A Quantum Bound State Description of Black Holes

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Black holes are embedded as bound states in a relativistic quantum theory of weakly coupled constituent fields. The constituents are immersed in a non-perturbative vacuum structure representing strong collective effects via in-medium modifications. The latter are parametrized in terms of condensates. Observables such as the constituent distribution, the constituent number and energy density are constructed, and calculated at the parton-level. Remarkably, even at this level, these observables receive only non-perturbative contributions. Furthermore, observables associated with the interior of black holes show simple scalings with the total constituent number. Finally, an explicit framework for calculating gauge corrections is presented.

I. INTRODUCTION

Qualifying the importance of space-time geometry relative to the fundamental status of Hilbert–space geometry gives rise to a prominent cultural clash across physics communities. The favoured perspective seems to prefer quantum mechanics as the sole fundamental framework, while space–time geometry appears to be a derived rather than a basic concept. The combination of quantum mechanics with causal structures, however, requires light–cones at its foundation. This potential tension is resolved by distinguishing the Minkowski light–cone in accordance with locality principles.

Within this framework, general relativity is an effective description derived from an underlying quantum theory of the gravitational field. As such its domain of validity is, a priori, restricted to deformations of the fundamental Minkowski light–cone. Beyond this domain, solutions of general relativity are either describing new ground states, thus qualifying the existence of a fundamental light–cone, or, in accordance with a distinguished causal structure, are mere geometrical artefacts void of a Hilbert–space connection.

In this article, we follow a more pragmatic approach and show how certain space–times can be, at least complementary, described as quantum bound states. The bound state description is explicitly developed in the case of Schwarzschild black holes. This example is particularly plausible since it admits a distinct spherical surface that is of fundamental significance. On the near side of the Schwarzschild surface, the causal structure is granted by light–cones that can be described as weakly deformed cousins of Minkowski light–cones. At the Schwarzschild surface, however, these deformations become strong. Accordingly, the interior of the Schwarzschild sphere might not be represented by a physical geometry, albeit a geometrical solution exists. The relation of this geometrical solution to quantum mechanics is unclear and it is doubtful that one can be established. Therefore, in the present work, it will be qualified as unphysical. As a consequence, the geometrical definition of a black hole is valid only on the near side of the Schwarzschild surface. Clearly, this definition cannot be fundamental.

In summary, the Schwarzschild surface can be considered as a boundary separating the interior of a quantum bound state from an exterior admitting an approximate geometrical description.

Following this logic, Dvali & Gomez pictured black holes based on a constituent portrait [1]. In this description, black holes are self–sustained bound states of $N_c$ gravitons having a characteristic wavelength $r_g$. At this scale, gravitons are weakly interacting individually with a dimensionless coupling $\alpha_g(r_g) \propto 1/N_c$, while experiencing a strong collective binding potential suggesting a mean field approach. The portrait pictured by Dvali & Gomez is remarkable, because all black hole quantities are given in terms of a single quantity, $N_c$, via simple scaling relations. Most importantly, quantum corrections to the portrait are suppressed by $1/N_c$, qualifying black holes as true quantum bound states inaccessible to semi–classical scrutiny. This quantum character of black holes is free of any paradoxes or seemingly mysterious properties arising in the invalidated semiclassical approach to black hole physics.

So far, the considerations leading to the appealing black hole portrait have been qualitative rather than quantitative. Compared to Witten’s description of...
baryons built from heavy quarks in the $1/N_c$-expansion \[\text{[1]},\] the problem at hand requires some sort of relativistic mean field approach in four dimensions. The purpose of this article and its successors is to provide an analytical framework for realising a relativistic mean field description of black holes (and other gravitational bound states) without relying on the Hartree ansatz. The mean field in this approach is represented by a non-trivial vacuum structure communicating the in-medium modifications related to the collective binding effects. This mimics the bound state description developed for quantum chromodynamics by Shifman, Vainshtein & Zakharov, where hadron properties at low energies are parametrized by certain vacuum condensates of quarks and gluons \[\text{[5]}\].

In this article, we present an embedding of the constituent portrait by Dvali & Gomez in relativistic quantum field theory. This includes the construction of observables such as the light–cone distribution of black hole constituents, their number density and energy density. The formalism is sufficiently potent for relating the number of microscopic black hole constituents to the mass measured by a classical observer at null infinity, thereby establishing the instrumental scaling law of previous phenomenological approaches. These results are based on a partonic description, albeit already non–perturbative due to the large–$N_c$ nature of black holes and the diagrammatic configurations associated with the observables. We outline how gauge corrections can be implemented in a practical way.

The study of graviton distributions is left for future work, since it is more demanding at the technical level, involving a bi–local composition of Ricci operators connected by a Wilson line. Here, for simplicity and conceptual clarity, observables are constructed from scalar fields solely.

II. PRELIMINARIES: CONSTITUENT NUMBER DENSITY

The occupation number density $n$ of a free scalar field $\Phi$ is proportional to the field intensity, $n \propto |\Phi(k)|^2$, corresponding to a Fourier–transform of the bi–local operator $O^{(0)}(x,y) := \Phi(x)\Phi(y)$.

The constituent number density in a state $|B\rangle$ is the momentum occupation $n(k)$ of that state, in the absence of any interactions. Expanding $|B\rangle$ in momentum eigenstates $|PQ_B\rangle$ and denoting the associated momentum wave function by $\mathcal{B}(P)$,

$$
\langle B | (\zeta^3 n)(k) | B \rangle = \int d^3P \ |\mathcal{B}(P)|^2 \times
$$

$$
\times \int \frac{d^3r}{(2\pi)^3} e^{ik \cdot r} \langle PQ_B | O^{(0)}(r,0) | PQ_B \rangle. \quad (1)
$$

Here, $Q_B$ denotes a quantum compatible list of state identifiers, or quantum numbers, and $\zeta(k) := 1/\sqrt{(2\pi)^3 2k^0(k)}$ is the characteristic size of a momentum eigenstate in phase space. Provided the bound state is in a regularised momentum eigenstate, $|B\rangle = |P_B Q_B; \vartheta\rangle$, the $P$–integration becomes trivial. The regularisation parameter $\vartheta$ normalises the variance around $P_B$.

Associating the number density of constituents with a space–time configuration requires an auxiliary current description for $|B\rangle$. This allows to interpret $O^{(0)}$ operationally as a measurement device. The corresponding diagnostic process is incorporated by the connected component of the time–ordered product $T \mathcal{J}(x) \mathcal{O}^{(0)}(y;r/2) \mathcal{J}(0)$, where $\mathcal{J}$ denotes the auxiliary current representing the quantum bound state, and $O^{(0)}(y;r/2) := \Phi(y + r/2)\Phi(y - r/2)$ in the absence of interactions. It can be shown that the fixed-order operator product can be replaced by a time–ordered product, $O^{(0)}(y;r/2) := T \Phi(y + r/2)\Phi(y - r/2)$, without changing the value of the constituent number density. Note that the origin of the coordinate system is chosen such that is makes the external scale $r$ of the diagnostic device manifest.

Diagrammatically, the diagnostic process corresponds to a triangular–like graph, with one vertex point–split to accommodate the diagnostic scale $r$. Connectedness requires either at least one correlation bridging the currents space–time locations, or a condensation process merging the two events. In principle, however, multiple connections are possible, depending on the auxiliary current and the ground state.

Since the constituent number is, in the absence of interactions, tied to the occupation number, a Fock–space description in agreement with the interpretation for $O^{(0)}(y;r/2)$ can be expected. The mode expansion of $\Phi$ can simply be inverted to give the annihilation and creation operators, using the hyper–surface independence of the symplectic product:

$$
a(k) = 2k_0(k)\zeta(k) \int d^3x \ e^{-ik \cdot x} \Phi(x). \quad (2)
$$

The distinction between annihilation and creation operators is made by the sign in the argument of the exponential. Following the logic of equal–time quantisation, $k_0$ is on–shell. In principle, this is a subtle point for massive quanta once interactions are included, because which mass to choose is neither obvious nor unique. For instance, the following choices are plausible: the bare mass, the physical mass, the renormalised mass in some prescription, none of which are equal. Moreover, the relation between these choices is only known after the theory has been solved. We are, however, ultimately interested in graviton correlations inside a black hole represented by $|B\rangle$, and, therefore, need not to worry about said subtleties.
The occupation number density $n$ with respect to the covariant momentum measure is given by $n(k) = d\mathcal{N}_c/(d^3k \, (\zeta k^2))$, where $\mathcal{N}_c$ is the total constituent number. With respect to this measure, $n(k) = (\zeta^{-1} a^1 a)(k)$.

In the context of gauge theories, the constituent description requires a suitable generalisation to accommodate the related gauge symmetry. A gauge–invariant definition, in accordance with \cite{1}, is given by inserting a suitable path–ordered exponential of the gauge field, or a Wilson line.

For gravitational interactions, the gauge field $\mathcal{G}$ is given by the connection $\mathcal{G}$, in components $\mathcal{G}_\mu = \Gamma_{\lambda\mu}$. Treating $\mathcal{G}$ as an external gauge field,

$$(-\Box + \mathcal{G} \cdot \partial) \mathcal{O}(y; r/2) = \delta^{(4)}(r).$$  

This equation can be solved by iteration, for a detailed derivation see Appendix A. The result is,

$$\mathcal{O}(y;r/2) = \mathcal{P} \exp \left(-\int dz^2 \mathcal{G}_\lambda(z)\right) \mathcal{O}^{(0)}(y;r/2),$$  

where $C$ denotes the contour given by the path $z : [0,1] \rightarrow \mathbb{R}^4$, $u \rightarrow z(u) := y - (1 - 2u)r/2$, and $\mathcal{P}$ refers to path ordering along this contour.

Equation (1) shows a simple relation between the constituent number density in the presence and the absence of external gauge fields, respectively, holding to all orders in $\mathcal{G}$, provided the characteristic diagnostic scale is close to the light–cone.

It should be noted that a diagrammatic expansion of \cite{1} can result in conflicts between the usual time ordering used for correlation functions and the path ordering defining Wilson lines. This can be avoided by the following prescription: the light–like curve is always constructed as a limiting process approaching the light–cone from its exterior.

The scientific objective can now be stated precisely: In this article, we calculate the constituent distribution

$$\mathcal{D}(r) := \int d^3k \, e^{-ik \cdot r} \langle B|(\zeta^3 n)(k)|B\rangle,$$  

where $|B\rangle$ could be any bound state, and, in particular, the black hole quantum bound state, to be constructed below\cite{3}. In explicit calculations presented below, we calculate the parton distribution of massless quanta constituting the black hole, the total constituent number and energy density inside the black hole. In section IV we show how gauge corrections can be taken into account based on existing external field methods. Gauge corrections to our results will be presented somewhere else.

### III. Auxiliary Current Description

We postulate the existence of a ground state $|\Omega\rangle$ that is distinguished from the perturbative vacuum in that it supports the creation of bound states $|B\rangle$ when appropriate auxiliary currents $J$ are operative. The operation consists of storing certain information that identifies the quantum bound state in $|\Omega\rangle$. Accordingly, the bound state is represented by a list $L_B$ comprising of quantum compatible identifiers. The composition of the currents in terms of elementary degrees of freedom is not relevant, as long as it allows to store $L_B$. In particular, if certain symmetries give rise to quantum numbers, the composition of the currents will be subject to these symmetries. As a consequence, any given list $L_B$ of quantum compatible identifiers allows for a huge degeneracy of auxiliary currents as read in devices. However, for any observable $O$, the ground state expectation value $\langle \Omega| T(J\Omega J) |\Omega\rangle$ should be independent of the concrete choice, as long as it represents $L_B$.

In general, $|B\rangle$ needs not to be in a momentum eigenstate. Suppose $|L\rangle$ is a momentum eigenstate corresponding to momentum $K$. The list $L = \{K, Q\}$ is the union of identifiers related to space–time isometries and internal symmetries. Expanding $|B\rangle$,

$$\langle B|J(x)|\Omega\rangle = \sum B^*(L)\langle L|J(x)|\Omega\rangle,$$  

where the integral is over energy–momentum and the sum over the quantum numbers collected in $L$. The overlap $\langle L|J(0)|\Omega\rangle = \Gamma_B \delta(Q, Q_B)$ is a priori unknown and parametrized by the constant $\Gamma_B$. Here, $\delta(Q, Q_B)$ denotes the appropriate collection of delta functions, restricting the quantum numbers in $Q \subset L$ to those characterising the bound state. Note that $\Gamma_B$ encodes structural information vital for gravity, as its definition employs a quantum bound state in the spectrum of gravity. Therefore, even at the parton–level, we expect to shed light on essential features of the full interaction theory. In terms of the non–perturbative parameter $\Gamma_B$,

$$|L\rangle = \Gamma_B^{-1} \int d^4x \, e^{iK \cdot x} \langle J(x)|\Omega\rangle.$$  

Hence, the auxiliary current description of the constituent distribution $\mathcal{D}(r)$, eq. (5), is given by

$$\mathcal{D}(r) = \Gamma_B^{-2} \int d^3P \, |B(P, Q_B)|^2 \int d^4x dy \, e^{iP \cdot (x+y)} \langle \Omega| T(J(x))O(y;r/2)J(0)|\Omega\rangle.$$  

Here, $O(y;r/2)$ denotes the gauge–invariant bilocal operator \cite{1}.
A. Analytical continuation of truncated observables

In principle, albeit not in the case under consideration, an important subtlety arises when an S-matrix description is sought after to complete the construction of observables coupled to auxiliary currents. Consider the annihilation of massless Φ–quanta into something represented by |Q⟩ in the one–graviton approximation, i.e. up to second order in the coupling GN. The corresponding Feynman amplitude is given by

$$\langle Q|\Phi(k)\Phi(p)\rangle^{(2)} = G_N^2 \left[ k^\mu p^\nu - (k \cdot p)\eta^\mu\nu \right] \Delta^{(0)}(k + p) \langle Q|T_{\mu\nu}(0)|\Omega\rangle.$$  \hspace{1cm} (9)

Here, Δ^{(0)} denotes the scalar part of the perturbative graviton propagator, and T is the energy–momentum tensor associated with the final state. The non-perturbative information encoded in the amplitude factorizes as usual, and the total cross sections does so accordingly, σ ∝ P × N, where P and N are rank–four Lorentz–tensors encoding the perturbative and non–perturbative contributions, respectively. In particular,

$$N_{\alpha\beta\mu\nu} = i \int d^4x \, e^{iq \cdot x} \langle \Omega|T(\alpha\beta(x)T_{\mu\nu}(0))|\Omega\rangle$$  \hspace{1cm} (10)

with q = k + p. Due to the conservation of energy and momentum, N is transversal in the following sense:

$$N_{\alpha\beta\mu\nu} = (q_\alpha q_\beta - q^2 \eta_{\alpha\beta})(q_\mu q_\nu - q^2 \eta_{\mu\nu})N(-q^2).$$  \hspace{1cm} (11)

The, at least in principle, experimentally accessible quantity is the imaginary part of N(−q^2) at positive values of −q^2 ≥ (q_0)^2 − |q|^2, the spectral density,

$$\rho(-q^2) \propto \text{Im } N(-q^2) \propto \sigma.$$  \hspace{1cm} (12)

Theoretically, N(−q^2) can be calculated at negative −q^2. Provided such a calculation can be done exactly, an analytical continuation to positive −q^2 could be performed, and the imaginary part be taken. This way, the spectral density could be predicted exactly and directly be compared to the measurable gravitational cross section. In practice, however, this is not possible. Neither the perturbative nor the non–perturbative contributions to N(−q^2) can be summed up, instead both have to be truncated with respect to some ordering scheme. In this situation, only the truncated result can be analytically continued from negative to positive −q^2. For each term in such an expansion the imaginary part at positive −q^2 is well–defined. Then we (have to) postulate that the approximated spectral density agrees with the true spectral density. This is by no means justified, as small corrections truncated in the approximation might become important after the analytical continuation [8].

This subtlety, however, bears no impact on the calculations presented below as we are interested in matrix elements that are connected to static properties of the bound state.

IV. PARTON–LEVEL RESULT

In this section the calculation of D(r) is presented, using J = Φ^N (N ≫ 1) as an auxiliary current representation of a black hole quantum bound state |B⟩, and O^{(0)}(y; r/2) as the diagnostic device probing the free constituent distribution inside the bound state at diagnostic scale r. Furthermore, the total number of black hole constituents Nc is related to its mass measured by a classical observer at null infinity. This establishes the central scaling relation at the core of the quantum black hole conjecture.

Unless otherwise mentioned, all calculations are performed in the limit MB → ∞, N → ∞, N/MB = const., where MB denotes the black hole mass. We shall refer to this prescription as double scaling limit. For finite MB and N, corrections in 1/N arise. This follows from simple combinatorics associated with the diagrams involved. Notice that these are exactly the corrections required to resolve the so–called black hole mysteries related to a semi–classical description [9] that is invalidated by their presence. From this point of view, the double scaling limit can be considered as a semi–classical approximation derived from a quantum bound state description of black holes.

A connected component in the time–ordered product TJ(x)O^{(0)}(y; r/2)J(0) requires N ≥ 2. Before appreciating the limit N ≫ 1, it is instructive to calculate D(r) for the minimal connected component, given by the diagrams shown in Figure 1.

[Diagram: Two connected components, one with |x⟩ and |y⟩ at r/2, and the other with |x⟩ and |y⟩ at r/2.]

FIG. 1. Diagrams contributing to the calculation of D(r) for N = 2. The first diagram represents a purely perturbative configuration, while the second represents an in–medium modification due to the nontrivial vacuum structure and is related to the condensation of space–time events originally located at x and 0.

In this case, the double scaling limit cannot be employed for obvious reasons. Instead, we assume that the mass of the bound state is large compared to the typical constituent energies. The purpose of this calculation is,
however, to stress the importance of perturbative contributions, which are non–vanishing only in this case. In the absence of gauge interactions involving $G$, the perturbative contribution is readily calculated:

$$D^{(0)}(r)|_{N=2} = 8 M_B^{-4} \Gamma_B^{-2} \Delta^{(0)}(r)$$

$$\int d^3P \cos (P \cdot r/2) |\mathcal{B}(P, Q_B)|^2,$$ (13)

where $\Delta^{(0)}(r)$ is the free $\Phi$–propagator carrying the light–cone correlation, and $|\mathcal{B}(P, Q_B)|$ denotes the on-shell momentum wave function associated with the state $|\mathcal{B}\rangle$.

There is a non–perturbative correction proportional to the dimension–2 condensate $(\Phi^2) \equiv \langle \Omega | \Phi^2(0) | \Omega \rangle$, corresponding to a condensation of the space–time events located at $x$ and $0$, given by

$$D^{(2)}(r)|_{N=2} = 8 M_B^{-4} \Gamma_B^{-2} (\Phi^2)$$

$$\int d^3P \cos (3P \cdot r/2) |\mathcal{B}(P, Q_B)|^2.$$ (14)

In this case, no light–cone singularity developed, since the light–cone correlation is effectively disconnected, as shown in Figure 1.

An important question arising from the $N = 2$ case is whether a purely perturbative contribution is generic and persistent for $N \gg 1$? The answer is no. If the connectivity between the space–time events $x$ and $0$ is increased by means of perturbative correlations (as opposed to condensation), then a contribution proportional to $\Delta^{(0)}(0)$ is inevitable, corresponding to a loop anchored at one of the auxiliary currents space–time location, as shown in Figures 2 and 3.

![FIG. 2. Purely perturbative contribution for $N = 4$. Both diagrams reduce to the same divergency class in the limit of large black hole mass. The divergence qualifies, however, as unphysical, since it contains no logarithmic admixture.](image)

These contributions contain quadratically divergent integrals (with no logarithmic admixture) and can, therefore, safely be ignored. Extracting the leading light–cone contribution,

$$D^{[2(N-2)]}(r) = 2(N/M_B)^4 \Gamma_B^{-2} (\Phi^{2(N-2)}) \Delta^{(0)}(r)$$

$$\int d^3P \cos (P \cdot r/2) |\mathcal{B}(P, Q_B)|^2,$$ (15)

Note that (15) is proportional to $(N/M_B)^4$ and, hence, is finite in the double scaling limit. Furthermore, $D^{[2(N-2)]}(r)/D^{(0)}(r)|_{N=2} \propto (\Phi^{2(N-2)})$, for $N \gg 1$. In fact, only $N > 2$ is required to establish this scaling, see Figure 4.

![FIG. 3. The generic situation corresponding to Figure 2 for $N$ scalar fields constituting the auxiliary current. Shown are $k$ condensate insertions and $l = N - k - 2 \neq 0$ loops connecting the space–time points $x$ and $0$. All diagrams with $l > 0$ vanish in the double scaling limit.](image)

As a result, the constituent distribution within a black hole quantum bound state, diagnosed on the light–cone, has no purely perturbative contribution and receives, even at the parton–level, contributions from constituent condensation supported by a non–perturbative ground state, provided $N > 2$.

While $N$ counts the field content of the auxiliary current $\mathcal{J}$, or its mass dimension, the constituent number, $\mathcal{N}_c$, is given by

$$\mathcal{N}_c := \int d^3q \zeta(q) \langle \mathcal{B}|n(q)|\mathcal{B}\rangle$$

$$= \int d^4q \zeta^{-2}(q) \int \frac{d^3r}{(2\pi)^3} e^{iq \cdot r} D^{[2(N-2)]}(r).$$ (16)

We find $\mathcal{N}_c = 8C N^4 \Gamma_B^{-2} (\Phi^{2(N-2)})$, where $C$ denotes a dimensionless constant. As to be expected, the total number of constituents diverges in the semi–classical limit. Note that while the same auxiliary description
could be used in the presence of gauge interactions, \( \mathcal{N}_c \) probes the physical constitution of the quantum bound state at a given resolution.

At the parton–level, we expect for any additive Observable \( \mathcal{M} \) the scaling \( \langle \mathcal{B} | \mathcal{M} | \mathcal{B} \rangle \propto \mathcal{N}_c \). For instance, consider the partons energy–momentum tensor, \( T_{\alpha\beta} = G_{\alpha\beta} \delta_{\mu} \phi \delta_{\nu} \Phi / 2 \), where \( G \) denotes the Wheeler–DeWitt metric. The expectation for the energy density inside the black hole is related to the diagrams depicted in Figure 5 and readily calculated to be

\[
E(x) = \frac{c}{2} (N/M_B)^2 |x(y)|^2 \Gamma_{B}^{-2}(\Phi^{2(N-1)}) .
\] (17)

Here, \( c \) is a dimensionless constant.

![Diagram](image)

**FIG. 5.** Diagrams contributing to the constituents energy density inside a black hole for a generic interpolating current. Only the first diagram is nontrivial. It corresponds to a constituent contribution in the double scaling limit.

Integrating the energy density over a spatial slice, we can establish a link between the microscopic and macroscopic description of black holes. Indeed,

\[
M_B^2 = \frac{1}{16} \frac{\Phi^{2(N-1)}}{\Phi^{2(N-2)}} \mathcal{N}_c^2 ,
\] (18)

As a reminder, \( \mathcal{N}_c \propto N^4 \), so \( M_B \propto N \), which is precisely the scaling found by Witten in the 1/N–expansion of baryons built from heavy quark fields.

Furthermore notice that the ratio of condensates becomes \( N \)-independent in the double scaling limit. Since we are describing black holes, it is natural to identify this ratio with the only fundamental scale in gravity, the Planck mass. This is also consistent with the findings of [1] and suggests that black hole masses should be quantised in units of the Planck mass.

**V. BEYOND PARTON–LEVEL**

There are various corrections to the free constituent result [15]. Perturbative graviton exchanges give rise to a series in the gravitational coupling strength. Non–perturbative contributions arise from a richer condensate structure due to graviton condensation [4].

This can easily be appreciated by imposing the following gauge:

\[
x^\lambda x^\sigma \Gamma_{\lambda\sigma}^\mu (x) = 0 ,
\] (19)

which is the exact analogue of the celebrated Fock–Schwinger gauge, originally proposed in electrodynamics. In gravitation it corresponds to the choice of a well–known coordinate neighbourhood called a (pseudo) Riemann normal coordinate system. Indeed, the Fock–Schwinger gauge is equivalent to \( x^\mu g_{\mu\nu}(x) = x^\mu g_{\mu\nu}(0) \), which in combination with \( g_{\mu\nu}(0) = \eta_{\mu\nu} \) defines a normal coordinate system anchored at 0. The geodesic interpretation is that straight lines through the origin parameterize geodesics in these coordinates.

The Fock–Schwinger gauge has the important consequence that the potential \( \mathcal{G} \) can directly be expressed in terms of the Ricci tensor. Since we are interested in an expansion in local operators, it is convenient to work with the following representation:

\[
\mathcal{G}_\mu(x) = -\frac{1}{3} x^\lambda R_{\lambda\mu}(0) + \cdots .
\] (20)

Terms suppressed in this expansion involve covariant derivatives and products of Riemann tensors. Although a closed formula for the Riemann normal coordinate expansion of \( \mathcal{G}(x) \) in local operators evaluated at the origin can be given, it suffices to work with [20] to illustrate the main idea.

Correlation functions involving auxiliary currents in a non–trivial ground state, like the diagnostic process \( \langle \Omega | T(\mathcal{O}_1 \mathcal{O}_2) | \Omega \rangle \) determining the constituent distribution, are determined by propagators involving the black hole’s constituents. However, unlike for the ordinary Feynman diagrams, these describe the propagation through a medium consisting of external graviton (and other constituent) fields.

Consider a \( \Phi \)–quantum emitted at the space–time point \( y \) and absorbed at \( x \). The propagator \( \Delta(x, y) \equiv i \langle \Omega | T(\Phi(x)\Phi(y)) | \Omega \rangle \) satisfies the standard equation involving a derivative coupling between \( \mathcal{G} \) and \( \Phi \). Assuming \( \mathcal{G} \) to be small as compared to the free propagation scale \( x - y \), and using equation of motion [3], \( \Delta(x, y) \) can be

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4 In this section, we sketch the general strategy for incorporating these corrections. Detailed calculations are presented elsewhere.
expanded as
\[
\Delta(x, y) = \sum_{n=0}^{\infty} \Delta^{(n)}(x, y),
\]
\[
\Delta^{(n)}(x, y) = \int d^4z_1 \cdots d^4z_n \, (-1)^n \Delta^{(0)}(x - z_1)
\times G \cdot \partial \Delta^{(0)}(z_n - y) \prod_{a=1}^{n-1} G \cdot \partial \Delta^{(0)}(z_a - z_{a+1}), \quad (21)
\]
where \(\Delta^{(0)}\) denotes the free propagator. This formula has a simple diagrammatic interpretation, shown in Figure 6.

FIG. 6. Diagrammatic representation of the scalar propagator in the external \(G\)-field. The constituent scatters zero, one, two, . . . times off the external potential \(G\), represented by the wavy lines. On the light–cone, the series of interactions can be summed up, resulting in a path–ordered exponential of the connection \(G\), in accordance with gauge invariance.

The qualification free here includes not only the absence of quantum interactions, but the absence of the medium representing the non–perturbative ground state, as well. Therefore, \(\Delta^{(0)}\) transforms invariant under space–time translations, while \(\Delta\) is non–invariant, since \(G\) depends on the space–time location. Given that \(G\) is external and tied to the ground state properties, this space–time dependence is fictitious when the averaged ground state structure is considered. There is, however, a second reason for breaking translation invariance. Namely, once we choose Fock–Schwinger gauge for evaluating \(21\), the origin of the Riemann normal coordinate neighbourhood is distinguished. But this is simply due to choosing a coordinate system and bears no physical significance, provided all calculations are performed in these coordinates.

Having a bookkeeping procedure in mind such as the operator product expansion, there might be situations where we are only interested in the operator \(R_{\mu \nu}\). Then, effectively, \(G_{\mu}(x) = -x^3 R_{\mu}(0)/3\). Other operators in the Riemann normal coordinate expansion of \(G\) cannot result in \(R\)-contributions to \(\Delta\). An elementary calculation gives
\[
\Delta^{(1)}(x, y) = \frac{-i}{96\pi^2} \langle R(0) \rangle \times \ln \left( \frac{y^2}{\delta^2} \right) - 1 - \frac{y^2 - (x - y)^2}{(x - y)^2} \left[ \ln \left( \frac{y^2 - (x - y)^2}{y^2} \right) - 1 \right]. \quad (22)
\]
Note that the result is exact up to operators of higher dimension which are not shown here. The Ricci condensate \(\langle R(0) \rangle\) originates from the condensation of \(G\). It is a rather complex ground state structure corresponding to a high graviton flux density, in fact to a certain resummation of infinitely many graviton contributions. This is how the external field formalism in Fock–Schwinger gauge makes the non–perturbative bound state description of black holes feasible.

As an example for a gauge correction to the constituent distribution, consider the diagram shown in Figure 7, which gives rise to a contribution proportional to the dimension–2\((N - 1)\) condensate \(\langle \Omega | \Phi^{(N-2)} R | \Omega \rangle\). The amplitude is given by
\[
A^{[2(N-2),2]}(P, r) = \frac{1}{96\pi^2} \int \frac{d^4x}{(2\pi)^4} \frac{d^4y}{(2\pi)^4} \, e^{-iP \cdot (x + y)} \Delta^{(0)}(x) \Delta^{(0)}(x - y) \left[ \ln \left( \frac{-(y_+)^2}{\delta^2} \right) - 2 \right] \langle \Phi^{N-2}(x) | \Phi^{N-2} R | 0 \rangle, \quad (23)
\]
where \(y_- := y - r/2\) and \(d\) denotes a remormalisation scale. Expanding \(\Phi^{N-2}(x)(\Phi^{N-2} R)(0)\) into local operators yields coefficients suppressed by powers of \(P^2\). Since the contribution to the constituent distribution is
\[
\delta C^{[2(N-2),2]}(r) = \Gamma_{\Phi}^{-2} \int d^3P \, |\mathcal{B}(P, Q_\Phi)|^2 \, A^{[2(N-2),2]}(P, r), \quad (24)
\]
the value of \(-P^2\) is approximately given by the black hole’s mass squared. Hence, the operator product expansion generates coefficients that are increasingly suppressed by powers of \(M_\Phi\) as higher derivatives of \(\Phi\) are considered in condensates. Therefore, the leading contribution is given by
\[
A^{[2(N-2),2]}(P, r) = \frac{1}{96\pi^2} \frac{1}{M_\Phi^{3/2}} \, |\mathcal{B}(P, Q_\Phi)|^2 \, \ln \left( \frac{-x^2}{\delta^2} \right) - 2 \right] \langle \Phi^{2(N-2)} R \rangle. \quad (25)
\]

Two remarks are in order. First, expanding the Wilson line connecting the space–time anchors of the diagnostic device on the light cone does not result in a \(\langle \Phi^{2(N-2)} R \rangle\) contribution. Second, while at the parton–level the connectivity between the interpolating currents space–time
VI. SUMMARY, CONCLUSIONS & OUTLOOK

We presented a quantum bound state description of black holes in a relativistic framework using weakly coupled constituent fields embedded in a non–perturbative vacuum structure that supports strong collective phenomena.

At this stage, the quantum state associated with a black hole is unknown. However, the memory devices employed in quantum mechanics store only information such as Casimir invariants and quantum compatible identifiers. These can be generated by operating with a plethora of auxiliary currents on the non–perturbative vacuum. This degeneracy is of no consequence, since observables, at least in principle, enjoy a perfect ignorance of it. In order to guide intuition, we had in mind a large system of weakly coupled quanta, for the sake of simplicity scalars, with no self–interactions and solely subject to gravitational attraction.

We constructed the bi–local operator measuring the constituents space–time distribution including its gauge–invariant completion on the light–cone. Here, the gauge symmetry is local Poincare invariance with the affine connection as the corresponding gauge potential.

The explicit calculation of the constituent distribution, the total constituent number and the energy density inside the black hole were presented at the parton–level. A detailed discussion concerning gauge corrections followed, but the complete calculation will be presented elsewhere.

Perhaps the most remarkable result is that said observables exclusively receive non–perturbative contributions in the double scaling limit, $M_B, N \to \infty : N/M_B = \text{const}$. The physical reason is quite transparent: Despite the fact that individual constituents are weakly interacting, there is a large collective binding effect. This leads to Hartree–like situations, where the mean field is given in terms of condensates with respect to the nontrivial ground state.

Furthermore we find the simple scaling relation

$$M_B^2 = \frac{1}{16} \left( \frac{\Phi(2(N-2))}{\Phi(2(N-1))} \right) \frac{N_c}{N^2}, \quad (26)$$

with $N_c \propto N^4$, where $N$ denotes the number of fields composing the auxiliary current density. On one hand, this shows the scaling $M_B \propto N$ in accordance with $[4]$. On the other hand, the ratio of condensates in (26) becomes $N$–independent in the double scaling limit. As it is of mass dimension two, and following the reasoning in $[1]$, it would be plausible to identify it with the Planck mass squared. This implies that black hole masses are quantised in terms of the Planck mass.

Let us stress that scaling relations like (26) should be universal, as long as the bound state mass is large compared to the typical constituent energies. Thus, these relations hold, in particular, for black holes. Within this description, at the parton–level, a black hole is an ordinary quantum bound state.

We are convinced that while the embedding of black holes in Hilbert spaces is tempting and plausible (both can be argued for given the existence of a surface that proves to be of fundamental significance), the framework presented here has a larger domain of applicability.

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Appendix A: Constituent density in external fields

For the sake of a self–contained presentation, in this appendix we derive the relation between the bi–local operator $\mathcal{O}$ representing the constituent occupation in the absence and presence of $\mathcal{G}$ to all orders in the derivative coupling on the light–cone. We follow [2].

The equation of motion (3) for the diagnostic device $\mathcal{O}$ can be solved iteratively. Including $n$ derivative couplings to the gauge connection $\mathcal{G}$, the associated bi–local operator at this level is given by

$$\mathcal{O}^{(n)}(y; r/2) = \int \sigma((z)_n) (-1)^n \mathcal{O}^{(0)}(y_+, z_1) \mathcal{G} \cdot \partial \mathcal{O}^{(0)}(z_n, y_--) \prod_{a \in I(n-1)} \mathcal{G} \cdot \partial \mathcal{O}^{(0)}(z_a, z_{a+1}), \quad (A1)$$
Here, \( y_\pm := y \pm r/2 \), \( I(n) \) denotes the index set \( \{1, \ldots, n\} \), 
\( \sigma(z) := d^4z \) and \( \sigma((z)_n) := \sigma(z_1) \cdots \sigma(z_n) \).

Fourier–transforming the free constituent number operator \( \mathcal{O}^{(0)} \),
\[
\mathcal{O}^{(n)}(y; r/2) = \frac{(-1)^n}{(2\pi)^3} \int \sigma(k_0, k_n) e^{i(k_n-k_a) y} e^{i(k_0+k_a) r} \int \sigma((k)_{n-1}) F(k_0, (k)_n) \prod_{a \in I(n)} k_a \cdot \mathcal{G}(k_{a-1} - k_a),
\]
where \( F \) denotes the usual propagator denominators for the specified momenta. Introducing the new momentum variables \( 2K := k_0 + k_n \), \( 2Q := k_0 - k_n \), which are Fourier–conjugated to \( y \) and \( r \), respectively, and \( q_a := k_{a-1} - k_a \), gives
\[
\mathcal{O}^{(n)}(y; r/2) = \frac{(-1)^n}{(2\pi)^4} \int \sigma(K) \sigma(Q) e^{i2Q y} e^{i2K r} \int \sigma((q)_n) \delta^{(4)}(Q - \sum_{a \in I(n)} q_a/2) F(K, Q, (q)_{n-1}) \prod_{b \in I(n)} \left( K + Q - \sum_{j=1}^b q_j \right) \cdot \mathcal{G}(q_b). \tag{A3}
\]
The scale \( r \) characterising the diagnostic process is an external scale and can be further qualified to simplify the expression for \( \mathcal{O}^{(n)}(y; r/2) \). A common qualification is to make it light–like and to extract the leading light–cone contribution to \( \mathcal{O}^{(n)}(y; r/2) \),
\[
\mathcal{O}^{(n)}(y; r/2) = \frac{(-1)^n n!}{(2\pi)^4} \int du_0 \prod_{a \in I(n)} du_a \delta^{(1)}(1 - u_0 - \sum_{b \in I(n)} u_b) \prod_{c \in I(n)} \int \sigma(q_c) \exp \left\{ i \sum_{d \in I(n)} q_d \cdot \left[ y - \left( 1 - 2 \sum_{i=1}^d u_i \right) r \right] \right\} \int \sigma(P) \exp (i2r \cdot P) \prod_{m \in I(n)} P \cdot \mathcal{G}(q_c)/(P^2)^{n+1},
\]
where Feynman parameters have been used. The Fourier–transformation \( P \rightarrow r \) requires regularisation. Employing the \( \overline{\text{MS}} \) scheme it is readily evaluated:
\[
(2\pi)^4 \frac{i^n}{n!} \prod_{a \in I(n)} r^{\lambda_a} \mathcal{O}^{(0)}(y; r). \tag{A4}
\]
Performing the \( u_0 \)–integration, we arrive at
\[
\mathcal{O}(y; r/2) = \mathcal{P} \exp \left( - \int dz^4 \mathcal{G}_\lambda(z) \right) \mathcal{O}^{(0)}(y; r/2),
\]
where \( \mathcal{C} \) denotes the contour given by the path \( z : [0, 1] \rightarrow \mathbb{R}^4, u \rightarrow z(u) := y - (1 - 2u) r, \) and \( \mathcal{P} \) refers to path ordering along this contour.

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