UPPER BOUNDS FOR BETTI NUMBERS OF MULTIGRADED MODULES

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Abstract. This paper gives a sharp upper bound for the Betti numbers of a finitely generated multigraded \( R \)-module, where \( R = k[x_1, \ldots, x_m] \) is the polynomial ring over a field \( k \) in \( m \) variables. The bound is given in terms of the rank and the first two Betti numbers of the module. An example is given which achieves these bounds simultaneously in each homological degree. Using Alexander duality, a bound is established for the total multigraded Bass numbers of a finite multigraded module in terms of the first two total multigraded Bass numbers.

Introduction

Let \( R = k[x_1, \ldots, x_m] \) be the polynomial ring over a field \( k \) in \( m \) variables with the standard \( \mathbb{Z}^m \) grading and \( L \) a finite \( \mathbb{Z}^m \) (multigraded) \( R \)-module. Much work has been done on establishing lower bounds for Betti numbers of \( L \), initially motivated by the Buchsbaum-Eisenbud-Horrocks conjecture for finite modules over regular local rings. This conjecture was shown to hold for \( R/I \) when \( I \) is a monomial ideal by Evans and Griffith [3] and generally for all multigraded modules by Charalambous [1] and Santoni [6].

On the other hand, little is known about upper bounds for the Betti numbers. The main result of this paper gives sharp upper bounds for the total Betti number of \( L \) in each homological degree in terms of the rank and the first two Betti numbers of the module \( L \).

Main Theorem. For \( i \geq 2 \), we have

\[
\beta_i(L) \leq \left( \frac{\beta_1(L)}{\beta_0(L) - \text{rank } L + i - 1} \right) \left( \frac{\beta_0(L) - \text{rank } L + i - 3}{i - 2} \right).
\]

These bounds are precisely the ranks of the free modules in the multigraded Buchsbaum-Rim complex from [2] (called there the Buchsbaum-Rim-Taylor complex). However, there seems to be no appropriate map that achieves comparison between the minimal free resolution of \( L \) and the Buchsbaum-Rim complex. Instead, we obtain our bound using the combinatorial structure of the (not necessarily minimal) free resolution defined by Tchernev [7]. We show that our bounds are sharp by giving a class of examples that attains them simultaneously in each homological degree. To achieve this, we make the generators of \( L \) and their multidegrees sufficiently generic, so that the Buchsbaum-Rim-Taylor complex is the minimal free resolution. In the last section, as a corollary to our main theorem, we give bounds for the total multigraded Bass numbers of multigraded modules in terms of the first two total multigraded Bass numbers, by using the Alexander duality functors defined by Miller [5].
1. Counting T-flats

A matroid \( M \) is a finite set \( S \) coupled with the nonempty collection of independent subsets \( I \) of \( S \) satisfying the following two properties:

1. If \( Y \in I \) and \( X \subseteq Y \) then \( X \in I \).
2. If \( X, Y \in I \), and \( |Y| = |X| + 1 \) then there is \( y \in Y \setminus X \) such that \( X \cup \{y\} \in I \).

A maximal independent set is a basis. The collection of all bases will be denoted by \( B(M) \). We say that a subset \( A \subseteq S \) is dependent if \( A \notin I \). A minimal dependent set is called a circuit. The collection of all circuits of \( M \) will be denoted by \( \mathcal{T}_0(M) \).

Notice that a matroid can be defined by specifying the set \( B(M) \) or the set \( \mathcal{T}_0(M) \).

The rank in \( M \) of \( A \subseteq S \) is the number \( r^M_A = \max\{|X| \mid X \in I \text{ and } X \subseteq A\} \).

A subset \( F \subseteq S \) is called a flat of the matroid \( M \) if it has the property that for every \( x \in S \setminus F \) we have \( r^M_{F\cup\{x\}} = r^M_F + 1 \). The dual matroid \( M^* \) is defined on the same finite set \( S \) as \( M \) with \( B(M^*) = \{S \setminus B \mid B \in B(M)\} \).

In the dual matroid, the rank of a set \( A \subseteq S \) is the number \( r^M_A = |A| - r^M_S + r^M_{S \setminus A} \).

The subset \( A \) is a T-flat of \( M \) precisely when the complement \( S \setminus A \) is a proper flat of the dual matroid \( M^* \) ([7], Definition 2.1.1). The level of a subset \( A \subseteq S \) is defined to be the number

\[ l_A = |A| - r^M_A - 1. \]

Notice that the level of a circuit is 0, hence the notation \( \mathcal{T}_0(M) \) for the collection of circuits. We will similarly denote the collection of T-flats of level \( k \) by \( \mathcal{T}_k(M) \).

**Lemma 1.1.** Let \( M \) be a matroid on a finite set \( S \). If \( |S| = n \) and \( r^M_S = r \) then

\[ |\mathcal{T}_k(M)| \leq \binom{n}{r + k + 1}. \]

**Proof.** First, rewrite the rank of the dual matroid as follows

\[ r^M_A = |A| - r^M_S + r^M_{S \setminus A} = |S| - r^M_S - l^M_{S \setminus A} - 1. \]

In the above notation, we see for a T-flat \( A \) of level \( k \), the rank of its complementary flat in the dual matroid will be

\[ r^M_{S \setminus A} = n - r - k - 1. \]

So by definition

\[ |\mathcal{T}_k(M)| = |\mathcal{F}_{n-r-k-1}(M^*)| \]

where \( \mathcal{F}_\rho(M) \) denotes the collection of all flats with rank \( \rho \) in the matroid \( M \). Since a subset has the property that \( r^M_A \leq |A| \), we see that the number of flats with rank \( n - r - k - 1 \) must be less than the total number of subsets of \( S \) with cardinality \( n - r - k - 1 \). Thus

\[ |\mathcal{T}_k(M)| = |\mathcal{F}_{n-r-k-1}(M^*)| \leq \binom{n}{n - r - k - 1} = \binom{n}{r + k + 1} \]

which verifies our bound for the number of T-flats of level \( k \). \( \square \)
2. Betti Numbers of Multigraded Modules

Let
\[ E \xrightarrow{\Phi} G \xrightarrow{\psi} L \xrightarrow{\phi} 0 \]
be a minimal finite free multigraded presentation of \( L \). We consider the field \( k \) as an \( R \)-module under the quotient map where we send each variable to 1. In this way, tensoring the free presentation with \( k \) gives the map
\[ E \otimes_R k \xrightarrow{\Phi \otimes \text{id}} G \otimes_R k = W \]
Let \( S = S \otimes 1 \) be a basis of \( E \otimes_R k \). Then the set map \( \phi : S \xrightarrow{\Phi \otimes \text{id}} W \)
defines a matroid \( M \) on \( S \) where the independent sets are precisely those subsets \( A \) of \( S \) whose image under \( \phi \) spans a vector space \( V_A \) of dimension \( |A| \). From this matroid Tchernev ([7], Section 2.2) constructs for each \( T \)-flat \( I \)
the vector space \( \mathcal{T}_I(\phi) \). Then the resolution of \( L \) from ([7], Definition 4.3)
has the form (with \( \lambda = l_S + 2 = |S| - r_M + 1 \)),
\[ T_*(\Phi, S) = 0 \rightarrow T_{\lambda}(\Phi, S) \xrightarrow{\Phi \otimes \psi} T_{\lambda-1}(\Phi, S) \rightarrow \cdots \rightarrow T_1(\Phi, S) \xrightarrow{\Phi \otimes \psi} T_0(\Phi, S) \rightarrow 0, \]
and the free \( R \)-modules are defined as
\[ T_i(\Phi, S) = \bigoplus_{I \in \mathcal{T}_i-2(M)} R \otimes_k T_i(\phi), \quad 2 \leq i \leq l_S + 2 \]
and
\[ T_0(\Phi, S) = G \quad \text{and} \quad T_1(\Phi, S) = E. \]

**Proof of Main Theorem.** Let \( E \xrightarrow{\Phi} G \xrightarrow{\psi} L \xrightarrow{\phi} 0 \) be a minimal finite free multigraded presentation of \( L \) so that \( \beta_0(L) = \text{rank}(G) \) and \( \beta_1(L) = \text{rank}(E) \). When \( i \geq 2 \) then
\[ \beta_i(L) \leq \text{rank}(T_i(\Phi, S)). \]
By definition
\[ \text{rank}(T_i(\Phi, S)) = \text{rank} \left( \bigoplus_{I \in \mathcal{T}_i-2(M)} R \otimes T_i(\phi) \right) \leq |\mathcal{T}_{i-2}(M)| \max_{I \in \mathcal{T}_{i-2}(M)} (\dim_k T_i(\phi)). \]
It follows from the definition ([7], Definition 2.2.3) that for each \( I \in \mathcal{T}_i(M) \) one has
\[ \dim T_i(\phi) \leq \dim S_k(V) = \binom{r + k - 1}{k}, \]
where \( S_k(V) \) denotes the \( k \)-th symmetric power of \( V = \text{Im} \phi \) and \( r = \dim_k(V) \). Therefore, by Lemma 1.1, we get
\[ \beta_i(L) \leq \binom{|S|}{r_M + i - 1} \binom{r_M + i - 3}{i - 2}. \]
Since
\[ |S| = \text{rank}(E) = \beta_1(L) \]
and
\[ r_M = \text{rank}(\phi) = \text{rank}(\Phi) = \text{rank} G - \text{rank} L = \beta_0(L) - \text{rank} L, \]
the bounds on Betti numbers of multigraded modules have been established.
Next, we give an example of a finite multigraded module that achieves these bounds simultaneously in each homological degree.

**Example 2.1.** Let $L$ be a finite multigraded module with a minimal presentation matrix of uniform rank. In the above setup, this is equivalent to saying that if the rank of the matrix $\phi$ is $r$ then the image of every $r$-element subset of $S$ has dimension $r$. Then the free $R$-modules described in [7] (Definition 4.2) are precisely those of the Buchsbaum-Rim-Taylor complex defined in [2] (Definition 4.3). In addition, choose the multidegrees of the generators of $E$ to be generic. In that case no two $T$-flats have the same multidegree. Therefore, the Buchsbaum-Rim-Taylor complex is the minimal free multigraded resolution of $L$. Thus for $i \geq 2$,

$$\beta_i(L) = \binom{|S|}{r+i-1} \binom{r+i-3}{i-2};$$

our bounds are simultaneously achieved.

### 3. Multigraded Bass numbers of multigraded modules

We write $\mu_i(p, L)$ for the $i^{th}$ Bass number of an $R$-module $L$ at a prime $p$. Since we are over a polynomial ring and $L$ is multigraded, there are only finitely many multigraded primes $p$ generated by $\{x_{j_1}, \ldots, x_{j_l}\}$ for $\{j_1, \ldots, j_l\} \subset \{1, \ldots, m\}$. Thus it makes sense to call the well-defined integer

$$\ast \mu_i(L) = \sum_{p:\text{multigraded prime ideal}} \mu_i(p, L)$$

the total multigraded $i^{th}$ Bass number of $L$.

**Theorem 3.1.** Let $R = \mathbb{k}[x_1, \ldots, x_m]$ be a polynomial ring over a field $\mathbb{k}$ with the standard grading, $L$ a finite multigraded $R$-module, then the bounds for $i \geq 2$ of the total multigraded Bass numbers of $L$ are

$$\ast \mu_i(L) \leq \left( \ast \mu_1(L) \right) \left( \ast \mu_0(L) + i - 3 \right).$$

**Proof.** For any $c$, $\ast \mu_i(L(-c)) = \ast \mu_i(L)$. Since $L$ is finitely generated and has a finite multigraded injective resolution, there is a $c$ so that $L(-c)$ only has non-zero degrees greater than or equal to 1 and so that the nonzero Bass numbers at each multigraded prime occur in only positive degree. Thus it suffices to establish our bound when $L$ satisfies these two conditions.

Let $a$ be the componentwise maximum of the degrees of the $0^{th}$ and $1^{st}$ Betti numbers of $L$. Thus $L$ is a positively $a$-determined module as defined in ([5], Definition 2.1) and the minimal free resolution will be positively $a$-determined ([5], Proposition 2.5).

We write $L^a$ for the Alexander dual of an $R$-module $L$ with respect to degree $a$ as defined in [5]. We will write $B_aL$ of an $R$-module for the quotient of $L$ by the submodule $\bigoplus_{b \not\leq a} L_b$ as in [5]. In the case when $L$ is generated in degrees greater than 1, we have that $B_aL^a = L^a$, since $L^a$ is bounded inside the interval $[0, a-1]$. Thus we will use Miller’s results for $B_aL^a$ and $L^a$ interchangeably throughout the remainder of this paper.

Let

$$0 \longrightarrow L \longrightarrow I \xrightarrow{A} J$$
be a minimal multigraded copresentation of $L$. Miller ([5], Theorem 4.5) shows that the matrix $\Lambda$ is also a minimal presentation matrix for a free resolution of $L^a$ after some appropriate degree shifts. In establishing bounds for the Betti numbers in the previous section, we obtain the coefficient matrix of the vector space complex, which is independent of the degrees of the original free modules. Thus the coefficient matrix of $\Lambda$ defines a representable matroid for the multigraded module $L^a$. One easily sees that the bounds established in the Main Theorem depend only on the presentation matrix. Since rank $L^a = 0$ we have for $i \geq 2$

$$\beta_i(L^a) \leq \left( \frac{\beta_1(L^a)}{\beta_0(L^a) + i - 1} \right) \left( \frac{\beta_0(L^a) + i - 3}{i - 2} \right)$$

Further, Miller ([5], Theorem 5.3) establishes the relation

$$\beta_i(L^a) = \mu_i(L^a) - \text{supp}(b)$$

where $0 \leq b \leq a \cdot \text{supp}(b)$. By summing over all possible degrees and since the nonzero Bass numbers at each multigraded prime occur in positive degrees, we see that $\mu_0(L) = \beta_0(L^a) =$ number of rows of $\Lambda$, $\mu_1(L) = \beta_1(L^a) =$ number of columns of $\Lambda$, and

$$\mu_i(L) = \beta_i(L^a)$$

Therefore, the bounds for the total multigraded Bass numbers have been established.

The following corollary generalizes this bound to all Bass numbers.

**Corollary 3.2.** Let $R = k[x_1, \ldots, x_m]$ be a polynomial ring over a field $k$ with the standard $\mathbb{Z}^m$ grading, $L$ a finite multigraded $R$-module and $p$ a prime ideal of $R$. Let $d = \dim R_p/p^*R_p$, where $p^*$ denotes the largest multigraded prime ideal of $R$ contained in $p$. Then, for $i \geq 2 + d$, the bounds for Bass numbers of $L$ are

$$\mu_i(p, L) \leq \left( \frac{\mu_1(L)}{\mu_0(L) + i - d - 1} \right) \left( \frac{\mu_0(L) + i - d - 3}{i - d - 2} \right).$$

**Proof.** Goto and Watanabe showed in [4] that

$$\mu_i(p, L) = \mu_{i-d}(p^*, L).$$

Clearly, $\mu_{i-d}(p^*, L) \leq \mu_{i-d}(L)$. Thus Theorem 3.1 establishes our bounds. \qed

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