Determination of the anomalous dimension of gluonic operators in deep inelastic scattering at $O(1/N_f)$

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Abstract. Using large $N_f$ methods we compute the anomalous dimension of the predominantly gluonic flavour singlet twist-2 composite operator which arises in the operator product expansion used in deep inelastic scattering. We obtain a $d$-dimensional expression for it which depends on the operator moment $n$. Its expansion in powers of $\epsilon = (4 - d)/2$ agrees with the explicit exact three loop $\overline{\text{MS}}$ results available for $n \leq 8$ and allows us to determine some new information on the explicit $n$-dependence of the three and higher order coefficients. In particular the $n$-dependence of the three loop anomalous dimension $\gamma_{gg}(a)$ is determined in the $C_2(G)$ sector at $O(1/N_f)$. 
1 Introduction.

Perturbative quantum chromodynamics, (QCD), is widely accepted as being the tool to describe high energy scattering processes involving the strong interactions. (For a review see, for example, [1].) This is primarily due to its property of asymptotic freedom which implies that as higher energies are attained the strength of the strong coupling decreases and perturbation theory becomes a more valid approximation. One of the current topics of interest is in constructing the perturbative series at a higher order than is presently known for the anomalous dimensions of the twist-2 operators which appear in the operator product expansion, which is fundamental to the mathematics underlying deep inelastic scattering. One reason that these are required is that they arise in the renormalization group equation for the evolution of the coefficient functions which are also required at a higher loop order. Indeed knowledge of the three loop operator dimensions and two loop coefficient functions will mean that the full evolution with energy scale can be performed at a new order. The anomalous dimensions have been studied at successive orders since the discovery that QCD was asymptotically free [2]. The original one loop calculations were carried out in [3] whilst two loop corrections were constructed later both for the flavour non-singlet and singlet cases in [4, 5, 6]. These results were expressed not only as a function of the colour group Casimirs and number of quark flavours, $N_f$, but also of the moment of the operator denoted by $n$. To two loops the expressions as a function of $n$ were reasonably straightforward to achieve. One requires this explicit dependence rather than, for example, a table of numerical values, in order to make contact with an alternative method of calculation. This is the splitting function approach of the DGLAP formalism, [7]. The relation between the two being determined by the Mellin transform and its inverse where the conjugate variable to $n$ is $x$ which corresponds to the momentum fraction carried by the parton. Recently substantial progress has been made towards the determination of the three loop terms for the operator dimensions. In [8, 9] exact expressions have been given for the even moments for the non-singlet, $(n \leq 10)$, and singlet, $(n \leq 8)$, operators. These calculations were performed by use of computer algebra and symbolic manipulation techniques to handle the huge number of Feynman diagrams which occur and to organise the resulting expressions for each graph. The corresponding coefficient functions are also known for the same values of $n$, [8, 9]. Therefore it remains to construct the full $n$-dependent result. Clearly this is a highly non-trivial but important challenge.

Some insight into the form of the operator dimensions has, however, already been provided through the large $N_f$ technique, [10, 11]. This is an alternative method to conventional perturbation theory where one can express all orders results in the coupling constant of a renormalization group function as a closed function of the spacetime dimension $d$. For QCD the method has extended the pioneering work for the $O(N)$ σ model developed in the series of articles [12, 13]. Through ideas in the critical renormalization group one can directly relate the coefficients in the $\epsilon$-expansion of this function or critical exponent, where $d = 4 - 2\epsilon$, to the explicit perturbative coefficients at the same order in powers of $1/N_f$. So, for example, using this approach an expression was derived for the $\beta$-function of QCD, [14], and the non-singlet twist-2 operator dimension at $O(1/N_f)$, [10]. The three loop coefficient of the latter, which was given explicitly as a function of $n$, agrees exactly with the full calculation of [8] for the even moments $2 \leq n \leq 10$. Likewise the extension of that work to the singlet case, [11], yielded similar exact agreement for the region of overlap, [8]. Clearly these analytic results will play an important role for checking the full $n$-dependent three loop result. However, the singlet $1/N_f$ calculation of [11] only concentrated on one of the operators which occurs. In perturbation theory there is mixing between two operators which are respectively predominantly fermionic and gluonic in nature, [8]. As the former was studied in [11] it remains to derive results for the latter type of
operator in order to complete the full leading order in $1/N_f$ computation. This is the main aim of this paper where we will discuss the technical issues required to obtain a result on a par with [13]. Although these are provided with the particular case of QCD in mind, the calculational algorithms to determine the values of the scalar integrals which arise are reasonably general and therefore applicable to the determination of the anomalous dimensions of similar singlet operators in other field theories in the $1/N_f$ method. Finally, we remark that although this paper is concerned with large $N_f$ methods for the operator dimensions, some insight into the $n$-dependence of the coefficient functions in large $N_f$ have already been provided in [13, 16].

The paper is organised as follows. We review the basic formalism and background to our problem in section 2. The details of the computation of the Feynman diagrams which occur in the large $N_f$ critical point formalism are provided in sections 3 and 4. These deal respectively with the QED and non-QED sectors of the operator dimension defined essentially by those graphs which do not and do vanish by Furry’s theorem. Section 5 is devoted to the derivation of explicit $n$-dependent results as well as several concluding remarks. The appendices respectively provide some basic integral results and an explicit three loop calculation to illustrate some of the internal cross checks which were necessary to validate the general recurrence relations we had to derive.

2 Background.

We begin by reviewing the necessary features of the perturbative structure of the anomalous dimensions of the twist-2 singlet operators we are interested in. These are defined to be, [3],

\[
\begin{align*}
O_q^{\mu_1 \cdots \mu_n} &= i^{n-1} S^{\gamma} I^{\mu_1} D^{\mu_2} \ldots D^{\mu_n} \bar{\psi}^{J} - \text{trace terms} \\
O_g^{\mu_1 \cdots \mu_n} &= \frac{1}{2} i^{n-2} S \text{tr} G^{\alpha} I^{\mu_1} D^{\mu_2} \ldots D^{\mu_{n-1}} G^{\alpha \nu} - \text{trace terms}
\end{align*}
\]  

(2.1)

where $\bar{\psi}^I$ is the quark field, $A^a_\mu$ is the gluon field, $D_\mu = \partial_\mu + T^a A^a_\mu$, $G^{\alpha \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$, $T^a$ are the colour group generators, $f^{abc}$ are the structure constants, $1 \leq i \leq N_c$, $1 \leq I \leq N_f$, $1 \leq a \leq (N_c^2 - 1)$ and $S$ denotes symmetrization in the Lorentz indices. In perturbation theory the canonical dimension of each of the operators is equivalent. Therefore they mix under renormalization which leads to a matrix of renormalization constants and associated matrix of anomalous dimensions. To fix notation we define the renormalization constants by

\[
O_i^{\text{ren}} = Z_{ij} O_j^{\text{bare}}
\]

(2.2)

where the anomalous dimensions, $\gamma_{ij}(a)$, are defined by

\[
\gamma_{ij}(a) = \begin{pmatrix} \gamma_{qq}(a) & \gamma_{qg}(a) \\ \gamma_{gq}(a) & \gamma_{gg}(a) \end{pmatrix}
\]

(2.3)

with $\gamma_{ij}(a) = \beta(a)(\partial/\partial a) \ln Z_{ij}$ and $\beta(a)$ is the usual renormalization group function governing the running of the coupling constant $a$. The entries in $\gamma_{ij}(a)$ depend on the colour group parameters, $N_f$ and $n$ and since it is the $1/N_f$ corrections which we will focus on we define the explicit form of each the entries as

\[
\begin{align*}
\gamma_{qq}(a) &= a_1 a + (a_2 N_f + a_22) a^2 + (a_3 N_f^2 + a_32 N_f + a_33) a^3 + O(a^4) \\
\gamma_{qg}(a) &= b_1 a + (b_2 N_f + b_22) a^2 + (b_3 N_f^2 + b_32 N_f + b_33) a^3 + O(a^4) \\
\gamma_{gq}(a) &= c_1 N_f a + c_2 N_f a^2 + (c_3 N_f^2 + c_32 N_f + c_33) a^3 + O(a^4) \\
\gamma_{gg}(a) &= (d_11 N_f + d_12) a + (d_21 N_f + d_22) a^2 + (d_31 N_f^2 + d_32 N_f + d_33) a^3 + O(a^4)
\end{align*}
\]

(2.4)
Clearly the $N_f$ dependence does not enter in the same fashion in each term. The explicit determination was carried out in \[4, 5, 6\] but we record for completeness only those which we require

\[
a_1 = C_2(R) \left[ 4S_1(n) - 3 - \frac{2}{n(n+1)} \right], \quad b_1 = -\frac{2(n^2 + n + 2)C_2(R)}{n(n^2 - 1)}
\]

\[
c_1 = -\frac{4(n^2 + n + 2)T(R)}{n(n+1)(n+2)}, \quad d_{11} = \frac{4}{3}T(R)
\]

\[
d_{12} = C_2(G) \left( 4S_1(n) - \frac{11}{3} - \frac{4}{n(n-1)} - \frac{4}{(n+1)(n+2)} \right)
\]

\[
a_{21} = T(R)C_2(R) \left[ \frac{2}{3} - \frac{80}{9}S_1(n) + \frac{16}{3}S_2(n) \right. \\
+ \left. \frac{8(11n^7 + 49n^6 + 5n^5 - 329n^4 - 514n^3 - 350n^2 - 240n - 72)}{9n^3(n+1)^3(n+2)^2(n-1)} \right]
\]

\[
b_{21} = -\frac{16C_2(R)T(R)}{3} \left[ \frac{1}{(n+1)^2} + \frac{(n^2 + n + 2)}{n(n^2 - 1)} \left( S_1(n) - \frac{8}{3} \right) \right]
\]

\[
d_{21} = C_2(R)T(R) \left( 4 + \frac{8(2n^6 + 5n^5 + n^4 - 10n^3 - 5n^2 - 4n - 4)}{(n-1)n^3(n+1)^3(n+2)} \right)
\]

\[
+ C_2(G)T(R) \left( -\frac{80}{9}S_1(n) + \frac{16}{3} + \frac{8(38n^4 + 76n^3 + 94n^2 + 56n + 12))}{9(n-1)n^2(n+1)^2(n+2)} \right) \quad (2.5)
\]

where the colour group Casimirs are defined by $T^aT^a = C_2(R)$, $tr(T^aT^b) = T(R)\delta^{ab}$ and $T^aT^b = C_2(G)\delta^{ab}$. The function $S_1(n)$ is defined by $S_1(n) = \sum_{r=1}^n 1/r^3$. These expressions (2.5) are determined in perturbation theory by inserting both operators, \[2,4\], in quark and gluon 2-point functions and deducing the pole with respect to the dimensional regulator before using \[\overline{\text{MS}}\] to define $Z_{ij}$. The procedure to find leading order $1/N_f$ information on $\gamma_{ij}(a)$ is similar and has been discussed in \[11, 13\]. Briefly one uses critical point renormalization group techniques to analyze the scaling behaviour of the operators we are interested in at a fixed point in the $\beta$-function of QCD. This is defined to be a non-trivial zero of the $d$-dimensional $\beta$-function and the value of the coupling is denoted by $a_c$. As has been observed in \[14\] the propagators take a particularly simple scaling form at $a_c$. Further, the anomalous dimension of each operator can be deduced in powers of $1/N_f$ by noting that there is a relationship between the critical exponent of the operator and the analogous anomalous dimension renormalization group function evaluated at $a_c$. In the $d$-dimensional case such a non-trivial zero exists and, moreover, is computable in a power series in $1/N_f$. For instance, from the 3-loop result, \[17, 18\], in the notation of \[19\],

\[
\beta(a) = \frac{(d-4)}{2}a + \left[ \frac{4}{3}T(R)N_f - \frac{11}{3}C_2(G) \right]a^2 \\
+ \left[ 4C_2(R)T(R)N_f + \frac{20}{3}C_2(G)T(R)N_f - \frac{34}{3}C_2^2(G) \right]a^3 + O(a^4) \quad (2.6)
\]

one deduces that as $N_f \to \infty$, omitting terms $O(e^5/N_f^3)$,

\[
a_c = \frac{3e}{4T(R)N_f} + \left[ \frac{33}{16}C_2(G)e - \left( \frac{27}{16}C_2(R) + \frac{45}{16}C_2(G) \right)e^2 \right. \\
+ \left( \frac{99}{64}C_2(R) + \frac{237}{128}C_2(G) \right)e^3 \\
+ \left( \frac{77}{64}C_2(R) + \frac{53}{128}C_2(G) \right)e^4 \right] \frac{1}{T^2(R)N_f^2} \quad (2.7)
\]

where we have included the contribution from the recently computed exact 4-loop $\overline{\text{MS}}$ $\beta$-function, \[19\].
The scaling behaviour of the quark and gluon propagators in momentum space at $a_c$ are then simply,

$$\psi(k) \sim \frac{A_k}{(k^2)^{\mu-\alpha}}, \quad A_{\mu\nu}(k) \sim \frac{B}{(k^2)^{\mu-\beta}} \left[ \eta_{\mu\nu} - (1-b) \frac{k_\mu k_\nu}{k^2} \right]$$

where $b$ is the covariant gauge parameter, $d = 2\mu$ and the exponents $\alpha$ and $\beta$ are defined as

$$\alpha = \mu - 1 + \frac{1}{2}\eta, \quad \beta = 1 - \eta - \chi$$

The quantity $\eta$ is the quark anomalous dimension which corresponds to the quark wave function renormalization at the critical coupling and $\chi$ is the anomalous dimension of the quark gluon vertex. The canonical dimensions are deduced by demanding that the $d$-dimensional action is dimensionless. This is contrary to perturbation theory where the canonical part is defined by ensuring the 4-dimensional action is dimensionless. The quantities $A$ and $B$ are momentum independent amplitudes. Using (2.8), various leading order results have been established such as the QCD $\beta$-function at $O(1/N_f)$, [1]. This encodes the coefficients of all powers of $\epsilon$ in the $O(1/N_f^2)$ term of (2.7). As the leading order values of $\eta, \chi$ and the amplitude combination $z = A^2B$ are required here, we note their explicit values are, [20],

$$\eta_1 = \frac{[(2\mu - 1)(\mu - 2) + \mu b]C_2(R)\eta_1^0}{(2\mu - 1)(\mu - 2)R}$$

$$\chi_1 = - \frac{[(2\mu - 1)(\mu - 2) + \mu b]C_2(R)\eta_1^0}{(2\mu - 1)(\mu - 2)R} - \frac{[2(2\mu - 1)C_2(G)\eta_1^0]}{2(2\mu - 1)(\mu - 2)R}$$

$$z_1 = \frac{\Gamma(\mu + 1)\eta_1^0}{2(2\mu - 1)(\mu - 2)R}$$

where, for example, $\eta = \sum_{i=1}^{\infty} \eta_i(\epsilon)/N_f^i$ and

$$\eta_1^0 = \frac{(2\mu - 1)(\mu - 2)\Gamma(2\mu)}{4\Gamma^2(\mu)\Gamma(\mu + 1)\Gamma(2 - \mu)}$$

In [1] it was these propagators which were used to analyse the fermionic operator $O_q$ and produce an expression which agreed with known results to three loops. This remark needs to be qualified, however, since the canonical dimensions of the fields at $a_c$ differ from the perturbative ones. Therefore the canonical dimensions of (2.1) are different and so they cannot mix under renormalization. In other words the mixing matrix at criticality is triangular at least at leading order in $1/N_f$ and both operators cease being twist-2 at $a_c$. Instead $O_q$ is twist-$(2 + O(\epsilon))$. Although this would appear to mean that the critical point large $N_f$ method fails to have any contact with perturbation theory which we are seeking, there is a relation between $\gamma_{ij}(a_c)$ and the scaling dimensions of (2.1) at $a_c$. This was established in [1] where it was noted that the eigen-anomalous dimensions of (2.3) evaluated at $a_c$ are in fact the dimensions of the operators (2.1) whose fields have the canonical dimensions given by (2.9). In other words in perturbation theory the information contained in the critical exponents relating to the dimensions of (2.1) at $a_c$ are given by two combinations of the perturbative coefficients listed in (2.7). To make these comments more transparent we return to $\gamma_{ij}(a)$ and deduce the eigen-anomalous dimensions at $a_c$ in powers of $1/N_f$. First, the eigenvalues for general $a$ are given by

$$\lambda_\pm(a) = \frac{1}{2}(\gamma_{qq} + \gamma_{gg}) \pm \frac{1}{2} \left[ (\gamma_{qq} - \gamma_{gg})^2 + 4\gamma_{qg}\gamma_{gg} \right]^{1/2}$$
However, expanding in powers of $a$ and retaining the same orders in $1/N_f$ with the definitions \[2.4\], we find

$$
\lambda_-(a) = \left(a_1 - \frac{b_{1c_1}}{d_{11}} \right) a + \left(a_{21} - \frac{b_{21c_1}}{d_{11}} \right) N_f a_1 + \left(a_{31} - \frac{b_{31c_1}}{d_{11}} \right) N_f^2 a^3 + O(N_f^3 a^4)
$$

$$
\lambda_+(a) = \left(d_{11} N_f + d_{12} + \frac{b_{1c_1}}{d_{11}} \right) a + \left(d_{21} + \frac{b_{21c_1}}{d_{11}} \right) N_f a_1
$$

$$
+ \left(d_{31} + \frac{b_{31c_1}}{d_{11}} \right) N_f^2 a^3 + O(N_f^3 a^4)
$$

(2.15)

Noting that $a_c$ is $O(1/N_f)$, it is easy to observe that $\lambda_-(a_c) = O(1/N_f)$ whereas $\lambda_+(a_c) = O(\epsilon)$ which bears out our remarks on the difference in twist at $a_c$. As the dimension of the predominantly fermionic operator has been discussed elsewhere, we record that the predominantly gluonic eigen-operator dimension is

$$
\lambda_+(a_c) = \frac{3d_{11}\epsilon}{4} + \frac{1}{T(R)N_f} \left[ \frac{3}{4} \left( d_{12} + \frac{b_{1c_1}}{d_{11}} + \frac{11}{4} C_2(G)d_{11} \right) \right] + \epsilon^2
$$

$$
+ \frac{9}{16} \left( d_{21} + \frac{b_{21c_1}}{d_{11}} - (3C_2(R) + 5C_2(G)) d_{11} \right) \epsilon^2
$$

$$
+ \frac{27}{64} \left( d_{31} + \frac{b_{31c_1}}{d_{11}} + \left( \frac{11}{3} C_2(R) + \frac{79}{18} C_2(G) \right) d_{11} \right) \epsilon^3 + O\left( \frac{\epsilon^4}{N_f^2} \right)
$$

(2.16)

We note that the location of $a_c$ is present in this expression because of a non-zero $d_{11}$ term at one loop in \[2.4\].

Two technical issues remain to be discussed which are fundamental to the calculation of $\lambda_+(a_c)$. The first relates to a property of the fixed point. At $a_c$ it was demonstrated by Hasenfratz and Hasenfratz, \[2\], that at $O(1/N_f)$ QCD is equivalent to a simpler lower dimensional model which is the non-abelian Thirring model, (NATM). This means that they lie in the same universality class and therefore critical exponents are the same in each model. In other words it does not matter which theory we study at $a_c$, the same $d$-dimensional expressions are obtained at leading order. Similar and more widely studied $d$-dimensional equivalences exist like, for example, that between the $O(N)$ $\sigma$ model and $O(N)$ $\phi^4$ theory. (See, for example, \[22\].) For completeness the respective lagrangians are

$$
L^\text{QCD} = i\bar{\psi}i(P\psi)i - \frac{(G_{\mu\nu})^2}{4\epsilon^2}
$$

$$
L^\text{NATM} = i\bar{\psi}i(P\phi)i - \frac{(A_\mu)^2}{2\lambda^2}
$$

(2.17)

(2.18)

where $\epsilon$ and $\lambda$ are the coupling constants which are dimensionless in 4 and 2 dimensions respectively. However, since the NATM has fewer interactions due to the absence of cubic and quartic gluon self-interactions it is more practical to use it. The effects of such interactions are recovered in diagrams with quark loops with respectively three and four gluon legs, \[21\]. Indeed this equivalence has already proved useful in establishing several results in large $N_f$ QCD, \[4\].

The second point concerns the regularization of the theory at the fixed point. Unlike perturbation theory we use a fixed non-regularizing dimension $d$ in the critical point approach and therefore the only sensible type of regulator which remains to us is an analytic one. This is introduced by shifting the gluon dimension $\beta$ to $\beta - \Delta$ where $\Delta$ is a dimensionless infinitesimal parameter, \[13\]. It plays a similar role to the $\epsilon$-regularizing parameter of dimensional regularization. Therefore when we insert $O_g$ in a gluon 2-point function and replace the lines by the critical propagators with a non-zero $\Delta$ the value of the diagram is no longer infinite but proportional to $1/\Delta$. As has been discussed in \[13, 14\] the residue of this simple pole is an $n$ and $d$-dependent function and it is this which contributes to the full value of $\lambda_+(a_c)$.
3 QED sector.

We now turn to the discussion of the technical issues involved in computing the critical exponent \( \lambda_+(a_c) \) beginning with some general considerations. As in a perturbative calculation we insert the composite operator \( O_g \) into a gluonic 2-point Green’s function and compute its divergent part with respect to the critical regularization. This is illustrated in fig. 1 and the total value of such contributions correspond to the renormalization of the bare operator whose critical exponent is \( \eta_{O_g} \). To obtain the full value one must also include the wave function renormalization of the fields present in the composite operator as in perturbation theory. Since these are gluons then the dimension is

\[
\lambda_+(a_c) = -\eta - \chi + \eta_{O_g} \quad (3.1)
\]

It is worth noting that because the operators (2.1) are physical the combination (3.1) is independent of the gauge choice even though each of the individual quantities of (3.1) depend on \( b \). Therefore we have chosen to perform the calculation in the Feynman gauge. From (2.1) and (2.1) the Feynman rule for the (bare) insertion which is quadratic in the gluon field, is, for an external momentum \( p \),

\[
\frac{1}{2}[1 + (-1)^n] \left[ \eta_{\mu\nu}(\Delta p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu)\Delta p + p^2 \Delta_\mu \Delta_\nu \right] (\Delta p)^{n-2} \quad (3.2)
\]

where \( \mu \) and \( \nu \) are the free indices which remain after contracting with the null vector \( \Delta_\mu \) of the light cone projection, \( \Box \). (Although we have used the same symbol for the analytic regularization there ought not to be any confusion in what follows as one is clearly a scalar object whilst the other is a Lorentz vector with the property \( \Delta^2 = 0 \).) The Feynman rule for the insertion with three or more gluon legs have been given elsewhere, \( \Box \). At the order in \( 1/N_f \) which we are considering there are only a few contributing graphs which are illustrated in fig. 2. There are no graphs with self energy insertions on internal lines since we are computing with propagators which have non-zero anomalous dimensions. Each of these graphs in fig. 2 will in principle have the same general Lorentz structure in its pole part which is

\[
\Gamma_{\mu\nu} = \left[ A\eta_{\mu\nu}(\Delta p)^2 + B(\Delta_\mu p_\nu + \Delta_\nu p_\mu)\Delta p + C\frac{p_\mu p_\nu}{p^2}(\Delta p)^2 + Dp^2 \Delta_\mu \Delta_\nu \right] (\Delta p)^{n-2} \quad (3.3)
\]

where \( A, B, C \) and \( D \) will depend on \( \mu, n \) and the regulator though it is only the residue of the simple pole in the latter which we are interested in. In principle we ought to compute each of these expressions for each of the integrals, which would require taking various projections on the explicit form of each graph. However we can minimize the amount of computation that needs to be performed by appealing to the renormalizability of QCD. As the theory is renormalizable then the sum of the contribution of the form (3.3) from each of the graphs of fig. 2 must be proportional to (3.2). The resulting coefficient of proportionality is then simply related to \( \eta_{O_g} \). Therefore it is much simpler to compute only one of the four quantities in (3.3). (However, we note that we have checked this assumption by explicit computation for the QED sector of fig. 2 which involves only the first two 2-loop graphs.) We have chosen to deduce \( A \) and it is a simple exercise to invert (3.3) to produce

\[
A = \frac{1}{2(\mu - 1)} \left[ \eta_{\mu\nu}(\Delta p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu)\Delta p + p^2 \Delta_\mu \Delta_\nu \right] (\Delta p)^{n-2} \Gamma_{\mu\nu} \quad (3.4)
\]

This projection on each of the four graphs can be simplified further by noting that renormalizability also requires that \( C \) is finite with respect to the regularization. In the same inversion which produces (3.4),

\[
C = \frac{\Delta_\mu \Delta_\nu}{2(\mu - 1)} \Gamma_{\mu\nu} \quad (3.5)
\]
Therefore the projection of each of the integrals we take to produce the divergent part with respect to the regularization is

\[
\frac{1}{2(\mu - 1)} \left[ \eta_{\mu\nu}(\Delta p)^2 - (\Delta_\mu p_\nu + \Delta_\nu p_\mu)\Delta p \right] (\Delta p)^{n-2}\Gamma_{\mu\nu}
\]  

(3.6)

This step substantially reduces the number of constituent integrals which would have to be considered in, for example, the final graph of fig. 2. A second simplification of the organization of the calculation is to examine the contributions from the sectors defined by the colour group Casimirs $C_2(R)$ and $C_2(G)$ separately. As the latter arises due to non-zero colour group structure constants we will refer to the former as the QED sector and concentrate on determining its contribution first. One reason for this is that there are various technical issues in the full calculation which we need to discuss and these are best illustrated in this case. In the $C_2(G)$ sector other techniques will be introduced which will assume these results.

For the remainder of the section we will discuss the first two 2-loop graphs of fig. 2 as only they involve $C_2(R)$. The first of these graphs is elementary to determine, primarily because it is equivalent to chain integrals in the language of [12]. However the presence now of an operator with a non-zero moment means that more general chain results are required. These have been given in appendix A as they have been widely used by other authors in this area, [23]. Therefore we merely record the total contribution from this graph to $\eta_{C_s}$ is

\[
\begin{align*}
&\frac{[\mu^2 n^2 + \mu^2 n - 2 \mu^2 - 9 \mu n^2 - 9 \mu n + 2 \mu - n^4 - 2 n^3 + 9 n^2 + 10 n] (\mu - 1) \Gamma(\mu) \Gamma(n + 2 - \mu) C_2(R) \eta^0}{[n(n + 1)(\mu - 2)(\mu - n - 1) \Gamma(3 - \mu) \Gamma(\mu + n + 1) T(R)]} \\
&\quad / n(n + 1)(\mu - 2)(\mu - n - 1) \Gamma(3 - \mu) \Gamma(\mu + n + 1) T(R)]
\end{align*}
\]

(3.7)

The computation of the second integral of fig. 2 is more tedious to determine. From (3.6) we treat the contributions from the $\eta_{\mu\nu}$ and $(\Delta_\mu p^\nu + \Delta_\nu p^\mu)$ projections separately. To proceed one contracts with either projection tensor and then takes the trace over the quark loop which leaves a set of two loop scalar integrals. The majority of these integrals can be calculated in the same way as those of the first graph of fig. 1 since they are elementary chains. However a set of two loop graphs remain which have the form illustrated in fig. 3 which clearly require another technique. We use a coordinate space notation for integrals, similar to [12], but the explicit form of that graph is given by

\[
G_{pq\alpha}(\alpha) = \int_{yz} \frac{(\Delta y)^p(\Delta z)^q(\Delta(y - z))^n}{y^2(x - y)^2 z^2(x - z)^2((y - z)^2)^\alpha}
\]

(3.8)

where $p, q$ and $n$ are integers and $\alpha$ is an arbitrary exponent. Here and elsewhere we will use the notation that a line in a graph which has a bracketed integer, $(p)$, beside it corresponds to a factor in the numerator of $p$ contractions of its vector with $\Delta_\mu$. Unbracketed symbols beside a line correspond to its propagator exponent. From the nature of the original integral the central $(y - z)$ line corresponds to the location of the operator insertion with moment $n$. It turns out that only several values for $p$ and $q$ are needed for $n$ arbitrary. These are $(p,q) = (0,0), (1,0), (1,1)$ and $(2,0)$. However, some of these are related by symmetry or by rewriting $\Delta y$ as $\Delta y = \Delta(y - z) + \Delta z$. Therefore for $n$ even

\[
G_{10n}(\alpha) = \frac{1}{2} G_{00n}(\alpha)
\]

(3.9)

and for arbitrary $n$

\[
G_{11n} = G_{20n}(\alpha) - \frac{1}{2} G_{00(n+2)}(\alpha)
\]

(3.10)

It therefore remains to compute $G_{00n}(\alpha)$ and $G_{20n}(\alpha)$ which is achieved by establishing a recurrence relation for the more general integrals where the $\Delta_\mu$ contractions of all numerator vectors
has not been performed. Then we will solve this for the cases we are interested in. It is best to illustrate this for the simpler \( G_{00n}(\alpha) \) case first. Then using Lorentz symmetry that integral will decompose into the form

\[
\left[ A(\alpha, n)x_{\mu_1} \ldots x_{\mu_n} + B(\alpha, n)[\eta_{\mu_1 \mu_2} x_{\mu_3} \ldots x_{\mu_n} + (\frac{1}{2}n(n - 1) - 1) \text{ terms}] + O(x^{n-4}) \right] \tag{3.11}
\]

In this and similar decompositions we will omit the overall power of the factor of \((x^2)\) as it plays no significant role in the derivation and can readily be restored by simple dimensional analysis. Here \( A(\alpha, n) \) and \( B(\alpha, n) \) are invariant amplitudes and our aim is to determine a recurrence relation involving \( A(\alpha, n) \) only. This is because after contracting (3.11) with \( \Delta^{\mu_1} \ldots \Delta^{\mu_n} \) only \( A(\alpha, n) \) remains which is the value of \( G_{00n}(\alpha) \). The amplitude \( B(\alpha, n) \) is necessary to achieve this as can be seen in the explicit derivation, though the terms in powers of \( x \) will not be necessary and these have been indicated by the order symbol. In the following construction of a relation for \( A(\alpha, n) \) we consider only the leading contribution in powers of \( x \) when we take two contractions. These are \( \eta^{\mu_n - 1\mu_1} \) and \( x^{\mu_n} \) and give respectively

\[
A(\alpha - 1, n - 2) = A(\alpha, n) + 2(\mu + n - 2)B(\alpha, n)
\]

\[
2G_{00(0-1)(1)}^{0111} = A(\alpha, n) + (n - 1)B(\alpha, n)
\tag{3.12}
\]

where \( G_{00n}^{0111}(\alpha) \) corresponds to the integral of fig. 3 but with the exponent of the top left line replaced by zero. In other words it is an elementary chain integral of the form which has arisen already. From these two expressions we deduce, for arbitrary \( \alpha \),

\[
G_{00(n-2)}(\alpha - 1) = \frac{4(\mu + n - 2)}{(n - 1)} G_{00(n-1)}^{0111}(\alpha) - \frac{(2\mu - 3 + n)}{(n - 1)} G_{00n}(\alpha)
\tag{3.13}
\]

from the decomposition the various integrals we require are \( G_{00n}(2\mu - p + \Delta) \) for \( p = 2, 3 \) and 4. Indeed once we have obtained one of these values the others will follow from elementary application of (3.13). It turns out that \( G_{00n}(2\mu - 3 + \Delta) \) is \( \Delta \)-finite. The justification for this will prove to be fairly general and will be used later in solving recurrence relations for other integrals. Essentially it follows from the fact that the dimension of the integral is greater than unity. We first recall the usual calculational technique used in \[12\] for computing integrals without a moment factor in their numerators but which are divergent and similar in form to fig. 3. The first step is to transform it to the momentum representation defined in \[2\]. In this one has to include factors arising from the Fourier transform. In particular one factor is related to the dimension of the integral. If this has the form \((p + \Delta)\) for a positive integer \( p \), then the corresponding factor is divergent with respect to \( \Delta \). It therefore remains only to compute the momentum representation integral for \( \Delta = 0 \). This is possible as the final form is related, for example by integration by parts, to known integrals like those of Chetyrkin and Tkachov, \[2\] denoted by \( ChT(\alpha, \beta) \), \[2\]. It follows therefore for this case that if the original integral has a dimension greater than unity then it is finite. For the integrals with a non-zero moment the argument is similar. Indeed we checked this reasoning by explicitly computing \( G_{00n}(2\mu - 3 + \Delta) \) for various values of low \( n \). For future reference we make some comment on the extension of this argument to other integrals not necessarily of the exponent structure of fig. 1. It may not always be the case that the corresponding \( n = 0 \) integral is \( \Delta \)-finite even though its dimension satisfies the above criterion. However, its moment partner may be. This is because the presence of a numerator factor like \( \Delta(y - z) \) alters the nature of the singularity which we are interested in. These remarks can be simply illustrated by considering the elementary function \( a_{n}(\alpha) \) which is common in all our calculations. For example, the function \( a_{0}(\mu - \Delta) \) is divergent in the limit \( \Delta \to 0 \), whereas for a non-zero \( n \) \( a_{n}(\mu - \Delta) \) is finite in the same limit. The presence of the non-zero moment appears to be regularizing the infinity of the \( n = 0 \) function and it is this which occurs
in the two and three loop integrals. Indeed throughout our application of recurrence relations we always have used this dimensional argument for \( n \neq 0 \) integrals to determine the terminating case. For all the cases we are interested in we checked the finiteness explicitly for at least one value of non-zero \( n \). The argument appears robust.

Therefore given this property we can deduce \( G_{00n}(2\mu - 4 + \Delta) \), for example, from (3.13) which now is reduced to a computation of chain integrals. Again we have checked this result by carrying out the explicit calculation for low \( n \) and found agreement. To complete the calculation of the second graph a recurrence relation for \( G_{20n}(\alpha) \) is deduced in the same way as for (3.13). The major difference here is that the decomposition analogous to (3.11) now has four terms to \( O(x^{(n-2)}) \). However there are various contractions which can be made from which one can deduce, for example, \( G_{20n}(\alpha) \). Finally the two projections of (3.6) can be performed and the contributing value of the second integral to \( \eta_{\partial_o} \) is

\[
- \left[ 8\mu^4 n^2 + 8\mu^4 n + 8\mu^4 + \mu^3 n^4 + 2\mu^3 n^3 - 11\mu^3 n^2 \\
- 12\mu^3 n - 24\mu^3 - 3\mu^3 n^4 - 6\mu^2 n^3 - 19\mu^2 n^2 - 16\mu^2 n \\
+ 24\mu^2 - \mu n^6 - 3\mu n^5 - 5\mu n^4 - 5\mu n^3 + 34\mu n^2 + 36\mu n \\
- 8\mu + 2n^6 + 6n^5 + 10n^4 + 10n^3 - 12n^2 - 16n \right] \\
\times \mu \Gamma(\mu - 1) \Gamma(n + 1 - \mu) [C_2(R) - C_2(G)/2] \eta_1^0 \\
/[n(n-1)(n+2)\Gamma(3 - \mu) \Gamma(\mu + n + 1) T(R)]
\]

(3.14)

We close this section with the remark that we have made use of the computer algebra and symbolic manipulation packages REDUCE, [23], and FORM, [24], in this and subsequent graph evaluations as there is a substantial amount of tedious algebra to handle.

4 \( C_2(G) \) sector.

We now detail the calculation of the remaining two graphs of fig. 2 each of which involves only the colour group factor \( C_2(G) \). The strategy is the same as in the \( C_2(R) \) sector. In other words we take the trace over the fermion loop and evaluate the pole part of the resulting bosonic integrals. The major effort in the calculation resides in the determination of the integrals which are not reducible to chains. First, we consider the two loop graph. The three gluon insertion Feynman rules can be found in, for example, [2]. It has a part which involves a sum over the operator moment and it is therefore not clear whether it will be possible to obtain a closed analytic expression for the contribution from this graph. Moreover, the distribution of the exponents around the two loop topology is different from that of fig. 3 and so a new recurrence relation will be needed. The integral is defined in fig. 4 where \( \alpha \) and \( \beta \) are arbitrary exponents and \( n \) and \( p \) are integers. In addition to the simple chain diagrams given by the graph of fig. 4 where each of the lines with a unit exponent is successively replaced by zero, we have to deal with those graphs where the exponents of each of the lines are also replaced by an exponent of \((-1)\). These integrals are easily related to a sum of chain diagrams by integration rules like that illustrated in fig. 5 which is simple to establish. As these resulting chain integrals are elementary to deal with we focus on the construction of a relation for \( H_{pn}(\alpha, \beta) \) and consider \( H_{0n}(\alpha, \beta) \) first. A clue on how to proceed comes from the case \( H_{0n}(\alpha, \beta) \) which has been determined in closed form in [24]. It is related to \( \text{ChT}(\mu - \alpha, \mu - \beta) \) of [12] after applying the momentum representation transformation of [12]. The presence of a non-zero moment \( n \) however, disrupts the construction of a closed form when the same algorithm is applied. Instead we apply a series of transformations to \( H_{0n}(\alpha, \beta) \). Noting that under a conformal transformation \( z^\mu \to z^\mu/z^2 \) and therefore \( \Delta z \to \Delta z/z^2 \), we apply a left conformal transformation on fig. 4 with the origin at the left external
point. The lines joining the upper vertex now have no moment dependence and so we can apply a transformation of \([12]\). Finally a momentum representation transformation leaves us with an integral proportional to the graph given in fig. 6. Again as the lower vertex has no lines joining it which have a moment dependence we can integrate by parts on it using, for example, the rule of \([23]\) and relate it to four integrals. Three of these are simple chains whilst the fourth is related to \(H_{00}(\alpha, \beta - 1)\) when one undoes the original transformations. The result therefore is, for arbitrary \(\alpha\) and \(\beta\),

\[
H_{00}(\alpha, \beta) = \frac{(2\mu - \alpha - \beta + n - 1)(\alpha + \beta - \mu)}{(\beta - 1)(\mu - \beta - 1)} H_{00}(\alpha, \beta - 1) - \frac{a(1)a(\beta + 2 - \mu)a_n(3\mu - \alpha - \beta - 2 + n)}{(\beta - 1)a(3 - \mu)a_n(2\mu - \alpha - 2 + n)} \times \left[ \frac{(\beta - 1)}{(\mu - \beta - 1)}\nu(2, \beta, 2\mu - \beta - 2) + \nu_{\text{non}}(\alpha, 2, 2\mu - \alpha - 2 + n) \right]
\]

(4.1)

For \(4.1\) to be practical we once again need to have a terminating value which will be an integral which is finite with respect to the regularization. Unlike in section 3 we require several distinct series of expressions such as \(H_{00}(\mu - 1 + \Delta, \mu - 2 + \Delta)\), \(H_{00}(2\mu - 3 + \Delta, \mu - 1 + \Delta)\) and \(H_{00}(2\mu - 3 + \Delta, \mu - 1 + \Delta)\). For the former the natural endpoint is \(H_{00}(\mu - 1 + \Delta, \mu - 1 + \Delta) = O(1)\). This has been verified for various values of small \(n\) and simple substitution reveals

\[
H_{00}(\mu - 1 + \Delta, \mu - 2 + \Delta) = -\frac{(2\mu - 3)(\mu - 1)\nu(1, 1, 2\mu - 2)}{2(n + 1)(\mu - 1 + n)} \frac{\Gamma(\mu - 1)\Gamma(n + 1)}{\Gamma(\mu + n)\Delta}
\]

(4.2)

which we have also checked for low \(n\). Of course we have only recorded the value of the \(O(1/\Delta)\) part of the integral since the contribution to \(\eta_{ij}\) only arises from the residue of the simple pole. One cannot always directly use \(4.1\) for each integral due to the fact that if, for instance, \(\beta = \mu - 1 + \Delta\) then a denominator factor of \(4.1\) is \(\Delta\). This would require the associated two loop integral to be computed to the next order in \(\Delta\) which would be tedious. Instead one can employ the identity

\[
H_{00}(\mu - 2 + \Delta, \mu - 1 + \Delta) = \sum_{r=0}^{n} \left( \frac{n}{r} \right)^{(-1)^r} H_{0r}(\mu - 1 + \Delta, \mu - 2 + \Delta)
\]

(4.3)

and insert the previous result. Elementary algebraic identities allow the finite sum to be determined, leaving

\[
H_{00}(\mu - 2 + \Delta, \mu - 1 + \Delta) = -\frac{(2\mu - 3)(\mu - 1)\nu(1, 1, 2\mu - 2)}{2(\mu - 2)\Gamma(\mu)\Delta} \left[ \frac{1}{(n + 1)} - \frac{\Gamma(\mu - 1)\Gamma(n + 1)}{\Gamma(\mu + n)} \right]
\]

(4.4)

For the other structures we note that by our earlier dimensional argument \(H_{00}(2\mu - 3 + \Delta, 1 + \Delta)\) and \(H_{00}(2\mu - 2 + \Delta, \Delta)\) are \(\Delta\)-finite and lead to the following results

\[
H_{00}(2\mu - 3 + \Delta, \Delta) = \frac{\nu_{00}(1, 2, 2\mu - 3)}{2(n + 1)\Gamma(\mu + n)\Delta}
\]

\[
H_{00}(2\mu - 4 + \Delta, 1 + \Delta) = \frac{(2\mu - 3)\nu(1, 1, 2\mu - 2)}{2(\mu - 2)(n + 1)\Gamma(\mu + n)\Delta} \left[ \frac{\Gamma(4 - \mu + n)}{\Gamma(3 - \mu)} - \Gamma(n + 2) \right]
\]

\[
H_{00}(2\mu - 2 + \Delta, -1 + \Delta) = -\frac{(\mu - 1)(2\mu - 3)\nu_{00}(1, 2, 2\mu - 2)}{2(n + 1)\Gamma(\mu + n)\Delta}
\]

(4.5)

Again we have verified these (and others) all explicitly for at least one non-zero value of \(n\). It turns out that the remaining integrals which we need of the form \(H_{00}(2\mu - p + \Delta, \mu - q + \Delta)\) for positive integers \(p\) and \(q\) are \(\Delta\)-finite.
To complete the construction of results for the topology of fig. 4 we note that applying the generalized moment integration by parts rule of [23] to \( H_{(p-1)(n+1)}(\alpha, \beta) \) gives the reduction formula, for arbitrary \( \alpha \) and \( \beta \),

\[
H_{pn}(\alpha, \beta) = \frac{(2\mu - \alpha - \beta + n - 1)}{(n + 1)} H_{(p-1)(n+1)}(\alpha, \beta)
- \frac{\alpha}{(n + 1)} \left[ H^{101}_{(p-1)(n+1)}(\alpha + 1, \beta) - H^{011}_{(p-1)(n+1)}(\alpha + 1, \beta) \right]
- \frac{\beta}{(n + 1)} \left[ H^{101}_{(p-1)(n+1)}(\alpha, \beta + 1) - H^{110}_{(p-1)(n+1)}(\alpha, \beta + 1) \right]
\]

(4.6)

where the superscripts on an \( H \)-function here refer to the exponents respectively of the top left, central and top right lines relative to the diagram of fig. 4. The result (4.6) allows us to reduce any integral \( H_{pn}(\alpha, \beta) \) with \( p > 0 \) down to ones of the form \( H_{0n}(\alpha, \beta) \) whose values are known from (4.11). Therefore one can now assemble all the pieces to both projections of the original two loop diagram.

For that part of the operator insertion which does not involve a finite sum over moments one obtains the contribution

\[
\left[ (4\mu^3 n^2 - 4\mu^3 - 10\mu^2 n^2 + 2\mu^2 n + 8\mu^2 + 8\mu n^2 - 4\mu n - 3\mu - 2n^2 + 2n) \Gamma(\mu + n) 
- (2\mu^2 n - 2\mu^2 - 3\mu n + 4\mu + n - 1) \Gamma(\mu + 1) \Gamma(n + 1) \right] \eta_1^0 C_2(G)
\]

\[
/ [2(2\mu - 1)(\mu - 1)(\mu - 2)(n - 1) \Gamma(\mu + n) n]
\]

(4.7)

However, the total of all pieces to that part which does involve the sum yields, ignoring colour factors and powers of \( z \) for the moment,

\[
- \frac{4(\mu - 1) \nu(1, 1, 2\mu - 2)}{(2\mu - 1) \Gamma(\mu)} \sum_{r=1}^{n-2} \frac{1}{r(r + 1) \Gamma(\mu + r + 1)} 
\times \left[ [4r(\mu - 1)^2 + 1] \Gamma(\mu + r + 1) 
- [r(2\mu - 1)(\mu - 1) + \mu] \Gamma(\mu) \Gamma(r + 2) \right]
\]

(4.8)

From the definition for \( S_l(n) \) and the result

\[
\sum_{r=1}^{n} \frac{\Gamma(r + a)}{\Gamma(r + b)} = \frac{1}{(a - b - 1)} \left[ \frac{\Gamma(a + 1)}{\Gamma(b)} - \frac{\Gamma(n + a + 1)}{\Gamma(n + b)} \right]
\]

(4.9)

then we deduce the closed form for (4.8) is

\[
- \frac{4(\mu - 1) \nu(1, 1, 2\mu - 2)}{(2\mu - 1) \Gamma(\mu)} \left[ (2\mu - 3)(2\mu - 1) \left( S_1(n) - \frac{(n + 1)}{n} \right) 
+ S_1(n) - \frac{(2n - 1)}{n(n - 1)} - 2 + [(2\mu - 1)(n - 1) + 1] \frac{\Gamma(n - 1) \Gamma(\mu)}{\Gamma(\mu + n - 1)} \right]
\]

(4.10)

Thus the total contribution from this two loop graph is

\[
\left[ \frac{2\mu(\mu - 1) S_1(n)}{(2\mu - 1)(\mu - 2)} + \frac{(2\mu n - n + 1) \Gamma(\mu + 1) \Gamma(n)}{2(2\mu - 1)(\mu - 1)(\mu - 2) \Gamma(\mu + n)} - \frac{(2\mu^2 n - 3\mu n + \mu + 2n)}{2(2\mu - 1)(\mu - 1)(\mu - 2)n} \right] \eta_1^0 C_2(G) / T(R)
\]

(4.11)

To complete our computation we now turn to the final graph of fig. 2, which is the hardest of the four to evaluate. As we are in effect performing two separate projections on the integral itself, we note that multiplying the graph by \( \Delta^\mu p^\nu \), for instance, allows us to immediately complete...
the integration for the quark loop that the $p^\nu$ contracts into. The resulting two loop integral is effectively completed since its constituent scalar integrals have already been evaluated for the third graph of fig. 2. A similar property arises for the $\eta_{\mu\nu}$ contraction of the second term of the operator insertion (3.3). As for the $\Delta^\mu p^\nu$ contraction there is also no new feature for this case. Instead we consider the remaining two terms of (3.3) in fig. 2 and their $\eta_{\mu\nu}$ projection. In taking the relevant traces one obtains scalar integrals which are now either simple chains or two loop integrals which have already been considered. In addition we obtain a new three loop topology whose most general form is given in fig. 7 and has a similar integral definition to (3.8). Here $p, q, n$ and $\gamma$ are integers with $\alpha$ and $\beta$ arbitrary exponents. We emphasise that here the line with exponent $(-\gamma)$ corresponds to a numerator factor of $[(y-z)^2]^\gamma$. Similar to the other $C_2(G)$ two loop chain type graphs there are a variety of three loop graphs of the form of fig. 7 but where one of the lines with unit exponent is replaced by $(-1)$ for $\gamma = 1, 2$ or 3. These can readily be reduced to at most two loop graphs already computed using integration rules similar to that of fig. 5.

Therefore since the structure of fig. 7 is new to the calculation we give some details of its treatment, though the algorithm is no different from those already established for the two loop cases. We consider the construction of a relation to deduce $C_{\alpha\beta\gamma}(\alpha, \beta, n)$ first and note that its decomposition into Lorentz invariants where the $\Delta^\mu$ vectors have been omitted is the same as in (3.11). However, due to the form of the present integral the contractions $\eta_{\mu\nu}x^{\mu\nu}$ and $x^{\mu\nu}$ yield respectively

$$A(\alpha - 1, \beta, n - 2) = A(\alpha, \beta, n) + 2(\mu + n - 2)B(\alpha, \beta, n)$$

(4.12)

$$A(\alpha, \beta, n) + (n-1)B(\alpha, \beta, n) = \frac{1}{2}[A(\alpha, \beta, n-1) + A(\alpha-1, \beta, n-1) - A(\alpha, \beta-1, n-1)]$$

(4.13)

Eliminating $B(\alpha, \beta, n)$ gives

$$A(\alpha - 1, \beta, n - 2) = -\frac{2\mu + n - 3}{(n-1)}A(\alpha, \beta, n) + \frac{\mu + n - 2}{(n-1)}[A(\alpha, \beta, n-1) + A(\alpha-1, \beta, n-1) - A(\alpha, \beta-1, n-1)]$$

(4.14)

This is not quite in a form which is practical since we require a relation between $A(\alpha, \beta, n)$ and say $A(\alpha - 1, \beta, n)$ where the former set can be set to zero upon reaching a terminating value. The presence of the final two terms in the square brackets leaves us an equation which will relate several integrals without allowing us to evaluate them individually. To eliminate these problem terms we consider the general integral of fig. 7 and apply a conformal left transformation to it in the language of [12]. Then integrating by parts on the central 4-vertex with the line with exponent $\beta$ as reference and applying the inverse conformal left transformation to the resulting five integrals, we obtain the general result, for arbitrary $\alpha$ and $\beta$,

$$C_{pq\gamma}(\alpha, \beta, n) = \frac{(2\mu - \alpha - \beta + 2 + n)}{(\alpha - \beta)}[C_{pq\gamma}(\alpha, \beta - 1, n) - C_{pq\gamma}(\alpha - 1, \beta, n)]$$

(4.15)

$$+ \frac{1}{(\alpha - \beta)}[C_{pq\gamma}^{021}(\alpha, \beta - 1, n) - C_{pq\gamma}^{120}(\alpha - 1, \beta, n)]$$

$$+ \frac{1}{(\alpha - \beta)}[C_{pq\gamma}^{021}(\alpha, \beta - 1, n) - C_{pq\gamma}^{120}(\alpha - 1, \beta, n)]$$

$$+ \frac{n}{(\alpha - \beta)}C_{pq\gamma}(\alpha - 1, \beta, n - 1)$$

where a similar notation to [14] is used but here the superscripts refer only to the exponents of the lines joining the top vertex. Although we are dealing with the leading term $A(\alpha, \beta, n)$
in (4.14) we can use (4.13) to eliminate the unnecessary terms since after the contraction with \( \Delta^{\mu_1} \ldots \Delta^{\mu_n} \) only \( A(\alpha, \beta, n) \) will survive and be equal in value to \( C_{00\gamma}(\alpha, \beta, n) \). Consequently after some elementary algebra which includes shifting exponents, we deduce, for arbitrary \( \alpha \) and \( \beta \),

\[
C_{00\gamma}(\alpha, \beta, n) = - \frac{(2\mu + n - 1)(2\mu - \alpha - \beta + n - 2)}{(n + 1)(\mu - \alpha - \beta - 2)} C_{00\gamma}(\alpha + 1, \beta, n + 2) \\
+ \frac{(\mu + n)(2\mu - 2\alpha + n - 3)}{(n + 1)(\mu - \alpha - \beta - 2)} C_{00\gamma}(\alpha + 1, \beta, n + 1) \\
+ \frac{2(\mu + n)}{(n + 1)(\mu - \alpha - \beta - 2)} [C_{00\gamma}^{02}(\alpha + 1, \beta - 1, n + 1) - C_{00\gamma}^{120}(\alpha, \beta, n + 1)]
\]

(4.16)

This differs in a minor respect from the two loop relation of (4.1) in that the first two terms on the right side are not at the same moment level. However, the dimension argument we use to find a terminating value is valid for all \( n \geq 1 \) and these terms will be \( \Delta \)-finite.

Relations are also required for \( C_{10\gamma}(\alpha, \beta, n) \), \( C_{11\gamma}(\alpha, \beta, n) \) and \( C_{20\gamma}(\alpha, \beta, n) \) and these are determined by the same method as for \( C_{00\gamma}(\alpha, \beta, n) \) though there are of course more invariant amplitudes which can appear. However, one constructs the result for \( C_{10\gamma}(\alpha, \beta, n) \) first since the decomposition of the other two relies on this lower level expression and \( C_{00\gamma}(\alpha, \beta, n) \). For completeness we quote the recurrence relations for each level in appendix A where again the integrals where a line is zero are easily computed. As an aid to an interested reader we record the explicit results of the recurrence relations for a set of representative integrals at each level, noting that we have used the obvious values such as \( C_{000}(2\mu - 3 + \Delta, \mu - 1 + \Delta, n) = O(1) \) for all \( n \geq 1 \) to terminate the relations. We found

\[
C_{003}(2\mu - 2 + \Delta, \mu - 1 + \Delta, n) = - \left[ \frac{[2(2\mu - 3)(\mu^2 - 1)\Gamma(\mu + n + 1)}{\Gamma(\mu + 2)\Gamma(n + 3]} a(2\mu - 2) \\
+ 2(\mu - 3)(\mu - 1)a_0(2\mu - 2)\Gamma(\mu + 2) \\
\times a(2\mu - 2)a_4(1)\mu/[4(2\mu - 1)^2(\mu - 2)] \\
/[(n + 2)(n + 1)\Gamma(\mu + n)\Gamma(\mu)\Delta] \right]
\]

(4.17)

\[
C_{100}(2\mu - 4 + \Delta, \mu - 2 + \Delta, n) = \left[ \frac{(\mu^2 + 2\mu n + 2\mu n^2 + 4\mu - 3n^2 - 12n - 8)(\mu - n - 2)}{(\mu - n - 3)a_0(2\mu - 2)\Gamma(\mu)} \\
+ (\mu - 1)^2(\mu - 2)(n + 2)a(2\mu - 2)\Gamma(\mu + n + 1) \\
\times (2\mu - 3)a(2\mu - 2)\Gamma^3(\mu)/[2(\mu - 1)^4(\mu - 2)] \\
/[(n + 3)(n + 2)(n + 1)\Gamma(\mu + n + 1)\Delta] \right]
\]

(4.18)

\[
C_{112}(2\mu - 3 + \Delta, \mu - 1 + \Delta, n) = \left[ \frac{[(2\mu^3 n^2 + 10\mu^3 n + 10\mu^3 - 3\mu^2 n^2 - 19\mu^2 n - 25\mu^2} \\
- \mu n^2 - \mu n + 7\mu + n^2 + 7n + 6)\Gamma(\mu + n + 3) \\
- (\mu^2 n + 2\mu n^2 + 4\mu n + \mu - n^2 - 3n - 2)(\mu - 1) \\
\times \Gamma(\mu)\Gamma(n + 5)]a(2\mu - 2) \\
+ (\mu^4 + 2\mu^3 n^2 + 8\mu^3 n + 4\mu^3 + 4\mu^2 n^4 + 36\mu^2 n^3 \right]
\]
\[
\begin{aligned}
+ 110\mu^2n^2 + 132\mu^2n + 53\mu^2 - 4\mu n^4 - 38\mu n^3 \\
- 124\mu n^2 - 162\mu n - 70\mu + n^4 + 10n^3 + 35n^2 \\
+ 50n + 24)(2\mu - 3)(\mu - n - 2)\alpha_n(2\mu - 2)\Gamma(\mu)
\end{aligned}
\] \\
x \frac{a(2\mu - 2)\Gamma^3(\mu)/[4(2\mu - 1)(\mu - 1)^4(\mu - 2)]}{[(n + 4)(n + 3)(n + 2)(n + 1)\Gamma(\mu + n + 2)\Delta]}
\] (4.19)

and
\[
C_{201}(2\mu - 4 + \Delta, \mu - 1 + \Delta, n) = \left[(\mu^3 + 4\mu^2n + 5\mu^2 + 5\mu n^2 + 13\mu n + 8\mu - 2n^2 \\
- 6n - 4)(n + 4)(n + 3)\Gamma(\mu)\Gamma(n + 2) \\
- 2(\mu n + 3\mu - 1)\Gamma(\mu + n + 3)\mu(n + 2)a(2\mu - 2) \\
- 4(\mu^3 + 2\mu^2n^2 + 10\mu^2n + 9\mu^2 + 2\mu n^4 + 20\mu n^3 \\
+ 69\mu n^2 + 95\mu n + 44\mu - n^4 - 10n^3 - 35n^2 - 50n \\
- 24)(\mu - n - 2)(\mu - n - 3)\alpha_n(2\mu - 2)\Gamma(\mu)
\right] \\
x \frac{a(2\mu - 2)\Gamma^3(\mu - 1)/[4(\mu - 1)(\mu - 2)(n + 4)]}{[(n + 3)(n + 2)(n + 1)\Gamma(\mu + n + 2)\Delta]}
\] (4.20)

With the results from these recurrence relations and the earlier two loop results we summarise this intermediate step in determining the value of the three loop graph by giving the contributions from the \(\eta^{\mu
\nu}\) projections of the first and third terms of the operator insertion, \([3,2]\). These are respectively
\[
\begin{aligned}
- \left[16\mu^6 - 16\mu^5n - 104\mu^5 + 4\mu^4n^2 + 60\mu^4n + 264\mu^4 + 16\mu^3n^2 - 80\mu^3n - 336\mu^3 - 21\mu^2n^2 \\
+ 51\mu^2n + 232\mu^2 + 11\mu n^2 - 20\mu n - 88\mu - 2n^2 + 4n + 16\Gamma(n + 2 - \mu)\Gamma(\mu - 1) \\
\times \eta^0C_2(G)/[8(2\mu - 1)(\mu - 1)^2(\mu - 2)(n + 2)(n + 1)(2 - \mu)\Gamma(\mu + n)T(R)]
\right] \\
- (4\mu - 1)\Gamma(\mu - 1)\Gamma(n + 1)\eta^0C_2(G)/[4(2\mu - 1)(\mu - 2)\Gamma(\mu + n)T(R)]
\end{aligned}
\] (4.21)

and
\[
\begin{aligned}
+ \left[16\mu^7n^2 + 48\mu^7n + 32\mu^7 + 32\mu^6n^3 - 56\mu^6n^2 - 360\mu^6n - 256\mu^6 + 16\mu^5n^4 - 160\mu^5n^3 \\
- 72\mu^5n^2 + 104\mu^5 + 852\mu^5 - 88\mu^4n^3 + 312\mu^4n^3 + 614\mu^4n^2 - 1518\mu^4n - 1530\mu^4 \\
+ 184\mu^3n^4 - 288\mu^3n^3 - 1145\mu^3n^2 + 1141\mu^3n + 1600\mu^3 - 182\mu^2n^4 + 108\mu^2n^3 \\
+ 973\mu^2n^2 - 401\mu^2n - 970\mu^2 + 83\mu n^4 + 4\mu n^3 - 385\mu n^2 + 34\mu n + 312\mu - 14n^4 \\
- 8n^3 + 54n^2 + 8n - 40\Gamma(\mu + n + 1)\Gamma(\mu + 1)\eta^0C_2(G) \\
/\left[8(2\mu - 1)(\mu - 1)^2(\mu - 2)(n + 2)(n + 1)(2 - \mu)\Gamma(\mu + n)\Gamma(T(R)]
\right] \\
- (2\mu^2 + 3\mu n - 3\mu - n + 1)\Gamma(\mu - 1)\Gamma(n + 1)\eta^0C_2(G)/[2(2\mu - 1)(\mu - 2)\Gamma(\mu + n)\Gamma(T(R)]
\end{aligned}
\] (4.22)
Therefore summing all the contributions from each of the projections the full contribution from the final graph of fig. 2 is simply

\[
+ \left[ 32\mu^7 n^2 + 32\mu^7 n + 32\mu^7 - 96\mu^6 n^2 - 96\mu^6 n - 160\mu^6 + 28\mu^5 n^4 + 56\mu^5 n^3 - 20\mu^5 n^2 \\
\quad - 48\mu^5 n + 324\mu^5 - 148\mu^4 n^4 - 296\mu^4 n^3 + 394\mu^4 n^2 + 542\mu^4 n - 338\mu^4 + 4\mu^3 n^6 \\
\quad + 12\mu^3 n^5 + 303\mu^3 n^4 + 586\mu^3 n^3 - 606\mu^3 n^2 - 897\mu^3 n + 188\mu^3 - 8\mu^2 n^6 - 24\mu^2 n^5 \\
\quad - 288\mu^2 n^4 - 536\mu^2 n^3 + 421\mu^2 n^2 + 685\mu^2 n - 50\mu^2 + 5\mu n^6 + 15\mu n^5 + 125\mu n^4 \\
\quad + 225\mu n^3 - 144\mu n^2 - 254\mu n + 4\mu n^3 - 8n^5 - 19n^4 - 33n^3 + 20n^2 + 36n \\
\times \Gamma(n + 1 - \mu)\Gamma(\mu - 1)\eta_1^0 C_2(G) \\
/ [8(2\mu - 1)(\mu - 2)(n + 2)(n^2 - 1)\Gamma(2 - \mu)\Gamma(\mu + n + 1)nT(R)]
\]

\[
- (2\mu n - n + 1)\Gamma(\mu - 1)\Gamma(n)\eta_1^0 C_2(G)/[2(2\mu - 1)(\mu - 2)\Gamma(\mu + n)T(R)]
\]

\[
- \left[ 32\mu^5 n^2 + 32\mu^5 n + 32\mu^5 - 144\mu^4 n^2 - 144\mu^4 n - 160\mu^4 + 236\mu^3 n^2 + 236\mu^3 n + 316\mu^3 \\
\quad - 4\mu^2 n^3 - 172\mu^2 n^2 - 160\mu^2 n - 298\mu^2 + 8\mu n^3 + 55\mu n^2 + 31\mu n + 130\mu - 4n^3 - 6n^2 \\
\quad + 6n - 20\mu n T(R)]\eta_1^0 C_2(G)/[8(2\mu - 1)(\mu - 1)^3(\mu - 2)(n + 2)(n^2 - 1)\Gamma(2 - \mu)\Gamma(\mu + n)nT(R)]
\]

\[
(4.23)
\]

5 Results and discussion.

We are now in a position to complete the calculation by adding the contributions from the four graphs of fig. 2 as well as including the gluon anomalous dimension in the Feynman gauge. Our final result is

\[
\lambda_{+,1}(a_c) = \frac{\left[ 8\mu^3 n^2 + 8\mu^3 n + 8\mu^3 + 2\mu^2 n^4 + 4\mu^2 n^3 - 22\mu^2 n^2 - 24\mu^2 n \\
\quad - 28\mu^2 - 6\mu n^4 - 12\mu n^3 + 14\mu n^2 + 20\mu n + 32\mu + 5n^4 \\
\quad + 10n^3 + n^2 - 4n - 12\Gamma(n + 2 - \mu)\Gamma(\mu - 1)\mu C_2(R)\eta_1^0 \\
/ [(\mu - 2)^2(n + 2)(n + 1)(n - 1)\Gamma(2 - \mu)\Gamma(\mu + n)nT(R)]
\]

\[
+ \frac{2\mu(\mu - 1)S_1(n)C_2(G)\eta_1^0}{(2\mu - 1)(\mu - 2)nT(R)}
\]

\[
- \left[ 32\mu^5 n^2 + 32\mu^5 n + 32\mu^5 - 144\mu^4 n^2 - 144\mu^4 n - 160\mu^4 + 4\mu^3 n^4 \\
\quad - 8\mu^3 n^3 + 240\mu^3 n^2 + 244\mu^3 n + 316\mu^3 + 16\mu^2 n^4 + 32\mu^2 n^3 \\
\quad - 180\mu^2 n^2 - 196\mu^2 n - 306\mu^2 - 20\mu n^4 - 40\mu n^3 + 59\mu n^2 \\
\quad + 79\mu n + 146\mu + 8n^3 + 6n^2 - 14n - 28\mu C_2(G)\eta_1^0 \\
/ [8(2\mu - 1)(\mu - 1)^3(\mu - 2)(n + 2)(n + 1)(n - 1)\Gamma(2 - \mu)\Gamma(\mu + n)nT(R)]
\]

\[
+ \left[ 32\mu^5 n^2 + 32\mu^5 n + 32\mu^5 + 8\mu^4 n^4 + 16\mu^4 n^3 - 120\mu^4 n^2 - 128\mu^4 n \\
\quad - 160\mu^4 - 32\mu^3 n^4 - 64\mu^3 n^3 + 160\mu^3 n^2 + 192\mu^3 n + 316\mu^3 + 48\mu^2 n^4 \\
\quad + 90\mu^2 n^3 - 78\mu^2 n^2 - 120\mu^2 n - 306\mu^2 - 31\mu n^4 - 62\mu n^3 + 31\mu n \\
\quad + 146\mu + 7n^4 + 14n^3 + 7n^2 - 28\Gamma(n + 2 - \mu)\Gamma(\mu - 1)\mu C_2(G)\eta_1^0 \\
/ [8(2\mu - 1)(\mu - 1)^2(\mu - 2)(n + 2)(n + 1)(n - 1)\Gamma(2 - \mu)\Gamma(\mu + n)nT(R)]
\]

(5.1)

where we have set \( \lambda_{+,i}(a_c) = \sum_{i=0}^{\infty} \lambda_{+,i}(a_c)/N_i \). One curious feature of this expression is that not all the \( \Gamma \)-function structures of the individual graphs survive. Specifically the terms involving...
the factor $\Gamma(\mu - 1)\Gamma(n)/\Gamma(\mu + n)$ which appears in say (4.21) have cancelled in (5.1). We can offer no explanation of this and it is not clear if it would be a feature of the analogous dimension in other (renormalizable) field theories. Moreover, the result is remarkably more compact than one would expect given the nature of (4.21) and (4.22).

It remains to check that (5.1) is in agreement with perturbation theory. Setting $\mu = 2 - \epsilon$ and Taylor series expanding to several orders it is straightforward to produce the exact $n$-dependence for the two loop eigenvalue (2.10). Moreover, expanding to three loops we obtain the following result for the combination of 3-loop entries of the mixing matrix,

$$d_{31} + \frac{b_{31}c_{1}}{d_{11}} = \frac{64(n^2 + n + 2)^2(S_{1}(n))^2C_{2}(R)}{3(n + 2)(n + 1)^2(n - 1)n^2T(R)} - \frac{64(10n^6 + 30n^5 + 109n^4 + 168n^3 + 155n^2 + 76n + 12)S_{1}(n)C_{2}(R)}{9(n + 2)(n + 1)^3(n - 1)n^3T(R)} - \frac{4[33n^{10} + 165n^9 - 32n^8 - 1118n^7 - 5807n^6 - 12815n^5 - 16762n^4 - 13800n^3 - 7112n^2 - 2112n - 288]C_{2}(R)/[27(n + 2)(n + 1)^4(n - 1)n^4T(R)]}{8(8n^6 + 24n^5 - 19n^4 - 78n^3 - 253n^2 - 210n - 96)S_{1}(n)C_{2}(G)} - \frac{27(n + 2)(n + 1)^2(n - 1)n^2T(R)}{2(8n^6 + 348n^5 + 848n^4 + 1326n^3 + 2609n^2 + 3414n^3 + 2632n^2 + 1088n + 192)C_{2}(G)/[27(n + 2)(n + 1)^3(n - 1)n^3T(R)]}$$

This needs to be compared with the first few non-zero moments of (11). To this end we have evaluated (5.2) for even $n$, $2 \leq n \leq 24$, and recorded the values as exact fractions in table 1. The coefficients with respect to each colour Casimir have been listed separately. The entries with $n \leq 8$ are in exact agreement with the explicit 3-loop MS expressions of (11) at this order in $1/N_f$. Therefore we believe (5.1) is correct. The provision of higher moments in table 1 is to be an aid to future checks on the explicit full analytic result as a function of $n$ when it becomes available. Ordinarily in the perturbative calculation one has an additional check on the result emanating from general considerations. For the case when $n = 2$ the singlet operators correspond to the energy momentum tensor which is a conserved (non-anomalous) physical current. Therefore its anomalous dimension vanishes to all orders in perturbation theory, [22]. Such a feature ought to be present in the large $N_f$ results and this indeed was apparent in the dimension of the fermionic operator of (11). However, in the gluonic case, as can be seen in table 1, the $n = 2$ entries are non-zero. This is not inconsistent with this principle. If one examines the combination of perturbative coefficients which vanishes at $n = 2$ one finds it is, at one loop, $(a_1 + b_1)$ and $(c_1 + d_{11}N_f + d_{12})$ with similar combinations at higher order. However the large $N_f$ combinations are $(a_1 - b_{1c1}/d_{11})$ and $(d_{12} + b_{1c1}/d_{11})$ respectively for the fermionic and gluonic operators. Therefore clearly the former ought to be zero at $n = 2$ but not the latter.

One feature of the form of the mixing matrix (2.4) is that the leading order series in $N_f$ for $\gamma_{gg}(a)$ depends only on the Casimir $C_2(R)$. This is clear from the one and two loop results of (8, 4, 4, 4) as well as the three loop results of (8) for $n \leq 8$. Therefore if we make the assumption that this is true for all $n$ at three loops then we can deduce the exact form of the coefficients $d_{31}$ in the $C_2(G)$ sector from (5.1). It is given by the last two terms of (5.2) which are each proportional to $C_2(G)$. Indeed if one examines the second column of table 1, one observes that the first four entries are the same as those which appear in (11), after noting that $n_f = 2T(R)N_f$.

As another motivation of this study is to provide a window into the structure of the operator dimension beyond what is currently available, it is a trivial exercise to expand (5.1) for example
to four loops to deduce

\[
d_{41} + \frac{b_{41} c_1}{d_{11}} = \frac{256(n^2 + n + 2)^2[S_3(n) + 2(S_1(n))^3]C_2(R)}{27(n + 2)(n + 1)^2(n - 1)n^2 T(R)}
\]
\[
- 256[10n^6 + 30n^5 + 109n^4 + 168n^3 + 155n^2 + 76n + 12]
\times (S_1(n))^2 C_2(R)/[27(n + 2)(n + 1)^3(n - 1)n^3 T(R)]
\]
\[
+ 256[37n^8 + 148n^7 + 697n^6 + 1573n^5 + 2087n^4 + 1725n^3 + 889n^2
\]
\[
+ 264n + 36]S_1(n)C_2(R)/[81(n + 2)(n + 1)^4(n - 1)n^4 T(R)]
\]
\[
+ \frac{256(n^2 + n + 2)^2 \zeta(3) C_2(R)}{9(n + 2)(n + 1)^2(n - 1)n^2 T(R)}
\]
\[
- 16[77n^{12} + 462n^{11} + 1481n^{10} + 3170n^9 + 12839n^8 + 37418n^7
\]
\[
+ 75483n^6 + 103718n^5 + 98592n^4 + 64176n^3 + 27672n^2
\]
\[
+ 7200n + 864] C_2(R)/[243(n + 2)(n + 1)^5(n - 1)n^5 T(R)]
\]
\[
+ \frac{32(9n^4 + 18n^3 + 79n^2 + 70n + 32]S_1(n))^2 C_2(G) + 512S_1(n)\zeta(3) C_2(G)}{27(n + 2)(n + 1)^4(n - 1)n^2 T(R)}
\]
\[
- 32[8n^8 + 32n^7 + 95n^6 + 173n^5 + 781n^4 + 1311n^3 + 1176n^2
\]
\[
+ 544n + 96]S_1(n)C_2(G)/[81(n + 2)(n + 1)^3(n - 1)n^3 T(R)]
\]
\[
- \frac{1024(n^2 + n + 1)\zeta(3) C_2(G)}{27(n + 2)(n + 1)(n - 1)n T(R)}
\]
\[
- 8[5n^{10} + 25n^9 + 202n^8 + 658n^7 - 2639n^6 - 10115n^5 - 17896n^4 - 18216n^3
\]
\[
- 11160n^2 - 3840n - 576]C_2(G)/[243(n + 2)(n + 1)^4(n - 1)n^4 T(R)]
\]

(5.3)

where \(\zeta(n)\) is the Riemann \(\zeta\)-function. To obtain five and higher order combinations of perturbative coefficients at this order in \(1/N_f\) one needs to first deduce the corresponding corrections to (2.7) which are encoded in the \(O(1/N_f)\) QCD \(\beta\)-function. Again if we make the assumption that \(b_{41}\) involves only \(C_2(R)\) then the terms proportional to \(C_2(G)\) in (5.3) determine exactly the \(C_2(G)\) part of \(d_{41}\).

Finally we make some concluding remarks. First the determination of the expression (5.1) completes the leading order analysis in the large \(N_f\) expansion of the unpolarized anomalous dimensions for twist-2 operators. To complete the full \(O(1/N_f)\) analysis of deep inelastic scattering one requires information on the corresponding coefficient functions. This has been achieved for the non-singlet case by various authors, [13, 14]. As far as we are aware though, the unpolarized singlet case has yet to be examined. Second, it ought to be possible to extend our present work to determine the anomalous dimensions of the gluonic twist-2 singlet operators of polarized scattering. We would hope to return to this in a future article.

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A Basic results.

We give here several elementary integration rules for simple chain integrals. Although they have appeared before (see, for example, [23]), we record the simplest cases here for completeness and also to fix our notation. First, we recall the chain result of [12], for arbitrary \( \alpha \) and \( \beta \),

\[
\int_y \frac{1}{(y^2)^\alpha ((x-y)^2)^\beta} = \frac{\nu(\alpha, \beta, 2\mu - \alpha - \beta)}{(x^2)^{\alpha+\beta-\mu}} \tag{A.1}
\]

where \( \nu(\alpha, \beta, \gamma) = \pi^\mu a(\alpha)a(\beta)a(\gamma) \) and \( a(\alpha) = \Gamma(\mu - \alpha)/\Gamma(\alpha) \) which is established by using Feynman parametrization. The more general moment type chain integral is given by

\[
\int_y \frac{(\Delta y)^p (\Delta (x-y))^q}{(y^2)^\alpha ((x-y)^2)^\beta} = \nu_{p\gamma q}(\alpha, \beta, 2\mu - \alpha - \beta + p + q)\frac{(\Delta x)^{p+q}}{(x^2)^{\alpha+\beta-\mu}} \tag{A.2}
\]

where \( \nu_{p\gamma q}(\alpha, \beta, \gamma) = \pi^\mu a_m(\alpha)a_n(\beta)a_p(\gamma) \), \( a_n(\alpha) = \Gamma(\mu - \alpha + n)/\Gamma(\alpha) \) and \( m, n \) and \( p \) are integers. With (A.2) one can, for example, build up two loop integrals which appear in the determination of the graph of fig. 3 where the exponent of the top left propagator is replaced by 0.

Next we record the explicit expressions for the three loop recurrence relations we have used, where \( \alpha \) and \( \beta \) are arbitrary and \( n \) is a strictly positive integer. Their derivation has been discussed in section 4.

\[
C_{10\gamma}(\alpha, \beta, n) = -\frac{2(2\mu + n)(2\mu - \alpha - \beta + n - 2)}{(n+1)(n+2)} C_{10\gamma}(\alpha + 1, \beta, n+2) \\
- \frac{(2\mu - 2\alpha + n - 3)}{(n+1)} C_{10\gamma}(\alpha + 1, \beta, n+1) \\
+ [C_{00\gamma}^{011}(\alpha + 1, \beta, n+1) + C_{00\gamma}(\alpha, \beta, n+1) - C_{00\gamma}^{101}(\alpha + 1, \beta, n+1)] \\
\times \frac{(2\mu - \alpha - \beta + n - 2)}{(n+1)} \\
- [C_{00\gamma}(\alpha + 1, \beta, n+2) + C_{00\gamma}^{011}(\alpha + 1, \beta, n+2) - C_{00\gamma}^{101}(\alpha + 1, \beta, n+2)] \\
\times \frac{(2\mu + n)(2\mu - \alpha - \beta + n - 2)}{(n+1)(n+2)} \\
- [C_{10\gamma}^{021}(\alpha + 1, \beta - 1, n+1) - C_{10\gamma}^{120}(\alpha, \beta, n+1) \\
+ C_{01\gamma}^{021}(\alpha + 1, \beta - 1, n+1) - C_{01\gamma}^{120}(\alpha, \beta, n+1)]/(n+1) \tag{A.3}
\]

\[
C_{11\gamma}(\alpha, \beta, n) = \frac{2(2\mu + n+1)(2\mu - \alpha - \beta + n - 2)}{(n+1)(n+2)} C_{11\gamma}(\alpha + 1, \beta, n+2) \\
- \frac{(2\mu - 2\alpha + n - 3)}{(n+1)} C_{11\gamma}(\alpha + 1, \beta, n+1) \\
+ [2C_{00\gamma}^{011}(\alpha + 1, \beta, n+2) - C_{00\gamma}^{011}(\alpha + 1, \beta, n+2)] \\
\times \frac{(2\mu - \alpha - \beta + n - 2)}{(n+1)(n+2)} \\
+ [C_{10\gamma}(\alpha, \beta, n+1) + C_{01\gamma}^{011}(\alpha + 1, \beta, n+1) - C_{01\gamma}^{101}(\alpha + 1, \beta, n+1)] \\
\times \frac{(2\mu - \alpha - \beta + n - 2)}{(n+1)} \\
- [C_{10\gamma}(\alpha + 1, \beta, n+2) + C_{01\gamma}^{011}(\alpha + 1, \beta, n+2) - C_{01\gamma}^{101}(\alpha + 1, \beta, n+2)]
\]
calculation of the result for C tensor decomposition of the corresponding integral without the null vector ∆struct recurrence relations we need to compute all of the invariant amplitudes which arise in the yn result which will agree with the general calculation contracting with the appropriate number of the se vectors will give us a σrelations for low values of nThroughout our discussion we have mentioned carrying out explicit checks on recurrence relations for low values of n. We give a detailed illustration of this here by summarizing the right side which are the same function on the left but with α replaced by α + 1 will be zero by the dimension argument.

B Explicit check for $C_{20}((2\mu - 4 + \Delta, \mu - 1 + \Delta, 1)$.

Throughout our discussion we have mentioned carrying out explicit checks on recurrence relations for low values of n. We give a detailed illustration of this here by summarizing the calculation of the result for $C_{20}((2\mu - 4 + \Delta, \mu - 1 + \Delta, 1)$. Unlike the procedure used to construct recurrence relations we need to compute all of the invariant amplitudes which arise in the tensor decomposition of the corresponding integral without the null vector ∆. At the end of this calculation contracting with the appropriate number of these vectors will give us a µ-dependent result which will agree with the general n-dependent expression evaluated at n = 1. Therefore if we label the indices of the y-propagator of fig. 7 as µ and ν and that of the u-propagator as σ, then the integral containing $C_{20}((2\mu - 4 + \Delta, \mu - 1 + \Delta, 1)$ decomposes into

$$A\eta_{\mu\nu}x_\sigma + B[\eta_{\mu\sigma}x_\nu + \eta_{\nu\sigma}x_\mu] + C\frac{x_\mu x_\nu x_\sigma}{x^2}$$

(B.1)

The three contractions $\gamma^\mu x^\sigma$, $\gamma^\mu x^\nu$ and $x^\mu x^\nu x^\sigma$ yield the respective combinations $(2\mu A + 2B + C)$, $(A + (2\mu + 1)B + C)$ and $(A + 2B + C)$. Using identities like $xy = \frac{1}{2}[x^2 + y^2 - (x - y)^2]$ we can rewrite the resulting integrals as a sum of scalar integrals. Again the majority of these are computable using the simple rules analogous to that of fig. 5. However, several basic results are necessary which it turns out have already been computed in the calculation of 14 and we merely quote these here

$$C_{00}((2\mu - 4 + \Delta, \mu - 1 + \Delta, 0) = -\frac{(\mu - 1)^2\nu^2(1, 1, 2\mu - 2)}{\Gamma(\mu)\Delta}$$

$$C_{00}((2\mu - 5 + \Delta, \mu - 1 + \Delta, 0) = -\frac{(\mu - 1)^2(2\mu^2 - 9\mu + 16)\nu^2(1, 1, 2\mu - 2)}{4\Gamma(\mu + 1)\Delta}$$

(B.2)
\[ C_{001}(2\mu - 4 + \Delta, \mu - 2 + \Delta, 0) = -\frac{(\mu - 1)^2(2\mu^2 - 5\mu + 4)\nu^2(1, 1, 2\mu - 2)}{4\Gamma(\mu + 1)\Delta} \]

Therefore the above combinations become

\[
2\mu A + 2B + C = -\frac{(2\mu - 7)(\mu - 1)(\mu - 2)(\mu - 3)\nu^2(1, 1, 2\mu - 2)}{6\Gamma(\mu + 2)\Delta} \\
A + (2\mu + 1)B + C = -\frac{(2\mu^4 + 11\mu^3 - 121\mu^2 + 374\mu - 324)(\mu - 1)\nu^2(1, 1, 2\mu - 2)}{48\Gamma(\mu + 2)\Delta} \tag{B.3} \\
A + 2B + C = -\frac{(6\mu^4 + 53\mu^3 - 333\mu^2 + 1408\mu - 1464)(\mu - 1)\nu^2(1, 1, 2\mu - 2)}{96\Gamma(\mu + 3)\Delta}
\]

Finally, solving for \(A, B\) and \(C\) the uncontracted integral is

\[
C_{201}(2\mu - 4 + \Delta, \mu - 1 + \Delta, 1) = -\left[ \eta_{\mu\nu}x_\sigma(26\mu^4 - 261\mu^3 + 541\mu^2 - 576\mu + 120) \\
+ (4\mu^4 + 36\mu^3 - 143\mu^2 + 168\mu - 56)(\mu - 3) \\
\times (\eta_{\mu\sigma}x_\nu + \eta_{\nu\sigma}x_\mu) \\
+ 2(2\mu^5 + 13\mu^4 + 22\mu^3 + 707\mu^2 - 1320\mu + 504) \\
\times \frac{x_\mu x_\nu x_\sigma}{x^2} \right] (\mu - 1)\nu(1, 1, 2\mu - 2) \\
/\left[ 96(2\mu - 1)\Gamma(\mu + 3)\Delta \right] \tag{B.4}
\]

whence the check on the recurrence relation we require is given by the coefficient of the \(x^\mu x^{\nu'} x^{\sigma'}\) term. It is in agreement with the value of (4.20) at \(n = 1\).
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| $n$ | $C_2(R)/T(R)$ coefficient | $C_2(G)/T(R)$ coefficient |
|-----|---------------------------|---------------------------|
| 2   | $-\frac{2716}{243}$      | $-\frac{4232}{243}$      |
| 4   | $-\frac{3765671}{607500}$ | $-\frac{757861}{60750}$ |
| 6   | $-\frac{373918478}{72930375}$ | $-\frac{26390948}{2083725}$ |
| 8   | $-\frac{261237619387}{54010152000}$ | $-\frac{420970849}{32145900}$ |
| 10  | $-\frac{1994718278948}{420260754375}$ | $-\frac{2752314359}{203762790}$ |
| 12  | $-\frac{26908165162400891}{3703299664286200}$ | $-\frac{2635361358193}{189919269540}$ |
| 14  | $-\frac{16449437567815379}{34897665777797600}$ | $-\frac{4616790551}{325001100}$ |
| 16  | $-\frac{33164191885312621883}{7028199711799968000}$ | $-\frac{3563129324283}{21083944328000}$ |
| 18  | $-\frac{7462146988179117961999}{157843925922893948379}$ | $-\frac{24946512233659}{16907213453300}$ |
| 20  | $-\frac{821627355747899257489}{174422358288281810000}$ | $-\frac{1340260211455733}{894010727610000}$ |
| 22  | $-\frac{228675683353467573965719}{481684211509665976097452}$ | $-\frac{648057715871521}{12614381628819}$ |
| 24  | $-\frac{45507171465596384735677583}{956698391661142985000000}$ | $-\frac{53567598174594947}{3463316806750000}$ |

Table 1. Coefficients of $[d_{31} + b_{31}c_1/d_{11}]$ as a function of moment.
Fig. 1. Operator insertion in gluon 2-point function.

Fig. 2. Leading order diagrams for $\lambda_+(a_c)$.

Fig. 3. Definition of $G_{pqn}(\alpha)$. 
Fig. 4. Definition of $H_{pn}(\alpha, \beta)$.

$$
\begin{bmatrix}
0 \\
1 (n) \\
z \\
y \\
x
\end{bmatrix} = \frac{\nu_{ab}(1,1,2\mu-2+n)}{(2\mu-2+n)} \left( \begin{array}{c}
\mu - 1 \\
\mu - 1 + n \\
\mu - 1 + n
\end{array} \right)
$$

Fig. 5. Example of elementary integral rule.

Fig. 6. Intermediate integral in calculation of $H_{pn}(\alpha, \beta)$.

Fig. 7. Definition of $C_{pq\gamma}(\alpha, \beta, n)$. 