NOT ALL GROUPS ARE LEF GROUPS, OR CAN YOU KNOW IF A GROUP IS INFINITE?

MELVYN B. NATHANSON

Abstract. This is an introduction to the class of groups that are locally embeddable into finite groups.

1. Finite or infinite?

A simple question: Do the finite subsets of a group tell us if the group is infinite? Assume that we can only see the finite subsets of a group, and, also, that we can determine if a finite subset is a subset of some finite group. This means that we can answer the following question. Let \( A \) be a finite subset of a group \( G \). Does there exist a finite group \( H \) and a partial homomorphism \( f : A \to H \) that is one-to-one. A partial homomorphism from a subset \( A \) of a group to a group \( H \) is a function \( f : A \to H \) such that, if \( a, b \in A \) and \( ab \in A \), then \( f(ab) = f(a)f(b) \). A one-to-one partial homomorphism is also called a local embedding. Of course, if the group \( G \) is finite, then, for every subset \( A \) of \( G \), the restriction of the identity homomorphism on \( G \) to the subset \( A \) is a local embedding into a finite group.

Does there exist an infinite group \( G \) such that every finite subset of \( G \) looks like (equivalently, can be partially embedded into) a subset of a finite group? Does there exist an infinite group \( G \) in which some finite subset of \( G \) is not also a subset of a finite group?

Theorem 3 answers the second question. The following example answers the first question. Let \( A \) be a nonempty finite subset of the infinite abelian group \( \mathbb{Z} \). Choose an integer

\[
m > \max\{ |a - b| : a, b \in A \} = \max(A) - \min(A).
\]

Consider the function \( f : A \to \mathbb{Z}/m\mathbb{Z} \) defined by \( f(a) = a + m\mathbb{Z} \) for all \( a \in A \). This is a partial homomorphism because it is the restriction of the canonical homomorphism \( a \mapsto a + m\mathbb{Z} \) from \( \mathbb{Z} \) to \( \mathbb{Z}/m\mathbb{Z} \). For \( a, b \in A \), we have \( f(a) = f(b) \) if and only if \( a \equiv b \pmod{m} \) if and only if \( m \) divides \( |a - b| \). The inequality \( |a - b| < m \) implies that \( f(a) = f(b) \) if and only if \( a = b \), and so \( f \) is a local embedding. Thus, every finite subset of the infinite group \( \mathbb{Z} \) can be embedded into a finite cyclic group. By looking only at finite subsets, we cannot decide if \( \mathbb{Z} \) is infinite.

Let us call a group \( G \) locally embeddable into finite groups, or an LEF group, if every finite subset of \( G \) can be embedded into a finite group. Vershik and Gordon \(^5\) introduced this definition, and obtained many fundamental results.

Here are two classes of LEF groups.

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Theorem 1. Every locally finite group is an LEF group. Every abelian group is an LEF group.

Proof. A group is locally finite if every finite subset generates a finite group. For such groups, the proof is immediate from the definition.

For abelian groups, the proof follows easily from the structure theorem for finitely generated abelian groups, and an easy modification of the preceding argument that $\mathbb{Z}$ is an LEF group. \qed

It is natural to ask: Is every infinite group an LEF group, or does there exist an infinite group that is not an LEF group?

2. Finitely presented groups

Let $W$ be a group with identity $e$, and let $X$ be a subset of $W$ that generates $W$. We assume that $e \notin X$. The length of an element $w \in W$ with $w \neq e$ is the smallest positive integer $k = \ell(w)$ such that there is a representation of $w$ in the form

$$w = x_1^{\varepsilon_1}x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k}$$

where

$$x_i \in X \text{ and } \varepsilon_i \in \{1, -1\} \text{ for } i = 1, \ldots, k.$$ (1)

We define $\ell(e) = 0$. Note that $\ell(w) = 1$ if and only if $w = x$ or $w = x^{-1}$ for some $x \in X$.

For every nonnegative integer $L$, we define the “closed ball”

$$B_L = \{w \in W : \ell(w) \leq L\}.$$ (2)

We have

$$B_0 = \{e\} \quad \text{and} \quad B_1 = \{e\} \cup \{x^\varepsilon : x \in X \text{ and } \varepsilon \in \{1, -1\}\}.$$ (3)

If the generating set $X$ is finite, then, for every $L$, the group $W$ contains only finitely many elements $w$ of length $\ell(w) \leq L$, and so the $B_L$ is a finite subset of $W$.

If $w \in B_L$, then $w$ satisfies (1) and (2) for some $k \leq L$. For all $j = 1, \ldots, k$, the partial product

$$w_j = x_1^{\varepsilon_1}x_2^{\varepsilon_2} \cdots x_j^{\varepsilon_j}$$

has length $\ell(w_j) \leq j \leq L$, and so $w_j \in B_L$. (We observe that if $\ell(w_j) < j$, then $\ell(w) < k$, which is absurd. Therefore, $\ell(w_j) = j$ for all $j \in \{1, \ldots, k\}$.) Let $w_0 = e$.

Note that $w = w_k$ and that

$$w_j = w_{j-1}x_j^{\varepsilon_j}$$

for all $j \in \{1, \ldots, k\}$. If $f : B_L \to H$ is a partial homomorphism, then

$$f(w) = f(w_{k-1}x_k^{\varepsilon_k}) = f(w_{k-1})f(x_k^{\varepsilon_k})$$

$$= f(w_{k-2}x_{k-1}^{\varepsilon_{k-1}}f(x_k^{\varepsilon_k}) = f(w_{k-2})f(x_{k-1}^{\varepsilon_{k-1}})f(x_k^{\varepsilon_k})$$

$$= \cdots$$

$$= f(x_1^{\varepsilon_1})f(x_2^{\varepsilon_2}) \cdots f(x_{k-1}^{\varepsilon_{k-1}})f(x_k^{\varepsilon_k})$$

For partial products in finite groups, see Nathanson [3].

Let $X$ be a nonempty set, and let $F(X)$ be the free group generated by $X$. Let $R$ be a nonempty subset of $F(X)$. The normal closure of $R$ in $F(X)$, denoted $N(R)$,
is the smallest normal subgroup of $F(X)$ that contains $R$. The subgroup $N(R)$ is generated by the set

\[ \{ wr^ew^{-1} : w \in F(X), \ r \in R, \ e \in \{1,-1\}\}. \]

A group $G$ is finitely presented if

\[ G = \langle X; R \rangle = F(X)/N(R) \]

where $F(X)$ is the free group generated by a finite set $X$ and the subgroup $N(R)$ is the normal closure of a finite subset $R$ of $F(X)$. If $\pi : F(X) \to G$ is the canonical homomorphism, then the set

\[ X^* = \pi(X) = \{ xN(R) : x \in X \} \]

generates $G$.

The following result is Proposition 1.10 in Pestov and Kwiatkowska [4].

**Theorem 2.** Let $G$ be a finitely presented infinite group. If $G$ is an LEF group, then $G$ contains a nontrivial proper normal subgroup. Equivalently, a finitely presented infinite simple group is not an LEF group.

**Proof.** Let $G = \langle X; R \rangle = F(X)/N$ be a finitely presented infinite group, where $F(X)$ is the free group generated by a finite set $X$, and $N = N(R)$ is the normal closure of a finite subset $R$ of $F(X)$. Let $e_F$ be the identity in $F(X)$. The identity in $G$ is $e_G = e_FN = N$. The canonical homomorphism $\pi : F(X) \to G$ is defined by $\pi(w) = wN$ for all $w \in F(X)$.

Choose an integer $L$ such that

\[ L \geq \max\{\ell(w) : w \in X \cup R\}. \]

The closed ball

\[ B_L = \{ w \in F(X) : \ell(w) \leq L \} \]

is a finite subset of $F(X)$. We have

\[ \{ e_F \} \cup X \cup X^{-1} \cup R \subseteq B_L. \]

The set

\[ A = \pi(B_L) \subseteq G \]

is a finite subset of $G$ that contains $X^* = \pi(X)$. Also, $e_G = \pi(e_F) = N \in A$.

If $G$ is an LEF group, then there exist a finite group $H$ and a local embedding $f$ of $A$ into $H$. Let $e_H$ be the identity in $H$. For all $x \in X$, we have $\pi(x) \in A$ and so $f\pi(x) \in H$.

By the universal property of a free group, there exists a unique homomorphism

\[ f^* : F(X) \to H \]

such that

\[ f^*(x) = f\pi(x) \]

for all $x \in X$. The subgroup

\[ N^* = \text{kernel}(f^*) \]

is a normal subgroup of $F(X)$. We shall prove that

(3) \[ N \subseteq N^* \subseteq F(X). \]
The diagram is

\[
\begin{array}{c}
\text{\(N^*\)} \\
\text{\(X\)} \\
\text{\(B_L\)} \\
\text{\(f\)} \\
\text{\(H\)} \\
\text{\(f\)} \\
\text{\(G\)}
\end{array}
\]

If \(N^* = F(X)\), then \(x \in N^*\) for all \(x \in X\). Because \(X \subseteq B_L\) and \(\pi(x) = xN \in A\), we have

\[
f(xN) = f\pi(x) = f^*(x) = e_H = f(N).
\]

Because \(f\) is one-to-one and \(f(xN) = f(N)\), it follows that \(\pi(x) = xN = N\) for all \(x \in X\). The set \(\pi(X)\) generates \(G\), and so \(G = \{N\}\) is the trivial group, which is absurd. Therefore, \(N^*\) is a proper normal subgroup of \(F(X)\).

Next we prove that \(N^*\) contains \(N\). Let \(r \in R\). There is a nonnegative integer \(k = \ell(r) \leq L\) such that

\[
r = \prod_{i=1}^{k} x_i^{\varepsilon_i}
\]

where \((x_i)_{i=1}^{k}\) is a sequence of elements of \(X\) and \((\varepsilon_i)_{i=1}^{k}\) is a sequence of elements of \(\{1, -1\}\).

Because \(r \in R \subseteq N\), we have \(rN = N\) and

\[
f^*(r) = f^* \left( \prod_{i=1}^{k} x_i^{\varepsilon_i} \right) = \prod_{i=1}^{k} f^*(x_i)^{\varepsilon_i} = \prod_{i=1}^{k} f\pi(x_i)^{\varepsilon_i}
\]

\[
= \prod_{i=1}^{k} f(x_iN)^{\varepsilon_i} = f \left( \prod_{i=1}^{k} x_i^{\varepsilon_i} N \right) = f(rN) = f(N) = e_H.
\]

Therefore, \(r \in N^*\). Because \(R \subseteq N^*\) and \(N^*\) is a normal subgroup of \(F(X)\), it follows that \(N^*\) contains \(N\), which is the normal closure of \(R\), and so \(N \subseteq N^*\).

Finally, if \(N = N^* = \text{kernel}(f^*)\), then \(G = F(X)/N = F(X)/N^*\) is isomorphic to a subgroup of the finite group \(H\), and so \(G\) is finite, which is absurd. Therefore, \(N\) is a proper subgroup of \(N^*\).

This proves relation (3). The correspondence theorem in group theory implies that \(N^*/N\) is a nontrivial proper normal subgroup of \(G\), and so \(G\) is not a simple group. It follows that no finitely presented infinite simple group is an LEF group. This completes the proof. \(\Box\)

**Theorem 3.** There exist infinite groups that are not LEF groups. In particular, the Thompson groups \(T\) and \(V\) are not LEF groups.

**Proof.** The Thompson groups \(T\) and \(V\) are finitely presented infinite simple groups (Cannon, Floyd, and Parry [2], Cannon and Floyd [1]). \(\Box\)
3. Open Problem

Theorem 3 suggests that “almost all” groups are LEF groups. It is an open problem to construct new classes of groups that are not LEF groups, and to understand why ”most finite subsets of most groups” are indistinguishable from subsets of finite groups.

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Lehman College (CUNY), Bronx, New York 10468
E-mail address: melvyn.nathanson@lehman.cuny.edu