Self-organization of collective oscillation in many body systems

Hidetoshi Morita\textsuperscript{1,2} and Kunihiko Kaneko\textsuperscript{2,3}

\textsuperscript{1} Present address: Faculty of Science and Engineering, Waseda University, Shinjuku-ku, Tokyo 169-8555, Japan
\textsuperscript{2} Department of Basic Science, The University of Tokyo, Komaba, Meguro-ku, Tokyo 153-8902, Japan
\textsuperscript{3} ERATO Complex Systems Biology Project, JST, Meguro-ku, Tokyo 153-8902, Japan

E-mail: morita@aoni.waseda.jp

Abstract. We briefly report the recent discovery of collective oscillation of macroscopic variables in a metastable state in some Hamiltonian systems, using the mean field XY model as an example. This periodic or quasiperiodic collective motion appears through Hopf bifurcation. The origin of the oscillation is explained, with self-consistent analysis of the distribution function, as the self-organization of a self-excited “swing” state through the mean field.

1. Introduction
Self-organization of macroscopic pattern in many body systems is an issue of common importance among astrophysics and condensed matter physics. In the former field, the formation of galaxy or some other huge structures that emerges through gravitational interaction among stars is an example of collective dynamics. In the latter field, on the other hand, the so-called dissipative structures far from equilibrium has gathered much attention of physicists. Here we note that such low-dimensional temporal (or spatio-temporal) pattern at a macroscopic level results from microscopic (molecular) high-dimensional chaos. Such low-dimensional motion, which we here refer to as collective motion, has been intensively studied \cite{1, 2, 3, 4}, but the study so far is restricted to dissipative systems.

Is such collective motion possible in Hamiltonian systems? According to thermodynamics, the equilibrium state is expected to be reached, where no macroscopic temporal pattern is observed. Thus the collective motion can exist only in the transient. The authors have actually discovered such collective motion in some Hamiltonian systems, which is observed in a metastable state during the relaxation to equilibrium\cite{7}. Here, we give a very short report on the essence of this recent result, focusing on a single model as an example.

2. Model
We adopt the Hamiltonian system of XY model with mean field coupling \cite{8, 9, 5}:

\[ H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} [1 - \cos(\theta_i - \theta_j)]. \] (1)
Note that this model can be regarded as the first Fourier component of the gravity sheet model. \(N\) pendula interact with each other through phase differences, rotating at a higher energy and librating at a lower energy. The thermal equilibrium state is analytically obtained in the thermodynamic limit \((N \to \infty)\); it is determined only by the total energy density \(U = \mathcal{H}/N\), and shows continuous phase transitions at \(U_c = 0.75\) (corresponding temperature \(T_c = 0.5\)).

As macroscopic variables, we consider the variance of momentum, \(T(t)\), and the modulus of the mean field, \(M(t)\), respectively defined as,

\[
T(t) \overset{\text{def}}{=} \frac{1}{N} \sum_{j=1}^{N} p_j(t)^2, \quad M(t)e^{i\phi(t)} \overset{\text{def}}{=} \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j(t)}. \quad (2)
\]

In equilibrium, they are nothing but the temperature and the magnetization of the system, respectively. Since (1) yields the constraint, \(2U = T(t) + 1 - M(t)^2\), we focus on the dynamics of \(M(t)\) in the following. Note that, using (2), the equations of motion are described as a single pendulum interacting with the mean field:

\[
\dot{\theta}_j = p_j, \quad \dot{p}_j = -M(t) \sin(\theta_j - \phi(t)). \quad (3)
\]

We assign initial conditions of \(\{\theta_i\}\) and \(\{p_i\}\) with only two macroscopic parameters: total energy density \(U\) and initial magnetization \(M_0 = M(0)\). The initial distribution is factorized into two parts, \(f_\theta^0(\theta, p) = f_\theta^0(\theta) f_p^0(p)\). First, \(f_\theta^0\) is set as a Boltzmann distribution determined with the temperature \(T_{eq}(M_0)\), where \(T_{eq}(M)\) is the equation of state in equilibrium:

\[
f_\theta^0(\theta; M_0) = \frac{1}{Z_\theta(M_0)} \exp \left[ \frac{M_0}{T_{eq}(M_0)} \cos \theta \right], \quad (4)
\]

where \(Z_\theta(M_0)\) is the normalization. Next, \(f_p^0\) is set as a Maxwellian determined with the temperature \(T_0(U, M_0) = 2U - 1 + M_0^2\), to fulfill the given \(U\):

\[
f_p^0(p; U, M_0) = \frac{1}{Z_p(U, M_0)} \exp \left[ -\frac{p^2}{2T_0(U, M_0)} \right], \quad (5)
\]

where \(Z_p(U, M_0)\) is the normalization. If \(M_0\) is set at the equilibrium value for the given \(U\), then \(T_{eq} = T_0\), and the above distribution yields the correct equilibrium one. Since \(T_{eq} \neq T_0\) in general, however, the initial state is set far away from equilibrium, while this initial distribution is smoothly connected to the equilibrium one on the \((U, M_0)\) plane.

3. Macroscopics
A typical time series of \(M(t)\) in the relaxation to equilibrium is shown in Figure 1. As shown, \(M(t)\) does not simply decay exponentially, but stays at a metastable state over a long time; the duration of this state increases linearly with \(N\) [7]. After such long interval, \(M(t)\) departs again from the plateau, and finally reaches the equilibrium value, fluctuating around it [10].

The close-up time series of \(M(t)\) in the metastable state (Figure 2) shows periodic oscillation. This oscillation is not due to the finiteness of \(N\); indeed, it is more apparent with increasing \(N\). Moreover, its power spectrum has a large peak at the corresponding frequency, which remains sharp with increasing \(N\) [7]. This is in strong contrast with thermal fluctuations around equilibrium, which reduces to zero in the thermodynamic limit. The periodic motion survives even in the thermodynamic limit.

Depending on the parameter \((U, M_0)\), \(M(t)\) takes other temporal patterns than the simple periodic motion. In the region farther from equilibrium, it shows \(T^2\) quasiperiodic motion, while in the region nearer to equilibrium, it behaves almost stationary.
Figure 1. A time series of $M(t)$. The dotted line is the equilibrium value. $N = 10^5$, $U = 0.67$ and $M_0 = 1$.

Figure 2. A time series of $M(t)$ in the metastable state. $U = 0.5$ and $M_0 = 0.8$.

Figure 3. The mean amplitude of $M(t)$ against $U$, in the vicinity of the bifurcation point $U_b = 0.42$, obtained by fitting the data. The broken line indicates $\sim (U - U_b)^{1/2}$. 25 samples for each $U$. $N = 10^5$, $M_0 = 0.8$.

Figure 4. A snapshot of the contour of the one-body distribution function, $f(\theta, p, t)$, at $t = 1006.1$ of Figure 2. The dash-dotted line indicates the separatrix.

Thus, with increasing $U$ for example, the temporal pattern of the macroscopic variable “bifurcates” as, stationary $\rightarrow$ periodic $\rightarrow$ quasiperiodic. Note that this is similar to the Hopf bifurcation from fixed point $\rightarrow$ limit cycle $\rightarrow$ torus, which is typically seen in low-dimensional dissipative dynamical systems. Hence it is suggested that the present bifurcation of the collective motion is described as that of low-dimensional dynamical systems, in particular by the Hopf bifurcation. Indeed, the amplitude of the oscillation shows the dependence of $(U - U_b)^{1/2}$ in the vicinity of the bifurcation (Figure 3), where $U_b$ is the critical energy, which verifies that the bifurcation from stationary to periodic motion is actually a Hopf type.

4. Microscopics
To search for the origin of the collective periodic motion, we study the microscopic dynamical system. Figure 4 shows a snapshot of the one-body distribution function $f(\theta, p, t)$ when the phase of the oscillation of $M(t)$ is zero. There is a pair of high density regions at $((\theta - \phi)/2\pi, p) \approx (\mp 0.3, \pm 1)$, besides the expected one at the center. The pair of peaks rotates
clockwise along the separatrix, keeping its localization. With this rotation of high density regions the oscillation of macroscopic variables appears. Now the problem is how this localized distribution is sustained.

Here, for comparison, we consider a population of independent pendula, parametrically driven with the periodic external force, $M_{\text{ext}}(t) = g + h \sin \Omega t$:

$$\dot{\theta}_j = p_j, \quad \dot{p}_j = -M_{\text{ext}}(t) \sin \theta_j,$$  \hspace{1cm} (6)

This system is equivalent to the original Hamiltonian system (3) if $M_{\text{int}}(t) = M_{\text{ext}}(t)$, where $M_{\text{int}}(t)$ is the internal mean field given by (2). The Poincaré section of (6) on $\Omega t \equiv 0 \pmod{2\pi}$ shows islands of 1:1 resonance at $(\theta/2\pi, p) \approx (\mp 0.3, \pm 1)$. Note that this resonance region agrees with the region of the concentrated densities in Figure 4.

On the basis of the above correspondence, we make a self-consistent explanation on the emergence of the collective periodic motion. Once a considerable number of elements exist in the islands of 1:1 resonance, $M(t)$ changes periodically in time. Then $M(t)$ in turn makes the elements to stay in the island stably. Hence the collective periodic motion is the state of self-excitation through the mean field, or self-excited “swing,” which is self-organized in the transient process to equilibrium.

The bifurcation from the stationary to the periodic state is also explained in the above picture. Note that an element requires not a little energy to stay in the island of 1:1 resonance. If the total energy is small, only a small number of elements can have such energy, which is too short to make $M(t)$ stable. With increasing the total energy, more and more elements are able to have enough energy to stay the island, which finally leads to stable periodic motion of $M(t)$.

5. Conclusion

Using the Hamiltonian system of the mean-field XY model, we have discovered the collective oscillation of the macroscopic variables. It is observed at the metastable state in the relaxation to equilibrium, the lifetime of which diverges in the thermodynamic limit. The oscillation appears through Hopf bifurcation, similarly to low-dimensional dissipative dynamical systems. Such temporal pattern results from the self-organization of the self-excited “swing” state.

Although we have focused on the mean field model here, the above results are preserved in long-ranged coupling models on $d$-dimensional lattices with the interaction strength decreasing as $r^{-\alpha}$ for $\alpha < d$, $r$ being the distance on the lattice [7]. Moreover, such collective oscillation is discussed to be universal for systems with long-ranged interaction and phase transition [7]. It is interesting to search for this type of collective motion in gravitational many-body systems also.

Acknowledgments

The authors thank Kensuke S. Ikeda and Yoshiyuki Y. Yamaguchi for discussion. This work was supported by a Grant-in-Aid for Scientific Research from MEXT Japan.

6. References

[1] K. Kaneko, Phys. Rev. Lett. 65, 1391 (1990), Physica D 55, 368 (1992).
[2] H. Chaté and P. Manneville, Prog. Theor. Phys. 87, 1 (1992).
[3] G. Perez and H. A. Cerdeira, Phys. Rev. A 46, 7492 (1992); N. Nakagawa and Y. Kuramoto, Prog. Theor. Phys. 89, 313 (1993); A. S. Pikovsky and J. Kurths, Phys. Rev. Lett. 72, 1644 (1994).
[4] T. Shibata and K. Kaneko, Phys. Rev. Lett. 81, 4116 (1998b); T. Shibata, T. Chawanya, and K. Kaneko, Phys. Rev. Lett. 82, 4424 (1999).
[5] M. Antoni and S. Ruffo, Phys. Rev. E 52, 2361 (1995).
[6] H. Morita and K. Kaneko, Europhys. Lett. 66, 198 (2004).
[7] H. Morita and K. Kaneko, submitted, cond-mat/0506261; in preparation.
[8] T. Konishi and K. Kaneko, J. Phys. A 25, 6283 (1992).
[9] S. Inagaki, Prog. Theor. Phys. 90, 577 (1993).

[10] This metastable state is different from the quasi-stationary state reported in [5]. See also, for example, recent papers; V. Latora, and A. Rapisarda, Physica D 193, 315 (2004); Y. Y. Yamaguchi et al., Physica A 337, 36 (2004); and references therein. The difference is discussed in [7].