action is induced by a real action of $H$ on $M$ by diffeomorphisms preserving $f$.

Namely the following theorem holds.

Theorem 2.2. Suppose $f \in \mathcal{F}(M)$ is such that its KR-graph $\Gamma_f$ contains a special vertex $v$, and $G^{\text{loc}}_v$ be the local stabilizer of $v$. Let also $H$ be a subgroup of $G^{\text{loc}}_v$, and $\mathcal{H} = \phi^{-1}(H)$ be a subgroup of $S_v(f)$ satisfying condition (C). Then there exists a section $s : H \to \mathcal{H}$ of the map $\phi$, i.e., the map $s$ is a homomorphism satisfying the condition $\phi \circ s = \text{id}_H$.  

Group actions which have the property of invariance of some partition of the surfaces are studied by Bolsinov and Fomenko [1], Braiov [2], Braiov and Kudryavtseva [3], Kudryavtseva [4], Maksymenko [11], Kudryavtseva and Fomenko [6, 7].

2.3. Structure of the paper. In Section 3 we recall the definitions and statements that will be used in the text. The topological structure of the atom $\mathcal{E}_{V,a}$ which corresponds to $V$ is described in Section 4. In section 5 we construct an $H$-action on the surface $M$.

3. Symmetries of homogeneous polynomials

Let $f_z : \mathbb{R}^2 \to \mathbb{R}$ be a homogeneous polynomial without multiple linear factors. Suppose the origin $0 \in \mathbb{R}^2$ is not a local extreme for $f_z$. Let also $\mathcal{L}(f_z)$ be a group of orientation preserving linear automorphisms $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f_z \circ h = f_z$. The following lemma holds:

**Lemma 3.1.** ([10], Section 6). After some linear change of coordinates one can assume that

1. if $\deg f_z = 2$, then the group $\mathcal{L}(f_z)$ consists of the linear transformations of the following form
   $$ \pm \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}, \quad a > 0,$$
   see [10] Section 6, case (B)];

2. if $\deg \geq 3$, then the group of $\mathcal{L}(f_z)$ is a finite cyclic subgroup of $SO(2)$, [10] Section 6, case (E)].

We will also need the following lemma:

**Lemma 3.2.** ([10], Corollary 7.4). Let $h : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ be a germ of a diffeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ at $0 \in \mathbb{R}^2$, and $T_0 h$ be its tangent map at $0 \in \mathbb{R}^2$. If $f_z \circ h = f_z$, then $f_z \circ T_0 h = f_z$.

**Proof.** For the sake of completeness we will recall a short proof from [10].

Assume that the polynomial $f_z$ is a homogeneous function of degree $k$, i.e., $f_z(tx) = t^k f_z(x)$ for $t \geq 0$ and $x \in \mathbb{R}^2$. Then

$$ f_z(x) = \frac{f_z(tx)}{t^k} = \frac{f_z(h(tx))}{t^k} = f_z \left( \frac{h(tx)}{t} \right) \xrightarrow{t \to 0} (f_z \circ T_0 h)(x). $$

Lemma 3.2 is proved. $\square$

4. Topological structure of the atom $\mathcal{E}_{V,a}$

Let $f$ be a smooth function from $\mathcal{F}(M)$, $\varepsilon_1 > 0$, $c \in \mathbb{R}$, and $V$ be a connected component of some critical level $f^{-1}(c)$ of $f$.

Let also $\mathcal{E}$ be a connected component of $f^{-1}([c-\varepsilon_1, c+\varepsilon_1])$, which contains $V$. Assume that the boundary $\partial \mathcal{E}$ consists of $n + k$ connected components $A_i$, $i = 1, 2, \ldots, n + k$, i.e., $\partial \mathcal{E} = \bigcup_{i=1}^n A_i \cup \bigcup_{j=1}^k A_{-j}$. Since $f \in \mathcal{F}(M)$, it follows that $f|_{\mathcal{E}}$ belongs to $\mathcal{F}(\mathcal{E})$, and so, by (B), $f|_{\mathcal{E}}$ takes a constant value at each connected component of the boundary $\partial \mathcal{E}$. Assume that $f(A_i) = c_i \in [c, c + \varepsilon_1]$, $i \geq 1$, and $f(A_i) = d_i \in [c - \varepsilon_1, c]$, $i \leq 1$. Put $c' = \min\{c_i\}$ and $d' = \max\{d_i\}$. Fix $a > 0$ such that $[c - a, c + a] \subset [d', c']$.

A connected component of $f^{-1}([c-a, c+a])$, which contains $V$ will be called an atom of $V$ and denoted by $\mathcal{E}_{V,a}$.

Let $H$ be a subgroup of $G_{v}^{lat}$ and $\mathcal{H} = \phi^{-1}(H) \subset \mathcal{S}_{V}(f)$. We will need the following lemma.
Lemma 4.1. Let $E_{V,a}$ be an atom of a special critical component $V$, $A$ be a connected component of $\partial E_{V,a}$, and $h \in H$. Assume that the group $H$ has property (C). If $h |_{E_{V,a}}(A) = A$, then $h$ preserves the orientation of $A$.

Proof. Fix a Riemannian metric $\langle \cdot, \cdot \rangle$ on $M$. Let $\nabla f$ be a gradient vector field of the function $f$ in this Riemannian metric. Let also $Q$ be a set of points $x \in A$ such that there exists an integral curve $c_x$ of $\nabla f$, which joins the point $x$ with some point $y_x \in V$. Then $Q$ is a union of open intervals in $A$, and the map $\psi : Q \to V$, $\psi(x) = y_x$ is an embedding. The image of $\psi(Q)$ is a cycle in $V$. So, the connected component $A$ of $\partial E_{V,a}$ defines the cycle $\gamma_A$ in $V$. Moreover the orientation of $A$ induces the orientation of $\gamma_A$ and vice versa, see [12].

Assume that $H$ has property (C). Let $h \in H$ and $E$ be a 2-dimensional element of $\Xi$ such that $h(E) = E$. Then by (C), $h(e) = e$ for all other $e \in \Xi$. In particular $h(\gamma_A) = \gamma_A$ and $h$ preserves orientation of $\gamma_A$. Then $h(A) = A$, and $h$ preserves orientation of $A$. 

\[ \Box \]

5. Proof of Theorem 2.2

Suppose $f \in F(M)$ is such that its KR-graph contains a special vertex $v$, $V = p_f^{-1}(v)$ be the corresponding special component of some level set, which corresponds to $v$, and $G_v^{\text{loc}}$ be the local stabilizer of $v$.

Let $H$ be a subgroup of $G_v^{\text{loc}}$ such that $H = \phi^{-1}(H)$ has property (C). We will construct a lifting of the $H$-action on $st(v)$ to the action $\Sigma : H \times M \to M$ of the group $H$ on the surface $M$.

By Lemma 2.1 there is an action $\sigma^0 : H \times V \to V$ of $H$ on the set of vertices of $V$ defined by the rule:

$$\sigma^0(g, z) = h(z),$$

where $h \in H$ is any diffeomorphism such that $\phi(h) = g$.

Step 1. Now we will extend the action $\sigma^0$ to the $H$-action $\sigma^1$ on the set of neighborhoods of vertices of $V$. Assume that the action $\sigma^0$ has $s$ orbits $V_r = \{ z_{r0}, z_{r1}, \ldots, z_{rk(r)} \}$ for some $k(r) \in \mathbb{N}$, $r = 1, 2, \ldots, s$, and let $V = \bigcup_{r=1}^s V_r$ be the union of vertices of $V$.

Then, by definition of the class $F(M)$, for each $r = 1, 2, \ldots, s$ there exists a chart $(U_{r0}, q_{r0})$ which contains $z_{r0}$ such that the map $f \circ q_{r0}^{-1} = f_r$ is a homogeneous polynomial without multiple linear factors. We can also assume that $q_{r0}(U_{r0}) \subset \mathbb{R}^2$ is a 2-disk with the center at $0 \in \mathbb{R}^2$ and radius $\varepsilon$, and the group $\mathcal{L}(f_r)$ has the properties described in Lemma 3.1. Fix any diffeomorphisms

\[ (5.1) \quad h_{r_i} \in H \quad \text{such that} \quad h_{r_i}(z_{r0}) = z_{r_i}, \quad i = 1, 2, \ldots, k(r), \]

and define charts $(U_{r_i}, q_{r_i})$ for the points $z_{r_i}$, $i = 1, 2, \ldots, k(r)$ in the following way:

- $U_{r_i} = h_{r_i}(U_{r0})$;
- the map $q_{r_i}$ is defined from the diagram:

\[ U_{r0} \xrightarrow{h_{r_i}} U_{r_i} \]

\[ q_{r0} \downarrow \quad q_{r_i} \]

\[ q_{r0}(U_{r0}) \]

i.e., $q_{r_i} = q_{r0} \circ h_{r_i}^{-1}$.

Reducing $\varepsilon$, we can assume that $U_{r_i} \cap U_{r_j} = \emptyset$ for $i \neq j$.

Thus the chart $(U_{r_i}, q_{r_i})$ is chosen so that the map $f \circ q_{r_i}^{-1} : q_{r0}(U_{r0}) \to \mathbb{R}$ is a homogeneous polynomial without multiple linear factors which coincides with given polynomial $f_r$ for the chart $(U_{r0}, q_{r0})$. We also put $U_r = \bigcup_{i=0}^{k(r)} U_{r_i}$, and $U = \bigcup_{r=1}^s U_r$. 

\[ \Box \]
Lemma 5.1. There exist a homomorphism $\lambda_1 : H \to \text{Diff}(U)$ and a monomorphism $\chi_1 : H \to \text{Diff}(U)$ such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\lambda_1} & \text{Diff}(U) \\
\phi \downarrow & & \downarrow \chi_1 \\
H & & \\
\end{array}
\]

Proof. (1) First we construct a map $\lambda_1$. Let $h \in H$ be such that $h(z_r) = z_j$ for some $i, j = 0, 1, \ldots, k(r)$ and $r = 1, 2, \ldots, s$. Let also $\gamma_h = q_j \circ h \circ q_i^{-1}$ be a diffeomorphism of $q_{r_0(U_r)}$. It is easy to see that the map $\gamma_h$ preserves the polynomial $f_r$. By Lemma 3.1, the tangent map $T_0\gamma_h$ also preserves the polynomial $f_r$, so $T_0\gamma_h \in \mathcal{L}(f_r)$. Define a linear map $A \in \mathcal{L}(f_r)$ as follows: if $\deg f_r = 2$, then, by Lemma 3.1,

\[
T_0\gamma_h = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}, \quad a \neq 0,
\]

and we set

\[
A_h = \text{sign}(a) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

If $\deg f_r \geq 3$, then by assumption and Lemma 3.1, $\mathcal{L}(f_r)$ is a cyclic subgroup of $\text{SO}(2)$. In this case we put

\[
A_h = T_0\gamma_h.
\]

We define the diffeomorphism $\lambda_1(h) \in \text{Diff}(U)$ by the rule:

\[
(5.2) \quad \lambda_1(h)|_{U_{\gamma_i}} = q_j^{-1} \circ A_h \circ q_i.
\]

(2) Now we prove that the map $\lambda_1$ is a homomorphism. Suppose $h_1, h_2 \in H$ are such that $h(z_r) = z_j$ and $h(z_{r_1}) = z_{r_1}$. By (5.2), we have

\[
\lambda_1(h_1)|_{U_{\gamma_i}} = q_j^{-1} \circ A_{h_1} \circ q_i, \quad \lambda_1(h_2)|_{U_{\gamma_i}} = q_{r_1}^{-1} \circ A_{h_2} \circ q_i.
\]

and

\[
\lambda_1(h_2) \circ \lambda_1(h_1)|_{U_{\gamma_i}} = q_{r_1}^{-1} \circ A_{h_2} \circ A_{h_1} \circ q_i.
\]

On the other hand, we have

\[
\lambda_1(h_2 \circ h_1) = q_{r_1}^{-1} \circ A_{h_2 \circ h_1} \circ q_i.
\]

It follows from the definition of the linear map $A_h$, that $A_{h_2 \circ h_1} = A_{h_2} \circ A_{h_1}$. Hence

\[
\lambda_1(h_2 \circ h_1) = \lambda_1(h_2) \circ \lambda_1(h_1).
\]

So, the map $\lambda_1$ is a homomorphism.

(3) Let $g \in H$ and $h \in H$ be such that $\phi(h) = g$. Then we define the map $\chi_1 : H \to \text{Diff}(U)$ by the rule

\[
\chi_1(g) = \lambda_1(h).
\]

Obviously that $\chi_1$ is a homomorphism. It remains to prove that the map $\chi_1$ is a monomorphism. It is sufficient to check that $\text{Ker}\chi_1 = \text{Ker}\psi$, i.e., $\lambda_1(h) = \text{id}_U$ iff $h$ trivially acts on the set of 2-dimensional elements of $\Xi$.

Suppose that $h$ trivially acts on the set of 2-dimensional elements of $\Xi$. By condition (C), $h$ trivially acts on set of vertices and edges of $V$. Since $h(z_{r_i}) = z_{r_i}$ for all $i = 0, 1, \ldots, k(r)$ and $r = 1, 2, \ldots, s$, it follows from (5.2) that $\lambda_1(h) = \text{id}_U$.

Suppose $h \in H$ is such that $\lambda_1(h) = \text{id}_U$. Then $h(e) = e$ for each edge $e$ of $V$, and $h$ preserves the orientation of $e$. Hence by Lemma 4.1, $h$ leaves invariant each connected component of $\partial \Xi_{V,e}$ with its orientation. Therefore $h$ trivially acts on the set of 2-dimensional elements of $\Xi$. }
Let $\sigma^1 : H \times U \to U$ be a map defined by the formula

$$\sigma^1(g, x) = \chi_1(g)(x), \quad x \in U.$$ 

Since $\chi_1$ is a homomorphism, it follows that $\sigma^1$ is an $H$-action on $U$.

**Step 2.** In this step we extend the action $\sigma^1$ to the $H$-action $\sigma$ on the atom $\mathcal{E}_{V,a}$.

We start with some preliminaries. Let $(U_{ri}, q_{ri})$ be the chart on $M$, which contains $z_{ri}$, defined above. The projection map $q_{ri}$ induces the map $Tq_{ri} : TU_{ri} \to Tq_{ri}(U_{ri})$ between tangent bundles of $U_{ri}$ and $q_{ri}(U_{ri}) \subset \mathbb{R}^2$. Fix a Riemannian metric $\langle \cdot, \cdot \rangle$ on $M$ such that the following diagram is commutative

$$
\begin{array}{ccc}
TU_{ri} & \xrightarrow{Tq_{ri}} & Tq_{ri}(U_{ri}) \\
\nabla f|_{U_{ri}} & \bigg\downarrow \nabla f_{|q_{ri}(U_{ri})} & \\
U_{ri} & \xrightarrow{q_{ri}} & q_{ri}(U_{ri})
\end{array}
$$

where $\nabla f$ and $\nabla f_{|q_{ri}}$ are gradient fields of $f$ and $f_{|q_{ri}}$ in Riemannian metrics on $M$ and on $\mathbb{R}^2$ respectively. Let also $G$ be the flow of $\nabla f$ on $M$.

**Another description of the diffeomorphism $\lambda_1(h)$.** Let $x \in U_{ri}$ be a point, $i = 0, 1, 2, \ldots, k(r)$, $r = 1, 2, \ldots, s$, and $y = \lambda_1(h)(x)$ be its image under $\lambda_1(h)$. Let also $\omega_x$ and $\omega_y$ be the trajectories of the gradient flow $G$ such that $x \in \omega_x$ and $y \in \omega_y$. Since $\lambda_1(h)$ preserves trajectories of the flow $G$ in $U$, it follows that $\lambda_1(h)(\omega_x \cap U_{ri}) = \omega_y \cap \lambda_1(h)(U_{ri})$. By definition of $\lambda_1(h)$ we have that $f(x) = f(y)$. In particular, if the trajectory $\omega_x$ intersects some edge $R$ of $V$ at some point $x'$, and $y' = \lambda_1(h)(x')$, then $y = f^{-1}(f(x)) \cap \omega_{y'}$, where $\omega_{y'}$ is the trajectory of $G$, which passes through the point $y'$. Namely the image of $x$ is uniquely defined by the image of the point $x'$.

By Lemma 2.1, the group $H$ acts on the set of all edges $R$ of $V$. Assume that this action has $u$ orbits $R_r = \{R_{r0}, R_{r1}, \ldots, R_{rn(u)}\}$ for some $n(u) \in \mathbb{N}$ and $r = 1, 2, \ldots, u$. We also put $R = \bigcup_{r=1}^{u} R_r$. For each edge $R_{ri}$ fix

(I) a $C^\infty$-diffeomorphism $\ell_{ri} : (-1, 1) \to R_{ri}$ such that restrictions $\ell_{ri}|_{(-1, 1-\varepsilon)}$ and $\ell_{ri}|_{(1-\varepsilon, 1)}$ are isometries,

where $\varepsilon$ is the radius of the disk $q_{ri}(U_{ri})$ defined in Step 1.

**Lemma 5.2.** There exist a homomorphism $\lambda_2 : H \to \text{Diff}(\mathcal{E}_{V,a})$ and a monomorphism $\chi_2 : H \to \text{Diff}(\mathcal{E}_{V,a})$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\lambda_2} & \text{Diff}(\mathcal{E}_{V,a}) \\
\phi \downarrow & & \downarrow \chi_2 \\
H & & 
\end{array}
$$

and $\lambda_2(h)|_{U} = \lambda_1(h)|_{U \cap \mathcal{E}_{V,a}}$.

**Proof.** Let $h \in \mathcal{H}$. We will extend the diffeomorphism $\lambda_1(h)$ to a diffeomorphism $\lambda_2(h)$ of the atom $\mathcal{E}_{V,a}$. Let $x \in \mathcal{E}_{V,a}$ be any point. If $x \in U_{ri}$ for some $i = 0, 1, \ldots, k(r)$, $r = 1, 2, \ldots, s$, then we put $\lambda_2(h)(x) := \lambda_1(h)(x)$.

Suppose that $x \not\in U$. Let $\omega_x$ be a trajectory of the flow $G$ passing through the point $x$. Then we have one of the following two cases: the trajectory $\omega_x$ either

(1) intersects some edge $R$ of $V$ at a point, say $y$, or
(2) converges to some vertex $z$ of $V$. 


In the case (1) let \( R' = h(R) \), \( \ell : (-1, 1) \to R \) and \( \ell' : (-1, 1) \to R' \) be maps, defined by (I) for \( R \) and \( R' \) respectively, and
\[
h' = \ell' \circ \ell^{-1} : R \xrightarrow{\ell^{-1}} (-1, 1) \xrightarrow{\ell'} R'.
\]
Let also \( y' = h'(y) \in R' \), \( \omega_{y'} \) be the trajectory of \( G \), which passes through \( y' \), and \( x' \) be a unique point in \( \omega_{y'} \) such that \( f(x) = f(x') \). Then we put \( \lambda_{2}(h)(x) = x' \).

Consider the case (2). Let \( U \) be the neighborhood of \( z \), defined in Step 1, \( z' = \lambda_{1}(h)(z) \) be the corresponding point in \( U' = \lambda_{1}(h)(U) \), \( \omega_{z'} \) be the trajectory of \( G \) such that \( \omega_{z'} \cap U' = \lambda_{1}(h)\omega_{z} \cap U \), and \( x' \) be a unique point in \( \omega_{z'} \) such that \( f(x) = f(x') \). In this case we define \( \lambda_{3}(h) \) by the rule: \( \lambda_{3}(h)(x) = x' \).

By definition \( \lambda_{2}(h)|_{U} = \lambda_{1}(h)|_{U \cap E_{V,a}} \). Let \( \chi_{2} : H \to \text{Diff}(E_{V,a}) \) be the map defined as follows: for \( g \in H \) and \( h \in H \) such that \( \phi(h) = g \), we put \( \chi_{2}(g) = \lambda_{2}(h) \). It is easy to check that the map \( \lambda_{2} \) is a homomorphism. Moreover \( \lambda_{2}(h) = \text{id}_{E_{V,a}} \) iff \( \lambda_{1}(h) = \text{id}_{U} \). Therefore \( \chi_{2} \) is a homomorphism.

Define the map \( \sigma : H \times E_{V,a} \to E_{V,a} \) by the rule
\[
\sigma(g, x) = \chi_{2}(g)(x).
\]

Since \( \chi_{2} \) is the homomorphism, it follows that the map \( \sigma \) is an \( H \)-action on the atom \( E_{V,a} \).

**Step 3.** In this step we extend the \( H \)-action \( \sigma \) on the atom \( E_{V,a} \) to the \( H \)-action on the surface \( M \). We start with some preliminaries. Let \( E \) be a set of 2-dimensional elements of \( \Xi \). By Lemma 2.1 the group \( H \) acts on the set \( E \). Assume that this action has \( y \) orbits \( E_{r} = \{ E_{r0}, E_{r1}, \ldots, E_{rk(r)} \} \), \( i = 0, 1, \ldots, k(r) \), and \( r = 1, 2, \ldots, y \). We also put \( E = \bigcup_{i=1}^{y} E_{r} \). Fix diffeomorphisms \( h_{ri} \in H \) such that \( h_{ri}(E_{ro}) = E_{ri} \).

Let \( Y_{r} = E_{r0} \cap f^{-1}([-a, -a/2] \cup [a/2, a]) \cap E_{V,a} \). Since \( v \) is a special vertex, it follows that the set \( Y_{r} \) is a cylinder. We put \( Y_{ri} = h_{ri}(Y_{r}) \), and \( Y = \bigcup_{r=1}^{y} \bigcup_{i=0}^{k(r)} Y_{ri} \).

We choose \( a > a \) such that the set \( E_{V,a_{1}} \) is also an atom of \( V \). Let
\[
Z_{r} = E_{r0} \cap f^{-1}([-a_{1}, a/2] \cup [a/2, a_{1}]) \cap E_{V,a_{1}}.
\]
By definition, we have that \( Y_{r} \subset Z_{r} \), and \( Z_{r} \) does not contain critical points of \( f \). We also put \( Z_{ri} = h_{ri}(Y_{r}) \) and \( Z = \bigcup_{r=1}^{y} \bigcup_{i=0}^{k(r)} Z_{ri} \).

![Figure 5.1](image)

**Figure 5.1.** The 2-dimensional component \( E_{r0} \), and its subsets \( Y_{r} \) and \( Z_{r} \).

Fix a vector field \( F \) on \( Z \) such that its orbits coincide with connected components of level sets of the restriction \( f|_{Z} \), and let \( F \) be the flow of \( F \). Then for each smooth function \( \alpha \in C^{\infty}(M) \) we can define the following map
\[
F_\alpha : M \to M, \quad F_\alpha(x) = F(x, \alpha(x)).
\]
Such maps have been studied in [8].
Since all orbits of $F$ are closed, it follows from [5] Theorem 19 that the map $F_\alpha$ is a diffeomorphism, iff the Lie derivative $F\alpha$ of $\alpha$ along $F$ satisfies the condition: $F\alpha > -1$. Moreover we have that $(F_\alpha)^{-1} = F_{\xi}$, where

$(5.3)$

$$\xi = -\alpha \circ F_\alpha^{-1}.$$

**Lemma 5.3.** For each $g \in H$ the map $\chi_2(g)$ extends to a diffeomorphism $\Sigma(g) \in S(f)$, so that the correspondence $g \mapsto \Sigma(g)$ is a homomorphism $\Sigma : H \to S(f)$.

**Proof.** We will need the following two lemmas.

**Lemma 5.4.** Let $g \in H$ and $h \in H$ be such that $\phi(h) = g$, and $h(E_{r_{i_0}}) = E_{r_{i_1}}$. Then there exists a unique $C^\infty$-function $\xi_{r_{i_1}} : Y_r \to \mathbb{R}$ such that

$$\chi_2(g)|_{Y_r} = h_{r_{i_1}}|_{Y_r} \circ F_{\xi_{r_{i_1}}} : Y_r \to Y_r.$$

In particular, the function $\xi_{r_{i_1}}$ depends only on $g$.

**Lemma 5.5.** The diffeomorphism $F_{\xi_{r_{i_1}}}$ extends to a diffeomorphism $w_{r_{i_1}} : E_{r_{i_0}} \to E_{r_{i_1}}$ such that $f \circ w_{r_{i_1}} = f$ on $E_{r_{i_0}}$.

We prove Lemma 5.4 and Lemma 5.5 bellow, and now we will complete Theorem 2.2.

Define a diffeomorphism $h_{r_{i_1}} : E_{r_{i_0}} \to E_{r_{i_1}}$ by the formula:

$$h_{r_{i_1}} = h_{r_{i_1}} \circ w_{r_{i_1}}.$$

Let $h \in H$. Define the diffeomorphism $\lambda_3(h)$ by the rule: if $h(E_{r_{i_1}}) = E_{r_{i_2}}$, then

$$\lambda_3(h)|_{E_{r_{i_1}}} = h_{r_{i_1}} \circ h_{r_{i_2}}^{-1} : E_{r_{i_1}} \xrightarrow{h_{r_{i_1}}^{-1}} E_{r_{i_0}} \xrightarrow{h_{r_{i_2}}} E_{r_{i_2}}.$$ 

It follows from Lemma 5.3 that $\lambda_3(h)$ coincides with $\lambda_2(h)$ on $Y$.

Now we will check that the correspondence $h \mapsto \lambda_3(h)$ is a homomorphism. Let $h_1$ and $h_2$ be homeomorphisms from $H$ such that $h_1(E_{r_{i_1}}) = E_{r_{i_2}}$ and $h_2(E_{r_{i_2}}) = E_{r_{i_3}}$. By definition $\lambda_3(h_1)|_{E_{r_{i_1}}} = h_{r_{i_2}} \circ h_{r_{i_1}}^{-1}$, and $\lambda_3(h_2)|_{E_{r_{i_2}}} = h_{r_{i_3}} \circ h_{r_{i_2}}^{-1}$. Then $\lambda_3(h_1)|_{E_{r_{i_1}}} \circ \lambda_3(h_2)|_{E_{r_{i_2}}} = h_{r_{i_3}} \circ h_{r_{i_2}}^{-1} \circ h_{r_{i_1}} \circ h_{r_{i_2}}^{-1} = h_{r_{i_3}} \circ h_{r_{i_1}}^{-1} = \lambda_3(h_2 \circ h_1)|_{E_{r_{i_1}}}$.

Hence, the map $\lambda_3$ is a homomorphism.

Let $g \in H$, and $h \in H$ be such that $\phi(h) = g$. By condition (C), $\lambda_3(h) = \text{id}$ iff $h(E_{r_{i_1}}) = E_{r_{i_2}}$ for some $r = 1, 2, \ldots, y$, and $i = 0, 1, \ldots, k(r)$. So the map $\chi_3 : H \to \text{Diff}(E)$, defined by $\chi_3(h) = \lambda_3(h)$, is a monomorphism.

Let $\sigma' : H \times E \to E$ be the map given by the formula

$$\sigma'(g, x) = \chi_3(g)(x), \quad x \in E.$$ 

Since $\chi_3$ is a homomorphism, it follows that $\sigma'$ is an $H$-action on $E$.

Hence, we define an $H$-action $\Sigma : H \times M \to M$ on $M$ by the rule:

$$\Sigma = \begin{cases} 
\sigma', & \text{on } H \times E, \\
\sigma, & \text{on } H \times \mathcal{E}_{V, \alpha}.
\end{cases}$$

Theorem 2.2 is proved.

**Proof of Lemma 5.4.** Due to [5] Lemma 4.12, for the diffeomorphism $h'_{r_{i_1}} = \chi_2^{-1}(g)|_{Y_r} \circ h_{r_{i_1}}|_{Y_r} : Y_r \to Y_r$ of the cylinder there exists a smooth function $\alpha_{r_{i_1}}$ such that $h'_{r_{i_1}} = F_{\alpha_{r_{i_1}}}$ whenever for each trajectory $\omega$ of $F$ we have that $h'_{r_{i_1}}(\omega) = \omega$ and $h'_{r_{i_1}}$ preserves orientation of $\omega$.

Let $\omega$ be a trajectory of $F$. It follows from condition (C) that $\omega = f^{-1}(t) \cap Y_r$ for some $t \in \mathbb{R}$. The sets $f^{-1}(t)$ and $Y_r$ are $h'_{r_{i_1}}$-invariant, so the set $f^{-1}(t) \cap Y_r$ is also $h'_{r_{i_1}}$-invariant. Hence, $h'_{r_{i_1}}(\omega) = \omega$ for all trajectories of $F$. Moreover, by Lemma 4.1 $h'_{r_{i_1}}$ preserves orientations of each orbit $\omega$ of $F$. Thus $h'_{r_{i_1}} = F_{\alpha_{r_{i_1}}}$, and due to (5.3) we put $\xi_{r_{i_1}} = -\alpha_{r_{i_1}} \circ F_{\alpha_{r_{i_1}}}^{-1}$. Lemma is proved.
Proof of Lemma 5.5. By the result of Seeley [14], the function $\xi_{ri}$ extends to some smooth function $\beta_{ri}^\prime$ on $E_{ri}$. It is easy to construct a function $\delta_r \in C^\infty(E_{r0}, [0, 1])$ which satisfies the following conditions:

1. $\delta_r = 1$ on $Y_r$,
2. $\delta_r = 0$ on some neighborhood of $Z_r \cap (f^{-1}(-a_1) \cup f^{-1}(a_1))$.
3. $F \delta_r = 0$, i.e., $\delta_r$ is constant along orbits of $F$,
4. the function $\beta_{ri}^\prime = \delta_r \beta_{ri}^\prime, i = 1, 2, \ldots, n$ satisfies the inequality $F \beta_{ri}^\prime |_{Z_r} > -1$.

Indeed, since $\beta_{ri}^\prime = \xi_{ri}$ on $Y_{ri}$ and $F_{\alpha_{ri}}$ is a diffeomorphism, it follows that $F \beta_{ri}^\prime > -1$ on $Y_{ri}$. Then there exists $b \in (a, a_1)$ such that $F \beta_{ri}^\prime > -1$ on $A_r = E_{r0} \cap f^{-1}([-b, a/2] \cup [a/2, b]) \cap E_{V,a}$. Let $\delta_r : E_{r0} \to [0, 1]$ be a smooth function such that $\delta_r = 1$ on $Y_r$, $\delta_r = 0$ on $E_{r0} \setminus A_r$, and $F \delta_r = 0$. Then $\delta F \beta_{ri}^\prime > -1$ on $E_{ri}$. Now the required diffeomorphism $w_{ri} : E_{r0} \to E_{r0}$ can be defined by the formula

$$w_{ri}(x) = \begin{cases} F \beta_{ri}, & x \in Z_r, \\ x, & x \in E_{r0} \setminus Z_r. \end{cases}$$

Lemma is proved.

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