A CONSTRUCTIVE APPROACH TO COHERENT SHEAVES VIA GABRIEL MONADS

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ABSTRACT. The ideal transform of a graded module $M$ is known to compute the module of twisted global sections of the sheafification of $M$ over a relative projective space. We introduce a second description motivated by the relative BGG-correspondence. However, our approach avoids the full BGG-correspondence by replacing the Tate resolution with the so-called purely linear saturation and the Castelnuovo-Mumford regularity with the (often enough much smaller) linear regularity. This paper provides elementary, constructive, and unified proofs that these two descriptions compute the (truncated) modules of twisted global sections. The main argument relies on an established characterization of Gabriel monads. As a byproduct we get Gröbner-basis-based algorithms for eliminating homogeneous coordinates which use the fast degrevlex ordering rather than the usually more expensive block orderings.

1. INTRODUCTION

We consider coherent sheaves over the projective space $\mathbb{P}_B^n \to \text{Spec } B$ for a suitable ring $B$. Any such coherent sheaf $\mathcal{F} \in \text{Coh} \mathbb{P}_B^n$ can be described by a graded module over the polynomial ring $S := B[x_0, \ldots, x_n]$. Even though this representation is not unique, among the different graded $S$-modules representing $\mathcal{F}$ there is the distinguished representative $H_0^\bullet(\mathcal{F}) := \bigoplus_{p \in \mathbb{Z}} H^0(\mathbb{P}_B^n, \mathcal{F}(p))$, the module of twisted global sections. In general the module of twisted global sections is not finitely generated, but any of its truncations $H^0_{\geq d}(\mathcal{F}) := (H_0^\bullet(\mathcal{F}))_{\geq d}$ is.

It is well-known that the ideal transform $D_m(M) := \lim_{\leftarrow \ell} \text{Hom}_\bullet(m^\ell, M)$ computes $H^0_0(\tilde{M})$, where $\tilde{M}$ is the sheafification of the graded $S$-module $M$ and $m = \langle x_0, \ldots, x_n \rangle \subset S$ is the irrelevant ideal (cf., e.g., [Vas98, §C.3]). This isomorphism implies $H_\bullet \simeq \lim_{\leftarrow \ell} \text{Ext}_\bullet^q(m^\ell, -)$ for the higher derived cohomology functors [BS98, 20.4.4].

Another description of the cohomology functors $H^q_\bullet$ arose from the BGG-correspondence [BGG78]. It is a triangle equivalence between the bounded derived category of $\text{Coh} \mathbb{P}_B^n$ (originally over a base field $B$) and the stable category of finitely generated graded $E$-modules over the exterior algebra $E$, the Koszul dual algebra of $S$. Since $E$ is a Frobenius algebra, this stable category is easily seen to be triangle equivalent to the homotopy category of so-called Tate complexes. A constructive description of the composition of these two triangle equivalences was given in [EFS03, DE02]. The treatment of the relative BGG-correspondence in [ES08] does not only describe the coherent sheaf cohomologies $H^q(\tilde{M}) = R^q\pi_*M$ as $B$-modules, but

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also provides a concrete realization of the direct image complex $R\pi_\ast \tilde{M}$. But in this approach even the computation of global sections $H^0(\tilde{M})$ in the relative case relies a priori on the full Tate resolution.

Abstractly, the category $\text{Coh} \mathbb{P}^n_B$ of coherent sheaves on $\mathbb{P}^n_B$ is equivalent to the Serre quotient category $\mathcal{A}/\mathcal{C}$ of the Abelian category $\mathcal{A}$ of finitely presented graded $S$-modules modulo a certain subcategory $\mathcal{C}$. The categories $\mathcal{A}$ and $\mathcal{C}$ can be replaced by their respective full subcategories of modules which vanish in degrees $< d$ (cf. Proposition 7.1). The $\mathcal{A}$-endofunctor $M \mapsto H^0_{\geq d}(\tilde{M})$ is a special case of what we call a Gabriel monad, which we characterized in [BLH13] by a short set of properties.

In this paper we verify that both the $(d$-truncated version of the) ideal transform $D_{m, \geq d}$ and the so-called purely linear saturation $S^{\geq d}$ satisfy these properties and hence compute the $(d$-truncated) module of twisted global sections. These are Theorems 4.2 and 6.7, respectively. Our constructive description of $S^{\geq d}$ avoids the above drawback of computing the full Tate resolution, while providing the $S$-module structure of $H^0_{\geq 0}(\tilde{M})$. Furthermore, the so-called linear regularity gives the precise number of recursive steps needed to achieve saturation. Since computing $S^{\geq d}$ relies on Gröbner bases over the exterior algebra $E$ of finite rank over $B$ the involved algorithms are often faster than the ones for the ideal transform. The latter involve Gröbner bases over polynomial ring $S$ of infinite rank over $B$.

For a coherent sheaf $\tilde{M}$ (e.g., the structure sheaf $\mathcal{O}_X \subset \mathcal{O}_{\mathbb{P}^n_B}$ of a subscheme $X \subset \mathbb{P}^n_B$) the direct image sheaf $H^0(\tilde{M}) = \pi_\ast \tilde{M}$ over $\text{Spec} B$ is the degree zero part of $H^0_{\geq 0}(\tilde{M})$ (which we compute using Remark 3.1). It corresponds to eliminating all $n + 1$ homogeneous coordinates from a graded free presentation of $\tilde{M}$ (e.g., the defining equations of $X \subset \mathbb{P}^n_B$).\(^1\) And the proof of correctness of this algorithm is now basically a categorical argument, another instance of the deep interconnection between abstract and algorithmic mathematics.

Two further applications which rely on computability of the Gabriel monad now become algorithmically accessible for the category $\text{Coh} \mathbb{P}^n_B$. If the Gabriel monad is constructive then the Serre quotient category $\mathcal{A}/\mathcal{C}$ is constructively Abelian once $\mathcal{A}$ is constructively Abelian [BLH14b, Appendix D].\(^2\) And in [BLH14c] we showed how and under what conditions the computability of the bivariate $\text{Hom}$ and $\text{Ext}^i$ functors in $\mathcal{A}/\mathcal{C}$ be reduced to the computability of $\text{Hom}$ and $\text{Ext}^i$ in $\mathcal{A}$ (modulo a directed colimit process if $i > 0$).

Section 2 fixes the notation from category theory used in this paper. Section 3 introduces some preliminaries of graded $S$-modules and includes various characterizations of saturated graded modules. Section 4 then proves that the ideal transform computes the Gabriel monad of coherent sheaves. In Section 5 we constructively describe the adjoint equivalence between the category of finitely presented graded $S$-modules and a category of certain linear complexes over the exterior algebra $E$. Section 6 introduces a constructive Gabriel monad for these linear

\(^1\)Before eliminating a subset of the affine coordinates the input has to first be homogenized in this subset (and after applying our algorithm de-homogenized). Even if the original affine input is coincidentally homogenous, a homogenization is still necessary!

\(^2\)The base $\text{Spec} B$ might even serve as the ambient space of a geometric quotient, e.g., if $B$ is the Cox ring of a toric variety.

\(^3\)However, the approach using Gabriel morphisms in [BLH14b] seems computationally faster.
complexes which via the adjoint equivalence of Section 5 computes the Gabriel monad for coherent sheaves. Finally, in Section 7 we discuss various models of \( \text{Coh} \mathbb{P}^n_B \) as a Serre quotient category equipped with a Gabriel monad.

2. Preliminaries on Serre Quotient Categories

A non-empty full subcategory \( \mathcal{C} \) of an Abelian category \( \mathcal{A} \) is called thick if it is closed under passing to subobjects, factor objects, and extensions. In this case the Serre quotient category \( \mathcal{A}/\mathcal{C} \) is a category with the same objects as \( \mathcal{A} \) and Hom-groups defined by the directed colimit

\[
\text{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) := \lim_{M' \leq M, N' \leq N, M'/M', N'/N' \in \mathcal{C}} \text{Hom}_\mathcal{A}(M', N'/N').
\]

The canonical functor \( \mathcal{D} : \mathcal{A} \to \mathcal{A}/\mathcal{C} \) is defined to be the identity on objects and maps a morphism \( \varphi \in \text{Hom}_\mathcal{A}(M, N) \) to its class in the directed colimit \( \text{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) \). The category \( \mathcal{A}/\mathcal{C} \) is Abelian and the canonical functor \( \mathcal{D} : \mathcal{A} \to \mathcal{A}/\mathcal{C} \) is exact. An object \( M \in \mathcal{A} \) is called \( \mathcal{C} \)-saturated if \( \text{Ext}^1_\mathcal{A}(C, M) \cong \text{Ext}^1_\mathcal{A}(C, M) \cong 0 \) for all \( C \in \mathcal{C} \), i.e., \( M \) has no nonzero subobjects in \( \mathcal{C} \) and every extension of \( M \) by an object \( C \in \mathcal{C} \) is trivial. Denote by Sat\( \mathcal{C}(\mathcal{A}) \subset \mathcal{A} \) the full subcategory of \( \mathcal{C} \)-saturated objects and by \( \iota : \text{Sat}_\mathcal{C}(\mathcal{A}) \hookrightarrow \mathcal{A} \) its full embedding. A sequence \( F \) in Sat\( \mathcal{C}(\mathcal{A}) \) is exact if and only if \( \iota(F) \) only has defects in \( \mathcal{C} \).

A thick subcategory \( \mathcal{C} \subset \mathcal{A} \) is called localizing if the canonical functor \( \mathcal{D} : \mathcal{A} \to \mathcal{A}/\mathcal{C} \) admits a right adjoint \( \mathcal{I} : \mathcal{A}/\mathcal{C} \to \mathcal{A} \), called the section functor of \( \mathcal{D} \). In this case, the image of \( \mathcal{I} \) is contained in Sat\( \mathcal{C}(\mathcal{A}) \) and \( \mathcal{A}/\mathcal{C} \xrightarrow{\mathcal{I}} \mathcal{I}(\mathcal{A}/\mathcal{C}) \hookrightarrow \text{Sat}_\mathcal{C}(\mathcal{A}) \) are equivalences of categories. The Hom-adjunction \( \text{Hom}_{\mathcal{A}/\mathcal{C}}(\mathcal{D}(M), \mathcal{D}(N)) \cong \text{Hom}_\mathcal{A}(M, (\mathcal{I} \circ \mathcal{D})(N)) \) allows to compute Hom-groups in \( \mathcal{A}/\mathcal{C} \) if they are computable in \( \mathcal{A} \) and the monad \( \mathcal{I} \circ \mathcal{D} \) is computable. In particular, this avoids computing the directed colimit in the definition of \( \text{Hom}_{\mathcal{A}/\mathcal{C}} \). We call any monad equivalent to \( \mathcal{I} \circ \mathcal{D} \) a Gabriel monad (of \( \mathcal{A} \) w.r.t. \( \mathcal{C} \)). The following theorem characterizes Gabriel monads.

**Theorem 2.1** ([BLH13, Thm. 3.6 (Thm. 4.6 in arXiv version)]). *Let \( \mathcal{C} \subset \mathcal{A} \) be a localizing subcategory of the Abelian category \( \mathcal{A} \) and \( \iota : \text{Sat}_\mathcal{C}(\mathcal{A}) \hookrightarrow \mathcal{A} \) the full embedding. An endofunctor \( \mathcal{W} : \mathcal{A} \to \mathcal{A} \) together with a natural transformation \( \eta : \text{Id}_\mathcal{A} \to \mathcal{W} \) is a Gabriel monad (of \( \mathcal{A} \) w.r.t. \( \mathcal{C} \)) if and only if the following five conditions hold:

(a) \( \mathcal{C} \subset \ker \mathcal{W} \),

(b) \( \mathcal{W}(\mathcal{A}) \subset \text{Sat}_\mathcal{C}(\mathcal{A}) \),

(c) the corestriction \( \text{co-}\text{res}_{\text{Sat}_\mathcal{C}(\mathcal{A})} \mathcal{W} \) of \( \mathcal{W} \) to \( \text{Sat}_\mathcal{C}(\mathcal{A}) \) is exact,

(d) \( \eta \mathcal{W} = \mathcal{W} \eta \), and

(e) \( \eta_\iota : \text{Id}_{\mathcal{A}|_{\text{Sat}_\mathcal{C}(\mathcal{A})}} \to \mathcal{W}|_{\text{Sat}_\mathcal{C}(\mathcal{A})} \) is a natural isomorphism.*

In Sections 4 and 6 we utilize this theorem to prove that certain functors are the Gabriel monad of the category of coherent sheaves on the relative projective space \( \mathbb{P}^n_B \), and thus compute the (truncated) module of twisted global sections. However, this theorem, abstract as it is, can be applied to categories of coherent sheaves of more general schemes.
3. Graded modules over the free polynomial ring

For the rest of the paper let $B$ denote a Noetherian commutative ring with 1, $V$ a free $B$-module of rank $n+1$, $W := V^* = \text{Hom}_B(V, B)$ its $B$-dual, and $x_0, \ldots, x_n$ a $B$-basis of $W$. Set 

$$S := \text{Sym}_B(W) = B[V] = B[x_0, \ldots, x_n]$$

to be the free polynomial ring over $B$ in the $n+1$ indeterminates $x_0, \ldots, x_n$. Setting $\deg(x_j) = 1$ turns $S$ into a positively graded ring $S = \bigoplus_{i \geq 0} S_i$ where $S_i$ is the set of homogeneous polynomials of degree $i$ in $S$. Define the irrelevant ideal 

$$\mathfrak{m} := S_{>0} = \langle x_0, \ldots, x_n \rangle \triangleleft S.$$ 

The isomorphism $B = S_0 \cong S/\mathfrak{m}$ endows $B$ with a natural graded $S$-module structure.

To make the statements of this paper constructive, the ring $S$ needs to have a Gröbner bases algorithm. This is the case if $B$ has effective coset representatives [AL94, §4.3], i.e., for every ideal $I \subset B$ we can determine a set $T$ of coset representatives of $B/I$, such that for every $b \in B$ we can compute a unique $t \in T$ with $b + I = t + I$.

We denote by $S\text{-mod}$ the category of (non-graded) finitely presented $S$-modules and by $S\text{-grmod}$ the category of finitely presented graded $S$-modules. Further denote by $S\text{-grmod}_{\geq d} \subset S\text{-grmod}$ the full subcategory of all modules $M$ with $M = M_{\geq d}$. Define the shift autoequivalence on $S\text{-grmod}$ by $M(i)_j := M_{i+j}$ for all $i \in \mathbb{Z}$; it induces an endofunctor on the subcategory $S\text{-grmod}_{\geq d}$ for $i \leq 0$.

Remark 3.1. We briefly describe an algorithm to compute the $i$-th homogeneous part of an $M \in S\text{-grmod}$. Such a module is realized on the computer as the cokernel of a free $S$-presentation $M \leftarrow \bigoplus_k S(g_k) \stackrel{\pi}{\leftarrow} \bigoplus_k S(r_k)$. The image of the graded submodule $\bigoplus_{k, i+g_k \geq 0} S_{\geq i+g_k}(g_k) \leq \bigoplus_k S(g_k)$ under $\pi$ is the graded submodule $\langle M_i \rangle \leq M$, which we compute as the kernel of the cokernel of the restricted map $M \leftarrow \bigoplus_{k, i+g_k \geq 0} S_{\geq i+g_k}(g_k)$. Computing a free $S$-presentation $\langle M_i \rangle \leftarrow \bigoplus_k S(i) \stackrel{\mathfrak{m}_i}{\leftarrow} \bigoplus_j S(r'_j)$ of $\langle M_i \rangle$ thus involves two successive syzygy computations as explained in [BLH11, (10) in the proof of Theorem 3.4]. To get a free $B$-presentation of $M_i$ we just need to tensor the last exact sequence with $B = S/\mathfrak{m}$ over $S$, which corresponds to extracting the degree 0 relations in the reduced Gröbner basis of the $S$-matrix of relations $\mathfrak{m}_i$.

3.1. Internal and external Hom functors. Let $M, N \in S\text{-grmod}$. Then the Hom-group $\text{Hom}_{S\text{-mod}}(M, N)$ of their underlying modules in $S\text{-mod}$ is again naturally graded. This induces internal Hom functors 

$$\text{Hom}_{\bullet} : S\text{-grmod}^{\text{op}} \times S\text{-grmod} \to S\text{-grmod}$$

in the category $S\text{-grmod}$ and

$$\text{Hom}_{\geq d} : S\text{-grmod}^{\text{op}}_{\geq d} \times S\text{-grmod}_{\geq d} \to S\text{-grmod}_{\geq d}.$$ 

in $S\text{-grmod}_{\geq d}$. These internal Hom functors are algorithmically computable if $B$ has effective coset representatives (cf., e.g., [AL94, §4.3] and [BLH11, §3.3]).
The (external) Hom-groups of the category $S$-grmod are finitely generated $B$-modules. They can be recovered as the graded part of degree 0 of the corresponding internal Hom’s:

$$\Hom_{S\text{-grmod}}(M, N) \cong \Hom_\bullet(M, N)_0,$$

$$\Hom_{S\text{-grmod}_{≥d}}(M, N) \cong \Hom_{≥d}(M, N)_0 \quad \text{for } d ≤ 0.$$  

In particular, $\Hom_{S\text{-grmod}}(S, M) ≅ M_0$ and $\Hom_{S\text{-grmod}_{≥d}}(S, M) ≅ M_0$ for $d ≤ 0$.

**Remark 3.2.** Applying $\Hom_\bullet(−, M)$ to the short exact sequence $S/m^\ell ⊆ S \Longleftrightarrow m^\ell$ yields

$$\eta_\ell^f : \Hom_\bullet(S/m^\ell, M) \to M \xrightarrow{\eta_\ell^f} \Hom_\bullet(m^\ell, M) \to \Ext_\bullet^1(S/m^\ell, M)$$

as part of the long exact contravariant $\Ext_\bullet$-sequence. We will repeatedly refer to this exact sequence as well as to the $\ell = 1$ case

$$\eta_1^f : \Hom_\bullet(B, M) \to M \xrightarrow{\eta_1^f} \Hom_\bullet(m, M) \to \Ext_\bullet^1(B, M).$$

### 3.2. Quasi-zero modules

Let $S$-grmod$^0$ denote the thick subcategory of **quasi-zero** modules, i.e., those with $M_{≥\ell} = 0$ for $\ell$ large enough. Analogously, we denote by $S$-grmod$^0_{≥d}$ the **localizing** (cf. Theorem 4.2) subcategory of quasi-zero modules in $S$-grmod$_{≥d}$.

**Remark 3.3.** For $M ∈ S$-grmod. Then for all $\ell ≥ 0$

(a) $\Tor_i^S(S/m^\ell, M)_\bullet ∈ S$-grmod$^0$ for all $i ≥ 0$.

(b) $\Ext_j^S(S/m^\ell, M) ∈ S$-grmod$^0$ for all $j ≥ 0$.

(c) $\Ext_j^d(m^\ell, M) ∈ S$-grmod$^0$ for all $j ≥ 1$.

**Proof.** The existence of a finitely generated free resolution of the first argument $S/m^\ell$ (and hence of $m^\ell$) implies that all the above derived modules lie in $S$-grmod. By applying $S/m^\ell ⊗_S −$ to a projective resolution of $M$ and $\Hom_\bullet(S/m^\ell, −)$ to an injective resolution of $M$ shows that the ideal $m^\ell ⊆ S$ annihilates $\Tor_i^S(S/m^\ell, M)_\bullet$ and $\Ext_j^d(S/m^\ell, M)$, which implies that they are also finitely generated $S/m^\ell$-modules, proving (a) and (b). The existence of the connecting isomorphisms $\Ext_j^d(m^\ell, M) ≅ \Ext_{j+1}^d(S/m^\ell, M)$ ($j ≥ 1$) finally implies (c). $\square$

**Remark 3.4.** The use of the nonconstructive injective resolution in the previous proof is an example of an admissible use of nonconstructive arguments in an otherwise constructive setup to prove statements which neither involve existential quantifiers nor disjunctions (so-called negative formulae): $\Ext_\bullet(S/m^\ell, M)$ has two descriptions. The nonconstructive one in the proof and the constructive one in which $\Hom(−, M)$ is applied to a finite free resolution of $S/m^\ell$. Although the isomorphism between the two descriptions is not constructive it is “good enough” for transferring the property we want to establish.

### 3.3. Regularity, linear regularity, and relation to $\text{Tor}$ and $\text{Ext}$

For convenience of the reader we recall the definition of the **Castelnuovo-Mumford regularity** in the relative case from [ES08, §2]. For any quasi-zero graded $S$-module $N$ define

$$\text{reg } N := \max\{d ∈ Z \mid N_d ≠ 0\}.$$
The regularity of the zero module is set to \(-\infty\). Then, for \(M \in S\text{-grmod}\) the \(S\)-module \(\text{Tor}_i^S(B, M)\) is quasi-zero and
\[
\text{reg } M := \max \{ \text{reg } \text{Tor}_i^S(B, M) - i \mid i = 0, \ldots, n + 1\}.
\]

Equivalently, one can define
\[
\text{reg } M := \max \{ \text{reg } \text{Tor}_i^S(B, M) + j \mid j = 0, \ldots, n + 1\}
\]
using the local cohomology modules \(H^j_m(M) = \lim_{\ell \to \infty} \text{Ext}_{S}^j(S/m^\ell, M)\) (cf. [ES08, Prop. 2.1]).\(^4\)

In fact only \(\ell = 1\) in this sequential colimit is relevant. To see this we need the following result, which we also use in the proof of our key Lemma 5.3.

**Lemma 3.5.** There exists a natural isomorphism
\[
\text{Tor}_i^S(B, M) \cong \text{Ext}_{S}^{n+1-i}(\wedge^{n+1}V, M).
\]

**Proof.** The Tor-Ext spectral sequence\(^5\) \(\text{Tor}_p^q(\text{Ext}_{S}^q(\wedge^{n+1}V, S), M) \Rightarrow \text{Ext}_{S}^{p+q}(\wedge^{n+1}V, M)\) collapses since \(\text{Ext}_{S}^{p+q}(\wedge^{n+1}V, S) = B\). \(\square\)

When \(B = k\) is a field this Lemma becomes the intrinsic and rather generalizable form of the equality between the graded Betti numbers \(\beta_{ij} := \text{Tor}_i^S(B, M)_j\) and the graded **Bass numbers**:
\[
\mu_{n+1-i,j-n-1} := \text{Ext}_{S}^{n+1-i}(\wedge^{n+1}V, M)_j.
\]

**Remark 3.6.** Lemma 3.5 and the noncanonical isomorphism \(\wedge^{n+1}V \cong B(n + 1)\) yield\(^6\)
\[
\text{reg } M = \max \{ \text{reg } \text{Ext}_{S}^j(B, M) + j \mid j = 0, \ldots, n + 1\}.
\]

The value of the following definition will start to become obvious in Proposition 3.8 in the next subsection.

**Definition 3.7.** Define the **linear regularity** of \(M \in S\text{-grmod}\) to be
\[
\text{linreg } M = \max \{ \text{reg } \text{Ext}_{S}^j(B, M) + j \mid j = 0, 1\}.
\]

Analogously, the **\(d\)-th truncated linear regularity** of \(M \in S\text{-grmod}_{\geq d}\) is defined by
\[
\text{linreg}_{\geq d} M = \max \{ \text{reg } \text{Ext}_{S}^j(B, M) + j \mid j = 0, 1\}.
\]

Note that \(\text{linreg} = \text{reg}\) on \(S\text{-grmod}^0\) and \(\text{linreg} \leq \text{reg}\) on \(S\text{-grmod}\). The motivation behind introducing \(\text{linreg}\) is that it offers a **sharp** upper bound in the saturation algorithms discussed below, where the use of the (often enough much larger) regularity would be a waste of computational resources.

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\(^4\)This definition clarifies the relation to two other regularity notions: The **geometric regularity** is defined by \(\text{g-reg } M := \max \{ \text{reg } H^j_m(M) + j \mid j = 1, \ldots, n + 1\}\) and the **regularity of the sheafification** \(\text{reg } \tilde{M} := \max \{ \text{reg } H^j_m(M) + j \mid j = 2, \ldots, n + 1\}\).

\(^5\)Cf. [Bar09] for a constructive treatment of such spectral sequences.
3.4. Saturated modules. The equivalent conditions (d) and (e) in the following proposition are computationally effective characterizations of saturated modules.

**Proposition 3.8.** For \( M \in S\text{-grmod} \) the following are equivalent:

(a) \( M \) is saturated w.r.t. \( S\text{-grmod}^0 \);
(b) \( \operatorname{Ext}_0^0(S/\mathfrak{m}^\ell, M) = 0 \) and \( \operatorname{Ext}_1^1(S/\mathfrak{m}^\ell, M) = 0 \) for all \( \ell \geq 0 \);
(c) The natural map \( \eta^\ell_M := \operatorname{Hom}_S(S \leftrightarrow \mathfrak{m}^\ell, M) : M \to \operatorname{Hom}_S(\mathfrak{m}^\ell, M) \) is an isomorphism for all \( \ell \geq 0 \);
(d) \( \operatorname{Ext}_0^0(B, M) = 0 \) and \( \operatorname{Ext}_1^1(B, M) = 0 \);
(e) The natural map \( \eta^\ell_M := \operatorname{Hom}_m(S \leftrightarrow \mathfrak{m}, M) : M \to \operatorname{Hom}_m(\mathfrak{m}, M) \) is an isomorphism.
(f) \( \operatorname{Tor}^S_n(B, M) = 0 \) and \( \operatorname{Tor}^S_n(B, M) = 0 \).
(g) \( \operatorname{linreg} M = -\infty \).

And if the base ring \( B \) is a field the above is also equivalent to:

(h) The projective dimension \( \text{pd} M \leq n - 1 \).

In the proof of this proposition, we use the following simple remark.

**Remark 3.9.** The kernel \( K \) of the epimorphism \( \mathfrak{m}^\ell \leftrightarrow \otimes^\ell \mathfrak{m} \) is concentrated in degree \( \ell \). To see this note that any homogeneous element in \( \otimes^\ell \mathfrak{m} \) of degree \( m > \ell \) which is the tensor product of monomials can be brought to the normal form \( x_{i_1} \otimes_S \cdots \otimes_S x_{i_{m-1}} \otimes_S x^{\mu} \) with \( i_1 \leq \cdots \leq i_{m-1} \leq \min \{ i \mid \mu_i \neq 0 \} \) and \( |\mu| = m - \ell + 1 \). This kernel \( K \) is free over \( B \) of rank \( (n+1)^\ell - \binom{n+\ell}{n} \) as the kernel of the \( B \)-epimorphism \( \text{Sym}^\ell W \leftrightarrow \otimes^\ell W \).

**Proof of Proposition 3.8.**

(b) \( \Leftrightarrow \) (c): The claim is obvious from the \( (\eta^\ell_M) \)-sequence in Remark 3.2.

(d) \( \Leftrightarrow \) (e): This is a special case of the equivalence (b) \( \Leftrightarrow \) (c) for \( \ell = 1 \).

(d) \( \Leftrightarrow \) (f): This is the statement of Lemma 3.5 for \( i = n+1 \) and \( i = n \).

(d) \( \Leftrightarrow \) (g): By definition of \( \operatorname{linreg} \).

(a) \( \Rightarrow \) (d): This follows directly from the definition of saturated objects (cf. Section 2), as \( B \in \mathcal{C} = S\text{-grmod}^0 \).

(e) \( \Rightarrow \) (c): Applying the \( \ell \)-th power of \( \operatorname{Hom}_m(S \leftrightarrow \mathfrak{m}, -) \) to \( M \) and taking the diagonal in the \( \ell \)-dimensional cube yields the isomorphism

\[
\varphi := M \cong \operatorname{Hom}_S(\otimes^\ell \mathfrak{m}, M)
\]

by the adjunction between \( \otimes \) and \( \operatorname{Hom}_S \). This isomorphism can be written as the composition

\[
\operatorname{Hom}_S(S \leftrightarrow \mathfrak{m}^\ell \leftrightarrow \otimes^\ell \mathfrak{m}, M) = \left( M \xrightarrow{\psi} \operatorname{Hom}_S(\mathfrak{m}^\ell, M) \xrightarrow{\sim} \operatorname{Hom}_S(\otimes^\ell \mathfrak{m}, M) \right).
\]

The homomorphism \( \chi \) is a monomorphism since \( \operatorname{Hom}_S \) is left exact and an epimorphism since \( \chi \circ \psi = \varphi \) is an isomorphism. Hence, \( \chi \) is isomorphism and thus \( \psi \) is an isomorphism.

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6Conditions (d) and (e) are in their use of Gröbner bases algorithmically equivalent. Computing them only involves the first two morphisms in the Koszul resolution of \( B \) (and then tensoring their duals with \( M \)). One might be tempted to expect that (d) is algorithmically superior to condition (f), which seem to involve an \( n + 1 \)-term resolution of either \( B \) or of \( M \). However, one can easily construct examples of \( M \in S\text{-grmod} \), where condition (f) is algorithmically superior, e.g., if the resolution of \( M \) terminates after few steps, long before reaching step \( n \).
It is clear that any $N \in S$-grmod$_0$ is an epimorphic image of $\bigoplus_{i \in I} (S/m^a_i)(b_i)$ for a finite set $I$ and suitable $a_i$ and $b_i$. Denote the kernel of $N \twoheadrightarrow \bigoplus_{i \in I} (S/m^a_i)(b_i)$ by $K$. Applying $\text{Hom}_\bullet(\cdot, M)$ to $N \twoheadrightarrow \bigoplus_{i \in I} (S/m^a_i)(b_i) \twoheadrightarrow K$ yields as parts of the long exact sequence

\[
\text{Hom}_\bullet(N, M) \hookrightarrow \text{Hom}_\bullet\left( \bigoplus_{i \in I} (S/m^a_i)(b_i), M \right),
\]

and

\[
\text{Hom}_\bullet(K, M) \rightarrow \text{Ext}_1^\bullet(N, M) \rightarrow \text{Ext}_1^\bullet\left( \bigoplus_{i \in I} (S/m^a_i)(b_i), M \right).
\]

The first part implies $\text{Hom}_\bullet(\cdot, M) = 0$ vanishing on $S$-grmod$_0$. In particular $\text{Hom}_\bullet(K, M) = 0$ since $K \in S$-grmod$_0$. Combining this and the second part implies that $\text{Ext}_1^\bullet(\cdot, M)$ vanishes on $S$-grmod$_0$.

(h) $\iff$ (b): If $B$ is a field then there exists a finite free (and not merely relatively free) presentation $M \leftarrow F_\bullet$ with $\text{Tor}_n^S(B, M)$ isomorphic to the head of $F_\bullet$. \hfill $\Box$

**Corollary 3.10.** For $M \in S$-grmod$_{\geq 0}$ the following are equivalent:

(a) $M$ is saturated w.r.t. $S$-grmod$_{\geq 0}$.

(b) $\text{Ext}_0^\geq 0(S/m^\ell, M) = 0$ and $\text{Ext}_1^\geq 0(S/m^\ell, M) = 0$ for all $\ell \geq 0$;

(c) The natural map $\eta_0^M : \text{Hom}_{\geq 0}(S \leftarrow m^\ell, M) : M \rightarrow \text{Hom}_{\geq 0}(m^\ell, M)$ is an isomorphism for all $\ell \geq 0$.

(d) $\text{Ext}_0^\geq 0(B, M) = 0$ and $\text{Ext}_1^\geq 0(B, M) = 0$;

(e) The natural map $\eta_1^M : \text{Hom}_{\geq 0}(S \leftarrow m, M) : M \rightarrow \text{Hom}_{\geq 0}(m, M)$ is an isomorphism.

(f) $\text{Tor}_n^S(B, M)_{\geq 0} = 0$ and $\text{Tor}_n^S(B, M)_{\geq 0} = 0$.

(g) $\text{linreg}_{\geq 0} M = -\infty$.

**4. Ideal Transforms**

Recall, the $m$-transform of $M \in S$-grmod is the (not necessarily finitely generated) graded $S$-module defined by the sequential colimit

\[
D_m := \lim_{\ell \geq 0} \text{Hom}_\bullet(m^\ell, -) : S\text{-grmod} \rightarrow S\text{-grMod}.
\]

On the full subcategory $S$-grmod$_{\geq d}$ the $d$-truncated $m$-transform

\[
D_{m, \geq d} := \lim_{\ell \geq d} \text{Hom}_{\geq d}(m^\ell, -) : S\text{-grmod}_{\geq d} \rightarrow S\text{-grmod}_{\geq d}
\]

is an endofunctor. This is a simple corollary of the following fact:
Remark 4.1. For each $M \in S\text{-grmod}_{\geq d}$ the sequential colimit defining the $m$-transform is finite, i.e., there exists an $\ell_{M,d} \geq d$ high enough such that the induced maps

$$\text{Hom}_{\geq d}(m^\ell, M) \to \text{Hom}_{\geq d}(m^{\ell+1}, M)$$

are isomorphisms for all $\ell \geq \ell_{M,d}$. In particular,

$$D_{m,\geq d}(M) = \text{Hom}_{\geq d}(m^{\ell_{M,d}}, M).$$

Proof. The short exact sequence $B(\ell)^{\oplus ?} \cong m^\ell / m^{\ell+1} \hookrightarrow m^\ell \twoheadrightarrow m^{\ell+1}$ induces for $M \in S\text{-grmod}$ the exact contravariant $\text{Ext}_\bullet$-sequence of which the first four terms are

$$\text{Hom}_\bullet(B, M)^{\oplus ?}(\ell) \hookrightarrow \text{Hom}_\bullet(m^\ell, M) \to \text{Hom}_\bullet(m^{\ell+1}, M) \to \text{Ext}_1^\bullet(B, M)^{\oplus ?}(\ell).$$

By Remark 3.3.(b) both $\text{Hom}_\bullet(B, M)$ and $\text{Ext}_1^\bullet(B, M)$ are quasi-zero. Hence, the truncated morphisms $\text{Hom}_{\geq d}(m^\ell, M) \to \text{Hom}_{\geq d}(m^{\ell+1}, M)$ become isomorphisms in $S\text{-grmod}_{\geq d}$ for high enough twists $\ell$. □

The natural transformation

$$\eta_M := \lim_{\ell \geq d} \big( \eta_M^\ell : M \to \text{Hom}_{\geq d}(m^\ell, M) \big) : M \to D_{m,\geq d}(M)$$

is induced by applying $\text{Hom}_{\geq d}(\_, M)$ to the embeddings $(S \hookrightarrow m^\ell)_{\geq d}$.

Theorem 4.2. The $d$-truncated $m$-transform $D_{m,\geq d}$ together with the natural transformation $\eta : \text{Id}_A \to D_{m,\geq d}$ is a Gabriel monad of $A = S\text{-grmod}_{\geq d}$ w.r.t. $C := S\text{-grmod}^0_{\geq d}$.

Before proving the theorem we state some simple facts about ideal transforms.

Remark 4.3.
(a) Any $N \in S\text{-grmod}^0$ vanishes in degrees greater than $\text{reg} N$. Thus,

$$\text{Hom}_\bullet(L_{\geq t}, N)_{\geq \text{reg} N+1-\ell} = 0$$

for all $\ell \in \mathbb{Z}$ and $L \in S\text{-grmod}$.

(b) The embedding $M_{\geq t} \hookrightarrow M$ induces (by simple degree considerations) an isomorphism

$$\text{Hom}_{\geq d}(L_{\geq t}, M_{\geq t}) \cong \text{Hom}_{\geq d}(L_{\geq t}, M) \quad \text{for all } t \leq \ell + d.$$

In particular, $D_{m,\geq d}(M_{\geq t}) \cong D_{m,\geq d}(M_{\geq t})$ for any $t \geq d$ and we are allowed to replace $M$ by any of its truncations.

(c) For $M \in S\text{-grmod}_{\geq d}$ take $t \geq d$ large enough such that the submodule $M_{\geq t}$ has no $S\text{-grmod}^0_{\geq d}$-torsion. Then $\text{Hom}_{\geq d}(m^\ell, M) \cong \text{Hom}_{\geq d}(m^\ell, M_{\geq t}) \cong \text{Hom}_{\geq d}(\otimes^\ell m, M_{\geq t})$ for all $\ell \geq t - d$ by (b) and Remark 3.9. In particular, after replacing $M$ by a high enough truncation we can always assume that

$$\text{Hom}_{\geq d}(m^\ell, M) \cong \text{Hom}_{\geq d}(\otimes^\ell m, M).$$

(d) Since the shift functor $(1) : S\text{-grmod}_{\geq d} \to S\text{-grmod}_{\geq d+1}, M \mapsto M(1), \varphi \mapsto \varphi(1)$ is (quasi-)inverse to the shift functor $(-1) : S\text{-grmod}_{\geq d+1} \to S\text{-grmod}_{\geq d}$ and $D_{m,\geq d} \circ (-1) = (-1) \circ D_{m,\geq d+1}$ we can restrict the following proofs to $D_{m,\geq 0}$. 

Proof of Theorem 4.2. We use Theorem 2.1. Due to Remark 4.3.(d) we only need to consider the case \( d = 0 \).

2.1.(a) \( C \subset \ker D_{m, \geq 0} \):

Applying Remark 4.3.(a) with \( L = S \) (and \( L_{\geq \ell} = S_{\geq \ell} = m^\ell \)) we conclude that \( D_m \) vanishes\(^7\) on \( S^{-\text{grmod}}_0 \) and \( D_{m, \geq 0} \) on \( S^{-\text{grmod}}_{\geq 0} \).

2.1.(b) \( D_{m, \geq 0}(A) \subset \text{Sat}_C(A) \):

For any \( M \in A \), the map
\[
\Hom_{m, \geq 0}(S \leftrightarrow m, D_{m, \geq 0}(M)) = \Hom_{m, \geq 0}(S \leftrightarrow m, \Hom_{m, \geq 0}(m^\ell, M))
\]
\[
= \Hom_{m, \geq 0}(S \leftrightarrow m, \Hom_{m, \geq 0}(\otimes m^\ell, M))
\]
\[
= \Hom_{m, \geq 0}(\otimes m^\ell, m \leftrightarrow \otimes m^{\ell+1}, M)
\]
\[
= \Hom_{m, \geq 0}(\otimes m^\ell, M) \to \Hom_{m, \geq 0}(\otimes m^{\ell+1}, M)
\]
\[
= \Hom_{m, \geq 0}(m^\ell, M) \to \Hom_{m, \geq 0}(m^{\ell+1}, M)
\]
is an isomorphism by Remark 4.1 proving statement (e) of Corollary 3.10. We have repeatedly used Remark 4.3.(c) and the adjunction between \( \otimes \) and \( \Hom_{m, \geq 0} \).

2.1.(c) \( G := \text{co-res}_{\text{Sat}_C(A)} D_{m, \geq 0} \) is exact:

Applying \( \Hom_{m, \geq 0}(m^\ell, -) \) to the short exact sequence \( L \leftrightarrow M \to N \) in \( S^{-\text{grmod}}_{\geq 0} \) yields the exact sequence
\[
\Hom_{m, \geq 0}(m^\ell, L) \leftrightarrow \Hom_{m, \geq 0}(m^\ell, M) \to \Hom_{m, \geq 0}(m^\ell, N) \to \Ext_{m, \geq 0}^1(m^\ell, L)
\]
as part of the long exact covariant \( \Ext_{m, \geq 0} \)-sequence. Since \( \Ext_{m, \geq 0}^1(m^\ell, L) \) is quasi-zero by Remark 3.3.(c) the sequence is exact up to defects in \( S^{-\text{grmod}}_{\geq 0} \).

2.1.(d) \( \eta D_{m, \geq 0} = D_{m, \geq 0} \eta \):

We repeatedly use the adjunction between \( \otimes \) and \( \Hom_{m, \geq 0} \) and Remark 4.1 to interchange the involved sequential colimits over \( \ell^\prime \) and \( \ell'' \) by a common \( \ell \geq \ell^\prime, \ell'' \), high enough to stabilize both colimits:
\[
\eta D_{m, \geq 0}(M) = \lim_{\ell^\prime \geq 0} \Hom_{m, \geq 0}(S \leftrightarrow m^\ell, \lim_{\ell'' \geq 0} \Hom_{m, \geq 0}(m^{\ell''}, M))
\]
\[
= \Hom_{m, \geq 0}(S \leftrightarrow m^\ell, \Hom_{m, \geq 0}(m^\ell, M))
\]
\[
= \Hom_{m, \geq 0}(S \leftrightarrow m^\ell) \otimes_S m^\ell, M)
\]
\[
= \Hom_{m, \geq 0}(m^\ell, m^\ell, M)
\]
\[
= \lim_{\ell^\prime \geq 0} \Hom_{m, \geq 0}(m^{\ell''}, \lim_{\ell^\prime \geq 0} \Hom_{m, \geq 0}(S \leftrightarrow m^\ell, M))
\]
\[
= D_{m, \geq 0}(\eta_M).
\]

The proof implicitly uses commuting diagrams of morphisms in \( S^{-\text{grmod}}_{\geq 0} \) to justify the equality signs.\(^8\)

---

\(^7\)For \( N \in C \) all modules in the sequential colimit defining \( D_{m, \geq 0}(N) \) vanish for \( \ell \geq \ell_{N, 0} := \reg N + 1 < \infty \).

\(^8\)We could have used the fact that \( D_{m, \geq 0} = \lim_{\ell \geq 0} \Hom_{m, \geq 0}(m^\ell, -) \) commutes with directed colimits. However, the general form of the second statement is not quite trivial [BS98, Coro. 3.4.11] (the directed colimit is called...
2.1.(e) \( \eta \) is a natural isomorphism:

Let \( M \in S\text{-}gr\text{mod}_{\geq 0} \) be saturated w.r.t. \( S\text{-}gr\text{mod}_0^{\geq 0} \). Applying \( \text{Hom}_{\geq 0}(\_, M) \) to the short exact sequence \( S/\mathfrak{m}^\ell \hookrightarrow S \hookrightarrow \mathfrak{m}^\ell \) yields

\[
\text{Hom}_{\geq 0}(S/\mathfrak{m}^\ell, M) \xrightarrow{\eta_M^{\ell}} \text{Hom}_{\geq 0}(\mathfrak{m}^\ell, M) \rightarrow \text{Ext}_{\geq 0}^1(S/\mathfrak{m}^\ell, M)
\]

since \( S/\mathfrak{m}^\ell \in S\text{-}gr\text{mod}_0^{\geq 0} \). In other words, \( \eta_M^{\ell} \) is an isomorphism for all \( \ell \).

\( \square \)

**Remark 4.4.** The saturation process of \( M \in S\text{-}gr\text{mod} \) conducted by \( D_m \) brings \( \text{linreg} \) to \( -\infty \), whereas \( \text{reg} \) is only brought down to the regularity of the sheafification.

## 5. Graded \( S \)-modules and linear \( E \)-complexes

In this section we describe how to translate the module structure of \( M \in S\text{-}gr\text{mod} \) into the structure of a linear complex \( R(M) \) over the exterior algebra \( E := \wedge V \), which is Koszul dual to \( S = \text{Sym} V^* \). This translation turns out to be functorial, algorithmic, and an adjoint equivalence of categories. We denote the category of finitely generated graded \( E \)-modules by \( E\text{-}gr\text{mod} \).

Let \( e_0, \ldots, e_n \) denote a \( B \)-basis \( V \) of which the indeterminates \( x_0, \ldots, x_n \) of \( S \) form the dual \( B \)-basis of \( W = V^* = \text{Hom}_B(V, B) \). We set \( \deg(e_i) = -1 \) for all \( i = 0, \ldots, n \).

### 5.1. The functor \( R \)

The \( B \)-linear maps

\[
\mu^i(x_j) : M_i \rightarrow M_{i+1}, m \mapsto x_j m, \quad \text{for } j = 0, \ldots, n, \text{ and } i \in \mathbb{Z}
\]

induced by the indeterminates \( x_j \) encode the graded \( S \)-module structure of an \( M \in S\text{-}gr\text{mod} \) (cf. Remark 3.1 for an algorithm to compute \( M_i \)).

**Example 5.1.** For \( S := B[x_0, x_1] \) consider the free \( S \)-module \( M := S = S(0) \) of rank 1. Each graded part \( M_i \) is a free \( B \)-module for which we fix a basis of monomials, e.g., \( M_0 = \langle 1 \rangle_B \), \( M_1 = \langle x_0, x_1 \rangle_B \), \( M_2 = \langle x_0^2, x_0 x_1, x_1^2 \rangle_B \). Then the matrices

\[
\begin{align*}
0 : & \quad \mu^0(x_0) = (1 \ 0), \quad \mu^0(x_1) = (0 \ 1), \\
1 : & \quad \mu^1(x_0) = (1 \ 0 \ 0), \quad \mu^1(x_1) = (0 \ 1 \ 0), \\
\vdots
\end{align*}
\]

represent the maps \( \mu^i(x_j) \).

Using the \( B \)-basis \( \langle e_0, \ldots, e_n \rangle \) of \( V \) define for each \( i \in \mathbb{Z} \) the map \( \mu^i \) as the composition

\[
\mu^i : \begin{cases} 
M_i & \rightarrow \text{End}_B(V) \otimes_B M_i \rightarrow V \otimes_B M_{i+1}, \\
m & \mapsto \text{id}_V \otimes m \mapsto \sum_{j=0}^n e_j \otimes x_j m.
\end{cases}
\]

(\text{direct limit in [BS98, Terminology 3.4.1])}. Note that although the ideal transform commutes with \textit{directed} colimits, it does not generally commute with arbitrary finite colimits, for otherwise it would be right exact and hence exact.
Using the natural isomorphism $\text{Hom}_B(M_i, V \otimes_B M_{i+1}) \cong \text{Hom}_{E, \text{grmod}}(E \otimes_B M_i, E \otimes_B M_{i+1})$ each $\mu^i$ can equally be understood as a map of graded $E$-modules

$$\mu^i : E \otimes_B M_i \rightarrow E \otimes_B M_{i+1},$$

where the $B$-module $M_j$ is considered as a graded $B$-module concentrated in degree $j$ and, therefore, $E \otimes_B M_j$ is generated by (a generating set of) $M_j$ in degree $j$.

For a better functorial behavior we replace $E$ by its $B$-dual [Lam99, §16C]

$$\omega_E := \text{Hom}_B(E, B) \cong \wedge W \cong \wedge^{n+1} W \otimes_B E$$

in the above maps. In particular, $\omega_E$ lives in the degree interval $0, \ldots, n+1$ and its socle $(\omega_E)_0$, which is naturally isomorphic to $B$, is concentrated in degree 0. We denote the distinguished generator of the socle corresponding to $1_B$ by $1_{\omega_E}$.

This change of language is justified by reinterpreting $\mu^i : M_i \rightarrow V \otimes_B M_{i+1}$ as a map $\mu^i : W \otimes_B M_i \rightarrow M_{i+1}$ using the adjunction

$$\text{Hom}_B(W \otimes_B X, Y) \cong \text{Hom}_B(X, \text{Hom}_B(W, Y)) \cong \text{Hom}_B(X, W^* \otimes_B Y).$$

The graded $E$-module $\omega_E \otimes_B M_j$ has (compared with $E \otimes_B M_j$) the advantage of having the $B$-module $M_j$ as its socle interpreted as a graded $E$-module concentrated in degree $j$.

The commutativity of $S$ implies that the composed map $\mu^{i+1} \circ \mu^i : \omega_E \otimes_B M_i \rightarrow \omega_E \otimes_B M_{i+2}$ is zero. Thus, the sequence of $\mu_i$’s yields the so-called $R$-complex (cf. [EFS03, §2] and [ES08, §2])

$$R(M) : \cdots \rightarrow \omega_E \otimes_B M_i \xrightarrow{\mu^i} \omega_E \otimes_B M_{i+1} \xrightarrow{\mu^{i+1}} \omega_E \otimes_B M_{i+2} \rightarrow \cdots$$

**Example 5.1 (continued).** For $M = S(0)$ we obtain the $R$-complex

$$0 \rightarrow \omega_E(0) \xrightarrow{(e_0 e_1)} \omega_E(-1)^2 \xrightarrow{(e_0 e_1)} \omega_E(-2)^3 \xrightarrow{(e_0 e_1)} \omega_E(-3)^4 \rightarrow \cdots$$

**Lemma 5.2 ([EFS03, Prop. 2.3]).** There exists a natural isomorphism

$$H^a(R(M))_{a+i} \cong \text{Tor}^S_a(B, M)_{a+i}.$$

*Proof.* The idea is to interpret the bigraded differential module $\omega_E \otimes_B M$ either as $R(M)$ or as the Koszul resolution of $B$ tensored with $M$ over $S$. \qed

Lemmas 5.2 and 3.5 imply the following lemma, an important technical insight for the rest of this paper.

**Lemma 5.3 (Key Lemma).** There exists a natural isomorphism

$$H^a(R(M))_{a+i} \cong \text{Ext}^1_{\ast}(\wedge^{n+1} V, M)_{a+i}.$$

Hence, there is a noncanonical isomorphism $H^a(R(M))_{a+n+1-j} \cong \text{Ext}^1_{\ast}(B, M)_{a-j}$.

Let $A$ be either $S$ or $E$. An epimorphism in $A$-grmod is said to be $B$-split if it splits as a morphism over $B$. A graded module $P \in A$-grmod is said to be relatively projective (with respect to $B$) if $\text{Hom}_S(P, -)$ if it sends $B$-split epis to surjections. Any module of the form

---

9It is again a free graded $E$-module which is *non*naturally isomorphic to $E(-n-1)$. 
\(A \otimes_B M\), where \(M\) is a \(B\)-module, is called relatively free (with respect to \(B\)). By [ES08, Proposition 1.1], an \(N \in \text{A-grmod}\) is relatively projective if and only if it is relatively free.

We call a complex \(C = C^*\) of graded \(E\)-modules linear if each \(C^i\) is relatively free (with respect to \(B\)) with socle concentrated in degree \(i\).\(^{10}\) The regularity of a linear complex \(C\) is defined as

\[
\text{reg } C := \sup\{a \in \mathbb{Z} \mid H^a(C) \neq 0\} \in \mathbb{Z} \cup \{-\infty, \infty\}.
\]

Lemma 5.2 connects the regularity of a graded module with that of its \(R\)-complex.

**Corollary 5.4.** For \(M \in \text{S-grmod}\) the equality \(\text{reg } M = \text{reg } R(M)\) holds.

These definitions allow us to describe the image of \(R\).

**Definition 5.5.** We denote by \(E\)-grlin the full subcategory of complexes \(C\) of graded \(E\)-modules satisfying

(a) \(C\) is linear;

(b) each \(C^i\) is finitely generated;

(c) \(C\) is left bounded;

(d) \(\text{reg } C < \infty\).

Finally denote by \(E\)-grlin\(^{>0}\) the full subcategory of complexes in \(E\)-grlin with \(C^{<0} = 0\).

**Proposition 5.6.** The construction \(R\) induces two fully faithful functors \(R : \text{S-grmod} \to E\)-grlin and \(R_{>0} : \text{S-grmod}_{>0} \to E\)-grlin\(^{>0}\) for all \(d \in \mathbb{Z}\).

**Proof.** As \(M \in \text{S-grmod}\) is finitely generated, \(R(M)\) is left bounded. By definition, each \(R(M)^i = \omega_E \otimes_B M_i\) is a finitely generated graded relatively free module with socle \(M_i\) concentrated in degree \(i\). Furthermore \(\text{reg } R(M) = \text{reg } M < \infty\) by Corollary 5.4.

A graded morphism \(\varphi : M \to N\) induces morphisms \(\varphi_i : M_i \to N_i\) for all \(i \in \mathbb{Z}\). Tensoring with \(\omega_E\) yields morphisms \(R(\varphi)^i : R(M)^i \to R(N)^i\). These morphisms are chain morphisms, as \(x_j \circ \varphi_i = \varphi_{i+1} \circ x_j\) and the \(\mu^i\) are induced by the \(x_j\) for all \(i \in \mathbb{Z}\) and all \(0 \leq j \leq n\).

Restricting \(R\) to \(\text{S-grmod}_{>0}\) corestricts to \(E\)-grlin\(^{>0}\) by construction. These functors are obviously faithful. The fullness \(R\) and \(R_{>0}\) follows directly from the below Proposition 5.7 and Corollary 5.8, respectively. \(\square\)

5.2. **The functor \(R\) induces an equivalence.** The functor \(R\) is an equivalence \(\text{S-grmod} \xrightarrow{\sim} E\)-grlin by [EFS03, Prop. 2.1]. In this section we explicitly construct the left adjoint quasi-inverse \(M\) of \(R\) and thus show constructively that \(R\) is an adjoint equivalence.

**Proposition 5.7.** There exists a functor \(M : E\)-grlin \(\to \text{S-grmod}\) such that \(M \dashv R\) is an adjoint equivalence of categories.

**Proof.** Let \((C, \mu) \in E\)-grlin.

For a preparatory step, assume that

(A) \(H^r(C)\) is the only nonvanishing cohomology (this implies that \(C^{<r} = 0\)).

\(^{10}\)and hence \(C^i\) is generated in degree \(i + n + 1\).
Consider \( \mu^r : W \otimes_B C^r \rightarrow C^r_{r+1} \) and extend \( \ker(\mu^r) \xrightarrow{\delta} W \otimes_B C^r \) to a map \( S \otimes_B \ker(\mu^r) \rightarrow S \otimes_B C^r \). Define \( M(C) \) as its cokernel (with relatively free presentation \( \pi_r : S \otimes_B C^r \rightarrow M(C) \)).

To justify the correctness of this preparatory step let \( M \in S_{-\text{grmod}} \) with \( M_{<r} = 0 \) and such that \( R(M) \) satisfies assumption (A). The natural isomorphism \( \sim^{-1} : M_r \sim M(R(M))_r : m \mapsto 1_S \otimes_B 1_{\omega_E} \otimes_B m \) identifies a minimal set of generators \( M \) with one of \( M(R(M)) \). The assumption (A) for \( R(M) \) is equivalent, by Lemma 5.2, to \( M \) being generated in degree \( r \) and having a relatively free resolution which is linear in the \( x_i \). In particular, the only relations involving the indeterminates \( x_i \) of the finite set of generators of \( M \) in \( M_r \) are linear relations. All these linear relations are encoded in the map \( R(M)^r \rightarrow R(M)^{r+1} \). The construction of \( M \) above just imposes these linear relations of the generators of \( M \) to the generators of \( R(M) \).

In particular, \( \delta \) induces an isomorphism \( \delta_M : M(R(M)) \rightarrow M \). Similarly, there exists an isomorphism \( \eta_C : C \rightarrow R(M(C)) \) for any \( C \in E_{-\text{grlin}} \) satisfying assumption (A).

For a general \((C, \mu) \in E_{-\text{grlin}}\), there is a bound \( r \) (e.g., any \( r > \text{reg}(C) \)) such that the preparatory step applies to \((C^{2r}, \mu^{2r})\). Then, we inductively define \( M(C) \) by decreasing the cohomological degree \( d \). Let \((C, \mu) \in E_{-\text{grlin}}\) be a complex and \( d < r \) such that \( M(C)^{\geq d+1} \) is defined by the induction hypothesis with relatively free presentation \( \pi_{d+1} : S \otimes_B (C_{d+1}^{d+1} \oplus \cdots \oplus C_r^r) \rightarrow M(C)^{\geq d+1} \). We define \( M(C)^{\geq d} \) as a pushout of the span of \( \beta \) and \( \gamma \) as defined as follows:

Let \( \alpha : S \otimes_B W \rightarrow S : p \otimes x_i \rightarrow x_i \) and \( \iota : C_{d+1}^{d+1} \hookrightarrow C_{d+1}^{d+1} \oplus \cdots \oplus C_r^r \) be the embedding in the direct sum. Now set \( \beta := \alpha \otimes_B C_d^d \) and \( \gamma := \pi_{d+1} \circ (S \otimes_B (\iota \circ \mu^d)) \) with common source \( S \otimes_B W \otimes C_d^d \) (recall, \( \mu^d : W \otimes B C_d^d \rightarrow C_{d+1}^{d+1} \)). This inductive step of the construction of \( M \) is the reverse construction of \( R \).

This equivalence of categories is an adjoint equivalence. We already have constructed the unit \( \eta \) and counit \( \delta \) as natural isomorphisms in the preparatory step. This unit and counit naturally extend into lower cohomological degrees using the natural \( B \)-isomorphisms \( C_i^r \sim M(C)_i : c \mapsto 1_S \otimes_B C_m^r \) and \( M_i \sim R(M)_i^r : m \mapsto 1_{\omega_E} \otimes_B m \). The triangle identities are easily verified. \( \square \)

**Corollary 5.8.** The restriction-corestriction \( M_{\geq d} : E_{-\text{grlin}}^{\geq d} \rightarrow S_{-\text{grmod}}^{\geq d} \) of \( M \) and the functor \( R_{\geq d} \) form an adjoint equivalence \( M_{\geq d} \dashv R_{\geq d} \).

### 5.3. Saturated linear complexes.

We now give a characterization of saturated linear complexes corresponding to the one we gave for graded modules.

**Definition 5.9.** The **linear regularity** of a linear complex \( C \in E_{-\text{grlin}} \) is defined as \( \text{linreg } C := \sup \{ a \in \mathbb{Z} \mid H^a(C)_{a+n+1} \neq 0 \text{ or } H^a(C)_{a+n} \neq 0 \} \in \mathbb{Z} \cup \{-\infty\} \).

We get a further characterization of \( E_{-\text{grlin}}^{0} \)-saturated linear complexes.

**Proposition 5.10.** A complex \( C \in E_{-\text{grlin}} \) is \( E_{-\text{grlin}}^{0} \)-saturated iff \( \text{linreg } C = -\infty \).

**Proof.** By Proposition 3.8, the module \( M(C) \) is \( S_{-\text{grmod}}^{0} \)-saturated if \( \text{Ext}^j(B, M(C)) = 0 \) for \( j \in \{0, 1\} \). This is equivalent to \( H^a(R(M(C)))_{a+n+1-j} = 0 \) for \( j \in \{0, 1\} \) by the key Lemma 5.3. The claim follows from \( C \cong R(M(C)) \). \( \square \)

The localizing subcategory \( S_{-\text{grmod}}^{0, d} \) of \( S_{-\text{grmod}}^{d} \) corresponds via the adjoint equivalence \( M \dashv R \) to the full localizing subcategory \( E_{-\text{grlin}}^{d, 0} \) of right bounded complexes in
leads to an algorithm for the saturation of graded \( \phi \) with \( C^{\geq \ell} = 0 \) for \( \ell \) large enough. A module \( M \in S\text{-}\text{grmod}_{\geq d} \) is then \( S\text{-}\text{grmod}_{\geq d} \)-saturated if and only if \( R(M) \) is \( E\text{-}\text{grlin}_{\geq d,0} \)-saturated, i.e., the adjoint equivalence \( M_{\geq d} \mapsto (R_{\geq d} : S\text{-}\text{grmod}_{\geq d} \to E\text{-}\text{grlin}_{\geq d}) \) restricts to an adjoint equivalence between the full subcategories of \( S\text{-}\text{grmod}_{\geq d} \)-saturated resp. \( E\text{-}\text{grlin}_{\geq d,0} \)-saturated objects.

The definition of the linear regularity of complexes in \( E\text{-}\text{grlin}_{\geq d} \) and the characterization of \( E\text{-}\text{grlin}_{\geq d,0} \)-saturated linear complexes is a little bit more subtle and is therefore deferred to the next section. The reason is that the lowest cohomology \( H^d(C) \) has to be treated separately.

6. Saturation of Linear Complexes

The ideal transform in Section 4 leads to an algorithm for the saturation of graded \( S \)-modules. In this section, we present an algorithm to saturate linear complexes. The adjoint equivalence \( M \mapsto R \) translates this to a second algorithm for the saturation of graded \( S \)-modules.

Proposition 5.10 indicates that one has to modify a linear complex \( C \) until \( H^a(C)_{a+n+1} = 0 \) and \( H^a(C)_{a+n} \neq 0 \). Our purely linear saturation is similar to that of the Tate resolution in that one truncates \( C \) in cohomological degree high enough and then computes a suitable part in lower cohomological degrees. In contrast to the Tate resolution, our approach remains in the category of linear complexes, as we do not take free presentations of kernels to compute the part of lower cohomological degree, but so-called purely linear kernels.

6.1. Purely linear kernels. Let \( C^i, C^{i+1} \in E\text{-}\text{grmod} \) be relatively free with socle concentrated in degree \( i \) and \( i + 1 \), respectively. We call a morphism \( \varphi : C^i \to C^{i+1} \) purely linear (of degree \( i \)) if its kernel vanishes in the top degree \( i + n + 1 \).

Definition 6.1. Let \( \varphi^i : C^i \to C^{i+1} \) be purely linear of degree \( i \). A purely linear morphism \( \kappa : K^i \to C^i \) of degree \( i - 1 \) with \( \varphi^i \circ \kappa = 0 \) is called a purely linear kernel if for any purely linear \( \lambda : L^i \to C^i \) of degree \( i - 1 \) with \( \varphi^i \circ \lambda = 0 \) there exists a unique morphism \( \psi : L^i \to K^i \) with \( \kappa \circ \psi = \lambda \).

Lemma 6.2. Each purely linear morphism has a purely linear kernel, which, by the universal property, is unique up to a unique isomorphism.

Proof. We denote the restriction of any morphism \( \beta \) to the graded part of degree \( i + n \) by \( \beta_{i+n} \).

Let \( \varphi^i : M^i \to M^{i+1} \) be purely linear of degree \( i \) and \( \nu : N^{i-1} \to M^i \) be its kernel. Denote by \( K^{i-1} := N_{i+n}^i \otimes_B E \) and by \( \lambda : K^{i-1} \to N^{i-1} \) the map induced by the identity on \( N_{i+n}^{i-1} \). We show that \( \kappa := \nu \circ \lambda : K^{i-1} \to M^i \) is a purely linear kernel of \( \varphi^i \).

By definition, \( K \) and \( M \) are relatively free generated in degree \( i + n \) and \( i + n + 1 \), respectively.

As \( \varphi^i \) is purely linear, \( N^{i-1} \) lives in the degree interval \( i, \ldots, i+n \). In particular, \( \nu_{i+n} \) is a kernel of \( \varphi^i_{i+n} \). By definition, \( \lambda_{i+n} : K_{i+n}^{i-1} \to N_{i+n}^{i-1} \) is an isomorphism and thus also \( \kappa_{i+n} \) is a kernel of \( \varphi^i_{i+n} \). In particular, the kernel of \( \kappa \) lives in the degree interval \( i-1, \ldots, i+n-1 \). Thus, \( \kappa \) is purely linear of degree \( i-1 \).

\(^{11}\) or, equivalently, freely generated in degree \( i + n + 1 \) and \( i + n + 2 \), respectively.
The composition $\varphi^i \circ \nu$ is zero and $\kappa$ factors over $\nu$ by construction. Thus, $\varphi^i \circ \kappa = 0$.

$$
\begin{align*}
\begin{array}{ccc}
K^{i-1} & \xrightarrow{\kappa} & M^i \\
\psi & \downarrow & \varphi^i \\
M^{i-1} & \xrightarrow{\varphi^{i-1}} & 0
\end{array}
\end{align*}
$$

To show the universal property of $\kappa$ let $\varphi^{i-1} : M^{i-1} \to M^i$ be purely linear with $\varphi^i \circ \varphi^{i-1} = 0$. From the universal property of $\kappa_{i+n}$ as a kernel, there exists a unique $\psi_{i+n} : M^{i-1}_{i+n} \to K^{i-1}_{i+n}$ with $\kappa_{i+n} \circ \psi_{i+n} = \varphi^{i-1}_{i+n}$, since $\varphi^{i-1}_{i+n} \circ \varphi^i_{i+n} = 0$. We define

$$
\psi := \psi_{i+n} \otimes_B E : M^{i-1} \cong M^{i-1}_{i+n} \otimes_B E \to K^{i-1} \cong K^{i-1}_{i+n} \otimes_B E, 
$$

which extends $\psi_{i+n}$ to a morphism of graded $E$-modules. Finally, $\varphi^{i-1} = \kappa \circ \psi$ since $\kappa_{i+n} \circ \psi_{i+n} = \varphi^{i-1}_{i+n}$ and $\varphi^{i-1}$ is uniquely determined by $\varphi^{i-1}_{i+n}$. \hfill \Box

Note that all steps in the proof of this last lemma are constructive.

We can now state the definition of linear regularity for complexes in $E$-grlin$^{d}$.

**Definition 6.3.** The $d$-th truncated linear regularity of $C \in E$-grlin$^{d}$ is as follows: If there exists an $a \in \mathbb{Z}_{>d}$ such that $H^a(C)_{a+n+1} \neq 0$ or $H^a(C)_{a+n} \neq 0$ then

$$
\text{linreg}_{\geq d} C := \sup \{a \in \mathbb{Z}_{>d} \mid H^a(C)_{a+n+1} \neq 0 \text{ or } H^a(C)_{a+n} \neq 0 \}. 
$$

Otherwise, if the lowest morphism $C^d \to C^{d+1}$ is a purely linear kernel (of $C^{d+1} \to C^{d+2}$) then

$$
\text{linreg}_{\geq d} C := -\infty \text{ else } \text{linreg}_{\geq d} C := d.
$$

**Corollary 6.4.** A complex $C \in E$-grlin$^{d}$ is $E$-grlin$^{d,0}$-saturated if and only if $\text{linreg}_{\geq d} C = -\infty$.

**Proof.** The claim follows from Proposition 5.10 and Corollary 3.10 (by Remark 4.3.(d) we only need to consider the case $d = 0$). \hfill \Box

**Proposition 6.5.** A $(C, \mu) \in E$-grlin$^{d}$, is $E$-grlin$^{d,0}$-saturated if and only if $\mu^i$ is the purely linear kernel of $\mu^{i+1}$ for all $i \geq d$.

**Proof.** By the proof of Lemma 6.2, a morphism $\kappa : K^{i-1} \to M^i$ is a purely linear kernel of a purely linear morphism $\varphi^i : M^i \to M^{i+1}$ of degree $i$ if and only if it is purely linear and $K^{i-1} \xrightarrow{\kappa} M^i \xrightarrow{\varphi^i} M^{i+1}$ is a complex which is exact in degrees $i+n$ and $i+n+1$. Now, the claim follows from the characterizations of saturated linear complexes in Corollary 6.4. \hfill \Box

6.2. **Saturation of a linear complex.** In this subsection we algorithmically saturate linear complexes by iteratively computing purely linear kernels.

Let $(C, \mu) \in E$-grlin$^{d}$ with regularity $r \in \mathbb{Z}$. Define the purely linear saturation (truncated in degree $d$) functor $S^{d} : E$-grlin$^{d} \to E$-grlin$^{d}$ as follows. The idea is to truncate the complex above the regularity and then to “saturate” it by purely linear kernels, more precisely: For cohomological degrees greater than the linear regularity $r = \text{linreg}_{\geq d} C$ define $S^{d+1}$ by setting $S^{d+1}(C, \mu) := (C^{d+1}, \mu^{d+1})$. Assume that $(D^{d}, \tau^{d}) = S^{d}(C, \mu)$ is defined for
some \( i > d \). Let \( \tau^{i-1} : D^{i-1} \to D^i \) be the purely linear kernel of \( \tau^i \). Define \( S^{\geq i-1}(C, \mu) \) by adding \( \tau^{i-1} \) to \( (D^{\geq i}, \tau^{\geq i}) \) in cohomological degree \( i - 1 \).\(^{12}\)

The morphism part \( S^{\geq d}(\varphi) \) for \( \varphi : (B, \mu_B) \to (C, \mu_C) \) in \( E^\text{-grlin}^{\geq d} \) is induced by the identity in high degrees. The universal property of the purely linear kernels implies a unique completion of the square

\[
\begin{array}{ccc}
S^{\geq d}(B)^\ell & \longrightarrow & S^{\geq d}(B)^{\ell+1} \\
\downarrow & & \downarrow \\
S^{\geq d}(C)^\ell & \longrightarrow & S^{\geq d}(C)^{\ell+1}
\end{array}
\]

and thus iteratively constructs the chain morphisms in lower degrees.

**Remark 6.6.** In the absolute case, i.e., when \( B \) is a field, the purely linear saturation is isomorphic to the subcomplex of the Tate resolution consisting of the direct summands of objects where the degree of the socle equals the cohomological degree. In the relative case it is isomorphic to the complex in cohomological degree \( 0 \) on the first page of the spectral sequence of the Tate complex\(^{13}\) (w.r.t. the filtration given by the multi-complex structure).

**Theorem 6.7.** Let \( A = E^\text{-grlin}^{\geq d} \) and \( C := E^\text{-grlin}^{\geq d,0} \). There exists a natural transformation \( \eta : \text{Id}_A \to S^{\geq d} \) such that the purely linear saturation \( S^{\geq d} \) truncated in degree \( d \) together with this natural transformation \( \eta \) is equivalent to the Gabriel monad of \( A \) w.r.t. \( C \).

**Proof.** First, we construct the natural transformation \( \eta_C \) for \( (C, \mu) \in E^\text{-grlin} \). Consider the cochain-isomorphism \( \eta_C : C^{\geq r} \to S^{\geq r}(C) \) induced by the identity for \( r > \text{reg} C \). Assume that \( \eta_C \) is lifted to a cochain morphism \( C^{\geq \ell+1} \to S^{\geq \ell+1}(C) \). The universal property of the purely linear kernels implies a completion of the square

\[
\begin{array}{ccc}
C^{\ell} & \longrightarrow & C^{\ell+1} \\
\downarrow \eta^\ell_C & & \downarrow \eta^{\ell+1}_C \\
S^{\geq d}(C)^\ell & \longrightarrow & S^{\geq d}(C)^{\ell+1}
\end{array}
\]

by a morphism \( \eta^\ell_C : C^{\ell} \to S^{\geq d}(C)^{\ell} \). Iteratively, we get the cochain-morphism \( \eta_C : C \to S^{\geq d}(C) \).

Now, we use Theorem 2.1 to show that \( S^{\geq d} \) together with \( \eta \) is equivalent to a Gabriel monad.

2.1.(a) \( C \subset \ker S^{\geq d} \):

As \( \eta_C \) is an isomorphism in high cohomological degrees, its kernel is contained in \( C \).

2.1.(b) \( S^{\geq d}(A) \subset \text{Sat}_C(A) \):

\( S^{\geq d}(C) \) has only trivial cohomologies above the regularity of \( C \). Below the regularity we use Proposition 6.5.

\(^{12}\)In particular, \( \text{linreg}^{\geq d} C = d + 1 \) is the precise count of recursive steps needed to achieve saturation.

\(^{13}\)More generally, \( E^\bullet_q (T^\bullet \bullet (M)) \cong R \left( H^\bullet_q (\check{M}) \right) \).
2.1. (c) \( G := \text{co-res}_{\text{Sat}}(A) \) \( S^{\geq d} \) is exact:
As \( S^{\geq d} \) is the identity on objects and morphism in high cohomological degree, applying it to a short exact sequence in \( A \) yields a new sequence with \( A \)-defects, which are bounded by the maximum of the regularities of said short exact sequence. Thus, the \( A \)-defects are contained in \( C \). In particular, this sequence is exact when considered in \( \text{Sat}_{\mathcal{A}}(C) \).

2.1. (d) \( \eta S^{\geq d} = S^{\geq d} \eta \):
Truncated at cohomological degree \( \ell \) above the regularity this is clear, since both natural transformations are induced by the identity. For lower degrees, this follows from the uniqueness of the universal morphism \( \psi \) in the definition of purely linear kernels.

2.1. (e) \( \eta \ell \) is a natural isomorphism:
Let \( C \in \mathcal{A} \) be \( C \)-saturated. We need to show that \( \eta_C \) is a cochain isomorphism. This is clear in high cohomological degrees, as \( S^{\geq d} \) is the identity on objects and morphism there. Assume that \( \eta_C \) restricted to \( C^{\geq \ell + 1} \to S^{\geq \ell + 1}(C) \) for some \( \ell \in \mathbb{Z} \) is a cochain isomorphism. Then, the morphism \( \eta_C^\ell \), from the completion of the square
\[
\begin{array}{ccc}
C^\ell & \longrightarrow & C^\ell + 1 \\
\downarrow \eta_C^\ell & & \downarrow \eta_C^{\ell + 1} \\
S^{\geq d}(C)^\ell & \longrightarrow & S^{\geq d}(C)^{\ell + 1}
\end{array}
\]
is an isomorphism, since both \( C \) and \( S^{\geq d}(C) \) are saturated and, by Proposition 6.5 \( C^\ell \) and \( S^{\geq d}(C)^\ell \) are purely linear kernels of \( \mu^{\ell + 1} \) and the morphism in cohomological degree \( \ell + 1 \) of \( S^{\geq d}(C) \), respectively. The uniqueness of purely linear kernels implies that \( \eta_C \) restricted to \( C^{\geq \ell} \to S^{\geq \ell}(C) \) is a cochain isomorphism, and so is \( \eta_C \) by induction. \( \square \)

We stress that the above functors \( M, M_{\geq d}, R, R_{\geq d} \), and \( S^{\geq d} \) are constructive functors between constructively Abelian categories. We furthermore note that computing the natural transformation \( \eta \) is constructive.

7. THE GABRIEL MONAD OF THE CATEGORY OF COHERENT SHEAVES

In this section we prove that for any \( d \in \mathbb{Z} \) the quotient category \( S\text{-grmod}_{\geq d}/S\text{-grmod}^0_{\geq d} \) is equivalent to the category \( \mathfrak{Coh} \mathbb{P}_B^{n} \) and that the corresponding Gabriel monad computes the (truncated) module of twisted global sections.

Proposition 7.1. \( \mathfrak{Coh} \mathbb{P}_B^{n} \simeq S\text{-grmod}_{\geq d}/S\text{-grmod}^0_{\geq d} \) for all \( d \in \mathbb{Z} \).

Proof. First, the definitions directly imply \( S\text{-grmod}^0 \cap S\text{-grmod}_{\geq d} = S\text{-grmod}^0_{\geq d} \). Second, a preimage of \( M \in S\text{-grmod}/S\text{-grmod}^0 \) under \( S\text{-grmod}_{\geq d} \to S\text{-grmod}/S\text{-grmod}^0 \) is given by \( M_{\geq d} \), since \( M \simeq M_{\geq d} \) in \( S\text{-grmod}/S\text{-grmod}^0 \). Now the second isomorphism theorem for Abelian categories [BLH14a, Prop. 3.2] implies the equivalence \( S\text{-grmod}_{\geq d}/S\text{-grmod}^0_{\geq d} \simeq S\text{-grmod}/S\text{-grmod}^0 \). The latter category is equivalent to \( \mathfrak{Coh} \mathbb{P}_B^{n} \) by [BLH14a, Coro. 4.2]. \( \square \)

A graded \( S \)-modules \( M \) is called quasi finitely generated if each truncation \( M_{\geq d} \) is finitely generated. We denote by \( S\text{-qfgrmod} \subset S\text{-grMod} \) the full subcategory of such modules. The
functor
\[ H^0_\ast : \mathsf{Coh} \mathbb{P}^n_B \to S\text{-qfgrmod} : \mathcal{F} \mapsto \bigoplus_{p \in \mathbb{Z}} H^0(\mathbb{P}^n_B, \mathcal{F}(p)) \]
computing the module of twisted global sections is right adjoint to the sheafification functor \( \mathsf{Sh} : S\text{-qfgrmod} \to \mathsf{Coh} \mathbb{P}^n_B, M \mapsto M \). This was proved by Serre in the absolute case \([\text{Ser}55, 59]\) and later by Grothendieck for the relative case.

Denote by \( \mathsf{Sh}_{\geq d} : S\text{-grmod}_{\geq d} \to \mathsf{Coh} \mathbb{P}^n_B \) the restriction of \( \mathsf{Sh} \) to \( S\text{-grmod}_{\geq d} \) and by
\[ H^0_{\geq d} : \mathsf{Coh} \mathbb{P}^n_B \to S\text{-grmod}_{\geq d} : \mathcal{F} \mapsto \bigoplus_{p \in \mathbb{Z}_{\geq d}} H^0(\mathbb{P}^n_B, \mathcal{F}(p)) \]
the functor computing the truncated module of twisted global sections. It follows that \( H^0_{\geq d} \) is the right adjoint of \( \mathsf{Sh}_{\geq d} \).

**Proposition 7.2.** The monad \( H^0_{\geq d} \circ \mathsf{Sh}_{\geq d} \) is a Gabriel monad of \( S\text{-grmod}_{\geq d} \) w.r.t. the localizing subcategory \( S\text{-grmod}^0_{\geq d} \). In particular, any such Gabriel monad computes the truncated module of global sections.

**Proof.** Let \( \mathcal{Q}_{\geq d} : S\text{-grmod}_{\geq d} \to S\text{-grmod}_{\geq d}/S\text{-grmod}_{\geq d}^0 \) be the canonical functor. The equivalence in Proposition 7.1 is constructed as a functor \( \alpha_{\geq d} : S\text{-grmod}_{\geq d}/S\text{-grmod}_{\geq d}^0 \to \mathsf{Coh} \mathbb{P}^n_B \) with \( \alpha_{\geq d} \circ \mathcal{Q}_{\geq d} \simeq \mathsf{Sh}_{\geq d} \). An easy calculation shows that a right adjoint of \( \mathcal{Q}_{\geq d} \) is given by \( \mathcal{I}_{\geq d} := H^0_{\geq d} \circ \alpha_{\geq d} \). In particular, \( \mathcal{I}_{\geq d} \circ \mathcal{Q}_{\geq d} \) is a Gabriel monad of \( S\text{-grmod}_{\geq d} \) w.r.t. \( S\text{-grmod}_{\geq d}^0 \) by \([\text{BLH}13, \text{Lemma } 4.3]\). Now the claim follows, as \( \mathcal{I}_{\geq d} \circ \mathcal{Q}_{\geq d} = H^0_{\geq d} \circ \alpha_{\geq d} \circ \mathcal{Q}_{\geq d} \simeq H^0_{\geq d} \circ \mathsf{Sh} \). \( \square \)

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