Perturbative approach to the dynamics of a trapped ion interacting with a light field

V Penna and F A Raffa

Politecnico di Torino, Dipartimento di Scienza Applicata e Tecnologia, Corso Duca degli Abruzzi 24, I-10129 Torino, Italy
E-mail: vittorio.penna@polito.it

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Abstract

We present a second-order perturbative analysis of the model describing a two-level trapped ion interacting with a travelling laser field, in the Lamb–Dicke regime. Unlike the customary approach, based on the interaction picture and the rotating wave approximation, we reduce the original Hamiltonian to a time-independent model, unitarily equivalent to a spin-boson model, which features a form well suited to apply the perturbation theory. We determine the first- and second-order corrections of eigenstates and eigenvalues. By varying the interaction parameters we identify four regimes characterized by different eigenvalue distributions (possibly including level doublets) and suggest the presence of an intermediate regime where the spectrum undergoes macroscopic changes. In this regime, where the perturbation approach does not hold, our reduced Hamiltonian is shown to reproduce the Jaynes–Cummings model.

Keywords: trapped ions, perturbation theory, spin-boson model

1. Introduction

Quantum systems formed by trapped ions interacting with a light field have been the subject of growing interest over the last two decades, both due to their manifold applications in fields such as quantum measurement, generation of nonclassical states, and quantum-information processing [1–3] and because they allow effective simulators of complex quantum phenomena to be engineered [4–7]. The intense theoretical work on these systems and the parallel experimental progress in ion trapping are thoroughly reviewed in [1, 8–10].

The distinctive feature of trapped-ion models is the time dependence introduced by the coupling with the light field. The resulting dynamics, which is known to be extremely complex but also very rich, has been explored by applying or combining different well-established analytic techniques and approximation schemes. The customary approach entails transforming the Hamiltonian of the system in the interaction picture and simplifying it through the rotating-wave approximation (RWA). Its study can be further simplified through the Lamb–Dicke (LD) approximation \( \ell/\lambda_L \ll 1 \) according to which the size \( \ell \) of the ion-confinement region is much smaller than the laser wavelength \( \lambda_L \). Applications of such an approach are well documented, e.g., in [11, 12], with reference to different ion–light resonance conditions, in the field of quantum computation [13, 14], and in the theoretical study of the atomic population inversion [15] and of the decoherence induced by measurement processes [16].

Non-standard approaches are reported as well. For instance, in [17] and [18], by resorting to appropriate unitary transformations, the exact diagonalization of the time-dependent Hamiltonian \( H_t \) describing the ion–light field interaction is obtained also beyond the LD approximation, while in [10] it is shown that a unitary transformation of this Hamiltonian leads to a mapping into the Jaynes–Cummings (JC) model, with no need to utilize the interaction picture and the RWA. For unrestricted values of the LD parameter this technique holds provided the physical parameters of the system (detuning, LD parameter, trap and Rabi frequencies) satisfy specific constraints. Interestingly, an analogous transformation scheme has been devised in the analysis of mesoscopic spin-boson systems of trapped ions in a phonon bath [19].

The reduction of Hamiltonian \( H_t \) to a unitarily equivalent time-independent model discussed in [10] is also the path we follow in this work. We resort to appropriate unitary transformations in order to eliminate in the ion–light interaction term both the time dependence, controlled by the laser frequency \( \omega_L \), and the exponential factors depending on
the trapped-ion position. The new Hamiltonian $H_\nu$ represents
a minimal model, not further reducible to a more elementary
form, in which the dependence on laser wave number $k_L$ (which
is proportional to the LD parameter $\eta$) is simply linear. It
is worth noting that this model can be proved to be equivalent
(up to a unitary transformation) to the well-known single-mode
spin-boson model which is thoroughly reviewed in [20] and
whose integrability has been recently analysed in [21].

The main aim of this paper consists in exploiting the linear
character of $H_\eta$ to diagonalize the latter, in the LD regime,
via the standard perturbation method in which $\eta$ assumes,
quite naturally, the role of the perturbation parameter. After
determining the first-order corrections of the eigenstates of
$H_\eta$, we show that first-order corrections in the eigenvalue
perturbation series are zero and calculate the second-order
corrections. We then focus on the resonant case where the
laser detuning vanishes. Despite this assumption, the resulting
spectrum can be applied are, in a sense, complementary in that they
lead straightforwardly to the JC model, and show that the
solution in which time evolution appears to be significantly
influenced by the structure of the eigenvalue distribution.

We also prove that applying the RWA to our Hamiltonian
leads straightforwardly to the JC model, and show that the
RWA regime and the regimes where the perturbation approach
can be applied are, in a sense, complementary in that they
hold in adjacent but non-overlapping regions of the parameter
space.

The layout of the paper is the following. In section 2 we
perform the reduction of the ion–light interaction model to
its minimal form. Section 3 is devoted to the application of
perturbation theory. The properties of the resulting spectrum
are discussed in section 4 while section 5 is focused on the
derivation of the JC model. Section 6 is devoted to concluding
remarks.

2. Minimal form of the trapped-ion model

We consider an ion trapped in a one-dimensional harmonic
potential which interacts with a single travelling light field.
The internal structure of the ion is represented by a two-level
system with ground and excited states $|g\rangle$ and $|e\rangle$, respectively,
whose energy difference is $\hbar \omega_0 = \hbar (\omega_e - \omega_g)$. The relevant
free Hamiltonian reads

$$H_1 = \hbar v \hat{n} + \hbar \omega_0 S_z,$$

where $v$ is the harmonic-potential frequency, $\hat{n} = a^\dagger a$ is the
number operator and boson operators $a, a^\dagger$ obey commutator $[a, a^\dagger] = 1$. Operator $S_z = \langle g | e \rangle \langle e | g \rangle / 2$ describes
the two-level system together with raising and lowering operators $S_+ = |e\rangle \langle g |$ and $S_- = (S_+)^\dagger$ satisfying the standard
commutators $[S_+, S_-] = 2S_z$, $[S_z, S_\pm] = \pm S_\pm$ of algebra
su(2). The coupling of the ion with a light field travelling in the $x$
direction is described by

$$H_{\text{int}} = \lambda [E_L e^{i(k_L x - \omega_L t)} S_+ + \text{h.c.}],$$

where $\lambda$ is the (real) coupling coefficient, $\hat{x}$ is the ion centre-of-
mass position operator, $E_L$ is the light (laser) amplitude, and
the standard formulas $k_L = \omega_L / c = 2\pi / \lambda_L$ hold. By resorting to
the canonical picture in terms of displacement and momentum operators

$$\hat{x} = \sqrt{\hbar/(2M\nu)}(a^\dagger + a), \quad \hat{p} = i\sqrt{\hbar M/2}(a^\dagger - a),$$

the interaction Hamiltonian $H_{\text{int}}$ can be expressed in an
equivalent form depending on the LD parameter $\eta$ with $k_L \hat{x} = \eta(a^\dagger + a)$ and

$$\eta = k_L \frac{\ell}{\sqrt{2}} = \sqrt{2} \frac{\nu}{\lambda_L}. \quad (1)$$

In formula (1) $\ell = \sqrt{\hbar/(M\nu)}$ represents the characteristic
length of the harmonic-confinement region and $M$ is the ion mass.
The trapped-ion total Hamiltonian then takes the form
(cf. e.g., [10, 11])

$$H_i = H_0 + H_{\text{int}} = \hbar \nu \hat{n} + \hbar \omega_0 S_z + K [e^{-i\omega_L t} e^{\eta(a^\dagger + a)} S_+ + \text{h.c.}], \quad (2)$$

where parameter $K = \lambda E_L$ has been introduced.

2.1. Transforming the Hamiltonian

The reformulation of Hamiltonian (2) in a form suitable for the
application of perturbation theory can be realized in two steps.
First, we resort to the unitary time-dependent transformation
$U_t = e^{i\nu \hat{x}}$, with $\varphi = \varphi(t) \in \mathbb{R}$, so that, by setting consistently $|\Psi(t)\rangle = U_t |\Phi(t)\rangle$, we can recast the Schrödinger problem
$i\hbar \delta_t |\Psi(t)\rangle = H_i |\Psi(t)\rangle$ of Hamiltonian (2) into the form

$$i\hbar \delta_t |\Phi(t)\rangle = (\hbar \omega_S + U_t^\dagger H U_t) |\Phi(t)\rangle.$$ 

This leads to the new (possibly time-dependent) model
Hamiltonian $H_1 = \hbar \nu \hat{n} + U_t^\dagger H U_t$. By using the customary
transformation formulas

$$e^{i\nu \hat{x}} S_e e^{-i\nu \hat{x}} = e^{i\nu S_e}, e^{i\nu \hat{x}} S_e e^{-i\nu \hat{x}} = e^{-i\nu S_e},$$

representing the action of $e^{i\nu \hat{x}}$ on the algebra-su(2) (generators
$S_e$, and setting $\varphi = -\omega_L t$, we obtain from $H_1$ the time-
dependent Hamiltonian

$$H_2 = \hbar \nu \hat{n} - \hbar \Delta S_z + K [e^{i\eta(a^\dagger + a)} S_+ + \text{h.c.}], \quad (3)$$

where $\Delta = \omega_L - \omega_0$ is the detuning of the system. It proves
now convenient to perform a second unitary transformation
$U_2 = e^{i\beta \hat{p}^2}$ with $\beta = k_L \hat{x}$ which removes the exponential factors
depending on $\eta (a^\dagger + a) \equiv k_L \hat{x}$ from $H_2$. Observing that in
$U_2^\dagger (\hbar \nu \hat{n}) U_2$ the term $\hat{p}^2$ undergoes the transformation

$$e^{-i\beta \hat{p}^2} e^{i\beta \hat{p}^2} = (\hat{p} + \hbar S_z \alpha \beta)^2.$$

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while harmonic term \( Mv^2x^2/2 \) is unchanged, we obtain the minimal form for the trapped-ion model
\[
H_0 = U_1^1 H_2 U_2 = \left( \hbar v n + \frac{h^2 k_L^2}{8M} \right) + 2KS_z - h\Delta S_z + \frac{hk_L}{M}S_xp.
\] (4)
The comparison with formula (3) (or (2)) clearly shows its considerably simpler form. If the eigenvalue problem \( H_0\{E(n, s)\} = E(n, s)\{E(n, s)\} \) is solved (\( n \) and \( s \) are suitable quantum numbers relevant to the non-interacting part of \( H_0 \)), then the general solution to the Schrödinger problem \( i\hbar\partial_t\{\xi(t)\} = H_0\{\xi(t)\} \) is represented by
\[
|\xi(t)\rangle = \sum_{(n, s)} C(n, s) e^{-iE(n, s)/\hbar} |E(n, s)\rangle,
\] (5)
where \( C(n, s) \) are arbitrary amplitudes. In this perspective, the advantage to deal with \( H_0 \) is evident since, unlike \( H_2 \) or \( H_2 \), \( H_0 \) features the linear form apt to reconstruct states \( |E(n, s)\rangle \) and their eigenvalues within the perturbation theory. This scenario \( \hbar k_L S_xp/M \) can be interpreted as the perturbation term since \( k_L \propto \eta \) assumes arbitrarily small values in the LD regime.

3. Perturbative scheme in the LD regime

We rewrite Hamiltonian (4) in the form \( H_0 = H_0 + \epsilon W \) where \( \epsilon W = (hk_L/M)S_xp \) represents the perturbation term. The natural choice for the perturbation parameter is the dimensionless ratio \( \epsilon = \ell/\hbar \) which, up to a constant factor, identifies with the LD parameter \( \eta \). Then potential \( W \) is found from
\[
W = \frac{hk_L}{\epsilon M} S_xp = \cdots = i\sqrt{2\pi} \hbar v (a^\dagger - a)S_z.
\] (6)
Formula (6), controlled by factor \( hv \), shows that \( W \) features the same energy scale as \( \hbar v \) in the unperturbed Hamiltonian \( H_0 \). Then the perturbative character of \( \epsilon W \) is ensured by the condition \( \epsilon \ll 1 \). In passing, we observe that the \( k_L^2 \)-dependent term in (4) is not involved in the perturbative scheme since, due to its constant character, it can be embedded in the unperturbed part \( H_0 \). According to the stationary perturbation theory \([22]\) we expand both the eigenstates and the relevant eigenvalues in powers of \( \epsilon \)
\[
E(n, s) = \sum_{k=0}^{\infty} \epsilon^k E_k(n, s),
\] (7)
\[
|E(n, s)\rangle = \sum_{k=0}^{\infty} \epsilon^k |E_k(n, s)\rangle,
\] (8)
where \( s = \pm \) is associated to the two-level states \( |+\rangle \equiv |e\rangle \), \( |-\rangle \equiv |g\rangle \) such that \( S_z |s\rangle = s|s\rangle/2 \). With this notation one also has \( S_{+}|\pm\rangle = 0 \), and \( S_{-}|\mp\rangle = |\pm\rangle \).

3.1. The unperturbed problem

In order to solve the unperturbed problem \( H_0|E_0(n, s)\rangle = E_0(n, s)|E_0(n, s)\rangle \), we write the eigenstates of \( H_0 \) in the form
\[
|E_0(n, s)\rangle = |n\rangle T^\dagger |f(s)\rangle,
\] (9)
where \( |n\rangle \) is the number state such that \( \hat{n}|n\rangle = n|n\rangle \), \( T \) is a unitary transformation, and \( T^\dagger |f(s)\rangle \) is a spinor state defined by
\[
T = e^{i\nu S_z}, \quad |f(s)\rangle = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ s \end{array} \right),
\] (10)
the latter satisfying the operator equations
\[
S_z |f(s)\rangle = \frac{s}{2} |f(s)\rangle, \quad S_x |f(s)\rangle = \frac{1}{2} |f(-s)\rangle.
\] (11)
The purely spin component \( 2KS_z - h\Delta S_z \) of the unperturbed Hamiltonian
\[
H_0 = \left( \hbar v n + \frac{h^2 k_L^2}{8M} \right) + 2KS_z - h\Delta S_z
\] is readily diagonalized by means of unitary transformation \( T^\dagger |f(s)\rangle \). Thus coincides with those of the problem \( CS_z |f(s)\rangle = \mu_s |f(s)\rangle \). Explicitly,
\[

\mu_s = \frac{s}{2} C = \frac{s}{2} \sqrt{\hbar^2 \Delta^2 + 4K^2}.
\]
Hence, the eigenvalues and the eigenstates of the unperturbed problem are given by
\[
E_0(n, s) = \frac{h^2 k_L^2}{8M} + \hbar v n + \frac{s}{2} \sqrt{\hbar^2 \Delta^2 + 4K^2},
\] (14)
\[
|E_0(n, s)\rangle = \frac{|n\rangle}{\sqrt{2}} \sum_{\sigma = \pm 1} \left( \cos \frac{\alpha}{2} - s - \sigma \sin \frac{\alpha}{2} \right) |\sigma\rangle.
\] (15)

3.2. First-order corrections

In view of the linear dependence of (6) from operators \( a \) and \( a^\dagger \), one easily shows that the first-order correction to the energy eigenvalues vanishes
\[
E_1(n, s) = \langle E_0(n, s)|W|E_0(n, s)\rangle = 0.
\]
As for the first-order contribution to states \( |E(n, s)\rangle \), we have
\[
|E_1(n, s)\rangle = \sum_{m \neq n} \sum_\mu \frac{|E_0(m, \mu)|}{E_0(n, s) - E_0(m, \mu)} |E_0(m, \mu)\rangle,
\]
in which, owing to the structure of unperturbed states (9),
\[
\langle E_0(m, \mu)|W|E_0(n, s)\rangle
= i\sqrt{2\pi} \hbar v |m\rangle (a^\dagger - a)|n\rangle \langle f(\mu)|TS_zT^\dagger |f(s)\rangle.
\] (16)
can therefore be included in the summations. The quantity (16) can be calculated thanks to the identities

\[
\begin{align*}
(f(s)|TST^\dagger f(s)) &= -\frac{i}{2} \sin \alpha \\
(f(-s)|TST^\dagger f(s)) &= \frac{i}{2} \cos \alpha
\end{align*}
\]

(17)

where transformation \( TST^\dagger = S_{\alpha} \cos \alpha - S_{\beta} \sin \alpha \) has been used. The final form of \(|E_1(n, s)|\) is thus found to be

\[
|E_1(n, s)| = \frac{\pi \hbar v}{\sqrt{2C}} \left[ \frac{\sin \alpha}{v} \left[ \sqrt{n + 1} [n + 1, s] + \sqrt{n} [n - 1, s] \right] \right]
\]

\[+ \frac{2K \sqrt{n + 1}}{-\hbar v + sC} [n + 1, -s] - \frac{2K \sqrt{n}}{\hbar v + sC} [n, -1, -s] \right]
\]

(18)

in which equations (13) have been used for \( \sin \alpha \) and \( \cos \alpha \), and the simplified notation \(|u, s]\rangle = |E_0(n + u, s)\rangle\rangle\) with \(u = -1, 0, +1\) has been introduced.

3.3. Second-order corrections and resonant case

Within the perturbation scheme based on states (9) the expression of the second-order correction to the energy can be shown to have the form

\[
E_2(n, s) = \sum_{n_0, n_\mu} \left| \langle E_0(n, \mu) \rangle W |E_0(n, s)\rangle^2 \right| - E_0(n, s) - E_0(n + 1, \mu).
\]

By utilizing equations (16) and (17), we find

\[
E_2(n, s) = 2\pi^2(hv)^2 \left[ \sum_{\mu} \frac{(n + 1)}{E_0(n, s) - E_0(n + 1, \mu)} \right]
\]

\[
+ \sum_{\mu} n \frac{\langle f(\mu)|TST^\dagger f(s)\rangle^2}{E_0(n, s) - E_0(n - 1, \mu)}
\]

\[\]

\[\frac{\pi^2(hv)^2}{2} \left[ -\frac{\sin \alpha}{hv} + \frac{(2n + 1)sC + hv \cos \alpha}{C^2 - \hbar^2 v^2} \right].
\]

(19)

The expression of such second-order corrections remarkably simplifies in the resonant case. Equations (14) and (19) show that, in this case,

\[
\Delta = 0 \Rightarrow \alpha = 0, C = 2K,
\]

leading to the second-order expression for the eigenvalues

\[
E(n, s) = E_0(n, s) + \epsilon^2 E_2(n, s)
\]

\[= h\nu + sK + \pi^2 \hbar v \epsilon^2 \left( \frac{2K + \hbar v (2n + 1) + sK}{4K^2 - \hbar^2 v^2} \right),
\]

(20)

where one should recall that

\[
\epsilon^2 = \frac{\hbar^2 k^2}{4\pi^2 Mv}, \quad K = \lambda E_L.
\]

(21)

Note that equations (21) contain the significant physical parameters of the model, namely, the ion mass \( M \), the trap frequency \( v \), the ion-field coupling constant \( \lambda \) and the laser wavenumber \( k_L \) and intensity \( E_L \). In view of equations (15) and (18) (note that for \( \alpha = 0 \) one has \( |E_0(n, s)\rangle = |n\rangle |f(s)\rangle \)), the corresponding eigenvector proves to be

\[
|E(n, s)\rangle \simeq |E_0(n, s)\rangle + \epsilon |E_1(n, s)\rangle = |n\rangle |f(s)\rangle
\]

\[+ \frac{\pi \hbar v}{\sqrt{2}} \left[ \sqrt{n + 1} [n + 1, -s] - \sqrt{n} [n - 1, -s] \right] \]{\hbar v + sC}
\]

(22)

For the sake of completeness, the second-order correction \(|E_2(n, s)\rangle\rangle\) is reported in appendix A.

4. Structure spectrum

A significant (totally equivalent) form of eigenvalues (20) is

\[
E_s(n, s) = \left[ h\nu \left( n + \frac{1}{2} \right) + sK \right] (1 + s\gamma) - \frac{\hbar v}{2}
\]

(23)

where the subscript * has been introduced to evidence its factorized form and

\[
\gamma = \frac{2\pi^2 \hbar v^2 K}{4K^2 - \hbar^2 v^2}.
\]

4.1. Spectrum properties

Expression (23) is interesting because it entails an eigenvalue distribution which, owing to \( \gamma \), is strongly dependent from the interplay of interaction parameters. Various properties can be evinced from eigenvalue (23).

First, it implicitly defines the parameter regions

\[
2K < h\nu, \quad 2K > h\nu,
\]

in which \( \gamma \) does not exhibit a diverging behaviour, and the value of \( K \) and \( h\nu \) are compatible with the perturbation approach. In fact, the quantity \( 1/(2K - h\nu) \) in \( \gamma \) makes it evident how the singularities emerging when \( K \) is excessively close to \( h\nu/2 \) must be avoided. Second, one easily shows that, in the limiting regimes where \( 2K \ll h\nu \) or \( 2K \gg h\nu \), \( E_s(n, s) \) is characterized by

\[
\gamma \simeq -\frac{2\pi^2 \epsilon^2}{h\nu}, \quad \gamma \simeq \frac{h\nu \pi^2 \epsilon^2}{2K},
\]

satisfying, in both cases, the inequality \( |\gamma| \ll 1 \).

Weak-interaction regime. With \( 2K \ll h\nu \) one finds

\[
E_s(n, s) \simeq h\nu \left[ \left( n + \frac{1}{2} \right) \left( 1 - \frac{2K^2 \epsilon^2 \gamma}{h\nu} \right) + sK \right] - \frac{\hbar v}{2},
\]

(24)

where the fine structure of \( E_s(n, s) \) is determined by \( sK \) in which \( s = \pm 1 \) causes, for each \( n \), the occurrence of a doublet of levels. In this formula the \( \epsilon \)-dependent term might be ignored considering that the perturbative factor \( \epsilon^2 \) in \( \gamma \) is further depressed by \( 2K/(h\nu) \ll 1 \). This further contribution thereby represents a sort of hyperfine correction to the spectrum.

Strong-interaction regime. The opposite case \( 2K \gg h\nu \) is more complex. One finds

\[
E_s(n, s) \simeq K \left[ s + \frac{h\nu}{2K} \left( 2n + 1 \right) + \pi^2 \hbar v \epsilon^2 \right] - \frac{\hbar v}{2}
\]

(25)

The spectrum features two bands corresponding to \( E_s(n^*, -1) \) and \( E_s(n^* + 1) \). For \( 0 \leq n \leq 2K/(h\nu) \) eigenvalues \( E_s(n^*, -1) \) are always smaller than \( E_0(0, +1) \). For \( n^* > 2K/(h\nu) \) the levels of the band \( s = -1 \) intercalate those of the band \( s = +1 \). In this case, a suitable choice of parameters allows the formation of interband doublets such that \( E_s(n^*, -1) \simeq E_s(n^* + 1) \) provided the condition

\[
n^* = n + 2K/(h\nu)
\]

is satisfied when \( 2K/(h\nu) = m + \delta \) is assumed where \( m \) is a positive integer, \( \delta \ll 1 \) and \( [x] \) means integer part of \( x \).
Then the separation between the levels of a given doublet is proportional to $\delta$. The fine structure of each band is represented by the $n$-dependent term in $E_n(n,s)$ while the hyperfine structure is that corresponding to the doublets derived from the superposition of the two bands. The perturbative factor $\epsilon^2$ is, in this case, further depressed by $\hbar v/(2K) \ll 1$.

Intermediate regime $\hbar v/(2K) \approx 1$. A third, significant situation is found when $K$ approaches the value $\hbar v/2$ but remains far enough from it to preserve the perturbative character of $\gamma$. In this case $\gamma$ can assume both positive and negative values implying that, in $E_n(n,s)$, the factor $(1 + s\gamma)$ can decrease ($s\gamma < 0$) or increase ($s\gamma > 0$) the level separation. Quantum number $s$ plays a central role in determining the change of the spectrum structure.

In particular, for $K \to (\hbar v/2)^-$ and $s = -1$ one finds

$$s\gamma \simeq \frac{2\pi^2 K^2 \epsilon^2 (-1)}{4K(2K - \hbar v)} = \frac{\pi^2 \epsilon^2 K}{|2K - \hbar v|} > 0,$$

where $|2K - \hbar v|$ can contrast the effect of the second-order term $\epsilon^2$ in $\gamma$ (note that $|2K - \hbar v|$ represents a quantity independent from perturbation parameter $\epsilon$ since it is fully unrelated to $\ell$ and $\lambda_L$). As a consequence, the separation between subsequent levels $E_n(n,-1)$ and $E_n(n+1,-1)$ tends to grow in that factor $(1 + s\gamma)$, which controls this separation, increases.

Instead, for $K \to (\hbar v/2)^+$ with $s = +1$, one finds that

$$s\gamma \simeq \frac{2\pi^2 K^2 \epsilon^2 \times 2K}{4K(2K - \hbar v)} = -\frac{\pi^2 \epsilon^2 K}{|2K - \hbar v|} < 0.$$

Again factor $\epsilon^2$ (depending on $\ell$ and $\lambda_L$) can be contrasted by a sufficiently small $|2K - \hbar v|$. Owing to the reduction of factor $(1 + s\gamma)$ ($\gamma < 0$) the series of eigenvalues $E_n(n,-1)$ shows a thicker level distribution. The same effects are observed in the case $K \to (\hbar v/2)^+$ which displays the decreasing (increasing) of the level separation for $s = -1$ ($s = +1$). Concluding, a peculiar behaviour seems to characterize the regime $K \approx \hbar v/2$ suggesting that the region around $K = \hbar v/2$ (excluded from the perturbative scheme) is the source of a macroscopic change of the spectrum structure. Crossing this region evidences how, for a given quantum number $s$, a rarefied level distribution evolves into a thicker one while a thick distribution evolves into a more rarefied one, depending on the direction of the crossing.

4.2. Time-dependent solutions of the Schrödinger problem relevant to $H_j$

The general solution to the original Schrödinger problem $i\hbar \partial_t |\Psi(t)\rangle = H_1 |\Psi(t)\rangle$ is easily found to be

$$|\Psi(t)\rangle = \sum_{(n,s)} c(n,s) e^{-iE(n,s)\hbar/t} |\psi_1(n,s)\rangle$$

where $|\psi_1(n,s)\rangle = U_1(t) U_2(E(n,s))$ and $|\xi(t)\rangle$ is defined by equation (5). $|\Psi(t)\rangle$ is the superposition of elementary time-dependent solutions $|\Psi(n,s)\rangle = e^{-iE(n,s)\hbar/t} |\psi_1(n,s)\rangle$, that satisfy the same Schrödinger problem. In the present context, $|\Psi_1(n,s)\rangle$ is the counterpart of the stationary states characterizing Schrödinger problems in which the Hamiltonian is time independent. Since they are in a one-to-one correspondence with states $|E(n,s)\rangle$ they form a time-dependent basis.

The explicit form of $U_1(t) U_2(E(n,s))$ is found by determining the action of such transformations. Since $U_2 = \exp\left[i\gamma(n^2 + aS_1)\right]$ one finds, to the first order in the perturbation parameter, that state (22) (with $|n,s\rangle = |n\rangle f(s)$) and $|n\rangle = \sqrt{n}/(n + 1)$ becomes

$$U_2 |E(n,s)\rangle = |n,s\rangle + \frac{2nK\hbar^2}{\sqrt{2}} \left[ a^\dagger(n,-s) - a(n,-s) - a^\dagger(n,s) + a(n,s) \right].$$

The action of $U_1(t) = \exp\left(-i\omega tS_1\right)$ involves only spinor states $|f(s)\rangle$ giving

$$U_1(t) |f(s)\rangle = |f(s)\rangle \cos(\omega t/2) - i |f(-s)\rangle \sin(\omega t/2).$$

An interesting aspect that deserves some comment is the considerable influence of the spectrum structure, discussed in subsection 4.1, on the general solution. In the regime $2K \ll \hbar v$ state $|\Psi(t)\rangle$ will have the form

$$|\Psi(t)\rangle = \sum_n [C(n,1) e^{i\delta K/n} |\psi_1(n,1)\rangle + C(n,-1) e^{-i\delta K/n} |\psi_1(n,-1)\rangle] e^{-\nu t},$$

showing how states $|\psi_1(n,\pm 1)\rangle$ of level doublets feature a slow internal dynamics in which frequencies $\pm K/\hbar$ could include corrections depending on $\epsilon^2$ (see equation (24)). A different decomposition characterizes the regime $2K \gg \hbar v$. We find

$$|\Psi(t)\rangle = |\Psi_1(t)\rangle + \sum_{m=\pm 1} C(n,m) |\psi_1(n,m)\rangle e^{-\nu t} + C(n,m+1) e^{-i\nu t} |\psi_1(n,m+1)\rangle e^{-\nu t},$$

where $m = \pm K/\hbar$ (showing again the presence of doublet states $|\psi_1(n,\pm 1)\rangle$ the (slow) internal dynamics of which is controlled by parameter $\delta$ (see the comment below equation (25)). State

$$|\Psi_1(t)\rangle = \sum_{n=\pm 1} C(n,\pm 1) e^{-i\nu t} |\psi_1(n,\pm 1)\rangle$$

involves singlet states of the low region of the spectrum exhibiting a fast evolving dynamics.

Concerning the two intermediate regimes where $2K \approx \hbar v$ (notice that $K \to \hbar v/2$, i.e., $K$ arbitrarily close to $\hbar v/2$ is forbidden) the general solution will be composed by two macroscopic components $|\Psi_1(t)\rangle$ relevant to spin states $s = \pm 1$ such that $|\Psi(t)\rangle = |\Psi_+(t)\rangle + |\Psi_-(t)\rangle$. According to our discussion on the spectrum structure, if the eigenvalue distribution for $s = \mp 1$ is thick then the one for $s = \mp 1$ will be rarefied. Suppose that such an effect is emphasized when $K$ approaches $\hbar v/2$ from below. If the dominating components of (26) involve eigenvalues $E(n,s)$ ranging in a finite interval of ‘energies’ around a given average energy, a natural situation when a state is prepared experimentally, then $|\Psi(t)\rangle$ will exhibit an almost continuous part corresponding to $|\Psi_+(t)\rangle$ superimposed to $|\Psi_-(t)\rangle$ formed by a few states $|\Psi_-(n,-1)\rangle$. Due to this very unbalanced situation we conjecture that
composition of $|\Psi(t)\rangle$ could be dramatically changed by a relatively small change of $K$. This will depend on the actual size of the region around $K = \hbar v/2$ that separates the opposite situations where $K$ is approaching $\hbar v/2$ from below and from above. In the latter case $|\Psi(t)\rangle$ is almost continuous while $|\Psi_+(t)\rangle$ involves a few states $|\Psi(n, +1)\rangle$. The possibly macroscopic character of this effect could be proven through an exact reconstruction of the spectrum around $K = \hbar v/2$ via numerical calculations.

5. The Jaynes–Cummings picture of $H_\eta$

In this section we show that the JC model naturally emerges from Hamiltonian $H_\eta$ in the parameter-space region where the RWA is viable. This region exactly corresponds to the intermediate regime $K \simeq \hbar v/2$ (and the case $K = \hbar v/2$ is admitted) where the perturbation theory cannot be implemented. This circumstance seems to be a distinctive feature of the model described by Hamiltonian (4)

$$H'_\eta = \hbar v n + \frac{\hbar^2 k_1^2}{8M} + 2K S_\sigma + \frac{\hbar k_0}{M} S_i p,$$

in which the resonance condition $\Delta = 0$ has been assumed. $H'_\eta$ can be expressed in an equivalent form by introducing the (non-standard) raising and lowering operators

$$D_+ = S_\sigma + i S_i, \quad D_- = S_\sigma - i S_i.$$

If $S_\sigma = D_\sigma$ is assumed, then $D_-, D_+$ and $D_\sigma$ are shown to satisfy commutators

$$[D_+, D_-] = \pm D_\sigma, \quad [D_+, D_\sigma] = \mp D_\sigma, \quad [D_-, D_\sigma] = \pm D_\sigma,$$

thereby providing an alternative (but equivalent) basis of generators for the algebra su(2). The link with the standard basis $S_i$ and $S_\sigma$ is realized by the unitary transformation $U^{+} S_\sigma U = D_\sigma, \sigma = \pm, \tilde{\sigma}$, discussed in appendix B. In this picture one has

$$H'_\eta = \hbar v n + \frac{\hbar^2 k_1^2}{8M} + 2K D_\sigma + \frac{\hbar k_0}{M} D_\sigma p,$$

(27)

with

$$D_\sigma = (D_+ - D_-)/(2i) = S_\sigma, \quad D_\sigma = (D_+ + D_-)/2 = S_i.$$

With respect to Hamiltonian $H_\eta$, the application of the RWA implies the condition

$$2K \simeq \hbar v,$$

$K$ and $\hbar v$ being the characteristic energies of the non-interacting part of $H_\eta$. In fact, it is well known that transforming Hamiltonian (27) in the interaction picture, the action of operators $D_\sigma a$ and $D_\sigma a^\dagger$ is negligible since they are multiplied by fast oscillating factors $\exp (\pm i(\nu + 2K/\hbar))$ while $D_\sigma a$ and $D_\sigma a^\dagger$ survive because, due to $K \simeq \hbar v/2$, the relevant oscillating factors $\exp (\pm i(\nu - 2K/\hbar))$ are essentially time independent. Then, in

$$D_\sigma p = \frac{1}{2} \sqrt{\frac{\hbar M v}{2}} (D_+ a^\dagger - D_+ a - D_- a^\dagger + D_- a),$$

the virtual processes corresponding to terms $D_\sigma a$ and $D_\sigma a^\dagger$ are negligible, so that the final form of $H_{IC}$ is

$$H_{IC} = \hbar v n + \frac{\hbar^2 k_2}{8M} + 2K D_\sigma + \Gamma (D_- a^\dagger + D_+ a),$$

(28)

with $\Gamma = -\pi \hbar v/e \sqrt{2}. Model (28)$ represents the JC form of the original Hamiltonian $H'_\eta$. In this form the latter is block-diagonal: specifically, with respect to the states $|\psi(n-1)\rangle \otimes U^\dagger |\nu\rangle$, $|\psi(n)\rangle \otimes U^\dagger |\nu\rangle$, the nth $2 \times 2$ block, $n \geq 1$, is

$$\begin{bmatrix} \hbar v(n-1)+K & \Gamma \sqrt{n} \\ -\Gamma \sqrt{n} & \hbar v n - K \end{bmatrix}.$$}

The latter reduces to $-K$ in the degenerate case $n = 0$. The nice algebraic properties of model (28) from which the block-diagonal structure is stemmed are reviewed in appendix C. The diagonalization of $H_{IC}$ is straightforward. The eigenvalues are

$$E_{IC}(n, \pm) = \hbar v \left(n - \frac{1}{2}\right) \pm \sqrt{\frac{(\hbar v - 2K)^2}{4} + \Gamma^2 n},$$

where only the lower sign applies for $n = 0$. Such an expression appears to be very different from eigenvalues (23). When $\hbar v = 2K$, eigenvalues $E_{IC}(n, s)$ exhibit a linear dependence on $\Gamma \propto \epsilon$ which might be interpreted as the signature of the transition from the $\hbar v = 2K > 0$ to the $\hbar v = -2K < 0$ regime.

The complementary character of the JC regime, where $\hbar v \simeq 2K$, with respect to regimes where $\hbar v > 2K$ or $\hbar v < 2K$ is confirmed when reformulating perturbed eigenstates $|E(n, s)\rangle$ in terms of operators $a, a^\dagger$ and $D_\sigma$. In appendix D we show that, in addition to $D_- a$ and $D_\sigma a$, the correction $|E_0(n, s)\rangle$ is generated by the action of $D_- a$ and $D_\sigma a^\dagger$ on $|E_0(n, s)\rangle$, namely, by the operators excluded from the JC picture.

6. Concluding remarks

We have studied the time-dependent model describing the ion–light interaction within a typical scheme which reduces the latter to a (unitarily equivalent) time-independent minimal form. The resulting Hamiltonian $H_\eta$ features a simple operator structure and a linear dependence on the LD parameter $\eta$ that have allowed us to diagonalize $H_\eta$ via the standard perturbation method.

In section 3 we have determined the second-order analytic expressions of the eigenstates and the eigenvalues in terms of the physically significant parameters of the model (see equation (21)). The eigenvalue formula has been used in section 4 to identify four independent regimes that correspond to different spectrum structures in which the eigenvalue distribution can be more or less dense and possibly includes level doublets. In particular, we have shown that the level density exhibits contrasting behaviours in the intermediate regimes characterized by $K$ ‘close’ to $\hbar v/2$: for $s = -1$, it is decreasing when approaching $\hbar v/2$ from below and increasing when approaching $\hbar v/2$ from above. The opposite effect characterizes the choice $s = +1$. We thus expect to observe considerable changes of the spectrum in the unexplored
neighbourhood of $\hbar/v/2$ where the perturbation scheme cannot be applied. In passing, we note that similar effects have been observed in simple few-mode bosonic models [24]. In this case the decreasing of the level separation has been related to the transition from stable to unstable regimes.

In the second part of section 4 we have reconstructed the general solution of the Schrödinger problem of model $H_t$ by superposing the elementary solutions of minimal model $H_R$. The information about the eigenvalue distribution has been used to investigate the structure of the general solution in the four regimes characterizing the eigenvalue spectrum. To complete our exploration of Hamiltonian $H_R$, in section 5 we have applied the RWA to $H_R$ reducing the latter to a JC model. Since this approximation is based on the condition $K \simeq \hbar/v/2$, then the RWA leads, quite naturally, to obtain information about the intermediate (unexplored) regime where the perturbation scheme cannot be applied. In particular, we have found that the modifications of the algebraic structure of $H_R$ caused by the RWA strongly influence the spectrum of $H_{IC}$ changing the $\epsilon^2$-dependence into the simpler dependence on $\epsilon$ for $K = \hbar/v/2$.

Future work will be focused on investigating the spectrum changes caused by the crossing of the RWA intermediate regime. The spectral properties of the non-resonant case $\Delta \neq 0$ will also be explored. This requires a more extended analysis supported by numerical calculations to get an exact description of the observed effects. The decreasing of the level separation in certain regimes also suggests the application of the continuous-variable picture in the RWA intermediate regime. This picture has been successfully applied [25, 26] to attractive bosonic lattices, characterized by a dense level distribution, to study the transition between different dynamical regimes.

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Appendix A. Eigenstate second-order corrections

The second-order corrections in $|E(n, s)| \simeq |E_0(n, s)| + \epsilon|E_1(n, s)| + \epsilon^2|E_2(n, s)|$ can be easily calculated by means of the standard formula

$$|E_2(n, s)| = \sum_{m \neq n} \sum_{r} \frac{|E_0(m, r)| W|E_1(n, s)|}{E_0(n, s) - E_0(m, r)} |E_0(m, r)|.$$

From

$$W|E_1(n, s)| = \sqrt{2\pi} \hbar \nu (a^+ - a) S_1;$$

$$\epsilon \pi \hbar \nu \left[ \frac{\sqrt{n + 1}[n + 1, -s]}{-\hbar v + 2sK} - \frac{\sqrt{n}[n - 1, -s]}{\hbar v + 2sK} \right]$$

$$= -\pi^2 \hbar^2 \nu^2 \left[ \frac{\sqrt{n + 1}[n + 1, -s]}{2(2sK - \hbar v)} - \frac{\sqrt{n}[n - 1, -s]}{2(2sK + \hbar v)} + (\ldots)|n, s| \right].$$

where the contribution of states $|n, s\rangle$ in $W|E_1(n, s)|$ can be neglected due to the condition $m \neq n$, one finds

$$|E_2(n, s)| = \pi^2 \hbar \nu \left[ \frac{a^2|n, s|}{2(2sK + \hbar v)} - \frac{(a^+)^2|n, s|}{2(2sK - \hbar v)} \right].$$

Appendix B. New basis of su(2)

The unitary transformation $U^1 S_a U = D_a$ connects $S_+$ and $S_\pm$ with $D_+$ and $D_\pm$, respectively. Its explicit definition is given by $U = \exp(\alpha S_y) \exp(\phi S_z)$ where $\alpha = \phi = \pi/2$. For example, with $S_- = S_y - i S_\pi$ one has

$$U^1 S_- U = \exp(-i\phi S_z) \exp(i\phi S_y)$$

$$= \epsilon \exp(-i\phi (S_y - i S_\pi)) \epsilon S_y$$

so that $U^1 S_- U = -\epsilon^2 \exp(-i\phi (S_y + i S_z)) \epsilon S_y$, for $\alpha = \pi/2$. Then the action of the $\phi$-dependent transformation gives

$$U^1 S_- U = -\epsilon^2 \exp(-i\phi (S_y + i S_z)) \epsilon S_y$$

$$= -[S_y \cos \phi - S_z \sin \phi + i(S_y \cos \phi + S_z \sin \phi)]$$

with $U^1 S_- U = S_y - i S_\pi = D_-$. for $\phi = \pi/2$.

Appendix C. Diagonalization scheme of the JC model

Hamiltonian (28) features a dynamical algebra again showing the structure of algebra su(2). Indeed, since

$$[D_+, a, D_- a^+] = 2D_2 N, \quad [D_+, a, N] = [D_- a^+, N] = 0,$$

where $N \equiv (\tilde{n} + D_+ + 1/2)$ denotes the quantum version of a constant of the motion relevant to Hamiltonian (28), one readily proves that the new operators

$$J_\pm = D_\pm, \quad J_\mp = \frac{1}{\sqrt{N}} D_+ a, \quad J_{\mp} = \frac{1}{\sqrt{N}} D_- a^+,$$

satisfy $[J_+, J_-] = 2J_z$ and $[J_z, J_{\pm}] = \pm J_{\mp}$. The latter can be seen as the generators of a more general algebra su(2) representing the dynamical algebra of the JC model. In fact, up to a constant term, Hamiltonian (28) can be expressed as a linear combination of $J_z$ and $J_{\pm}$

$$H_{IC} = \hbar v(N - 1/2) + (2K - \hbar v)J_z + \Gamma \sqrt{N}(J_+ + J_-).$$

The algebra su(2) generated by $J_z$ and $J_{\pm}$ turns out to be a special case of the su(2) representation for generalized JC models reported in [23].

Appendix D. Operators generating the first-order correction $|E_1(n, s)|$

Eigenstates (22), including the first-order correction, can be written as

$$|E(n, s)| = |n\rangle |f(s)\rangle - \frac{i\pi \hbar v e^{R(n, s)}}{\sqrt{2(h^2 v^2 - 4K^2)}}.$$
with

$$|R(n, s)\rangle = (\hbar v + s2K)a^\dagger |n, -s\rangle + (\hbar v - s2K)a|n, -s\rangle.$$  

Owing to the identities (11), $S_x|f(s)\rangle = (s/2)|f(s)\rangle$ and $S_z|f(s)\rangle = (1/2)|f(-s)\rangle$, state $|R(n, s)\rangle$ reduces to

$$|R(n, s)\rangle = 2\hbar v(a^\dagger + a)S_z|n, s\rangle + 4sK(a^\dagger - a)S_x|n, s\rangle$$

$$= 2\hbar v(a^\dagger + a)S_z|n, s\rangle + 8K(a^\dagger - a)S_x|n, s\rangle$$

$$= 2\hbar v(a^\dagger + a)S_z|n, s\rangle + 4Ki(a^\dagger - a)S_z|n, s\rangle.$$

Then $|R(n, s)\rangle$ can be expressed as

$$|R(n, s)\rangle = -i[\hbar v - 2K](D_+ a^\dagger + D_- a)|n, s\rangle$$

$$- i[\hbar v + 2K](D_+ a - D_- a^\dagger)|n, s\rangle,$$

showing that, in addition to operators $D_+ a$ and $D_- a^\dagger$ characterizing the JC Hamiltonian, the first-order corrections of state $|E(n, s)\rangle$ must include operators $D_+ a$ and $D_- a^\dagger$ occurring in model $H_0$ through the interaction term $D_+ p$ (see equation (27)).

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