Deformation of line bundles on coisotropic subvarieties.

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Abstract

We prove a criterion stating when a line bundle on a smooth coisotropic subvariety \( Y \) of a smooth variety \( X \) with algebraic Poisson structure, admits a first order deformation quantization.

1 Introduction

In this paper, partially motivated by [BG], we consider a smooth algebraic variety \( X \) with an algebraic Poisson structure \( P \in H^0(X, \Lambda^2 T_X) \) and a smooth subvariety \( Y \subset X \). Any line bundle \( L \) on \( Y \) defines a sheaf of \( \mathcal{O}_X \)-modules. Since \( P \) gives a first order deformation \( \mathcal{A} \) of the structure sheaf \( \mathcal{O}_X \), it is natural to ask when \( L \) can be deformed to an \( \mathcal{A} \)-module \( L \). In fact, we consider a slightly more general first order non-commutative deformation \( \mathcal{A} \) which also depends on a class in \( H^1(X, T_X) \).

Based on the standard formalism of hypercohomology and a version of the deformation complex of Gerstenhaber and Schack, cf. [GS] and [FMY], one expects three obstruction classes which should vanish if such an \( \mathcal{L} \) is to exist: these belong to the groups \( H^0(Y, \Lambda^2 N) \), \( H^1(Y, N) \) and \( H^2(Y, \mathcal{O}_Y) \), respectively, where \( N \) is the normal bundle of \( Y \) in \( X \). The first obstruction is local in nature, and according to loc. cit. its vanishing simply means that \( Y \) should be coisotropic with respect to \( P \), i.e. the natural projection of \( P \) to \( \Lambda^2 N \) should be zero. We impose this assumption on \( Y \) throughout this paper and show in Section 3.2 that in this case \( L \) always exists Zariski locally.

Next we consider the obstruction class in \( H^1(Y, N) \) formulating, cf. Theorem 7 of Section 3.3, a precise condition on \( c_1(L) \in H^1(Y, \Omega^1_Y) \) which guarantees existence of a global deformation \( \mathcal{L} \). In general, it will only be a twisted sheaf of \( \mathcal{A} \)-modules; one has an honest sheaf of modules if and only if a further class in \( H^2(Y, \mathcal{O}_Y) \) vanishes (similarly, coherent sheaves on a general algebraic variety \( X \) may be twisted by a class in \( H^2(X, \mathcal{O}_X) \)). In Section 4 we assume that \( H^2(Y, \mathcal{O}_Y) \) is trivial, but this is mostly for convenience since one could work with twisted sheaves of modules instead.

When \( X \) is algebraic symplectic, \( Y \) is Lagrangian and the class in \( H^1(X, T_X) \) vanishes, i.e. \( \mathcal{A} = \mathcal{O}_X \oplus \epsilon \mathcal{O}_X \), our condition on \( L \) simply says that \( 2c_1(L) = c_1(K_Y) \) in \( H^1(Y, \Omega^1_Y) \), cf. Section 5.
If $X$ is symplectic but $Y$ just coisotropic, we can consider the standard null foliation $T_F \subset T_Y$ obtained by applying the Poisson bivector to the normal bundle of $Y$. In this case, let $L_1$ be a line bundle on $Y$ admitting a first order deformation, and $L_2, M$ two line bundles such that $L_2 = M \otimes_{\mathcal{O}_Y} L_1$. We show that $L_2$ admits a first order deformation if and only if $M$ has a partial algebraic connection along the null foliation. A similar statement is expected for second order deformations if the partial connection on $M$ is flat (we prove the “only if” part). These results are explained in Section 4.2 and 4.3, cf. Theorems 10 and 12.

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## 2 Generalities

### 2.1 First cohomology of a complex.

Let $\mathcal{K} = \{\mathcal{K}^0 \to \mathcal{K}^1 \to \mathcal{K}^2 \to \ldots\}$ be a complex of sheaves on $X$ concentrated in positive degrees. We briefly recall one interpretation of the hypercohomology group $H^1(\mathcal{K})$. Let $\mathcal{H}^i = \mathcal{Z}^i/\mathcal{B}^i$ be the cohomology sheaves of $\mathcal{K}$. The standard spectral sequence yields

$$0 \to H^1(X, \mathcal{H}^0) \to \mathbb{H}^1(X, \mathcal{K}) \to H^0(X, \mathcal{H}^1) \to H^2(X, \mathcal{H}^0) \to \ldots$$

A class in $H^0(X, \mathcal{H}^1)$ is represented by an open covering $\{U_i\}_{i \in I}$ and sections $\alpha_i \in \Gamma(U_i, \mathcal{Z}^1)$ such that on $U_i \cap U_j$

$$\alpha_i - \alpha_j = d\beta_{ij}, \quad \beta_{ij} \in \Gamma(U_i \cap U_j, \mathcal{K}^0)$$

By definition the elements $d\beta_{ij}$ satisfy the cocycle condition on triple intersections and on $U_i \cap U_j \cap U_k$ the expression $\gamma_{ijk} = \beta_{ij} + \beta_{jk} + \beta_{ki}$ will be a section in $\Gamma(U_i \cap U_j \cap U_k, \mathcal{K}^0)$; which gives a class in $H^2(X, \mathcal{H}^0)$ as in the sequence above. If this class is zero, refining $\{U_i\}$ if necessary we can adjust $\beta_{ij}$ by adding an element of $\Gamma(U_i \cap U_j, \mathcal{H}^0)$ to ensure

$$\beta_{ij} + \beta_{jk} + \beta_{ki} = 0$$

on $U_i \cap U_j \cap U_k$. Thus, a class in $H^1(\mathcal{K})$ is represented by a covering $U_i$, sections $\alpha_i \in \Gamma(U_i, \mathcal{Z}^1)$ and $\beta_{ij} \in \Gamma(U_i \cap U_j, \mathcal{K}^0)$ such that $\alpha_i - \alpha_j = d\beta_{ij}$ on $U_i \cap U_j$ and $\beta_{ij} + \beta_{jk} + \beta_{ki} = 0$ on $U_i \cap U_j \cap U_k$. For $\{\beta_i \in \Gamma(U_i, \mathcal{K}^0)\}_{i \in I}$ the collection $\alpha'_i = \alpha_i + d\beta_i$ and $\beta'_{ij} = \beta_{ij} + \beta_i - \beta_j$ represents the same class.

### 2.2 Three examples.

Let $F_1, \ldots, F_n, G$ be sheaves of $\mathcal{O}_X$-modules. We will denote by $\mathcal{D}(F_1 \times \ldots \times F_n, G)$ the sheaf of algebraic differential operators (of finite order in each of the $n$ variables). It has an increasing filtration by subsheaves $\mathcal{D}^k(F_1 \times \ldots \times F_n, G)$ of operators which have total order $\leq k$. The $\mathcal{O}_X$-module structure on $G$ gives an $\mathcal{O}_X$-module structure on differential operators. If $F_1, \ldots, F_n$ and $G$ are coherent then so is $\mathcal{D}^k(F_1 \times \ldots \times F_n, G)$. We will also
write $F^{\times n}$ for the $n$-fold cartesian product $F \times \ldots \times F$; and $\mathcal{D}_0(\mathcal{O}_X^{\times n}, G)$ for the sheaf of differential operators which vanish if either of the arguments is a constant.

The three examples described below are sheafifications of a deformation complex considered by Gerstenhaber and Schack, cf. [GS].

**Example 1.** Set $\mathcal{K}^i(X) = \mathcal{D}_0(\mathcal{O}_X^{\times (i+1)}, \mathcal{O}_X)$ with the Hochschild differential. Then $H^1(\mathcal{K}(X)) = H^1(X, T_X) \oplus H^0(X, \Lambda^2 T_X)$ parameterizes flat deformations $\mathcal{A}$ of $\mathcal{O}_X$ over $\mathbb{C}[\epsilon]/\epsilon^2$, which locally split as $\mathcal{O}_X \oplus \epsilon \mathcal{O}_X$. Locally these are given by $\alpha_i \in \Gamma(U_i, \mathcal{D}_0(\mathcal{O}_X \times \mathcal{O}_X, \mathcal{O}_X))$ which satisfy the cocycle condition $da_i(f, g, h) = a_i(fg, h) - a_i(f, gh) + a_j(f, g)h - f a_i(g, h) = 0$ and on double intersections

$$(\alpha_i - \alpha_j)(fg) = d\beta_{ij}(fg) = \beta_{ij}(fg) - f \beta_{ij}(g) - \beta_{ij}(f)g.$$ 

In addition, on triple intersections $U_i \cap U_j \cap U_k$ we require $\beta_{ij} + \beta_{jk} = \beta_{ik}$. Write each $a_i(f, g)$ as a sum of its symmetric and antisymmetric part:

$$\alpha_i(f, g) = \alpha_i^+(f, g) + \alpha_i^-(f, g) = \frac{1}{2}(\alpha_i(f, g) + \alpha_i(g, f)) + \frac{1}{2}(\alpha_i(f, g) - \alpha_i(g, f))$$

Since $d\beta_{ij}$ is symmetric in $f, g$, we see that $\alpha_i^- = \alpha_j^-$ on $U_i \cap U_j$. Moreover, the cocycle condition implies that $\alpha_i^-$ is a first order operator in each of the arguments. Since $\alpha_i^-$ also vanish on constant functions, they glue into a section $\alpha^- \in H^0(X, \Lambda^2 T_X)$.

As for the symmetric part, since each $U_i$ is smooth, we can write $\alpha_i^+ = d\beta_i$ for some $\beta_i \in \Gamma(U_i, \mathcal{D}_0(\mathcal{O}_X, \mathcal{O}_X))$. Then on every double intersection $\beta_i - \beta_j = \beta_{ij}$ is a derivation of $\mathcal{O}_X$. This defines a class in $H^1(X, T_X)$. Conversely, if $\beta_{ij}$ are vector fields on $U_i \cap U_j$ representing a class in $H^1(X, T_X)$ then $a_0 + \epsilon a_1 \mapsto a_0 + \epsilon(a_1 + \beta_{ij}(a_0))$ give transition functions which allow to glue the first order deformations $\mathcal{O}_{U_i} \oplus \epsilon \mathcal{O}_{U_i}$ with cocycle $\alpha^-$, into a sheaf of algebras $\mathcal{A}$.

We also observe that $H^0(\mathcal{K}(X)) = H^0(X, T_X)$ classifies those automorphisms of $\mathcal{A}$ which restrict to the identity $\text{mod}(\epsilon)$.

**Example 2.** If $F$ is a coherent sheaf on $X$ consider $\mathcal{K}(F)$ with $\mathcal{K}^i(F) = \mathcal{D}_0(\mathcal{O}_X^{\times i}, \mathcal{D}(F, F))$ with the Hochschild differential corresponding to the natural $\mathcal{O}_X$-bimodule structure on $\mathcal{D}(F, F)$, see [We]. Then $H^1(\mathcal{K}(F)) = \text{Ext}^1(F, F)$ parameterizes flat deformations $\mathcal{F}$ of $F$ to a module over $\mathcal{O}_X[\epsilon]/\epsilon^2$, and $H^0(\mathcal{K}(F)) = \text{Hom}_X(F, F)$ can be identified with automorphisms of $\mathcal{F}$ which restrict to the identity $\text{mod}(\epsilon)$.

**Example 3.** The $\mathcal{O}_X$-module structure on $F$ gives a morphism of $\mathcal{O}_X$-bimodules $\mathcal{O}_X \to \mathcal{D}(F, F)$ hence a morphism of complexes $\mathcal{K}(X) \to \mathcal{K}(F)$. Set $\mathcal{K}(X, F) = \text{Cone}(\mathcal{K}(X) \to \mathcal{K}(F))(-1)$ so that $\mathcal{K}^i(X, F) = \mathcal{D}_0(\mathcal{O}_X^{\times (i+1)}, \mathcal{O}_X) \oplus \mathcal{D}_0(\mathcal{O}_X^{\times i}, \mathcal{D}(F, F))$. By the previous subsection and [GS] $H^1(\mathcal{K}(X, F))$ corresponds to isomorphism classes of flat deformations of $(\mathcal{O}_X, F)$ to a pair $(\mathcal{A}, \mathcal{F})$ where $\mathcal{F}$ is a left $\mathcal{A}$-module. Observe that in loc. cit. the deformation complex has three factors. The extra factor would correspond in our case to deforming the algebra structure of $\mathcal{D}(F, F)$ as well. Since in this paper we will not consider such deformations we omit the third factor of the Gerstenhaber-Schack deformation complex.

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A short exact sequence of complexes $0 \to \mathcal{K}(F) \to \mathcal{K}(X,F) \to \mathcal{K}(X) \to 0$ induces a long exact sequence of cohomology

$$\cdots \to H^1(X,\mathcal{K}(F)) \to H^1(X,\mathcal{K}(X,F)) \to H^1(X,\mathcal{K}(X)) \to H^2(X,\mathcal{K}(F)) \to \cdots$$

Therefore, given an isomorphism class of $\mathcal{A}$, one expects that a coherent sheaf $F$ admits a deformation to a left $\mathcal{A}$-module $\mathcal{F}$ if and only if a certain class $c \in H^2(X,\mathcal{K}(F))$ vanishes.

The purpose of this note is to make this vanishing condition explicit in a particular case.

### 3 Line bundles on coisotropic subvarieties

From now on we fix a closed embedding $\eta : Y \to X$ of a smooth subvariety, a first order deformation $\mathcal{A}$ of $\mathcal{O}_X$ with a class $(\kappa, \frac{1}{2}p) \in H^1(X,T_X) \oplus H^0(X,\Lambda^2 T_X)$, and a line bundle $L$ on $Y$. Any vector bundle on $Y$ may also be viewed as $\mathcal{O}_X$-module by applying $\eta_*$ and in such a case we abuse notation by dropping $\eta_*$ to make the formulas more readable. We would like to state explicitly when $L$ admits a deformation to a left $\mathcal{A}$-module. Write $T_Y, N, I$ for the tangent bundle, normal bundle and the ideal sheaf of $Y$, respectively.

#### 3.1 Cohomology

We observe that the cohomology sheaves of $\mathcal{H}^p(\mathcal{K}(X))$ are given by $\Lambda^{p+1}T_X$ due to the Hochschild-Kostant-Rosenberg isomorphism.

**Proposition 1** The $p$-th cohomology sheaf $\mathcal{H}^p(\mathcal{K}(L))$ is isomorphic to $\Lambda^p N$.

For $p = 0,1$ we can make it explicit. First, $\mathcal{H}^0(\mathcal{K}(F)) = \text{Hom}_{\mathcal{O}_X}(F,F)$ for any sheaf $F$ of $\mathcal{O}_X$-modules by definition. In our case this gives $\text{End}_{\mathcal{O}_Y}(L) = \mathcal{O}_Y$. For $p = 1$ a section of $\mathcal{H}^1$ is represented locally by a map $\alpha_L : \mathcal{O}_X \otimes \mathcal{C} L \to L$ which satisfies

$$\alpha_L(fg,l) = \alpha_L(f,gl) + f\alpha_L(g,l).$$

This immediately gives $\alpha_L|_{I^2 \otimes \mathcal{C} L} = 0$. Restricting $\alpha_L$ to $(I/I^2) \otimes \mathcal{C} L$ we see that for a local section $x$ on $I$ and a local section $f$ of $\mathcal{O}_X$ we have

$$f\alpha_L(x,l) = \alpha_L(fx,l) = \alpha_L(xf,l) = \alpha_L(x,fl)$$

where the first and the third equalities follow from the cocycle condition. Recalling that $I/I^2 = N^\vee$ we can view the restriction of $\alpha_L$ as an $\mathcal{O}_Y$-bilinear map $N^\vee \otimes \mathcal{C} L \to L$. This gives a morphism $\mathcal{H}^1(L) \to N$. To show that it is an isomorphism it suffices to restrict to an affine open subset on which $L \simeq \mathcal{O}_Y$ and $Y$ is given by vanishing of a regular sequence; then the isomorphism follows from the Koszul complex. The spectral sequence of hypercohomology for $\mathcal{K}(L)$ gives in lower degrees

$$0 \to H^1(Y,\mathcal{O}_Y) \to H^1(\mathcal{K}(L)) \to H^0(Y,N) \to H^2(Y,\mathcal{O}_Y) \to H^2(\mathcal{K}(L)) \to H^1(Y,N) \to \cdots$$
\[ 0 \to \hat{H}^2(\mathcal{K}(L)) \to H^2(\mathcal{K}(L)) \to H^0(Y, \Lambda^2 N) \to \ldots \]

where \( \hat{H}^2(\mathcal{K}(L)) \) is a subspace of \( H^2(\mathcal{K}(L)) \) which may be defined through the second line. Therefore existence of a deformation \( \mathcal{L} \) of \( L \) is equivalent to the vanishing of a certain class \( c \in H^2(\mathcal{K}(L)) \) and the latter can be split as a chain of conditions:

- the image of \( c \) in \( H^0(Y, \Lambda^2 N) \) is zero, and (2)
- the image of \( c \) in \( H^1(Y, N) \) is zero, and (3)
- the image of \( c \) in \( \text{Coker}(H^0(Y, N) \to H^2(Y, \mathcal{O}_Y)) \) is zero (4)

Note that (3) can be formulated only if (2) holds, and (4) can be formulated only (2), (3) hold. The following lemma follows immediately from the definitions

**Lemma 2** The image of \((\kappa, \frac{1}{2} P) \in H^1(\mathcal{K}(X))\) under the composition

\[ H^1(\mathcal{K}(X)) \to H^2(\mathcal{K}(L)) \to H^0(Y, \Lambda^2 N) \]

is equal to the projection of \( \frac{1}{2} P \in H^0(X, \Lambda^2 T_X) \) to \( H^0(Y, \Lambda^2 N) \). In other words, (2) holds if and only if \( Y \) is coisotropic with respect to the bivector field \( P \).

Throughout the rest of the paper we will assume that \( Y \) is coisotropic in \( X \). In this case, there is a well-defined projection of \( P \) to \( H^0(Y, N \otimes_{\mathcal{O}_Y} T_Y) \) which we denote again by \( P \).

### 3.2 The affine case.

When \( X = \text{Spec}(A) \) and \( Y = \text{Spec}(B) \) are affine the conditions (3) and (4) become trivial. Since \( H^1(X, T_X) \) is trivial we can assume that \( \alpha_X(f, g) = \frac{1}{2} P(df, dg) \). For \( I = \text{Ker}(A \to B) \) the \( B \)-module \( N^V = I/I^2 \) is isomorphic to the global sections of the conormal bundle to \( Y \) in \( X \). In this subsection we write \( A, B, \Omega_A \) and \( \Omega_B \) for the global sections of the sheaves \( \mathcal{O}_X, \mathcal{O}_Y, \Omega^1_X|_Y \) and \( \Omega^1_Y \), respectively. In particular, there is a short exact sequence of \( B \)-modules

\[ 0 \to N^V \to \Omega_A \to \Omega_B \to 0 \]

**Theorem 3** For any \( Y = \text{Spec}(B) \subset X = \text{Spec}(A) \) and \( L \) as above there exists \( \alpha_L : A \otimes_C L \to L \) such that

\[ \alpha_L(aa', l) - \alpha_L(a, a'l) + \alpha_X(a, a')l - a\alpha_L(a', l) = 0 \] (5)

Any such \( \alpha_L \) vanishes on \( I^2 \otimes_C L \). Moreover, it may be taken in the form

\[ \alpha_L(a, l) = \psi(a)l + \rho(da, l) \]

where

\[ \psi \in \mathcal{D}^0_0(A, \mathcal{D}^0(L, L)) = \mathcal{D}^0_0(A, B); \quad \rho \in \mathcal{D}^1_0(A, \mathcal{D}^1(L, L)) = \text{Hom}_B(\Omega_A, \mathcal{D}^1(L, L)) \]

and both \( \psi \) and \( \rho \), in addition to vanishing on the constants, also vanish on \( I^2 \subset A \).
Proof. The vanishing on $I^2 \otimes_C L$ is an immediate consequence of the equation imposed on $\alpha_L$ and the coisotropness condition $\alpha_X(I, I) \subset I$. Substituting the expression for $\alpha_L$ in terms of $\psi$ and $\rho$ and using $d(aa') = ad(a') + d(a)a'$ we get:

\[
(\psi(aa') - \psi(a)a' - a\psi(a'))l + \frac{1}{2}P(da, da')l + (a'\rho(da, l) - \rho(da, a'l)) = 0
\]

By assumption $\psi(1) = 0$ and $\psi$ has order 2, while $\rho$ has order 1 in the $l$ variable. Therefore, if $\sigma_\psi \in \text{Hom}_B(\text{Sym}^2\Omega_A, B)$ and $\sigma_\rho \in \text{Hom}_B(\Omega_A \otimes_B \Omega_B, B)$ are the corresponding principal symbols, then we must ensure that

\[
\sigma_\psi(da \otimes da') + \frac{1}{2}P(da, da') + \sigma_\rho(da, d(a'|_Y)) = 0
\]

in $B$. Observe that all three terms may be viewed as $B$-linear homomorphisms

\[
\Omega_A \otimes_B \Omega_A \to B.
\]

The third term vanishes on $\Omega_A \otimes_B N^\vee$ by its definition. Hence if $\sigma_\psi$ is known, existence of $\sigma_\rho$ is equivalent to the condition

\[
(\sigma_\psi + \frac{1}{2}P)|_{\Omega_A \otimes_B N^\vee} = 0.
\]

Since $Y$ is coisotropic, $P$ vanishes on $N^\vee \otimes_B N^\vee$ and therefore we should look for $\sigma_\psi$ in the submodule $S \subset \text{Hom}_B(\text{Sym}^2\Omega_A, B)$ of homomorphisms which vanish on $\text{Sym}^2N^\vee$. For this submodule we can write a short exact sequence

\[
0 \to \text{Hom}_B(\text{Sym}^2\Omega_B, B) \to S \to \text{Hom}_B(\Omega_B \otimes_B N^\vee, B) \to 0
\]

Since $Y$ is affine and smooth, this sequence splits, and the image of $-\frac{1}{2}P$ in its quotient term may be lifted to some $\sigma_\psi \in S \subset \text{Hom}_B(\text{Sym}^2\Omega_A, B)$. This will ensure the vanishing condition for $\sigma_\psi + \frac{1}{2}P$ and therefore existence of $\sigma_\rho$. Finally, since $X$, $Y$ are affine and smooth, symbols can be lifted to differential operators. This finishes the proof. $\square$

**Theorem 4** Let $\alpha_L$ and $\alpha'_L$ be two maps $(A/I^2) \otimes_C L \to L$ satisfying the first equation of the previous theorem. Then $(\alpha_L - \alpha'_L)$ vanishes on $(I/I^2) \otimes_C L$ if and only if there exists $C$-linear $\beta : L \to L$ such that

\[
\alpha_L(a, l) - \alpha'_L(a, l) = \beta(al) - a\beta(l).
\]

This condition means precisely that the two left $A$-module structures on $\mathcal{L} = L \oplus \epsilon L$ defined by $\alpha_L$ and $\alpha'_L$, respectively, are equivalent via the isomorphism

\[
l_1 + \epsilon l_2 \mapsto l_1 + \epsilon(\beta(l_1) + l_2).
\]

Moreover, if both $\alpha_L$ and $\alpha'_L$ are bidifferential operators as in the previous theorem then $\beta \in D^2(L, L)$.
First we need the following lemma

**Lemma 5** Let \( R : A \otimes_C L \to L \) be a \( C \)-linear map. Then
\[
R(a,l) = \beta(al) - a\beta(l)
\]
for some \( \beta \in \text{Hom}_C(L,L) \) if and only if \( R \) vanishes on \( I \otimes_C A \) and also satisfies
\[
R(ab,l) - R(a,bl) + aR(b,l) = 0
\]

**Proof.** Then “only if” part is obvious. Suppose that \( R \) vanishes on \( I \otimes L \), i.e. descends to a linear map \( B \otimes L \to L \) which we denote again by \( R \). Then the equation imposed on \( R \) means that \( R \) gives a 1-cocycle in the Hochschild complex of the \( B \)-bimodule \( \text{End}_C(L) \). By Lemma 9.1.9 of [We] we have
\[
HH^i(B, \text{End}_C(L)) = \text{Ext}^i_B(L, L).
\]
Since \( L \) is projective over \( B \) we have \( \text{Ext}^i_B(L, L) = 0 \). Therefore, \( R \) is a coboundary, which means precisely \( R(al) = \beta(al) - a\beta(l) \) for \( \beta \in \text{End}_C(L) \).

**Proof of the theorem.** The three-term equation of the previous lemma obviously holds for \( \alpha_L - \alpha'_L \). Hence by previous lemma a required \( \xi \) exists but apriori it may be just a linear map. However, by our choice of \( \alpha_L, \alpha'_L \) the difference is a bidifferential operator which has order \( \leq 1 \) in \( l \). Therefore \( \beta(l) \) is a differential operator of order \( \leq 2 \).

Finally, we would like to identify those maps \( \gamma : N^\vee \otimes_C L \to L \) which locally extend to \( \alpha_L \) satisfying (5). Observe that this equation implies
\[
\gamma(ax,l) + \alpha_X(a,x)l - a\gamma(x,l) = 0; \quad a \in A, x \in N^\vee, l \in L
\]
\[
\gamma(xa',l) - \gamma(x,a'l) + \alpha_X(x,a')l = 0; \quad x \in N^\vee, a' \in A, l \in L
\]

**Proposition 6** Any \( \gamma : N^\vee \otimes_C B \to B \) satisfying (6) and (7) extends to \( \alpha_L : (A/I^2) \otimes_C L \to L \) satisfying (5). Moreover, \( \alpha_L \) may be taken in the form \( \psi(a)l + \rho(da,l) \) if and only if \( \gamma \) has the form \( \psi'(x)l + \rho'(x,l) \) where \( \psi' \in \mathcal{D}^1(N^\vee, B) \) and \( \rho' \in \text{Hom}_B(I, \mathcal{D}^1(L, L)) \).

**Proof.** Observe that the conormal sequence \( 0 \to N^\vee \to \Omega_A \to \Omega_B \to 0 \) admits a \( B \)-linear splitting since its terms are projective \( B \)-modules. Let \( p, \) resp. \( q, \) be the projectors \( \Omega_A \to \Omega_A \) such that their images are identified with \( N^\vee \) and \( \Omega_B \), respectively.
\[
\alpha_B(a,l) = \gamma(p(da),l) + t(a,l)
\]
where \( t \in \mathcal{D}^1_0(A, \mathcal{D}^1(L, L)) \) is such that \( t(a,bl) - bt(a,l) = \alpha_X(q(da), q(db))l \). Such \( t \) may be found e.g. by choosing a connection on \( L \), which is possible on any smooth affine variety.

An easy computation shows that (6) and (7) imply (5), and that \( \gamma \) of the form \( \psi'(x)b + \rho'(x,b) \) gets lifted to \( \alpha_L \) of the form \( \psi(a)b + \rho(da,b) \). Conversely, if \( \alpha_L \) is represented in such a form then we can take \( \psi' \) and \( \rho' \) to be the restrictions of \( \psi \) and \( \rho \) to \( N^\vee = I/I^2 \subset A/I^2 \), respectively. For the orders of these operators, we observe that whenever \( I \) annihilates an \( A \)-module \( M \), any degree \( \leq k \) operator \( A \to M \) restricts to a degree \( \leq (k - 1) \) operator \( I \to M \) by an easy induction involving the definition of degree.
3.3 The obstruction in $H^1(Y,N)$.  

We consider the general situation when the first order deformation $A$ of $\mathcal{O}_X$ is non split, i.e. both classes $P \in H^0(X, \Lambda^2 T_X)$ and $\kappa \in H^1(X, T_X)$ are nonzero. Thus, we have an open covering $\{U_i\}$ of $X$ and $A|_{U_i} \simeq \mathcal{O}_X \oplus \epsilon \mathcal{O}_X$, while the collection of vector fields $\beta_{X,ij}$ on $U_i \cap U_j$, representing the class $\kappa$, gives transition functions between the two trivializations of $A|_{U_i \cap U_j}$: $f_0 + \epsilon f_1 \mapsto f_0 + \epsilon (f_1 + \beta_{X,ij}(f_0))$. The Leibniz rule for $\beta_{X,ij}$ ensures that the transition function agrees with the product

$$(f_0 + \epsilon f_1) \ast (g_0 + \epsilon g_1) = f_0 g_0 + \epsilon (f_0 g_1 + f_1 g_0 + \alpha_X(f_0, g_0))$$

where $\alpha_X(f_0, g_0) = \frac{1}{2} P(df_0, dg_0)$. We will also assume that $N$ and $L$ are trivial on each $Y \cap U_i$.

In this setup, we would like to find a condition which guarantees existence of a collection $\{\alpha_{L,i}, \beta_{L,ij}\}$ where

- $\alpha_{L,i} \in \Gamma(U_i, D(\mathcal{O}_X \otimes \mathcal{C} L, L))$ satisfies the condition
  $$\alpha_{L,i}(f, g, l) - \alpha_{L,i}(f, gl) + \alpha_X(f, g)l - f \alpha_{L,i}(g, l) = 0$$
  which means that $\mathcal{L} = (L \oplus \epsilon L)|_{U_i}$ is a module over $A|_{U_i}$ with respect to the module structure
  $$(f_0 + \epsilon f_1) \ast (l_0 + \epsilon l_1) = f_0 l_0 + \epsilon (f_1 l_0 + f_0 l_1 + \alpha_{L,i}(f_0, l_0))$$

- $\beta_{L,ij} \in \Gamma(U_i \cap U_j, D(L, L))$ satisfy
  $$\alpha_{L,i}(f, l) - \alpha_{L,j}(f, l) = \beta_{L,ij}(fl) - f \beta_{L,ij}(l) - \beta_{X,ij}(f)l$$
  which means that the transition functions $(l_0 + \epsilon l_1) \mapsto (l_0 + \epsilon (l_1 + \beta_{L,ij}(l_0)))$ agree with the module structure.

- $\beta_{L,ij}$ satisfy the cocycle condition on triple intersections, which guarantees that the modules over $U_i$ may be glued into a left $A$-module $\mathcal{L}$.

**Theorem 7** Let $at(N) \in H^1(Y, End(N) \otimes \Omega_Y^1)$ be the Atiyah class of $N$. Then existence of $(\alpha_{L,i}, \beta_{L,ij})$ satisfying the first two of the three conditions stated above, is equivalent to the equation in $H^1(Y, N)$:

$$[- at(N) + 2Id_N \otimes c_1(L)] \cup P + \pi = 0$$

where $\pi$ stands is the image of $\kappa \in H^1(X, T_X)$ in $H^1(Y, N)$ and $(\epsilon) \cup P$ stands for the Yoneda product of a class in $H^1(Y, End(N) \otimes \Omega_Y^1)$ $\simeq Ext_Y^1(N \otimes T_Y, N)$ with the image of $P \in H^0(X, \Lambda^2 T_X)$ in $H^0(Y, N \otimes T_Y)$.  

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Proof. By Theorem 3 we can always find $\alpha_{L,i}$ satisfying the first equation. To find $\beta_{L,ij}$ with
\[
\alpha_{L,i}(f, l) - \alpha_{L,j}(f, l) + \beta_{X,ij}(f)l = \beta_{L,ij}(fl) - f\beta_{L,ij}(l),
\]
first observe that existence of $\beta_{L,ij}$ does not depend on the choice of $\alpha_{L,i}$ since any other choice will be given by adding $\eta_i(f, l)$ such that $\eta_i(fg, l) - \eta_i(f, gl) + f\eta_i(g, l) = 0$. We have assumed that $L$ is identified with $O_Y$ on $U_i$ and hence for any section $l$ of $L$ on $U_i$ we can find a $\mathbb{C}$-linear splitting $l \mapsto \hat{l}$ of the surjection $O_X \to O_Y \simeq L$ on $U_i$. Then
\[
\eta_i(f, l) = \eta_i(f, \hat{l} \cdot 1) = \eta_i(f\hat{l}, 1) - f\eta_i(\hat{l}, 1)
\]
tells us that if $\{\beta_{L,ij}\}$ solve (8) for $\alpha_{L,i}$ then $\beta_{L,ij} + \eta_i(\hat{l}, 1) - \eta_j(\hat{l}, 1)$ solve the same equation for $\alpha_{L,i} + \eta_i$.

Denote by $R(f, l)$ the right hand side of (8). If $R|_{I \otimes L} = 0$ then we can apply Lemma 5 and find $\beta_{L,ij}$. It remains to establish whether we can replace $\alpha_{L,i}$ by $\alpha_{L,i} + \eta_i$ so that the right hand side of (8) vanishes on $I \otimes_C L$. In other words, we would like to have equality of maps $I \otimes_C L \to L$:
\[
\alpha_{L,j}(f, l) - \alpha_{L,i}(f, l) + \beta_{X,ij}(f)l = \eta_i(f, l) - \eta_j(f, l)
\]
Since $Y$ is coisotropic, each term on the left hand side vanishes on $I^2 \otimes_C L$. Therefore recalling $I/I^2 \simeq N^\vee$ we can view the above equality as equality of functions on $N^\vee \otimes_C L$. We observe that if $\eta_i$ are found as functions on $N^\vee \otimes_C L$ we can always extend them to $(O_X/I^2) \otimes_C L$ as in Proposition 6.

We will show that the left hand side is an $O_Y$-bilinear map $N^\vee \otimes L \to L$ for a particular choice of $\alpha_{L,i}$. The condition $\eta_i(fg, l) - \eta_i(f, gl) + f\eta_i(g, l) = 0$ implies that $\eta_i$ also must be $O_Y$-bilinear on $N^\vee \otimes_C L$. Hence existence of $\{\eta_i\}$ will be equivalent to vanishing of a class in $H^1(Y, N)$.

To calculate the class in $H^1(Y, N)$ explicitly, let $x_1^i, \ldots, x_r^i$ be the basis of $N^\vee|_{U_i}$ and $e^i$ a section spanning $L|_{U_i}$. On $N^\vee \otimes_C L$ we can set $\alpha_{L,i}(x^i_s, e^i) = 0$ which implies
\[
\alpha_{L,i}(\sum_s a_s x^i_s, b e^i) = (\sum_s \alpha_X(x^i_s, a_s)b + 2\alpha_X(\sum_s a_s x^i_s, b))e^i.
\]
On a double intersection we have $x^i_s = \sum_s A^{ij}_{sp} x^j_p$ where $A^{ij}$ is the transition matrix. Similarly $e^i = B^{ij} e^j$. Rewriting $\alpha_{L,j}$ in the basis $x^i_s$ we find that
\[
(\alpha_{L,j} - \alpha_{L,i})(\sum_s a_s x^i_s, b e^i) = (\sum_s a_s \alpha_X(x^i_s, A^{ij}_{sp} A^{ij}_{pr}) + 2\alpha_X(\sum_s a_s x^i_s, B^{ij})B^{ij}))b e^i
\]
By a similar calculation involving lifts of vector fields to elements of Atiyah algebras we find that $dA^{ij} \cdot A^{il}$ represents minus the Atiyah class of $N$ and $dB^{ij} \cdot B^{il}$ the first Chern class of $L$. It is clear that the term $\beta_{X,ij}(f)l$ in (8) represents the class $\pi$ as in the statement of the theorem. This finishes the proof. \qed
Remark. Even if the class in $H^1(Y, N)$ vanishes and $\beta_{L,ij}$ exist, they may not satisfy the cocycle condition on triple intersections. However, equation (3) implies that on $U_i \cap U_j \cap U_k$ the expression $\beta_{L,ij} + \beta_{L,jk} + \beta_{L,ki}$ is $O_Y$-linear and thus defines a class in $H^2(Y, O_Y)$. The vanishing of this class, or a weaker condition (4), is needed to ensure that $L$ exists.

Corollary 8 Let $Y$ be a coisotropic smooth subvariety in $X$ with $H^2(Y, O_Y) = 0$, $A$ a first order deformation of $O_X$ with class $(\kappa, P)$, and $L$ a line bundle on $Y$ such that

$$[ - at(N) + 2 Id_N \otimes c_1(L) ] \cup P + \pi = 0$$

in $H^1(Y, N)$. Then $L$ admits a first order deformation $L$ to a left $A$-module. If $H^1(Y, O_Y) = 0$ the set of isomorphism classes of such $L$ is parameterized by $H^0(Y, N)$. In general, the group of automorphisms (restricting to the identity mod($\epsilon$)) of each $L$ is isomorphic to $H^0(Y, O_Y)$.

Proof. For the isomorphism classes we recall the sequence (1). By section 3.1

$$\mathbb{H}^0(K(L)) = H^0(Y, O_Y), \quad \mathbb{H}^1(K(L)) = H^0(Y, N)$$

Recall that $\mathbb{H}^0(K(X)) = H^0(X, T_X)$. It follows from the definitions that $\mathbb{H}^1(K(X, L))$ is the vector space of all pairs $\partial_L, \partial$, where $\partial_L \in D^1(L, L)$ and $\partial$ is an extension of the symbol of $\partial_L$ to a vector field on $X$. It follows that

$$0 \to \mathbb{H}^0(K(L)) \to \mathbb{H}^0(K(X, L)) \to \mathbb{H}^0(K(X)) \to 0$$

is exact. Therefore

$$0 \to \mathbb{H}^1(K(L)) \to \mathbb{H}^1(K(X, L)) \to \mathbb{H}^1(K(X)) \to \ldots$$

is also exact, and a lift of any element in $\mathbb{H}^1(K(X))$ is well defined up to an element of $\mathbb{H}^1(K(L)) = H^0(Y, N)$. More explicitly, a section of $H^0(Y, N)$ restricted to $U_i$ may be lifted to a derivation $\delta_i : O_X \to O_Y$ and we can adjust $\alpha_{L,i}(a, l)$ replacing it by $\alpha_{L,i}(a, l) + \delta_i(a)l$. On a double intersection the difference $\zeta_{ij} = \delta_i - \delta_j$ is a derivation $O_Y \to O_Y$ which we can view as an operator from $L$ to itself, since $L$ is trivialized on $U_i \cap U_j$. Thus the data $(\alpha_{L,i}, \beta_{L,ij})$ will be replaced by the data $(\alpha_{L,i} + \delta_i \cdot Id_L, \beta_{L,ij} + \zeta_{ij})$.

To prove the assertion about automorphisms of $L$: let $x$ be such automorphism, then $Id_L - x$ is an endomorphism of $L$ which takes values on $L/eL \simeq L$ and vanishes on $eL \simeq L$, i.e. a morphism of sheaves $L \to L$. It is easy to see that such morphism must be $O_X$-linear, i.e. given by an element of $H^0(Y, O_Y)$.

Remark It follows from the definitions that for a deformation $A$ constructed from a pair $(\kappa, {1 \over 2}P)$ the deformation $A^{op}$ corresponds to the pair $(\kappa, -{1 \over 2}P)$. Therefore a line bundle $L$ admits a deformation to a right $A$-module precisely when

$$[ - at(N) + 2 Id_N \otimes c_1(L) ] \cup P - \pi = 0.$$
4 The case $\kappa = 0$.

In this section we assume that $\kappa = 0$, i.e. there exists a global splitting $\mathcal{A} = \mathcal{O}_X \oplus \epsilon \mathcal{O}_X$, and that $H^1(Y, \mathcal{O}_Y) = H^2(Y, \mathcal{O}_Y) = 0$. The last condition automatically ensures (\[\text{4}\]).

Remark. With some minor modifications, the arguments below can be adjusted to the slightly more general case when $\pi = 0$. This means that vector fields $\beta_{X,ij}$ representing the class $\kappa$, may be chosen to satisfy $\beta_{X,ij}(I) \subset I$. We leave the details to the motivated reader.

4.1 Equivalence classes via a global operator.

Assume that $L$ admits a (non-split) first order deformation $\mathcal{L}$ to a left $\mathcal{A}$-module. Embedding $I \subset \mathcal{O}_X \subset A$ we get a globally defined map $\gamma : I \otimes \mathbb{C} L \to L$ given by

$$x \ast l = 0 + \epsilon \gamma(x, l)$$

Repeating the reasoning of Section 3.2 we see that $\gamma$ descends to $(I/I^2) \otimes \mathbb{C} L \simeq N^\vee \otimes \mathbb{C} L$ and satisfies (\[\text{6}\]) and (\[\text{7}\]). However, since

$$\mathcal{D}^1(N^\vee, \mathcal{O}_Y) \cap Hom_{\mathcal{O}_Y}(N^\vee, \mathcal{D}^1(L, L)) = Hom_{\mathcal{O}_Y}(I/I^2, \mathcal{O}_Y) \simeq N,$$

the splitting $\psi(x)l + \rho(x, l)$ will in general exist only on the open sets $U_i$ but not globally. Thus, we can only say that $\gamma_i$ glue into a global section

$$\gamma \in \Gamma(Y, \mathcal{D}^1(N^\vee \times L, L)),$$

i.e. $\gamma$ has the total order $\leq 1$ in its two arguments.

**Proposition 9** Suppose a line bundle $L$ on $Y$ satisfies the condition on $c_1(L)$ stated in Theorem \[\text{7}\]. In the assumption of this section, the set of equivalence classes of $\mathcal{A}$-modules $\mathcal{L}$ deforming $L$ is in bijective correspondence with the set of globally defined differential operators

$$\gamma \in \Gamma(Y, \mathcal{D}^1(N^\vee \times L, L)),$$

satisfying (\[\text{6}\]) and (\[\text{7}\]).

Proof. We have seen above that any $\mathcal{L}$ leads to $\gamma(x, l)$ as in the statement of the theorem. Conversely, suppose that $\gamma(x, l)$ exists. Taking an affine open covering $\{U_i\}$ and using the first formula in the proof of Proposition \[\text{6}\] we can extend $\gamma|_{U_i}$ to an operator $\alpha_L(a,l) : \mathcal{O}_X \otimes L \to L$ defined on $U_i$ and satisfying (\[\text{5}\]). On double intersections $U_i \cap U_j$ the two operators $\alpha_{L,i}$ and $\alpha_{L,j}$ both extend $\gamma|_{U_i \cap U_j}$ thus their difference satisfies the condition of Theorem \[\text{4}\] and we can find appropriate transition functions $\beta_{L,ij}$ which automatically satisfy the cocycle condition on triple intersections due to the assumption $H^2(Y, \mathcal{O}_Y) = 0$. This shows that the map from equivalence classes to the set of $\gamma$ is onto.
To show that this map is also injective, assume that two deformations \( \mathcal{L} \) and \( \mathcal{\hat{L}} \) are given. We can choose a common refinement of the open coverings on which these deformations split, and assume that they are given by the data \( \{ \alpha_{L,i}, \beta_{L,ij} \} \) and \( \{ \hat{\alpha}_{L,i}, \hat{\beta}_{L,ij} \} \), respectively. By assumption, on each \( U_i \) both \( \alpha_{L,i} \) and \( \hat{\alpha}_{L,i} \) extend \( \gamma(x,l) \) on \( U_i \) and invoking Theorem 4 again we can find \( \beta_{L,i} \in \Gamma(U_i, \mathcal{D}^2(L, L)) \) which allows to change the splitting of \( \hat{\mathcal{L}} \) in such a way that \( \alpha_{L,i} = \hat{\alpha}_{L,j} \). Then on double intersections both \( \beta_{L,ij} \) and \( \hat{\beta}_{L,ij} \) solve the equation 

\[
(\alpha_{L,i} - \alpha_{L,j})(f, l) = \beta_{L,ij}(fl) - f\beta_{L,ij}(l)
\]

to \( \beta_{L,ij}(l) \). Hence the difference \( \beta_{L,ij} \) and \( \hat{\beta}_{L,ij} \) is \( O_Y \)-linear, i.e. given by multiplication of \( l \) by a regular function \( \tilde{\beta}_{L,ij} \). By definition such functions satisfy the cocycle condition on triple intersections. By our assumption \( H^1(Y, \mathcal{O}_Y) = 0 \) and we can find \( \tilde{\beta}_{L,i} \in \Gamma(U_i, \mathcal{O}_Y) \) such that \( \tilde{\beta}_{L,ij} = \tilde{\beta}_{L,i} - \tilde{\beta}_{L,j} \) on \( U_i \cap U_j \). Using \( \tilde{\beta}_{L,i} \) to adjust the splitting of \( \hat{\mathcal{L}}|_{U_i} \) one more time, we achieve \( \beta_{L,ij} = \hat{\beta}_{L,ij} \). This means that the deformations \( \mathcal{L} \) and \( \mathcal{\hat{L}} \) are equivalent. \( \square \).

4.2 The case of non-degenerate bivector.

In this subsection we assume that the bivector \( P \) is non-degenerate, i.e. gives an isomorphism \( \Omega_X \to T_X \). If in addition the Schouten-Nijenhuis bracket \( \{ P, P \} \) vanishes, this means that \( X \) has algebraic symplectic structure (but we only need this condition when discussing the second order deformations).

By non-degeneracy the restriction of \( P \) to \( Y \) embeds \( N^\circ \) as a subbundle into \( T_Y \). We will denote the image by \( T_F \) and call it the null foliation subbundle. Coisotropness of \( Y \) means that \( T_F \) is involutive, i.e. a sheaf of Lie subalgebras with respect to the bracket of vector fields. We define the null foliation Atiyah algebra \( \mathcal{A}_n(L) \) to be the preimage of \( T_F \subset T_Y \) in \( \mathcal{D}^1(L, L) \) with respect to the symbol map \( \sigma : \mathcal{D}^1(L, L) \to T_Y \), i.e. first order operators from \( L \) to itself with symbol in \( T_F \). Thus we have an extension

\[
0 \to \mathcal{O}_Y \to \mathcal{A}_n(L) \to T_F \to 0
\]

Note that \( \mathcal{A}_n(L) \) is a sheaf of Lie algebras with respect to the commutator of differential operators. Applying the isomorphism \( N^\circ \cong T_F \) we can view \( \gamma \) as a map \( T_F \to \mathcal{D}^1(L, L) \) and equations (6) and (7) - (6) become

\[
\gamma(ax, l) - a\gamma(x, l) = 1/2 x(a)l
\]

(9)

\[
\gamma(x, al) - a\gamma(x, l) = x(a)l.
\]

(10)

The second equation simply says that \( \sigma \circ \gamma = Id_{T_F} \). However, \( \gamma \) is not \( \mathcal{O}_Y \)-linear, as can be seen from the first equation. The meaning of the first equation can be seen from the following theorem

**Theorem 10** For a non-degenerate \( P \) the following conditions on the line bundle \( L \) are equivalent:
The equation
\[- at(N) + 2Id_N \otimes c_1(L) \cup P = 0;\]
holds in \(H^1(Y, N);\)

2. There exists \(\mathbb{C}\)-linear splitting \(\gamma : T_F \to \text{At}_n(L)\) satisfying (9) and (10).

3. There exists an anti-involution \(\partial \mapsto \partial^*\) on \(\text{At}_n(L)\) such that
\[\sigma(\partial^*) = -\sigma(\partial), \quad f^* = f, \quad (f\partial)^* = \partial^* f\]
where \(f \in \mathcal{O}_Y \subset \text{At}_n(L)\).

If \(H^2(Y, \mathcal{O}_Y) = 0\) then either of these conditions is equivalent to existence of a first order deformation \(\mathcal{L}\) of \(L\). If in addition \(H^1(Y, \mathcal{O}_Y) = 0\) the equivalence class of \(\mathcal{L}\) is uniquely determined by \(\gamma\).

Proof. Since (9) and (10) are simply reformulations of (6) and (7) the equivalence 1 \(\Leftrightarrow\) 2 is essentially proved in Proposition 9 (note that the vanishing of a class in \(H^2(Y, \mathcal{O}_Y)\) is irrelevant to this equivalence).

To prove equivalence 2 \(\Leftrightarrow\) 3 first assume that \(\gamma\) exists and define the *-involution to be +1 on \(\mathcal{O}_Y \subset \text{At}_n(L)\) and -1 on the image of \(\gamma\). Conversely, if the *-involution exists then its \((-1)\)-eigensheaf projects isomorphically onto \(T_F\) and there is a unique \(\gamma\) such that \(\gamma\sigma\) is the projection on the \((-1)\)-eigensheaf. A direct easy computation shows that the conditions imposed on \(\gamma\) and * are equivalent. \(\square\)

Suppose that the first order deformation \(\mathcal{A} = \mathcal{O}_X \oplus \epsilon \mathcal{O}_X\) extends to a second order deformation \(\mathcal{A}'\) over \(\mathbb{C}[\epsilon]/\epsilon^3\). Then on affine open subsets the product in \(\mathcal{A}'\) will be given by
\[f \ast g = fg + \epsilon \alpha_X(f, g) + \epsilon^2 \alpha'_X(f, g)\]
with the usual associativity condition
\[\alpha_X(a, \alpha_X(b, c)) - \alpha_X(\alpha_X(a, b), c) = d\alpha'_X(a, b, c)\]
By the explicit formula of Section 1.4.2 of [Ko] the locally defined operator \(\alpha'_X(f, g)\) may be taken symmetric (after a local choice of an algebraic connection on the tangent bundle).

Proposition 11 In the notation the previous theorem, assume that \(L\) admits a second order deformation to a left \(\mathcal{A}'\)-module \(\mathcal{L}'\). Then the operator \(\gamma\) agrees with the Lie brackets:
\[\gamma(\partial_1), \gamma(\partial_2) - \gamma([\partial_1, \partial_2]) = 0\]

Proof. Locally the second order deformation \(\mathcal{L}'\) is given by
\[a \ast l = al + \epsilon \alpha_L(a, l) + \epsilon^2 \alpha'_L(a, l)\]
for some bidifferential operator \(\alpha'_L : \mathcal{O}_X \times L \to L\). The usual associativity equation reads
\[\alpha_L(\alpha_X(a_1, a_2), l) - \alpha_L(a_1, \alpha_L(a_2, l)) + \alpha'_X(a_1, a_2) = \alpha'_L(a_1, a_2) + a_1 \alpha'_L(a_2, l) - \alpha'_L(a_1a_2, l)\] (11)
If \(a_1, a_2\) are sections in \(I\) then the first two terms on the right disappear. Antisymmetrizing in \(a_1\) and \(a_2\) and using the fact that \(2\alpha_X : I \times I \to I\) descends to the bracket of vector fields on \(T_F \simeq I/I^2\), we obtain the result. \(\square\)
4.3 Deformations and connections.

We keep the previous assumptions running: $\kappa = 0$, $H^1(Y, \mathcal{O}_Y) = H^2(Y, \mathcal{O}_Y) = 0$ and $P$ is non-degenerate.

**Theorem 12** Let $L_1, L_2$ be two line bundles on $Y$ admitting first order deformations $\mathcal{L}_1, \mathcal{L}_2$ corresponding to the operators $\gamma_1 : T_F \otimes L_1 \to L_1$ and $\gamma_2 : T_F \otimes L_2 \to L_2$. Then the bundle $M = \mathcal{H}om_{\mathcal{O}_Y}(L_1, L_2)$ admits a partial algebraic connection $\gamma_M : T_F \otimes M \to M$ defined by

$$\gamma_M(\partial, \phi)(l_1) = \gamma_2(\partial, \phi_2(l_1)) - \phi(\gamma_1(\partial, l_1))$$

Conversely, if $M$ is a bundle with a partial connection $\gamma : T_F \times M \to M$ and $L_1$ is a line bundle admitting a first order deformation $\mathcal{L}_1$ then the line bundle $L_2 = M \otimes_{\mathcal{O}_Y} L_1$ also admits a first order deformation $\mathcal{L}_2$ with $\gamma_2 : T_F \times L_2 \to L_2$ defined by the formula

$$\gamma_2(\partial, m \otimes l_1) = \gamma_M(\partial, m) \otimes l_1 + m \otimes \gamma_1(\partial, l_1).$$

In addition, suppose that $\mathcal{A}$ extends to a second order deformation $\mathcal{A}'$. If $\mathcal{L}_1, \mathcal{L}_2$ admit second order deformations to left $\mathcal{A}'$-modules $\mathcal{L}'_1, \mathcal{L}'_2$ then the partial connection $\gamma_M$ is flat, i.e. its curvature in $H^0(Y, \Lambda^2 T_F^\vee)$ is zero.

**Proof.** One needs to show that the first formula indeed defines an $\mathcal{O}_Y$-linear operator and that the second formula is well-defined on the tensor product over $\mathcal{O}_Y$, i.e. $\gamma_2(m \otimes al) = \gamma_2(ma \otimes l)$. These assertions and the assertions about first order deformations follow from (9) and (10) by a straightforward computation.

For the second order deformation, let $\partial_1, \partial_2$ be two sections of $T_F$. We need to show that

$$\gamma_M(\partial_1, \gamma_M(\partial_2, \phi)) - \gamma_M(\partial_2, \gamma_M(\partial_1, \phi)) - \gamma_M([\partial_1, \partial_2], \phi) = 0$$

for any section $\phi$ of $M = \mathcal{H}om_{\mathcal{O}_Y}(L_1, L_2)$. This follows immediately from the formula of previous proposition and definition of $\gamma_M$. □

**Remark.** We also expect that, conversely, if $\gamma_M$ is a flat algebraic connection along the null foliation and $L_1$ admits a second order deformation, then $L_2 = M \otimes L_1$ also admits a second order deformation. In fancier terms, the category of second order deformations of line bundles on $Y$ should be a gerbe over the Picard category of line bundles with a flat algebraic connection along the null-foliation.

5 The Lagrangian case.

In this section we assume that the bivector $P \in H^0(X, \Lambda^2 T_X)$ is non-degenerate everywhere on $X$ and that $Y$ is Lagrangian, i.e. its dimension is half the dimension of $X$. Since at this moment we work with first order deformations, we will not need the condition that the
algebraic 2-form defined by $P$ is closed. The restriction of $P$ to $Y$ defines an isomorphism $N \simeq \Omega^1_Y$. Therefore we can write

$$\text{at}(N) \in H^1(Y, \text{End}(N) \otimes \Omega^1_Y) = H^1(Y, \Omega^1 \otimes T_Y \otimes \Omega^1_Y)$$

where we agree that the first two factors in the last expression represent $\text{End}(\Omega^1_Y)$.

**Corollary 13** If $Y \subset X$ is Lagrangian, $H^2(Y, \mathcal{O}_Y) = 0$ and $L$ is a line bundle on $Y$ then $L$ admits a first order deformation to a left $\mathcal{A}$-module is and only if

$$-c_1(K_Y) + 2c_1(L) + \kappa = 0$$

in $H^1(Y, \Omega^1)$. In particular, if $\kappa = 0$ the deformation exists if and only if $2c_1(L) = c_1(K_Y)$.

**Proof.** Since in the Lagrangian case $P \in H^0(Y, N \otimes T_Y) \simeq H^0(Y, \Omega^1_Y \otimes T_Y)$ is simply the canonical identity element, the cup product $[- \text{at}(N) + 2\text{Id}_N \otimes c_1(L)] \cup P$ of the Theorem (7) simply amounts to the contraction of a class in $H^1(Y, \Omega^1 \otimes T_Y \otimes \Omega^1_Y)$ in the last two factors. For $2\text{Id}_N \otimes c_1(L)$ this immediately gives $2c_1(L)$. For $\text{at}(N)$ this would give $c_1(N) = c_1(\Omega^1_Y) = c_1(K_Y)$ by one of the equivalent definitions of $c_1$ if we were contracting in the first two factors. However, by Proposition 2.1.1 in [Ka] the Atiyah class $\text{at}(N) = \text{at}(\Omega^1_Y) = -\text{at}(T_Y)$ is symmetric with respect to the first and the third factors of $\Omega^1 \otimes T_Y \otimes \Omega^1_Y$. This finishes the proof. $\square$

We also remark that in the Lagrangian case the null foliation bundle $T_F$ is equal to the full tangent bundle $T_Y$, and the partial connections considered in Section 4.3 become connections in the usual sense.

**References**

[BG] Baranovsky, V.; Ginzburg, V.: Gerstenhaber-Batalin-Vilkoviski structures on coisotropic intersections, to appear in Math. Res. Lett., also preprint arXiv:0907.0037.

[CF] Cattaneo A., Felder, G.: Relative formality theorem and quantisation of coisotropic submanifolds. Adv. Math. 208 (2007), no. 2, 521–548.

[GS] Gerstenhaber, M.; Schack S.D.: On the cohomology of an algebra morphism. J. Algebra 95 (1985), no. 1, 245–262.

[Ka] Kapranov, M: Rozansky-Witten invariants via Atiyah classes. Compositio Math. 115 (1999), no. 1, 71–113.

[Ko] Kontsevich, M.: Deformation quantization of Poisson manifolds. Lett. Math. Phys. 66 (2003), no. 3, 157–216.
[FMY] Frégier, Y.; Markl, M.; Yau, D.: The $L_\infty$-deformation complex of diagrams of algebras. New York Journal of Math. 15 (2009), 353-392.

[We] Weibel, C.: An introduction to homological algebra. Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994.

[Ye] Yekutilei, A.: Deformation quantization in algebraic geometry. Adv. Math. 198 (2005), no. 1, 383–432.

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