Diffeomorphisms of the Klein Bottle and Projective Plane

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ABSTRACT

We calculate the Riemann curvature tensor and sectional curvature for the Lie group of volume-preserving diffeomorphisms of the Klein bottle and projective plane. In particular, we investigate the sign of the sectional curvature, and find a possible disagreement with a theorem of Lukatskii. We suggest an amendment to this theorem.
1 Introduction

The theory of diffeomorphisms on two-dimensional manifolds can be applied to ideal fluid dynamics, as well as to relativistic membranes. In the former case, we can see a diffeomorphism on a two-manifold $\mathcal{M}$ as an infinitely differentiable mapping of one fluid configuration onto another. The set of possible diffeomorphisms forms an infinite-dimensional Lie group manifold, denoted by $\text{Diff}\mathcal{M}$, upon which we choose the identity to be an arbitrary configuration of the fluid. The fluid flow is then given by a curve on $\text{Diff}\mathcal{M}$ which is parameterized by time. In particular, diffeomorphisms which preserve the volume element on $\mathcal{M}$ form a Lie subgroup of $\text{Diff}\mathcal{M}$, denoted by $\text{SDiff}\mathcal{M}$. These are related to the flow of an incompressible fluid.

With the application of a least-action principle for the fluid, the curve on $\text{SDiff}\mathcal{M}$ can be shown to be a geodesic \cite{1}. As the curvature of the space affects the deviation of nearby geodesics, calculation of the curvature tensor for $\text{SDiff}\mathcal{M}$ will give information about the behaviour of the fluid on $\mathcal{M}$. More precisely, a negative sectional curvature will cause geodesics to diverge, while if it is positive, they will remain in the same neighbourhood. In this way, it is possible to predict whether or not two initially similar fluid flows will remain similar, i.e. whether a flow is “stable”.

Arnold’s fundamental paper \cite{2} calculates the curvature of the volume-preserving diffeomorphism group for the torus, $\text{SDiff}T^2$. The calculation has been repeated for the two-sphere by Lukatskii \cite{3} (see also \cite{4}), for the three-sphere by Dowker \cite{5} and for the flat Riemann surfaces by Wolski and Dowker \cite{6}, \cite{7}. In this paper, we apply Arnold’s calculation to the group of volume-preserving diffeomorphisms of the Klein bottle and projective plane.
2 Formalism

The generators of volume-preserving diffeomorphisms on a locally Euclidean two-manifold $\mathcal{M}$ have the form

$$v_k = \varepsilon^{ab} \partial_b \phi_k \partial_a$$

(1)

where $\phi_k$ is some scalar function on $\mathcal{M}$. A complete set of diffeomorphisms on $\mathcal{M}$ given by a complete set of functions $\phi_k$ is a Lie algebra, i.e.

$$[v_k, v_l] = \sum_m f^{m}_{kl} L^m$$

(2)

where the $f^m_{kl}$ are the structure constants. This is the Lie algebra of the infinite-dimensional Lie group $\text{SDiff}\mathcal{M}$.

The kinetic energy of the fluid on $\mathcal{M}$ is used to define the metric at the identity of $\text{SDiff}\mathcal{M}$.

The kinetic energy is invariant under the right action of the diffeomorphism group, since it is independent of the fluid configuration.

With the intention of calculating the curvature of $\text{SDiff}\mathcal{M}$, we need an expression for the covariant derivative. We follow the approach of Milnor [8] (though in our own notation). If the metric is right-invariant, then for right-invariant fields $\tilde{v}_k$, equal to $v_k$ at the identity, we have

$$\nabla_{\tilde{v}_k} \langle \tilde{v}_l, \tilde{v}_m \rangle = 0$$

(6)
so that
\[ \langle \nabla_{\tilde{v}_k} \tilde{v}_l, \tilde{v}_m \rangle + \langle \tilde{v}_l, \nabla_{\tilde{v}_k} \tilde{v}_m \rangle = 0 \] (7)

From now on we drop the tilde and designate the right-invariant fields on SDiff\(M\) by \(v_k\). If we use the torsionless definition of the covariant derivative:
\[ [v_k, v_l] = \nabla_{v_k} v_l - \nabla_{v_l} v_k \] (8)
then from (8) we can derive, by permuting \(k, l\) and \(m\):
\[ 2 \langle v_k, \nabla_{v_l} v_m \rangle = \langle v_k, [v_l, v_m] \rangle + \langle v_m, [v_k, v_l] \rangle - \langle v_l, [v_m, v_k] \rangle \] (9)

To find an explicit expression for \(\nabla_{v_l} v_m\), Arnold [2] defines the vectors \(B(v_k, v_l)\) such that
\[ \langle v_k, [v_l, v_m] \rangle = \langle v_m, B(v_k, v_l) \rangle \] (10)

We then obtain
\[ 2 \nabla_{v_l} v_m = [v_l, v_m] - B(v_l, v_m) - B(v_m, v_l) \] (11)
due to the completeness of the set \(\{v_k\}\). The \(B\) vectors can be found by expanding on the basis \(\{v_k\}\):
\[ B(v_k, v_l) = \sum_m b^m_{kl} v_m \] (12)
and then using the definition (10) to calculate the \(b^m_{kl}\). Note that (11) holds only for right-invariant fields.

As we have, from (2) and (4), the Lie bracket and metric at the identity, we can therefore calculate the covariant derivative at the identity. The realization that, for right-invariant fields, the structure constants in (2) are the same at every point on the Lie group manifold, then allows us to calculate the second derivative, since the \(f^m_{kl}\) are constant with respect to the covariant derivative.

The Riemann curvature tensor is defined by
\[ R(v_k, v_l)v_m = \nabla_{[v_k, v_l]}v_m - [\nabla_{v_k}, \nabla_{v_l}]v_m \] (13)

3
and its covariant components are

\[ R_{\nu_k, \nu_l, \nu_m, \nu_n} = \langle \nabla_{[\nu_k, \nu_l]} \nu_m - [\nabla_{\nu_k}, \nabla_{\nu_l}] \nu_m, \nu_n \rangle \]  

(14)

Additionally, in this paper we evaluate the normalized sectional curvature, and use the general definition:

\[ C_{\nu_k, \nu_l} = \frac{R_{\nu_k, \nu_l, \nu_k, \nu_l}}{\langle \nu_k, \nu_k \rangle \langle \nu_l, \nu_l \rangle - \langle \nu_k, \nu_l \rangle^2} \]  

(15)

The sign of the above quantity is important, since it affects the stability of the fluid flow generated by \( \nu_k \) and \( \nu_l \). Lukatskii [9] has published a theorem which states that on a locally Euclidean manifold, the sectional curvature defined by any two flows, of which at least one is stationary, is non-positive, where a stationary flow is one for which the fluid velocity field is time-independent. It can be shown (e.g. [10]) that the fluid velocity satisfies

\[ \frac{\partial \nu}{\partial t} = -\nabla_{\nu} \nu \]  

(16)

where our definition of \( \nabla_{\nu} \nu \) is different to that in [10] as we have defined \( \nabla \) to be a vector on \( \text{SDiff} \mathcal{M} \). Using (11) leads to the equation for vectors \( \nu \) which generate stationary flows:

\[ B(\nu, \nu) = 0 \]  

(17)

3 The Curvature of \( \text{SDiff}T^2 \)

In calculating the curvature of \( \text{SDiff}(\text{Klein bottle}) \), we use many of Arnold’s results [2] for the torus. In this section, we quote these results.

Arnold considers a \( 2\pi \times 2\pi \) torus, constructed by making the identifications

\[ (x, y) = (x + 2\pi, y) \]  

(18)

\[ (x, y) = (x, y + 2\pi) \]  

(19)
on a plane. The basis functions are chosen to be the eigenfunctions of the laplacian, i.e.
\[ \phi_k = e^{i k \cdot x} \]  
where \( k \) is a two-vector
\[ \mathbf{k} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \]
and \( k_1 \) and \( k_2 \) are integers, so that the \( \phi_k \) are invariant under (18) and (19). The diffeomorphism generators are therefore
\[ L_k = \epsilon^{ab} \partial_b e^{i k \cdot x} \partial_a \]  
and, from (14), we have the scalar product
\[ \langle L_k, L_l \rangle = 4\pi^2 k^2 \delta_{k+1} \]
where we define \( \delta_a \equiv \delta_{a,0} \)

The Lie bracket is simply the commutator of the derivatives (22):
\[ \left[ L_k, L_l \right] = \epsilon^{ab} \partial_b \phi_k \partial_a (\epsilon^{cd} \partial_d \phi_l \partial_c) - \epsilon^{ab} \partial_b \phi_l \partial_a (\epsilon^{cd} \partial_d \phi_k \partial_c) \]
\[ = (\mathbf{k} \times \mathbf{l}) L_{k+l} \]  
where \( \mathbf{k} \times \mathbf{l} \) is the 2-dimensional cross product \( k_1 l_2 - k_2 l_1 \). Inserting (23) and (25) in (10) we obtain the expansion coefficients for \( B(L_k, L_l) \):
\[ b_{mn}^{kl} = (\mathbf{m} \times \mathbf{l}) \frac{k^2}{m^2} \delta_{k+1-m} \]  
so that
\[ B(L_k, L_l) = (\mathbf{k} \times \mathbf{l}) \frac{k^2}{(k+1)^2} L_{k+l} \]  

The calculation of the Riemann tensor is now straightforward. We simplify the notation by defining, for use in the next section:
\[ f_{k,l} = \mathbf{k} \times \mathbf{l} \]
\[ g_{k,l} = \frac{k^2}{(k+1)^2} (k \times l) \]  
\[ h_{k,l} = \frac{1}{2} (f_{k,l} - g_{k,l} - g_{l,k}) \]  
\[ t_{k,l,m} = f_{k,l} h_{k+l,m} - h_{l,m} h_{k,l+m} + h_{k,m} h_{l,k+m} \]  

Arnold’s results can then be summarized:

\[ \nabla_{L_k} L_l = h_{k,l} L_{k+l} \]  
\[ \nabla_{L_k} \nabla_{L_l} L_m = h_{l,m} h_{k,l+m} L_{k,l+m} \]  
\[ R(L_k, L_l) L_m = t_{k,l,m} L_{k+l+m} \]  

where we have used the definition (13). The components of the Riemann tensor become

\[ R_{L_k, L_l, L_m, L_n} = 4\pi^2 n^2 t_{k,l,m} \delta_{k+l+m+n} \]  

Arnold shows (though we will not go into detail here – the proof simply involves two-vector algebra) that the components can be written

\[ R_{L_k, L_l, L_m, L_n} = 4\pi^2 \left[ \frac{(k \times m)^2 (l \times n)^2}{|k+m||l+n|} - \frac{(k \times n)^2 (l \times m)^2}{|k+n||l+m|} \right] \delta_{k+l+m+n} \]  

4 The Curvature of SDiff(Klein Bottle)

The Klein bottle can be seen as a Möbius band with periodic boundary conditions. A \(2\pi \times \pi\) Klein bottle can be constructed from the \(2\pi \times 2\pi\) plane by making the identifications

\[ (x, y) = (x + 2\pi, y) \]  
\[ (x, y) = (2\pi - x, y + \pi) \]  

Due to the reflection (38), this is a non-orientable manifold, although it has the same intrinsic geometry as the torus. Any diffeomorphism on the manifold must be
invariant under the above identifications – this places a restriction on the possible
diffeomorphism generators. As Pope and Romans [11] have stated, the allowed
generators are

\[ e_k = L_k - (-)^{k_2} L_{\overline{k}} \]  

(39)

where

\[ \overline{k} = \left( \begin{array}{c} -k_1 \\ k_2 \end{array} \right) \]  

(40)

and the \( L_k \) are the diffeomorphism generators on the torus. The \( e_k \) have the required properties

\[ e_k(x, y) = e_k(x + 2\pi, y) \]  

(41)

\[ e_k(x, y) = e_k(2\pi - x, y + \pi) \]  

(42)

and are hence invariant under (37) and (38). Note that (39) also generate a subgroup of diffeomorphisms on the torus.

Although there is no continuous volume-form on the Klein bottle, there is no
problem with integrating a scalar field over this manifold, and we can obtain the
metric in the same way as for the torus. In the fluid dynamical interpretation, this
is just the kinetic energy of the fluid on the Klein bottle, which is well-defined. The
metric at the identity on \( \text{SDiff}(\text{Klein bottle}) \) is

\[
\langle e_k, e_l \rangle = \int_{x=0}^{2\pi} \int_{y=0}^{\pi} dx \ dy \ \nabla [e^{ik \cdot x} - (-)^{k_2} e^{i\overline{k} \cdot x}] \cdot \nabla [e^{i1 \cdot x} - (-)^{l_2} e^{i\overline{1} \cdot x}] 
\]

(43)

\[
= \frac{1}{2} \int_{x=0}^{2\pi} \int_{y=0}^{\pi} dx \ dy \ \nabla [e^{ik \cdot x} - (-)^{k_2} e^{i\overline{k} \cdot x}] \cdot \nabla [e^{i1 \cdot x} - (-)^{l_2} e^{i\overline{1} \cdot x}] 
\]

(44)

\[
= 4\pi^2 k^2 \delta_{k+1} - (-)^{k_2} \delta_{\overline{k}+1} 
\]

(45)

The Lie bracket can be derived using linearity:

\[
[e_k, e_l] = [L_k, L_l] - (-)^{k_2} [L_{\overline{k}}, L_l] - (-)^{l_2} [L_k, L_{\overline{l}}] + (-)^{k_2+l_2} [L_{\overline{k}}, L_{\overline{l}}] 
\]

(46)

\[
= (k \times 1)e_{k+l} - (-)^{k_2} (\overline{k} \times 1)e_{\overline{k}+l} 
\]

(47)
To calculate $B(e_k, e_l)$, we expand on the set $\{e_k\}$. From (39), we see that
\[ e_k = -(-)^k e_k \] (48)
so that $e_k = 0$ for $k_1 = 0, k_2$ even, and that there is no need to sum over negative values of $k_1$ in an expansion. Alternatively, we can sum over all values of $k_1$ and $k_2$:
\[ B(e_k, e_l) = \sum_{m_1 = -\infty}^{\infty} \sum_{m_2 = -\infty}^{\infty} b^{m}_{kl} e_m \] (49)
and then impose the condition
\[ b^{m}_{kl} = -(-)^{m_2} b_{kl} \] (50)
without losing any generality. Using (40) together with (50), we arrive at
\[ B(e_k, e_l) = k^2 \left[ \frac{(k + 1)}{(k + 1)^2} e_{k+l} - (-)^{k_2} \frac{(k + 1)}{(k + 1)^2} e_{k+l} \right] \] (51)
Writing these results in terms of the notation of section 3, we have
\[ [e_k, e_l] = f_{k,l} e_{k+l} - (-)^{k_2} f_{k,l} e_{k+l} \] (52)
\[ B(e_k, e_l) = g_{k,l} e_{k+l} - (-)^{k_2} g_{k,l} e_{k+l} \] (53)
It follows that
\[ \nabla e_k e_l = h_{k,l} e_{k+l} - (-)^{k_2} h_{k,l} e_{k+l} \] (54)
\[ \nabla e_k e_l e_m = h_{l,m} h_{k,l+m} e_{k+l+m} - (-)^{k_2} h_{l,m} h_{k,l+m} e_{k+l+m} \]
\[ -(-)^{l_2} h_{l,m} h_{k,l+m} e_{k+l+m} + (-)^{k_2+l_2} h_{l,m} h_{k,l+m} e_{k+l+m} \] (55)
\[ \nabla [e_k, e_l] e_m = f_{k,l} h_{k,l+m} e_{k+l+m} - (-)^{k_2} f_{k,l} h_{k,l+m} e_{k+l+m} \]
\[ -(-)^{l_2} f_{k,l} h_{k,l+m} e_{k+l+m} + (-)^{k_2+l_2} f_{k,l} h_{k,l+m} e_{k+l+m} \] (56)
The Riemann tensor for the group of volume-preserving diffeomorphisms of the Klein bottle is therefore
\[ R(e_k, e_l) e_m = t_{k,l,m} e_{k+l+m} - (-)^{k_2} t_{k,l,m} e_{k+l+m} \]
\[ -(-)^{l_2} t_{k,l,m} e_{k+l+m} + (-)^{k_2+l_2} t_{k,l,m} e_{k+l+m} \] (57)
Its components become, using the metric (15):

\[ R_{e_k,e_l,e_m,e_n} = T_{k,l,m,n} - (-)^{k_2} T_{k,l,m,n} - (-)^{l_2} T_{k,l,m,n} - (-)^{m_2} T_{k,l,m,n} \]
\[ + (-)^{k_2+l_2} T_{k,l,m,n} + (-)^{l_2+m_2} T_{k,l,m,n} + (-)^{k_2+m_2} T_{k,l,m,n} \]
\[ - (-)^{k_2+l_2+m_2} T_{k,l,m,n} \]  \hspace{1cm} (58)

where the \( T_{k,l,m,n} \) denote the components of the curvature for the torus (36). This can be more compactly written by using \( T_{k,l,m,n} = T_{k,l,m,n} \) and \( k_2 + l_2 + m_2 + n_2 = 0 \) for \( R_{k,l,m,n} \) non-zero:

\[ R_{e_k,e_l,e_m,e_n} = \frac{1}{2} \sum_{\mu,\nu,\rho,\sigma} \mu^{k_2+1} \nu^{l_2+1} \rho^{m_2+1} \sigma^{n_2+1} T_{(\mu k_1,k_2),(\nu l_1,l_2),(\rho m_1,m_2),(\sigma n_1,n_2)} \]  \hspace{1cm} (59)

where the Greek variables take the values 1 or \(-1\). In this form the symmetries

\[ R_{e_k,e_l,e_m,e_n} = R_{e_m,e_n,e_k,e_l} = -R_{e_l,e_k,e_m,e_n} = -R_{e_k,e_l,e_m,e_n} \]  \hspace{1cm} (60)

are apparent.

5 The Sectional Curvature

We now proceed to calculate the sectional curvature for a pair of real diffeomorphism generators on the Klein bottle. Consider the vectors

\[ \xi_k = (L_k + L_{-k}) - (-)^{k_2}(L_k + L_{-k}) \]  \hspace{1cm} (61)

By the linearity of the Riemann tensor, we can write

\[ R_{\xi_k,\xi_l,\xi_k,\xi_l} = \sum_{\mu,\nu,\rho,\sigma} R_{e_{\mu k},e_{\nu l},e_{\rho k},e_{\sigma l}} \]  \hspace{1cm} (62)

Using the symmetries (60), this can be expressed as

\[ R_{\xi_k,\xi_l,\xi_k,\xi_l} = 4R_{e_k,e_l,e_k,e_l} + 4R_{e_k,e_l,e_{-k},e_{-l}} + 2R_{e_k,e_l,e_k,e_{-l}} + 2R_{e_k,e_l,e_{-k},e_l} \]
\[ + 2R_{e_k,e_{-l},e_k,e_{-l}} + 2R_{e_k,e_{-l},e_{-k},e_l} + 2R_{e_k,e_{-l},e_{-k},e_{-l}} \]  \hspace{1cm} (63)
Since $T_{k,l,m,n}$ is non-zero only if $k + l + m + n = 0$, all terms in $R_{\xi_k,\xi_l,\xi_k,\xi_l}$ will vanish unless $k$ and $l$ obey the relations

$$\mu k_1 + \nu l_1 + \rho k_1 + \sigma l_1 = 0 \quad (64)$$
$$\alpha k_2 + \beta l_2 + \gamma k_2 + \delta l_2 = 0 \quad (65)$$

Therefore, the only terms which always contribute are those for which $\mu + \rho = \nu + \sigma = \alpha + \gamma = \beta + \delta = 0$. By inspection of (63), all of these are included in

$$2R_{e_k,e_l,e_k,\bar{e}l} + 2R_{e_k,\bar{e}l,e_k,e_l}$$

where, from (58)

$$R_{e_k,e_l,e_k,\bar{e}l} = T_{k,l,-k,-l} + T_{k,l,-\bar{k},-l} \quad (66)$$
$$R_{e_k,\bar{e}l,e_k,e_l} = T_{k,-l,-k,l} + T_{k,-l,-\bar{k},l} \quad (67)$$

and all other terms vanish when $k$ and $l$ obey no special relations of the form (64), (65). Using (34) to calculate these terms, we arrive at

$$R_{\xi_k,\xi_l,\xi_k,\xi_l} = -8\pi^2 \left[ \frac{(k \times l)^4}{(k + l)^2} + \frac{(\bar{k} \times \bar{l})^4}{(\bar{k} + \bar{l})^2} + \frac{(k \times \bar{\bar{l}})^4}{(k - \bar{l})^2} + \frac{(\bar{k} \times l)^4}{(\bar{k} - l)^2} \right] \quad (68)$$

Using (13) to give us the orthogonality relations $\langle \xi_k, \xi_k \rangle = 8\pi^2k^2$, $\langle \xi_l, \xi_l \rangle = 8\pi^2l^2$, $\langle \xi_k, \xi_l \rangle = 0$, together with (13), we express the sectional curvature in a form similar to that of Arnold [2]:

$$C_{\xi_k,\xi_l} = -\frac{(k^2 + l^2)}{16\pi^2} \left[ \sin^2 \alpha \sin^2 \beta + \sin^2 \overline{\alpha} \sin^2 \overline{\beta} \right] \quad (69)$$

where $\alpha$, $\beta$, $\overline{\alpha}$ and $\overline{\beta}$ are the angles between $k$ and $l$, $(k + l)$ and $(k - l)$, $\bar{k}$ and $l$ and $(\bar{k} + l)$ and $(\bar{k} - l)$ respectively. For the general case, therefore, the sectional curvature is non-positive.

It remains for us to calculate the sectional curvature for $\xi_k$ and $\xi_l$ such that $k$ and $l$ have a non-trivial relationship of the form (64), (65). As an example, we consider the case $k_2 = l_2$, $k_1 \neq \pm l_1$, where $k_1, l_1, k_2, l_2 \neq 0$. 


In addition to the terms already considered, there is also a contribution from

$$2R_{ek,e-l,ek,e-l} = 2T_{k,-l,k,-l} + 2T_{k,-l,k,-l}$$

giving the result

$$C_{\xi_k,\xi_l} = \frac{1}{8\pi^2} \left[ \frac{2(k \times \overrightarrow{1})^2(1 \times \overrightarrow{1})^2}{(k + \overrightarrow{1})^2} - 2 \frac{(k \times 1)^4}{(k - 1)^2} - 2 \frac{(k \times \overrightarrow{1})^4}{(\overrightarrow{k} - 1)^2} - \frac{(k \times 1)^4}{(k + 1)^2} - \frac{(\overrightarrow{k} \times 1)^4}{(\overrightarrow{k} + 1)^2} \right]$$

(70)

Calculation in individual cases shows the sign of this quantity to be indefinite.

Using (10), we see that

$$\langle \xi_k, [\xi_l, e_m] \rangle = \langle e_m, B(\xi_k, \xi_l) \rangle$$

(71)

and it can be shown that

$$B(\xi_k, \xi_k) = 0$$

(72)

so that $\xi_k$ generates a stationary flow. As the Klein bottle is locally Euclidean, it seems that Lukatskii’s theorem should apply. In the general case, there is no disagreement – however, the special case (70) can give a positive sectional curvature.

The vectors $\xi_k$ also generate diffeomorphisms on the torus, and we can derive the curvature with respect to these simply by using linearity. The result is

$$R_{\xi_k,\xi_l,\xi_k,\xi_l} = \sum_{\mu,\nu,\rho,\sigma} R_{e_{\mu k},e_{\nu l},e_{\rho k},e_{\sigma l}}$$

(73)

where

$$R_{e_{k1,\nu l},e_{\rho m},e_{\sigma n}} = \sum_{\mu,\nu,\rho,\sigma} \mu^{k_2+1} \nu^{l_2+1} \rho^{m_2+1} \sigma^{n_2+1} T_{(\rho k_2),(\nu l_2),(\rho m_2),(\nu n_2)}$$

(74)

i.e. the same as for the Klein bottle apart from the absence of the factor $1/2$. As the form of the scalar product is also the same, the sectional curvature will have the same sign. Furthermore, from

$$\langle \xi_k, [\xi_l, L_m] \rangle = \langle L_m, B(\xi_k, \xi_l) \rangle$$

(75)
we can show that \( B(\xi_k, \xi_k) = 0 \), so that the \( \xi_k \) also generate stationary flows on the torus. We have therefore found a pair of diffeomorphism generators on the torus which disobey Lukatskii’s theorem. We now investigate this theorem in detail.

6 Lukatskii’s Theorem

Lukatskii [9] gives the following formula for the (unnormalized) sectional curvature with respect to the volume-preserving diffeomorphism generators \( u \) and \( v \):

\[
R_{u,v,u,v} = -\frac{1}{4} \left[ h(\chi(u, v), \chi(u, v)) + h(\chi(u, u), \chi(v, v)) \right]
\]

(76)

\[
h(w, x) \equiv \langle \hat{w}, \hat{x} \rangle
\]

(77)

\[
\chi(u, v) \equiv D_u v + D_v u
\]

(78)

where \( \hat{w} \) is the projection of a vector \( w \) on \( \text{Diff}_M \) onto the space of vectors orthogonal to \( \text{SDiff}_M \), and

\[
D_u v = u^a \partial_a v^b \partial_b
\]

(79)

for \( u = u^a \partial_a \)

We restrict our considerations to the torus. To calculate the projections, we introduce generators of \( \text{Diff}_M \):

\[
w_{k,a} = e^{ik \cdot x} \partial_a
\]

(80)

with the scalar product

\[
\langle w_{k,a}, w_{l,b} \rangle = 4\pi^2 \delta_{ab} \delta_{k+1}
\]

(81)

This is just the extension of (3) to non-volume-preserving diffeomorphisms.

We denote the vectors orthogonal to the \( L_k \) by \( M_k \). We have

\[
L_k = i\epsilon^{ab} k_b w_{k,a}
\]

(82)

\[
M_k = i\delta^{ab} k_b w_{k,a}
\]

(83)

\[
\langle L_k, M_l \rangle = 0
\]

(84)
The $L_k$ and $M_k$ span $\text{Diff} \mathcal{M}$:

$$w_{k,a} = (\epsilon_{ab} L_k + \delta_{ab} M_k) \frac{l_b}{ik^2}$$

(85)

Using (79), we obtain

$$\chi(L_k, L_l) = i(k \times l) \epsilon^{ab} (l_b - k_b) w_{k+l,a}$$

(86)

Projecting onto the space orthogonal to $\text{SDiff} \mathcal{M}$ by considering the second term in (85), we arrive at

$$\hat{\chi}(L_k, L_l) = 2 \frac{(k \times l)^2}{(k+1)^2} M_{k+l}$$

(87)

Lukatskii states that if $u$ or $v$ generates a stationary flow, the second term in (76) vanishes. This is certainly true for the vectors $L_k + L_{-k}$ considered by Arnold, and it is easy to show that the use of (76) together with (87) for these vectors produces his results. However, if we consider $e_k = L_k - (-)^k L_{-k}$ on the torus then we obtain, by linearity:

$$\hat{\chi}(e_k, e_l) = 2 \frac{(k \times l)^2}{(k+1)^2} \left[ M_{k+l} + (-)^{k_2+l_2} M_{k+l} \right] - 2 \frac{(k \times l)^2}{(k+1)^2} \left[ (-)^{k_2} M_{k+l} + (-)^{l_2} M_{k+l} \right]$$

(88)

so that

$$\hat{\chi}(e_k, e_k) = -4(-)^{k_2} \frac{(k \times l)^2}{(k+1)^2} M_{k+l}$$

(89)

giving

$$\langle \hat{\chi}(e_k, e_k), \hat{\chi}(e_l, e_l) \rangle = 64 \pi^2 \frac{(k \times l)^2 (1 \times l)^2}{(k+1)^2} \delta_{k+l+1, k+l}$$

(90)

We see that this term is non-zero if $k + l + 1 = 0$, i.e. if $k_2 + l_2 = 0$. As $B(e_k, e_k) = 0$, the $e_k$ generate stationary flows, and we deduce that Lukatskii’s statement is incorrect. Note, however, that for most $k$ and $l$, the term vanishes, and Lukatskii’s theorem holds.

For $k_2 + l_2 = 0$, $k_1 \neq \pm l_1$ and $k_1, l_1, k_2, l_2 \neq 0$, the other term in (76) is

$$\langle \hat{\chi}(e_k, e_l), \hat{\chi}(e_k, e_l) \rangle = 32 \pi^2 \left[ \frac{(k \times l)^4}{(k+1)^2} + \frac{(k \times l)^4}{(k+1)^2} \right]$$

(91)

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We suspect further that there is a misprint in [9], and that (76) should be

\[
R_{u,v,u,v} = -\frac{1}{4} \left[ h(\chi(u, v), \chi(u, v)) - h(\chi(u, u), \chi(v, v)) \right] \tag{92}
\]

Then

\[
R_{e_k,e_l,e_k,e_l} = 8\pi^2 \left[ 2 \frac{(k \times \bar{k})^2 (1 \times \bar{l})^2}{(k + \bar{k})^2} \right. - \left. \frac{(k \times 1)^4}{(k + 1)^2} \right] \left. - \frac{(k + \bar{k})^2}{(k + 1)^2} \right] \tag{93}
\]

which agrees with the result derived by using the linearity of the Riemann tensor.

Similarly, it can be shown that (92) produces the correct results for a general linear combination of the \( L_k \). In particular, it can be used to derive the expression (70) for the real flows \( \xi_k \) and \( \xi_l \), where \( k_2 = l_2 \).

Using the first term in (85), we see that

\[
P[DL_k L_l] = (k \times l) \cdot (1 \times \bar{l}) L_{k+l} \tag{94}
\]

where \( P[ ] \) denotes the projection onto the Lie algebra of SDiff\( \mathcal{M} \). This gives

\[
P[DL_k L_l] = \nabla_{L_k} L_l \tag{95}
\]

By linearity, this holds for all vectors. In fact, Nakamura et al [10] use \( D_u v \) as the definition of \( \nabla_u v \). The calculations are unaffected since only the projection onto SDiff\( \mathcal{M} \) is used. (16) gives us

\[
\frac{\partial v}{\partial t} = -P[D_v v] \tag{96}
\]

If \( D_v v = 0 \), then \( v \) certainly generates a stationary flow, and Lukatskii’s theorem holds, since \( \hat{\chi}(v, v) = 0 \). However, in general, we only require that \( P[D_v v] = 0 \), for which \( \hat{\chi}(v, v) \) does not necessarily vanish. Lukatskii’s theorem therefore only works for a subclass of stationary flows.
# The Projective Plane

The projective plane can be constructed from a $2\pi \times 2\pi$ plane by making two reflections:

\[
(x, y) = (x + \pi, \pi - y) \quad (97)
\]

\[
(x, y) = (\pi - x, y + \pi) \quad (98)
\]

As stated in [11], generators which are invariant under these identifications have the form

\[
f_k = (L_k + L_{-k}) - (-)^{k_1+k_2}(L_\pi + L_{-\pi}) \quad (99)
\]

As the calculation is similar to that for the Klein bottle, we do not go into detail. The components of the Riemann tensor for the group of diffeomorphisms on the projective plane are

\[
R_{f_k,f_l,f_m,f_n} = \frac{1}{4} \sum_{\alpha,\beta,\gamma,\delta} \sum_{\mu,\nu,\rho,\sigma} [(\alpha\mu)^{k_1+k_2+1}(\beta\nu)^{l_1+l_2+1}(\gamma\rho)^{m_1+m_2+1}(\delta\sigma)^{n_1+n_2+1} T_{(\mu k_1,\alpha k_2),(\nu l_1,\beta l_2),(\rho m_1,\gamma m_2),(\sigma n_1,\delta n_2)}] \quad (100)
\]

The sectional curvature with respect to $f_k$ and $f_l$ is given exactly by (69) in the general case. In special cases of the form (64) and (65) the sign of the sectional curvature is indefinite.

Alternatively, a curved version of the projective plane can be formed by identifying antipodal points on a sphere, ie:

\[
(\theta, \phi) = (\pi - \theta, \phi + \pi) \quad (101)
\]

On this manifold, the spherical harmonics $Y_{lm}(\theta, \phi)$ are a complete set of basis functions. The diffeomorphism generators can therefore be constructed from

\[
L_{lm} = \frac{1}{\sin \theta} \epsilon^{ab} \partial_b Y_{lm} \partial_a \quad (102)
\]
These are invariant under (101) if and only if \( l \) is odd.

By symmetry, we have

\[
\langle L_{l_1 m_1}, L_{l_2 m_2} \rangle_{\text{projective plane}} = \frac{1}{2} \langle L_{l_1 m_1}, L_{l_2 m_2} \rangle_{\text{sphere}} \tag{103}
\]

Using this together with the expressions for the Lie bracket and covariant derivative derived by Arakelyan and Savvidy [12] for the sphere:

\[
[L_{l_1 m_1}, L_{l_2 m_2}] = \sum_{l_3, m_3} G_{l_1 m_1 l_2 m_2}^{l_3 m_3} L_{l_3 m_3} \tag{104}
\]

\[
\nabla_{L_{l_1 m_1}} L_{l_2 m_2} = \sum_{l_3, m_3} \Gamma_{l_1 m_1 l_2 m_2}^{l_3 m_3} L_{l_3 m_3} \tag{105}
\]

\[
G_{l_1 m_1 l_2 m_2}^{l_3 m_3} = \Gamma_{l_1 m_1 l_2 m_2}^{l_3 m_3} = 0, \ l_1 + l_2 + l_3 \text{ even} \tag{106}
\]

we see that we can calculate the curvature of SDiff(projective plane) simply by considering the curvature with respect to the \( L_{l m} \), where \( l \) is odd, on the sphere. For an expression for the curvature, we refer to [12].
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