Hypothesis Testing of One Sample Mean Vector in Distributed Frameworks

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Abstract

Distributed frameworks are widely used to handle massive data, where sample size $n$ is very large, and data are often stored in $k$ different machines. For a random vector $X \in \mathbb{R}^p$ with expectation $\mu$, testing the mean vector $H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$ for a given vector $\mu_0$ is a basic problem in statistics. The centralized test statistics require heavy communication costs, which can be a burden when $p$ or $k$ is large. To reduce the communication cost, distributed test statistics are proposed in this paper for this problem based on the divide and conquer technique, a commonly used approach for distributed statistical inference. Specifically, we extend two commonly used centralized test statistics to the distributed ones, that apply to low and high dimensional cases, respectively. Comparing the power of centralized test statistics and the distributed ones, it is observed that there is a fundamental tradeoff between communication costs and powers of the tests. This is quite different from the application of the divide and conquer technique in many other problems such as estimation, where the associated distributed statistics can be as good as the centralized ones. Numerical results confirm the theoretical findings.

Key words: mean test, large and high dimension, divide and conquer, power function.
1 Introduction

In recent years, massive data are commonly encountered in many applications and are often stored in \( k \) different machines in a distributed system. Developing communication-efficient statistical methods has attracted a lot of attention recently. The divide and conquer technique is a popular method in distributed frameworks, where one constructs a statistic or an estimator using data in each machine, and then transmits them to the hub to get a pooled one. The divide and conquer technique has been applied successfully in many problems, including regression and classification, hypothesis testing, confidence intervals, principal eigenspaces analysis, linear discriminant analysis, and many others (Zhang et al., 2013; Hsieh et al., 2014; Zhang et al., 2015; Lin et al., 2017; Szabó and Van Zanten, 2019; Battey et al., 2018; Guo et al., 2019; Chen and Peng, 2018; Jordan et al., 2019; Fan et al., 2019; Tian and Gu, 2017; Li and Zhao, 2020; Dobriban and Sheng, 2021, etc.).

In many problems such as point estimations, the statistic constructed by the divide and conquer technique can be as efficient as the centralized one, when the number of machines \( k \) is not too large, usually much smaller than \( \sqrt{n} \) (Chen and Xie, 2014; Zhang et al., 2013, 2015; Jordan et al., 2019; Volgushev et al., 2019; Chen and Peng, 2018, etc.). Although the divide and conquer technique has been used successfully in many problems in the distributed system, its application to hypothesis testing of the mean vector for massive data has not been studied.

For a variable \( X \in \mathbb{R}^p \) with \( E(X) = \mu \), the hypothesis testing on the mean vector \( \mu \) is a basic problem in statistics, playing a critical role in many applications such as quality control, environmental science, geography, medicine, education, social sciences, and many others. For example, in quality control, to test a batch of products is qualified or not, one needs to consider the test problem: \( H_0 : \mu = \mu_0 \) vs \( H_1 : \mu \neq \mu_0 \), where \( \mu \in \mathbb{R}^p \) is the mean of the products considered and \( \mu_0 \in \mathbb{R}^p \) is the standard (Edward Jackson, 1985; Ye et al., 2002, etc.). In addition, to analyze the difference of paired samples and the effect of treatments with matched samples (Davison, 1992; Rubin, 2006; Haug et al., 2011, etc.), it is common to transform the problem into a one-sample mean test problem \( H_0 : \mu = 0 \) vs \( H_1 : \mu \neq 0 \), where \( \mu \in \mathbb{R}^p \) is the mean of the difference of the paired samples, and rejecting \( H_0 \) implies that there exist treatment effects.
In this paper, we consider the hypothesis testing problem

\[ H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0, \tag{1.1} \]

where \( \mu_0 = (\mu_{01}, \cdots, \mu_{0p})^\top \) is a given vector. Supposing that \( \{X_i\}_{i=1}^n \) are \( p \)-dimensional random vectors that are \( i.i.d. \) from the normal distribution \( \mathcal{N}_p(\mu, \Sigma) \) with the unknown covariance matrix \( \Sigma \), satisfying \( p < n \), the classical test statistic for this problem is the Hotelling \( T^2 \) statistic (Anderson, 1984), which is a centralized one, defined as follows

\[ T^2_{cen,n} = (n - 1)(\bar{X} - \mu_0)^\top \hat{\Sigma}^{-1}(\bar{X} - \mu_0), \]

where \( \bar{X} = n^{-1} \sum_{i=1}^n X_i \) is the sample mean and \( \hat{\Sigma} = \sum_{i=1}^n (X_i - \bar{X})(X_i - \bar{X})^\top / n \) is the sample covariance matrix. However, when the sample covariance matrix is nearly singular, the power of the Hotelling \( T^2 \) test decreases significantly even when \( p < n \), (Bai and Saranadasa, 1996). Many extensions of the Hotelling \( T^2 \) test to high dimensional cases have been proposed in the literature. For example, Bai and Saranadasa (1996) considered the case of \( p/n \to r \in (0, 1) \), Srivastava and Du (2008) investigated the case of \( p = O(n^{\zeta}) \) with \( \zeta \in (1/2, 1) \), and Lee et al. (2012) studied the case of \( p/n \to r > 0 \). In addition, many authors considered the case where \( p \) can be much larger than \( n \) and the distribution is not Gaussian (Chen et al., 2010; Wang et al., 2015; Srivastava et al., 2016; Xu et al., 2016; Dong et al., 2016; Chakraborty et al., 2017; Li et al., 2020, etc.). For example, the test statistic in Wang et al. (2015) has significant gains for heavy-tailed multivariate distributions.

Although there are many works on the problem (1.1), to the best of our knowledge, this problem has not been studied in a distributed framework, when massive data are collected and are stored in \( k \) different machines. When \( p < n \), the classical Hotelling \( T^2 \) test can be computed by transmitting \( k \) matrices of the size \( p \times p \), as shown in Section 2.1, of which the communication cost can be expensive when \( p \) or \( k \) is large. Taking this into account, we propose a distributed version of the Hotelling \( T^2 \) test statistic that is communication-efficient, based on the idea of the divide and conquer technique, and derive the asymptotical distribution of the distributed test statistic under \( H_0 \). Furthermore, we compare the power functions of the centralized Hotelling \( T^2 \) test with the distributed one.

When \( p \) is much larger than \( n \), as an illustration, we extend the centralized test statis-
tic of Wang et al. (2015), proposing a distributed test statistic, deriving its asymptotic distribution under $H_0$, and comparing the power functions of the distributed test with the centralized one. Theoretical results show that the powers of the distributed test statistics are decreasing functions of $k$, indicating that there is a fundamental tradeoff between communication cost and efficiency. This is different from the existing results in other settings such as point estimations, where the distributed statistics can be as good as the centralized ones.

The rest of the paper is organized as follows. In Section 2, a distributed test statistic is constructed by applying the divide and conquer technique to the Hotelling $T^2$ test, and the asymptotic distribution under $H_0$ is developed and the power function of the distributed test statistic is compared with the centralized one. In Section 3, we consider the high dimensional case, constructing a distributed test statistic by applying the divide and conquer technique to the test statistic of Wang et al. (2015); the asymptotic distribution under $H_0$ and its power are studied. Simulation results and real data analysis are reported in Section 4 and a brief discussion is presented in Section 5. All the proofs are presented in Section 6.

2 One Sample Mean Test in Distributed Frameworks

2.1 Computation of the centralized Hotelling $T^2$ test in distributed frameworks

Since data are stored in $k$ different machines, we denote by $S_l$ the set of indices of the observations on the $l$-th machine, that is, $S_l = \{i : X_i$ is stored in the $l$-th machine$\}$, $l = 1, \cdots, k$. Let $n_l = |S_l|$, the sample size on the $l$-th machine. In the distributed framework, constructing the centralized test statistic is still feasible by transmitting some matrices of size $p \times p$, as shown in Algorithm 1.

Suppose that $\{X_i\}_{i=1}^n$ are i.i.d. random variables following $N_p(\mu, \Sigma)$. Under $H_0$, according to the relationship between Hotelling $T^2$ distribution and the $F$-distribution, we have

$$\frac{n-p}{(n-1)p}T^2_{cen,n} \sim F(p, n-p),$$

where $F(p, n-p)$ denotes the $F$-distribution with the degrees of freedom $p$ and $n-p$. 


Algorithm 1: Computing the Centralized Hotelling $T^2$ in a Distributed Framework

1: For $l = 1, \cdots, k$, compute the local sample mean $\overline{X}_l$ and the local second moment $A_l$ using observations in the $l$-th machine where $\overline{X}_l = n_l^{-1}\sum_{i \in S_l} X_i$ and $A_l = n_l^{-1}\sum_{i \in S_l} X_i X_i^\top$.

2: Transmit the $\overline{X}_l$’s and $A_l$’s to the central hub, and compute the global sample mean and the global sample covariance matrix denoted as $\overline{X} = n^{-1}\sum_{k} n_l \overline{X}_l$ and $\hat{\Sigma} = n^{-1}\sum_{l=1}^k n_l A_l - \overline{X} \overline{X}^\top$, respectively.

3: Define the centralized Hotelling $T^2$ test statistic $T^2_{cen,n} = \frac{(n-1)(\overline{X} - \mu_0)^\top \hat{\Sigma}^{-1}(\overline{X} - \mu_0)}{\frac{n - p}{(n-1) p} T^2_{cen,n}} > F_{1-\alpha}(p, n-p)$.

Given the significant level $\alpha \in (0, 1)$, the rejection region for Hotelling $T^2$ test is denoted as

$$C_1 = \left\{ T^2_{cen,n} : \frac{n - p}{(n-1) p} T^2_{cen,n} > F_{1-\alpha}(p, n-p) \right\},$$

where $F_{1-\alpha}(p, n-p)$ is $1 - \alpha$ quantile of $F(p, n-p)$. Clearly, to obtain centralized test statistic $T^2_{cen,n}$, one needs to transfer $p \times p$ matrices $A_l$’s to the hub, which can be a burden when $p$ or $k$ is large. To reduce the communication cost, we consider a distributed test statistic.

### 2.2 Extending Hotelling $T^2$ test in distributed frameworks

For simplicity of notations, assume that data are randomly and evenly distributed in $k$ machines, that is

$$n_1 = \cdots = n_k = n/k,$$

where $n_l$ is the sample size in the $l$-th machine, $l = 1, \cdots, k$. We propose the following distributed test statistic using the divide and conquer technique, by computing first the Hotelling $T^2$ test statistic with data in each machine, and then pooling them together. Specifically, the distributed test statistic is defined as follows

$$T^2_{dis,n} = \frac{1}{k \sqrt{p}} \sum_{l=1}^k T^2_l,$$

where $T^2_l = (n_l - 1)(\overline{X}_l - \mu_0)^\top \hat{\Sigma}_l^{-1}(\overline{X}_l - \mu_0)$ is the local Hotelling $T^2$ statistic based on the data in the $l$-th machine, and $\hat{\Sigma}_l$ and $\overline{X}_l$ are the local sample covariance matrix and the local sample mean, respectively. Note that $T^2_{dis,n}$ is communication efficient, requiring only the transmission of the scalars $T_l$’s to the hub. Under $H_0$, it is shown in the Theorem that $T^2_{dis,n}$ converges to a normal distribution. Assume that $k$ and $p$ are functions of $n$.\[\square\]
that is, $k = k_n$ and $p = p_n$, and the following conditions hold.

\begin{itemize}
  \item[(A1)] Let $\gamma_n = pk/n$. Assume that $\gamma_n < 1$ and that $\gamma_n \to r \in [0, 1)$, as $n \to \infty$. Moreover, assume that $n_l = n/k \to \infty, l = 1, \ldots, k$, as $n \to \infty$.
\end{itemize}

Condition (A1) requires that $k = O(n/p)$, where both $k$ and $p$ can diverge with $n$.

**Theorem 1.** Suppose that $X_i$'s are i.i.d. from $N_p(\mu, \Sigma)$ and that condition (A1) holds.

Under $H_0$, as $k \to \infty$, it holds that

$$
\sqrt{k} \left( T_{\text{dis}, n}^2 - \frac{(n/k - 1)\sqrt{p}}{n/k - p - 2} \right) \overset{d}{\to} N \left( 0, \frac{2}{(1-r)^3} \right). 
$$

(2.1)

Based on Theorem 1, the rejection region for the distributed test can be written as follows,

$$
C_2 = \left\{ T_{\text{dis}, n}^2 : \left( \frac{(1-r)^3k}{2} \right)^{1/2} \left| T_{\text{dis}, n}^2 - \frac{(n/k - 1)\sqrt{p}}{n/k - p - 2} \right| > z_{1-\alpha/2} \right\},
$$

where $z_{1-\alpha/2}$ is the $1 - \alpha/2$ quantile of standard normal distribution. Compare the distributed test statistic with the centralized one, there are two observations. (1) To compute the centralized test statistic, one needs to transmit $k$ matrices of size $p \times p$ to the hub, while the distributed one only requires the transmission of $k$ scalars to the hub, having great advantages in communication cost. (2) For the distributed test statistic, we get a normal distribution asymptotically, which differs from the centralized one.

### 2.3 The power of the distributed test

In this section, we compare the powers of the distributed test statistic and the centralized one under $H_1$. Denote by $\psi_n^{\text{cen}} = P(C_1|H_1)$ and $\psi_n^{\text{dis}} = P(C_2|H_1)$ the power functions of the centralized test and the distributed one, respectively. Let $\alpha$ be a significance level of the test and let $\Delta = (\mu - \mu_0)^\top \Sigma^{-1} (\mu - \mu_0)$. Denote

$$
\phi_{\Delta, \gamma_n} = 1 - \Phi \left( A_{1n} z_{1-\alpha/2} - A_{2n} \sqrt{(1 - \gamma_n)/2} \right) + \Phi \left( A_{1n} z_{1-\alpha/2} - A_{2n} \sqrt{(1 - \gamma_n)/2} \right)
$$

with

$$
A_{1n} = \left( \frac{\gamma_n}{\gamma_n + 2\Delta + \Delta^2} \right)^{1/2}, \quad A_{2n} = \left( \frac{n\Delta^2}{\gamma_n + 2\Delta + \Delta^2} \right)^{1/2}
$$
where \( \Phi(\cdot) \) denotes the cumulative distribution function of standard normal distribution. The properties of \( \psi_{\text{cen}} \) and \( \psi_{\text{dis}} \) are as follows. In the following descriptions, notations \( a_n \gg b_n \) and \( a_n \ll b_n \) mean \( a_n/b_n \to \infty \) and \( a_n/b_n \to 0 \) respectively, as \( n \to \infty \).

**Theorem 2.** (1) If \( \Delta \gg \sqrt{p/n} \), then \( \psi_{\text{cen}} \to 1 \) when \( n \to \infty \). (2) Assuming that condition (A1) holds and that \( k \to \infty \), then \( \psi_{\text{dis}} - \phi_{\Delta, \gamma_n} \to 0 \).

The proof of Theorem 2 is given in Appendix. From the definition of \( \phi_{\Delta, \gamma_n} \), it is easy to see that \( \phi_{\Delta, \gamma_n} \to 1 \), when \( A_{1n} \to 0 \) or \( A_{2n} \to \infty \). The following corollary clarifies conditions of \( \psi_{\text{dis}} \to 1 \).

**Corollary 1.** Assume that condition (A1) holds and that \( k \to \infty \). Then \( A_{1n} \to 0 \) when \( \Delta \gg \gamma_n \), and \( A_{2n} \to \infty \) when \( \Delta \gg \sqrt{kp/n} \). Consequently, \( \psi_{\text{dis}} \to 1 \) as \( \Delta \gg \sqrt{kp/n} \).

**Proof.** The conclusions in Corollary 1 is derived by discussing the order of \( \Delta \) and \( \gamma_n \). First, we prove the conclusion on \( A_{1n} \). From the definition of \( A_{1n} \), it holds that \( A_{1n} \to 0 \) when \( \Delta \gg \gamma_n \) or \( \Delta^2 \gg \gamma_n \). Because \( \gamma_n < 1 \), we see that \( A_{1n} \to 0 \) when \( \Delta \gg \gamma_n \). Second, we prove the conclusion on \( A_{2n} \). Recall that

\[
A_{2n} = \left( \frac{n\Delta}{\gamma_n/\Delta + \Delta + 2} \right)^{1/2}.
\]

When \( \Delta \gg \gamma_n \), it is seen that \( A_{2n} \gg \sqrt{kp} \), due to \( kp < n \). Moreover, when \( \Delta \) and \( \gamma_n \) have the same order, \( A_{2n} \) has the same order as \( \sqrt{kp} \). Due to \( k \to \infty \), we see that \( A_{2n} \to \infty \), when \( \gamma_n = O(\Delta) \). When \( \Delta \ll \gamma_n \), it is easy to see that

\[
A_{2n} = \left( \frac{n\Delta^2}{\gamma_n} \right)^{1/2} (1 + o(1)),
\]

and consequently that \( A_{2n} \to \infty \), when \( \Delta \gg \sqrt{\gamma_n/n} \), i.e. \( \Delta \gg \sqrt{kp/n} \). Combining the argument together, we have \( A_{2n} \to \infty \), when \( \Delta \gg \sqrt{kp/n} \). □

To get some insights, we make a plot of the function \( \phi_{\Delta, \gamma_n} \) with \( n = 10^5 \), \( p = 10^3 \) and \( \alpha = 0.05 \) for different values of \( k \) in Figure 1. It is seen that \( \phi_{\Delta, \gamma_n} \) is an increasing function of \( \Delta \) and a decreasing one of \( k \). And from Theorem 2 and Corollary 1, we notice the power of the distributed test is lower than the centralized one.
Recall that $\psi_{n}^{\text{dis}} \to 1$ as $\Delta \gg \sqrt{kp/n}$ and that $\psi_{n}^{\text{cen}} \to 1$ as $\Delta \gg \sqrt{p/n}$. Then the asymptotic relative efficiency of the distributed test statistic to the centralized one can be measured by the ratio $(\sqrt{p/n})/(\sqrt{kp/n}) = \sqrt{1/k}$.

3 One Sample Mean Test in High-dimensional Cases in Distributed Frameworks

Hotelling $T^2$ test statistic is not well defined when the dimension is larger than the sample size. Moreover, the distribution of Hotelling $T^2$ test statistic depends on the normal distribution heavily. In this section, we apply the divide and conquer approach to the test statistics that are applicable to general distributions and high dimensional settings.

3.1 Review of some extensions of Hotelling $T^2$ statistic in high dimensional cases

Although Hotelling $T^2$ test is a classical one, it has serious limitations when the dimension $p$ is comparable to, or larger than the sample size $n$. Obviously, the test statistic is not well defined for $p > n$ because of the singularity of the sample covariance matrix. Moreover, Hotelling $T^2$ test is also known to perform poorly when $p/n \to r \in (0, 1)$. For example, Dempster (1958) proposed the non-exact test in high dimension. Bai and Saranadasa (1996) further found that even when Hotelling $T^2$ statistic is well defined, Dempster’s
non-exact test is more powerful than the Hotelling $T^2$ for $p/n \rightarrow r \in (0, 1)$.

To overcome the limitations of Hotelling $T^2$ test. Many test statistics are developed by replacing the inverse of sample covariance matrix in Hotelling $T^2$ statistic with a diagonal matrix or the identity matrix. These statistics are applicable to high-dimensional cases and to more general distributions beyond normal. For example, Bai and Saranadasa (1996) assumed the following model: $X_i = \Lambda W_i + \mu_i$, for $i = 1, \cdots, n$, where $\Lambda$ is a $p \times m$ non-random matrix for some $m \geq p$ such that $\Lambda \Lambda^\top = \Sigma$, and $\{W_i\}_{i=1}^n$ are $m$-dimensional vectors satisfying $E(W_i) = 0$ and $\text{cov}(W_i) = I_m$, the $m \times m$ identity matrix. And they assumed $p/n \rightarrow r > 0$, that is, $p$ can be comparable to $n$ or larger than $n$. This model includes the normal distribution as a special case. Instead of using Mahalanobis distance, Bai and Saranadasa (1996) constructed the test statistic based on Euclidean distance as follows,

$$T_{BS} = n(\bar{X} - \mu_0)^\top(\bar{X} - \mu_0) - \frac{n}{n-1} \text{Tr}(\hat{\Sigma}),$$

where $\bar{X}$ is the sample mean. Srivastava and Du (2008) noticed that the statistic $T_{BS}$ is not scale-invariant, which may reduce the power when the scales of different components of the random vector $X_i$ are quite different. As a remedy, they proposed the following statistic $T_{SD} = (\bar{X} - \mu_0)^\top D^{-1}(\bar{X} - \mu_0)$, where $D = \text{diag}(n\hat{\sigma}_{11}/(n-1), \cdots, n\hat{\sigma}_{pp}/(n-1))$, consisting of the diagonal elements of the matrix $n\hat{\Sigma}/(n-1)$ with $\hat{\Sigma}$ denoting the sample covariance matrix. In addition, it is assumed by Srivastava and Du (2008) that $n = O(p^\zeta)$ for some $1/2 < \zeta \leq 1$ with normality assumption. Furthermore, Srivastava (2009) relaxed the normality assumption, and assumed $n = O(p^\zeta)$, $0 < \zeta \leq 1$. Chen et al. (2011) proposed another test statistic

$$T_{RHT} = (\bar{X} - \mu_0)^\top(n\hat{\Sigma}/(n-1) + \lambda I)^{-1}(\bar{X} - \mu_0), \quad \text{for } \lambda > 0,$$

under the setting $p/n \rightarrow r \in (0, \infty)$, stabilizing the inverse of the sample covariance matrix. As an improvement to the work of Bai and Saranadasa (1996), Chen et al. (2010) proposed a statistic by removing the quadratic terms in $(\bar{X} - \mu_0)^\top(\bar{X} - \mu_0)$, which relaxed the restrictions on $p$ and $n$ greatly. Wang et al. (2015) considered robust testing procedures which are powerful to distributions with heavy-tailed and to data with outlying observations. In the following subsection 3.2, as an example for illustration, we apply the divide and conquer method to the test statistic of Wang et al. (2015).
3.2 Nonparametric multivariate tests in a distributed system

To handle heavy-tailed distributions, Wang et al. (2015) assumed that $X_i$’s are i.i.d. variables following an elliptical distribution. Specifically,

$$X_i = \mu + \epsilon_i, \quad \epsilon_i = \Gamma R_i U_i, \quad 1 \leq i \leq n,$$

where $\Gamma$ is a $p \times p$ matrix, $U_i$ is a random vector uniformly distributed on the unit sphere in $\mathbb{R}^p$, and $R_i$ is a nonnegative random variable independent of $U_i$. For the hypothesis testing (1.1), the centralized test statistic in Wang et al. (2015) is as follows

$$G_{cen,n} = \sum_{i=1}^{n} \sum_{j=1}^{i-1} Z_i^\top Z_j,$$

where $Z_i = (X_i - \mu_0)/\|X_i - \mu_0\|$ if $X_i \neq \mu_0$, $Z_i = 0$ if $X_i = \mu_0$, and $\|X_i\|$ denotes the $L_2$ norm of $X_i$. When data are stored at random in $k$ machines, $G_{cen,n}$ can be computed as follows. For $l = 1, \ldots, k$, we compute local sample sum $B_l = \sum_{i \in S_l} Z_i$ and transmit $B_l$’s to the central hub. Then $G_{cen,n} = [(\sum_{l=1}^{k} B_l)^\top (\sum_{l=1}^{k} B_l) - n]/2$.

However, when $p$ is large and communication cost is prohibitively heavy, we are interested in the following distributed version of the statistic

$$G_{dis,n} = \sum_{l=1}^{k} \sum_{i,j \in S_l, j < i} Z_i^\top Z_j := \sum_{l=1}^{k} G_l,$$

where $G_l = \sum_{i,j \in S_l, j < i} Z_i^\top Z_j \in \mathbb{R}$ and $S_l$ is the indices of observations in $l$-th machine. To obtain $G_{dis,n}$, only scalars $G_l$’s need to be transmitted to the hub. Comparison of the powers of $G_{cen,n}$ and $G_{dis,n}$ is an interesting problem. To derive the asymptotic properties of $G_{cen,n}$ and $G_{dis,n}$, we need the following conditions, where $\text{Tr}(\cdot)$ denotes the trace and $\Sigma = \text{cov}(X_i)$.

(A2) Let $a_{m,\Sigma} = \text{Tr}(\Sigma^m)$ for any positive integer $m$, and $a_{\max}$ and $a_{\min}$ be the maximum and minimum eigenvalues of $\Sigma$, respectively. Assume that (i) $a_{4,\Sigma} = o(a_{2,\Sigma}^2)$; (ii) $(a_{1,\Sigma}/a_{2,\Sigma}^2) \exp \left\{ -a_{2,\Sigma}^2/(128p^2a_{\max}^2) \right\} = o(1)$; (iii) $a_{\max} = o(a_{1,\Sigma})$; (iv) $\exp \left\{ -a_{1,\Sigma}^2/(256p^2a_{\max}^2) \right\} = o(\min\{a_{\max}/a_{1,\Sigma}, a_{\min}/a_{\max}\})$.

(A3) For some $0 < \delta < 1$, $\|\mu - \mu_0\|^{2\delta} E(\|\epsilon_i\|^{-2-2\delta}) = o(E^2(\|\epsilon_i\|^{-1}))$. 

(A4) \[ \| \mu - \mu_0 \|^2 E(\| \epsilon_i \|^2) = o \left( \min \{ a_{2, \Sigma} / (n a_{\max} a_{1, \Sigma}), a_{2, \Sigma}^{1/2} / (n^{1/2} a_{1, \Sigma}) \} \right). \]

These conditions are exactly the conditions (C1)–(C6) in Wang et al. (2015). Let

\[ A_\epsilon = E((I_p - \epsilon_i \epsilon_i^T / \| \epsilon_i \|^2) / \| \epsilon_i \|), \quad B_\epsilon = E(\epsilon_i \epsilon_i^T / \| \epsilon_i \|^2). \]

Wang et al. (2015) proved the following conclusion on the centralized test statistic \( G_{\text{cen}, n} \).

**Lemma 1.** (Wang et al. (2015)) Assuming the conditions (i) and (ii) in (A2), under \( H_0 \), as \( n, p \to \infty \), it holds that

\[ G_{\text{cen}, n} / \sqrt{n(n - 1) \text{Tr}(B_\epsilon^2) / 2} \overset{d}{\to} N(0, 1). \]

And under \( H_1 \), if the conditions (A2)–(A4) hold, as \( n, p \to \infty \),

\[ [G_{\text{cen}, n} - n(n - 1)(\mu - \mu_0)^\top A_\epsilon^2(\mu - \mu_0)(1 + o(1)) / 2] / \sqrt{n(n - 1) \text{Tr}(B_\epsilon^2) / 2} \overset{d}{\to} N(0, 1). \]

To discuss the asymptotic properties of the distributed statistic \( G_{\text{dis}, n} \), we assume

(4') \[ \| \mu - \mu_0 \|^2 E(\| \epsilon_i \|^2) = o \left( \min \{ k a_{2, \Sigma} / (n a_{\max} a_{1, \Sigma}), (k a_{2, \Sigma})^{1/2} / (n^{1/2} a_{1, \Sigma}) \} \right). \]

It is seen that (A4') is weaker than (A4). The asymptotic distribution of \( G_{\text{dis}, n} \) is as follows.

**Theorem 3.** Let \( k = k_n \) and \( p = p_n \) be constants depending on \( n \) and set \( n_i = n / k \). Assume that \( \min \{ k, n_i, p \} \to \infty \) as \( n \to \infty \).

1. Under \( H_0 \), assuming that conditions (i) and (ii) in (A2) hold, as \( n \to \infty \), it follows that

\[ G_{\text{dis}, n} / \sqrt{n(n/k - 1) \text{Tr}(B_\epsilon^2) / 2} \overset{d}{\to} N(0, 1). \]

2. Under \( H_1 \), assuming that conditions (A2), (A3), and (A4') hold, as \( n \to \infty \), we have

\[ G_{\text{dis}, n} - n(n/k - 1)(\mu - \mu_0)^\top A_\epsilon^2(\mu - \mu_0)(1 + o(1)) / 2 \overset{d}{\to} N(0, 1). \]

Under the condition (i) in (A2), when \( \min \{ n_i, p \} \to \infty \), following Chen et al. (2010) and Wang et al. (2015), \( \text{Tr}(B_\epsilon^2) \) can be estimated using only the data in the first machine,
specifically,

\[
\hat{\text{Tr}}(\hat{B}_2^2) = -\frac{n_1}{(n_1 - 2)^2} + \frac{(n_1 - 1)}{n_1(n_1 - 2)^2} \text{Tr}\left\{ \left( \sum_{j \in S_1} Z_j Z_j^\top \right)^2 \right\} \\
+ \frac{1 - 2n_1}{n_1(n_1 - 1)} \bar{Z}^*^\top \left( \sum_{j \in S_1} Z_j Z_j^\top \right) \bar{Z}^* + \frac{2}{n_1} \| \bar{Z}^* \|^2 + \frac{(n_1 - 2)^2}{n_1(n_1 - 1)} \| \bar{Z}^* \|^4,
\]

where \( \bar{Z}^* = (n_1 - 2)^{-1} \sum_{i \in S_1} Z_i \). We compare the centralized test with the distributed one in terms of the asymptotic relative efficiency. Let \( \eta_p = (\mu - \mu_0)^\top A^2(\mu - \mu_0)(1 + o(1))/\sqrt{2 \text{Tr}(\hat{B}_2^2)} \). Theorem 3 implies that under \( H_1 \), the distributed test with sample size \( n \) has the power

\[
\psi_n^{\text{dis}} = 1 - \Phi(z_{1 - \alpha/2} - \sqrt{n(n/k - 1)}\eta_p) + \Phi(z_{\alpha/2} - \sqrt{n(n/k - 1)}\eta_p).
\]

On the other hand, by Wang et al. (2015), the centralized statistic \( G_{\text{cen},N} \) with sample size \( N \) has the power

\[
\psi_N^{\text{cen}} = 1 - \Phi(z_{1 - \alpha/2} - \sqrt{N(N - 1)}\eta_p) + \Phi(z_{\alpha/2} - \sqrt{N(N - 1)}\eta_p).
\]

Then we describe the asymptotic relative efficiency of the distributed test to the centralized one by the ratio of the sample sizes \( N/n \) such that \( \psi_N^{\text{cen}} = \psi_n^{\text{dis}} \), that is, \( N/n = 1/\sqrt{k} \). Thus, the distributed test is less efficient than that of the centralized one.

Combining the results in Section 2 and 3 together we derive the following conclusions.

For the distributed test statistic constructed using the divide and conquer method, there is a tradeoff between the efficiency represented by the power and the communication cost. On the other hand, when point estimations are concerned, many works have shown that in terms of means square error (MSE), the estimators constructed using divide and conquer can be the same order as that of the centralized estimators (Chen and Xie, 2014; Zhang et al., 2015; Jordan et al., 2019; Volgushev et al., 2019; Chen and Peng, 2018, etc.).
4 Numerical Experiments

4.1 Simulation results

We illustrate the performance of the distributed tests and compare them with that of the centralized ones by simulations. For any integer \( m \), denote by \( \mathbf{0}_m \) the vector of zero in \( \mathbb{R}^m \), and \( \mathbf{1}_m \) the vectors in \( \mathbb{R}^m \) with elements 1. Let \( \mu_0 = \mathbf{0}_p \), and consider the following hypothesis testing problem

\[
H_0: \mu = \mu_0 \quad \text{vs} \quad H_1: \mu \neq \mu_0.
\]

For both distributed and centralized tests, we repeat 500 times to compute the type I error and the means of the power.

**Example 1.** We consider the case of \( pk < n \). Let \( \{X_i\}_{i=1}^n \) be i.i.d. \( p \)-dimensional random vectors from the normal distribution \( N_p(\mu, \Sigma) \), where \( E(X_i) = \mu \in \mathbb{R}^p \) and \( \text{cov}(X_i) = \Sigma \in \mathbb{R}^{p \times p} \). With the notation \( \tilde{\Sigma} = (\tilde{\sigma}_{ij}) \) with \( \tilde{\sigma}_{ij} = 0.5^{|i-j|} \), we assume that \( \mu \) and \( \Sigma \) are generated from the following four cases: (1) \( \mu = (c_1^\top, \mathbf{0}_p^\top)^\top \) and \( \Sigma = I_p \); (2) \( \mu = c \mathbf{1}_p \) and \( \Sigma = I_p \); (3) \( \mu = (c_1^\top, \mathbf{0}_{p-2}^\top)^\top \) and \( \Sigma = \tilde{\Sigma} \); (4) \( \mu = c \mathbf{1}_p \) and \( \Sigma = \tilde{\Sigma} \), where \( c \) is a constant, and different values of \( c \) will be considered.

We set \( p = 50 \) and 100 respectively. For the distributed test, we set \( k = 30, 50, \) and 70, respectively, and split the data randomly into \( k \) machines with equal sizes. For each setting, we estimate the sizes of the test by the frequency of rejection in 500 replicas. Simulation results on the type I error for \( k = 30 \) are presented in Table 1, and those of \( k = 50 \) and \( k = 70 \) are similar and are omitted.

| \( n \)   | \( \Sigma = I_p \) \( p = 50 \) | \( \Sigma = I_p \) \( p = 100 \) | \( \Sigma = \tilde{\Sigma} \) \( p = 50 \) | \( \Sigma = \tilde{\Sigma} \) \( p = 100 \) |
|----------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| 5000     | 0.060                            | 0.058                            | 0.056                            | 0.056                            |
| 8000     | 0.048                            | 0.052                            | 0.048                            | 0.050                            |
| 10000    | 0.052                            | 0.052                            | 0.048                            | 0.052                            |

Table 1: Type I error of the distributed test under \( H_0 \) for \( k = 30 \) with significant level \( \alpha = 0.05 \) for Example 1.

The simulation results with \( n = 10000 \) and \( p = 50 \) for different \( k \) are presented in Figure 2 and results on \( p = 100 \) are similar and are omitted. From Figure 2 it is seen...
that the power of distributed test increases to 1, as $c$ increases. In addition, the distributed test is less powerful than the centralized one, which coincides with the theoretical results.

![Figure 2: Comparison of the powers of the centralized test with those of the distributed one in Example 1 for different $k$ with $n = 10000$ and $p = 50$.](image)

**Example 2.** We consider the high-dimensional cases where $p > n$. Let $\{X_i \in \mathbb{R}^p, 1 \leq i \leq n\}$ be i.i.d. sample with the mean $E(X_i) = \mu$ and covariance $\Sigma = (\sigma_{ij})$. Let $p = 1000$ and $n = 900$. Moreover, we take $k = 10, 20$ and $30$, and split the data randomly into $k$ machines such that each machine has samples of size $n/k$, which is much smaller than $p$. Let $\mu = (c1_{20}^T, 0_{p-20}^T)^T$ with $c = 0, 0.25, 0.35, 0.45$, respectively, and $c = 0$ implies that $H_0$ is true. $\Sigma$ is taken as either $\Sigma_1$ or $\Sigma_2$, defined as $\Sigma_1 = (\sigma_{1,ij})$ with $\sigma_{1,ij} = 0.8|\!|i-j|\!|$, and $\Sigma_2 = (\sigma_{2,ij})$ with $\sigma_{2,ii} = 1$ and $\sigma_{2,ij} = 0.2$ for $i \neq j$. We consider two cases: (1) $X_i$ follows the normal distribution $N_p(\mu, \Sigma)$. (2) $X_i$ follows the multivariate $t$ distribution $t_\nu(\mu, \Sigma)$ with the degrees of freedom $\nu = 3$.

Simulation results are presented in Table 2. From Table 2, it is seen that the type I error is controlled well for both methods. In addition, we see that the powers of the
\( X_i \sim N(\mu, \Sigma) \quad X_i \sim t_3(\mu, \Sigma) \)

| \(c\) | \(\Sigma = \Sigma_1\) | \(\Sigma = \Sigma_2\) |
|------|----------------|----------------|
|      | \(k = 10\)   | \(k = 20\)   |
|      | \(k = 30\)   | \(k = 30\)   |
| 0    | 0.058 0.038 0.048 0.050 | 0.050 0.046 0.038 0.030 |
| 0.25 | 1 0.926 0.718 0.538 | 1 0.928 0.740 0.588 |
| 0.35 | 1 0.996 0.984 | 1 1 0.996 0.980 |
| 0.45 | 1 1 1 | 1 1 1 1 |

Table 2: Type I error (i.e. \(c = 0\)) and the powers of the centralized test and distributed one for Example 2, where ‘Cen’ stands for the centralized test and the columns with different values of \(k\) are of the distributed test.

distributed test statistics decrease as \(k\) grows and is lower than those of the centralized ones, which coincides with the theoretical findings.

### 4.2 Real data

We compare the distributed and centralized tests on the Statlog (Landsat Satellite) dataset from the UCI machine learning repository \([\text{Dua and Graff, 2017}]\), a multi-spectral data having 36 covariates and 6 class labels. The covariates contain the pixel values of the four spectral bands of each of the 9 pixels in the \(3 \times 3\) neighbourhood in a satellite image, and the response is the classification label of the central pixel. This data have been analyzed by several authors \([\text{Tang et al., 2005; Kim and Ghahramani, 2012, etc.}]\).

In multi-spectral data analysis, predictors in each class are generally viewed as normal vectors in the literature of geography and biology \([\text{Pickup et al., 1993; Bauer et al., 2011; Natsagdorj et al., 2017, etc.}]\). For this data, classes 3 and 7 represent grey soil and very damp grey soil with the sample size being 1357 and 1508, respectively. After applying the Box-Cox transformations on variables of indices 9, 24, 35, and 36, we find that the data in each class are approximately normal. Denoting by \(\mu_g\) the mean of class \(g\) with \(g \in \{3, 7\}\), we aim to test \(H_0 : \mu_3 - \mu_7 = 0\).

We select \(n = 1000\) observations at random for each class, denoted as \(\{X_{gi}, i = 1, \ldots, n, g = 3, 7\}\), and construct the observations \(Z_i = X_{3i} - X_{7i}, 1 \leq i \leq n\). The problem we considered becomes \(H_0 : E(Z_i) = 0\). We apply the centralized Hotelling \(T^2\)
test and the distributed one with \( k = 15 \) and 25, respectively, and compute the \( p \)-values. Repeating the procedure 100 times, the results of the average of the \( p \)-values are presented in the column associated with \( \delta = 0 \) in Table 3.

Furthermore, to compare the powers of the centralized test with those of the distributed one, we shift the means such that two populations are close to each other. Reminding that there are 357 and 508 observations left for classes 3 and 7, respectively, then for \( g = 3 \) and 7, \( \mu_g \) can be estimated by the sample mean of these remaining observations in class \( g \), denoted as \( \bar{\mu}_g \), which is independent of \( Z_i \)'s. Then we shift the mean \( \mu_3 \) of class 3 to \( \mu_3 - \delta(\bar{\mu}_3 - \bar{\mu}_7) \) with \( 0 \leq \delta < 1 \), which is equivalent to shifting \( Z_i \) to \( Z_{i,\delta} = Z_i - \delta(\bar{\mu}_3 - \bar{\mu}_7) \). Clearly, when \( \mu_3 \neq \mu_7 \), distinguishing two classes becomes more difficult for large \( \delta \). We repeat the procedure for 100 times and compute the average of the \( p \)-values. The results are presented in Table 3. It is seen that the centralized test performs better when \( \delta \) is close to 1.

| Methods       | \( \delta \) |
|---------------|--------------|
|               | 0  | 0.3 | 0.6 | 0.7 | 0.8 | 0.9 |
| Centralized   | 0  | 0   | 0   | 0   | 0   | 1.4e-8 |
| Distributed   | \( k = 15 \) | 0   | 0   | 0   | 2.2e-6 | 1.6e-2 |
|               | \( k = 25 \) | 0   | 5.6e-5 | 3.2e-3 | 0.0649 | 0.1474 |

Table 3: The average of \( p \)-values over 100 replicas, where 1.4e-8 represents \( 1.4 \times 10^{-8} \) and other quantities are defined similarly.

5 Discussion

In this paper, we consider the one sample mean testing in distributed frameworks, extending Hotelling \( T^2 \) test and the statistic of Wang et al. (2015), respectively. Obviously, the divide and conquer technique can be applied to many other statistics for one sample mean test, including those, are developed for high dimensional cases. For distributed tests constructed from the divide and conquer technique, it is observed that there is a fundamental tradeoff between the communication costs and the powers of the test.
6 Appendix

6.1 Proof of Theorem 1

Recalling that $T^2_{l} \sim T^2(p, n_l - 1)$ and using the connection between Hotelling $T^2$ distribution and the $F$-distribution, we have

$$\frac{n_l - p}{(n_l - 1)p} T^2_l \sim F(p, n_l - p), \quad l = 1, \cdots, k.$$ 

Then, it is easy to derive that

$$E(T^2_l / \sqrt{p}) = \frac{(n_l - 1)\sqrt{p}}{n_l - p - 2}, \quad \text{var}(T^2_l / \sqrt{p}) = \frac{2(n_l - 1)^2(n_l - 2)}{(n_l - p - 2)^2(n_l - p - 4)}.$$ 

We now establish the asymptotic normality of $\sqrt{k}T^2_{\text{dis}, n}$ by verifying the following two conditions (B1) and (B2) required by the Lyapunov central limit theorem (Billingsley, 2008).

(B1) $\text{var}(T^2_l / \sqrt{p})$ is finite.

(B2) Lyapunov condition: for some $\delta > 0$,

$$\lim_{k \to \infty} \frac{1}{s_k^{2+\delta}} \sum_{l=1}^{k} E \left[ \left| \frac{T^2_l / \sqrt{p} - E(T^2_l / \sqrt{p})}{\sqrt{p}} \right|^{2+\delta} \right] = 0,$$

where $s_k = \left[ \sum_{l=1}^{k} \text{var}(T^2_l / \sqrt{p}) \right]^{1/2}.$

It is easy to see that (B1) holds. In fact, it follows from (A1) that

$$\text{var}(\sqrt{k}T^2_{\text{dis}, n}) = \text{var}(T^2_l / \sqrt{p}) = \frac{2(n_l - 1)^2(n_l - 2)}{(n_l - p - 2)^2(n_l - p - 4)} \to \frac{2}{(1 - r)^3}.$$ 

Thus $\sqrt{k}T^2_{\text{dis}, n}$ has finite variance as $r \in [0, 1)$.

We now verify condition (B2) for $\delta = 2$. For a positive integer $\delta$, denote order-$\delta$ central moment of distribution $F(p, n_l - p)$ as $a_\delta$. Recalling that $(n_l - p)(n_l - 1)p^{-1}T^2_l \sim F(p, n_l - p)$, then the order-4 central moment of $T^2_l / \sqrt{p}$ can be written as

$$E \left[ \left| T^2_l / \sqrt{p} - E(T^2_l / \sqrt{p}) \right|^4 \right] = \left[ \frac{(n_l - 1)\sqrt{p}}{n_l - p} \right]^4 a_4.$$
Moreover, due to the fact that the kurtosis $K$ of $F(p, n_l - p)$ equals $a_4/a_2^2$, it follows that

$$a_4 = Ka_2^2 = K \left[ \frac{2(n_l - p)(n_l - 2)}{p(n_l - p - 2)(n_l - p - 4)} \right]^2.$$  

When $n_l - p > 8$, it holds that

$$K = \frac{12\{p[5(n_l - p) - 22](n_l - 2) + (n_l - p - 4)(n_l - p - 2)^2\}}{p(n_l - p - 6)(n_l - p - 8)(n_l - 2)} + 3.$$  

Under condition (A1), we have $K = 3 + O(1/p)$ and consequently

$$E \left[ |T_l^2/\sqrt{p} - E(T_l^2/\sqrt{p})|^4 \right] = O(1).$$

Thus,

$$\frac{1}{s_k} \sum_{l=1}^k E \left[ |T_l^2/\sqrt{p} - E(T_l^2/\sqrt{p})|^4 \right] = O(1/k).$$

This verifies the condition (B2). As $k \to \infty$, the conclusion is derived by the Lyapunov central limit theorem. This completes the proof.

### 6.2 Proof of Theorem 2

(1) We first prove the conclusions on the distributed test statistic.

Recall the asymptotic distribution (2.1) of $\sqrt{k}T_{dis,n}^2$ under $H_0$. Let

$$E_0 = \frac{(n_l - 1)\sqrt{p}}{n_l - p - 2}, \quad V_0^2 = \frac{2}{(1 - r)^3}. \quad (6.1)$$

Under $H_1$, the power function of distributed test has the following form,

$$\psi_{dis} = P \left( V_0^{-1} \left| \sqrt{k} \left( T_{dis,n}^2 - E_0 \right) \right| > z_{1-\frac{\alpha}{2}} \left| H_1 \right\right),$$

where $T_{dis,n}^2 = k^{-1} \sum_{l=1}^k T_l^2/\sqrt{p}$. Under $H_1$, $(n_l - p)[(n_l - 1)p]^{-1}T_l^2$ follows a non-central $F$-distribution $F(p, n_l - p, n_l \Delta)$ for $l = 1, \ldots, k$. Thus it follows that

$$E(T_l^2) = \frac{(n_l - 1)(p + n_l \Delta)}{n_l - p - 2}, \quad \text{var}(T_l^2) = 2(n_l - 1)^2 \frac{(p + n_l \Delta)^2 + (p + 2n_l \Delta)(n_l - p - 2)}{(n_l - p - 2)^2(n_l - p - 4)}.$$
Under condition (A1), it holds that, as \( n \to \infty \),
\[
\frac{\text{var}(T^2_i)}{p + n_l \Delta} \to \frac{2(\Delta^2 + 2\Delta + r)}{(r + \Delta)(1 - r)^3}.
\]
And we denote
\[
V_1^2 = \frac{2(\Delta^2 + 2\Delta + r)}{(r + \Delta)(1 - r)^3}.
\]
Recall \( n_l = n/k, l = 1, \ldots, k \). Let \( b_{n,\Delta} = \sqrt{p/(p + n_l \Delta)} \) and define
\[
\widetilde{T}_{\text{dis},n}^2 := b_{n,\Delta} T_{\text{dis},n}^2 = \frac{1}{k} \sum_{l=1}^{k} \frac{T^2_i}{\sqrt{p + n_l \Delta}},
\]
which is a sum of i.i.d. random variables. The normality of \( \widetilde{T}_{\text{dis},n}^2 \) can be derived by verifying the Lyapunov condition: for some \( \delta > 0 \),
\[
\lim_{k \to \infty} \frac{1}{\tilde{s}_k^{2+\delta}} \sum_{l=1}^{k} E \left[ \left| \frac{T^2_i}{\sqrt{p + n_l \Delta}} - E \left( \frac{T^2_i}{\sqrt{p + n_l \Delta}} \right) \right|^{2+\delta} \right] = 0, \tag{6.2}
\]
where \( \tilde{s}_k = \left[ \sum_{l=1}^{k} \text{var}(T^2_i/\sqrt{p + n_l \Delta}) \right]^{1/2} \). For \( \delta = 2 \), we have \( \tilde{s}_k^4 = k^2 V_1^4(1 + o(1)) \), as \( n \to \infty \). Recall that \( (n_l - p)(n_l - 1)p^{-1} T^2_i \) follows a non-central F-distribution. Then following Patnaik (1949), the first four moments about the origin of the non-central F-distribution, we can derive that \( E(|T^2_i/\sqrt{p + n_l \Delta} - E(T^2_i/\sqrt{p + n_l \Delta})|^4) = O(1) \), as \( n \to \infty \). Then we have
\[
\tilde{s}_k^4 \sum_{l=1}^{k} E \left[ \left| \frac{T^2_i}{\sqrt{p + n_l \Delta}} - E \left( \frac{T^2_i}{\sqrt{p + n_l \Delta}} \right) \right|^4 \right] = O(1/(kV_1^4)).
\]
This verifies \( \text{(6.2)} \). Then as \( k \to \infty \), we obtain the asymptotic normality
\[
\sqrt{k} \left( \widetilde{T}_{\text{dis},n}^2 - E_1 \right) \xrightarrow{d} N \left( 0, V_1^2 \right),
\]
where
\[
E_1 = \frac{E(T^2_i)}{\sqrt{p + n_l \Delta}} = \frac{(n_l - 1)(p + n_l \Delta)}{\sqrt{p}(n_l - p - 2)}. \tag{6.3}
\]
Under \( H_1 \), as \( \min\{k, n\} \to \infty \), the power function of the distributed test has the following
form,
\[
\psi_{\text{dis}}^n = P \left( \sqrt{k} \left( T_{\text{dis},n}^2 - E_0 \right) > z_{1-\alpha} \middle| H_1 \right)
\]
\[
= P \left( V_1^{-1} \sqrt{k} \left( \tilde{T}_{\text{dis},n}^2 - E_1 \right) + \sqrt{k} \left( E_1 - b_{n,\Delta} E_0 \right) > V_1^{-1} V_0 b_{n,\Delta} z_{1-\alpha} \middle| H_1 \right)
\]
\[
\rightarrow 1 - \Phi \left( V_1^{-1} V_0 b_{n,\Delta} z_{1-\alpha} - V_1^{-1} \sqrt{k} \left( E_1 - b_{n,\Delta} E_0 \right) \right)
\]
\[
+ \Phi \left( V_1^{-1} V_0 b_{n,\Delta} z_{1-\alpha} - V_1^{-1} \sqrt{k} \left( E_1 - b_{n,\Delta} E_0 \right) \right).
\]  (6.4)

Substituting (6.1) and (6.3) into (6.4), the right hand side of (6.4) can be denoted as
\[
\phi_{\Delta,\gamma_n} = 1 - \Phi \left( A_1 z_{1-\alpha} - A_2 \sqrt{\left( 1 - \gamma_n / 2 \right) / 2} \right) + \Phi \left( A_1 z_{1-\alpha} - A_2 \sqrt{\left( 1 - \gamma_n / 2 \right) / 2} \right),
\]
where
\[
A_1 = \left( \frac{\gamma_n}{\gamma_n + 2\Delta + \Delta^2} \right)^{1/2}, \quad A_2 = \left( \frac{n\Delta^2}{\gamma_n + 2\Delta + \Delta^2} \right)^{1/2}.
\]

Then we have \( \psi_{\text{dis}}^n - \phi_{\Delta,\gamma_n} \rightarrow 0. \)

(2) We prove the conclusions on the centralized test statistic.

First denoting
\[
\tilde{T}_{\text{cen},n}^2 := \frac{n - p}{(n - 1)p} T_{\text{cen},n}^2,
\]
the power function of the centralized test under \( H_1 \) can be written as
\[
\psi_{\text{cen}}^n = P \left( \tilde{T}_{\text{cen},n}^2 > F_{1-\alpha}(p, n-p) \middle| H_1 \right).
\]  (6.5)

Since \( \tilde{T}_{\text{cen},n}^2 \) follows the non-central \( F \)-distribution \( F(p, n-p, n\Delta) \) under \( H_1 \), it is easy to see that
\[
E \left( \tilde{T}_{\text{cen},n}^2 \right) = \frac{n - p}{(n - p - 2)p} (p + n\Delta),
\]
\[
\text{var} \left( \tilde{T}_{\text{cen},n}^2 \right) = \frac{2(n-p)^2}{(n-p-2)^2} \left[ \frac{(p + n\Delta)^2}{(n-p-4)p^2} + \frac{(n-p-2)(p + 2n\Delta)}{(n-p-4)p^3} \right].
\]

Then denote
\[
\tilde{V} := \left[ \text{var} \left( \tilde{T}_{\text{cen},n}^2 \right) \right]^{1/2}
\]
as the standard deviation of \( F(p, n-p, n\Delta) \).
Thus the power of centralized test statistic \( \psi_{cen} \) in (6.5) can be written as

\[
\psi_{cen} = P\left( \tilde{V}^{-1} \tilde{T}^2_{cen,n} > \tilde{V}^{-1} F_{1-\alpha}(p, n - p) \bigg| H_1 \right).
\]

As \( n \to \infty \), a variable following \( F(p, n - p) \) can be denoted as \( 1 + O_p(1/\sqrt{p}) \). Therefore \( F_{1-\alpha}(p, n - p) \) can be written as \( 1 + c/\sqrt{p} \), where \( c \) is a bounded constant. Then when \( \Delta \gg \sqrt{p}/n \), one can verify easily that,

\[
E \left( \tilde{V}^{-1} \tilde{T}^2_{cen,n} \right) \gg \tilde{V}^{-1} F_{1-\alpha}(p, n - p),
\]

and that the variance of \( \tilde{V}^{-1} \tilde{T}^2_{cen,n} \) equals 1. Therefore, the centralized test power \( \psi_{cen} \) tends to 1, as \( n \to \infty \). ■

### 6.3 Proof of Theorem 3

According to Wang et al. (2015), it follows that

\[
\text{var}(G_l) = n_l(n_l - 1) \text{Tr}(B^2_\epsilon)/2 := V^2_{nl}, \quad l = 1, \ldots, k.
\]

Denote

\[
E_{nl} = n_l(n_l - 1)(\mu - \mu_0)^\top A^2_\epsilon(\mu - \mu_0)(1 + o(1))/2.
\]

(1) We prove the conclusion under \( H_0 \).

Under \( H_0 \), \( G_l = \sum_{i,j \in S_l, j<i} Z_i^\top Z_j, 1 \leq l \leq k \) are i.i.d. variables, satisfying \( E(G_l) = 0 \) and \( \text{var}(G_l) = V^2_{nl} \). Recall that \( p = p_n, k = k_n, \) and \( n_l = n/k \) are functions of \( n \). Then \( \{G_l, 1 \leq l \leq k, n_l = 1, 2, \cdots \} \) is a double array. To derive the asymptotic normality of

\[
\frac{G_{\text{dis},n}}{\sqrt{kV_{nl}}} = \frac{1}{\sqrt{k}} \sum_{l=1}^{k} \frac{G_l}{V_{nl}}.
\]

we apply the central limit theorem of double array (Anderson, 1984). To this end, it is sufficient to check the following condition:

\[
\sum_{l=1}^{k} E \left( \left| \frac{G_l}{V_{nl}} \right|^{2+\delta} \right) = o(S^2_k), \quad n \to \infty, \quad (6.6)
\]
for some $\delta > 0$, where $S_k^2 = \sum_{i=1}^k \text{var}(G_i/V_{ni}) = k$. We will verify (6.6) with $\delta = 2$, that is,

$$E \left( \frac{|G_i|^4}{V_{ni}} \right) = o(k), \quad n \to \infty. \quad (6.7)$$

It is sufficient to check (6.7) for $i = 1$, since $G_i$’s are i.i.d. variables. For simplicity, we denote the indices of the observations in the first machine as $\{1, 2, \cdots, n_1\}$. Denote $G_1 = \sum_{i=2}^{n_1} Y_i$, where $Y_i = \sum_{j=1}^{i-1} Z_i^T Z_j$. Noting that $E(Z_i) = 0$ and $E(Z_i^T Z_j) = 0$ for $i \neq j$ under $H_0$, and consequently that $E(Y_{i_1} Y_{i_2} Y_{i_3} Y_{i_4}) = 0$ for any indices $i_1 \neq i_2 \neq i_3 \neq i_4$, then it follows that

$$E(|G_1|^4) = \sum_{i_1=2}^{n_1} E(Y_{i_1}^4) + \sum_{i_1, i_2=2, \atop i_1 \neq i_2}^{n_1} E(Y_{i_1} Y_{i_2}^3) + \sum_{i_1, i_2=2, \atop i_1 \neq i_2}^{n_1} E(Y_{i_1}^2 Y_{i_2}^2) + \sum_{i_1, i_2, i_3=2, \atop i_1 \neq i_2 \neq i_3}^{n_1} E(Y_{i_1} Y_{i_2} Y_{i_3}^2).$$

Under the conditions $(i)$ and $(ii)$ in (A2), following Wang et al. (2015), we have $\sum_{i=1}^{n_1} E(Y_{i_1}^4)/V_{n_1} \to 0$, as $n_1, p \to \infty$. Noting that $Z_i Z_j$ and $Z_i Z_s$ for any $i \neq j \neq s$ follow the same distribution, by applying the Cauchy-Schwartz inequality, it follows that

$$\sum_{i_1, i_2=2, \atop i_1 \neq i_2}^{n_1} E(Y_{i_1} Y_{i_2}^3) = 6 \sum_{i_1 < i_2} (i_1 - 1) E\{Z_1^T Z_2 Z_3^T Z_3 (Z_2^T Z_3)^2\} \leq 6 \sum_{i_1 < i_2} (i_1 - 1) E\{(Z_2^T Z_3)^4\} \leq 6 n_1^3 E\{(Z_2^T Z_3)^4\},$$

$$\sum_{i_1, i_2=2, \atop i_1 \neq i_2}^{n_1} E(Y_{i_1}^2 Y_{i_2}^2) = 8 \sum_{i_1 < i_2} (i_1 - 1) (i_2 - 2) E\{Z_1^T Z_2 Z_1^T Z_3 Z_2^T Z_4 Z_3^T Z_4\} + 4 \sum_{i_1 < i_2} (i_1 - 1) \left[ E\{(Z_1^T Z_2)^2 (Z_3^T Z_4)^2\} + E\{(Z_1^T Z_2)^2 Z_1^T Z_3 Z_2^T Z_3\} + E\{(Z_1^T Z_2)^2 (Z_2^T Z_3)^2\} \right] \leq C n_1^4 E\{(Z_1^T Z_2)^4\},$$

for some constant $C > 0$,

$$\sum_{i_1, i_2, i_3=2, \atop i_1 \neq i_2 \neq i_3}^{n_1} E(Y_{i_1} Y_{i_2} Y_{i_3}^2) = 8 \sum_{i_1 < i_2 < i_3} (i_1 - 1) E\{Z_1^T Z_2 Z_3^T Z_2 Z_4^T Z_1 Z_1^T Z_3\} \leq 8 n_1^4 E\{(Z_1^T Z_2)^4\}.$$

Under the conditions $(i)$ and $(ii)$ in (A2), and by Lemma 1 in Wang et al. (2015), we have
\[ E\{(Z_1^TZ_2)^4\} = O(E^2\{(Z_1^TZ_2)^2\}) = O(\text{Tr}^2(B^2)). \] Then as \(\min\{n_t, p, k\} \to \infty\),

\[ E \left( \left| \frac{G_1}{V_{n_t}} \right|^4 \right) = O(1) = o(k). \]

Therefore, \((6.7)\) holds. Then by central limit theorem (CLT) of a double array, as \(\min\{n_t, k, p\} \to \infty\), we have

\[ \frac{G_{\text{dis,n}}}{\sqrt{n(n/k - 1)\text{Tr}(B_n^2)/2}} = \frac{1}{\sqrt{k}} \sum_{l=1}^{k} \frac{G_l}{V_{n_t}} \overset{d}{\to} N(0, 1). \]

(2) Under \(H_1\), following [Wang et al. (2015)], it holds that

\[ G_l = \sum_{i,j \in S_l, j < i} \left\{ \frac{\epsilon_i}{\|\epsilon_i\|} + \frac{\epsilon_i + \mu - \mu_0}{\|\epsilon_i + \mu - \mu_0\|} - \frac{\epsilon_i}{\|\epsilon_i\|} \right\}^\top \left\{ \frac{\epsilon_j}{\|\epsilon_j\|} + \frac{\epsilon_j + \mu - \mu_0}{\|\epsilon_j + \mu - \mu_0\|} - \frac{\epsilon_j}{\|\epsilon_j\|} \right\} = G_{l,1} + G_{l,2} + G_{l,3}, \]

where

\[ G_{l,1} = \sum_{i,j \in S_l, j < i} \frac{\epsilon_i^\top \epsilon_j}{\|\epsilon_i\|\|\epsilon_j\|}, \quad G_{l,2} = \sum_{i,j \in S_l, j \neq i} \frac{\left( \frac{\epsilon_i + \mu - \mu_0}{\|\epsilon_i + \mu - \mu_0\|} - \frac{\epsilon_i}{\|\epsilon_i\|} \right)^\top \epsilon_j}{\|\epsilon_j\|}, \]

\[ G_{l,3} = \sum_{i,j \in S_l, j < i} \left( \frac{\epsilon_i + \mu - \mu_0}{\|\epsilon_i + \mu - \mu_0\|} - \frac{\epsilon_i}{\|\epsilon_i\|} \right)^\top \left( \frac{\epsilon_j + \mu - \mu_0}{\|\epsilon_j + \mu - \mu_0\|} - \frac{\epsilon_j}{\|\epsilon_j\|} \right). \]

By the conclusion (1) in this subsection, we have

\[ \frac{1}{\sqrt{k}} \sum_{l=1}^{k} \frac{G_{l,1}}{V_{n_t}} \overset{d}{\to} N(0, 1). \]

Moreover, by conditions (A1) and (A4') in Section 3.2, and Lemma 1, 4, and 5 of [Wang et al. (2015)], we have \(E(G_{l,2}) = 0\) and \(\text{var}(G_{l,2}/V_{n_t}) = o(1)\). As a result, it follows that \(\text{var}\left( \sum_{l=1}^{k} G_{l,2}/(\sqrt{k}V_{n_t}) \right) = o(1)\), that is, \(\sum_{l=1}^{k} G_{l,2}/(\sqrt{k}V_{n_t}) = o_p(1)\). For \(G_{l,3}\),

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Then it follows that

\[ G_{l,31} = \frac{n_l(n_l - 1)}{2} E \left( \frac{\epsilon_1 + \mu - \mu_0}{\| \epsilon_1 + \mu - \mu_0 \|} \right)^\top E \left( \frac{\epsilon_2 + \mu - \mu_0}{\| \epsilon_2 + \mu - \mu_0 \|} \right), \]

\[ G_{l,32} = \sum_{i,j \in S_l, j \neq i} E \left( \frac{\epsilon_i + \mu - \mu_0}{\| \epsilon_i + \mu - \mu_0 \|} \right)^\top \left\{ \frac{\epsilon_j + \mu - \mu_0}{\| \epsilon_j + \mu - \mu_0 \|} - \frac{\epsilon_j}{\| \epsilon_j \|} \right\} - E \left( \frac{\epsilon_j + \mu - \mu_0}{\| \epsilon_j + \mu - \mu_0 \|} \right), \]

\[ G_{l,33} = \sum_{i,j \in S_l, j < i} \left\{ \frac{\epsilon_i + \mu - \mu_0}{\| \epsilon_i + \mu - \mu_0 \|} - \frac{\epsilon_i}{\| \epsilon_i \|} \right\} - E \left( \frac{\epsilon_j + \mu - \mu_0}{\| \epsilon_j + \mu - \mu_0 \|} \right). \]

Then it follows that

\[ G_{l,3} = \sum_{i,j \in S_l, j \neq i} \left[ E \left( \frac{\epsilon_i + \mu - \mu_0}{\| \epsilon_i + \mu - \mu_0 \|} \right) + \left\{ \frac{\epsilon_i + \mu - \mu_0}{\| \epsilon_i + \mu - \mu_0 \|} - \frac{\epsilon_i}{\| \epsilon_i \|} \right\} - E \left( \frac{\epsilon_i + \mu - \mu_0}{\| \epsilon_i + \mu - \mu_0 \|} \right) \right]^\top \]

\[ \times \left[ E \left( \frac{\epsilon_j + \mu - \mu_0}{\| \epsilon_j + \mu - \mu_0 \|} \right) + \left\{ \frac{\epsilon_j + \mu - \mu_0}{\| \epsilon_j + \mu - \mu_0 \|} - \frac{\epsilon_j}{\| \epsilon_j \|} \right\} - E \left( \frac{\epsilon_j + \mu - \mu_0}{\| \epsilon_j + \mu - \mu_0 \|} \right) \right] = G_{l,31} + G_{l,32} + G_{l,33}. \]

By conditions (A2), (A3), and (A4’) in Section 3.2, Lemma 4 and 6 of Wang et al. (2013), we have \( G_{l,31} = E_{n_l} \), which is a constant. Also, it is easy to see \( E(G_{l,32}) = E(G_{l,33}) = 0 \), \( \text{var}(G_{l,32}/V_{n_l}) = o(1) \), and \( \text{var}(G_{l,33}/V_{n_l}) = o(1) \). Therefore, \( \sum_{l=1}^k (G_{l,3} - E_{n_l})/(\sqrt{k}V_{n_l}) = o_p(1) \). Then we have

\[ \frac{G_{\text{dis},n} - kE_{n_l}}{\sqrt{k}V_{n_l}} = \frac{1}{\sqrt{k}} \sum_{l=1}^k (G_l - E_{n_l})/V_{n_l} = \frac{1}{\sqrt{k}} \sum_{l=1}^k G_{l,1}/V_{n_l} + o_p(1). \]

Then by CLT and Slutsky’s theorem, the asymptotic distribution of \( G_{\text{dis},n} \) can be derived.

\[ \blacksquare \]

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