Inference Functions for Semiparametric Models

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Abstract

The paper discusses inference techniques for semiparametric models based on suitable versions of inference functions. The text contains two parts. In the first part, we review the optimality theory for non-parametric models based on the notions of path differentiability and statistical functional differentiability. Those notions are adapted to the context of semiparametric models by applying the inference theory of statistical functionals to the functional that associates the value of the interest parameter to the corresponding probability measure. The second part of the paper discusses the theory of inference functions for semiparametric models. We define a class of regular inference functions, and provide two equivalent characterizations of those inference functions: One adapted from the classic theory of inference functions for parametric models, and one motivated by differential geometric considerations concerning the statistical model. Those characterizations yield an optimality theory for estimation under semiparametric models. We present a necessary and sufficient condition for the coincidence of the bound for the concentration of estimators based on inference functions and the semiparametric Cramèr-Rao bound. Projecting the score function for the parameter of interest on specially designed spaces of functions, we obtain optimal inference functions. Considering estimation when a sufficient statistic is present, we provide an alternative justification for the conditioning principle in a context of semiparametric models. The article closes with a characterization of when the semiparametric Cramèr-Rao bound is attained by estimators derived from regular inference functions.

Key words: Estimating functions, Quasi estimating functions; Quasi inference functions; Statistical functional differentiability; Statistical differential geometry; Non-parametric models.

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1 Introduction

In this article, we revise the classic optimality theory for non- and semiparametric models. A range of notions of path differentiability and tangent spaces are introduced and their inter-relations studied. Next, it is studied some concepts of statistical functional differentiability. Here the differentiability is considered relatively to a pointed cone contained in the tangent space and not relatively to the whole tangent space, as is currently in the literature. These cones are referred to as the tangent cones. The optimality theory of differentiable functionals is reviewed next. Again, the results are stated relative to the tangent cone and not with respect to the whole tangent space, as is usual. The estimation of the interest parameter of semiparametric models is studied by applying the optimality theory to a specially designed functional called the interest parameter functional, which associates to any probability measure in the model in play the value of the interest parameter associated to it. We will consider an increasing range of tangent cones. Here, the larger is the tangent cone used, the sharper is the bound for the concentration of regular estimators obtained. However, too large tangent cones may imply that the interest parameter functional is differentiable only under somehow stringent regularity conditions on the model. We show how the imposition of such conditions usually done in the literature can be avoided by using adequate choices of tangent cones. The bound for the concentration for regular inference sequences obtained with this choice of the tangent cone is referred to as the semiparametric Cramér-Rao bound.

2 Path and Functional Differentiability

We consider in this section some aspects of the general theory of non-parametric statistical models which will be useful for the theory of semiparametric models. The key notions introduced here are the path differentiability, the associated concept of tangent spaces and tangent sets, and the notions of functional differentiability.

In section 2.1 we study a range of concepts of path differentiability and comparisons of those notions are provided. An important point there is the equivalence between the Hellinger differentiability, often used in the literature (see Bickel et al., 1993), and the weak differentiability (see Pfanzagl, 1982, 1985 and 1990). Two auxiliary notions of path differentiability are introduced: strong and mean differentiability. It is proved that weak (or Hellinger differentiability) is an intermediate notion of path differentiability, weaker than strong differentiability and stronger than mean differentiability. A new notion of path differentiability, called essential differentiability, is introduced. We will interpret the tangents of essential differentiable paths as score functions of one dimensional “regular submodels” in the classical sense. Since the essential differentiability is weaker than the other notions provided, this interpretation extends immediately to all the other path
differentiability notions considered.

In section 2.2 some differentiability notions of functionals are studied. In the approach given a cone contained in the tangent set (i.e., the class of tangents of differentiable paths) is chosen and the differentiability of the functional in question will be defined relatively to this cone (termed tangent cone). Alternative notions of functional differentiability are given by adopting different notions of path differentiability and/or using different tangent cones. As we will see, the stronger the path differentiability notion used and the smaller is the tangent cone, the weaker the notion of differentiable functionals induced, in the sense that more statistical functionals are differentiable. We provide next some lower bounds for the concentration of “regular” sequences of estimators for a differentiable functional under a repeated sampling scheme. The weaker the path differentiability required and the larger the tangent cone adopted, the sharper are the bounds obtained. The theory will be applied to estimation in semiparametric models in section 2.3.

2.1 Differentiable paths

The main purpose of this section is to introduce the mathematical machinery necessary to extend the notion of score function, classically defined for parametric models, to a context where no (or only a partial) finite dimensional parametric structure is assumed. The key idea here is to consider one-dimensional submodels of the family \( P \) of probability measures (typically infinite dimensional). These submodels will be called paths. Following the steps of Stein (1956), one should consider a class of submodels (or paths) sufficiently regular in order to have a score function well defined and well behaved for each submodel, in the sense that, at least, each score function should be unbiased (i.e., have expectation zero) and have finite variance. Stein’s idea is to use the worst possible regular submodel to assess the difficulty of statistical inference procedures for the entire family \( P \). Evidently, if the class of ”regular submodels” is too small, no sensible results are to be expected from that procedure. On the other hand, if the class of ”regular submodels” is too large, the Stein’s procedure can become intractable or no simplification is really gained, which is not in the spirit of the method proposed. Hence, when applying the Stein procedure it is our task to find a class of ”regular submodels” with the adequate size.

The idea of ”regular submodel” mentioned will be formalised by introducing the notion of path differentiability. A range of concepts of path differentiability are studied in this section, all of them fulfilling the minimal requirement for a ”regular submodel”, i.e., the score functions of the differentiable paths (viewed as submodels) will be automatically well defined, unbiased and possess finite variances. The strongest notion of path differentiability considered is the \( L^\infty \) differentiability (or pointwise differentiability) and the weakest notion is the essential differentiability. It will turn out that a notion of path differentiability called “Hellinger differentiability” (or “weak differentiability”) is
the weakest notion that captures some important essential statistical properties of the model $\mathcal{P}$. Another distinguished notion considered is the $L^2$ differentiability which will involve calculations with Hilbert spaces, simplifying all the computations required. The $L^2$ differentiability coincides with the Hellinger differentiability in most of the examples considered in this thesis. It turns that the $L^2$ differentiability will be useful in the theory of inference functions.

This section is organised as follows. Subsection 2.1.1 studies the basic notion of path differentiability and some general properties of differentiable paths. Some specific concepts of differentiability are introduced in the subsections 2.1.2, 2.1.3 and 2.1.4 where weak or Hellinger, $L^q$ and essential differentiability are studied, respectively. The associated notions of tangent sets and tangent spaces are discussed in subsection 2.1.5.

2.1.1 General definition of path differentiability

We give next a more precise definition of the terms ”submodel” and ”regular submodel” informally used in the previous discussion. Recall that we were interested in defining a one-dimensional submodel contained in the family $\mathcal{P}$ for which the score function would be well defined and well behaved.

Let us consider a subset $V$ of $[0, \infty)$ which contains zero and for which zero is an accumulation point. The set $V$ will play the role of the parameter space in the ”submodel” we define. Typical examples are: $[0, \epsilon)$ for some $\epsilon > 0$ and $\{1/n : n \in N\} \cup \{0\}$. A mapping from $V$ into $\mathcal{P}^*$ assuming the value $p \in \mathcal{P}^*$ at zero is said to be a path converging to $p$. Here the image of $V$ under a path plays the role of the ”submodel” of $\mathcal{P}$ and the path acts as a one-dimensional parametrisation of the ”submodel”. It is convenient to represent a path by a generalised sequence $\{p_t\}_{t \in V} = \{p_t\}$, where for each $t \in V$, $p_t \in \mathcal{P}^*$ is the value of the path at $t$.

We introduce next the notion of differentiability which will enable us to formalise more precisely what in the Stein program is the class of ”regular submodels”. A path $\{p_t\}_{t \in V}$ (converging to $p$) is differentiable at $p \in \mathcal{P}^*$ if for each $t \in V$ we have the representation

$$p_t(\cdot) = p(\cdot) + tp(\cdot)\nu(\cdot) + tp(\cdot)r_t(\cdot)$$

for a certain $\nu(\cdot) \in L^2_0(p)$, and

$$r_t \longrightarrow 0, \quad \text{as } t \downarrow 0.$$  

The convergence in (2) is in some appropriate sense to be specified later. In fact, in the next subsections we explore several notions of path differentiability by introducing alternative definitions for that convergence. The term $r_t$ in (1) will be referred to as the remainder term.
The function $\nu : \mathcal{X} \rightarrow \mathbb{R}$ given in (1) is said to be the tangent associated to the differentiable path $\{p_t\}$. Here the tangent plays the role of the score function of the submodel parametrised by $t \in V$ at $p_0 = p$. To see the analogy with the score function suppose that the convergence of $r_t$ in (2) is in the sense of the pointwise convergence. In that case the tangent coincides with the score function of the submodel associated with the differentiable path $\{p_t\}$ at $p_0 = p$. In the general case, where the convergence of $r_t$ is not necessarily pointwise convergence, the general chain rule for differentiation of functions in metric spaces (see Dieudonné, 1960) can often be applied to justify our interpretation of the tangent. We stress that according to our definition, the tangent of a differentiable path (or alternatively the score of a regular submodel) has automatically finite variance and mean zero (i.e. it is in $L_2^0(p)$).

Before embracing the study of notions of differentiability generated by some specific definitions of the convergence of $r_t$, we give a useful and trivial general property of remainder terms of differentiable paths. Suppose that a path $\{p_t\}$ is differentiable at $p \in \mathcal{P}^*$ with representation given by (1), with $\nu \in L_2^0(p)$. Then we have, for each $t \in V$

$$r_t(\cdot) = \frac{p_t(\cdot)}{tp(\cdot)} - \nu(\cdot)$$

(3)

and

$$\int_{\mathcal{X}} r_t(x)p(x)\lambda(dx) = \int_{\mathcal{X}} \left\{ \frac{p_t(x)}{tp(x)} - \nu(x) \right\} p(x)\lambda(dx) = 0.$$

2.1.2 Hellinger and weak path differentiability

Most of the estimation theory for non- and semi-parametric models found in the literature (see Bickel et al., 1993 and references therein) is developed using the notion of Hellinger differentiability studied next. This notion appears in the literature in two equivalent forms: weak differentiability (see Pfanzagl 1982, 1985 and 1990) and Hellinger differentiability (see Hájeck, 1962, LeCam, 1966 and Bickel et al., 1993). This notion of differentiability plays a central role in the theory presented because it enables us to grasp some essential statistical properties of the models considered. For instance, the Hellinger differentiability is equivalent to local asymptotic normality of the submodel defined by the path. Moreover, the Hellinger differentiability is used in the so called convolution theorem, which gives a bound for the concentration of a rich class of estimators (the regular asymptotic linear estimators).

We begin by introducing the weak differentiability which is in the general form of path differentiability formulated before. A path $\{p_t\}_{t \in V}$ is weakly differentiable at $p \in \mathcal{P}^*$ if there exist $\nu \in L_2^0(p)$ and a generalised sequence of functions $\{r_t\}_{t \in V}$ such that for each
\( t \in V \)

\[
p_t(\cdot) = p(\cdot) + tp(\cdot) \nu(\cdot) + tp(\cdot) r_t(\cdot)
\]

and

\[
\frac{1}{t} \int_{\{x : |r_t(x)| > 1\}} |r_t(x)| p(x) \lambda(dx) \to 0, \quad \text{as } t \to 0 , \quad (4)
\]

\[
\int_{\{x : |r_t(x)| \leq 1\}} |r_t(x)|^2 p(x) \lambda(dx) \to 0, \quad \text{as } t \to 0 . \quad (5)
\]

In other words, \( \{p_t\} \) is weakly differentiable if it is differentiable according to the general definition of path differentiability with the convergence of the generalised sequence \( \{r_t\} \) given by (4) and (5).

Let us introduce now the Hellinger differentiability of paths. The key idea in this approach is to characterise the family \( \mathcal{P} \) of probability measures by the class of square roots of the densities, instead of the densities. The advantage of this alternative characterisation is that the square roots of the densities are in the Hilbert space

\[
L^2(\lambda) = \left\{ f : \mathcal{X} \to \mathbb{R} : \int_{\mathcal{X}} f^2(x) \lambda(dx) < \infty \right\}.
\]

In this way the statistical model in play is naturally embedded into a space with a rich mathematical structure. Using the usual topology of \( L^2(\lambda) \) one defines the differentiability of paths in the sense of Fréchet (or in this case, since the domain of the path is contained in \( \mathbb{R} \), the equivalent notions of Hadamard and Gateaux differentiability could be used also). The precise definition of Hellinger differentiability is the following. A path \( \{p_t\}_{t \in V} \) is \textit{Hellinger differentiable} at \( p \in \mathcal{P}^* \) if there exists a generalised sequence \( \{s_t\}_{t \in V} \) in \( L^2_0(p) \) converging to zero as \( t \downarrow 0 \), \textit{i.e.}

\[
\|s_t\|_p \to 0, \quad \text{as } t \downarrow 0 \quad (6)
\]

and \( \nu \in L^2_0(p) \) such that

\[
p_t^{1/2}(\cdot) = p^{1/2}(\cdot) + tp^{1/2}(\cdot) \frac{1}{2} \nu(\cdot) + tp^{1/2}(\cdot) s_t(\cdot) . \quad (7)
\]

The factor \( \frac{1}{2} \) in the second term of the right side of (7) will serve to accommodate with the other notions of differentiability. Note that each \( s_t \) is in fact in \( L^2_0(p) \). For, from (7)

\[
s_t(\cdot) = \frac{p_t^{1/2}(\cdot) - p^{1/2}(\cdot)}{tp^{1/2}(\cdot)} - \frac{\nu(\cdot)}{2} . \quad (8)
\]

Since \( \int_{\mathcal{X}} \left\{ \frac{p_t^{1/2}(x)}{p^{1/2}(x)} \right\}^2 p(x) \lambda(dx) = \int_{\mathcal{X}} p_t(x) \lambda(dx) = 1 < \infty \), we have that \( p_t^{1/2}(\cdot)/p^{1/2}(\cdot) \in L^2(p) \), and hence

\[
\frac{p_t^{1/2}(x)}{tp^{1/2}(x)} = \frac{1}{t} \left\{ \frac{p_t^{1/2}(x)}{p^{1/2}(x)} - 1 \right\} \in L^2(p),
\]

since \( \left\{ \frac{p_t^{1/2}(x)}{p^{1/2}(x)} - 1 \right\} \to 0 \) as \( t \to 0 \).
Proposition 1 A path \( \{p_t\} \) is Hellinger differentiable if and only if \( \{p_t\} \) is weak differentiable.

Proof: See Pfanzagl (1985).

2.1.3 \( L^q \) path differentiability

We study next a useful range of path differentiability notions. These notions will serve us to graduate how strong is the Hellinger or weak differentiability; and they will be used auxiliary in the calculation of the weak tangents of weak differentiable paths. In spite of the secondary role these differentiability notions play in our development, they are important in the general theory of differentiability of statistical functionals, in particular in the theory of von Mises functionals. The \( L^2 \) differentiability defined below will be useful when studying the use of inference functions for semiparametric models.

The main idea here is to consider the \( L^q \) convergence for the generalised sequence \( \{r_t\} \) appearing in the definition of differentiable paths. The precise definition is the following. A path \( \{p_t\}_{t \in V} \subseteq \mathcal{P}^* \) is \( L^q \) differentiable at \( p \in \mathcal{P}^* \), for \( q \in [1, \infty] \), if there exist \( \nu \in L^2_0(p) \) and a generalised sequence \( \{r_t\} \) in \( L^q(p) \) such that for each \( t \in V \),

\[
p_t(\cdot) = p(\cdot) + tp(\cdot)\nu(\cdot) + tp(\cdot)r_t(\cdot)
\]

and

\[
\|r_t\|_{L^q(p)} \to 0, \quad \text{as} \; t \downarrow 0.
\]

The following proposition relates the notions of \( L^q \) path differentiability.

Proposition 2 Consider \( r, q \in [1, \infty] \) such that \( r \leq q \). If a path is \( L^q \) differentiable at \( p \in \mathcal{P}^* \), then it is also \( L^r \) differentiable at \( p \) with the same tangent.

Proof: The proposition follows immediately from the fact that convergence in \( L^q(p) \) implies convergence in \( L^r(p) \). □

There are two distinguished cases of \( L^q \) path differentiability: strong and mean differentiability corresponding to \( L^2 \) and \( L^1 \) differentiability respectively. The \( L^1 \) differentiability is remarkable because it is the weakest notion of differentiability found in the literature, and the \( L^2 \) differentiability distinguish itself because the \( L^2 \) spaces, when endowed with the natural inner product, are Hilbert spaces, which simplifies significantly the calculations.
We study next the relation between weak and $L^q$ path differentiability. As we will see in the propositions 3 and 4 given above, weak differentiability is an intermediate notion of path differentiability between $L^2$ and $L^1$ differentiability.

**Proposition 3** If a path is $L^2$ differentiable at $p \in \mathcal{P}^*$, then it is weakly (or Hellinger) differentiable at $p$, with the same tangent.

**Proof:** Let $\{p_t\}$ be a differentiable path in the $L^2$ sense with representation (9) and $\|r_t\|_{L^2(p)} \to 0$ as $t \downarrow 0$. We show that the path $\{p_t\}$ fulfills the conditions (4) and (5) for the convergence of the remainder term in the sense of the weak path differentiability. For,

$$\frac{1}{t} \int_{\{x: |r_t(x)| > 1\}} |r_t(x)| p(x) \lambda(dx) \leq \frac{1}{t} \int_{\{x: |r_t(x)| > 1\}} t|r_t(x)| |r_t(x)| p(x) \lambda(dx)$$

$$= \int_{\{x: |r_t(x)| > 1\}} |r_t(x)|^2 p(x) \lambda(dx)$$

$$\leq \int_{\mathcal{X}} |r_t(x)|^2 p(x) \lambda(dx)$$

$$= \|r_t\|_p^2 \to 0, \text{ as } t \downarrow 0.$$  

Hence $\{r_t\}$ satisfies (4). On the other hand,

$$\int_{\{x: |r_t(x)| \leq 1\}} |r_t(x)|^2 p(x) \lambda(dx) \leq \int_{\mathcal{X}} |r_t(x)|^2 p(x) \lambda(dx) = \|r_t\|_p^2 \to 0, \text{ as } t \downarrow 0.$$  

Hence $\{r_t\}$ satisfies (5). We conclude that $\{p_t\}$ is differentiable in the weak sense with tangent $\nu$. $\square$

**Proposition 4** If a path is weakly (or Hellinger) differentiable at $p \in \mathcal{P}^*$, then it is $L^1$ differentiable (or differentiable in mean) at $p$, with the same tangent.

**Proof:** Take a path $\{p_t\}$ weakly differentiable at $p$ with tangent $\nu$. There exists a generalised sequence of functions $\{r_t\}$ satisfying (4) and (5), such that for all $t \in \mathcal{V}$,

$$p_t(\cdot) = p(\cdot) + tp(\cdot)\nu(\cdot) + tp(\cdot) r_t(\cdot).$$

Note that (4) implies that

$$\int_{\{x: |r_t(x)| > 1\}} |r_t(x)| p(x) \lambda(dx) \to 0, \text{ as } t \downarrow 0, \quad (11)$$
and (5) implies that
\[
\int_{\{x: |r_t(x)| \leq 1\}} |r_t(x)| p(x) \lambda(dx) \to 0, \text{ as } t \downarrow 0. \tag{12}
\]

For, (5) is equivalent to $L^2(p)$ convergence of $s_t(\cdot) := r_t(\cdot) \chi_{\{x: |r_t(x)| \leq 1\}}(\cdot)$ to zero. From (10),
\[
\int_{\{x: |r_t(x)| \leq 1\}} |r_t(x)| p(x) \lambda(dx) = \|s_t\|_{L^1(p)} \leq \|s_t\|_{L^2(p)} \to 0.
\]

Combining (11) and (12) we obtain
\[
\int_X |r_t(x)| p(x) \lambda(dx) = \int_{\{x: |r_t(x)| \leq 1\}} |r_t(x)| p(x) \lambda(dx)
+ \int_{\{x: |r_t(x)| > 1\}} |r_t(x)| p(x) \lambda(dx) \to 0, \text{ as } t \downarrow 0.
\]

We conclude that $\{r_t\}$ is $L^1$ differentiable at $p$ with tangent $\nu$. \hfill \square

2.1.4 Essential path differentiability

We study next the weakest notion of path differentiability considered in this text. A path $\{p_t\}_{t \in V} \subseteq P^*$ is essential differentiable at $p \in P^*$ if there exists $\nu \in L^2_0(p)$ and a generalised sequence $\{r_t: X \to \mathbb{R}\}_{t \in V}$ of $(\mathcal{A}, \mathcal{B}(\mathbb{R}))$-measurable functions such that for each $t \in V$,
\[
p_t(\cdot) = p(\cdot) + tp(\cdot)\nu(\cdot) + tp(\cdot)r_t(\cdot) \tag{13}
\]

and for any sequence $\{k_n\}_{n \in N} \subseteq V$ such that $k_n \to 0$ as $n \to \infty$ there is a subsequence $\{k_{i}\}_{i \in N} \subseteq \{k_n\}_{n \in N}$ such that $r_{k_i}(\cdot) \to 0$ $p$-almost surely as $i \to \infty$.

We show next that essential differentiability is weaker than differentiability in mean which, in view of propositions 3 and 4 implies that the essential differentiability is the weakest notion of path differentiability considered here.

**Proposition 5** If a path is $L^1$ differentiable, then it is essential differentiable, with the same tangent.

**Proof:** The generalised sequence $\{r_t\}_{t \in V}$ is Cauchy, because it converges in $L^1$ to zero. Using theorem 3.12 in Rudin (1987, page 68) the essential differentiability follows. \hfill \square
The following scheme represents the interrelation between the various notions of path differentiability considered.

\[
\begin{align*}
L^\infty & \text{ differentiability} \\
\downarrow & \\
L^p & \text{ differentiability} \\
\downarrow & \\
L^q & \text{ differentiability} \\
\downarrow & \\
L^2 & \text{ differentiability} \\
\downarrow & \\
\text{Weak differentiability} & \iff \text{Hellinger differentiability} \\
\downarrow & \\
L^1 & \text{ differentiability} \\
\downarrow & \\
\text{essential differentiability}
\end{align*}
\]

Here \(2 < p < r < \infty\).

2.1.5 Tangent spaces and tangent sets

Re-taking the Stein approach, the notion of differentiable path formalised the idea of "regular one-dimensional submodel", the tangent of a differentiable path playing the role of the score function of these submodels. Here we elaborate the notion of tangent set which is the class of all possible tangents of differentiable paths. This will be useful to work with the idea of "worst possible case" contained informally in the Stein method, and to specify global properties common to all the scores of "regular one-dimensional submodels". For technical reasons we need in fact to work in many situations with the smallest closed subspace containing the tangent set, which is called the tangent space.

In the next section we will define a notion of differentiability for statistical functionals. There the tangent set will play the role of "test functions", analogous to the role of test functions when one defines the differentiability of tempered distributions (see Rudin, 1973). The notion of tangent space plays a crucial role when studying the theory of models with nuisance parameters. There we will need to obtain a component of a partial score function orthogonal (in the \(L^2\) sense, \textit{i.e.} uncorrelated) to the scores of a model obtained by fixing the parameter of interest and letting the nuisance parameter vary. This component of the partial score function is obtained by orthogonal projection of the score function onto the orthogonal complement of the tangent space (or nuisance tangent space).
space as we will call the tangent space of the submodel we mentioned). It will then be comfortable to work with a closed subspace of $L^2$. We remark that it can be proved that the tangent set is a pointed cone, but in general not even a vector space. Therefore the necessity to introduce the notion of tangent space as given here.

The formal definition of tangent space and tangent set depends on the notion of path differentiability one uses. We give next a general definition of tangent set and tangent space which will be made precise when we specify the notion of path differentiability we use. Suppose we adopt a certain definition of path differentiability according to which a differentiable path at $p \in P^*$, say $\{p_t\}$, has representation, for each $t \in V$,

$$p_t(\cdot) = p(\cdot) + tp(\cdot)\nu(\cdot) + tp(\cdot)r_t(\cdot) \quad (14)$$

and

$$r_t \longrightarrow 0, \quad t \downarrow 0, \quad (15)$$

where the convergence in (15) is in a certain sense known. Then the tangent set of $P$ at $p \in P^*$ is the class

$$T^o(p) = T^o(p, \mathcal{P}) = \left\{ \nu \in L^2_0(p) : \exists V, \{p_t\}_{t \in V} \subseteq \mathcal{P}^*, \{r_t\}_{t \in V}, \text{ such that } \forall t \in V, (14) \text{ and } (15) \text{ hold } \right\}.$$

The tangent space of $P$ at $p \in P^*$ is given by

$$T(p) = T(p, \mathcal{P}) = cl_{L^2_0(p)}[span\{T^o(p, \mathcal{P})\}].$$

Since the tangent sets and spaces depend on the notion of path differentiability adopted, we speak of $L^q$ (for $q \in [1, \infty]$), weak (or Hellinger) tangent sets and tangent spaces. When necessary we use the notation $T^W$ for the weak tangent space. The $L^q$ tangent spaces are represented by $T^q$ and the essential tangent spaces by $T^e$.

The following proposition relates the notions of tangent sets and tangent spaces given.

**Proposition 6** For each $p \in \mathcal{P}^*$ and for $2 < q < r < \infty$ we have:

$$\Big( T^q(p) \subseteq T^r(p) \subseteq T^2(p) \subseteq T^W(p) \subseteq T^1(p) \subseteq T^e(p) \Big).$$

**Proof:** Straightforward from the interrelations between the notions of path differentiability. \[\square\]

We close this section with two examples of the calculation of tangent spaces.
Example 1 (Full tangent spaces of a large class of distributions) Consider the class $\mathcal{P}$ of all distributions in $\mathbb{R}$ dominated by the Lebesgue measure with continuous density (with respect to the Lebesgue measure) and with support (of the density) equal to the whole real line. Denote the class of densities of $\mathcal{P}$ by $\mathcal{P}^*$. We calculate the tangent space of $\mathcal{P}$ at each $p \in \mathcal{P}^*$.

Take an arbitrary element $\nu$ of $C_b \cap L^2_0(p)$. Here $C_b$ denotes the class of continuous compact supported functions from $\mathbb{R}$ to $\mathbb{R}$. It is a classical result of analysis that $C_b$ is dense in $L^2(p)$ (see Rudin, 1966), hence $C_b \cap L^2_0(p)$ is dense in $L^2_0(p)$. We show that $\nu \in T^0(p,\mathcal{P})$ (for any notion of tangent sets defined before). Consider the path $\{p_t\}$ given for $t \in [0, \infty)$ small enough, by

$$p_t(\cdot) = p(\cdot) + tp(\cdot)\nu(\cdot). \quad (16)$$

We claim that for $t$ sufficiently small, $p_t \in \mathcal{P}^*$, which implies that $\nu \in T^0(p,\mathcal{P})$. It suffices to verify that $p_t$ is positive and integrates $1$. For $t$ small $p_t$ is positive because $\nu$ is bounded and $p$ is bounded in the support of $\nu$, hence the second term in the right hand of (16) is smaller than $p$ (for $t$ small). That $p_t$ integrates $1$ follows from the fact that $\nu$ has expectation zero (with respect to $p$). \hfill $\square$

It is not surprising that the previous enormous class of distributions possesses a “full” tangent space. The next example show that this could be the case even in families where we have a lot of information about the distributions of the family.

Example 2 (Full tangent space for families with information on the moments) Consider the class $\mathcal{P}$ of all distributions in $\mathbb{R}$ dominated by the Lebesgue measure with continuous density (with respect to the Lebesgue measure) and with support (of the density) equal to the whole real line. Suppose further that the moments of all orders exist and that there exist a $\delta > 0$, a $k \in \mathbb{N}$ and the constants $m_1, \ldots, m_k$ such that for each $i \in \{1, \ldots, k\}$ the moment of order $i$ is contained in the open interval $(m_i - \delta, m_i + \delta)$. I claim that the tangent space of $\mathcal{P}$ at any $\mathcal{P}^*$ is $L^2_0(p)$. The proof follows the same line of the argument as given in the previous example. Take a path as in (16) with $\nu \in C_b \cap L^2_0(p)$. For $t$ sufficiently small, $p_t$ will be positive, integrate to one, possess finite moments of all orders, and the moments of order $i$, for $i \leq k$ will be contained in the interval $(m_i - \delta, m_i + \delta)$. \hfill $\square$
2.2 Functional differentiability

2.2.1 Definition and first properties of functional differentiability

We consider in this section a functional $\phi : P^* \rightarrow IR^q$ (for some $q \in \mathbb{N}$) which will play the role of a parameter of interest that we want to estimate. Typical examples are the mean and the second moment functionals defined by $\phi(p) = \int_X xp(x)\lambda(dx)$ and $\phi(p) = \int_X x^2p(x)\lambda(dx)$ respectively. An important non trivial example for the theory of semiparametric models is the interest parameter functional defined next and studied in detail in section 2.3.

Example 3 Semiparametric models

Suppose that the family $P^*$ of probability densities with respect to a measure $\lambda$ can be represented in the form

$$P^* = \{p(\cdot; \theta, z) : \theta \in \Theta \subseteq IR^q, z \in \mathcal{Z}\}.$$ 

Here it is assumed that the mapping $(\theta, z) \mapsto p(\cdot; \theta, z)$ is a bijection between $\Theta \times \mathcal{Z}$ and $P^*$. The interest parameter functional $\phi : P^* \rightarrow IR^q$ is defined, for each $p(\cdot; \theta, z) \in P^*$, by

$$\phi\{p(\cdot; \theta, z)\} = \theta.$$ 

We introduce next a notion of functional differentiability that will enable us to develop a theory of estimation for the functional $\phi$. Let $p$ be a fixed element of $P^*$. Consider a non empty subset $\mathcal{T}(p)$ of the tangent space at $p$. A functional $\phi : P^* \rightarrow IR^q$ is said to be differentiable at $p \in P^*$ with respect to $\mathcal{T}(p)$ if there exists a function $\phi^*_p : \mathcal{X} \rightarrow IR^q$, such that $\phi^*_p \in \{L^2(p)\}^q$ and for each $\nu \in \mathcal{T}(p)$ there is a differentiable path $\{p_t\}$ with tangent $\nu$ and

$$\frac{\phi(p_t) - \phi(p)}{t} \rightarrow <\phi^*_p, \nu>_p, \quad \text{as } t \downarrow 0. \quad (17)$$

Here $<\phi^*_p, \nu>_p$ is the vector with components given by the inner product of the $q$ components of $\phi^*_p$ and $\nu$. The function $\phi^*_p : \mathcal{X} \rightarrow IR$ is said to be a gradient of the functional $\phi$ at $p$ (with respect to $\mathcal{T}(p)$). Note that $\phi^*_p$ depends on the point $p$ at which we study the differentiability of the functional $\phi$. If a functional $\phi$ is differentiable at each $p \in P^*$ we say that $\phi$ is differentiable.

Since the definition of functional differentiability depends on the notion of path differentiability, we speak of $L^\infty$, $L^p$, strong ($L^2$), weak, mean ($L^1$) and essential functional
differentiability. When necessary we superpose a symbol indicating the notion of path differentiability in play. When we are speaking generically or when it is clear from the context which notion of path differentiability is in play, we just use the notation \( \phi^*_p \) for the gradient and \( T^0(p, \mathcal{P}^*) = T^0(p), T(p, \mathcal{P}^*) = T(p) \) for the tangent set and the tangent space of \( \mathcal{P}^* \) at \( p \) respectively.

Note that the notion of functional differentiability introduced here involves a subset \( \mathcal{T}(p) \) of the tangent space and not necessarily the whole tangent space as is current in the literature. This will give much more flexibility to the estimation theory developed. Clearly the smaller is the class \( \mathcal{T}(\mathcal{P}) \) (or the stronger is the notion of path differentiability) used, the weaker is the related functional differentiability. On the other hand, the larger is the class \( \mathcal{T}(p) \), the sharper will be the results of the estimation theory related, in the sense that the bounds for the lower asymptotic variance will be larger or the optimality results will include more estimating sequences. In this sense the ideal would be to choose the larger \( \mathcal{T}(p) \) (and the stronger path differentiability) that makes differentiable the functional under study. Of course, we will have to require some mathematical properties for the classes \( \mathcal{T}(p) \) in order to obtain a notion of functional differentiability useful for the estimation theory of differentiable functionals. For instance, it will be assumed through (and silently) that \( \mathcal{T}(p) \) is a pointed cone (i.e. if \( \nu \in \mathcal{T}(p) \), then for each \( \alpha \in \mathbb{R}_+ \cup \{0\} \), \( \alpha \nu \in \mathcal{T}(p) \)). We will refer form now on to \( \mathcal{T}(p) \) as the tangent cone. It will be necessary sometimes to require the tangent cones to be convex.

We consider next a trivial example that illustrates the mechanics of the functional differentiability.

**Example 4 (Mean functional)** Let \( \lambda \) be a \( \sigma \)-finite measure defined on a measurable space \((\mathcal{X}, \mathcal{A})\). Consider a family of probability measures \( \mathcal{P} \) on \((\mathcal{X}, \mathcal{A})\) dominated by \( \lambda \) given by the following representation

\[
\mathcal{P} = \left\{ \frac{dP}{d\lambda}(\cdot) = p(\cdot) : \quad (19) - (22) \text{ hold} \right\}. \tag{18}
\]

The conditions to define \( \mathcal{P} \) are

\[
\forall x \in \mathcal{X}, \quad p(x) > 0; \tag{19}
\]

\[
\int_{\mathcal{X}} p(x) \lambda(dx) = 1; \tag{20}
\]

\[
p \text{ is continuous}; \tag{21}
\]

\[
\int_{\mathcal{X}} x^2p(x) \lambda(dx) \in \mathbb{R}_+. \tag{22}
\]
We denote the class of densities of the elements of $\mathcal{P}$ with respect to $\lambda$ by $\mathcal{P}^\ast$. Define the functional $M : \mathcal{P}^\ast \rightarrow \mathbb{R}$ by, for each $p \in \mathcal{P}^\ast$

$$M(p) = \int_X x p(x) \lambda(dx).$$

We prove that $M$ is a differentiable functional with respect to the $L^2$ tangent space. As we have seen in the previous section the tangent space of $\mathcal{P}$ at any $p \in \mathcal{P}^\ast$ is the whole space $L^2_0(p)$.

Take $p \in \mathcal{P}^\ast$ fixed and an arbitrary $L^2$-differentiable path at $p$, say $\{p_t\}_{t \in V}$, with representation given by for each $t \in V$

$$p_t(\cdot) = p(\cdot) + tp(\cdot) \nu(\cdot) + tp(\cdot) r_t(\cdot),$$

where $\nu \in L^2_0(p)$, $\{r_t\} \subset L^2_0(p)$ and $r_t \xrightarrow{L^2_0(p)} 0$ as $t \downarrow 0$. We have,

$$\frac{M(p_t) - M(p)}{t} = \frac{\int_X x p_t(x) \lambda(dx) - \int_X x p(x) \lambda(dx)}{t} \xrightarrow{t \downarrow 0} <\nu(\cdot), (\cdot)>_p + <r_t(\cdot), (\cdot)>_p = <\nu(\cdot), (\cdot)>_p,$$

The last convergence comes from the continuity of the inner product and the $L^2(p)$ convergence of the path remainder term to zero.

Define the function

$$M^\ast_p(\cdot) = (\cdot) - \int_X x p(\cdot) \lambda(dx).$$

Clearly, $M^\ast_p$ is in $L^2_0(p)$ and

$$<\nu, M^\ast_p>_p = <\nu(\cdot), (\cdot) - \int_X x p(x) \lambda(dx)>_p = <\nu(\cdot), (\cdot)>_p.$$

Since (24) and (23) hold for any $L^2$ differentiable path, we conclude that $M$ is differentiable with respect to the $L^2$ tangent set and $M^\ast_p$ is a gradient of $M$. An argument based on subsequences (c.f. Labouriau, 1998) yields the differentiability of $M$ with respect to the essential tangent set, i.e. the mean functional is differentiable with is the strongest sense we can define in our setup.

Note that in this example (17) holds for any differentiable path with tangent $\nu \in \mathcal{P}(p)$. However, according to our definition of functional differentiability it would be enough if the condition (17) holds for one path with tangent $\nu$. $\Box$
Let \( \phi : \mathcal{P}^* \to \mathbb{R}^q \) be a differentiable functional at \( p \in \mathcal{P}^* \) with gradient \( \phi_p^* : \mathcal{X} \to \mathbb{R} \). It follows immediately from the definition of gradient that a function \( \phi_p^* : \mathcal{X} \to \mathbb{R} \) in \( L_0^2(a) \) is also a gradient of \( \phi \) at \( p \) if and only if,

\[
\forall \nu \in \mathcal{T}(p), \quad <\nu, \phi_p^*>_p = <\nu, \phi_p^*>_p .
\] (25)

We conclude from the remark above that if \( \phi_p^* \) is a gradient of \( \phi \) at \( p \) and \( \xi \in \{ \mathcal{T}(p) \}^\perp \) (i.e. \( \xi \) is in the orthogonal complement of the tangent space with respect to \( L_0^2(p) \)), then \( \phi_p^* + \xi \) is also a gradient of \( \phi \) at \( p \). Hence, in general the gradient of a differentiable functional is not unique.

A gradient \( \phi_p^* \) of a differentiable functional at \( p \in \mathcal{P}^* \) is said to be a canonical gradient if \( \phi_p^*(\cdot) \in \mathcal{T}(p) \). Here \( \mathcal{T}(p) \) denotes the \( L^2 \) closure of the space spanned by \( \mathcal{T}(p) \). The following proposition shows that there exists only one canonical gradient (apart from almost surely equal functions) and gives a recipe to compute the canonical gradient, namely by orthogonal projecting any gradient onto \( \mathcal{T}(p) \). We will see that the canonical gradient plays a crucial role in the theory of estimation of functionals.

**Proposition 7** Let \( \phi : \mathcal{P}^* \to \mathbb{R} \) be a differentiable functional at \( p \in \mathcal{P}^* \). If \( \phi_p^* : \mathcal{X} \to \mathbb{R}^q \) is a gradient of \( \phi \) at \( p \), then the vector formed by the orthogonal projection of components of \( \phi_p^* \) onto \( \mathcal{T}(p) \), say

\[
(\prod \{\phi_{1p}^*|\mathcal{T}(p)\}, \ldots, \prod \{\phi_{qp}^*|\mathcal{T}(p)\})^T,
\]

is also a gradient of \( \phi \) at \( p \). Furthermore, if \( \phi_p^* \) is another gradient of \( \phi \) at \( p \), then

\[
(\prod \{\phi_{1p}^*|\mathcal{T}(p)\}, \ldots, \prod \{\phi_{qp}^*|\mathcal{T}(p)\})^T = (\prod \{\phi_{1p}^*|\mathcal{T}(p)\}, \ldots, \prod \{\phi_{qp}^*|\mathcal{T}(p)\})^T,
\]

\( p \) almost surely.

**Proof:** We prove the proposition for the case where \( q = 1 \). The same argument applied componentwisely proves the case for \( q \in \mathbb{N} \), but with a more notation. From the projection theorem we have the following orthogonal decomposition

\[
\phi_p^* = \prod \{\phi_{1p}^*|\mathcal{T}(p)\} + \prod \{\phi_{p}^*|\mathcal{T}^\perp(p)\} .
\]

Here \( \mathcal{T}^\perp(p) \) is the orthogonal complement of \( \mathcal{T}(p) \) in \( L_0^2(p) \). Hence

\[
\prod \{\phi_{p}^*|\mathcal{T}(p)\} = \phi_p^* - \prod \{\phi_{p}^*|\mathcal{T}^\perp(p)\} .
\]

Since \( \prod \{\phi_{p}^*|\mathcal{T}^\perp(p)\} \) is orthogonal to \( \mathcal{T}(p) \), we conclude from (25) that \( \prod \{\phi_{p}^*|\mathcal{T}(p)\} \) is a gradient.
Reasoning analogously we conclude that if $\phi^*$ is another gradient of $\phi$ at $p$, then

$$\prod\{\phi^*_p|\mathcal{T}(p)\} = \phi^*_p - \prod\{\phi^*_p|\mathcal{T}^\perp(p)\}.$$ 

is a gradient of $\phi$ at $p$. From (25), for all $\nu \in \overline{T}(p)$

$$\langle \prod\{\phi^*_p|\mathcal{T}(p)\}, \nu \rangle_p = \langle \prod\{\phi^*_p|\mathcal{T}(p)\}, \nu \rangle_p$$

and hence, for all $\nu \in \overline{T}(p)$,

$$\langle \prod\{\phi^*_p|\mathcal{T}(p)\} - \prod\{\phi^*_p|\mathcal{T}(p)\}, \nu \rangle_p = 0.$$ (26)

In particular (26) holds for

$$\nu = \prod\{\phi^*_p|\mathcal{T}(p)\} - \prod\{\phi^*_p|\mathcal{T}(p)\},$$

which yields

$$\| \prod\{\phi^*_p|\mathcal{T}(p)\} - \prod\{\phi^*_p|\mathcal{T}(p)\} \|^2_{L^2(p)} = 0.$$ 

We conclude that $\prod\{\phi^*_p|T(p, \mathcal{P})\} = \prod\{\phi^*_p|T(p, \mathcal{P})\}$ almost surely.

\[ \square \]

**Example 5 (Mean functional continued)** It can be shown that the tangent space of the model $\mathcal{P}$ given by (18) at each $p \in \mathcal{P}^*$ is the whole space $L^2_0(p)$. Hence the gradient calculated in example 4 is the canonical gradient. Moreover, the canonical gradient is the only possible gradient for the mean functional. Note that if we drop the condition that requires the existence of the variance of $p$ (i.e. condition (22)), then $M^\bullet_p$ is no longer a gradient (because it is not in $L^2$) and $M$ is not differentiable at $p$.

\[ \square \]

We consider next a proposition given trivial (but useful) rules for calculating gradients of “composed” gradients.

**Proposition 8** Let $\Psi, \phi : \mathcal{P} \rightarrow \mathbb{R}^q$ be two differentiable functionals with (canonical) gradient at $p \in \mathcal{P}^*$ $\Psi^\bullet$ and $\phi^\bullet$ respectively. Let $g : \mathbb{R}^q \rightarrow \mathbb{R}^q$ be a differentiable function.

i) For all $a, b \in \mathbb{R}^q$, $a \Psi + b \phi$ is differentiable at $p$ and its (canonical) gradient is given by $a \Psi^\bullet + b \phi^\bullet$ ($a \Psi^\bullet + b \phi^\bullet$).

ii) $g \circ \phi$ is differentiable at $p$ functional with gradient $\nabla g\{\phi(p)\}\{\phi^\bullet(\cdot)\}^T$. If $\phi^\bullet$ is the canonical gradient of $\phi$ then $\nabla g\{\phi(p)\}\{\phi^\bullet(\cdot)\}^T$ is the canonical gradient of $g \circ \phi$. 

Proof:

(i) Straightforward.

(ii) We give next the proof for the case where \( q = 1 \). The general case is obtained in a similar way. Take an arbitrary differentiable path \( \{p_t\} \) with tangent \( \nu \). Define \( \xi(t) = \phi(p_t) \), we have

\[
\frac{\phi(p_t) - \phi(p)}{t} \to <\nu, \phi^* >_p = \xi'(0).
\]

Now,

\[
\frac{(g \circ \phi)(p_t) - (g \circ \phi)(p)}{t} \to (g \circ \xi)'(0) = g(\xi(0)) <\nu, \phi^* >_p = <\nu, \phi(p) \phi^* >_p.
\]

\[\square\]

2.2.2 Asymptotic bounds for functional estimation

We study next some results concerning the estimation of a differentiable statistical functional under repeated sampling. These results will illustrate the importance of the canonical gradient and will guide the choice of the notion of path differentiability and tangent cone to be used.

We start by defining sequences of estimators for a given differentiable functional \( \phi : \mathcal{P} \to \mathbb{R}^q \) (with respect to some tangent cones \( T(p) \)) based on samples. A sequence of functions \( \{\hat{\phi}_n\}_{n \in \mathbb{N}} = \{\hat{\phi}_n\} \) such that for each \( n \in \mathbb{N} \), \( \hat{\phi}_n : X^n \to \mathbb{R}^q \) is \((\mathcal{A}^n, \mathcal{B}(\mathbb{R}^q))\)-measurable is said to be an estimating sequence. Next we introduce two notions of regularity of estimating sequences often found in the literature. An estimating sequence \( \{\hat{\phi}_n\} \) is said to be weakly regular (for estimating \( \phi \), with respect to the choice of tangent cones made) if for each \( p \in \mathcal{P}^* \) and each \( \nu \in \mathcal{T}(p) \) there exists a differentiable path \( \{p_{n^{-1/2}}\}_{n \in \mathbb{N}} \) converging to \( p \) and with domain \( V = \{n^{-1/2} : n \in \mathbb{N}\} \), for which

\[
\sqrt{n}(\phi(p_{n^{-1/2}}) - \phi(p)) \to \int_X \phi^*(x,p)\nu(x)p(x)\lambda(dx)
\]

and there exists a probability distribution \( L_{p\nu} \) (not depending on the path) such that

\[
\mathcal{L}_{p_{n^{-1/2}}} \left[ \sqrt{n}(\hat{\phi}_n(\cdot)\phi(p)) \right] \xrightarrow{D} L_{p\nu}.
\]

If the distributions \( L_{p\nu} \) above do not depend on the tangent \( \nu \), then we say that \( \{\hat{\phi}_n\}_{n \in \mathbb{N}} \) is regular.
An important class of estimating sequences are the asymptotic linear sequences defined next. An estimating sequence \( \hat{\phi}_n \) is said to be asymptotic linear (for estimating \( \phi \)) if there exists a function \( IC_\phi: \mathcal{X} \times \mathcal{P}^* \rightarrow \mathbb{R} \) such that for each \( p \in \mathcal{P}^* \), the function \( IC_\phi(\cdot; p): \mathcal{X} \rightarrow \mathbb{R} \) is in \( L^2_\phi(p) \) and for each \( n \in \mathbb{N} \) given a sample \( x = (x_1, \ldots, x_n) \) of size \( n \), \( \hat{\phi}_n \) admits the following representation

\[
\hat{\phi}_n(x) = \phi(p) + \frac{1}{n} \sum_{i=1}^{n} IC_\phi(x_i; p) + o_{p^n}(n^{-1/2}).
\]  

(27)

The function \( IC_\phi \) is called the influence function of \( \phi \). The representation (27) can be re-written as

\[
\sqrt{n} \left\{ \hat{\phi}_n - \phi(p) \right\} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} IC_\phi(x_i; p) + o_{p^n}(1).
\]

From the central limit theorem and the Slutsky theorem

\[
\sqrt{n} \left\{ \hat{\phi}_n - \phi(p) \right\} \xrightarrow{D} N[0, \text{Cov}_p\{IC_\phi(\cdot; p)\}],
\]

where

\[
\text{ov}_p\{IC_\phi(\cdot; p)\} = \int_{\mathcal{X}} IC_\phi(x; p)IC_\phi^T(x; p)p(x)\lambda(dx).
\]  

(28)

**Theorem 1** Let \( \{\hat{\phi}_n\} \) be an asymptotic linear estimating sequence with influence function \( IC \). Suppose that for each \( p \in \mathcal{P}^* \) the tangent cone is given by \( \mathcal{T}(p) = T^0(p, \mathcal{P}^*) \). Then, \( \{\hat{\phi}_n\} \) is regular if and only if for all \( p \in \mathcal{P}^* \), \( \phi \) is differentiable at \( p \) (with respect to \( \mathcal{T}(p) \)) and \( IC(\cdot; p) \) is a gradient of \( \phi \) at \( p \).

**Proof:** See Pfanzagl (1990) for the case where \( q = 1 \) or Bickel et al. (1995). \( \Box \)

The theorem above identifies (influence functions of) regular asymptotic linear sequences of estimators for estimating the functional \( \phi \) with the gradients of \( \phi \). The covariance, \( \int \phi_p^\star(x)\phi_p^\star(x)^T p(x)\lambda(dx) \), of a gradient \( \phi_p^\star \) of \( \phi \) is the asymptotic covariance of the corresponding regular asymptotic linear estimating sequence (under \( p \)) with influence function \( \phi_p^\star \). On the other hand, since the components of the canonical gradient \( \phi^\star \) of \( \phi \) are the orthogonal projection of the components of any gradient onto the tangent space, we have for a given gradient \( \phi_p^\star \) and for all \( p \in \mathcal{P}^* \)

\[
\phi_p^\star(\cdot) = \phi_p^\star(\cdot) + R(\cdot; p),
\]
for some $R(\cdot; p) \in \{T^1(p; \mathcal{P})\}^q$. A standard argument yields then that, for all $p \in \mathcal{P}^*$,
\[
\int_X \left\{ \phi_p^*(x) \right\}^T p^*(x) \lambda(dx) \;
\leq \; \int_X \left\{ \phi_p^*(x) \right\}^T p(x) \lambda(dx),
\]
with inequality in the sense of the Löwner partial order of matrices. That is, the covariance of the canonical gradient is a lower bound for the asymptotic covariance of regular asymptotic linear estimating sequences. Moreover, only an asymptotic linear estimating sequence with influence curve equal to the canonical gradient achieves this bound. We say that an asymptotic linear estimating sequence is optimal if, for each $p \in \mathcal{P}^*$, its influence function is the canonical gradient of $\phi$. The bound (29) is sometimes called the semiparametric Cramér-Rao bound.

In spite of the elegance of this theory, some care should be observed in applying it. Firstly, there is a certain degree of arbitrariness in choosing only the class of regular asymptotic linear estimating sequences. When restricting to that class one can discard many interesting sequences. This criticism applies, of course, to any optimality approach. A second, more specific criticism is the following: It occurs very often that the tangent space of large (semi- or non-parametric models) is the whole space $L^2_0$ (see the examples at the end of the section on tangent spaces). In those cases, due to the uniqueness of the canonical gradient, each differentiable functional possesses only one gradient. We conclude from the previous discussion that then there is only one possible influence function and hence all regular asymptotic linear estimating sequences are asymptotically equivalent (as far as the asymptotic variance is concerned). Therefore an optimality theory for regular asymptotic linear estimators is meaningless for the models with tangent spaces equal to the whole $L^2_0$. We refine next the optimality theory for functional estimation.

It is convenient to introduce the following notation. Given a differentiable functional $\phi$ with respect to the tangent cones $\{ T(p) : p \in \mathcal{P}^* \}$ and with canonical gradient $\phi^* (\cdot, p)$ at each $p \in \mathcal{P}^*$, denote $\int_X \phi_p^*(x) \phi_p^*(x)^T p(x) \lambda(dx)$ by $I_\phi(p)$. That is $I_\phi(p)$ is the covariance matrix of the canonical gradient. A weakly regular estimating sequence $\{ \hat{\phi}_n \}$ is asymptotically of constant bias at $p \in \mathcal{P}^*$ if for each $\nu, \eta \in T(p)$
\[
\int xdL_{\nu}(x) = \int xdL_{\eta}(x) \in \mathbb{R}^q.
\]
In particular, any regular estimating sequence is asymptotically of constant bias.

**Theorem 2 (van der Vaarts extended Cràmer-Rao theorem)** Let $\phi : \mathcal{P}^* \rightarrow \mathbb{R}^q$ be a differentiable at $p \in \mathcal{P}^*$ with respect to $T(p) \subseteq T(p)$. Suppose that the sequence $\{ \hat{\phi}_n \}$ is weakly regular and asymptotically of constant bias at $p \in \mathcal{P}^*$. Suppose also that the covariance matrix of $L_{\rho0}$ exists. Then
\[
\text{Cov}(L_{\rho0}) \geq I_\phi(p),
\]
(30)
where the symbol "\( \geq \)" is understood in the sense of the L"owner partial order of matrices. Moreover, the equality in (30) occurs only if

\[
\sqrt{n}\{\hat{\phi}_n - \phi(p)\} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \phi_p^*(x_j) + o_P(1) .
\]  

(31)

**Proof:** See van der Vaart (1980).

We see from the theorem above that the larger are the tangent cones \( \mathcal{T}(p) \) used, the sharper are the inequalities (30). Small tangent cones make more likely the differentiability of the functional but can make also the bound in (30) unattainable.

Another important optimality result in the theory of estimation of functionals is the convolution theorem, which we give the following version.

**Theorem 3 (Convolution theorem)** Suppose that \( \mathcal{T}(p) \) is convex and \( \phi: \mathcal{P}^* \rightarrow \mathbb{R}^q \) differentiable at \( p \in \mathcal{P}^* \) with respect to \( \mathcal{T}(p) \). Then any limiting distribution \( L_p \) of a regular estimating sequence for \( \phi \) at \( p \) satisfies

\[
L_p = N(0, I_{\phi}(p)) * M,
\]

(32)

where \( M \) is a probability measure on \( \mathbb{R}^q \).

**Proof:** See Pfanzagl (1990) for the case where \( q = 1 \) and \( \mathcal{T}(p) = \mathcal{T}(p) \) and van der Vaart (1980) for the general case.

The expression (32) shows that, under the assumptions of the convolution theorem, a regular estimating sequence cannot possess asymptotic covariance smaller than the \( L^2(p) \) squared norm of the canonical gradient. This provides an extension of the interpretation of the optimality theory for regular asymptotic linear estimating sequences. In fact, even when the tangent cone is is the whole \( L^2_0 \), the “optimal” regular asymptotic linear estimating sequence attains the bound for the concentration of regular estimating sequences given by the convolution theorem, provide the functional is differentiable. An advantage of the version of the convolution theorem presented is that we need not to work with the whole tangent space but with a convex cone of it. This can be useful when the functional in study is not differentiable or when the calculation of the (weak) tangent space is not feasible.

We close this section presenting a theorem that gives a minimax approach to the problem of estimation of functionals. A function \( l: \mathbb{R}^q \rightarrow \mathbb{R} \) is sad to be bowl-shaped if \( l(0) = 0, l(x) = l(-x) \) and for all \( k \in \mathbb{R}, \{x : l(x) \leq k\} \) is convex.

\(^{1}\)That is \( A \geq B \) means that \( A - B \) is positive definite.
Theorem 4 (Local asymptotic minimax theorem) Suppose that for each $p \in \mathcal{P}^*$, $\mathcal{T}(p) \subseteq \mathcal{W}(p)$ is convex and $\phi : \mathcal{P}^* \rightarrow \mathbb{R}^d$ differentiable at $p \in \mathcal{P}^*$ with respect to $\mathcal{T}(p)$. Then

i) For any sequence of estimators which is weakly regular at $p$ and bowl-shaped loss function $l$

$$\sup_{\nu \in \mathcal{T}(p)} \int l(x) dF_{\nu}(x) \geq \int l(x) dN(0, I_\phi(p))(x).$$

(33)

ii) For any bowl-shaped loss function $l$ and any estimating sequence $\{\hat{\phi}_n\}$,

$$\lim_{c \to \infty} \lim_{n \to \infty} \sup_{Q \in H_n(p, c)} E_Q\{l[\sqrt{n}\{\hat{\phi}_n - \phi(Q)\}]\} \geq \int l(x) dN(0, I_\phi(p))\lambda(dx),$$

(34)

where $H_n(p, c) := \{Q \in \mathcal{P} : n \int \{dQ^{1/2}(x) - p^{1/2}(x)\}^2 \lambda(dx)\}$ is the interception between $\mathcal{P}$ and the ball constructed with the Hellinger distance of center $p$ and radius $n^{-1/2}$.

Proof: See van der Vaart (1980).

Note that from part i) one can obtain a bound for the concentration of weakly regular estimating sequences based on the canonical gradient, provided $\phi$ is differentiable with respect to some convex tangent cones. In particular, if there exist an optimal asymptotic linear estimating sequences and the assumptions of the theorem hold (i.e. differentiability of $\phi$ and convexity of the tangent cone), then the bound for weak regular estimating sequences given by (33) is attained by this regular asymptotic linear estimating sequence. In this way, in the case where the tangent space is the whole $L^2_0$, the optimality of the (unique) regular asymptotic linear estimating sequence can be justified. The bound of the second part of the theorem above holds for the whole class of estimators, however it is in general not attainable.
2.3 Asymptotic bounds for semiparametric models

We consider a family of distributions $P$ dominated by a $\sigma$-finite measure $\lambda$ with representation

$$P^* = \left\{ \frac{dP_{\theta z}}{d\lambda}(\cdot) = p(\cdot; \theta, z) : \theta \in \Theta \subseteq \mathbb{R}^q, z \in \mathcal{Z} \right\}.$$ 

Here $\theta$ is a $q$-dimensional interest parameter and $z$ is a nuisance parameter of arbitrary nature. We assume that $\Theta$ is open and that the mapping $(\theta, z) \mapsto p(\cdot; \theta, z)$ is a bijection between $\Theta \times \mathcal{Z}$ and $P^*$. The interest parameter functional $\phi : P^* \rightarrow \mathbb{R}^q$ is defined, for each $p(\cdot; \theta, z) \in P^*$, by

$$\phi\{p(\cdot; \theta, z)\} = \theta.$$ 

We will consider the differentiability of the interest parameter functional $\phi$ for a range of tangent cones.

Recall that we assumed that for each $(\theta_0, z_0) \in \Theta \times \mathcal{Z}$,

$$\forall x \in \mathcal{X}, \ p(x; \theta_0, z_0) > 0,$$

that the partial score function

$$l(x; \theta_0, z_0) = \frac{\nabla p(x; \theta, z_0)|_{\theta = \theta_0}}{p(x; \theta_0, z_0)} = (l_1(x; \theta_0, z_0), \ldots, l_q(x; \theta_0, z_0))^T$$

is $\lambda$-almost everywhere well defined and that for $i = 1, \ldots, q$,

$$l_i(x; \theta_0, z_0) \in L^2_0(P_{\theta_0, z_0}).$$

Let us consider a fixed $(\theta, z) \in \Theta \times \mathcal{Z}$ at which we will study the differentiability of $\phi$. For notational simplicity we denote $p(\cdot; \theta, z)$ by $p(\cdot)$.

The first tangent cone we consider is

$$T_1(p) = span\{l_i(x; \theta, z) : i = 1, \ldots, q\}.$$ 

Take $\nu \in T_1(p)$. There exists $\alpha \in \mathbb{R}^q$ such that $\nu(\cdot) = \nu^T(\cdot; \theta, z)\alpha$. Define (for $t$ small enough) the path

$$p_t(\cdot) = p(\cdot; \theta + t\alpha, z).$$ 

Clearly, there exists $\{r_t\}$ such that

$$\nu^T(\cdot; \theta, z)\alpha = \frac{p(\cdot; \theta + t\alpha, z) - p(\cdot)}{tp(\cdot)} + r_t(\cdot), \quad (35)$$
with \( r_i(\cdot) \to 0 \) \( \lambda \)-almost everywhere. Hence the path \( \{p_t\} \) is \( (L^\infty) \) differentiable with tangent \( l^T(\cdot; \theta, z)\alpha \). Moreover,

\[
\frac{\phi(p_t) - \phi(p)}{t} = \frac{\theta + t\alpha - \theta}{t} = \alpha.
\]

Defining \( \phi^*_p(\cdot) = \text{Cov}_{\theta z}^{-1}\{l(\cdot; \theta, z)\} l(\cdot; \theta, z) \) we obtain,

\[
\int_X \phi^*_p(x) \nu(x)p(x) \lambda(dx) = \int_X \text{Cov}_{\theta z}^{-1}\{l(\cdot; \theta, z)\} l(x; \theta, z) l^T(x; \theta, z) \alpha p(x) \lambda(dx)
\]

\[
= \alpha = \lim_{t \to 0} \frac{\phi(p_t) - \phi(p)}{t}.
\]

We conclude that \( \phi \) is differentiable at \( p \) with respect to \( T_1(p) \). Moreover,

\[
\text{Cov}_{\theta z}(l(\cdot; \theta, z))^{-1}l(\cdot; \theta, z)
\]

is the canonical gradient of \( \phi \). Note that we used (in (35)) implicitly the \( L^\infty \) path differentiability, however the argument presented holds for any weaker path differentiability. For, note that the essential point is that we identify (through (35)) any element of the tangent cone \( T_1(p) \) with a \( L^\infty \) differentiable path. If we adopt a path differentiability weaker than the \( L^\infty \) differentiability, then the \( L^\infty \) differentiable paths identified with the elements of the tangent cone would be differentiable in the current sense also and the differentiability of the functional \( \phi \) follows from the argument presented above.

The efficient scores \( I_{\phi}(p) \) (i.e. the correlation matrix of the canonical gradient of \( \phi \) at \( p \)) is the inverse of the correlation matrix of the score function \( l(\cdot; \theta z) \). The bounds for the asymptotic variance obtained with this naive choice of tangent cones are not attainable in general. This will be apparent from the development presented next where sharper bounds will be presented.

We introduce the notion of nuisance tangent space that plays a fundamental rule in the estimation theory in semiparametric models. For each \( \theta_0 \in \Theta \) consider the submodels

\[
\mathcal{P}_{\theta_0}^* = \{p(\cdot; \theta_0, z) : z \in \mathcal{Z}\}.
\]

The nuisance tangent set at \( (\theta, z) \in \Theta \times \mathcal{Z} \), \( T_N^0(\theta, z) \), is the tangent set of \( \mathcal{P}_{\theta}^* \), i.e. \( T_N^0(\theta, z) = T^0(p, \mathcal{P}_{\theta}^*) \). The closure of the space spanned by the nuisance tangent set is called the nuisance tangent space and denoted by \( T_N(\theta, z) \). Here we do not specify the notion of path differentiability adopted, but when necessary a symbol will be superimposed.

An alternative for the tangent cone better than \( T_1(p) \) is

\[
T_2(p) = \text{span}\{l_i(x; \theta, z) : i = 1, \ldots, q\} \cup T^0_N(\theta, z).
\]
We show next that \( \phi \) is differentiable with respect to \( T_2(p) \), no matter which notion of path differentiability we use. Consider a \( \nu \in T_0^0(p) \subseteq T_2(p) \). There is a differentiable path \( \{ p_t \} \) contained in \( \mathcal{P}_0^* \) with tangent \( \nu \). Since for each \( t, p_t \in \mathcal{P}_0^* \), \( \phi(p_t) = \theta = \phi(p) \) and

\[
\frac{\phi(p_t) - \phi(p)}{t} = 0.
\]

From the definition of functional differentiability, any gradient \( \phi_p^* \) of \( \phi \) should satisfy, for each \( \nu \in T_0^0(\theta, z) \),

\[
0 = \lim_{t \to 0} \frac{\phi(p_t) - \phi(p)}{t} = \int_X \phi_p^*(x) \nu(x) p(x) \lambda(dx).
\]

On the other hand, the argument presented in the case of the tangent cone be \( T_1(p) \) implies that, if \( \nu \in \text{span}\{l_i(x; \theta, z) : i = 1, \ldots, q\} \), say \( \nu(\cdot) = l(\cdot; \theta, z)^T \alpha \), for some \( \alpha \in \mathbb{R}^q \), then any gradient \( \phi_p^* \) of \( \phi \) satisfies,

\[
\alpha = \int_X \phi_p^*(x) \nu(x) p(x) \lambda(dx).
\]

Clearly, the conditions (36) and (37) are sufficient to ensure that \( \phi_p^* \) is a gradient of \( \phi \). From these considerations, a natural candidate for being a gradient of \( \phi \) is the (standardised) projection of the score function onto the orthogonal complement of the nuisance tangent space. Formally, define the function \( l^E : X \times \Theta \times Z \to \mathbb{R}^q \) by, for each \( (\theta, z) \in \Theta \times Z \), \( l^E(\cdot; \theta, z) = (l^E_1(\cdot; \theta, z), \ldots, l^E_q(\cdot; \theta, z))^T \) where, for \( i = 1, \ldots, q \),

\[
l^E_i(\cdot; \theta, z) = \prod(l_i(\cdot; \theta, z)|T_N^1(\theta, z)).
\]

Here \( \prod(g|A) \) is the orthogonal projection of \( g \in L_0^2(P_{\theta z}) \) onto \( A \subseteq L_0^2(P_{\theta z}) \). Moreover, \( T_N^1(\theta, z) \) is the orthogonal complement of \( T_N(\theta, z) \) in \( L_0^2(P_{\theta z}) \). The function \( l^E \) is called the efficient score function and we define the efficient score by

\[
J(\theta, z) = \int_X l^E(x; \theta, z) l^E(x; \theta, z)^T p(x) \lambda(dx).
\]

Define

\[
\phi_p^*(\cdot) = J(\theta, z)^{-1} l^E(x; \theta, z).
\]

Clearly \( \phi_p^* \) satisfies (36) and (37). We conclude that \( \phi_p^* \) is a gradient of \( \phi \). Moreover, \( \phi_p^* \) is the canonical gradient (with respect to \( T_2(p) \)), since \( \phi_p^* \) is in the closure of the span of the tangent cone.
Note that choosing $\mathcal{T}_2(p)$ as the tangent cone, the functional $\phi$ is still differentiable and we obtain a bound related with the extended Cramér-Rao inequality sharper than the bound obtained with $\mathcal{T}_1(p)$. However, since the $\mathcal{T}_2(p)$ is not necessarily convex, it is impossible to use the convolution theorem and the local minimax theorem.

A third alternative for the tangent cone is
\[
\mathcal{T}_3(p) = \text{span}\{l_i(x; \theta, z) : i = 1, \ldots, q\} + T_N^0(\theta, z)
\]
\[
= \left\{l(\cdot; \theta, z)^T \alpha + \eta(\cdot) : \alpha \in \mathbb{R}^q, \eta \in T_N^0(\theta, z)\right\}.
\]
Clearly $\mathcal{T}_3(p)$ is convex, however the functional $\phi$ is not necessarily differentiable. We introduce next an additional assumption in the model that will make $\phi$ differentiable.

Suppose that for each $\alpha \in \mathbb{R}^q$ and each $\eta \in T_N^0(\theta, z)$ there exists a generalised sequence $\{z_t\} = \{z_t(\theta, z)\}$ such that $\{p_t\} \subset \mathcal{P}^*$, given by
\[
p_t(\cdot) = p(\cdot; t\alpha + \theta, z_t)
\]
is a differentiable path with tangent $l^T(\cdot; \theta, z)\alpha + \eta(\cdot)$. This assumption can be found often in the literature in an implicit form (see for instance Pfanzagl, 1990, page 17, for the case where $q = 1$). We prove differentiability of $\phi$ at $p$ with respect to $\mathcal{T}_3(p)$ under (38).

Given $\nu(\cdot) = l(\cdot; \theta, z)^T \alpha + \eta(\cdot) \in \mathcal{T}_3(p)$, and taking a path $\{p_t\}$ as in (38) we obtain
\[
\frac{\phi(p_t) - \phi(p)}{t} = \frac{t\alpha + \theta - \theta}{t} = \alpha.
\]
On the other hand,
\[
\int_{X} J^{-1}(\theta, z) l^E(x; \theta, z) \nu(x) p(x) \lambda(dx) = J^{-1}(\theta, z) \int_{X} l^E(x; \theta, z) l^T(x; \theta, z) p(x) \lambda(dx) \alpha
\]
\[
+ J^{-1}(\theta, z) \int_{X} l^E(x; \theta, z) \eta(x) p(x) \lambda(dx)
\]
\[
= \alpha = \lim_{t \to 0} \frac{\phi(p_t) - \phi(p)}{t}.
\]
Hence $\phi$ is differentiable at $p$ with respect to $\mathcal{T}_3(p)$ and
\[
\phi^*_p(\cdot) = J^{-1}(\theta, z) l^E(\cdot; \theta, z)
\]
is the canonical gradient. In other words, we obtained the same canonical gradient of $\phi$ if we work with $\mathcal{T}_2(p)$ or $\mathcal{T}_3(p)$ and consequently the extended Cramér-Rao bound is also the same with the two choices of tangent cone. Note that $\mathcal{T}_3(p)$ is convex hence we can use the convolution and the local asymptotic minimax theorems. This provides an additional justification of the extended Cramér-Rao bound (via convolution theorem).
and a optimality theory involving a larger class of estimators, namely the weakly regular asymptotic linear estimating sequences (as in the first part of the local asymptotic minimax theorem) or even arbitrary estimating sequences (as in the second part of the local asymptotic minimax theorem). However, we pay a price for these improvements, we have to introduce regularity conditions on the model in order to obtain the differentiability of the interest parameter functional.

It is current in the literature to take the whole (weak or Hellinger) tangent set as the tangent cone, assume that the tangent set is equal to $T_\delta(p)$ and use (implicitly) assumptions equivalent to (38) (see Pfanzagl, 1990 page 17). The strength of the approach based on tangent cones, and not necessarily on the whole tangent set, is that it allow us to graduate the regularity conditions. We can avoid the assumptions mentioned above in the difficult cases or take full advantage of them in the sufficiently regular cases. The approach based on tangent cones allow us to treat the cases where the tangent set is difficult (or virtually impossible) to calculate.

We conclude the section with a comment regarding reparametrisations. Suppose that we reparametrise the model by considering the interest parameter $g(\theta)$ instead of $\theta$. Here $g$ is a one-to-one differentiable application from $\mathbb{R}^q$ to $\mathbb{R}^q$. The interest parameter functional becomes $g \circ \psi(P_{\theta z}) = g(\theta)$. An application of the proposition 8 and the chain rule shows that if an estimating sequence $\{\hat{\theta}_n\}$ attains the semiparametric Cramèr-Rao bound for estimating $\theta$ then the transformed sequence $\{g(\hat{\theta}_n)\}$ attains the Cramèr-Rao bound for estimating $g(\theta)$. 
3 Estimating and Quasi Inference Functions

In this section the theory of inference functions for models with nuisance parameters is studied. The basic definitions and properties of inference functions are given in section 3.1. There a related notion called quasi estimating function is also introduced. Quasi inference functions are essentially functions of the observations, the interest parameter and (different from the inference functions) of the nuisance parameter. They will provide a way to formalise in a more clear way the theory of inference function and relate inference functions with regular asymptotic linear estimators. In order to construct an optimality theory for inference functions, we define a class of what we call regular inference functions. Two alternative (and equivalent) characterisations of the regular estimating functions are provided in the subsections 3.1.2 and 3.1.3. The second characterisation is motivated by differential geometric considerations concerning the statistical model (inspired by Amari and Kawanabe, 1996).

The characterisations referred to are used to derive an optimality theory in section 3.2. A necessary and sufficient condition for the coincidence of the bound for the concentration of estimators based on estimating functions and the semiparametric Cramér-Rao bound is provided in subsection 3.2.3. This condition says essentially that the nuisance tangent space should not depend on the nuisance parameter.

The last section contains some complementary material. Subsection 3.3.1 studies a technique for obtaining optimal inference functions when the likelihood function can be decomposed in certain way. In this way an alternative justification for the so called principle of conditioning will be provided. A generalisation of the notion of inference function is introduced in subsection 3.3.2. The section closes with a result that will allow us to characterise when the semiparametric Cramér-Rao bound is attained by estimators derived from regular inference functions.

3.1 Estimating functions and quasi- inference functions: basic definitions and properties

3.1.1 Inference and quasi-inference functions

A function $\Psi : \mathcal{X} \times \Theta \longrightarrow \mathbb{R}^q$ such that for each $\theta \in \Theta$, the associated function $\Psi(\cdot; \theta, z) : \mathcal{X} \longrightarrow \mathbb{R}^q$ is measurable, is termed an inference function. Estimating functions are used to define sequences of estimators for the parameter of interest $\theta$ in the following way. Under a repeated independent sample scheme, given a sample $x = (x_1, \ldots, x_n)^T$ of size $n$ of the (unknown) distribution $P_{\theta z} \in \mathcal{P}$, define $\hat{\theta}_n$ implicitly by the solution of the equation

$$\sum_{i=1}^{n} \Psi(x_i; \hat{\theta}_n) = 0 .$$

(39)
Under regularity conditions each $\hat{\theta}_n$ is well defined and the sequence $\{\hat{\theta}_n\}$ is consistent (for estimating $\theta$) and asymptotically normally distributed. We explore this fact to construct an optimality theory.

We introduce next a notion related to inference functions. A function $\Psi : \mathcal{X} \times \Theta \times \mathcal{Z} \rightarrow \mathbb{R}^q$, of the parameters and the observations, such that for each $\theta \in \Theta$ and each $z \in \mathcal{Z}$, the function $\Psi(\cdot; \theta, z) : \mathcal{X} \rightarrow \mathbb{R}^q$ is measurable is called a \textit{quasi-inference function}. Each inference function can be naturally identified with a quasi-inference function by making it correspond to a suitable quasi-inference function constant on the nuisance parameter. We make no distinction between inference functions and the corresponding quasi-inference functions. This abuse of language causes, in general, no risk of ambiguity.

A quasi-inference function $\Psi : \mathcal{X} \times \Theta \times \mathcal{Z} \rightarrow \mathbb{R}^q$ such that the conditions (40)-(44) below are satisfied is said to be a \textit{regular quasi-inference function}. The conditions are, with $\psi_i$ denoting the $i^{th}$ component of $\Psi$ and for all $\theta_0 \in \Theta$, all $z \in \mathcal{Z}$ and all $i, j \in \{1, ..., p\}$,

$$\psi_i(\cdot; \theta_0, z) \in L^2_0(P_{\theta_0 z}); \quad (40)$$

the partial derivative with respect to $\theta$ is well defined (almost everywhere), \textit{i.e.}

$$\frac{\partial}{\partial \theta_j} \psi_i(\cdot; \theta, z) \bigg|_{\theta = \theta_0} \text{ exists;} \quad (41)$$

the order of differentiation with respect to $\theta$ and integration can be exchanged in the following sense

$$\frac{\partial}{\partial \theta_j} \int_{\mathcal{X}} \psi_i(x; \theta, z)p(x; \theta, z)\lambda(dx)\bigg|_{\theta = \theta_0} = \int_{\mathcal{X}} \frac{\partial}{\partial \theta_j} \psi_i(x; \theta, z)p(x; \theta, z)\bigg|_{\theta = \theta_0} \lambda(dx); \quad (42)$$

the following $q \times q$ matrix is nonsingular

$$\text{E}_{\theta z} \{\nabla_\theta \Psi(\cdot; \theta, z)\} = \left[ \int_{\mathcal{X}} \frac{\partial}{\partial \theta_j} \psi_i(x; \theta, z) \bigg|_{\theta = \theta_0} p(x; \theta_0 z)\lambda(dx) \right]_{i, j = 1, ..., q}; \quad (43)$$

and

$$\text{E}_{\theta z}[\Psi(\cdot; \theta_0, z)\Psi^T(\cdot; \theta_0, z)] = \left[ \int_{\mathcal{X}} \psi_i(x; \theta_0, z)\psi_j(x; \theta_0, z)p(x; \theta_0 z)\lambda(dx) \right]_{i, j = 1, ..., q} \quad (44)$$

is positive definite.

It is presupposed that the parametric partial score function is a regular quasi-inference function.

A regular quasi-inference function that does not depend on the nuisance parameter $z$ is said to be a \textit{regular inference function}.
3.1.2 First characterisation of regular inference functions

In this section we give a characterisation of the class of regular inference functions.

**Proposition 9** Let $\Psi : \mathcal{X} \times \Theta \times \mathcal{Z} \to \mathbb{R}^q$ be a regular quasi-inference function with components $\psi_1, \ldots, \psi_q$. For all $(\theta, z) \in \Theta \times \mathcal{Z}$ and $i \in \{1, \ldots, q\}$,

$$
\psi_i(\cdot; \theta, z) \in T^\perp_N(\theta, z).
$$

Here and in the rest of this text $T^\perp_N(\theta, z) = 2T^0_N(\theta, z)$ and $T^\perp_N(\theta, z)$ is the orthogonal complement of the nuisance tangent space $2T^0_N(\theta, z)$ in $L^2_0(P_{\theta z})$.

**Proof:** Take $(\theta, z) \in \Theta \times \mathcal{Z}$ and $i \in \{1, \ldots, k\}$ fixed and $\nu \in T^0_N(\theta, z)$ arbitrary. We prove that $\nu$ and $\psi_i(\cdot; \theta)$ are orthogonal in $L^2_0(P_{\theta z})$. This implies the proposition, because of the continuity of the inner product.

Let $\{p_t\}_{t \in V}$ be a differentiable path at $(\theta, z)$ with tangent $\nu$ and remainder term $\{r_t\}_{t \in V}$. Using (1), for each $t \in V$,

$$
\langle \nu(\cdot), \psi_i(\cdot; \theta, z) \rangle_{\theta z} = \langle [\{p_t(\cdot) - p(\cdot; \theta, z)\}] / t, \psi_i(\cdot; \theta, z) \rangle_{\theta z}
$$

$$
= \frac{1}{t} \int_{\mathcal{X}} \psi_i(x; \theta, z) p_t(x) d\mu(x) - \frac{1}{t} \int_{\mathcal{X}} \psi_i(x; \theta, z) p(x; \theta, z) d\mu(x)
$$

$$
- \langle r_t(\cdot), \psi_i(\cdot; \theta, z) \rangle_{\theta z}
$$

$$
= - \langle r_t(\cdot), \psi_i(\cdot; \theta, z) \rangle_{\theta z}.
$$

Since $r_t \overset{L^2(P_{\theta z})}{\to} 0$, from the continuity of the inner product, we conclude that

$$
\langle \nu(\cdot), \psi_i(\cdot; \theta, z) \rangle_{\theta z} = 0.
$$

If the quasi-inference function does not depend on the nuisance parameter (i.e. it corresponds to a genuine inference function), then we can obtain a sharper result.

**Proposition 10** Let $\Psi : \mathcal{X} \times \Theta \to \mathbb{R}^q$ be a regular inference function with components $\psi_1, \ldots, \psi_q$. For all $\theta \in \Theta$ and $i \in \{1, \ldots, q\}$,

$$
\psi_i(\cdot; \theta) \in \bigcap_{z \in \mathcal{Z}} T^\perp_N(\theta, z).
$$
In fact, the proposition above holds for the class of quasi-estimating functions with expectation invariant with respect to the nuisance parameter.

Proof: Take $\theta \in \Theta$ and $i \in \{1, \ldots, k\}$ fixed and arbitrary $\xi \in Z$ and $\nu \in T^0_N(\theta, \xi)$. We prove that $\nu$ and $\psi_i(\cdot; \theta)$ are orthogonal in $L^2(P_\theta z)$.

Let $\{p_t\}_{t \in V}$ be a differentiable path at $(\theta, z)$ with tangent $\nu$ and remainder term $\{r_t\}_{t \in V}$. Using (1), for each $t \in V$,

$$\langle \nu(\cdot), \psi_i(\cdot; \theta, z) \rangle_{\theta z} = -\langle r_t(\cdot), \psi_i(\cdot; \theta, z) \rangle_{\theta z}.$$ 

Since $r_t \xrightarrow{L^2(P_\theta z)} 0$, from the continuity of the inner product, we conclude that

$$\langle \nu(\cdot), \psi_i(\cdot; \theta, z) \rangle_{\theta z} = 0.$$ 

\(\Box\)

3.1.3 Amari-Kawanabe’s geometric characterisation of regular inference functions

We present in this section a variant of the geometric theory of estimating functions for semiparametric models given in Amari and Kawanabe (1996). The development presented is closely connected with the theory given in that paper, however it is not exactly the same. We point out the most remarkable differences at the end of the section.

Take $(\theta, z) \in \Theta \times Z$ fixed. Given $a \in L^2_0(P_\theta z)$ and $z_* \in Z$ denote $p(\cdot, \theta, z)$ and $p(\cdot, \theta, z_*)$ by $p(\cdot)$ and $p_*(\cdot)$ respectively and define the $m$-parallel transport of $a$ from $z$ to $z_*$ by

$$a^{(m)}(\cdot)_{z_*} = \frac{p(\cdot)}{p_*(\cdot)} a(\cdot).$$

If $a$ posses a finite expectation under $P_{\theta z}$, we define the $e$-parallel transport of $a$ from $z$ to $z_*$ by

$$a^{(e)}(\cdot) = a(\cdot) - \int_X a(x)p(x; \theta, z_*) \lambda(dx).$$

The basic properties of the $m$- and $e$-parallel transport are given next.

Proposition 11 We have for each $z, z_* \in Z$ and each $a, b \in L^2_0(p) \cap L^2_0(p_*)$:

$$\int_X a^{(m)}(b(x))_{z_*} \lambda(dx) = \int_X a^{(e)}(b(x))_{z_*} \lambda(dx) = 0;$$

(45)
\[ \langle a, \Pi_z^{z_\ast} b \rangle_{\theta z_\ast} = \langle a, b \rangle_{\theta z}; \quad (46) \]

\[ \langle \Pi_z^{z_\ast} a, \Pi_z^{z_\ast} b \rangle_{\theta z_\ast} = \langle a, b \rangle_{\theta z}; \quad (47) \]

and

\[ \Pi_z^{z_\ast} \Pi_z^{z_\ast} a(\cdot) = \Pi_z^{z_\ast} \Pi_z^{z} a(\cdot) = \Pi_z^{z} a(\cdot) = a(\cdot). \quad (48) \]

**Proof:** Straightforward from the definition of \( m \)-parallel transport. \( \square \)

The parallel transports defined above have their origin in differential geometric considerations for statistical parametric models (\( \alpha \)-connections). We will not enter in details of the geometric theory for semiparametric models, but refer instead to Amari and Kawanabe (1996) for an informal discussion. The parallel transports permit us to change the inner product (see (46)), i.e. it permits us to move from one \( L^2 \) space to another, keeping to certain extent the structure given by the inner product of the first space. For instance the \( L^2 \) orthogonality (i.e. noncorrelation) is preserved after \( m \)-parallel transporting. From the statistical viewpoint the \( e \)- and the \( m \)-parallel transport corresponds to correcting for the mean and correcting for the distribution, respectively, when we move from one \( L^2 \) space to another.

The following class of functions will be of interest in the theory of inference functions,

\[
F_{IA}(\theta, z) = \left\{ r \in T_N^{\perp}\theta z_\ast : \forall z_\ast \in \mathcal{Z} \text{ and } \forall \nu_\ast \in T_N(\theta, z_\ast), \quad \langle \Pi_z^{z_\ast} \nu_\ast, r \rangle_{L^2(P_{\theta z})} = 0 \right\}. 
\]

When the \( e \)-parallel transport is well defined one can use alternatively the relation \( \langle \nu_\ast, \Pi_z^{z_\ast} r \rangle_{L^2(P_{\theta z})} = 0 \) instead of \( \langle \Pi_z^{z_\ast} \nu_\ast, r \rangle_{L^2(P_{\theta z})} = 0 \). Informally, \( F_{IA} \) is the class of functions \( r \) in \( T_N^{\perp}\theta z_\ast \) such that \( r \) corrected for the mean or corrected for the distribution is orthogonal to each \( T_N(\theta, z_\ast) \) under \( P_{\theta z_\ast} \) (for \( z_\ast \) running in the whole \( \mathcal{Z} \)).

**Proposition 12** For each \((\theta, z) \in \Theta \times \mathcal{Z}\), \( F_{IA}(\theta, z) \) is a closed subspace of \( L^2_0(P_{\theta z}) \).

**Proof:** The linearity and the continuity of \( \langle \Pi_z^{z_\ast} \nu_\ast, (\cdot) \rangle_{L^2(P_{\theta z})} \) implies that \( F_{IA}(\theta, z) \) is a vector subspace and a closed set in \( L^2(P_{\theta z_\ast}) \), respectively. \( \square \)

The following proposition gives a characterisation of regular estimating functions in terms of the classes of functions \( F_{IA} \)'s.
**Proposition 13** Given a regular inference function $\Psi$ with components $\psi_1, \ldots, \psi_q$, we have, for $i = 1, \ldots, q$, for all $\theta \in \Theta$ and all $z \in Z$,

$$\psi_i(\cdot; \theta) \in F_{IA}(\theta, z).$$

**Proof:** Take $i = 1, \ldots, q$, $\theta \in \Theta$ and all $z \in Z$ fixed. Given any $z_\ast \in Z$ and $\nu_\ast \in T_N(\theta, z_\ast)$ we have from proposition 10 that $\psi_i(\cdot; \theta) \in T_{\nu_\ast}^\perp N(\theta, z_\ast)$ and then

$$\langle (m) \nu_\ast, \psi_i(\cdot; \theta) \rangle_{L^2(P_{\theta z_\ast})} = 0.$$

Since $z_\ast$ was chosen arbitrarily, $\psi_i(\cdot; \theta) \in F_{IA}(\theta, z)$. \hfill $\Box$

The proposition above can be easily sharpened making the components of the regular inference functions belong to the intersection (over the nuisance parameter) of the $F_{IA}$'s. However the following theorem shows that this is in fact not necessary, since in fact $F_{IA}$ does not depend on the nuisance parameter. We will use sometimes the notation $F_{IA}(\theta)$.

**Proposition 14** For all $\theta \in \Theta$ and all $z \in Z$ we have,

$$F_{IA}(\theta, z) = \cap_{z_\ast \in Z} T_{N}(\theta, z_\ast).$$

Here $T_{N}(\theta, z_\ast)$ denotes the orthogonal complement of $T_N(\theta, z_\ast)$ in $L^2_0(P_{\theta z_\ast})$.

**Proof:**

\[ \subseteq \] Take $\eta \in F_{IA}(\theta, z)$, $z_\ast \in Z$ arbitrary and $\nu_\ast \in T_N(\theta, z_\ast)$. Applying (46) yields

$$\langle \nu_\ast, \eta \rangle_{L^2(P_{\theta z_\ast})} = \langle (m) \nu_\ast \nu_\ast, \eta \rangle_{L^2(\nu_\ast z_\ast)} = 0.$$

Hence $\eta \in T_{N}(\theta, z_\ast)$. Since $z_\ast$ was choose arbitrarily in $Z$, $\eta \in \cap_{z_\ast \in Z} T_{N}(\theta, z_\ast)$.

\[ \supseteq \] Take an arbitrary $z_\ast \in Z$, $\eta \in \cap_{z_\ast \in Z} T_{N}(\theta, z_\ast)$ and $\nu_\ast \in T_N(\theta, z_\ast)$. Using (46) we obtain

$$\langle (m) \nu_\ast, \eta \rangle_{L^2(\nu_\ast z_\ast)} = \langle \nu_\ast, \eta \rangle_{L^2(\nu_\ast z_\ast)} = 0.$$

Since $z_\ast$ is arbitrary, $\eta \in F_{IA}(\theta, z)$. \hfill $\Box$

The proposition 14 shows that the characterisation of regular inference functions obtained here is equivalent to what we obtained in the last section. We remark that the characterisation based on the intersection of the nuisance tangent spaces can be found in Jørgensen and Labouriau (1998) and the characterisation based on parallel transports...
(i.e. based on $F_{IA}$) is a variant of the results of Amari and Kawanabe (1996). The main difference of the variant presented here and the original formulation in Amari and Kawanabe (1996) is that here we define via the $m$-parallel transport and there $F_{IA}$ is constructed through $e$-parallel transport. Both formulations are equivalent from this point of view, provided the $e$-parallel transport is well defined. Moreover, when defining via the $m$-parallel transport the class $F_{IA}$ is automatically a closed subspace in $L_0^2$.

3.2 Optimality theory for estimating functions

3.2.1 Classic optimality theory

Given a regular (estimating) quasi-inference function $\Psi$ we define the Godambe information of $\Psi$ by

$$J_{\Psi}(\theta, z) = E_{\theta z}\{\nabla_\theta \Psi(\cdot; \theta, z)\}E_{\theta z}\{\Psi(\cdot; \theta, z)\Psi^T(\cdot; \theta, z)\}^{-1}E_{\theta z}\{\nabla_\theta \Psi(\cdot; \theta, z)\}^T.$$

Here

$$S_{\Psi}(\theta, z) := E_{\theta z}\{\nabla_\theta \Psi(\cdot; \theta, z)\} \text{ and } V_{\Psi}(\theta z) := E_{\theta z}\{\Psi(\cdot; \theta, z)\Psi^T(\cdot; \theta, z)\}$$

are called the sensibility and the variability of $\Psi$ (at $(\theta, z)$), respectively.

Using standard arguments based on a Taylor expansion of $\Psi$ it can be shown that under some additional regularity conditions (each $\psi_i$ twice continuous differentiable with respect to each component of $\theta$, for instance) a sequence $\{\hat{\theta}_n\}$ of roots of a regular inference functions is asymptotically normally distributed with asymptotic variance given by $J_{\Psi}^{-1}(\theta, z)$, provided $\{\hat{\theta}_n\}$ is weakly consistent. (see Jørgensen and Labouriau, 1995 for conditions for consistency and asymptotic normality). Hence, we say that a regular inference function $\Psi$ is optimal when for all $\theta \in \Theta$, for all $z \in Z$ and for each regular inference function $\Phi$,

$$J_{\Phi}(\theta, z) \leq J_{\Psi}(\theta, z).$$

Here "$\leq$" is understood in the sense of the Löwner partial order of matrices given by the positive definiteness of the difference.

In the literature of inference functions it is customary to say that it is possible to justify the use of some estimators using finite sample arguments via inference functions and the Godambe estimation (see the articles of Godambe referred to). The argument used there is that the Godambe information is a quantity that should be maximised when using inference functions. We do not share this point of view. The inference functions themselves are not the object of our direct interest. Our concern with inference functions
is only through the estimators (or inferential procedures) associated with them. Hence one should judge inference functions only through the properties of such inferential procedures. In fact, apart from the asymptotic variance, there are no clear connections between the Godambe information and the (asymptotic or finite sample) properties of the estimators associated with regular inference functions.

We say that two regular quasi-inference functions, \( \Psi, \Phi : \mathcal{X} \times \Theta \times \mathcal{Z} \rightarrow \mathbb{R}^k \), are equivalent if, for each \( \theta \in \Theta \) and \( z \in \mathcal{Z} \) there exists a \( k \times k \) matrix with full rank \( K(\theta, z) \), such that

\[
\Psi(x; \theta, z) = K(\theta, z)\Phi(x, \theta, z), \quad P_{\theta z} \text{ a.s.}
\]

We stress that \( K(\theta, z) \) must not depend on the observation \( x \). Clearly, two equivalent inference functions have the same roots almost surely and hence produce essentially the same estimators, i.e. they are equivalent from the statistical point of view. Moreover, it is easy to see that two equivalent quasi-inference functions share the same Godambe information for each value of the parameters.

### 3.2.2 Lower bound for the asymptotic covariance of estimators obtained through inference functions

We define the information score function, \( l^I : \mathcal{X} \times \Theta \times \mathcal{Z} \rightarrow \mathbb{R}^q \), by the orthogonal projection of the partial score function, \( l \) onto \( F_{IA}(\theta) \). More precisely, for each \( \theta \in \Theta \) and \( z \in \mathcal{Z} \), the \( i \)th component of the information score function \( (i = 1, \ldots, q) \) at \( (\theta, z) \) is given by

\[
l^I_i(\cdot; \theta, z) = \Pi\{l(\cdot; \theta, z)|F_{IA}(\theta)\},
\]

The space spanned by the components \( l^I_1, \ldots, l^I_q \) of the information score function at \( (\theta, z) \in \Theta \times \mathcal{Z} \) is denoted by \( E(\theta, z) \), i.e.

\[
E(\theta, z) = \text{span}\{l^I_i(\cdot; \theta, z) : i = 1, \ldots, q\}.
\]

Note that \( E(\theta, z) \) is a closed (since it is finite-dimensional vector space) subspace of \( L_0^2(P_{\theta z}) \). Hence given any regular inference function \( \Psi : \mathcal{X} \times \Theta \times \mathcal{Z} \rightarrow \mathbb{R}^q \) with components \( \psi_1, \ldots, \psi_q \) we have, for all \( \theta \in \Theta, z \in \mathcal{Z} \) and \( i \in \{1, \ldots, q\} \) the orthogonal decomposition

\[
\psi_i(\cdot; \theta, z) = \psi^A_i(\cdot; \theta, z) + \psi^I_i(\cdot; \theta, z),
\]

(49)

where \( \psi^I_i(\cdot; \theta, z) \in E(\theta, z) \) and \( \psi^A_i(\cdot; \theta, z) \in A(\theta, z) := E^\perp(\theta, z) \). Here \( A(\theta, z) \) is the orthogonal complement of \( E(\theta, z) \) in \( L_0^2(P_{\theta z}) \). The decomposition above induces the following decomposition of each regular quasi-inference function

\[
\Psi(\cdot; \theta, z) = \Psi^A(\cdot; \theta, z) + \Psi^I(\cdot; \theta, z),
\]

(50)
where the components \( \psi^A_i (\cdot \theta, z), \ldots, \psi^A_q (\cdot \theta, z) \) of \( \Psi^A \) at \((\theta, z)\) are in \( A(\theta, z) \) and the components \( \psi_j^I (\cdot \theta, z), \ldots, \psi_j^I (\cdot \theta, z) \) of \( \Psi^I \) at \((\theta, z)\) are in \( E(\theta, z) \).

We show next that taking the “component” \( \Psi^I \) of a regular (quasi-) inference function improves the Godambe information. However, at this stage a technical difficulty appears, the function \( \Psi^I \) is not necessarily a regular quasi-inference function, and hence does not necessarily possess a well-defined Godambe information. For this reason we introduce next an extension of the notion of sensitivity, and consequently of Godambe information, which will make us able to speak of Godambe information of some non-regular (quasi) inference functions. To motivate our extended notion of sensitivity, consider a regular alternative form that will suggest the extension one should define. For each \((\theta, z) \in \Theta \times \mathcal{Z}\) and each \(i,j \in \{1, \ldots, q\}\) we have

\[
0 = \frac{\partial}{\partial \theta_i} \int_X \psi_j(x; \theta, z)p(x; \theta, z)d\mu(x) \tag{51}
\]

(differentiating under the integral sign)

\[
= \int_X \frac{\partial}{\partial \theta_i} \{\psi_j(x; \theta, z)p(x; \theta, z)\} d\lambda(x)
\]

\[
= \int_X \frac{\partial}{\partial \theta_i} \{\psi_j(x; \theta, z)\} p(x; \theta, z)d\lambda(x) + \int_X \psi_j(x; \theta, z) \frac{\partial}{\partial \theta_i} \{p(x; \theta, z)\} d\lambda(x).
\]

Hence

\[
\int_X \frac{\partial}{\partial \theta_i} \{\psi_j(x; \theta, z)\} p(x; \theta, z)d\lambda(x)
\]

\[
= -\int_X \psi_j(x; \theta, z) \frac{\partial}{\partial \theta_i} \{p(x; \theta, z)\} d\lambda(x)
\]

\[
= \int_X \psi_j(x; \theta, z) l_i(x; \theta, z)p(x; \theta, z)d\lambda(x) = -\langle \psi_j (\cdot \theta, z), l_i (\cdot \theta, z) \rangle_{\theta z}
\]

(decomposing \( l_i = l_i^A + l_i^I \) with \( l_i^I \in F_{IA} \) and \( l_i^A \in T_N \))

\[
= -\langle \psi_j (\cdot \theta, z), l_i^I (\cdot \theta, z) \rangle_{\theta z} - \langle \psi_j (\cdot \theta, z), l_i^A (\cdot \theta, z) \rangle_{\theta z}
\]

(Since \( \psi_j \in F_{IA} \) and \( l_i^A \) orthogonal \( F_{IA} \))

\[
= -\langle \psi_j (\cdot \theta, z), l_i^I (\cdot \theta, z) \rangle_{\theta z}
\]

(decomposing \( \psi_j = \psi_j^A + \psi_j^I \) and using the orthogonality of \( l_i^I \) and \( \psi_j^A \))

\[
= -\langle \psi_j^I (\cdot \theta, z), l_i^I (\cdot \theta, z) \rangle_{\theta z}.
\]

We conclude that the sensitivity of \( \Psi \) at \((\theta, z)\) is given by

\[
S_{\Psi}(\theta, z) = \left[ -\langle \psi_j^I (\cdot \theta, z), l_i^I (\cdot \theta, z) \rangle_{\theta z} \right]_{j=1, \ldots, k} \tag{52}
\].
Here $[a_{ij}]_{i=1,...,k}^{j=1,...,k}$ denotes the matrix formed by $a_{ij}$’s with $i$ indexing the columns and $j$ indexing the lines.

We define the **extended sensitivity** (or simply the **sensitivity**) of $\Psi$ by the matrix in the right-hand side of (52). The (extended) Godambe information is defined in the same way we did before but using the extended sensitivity instead of the sensitivity. Note that both, the standard and the extended, versions of the sensitivity (and the Godambe information) coincide in the case where $\Psi$ is regular. Moreover, the extended sensitivity is defined for each quasi-inference function whose components are in $L_2^0$, not only for regular inference functions. According to the new definition both $\Psi$ and $\Psi^I$ possess the same sensitivity.

**Proposition 15** Given a regular inference function $\Psi$, for all $\theta \in \Theta$ and all $z \in Z$,

$$J_\Psi(\theta, z) \leq J_{\Psi^I}(\theta, z).$$

**Proof:** For each $\theta \in \Theta$ and $z \in Z$,

$$J_{\Psi}^{-1}(\theta, z) = S_{\Psi}^{-1}(\theta, z)V_{\Psi}(\theta, z)S_{\Psi}^{-T}(\theta, z)$$

$$= S_{\Psi}^{-1}(\theta, z)\{V_{\Psi}(\theta, z) + V_{\Psi^A}(\theta, z)\}S_{\Psi}^{-T}(\theta, z)$$

$$= S_{\Psi}^{-1}(\theta, z)V_{\Psi}(\theta, z)S_{\Psi}^{-T}(\theta, z) + S_{\Psi}^{-1}(\theta, z)V_{\Psi^A}(\theta, z)V_{\Psi}^{-1}(\theta, z)S_{\Psi}^{-1}(\theta, z)$$

$$\geq S_{\Psi}^{-1}(\theta, z)V_{\Psi}(\theta, z)S_{\Psi}^{-T}(\theta, z) = J_{\Psi^I}^{-1}(\theta, z).$$

$\square$

The following proposition gives further properties of regular inference functions, which will allow us to establish an upper bound for the Godambe information.

**Proposition 16** Given a regular inference function $\Psi$, for all $\theta \in \Theta$ and all $z \in Z$, we have:

(i) $\Psi^I \sim I^I$;

(ii) $\text{span}\{\Psi^I_i(\cdot; \theta, z) : i = 1, \ldots, k\} = E(\theta, z)$;

(iii) $J_{\Psi^I}(\theta, z) = J_{I^I}(\theta, z)$.

**Proof:** Take $\theta \in \Theta$ and $z \in Z$ fixed.
(i) Assume without loss of generality that the components of the efficient score function $l_1^I(\cdot; \theta, z), \ldots, l_q^I(\cdot; \theta, z)$ are orthonormal in $L_0^2(P_{\theta z})$. For each $i \in \{1, \ldots, q\}$, expanding $\psi_i(\cdot; \theta, z)$ in a Fourier series with respect to a basis whose first $q$ elements are $l_1^I(\cdot; \theta, z), \ldots, l_q^I(\cdot; \theta, z)$ one obtains

$$
\psi_i(\cdot; \theta, z) = \langle l_1^I(\cdot; \theta, z), \psi_i(\cdot; \theta, z) \rangle_{\theta z} l_1^I(\cdot; \theta, z) + \cdots + \langle l_q^I(\cdot; \theta, z), \psi_i(\cdot; \theta, z) \rangle_{\theta z} l_q^I(\cdot; \theta, z) + \psi_A^I(\cdot; \theta, z).
$$

That is,

$$
\psi^I_i(\cdot; \theta, z) = \langle l_1^I(\cdot; \theta, z), \psi_i(\cdot; \theta, z) \rangle_{\theta z} l_1^I(\cdot; \theta, z) + \cdots + \langle l_q^I(\cdot; \theta, z), \psi_i(\cdot; \theta, z) \rangle_{\theta z} l_q^I(\cdot; \theta, z).
$$

Moreover, for $j = 1, \ldots, q$

$$
\langle l_j^I(\cdot; \theta, z), \psi_i(\cdot; \theta, z) \rangle_{\theta z} = -\int_X \left\{ \frac{\partial}{\partial \theta_j} \psi_i(x; \theta, z) \right\} p(x; \theta, z) d\lambda(x).
$$

We conclude from (53) and (54) that $\Psi^I(\cdot; \theta, z) = S_\Psi(\theta, z) l^I(\cdot; \theta, z)$, which means that $\Psi^I$ and $l^I$ are equivalent.

(ii) From the previous discussion $\text{span}\{\psi_i^I(\cdot; \theta, z) : i = 1, \ldots, q\}$ is the space spanned by $-S_\Psi(\theta, z) l^I(\cdot; \theta, z)$ which is the span of $\{l_i^I(\cdot; \theta, z) : i = 1, \ldots, q\}$, since the sensitivity by assumption is of full rank.

(iii) Straightforward.

A consequence of the two last proposition is that $J_l^I$ is an upper bound for the Godambe information of regular quasi inference functions. This upper bound is attained by any (if any exists) extended regular inference functions with components in $E$. In particular if $l^I$ is a regular (quasi-) inference function, then it is an optimal (quasi-) inference function.

### 3.2.3 Attainability of the semiparametric Cramér-Rao bound

We study in this section the attainability of the semiparametric Cramér-Rao bound through regular inference function. More precisely, we give a necessary and sufficient condition for the coincidence of the semiparametric Cramér-Rao bound and the bound given in the previous section for the asymptotic variance of estimators derived from regular inference functions.
Let us consider the interest parameter functional \( \Phi : \mathcal{P}^* \rightarrow \Theta \) given by, for each \((\theta, z) \in \Theta \times \mathcal{Z}\),
\[
\Phi(p(\cdot; \theta, z)) = \theta.
\]
As shown in chapter 2, the functional \( \Phi \) is differentiable at each \( p(\cdot; \theta, z) := p(\cdot) \in \mathcal{P}^* \), with respect to the tangent cone \( \mathcal{T}(p) = T_N(\theta, z) \cup \text{span}\{l_i(\cdot; \theta, z) : i = 1, \ldots, q\} \).

Here we adopt the \( L^2 \) path differentiability, since this is the path differentiability used to characterise the class of regular estimating functions. We stress that the theory of functional differentiability used here is compatible with any notion of path differentiability stronger than (or equal to) the weak path differentiability, in particular the \( L^2 \) path differentiability is allowed. Moreover, in the examples we have in mind (\( L^2 \)-restricted models) the notions of weak and \( L^2 \) path differentiability coincide. Take a fixed \( p(\cdot; \theta, z) = p(\cdot) \in \mathcal{P}^* \).

Consider the function
\[
\phi_p^*(\cdot) = \text{Cov}_p(l^I)^{-1} l^I(\cdot; \theta, z).
\]

The following lemma will allow us to connect the optimality theory for inference functions with the semiparametric Cramér-Rao lower bound.

**Lemma 1** The function \( \phi_p^*(\cdot) \) is a gradient of \( \Phi \) at \( p \).

**Proof:** A little of reflection reveals that it is enough to verify that
\[
\forall \nu \in T_N^0(\theta, z), \quad \int_X \phi_p^*(x) \nu(x) p(x) \lambda(dx) = 0 \tag{55}
\]
and
\[
\int_X \phi_p^*(x) l^T(x; \theta, z) p(x) \lambda(dx) = I_q, \tag{56}
\]
where \( I_q \) is the \( q \times q \) identity matrix.

Take \( \nu \in T_N^0(\theta, z) \). Since each component of \( u^I(\cdot; \theta, z) \) is in \( F_{IA}(\theta, z) \subseteq T_N^1(\theta, z) \), condition (55) holds. On the other hand,
\[
\int_X \phi_p^*(x) l^T(x; \theta, z) p(x) \lambda(dx) = \int_X \text{Cov}_p(l^I)^{-1} l^I(x; \theta, z) l^T(x; \theta, z) p(x) \lambda(dx)
= \text{Cov}_p(l^I)^{-1} \int_X l^I(x; \theta, z) l^I(x; \theta, z) p(x) \lambda(dx)
= I_q,
\]
that is the condition (56) holds. We conclude that \( \Phi_p^* \) is a gradient of \( \Phi \) at \( p \) with respect to the tangent cone \( \mathcal{T}(p) \).

\( \square \)
According to the lemma above $\Phi^\bullet_p$ is a gradient of $\Phi$ at $p$, but not necessarily the canonical gradient. In fact the canonical gradient of the functional $\Phi$ at $p$ with respect to $\mathcal{T}(p)$ is

$$\Phi^\bullet_p(\cdot) = J^{-1}(\theta, z)l^E(\cdot; \theta, z),$$

where $l^E(\cdot; \theta, z)$ is the efficient score function at $(\theta, z)$ and $J^{-1}(\theta, z)$ is the covariance matrix of $l^E(\cdot; \theta, z)$ under $P_{\theta z}$ (see chapter 2). The unicity of the canonical gradient implies that $\Phi^\bullet_p$ is the canonical gradient if and only if it is equal to $\Phi^\star_p$ and this occurs if and only if $T^\perp_N(\theta, z) = F_{IA}(\theta)$. The covariance of $\Phi^\bullet_p(\cdot)$ (under $P_{\theta z}$), that is $J^{-1}(\theta, z)$, gives the semiparametric Cramér-Rao lower bound. On the other hand, the lower bound for the asymptotic covariance of estimators obtained from regular inference functions is the covariance (under $P_{\theta z}$) of $\Phi^\bullet_p$. We conclude that the following result holds.

**Theorem 5** The semiparametric Cramér-Rao lower bound coincides with the bound for the asymptotic covariance of estimators defined through regular inference functions at $(\theta, z) \in \Theta \times Z$ if and only if

$$T^\perp_N(\theta, z) = F_{IA}(\theta)(= \cap_{z \in Z} T^\perp_N(\theta, z)).$$

The theorem above implies that inference functions produce efficient estimators only if the orthogonal complement of the nuisance tangent space does not depend on the nuisance parameter.

### 3.3 Further aspects

#### 3.3.1 Optimal inference functions via conditioning

We present in this section some results which allow us to compute optimal inference functions in many practical situations. The results will be in accordance with the so-called conditioning principle. For the sake of simplicity we study here only models with a one-dimensional parameter of interest.

We study the situation where we have a likelihood factorisation of the following form. Suppose that there exists a statistic $T = t(X)$ such that, for all $\theta \in \Theta$ all $z \in Z$ and all $x \in \mathcal{X},$

$$p(x; \theta, z) = f_t(x; \theta)h\{t(x); \theta, z\}. \tag{57}$$

**Theorem 6** Assume that there exists a statistic $T$ such that one has the decomposition (57). Moreover, suppose that the class $\{P_{\theta z} : z \in Z\}$, where $P_{\theta z}$ is the distribution of $T(x)$ under $P_{\theta z}$ (i.e. $X \sim P_{\theta z}$), is complete. Then the regular inference function given by

$$\Psi(x; \theta) = \frac{\partial}{\partial \theta} \log f_t(x; \theta), \forall x \in \mathcal{X}, \forall \theta \in \Theta \tag{58}$$
is optimal. Moreover, if $\Phi$ is also an optimal inference function then $\Phi$ is equivalent to $\Psi$.

The theorem above gives an alternative justification for the use of conditional inference.

The following technical (and trivial) lemma will be the kernel of the proofs that follow. But first it is convenient to introduce the following notation. Given a regular inference function $\Psi : \mathcal{X} \times \Theta \rightarrow \mathbb{R}^k$, we define

$$\tilde{\Psi}(x; \theta) = \Psi(x; \theta, z) \frac{\Psi'(\theta)}{E_{\theta z}\{\Psi'(\theta)\}},$$

which is called the standardised version of $\Psi$. Here $\Psi'(\theta) = \nabla_\theta \Psi(\theta)$. Along this section we denote the class of all regular inference functions by $G$.

**Lemma 2** For each regular inference function $\Psi$ and $\Phi : \mathcal{X} \times \Theta \rightarrow \mathbb{R}$ and each $\theta \in \Theta$ and $z \in \mathcal{Z}$, the following assertions hold:

1. $E_{\theta z}\{\Psi(\theta)l(\theta; z)\} = \frac{E_{\theta z}\{\Psi(\theta)\Psi'(\theta)\}}{E_{\theta z}\{\Psi'(\theta)\}} = -1,$

where $l(\theta; z)$ is the partial score function at $(\theta; z)$;

2. $E_{\theta z}\{\tilde{\Phi}^2(\theta)\} = E_{\theta z}\{(\tilde{\Phi}(\theta) - \tilde{\Psi}(\theta))^2\} + 2E_{\theta z}\{\tilde{\Phi}(\theta)\tilde{\Psi}(\theta)\} - E_{\theta z}\{\tilde{\Psi}^2(\theta)\}$.

**Proof:** Since $\Psi$ is unbiased, one has

$$\int \Psi(x; z) p(x; \theta, z) d\mu(x) = 0.$$  

Differentiating the expectation above with respect to $\theta$ and interchanging the order of differentiation and integration, we obtain

$$E_{\theta z}\left\{\frac{\partial}{\partial \theta} \Psi(\theta)\right\} + E_{\theta z}\{\Psi(\theta)l(\theta; z)\} = 0$$

which is equivalent to the first part of the lemma. The second part is straightforward. \qed

The following lemma gives a useful tool for computing optimal inference functions.
Lemma 3 Assume the previous regularity conditions. Consider two functions $A: \Theta \rightarrow \mathbb{R}\backslash\{0\}$ and $R: \mathcal{X} \times \Theta \times Z \rightarrow \mathbb{R}$. Suppose that, for each regular inference function $\Phi$, one has, for each $\theta \in \Theta$ and $z \in Z$,

$$\int R(x; \theta, z) \Phi(x; \theta) p(x; \theta, z) d\mu(x) = 0.$$  

If a regular inference function $\Psi$ can be written in the form, for all $\theta \in \Theta$,

$$\Psi(x; \theta) = A(\theta) l(x; \theta, z) + R(x; \theta, z),$$  \hspace{1cm}(59)

for $x P_{\theta z}$-almost surely, ($\Psi$ does not depend on $z$ even though $l$ and $R$ do), then $\Psi$ is optimal. Furthermore, a regular inference function $\Phi$ is optimal if and only if for all $(\theta, z) \in \Theta Z$,

$$\Phi(\theta) = \tilde{\Psi}(\theta), \text{ for } x P_{\theta z} \text{ almost surely,}$$

provided that there exists a decomposition as (59) above.

**Proof:** Take an arbitrary $(\theta, z) \in \Theta \times Z$. Given $\Phi \in \mathcal{G}$ one has

$$E_{\theta z}\{\Phi(\theta) \tilde{\Psi}(\theta)\} = E_{\theta z}\left[\frac{\Phi(\theta) A(\theta) l(\theta, z) + \tilde{\Phi}(\theta) R(\cdot; \theta, z)}{E_{\theta z}\{\Phi(\theta)\} E_{\theta z}\{\tilde{\Phi}(\theta)\}}\right]$$  \hspace{1cm}(60)

$$= \frac{A(\theta)}{E_{\theta z}\{\tilde{\Phi}(\theta)\} E_{\theta z}\{\Phi(\theta)\}} E_{\theta z}\{\Phi(\theta)\}$$

$$= -\frac{A(\theta)}{E_{\theta z}\{\tilde{\Phi}(\theta)\}}.$$

Hence the value of $E_{\theta z}\{\Phi(\theta) \tilde{\Psi}(\theta)\}$ does not depend on $\Phi$, in particular,

$$E_{\theta z}\{\tilde{\Phi}(\theta) \tilde{\Psi}(\theta)\} = E_{\theta z}\{\tilde{\Psi}^2(\theta)\} > 0.$$

On the other hand, from (ii) of Lemma 2 one has

$$E_{\theta z}\{\tilde{\Phi}^2(\theta)\} = E_\theta\{[\tilde{\Phi}(\theta) - \tilde{\Psi}(\theta)]^2\} + 2E_{\theta z}\{\tilde{\Phi}(\theta) \tilde{\Psi}(\theta)\} - E_{\theta z}\{\tilde{\Psi}^2(\theta)\}$$  \hspace{1cm}(61)

$$\geq E_{\theta z}\{\tilde{\Psi}^2(\theta)\},$$

for each $\Phi \in \mathcal{G}$. Thus, $\forall \theta \in \Theta, \forall z \in Z, \forall \Phi \in \mathcal{G}$,

$$J_\Phi(\theta, z) = \frac{1}{E_{\theta z}\{\Phi^2(\theta)\}} \leq \frac{1}{E_{\theta z}\{\tilde{\Psi}^2(\theta)\}} = J_\psi(\theta, z).$$  \hspace{1cm}(62)
We conclude that $\Psi$ is optimal. For the second part of the theorem, note that one has equality in (61), and hence in (62), if and only if $\forall \theta \in \Theta, \forall z \in Z, E_{\theta_z}[\{\tilde{\Phi}(\theta) - \tilde{\Psi}(\theta)\}^2] = 0$. That is, if a regular inference function $\Phi$ is optimal then $\tilde{\Phi}(\cdot; \theta) = \tilde{\Psi}(\cdot; \theta) P_{\theta_z}$-a.s., $\forall \theta \in \Theta, \forall z \in Z$.

We can prove now the main theorem of this section.

**Proof**: (of theorem 6) Take $\theta \in \Theta$ and $z \in Z$ fixed. From (57),

$$l(x; \theta, z) = \frac{\partial}{\partial \theta} \log p(x; \theta, z) = \frac{\partial}{\partial \theta} \log f_t(x; \theta) + \frac{\partial}{\partial \theta} \log h(x; \theta, z).$$

(63)

We apply Theorem 3 to prove that $\psi$ is a (“unique”) optimal inference function. More precisely, defining $A(\theta) = 1$ and $R(x; \theta, z) = -\frac{\partial}{\partial \theta} \log h\{t(x); \theta, z\}/\partial \theta$, and using (63) we can write $\Psi$ in the form

$$\Psi(x; \theta) = \frac{\partial}{\partial \theta} \log f_t(\cdot; \theta) = A(\theta)l(x; \theta, z) + R(x; \theta, z).$$

According to lemma 3, if $R$ is orthogonal to every regular inference function, then $\Psi$ is optimal, moreover $\Psi$ is the unique optimal inference function, apart from equivalent inference functions.

Take an arbitrary regular inference function $\phi$. We show that $\phi$ and $R$ are orthogonal. Note that for each $z \in Z$,

$$0 = \int \phi(x; \theta)p(x; \theta, z)d\mu(x) = \int \phi(x; \theta)f_t(x; \theta)h\{t(x); \theta, z\}d\mu(x).$$

On the other hand $E_{\theta_z}(\phi|T) = \int \phi(x; \theta)f_t(x; \theta)d\mu(x)$, which is independent of $z$. We write $E_\theta(\phi|T)$ for $E_{\theta_z}(\phi|T)$, and we have $E_{\theta_z}\{E_\theta(\phi|T)\} = 0$. Since $T$ is complete, $E_\theta(\phi|T) = 0$, $P_{\theta_z}$ almost surely. We have then,

$$E_{\theta_z}\{\phi(\theta)R(\theta, z)\} = E_{\theta_z}\{R(\theta, z)E_\theta(\phi|T)\} = 0.$$

3.3.2 Generalised inference functions

A quasi inference function is said to be an *generalised inference function* if it is equivalent to an inference function. More precisely, a quasi inference function $\Psi : \mathcal{X} \times \Theta \times Z \to \mathbb{R}^q$ is an generalised inference function if for each $(\theta, z) \in \Theta \times Z$ there exists a non-singular $q \times q$ matrix $A(\theta, z)$ and a measurable function $\Phi(\cdot, \theta) : \mathcal{X} \to \mathbb{R}^q$ such that

$$\Psi(x; \theta, z) = A(\theta, z)\Phi(x, \theta),$$
for \( x \lambda \)-almost everywhere. If \( \Phi \) is a regular inference function, then \( \Psi \) is said to be a regular generalised inference functions.

Generalised inference functions are used for estimating the interest parameter in the following way. Given a sample \( x = (x_1, \ldots, x_n)^T \) of size \( n \) of a unknown probability measure of the model, define the estimator \( \hat{\theta}_n \) implicitly by the solution of the following equation

\[
0 = \sum_{i=1}^{n} \Psi(x_i; \hat{\theta}_n; z) = A(\hat{\theta}_n, z) \sum_{i=1}^{n} \Phi(x_i; \hat{\theta}_n),
\]

which is equivalent to

\[
\sum_{i=1}^{n} \Phi(x_i; \hat{\theta}_n) = 0.
\]

In other words, for each generalised inference function there is an inference function that yields the same estimating sequence. Generalised inference functions are just a tool that will simplify some formalisations. Examples of generalised inference functions are the efficient estimating function of most of the regular semiparametric models. The following result will be useful latter on.

**Proposition 17** If for each \((\theta, z) \in \Theta \times Z\), \( F_{IA}(\theta) = T_N^{\perp}(\theta, z) \), and the efficient score function is a generalised estimating function, then the inference function equivalent to the efficient score function attains the semiparametric Cramèr-Rao bound.

**Proof:** Take an arbitrary \((\theta, z) \in \Theta \times Z\). Since \( F_{IA}(\theta) = T_N^{\perp}(\theta, z) \), the efficient score function \( l^E \) coincides with the information score function \( l^I \) at \((\theta, z)\). Hence the extended sensibility of the efficient score function (at \((\theta, z)\)) is the \( q \times q \) identity matrix and the Godambe information of \( l^E \) at \((\theta, z)\) is

\[
J_{l^E}(\theta, z) = Cov_{\theta, z}(l^E),
\]

which is the semiparametric Cramèr-Rao bound. If \( \Phi \) is an inference function equivalent to the efficient score function then its Godambe information is equal to the Godambe information of the efficient score function, that is \( \Phi \) attains the semiparametric Cramèr-Rao bound at \((\theta, z)\). The proof follows now from the fact that \((\theta, z)\) was chosen arbitrarily. \( \square \)
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