A Deep Dive into $f(R)$ Gravity Theory

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Abstract

In this paper we have derived the behavior of deceleration parameter with respect to redshift in context of $f(R)$ gravity in vacuum using Taylor expansion of derivative of action. Here we have obtained that the two first terms in Taylor expansion may describe the late time acceleration which is appeared by SNeIa without need of dark energy and dark matter. Also we have derived that any other terms higher than $z$ in Taylor expansion may describe main inflationary epoch in the early Universe. We have shown that $f(R)$ gravity may cover all the dynamical history of the Universe from the beginning to the late time accelerating phase transition.
The recent data coming from the luminosity distance of SuperNovae Ia (SNeIa)\cite{1}, wide galaxy surveys\cite{2} and the anisotropy of cosmic microwave background radiation\cite{3,4} suggest that the Universe is undergoing an accelerating expansion. Several candidates, responsible for this expansion, have been proposed in the literature, in particular, dark energy models and modified gravity. The main problem of dark energy models are understanding their nature, since they are introduced as ad hoc gravity sources in a well define models of gravity. In this context, modified theories of gravity, such as f(R) gravity\cite{5,6,7}, has been verified in an attempt to explain the late-time accelerated expansion of the Universe. These theories are also referred to as ’extended theories of gravity’, since they naturally generalize General Relativity. There has been predicted that the universe might have appeared from an inflationary phase in the past. It is also believed that the present universe is passing through a phase of the cosmic acceleration. While a series of cosmological models were proposed in Einsteins gravity with early inflationary scenario are working well, the mystery of passing universe through an accelerated phase of expansion is an interesting open question that is yet to be understood. There are many interesting works have done in the context of f(R) gravity on unifying late time accelerating behavior and early inflationary behavior of the Universe\cite{8}, where they have shown that the late time acceleration is in continues of early time inflation for specific models of f(R) gravity. Also, their vacuum solutions are obtained for constant Ricci scalar while it is possible to derive non constant curvature scalar vacuum solutions in f(R) gravity theories. In this letter we show easily that f(R) gravity theories are the most powerful approach to describe all the dynamical behavior of the Universe in one scenario. Here we look to the story of f(R) gravity from a different angle. Actually we consider vacuum solutions of f(R) gravity. But there are two main difference with other vacuum solutions. The first one is that we do not assume constant scalar curvature to obtain vacuum solution. The second one is that we do not consider any specific model of f(R) gravity action. Vacuum solutions of modified f(R) gravity would like to explain all the phase transition of cosmological parameters like deceleration parameter without need of dark companion of the Universe, alone by pure geometry. The action of modified theory of gravity is given by
\begin{equation}
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa} f(R) + L_m \right],
\end{equation}
where \(L_m\) is the matter action such as radiation, baryonic matter, dark matter and so on which we do not consider in field equation. In this work, we consider the flat Friedmann
Robertson Walker, (FRW) background, so that the gravitational field equations for $f(R)$ modified gravity are provided by the following form

$$-3\ddot{a}f' + 3\frac{\dot{a}}{a}\dot{R}f'' + \frac{1}{2}f = 0,$$

(2)

$$[\ddot{a} + 2\frac{\dot{a}^2}{a^2}]f' - 2\frac{\dot{a}}{a}\dot{R}f'' - \dot{R}^2 f''' - \ddot{R}f'' - \frac{1}{2}f = 0.$$  

(3)

where the overdot denotes a derivative with respect $t$, $a(t)$ is the scale factor and $H = \dot{a}(t)/a(t)$ is the Hubble parameter. Eliminating $f$ between Eqs. (2) and (3) obtains:

$$-2[\ddot{a} - (\frac{\dot{a}}{a})^2]f' + \frac{\dot{a}}{a}\dot{R}f'' - \ddot{R}f'' - \dot{R}^2 f''' = 0.$$  

(4)

Which can be changed to the form of:

$$\ddot{F} - H\dot{F} + 2\dot{H}F = 0,$$

(5)

where $F = df/dR$. Eq. (5) is a second order differential equation of $F$ with respect to time, in which both of $F$ and $H$ are undefined. The usual method to solve Eq. (5) is based on definition of $f(R)$. But in this paper we would like to replace the variable of Eq. (5) by another cosmic parameter which named redshift, $z$. Each redshift, $z$ has an associated cosmic time $t$ (the time when objects observed with redshift $z$ emitted their light), so we can replace all the differentials with respect to $t$ by $z$ via:

$$\frac{da}{dt} = \frac{dz}{dz} \frac{da}{dz},$$  

(6)

where we use $1 + z = a_0/a$, and we consider $a_0 = 1$, in the present time. Now, we can replace the variable of Eq. (5) from $t$ to $z$ by using Eq. (6), and we obtain a first order differential equation for $H^2$ with respect to $z$ as:

$$\frac{d}{dz}H(z)^2 = P(z)H(z)^2,$$

(7)

where $P(z)$ is a function with respect to $z$, which depend on definition of $F$ with respect to $z$ as:

$$P(z) = \frac{2(1 + z)(d^2F/dz^2) + 4(dF/dz)}{2F - (1 + z)(dF/dz)}.$$  

(8)

The deceleration parameter, $q$ in cosmology is a dimensionless measure of the cosmic acceleration of the expansion of space in a FRW universe. It is defined by:

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2}.$$  

(9)
here we change the variable of Eq. (9) from \( t \) to \( z \) to have evolution of deceleration parameter with respect to redshift as:

\[
q(z) = -1 + \frac{1 + z}{2H(z)^2} \frac{dH(z)^2}{dz},
\]

\[
= -1 + \frac{1 + z}{2} P(z),
\]

which only depends on \( P(z) \) for vacuum solution. Since we would like to consider a general behavior of \( q(z) \) due to the model independent \( F(z) \), we use Taylor expansion of \( F(z) \) around the present value of redshift, \( z = 0 \) as:

\[
F(z) = \sum_{n=0}^{m} F_n z^n
\]

(12)

where

\[
F_n = \frac{1}{n!} \frac{d^n F}{dz^n} \bigg|_{z=0},
\]

(13)

are some constants at \( z = 0 \). Now to calculate deceleration parameter, with some algebra we put Taylor expansion of Eq. (12) in Eq. (8), then we put the result in Eq. (11) to find:

\[
q(z) = -1 + \frac{1 + z}{2} \left( \sum_{n=0}^{m} n F_n z^{n-2} [(n+1)z + n - 1] \right)
\]

\[
\sum_{n=0}^{m} F_n z^{n-1} [(2-n)z - n]
\]

(14)

which is the model independent deceleration parameter in a redshift based \( f(R) \) gravity.

The present time deceleration parameter for a general \( F(z) \) obtains as:

\[
q_0 \equiv q(z = 0) = -1 + \frac{\alpha + \beta}{1 - \alpha/2},
\]

(15)

where \( \alpha = F_1/F_0, \beta = F_2/F_0 \) which shows that the present value of deceleration parameter depends on, only the first two terms coefficients of Taylor expansion of \( F(z) \). Since the present value of deceleration parameter should be negative, \( q_0 < 0 \), \( \alpha \) and \( \beta \) will constrained by \( 3\alpha/2 < 1 - \beta \). In the early time, \( z \to \infty \) the only remained terms in the numerator and denominator of the right hand side of Eq. (14) are the terms with higher powers in \( z \) as:

\[
q_\infty(m) \equiv q(z \to \infty) = -1 + \frac{m(m+1)}{2-m},
\]

(16)

which is independent of coefficients of Taylor expansion of \( F(z) \). Here we see that \( q_\infty \) depends on the number of the sentences in Taylor expansion. For example if \( m = 0 \) (one
term in Taylor expansion), we will have a constant modification, which will recover de’ Sitter space, \( q_\infty(0) = -1 \). If \( m = 1 \) (two terms in Taylor expansion), the value of deceleration parameter is \( q_\infty(1) = +1 \), which is consistent with decelerating Universe after inflationary epoch. If \( m = 2 \) (three terms in Taylor expansion), we have \( q_\infty(2) \rightarrow \pm \infty \), which is an unstable value for deceleration parameter. Eventually for any values of \( m \geq 3 \), the early time value of deceleration parameter is a negative number, \( q_\infty(m \geq 3) < -1 \), which shows a continues inflationary behavior. One can show that adding any extra term higher than first order term in Taylor expansion can make an unstable epoch in evolution of deceleration parameter. But location of this instability which we may call it, inflation, depends on the value of coefficients of Taylor expansion. Dependency of \( q_\infty \) on the number of sentences in Taylor expansion of \( F(z) \) is plotted in Fig. (1). As a very simple but phenomenological case we can assume there are four nonzero terms in Taylor expansion as:

\[
F(z) = F_0(1 + \alpha z + \beta z^2 + \gamma z^3),
\]

where \( \gamma = F_3/F_0 \). Depend on choosing expansion coefficients we can see evolutionary behavior of deceleration parameter with respect to redshift. There has plotted three different cases for early time period in Figs. (2-4). The common result of three above figures is that, there is a long era between early time and late time evolution of deceleration parameter in which \( q(z) \) is equal to +1 in each figure. This period is called radiation dominated era. Before that in higher \( z \)'s there are different instabilities. The other common point is that they desire to have an acceleration phase transition point in late time Universe. In Fig. (2)
\( \beta = \pm 10^{-12} \) and \( \gamma = 0 \). For the positive sign (solid line) of \( \beta \) the Universe may begin from an infinite decelerating state which is continued to the end of radiation dominated era. For the negative case (dashed line) the Universe may begin from an infinite accelerating state which is continued to the beginning of radiation dominated era. There is one acceleration transition point in early time in this latter case. In Fig. (3) we choose \( \beta = 0 \) and \( \gamma = \pm 10^{-25} \), then for the positive sign (solid line) the Universe may begin from a finite accelerating state. But it is faced to an instability in a identified redshift which is called inflation. Therefor deceleration parameter goes to \(-\infty\) and then comes back to a finite value from \(+\infty\). For the negative case (dashed line) the Universe again may begin from the same infinite accelerating state which goes to the transition point in a continuous state up to join to beginning of the radiation era. In Fig (4) there are two cases of composition of values of coefficients. In the first one \( \beta = +10^{-12} \) and \( \gamma = -10^{-25} \) (solid line) which shows that the Universe may begin from a finite accelerating state and then goes to a finite maximum deceleration value and then comes back to radiation era. There could not be an inflationary period in this case. While in the other case \( \beta = -10^{-12} \) and \( \gamma = -10^{-25} \) (dashed line) there could be an stable inflation. Here we saw that smallness of the coefficients \( \beta \) and \( \gamma \) may move any inflationary behavior to early Universe. However this example of composition of expansion of coefficients are nor realistic but will make this sense that we can find a close combination which might be compatible with number of e-folding and other early time cosmological constraints. The other main late time cosmological constraint which is tested for the above simple example
FIG. 3. Evolution of deceleration parameter which is introduced in Eq. (17) for $\beta = 0$, $\gamma > 0$ solid line and $\gamma < 0$ dashed line.

FIG. 4. Evolution of deceleration parameter which is introduced in Eq. (17) for the case $\beta > 0$ and $\gamma < 0$ solid line and the case $\beta < 0$ and $\gamma < 0$ dashed line.

is SNeIa distance module with respect to redshift constraint. In this comparison we used the Union2 data set [9] which is provide 557 SNeIa specifications. According to the value of numerical results which was taken in comparison with SNeIa data set from usual $\chi^2$ algorithm, we obtain that for $\alpha = 0.41$, $h = 0.7$ and no need to dark companions of the Universe, $\chi^2 = 546.48$ and the ratio of chi square error to the number of freedom is about 0.98. The result of this calculation is shown in Fig. (5). According to the obtained value for of parameter $\alpha$, evolution of deceleration parameter for low redshift Universe is plotted in Fig. (6). Other cosmological parameters such as jerk, snap and lerk is calculated for this combination of coefficients. The obtained values of present cosmological parameters
FIG. 5. Accordance between Union2 SNeIa data and late time approximation of Eq. \((\ref{eq:17})\) for 
\(\alpha = 0.41\) and \(h = 0.7\).

![Graph showing Accordance between Union2 SNeIa data and late time approximation of Eq. \(\alpha = 0.41\) and \(h = 0.7\).]

FIG. 6. Late time evolution of deceleration parameter for \(\alpha = 0.41\) and \(h = 0.7\).

\((q_0, j_0, s_0, l_0)\) are \((-0.48, 0.37, -0.03, 0.71)\) while their values at the beginning of radiation era
\((q_r, j_r, s_r, l_r)\) obtain as \((1, 3, -15, 105)\). The same values of present cosmological parameters
from \(\Lambda CDM\) for \(\Omega_M = 0.3\) and \(h = 0.72\) is \((-0.55, 1.0, -0.35, 3.11)\). In this letter we have
considered a general effect of changing the gravitational action on cosmological deceleration
parameter, from \(R\) to \(f(R)\). Here we have obtained that all the cosmic history may describe
by knowing effect of any sentences in Taylor expansion of a generic function \(F(z)\). The
first term which is a constant will recover a de' Sitter Universe, while the first two terms
have cover all dynamical properties of the Universe from radiation era to late accelerating
Universe. No one extra term will affect on the present value of deceleration parameter.
Adding the third term to the last two terms may make an inflationary behavior, in which
deceleration parameter may goes back to a huge positive or negative amount. Adding any
other terms to the last three terms may change the value of deceleration parameter from an
unstable value to a stable but negative value in very early Universe. It means that in \( f(R) \)
theory of gravity in the absence of matter, the value of deceleration parameter was not a
positive value which denotes there was a low inflationary era before main inflation. Adding
more terms to the last four terms do not change whole the story, but will vary the value of
deceleration parameter to a larger value in negative direction. Thus for an infinite number
of terms in Taylor expansion of \( F(z) \) the value of deceleration parameter will goes to \(-\infty\).
It means the Universe is born in a main inflationary state. Then if the number of Taylor
expansion sentences should be finite whether the Universe was born in an inflationary state.
However the value of expansion coefficients may make some fluctuations between the first
moment of creation and the beginning of radiation era in \( f(R) \) gravity theory.

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