THE BENJAMIN–FEIR INSTABILITY IN THE INFINITE DEPTH

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Abstract. We prove that a Stokes’ periodic wave of sufficiently small amplitude, traveling under gravity at the free surface of a two dimensional, infinitely deep, and irrotational flow, is spectrally unstable to slow modulation, rigorously justifying Benjamin and Feir’s formal argument.

1. Introduction

Stokes in his great 1847 paper [32] (see also [33]) made significant contributions to periodic waves at the free surface of a two dimensional, infinitely deep, and irrotational flow of an incompressible inviscid fluid, under the influence of gravity, traveling in permanent form at a constant velocity. For instance, he successfully approximated the equations when the motion is small. The existence of Stokes waves was rigorously established in the 1920s for small amplitude [22, 27], and in the early 1960s for large amplitude [20, 21]. Thus it came as a surprise in the mid 1960s when Benjamin and Feir [2, 3] and Whitham [35] discovered that a Stokes wave in sufficiently deep water, so that (the wave number) \times (the fluid depth) > 1.363\ldots, is unstable to slow modulation – namely, the Benjamin–Feir or modulational instability. Corroborating results arrived about the same time, but independently, by Lighthill [23], Zakharov [36], Ostrovsky [30], Benney and Newell [4], and many others. To quote Zakharov and Ostrovsky [37], “the idea was emerging when the time was indeed ripe.” Modulational instability occurs in numerous physical situations from fluids to optics to plasmas, and it plays a crucial role in several wave phenomena of interest from envelop solitons and shocks to rogue waves [37].

In the 1990s, Bridges and Mielke [5] addressed the spectral instability of a Stokes wave of sufficiently small amplitude in the finite depth, rigorously justifying the formal arguments of [2, 35] and others in a functional analytic setting. Some fundamental issues remain open, however, such as: (1) the spectral instability in the infinite depth, (2) the spectrum away from the origin of the complex plane, and (3) the nonlinear stability and instability. Recently, Nguyen and Strauss [28] proved (1). Here we offer a simpler alternative proof of (1), and address (2). Our approach is potentially useful for (3) and other, nonlinear dispersive waves.

Bridges and Mielke [5] remarked that their proof breaks down in the infinite depth because the discrete spectrum of the linear operator for the Stokes wave problem, in a bounded domain in two dimensions when the fluid depth is finite, becomes continuous in an unbounded domain as the depth increases infinitely, whence their center manifold reduction is not applicable. In an irrotational flow, on the other hand, one can reformulate the water wave problem in two dimensions, in the finite or infinite depth, in terms of quantities at the fluid surface, so that there is no continuous spectrum of the linear operator for the associated Stokes wave problem, in a bounded domain in one dimension. This comes at a price, however, and the resulting equations become nonlocal.

We make a conformal mapping from a lower half plane to the fluid region, suggested by [12] and others, and reformulate the physical problem as nonlinear pseudo-differential equations in one dimension (see [2, 11]). There are other ways to reformulate, for instance, introducing the Dirichlet–to–Neumann operator for the nontrivial fluid surface (see [28] and references therein). We prefer the conformal mapping approach because the resulting equations involve the Hilbert transform.

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for the trivial fluid surface alone, which is explicit in Fourier analysis, and quadratic polynomial nonlinearities (and no higher order terms). This greatly simplifies perturbation analysis calculations. Also, the associated Stokes wave problem (see (3.1)) has been analytically well studied (see [10], for instance, and references therein). Particularly, the solutions depend real analytically on small values of the amplitude parameter.

In recent years, the author and her collaborators [6, 14–16] (see also [7]) have worked out spectral perturbation analysis in the vicinity of the origin of the complex plane for small values of the modulation parameter, determining stability and instability for a large class of nonlinear dispersive equations, permitting nonlocal operators, for which the periodic Evans function and other ODE techniques are not applicable. When the modulation parameter is zero, the linearized operator of the water wave problem about a Stokes wave of sufficiently small amplitude possesses four eigenvalues at the origin (see Lemma 2), and as the parameter value increases, the eigenvalues may enter the right half plane, resulting in modulational instability. Unfortunately, the linearized operator of (2.11) is not continuously differentiable at the zero modulation parameter, whence the analytic perturbation of [14, 16] and others seems not applicable. The operators in [28], and also [5] in the finite depth, by contrast, depend analytically on the modulation parameter, whereby amenable to the argument of [14, 16] and others.

Our approach instead takes advantage of that the linearized operator of (2.11) depends analytically on small values of the amplitude parameter for any nonzero value of the modulation parameter. When the amplitude parameter is zero, two simple and purely imaginary eigenvalues in the vicinity of the origin depend analytically on the modulation parameter, say, \( p \) and collide at \( \frac{1}{2}ip \) to leading order for \( p \neq 0, \ll 1 \) (see (5.3)); as the amplitude parameter value increases, they may leave the imaginary axis, resulting in modulational instability. Two other eigenvalues near the origin are not continuously differentiable at \( p = 0 \); on the other hand, they are away from other eigenvalues and from each other for \( p \neq 0, \ll 1 \) (see (5.4)), whence they will remain on the imaginary axis for small values of the amplitude parameter. For such \( p \neq 0, \ll 1 \), the simple eigenvalues near \( \frac{1}{2}ip \) and the eigenfunctions depend analytically on small values of the amplitude parameter, and we make analytic perturbation to demonstrate modulational instability. Our approach is potentially useful for other equations whose dispersion relation is not smooth, so that one is unable to make analytic perturbation with respect to the modulation parameter, for instance, the Kadomtsev–Petviashvili equations.

Also, our approach permits eigenvalues away from the origin of the complex plane. The proofs of [5, 28], by contrast, make strong use of spectral information at the origin. The linear operator of the water wave problem possesses infinitely many collisions of purely imaginary eigenvalues away from the origin (see (7.1)), and numerical computations in the 1980s (see [24, 25] among others) suggested that they lead to spectral instability for small values of the amplitude parameter. To the contrary, we make analytic perturbation and demonstrate spectral stability away from the origin up to the quadratic order of infinitesimally small values of the amplitude parameter. Thus modulational instability dominates for Stokes waves of sufficiently small amplitude, and spectral instability away from the origin can occur when the amplitude is not infinitesimally small. Recent numerical computations (see [1], for instance) bear this out. In the finite depth, by contrast, a colliding eigenvalue away from the origin leads to a new kind of unstable Stokes waves [17].

The well-posedness of the Cauchy problem for (2.11) has been rigorously established [13], and one may attempt to promote modulational instability to spatially localized and temporally exponentially growing solutions of (2.11). See [18] for some classes of nonlinear dispersive equations. This is an interesting direction of future investigation.
2. **The water wave problem in conformal coordinates**

The water wave problem, in the simplest form, concerns the wave motion at the free surface of a two dimensional, infinitely deep, and irrotational flow of an incompressible inviscid fluid, under the influence of gravity, without the effects of surface tension. Although an incompressible fluid can have variable density, we assume for simplicity unit density. Suppose for definiteness that in Cartesian coordinates, the \( x \) axis points in the direction of wave propagation, and the \( y \) axis vertically upward. Suppose that the fluid occupies a region \( \Omega(t) \) in the \((x, y)\) plane at time \( t \), bounded above by a free surface \( \Gamma(t) \).

Let \( \mathbf{u}(x, y, t) \) denote the velocity of the fluid at the point \((x, y)\) and time \( t \), and \( P(x, y, t) \) the pressure. They satisfy the Euler equations for an incompressible fluid

\[
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla P = (0, -g) \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in} \ \Omega(t),
\]

where \( g > 0 \) is the constant of gravitational acceleration, subject to the boundary condition

\[
\mathbf{u}(x, y, t) \to 0 \quad \text{as} \ y \to -\infty.
\]

The kinematic and dynamic boundary conditions at the fluid surface

\[
\partial_t + \mathbf{u} \cdot \nabla \text{ is tangential to } \bigcup_t \Gamma(t) \quad \text{and} \quad P = \text{const.} \quad \text{at} \ \Gamma(t)
\]

state, respectively, that each fluid particle at the surface remains there for all time, and that the pressure at the fluid surface equals that of the air.

In an irrotational flow, \( \nabla \times \mathbf{u} = 0 \). Let \( \mathbf{u} = \nabla \phi \), where \( \phi(x, y, t) \) is a velocity potential, and the latter equation of \( 2.1 \) implies that

\[
\nabla^2 \phi = 0 \quad \text{in} \ \Omega(t).
\]

Substituting in the former equation of \( 2.1 \), we make a straightforward calculation and recall the latter equation of \( 2.3 \) to arrive at

\[
\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + gy = b(t) \quad \text{at} \ \Gamma(t),
\]

where \( b \) is arbitrary. Since \( \phi \) is determined up to addition by a constant at each instance of time, we may assume without loss of generality that \( b(t) = 0 \) for all \( t \). Also, we may assume that

\[
\phi(x, y, t) \to 0 \quad \text{as} \ y \to -\infty.
\]

Additionally, we assume that \( \Omega(t) \) approaches a lower half plane of \( \mathbb{R}^2 \) and \( \phi(x, y, t) \to 0 \) as \( |x| \to \infty \), or else we assume that \( \Omega \) and \( \phi \) are periodic in \( x \). Translating the free surface along the \( y \) axis, if necessary, we may assume without loss of generality that \( \Gamma(t) \) approaches the \( x \) axis as \( |x| \to \infty \), or else it is of mean zero, for instance, over one period.

We proceed as in [12] and others, to reformulate \( 2.1 \), the former equation of \( 2.3 \), \( 2.5 \) and \( 2.6 \) and, hence, \( 2.1 \), \( 2.3 \), \( 2.5 \), \( 2.6 \), in conformal coordinates. In what follows, we identify \( \mathbb{R}^2 \) with \( \mathbb{C} \) whenever it is convenient to do so.

Suppose that

\[
(x + iy)(\alpha + i\beta, t)
\]

conformally maps \( \mathbb{C}_- := \{ \alpha + i\beta \in \mathbb{C} : \beta < 0 \} \) to \( \Omega(t) \), and

\[
(x + iy)(\alpha + i\beta, t) - (\alpha + i\beta) \to 0 \quad \text{as} \ \beta \to -\infty.
\]

Suppose that \( 2.7 \) extends to map \( \{ \alpha + 0i : \alpha \in \mathbb{R} \} \) to \( \Gamma(t) \). We assume that \((x + iy)(\alpha + i\beta, t) - (\alpha + i\beta) \in L^2(\mathbb{R}) \) for any \( \beta \in (-\infty, 0) \), or else \((x + iy)(\alpha + i\beta, t) - (\alpha + i\beta) \) is a periodic function of
\(\alpha.\) Since \((x + iy)(\alpha + i\beta, t) - (\alpha + i\beta)\) is holomorphic in \(C_-,\) by hypothesis, a Titchmarsh theorem (see [34], for instance) asserts that
\[
(2.8) \quad (x + iy)(\alpha + 0i, t) = \alpha + (H + i)y(\alpha + 0i, t).
\]
Here and elsewhere, \(H\) denotes the Hilbert transform, defined in Fourier analysis as
\[
(2.9) \quad \hat{H}f(k) = -i\text{sgn}(k)f(k),
\]
where the circumflex means the Fourier transform. Thus
\[
\Gamma(t) = \{(\alpha + Hy(\alpha, t), y(\alpha, t)) : \alpha \in \mathbb{R}\}
\]
in conformal coordinates. Let, by abuse of notation,
\[
(\phi + i\psi)(\alpha + i\beta) = (\phi + i\psi)((x + iy)(\alpha + i\beta, t), t),
\]
where \(\psi\) is a harmonic conjugate of \(\phi.\) Thus \(\phi(\alpha, t)\) is a velocity potential at the fluid surface in conformal coordinates. We assume that \((\phi + i\psi)(\alpha + i\beta, t) \in L^2(\mathbb{R})\) for any \(\beta \in (-\infty, 0),\) or else it is a periodic function of \(\alpha.\) Since \(\phi + i\psi\) is holomorphic in \(C_-\) by (2.4) and \((\phi + i\psi)(\alpha + i\beta, t) \to 0\) as \(\beta \to -\infty\) by (2.4), a Titchmarsh theorem [34] asserts that
\[
(2.10) \quad (\phi + i\psi)(\alpha + 0i, t) = (1 - iH)\phi(\alpha + 0i, t).
\]
Substituting (2.8) and (2.10) in the former equation of (2.3) and (2.5), we use the chain rule and make a straightforward calculation to arrive at
\[
(2.11) \quad (1 + H\partial_\alpha y)\partial_t y - \partial_\alpha yH\partial_t y - H\partial_\alpha \phi = 0,
\]
\[
(1 + \partial_\alpha y)\partial_t \phi - \partial_\alpha \phi H\partial_t y + H(\partial_\alpha y\partial_t \phi - \partial_t y\partial_\alpha \phi) + g(y + yH\partial_\alpha y + H(y\partial_\alpha y)) = 0.
\]
See [12], for instance, for details. Conversely, a solution of (2.11) gives rise to a solution of (2.1)–(2.3), provided that
\[
(2.12) \quad (1 + H\partial_\alpha y)^2 + (\partial_\alpha y)^2(\alpha, t) \neq 0 \quad \text{for any } \alpha \in \mathbb{R}.
\]
The former states that the fluid surface does not intersect itself, and the latter ensures that (2.7) is well defined throughout \(C_- \cup (\mathbb{R} + 0i).\) One can solve the Cauchy problem for (2.11) at least for short time even when the former of (2.12) fails to hold, although the solution is physically unrealistic [13]. For Stokes waves, the latter of (2.12) implies that there are no stagnation points, where \(\nabla_{(x,y)}\phi = 0,\) throughout the fluid region. Clearly, (2.12) holds when \(y\) is small.

An integration of the former equation of (2.11) leads to that
\[
\int y(1 + H\partial_\alpha y) \, d\alpha = \text{const.} \quad \text{for all } t.
\]
In the finite depth, we replace (2.6) by
\[
\partial_y\phi(x, -h, t) = 0 \quad \text{for some } h > 0,
\]
and proceed in like manner, but with suitable modifications to accommodate the effects of a rigid flat bottom, to arrive at
\[
(2.13) \quad (1 + T_h\partial_\alpha y)\partial_t y - \partial_\alpha yT_h\partial_t y - H_h\partial_\alpha \phi = 0,
\]
\[
(1 + T_h\partial_\alpha y)\partial_t \phi - \partial_\alpha \phi T_h\partial_t y + H_h(\partial_\alpha y\partial_t \phi - \partial_t y\partial_\alpha \phi) + g(y + yT_h\partial_\alpha y + T_h(y\partial_\alpha y)) = 0,
\]
where
\[
(2.14) \quad \hat{H}_h f(\xi) = -i\tanh(kh)\hat{f}(k) \quad \text{and} \quad \hat{T}_h f(k) = -i\coth(kh)\hat{f}(k).
\]
See [11], for instance, for details. We remark that \(H_h, T_h \to H\) as \(h \to \infty.\)
3. Analytic bifurcation of Stokes waves

By a Stokes wave, we mean a periodic, traveling wave solution of (2.11). Suppose that \( y \) and \( \phi \) are \( 2\pi/k \) periodic functions of \( \alpha - ct \), where \( c \neq 0, \in \mathbb{R} \) denotes the velocity of the wave, and \( k > 0 \) the wave number. We proceed to a moving coordinate, changing \( \alpha - ct \) by \( \alpha \), whereby \( t \) disappears. The result becomes

\[
(3.1) \quad c^2 \mathcal{H}y' = g(y + y\mathcal{H}y' + \mathcal{H}(yy')) \quad \text{and} \quad \phi' = c\mathcal{H}y',
\]

where the prime means differentiation with respect to \( \alpha \). In the finite depth,

\[
(3.2) \quad c^2 \mathcal{T}_h y' = g(y + y\mathcal{T}_h y' + \mathcal{T}_h(yy')) \quad \text{and} \quad \phi' = c\mathcal{H}_h y'.
\]

Observe that (3.1) remains invariant under

\[
\alpha \mapsto \alpha + \alpha_0 \quad \text{and} \quad \alpha \mapsto -\alpha
\]

for any \( \alpha_0 \in \mathbb{R} \), whence we may assume that \( y \) is even, and \( \phi \) is odd by the latter equation of (3.1).

Observe that (3.1) remains invariant under

\[
y(\alpha) \mapsto k^{-1}y(k\alpha), \quad \phi(\alpha) \mapsto k^{-1/2}\phi(k\alpha) \quad \text{and} \quad c^2 \mapsto k^{-1}c^2
\]

for any \( k > 0 \), whence we may assume without loss of generality that \( k = 1 \). Thus \( y \) and \( \phi \) are \( 2\pi \) periodic. Moreover, we may assume without loss of generality that \( g = 1 \). Thus \( c(>0) \) is the Froude number. In the finite depth, (3.2) does not possess scaling invariance. Rather, the instability result of [5, Section 11.1] and others depends on the wave number of a Stokes wave.

The existence of small amplitude solutions of (3.1) may be rigorously established by analytic theory of local bifurcation.

**Proposition 1** (Existence of Stokes waves of sufficiently small amplitude). There exists a curve of nontrivial solutions of (3.1) in \( H^1(T) \times H^1(T) \times \mathbb{R} \), denoted \( y(\alpha; \varepsilon), \phi(\alpha; \varepsilon) \) and \( c(\varepsilon) \) for \( \varepsilon \in \mathbb{R} \) and \( |\varepsilon| \ll 1 \), which admits a real analytic reparametrization at each \( \varepsilon \); \( y \) and \( \phi \) are \( 2\pi \) periodic and real analytic functions of \( \alpha \), \( y \) is even and \( \phi \) is odd in \( \alpha \), and \( \varepsilon \) is even in \( \varepsilon \). Moreover,

\[
(3.3a) \quad y(\alpha; \varepsilon) = \varepsilon \cos \alpha + \varepsilon^2 \left( \cos 2\alpha - \frac{1}{2} \right) + \frac{3}{2} \varepsilon^3 \cos 3\alpha + O(\varepsilon^4),
\]

\[
(3.3b) \quad \phi(\alpha; \varepsilon) = \varepsilon \sin \alpha + \varepsilon^2 \sin 2\alpha + \varepsilon^3 \left( \frac{3}{2} \sin 3\alpha + \frac{1}{2} \sin \alpha \right) + O(\varepsilon^4)
\]

and

\[
(3.3c) \quad c(\varepsilon) = 1 + \frac{1}{2} \varepsilon^2 + O(\varepsilon^4)
\]

as \( |\varepsilon| \to 0 \). There are no other nontrivial solutions of (3.1) in the vicinity of the solution curve in \( H^1_{\text{even}}(T) \times H^1_{\text{odd}}(T) \times \mathbb{R} \).

**Proof.** Details can be found in [9, Section 11.1] and references therein, whence we merely hit the main points.

We begin by recording that if \( y \in H^1(T) \) solves the former equation of (3.1) for some \( c \neq 0, \in \mathbb{R} \) and if \( c^2 - 2y(\alpha) > 0 \) for all \( \alpha \in T \) then \( y \) is a real analytic function and \( (1 + \mathcal{H}y')^2 + (y')^2)(\alpha) \neq 0 \) for all \( \alpha \in T \) [9, Theorem 3.1]. By the way, if \( c^2 - 2\max_{\alpha \in T} y(\alpha) = 0 \) then \( y \) makes the celebrated extreme wave [33], for which \( ((1 + \mathcal{H}y')^2 + (y')^2)(0) = 0 \), say. Thus the wave crest is a stagnation point.

Let

\[
G(\mu, y) = \mathcal{H}y' - \mu(y + y\mathcal{H}y' + \mathcal{H}(yy')) : \mathbb{R} \times H^1(T) \to L^2(T),
\]
and it is well defined by (2.9) and a Sobolev inequality. Note that if
\begin{equation}
(3.5) \quad G(\mu, y) = 0, \quad \text{where } (\mu, y) \in (0, \infty) \times H^1_{\text{even}}(\mathbb{T}),
\end{equation}
then \(y\) is a critical point of
\[
\int_{-\pi}^{\pi} \frac{1}{2} (y_H' - \mu y^2 (1 + H y')) \, d\alpha, \quad \text{subject to } \int_{-\pi}^{\pi} y (1 + H y') \, d\alpha = \text{const}.
\]

Clearly, (3.4) is Fréchet continuously differentiable. Moreover, since all its Fréchet derivatives of orders \(\geq 3\) are zero everywhere, (3.4) is a real analytic operator.

Note that \(G(\mu, 0) = 0\) for any \(\mu \in (0, \infty)\). Note that
\[
\partial_y G(\mu, 0) y = H y' - \mu y = 0
\]
admits a nontrivial solution in \(H^1_{\text{even}}(\mathbb{T})\) if and only if
\[
\mu = n \quad \text{and} \quad y(\alpha) = \cos n\alpha, \quad \text{where } n \in \mathbb{N}.
\]
Indeed, thanks to the variational structure, for each \(n \in \mathbb{N}\), \((n, 0)\) is a bifurcation point of (3.5) [9, Theorem 4.1], and for some \(\varepsilon_n > 0\) for some \(O_n \subset (0, \infty) \times H^1_{\text{even}}(\mathbb{T})\), where \((n, 0) \in O_n\) and \(O_n\) is open, there exists a unique, real analytic function \((\mu_n, y_n)(\varepsilon) : (-\varepsilon_n, \varepsilon_n) \to O_n\), where \((\mu_n, y_n)(0) = (n, 0)\) and
\[
\int_{-\pi}^{\pi} y_n(\alpha; \varepsilon) \cos n\alpha \, d\alpha = 0
\]
for any \(\varepsilon \in (-\varepsilon_n, \varepsilon_n)\),
such that \((\mu, y) \in O_n\) is a nontrivial solution of (3.3) if and only if
\begin{equation}
(3.6) \quad (\mu, y) = (\mu_n(\varepsilon), \varepsilon (\cos n\alpha + y_n(\alpha; \varepsilon))) \quad \text{for some } \varepsilon \in (-\varepsilon_n, \varepsilon_n) \setminus \{0\}
\end{equation}
[10, Theorem 11.1.1]. Thanks to the symmetry of (3.4), the solution curve \(\{ (\mu_n(\varepsilon), \varepsilon (\cos n\alpha + y_n(\alpha; \varepsilon))) : \varepsilon \in (-\varepsilon_n, \varepsilon_n) \} \subset \mathbb{R} \times H^1_{\text{even}}(\mathbb{T})\) is symmetric about the \(\mu\) axis.

Thanks to the symmetry of (3.4), we may restrict the attention to \(n = 1\). Substituting (3.6) in (3.5) and differentiating with respect to \(\varepsilon\) and evaluating \(\varepsilon = 0\) repeatedly, we find (3.3a) and (3.3b), and (3.3c) by the latter equation of (3.1). Since \(\phi\) is determined up to addition by a constant, we may assume without loss of generality that \(\int_{-\pi}^{\pi} \phi(\alpha; \varepsilon) \, d\alpha = 0\).

We record for future usefulness that if \((\mu, y) \in (0, \infty) \times H^1_{\text{even}}(\mathbb{T})\) and \(1 - 2\mu y(\alpha) > 0\) for all \(\alpha \in \mathbb{T}\), then \(\partial_y G(\mu, y) : H^1(\mathbb{T}) \to L^2(\mathbb{T})\) is a Fredholm operator with index 0 [8]. See also [10, Section 10.5].

4. The modulational instability problem

Let \((y, \phi)(\alpha; \varepsilon)\) and \(c(\varepsilon)\), for some \(\varepsilon \in \mathbb{R}\) and \(|\varepsilon| \ll 1\), denote a Stokes wave of sufficiently small amplitude, whose existence follows from Proposition 11. It makes a stationary and \(2\pi\) periodic solution of
\[
(1 + H \partial_\alpha y) \partial_t y - \partial_\alpha y H \partial_\alpha y - c \partial_\alpha y - H \partial_\alpha \phi = 0,
\]
\[
(1 + H \partial_\alpha y) \partial_t \phi - \partial_\alpha \phi H \partial_\alpha y + H (\partial_\alpha y \partial_\alpha \phi - \partial_\alpha y \partial_\alpha \phi) - c \partial_\alpha \phi + y + y H \partial_\alpha y + H (y \partial_\alpha y) = 0,
\]
and we address its spectral stability and instability to slow modulation.

Linearizing about \((y, \phi)(\varepsilon)\), and evaluating \(c = c(\varepsilon)\), we arrive at
\begin{equation}
\mathcal{J}(\varepsilon) \partial_t v = \mathcal{L}(\varepsilon) v,
\end{equation}
where
\begin{equation}
\mathcal{J}(\varepsilon), \mathcal{L}(\varepsilon) : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \subset L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to L^2(\mathbb{R}) \times L^2(\mathbb{R}),
\end{equation}
(4.3a) \[ \mathcal{J}(\varepsilon) = \begin{pmatrix} (1 + \mathcal{H}y(\varepsilon)') - y(\varepsilon)'\mathcal{H} & 0 \\ -(\mathcal{H}\phi(\varepsilon)' + \phi(\varepsilon)'\mathcal{H}) & (1 + \mathcal{H}y(\varepsilon)') + \mathcal{H}y(\varepsilon)' \end{pmatrix} \]

and

(4.3b) \[ \mathcal{L}(\varepsilon) = \begin{pmatrix} c(\varepsilon)\partial_\alpha & \mathcal{H}\partial_\alpha \\ -((1 + \mathcal{H}y(\varepsilon)') + y(\varepsilon)'\mathcal{H}\partial_\alpha + \mathcal{H}\partial_\alpha y(\varepsilon)) & c(\varepsilon)\partial_\alpha \end{pmatrix}. \]

Note that

(4.4) \[ \mathcal{J}(\varepsilon), \mathcal{L}(\varepsilon) : H^1_{\text{loc}}(\mathbb{R}) \times H^1_{\text{loc}}(\mathbb{R}) \subset L^2_{\text{loc}}(\mathbb{R}) \times L^2_{\text{loc}}(\mathbb{R}) \to L^2_{\text{loc}}(\mathbb{R}) \times L^2_{\text{loc}}(\mathbb{R}), \]

where \( L^2_{\text{loc}}(\mathbb{R}) \) consists of uniformly locally \( L^2(\mathbb{R}) \) functions, and \( H^1_{\text{loc}}(\mathbb{R}) \) consists of \( L^2_{\text{loc}}(\mathbb{R}) \) functions whose derivatives are in \( L^2_{\text{loc}}(\mathbb{R}) \). See [25], for instance, and references therein for details. Seeking a solution of (4.1) of the form \( \psi(\alpha, t) = e^{\lambda t}\psi(\alpha) \), where \( \lambda \in \mathbb{C} \), we arrive at

(4.5) \[ \lambda \mathcal{J}(\varepsilon) \psi = \mathcal{L}(\varepsilon) \psi. \]

We say that \( (y, \phi)(\varepsilon) \) is spectrally unstable if (4.5) admits a nontrivial solution in \( L^2(\mathbb{R}) \times L^2(\mathbb{R}) \) for some \( \lambda \in \mathbb{C} \) and \( \text{Real}(\lambda) > 0 \).

Clearly, (4.2) (or (4.4)) and (4.3) are analytic with respect to \( \alpha \). Since \( y(\varepsilon), \phi(\varepsilon) \) and \( c(\varepsilon) \) are real analytic functions of \( \varepsilon \) for \( \varepsilon \in \mathbb{R} \) and \( |\varepsilon| \ll 1 \) (see Proposition 1), (4.2) (or (4.4)) and (4.3) depend analytically on \( \varepsilon \). We use (3.3) and calculate that

(4.6a) \[ \mathcal{J}(\varepsilon) = \mathbf{I} + \varepsilon \begin{pmatrix} \cos \alpha + \sin \alpha \mathcal{H} & 0 \\ -(\cos \alpha \mathcal{H} + \mathcal{H} \cos \alpha) & \cos \alpha - \mathcal{H} \sin \alpha \end{pmatrix} \]

\[ + 2\varepsilon^2 \begin{pmatrix} \cos 2\alpha + \sin 2\alpha \mathcal{H} & 0 \\ -(\cos 2\alpha \mathcal{H} + \mathcal{H} \cos 2\alpha) & \cos 2\alpha - \mathcal{H} \sin 2\alpha \end{pmatrix} \]

\[ + \frac{1}{2} \varepsilon^3 \begin{pmatrix} 9(\cos 3\alpha + \sin 3\alpha \mathcal{H}) & 0 \\ -(\cos 3\alpha \mathcal{H} + \mathcal{H} \cos 3\alpha) & 9(\cos 3\alpha - \mathcal{H} \sin 3\alpha) \end{pmatrix} + O(\varepsilon^4) \]

and

(4.6b) \[ \mathcal{L}(\varepsilon) = \begin{pmatrix} \partial_\alpha & \mathcal{H}\partial_\alpha \\ -1 & 0 \end{pmatrix} - \varepsilon \begin{pmatrix} \cos \alpha + \cos \alpha \mathcal{H}\partial_\alpha + \mathcal{H}\partial_\alpha \cos \alpha & 0 \\ 0 & \cos \alpha + \mathcal{H}\partial_\alpha \cos \alpha \end{pmatrix} \]

\[ + \varepsilon^2 \left( \begin{pmatrix} \frac{\partial_\alpha}{\mathcal{H}\partial_\alpha} & \mathcal{H}\partial_\alpha \\ \frac{1}{\mathcal{H}\partial_\alpha} & 0 \end{pmatrix} - 2 \begin{pmatrix} \cos 2\alpha + \cos 2\alpha \mathcal{H}\partial_\alpha + \mathcal{H}\partial_\alpha \cos 2\alpha & 0 \\ 0 & \cos 2\alpha + \mathcal{H}\partial_\alpha \cos 2\alpha \end{pmatrix} \right) \]

\[ - \frac{3}{2} \varepsilon^3 \begin{pmatrix} 3 \cos 3\alpha + \cos 3\alpha \mathcal{H}\partial_\alpha + \mathcal{H}\partial_\alpha \cos 3\alpha & 0 \\ 0 & 3 \cos 3\alpha + \mathcal{H}\partial_\alpha \cos 3\alpha \end{pmatrix} + O(\varepsilon^4) \]

as \( |\varepsilon| \to 0 \). Since

\[ \mathcal{J}(\varepsilon) = \mathbf{I} + O(\varepsilon) : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to L^2(\mathbb{R}) \times L^2(\mathbb{R}) \]

(or \( L^2_{\text{loc}}(\mathbb{R}) \times L^2_{\text{loc}}(\mathbb{R}) \to L^2_{\text{loc}}(\mathbb{R}) \times L^2_{\text{loc}}(\mathbb{R}) \)) is invertible for \( \varepsilon \in \mathbb{R} \) and \( |\varepsilon| \ll 1 \), \( (y, \phi)(\varepsilon) \) is spectrally stable if and only if the \( L^2(\mathbb{R}) \times L^2(\mathbb{R}) \) spectrum of \( (\mathcal{J}^{-1}\mathcal{L})(\varepsilon) \) intersects the open, right half plane of \( \mathbb{C} \).

Observe that (4.5) remains invariant under

\[ \lambda \mapsto \lambda^* \quad \text{and} \quad \psi \mapsto \psi^*, \]

where \( \psi^* \) is the complex conjugate of \( \psi \) and \( \lambda^* \) is the complex conjugate of \( \lambda \).
where the asterik means complex conjugation, and under
\[ \lambda \mapsto -\lambda \quad \text{and} \quad \alpha \mapsto -\alpha, \quad c \mapsto -c. \]
Together, the spectrum of \((J^{-1}L)(\varepsilon)\) is symmetric about the real and imaginary axes. Therefore, \((y, \phi)(\varepsilon)\) is spectrally unstable if and only if the spectrum of \(J^{-1}L\) is not contained in the imaginary axis.

Since \(y(\varepsilon), \phi(\varepsilon) \notin L^2(\mathbb{R}) \) (but in \(L^2_{\text{loc}}(\mathbb{R})\)), a nontrivial solution of (4.5) is not bounded in \(L^2(\mathbb{R}) \times L^2(\mathbb{R})\) (indeed, not in \(L^p(\mathbb{R}) \times L^p(\mathbb{R})\) for all \(p \in [1, \infty)\)). Rather, it is at best in \(L^2_{\text{loc}}(\mathbb{R}) \times L^2_{\text{loc}}(\mathbb{R})\). See [31, Section 8.16], for instance, for details; see also [7] and references therein. Thus the \(L^2(\mathbb{R}) \times L^2(\mathbb{R})\) spectrum of \((J^{-1}L)(\varepsilon)\) possesses no eigenvalues. Rather, it consists of essential spectrum. Since
\[
y \mapsto c(\varepsilon)^2 \mathcal{H} \partial_\alpha y - (1 + \mathcal{H} y(\varepsilon)^1 + y(\varepsilon) \mathcal{H} \partial_\alpha + \mathcal{H} \partial_\alpha y(\varepsilon)) y : H^1(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})
\]
(see the former equation of (3.1)) is elliptic for \(\varepsilon \in \mathbb{R}\) and \(|\varepsilon| \ll 1\) [10, Section 10.5], and so is \(\mathcal{L}(\varepsilon) : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \subset L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to L^2(\mathbb{R}) \times L^2(\mathbb{R}),\) or \(H^1_{\text{loc}}(\mathbb{R}) \times H^1_{\text{loc}}(\mathbb{R}) \subset L^2_{\text{loc}}(\mathbb{R}) \times L^2_{\text{loc}}(\mathbb{R}) \to L^2_{\text{loc}}(\mathbb{R}) \times L^2_{\text{loc}}(\mathbb{R}),\) and since \(J(\varepsilon) : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to L^2(\mathbb{R}) \times L^2(\mathbb{R}),\) or \(L^2_{\text{loc}}(\mathbb{R}) \times L^2_{\text{loc}}(\mathbb{R}) \to L^2_{\text{loc}}(\mathbb{R}) \times L^2_{\text{loc}}(\mathbb{R}),\) is invertible,
\[
spec_{L^2(\mathbb{R}) \times L^2(\mathbb{R})} (J^{-1}L)(\varepsilon) = spec_{L^2_{\text{loc}}(\mathbb{R}) \times L^2_{\text{loc}}(\mathbb{R})} (J^{-1}L)(\varepsilon).
\]
See [26, Appendix A1], for instance, for details. We remark that the left side is essential spectrum whereas the right side is point spectrum.

We advocate a Bloch wave or Floquet approach to characterize the \(L^2(\mathbb{R}) \times L^2(\mathbb{R})\) or \(L^2_{\text{loc}}(\mathbb{R}) \times L^2_{\text{loc}}(\mathbb{R})\) spectrum of \(J^{-1}L(\varepsilon)\) conveniently. Details can be found in [7,16] and references therein, whence we merely hit the main points.

For \(v \in L^2(\mathbb{R})\), let
\[
v(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-1/2}^{1/2} \left( \sum_{n \in \mathbb{Z}} \hat{v}(n + p)e^{i\alpha n} \right) e^{ip\alpha} dp =: \int_{-1/2}^{1/2} v(\alpha; p)e^{ip\alpha} dp.
\]
Namely, \(p \in [-1/2, 1/2]\) is the Bloch frequency or Floquet exponent, and \(v(p)\) is the Bloch wave. It is well defined in the Schwartz class by the Fubini theorem and the dominated convergence theorem, and it extends to \(L^2(\mathbb{R})\) by a density argument. Note that \(v(p) \in L^2(\mathbb{T})\) for each \(p \in [-1/2, 1/2]\).

The Parseval theorem asserts that
\[
\|v\|^2_{L^2(\mathbb{R})} = \|\hat{v}\|^2_{L^2(\mathbb{T})} = \int_{-1/2}^{1/2} \|v(p)\|^2_{L^2(\mathbb{T})} dp.
\]
Thus \(v \mapsto v(p)\) is an isomorphism of \(L^2(\mathbb{R})\) and \(L^2([-1/2, 1/2]; L^2(\mathbb{T}))\). Let \(\mathcal{M} : \text{dom}(\mathcal{M}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})\) denote a Fourier multiplier operator, defined as
\[
\widehat{\mathcal{M}v}(k) = m(k)\hat{v}(k) \quad \text{for a suitable } m,
\]
for instance, [28,9]. A straightforward calculation reveals that
\[
(\mathcal{M}v)(p) = e^{-ip\alpha} \mathcal{M}e^{ip\alpha}v(p) =: \mathcal{M}(p)v(p),
\]
where \(v \in L^2(\mathbb{R})\) and \(p \in [-1/2, 1/2]\). Namely, \(\mathcal{M}(p)\) is the Bloch operator. Note that \(\mathcal{M}(p) : \text{dom}(\mathcal{M}(p)) \subset L^2(\mathbb{T}) \to L^2(\mathbb{T})\) for each \(p \in [-1/2, 1/2]\). Likewise,
\[
(fv)(p) = e^{-ip\alpha} f e^{ip\alpha}v(p) \quad \text{for a suitable function } f,
\]
where \(v \in L^2(\mathbb{R})\) and \(p \in [-1/2, 1/2]\). Thus let
\[
(4.7) \quad J(\varepsilon, p) = e^{-ip\alpha} J(\varepsilon)e^{ip\alpha} \quad \text{and} \quad \mathcal{L}(\varepsilon, p) = e^{-ip\alpha} \mathcal{L}(\varepsilon)e^{ip\alpha},
\]
where \(J(\varepsilon)\) and \(\mathcal{L}(\varepsilon)\) are in [4,3] and \(p \in [-1/2, 1/2]\). Note that
\[
(4.8) \quad J(\varepsilon, p), \mathcal{L}(\varepsilon, p) : H^1(\mathbb{T}) \times H^1(\mathbb{T}) \subset L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to L^2(\mathbb{T}) \times L^2(\mathbb{T}),
\]
for each \( p \). Moreover,

\[
(4.9) \quad \text{spec}_{L^2(\mathbb{R}) \times L^2(\mathbb{R})}(J^{-1}L)(\varepsilon) = \bigcup_{p \in [-1/2, 1/2]} \text{spec}_{L^2(\mathbb{T}) \times L^2(\mathbb{T})}(J^{-1}L)(\varepsilon, p).
\]

See [26] Appendix A, for instance, for details. Therefore, \((y, \phi)(\varepsilon)\) is spectrally unstable if and only if the \(L^2(\mathbb{T}) \times L^2(\mathbb{T})\) spectrum of \((J^{-1}L)(\varepsilon, p)\) is not contained in the imaginary axis for some \( p \in [-1/2, 1/2] \). Thus we turn the attention to

\[
(4.10) \quad \lambda J(\varepsilon, p) v = L(\varepsilon, p) v, \quad \text{where } \lambda \in \mathbb{C} \quad \text{and} \quad v \in L^2(\mathbb{T}) \times L^2(\mathbb{T})
\]

for \( p \in [-1/2, 1/2] \), or alternatively,

\[
\lambda J(\varepsilon, 0)(e^{ip\alpha} v) = L(\varepsilon, 0)(e^{ip\alpha} v).
\]

Indeed, \((J^{-1}L)(\varepsilon, 0) : e^{ip\alpha} L^2(\mathbb{T}) \times e^{ip\alpha} L^2(\mathbb{T}) \rightarrow e^{ip\alpha} L^2(\mathbb{T}) \times e^{ip\alpha} L^2(\mathbb{T})\) for each \( p \in [-1/2, 1/2] \). Since

\[
y \mapsto c(\varepsilon)^2 \mathcal{H}\partial_\alpha y - (1 + \mathcal{H}y(\varepsilon)' + y(\varepsilon)\mathcal{H}\partial_\alpha + \mathcal{H}\partial_\alpha y(\varepsilon))y : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})
\]

(see the former equation of \((3.1)\)) is a Fredholm operator with index 0 for \( \varepsilon \in \mathbb{R} \) and \(|\varepsilon| \ll 1 \) [8, 10], Section 10.5], and so is \( L(\varepsilon, 0) : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T}) \times L^2(\mathbb{T}), \) and since

\[
J(\varepsilon, 0) = 1 + O(\varepsilon) : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T}) \times L^2(\mathbb{T})
\]

is invertible, the \(L^2(\mathbb{T}) \times L^2(\mathbb{T})\) spectra of \((J^{-1}L)(\varepsilon, 0)\) and, hence, \((J^{-1}L)(\varepsilon, p)\) for any \( p \in [-1/2, 1/2] \) consist of eigenvalues with finite multiplicities. Thus \((4.9)\) allows us to parametrize the \(L^2(\mathbb{R}) \times L^2(\mathbb{R})\) essential spectrum of \((J^{-1}L)(\varepsilon)\) by the one parameter family of \(L^2(\mathbb{T}) \times L^2(\mathbb{T})\) point spectra of \((J^{-1}L)(\varepsilon, p)\).

We remark that \((4.8)\) and \((4.7)\) depend analytically on \( \varepsilon \) for \( \varepsilon \in \mathbb{R} \) and \(|\varepsilon| \ll 1 \) for each \( p \in [-1/2, 1/2] \).

Observe that \((4.10)\) remains invariant under

\[
\lambda \mapsto \lambda^*, \quad v \mapsto v^* \quad \text{and} \quad p \mapsto -p.
\]

Thus it suffices to take \( p \in [0, 1/2] \).

Note that \( p = 0 \) amounts to same period perturbations as \( y(\varepsilon) \) and \( \phi(\varepsilon) \), and \( p \neq 0, \ll 1 \) amounts to long wavelength perturbations, whose effects are to vary wave properties over a large spatial scale. Moreover, \(|\lambda| \ll 1 \) is to vary wave properties over a large temporal scale. Thus we say that \((y, \phi)(\varepsilon)\) is modulationally unstable if \((4.10)\) admits a nontrivial solution for some \(|\lambda| \ll 1 \) and \( \text{Real}(\lambda) \neq 0 \) for some \( p \neq 0, \ll 1 \).

**Lemma 2** (Spectrum of \((J^{-1}L)(\varepsilon, 0)\) at the origin). When \( \varepsilon \in \mathbb{R}, \ |\varepsilon| \ll 1 \) and \( p = 0 \), zero is an eigenvalue of \((4.10)\) with algebraic multiplicity four and geometric multiplicity two.

The proof is in Appendix [A].

Thus one may attempt to examine if the four eigenvalues of \((J^{-1}L)(\varepsilon, p)\) at \( 0 \in \mathbb{C} \), where \( p = 0 \), enter the right half plane as \( p \) increases, resulting in modulational instability. Unfortunately, \((4.8)\) and \((4.7)\) are not continuously differentiable in \( p \) at \( p = 0 \), whence analytic perturbation of \((4.1)\) and others is not applicable. Indeed, if it were to depend analytically on \( p \) at \( p = 0 \) then, for instance,

\[
J(\varepsilon, p) = J(\varepsilon, 0) + ip[J(\varepsilon, 0), \alpha] - \frac{1}{2}p^2[[J(\varepsilon, 0), \alpha], \alpha] + O(p^3)
\]

as \( p \rightarrow 0 \). But \([\mathcal{H}, \alpha] \), \([\mathcal{H}, \alpha, \alpha] \), \ldots are not well defined. Instead, we take advantage of that \((4.8)\) and \((4.7)\) depend analytically on \( \varepsilon \) for \( \varepsilon \in \mathbb{R} \) and \(|\varepsilon| \ll 1 \).
Thus there are four eigenvalues of $\mathcal{L}(0, p)$ in the vicinity of the imaginary axis. This is the subject of investigation here. Since the spectrum of $\mathcal{L}(0, p)$ is symmetric about the real and imaginary axes for all $p \neq 0$, $\ll 1$, a necessary condition of spectral instability is that a pair of eigenvalues collide on the imaginary axis.

Note that $\lambda(0 + 0, \pm) = \lambda(\mp 1 + 0, \pm) = 0$ and $\lambda(n + p, \pm) \neq 0$ otherwise.

Thus there are four eigenvalues of $(\mathcal{J}^{-1}\mathcal{L})(0, 0)$ at $0 \in \mathbb{C}$ (see also Lemma 2). Since the eigenvalues vary continuously with $p$, $\lambda(0 + p, \pm)$ and $\lambda(\mp 1 + p, \pm)$ are the four eigenvalues of $(\mathcal{J}^{-1}\mathcal{L})(0, p)$ in the vicinity of $0 \in \mathbb{C}$ for $p \neq 0, \ll 1$. Note that $\lambda(\mp 1 + p, \pm)$ depend analytically on $p$ and

$$\lambda(\mp 1 + p, \pm) = \frac{i}{2}p + O(p^2) \quad \text{as} \quad p \to 0.
$$

Thus $\lambda(\mp 1 + p, \pm)$ collide at $\frac{i}{2}p$ to the order of $p$ for $p \neq 0, \ll 1$, and they may result in modulational instability for $\varepsilon \in \mathbb{R}$ and $|\varepsilon| \neq 0, \ll 1$. By contrast,

$$\lambda(0 + p, \pm) = i(\pm \sqrt{p} + p)
$$

are continuous but not continuously differentiable in $p$ at $p = 0$. On the other hand, $\lambda(0 + p, \pm)$ are away from $\lambda(\mp 1 + p, \pm)$ and from each other for $p \neq 0, \ll 1$, and they will remain on the imaginary axis for $|\varepsilon|$ sufficiently small. Thus we restrict the attention to (4.10) when $\lambda$ is in the vicinity of $\frac{i}{2}p$ for such $p \neq 0, \ll 1$ for $\varepsilon \in \mathbb{R}$ and $|\varepsilon| \neq 0, \ll 1$.

Therefore, for $p \neq 0, \ll 1$, let

$$\text{spec}_{L^2(\mathbb{T}) \times L^2(\mathbb{T})}(\mathcal{J}^{-1}\mathcal{L})(0, p) = \Sigma(0, p) \bigcup \Sigma'(0, p),
$$

where $\Sigma(0, p) = \{\lambda(\mp 1 + p, \pm)\}$ and $\Sigma(0, p) \bigcap \Sigma'(0, p) = \emptyset$.

Thus the spectrum of $(\mathcal{J}^{-1}\mathcal{L})(0, p)$ separates into two simple eigenvalues in the vicinity of $\frac{i}{2}p$ and others away from $\frac{i}{2}p$. Since (4.3) and (4.7) depend analytically on $\varepsilon$ for $\varepsilon \in \mathbb{R}$ and $|\varepsilon| \ll 1$, and since $\mathcal{J}(\varepsilon, p) : L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to L^2(\mathbb{T}) \times L^2(\mathbb{T})$ is invertible, whence

$$\|\mathcal{J}^{-1}\mathcal{L}(\varepsilon, p) - \mathcal{J}^{-1}\mathcal{L}(0, p)\|_{L^2(\mathbb{T}) \times L^2(\mathbb{T}) \to L^2(\mathbb{T}) \times L^2(\mathbb{T})} = O(\varepsilon)
$$

as $\varepsilon \to 0$, analytic perturbation theory (see [10] Theorem 7.3.1, for instance) asserts that for $p \neq 0, \ll 1$ so that (5.5) holds, for $\varepsilon \in \mathbb{R}$ and $|\varepsilon| \ll 1$,

$$\text{spec}_{L^2(\mathbb{T}) \times L^2(\mathbb{T})}(\mathcal{J}^{-1}\mathcal{L})(\varepsilon, p) = \Sigma(\varepsilon, p) \bigcup \Sigma'(\varepsilon, p),
$$

where $\Sigma(\varepsilon, p)$ consists of two eigenvalues near $\frac{i}{2}p$ and $\Sigma(\varepsilon, p) \bigcap \Sigma'(\varepsilon, p) = \emptyset$.

We remark that the eigenvalues in $\Sigma(\varepsilon, p)$ are simple for $\varepsilon \in \mathbb{R}$ and $|\varepsilon| \ll 1$. 

When $\varepsilon = 0$, (4.10) becomes

$$(5.1) \quad \lambda \mathbf{v} = \mathcal{L}(0, p) \mathbf{v} = e^{-ip\alpha} \left( \frac{\partial}{\partial \alpha} \begin{pmatrix} \mathcal{H} & \mathcal{J} \\ -1 & \mathcal{H} \end{pmatrix} \right) e^{ip\alpha} \mathbf{v},$$

and a Fourier analysis calculation reveals that

$$(5.2) \quad \lambda(n + p, \pm) = i(n + p \pm \sqrt{n + p}) \quad \text{and} \quad \mathbf{v}(n + p, \pm) = \left( \mp i \sqrt{\frac{n + p}{1}} \right) e^{i\alpha},$$

where $n \in \mathbb{Z}$ and $p \in [0, 1/2]$, are the $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ eigenvalues and eigenfunctions of $(\mathcal{J}^{-1}\mathcal{L})(0, p)$. All the eigenvalues lie in the imaginary axis. Thus the Stokes wave of zero amplitude is spectrally stable. As $|\varepsilon|$ increases, the eigenvalues move around in $\mathbb{C}$ and they may leave the imaginary axis, resulting in spectral instability. This is the subject of investigation here. Since the spectrum of $(\mathcal{J}^{-1}\mathcal{L})(\varepsilon, \pm p)$ is symmetric about the real and imaginary axes for all $\varepsilon \in \mathbb{R}$ and $|\varepsilon| \ll 1$ for all $p \in [0, 1/2]$, a necessary condition of spectral instability is that a pair of eigenvalues collide on the imaginary axis.
In the finite depth, we proceed in like manner, and
\[
\lambda(n+p, \pm) = i(\sqrt{g} \tanh h(n+p) \pm \sqrt{g(n+p) \tanh(h(n+p))})
\]
where \(n \in \mathbb{Z}\) and \(p \in [0, 1/2]\), are the eigenvalues of the linearized operator about zero amplitude. Note that \(\lambda(0 + 0, \pm) = \lambda(\mp 1 + 0, \pm) = 0\). Also note that when \(p \neq 0, \ll 1\), \(\lambda(\pm 1 + p, \mp)\) are simple eigenvalues colliding at the order of \(p\), whereby they may result in modulational instability as the amplitude parameter varies, whereas \(\lambda(0 + p, \pm)\) are away from other eigenvalues, and from each other, remaining on the imaginary axis for sufficiently small amplitude. Bridges and Mielke \cite{bridges1989} make use of it in their proof.

**Notation.** For \(\left(\frac{u_1}{v_1}, \frac{u_2}{v_2}\right) \in L^2(\mathbb{T}) \times L^2(\mathbb{T})\), let
\[
\left\langle \left(\frac{u_1}{v_1}, \frac{u_2}{v_2}\right), \left(\frac{u_1}{v_1}, \frac{u_2}{v_2}\right) \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} (u_1 u_2^* + v_1 v_2^*)(\alpha) \, d\alpha.
\]

**Lemma 3** (Spectrum of \((J^{-1}L)(0,p)\) near \(ip\)). If \(\lambda = \frac{1}{2}ip + O(p^2)\) for \(p \neq 0, \ll 1\), then
\[
\ker(L(0,p) - \lambda I) = \text{span}\{v_\pm\}, \quad \text{where } v_\pm := \left(\frac{1(\mp 1 + \frac{1}{2}p)}{\mp 1 + \frac{1}{2}p}\right) e^{\mp i\alpha} + O(p^2)
\]
are linearly independent, and
\[
\ker(L(0,p) - \lambda I)^\dagger = \text{span}\{w_\pm\}, \quad \text{where } w := \left(\frac{-1}{\mp 1 + \frac{1}{2}p}\right) e^{\mp i\alpha} + O(p^2).
\]
Here and elsewhere, the dagger means the adjoint. Moreover,
\[
(L(0,p) - \lambda I)v = f, \quad \text{where } f \in L^2(\mathbb{T}) \times L^2(\mathbb{T}),
\]
is solvable, provided that \(\langle f, w_\pm \rangle = 0\).

**Proof.** Clearly, (5.8a) holds, where \(v_\pm = v(\mp 1 + p, \pm)\) (see (5.2)), and (5.8b) holds because
\[
(L_0(p) - \lambda I)^\dagger = \sigma (L_0(p) - \lambda I) \sigma, \quad \text{where } \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
The solution condition for \((L(0,p) - \lambda I)\) comes from the Fredholm alternative. \qed

6. **Analytic perturbation in the amplitude parameter**

For \(p \neq 0, \ll 1\), so that (5.5) holds, for \(\varepsilon \in \mathbb{R}\) and \(|\varepsilon| \ll 1\), we turn the attention to (4.10), where \(\lambda \in \Sigma(\varepsilon, p) \subset \mathbb{C}\) is in the vicinity of \(\frac{1}{2}ip\). Since (4.8) and (4.7) depend analytically on \(\varepsilon\) and since the eigenvalues in \(\Sigma(\varepsilon, p)\) are simple, analytic perturbation theory for linear operators (see [10] Section 7.6, for instance) ensures that the eigenvalues and the eigenfunctions depend analytically on \(\varepsilon\). See also [29] for a direct proof. Thus we write
\[
\lambda = \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \varepsilon^3 \lambda_3 + O(\varepsilon^4) \quad \text{and} \quad v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 + O(\varepsilon^4)
\]
as \(\varepsilon \to 0\), where
\[
\lambda_0 = \frac{1}{2}ip + O(p^2) \quad \text{as } p \to 0,
\]
and
\[
\lambda = \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \varepsilon^3 \lambda_3 + O(\varepsilon^4)
\]
as \(\varepsilon \to 0\), where
\[
\lambda_0 = \frac{1}{2}ip + O(p^2) \quad \text{as } p \to 0.
\]
\( \lambda_1, \lambda_2, \cdots \in \mathbb{C} \) and \( v_0, v_1, v_2, \cdots \in L^2(\mathbb{T}) \times L^2(\mathbb{T}) \) are to be determined. Substituting in (4.10), we arrive at

\[
(6.2) \quad (\lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \cdots)(I + \varepsilon J_1(p) + \varepsilon^2 J_2(p) + \cdots)(v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots) = (L_0(p) + \varepsilon L_1(p) + \varepsilon^2 L_2(p) + \cdots)(v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots),
\]

where

\[
J_k(p) = e^{-ip\alpha} J_k e^{ip\alpha} \quad \text{and} \quad L_k(p) = e^{-ip\alpha} L_k e^{ip\alpha},
\]

and \( J_k \) and \( L_k \) are in (4.6). We advocate Rayleigh-Schrödinger perturbation theory in quantum mechanics and solve (6.2) successively. This is reminiscent of spectral perturbation analysis of [6] and others with respect to \( p \).

At the order of 1, (6.2) becomes

\[
\lambda_0 v_0 = L_0(p)v_0,
\]

and Lemma 3 implies that

\[
v_0 = c_+ v_+ + c_- v_- \quad \text{for some } c_+ \in \mathbb{C},
\]

where \( v_\pm \) are in (5.8a).

At the order of \( \varepsilon \), we gather that

\[
\lambda_1 v_0 = (L_0(p) - \lambda_0 I)v_1 + (L_1(p) - \lambda_0 J_1(p))v_0.
\]

This is solvable, by the Fredholm alternative, provided that \( (L_0(p) - \lambda_0 I)v_1, w_\pm \) is zero, where \( w_\pm \) are in (5.8b), whence, suppressing \( p \) for simplicity of notation,

\[
(6.3) \quad \lambda_1 \left(\begin{array}{c}
\langle v_+, w_+ \rangle \\
\langle v_+, w_- \rangle
\end{array}\right) \left(\begin{array}{c}
\langle v_-, w_+ \rangle \\
\langle v_-, w_- \rangle
\end{array}\right) \left(\begin{array}{c}
c_+
\\
c_-
\end{array}\right) = \left(\begin{array}{c}
\langle (L_1 - \lambda_0 J_1)v_+, w_+ \rangle \\
\langle (L_1 - \lambda_0 J_1)v_-, w_- \rangle
\end{array}\right) \left(\begin{array}{c}
c_+
\\
c_-
\end{array}\right).
\]

A straightforward calculation reveals that (see Appendix B for details)

\[
(L_1(p) - \lambda_0 J_1(p))v_\pm = \left(\begin{array}{c}
\mp\frac{1}{2} \pm i(1 \mp p) \\
\mp i(1 \mp p)
\end{array}\right) + \left(\begin{array}{c}
0
\\
\pm 2i(1 \mp p)
\end{array}\right) e^{\pm 2i\alpha} + O(p^2),
\]

whence \( \langle L_1(p) - \lambda_0 J_1(p), v_\pm, w_\pm \rangle = 0 + O(p^2) \). Also,

\[
\langle v_\pm, w_\pm \rangle = \pm i(2 \mp p) + O(p^2) \quad \text{and} \quad \langle v_\pm, w_\mp \rangle = 0 + O(p^2).
\]

Therefore, (6.3) becomes

\[
(6.5) \quad \lambda_1 = 0 + O(p^2)
\]

and

\[
(6.6) \quad v_1 = -(L_0(p) - \lambda_0 I)^{-1}(L_1(p) - \lambda_0 J_1(p))(c_+ v_+ + c_- v_-).
\]

A straightforward calculation reveals that (see Appendix B for details)

\[
-(L_0(p) - \lambda_0 I)^{-1}(L_1(p) - \lambda_0 J_1(p))v_+ = \left(\begin{array}{c}
\frac{i(1 - \frac{1}{2} q)}{1 - \frac{3}{2} q}
\\
\frac{-2i(1 - p)}{2 - \frac{3}{2} q}
\end{array}\right) e^{-2i\alpha} + O(p^2)
\]

and

\[
-(L_0(p) - \lambda_0 I)^{-1}(L_1(p) - \lambda_0 J_1(p))v_- = \left(\begin{array}{c}
\frac{-i(1 + \frac{3}{2} q)}{1 + \frac{3}{2} q}
\\
\frac{2i(1 + p)}{2 + \frac{3}{2} q}
\end{array}\right) e^{2i\alpha} + O(p^2).
\]

Of course, (6.6) is determined up to addition by an element of ker\((L_0(p) - \lambda_0 I)\). Any element in \( \text{span}\{v_\pm\} \), however, is to redefine \( v_0 \). Thus (6.6) can be uniquely determined, orthogonal to \( v_\pm \).

To proceed, at the order of \( \varepsilon^2 \), we gather that

\[
\lambda_2 v_0 + \lambda_1 (v_1 + J_1(p)v_0) = (L_0(p) - \lambda_0 I)v_2 + (L_1(p) - \lambda_0 J_1(p))v_1 + (L_2(p) - \lambda_0 J_2(p))v_0.
\]
This is solvable, likewise, by the Fredholm alternative, provided that $\langle (L_0(p) - \lambda_0 I)v_2, w_+ \rangle = 0$, whence recalling (6.5) and (6.6) and suppressing $p$ for simplicity of notation,
\[(6.7)\]
\[\lambda_2 \begin{pmatrix} \langle v_+, w_+ \rangle \\ \langle v_+, w_- \rangle \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = \begin{pmatrix} \langle (-L_1 - \lambda_0 J_1)(L_0 - \lambda_0)^{-1}(L_1 - \lambda_0 J_1) + (L_2 - \lambda_0 J_2) \rangle v_+, w_+ \\ \langle (-L_1 - \lambda_0 J_1)(L_0 - \lambda_0)^{-1}(L_1 - \lambda_0 J_1) + (L_2 - \lambda_0 J_2) \rangle v_-, w_- \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix}.
\]
A straightforward calculation reveals that (see Appendix B for details)
\[-(L_1 - \lambda_0 J_1)(L_0 - \lambda_0)^{-1}(L_1 - \lambda_0 J_1)v_+ = \begin{pmatrix} p \alpha \\ (3 + 2p) \alpha \end{pmatrix} e^{i\alpha} + \begin{pmatrix} -p \alpha/2 \\ (1 + 2p) \alpha \end{pmatrix} e^{-i\alpha} + \begin{pmatrix} 0 \\ -(6 + 8p) \alpha \end{pmatrix} e^{3i\alpha} + O(p^2),
\[-(L_1 - \lambda_0 J_1)(L_0 - \lambda_0)^{-1}(L_1 - \lambda_0 J_1)v_- = \begin{pmatrix} -p \alpha/2 \\ (3 - 5p) \alpha \end{pmatrix} e^{-i\alpha} + \begin{pmatrix} 0 \\ -(1 + 2p) \alpha \end{pmatrix} e^{i\alpha} + \begin{pmatrix} 0 \\ (6 - 8p) \alpha \end{pmatrix} e^{-3i\alpha} + O(p^2)
\]
and
\[(L_2 - \lambda_0 J_2)v_\pm = \begin{pmatrix} -p/2 + 3p/2 \\ (3 + 3p) \alpha \end{pmatrix} e^{+i\alpha} + \begin{pmatrix} p/2 + 1 + 3p/2 \\ i(3 + 3p) \alpha \end{pmatrix} e^{-i\alpha} + \begin{pmatrix} 0 \\ i(3 + 3p) \alpha \end{pmatrix} e^{+3i\alpha} + O(p^2).
\]
Recall (6.3), and (6.7) becomes
\[
\begin{pmatrix} -1 + 2q + O(p^2) - (2 - p + O(p^2))\lambda_2 i \\ 1 - p + O(p^2) \end{pmatrix} \begin{pmatrix} 1 - p + O(p^2) \\ -(1 + 3/2q + O(p^2) + (2 + p + O(p^2))\lambda_2 i \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix} = 0,
\]
whence
\[
\lambda_2 = \pm \frac{3}{4} p - i p + O(p^2),
\]
implies modulational instability.

We summarize our conclusion.

**Theorem 4** (The Benjamin–Feir instability). A Stokes wave of sufficiently small amplitude in the infinite depth is modulationally unstable.

7. Remarks on the spectrum away from the origin

Recall (5.2), and a straightforward calculation reveals that
\[(7.1a)\]
\[\lambda(n^2, +) = \lambda((n + 1)^2, -) = i(n^2 + n)
\]
and
\[(7.1b)\]
\[\lambda((n - 1/2)^2, +) = \lambda((n + 1/2)^2, -) = i(n^2 - 1/4)
\]
for any $n \in \mathbb{N}$. Thus when $\varepsilon = 0$, there are infinitely many collisions of pairs of purely imaginary eigenvalues of $J^{-1}L$. By the way, thanks to the symmetry of the spectrum, it suffices to consider eigenvalues whose imaginary part is positive. For $\varepsilon \in \mathbb{R}$ and $|\varepsilon| \ll 1$, we proceed as in the previous sections and solve (6.2) and, hence, (1.10), where $\lambda$ is close to (7.11). When $\varepsilon = 0$ and $p \neq 0$, $\ll 1$, let
\[(7.2)\]
\[\lambda_0 = i(n^2 + n) + O(p) \quad \text{or} \quad i(n^2 - 1/4) + O(p)
\]denote a simple eigenvalue of $J^{-1}L$ near (7.11).
At the order of 1, we gather that
\[ \lambda_0 v_0 = \mathcal{L}_0(p)v_0, \]
and
\[ v_0 = c_+ v_+ + c_- v_- \quad \text{for some } c_\pm \in \mathbb{C}, \]
where
\[ (7.3a) \quad v_+ := \left( \frac{-in}{1} \right) e^{in^2\alpha} + O(p) \quad \text{and} \quad v_- := \left( \frac{i(n+1)}{1} \right) e^{-i(n+1)^2\alpha} + O(p) \]
for \( \lambda_0 = i(n^2 + n) + O(p) \); and
\[ (7.3b) \quad v_+ := \left( \frac{-i(n-1/2)}{1} \right) e^{i(n^2-n)\alpha} + O(p) \quad \text{and} \quad v_- := \left( \frac{i(n+1/2)}{1} \right) e^{i(n^2+n)\alpha} + O(p) \]
for \( \lambda_0 = i(n^2 - 1/4) + O(p) \).
At the order of \( \varepsilon \),
\[ \lambda_1 v_0 = (\mathcal{L}_0(p) - \lambda_0 1)v_1 + (\mathcal{L}_1(p) - \lambda_0 1)(p)v_0. \]
This is solvable, by the Fredholm alternative, provided that
\[ \lambda_1 \langle v_0, w_\pm \rangle = \langle (\mathcal{L}_1(p) - \lambda_0 1)(p)v_0, w_\pm \rangle, \]
where
\[ (7.4a) \quad w_+ := \left( \frac{-1}{-in} \right) e^{in^2\alpha} + O(p) \quad \text{and} \quad w_- := \left( \frac{-1}{i(n+1)} \right) e^{-i(n+1)^2\alpha} + O(p) \]
for \( \lambda_0 = i(n^2 + n) + O(p) \); and
\[ (7.4b) \quad w_+ := \left( \frac{-1}{-i(n-1/2)} \right) e^{i(n^2-n)\alpha} + O(p) \quad \text{and} \quad w_- := \left( \frac{-1}{i(n+1/2)} \right) e^{i(n^2+n)\alpha} + O(p) \]
for \( \lambda_0 = i(n^2 - 1/4) + O(p) \).

When \( \lambda_0 = i(n^2 + n) + O(p) \), \( 7.4a \) is supported at the wave numbers \( n^2 \) and \( (n+1)^2 \), whereas \( (\mathcal{L}_1(p) - \lambda_0 1)(p)v_\pm \) are supported at the wave numbers \( n^2 \pm 1 \) and \( (n+1)^2 \pm 1 \) by \( 4.6 \) and \( 7.3a \). Since \((n+1)^2 - n^2 \geq 3\) for any \( n \in \mathbb{N} \),
\[ \langle (\mathcal{L}_1(p) - \lambda_0 1)(p)v_s, w_s' \rangle = 0, \quad \text{where } s, s' = \pm 1. \]
In case \( \lambda_0 = i(n^2 - 1/4) + O(p) \), likewise, \( 7.4b \) is supported at the wave numbers \( n^2 \pm n \), whereas \( (\mathcal{L}_1(p) - \lambda_0 1)(p)v_\pm \) at \( (n^2 \pm n) \pm 1 \) by \( 4.6 \) and \( 7.3a \). Since \((n^2 + n) - (n^2 - n) \geq 2\) for \( n \in \mathbb{N} \),
\[ \langle (\mathcal{L}_1(p) - \lambda_0 1)(p)v_s, w_s' \rangle = 0, \quad \text{where } s, s' = \pm 1. \]
Therefore,
\[ \lambda_1 = 0 + O(p) \]
and \( v_1 = -(\mathcal{L}_0(p) - \lambda_0 1)^{-1}(\mathcal{L}_1(p) - \lambda_0 1)(p)v_0. \)
To proceed, at the order of \( \varepsilon^2 \), we gather that
\[ \lambda_2 v_0 + \lambda_1 (v_1 + \mathcal{J}_1(p)v_0) = (\mathcal{L}_0(p) - \lambda_0 1)v_2 + (\mathcal{L}_1(p) - \lambda_0 1)(p)v_1 + (\mathcal{L}_2(p) - \lambda_0 1)(p)v_0. \]
This is solvable, likewise, provided that
\[ \lambda_2 \langle v_0, w_\pm \rangle = \langle (\mathcal{L}_1 - \lambda_0 1)(\mathcal{L}_0 - \lambda_0)^{-1}(\mathcal{L}_1 - \lambda_0 1)v_0, w_\pm \rangle + \langle (\mathcal{L}_2 - \lambda_0 1)(p)v_0, w_\pm \rangle. \]
The right side vanishes except when
\[ \lambda_0 = \frac{3}{4} i + O(p), \quad \text{where } \lambda \left(0 + \frac{1}{4}, +\right) = \lambda \left(2 + \frac{1}{4}, -\right) = \frac{3}{4} i. \]
Thus let
\[ v_+ := \left( -\frac{1}{2}i \right) + O(p) \quad \text{and} \quad v_- := \left( \frac{3}{2}i \right) e^{2i\alpha} + O(p) \];
also,
\[ w_+ := \left( -\frac{1}{2}i \right) + O(p) \quad \text{and} \quad w_- := \left( -\frac{1}{2}i \right) e^{2i\alpha} + O(p), \]
and we calculate
\[ \langle v_+, w_+ \rangle = i + O(p), \quad \langle v_-, w_- \rangle = -3i + O(p) \quad \text{and} \quad \langle v_\pm, w_\mp \rangle = 0 + O(p). \]
A straightforward calculation reveals that (see Appendix B for details)
\[
(L_1(1/4 + p) - \lambda_0 J_1(1/4 + p))v_+ = \left( 0 \right. - \frac{1}{2}i \left. \right) e^{i\alpha} - \frac{1}{4} \left( \frac{3}{2}i \right) e^{-i\alpha} + O(p),
\]
\[
(L_1(1/4 + p) - \lambda_0 J_1(1/4 + p))v_- = \left( 0 \right. - \frac{1}{2}i \left. \right) e^{3i\alpha} + 9 \left( \frac{1}{2}i \right) e^{i\alpha} + O(p),
\]
and
\[
-(L_0 - \lambda_0 I)^{-1}(L_11 - \lambda_0 J_1)v_+ = -\left( \frac{1}{4} \right. - \frac{1}{2}i \left. \right) e^{3i\alpha} + 3\left( \frac{1}{2}i \right) e^{2i\alpha} + O(p),
\]
\[
-(L_0 - \lambda_0 I)^{-1}(L_1 - \lambda_0 J_1)v_- = \left( \frac{3}{2}i \right) e^{3i\alpha} + 9 \left( \frac{1}{2}i \right) e^{i\alpha} + O(p).
\]
Also,
\[
-(L_1 - \lambda_0 J_1)(L_0 - \lambda_0)^{-1}(L_1 - \lambda_0 J_1)v_+ = \left( \frac{0}{3}i \right) e^{2i\alpha} - \frac{5}{16} \left( \frac{3}{2}i \right) e^{i\alpha} - \frac{1}{8} \left( \frac{3}{i} \right) + \left( \frac{0}{-\frac{1}{2}i(2 + \frac{1}{4})} \right) e^{3i\alpha} + O(p),
\]
\[
-(L_1 - \lambda_0 J_1)(L_0 - \lambda_0)^{-1}(L_1 - \lambda_0 J_1)v_- = \left( \frac{3}{2}i(1 + \frac{1}{4}) \right) e^{4i\alpha} + \frac{13}{4} \left( -i \right) \left( 3 + \frac{1}{7} \right) + \left( \frac{0}{-\frac{1}{2}i(2 + \frac{1}{4})} \right) e^{2i\alpha} + \left( \frac{3}{2}i \right) e^{2i\alpha} + \frac{27}{16} \left( \frac{3}{i} \right) + O(p),
\]
and
\[
(L_2 - \lambda_0 J_2)v_+ = \left( \frac{1}{10} \right. - \frac{1}{2}i(2 + \frac{1}{4}) \left. \right) e^{2i\alpha} - \frac{1}{2} \left( \frac{3}{i} \right) e^{-2i\alpha} + O(p),
\]
\[
(L_2 - \lambda_0 J_2)v_- = \left( \frac{9}{4} \right. - \frac{3}{2}i(4 + \frac{1}{4}) \left. \right) e^{4i\alpha} + \frac{9}{4} \left( \frac{1}{2}i \right) + O(p).
\]
We calculate that
\[ \langle -(L_1 - \lambda_0 J_1)(L_0 - \lambda_0)^{-1}(L_1 - \lambda_0 J_1) + (L_2 - \lambda_0 J_2)v_+, v_- \rangle = 0, \]
whence $\lambda_2$ is purely imaginary (see (6.7)), implying spectral stability.

We summarize our conclusion.

**Theorem 5** (Spectral stability away from the origin). A Stokes wave of sufficiently small amplitude in the infinite depth is spectrally stable away from the origin of $\mathbb{C}$ up to the quadratic order of infinitesimally small values of the amplitude parameter.
Differentiating (3.1) (where \( g = 1 \)) with respect to \( \alpha \) and evaluating \( y = y(\varepsilon) \) and \( \phi = \phi(\varepsilon) \),
\[
c(\varepsilon)^2 \mathcal{H} \partial \alpha y(\varepsilon)' = (1 + \mathcal{H} y(\varepsilon)') + y(\varepsilon) \mathcal{H} \partial \alpha + \mathcal{H} \partial \alpha y(\varepsilon))y(\varepsilon)' \quad \text{and} \quad \partial \alpha \phi(\varepsilon)' = c(\varepsilon) \mathcal{H} \partial \alpha y(\varepsilon)',
\]
whence we infer from (4.3) that
\[
\mathcal{L}(\varepsilon, 0)v_1 := \mathcal{L}(\varepsilon, 0) \begin{pmatrix} y(\varepsilon)' \\ \phi(\varepsilon)' \end{pmatrix} = 0 = \partial \mathcal{J}(\varepsilon, 0)v_1.
\]
Differentiating (3.1) with respect to \( \varepsilon \) and evaluating \( y = y(\varepsilon) \) and \( \phi = \phi(\varepsilon) \), likewise,
\[
\mathcal{L}(\varepsilon, 0)v_2 := \mathcal{L}(\varepsilon, 0) \begin{pmatrix} \partial \varepsilon y(\varepsilon) \\ \partial \varepsilon \phi(\varepsilon) \end{pmatrix} = -\partial \varepsilon c(\varepsilon) \begin{pmatrix} y(\varepsilon)' \\ \phi(\varepsilon)' \end{pmatrix} = -\partial \varepsilon c(\varepsilon) \mathcal{J}(\varepsilon, 0)v_1.
\]
Moreover, a straightforward calculation reveals that
\[
\mathcal{L}(\varepsilon, 0)v_3 := \mathcal{L}(\varepsilon, 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 = \mathcal{J}(\varepsilon, 0)v_3
\]
and
\[
\mathcal{L}(\varepsilon, 0)v_4 := \mathcal{L}(\varepsilon, 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 + 2\mathcal{H}y'(\varepsilon) \end{pmatrix} = \mathcal{J}(\varepsilon, 0)v_3.
\]
To recapitulate,
\[
v_1, v_3 \in \ker(\mathcal{J}^{-1}\mathcal{L})(\varepsilon, 0), \quad v_2, v_4 \in \ker(\mathcal{J}^{-1}\mathcal{L})^2(\varepsilon, 0) \quad \text{and} \quad v_2, v_4 \notin \ker(\mathcal{J}^{-1}\mathcal{L})(\varepsilon, 0).
\]
Clearly, \( v_1 \) and \( v_3 \) are linearly independent. When \( \varepsilon \in \mathbb{R} \) and \( |\varepsilon| \ll 1 \), since the kernel of
\[
y \mapsto c(\varepsilon)^2 \mathcal{H} \partial \alpha y - (1 + \mathcal{H} y(\varepsilon)') + y(\varepsilon) \mathcal{H} \partial \alpha + \mathcal{H} \partial \alpha y(\varepsilon))y : L^2(\mathbb{T}) \times L^2(\mathbb{T})
\]
is one dimensional and spanned by \( y(\varepsilon)' \) and, hence, \( \ker(\mathcal{J}^{-1}\mathcal{L})(\varepsilon, 0) \) is at most two dimensional, we deduce that \( \ker(\mathcal{J}^{-1}\mathcal{L})(\varepsilon, 0) = \text{span}\{v_1, v_3\} \). Moreover, \( \ker(\mathcal{J}^{-1}\mathcal{L})^2(\varepsilon, 0) \setminus \ker(\mathcal{J}^{-1}\mathcal{L})(\varepsilon, 0) = \text{span}\{v_2, v_4\} \) by the Fredholm alternative.

**Appendix B. Assorted calculations**

Recall (4.7), (4.6) and (6.1), and we calculate that
\[
(\mathcal{L}_0(p) - \lambda_0 \mathbb{I})e^{i\alpha} = e^{-ip\alpha} \begin{pmatrix} \partial \alpha - \frac{1}{2}pi \\ \partial \alpha - \frac{1}{2}pi \end{pmatrix} \begin{pmatrix} \mathcal{H} \partial \alpha \\ \mathcal{H} \partial \alpha \end{pmatrix} e^{ip\alpha} e^{i\alpha} + O(p^2)
\]
whence
\[
(\mathcal{L}_0(p) - \lambda_0 \mathbb{I})^{-1}e^{i\alpha} = 1 + \frac{n + \text{sgn}(n + p)}{n^2 - |n - n|^2} \begin{pmatrix} (n + \frac{1}{2}pi) \\ (n + \frac{1}{2}pi) \end{pmatrix} e^{i\alpha} + O(p^2).
\]
Recall (4.7) and (4.6), and we calculate that
\[
\mathcal{L}_1(p)e^{i\alpha} = e^{-ip\alpha} \begin{pmatrix} 0 \\ -(\cos \alpha (1 + \mathcal{H} \partial \alpha) + \mathcal{H} \partial \alpha \cos \alpha) \end{pmatrix} e^{ip\alpha} e^{i\alpha}
\]
whence
\[
\mathcal{L}_1(p)e^{i\alpha} = e^{-ip\alpha} \begin{pmatrix} 0 \\ 0 \end{pmatrix} e^{ip\alpha} e^{i\alpha}
\]
and

$$J_1(p)e^{i\alpha} = e^{-ip\alpha} \begin{pmatrix} \cos \alpha + \sin \alpha \mathcal{H} & 0 \\ - (\cos \alpha \mathcal{H} + \mathcal{H} \cos \alpha) & \cos \alpha - \mathcal{H} \sin \alpha \end{pmatrix} e^{ip\alpha} e^{i\alpha}$$

$$= \begin{pmatrix} \frac{1}{2} e^{i(n+1)\alpha} (1 - \text{sgn}(n+p)) & 0 \\ \frac{1}{2} e^{i(n-1)\alpha} (1 + \text{sgn}(n+p)) & \frac{1}{2} e^{i(n+1)\alpha} (1 + \text{sgn}(n+1+p)) + \frac{1}{2} e^{i(n-1)\alpha} (1 - \text{sgn}(n+1+p)) \end{pmatrix} e^{i\alpha}.$$  

Continuing,

$$L_2(p)e^{i\alpha} = e^{-ip\alpha} \begin{pmatrix} \frac{1}{2} \partial_\alpha & 0 \\ \frac{1}{2} \partial_\alpha & \frac{1}{2} \partial_\alpha \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ - \frac{1}{2} e^{i(n+2)\alpha} (2 + |n+p| + |n+2+p|) & 0 \end{pmatrix} e^{i\alpha}$$

and

$$J_2(p)e^{i\alpha} = e^{-ip\alpha} \begin{pmatrix} 2(\cos 2\alpha + \sin 2\alpha \mathcal{H}) & 0 \\ -2(\cos 2\alpha \mathcal{H} + \mathcal{H} \cos 2\alpha) & 2(\cos 2\alpha - \mathcal{H} \sin 2\alpha) \end{pmatrix} e^{ip\alpha} e^{i\alpha}$$

$$= \begin{pmatrix} e^{i(n+2)\alpha} (1 - \text{sgn}(n+p)) + e^{i(n-2)\alpha} (1 + \text{sgn}(n+2+p)) & 0 \\ ie^{i(n+2)\alpha} (\text{sgn}(n+p) + \text{sgn}(n+2+p)) + e^{i(n-2)\alpha} (1 - \text{sgn}(n+2+p)) & e^{i(n+2)\alpha} (1 + \text{sgn}(n+2+p)) + e^{i(n-2)\alpha} (1 - \text{sgn}(n-2+p)) \end{pmatrix} e^{i\alpha}.$$  

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