Traveling wave oscillatory patterns in a signed Kuramoto-Sivashinsky equation with absorption

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Abstract

In this paper, a proof of a conjecture raised in [9] concerning existence and global uniqueness of an asymptotically stable periodic orbit in a fourth-order piecewise linear ordinary differential equation is presented. The fourth-order equation comes from the study of traveling wave patterns in a signed Kuramoto-Sivashinsky equation with absorption. The proof is threefold. First, the problem of solving for the periodic orbit is transformed into a zero finding problem on \( \mathbb{R}^4 \), which is solved with a computer-assisted proof based on Newton’s method and the contraction mapping theorem. Second, the rigorous bounds about the periodic orbit in phase space are combined with the theory of discontinuous dynamical systems to prove that the orbit is asymptotically stable. Finally, global uniqueness is proved using a result from number theory.

Key words. Traveling wave patterns, Kuramoto-Sivashinsky model, discontinuous dynamical systems, periodic orbits, computer-assisted proofs

1 Introduction

The Kuramoto-Sivashinsky equation

\[
  u_t + \nabla^4 u + \nabla^2 u + \frac{1}{2} |\nabla u|^2 = 0,
\]

(1)

where \( \nabla^2 \) is the Laplace operator and \( \nabla^4 \) is the biharmonic operator, is a fourth-order semilinear parabolic PDE which was originally introduced to model flame front propagation and later became a popular model to analyze weak turbulence or spatiotemporal chaos [3, 13, 14, 15, 17, 22]. In an attempt to study extinction phenomena, Galaktionov and Svirshchevskii consider in [9] a modification of (1), namely the signed KS equation with absorption

\[
  u_t + \nabla^4 u + \text{sign}(u) - |\nabla u|^2 = 0.
\]

(2)

Considering equation (2) on the real line (i.e. \( u = u(\xi, t) \), with \( \xi \in \mathbb{R} \) and \( t \geq 0 \)), and plugging the traveling wave ansatz \( u(\xi, t) = f(y) \) (with \( y \overset{\text{def}}{=} \xi - ct \)) in (2) leads to the problem

\[
  -cf'(y) + f'''(y) + \text{sign}(f(y)) - (f''(y))^2 = 0, \quad y \in (0, \infty) \text{ and } f(0) = 0.
\]

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Following the approach of [9], keeping the two leading terms leads to

\[ f'''(y) + \text{sign} f(y) = 0, \quad y \in (0, \infty) \text{ and } f(0) = 0. \]  

(3)

Changing coordinates \((y, f(y)) \rightarrow (s, \varphi(s))\) via

\[ f(y) = y^4 \varphi(s), \quad \text{with} \quad s \overset{\text{def}}{=} \ln y \]

and plugging the transformation in (3) leads to the fourth-order piecewise linear ordinary differential equation

\[ \varphi'''(s) + 10\varphi''(s) + 35\varphi'(s) + 50\varphi(s) + \text{sign} (\varphi(s)) = 0. \]

(4)

The purpose of the present paper is to prove Conjecture 3.2 on page 150 of [9], which we now state as a theorem.

**Theorem 1.1.** Equation (4) has a unique nontrivial asymptotically stable periodic solution.

The periodic solution of Theorem 1.1 is portrayed in Figure 1 and the corresponding traveling wave pattern \(u(\xi, t) = f(\xi - ct) = (\xi - ct)^4 \varphi(\ln(\xi - ct))\) is plotted in Figure 2. Note that we set \(f(\xi - ct) = 0\) for \(\xi - ct \leq 0\), and that we did not solve for the wave speed \(c\).

![Figure 1: Profile of the periodic solution of Theorem 1.1 (left) and the corresponding orbit in the phase space (right).](image1)

![Figure 2: Different snapshots of \(u(\xi, t) = f(\xi - ct) = (\xi - ct)^4 \varphi(\ln(\xi - ct))\).](image2)

The proof of Theorem 1.1 has three parts. The first part of the proof (existence) is presented in Section 2, where the problem of finding the periodic solution \(\varphi(s)\) of (4) is transformed (via the
symmetry argument of Lemma 2.1 into a zero finding problem \( F(a) = 0 \) where \( F : \mathbb{R}^4 \to \mathbb{R}^4 \) is defined in (12). Proving the existence of \( \tilde{a} \in \mathbb{R}^4 \) such that \( F(\tilde{a}) = 0 \) is done with a computer-assisted proof based on a Newton-Kantorovich type theorem (Theorem 2.2). The second part of the proof is presented in Section 3, where the rigorous enclosure of the periodic solution is combined with the theory of discontinuous dynamical systems to prove that the orbit is asymptotically stable. The final part of the proof is presented in Section 4, where a result from number theory (Hurwitz’s Theorem) is combined with the contracting dynamics of (4) to prove that the periodic orbit is globally unique. These three parts conclude the proof of Theorem 1.1

2 Existence: a computer-assisted proof

In this section, we prove the existence of a periodic solution \( \varphi(s) \) of (4). To achieve this goal, we reformulate this into a zero finding problem \( F(a) = 0 \) defined on \( \mathbb{R}^4 \). Proving the existence of a solution is done by verifying the hypotheses of Theorem 2.2 with the help of the digital computer and interval arithmetic (e.g. see (19, 21)).

We begin by making the change of variables \( x_1 = \varphi, x_2 = \varphi', x_3 = \varphi'' \) and \( x_4 = \varphi''' \) to rewrite the fourth-order equation (4) as the system

\[
\dot{x} = Mx + g(x) \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -24 & -50 & -35 & -10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\text{sign}(x_1) \end{pmatrix}. \tag{5}
\]

Equation (5) is a piecewise smooth dynamical system and changes rule as \( x = (x_1, x_2, x_3, x_4) \) goes through the switching manifold defined by

\[ \Sigma \equiv \{ x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = 0 \}. \]

The switching manifold \( \Sigma \) separates the phase space \( \mathbb{R}^4 \) into the two regions \( \mathcal{R}_+ \) and \( \mathcal{R}_- \) defined by \( \mathcal{R}_+ \equiv \{ x \in \mathbb{R}^4 : x_1 \geq 0 \} \) and \( \mathcal{R}_- \equiv \{ x \in \mathbb{R}^4 : x_1 \leq 0 \} \). Denoting \( b \equiv (0, 0, 0, -1) \), system (5) can then be written as

\[
\dot{x} = \begin{cases} f_+(x) \equiv Mx + b, & x \in \mathcal{R}_+ \\ f_-(x) \equiv Mx - b, & x \in \mathcal{R}_- \end{cases}. \tag{6}
\]

Given \( b_0 \in \{ \pm b \} \), the unique solution of \( \dot{x} = Mx + b_0 \), \( x(0) = x_0 \in \mathbb{R}^4 \) is given by

\[
t \mapsto e^{Mt}x_0 + e^{Mt} \int_0^t e^{-Ms}b_0 \, ds = e^{Mt}(x_0 + M^{-1}b_0) - M^{-1}b_0. \tag{7}
\]

Note that

\[
M^{-1} = \begin{pmatrix} -\frac{50}{24} & -\frac{35}{24} & -\frac{10}{24} & -\frac{1}{24} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]

Moreover, \( M = PDP^{-1}, e^{Mt} = Pe^{Dt}P^{-1} \) where \( e^{Dt} = \text{diag}(e^{-4t} \quad e^{-3t} \quad e^{-2t} \quad e^{-t}) \) and

\[
P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -4 & -3 & -2 & -1 \\ 16 & 9 & 4 & 1 \\ -64 & -27 & -8 & -1 \end{pmatrix} \quad \text{and} \quad P^{-1} = \begin{pmatrix} -1 & -\frac{11}{6} & -1 & -\frac{1}{5} \\ 4 & 7 & 2 & \frac{1}{2} \\ -6 & -\frac{19}{3} & -4 & -\frac{2}{3} \\ 4 & 13 & 3 & \frac{1}{6} \end{pmatrix}. \tag{8}
\]
We now introduce a result which exploits the symmetry of the problem and establishes a mechanism to obtain a periodic solution of (6).

Lemma 2.1. If there exist \( L > 0 \) and a solution \( \phi : [0, L] \to \mathbb{R}^4 \) of \( \dot{x} = f_+(x) \) with

\[
\phi(L) = -\phi(0) \tag{9}
\]

and

\[
\phi([0, L]) \subset \mathcal{R}_+, \tag{10}
\]

then \( \phi(0), \phi(L) \in \Sigma \) and \( \Gamma : [0, 2L] \to \mathbb{R}^4 \) defined by

\[
\Gamma(t) \defeq \begin{cases} 
\phi(t), & t \in [0, L] \\
-\phi(t - L), & t \in [L, 2L]
\end{cases}
\tag{11}
\]

is a 2L-periodic solution of (6).

Proof. First, \( \phi([0, L]) \subset \mathcal{R}_+ \) implies that \( \phi(0))_1, (\phi(L))_1 \geq 0 \) and then \( 0 \leq (\phi(L))_1 = - (\phi(0))_1 \leq 0 \). Hence, \( (\phi(0))_1 = (\phi(L))_1 = 0 \), that is \( \phi(0), \phi(L) \in \Sigma \).

Now, for \( t \in [L, 2L] \), \( \psi(t) \defeq -\phi(t - L) \) solves \( \dot{x} = f_-(x) \), as \( \psi'(t) = -\phi'(t - L) = M(-\phi(t - L)) - b = M\psi(t) - b = f_-(\psi(t)) \). Also \( \phi([0, L]) \subset \mathcal{R}_+ \) implies that \( \psi([L, 2L]) \subset \mathcal{R}_- \). Moreover, \( \psi(L) = -\phi(0) = \phi(L) = -\psi(2L) \).

By definition of \( \Gamma(t) \) in (11), \( \Gamma([0, L]) \subset \mathcal{R}_+, \Gamma([L, 2L]) \subset \mathcal{R}_- \) and \( \phi(L) = -\phi(L - L) = -\phi(0) \). Hence \( \Gamma \) is continuous at \( t = L \). Finally, since \( \Gamma(0) = \phi(0) = -\phi(L) = -\phi(2L - L) = \Gamma(2L) \), we conclude that \( \Gamma(t) \) is a 2L-periodic orbit of (6).

To find the segment of the orbit \( \phi : [0, L] \to \mathbb{R}^4 \) solving \( \dot{x} = f_+(x) = Mx + b \) as in Lemma 2.1 we use formula (7), impose that the segment begins in the switching manifold (i.e. \( \phi(0) \in \Sigma \)) and that \( \phi(L) = -\phi(0) \). Note that if \( \phi(0) \in \Sigma \), then \( \phi(0) = (0, a_2, a_3, a_4) \) for some \( a_2, a_3, a_4 \in \mathbb{R} \). Using (7), the condition \( \phi(L) = -\phi(0) \) reduces to solve

\[
0 = \phi(0) + \phi(L)
= \phi(0) + e^{ML}\phi(0) + e^{ML} \int_0^L e^{-Ms} b \ ds
= P(I + e^{DL}) P^{-1} \begin{pmatrix} 0 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} + \int_0^L P e^{D(L-s)} P^{-1} b \ ds.
\]

Denote \( a \defeq (L, a_2, a_3, a_4) \) and let

\[
F(a) = F \begin{pmatrix} L \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} \defeq P \left( e^{DL} + I \right) P^{-1} \begin{pmatrix} 0 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} + \int_0^L P e^{D(L-s)} P^{-1} b \ ds. \tag{12}
\]

To prove the existence of a periodic solution \( \varphi(s) \) of (4), it is sufficient to prove the existence of a zero of \( F : \mathbb{R}^4 \to \mathbb{R}^4 \) defined in (12) and then to verify the extra condition (10). While the rigorous verification of (10) is done a-posteriori using interval arithmetics, the existence of a zero
of $F$ is done using the \textit{radii polynomial approach} (e.g. see \cite{25, 18}) which is essentially the Newton-Kantorovich theorem (e.g. see \cite{20}). We now introduce this approach for a general $C^2$ map defined on $\mathbb{R}^n$. Endow $\mathbb{R}^n$ with the supremum norm $\|a\|_\infty = \max_{i=1,...,n} |a_i|$ and denote by $\overline{B}_r(b) \defeq \{ a \in \mathbb{R}^n \mid \|a-b\|_\infty \leq r \} \subset \mathbb{R}^n$ the closed ball of radius $r$ and centered at $b$.

**Theorem 2.2.** Let $F: \mathbb{R}^n \to \mathbb{R}^n$ be a $C^2$ map. Consider $\bar{a} \in \mathbb{R}^n$ (typically a numerical approximation with $F(\bar{a}) \approx 0$). Assume that the Jacobian matrix $DF(\bar{a})$ is invertible and let $A \defeq DF(\bar{a})^{-1}$. Let $Y_0 \geq 0$ be any number satisfying
\begin{equation}
\|AF(\bar{a})\|_\infty \leq Y_0. \tag{13}
\end{equation}
Given a positive radius $r_* > 0$, let $Z_2 = Z_2(r_*)$ be any number satisfying
\begin{equation}
\sup_{a \in \overline{B}_{r_*}(\bar{a})} \left( \max_{1 \leq i \leq n} \sum_{1 \leq k,m \leq n} A_{ij} D_{km}^2 F_j(a) \right) \leq Z_2. \tag{14}
\end{equation}

Define the radii polynomial by
\begin{equation}
p(r) \defeq Z_2 r^2 - r + Y_0. \tag{15}
\end{equation}
If there exists $r_0 \in (0, r_*)$ with $p(r_0) < 0$, then there exists a unique $\bar{a} \in \overline{B}_{r_0}(\bar{a})$ such that $F(\bar{a}) = 0$.

**Proof.** Let $r \leq r_*$ and consider $c \in \overline{B}_{r}(\bar{a})$. Applying the Mean Value Inequality and using (14),
\begin{equation}
\|A(DF(c) - DF(\bar{a}))\|_\infty \leq \sup_{h \|h\|_\infty = 1} \sup_{a \in \overline{B}_{r}(\bar{a})} \|AD^2 F(a) h\|_\infty \|w - \bar{a}\|_\infty \leq Z_2 r, \tag{16}
\end{equation}
where $\|\cdot\|_\infty$ denotes matrix norm. Define the \textit{Newton-like operator} $T: \mathbb{R}^n \to \mathbb{R}^n$ by $T(a) = a - AF(a)$. Since $A$ is invertible, $F(\bar{a}) = 0$ if and only if $T(\bar{a}) = \bar{a}$. Let $r_0 > 0$ be such that $p(r_0) < 0$. Hence $Z_2 r_0^2 + Y_0 < r_0$ and $Z_2 r_0 + \frac{Y_0}{r_0} < 1$. Since $Y_0, Z_2 \geq 0$, one gets that
\begin{equation}
Z_2 r_0 < 1. \tag{17}
\end{equation}
For any $a \in \overline{B}_{r_0}(\bar{a})$, apply (16) to get
\begin{equation}
\|DT(a)\|_\infty = \|I - ADF(a)\|_\infty = \|A[DF(\bar{a}) - DF(a)]\|_\infty \leq Z_2 r_0 < 1.
\end{equation}
Hence,
\begin{align*}
\|T(a) - \bar{a}\|_\infty &\leq \|T(a) - T(\bar{a})\|_\infty + \|T(\bar{a}) - \bar{a}\|_\infty \\
&\leq \sup_{c \in \overline{B}_{r_0}(\bar{a})} \|DT(c)\|_\infty \|a - \bar{a}\|_\infty + \|AF(\bar{a})\|_\infty \\
&\leq (Z_2 r_0) r_0 + Y_0 < r_0.
\end{align*}
Then $T$ maps $\overline{B}_{r_0}(\bar{a})$ into itself. Finally, given $a_1, a_2 \in \overline{B}_{r_0}(\bar{a})$ combine (17) with the Mean Value Inequality to get
\begin{equation}
\|T(a_1) - T(a_2)\|_\infty \leq \sup_{c \in \overline{B}_{r_0}(\bar{a})} \|DT(c)\|_\infty \|a_1 - a_2\|_\infty \leq (Z_2 r_0) \|a_1 - a_2\|_\infty \leq \kappa \|a_1 - a_2\|_\infty,
\end{equation}
where $\kappa \defeq Z_2 r_0 < 1$. Then, by the Contraction Mapping Theorem, $T$ has a unique fixed point $\bar{a} \in \overline{B}_{r_0}(\bar{a})$. It follows from the invertibility of $A$ that $\bar{a}$ is the unique zero of $F$ in $\overline{B}_{r_0}(\bar{a})$. \qed
We now apply Theorem 2.2 to prove the existence of a zero of $F$ defined in (12). This begins by computing an approximate solution. Applying Newton’s method, we find an approximate zero of $F$ given by

$$
\tilde{a} = \begin{pmatrix} 1.418316134968973 \\ 2.45235091866104 \times 10^{-2} \\ 8.358590910573891 \times 10^{-3} \\ -4.883983455701284 \times 10^{-2} \end{pmatrix}.
$$

(18)

Then, using INTLAB (see [21]) we compute rigorous enclosures of $DF(\tilde{a})$ and $A \overset{\text{def}}{=} DF(\tilde{a})^{-1}$. We then verify rigorously that $Y_0 \overset{\text{def}}{=} 7.4 \times 10^{-15} \geq \|AF(\tilde{a})\|_\infty$, which settles the computation of the bound (13).

The next bound to compute is $Z_2$ satisfying (14). The only non zero second partial derivatives are the terms $\frac{\partial^2 F_j}{\partial a_1 \partial a_3}$ for $j, k \in \{1, 4\}$, where we note that by Clairaut’s theorem $\frac{\partial^2 F_j}{\partial a_1 \partial a_1} = \frac{\partial^2 F_j}{\partial a_3 \partial a_3}$ for $k \in \{2, 4\}$, $j \in \{1, 4\}$. Hence, we can write the bound (14) as

$$
Z_2(r_*) \geq \max_{1 \leq i \leq 4} \left\{ \sup_{c \in B_{r_*}(\tilde{a})} \left( \left| \sum_{1 \leq j \leq 4} A_{i,j} \frac{\partial^2 F_j}{\partial a_1 \partial a_2} (c) \right| + 2 \left| \sum_{1 \leq j \leq 4} A_{i,j} \frac{\partial^2 F_j}{\partial a_3 \partial a_2} (c) \right| + 2 \left| \sum_{1 \leq j \leq 4} A_{i,j} \frac{\partial^2 F_j}{\partial a_3 \partial a_3} (c) \right| \right) \right\}.
$$

(19)

Choosing $r_* = 0.01$ we use interval arithmetic to obtain that $Z_2 \overset{\text{def}}{=} 41$ satisfies (19).

Therefore, for $r \leq r_*$ the radii polynomial is given by

$$
p(r) = 41r^2 - r + 7.4 \times 10^{-15}.
$$

Using interval arithmetic, we show that for every $r_0 \in [7.5 \times 10^{-15}, 0.01], p(r_0) < 0$. By Theorem 2.2 there exists a unique zero $\tilde{a}$ of $F$ in $B_{r_0}(\tilde{a})$. Denote $\tilde{a} = (\tilde{L}, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4)$. Then since $|\tilde{L} - \tilde{a}_1| \leq \|\tilde{a} - \tilde{a}\|_\infty \leq 7.5 \times 10^{-15}$ and $\tilde{a}_1 = 1.418316134968973$, we conclude that $\tilde{L} > 0$. By construction,

$$
\bar{\phi}(t) \overset{\text{def}}{=} e^{Mt} \left( \tilde{\phi}(0) + M^{-1}b \right) - M^{-1}b = e^{Mt} \begin{pmatrix} 0 \\ \tilde{a}_2 \\ \tilde{a}_3 \\ \tilde{a}_4 \end{pmatrix} + M^{-1}b
$$

(20)

defines a solution $\bar{\phi} : [0, \tilde{L}] \rightarrow \mathbb{R}^4$ of $\dot{x} = f_+ (x)$ with $\bar{\phi} (\tilde{L}) = -\tilde{\phi} (0)$. The last hypothesis which needs to be verified to apply Lemma 2.1 is the condition (10), that is $\tilde{\phi}([0, \tilde{L}]) \subset \mathcal{R}_+$. Using a MATLAB program using INTLAB, we consider a uniform time mesh (of size 300) of the time interval $[0, \tilde{L}]$, that is $0 = t_0 < t_1 < \cdots < t_{300} = \tilde{L}$. For each mesh interval $I_k = [t_{k-1}, t_k]$ ($k = 1, \ldots, 300$), the code computes an interval enclosure of $\tilde{\phi}(I_k)$ using formula (20). Then the code verifies that $\tilde{\phi}_1(t) > 0$ for all $t \in I_k$ and $k = k_1, \ldots, k_2$ for some $1 < k_1 < k_2 < 300$. This implies that $\tilde{\phi}([t_{k_1-1}, t_{k_2}]) \subset \mathcal{R}_+$. Afterward, it verifies that $\tilde{\phi}_1(t) > \tilde{\phi}_2(t) > 0$ for all $t \in I_k$ and $k = 1, \ldots, k_1 - 1$. Hence $\tilde{\phi}_1(t)$ is strictly increasing over the interval $[0, t_{k_1-1}]$, and since $\tilde{\phi}_1(0) = 0$, it follows that $\tilde{\phi}_1(t) > 0$ for all $t \in (0, t_{k_1-1}]$, that is $\tilde{\phi}([0, t_{k_1-1}]) \subset \mathcal{R}_+$. Similarly, the code verifies that $\tilde{\phi}_1(t) = \tilde{\phi}_2(t) < 0$ for all $t \in I_k$ and $k = k_2 + 1, \ldots, 300$. Hence $\tilde{\phi}_1(t)$ is strictly decreasing over the interval $[t_{k_2}, \tilde{L}]$, and since $\tilde{\phi}_1(\tilde{L}) = 0$, it follows that $\tilde{\phi}_1(t) > 0$ for all $t \in [t_{k_2}, \tilde{L}]$, that is $\tilde{\phi}([t_{k_2}, \tilde{L}]) \subset \mathcal{R}_+$. We conclude that

$$
\tilde{\phi}([0, \tilde{L}]) = \tilde{\phi}([0, t_{k_1-1}]) \cup \tilde{\phi}([t_{k_1-1}, t_{k_2}]) \cup \tilde{\phi}([t_{k_2}, \tilde{L}]) \subset \mathcal{R}_+.
$$
This implies that $X e$ saltation matrices the monodromy matrix $X$ stable using the theory of discontinuous dynamical systems (e.g. see \[5\]). We do this by computing that $\tilde{\phi}$.

In this section, we demonstrate that the $2 \times 3$ Asymptotic stability

Denote by $\tilde{\phi}$ from

$x$ initial condition

$\phi$ $\tilde{\phi}$: \[ \phi(t), \quad t \in [0, \bar{L}] \] \[ -\tilde{\phi}(t - \bar{L}), \quad t \in [\bar{L}, 2\bar{L}] \] \[ \tilde{\Gamma}(t) \] (21)

is a $2\bar{L}$-periodic solution of \[6\]. All the computational steps described in this section are carried in the MATLAB program $\textbf{proof.m}$ available at \[1\].

3 Asymptotic stability

In this section, we demonstrate that the $2\bar{L}$-periodic orbit $\tilde{\Gamma}(t)$ defined in \[21\] is asymptotically stable using the theory of discontinuous dynamical systems (e.g. see \[5\]). We do this by computing the monodromy matrix $X(2\bar{L})$ of $\tilde{\Gamma}$ and show that all its nontrivial Floquet multipliers have modulus less than one.

Define $h : \mathbb{R}^4 \to \mathbb{R}$ by $h(x) \equiv x_1$ so that the switching manifold is given by

$$\Sigma = \{ x \in \mathbb{R}^4 : h(x) = 0 \}.$$ 

Denote by $\tilde{\phi}(1) \equiv \tilde{\phi}(\bar{L}) = (0, -\bar{a}_2, -\bar{a}_3, -\bar{a}_4)^T$ the point at which $\tilde{\Gamma}$ crosses $\Sigma$ coming from $\mathcal{R}_+$ and entering in $\mathcal{R}_-$. Similarly, denote by $\tilde{\phi}(2) \equiv (0, \bar{a}_2, \bar{a}_3, \bar{a}_4)^T$ the point at which $\tilde{\Gamma}$ crosses $\Sigma$ coming from $\mathcal{R}_-$ and entering in $\mathcal{R}_+$. For that reason, $\tilde{\Gamma}$ is called a crossing periodic orbit (e.g. see \[6\], \[16\]).

Denote by $T = 2\bar{L}$ the period of $\tilde{\Gamma}$, $\bar{t} \equiv \bar{L}$, and denote by $\Phi(t,x_0)$ the solution of \[6\] at time $t$ with initial condition $x_0$. Then (i.e. see \[7\]) the monodromy matrix is given by

$$X(T) = X(T, \bar{t})S_{+-}(\bar{t})X(\bar{t}, 0)S_{+-}(\bar{a}(1))$$

where

$$S_{+-}(\bar{a}(1)) \equiv I + \left( \frac{(f_+ - f_-)}{\nabla h^T \cdot f_+} \nabla h^T \right) \left( \bar{a}(1) \right)$$

$$S_{+-}(\bar{a}(2)) \equiv I + \left( \frac{(f_+ - f_-)}{\nabla h^T \cdot f_-} \nabla h^T \right) \left( \bar{a}(2) \right)$$

are called the saltation matrices, and where the fundamental matrix solutions $X(t,0)$ and $X(t,\bar{t})$ satisfy

$$\dot{X}(t,0) = Df_{-}(\Phi(t, \bar{a}(1)))X(t,0) = MX(t,0), \quad \text{for } t \in [0, \bar{L}], \quad \text{with } X(0,0) = I$$

$$\dot{X}(t, \bar{L}) = Df_{+}(\Phi(t, \bar{a}(1)))X(t, \bar{L}) = MX(t, \bar{L}), \quad \text{for } t \in [\bar{L}, 2\bar{L}], \quad \text{with } X(\bar{L}, \bar{L}) = I.$$ 

This implies that $X(t,0) = e^{Mt}$ and therefore $X(\bar{t},0) = X(\bar{L},0) = e^{M\bar{L}}$. Similarly, $X(t, \bar{t}) = e^{M(t-L)}$ and then $X(T, \bar{t}) = X(2\bar{L}, \bar{L}) = e^{M\bar{L}}$.

Since $\nabla h = (1, 0, 0, 0)^T$ we obtain that $\nabla h^T \cdot f_+(\bar{a}(1)) = -\bar{a}_2$ and $\nabla h^T \cdot f_-(\bar{a}(2)) = \bar{a}_2$. Simple computations yield

$$S_{+-}(\bar{a}(1)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{2}{\bar{a}_2} & 0 & 0 & 1 \end{pmatrix} = S_{+-}(\bar{a}(2)).$$
Hence, the monodromy matrix is given by

$$X(2\tilde{L}) = e^{M\tilde{L}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{2}{\tilde{a}_2} & 0 & 0 & 1 \end{pmatrix} e^{M\tilde{L}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{2}{\tilde{a}_2} & 0 & 0 & 1 \end{pmatrix}. \quad (23)$$

Using interval arithmetics and that $|\tilde{L} - \bar{a}_1|, |\tilde{a}_2 - \bar{a}_2| \leq 7.5 \times 10^{-15}$, we compute rigorously an interval enclosure of (23) and using the rigorous computational method from [2] we prove that the spectrum $\sigma(X(2\tilde{L}))$ of $X(2\tilde{L})$ satisfies

$$\sigma(X(2\tilde{L})) \subset \bigcup_{i=1}^{4} B_i$$

where

\begin{align*}
B_1 &\equiv [0.99999989798820, 1.00000010201179] \\
B_2 &\equiv [0.05862265751705, 0.05862286154064] \\
B_3 &\equiv [0.99999959034712, 0.00000079437071] \\
B_4 &\equiv [0.00001170839977, 0.00001191242336].
\end{align*}

This rigorous computation is carried in the MATLAB program Proof.m available at [1]. From this, we conclude that three Floquet multipliers of $\tilde{\Gamma}$ have modulus strictly less than one. This concludes the proof that the $2\tilde{L}$-periodic orbit $\tilde{\Gamma}(t)$ defined in (21) is asymptotically stable.

### 4 Global uniqueness

In this section, we combine the contracting dynamics of (4) on each sides of $\Sigma$ with a result from number theory (Hurwitz’s Theorem) to prove that the periodic orbit is globally unique.

For the rest of this section, we denote by $\tilde{x}(t) \equiv \tilde{\Gamma}(t)$ the periodic solution of (6) as defined in (21) and denote its period by $p_1 \equiv 2\tilde{L}$. Suppose that $x(t)$ is another non trivial solution of (6) with period $p_2$. Denote

$$\tilde{\Gamma} \equiv \{\tilde{x}(t) : t \in [0,p_1]\} \quad \text{and} \quad \Gamma \equiv \{x(t) : t \in [0,p_2]\}$$

and define the distance between the periodic orbits by

$$d \equiv \inf_{\tilde{y} \in \tilde{\Gamma}, y \in \Gamma} \|\tilde{y} - y\|. \quad (24)$$

To prove that there is a globally unique periodic orbit, we show that $d = 0$.

Denote the first components of $x(t)$ and $\tilde{x}(t)$ by $\varphi(t)$ and $\tilde{\varphi}(t)$, respectively. Let $\Upsilon(t) \equiv \varphi(t) - \tilde{\varphi}(t)$. Then, for all $t$ such that $\varphi(t)$ and $\tilde{\varphi}(t)$ have the same sign, or equivalently, such that $x(t)$ and $\tilde{x}(t)$ are on the same side of $\Sigma$, $\Upsilon(t)$ satisfies the homogeneous fourth-order equation

$$\Upsilon^{(4)}(t) + 10\Upsilon^{(3)}(t) + 35\Upsilon''(t) + 50\Upsilon'(t) + 24\Upsilon(t) = 0. \quad (25)$$

The solution to the system of equations corresponding to (25) is given by $z(t) \equiv x(t) - \tilde{x}(t)$, for $t$ as mentioned above. More explicitly, $z(t)$ is given by

$$z(t) = e^{Mt}z(0) \quad (26)$$
with $M$ given in $\Box$. We will construct a sequence of times that goes to infinity such that both periodic solutions $x(t), \tilde{x}(t)$ lie on the same side of the switching manifold $\Sigma$ for these times. Before proving the existence of such a sequence of times we will first prove that the trajectory of all orbits crosses $\Sigma$.

**Lemma 4.1.** For every solution $\varphi(t)$ of (4) there exists a $t_0$ such that $\varphi(t_0) = 0$.

*Proof.* By contradiction, suppose it is not the case. Then either $\varphi(t) > 0$ or $\varphi(t) < 0$ for all $t$ in the domain of $\varphi$. Without loss of generality, suppose $\varphi(t) > 0$, for all $t$. Let $\Psi(t) \stackrel{\text{def}}{=} \varphi(t) + \frac{1}{24}$. Then $\Psi(t)$ satisfies

$$\Psi(4) + 10\Psi(3) + 35\Psi'' + 50\Psi' + 24\Psi = 0.$$  

Denoting $y(t) = (\Psi(t), \Psi'(t), \Psi''(t), \Psi'''(t))^T$, we get that $\dot{y} = My$, where $M$ is defined in $\Box$. The explicit solution is given by $y(t) = e^{Mt}y(0)$ and it satisfies $y(t) \to 0 \in \mathbb{R}^4$ for all initial conditions $y(0) \in \mathbb{R}^4$, since the eigenvalues of $M$ are given by $-1, -2, -3, -4$. In particular $y_1(t) = \Psi(t) = \varphi(t) + \frac{1}{24} \to 0 \in \mathbb{R}$ as $t \to \infty$. This implies that $\varphi(t) \to -\frac{1}{24}$ as $t \to \infty$. This contradicts our hypothesis that $\varphi(t) > 0$. \hfill \Box

From Lemma 4.1 we can assume without loss of generality that the periodic solutions satisfy $x(0), \tilde{x}(0) \in \Sigma$.

The following result helps characterizing the behaviour of a periodic orbit when it intersects $\Sigma$.

**Lemma 4.2.** Let $x_0 \in \Sigma$ and assume that $\Phi(t, x_0)$ is a periodic orbit of (6). Then for every $\varepsilon > 0$, $\Phi([0, \varepsilon], x_0) \not\subset \Sigma$ and $\Phi([-\varepsilon, 0], x_0) \not\subset \Sigma$. In other words, when a periodic orbit intersects $\Sigma$, it must cross it (i.e. it cannot “slide” on it).

*Proof.* The proof uses the theory of Filippov systems (e.g. see $\Box$). Define the sliding manifold as

$$\Sigma_s = \{ x \in \Sigma : (\nabla h^T(x) \cdot f_+(x)) (\nabla h^T(x) \cdot f_-(x)) \leq 0 \} = \{ x \in \mathbb{R}^4 : x_2^2 \leq 0 \} = \{ x \in \mathbb{R}^4 : x_1 = x_2 = 0 \}.$$  

Since $x_0$ is a periodic point, it cannot be a fixed point. The three fixed points of (6) are all in the switching manifold and are given by $(0, 0, 0, \pm \frac{1}{4})$ and $(0, 0, 0, 0)$. It is known that if $x_0 \in \Sigma \setminus \Sigma_s$, then the orbit $\Phi(t, x_0)$ crosses transversally the switching manifold $\Sigma$ at $t = 0$. Hence, we only need to prove the result for $x_0 \in \Sigma_s$. Denote $x_0 = (0, 0, 0, \beta)$. Arguing by contradiction, assume the existence of $\varepsilon_0 > 0$ such that $\Phi([0, \varepsilon_0], x_0) \subset \Sigma_s$. Recalling (6), note that $(f_+(x_0))_2 = (f_-(-x_0))_2 = \alpha$. If $\alpha \neq 0$, then there exists $\delta_0 \in (0, \varepsilon_0)$ such that $\Phi(\delta_0, x_0)_2 \neq 0$, that is $\Phi(\delta_0, x_0) \notin \Sigma_s$, a contradiction. Hence $\alpha = 0$ and therefore $x_0 = (0, 0, 0, \beta) \in \Sigma_s$. In this case $(f_+(x_0))_3 = (f_-(-x_0))_3 = \beta$. If $\beta \neq 0$, then there exists $\delta_0 \in (0, \varepsilon_0)$ such that $\alpha_1 \stackrel{\text{def}}{=} (\Phi(\delta_0, x_0))_3 \neq 0$ with $\Phi(\delta_0, x_0) \in \Sigma_s$. Denote $y_0 = \Phi(\delta_0, x_0) = (0, 0, \alpha_1, \beta_1) \in \Sigma_s$. Then there exists $t_0 \in (0, \varepsilon_{0} - \delta_0)$ such that $\Phi(t_0, y_0) \notin \Sigma_s$, since $(f_+(y_0))_2 = (f_-(y_0))_2 = \alpha_1 \neq 0$. But $\Phi(t_0, y_0) = \Phi(t_0 + \delta_0, x_0) \subset \Phi((0, \varepsilon_0], x_0) \subset \Sigma_s$ which leads to a contradiction. This implies that $\beta = 0$ and therefore $\Sigma_s = \{(0, 0, 0, 0)\}$. Since $x_0 \in \Sigma_s$ then we conclude that $x_0 = (0, 0, 0, 0)$ is a fixed point, which contradicts the fact that $x_0$ is a periodic point. \hfill \Box

Combining Lemma 4.1 and Lemma 4.2, we can assume that the periodic solutions satisfy $x(0), \tilde{x}(0) \in \Sigma$ and that they both cross transversally $\Sigma$ at $t = 0$. Assume moreover that both periodic orbits $x(t), \tilde{x}(t)$ enter the region $\mathcal{R}_+$ for $t > 0$ small enough.

We are ready to prove global uniqueness of the periodic solution.
Proposition 4.3. The orbit $\tilde{\Gamma}$ defined in (21) is the unique periodic orbit of (16).

Proof. We begin by proving unicity among periodic solutions for which the periods are rationally related, then deal with the case of irrational periods afterwards.

Assume $p_1, p_2 \in \mathbb{Q}$. Let $t_0 > 0$ such that $\phi(t_0) > 0$ and $\tilde{\phi}(t_0) > 0$. See Figure 3 for a geometric interpretation. Then there exist $k_1, k_2 \in \mathbb{N}$ such that $k_1 p_1 = k_2 p_2$. Let $\bar{t}_n \equiv t_0 + nk_1 p_1$ for $n \in \mathbb{N}$. Combining the periodicity of $x, \tilde{x}, (24)$ and (26) leads to

$$\bar{d} \leq \|x(t_0) - \tilde{x}(t_0)\| = \|x(\bar{t}_n) - \tilde{x}(\bar{t}_n)\| = \|z(\bar{t}_n)\| \leq \|e^{Mt_0}\|\|z(0)\| \xrightarrow{n \to \infty} 0,$$

which implies that $\bar{d} = 0$.

Assume now that $p_1 \in \mathbb{I}$ or $p_2 \in \mathbb{I}$, and let $\bar{t} > 0$ be the first time one of the two periodic solutions intersects $\Sigma$ (which exists by periodicity). See Figure 4 for a geometric interpretation.

Then for all $k, \ell \in \mathbb{N}$ such that

$$I_{k, \ell} \overset{\text{def}}{=} (kp_1, kp_1 + \bar{t}) \cap (\ell p_2, \ell p_2 + \bar{t}) \neq \emptyset \quad (27)$$

both of our periodic solutions are on the same side of $\Sigma$. We now show the existence of an infinite sequence of non empty intervals $\{I_{k_i, \ell_i}\}_{i \geq 0}$ of the form (27) with $\ell_i, k_i \to \infty$ as $i \to \infty$. To prove this we call upon a result from number theory, namely Hurwitz’s Theorem (e.g. see [10, 12]), which states that for any irrational number $\xi$ there are relatively prime numbers $k$ and $\ell$ going to infinity such that

$$|\xi - \frac{\ell}{k}| < \frac{1}{\sqrt{5}k^2}.$$
Using this theorem we obtain that there exists a sequence \( \{(k_i, \ell_i)\}_{i=1}^{\infty} \) tending to infinity such that,

\[
|k_ip_1 - \ell_ip_2| = k_ip_2|\frac{p_1}{p_2} - \frac{\ell_i}{k_i}| < \frac{p_2}{\sqrt{5k_i}} \quad \text{as } i \to \infty.
\]

Hence, there exists \( i_0 \) such that \( |k_ip_1 - \ell_ip_2| < \bar{t} \) for all \( i \geq i_0 \). Hence, \( I_{k_i,\ell_i} \neq \emptyset \) for all \( i \geq i_0 \). For each \( i \geq i_0 \), let \( t_i \in I_{k_i,\ell_i} \). We conclude that

\[
\bar{d} \leq \|x(t_i) - \tilde{x}(t_i)\| = \|z(t_i)\| \leq \|e^{Mt_i}\|\|z(0)\| \quad \text{as } i \to \infty,
\]

which implies that \( \bar{d} = 0 \).

In all cases, \( \bar{d} = 0 \) and therefore \( \Gamma = \tilde{\Gamma} \), which completes the proof.

\[ \square \]

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