Proofs of some conjectures of Chan-Mao-Osburn on Beck’s partition statistics

Liuxin Jin, Eric H. Liu and Ernest X.W. Xia

1Department of Mathematics, Jiangsu University, Jiangsu, Zhenjiang, 212013, P. R. China
2School of Statistics and Information, Shanghai University of International Business and Economics, Shanghai, 201620, P. R. China
3School of Mathematical Sciences, Suzhou University of Science and Technology, Suzhou, 215009, Jiangsu Province, P. R. China

Email: liuxj@ujs.edu.cn, liuhai@suibe.edu.cn, ernestxwxia@163.com

Abstract. Recently, George Beck introduced two partition statistics $NT(m, j, n)$ and $M_\omega(m, j, n)$, which denote the total number of parts in the partition of $n$ with rank congruent to $m$ modulo $j$ and the total number of ones in the partition of $n$ with crank congruent to $m$ modulo $j$, respectively. Andrews proved a congruence on $NT(m, 5, n)$ which was conjectured by Beck. Very recently, Chan, Mao and Osburn established a number of Andrews-Beck type congruences and posed several conjectures involving $NT(m, j, n)$ and $M_\omega(m, j, n)$. Some of those conjectures were proved by Chern and Mao. In this paper, we confirm the remainder three conjectures of Chan-Mao-Osburn and two conjectures due to Mao. We also present two new conjectures on $M_\omega(m, j, n)$ and $NT(m, j, n)$.

Keywords: partition statistics, Andrews-Beck type congruences, rank, crank, partition.

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1 Introduction

A partition $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ of a positive integer $n$ is a sequence of positive integers such that $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_k > 0$ and $\pi_1 + \pi_2 + \cdots + \pi_k = n$. The $\pi_i$ are called the parts of the partition [1]. We shall write $\pi \vdash n$ to denote $\pi$ is a partition of $n$. We also use $\#(\pi)$ and $\lambda(\pi)$ to denote the number of parts of $\pi$ and the largest part of $\pi$, respectively. As usual, let $p(n)$ denote the number of partitions of $n$ and set $p(0) = 1$. In the theory of partition, one of the most well-known results is achieved by Ramanujan [10]. In 1919, he found that for $n \geq 0$,

\begin{align*}
    p(5n + 4) &\equiv 0 \pmod{5}, \\
    p(7n + 5) &\equiv 0 \pmod{7}, \\
    p(11n + 6) &\equiv 0 \pmod{11}.
\end{align*}

In 1944, to give combinatorial interpretations of Ramanujan’s congruences, Dyson [10] defined the
rank of a partition to be the largest part of the partition minus the number of parts, namely,
\[
\text{rank}(\pi) := \lambda(\pi) - \#(\pi).
\]
For example, the rank of the partition \(4 + 2 + 2 + 2 + 1\) is \(4 - 5 = -1\). Let \(N(m, j, n)\) count the number of partitions of \(n\) with rank congruent to \(m\) modulo \(j\). Dyson \[10\] also conjectured that for \(0 \leq m \leq 4\)
\[
N(m, 5, 5n + 4) = \frac{p(5n + 4)}{5},
\]
and for \(0 \leq m \leq 6\),
\[
N(m, 7, 7n + 5) = \frac{p(7n + 5)}{7}.
\]

In 1954, Atkin and Swinnerton-Dyer \[5\] proved Dyson’s conjectures (1.1) and (1.2). Therefore Dyson’s rank gives combinatorial interpretations for Ramanujan’s first two congruences. Unfortunately, it turned out that Dyson’s rank fails to explain Ramanujan’s third congruence modulo 11 combinatorially. So Dyson conjectured the existence of an unknown partition statistic, which he whimsically called “the crank” to explain the third congruences modulo 11. In 1988, Andrews and Garvan \[4\] finally found the actual crank. For a partition \(\pi\), let \(\omega(\pi)\) denote the number of ones, and \(\mu(\pi)\) the number of parts larger than \(\omega(\pi)\). Then, the crank of \(\pi\) is defined as follows
\[
\text{crank}(\pi) := \begin{cases} 
\lambda(\pi), & \text{if } \omega(\pi) = 0, \\
\mu(\pi) - \omega(\pi), & \text{otherwise}. 
\end{cases}
\]

Recently, Andrews \[2\] mentioned that George Beck introduced two partition statistics \(N_T(m, j, n)\) and \(M_\omega(m, j, n)\), which denote the total number of parts in the partition of \(n\) with rank congruent to \(m\) modulo \(j\) and the total number of ones in the partition of \(n\) with crank congruent to \(m\) modulo \(j\), respectively, i.e.,
\[
N_T(m, j, n) = \sum_{\substack{\pi \vdash n, \\
\text{rank}(\pi) \equiv m \pmod{j}}} \#(\pi)
\]
and
\[
M_\omega(m, j, n) = \sum_{\substack{\pi \vdash n, \\
\text{crank}(\pi) \equiv m \pmod{j}}} \omega(\pi).
\]
Andrews \[2\] proved the following Andrews-Beck type congruence which was conjectured by Beck
\[
\sum_{m=1}^{4} mN_T(m, 5, 5n + 1) \equiv \sum_{m=1}^{4} mN_T(m, 5, 5n + 4) \equiv 0 \pmod{5}.
\]

Motivated by Andrews’s work, Chern \[7, 8, 9\] proved some identities involving the weighted rank and crank moments and established a number of new Andrews-Beck type congruences on \(N_T(m, j, n)\) and \(M_\omega(m, j, n)\). For example, Chern \[7\] proved that for \(n \geq 0\),
\[
\sum_{m=1}^{4} mM_\omega(m, 5, 5n + 4) \equiv 0 \pmod{5}. \quad (1.3)
\]
In a recent paper, Lin, Peng and Toh \[13\] considered the generalized crank defined by Fu and Tang \[11\] for \(k\)-colored partitions and derived a number of Andrews-Beck type congruences. Very recently, Chan, Mao and Osburn \[6\] proved three variations of Andrews-Beck type congruences and posed a
number of conjectures on $NT(m, j, n)$ and $M_\omega(m, j, n)$. Those conjectures on Andrews-Beck type congruences of $NT(m, j, n)$ and $M_\omega(m, j, n)$ were proved by Chern \cite{9}. Later, Mao \cite{14} proved the following two identities on $NT(m, j, n)$ which were conjectured by Chan, Mao and Osburn \cite{6}:

$$\sum_{n=0}^\infty (NT(1, 7, 7n + 5) - NT(6, 7, 7n + 5) + 3NT(2, 7, 7n + 5) - 3NT(5, 7, 7n + 5)) q^n$$

$$= -7 \frac{(q^3, q^4, q^7, q^7; q^7)_\infty}{(q, q^2, q^3, q^6, q^6; q^7)_\infty}$$

and

$$\sum_{n=0}^\infty (NT(17, 7n + 4) - NT(6, 7, 7n + 4) + 2NT(3, 7, 7n + 4) - 2NT(4, 7, 7n + 4)) q^n$$

$$= -7 \frac{(q^3, q^4, q^4, q^7, q^7, q^7; q^7)_\infty}{(q, q^2, q^2, q^5, q^5, q^6, q^6; q^7)_\infty}$$

where here and throughout the rest of the paper, we adopt the standard $q$-series notation

$$(a; q)_\infty = \prod_{n=0}^\infty (1 - aq^n)$$

and for each positive integer $k$,

$$(a_1, a_2, \ldots, a_k; q)_\infty = (a_1; q)_\infty(a_2; q)_\infty \cdots (a_k; q)_\infty.$$

In another paper, Mao \cite{15} also gave several conjectures on $NT(m, j, n)$ and $M_\omega(m, j, n)$.

The aim of the paper is to confirm the remainder three conjectures of Chan-Mao-Osburn \cite{6} and two conjectures due to Mao \cite{15} on some relations involving $NT(m, 5, n)$ and $M_\omega(m, 5, n)$.

**Theorem 1.1** We have

$$\sum_{n=0}^\infty (NT(1, 5, 5n + 4) - NT(4, 5, 5n + 4)$$

$$+ 2M_\omega(2, 5, 5n + 4) - 2M_\omega(3, 5, 5n + 4)) q^n = -5 \frac{(q^3; q^3)_\infty^4}{(q; q)_\infty}$$

(1.4)

and for $n \geq 0$,

$$M_\omega(2, 5, 5n + 4) - M_\omega(3, 5, 5n + 4) = 2NT(1, 5, 5n + 4) - 2NT(4, 5, 5n + 4).$$

(1.5)

**Theorem 1.2** For $n \geq 0$,

$$M_\omega(1, 5, 5n + 4) - M_\omega(4, 5, 5n + 4) = 2M_\omega(3, 5, 5n + 4) - 2M_\omega(2, 5, 5n + 4).$$

(1.6)

**Theorem 1.3** For $n \geq 0$,

$$M_\omega(1, 5, 5n + 2) - M_\omega(4, 5, 5n + 2) = 2NT(3, 5, 5n + 2) - 2NT(2, 5, 5n + 2).$$

(1.7)

**Theorem 1.4** For $n \geq 0$,

$$M_\omega(2, 5, 5n + 1) - M_\omega(3, 5, 5n + 1) = NT(2, 5, 5n + 1) - NT(3, 5, 5n + 1).$$

(1.8)

Identities (1.4), (1.6) and (1.7) were first conjectured by Chan, Mao and Osburn \cite{6} and (1.5) and (1.8) were conjectured by Mao \cite{15}. Moreover, identity (1.6) implies (1.3).
2 Preliminaries

In this section, we present several lemmas which will be used to prove the main results of this paper.

The following lemma was given by Garvan [12].

Lemma 2.1 [12] (3.1) Let $\zeta = e^{2\pi i/5}$. For $m = 1, 2$,

$$\left(\frac{q}{q_1}\right)_\infty = A(q^5) - (\zeta^m + \zeta^{-m})qB(q^5)$$

$$+ (\zeta^{2m} + \zeta^{-2m})q^2C(q^5) - (\zeta^m + \zeta^{-m})q^3D(q^5),$$

where

$$A(q) = \frac{(q^2, q^3, q^5; q^5)_\infty}{(q, q^4, q^6; q^6)_\infty},$$

$$B(q) = \frac{(q^5; q^5)_\infty}{(q, q^4, q^6; q^6)_\infty},$$

$$C(q) = \frac{(q^5; q^5)_\infty}{(q^2, q^4, q^6; q^6)_\infty},$$

$$D(q) = \frac{(q, q^4, q^5; q^5)_\infty}{(q^2, q^4, q^6; q^6)_\infty}.$$

Lemma 2.2 We have

$$\sum_{n=0}^{\infty} \frac{q^n}{1 - q^{5n+1}} - \sum_{n=0}^{\infty} \frac{q^{4n+3}}{1 - q^{5n+4}} = \frac{(q^2, q^3, q^5; q^5)_\infty}{(q, q^4, q^6; q^6)_\infty},$$

$$\sum_{n=0}^{\infty} \frac{q^{2n+1}}{1 - q^{5n+3}} - \sum_{n=0}^{\infty} \frac{q^{3n+1}}{1 - q^{5n+2}} = 0,$$  

$$\sum_{n=0}^{\infty} \frac{q^n}{1 - q^{5n+2}} - \sum_{n=0}^{\infty} \frac{q^{4n+2}}{1 - q^{5n+3}} = \frac{(q^5; q^5)_\infty}{(q, q^4, q^6; q^6)_\infty},$$

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{1 - q^{5n+2}} - \sum_{n=0}^{\infty} \frac{q^{3n+1}}{1 - q^{5n+3}} = \frac{(q^2, q^4, q^5; q^6)_\infty}{(q^2, q^4, q^6; q^6)_\infty},$$

$$\sum_{n=0}^{\infty} \frac{q^n}{1 - q^{5n+3}} - \sum_{n=0}^{\infty} \frac{q^{4n+1}}{1 - q^{5n+2}} = \frac{(q^5, q^5; q^5)_\infty}{(q^2, q^4, q^6; q^6)_\infty},$$

$$\sum_{n=0}^{\infty} \frac{q^{2n+1}}{1 - q^{5n+4}} - \sum_{n=0}^{\infty} \frac{q^{3n}}{1 - q^{5n+1}} = 0,$$  

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{1 - q^{5n+1}} - \sum_{n=0}^{\infty} \frac{q^{3n+2}}{1 - q^{5n+4}} = \frac{(q, q^4, q^5; q^5)_\infty}{(q^2, q^4, q^6; q^6)_\infty}.$$  

Proof. Let $r, s, t$ be integers with $1 \leq r, s \leq 4$ and $0 \leq t \leq 4$. It is easy to check that

$$\sum_{n=0}^{\infty} \frac{q^{rn+t}}{1 - q^{5n+s}} - \sum_{n=0}^{\infty} \frac{q^{(5-r)n+5+t-r-s}}{1 - q^{5n+s}} = \sum_{n=0}^{\infty} \frac{q^{rn+t}}{1 - q^{5n+s}} - \sum_{n=1}^{\infty} \frac{q^{(5-r)(n-1)+5+t-r-s}}{1 - q^{5n-s}}$$
Combining (2.12) and (2.13) yields
\[ \sum_{n=0}^{\infty} \frac{q^{rn+t}}{1-q^{5n+s}} - \sum_{n=-\infty}^{-1} \frac{q^{(5-r)(-n-1)+5+t-r-s}}{1-q^{-5n}} \]
\[ = \sum_{n=0}^{\infty} \frac{q^{rn+t}}{1-q^{5n+s}} + \sum_{n=-\infty}^{-1} \frac{q^{rn+t}}{1-q^{5n+s}} \]
\[ = \sum_{n=-\infty}^{\infty} \frac{q^{rn+t}}{1-q^{5n+s}}. \]  

(2.12)

It follows from [3, Lemma 4.4.2, p. 117] that
\[ \sum_{n=-\infty}^{\infty} \frac{q^{ni}}{1-q^{5n+j}} = \frac{(q^{i+j}, q^{5-i-j}, q^5, q^5; q^5)_{\infty}}{(q^5, q^3, q^3-1, q^5-1, q^5; q^5)_{\infty}}. \]  

(2.13)

Combining (2.12) and (2.13) yields
\[ \sum_{n=0}^{\infty} \frac{q^{(5-r)n+5+t-r-s}}{1-q^{5n+5-s}} = \frac{q^r(q^{r+s}, q^{5-r-s}, q^5, q^5; q^5)_{\infty}}{(q^r, q^s, q^{5-r}, q^{5-s}; q^5)_{\infty}}. \]  

(2.14)

The proofs below make frequent use of (2.14).

To prove (2.3), use (2.14) with \( r = 1, s = 1 \) and \( t = 0 \).

To prove (2.4), use (2.14) with \( r = 2, s = 3 \) and \( t = 1 \).

To prove (2.5), use (2.14) with \( r = 1, s = 2 \) and \( t = 0 \).

To prove (2.6), use (2.14) with \( r = 2, s = 2 \) and \( t = 0 \).

To prove (2.7), use (2.14) with \( r = 1, s = 3 \) and \( t = 0 \).

To prove (2.8), use (2.14) with \( r = 2, s = 3 \) and \( t = 0 \).

To prove (2.9), use (2.14) with \( r = 1, s = 4 \) and \( t = 1 \).

To prove (2.10), use (2.14) with \( r = 2, s = 4 \) and \( t = 0 \).

To prove (2.11), use (2.14) with \( r = 2, s = 2 \) and \( t = 1 \).

The proof of Lemma (2.2) is complete.

Lemma 2.3 We have
\[ \sum_{n=1}^{\infty} \frac{q^n + q^{2n} - q^{3n} - q^{4n}}{1-q^{5n}} = \frac{2(q^2, q^3, q^5; q^5)_{\infty}^2}{5(q, q^4, q^5; q^5)_{\infty}^3} - \frac{2}{5}. \]  

(2.15)

and
\[ \sum_{n=1}^{\infty} \frac{q^n - 2q^{2n} + 2q^{3n} - q^{4n}}{1-q^{5n}} = \frac{(q^2, q^3, q^5; q^5)_{\infty}^2}{10(q, q^4, q^5; q^5)_{\infty}^3} + \frac{7q(q^4, q^5; q^5)_{\infty}^2}{10(q^2, q^4, q^5; q^5)_{\infty}^3} - \frac{1}{10}. \]  

(2.16)

Proof. It is easy to verify that for \( 1 \leq m \leq 4 \),
\[ \sum_{n=1}^{\infty} \frac{q^{mn}}{1-q^{5n}} = \sum_{n=1}^{\infty} q^{mn} \sum_{j=0}^{\infty} q^{5nj} \]
Theorem 3.1

We have

\[
= \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} q^{(5j+m)n} = \sum_{j=0}^{\infty} \frac{q^{5j+m}}{1 - q^{5j+m}} \quad (2.17)
\]

Thanks to (2.17),

\[
\sum_{n=1}^{\infty} \frac{q^n + q^{2n} - q^{3n} - q^{4n}}{1 - q^{5n}} = \sum_{j=0}^{\infty} \left( \frac{q^{5j+1}}{1 - q^{5j+1}} - \frac{q^{5j+4}}{1 - q^{5j+4}} + \frac{q^{5j+2}}{1 - q^{5j+2}} - \frac{q^{5j+3}}{1 - q^{5j+3}} \right). \quad (2.18)
\]

It is easy to check that

\[
\sum_{j=0}^{\infty} \left( \frac{q^{5j+1}}{1 - q^{5j+1}} - \frac{q^{5j+4}}{1 - q^{5j+4}} \right) = \sum_{j=0}^{\infty} \left( \frac{q^{5j+6}}{1 - q^{5j+6}} - \frac{q^{5j-1}}{1 - q^{5j-1}} \right) - 1 \quad (2.19)
\]

and

\[
\sum_{j=0}^{\infty} \left( \frac{q^{5j+2}}{1 - q^{5j+2}} - \frac{q^{5j+3}}{1 - q^{5j+3}} \right) = \sum_{j=0}^{\infty} \left( \frac{q^{5j+7}}{1 - q^{5j+7}} - \frac{q^{5j-2}}{1 - q^{5j-2}} \right) - 1. \quad (2.20)
\]

It follows from [13] Lemma 2.4, (2.11) and (2.12)] that

\[
\sum_{j=0}^{\infty} \left( \frac{q^{5j+6}}{1 - q^{5j+6}} - \frac{q^{5j-1}}{1 - q^{5j-1}} \right) = \frac{1}{10} \left( \frac{3(q^2, q^3; q^5)_\infty^2}{(q, q^4; q^5)_\infty^3} + q \left( \frac{q^2, q^4, q^5; q^5}_\infty^3 \right)^2 + 7 \right) \quad (2.21)
\]

and

\[
\sum_{j=0}^{\infty} \left( \frac{q^{5j+7}}{1 - q^{5j+7}} - \frac{q^{5j-2}}{1 - q^{5j-2}} \right) = \frac{1}{10} \left( \frac{(q^2, q^3, q^5; q^5)_\infty^2}{(q, q^4; q^5)^3_\infty} - 3q \left( \frac{q^2, q^4, q^5; q^5}_\infty^3 \right)^2 + 9 \right) \quad (2.22)
\]

Combining (2.18) and (2.22), we arrive at (2.15). This completes the proof of Lemma 2.3.

\[\blacksquare\]

3 The generating functions for \(M_\omega(a, 5, n)\)

In this section, we establish the generating functions for \(M_\omega(a, 5, n)\).

Theorem 3.1 We have

\[
\sum_{n \geq 0} M_\omega(0, 5, n) q^n = \frac{3}{5} D(q^5) (-3R_1(q) + 2R_2(q) + 2R_3(q) - 3R_4(q) + 2R_5(q) - 2S(q))
\]

\[
+ \frac{q^2}{5} C(q^5) (-2R_1(q) + 3R_2(q) + 3R_3(q) - 2R_4(q) - 2R_5(q) + 2S(q))
\]

\[
+ \frac{q}{5} B(q^5) (4R_1(q) - R_2(q) - R_3(q) + 4R_4(q) - 6R_5(q) + 6S(q))
\]

\[
+ \frac{1}{5} A(q^5) (-R_1(q) - R_2(q) - R_3(q) - R_4(q) + 4R_5(q) - 4S(q)) + T(q), \quad (3.1)
\]

\[
\sum_{n \geq 0} M_\omega(1, 5, n) q^n = \frac{3}{5} D(q^5) (2R_1(q) + 2R_2(q) - 3R_3(q) + 2R_4(q) - 3R_5(q) + 3S(q))
\]

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\[ \sum_{n \geq 0} M_\omega(2, 5, n) q^n = \frac{q^3}{5} D(q^5) (2R_1(q) - 3R_2(q) + 2R_3(q) - 3R_4(q) + 2R_5(q) - 2S(q)) + \frac{q^2}{5} C(q^5) (3R_1(q) - 2R_2(q) - 2R_3(q) - 2R_4(q) - 2R_5(q) + 2S(q)) + \frac{q}{5} B(q^5) (-R_1(q) - R_2(q) + 4R_3(q) - 6R_4(q) + 4R_5(q) - 4S(q)) + \frac{1}{5} A(q^5) (-R_1(q) - R_2(q) - R_3(q) + 4R_4(q) - R_5(q) + S(q)) + T(q), \quad (3.2) \]

\[ \sum_{n \geq 0} M_\omega(3, 5, n) q^n = \frac{q^3}{5} D(q^5) (-3R_1(q) + 2R_2(q) - 3R_3(q) + 2R_4(q) + 2R_5(q) - 2S(q)) + \frac{q^2}{5} C(q^5) (-2R_1(q) - 2R_2(q) - 2R_3(q) + 3R_4(q) + 3R_5(q) - 3S(q)) + \frac{q}{5} B(q^5) (4R_1(q) - 6R_2(q) + 4R_3(q) - R_4(q) - R_5(q) + S(q)) + \frac{1}{5} A(q^5) (-R_1(q) + 4R_2(q) - R_3(q) - R_4(q) - R_5(q) + S(q)) + T(q), \quad (3.3) \]

\[ \sum_{n \geq 0} M_\omega(4, 5, n) q^n = \frac{q^3}{5} D(q^5) (2R_1(q) - 3R_2(q) + 2R_3(q) + 2R_4(q) - 3R_5(q) + 3S(q)) + \frac{q^2}{5} C(q^5) (-2R_1(q) - 2R_2(q) + 3R_3(q) + 3R_4(q) - 2R_5(q) + 2S(q)) + \frac{q}{5} B(q^5) (-6R_1(q) + 4R_2(q) - R_3(q) - R_4(q) + 4R_5(q) - 4S(q)) + \frac{1}{5} A(q^5) (4R_1(q) - R_2(q) - R_3(q) - R_4(q) - R_5(q) + S(q)) + T(q), \quad (3.4) \]

where \( A(q), B(q), C(q), D(q) \) are defined by (2.2) and

\[ R_i(q) = \sum_{n=1}^{\infty} \frac{q^n i}{1 - q^{5n}}, \quad S(q) = \sum_{n=1}^{\infty} \frac{q^{n+1}}{1 - q^{n+1}}, \quad T(q) = \frac{q}{5(1 - q)(q; q)_{\infty}}. \quad (3.6) \]

Proof. Chern [7] (3.2)] proved that

\[ \sum_{n \geq 0} \sum_{\lambda \vdash n} \omega(\lambda) z^{crank(\lambda)} q^n = \frac{(q; q)_{\infty}}{(zq, zdq; q)_{\infty}} \sum_{n \geq 1} \left( \frac{q^n z}{1 - q^n z} + \frac{q^{n+1} z}{1 - q^{n+1} z} \right). \quad (3.7) \]

By (3.7) and the definition of \( M_\omega(5, 5, n) \),

\[ \sum_{n \geq 0} M_\omega(5, 5, n) q^n = \frac{1}{5} \sum_{j=0}^{4} \zeta^{-bj} \frac{(q; q)_{\infty}}{(\zeta_j q; q)_{\infty} (\zeta_j^2 q; q)_{\infty}} \left( \sum_{n=1}^{\infty} \frac{\zeta^{-j q^n}}{1 - q^n \zeta^{-j}} - S(q) \right). \]

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by

If we extract those terms in which the power of the

Proof of Theorem 1.1.
The objective of this section is to prove Theorems 1.1–1.4.

4 Proofs of Theorems 1.1–1.4

Very recently, Mao [14, Theorem 1.1] proved the following identity:

\[ S(q) = \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} = \sum_{j=1}^{\infty} \frac{q^{-j} q^n}{1-q^{-j} q^n} - S(q), \] (4.4)

where \( S(q) \) and \( T(q) \) are defined by (3.8) and \( \zeta = e^{2\pi i/5} \). Moreover, it is easy to check that

\[ \sum_{n=1}^{\infty} \frac{\zeta^{-j} q^n}{1-\zeta^{-j} q^n} = \sum_{n=1}^{\infty} \frac{q^{5n}}{1-q^{5n}} + \sum_{n=1}^{\infty} \frac{q^n}{1-q^{5n}} + \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{5n}} + \sum_{n=1}^{\infty} \frac{q^{4n}}{1-q^{5n}}. \] (3.9)

Setting \( b = 0, 1, 2, 3, 4 \) in (3.8) and substituting (2.1) and (3.9) into (3.8), we arrive at (3.1)–(3.5), respectively. This completes the proof of Theorem 3.1.

4 Proofs of Theorems 1.1–1.4

The objective of this section is to prove Theorems 1.1–1.4.

Proof of Theorem 1.1. In light of (3.3) and (3.4),

\[ \sum_{n=0}^{\infty} (M_\omega(2, 5, n) - M_\omega(3, 5, n)) q^n = q^3 D(q^3) (R_1(q) - R_2(q) + R_3(q) - R_4(q)) + q^2 C(q^5) (R_1(q) - R_4(q)) + qB(q^5) (-R_1(q) + 2R_2(q) - 2R_3(q) + R_4(q)) - A(q^3) (R_2(q) - R_3(q)). \] (4.1)

If we extract those terms in which the power of \( q \) is congruent to 4 modulo 5 in (4.1), then divided by \( q^3 \) and replace \( q^3 \) by \( q \), we arrive at

\[ \sum_{n=0}^{\infty} (M_\omega(2, 5, 5n+4) - M_\omega(3, 5, 5n+4)) q^n = \]

\[ = D(q) \left( \sum_{n=0}^{\infty} \frac{q^n}{1-q^{5n+1}} - \sum_{n=0}^{\infty} \frac{q^{4n+3}}{1-q^{5n+4}} \right) - D(q) \left( \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1-q^{5n+3}} - \sum_{n=0}^{\infty} \frac{q^{3n+1}}{1-q^{5n+2}} \right) \]

\[ + C(q) \left( \sum_{n=0}^{\infty} \frac{q^n}{1-q^{5n+2}} - \sum_{n=0}^{\infty} \frac{q^{4n+2}}{1-q^{5n+3}} \right) - B(q) \left( \sum_{n=0}^{\infty} \frac{q^n}{1-q^{5n+3}} - \sum_{n=0}^{\infty} \frac{q^{4n+1}}{1-q^{5n+2}} \right) \]

\[ + 2B(q) \left( \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1-q^{5n+4}} - \sum_{n=0}^{\infty} \frac{q^{3n+1}}{1-q^{5n+3}} \right) - A(q) \left( \sum_{n=0}^{\infty} \frac{q^{2n}}{1-q^{5n+2}} - \sum_{n=0}^{\infty} \frac{q^{3n+1}}{1-q^{5n+3}} \right). \] (4.2)

Thanks to (2.8)–(2.11) and (4.2),

\[ \sum_{n=0}^{\infty} (M_\omega(2, 5, 5n+4) - M_\omega(3, 5, 5n+4)) q^n = -2 \frac{(q^5: q^5)^4}{(q: q)^{\infty}} \] (4.3)

Very recently, Mao [14, Theorem 1.1] proved the following identity:

\[ \sum_{n=0}^{\infty} (NT(1, 5, 5n+4) - NT(4, 5, 5n+4)) q^n = -\frac{(q^5: q^5)^4}{(q: q)^{\infty}}. \] (4.4)
Theorem 1.1 follows from (4.3) and (4.4).

Now, we turn to prove Theorem 1.2.

Proof of Theorem 1.2 In view of (3.2) and (3.5),
\[ \sum_{n=0}^{\infty} (M_\omega(1,5,n) - M_\omega(4,5,n))q^n = q^3D(q^5)(R_2(q) - R_3(q)) + q^2C(q^5)(R_1(q) - R_4(q) + R_2(q) - R_3(q)) + qB(q^5)(R_1(q) - R_4(q) - R_2(q) + R_3(q)) - A(q^5)(R_1(q) - R_4(q)). \] (4.5)

Extracting those terms in which the power of \( q \) is congruent to 4 modulo 5 in (4.5), then dividing by \( q^4 \) and replacing \( q^5 \) by \( q \), we deduce that
\[ \sum_{n=0}^{\infty} (M_\omega(1,5,5n+4) - M_\omega(4,5,5n+4))q^n \]
\[ = D(q) \left( \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1 - q^{5n+3}} - \sum_{n=0}^{\infty} \frac{q^{3n+1}}{1 - q^{5n+2}} \right) + C(q) \left( \sum_{n=0}^{\infty} \frac{q^n}{1 - q^{3n+2}} - \sum_{n=0}^{\infty} \frac{q^{4n+2}}{1 - q^{5n+3}} \right) 
+ B(q) \left( \sum_{n=0}^{\infty} \frac{q^n}{1 - q^{3n+4}} - \sum_{n=0}^{\infty} \frac{q^{4n+1}}{1 - q^{5n+4}} \right) - A(q) \left( \sum_{n=0}^{\infty} \frac{q^n}{1 - q^{5n+1}} - \sum_{n=0}^{\infty} \frac{q^{4n}}{1 - q^{5n+2}} \right). \] (4.6)

It follows from (2.4), (2.5), (2.7) – (2.10) and (4.6) that
\[ \sum_{n=0}^{\infty} (M_\omega(1,5,5n+4) - M_\omega(4,5,5n+4))q^n = \frac{(5; q^5)^4}{(q; q)^{\infty}}. \] (4.7)

Identity (1.6) follows from (4.3) and (4.7) and the proof of Theorem 1.2 is complete.

Next, we present a proof of Theorem 1.3.

Proof of Theorem 1.3 If we pick out those terms in which the power of \( q \) is congruent to 2 modulo 5 in (4.5), then divided by \( q^2 \) and replace \( q^5 \) by \( q \), we obtain
\[ \sum_{n=0}^{\infty} (M_\omega(1,5,5n+2) - M_\omega(4,5,5n+2))q^n \]
\[ = D(q) \left( \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1 - q^{5n+2}} - \sum_{n=0}^{\infty} \frac{q^{3n+2}}{1 - q^{5n+3}} \right) + C(q) \sum_{n=1}^{\infty} \frac{q^n + q^{2n} - q^{3n} - q^{4n}}{1 - q^{5n}} 
+ B(q) \left( \sum_{n=0}^{\infty} \frac{q^n}{1 - q^{3n+1}} - \sum_{n=0}^{\infty} \frac{q^{4n+3}}{1 - q^{5n+4}} \right) - B(q) \left( \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1 - q^{5n+3}} - \sum_{n=0}^{\infty} \frac{q^{3n+1}}{1 - q^{5n+2}} \right) 
- A(q) \left( \sum_{n=0}^{\infty} \frac{q^n}{1 - q^{5n+2}} - \sum_{n=0}^{\infty} \frac{q^{4n+2}}{1 - q^{5n+3}} \right). \] (4.8)
It follows from (2.2), (2.7), (2.10), and (4.8) that
\[
\sum_{n=0}^{\infty} (M_\omega(1, 5, 5n + 2) - M_\omega(4, 5, 5n + 2))q^n = \frac{4}{5} \frac{(q, q^4; q^5)_\infty^2 (q^5; q^5)_\infty^3}{(q^2, q^4; q^5)_\infty^3} + \frac{2}{5} \frac{(q^2, q^3; q^5)_\infty (q^5; q^5)_\infty^3}{(q, q^4; q^5)_\infty^3} - \frac{2}{5} \frac{(q^5; q^5)_\infty}{(q, q^4; q^5)_\infty}, \quad (4.9)
\]
Mao [14, Theorem 1.2] also proved that
\[
2 \sum_{n=0}^{\infty} (NT(2, 5, 5n + 2) - NT(3, 5, 5n + 2))q^n = -\frac{2}{5} \frac{(q^2, q^3; q^5)_\infty (q^5; q^5)_\infty^3}{(q, q^4; q^5)_\infty^3} + \frac{2}{5} \frac{(q^5; q^5)_\infty}{(q^2, q^4; q^5)_\infty} - \frac{4}{5} \frac{(q, q^4; q^5)_\infty^2 (q^5; q^5)_\infty^3}{(q, q^4; q^5)_\infty^3}, \quad (4.10)
\]
which yields (1.7) after combining (4.9). The proof of Theorem 1.3 is complete.

Finally, we give a proof of Theorem 1.4.

Proof of Theorem 1.4 Extracting those terms in which the power of $q$ is congruent to 1 modulo 5 in (4.11), then dividing by $q^4$ and replacing $q^5$ by $q$, we arrive at
\[
\sum_{n=0}^{\infty} (M_\omega(2, 5, 5n + 1) - M_\omega(3, 5, 5n + 1))q^n = D(q) \left( \sum_{n \geq 0} \frac{q^{n+1}}{1 - q^{5n+3}} - \sum_{n \geq 0} \frac{q^{4n+2}}{1 - q^{5n+2}} \right) - D(q) \left( \sum_{n \geq 0} \frac{q^{2n+2}}{1 - q^{5n+4}} - \sum_{n \geq 0} \frac{q^{3n+1}}{1 - q^{5n+1}} \right) + C(q) \left( \sum_{n \geq 0} \frac{q^{n+1}}{1 - q^{5n+4}} - \sum_{n \geq 0} \frac{q^{4n+1}}{1 - q^{5n+1}} \right) - B(q) \sum_{n \geq 1} \frac{q^n - 2q^{2n} + 2q^{3n} - q^{4n}}{1 - q^{5n}} - A(q) \left( \sum_{n \geq 0} \frac{q^{2n+1}}{1 - q^{5n+3}} - \sum_{n \geq 0} \frac{q^{3n+1}}{1 - q^{5n+2}} \right). \quad (4.11)
\]
Thanks to (2.2), (2.3), (2.7) – (2.9), (2.16) and (4.11),
\[
\sum_{n=0}^{\infty} (M_\omega(2, 5, 5n + 1) - M_\omega(3, 5, 5n + 1))q^n = \frac{(q^5; q^5)_\infty}{10(q, q^4; q^5)_\infty^2} - \frac{(q^2, q^3; q^5)_\infty (q^5; q^5)_\infty^3}{10(q, q^4; q^5)_\infty^3} + \frac{13(q, q^4; q^5)_\infty^2 (q^5; q^5)_\infty^3}{10(q, q^4; q^5)_\infty^3}, \quad (4.12)
\]
Mao [14, Theorem 1.1, (1.12)] proved that
\[
\sum_{n=0}^{\infty} (NT(2, 5, 5n + 1) - NT(3, 5, 5n + 1))q^n = \frac{(q^5; q^5)_\infty}{10(q, q^4; q^5)_\infty^2} - \frac{(q^2, q^3; q^5)_\infty (q^5; q^5)_\infty^3}{10(q, q^4; q^5)_\infty^3} + \frac{13(q, q^4; q^5)_\infty^2 (q^5; q^5)_\infty^3}{10(q, q^4; q^5)_\infty^3}, \quad (4.13)
\]
which yields (1.8) after combining (4.12). This completes the proof of Theorem 1.4. \qed
5 Concluding Remarks

As seen in Introduction, the two partition statistics $NT(m, j, n)$ and $M_\omega(m, j, n)$ introduced by Beck have received a lot of attention in recent years. In particular, Chan, Mao and Osburn [6] posed several conjectures on $NT(m, j, n)$ and $M_\omega(m, j, n)$ and some of them have been proved by Chern [9] and Mao [14]. In this paper, we confirm the remainder three conjectures of Chan-Mao-Osburn [6] and two conjectures given by Mao [15] on the relations between $NT(m, j, n)$ and $M_\omega(m, j, n)$ by proving some $q$-series identities. From the results of this paper, we can obtain some congruences. For example, by (4.3), (4.7) and (4.9), we see that for $n \geq 0$,

$$M_\omega(2, 5, 5n + 4) \equiv M_\omega(3, 5, 5n + 4) \pmod{2},$$  \hfill (5.1)
$$M_\omega(1, 5, 5n + 2) \equiv M_\omega(4, 5, 5n + 2) \pmod{2},$$  \hfill (5.2)
$$M_\omega(1, 5, 5n + 4) \equiv M_\omega(4, 5, 5n + 4) \pmod{4}. \hfill (5.3)$$

It is interesting to give combinatorial proofs of (5.1)–(5.3).

Computer evidence suggests that the following two conjectures might hold.

**Conjecture 5.1** Let $0 \leq i < j \leq 4$ be integers. If $i + j \neq 5$, then

$$\lim_{n \to \infty} \frac{\# \{k | M_\omega(i, 5, k) \equiv M_\omega(j, 5, k) \pmod{2}, 1 \leq k \leq n \}}{n} = \frac{1}{2}. \hfill (5.4)$$

Moreover,

$$\lim_{n \to \infty} \frac{\# \{k | M_\omega(1, 5, k) \equiv M_\omega(4, 5, k) \pmod{2}, 1 \leq k \leq n \}}{n} = \frac{3}{10}. \hfill (5.5)$$

and

$$\lim_{n \to \infty} \frac{\# \{k | M_\omega(2, 5, k) \equiv M_\omega(3, 5, k) \pmod{2}, 1 \leq k \leq n \}}{n} = \frac{2}{5}. \hfill (5.6)$$

**Conjecture 5.2** Let $0 \leq i < j \leq 4$ be integers. Then

$$\lim_{n \to \infty} \frac{\# \{k | NT(i, 5, k) \equiv NT(j, 5, k) \pmod{2}, 1 \leq k \leq n \}}{n} = \frac{1}{2}. \hfill (5.7)$$

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