ON VIETORIS–RIPS COMPLEXES OF PLANAR CURVES

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Abstract. A Vietoris–Rips complex is a way to thicken a (possibly discrete) metric space into a larger space containing more topological information. We prove that if $C$ is a convex closed differentiable curve in the plane such that the convex hull of $C$ contains the evolute of $C$, then the homotopy type of a Vietoris–Rips complex of a subset of $n$ points from $C$ can be computed in time $O(n \log n)$. Furthermore, we show how to compute the $k$-dimensional persistent homology of these $n$ points in running time $O(n^2(k + \log n))$, which is nearly quadratic in the number of vertices $n$. This improves upon the traditional persistent homology algorithm, which is cubic in the number of simplices of dimension at most $k+1$, and hence of running time $O(n^{3(k+2)})$ in the number of vertices $n$. We ask if there are other geometric settings in which computing persistent homology is (say) quadratic or cubic in the number of vertices, instead of in the number of simplices.

1. Introduction

Given only a finite sample from a metric space, what properties of the space can one recover from the finite sample? Vietoris–Rips complexes are a commonly used tool in applied topology in order to recover the homotopy type, homology groups, or persistent homology of a space from a finite sample [9, 10, 13, 14, 16, 17, 18, 19, 23]. Given a metric space $X$ and a scale parameter $r \geq 0$, the Vietoris–Rips simplicial complex $VR(X; r)$ has a simplex every finite subset of $X$ of diameter at most $r$.

It can in general be expensive to compute the homotopy type or persistent homology of a Vietoris–Rips complex. Indeed, let $n = |X|$ be the number of points in a finite metric space $X$. Computing the $k$-dimensional persistent homology of $VR(X; r)$ as $r$ increases is cubic in the number of simplices of dimension at most $k+1$, and hence of running time $O((n^{k+2})) = O(n^{3(k+2)})$ in the number of vertices $n$. In this paper we show that if $X$ is sampled from a convex closed differentiable curve in the plane whose convex hull contains its evolute, then the $k$-dimensional homology of $VR(X; r)$ can be computed in running time $O(n^2(k + \log n))$, which is nearly quadratic in the number of vertices $n$. We hope this is a first step towards identifying more general geometric settings in which computing persistent homology is (say) quadratic or cubic in the number of vertices, instead of in the number of simplices.

Our main results follow. The statement of Theorem 1.1 relies on evolutes and cyclic graphs. The evolute of a curve is the envelope of the normals, or equivalently, the locus of the centers of curvature. A cyclic graph is a combinatorial abstraction of the 1-skeleton of a Vietoris–Rips complex built on a subset of the circle; a precise definition is given in Section 3.

Theorem 1.1. Given a strictly convex closed differentiable planar curve $C$ equipped with the Euclidean metric, the 1-skeleton of $VR(C; r)$ is a cyclic graph for all $r \geq 0$ if and only if the convex hull of $C$ contains the evolute of $C$.

Corollary 1.2. There is an $O(n \log n)$ algorithm for determining the homotopy type of $VR(X; r)$, where $X$ is a sample of $n$ points from a strictly convex closed differentiable planar curve whose convex hull contains its evolute, and where $r \geq 0$.

Theorem 1.3. There is an $O(n^2(k + \log n))$ algorithm for determining the $k$-dimensional persistent homology of $VR(X; r)$, where $X$ is a sample of $n$ points from a strictly convex closed differentiable planar curve whose convex hull contains its evolute.

We emphasize that even though $X$ is planar, the homotopy type of $VR(X; r)$ in Corollary 1.2 and Theorem 1.3 can be surprising (an odd sphere $S^{2k+1}$ for any $k \geq 0$, or a wedge of even spheres $V^m S^{2k}$ for any $m \geq 0$ and $k \geq 0$); see Figure 1.

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Key words and phrases. Persistent homology, Vietoris–Rips complex, convex curve, evolute, computational complexity.

We would like to thank Bei Wang for asking about the computational complexity of computing the persistent homology of points from the circle.

1For example, computing the 3-dimensional persistent homology of a Vietoris–Rips complex of $n$ points is cubic in the number of simplices, but of order $O(n^{15})$ in the number of vertices $n$. 

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Figure 1. A possible evolution of the homotopy types of VR\((X;r)\), for \(X\) a subset of a convex differentiable planar curve containing its evolute, as the scale \(r\) increases (horizontal axis). The homotopy type is either a single odd sphere \(S_{2k+1}\), or a wedge sum of even spheres \(\bigwedge^m S_{2k}\) for some \(m \geq 0\). The vertical axis gives a cartoon of how \(m\) might vary with \(r\): the number \(m\) of \(2k\)-spheres is a non-decreasing function of \(r\) for \(2k = 0\), but otherwise \(m\) need not be a monotonic function of \(r\) for \(2k \geq 2\).

When given a finite planar subset \(X \subseteq \mathbb{R}^2\), in order to understand the “shape” of \(X\) one would often compute its Čech or alpha complex [23] instead of computing its Vietoris–Rips complex. Indeed, by the nerve lemma the Čech and alpha complexes will have milder homotopy types. However, in higher-dimensional Euclidean space, it becomes prohibitively difficult to compute a Čech or alpha complex, and hence Vietoris–Rips complexes are frequently used. Our work motivates the following question: Are there other geometric contexts where computing the persistent homology of the Vietoris–Rips complex of a sample of \(n\) points can be similarly improved, from cubic in the number of simplices to a low-degree polynomial in \(n\)?

**Question 1.4.** For \(X \subseteq \mathbb{R}^2\) arbitrary, is there a cubic or near-quadratic algorithm in the number of vertices \(n = |X|\) for determining the \(k\)-dimensional persistent homology of VR\((X;r)\)?

**Question 1.5.** For \(X \subseteq \mathbb{R}^3\) arbitrary, what is the computational complexity of computing the \(k\)-dimensional persistent homology of VR\((X;r)\) in terms of the number of vertices \(n = |X|\)?

**Question 1.6.** For \(C\) a sufficiently nice curve in \(\mathbb{R}^d\), what is the computational complexity of computing the \(k\)-dimensional persistent homology of a subset of \(n\) points from \(C\)?

See the conclusion section for some initial ideas towards Question 1.6.

Another motivating question behind this work is the following. Given a planar subset \(X \subseteq \mathbb{R}^2\), is the Vietoris–Rips complex VR\((X;r)\) necessarily homotopy equivalent to a wedge of spheres for all \(r \geq 0\)? See [6, Problem 7.3] and [27, Section 2, Question 5]. Some evidence towards this conjecture is contained in [16] and [6]. Our results show that the conjecture is true in the limited case where \(X\) is a subset of a strictly convex closed differentiable curve whose convex hull contains its evolute.

2. Related Work

Vietoris–Rips complexes were invented independently by Vietoris for use in algebraic topology [43], and by Rips for use in geometric group theory [29]. Indeed, Rips proved that if a group \(G\) equipped with the word metric is \(\delta\)-hyperbolic, then VR\((G;r)\) is contractible for \(r \geq 4\delta\). An important theorem by Hausmman [31] states that if \(M\) is a Riemannian manifold, then VR\((M;r)\) is homotopy equivalent to \(M\) for scale parameters \(r\) sufficiently small (depending on the curvature of \(M\)). This theorem has been extended by Latschev [33] to state that if \(X\) is a (possibly finite) metric space that is sufficiently close to \(M\) in the Gromov–Hausdorff distance, then VR\((X;r)\) is still homotopy equivalent to \(M\).

Hausmann’s and Latschev’s theorems form the theoretical basis for more recent applications of Vietoris–Rips complexes in applied and computational topology [23, 13, 14]. There are by now a wide variety of reconstruction guarantees—one can use Vietoris–Rips complexes to recover a wide variety of topological properties, such as homotopy type, homology, or fundamental group, from a finite subset
drawn from some unknown underlying shape [9, 10, 13, 14, 16, 17, 18, 19, 23]. Several algorithms exist in order for approximating the persistent homology of a Vietoris–Rips complex filtration in a more computationally efficient manner [12, 20, 21, 37, 42].

This paper relies upon and builds upon cyclic graphs, and on the known homotopy types of the Vietoris–Rips complex of the circle [1, 2, 3, 4, 5, 7, 41]. Section 4 of our paper can be viewed as a generalization of [5] from ellipses to a much broader class of planar curves.

3. Preliminaries

We set notation for topological and metric spaces, simplicial complexes, persistent homology, cyclic graphs, convex curves, and evolves. See [8, 30, 32] for background on topological spaces, simplicial complexes, and homology, and [23] for background on Vietoris–Rips complexes and persistent homology.

**Topological spaces.** A topological space is a set \( X \) equipped with a collection of subsets of \( X \), called open sets, such that any union of open sets is open, any finite intersection of open sets is open, and both \( X \) and the empty set are open. For \( X \) a topological space and \( Y \subseteq X \) a subset, we denote the interior of \( Y \) by \( \text{int}(Y) \) and the boundary of \( Y \) by \( \partial Y \). Let \( I = [0,1] \) denote the unit interval. We let \( S^k \) denote the \( k \)-dimensional sphere and \( \vee^m S^k \) denote the \( m \)-fold wedge sum of \( S^k \) with itself. We write \( X \simeq Y \) to denote that spaces \( X \) and \( Y \) are homotopy equivalent, which roughly speaking means that “they have the same shape up to bending and stretching”.

**Metric spaces.** A metric space is a set \( X \) equipped with a distance function \( d: X \times X \to \mathbb{R} \) satisfying certain properties: nonnegativity, symmetry, the triangle inequality, and the identity of indiscernibles (\( d(x,x') = 0 \) if and only if \( x = x' \)). Given a point \( x \in X \) and a radius \( r > 0 \), we let \( B_X(x,r) = \{ y \in X \mid d(x,y) < r \} \) denote the open ball with center \( x \) and radius \( r \). Given a metric space \( X \), a point \( x \in X \), and a set \( Y \subseteq X \), we define \( d(x,Y) = \inf\{d(x,y) \mid y \in Y\} \).

**Simplicial complexes.** A simplex is a generalization of the notion of a vertex, edge, triangle, or tetrahedron to arbitrary dimensions. Formally, given \( k+1 \) points \( x_0, x_1, \ldots, x_k \) in general position, a simplex of dimension \( k \) (a \( k \)-simplex) is the smallest convex set containing them. A simplicial complex \( K \) on a vertex set \( X \) is a collection of subsets (simplices) of \( X \), including each element of \( X \) as a singleton, such that if \( \sigma \in K \) is a simplex and \( \tau \subseteq \sigma \) is a face of \( \sigma \), then also \( \tau \in K \). We do not distinguish between abstract simplicial complexes (which are combinatorial) and their geometric realizations (which are topological spaces).

**Vietoris–Rips complexes.** A Vietoris–Rips complex \( \text{VR}(X;r) \) is a simplicial complex, defined from a metric space \( X \) and distance \( r \geq 0 \), by including as a simplex every finite set of points in \( X \) that has a diameter at most \( r \) [31]. Said differently, the vertex set of \( \text{VR}(X;r) \) is \( X \), and \( \{x_0, x_1, \ldots, x_k\} \) is a simplex when \( d(x_i, x_j) \leq r \) for all \( 0 \leq i, j \leq k \).

**Homology and persistent homology.** Given a topological space \( Y \) and an integer \( k \geq 0 \), the homology group \( H_k(Y) \) measures the independent “\( k \)-dimensional holes” in \( Y \) (roughly speaking). For example, \( H_0(Y) \) measures the number of connected components, \( H_1(Y) \) measures the loops, and \( H_2(Y) \) measures the “2-dimensional voids” in \( Y \).

Given an increasing sequence of spaces \( Y_1 \subseteq Y_2 \subseteq \ldots \subseteq Y_M \), persistent homology is a way to “track the holes” as the spaces get larger. A common choice for applications is \( Y_t = \text{VR}(X;r_t) \), for \( X \) a metric
space and for \( r_1 < r_2 < \ldots < r_M \) an increasing sequence of scale parameters. We apply the homology functor (with coefficients in a field) in order to get a sequence of vector spaces \( H_k(Y_1) \to H_k(Y_2) \to \ldots \to H_k(Y_M) \), which decomposes into a collection of 1-dimensional interval summands. Each interval corresponds to a \( k \)-dimensional topological feature that is born and dies at the start and endpoints of the interval. Our algorithm for persistent homology in Section 5 works simultaneously for all choices of field coefficients. It also works for integer coefficients (which are much more subtle \cite{38}), or for persistent homotopy \cite{25, 34}, since all of the spaces that appear in our context are homotopy equivalent to wedges of spheres, with controllable maps in-between.

**Cyclic graphs and clique complexes.** A directed graph \( G = (X, E) \) consists of a set of vertices \( X \) and edges \( E \subseteq X \times X \), where no loops, multiple edges, or edges oriented in opposite directions are allowed. A cyclic graph \cite{2, 5} is a directed graph in which the vertex set is equipped with a counterclockwise cyclic order, such that whenever we have three cyclically ordered vertices \( x \prec y \prec z \prec x \) and a directed edge \( x \to z \), then we also have the directed edges \( x \to y \) and \( y \to z \). We say a cyclic graph \( G \) is finite if its vertex set is finite, and otherwise \( G \) is infinite. For example, in the proof of Theorem 1.1, when \( r > 0 \) we will show that the 1-skeleton of \( VR(C, r) \) is an infinite cyclic graph (recall \( C \) is a curve). We give more preliminaries on cyclic graphs in Section 5, for use in the proof of Theorem 1.3.

![Figure 3. Four example cyclic graphs. The homotopy types of their clique complexes, from left to right, are \( S^1 \), \( S^2 \), \( \sqrt{2} S^2 \), and \( S^3 \) (since, as will be explained in Section 5, the winding fractions are \( \frac{1}{4}, \frac{1}{3}, \frac{1}{3}, \) and \( \frac{2}{5} \), with the graphs having 1, 2, 3, and 1 periodic orbit(s)).](image)

The clique complex of a (directed or undirected) graph \( G \) (with no loops or multiple edges), denoted \( \text{Cl}(G) \), is the largest simplicial complex that contains \( G \) as its 1-skeleton. Note, for example, that a Vietoris–Rips complex is the clique complex of its 1-skeleton.

**Strictly convex curves.** A set \( Y \subseteq \mathbb{R}^2 \) is strictly convex if for all \( y, y' \in Y \) and \( t \in (0, 1) \), the point \( ty + (1-t)y' \) is in the interior of \( Y \). We say a curve \( C \subseteq \mathbb{R}^2 \) is strictly convex if \( C = \partial Y \) for some strictly convex set \( Y \subseteq \mathbb{R}^2 \). If \( L \) is a line and \( C \) is strictly convex curve in \( \mathbb{R}^2 \) that intersect transversely, then \( L \) and \( C \) intersect in either zero or two points; see for example \cite{11, 22, 28, 40}.

**Tangent vectors, normal vectors, and evolutes.** Let \( \alpha : I \to \mathbb{R}^2 \) be a differentiable curve in the plane. Then \( \alpha'(t) \) is the tangent vector to \( \alpha \) at time \( t \), and we denote the corresponding unit tangent vector by \( T(\alpha(t)) = \frac{\alpha'(t)}{||\alpha'(t)||} \). The unit normal vector, which is perpendicular to \( T(\alpha(t)) \) and points in the direction the curve is turning, is given by \( n(\alpha(t)) = \frac{T'(\alpha(t))}{||T'(\alpha(t))||} \).

The curvature of \( \alpha \) at time \( t \) is \( \kappa(t) = \frac{||T'(\alpha(t))||}{||\alpha'(t)||} \), with corresponding radius of curvature \( r(t) = \frac{1}{\kappa(t)} \). The center of curvature is the point on the inner normal line to \( \alpha(t) \) at distance equal to the radius of curvature away, given by \( x_c(t) = \alpha(t) + \frac{1}{\kappa(t)} n(\alpha(t)) \). The evolute of a curve is the envelope of the normals, or equivalently, the set of all centers of curvature.

**Critical Points, Extrema, and Monotonicity.** Let \( f : Y \to \mathbb{R} \) be a differentiable real-valued function. A point \( x \in \text{int}(Y) \) is a critical point of \( f \) if \( f'(x) = 0 \). The function \( f \) has an absolute maximum at a point \( x \) if \( f(x) \geq f(y) \) for all \( y \in Y \), and an absolute minimum at \( x \) if \( f(x) \leq f(y) \) for all \( y \in Y \). The function \( f \) has a local maximum at a point \( x \in Y \) if \( f(x) \geq f(y) \) for all \( y \) in some open set containing \( x \), and similarly for a local minimum. For \( Y \subseteq \mathbb{R} \), a function \( f : Y \to \mathbb{R} \) is said to be monotonic if we have \( f(x) \leq f(y) \) for all \( x, y \in Y \) with \( x \leq y \), or alternatively, \( f(x) \geq f(y) \) for all \( x \leq y \).
Throughout this section, let $C$ be a strictly convex closed differentiable planar curve equipped with the Euclidean metric. We provide an example before giving the proof of Theorem 1.1.

**Example 4.1.** Let $C = \{(a \cos(t), b \sin(t)) \mid t \in \mathbb{R}\} \subseteq \mathbb{R}^2$ be an ellipse, where we assume $a > b > 0$. The evolute of $C$ is given by the points $\left\{ \left(\frac{d^2 - b^2}{a^2} \cos^3 t, \frac{d^2 - a^2}{b^2} \sin^3 t \right) \right\} \subseteq \mathbb{R}^2$. One can show that the convex hull of $C$ contains its evolute if and only if $\frac{a}{b} \leq \sqrt{2}$. Indeed, for one direction, note that the point $(0, \frac{b^2 - a^2}{b^2})$ on the evolute (corresponding to $t = \frac{\pi}{2}$) is contained in the convex hull of $C$ if and only if $-b \leq \frac{b^2 - a^2}{b^2}$, meaning $a^2 \leq 2b^2$, or equivalently $\frac{a}{b} \leq \sqrt{2}$. Hence the convex hull of $C$ contains its evolute if and only if $C$ is an “ellipse of small eccentricity” [5], showing that our work generalizes [5] to a broader class of curves.

![Figure 4. Evolutes of ellipses](image)

Let $C$ be a strictly convex closed differentiable curve in the plane.

**Definition 4.2.** Define the continuous function $h: C \to C$ which maps a point $p \in C$ to the unique point in the intersection of the normal line to $C$ at $p$ with $C \setminus \{p\}$.

**Lemma 4.3.** The map $h: C \to C$ is of degree one, which implies that $h$ is surjective.

**Proof.** Suppose that $C$ is parametrized by $\alpha: S^1 \to C$. Write $h(\alpha(e^{2\pi it})) = \alpha(e^{2\pi it}+f(e^{2\pi it}))$ where $f: S^1 \to [0, 2\pi)$ is a continuous function, which is possible since $h \circ \alpha$ and $\alpha$ never intersect. Then we can do a “straight-line” homotopy from $f$ down to the zero function to show that $h \circ \alpha$ and $\alpha$ are homotopy equivalent. Indeed, consider the homotopy $H: S^1 \times I \to C$ defined by $H(e^{2\pi it}, s) = \alpha(e^{2\pi it}+sf(e^{2\pi it}))$, in which $H(\cdot, 0) = \alpha$ and $H(\cdot, 1) = h \circ \alpha$. Since these maps are homotopy equivalent, it follows\(^2\) that the winding number of $h \circ \alpha$ is equal to the winding number of $\alpha$, which is 1. Hence $h$ is surjective.

For $p \in C$, define the function $d_p: C \to \mathbb{R}$ by $d_p(q) = d(p, q)$, where $d(p, q)$ is the Euclidean distance between $p, q$.

**Lemma 4.4.** Let $p \in C$. Then a point $q \in C$ is a critical point of the function $d_p: C \to \mathbb{R}$ if and only if $q = p$ or $h(q) = p$.

**Proof.** It is clear that $p$ is a global minimum of $d_p$, and therefore we may restrict attention to $q \neq p$. Consider an arbitrary point $p = (p_1, p_2) \in C \subseteq \mathbb{R}^2$. Let $\alpha: I \to C$ be a parametrized curve in $C$. Note that $d_p(\alpha(t)) = \sqrt{(p_1 - \alpha_1(t))^2 + (p_2 - \alpha_2(t))^2}$, for all $t \in I$. This gives us

$$
\frac{d}{dt} d_p(\alpha(t)) = \frac{-\alpha'_1(t)(p_1 - \alpha_1(t)) - \alpha'_2(t)(p_2 - \alpha_2(t))}{\sqrt{(p_1 - \alpha_1(t))^2 + (p_2 - \alpha_2(t))^2}} = \frac{-\alpha'(t) \cdot (p - \alpha(t))}{\|p - \alpha(t)\|}.
$$

\(^2\) A more general statement, which follows from the same proof, is that if two maps $\alpha, \tilde{\alpha}: S^1 \to S^1$ satisfy $\alpha(p) \neq \tilde{\alpha}(p)$ for all $p \in S^1$, then the winding numbers of $\alpha$ and $\tilde{\alpha}$ are equal.
Therefore, the only critical points of \(d_p\) away from \(p\) are when \(\frac{d}{dt}d_p(\alpha(t)) = 0\), i.e. \(\alpha'(t) \cdot (p - \alpha(t)) = 0\). These are precisely the points where the tangent line to \(\alpha(t)\) is perpendicular to the line between \(\alpha(t)\) and \(p\). Therefore, the critical points of \(d_p\) occur at \(p\) and all \(q \in C\) such that \(h(q) = p\). \(\square\)

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \(C\) be a strictly convex closed differentiable planar curve equipped with the Euclidean metric. We must show that the 1-skeleton of \(\text{VR}(C; \epsilon)\) is a cyclic graph for all \(\epsilon \geq 0\) if and only if the convex hull of \(C\) contains the evolute of \(C\). We will show that the following are equivalent.

(i) The evolute of \(C\) is contained in the convex hull \(\text{conv}(C)\).
(ii) Function \(h: C \to C\) is injective.
(iii) For all points \(p \in C\), the distance function \(d_p\) has exactly two critical points.
(iv) For all points \(p \in C\), the distance function \(d_p\) is monotonic along the two arcs in \(C\) from \(p\) to the global maximum of \(d_p\).
(v) The intersections \(B_{\epsilon^2}(p, r)\cap C\) are connected for all \(p \in C\) and \(r > 0\).
(vi) The 1-skeleton of \(\text{VR}(C; \epsilon)\) is a cyclic graph for all \(\epsilon \geq 0\).

(i) \(\Leftrightarrow\) (ii). The intuition (before the proof) is as follows. For an example, see Figure 4. If \(\alpha: I \to C\) is a curve moving in the counterclockwise direction at \(t \in I\) if and only if the center of curvature \(x_{\alpha}(t)\) is in \(\text{conv}(C)\). For the proof, we first note that a continuous map \(h: C \to C\) of degree one is injective if and only if for any continuous function \(\alpha: I \to C\), the orientations of \(\alpha(t)\) and \(h(\alpha(t))\) match for all times \(t\). The result then follows from Lemma C.4, which says that \(\alpha(t)\) and \(h(\alpha(t))\) have matching orientations for all \(t\) if and only if the evolute of \(C\) is contained in the convex hull \(\text{conv}(C)\).

(ii) \(\Leftrightarrow\) (iii). Note that \(h\) is injective if and only if for each \(p \in C\), there is a unique point \(q \in C\) with \(h(q) = p\), which by Lemma 4.4 is equivalent to \(d_p\) having exactly two critical points (a global minimum \(p\), and the point \(q\) satisfying \(h(q) = p\) as a global maximum).

(iii) \(\Leftrightarrow\) (iv). For (iii) \(\Rightarrow\) (iv), note that for all \(p \in C\), compactness implies that \(d_p\) has at least two extrema (a global minimum at \(p\), and a global maximum). If each \(d_p\) has exactly two critical points, then each \(d_p\) must have exactly two extrema (as every extremum is a critical point). Therefore (iv) is satisfied.

We remark that the proof of (iv) \(\Rightarrow\) (iii) is subtle (see Figure 5), but we proceed regardless. Suppose that \(d_p\) has more than two critical points for some \(p \in C\); we must find some point \(\tilde{p} \in C\) such that \(d_{\tilde{p}}\) does not satisfy the monotonicity property in (iv). The assumption on \(p\) means that there exists some critical point \(q\) of \(d_p\) that is neither the global minimum nor maximum of \(d_p\). Since \(q\) is a critical point we can find a curve \(\alpha: (-\delta, \delta) \to C\) with \(\alpha(0) = q\) and \(\frac{d}{dt}d_p(\alpha(t)) = 0\) for \(t = 0\).

![Figure 5](image.png)

**Figure 5.** For a single point \(p\), it may be that \(d_p\) has more critical points than extrema. Indeed, see the half-circle-half-ellipse example for \(C\) above, with the specific point \(p\) as pictured. We have three critical points and only two extrema of \(d_p\) (note that \(q\) is a critical point of \(d_p\) that is not an extremum). Nevertheless, (iv) \(\Rightarrow\) (iii) still holds since for other points \(\tilde{p} \in C\), we have more than two extrema of \(d_{\tilde{p}}\).

We claim that there is a point \(\tilde{p} \in C\) arbitrarily close to \(p\) such that

(a) \(\frac{d}{dt}d_{\tilde{p}}(\alpha(t)) < 0\) for \(t = 0\), and
Proof of Corollary 1.2. We provide an algorithm for determining the cyclic ordering of the points in \( C \) to determine the cyclic ordering of the points in \( (C; t) \) from their coordinates in \( \mathbb{R}^2 \); this algorithm does not require knowledge of \( C \). The result then follows since given a cyclic graph \( G \) with \( n \) vertices, there exists an algorithm for determining the monotonicity property in (iv), and hence we have completed the proof of (iii) \( \Leftrightarrow \) (iv).

(iv) \( \Leftrightarrow \) (v). Given a fixed point \( p \in C \), note that \( B(p, r) \cap C \) is connected for all \( r > 0 \) if and only if \( d_p \) satisfies the monotonicity property in (iv).

(v) \( \Leftrightarrow \) (vi) The final equivalence follows from [7, Definition 4.10 and Lemma 4.11].

We can now also prove Corollary 1.2.

Proof of Corollary 1.2. We provide an \( O(n \log n) \) algorithm for determining the homotopy type of \( VR(X; r) \), where \( X \) is a sample of \( n \) points from a strictly convex differentiable planar curve \( C \) whose convex hull contains its evolute, and where \( r \geq 0 \). By Theorem 1.1 we know that the 1-skeleton of \( VR(C; r) \) is a cyclic graph, and it follows that the same is true of \( VR(X; r) \) for any \( X \subseteq C \). We use the \( O(n \log n) \) algorithm in Section A to determine the cyclic ordering of the points in \( X \) (along \( C \)) from their coordinates in \( \mathbb{R}^2 \); this algorithm does not require knowledge of \( C \). The result then follows since given a cyclic graph \( G \) with \( n \) vertices, there exists an \( O(n \log n) \) algorithm for determining the homotopy type of the clique complex of \( G \). Indeed, this is contained in [3, Theorem 5.7 and Corollary 5.9], in which the algorithm is stated for a Vietoris–Rips complex of points on the circle, though it holds more generally for the clique complex of any finite cyclic graph. The algorithm proceeds by removing dominated vertices, without changing the homotopy type of the clique complex, until arising at a minimal regular configuration (in which each vertex has the same number of outgoing neighbors, such as the two cyclic graphs in the middle of Figure 3). One can read off the homotopy type from this regular configuration.

Remark 4.5. The algorithm in Corollary 1.2 is furthermore of linear running time \( O(n) \) if the vertices are provided in cyclic order.

5. PROOF OF THEOREM 1.3 ON AN ALGORITHM FOR PERSISTENT HOMOLOGY

Before proving Theorem 1.3, we provide some further background on cyclic graphs and their associated dynamical systems. We then give the algorithm for computing even-dimensional persistent homology, followed by the algorithm for odd homological dimensions. Our algorithms are more general than the statement of Theorem 1.3: given any increasing sequence of cyclic graphs \( G_1 \subseteq G_2 \subseteq \ldots \subseteq G_M \) on the same vertex set \( X \), we show how to compute the persistent homology of the resulting increasing sequence of clique complexes \( Cl(G_1) \subseteq Cl(G_2) \subseteq \ldots \subseteq Cl(G_M) \). Theorem 1.3 then follows since, as we explain in Section A, given a sample \( X \) of \( n \) points from some strictly convex closed differentiable planar curve \( C \) whose convex hull contains its evolute, it is easy to determine the cyclic graph structure on the 1-skeleton of \( VR(X; r) \) even without knowledge of \( C \).

Cyclic dynamical systems and winding fractions. Let \( G \) be a finite cyclic graph with vertex set \( X \). The associated cyclic dynamical system is generated by the dynamics \( f : X \to X \), where we assign \( f(x) \) to be the vertex \( y \) with a directed edge \( x \to y \) that is counterclockwise farthest from \( x \) (else \( f(x) = x \) if \( x \) is not the source vertex of any directed edges in \( G \)). Since \( X \) is finite, the dynamical system \( f : X \to X \) necessarily has at least one periodic orbit. By [5, Lemma 2.3] or [7, Lemma 3.4], every periodic orbit of \( f \) has the same length \( \ell \) and winding number \( \omega \) (the number of times a periodic orbit \( x \to f(x) \to f^2(x) \to \ldots \to f^\ell(x) = x \) wraps around the cyclic ordering on \( X \)). We define the winding fraction of \( G \) to be \( \text{wf}(G) = \frac{\omega}{\ell} \).

Let \( P \) be the number of periodic orbits in a finite cyclic graph \( G \). By [5, Proposition 4.1], we have

\[
\text{Cl}(G) \simeq \begin{cases} S^{2k+1} & \text{if } \frac{k+1}{2k+1} < \text{wf}(G) < \frac{k+1}{2k+3} \\
\sqrt{p-1} S^{2k} & \text{if } \text{wf}(G) = \frac{k}{2k+1} \end{cases}
\]

for some \( k \in \mathbb{N} \).
we update the algorithm. Since the 2
new step
x
a prior smaller step from
P
edge. Then,

we started, after looping

P
periodic orbit, we update

f
B
,B
2
k
X
E
k
X

n
of clique complexes. We let
k
directed edges, and

k
the number of periodic orbits of winding fraction

winding fraction
k

B
, and increment
P
acters that are periodic with winding fraction
2
k
, initialized to be 0 (again unless
k = 0). Let
f
be an array of
n
integers, where
f[i] = j when the cyclic dynamical system maps the
i-th vertex to the
j-th vertex. Initialize
f
so that
f[i] = i for all
i. For each new edge, in order of appearance, we must update whether some vertices are periodic with winding fraction
k
2
k+1
If the source vertex of the new edge is periodic (look it up in
P
), walk along
f
, marking every vertex along this (old) periodic orbit as non-periodic in
B. Decrement
P. Now, update
f
to add the new edge. Walk
2k + 1 steps along
f
starting from the source vertex of the new edge. If we get back to where we started, after looping
k
times around, then re-walk along this newly found periodic orbit, marking all the vertices as periodic in
B, and increment
P. We’re now done updating
f,
B,
and
P
for the new edge. Then,
P − 1 is the number of
2k
-dimensional spheres in the homotopy type of
Cl(G). Thus, we know what the homology groups are. See Section
B
for an explanation of how to recover not only the homology at each stage, but also the persistent homology (i.e., the maps between homology groups induced by inclusions).

We now explain why this algorithm works. The algorithm has a loop invariant that
f,
B,
and
P
are correct. When we add a new larger step
x \mapsto f(x) to the cyclic dynamics
f, we are implicitly removing a prior smaller step from
x. This prior step was either part of a periodic orbit or not. If it was on a periodic orbit, we update
f,
B,
and
P
to account for destroying this periodic orbit. When we add the new step
x \mapsto f(x), it either creates a new periodic orbit or not. We check to see if it does, and, if so, we update
f,
B,
and
P
accordingly. Thus, the loop invariant is maintained throughout the execution of the algorithm. Since the
2k
-dimensional homology of the clique complex of
G is determined purely by the number of periodic orbits of winding fraction
k
2
k+1
, this algorithm produces the correct homology at each stage.

Pseudocode for even-dimensional homology. The following pseudocode for the even-dimensional persistent homology algorithm described above accepts as inputs the vertex set
X
, the sorted list
E
of directed edges, and
k
. It computes the
2k
-dimensional persistent homology of the increasing sequence of clique complexes. We let
n = |X|.

function computeEvenDimensionalPersistentHomology(X, E, k):
    set numPeriodicOrbits to 0 (unless
k = 0)
    set isPeriodic to an array of length
n
, filled with 0’s (unless
k = 0)
    set
f
to an array of length
n
, where the
i
-th entry is
i

set edges to a sorted list of all edges between points in
X
for edge in edges:
if isPeriodic[edge.sourceVertex]:
    walk along the periodic orbit, marking each vertex nonperiodic
    numPeriodicOrbits -= 1
edit f to add the new edge
walk around f, starting at edge.sourceVertex, taking 2k+1 steps
if we returned to edge.sourceVertex, and looped around k times:
    walk along the new periodic orbit, marking each vertex periodic
    numPeriodicOrbits += 1
print the number (numPeriodicOrbits-1) of 2k-spheres

See Section B for an explanation of how to recover not only the homology at each stage, but also the persistent homology.

Odd-dimensional homology. If we’re interested in 2k + 1 dimensional homology, run the algorithm for even-dimensional homology in dimensions 2k and 2k + 2. There is only ever at most one 2k + 1-dimensional sphere. This sphere is born when the last periodic orbit with winding fraction \( \frac{k}{2k+1} \) is destroyed (at which point the winding fraction first exceeds \( \frac{k}{2k+1} + 1 \), and this sphere dies when the first periodic orbit with winding fraction \( \frac{k+1}{2k+3} \) is created.

Computational complexity. Computing the list of edges in sorted order takes \( O(n^2 \log(n^2)) = O(n^2 \log n) \) time. Walking along the length of a periodic orbit takes \( O(k) \) time. We walk along the length of a periodic orbit a constant number of times for each edge. Since there are \( O(n^2) \) edges, this takes \( O(n^2 k) \) time. Thus, the total runtime is \( O(n^2 (k + \log n)) \).

6. Conclusion

We have shown that the convex hull of a strictly convex closed differentiable planar curve \( C \) contains the evolute of \( C \) if and only if the 1-skeleton of \( \text{VR}(C; r) \) is a cyclic graph for all \( r \geq 0 \). As a consequence, if \( X \) is any finite set of \( n \) points from \( C \), then the homotopy type of \( \text{VR}(X; r) \) can be computed in time \( O(n \log n) \). Furthermore, we give an \( O(n^2 (k + \log n)) \) time algorithm for computing the \( k \)-dimensional persistent homology of \( \text{VR}(X; r) \) (or more generally, the persistent homology of an increasing sequence of cyclic graphs). This is significantly faster than the traditional persistent homology algorithm, which is cubic in the number of simplices of dimension at most \( k + 1 \), and hence of running time \( O(n^3 (k+2)) \) in the number of vertices \( n \).

Important questions motivated by our work are as follows. For \( X \subseteq \mathbb{R}^2 \), is there a cubic or near-quadratic algorithm in the number of vertices \( n = |X| \) for determining the \( k \)-dimensional persistent homology of \( \text{VR}(X; r) \)? This would be more likely if the conjecture that the Vietoris–Rips complex \( \text{VR}(X; r) \) of any planar subset \( X \subseteq \mathbb{R}^2 \) is homotopy equivalent to a wedge sum of spheres were true (see [6, Problem 7.3] and [27, Section 2, Question 5]). As for one dimension higher, what is the computational complexity of computing the \( k \)-dimensional persistent homology of \( \text{VR}(X; r) \) in terms of the number of vertices \( n = |X| \) when \( X \subseteq \mathbb{R}^3 \)?

We end with some initial thoughts on Question 1.6. Let \( C \) be a closed curve in \( \mathbb{R}^d \). If the intersections \( \mathcal{B}_{2^d}(p, r) \cap C \) are connected for all \( p \in C \) and \( r > 0 \), then the 1-skeleton of \( \text{VR}(C; r) \) will be cyclic for all \( r \). The ideas in Section 5 would then provide fast algorithms for computing the homotopy type and persistent homology of \( \text{VR}(X; r) \) whenever \( X \) is a finite subset of \( C \). Furthermore, even if the 1-skeleton of \( \text{VR}(C; r) \) is not cyclic for all \( r \geq 0 \), it will be cyclic for sufficiently small \( r \leq r_C \), where \( r_C \) is some constant depending on \( C \). This will allow the persistent homology algorithm in Section 5 to be applied in the regime when the scale \( r \leq r_C \) is small.

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Appendix A. From Euclidean points to a cyclic graph

Given a sample $X$ of $n$ points from a strictly convex differentiable planar curve $C$ whose convex hull contains its evolute, it is easy to determine the cyclic graph structure on the 1-skeleton of $\text{VR}(X; r)$ even without knowledge of $C$.

Determining the cyclic order of the points. Given the $n$ points $X$ on $C$, we can place them in cyclically sorted order as follows. Let $\overline{x} \in \mathbb{R}^2$ be the average of every point in $X$ (see Figure 7(left)); since $C$ is strictly convex it follows that $\overline{x} \in \text{int}(\text{conv}(C))$. Translate $X$ so that $\overline{x}$ is at the origin in $\mathbb{R}^2$, and then compute the polar coordinates of all of the (translated) points in $X$. The angles from these polar coordinates allow us to cyclically order the points of $X$ in time $O(n \log n)$.

![Figure 7](image1)

Figure 7. (Left) The average of the points in $X$, namely the red point $\overline{x}$, is contained inside the curve $C$. Hence we can use the black vectors to determine the cyclic ordering. (Right) We identify that the new edge, the dashed edge, must be oriented from bottom-right to top-left.

Determining the direction of each edge. Given the $n$ points $X$ on $C$, suppose we are adding the next shortest undirected edge $\{x, y\}$ and need to determine whether this edge is oriented $x \rightarrow y$ or $y \rightarrow x$.

First, we add another array to the algorithm in Section 5, to keep track of the degree of each vertex. Indeed, if any vertex $v$ has full degree $n - 1$, then the Vietoris-Rips complex is contractible (it is a cone with $v$ as its apex), and so we’re trivially done. So, consider the case where no vertex has degree $n - 1$.

Going counterclockwise from $x$, all the other vertices in $X$ fall into three categories: first the vertices $v$ with an edge $x \rightarrow v$, then the vertices that are not connected to $x$, then the vertices $v$ with an edge $v \rightarrow x$, before finally getting back to $x$ (see Figure 7(right)). To determine the direction on the new edge $\{x, y\}$, note that since the graph is cyclic, $y$ will be adjacent (in the cyclic order) to either a vertex $v$ from the first category ($x \rightarrow v$) or to a vertex $v$ from the third category ($v \rightarrow x$). If $y$ is adjacent exclusively to a vertex $v$ with $x \rightarrow v$, then the direction on our new edge is $x \rightarrow y$. If $y$ is adjacent exclusively to a vertex $v$ with $v \rightarrow x$, then the direction on our new edge is $y \rightarrow x$. Finally, if $y$ is adjacent to a vertex of each type, then after adding the new edge $\{x, y\}$ necessarily $x$ has degree $n - 1$, meaning that the Vietoris-Rips complex is contractible.

Appendix B. Determining the persistent homology maps

In Section 5 we provide an algorithm to compute the homology groups (and indeed the homotopy types) of an increasing sequence of clique complexes of cyclic graphs $\text{Cl}(G_1) \subseteq \text{Cl}(G_2) \subseteq \ldots \subseteq \text{Cl}(G_M)$. From these computations, it is not difficult to determine the persistent homology information, i.e., the maps on homology induced by inclusions. Indeed, if $\text{Cl}(G_i)$ and $\text{Cl}(G_{i+1})$ are not (wedge sums of) spheres of the same dimension, then necessarily the induced map on homology is trivial in all homological dimensions. Furthermore, if $\text{Cl}(G_i)$ and $\text{Cl}(G_{i+1})$ are each homotopy equivalent to $S^{2k+1}$, then it follows from [2, Proposition 4.9] that the induced map $\text{Cl}(G_i) \hookrightarrow \text{Cl}(G_{i+1})$ is a homotopy equivalence.

For the case of even-dimensional homology, we rely on [5, Proposition 4.2]. Each interval in the $2k$-dimensional persistent homology barcode will be labeled with a periodic orbit of winding fraction $\frac{1}{2k+1}$. Upon adding a new directed edge with source vertex $x$, there are four cases in the algorithm for even-dimensional homology in Section 5.
Lemma C.1. Let \( C \) be a convex curve in \( \mathbb{R}^2 \), and let \( \alpha : I \to C \subseteq \mathbb{R}^2 \) with \( \alpha \) injective is said to be positively (resp. negatively) oriented if it is moving in the counterclockwise (resp. clockwise) direction around \( C \). It follows from Lemma C.1 that \( (\alpha'(t), n(\alpha(t))) \) is a positive (resp. negative) basis for \( \mathbb{R}^2 \), for all \( t \in I \). The curve \( \beta : I \to C \) is said to have a matching orientation to \( \alpha \) when the basis \( (\beta'(t), n(\beta(t))) \) has the same sign as \( (\alpha'(t), n(\alpha(t))) \) [22].

Recall from Section 3 that for \( \alpha : I \to \mathbb{R}^2 \) a differentiable curve in the plane, we define the inner normal vector \( n(t) \), the curvature \( \kappa(t) \), and the center of curvature \( c(t) = \alpha(t) + \frac{1}{\kappa(t)} n(t) \).

Lemma C.1. Let \( C \) be a convex curve in \( \mathbb{R}^2 \), and let \( \alpha : I \to C \) be a differentiable curve moving in the counterclockwise (resp. clockwise) direction around \( C \). Then \( (\alpha'(t), n(\alpha(t))) \) is a positive (resp. negative) basis for \( \mathbb{R}^2 \).

Proof. Definition 2.1.2 and Theorem 2.4.2 of [39] show that the signed curvature \( n(\alpha(t)) \cdot \alpha'(t) \) never changes for \( C \) convex, which implies that the orientation on the basis \( (\alpha'(t), n(\alpha(t))) \) never changes signs.

Lemma C.2. Let \( C \) be a differentiable convex curve in \( \mathbb{R}^2 \), and let \( \alpha : I \to C \) be differentiable. The line through \( \alpha(t) \) and \( x_\alpha(t) \) intersects \( C \) at a unique other point, which we denote by \( \beta(t) = h(\alpha(t)) \). If we define \( s : I \to \mathbb{R} \) to satisfy \( \beta(t) = \alpha(t) + s(t)n(\alpha(t)) \), then the function \( s \) is differentiable.

Proof. We employ the implicit function theorem. Define \( d^\pm : \mathbb{R}^2 \to \mathbb{R} \) to be the signed distance to the curve \( C \), namely

\[
d^\pm(y) = \begin{cases} d(x, C) & \text{if } x \in \text{conv}(C) \\ -d(x, C) & \text{otherwise}. \end{cases}
\]

Since \( C \) is a smooth and complete manifold, it follows from Section 3 of [35] that \( d^\pm \) is differentiable on an open neighborhood of \( C \), with derivative

\[
\nabla d^\pm(x) = \begin{cases} \frac{x-y}{||x-y||} & \text{if } x \in \text{int}(\text{conv}(C)) \\ \frac{y-x}{||x-y||} & \text{if } x \notin \text{conv}(C) \\ n(x) & \text{if } x \in C, \end{cases}
\]

where \( y \in C \) is the unique closest point on \( C \) to \( x \). Related references include [15, 24, 26, 36, 44]. Furthermore, define \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) via \( g(t, s) = \alpha(t) + sn(\alpha(t)) \), and define \( f : \mathbb{R}^2 \to \mathbb{R} \) by \( f = d^\pm \circ g \). Note that \( g \) is differentiable since \( C \) is, and hence \( f \) is differentiable as the composition of \( d^\pm \) with \( g \).

Pick \( t_0, s_0 \) such that \( f((t_0, s_0)) = 0 \); hence \( g(s_0, t_0) \in C \). In order to apply the implicit function theorem we need to show that the Jacobian of \( f \) with respect to \( s \) is invertible at \( (t_0, s_0) \); this is equivalent to showing that \( \frac{\partial f}{\partial s}(t_0, s_0) \neq 0 \). Using the chain rule for \( f = d^\pm \circ g \), we compute

\[
\frac{\partial f}{\partial s}(t_0, s_0) = \nabla d^\pm(\alpha(g(t_0, s_0)))^T : \frac{\partial g}{\partial s}(s_0, t_0) = n(g(t_0, s_0))^T : n(\alpha(t_0)).
\]

In homological dimension 0 we are using reduced homology.
Suppose for a contradiction that the vectors $\alpha'(g(t_0, s_0))$ and $n(\alpha(t_0))$ were parallel. Then the normal to $C$ at $\alpha(t_0)$ and the tangent to $C$ at $g(t_0, s_0)$ would be the same line (they have the same direction vectors, and both pass through $g(t_0, s_0)$). This would mean that $\alpha(t_0)$ lives on the tangent line to $C$ at $g(t_0, s_0)$ and that $\alpha'(t_0)$ is perpendicular to $\alpha'(g(t_0, s_0))$, contradicting convexity. Hence it must be that $\alpha'(g(t_0, s_0))$ and $n(\alpha(t_0))$ are not parallel, and therefore $\frac{\partial}{\partial t}(t_0, s_0) \neq 0$.

It then follows from the implicit function theorem that there exists an open set $U$ about $t \in \mathbb{R}$, and a differentiable function $s : U \to \mathbb{R}$, such that $s(t_0) = s_0$ and $f(t, s(t)) = 0$ for all $t \in U$. This gives the differentiability of $s : I \to \mathbb{R}$, as desired.

**Lemma C.3.** Let $C$ be a differentiable convex curve in $\mathbb{R}^2$, and let $\alpha : I \to C$ be a differentiable curve moving in the counterclockwise direction about $C$. Let $L(t)$ be the line through $\alpha(t)$ and $x_\alpha(t)$, namely $L(t) = \{\alpha(t) + sn(t) \mid s \in \mathbb{R}\}$. Suppose that $\beta(t) = \alpha(t) + sn(\alpha(t))$ is an arbitrary differentiable curve with $\beta(t) \in L(t) \setminus \{x_\alpha(t)\}$ for all $t \in I$. Then $(\beta'(t), n(\alpha(t)))$ is a positive basis for $\mathbb{R}^2$ if and only $\beta(t)$ is in the same connected component of $L(t) \setminus \{x_\alpha(t)\}$ as $\alpha(t)$.

**Proof.** We claim that when $s(t) = \frac{1}{\kappa(t)}$, we have $\alpha'(t) \cdot \beta'(t) = 0$. From the Frenet-Serret formulas [22], we have

$$n'(\alpha(t)) = \|\alpha'(t)\| \left( -\kappa(t) \left( \frac{\alpha'(t)}{\|\alpha'(t)\|} \right) + \tau(t)B(t) \right) = -\kappa(t)\alpha'(t),$$

where the last equality follows since the torsion term is $\tau(t) = 0$ for all curves in $\mathbb{R}^2$. Note that when $s(t) = \frac{1}{\kappa(t)}$, we have

$$\alpha'(t) \cdot \beta'(t) = \alpha'(t) \cdot \left( \alpha'(t) + s' n(\alpha(t)) + s n'(\alpha(t)) \right)$$

$$= \|\alpha'(t)\|^2 + \frac{1}{\kappa(t)} \alpha'(t) \cdot n'(\alpha(t)) \quad \text{since } \alpha'(t) \text{ and } n(\alpha(t)) \text{ are orthogonal}$$

$$= \|\alpha'(t)\|^2 - \frac{1}{\kappa(t)} \|\alpha'(t)\| \|n'(\alpha(t))\| \quad \text{by (2)}$$

$$= \|\alpha'(t)\| \left( \|\alpha'(t)\| - \frac{1}{\kappa(t)} \|n'(\alpha(t))\| \right)$$

$$= 0 \quad \text{by (2)}.$$

Now, let’s consider the case where $\alpha(t)$ and $\beta(t)$ are on the same connected component of $L(t) \setminus \{x_\alpha(t)\}$. Since $\alpha'(t)$ is perpendicular to $n(\alpha(t))$, it suffices to show that the dot product $\alpha'(t) \cdot \beta'(t)$ is positive. Since $\alpha(t)$ and $\beta(t)$ are on the same connected component, we know that $s(t) < \frac{1}{\kappa(t)}$. Since

$$\alpha'(t) \cdot \beta'(t) = \|\alpha'(t)\|^2 - \frac{1}{\kappa(t)} \|\alpha'(t)\| \|n'(\alpha(t))\| = 0,$$

it must be the case that for for $s(t) < \frac{1}{\kappa(t)}$ we have

$$\alpha'(t) \cdot \beta'(t) = \|\alpha'(t)\|^2 - \|\alpha'(t)\| \|n'(\alpha(t))\| > 0.$$ 

Finally, consider the case where $\alpha(t)$ and $\beta(t)$ are not on the same connected component of $L(t) \setminus \{x_\alpha(t)\}$. So $s(t) > \frac{1}{\kappa(t)}$. This gives us that

$$\alpha'(t) \cdot \beta'(t) = \|\alpha'(t)\|^2 - \|s(t)\| \|\alpha'(t)\| \|n'(\alpha(t))\| < 0.$$

**Lemma C.4.** Let $C$ be a convex curve in $\mathbb{R}^2$, and let $\alpha : I \to C$ be a differentiable curve moving in the counterclockwise direction about $C$. The line through $\alpha(t)$ and $x_\alpha(t)$ intersects $C$ at a unique point of $C \setminus \{\alpha(t)\}$, which we denote by $\beta(t) = h(\alpha(t))$. Then $\alpha$ and $\beta$ have matching orientations at time $t$ if and only if $x_\alpha(t) \in \text{conv}(C)$.

**Proof.** Note that, by definition of convexity, $C$ lies completely on one side of its tangent lines. A vector $v \in \mathbb{R}^2$ with its tail placed at $p \in C$ points toward the interior of $C$ if it is completely contained on the same side of the tangent line to $C$ at $p$ as $C$. For such a vector, there exists some constant $c > 0$ such that $cv$ intersects $C \setminus \{p\}$ at a unique point $\tilde{p}$. If we instead place the tail of $v$ at $\tilde{p}$, then $v$ is on the side of the tangent line to $C$ at $\tilde{p}$ that does not contain $C$. Thus, $v$ points toward the exterior of $C$ at $\tilde{p}$. By definition, the unit normal vector to $C$ at each $p \in C$ points to the interior of $C$. This means

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4In Lemma C.2 we assume that $\beta$ has image in $C$; that is not necessary here.
that \( n(\alpha(t)) \) points to the interior of \( C \) when its tail is placed at \( \alpha(t) \), and to the exterior of \( C \) when its tail is placed at \( \beta(t) \). In summary, when their tails are placed at \( \beta(t) \), both of the vectors \( -n(\alpha(t)) \) and \( n(\beta(t)) \) point toward the interior of \( C \).

Define \( s: I \to \mathbb{R} \) to satisfy \( \beta(t) = \alpha(t) + s(t)n(\alpha(t)) \); note that \( s \) is differentiable by Lemma C.2. Suppose \( x_\alpha(t) \in \text{conv}(C) \). So \( 0 < \frac{1}{n(t)} < s(t) \). From Lemma C.3, \( (\beta'(t), n(\alpha(t))) \) is a negative basis for \( \mathbb{R}^2 \), and hence \( (\beta'(t), -n(\alpha(t))) \) is a positive basis for \( \mathbb{R}^2 \). So it must be the case that \( (\beta'(t), -n(\alpha(t))) \) and \( (\beta'(t), n(\beta(t))) \) are bases of the same sign. Thus, \( \alpha \) and \( \beta \) have matching orientations.

Next, suppose that \( x_\alpha \notin \text{conv}(C) \). So \( s(t) < \frac{1}{n(t)} \). From Lemma C.3, \( (\beta'(t), n(\alpha(t))) \) is a positive basis. However, at \( \beta(t) \), the vector \( n(\alpha(t)) \) points outward while \( n(\beta(t)) \) must point inward. Thus, \( (\beta'(t), n(\beta(t))) \) is a negative basis, and so \( \alpha \) and \( \beta \) do not have matching orientations. \( \Box \)

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