The groups of Richard Thompson and complexity

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Dedicated with gratitude to John L. Rhodes on his 65th birthday.

Abstract

We prove new results about the remarkable infinite simple groups introduced by Richard Thompson in the 1960s. We give a faithful representation in the Cuntz C*-algebra. For the finitely presented simple group \( V \) we show that the word-length and the table size satisfy an \( n \log n \) relation. We show that the word problem of \( V \) belongs to the parallel complexity class \( \text{AC}^1 \) (a subclass of \( P \)), whereas the generalized word problem of \( V \) is undecidable.

We study the distortion functions of \( V \) and show that \( V \) contains all finite direct products of finitely generated free groups as subgroups with linear distortion. As a consequence, up to polynomial equivalence of functions, the following three sets are the same: the set of distortions of \( V \), the set of Dehn functions of finitely presented groups, and the set of time complexity functions of nondeterministic Turing machines.

1 Introduction

In [40] Thompson constructed a simple finitely presented infinite group, one of the most remarkable groups ever found. He denoted it by \( \text{Pa}(\omega^2) \) and by \( \text{Ft}(K) \) in [11], by \( \hat{V} \) in [40], and by \( \mathcal{C}' \) in [20]; in [4] it is denoted by \( V \), and we will follow that convention, which has been widely adopted. We will also use the uncountable Thompson group \( G_{2,1} \) (following the notation of [36]). The proofs of the main properties of \( V \) were first outlined in [11] and can be found in detail in [4], or in [14] (where it is called \( G_{2,1} \), as part of an infinite family of finitely presented simple groups).

Thompson defined his groups as permutation groups of certain sets of infinite words over the alphabet \( \{0,1\} \). We will follow [36], and indirectly [14], and define \( V \) by partial bijections of the free monoid \( \{a,b\}^\ast \). The advantage of this definition is that partial actions on finite words enable us to define algorithmic problems and their complexity. From now on, “word” will mean “finite word”.

Our setting for the Thompson groups requires a number of elementary definitions and facts. Almost all of these concepts are standard (the literature on Thompson groups suffers from idiosyncratic terminology, which can usually be avoided). Since the Thompson groups are based on partial actions, we have to choose a side for the actions. We choose to act on the

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left. The main advantage of this choice will turn out to be the connection between Thompson groups acting on the left, and prefix codes. The literature on codes greatly prefers prefix codes over suffix codes, and we hope that this choice improves readability.

Let \( A \) be a finite alphabet. The set of all words over \( A \) (including the empty word \( \varepsilon \)) is denoted by \( A^* \). Concatenation of two words \( u, v \in A^* \) is denoted by \( u \cdot v \) or simply \( uv \); \( A^* \) is a monoid under the concatenation operation. For two sets \( X_1, X_2 \subseteq A^* \), we denote their concatenation by \( X_1 X_2 \) or by \( X_1 \cdot X_2 \), defined by \( X_1 X_2 = \{ x_1 x_2 \in A^* : x_1 \in X_1, x_2 \in X_2 \} \). From now on we assume that the alphabet \( A \) has at least two letters.

A right ideal of \( A^* \) is defined to be a subset \( R \subseteq A^* \) such that \( R \cdot A^* \subseteq R \) (i.e., \( R \) is closed under multiplication by any word in \( A^* \) on the right).

For two words \( u, v \in A^* \), we say that \( u \) is a prefix of \( v \) iff \( v = ux \) for some \( x \in A^* \); we also write \( u \geq_{\text{pref}} v \) or \( v \leq_{\text{pref}} u \); this is a partial order, related to set inclusion by the fact that \( v \leq_{\text{pref}} u \) iff \( vA^* \subseteq uA^* \). We say that \( u \) is a strict prefix of \( v \) (and write \( u >_{\text{pref}} v \)) iff \( u \geq_{\text{pref}} v \) and \( u \neq v \). We say that \( u \) and \( v \) are prefix-comparable iff \( v \leq_{\text{pref}} u \) or \( u \leq_{\text{pref}} v \); we denote this by \( u \preccurlyeq_{\text{pref}} v \). A prefix code over \( A \) is defined to be a subset \( C \) of \( A^* \) such that no element of \( C \) is a strict prefix of another element of \( C \). The monograph [1] is an excellent reference for the material on prefix codes that we use here. By definition, a maximal prefix code over an alphabet \( A \) is a prefix code over \( A \) which is not a strict subset of any other prefix code over \( A \).

For a right ideal \( R \) of \( A^* \), a set \( \Gamma \subseteq R \) is called a set of right-ideal generators of \( R \) (or just “generators”) iff \( R = \Gamma \cdot A^* \). One can prove (see Lemma 8.1) that any right ideal \( R \) of \( A^* \) has a unique minimal (under inclusion) set of right-ideal generators, and this set of generators is a prefix code. Hence, prefix codes could be called “right-ideal bases”. Moreover, since the prefix code of a right ideal is unique, right ideals of \( A^* \) and prefix codes over \( A \) are in one-to-one correspondence (see Lemma 8.1 in Appendix A1 for proofs).

A right ideal \( R \) of \( A^* \) is said to be finitely generated iff the prefix code corresponding to \( R \) is finite. A right ideal \( R \) of \( A^* \) is called essential iff \( R \) has a non-empty intersection with every right ideal of \( A^* \). (This is \( C^* \)-algebra terminology.) Note that a right ideal \( R \) of \( A^* \) is essential if its prefix code is maximal; this is also equivalent to saying that \( R \) is a right ideal such that for every \( u \in A^* \) there is \( x \in A^* \) such that \( ux \in R \) (see Lemma 8.1). Clearly, if a right ideal \( J \) contains an essential right ideal of \( A^* \), then \( J \) is essential too.

By definition, a right-ideal homomorphism of \( A^* \) is a function \( \varphi : R_1 \to R_2 \) such that \( R_1 \) and \( R_2 \) are right ideals of \( A^* \), and such that for all \( u \in R_1 \) and all \( x \in A^* \): \( \varphi(u) \cdot x = \varphi(ux) \). A right-ideal isomorphism of \( A^* \) is a bijective right-ideal homomorphism.

One can prove (see Lemma 8.2) that the set of all right-ideal homomorphisms (or isomorphisms) of \( A^* \) is in one-to-one correspondence with the set of all functions (respectively bijections) between prefix codes of \( A^* \). For a right-ideal isomorphism \( \varphi : P_1 A^* \to P_2 A^* \), where \( P_1 \) and \( P_2 \) are prefix codes, the restriction \( \tau_\varphi : P_1 \to P_2 \) is a bijection, and \( \tau_\varphi \) determines \( \varphi \) uniquely. Following Thompson, the restriction \( \tau_\varphi : P_1 \to P_2 \) of \( \varphi \) will be called the table of \( \varphi \), and will be used to represent \( \varphi \) by a traditional (finite) function table. (In [14] and [36] this was called the “symbol of \( \varphi \)”). The maximal prefix code \( P_1 \) is called the domain code of \( \varphi \), and \( P_2 \) is called the image code or range code of \( \varphi \).

By definition, an extension of a right-ideal isomorphism \( \varphi : R_1 \to R_2 \) is a right-ideal isomorphism \( \Phi : J_1 \to J_2 \) where \( J_1, J_2 \) are right ideals such that \( R_1 \subseteq J_1 \), \( R_2 \subseteq J_2 \), and \( \Phi \) agrees with \( \varphi \) on \( R_1 \) (i.e., \( \Phi(x) = \varphi(x) \) for all \( x \in R_1 \)). In that case we also call \( \varphi \) a restriction.
of \( \Phi \). The extension, and the restriction, are called strict iff \( \varphi \neq \Phi \).

A right-ideal isomorphism is said to be \textit{maximal} iff it has no strict extension in \( A^* \); it is called \textit{extendable} otherwise. We denote the maximum extension of \( \varphi \) by \( \max \varphi \); we will prove in Lemma \[2.1\] that the \textit{maximum} extension of an isomorphism between essential right ideals is unique (for this uniqueness, it is necessary that the ideals be essential).

Most of the above concepts can be pictured, using trees. The monoid \( A^* \) can be described by the Cayley graph of the right regular representation of \( A^* \) relative to the generating set \( A \). We will simply call this the \textit{tree of} \( A^* \). It is an infinite tree rooted at the empty word \( \varepsilon \). Every vertex has \( |A| \) children. Every subset of \( A^* \) is pictured as a set of vertices of this infinite tree. A prefix code is pictured as a set of vertices, no two of which lie on a same directed path from the root. A finite prefix code is maximal iff it is a prefix code that forms a “cut” in the tree (i.e., a set of vertices whose removal disconnects the root of the tree from all the “ends” of the tree). Infinite maximal prefix codes are harder to visualize, but the following concept is useful:

For any prefix code \( P \subset A^* (P \neq \emptyset) \), the \textit{prefix tree} of \( P \) is defined to be the subtree of the tree of \( A^* \), whose vertex subset consists of all the prefixes of words in \( P \) (and whose root is still \( \varepsilon \)). Hence, the set of leaves of this subtree is \( P \). We have the following general facts about non-empty subsets \( P \subset A^* \) (finite or infinite):

- \( P \) is a prefix code iff \( P \) is the set of leaves of a subtree of the tree of \( A^* \).
- A prefix code \( P \) is maximal iff every non-leaf vertex of the prefix tree of \( P \) has exactly \( |A| \) children (in the prefix tree of \( P \)).

A right ideal of \( A^* \) is the same thing as an order ideal relative to the prefix order \( \leq_{\text{pref}} \) in \( A^* \). A (maximal) prefix code in \( A^* \) is the same thing as a (maximal) anti-chain relative to the prefix order \( \leq_{\text{pref}} \). Right-ideal isomorphisms are the same thing as prefix-order isomorphisms between prefix-order ideals. So, our discussion could also be carried out in partial-order terminology.

Prefix codes are well known; see e.g. [3], [4]. They are not only of mathematical interest but are used in practice (e.g., in text compression by Huffman coding, and for error correcting codes).

\textbf{Example 1.1} — \textit{Some infinite maximal prefix codes}

Infinite maximal prefix codes can be extremely complex. Here are some examples.

1. For any fixed infinite sequence \((a_1, a_2, \ldots, a_{n-1}, a_n, \ldots) \in A^\omega \) one can build an infinite maximal prefix code as follows. For any \( a \in A \), let \( \sigma(a) \in A \) be another letter \((\neq a)\) chosen in \( A \). Consider the code \( P = \{a_1 \ldots a_{n-1} \sigma(a_n) : n \geq 1\} \). Such infinite prefix codes have “one infinite path-shaped end”.

2. \textit{Combination of prefix codes:} Let \( X = \{x_i : i \in I\} \) be a prefix code, and let \((Q_i : i \in I)\) be a family of prefix codes, with \(|X| = |I| \) (finite or infinite). Then \( \bigcup_{i \in I} x_i Q_i \) is a prefix code, which is maximal if \( X \) and each \( Q_i \) are maximal. This enables us to construct maximal prefix codes with any number of “infinite path-shaped ends”.

3. An infinite maximal prefix code does not need to have any “path-shaped ends”; instead, it could have any number of infinite “tree-shaped ends”. For example, consider the following code over \( \{a, b\} \): \( P = \{a^2, b^2\}^* \cdot \{ab, ba\} \). By looking at the prefix tree of \( P \) it is easy to see that \( P \) is a prefix code (all the words in \( P \) are leaves of the prefix tree) and that it is maximal (all non-leaves have two children). Another example of a similar infinite maximal prefix code is \( \{a^2, ab, b^2\}^* \cdot \{ba\} \). See [4] for more examples.
The Thompson groups

Before defining the Thompson groups we prove a few facts about isomorphisms of right ideals of \( A^* \). Proposition 2.1 and Lemmas 2.2 and 2.5 appear in Thompson’s work and in [36], with a similar content but a different formalism. Lemmas 2.3 and 2.4 are new.

**Proposition 2.1** An isomorphism between essential right ideals of \( A^* \) has a unique maximum extension.

Equivalently, if two isomorphisms \( \varphi_1, \varphi_2 \) between essential right ideals agree on an essential right ideal then \( \varphi_1 \) and \( \varphi_2 \) have the same maximum extension.

**Proof.** Let \( \varphi : P_1 A^* \to P_2 A^* \) be an isomorphism of essential right ideals, where \( P_1 \) and \( P_2 \) are maximal prefix codes. If \( \varphi(x) \) is not defined for some \( x \in A^* \) then (by Lemma 8.1 (4)), there exists \( p \in P_1 \) with \( x >_{\text{pref}} p \). Let \( p_x \) be the first element in the lexicographic order (assuming we have chosen a fixed total order for the finite alphabet \( A \)) such that \( p_x \in P_1 \) and \( x >_{\text{pref}} p_x = x u_x \) (for some \( u_x \in A^* \)). Then \( p_x \) and \( u_x \) are uniquely determined by \( x \) and \( \varphi \).

If there is an extension \( \Phi \) of \( \varphi \) such that \( \Phi(x) \) is defined, then \( \varphi(p_x) = \Phi(p_x) = \Phi(x) u_x \). Hence, \( \Phi(x) \) is uniquely determined by \( x \) and \( \varphi \).

It follows from this the union of extensions of \( \varphi \) is a well defined extension too. Thus, we can take the union of all extensions of \( \varphi \) to obtain the maximum extension of \( \varphi \). □

The next two Lemmas give useful characterizations of extendability and maximality of right-ideal isomorphisms of essential right ideals. They will be used in the next section.

**Lemma 2.2** Let \( \varphi : P_1 A^* \to P_2 A^* \) be an isomorphism of essential right ideals, where \( P_1 \) and \( P_2 \) are finite maximal prefix codes. Then \( \varphi \) is extendable iff there are \( x_0, y_0 \in A^* \) such that for every letter \( \alpha \in A \): \( x_0 \alpha \in P_1, y_0 \alpha \in P_2 \), and \( \varphi(x_0 \alpha) = y_0 \alpha \).

(If this condition holds, \( \varphi \) can be extended by mapping \( x_0 \) to \( y_0 \).)

**Proof.** If \( x_0 \alpha \in P_1, y_0 \alpha \in P_2 \) and \( \varphi(x_0 \alpha) = y_0 \alpha \) for every letter \( \alpha \in A \), then \( \varphi \) can be extended by defining \( \varphi(x_0) \) to be \( y_0 \). The prefix code of the domain then becomes \( P_1 \cup \{x_0\} - x_0 A \), and the prefix code of the range becomes \( P_2 \cup \{y_0\} - y_0 A \).

Conversely, suppose \( \varphi \) can be strictly extended to \( \Phi \). Consider a word \( x_0 \) on which \( \varphi \) is not defined, but on which \( \Phi \) is defined.

Case 1: If for all \( \alpha \in A \), \( \varphi(x_0 \alpha) \) is defined, i.e., \( x_0 \alpha \in P_1 A^* \), then actually \( x_0 \alpha \in P_1 \) (since \( \varphi \) is not defined on any strict prefix of \( x_0 \alpha \)). Also, \( \varphi(x_0 \alpha) = \Phi(x_0 \alpha) = \Phi(x_0) \alpha \). So we pick \( y_0 \) to be \( \Phi(x_0) \). Then \( y_0 \alpha \in P_2 A^* \), for all \( \alpha \in A \); but actually, \( y_0 \alpha \in P_2 \) (since \( \varphi^{-1} \) is not defined on any strict prefix of \( y_0 \alpha \)). Now \( x_0 \) and \( y_0 \) satisfy the properties of the Lemma.

Case 2: If for some \( \alpha \in A \), \( \varphi(x_0 \alpha) \) is not defined, we replace \( x_0 \) by \( x_0 \alpha \) and continue the reasoning. Eventually, we reach case 1, since \( P_1 \) is finite and maximal. □

Note that the Lemma is not always true for infinitely generated essential right ideals. For example, let \( A = \{a, b\} \), consider the maximal prefix code \( P_1 = P_2 = \{a^nb : n \in \mathbb{N}\} \), and let \( \varphi \) be the identity map on \( P_1 A^* = \{a^nb : n \in \mathbb{N}\} \cdot \{a, b\}^* = \{a, b\}^* - \{a\}^* \). Then \( \varphi \) can obviously be extended to the identity map on \( \{a, b\}^* \), but there is no word \( x \) such that \( xa, xb \in P_1 \).

The following Lemma gives a criterion for extendability in the general case.
We will use the following notation: For any set $x$, consider a word $\phi$ extended by defining the image of $x$.

**Claim:** By Lemma 8.5 in Appendix A1, prefix codes are used (instead of the alphabet $A$, which is a very special maximal prefix code).

One sees that the general Lemma differs from the finite case by the fact that all possible maximal prefix codes are used (instead of the alphabet $A$, which is a very special maximal prefix code).

We will use the following notation: For any set $L \subseteq A^*$ and any word $x \in A^*$, we define

$$\tau_L = \{ w \in A^* : xw \in L \}.$$ 

**Proof.** If the condition in the Lemma holds (i.e., $\phi(x_0q) = y_0q$ for all $q \in Q$), then $\phi$ can be extended by defining the image of $x_0w$ to be $y_0w$ (for all $w \in A^*$).

Conversely, suppose $\phi$ can be extended to an isomorphism of essential right ideals $\Phi$. Consider a word $x_0$ on which $\phi$ is not defined, but on which $\Phi$ is defined, and suppose $\Phi(x_0) = y_0$. Define $Q$ as follows:

$$Q = \{ w \in A^* : x_0w \in P_1 \} = \overline{x_0}P_1.$$

By Lemma 8.5 in Appendix A1, $Q$ is a maximal prefix code.

**Claim:** $\overline{x_0}P_1 = \overline{y_0}P_2$ ( = $Q$). Hence, $y_0q \in P_2$ for all $q \in Q$. Indeed, $w \in \overline{x_0}P_1$ iff $x_0w \in P_1$, iff $\phi(x_0w) \in P_2$. Moreover, $\phi(x_0w) = \Phi(x_0w) = \Phi(x_0)w = y_0w$. Hence, $w \in \overline{x_0}P_1$ iff $y_0w \in P_2$; the latter holds iff $w \in \overline{y_0}P_2$. This proves the Claim.

Finally, for any $q \in Q$, $\phi(x_0q) = \Phi(x_0q) = \Phi(x_0)q = y_0q$. \Box

**Lemma 2.4** Let $\phi : P_1A^* \rightarrow P_2A^*$ be as in the previous Lemma; then the maximum extension of $\phi$ can be obtained as follows. There are two maximal prefix codes $\{x_i : i \in I\}, \{y_i : i \in I\} \subseteq A^*$ (for an index set $I \subseteq N$), such that

(a) for each $i \in I$ there is a maximal prefix code $Q_i$ such that $x_iQ_i \subseteq P_1$, $y_iQ_i \subseteq P_2$, and for all $q \in Q_i$, $\phi(x_iq) = y_iq$;

(b) the sets $x_iQ_i$ and $y_iQ_i$ are $\subseteq$-maximal.

Then $\Phi$, defined by $x_i \mapsto y_i$ (for all $i \in I$), is the maximum extension of $\phi$.

Figure 1 below gives the tree picture representing the prefix code $P_1'$ of $\Phi : P_1'A^* \rightarrow P_2'A^*$, the prefix code $P_1$ of $\phi : P_1A^* \rightarrow P_2A^*$, an element $x_i \in P_1'$, and the prefix code $Q_i$ used to extend $\phi$ to $\Phi$, at $x_i$. To extend $\phi$ to $\Phi$, we need a prefix code $Q_i = \overline{x_i}P_1$ for each $x_i \in P_1'$. In this picture, $x_i$ should be viewed as the root of the tree for $Q_i$. 

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Then $\phi$ that $\Phi$ is the maximum extension of $\phi$ such that $\Phi$ is a prefix code.

Proof of the Claim: If, by contradiction, the Claim is false then there is a non-trivial maximal $\Phi$ can be extended then, by Lemma 2.3, the words $x_i, y_i$ and the sets $Q_i$ exist, satisfying the claimed properties (a) and (b), and the map $\Phi$ obtained is an extension of $\varphi$. We have to show that $\Phi$ is the maximum extension of $\varphi$.

After each set $x_iQ_i$ has been replaced by $x_i$ (and each $y_iQ_i$ by $y_i$) in the extension process, the domain code of $\Phi$ is $\{x_i : i \in I\}$ and the image code of $\Phi$ is $\{y_i : i \in I\}$.

The following Claim implies, by Lemma 2.3, that $\Phi$ cannot be extended.

Claim. The sets $\{x_i : i \in I\}, \{y_i : i \in I\}$ have no strict subset of the form $xP$, respectively $yP$, with $\Phi(xp) = yp$ for all $p \in P$ (where $P$ is a maximal prefix code with more than one element).

Proof of the Claim: If, by contradiction, the Claim is false then there is a non-trivial maximal prefix code $P$ such that $xP = \{x_j : j \in J\} = \{xp_j : j \in J\}$ and $yP = \{y_j : j \in J\} = \{yp_j : j \in J\}$, for some $J \subseteq I$. Then $\bigcup_{j \in J} xp_jQ_j = x \bigcup_{j \in J} p_jQ_j$; moreover, $\bigcup_{j \in J} p_jQ_j = Q$ is a maximal prefix code (by construction (2) in Example 1.1), with $xQ \subseteq P_1$ and $yQ \subseteq P_2$. Now, $xQ$ contains $x_{j_1}Q_{j_1} \cup x_{j_2}Q_{j_2} \cup \ldots$ (for some $j_1, j_2, \ldots \in J \subseteq I$ with $j_1 \neq j_2$), which contradicts $\subseteq$-maximality (assumption (b)).

Lemma 2.5 Let $\varphi_1$ and $\varphi_2$ be right-ideal isomorphisms between essential right ideals of $A^*$. Then $\varphi_1$, $\varphi_2$, have restrictions $\varphi'_1$, respectively $\varphi'_2$, such that the range of $\varphi'_1$ is equal to the domain of $\varphi'_2$.

If the domain and ranges of $\varphi_1$ and $\varphi_2$ are finitely generated then so are the domains and ranges of $\varphi'_1$ and $\varphi'_2$. 

Remark. (0) In Lemma 2.4 we allow $Q_i$ to consist of just the empty word.

(1) $\subseteq$-Maximality of the sets $x_iQ_i$ and $y_iQ_i$ ($i \in I$) is defined as follows: Suppose there exist a maximal prefix code $Q$ and words $x, y$ such that $xQ \subseteq P_1$, $yQ \subseteq P_2$, and for all $q \in Q$: $\varphi(xq) = yq$. Then $x = x_j$, $y = y_j$, $Q = Q_j$.

(2) When $\varphi$ is already maximum then the Lemma holds with $\{x_i : i \in I\} = P_1$, $\{y_i : i \in I\} = P_2$ and $Q_i = \{\varepsilon\}$ for all $i \in I$ (where $\varepsilon$ denotes the empty string).

(3) The sets $x_iQ_i$ (for $i \in I$) are two-by-two disjoint (and similarly for the sets $y_iQ_i$). So all the extensions, from $x_iQ_iA^*$ to $x_iA^*$, as $i$ ranges over $I$, are independent (for different $i$’s), and can be viewed as being carried out in parallel.

Proof of Lemma 2.4. If $\varphi$ is already maximal there is nothing to prove (by Remark (2)). If $\varphi$ can be extended then, by Lemma 2.3, the words $x_i, y_i$ and the sets $Q_i$ exist, satisfying the claimed properties (a) and (b), and the map $\Phi$ obtained is an extension of $\varphi$. We have to show that $\Phi$ is the maximum extension of $\varphi$.

After each set $x_iQ_i$ has been replaced by $x_i$ (and each $y_iQ_i$ by $y_i$) in the extension process, the domain code of $\Phi$ is $\{x_i : i \in I\}$ and the image code of $\Phi$ is $\{y_i : i \in I\}$.

The following Claim implies, by Lemma 2.3, that $\Phi$ cannot be extended.

Claim. The sets $\{x_i : i \in I\}, \{y_i : i \in I\}$ have no strict subset of the form $xP$, respectively $yP$, with $\Phi(xp) = yp$ for all $p \in P$ (where $P$ is a maximal prefix code with more than one element).

Proof of the Claim: If, by contradiction, the Claim is false then there is a non-trivial maximal prefix code $P_1$ such that $xP_1 = \{x_j : j \in J\} = \{xp_j : j \in J\}$ and $yP_1 = \{y_j : j \in J\} = \{yp_j : j \in J\}$, for some $J \subseteq I$. Then $\bigcup_{j \in J} xp_jQ_j = x \bigcup_{j \in J} p_jQ_j$; moreover, $\bigcup_{j \in J} p_jQ_j = Q$ is a maximal prefix code (by construction (2) in Example 1.1), with $xQ \subseteq P_1$ and $yQ \subseteq P_2$. Now, $xQ$ contains $x_{j_1}Q_{j_1} \cup x_{j_2}Q_{j_2} \cup \ldots$ (for some $j_1, j_2, \ldots \in J \subseteq I$ with $j_1 \neq j_2$), which contradicts $\subseteq$-maximality (assumption (b)). 

Fig. 1: Extendability condition
As mentioned before, Thompson showed the following:

The first two facts are not obvious at all (see [40], [14], [4]; the third fact is straightforward (but also remarkable).

Associativity follows from this. Hence, the maximum extension of the composition of \(\psi\) and \(\varphi\), where \(\psi\) is applied first).

The intersection of two right ideals \(Q_1, R_2\), is obviously a right ideal, and \(Q_1 \cap R_2\) is essential if \(Q_1, R_2\) are essential (Lemma 8.3). Also, \(\varphi^{-1}(S')\) and \(\varphi'_2(S')\) are essential, by Lemma 8.4. The other properties of the Lemma are straightforward.

If \(R_1, Q_1, R_2, Q_2\) are finitely generated right ideals then \(S' = Q_1 \cap R_2\) is finitely generated, by Lemma 8.3. Moreover, \(R'_1 = \varphi^{-1}(S')\) and \(Q'_2 = \varphi_2(S')\) are finitely generated since \(\varphi_1\) and \(\varphi_2\) are isomorphisms. □

The following definition of the Thompson groups is very close to the definition of Scott [36] (and indirectly, to the definition in [14]). The tree representation of codes connects this definition and the definition by action on finite trees used in [14]. Although the Thompson groups are traditionally defined with the alphabet \(\{0,1\}\), we prefer to use \(\{a,b\}\) (the symbols “0” and “1” have too many meanings already).

Definition 2.6 The Thompson group \(V\) is the partial action group on \(\{a,b\}^*\) consisting of all maximal isomorphisms between finitely generated essential right ideals of \(\{a,b\}^*\).

The Thompson group \(G_{2,1}\) is the partial action group on \(\{a,b\}^*\) consisting of all maximal isomorphisms between essential right ideals of \(\{a,b\}^*\).

Multiplication: For \(\varphi, \psi \in G_{2,1}\) (or \(\in V\)), the product \(\varphi \cdot \psi\) is \(\max(\varphi \circ \psi)\) (i.e., the maximum extension of the composition of \(\psi\) and \(\varphi\), where \(\psi\) is applied first).

We will usually write just \(\varphi \psi\) for \(\varphi \cdot \psi\). One can check easily that \(\varphi^{-1} \varphi = \varphi \varphi^{-1} = 1\) (the identity map on \(\{a,b\}^*\)). Maximum extension is needed for this to be true; without taking the maximum extension, the composite \(\varphi^{-1} \circ \varphi\) is the restriction of \(1\) to the domain of \(\varphi\), and \(\varphi \circ \varphi^{-1}\) is the restriction of \(1\) to the image of \(\varphi\). It is also easy to check that for any (not necessarily maximal) isomorphisms of essential right ideals, \(\max(\psi \circ \varphi) = \max(\max(\psi \circ \max) \varphi)\).

Associativity follows from this. Hence, \(G_{2,1}\) and \(V\) are groups.

In connection with the definition of the Thompson groups it is natural to introduce the following terminology: Two isomorphisms \(\varphi\) and \(\psi\) between essential right ideals of \(A^*\) are congruent iff \(\varphi\) and \(\psi\) have the same maximum extension (\(\max \varphi = \max \psi\)).

As mentioned before, Thompson showed the following:

- \(G_{2,1}\) and \(V\) are simple;
- \(V\) is finitely presented;
- \(V\) contains all finite groups as subgroups.

The first two facts are not obvious at all (see [10], [14], [1]); the third fact is straightforward (but also remarkable).
Remark on other Thompson groups: Richard Thompson defined subgroups of $V$ that are of great interest. One of them, denoted $\mathbb{P}$ in [31], $\mathbb{P}'$ in [20], and $F$ in [4], is defined as

Example — Another finitely generated group containing all finite groups:

Let $\mathcal{G}_Z$ be the set of all permutations of $\mathbb{Z}$ (the integers), and let $\mathcal{G}_{\mathbb{Z}^\infty}$ be the set of all finitary permutations of $\mathbb{Z}$ (i.e., permutations that fix all but a finite number of integers). Let $\sigma: z \in \mathbb{Z} \mapsto z + 1$ be the right-shift function. Then the group $G = \langle \mathcal{G}_{\mathbb{Z}^\infty} \cup \{\sigma\} \rangle$ (i.e., the subgroup of $\mathcal{G}_Z$ generated by $\mathcal{G}_{\mathbb{Z}^\infty}$ and $\sigma$) contains all finite groups. Moreover, $G$ is generated by $\sigma$ and the transposition $\tau_{(0,1)}$, so $G$ is finitely generated. This group has been known for a long time; it is less well known, but easy to prove that $\langle \mathcal{G}_{\mathbb{Z}^\infty} \cup \{\sigma\} \rangle$ is a subgroup of $V$.

To show that $G = \langle \mathcal{G}_{\mathbb{Z}^\infty} \cup \{\sigma\} \rangle$ is a subgroup of $V$ we use a one-to-one correspondence between $\mathbb{Z}$ and the maximal prefix code $a^*ab \cup b^*ba \subset \{a, b\}^*$, defined by

$$z \mapsto \begin{cases} a^{-z}ab & \text{if } z \leq 0 \\ b^za & \text{if } z > 0 \end{cases}$$

It follows that $\tau_{(0,1)}$ is represented by the following element of $G_{2,1}$: $ab \mapsto ba$, $ba \mapsto ab$, and $\tau_{(0,1)}$ is the identity elsewhere on $a^*ab \cup b^*ba$. The shift $\sigma$ is represented by $a^{-z}ab \mapsto a^{-(z+1)}ab$ (for all $z < 0$), $ab \mapsto ba$, and $bb^za \mapsto bb^{z+1}a$ (for all $z \geq 0$). Here we just indicate how the maps are defined on the maximal prefix code; the definition on the corresponding essential right ideal follows automatically.

Maximum extension of these two maps reveals that they actually belong to $V$. The map representing $\tau_{(0,1)}$ is easy to extend to $aa \mapsto aa$, $bb \mapsto bb$, $ab \mapsto ba$, $ba \mapsto ab$. (Again, we just indicate the map on a maximal prefix code.) The shift can be extended to $aa \mapsto a$, $ab \mapsto ba$, $b \mapsto bb$ (as defined on maximal prefix codes); we are using Lemma 2.3.

Notice that the representation of the right-shift $\sigma$ above is (the inverse of) the generator of $V$, called “$A$” in [4] (see also the remarks below on other Thompson groups). This provides a nice interpretation of the generator “$A$”.

Remark on partial actions: It would not be correct to say that $G_{2,1}$ and $V$ act on $\{a, b\}^*$ by partial maps, since in addition to the composition of the partial maps we also take the maximum extension. What we have here is a partial action (see [18]), as opposed to an (ordinary) action by partial maps. Here, with each element $g$ of the group one associates a partial transformation $\tau(g)$ on some chosen set such that: For the identity of $G$ we have $\tau(1_G) = 1$; for all $g \in G$, $\tau(g^{-1}) = \tau(g)^{-1}$; and for all $g_1, g_2 \in G$, $\tau(g_1) \circ \tau(g_2) \subseteq \tau(g_1g_2)$.

On the other hand, Thompson [31] used an ordinary action, by total permutations, but on infinite words. But from a computational point of view, partially defined operations are common and easy to deal with, whereas infinite objects pose problems (e.g., it is not clear how one should define complexity of computations on infinite words); for that reason we will not use Thompson’s original definition. Higman [13] defined $V$ by total functions on larger algebras, that contain $\{a, b\}^*$. But Higman’s actions are uniquely determined by the partial action on $\{a, b\}^*$ (and are in fact the same as the partial actions in [36], which themselves are the same as ours, up to terminology).

Remark on other Thompson groups: Richard Thompson defined subgroups of $V$ that are of great interest. One of them, denoted $\mathbb{P}$ in [31], $\mathbb{P}'$ in [20], and $F$ in [4], is defined as
follows: $F$ is the subgroup of $V$ consisting of all maximal right-ideal isomorphisms of $\{a,b\}^*$ that preserve the dictionary order.

The dictionary order on $\{a,b\}^*$ (with $a < b$) is a very classical concept and is defined as follows: For $x,y \in \{a,b\}^*$ we have $x \leq_{\text{dict}} y$ if $x$ is a prefix of $y$ or there exists $p \in \{a,b\}^*$ such that $x \in pa\{a,b\}^*$ and $y \in pb\{a,b\}^*$ (so $p$ is the maximal common prefix of $x$ and $y$). One can easily verify that this is a total order, compatible with concatenation on the right for non-prefix comparable words (i.e., $x \leq_{\text{dict}} y$ and $x, y$ prefix incomparable implies $xw \leq_{\text{dict}} yw$).

In the tree picture of $\{a,b\}^*$, if $x,y$ are prefix incomparable then we have: $x <_{\text{dict}} y$ iff $x$ is in a tree branch that is more to the left than the tree branch containing $y$. We will say that a map $\varphi : \{a,b\}^* \rightarrow \{a,b\}^*$ preserves the dictionary order iff the following holds for all $x_1, x_2 \in \{a,b\}^*$: if $x_1 \leq_{\text{dict}} x_2$ and if $x,y$ are prefix incomparable then $\varphi(x_1) \leq_{\text{dict}} \varphi(x_2)$. One can prove easily that if an isomorphism between two essential right ideals preserves the dictionary order then its maximum extension also preserves the dictionary order. Thompson proved that $F$ is a finitely presented group, whose commutator is simple and of finite index.

### 3 Word length in the Thompson group $V$

Let $\Delta$ be a finite set of generators of $V$.

**Definition 3.1** For every element $g \in V$, the word length of $g$ (over the generating set $\Delta$) is the length of a shortest word $\in (\Delta^\pm 1)^*$ that represents $g$.

The word length of $g$ is denoted by $|g|_\Delta$.

It is easy to prove that if $\Omega$ is another finite set of generators of $V$ then $|g|_\Omega \leq C_{12} \cdot |g|_\Delta$ and $|g|_\Delta \leq C_{21} \cdot |g|_\Omega$, where $C_{12}, C_{21} > 0$ depend on $\Delta$ and $\Omega$, but not on $g$.

Since the elements of $V$ are functions, there is another size measure for elements of $V$. In the following we will denote the (finite of infinite) cardinality of a set $X$ by $|X|$.

**Definition 3.2** For a right-ideal isomorphism $\varphi : P_1A^* \rightarrow P_2A^*$, where $P_1$ and $P_2$ are finite maximal prefix codes, the restriction $P_1 \rightarrow P_2$ of $\varphi$ is called the **table** of $\varphi$. (Recall that this restriction is a bijection.)

We define $\|\varphi\|$ to be $|P_1| (= |P_2|)$; we call this the **table size** of $\varphi$.

For an element $g \in V$, the table size $\|g\|$ of $g$ is defined to be the table size of the maximally extended right-ideal isomorphism that represents $g$.

The following lemmas will be useful when we study the table size of right-ideal isomorphisms.

**Lemma 3.3** Let $P, Q, R \subseteq A^*$ be such that $PA^* \cap QA^* = RA^*$, and $R$ is a prefix code. Then $R \subseteq P \cup Q$.

As a consequence, the intersection of two finitely generated right ideals is finitely generated.

**Proof.** The Lemma has a simple and intuitive interpretation in terms of prefix trees. We’ll give a formal proof, which is almost as simple.

For any $r \in R$ there exist $p \in P, q \in Q$ and $v, w \in A^*$ such that $r = pv = qw$. Hence $p$ and $q$ are prefix-comparable. Let us assume $p \geq_{\text{pref}} q = px$, for some $x \in A^*$ (the other case is handled the same way). Hence $q = px \in PA^* \cap QA^* = RA^*$, and $q$ is a prefix of $r = qw$. Since $R$ is a prefix code, $r = q$, hence $r \in Q$. □
Lemma 3.4  Let \( P, Q \subseteq A^* \) be maximal prefix codes such that \( PA^* \subseteq QA^* \). Then \( |Q| \leq |P| \).

Proof. For every \( p \in P \) there is \( q \in Q \) such that \( p \leq_{\text{pref}} q \). In fact, this correspondence \( p \mapsto q \) is a function; indeed, if there are \( q_1, q_2 \in Q \) such that \( p \leq_{\text{pref}} q_1 \) and \( p \leq_{\text{pref}} q_2 \) then \( p = q_1 x_1 = q_2 x_2 \) (for some \( x_1, x_2 \in A^* \)), hence \( q_1 \) and \( q_2 \) are prefix-comparable. This implies \( q_1 = q_2 \) since \( Q \) is a prefix code.

Moreover, this map is surjective. Indeed, let \( q \in Q \). If \( q \in PA^* \) then there exists a prefix of \( q \) in \( P \), hence an inverse. If \( q \notin PA^* \) then the inverse exists by Lemma 8.1 (4).

Since there is a surjective function \( P \to Q \), the result follows. \( \square \)

Proposition 3.5  For any right-ideal isomorphisms \( \varphi_2 \) and \( \varphi_1 \) between essential right ideals of \( A^* \):

\[
\| \max \varphi_1 \| \leq \| \varphi_1 \|
\]

and

\[
\| \varphi_2 \cdot \varphi_1 \| \leq \| \varphi_2 \circ \varphi_1 \| \leq \| \varphi_2 \| + \| \varphi_1 \|
\]

Proof. The fact that \( \| \max \varphi_1 \| \leq \| \varphi_1 \| \) follows directly from Lemma 3.3.

Let \( \varphi_1 : P_1 A^* \to P_1' A^* \) and \( \varphi_2 : P_2 A^* \to P_2' A^* \), where \( P_1, P_1', P_2, P_2' \) are maximal prefix codes, and \( \| \varphi_1 \| = \| P_1 \| = \| P_1' \|, \| \varphi_2 \| = \| P_2 \| = \| P_2' \| \). Then the domain of the functional composite \( \varphi_2 \circ \varphi_1 \) is a right ideal \( RA^* \) where \( R \) is a maximal prefix code; hence, \( \| \varphi_2 \circ \varphi_1 \| = \| R \| \).

Moreover, \( RA^* = \varphi_1^{-1}(P_1'A^* \cap P_2'A^*) \). By Lemma 3.3, \( P_1'A^* \cap P_2'A^* = SA^* \) for some maximal prefix code \( S \) such that \( \| S \| \leq \| P_1' \| + \| P_2 \| = \| \varphi_1 \| + \| \varphi_2 \| \). Since \( RA^* \) is the domain of \( \varphi_2 \circ \varphi_1 \), \( \varphi_1 \) is defined everywhere on \( RA^* \); and since \( SA^* \subseteq P_1'A^* \) (which is the domain of \( \varphi_1^{-1} \)), \( \varphi_1^{-1} \) is defined everywhere on \( SA^* \). Thus, \( \varphi_1^{-1} \) is a bijection from \( SA^* \) onto \( RA^* \), hence by Lemma 8.2 \( |S| = |R| = \| \varphi_2 \circ \varphi_1 \| \). It follows that \( |R| \leq |P_1| + |P_2| \). \( \square \)

Lemma 3.6  Let \( P \subseteq A^* \) be a finite maximal prefix code. Then any word in \( P \) has length at most \( \frac{|P| - 1}{|A| - 1} \). In particular, when the alphabet has 2 letters, the length is at most \( |P| - 1 \).

Proof. Consider the prefix tree of \( P \), which has \( |P| \) leaves. Let the number of non-leaves be \( N \). Then \( N = \frac{|P| - 1}{|A| - 1} \), as can easily be shown by induction on \( N \). The length of a word in \( P \) is equal to the length of the path from the root to the leaf labeled by this word; such a path has length at most \( N \). \( \square \)

As a consequence of Lemmas 3.5 and 3.6 we have:

Corollary 3.7  Let \( \Delta \) be a fixed finite generating set of \( V \). If \( \varphi \in V - \{1\} \) is described by a word of length \( n \) over \( \Delta^\pm \), then the table size satisfies \( \| \varphi \| \leq C_\Delta n \) (where \( C_\Delta = \max\{\| \delta \| : \delta \in \Delta \} \)).

Similarly, the length of the longest word in the table of \( \varphi \) is \( \leq C_\Delta n \).

We will prove that the two size measures (namely word length and table size) on elements of \( V \) are closely related. This similar to what happens in the symmetric groups \( S_k \), concerning the relation between \( k \) and the word length of permutations (over a bounded number of generators, with bound independent of \( k \)).
Theorem 3.8  The table size and word size of an element \( g \in V \) are related as follows:

1. There are \( c_\Delta, c'_\Delta > 0 \) (depending on the choice of \( \Delta \)) such that for all \( g \in V - \{1\} \):

\[
  c'_\Delta \|g\| \leq |g|_\Delta \leq c_\Delta \|g\| \cdot \log_2 \|g\|.
\]

2. For almost all \( g \in V \),

\[
  |g|_\Delta > \|g\| \cdot \log_2 |\Delta| \|g\|.
\]

“Almost all” means here that in the set \( \{g \in V : \|g\| = n\} \), the subset that does not satisfy the above inequality has a proportion that tends to 0 exponentially fast as \( n \to \infty \).

Inequality (2) shows that up to big-O, the function \( x \cdot \log x \) is the best possible upper bound in terms of \( \|g\| \).

However, although inequality (2) holds for “almost all” \( g \in V \), it also fails to hold for infinitely many \( g \in V \); for example, we will see in Proposition 3.10 below that for all \( g \in F \),

\[
  |g|_\Delta < c \|g\|.
\]

Proof of (2). The proof is a counting argument. The number of maximal prefix codes of cardinality \( n \) over the alphabet \( \{a, b\} \) is the Catalan number \( C_{n-1} \) (see the beginning of our Appendix A1).

If we count only elements of \( V \) with domain code \( \{a^i b : i = 0, 1, \ldots, n-2\} \cup \{a^{n-1}\} \), and an arbitrary fixed range code of cardinality \( n \), the number of elements of \( V \) obtained is \( \geq n(n-2)(n-2)! \). Note that the number is not \( n! \) because we want to make sure to count only maximal right-ideal isomorphisms; therefore, if we choose to map \( a^{n-1} \) to some word \( ua \), we cannot map \( a^{n-2}b \) to \( ub \), respectively \( ua \); thus only \( n-2 \) choices exist for the image of \( a^{n-2}b \).

Asymptotically, however, \( n(n-2)(n-2)! \) and \( n! \) are equivalent. Hence, the number of elements \( g \in V \) with \( \|g\| = n \) is at least \( C_{n-1} n! = \frac{\sqrt{2}(n-1)^{n-1}}{(n-1)!(n-2)!} n! \). By Stirling’s formula, this is equal to \( 2^{-1/2}(4/e)^{n-1}(n-1)^{n-1}(1+\varepsilon(n)) \) (where \( \lim_{n \to \infty} \varepsilon(n) = 0 \)).

For any \( \ell \), the number of words over \( \Delta^{\pm} \) of length \( \leq \ell \) is \( \leq \delta^{\ell+1} \), where \( \delta = 2 |\Delta| \). Hence, in \( V \) we have: The ratio of the number of elements that have word length \( \ell \leq n \cdot \log_\delta n \), over the number of elements that have table size \( n \), is less than

\[
  n^n \cdot 2^{1/2}(e/4)^{n-1}(n-1)^{1-n}(1+\varepsilon_1(n)) = 2^{1/2}(e/4)^{n-1}n (\frac{n}{n-1})^{n-1} \cdot (1+\varepsilon_1(n)) = 2^{1/2}(e/4)^{n-1}n e \cdot (1+\varepsilon_2(n)).
\]

This ratio tends to 0 exponentially fast as \( n \to \infty \) (since \( e < 4 \)). 

Proof of (1). We proved the first inequality of (1) already in Corollary 3.7. The proof of the second inequality consists of three steps. In summary:

1. We give a canonical factorization of any element of \( V \), as a right-ideal automorphism and two elements of \( F \).
2. We show that all elements of \( F \) have linearly bounded word length.
3. We prove that the word length of right-ideal automorphisms is \( \leq c \|g\| \cdot \log \|g\| \).

(1.1) Canonical factorization. At the end of Section 1 we already mentioned the subgroup \( F \), which consists of the elements of \( V \) that preserve the dictionary order of \( \{a, b\}^* \).
A right-ideal automorphism of a finitely generated essential right ideal \( P \{a, b\}^* \) (where \( P \) is a maximal prefix code) has a table whose domain code and range code are the same (namely \( P \)); the table gives a permutation of \( P \).

Contrary to a first impression, the set of right-ideal automorphisms (of all essential right ideals) is not a group, and it is not closed under restriction nor under extension.

**Proposition 3.9** For every \( n \geq 1 \) let us fix one maximal prefix code \( S_n \) of cardinality \( n \). Then for every \( g \in V - \{1\} \) there exist unique elements \( \alpha_g, \beta_g, \pi_g \in V \) such that

\[
g = \beta_g \pi_g \alpha_g,
\]

\( \alpha_g \) and \( \beta_g \) belong to \( F \),

\( \pi_g \) is an automorphism whose table is a permutation of \( S_{||g||} \).

Moreover, \( ||\alpha_g||, ||\beta_g||, ||\pi_g|| \leq ||g|| \).

**Proof.** Let \( ||g|| = n \). Consider a maximal right-ideal isomorphism that represents \( g \), and let \( \varphi : P_1 \rightarrow P_2 \) be its table, where \( P_1, P_2 \) are maximal prefix codes, \( |P_1| = |P_2| = n \).

We define \( \alpha_g \) by mapping \( P_1 \) in an order-preserving way bijectively onto \( S_n \); in other words, the table of \( \alpha_g \) is obtained by taking the elements of \( P_1 \) in increasing dictionary order as the domain, and by taking the elements of \( S_n \) in increasing dictionary order as the corresponding range. Similarly, \( \beta_g \) is defined by mapping \( S_n \) in an order-preserving way bijectively onto \( P_2 \). Uniqueness of \( \alpha_g \) and \( \beta_g \) follows from the fact that once the domain and image codes of elements of \( F \) are specified, the elements are uniquely determined. Finally, we simply let \( \pi_g = \beta_g^{-1} \varphi \alpha_g^{-1} \); hence, \( \pi_g \) is also uniquely determined. \( \square \)

The idea of a factorization of the above type appears in proofs of Thompson’s [11], where he uses the family of maximal prefix codes \( S_n = \{a^i b : 0 \leq i \leq n - 1\} \cup \{a^{n-1}\} \). This family has the following nice property (which Thompson does not mention or use, however): If one only considers automorphisms whose domain (and range) code is of the form \( S_n = \{a^i b : 0 \leq i \leq n - 2\} \cup \{a^{n-1}\} \) (for \( n \in \mathbb{N} \)), then this particular set of automorphisms (as \( n \) ranges over all integers \( > 1 \)) is a subgroup of \( V \), and this set of automorphisms is closed under extension and restriction. The only other family of maximal prefix codes with this property is \( \{b^i a : 0 \leq i \leq n - 2\} \cup \{b^{n-1}\} \). The reason is that for those two families of prefix codes there is only one place in a domain (and range) code, where extension of some automorphisms is possible (namely just above the deepest point in the prefix tree, where \( \{a^{n-1}, a^{n-2}b\} \) will be replaced by \( \{a^{n-2}\} \)); restriction (subject to the constraint that the code of the restriction should belong to this particular family of prefix codes) is also only possible at one place (again, at the deepest point in the prefix tree).

The above two families of prefix codes have a disadvantage for us: The depth of the prefix tree of such an \( S_n \) is \( n - 1 \); as a consequence we would get \( ||g||^2 \) in our theorem, instead of \( ||g|| \cdot \log ||g|| \). So, we will use the following family of codes, that have logarithmic depth.

As the domain code and range code of the table of \( \pi_g \) we pick a set \( S_n \subset \{a, b\}^{k-1} \cup \{a, b\}^k \), where \( |S_n| = n = ||g|| \), and \( k = \lceil \log_2 n \rceil \) (equivalently, \( 2^{k-1} < n \leq 2^k \)). When \( n = 2^k \), \( S_n = \{a, b\}^k \). When \( n < 2^k \), we choose \( S_n \) as in the figure below, representing the prefix tree of the maximal prefix code \( S_n \). The horizontal lines indicate the leaves of the prefix tree (i.e., the elements of \( S_n \)); the higher one of the two lines pictures the vertices at depth \( k - 1 \) (i.e., the
elements of $S_n$ in $\{a, b\}^{k-1}$, of which there are $2^k - n$), the lower horizontal line pictures the vertices at depth $k$ (i.e., the elements of $S_n$ in $\{a, b\}^{k}$, of which there are $2n - 2^k$).

Fig. 2: The maximal prefix code $S_n$

\((1.2)\) **Word length in** $F$. The subgroup $F$ is generated by the following two elements of $F$ (Thompson [40], [41]):

\[
\sigma = \begin{bmatrix} a^2 & ab & b \\ a & ba & b^2 \end{bmatrix}
\]

\[
\theta = \begin{bmatrix} a & ba^2 & bab & b^2 \\ a & ba & b^2a & b^3 \end{bmatrix}
\]

In [4], our $\sigma$ is called $A^{-1}$ (we changed the notation because we use $A$ to denote the alphabet); in Section 2 we saw that $\sigma$ can be interpreted as the shift operator on $\mathbb{Z}$. In [4], our $\theta$ is called $B^{-1}$.

**Proposition 3.10** For every $g \in F$ we have $\|g\|_{\{\sigma, \theta\}} < 4 \|g\|$.

In words: $F$ has linearly bounded word length.

**Proof.** Let $g \in F$ be represented by a table $\varphi : R \to S$, let $n + 1 = \|g\| = \|\varphi\|$, and let $X_i = \sigma^{-i+1}\theta^{-1}\sigma^{-i+1}$ for all $i \geq 1$, and $X_0 = \sigma^{-1}$. Cannon, Floyd and Parry [4] (Theorem 2.5, page 223) prove that

\[
g = X_0^{b_0}X_1^{b_1} \ldots X_i^{b_i}X_n^{-a_n} \ldots X_1^{-a_1}X_0^{-a_0}
\]

where $b_\ell$ ($0 \leq \ell \leq n$) is the length of the longest path in the prefix tree of $S$, subject to the following conditions:

- the path consists only of left-edges,
- the start vertex of the path is leaf number $\ell$ (the leaves are numbered from 0 through $n$),
- the end vertex of the path does not have a label in $b^*$ ($\subset \{a, b\}^*$).

Similarly, one defines $a_\ell$ ($0 \leq \ell \leq n$) for the prefix tree of $R$. (One observes that for the rightmost leaf $a_n = b_n = 0$, so the above expression could be simplified; but that doesn’t matter.)

By replacing each $X_i$ we obtain

13
\[ g = \sigma^{-b_0} \theta^{-b_1} \sigma \theta^{-b_2} \sigma \theta^{-b_3} \ldots \sigma \theta^{-b_{n-1}} \sigma \theta^{-b_n} a_n \sigma \theta^{a_{n-1}} \sigma \ldots \theta^{a_3} \sigma \theta^{a_2} \sigma \theta^{a_1} \sigma^{a_0} \]

hence, \( |g|_{(\sigma, \theta)} < \sum_{i=0}^{n} b_i + n + \sum_{i=0}^{n} a_i + n. \)

By the definition of \( b_i \), \( \sum_{i=0}^{n} b_i \) is less than the number of left-edges in the prefix tree of \( S \).

In a prefix tree over the alphabet \( \{a, b\} \) there is an equal number of left-edges and right-edges (since every vertex has 0 or 2 children). Moreover, the total number \( e \) of edges in a prefix tree with \( n + 1 \) leaves satisfies \( e = 2n \) (since such a tree has \( n \) interior vertices, and each interior vertex corresponds to two edges, and vice versa). Therefore, \( \sum_{i=0}^{n} b_i < n \) (and similarly, \( \sum_{i=0}^{n} a_i < n \)). The result follows. \( \square \)

(1.3) **Word length of right-ideal automorphisms.** Let us prove the claimed bound on the word length of all right-ideal automorphisms. Let \( \pi : S \rightarrow S \) be the table of any automorphism, where \( S \) is any finite maximal prefix code, and where \( \pi \) is a permutation of \( S \) (so here \( S \) is not necessarily of the form \( S_n \)). It is well known that every permutation of a finite set \( \{1, 2, \ldots, n\} \) can be expressed as the composition of \( \leq 3n \) transpositions of the form \( (1|i) \) with \( i \in \{1, 2, \ldots, n\} \). Indeed, we can first take disjoint cycles; and for a cycle we have \( (x_1|x_2|x_3|\ldots|x_{r-1}|x_r) = (x_1|x_r)(x_1|x_{r-1})\ldots(x_1|x_3)(x_1|x_2) \); finally, for a transposition, \( (i|j) = (1|i)(1|j)(1|i) \). Recall that all our functions and permutations are applied on the left of the argument.

We will write the automorphism \( \pi \) as the product of \( \leq 3 \|\pi\| \) transpositions of the form \( (a^k|w) \) with \( a^k, w \in S \); in particular, \( w \notin a^* \) (\( \subset \{a, b\}^* \)), and \( k \geq 1 \).

The transposition \( (a^k|w) \) is defined as follows. Let \( j (0 \leq j < k) \) be such that \( a^j b \) is a prefix of \( w \); such a \( j \) exists (and is unique) since \( w \notin a^* \) and since \( w \) is not prefix-comparable with \( a^k \). So \( w \) can be written as \( w = a^j bv \), for some \( v \in \{a, b\}^* \). Then \( (a^k|w) \) is defined by the table

\[
(a^k|w) = \left[ \begin{array}{cccc}
a^k & w & a^j b & a^j b p \ell \\
w & a^k & a^j b & a^j b p \ell \\
0 \leq i < k & p > v & i \neq j & p \ell \notin v \\
\end{array} \right]
\]

Here the range of \( i \) is \( 0 \leq i \leq k - 1 \) and \( i \neq j \). The word \( p \) ranges over all strict prefixes of \( v \); the notation \( x > y \) is short for \( x >_{\text{pref}} y \) and means that \( x \) is a strict prefix of \( y \) (as defined in the Introduction); \( \ell \in \{a, b\} \) is such that \( p \ell \) is \textit{not} a prefix of \( v \). For every strict prefix \( p \) there will be exactly one letter \( \ell \) such that \( p \ell \) is not a prefix of \( v \).

In our canonical factorization \( g = \beta_y \pi_g \alpha_g \), the automorphism \( \pi_g \) has a table which is a permutation of the maximal prefix code \( S_n \), where \( n = \|g\| \). We saw that all words in \( S_n \) have length \( \leq \lceil \log_2 n \rceil \). Also, \( \|\pi_g\| = |S_n| = n \). So, when we factor \( \pi_g \) as \( \leq 3n \) transpositions of the form \( (a^k|w) \), the parameters \( k \) and \( w \) satisfy \( k, |w| \leq \lceil \log_2 n \rceil \).

Therefore the next Lemma will complete the proof of Theorem 3.8.

**Lemma 3.11** Every transposition \( (a^k|w) \) has word length \( \leq c \cdot (k + |w|) \leq 2c \lceil \log_2 n \rceil \) over some finite set of generators of \( V \) (for some constant \( c > 0 \)).

**Proof.** We will use the following generators of \( V \): \( \sigma \) and \( \theta \) (the generators of \( F \) used before), and
\[ \gamma_1 = \begin{bmatrix} a^2 & aba & ab^2 & b \\ a^2 & ba & ab & b^2 \end{bmatrix} \]

\[ \gamma_2 = \begin{bmatrix} a^2 & aba & ab^2 & b \\ a^2 & ab & ba & b^2 \end{bmatrix} \]

\[ \delta = \begin{bmatrix} a^3 & ab & a^2 b & b \\ a^2 & ab & ba & b^2 \end{bmatrix} \]

\[ (a^2|ab) = \begin{bmatrix} a^2 & ab & b \\ ab & a^2 & b \end{bmatrix} \]

\[ (a^2|aba) = \begin{bmatrix} a^2 & aba & ab^2 & b \\ aba & a^2 & ab^2 & b \end{bmatrix} \]

\[ (ab|b) = \begin{bmatrix} a^2 & ab & b \\ a^2 & b & ab \end{bmatrix} \]

\[ (a|b) = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \]

**Case 1:** The transposition \((a^k|w)\) is such that \(w \in b\{a,b\}^*\) (i.e., \(w\) starts with \(b\)).

Recall that \(k,|w| \in [1, \lceil \log_2 n \rceil]\). We will eliminate Case 1 by showing: If \((a^k|w)\) is conjugated by at most \(k + |w|\) generators, a transposition of the form \((a^K|az)\) is obtained, where \(K + |az| \leq k + |w| + 1\). Indeed, we have:

\[ (1.1) \quad \sigma^{-1} \cdot (a^k|b^h av) \cdot \sigma = (a^{k+1}|b^{h-1}av) \quad \text{for all} \quad h \geq 2, \quad v \in \{a, b\}^*, \quad k \geq 1; \]

\[ (1.2) \quad \sigma^{-1} \cdot (a^k|bav) \cdot \sigma = (a^{k+1}|abv) \quad \text{for all} \quad v \in \{a, b\}^*, \quad k \geq 1; \]

\[ (1.3) \quad \sigma^{-1} \cdot (a^k|b^h) \cdot \sigma = (a^{k+1}|b^{h-1}) \quad \text{for all} \quad h \geq 2, \quad k \geq 1; \]

\[ (1.4) \quad (ab|b) \cdot (a^k|b) \cdot (ab|b) = (a^k|ab) \quad \text{for all} \quad k \geq 2. \]

(When \(k = 1, (a|b)\) is a generator.)

For the detailed verification of (1.1), (1.2), (1.3), and (1.4), see Appendix A2.

If the transposition \((a^k|w)\) is in situation (1.1), i.e., \(w = b^h av\) with \(h \geq 2\), then after \(h - 1\) conjugations we are in situation (1.2). In situation (1.2), one conjugation gets us out of Case 1. If the transposition \((a^k|w)\) is in situation (1.3), i.e., \(w = b^h\) with \(h \geq 2\), then after \(h - 1\) conjugations we are in situation (1.4). In situation (1.4), one conjugation gets us out of Case 1. Thus we have eliminated Case 1.

**Case 2:** The transposition \((a^k|w)\) is such that \(w \in a\{a, b\}^*\) (i.e., \(w\) starts with \(a\)).
Since \( a^k \) and \( w \) belong to the same maximal prefix code they are not prefix-comparable; therefore, \( w \) is of the form \( w = a^j b^h v \) where \( k > j \geq 1 \) (hence \( k \geq 2 \)), \( h \geq 1 \), \( v \in \{a, b\}^* - b\{a, b\}^* \) (\( v \) does not start with \( b \)).

We have:

(2.1) \( \text{If } (k >) \ j \geq 2, \ h \geq 1, \ \text{then } \sigma \cdot (a^k | a^j b^h v) \cdot \sigma^{-1} = (a^{k-1} | a^{j-1} b^h v). \)

(2.2) \( \text{If } j = 1, \ k \geq 3, \ h \geq 1, \ \text{then } \delta \cdot (a^k | a b^h v) \cdot \delta^{-1} = (a^{k-1} | a b^h v). \)

(2.3) \( \text{If } j = 1, \ k = 2, \ h \geq 2, \ \text{then } \gamma_1 \cdot (a^k | a b^h v) \cdot \gamma_1^{-1} = (a^2 | a b^{h-1} v). \)

(2.4) \( \text{If } j = 1, \ k = 2, \ h = 1, \ |v| \geq 2 \) (so \( v = au \) for some \( u \in \{a, b\} \{a, b\}^* \)), then

\[ \gamma_2 \cdot (a^2 | abau) \cdot \gamma_2^{-1} = (a^2 | abu). \]

(2.5) \( \text{If } j = 1, \ k = 2, \ h = 1, \ |v| \leq 1, \ \text{then } (a^2 | aba) \text{ and } (a^2 | ab) \text{ are generators.} \)

The detailed verification of (2.1) – (2.4) appears in Appendix A2.

If the transposition \( (a^k | w) \) is in situation (2.1), then after \( j - 1 \) steps we reach situation (2.2); in each step, both \( k \) and \( |w| \) decrease. If the transposition \( (a^k | w) \) is in situation (2.2), then after \( k - 2 \) steps (during which \( k \) keeps decreasing while \( w \) remains unchanged) we reach situation (2.3). If the transposition \( (a^2 | w) \) is in situations (2.3) or (2.4), we remain in situations (2.3) or (2.4) as long as \( w \) is of the form \( ab^h v \) (with \( h \geq 2 \)) or \( abau \); at each step, \( w \) becomes shorter. Eventually we reach case (2.5).

This completes the proof of Lemma 3.11 and hence of Theorem 3.8. \( \square \)

4 The word problem of \( V \)

It is fairly obvious from the representation of \( V \) by partial functions that the word problem is decidable. In this section we show that the word problem of \( V \) has low complexity.

Since \( V \) is finitely generated, we can consider the word problem of this group; let \( \Delta \) be a finite set of generators of \( V \). As an input for the word problem we consider a string over the alphabet \( \Delta \); moreover, given \( s \) and \( t = sz \in A^* \), an \( AC^0 \) circuit can output \( z \).

One can prove much more detailed and stronger complexity results. For this we will use the parallel complexity classes \( AC^0 \) and \( AC^1 \), which are subclasses of \( P \). In short, \( AC^k \) (for \( k \in \mathbb{N} \)) consists of those problems whose output can be computed by acyclic boolean circuits (where the gates have unbounded finite fan-in), of depth \( O((\log n)^k) \), and of size polynomial in \( n \) (where \( n \) is the input length). See [42, 43] and [44] for details.

It is well known that the prefix relation between two words can be checked by an \( AC^0 \) circuit; moreover, given \( s \) and \( t = sz \in A^* \), an \( AC^0 \) circuit can output \( z \).
Let us define the problems precisely.

(1) **Composition problem**

**Input:** Two isomorphisms between essential right ideals $\varphi : P_1 A^* \to Q_1 A^*$, and $\psi : P_2 A^* \to Q_2 A^*$, where $P_1, Q_1, P_2, Q_2$ are maximal prefix codes; $\varphi$ is described by its finite table $\{(p, \varphi(p)) : p \in P_1\} \subseteq P_1 \times Q_1$, and similarly for $\psi$.

**Output:** The composite $\psi \circ \varphi$, described by a finite table.

(2) **Word problem of $V$**

**Input:** A sequence $(\psi_1, \ldots, \psi_n)$ (where $n \in \mathbb{N}$ is also variable) of isomorphisms between essential right ideals $\psi_i : P_i A^* \to Q_i A^*$, where $P_i, Q_i$ are maximal prefix codes ($i = 1, \ldots, n$); each $\psi_i$ is described by its finite table $\{(p, \psi_i(p)) : p \in P_i\} \subseteq P_i \times Q_i$.

**Output:** “Yes” if $\psi_1 \cdot \ldots \cdot \psi_n = 1$ (i.e., the product in $V$ is the identity map on $A^*$); “no” otherwise.

Note that this definition of the word problem is consistent with the usual definition. Since $V$ is finitely generated the usual definition of the word problem can be applied to any finite set of generators. By Corollary 3.7 the length of the description of $(\psi_1, \ldots, \psi_n)$ by finite tables is linearly bounded by the length of $(\psi_1, \ldots, \psi_n)$ as words over a fixed finite set of generators of $V$. Moreover, the description of the input of the word problem by tables is not significantly more compact than a description by a word over generators, by Theorem 3.8.

**Theorem 4.1** (1) The composition problem (for two isomorphisms between essential right ideals) is in $\text{AC}^0$.

(2) The word problem of $V$ is in $\text{AC}^1$.

**Proof.** (1) Let $\varphi = \{(x_i, y_i) : i = 1, \ldots, n\}$ and $\psi = \{(u_j, v_j) : j = i, \ldots, m\}$. For an $\text{AC}^0$ algorithm, we consider all $nm$ individual composites $(x_i, y_i) \circ (u_j, v_j)$ in parallel. Each composite $(x_i, y_i) \circ (u_j, v_j)$ can be computed by a constant-depth circuit $C_{i,j}$ by checking the prefix-relation between $x_i$ and $v_j$; if $x_i$ and $v_j$ are not prefix-comparable, the circuit $C_{i,j}$ outputs 0; if $x_i = v_j z \preceq_{\text{pref}} v_j$ (for some $z \in A^*$), the circuit $C_{i,j}$ outputs $(u_j z, y_i)$; if $x_i \succeq_{\text{pref}} v_j = x_i z$ (for some $z \in A^*$), the circuit $C_{i,j}$ outputs $(u_j, y_i z)$. The size of $C_{i,j}$ is $O(\max\{|x_i|, |v_j|\})$, hence $O(n)$ by Lemma 3.6.

Finally, the overall circuit consists of all the $C_{i,j}$ ($1 \leq i \leq n, 1 \leq j \leq m$), and an additional layer which masks outputs that are 0, and just outputs the set of non-zero outputs of the circuits $C_{i,j}$. The overall size of the circuit is

$$O(n \cdot m \cdot \max\{|x_i|, |y_i|, |v_j|, |u_j| : 1 \leq i \leq n, 1 \leq j \leq m\}),$$

which corresponds to big-O of the cube of the input length.

(2) We are given a sequence $(\psi_1, \ldots, \psi_n)$ of isomorphisms between essential right ideals; let $N$ be the total length of this input (when each $\psi_i$ is described by a finite table). We compose the $\psi_i$ two-by-two in parallel ($\psi_1$ with $\psi_2$, $\psi_3$ with $\psi_4$, etc.); then we compose the resulting $[n/2]$ functions two-by-two, then the resulting $[n/4]$, etc.; we obtain a “composition tree” of depth $[\log n]$, with $n - 1$ nodes. By Proposition 3.8 and Lemma 3.6 the intermediate and the final composites have finite tables of space $O(N^2)$ each. By (1) above, each node of the composition tree can be implemented by a constant-depth circuit of size $O((N^2)^3)$. So the total size of the circuit is $O(nN^6)$, and the depth is $O(\log n)$.
Finally, once we have a finite table for the composite \( \psi_1 \ldots \psi_n \) we can check easily (in constant depth) whether this function is a partial identity. \( \square \)

Since the word problem of \( V \) is in \( \text{AC}^1 \) it follows (see [42]) that it is also in \( \text{DSpace}(\log^2 n) \).

We saw already that the word problem of \( V \) is in \( \text{P} \). The next proposition gives more precise bounds on the time complexity; the bounds for deterministic and nondeterministic time are still rather crude. Here, \( \text{“coNTTime”, “NTime”, and “DTIME” refer to the usual time complexity classes (co-nondeterministic, nondeterministic, and deterministic time complexity, respectively; see [12]).} \)

**Proposition 4.2** The word problem of \( V \) is in \( \text{coNTime}(n) \), in \( \text{NTime}(n^2) \), and in \( \text{DTIME}(n^3) \).

**Proof.** Let \( \Delta \) be a finite generating set of \( V \), let \( w \in (\Delta^{\pm 1})^* \), let \( n = |w| \), and let \( \varphi_w \) be the right-ideal isomorphism obtained by composing the generators as they appear in \( w \), without taking maximum extensions. Obviously, \( w = 1 \) in \( V \) iff \( \varphi_w \) is a partial identity. Recall (Corollary 3.7) that \( \|\varphi_w\| \leq C_\Delta n \), and that the length of the longest word in the table of \( \varphi_w \) is \( \leq C_\Delta n \). Hence the sum of the lengths of the words in the table of \( \varphi_w \) is \( \leq C_\Delta^2 n^2 \).

1. The word problem of \( V \) is in \( \text{coNTime}(n) \) iff the negation of the word problem is in \( \text{NTime}(n) \). To prove the latter we consider a nondeterministic two-tape Turing machine, one tape containing the input \( w \); on the other tape the machine guesses a word \( z \in \{a,b\}^* \) such that \( z \) belongs to the domain code of \( \varphi_w \) (so \( |z| \leq C_\Delta n \)), and \( \varphi_w(z) \neq z \). Such a word \( z \) exists iff \( w \neq 1 \) in \( V \). Moreover, \( |z| \leq C_\Delta n \), hence it takes only linear time to write \( z \) on the second tape.

   On tape 1 the machine also makes a copy of \( z \).

   Next, the machine reads \( w \) from right to left, and applies the generators to the word \( (\in \{a,b\}^*) \) on tape 2 (initially that word is \( z \)). Each application of a generator changes a prefix of bounded length of the word on tape 2. (This bound is the length of the longest word in the table of any generator \( \in \Delta \)). Hence, applying one generator takes only a bounded amount of time, and the total time to apply all the generators in \( w \) is linear in \( |w| \); hence the time is \( \leq cn \) for some constant \( c \). The final content of tape 2 will be \( \varphi_w(z) \).

   Finally, the machine compares the original copy of \( z \) (saved on tape 1) with \( \varphi_w(z) \); this takes linear time. It accepts if these two words are different.

2. To prove that the word problem of \( V \) is in \( \text{NTime}(n^2) \) we consider a nondeterministic Turing machine which guesses the table of a partial identity map, and then verifies that the guess is indeed the table of \( \varphi_w \). The machine accepts \( w \) if the verification succeeds. Such an accepting computation exists iff \( w = 1 \) in \( V \).

   More precisely, the Turing machine guesses words \( x_i \in \{a,b\}^* \), each of length \( |x_i| < \|\varphi_w\| (\leq C_\Delta n) \), in such a way that \( \{x_i : 1 \leq i \leq \|\varphi_w\| \leq C_\Delta n \} \) is a maximal prefix code. This is done by means on a nondeterministic depth-first search in the tree of \( \{a,b\}^* \). This search starts at the root of the tree and proceeds like an ordinary depth-first search, except that we guess where the leaves of the prefix tree of the code \( \{x_i : 1 \leq i \leq \|\varphi_w\| \} \) are; each time we reach such a leaf we write down the corresponding word \( x_i \). Since the set of the guessed \( x_i \) consists of the leaves of a subtree of \( \{a,b\}^* \), it is a prefix code; since the depth-first search will not end until it has visited all the leaves of a tree, this prefix code is maximal. The running time of the search is
proportional to the number of leaves, which is $\|\varphi_w\|$ (in an accepting computation). The time to write down all the $x_i$ is $< \|\varphi_w\|^2$ (since $|x_i| < \|\varphi_w\|$ in case of acceptance). Thus, we can guess the table of a partial identity of table size $\|\varphi_w\|$ in time $O(n^2)$.

Next, the Turing machine verifies, deterministically, that $\varphi_w(x_i) = x_i$ for each $i$. This is done in the same way as in (1), and takes linear time for each $x_i$. Hence the total time of a successful verification is $O(n^2)$.

(3) Let us first look at the deterministic time complexity of composition (without maximum extension) of two elements $\varphi, \psi \in V$, described by their tables. Let $\varphi = \{(s_i, t_i) : i = 1, \ldots, \ell\}$, hence by Lemma 3.6 $|t_i|, |s_i| < \|\varphi\| = \ell$. Let $\psi = \{(u_j, v_j) : j = 1, \ldots, m\}$, hence $|v_j|, |u_j| < \|\psi\| = m$. Suppose the two tables are written on two different tapes of a deterministic Turing machine. The machine can then write the set $\{(u_j, v_j) \circ (s_i, t_i) : 1 \leq i \leq \ell, 1 \leq j \leq m\}$ on a third tape and reduce each term $(u_j, v_j) \circ (s_i, t_i)$ to either 0 (in which case it is removed from the set), or to a new term of the from $(x_k, y_k)$. This takes time $\leq c_1 \|\varphi\|^2 \|\psi\|^2$ for some constant $c_1$.

Finally, we can compute the table of $\varphi_w = d_n \ldots d_k \ldots d_2 d_1$ (with $d_k \in \Delta^{\pm 1}$) by composing from right to left. Recall that $C_\Delta$ be the maximum table size of any generator in $\Delta$. Inductively, assume the table of $d_k \ldots d_2 d_1$ is $\psi$, with $\|\psi\| \leq C_\Delta k$. Then the table for $d_{k+1} \psi$ can be computed in time $\leq c_1 \|d_{k+1}\|^2 \|\psi\|^2 \leq c_1 C_\Delta^2 C_\Delta^3 k^2$. So the total time to compute the table of $w$ is $\leq c_1 C_\Delta^4 \sum_{i=1}^n k_i^2 = O(n^3)$. $\square$

5 Generalized word problems of $V$

Generalized word problems of a group $G$ ask about membership of elements of $G$ in specified subgroups; the word problem is the special case obtained by taking the trivial subgroup \{1\}. Let $G$ be a group with finite generating set $A$. If $S \subseteq G$ and $w \in (A^{\pm 1})^*$, we write $w \in_G S$ if the element of $G$ represented by $w$ is in $S$. If $x_1, \ldots, x_n \in (A^{\pm 1})^*$, then $\langle x_1, \ldots, x_n \rangle_G$ denotes the subgroup of $G$ generated by the elements of $G$ represented by the words $\{x_1, \ldots, x_n\}$. If $S \subseteq G$ then $\langle S \rangle_G$ denotes the subgroup of $G$ generated by $S$.

By definition, the **generalized word problem** of a fixed group $G$ with a fixed finite generating set $A$, is specified as follows:

**INPUT:** A finite set $X \subset (A^{\pm 1})^*$, and an additional “test word" $y \in (A^{\pm 1})^*$.

**QUESTION:** $y \in_G \langle X \rangle_G$ (i.e., does the element of $G$ represented by the word $y$ belong to the subgroup of $G$ generated by the elements represented by the words in $X$)?

We consider two special forms of the generalized word problem.

(1) The generalized word problem of a fixed group $G$, with a fixed finite generating set $A$ and a fixed subgroup generating set $X = \{x_1, \ldots, x_k\} \subset (A^{\pm 1})^*$, is specified as follows:

**INPUT:** A word $y \in (A^{\pm 1})^*$.

**QUESTION:** $y \in_G \langle x_1, \ldots, x_k \rangle_G$?

(2) The generalized word problem of a fixed group $G$ with a fixed finite generating set $A$ and a fixed “test word" $y \in (A^{\pm 1})^*$, is specified as follows:

**INPUT:** A finite set $X \subset (A^{\pm 1})^*$.

**QUESTION:** $y \in_G \langle X \rangle_G$?
We will use the next few facts to obtain undecidability results for the generalized word problem of \( V \) and its special versions.

Graham Higman proved that Thompson’s group \( V \) contains a subgroup isomorphic to \( \text{FG}_2 \) (the free group on two generators). A proof appears in the Appendix of [37].

Thompson [41] proved that if \( G \) is a subgroup of \( V \) then \( V \) also contains a subgroup isomorphic to the direct product \( G \times G \). Hence, \( V \) contains all finite direct powers of \( G \). Thompson [41] (and also Higman [14]) mention that this can be generalized to any finite direct product and to countably infinite direct sums. The direct sum of a family of groups \( (G_i : i \in I) \) is defined to be the subgroup of the direct product \( \prod_{i \in I} G_i \) consisting of the sequences that have only a finite number of non-identity components.

As a consequence of these results of Thompson and Higman we obtain: \( V \) contains a subgroup isomorphic to \( \text{FG}_2 \times \text{FG}_2 \).

We will also need to talk about “uniform word problems”, and related problems for Turing machines.

- The **uniform word problem** for groups is specified as follows.
  **Input:** A finite presentation \( \langle B, R \rangle \) of a group, and a word \( w \in (B\pm1)^* \) (where letters in the alphabet \( B \) are encoded over some fixed finite alphabet).
  **Question:** \( w =_{\langle B, R \rangle} 1 \)? i.e., is the element of the group \( \langle B, R \rangle \) represented by \( w \) the identity?

- The **uniform word problem for a fixed word** \( w_0 \in (B_0\pm1)^* \) (where \( B_0 \) is a fixed finite alphabet), is specified as follows.
  **Input:** A finite presentation \( \langle B, R \rangle \), with \( B_0 \subseteq B \) (where letters in the alphabet \( B \) are encoded over some fixed finite alphabet).
  **Question:** \( w_0 =_{\langle B, R \rangle} 1 \)?

- The **acceptance problem** is specified as follows.
  **Input:** A Turing machine \( M \) and a word \( w \) over the tape alphabet of \( M \) (where letters and states of \( M \) are encoded over some fixed finite alphabet).
  **Question:** Does \( M \) accept \( w \)?

- The **fixed-word acceptance problem** for a fixed word \( w_0 \in B_0^* \) (where \( B_0 \) is a fixed finite alphabet), is specified as follows.
  **Input:** A Turing machine \( M \) whose tape alphabet contains \( B_0 \) (such that letters and states of \( M \) are encoded over some fixed finite alphabet).
  **Question:** Does \( M \) accept \( w_0 \)?

**Lemma 5.1** There are infinitely many words \( w_0 \) such that the uniform word problem with fixed word \( w_0 \) is undecidable.

**Proof.** By Rice’s theorem, the acceptance problem for any fixed word \( w_0 \) (and variable Turing machine \( M \)) is undecidable. By Boone’s Lemma (see Lemma 12.7 in [33]), there is a computable function with linear time complexity which maps \( (w, M) \) (where \( M \) is a Turing machine and \( w \) is a word over the tape alphabet of \( M \)) to a finite presentation \( G_M = \langle B_M, R_M \rangle \) and a word \( \rho(w) \in (B_M^{\pm1})^* \) such that: \( M \) accepts \( w \) iff \( \rho(w) =_{G_M} 1 \).

Hence, for any fixed word \( \rho(w_0) \), the fixed-word uniform word problem of \( G_M \) is undecidable.

\( \square \)
Proposition 5.2  For $V$ (with any fixed finite generating set $\Delta$), the generalized word problem is undecidable.

The first special form of the generalized word problem is undecidable for some fixed subgroups.
The second special form of the generalized word problem is undecidable for some fixed non-empty test words.

Proof.  We saw that $V$ contains $\text{FG}_2 \times \text{FG}_2$. Therefore we can apply Mikhailova’s theorem \[25\] (see Theorem IV.4.3 in \[19\]), which states that the generalized word problem, as well as its first special form (for some fixed finite subgroup generator sets $X$), are undecidable for $\text{FG}_2 \times \text{FG}_2$.

Moreover, it follows from the proof of Mikhailova’s theorem (see Lemma IV.4.2 in \[19\]) that the second special form of the generalized word problem (for some fixed test words) of $\text{FG}_2 \times \text{FG}_2$ is also undecidable. Indeed, this proof gives a reduction of the uniform word problem “is $w =_H 1$?” (where $H$ is a variable finitely presented group, and $w$ is a variable word) to the generalized word problem of $\text{FG}_2 \times \text{FG}_2$, with some subgroup generating set $L_H$ (where $L_H$ is finite). We saw in Lemma \[7.1\] that the uniform word problem for a fixed word (but variable finite presentations) is undecidable.

Hence, the generalized word problem and its two special forms are undecidable for any group containing $\text{FG}_2 \times \text{FG}_2$, and in particular for $V$. $\square$

Other decision problems

Graham Higman (Theorem 9.3 in \[14\]) proved that the conjugacy problem and the order problem of $V$ are decidable. See also \[38\], \[37\].

Open problem: Is the generation problem of $V$ decidable? (The generation problem is specified as follows: The input is a finite set $\Gamma$ of elements of $V$, given by their tables; the question is whether $\Gamma$ generates all of $V$.)

6  Distortion

We just saw that in some (in fact, infinitely many) cases, the first special form of the generalized word problem of $V$ (for a fixed subgroup) is undecidable. This leads to the question: What can the complexity of this problem be when the problem is decidable?

In relation to the first special form of the generalized word problem (for a fixed subgroup), the “Cayley graph distortion function” will play the role of inherent (group-theoretic) complexity. We will see that it is closely connected with nondeterministic time complexity. Let $G$ be a fixed group with fixed finite generating set $A$, and let $X \subset (A^{\pm 1})^*$ be a fixed finite set, generating a subgroup $H = \langle X \rangle_G$. If $y \in (A^{\pm 1})^*$ is such that $y \in_G \langle X \rangle_G$, then $y$ has two lengths, namely, one over the alphabet $A^{\pm 1}$ and one over $X^{\pm 1}$. How the two lengths are related is an important question, first addressed by Gromov \[10\].

Definition 6.1  (1) For a group $G$ with finite generating set $A$, and an element $g \in G$, the $A$-length of $g$, denoted by $|g|_A$ is the minimum length of any word $y \in (A^{\pm 1})^*$ that represents $g$.

In the Cayley graph $\Gamma(G, A)$ of $G$ with generating set $A$, $|g|_A$ is the distance from the root (i.e., the identity element of $G$) of the element $g$.
(2) Let $X$ be a finite subset of $(A^{\pm 1})^*$ and let $H = \langle X \rangle_G$ be the subgroup of $G$ generated by $X$. If $y \in_G \langle X \rangle_G$, we define the $X$-length, denoted by $|y|_X$, to be the length of a shortest sequence $(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ of elements of $X^{\pm 1}$ such that $y =_G x_{i_1}x_{i_2}\cdots x_{i_k}$.

In the Cayley graph $\Gamma(G,A)$, the subgroup $H$ with generating set $X$ generates a path-subgraph, whose vertex set is $H$, and whose edges correspond to $X$-labeled paths between vertices in $H$. This path-subgraph is the image of a path-embedding of the Cayley graph $\Gamma(H,X)$ into $\Gamma(G,A)$. (If $X \subseteq A$, then $\Gamma(H,X)$ is a subgraph of $\Gamma(G,A)$.)

The Cayley graph distortion function (we’ll call it simply “distortion function”) describes the relation between the two lengths of an element $y$ of the subgroup $\langle X \rangle_G$ of $G$.

**Definition 6.2** Let us fix a group $G$ with finite generating set $A$, and a finite set $X \subset (A^{\pm 1})^*$. A non-decreasing function $f : \mathbb{N} \rightarrow \mathbb{N}$ is a distortion function of the subgroup $H = \langle X \rangle_G$ in $G$ iff for all elements $g \in \langle X \rangle_G$ we have: $|g|_X \leq f(|g|_A)$.

Note that according to our definition, distortion functions are non-decreasing (i.e., if $m \leq n$ then $f(m) \leq f(n)$). The minimum distortion function for a given $G$, $A$, and $X$ (as above) is called “the” distortion function.

Two functions $f_1$ and $f_2 : \mathbb{N} \rightarrow \mathbb{N}$ are said to be equivalent (or more precisely, linearly equivalent) iff there exist positive constants $c_0$, $C_{12}$, $c_{12}$, $C_{21}$, $c_{21}$ such that for all $n \geq c_0$: $f_1(n) \leq C_{12}f_2(c_{12}n)$ and $f_2(n) \leq C_{21}f_1(c_{21}n)$. In big-O notation, $f_1$ and $f_2$ are linearly equivalent iff $f_1(n) = O(f_2(O(n)))$ and $f_2 = O(f_1(O(n)))$.

The concept of distortion was formally introduced by Gromov [10]. However, Gromov’s definition used an additional factor $n$; so, Gromov’s distortion is constant when ours is linear. We follow Ol’shanskii and Sapir [30], whose theorem connecting distortion to Dehn functions (stated below) makes Gromov’s version of the definition look less well motivated.

It is easy to see the following: When one changes the generating set $A$ of $G$ to another finite generating set (of the same group $G$), and one changes the set $X$ that generates the subgroup $H = \langle X \rangle_G$ to another finite set (that generates the same subgroup $H$), then the minimum distortion does not change (up to linear equivalence); see e.g., [8]. Hence for finitely generated groups the distortion depends only on the groups $G$ and $H \subseteq G$ (up to linear equivalence).

The distortion cannot be less than linear, since (up to finite change of generators) one can increase $A$ so that $X \subseteq A$. When the distortion is linear one also says that “there is no distortion”, or that $H$ is “isometrically embedded” in $G$.

In [2], [29] and [3] it was proved that the Higman embedding theorem for semigroups, respectively groups, can be strengthened in such a way that the distortion is linear. Another important result about distortion functions is the following theorem of Ol’shanskii and Sapir [30]: The set of distortion functions of finitely generated subgroups of $FG_2 \times FG_2$ coincides (up to linear equivalence) with the set of all Dehn functions of finitely presented groups. Guba and Sapir [12] proved that for any integer $d \geq 2$ the Thompson group $F$ (in the notation of [11]) has a subgroup with distortion $\geq n^d$ (up to linear equivalence).

**Theorem 6.3** The set of distortion functions of the finitely generated subgroups of the Thompson group $V$ contains (up to linear equivalence) the set of all Dehn functions of finitely presented groups.
Proof. This is an immediate consequence of the Ol’shanskii-Sapir theorem and Theorem 6.4.

\[ \square \]

**Theorem 6.4** (1) Thompson’s group \( V \) contains a subgroup isomorphic to the free group \( \mathbb{F}_2 \) with linear distortion.

(2) The group \( V \) also contains a subgroup isomorphic to \( \mathbb{F}_2 \times \mathbb{F}_2 \) with linear distortion.

**Proof of (1).** We start out from Graham Higman’s result (mentioned earlier), that Thompson’s group \( V \) contains a subgroup isomorphic to \( \mathbb{F}_2 \). In the Appendix of [37] it is shown that the following two elements of \( V \) generate a free group:

\[ \alpha = \begin{bmatrix} a & b^3 & b^2ab & b^2a^3 & b^2a^2b & ba \\ b^4a & b^3a & b^2a & a & ba & b^5 \end{bmatrix} \]

\[ \beta = \begin{bmatrix} b & ab & a^2b^2 & a^2ba^2 & a^2bab & a^3 \\ a^3ba & a^3b^2 & a^2b & b & ab & a^4 \end{bmatrix} \]

Our main goal now is to show that the free subgroup generated by \( \{ \alpha, \beta \} \) has linear distortion in \( V \).

Let \( \mu \) be any element of \( \langle \alpha, \beta \rangle \), with \( \mu \neq 1 \). If we write \( \mu \) in such a way that there are no cancellations then \( \mu \) has one of the following expressions:

- **Case** \( \alpha \alpha \): \( \mu = \alpha^{k_m} \beta^{k_{m-1}} \cdots \beta^{k_2} \alpha^{h_1} \alpha^h \beta^{k_1} \alpha^1 \),
- **Case** \( \beta \alpha \): \( \mu = \beta^{h_m} \alpha^{h_{m-1}} \cdots \beta^{h_2} \alpha^{k_1} \alpha^h \beta^{k_1} \alpha^1 \),
- **Case** \( \alpha \beta \): \( \mu = \alpha^{k_m} \beta^{k_{m-1}} \cdots \beta^{k_2} \alpha^{h_1} \alpha^h \beta^{k_1} \alpha^1 \),
- **Case** \( \beta \beta \): \( \mu = \beta^{h_m} \alpha^{h_{m-1}} \cdots \beta^{h_2} \alpha^{k_1} \alpha^h \beta^{k_1} \alpha^1 \).

with \( h_m, k_{m-1}, h_{m-1}, \ldots, k_2, h_2, k_1 \in \mathbb{Z} - \{0\} \), \( k_m \in \mathbb{Z} - \{0\} \) in cases \( \beta \alpha \) and \( \beta \beta \), \( k_m = 0 \) in cases \( \alpha \alpha \) and \( \alpha \beta \), \( h_1 \in \mathbb{Z} - \{0\} \) in cases \( \alpha \alpha \) and \( \alpha \beta \), \( h_1 = 0 \) in cases \( \alpha \beta \) and \( \beta \beta \).

Since \( \langle \alpha, \beta \rangle \) is a free group and \( \mu \) is reduced, the \( \{ \alpha, \beta \} \)-length of \( \mu \) is \( |\mu|_{\{\alpha, \beta\}} = |k_m| + |h_m| + \cdots + |k_1| + |h_1| \).

In order to prove that the subgroup \( \langle \alpha, \beta \rangle \) of \( V \) has linear distortion, we need to show that the minimum length of \( \mu \) over some fixed finite generating set \( \Delta \) of \( V \) is \( |\mu|_\Delta \geq c \cdot |\mu|_{\{\alpha, \beta\}} \) (for some constant \( c > 0 \) depending only on the chosen set of generators \( \Delta \) of \( V \)).

We will use the following method. We will show that for any \( \mu \) there is a word \( y \in \{a, b\}^* \) such that

**Case** \( (*\alpha) \). If \( \mu \) is in cases \( \alpha \alpha \) or \( \beta \alpha \), then \( y \) satisfies:

- \( \mu(a) = ya \),
- \( \mu(ba) = yb \),
- \( |y| > |\mu|_{\{\alpha, \beta\}} \).

**Case** \( (*\beta) \). If \( \mu \) is in cases \( \alpha \beta \) or \( \beta \beta \), then \( y \) satisfies:

- \( \mu(b) = ya \),
- \( \mu(ab) = yb \),
- \( |y| > |\mu|_{\{\alpha, \beta\}} \).
The existence of $y$, as above, implies that the table of the maximum extension of $\mu$ contains an entry of length $> |\mu|_{(\alpha, \beta)}$. Indeed, $ya$ and $yb$ are such entries (in the range code of $\max \mu$); by Lemma 22, $ya$ and $yb$ remain in the table when $\mu$ is maximally extended, since $ya$ and $yb$ are the images of $a$, resp. $ba$ (or $b$ resp. $ab$).

Now, by Lemma 3.6, the table size of the maximum extension of $\mu$ satisfies $\|\max \mu\| > |\mu|_{(\alpha, \beta)}$. Also, by Proposition 3.5, $\|\max \mu\| \leq \|\mu\|$.

By Corollary 3.7, $\|\max \mu\| \leq \|\mu\| \leq C_\Delta |\mu|_\Delta$, therefore by the above, $|\mu|_{(\alpha, \beta)} < C_\Delta |\mu|_\Delta$.

So the proof of Theorem 6.4 will be complete once we prove the following two claims, which show that the appropriate $y$ exists.

Claim $(\ast\alpha)$. Let $\mu$ be as above, according to cases $\alpha\alpha$ or $\beta\alpha$. Then,

$$\mu(a) = (w_m v_m^{[k_m]-1} t_m u_m^{[h_m]-1} w_{m-1} \ldots w_i v_i^{[k_i]-1} t_i u_i^{[h_i]-1} \ldots w_1 v_1^{[k_1]-1} t_1 u_1^{[h_1]-1} w_0$$

where:

$$w_0 = \begin{cases} ba & \text{if } h_1 > 0 \\ a^2 & \text{if } h_1 < 0 \end{cases} \quad (\text{Note that } w_0 \text{ ends in } a.)$$

For $m - 1 \geq i \geq 1$: $w_i = \begin{cases} ba^3 & \text{if } h_{i+1} > 0, k_i > 0 \\ ba^2 b & \text{if } h_{i+1} > 0, k_i < 0 \\ a^4 & \text{if } h_{i+1} < 0, k_i > 0 \\ a^3 b & \text{if } h_{i+1} < 0, k_i < 0 \end{cases}$

In case $\beta\alpha$: $w_m = \begin{cases} a^3 & \text{if } k_m > 0 \\ a^2 b & \text{if } k_m < 0 \end{cases}$

For $m - 1 \geq i \geq 1$, or $i = m$ in case $\beta\alpha$: $t_i = \begin{cases} bab^2 & \text{if } h_i > 0, k_i > 0 \\ a^2 b^2 & \text{if } h_i > 0, k_i < 0 \\ baba & \text{if } h_i < 0, k_i > 0 \\ a^2 b a & \text{if } h_i < 0, k_i < 0 \end{cases}$

In case $\alpha\alpha$: The factor $(w_m v_m^{[k_m]-1})$ is absent, and $t_m = \begin{cases} b^3 & \text{if } h_m > 0 \\ b^2 a & \text{if } h_m < 0 \end{cases}$

For $m \geq i \geq 1$: $u_i = \begin{cases} a & \text{if } h_i > 0 \\ b & \text{if } h_i < 0 \end{cases}$ $v_i = \begin{cases} a & \text{if } k_i > 0 \\ b & \text{if } k_i < 0 \end{cases}$

We also have:

$$\mu(ba) = (w_m v_m^{[k_m]-1} t_m u_m^{[h_m]-1} w_{m-1} \ldots w_i v_i^{[k_i]-1} t_i u_i^{[h_i]-1} \ldots w_1 v_1^{[k_1]-1} t_1 u_1^{[h_1]-1} W_0$$

where:

$$W_0 = \begin{cases} b^2 & \text{if } h_1 > 0 \\ ab & \text{if } h_1 < 0 \end{cases} \quad (\text{Note that } W_0 \text{ ends in } b.)$$

and all other $w_i$, $v_i$, $t_i$, and $u_i$ are the same as for $\mu(a)$.

Proof of Claim $(\ast\alpha)$. The proof goes by induction on the number of exponents $k_m, h_m, \ldots, k_1, h_1$. For the details, see Appendix A3.

Claim $(\ast\beta)$. Let $\mu$ be as above, according to cases $\alpha\beta$ or $\beta\beta$. Then,

$$\mu(b) = (w_m v_m^{[k_m]-1} t_m u_m^{[h_m]-1} w_{m-1} \ldots w_i v_i^{[k_i]-1} t_i u_i^{[h_i]-1} \ldots w_1 v_1^{[k_1]-1} t_1$$

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where:
\[
t_1 = \begin{cases} 
  ba & \text{if } k_1 > 0 \\
  a^2 & \text{if } k_1 < 0 
\end{cases} \quad (\text{Note that } t_1 \text{ ends in a.})
\]
and all other \( w_i, v_i, t_i, \) and \( u_i \) are the same as for \( \mu(a) \) in Claim \((\ast \alpha)\).

We also have:
\[
\mu(ab) = (w_m v_m^{[k_m]-1}) t_m w_m^{[h_m]-1} w_{m-1} \ldots w_i v_i^{[k_i]-1} t_i u_i^{[h_i]-1} \ldots w_1 v_1^{[k_1]-1} T_1
\]
where:
\[
T_1 = \begin{cases} 
  b^2 & \text{if } k_1 > 0 \\
  ab & \text{if } k_1 < 0 
\end{cases} \quad (\text{Note that } T_1 \text{ ends in } b.)
\]
and all other \( w_i, v_i, t_i, \) and \( u_i \) are the same as for \( \mu(b) \).

**Proof of Claim \((\ast \beta)\).** The proof goes by induction on the number of exponents \( k_m, h_m, \ldots, k_1 \) in \( \mu \). For details, see Appendix A3.

This completes the proof of part (1) of Theorem 6.4.

**Proof of (2).** For each element \( \varphi \in V \) we consider two elements \( \varphi_a, \varphi_b \in V \) defined as follows:

\[
\varphi_a(ax) = a \varphi(x), \quad \text{for all } x \text{ in the domain of } \varphi; \\
\varphi_a(b) = b; \\
\varphi_a(a) = a; \\
\varphi_a(bx) = b \varphi(x), \quad \text{for all } x \text{ in the domain of } \varphi.
\]

Then from the generators \( \alpha, \beta \) of \( \text{FG}_2 \) seen in the proof of (1), we obtain a set \( \{\alpha_a, \beta_a, \alpha_b, \beta_b\} \) which generates a subgroup of \( V \) isomorphic to \( \text{FG}_2 \times \text{FG}_2 \). The proof that the distortion of \( \langle \alpha, \beta \rangle \) in \( V \) is linear is very similar to the proof for the distortion of \( \langle \alpha, \beta \rangle \) in \( V \).

Any element of \( \langle \alpha_a, \beta_a, \alpha_b, \beta_b \rangle \) can be put in the form \( \varphi_a \psi_b \), for some \( \varphi, \psi \in V \). Looking at cases \((\ast \alpha)\) and \((\ast \beta)\) for both \( \varphi \) and \( \psi \) (in the proof of part (1)), we obtain four cases. In each case we find that there are words \( y, z \in \{a, b\}^* \) such that

- **Case \((\ast \alpha, \ast \alpha)\)**
  \[
  \varphi_a \psi_b(aa) = a \varphi(a) = aya, \quad \varphi_a \psi_b(aba) = a \varphi(ba) = ayb, \quad \text{and } |y| > |\varphi|_{\{\alpha, \beta\}}; \\
  \varphi_a \psi_b(ba) = b \psi(a) = bza, \quad \varphi_a \psi_b(bba) = b \psi(ba) = bzb, \quad \text{and } |z| > |\psi|_{\{\alpha, \beta\}}.
  \]

- **Case \((\ast \alpha, \ast \beta)\)**
  \[
  \varphi_a \psi_b(aa) = a \varphi(a) = aya, \quad \varphi_a \psi_b(aba) = a \varphi(ba) = ayb, \quad \text{and } |y| > |\varphi|_{\{\alpha, \beta\}}; \\
  \varphi_a \psi_b(bb) = b \psi(b) = bza, \quad \varphi_a \psi_b(bab) = b \psi(ab) = bzb, \quad \text{and } |z| > |\psi|_{\{\alpha, \beta\}}.
  \]

- **Case \((\ast \beta, \ast \alpha)\)**
  \[
  \varphi_a \psi_b(ab) = a \varphi(b) = aya, \quad \varphi_a \psi_b(aab) = a \varphi(ab) = ayb, \quad \text{and } |y| > |\varphi|_{\{\alpha, \beta\}}; \\
  \varphi_a \psi_b(ba) = b \psi(a) = bza, \quad \varphi_a \psi_b(bba) = b \psi(ba) = bzb, \quad \text{and } |z| > |\psi|_{\{\alpha, \beta\}}.
  \]

- **Case \((\ast \beta, \ast \beta)\)**
  \[
  \varphi_a \psi_b(ab) = a \varphi(b) = aya, \quad \varphi_a \psi_b(aab) = a \varphi(ab) = ayb, \quad \text{and } |y| > |\varphi|_{\{\alpha, \beta\}}; \\
  \varphi_a \psi_b(bb) = b \psi(b) = aza, \quad \varphi_a \psi_b(bab) = b \psi(ab) = bzb, \quad \text{and } |z| > |\psi|_{\{\alpha, \beta\}}.
  \]

Note that by Lemma 7 (page 300) of [36], there is an isomorphism between \( \langle \alpha, \beta \rangle \) and \( \langle \alpha_a, \beta_a \rangle \), mapping generators to generators. Hence \( |\varphi|_{\{\alpha, \beta\}} = |\varphi_a|_{\{\alpha_a, \beta_a\}} \), and \( |\psi|_{\{\alpha, \beta\}} = |\psi_b|_{\{\alpha_b, \beta_b\}} \). Also,
the generators $\alpha_b, \beta_b$ cannot occur in a shortest word over $\{\alpha_a, \beta_a, \alpha_b, \beta_b\}$ representing $\varphi_a$; thus $|\varphi|_{\{\alpha, \beta\}} = |\varphi|_{\{\alpha_a, \beta_a, \alpha_b, \beta_b\}}$. Similarly, $|\psi|_{\{\alpha, \beta\}} = |\psi|_{\{\alpha_a, \beta_a, \alpha_b, \beta_b\}}$.

The existence of $y$ and $z$ as above implies (as in part (1)), that the table of the maximum extension of $\varphi_a \psi_b$ contains an entry of length $|\varphi_a|_{\{\alpha_a, \beta_a, \alpha_b, \beta_b\}}$, and an entry of length $|\psi_b|_{\{\alpha_a, \beta_a, \alpha_b, \beta_b\}}$; hence, by Lemma 3.6, the table size of the maximum extension of $\varphi_a \psi_b$ satisfies

$$\|\text{max } \varphi_a \psi_b\| \geq \max\{|\varphi_a|_{\{\alpha_a, \beta_a, \alpha_b, \beta_b\}}, |\psi_b|_{\{\alpha_a, \beta_a, \alpha_b, \beta_b\}}\} > \frac{1}{2} (|\varphi_a|_{\{\alpha_a, \beta_a, \alpha_b, \beta_b\}} + |\psi_b|_{\{\alpha_a, \beta_a, \alpha_b, \beta_b\}}).$$

Moreover, by Corollary 3.7, $\|\text{max } \varphi_a \psi_b\| \leq |\varphi_a \psi_b| \leq C \max|\varphi_a \psi_b|$. Therefore,

$$|\varphi_a \psi_b|_{\Delta} > c (|\varphi_a|_{\{\alpha_a, \beta_a, \alpha_b, \beta_b\}} + |\psi_b|_{\{\alpha_a, \beta_a, \alpha_b, \beta_b\}}),$$

for some constant $c > 0$. Obviously, $|\varphi_a|_{\{\alpha_a, \beta_a, \alpha_b, \beta_b\}} + |\psi_b|_{\{\alpha_a, \beta_a, \alpha_b, \beta_b\}} \geq |\varphi_a \psi_b|_{\{\alpha_a, \beta_a, \alpha_b, \beta_b\}}$. Thus,

$$|\varphi_a \psi_b|_{\Delta} > c |\varphi_a \psi_b|_{\{\alpha_a, \beta_a, \alpha_b, \beta_b\}},$$

which completes the proof of part (2) of Theorem 6.4 $\square$

The following is of independent interest.

**Proposition 6.5** If $G$ is a finitely generated subgroup of $V$ with a distortion function $\delta$ then $V$ also contains a subgroup isomorphic to the direct product $G \times G$, with distortion function linearly equivalent to $n \mapsto \delta(n \log n)$.

**Proof.** Recall that for each element $\varphi \in G \subseteq V$ we considered the two elements $\varphi_a, \varphi_b \in V$ defined as follows:

$$\varphi_a(ax) = a \varphi(x), \; \text{for all } x \text{ in the domain of } \varphi;$$
$$\varphi_a(b) = b;$$
$$\varphi_b(a) = a;$$
$$\varphi_b(ax) = b \varphi(x), \; \text{for all } x \text{ in the domain of } \varphi.$$

Let $\Delta$ be a finite generating set of $V$, and let $\Gamma$ be a finite generating set of $G$. For $\varphi \in G$ let $n = |\varphi|_\Delta$. Then $|\varphi|_\Gamma \leq \delta(n)$. Let $\Delta_a = \{\theta_a : \theta \in \Delta\}$, and $\Delta_b = \{\theta_b : \theta \in \Delta\}$. Then $\Delta_a \cup \Delta_b$ is a finite generating set of a subgroup isomorphic to $V \times V$. Similarly, $G \times G$ is generated by $\Gamma_a \cup \Gamma_b$. Moreover, for any $(\varphi_a, \psi_b) \in G \times G$ we have:

$$|(\varphi_a, \psi_b)|_{\Gamma_a \cup \Gamma_b} = |\varphi_a \psi_b|_{\Gamma_a \cup \Gamma_b} = |\varphi_a|_{\Gamma_a} + |\psi_b|_{\Gamma_b} \leq \delta(|\varphi_a|_\Delta_a) + \delta(|\psi_b|_\Delta_b) \leq 2 \cdot \delta(|\varphi_a \psi_b|_{\Delta_a \cup \Delta_b}).$$

The latter inequality uses the fact that distortion functions are non-decreasing, by definition. Moreover we have

$$|\varphi_a \psi_b|_{\Delta_a \cup \Delta_b} \leq |\varphi_a|_{\Delta_a \cup \Delta_b} + |\psi_b|_{\Delta_a \cup \Delta_b} = |\varphi|_\Delta + |\psi|_\Delta.$$

The last equality holds by Lemma 7 (page 300) of [36], as we observed in the proof of part (2) of Theorem 6.4. Next, by Theorem 3.8,

$$|\varphi|_\Delta + |\psi|_\Delta \leq C \Delta \left(\|\varphi\| \log_2 \|\varphi\| + \|\psi\| \log_2 \|\psi\|\right).$$

CLAIM: For all $\varphi, \psi \in V$, $\|\varphi \psi\| = \|\varphi\| + \|\psi\|$. Proof of the Claim: Suppose the tables for $\varphi, \psi$, in maximally extended form are

$$\varphi = \begin{bmatrix} x_1 & \ldots & x_m \\ y_1 & \ldots & y_m \end{bmatrix}, \; \psi = \begin{bmatrix} u_1 & \ldots & u_n \\ v_1 & \ldots & v_n \end{bmatrix},$$

Then $\varphi_a = \begin{bmatrix} ax_1 & \ldots & ax_m \\ ay_1 & \ldots & ay_m \end{bmatrix}, \; \psi_b = \begin{bmatrix} a & bu_1 & \ldots & bu_n \\ a & bv_1 & \ldots & bv_n \end{bmatrix}$, and
ϕₐψₖ = \[
\begin{bmatrix}
ax₁ & \ldots & axₘ & bu₁ & \ldots & buₙ \\
ay₁ & \ldots & ayₘ & bv₁ & \ldots & bvₙ
\end{bmatrix}.
\]

It is easy to see that the latter table is in maximally extended from. Indeed, no pair \(axᵢ, buⱼ\) can lead to extension; and since the original tables of \(ϕ\) and \(ψ\) were maximally extended already, no extension can happen among pairs \(axᵢ, axⱼ\) or pairs \(buᵢ, buⱼ\). This proves the Claim.

By the Claim, and the inequalities proved just before the Claim, we obtain:
\[
|ϕₐψₖ|_{Δₐ∪Δₖ} ≤ C_{Δ} \|ϕₐψₖ\| \log₂ \|ϕₐψₖ\|
\]

By the first inequality of Theorem 3.8 this implies \(|ϕₐψₖ|_{Δₐ∪Δₖ} \leq c \|ϕₐψₖ\| \Delta \log₂ |ϕₐψₖ|\Delta\) for some constant \(c > 0\). We already proved \(|(ϕₐ, ψₖ)|_{Γₐ∪Γₖ} \leq 2 \cdot \delta |(ϕₐψₖ)|_{Δₐ∪Δₖ}\). Therefore we have
\[
|(ϕₐ, ψₖ)|_{Γₐ∪Γₖ} \leq 2 \cdot \delta(C \|ϕₐψₖ\| \Delta \log₂ |ϕₐψₖ|\Delta)
\]

for some constant \(C > 0\). This proves the Proposition. \(\Box\)

We will need the following definitions:

**Definition 6.6** Two functions \(f_₁ : \mathbb{N} \to \mathbb{N}\) and \(f₂ : \mathbb{N} \to \mathbb{N}\) are **polynomially equivalent** iff there exist positive constants \(c₀, C₁₂, c₁₂, C₂₁, c₂₁, d₁, d₂, d₃, d₄\) such that for all \(n \geq c₀\):
\[
|f₁(n)| ≤ C₁₂ \cdot (f₂(c₁₂ n^{d₁})^{d₃}) \quad \text{and} \quad |f₂(n)| ≤ C₂₁ \cdot (f₁(c₂₁ n^{d₂})^{d₄}).
\]

In big-O notation this means \(f₁(n) = O(f₂(O(n^{O(1)}))^{O(1)})\) and \(f₂(n) = O(f₁(O(n^{O(1)})))^{O(1)}\).

**Definition 6.7** A function \(f : \mathbb{N} \to \mathbb{N}\) is **superadditive** iff for all \(n, m\):
\[
f(n + m) ≥ f(n) + f(m).
\]

**Proposition 6.8** Let \(t : \mathbb{N} \to \mathbb{N}\) be a non-decreasing function such that \(t(n) \geq n\) for all \(n\). Then there is a superadditive function \(T : \mathbb{N} \to \mathbb{N}\) such that for all \(n\),
\[
t(n) ≤ T(n) ≤ n \cdot t(n).
\]

Hence, every non-decreasing function which is larger than the identity function is polynomially equivalent to a non-decreasing superadditive function.

**Proof.** Given the function \(t\) we define the desired superadditive function \(T\) by
\[
T(n) = \max\{ \sum_{i=1}^{k} t(nᵢ) : k \geq 1, (n₁, \ldots, nₖ) ∈ (\mathbb{N} - \{0\})^k, \sum_{i=1}^{k} nᵢ = n \}.
\]

In other words, \((n₁, \ldots, nₖ)\) is any partition of \(n\); we define \(T(n)\) to be \(\sum_{i=1}^{k} t(nᵢ)\), maximized over all partitions of \(n\). It is straightforward to verify that \(T\) is superadditive and that \(t(n) ≤ T(n) ≤ n \cdot t(n)\). \(\Box\)

Theorem 6.3 gives us a large subset of the set of distortion functions of \(V\), namely the set of all Dehn functions. Moreover, we know from [35] that for any time complexity function of a nondeterministic Turing machine, its fourth power is linearly equivalent to a Dehn function (if it is superadditive). By Proposition 6.8 we can always assume that our functions are superadditive (up to polynomial equivalence).

Corollary 6.11 below will give a kind of converse to Theorem 6.3.
Lemma 6.9  If a finitely generated subgroup $H$ of a finitely generated group $G$ has distortion function $\delta$, and if the word problem of $G$ has nondeterministic time complexity $T(\cdot)$, then the generalized word problem of $H$ in $G$ has nondeterministic time complexity bounded by a function linearly equivalent to $T(\delta(\cdot))$.

Moreover, if $T$ is the time complexity of a nondeterministic Turing machine and $\delta$ is the distortion of a subgroup of $G$ then $T(\delta(\cdot))$ is linearly equivalent to the time complexity of a nondeterministic Turing machine.

Proof. Suppose $G$ has a finite generating set $A$ and $H$ has a finite generating set $\Delta \subset (A^{\pm 1})^*$. Let $w \in (A^{\pm 1})^*$ of length $|w| = n$ be an input to the generalized word problem. We guess a word $z \in (\Delta^{\pm 1})^*$ of length $|z| \leq \delta(n)$ such that $w = z$ in $G$. Such a $z$ exists iff the answer to the generalized word problem is “yes”. It takes time $O(\delta(n))$ to guess $z$.

To check correctness of the guessed $z$ we solve the word problem “Is $w = z$ in $G$?”, in nondeterministic time linearly equivalent to $T(n + \delta(n))$, which is linearly equivalent to $T(\delta(\cdot))$. $\square$

If we let $G$ be $V$, the above and Proposition 4.2 give us a nondeterministic Turing machine with time complexity linearly equivalent to $\delta(\cdot)^2$, for solving the generalized word problem of $H$ in $V$. Therefore we have:

Proposition 6.10  Every distortion of $V$ is linearly equivalent to the square-root of a nondeterministic time complexity function.

In summary, we have the following corollary.

Corollary 6.11  The following classes of functions are polynomially equivalent:

- Time complexity functions of nondeterministic Turing machines.
- Dehn functions of finitely presented groups.
- Distortions in the Thompson group $V$.

The polynomial equivalences between the above three classes of functions actually have “uniform degree”; this means that the degrees $d_1, d_2, d_3, d_4$ that appear in the polynomial equivalences between functions (in Definition 6.6) are the same for all pairs of functions.

Questions: (1) Of course (by Ol’shanskii and Sapir’s theorem [30], and by the results of [35]), Corollary 6.11 also holds for $\text{FG}_2 \times \text{FG}_2$. For what other groups does the Corollary hold?

Some linear groups are candidates; it is well known that $\text{SL}_4(\mathbb{Z})$ has $\text{FG}_2 \times \text{FG}_2$ as a subgroup (see pp. 41-42 in [26], and [27]), but one would also need to study the distortion of linear groups as subgroups of other linear groups.

(2) Are the distortions of $V$ linearly (rather than just polynomially) equivalent to the Dehn functions of finitely presented groups? (By [30], $\text{FG}_2 \times \text{FG}_2$ has this stronger property.)
7 Representation of the Thompson groups in algebras

We will show that the Thompson groups are subgroups of Cuntz C*-algebras. Those algebras can be defined as the completion of quotient algebras of the polycyclic monoid.

The polycyclic monoid on a generating set $A$ is an inverse monoid with zero $0$, defined by an inverse monoid presentation with relations

$$\{a^{-1}a = 1 : a \in A\} \cup \{\alpha^{-1}\beta = 0 : \alpha, \beta \in A \text{ with } \alpha \neq \beta\}.$$

It follows from this presentation that every element of $PC(A)$, other than $1$ and $0$, is of the form $yx^{-1}$ with $y, x \in A^*$. On the other hand, $u^{-1}v = 0$ if $u, v \in A^*$ are not prefix-comparable; and if $u \geq_{\text{pref}} v = uz$, then $u^{-1}v = z \in A^*$; if $v \geq_{\text{pref}} u = vw$, then $u^{-1}v = w^{-1} \in (A^{-1})^*$.

For the definition of “inverse monoid” and more information on these monoids, see e.g. the monograph [16]. Polycyclic monoids were introduced in [28] (and were re-invented in [5]). The papers [21], [15] and [17] give interesting applications of polycyclic monoids.

We assume that the alphabet is $A = \{a, b\}$, and in order to represent $V$ we first consider the monoid algebra of $PC(a, b)$ over any field $\mathbb{K}$

$$\mathbb{K}[PC(a, b)] = \{ \sum_{i=1}^{n} \kappa_i y_i x_i^{-1} : n \in \mathbb{N}, \ y_i x_i^{-1} \in PC(a, b) \text{ and } \kappa_i \in \mathbb{K} \text{ for all } i = 1, \ldots, n \} \cup \{0\}.$$

Before embedding $V$ in a Cuntz algebra, we will represent $V$ as a subgroup of the multiplicative part of a quotient algebra of $\mathbb{K}[PC(a, b)]$. Let us look at an example before going into details.

**Example:** Consider the two elements $C, B$ of $V$ given by tables

$$B = \begin{bmatrix} a & ba & b^2a & b^3 \\ a & ba^2 & bab & b^2 \end{bmatrix}, \quad C = \begin{bmatrix} a^2 & ab & bab & ba^2 & b^2 \\ a & ba & b^3a & b^2a & b^4 \end{bmatrix}. $$

They will be represented by elements of $\mathbb{K}[PC(a, b)]$ as follows:

- $B$ is represented by $aa^{-1} + ba^2a^{-1}b^{-1} + baba^{-1}b^{-2} + b^2b^{-3}$,
- $C$ is represented by $aa^{-2} + bab^{-1}a^{-1} + b^2ab^{-1}a^{-1}b^{-1} + b^2aa^{-2}b^{-1} + b^4b^{-2}$.

One observes that the composite $C \circ B$ is then represented by the product of the corresponding elements of $\mathbb{K}[PC(a, b)]$:

$$aa^{-2} + bab^{-1}a^{-1} + b^2ab^{-1}a^{-1}b^{-1} + b^2aa^{-2}b^{-1} + b^4b^{-2} \cdot \ (aa^{-1} + ba^2a^{-1}b^{-1} + baba^{-1}b^{-2} + b^2b^{-3})$$

$$= aa^{-2} + bab^{-1}a^{-1} + b^3aa^{-1}b^{-2} + b^2aa^{-1}b^{-1} + b^4b^{-2},$$

where we applied the relations of $PC(a, b)$ and omitted the terms that are $0$.

The maximum extension of $C \circ B$ turns out to be $A = \begin{bmatrix} a^2 & ab & b \\ a & ba & b^2 \end{bmatrix}$. The $\mathbb{K}[PC(a, b)]$-representation of $A$ can be obtained from the $\mathbb{K}[PC(a, b)]$-representation of $C \circ B$ by repeatedly applying the relation $aa^{-1} + bb^{-1} = 1$, as follows:

$$aa^{-2} + bab^{-1}a^{-1} + b^2aa^{-1}b^{-2} + b^2aa^{-1}b^{-1} + b^4b^{-3}$$

$$= aa^{-2} + bab^{-1}a^{-1} + b^3(aa^{-1} + bb^{-1})b^{-2} + b^2aa^{-1}b^{-1}$$

$$= aa^{-2} + bab^{-1}a^{-1} + b^2bb^{-2} + b^2aa^{-1}b^{-1}$$

$$= aa^{-2} + bab^{-1}a^{-1} + b^2(bb^{-1} + aa^{-1})b^{-1}$$

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= aa^{-2} + bab^{-1}a^{-1} + b^2b^{-1}.

This ends the Example. We will now formalize this for $V$, then generalize the representation to $G_{2,1}$, and finally prove properties.

For an algebra $A$ and a set $X \subseteq A$ we write $\langle \langle X \rangle \rangle$ for the ideal generated by $X$ in $A$. We now take the following quotient algebra:

\[ C_V = \mathbb{K}[PC(a, b)]/I_V \]

where

\[ I_V = \langle \langle a^{-1} + bb^{-1} - 1 \rangle \rangle. \]

We can interpret the ideal $I_V$ as a term rewrite system with the two rules, $a a^{-1} + bb^{-1} \rightarrow 1$ and $1 \rightarrow a a^{-1} + bb^{-1}$. The ideal $I_V$ and the corresponding rewrite system interpretation are inspired from Lemma 2.2, which tells us how to find maximum extensions of right-ideal isomorphisms.

Note that in the quotient algebra $C_V$ we have $\sum_{q \in Q} q^{-1} = 1$ for any finite maximal prefix code $Q$. Hence, in the above example, the sum representing $C \circ B$ could have been maximally extended in one step as follows:

\[
\begin{align*}
aa^{-2} + bab^{-1}a^{-1} + b^2aa^{-1}b^{-2} + b^2a^{-1}b^{-1} + b^4b^{-3} \\
= aa^{-2} + bab^{-1}a^{-1} + b^2(baa^{-1}b^{-1} + aa^{-1} + b^2b^{-2})b^{-1} \\
= aa^{-2} + bab^{-1}a^{-1} + b^2b^{-1}
\end{align*}
\]

where \( \{ba, a, b^2\} \) is a maximal prefix code.

Representing $G_{2,1}$ by algebras is more complicated. For a right-ideal isomorphism $\varphi$ between essential right ideals we want the representation $\sum_{x \in P} \varphi(x) x^{-1}$, where $P \subset A^*$ is the (possibly infinite) domain code of $\varphi$. Then for any $w \in PA^*$ we have:

\[ \varphi(w) = \sum_{x \in P} \varphi(x) x^{-1} w \cap A^*. \]

The sum $\sum_{x \in P} \varphi(x) x^{-1}$ belongs to an algebra consisting of possibly infinite sums over the monoid $PC(a, b)$ over any field $\mathbb{K}$; but we need to restrict the infinite sums in order to get the desired algebraic properties. The following property of sums will guarantee that our algebra is closed under multiplication, but further restrictions will be needed.

**Definition 7.1** We call a relation $S \subseteq A^* \times A^*$ finite-to-finite iff for any $x \in A^*$ there are only finitely many $y \in A^*$ such that $(x, y) \in S$, and for any $y \in A^*$ there are only finitely many $x \in A^*$ such that $(x, y) \in S$.

As a preliminary step we start out with the following algebra.

\[ B_\infty = \{ \sum_{i \in I} \kappa_i y_i x_i^{-1} : I \subseteq \mathbb{N}, \ y_i x_i^{-1} \in PC(a, b) \text{ and } \kappa_i \in \mathbb{K} \text{ for all } i \in I, \text{ and the relation } \{(y_i, x_i) : i \in I \} \text{ is finite-to-finite} \} \cup \{0\} \]

It is obvious that $B_\infty$ is closed under addition. In Lemma 11.1 we’ll prove that $B_\infty$ is closed under multiplication.
We define the set of **unary sums** to be the subset \( U_\infty \) of \( B_\infty \) consisting of sums of the form \( \sum_{i \in I} y_i x_i^{-1} \) with the following properties:

- All the coefficients in the sum are equal to 1.
- The sets \( \{ x_i : i \in I \} \) and \( \{ y_i : i \in I \} \) are maximal prefix codes, such that the indexing \( i \in I \mapsto x_i \) and the indexing \( i \in I \mapsto y_i \) are bijective functions.

The set \( U_V \) (\( \subset \mathbb{K}[PC(a,b)] \)) is defined in a similar way, by taking the index sets \( I \) to be finite in the above definition.

We also define the sets of **partial unary sums**, \( U_\infty^{\text{part}} \) and \( U_V^{\text{part}} \), by just requiring \( \{ x_i : i \in I \} \) and \( \{ y_i : i \in I \} \) to be prefix codes (not necessarily maximal), while the rest of the definitions is kept unchanged.

We will prove later (Lemma 11.4) that the sets \( U_\infty \), \( U_V \), \( U_\infty^{\text{part}} \) and \( U_V^{\text{part}} \) are closed under multiplication, so they are monoids.

We define the algebra \( A_\infty \) as the subalgebra of \( B_\infty \) generated as an algebra by \( U_\infty^{\text{part}} \):

\[
A_\infty = \langle U_\infty^{\text{part}} \rangle.
\]

Note that \( \mathbb{K}[PC(a,b)] \) is the subalgebra of \( B_\infty \) generated by \( U_V^{\text{part}} \). Since \( U_\infty^{\text{part}} \) is closed under multiplication (as we will prove in Lemma 11.4), every element of \( A_\infty \) is a linear combination of elements of \( U_\infty^{\text{part}} \); in other words, \( A_\infty \) is the monoid algebra of the monoid \( U_\infty^{\text{part}} \).

The algebra \( A_\infty \) can also be characterized as follows (as will be proved in Lemma 11.2):

1. The relation \( S = \{(x_i,y_i) : i \in I \} \) is bounded finite-to-finite. This means that there exists \( n_0 \) such that for every \( x_{i_0} \in \{ x_i : i \in I \} \), \( |S(x_{i_0})| = |\{ y_j : (x_{i_0}, y_j) \in S \}| \leq n_0 \), and for every \( y_{j_0} \in \{ y_i : i \in I \} \), \( |S^{-1}(y_{j_0})| = |\{ x_i : (x_i, y_{j_0}) \in S \}| \leq n_0 \). (i.e., there is a bound on the cardinalities of all the sets \( S(x_i) \) and \( S^{-1}(y_i) \) as \( i \) ranges over \( I \)).
2. In \( \{ x_i : i \in I \} \) and in \( \{ y_i : i \in I \} \), all \( >_{\text{pref}} \)-chains have bounded length.
3. The set \( \{ \kappa_i : i \in I \} \) is finite.

Finally, in order to embed \( G_{2,1} \) we consider the quotient algebra

\[
C_\infty = A_\infty / I_\infty
\]

where the ideal \( I_\infty \) is defined by

\[
I_\infty = \langle \sum_{i \in I} y_i (\sum_{q \in Q} q q^{-1} - 1) x_i^{-1} : I \subseteq \mathbb{N}, \{ x_i : i \in I \} \text{ and } \{ y_i : i \in I \} \text{ are maximal prefix codes,} \>
\]

\[
i \in I \mapsto x_i \in \{ x_i : i \in I \} \text{ and } i \in I \mapsto y_i \in \{ y_i : i \in I \} \text{ are bijections,} \>
\]

\[
\text{and } Q_i \text{ is a maximal prefix code of } A^* \text{ (for every } i \in I \text{ )} \rangle.
\]

Note that \( \bigcup_{i \in I} x_i Q_i \) and \( \bigcup_{i \in I} y_i Q_i \) are maximal prefix codes, by construction (2) in Example 11.1, hence \( \sum_{i \in I} \sum_{q \in Q} y_i q q^{-1} x_i^{-1} \in U_\infty^{\text{part}} \), and thus \( I_\infty \subset A_\infty \). We can interpret the ideal \( I_\infty \) as a **generalized term rewrite system** with the rules \( \sum_{q \in Q} q q^{-1} \to 1 \) and \( 1 \to \sum_{q \in Q} q q^{-1} \), where \( Q \) ranges over all maximal prefix codes of \( A^* \). This is a generalized rewrite system, in
the sense that these rules are applied to infinite sums, and we allow infinitely many rules to be applied “in parallel”; i.e., infinitely many rules of the form \( \sum_{q \in \mathbb{Q}} q q^{-1} \Rightarrow 1 \) \( (i \in I) \) are applied in infinitely many non-overlapping locations in the infinite sum.

The ideals \( I_V \) and \( I_\infty \) (and the corresponding rewrite system interpretation) are inspired by Lemmas 2.2, 2.3 and 2.4 (especially Remark 3), which tell us how to find maximum extensions of right-ideal isomorphisms.

The congruence class of \( \sum_{i \in I} \kappa_i y_i x_i^{-1} \) is \( \sum_{i \in I} \kappa_i y_i x_i^{-1} + I_V \), respectively \( \sum_{i \in I} \kappa_i y_i x_i^{-1} + I_\infty \). Two elements of \( \mathbb{K}[PC(a, b)] \) or \( \mathcal{A}_\infty \) that belong to the same congruence class are called congruent.

We will obtain the following representations of \( V \) and \( G_{2,1} \) as subgroups of the multiplicative part of \( C_V \), respectively \( C_\infty \):

**Theorem 7.2** The Thompson group \( V \) is isomorphic to the subgroup \( U_V/I_V \) of the multiplicative part of the algebra \( C_V = \mathbb{K}[PC(a, b)]/I_V \).

The Thompson group \( G_{2,1} \) is isomorphic to the subgroup \( U_\infty/I_\infty \) of the multiplicative part of the algebra \( C_\infty = A_\infty/I_\infty \).

The fact that only the elements 0 and 1 of the field are used in this representation is reminiscent of the regular representation by permutation matrices.

All the algebras above are \( * \)-algebras, by defining \( (\sum_{i \in I} \kappa_i y_i x_i^{-1})^* = \sum_{i \in I} \kappa_i^* x_i y_i^{-1} \) and taking the field \( \mathbb{K} \) to be \( \mathbb{C} \) (the complex numbers, where \( \ast \) denotes conjugation). Then for all algebra elements \( s, t \) and scalars \( \kappa \in \mathbb{C} : (s^*)^* = s, \ (\kappa s + t)^* = \kappa s^* + t^*, \ (st)^* = t^* s^* \).

**Remark 7.3** – Connection with the Cuntz algebras

The use of the polycyclic monoid and the relations \( \sum_{q \in \mathbb{Q}} q q^{-1} = 1 \) means that the Cauchy completion of our algebra \( C_V \) (defined for any alphabet \( A \)) is the Cuntz \( C^\ast \)-algebra \( \mathcal{O}_{|A|} \). These \( C^\ast \)-algebras were first introduced by Dixmier \( [7] \), then studied by Cuntz \( [5] \) who proved many remarkable properties, many of which are reminiscent of the properties of \( V \) itself (e.g., that \( \mathcal{O}_{|A|} \) is a simple algebra). See also \( [6, 31, 32] \), where connections between \( C^\ast \)-algebras and inverse monoids and, in particular, the relation between the Cuntz algebras and the polycyclic monoid, are exposited.

In summary, from this and Theorem 7.2 we obtain:

**Corollary 7.4** The Thompson group \( V \) is a subgroup of the multiplicative part of the Cuntz-Dixmier \( C^\ast \)-algebra \( \mathcal{O}_2 \). More generally, the Thompson-Higman group \( G_{n,1} \) \( (n \in \mathbb{N}, n \geq 2) \) is a subgroup of the multiplicative part of the Cuntz \( C^\ast \)-algebra \( \mathcal{O}_n \).

We outline the proof of Theorem 7.2. First we list some lemmas. Other lemmas that play an indirect role, as well as the proofs of all the lemmas, are given in Appendix A4.

- **Lemma 11.3** There is a one-to-one correspondence between (1) the set of all isomorphisms between (essential) right ideals of \( \{a, b\}^* \), and (2) the set \( U_\infty^{\text{part}} \) (respectively \( U_\infty \))

\[
\Sigma : \varphi \mapsto \sum_{x \in \text{Dom}(\varphi)} \varphi(x) x^{-1} \in U_\infty^{\text{part}}.
\]

\( ^1 \) I owe the observation of the connection between the above algebra \( C_V \) and the Cuntz algebras to John Meakin \( [22] \).
Its inverse is

$$\Phi : \sum_{i \in I} y_i x_i^{-1} \in U_\infty^{\text{part}} \mapsto (\varphi : \{x_i : i \in I\} A^* \to \{y_i : i \in I\} A^*)$$

where $\varphi$ is defined by $\varphi(x) = \sum_{i \in I} y_i x_i^{-1} x \cap A^*$.

Similarly, there is a one-to-one correspondence between (1) the set of all isomorphisms between finitely generated (essential) right ideals of $\{a, b\}^*$, and (2) the set $U_\infty^{\text{part}}$ (respectively $U_V$).

- **Lemma 11.4** The sets $U_\infty, U_V, U_\infty^{\text{part}}$ and $U_V^{\text{part}}$ are closed under multiplication. We have the following formula for the multiplication in $U_\infty^{\text{part}}$:

$$\sum_{j \in J} y_j x_j^{-1} \cdot \sum_{i \in I} v_i u_i^{-1} =$$

$$\sum_{j \in \text{dom} f \cap \text{img}} y_j u_{f(j)}^{-1} \sum_{j \in \text{dom} f - \text{img}} y_j z_j^{-1} u_{f(j)}^{-1} + \sum_{i \in \text{dom} g - \text{img} f} y_{g(i)} t_i u_i^{-1}$$

where $f : J \to I$ and $g : I \to J$ are partial functions defined by $x_j \leq_{\text{pref}} v_{f(j)}$ and $x_j = v_{f(j)} z_j$; similarly, $x_{g(i)} \geq_{\text{pref}} v_i$ and $v_i = x_{g(i)} t_i$.

- **Lemma 11.5** The one-to-one correspondence $\Phi$ of Lemma 11.3 is a homomorphism, i.e., for all $\sigma_2, \sigma_1 \in U_\infty^{\text{part}}$:

$$\Phi(\sigma_2 \cdot \sigma_1) = \Phi(\sigma_2) \circ \Phi(\sigma_1).$$

- **Lemma 11.6** The one-to-one correspondence $\Sigma$ of Lemma 11.3 respects the congruence relations on the set $U_\infty$ (induced by $I_\infty$) and on the set of all isomorphisms between essential right ideals. In other words, two isomorphisms between essential right ideals, $\varphi_1$ and $\varphi_2$ are congruent (i.e., they have the same maximum extension) iff $\Sigma(\varphi_1)$ and $\Sigma(\varphi_2)$ are congruent (relative to the ideal $I_\infty$). A similar fact holds for $U_V$.

**Proof of Theorem 7.2.** We prove the theorem for $G_{2.1}$; for $V$ the proof is similar. By Lemma 11.3 there is a one-to-one correspondence $\Sigma$ from the set of all isomorphisms between essential right ideals, to the set $U_\infty$. By Lemma 11.6 this one-to-one correspondence respects the congruence relation of the set $U_\infty$ and the congruence relation of the set of all isomorphisms between essential right ideals. Therefore $\Sigma$ determines a one-to-one correspondence between $G_{2.1}$ and $U_\infty/I_\infty$.

Moreover, $G_{2.1}$ and $U_\infty/I_\infty$ are isomorphic as groups: Any subidentity isomorphism of the form $pw \in PA^* \to pw \in PA^*$ (with $p$ ranging over a maximal prefix code $P$, $w$ ranging over $A^*$) is mapped by $\Sigma$ to $\sum_{p \in P} p p^{-1}$, which is congruent to 1 modulo $I_\infty$. Also, for any isomorphism $\varphi : P_1 A^* \to P_2 A^*$ between essential right ideals, the one-to-one correspondence $\Sigma$ maps the inverse $\varphi^{-1}$ to $\sum_{p_2 \in P_2} \varphi^{-1}(p_2) p_2 = \sum_{p_1 \in P_1} p_1 \varphi(p_1)^{-1}$, which is congruent to the inverse of $\Sigma(\varphi)$. Indeed, $\sum_{i \in I} y_i x_i^{-1} \cdot \sum_{i \in I} x_i y_i^{-1} = \sum_{i \in I} y_i y_i^{-1}$, and $\sum_{i \in I} y_i y_i^{-1} + I_\infty = 1 + I_\infty$. Finally, the product $\Sigma(\psi) \Sigma(\varphi)$ is congruent to $\Sigma(\psi \varphi)$, as we saw in Lemma 11.5. □
8 Appendix A1

In this appendix we present basic information about prefix codes and right ideals of free monoids.

A combinatorial observation: The number of maximal prefix codes of cardinality \( n \) over the alphabet \( \{a, b\} \) is \( C_{n-1} \), where \( C_n = \frac{1}{n+1} \binom{2n}{n} \) (the classical Catalan number). This is proved by counting binary trees (see e.g. [39], Chapter 5). Asymptotically, \( C_n = \frac{4^n}{\sqrt{\pi n^3}} (1 + \varepsilon(n)) \), with \( \lim_{n \to \infty} \varepsilon(n) = 0 \).

So, instead of defining the elements of the Thompson group \( V \) as “maximum” bijections between maximal prefix codes over the alphabet \( \{a, b\} \), one could define \( V \) as bijections between other combinatorial objects, parameterized by a positive integer \( n \), such that the number of objects of “size” \( n \) is the Catalan number (see e.g. [39], Section 5.11).

The next lemmas give some elementary properties of prefix codes.

Lemma 8.1

1. \( R \) is a right ideal of \( A^* \) iff there exists a prefix code \( P \) over \( A \) such that \( R = PA^* \).

2. For a right ideal \( R \) the prefix code \( P \) such that \( R = PA^* \) is unique.

3. \( R \) is an essential right ideal of \( A^* \) iff \( R \) is a right ideal such that
   \[
   (\forall u \in A^*)(\exists x \in A^*) \; ux \in R
   \]
   (i.e., in the terminology of [36], \( R \) is “inescapable”).

4. \( P \) is a maximal prefix code of \( A^* \) iff \( P \) is a prefix code and
   \[
   (\forall u \in A^* - PA^*)(\exists x \in A^*) \; ux \in P.
   \]

Proof of (1): \([-\rightarrow]\) Obviously, \( PA^* \) is a right ideal.

\([\rightarrow]\) We claim that for any right ideal \( R \subseteq A^* \) we have:
\[
P = R - RA \quad \text{is a prefix code such that} \quad R = PA^*.
\]

Obviously, \( PA^* \subseteq RA^* \subseteq R \), since \( R \) is a right ideal. Conversely, let us show that \( R \subseteq PA^* \).

For any \( r \in R \), let \( p \) be the shortest prefix of \( r \) that belongs to \( R \). Since \( r \) itself is in \( R \), \( p \) exists, and by definition, \( p \in R \).

Also, \( r = px \) for some \( x \in A^* \), since \( p \) is a prefix of \( r \). Finally, \( p \notin RA \) (otherwise, we would have \( p = r'a \) for some \( r' \in R \), \( a \in A \), which would imply that \( p \) is not the shortest prefix that \( r \) has in \( R \)). Thus, \( p \in P \). Since \( r = px \), we have \( R \subseteq PA^* \).

To show that \( P \) is a prefix code, let \( p, p' \in P \) and suppose \( p' \) is a prefix of \( p \): \( p = p'x \), for some \( x \in A^* \). If \( x \) is not empty then \( p \in RA \), contradicting the assumption \( p \in P \) (= \( R - RA \)). Thus, the words in \( P \) that are prefixes of each other are equal to each other.

Proof of (1'): If \( P_1A^* = P_2A^* \) for two prefix codes \( P_1, P_2 \), then for every \( p_1 \in P_1 \) there exists \( p_2 \in P_2 \) such that \( p_1 = p_2x \) (for some \( x \in A^* \)). Also, there is \( p'_1 \in P_1 \) such that \( p_2 = p'_1y \) (for some \( y \in A^* \)). Hence \( p_1 = p'_1xy \), which implies \( x = y = \varepsilon \) (the empty word), since \( P_1 \) is a prefix code. Thus, \( p_1 = p_2 \in P_2 \). Therefore, \( P_1 \subseteq P_2 \). Similarly, \( P_2 \subseteq P_1 \), so \( P_1 = P_2 \).

Proof of (2). If \( R \) is essential then \( R \) intersects any right ideal, in particular \( uA^* \). Thus, \( R \cap uA^* \neq \emptyset \), hence \( ux \in R \) for some \( x \in A^* \).
Conversely, consider any right ideal $PA^*$. For any $u \in P$, $ux \in R$ for some $x \in A^*$, hence $R$ intersects $uA^*$, hence $R$ intersects $PA^*$.

**Proof of (3).** Let $P$ be the prefix code corresponding to the right ideal $R$. By definition, $P$ is a maximal prefix code iff $(\forall u \in A^* - P): P \cup \{u\}$ is not a prefix code.

If $PA^*$ is essential then (by (2)) for any $u \in A^* - P$, $uA^* \cap PA^* \neq \emptyset$, hence $(\exists p \in P)(\exists x, y \in A^*) ux = py$. Hence, $u \leq_{\text{pref}} p$ or $p \leq_{\text{pref}} u$. Therefore, since $u$ is prefix-comparable to some word in $P$, $P \cup \{u\}$ is not a prefix code. Thus, $P$ is maximal.

Conversely, if $P$ is maximal and $u \in A^*$, let us show that $ux \in PA^*$ for some $x \in A^*$. If $u \in P$ this is obviously true (taking $x = \varepsilon$). If $u \not\in P$, $P \cup \{u\}$ is not a prefix code, by maximality of $P$, hence $u$ either has a prefix $p$ for some $p \in P$, hence $u = py \in PA^*$ (for some $y \in A^*$), or $u$ is a prefix of some $p \in P$; then $ux = p \in PA^*$ (for some $x \in A^*$).

**Proof of (4).** Let $P$ be a maximal prefix code. By (1) – (3) of this Lemma, for any $u \in A^* - PA^*$ there is $x \in A^*$ such that $ux \in PA^*$. Then, replacing $x$ by the shortest prefix $x'$ of $x$ for which $ux' \in PA^*$, we have $ux' \in P$.

The converse is immediate form (2). $\square$

**Lemma 8.2** Let $\varphi : P_1A^* \to P_2A^*$ be a right-ideal isomorphism, where $P_1, P_2$ are prefix codes. Then $\varphi$ maps $P_1$ bijectively onto $P_2$.

**Proof.** Since $\varphi$ is injective on $P_1A^*$ it is also injective on $P_1$. Let us show that $\varphi(P_1) \subseteq P_2$. If $p_1 \in P_1$ then $\varphi(p_1) = p_2w$ for some $p_2 \in P_2$, $w \in A^*$. Also, $p_1 = \varphi^{-1}(p_2w) = \varphi^{-1}(p_2)w = p_1'vw$, for some $v \in A^*$. Since $P_1$ is a prefix code it follows that $p_1 = p_1'$ and hence $w = \varepsilon$. Thus, $\varphi(p_1) = p_2 \in P_2$. A similar reasoning, applied to $\varphi^{-1}$, implies that $\varphi^{-1}(P_1) \subseteq P_2$, hence $P_1 \subseteq \varphi(P_2)$. $\square$

**Lemma 8.3** The intersection of two essential right ideals $R_1, R_2$ of $A^*$ is an essential right ideal of $A^*$.

**Proof.** Let $R_1, R_2$ be two essential right ideals of $A^*$. The intersection $R_1 \cap R_2$ is obviously a right ideal. By Lemma 8.1 (2), for all $u \in A^*$ there is $x \in A^*$ such that $ux \in R_1$ (since $R_1$ is essential). Moreover, applying Lemma 8.1 (2) to $R_2$: for $ux$ there is $y \in A^*$ such that $uxy \in R_2$. Since $R_1$ is a right ideal and since $ux \in R_1$, we also have $uxy \in R_1$. Thus, for all $u \in A^*$ there is $z = xy \in A^*$ such that $uxy \in R_1 \cap R_2$, which implies that $R_1 \cap R_2$ is essential. $\square$

**Lemma 8.4** Let $\varphi : R \to Q$ be an isomorphism between essential right ideals, and let $\varphi' : R' \to Q'$ be a restriction of $\varphi$ to right subideals $R' \subset R$, $Q' \subset Q$, with $\varphi'(R') = Q'$. Then we have: $R'$ is essential iff $Q'$ is essential.

**Proof.** Assume $Q'$ is essential. Let $u$ be any word over $A^*$. Since $R$ is essential, there exists $x \in A^*$ such that $ux \in R$; hence $\varphi(ux) \in Q$. Since $Q'$ is essential, there exists $y \in A^*$ such that $\varphi(ux) y \in Q'$; hence $\varphi^{-1}(\varphi(ux) y) \in R'$. Moreover, $\varphi^{-1}(\varphi(ux) y) = ux \varphi^{-1}(y)$. Thus, every word $u \in A^*$ is the prefix of a word in $R'$, which implies that $R'$ is essential.

The proof in the other direction is symmetric to this proof, since $\varphi$ is an isomorphism. $\square$
Lemma 8.5  Assume $Q$ is a prefix code of $A^*$, and let $x \in A^*$. Then $\overline{xQ} = \{y \in A^* : xy \in Q\}$ is either the empty set or a prefix code. If $Q$ is a maximal prefix code of $A^*$ then $\overline{xQ}$ is maximal too.

Proof.  If $y_1, y_2 \in \overline{xQ}$ are prefix-comparable then $xy_1, xy_2$ will also be prefix-comparable, contradicting the fact that $Q$ is a prefix code.

To show maximality of $\overline{xQ}$ if $Q$ is maximal, consider any word $z \in A^*$; we want to show that $z$ is prefix-comparable with some element of $\overline{xQ}$. Since $Q$ is a maximal prefix code, $xz$ is prefix-comparable with some $q \in Q$.

- If $xz$ is a prefix of $q$ then $xzt = q$ for some $t \in A^*$, so $z$ is a prefix of $zt \in \overline{xQ}$.
- If $q$ is a prefix of $xz$ then no other element of $Q$ is a prefix of $xza$. Either (case 1), $q$ is a prefix of $x$ or (case 2), $q = xp$ for some $p \in \overline{xQ}$ such that $p$ is a prefix of $z$; hence in case 2, $z$ is prefix-comparable to an element of $\overline{xQ}$. In case 1, $\overline{xQ} = \emptyset$; indeed, $q$ is a prefix of $x$ (so $\overline{x\{q\}} = \emptyset$), and no other element of $Q$ is prefix-comparable with $x$ ($Q$ being a prefix code), hence $\overline{x\{q'\}} = \emptyset$ for all $q' \in Q - \{q\}$.  □
9 Appendix A2

In this appendix we give details of the proof of Lemma 3.11. If $x, y \in \{a, b\}^*$, we abbreviate $x >_{\text{pref}} y$ to $x > y$; recall that this means that $x\{a, b\}^*$ strictly contains $y\{a, b\}^*$, i.e., $x$ is a strict prefix of $y$. Similarly, $x \not{\geq} y$ means that $x$ is not a prefix of $y$.

**Fact A2.1**

\begin{align*}
\sigma^{-1} \cdot (a^k|b^h\alpha v) \cdot \sigma &= (a^{k+1}|b^{h-1}\alpha v) \quad \text{for all } h \geq 2, \; v \in \{a, b\}^*, \; k \geq 1; \\
\sigma^{-1} \cdot (a^k|\alpha v) \cdot \sigma &= (a^{k+1}|\alpha b) \quad \text{for all } v \in \{a, b\}^*, \; k \geq 1; \\
\sigma^{-1} \cdot (a^k|\alpha h) \cdot \sigma &= (a^{k+1}|\alpha b^{h-1}) \quad \text{for all } h \geq 2, \; k \geq 1; \\
(ab|b) \cdot (a^k|b) \cdot (ab) &= (a^k|ab) \quad \text{for all } k \geq 2.
\end{align*}

**Proof.** Verification of (1.1): When $h \geq 2, \; v \in \{a, b\}^*, \; k \geq 1$, then

\[
\sigma^{-1} \cdot (a^k|b^h\alpha v) \cdot \sigma = \sigma^{-1} \cdot \left[ \begin{array}{cccc} a^k & b^h & \alpha v \\ a^k & b^h & \alpha v \\ 1 \leq i < k & 1 \leq j < h & u > v, \; u \notin 2v & \end{array} \right] \cdot \left[ \begin{array}{ccc} a^2 & ab & b \\ a & b^2 \\ 1 \leq j < h-1 & \end{array} \right] = (a^{k+1}|b^{h-1}\alpha v).
\]

- Verification of (1.2): When $v \in \{a, b\}^*, \; k \geq 1$, then

\[
\sigma^{-1} \cdot (a^k|\alpha v) \cdot \sigma = \sigma^{-1} \cdot \left[ \begin{array}{cccc} a^k & \alpha v \\ a^k & \alpha v \\ 1 \leq i < k & u > v, \; u \notin 2v & \end{array} \right] \cdot \left[ \begin{array}{ccc} a^2 & ab & b \\ a & b^2 \\ 1 \leq j < h-1 & \end{array} \right] = (a^{k+1}|\alpha b).
\]

- Verification of (1.3): When $h \geq 2, \; k \geq 1$, then

\[
\sigma^{-1} \cdot (a^k|\alpha h) \cdot \sigma = \sigma^{-1} \cdot \left[ \begin{array}{cccc} a^k & \alpha h \\ a^k & \alpha h \\ 1 \leq i < k & \end{array} \right] \cdot \left[ \begin{array}{ccc} a^2 & ab & b \\ a & b^2 \\ 2 \leq i \leq k & \end{array} \right] = (a^{k+1}|\alpha b).
\]
\[ \sigma^{-1} \cdot \left[ \begin{array}{cccc} a^k & b^h & a^i b & b^i a \\ b^h & a^k & a^i b & b^i a \end{array} \right] \cdot \left[ \begin{array}{cc} a^2 & ab \\ ba & b^2 \end{array} \right] \]

\[ = \left[ \begin{array}{cc} a & ba \\ a^2 & ab^2 \end{array} \right] \cdot \left[ \begin{array}{cccc} a^{k+1} & b^{h-1} & a^i b & ab^i a \\ b^{h-1} & a^{k+1} & a^i b & ab^i a \end{array} \right] \]

\[ = \left( a^{k+1} | b^{h-1} \right). \]

**Verification of (1.4):** When \( k \geq 2 \), then

\[ (ab|b) \cdot (ab|b) \cdot (ab|b) \]

\[ = (ab|b) \cdot \left[ \begin{array}{cccc} a^k & b & a^i b \\ b & a^k & a^i b \\ a^i b & ab & b \end{array} \right] \cdot \left[ \begin{array}{cc} a^2 & ab \\ ba & b \end{array} \right] \]

\[ = \left[ \begin{array}{cccc} a^2 & ab \\ a^2 & ba \\ a^2 & ab \end{array} \right] \cdot \left[ \begin{array}{cccc} a^k & ab & a^i b & b \\ b & a^k & a^i b & ab \\ 0 \leq i \leq k-1 & 1 \leq j \leq h-1 & p \geq v, p \not\preceq v \end{array} \right] \]

\[ = (a^k|ab). \]

This proves Fact A2.1. \( \square \)

**Fact A2.2**

(2.1) If \((k > j) \geq 2, h \geq 1\), then \(\sigma \cdot (a^k|a^j b^h v) \cdot \sigma^{-1} = (a^{k-1}|a^j b^{h-1} v)\).

(2.2) If \(j = 1, k \geq 3, h \geq 1\), then \(\delta \cdot (a^k|a b^h v) \cdot \delta^{-1} = (a^{k-1}|a b^h v)\).

(2.3) If \(j = 1, k = 2, h \geq 2\), then \(\gamma_1 \cdot (a^2|a b^h v) \cdot \gamma_1^{-1} = (a^2|a b^{h-1} v)\).

(2.4) If \(j = 1, k = 2, h = 1, |v| \geq 2\) (so \(v = au\) for some \(u \in \{a, b\} \{a, b\}^*\)), then \(\gamma_2 \cdot (a^2|a b a u) \cdot \gamma_2^{-1} = (a^2|a b u)\).

**Proof.** Verification of (2.1): If \((k > j) \geq 2, h \geq 1\), then

\[ \sigma \cdot (a^k|a^j b^h v) \cdot \sigma^{-1} \]

\[ = \sigma \cdot \left[ \begin{array}{cccccc} a^k & a^j b^h v & a^j b & a^j b^h v & a^j b^h v & a^j b^h v & \ldots \\ a^j b^h v & a^k & a^j b & ab^j a & a^j b^h p & a^j b^h p & \ldots \\ 0 \leq i \leq k-1 & 1 \leq j < h & p \geq v & p \not\preceq v \end{array} \right] \cdot \left[ \begin{array}{cc} a & ba \\ ba & b^2 \\ a^2 & ab \end{array} \right] \]

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\[ \begin{bmatrix} a^2 & ab & b \\ a & ba & b^2 \end{bmatrix} \cdot \begin{bmatrix} a^{k-1} & a^{i-1} & a^{i+1} & a^{j-1} & a^{j+1} & a^{i-1}p & ba & bb \\ a^{j-1} & a^{k-1} & a^{i-1} & a^{i+1} & a^{j-1} & a^{j+1} & a^{i-1}p & ba & bb \\ 0 \leq i \leq k-2 & 0 \leq i \leq k-2 & 0 \leq i \leq k-2 & 0 \leq i \leq k-2 & 0 \leq i \leq k-2 & 0 \leq i \leq k-2 & 0 \leq i \leq k-2 & 0 \leq i \leq k-2 & 0 \leq i \leq k-2 \end{bmatrix} \]

\[ = (a^{k-1}|a^{j-1}b^h v). \]

- Verification of (2.2): If \( j = 1 \), \( k \geq 3 \), \( h \geq 1 \), then
  \[ \delta \cdot (a^k|ab^h v) \cdot \delta^{-1} \]
  (The column “ab^r a” is absent if \( h = 1 \).)

\[ = \delta \cdot \begin{bmatrix} a^k & ab^h v & a^{i}b & a^{i+1} & a^{j-1} & a^{j+1} & a^{i-1}p & ba & bb \\ ab^h v & a^k & a^{i}b & a^{i+1} & a^{j-1} & a^{j+1} & a^{i-1}p & ba & bb \\ 2 \leq i \leq k-2 & 2 \leq i \leq k-2 & 2 \leq i \leq k-2 & 2 \leq i \leq k-2 & 2 \leq i \leq k-2 & 2 \leq i \leq k-2 & 2 \leq i \leq k-2 & 2 \leq i \leq k-2 & 2 \leq i \leq k-2 \end{bmatrix} \cdot \begin{bmatrix} a^2 & ba & ab & b^2 \\ a^3 & a^2b & ab & b \\ 1 \leq r \leq h & p > v, p \notin v \\
\]

\[ = \begin{bmatrix} a^3 & a^2b & ab & b \\ a^2 & ba & ab & b^2 \end{bmatrix} \cdot \begin{bmatrix} a^{k-1} & ab^h v & a^{i}b & a^{j-1} & a^{j+1} & a^{i-1}p & ba & bb \\ ab^h v & a^k & a^{i}b & a^{j-1} & a^{j+1} & a^{i-1}p & ba & bb \\ 2 \leq i \leq k-2 & 2 \leq i \leq k-2 & 2 \leq i \leq k-2 & 2 \leq i \leq k-2 & 2 \leq i \leq k-2 & 2 \leq i \leq k-2 & 2 \leq i \leq k-2 & 2 \leq i \leq k-2 & 2 \leq i \leq k-2 \end{bmatrix} \]

\[ = (a^{k-1}|ab^h v). \]

- Verification of (2.3): If \( j = 1 \), \( k = 2 \), \( h \geq 2 \), then
  \[ \gamma_1 \cdot (a^2|ab^h v) \cdot \gamma_1^{-1}. \]
  (The column “ab^r p^\ell” is absent if \( v \) is empty.)

\[ = \gamma_1 \cdot \begin{bmatrix} a^2 & ab^h v & ab^r a & ab^r p^\ell & ba & bb \\ ab^h v & a^2 & ab^r a & ab^r p^\ell & ba & bb \\ 1 \leq r \leq h & p > v, p \notin v \\
\]

\[ = \begin{bmatrix} a^2 & aba & ab^2 & b \\ a^2 & ba & ab^2 & b^2 \end{bmatrix} \cdot \begin{bmatrix} a^2 & ab^{h-1} v & ab^r a & ab^r p^\ell & ba & bb \\ ab^{h-1} v & a^2 & ab^r a & ab^r p^\ell & ba & bb \\ 1 \leq r \leq h-2 & 1 \leq r \leq h-2 \end{bmatrix} \]

\[ = (a^2|ab^{h-1} v). \]

- Verification of (2.4): If \( j = 1 \), \( k = 2 \), \( h = 1 \), \(|v| \geq 2 \) (so \( v = au \) for some \( u \in \{a, b\} \{a, b\}^* \)), then \( \gamma_2 \cdot (a^2|abau) \cdot \gamma_2^{-1} = (a^2|abu) \).

  Case (2.4.a): \( v = aaw \), for some \( w \in \{a, b\}^* \).
\[ \gamma_2 \cdot (a^2|aba^2 w) \cdot \gamma_2^{-1} \]  
\[
= \gamma_2 \cdot \left[ \begin{array}{cccccc}
    a^2 & ab^2 & ab & ab^2 & aba^2 p\ell & b \\
apa^2 w & a^2 & abab & ab^2 & aba^2 p\ell & b \\
\end{array} \right] \cdot \left[ \begin{array}{cccccc}
    a^2 & ab & ba & b^2 \\
\end{array} \right] 
\]
\[
= \left[ \begin{array}{cccccc}
    a^2 & ab & ab^2 & b \\
apa^2 w & a^2 & abab & ab^2 & aba^2 p\ell & ab^2 & b \\
\end{array} \right] \cdot \left[ \begin{array}{cccccc}
    a^2 & abw & ab^2 & abap\ell & ba & bb \\
\end{array} \right] 
\]
\[
= \left[ \begin{array}{cccccc}
    a^2 & abw & ab^2 & abap\ell & ba & bb \\
\end{array} \right] 
\]
\[
= a^2|abaw). 
\]

Case (2.4.b): \( v = abw \), for some \( w \in \{a,b\}^* \).

\[ \gamma_2 \cdot (a^2|abaw) \cdot \gamma_2^{-1} \]  
\[
= \gamma_2 \cdot \left[ \begin{array}{cccccc}
    a^2 & abw & ab^2 & abw & aba^2 p\ell & b \\
\end{array} \right] \cdot \left[ \begin{array}{cccccc}
    a^2 & ab & ba & b^2 \\
\end{array} \right] 
\]
\[
= \left[ \begin{array}{cccccc}
    a^2 & abw & ab^2 & abw & aba^2 p\ell & b \\
\end{array} \right] \cdot \left[ \begin{array}{cccccc}
    a^2 & abw & ab^2 & abw & aba^2 p\ell & ab^2 & b \\
\end{array} \right] 
\]
\[
= \left[ \begin{array}{cccccc}
    a^2 & abw & ab^2 & abw & abap\ell & ab^2 & abp\ell & ba & bb \\
\end{array} \right] 
\]
\[
= (a^2|abbw). 
\]

This proves Fact A2.2. □
10 Appendix A3

In this appendix we give details of the proof of Theorem 6.4

CASE $\alpha\alpha$: $\mu = \alpha_{h_1}^m \beta_{k_i}^{m-1} \ldots \beta_{k_1}^1 \alpha_{h_1} \ldots \alpha_{h_2}^2 \beta_{k_1} \alpha_{h_1}$,

CASE $\beta\alpha$: $\mu = \beta_{k_i}^m \alpha_{h_1} \beta_{k_i}^{m-1} \ldots \beta_{k_1}^1 \alpha_{h_1} \ldots \alpha_{h_2}^2 \beta_{k_1} \alpha_{h_1}$,

Claim (*$\alpha$). Let $\mu$ be as above, according to cases $\alpha\alpha$ or $\beta\alpha$. Then,

$\mu(a) = (w_m v_m^{k_{m-1}}) t_m u_m^{h_{m-1}} w_{m-1} \ldots w_i v_i^{k_{i-1}} t_i u_i^{h_{i-1}} \ldots w_j v_j^{k_{j-1}} t_j u_j^{h_{j-1}} w_0$

where:

$w_0 = \begin{cases} ba & \text{if } h_1 > 0 \\ a^2 & \text{if } h_1 < 0 \end{cases}$

For $m - 1 \geq i \geq 1$: $w_i = \begin{cases} ba^3 & \text{if } h_{i+1} > 0, k_i > 0 \\ ba^2b & \text{if } h_{i+1} > 0, k_i < 0 \\ a^4 & \text{if } h_{i+1} < 0, k_i > 0 \\ a^3b & \text{if } h_{i+1} < 0, k_i < 0 \end{cases}$

In case $\beta\alpha$: $w_m = \begin{cases} a^3 & \text{if } k_m > 0 \\ a^2b & \text{if } k_m < 0 \end{cases}$

For $m - 1 \geq i \geq 1$, or $i = m$ in case $\beta\alpha$: $t_i = \begin{cases} bab^3 & \text{if } h_i > 0, k_i > 0 \\ a^2b^2 & \text{if } h_i > 0, k_i < 0 \\ baba & \text{if } h_i < 0, k_i > 0 \\ a^2ba & \text{if } h_i < 0, k_i < 0 \end{cases}$

In case $\alpha\alpha$: The factor $(w_m v_m^{k_{m-1}})$ is absent, and $t_m = \begin{cases} b^3 & \text{if } h_m > 0 \\ b^2a & \text{if } h_m < 0 \end{cases}$

For $m \geq i \geq 1$: $u_i = \begin{cases} a & \text{if } h_i > 0 \\ b & \text{if } h_i < 0 \end{cases}$, $v_i = \begin{cases} a & \text{if } k_i > 0 \\ b & \text{if } k_i < 0 \end{cases}$

We also have:

$\mu(ba) = (w_m v_m^{k_{m-1}}) t_m u_m^{h_{m-1}} w_{m-1} \ldots w_i v_i^{k_{i-1}} t_i u_i^{h_{i-1}} \ldots w_j v_j^{k_{j-1}} t_j u_j^{h_{j-1}} W_0$

where:

$W_0 = \begin{cases} b^2 & \text{if } h_1 > 0 \\ ab & \text{if } h_1 < 0 \end{cases}$

and all other $w_i$, $v_i$, $t_i$, and $u_i$ are the same as for $\mu(a)$.

Proof of Claim (*$\alpha$). The proof goes by induction on the number of exponents $(k_m, h_m, \ldots, k_1, h_1 \in \mathbb{Z} - \{0\})$.

Base of the Induction: A straightforward induction on $h_1$ shows that for all $h_1 \in \mathbb{Z} - \{0\}$,

$\alpha^{h_1}(a) = \begin{cases} b^3a^{h_1-1}ba & \text{if } h_1 > 0 \\ b^2ab^{h_1-1}a^2 & \text{if } h_1 < 0 \end{cases}$

$\alpha^{h_1}(ba) = \begin{cases} b^3a^{h_1-1}b^2 & \text{if } h_1 > 0 \\ b^2ab^{h_1-1}ab & \text{if } h_1 < 0 \end{cases}$
So Claim \((\ast \alpha)\) holds when we just have one non-zero exponent.

**Inductive Step:** We assume \(h_1 \neq 0\).

**Case** \(\alpha \alpha\): We consider the case where \(\mu\) is of the form \(\alpha \alpha\) (with, \(k_m = 0, h_m \neq 0\)).

Assume \(h_m, \ldots, k_1, h_1\) are non-zero, and \(k_m = 0\). By induction, \(\mu(a)\) and \(\mu(ba)\) are of the form
\[
\mu(a) = t_m u_m^{h_m-1} w_{m-1} \ldots w_1 v_1^{k_1} t_1 u_1^{h_1-1} w_0
\]
\[
\mu(ba) = t_m u_m^{h_m-1} w_{m-1} \ldots w_1 v_1^{k_1} t_1 u_1^{h_1-1} W_0
\]
with \(t_m = \begin{cases} b^3 & \text{if } h_m > 0 \\ b^2a & \text{if } h_m < 0 \end{cases}\)

(1) If \(h_m > 0\) and another \(\alpha\) is applied to \(\mu(a)\) or \(\mu(ba)\), then \(t_m = b^3\); by looking at the entry \(b^3\) in the table of \(\alpha\) we obtain
\[
\alpha \mu(a) = t_m u_m^{h_m+1} w_{m-1} \ldots w_1 v_1^{k_1} t_1 u_1^{h_1-1} w_0 \quad (\text{respectively } W_0 \text{ instead of } w_0 \text{ for } \alpha \mu(ba)).
\]

Also, \(|h_m| + 1 - 1 = |h_m + 1| - 1\) when \(h_m > 0\), so we have verified the induction hypothesis.

(2) If \(h_m < 0\) and another \(\alpha^{-1}\) is applied to \(\mu(a)\) or \(\mu(ba)\), then \(t_m = b^2a\); the entry \(b^2a\) in the range-row of the table of \(\alpha\) yields then
\[
\alpha^{-1} \mu(a) = t_m u_m^{h_m+1} w_{m-1} \ldots w_1 v_1^{k_1} t_1 u_1^{h_1-1} w_0 \quad (\text{respectively } W_0 \text{ instead of } w_0 \text{ for } \alpha^{-1} \mu(ba)).
\]

When \(h_m < 0, |h_m| + 1 - 1 = |h_m - 1| - 1\), so we have verified the induction hypothesis.

(3) If \(\beta\) is applied to \(\mu(a)\) or to \(\mu(ba)\), and \(h_m > 0\), we use the fact that \(t_m (= b^3)\) starts with \(b\). The entry \(b\) in the table of \(\beta\) implies that the leftmost \(b\) in \(t_m\) is replaced by \(a^3ba\). So, \(\mu(a)\) (or \(\mu(ba)\)) becomes
\[
\beta(\mu(a)) \quad (\text{or } \beta(\mu(ba)))
\]
\[
= a^3ba b^2 u_m^{h_m-1} w_{m-1} \ldots w_1 v_1^{k_1} t_1 u_1^{h_1-1} w_0 \quad (\text{respectively } W_0).
\]
\[
= w_m' v_m'^{h_m-1} u_m'^{h_m-1} w_{m-1} \ldots w_1 v_1^{k_1} t_1 u_1^{h_1-1} w_0 \quad (\text{respectively } W_0).
\]

Indeed, now \(k_m\) becomes \(1\); when \(k_m > 0\) we have for the new factors \(w_m' = a^3\) (and \(v_m = b\), \(b = 1\)); moreover, when \(k_m > 0\) and \(h_m > 0\), the new value of \(t_m\) is \(t_m' = ab^2\). So we have verified the induction hypothesis.

If \(h_m < 0\), we obtain
\[
\beta(\mu(a)) \quad (\text{or } \beta(\mu(ba)))
\]
\[
= a^3ba ba u_m^{h_m-1} w_{m-1} \ldots w_1 v_1^{k_1} t_1 u_1^{h_1-1} w_0 \quad (\text{respectively } W_0).
\]
\[
= w_m' v_m'^{h_m-1} u_m'^{h_m-1} w_{m-1} \ldots w_1 v_1^{k_1} t_1 u_1^{h_1-1} w_0 \quad (\text{respectively } W_0).
\]

Indeed, now \(k_m = 1\); when \(k_m > 0\) we have \(w_m' = a^3\) (and \(v_m = b\), but \(|k_m| - 1 = 0\) here); moreover, when \(k_m > 0\) and \(h_m < 0\), \(t_m' = bab\). So we have verified the induction hypothesis.

(4) If \(\beta^{-1}\) is applied to \(\mu(a)\) or to \(\mu(ba)\), we again use the fact that \(t_m\) starts with \(b\). The entry \(b\) in the range-row of the table of \(\beta\) implies that the leftmost \(b\) in \(t_m\) is replaced by \(a^2ba^2\).

If \(h_m > 0, \mu(a)\) (or \(\mu(ba)\)) becomes
\[
\beta^{-1}(\mu(a)) \quad (\text{or } \beta^{-1}(\mu(ba)))
\]
\[
= a^2ba^2 b^2 u_m^{h_m-1} w_{m-1} \ldots w_1 v_1^{k_1} t_1 u_1^{h_1-1} w_0 \quad (\text{respectively } W_0).
\]
Indeed, now, \( k_m = -1 \), and when \( k_m > 0 \) we have \( w'_m = a^2 b \) (and \( v_m = a \), but \( |k_m| - 1 = 0 \) here); moreover, when \( k_m < 0 \) and \( h_m > 0 \), \( t'_m = a^2 b^2 \). So we have verified the induction hypothesis.

If \( h_m < 0 \), \( \mu(a) \) (or \( \mu(ba) \)) becomes
\[
\beta^{-1}(\mu(a)) \quad (or \quad \beta^{-1}(\mu(ba)) )
\]
\[
= a^2 ba^2 ba v_m^{h_m-1} w_m - 1 \cdots w_1 v_1^{k_1-1} t_1 u_1^{h_1-1} w_0 \quad (respectively \ W_0 ).
\]
\[
= w'_m v_m^{k_m-1} t_m u_m^{h_m-1} w_m - 1 \cdots w_1 v_1^{k_1-1} t_1 u_1^{h_1-1} w_0 \quad (respectively \ W_0 ).
\]
Indeed, now, \( k_m = -1 \), and when \( k_m > 0 \) we have \( w'_m = a^2 b \) (and \( v_m = a \), but \( |k_m| - 1 = 0 \) here); moreover, when \( k_m < 0 \) and \( h_m < 0 \), \( t'_m = a^2 ba \).

This completes the verification of the induction hypothesis in the case where \( \mu \) is of the form \( \alpha \).

**Case \( \beta \alpha \):** We consider the case where \( \mu \) is of the form \( \beta \alpha \) (with \( h_{m+1} = 0, k_m \neq 0 \)). By induction, \( \mu(a) \) (and \( \mu(ba) \)) are of the form
\[
\mu(a) = w_m v_m^{k_m-1} t_m u_m^{h_m-1} w_m - 1 \cdots w_1 v_1^{k_1-1} t_1 u_1^{h_1-1} w_0
\]
\[
\mu(ba) = w_m v_m^{k_m-1} t_m u_m^{h_m-1} w_m - 1 \cdots w_1 v_1^{k_1-1} t_1 u_1^{h_1-1} W_0
\]
with \( w_m = \begin{cases} a^3 & \text{if } k_m > 0 \\ a^2 b & \text{if } k_m < 0 \end{cases} \)

(5) If \( \alpha \) is applied to the string \( \mu(a) \) or \( \mu(ba) \), we look at the entry \( a \) in the table of \( \alpha \) (since in all case, \( w_m \) starts with \( a \)).

If \( k_m > 0 \), we use the entry \( a \) in the table of \( \alpha \) (since \( w_m = a^3 \), and \( \mu(a) \) (or \( \mu(ba) \)) becomes
\[
\alpha(\mu(a)) \quad (or \quad \alpha(\mu(ba)) )
\]
\[
= b^4 a^2 v_m^{k_m-1} t_m u_m^{h_m-1} w_m - 1 \cdots w_1 v_1^{k_1-1} t_1 u_1^{h_1-1} w_0 \quad (respectively \ W_0 )
\]
\[
= t_m u_m^{h_m+1} w_m v_m^{k_m-1} t_m u_m^{h_m-1} w_m - 1 \cdots w_1 v_1^{k_1-1} t_1 u_1^{h_1-1} w_0 \quad (respectively \ W_0 )
\]
Indeed, now \( h_{m+1} = 1 \); when \( h_{m+1} > 0 \) we have \( t'_m + 1 = b^2 \) (and \( u_m = a \), but \( |h_{m+1}| - 1 = 0 \) here anyway); moreover, when \( h_{m+1} = 0 \) and \( k_m > 0 \), \( w'_m = ba^3 \). So we have verified the induction hypothesis.

If \( k_m < 0 \), we use again the entry \( a \) in the table of \( \alpha \) (now, \( w_m = a^2 b \), and \( \mu(a) \) (or \( \mu(ba) \)) becomes
\[
\alpha(\mu(a)) \quad (or \quad \alpha(\mu(ba)) )
\]
\[
= b^4 a ab v_m^{k_m-1} t_m u_m^{h_m-1} w_m - 1 \cdots w_1 v_1^{k_1-1} t_1 u_1^{h_1-1} w_0 \quad (respectively \ W_0 )
\]
\[
= t_m u_m^{h_m+1} w_m v_m^{k_m-1} t_m u_m^{h_m-1} w_m - 1 \cdots w_1 v_1^{k_1-1} t_1 u_1^{h_1-1} w_0 \quad (respectively \ W_0 )
\]
Indeed, now \( h_{m+1} = 1 \); when \( h_{m+1} > 0 \) we have \( t'_m + 1 = b^2 \) (and \( u_m = a \), but \( |h_{m+1}| - 1 = 0 \) here anyway); moreover, when \( h_{m+1} > 0 \) and \( k_m < 0 \), \( w'_m = ba^2 b \). So we have verified the induction hypothesis.

(6) If \( \alpha^{-1} \) is applied to the string \( \mu(a) \) or \( \mu(ba) \), we look at the entry \( a \) in the table of \( \alpha \) (since in all case, \( w_m \) starts with \( a \)).

If \( k_m > 0 \), we use the entry \( a \) in the range-row of the table of \( \alpha \) (since \( w_m = a^3 \), and \( \mu(a) \) (or \( \mu(ba) \)) becomes
Indeed, now \( h_{m+1} = -1 \); when \( h_{m+1} < 0 \) we have \( t'_{m+1} = b^2a \) (and \( u_{m+1} = b \), but \( |h_{m+1}| - 1 = 0 \) here anyway); moreover, when \( h_{m+1} < 0 \) and \( k_m > 0 \), \( w_m = a^4 \). So we have verified the induction hypothesis.

If \( k_m < 0 \), we use again the entry \( a \) in the table of \( \alpha \) (now, \( w_m = a^2b \), and \( \mu(a) \) (or \( \mu(ba) \)) becomes

\[
\alpha^{-1}(\mu(a)) \quad (or \quad \alpha^{-1}(\mu(ba)) \quad = b^2a^2 v_m |_{k_m=1} t_m u_m |_{h_m=1} w_{m-1} \ldots w_1 v_1 |_{k_1=1} t_1 u_1 |_{h_1=1} w_0 \quad (\text{respectively } W_0)
\]

Indeed, now \( h_{m+1} = -1 \); when \( h_{m+1} < 0 \) we have \( t'_{m+1} = b^2a \) (and \( u_{m+1} = b \), but \( |h_{m+1}| - 1 = 0 \) here anyway); moreover, when \( h_{m+1} < 0 \) and \( k_m < 0 \), \( w_m = a^2b \). So we have verified the induction hypothesis.

(7) If \( k_m > 0 \) and another \( \beta \) is applied to \( \mu(a) \) or \( \mu(ba) \), we look at the entry \( a^3 \) in the table of \( \beta \) (since \( w_m = a^3 \)). Then and \( \mu(a) \) (or \( \mu(ba) \)) becomes

\[
\beta(\mu(a)) \quad (or \quad \beta(\mu(ba)) \quad = w_m v_m |_{k_m=1} t_m u_m |_{h_m=1} w_{m-1} \ldots w_1 v_1 |_{k_1=1} t_1 u_1 |_{h_1=1} w_0 \quad (\text{respectively } W_0)
\]

When \( k_m > 0 \), \( v_m = a \) and \( |k_m| + 1 - 1 = |k_m + 1| - 1 \), so we have verified the induction hypothesis.

(8) If \( k_m < 0 \) and another \( \beta^{-1} \) is applied to \( \mu(a) \) or \( \mu(ba) \), we look at the entry \( a^2b \) in the range-row of the table of \( \beta \) (since \( w_m = a^2b \)). Then and \( \mu(a) \) (or \( \mu(ba) \)) becomes

\[
\beta^{-1}(\mu(a)) \quad (or \beta^{-1}(\mu(ba)) \quad = w_m v_m |_{k_m=1} t_m u_m |_{h_m=1} w_{m-1} \ldots w_1 v_1 |_{k_1=1} t_1 u_1 |_{h_1=1} w_0 \quad (\text{respectively } W_0)
\]

When \( k_m < 0 \), \( v_m = b \) and \( |k_m| - 1 - 1 = |k_m + 1| - 1 \), so we have verified the induction hypothesis.

This completes the proof of Claim (\( *\alpha \)).

\[ \square \]

CASE \( \alpha \beta \): \( \mu = \alpha^h \beta^{k_{m-1}} \ldots \beta^{k_i} \alpha^{k_i} \ldots \alpha^{k_2} \beta^{k_1} \).

CASE \( \beta \alpha \): \( \mu = \beta^{k_{m-1}} \alpha^{h_{m-1}} \ldots \beta^{k_i} \alpha^{k_i} \ldots \alpha^{k_2} \beta^{k_1} \).

Claim (\( *\beta \)). Let \( \mu \) be as above, according to cases \( \alpha \beta \) or \( \beta \alpha \). Then,

\[
\mu(b) = (w_m v_m |_{k_m=1} t_m u_m |_{h_m=1} w_{m-1} \ldots w_1 v_1 |_{k_1=1} t_1 u_1 |_{h_1=1} w_0)
\]

where:

\[
t_1 = \begin{cases} 
ba & \text{if } k_1 > 0 \\
 a^2 & \text{if } k_1 < 0
\end{cases}
\]

and all other \( w_i, v_i, t_i, \) and \( u_i \) are the same as for \( \mu(a) \) in Claim (\( *\alpha \)).

We also have:

\[
\mu(ab) = (w_m v_m |_{k_m=1} t_m u_m |_{h_m=1} w_{m-1} \ldots w_1 v_1 |_{k_1=1} t_1 u_1 |_{h_1=1} w_0)
\]
where:

\[
T_1 = \begin{cases} 
  b^2 & \text{if } k_1 > 0 \\
  ab & \text{if } k_1 < 0 
\end{cases}
\]

and all other \( w_i, v_i, t_i, \) and \( u_i \) are the same as for \( \mu(b) \).

**Proof of Claim (\( \ast \beta \)).** The proof goes by induction on the number of non-zero exponents \( k_m, h_m, \ldots, k_1 \) in \( \mu \). By assumption, in Claim (\( \ast \beta \)) we have \( k_1 \neq 0 \).

**Base of the Induction:** A straightforward induction on \( k_1 \) shows that for all \( k_1 \in \mathbb{Z} - \{0\} \),

\[
\beta(b) = \begin{cases} 
  a^3a^{|k_1|−1}ba & \text{if } k_1 > 0 \\
  a^2bb^{|k_1|−1}a^2 & \text{if } k_1 < 0 
\end{cases}
\]

\[
\beta(ab) = \begin{cases} 
  a^3a^{|k_1|−1}b^2 & \text{if } k_1 > 0 \\
  a^2bb^{|k_1|−1}ab & \text{if } k_1 < 0 
\end{cases}
\]

So Claim (\( \ast \alpha \)) holds when we just have one non-zero exponent.

**Inductive Step.**

The proof of the inductive step is very similar to the proof of the inductive step of Claim (\( \ast \alpha \)). The results are the same too, except for \( t_1 \) and \( T_1 \), which are dealt with in the base of the induction.

The proof of Claim (\( \ast \beta \)) completes the proof of Theorem 6.4. \( \square \)

### 11 Appendix A4

In this appendix we give the proofs of the Lemmas related to Theorem 7.2, about the representation of Thompson groups in algebras.

**Lemma 11.1** \( \mathcal{B}_\infty \) is closed under multiplication.

**Proof.** The condition that \( \{(y_i, x_i) : i \in I\} \) is finite-to-finite will guarantee that multiplication in \( \mathcal{B}_\infty \) is well defined (i.e., no infinite sums in \( \mathbb{K} \) are used). Indeed, let \( \sum_{j \in J} \alpha_j v_j u_j^{-1}, \sum_{k \in K} \beta_k t_k s_k^{-1} \in \mathcal{B}_\infty \), and let

\[
\sum_{i \in I} \kappa_i y_i x_i^{-1} = \sum_{j \in J} \alpha_j v_j u_j^{-1} \cdot \sum_{k \in K} \beta_k t_k s_k^{-1}.
\]

Then, for any fixed \( i \in I \) the coefficient of \( y_i x_i^{-1} \) is

\[
\kappa_i = \sum \{ \alpha_j \beta_k : j \in J, k \in K \text{ are such that } y_i x_i^{-1} = v_j u_j^{-1} t_k s_k^{-1} \}.
\]

If \( y_i x_i^{-1} = v_j u_j^{-1} t_k s_k^{-1} \) then we have the following two possibilities:

- **[case 1]** \( y_i x_i^{-1} = v_j r_k s_k^{-1} \), if \( t_k = u_j r_k \leq \text{pref } u_j \), for some \( r_k \in A^* \), or
- **[case 2]** \( y_i x_i^{-1} = v_j (s_k w_j)^{-1} \), if \( t_k \geq \text{pref } u_j = t_k w_j \), for some \( w_j \in A^* \).

In both cases, there are only finitely many ways to choose \( v_j \) for a given \( y_i \) (since \( v_j \) is a prefix of \( y_i \)). Hence, there are only finitely many ways to choose \( u_j \) (by the finite-to-finite property of the sum \( \sum \alpha_j v_j u_j^{-1} \)).

In case 1, \( r_k \) can be chosen in a finite number of ways (being a suffix of \( y_i \)), hence there are only finitely many choices for \( t_k \). Hence, there are only finitely many choices for \( s_k \) (by the finite-to-finite property of the sum \( \sum \beta_k t_k s_k^{-1} \)).
In case 2, \( t_k \) is a prefix of \( u_j \), and there were only finitely many possible choices for \( t_k \) (since there are only finitely many choices for \( u_j \) in both cases, as we saw). By the the finite-to-finite condition of the sum \( \sum \beta_k t_k s_k^{-1} \), there will only be finitely many choices for \( s_k \).

In summary, if we fix just \( y_i \) (irrespective of what \( x_i \) might be), \( y_i x_i^{-1} \) has only a finite number of factorizations of the form \( y_i x_i^{-1} = v_j u_j^{-1} t_k s_k^{-1} \) (for a fixed \( \sum_{j \in J} \alpha_j v_j u_j^{-1} \) and \( \sum_{k \in K} \beta_k t_k s_k^{-1} \in B_\infty \)). Therefore, \( \kappa_i = \sum \{ \alpha_j \beta_k : j, k \text{ etc.} \} \) is a finite sum, hence \( \kappa_i \) is well defined.

The above also implies that \( y_i \) determines a finite number of possibilities for \( x \) such that \( y_i x_i^{-1} \in \{ y_i x_i^{-1} : i \in I \} \) (for a fixed \( \sum_{j \in J} \alpha_j v_j u_j^{-1}, \sum_{k \in K} \beta_k t_k s_k^{-1} \in B_\infty \)). Hence, \( y_i \) determines a finite number of possibilities for the value of \( x_i \). In a similar way one proves that, given \( x_i \), there are only finitely many choices for \( y \) such that \( y x_i^{-1} \in \{ y_i x_i^{-1} : i \in I \} \). Hence the finite-to-finite property is preserved under multiplication. \( \square \)

**Lemma 11.2**  The algebra \( \mathcal{A}_\infty \) consists of the elements \( \sum_{i \in I} \kappa_i y_i x_i^{-1} \) of \( B_\infty \) that have the following three properties:

1. The relation \( S = \{ (x_i, y_i) : i \in I \} \) is bounded finite-to-finite (i.e., there is a bound on the cardinalities of all the sets \( S(x_i) \) and \( S^{-1}(y_i) \) as \( i \) ranges over \( I \)).
2. In \( \{ x_i : i \in I \} \) and in \( \{ y_i : i \in I \} \), all \( \succ_{\text{pref}} \)-chains have bounded length.
3. The set \( \{ \kappa_i : i \in I \} \) is finite (i.e., only finitely many different coefficients occur).

**Proof.** Properties (1), (2) and (3) are straightforward consequences of the definition of \( \mathcal{A}_\infty \).

Conversely, suppose \( \sum_{i \in I} \kappa_i y_i x_i^{-1} \in B_\infty \) satisfies (1), (2) and (3). Let \( n_{1,x}, n_{1,y}, n_{2,x} \) and \( n_{2,y} \) be the bounds that occur in properties (1) and (2); \( n_{2,x} \) is the maximum length of a \( \succ_{\text{pref}} \)-chain in \( \{ x_i : i \in I \} \), and similarly for \( n_{2,y} \); \( n_{1,x} = \max \{|S(x_i)| : i \in I \} \), and \( n_{1,y} = \max \{|S^{-1}(y_i)| : j \in I \} \).

We want to prove by induction on \( n_{1,x} + n_{1,y} + n_{2,x} + n_{2,y} \) that \( \sum_{i \in I} \kappa_i y_i x_i^{-1} \) is a finite linear combination of elements of \( U_{\text{part}}^{\infty} \).

Base of the induction: If \( n_{2,x} = n_{2,y} = 1 \) then \( \{ x_i : i \in I \} \) and \( \{ y_i : i \in I \} \) are prefix codes. If in addition \( n_{1,x} = n_{1,y} = 1 \) then the relation \( S \) is injective, hence \( \sum_{i \in I} y_i x_i^{-1} \in U_{\text{part}}^{\infty} \). It follows that \( \sum_{i \in I} \kappa_i y_i x_i^{-1} \) is a finite linear combination of elements of \( U_{\text{part}}^{\infty} \), since \( \{ \kappa_i : i \in I \} \) is finite.

Inductive steps: If \( n_{2,x} > 1 \), let \( P = \{ x_i : i \in J \} \subseteq \{ x_i : i \in I \} \) (\( J \subseteq I \)) be a prefix code which is \( \subseteq \)-maximal in the set \( \{ x_i : i \in I \} \). Then \( \sum_{i \in J} \kappa_i y_i x_i^{-1} = \sum_{i \in J} \kappa_i y_i x_i^{-1} + \sum_{i \in I - J} \kappa_i y_i x_i^{-1} \). Since \( P \) is maximal within \( \{ x_i : i \in I \} \), the longest \( \succ_{\text{pref}} \)-chain in \( \{ x_i : i \in I - J \} \) is strictly shorter than the longest \( \succ_{\text{pref}} \)-chain in \( \{ x_i : i \in I \} \). So, the number \( n_{2,x} \) has strictly decreased for \( \sum_{i \in I - J} \kappa_i y_i x_i^{-1} \), while the other three numbers did not increase. Hence, by induction, \( \sum_{i \in I - J} \kappa_i y_i x_i^{-1} \in \mathcal{A}_\infty \).

For the sum \( \sum_{i \in J} \kappa_i y_i x_i^{-1} \), the number \( n_{2,x} = 1 \); if for this sum the number \( n_{2,y} > 1 \), we take a prefix code \( Q = \{ y_i : i \in H \} \subseteq \{ y_i : i \in I \} \) which is \( \subseteq \)-maximal in \( \{ y_i : i \in I \} \). Then \( \sum_{i \in H} \kappa_i y_i x_i^{-1} = \sum_{i \in H} \kappa_i y_i x_i^{-1} + \sum_{i \in J - H} \kappa_i y_i x_i^{-1} \). Then for \( \sum_{i \in J - H} \kappa_i y_i x_i^{-1} \), the number \( n_{2,y} \) (i.e., the maximum length of a \( \succ_{\text{pref}} \)-chain in \( \{ y_i : i \in J - H \} \)) has strictly decreased, while the other three numbers did not increase. Hence, by induction, \( \sum_{i \in J - H} \kappa_i y_i x_i^{-1} \in \mathcal{A}_\infty \).

For the sum \( \sum_{i \in H} \kappa_i y_i x_i^{-1} \), the numbers \( n_{2,x} = n_{2,y} = 1 \) (since both \( \{ x_i : i \in H \} \) and \( \{ y_i : i \in H \} \)) are prefix codes). If for this sum \( n_{1,x} > 1 \) or \( n_{1,y} > 1 \), we write \( \sum_{i \in H} \kappa_i y_i x_i^{-1} \) as
Lemma 11.3 There is a one-to-one correspondence between (1) the set of all isomorphisms
between (essential) right ideals of \( \{a, b\}^* \), and (2) the set \( \mathcal{U}_\infty^{\text{part}} \) (respectively \( \mathcal{U}_\infty \)).

Similarly, there is a one-to-one correspondence between (1) the set of all isomorphisms
between finitely generated (essential) right ideals of \( \{a, b\}^* \), and (2) the set \( \mathcal{U}_V^{\text{part}} \) (respectively \( \mathcal{U}_V \)).

Proof. We give the proof for \( \mathcal{U}_\infty^{\text{part}} \) (and for \( \mathcal{U}_\infty \)); for \( \mathcal{U}_V^{\text{part}} \) and \( \mathcal{U}_V \) it is similar. The correspondence map is

\[
\Sigma : \varphi \mapsto \sum_{x \in P_1} \varphi(x) x^{-1}
\]

where \( P_1 \) (the domain code of \( \varphi \)) is a prefix code. This map is clearly onto \( \mathcal{U}_\infty^{\text{part}} \). Injectiveness of \( \Sigma \) follows from the fact that the map defined below, is the inverse of \( \Sigma \):

\[
\Phi : \sum_{i \in I} y_i x_i^{-1} \in \mathcal{U}_\infty^{\text{part}} \mapsto (\varphi : \{x_i : i \in I\} A^* \to \{y_i : i \in I\} A^*)
\]

where \( \varphi \) is defined by \( \varphi(x) = \sum_{i \in I} y_i x_i^{-1} x \cap A^* \). Equivalently, \( \varphi(x) = y_j w \) if \( x = x_j w \) for some \( j \in I \), \( w \in A^* \); \( \varphi(x) \) is undefined otherwise. Then, since \( \{x_i : i \in I\} \) and \( \{y_i : i \in I\} \) are prefix codes, \( \varphi \) is an isomorphism between right ideals (which are essential if \( \sum_{i \in I} y_i x_i^{-1} \in \mathcal{U}_\infty \)).

\( \square \)

Lemma 11.4 The sets \( \mathcal{U}_\infty, \mathcal{U}_V, \mathcal{U}_\infty^{\text{part}} \) and \( \mathcal{U}_V^{\text{part}} \) are closed under multiplication.

Proof. Let \( \sigma_2 = \sum_{j \in J} y_j x_j^{-1}, \sigma_1 = \sum_{i \in I} v_i u_i^{-1} \in \mathcal{U}_\infty^{\text{part}} \). In the product \( \sigma_2 \sigma_1 \), each term \( y_j x_j^{-1}.v_i u_i^{-1} \) falls into one of the following four cases:

(0) \( x_j \) and \( v_i \) are not prefix-comparable: Then \( y_j x_j^{-1}.v_i u_i^{-1} = 0 \).

(1) \( x_j <_{\text{pref}} v_i \) : Since \( \{v_i : i \in I\} \) is a prefix code there will be at most one \( v_i \) for a given \( x_j \), such that we are in this case; so, there is a partial function \( f : j \in J \mapsto f(j) \in I \) such that \( x_j <_{\text{pref}} v_{f(j)} \). The domain of \( f \) is \( \text{dom} f = \{j \in J : x_j <_{\text{pref}} v_i \text{ for some } i \in I\} \). For every \( j \in \text{dom} f \) there is (a unique) \( z_j \in A^* \) such that \( x_j = v_{f(j)} z_j \). Now, \( y_j x_j^{-1}.v_i u_i^{-1} = y_j z_j^{-1} u_{f(j)}^{-1} \).

(2) \( x_j >_{\text{pref}} v_i \) : Since \( \{x_j : j \in J\} \) is a prefix code there will be at most one \( x_j \) for a given \( v_i \), such that we are in this case; so, there is a partial function \( g : i \in I \mapsto g(i) \in J \) such that \( x_j >_{\text{pref}} v_i \). The domain of \( g \) is \( \text{dom} g = \{i \in I : x_j >_{\text{pref}} v_i \text{ for some } j \in J\} \). Then there is (a unique) \( t_i \in A^* \) such that \( v_i = x_{g(i)} t_i \). Now, \( y_j x_j^{-1}.v_i u_i^{-1} = y_{g(i)} t_i u_i^{-1} \).

(3) \( x_j = v_i \) : Then \( y_j x_j^{-1}.v_i u_i^{-1} = y_j u_i^{-1} \). Moreover, for a given \( v_i \) there is at most one \( x_j \) such that \( x_j = v_i \). The same reasoning as in the two previous cases applies here; we can write \( i = f(j) \) and \( j = g(i) \), i.e., this case corresponds to \( j \in \text{dom} f \cap \text{img} \), or equivalently, \( i \in \text{dom} g \cap \text{im} f \). Now, \( y_j x_j^{-1}.v_i u_i^{-1} = y_j u_{f(j)}^{-1} \).

This yields the following formula for the multiplication in \( \mathcal{U}_\infty^{\text{part}} \):

\[
\sum_{j \in J} y_j x_j^{-1}.\sum_{i \in I} v_i u_i^{-1} = \sum_{j \in \text{dom} f \cap \text{img}} y_j u_{f(j)}^{-1} + \sum_{j \in \text{dom} f - \text{img}} y_j z_j^{-1} u_{f(j)}^{-1} + \sum_{i \in \text{dom} g - \text{im} f} y_{g(i)} t_i u_i^{-1}
\]
Let us check that the domain code \( \{u_{f(j)}z_j : j \in \text{dom} f\} \cup \{u_i : i \not\in \text{im} f\} \) of \( \sigma_2 \sigma_1 \) is a prefix code, which is maximal if \( \sigma_2, \sigma_1 \in U_\infty \). (For the image code \( \{y_{g(i)}t_i : i \in \text{dom} g\} \cup \{y_j : j \not\in \text{im} g\} \) of \( \sigma_2 \sigma_1 \) the proof is similar.) Indeed, for each \( i \in \text{im} f \) (this corresponds to cases (1) and (3)), the set
\[
Z_i = \{z_j : f(j) = i\} = \overline{v_i}\{x_j : j \in J\}
\]
is a prefix code of \( A^* \), which is maximal if \( \{x_j : j \in J\} \) is a maximal prefix code; see Lemma 8.5 for the notation \( \overline{v_i}\{\ldots\} \). Similarly, for each \( j \in \text{im} g \) (corresponding to cases (2) and (3)), the set \( T_j = \{t_i : g(i) = j\} = \overline{v_j}\{v_i : i \in I\} \) is a prefix code of \( A^* \), which is maximal if \( \{v_i : i \in I\} \) is a maximal prefix code. This follows from Lemma 8.5 since \( \{x_j : j \in J\} \) and \( \{v_i : i \in I\} \) are prefix codes.

Now, observe that \( \{u_{f(j)}z_j : j \in \text{dom} f\} \cup \{u_i : i \not\in \text{im} f\} = \bigcup_{i \in I} u_iZ_i \). By construction (2) in Example 14 this is a prefix code, which is maximal if \( \{u_i : i \in I\} \) and each \( Z_i \) are maximal (moreover, each \( Z_i \) will be maximal if \( \{x_j : j \in J\} \) is maximal). Similarly, \( \{y_{g(i)}t_i : i \in \text{dom} g\} \cup \{y_j : j \not\in \text{im} g\} = \bigcup_{j \in J} y_jT_j \) is a prefix code, which is maximal if \( \{y_j : j \in J\} \) and each \( T_j \) are maximal (moreover, each \( T_j \) will be maximal if \( \{v_i : i \in I\} \) is maximal).

Now, from the multiplication formula and the fact that the domain and image codes in that formula are indeed prefix codes, it follows that \( U_\infty^{\text{part}} \) is closed under multiplication. Since the domain and image codes in the formula are maximal prefix codes if the sums are in \( U_\infty \), it follows that \( U_\infty \) is also closed under multiplication. For \( U_V^{\text{part}} \) and \( U_V \) the proofs are similar.

\( \square \)

**Lemma 11.5**  The one-to-one correspondence \( \Phi \) of Lemma 11.3 is a homomorphism, i.e., for all \( \sigma_2, \sigma_1 \in U_\infty \):  
\( \Phi(\sigma_2 \cdot \sigma_1) = \Phi(\sigma_2) \circ \Phi(\sigma_1) \).

**Proof.**  Let \( \sigma_2 = \sum_{j \in J} y_jx_j^{-1}, \sigma_1 = \sum_{i \in I} v_iu_i^{-1}, \) and \( \pi = \sum y_jz_j^{-1}u_f(j) + \sum y_jz_j^{-1}u_f(j) + \sum y_{g(i)}t_iu_i^{-1}, \) as in the multiplication formula of Lemma 11.4. We also saw in Lemma 11.4 that  
\( D = \{u_{f(j)}z_j : j \in \text{dom} f\} \cup \{u_i : i \not\in \text{im} f\} \) is a prefix code of \( A^* \) (which is maximal if \( \sigma_2, \sigma_1 \in U_\infty \)).

It is straightforward to check that for all \( x \in DA^* \), \( \Phi(\pi)(x) = (\Phi(\sigma_2) \circ \Phi(\sigma_1))(x) \), and both sides are defined.

For all \( x \not\in DA^* \), \( \Phi(\pi)(x) \) is undefined; so, to complete the proof we still must show that \( (\Phi(\sigma_2) \circ \Phi(\sigma_1))(x) \) is undefined when \( x \not\in DA^* \). Note that \( x \not\in DA^* \) iff \( x >_{\text{pref}} d \) for some \( d \in D \), or \( x \) is not prefix-comparable with any word in \( D \). More technically, \( \Phi(\sigma_2)(\Phi(\sigma_1)(x)) \) is undefined iff for all \( i \in I, j \in J \): either we have \( v_iu_i^{-1} x \not\in A^* \), or we have \( v_iu_i^{-1} x = s \in A^* \) but \( y_jx_j^{-1} s \not\in A^* \).

**Case 1:** \( x >_{\text{pref}} d \) for some \( d \in D \).

Let \( \{d_k : k \in K\} \subset D \) be the elements of \( D \) that have \( x \) as a prefix; all other elements of \( D \) are prefix-incomparable with \( x \) (since \( D \) is a prefix code).

- If \( x >_{\text{pref}} d_k \) and \( d_k \) is of the form \( u_i \) (with \( v_i = x_j \) or \( v_i <_{\text{pref}} x_j \)), or if \( d_k \) is of the form \( u_i z_j \) and \( x >_{\text{pref}} u_i = x s_i \) (for some non-empty word \( s_i \in A^* \)), then \( v_iu_i^{-1} = x = v_i s_i^{-1} \not\in A^* \).

- If \( d_k \) is of the form \( u_i z_j \) (with \( v_i >_{\text{pref}} x_j = v_i z_j \)), and \( u_i >_{\text{pref}} x = u_is \) >_{\text{pref}} u_i z_j, then \( u_iu_i^{-1} \cdot x = v_i s_i, \) and \( y_jx_j^{-1} \cdot u_is = y_jz_j^{-1}s_i = \emptyset \) (since \( s_i \) and \( z_j \) are not prefix-comparable).

So, in all these cases, \( \Phi(\sigma_2)(\Phi(\sigma_1)(x)) \) is undefined.

**Case 2:** \( x \) is not prefix-comparable with any word in \( D \).
If $x$ is not prefix-comparable with any word $u_i$ then $\Phi(\sigma_1)(x)$ is undefined ($v_iu_i^{-1} \cdot x = 0$ for all $i$).

If $x \geq_{\text{pref}} u_i$ for some $u_i$ then $x$ is prefix-comparable with some word in $D_i$ but this cannot happen in Case 2.

If $x <_{\text{pref}} u_i$ for some $u_i$ but $x$ is not prefix-comparable with any word $u_i z_j \in D_i$ then for those words $u_i$ we have: $x = u_i s_i$ where $s_i$ is not prefix-comparable with $z_j$. Then $v_i u_i^{-1} x = v_i s_i \in A^*$, but $y_j x_j^{-1} v_i s_i = y_j (v_i z_j)^{-1} v_i s_i = y_j z_j^{-1} s_i = 0$ (since $s_i$ is not prefix-comparable with $z_j$).  

It will be useful to extend the definition of the map $\Phi$ of Lemma \ref{lem11.3} to all of $A_\infty$. First some notation: For a set $S$ and a field $K$ we denote the set of all finite $K$-multisets over $S$ by $K[S]$; such a multiset has the form $\{\kappa_j s_j : j = 1, \ldots, n\}$, with $n \in \mathbb{N}$, $\kappa_j \in K$ and $s_j \in S$. For any $\sum_{i \in I} \kappa_i y_i x_i^{-1} \in A_\infty$ we define

$$\Phi(\sum_{i \in I} \kappa_i y_i x_i^{-1}) = (\varphi : \{x_i : i \in I\} A^* \to K[\{y_i : i \in I\} A^*])$$

where $\varphi$ is defined by

$$\varphi(x) = \{\kappa_i y_i x_i^{-1} : x : i \in I\} \cap K[A^*] = \{\kappa_i y_i w_i : i \in I, x_i \geq_{\text{pref}} x = x_i w_i\}.$$

**Lemma 11.6**  The one-to-one correspondence $\Sigma$ of Lemma \ref{lem11.3} respects the congruence relations on the set $U_\infty$ (induced by $I_\infty$) and on the set of all isomorphisms between essential right ideals (i.e., two isomorphisms $\varphi_1$ and $\varphi_2$ between essential right ideals are congruent iff $\Sigma(\varphi_1)$ and $\Sigma(\varphi_2)$ are congruent). A similar fact holds for $U_V$.

**Proof.** We only prove this for for $U_\infty$; for $U_V$ it is similar. Suppose $\varphi_1$ and $\varphi_2$ are congruent, i.e., $\varphi_2$ can be obtained from $\varphi_1$ by extensions and restrictions. Then, by Lemmas \ref{lem2.3} and \ref{lem2.4} any extension or restriction of $\varphi_1$ can be carried out by repeatedly (perhaps infinitely often) using maximal prefix codes $Q_i$ as in Lemma \ref{lem2.4} This corresponds precisely to applying (perhaps infinitely many) relations of the form $\sum_{q \in Q_i} q q^{-1} \to 1$ for an extension, or with “$\to$” replaced by “$\leftarrow$” for a restriction. Hence $\Sigma(\varphi_1)$ and $\Sigma(\varphi_2)$ are congruent modulo $I_\infty$.

Conversely, if $\Sigma(\varphi_1)$ and $\Sigma(\varphi_2)$ are congruent then their difference is an element of $I_\infty$, i.e., $\Sigma(\varphi_2) - \Sigma(\varphi_1)$ is a linear combination of elements of the form

$$\sum_{j \in I} v_j u_j^{-1} \cdot \sum_{i \in I} y_i (\sum_{q \in Q_i} q q^{-1} - 1) x_i^{-1} \cdot \sum_{k \in K} t_k s_k^{-1},$$

where $\{x_i, i \in I\}, \{y_i, i \in I\}$, and every $Q_i$ are maximal prefix codes, and where $\sum_{j \in J} v_j u_j^{-1}$, $\sum_{k \in K} t_k s_k^{-1} \in U_\text{part}$. Since $\{x_i, i \in I\}, \{y_i, i \in I\}$, and every $Q_i$ are maximal prefix codes, it follows that $\bigcup_{i \in I} x_i Q_i$ and $\bigcup_{i \in I} y_i Q_i$ are maximal prefix codes, by construction (2) of Example \ref{ex11.1}. It is straightforward to check that $\sum_{i \in I} y_i (\sum_{q \in Q_i} q q^{-1} - 1) x_i^{-1}$ acts as the empty map on the essential right ideal $\bigcup_{i \in I} x_i Q_i A^*$, i.e., for every $x = x_i q_i w$ (with $i_0 \in I, q_0 \in Q_{i_0}$ and $w \in A^*$), $\sum_{i \in I} y_i (\sum_{q \in Q_i} q q^{-1} - 1) x_i^{-1} \cdot x = 0$. Therefore, $\sum_{i \in I} y_i (\sum_{q \in Q_i} q q^{-1} - 1) x_i^{-1} \cdot \sum_{k \in K} t_k s_k^{-1}$ acts as the empty map on the set

$$\Phi(\sum_{k \in K} t_k s_k^{-1})^{-1}(\bigcup_{i \in I} x_i Q_i A^*) = \Phi(\sum_{k \in K} s_k t_k^{-1})(\bigcup_{i \in I} x_i Q_i A^*).$$

This set is a right ideal (being the image of the right-ideal isomorphism $\Phi(\sum_{k \in K} s_k t_k^{-1})$), but not necessarily an essential right ideal. Let $RA^*$ be any essential right ideal containing $\Phi(\sum_{k \in K} s_k t_k^{-1})\bigcup_{i \in I} x_i Q_i A^*$). Then $\sum_{i \in I} y_i (\sum_{q \in Q_i} q q^{-1} - 1) x_i^{-1} \cdot \sum_{k \in K} t_k s_k^{-1}$ acts as the empty map on $RA^*$, and so does $\sum_{j \in J} v_j u_j^{-1} \cdot \sum_{i \in I} y_i (\sum_{q \in Q_i} q q^{-1} - 1) x_i^{-1} \cdot \sum_{k \in K} t_k s_k^{-1}$.
Thus, \( \varphi_1 \) and \( \varphi_2 \) agree on the essential right ideal \( RA^* \). Therefore \( \max \varphi_1 = \max \varphi_2 \) by uniqueness of the maximum extension (Lemma 2.1).

\[ \square \]

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