Results on vertex-edge and independent vertex-edge domination

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Abstract
Given a graph \( G = (V, E) \), a vertex \( u \in V \) ve-dominates all edges incident to any vertex of \( N_G[u] \). A set \( S \subseteq V \) is a ve-dominating set if for all edges \( e \in E \), there exists a vertex \( u \in S \) such that \( u \) ve-dominates \( e \). Lewis (Vertex-edge and edge-vertex parameters in graphs. Ph.D. thesis, Clemson, SC, USA, 2007) proposed a linear time algorithm for ve-domination problem for trees. In this paper, we have constructed an example where the algorithm proposed by Lewis, fails. We have proposed linear time algorithms for ve-domination and independent ve-domination problem in block graphs, which is a superclass of trees. We have also proposed a linear time algorithm for weighted ve-domination problem in trees. We have also proved that finding minimum ve-dominating set is NP-complete for undirected path graphs. Finally, we have characterized the trees with equal ve-domination and independent ve-domination number.

Keywords Vertex-edge domination · Independent vertex-edge domination · NP-completeness

1 Introduction

Domination and its variants are one of the classical problems in graph theory. Let \( G = (V, E) \) be a graph and \( N_G(v) \) (or \( N_G[v] \)) be the open (respectively, closed) neighborhood of \( v \) in \( G \). A set \( D \subseteq V \) is called a dominating set of a graph \( G = (V, E) \)
if $|N_G[v] \cap D| \geq 1$ for all $v \in V$. Our goal is to find a dominating set of minimum cardinality which is known as domination number of $G$ and denoted by $\gamma(G)$. For details the readers are referred to Haynes et al. (1998b, a).

In this paper, we have studied one variant of domination problem, namely vertex-edge domination problem, also known as ve-domination problem. Given a graph $G = (V, E)$, a vertex $u \in V$ ve-dominates all edges incident to any vertex of $N_G[u]$. A set $S \subseteq V$ is a vertex-edge dominating set (or simply a ve-dominating set) if for all edges $e \in E$, there exists a vertex $u \in S$ such that $u$ ve-dominates $e$. The minimum cardinality among all the ve-dominating sets of $G$ is called the vertex-edge domination number (or simply ve-domination number), and is denoted by $\gamma_{ve}(G)$. A set $S$ is called an independent ve-dominating set if $S$ is both an independent set and a ve-dominating set. The independent ve-domination number of a graph $G$ is the minimum cardinality of an independent ve-dominating set and is denoted by $i_{ve}(G)$.

The vertex-edge domination problem was introduced by Peters (1986) in his PhD thesis in 1986. However, it did not receive much attention until Lewis (2007) in 2007 introduced some new parameters related to it and established many new results in his PhD thesis. In his PhD thesis, Lewis has given some lower bound on $\gamma_{ve}(G)$ for different graph classes like connected graphs, $k$-regular graphs, cubic graphs etc. On the algorithmic side, Lewis has also proved that the ve-domination problem is NP-Complete for bipartite, chordal, planar and circle graphs, and independent ve-domination problem is NP-Complete even when restricted to a bipartite and chordal graph. Also, approximation algorithm and approximation hardness results are proved in Lewis (2007). In Lewis et al. (2010), the authors have characterized the trees with equal domination and vertex-edge domination number. In Krishnakumari et al. (2014), both upper and lower bounds on the ve-domination number of a tree have been proved. Some upper bounds on $\gamma_{ve}(G)$ and $i_{ve}(G)$ and some relationship between a ve-domination number and other domination parameters have been proved in Boutrig et al. (2016). In Żyliński (2019), Żyliński has shown that for any connected graph $G$ with $n \geq 6$, $\gamma_{ve}(G) \leq n/3$. Other variations of ve-dominations have also been studied in literature (Boutrig and Chellali 2018; Krishnakumari et al. 2017).

In Lewis (2007), Lewis proposed a linear time algorithm for solving the ve-domination problem for trees. Basically, he proposed a linear time algorithm for solving weighted distance-3 domination problem for trees. A set $D \subseteq V$ is called a distance-3 dominating set of a graph $G = (V, E)$ if every vertex in $V$ is at most distance 3 from some vertex in $D$. In weighted distance-3 domination problem, given a weighted graph, our goal is to find a distance-3 dominating set with minimum weight. The algorithm proposed by Lewis reduces an instance of ve-domination problem for trees into an instance of weighted distance-3 domination problem for trees in linear time. This reduction takes a weighted tree $T = (V, E)$ as an input and outputs a new weighted tree $T' = (V', E')$ by subdividing each edge of $E$ once by introducing a new vertex having $\infty$ weight and weights of all the original vertices remain 1. Clearly $|E'| = 2 \cdot |E|$ and $|V'| = |V| + |E|$ and hence reduction algorithm takes linear time. Lewis claimed that the set of vertices that forms a minimum distance-3 dominating set in the weighted tree $T'$, also forms a minimum ve-dominating set of $T$. But, we have found a counterexample of this claim. In Fig. 1, it is easy to see that $\gamma_{ve}(T) = 2$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{counterexample.png}
\caption{Counterexample to Lewis' claim.}
\end{figure}
Now, in the new weighted tree $T' = (V', E')$, it is not possible to find any distance-3 dominating set whose weight is 2.

This motivates us to study the ve-domination problem in trees and other graph classes. The rest of the paper is organized as follows. Sections 2 and 3 deal with block graphs, where we have proposed a linear time algorithm for finding minimum ve-dominating set and minimum independent ve-dominating set. In Sect. 4 we have proposed a linear time algorithm for solving the weighted ve-domination problem in trees. Section 5 deals with NP-completeness of this problem in undirected path graphs. In Sect. 6, we have characterized the trees having equal ve-domination number and independent ve-domination number. Finally, Sect. 7 concludes the paper.

2 VE-domination in block graph

In this section, we propose a linear time algorithm to find a minimum ve-dominating set of a given block graphs. A vertex $v \in V$ is called a cut vertex of $G = (V, E)$ if removal of $v$ increases the number of components in $G$. A maximal connected induced subgraph without a cut vertex of $G$ is called a block of $G$. A graph $G$ is called a block graph if each block of $G$ is a complete subgraph. Note that, the intersection of two distinct blocks can contain at most one vertex. Two blocks are called adjacent if they contain a common cut vertex of $G$. The blocks that contain exactly one cut vertex are called end blocks. The distance between two blocks $B_i$ and $B_j$ is defined as $dist_G(B_i, B_j) = \max\{dist(u, v) | u \in B_i, v \in B_j\} - 1$. The distance between a vertex $v$ and a block $B$ of a block graph $G$ is denoted as $dist_G(v, B) = \max\{dist_G(v, u) | u \in B\} - 1$.

Our proposed algorithm is a labelling based algorithm. Let $G = (V, E)$ be a block graph where each vertex $v \in V$ is associated with a label $l(v)$ and each edge $e = (xy) \in E$ is associated with a label $m(xy)$, where $l(v), m(xy) \in \{0, 1\}$. We call such graph a labelled block graph.

**Definition 1** Given a labelled block graph $G = (V, E)$ with labels $l$ and $m$, an optional ve-dominating set is a subset $D \subseteq V$ such that

(i) if $l(v) = 1$, then $v \in D$,
(ii) $D$ ve-dominates every edge $e = xy$ with $m(e) = 1$. 

Now, in the new weighted tree $T' = (V', E')$, it is not possible to find any distance-3 dominating set whose weight is 2.

This motivates us to study the ve-domination problem in trees and other graph classes. The rest of the paper is organized as follows. Sections 2 and 3 deal with block graphs, where we have proposed a linear time algorithm for finding minimum ve-dominating set and minimum independent ve-dominating set. In Sect. 4 we have proposed a linear time algorithm for solving the weighted ve-domination problem in trees. Section 5 deals with NP-completeness of this problem in undirected path graphs. In Sect. 6, we have characterized the trees having equal ve-domination number and independent ve-domination number. Finally, Sect. 7 concludes the paper.
The **optional ve-domination number**, denoted by $\gamma_{opve}(G)$, is the minimum cardinality among all optional ve-dominating sets of $G$.

Note that, an optional ve-dominating set $D$ may contain some vertices $v$ such that $l(v) = 0$ but $D$ must contain all vertices $u$ such that $l(u) = 1$. Hence, if $l(v) = 0$ for all $v \in V$ and $m(e) = 1$ for all $e \in E$, then the minimum optional ve-dominating set is basically a minimum ve-dominating set of $G$. Therefore, we propose an algorithm that outputs a minimum optional ve-dominating set of a labelled block graph $G$ with labels $l(v) = 0$ for each $v \in V$ and $m(e) = 1$ for each $e \in E$. Next, we present the outline of the algorithm.

Let $B_0$ be an end block of a block graph $G = (V, E)$. Since block graph has tree-like structures, we can view $G$ as a graph rooted at the end block $B_0$. The **height** of $G$ is defined as $h(G) = \max\{\text{dist}_G(B_0, B)|B \text{ is end block of } G\}$. The **level** of a block $B$ is defined as the distance between $B$ and $B_0$, i.e., $\text{dist}(B_0, B)$. At each step, the algorithm processes one of the end blocks at the maximum level. Based on the number and configuration of edges having label 1 in an end block $B$, we decide whether to take some vertices from $B$ in the optional ve-dominating set or not and then delete that block $B$ (except the cut vertex) from $G$. We also modify the labels of some of the vertices and edges of the new graph. In the next iteration, we process another end block, and the process continues till we are left with the root block $B_0$. For the root block, we directly calculate the optional ve-dominating set. The outline of the algorithm is given in Algorithm 1. In Algorithm 1, for an end block $B$, $t_B$ denotes the number of edges with $m(xy) = 1$ in $B$ (see Fig. 2), and $P$ denotes the set of non-cut vertices of $B$, i.e., $P = V(B) \setminus \{c\}$, where $V(B)$ is the set of vertices of $B$ and $c$ is the cut vertex in $B$. Also let $f(c)$ denote the unique cut vertex of $G$ in $N_G[c]$ which has the minimum distance from $B_0$. Note that $f(c) = c$ if and only if $c$ is the cut vertex of $B_0$.

Before proving the correctness of Algorithm 1, we make the following observations which follow from the modification of the labels.

**Observation 2** After each iteration, if $l(v) = 1$ for some $v \in V$, then $m(xy) = 0$ for all edges incident to any vertex $x \in N_G(v)$.

**Observation 3** After each iteration, in any block $B$, the set of edges with label 1 forms a clique.
Algorithm 1 Min_Optional_VEDom(G)

Input: A labelled block graph $G = (V, E)$ with $l(v) = 0$, for all $v \in V$ and $m(e) = 1$, for all $e \in E$

Output: Minimum optional ve-dominating set $S$ of $G$

1: $S = \emptyset$
2: for $i = h(G)$ to 1 do
3:   Consider all blocks at level $i$
4:   while $(\exists B$ such that $(t_B \geq 2) \lor ((t_B = 1) \land (e \text{ is not incident to } c \text{ and } m(e) = 1)))$ do
5:     $l(c) = 1$
6:     $G = G \setminus P$
7:     $m(xy) = 0, \forall x \in N_G(c)$
8:   while $(\exists B$ such that $t_B = 1)$ do
9:     $l(f(c)) = 1$
10:    $G = G \setminus P$
11:    $m(xy) = 0, \forall x \in N_G(f(c))$
12:   while $(\exists B$ such that $t_B = 0)$ do
13:     $S = S \cup \{x | x \in P \text{ and } l(x) = 1\}$
14:    $G = G \setminus P$
15: if $h(G) = 0$ then
16:   if $t_B > 0$ then
17:     $S = S \cup \{v\}$ for some $v \in V$
18:   else
19:     $S = S \cup \{v | v \in G \land l(v) = 1\}$
20: Return $S$

The following lemma gives the correctness of the above algorithm in all the intermediate iterations.

Lemma 4 Let $G$ be a labelled block graph after an iteration. Also, let the root block of $G$ be $B_0$, and $B$ be another end block at maximum level. Also assume that $P = V(B) \setminus \{c\}$, where $c$ is the cut vertex of $B$ and $t_B$ denotes the number of edges with label 1 in $B$. Then the followings are true.

(a) If $t_B \geq 2$, then $\gamma_{opve}(G) = \gamma_{opve}(G')$, where $G'$ is a new block graph obtained by removing all vertices of $P$ and the labels of $G'$ are obtained by relabelling $c$ as $l(c) = 1$ and all edges $xy$ as $m(xy) = 0$ for all $x \in N_G'(c)$.

(b) If $t_B = 1$ and the edge $e$, with $m(e) = 1$, is not incident to $c$, then $\gamma_{opve}(G) = \gamma_{opve}(G')$, where $G'$ is a new block graph obtained by removing all vertices of $P$ and the labels of $G'$ are obtained by relabelling $c$ as $l(c) = 1$ and all edges $xy$ as $m(xy) = 0$ for all $x \in N_G'(c)$.

(c) Let for all end blocks at maximum level conditions in (a) and (b) do not hold. If $t_B = 1$ and the edge $e$, with $m(e) = 1$, is incident to $c$, then $\gamma_{opve}(G) = \gamma_{opve}(G')$, where $G'$ is a new block graph obtained by removing all vertices of $P$ and the labels of $G'$ are obtained by relabelling $f(c)$ as $l(f(c)) = 1$ and all edges $xy$ as $m(xy) = 0$ for all $x \in N_G'(f(c))$.

(d) If $t_B = 0$ and $B - \{c\}$ has $k$ vertices with label 1, then $\gamma_{opve}(G) = \gamma_{opve}(G') + k$, where $G'$ is new block graph obtained from $G$ by deleting all vertices of $P$ and the labels of $G'$ remain same as in $G$.

Proof (a) Let $S$ be a minimum optional ve-dominating set of $G$. Note that, by Observation 2, in this case $l(v) = 0$ for all $v \in V(B)$. Also, by Observation 3, in this
Lemma 5 Let $G$ be a labelled block graph with only one block $B$. If $t_B \geq 1$, then $\gamma_{opve}(G) = 1$. Otherwise, $\gamma_{opve}(G) = k$, where $k$ is the number of vertices in $B$ with $l(v) = 1$.  

The following lemma calculates the optional ve-dominating set of the root block $B_0$ at the last iteration.

Lemma 5 Let $G$ be a labelled block graph with only one block $B$. If $t_B \geq 1$, then $\gamma_{opve}(G) = 1$. Otherwise, $\gamma_{opve}(G) = k$, where $k$ is the number of vertices in $B$ with $l(v) = 1$.  

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Proof If $t_B \geq 1$, then $B$ does not have any vertex with label-1. So, we need at least one vertex from the block $B$ to ve-dominate all the edges with $m(e) = 1$ and only one vertex is sufficient to ve-dominate all the edges. Hence, $\gamma_{opv}(G) = 1$.

If $t_B = 0$, then none of the edges needs to be ve-dominated. So, all the vertices with $l(v) = 1$ forms an optional ve-dominating set and there are $k$ many such vertices. Hence, $\gamma_{opv}(G) = k$. 

Lemmas 4 and 5 shows that the output of Algorithm 1 is minimum optional ve-dominating set. At each iteration, we are taking $O(deg(c))$ time. Hence the running time of Algorithm 1 is $O(n + m)$. Thus, we have the following theorem.

Theorem 6 The ve-domination problem can be solved in linear time for block graphs.

3 Independent VE-domination in block graph

In this section, we focus on the independent ve-domination problem. Algorithm 1 outputs a minimum ve-dominating set of a given block graph $G$. This set is not always independent as it may contain more than one vertices from the same block. So, here we propose a linear-time algorithm to find a minimum independent ve-dominating set of a given block graph. This algorithm is similar to the one discussed in Sect. 2, but with suitable modification to ensure the output is an independent set.

In Algorithm 1, whenever a vertex is labelled 1, we relabel the edges that are incident to its neighbours, but we are not relabelling any vertex in its neighbourhood. So, the neighbouring vertices can still be selected in the ve-dominating set in subsequent steps. But, since we want an independent ve-dominating set, whenever a vertex $v$ is labelled 1, we relabel all the vertices in $N_G(v)$ as $-1$ to ensure that these vertices are not selected in subsequent steps. Hence, in this case, the labelling of the graph is defined as follows: each vertex $v \in V(G)$ is associated with a label $l(v)$ and each edge $e = (xy) \in E(G)$ is associated with a label $m(xy)$, where $l(v) \in \{-1, 0, 1\}$ and $m(xy) \in \{0, 1\}$. Next, we define an optional independent ve-dominating set and optional independent ve-domination number as follows:

Definition 7 For a given labelled block graph $G = (V, E)$, with labels $l$ and $m$, an optional independent ve-dominating set is an independent subset $D \subseteq V$ such that

(i) if $l(v) = 1$, then $v \in D$,

(ii) if $l(v) = -1$, then $v \notin D$,

(iii) $D$ ve-dominates every edge $e$ with $m(e) = 1$.

The optional independent ve-domination number, denoted by $i_{opv}(G)$, is the minimum cardinality among all optional independent ve-dominating sets of $G$.

We propose an algorithm that outputs a minimum optional independent ve-dominating set of a labelled block graph $G$ with labels $l(v) = 0$ for each $v \in V$ and $m(xy) = 1$ for each $xy \in E$ because with these labelling, a minimum optional independent ve-dominating set of $G$ is indeed a minimum independent ve-dominating set of $G$. 

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At each step, the proposed algorithm processes one of the end blocks at the maximum level. Based on the number of edges having label 1 and label of the cut vertex, say \( c \), corresponding to this end block \( B \), we decide which vertex to include in the optional independent ve-dominating set and then delete the block \( B \) (except \( c \)). We also modify the labels of some of the vertices and edges of the new smaller graph. In the next iteration, we process another end block at the maximum level, and the process continues till we are left with the root block \( B_0 \). For the root block, we directly calculate the optional independent ve-dominating set by taking one of the end vertices of an edge with label 1, if any such edge exists. The outline of the algorithm is given in Algorithm 2. In this algorithm, we use a letter \( Q \) to denote the set of non-cut vertices of \( B \) which are incident to some edge \( e \) having label 1, i.e., \( Q = \{ v, t | v, t \in V(B) \text{ and } m(vt) = 1 \} \), where \( V(B) \) is the set of vertices of \( B \). All other notations used in Algorithm 2 are same as those used in Algorithm 1.

Now, we prove the correctness of Algorithm 2. First note that, Observations 2 and 3 still hold for Algorithm 2. We make some more observations which follows from the modification of labels.

**Observation 8** After each iteration, if \( l(v) = 1 \) for some \( v \in V \), then \( l(u) = -1 \) for all \( u \in N_G(v) \).

**Observation 9** After each iteration, if \( l(v) = -1 \) for some \( v \in V \), then \( m(uv) = 0 \) for all \( u \in N_G(v) \).

**Observation 10** Let \( B \) be an end block at the last level, and \( B \) contain at least one edge with label 1. Then for every vertex \( v \in V(B) \), if all edges of \( B \) incident to \( v \) are of label 0, then \( l(v) = -1 \).

The following lemma gives the correctness of the above algorithm in all the intermediate iterations.

**Lemma 11** Let \( G \) be a labelled block graph after an iteration. Also, let the root block of \( G \) be \( B_0 \) and \( B \) be another end block at maximum level. Also assume that \( P = V(B) \setminus \{ c \} \), where \( c \) is the cut vertex of \( B \), \( Q \) denotes the set of non-cut vertices of \( B \) which are end points of some edge \( e \) with label 1, i.e., \( Q = \{ v, w | v, w \in V(B) \text{ and } m(vw) = 1 \} \) and \( t_B \) denotes the number of edges with label 1 in \( B \). Then followings are true.

(a) If \( t_B \geq 2 \) and \( l(c) = 0 \), then \( i_{opve}(G) = i_{opve}(G') \), where \( G' \) is new block graph obtained by removing all vertices of \( P \) and labels of \( G' \) are obtained by relabelling \( c \) as \( l(c) = 1 \), all vertices \( v \in N_G(c) \) as \( l(v) = -1 \) and all edges \( xy \) as \( m(xy) = 0 \) for all \( x \in N_G(c) \).

(b) If \( t_B \geq 1 \) and \( l(c) = -1 \), then \( i_{opve}(G) = i_{opve}(G') + 1 \), where \( G' \) is new block graph obtained by removing all vertices of \( P \).

(c) Let conditions in (a) and (b) do not hold for all end blocks at maximum level. If \( t_B = 1 \) and \( l(f(c)) \neq -1 \), then \( i_{opve}(G) = i_{opve}(G') \), where \( G' \) is new block graph obtained from \( G \) by removing all vertices of \( P \) and labels of \( G' \) are obtained by relabelling \( f(c) \) as \( l(f(c)) = 1 \), all vertices \( v \in N_G(f(c)) \) as \( l(v) = -1 \) and all edges \( xy \) as \( m(xy) = 0 \) for all \( x \in N_G(f(c)) \).
Algorithm 2 MIN_OPT_IVEDom(G)

Input: A labelled block graph $G = (V, E)$ with $l(v) = 0$, for all $v \in V$ and $m(xy) = 1$, for all $xy \in E$

Output: Minimum optional independent ve-dominating set $S$ of $G$

1: $S = \phi$
2: for $i = h(G) \to 1$ do
3:   Consider all blocks at level $i$
4:   while $\exists B$ such that $(t_B \geq 2 \text{ AND } l(c) = 0 \text{ OR } (t_B \geq 1 \text{ AND } l(c) = -1))$ do
5:     if $l(c) = 0$ then
6:        $l(c) = 1$
7:        $G = G \setminus P$
8:        $m(xy) = 0, \forall x \in N_G(c)$
9:        $l(v) = -1, \forall v \in N_G(c)$
10:    else
11:       Take any $w$ from $Q$
12:       $l(w) = 1$
13:       $S = S \cup \{w\}$
14:       $m(xy) = 0, \forall x \in N_G(w)$
15:       $l(v) = -1, \forall v \in N_G(w)$
16:       $G = G \setminus P$
17:   while $\exists B$ such that $(t_B = 1 \text{ AND } l(c) = 0)$ do
18:      if $l(f(c)) = -1$ then
19:         $l(c) = 1$
20:         $G = G \setminus P$
21:         $m(xy) = 0, \forall x \in N_G(c)$
22:         $l(v) = -1, \forall v \in N_G(c)$
23:      else
24:         $l(f(c)) = 1$
25:         $G = G \setminus P$
26:         $m(xy) = 0, \forall x \in N_G(f(c))$
27:         $l(v) = -1, \forall v \in N_G(f(c))$
28:      while $\exists B$ such that $t_B = 0$ do
29:         $S = S \cup \{x| x \in P \text{ and } l(x) = 1\}$
30:      $G = G \setminus P$
31:   if $h(G) = 0$ then
32:      if $t_B > 0$ then
33:         $S = S \cup \{v\}$ such that $m(vu) = 1$ and Return $S$
34:      else
35:         $S = S \cup \{v| v \in G \text{ AND } l(v) = 1\}$ and Return $S$

(d) Let conditions in (a) and (b) do not hold for all end blocks at maximum level. If $t_B = 1$ but $l(f(c)) = -1$, then $i_{opve}(G) = i_{opve}(G')$, where $G'$ is new block graph obtained from $G$ by removing all vertices of $P$ and labels of $G'$ are obtained by relabelling $c$ as $l(c) = 1$, all vertices $v \in N_{G'}(c)$ as $l(v) = -1$ and all edges $xy$ as $m(xy) = 0$ for all $x \in N_{G'}(c)$.

(e) If $t_B = 0$ and $B \setminus \{c\}$ has $k$ many vertices with label $1$, then $i_{opve}(G) = i_{opve}(G') + k$, where $G'$ is new block graph obtained by removing all vertices of $P$.

Proof (a) Let $S$ be a minimum optional independent ve-dominating set of $G$. Note that, by Observation 2, in this case $l(v) = 0$ or $l(v) = -1$ for all $v \in V(B)$. But by Observation 9, there must be at least one vertex $v$ in $P$ with $l(v) \neq -1$. This implies $l(v) = 0$ for some $v \in P$. Also, by Observation 3, in this case, there exists an edge $e$ in the block $B$ such that $m(e) = 1$ and $e$ is not incident to $c$. 

\[\text{Proof}\]
Hence, $S$ contains at least one vertex, say $v$, from $V(B)$. As $S$ is an independent set, it contains at most one vertex from $V(B)$. Since $l(c) = 0$, by Observation 8, $l(x) \neq 1$ for all $x \in N_G(c)$. Next, we claim that if $v \in P \cap S$, then there exists a minimum optional independent ve-dominating set of $G$ that does not contain any vertex from $B_1 \setminus \{c\}$, where $B_1$ is the parent block of $B$. If $S$ contains a vertex $w \in V(B_1) \setminus \{c\}$, then there exists some edges that are ve-dominated only by $w$. If all such edges are incident to some vertex of $B_1$, then $(S \setminus \{v, w\}) \cup \{c\}$ is also an optional independent ve-dominating set. This contradicts the minimality of $S$. Otherwise, let $yz$ be an edge that is ve-dominated only by $w$ and $yz \notin B_1$. Then $(S \setminus \{w, v\}) \cup \{y, c\}$ is also a minimum optional independent ve-dominating set of $G$. Hence, without loss of generality, let us assume that $S$ is a minimum optional independent ve-dominating set of $G$ that does not contain any vertex from $B_1 \setminus \{c\}$. In this case, $S' = (S \setminus \{v\}) \cup \{c\}$ is also a minimum optional independent ve-dominating set of $G$. It is easy to see that $S'$ is an optional independent ve-dominating set of $G'$ containing $c$. Hence, $i_{\text{opve}}(G') \leq i_{\text{opve}}(G)$.

Conversely, let $S'$ be a minimum optional independent ve-dominating set of $G'$. Note that $c \in S'$ as $l(c) = 1$. Let $e$ be any edge of $G$ with $m(e) = 1$. If $e \notin B$, then obviously some $v \in S'$ ve-dominates $e$. If $e \in B$, then $c$ ve-dominates $e$. Also $S'$ is independent set. So, $S'$ is also optional independent ve-dominating set of $G$. Hence, $i_{\text{opve}}(G) \leq i_{\text{opve}}(G')$. Therefore, in this case, we have $i_{\text{opve}}(G) = i_{\text{opve}}(G')$.

(b) Let $S$ be a minimum optional independent ve-dominating set of $G$. Note that, by Observation 2, in this case, $l(v) \neq 1$ for all $v \in V(B)$. Using Observation 9, as above, we have all the vertices of $Q$ have label 0. Hence, to ve-dominate the label 1 edges of block $B$, $S$ must contain exactly one vertex $v$ from $V(B)$. Also, since $l(c) = -1$, $v \neq c$. Again by Observation 9, all edges incident to $c$ is labelled 0. So, $v$ does not ve-dominate any label 1 edge of $G'$. Thus, $S' = S \setminus \{v\}$ is an optional independent ve-dominating set of $G'$. Hence, $i_{\text{opve}}(G') \leq i_{\text{opve}}(G) - 1$. Conversely, let $S'$ be a minimum optional independent ve-dominating set of $G'$. Note that $c \notin S'$ as $l(c) = -1$. Let $e$ be any edge of $G$ with $m(e) = 1$. If $e \notin B$, then obviously some $v \in S'$ ve-dominates $e$. If $e \in B$, then it is not incident to $c$ (by Observation 9). Hence it can not be ve-dominated by any vertex $u \in S'$. So, to ve-dominate $e$, we must select a vertex, say $v$, from $P$. Hence $S = S' \cup \{v\}$ is an optional independent ve-dominating set of $G$. Hence, $i_{\text{opve}}(G) \leq i_{\text{opve}}(G') + 1$. Therefore, in this case, we have $i_{\text{opve}}(G) = i_{\text{opve}}(G') + 1$.

(c) Let $S$ be a minimum optional independent ve-dominating set of $G$. Since condition (b) does not hold, $l(c) = 0$. Hence, the only edge $e$ with $m(e) = 1$ of block $B$ is incident to $c$, otherwise, by Observation 10, $l(c) = -1$. Note that $l(f(c)) \neq -1$ implies none of the neighbours of $f(c)$ is labelled 1. Also, $S$ can contain at most one vertex from $V(B)$. But if there exists a $v \in V(B) \cap S$, then $S' = (S \setminus \{v\}) \cup \{f(c)\}$ is also an optional independent ve-dominating set of $G$ because all the label 1 edges which are ve-dominated by $v$ are also ve-dominated by $f(c)$ and $S'$ is an independent set. The arguments for independence of $S'$ is similar to the arguments in the proof of part (a). So, without loss of generality, assume that $S \cap V(B) = \emptyset$ and $f(c) \in S$. It is easy check that $S$ is an optional independent ve-dominating set of $G'$. Hence, $i_{\text{opve}}(G') \leq i_{\text{opve}}(G)$.
Conversely, let \( S' \) be a minimum optional independent ve-dominating set of \( G' \). Note that \( f(c) \in S' \) as \( l(f(c)) = 1 \). So, all the edges incident to \( c \) are ve-dominated by \( f(c) \). In the block \( B \), only one edge is labelled 1 and this edge is incident to \( c \) (using similar argument as above). So, it is ve-dominated by \( f(c) \) and \( S' \) is an optional ve-dominating set of \( G \). Hence, \( i_{opve}(G) \leq i_{opve}(G') \). Therefore, in this case, we have \( i_{opve}(G') = i_{opve}(G) \).

(d) Let \( S \) be a minimum optional independent ve-dominating set of \( G \). Since condition (b) does not hold, \( l(c) = 0 \). Hence, by Observation 8, \( l(v) \neq 1 \) for all \( v \in N_G(c) \). And the only edge \( e \) with \( m(e) = 1 \) of block \( B \) must be incident to \( c \) (using similar argument as above). Also \( S \) can contain at most one vertex from \( V(B) \) as it is an independent set. If there exists \( v \in P \cap S \), then \( S' = (S\setminus\{v\}) \cup \{c\} \) is also an optional independent ve-dominating set of \( G \) (all the edges ve-dominated by \( v \) is also ve-dominated by \( c \) and \( S' \) is an independent set by the same argument as above). So, without loss of generality, assume that \( S \cap P = \emptyset \) and \( c \in S \). It is easy to check that \( S \) is an optional independent ve-dominating set of \( G' \). Hence, \( i_{opve}(G') \leq i_{opve}(G) \).

Conversely, let \( S' \) be a minimum optional independent ve-dominating set of \( G' \). Note that \( c \in S' \) as \( l(c) = 1 \). So, all the edges of \( B \) is ve-dominated by \( c \). So, \( S' \) is optional independent ve-dominating set of \( G \). Hence \( i_{opve}(G) \leq i_{opve}(G') \). Therefore, in this case, we have \( i_{opve}(G') = i_{opve}(G) \).

(e) Let \( S \) be a minimum optional independent ve-dominating set of \( G \). Clearly, \( S \) can contain at most one vertex from block \( B \). So, \( k \) can either be 0 or 1. If \( k = 0 \) and \( S \) contains a vertex \( v \) from \( P \), then \( S' = (S\setminus\{v\}) \cup \{c\} \) is a minimum optional independent ve-dominating set of \( G \) which does not contain any vertex from \( P \).

If \( k = 1 \) and \( q \) be the vertex in \( P \) with \( l(q) = 1 \), then \( S \) is a minimum optional independent ve-dominating set of \( G \) that does not contain any vertex from \( P\setminus\{q\} \).

Let \( S' = S\setminus\{q\} \). It is easy to verify that, in both the cases, \( S' \) is an optional independent ve-dominating set of \( G' \). Therefore, \( i_{opve}(G') \leq i_{opve}(G) - k \).

Conversely, let \( S' \) be a minimum optional independent ve-dominating set of \( G' \). If \( k = 0 \), then \( S' \) is an optional independent ve-dominating set of \( G \). And if \( k = 1 \), then \( S' \cup \{q\} \) is an optional independent ve-dominating set of \( G \), where \( q \in P \) and \( l(q) = 1 \) in \( G \). Hence, \( i_{opve}(G) \leq i_{opve}(G') + k \). Therefore, in this case, we have \( i_{opve}(G) = i_{opve}(G') + k \) \( \square \)

Next lemma calculates the optional independent ve-dominating set of the root block \( B_0 \) at the last iteration.

**Lemma 12** Let \( G \) be a labelled block graph with only one block \( B \). If \( t_B \geq 1 \), then \( i_{opve}(G) = 1 \). Otherwise, \( i_{opve}(G) = k \), where \( k \) is the number of vertices in \( B \) with \( l(v) = 1 \).

**Proof** If \( t_B \geq 1 \) then \( B \) does not have any vertex with label 1. So, we need at least one vertex from block \( B \) to ve-dominate all the edges with \( m(e) = 1 \) and only one vertex is sufficient to ve-dominate all the edges. Here, to maintain the independence of the optional ve-dominating set, we select a vertex \( v \) such that \( l(v) \neq -1 \). Hence, \( i_{opve}(G) = 1 \).
The sets \{a, j\}, \{c, d, e\}, \{e, f, g\} and \{f, g, j\} are 0-ve-dominating set, 1-ve-dominating set, 2-ve-dominating set and ve-set of \(T_a\), respectively.

If \(l_B = 0\), then none of the edges needs to be ve-dominated. So, all the vertices with \(l(v) = 1\) forms an optional independent ve-dominating set. Hence, \(i_{\text{opve}}(G) = k\).

Note that, the block can have at most one vertex with label 1, i.e., \(k \leq 1\).

Lemmas 11 and 12, shows that the output of Algorithm 2 is minimum optional independent ve-dominating set. At each iteration, we are taking \(O(\text{deg}(c))\) time. Hence the running time of Algorithm 2 is \(O(n + m)\). Thus, we have the following theorem.

**Theorem 13** The independent ve-domination problem can be solved in linear time for block graphs.

### 4 Weighted VE-domination in trees

In this section, we study the weighted version of the vertex-edge domination problem for trees. The weighted ve-domination problem is defined as follows: given a graph \(G = (V, E)\) with each vertex assigned some positive weights, find a ve-dominating set with minimum total weight. We propose dynamic programming based linear time algorithm for solving the weighted ve-domination problem in trees.

Let \(T_r = (V, E)\) be a rooted tree rooted at a vertex \(r\). We define four parameters on \(T_r\), which will be used to compute the minimum weighted ve-dominating set of \(T_r\). For a set \(D \subseteq V\), let \(\text{dist}(r, D)\) denote the distance between the root \(r\) and the set \(D\) and is defined as \(\min_{v \in D} \{\text{dist}(r, v)\}\). For \(i = 0, 1, 2\), a subset \(D\) is called an \(i\)-ve-dominating set of \(T_r\) if \(D\) is a ve-dominating set of \(T_r\) and \(\text{dist}(r, D) = i\). Let the minimum weight of an \(i\)-ve-dominating set of \(T_r\) be \(\gamma_{\text{ve}}^i(T_r)\) for \(0 \leq i \leq 2\). Further, a subset \(D\) is called a ve-set of \(T_r\) if \(D\) is a ve-dominating set of \(T_r - \{r\}\) but not a ve-dominating set of \(T_r\) and the minimum weight of a ve-set of \(T_r\) is denoted by \(\gamma_{\text{ve}}(T_r)\). Refer Fig. 3 for example. Since, every ve-dominating set of \(T_r\) is either a 0-ve-dominating set or 1-ve-dominating set or 2-ve-dominating set, we have the following lemma.

**Lemma 14** For any rooted tree \(T_r\), \(\gamma_{\text{ve}}(T_r) = \min\{\gamma_{\text{ve}}^0(T_r), \gamma_{\text{ve}}^1(T_r), \gamma_{\text{ve}}^2(T_r)\}\).

The following lemmas show how to update these newly defined parameters in the bottom-up approach.

**Lemma 15** Let \(T_v\) be a weighted rooted tree with the root \(v\) having only one child \(u\) and \(T_u\) be the subtree rooted at \(u\). Then the following statements are true.
(a) \( \gamma_{ve}^0(T_v) = w(v) + \min\{\gamma_{ve}(T_u), ve(T_u)\} \)

(b) \( \gamma_{ve}^1(T_v) = \gamma_{ve}^0(T_u) \)

(c) \( \gamma_{ve}^2(T_v) = \gamma_{ve}^1(T_u) \)

(d) \( ve(T_v) = \gamma_{ve}^2(T_u) \)

**Proof** (a) Let \( S \) be a 0-ve-dominating set of \( T_v \) of weight \( \gamma_{ve}^0(T_v) \). Clearly, \( v \in S \).

Let \( S' = S \setminus \{v\} \). If all edges of \( T_u \) are ve-dominated by \( S' \), then \( S' \) is a ve-dominating set of \( T_u \). Otherwise, assume that, there exists an edge \( e \in T_u \) which is not ve-dominated by \( S' \). Since, \( S \) is a ve-dominating set of \( T_v \), it follows that \( e \) must be incident to \( u \). Hence, \( S' \) is a ve-dominating set of \( T_u - \{u\} \) but not a ve-dominating set of \( T_u \). Therefore, \( S' \) is a ve-set of \( T_u \). So, we have, \( \gamma_{ve}^0(T_v) \geq w(v) + \min\{\gamma_{ve}(T_u), ve(T_u)\} \).

Conversely, if \( \min\{\gamma_{ve}(T_u), ve(T_u)\} = \gamma_{ve}(T_u) \) and \( S' \) is a ve-dominating set of \( T_u \) of weight \( \gamma_{ve}(T_u) \), then \( S' \cup \{v\} \) is a ve-dominating set of \( T_v \). On the other hand, if \( \min\{\gamma_{ve}(T_u), ve(T_u)\} = ve(T_u) \) and \( S' \) is a ve-set of \( T_u \) of weight \( ve(T_u) \), then also \( S' \cup \{v\} \) is a ve-dominating set of \( T_v \). So, we have, \( \gamma_{ve}^0(T_v) \leq w(v) + \min\{\gamma_{ve}(T_u), ve(T_u)\} \).

(b) Let \( S \) be a 1-ve-dominating set of \( T_v \) of weight \( \gamma_{ve}^1(T_v) \). Clearly, \( u \in S \) and \( S \) is also a ve-dominating set of \( T_u \). Hence, \( S \) is a 0-ve-dominating set of \( T_u \). So, we have, \( \gamma_{ve}^1(T_v) \geq \gamma_{ve}^0(T_u) \).

Conversely, let \( S \) be a 0-ve-dominating set of \( T_u \) of weight \( \gamma_{ve}^0(T_u) \). Clearly, \( u \in S \) and \( S \) is also a ve-dominating set of \( T_u \). Hence, \( S \) is a 1-ve-dominating set of \( T_v \).

So, we have, \( \gamma_{ve}^0(T_v) \leq \gamma_{ve}^0(T_u) \).

(c) Let \( S \) be a 2-ve-dominating set of \( T_v \) of weight \( \gamma_{ve}^2(T_v) \). Hence, at least one child of \( u \) belongs to \( S \). Clearly, \( S \) is also a ve-dominating set of \( T_u \). Hence, \( S \) is a 1-ve-dominating set of \( T_u \). So, we have, \( \gamma_{ve}^2(T_v) \geq \gamma_{ve}^1(T_u) \).

Conversely, let \( S \) be a 1-ve-dominating set of \( T_u \) of weight \( \gamma_{ve}^1(T_u) \). Hence, at least one child of \( u \) belongs to \( S \). Clearly, \( S \) is also a ve-dominating set of \( T_v \). Hence, \( S \) is a 2-ve-dominating set of \( T_v \). So, we have, \( \gamma_{ve}^2(T_v) \leq \gamma_{ve}^1(T_u) \).

(d) Let \( S \) be a ve-set of \( T_v \) of weight \( ve(T_v) \). Note that, \( dist(v, S) = 3 \) as \( S \) is not a ve-dominating set of \( T_v \) but is a ve-dominating set of \( T_v - \{v\} \), i.e., \( T_u \). So, \( dist(u, S) = 2 \). Hence, \( S \) is a 2-ve-dominating set of \( T_v \). So, we have, \( ve(T_v) \geq \gamma_{ve}^2(T_u) \).

Conversely, let \( S \) be a 2-ve-dominating set of \( T_u \) of weight \( \gamma_{ve}^2(T_u) \). Clearly, \( S \) does not ve-dominate the edge \( vu \). Hence, \( S \) is not a ve-dominating set of \( T_v \). Hence, \( S \) is a ve-set of \( T_v \). So, we have, \( ve(T_v) \leq \gamma_{ve}^2(T_u) \). \( \square \)

**Lemma 16** Let \( T'_v \) and \( T'_u \) be two rooted trees and \( T_v \) be the rooted tree obtained by joining an edge between \( v \) and \( u \) in the disjoint union of \( T'_v \) and \( T'_u \). Then the following statements are true.

(a) \( \gamma_{ve}^0(T_v) = \gamma_{ve}^0(T'_v) + \min\{\gamma_{ve}(T'_u), ve(T'_u)\} \)

(b) \( \gamma_{ve}^1(T_v) = \min\left\{ \gamma_{ve}^1(T'_v) + \gamma_{ve}(T'_u), \left( \min\{\gamma_{ve}(T'_u), ve(T'_u)\}\right)^2\right\} \)

(c) \( \gamma_{ve}^2(T_v) = \gamma_{ve}^2(T'_v) + \gamma_{ve}^1(T'_u) \)

(d) \( ve(T_v) = \min\left\{ \gamma_{ve}^2(T'_v) + \gamma_{ve}^1(T'_u), \left( ve(T'_v) + \min\{\gamma_{ve}(T'_u), \gamma_{ve}(T'_u)\}\right)^2\right\} \)
Proof (a) Let $S$ be a 0-ve-dominating set of $T_v$ of weight $\gamma_{ve}^0(T_v)$. Clearly, $v \in S$. Let $S = S_v \cup S_u$, where $S_v$ and $S_u$ are the vertices of $T'_v$ and $T'_u$, respectively. Note that, $S_v$ is a ve-dominating set of $T'_v$ containing $v$. If every edge of $T'_u$ are ve-dominated by $S_u$, then $S_u$ is a ve-dominating set of $T'_u$. Otherwise, assume that, there exists an edge $e \in T'_u$ which is not ve-dominated by $S_u$. Since, $S$ is a ve-dominating set of $T_v$, it follows that $e$ must be incident to $u$. Hence, $S_u$ is a ve-dominating set of $T'_u - \{u\}$ but not a ve-dominating set of $T'_u$. Therefore, $S_u$ is a ve-set of $T'_u$. So, we have, $\gamma_{ve}(T'_u) = \gamma_{ve}(T'_u) + \min\{\gamma_{ve}(T'_u), ve(T'_u)\}$.

Conversely, let $S_v$ be a 0-ve-dominating set of $T'_v$ of weight $\gamma_{ve}(T'_v)$ and $S_u$ be either a ve-dominating set of $T'_u$ of weight $\gamma_{ve}(T'_u)$ or a ve-set of $T'_u$ of weight $ve(T'_u)$. Then, it is easy to verify that $S = S_v \cup S_u$ is a 0-ve-dominating set of $T_v$. So, we have, $\gamma_{ve}(T'_v) = \gamma_{ve}(T'_v) + \min\{\gamma_{ve}(T'_v), ve(T'_v)\}$.

(b) Let $S$ be a 1-ve-dominating set of $T_v$ of weight $\gamma_{ve}^1(T_v)$ and let $S = S_v \cup S_u$, where $S_v$ and $S_u$ are the vertices of $T'_v$ and $T'_u$, respectively. Clearly, $v \notin S_u$. So, $S_v$ either a 1-ve-dominating set or a 2-ve-dominating set or a ve-set of $T'_v$. If $S_v$ is a 1-ve-dominating set, then the edge $vu$ is dominated by $S_v$. So, $S_u$ is a ve-dominating set of $T'_u$. If $S_v$ is a 2-ve-dominating set, then $u \in S_v$. So, $S_u$ is a 0-ve-dominating set of $T'_u$. And finally, if $S_v$ is a ve-set of $T'_v$, then also $u \in S_u$. So, $S_u$ is a 0-ve-dominating set of $T'_u$. Hence, $\gamma_{ve}^1(T_v) = \min\{\gamma_{ve}^1(T'_v), (\gamma_{ve}(T'_v) + \gamma_{ve}(T'_v))\}$.

Conversely, let $S_v$ be a 1-ve-dominating set of $T'_v$ of weight $\gamma_{ve}^1(T'_v)$ and $S_u$ be a ve-dominating set of $T'_u$ of weight $\gamma_{ve}(T'_u)$. Then $S = S_v \cup S_u$ is a 1-ve-dominating set of $T_v$ because all the edges incident to the vertex $v$ are ve-dominated by $S_v$ as the minimum distance between $v$ and $S_v$ is 1. Also all other edges are ve-dominated either by $S_v$ or by $S_u$ and the distance between $S$ and $v$ remains 1. Also, if $S_v$ is either a 2-ve-dominating set of $T'_v$ of weight $\gamma_{ve}^2(T'_v)$ or a ve-set of $T'_v$ of weight $ve(T'_v)$ and $S_u$ is a 0-ve-dominating set of $T'_u$ of weight $\gamma_{ve}^0(T'_u)$, then also $S = S_v \cup S_u$ is a 1-ve-dominating set of $T_v$. So, we have, $\gamma_{ve}^1(T_v) = \min\{\gamma_{ve}^1(T'_v) + \gamma_{ve}(T'_v)\}$.

(c) Let $S$ be a 2-ve-dominating set of $T_v$ of weight $\gamma_{ve}^2(T_v)$ and let $S = S_v \cup S_u$, where $S_v$ and $S_u$ are the vertices of $T'_v$ and $T'_u$, respectively. Clearly $u \notin S_u$. Since, no edge of $T'_v$ is ve-dominated by $S_u$, $S_v$ is a 2-ve-dominating set of $T'_v$. Note that, the edge $vu$ is not ve-dominated by $S_u$, rather it is ve-dominated by $S_u$. So, $S_u$ is a 1-ve-dominating set of $T'_u$. Hence, $\gamma_{ve}^2(T_v) = \gamma_{ve}^2(T'_u) + \gamma_{ve}(T'_u)$. Conversely, $S_v$ be a 2-ve-dominating set of $T'_v$ of weight $\gamma_{ve}^2(T'_v)$ and $S_u$ be a 1-ve-dominating set of $T'_u$ of weight $\gamma_{ve}^1(T'_u)$. Then, it is easy to verify that $S = S_v \cup S_u$ is a 2-ve-dominating set of $T_v$. So, we have, $\gamma_{ve}^2(T_v) = \gamma_{ve}^2(T'_v) + \gamma_{ve}(T'_v)$.

(d) Let $S$ be a ve-set of $T_v$ of weight $ve(T_v)$ and let $S = S_v \cup S_u$, where $S_v$ and $S_u$ are the vertices of $T'_v$ and $T'_u$, respectively. Clearly $u \notin S_u$. If all edges of $T'_v$ and $T'_u$ are ve-dominated by $S$, then $S_v$ and $S_u$ are 2-ve-dominating sets of $T'_v$ and $T'_u$, respectively. If some edges in $T'_v$ are not ve-dominated by $S$, then these edges must be adjacent to $v$. So, $S_v$ is a ve-set of $T'_v$. In this case, $S_u$
must be either a 1-ve-dominating set or a 2-ve-dominating set of $T'_u$. Hence, 
\[ ve(T_v) \geq \min \left\{ \left( \gamma^2_{ve}(T'_v) + \gamma^2_{ve}(T'_u) \right), \left( ve(T'_v) + \min\{\gamma^1_{ve}(T'_u), \gamma^2_{ve}(T'_u)\} \right) \right\}. \]

Conversely, let $S_v$ be a 2-ve-dominating set of $T'_v$ of weight $\gamma^2_{ve}(T'_v)$ and $S_u$ be a 2-ve-dominating set of $T'_u$ of weight $\gamma^2_{ve}(T'_u)$. Then, all edges of $T_v$ expect $vu$ are ve-dominated by $S = S_v \cup S_u$. So, $S$ is a ve-set of $T_v$. Also, if $S_v$ is a ve-set of $T'_v$ of weight $ve(T'_v)$ and $S_u$ is either a 1-ve-dominating set of $T'_u$ of weight $\gamma^1_{ve}(T'_u)$ or a 2-ve-dominating set of $T'_u$ of weight $\gamma^2_{ve}(T'_u)$, then also $S = S_v \cup S_u$ is a ve-set of $T_v$ as there exists at least one edge $vw$ incident to $v$ that is not ve-dominated by $S$ but all edges not incident to $v$ are ve-dominated by $S$ (Using $S_v$ or $S_u$). Hence, 
\[ ve(T_v) \leq \min \left\{ \left( \gamma^2_{ve}(T'_v) + \gamma^2_{ve}(T'_u) \right), \left( ve(T'_v) + \min\{\gamma^1_{ve}(T'_u), \gamma^2_{ve}(T'_u)\} \right) \right\}. \]

\[ \square \]

A support vertex $v$ is a vertex in a tree such that $v$ is adjacent to some leaf vertices. Next, we describe the algorithm for finding minimum weighted ve-dominating set of a tree. The input to the algorithm is a weighted tree $T_r = (V, E)$ rooted at a vertex $r$ and a weight $w(v)$ associated with every vertex $v \in V$. First, we assign a four tuple $L(v) = (l^0_v, l^1_v, l^2_v, l^{ve}_v)$ to each non-leaf vertex of $T_r$ as follows:

- if $v$ is a support vertex, then $L(v) = (w(v), w(c), \infty, 0)$, where $c$ is a child of $v$ having minimum weight.
- if $v$ is not a support vertex, then $L(v) = (\infty, \infty, \infty, \infty)$.

At each iteration, we process a non-leaf vertex in the reverse BFS order to ensure that before processing any vertex $v$, all children of $v$ must be processed. After the $i$th iteration, let $T'_v$ denote the subtree rooted at $v$ such that all its descendant are processed vertices. Since, we are not processing any leaf in the algorithm, we consider the leafs to be processed before the first iteration. Note that, the root vertex $v$ of $T'_v$ may not be a processed vertex. After the $i$th iteration, $l^j_v$ represents the value of $\gamma^j_{ve}(T'_v)$ for each $j = 0, 1, 2$ and $l^{ve}_v$ represents the value of $ve(T'_v)$. While processing a vertex $u$, we actually explore the edge $vu$ and update $L(v)$, where $v$ is the parent of $u$. If $L(v) = (\infty, \infty, \infty, \infty)$, then we update $L(v)$ based on Lemma 15, otherwise we apply Lemma 16. We repeat the process till all the vertices, except the root $r$, are processed. Hence, the total number of iterations is $t = |V| - |L| - 1$, where $L$ is the set of leaves of $T_r$ if $r$ is not a leaf, otherwise $t = |V| - |L|$. Clearly, $T'_r$ is nothing but the rooted tree $T_r$. Hence, at the end of the algorithm, $l^j_v$ represents the value of $\gamma^j_{ve}(T_r)$ for all $j = 0, 1, 2$. Then, we compute $\gamma_{ve}(T)$ by using Lemma 14. The outline of the algorithm is given below in Algorithm 3.

Note that the initialization in line 2–6 takes $O(n)$ time, where $n$ is the number of vertices. The updates in line 9 and 11 takes constant time. Hence, the for loop in line 7–11 takes also $O(n)$ time. Therefore, we have the following theorem.

**Theorem 17** The weighted ve-domination problem for trees can be solved in linear time.
5 Undirected path graphs

In this section, we prove that the ve-domination problem for undirected path graphs is NP-complete by showing a polynomial-time reduction from 3-dimensional matching problem, which is a well-known NP-complete problem (Garey and Johnson 1990).

A graph $G$ is called an undirected path graph if $G$ is the intersection graphs of a family of paths of a tree. In Gavril (1975), Gavril proved that a graph $G = (V, E)$ is an undirected path graph if and only if there exists a tree $T$ whose vertices are the maximal cliques of $G$ and the set of all maximal cliques containing a particular vertex $v$ of $V$ forms a path in $T$. This tree is called the clique tree of the undirected path graph $G$. The 3-dimensional matching problem is as follows: given a set $M \subseteq U \times V \times W$, where $U$, $V$ and $W$ are disjoint set with $|U| = |V| = |W| = q$, does $M$ contain a matching $M'$, i.e., a subset $M' \subseteq M$ such that $|M'| = q$ and no two elements of $M'$ agree in any coordinate?

**Theorem 18** The ve-domination problem is NP-complete for undirected path graphs.

**Proof** It is easy to see that ve-domination problem is in NP. Now, we describe polynomial reduction form 3-dimensional matching problem to ve-domination problem in undirected path graph. Let $U = \{u_r | 1 \leq r \leq q\}$, $V = \{v_s | 1 \leq s \leq q\}$, $W = \{w_t | 1 \leq t \leq q\}$, and $M = \{m_i = (u_r, v_s, w_t) | 1 \leq i \leq p, u_r \in U, v_s \in V, w_t \in W\}$ be an instance of 3-dimensional matching problem. Now we construct a tree $T$ having $8p + 6q + 1$ vertices that becomes the clique tree of an undirected path graph $G$. The vertices of the tree $T$ are maximal cliques of $G$. The vertex set and the edge set are as follows:

For $1 \leq i \leq p$, each $m_i = (u_r, v_s, w_t) \in M$ corresponds to 8 cliques which are vertices of $T$, namely $\{A_i, B_i, C_i, D_i\}, \{A_i, B_i, D_i, F_i\}, \{C_i, D_i, G_i\}, \{A_i, B_i, E_i\}, \{C_i, G_i, K_i\}, \{A_i, E_i, H_i\}, \{B_i, E_i, I_i\}$, and $\{B_i, I_i, J_i\}$. These vertices depend only on the triple $m_i$ itself but not on the elements within the triple. These eight vertices induces a subtree corresponding to $m_i$ as illustrated in Fig. 4. Further, for each $u_r \in U$, $1 \leq r \leq q$, we take two cliques $\{R_r\}$ and $\{\{A_i | u_r \in m_i\} | R_r, X_r\}$ which are vertices of $T$ forming a subtree as shown in Fig. 4. Similarly, for each $v_s \in V$, $1 \leq s \leq q$. 

\begin{algorithm}
\caption{Min\_Weighted\_VEDom($T$, $w$)}
\begin{algorithmic}[1]
\Require A rooted tree $T_r = (V, E)$ and the weight function $w(\cdot)$ on $V$
\Ensure Minimum weight of a ve-dominating set of $T_r$
\State 1: Let $\delta$ be the reverse BFS ordering of $V$
\State 2: for (each internal vertex $u$) do
\State 3: \quad if ($u$ is support vertex) then
\State 4: \quad \quad $L(u) = (w(u), w(c), \infty, 0)$ \quad \Comment$c$ is the child of $u$ having minimum weight\n\State 5: \quad else
\State 6: \quad \quad $L(u) = (\infty, \infty, \infty, \infty)$
\State 7: for (each non-leaf vertex $u$ in $\delta$ except $r$) do
\State 8: \quad if $L(v) = (\infty, \infty, \infty, \infty)$ then \Comment$v$ is the parent of $u$
\State 9: \quad \quad Update $L(v)$ using Lemma 15
\State 10: \quad else
\State 11: \quad \quad Update $L(v)$ using Lemma 16
\State 12: Return $(\min(l^0, l^1, l^2))$.
\end{algorithmic}
\end{algorithm}
and \( w_t \in W, 1 \leq t \leq q \), we add the cliques \( \{S_i\} \cup \{B_i| v_t \in m_i\}, \{S_y, Y_k\} \) and \( \{T_i\} \cup \{C_i| w_t \in m_i\}\), respectively to the tree \( T \) as shown in Fig. 4. Finally, \( \{A_i, B_i, C_i| 1 \leq i \leq p\} \) is the last vertex of tree \( T \). The construction of \( T \) is illustrated in Fig. 4.

Hence, \( T \) is the clique tree of the undirected path graph \( G \) whose vertex set is

\[
\{A_i, B_i, C_i, D_i, E_i, F_i, G_i, H_i, I_i, J_i, K_i| 1 \leq i \leq p\} \cup \{R_j, S_j, T_j, X_j, Y_j, Z_j : 1 \leq j \leq q\}.
\]

**Claim 19** The graph \( G \) has a ve-dominating set of size \( 2p + q \) if and only if 3-dimensional matching has a solution.

**Proof** Let \( D \) be a ve-dominating set of \( G \) of size \( 2p + q \). For any \( i \in \{1, 2, \ldots, p\} \), the only way to ve-dominate the edge-set of the subgraph induced by the vertex set \( \{A_i, B_i, C_i, D_i, E_i, F_i, G_i, H_i, I_i, J_i, K_i\} \) corresponding to \( m_i \) with two vertices is to choose \( D_i \) and \( E_i \). Hence, to ve-dominate the edge-set of that induced subgraph by any larger vertex set, at least three vertices has to be taken. Note that, the set \( \{A_i, B_i, C_i\} \) ve-dominates the edge-set of that induced subgraph. So, without loss of generality, we assume that \( D \) consists of \( A_i, B_i, C_i \) for \( t \) many \( m_i \)’s and \( D_i, E_i \) for \( (p-t) \) many other \( m_i \)’s. Also to ve-dominate the edges of the form \( R_j, X_j, S_yY_s \) and \( T_tZ_t \), \( D \) contains at least \( \max\{3(q-t), 0\} \) many vertices (namely, \( R_j \) or \( X_j, S_y \) or \( Y_k, T_t \) or \( Z_t \)). Hence, we have,

\[
2p + q = |D| \geq 3t + 2(p-t) + 3(q-t) = 2p + 3q - 2t.
\]

So, \( t \geq q \), i.e. \( D \) must contain at least \( q \) many \( A_i, B_i, C_i \). Picking the corresponding \( m_i \)’s form a matching \( M' \) of size \( q \).
Hence, the ve-domination problem is NP-complete for undirected path graph.

6 Trees with equal $γ_{ve}$ and $i_{ve}$

For every graph $G$, the independent ve-domination number is obviously at least as large as the ve-domination number. In this section, we characterize the trees for which these two parameters $γ_{ve}$ and $i_{ve}$ are equal. We start with some pertinent definitions.

**Definition 20** An atom $A$ is a tree with at least 3 vertices with a vertex, say $c$, designated as center of the atom such that distance of every vertex from $c$ is at most 2.

We denote an atom along with its center by $(A, c)$. Note that the center $c$ ve-dominates all edges of the atom $A$. Next, we define an operation for joining two atoms to construct another tree.

**Definition 21** Let $(A', c')$ and $(A, c)$ be two atoms along with their centers $c'$ and $c$, respectively. For some $i, j \in \{0, 1, 2\}$, we define $(A', c') (i - j)$-join $(A, c)$ as the addition of an edge $(x_{c'}, x_c)$ between the vertices $x_{c'} \in V(A')$ and $x_c \in V(A)$ such that $\text{dist}_{A'}(c', x_{c'}) = i$ and $\text{dist}_{A}(c, x_c) = j$.

Let $S \subseteq V$. We say that a vertex $v \in S$ has a private edge $e \in E$ with respect to $S$ if $e$ is ve-dominated only by $v$. An edge $e \in E$ is said to be pendant if one of its vertices is a pendant vertex. An edge $e = xy$ is called a distance-1 edge of $v$ if $\min\{\text{dist}(v, x), \text{dist}(v, y)\} = 1$. An edge $e$ is called a distance-1 private edge of $v \in S$ with respect to the set $S$ if $e$ is a private edge of $v$ with respect to $S$ and $e = xy$ is a distance-1 edge of $v$. Next, we give the recursive definition of a family of trees, say $\mathcal{T}$, using the notion of atom and $(i - j)$-join (Figs. 5, 6, 7).

**Definition 22** The recursive definition of the family $\mathcal{T}$ of trees is as follows:

1. Every atom $(A, c) \in \mathcal{T}$ and
2. Let $T' \in \mathcal{T}$ and $(A', c')$ be an atom in $T'$ and $S'$ be the set of all atom centers in $T'$. Then $T \in \mathcal{T}$ if and only if $T$ is obtained by joining an edge between $x_c \in (A, c)$ and $x_{c'} \in (A', c')$ satisfying one of the following rules:

   (i) $(A', c') (0 - 1)$-join $(A, c)$:
   
   (a) $c'$ has a neighbour $y$ such that all edges incident to $y$, except $yc'$, are pendant edges and $c$ has no distance-1 edge.
   
   (b) $c'$ has distance-1 private edges with respect to $S'$ and $c$ has at least one distance-1 edge which is not incident to $x_c$.
   
   (c) $c'$ has no distance-1 private edge with respect to $S'$ and $c$ has at least one distance-1 edge which is not incident to $x_c$.

   (ii) $(A', c') (1 - 0)$-join $(A, c)$:
(a) $c'$ has a neighbour $y$ such that all edges incident to $y$, except $c'y$, are pendant as well as private edges of $c'$ with respect to $S'$ and $c$ has distance-1 edges.
(b) $c'$ has a neighbour $y$ ($\neq xc'$) such that $y$ is a pendant vertex and $c$ has distance-1 edges.
(iii) $(A', c') (1 - 1)$-join $(A, c)$:
$c'$ has a private edge with respect to $S'$ and $c$ has at least one distance-1 edge which is not incident to $xc$.
(iv) $(A', c') (2 - 1)$-join $(A, c)$:
$c'$ has a private edge with respect to $S'$ and $c$ has at least one distance-1 edge which is not incident to $xc$.

Now, we show that if $T \in \mathcal{T}$, then \textit{ve-domination number} and \textit{independent ve-domination number} are same.

\begin{lemma}
Let $T \in \mathcal{T}$. If $T$ is obtained from $T' \in \mathcal{T}$ by joining the atom $(A, c)$ with an atom $(A', c')$ of $T'$ satisfying the joining rules of Definition 22, then $\gamma_{ve}(T) = \gamma_{ve}(T') + 1$.
\end{lemma}

\begin{proof}
Let $S'$ be a minimum ve-dominating set of $T'$. Since no vertex of $T'$ can ve-dominate all edges of $(A, c)$, it is easy to verify that $S' \cup \{c\}$ is a ve-dominating set of $T$. Hence, $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Next, we show that for each of the joining rules of Definition 22, $\gamma_{ve}(T) \geq \gamma_{ve}(T') + 1$.

\textbf{(0 - 1)(a)-join:} Let $S$ be a minimum ve-dominating set of $T$. Note that $S$ contains exactly one vertex of $(A, c)$, say $p$, to ve-dominate all edges of $y$.
(A, c). Suppose \(x_1, x_2, \ldots, x_t\) are the pendant vertices adjacent to \(y\).
So, to ve-dominate these pendant edges, \(S\) must contain one vertex, say \(q\), from \(\{x_1, x_2, \ldots, x_t, y, c'\}\).
Since the set of edges that are ve-dominated by \(\{p, q\}\) can also be ve-dominated by \(\{c, c'\}\), \(S'' = (S\setminus\{p, q\}) \cup \{c, c'\}\) is also a minimum ve-dominated set of \(T\).
It is easy to verify that \(S''\setminus\{c\}\) is a ve-dominated set of \(T'\). Hence \(\gamma_{ve}(T) \geq \gamma_{ve}(T') + 1\).

**\((0 - 1) (b)\)-join:** Let \(S\) be a minimum ve-dominated set of \(T\). If \(S\) contains exactly one vertex from \((A, c)\), say \(p\), then clearly \(S\setminus\{p\}\) is a ve-dominated set of \(T'\). So, assume that, \(S\) contains two vertices, say \(p\) and \(q\), from \((A, c)\). Since the set of edges that are ve-dominated by \(\{p, q\}\) can also be ve-dominated by \(\{c, c'\}\), \(S'' = (S\setminus\{p, q\}) \cup \{c, c'\}\) is also a minimum ve-dominated set of \(T\). Now, it is easy to verify that \(S''\setminus\{c\}\) is a ve-dominated set of \(T'\). Hence \(\gamma_{ve}(T) \geq \gamma_{ve}(T') + 1\).

**\((0 - 1) (c)\)-join:** Same as **\((0 - 1) (b)\)-join**.

**\((1 - 0) (a)\)-join:** Let \(S\) be a minimum ve-dominated set of \(T\). Note that \(S\) contains exactly one vertex of \((A, c)\), say \(p\), to ve-dominate all edges of \((A, c)\). Suppose \(x_1, x_2, \ldots, x_t\) are the pendant vertices adjacent to \(y\).
So, to ve-dominate these pendant edges, \(S\) must contain one vertex, say \(q\), from \(\{x_1, x_2, \ldots, x_t, y, c'\}\). Since the set of edges that are ve-dominated by \(\{p, q\}\) can also be ve-dominated by \(\{c, c'\}\), \(S'' = (S\setminus\{p, q\}) \cup \{c, c'\}\) is also a minimum ve-dominated set of \(T\). It is easy to verify that \(S''\setminus\{c\}\) is a ve-dominated set of \(T'\). Hence \(\gamma_{ve}(T) \geq \gamma_{ve}(T') + 1\).

**\((1 - 0) (b)\)-join:** Let \(S\) be a minimum ve-dominated set of \(T\). Note that \(S\) contains exactly one vertex of \((A, c)\), say \(p\), to ve-dominate all edges of \((A, c)\). Since \(p\) cannot ve-dominate the edge \(c'\), \(S\) must contain a vertex, say \(q\), to ve-dominate this edge. Note that \(q \in N[c']\) because \(y\) is leaf vertex and \(q\) also ve-dominate the edge \(c'x_c\). Since the set of edges that are ve-dominated by \(p\) can also be ve-dominated by \(c, c'\), \(S'' = (S\setminus\{p\}) \cup \{c, c'\}\) is also a minimum ve-dominated set of \(T\). It is easy to see that \(S''\setminus\{c\}\) is a ve-dominated set of \(T'\). Hence \(\gamma_{ve}(T) \geq \gamma_{ve}(T') + 1\).

**\((1 - 1)\)-join:** Let \(S\) be a minimum ve-dominated set of \(T\). If \(S\) contains exactly one vertex from \((A, c)\), say \(p\), then clearly \(S\setminus\{p\}\) is a ve-dominated set of \(T'\). So, assume that, \(S\) contains two vertices, say \(p\) and \(q\), from \((A, c)\). Since the set of edges that are ve-dominated by \(\{p, q\}\) can also be ve-dominated by \(\{c, c'\}\), \(S'' = (S\setminus\{p, q\}) \cup \{c, c'\}\) is also a minimum ve-dominated set of \(T\). Now, it is easy to verify that \(S''\setminus\{c\}\) is a ve-dominated set of \(T'\). Hence \(\gamma_{ve}(T) \geq \gamma_{ve}(T') + 1\).

**\((2 - 1)\)-join:** Same as **\((1 - 1)\)-join**.

Hence, if \(T\) is obtained from \(T' \in \mathcal{T}\) by joining the atom \((A, c)\) with an atom \((A', c')\) of \(T'\) satisfying the joining rules of Definition 22, then \(\gamma_{ve}(T) = \gamma_{ve}(T') + 1\).

\(\square\)
Lemma 24 If $T \in \mathcal{T}$, then the set of all atom centers of $T$ forms a minimum ve-dominating set of $T$.

Proof We prove this by induction on the number of atoms in $T$. Clearly, when $T$ is an atom, the hypothesis is true. Let $T \in \mathcal{T}$ be a tree containing $k$ atoms and $T$ is obtained from $T' \in \mathcal{T}$ by joining the atom $(A, c)$ with an atom $(A', c')$ of $T'$ satisfying the joining rules. Let $S$ and $S'$ be the atom centers of $T$ and $T'$, respectively. Clearly, $S = S' \cup \{c\}$. By induction hypothesis, $S'$ is a minimum ve-dominating set of $T'$. It is easy to verify that $S$ is a ve-dominating set of $T$. By Lemma 23, it follows that $S$ is also a minimum ve-dominating set of $T$. □

Theorem 25 For all $T \in \mathcal{T}$, $\gamma_{ve}(T) = i_{ve}(T)$.

Proof By Lemma 24, the set of atom centers, say $S$, is also a minimum ve-dominating set. By the definition of $\mathcal{T}$, the distance between any two atom centers in $T$ is at least 2. Hence, $S$ is also an independent set. Hence, $\gamma_{ve}(T) = i_{ve}(T)$ for all $T \in \mathcal{T}$. □

To show the converse of Theorem 25, we first prove following lemma that allow us to construct a minimum independent ve-dominating set of a tree having some desirable properties.

Lemma 26 For any tree $T$ $(n \geq 3)$, there exists a minimum independent ve-dominating set which does not contain any leaf.

Proof Let $S$ be a minimum independent ve-dominating set of $T$. If $S$ does not contain any leaf, then we are done. Otherwise assume that $S$ contains a leaf, say $x$. Let the neighbour of $x$ is $y$ and $N_T(y) = \{z_1, z_2, \ldots, z_q, x\}$. Since $S$ is an independent set, $y \notin S$. Also, none of the $z_i$ are in $S$, because if any of $z_i \in S$, then $S\backslash\{x\}$ is also an independent ve-dominating set. Hence, $(S\backslash\{x\})\cup\{y\}$ is another minimum independent ve-dominating set of $T$. Repeating this process, we can form a minimum independent ve-dominating set of $T$ which does not contain any leaf. □

Now, we are ready to show that the converse of Theorem 25 is also true. Let $T_r$ be a tree rooted at a vertex $r$. The level of a vertex $v$, denoted by $l(v)$, in $T_r$ is the length of the unique path from the root $r$ to $v$. The height of $T_r$ is the maximum of the levels of vertices of $T_r$. The parent of a vertex $v$ is the vertex adjacent to $v$ on the path to the root and it is denoted by $p(v)$. If $u$ is the parent of $v$, then $v$ is a child of $u$ and it is denoted by $child(u)$. Two vertices are called siblings if they belong to same parent.

Theorem 27 If $\gamma_{ve}(T) = i_{ve}(T)$ for a tree $T$ with $n \geq 3$, then $T \in \mathcal{T}$.

Proof We prove this by induction on $k$, the size of minimum (independent) ve-dominating set. For base case, when $\gamma_{ve}(T) = i_{ve}(T) = 1$, then $T$ is an atom. Hence $T \in \mathcal{T}$. As induction hypothesis, let us assume that if $\gamma_{ve}(T) = i_{ve}(T) = k - 1$, then $T \in \mathcal{T}$. Note that, by Lemma 24, the set of all atom centers of $T$ forms a minimum (independent) ve-dominating set of $T$.

For our convenience, we assume that $T$ is a tree rooted at a vertex $r$ having height $h$. Let $S$ be a minimum (independent) ve-dominating set of $T$ such that $|S| = k$.
By Lemma 26, we also assume that $S$ has no vertices from $h^{th}$-level. Moreover, we also assume that $S$ is a minimum (independent) ve-dominating set of $T$ such that level of any vertex of $S$ cannot be decreased without violating independence and ve-dominating properties. Consider a vertex $s \in S$ that has maximum level and has a descendant at $h^{th}$-level. Clearly, $l(s) = h - 1$ or $l(s) = h - 2$. Let $s'$ be the nearest vertex to $s$ such that $s' \in S$. Obviously, $\text{dist}_T(s, s') \in \{2, 3, 4\}$ because, otherwise, $S$ would not be an independent ve-dominating set. Next we show that there exists an edge $e$ in $T$ whose removal forms another tree $T'$ such that $S\setminus \{s\}$ is a minimum (independent) ve-dominating set of $T'$ and an atom $(A, c)$ whose center is $s$. Now by induction hypothesis, $T' \in \mathcal{T}$. Finally we show that there exists an atom $(A', c')$ in $T'$ such that $T$ can be formed by joining the edge $e$ which satisfies the rules defined in Definition 22. The proof is divided into three cases according to the distance between $s$ and $s'$. Further, each case is divided according to the levels $l(s)$ and $l(s')$.

**Case (A): $\text{dist}_T(s, s') = 2$**

**Case (1): $l(s) = h - 1$ and $l(s') = h - 1$**

This case is not possible because $(S\setminus \{s, s'\}) \cup \{p(s)\}$ is a ve-dominating set of size less than $k$. This contradicts the minimality of $S$.

**Case (2): $l(s) = h - 1$ and $l(s') = h - 3$**

Since $l(s')$ cannot be decreased, either there exists $s'' \in S$ such that $l(s'') \leq l(s')$ and $\text{dist}_T(s', s'') = 2$ or $s'$ has a distance-1 pendant private edge with respect to $S$. In the first case, note that $S\setminus \{s'\}$ becomes a minimum ve-dominating set contradicting the minimality of $S$. Moreover, if all edges incident to $s'$ are pendant private edges with respect to $S$, then $(S\setminus \{s, s'\}) \cup p(s)$ becomes a ve-dominating set contradicting the minimality of $S$. Hence let us assume that $s'$ has a distance-1 pendant private edge with respect to $S$. Let $pq$ be that edge with $q$ as its pendant vertex. Let us remove the edge $s'pq(s)$ to form a tree $T'$ rooted at $r$ and an atom $(A, c)$ which is the subtree rooted at $p(s)$. Clearly, $S\setminus \{s\}$ is a minimum (independent) ve-dominating set of $T'$. By induction hypothesis, $T' \in \mathcal{T}$. Let the edge $pq$ be dominated by the vertex $c'$ which is the center of atom $(A', c')$ in $T'$.

**Case (i): $c' = p$**

In this case, let us consider the atom center of $(A, c)$ as $c = p(s)$. The edge $s'pq(s)$ satisfies the joining rule $(1 - 0)(b)$ between $(A', c')$ and $(A, c)$.

**Case (ii): $c' = s'$**

In this case, let us consider the atom center of $(A, c)$ as $c = s$. The edge $s'p(s)$ satisfies the joining rule $(0 - 1)(a)$ between $(A', c')$ and $(A, c)$.

**Case (3): $l(s) = h - 2$ and $l(s') = h - 2$**

**Case (a): $s$ has a sibling $s' \in S$ such that all edges incident to $s'$, except $p(s)s'$, are pendant**

Note that there is exactly one such sibling of $s$. Otherwise, we can remove all such siblings of $s$ from $S$ and include $p(s)$ into $S$ to form a ve-dominating set of $T$ which contradicts the minimality of $S$. Let us remove the edge $p(s)s$ to form a tree $T'$ rooted at $r$ and an atom $(A, c)$ which is the subtree rooted at $s$. Clearly, $S\setminus \{s\}$ is a minimum (independent) ve-dominating set of $T'$. By induction hypothesis, $T' \in \mathcal{T}$. Let the pendant...
edges incident to $s'$ be dominated by the vertex $c'$ which is the center of atom $(A', c')$ in $T'$.

**Case (i):** $c' = s'$

In this case, let us consider the atom center of $(A, c)$ as $c = s$. The edge $p(s)s$ satisfies the joining rule $(1 - 0)(b)$ between $(A', c')$ and $(A, c)$.

**Case (ii):** $c' = p(s')$:

Let us construct an alternating sequence of vertices of $S$ and atom centers of $T'$ as follows: start with $s'$ and $c'$. Now, there is an edge $e'$ such that $e'$ is private to $c'$ but not dominated by $s'$. If such an edge does not exist, we can shift the atom center $c'$ to $s'$. Let the edge $e'$ be dominated by another vertex $s_1 \in S$. Corresponding to this $s_1$, there exists another atom center $c_1$ which has a private edge $e_1$ but not dominated by $\{s', s_1\}$. Now to dominate this $e_1$, we get another vertex $s_2 \in S$ and corresponding to the $s_2$, we get another $c_2$ and $e_2$ with same properties. We continue this process till we get an atom center $c_t$ whose all private edges are dominated by $\{s', s_1, \ldots, s_t\}$. Such a sequence ending with an atom center always exists because otherwise cardinality of $S$ would be greater than the cardinality of atom centers of $T'$, contradicting the minimality of $S$. Now we shift the atom centers $c', c_1, \ldots, c_t$ by one edge towards the direction of $s'$. The relative joining rules will be unchanged between the atom centers. After this shifting, the atom center $c'$ is now at $s'$. Hence, this case can be reduced to Case(A)(3)(a)(i).

**Case (b):** $s$ has no sibling $s' \in S$ such that all edges incident to $s'$, except $p(s')s'$, are pendant

In this case, $s'$ has a distance-1 pendant private edge with respect to $S$. Let that edge be $pq$ with $q$ as its pendant vertex. Let us remove the edge $p(s)$ to form a tree $T'$ rooted at $r$ and an atom $(A, c)$ which is the subtree rooted at $s$. Clearly, $S\setminus\{s\}$ is a minimum (independent) ve-dominating set of $T'$. By induction hypothesis, $T' \in \mathcal{T}$. Let the edge $pq$ be dominated by the vertex $c'$ which is the center of atom $(A', c')$ in $T'$.

**Case (i):** $c' = p$

Note that, in $T'$, the only possible way to join the atom $(A', c')$ is by using $(0 - 1)(a)$ joining rule. This implies that $p(s')$ must have a neighbour $y$ such that all edges incident to $y$, except $p(s')y$, are pendant edges. Since $S$ is an independent ve-dominating set, $y$ must belong to $S$. This contradicts our assumption that $s$ has no sibling $s' \in S$ such that all edges incident to $s'$, except $p(s')s'$, are pendant.

**Case (ii):** $c' = s'$:

In this case, let us consider the atom center of $(A, c)$ as $c = s$. The edge $p(s)s$ satisfies the joining rule $(1 - 0)(a)$ between $(A', c')$ and $(A, c)$.

**Case (4):** $l(s) = h - 2$ and $l(s') = h - 4$

In this case, $s'$ must have distance-0 and/or distance-1 pendant private edges with respect to $S$. 

\[ \square \]
Case (a): $s'$ has a distance-1 pendant private edge with respect to $S$

Let $pq$ be that edge with $q$ as its pendant vertex. Let us remove the edge $s'p(s)$ to form a tree $T'$ rooted at $r$ and an atom $(A, c)$ which is the subtree rooted at $p(s)$. Clearly, $S\backslash \{s\}$ is a minimum (independent) ve-dominating set of $T'$. By induction hypothesis, $T' \in \mathcal{T}$. Let the edge $pq$ be dominated by the vertex $c'$ which is the center of atom $(A', c')$ in $T'$.

**Case (i): $c' = p$**

In this case, let us consider the atom center of $(A, c)$ as $c = s$. The edge $s'p(s)$ satisfies the joining rule $(1 - 1)$ between $(A', c')$ and $(A, c)$.

**Case (ii): $c' = s'$** In this case, let us consider the atom center of $(A, c)$ as $c = s$. The edge $s'p(s)$ satisfies the joining rule $(0 - 1)(b)$ between $(A', c')$ and $(A, c)$.

Case (b): $s'$ has no distance-1 pendant private edge with respect to $S$

In this case, all the pendant private edges are incident to $s'$. Let $q$ be a pendant vertex adjacent to $s'$. Let us remove the edge $s'p(s)$ to form a tree $T'$ rooted at $r$ and an atom $(A, c)$ which is the subtree rooted at $p(s)$. Clearly, $S\backslash \{s\}$ is a minimum (independent) ve-dominating set of $T'$. By induction hypothesis, $T' \in \mathcal{T}$. Let the edge $s'q$ be dominated by the vertex $c'$ which is the center of atom $(A', c')$ in $T'$.

**Case (i): $c' = p(s')$**

In this case, let us consider the atom center of $(A, c)$ as $c = s$. The edge $s'p(s)$ satisfies the joining rule $(1 - 1)$ between $(A', c')$ and $(A, c)$.

**Case (ii): $c' = s'$**

In this case, let us consider the atom center of $(A, c)$ as $c = s$. The edge $s'p(s)$ satisfies the joining rule $(0 - 1)(c)$ between $(A', c')$ and $(A, c)$.

Case (B): $\text{dist}_T(s, s') = 3$

**Case (1): $l(s) = h - 1$**

This case is not possible because in this case we can decrease $l(s)$. This contradicts the choice of $S$.

**Case (2): $l(s) = h - 2$ and $l(s') = h - 3$**

In this case, $s'$ must have distance-0 and/or distance-1 pendant private edges with respect to $S$.

**Case (a): $s'$ has a distance-1 pendant private edge with respect to $S$**

Let $pq$ be the distance-1 pendant private edge with $q$ as its pendant vertex. Let us remove the edge $p(s')p(s)$ to form a tree $T'$ rooted at $r$ and an atom $(A, c)$ which is the subtree rooted at $p(s)$. Clearly, $S\backslash \{s\}$ is a minimum (independent) ve-dominating set of $T'$. By induction hypothesis, $T' \in \mathcal{T}$.

Let the edge $pq$ be dominated by the vertex $c'$ which is the center of atom $(A', c')$ in $T'$.

**Case (i): $c' = s'$**

In this case, let us consider the atom center of $(A, c)$ as $c = s$. The
edge \( p(s')p(s) \) satisfies the joining rule \((1 - 1)\) between \((A', c')\) and \((A, c)\).

**Case (ii):** \( c' = p \)

In this case, let us consider the atom center of \((A, c)\) as \( c = s \). The edge \( p(s')p(s) \) satisfies the joining rule \((2 - 1)\) between \((A', c')\) and \((A, c)\).

**Case (b):** \( s' \) does not have any distance-1 pendant private edge with respect to \( S \)

In this case, all the pendant private edges are incident to \( s' \). Let \( q \) be a pendant vertex adjacent to \( s' \). Let us remove the edge \( p(s')p(s) \) to form a tree \( T' \) rooted at \( r \) and an atom \((A, c)\) which is the subtree rooted at \( p(s) \). Clearly, \( S\setminus\{s\} \) is a minimum (independent) ve-dominating set of \( T' \). By induction hypothesis, \( T' \in \mathcal{T} \). Let the edge \( s'q \) be dominated by the vertex \( c' \) which is the center of atom \((A', c')\) in \( T' \).

**Case (3):** \( l(s) = h - 2 \) and \( l(s') = h - 5 \)

**Case (a):** \( s' \) has a distance-1 pendant private edge with respect to \( S \)

Let \( pq \) be the distance-1 pendant private edge with \( q \) as its pendant vertex. Let us remove the edge \( p(p(s))p(s) \) to form a tree \( T' \) rooted at \( r \) and an atom \((A, c)\) which is the subtree rooted at \( p(s) \). Clearly, \( S\setminus\{s\} \) is a minimum (independent) ve-dominating set of \( T' \). By induction hypothesis, \( T' \in \mathcal{T} \). Let the edge \( pq \) be dominated by the vertex \( c' \) which is the center of atom \((A', c')\) in \( T' \).

**Case (i):** \( c' = s' \)

In this case, let us consider the atom center of \((A, c)\) as \( c = s \). The edge \( p(p(s))p(s) \) satisfies the joining rule \((1 - 1)\) between \((A', c')\) and \((A, c)\).

**Case (ii):** \( c' = p \)

In this case, let us consider the atom center of \((A, c)\) as \( c = s \). The edge \( p(p(s))p(s) \) satisfies the joining rule \((2 - 1)\) between \((A', c')\) and \((A, c)\).

**Case (b):** \( s' \) does not have any distance-1 pendant private edge with respect to \( S \)

Let us remove the edge \( p(p(s))p(s) \) to form a tree \( T' \) rooted at \( r \) and an atom \((A, c)\) which is the subtree rooted at \( p(s) \). Clearly, \( S\setminus\{s\} \) is a minimum (independent) ve-dominating set of \( T' \). By induction hypothesis, \( T' \in \mathcal{T} \). Let the edge \( s'p(p(s)) \) be dominated by the vertex \( c' \) which is the center of atom \((A', c')\) in \( T' \).
Case (C): $\text{dist}_T(s, s') = 4$

Case (1): $l(s) = h − 1$
This case is not possible because in this case we can decrease $l(s)$. This contradicts the choice of $S$.

Case (2): $l(s) = h − 2$ and $l(s') = h − 2$

Case (a): $s'$ has a distance-1 pendant private edge with respect to $S$
Let $pq$ be the distance-1 pendant private edge with $q$ as its pendant vertex. Let us remove the edge $p(p(s))p(s)$ to form a tree $T'$ rooted at $r$ and an atom $(A, c)$ which is the subtree rooted at $p(s)$. Clearly, $S\{s\}$ is a minimum (independent) ve-dominating set of $T'$. By induction hypothesis, $T' \in \mathcal{T}$. Let the edge $pq$ be dominated by the vertex $c'$ which is the center of atom $(A', c')$ in $T'$.

Case (i): $c' = s'$
In this case, let us consider the atom center of $(A, c)$ as $c = s$. The edge $p(p(s))p(s)$ satisfies the joining rule $(1 - 1)$ between $(A', c')$ and $(A, c)$.

Case (ii): $c' = p(s')$ or $c' = \text{child}(s') \neq p(p(s))$
In this case, let us consider the atom center of $(A, c)$ as $c = s$. The edge $p(p(s))p(s)$ satisfies the joining rule $(2 - 1)$ between $(A', c')$ and $(A, c)$.

Case (3): $l(s) = h − 2$ and $l(s') = h − 4$

Case (a): $s'$ has a distance-1 pendant private edge with respect to $S$
Let $pq$ be the distance-1 pendant private edge with $q$ as its pendant vertex. Let us remove the edge $p(p(s))p(s)$ to form a tree $T'$ rooted at $r$ and an atom $(A, c)$ which is the subtree rooted at $p(s)$. Clearly, $S\{s\}$ is a minimum (independent) ve-dominating set of $T'$. By induction hypothesis, $T' \in \mathcal{T}$. Let the edge $pq$ be dominated by the vertex $c'$ which is the center of atom $(A', c')$ in $T'$.
Hence,

Combining Theorem 25 and Theorem 27, we have the main result of this section.

**Case (i):** $c' = s'$
In this case, let us consider the atom center of $(A, c)$ as $c = s$. The edge $p(p(s))p(s)$ satisfies the joining rule $(2 - 1)$ between $(A', c')$ and $(A, c)$.

**Case (ii):** $c' = p$
Note that, in $T'$, the only possible way to join the atom $(A', c')$ is by using $(0 - 1)(a)$ joining rule. This implies that $p(s')$ must have a neighbour $y$ such that all edges incident to $y$, except $p(s')y$, are pendant edges and $p(s')$ is also the center of another atom $(A'', c'')$.

Also, let us consider the atom center of $(A, c)$ as $c = s$. The edge $p(p(s))p(s)$ satisfies the joining rule $(1 - 1)$ between $(A'', c'')$ and $(A, c)$.

**Case (b):** $s'$ does not have any distance-1 pendant private edge with respect to $S$
In this case, all the pendant private edges are incident to $s'$. Let $q$ be a pendant vertex adjacent to $s'$. Let us remove the edge $p(p(s))p(s)$ to form a tree $T'$ rooted at $r$ and an atom $(A, c)$ which is the subtree rooted at $p(s)$. Clearly, $S\setminus\{s\}$ is a minimum (independent) ve-dominating set of $T'$. By induction hypothesis, $T' \in \mathcal{T}$. Let the edge $s'q$ be dominated by the vertex $c'$ which is the center of atom $(A', c')$ in $T'$.

**Case (i):** $c' = s'$
In this case, let us consider the atom center of $(A, c)$ as $c = s$. The edge $p(p(s))p(s)$ satisfies the joining rule $(2 - 1)$ between $(A', c')$ and $(A, c)$.

**Case (ii):** $c' = p(s')$
In this case, let us consider the atom center of $(A, c)$ as $c = s$. The edge $p(p(s))p(s)$ satisfies the joining rule $(1 - 1)$ between $(A', c')$ and $(A, c)$.

**Case (4):** $l(s) = h - 2$ and $l(s') = h - 6$
Let us remove the edge $p(p(s))p(s)$ to form a tree $T'$ rooted at $r$ and an atom $(A, c)$ which is the subtree rooted at $p(s)$. Clearly, $S\setminus\{s\}$ is a minimum (independent) ve-dominating set of $T'$. By induction hypothesis, $T' \in \mathcal{T}$. Let the edge $p(p(s))p(p(s))$ be dominated by the vertex $c'$ which is the center of atom $(A', c')$ in $T'$.

**Case (i):** $c' = s$
In this case, let us consider the atom center of $(A, c)$ as $c = s$. The edge $p(p(s))p(s)$ satisfies the joining rule $(2 - 1)$ between $(A', c')$ and $(A, c)$.

**Case (ii):** $c' = p(p(p(s)))$
In this case, let us consider the atom center of $(A, c)$ as $c = s$. The edge $p(p(s))p(s)$ satisfies the joining rule $(1 - 1)$ between $(A', c')$ and $(A, c)$.

Therefore, in all the cases, we have shown that the tree $T$ is formed by joining an atom $(A, c)$ to an atom of a tree $T' \in \mathcal{T}$ satisfying the joining rules of Definition 22. Hence, $T \in \mathcal{T}$.

Combining Theorem 25 and Theorem 27, we have the main result of this section.
**Theorem 28** For a tree $T$ with at least 3 vertices, $\gamma_{ve}(T) = i_{ve}(T)$ if and only if $T \in \mathcal{T}$.

**7 Conclusions**

We have proposed linear time algorithms for solving minimum ve-domination problem and minimum independent ve-domination problem in block graphs. We have also solved the weighted version of ve-domination in trees. Further, we have strengthened the hardness result by showing that this problem remains NP-complete for undirected path graphs, an important subgraph of chordal graphs. Finally, we have characterized the trees with equal ve-domination and independent ve-domination number. It would be interesting to study this problem in other subclasses like interval graphs, directed path graphs etc. Also characterization of graphs having equal ve-domination parameters is another interesting problem.

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