Algebraic quantum field theory for particles with structure

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Abstract: Conventional quantum field theory is a method for studying structureless elementary particles, their interaction with each other and with their environment. Non-elementary particles, on the other hand, are those with internal structure or particles that are made up of elementary constituents like the nucleons, which contain quarks and gluons. We introduce a structure-inclusive algebraic formulation of quantum field theory that could handle such particles in which orthogonal polynomials play a central role. For simplicity, we consider non-elementary scalar particles in 3+1 Minkowski space-time and, in an appendix, we treat spinors having structure in 1+1 space-time. The aim of this short exposé is to motivate further studies and research using this approach.

Keywords: quantum field theory, non-elementary particles, orthogonal polynomials, tridiagonal representation approach

1. Introduction

Conventional quantum field theory (QFT) was developed to describe structureless elementary particles, their interaction with each other and with their environment [1-4]. An example is the most successful theory that accounts for the electromagnetic interaction of electrons with photons, called quantum electrodynamics (QED) [5-7]. However, in its early days, QFT did not succeed in describing the interaction of nucleons even at low energies because they are not elementary. It was later replaced by the more successful quantum chromodynamics (QCD), which is the QFT for quarks and gluons as structureless elementary particles [8,9]. In one of its representations, QFT is visualized by diagrams known as the Feynman diagrams that consist of points (vertices) connected by lines (propagators) [10,11]. The lines represent free propagation of elementary particles and the points represent the interaction among particles meeting at those points.

If the elementary particle has a structure* of an infinitesimal size, it is then believed that conventional QFT could still be used successfully at relatively low energies. It is only at higher energies that hidden structural effects become significant. Consequently, we introduce a QFT for particles that may or may not be elementary. That is, particles with internal structure or particles that are built from elementary constituents. It will become evident in the text that the proposed theory has a clear and strong algebraic underpinning. However, it is fundamentally and technically different from that which is commonly known in the mathematics/physics literature as Algebraic Quantum Field Theory (AQFT). A brief and recent description of AQFT can be found in [12] and references cited therein. We refer to the theory introduced here as “Structural Algebraic Quantum Field Theory” (SAQFT)† and for simplicity we consider (in Section 2) scalar non-elementary particles in 3+1 Minkowski space-time. Moreover, in an Appendix, we give a brief mathematical depiction of SAQFT for the Dirac spinor with structure but in 1+1 space-time. The presentation here is elementary and requires basic knowledge in QFT and orthogonal polynomials [13-16]. The objective of this brief introductory study is to

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* Particle structure is a finite or infinite set of discrete but spatially confined configurations/states.
† An easy pronunciation of SAQFT would be “sak FT”
provide motivation for further investigations and advanced research using this structural algebraic approach.

2. Scalar particles in SAQFT

In the relativistic units $\hbar = c = 1$, the Klein-Gordon quantum field in 3+1 Minkowski space-time is represented in SAQFT by the following Fourier expansion in the energy

$$
\Psi(t,\vec{r}) = \int_{\Omega} e^{-i\omega t} \psi(E,\vec{r}) a(E) dE + \sum_{j=0}^{N} e^{-i\omega_j t} \psi_j(\vec{r}) a_j.
$$

(1)

The integral represents the continuous energy spectrum of the particle whereas the sum represents the discrete spectrum (i.e., the particle structure resolved in the energy domain). The latter is a new addition to conventional QFT that could have a positive impact on the treatment of complex systems. The structure in (1) consists of $N+1$ discrete channels whereas $\Omega$ consists generally of several disconnected but continuous channels (called energy bands or energy intervals). For simplicity, we take $\Omega$ to stand for the single energy interval $E^2 \geq M^2$ and take $0 \leq E_j^2 < M^2$, where $M$ is the rest mass of the particle. The objects $a(E)$ and $a_j$ are field operators (the vacuum annihilation operators). They satisfy the following conventional commutation relations [1-4]

$$
[a(E), a^\dagger(E')] := a(E)a^\dagger(E') - a^\dagger(E')a(E) = \delta(E - E') , \quad [a_i, a_j^\dagger] = \delta_{ij}.
$$

(2)

All other commutators among $a(E)$, $a^\dagger(E)$, $a_j$, and $a_j^\dagger$ vanish.

The continuous Fourier energy components $\psi(E,\vec{r})$ in (1) has an extended spatial range whereas the discrete component $\psi_j(\vec{r})$ has a short-range and is confined in space. They are written as the following pointwise convergent series

$$
\psi(E,\vec{r}) = \sum_{n=0}^{\infty} f_n(E) \phi_n(\vec{r}) = f_0(E) \sum_{n=0}^{\infty} p_n(z) \phi_n(\vec{r}) ,
$$

(3a)

$$
\psi_j(\vec{r}) = \sum_{n=0}^{\infty} g_n(E_j) \phi_n(\vec{r}) = g_0(E_j) \sum_{n=0}^{\infty} p_n(z_j) \phi_n(\vec{r}) .
$$

(3b)

where $z$ is an energy parameter to be determined and $\{f_n, g_n\}$ are real expansion coefficients which are written as $f_n = f_0 p_n$ and $g_n = g_0 p_n$ making $p_0 = 1$. $\{\phi_n(\vec{r})\}$ is a complete set of functions that satisfy the following differential equation

$$
-\vec{\nabla}^2 \phi_n(\vec{r}) = \alpha_n \phi_n(\vec{r}) + \beta_{n-1} \phi_{n-1}(\vec{r}) + \beta_n \phi_{n+1}(\vec{r}) ,
$$

(4)

where $\vec{\nabla}^2$ is the three dimensional Laplacian and $\{\alpha_n, \beta_n\}$ are real constants that are independent of $z$ such that $\beta_n \neq 0$ for all $n$. Using Eq. (4), the free Klein-Gordon wave equation, $\left( \partial_t^2 - \vec{\nabla}^2 + M^2 \right) \Psi(t,\vec{r}) = 0$, becomes the following algebraic relation

$$
z p_n(z) = \alpha_n p_n(z) + \beta_{n-1} p_{n-1}(z) + \beta_n p_{n+1}(z) ,
$$

(5)
for \( n = 1, 2, 3, \ldots \) and with \( z = E^2 - M^2 \). This is a symmetric three-term recursion relation that makes \( \{ p_n(z) \} \) a sequence of orthogonal polynomials in \( z \) with the two initial values \( p_0(z) = 1 \) and \( p_1(z) = (z - \alpha_0) / \beta_0 \). Due to Favard theorem (a.k.a. the spectral theorem, see Section 2.5 in [16]), these polynomials satisfy the following general orthogonality relation [13-16]

\[
\int_\Omega \rho(z) p_n(z) p_m(z) \, dz + \sum_{j=0}^{N} \xi(z_j) p_n(z_j) p_m(z_j) = \delta_{n,m},
\]

where \( \rho(z) \) is the continuous component of the weight function and \( \xi(z_j) \) is the discrete component. These weight functions are positive definite and will be determined below in terms of \( f_0(E) \) and \( g_0(E_j) \), respectively. The fundamental algebraic relation (5), which is equivalent to the Klein-Gordon wave equation, is the reason behind the algebraic setup of the theory and for which we qualify this QFT as algebraic. In fact, postulating the three-term recursion relation (5) eliminates the need for specifying a wave equation. Furthermore, once the set of orthogonal polynomials \( \{ p_n(z) \} \) is given then all physical properties of the corresponding system are determined. Thus, a physical process in SAQFT is equivalent to calculating the change from the set \( \{ p_n(z) \} \) to another set \( \{ p'_n(z') \} \) due to this process.

In conventional QFT, the quantum field (1) is expressed as Fourier expansion in the linear momentum \( \vec{k} \)-space not in the energy space. That is, \( \Psi(t, \vec{r}) \) is written as the integral

\[
\int e^{-iE_i \vec{k} \cdot \vec{r}} a(\vec{k}) \frac{d^3k}{(2\pi)^3 2E},
\]

where \( \vec{k}^2 = E^2 - M^2 \) and giving \( \psi(E, \vec{r}) \propto e^{i\vec{k} \cdot \vec{r}} \). One can show that \( e^{i\vec{k} \cdot \vec{r}} \) could be written as an infinite series having the same form as Eq. (3a) by using the relation

\[
e^{i\vec{k} \cdot \vec{r}} = \cos(\vec{k} \cdot \vec{x}) + i \sin(\vec{k} \cdot \vec{x}) = \sqrt{2} e^{-z^2/2} \sum_{n=0}^{\infty} (-1)^n \left[ \frac{H_{2n}(z)}{\sqrt{2^{2n} (2n)!}} \phi_{2n}(x) + i \frac{H_{2n+1}(z)}{\sqrt{2^{2n+1} (2n+1)!}} \phi_{2n+1}(x) \right],
\]

where \( z = k / \lambda \), \( H_m(z) \) is the Hermite polynomial, \( \lambda \) is a real scale parameter, and \( \phi_n(x) = e^{-z^2/2} \left[ H_m(\lambda x) / \sqrt{2^m m!} \right] \). Therefore, \( p_m(z) = H_m(z) / \sqrt{2^m m!} \) with \( \alpha_m = 0 \), \( \beta_m = (m+1)/2 \), and \( f_0(E) \propto e^{-z^2/2} \). An equivalent expansion could also be written in terms of products of the Gegenbauer (ultra-spherical) polynomials \( \{ C_n^\nu(z) \} \) and the Bessel function with discrete index \( J_{n+\nu}(\lambda x) \) as follows, see Eq. (4.8.3) in Ref. [16],

\[
e^{i\vec{k} \cdot \vec{r}} = 2^\nu \Gamma(\nu)(\lambda x)^{-\nu} \sum_{n=0}^{\infty} i^n (n+\nu) C_n^\nu(z) J_{n+\nu}(\lambda x),
\]

giving \( p_n(z) \propto C_n^\nu(z) \) and \( \phi_n(x) \propto i^n (n+\nu)(\lambda x)^{-\nu} J_{n+\nu}(\lambda x) \).

Now, the conjugate quantum field \( \bar{\Psi}(t, \vec{r}) \) in SAQFT is obtained from (1) by complex conjugation and the replacement \( \phi_n(\vec{r}) \mapsto \bar{\phi}_n(\vec{r}) \) where

\[
\langle \phi_n(\vec{r}) | \bar{\phi}_m(\vec{r}) \rangle = \langle \bar{\phi}_n(\vec{r}) | \phi_m(\vec{r}) \rangle = \delta_{n,m},
\]

\[
-3-
\]
The first is the orthogonality relation\(^\dagger\) and the second is the completeness statement. Therefore, we write \(\bar{\Psi}(t,\vec{r})\) as follows

\[
\bar{\Psi}(t,\vec{r}) = e^{iE^2} \bar{\varphi}(E,\vec{r}) a^\dagger(E) dE + \sum_{j=0}^{N} e^{iE_j} \bar{\varphi}_j(\vec{r}) a_j^\dagger.
\]

where the components \(\varphi(E,\vec{r})\) and \(\varphi_j(\vec{r})\) are identical to (3) but with \(\phi_n(\vec{r}) \mapsto \bar{\phi}_n(\vec{r})\). Using the commutators (2) of the field operators \(a(E)\) and \(a_j\), we can write

\[
\left[ \Psi(t,\vec{r}), \bar{\Psi}(t',\vec{r}') \right] = \sum_{n,m=0}^{\infty} \phi_n(\vec{r}) \bar{\phi}_m(\vec{r}') \times \\
\left[ \int_{\Omega} e^{-iE(t-t')} f_0^2(E) p_{n}(z) p_m(z) dE + \sum_{j=0}^{N} e^{-iE_j(t-t')} g_0^2(E_j) p_{n}(z_j) p_m(z_j) \right].
\]

The general orthogonality (6) and the completeness (8b) turn Eq. (10) with \(t = t'\) into

\[
\left[ \Psi(t,\vec{r}), \bar{\Psi}(t,\vec{r}) \right] = \delta^3(\vec{r} - \vec{r}'),
\]

where we took \(f_0^2(E) dE = \rho(z) dz\) and \(g_0^2(E_j) = \xi(z_j)\), which also imply positivity of the two weight functions. Moreover, it is straightforward to write

\[
\left[ \Psi(t,\vec{r}), \bar{\Psi}(t,\vec{r}) \right] = \left[ \bar{\Psi}(t,\vec{r}), \bar{\Psi}(t,\vec{r}) \right] = 0.
\]

In the canonical quantization of fields, the canonical conjugate to \(\Psi(t,\vec{r})\) is written as \(\Pi(t,\vec{r})\) and they satisfy the following equal time commutation relations [1-4]

\[
\left[ \Psi(t,\vec{r}), \Psi(t,\vec{r}') \right] = \left[ \Pi(t,\vec{r}), \Pi(t,\vec{r}') \right] = 0,
\]

\[
\left[ \Psi(t,\vec{r}), \Pi(t,\vec{r}') \right] = i\delta^3(\vec{r} - \vec{r}').
\]

Therefore, we obtain the following identification: \(\Pi(t,\vec{r}) = i\bar{\Psi}(t,\vec{r})\). Moreover, in analogy with conventional QFT [1-4], we can write Eq. (10) as

\[
\left[ \Psi(t,\vec{r}), \bar{\Psi}(t',\vec{r}') \right] = \Delta(t-t',\vec{r} - \vec{r}'),
\]

where the singular function \(\Delta(t-t',\vec{r} - \vec{r}')\) in SAQFT reads as follows

\[
\Delta(t-t',\vec{r} - \vec{r}') = \\
\sum_{n,m=0}^{\infty} \phi_n(\vec{r}) \bar{\phi}_m(\vec{r}') \times \\
\left[ \int_{\Omega} e^{-iE(t-t')} \rho(z) p_{n}(z) p_m(z) dz + \sum_{j=0}^{N} e^{-iE_j(t-t')} \xi(z_j) p_{n}(z_j) p_m(z_j) \right].
\]

Moreover, Eq. (11) and Eq. (14) give \(\Delta(0,\vec{r} - \vec{r}') = \delta^3(\vec{r} - \vec{r}')\).

\(^\dagger\) The analysis in the present work does not require the orthogonality (8a).
Now, we can define the real (neutral) scalar non-elementary particle by the quantum field
\[ \Phi(t, \vec{r}) = \frac{1}{\sqrt{2}} \left[ \Psi(t, \vec{r}) + \phi(t, \vec{r}) \right] \]
with \( \phi(t, \vec{r}) = \phi(t, \vec{r}) \). On the other hand, the complex (charged) scalar non-elementary particle is defined by the positive-energy quantum field
\[ \Phi(t, \vec{r}) = \frac{1}{\sqrt{2}} \left[ \Psi(t, \vec{r}) + \Psi\dagger(t, \vec{r}) \right] . \] (16a)
\( \Psi(t, \vec{r}) \) is identical to (1) but with \( a(E) \mapsto a(E) \) and \( a_j \mapsto a_j^\dagger \) such that \( [a_j(E), a_j(E')] = \delta_{j,j'} \delta(E - E') \) and \( [a_j^\dagger, a_j^\dagger] = \delta_{j,j'} \delta^{ij} \) where \( r \) and \( r' \) stand for ±. The corresponding charged scalar antiparticle is represented by the following negative-energy quantum field
\[ \Phi(t, \vec{r}) = \frac{1}{\sqrt{2}} \left[ \Psi(t, \vec{r}) + \Psi\dagger(t, \vec{r}) \right] . \] (16b)

For this scalar particle, the Feynman propagator \( \Delta(t - t', \vec{r} - \vec{r'}) \) between the two space-time points \( (t, \vec{r}) \) and \( (t', \vec{r'}) \) is constructed by combining the following two processes [1-4]:

1. The creation of a particle from the vacuum \( \langle 0 | \Psi(t, \vec{r}) \rangle \) at \( (t, \vec{r}) \) and annihilating it later \( (t' > t) \) back into the vacuum at \( (t', \vec{r'}) \).
2. The conjugate process of creating an antiparticle from the vacuum at \( (t', \vec{r'}) \) then annihilating it later \( (t > t') \) at \( (t, \vec{r}) \).

That is,
\[ \Delta(t - t', \vec{r} - \vec{r'}) = \langle 0 | T \left( \Phi(t', \vec{r})\Phi(t, \vec{r}) \right) | 0 \rangle = \langle 0 | \Phi(t', \vec{r})\Phi(t, \vec{r}) | 0 \rangle \delta(t - t') + \langle 0 | \Phi(t, \vec{r})\Phi(t', \vec{r}) | 0 \rangle \delta(t - t') \] (17)
where \( T \) is the time ordering operator and \( \delta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \). With the free propagator being determined, one needs to identify the type of interaction to account for the behavior of the scalar particle when coupled to its environment. Without such an interaction, the internal structure of the non-elementary particle (summation part of the quantum field) has no bearing on its free motion. Only in the presence of interaction will we observe the added effect of the internal structure. Moreover, the type and extent of such an effect will certainly depend on the nature of the interaction (electromagnetic, nuclear, gravitational, etc.). One way to incorporate the interaction of the particle with an external field is by using the gauge invariant minimal coupling scheme where the 4-gradient \( \partial_\mu + iqA_\mu \) in the wave equation is replaced by \( \partial_\mu + iqA_\mu + iqA_\mu \) with \( q \) being the coupling parameter and \( (A_\mu, \vec{A}) \) the external 4-vector field.

It was shown elsewhere [17,18] that electromagnetic interaction (of which, the Dirac-Coulomb problem is an example) is associated with the two-parameter Meixner-Pollaczek polynomial \( P_\mu^\theta(z, \theta) \) where \( \mu > 0 \) and \( 0 < \theta < \pi \). This polynomial is known to have only a continuous spectrum. That is, the summation part in the orthogonality (6) is absent. Consequently, the internal structure of the corresponding particle has a null effect. In this case, SAQFT is equivalent to the conventional QFT; both leading to QED. In the following section, we present a simple nontrivial example; a type of interaction where the particle structure will have an effect on the outcome.
3. Scalar SAQFT example

As a simple but nontrivial example of scalar SAQFT, we consider a non-elementary scalar particle in 1+1 Minkowski space-time whose structure is associated with the three-parameter continuous dual Hahn polynomial \( S^\mu_n(z;\sigma,\tau) \). The properties of this orthogonal polynomial that are relevant to our study are given in Appendix A. We consider here the special case where \( \mu < 0 \) and \( \sigma = \tau > -\mu \). From the formulation given above, it is obvious that the particle structure is determined by the recursion relation (5) \textit{vis-à-vis} its recursion coefficients \( \{\alpha_n, \beta_n\} \).

The symmetric three-term recursion relation for the orthonormal version of the polynomial \( S^\mu_n(z;\sigma,\sigma) \) is shown as Eq. (A6) in the Appendix giving

$$
\begin{align*}
\alpha_n &= (n + \mu + \sigma)^2 + n(n + 2\sigma - 1) - \mu^2, \\
\beta_n &= -(n + \mu + \sigma)\sqrt{(n+1)(n+2\sigma)}.
\end{align*}
$$

(18a, 18b)

The physical effect of this internal structure becomes evident when we introduce the interaction, which is portrayed by the differential equation satisfied by \( \phi_n(x) \). We choose

$$
\phi_n(x) = \frac{\lambda \Gamma(n+1)}{\sqrt{\Gamma(n+2\sigma)}} y^\sigma e^{-\lambda x/2} L_n^{2\sigma-1}(y).
$$

(19)

where \( y = e^{-\lambda x} \) and \( L_n^{2\sigma-1}(y) \) is the Laguerre polynomial. The scale parameter \( \lambda \) is real and positive with inverse length dimension. It could be considered as measure of the size of the structure or range of its coupling in the interaction. The orthogonality of the Laguerre polynomials shows that \( \{\phi_n(x)\} \) is an orthonormal set: \( \phi_n(x) = \phi_n(x) \). Using the differential equation, differential property and recursion relation of the Laguerre polynomials, we obtain the differential equation associated with the structure of this scalar particle that replaces the free wave equation (4). It reads

$$
\begin{bmatrix}
-1 & \frac{d^2}{dx^2} + W(x)
\end{bmatrix} \phi_n(x) = \alpha_n \phi_n(x) + \beta_{n-1}\phi_{n-1}(x) + \beta_n \phi_{n+1}(x),
$$

(20)

where \( W(x) \) is a manifestation of the interaction, which is associated with the structure of the particle corresponding to \( S^\mu_n(z;\sigma,\sigma) \). It is also associated with \( \phi_n(x) \) and it reads

$$
W(x) = \frac{1}{4} e^{-2\lambda x} + \left(\mu - \frac{1}{2}\right) e^{-\lambda x}.
$$

(21)

Moreover, \( z = (E^2 - M^2)/\lambda^2 \) and the size of the structure is equal to \( N+1 \), where \( N \) is the largest integer less than or equal to \( -\mu \). The rest of the objects needed to determine the quantum fields and propagators are the continuous and discrete components of the weight functions \( \rho(z) \) and \( \xi(z) \) in addition to the spectrum \( \{z_j\} \) of the discrete structure. These are given in Appendix A by Eq. (A2), Eq. (A4) and Eq. (A5), respectively.
The interaction in this system is a physical process whereby the set of parameters \( \{E, \lambda, \mu, \sigma\} \) that defines the system is changed to \( \{E', \lambda', \mu', \sigma'\} \). That is, \( S^n_{\nu}(z; \sigma, \sigma') \mapsto S^n'_{\nu}(z'; \sigma', \sigma') \), which could alter the size and nature of the particle’s structure. For example, if \( \mu = -3.7 \mapsto \mu' = -3.2 \) while \( \lambda \) remains the same then the structure will maintain its size (\( N + 1 = 4 \)) but the levels of the structure will change from \( E_j = M^2 - \lambda^2 \left( j - 3.7 \right)^2 \) to \( E'_j = M^2 - \lambda^2 \left( j - 3.2 \right)^2 \) where \( j = 0, 1, 2, 3 \). However, if \( \mu \) changes to, say \( \mu' = -1.5 \), then not only the levels will change but also the size of the structure (i.e., the nature of the particle itself) changes from 4 to 2. Moreover, if \( \mu' \) becomes positive then the entire structure disappears signifying a total decay of the particle’s constituents or structure due to the process.

4. Conclusions and comments

In this work, we posed the question: Why doesn’t conventional QFT work at low energies for particles with infinitesimal structure (e.g., the nucleons)? We suggested a simple answer that it is because of a missing piece; the particle structure itself. Consequently, we introduced an algebraic version of QFT that accommodates particles with internal structure, which is resolved in the energy. The theory of orthogonal polynomials plays a critical role in the formulation. It is hoped that the brief introduction of the theory given here will motivate further studies using this approach towards a more effective and generalized QFT. Incorporating the particle structure may bring new elements into the theory that could be exploited to tackle some of the persistent difficulties in conventional QFT. We believe that these new elements may have a positive impact on the renormalization program. As illustration, we presented in Section 2 the SAQFT for scalar particles in 3+1 space-time and gave in Section 3 a simple nontrivial model of the same but in 1+1 space-time.

In Appendix B, we give a rough and brief mathematical presentation of SAQFT for spinor particles with structure in 1+1 Minkowski space-time. It is understood that spinors in such space-time do not carry physical significance. However, the aim is to provide the framework and tools for further rigorous treatment. For completeness, one also needs to provide the massless vector field (e.g., the electromagnetic potential) and massive vector meson in the SAQFT formulation.

One of the remaining tasks in SAQFT (as presented here) is to establish the relativistic invariance of its physical objects and covariance of its elements under the space-time Poincaré transformation (Lorentz rotation + translation). It is conceivable that results from such an endeavor may alter and/or improve on the development of the theory as presented in this work.

Finally, we conclude with the following remarks:

- The SAQFT presented in this work does not require a wave equation to be specified. In fact, the formulation of the theory is built upon four defining postulates:

  (i) The quantum field and its conjugate resolved in the energy domain, which for scalar particles are given by Eq. (1) and Eq. (9), respectively.

  (ii) The completeness of spatial functions used in the expansion series, which for scalar particles is given by Eq. (8).
(iii) The creation/annihilation operators, which for scalar particles satisfy the algebra given by the commutation relations (2).

(iv) The orthogonal polynomials satisfying the three-term recursion relation (5) and orthogonality (6).

The corresponding postulates in the case of spinors are shown in Appendix B by Eq. (B2) and Eq. (B7) for the quantum fields, and by the anti-commutation algebra (B3) for the creation/annihilation operators.

- One may suggest a modification of the conventional QFT to incorporate the particle structure in a simpler manner without going through the sumptuous construction of SAQFT involving orthogonal polynomials. That is, the conventional quantum field for scalar particles could be redefined as follows

\[ \tilde{\Psi}(t, \vec{r}) = \int e^{-iE\hat{t} + i\hat{k} \cdot \vec{r}} a(\hat{k}) \frac{d^3k}{(2\pi)^3 2E} + \sum_{j=0}^{N} e^{-iE_j \hat{t} + i\hat{k_j} \cdot \vec{r}} a_j. \]  

(22)

However, this will result in a quantum field that is “over-complete”. Namely, \( \|\tilde{\Psi}(t, \vec{r})\| > 1 \) because the continuous spectrum is already complete since \( e^{i\hat{k} \cdot \vec{r}} \| = 1 \). On the other hand, the orthogonality (6) of the polynomials in SAQFT gives the completeness of the quantum field \( \Psi(t, \vec{r}) \) as a sum of the continuous and discrete spectra that could be written symbolically as follows

\[ \|\Psi(t, \vec{r})\| = \|\nu(E, \vec{r})\| + \sum_{j=0}^{N} \|\nu_j(\vec{r})\| = 1. \]  

(23)

Another way to show the inadequacy of the representation (22) is to note that, in the language of SAQFT, the expansion of \( e^{i\hat{k} \cdot \vec{r}} \) is associated with the Hermite polynomial (or the Gegenbauer polynomial) whose entire spectrum is continuous leaving no room for including an internal discrete structure; cf. Eq. (7).

- If the set of orthogonal polynomial \( \{ p_n(z) \} \) is endowed only with a discrete spectrum, then the continuous integral in the definition of the quantum field (1) does not appear and the orthogonality (6) consists only of the summation part. Examples of such polynomials include the Meixner, Charlier, dual Hahn, and the Racah polynomials [19]. We conjecture that such systems might constitute an alternative to the traditional approach that accounts for the confinement of particles like the quarks. They could also be used in the treatment of point contact (zero range) interactions (those where massless gauge fields are absent).

- All physically relevant orthogonal polynomials with a continuous spectrum that are compatible with SAQFT must have a sinusoidal asymptotic behavior. Specifically, in the limit as \( n \to \infty \), we require that \( p_n(z) \) takes the following form

\[ p_n(z) \approx \frac{1}{n^\nu \sqrt{\rho(z)}} \cos\left[n^\nu \phi(z) + \delta(z)\right], \]  

(24)

where \( \kappa \) and \( \nu \) are positive real parameters, \( \phi(z) \) is an entire function, and \( \delta(z) \) is the scattering phase shift. If \( \nu \to 0 \) then \( n^\nu \to \ln(n) \). Only under this asymptotic condition, will the series (3a) produce oscillatory scattering states at the boundaries of space (see, for example, [17,20-23]). For a rigorous discussion about the connection between the
asymptotics of such polynomials and scattering, one may consult [24-26] and references therein. Fortunately, all of the many known hypergeometric orthogonal polynomials that appear abundantly in the physics literature do meet this requirement. They include, but not limited to, the polynomials in the Askey scheme [19] such as the Wilson, continuous Hahn, continuous dual Hahn, Meixner-Pollaczek, Jacobi, Laguerre, Gegenbauer, Chebyshev, Hermite, etc.

- In conventional QFT, a physical process could be accounted for by summing all Feynman diagrams occurring within the process up to a given order. An equivalent scheme in SAQFT is to determine the change in the associated orthogonal polynomials \( \{ p_n(z) \} \) resulting from the physical process. Such change could be evaluated perturbatively in one of the physical parameters that appear in the recursion coefficients \( \{ \alpha_n, \beta_n \} \) or in \( z \). For example, in Section 3, we could take \( |z/M| << 1 \) or take \( \sigma + \mu << 1 \), and so on.

- A more general differential equation satisfied by the complete set of functions \( \{ \phi_n(\bar{r}) \} \) that still maintain the same algebraic structure of the theory is

\[
D\phi_n(\bar{r}) = \omega(\bar{r})\left[ (\alpha_n - z)\phi_n(\bar{r}) + \beta_{n+1}\phi_{n+1}(\bar{r}) + \beta_n\phi_{n+1}(\bar{r}) \right],
\]

where \( D \) is the wave operator (e.g., the Klein-Gordon, Dirac, Dirac-Coulomb, etc.) and \( \omega(\bar{r}) \) is an entire function that does not vanish locally. This equation should be considered as a generalization of Eq. (4) for scalars or Eq. (B5) for spinors. For example, in Eq. (20), \( D = -\frac{1}{\lambda^2} \frac{d^2}{dx^2} + W(x) \) and \( \omega(\bar{r}) = 1 \).

- The linear momentum \( \vec{k} \) (where \( \vec{k}^2 = E^2 - M^2 \)) is an essential variable in the formulation of conventional QFT, which is usually written in the \( k \)-space representation. On the other hand, \( \vec{k} \) does not appear explicitly in SAQFT. This is due to the tridiagonal structure of the fundamental differential equation for \( \phi_n(\bar{r}) \). In fact, using (4) or (20) in the Klein-Gordon wave equation gives \( \vec{k}^2 \) as one of the eigenvalues\(^8\) of the following infinite tridiagonal symmetric matrix

\[
\begin{pmatrix}
\alpha_0 & \beta_0 & & & \\
\beta_0 & \alpha_1 & \beta_1 & & \\
& \beta_1 & \alpha_2 & \beta_2 & \\
& & \beta_2 & \alpha_3 & \beta_3 \\
& & & \times & \times \\
& & & \times & \times \\
& & & \times & \times
\end{pmatrix}.
\]

- For special systems where the continuous and discrete channels are independent, SAQFT formulation could be extended by choosing two distinct sets of functions in the expansion of the continuous channel kernel \( \psi(E, \bar{r}) \) and discrete channel kernel \( \psi_j(\bar{r}) \). That is, we

\(^8\) Let \( \Sigma \) be the matrix whose elements are \( \langle \phi_i | \phi_n \rangle \), then \( \vec{k}^2 \) is a generalized eigenvalue in the matrix equation \( Q|\cdot\rangle = \vec{k}^2 \Sigma |\cdot\rangle \) where \( Q \) is the tridiagonal symmetric matrix (26). If \( \vec{\phi}_n = \phi_n \), then \( \Sigma_{n,m} = \delta_{n,m} \) and \( Q|\cdot\rangle = \vec{k}^2 |\cdot\rangle \).

- •
could still expand $\psi(E, \vec{r})$ in the same set $\{\phi_n(\vec{r})\}$ as given by (3a) but expand $\psi_j(\vec{r})$ in another independent set $\{\chi_n(\vec{r})\}$ by rewriting (3b) as

$$\psi_j(\vec{r}) = \sum_{n=0}^{\infty} g_n(E_j) \chi_n(\vec{r}) = g_0(E_j) \sum_{n=0}^{\infty} q_n(z_j) \chi_n(\vec{r}),$$  \hspace{1cm} (27)

where $\{q_n(z_j)\}$ is a set of orthogonal polynomials having only discrete spectrum, whereas $\{p_n(z)\}$ has a continuous spectrum only. Moreover, $\sum_{n=0}^{\infty} \chi_n(\vec{r}) \overline{\chi}_n(\vec{r}) = \sum_{n=0}^{\infty} \overline{\chi}_n(\vec{r}) \chi_n(\vec{r}) = \delta^3(\vec{r} - \vec{r}')$ and $\langle \chi_n(\vec{r}) | \overline{\chi}_m(\vec{r}) \rangle = \langle \overline{\chi}_n(\vec{r}) | \chi_m(\vec{r}) \rangle = \delta_{n,m}$. This allows us to use two different set of function, one is suitable for the continuous channel and the other is more appropriate for the discrete channel (structure). The wave equation (4) or its generalization (25) for $\{\chi_n(\vec{r})\}$ maintains the same tridiagonal form but possibly with different recursion coefficients $\{\alpha_n, \beta_n\}$. Furthermore, we obtain

$$\left[\Psi(t,\vec{r}), \overline{\Psi}(t',\vec{r}')\right] = \Delta(t-t',\vec{r} - \vec{r}') = \frac{1}{2} \sum_{n,m=0}^{\infty} \phi_n(\vec{r}) \overline{\phi}_m(\vec{r}) e^{-ix(t-t')} \rho(z) p_n(z) p_m(z) dz + \frac{1}{2} \sum_{n,m=0}^{\infty} \chi_n(\vec{r}) \overline{\chi}_m(\vec{r}) \sum_{j=0}^{N} e^{-ix(t-t')} \xi(z_j) q_n(z_j) q_m(z_m)$$  \hspace{1cm} (28)

giving $\Delta(0,\vec{r} - \vec{r}') = \delta^3(\vec{r} - \vec{r}')$.

- The tridiagonal representation approach (TRA) is an algebraic method for solving linear ordinary differential equations [27,28]. It has been used successfully in the solution of the wave equation in quantum mechanics resulting in a larger class of exactly solvable problems (see, for example, [29] and references therein). The SAQFT presented here could be viewed as an application of the TRA in QFT.

### Appendix A: The continuous dual Han polynomial

The orthonormal version of this three-parameter polynomial reads as follows

$$S^\mu_n(x^2; \sigma, \tau) = \sqrt{\frac{(\mu+\sigma)_{n}(\mu+\tau)_{n}}{n!(\sigma+\tau)^n}} 3F_2 \left( \begin{array}{c} -n, \mu+ix, \mu-ix \\ \mu+\sigma, \mu+\tau \end{array} \right | 1 \right),$$  \hspace{1cm} (A1)

where $3F_2 \left( \begin{array}{c} a, b, c \\ d, e \end{array} \right | x \right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n} \frac{x^n}{n!}$ is the generalized hypergeometric series and $(a)_n = a(a+1)(a+2)...(a+n-1) = \frac{\Gamma(n+a)}{\Gamma(a)}$ is the Pochhammer symbol (shifted factorial). Moreover, if $\text{Re}(\mu, \sigma, \tau) > 0$ with non-real parameters occurring in conjugate pairs then this is a polynomial in $x^2$ which is orthogonal with respect to the measure $\rho^\mu(x; \sigma, \tau) dx$ whose normalized form reads as follows

$$\rho^\mu(x; \sigma, \tau) = \frac{1}{2\pi} \frac{\Gamma(\mu + ix)\Gamma(\sigma + ix)\Gamma(\tau + ix)/\Gamma(2ix)}{\Gamma(\mu + \sigma)\Gamma(\mu + \tau)\Gamma(\sigma + \tau)}$$  \hspace{1cm} (A2)
That is, \( \int_0^\infty S_n^\mu(x^2;\sigma,\tau)S_m^\mu(x^2;\sigma,\tau)\rho^\mu(x;\sigma,\tau)\,dx = \delta_{nm} \). However, if the parameters are such that \( \mu < 0 \) and \( \sigma + \mu, \tau + \mu \) are positive or a pair of complex conjugates with positive real parts, then the polynomial will have a continuous spectrum as well as a finite size discrete spectrum and the polynomial satisfies the following generalized orthogonality relation

\[
\int_0^\infty \rho^\mu(x;\sigma,\tau)S_n^\mu(x^2;\sigma,\tau)S_m^\mu(x^2;\sigma,\tau)\,dx + \sum_{j=0}^N \xi^\mu(x_j;\sigma,\tau)S_n^\mu(x_j^2;\sigma,\tau)S_m^\mu(x_j^2;\sigma,\tau) = \delta_{nm}.
\]  

(A3)

where \( N \) is the largest integer less than or equal to \( -\mu \) and

\[
\xi^\mu(x_j;\sigma,\tau) = 2(-1)^{1+j} \frac{(j + \mu)(\mu + \sigma)(\mu + \tau)}{j!} \frac{(2\mu)}{(\mu - \sigma + 1)(\mu - \tau + 1)} \frac{\Gamma(\sigma - \mu)}{\Gamma(\sigma + \tau)} \frac{\Gamma(\tau - \mu)}{\Gamma(1 - 2\mu)}.
\]  

(A4)

The asymptotics \((n \to \infty)\) of \( S_n^\mu(x^2;\sigma,\tau) \), which could be found in the Appendix of Ref. [21], vanishes if \( \mu + i\chi = -j \), where \( j = 0, 1, 2, \ldots, N \). Thus, the spectrum formula associated with this polynomial is

\[
x_j^2 = -(j + \mu)^2.
\]  

(A5)

Moreover, these polynomials satisfy the following symmetric three-term recursion relation

\[
-x^2 S_n^\mu(x^2;\sigma,\tau) = \left[(n + \mu + \sigma)(n + \mu + \tau) + n(n + \sigma + \tau - 1) - \mu^2 \right] S_n^\mu(x^2;\sigma,\tau)
\]

\[+ \sqrt{n(n + \sigma + \tau - 1)(n + \mu + \sigma - 1)(n + \mu + \tau - 1)} S_{n-1}^\mu(x^2;\sigma,\tau)
\]

\[+ \sqrt{(n + 1)(n + \sigma + \tau)(n + \mu + \sigma)(n + \mu + \tau)} S_{n+1}^\mu(x^2;\sigma,\tau)
\]  

(A6)

Appendix B: Dirac spinor in SAQFT

Although spinors do not carry physical significance in 1+1 Minkowski space-time, we provide in this Appendix a brief mathematical depiction of the 2-component quantum fields \( \Psi^{\pm}(t,x) \) for an artificial Dirac spinor in SAQFT. The hope is that a more suitable, rigorous and extended treatment could be developed using the basic framework provided here.

The two Dirac gamma matrices in this space-time could be written in terms of the \( 2 \times 2 \) Pauli matrices as \( \gamma^0 = \sigma_3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \) and \( \gamma^1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \). The corresponding free Dirac equation for \( \Psi^{\pm}(t,x) \) reads

\[
\begin{pmatrix}
    i\partial_t - M & -\partial_x \\
    \partial_x & i\partial_t + M
\end{pmatrix}
\begin{pmatrix}
    \Psi_+^{\pm}(t,x) \\
    \Psi_-^{\pm}(t,x)
\end{pmatrix} = 0.
\]  

(B1)

The two components of the quantum spinor field are written as

\[
\Psi_\pm(t,x) = \int e^{iE_j}\psi_\pm(E,x) b_\pm(E)\,dE + \sum_{j=0}^N e^{-iE_j}\psi_\pm^{\dagger}(x)b_\pm^{\dagger}.
\]  

(B2a)
\[ \Psi^\dagger_\pm(t,x) = \int e^{-iEt} \eta^\dagger_\pm(E,x) b_\pm(E) dE + \sum_{j=0}^N e^{-iE_j t} \eta^\dagger_\pm(x) b^j_\pm . \]  

(B2b)

The creation and annihilation operators satisfy the following anti-commutation relations

\[ \{ b_s(E), b_{s'}(E') \} = b_s(E) b_{s'}(E') + b_{s'}(E') b_s(E) = \delta_{s,s'} \delta(E - E') , \quad \{ b^j_s, (b^{j'}_{s'})^\dagger \} = \delta_{s,s'} \delta^{j,j'} . \]  

(B3)

where \( s \) and \( s' \) stand for the \( \uparrow \downarrow \) spins. All other anti-commutators vanish:

\[ \{ b_s(E), b_s(E') \} = 0 , \quad \{ b^j_s, b^{j'}_{s'} \} = 0 , \quad \text{and} \quad \{ b_s(E), b^j_{s'} \} = \{ b^j_s(E), b^j_{s'} \} = 0 . \]

The Fourier energy components are written as the following series

\[ \psi^\dagger_i(E,x) = \sum_{n=0}^\infty f^\dagger_n(E) \phi^\dagger_n(x) = f^\dagger_0(E) \sum_{n=0}^\infty p^\dagger_n(z) \phi^\dagger_n(x) , \]  

(B4a)

\[ \psi^{i,j}(x) = \sum_{n=0}^\infty g^i_n(E) \phi^j_n(x) = g^i_0(E) \sum_{n=0}^\infty p^i_n(z) \phi^j_n(x) . \]  

(B4b)

where \( r \) stands for the \pm component. The energy parameter \( z \) is to be determined and \( \{ \phi^\dagger_n(x) \} \) is a complete set of square integrable functions. Substituting (B4) into (B2), the coupled free Dirac equation (B1) could be written as

\[ r \left( \frac{d}{dx} \phi^\dagger_n(x) \right) = (\alpha_n - rM) \phi^\dagger_n(x) + \beta_{n-1} \phi_{n-1}^\dagger(x) + \beta_n \phi_{n+1}^\dagger(x) , \]  

(B5)

provided that \( \{ p^{\dagger n}_n(z) \} \) is a sequence of orthogonal polynomials in \( z \) (with \( z = E \)) that satisfy the same symmetric three-term recursion relation (5) but with the initial values: \( p^{\dagger n}_0(z) = 1 \) and \( p^{\dagger n}_n(z) = (z - \alpha_n) / \beta_n \) whereas \( p^{\dagger n}_0(z) = 0 \) and \( p^{\dagger n}_1(z) = 1 \).** Moreover, \( \{ p^{\dagger n}_n(z) \} \) are also required to fulfill the general orthogonality relation (6) but with \( \{ \rho(z), \xi(z) \} \mapsto \{ \rho^{\dagger n}(z), \xi^{\dagger n}(z) \} \).

The conjugate quantum field \( \overline{\Psi}^{\dagger\pm}(t,x) \) is obtained from (B2) by the maps \( \Psi^{\dagger\pm}_\pm \mapsto (\Psi^{\dagger\pm}_\pm)^\dagger \) and \( \phi_n^\dagger \mapsto \overline{\phi}_n^\dagger \) where

\[ \left\langle \phi^\dagger_n(x) \overline{\phi}^\dagger_m(x) \right\rangle = \left\langle \overline{\phi}^\dagger_n(x) \phi^\dagger_m(x) \right\rangle = \delta_{n,m} \delta^\dagger_{n,m} , \]  

(B6a)

\[ \sum_{n=0}^\infty \phi^\dagger_n(x) \overline{\phi}^\dagger_n(x') = \sum_{n=0}^\infty \overline{\phi}^\dagger_n(x) \phi^\dagger_n(x') = \delta^\dagger_{n,m} \delta(x - x') . \]  

(B6b)

Therefore, we write \( \overline{\Psi}^{\dagger\pm}(t,x) \) as follows

** Therefore, these two sets of orthogonal polynomials, \( \{ p^{\dagger n}_n(z) \} \), satisfy the following Wronskian-like relation:

\[ \frac{\beta_n}{\beta_0} \left[ p^{\dagger n}_n(z) p^{\dagger m}_{n+1}(z) - p^{\dagger n}_n(z) p^{\dagger m}_{n+1}(z) \right] = 1 . \]
\[
\overline{\Psi}_r(t,x) = \int_{\Omega} e^{i\mathcal{H}_r(E,x)\xi} b^\dagger_r(E) dE + \sum_{j=0}^N e^{i\mathcal{H}_r^{ij}(x)} (b_j^\dagger)^j.
\]  \hspace{1cm} (B7)

where the components \(\overline{\Psi}_r^{ij}(E,x)\) and \(\overline{\Psi}_r^{ij}(x)\) are identical to (B4) but with \(\phi^\dagger_n(x) \mapsto \overline{\phi}_n^\dagger(x)\). Using the anti-commutators (B3), we can write

\[
\left\{\Psi'_r(t,x), \overline{\Psi}'_r(t',x')\right\} = \delta_{r,r'} \sum_{n,m=0}^\infty \phi^{\dagger}_n(x) \phi_m^\dagger(x') \times
\]

\[
\left[ e^{-iE(t-t')} \rho'(z) p^+_n(z) p^+_m(z) dz + \sum_{j=0}^N e^{-i\mathcal{H}_r^{ij}(x)} \xi'(z_j) p^+_n(z_j) p^+_m(z_j) \right]
\]

where we wrote \([f_0'(E)]^2 dE = \rho'(z) dz\) and \([g_0'(E)]^2 = \xi'(z_j)\). As in the conventional QFT, this defines the singular distribution \(\Delta_{r,r'}(t-t',x-x')\):

\[
\left\{\Psi'_r(t,x), \overline{\Psi}'_r(t',x')\right\} = \delta_{r,r'} \Delta_{r,r'}(t-t',x-x').
\]  \hspace{1cm} (B9)

Using the orthogonality (6) of the polynomials \(\left\{p^+_n(z)\right\}\) and the completeness (B6b) of the set \(\left\{\phi^\dagger_n(x)\right\}\), this equation with \(t=t'\) becomes

\[
\left\{\Psi'_r(t,x), \overline{\Psi}'_r(t',x')\right\} = \delta_{r,r'} \delta(x-x').
\]  \hspace{1cm} (B10)

Equations (B9) and (B10) give \(\Delta_{r,r'}(0,x-x') = \delta_{r,r'} \delta(x-x')\). Moreover, it is straightforward to write

\[
\left\{\Psi'_r(t,x), \Psi'_r(t,x')\right\} = \left\{\overline{\Psi}'_r(t,x), \overline{\Psi}'_r(t,x')\right\} = 0.
\]  \hspace{1cm} (B11)

Therefore, the canonical conjugate to the spinor quantum field \(\Psi'_r(t,x)\) is \(\Pi'_r(t,x) = i\overline{\Psi}'_r(t,x)\). Additionally, and as done for the scalar particle, we can define the positive-energy non-elementary spinor particle by the quantum field

\[
\chi_+(t,x) = \frac{1}{\sqrt{2}} \left[ \Psi'_r(t,x) \pm \Psi'_r(t,x)^\dagger \right].
\]  \hspace{1cm} (B12a)

The \(\pm\) within the square bracket comes from the multiplication (on the right) of the 2-component spinor \(\Psi'_r(t,x)\) by the Dirac matrix \(\gamma^0\). On the other hand, the corresponding anti-particle is represented by the negative-energy quantum field

\[
\chi_-(t,x) = \frac{1}{\sqrt{2}} \left[ \Psi'_r(t,x) \pm \Psi'_r(t,x)^\dagger \right].
\]  \hspace{1cm} (B12b)

Consequently, the associated Feynman propagator is obtained as follows

\[
\Delta^F_{r,r'}(t-t',x-x') = \langle 0 | T \left( \chi_-(t',x'), \chi_+(t,x) \right) | 0 \rangle =
\]

\[
\langle 0 | \overline{\chi}_+(t',x') \chi_+(t,x) | 0 \rangle \Theta(t-t') - \langle 0 | \chi_+(t,x) \overline{\chi}_+(t',x') | 0 \rangle \Theta(t-t')
\]  \hspace{1cm} (B13)

Note the minus sign in the middle of the second line of this propagator due to anti-commutation of the field operators.
We end this Appendix by noting that the spinor formulation of SAQFT presented above could have been made more sumptuous by choosing the set of square integrable functions \( \{\phi_n^+ (x)\} \) to be spin dependent as \( \{\phi_n^{s+} (x)\} \). In that case, some of the equations/relations above should be modified as follows:

\[
\psi^s_r (E, x) = \sum_{n=0}^{\infty} f_n^s (E) \phi_n^{s+} (x) = f_0^s (E) \sum_{n=0}^{\infty} p_n^s (z) \phi_n^{s+} (x), \tag{B4a*}
\]

\[
\psi^s_{r'} (x) = \sum_{n=0}^{\infty} g_n^s (E_j) \phi_n^{s+} (x) = g_0^s (E_j) \sum_{n=0}^{\infty} p_n^s (z) \phi_n^{s+} (x). \tag{B4b*}
\]

\[
r \left[ \frac{d}{dx} \phi^{s-\sigma} (x) \right] = \left( \alpha_n^s - rM \right) \phi_n^{s+} (x) + \beta_{n-1}^s \phi_n^{s-} (x) + \beta_n^s \phi_n^{s+} (x), \tag{B5*}
\]

\[
\left\{ \phi_n^{s+} (x) \phi_n^{s-} (x') \right\} = \left\{ \phi_n^{s+} (x') \phi_n^{s-} (x) \right\} = \delta_{s,s'} \delta_{r,r'} \delta_{n,m}, \tag{B6a*}
\]

\[
\sum_{n=0}^{\infty} \phi_n^{s+} (x) \phi_n^{s-} (x') = \sum_{n=0}^{\infty} \phi_n^{s+} (x') \phi_n^{s-} (x) = \delta_{s,s'} \delta_{r,r'} \delta (x-x'), \tag{B6b*}
\]

\[
\left\{ \Psi_r^s (t, x), \overline{\Psi}_{r'}^{s'} (t', x') \right\} = \delta_{s,s'} \sum_{n=0}^{\infty} \phi_n^{s+} (x) \phi_n^{s-} (x') \times \left[ e^{-iE(t-t')} \rho^s (z) p_n^s (z) p_n^{s'} (z) dz + \sum_{j=0}^{N} e^{-iE_j (t-t')} \xi_j^s (z_j) p_n^s (z_j) p_n^{s'} (z_j) \right] \tag{B8*}
\]

Moreover, the polynomials \( \{p_n^{s+} (z)\} \) must satisfy the symmetric three-term recursion relation (5) but with \( \{\alpha_n, \beta_n\} \mapsto \{\alpha_n^{s+}, \beta_n^{s+}\} \) and with the initial values, \( p_0^s (z) = 1 \) and \( p_1^s (z) = (z-\alpha_0^s)/\beta_0^s \).

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