Finite-dimensional irreducible modules of the Bannai–Ito algebra at characteristic zero

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Abstract
Assume that \( \mathbb{F} \) is algebraically closed with characteristic 0. A central extension \( \mathcal{BI} \) of the Bannai–Ito algebras is a unital associative \( \mathbb{F} \)-algebra generated by \( X, Y, Z \), and the relations assert that each of
\[
\{X, Y\} - Z, \quad \{Y, Z\} - X, \quad \{Z, X\} - Y
\]
is central in \( \mathcal{BI} \). In this paper, we classify the finite-dimensional irreducible \( \mathcal{BI} \)-modules up to isomorphism. As we will see, the elements \( X, Y, Z \) are not always diagonalizable on finite-dimensional irreducible \( \mathcal{BI} \)-modules.

Keywords Bannai–Ito algebra · Irreducible modules · Universal property

Mathematics Subject Classification 16G30 · 33D45

1 Introduction

Throughout this paper, we adopt the following notations: let \( \mathbb{F} \) denote a field and let \( \text{char} \mathbb{F} \) denote the characteristic of \( \mathbb{F} \). Let \( \mathbb{Z} \) denote the ring of integers. Let \( \mathbb{N} \) denote the set of nonnegative integers. Recall that the anticommutator \( \{X, Y\} \) of two elements \( X, Y \) in an algebra is defined by \( \{X, Y\} = XY + YX \).

In this paper, we consider a central extension of the Bannai–Ito algebras called the universal Bannai–Ito algebra and denoted by \( \mathcal{BI} \). The algebra \( \mathcal{BI} \) is a unital associative \( \mathbb{F} \)-algebra defined by generators and relations. The generators are \( X, Y, Z \), and the relations state each of
\[
\{X, Y\} - Z, \quad \{Y, Z\} - X, \quad \{Z, X\} - Y
\]
commutes with $X$, $Y$, $Z$. The concept of central extensions comes from [20]. The Bannai–Ito algebras are the case $q = -1$ of the Askey–Wilson algebras [21,23], and the corresponding orthogonal polynomials were first known in [1]. In [22], the Bannai–Ito algebras were reintroduced to connect the Dunkl shift operators and the Bannai–Ito polynomials. Recently, the Bannai–Ito algebras and their representation theory have been explored in many other subjects such as the additive DAHA of type $(\mathbb{C}_1^+, \mathbb{C}_1)$ [14,16], the Lie superalgebra $\mathfrak{osp}(1|2)$ [2,9], the Racah algebras [13,17] and the Brauer algebra [9]. The realizations of the Bannai–Ito algebras via Dunkl harmonic analysis on the two spheres were displayed in [6,11,12]. For generalizations of the Bannai–Ito algebras, please refer to [3,4,7,10].

Assume that $\mathbb{F}$ is algebraically closed with $\text{char } \mathbb{F} = 0$. It was falsely claimed in [15, Lemma 5.9] that $X$, $Y$, $Z$ are diagonalizable on each finite-dimensional irreducible $\mathfrak{BI}$-module and the mistake was used to classify even-dimensional irreducible $\mathfrak{BI}$-modules [15, Theorem 6.15] and odd-dimensional irreducible $\mathfrak{BI}$-modules [15, Theorem 7.5]. We display the following two examples to pinpoint the failure of [15, Theorem 6.15] and [15, Theorem 7.5], respectively:

**Example 1.1** It is routine to verify that there exists a four-dimensional $\mathfrak{BI}$-module $E$ that has an $\mathbb{F}$-basis $\{u_i\}_{i=0}^3$ with respect to which the matrices representing $X$, $Y$, $Z$ are

\[
\begin{pmatrix}
-\frac{1}{2} & 0 & 0 & 0 \\
1 & -\frac{1}{2} & 0 & 0 \\
0 & 1 & \frac{3}{2} & 0 \\
0 & 0 & 1 & -\frac{5}{2}
\end{pmatrix}, \quad
\begin{pmatrix}
-\frac{3}{2} & 1 & 0 & 0 \\
0 & \frac{1}{2} & 4 & 0 \\
0 & 0 & \frac{1}{2} & -3 \\
0 & 0 & 0 & -\frac{3}{2}
\end{pmatrix}, \quad
\begin{pmatrix}
-\frac{3}{2} & -1 & 0 & 0 \\
-1 & \frac{1}{2} & 4 & 0 \\
0 & 1 & -\frac{3}{2} & 3 \\
0 & 0 & -1 & \frac{1}{2}
\end{pmatrix},
\]

respectively. More precisely,

$$
\{X, Y\} = Z + 4, \quad \{Y, Z\} = X + 4, \quad \{Z, X\} = Y + 2
$$

on the $\mathfrak{BI}$-module $E$. It is straightforward to check that the minimal polynomials of $X$, $Y$, $Z$ on $E$ are

\[
\left(x - \frac{3}{2}\right) \left(x + \frac{1}{2}\right)^2 \left(x + \frac{5}{2}\right),
\]

\[
\left(x - \frac{1}{2}\right)^2 \left(x + \frac{3}{2}\right)^2,
\]

\[
\left(x - \frac{3}{2}\right) \left(x + \frac{1}{2}\right)^2 \left(x + \frac{5}{2}\right),
\]

respectively. Therefore, none of $X$, $Y$, $Z$ is diagonalizable on $E$. To see the irreducibility of $E$, we suppose that $W$ is any nonzero $\mathfrak{BI}$-submodule of $E$ and show that $W = E$. The element $Y$ has exactly two eigenvalues $-\frac{3}{2}$ and $\frac{1}{2}$ in $E$, and the $-\frac{3}{2}$- and $\frac{1}{2}$-eigenspaces of $Y$ in $E$ are of dimension 1 spanned by

$$
u_0, \quad \nu_0 + 2\nu_1,
$$
Finite-dimensional irreducible $\mathfrak{B}\mathfrak{J}$-modules at $F = 0$

respectively. Since $W$ is nonzero, $-\frac{3}{2}$ or $\frac{1}{2}$ is an eigenvalue of $Y$ in $W$. Therefore, $W$ contains $u_0$ or $u_0 + 2u_1$. If $u_0 + 2u_1 \in W$, then

$$u_0 = \frac{1}{3}(X - Z)(u_0 + 2u_1) \in W.$$ 

By these comments, $u_0 \in W$. The $\mathfrak{B}\mathfrak{J}$-module $E$ is generated by $u_0$, so $W = E$, a counterexample to [15, Theorem 6.15].

**Example 1.2** It is routine to verify that there exists a five-dimensional $\mathfrak{B}\mathfrak{J}$-module $O$ that has an $F$-basis with respect to which the matrices representing $X$, $Y$, $Z$ are

\[
\begin{pmatrix}
-\frac{1}{2} & 0 & 0 & 0 & 0 \\
1 & -\frac{1}{2} & 0 & 0 & 0 \\
0 & 1 & \frac{3}{2} & 0 & 0 \\
0 & 0 & 1 & -\frac{5}{2} & 0 \\
0 & 0 & 0 & 1 & \frac{7}{2}
\end{pmatrix},
\begin{pmatrix}
-\frac{3}{2} & 4 & 0 & 0 & 0 \\
0 & \frac{1}{2} & -2 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 6 & 0 \\
0 & 0 & 0 & -\frac{3}{2} & -12 \\
0 & 0 & 0 & 0 & \frac{5}{2}
\end{pmatrix},
\begin{pmatrix}
\frac{3}{2} & -4 & 0 & 0 & 0 \\
-1 & -\frac{5}{2} & -2 & 0 & 0 \\
0 & 1 & \frac{3}{2} & -6 & 0 \\
0 & 0 & -1 & -\frac{5}{2} & -12 \\
0 & 0 & 0 & 1 & \frac{3}{2}
\end{pmatrix},
\]

respectively. More precisely,

$$\{X, Y\} = Z + 4, \quad \{Y, Z\} = X - 8, \quad \{Z, X\} = Y - 4$$

on the $\mathfrak{B}\mathfrak{J}$-module $O$. It is routine to check that the minimal polynomials of $X$, $Y$, $Y$ on $O$ are

\[
\left(x - \frac{7}{2}\right)\left(x - \frac{3}{2}\right)\left(x + \frac{1}{2}\right)^2\left(x + \frac{5}{2}\right),
\left(x - \frac{5}{2}\right)\left(x - \frac{1}{2}\right)^2\left(x + \frac{3}{2}\right)^2,
\left(x - \frac{3}{2}\right)^2\left(x + \frac{1}{2}\right)^2\left(x + \frac{5}{2}\right),
\]

respectively. Therefore, none of $X$, $Y$, $Z$ is diagonalizable on $O$. Similar to Example 1.1, to see the irreducibility of $O$ we suppose that $W$ is any nonzero $\mathfrak{B}\mathfrak{J}$-submodule of $O$ and show that $W = O$. The element $Y$ has exactly three eigenvalues $-\frac{3}{2}, \frac{1}{2}, \frac{5}{2}$ in $O$, and the $-\frac{3}{2}$-, $\frac{1}{2}$-, $\frac{5}{2}$-eigenspaces of $Y$ in $O$ are of dimension 1 spanned by

$$u_0, \quad u_0 + \frac{1}{2}u_1, \quad u_0 + u_1 - u_2 - \frac{1}{3}u_3 + \frac{1}{9}u_4,$$

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respectively. Since $W$ is nonzero, $-\frac{3}{2}, \frac{1}{2}$ or $\frac{5}{2}$ is an eigenvalue of $Y$ in $W$. Therefore, $W$ contains $u_0, u_0 + \frac{1}{2}u_1$ or $u_0 + u_1 - u_2 - \frac{1}{3}u_3 + \frac{1}{9}u_4$. If $u_0 + \frac{1}{2}u_1 \in W$, then

$$u_0 = \frac{1}{6}(Z - X + 6)\left(u_0 + \frac{1}{2}u_1\right) \in W.$$ 

If $u_0 + u_1 - u_2 - \frac{1}{3}u_3 + \frac{1}{9}u_4 \in W$, then

$$u_0 = -\frac{1}{4}(3X + Z)\left(u_0 + u_1 - u_2 - \frac{1}{3}u_3 + \frac{1}{9}u_4\right) \in W.$$ 

By these comments, $u_0 \in W$. The $\mathcal{B}\mathcal{J}$-module $O$ is generated by $u_0$, so $W = O$, a counterexample to [15, Theorem 7.5].

The main result of this paper is to give a complete classification of finite-dimensional irreducible $\mathcal{B}\mathcal{J}$-modules, which answers the first open question listed in [5, §11]. The proof idea originates from [18]. We mentioned earlier that [15] contains a mistake. Additionally, the same mistake was made in the case of the Racah algebras [8]. Bockting-Conrad and the present author write a paper [19] to correct it in greater detail. The results of [13,17] reveal that the universal Racah algebra $\mathcal{R}$ is isomorphic to an $\mathbb{F}$-subalgebra of $\mathcal{B}\mathcal{J}$. As an application of [19] and our result, the lattices of $\mathcal{R}$-submodules of finite-dimensional irreducible $\mathcal{B}\mathcal{J}$-modules are classified in [16].

The paper is organized as follows: in Sect. 2, we state the classification of irreducible $\mathcal{B}\mathcal{J}$-modules that have even and odd dimensions in Theorem 2.5 and Theorem 2.8, respectively. In Sect. 3, we introduce our main tool, an infinite-dimensional $\mathcal{B}\mathcal{J}$-module. In Sect. 4, we establish the necessary and sufficient conditions for the irreducibility of even-dimensional $\mathcal{B}\mathcal{J}$-modules. In Sect. 5, we study the isomorphism classes of even-dimensional $\mathcal{B}\mathcal{J}$-modules. In Sect. 6, we give a proof for Theorem 2.5. Theorem 2.8 follows by a similar argument.

2 Statement of results

Definition 2.1 The universal Bannai–Ito algebra $\mathcal{B}\mathcal{J}$ is a unital associative $\mathbb{F}$-algebra generated by $X, Y, Z$, and the relations assert that each of the following elements commutes with $X, Y, Z$:

$$\{X, Y\} - Z, \quad \{Y, Z\} - X, \quad \{Z, X\} - Y.$$ (1)

For notational convenience, we define $\kappa, \lambda, \mu$ to be the central elements (1)--(3) of $\mathcal{B}\mathcal{J}$, respectively.

Lemma 2.2 The algebra $\mathcal{B}\mathcal{J}$ is generated by $X, Y, \kappa$. 

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Proof By (1), the element $Z$ can be expressed in terms of $X, Y, \kappa$. Combined with Definition 2.1, the lemma follows. □

Lemma 2.3 The algebra $\mathfrak{B}\mathfrak{J}$ has a presentation given by generators $X, Y, \kappa, \lambda, \mu$ and relations

$$Y^2X + 2YXY + XY^2 - X = 2\kappa Y + \lambda,$$
$$X^2Y + 2XYX + YX^2 - Y = 2\kappa X + \mu,$$
$$\lambda X = X\lambda, \quad \mu X = X\mu, \quad \kappa X = X\kappa,$$
$$\lambda Y = Y\lambda, \quad \mu Y = Y\mu, \quad \kappa Y = Y\kappa,$$
$$\lambda\kappa = \kappa\lambda, \quad \mu\kappa = \kappa\mu.$$ (8)

Proof Rewrite (1) as

$$Z = \{X, Y\} - \kappa.$$ (9)

Relations (4) and (5) follow by substituting (9) into (2) and (3), respectively. Relations (6)–(8) reformulate the commutation of $\kappa, \lambda, \mu$ with $X, Y, Z$. □

Proposition 2.4 For any scalars $a, b, c \in \mathbb{F}$ and any odd integer $d \geq 1$, there exists a $(d + 1)$-dimensional $\mathfrak{B}\mathfrak{J}$-module $E_d(a, b, c)$ satisfying the following conditions:

(i) There exists an $\mathbb{F}$-basis $\{v_i\}_{i=0}^d$ for $E_d(a, b, c)$ with respect to which the matrices representing $X$ and $Y$ are

$$\begin{pmatrix}
\theta_0 \\
1 \\
\theta_1 \\
1 \\
\ddots \\
\ddots \\
0 \\
1 \\
\theta_d
\end{pmatrix}, \quad \begin{pmatrix}
\varphi_0 \\
\varphi_1 \\
\varphi_2 \\
\ddots \\
\ddots \\
\varphi_d
\end{pmatrix},$$

respectively, where

$$\theta_i = \frac{(-1)^i(2a - d + 2i)}{2} (0 \leq i \leq d),$$
$$\theta_i^* = \frac{(-1)^i(2b - d + 2i)}{2} (0 \leq i \leq d),$$
$$\varphi_i = \begin{cases}
i(d - i + 1) & \text{if } i \text{ is even}, \\
c^2 - \frac{(2a + 2b - d + 2i - 1)^2}{4} & \text{if } i \text{ is odd}
\end{cases} (1 \leq i \leq d).$$

(ii) The elements $\kappa, \lambda, \mu$ act on $E_d(a, b, c)$ as scalar multiplication by

$$c^2 - a^2 - b^2 + \frac{(d + 1)^2}{4},$$
respectively.

**Proof** It is straightforward to verify the proposition using Lemma 2.3.

Recall that \(\{\pm 1\}\) is a group under multiplication and the group \(\{\pm 1\}^2\) is isomorphic to the Klein 4-group. Observe that there exists a unique \(\{\pm 1\}^2\)-action on \(\mathcal{B}\mathcal{J}\) such that each \((\varepsilon, \varepsilon') \in \{\pm 1\}^2\) acts on \(\mathcal{B}\mathcal{J}\) as an \(\mathbb{F}\)-algebra automorphism in the following way (Table 1):

| \(u\)             | \(X\) | \(Y\) | \(Z\) |
|-------------------|-------|-------|-------|
| \(u^{(1,1)}\)     | \(X\) | \(Y\) | \(Z\) |
| \(u^{(1,-1)}\)    | \(X\) | \(-Y\)| \(-Z\) |
| \(u^{(-1,1)}\)    | \(-X\)| \(Y\) | \(-Z\) |
| \(u^{(-1,-1)}\)   | \(-X\)| \(-Y\)| \(Z\) |

Let \(V\) denote a \(\mathcal{B}\mathcal{J}\)-module. For any \((\varepsilon, \varepsilon') \in \{\pm 1\}^2\), we define \(V^{(\varepsilon, \varepsilon')}\) to be the \(\mathcal{B}\mathcal{J}\)-module obtained by twisting the \(\mathcal{B}\mathcal{J}\)-module \(V\) via \((\varepsilon, \varepsilon')\). The classification of even-dimensional irreducible \(\mathcal{B}\mathcal{J}\)-modules is as follows:

**Theorem 2.5** Assume that \(\mathbb{F}\) is algebraically closed with \(\text{char } \mathbb{F} = 0\). Let \(d \geq 1\) denote an odd integer. Let \(EM_d\) denote the set of all isomorphism classes of irreducible \(\mathcal{B}\mathcal{J}\)-modules that have dimension \(d + 1\). Let \(EP_d\) denote the set of all \((a, b, c) \in \mathbb{F}^3\) that satisfy

\[
a + b + c, -a + b + c, a - b + c, a + b - c \notin \left\{ \frac{d - 1}{2} - i \bigg| i = 0, 2, \ldots, d - 1 \right\}.
\]

Define an action of the abelian group \(\{\pm 1\}^3\) on \(EP_d\) by

\[
(a, b, c)^{(1, -1, 1)} = (a, -b, c),
\]

\[
(a, b, c)^{(1, -1, -1)} = (a, b, c),
\]

\[
(a, b, c)^{(1, -1, 1)} = (a, b, c),
\]

for all \((a, b, c) \in EP_d\). Let \(EP_d/\{\pm 1\}^3\) denote the set of the \(\{\pm 1\}^3\)-orbits of \(EP_d\). For \((a, b, c) \in EP_d\), let \(\{a, b, c\}\) denote the \(\{\pm 1\}^3\)-orbit of \(EP_d\) that contains \((a, b, c)\).
Then, there exists a bijection $\mathcal{E} : (\pm 1)^2 \times \mathbb{F}P_d/(\pm 1)^3 \to \mathbb{E}M_d$ given by

$((\epsilon, \epsilon'), [a, b, c]) \mapsto \text{the isomorphism class of } E_d(a, b, c)^{\langle\epsilon, \epsilon'\rangle}$

for all $(\epsilon, \epsilon') \in \{\pm 1\}^2$ and all $[a, b, c] \in \mathbb{F}P_d/(\pm 1)^3$.

We now turn our attention to the odd-dimensional irreducible $\mathfrak{B}\mathcal{J}$-modules.

**Proposition 2.6** For any scalars $a, b, c \in \mathbb{F}$ and any even integer $d \geq 0$, there exists a $(d + 1)$-dimensional $\mathfrak{B}\mathcal{J}$-module $O_d(a, b, c)$ satisfying the following conditions:

(i) There exists an $\mathbb{F}$-basis for $O_d(a, b, c)$ with respect to which the matrices representing $X$ and $Y$ are

$$
\begin{pmatrix}
\theta_0 & \theta_1 & 0 \\
1 & \theta_1 & \theta_2 \\
0 & \ldots & 1 & \theta_d
\end{pmatrix},
\begin{pmatrix}
\varphi_0 & \varphi_1 & \varphi_2 & 0 \\
\theta_0^* & \varphi_1^* & \varphi_2^* & \ldots \\
0 & \theta_1^* & \varphi_2^* & \ldots \\
0 & \ldots & \theta_d^*
\end{pmatrix},
$$

respectively, where

$$
\theta_i = \frac{(-1)^i (2a - d + 2i)}{2} \quad (0 \leq i \leq d),
$$

$$
\theta_i^* = \frac{(-1)^i (2b - d + 2i)}{2} \quad (0 \leq i \leq d),
$$

$$
\varphi_i = \begin{cases}
\frac{i(d + 1 - 2i - 2a - 2b - 2c)}{(i - d - 1)(d + 1 - 2i - 2a - 2b + 2c)} & \text{if } i \text{ is even}, \\
\frac{(i - d)(d + 1 - 2i - 2a - 2b + 2c)}{2} & \text{if } i \text{ is odd}
\end{cases} \quad (1 \leq i \leq d).
$$

(ii) The elements $\kappa, \lambda, \mu$ act on $O_d(a, b, c)$ as scalar multiplication by

$$2ab - c(d + 1),$$

$$2bc - a(d + 1),$$

$$2ca - b(d + 1),$$

respectively.

**Proof** It is straightforward to verify the proposition using Lemma 2.3.

The following result is concerning the $\mathfrak{B}\mathcal{J}$-modules $O_d(a, b, c)^{\langle\epsilon, \epsilon'\rangle}$ for all $(\epsilon, \epsilon') \in \{\pm 1\}^2$. To prove this, one may follow the proof of Theorem 5.3.

**Theorem 2.7** Suppose that the $\mathfrak{B}\mathcal{J}$-module $O_d(a, b, c)$ is irreducible. Then, the following hold:
(i) The $\mathcal{B}J$-module $O_d(a, b, c)^{(1, -1)}$ is isomorphic to $O_d(a, -b, -c)$.
(ii) The $\mathcal{B}J$-module $O_d(a, b, c)^{(-1, 1)}$ is isomorphic to $O_d(-a, b, -c)$.
(iii) The $\mathcal{B}J$-module $O_d(a, b, c)^{(-1, -1)}$ is isomorphic to $O_d(-a, -b, c)$.

The classification of odd-dimensional irreducible $\mathcal{B}J$-modules is as follows:

**Theorem 2.8** Assume that $\mathbb{F}$ is algebraically closed with $\text{char } \mathbb{F} = 0$. Let $d \geq 0$ denote an even integer. Let $\text{OM}_d$ denote the set of all isomorphism classes of irreducible $\mathcal{B}J$-modules that have dimension $d + 1$. Let $\text{OP}_d$ denote the set of all $(a, b, c) \in \mathbb{F}^3$ that satisfy

$$a + b + c, \ a - b - c, \ -a + b - c, \ -a - b + c \notin \left\{ \frac{d + 1}{2} - i \mid i = 2, 4, \ldots, d \right\}.$$ 

Then, there exists a bijection $\mathcal{O} : \text{OP}_d \to \text{OM}_d$ given by

$$(a, b, c) \mapsto \text{the isomorphism class of } O_d(a, b, c)$$

for all $(a, b, c) \in \text{OP}_d$.

The proofs for Theorems 2.5 and 2.8 are similar and tedious. Thus, the rest of this paper is devoted to the proof of Theorem 2.5 and the proof of Theorem 2.8 is omitted.

3 An infinite-dimensional $\mathcal{B}J$-module and its universal property

The following notations are used throughout the rest of this paper: We let $a, b, c, \delta$ be any scalars in $\mathbb{F}$ and let $d \geq 1$ be an odd integer. We let $\{v_i\}_{i=0}^d$ denote the $\mathbb{F}$-basis for $E_d(a, b, c)$ from Proposition 2.4(i). We adopt the following parameters associated with $a, b, c, \delta$:

$$\theta_i = \frac{(-1)^i (2a - \delta + 2i)}{2} \text{ for all } i \in \mathbb{Z},$$

$$\theta_i^* = \frac{(-1)^i (2b - \delta + 2i)}{2} \text{ for all } i \in \mathbb{Z},$$

$$\phi_i = \begin{cases} 
    i(\delta - i + 1) \\
    c^2 - \frac{(2b - 2a - \delta + 2i - 1)^2}{4} 
\end{cases} \text{ for all } i \in \mathbb{Z},$$  

$$\varphi_i = \begin{cases} 
    i(\delta - i + 1) \\
    c^2 - \frac{(2a + 2b - \delta + 2i - 1)^2}{4} 
\end{cases} \text{ for all } i \in \mathbb{Z},$$

$$\omega = c^2 - a^2 - b^2 + \frac{(\delta + 1)^2}{4},$$

$$\omega^* = a^2 - b^2 - c^2 + \frac{(\delta + 1)^2}{4},$$
Finite-dimensional irreducible $\mathfrak{B} \mathfrak{J}$-modules at $\mathbb{F} = 0$

\[
\omega^\Diamond = b^2 - c^2 - a^2 + \frac{(\delta + 1)^2}{4}. \tag{16}
\]

**Proposition 3.1** There exists a $\mathfrak{B} \mathfrak{J}$-module $M_{\delta}(a, b, c)$ satisfying the following conditions:

(i) There exists an $\mathbb{F}$-basis $\{m_i\}_{i=0}^\infty$ for $M_{\delta}(a, b, c)$ with respect to which the matrices representing $X$ and $Y$ are

\[
\begin{pmatrix}
\theta_0 & \theta_1 & 0 \\
1 & \theta_2 & \\
. & . & . \\
0 & . & .
\end{pmatrix}, \quad \begin{pmatrix}
\theta_0^* & \varphi_1 & \varphi_2 & 0 \\
\theta_1^* & \theta_2^* & . & . \\
0 & . & . & .
\end{pmatrix},
\]

respectively.

(ii) The elements $\kappa, \lambda, \mu$ act on $M_{\delta}(a, b, c)$ as scalar multiplication by $\omega, \omega^*, \omega^\Diamond$, respectively.

**Proof** It is routine to verify the proposition using Lemma 2.3. \hfill \Box

Throughout the rest of this paper, we let $\{m_i\}_{i=0}^\infty$ denote the $\mathbb{F}$-basis for $M_{\delta}(a, b, c)$ from Proposition 3.1(i).

**Lemma 3.2** For any integers $i, j$ with $0 \leq i \leq j$, the following equation holds:

\[ m_{j+1} = \prod_{h=i}^j (X - \theta_h)m_i. \]

**Proof** Immediate from Proposition 3.1(i). \hfill \Box

We shall give an alternative description for the $\mathfrak{B} \mathfrak{J}$-module $M_{\delta}(a, b, c)$. To do this, we begin with a Poincaré–Birkhoff–Witt basis for $\mathfrak{B} \mathfrak{J}$.

**Lemma 3.3** The elements

\[ X^i Z^j Y^k \mu^r \lambda^s \kappa^t \text{ for all } i, j, k, r, s, t \in \mathbb{N} \]

are an $\mathbb{F}$-basis for $\mathfrak{B} \mathfrak{J}$.

**Proof** Recall from [17, Theorem 3.4] that

\[ X^i Y^j Z^k \mu^r \lambda^s \mu^t \text{ for all } i, j, k, r, s, t \in \mathbb{N} \tag{17} \]

form an $\mathbb{F}$-basis for $\mathfrak{B} \mathfrak{J}$. By Definition 2.1, there exists a unique $\mathbb{F}$-algebra automorphism of $\mathfrak{B} \mathfrak{J}$ that sends $X, Y, Z, \kappa, \lambda, \mu$ to $X, Z, Y, \mu, \lambda, \kappa$, respectively. The lemma follows by applying the automorphism to (17). \hfill \Box
Let $I_\delta(a, b, c)$ denote the left ideal of $\mathcal{B}J$ generated by

\[
Y - \theta_0^*, \quad (Y - \theta_1^*)(X - \theta_0) - \varphi_1, \quad \kappa - \omega, \quad \lambda - \omega^*, \quad \mu - \omega^\circ.
\]

(18)  
(19)  
(20)

**Lemma 3.4** For all $n \in \mathbb{N}$, the following hold:

(i) $YX^n + I_\delta(a, b, c)$ is an $F$-linear combination of $X^i + I_\delta(a, b, c)$ for all $0 \leq i \leq n$.

(ii) $ZX^n + I_\delta(a, b, c)$ is an $F$-linear combination of $X^i + I_\delta(a, b, c)$ for all $0 \leq i \leq n + 1$.

(iii) $Z^n + I_\delta(a, b, c)$ is an $F$-linear combination of $X^i + I_\delta(a, b, c)$ for all $0 \leq i \leq n$.

**Proof** (i) Proceed by induction on $n$. Since $I_\delta(a, b, c)$ contains (18), it is true when $n = 0$. Since $I_\delta(a, b, c)$ contains (18) and (19), it is true when $n = 1$. Now suppose $n \geq 2$. Right multiplying either side of (5) by $X^n - 2$ yields that

\[
X^2YX^{n-2} + 2XYX^{n-1} + YX^n - YX^{n-2} = 2X^{n-1}\kappa + X^{n-2}\mu.
\]

Since $I_\delta(a, b, c)$ contains (20), it follows that $YX^n$ is congruent to

\[
YX^{n-2} - X^2YX^{n-2} - 2XYX^{n-1} + 2\omega X^{n-1} + \omega^\circ X^{n-2}
\]

modulo $I_\delta(a, b, c)$. By induction hypothesis, the term (21) is congruent to an $F$-linear combination of $X^i$ for all $0 \leq i \leq n$ modulo $I_\delta(a, b, c)$. Therefore, (i) follows.

(ii) Right multiplying either side of (1) by $X^n$ yields that

\[
ZX^n = XYX^n + YX^{n+1} - X^n\kappa \pmod{I_\delta(a, b, c)}.
\]

Since $I_\delta(a, b, c)$ contains (20) and by (i), the right-hand side of (22) is congruent to an $F$-linear combination of $X^i$ for all $0 \leq i \leq n + 1$ modulo $I_\delta(a, b, c)$. Therefore, (ii) follows.

(iii) Proceed by induction on $n$. There is nothing to prove for $n = 0$. Suppose $n \geq 1$. By induction hypothesis, $Z^n$ is congruent to an $F$-linear combination of

\[
ZX^i \quad \text{for all } 0 \leq i \leq n - 1
\]

modulo $I_\delta(a, b, c)$. By (ii), each of (23) is congruent to an $F$-linear combination of $X^k$ for all $0 \leq k \leq n$ modulo $I_\delta(a, b, c)$. Therefore, (iii) follows.

\[\square\]

**Lemma 3.5** The $F$-vector space $\mathcal{B}J/I_\delta(a, b, c)$ is spanned by

\[
X^i + I_\delta(a, b, c) \quad \text{for all } i \in \mathbb{N}.
\]
Proof By Lemma 3.3, the elements
\[ X^i Z^j Y^k \mu^r \lambda^s \kappa^t + I_\delta(a, b, c) \] for all \( i, j, k, r, s, t \in \mathbb{N} \)
span \( \mathfrak{B} / I_\delta(a, b, c) \). Since \( I_\delta(a, b, c) \) contains (18) and (20), it follows that
\[ X^i Z^j + I_\delta(a, b, c) \] for all \( i, j \in \mathbb{N} \) (24)
span \( \mathfrak{B} / I_\delta(a, b, c) \). Applying Lemma 3.4(iii), each of (24) is an \( \mathbb{F} \)-linear combination of \( X^k + I_\delta(a, b, c) \) for all \( k \in \mathbb{N} \). The lemma follows. \( \Box \)

Theorem 3.6 There exists a unique \( \mathfrak{B} \)-module homomorphism
\[ \Phi : \mathfrak{B} / I_\delta(a, b, c) \rightarrow M_\delta(a, b, c) \]
that sends \( 1 + I_\delta(a, b, c) \) to \( m_0 \). Moreover, \( \Phi \) is an isomorphism.

Proof Consider the \( \mathfrak{B} \)-module homomorphism \( \Psi : \mathfrak{B} \rightarrow M_\delta(a, b, c) \) that sends \( 1 \) to \( m_0 \). By Proposition 3.1(i), (18) and (19) are in the kernel of \( \Psi \). By Proposition 3.1(ii), each of (20) is in the kernel of \( \Psi \). It follows that \( I_\delta(a, b, c) \) is contained in the kernel of \( \Psi \). Hence, \( \Psi \) induces a \( \mathfrak{B} \)-module homomorphism \( \mathfrak{B} / I_\delta(a, b, c) \rightarrow M_\delta(a, b, c) \) that maps \( 1 + I_\delta(a, b, c) \) to \( m_0 \). The existence of \( \Phi \) follows. Since the \( \mathfrak{B} \)-module \( \mathfrak{B} / I_\delta(a, b, c) \) is generated by \( 1 + I_\delta(a, b, c) \), the uniqueness follows.

By Proposition 3.1(i), the homomorphism \( \Phi \) maps
\[ \prod_{h=0}^{i-1} (X - \theta_h) + I_\delta(a, b, c) \] (25)
to \( m_i \) for all \( i \in \mathbb{N} \). Since the vectors \( \{m_i\}_{i \in \mathbb{N}} \) are linearly independent, it follows that (25) for all \( i \in \mathbb{N} \) are linearly independent. By Lemma 3.5, the cosets (25) for all \( i \in \mathbb{N} \) span \( \mathfrak{B} / I_\delta(a, b, c) \), and hence, these cosets form an \( \mathbb{F} \)-basis for \( \mathfrak{B} / I_\delta(a, b, c) \). Since \( \Phi \) maps an \( \mathbb{F} \)-basis for \( \mathfrak{B} / I_\delta(a, b, c) \) to an \( \mathbb{F} \)-basis for \( M_\delta(a, b, c) \), it follows that \( \Phi \) is an isomorphism. \( \Box \)

As a consequence of Theorem 3.6, the \( \mathfrak{B} \)-module \( M_\delta(a, b, c) \) has the following universal property:

Proposition 3.7 If \( V \) is a \( \mathfrak{B} \)-module which contains a vector \( v \) satisfying
\[ Yv = \theta_0^* v, \]
\[ (Y - \theta_1^*)(X - \theta_0)v = \varphi_1 v, \]
\[ \kappa v = \omega v, \quad \lambda v = \omega^* v, \quad \mu v = \omega^\circ v, \]
then there exists a unique \( \mathfrak{B} \)-module homomorphism \( M_\delta(a, b, c) \rightarrow V \) that sends \( m_0 \) to \( v \).
From now on until the end of this paper, we set $\delta = d$. Let

$$N_d(a, b, c)$$

denote the $X$-cyclic $\mathbb{F}$-subspace of $M_d(a, b, c)$ generated by $m_{d+1}$.

**Lemma 3.8** $N_d(a, b, c)$ is a $\mathcal{B}\mathcal{J}$-submodule of $M_d(a, b, c)$ with the $\mathbb{F}$-basis $\{m_i\}_{i=d+1}^\infty$.

**Proof** Let $N$ denote the $\mathbb{F}$-subspace of $M_d(a, b, c)$ spanned by $\{m_i\}_{i=d+1}^\infty$. It follows from Lemma 3.2 that

$$(X - \theta_i)m_i = m_{i+1} \quad \text{for all } i \geq d + 1.$$ 

Hence, $N$ is an $X$-invariant $\mathbb{F}$-subspace of $N_d(a, b, c)$. It follows from the construction of $N_d(a, b, c)$ that $N = N_d(a, b, c)$. Therefore, $N_d(a, b, c)$ has the $\mathbb{F}$-basis $\{m_i\}_{i=d+1}^\infty$. From Proposition 3.1(i), we see that

$$(Y - \theta_i^*)m_i = \phi_im_{i-1} \quad \text{for all } i \geq d + 1.$$ 

By (13), the scalar $\varphi_{d+1} = 0$ under the setting $\delta = d$. Hence, $N_d(a, b, c)$ is $Y$-invariant. By Proposition 3.1(ii), the element $\kappa$ acts on $N_d(a, b, c)$ as scalar multiplication. Therefore, $N_d(a, b, c)$ is a $\mathcal{B}\mathcal{J}$-module by Lemma 2.2. The lemma follows.

**Lemma 3.9** There exists a unique $\mathcal{B}\mathcal{J}$-module isomorphism

$$M_d(a, b, c)/N_d(a, b, c) \rightarrow E_d(a, b, c)$$

that sends $m_i + N_d(a, b, c)$ to $v_i$ for all $0 \leq i \leq d$.

**Proof** By Lemma 3.8, the quotient space $M_d(a, b, c)/N_d(a, b, c)$ is a $(d + 1)$-dimensional $\mathcal{B}\mathcal{J}$-module with the $\mathbb{F}$-basis

$$\{m_i + N_d(a, b, c)\}_{i=0}^d.$$  

Comparing Propositions 2.4(i) and 3.1(i), the matrices representing $X$ and $Y$ with respect to the $\mathbb{F}$-basis $\{v_i\}_{i=0}^d$ for $E_d(a, b, c)$ are identical with the matrices representing $X$ and $Y$ with respect the $\mathbb{F}$-basis (26) for $M_d(a, b, c)/N_d(a, b, c)$, respectively. By Propositions 2.4(ii) and 3.1(ii), the actions of $\kappa$ on $E_d(a, b, c)$ and $M_d(a, b, c)/N_d(a, b, c)$ are scalar multiplication by the same scalar $\omega$. Therefore, the lemma follows by Lemma 2.2.

**Proposition 3.10** If there is a $\mathcal{B}\mathcal{J}$-module homomorphism $M_d(a, b, c) \rightarrow V$ that sends $m_0$ to $v$ and

$$\prod_{i=0}^d (X - \theta_i)v = 0,$$  

then there exists a $\mathcal{B}\mathcal{J}$-module homomorphism $E_d(a, b, c) \rightarrow V$ that sends $v_0$ to $v$.
Proof Denote by $\varrho$ the $\mathfrak{B}\mathfrak{J}$-module homomorphism $M_d(a, b, c) \to V$. It follows from Lemma 3.2 that

$$m_{d+1} = \prod_{i=0}^{d} (X - \theta_i)m_0$$

Combined with (27), this yields that $m_{d+1}$ lies in the kernel of $\varrho$. Hence, $N_d(a, b, c)$ is contained in the kernel of $\varrho$. By Lemma 3.8, the homomorphism $\varrho$ induces a $\mathfrak{B}\mathfrak{J}$-module homomorphism $M_d(a, b, c)/N_d(a, b, c) \to V$ that sends $m_0 + N_d(a, b, c)$ to $v$. Combined with Lemma 3.9, the proposition follows. □

4 The conditions for the $\mathfrak{B}\mathfrak{J}$-module $E_d(a, b, c)$ as irreducible

The goal of this section is to establish the necessary and sufficient conditions for $E_d(a, b, c)$ to be irreducible in terms of the parameters $a, b, c, d$. In this section, we set

$$w_i = \prod_{h=0}^{i-1} (X - \theta_{d-h})v_0 \quad (0 \leq i \leq d). \quad (28)$$

Lemma 4.1 If the $\mathfrak{B}\mathfrak{J}$-module $E_d(a, b, c)$ is irreducible, then the following conditions hold:

(i) $\text{char } \mathbb{F} \nmid i$ for all $i = 2, 4, \ldots, d - 1$.

(ii) $a + b + c, a + b - c \notin \left\{ \frac{d-1}{2} - i \mid i = 0, 2, \ldots, d - 1 \right\}$.

Proof Suppose there is an integer $i$ with $1 \leq i \leq d$ such that $\varphi_i = 0$. By Proposition 2.4, the $\mathbb{F}$-subspace $W$ of $E_d(a, b, c)$ spanned by $\{v_h\}_{h=i}^{d}$ is invariant under $X, Y, \kappa$. It follows from Lemma 2.2 that $W$ is a $\mathfrak{B}\mathfrak{J}$-submodule of $E_d(a, b, c)$, a contradiction to the irreducibility of $E_d(a, b, c)$. Therefore, $\varphi_i \neq 0$ for all $1 \leq i \leq d$, which is equivalent to (i) and (ii) by (13). The lemma follows. □

Lemma 4.2 $\{w_i\}_{i=0}^{d}$ is an $\mathbb{F}$-basis for $E_d(a, b, c)$.

Proof It follows from Proposition 2.4(i) that

$$v_i = \prod_{h=0}^{i-1} (X - \theta_h)v_0 \quad (0 \leq i \leq d).$$

Comparing with (28), the lemma follows. □
Proposition 4.3  The \( \mathfrak{B}\mathfrak{J} \)-module \( E_d(a, b, c) \) is isomorphic to \( E_d(-a, b, c) \). Moreover, the matrices representing \( X \) and \( Y \) with respect to the \( \mathbb{F} \)-basis \( \{w_i\}_{i=0}^d \) for \( E_d(a, b, c) \) are

\[
\begin{pmatrix}
\theta_d & 0 \\
1 & \theta_d-1 \\
& & \ddots \\
& & & 1 & \theta_0 \\
0 & & & & & \\
\end{pmatrix}, \quad \begin{pmatrix}
\theta_0^* & \phi_1 & \phi_2 & \cdots & 0 \\
1 & \theta_1^* & \phi_2 & \cdots & 0 \\
& & \ddots & \ddots & \ddots \\
& & & \theta_d^* & 0 \\
0 & & & & & \\
\end{pmatrix}, \quad (29)
\]

respectively.

Proof  By Proposition 2.4(i), there exists an \( \mathbb{F} \)-basis \( \{u_i\}_{i=0}^d \) for \( E_d(-a, b, c) \) with respect to which the matrices representing \( X \) and \( Y \) are equal to the matrices (29).

By Lemma 4.2, it suffices to show that there is a \( \mathfrak{B}\mathfrak{J} \)-module homomorphism \( E_d(a, b, c) \to E_d(-a, b, c) \) that sends \( w_i \) to \( u_i \) for all \( 0 \leq i \leq d \).

Observe that \( Yu_0 = \theta_0^* u_0 \) and a direct calculation yields that

\[
(Y - \theta_1^*)(X - \theta_0)u_0 = \varphi_1 u_0.
\]

By Proposition 2.4(ii), the elements \( \kappa, \lambda, \mu \) act on \( E_d(-a, b, c) \) as scalar multiplication by \( \omega, \omega^*, \omega^\circ \), respectively. According to Proposition 3.7, there exists a unique \( \mathfrak{B}\mathfrak{J} \)-module homomorphism \( M_d(a, b, c) \to E_d(-a, b, c) \) that sends \( m_0 \) to \( u_0 \). Using (29) yields that

\[
\prod_{i=0}^d (X - \theta_i)u_0 = 0.
\]

Hence, there exists a \( \mathfrak{B}\mathfrak{J} \)-module homomorphism

\[
E_d(a, b, c) \to E_d(-a, b, c)
\]

that maps \( v_0 \) to \( u_0 \) by Proposition 3.10. Using (28) yields that (30) sends \( w_i \) to \( u_i \) for all \( 0 \leq i \leq d \). The proposition follows.

Lemma 4.4  If the \( \mathfrak{B}\mathfrak{J} \)-module \( E_d(a, b, c) \) is irreducible, then the following conditions hold:

(i) \( \text{char} \mathbb{F} \not| \ i \) for all \( i = 2, 4, \ldots, d - 1 \).

(ii) \( a + b + c, -a + b + c, a - b + c, a + b - c \notin \left\{ \frac{d - 1}{2} - i \mid i = 0, 2, \ldots, d - 1 \right\} \).

Proof  By Proposition 4.3, the \( \mathfrak{B}\mathfrak{J} \)-modules \( E_d(a, b, c) \) and \( E_d(-a, b, c) \) are isomorphic. Hence, the lemma follows by applying Lemma 4.1 to \( E_d(a, b, c) \) and \( E_d(-a, b, c) \).
Consider the operators

\[ R = \prod_{h=1}^{d} (Y - \theta_h^n), \]

\[ S_i = \prod_{h=1}^{d-i} (X - \theta_{d-h+1}) \quad (0 \leq i \leq d). \]

It follows from Proposition 2.4(i) that \( Rv \) is a scalar multiple of \( v_0 \) for all \( v \in E_d(a, b, c) \). Thus, for any integers \( i, j \) with \( 0 \leq i, j \leq d \) there exists a unique \( L_{ij} \in \mathbb{F} \) such that

\[ RS_i v_j = L_{ij} v_0. \]  

Using Proposition 2.4(i) yields that

\[ L_{ij} = 0 \quad (0 \leq i < j \leq d), \]  

\[ L_{ij} = (\theta_i - \theta_{j-1}) L_{i,j-1} + L_{i-1,j-1} \quad (1 \leq j \leq i \leq d). \]

It follows from Proposition 4.3 that

\[ L_{i0} = \prod_{h=1}^{i} (\theta_0^n - \theta_{d-h+1}) \prod_{h=1}^{d} \phi_h \quad (0 \leq i \leq d). \]

Solving the recurrence relation (33) with the initial condition (34) yields that

\[ L_{ij} = \begin{cases} 
\prod_{h=1}^{i-j} (\theta_0^n - \theta_{d-h+1}) \prod_{h=1}^{d-i} \phi_h \prod_{h=1}^{\lceil \frac{i}{2} \rceil} \varphi_{2h-1} \prod_{h=1}^{\lfloor \frac{i}{2} \rfloor} \varphi_{2(\lfloor \frac{j}{2} \rfloor - h+1)} & \text{if } i \text{ is odd or } j \text{ is even}, \\
0 & \text{otherwise}
\end{cases} \]

for all \( 0 \leq j \leq i \leq d \).

**Theorem 4.5** The \( \mathfrak{B}_3 \)-module \( E_d(a, b, c) \) is irreducible if and only if the following conditions hold:

(i) \( \text{char } \mathbb{F} \nmid i \) for all \( i = 2, 4, \ldots, d - 1 \).

(ii) \( a + b + c, -a + b + c, a - b + c, a + b - c \notin \left\{ \frac{d-1}{2} - i \mid i = 0, 2, \ldots, d - 1 \right\} \).

**Proof** \((\Rightarrow)\): Immediate from Lemma 4.4.

\((\Leftarrow)\): Let \( W \) denote any nonzero \( \mathfrak{B}_3 \)-submodule of \( E_d(a, b, c) \). It suffices to show that \( W = E_d(a, b, c) \). Pick a nonzero vector \( w \in W \). Since \( W \) is invariant under \( X \) and \( Y \), it follows that

\[ RS_i w \in W \quad (0 \leq i \leq d). \]
Since \( \{v_i\}_{i=0}^d \) is an \( \mathbb{F} \)-basis for \( E_d(a, b, c) \), there are \( a_i \in \mathbb{F} \) for \( 0 \leq i \leq d \) such that
\[
w = \sum_{i=0}^d a_i v_i.
\]

Using (31) yields that
\[
RS_i w = \left( \sum_{j=0}^d L_{ij} a_j \right) v_0 \quad (0 \leq i \leq d).
\]

Recall the parameters \( \{\phi_i\}_{i \in \mathbb{Z}} \) and \( \{\varphi_i\}_{i \in \mathbb{Z}} \) from (12) and (13). By the conditions (i) and (ii), the parameters \( \varphi_i \neq 0 \) and \( \phi_i \neq 0 \) for all \( 1 \leq i \leq d \). Let \( L \) denote the square matrix indexed by \( 0, 1, \ldots, d \) with \( (i, j) \)-entry as \( L_{ij} \) for all \( 0 \leq i, j \leq d \). By (32), the \( (d+1) \times (d+1) \) matrix \( L \) is lower triangular. By (35), the diagonal entries of \( L \) are
\[
L_{ii} = \prod_{h=1}^{d-i} \phi_h \prod_{h=1}^{i} \varphi_i \neq 0 \quad (0 \leq i \leq d).
\]

Therefore, the matrix \( L \) is nonsingular. Since \( w \) is nonzero, at least one of \( \{a_i\}_{i=0}^d \) is nonzero. Hence, there exists an integer \( i \) with \( 0 \leq i \leq d \) such that
\[
\sum_{j=0}^d L_{ij} a_j \neq 0.
\]

Combined with (36) and (37), this yields that \( v_0 \in W \). Since the \( \mathbb{B} \mathbb{J} \)-module \( E_d(a, b, c) \) is generated by \( v_0 \), it follows that \( W = E_d(a, b, c) \). The result follows. \( \square \)

5 The isomorphism class of the \( \mathbb{B} \mathbb{J} \)-module \( E_d(a, b, c) \)

In Proposition 4.3, we saw that the \( \mathbb{B} \mathbb{J} \)-module \( E_d(a, b, c) \) is isomorphic to \( E_d(-a, b, c) \). In this section, we study the isomorphism class of the \( \mathbb{B} \mathbb{J} \)-module \( E_d(a, b, c) \) in further detail.

**Proposition 5.1** The \( \mathbb{B} \mathbb{J} \)-module \( E_d(a, b, c) \) is isomorphic to \( E_d(a, b, -c) \).

**Proof** By Proposition 2.4(i), there are \( \mathbb{F} \)-bases for \( E_d(a, b, c) \) and \( E_d(a, b, -c) \) with respect to which the matrices representing \( X \) and \( Y \) are the same. By Proposition 2.4(ii), the actions of \( \kappa \) on \( E_d(a, b, c) \) and \( E_d(a, b, -c) \) are multiplication by the same scalar \( \omega \). It follows from Lemma 2.2 that \( E_d(a, b, c) \) is isomorphic to \( E_d(a, b, -c) \). \( \square \)

**Proposition 5.2** If the \( \mathbb{B} \mathbb{J} \)-module \( E_d(a, b, c) \) is irreducible, then the \( \mathbb{B} \mathbb{J} \)-module \( E_d(a, b, c) \) is isomorphic to \( E_d(a, -b, c) \).
Proof By Proposition 2.4(i), there exists an \( F \)-basis \( \{ u_i \}_{i=0}^d \) for \( E_d(a, -b, c) \) with respect to which the matrices representing \( X \) and \( Y \) are

\[
\begin{pmatrix}
\theta_0 & \theta_1 & 0 \\
1 & \theta_2 & \ddots \\
0 & \ddots & \ddots & \theta_d \\
\end{pmatrix},
\begin{pmatrix}
\theta_0^* & \phi_d & \phi_{d-1} & 0 \\
\phi_d & \theta_{d-1}^* & \phi_{d-1} & \ddots \\
\phi_{d-2} & \ddots & \ddots & \ddots & \phi_1 \\
0 & \ddots & \ddots & \theta_0^* \\
\end{pmatrix},
\]

respectively. Since the \( \mathfrak{B}\mathfrak{J} \)-module \( E_d(a, b, c) \) is irreducible, it follows from Theorem 4.5 that \( \phi_i \neq 0 \) for all \( 1 \leq i \leq d \). Hence, we may set

\[
v = \sum_{i=0}^d \prod_{h=1}^i \frac{\theta_0^* - \theta_{d-h+1}^*}{\phi_{d-h+1}} u_i.
\]

A direct calculation yields that \( Yv = \theta_0^*v \) and

\[
(Y - \theta_1^*)(X - \theta_0)v = \varphi_1 v.
\]

By Proposition 2.4(ii), the elements \( \kappa, \lambda, \mu \) act on \( E_d(a, -b, c) \) as scalar multiplication by \( \omega, \omega^*, \omega^\diamond \), respectively. According to Proposition 3.7, there exists a unique \( \mathfrak{B}\mathfrak{J} \)-module homomorphism \( M_d(a, b, c) \to E_d(a, -b, c) \) that sends \( m_0 \) to \( v \). Using (38) yields that

\[
\prod_{i=0}^d (X - \theta_i)v = 0.
\]

By Proposition 3.10, there exists a \( \mathfrak{B}\mathfrak{J} \)-module homomorphism

\[
E_d(a, b, c) \to E_d(a, -b, c)
\]

that sends \( v_0 \) to \( v \). Since the \( \mathfrak{B}\mathfrak{J} \)-module \( E_d(a, b, c) \) is irreducible, it follows from Theorem 4.5 that the \( \mathfrak{B}\mathfrak{J} \)-module \( E_d(a, -b, c) \) is irreducible. Hence, the homomorphism (39) is onto. Since both of \( E_d(a, b, c) \) and \( E_d(a, -b, c) \) are of dimension \( d+1 \), it follows that (39) is an isomorphism. The proposition follows.

We end this section with a brief summary of Propositions 4.3, 5.1 and 5.2.

Theorem 5.3 If the \( \mathfrak{B}\mathfrak{J} \)-module \( E_d(a, b, c) \) is irreducible, then the \( \mathfrak{B}\mathfrak{J} \)-module \( E_d(a, b, c) \) is isomorphic to \( E_d(-a, b, c), E_d(a, -b, c) \) and \( E_d(a, b, -c) \).

6 Proof of Theorem 2.5

In this section, we are devoted to the proof of Theorem 2.5.
Lemma 6.1 Assume that \( \mathbb{F} \) is algebraically closed. If \( V \) is a finite-dimensional irreducible \( \mathcal{B}\mathcal{J} \)-module, then each central element of \( \mathcal{B}\mathcal{J} \) acts on \( V \) as scalar multiplication.

Proof Immediate from Schur’s lemma. \( \square \)

Lemma 6.2 For any \( i \in \mathbb{Z} \), the following hold:

(i) \( \theta_{i+1} + \theta_{i-1} = -2\theta_i \).
(ii) \( \theta_{i+1}\theta_{i-1} = (\theta_i - 1)(\theta_i + 1) \).

Proof It is routine to verify the lemma using (10). \( \square \)

Theorem 6.3 Assume that \( \mathbb{F} \) is algebraically closed with \( \text{char} \mathbb{F} = 0 \). If \( V \) is a \((d+1)\)-dimensional irreducible \( \mathcal{B}\mathcal{J} \)-module, then there exist \( a, b, c \in \mathbb{F} \) and \( (\varepsilon, \varepsilon') \in \{\pm 1\}^2 \) such that the \( \mathcal{B}\mathcal{J} \)-module \( E_d(a, b, c)^{(\varepsilon, \varepsilon')} \) is isomorphic to \( V \).

Proof Given any scalar \( \alpha \in \mathbb{F} \), we define

\[
\vartheta_i(\alpha) = (-1)^i(\alpha + i) \quad \text{for all} \quad i \in \mathbb{Z}.
\]

Since \( \mathbb{F} \) is algebraically closed and \( V \) is finite-dimensional, there exists an eigenvalue \( \alpha \) of \( X \) in \( V \). Since \( \text{char} \mathbb{F} = 0 \), for any distinct integers \( i, j \) the scalars \( \vartheta_i(\alpha) \) and \( \vartheta_j(\alpha) \) are equal if and only if \( i + j = -2\alpha \). Since there are at most \( d+1 \) distinct eigenvalues of \( X \) in \( V \), there exists an integer \( j \) such that \( \vartheta_j(\alpha) \) is an eigenvalue of \( X \), but \( \vartheta_{j-1}(\alpha) \) is not an eigenvalue of \( X \) in \( V \). Set

\[
\varepsilon = (-1)^j, \quad a = \alpha + j + \frac{d}{2}.
\]

Similarly, there is a scalar \( \beta \in \mathbb{F} \) and an integer \( k \) such that \( \vartheta_k(\beta) \) is an eigenvalue of \( Y \), but \( \vartheta_{k-1}(\beta) \) is not an eigenvalue of \( Y \) in \( V \). We set

\[
\varepsilon' = (-1)^k, \quad b = \beta + k + \frac{d}{2}.
\]

Under the settings, we have

\[
\begin{align*}
\varepsilon\theta_i &= \vartheta_{i+j}(\alpha) \quad \text{for all} \quad i \in \mathbb{Z}, \quad (40) \\
\varepsilon'\theta_i^* &= \vartheta_{i+k}(\beta) \quad \text{for all} \quad i \in \mathbb{Z}. \quad (41)
\end{align*}
\]

By Lemma 6.1, the element \( \kappa \) acts on \( V^{(\varepsilon, \varepsilon')} \) as scalar multiplication. Since \( \mathbb{F} \) is algebraically closed, there exists a scalar \( c \in \mathbb{F} \) such that the action of \( \kappa \) on \( V^{(\varepsilon, \varepsilon')} \) is the scalar multiplication by

\[
\omega = c^2 - a^2 - b^2 + \frac{(d+1)^2}{4}.
\]
To prove the theorem, it suffices to show that there exists a $\mathfrak{B}J$-module isomorphism from $E_d(a, b, c)$ into $V^{(\varepsilon, \varepsilon')}$.

Given any element $S \in \mathfrak{B}J$ and any $\theta \in \mathbb{F}$, we let

$$V_S^{(\varepsilon, \varepsilon')} (\theta) = \{ v \in V^{(\varepsilon, \varepsilon')} | Sv = \theta v \}. \tag{5}$$

Pick any $v \in V_X^{(\varepsilon, \varepsilon')} (\theta_0)$. Applying $v$ to (5) yields that

$$(X^2 + 2\theta_0 X + \theta_0^2 - 1)Yv = (2\theta_0 \omega + \mu)v. \tag{42}$$

By Lemma 6.2, the left-hand side of (42) is equal to

$$(X - \theta_{-1})(X - \theta_1)Yv. \tag{43}$$

Left multiplying either side of (42) by $(X - \theta_0)$, we obtain that

$$(X - \theta_{-1})(X - \theta_0)(X - \theta_1)Yv = 0. \tag{44}$$

By Table 1 and (40), the scalar $\theta_{-1}$ is not an eigenvalue of $X$ in $V^{(\varepsilon, \varepsilon')}$. It follows that

$$(X - \theta_0)(X - \theta_1)Yv = 0. \tag{45}$$

In other words, $(X - \theta_1)Yv \in V_X^{(\varepsilon, \varepsilon')} (\theta_0)$. This shows that $V_X^{(\varepsilon, \varepsilon')} (\theta_0)$ is invariant under $(X - \theta_1)Y$. Since $\mathbb{F}$ is algebraically closed, there exists an eigenvector $u$ of $(X - \theta_1)Y$ in $V_X^{(\varepsilon, \varepsilon')} (\theta_0)$. Similarly, there exists an eigenvector $w$ of $(Y - \theta_1^*)X$ in $V_Y^{(\varepsilon, \varepsilon')} (\theta_0^*)$.

Define

$$u_i = \prod_{h=0}^{i-1} (Y - \theta_h^*)u \quad \text{for all } i \in \mathbb{N}, \tag{43}$$

$$w_i = \prod_{h=0}^{i-1} (X - \theta_h)w \quad \text{for all } i \in \mathbb{N}. \tag{44}$$

We proceed by induction to show that

$$(X - \theta_i)u_i \in \text{span}_\mathbb{F}\{u_0, u_1, \ldots, u_{i-1}\} \quad \text{for all } i \in \mathbb{N}. \tag{45}$$

Since $u$ is an eigenvector of $(X - \theta_1)Y$ in $V_X^{(\varepsilon, \varepsilon')} (\theta_0)$, the claim is true for $i = 0, 1$. Now suppose that $i \geq 2$. Applying $u_{i-2}$ to (4), we obtain that

$$(Y^2 X + 2YXY + XY^2 - X - 2\omega Y)u_{i-2} = \lambda u_{i-2}. \tag{46}$$
By Lemma 6.1, the right-hand side of (46) is a scalar multiple of \( u_{i-2} \). Applying induction hypothesis and (43), the left-hand side of (46) is equal to

\[
(\theta_{i-2} + 2\theta_{i-1} + X)u_i
\]

plus an \( \mathbb{F} \)-linear combination of \( u_0, u_1, \ldots, u_{i-1} \). By Lemma 6.2(i), the term (47) is equal to \((X - \theta_i)u_i\). Combining the above comments, the result (45) follows. Next, we show that \( \{u_i\}_{i=0}^d \) is an \( \mathbb{F} \)-basis for \( V(\varepsilon, \varepsilon') \). Suppose on the contrary that there is an integer \( j \) with \( 0 \leq j \leq d - 1 \) such that \( u_{j+1} \) is an \( \mathbb{F} \)-linear combination of \( u_0, u_1, \ldots, u_j \). Let \( W \) denote the \( \mathbb{F} \)-subspace of \( V \) spanned by \( u_0, u_1, \ldots, u_j \). Observe that \( W \) is \( Y \)-invariant by (43) and \( X \)-invariant by (45). It follows from Lemma 2.2 that \( W \) is a nonzero \( \mathfrak{B} \mathfrak{J} \)-submodule of \( V(\varepsilon, \varepsilon') \). Since the \( \mathfrak{B} \mathfrak{J} \)-module \( V(\varepsilon, \varepsilon') \) is irreducible, this implies that \( W = V(\varepsilon, \varepsilon') \). However, \( W \) is of dimension less than or equal to \( d \), which contradicts that the dimension of \( V(\varepsilon, \varepsilon') \) is \( d + 1 \). Therefore, \( \{u_i\}_{i=0}^d \) is an \( \mathbb{F} \)-basis for \( V(\varepsilon, \varepsilon') \). By similar arguments, we have

\[
(Y - \theta_i^*)w_i \in \text{span}_{\mathbb{F}}\{w_0, w_1, \ldots, w_{i-1}\} \quad \text{for all} \quad i \in \mathbb{N}
\]

and \( \{w_i\}_{i=0}^d \) is an \( \mathbb{F} \)-basis for \( V(\varepsilon, \varepsilon') \).

By (45), the matrix representing \( X \) with respect to the \( \mathbb{F} \)-basis \( \{u_i\}_{i=0}^d \) for \( V(\varepsilon, \varepsilon') \) is an upper triangular matrix in which the diagonal entries are \( \{\theta_i\}_{i=0}^d \). Applying the Cayley–Hamilton theorem yields that

\[
\prod_{i=0}^d (X - \theta_i)w_0 = 0.
\]

Hence, \( X w_d = \theta_d w_d \) by (44) and the matrix representing \( X \) with respect to the \( \mathbb{F} \)-basis \( \{w_i\}_{i=0}^d \) for \( V(\varepsilon, \varepsilon') \) is

\[
\begin{pmatrix}
\theta_0 & & & & 0 \\
1 & \theta_1 & & & \\
& 1 & \theta_2 & & \\
& & \ddots & \ddots & \\
0 & & & 1 & \theta_d
\end{pmatrix}
\]

By (48), the matrix representing \( Y \) with respect to the \( \mathbb{F} \)-basis \( \{w_i\}_{i=0}^d \) for \( V(\varepsilon, \varepsilon') \) is upper triangular with diagonal entries \( \{\theta_i^*\}_{i=0}^d \) and let \( \{\varphi_i^*\}_{i=1}^d \) denote its superdiagonal
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entries as follows:

$$
\begin{pmatrix}
\theta_0^* & \varphi_1' & \varphi_2' & \cdots & * \\
\theta_1^* & \theta_2^* & \cdots & & \\
\theta_2^* & & \cdots & & \\
0 & & & \theta_d^* & \\
\end{pmatrix}
$$

Applying $w_{i-1}$ to either side of (5) and comparing the coefficients of $w_i$, we obtain that

$$
\varphi_{i+1}' + 2\varphi_i' + \varphi_{i-1}' = 2\omega - (3\theta_i + \theta_{i-1})\theta_i^* - (\theta_i + 3\theta_{i-1})\theta_{i-1}^* \quad (1 \leq i \leq d),
$$

(50)

where $\varphi_0'$ and $\varphi_d'$ are interpreted as zero. It is straightforward to verify that $\varphi_i' = \varphi_i$ for all $1 \leq i \leq d$ satisfy the recurrence relation (50). Since $\text{char } F = 0$, the corresponding homogeneous recurrence relation

$$
\sigma_{i+1} + 2\sigma_i + \sigma_{i-1} = 0 \quad (1 \leq i \leq d)
$$

with the initial values $\sigma_0 = 0$ and $\sigma_d+1 = 0$ has the unique solution $\sigma_i = 0$ for all $0 \leq i \leq d+1$. Thus, $\varphi_i' = \varphi_i$ for all $1 \leq i \leq d$. So far we have seen that

$$
Y w_0 = \theta_0^* w_0,
$$

(51)

$$
(Y - \theta_1^*)(X - \theta_0)w_0 = \varphi_1 w_0,
$$

(52)

$$
\kappa w_0 = \omega w_0.
$$

(53)

Applying $w_0$ to either side of (4) and simplifying the resulting equation by (52), it yields that

$$
\lambda w_0 = \omega^* w_0.
$$

(54)

Similarly, we have $\mu u_0 = \omega^* u_0$. It follows from Lemma 6.1 that

$$
\mu w_0 = \omega^* w_0.
$$

(55)

In light of (51)–(55), it follows from Proposition 3.7 that there exists a unique BI-module homomorphism $M_d(a, b, c) \to V^{(\epsilon, \epsilon')}$ that sends $m_0$ to $w_0$. Combined with (49), there exists a BI-module homomorphism

$$
E_d(a, b, c) \to V^{(\epsilon, \epsilon')}
$$

(56)

that sends $v_0$ to $w_0$ by Proposition 3.10. Since the BI-module $V^{(\epsilon, \epsilon')}$ is irreducible, the homomorphism (56) is onto. Since both of $E_d(a, b, c)$ and $V^{(\epsilon, \epsilon')}$ are of dimension $d + 1$, it follows that (56) is an isomorphism. The result follows. □
Lemma 6.4 For any \((\varepsilon, \varepsilon') \in \{\pm 1\}^2\), the traces of \(X\) and \(Y\) on the \(B\mathfrak{J}\)-module \(E_d(a, b, c)^{(\varepsilon, \varepsilon')}\) are

\[-\varepsilon \cdot \frac{d + 1}{2}, \quad -\varepsilon' \cdot \frac{d + 1}{2},\]

respectively.

**Proof** It is straightforward to verify the lemma using Proposition 2.4(i) and Table 1. \(\Box\)

Lemma 6.5 The elements \(\kappa + \mu, \lambda + \kappa, \mu + \lambda\) act on the \(B\mathfrak{J}\)-module \(E_d(a, b, c)\) as scalar multiplication by

\[-2 \left( a + \frac{d + 1}{2} \right) \left( a - \frac{d + 1}{2} \right),\]

\[-2 \left( b + \frac{d + 1}{2} \right) \left( b - \frac{d + 1}{2} \right),\]

\[-2 \left( c + \frac{d + 1}{2} \right) \left( c - \frac{d + 1}{2} \right),\]

respectively.

**Proof** It is straightforward to verify the lemma using Proposition 2.4(ii). \(\Box\)

**Proof of Theorem 2.5** By Theorems 4.5 and 5.3, the function \(\mathcal{E}\) is well defined. By Theorem 6.3, the function \(\mathcal{E}\) is surjective. Since any element of \(B\mathfrak{J}\) has the same trace and any central element of \(B\mathfrak{J}\) acts as the same scalar on the isomorphic finite-dimensional irreducible \(B\mathfrak{J}\)-modules, it follows from Lemmas 6.4 and 6.5 that \(\mathcal{E}\) is injective. The result follows. \(\Box\)

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References

1. Bannai, E., Ito, T.: Algebraic Combinatorics I: Association Schemes. Benjamin/Cummings, Menlo Park (1984)
2. Baseilhac, P., Genest, V.X., Vinet, L., Zhedanov, A.: An embedding of the Bannai–Ito algebra in \(U(\mathfrak{osp}(1, 2))\) and \(-1\) polynomials. Lett. Math. Phys. 108, 1623–1634 (2018)
3. De Bie, H., De Clercq, H.: The \(q\)-Bannai–Ito algebra and multivariate \((−q)\)-Racah and Bannai–Ito polynomials, arXiv:1902.07883
4. De Bie, H., Genest, V.X., Tsujimoto, S., Vinet, L., Zhedanov, A.: The Bannai–Ito algebra and some applications. J. Phys. Conf. Ser. 597(012001), 16 (2015)
5. De Bie, H., Genest, V.X., Tsujimoto, S., Vinet, L., Zhedanov, A.: The Bannai–Ito algebra and some applications. J. Phys. Conf. Ser. 597(012001), 16 (2015)
6. De Bie, H., Genest, V.X., Vinet, L.: A Dirac–Dunkl equation on \(S^2\) and the Bannai–Ito algebra. Commun. Math. Phys. 344, 447–464 (2016)
7. De Bie, H., Genest, V.X., Vinet, L.: The \(\mathbb{Z}_2^n\) Dirac–Dunkl operator and a higher rank Bannai–Ito algebra. Adv. Math. 303, 390–414 (2016)
Finite-dimensional irreducible $\mathfrak{B}_3$-modules at $F = 0$

8. Bu, L., Hou, B., Gao, S.: The classification of finite-dimensional irreducible modules of the Racah algebra. Commun. Algebra 47, 1869–1891 (2019)

9. Crampé, N., Frappat, L., Vinet, L.: Centralizers of the superalgebra $\mathfrak{osp}(1|2)$: the Brauer algebra as a quotient of the Bannai–Ito algebra, arXiv:1906.03936

10. Genest, V.X., Lapointe, L., Vinet, L.: $\mathfrak{osp}(1, 2)$ and generalized Bannai–Ito algebras. Trans. Am. Math. Soc. (2018). https://doi.org/10.1090/tran/7733. (to appear in print)

11. Genest, V.X., Vinet, L., Zhedanov, A.: The Bannai–Ito algebra and a superintegrable system with reflections on the two-sphere. J. Phys. A Math. Theor. 47(205202), 13 (2014)

12. Genest, V.X., Vinet, L., Zhedanov, A.: A Laplace–Dunkl equation on $S^2$ and the Bannai–Ito algebra. Commun. Math. Phys. 336, 243–259 (2015)

13. Genest, V.X., Vinet, L., Zhedanov, A.: Embeddings of the Racah algebra into the Bannai–Ito algebra. SIGMA 11(050), 11 (2015)

14. Genest, V.X., Vinet, L., Zhedanov, A.: The non-symmetric Wilson polynomials are the Bannai–Ito polynomials. Proc. Am. Math. Soc. 144, 5217–5226 (2016)

15. Hou, B., Wang, M., Gao, S.: The classification of finite-dimensional irreducible modules of Bannai/Ito algebra. Commun. Algebra 44, 919–943 (2016)

16. Huang, H.-W.: Finite-dimensional modules of the Racah algebra and the additive DAHA of type $(C_1^\vee, C_1)$. submitted

17. Huang, H.-W.: The Racah algebra as a subalgebra of the Bannai–Ito algebra, submitted

18. Huang, H.-W.: Finite-dimensional irreducible modules of the universal Askey–Wilson algebra. Commun. Math. Phys. 340, 959–984 (2015)

19. Huang, H.-W., Bockting-Conrad, S.: Finite-dimensional irreducible modules of the Racah algebra at characteristic zero. SIGMA 16, 018, 17 (2020)

20. Terwilliger, P.: The universal Askey–Wilson algebra. SIGMA 7(069), 24 (2011)

21. Terwilliger, P., Vidunas, R.: Leonard pairs and the Askey–Wilson relations. J. Algebra Appl. 3, 411–426 (2004)

22. Tsujimoto, S., Vinet, L., Zhedanov, A.: Dunkl shift operators and Bannai–Ito polynomials. Adv. Math. 229, 2123–2158 (2012)

23. Zhedanov, A.: “Hidden symmetry” of Askey–Wilson polynomials. Teoreticheskaya i Matematicheskaya Fizika 89, 190–204 (1991). English transl.: Theoretical and Mathematical Physics, 89:1146–1157, 1991

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