Chaotic Quantization: Maybe the Lord plays dice, after all?

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Abstract. We argue that the quantized non-Abelian gauge theory can be obtained as the infrared limit of the corresponding classical gauge theory in a higher dimension. We show how the transformation from classical to quantum field theory emerges, and calculate Planck's constant from quantities defined in the underlying classical gauge theory.

1 Introduction

The question, how gravitation and quantum mechanics can be merged into a consistent unified theory of all fundamental interactions, is still open. Logically, either general relativity (GR), or quantum mechanics (QM), or possibly both, will have to be replaced by a different theory at a more fundamental level. The almost universally accepted notion is that it is GR which needs to be replaced, while QM presumably provides a truly fundamental description of nature. Superstring theory, describing our four-dimensional space-time as the low-energy limit of a ten- or eleven-dimensional theory, is widely accepted as the most promising approach, but neither the precise form nor the full content of this theory is entirely understood at the present time.

What about the other option, considering QM as the low-energy limit of a more fundamental theory? This question has been raised by 't Hooft, who conjectured that quantum mechanics can logically arise as the low-energy limit of a microscopically deterministic, but dissipative theory [1,2]. Explicit, but highly simplified examples for such a mechanism have been constructed [3,4]. In a recent publication [5] with Biró and Matinyan, we showed how (Euclidean) quantum field theory can emerge in the infrared limit of a higher-dimensional, nonlinear classical field theory (Yang-Mills theory). We called this phenomenon chaotic quantization to distinguish it from the formal technique named stochastic quantization [6], not realizing that this term was already introduced by C. Beck several years earlier [7] for essentially the same mechanism. What is special about Yang-Mills fields, however, is that they “quantize themselves”, as we shall discuss below. In Sect. 2, we introduce the concept of chaotic quantization of a system with one degree of freedom, which we extend to field theory in Sect. 3. In Sect. 4 we review the chaotic properties of classical Yang-Mills theory, before we analyze their chaotic self-quantization in Sect. 5. In the final section, we enumerate and discuss several open problems.
2 Chaotic quantization

A classical physical system encodes much more information than the analogous quantized system. Consider, for instance a point particle in one dimension. The classical system is defined by the pair of coordinates \((x, p)\), implying that every point in the continuous phase space represents a different state. For the quantum system, on the other hand, the uncertainty relation limits the localization of the state in phase space to the finite element \(\Delta x \Delta p \sim \hbar\). This observation suggests that it may be useful to consider classical systems, whose internal dynamics results in a self-afflicted loss of information.

Deterministically chaotic systems satisfy this condition. For such a system, the rate of information loss is encoded in the Lyapunov exponent \(\lambda\), defined as the logarithmic rate of divergence between neighboring trajectories:

\[
|x_1(t) - x_2(t)| \sim e^{\lambda t} \quad (\lambda > 0)
\]  

or equivalently in the eigenvalue \(\gamma\) of the Perron-Frobenius operator, defined as the logarithmic rate of convergence of the phase space density to its stationary limit:

\[
\rho(x, t) \to \rho_{\text{lim}}(x) + \rho'(x)e^{-\gamma t} \quad (\gamma > 0).
\]  

Before we pursue this idea further, let us recall the method of stochastic quantization [6]. Consider a quantum field \(\phi(x)\) in Euclidean space with action \(S[\phi]\). The domain of \(\phi\) is formally extended into a fifth dimension denoted by \(\tau\). If the field \(\phi(x, \tau)\) obeys the Langevin-type equation

\[
\frac{\partial}{\partial \tau}\phi(x, \tau) = -\frac{\delta S}{\delta \phi}(x, \tau) + \xi(x, \tau),
\]  

where \(\xi(x, \tau)\) represents local white noise defined by the moments

\[
\langle \xi(x, \tau) \rangle = 0, \quad \langle \xi(x, \tau)\xi(x', \tau') \rangle = 2\delta(x - x')\delta(\tau - \tau'),
\]  

then the long-time average of any physical observable converges to the quantum mechanical vacuum expectation value:

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T d\tau \mathcal{O}[\phi(x, \tau)] = \langle \mathcal{O}[\phi(x)] \rangle_{\text{QM}}.
\]  

Beck’s [8] suggestion was to replace the artificial white noise \(\xi\) with the “noise” generated by a deterministic, but chaotic (more precisely: \(\varphi\)-mixing [9]) process.

Following Beck [10,11], let us start by considering a dynamical system with two variables \(x, y\), which evolves in discrete time steps of length \(\tau\). We denote the state of the system at \(t_n = n\tau\) as \((x_n, y_n)\). We are interested in the dynamics in the “physical” variable \(y\), if the motion in \(x\) is chaotic on short time scales. We define the evolution of the system as follows:

\[
(x_{n+1}, y_{n+1}) = f(x_n, y_n) = (T(x_n), \lambda y_n + \tau^{1/2} x_n).
\]
Here the map $T$ is assumed to be $\phi$-mixing [9]. Equation (6) can be considered as the stroboscopic map of the differential equation

$$\dot{y} = -\gamma y + \tau^{1/2} \sum_{n=1}^{\infty} x_{n-1}\delta(t-n\tau),$$

with $x_{n+1} = T(x_n)$ and $\lambda = \exp(-\gamma\tau)$. Obviously, the variables $\{x_n\}$ take the role of the noise in this equation.

The Langevin equation (7) is equivalent to the Perron-Frobenius equation for the evolution of the phase space density of an ensemble of systems:

$$
\rho_{n+1}(x', y') = \sum_{(x, y) \in f^{-1}(x', y')} \frac{\rho_n(x, y)}{\lambda|\partial T/\partial x|} = \sum_{x \in T^{-1}(x')} \frac{\rho_n(x, (y' - \tau^{1/2}x)/\lambda)}{\lambda|\partial T/\partial x|}.
$$

Expanding (8) into powers of $(\gamma\tau)^{1/2} \equiv \bar{\tau}^{1/2}$, taking the limit $\tau \to 0$, and interpolating $\rho_n(x, y)$ to a function $\rho(x, y, t)$ which depends continuously upon time,

$$\rho(x, y, t) = \varphi(x, y, t) + \bar{\tau}^{1/2} a(x, y, t) + \bar{\tau} b(x, y, t) + \bar{\tau}^{3/2} c(x, y, t) + \cdots,$$

yields a set of coupled equations for the coefficient functions:

$$\varphi(x', y, t) = \sum_{x \in T^{-1}(x')} \frac{1}{|\partial T/\partial x|} \varphi(x, y, t)$$

$$a(x', y, t) = \sum_{x \in T^{-1}(x')} \frac{1}{|\partial T/\partial x|} \left( a(x, y, t) - x \frac{\partial}{\partial y} \varphi(x, y, t) \right)$$

etc.

Being interested in the dynamics of the physical variable $y$ only, we define projected functions

$$p_0(y, t) = \int dx \varphi(x, y, t); \quad \alpha(y, t) = \int dx a(x, y, t); \quad \text{etc.}$$

For complete maps $T$, it is possible to show that

$$f(x', y, t) = \sum_{x \in T^{-1}(x')} \frac{g(x, y, t)}{|\partial T/\partial x|}$$

for all $y$ and $t$ implies

$$\int dx \ f(x, y, t) = \int dx \ g(x, y, t).$$

The first equation in the expansion of the Perron-Frobenius equation in powers of $\bar{\tau}^{1/2}$ then becomes a tautology, while the second one takes the form

$$\frac{\partial}{\partial y} \int dx \ \varphi(x, y, t) = 0.$$
The desired dynamical equation for the phase space density $p_0(y, t)$ is obtained as the third equation (at order $\bar{\tau}$):

$$\frac{\partial}{\partial y} \int dx \, x \, a(x, y, t) = \frac{\partial}{\partial y} (y \, p_0(y, t)) + \frac{1}{2} \frac{\partial^2}{\partial y^2} \int dx \, x^2 \varphi(x, y, t) - \frac{\partial p_0}{\partial t}.$$  \hspace{1cm} (15)

If $h(x)$ is an invariant of the map $T$, $\varphi(x, y, t) = h(x) \, p_0(y, t)$ is a solution of the first of the set of equations (10) for any function $p_0(y, t)$, and (14) requires $\langle x \rangle \equiv \int dx \, x \, h(x) = 0$. Finally, (15) turns into

$$\frac{\partial}{\partial t} p_0(y, t) = \frac{\partial}{\partial y} (y \, p_0(y, t)) + \frac{\langle x^2 \rangle}{2} \frac{\partial^2}{\partial y^2} p_0(y, t) - \frac{\partial p_0}{\partial t}.$$  \hspace{1cm} (16)

The left-hand side and the first two terms on the right-hand side have the form of a Fokker-Planck (FP) equation; the last term on the right-hand side represents a source term. For symmetric maps, $T(x) = T(-x)$, it is easy to see that the source term vanishes, and the projected phase space density $p_0(y, t)$ satisfies a homogeneous FP equation. The first correction in $\bar{\tau}^{1/2}$, $\alpha(y, t)$, also satisfies a FP equation, but in this case, the source term does not vanish. In other words, the deviations in the equation for the exact projected phase space density $p(y, t) = \int dx \, \rho(x, y, t)$ from a FP equation are proportional to $\bar{\tau}^{1/2}$.

An explicit solution can be found for, e.g., the Ulam map $T(x) = 1 - 2x^2$ with $x \in [-1, 1]$. The stationary solution of the FP equation is:

$$p_0(y, t) \equiv p_0(y) = \left( \frac{2}{\pi} \right)^{1/2} e^{-2y^2};$$  \hspace{1cm} (17)

and the full solution written as a power series in $\bar{\tau}^{1/2}$ is:

$$p(y, t) = p_0(y) \left[ 1 + \bar{\tau}^{1/2} \left( 2y - \frac{8}{3} y^3 + O(\bar{\tau}) \right) \right].$$  \hspace{1cm} (18)

To obtain the Euclidean Schrödinger equation, instead of the FP equation, we need to introduce a potential $V(y)$ and rescale the auxiliary variable $x$ according to [7]

$$T(x) \rightarrow T(x) \, e^{\bar{\tau}^2 V(y) / \hbar} \quad y_{n+1} = \lambda y_n + \bar{\tau}^{1/2} x_n \rightarrow y_{n+1} = \lambda y_n + \left( \frac{\hbar \bar{\tau}}{m \sigma^2} \right)^{1/2} x_n,$$  \hspace{1cm} (19)

where $\sigma^2 = \langle x^2 \rangle \equiv \int dx \, x^2 h(x)$. Identifying $p_0$ with the Euclidean wavefunction $\psi$, we find that it satisfies the imaginary-time Schrödinger equation:

$$\hbar \frac{\partial}{\partial t} \psi(y, t) + \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial y^2} + V(y) \right] \psi(y, t) \equiv S_E \psi(y, t) = 0.$$  \hspace{1cm} (20)

The correction linear in $\bar{\tau}^{1/2}$ satisfies the same equation, but with an additional source term:

$$S_E \alpha(y, t) = \left( \frac{\hbar}{2m} \right)^{3/2} \frac{\partial^3}{\partial y^3} \psi(y, t).$$  \hspace{1cm} (21)
The complete wavefunction is \( \tilde{\psi} = \psi + \bar{\tau}^{1/2} \alpha + \cdots \), which approaches the Schrödinger wavefunction in the limit \( \bar{\tau} = \gamma \tau \to 0 \).

The important insight to take away from this derivation is that an appropriate chaotic process can serve as the source of the random noise required for the stochastic quantization of a dynamical system if the time scale on which the chaotic process randomizes is sufficiently short, so that the corrections are negligible.

### 3 Extension to field theories

Can this mechanism of quantizing classical systems be generalized to fields \( \Phi(x,t) \) with \( x \) being a point in three-dimensional space? How this can be done is most easily seen, when the field theory is defined on a lattice, rather than a spatial continuum. Then all one needs to do is introduce a map \( T \) together with some internal space \( \{\xi^i\} \) at each lattice point \( x_i \). Beck has proposed to define the evolution law including a nearest neighbor coupling [7]

\[
\xi^i_{n+1} = (1 - g)T \left( \xi^i_n \right) + \frac{g}{2d} \sum_{\nu=1}^{d} \left( \xi^{i+\nu}_n + \xi^{i-\nu}_n \right),
\]

which has the continuum limit

\[
\xi^{(x)}_{n+1} = T \left( \xi^{(x)}_n \right) + \frac{g'}{2d} \nabla^2 \xi^{(x)}_n,
\]

with appropriate coupling constants \( g, g' \). We will not follow this route further here and refer to Beck’s recent monograph [12].

A more natural approach consists in identifying the local internal map space with a compact Lie group \( G_x \). The simplest realization of this idea is the SU(2) gauge theory, i.e., the Yang-Mills field theory. In this case, the internal degrees of freedom (color) of the gauge field provide the local space for the chaotic map, and the nonlinear dynamics of the gauge field uniquely defines the map. As we shall see below, there is no need to introduce new degrees of freedom beyond those provided by the gauge field (in one additional dimension) itself [5]. Before exploring this idea in more detail, however, we need to review what is known about the chaotic dynamics of Yang-Mills fields.

### 4 Interlude: Chaotic properties of Yang-Mills fields

The chaotic nature of classical non-Abelian gauge theories was first recognized twenty years ago [13,14]. Over the past decade, extensive numerical solutions of spatially varying classical non-Abelian gauge fields on the lattice have revealed that the gauge field has positive Lyapunov exponents that grow linearly with the energy density of the field configuration and remain well-defined in the limit of small lattice spacing \( a \) or weak-coupling [15,16]. More recently, numerical studies have shown that the \((3 + 1)\)-dimensional classical non-Abelian lattice gauge
theory exhibits global hyperbolicity. This conclusion is based on calculations of the complete spectrum of Lyapunov exponents [17] and on the long-time statistical properties of local fluctuations of the Kolmogorov-Sinai (KS) entropy in the classical SU(2) gauge theory [18].

It is useful to note some important relationships between ergodic and periodic orbits for globally hyperbolic dynamical systems. The ergodic Lyapunov exponents \( \lambda_{r,i} \) are obtained by numerical integration of a randomly chosen ergodic trajectory, denoted by its origin \( r \). In a Hamiltonian hyperbolic dynamical system with \( d \) degrees of freedom the sum of its \( d-1 \) positive ergodic Lyapunov exponents is obtained as the ergodic mean of the local expansion rate:

\[
\lim_{t \to \infty} h_r(t) \equiv \lim_{t \to \infty} \frac{1}{t} \int_0^t \chi(x(t')) \, dt' = \sum_{i=1}^{d-1} \lambda_{r,i} = h_{KS}.
\] (24)

Here \( h_{KS} \) denotes the Kolomogorov-Sinai entropy and

\[
\chi(x(t)) = \frac{d}{dt} \ln \det \left( \frac{\partial x(t)}{\partial x(0)} \right)
\] (25)

is the local rate of expansion along the trajectory \( x(t) \). Due to the equidistribution of periodic orbits in phase space it is possible to evaluate the ergodic mean in (24) by weighted sums over periodic orbits \( p \). In fact, for hyperbolic systems the thermodynamic formalism allows to express certain invariant measures on phase space in terms of averages over periodic orbits [19,20]. Labeling periodic orbits by \( p \) (rather than a starting point), and denoting their periods and positive Lyapunov exponents by \( T_p \) and \( \lambda_{p,i} \), respectively, the connection between the positive ergodic Lyapunov exponents and those of periodic orbits is given by:

\[
h_{KS} = \lim_{t \to \infty} \frac{\sum_{t \leq T_p \leq t+\epsilon} \left( \sum_{i=1}^{d-1} \lambda_{p,i} \right) \exp \left( -\sum_{i=1}^{d-1} \lambda_{p,i} T_p \right)}{\sum_{t \leq T_p \leq t+\epsilon} \exp \left( -\sum_{i=1}^{d-1} \lambda_{p,i} T_p \right)}
\] (26)

where \( \epsilon > 0 \) is a small number. The topological pressure \( P(\beta) \) is a useful tool for analyzing invariant measures on phase space in terms of periodic orbits. This function can be expressed as

\[
P(\beta) = \lim_{t \to \infty} \frac{1}{t} \ln \sum_{t \leq T_p \leq t+\epsilon} \exp \left( -\beta \sum_{i=1}^{d-1} \lambda_{p,i} T_p \right).
\] (27)

The relation (26) then follows from (24) and from the identity \(-P'(1) = h_{KS}\).

In order to apply this formalism to the Yang-Mills field, the gauge theory needs to be formulated as a Hamiltonian system on a spatial lattice. How to do this is well known: The Hamiltonian lattice gauge theory was formulated by Kogut and Susskind [21] in order to study the nonperturbative properties of nonabelian gauge theories, such as quark confinement. Denoting the lattice
spacing by \( a \) and the nonabelian coupling constant by \( g \), the Hamiltonian for the SU(2) gauge theory is given by

\[
H = \frac{g^2}{2a} \sum_{x,i} \text{tr} E^2_{x,i} + \frac{2}{g^2 a} \sum_{x,ij} (2 - \text{tr} U_{x,ij})
\]  

Here the \( U_{x,i} \in SU(2) \) are called the link variables at point \( x \) in the coordinate direction \( i \), and the \( E_{x,i} \in \text{LSU}(2) \) denote the color-electric field strengths components. \( U_{x,ij} \) denotes the plaquette product of four link variables, starting and ending at \( x \) and circumscribing the elementary square in the \( i, j \) directions. The chaotic nature of this theory was demonstrated [15] by numerical simulations, and Gong [17] obtained the complete Lyapunov spectrum for lattice volumes \( L^3 \) with \( L = 1, 2, 3 \). Bolte et al. [18] extended these calculations to the lattices of size \( L = 4, 6 \).

\[\text{Fig. 1. Distribution of numerically obtained ergodic Lyapunov exponents for a classical } SU(2) \text{ gauge theory on lattices of size } L = 2, 4, 6. \text{ The index } i \text{ numbers the Lyapunov exponents and the abscissa is scaled with } L^3. \text{ The energy per plaquette was chosen as } 1.8/g^2a.\]

For sufficiently long trajectories and fixed energy per lattice site the Lyapunov spectrum shows a unique shape, independent of the lattice size, as shown in Fig.1. Indeed, for a completely hyperbolic system, physical intuition dictates that the KS entropy \( h_{KS} \) is an extensive quantity. For this to be true, the sum over all positive Lyapunov exponents must scale like the lattice volume \( L^3 \) and
the shape of the distribution of Lyapunov exponents must be independent of $L$. Figure 1 confirms this expectation.

![Figure 1](image1.png)

**Fig. 2.** Top: The distribution of the sum of local expansion rates $h_r(t_s)$ for $L=4$ and a short sampling time $t_s = 0.1$, together with a Gaussian fit. Bottom: The autocorrelation function $a(\tau)$ for this distribution.

The KS entropy is a *global* property of the dynamical system. The next step of detail of the ergodic nature of the system is provided by the fluctuations in the quasi-local average of $\chi(x(t))$. These fluctuations are obtained by integrating (25) up to a sampling time $t_s$. For sampling times $t_s$ much longer than the largest correlation time one expects that observables sampled along ergodic trajectories exhibit Gaussian fluctuations about their ergodic mean. Waddington [22] has shown that for Anosov systems (i.e., fully hyperbolic systems on compact phase spaces) the probability distribution for $h_r(t_s)$ is Gaussian with mean $h_{KS}$:

$$P[h_r(t_s)] \rightarrow \exp \left( -\frac{(h_r(t_s) - h_{KS})^2}{2\Delta h_r(t_s)^2} \right) \quad \text{for} \quad t_s \rightarrow \infty . \quad (29)$$

and square variance proportional to $P''(1)$:

$$\Delta h_r(t) \rightarrow \sqrt{P''(1)/t} \quad \text{for} \quad t \rightarrow \infty . \quad (30)$$
The variance (30) can be related to the autocorrelation function
\[
a(\tau) = \langle \chi(x(\tau)) \chi(x(0)) \rangle - h_{KS}^2
\]
of the local ergodic Lyapunov exponents, through the relation
\[
t(\Delta h_r(t))^2 = \int_{-t}^{+t} \left( t - \frac{t}{t} \right) a(\tau) \, d\tau \rightarrow P''(1) .
\]

Figure 2 (top) shows the distribution of sampled valued of \( h_r(t_s) \), (obtained on a single, very long trajectory) for a short sampling interval \( t_s = 0.1 \) for a \( L = 4 \) lattice. The bottom part of Fig. 2 shows the autocorrelation function \( a(\tau) \) obtained for the same trajectory. A numerical fit of the function indicates that \( a(\tau) \) decays exponentially for large times. The value predicted for \( \Delta h_r \) by (30,32) is in excellent agreement with the value obtained by a Gaussian fit to the sampled distribution.

![Figure 3. \( \Delta h_r(t_s) \), scaled with \((t_s/L^3)^{1/2}\), as a function of \( t_s \).](image)

One can also study how the width of the Gaussian scales with the lattice size \( L \). To a very good approximation one finds that it is proportional to \( L^{3} \). If one includes the dependence on the sampling time, the variance of \( h_r \) scales like \( L^{3}/t_s \) (see Fig. 3). As the mean value \( h_{KS} \) scales like \( L^{3} \), this result confirms the Gaussian nature of the fluctuations: \( \Delta h_r/h_{KS} \sim (L^{3}t_s)^{-1/2} \).

These numerically obtained results provide strong (although not mathematically conclusive) evidence that the SU(2) lattice gauge theory is a strongly chaotic (Anosov, \( \varphi \)-mixing) system with properties required for the formalism of Sect. 2.
5 Yang-Mills fields quantize themselves

The results discussed in the previous section imply that correlation functions of physical observables decay rapidly, and that long-time averages of observables for a single initial gauge field configuration coincide with their microcanonical phase-space average, up to Gaussian fluctuations which vanish in the long observation time limit as $t_s^{-1/2}$. Since the relative fluctuations of extensive quantities scale as $L^{-3/2}$, the microcanonical (fixed-energy) average can be safely replaced by the canonical average when the spatial volume probed by the observable becomes large. In the following we discuss the hierarchy of time and length scales on which this transformation occurs [5].

According to the cited results, the classical non-Abelian gauge field self-thermalizes on a finite time scale $\tau_{eq}$ given by the ratio of the equilibrium entropy and the KS-entropy, which determines the growth rate of the course-grained entropy:

$$\tau_{eq} = S_{eq}/h_{KS}.$$  \hspace{1cm} (33)

At weak coupling, the KS-entropy for the $(3+1)$-dimensional SU(2) gauge theory scales as

$$h_{KS} \sim g^2 E \sim g^2 T(L/a)^3,$$  \hspace{1cm} (34)

where $E$ is the total energy of the field configuration and $T$ is the related temperature related to $E$ by (for SU($N_c$))

$$\epsilon = \frac{E}{L^2} = 2(N_c^2 - 1) \frac{T}{a^3} + \mathcal{O}(g^2).$$  \hspace{1cm} (35)

The equilibrium entropy of the lattice is independent of the energy and proportional to the number of degrees of freedom of the lattice: $S_{eq} \sim (L/a)^3$. The time scale for self-equilibration is thus given by

$$\tau_{eq} \sim \frac{E}{h_{KS} T} \sim (g^2 T)^{-1}.$$  \hspace{1cm} (36)

When one is interested only in long-term averages of observables, it is thus sufficient to consider the thermal classical gauge theory on a three-dimensional spatial lattice. Furthermore, on time scales $t \gg \tau_{eq}$, the Yang-Mills field generates a random Gaussian process, which is required for chaotic quantization.

The dynamic properties of thermal non-Abelian gauge fields at such long distances have been studied in much detail [23–25]. While these studies have been made exclusively for the thermal quantum field theory, their results are readily transcribed to the thermal classical gauge field with a lattice cutoff. The real-time dynamics of the gauge field at long distances and times can be described, at leading order in the coupling constant $g$, by a Langevin equation

$$\sigma_c \frac{\partial A}{\partial t} = -D \times B + \xi,$$  \hspace{1cm} (37)
where $D$ is the gauge covariant spatial derivative, $B = D \times A$ is the magnetic field strength, and $\xi$ denotes Gaussian distributed (white) noise with zero mean and variance

$$
\langle \xi_i(x,t) \xi_j(x',t') \rangle = 2\sigma_c T \delta_{ij} \delta^3(x-x') \delta(t-t').
$$

Here $\sigma_c$ denotes the color conductivity [26] of the thermal gauge field which is determined by the ratio $\omega^2/\gamma$ of the plasma frequency $\omega$ and the damping rate $\gamma$ of a thermal gauge field excitation. In the classical field theory with a lattice cutoff, the color conductivity scales as

$$
\sigma_c \sim \left( a \ln[d_{mag}/d_{el}] \right)^{-1}.
$$

This relation implies that the color conductivity is an ultraviolet sensitive quantity, which depends on the lattice cutoff $a$.

We now consider observers measuring physical quantities on long time and distance scales ($t, L \gg a, (g^2 T)^{-1}$). The random process defined by the Langevin equation (37) generates three-dimensional field configurations with a probability distribution $P[A]$ determined by the Fokker-Planck equation

$$
\sigma \frac{\partial}{\partial t} P[A] = \int d^3x \frac{\delta}{\delta A} \left( T \frac{\delta P}{\delta A} + \frac{\delta W}{\delta A} P[A] \right),
$$

where $W[A]$ denotes the magnetic energy functional

$$
W[A] = \int d^3x \frac{1}{2} B(x)^2.
$$

Any non-static excitations of the magnetic sector of the gauge field, i.e. magnetic fields $B(k)$ not satisfying $k \times B = 0$, die away rapidly on a time scale of order $\sigma_c/k^2$, where $k$ denotes the wave vector of the field excitation. Long-term averages are determined by the static magnetic field sector weighted by the stationary solution of the FP equation (40):

$$
P_0[A] = e^{-W[A]/T}.
$$

The observer measures

$$
\langle O[A] \rangle = \int D A O[A] e^{-W[A]/T}.
$$

The magnetic field

$$
B_i = \frac{1}{2} \epsilon_{ijk} f^{jk} \equiv \sqrt{a} \epsilon_{ijk} f^{jk}
$$

defines a three dimensional field strength tensor $f^{jk}$, and $W/T$ can be identified with the three-dimensional action $S_3$ measured in units of Planck’s constant $\hbar = a T [5]$

$$
\frac{W}{T} = \frac{S_3}{\hbar} = -\frac{1}{4\hbar} \int dx_3 \int d^2x f^{ik} f_{ik}.
$$
The rescaling of the gauge field strength by the fundamental length scale $a$ is required for dimensional reasons. The same rescaling also determines the three-dimensional coupling constant to be

$$g_3^2 = \frac{g^2}{a} = \frac{g^2 T}{\hbar_3}.$$  (46)

The central result of this section is that the highly excited classical $(3 + 1)$-dimensional Yang-Mills theory reduces to a vacuum quantum Yang-Mills theory in three Euclidean dimensions for an observer who is only interested in physics at long distance and time scales, with vacuum expectation values of the standard form:

$$\langle O[A] \rangle_3 = \int D A O[A] e^{-S_3[A]/\hbar}.$$  (47)

Planck’s constant is determined by two microscopic quantities of the “fundamental” theory, $a$ and $T$:

$$\hbar = a T.$$  (48)

The existence of both, a fundamental length scale and an energy scale, is critical to the emergence of a constant with the dimensions of an action.

Some noteworthy comments:

- It is important to note that the dimensional reduction is not induced by a compactification of the time coordinate, either in real or imaginary time. We have not assumed a thermal ensemble of gauge fields in the original Minkowski space theory, and the random solution of the $(3 + 1)$-dimensional classical field theory does not satisfy periodic boundary conditions in imaginary time. The effective dimensional reduction is not caused by a discreteness of the excitations with respect to the time-like dimension, but by the dissipative nature of the $(3 + 1)$-dimensional dynamics. Magnetic field configurations satisfying $D \times B = 0$ can be thought of as low-dimensional attractors of the dissipative motion, and the chaotic dynamical fluctuations of the gauge field around the attractor can be consistently interpreted as quantum fluctuations of a vacuum gauge field in 3-dimensional Euclidean space.
- The dimensional reduction by chaotic fluctuations and dissipation does not occur in scalar field theories, because there is no dynamical sector that survives long-time averaging. Quasi-thermal fluctuations generate a dynamical mass for the scalar field and thus eliminate any arbitrarily slow field modes. An exception may be the case where the excitation energy of the scalar field is just right to put the quasi-thermal field at the critical temperature of a second-order phase transition, where arbitrarily slow modes exist as fluctuations of the order parameter. In the case of gauge fields, the transverse magnetic sector is protected by the gauge symmetry, and it is this sector which survives the time average without any need for fine-tuning of the microscopic theory.
- Generalizing the results of Sect. 2 and 4, we expect corrections to the vacuum quantum field theory in three dimensions to be of the order of the relative
fluctuations of the KS entropy within a spatial region of size $\Delta x$:

$$\frac{\Delta h_r}{\hbar_{KS}} \sim (g^2 T \Delta x)^{-2} \sim \left( \frac{a}{g^2 \hbar \Delta x} \right)^2.$$  

(49)

If, e.g., $a$ is of the order of the Planck length and $\Delta x$ is any physically accessible length scale, the corrections to the dynamics of the quantized Yang-Mills field will be exceedingly small, indeed.

6 Open problems

Our example for the chaotic quantization of a three-dimensional gauge theory in Euclidean space raises a number of questions:

1. Does the principle of chaotic quantization generalize to higher dimensions, in particular, to quantization in four dimensions?
2. Can the method be extended to describe field quantization in Minkowski space?
3. Can gravity be included in this framework? Does the nonlinearity of classical general relativity provide a source of random noise at short distances, allowing for an effective quantum theory to emerge on long distance and time scales?

The first question is most easily addressed. As long as globally hyperbolic classical field theories can be identified in higher dimensions, our proposed mechanism should apply. Although we do not know of any systematic study of discretized field theories in higher dimensions, a plausibility argument can be made that Yang-Mills fields exhibit chaos in $(4+1)$ dimensions. For this purpose, we consider the infrared limit of a spatially constant gauge potential [13,14]. For the $SU(N_c)$ gauge field in $(D+1)$ dimensions in the $A_0 = 0$ gauge, there are $3(N_c^2 - 1)$ interacting components of the vector potential and $3(N_c^2 - 1)$ canonically conjugate momenta, which depend only on the time coordinate. The remaining gauge transformations and Gauss' law allow to eliminate $2(N_c^2 - 1)$ degrees of freedom.

Next, rotational invariance in $D$ dimensions permits to reduce the number of dynamical degrees of freedom by twice the number of generators of the group $SO(D)$, i.e. by $D(D - 1)$. This leaves a $(D - 1)(2N_c^2 - 2 - D)$-dimensional phase space of the dynamical degrees of freedom and their conjugate momenta. For the dynamics to be chaotic, this number must be at least three. For the simplest gauge group SU(2), this condition permits infrared chaos in $2 \leq D \leq 5$ dimensions, including the interesting case $D = 4$. Higher gauge groups are needed to extend the chaotic quantization scheme to gauge fields in $D > 5$ dimensions. Of course, this reasoning does not establish full chaoticity of the Yang-Mills field in these higher dimensions, it only indicates the possibility. Numerical studies will be required to establish the presence of strong chaos in these classical field theories.

The second question is more difficult. A formal answer could be that the Minkowski-space quantum field theory can (and even must) be obtained by
analytic continuation from the Euclidean field theory. Any observable in the Minkowski space theory that can be expressed as a vacuum expectation value of field operators can be obtained in this manner. If this argument appears somewhat unphysical, one might consider a completely different approach, beginning with a chaotic classical field theory defined in (3+2) dimensions. Field theories defined in spaces with two time-like dimensions were first proposed by Dirac in the context of conformal field theory [27] and have recently been considered as generalizations of superstring theory [28]. In this case, one time dimension is effectively eliminated by gauge fixing.

In the absence of similar explicit constraints, field theories with two time-like dimensions, even if the second time direction is compact, exhibit unphysical properties, such as a lack of causality and unitarity [29]. E.g., the Coulomb potential of a point charge in the presence of a second, curled up time-like dimension with period $L$ is complex:

$$V(r) = \frac{\alpha}{r} \left( 1 + 2 \sum_{n=1}^{\infty} e^{-2\pi n r / L} \right).$$

(50)

The lack of causality is closely related to the problem of the existence of time-like closed loops, which confuse the distinction between past and future. These problems are avoided, if the second timelike dimension is “thermal”, i.e. if it is compact in the imaginary time direction. The factor $i$ then disappears from the exponent of (50), and the corrections to the Coulomb law are real and exponentially suppressed at large distances. In the context of the mechanism discussed in Sect. 5, the physical time dimension may be defined as the coordinate orthogonal to the total 5-momentum vector $P^\mu$ of the initial field configuration. Whether this reasoning applies to the case, where the “thermal” field theory is really an ergodic one, remains to be confirmed, but it is quite plausible.

In the presence of two time-like dimensions, the “energy” becomes a two-component vector $E$, which is a part of the $(D+2)$-dimensional energy-momentum vector. If we select an initial field configuration with energy $E n$, where $n$ is a two-dimensional unit vector, this choice defines a preferred time-like direction $t_n$, in which the field thermalizes. Conservation of the energy-momentum vector ensures that the total energy component orthogonal to $n$ always remains zero. The choice of an initial field configuration corresponds to a spontaneous breaking of the global $SO(D, 2)$ symmetry down to a global $SO(D, 1)$ symmetry. Whether this process leads to an effective quantum field theory in the $(D+1)$-dimensional Minkowski space, remains to be investigated.

Finally, what about gravity? One reason, why this question is difficult to answer, is that little is known about the properties of general relativity as a dynamical system. Due to its different gauge group structure, gravity has more “capacity to resist” chaos than the Yang-Mills fields. Local invariance against coordinate transformations is incompatible with the concept of Hamiltonian dynamics, which has a preferred time direction. The Hamiltonian version of general relativity [30] used in cosmology (mixmaster universe) reflects this exceptional situation in GR. Even the most basic definitions of deterministic chaos are not
directly applicable to general relativity and require appropriate generalizations. Many special, chaotic solutions of Einstein’s equations have been found \[31,32\]. Most important among these is the eternally oscillating chaotic behavior discovered in \[33\] for the generic solution of the vacuum Einstein equations in the vicinity of a spacelike singularity, which has the character of deterministic chaos. However, it is not known whether the generic solution exhibits chaos, as in the case of the Yang-Mills theory. It is not even clear what the proper framework for a systematic numerical study of this question would be. For example, the Lyapunov exponents, which were so effectively used in Yang-Mills theory, are coordinate dependent in GR due to the general covariance of Einstein’s equations against coordinate transformations.

What is clear, is that general relativity shares many of the properties, which allow nonabelian gauge theories to chaotically quantize themselves: Einstein’s equations are strongly nonlinear and have a large set of gauge invariances which could guarantee the survival of a dynamical sector at long distances in the presence of quasi-thermal noise. Under such conditions, GR may not even require a short-distance cutoff, because the thermal Schwarzschild radius \(2GT\) defines an effective limit to short-distance dynamics which can couple to the dynamics at large distance scales. Applying the relation (48) determining \(\hbar\) for the Yang-Mills field, one might conjecture that the analogous relation for gravity has the form \(\hbar \sim GT^2\). If the temperature parameter \(T\) were of the order of the Planck mass, this relation would yield the observed magnitude of \(\hbar\).

However, it is not clear whether the same mechanism – chaos, or exponential growth of sensitivity to initial conditions – which causes information loss in the dynamics of Yang-Mills fields, must also operate in general relativity. ’t Hooft has speculated that microscopic black hole formation may be the mechanism that causes the loss of information in the case of gravity \[1\]. Again, a much better understanding of the structure of generic solutions of Einstein’s equations must be a prerequisite to an exploration of these interesting questions.

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