Differential calculi on noncommutative bundles

Markus J. Pflaum, Peter Schauenburg
pflaum@rz.mathematik.uni-muenchen.de,
schauen@rz.mathematik.uni-muenchen.de

Mathematisches Institut der Universität München
Theresienstraße 39
80333 München 2, Germany
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Abstract

We introduce a category of noncommutative bundles. To establish geometry in this category we construct suitable noncommutative differential calculi on these bundles and study their basic properties. Furthermore we define the notion of a connection with respect to a differential calculus and consider questions of existence and uniqueness. At the end these constructions are applied to basic examples of noncommutative bundles over a coquasitriangular Hopf algebra.

keywords: coquasitriangular Hopf algebra, smash product, noncommutative bundles, noncommutative differential calculus, noncommutative connection

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Introduction

The notion of a principal fiber bundle is crucial for geometry and gauge theory. Because of its importance and its far reaching applications people have generalized it to the noncommutative setting in one way or another (see Schneider [13], Pflaum [7, 9], Schauenburg [8, 11], Brzezinski and Majd [1]). In particular, [13] is based on the fact that the noncommutative version of an affine principal fiber bundle is a Hopf-Galois extension. Smash products of a Hopf algebra with a module algebra are important examples of such extensions. In [7, 9] smash products (or more generally crossed products, which are smash products with an additional twisting) play the role of local trivializations of fiber bundles with quantum structure groups. When we view a smash product $A\# H$ of a Hopf algebra $H$ with an $H$-module algebra $A$ as a noncommutative principal fiber bundle, the Hopf algebra $H$ is regarded as the ‘noncommutative function space’ of the quantum structure group, the algebra $A$ is the space of ‘functions’ on the base quantum space, and $A\# H$ itself the algebra of ‘functions’ on the quantum fiber bundle.

We will consider noncommutative differential calculi on smash product ‘bundles’. Our background for noncommutative differential calculus is based on the approaches of Woronowicz [17] and Manin [4]. In particular, the modules of differential forms on a quantum space are to be considered as an additional part of the structure of such a space, which is not determined by the function space alone, unlike in the classical case, where there is a canonical functor from commutative algebras of functions to spaces of differential forms. Whenever the quantum spaces are endowed with additional structures (like the one of a quantum group), these have to be compatible with the differentiable structure.

In this spirit we will construct differential calculi on smash products which are compatible with their algebraic structures. These calculi are examples for the theories in [1, 3, 11], and they are local differential calculi for the theory in [7, 9].

Let us summarize the content of our paper. In the first section we recall the definition of smash products and explain how they are noncommutative analogues of principal fiber bundles. In the following sections the concept of first order differential calculi is introduced, and we construct such a calculus for smash products in a canonical way. Later, we lift this first order calculus to a higher order one. In section 4 connections on the noncommutative fiber bundles are defined and their basic structure is studied. We close our article with an important example.
1 Smash products as noncommutative fiber bundles

If not specified otherwise, all algebras and Hopf algebras are supposed to be defined over the field \( k \) and possess a unit 1. We denote the comultiplication and counit of a Hopf algebra \( H \) by \( \Delta : H \to H \otimes H \) and \( \varepsilon : H \to k \). Recall the axioms of a Hopf algebra: \( \Delta \) is supposed to be an algebra map satisfying the coassociativity condition \( (\Delta \otimes 1_H)\Delta = (1_H \otimes \Delta)\Delta \), and \( \varepsilon \) is a counit, \( (\varepsilon \otimes 1_H)\Delta = (1_H \otimes \varepsilon)\Delta = 1_H \). Furthermore, \( H \) has to have an antipode \( S : H \to H \), which, however, we will not need explicitly. We will view noncommutative algebras loosely as the algebras of “functions” on “noncommutative” or “quantum spaces”. More precisely we have a duality (equivalence of categories reversing arrows) between the categories of quantum spaces and noncommutative algebras. In this picture, a Hopf algebra is viewed as the algebra of functions on a quantum group, the comultiplication \( \Delta \) corresponding by duality to the multiplication in the group.

Let \( h \in H \) be an element in a Hopf algebra. The image \( \Delta(h) \) of \( h \) under comultiplication can in general only be written as some finite sum \( \sum h'_i \otimes h''_i \in H \otimes H \) with \( h'_i, h''_i \in H \). To simplify calculations it is customary to use Sweedler’s notation [14, Sec. 1.2]: We write \( \Delta(h) =: \sum(h) h_{(1)} \otimes h_{(2)} \) where the individual symbols \( h_{(1)} \) and \( h_{(2)} \) cannot be used separately, but only make sense in bilinear expressions containing both of them. By coassociativity we can write \( (\Delta \otimes 1_H)\Delta(h) = (1_H \otimes \Delta)\Delta(h) = \sum(h) h_{(1)} \otimes h_{(2)} \otimes h_{(3)} \) etc. A right \( H \)-comodule is a vector space \( V \) with a structure map or coaction \( \rho : V \to V \otimes H \) satisfying coassociativity \( (\rho \otimes 1_H)\rho = (1_V \otimes \Delta)\rho \) and \( (1_V \otimes \varepsilon)\rho = 1_V \). A left \( H \)-comodule \( V \) has a structure map \( \lambda : V \to H \otimes V \) with analogous properties. We write \( \rho(v) = \sum(v) v_{(0)} \otimes v_{(1)} \), \( \lambda(v) = \sum(v) v_{(-1)} \otimes v_{(0)} \) etc. In complicated formulas we will sometimes even omit the summation symbols in Sweedler’s notation. Our general reference for Hopf algebra theory is SWEEDLER [14].

If \( V \) is a right (left) \( H \)-comodule, we define \( V^{coH} := \{ v \in V | v_{(0)} \otimes v_{(1)} = v \otimes 1 \} \) (resp. \( V^{coH} := \{ v \in V | v_{(-1)} \otimes v_{(0)} = 1 \otimes v \}) \) to be the subset of right (resp. left) coinvariant elements.

Let \( R \) be an algebra and a (right) \( H \)-comodule algebra if the structure map \( \rho : R \to R \otimes H \) is an algebra map, that is \( \rho(xy) = \rho(x)\rho(y) \) holds in the algebra \( R \otimes H \). In the language of quantum spaces this coaction corresponds to an action of a quantum group \( G \) on a noncommutative space \( X \). The subalgebra \( R^{coH} \) of coinvariant elements should be viewed as the algebra of functions on the quotient \( X/G \) of \( X \) under the action of \( G \).

An important example of a comodule algebra is the smash product of an algebra \( A \) with a Hopf algebra \( H \) that acts on \( A \). Let us review the construction.

**Definition 1.1** Let \( H \) be a Hopf algebra with counit \( \varepsilon \) and \( A \) an algebra. An \( H \)-action on \( A \) is given by a linear map \( H \otimes A \to A \) such that
(i) \( h \cdot (ab) = \sum_{(h)} \left( h^{(1)} \cdot a \right) \left( h^{(2)} \cdot b \right), \quad h \in H, \quad a, b \in A \)

(ii) \( 1 \cdot a = a, \quad a \in A \)

(iii) \( h \cdot 1 = \varepsilon(h) 1, \quad h \in H \)

(iv) \( g \cdot (h \cdot a) = (gh) \cdot a \quad g, h \in H, \quad a \in A \)

In this case \( A \) is also called an \( H \)-module algebra.

Example 1.2 Let \( H \otimes A \longrightarrow A \) be given by \( h \otimes a \longmapsto ha = \varepsilon(h)a \). Then \( H \) is said to act trivially on \( A \).

For an important example of a nontrivial action we refer to section 5.

Theorem and Definition 1.3 Let \( H \) act on the algebra \( A \). Then the mapping

\[
m : (A \otimes H) \otimes (A \otimes H) \longrightarrow (A \otimes H),
\]

\[
(a \otimes h, b \otimes g) \longmapsto \sum_{(h)} a \left( h^{(1)} \cdot b \right) \otimes h^{(2)} g
\]

defines a product on the linear space \( A \otimes H \), such that \( (A \otimes H, m) \) becomes an algebra with unit \( 1 \otimes 1 \). We simply write \( A \# H \) for this algebra and call it the smash product of \( A \) and \( H \). Through the mapping

\[
\rho : A \otimes H \longrightarrow (A \otimes H) \otimes H,
\]

\[
a \otimes h \longmapsto \sum_{(h)} \left( a \otimes h^{(1)} \right) \otimes h^{(2)}
\]

\( A \# H \) becomes an \( H \)-right comodule algebra. We sometimes write \( a \# h := a \otimes h \).

Proof: see Montgomery [5], Lemma 7.1.2, where a more general statement is proven.

Let us fix a Hopf-algebra \( H \). If \( B \) is another \( H \)-module algebra and \( f : A \longrightarrow B \) an algebra map such that

\( f(ha) = hf(a), \quad a \in A \)

we call \( f \) an \( H \)-module algebra map. Now consider the the mapping

\[
f \# 1_H : A \# H \longrightarrow B \# H, \quad a \otimes h \longmapsto f(a) \otimes h.
\]

It is easily seen that \( f \# 1_H \) preserves the product on \( A \# H \), and that the class of all smash products \( A \# H \) together with maps \( f \# 1_H \) form a category, which we call the category of \( H \)-smash products.
Definition 1.4 The category dual to the one of $H$ smash products is called the category of noncommutative $H$-bundles or quantum $H$-bundles.

In the duality between quantum spaces and noncommutative algebras, the (cartesian) product of spaces corresponds to the tensor product of algebras. Thus, the tensor product algebra $A \otimes H$ is the direct generalization of the function algebra on a trivial (i.e., product) fiber bundle with structure quantum group corresponding to $H$ and base quantum space represented by $A$. This is the special case of Theorem 1.3. that results if we take the trivial action of $H$ on $A$. Taking a nontrivial action results in a certain twisting of this “trivial bundle” which is natural to the noncommutative setting. An even more general construction is also well known (see Montgomery [5]), namely, crossed products involving a (weak) action of $H$ on $A$ and a cocycle. We will concentrate on the smash product since for this particular case we will be able to construct natural differential calculi in section 2.

In the sequel we will frequently use the notion of an $^H_A$-Hopf module. For the convenience of the reader we will give its definition.

Let $H$ be a Hopf-algebra and $A$ be an left $H$-comodule algebra with coaction $\lambda$. An left $A$-module $V$ which is at the same time an left $H$-comodule with structure map $\lambda$ is called an $^H_A$-Hopf module, if

$$\lambda(av) = \lambda a \cdot \lambda v, \quad a \in A, \quad v \in V.$$  \hspace{1cm} (1)

Analogously $^H_A$-Hopf modules are defined, and, for $B$ a right $H$-comodule algebra, $^A_B$-Hopf modules and $^H_B$-Hopf modules. In particular, all four definitions make sense for $A = B = H$. If $V$ is a left and right $H$-module and a left and right $H$-comodule with structure maps $\lambda$ and $\rho$ such that

$$ (av)b = a(vb) \quad a \in A, \quad b \in B, \quad v \in V $$  \hspace{1cm} (2)

$$ (1_H \otimes \rho) \circ \lambda = (\lambda \otimes 1_H) \circ \rho $$  \hspace{1cm} (3)

holds, and all the four Hopf module compatibility conditions are satisfied, then we call $V$ an $^H_H$-Hopf module.
2 First order differential calculi

Definition 2.1 A first order differential calculus for an algebra \( A \) is an \( A \)-bimodule \( M \) together with a derivation \( d : A \rightarrow M \), such that \( M \) is generated by \( d(A) \) as an \( A \)-bimodule. A morphism between first order differential calculi \( (A, M, d) \) and \( (B, N, \delta) \) is given by a pair \((f, F)\), such that the following conditions are fulfilled:

(i) \( f : A \rightarrow B \) is a morphism of algebras.

(ii) \( F : M \rightarrow N \) is an \( A \)-bimodule map, where \( f \) defines the \( A \)-bimodule structure on \( N \).

(iii) The diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{d} & & \downarrow{\delta} \\
M & \xrightarrow{F} & N
\end{array}
\]

commutes.

An algebra map \( f : A \rightarrow B \) is called (once) differentiable, if a map \( F : M \rightarrow N \) exists, such that the pair \((f, F)\) is a morphism of first order differential calculi.

If \((A, M, d)\) is a differential calculus, we usually write \( \Omega^1(A) \) instead of \( A \) and call it the space of 1-forms. A map \( F \) as above is uniquely determined by \( f \), and we will frequently denote it by \( \Omega^1(f) : \Omega^1(A) \rightarrow \Omega^1(B) \).

Note that if \( \Omega^1(A) \) is a first order differential calculus, then \( \Omega^1(A) \) is generated by \( d(A) \) as a left (or right) \( A \)-module. Indeed, every \( \omega \in \Omega^1(A) \) has, by definition, the form \( \omega = \sum x_i d(y_i) z_i \) for some \( x_i, y_i, z_i \in A \), and consequently \( \omega = \sum x_i d(y_i z_i) - \sum x_i y_i d(z_i) \in A d(A) \).

Theorem 2.2 (Tensor product of differential calculi) Assume to be given two differential calculi \( d : A \rightarrow M \) and \( \delta : B \rightarrow N \) over \( k \)-algebras \( A \) and \( B \). Then \( D : A \otimes B \rightarrow M \otimes B \oplus A \otimes N, \ a \otimes b \mapsto da \otimes b + a \otimes \delta b \) is a differential calculus on the tensor product algebra \( A \otimes B \).
Proof: Obviously $A \otimes B \rightarrow M \otimes B \oplus A \otimes N$ is an $A \otimes B$-bimodule in a natural way, and $D$ a $k$-linear map. $D$ is a derivation by the following equation.

\[
D((a \otimes b)(a' \otimes b')) = D(aa' \otimes bb') = d(aa') \otimes bb' + a a' \otimes \delta(bb') = \\
= \quad da \cdot a' \otimes bb' + a a' \otimes \delta b \cdot b' + a \cdot da' \otimes bb' + a a' \otimes b \cdot \delta b' = \\
= \quad D(a \otimes b) \cdot (a' \otimes b') + (a \otimes b) \cdot D(a' \otimes b') , \quad a \in A, \ b \in B.
\]

As the image $D(A \otimes B)$ generates $M \otimes B \oplus A \otimes N$, the theorem is shown. □

If we are given differential calculi $\Omega^1(A)$ and $\Omega^1(B)$ on $A$ resp. $B$ we consider the above defined differential calculus on $A \otimes B$ as the natural one and will always denote it by $\Omega^1(A \otimes B)$. This choice corresponds to a well known fact in the classical case: the decomposition of $\Omega^1(A \otimes B)$ into a direct sum is the analog of the decomposition of the cotangent space of a product manifold as the direct sum of the cotangent spaces of its factors.

If we are given a differential calculus on a Hopf algebra, the differential and Hopf structure should fit together. What we mean by that precisely is explained in the following definition.

Definition 2.3 Let $H$ be a Hopf algebra. A differential calculus $(H, \Omega^1(H), d)$ is called a Hopf differential calculus, if the comultiplication $\Delta : H \rightarrow H \otimes H$ is differentiable.

Proposition 2.4 A differential calculus $(H, \Omega^1(H), d)$ on a Hopf algebra $H$ is a Hopf differential calculus if and only if $\Omega^1(H)$ is an $H$-$H$-Hopf module such that $d$ is bicolinear.

That is, a first order differential calculus on a Hopf algebra is a Hopf differential calculus if and only if it is bicovariant in the sense of Woronowicz [17].

Proof: We have $\Omega^1(H \otimes H) = \Omega^1(H) \otimes H \oplus H \otimes \Omega^1(H)$.

If $\Delta$ is differentiable there exists an $H$-bimodule map $\Delta^1 : \Omega^1(H) \rightarrow \Omega^1(H \otimes H)$ such that the diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\Delta} & H \otimes H \\
\downarrow d & & \downarrow d \otimes 1 + 1 \otimes d \\
\Omega^1(H) & \xrightarrow{\Delta^1} & \Omega^1(H) \otimes H \oplus H \otimes \Omega^1(H)
\end{array}
\]
commutes. Now define $\Delta^1_l = pr_l \circ \Delta^1$, $\Delta^1_r = pr_r \circ \Delta^1$, where $pr_l$ (resp. $pr_r$) are the projections from $\Omega^1(H \otimes H)$ onto $H \otimes \Omega^1(H)$ (resp. $\Omega^1(H) \otimes H$). $pr_l$, $pr_r$ and $\Delta^1$ are $H$-bimodule maps, and therefore so are $\Delta^1_l$ and $\Delta^1_r$. We have for $h \in H$

$$
(1 \otimes \Delta^1_l) \Delta^1_l(dh) = \sum_{(h)} (1 \otimes \Delta^1_l)(h(1) \otimes dh(2))
= \sum_{(h)} h(1) \otimes h(2) \otimes dh(3) \quad (5)
$$

$$
(\Delta \otimes 1) \Delta^1_l(dh) = \sum_{(h)} (\Delta \otimes 1)(h(1) \otimes dh(2))
= \sum_{(h)} h(1) \otimes h(2) \otimes dh(3), \quad (6)
$$

which gives

$$
(1 \otimes \Delta^1_l) \Delta^1_l = (\Delta \otimes 1) \Delta^1_l, \quad (7)
$$
as $\Omega^1(H)$ is generated by elements of the form $dh$, $h \in H$. The same argument proves the corresponding equation for $\Delta^1_r$

$$
(\Delta^1_l \otimes 1) \Delta^1_r = (1 \otimes \Delta) \Delta^1_r \quad (8)
$$

and the equations

$$
(\varepsilon \otimes 1) \Delta^1_l = 1, \quad (9)
$$

$$
(1 \otimes \varepsilon) \Delta^1_r = 1. \quad (10)
$$

The compatibility condition

$$
(\Delta^1_l \otimes 1) \Delta^1_r = (1 \otimes \Delta^1_r) \Delta^1_l \quad (11)
$$
is also easily verified:

$$
(\Delta^1_l \otimes 1) \Delta^1_r(dh) = \sum_{(h)} h(1) \otimes dh(2) \otimes h(3) = (1 \otimes \Delta^1_r)\Delta^1_l(dh) \quad (12)
$$

for $h \in H$, and $\Omega^1(H)$ is an $H^H_H$-Hopf $H^H_H$-module since the left and right comodule structure maps are $H$-linear. Furthermore the above diagram entails $d$ being $H$-colinear, and one direction of the proposition is shown.

Let us suppose now $\Omega^1(H)$ is an $H$-comodule with structure maps $\Delta^1_l : \Omega^1(H) \rightarrow H \otimes \Omega^1(H)$ and $\Delta^1_r : \Omega^1(H) \rightarrow \Omega^1(H) \otimes H$. Define

$$
\Delta^1 : \Omega^1(H) \rightarrow \Omega^1(H \otimes H)
\alpha \mapsto \Delta^1_l(\alpha) + \Delta^1_r(\alpha).
$$
As $\Delta^1_l$ and $\Delta^1_r$ are $H$-bimodule maps, the same holds for $\Omega^1$. The commutativity of the above diagram is easily seen:

$$\Delta^1 d(h) = \Delta^1_l d(h) + \Delta^1_r d(h) = (d \otimes 1)\Delta(h) + (1 \otimes d)\Delta(h) = (d \otimes 1 + 1 \otimes d)\Delta(h).$$

Therefore $(\Delta, \Delta^1)$ is a morphism of differential calculi. $\square$

Assume we are given a differential calculus $\delta : A \rightarrow \Omega^1(A)$ on the algebra $A$ and a bicovariant differential calculus $d : H \rightarrow \Omega^1(H)$ on the Hopf algebra $H$. Suppose further that $H$ acts on $A$. The question now is whether one can construct canonically a differential calculus on the smash product $A \# H$. Recall that a smash product is a twisted version of an ordinary tensor product, which in turn corresponds to a product manifold or in this case trivial bundle. We also have a natural notion of a tensor product differential calculus defined on a tensor product of algebras. It will thus be natural to construct our differential calculus on the “bundle” $A \# H$ as a twisting in some sense of the natural tensor product calculus on the “trival bundle”. In particular, our $\Omega^1(A \# H)$ will have the same direct sum decomposition as $\Omega^1(A \otimes H)$, only the module structures will have to be modified. This is possible, if $H$ acts on $\Omega^1(A)$:

**Definition 2.5** Let $A$ be an $H$-module algebra and $d : A \rightarrow \Omega^1(A)$ a first order differential calculus. We say that $H$ acts on $\Omega^1(A)$, or that $\Omega^1(A)$ is an $H$-module differential calculus, if there is an $H$-module structure on $\Omega^1(A)$ such that $d$ is an $H$-module map and

$$h \cdot (a\beta) = \sum_{(h)} \left(h(1) \cdot a \right) \left(h(2) \cdot \beta\right)$$

for $h \in H$, $a \in A$ and $\beta \in \Omega^1(A)$.

**Remark 2.6**

(i) Let $A$ and $\Omega^1(A)$ be as in the definition. An $H$-action on $\Omega^1(A)$ is uniquely determined, if it exists. Indeed, any $\omega \in \Omega^1(A)$ is a linear combination of elements $ada'$ with $a, a' \in A$, and we have

$$h \cdot (ada') = (h(1) \cdot a)(h(2) \cdot da') = (h(1) \cdot a)d(h(2) \cdot a)$$

if (13) holds and $d$ is $H$-linear.

(ii) If $\Omega^1(A)$ is an $H$-module differential calculus, then a right handed analog of (13) also holds: we have

$$h \cdot (\beta a) = \sum_{(h)} (h(1) \cdot \beta)(h(2) \cdot a)$$

for $h \in H$, $a \in A$ and $\beta \in \Omega^1(A)$. 

9
Assume we are given an \( H \)-module differential calculus on \( A \). The differential calculus on \( A\#H \) is then defined by the following terms:

\[
\Omega^1(A\#H) := \Omega^1(A) \otimes H \oplus A \otimes \Omega^1(H) \quad \text{(differential forms)} \quad (14)
\]

\[
D : A\#H \longrightarrow \Omega^1(A\#H) \quad \text{(derivative)}
\]

\[
a \otimes h \longmapsto \delta a \otimes h + a \otimes dh
\]

\[
(A\#H) \otimes \Omega^1(A\#H) \longrightarrow \Omega^1(A\#H) \quad \text{(left \( A\#H \)-action)}
\]

\[
(a \otimes h, a \otimes g + b \otimes \xi) \longmapsto a(h_{(1)} \cdot \alpha) \otimes h_{(2)}g + a(h_{(1)} \cdot b) \otimes h_{(2)}\xi
\]

\[
\Omega^1(A\#H) \otimes (A\#H) \longrightarrow \Omega^1(A\#H) \quad \text{(right \( A\#H \)-action)}
\]

\[
(\alpha \otimes g + b \otimes \xi, a \otimes h) \longmapsto \alpha(g_{(1)} \cdot a) \otimes g_{(2)}h + b(\xi_{(-1)} \cdot a) \otimes \xi_{(0)}h.
\]

It is a standard calculation to show that the last two of these maps really are actions. We prove the Leibniz-rule for \( D \) by the following equation.

\[
D((a\#g)(b\#h))
\]

\[
= \sum_{(g)} \delta(a(g_{(1)} \cdot b)) \otimes g_{(2)}h + a(g_{(1)} \cdot b) \otimes d(g_{(2)}h)
\]

\[
= \sum_{(g)} \left( \delta(a)(g_{(1)} \cdot b) + a(g_{(1)} \cdot b) \right) \otimes g_{(2)}h + a(g_{(1)} \cdot b) \otimes \left( d(g_{(2)}h) + g_{(2)}dh \right)
\]

\[
(\delta a \otimes g + a \otimes dg)(b\#h) + (a\#g)(\delta b \otimes h + b \otimes dh)
\]

\[
= D(a\#g)(b\#h) + (a\#g)D(b\#h)
\]

for \( g, h \in H \) and \( a, b \in A \). \( D(A\#H) \) generates \( \Omega^1(A\#H) \) as an \( A\#H \)-bimodule. For the proof of this it suffices to show that \( a \delta b \otimes h \) and \( a \otimes g dh \) are in the submodule of \( \Omega^1(A\#H) \) generated by \( D(A\#H) \). But this is clear from

\[
(a \otimes 1) D(b \otimes 1) (1 \otimes h) = (a \otimes 1)(\delta b \otimes 1)(1 \otimes h) = a \delta b \otimes h,
\]

\[
(a \otimes g) D(1 \otimes h) = (a \otimes g)(1 \otimes dh) = \varepsilon(g_{(1)}) a \otimes g_{(2)} dh = a \otimes g dh.
\]

The above considerations prove the main part of the next theorem.

**Theorem 2.7** Let \( \delta : A \longrightarrow \Omega^1(A) \) and \( d : H \longrightarrow \Omega^1(H) \) be differential calculi on the algebra \( A \) resp. Hopf algebra \( H \) and let \( H \) act on \( \Omega^1(A) \). Further denote by \( i : A \longrightarrow A\#H \) the map \( a \mapsto a \otimes 1 \). Then there exists a \( A\#H \)-Hopf\(_{A\#H}^H \)-module \( \Omega^1(A\#H) \), a derivation \( D : A\#H \longrightarrow \Omega^1(A\#H) \) and a monomorphism \( i : \Omega^1(A) \longrightarrow \Omega^1(A\#H) \) such that the diagram
commutes, and \( D : A \# H \longrightarrow \Omega^1(A \# H) \) is a differential calculus. We call it the standard differential calculus on \( A \# H \).

**Proof:** Let \( D : A \# H \longrightarrow \Omega^1(A \# H) \) be the differential calculus on \( A \# H \) defined through the formulas (14) to (17). As

\[
i \circ d(a) = da \otimes 1 = D(a \otimes 1) = D \circ \iota(a),
\]

(22)

the above diagram commutes. \( \Omega^1(A \# H) \) is a right \( H \)-comodule with coaction

\[
\rho : \Omega^1(A \# H) \longrightarrow \Omega^1(A \# H) \otimes H
\]

\[
\rho(\alpha \otimes g + a \otimes \gamma) \mapsto \sum_{(g)} \alpha \otimes g_{(1)} \otimes g_{(2)} + \sum_{(\gamma)} a \otimes \gamma_{(0)} \otimes \gamma_{(1)},
\]

\( a \in A, \ g \in H, \ \alpha \in \Omega^1(A), \ \gamma \in \Omega^1(H) \).

By an easy calculation which is left to the reader it is shown that the right \( A \# H \)-action and the right \( H \)-coaction are compatible. So the theorem is proven. \( \square \)

Apart from the \( H \)-comodule structure on \( \Omega^1(A \# H) \), the standard first order differential calculus has an additional structure. To understand it we formulate an analog of Proposition 2.4 for differential calculi on comodule algebras. As the most important example for the comodule algebra \( R \) in the following theorem think of a smash-product \( A \# H \).

**Theorem 2.8** Let \( R \) be a right \( H \)-comodule algebra and \( \Omega^1(R), \ \Omega^1(H) \) first order differential calculi such that the comultiplication of \( H \) is differentiable. Then \( \rho : R \rightarrow R \otimes H \) is differentiable if and only if the following conditions are satisfied:

(i) \( \Omega^1(R) \) is a \( R \text{Hopf}_R^H \)-module such that \( d : R \rightarrow \Omega^1(R) \) is \( H \)-colinear.

(ii) There is a map \( \pi : \Omega^1(R) \rightarrow R \boxtimes \Omega^1(H) \) of \( R \text{Hopf}_R^H \)-modules making the diagram

\[
\begin{array}{ccccc}
R & \xrightarrow{\cong} & R \Box H \\
\downarrow{d} & & \downarrow{R \Box d} \\
\Omega^1(R) & \xrightarrow{\pi} & R \Box \Omega^1(H)
\end{array}
\]

(23)

commute.
Here \( V \square W := \{ \sum v_i \otimes w_i \in V \otimes W | \sum v_i(0) \otimes v_i(1) \otimes w_i = \sum v_i \otimes w_i(-1) \otimes w_i(0) \} \) denotes the cotensor product of a right \( H \)-comodule \( V \) and a left \( H \)-comodule \( W \). Note that \( R_H \Omega^1(H) \) is indeed a \( R \)-Hopf \( H \)-module with

\[
\rho(\sum r_i \otimes \gamma_i) = \sum r_i \otimes \gamma_i(0) \otimes \gamma_i(1) \]

\[
x \sum r_i \otimes \gamma_i = \sum x(0) r_i \otimes x(1) \gamma_i \]

\[
(\sum r_i \otimes \gamma_i)x = \sum r_i x(0) \otimes \gamma_i x(1) \]

for \( \sum r_i \otimes \gamma_i \in R_H \Omega^1(H) \) and \( x \in R \).

**Proof:** We have \( \Omega^1(R \otimes H) = \Omega^1(R) \otimes H \oplus R \otimes \Omega^1(H) \) with projections \( \text{pr}_\ell \) onto \( R \otimes \Omega^1(H) \) and \( \text{pr}_r \) onto \( \Omega^1(R) \otimes H \).

If \( \rho \) is differentiable, there exists \( \rho^1 : \Omega^1(R) \rightarrow \Omega^1(R \otimes H) \) making

\[
\begin{array}{ccc}
R & \xrightarrow{\rho} & R \otimes H \\
\downarrow{d} & & \downarrow{d \otimes 1 + 1 \otimes d} \\
\Omega^1(R) & \xrightarrow{\rho^1} & (\Omega^1(R) \otimes H) \oplus (R \otimes \Omega^1(H))
\end{array}
\]

commute. Define \( \rho^1_r := \text{pr}_r \delta^1 : \Omega^1(R) \rightarrow \Omega^1(R) \otimes H \) and \( \pi_0 := \text{pr}_r \rho^1 : \Omega^1(R) \rightarrow R \otimes \Omega^1(H) \). As in Proposition 2.4 we conclude that \( \rho^1_r \) makes \( \Omega^1(R) \) an \( R \)-Hopf \( H \)-module with \( d \) a colinear map. Moreover, we have, for \( r \in R \),

\[
(1 \otimes \Delta^1)\pi_0 d(r) = (1 \otimes \Delta^1)(r(0) \otimes dr(1)) = (\rho^1_r \otimes 1)(r(0) \otimes dr(1)) = (\rho^1_r \otimes 1)\pi_0 dr
\]

Since elements of the form \( dr \) for \( r \in R \) generate \( \Omega^1(R) \), this gives \( (\rho^1_r \otimes 1)\pi_0 = (1 \otimes \Delta^1)\pi_0 \), and thus the image of \( \pi_0 \) lies in \( R_H \Omega^1(H) \).

As in 2.4 \( \pi \) and \( \rho^1_r \) are \( R \)-\( R \)-linear maps since \( \rho^1_r \) and \( \text{pr}_r, \text{pr}_\ell \) are.

Conversely, if \( \rho^1_r : \Omega^1(R) \rightarrow \Omega^1(R) \otimes H \) making \( \Omega^1(R) \) a Hopf module and \( d \) colinear, and an \( R \)-\( R \)-linear map \( \pi : \Omega^1(R) \rightarrow R_H \Omega^1(H) \) making (23) commute exist, we can define \( \rho^1 \) by \( \rho^1(\omega) = \rho^1_r(\omega) + \pi(\omega) \).

\[ \square \]

**Theorem 2.9** Let \( A \) be an \( H \)-module algebra, \( \Omega^1(A) \) a first order differential calculus on which \( H \) acts, and \( \Omega^1(H) \) a bicovariant first order differential calculus.
Then the comodule algebra structure of \( A\#H \) is once differentiable with respect to the standard differential calculus \( \Omega^1(A\#H) \).

**Proof:** We already know from 2.7 that a suitable right comodule structure on \( \Omega^1(A\#H) \) exists. The map \( \pi \), if it exists, is the unique map sending \( xdy \) in \( \Omega^1(A\#H) \) to \( \sum_{(x),(y)} x(0)y(0) \otimes x(1)dy(1) \). We claim that the composition of \( \text{pr}_1 : \Omega^1(A\#H) \to A \otimes \Omega^1(H) \) with the canonical isomorphism

\[
\beta : A \otimes \Omega^1(H) \to (A\#H) \square \Omega^1(H)
\]

\[
a \otimes \omega \mapsto a(0)\#\omega_{(-1)} \otimes \omega(0)
\]

has this property: indeed, for \( a, b \in A \) and \( g, h \in H \) we have

\[
\beta \text{pr}_1 \left((a\#g)d(b\#h)\right) = \beta \left(a(g_1 \cdot b) \otimes g(2)dh\right)
\]

\[
= a(g_1 \cdot b)\#g(2)h_1 \otimes g(3)d(h_2)
\]

\[
= (a\#g)(0)(b\#h)(0) \otimes (a\#g)(1)d((b\#h)(1))
\]

\[\square\]
3 Higher order differential calculi

Definition 3.1 A differential graded algebra consists of an algebra $A^\bullet = \bigoplus_{i \geq 0} A^i$ with product $\wedge$ and a differential $d : A^\bullet \to A^\bullet$ such that

(i) $A^i \wedge A^j \subset A^{i+j}$ for $i, j \geq 0$.

(ii) $d^2 = 0$ and $d(A^i) \subset A^{i+1}$ for $i \geq 0$.

(iii) $d(\omega \wedge \nu) = \omega \wedge d\nu + (-1)^i \omega \wedge d\nu$ for $\omega \in A^i$ and $\nu \in A^j$.

Let $A$ be an algebra. A higher order differential calculus on $A$ is a differential graded algebra $\Omega^\bullet (A)$ with $\Omega^0(A) = A$ such that for all $n \in \mathbb{N}$ the vector space $\Omega^n(A)$ is spanned by elements of the form $a_0 da_1 \wedge \ldots \wedge da_n$ for $a_i \in A$.

Assume that $\Omega^\bullet (A)$ and $\Omega^\bullet (B)$ are higher order differential calculi on algebras $A, B$. We call an algebra homomorphism $f : A \to B$ differentiable, if there is a morphism $\Omega^\bullet (f) : \Omega^\bullet (A) \to \Omega^\bullet (B)$ of differential graded algebras with $\Omega^0(f) = f$. In other words $\Omega^\bullet (f) : \bigoplus \Omega^n(A) \to \bigoplus \Omega^n(B)$ is an algebra map defined by a collection of maps $\Omega^n(f) : \Omega^n(A) \to \Omega^n(B)$ such that $d \Omega^n(f) = \Omega^{n+1}(f) d$.

Lemma 3.2 If $f : A \to B$ is differentiable with respect to the higher order differential calculi $\Omega^\bullet (A)$ and $\Omega^\bullet (B)$, then the corresponding $\Omega^\bullet (f)$ is unique.

Proof: Obvious, since $\Omega^\bullet (A)$ is generated as an algebra by $\Omega^0(A) = A$ and $dA$. □

As an example for the concept of differentiability, we again consider differential calculi on Hopf algebras:

Lemma 3.3 Let $H$ be a bialgebra and $\Omega^\bullet (H)$ a higher order differential calculus on $H$. The comultiplication $\Delta$ of $H$ is differentiable if and only if $\Omega^\bullet (H)$ has the structure of a differential graded bialgebra with $\Omega^\bullet (H) = H$ as bialgebras.

For a proof see [12].

We need some notation for the graded comultiplication of the differential graded bialgebra $\Omega^\bullet (H)$. For $\omega \in \Omega^\bullet (H)$ we write

$$ \sum_{[\omega]} \omega_{[1]} \otimes \omega_{[2]} := \Omega^\bullet (\Delta)(\omega) \quad (24) $$
Denoting by \( p^{ij} : \Omega^{i+j}(H \otimes H) \to \Omega^i(H) \otimes \Omega^j(H) \) the projection, we define for \( \omega \in \Omega^n(H) \) and \( i + j = n \)

\[
\sum_{\omega} \omega_{<1,i>} \otimes \omega_{<2,j>} := p^{ij} \Omega^n(\Delta)(\omega)
\]

so that

\[
\Omega^n(\Delta)(\omega) = \sum_{i+j=n} \omega_{<1,i>} \otimes \omega_{<2,j>}.
\]

To simplify notation we sometimes omit the summation symbols \( \sum \) and \( \sum \).

**Definition 3.4** Let \( A \) be an \( H \)-module algebra and \( \Omega^\bullet(A) \) a differential calculus. We say that \( H \) acts on \( \Omega^\bullet(A) \), if there are module structures \( H \otimes \Omega^n(A) \to \Omega^n(A) \) making \( \Omega^\bullet(A) \) a module algebra such that \( d : \Omega^n(A) \to \Omega^{n+1}(A) \) are \( H \)-linear maps.

**Theorem and Definition 3.5** Let \( A \) be an \( H \)-module algebra, \( \Omega^\bullet(A) \) a higher order differential calculus on which \( H \) acts, and \( \Omega^\bullet(H) \) a differential calculus with respect to which \( \Delta : H \to H \otimes H \) is differentiable. Let us define

\[
\Omega^n(A \# H) := \bigoplus_{i=0}^n \Omega^i(A) \# \Omega^{n-i}(H)
\]

for all \( n \geq 0 \), and set

\[
d(\omega \# \gamma) = d\omega \# \gamma + (-1)^i \omega \# d\gamma
\]

\[
(\omega \# \gamma) \land (\nu \# \gamma') = (-1)^{jk} \omega \land (\gamma_{(-1)} \cdot \nu) \# \gamma(0) \gamma'
\]

for \( \omega \in \Omega^i(A), \gamma \in \Omega^j(H) \) and \( \nu \in \Omega^k(A) \). Then \( \Omega^\bullet(A \# H) \) is a differential calculus with respect to which \( \rho : A \# H \to A \# H \otimes H \) is differentiable.

**Proof:** To show the associativity of the product we compute for \( \omega \in \Omega^i(A), \omega' \in \Omega^j(A), \omega'' \in \Omega^{i''}(A), \gamma \in \Omega^j(H), \gamma' \in \Omega^j(H) \) and \( \gamma'' \in \Omega^{j''}(H) \):

\[
((\omega \# \gamma) \land (\omega' \# \gamma')) \land (\omega'' \# \gamma'')
\]

\[
= (-1)^{ji} \omega \land (\gamma_{(-1)} \cdot \omega') \# \gamma(0) \gamma'
\]

\[
= (-1)^{ji+j''} \omega \land (\gamma_{(-1)} \cdot \omega'') \land (\gamma(0) \# \gamma''(0) \gamma')
\]

\[
= (-1)^{ji+j''} \omega \land (\gamma_{(-1)} \cdot \omega'') \# \gamma(0) \gamma''(0) \gamma'
\]

\[
= (-1)^{ji''} \omega \# \gamma \land (\omega' \land (\gamma_{(-1)} \cdot \omega'') \# \gamma(0) \gamma'')
\]

\[
\]

\[
= (\omega \# \gamma) \land ((\omega' \# \gamma') \land (\omega'' \# \gamma'')).
\]
Now we have to prove that \( \rho : A\#H \rightarrow A\#H \otimes H \) is differentiable. Let us define \( \Omega^\bullet(\rho) : \Omega^\bullet(A\#H) \rightarrow \Omega^\bullet(A\#H \otimes H) \) by \( \Omega^\bullet(\rho)(\omega \# \gamma) = \omega \# \gamma_1 \otimes \gamma_2 \). To see that \( \Omega^\bullet(\rho) \) is an algebra map, we compute for \( \omega, \omega', \gamma \) and \( \gamma' \) as above:

\[
\Omega^\bullet(\rho)((\omega \# \gamma) \wedge (\omega' \# \gamma')) = (1)^{ij'} \Omega^\bullet(\rho)(\omega \wedge (\gamma_{(-1)} \cdot \omega') \# \gamma_0 \wedge \gamma')
\]

\[
= \sum_{k + \ell = j} (1)^{i+j+k} \omega \wedge (\gamma_{(1,k}) \wedge (-1) \cdot \omega') \# \gamma_{(0)} \wedge \gamma' \wedge \gamma_{<1,k'>} \otimes \gamma_{<2,l'>} \wedge \gamma'_{<2,l'>}
\]

Since we have \( d\Omega(\Delta) = \Omega(\Delta)d \) for \( \Omega(\Delta) : \Omega(H) \rightarrow \Omega(H) \otimes \Omega(H) \), it follows that

\[
\Omega^{i+j+1}(\rho)d(\omega \# \gamma) = \Omega^{i+j+1}(\rho)(d\omega \# \gamma + (-1)^i \omega \# d\gamma)
\]

\[
= (i : \gamma_{[1]} \otimes \gamma_{[2]} + (-1)^i \omega \# (d\gamma_{[1]} \otimes (d\gamma)_{[2]})
\]

\[
= d(\omega \# \gamma_{[1]} \otimes \gamma_{[2]}) + (-1)^i \gamma \# d(\gamma_{[1]} \otimes \gamma_{[2]})
\]

\[
= d(\omega \# \gamma_{[1]} \otimes \gamma_{[2]})
\]

for \( \omega, \gamma \) as above, and thus \( d\Omega(\rho) = \Omega(\rho)d \). \( \square \)

Note that the module \( \Omega^1(A\#H) \) of differential 1-forms in \( \Omega \) coincides with the one defined earlier in section [2].

To describe the structure of the modules of differential forms on \( \Omega(A\#H) \) we introduce a general construction of a Hopf module over \( A\#H \) as a semidirect product:

**Definition 3.6** Let \( A \) be an \( H \)-module algebra and \( N \) an \( A\)-\( A \)-bimodule. An \( H \)-action on \( N \) is an \( H \)-module structure on \( N \) such that

\[
h \cdot (ana') = \sum_{(h)} (h_{(1)} \cdot a)(h_{(2)} \cdot n)(h_{(3)} \cdot a')
\]
holds for all \( h \in H, n \in N, a, a' \in A \).

If \( N \) is an \( A\text{-}A \)-bimodule on which \( H \) acts, and \( \Gamma \) is a \( H^H \text{Hopf}^H \)-module, then \( N \# \Gamma := N \otimes \Gamma \) is a \( A \# H \text{Hopf}^H \)-module with structure maps

\[
(a \# g)(n \# h) = a(g_1 \cdot n) \# g_2 h \\
(n \# h)(a \# g) = n(h_1 \cdot a) \# h_2 g \\
\rho(n \# h) = n \# h_1 \otimes h_2
\]

for \( g, h \in H, a \in A \) and \( n \in N \).

Note that with this definition we can write

\[
\Omega^n(A \# H) = \bigoplus_{i=1}^{n} \Omega^i(A) \# \Omega^{n-i}(H)
\]

as an \( A \# H \text{Hopf}^H \)-module.
4 Connections

Let us motivate our concept of a noncommutative connection. In the classical case a connection on a fiber bundle corresponds to a splitting of the tangent bundle of the fiber bundle into horizontal and vertical tangent vectors. But a connection also gives rise to a splitting for the space of cotangent vectors. More precisely we have a natural notion of horizontal differential 1-forms on any smooth fiber bundle $P$, i.e. we have an exact sequence

$$0 \rightarrow \Omega^1_{h}(P) \rightarrow \Omega^1(P) \rightarrow \Omega^1(P)/\Omega^1_{h}(P) \rightarrow 0,$$

where $\Omega^1_{h}(P)$ is the pull-back bundle of $\Omega^1(M)$ to $P$. A connection on $P$ now gives a splitting of this sequence (into horizontal and vertical parts). In Theorem 4.1 we will give the noncommutative analog of this sequence and later on define certain notions of a noncommutative connection as different kinds of splittings of this sequence.

Suppose $R$ is an $H$-comodule algebra and $\Omega(H)$, $\Omega(R)$ are differential calculi with respect to which $\Delta : H \rightarrow H \otimes H$ and $\rho : R \rightarrow R \otimes H$ are differentiable. Then we have seen that there is a map $\pi^1 : \Omega^1(R) \rightarrow R \odot H \Omega^1(H)$ of $R\text{Hopf}^H$-modules induced by the $\Omega(H)$-comodule structure of $\Omega(R)$. Suppose that $A := R^{co}H$ is equipped with a differential calculus $\Omega(A)$ such that the inclusion map of $A$ into $R$ is differentiable. Then the map $\iota : \Omega^1(A) \rightarrow \Omega^1(R)$ together with the multiplication in $\Omega(R)$ induces maps

$$m_\ell : R \otimes \Omega^1(A) \rightarrow \Omega^1(R)$$

$$r \otimes \omega \mapsto r \iota(\omega)$$

$$m_r : \Omega^1(A) \otimes R \rightarrow \Omega^1(R)$$

$$\omega \otimes r \mapsto \iota(\omega)r$$

Speaking in a somewhat loose language the spaces $R \otimes \Omega^1(A)$ and $\Omega^1(A) \otimes R$ both correspond naturally to the space of horizontal forms on $R$. That we have a left and right version is a special feature of the noncommutative setting.

Let us now investigate the maps $m_\ell$ and $m_r$ in the special case of $R = A \# H$ with the differential calculus constructed in section 4, eqns. (14) to (17).

**Theorem 4.1** Let $A$ be a left $H$-module algebra, $\Omega(H)$ a differential calculus with respect to which $\Delta : H \rightarrow H \otimes H$ is differentiable, and $\Omega(A)$ a differential calculus on which $H$ acts. Let $\Omega(A \# H)$ be the smash product differential calculus. Then we have the following short exact sequences in $A \# H \mathcal{M}^H_{A \# H}$:

$$0 \rightarrow (A \# H) \otimes \Omega^1(A) \xrightarrow{m_\ell} \Omega^1(A \# H) \xrightarrow{\iota} (A \# H) \odot H \Omega^1(H) \rightarrow 0$$
0 \to \Omega^1(A) \otimes (A\#H) \xrightarrow{m_r} \Omega^1(A\#H) \xrightarrow{\pi^1} (A\#H) \sqcup \Omega^1(H) \to 0.

Note that these two short exact sequences are noncommutative analogues of (26).

The proof of the Theorem will be immediate from the following lemma:

**Lemma 4.2** We have isomorphisms

\[
\begin{align*}
\alpha_r : \Omega^1(A) \otimes (A\#H) & \to \Omega^1(A)\#H \\
\omega \otimes a\#h & \mapsto \omega a\#h \\
\alpha_\ell : (A\#H) \otimes \Omega^1(A) & \to \Omega^1(A)\#H \\
a\#h \otimes \omega & \mapsto a(h_{(1)} \cdot \omega)\#h_{(2)} \\
\beta : A\#\Omega^1(H) & \to (A\#H) \sqcup \Omega^1(H) \\
a\#\gamma & \mapsto a\#\gamma_{(-1)} \otimes \gamma_{(0)}
\end{align*}
\]

making the diagrams
To show commutativity of the three triangles we compute

\[ m_r(\omega \otimes a\#h) = (\omega\#1)(a\#h) = \omega a\#h = \iota_1 \alpha_r(\omega \otimes a\#h) \]

\[ m_\ell(a\#h \otimes \omega) = (a\#h)(\omega\#1) = a(h_{(1)} \cdot \omega)\#h_{(2)} = \iota_1 \alpha_\ell(a\#h \otimes \omega) \]

\[ \beta \text{pr}_2(\omega\#h + a\#\gamma) = \beta(a\#\gamma) = a\#\gamma(-1) \otimes \gamma(0) = \pi^1(\omega\#h + a\#\gamma). \]

It is easy to check that \( \alpha_r, \alpha_\ell \) and \( \beta \) are isomorphisms with \( \alpha_r^{-1}(\omega\#h) = \omega \otimes 1\#h \), \( \alpha_\ell^{-1}(\omega\#h) = 1\#S^{-1}(h_{(1)}) \cdot \omega \otimes h_{(2)} \) and \( \beta^{-1}(\sum a_i\#h_i \otimes \gamma_i) = \sum a_i \#\varepsilon(h_i)\gamma_i. \)

Note that the lemma actually shows that the two short exact sequences in theorem 4.1 are just the sequence

\[ 0 \to \Omega^1(A)\#H \to \Omega^1(A\#H) \to A\#\Omega^1(H) \to 0 \]

in disguise. Nevertheless the complicated statement in 4.1 is more natural, as it uses only maps whose existence is a consequence of differentiability assumptions and not of the specific construction of \( \Omega^1(A\#H) \). In fact, the exact sequence above can also be studied in more general cases, [11, 10].

It is obvious that the module of vector fields with respect to some chosen differential calculus on an algebra should be the dual module of the module of differential forms. In our noncommutative setting we have to make a choice with respect to which side we want to dualize our bimodules of differential forms. We make our choice so that it is natural to write a differential form to the left of a vector field it is evaluated on, as it is common in differential geometry.

**Definition 4.3** Let \( A \) be an algebra. We denote the functor

\[ A \mathcal{M}_A \ni N \mapsto \text{Hom}_{A-}(N, A) \in A \mathcal{M}_A \]

assigning to an \( A\)-\( A \)-bimodule the bimodule of left linear maps to \( A \) by \( ^*(-) \).

Let \( A \) be an algebra and \( \Omega^1(A) \) a first order differential calculus on \( A \). The module of right vector fields on \( A \) (with respect to the chosen differential calculus) is \( \mathcal{X}_r(A) := \text{Hom}_{A-}(\Omega^1(A), A) =: ^*\Omega^1(A) \)

We shall write \( \langle \omega, X \rangle \in A \) for the evaluation of \( \omega \in \Omega^1(A) \) on \( X \in \mathcal{X}_r(A) \).
We recall the definition of the Lie algebra of a bicovariant first order differential calculus on a Hopf algebra from Woronowicz [17]. To motivate it note that classically the Lie algebra of a Lie group is the space of left invariant vector fields, which is dual to the space of left invariant differential one forms. We use the notation described in [12].

**Definition 4.4** Let $H$ be a Hopf algebra and $\Omega^1(H)$ a first order differential calculus on $H$ with respect to which the comultiplication is differentiable. Then we denote by $\mathfrak{h} = \text{Hom}_k^{\co H\Omega^1(H), k}$ the Lie algebra of $H$ with respect to $\Omega^1(H)$. Assume that $\co H\Omega^1(H)$ is finite dimensional and $H$ has a bijective antipode. Then we equip $\mathfrak{h}$ with the right comodule structure defined by $\langle \gamma, X(0) \rangle X(1) = \langle \gamma(0), X \rangle S^{-1}(\gamma(1))$ for all $X \in \mathfrak{h}$ and $\gamma \in \co H\Omega^1(H)$. This is the comodule structure of $\mathfrak{h}$ as the right dual of $\co H\Omega^1(H)$ in the category of right $H$-comodules, that is, the unique comodule structure for which the evaluation map $\co H\Omega^1(H) \otimes \mathfrak{h} \to k$ is colinear.

In the following we assume $\Omega^1(H)$ being finite, that means $\co H\Omega^1(H)$ is finite dimensional. Note that the map $k \to \mathfrak{h} \otimes \co H\Omega^1(H)$ defined by $1_k \mapsto \sum x_i \otimes x_i$, where $x_i$ is a basis of $\mathfrak{h}$ and $x^i$ the dual basis, is also colinear.

**Lemma 4.5** $A\#\Omega^1(H)$ is a free left $A\#H$-module on the vector subspace of all $1\#\gamma$ for $\gamma \in \co H\Omega^1(H)$; that is, the map
\[
A\#H \otimes \co H\Omega^1(H) \to A\#\Omega^1(H)
\]
\[a\#h \otimes \gamma \mapsto (a\#h)(1\#\gamma) = (a\#h\gamma)
\]
is an isomorphism.

**Proof:** By the structure theorem on Hopf modules, Sweedler [14], we have
\[
H \otimes \co H\Omega^1(H) \cong \Omega^1(H)
\]
\[h \otimes \gamma \mapsto h\gamma
\]
\[
\gamma(-2) \otimes S(\gamma(-1))\gamma(0) \leftrightarrow \gamma
\]
whence an inverse for the map of the lemma is given by
\[
A\#\Omega^1(H) \ni a\#\gamma \mapsto (a\#\gamma(-2)) \otimes S(\gamma(-1))\gamma(0) \in A\#H \otimes \co H\Omega^1(H).
\]

**Definition and Lemma 4.6** Let $X \in \mathfrak{h}$. The fundamental vector field $\mathbf{X}$ on $A\#H$ associated to $X$ is the unique right vector field $\mathbf{X} \in \mathcal{X}_r(A\#H)$ vanishing on $\Omega^1(A)\#H$ and satisfying $\langle 1\#\gamma, \mathbf{X} \rangle = \langle \gamma, X \rangle 1_A \# H$ for all $\gamma \in \co H\Omega^1(H)$. We have $\langle \omega \# h + a\#\gamma, \mathbf{X} \rangle = (a\#\gamma(-2))(S(\gamma(-1))\gamma(0), X)$ for all $a \in A$, $\omega \in \Omega^1(A)$, $\gamma \in \Omega^1(H)$ and $h \in H$.  

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The first condition in the definition can be interpreted as saying that $\overline{X}$ is a vertical vector field. The space of vertical vector fields is dual (as an $A$-module) to $A\#\Omega^1(H)$, and the second condition in the definition describes a natural embedding of $\mathfrak{h}$ into this dual space.

**Proof:** Since $\Omega^1(A\#H) \cong \Omega^1(A)\#H \oplus A\#\Omega^1(H)$ as left $A\#H$-modules, giving a right vector field on $A\#H$ vanishing on $\Omega^1(A)\#H$ is equivalent to giving a left $A\#H$-linear map $f : A\#\Omega^1(H) \to A\#H$. By the preceeding Lemma we know that $A\#\Omega^1(H)$ is a left $A\#H$-free module over the vector subspace of all $1\#\gamma$ for $\gamma \in \Gamma^H\Omega^1(H)$, whence there is a unique such map $f$ with $f(1\#\gamma) = \langle \gamma, X \rangle \cdot 1_{A\#H}$ for $\gamma \in \Gamma^H\Omega^1(H)$. For the general formula, we compute

$$\langle \omega\#h + a\#\gamma, X \rangle = \langle a\#\gamma, X \rangle = \langle (a\#\gamma(-2))(1\#S(\gamma(-1))\gamma(0)), X \rangle = \langle a\#\gamma(-2), 1\#S(\gamma(-1))\gamma(0), X \rangle = \langle a\#\gamma(-2), S(\gamma(-1))\gamma(0), X \rangle$$

for $a \in A$ and $\gamma \in \Omega^1(H)$. \hfill $\square$

**Definition 4.7** Let $A$ be an algebra and $\Omega^\bullet(A)$ a differential calculus on $A$. Let $V$ be a vector space. The space of $V$-valued differential $n$-forms on $A$ is $V \otimes \Omega^n(A)$.

A $V$-valued differential $n$-form $\omega$ on $A$ is a possibly indecomposable tensor in $V \otimes \Omega^n(A)$. To simplify calculations, we shall nevertheless frequently write formally $\omega = \omega_V \otimes \omega_A$. The usage of this symbolic notation is similar to that of Sweedler's notation.

**Definition 4.8** A $V$-valued differential 0-form on $A$ will be called a $V$-valued function. For $v \in V$ we let $\text{const}_v := v \otimes 1_A \in V \otimes A$.

Let $\omega$ be a $V$-valued differential 1-form and $X \in \mathcal{X}_r(A)$ a right vector field. Then we put

$$\langle \omega, X \rangle := \omega_V \langle \omega_A, X \rangle,$$

so that $\langle \omega, X \rangle$ is a $V$-valued function.

Let $\omega$ be a $V$-valued differential $n$-form. If $f : \Omega^n(A) \to M$ is a map, we abbreviate

$$f(\omega) := \omega_V \otimes f(\omega_A) = (id_V \otimes f)(\omega).$$

If moreover $\phi \in V^*$, then we put

$$\langle \phi, \omega \rangle := \langle \phi, \omega_V \rangle \omega_A \in \Omega^n(A).$$
**Definition 4.9** Let $A$ be an $H$-comodule algebra and $\Omega^\ast(A)$ an $H$-colinear differential calculus on $A$. Let $V$ be an $H$-comodule. A $V$-valued differential $n$-form $\omega$ is said to be invariant, if we have $\omega \in (V \otimes \Omega^n(A))^\text{co}H$, where the $H$-comodule structure on the tensor product is the codiagonal one.

Thus $\omega$ is invariant if and only if

$$\omega_V(0) \otimes \omega_A(0) \otimes \omega_V(1) \omega_A(1) = \omega \otimes 1$$

Reasoning as in SCHNEIDER [13, Lemma 3.1], this holds iff

$$\omega_V \otimes \omega_A(0) \otimes \omega_A(1) = \omega_V(0) \otimes S(\omega_V(1)).$$

Now we are ready to give the natural definitions for connections and connection one forms by imitating the classical situation. As announced in the introduction to this section, a connection, classically a splitting of the short exact sequence (26), generalizes naturally to a splitting of the sequence (27). As for the connection 1-forms, we have provided enough notations in the preceding definitions to be able to just copy literally the classical formula.

**Definition 4.10** Let $A$ be a left $H$-module algebra, $\Omega(H)$ a differential calculus for which $\Delta : H \to H \otimes H$ is differentiable, and $\Omega(A)$ a differential calculus on which $H$ acts.

(i) A left connection is a map $c_\ell : (A\# H)_H \Omega^1(H) \to \Omega^1(A\# H)$ in $A\# H \mathcal{M}^H$ with $\pi_1 c_\ell = \text{id}$.

(ii) A right connection is a map $c_r : (A\# H)_H \Omega^1(H) \to \Omega^1(A\# H)$ in $\mathcal{M}^H_{A\# H}$ with $\pi_1 c_r = \text{id}$.

(iii) A two-sided connection is a map which is both a left and a right connection.

(iv) A connection 1-form is an element $\phi \in (\mathfrak{h} \otimes \Omega^1(A\# H))^\text{co} H$ satisfying $\langle \phi, X \rangle = \text{const}_X$ for all $X \in \mathfrak{h}$.

**Lemma 4.11** There exists a canonical two-sided connection on $A\# H$, given by

$$A\# H \Box \Omega^1(H) \xrightarrow{\beta^{-1}} A\# \Omega^1(H) \triangleleft \Omega^1(A\# H),$$

that is $c(\sum a_i \# h_i \otimes \gamma_i) = \sum a_i \varepsilon(h_i) \# \gamma_i$. 

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Lemma 4.12 Let \( \{x_i\} \) be a basis of \( \mathfrak{h} \) and let \( \{x^i\} \) denote the dual basis of \( {}^\text{co}H\Omega^1(H) \). Then an invariant \( \mathfrak{h} \)-valued 1-form \( \phi \) on \( A\#H \) is a connection 1-form if and only if \( \pi^1(\phi) = \sum x_i \otimes 1_{A\#H} \otimes x^i \).

Proof: Write \( \phi = \varphi + \omega \) with \( \varphi \in A\#\Omega^1(H) \) and \( \omega \in \Omega^1(A)\#H \). We have \( \langle \phi, X \rangle = \langle \varphi, X \rangle \), and thus it suffices to show that \( \pi^1(\phi) = \sum x_i \otimes 1_{A\#H} \otimes x^i \).

If \( \varphi = \sum x_i \otimes 1\#x^i \), then clearly \( \langle \varphi, X \rangle = \sum x_i \otimes \langle 1\#x^i, X \rangle = \sum x_i \otimes \langle x^i, X \rangle \cdot 1_R = X \otimes 1_R \).

Conversely, assume that \( \langle \varphi, X \rangle = X \otimes 1 \) for all \( X \in \mathfrak{h} \). Write \( \pi^1(\phi) = \sum x_i \otimes t_i \) with \( t_i \in A\#\Omega^1(H) \). By 4.5, we can write \( t_i = \sum r_{ij}(0) \otimes r_{ij}(1) x^j \) with \( r_{ij} \in A\#H \). It follows that

\[
\sum r_{k\ell} \langle x^j, x_\ell \rangle = \sum \langle r_{k\ell}(1\#x^j), x_\ell \rangle = \sum \langle x^k, x_i \otimes r_{ij}(1\#x^j), x_\ell \rangle = \langle x^k, \langle \varphi, x_\ell \rangle \rangle = \langle x^k, x_\ell \otimes 1_R \rangle = \rho_{k\ell} \cdot 1_R
\]

for all \( k, \ell \), and thus \( \varphi = \sum x_i \otimes 1\#x^i \).

Theorem 4.13 There is a bijection between connection 1-forms and left connections.

Proof: We identify left connections with left \( R \)-linear and right \( H \)-colinear maps \( A\#\Omega^1(H) \to \Omega^1(A\#H) \).

Then we can describe the claimed bijection explicitly as follows: Given a left connection \( c \), we put \( \phi_c = \sum x_i \otimes c(1\#x^i) \).

Obviously \( \pi^1(\phi_c) = \sum x_i \otimes 1 \otimes x^i \), so that \( \phi_c \) is a connection 1-form if we verify that it is invariant: \( \rho(\phi_c) = \sum x_{i(0)} \otimes c(1\#x_{i(0)}^i) \otimes x_{i(1)} \cdot 1_R = \sum x_{i(0)} \otimes c(1\#x^i) \) because \( c \) is colinear as well as the map \( k \ni 1 \mapsto x_i \otimes x^i \in \mathfrak{h} \otimes \mathfrak{h}^* \).

Given a connection 1-form \( \phi = \phi_H \otimes \phi_R \), put

\[
c_\phi(a\#\gamma) = (a\#\gamma_{(-2)})(S(\gamma(-1))\gamma(0), \phi) = (a\#\gamma_{(-2)})(S(\gamma(-1))\gamma(0), \phi_H) \phi_R
\]

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Let us check that \( c_\phi \) is a connection. It is left \( R \)-linear:

\[
c_\phi((x\#h)(a\#\gamma)) = c_\phi(x(h(1) \cdot a)\#h(2)\gamma) \\
= (x(h(1) \cdot a)\#h(2)\gamma)\langle S(h(3)\gamma(1))h(4)\gamma(0), \phi \rangle \\
= (x(h(1) \cdot a)\#h(2)\gamma)\langle S(\gamma(1))\gamma(0), \phi \rangle \\
= (x\#h)(a\#\gamma(1))\langle S(\gamma(1))\gamma(0), \phi \rangle \\
= (x\#h)c_\phi(a\#\gamma)
\]

and \( H \)-colinear:

\[
\rho c(a\#\gamma) = (a\#\gamma(3))\langle S(\gamma(1))\gamma(0), \phi_h \rangle \phi_R(0) \otimes \gamma(-2)\phi_R(1) \\
= (a\#\gamma(3))\langle S(\gamma(1))\gamma(0), \phi(0) \rangle \phi_R \otimes \gamma(-2)S(\phi(1)) \\
= (a\#\gamma(4))\langle S(\gamma(1))\gamma(0), \phi_h(0) \rangle \phi_R \otimes \gamma(-3)S(\gamma(-2))\gamma(1) \\
= (a\#\gamma(-2))\langle S(\gamma(1))\gamma(0), \phi_h(0) \rangle \phi_R \otimes \gamma(1) \\
= c(a\#\gamma(0)) \otimes \gamma(1).
\]

Finally

\[
\pi^1 c_\phi(a\#\gamma) = \pi^1((a\#\gamma(-3))\langle S(\gamma(-1))\gamma(0), \phi \rangle) \\
= (a\#\gamma(-2))\langle S(\gamma(-1))\gamma(0), \pi^1(\phi) \rangle \\
= (a\#\gamma(-2))\langle S(\gamma(-1))\gamma(0), x_i(1\#x^i) \rangle \\
= (a\#\gamma(-2))(1\#S(\gamma(-1))\gamma(0)) \\
= a\#\gamma.
\]

In this way we have assigned to every connection a connection 1-form, and vice versa. It remains to verify that the maps thus constructed are inverse to each other. But this follows from

\[
\phi_\phi = \sum x_i \otimes c_\phi(1\#x^i) \\
= \sum x_i \otimes \langle x^i, \phi_h \rangle \phi_R \\
= \phi_h \otimes \phi_R = \phi
\]

and

\[
c_{\phi_c}(a\#\gamma) = (a\#\gamma(-2))\langle S(\gamma(-1))\gamma(0), \phi_c \rangle \\
= (a\#\gamma(-2))\langle S(\gamma(-1))\gamma(0), x_i \rangle \phi(1\#x^i) \\
= (a\#\gamma(-2))\phi(1\#S(\gamma(-1))\gamma(0)) \\
= c((a\#\gamma(-2))(1\#S(\gamma(-1))\gamma(0))) \\
= c(a\#\gamma).
\]

\[\square\]
Remark 4.14 Surprisingly, there is also a bijection between right connections and connection 1-forms (and consequently also between left and right connections). It is given by

\[ c_\phi(a\#\gamma) = \langle \gamma(0) S^{-1}(\gamma(0)), \phi \rangle (a\#\gamma(-2)) \]

and put \( \phi_c = \sum x_i \otimes c(1\#x^i) \).

We omit the proof, which is similar to that of the preceding theorem.

Theorem 4.15 The set \( \mathcal{A} \) of connection 1-forms on \( A\#H \) is an affine space with translation space isomorphic to the space \( \mathfrak{h} \otimes \Omega^1(A) \) of \( \mathfrak{h} \)-valued differential 1-forms on the base quantum space \( A \).

Proof: We embed \( \mathfrak{h} \otimes \Omega^1(A) \) into \( \mathfrak{h} \otimes \Omega^1(A\#H) \) via

\[ j : \mathfrak{h} \otimes \Omega^1(A) \ni X \otimes \omega \longmapsto X(0) \otimes \omega \# S(X(1)) \in \mathfrak{h} \otimes \Omega^1(A\#H) \]

and claim that \( \mathcal{A} \) is an affine space with translation space \( \text{Im}(j) \).

We have \( \text{Im}(j) \subset (\mathfrak{h} \otimes \Omega^1(A\#H))^{\text{co } H} \), since

\[ \rho(X(0) \otimes \omega \# S(X(1))) = X(0) \otimes \omega \# S(X(3)) \otimes X(1) S(X(2)) = X \otimes \omega \# S(X(1)) \]

Moreover, \( \pi^1 j(t) = 0 \) for all \( t \in \mathfrak{h} \otimes \Omega^1(A) \) and thus \( \phi + j(t) \in \mathcal{A} \) whenever \( \phi \in \mathcal{A} \), by \ref{th:coclosed}

Conversely, let \( \phi, \phi' \in \mathcal{A} \) and put \( \theta = \phi' - \phi \). Since \( \pi^1(\phi') - \pi^1(\phi) \) by \ref{th:coclosed} we have \( \pi^1(\theta) = 0 \), so that \( \theta \in \mathfrak{h} \otimes \Omega^1(A)\#H \). Let us write \( \theta = \sum x_{ij} \otimes \omega_i \# h_{ij} \) with the \( \omega_i \in \Omega^1(A) \) linearly independent, \( x_{ij} \in \mathfrak{h} \) and \( h_{ij} \in H \). Then \( \theta \in (\mathfrak{h} \otimes \Omega^1(A\#H))^{\text{co } H} \) (which follows from the fact that both \( \phi \) and \( \phi' \) are invariant) implies \( \sum x_{ij} \otimes h_{ij} \in (\mathfrak{h} \otimes H)^{\text{co } H} \) for all \( i \). This means \( \sum x_{ij} \otimes h_{ij} \otimes 1 = \sum x_{ij(0)} \otimes h_{ij(1)} \otimes x_{ij(1)} h_{ij(2)} \). Now we conclude

\[ \sum x_{ij(0)} \otimes h_{ij} \otimes S(x_{ij(1)}) = \sum x_{ij(0)} \otimes h_{ij(1)} \otimes S(x_{ij(1)} x_{ij(2)} h_{ij(2)}) = \sum x_{ij} \otimes h_{ij(1)} \otimes h_{ij(2)} \]

which entails

\[ j(\sum x_{ij} \otimes (h_{ij} \otimes \omega_i) = \sum x_{ij(0)} \otimes \omega_i \otimes S(x_{ij(1)}) e(h_{ij}) = \sum x_{ij} \otimes \omega_i \otimes h_{ij(2)} \varepsilon(h_{ij(1)}) = \sum x_{ij} \otimes \omega_i \otimes h_{ij} = \theta \]

and in particular \( \theta \in \text{Im}(j) \). \( \square \)
5 A class of examples

For the following definition of a coquasitriangular or braided bialgebra see Larson and Towber [3] and [8].

Definition 5.1 A coquasitriangular bialgebra \((H, r)\) consists of a bialgebra \(H\) and a convolution invertible map \(r : H \otimes H \to k\) satisfying

\[
\sum_{(g),(h)} r(g(1) \otimes h(1))g(2)h(2) = \sum_{(g),(h)} h(1)g(1)r(g(2) \otimes h(2))
\]

\[
r(f \otimes gh) = \sum_{(f)} r(f(1) \otimes h)r(f(2) \otimes g)
\]

\[
r(fg \otimes h) = \sum_{(h)} r(f \otimes h(1))r(g \otimes h(2)).
\]

for \(f, g, h \in H\).

Convolution invertible means that there is a map \(\tau : H \otimes H \to k\) satisfying

\[
\tau(g(1) \otimes h(1))r(g(2) \otimes h(2)) = \varepsilon(g)\varepsilon(h) = r(g(1) \otimes h(1))\tau(g(2) \otimes h(2)).
\]

It is an important property of coquasitriangular bialgebras that all their comodules are also modules in a natural way, cf. [8].

Lemma 5.2 Let \((H, r)\) be a coquasitriangular bialgebra. Then there is a monoidal (that is, tensor product preserving) functor

\[
H\mathcal{M} \to H\mathcal{M}
\]

assigning to a left \(H\)-comodule \(V\) the vector space \(V\) with the \(H\)-module structure defined by

\[
h \cdot v = v(0)r(v(-1) \otimes h)
\]

Proof: The equation really defines an \(H\)-module, for

\[
g \cdot (h \cdot v) = g \cdot v(0)r(v(-1) \otimes h)
\]

\[
= v(0)r(v(-1) \otimes g)r(v(-2) \otimes h)
\]

\[
= v(0)r(v(-1) \otimes gh)
\]

\[
= (gh) \cdot v
\]
Moreover, the functor thus defined maps the tensor product of two comodules \( V \) and \( W \) to the tensor product of modules:

\[
    h \cdot (v \otimes w) = v(0) \otimes w(0)r(v(-1)w(-1) \otimes h)
    = v(0)r(v(-1) \otimes h(1)) \otimes w(0)r(w(-1) \otimes h(2))
    = h(1) \cdot v \otimes h(2) \cdot w
\]

holds for \( v \in V \), \( w \in W \) and \( h \in H \). \( \square \)

Assume we are given a left \( H \)-comodule algebra \( A \). Then lemma 5.2 entails that we can make \( A \) a left \( H \)-module algebra by setting \( h \cdot a = a(0)r(a(-1) \otimes h) \). We can then form the smash product algebra \( A\#H \), whose multiplication is given by

\[
    (a\#g)(b\#h) = a \cdot r(b(-1) \otimes g(1))b(0)\#g(2)h
\]

We can read this as a commutation relation. \( A \) and \( H \) can be identified with subalgebras of \( A\#H \) via \( A \ni a \mapsto a\#1 \in A\#H \) and \( H \ni h \mapsto 1\#h \in A\#H \). Elements \( a \in A \) and \( h \in H \) satisfy the commutation relation

\[
    ha = r(a(-1) \otimes h(1))a(0)h(2)
\]

in \( A\#H \).

Now assume in addition that we are given a left covariant first order differential calculus \( \Omega^1(A) \) on \( A \) and a bicovariant first order differential calculus \( \Omega^1(H) \) on \( H \). Again by lemma 5.2, \( H \) acts on \( \Omega^1(A) \) by \( h \cdot \omega = \omega(0)r(\omega(-1) \otimes h) \), and we can form the standard differential calculus \( \Omega^1(A\#H) \). The \( A\#H-A\#H \)-bimodule structure on \( \Omega^1(A\#H) \) is described by

\[
    (a\#g)(\omega\#h) = a \cdot r(\omega(-1) \otimes g(1))\omega(0)\#g(2)h
    (a\#g)(b\#\gamma) = a \cdot r(b(-1) \otimes g(1))b\#g(2)\gamma
    (\omega\#h)(a\#g) = \omega \cdot r(a(-1) \otimes h(1))a(0)\#h(2)g
    (b\#\gamma)(a\#g) = b \cdot r(a(-1) \otimes \gamma(-1))a(0)\#\gamma(0)g
\]

for \( a, b \in A, \ g, h \in H, \ \omega \in \Omega^1(A) \) and \( \gamma \in \Omega^1(H) \). From this we can read off commutation relations between differentials on \( A \) and elements of \( H \), and differentials on \( H \) and elements in \( A \):

\[
    (dh)a = r(a(-1) \otimes h(1))a(0)(dh(2))
    h(da) = r(a(-1) \otimes h(1))(da(0))h(2)
\]

The second equation is not useful if we want to simplify expressions in \( \Omega^1(A\#H) \) by moving all the differentials to the right and all the algebra elements to the left. We can replace it by

\[
    (da)h = \overline{\eta}(a(-1) \otimes h(1))h(2)(da(0))
\]
Finally we will become more explicit by recalling examples of coquasitriangular Hopf algebras (see Faddeev, Reshetikhin and Takhtajan [3]). Let \((R_{ij}^{kl}) \in M_{N^2}(k)\) an invertible solution of the matrix quantum Yang-Baxter equation. We write \(\hat{R}_{ij}^{kl} = R_{ji}^{kl}\).

The FRT algebra \(A(R)\) is defined by generators \((T_{ij}^k| i, j = 1, \ldots, N^2)\) and relations \(\hat{R}_{ij}^{kn}T_m^kT_n^l = T_k^iT_j^l\hat{R}_{im}^{kn}\). It is a coquasitriangular bialgebra with \(\Delta(T_i^j) = \sum T_k^iT_j^l\) and with \(r : A(R) \otimes A(R) \to k\) defined by \(r(T_j^i \otimes T_k^l) = \gamma R_{jk}^{il}\), where \(\gamma\) is a nonzero parameter in \(k\). The convolution inverse of \(r\) is given by \(r(T_i^j \otimes T_k^l) = (R_{jk}^{il})^{-1}\).

The quantum groups \(SL_q(N), SO_q(N), SU_q(N), Sp_q(N), SO_q(3,1)\) are all quotients of FRT algebras \(A(R)\). To ensure that \(r : A(A) \otimes A(R) \to k\) induces a well defined map \(r : H \otimes H \to k\), one is forced to make specific choices of the parameter \(\gamma\). Such choices suitable for the examples listed above can be found in Weixler [15, Sec.3.2].

Now assume that we are given an algebra \(A\) generated by \(\{x^i\} \subset A\) which is a left \(H\)-comodule algebra via \(\rho(x^i) = T_j^i \otimes x^j\) (where we have adopted the summation convention). Then \(A\) is at the same time a left \(H\)-module algebra by \(T_j^i \cdot x^k = \gamma R_{jk}^{il}x^l\).

The algebra \(A\#H\) is generated by \(\{x^i\} \cup \{T_j^i\}\) subject to the relations satisfied by the \(x^i \in A\) and those satisfied by the \(T_j^i \in H\), plus the additional relations

\[
T_j^ix^k = \gamma R_{mk}^{li}x^mT_j^l
\]

The usual differential calculi considered on algebras \(A\) and \(H\) as described above are such that the left \(A\)-module \(\Omega^1(A)\) and the left \(H\)-module \(\Omega^1(H)\) are generated by \(\{dx^i\}\) and \(\{dT_j^i\}\), respectively (see Wess and Zumino [13], where this idea appears first). Then the bimodule \(\Omega^1(A\#H)\) is generated by the differentials \(dx^i\) and \(dT_j^i\) subject to the relations satisfied by the \(dx^i\) and the \(x^i\) in \(\Omega^1(A)\), those satisfied by the \(dT_j^i\) and the \(T_j^i\) in \(\Omega^1(H)\), plus the additional relations

\[
(dT_j^i)x^k = \gamma R_{mk}^{li}x^mT_j^l
\]

\[
(dx^i)T_j^i = \gamma^{-1}(R^{-1})_{ik}^lT_j^l(dx^k)
\]

In particular, the left \(A\#H\)-module \(\Omega^1(A\#H)\) is generated by the differentials \(dx^i\) and \(dT_j^i\).
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