A Conjecture on Zero-sum 3-magic Labeling of 5-regular Graphs

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Abstract

In this paper, we obtained that every 5-regular graph admits a zero-sum 3-magic labeling, which give an affirmative answer to a conjecture proposed by Saieed Akbari, Farhad Rahmati and Sanaz Zare in Electron. J. Combin..

Key Words: zero-sum magic labeling; degree sequence; 1-factor
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1. Introduction

Graph considered here are all finite and undirected with vertex set \( V(G) \) and edge set \( E(G) \). A multigraph is a graph with multiple edges. If every vertex in a graph has the same degree \( r \) then this graph is referred to as a \( r \)-regular graph. A matching \( M \) in \( G \) is a set of independent edges, and \( |M| \) denotes the number of edges in \( M \). A factor of a graph \( G \) is a spanning subgraph of \( G \). A \( k \)-factor of \( G \) is a factor of \( G \) that is \( k \)-regular. Thus a 1-factor of \( G \) is a matching that saturates all vertices of \( G \), and is called a perfect matching of \( G \). A mapping \( l : E(G) \to A \), where \( A \) is an abelian group which written additively, is called a labeling of the graph \( G \). Given a labeling \( l \) of the graph \( G \), the symbol \( s(v) \), which represents the sum of the labels of edges incident with \( v \), is defined to be \( s(v) = \sum_{uv \in E(G)} l(uv) \), where \( v \in V(G) \). For every positive integer \( h \geq 2 \), a graph \( G \) is said to be zero-sum \( h \)-magic if there is an edge labeling from \( E(G) \) into \( \mathbb{Z}_h \setminus \{0\} \) such that \( s(v) = 0 \) for every vertex \( v \in V(G) \). The null set of a graph \( G \), denoted by \( N(G) \), is the set of all natural numbers \( h \in \mathbb{N} \) such that \( G \) admits a zero-sum \( h \)-magic labeling.

Recently, Saieed Akbari, Farhad Rahmati and Sanaz Zare obtained the following interesting results about magic labeling of regular graphs.

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Theorem 1.1 [1] Let $G$ be an $r$-regular graph ($r \geq 3$, $r \neq 5$). If $r$ is even, then $N(G) = N$, otherwise $N \setminus \{2, 4\} \subseteq N(G)$. Furthermore, if $r$ ($r \neq 5$) is odd and $G$ is a 2-edge connected $r$-regular graph, then $N(G) = N \setminus \{2\}$.

They also proposed the following conjecture in [1].

Conjecture Every 5-regular graph admits a zero-sum 3-magic labeling.

In this paper, we give an affirmative answer to this conjecture. The following lemma is essential in the proof of the conjecture.

Lemma 1.1 [2] Let $G$ be a graph of even order with degree sequence $d=(d_1, d_2, \ldots, d_n)$. If $\tilde{d}=(d_1-1, d_2-1, \ldots, d_n-1)$ is also a degree sequence of some graph, then $G$ has a 1-factor.

More information and related references concerning magic labeling of graphs can be seen in [1].

2. Main Results

In this section, we will give a proof of the Conjecture.

If a graph $G$ has vertices $v_1, v_2, \ldots, v_n$, the sequence $d=(d_1, d_2, \ldots, d_n)$ is called the degree sequence of $G$, where $d_i = d(v_i)$ for $i = 1, 2, \ldots, n$. A nonincreasing and nonnegative integer sequence $d=(d_1, d_2, \ldots, d_n)$ is graphical if there is a simple graph with degree sequence $d$. It is obvious that the conditions $d_i \leq n-1$ for all $i$, and $\sum_{i=1}^{n} d_i$ being even are necessary for a sequence to be graphical. Firstly, the following lemma will be obtained.

Lemma 2.1 Let $n$ be a positive even number, and $d=(d_1, d_2, \ldots, d_n)$ be a sequence of nonnegative integers. If $d_1 = d_2 = \ldots = d_n = 5$ and $n \geq 6$, or $d_1 = d_2 = \ldots = d_n = 4$ and $n \geq 6$, then $d$ is graphical.

Proof For convenience, we let $G_n$ denote the corresponding graph related to the sequence $d=(d_1, d_2, \ldots, d_n)$.

Firstly, we prove that if $d_1 = d_2 = \ldots = d_n = 5$ and $n \geq 6$ then $d$ is graphical. The proof is by induction on $n$. If $n = 6$, then it is a obvious result since the complete graph $K_6$ being the graph $G_6$ with degree sequence $(5, 5, 5, 5, 5, 5)$. When $n = 8$, the corresponding graph $G_8$ is obtained from $G_6$ through the following construction. Let $V(G_6) = \{v_1, v_2, \ldots, v_6\}$. Firstly, we add two new vertices $v_7$ and $v_8$ to $G_6$, and add an edge connecting $v_7$ and $v_8$. Secondly, we select, in $G_6$, two different matchings $M_1$ and $M_2$ with $M_1 \cap M_2 = \emptyset$ and $|M_1|=|M_2|=2$. Deleting the four edges in $M_1 \cup M_2$ from $G_6$, and connecting the four vertices in $M_1$ to $v_7$, the other four vertices in $M_2$ to $v_8$, we get the graph $G_8$ with
degree sequence \((5, 5, 5, 5, 5, 5, 5, 5)\). Now, suppose that \(n = 2(k + 1) \geq 10\). By induction hypothesis the \(2k\)-elements sequence \((5, 5, \ldots, 5)\) is graphical and the corresponding graph is \(G_{2k}\). So the graph \(G_{2(k+1)}\) can be obtained from \(G_{2k}\) through the same procedure as that of \(G_6\) to \(G_8\), and the proof is complete.

As for the case \(d_1 = d_2 = \ldots = d_n = 4\) and \(n \geq 6\), we also through the induction on \(n\). If \(n = 6\), then it is an easy work to find a 4-regular graph \(G_6\) with degree sequence \((4, 4, 4, 4, 4, 4)\). When \(n = 8\), the corresponding graph \(G_8\) is obtained from \(G_6\) through the following operation. Let \(V(G_6) = \{v_1, v_2, \ldots, v_6\}\). Firstly, we add two new vertices \(v_7\) and \(v_8\) to \(G_6\), and select, in \(G_6\), two different matchings \(M_1\) and \(M_2\) with \(M_1 \cap M_2 = \emptyset\) and \(|M_1| = |M_2| = 2\). Deleting the four edges in \(M_1 \cup M_2\), and connecting the four vertices in \(M_1\) to \(v_7\), the other four vertices in \(M_2\) to \(v_8\), we get the graph \(G_8\) with degree sequence \((4, 4, 4, 4, 4, 4, 4, 4)\). Now, suppose that \(n = 2(k + 1) \geq 10\). By induction hypothesis the \(2k\)-elements sequence \((4, 4, \ldots, 4)\) is graphical and the corresponding graph is \(G_{2k}\). So the graph \(G_{2(k+1)}\) can be obtained from \(G_{2k}\) through the same procedure as that of \(G_6\) to \(G_8\), and the proof is complete. \(\square\)

**Theorem 2.1** Every 5-regular graph admits a zero-sum 3-magic labeling.

**Proof** It is obvious that every 5-regular graph \(G\) is of even order since \(2 \cdot E(G) = 5 \cdot V(G)\). For \(|V(G)| < 6\), the correctness of the theorem is easily to verify. When \(|V(G)| \geq 6\), according to the Lemma 1.1 and Lemma 2.1 we can get that every 5-regular graph \(G\) contains a 1-factor. So, labeling the edges in the 1-factor with \(2 \in \mathbb{Z}_3 \setminus \{0\}\) and the remaining edges with \(1 \in \mathbb{Z}_3 \setminus \{0\}\), we will get a zero-sum 3-magic labeling of the 5-regular graph. \(\square\)

The following theorem can be easily deduced from the Theorem 1.1 and Theorem 2.1.

**Theorem 2.2** Let \(G\) be an \(r\)-regular graph with \(r \geq 3\). If \(r\) is even, then \(N(G) = \mathbb{N}\), otherwise \(N \setminus \{2, 4\} \subseteq N(G)\). Furthermore, if \(r\) is odd and \(G\) is a 2-edge connected \(r\)-regular graph, then \(N(G) = \mathbb{N} \setminus \{2\}\).

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