On Connes’ new principle of general relativity
Can spinors hear the forces of spacetime?

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Abstract

Connes has extended Einstein’s principle of general relativity to noncommutative geometry. The new principle implies that the Dirac operator is covariant with respect to Lorentz and internal gauge transformations and the Dirac operator must include Yukawa couplings. It further implies that the action for the metric, the gauge potentials and the Higgs scalar is coded in the spectrum of the covariant Dirac operator. This universal action has been computed by Chamseddine & Connes, it is the coupled Einstein-Hilbert and Yang-Mills-Higgs action. This result is rederived and we discuss the physical consequences.

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The dynamical variable of gravity is the metric on spacetime. Einstein used the matrix $g^{\mu\nu}(x)$ of the metric $g$ with respect to a coordinate system $x^\mu$ to parameterize the set of all metrics on a fixed spacetime $M$. The coordinate system being unphysical, Einstein required his field equations for the metric to be covariant under coordinate transformations, the principle of general relativity. Elie Cartan used tetrads, repères mobiles, to parameterize the set of all metrics. This parameterization allows to generalize the Dirac operator $\mathcal{D}$ to curved spacetimes and also reformulates general relativity as a gauge theory under the Lorentz group. Connes [1] goes one step further by relating the set of all metrics to the set of all Dirac operators. The Einstein-Hilbert action, from this point of view, is the Wodzicki residue of the second inverse power of the Dirac operator $\mathcal{D}$ and is computed most conveniently from the second coefficient of the heat kernel expansion of the Dirac operator squared. The heat kernel expansion $[3]$ is an old friend $[4]$ from quantum field theory in curved spacetime, from its formal relation to the one-loop effective action

$$S_{\text{eff}} = \text{tr} \log(\mathcal{D}/\Lambda), \quad \Lambda \text{ a cut off.} \tag{1}$$

This relation has been used by Sakharov $[5]$ to induce gravity from quantum fluctuations, leading however to a negative Newton constant $[6]$.

By generalizing the metric, the Dirac operator plays a fundamental role in noncommutative geometry. To describe Yang-Mills theories, Connes considers the product of spacetime and internal space in this new geometry, a natural point of view because the fermionic mass matrix qualifies as Dirac operator on internal space. Now comes the first miracle: in the same sense that Minkowskian geometry forces the electric field to be accompanied by a magnetic field, noncommutative geometry forces certain Yang-Mills fields to be accompanied by a symmetry breaking Higgs field $[7]$. This miracle takes place only in a tiny class of Yang-Mills-Dirac theories $[8]$ and the second miracle is that the standard model of electro-weak and strong forces is in this class.

Now we are ready for the third miracle. It extends Einstein’s principle of general relativity to noncommutative geometry. To generalize the Dirac operator from flat to curved spacetime (locally), it is sufficient to write the Dirac operator first in flat spacetime but with respect to noninertial coordinates. A straightforward calculation produces the covariant Dirac operator that contains the spin connection $\omega$. Although of vanishing curvature, $\omega$ contains a lot of physics, e.g. the centrifugal and Coriolis accelerations in the coordinates of the rotating disk, the quantum interference pattern of neutrons $[9]$ in oscillating coordinates. Then, the generalization to curved space is easy where $\omega$ describes the (minimal) coupling of the spinor to the gravitational field. From this point of view, the covariant Dirac operator is obtained by acting with the diffeomorphism group on the flat Dirac operator. But the diffeomorphism group is just the automorphism group of the associative (and commutative) algebra $C^\infty(M)$ representing spacetime in the new geometry. On the other hand, the product of spacetime and internal space is represented in this geometry by the tensor product of $C^\infty(M)$ with a matrix.
algebra. Its automorphism group is the semi-direct product of the diffeomorphisms and the group of gauge transformations, the diffeomorphisms are the outer, the gauge transformations are the inner automorphisms. And what do we get when this entire automorphism group acts on the flat Dirac operator? We get the completely covariant Dirac operator containing the spin connection, the gauge connection and the Higgs \[10\]. In other words, we get the minimal couplings of the Dirac spinor to the gravitational and Yang-Mills fields and its Yukawa couplings to the Higgs field. In Connes’ words, the Higgs and Yang-Mills fields are noncommutative fluctuations of the metric. (Abelian Yang-Mills theories do not have such fluctuations.) Accordingly, Connes generalizes Einstein’s principle of general relativity by postulating that only the intrinsic properties of the covariant Dirac operator be relevant for physics. Here intrinsic means invariant under automorphisms. Thus, these properties must concern the spectrum only.

So far we have only the kinematics of the metric (and its fluctuations). To get its dynamics, Einstein developed the full power of the principle of general relativity and derived the Einstein-Hilbert action. This is what Chamseddine & Connes \[11\] now did also for the fluctuations of the metric, the Yang-Mills and Higgs fields. The fourth miracle is that this action comes out to be the Einstein-Hilbert action accompanied by the Yang-Mills action, by the covariant Klein-Gordon action and by the symmetry breaking Higgs potential.

In even dimensions, the spectrum of the Dirac operator is even and it is sufficient to consider the positive part of the spectrum which in the Euclidean is conveniently characterized by its distribution function \(S_\Lambda\) equal to the number of eigenvalues smaller than the positive real \(\Lambda\),

\[
S_\Lambda = \text{tr} f(D/\Lambda).
\]

Here \(f(u)\) denotes the characteristic function of the unit interval. Instead, if \(f\) was the logarithm, this trace, after a proper renormalization, would be Sakharov’s induced gravity action.

## 1 The Dirac operator

The starting point of Chamseddine & Connes’ action calculation \[11\] is the covariant Dirac operator \(D\) of the standard model. \(D\) acts on a multiplet of Weyl spinors. These are therefore dynamical fields coupled minimally to fixed, dynamical fields, the gravitational, the Yang-Mills and the Higgs fields.

\[
D = \left( i[\bar{\varphi} \otimes 1_L + e^{\mu} \gamma^i \otimes \rho_L(A_\mu)]_{\gamma_5 \otimes \Phi^*} i[\bar{\varphi} \otimes 1_R + e^{\mu} \gamma^i \otimes \rho_R(A_\mu)]_{\gamma_5 \otimes \Phi} \right),
\]

with

\[
\bar{\varphi} = e^{\mu} \gamma^i \left( \frac{\partial}{\partial x^\mu} + \frac{1}{4} \omega_{ab} \gamma^{ab} \right),
\]

\(e^{\mu}(x)\partial/\partial x^\mu\) is an orthonormal frame of the tangent bundle of an compact, Euclidean spacetime \(M\), \(\omega\) the torsionless spin connection with respect to this frame, \(A\) is the Yang-Mills field, the
connection of the internal Lie algebra $g$, $\rho_L$ is the unitary representation of $g$ on the Hilbert space $\mathcal{H}_L$ of left-handed spinors, $1_L$ is the identity on $\mathcal{H}_L$, likewise for the right-handed spinors, $\cdot_R$. $\Phi$ is the scalar multiplet. It must necessarily be a subrepresentation of $\mathcal{H}_L^* \otimes \mathcal{H}_R \oplus \mathcal{H}_L \otimes \mathcal{H}_R^*$.

Our conventions for Dirac matrices are:

\[
\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab}1_4, \quad \gamma^{ab} = \frac{1}{2} [\gamma^a, \gamma^b], \quad \gamma^{\mu} = \gamma^\mu, \\
\eta^{ab} = \text{diag}(+1, +1, +1, +1), \quad \gamma_2^2 = 1_4, \quad \gamma_5^5 = \gamma_5.
\] (5)

Let us spell out the representations for the standard model. The group is $SU(2) \times U(1) \times SU(3)$,

\[
\mathcal{H}_L = (C^2 \otimes C^N \otimes C^3) \oplus (C^2 \otimes C^N), \\
\mathcal{H}_R = ((C \oplus C) \otimes C^N \otimes C^3) \oplus (C \otimes C^N).
\] (6) (7)

The first factor denotes weak isospin, the second $N$ generations, $N = 3$, and the third denotes colour triplets and singlets. We take as basis

\[
\begin{pmatrix} u \\ d \\ c \\ s \\ t \\ b \\ e \\ \mu \\ \tau \end{pmatrix}_L, \quad \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}_L
\] (8)

for $\mathcal{H}_L$ which is $24$ dimensional and

\[
\begin{pmatrix} u_R \\ d_R \\ c_R \\ s_R \\ t_R \\ b_R \\ e_R \\ \mu_R \\ \tau_R \end{pmatrix}
\] (9)

for $\mathcal{H}_R$ of $21$ dimensions. The fermionic mass matrix is

\[
\mathcal{M} = \begin{pmatrix} M_u & 0 & 0 \\ 0 & M_d & 0 \\ 0 & 0 & M_e \end{pmatrix} \otimes 1_3
\] (10)

with

\[
M_u := \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \quad M_d := C_{KM} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}, \quad M_e := \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}
\] (11)

where $C_{KM}$ denotes the Cabbibo-Kobayashi-Maskawa matrix. Let $\varphi = (\varphi_1, \varphi_2)^T$ be the complex scalar doublet, its embedding in $\mathcal{H}_L^* \otimes \mathcal{H}_R \oplus \mathcal{H}_L \otimes \mathcal{H}_R^*$ is given by

\[
\Phi = \frac{1}{v} \begin{pmatrix} \varphi_1 M_u & -\varphi_2 M_d \\ \varphi_2 M_u & \varphi_1 M_d \end{pmatrix} \otimes 1_3 \begin{pmatrix} 0 \\ 0 \\ -\varphi_2 M_e \end{pmatrix},
\] (12)

with $v$ denoting the vacuum expectation value. Finally our conventions for hypercharges are:

\[
y(u_L) = y(d_L) = \frac{1}{6}, \quad y(u_R) = \frac{2}{3}, \quad y(d_R) = -\frac{1}{3}, \\
y(\nu_L) = y(e_L) = -\frac{1}{2}, \quad y(e_R) = -1,
\] (13)

\[
y(\varphi_1) = y(\varphi_2) = -\frac{1}{2}.
\]
2 The Dirac operator squared

Since the trace of the characteristic function only counts eigenvalues, we have

\[ S_\Lambda := \text{tr} f(\mathcal{D}/\Lambda) = \text{tr} f((\mathcal{D}/\Lambda)^2). \] (14)

Although the Dirac operator is only a first order differential operator, the computation of its square is either very long or subtle. The result is the well known Lichnerowicz formula \[ \text{[12]} \]. In our case it yields:

\[ \mathcal{D}^2 = -\Delta + E, \] (15)

and its trace can be computed using the heat kernel technique \[ \text{[3]} \]. \( \Delta \) is the covariant Laplace operator

\[ \Delta = g^{\mu\nu} \left( \frac{\partial}{\partial x^\mu} 1_4 \otimes 1_H + \frac{1}{4} \omega_{ab\mu} \gamma^{ab} \otimes 1_H + 1_4 \otimes \rho(A_\mu) \right) \delta^\nu_\mu - \Gamma^\nu_\mu 1_4 \otimes 1_H \]
\times \left[ \frac{\partial}{\partial x^\nu} 1_4 \otimes 1_H + \frac{1}{4} \omega_{abc} \gamma^{ab} \otimes 1_H + 1_4 \otimes \rho(A_\nu) \right] \] (16)

with the fermionic representation \( \rho := \rho_L \oplus \rho_R \) on \( \mathcal{H} := \mathcal{H}_L \oplus \mathcal{H}_R \) and \( \Gamma \) are the Christoffel symbols of the spin connection. \( E \), for endomorphism, is a zero order operator, that is a matrix of size \( 4 \dim \mathcal{H} \) whose entries are functions constructed from the adynamical bosonic fields and their first and second derivatives,

\[ E = \frac{1}{2} \left[ e^\mu_c e^\nu_d \gamma^{cd} \otimes 1_H \right] \mathcal{R}_{\mu\nu} + \left( \begin{array}{cc} 1_4 \otimes \Phi \Phi^* & -i\gamma_5 \gamma^\mu (D_\mu \Phi)^* \\ -i\gamma_5 \gamma^\mu (D_\mu \Phi) & 1_4 \otimes \Phi^* \Phi \end{array} \right). \] (17)

\( \mathcal{R} \) is the total curvature, a 2-form with values in the (Lorentz \( \oplus \) internal) Lie algebra represented on (spinors \( \otimes \mathcal{H} \)). It contains the curvature 2-form \( R = d\omega + \frac{1}{2}[\omega, \omega] \) and the field strength 2-form \( F = dA + \frac{1}{2}[A, A] \), in components

\[ \mathcal{R}_{\mu\nu} = \frac{1}{4} R_{ab\mu\nu} \gamma^{ab} \otimes 1_H + 1_4 \otimes \rho(F_{\mu\nu}). \] (18)

An easy calculation shows that the first term in equation (17) produces the curvature scalar that we also (!) denote by \( R \),

\[ \frac{1}{2} \left[ e^\mu_c e^\nu_d \gamma^{cd} \right] \frac{1}{4} R_{ab\mu\nu} \gamma^{ab} = \frac{1}{4} R 1_4. \] (19)

In our conventions, the curvature scalar is positive on spheres. Finally \( \mathcal{D} \) is the covariant derivative appropriate to the representation of the scalars. For more details on our conventions the reader is referred to \[ \text{[13]} \]. Note that the decomposition \( \mathcal{D}^2 = -\Delta + E \) is simple thanks to the presence of \( \gamma_5 \) in \( \mathcal{D} \). This \( \gamma_5 \) is deeply rooted in noncommutative geometry.

We close this section with a remark on the powers of \( \Phi \). Here we need two, later we will also meet four powers. In general, they are cumbersome to compute. For the standard model,
there is a trick that comes from its noncommutative formulation. Let us denote by $X \in su(2)$ an element of weak isospin and by $\rho_{Lw}$ its representation on $\mathcal{H}_L$. This representation can be extended to a representation of the quaternions $\mathbb{H}$ as involution algebra. $\mathbb{H}$ contains $su(2)$ as the Lie algebra of its group of unitaries. A quaternion $\phi$ is parameterized by two complex numbers, $\varphi_1$ and $\varphi_2$,

$$
\phi = \left( \varphi_1, -\bar{\varphi}_2 \right) \in \mathbb{H}.
$$

(20)

Then, the embedding $[12]$ of the scalar doublet $\varphi$ in $\mathcal{H}_L^* \otimes \mathcal{H}_R \oplus \mathcal{H}_L \otimes \mathcal{H}_R'$, which is nothing but the Yukawa couplings, takes the form of a matrix product,

$$
\Phi = \rho_{Lw}(\phi) \mathcal{M}/v,
$$

(21)

and the higher powers of $\Phi$ follow easily from the identity

$$
\phi^* \phi = \phi \phi^* = (|\varphi_1|^2 + |\varphi_2|^2)1_2 = |\varphi|^2 1_2.
$$

(22)

3 The trace

Asymptotically, for large $\Lambda$, the distribution function of the spectrum is given in terms of the heat kernel expansion $[9]$:

$$
S_\Lambda = \text{tr} f(D^2/\Lambda^2) = \frac{1}{16\pi^2} \int_M [\Lambda^4 f_0 a_0 + \Lambda^2 f_2 a_2 + f_4 a_4 + \Lambda^{-2} f_6 a_6 + \ldots] \sqrt{\text{det} g} \, d^4 x,
$$

(23)

with

$$
\begin{align*}
  f_0 &= \int_0^\infty u f(u) \, du = \frac{1}{2}, \\
  f_2 &= \int_0^\infty f(u) \, du = 1, \\
  f_4 &= f(0) = 1, \\
  f_6 &= f'(0) = 0, \\
  f_8 &= f''(0) = 0, \ldots
\end{align*}
$$

(24-28)

The $a_j$ are the coefficients of the heat kernel expansion of the Dirac operator squared,

$$
\begin{align*}
  a_0 &= \text{tr} (1_4 \otimes 1_\mathcal{H}), \\
  a_2 &= \frac{1}{6} R \text{tr} (1_4 \otimes 1_\mathcal{H}) - \text{tr} E, \\
  a_4 &= \frac{1}{12} R^2 \text{tr} (1_4 \otimes 1_\mathcal{H}) - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} \text{tr} (1_4 \otimes 1_\mathcal{H}) + \frac{1}{180} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \text{tr} (1_4 \otimes 1_\mathcal{H}) \\
  &\quad + \frac{1}{12} \text{tr} (R_{\mu\nu} R^{\mu\nu}) - \frac{1}{6} R \text{tr} E + \frac{1}{2} \text{tr} E^2 + \text{surface terms}.
\end{align*}
$$

(29-31)

Let us first check the normalization $16\pi^2$ of equation (23). We take $M$ to be the flat 4-torus with unit radii, $\mathcal{H}_L = \mathbb{C}$, $\mathcal{H}_R = 0$ and $A = \varphi = 0$. Denote by $\psi_B$, $B = 1, 2, 3, 4$, the four
components of the spinor. The Dirac operator is
\[
\varphi = \begin{pmatrix}
i\partial/\partial x^0 & 0 & -\partial/\partial x^3 & -\partial/\partial x^1 + i\partial/\partial x^2 \\
0 & i\partial/\partial x^0 & -\partial/\partial x^1 - i\partial/\partial x^2 & 0 \\
\partial/\partial x^3 & \partial/\partial x^1 - i\partial/\partial x^2 & -i\partial/\partial x^0 & 0 \\
\partial/\partial x^1 + i\partial/\partial x^2 & -\partial/\partial x^3 & 0 & -i\partial/\partial x^0
\end{pmatrix}.
\] (32)

After a Fourier transform
\[
\psi_B(x) = \sum_{j_0, \ldots, j_3} \hat{\psi}_B(j_0, \ldots, j_3) \exp(-ij_jx^\mu), \quad B = 1, 2, 3, 4
\] (33)
the eigenvalue equation \(\varphi\psi = \lambda\psi\) reads
\[
\begin{pmatrix}
j_0 & 0 & ij_3 & ij_1 + j_2 \\
0 & j_0 & ij_1 - j_2 & -ij_3 \\
-ij_3 & -ij_1 - j_2 & -j_0 & 0 \\
-ij_1 + j_2 & ij_3 & 0 & -j_0
\end{pmatrix}
\begin{pmatrix}
\hat{\psi}_1 \\
\hat{\psi}_2 \\
\hat{\psi}_3 \\
\hat{\psi}_4
\end{pmatrix} = \lambda
\begin{pmatrix}
\hat{\psi}_1 \\
\hat{\psi}_2 \\
\hat{\psi}_3 \\
\hat{\psi}_4
\end{pmatrix}.
\] (34)
Its characteristic equation is \(|\lambda^2 - (j_0^2 + j_1^2 + j_2^2 + j_3^2)|^2 = 0\) and for fixed \(j_\mu\), each eigenvalue \(\lambda = \pm \sqrt{j_0^2 + j_1^2 + j_2^2 + j_3^2}\) has multiplicity two. Therefore asymptotically for large \(\Lambda\) there are \(4B_4\Lambda^4\) eigenvalues (counted with their multiplicity) whose absolute values are smaller than \(\lambda\), \(\lambda > 0\). \(B_4 = \pi^2/2\) denotes the volume of the unit ball in \(\mathbb{R}^4\) and
\[
S_\Lambda = 4\frac{1}{2}\pi^2\Lambda^4 = \frac{1}{16\pi^4}\Lambda^4\frac{1}{2}(2\pi)^4.
\] (35)

The computation of the Chamseddine-Connes action \(S_\Lambda\) for the Dirac operator of the standard model is straightforward. We give a few intermediate steps, a full account can be found in [14].

\[
a_0 = 4 \dim \mathcal{H},
\]
(36)
\[
\text{tr} E = \dim \mathcal{H} R + 8 \text{tr} \Phi^*\Phi = \dim \mathcal{H} R + 8L|\varphi/v|^2,
\]
(37)
\[
L := 3 \text{tr} (M_u^*M_u) + 3 \text{tr} (M_d^*M_d) + \text{tr} (M_e^*M_e)
\]
\[
= 3(m_t^2 + m_c^2 + m_u^2 + m_b^2 + m_s^2 + m_d^2) + m_t^2 + m_\mu^2 + m_e^2,
\]
(38)
\[
a_2 = \frac{1}{6} \dim \mathcal{H} R - \dim \mathcal{H} R - 8L|\varphi/v|^2 = -\frac{1}{3} \dim \mathcal{H} R - 8L|\varphi/v|^2,
\]
(39)
\[
\text{tr} \gamma^{ab}\gamma^{cd} = 4 \left[\eta^{ad}\eta^{bc} - \eta^{ac}\eta^{bd}\right],
\]
(40)
\[
\text{tr} R_{\mu\nu}\mathcal{R}^{\mu\nu} = -\frac{1}{2} \dim \mathcal{H} R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4 \text{tr} (F_{\mu\nu})^*\rho(F^{\mu\nu}),
\]
(41)
\[
\text{tr} E^2 = \frac{1}{4} \dim \mathcal{H} R^2 + 2 \text{tr} (F_{\mu\nu})^*\rho(F^{\mu\nu})
\]
\[
+ 8L_2|\varphi/v|^4 + 8L(D_\mu\varphi/v)^*(D^\mu\varphi/v) + 4L|\varphi/v|^2 R,
\]
(42)
\[
L_2 := 3 \text{tr} [M_u^*M_u]^2 + 3 \text{tr} [M_d^*M_d]^2 + \text{tr} [M_e^*M_e]^2
\]
\[
= 3(m_t^4 + m_c^4 + m_u^4 + m_b^4 + m_s^4 + m_d^4) + m_t^4 + m_\mu^4 + m_e^4.
\]
(43)

Using the Weyl tensor,
\[
C_{\mu\nu\rho\sigma} := R_{\mu\nu\rho\sigma} - \frac{1}{2}(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma}) + \frac{1}{8}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R,
\]
(44)
we can assemble all higher derivative gravity terms in $a_4$ to form the square of the Weyl tensor

$$C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2 = 2 R_{\mu\nu} R^{\mu\nu} - \frac{2}{3} R^2 + \text{surface term},$$

(45)

because $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2$ is proportional to the Euler characteristic of $M$. Then, up to this surface term, we have

$$-\frac{1}{360} \dim \mathcal{H} \left[ 7 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + 8 R_{\mu\nu} R^{\mu\nu} - 5 R^2 \right] = -\frac{1}{20} \dim \mathcal{H} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}.$$

(46)

Finally we have up to surface terms,

$$a_4 = -\frac{1}{20} \dim \mathcal{H} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{2}{3} \text{tr} \rho(F_\mu)\rho(F^{\mu}) + 4 L_2 |\phi/v|^4 + 4 L(D_\mu \phi/v)(D^\mu \phi/v) + \frac{\nu}{2} L |\phi/v|^2 R.$$

(47)

4 The bare action

The Chamseddine-Connes action $S_\Lambda$ is seen to be the combined Einstein-Hilbert and Yang-Mills-Higgs actions of the standard model plus a higher derivative gravity term plus the conformal scalar-gravity coupling. By normalizing the Higgs field $\phi$ and the Yang-Mills fields $A_j^{(\mu)}$, $j = 3, 2, 1$ for $su(3)$, $su(2)$ and $u(1)$, we rewrite the Lagrangian in its conventional Euclidean form,

$$\mathcal{L}_\Lambda = -m_P'^2/(16\pi) R + \mathcal{L}_C$$

$$+ 1/(2g_3^2) \text{tr} F_\mu^{(3)} F^\mu_{(3)\mu} + 1/(2g_2^2) \text{tr} F_\mu^{(2)} F^\mu_{(2)\mu} + 1/(4g_1^2) F_\mu^{(1)} F^\mu_{(1)\mu}$$

$$+ \frac{1}{2} (D_\mu \phi)^* D^\mu \phi + \lambda |\phi|^4 - \frac{1}{2} \mu^2 |\phi|^2$$

$$- a C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \frac{1}{12} |\phi|^2 R.$$  

(48)

Note the correct sign of the following terms: Newton’s constant $m_P'^2/(16\pi)$ is positive, the three gauge couplings $g_3$, $g_2$, $g_1$ are real (positive), the kinetic term of the Higgs field is positive, the Higgs self coupling $\lambda$ is positive, and the symmetry breaking term $\mu$ is real (positive). For the standard model with $N = 3$ generations and $\dim \mathcal{H} = 45$ we get,

$$g_3^2 = g_2^2 = 3\pi^2/N = \pi^2,$$

(49)

$$g_1^2 = 9\pi^2/(5N) = \frac{3}{5} \pi^2,$$

(50)

$$\lambda = \pi^2 L_2 / L^2 = \pi^2(1 - 2m_b^2/m_t^2) + O(m_c^2/m_t^2),$$

(51)

$$\mu^2 = 2\Lambda^2,$$

(52)

$$a = \dim \mathcal{H}/(320\pi^2) = 9/(64\pi^2).$$

(53)

Before identifying Newton’s constant and the cosmological constant $\Lambda_C$ we have to shift the Higgs field by its vacuum expectation value, $|\phi| = v = \mu/(2\sqrt{\lambda})$. This shift changes $m_P'$ and $\Lambda_C$ into

$$m_P'^2 = \frac{1}{3\pi}(\dim \mathcal{H} - 2 L^2/L_2) \Lambda^2 = \frac{43}{3\pi} \Lambda^2 + O(m_b^2/m_t^2) \Lambda^2,$$

(54)

$$\Lambda_C = \frac{1}{8\pi^2}(\dim \mathcal{H} - 2 L^2/L_2) \Lambda^4 = \frac{43}{8\pi^2} \Lambda^4 + O(m_b^2/m_t^2) \Lambda^4.$$  

(55)
5 The soft action

Equations (49,50) tell us $g_3 = g_2$ and $\sin^2 \theta_w = \frac{3}{8}$, like in $SU(5)$ grand unification. Naturally, we interpret these relations to hold at very high energy $10^{15}$ GeV. But this means that we have to swallow two assumptions. The first is the big desert, from $10^3$ GeV up to $10^{15}$ GeV, the standard model remains valid without modifications, in particular no new particles. The second is that perturbative quantum field theory gets through the big desert without collapsing. Note that these two assumptions imply [15] that the Higgs mass is constrained to an interval, $160 \text{ GeV} < m_H < 200 \text{ GeV}$ for the gauge couplings as measured in 1979. These assumptions also imply [16] that the above relations evolve down to $g_3 = 1.8$ and $\sin^2 \theta_w = 0.21$ at 5 GeV. In the beginning of the eighties, experiments were compatible with these values and grand unification was en vogue. Today with $\sin^2 \theta_w = 0.2315 \pm 0.0005$ experimentally, $\frac{3}{8}$ is less attractive. Moreover $g_3 = \pi$ from equation (49) excludes perturbation theory. Accordingly, we don’t attach much attention to the numerical values of the coefficients in the Lagrangian, but we do acknowledge that we get all necessary terms and with coefficients of the right sign. This attitude is supported by the observation that the Chamseddine-Connes action $S_{\Lambda} = \text{tr} f(D^2/\Lambda^2)$ is universal with respect to the choice of the characteristic function of the unit interval $f$. Indeed for any positive, sufficiently regular function $f$, we get the desired Lagrangian with correct signs and involving — for large $\Lambda$ — three additional, arbitrary positive constants $f_0, f_2, f_4$. Note that $f = \log$ of induced gravity is neither positive nor sufficiently regular. Allowing $f_4 \neq 1$ solves the problem $g_3 = \pi$, keeping the grand unification flavor $g_3 = g_2$ and $\sin^2 \theta_w = \frac{3}{8}$. This is no surprise, the Yang-Mills action with its coupling constants is still obtained from one trace over all fermions. At this point, we emphasize that taking due account of the mass matrix, the fermionic Hilbert space of the standard model has four irreducible pieces, the quarks and the three lepton generations. Accordingly, we will soften $S_{\Lambda}$ further by taking independent traces in each irreducible piece. This amounts to introducing the ‘noncommutative coupling’ $z$, a positive operator on the Hilbert space, that commutes with the representation, with the Dirac operator and with the chirality. For the standard model, $z$ is a constant matrix involving four positive numbers $x, y_1, y_2, y_3$. With respect to the basis (8,9), $z$ takes the form

$$z = 1_4 \otimes \begin{pmatrix} x/3 1_2 \otimes 1_N \otimes 1_3 & 0 & 0 & 0 \\ 0 & 1_2 \otimes y & 0 & 0 \\ 0 & 0 & x/3 1_2 \otimes 1_N \otimes 1_3 & 0 \\ 0 & 0 & 0 & y \end{pmatrix}, \quad y := \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_N \end{pmatrix}, \quad (56)$$

and the softened Chamseddine-Connes action is

$$S_{\Lambda} = \text{tr} f(zD^2/\Lambda^2). \quad (57)$$

We repeat that the form of the Lagrangian (48) remains unchanged and its couplings now read:

$$g_3^{-2} = \frac{1}{9\pi^2 f_4 N x}, \quad (58)$$
\[
\begin{align*}
g_2^{-2} &= \frac{1}{12f_4^2} f_4 (Nx + \text{tr} y), \\
g_1^{-2} &= \frac{1}{12f_4^2} f_4 \left( \frac{11}{9} Nx + 3 \text{tr} y \right), \\
\lambda &= \pi^2 L_2 / f_4 L_2 \\
L &= x \text{tr} M_u^u M_u + x \text{tr} M_d^d M_d + \text{tr} y M_e^e M_e \\
&= x(m_t^2 + m_c^2 + m_u^2 + m_s^2 + m_b^2 + m_d^2) + y_3 m_r^2 + y_2 m_{\mu}^2 + y_1 m_{\nu}^2, \\
L_2 &= x \text{tr} [M_u^u M_u]^2 + x \text{tr} [M_d^d M_d]^2 + \text{tr} y [M_e^e M_e]^2 \\
&= x(m_t^4 + m_c^4 + m_u^4 + m_s^4 + m_b^4 + m_d^4) + y_3 m_r^4 + y_2 m_{\mu}^4 + y_1 m_{\nu}^4, \\
\mu^2 &= 2 f_2 / f_4^2, \\
m_P^2 &= \frac{1}{3\pi} f_2 \left[ (4Nx + 3 \text{tr} y) - \frac{2L_2^2}{L_2} \right] \Lambda^2, \\
\Lambda_C &= \frac{1}{4\pi^2} \left[ f_0 (4Nx + 3 \text{tr} y) - \frac{f_2^2 L_2^2}{f_4 L_2} \right] \Lambda^4, \\
a &= f_4 \frac{4Nx + 3 \text{tr} y}{320\pi^2}. 
\end{align*}
\]

We recover the stiff values, equations (58-55), by taking for \( f \) the characteristic function of the unit interval: \( f_0 = 1/2, \ f_2 = 1, \ f_4 = 1 \), and by taking for \( z \) the identity: \( x = 3, \ y = 1_N \). The soft values avoid the problems of non perturbative gauge couplings and a huge cosmological constant. They can still not be interpreted at low energies because the weak angle,

\[
\sin^2 \theta_w = \frac{g_2^{-2}}{g_1^{-2} + g_2^{-2}} = \frac{Nx + \text{tr} y}{20/9 Nx + 4 \text{tr} y},
\]

is constrained by 0.25 < \( \sin^2 \theta_w < 0.45 \) and even in the soft version, we have to swallow the big desert. This is the subject of the next section.

6 Renormalization flow

In this section, we combine the soft constraints on the couplings (58-55) coming from noncommutative geometry with the renormalization flow coming from the short distance divergence of quantum corrections. Noncommutative geometry does to spacetime what quantum mechanics does to phase space: it introduces an uncertainty principle. Below the scale \( \hbar \), points of phase space are not resolved. Below the scale \( 1/\Lambda \), points of spacetime are not resolved and noncommutative geometry scoffs at short distance divergences. We adopt the philosophy that the constraints on the couplings are valid at the momentum cut off \( \Lambda \). Below this energy scale, we trust in the continuum approximation and the couplings should run according to Wilson.

We consider the energy dependence of the couplings \( g(t) \), \( t := \log E/\Lambda \) perturbatively in the one loop approximation and neglect threshold effects. We also work in flat spacetime only: although the gravitational part of the Chamseddine-Connes action includes a curvature square term it is non-renormalizable \[17\] and it contains negative modes \[18\]. With these
simplifications, the evolution of the gauge couplings decouples and is simply logarithmic:

\[
\frac{dg}{dt} =: \beta_g, \quad \beta_g = \frac{1}{16\pi^2} c_g g^3, \quad 4\pi g^{-2}(t) = 4\pi g^{-2}(0) - \frac{c_g}{2\pi^2} t. \tag{69}
\]

For the standard model with \( N = 3 \) generations, the \( \beta \) functions of the three gauge couplings are in this approximation given by \([20]\)

\[
c_3 = -7, \quad c_2 = -\frac{19}{6}, \quad c_1 = \frac{41}{6}. \tag{70}
\]

The first question we face is: is there a value \( \Lambda \), for which the noncommutative constraints on the three gauge couplings are met at \( E = \Lambda \) with the experimental initial conditions \([19]\), \( g_3 = 1.207, \; g_2 = 0.6507, \; g_1 = 0.3575 \) at \( E = m_Z \). The answer is affirmative, \( \Lambda = 10^{10} \) GeV, \( g_3(0) = 0.65, \; g_2(0) = 0.57 \) with \( f_4 x = 69.5 \) and \( f_4 y = 159.5 \). This value is to be compared to \( \Lambda = 10^{13} - 10^{17} \) GeV, \( g_3(0) = g_2(0) = 0.52 - 0.56 \) in the stiff case where the three gauge coupling constraints cannot be fit by one single \( \Lambda \).

If we want the perturbative calculation to make sense, we must require that — in the huge energy range from 100 MeV, where strong forces supposedly show confinement, all the way up to the noncommutative cut off \( \Lambda \) — all couplings remain positive and all dimensionless couplings must remain smaller than unity. In the following, we neglect all fermion masses with respect to the noncommutative cut off \( \Lambda \) — all couplings remain positive and all dimensionless couplings cannot be fit by one single \( \Lambda \).

We integrate the differential equations for \( \lambda(t) \) and \( g_i(t) \) numerically with ‘initial values’

\[
g_i(\log m_Z/\Lambda) = \frac{m_t}{v} = \frac{1}{2} g_2(\log m_Z/\Lambda) \frac{m_t}{m_W}, \tag{74}
\]

\[
\lambda(0) = \frac{1}{3} g_3^2(0). \tag{75}
\]

from the noncommutative constraint \([61]\). With \( m_W = 80 \) GeV and \( m_t = 180 \pm 12 \) GeV we get the Higgs mass (at the pole)

\[
m_H = 4\sqrt{2} \frac{\sqrt{\lambda(\log m_Z/\Lambda)}}{g_2(\log m_Z/\Lambda)} m_W = 235 \pm 3 \) GeV. \tag{76}
\]
Note that in presence of the noncommutative constraints (58) and (61), $\lambda = \frac{1}{3}g_3^2$, the hypothesis of the big desert implies the validity of perturbation theory throughout this desert.

7 Connes-Lott versus Chamseddine-Connes

In flat spacetime, the Connes-Lott (CL) and the Chamseddine-Connes (CC) actions coincide up to a possible cosmological constant. While the constraints on the gauge group and on the fermion and Higgs representations are identical in both approaches, this is not so for the constraints on the coupling constants. Let us recall the constraints on the couplings from Connes-Lott [21],

$$g_3^{-2} = \frac{4}{3}N\tilde{x}, \quad (77)$$

$$g_2^{-2} = Nx + y_1 + y_2 + y_3, \quad (78)$$

$$g_1^{-2} = Nx + \frac{2}{3}N\tilde{x} + \frac{1}{7}(y_1 + y_2 + y_3) + \frac{1}{2}N\tilde{y}, \quad (79)$$

$$\lambda = \frac{1}{16}K/L^2, \quad (80)$$

$$L = x\text{tr} M_u^*M_u + x\text{tr} M_d^*M_d + yM_e^*M_e, \quad (81)$$

$$K = \frac{2}{3}x\text{tr} [M_u^*M_u]^2 + \frac{2}{3}x\text{tr} [M_d^*M_d]^2 + \frac{2}{3}yM_e^*[M_e^*M_e]^2$$

$$\quad + x\text{tr} [M_uM_u^*M_dM_d^*] + \frac{1}{7}\left[\frac{1}{Nx + \text{tr}y} + \frac{1}{Nx + \text{tr}y/2 + N\tilde{y}/2}\right], \quad (82)$$

$$\mu^2 = K/L, \quad (83)$$

$$\Lambda_C = 0. \quad (84)$$

The differences have two origins. The first is: while the CL action is the square of a 2-form that brings in its junk, the CC action is computed from 1-forms only, the connections in the covariant Dirac operator. 1-forms do not carry junk. This accounts for the difference between $L_2$ and $K$ and for the different Higgs masses $\sqrt{2}\mu$. The second origin is more conceptual. The CL scheme affords the possibility to use two Dirac operators, the usual covariant one in the Dirac action for the fermions, and another one to build the bosonic action. This other one is covariant as well, but its couplings between fermions and gauge bosons break charge conjugation. The bosonic part of the CL action is a Dixmier trace over all fermions and anti-fermions of the square of the curvature. This trace of course leaves no trace of charge conjugation violation but allows for more flexible gauge couplings, i.e. two additional positive parameters, $\tilde{x}$ and $\tilde{y}$. As explained in the introduction, Connes’ extended principle of relativity forbids the use of the charge conjugation violating Dirac operator. One may of course choose to use only the symmetric Dirac operator also in the CL approach. Then, the constraints on the gauge couplings become as in CC [22].

What are the physical consequences of the constraints? From the CC action we get $\frac{1}{4} < \sin^2 \theta_w < \frac{9}{20}$ whereas today experimentally $\sin^2 \theta_w = 0.2315 \pm 0.0005$ at the $Z$ mass forcing upon us a cut off $\Lambda$ of at least $10^{10}$ GeV and the big desert. Trusting perturbative quantum
field theory extrapolated all the way up to science fiction energies we get a Higgs mass (at the pole) \( m_H = 235 \pm 3 \text{ GeV} \) for \( \Lambda = 10^{10} \text{ GeV} \), \( m_H = 221 \pm 5 \text{ GeV} \) for \( \Lambda = 10^{17} \text{ GeV} \).

The CL action on the other hand implies \( 0 < \sin^2 \theta_w < \frac{8}{15} \) and can well live without big quantum corrections, the Higgs mass is determined essentially by the top mass, \( m_H = 298 \pm 21 \text{ GeV} \) if \( m_t = 180 \pm 12 \text{ GeV} \). The cut off \( \Lambda \) can be taken of the order of the Higgs or top mass where it should be visible experimentally soon.

8 Outlook

We conclude with a list of open questions and wishful thinking.

- Let \( \mathcal{A} \) be the associative algebra \( \mathcal{C}^\infty(M) \otimes (\mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C})) \). This is the algebra of the internal space in the standard model. \( \mathcal{A} \) fits well with experiment, but \( \mathcal{A} \) is ugly.

- The basic variable is the Dirac operator acting on fermions. The fermions must define a representation of an associative algebra and are constrained by the axioms of noncommutative geometry, i.e. of spectral triples [1]. These axioms still leave many choices [24], one of which the quarks and leptons of the standard model with their mass matrix taken from experiment. Of course, we want an explanation for this choice. We have softened the action by allowing arbitrary positive parameters, \( f_0, f_2, f_4 \) from the function \( f \) and \( x, y_1, y_2, y_3 \) from the element \( z \) of the commutant. These parameters poised the trace just as gauge couplings in a Yang-Mills theory. We also want an explanation for these numbers.

- To define the Dirac operator in Riemannian geometry, the spin group is essential. There is no generalization of the spin group to noncommutative geometry yet. According to Connes [4], this should be a quantum group and it should help us to get a handle on the arbitrary choices above [23].

- Noncommutative geometry grew out of quantum mechanics. Noncommutative geometry unifies gravity with the subnuclear forces. We expect noncommutative geometry to reconcile gravity with quantum field theory and perhaps at the same time Connes-Lott and Chamseddine-Connes.

- Minkowskian geometry explains the magnetic field, Riemannian geometry explains gravity. Both geometries have operated revolutions on spacetime that today are well established experimentally: the loss of absolute time and the loss of universal time. How can we observe the noncommutative nature of time, its uncertainty or ‘fuzziness’, and what is its characteristic scale \( 1/\Lambda \)? A hand waving argument combining the Schwarzschild radius with Heisenberg’s uncertainty relation indicates \( \Lambda < m_P \).
• So far noncommutative geometry is developed in Euclidean, compact spacetimes, so ‘Wick rotation’ and 3+1 split remain to be understood [25]. After this, we expect noncommutative geometry to change our picture of black holes in a similar fashion that Heisenberg’s uncertainty relation has cured the Coulomb singularity of the hydrogen atom. Also our picture of the big bang, cosmology and the origin of time is expected to be revised [26].

Planetary motion has degraded circles to epicycles and dismissed them all together in favour of ellipses. Particle physics is about to dismiss Riemannian geometry in favour of noncommutative geometry and the question is, what dynamics is behind these new ellipses?

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