A $\ell_1$-PREDUAL WHICH IS NOT ISOMETRIC TO A QUOTIENT OF $C(\alpha)$

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Abstract. About twenty years ago Johnson and Zippin showed that every separable $L_1(\mu)$-predual was isometric to a quotient of $C(\Delta)$, where $\Delta$ is the Cantor set. In this note we will show that the natural analogue of the theorem for $\ell_1$-preduals does not hold. We will show that there are many $\ell_1$-preduals which are not isometric to a quotient of any $C(K)$-space with $K$ a countable compact metric space. We also prove some general results about the relationship between $\ell_1$-preduals and quotients of $C(K)$-spaces with $K$ a countable compact metric space.

About 20 years ago Johnson and Zippin [J-Z] proved the following theorem.

Theorem. Suppose that $X$ is a separable $L_1(\mu)$-predual, then there is subspace $Y$ of $C(\Delta)$, where $\Delta$ is the Cantor set, such that $X$ is isometric to $C(\Delta)/Y$.

Because the space $X$ might be $C(\Delta)$, $C(\Delta)$ is the smallest $L_1(\mu)$-predual that one could use for such a result. If we consider the class of $\ell_1$-preduals then it is conceivable that a smaller space might be sufficient although the space would need to depend on some measurement of the size of the $\ell_1$-predual. A natural class of spaces to consider is the spaces $C(\alpha)$ where $\alpha$ is a countable ordinal. (This is the same as the class of $C(K)$-spaces with $K$ a countable compact Hausdorff space by a classical result of Mazurkiewicz and Sierpiński, [M-S].) Thus one can consider the following question.

Question. If $X$ is an $\ell_1$-predual is there a countable ordinal $\alpha$ such that $X$ is isometric (isomorphic) to a quotient of $C(\alpha)$?

We will show that the isometric question has a negative answer and prove some technical results which are useful for deciding whether an $\ell_1$-predual is isometric to a quotient of $C(\alpha)$. The isomorphic question remains open and at the end of the paper we discuss some variants of the isomorphic problem. Note also that we consider only $X$ for which $X^*$ is isometric to $\ell_1$ because even the Johnson–Zippin result is false for isomorphic $\ell_1$-preduals, [B-D].

Throughout this paper $\ell_1$-predual will mean a Banach space with dual isometric to $\ell_1$. If $\alpha$ is an ordinal $C(\alpha)$ will denote the space of continuous functions on the ordinals less than or equal to $\alpha$ with the order topology. If $A$ is a subset of a Banach space, $[A]$ is the norm closed linear span of $A$. Notation and standard
results from Banach space theory may be found in the books of Lindenstrauss and Tzafriri, [L-T], Dunford and Schwartz, [D-S], and Diestel, [D].

We begin with some technical results about basic sequences equivalent to the usual $\ell_1$ basis which are contained in dual spaces.

**Lemma 1.** Let $Z$ be a Banach space with separable dual and let $Y$ be a subspace of $Z^*$ which is isomorphic to $\ell_1$ with normalized $\ell_1$-basis $(y_n)$. If $\{y_n\}^{w^*} \subset Y$ then $Y$ is $w^*$-closed in $Z^*$.

*Proof.* By the Krein-Smulian Theorem, [D-S, Theorem V.5.7], it is sufficient to show that $B_Y = \{ y \in Y : \|y\| \leq 1 \}$ is $w^*$-closed. Let $(z^*_n)$ be a sequence in $B_Y$ with $w^*$-limit $z^*$. We will show that $z \in Y$. Let $K=\{y_n\}^{w^*}$ and let $\lambda \geq 1$ satisfy
\[ \lambda \| \sum a_n y_n \| \geq \sum |a_n| \] for all finite sequences $(a_n)$. We know that $\overline{\text{co} \{ \pm y_n \}} \supset B_Y$.

Because $z^* \in \overline{B_Y^{w^*}} \subset \text{co} \{ \pm K^{w^*} \}$, by the Choquet, [D, p 154], and Milman Theorems, [D-S, Lemma V.8.5], $z^*$ is represented by a probability measure $\mu$ supported on $\text{Ext} \overline{\text{co} \{ \pm K^{w^*} \}}$. If $z^* \notin Y$ then by the Hahn-Banach Theorem there exists $z^{**} \in Z^{**}$ such that $z_{jY}^* = 0$, $\|z^{**}\| = 1$, and $z^{**}(z^*) > 0$. $z^{**}$ is the $w^*$-limit of a sequence in $Z$, $(z_j)$, with $\|z_j\| \leq 1$ for all $j$. Thus
\[ z^{**}(z^*) = \lim_j z^*(z_j) = \lim_j \int_{\pm \lambda K} z_j d\mu. \]

Since $K \subset Y$, $z^{**}(k) = 0$ for all $k \in K$. Hence $z_j(\lambda k) \to 0$ for all $k \in K$. Therefore $(z_j)$ is a uniformly bounded sequence converging pointwise to 0 on $\pm \lambda K$ and it follows from the Bounded Convergence Theorem that $z^{**}(z^*) = 0$. □

The next lemma tells us that the $w^*$-closure of the $\ell_1$-basis is the only thing that is important.

**Lemma 2.** Suppose that $X$ and $Y$ are separable Banach spaces and that $(x^*_n)$ and $(y^*_n)$ are normalized sequences in $X^*$ and $Y^*$, respectively, which are equivalent to the usual unit vector basis of $\ell_1$ and for which $[x^*_n]^{w^*} = [x^*_n]$ and $[y^*_n]^{w^*} = [y^*_n]$. Suppose that the basis to basis map $\phi$ of $[x^*_n]$ onto $[y^*_n]$, i.e., $\phi(\sum a_n x^*_n) = \sum a_n y^*_n$, is a $w^*$-homeomorphism of the $w^*$-closure of $\{x^*_n\}$ onto the $w^*$-closure of $\{y^*_n\}$.

*Proof.* Only the $w^*$-continuity needs to be proved. By passing to quotients we may assume that $(x^*_n)$ and $(y^*_n)$ are bases for the duals of $X$ and $Y$, respectively. If we identify $X$ and $Y$ with the $w^*$-continuous affine symmetric functions on $B_X^*$ and $B_Y^*$, respectively, we need only show that the linear extension of a $w^*$-continuous linear function on $K(X) = \lambda^{-1}[x^*_n]^{w^*} \cup \{0\}$, where $(x^*_n)$ is $\lambda$ equivalent to the usual basis of $\ell_1$, is $w^*$-continuous on $B_X^*$. Let $\Phi$ be the map defined by $(\Phi f)(x^*) = f(\phi(x^*)))$ for $x^* \in K(X)$ and let $J_X$ and $J_Y$ be the evaluation maps from $X$ into $C(K(X))$ and from $Y$ into $C(K(Y))$, respectively. $(K(Y)$ is defined analogously to
Example 1. Let $(x_n)$ be a sequence of points in the open unit square in $\mathbb{R}^2$ such that $K = \{t_n\} = \{t_n\} \cup \{(t,0) : 0 \leq t \leq 1\}$. Let $X = \{f \in C(K) : f(t,0) = tf((0,0)) + (1-t)f((1,0))\}$. Clearly $X$ is isomorphic to $c_0$ and the $\ell_1$-basis for the dual is the evaluations at the $t_n$'s and at $(0,0)$ and $(1,0)$. We claim that $X$ is isometric to a quotient of $C(\omega \cdot 2)$. Indeed, let $t_{n,1}$ denote the first coordinate of $t_n$ for all $n$. Define $\phi(t_{n,1} \delta_n + (1-t_{n,1}) \delta_{\omega+n}, \phi(\delta_{(0,0)})) = \delta_{\omega}$, and $\phi(\delta(1,0)) = \delta_{\omega-2}$. It is easy to verify that $\phi$ extends to be a $w^*$-continuous map from the closure of $X^*$ onto the closure of the image. Clearly the span of the range of $\phi$ contains the $w^*$-closure of the range of $\phi$. Thus $X$ is a quotient of $C(\omega \cdot 2)$ by the proposition.

Next we will note some simple facts about basic sequences in the dual of a $C(K)$-space which are 1-equivalent to the usual basis of $\ell_1$.

Lemma 4. If $(\mu_n)$ is a sequence of mutually singular (non-zero) measures in $C(K)^*$, $K$ compact metric, which converges $w^*$ to $\mu$ then

$$\text{supp } \mu \subset \{k \in K : \text{there exist } k_n \in \text{supp } \mu_n \text{ with } k_n = k\}.$$ 

If the measures in Lemma 4 are all atomic then we may replace the support of the measures by the set of points where the measure is non-zero and the result remains true. The proof of the lemma is straight-forward so we leave it to the reader.
Corollary 5. If $K$ is a countable compact metric space and $(\mu_n)$ is a sequence of mutually singular measures in $C(K)^*$ with $w^*$-limit $\mu$, then $\text{supp } \mu \subset K^{(1)}$, the first derived set of $K$.

Proof. This follows immediately from Lemma 4. \qed

Now let $X$ be a $\ell_1$-predual and let $(e_n)$ be the $\ell_1$-basis for the dual. We want to define a system of derived sets $N^{(\alpha)} = N^{(\alpha)}((e_n)_{n \in \mathbb{N}})$ of $\mathbb{N}$ that has properties similar to those of the Szlenk sets for the $w^*$-closure $K$ of the sequence $(e_n)$. (See [S] for the definition and properties of the Szlenk sets.)

Definition. Let $N^{(0)} = \mathbb{N}$ and if $N^{(\alpha)}$ has been defined, let

$$N^{(\alpha+1)} = \{ n \in N^{(\alpha)} : \text{there exists an infinite } M \subset N^{(\alpha)}$$

such that $(w^* \lim_{j \in M} e_j)(n) \neq 0 \}.$

If $\alpha$ is a limit ordinal define $N^{(\alpha)} = \cap_{\beta < \alpha} N^{(\beta)}$.

Proposition 6. Let $K$ be a countable compact metric space. If $(\mu_n)$ is a sequence of norm one measures in $C(K)^*$, $\mathcal{K}$ a countable compact metric space, which are $1$-equivalent to the unit vector basis of $\ell_1$ and $[\mu_n]$ is $w^*$-closed, then $\{N^{(\alpha)}((\mu_n)_{n \in \mathbb{N}}) : \alpha < \alpha_0 \}$ is a strictly decreasing family of subsets of $\mathbb{N}$, where $\alpha_0$ is the smallest ordinal $\alpha$ such that $N^{(\alpha)} = \emptyset$.

Proof. By Corollary 5 we have that if $(\mu_j)_{j \in M} \text{ converges to a non-zero measure } \mu = \sum a_k \mu_k$, then $\text{supp } \mu \subset K^{(1)}$. Therefore if $a_n \neq 0$ then $n \in N^{(1)}$, and $\text{supp } \mu \subset K^{(1)}$ implies that $\text{supp } \mu_n \subset K^{(1)}$ because the measures $\mu_k$ are mutually singular. Hence for all $n \in N^{(1)}$, $\text{supp } \mu_n \subset K^{(1)}$. A simple induction argument completes the proof. \qed

Using this proposition it is very easy to find $\ell_1$-preduals which are not isometric to a quotient of any $C(\alpha)$.

Example 2. Let $X = \{(a_n) \subset \mathbb{R} : \lim_{j \to \infty} a_j = \sum_{n \in \mathbb{N}} \frac{a_n}{2^n}\}.$ Note that $X$ is a codimension 1 subspace of $c = C(\omega)$. Thus $X$ is isomorphic to $c_0$. We claim that the evaluation functionals defined by $e_n((a_j)) = a_n$ are a basis for the dual $1$-equivalent to the usual $\ell_1$-basis. To do this for each $k \in \mathbb{N}$ we will exhibit a copy of $\ell_k^\perp$ which norms the span of the first $k$ of these functionals. Fix $K > k$ and let

$$f_j(n) = \begin{cases} 1, & \text{if } n = j \\ 0, & \text{if } n < K \text{ and } n \neq j \\ -2^{-j}, & \text{if } n = K \\ 2^{-j}, & \text{if } n > K. \end{cases}$$

for $j = 1, 2, \ldots, k$. Obviously $\lim_{n \to \infty} f_j(n) = 2^{-j} = \sum_{n \in \mathbb{N}} 2^{-n}f_j(n)$. Thus $(f_j)_{j \leq k}$ is a sequence in $X$ $1$-equivalent to the unit vectors in $\ell_k^\perp$ and biorthogonal to the first $k$ evaluation functionals.

We claim that $X$ is not isometric to a quotient of $C(\alpha)$. Indeed, the $w^*$-limit in $X^*$ of $(e_j) = \sum_{n \in \mathbb{N}} \frac{e_n}{2^n}$. Hence $N^{(\alpha)}((e_n)_{n \in \mathbb{N}}) = \mathbb{N}$ for all $\alpha$. By Propositions 3 and 6,
$X^*$ is not isometric to a $w^*$-closed subspace of $C(K)^*$ for any countable compact metric space $K$, consequently, $X$ is not isometric to a quotient of any $C(\alpha)$.

The point here is that in this case one really needs the Cantor set or some other uncountable compact metric space to support the measures. If we relax the requirement from isometric to a quotient to $(1 + \epsilon)$-isomorphic to a quotient of $C(\alpha)$, then we can embed the basis of $X^*$ in $C(\omega^2)^*$.

Fix $N$ and define $\phi(e_n) = \delta_{\omega\cdot n}$ for $n \leq N$ and

$$\phi(e_n) = (1 - \frac{1}{2^{n-1}})^{-1}\left[\sum_{j\leq N} \frac{\delta_{\omega\cdot (j-1)+n}}{2^j} + \sum_{N+1 \leq j < n} \frac{\phi(e_j)}{2^j}\right]$$

for $n > N$. It follows from Lemma 2 that $\phi$ can be extended to a $w^*$-isomorphism of $X^*$ into $C(\omega^2)$.

The example provides a trivial way of producing more such examples. Indeed, take any $\ell_1$-predual and construct an isomorph by adding the example as a $\ell_\infty$ direct summand to get another $\ell_1$-predual which is not isometric to a quotient of any $C(\alpha)$. There are probably much more interesting and subtle examples that can be constructed by using the behavior of the $w^*$-topology in the example.

As we noted earlier it is unknown whether every $\ell_1$-predual is isomorphic to a quotient of $C(\alpha)$. In addition there are several variants of this question which are open. Below $X$ is a $\ell_1$-predual.

**Question 1.** Given $\epsilon > 0$ does there exist $\alpha < \omega_1$ and a $w^*$-continuous into isomorphism $S$ of $X^*$ into $C(\alpha)^*$ with $\|S\|\|S^{-1}\|_1 < 1 + \epsilon$? In other words is $X$ $(1 + \epsilon)$-isomorphic to a quotient of $C(\alpha)$?

**Question 2.** Is there an $\alpha < \omega_1$ and a quotient $Y$ of $C(\alpha)$ such that $Y^*$ is isometric to $\ell_1$ and $X$ is isomorphic to $Y$?

A positive answer to Question 1 or 2 would of course answer the isomorphic question. A positive answer to Question 1 would imply that there is an $\alpha$ that works for all $\epsilon$ simultaneously, that is, there is an $\alpha$ so that $X$ is almost isometric to a quotient of $C(\alpha)$. Such a result should provide very useful information of the structure of $\ell_1$-preduals. Let us also note that very little is known about quotients of $C(\alpha)$. In particular it is unknown whether they must be $c_0$-saturated. ($Y$ is $c_0$-saturated if every subspace contains $c_0$.)

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