Moduli Space for Conifolds as Intersection of Orthogonal D6 branes

Robert de Mello Koch\textsuperscript{b}, Kyungho Oh\textsuperscript{a} and Radu Tatar\textsuperscript{b}

\textsuperscript{a} Dept. of Mathematics, University of Missouri-St. Louis, St. Louis, MO 63121, USA
oh@math.umsl.edu
\textsuperscript{b} Dept. of Physics, Brown University, Providence, RI 02912, USA
robert,tatar@het.brown.edu

Abstract

We show that a system of parallel D3 branes near a conifold singularity can be mapped onto an intersecting configuration of orthogonal branes in type IIA string theory. Using this brane configuration, we analyze the Higgs moduli space of the associated field theory. The dimension of the Higgs moduli space is computed from a geometrical analysis of the conifold singularity. Our results provide evidence for an extended s-rule. In addition, a discrepancy between the prediction of the brane configuration and the result obtained from a geometrical analysis is noted. This discrepancy can be traced back to worldsheet instanton effects.
1 Introduction

In the last few years it has become clear that gauge theory and gravity are complementary descriptions of a single theory. Insights which have clarified and motivated these important ideas have largely been developed using the solitonic brane solutions of M theory and string theory. In particular, configurations containing NS5 fivebranes and D branes in string theory provide a useful way of studying supersymmetric gauge field theory in different dimensions and with different amounts of unbroken supersymmetries (see [8] for a detailed review and with a complete set of references up to February 1998).

As an example of the power of the brane approach, we mention that these brane configurations provide techniques which may be used to derive large classes of Seiberg dualities (for $\mathcal{N} = 1$ theories) and to obtain exact results after lifting to M theory (for both $\mathcal{N} = 1, 2$ theories)[4]. Interesting recent work in this field has focused on the study of D branes which are finite in one direction [11, 12, 13, 14, 15, 16, 17, 18, 27]. Another variant of this approach is provided by Brane Boxes which allow the study of D branes which are finite in two spacetime direction [19, 20, 21, 22, 23, 24]. These studies have provided new insights into finite gauge theories and chiral four dimensional gauge theories among other things.

The use of brane configurations to describe the field theory realized on the world-volume of D branes on orbifold singularities has been given in [28, 2] (see [22] for a connection with Brane Box Models). The study of these brane configurations is interesting because they provide simple examples of field theories with a reduced amount of supersymmetry. Recently, in an extension of these ideas, D branes on non-orbifold singularities have been considered. The conifold singularity has been analyzed in [30] where an infrared theory on the worldvolume of D3 branes was proposed. This provides a novel extension of the original AdS-CFT correspondence to a case with reduced supersymmetry and thus where the compact space is not even locally $S^5$. Other results for the case of non-orbifold singularities and their connection to field theories in three and four dimensions have been obtained in [31, 32, 33, 36, 37, 38, 39].

Motivated by these preliminary studies, a more systematic way of studying D branes in the presence of conifold singularities has been developed in [1, 2]. These authors have exploited the fact that the conifold singularity is dual to a system of perpendicular NS5 fivebranes intersecting over a 3+1 dimensional world-volume.

In this paper we use a different approach to explore a system of D3 branes in the geometry given by a general conifold $xy = v^m w^n$. In a recent paper [2] the supergravity solution for intersecting branes [1] was used to give a heuristic but rather explicit map from brane configurations to conifolds. Motivated by this observation, we obtain new brane configurations together with the corresponding conifolds. In particular, we are
able to describe a system of D3 branes at a conifold singularity in terms of D4 branes in the presence of \( m \) D6 branes (spanning the 0123789 directions) and \( n \) D6' branes (spanning the 0123457 directions). The study of this brane configuration is interesting not only because it provides new and more general maps between brane configurations and conifolds, but also because it involves fascinating worldsheet instanton effects. These effects are non-perturbative in \( l_s \) and are still important in the limit \( g_s \to 0 \). They give rise to a long ranged repulsive interaction between D4 branes stretched between a D6 and D6' brane \([8]\). When constructing brane configurations, one imposes this rule in much the same way that the \( s \) rule is imposed. One of the issues which we address is a possible geometric explanation for this rule, which is motivated by the natural emergence of the \( s \) rule from purely geometrical considerations. A generalization of the result obtained in \([4]\) for a single type of D6 brane shows that the configuration involving both D6 and D6' branes gives rise to the geometry with a singularity of the form \( xy = v^m w^n \). We will argue below that our approach yields the correct dimension of the Higgs moduli space. In this way, we are able to generalize the result of \([4]\) where the dimension of the Higgs moduli space was calculated by blowing-up the \( A_{n-1} \) singularity and then counting the multiplicities of the \( P^1 \) curves appearing in the resolved surface. In our case the details are more involved because we need to resolve the \( A_{m-1,n-1} \) singularity which is a singularity of the form \( xy = v^m w^n \).

Our results in the case of configurations involving only D6 or D6' branes show complete agreement between the geometrical results and those predicted by the brane configurations. Our geometrical analysis provides evidence that the \( s \)-rule continues to hold when the D6 and NS5 branes are are at any non-zero angle with respect to each other. To the best of our knowledge, this is a new result. In the case of configurations involving D6 and D6' branes we find an interesting discrepancy between the geometrical results as compared to those predicted by the brane configurations. This is rather unexpected in light of previous results involving only D6 branes \([4, 9]\). The discrepancy can be traced back to the repulsive interaction between D4 branes stretched between D6 and D6' branes. Euclidean fundamental strings (i.e. worldsheet instantons) are responsible for this repulsive interactions. Our geometric model is apparently not corrected by these instanton effects. Maybe the geometry must be modified to reflect the instanton effects. We leave appropriate treatments of these effects in the M theory framework as an interesting open question.

The remainder of the paper is organized as follows: In section 2 we begin with an explanation of the set-up of our problem. Our starting point is the observation that there is a direct connection between the metric of a 3-brane at a conifold (obtained in equation (5.13) of \([4]\)) and the metric obtained from the supergravity solution for intersecting branes (described in equation (20) of \([6]\)), obtained by performing a dimensional reduction. We then show that by using (20) of \([6]\) we can obtain a configuration with two orthogonal D6 branes as opposed to the configuration with two orthogonal KK5
monopoles (which is of course connected by a T-duality to a configuration with two orthogonal NS5 branes). In [2] the metric for two orthogonal NS5 branes was shown to be the metric near the conifold singularity. This suggests that the metric for two orthogonal D6 branes is also the metric near the conifold singularity. This is not surprising and could have been anticipated from the results of [4] where it was demonstrated that the presence of \( m \) D6 branes leads naturally to the geometry \( xy = v^m \). The Higgs moduli space is discussed using a brane configuration in type IIA theory in section 3. The brane configuration consists of D4 branes suspended between NS5 branes, in the presence of D6 and D6′ branes. The dimension of the Higgs moduli space is computed by allowing the D4 branes to break into segments suspended between NS5, D6 and D6′ branes. There are two distinct “s-rules” that must be enforced in order to obtain the correct result. The first s-rule constrains the number of D4 branes stretched between NS5 and D6 branes. The second s-rule is directly related to the worldsheet instanton effects that we mentioned above. It constrains the number of D4 branes that can be stretched between D6 and D6′ branes, in order to have a stable brane configuration. Section 4 contains a discussion of relevant field theory results and their implications. In this section, we also show how to solve the simplest quadratic threefold singularity \( xy = vw \). In section 5 we solve the \( A_{-1,1} \) singularity which corresponds to a configuration with D6′ branes and no D6 branes. In field theory this corresponds to an \( \mathcal{N} = 1 \) theory with fundamental flavors and no superpotential. Our results on the solution of the most general \( A_{m-1,n-1} \) singularity are presented in section 6.

2 Branes at threefold singularities

A useful approach to the study of conifolds has been developed in [1, 2] where a conifold was mapped into a set of intersecting NS5 and NS5′ branes. The conifold is described in terms of two degenerating tori which vary over a \( \mathbb{P}^1 \) base. The pair of orthogonal NS5 branes is obtained by performing two T dualities. The first and second T dualities are performed along a cycle of the first and second tori. As discussed in [1, 2], when the NS5 branes have three of their five dimensions common, they give rise to conifold singularities. The argument of [2] is directly relevant to our study, and it is worth recalling some key points. These authors consider a configuration involving NS5 branes (spanning the 012389 directions), NS5′ branes (spanning the 012345 directions) and D3 branes (spanning the 0126 directions). The supergravity metric for this brane configuration resembles the metric for a 3-brane at a conifold singularity [2]. Our arguments will make use of the metric describing two Kaluza-Klein monopoles with a five dimensional common worldvolume in 11 dimensions (see (20) of [4]). From the results of [2], after removing the contribution of the D3 brane to the metric we have

\[
ds^2 = ds_{0123}^2 + H_5'^2d\sigma_5^2 + H_5d\sigma_8^2 + H_5H'_5d\sigma_5^2 + (H_5H'_5)^{-1}(ds_6 + A_1ds_4 + B_1ds_8)^2.
\] (2.1)
Compare this to equation (20) of [6]

\[ ds^2 = ds_{0123}^2 + ds_{10}^2 + H_1 ds_{45}^2 + H_1 H_2 ds_8^2 + (H_1 H_2)^{-1} (ds_6 + A_1 ds_4 + B_1 ds_8)^2 \]  

(2.2)

where we have changed the notation in order to compare with the result of [2]. A total of 6 gauge fields are considered in [6]; however, four of these can be gauged to zero. All the harmonic functions in (2.1) and (2.2) depend only on the \( x^7 \) coordinate. A key observation is that (2.1) can be obtained from (2.2) by reduction along \( x^{10} \). If we reduce along the \( x^6 \) direction, the resulting metric describes the intersection between a D6 brane (spanning the 123789 directions) and a D6′ brane (spanning the 123457 directions). So the configuration with D6(123789) and D6′(123457) and the one obtained from NS5-NS5′ after a T-duality are both obtained from the same solution in 11 dimensions; the only difference is that the dimensional reduction is performed on different directions in the two cases. This relationship between the two configurations, which has been argued at the level of supergravity, is related to a duality between the two configurations at the level of the full string theory. Both brane configurations correspond to a single brane configuration from the point of view of M theory. The M theory set up includes Kaluza Klein monopoles. To obtain the IIA brane configuration which contains D6 branes, we obtain IIA string theory from M theory by taking the strong coupling eleventh dimension to be transverse to the M theory Kaluza Klein monopoles. To obtain the IIA configuration which contains Kaluza Klein monopoles, we take the strong coupling direction to be parallel to the M theory Kaluza Klein monopoles. Thus from the point of view of M theory, the two configurations are related by a flip of the strong coupling direction [40, 41] with another direction so that the equivalence of these two configurations follows from eleven dimensional Lorentz invariance.

This shows that system of D6 and D6′ branes gives a geometry which is similar to the geometry obtained from the NS5-NS5′ configuration. An important result of [6] is that by T-dualizing a configuration with NS5 - NS5′ branes one maps the metric to the metric of a conifold singularity:

\[ xy = vw \]  

(2.3)

Our previous observation, lead us to the conclusion that the metric for a configuration with D6 and D6′ branes can be mapped to (2.3). With hindsight, we see that this geometry could have been anticipated from the results of [4]. There it was shown that the presence of a D6 brane induces a change in the geometry of the form \( xy = v \), whilst the presence of a D6′ brane would a change in the geometry of the form \( xy = w \). In view of these results, it is natural to expect that one obtains \( xy = vw \) in the presence of a system of D6 - D6′ branes. Our discussion above shows that this is indeed the geometry which appears. In [4] the geometric structure corresponding to a system of \( n \) D6 branes on top of each other, by using results of [6]. The geometric structure is that of a Taub - NUT space, which is a hyperKähler manifold with three complex structures. One of the complex structures is \( xy = v^n \) i.e. the \( A_{n-1} \) singularity. In our case, we do not have any
localized solution for the system of D6 - D6' branes so we have been unable to obtain the complex structure directly from a specific metric. This is why we have used the previous correspondence with a configuration of NS5 and NS5' branes to identify the metric.

The supersymmetry allows us to introduce in the final configuration two types of NS5 branes, one in directions 012345 denoted by NS5 and one in the 012389 directions denoted by NS5'. In this way obtain a standard configuration for $N = 1$ supersymmetric field theories in 4 dimensions. As usual strings between D4 branes give the gauge group which is $SU(N)$ for a stack of N D4 branes, strings between the D6 brane and D4 branes give one quark, denoted by $A$ and the strings between the D6' brane and the D4 branes give a second quark denoted by $B$.

The general case can be considered by using the following brane configuration: Take $N_f$ parallel D6-branes extended in the $(x_0, x_1, x_2, x_3, x_7, x_8, x_9)$ directions, located at $d_i$ in $v$-plane and $N'_f$ parallel D6'-branes extended in the $(x_0, x_1, x_2, x_3, x_4, x_7)$ directions, located at $e_i$ in $w$-plane. In what follows, we consider M theory on $R^{10} \times S^1$. This is equivalent to Type IIA on $R^{10}$, with the $U(1)$ gauge symmetry of Type IIA being associated in M theory with the rotations of the $S^1$. Via T and S dualities, as argued above, the D4 branes will live on a threefold given by

$$xy = \prod_{i=1}^{N_f} (v - d_i) \prod_{i=1}^{N'_f} (w - e_i),$$  \hspace{1cm} (2.4)$$

replacing the flat $(x_4, x_5, x_6, x_7, x_8, x_{10})$ space due the presence of D6 and D6' branes. Now, introduce two NS5 branes in the $(x_0, x_1, x_2, x_3, x_4, x_5)$ directions and suspend $N_c$ D4 branes between them. The equation of the Seiberg-Witten curve is given by

$$w = 0$$

$$x + y = B_{N_c}(v, u_k).$$  \hspace{1cm} (2.5) \hspace{1cm} (2.6)$$

A more general form for the conifold can again be read from the results of [1, 3] as:

$$xy = v^m w^n$$  \hspace{1cm} (2.7)$$

which can again be viewed as an orbifold of a $C^*$ fibration over the $C^2$ parameterized by $v, w$ by $Z_m \times Z_n$. The T-dual configuration now contains $m$ NS5 branes spanning the 023457 directions, $n$ NS5 branes spanning the 023789 directions. After a further T and S duality, we obtain $m$ D6 branes filling the 0123789 direction and $n$ D6' branes filling the 0123457 directions. Finally, after introducing the NS5 and NS5' brane, we obtain $m$ quarks $A_i$ from strings stretching between the D4 and D6 branes and $n$ quarks $B_j$ from strings stretching between the D4 and D6' branes. This is to be compared with the usual case when one has only one type of D6 branes, giving only quarks of type $A$.  \hspace{1cm} (2.7)
3 Higgs Moduli space from Brane Configurations

In this section we review the description of the Higgs branch in the type IIA picture. To reach the Higgs branch we allow the D4 branes to break on the D6 and D6' branes. After the breaking, there will be D4 branes suspended between the D6 and D6' branes. There are three distinct types of suspended D4 branes possible: a D4 brane suspended between a pair of D6 branes, a D4 brane suspended between a pair of D6' branes and a D4 branes suspended between a D6 and a D6' brane. In addition to these, there are D4 branes between D6 and NS5 branes and D4 branes between D6' branes and NS5 branes. The collective coordinates of a D4-brane suspended between a pair of D6-branes consists of two complex parameters built from the $x^7, x^8, x^9$ coordinate of the D4 together with the gauge field component $A_6$ corresponding to the compact $x^6$ coordinate. Similarly, the location of a D4-brane between a pair D6'-branes is parameterized by two complex parameters built from the $x^4, x^5, x^7$ coordinate together with the gauge field component $A_6$. The location of a D4-brane between a D6 brane and a D6' brane is parameterized by a single complex coordinate built from the $x^7$ coordinate together with the gauge field component $A_6$. Finally, the location of a D4-brane between a D6' and an NS5 brane is parameterized by one complex parameters built from the $x^8, x^9$ coordinate of the brane. A D4 brane that is suspended between an NS5 brane and a D6 brane is not free to move.

When computing the dimension of moduli space, it is crucial that one impose the s-rule which restricts the number of D4 branes stretched between an NS5 brane and a D6 brane to one. For the configuration that we are studying there is an additional non-perturbative effect due to worldsheet instantons, giving rise to a long range force between two D4 branes suspended between a D6 brane and a D6' brane. This long-range force drives the D4 branes to infinity in the $x^7$ direction [8]. Note that this effect is non-perturbative in $l_s$ and continues to remain important for the dynamics at arbitrarily weak string coupling.

We can now proceed to calculate the dimension of the Higgs branch. In order to have be able to compare with our results of section 6 we are going to discuss two cases, one with parallel NS branes in (12345) directions and one with rotated NS branes in the (4589) plane.

3.1 Parallel NS Branes in the (12345) Directions

Consider the case when we have $N_f$ D6' branes. The theory has $\mathcal{N} = 1$ supersymmetry. The matter content of the theory includes $N_f$ fundamental flavors $B$ (and their corresponding anti-fundamentals $\bar{B}$) as well as an adjoint field $\Phi$. In general, one expects
a superpotential of the form $\sin(\theta) \tilde{B}\Phi B$ where $\theta$ is the angle between the D6 branes and the NS5 branes. In the case that we are considering, the D6' branes are parallel to the NS5 branes so that the superpotential vanishes. The dimension of the Higgs moduli space is $2N_fN_c$. Consider now the case of $N_f$ D6 branes. The brane configuration realizes a theory with $\mathcal{N} = 2$ supersymmetry with a matter content that includes $N_f$ hypermultiplets in the fundamental representation. It is well known that the dimension of the Higgs moduli space is $2N_fN_c - 2N_c^2$.

Let us turn now to a general configuration consisting of $n$ unrotated D6' branes and $m$ rotated D6' branes (i.e. D6 branes). There are $N_c$ D4 branes suspended between the leftmost NS5 brane and the first D6' brane. These D4 branes make a contribution of $N_c$ to the total (complex) dimension of the Higgs moduli space. There are $N_c$ D4 branes suspended between the D6' branes which give a contribution of $2(n - 1)N_c$ to the total complex dimension. Counting the number of D4 branes between the D6 branes is a little more subtle because one has to correctly enforce the s-rule. The s-rule places restrictions on the D4 branes connecting the rightmost NS5 brane and the D6 branes. In addition, the worldsheet instanton effects imply that a stable brane configuration is only obtained after restricting to a single D4 brane between the rightmost D6' brane and each of the D6 branes. Summing these contributions leads to a complex dimension of

$$2(m - 1) + 2(m - 3) + \cdots 2(m - 2N_c + 1) = 2N_c(m - N_c).$$

(3.1)

The last thing contribution to the dimension of the Higgs moduli space comes from the D4 branes suspended between the rightmost D6' and the D6 branes. This contribution is $N_c$. The final result is that the complex dimension of the Higgs moduli space is

$$2(n - 1)N_c + 2N_c(m - N_c) + 2N_c = 2(n + m)N_c - 2N_c^2 = 2N_fN_c - 2N_c^2.$$  (3.2)

A comment is in order. Equation (3.2) shows that the dimension of the Higgs moduli space is the same as the dimension computed in the $\mathcal{N} = 2$ theory. This is explained by noting that we can break the gauge symmetry by first introducing the D6 branes and only then introducing the D6' branes to break the $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$. So, after introducing the D6 brane, the gauge symmetry is completely broken in the $\mathcal{N} = 2$ supersymmetric theory. The field theory calculations tell us that the dimension of the Higgs moduli space is $2mN_c - 2N_c^2$. By introducing further flavors in the form of D6' branes they will not break any more the gauge group and their contribution to the Higgs moduli space is $2mN_c$. Therefore, the field theory calculations give the result $2mN_c - 2N_c^2 + 2mN_c$ i.e. just the one of equation (3.2).

The quaternionic dimension of the Higgs moduli space is

$$N_c(n + m) - N_c^2.$$  (3.3)
3.2 Parallel NS Branes Rotated in the (4589) Plane

Let us now discuss the case of \( n \) unrotated D6' branes and \( m \) rotated D6' branes (i.e. D6 branes) together with \( N_c \) D4 branes suspended between two parallel NS5 branes which are now at an angle in the (4589) plane.

In this case, the D4 branes stretched between NS5 branes and D6 and D6' branes are constrained by the s-rule which restricts the number of D4 branes suspended between an NS5 brane and a D6 brane and an NS5 brane and a D6' brane to one. There are several pieces of evidence for this extended s-rule. The neat fit of the brane description with the field theory is only possible when this s-rule is enforced. The D4 branes suspended between a D6 brane and a rotated NS5 brane or between a D6' and a rotated NS5 cannot move between them because they are extended in different directions. So these D4 branes would necessarily be on top of each other. This is a singular situation which would presumably break supersymmetry as in the case of D4 branes between perpendicular NS5 and D6 branes. We have found further evidence for the extended s-rule. This additional evidence comes from considering the M theory curve. In particular we count the number of complex spheres which decouple from the curve and are free to slide along the D6 branes or along the D6' branes.

Worldsheet instanton effect still play an important role in the dynamics of D4 branes suspended between D6 and D6' branes. The contribution from the D4 branes suspended between D6' branes is given by taking the s-rule between the leftmost NS5 brane and the D6' branes into account i.e.

\[
2 \sum_{i=1}^{N_c} i + 2N_c(n - N_c - 1). \tag{3.4}
\]

The contribution of the D4 branes suspended between D6 branes is obtained by restricting to a single D4 brane between the rightmost D6' brane and each of the D6 branes and in addition, by considering the s-rule between the rightmost NS5 brane and the D6' brane. The result is

\[
2(m - 1) + 2(m - 3) + \cdots + 2(m - 2N_c + 1) = 2N_c(m - N_c). \tag{3.5}
\]

By adding the contribution from the D4 branes between D6' and D6 branes, the complex dimension of the Higgs moduli space is

\[
N_c(N_c + 1) + 2N_c(n - N_c) + 2N_c(m - N_c) + N_c - 2N_c = 2N_c N_f - 3N_c^2 \tag{3.6}
\]

The quaternionic dimension is \( N_c(n + m) - 3/2N_c^2 \).

For example, consider a configuration with two D6 branes and two D6' branes from left to right in addition to NS and NS' branes. If we start with a first D4 brane, we
break it between the left NS and the first D6 brane (contribution 0), between the two D6 brane (contribution 2), between the second D6 and the first D6' brane (contribution 1), between the two D6' branes (contribution 2) and between the second D6' brane and the right NS brane (contribution 0). So the total dimension is 5, which is equal to $2 \times 1 \times 4 - 3 \times 1^2 = 5$. If we want to insert a second D4 brane, we encounter several restrictions. We need firstly to suspend it between the left NS brane and the second D6 brane because of the s-rule between the left NS and the first D6 brane. The worldsheet instanton effect tells us that the D4 brane has then to be broken next between the second D6 brane and the second D6' brane and then between the second D6' brane and the right NS brane. But we already have one D4 brane between these two and is impossible to have a second one. Therefore it is impossible to insert a second D4 brane. We can have only one D4 brane for this configuration. The same discussion goes for more complicated configurations.

The following observation will help our understanding of the field theory calculation. The D4 branes break on the $m$ D6 branes as in any $\mathcal{N} = 2$ theory with NS(12345) branes and perpendicular D6 branes, the role of the right NS branes being played by the leftmost D6' brane. The worldsheet instanton effect plays the role of the s-rule, limiting the number of D4 branes between the D6 branes and the rightmost to one. The D4 branes break on the $n$ D6' branes as in any $\mathcal{N} = 1$ theory with NS, NS' branes and D6 branes parallel to the NS' branes. The role of the NS' brane is played by the first $N_c$ D6 branes because we can have only one D4 brane between them and the leftmost D6' brane. This is the case obtained if we decide to break the D4 branes first on the D6 branes. If we decide to break the D4 branes firstly on the D6' branes, the they do it on D6' as in $\mathcal{N} = 2$ and on D6 as in $\mathcal{N} = 1$ theory.

We want to turn to field theory calculations. Now we need to take care of the order in which we introduce flavor given by the D6 and the D6' branes. There is a superpotential coupling the adjoint field to both types of flavor given by the D6 branes and the D6' branes. To calculate the dimension of the moduli space we need to make the following observation. If we introduce only D6 branes or only D6' branes, the theory would be $\mathcal{N} = 2$ supersymmetric. Let us consider that we have $m$ flavors given by $m$ D6 branes that break completely the gauge group. The dimension of the moduli space is $2mN_c - 2N_c^2$. Now if we introduce $n$ D6' branes, the supersymmetry would be broken to $\mathcal{N} = 1$. The contribution to the dimension of the moduli space is $2nN_c - N_c^2$ where we used the fact that for $\mathcal{N} = 2$ matter multiplets there are two complex scalars eaten in Higgs mechanism whereas for $\mathcal{N} = 1$ matter multiplets there is only one. By adding the two contributions to the Higgs moduli space we obtain $2(m+n)N_c - N_c^2$ or $2N_fN_c - 3N_c^2$. Therefore, the results from the field theory and the brane configuration agree.
4 Resolution of the Singularity and the Higgs Branch

As mentioned in the last section, the transition to the Higgs branch occurs when the fivebrane intersects with the D6-branes. This is possible when $e_i = d_i = 0$ and the Seiberg-Witten curve passes through the singular point $x = y = v = w = 0$. Under these conditions the term $B_{N_c}(v, u_k)$ factorizes into

$$B_{N_c}(v, u_k) = v^r (v^{N_{c-r}} + \cdots + u^N_{N_{c-r}}),$$

where $r > 0$ and $u^N_{N_{c-r}} \neq 0$. Notice that the threefold will be of the form

$$f(x, y, v, w) := xy - v^{N_{c-r}} w^{N_{c-r}} = 0.$$  

As a warm-up exercise, in this section, we show how to resolve a quadratic threefold singularity

$$xy = vw.$$  

As explained below, the resolution is not unique. The relationship between the different resolutions is discussed.

- **Type A Blow-up**

  First, we blow up the $x - y - v - w$ space at $x = v = 0$ by replacing the $x - y - v - w$ space by a union of two spaces - coordinatized by $(x, y, \tilde{v}, w)$ and $(\tilde{x}, y, v, w)$ - which are mapped to the $x - y - v - w$ space by $(x, y, v, w) = (x, y, x\tilde{v}, w) = (\tilde{x}v, y, v, w)$. The $x - y - \tilde{v} - w$ and the $\tilde{x} - y - v - w$ spaces are glued together by the relation $\tilde{x}\tilde{v} = 1$ and $v = x\tilde{v}$. The equation $xy = vw$ becomes $x(y - \tilde{v}w)$ in the $x - y - \tilde{v} - w$ space and $v(\tilde{x}y - w)$ in the $\tilde{x} - y - v - w$ space. If we ignore the piece described by $x = 0$ and $v = 0$, which is mapped to the $y - w$ plane $x = v = 0$, we obtain a union of two smooth threefolds - $U_1 = \{y = \tilde{v}w\}$ in the $x - y - \tilde{v} - w$ space and $U_2 = \{\tilde{x}y = w\}$ in the $\tilde{x} - y - v - w$ space. The threefolds $U_1$ and $U_2$ are coordinatized by $(x, \tilde{v}, w)$ and $(\tilde{x}, y, v)$ respectively and glued together by the condition $\tilde{x}\tilde{v} = 1$, $v = x\tilde{v}$ and $y = \tilde{v}w$. Thus we obtain a smooth threefold. This threefold is mapped onto the original singular $A_{0,0}$ threefold $xy = vw$: $(x, y, v, w) = (x, \tilde{v}w, x\tilde{v}, w)$ on $U_1$ and $(x, y, v, w) = (\tilde{x}v, y, v, \tilde{x}y)$ on $U_2$. The inverse image of the singular point $x = y = v = w = 0$ is described by $x = w = 0$ in $U_1$ and by $y = v = 0$ in $U_2$. It is coordinatized by $\tilde{v}$ and $\tilde{x}$ which are related by $\tilde{v}\tilde{x} = 1$ and thus, is a projective line $\mathbf{P}^1$.

- **Type B Blow-up**

  Second, we blow up the $x - y - v - w$ space at $x = w = 0$ by replacing the $x - y - v - w$ space by a union of two spaces - coordinatized by $(x, y, v, \tilde{w})$ and $(\tilde{x}, y, v, w)$ - which are
mapped to the $x-y-v-w$ space by $(x, y, v, w) = (x, y, v, x\bar{w}) = (\bar{x}w, y, v, w)$. If we go through the same process, then we will obtain a union of two smooth threefolds - $V_1 = \{y = v\bar{w}\}$ in the $x-y-v-\bar{w}$ space and $V_2 = \{\bar{x}y = v\}$ in the $\bar{x} - y - v - w$ space. The threefolds $V_1$ and $V_2$ are coordinatized by $(x, v, \bar{w})$ and $(\bar{x}, y, w)$ respectively and glued together by $\bar{w}\bar{x} = 1$, $w = x\bar{w}$ and $y = v\bar{w}$. Thus we obtain a smooth threefold. This threefold is mapped onto the original singular $A_{0,0}$ threefold $xy = vw$: $(x, y, v, w) = (x, v\bar{w}, v, x\bar{w})$ on $V_1$ and $(x, y, v, w) = (\bar{x}w, y, \bar{x}y, w)$ on $V_2$. The inverse image of the singular point $x = y = v = w = 0$ is described by $x = v = 0$ in $V_1$ and by $y = w = 0$ in $V_2$. It is coordinatized by $\bar{w}$ and $\bar{x}$ which are related by $\bar{w}\bar{x} = 1$, and thus is a projective line $\mathbb{P}^1$.

In this special case, the resolved threefold we obtained in Type A blow-up and Type B blow-up are isomorphic. This is not true in general. Note that there is no isomorphism between the type A and type B blowup which will commute with the maps to the original singular variety. Two resolved threefold are related by a birational transformation, called a flop. The following diagram compares Type A and Type B blow-ups.

In this example, the flop is just a result of changing the role of $v$ and $w$, which corresponds to the exchange of $D6$ and $D6'$ branes.

5 Resolution of $A_{-1,n-1}$ singularity and Higgs Branch

Before discussing the resolution of the singularity, it is helpful to make an observation regarding the mass of the adjoint field. Usually the mass of the adjoint field is connected to the angle between the NS5 branes when the D6 branes are not rotated. However, the mass of the adjoint field is not a function of the angle between NS5 branes alone:
it also depends on the angle between D6 branes and NS5 branes. The superpotential is also a function of the angles between D6 and NS5 branes. The mass of the adjoint can be defined by measuring the strength of the quartic superpotential of the quarks obtained after integrating the massive adjoint out. In this way we see that if we have both superpotential terms and terms proportional to $\Phi^2$, then the mass of the adjoint is a function of the relative angle between the D6 branes and the NS5 branes. In this section we consider the case involving NS5 branes and D6' branes which are parallel. This means that the mass of the adjoint is zero and there is no superpotential in the $\mathcal{N} = 1$ theory.

We now turn to the problem of resolving the singularity of a threefold $\mathcal{X} \subset \mathbb{C}^4$ given by

$$\mathcal{X} : xy = w^n.$$  \hspace{1cm} (5.3)

which is a singularity of type $A_{-1,n-1}$.

We blow up $\mathbb{C}^4$ -coordinatized by $(x, y, v, w)$ along the plane $y = w = 0$ by replacing the $x - y - v - w$ space (i.e. $\mathbb{C}^4$) by a union of two spaces - coordinatized by $(x, y, v, \tilde{w})$ and $(x, \tilde{y}, v, w)$ - which are mapped to the $x - y - v - w$ space by $(x, y, v, w) = (x, y, v, y\tilde{w}) = (x, \tilde{y}w, v, w)$. This corresponds to surgery along the plane $y = w = 0$ which replaces the plane by the product of a plane and a projective line $\mathbb{P}^1$. The $x - y - v - \tilde{w}$ and the $x - \tilde{y} - v - w$ spaces are glued together by the condition $\tilde{y}\tilde{w} = 1$ and $w = y\tilde{w}$. Consider first the simplest type of singularity i.e. $A_{-1,1}$. The equation $xy = w^2$ becomes $y(x - \tilde{w}^2y)$ in the $x - y - v - \tilde{w}$ space and $w(xy - w)$ in the $x - \tilde{y} - v - w$ space. If we ignore the piece described by $y = 0$ and $w = 0$ which is mapped to the $x - v$ plane $y = w = 0$, we obtain a union of two smooth threefolds $W_1 = \{x = \tilde{w}^2y\}$ in the $x - y - \tilde{v} - w$ space and $W_2 = \{xy = w\}$ in the $x - \tilde{y} - v - w$ space. The threefolds $W_1$ and $W_2$ are coordinatized by $(y,v,\tilde{w})$ and $(x,\tilde{y},v)$ respectively and glued together by $\tilde{w}y = 1$, $x = y\tilde{w}^2$ and $w = xy$. Thus we obtain a smooth threefold. This threefold is mapped onto the original singular threefold $xy = w^2$: $(x,y,v,w) = (\tilde{w}^2y,y,v,y\tilde{w})$ on $W_1$ and $(x,y,v,w) = (x,xy^2,v,xy)$ on $W_2$. The inverse image of the singular line $x = y = w = 0$ is described by $y = 0$ in $W_1$ and by $x = 0$ in $W_2$. It is coordinatized by $\tilde{w}, v$ and $\tilde{y}, v$ which are related by $\tilde{w}y = 1$, and thus is a product $\mathbb{P}^1 \times \mathbb{A}^1$ of a projective line and an affine line.

If we started with a higher $A_{-1,n-1}$ singularity, the equation $xy = w^n$ becomes $x = y^{n-1}\tilde{w}^n$ in the $x - y - v - \tilde{w}$ space and $xy = w^{n-1}$ in the $x - \tilde{y} - v - w$ space (again ignoring the trivial piece $y = 0$ and $w = 0$). This is smooth in the $x - y - v - \tilde{w}$ space but has an $A_{-1,n-2}$ singularity at $x = \tilde{y} = v = w = 0$ in the $x - \tilde{y} - v - w$ space.

---

*We would like to thank Amihay Hanany and David Kutasov for useful discussions on this matter. See [8] for a complete discussion.
Thus the threefold is not yet resolved but it has become less singular. We can further decrease $n - 1$ by one by blowing up the $\tilde{y} - w$ plane at $\tilde{y} = w = 0$. Iterating this process, we can finally resolve the singular $A_{-1,n-1}$ singularity. It is straightforward to see that the resolved space is covered by $n$ three-space $V_1, V_2, V_3, \ldots, V_n$ with coordinates $(y_1, w_1, v) = (y, \tilde{w}, v)$, $(y_2 = \tilde{y}, w_2, v), (y_3, w_3, v), \ldots, (y_n, w_n = x, v)$ which are mapped to the singular $A_{-1,n-1}$ threefold by

$$V_j \ni (y_j, w_j, v) \mapsto \begin{cases} x = y_j^{n-j} w_j^{n+1-j} \\ y = y_j^j w_j^{j-1} \\ v = v \\ w = y_j w_j \end{cases} \quad (5.4)$$

The three-spaces $V_j$ are glued together by $w_j y_j+1 = 1$ and $y_j w_j = y_j+1 w_j+1$. The map onto the singular $A_{-1,n-1}$ threefold is isomorphic except at the inverse image of the singular line $x = y = w = 0$. The inverse image consists of $n - 1$ $\mathbb{P}^1 \times A^1$'s $D_1, D_2, \ldots, D_{n-1}$ where $D_j$ is the locus of $y_j = 0$ in $V_j$ and $w_{j+1} = 0$ in $V_{j+1}$, and is coordinatized by $w_j$ and $y_{j+1}$ that are related by $w_j y_{j+1} = 1$. $D_j$ and $D_k$ do not intersect unless $k = j \pm 1$, and $D_{j-1}$ and $D_j$ intersect transversely at $y_j = w_j = 0$. Inside each of the $D_j$, the projective line $\mathbb{P}^1$ defined by $v = 0$ is the inverse image of the origin $x = y = v = w = 0$ which will be denoted by $C_j$. We denote the resolved threefold by $\tilde{X}$ and the map onto the singular threefold, which is described in (5.4), by $\sigma$.

Thus, we have

$$\sigma: \tilde{X} = V_1 \bigsqcup_{y_1 y_2 = 1} V_2 \bigsqcup_{y_2 y_3 = 1} \cdots \bigsqcup_{y_{n-1} y_n = 1} V_n \rightarrow X \quad (5.5)$$

$$\sigma^{-1}(0) = C_1 \cup C_2 \cup \cdots \cup C_{n-1} \quad (5.6)$$

and the map $\sigma$ is onto and isomorphic outside the line defined by $y = w = 0$ on $X$.

Consider a Seiberg-Witten curve on the singular threefold $X$ given by

$$x + y = v^r \quad (5.7)$$

$$w = 0. \quad (5.8)$$

We would like to study the total transform of the curve on the resolved threefold, which is the inverse image (in an algebraic sense) of the curve under the map $\sigma$. On the $j$-th patch $V_j$, the equation of the curve will be

$$y_j^{n-j} w_j^{n+1-j} + y_j^n w_j^{j-1} = v^r \quad (5.9)$$

$$y_j w_j = 0. \quad (5.10)$$
Thus by plugging (5.10) into (5.9) we obtain
\[ y = v^r, \quad yw_1 = 0 \quad \text{for} \quad j = 1 \\
\]
\[ v^r = 0, \quad y_j w_j = 0 \quad \text{for} \quad 2 \leq j \leq n - 1 \\
\]
\[ x = v^r, \quad y_n x = 0 \quad \text{for} \quad j = n. \]  

Thus on the \( j \)-th patch \( V_j \) for \( 2 \leq j \leq n - 1 \), the total transform of the curve consists of \( r \)-multiple copies of \( C_j \cap V_j \), which is given by \( v^r = 0, y_j = 0 \) and \( r \)-multiple copies of \( C_{j-1} \cap V_j \), which is given by \( v^r = 0, w_j = 0 \). On \( V_1 \), the total transform consists of \( r \)-multiples of \( C_1 \cap V_1 \) and a curve defined by \( y - v^r = w_1 = 0 \). On \( V_n \), the total transform consists of \( r \)-multiple copies of \( C_{n-1} \cap V_n \) and a curve defined by \( x - v^r = y_n = 0 \). Thus the total transform consists of \( r \)-multiple copies of \( C_1, C_2, \ldots, C_{n-1} \) and two irreducible curves \( C_L \) and \( C_R \) which meet \( C_1 \) and \( C_{n-1} \) tangentially but in transversal direction. Since each \( C_j \) is isomorphic to \( \mathbb{P}^1 \) and the motion between \( D6' \) branes are parameterized by these \( \mathbb{P}^1 \)'s, the quaternionic dimension of the \( r \)-th Higgs branch will be \( r(n - 1) \). If we now add the contribution from the \( D4 \) branes ending on both \( \text{NS5} \) branes (which is not counted in the above derivation) the total dimension is \( rn \). The Higgs branch is depicted in Figure 2.

\[
2rn + n^2 - n^2 = 2rn \tag{5.12}
\]
i.e. a \( rn \) quaternionic dimension.

We thus have perfect agreement with the field theory results.

\section{Resolution of \( A_{m-1,n-1} \) singularity}

A complex 3-dimensional hypersurface singularity defined by

\[
f(x, y, v, w) := xy - v^m w^n = 0 \tag{6.13}
\]
in \( \mathbb{C}^4 \) will be called a type \( A_{m-1,n-1} \) singularity. For \( m > 1 \) and \( n > 1 \), the \( A_{m-1,n-1} \) singularity will be singular along the union of two complex lines \( x = y = v = 0 \) and \( x = y = w = 0 \). This is easily seen by noting that there is no well-defined normal vector to the threefold \( xy = v^m w^n \) along these lines while outside of these lines either \( \partial f/\partial x \) or \( \partial f/\partial y \) will provide a normal vector to the threefold. Thus the singularity is not isolated unless \( m = n = 1 \).

We can resolve the \( A_{m-1,n-1} \) singularity by successive (small) blow-ups. However there are many possible desingularizations which are smooth threefolds mapping surjectively onto the threefold \((6.13)\) and isomorphically over the smooth points of \((6.13)\). Hence they are birational and related to each other by flop transitions.

We now consider the problem of resolving the \( A_{m-1,n-1} \) singularity for higher \( m \) and \( n \). We first use Type A blow-up i.e. blowing up the \( x - y - v - w \) space at \( x = v = 0 \). Then the equation \( xy = v^m w^n \) becomes \( y = x^{m-1} v w^n \) in the \( x - y - v - w \) space and \( \tilde{x}y = v^{m-1} w^n \) in the \( \tilde{x} - y - v - w \) space (once again, ignoring the trivial piece \( x = 0 \) and \( v = 0 \)). It is smooth in the \( x - y - v - w \) space but has an \( A_{m-2,n-1} \) singularity at \( \tilde{x} = y = v = w = 0 \) in the \( \tilde{x} - y - v - w \) space. Thus the threefold is not yet resolved, but it has become less singular. We can further decrease \( m - 1 \) by one by blowing up the \( \tilde{x} - v \) plane at \( \tilde{x} = v = 0 \). Iterating this process, we arrive the singularity of type \( A_{1,m-1} \) in \( x_m - y - v - w \) space. Now by applying Type B blow-up successively i.e. blowing up the \( x_m - y - v - w \) space at \( x_m = w = 0 \), we can finally resolve the singularity. It is straightforward to see that the resolved space is covered by \( n + m \) three-spaces \( U_1, U_2, U_3, \ldots, U_m, V_1, V_2, \ldots, V_n \) with coordinates \((x_1, v_1, w) = (x, v/x, w), (x_2, v_2, w) = (x v/x_1, w, x v/x_1), \ldots, (x, \cdot, \cdot) = (w, \cdot, \cdot)\)

\[
\begin{align*}
U_i \ni (x_1, v_1, w) &\mapsto \begin{cases} 
 x = x_i^{i-1} \\
 y = x_i^{m-i} v_i^{m+1-i} w^n \\
 v = x_i v_i \\
 w = w
\end{cases} \\
V_j \ni (x_m+j, w_j, v) &\mapsto \begin{cases} 
 x = x_{m+j}^{j-1} w_j^n \\
 y = x_{m+j}^{n-j} w_j^{n+1-j} \\
 v = v \\
 w = x_{m+j} w_j.
\end{cases}
\end{align*}
\]

The three-spaces \( U_i \) are glued together by \( v_i x_{i+1} = 1 \) and \( x_i v_i = x_{i+1} v_{i+1} \) and the three-spaces \( V_j \) are glued together by \( w_j x_{m+j+1} = 1 \) and \( x_{m+j} w_j = x_{m+j+1} w_{j+1} \). Finally, the
three-spaces $U_m$ and $V_1$ are glued together by $x_{m+1} v_m = 1$, $w = w_1 x_{m+1}$, $v = v_m x_m$.

We denote the resolved threefold by $\tilde{X}$ and the map onto the singular threefold by $\sigma$ which is described by (6.14).

$\sigma$ maps the resolved threefold onto the singular $A_{m-1,n-1}$ threefold isomorphically outside the singular locus, which is a union of two lines defined by $x = y = v = 0$ and $x = y = w = 0$ on $\mathcal{X}$. We will now consider the exceptional loci of $\sigma$ i.e. the set of points where $\sigma$ is not one-to-one. There are three types of exceptional loci corresponding to the movement of D4 branes between two D6' branes, between two D6 branes and finally between D6' and D6 branes.

The inverse image of the singular line $x = y = v = 0$ consists of $(m - 1) \mathbb{P}^1 \times \mathbb{A}^1$'s $B_1, B_2, \ldots, B_{m-1}$ where $B_i$ is the locus of $x_i = 0$ in $U_i$ and $v_{i+1} = 0$ in $U_{i+1}$, and is coordinatized by $v_i$ and $x_{i+1}$ that are related by $v_i x_{i+1} = 1$. $B_i$ and $B_j$ do not intersect unless $j = i \pm 1$, and $B_{i-1}$ and $B_i$ intersect transversely at $x_i = v_i = 0$. Inside of each $B_i$, the projective line $\mathbb{P}^1$ defined by $w = 0$ is the inverse image of the origin $x = y = v = w = 0$ which will be denoted by $A_i$. These $\mathbb{P}^1$'s correspond to the D4 branes suspended between two D6' branes in the Higgs branch.

The inverse image of the singular line $x = y = w = 0$ consists of $(n - 1) \mathbb{P}^1 \times \mathbb{A}^1$'s $D_1, D_2, \ldots, D_{n-1}$ where $D_j$ is the locus of $x_{m+j} = 0$ in $V_j$ and $w_{j+1} = 0$ in $V_{j+1}$, and is coordinatized by $w_j$ and $x_{m+j+1}$ that are related by $w_j x_{m+j+1} = 1$. $D_j$ and $D_k$ do not intersect unless $k = j \pm 1$, and $D_{j-1}$ and $D_j$ intersect transversely at $y_j = w_j = 0$. Inside of each $D_j$, the projective line $\mathbb{P}^1$ defined by $v = 0$ is the inverse image of the origin $x = y = v = w = 0$ which will be denoted by $C_j$. These $\mathbb{P}^1$'s correspond to the D4 branes between two D6 branes in the Higgs branch.

Finally there is a special exceptional divisor $\mathbb{P}^1$, denoted by $E$, which is given by $x_m = w = 0$ in $U_m$ and $w_1 = v = 0$ in $V_1$ in the sense that $E$ does not move while $A_i$ and $C_j$ can move in a family $\mathbb{P}^1 \times \mathbb{A}^1$ of $\mathbb{P}^1$'s. $E$ corresponds to the D4 branes suspended between D6 and D6' branes in the Higgs branch.

Thus we have

$$
\sigma : \tilde{X} \longrightarrow \mathcal{X} \quad \quad \quad \quad (6.16)
$$

$$
\sigma^{-1}(0) = A_1 \cup A_2 \cup \cdots \cup A_{m-1} \cup E \cup C_1 \cup C_2 \cup \cdots \cup C_{n-1}. \quad \quad \quad (6.17)
$$

Consider now, a Seiberg-Witten curve on the singular threefold $\mathcal{X}$ in a general position in $v - w$ space given by

$$
x + y = (v + \mu w)^r \quad \quad (6.18)
$$
where \( \mu \neq 0, \infty \) is a constant. In the conventions described above equation (5.3) this would correspond to a configuration with parallel NS5 branes, rotated at angle \( \mu = tan(\theta) \) with respect to the D6’ branes.

We would like to study the total transform of the curve on the resolved threefold, which is the inverse image (in an algebraic sense) of the curve under the map \( \sigma \). On the \( i \)-th patch \( U_i \), the equation of the curve will be

\[
x_i^i v_i^{i-1} + x_i^{m-i} v_i^{m+1-i} w^n = (x_i v_i + \mu w)^r
\]

\[
\mu v - w = 0.
\]

By plugging the second equation into the first equation, we have

\[
x_i^i v_i^{i-1} + \mu^n x_i^{m+n-i} v_i^{m+n+1-i} = (1 + \mu^2)^r x_i^r v_i^r.
\]

The equation (6.22) will factorize into

\[
x_i^i v_i^{i-1} (1 + \mu^n x_i^{m+n-2i} v_i^{m+n-2i+2} - (1 + \mu^2)^r x_i^{r-i} v_i^{r+1-i}) = 0, \quad i = 1, \ldots, r,
\]

\[
x_i^i v_i^{i-1} (x_i^{i-r} v_i^{i-1-r} + \mu^n x_i^{m+n-r-i} v_i^{m+n-r-1} - (1 + \mu^2)^r) = 0, \quad i = r + 1, \ldots, m. \quad (6.23)
\]

Thus on each \( i \)-th patch \( U_i \) for \( i = 1, \ldots, r \), the total transform of the curve consists of \( i \)-multiple copies of \( A_i \cap U_i \), which is given by \( x_i^i = 0, w = 0 \) and \((i-1)\)-multiple copies of \( A_{i-1} \cap U_i \), which is given by \( v_i^{i-1} = 0, w = 0 \). On each \( i \)-th patch \( U_i \) for \( i = r + 1, \ldots, m \), the total transform consists of \( r \)-multiples of \( A_i \cap U_i \) and a curve defined by \( x_i^i = w = 0 \) and \( r \)-multiples of \( A_{i-1} \cap U_i \) and a curve defined by \( v_i^r = w = 0 \).

On the other hand, the equation of the curve on the \( j \)-th patch \( V_j \) will be

\[
x_{m+j}^j w_j^{j-1} v^m + x_{m+j}^{n-j} w_j^{n+1-j} = (v + \mu x_{m+j} w_j)^r
\]

\[
\mu v - x_{m+j} w_j = 0.
\]

Thus by plugging (6.23) into (6.24) the equation can be rewritten as

\[
\mu^{-m} x_{m+j}^{m+j} w_j^{m+j-1} + x_{m+j}^{n-j} w_j^{n+1-j} = (\mu^{-1} + \mu)^r x_{m+j}^r w_j^r.
\]

This will factorize into

\[
x_{m+j}^r w_j^r ((\mu^{-1} + \mu)^r - \mu^{-m} x_{m+j}^{m+j-r} w_j^{m+j-1-r} - x_{m+j}^{n-j-r} w_j^{n+1-j-r}) = 0
\]

for \( j = 1, \ldots, n-r \),

\[
x_{m+j}^{n-j} w_j^{n+1-j} (\mu^{-m} x_{m+j}^{m-n+2j} w_j^{m-n+2j-2} + 1 - (\mu^{-1} + \mu)^r x_{m+j}^{r-n+j} w_j^{r-n-1+j}) = 0
\]

for \( j = n-r + 1, \ldots, n \). \quad (6.27)
This is basically the same set of the equations as in (6.22). Thus on each \( j \)-th patch \( V_j \) for \( j = 1, \ldots, n - r \), the exceptional component of the curve consists of \( r \)-multiple copies of \( C_j \cap V_j \) defined by \( x_{m+j}^r = 0, v = 0 \) and \( r \)-multiple copies of \( C_{j-1} \cap V_j \) defined by \( w_j^r = 0, v = 0 \). On each \( j \)-th patch \( V_j \) for \( j = n - r + 1, \ldots, n \), the exceptional components consist of \((n-j)\)-multiples of \( C_j \cap V_j \) defined by \( x_{m+j}^{n+1-j} = v = 0 \) and \((n+1-j)\)-multiples of \( C_{j-1} \cap V_j \) defined by \( w_j^{n+1-j} = v = 0 \).

By counting the multiplicities of the complex spheres above, we have verified that the s-rule between the D6 and NS5 branes and between the D6′ and NS5 branes is naturally encoded in the geometry. This provides a non-trivial check of this s-rule.

The quaternionic dimension of the \( r \)-th Higgs branch will be

\[
\sum_{i=1}^{r} i + r(m - r - 1) + r(n - r) + \sum_{i=1}^{r-1} i = r(m + n - r - 1).
\]

We need to add \( r/2 \) from the contribution of the D4 branes between the rightmost D6′ brane and the D6 branes to obtain:

\[
r(m + n - r - 1/2).
\]

A crucial observation is in order here. If we compare the results of (3.6) and (6.29), they do not agree. What is the origin of the mismatch between the result obtained by considering the dynamics of the brane configuration and the result obtained from a study of the geometry? This discrepancy can be traced back to the rule (i.e. long ranged repulsion) between the D4 branes suspended between D6 and D6′ branes. This long ranged repulsion is a worldsheet instanton correction which gives an important contribution to the dynamics of the theory. The geometry is apparently not corrected by this instanton effect\( ^{†}\). As explained in [33, 34], these instantons correspond to Euclidean membranes wrapping a \( \mathbb{P}^1 \) and an interval in the \( x_7 \) direction between a pair of D4 branes. These branes can be reinterpreted as fundamental strings with a rectangular worldsheet defined by the D6, D6′ and D4 branes. As discussed in [35] and as pointed out to us by Uranga, the Euclidean string transforms under three dimensional mirror symmetry to an ADS instanton, which is known to lift the Coulomb branch. In this way, we see that the instanton (corresponding to Euclidean fundamental strings or membranes) has a clear affect on the Higgs moduli space. This effect can not be reproduced from a study of the equations for the geometry. The way in which M theory accounts for these instanton corrections is an interesting open problem.

\( ^{†}\)We thank Angel Uranga for a very important discussion regarding this mismatch between the brane configuration result and the result obtained by geometrical arguments, due to the instanton effects.
Consider now, the Seiberg-Witten curve after moving to the special position in $v-w$-space reached by setting $\mu = 0$ in (6.18). The equation is be given by
\begin{align}
x + y &= v^r \\
w &= 0.
\end{align}

We would like to study the total transform of the curve on the resolved threefold, which is the inverse image (in an algebraic sense) of the curve under the map $\sigma$. If we again use the convention discussed after equation (5.3), we see that this case corresponds to the situation when the NS5 branes are parallel to the D6' branes. The mass of the adjoint field is zero.

On the $i$-th patch $U_i$, the equation of the curve will be
\begin{align}
x^i_{i-1} + x^{m-i}v^{m+1-i}w^n &= x^rv_r \\
w &= 0.
\end{align}

Thus these equations will reduce to
\begin{align}
x^i_{i-1} - x^rv_r &= 0 \\
w &= 0.
\end{align}

The equation (6.34) will factorize into
\begin{align}
x^i_{i-1}(1 - x^rv_r) &= 0 \quad i = 1, \ldots, r \\
x^rv_r(x^r_{r-1} - 1) &= 0 \quad i = r + 1, \ldots, m.
\end{align}

Thus on each $i$-th patch $U_i$ for $i = 1, \ldots, r$, the total transform of the curve consists of $i$-multiple copies of $A_i \cap U_i$, which is given by $x^i_{i-1} = 0, w = 0$ and $(i-1)$-multiple copies of $A_{i-1} \cap U_i$, which is given by $v^{i-1}_r = 0, w = 0$. On each $i$-th patch $U_i$ for $i = r + 1, \ldots, m$, the total transform consists of $r$-multiples of $A_i \cap U_i$ and a curve defined by $x^i_r = w = 0$ and $r$-multiples of $A_{i-1} \cap U_i$ and a curve defined by $v^r_r = w = 0$.

On the other hand, the equation of the curve on the $j$-th patch $V_j$ will be
\begin{align}
x^j_{j-1}v^j_{j-1}v^m + x^{n-j}w^{n+1-j} &= v^r \\
x_{m+j}w_j &= 0.
\end{align}

Thus by plugging (6.38) into (6.37) the equation can be rewritten as
\begin{align}
x_{m+1}v^m &= v^r, \quad x_{m+1}w_1 = 0 \quad \text{for} \quad j = 1 \\
v^r &= 0, \quad x_{m+j}w_j = 0 \quad \text{for} \quad 2 \leq j \leq n - 1 \\
w_n &= v^r, \quad x_{m+n}w_n = 0 \quad \text{for} \quad j = n.
\end{align}
Thus on the each $j$-th patch $V_j$ for $2 \leq j \leq n - 1$, the exceptional components of the total transform consist of $r$-multiple copies of $C_j \cap V_j$ given by $v^r = 0, x_{m+j} = 0$ and $r$-multiple copies of $C_{j-1} \cap V_j$ given by $v^r = 0, w_j = 0$. On $V_1$, the total transform consists of $r$-multiples of $C_1 \cap V_1$ given by $x_{m+1} = v^r = 0$, $r$-multiples of $E \cap V_1$ given by $v^r = w_1 = 0$, and a curve defined by $x_{m+1} v^{m-r} - 1 = w_1 = 0$. On $V_n$, the total transform consists of $r$-multiple copies of $C_{n-1} \cap V_n$ given by $w_n = v^r = 0$ and a curve given by $x_{m+n} = y - v^r = 0$. The quaternionic dimension of the $r$-th Higgs branch will thus be

$$\sum_{i=1}^{r} i + r(m - r - 1) + r(n - 1) = r(m + n - \frac{r + 3}{2}).$$  \hspace{1cm} (6.40)

We again see that the result obtained from the brane configuration does not agree with the result obtained from geometrical arguments. This can again be traced back to contributions from worldsheet instantons that are not accounted for in the curve equations. It would be very interesting to see how these instanton effects could be incorporated in the theory.

**Acknowledgments** We would like to thank Edward Witten, David Morrison, Amihay Hanany and David Kutasov for important comments on our work. We are grateful to Angel Uranga for discussions which helped to clarify different aspects of our work and for his crucial suggestions. We would like to think our referee for thoughtful comments on the previous version of this paper.
References

[1] A. M. Uranga, *Brane Configurations for Branes at Conifolds*, hep-th/9811004.

[2] K. Dasgupta and S. Mukhi, *Brane Constructions, Conifolds and M-theory*, hep-th/9811139.

[3] M. Bershadsky, C. Vafa and V. Sadov, *D Strings on D Manifolds*, Nucl. Phys. B463 (1996) 398, hep-th/9510223.

[4] E. Witten, *Solutions of Four-Dimensional Field Theories via M Theory*, Nucl. Phys. B500 (1997) 3, hep-th/9703166.

[5] G.W. Gibbons and P. Rychenkova, *HyperKähler Quotient Construction of BPS Monopole Moduli Spaces*, hep-th/9608083.

[6] E. Bergshoeff, M. de Roo, E. Eyras, B. Janssen and J.P. van der Schaar, *Intersections involving waves and monopoles in eleven dimensions*, Class. Quant. Grav. 14 (1997) 2757, hep-th/9704120.

[7] S. Elitzur, A. Giveon and D. Kutasov, *Branes and N=1 Duality in String Theory*, Phys. Lett. B400 (1997) 269, hep-th/9702014.

[8] A. Giveon and D. Kutasov, *Brane Dynamics and Gauge Theory*, hep-th/9802067.

[9] K. Hori, H. Ooguri and Y. Oz, *Strong Coupling Dynamics of Four-Dimensional N=1 Gauge Theories from M Theory Fivebrane*, hep-th/9706082, Adv. Theor. Math. Phys. 1 (1998) 1.

[10] A. M. Uranga, *Towards Mass Deformed N=4 SO(n) and Sp(k) gauge configurations*, hep-th/9803054, Nucl. Phys. B526 (1998) 241.

[11] A. Hanany, M. J. Strassler and A. M. Uranga, *Finite Theories and Marginal Operators on the Brane*, hep-th/9803086, J. High Energy Phys. 06 (1998) 011.

[12] C. Ahn, K. Oh and R. Tatar, *Comments on SO/Sp Gauge Theories from Brane Configurations with an O6 Plane*, hep-th/9803197, to appear in Physical Review D.

[13] J. Park, *M-theory realization of a N=1 supersymmetric chiral gauge theory in four dimensions*, hep-th/9805029.

[14] K. Hori, *Consistency conditions for Fivebrane in M Theory on R^5/Z_2 Orbifold*, hep-th/9805141, *Branes and Electric-Magnetic Duality in Supersymmetric QCD*, hep-th/9805142.
[15] K. Landsteiner, E. Lopez and D. A. Lowe, *Supersymmetric Gauge Theories from Branes and Orientifold Six-planes*, hep-th/9805158, J.High Energy Phys. **07** (1998) 011.

[16] J. Lee and S. J. Sin, *More on S duality in MQCD*, hep-th/9806170.

[17] E. Caceres and P. Pasanen, *M Theory Fivebrane Wrapped on Curves for Exceptional Groups*, hep-th/9806224.

[18] E. Lopez and B. Ormsby, *Duality for SUxSO via Branes*, hep-th/9808123.

[19] A. Hanany and A. Zaffaroni, *On the Realization of Chiral Four-Dimensional Gauge Theories using Branes*, hep-th/9801134, Nucl.Phys. **B529** (1998) 180.

[20] E. G. Gimon and M. Gremm, *A Note on Brane Boxes at Finite String Coupling*, hep-th/9803033, Phys.Lett. **B433** (1998) 318.

[21] A. Armoni and A. Brandhuber, *Comments on (Non-)Chiral Gauge Theories and Type IIB Branes*, hep-th/9803186, Phys.Lett. **B438** (1998), 261.

[22] A. Hanany and A. M. Uranga, *Brane Boxes and Branes on Singularities*, hep-th/9805133, J.High Energy Phys. **05** (1998) 013.

[23] L. Randall, Y. Shirman and R. von Unge, *Brane Boxes: Bending and Beta Functions*, hep-th/9806092, Phys. Rev. **D58**, (1998) 105005.

[24] R. G. Leigh, M. Rozali, *The Large N Limit of (2,0) Superconformal Field Theory*, Phys.Lett. **B431** (1998) 311, hep-th/9803068.

[25] A. Karch, D. Lust and D. J. Smith, *Equivalence of Geometric Engineering and Hanany-Witten via Fractional Branes*, Nucl.Phys. **B533** (1998) 348, hep-th/9803232.

[26] A. Karch, D. Lust and A. Miemiec, *N=1 Supersymmetric Gauge Theories and Supersymmetric 3-cycles*, hep-th/9810254.

[27] A. Karch, *Field Theory Dynamics from Branes in String Theory*, hep-th/9812072.

[28] M. R. Douglas, B. R. Greene and D. R. Morrison, *Orbifold Resolution by D-branes*, hep-th/9704151 Nucl.Phys. **B506** (1997) 84.

[29] M. R. Douglas and G. Moore, *D-branes, Quivers and ALE Instantons*, hep-th/9603167.

[30] I. R. Klebanov and E. Witten, *Superconformal Field Theory on Threebranes at a Calabi-Yau Singularity*, hep-th/9807080.
[31] B. S. Acharya, J. M. Figueroa-O’Farrill, C. M. Hull and B. Spence, Branes at Conical Singularities and Holography, hep-th/9808014.

[32] G.W.Gibbons and P.Rychenkova, Cones, Tri-Sasakian Structures and Superconformal Invariance, hep-th/9809158.

[33] D. R. Morrison and M. R. Plesser, Non-Spherical Horizons, I, hep-th/9810201.

[34] J. de Boer, K. Hori, H. Ooguri and Y. Oz, Kahler Potential and Higher Derivative Terms from M Theory Fivebrane, Nucl. Phys. B518 (1998) 173, hep-th/9711143.

[35] O. Aharony and A. Hanany, Branes, Superpotentials and Superconformal Fixed Points, Nucl.Phys. B504 (1997) 239, hep-th/9704170.

[36] M. J. Strassler, On Renormalization Group Flows and Exactly Marginal Operators in Three Dimensions, hep-th/9810223.

[37] K. Oh and R. Tatar, Three Dimensional SCFT from M2 Branes at Conifold Singularities, hep-th/9810244.

[38] C. P. Boyer and K. Galicki, 3-Sasakian Manifolds, hep-th/9810250.

[39] E. Lopez, A Family of $N=1$ $SU(N)^k$ Theories from Branes at Singularities, hep-th/9812025.

[40] R. de Mello Koch and J. P. Rodrigues, Duality and Light Cone Symmetries of the Equations of Motion, hep-th 9709089, Phys. Lett. B432 (1998) 83.

[41] R. Dijkgraaf, E. Verlinde and H. Verlinde, Matrix String Theory, hep-th 9703030, Nucl. Phys. B500 (1997) 43.