Feedback linearization of singularly perturbed systems based on canonical similarity transformations

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Abstract. This paper discusses the problem of feedback linearization of a singularly perturbed system in a state-dependent coefficient form. The result is based on the introduction of a canonical similarity transformation. The transformation matrix is constructed from separate blocks for fast and slow part of an original singularly perturbed system. The transformed singular perturbed system has a linear canonical form that significantly simplifies a control design problem. Proposed similarity transformation allows accomplishing linearization of the system without considering the virtual output (as it is needed for normal form method), a technique of a transition from phase coordinates of the transformed system to state variables of the original system is simpler. The application of the proposed approach is illustrated through example.

1. Introduction

Constantly growing requirements to the quality and efficiency of control systems lead to taking into account various factors (nonlinearity, small parameters, uncertainties, etc.) in the mathematical description of real processes and systems. Instead of one-dimensional linear models, multidimensional nonlinear models appear, new methods for the synthesis of nonlinear systems are being developed and improved. The feedback linearization (FL) method has been engaged in various nonlinear systems. The idea of this method consists in converting the original nonlinear system into a linear one by means of feedback. There are few ways for FL design, the most common methods are the normal form method, method of canonical similarity transformation for the nonlinear system in a state-dependent coefficient (SDC) form, a method of transformation to a quasi-canonical form for the affine system [1-4].

There are numerous publications devoted to the analysis and synthesis of singular perturbed (SP) systems, some of which are reviewed in [5-7]. A recent review [6] includes more than 500 references and demonstrates a growing interest of researchers in nonlinear SP systems and their applications. The problem of nonlinear SP systems control using FL is studied in [8-12]. In [8] a diffeomorphism is proposed that is independent of the small singular perturbation parameter. Moreover, this diffeomorphism should satisfy the slow-fast dynamics separation condition. In [9] a new diffeomorphism is introduced, which does not require compliance with the dynamics separation conditions and linearizes not only the whole system but also the slow and fast models separately.

Both of the above mentioned papers can be attributed to the same approach, which used the normal form technique. Article [10] introduces another approach that is based on FL of the so-called "slow"
subsystem. According to this indirect approach, the original FL problem of the nonlinear SP system is reduced to the simpler problem of FL of an unperturbed slow subsystem. Restriction of the result in [10,11] is the class of nonlinear SP systems that are linear in control input and fast state variables. In [12] the approximate FL method is proposed for the class of the SP system with a nonlinear equation for slow state variables in a general form and an equation for a fast state variables system that is affine in the control. The transformation diffeomorphism is obtained through FL of slow and fast state variables system parts separately. This way, fast FL control is chosen so that the dynamics separation condition would be met.

In this paper, the problem of FL control is considered for the class of the nonlinear SP system in a SDC form. Let us propose a FL method for this class of SP systems based on canonical similarity transformation. The paper is organized as follows: Section 2 contains the formulation of the FL problem for the SP systems in the SDC form; Section 3 discusses the process of synthesis of the canonical similarity transformation for FL; an example of using the proposed FL method for the SP system is shown in Section 4; conclusion with the main findings and acknowledgements is shown in Section 5 and Section 6 respectively.

2. Statement of the Problem
In the formulation of the problem, let us consider a continuous SP system in the SDC form:

\[
E(\varepsilon)x(t) = A(x)x + B(x)u, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^l,
\]

where \(x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}\) are slow and fast state variables correspondingly (\(n_1 + n_2 = n\)); \(u \in \mathbb{R}^l\) is control input; \(A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2\) are state dependent matrices of the corresponding dimensions; \(I_1, I_2\) are unit matrices of the corresponding dimensions; \(\varepsilon > 0\) is a small parameter.

The goal is to find a non-singular similarity transformation:

\[
\begin{pmatrix}
    z_1 \\
    z_2
\end{pmatrix} = T(x)x = \begin{pmatrix}
    T_1(x_1) & 0 \\
    0 & T_2(x_1, x_2)
\end{pmatrix} \begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix},
\]

by means of which system (1) can be transformed to a quasi-linear canonical form:

\[
E(\varepsilon)\dot{z} = \tilde{A}(z)z + \tilde{B}(z)u,
\]

where

\[
\tilde{A}(z) = TAT^{-1} + E(\varepsilon)TT^{-1} = \begin{pmatrix}
    T_1A_{11}T_1^{-1} + T_1T_1^{-1} & T_1A_{12}T_2^{-1} \\
    T_2A_{21}T_1^{-1} & T_2A_{22}T_2^{-1} + \varepsilon T_2T_2^{-1}
\end{pmatrix},
\]

\[
\tilde{B}(z) = TB = \begin{pmatrix}
    T_1B_1 \\
    T_2B_2
\end{pmatrix}.
\]

Let us construct transformation (2) in such a way that the two conditions are satisfied:

1. Matrix \(\tilde{A}(z) = A_C(z) + \varepsilon A_D(z)\), where \(A_C(z)\) is a Frobenius matrix with the last linearly independent row, i.e.:
\[
A_c(z) = \begin{pmatrix}
J_1 & E_{12} \\
E_{21} & J_2
\end{pmatrix},
J_1 = \begin{pmatrix} 0 & I_1 \\ 0 & 0 \end{pmatrix},
J_2(z) = \begin{pmatrix} 0 & I_2 \\ a_{21}(z) & a_{22}(z) \ldots a_{2n_2}(z) \end{pmatrix},
E_{12} = e_{n_1} e_{n_1}^T, E_{21}(z) = \begin{pmatrix} 0 \\ a_{11}(z) \ldots a_{1n_1}(z) \end{pmatrix},
\]

Here \( e_{n_1} \) is a column \( n \)-vector, \( i \)-th element of which is 1, and all the other – 0, (when \( i = n \) let us write \( e_{n_1} = e_n \)).

2. Matrix \( \tilde{B}(z) \) is a unit vector: \( \tilde{B}(z) = T(x)B(x) = (0 \ e_{n_1})^T \).

3. Synthesis of the canonical similarity transformation

Let us proceed to synthesis of transformation (2) based on conditions 1 and 2.

**Assumption 1.** The part of system (1) that describes fast motions can be represented in the form:

\[
\dot{\mathbf{e}} = \mathbf{A}_2(\mathbf{x})\mathbf{e} + \mathbf{B}_2(\mathbf{x})\mathbf{u}, \quad \mathbf{A}_2(\mathbf{x})\mathbf{e} = \mathbf{A}_{21}(\mathbf{x})\mathbf{e}_1 + \mathbf{A}_{22}(\mathbf{x})\mathbf{e}_2,
\]

and matrices \( \mathbf{A}_{22}(\mathbf{x}) \) and \( \mathbf{A}_2(\mathbf{x}) \) are continuous, differentiable and have bounded derivatives. Pair \((\mathbf{A}_{22}(\mathbf{x}), \mathbf{A}_2(\mathbf{x}))\) is controllable, i.e. \( \exists \alpha > 0 \) that:

\[
|\det(W_2(\mathbf{x}))| > \alpha > 0, \quad \mathbf{x} \in \mathbb{R}^n, \quad t \geq t_0,
\]

\[
W_2(\mathbf{x}) = (\mathbf{B}_2(\mathbf{x}) \mathbf{A}_{22}(\mathbf{x}) \mathbf{B}_2(\mathbf{x}) \ldots \mathbf{A}_{2n_2}(\mathbf{x}) \mathbf{B}_2(\mathbf{x})).
\]

Let us introduce generating vector \( T_2(\mathbf{x}_1, \mathbf{x}_2) \) for fast state variables, defined later, and let us assume that (here and below the subscript in square brackets shows the number of the vector element):

\[
\dot{z}_{2[1]} = T_{2[1]} \mathbf{x}_2,
\]

\[
\dot{z}_{2[1]} = \mathbf{e}^{-1}(z_{2[2]} + T_{2[1]} \mathbf{B}_2(\mathbf{x}) + \dot{T}_{2[1]} \mathbf{x}_2), \quad z_{2[2]} = T_{2[1]} \mathbf{A}_{22}(\mathbf{x}) \mathbf{x}_2,
\]

\[
\dot{z}_{2[2]} = \mathbf{e}^{-1}(z_{2[3]} + T_{2[1]} \mathbf{A}_{22}(\mathbf{x}) \mathbf{x}_2 + \dot{T}_{2[1]} \mathbf{A}_{22}(\mathbf{x}) \mathbf{x}_2 + \dot{T}_{2[1]} \mathbf{A}_{22}(\mathbf{x}) \mathbf{x}_2), \quad z_{2[3]} = T_{2[1]} \mathbf{A}_{22}(\mathbf{x}) \mathbf{x}_2,
\]

\[
\dot{z}_{2[n_2]} = \mathbf{e}^{-1}(T_{2[1]} \mathbf{A}_{22}(\mathbf{x}) \mathbf{x}_2 + T_{2[1]} \mathbf{A}_{22}(\mathbf{x}) \mathbf{x}_2) + \dot{T}_{2[1]} \mathbf{A}_{22}(\mathbf{x}) \mathbf{x}_2 + T_{2[1]} \mathbf{A}_{22}(\mathbf{x}) \mathbf{x}_2.
\]

Let us suppose:

\[
T_{2[1]} \mathbf{A}_{22}(\mathbf{x}) \mathbf{B}_2 = 0, \quad i = 0, n_2 - 2, \quad T_{2[1]} \mathbf{A}_{22}(\mathbf{x}) \mathbf{B}_2 \equiv 0.
\]

Taking (6) into account, relations (5) can be rewritten in the form:

\[
\dot{\mathbf{e}} = \mathbf{e} T_{2[1]} \mathbf{x}_2,
\]

\[
\dot{\mathbf{e}} = \mathbf{e} T_{2[1]} \mathbf{x}_2 + \mathbf{e} T_{2[1]} \mathbf{x}_2 + \mathbf{e} T_{2[1]} \mathbf{x}_2.
\]

\[
\dot{\mathbf{e}} = \mathbf{e} T_{2[1]} \mathbf{x}_2 + \mathbf{e} T_{2[1]} \mathbf{x}_2 + \mathbf{e} T_{2[1]} \mathbf{x}_2.
\]

From (7) let us obtain the similarity transformation matrix: \( T_2(\mathbf{x}) \mathbf{x}_2, \ \mathbf{x}_2 = T_2^{-1}(\mathbf{x}_1, \mathbf{z}_2) \mathbf{z}_2 \).
\[ T_2(x) = \begin{pmatrix} T_{2[1]}(x) \\ T_{2[1]}(x)\overline{A}_{22} \\ \vdots \\ T_{2[1]}(x)\overline{A}_{22}^{n-1} \end{pmatrix}. \] (8)

Applying (8) to (4), let us obtain:

\[ \varepsilon \dot{z}_2(t) = (A_{2c} + A_{2f})z_2 + \varepsilon T_2 T_2^{-1} + B_{2c} u, \]

\[ A_{2c} = \begin{pmatrix} 0 & I_2 \\ 0 & 0 \end{pmatrix}, \quad A_{2f} = \begin{pmatrix} 0 & 0 \\ T_{2[1]}\overline{A}_{22}^{n-1} \end{pmatrix}, \quad B_{2c} = \epsilon u. \] (9)

Let us note that condition (6) leads to \( B_{2c} = T_2 B_2 = \epsilon u, \) so one can determine \( T_{2[1]} : \)

\[ T_{2[1]} = \epsilon u W^{-1}(x). \] (10)

The FL control for canonical system (9) can be defined as:

\[ u = (K_2 + G_2)z_2 + G_{2[1]} u_s, \quad K_2 = -T_{2[1]}\overline{A}_{22}^{n-1} T_2^{-1}, \] (11)

where \( G_2 \) is chosen so that \( \text{Re} \lambda(A_{2c} + B_{2c}G_2) < 0 \) (\( \lambda \) are eigenvalues).

**Assumption 2.** There is representation \( A_{12}(x_1,x_2)x_2 = \overline{A}_{12}(x_1,x_2)T_{2[1]}(x_1,x_2)x_2, \) and matrix coefficients \( B_i(x_1,x_2) \mid_{x_2 = T_2^{-1} x_2} \), \( A_{11}(x_1,x_2) \mid_{x_2 = T_2^{-1} x_2} \) and \( \overline{A}_{12}(x_1,x_2) \mid_{x_2 = T_2^{-1} x_2} \) are independent of \( z_{2[1]} \).

If Assumption 2 holds, then applying (8), (11) to the system (1), one obtains:

\[ \dot{x}(t) = A_{11}(x_1, z_2) x_1 + A_{12}(x_1, z_2) z_{2[1]} + B_1(x_1, z_2)((K_2 + G_2)z_2 + G_{2[1]} u_s), \]

\[ \varepsilon \dot{z}_2(t) = (A_{2c} + B_{2c}G_2)z_2 + B_{2c}G_{2[1]} u_s + O(\varepsilon). \]

From the last expression for \( \varepsilon = 0 \), let us get a slow subsystem:

\[ \dot{x}_1(t) = A_{11}(x_1, z_2) x_1 + B_1(x_1) u_s, \quad z_2 = (u_s \ 0 \ ... \ 0)^T, \]

\[ A_{11}(x_1) = A_{11}(x_1,0), \quad B_1(x_1) = (\overline{A}_{12}(x_1,0) + B_1(x_1,0)((K_2 + G_2) + G_{2[1]}). \] (12)

Now let us define similarity transformation \( z_i = T_i(x_i) x_i, \) \( x_i = T_i^{-1}(z_i) z_i \), applying which to (12), one obtains the slow canonical system:

\[ \dot{z}_i(t) = A_{1c}(z_i) z_i + B_{1c} u_s, \]

\[ A_{1c} = \begin{pmatrix} 0 \\ a_{11} & I_1 \\ a_{12} \ldots a_{1n_i} \end{pmatrix}, \quad B_{1c} = \epsilon u. \] (13)

Linearizing control for the slow canonical system (13):

\[ u_c(z_i) = K_1(z_i) z_i + v(z), \quad K_1 = -(a_{11}(z_i) \ a_{12}(z_i) \ ... \ a_{1n_i}(z_i)). \] (14)

The design procedure of canonical transformation \( z_i = T_i(x_i) x_i \) is described in detail in [13]. Let us write only the final result:
\[ T_i = \{ T_{i|i} \}, \quad i = 1, n_1, \quad T_{i|i} = e_{n_i} W_i^{-1}, \]
\[ T_{i|i+1} = \frac{d}{dt}(T_{i|i}) + C_i^1 \frac{d}{dt} T_{i|i} L_i + \ldots + T_{i|i} L_i, \]
\[ W_i = (w_1, w_2, \ldots, w_n), \quad k = 0, n - 1, \]
\[ w_{k+1} = L_1 B_1 - C_k^1 \frac{d}{dt}(L_{k-1} B_1) + C_k^2 \frac{d}{dt}(L_{k-1} B_2) + \ldots + (-1)^k \frac{d}{dt} B_k, \]

where \( C_k^j \) – binomial coefficients \((k, j)\).

Relations (8), (15) define blocks of the canonical similarity transformation (2).

4. Example

Let us consider the system as follows [9]:
\[ \dot{x}_{1|1} = x_{1|1}^2 + x_{2|2} + x_{2|2}^2, \]
\[ \dot{x}_{1|2} = \sin(x_{1|1}) + x_{2|2} + x_{2|2} x_{2|2}^2 + u, \]
\[ \varepsilon \dot{x}_{2|1} = (1 + 0.5 \sin x_{1|1})(-x_{1|1} + x_{2|2}), \]
\[ \varepsilon \dot{x}_{2|2} = x_{2|2}^2 + x_{2|2}^2 + u. \]

First let us rewrite a fast subsystem according to Assumption 2:
\[ \varepsilon \dot{x}_2 = \begin{pmatrix} -fx_{1|1} x_{2|2}^{-1} & f \\ x_{2|2}^2 x_{2|2}^{-1} & x_{2|2} \end{pmatrix} x_2 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad f = (1 + 0.5 \sin x_{1|1}), \]
and find the controllability matrix:
\[ W_2 = \begin{pmatrix} 0 & f \\ 1 & x_{2|2} \end{pmatrix}. \]

From (8), (10), let us obtain the transformation matrix:
\[ T_2(x) = \begin{pmatrix} f^{-1} & 0 \\ x_{1|1} x_{2|2}^{-1} & 1 \end{pmatrix}. \]

and the corresponding FL control from (11):
\[ u(x, z) = x_{1|1} z_{2|2} x_{2|2}^{-1} - x_{2|2}^2 - (z_{2|2} + x_{1|1})^2 + \]
\[ + G_{2|1}(f z_{2|1} - u_0(x)) + G_{2|2} z_{2|2}. \]

Applying transformation (17) and control (18) to system (16), one obtains:
\[ \dot{x}_{1|1} = x_{1|1}^2 + x_{1|1} + x_{1|1} + z_{2|2}, \]
\[ \dot{x}_{1|2} = \sin(x_{1|1}) + x_{1|2} + z_{2|2}^2 + f z_{2|1} + u, \]

letting \( \varepsilon \to 0 \), let us obtain a slow subsystem:
\[ 
\begin{align*}
\dot{x}_{[1]} &= x_{[1]}^2 \sin(x_{[1]} - 1) + x_{[1]} + x_{[2]}, \\
\dot{x}_{[2]} &= \sin(x_{[1]}) + x_{[1]} \sin(x_{[1]} - 1) - x_{[2]}^2 + u_s, \\
\Rightarrow \dot{x}_i &= \left( \begin{array}{ccc}
x_{[1]} + 1 & 1 & 1 \\
x_{[1]}x_{[2]} - x_{[1]} & \sin(x_{[1]}x_{[2]} - x_{[2]} - 1) & \end{array} \right) x_i + \left( \begin{array}{c}
0 \\
1 \\
\end{array} \right) u_s, \\
\end{align*}
\]

Solving the FL problem for slow subsystem (19), let us determine the transformation matrix:
\[
T_i(x) = \left( \begin{array}{cc}
1 & 0 \\
x_{[1]} + 1 & 1 \\
\end{array} \right)
\]

and slow FL control:
\[
\begin{align*}
u_s &= -(x_{[2]}(2x_{[1]} + 1) - \sin(x_{[1]}x_{[2]} + 1) + x_{[1]}x_{[2]} - x_{[1]} - x_{[2]} = -x_{[1]} - x_{[2]} + \sin(x_{[1]}x_{[2]} - x_{[2]} - 1)z_{[1]} - \\

v_s &= G_{[1]}z_{[1]} + G_{[2]}z_{[2]} - x_{[1]}x_{[2]} - x_{[1]}x_{[2]} - x_{[1]} - x_{[2]} = x_{[2]}^2 + x_{[1]} - x_{[2]}.
\end{align*}
\]

The complete FL control is obtained by substituting (21) in (18):
\[
\begin{align*}
u(x, z) &= x_{[1]}x_{[2]}z_{[1]} - x_{[1]} - (z_{[2]} - x_{[1]}x_{[2]} - x_{[1]} - x_{[2]} + \\
&\quad +G_{[1]}z_{[1]} + G_{[2]}z_{[2]} - x_{[1]}x_{[2]} + z_{[1]} - x_{[2]} = x_{[2]}^2 + x_{[1]} - x_{[2]} - x_{[1]} - x_{[2]} + \\
&\quad +G_{[1]}z_{[1]} + G_{[2]}z_{[2]} + G_{[1]}z_{[1]} + G_{[2]}z_{[2]} - x_{[1]}x_{[2]} - x_{[1]} - x_{[2]} = x_{[1]} + x_{[2]}.
\end{align*}
\]

The similarity transformation matrix is constructed using (17), (20):
\[
T(x) = \left( \begin{array}{ccc}
(1 + 0.5 \sin(x_{[1]}))^{-1} & 0 & 0 & 0 \\
x_{[1]}x_{[2]} & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & x_{[1]} + 1 & 1 \\
\end{array} \right)
\]

5. Conclusion
The FL method for nonlinear SP systems in the SDC form is proposed as a basis of canonical similarity transformations. The advantage of the presented method consists in simplifying the FL problem for the original nonlinear SP system by means of decomposition for a canonical similarity transformation for a fast and slow part of the original system. The transformed SP system has a linear canonical form (within the accuracy of $O(\varepsilon)$) that significantly simplifies a control design problem.

Proposed similarity transformation allows accomplishing feedback linearization of a system without considering the virtual output (as it is needed for a normal form method); the technique of the transition from phase coordinate of the transformed system to state variables of the original system is simpler. Such similarity transformation can be applied to nonlinear time-varying continuous and discrete MIMO systems. This fact allows using the proposed method for FL control design for these kinds of singular systems. It will be the subject of further research.

It must also be noted that canonical similarity transformation and FL control do not depend on the perturbation parameter that provides the system with robustness property with regard to a perturbation parameter.
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