Casimir interactions from multiple scatterings in dielectric media

Giuseppe Bimonte$^{1,2}$, Thorsten Emig$^3$

$^1$Dipartimento di Fisica E. Pancini, Universitá di Napoli Federico II, Complesso Universitario di Monte S. Angelo, Via Cinthia, I-80126 Napoli, Italy

$^2$INFN Sezione di Napoli, I-80126 Napoli, Italy

$^3$Laboratoire de Physique Théorique et Modèles Statistiques, CNRS UMR 8626, Bât. 530, Université Paris-Sud/Saclay, 91405 Orsay cedex, France

Fluctuation induced forces are a hallmark of the interplay of fluctuations and geometry. Recent measurements of Casimir forces have provided evidence of an intricate modification of quantum fluctuations of the electromagnetic field in complex geometries. Here we demonstrate the existence of a family of exact representations of Casimir interactions between bodies of arbitrary shape and material composition, admitting an expansion as a sequence of inter- and intra-body multiple wave scatterings. We show that interactions in complex geometries can be understood and computed within current experimental resolution from typically a few wave scatterings, notably without any a-priori knowledge of the scattering amplitudes of the bodies. Applications to various shapes composed of different materials offer novel insights into the emergence of Casimir forces from the induction of currents and diffraction of waves. Our results provide a systematic analytical framework for exploring Casimir effects in previously theoretically inaccessible configurations.
Introduction

A striking feature of quantum electrodynamics is that quantum fluctuations of the electromagnetic (EM) field can interact with matter to produce measurable long range interactions between atomic particles and consequently macroscopic bodies. A spectacular manifestation of this interaction is the attractive force between two perfectly conducting plates predicted by Casimir\textsuperscript{1}. The connection between an atomistic description and non-ideal macroscopic dielectric materials was established by Lifshitz who considered random currents within the interacting bodies to obtain the Casimir force between planar bodies\textsuperscript{2}. In the past decades, this approach has been the core theory for interpreting most of the precision measurements of Casimir interactions between various materials and surface shapes which were enabled by an enormous progress in force sensing techniques and the fabrication of nano-structures\textsuperscript{3–11}. Importantly, measurements almost always involve bodies with non-planar surfaces, either to avoid difficulties in parallel alignment or to intentionally detect geometry effects. Consequently, Lifshitz’s planar surface model is combined with the proximity force approximation (PFA) to obtain a force estimate by decomposing the surfaces into pairs of small and parallel patches\textsuperscript{12}. A breakdown of this approach is expected theoretically with increasing separation as it ignores complex diffraction effects and the non-additivity of Casimir forces, and hence their complex geometry dependence. Indeed, recent experiments have demonstrated large deviations from PFA\textsuperscript{13–15}, making theoretical formulations for a precise force computation highly desirable. This is all the more true as for potential applications in nano-mechanical systems the interplay between material properties and geometry needs to become predictable.
To date, enormous efforts have been put forward by many groups to develop theoretical and numerical methods that can cope with more general surface shapes\textsuperscript{16,17}. Specifically, the scattering method\textsuperscript{18–20}, originally devised for mirrors\textsuperscript{21,22}, expresses the interaction between dielectric bodies in terms of their scattering amplitude, known as T-operator. While this approach has enabled most of recent theoretical progress, the T-operator is known only for highly symmetric bodies, such as sphere and cylinder, or for a few perfectly conducting shapes\textsuperscript{23}, practically exhausting this method. In principle, the approach can be augmented by advanced numerical methods, for example for gratings\textsuperscript{24}, but they can be limited by computational power required for convergence. A more fundamental limitation is that interlocked geometries evade this method due to lack of convergence of the mode expansion\textsuperscript{15}. For gently curved surfaces a gradient expansion yields first order corrections to the PFA\textsuperscript{25,26}. The theoretical treatment of non-ideal materials with sharp surface features, such as used in atomic force microscopy or fabricated by lithographical techniques, is beyond the scope of existing methods. A notable exception is a more recent, powerful fully numerical scheme based on a boundary element method (SCUFF-EM) for computing the interaction of fluctuating surface currents\textsuperscript{27,28}. It is believed that this approach can provide in principle the exact force for arbitrary shapes, with computational power the only but practically important limiting factor\textsuperscript{15}. Moreover, this method depends on a suitable refinement of the surface mesh for a broad band of relevant wave lengths, making discretization error estimates challenging. To our knowledge, complementary, not fully numerical methods with comparably broad application range do not exist to date.

The main aim of our work is to overcome the difficulties residing in the existing scattering
method: The lack of knowledge of T-operators for arbitrary shapes and the limited convergence of the corresponding partial wave expansions. The idea of our approach is to describe the statistical properties of the fluctuating EM field in the presence of material bodies by treating the back and forth scatterings of waves between different objects on an equal footing as the scatterings within an isolated object, without making use of the concept of a T-operator. In simple words, a wave propagates freely between successive scattering points on the surfaces, no matter if the points belong to different objects or the same object. This natural physical picture gives rise to an expansion in the number of scattering points, where each scattering is treated exactly without resorting to any expansion in a partial wave basis. Motivation for this approach comes from the observation that contributions to the Casimir interaction from scatterings between points belonging to different objects decrease rapidly in the number of scattering points. For example, for parallel plates a two-scattering approximation reproduces Casimir’s original result within 8% \(^29\) and it is accurate to within even 1% for less aligned geometries \(^23\). There is no reason to expect that an expansion with respect to scattering points belonging to the same object is less effective. In fact, in a seminal work Balian and Duplantier demonstrated the very existence and convergence of a multiple scattering expansion (MSE) for Casimir forces which however is limited to perfect conductors \(^{30,31}\).

More than four decades later, here we demonstrate the existence of a MSE for Casimir interactions for arbitrary dissipative magneto-dielectric materials. Our physically intuitive description uses the fluctuation-dissipation theorem \(^{32}\) to express the equal-time statistical correlations of the EM field in terms of a class of surface scattering operators which provide the probability distribution of tangential fields at the bodies’ surfaces which can be interpreted as surface currents and
are in one-to-one correspondence with the macroscopic EM field configurations exterior and interior to the bodies. Validation examples show rapid convergence in the number of scatterings even at short surface separations. We apply our MSE to a planar gold surface and a silicon wedge to demonstrate effective performance for notoriously difficult to treat materials with sharp edges, revealing fast convergence with precise error estimates, a breakdown of PFA with increasing sharpness, anomalously large thermal effects at short distances, a large influence of optical properties of materials and good agreement with fully numerical SCUFF-EM estimates for selected geometry. Our work represents a powerful approach to substantially extend accurate predictions of Casimir forces to materials and shapes for which only computationally intensive fully numerical methods were available.

Results

Multiple scattering of electromagnetic waves at dielectric surfaces. We consider Casimir forces in a configuration of \( N \) material bodies with dielectric and magnetic permittivities \( \epsilon_\sigma \) and \( \mu_\sigma \) (\( \sigma = 1, \ldots, N \)) in thermal equilibrium at temperature \( T \). The bodies are bounded by surfaces \( S_\sigma \) which can be of arbitrary shape and separate their bulk from the surrounding homogeneous medium with dielectric and magnetic permittivities \( \epsilon_0 \) and \( \mu_0 \). Our description of the Casimir interaction among these bodies can be summarized as follows. We first express the Casimir force \( F_\sigma \) on one of the bodies, labelled by \( \sigma \), as the integral of the expectation value of the electromagnetic (EM) stress tensor at discrete Matsubara imaginary frequencies \( \xi = i \omega \) with \( \xi = \xi_n = 2\pi n k_B T / \hbar \) with \( n = 0, 1, \ldots \), over the surface \( S_\sigma \) using the fluctuation-dissipation theorem. A divergence
Figure 1: **Multiple scattering expansion.** Diagrammatic representation of contributions to the MSE, shown in panel a for the scattering Green function $\Gamma (r, r')$ of a single body with source point $r'$ and observation point $r$, and in panel b for the Casimir energy between two bodies. In the displayed examples, lines with arrows represent free propagation between surface points of the same body (blue lines) and to external points or between surface points of different bodies (magenta lines). Each free propagation between two surface points, followed by a scattering, is described by a surface operator $K_{\sigma \sigma'}$. The bodies have dielectric and magnetic permittivities $\epsilon_1, \mu_1$ and $\epsilon_2, \mu_2$, respectively, and they are surrounded by a medium with permittivities $\epsilon_0, \mu_0$. $G_0$ is the free Green tensor of the surrounding medium, and $M$ described the tangential surface components of the incident field generated by a source at position $r'$. 
in the surface integral, originating from the empty space stress tensor and hence unrelated to the Casimir force, is readily removed by replacing the $N$-body EM Green tensor $G$ by the scattering Green tensor $\Gamma(r, r') = G(r, r') - G_0(r, r')$ where $G_0$ is the empty space Green tensor for a homogeneous medium with contrast $\epsilon_0, \mu_0$ (see Methods). Physically, $\Gamma(r, r')$ describes the modification of the EM field at position $r$, due to the presence of the bodies, when it is generated by a source at position $r'$. This naturally implies to construct $\Gamma$ from the surface fields which are induced by an external source at the bodies. However, the primary current induced directly by the source induces in turn a secondary current, which induces again higher order currents, leading to an infinite sequence of induction processes. As we now demonstrate, an exact mathematical description of these processes is provided by our multiple scattering expansions (MSE) for $\Gamma$. While Green functions have been constructed in terms of surface currents, the existence of MSE between magneto-dielectric bodies is not obvious, particularly for Casimir interactions, and to the best of our knowledge had been demonstrated only for perfect electric conductors $^{29-31}$. The MSE is based on surface integral equations that determine the tangential electric and magnetic fields at the surfaces $S_\sigma$ which can be considered as magnetic surface currents $m_\sigma$ and electric surface currents $j_\sigma$, acting as equivalent sources for the scattered field $^{33}$. This can be viewed as a mathematical reformulation of Huygens principle. From the uniqueness of an EM field in a region specified by sources within the region and the tangential components of the field over the boundary of the region, one can construct the total EM field $(E, H)$ separately in the region external to the bodies, and in the $N$ interior regions of the bodies. When doing so, one can vary the field outside a given region at will as long as the surface currents are adjusted according to the jump conditions $j = n \times (H_+ - H_-), \quad m = -n \times (E_+ - E_-)$
where \( n \) is the surface normal pointing to the outside and the label \(+(-)\) indicates the value when surface is approached from the outside (inside). To proceed, we make the choice that the field outside a given region vanishes as this allows us to replace the magneto-dielectric media outside the region by the medium inside the region, so that the surface currents on the boundary of the region radiate in homogenous unbounded space. Hence the field can be expressed in the interior of the bodies as the surface integral 

\[
(E^{(\sigma)}, H^{(\sigma)})(r) = \int_{S_\sigma} ds_u G_\sigma(r, u)(j_{\sigma-}, m_{\sigma-})(u) \text{ where } G_\sigma \text{ in the free Green tensor in a medium with permittivities } \epsilon_\sigma, \mu_\sigma, \text{ and } j_{\sigma+} = -n_\sigma \times H_- , m_{\sigma-} = n_\sigma \times E_- \text{ are the tangential fields when } S_\sigma \text{ is approached from the inside of the bodies.}
\]

Exterior to the bodies the field 

\[
(E(0), H(0))(r) = \int dr' G_0(r, r')(J, M)(r') + \sum_{\sigma=1}^N \int_{S_\sigma} ds_u G_0(r, u)(j_{\sigma+}, m_{\sigma+})(u) \text{ where now } j_{\sigma+} = n_\sigma \times H_+ , m_{\sigma+} = -n_\sigma \times E_+ \text{ are the tangential fields when } S_\sigma \text{ is approached from the outside of the bodies and we assumed an external source of electric and magnetic currents } (J, M) \text{ outside the bodies to generate the incident field.}
\]

Surface integral equations for the surface fields follow by taking advantage of the property of the surface integrals that they are also defined when \( r \) is located on the surfaces and their corresponding value is the average of the limits taken from the inside and the outside \(^{34}\), and that one of the two limits vanishes by construction, leading to

\[
(m_{\sigma-}, -j_{\sigma-})(u) = 2n_\sigma(u) \times (E^{(\sigma)}, H^{(\sigma)})(u) \text{ and } (m_{\sigma+}, -j_{\sigma+})(u) = -2n_\sigma(u) \times (E(0), H(0))(u)
\]

for \( u \) located on surface \( S_\sigma \). Finally, to couple the interior and exterior solutions, we impose the usual continuity conditions on the tangential components of \( (E, H) \) at the interfaces between different media, leading to one unique set of surface currents \( (j_\sigma, m_\sigma) \equiv (j_{\sigma+}, m_{\sigma+}) = -(j_{\sigma-}, m_{\sigma-}) \).

The \( 4N \) surface integral equations constitute an overdetermined system for the \( 2N \) surface currents or tangential surface fields. Existence of a unique solution requires that only \( 2N \) equations are in-
dependent, agreeing with the number of constraints imposed by the continuity of the tangential field components. The additional $2N$ constraints, implicitly fulfilled by construction of the fields, must account for the unique relation between the tangential components of the electric and magnetic fields on both sides of the surfaces as specification of one of the two components determines a unique solution to the exterior and interior problems. For this reason, a consistent set of $2N$ integral equations with a unique solution can be obtained by taking linear combinations of the set of $2N$ equations involving $(E^{(σ)}, H^{(σ)})$ and the corresponding set involving $(E^{(0)}, H^{(0)})$ but not by considering only one of the two sets as this would ignore the coupling of the interior and exterior fields. In general, one can choose $4N$ suitable coefficients which form $2N$ diagonal $2 \times 2$ matrices $C^i_σ, C^e_σ$ acting on the two field components of the interior and exterior integral equations. To interpret the integral equations as successive scatterings, we introduce the surface scattering operators (SSOs) $K_{σσ'}(u, u')$ which describe free propagation from $u'$ on surface $S_{σ'}$ to $u$ on surface $S_σ$ and scattering at point $u$

$$K_{σσ'}(u, u') = 2 P(C^i_σ + C^e_σ)^{-1} n_σ(u) \times \left[ δ_{σσ'} C^i_σ G_σ(u, u') - C^e_σ G_0(u, u') \right], \quad P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(1)

acting on electric and magnetic tangential surface fields at $u'$ ($δ_{σσ'}$ is the Kronecker delta). With these SSOs the surface currents are determined in terms of the external source $(J, M)$ by the Fredholm integral equations

$$\sum_{σ' = 1}^{N} \int_{S_{σ'}} ds_{u'} \left[ 1 - K_{σσ'}(u, u') \right] \left( J_{σ'} \right) (u') = \int dr M_σ(u, r) (J_M)(r)$$

(2)

with

$$M_σ(u, r) = -2 P(C^i_σ + C^e_σ)^{-1} C^e_σ n_σ(u) \times G_0(u, r)$$

(3)
(For details on the derivation of the surface integral equations see Methods.) As we shall see, the choice of coefficients $C^i_\sigma$, $C^e_\sigma$ provides a powerful tool to engineer convergence of the MSE. Computation of the SSO requires integration of the free space Green tensor in homogenous media over the bodies surfaces which can be performed analytically in important cases, as we demonstrate below. Contributions to the Casimir energy from scatterings between remote surface positions are exponentially damped with distance as we need to consider the Green tensor only for purely imaginary frequencies.

The scattering Green tensor is determined by the field generated by the surface currents, and hence $\Gamma(r, r') = \int_{S_u} ds_u \int_{S_u'} ds_{u'} G_0(r, u)((1 - \kappa)^{-1}(u, u')M(u', r')$ where the integration extends over all surfaces $S_\sigma$ and a summation over all surface labels $\sigma$ is understood. The existence of a MSE follows from the Fredholm type of the operator $(1 - \kappa)^{-1}$ that permits an expansion in powers of $\kappa$ and hence in the number of scatterings, as illustrated for one body in Fig. 1(a). The regularized stress tensor involves only $\Gamma^{[33]}$, and it can be shown that the Casimir force on body $\sigma$ is determined solely by the SSO, expressed as a sum over Matsubara frequencies $\xi_n$ by $F_\sigma = k_B T \sum_{n=0}^{\infty} \text{Tr}[(1 - \kappa)^{-1}\nabla_{r_\sigma}\kappa]$ where $\nabla_{r_\sigma}$ is the gradient with respect to the position of the body, and the bare Casimir energy assumes the simple expression $E = k_B T \sum_{n=0}^{\infty} \text{Tr} \log(1 - \kappa)$, (where the primed sum gives a weight of 1/2 to the $n = 0$ term). Here the trace Tr involves a sum over vector indices of the electric and magnetic components and an integration over all surfaces. To gain insight into the structure of the MSE for the Casimir energy, we consider two bodies. After subtracting the self-energies, arising from isolated scatterings on a single body, the energy is
expressed in terms of four SSO as

\[ E = k_B T \sum_{n=0}^{\infty} \text{Tr} \log \left[ \mathbb{1} - (\mathbb{1} - K_{11})^{-1} K_{12} (\mathbb{1} - K_{22})^{-1} K_{21} \right]. \]  

(4)

We note that this formula provides the exact representation of the Casimir energy for all allowed choices of the coefficients \( C_i^e, C_e^e \) (see also next section). After expanding both the logarithm and the inverse operators in powers of the SSOs we obtain the MSE which involves at least one scattering on each body with closed paths going from body 1 to body 2 and back (\( K_{12} \) and \( K_{12} \)), possibly multiple times, and with an arbitrary number (including zero) of scatterings on each body (\( K_{11} \) and \( K_{22} \)), as illustrated in Fig. 1b. Comparison with scattering approaches relying on the knowledge of the bodies T-matrix shows that our MSE constructs the T-matrix in the number of scatterings on individual bodies by expanding \( (\mathbb{1} - K_{\sigma\sigma})^{-1} \), treating scatterings inside individual bodies and between them on an equal footing.

Previously, scatterings of EM waves at dielectric media have been described in terms of electric and magnetic surface currents for real frequencies, revealing sometimes poor convergence of expansions in the number of scatterings. However, since Casimir interactions can be formulated in terms of correlations of the EM field for purely imaginary frequencies, the exponential decay of Green tensors in separation can be expected to lead to rather fast convergence of the MSE for the scattering the Green function and the Casimir energy. This had been demonstrated only for perfect electric conductors, bases on a MSE that ignores the coupling between electric and magnetic surface currents\[^{30}\]. One remarkable property of this previous approach, the cancellation of an odd overall number of scatterings, is explained in retrospect by our general MSE by the observation that ignorance of the coupling leads to SSO with opposite signs for the electric and
magnetic components.

**Equivalent formulations.** With different interior coefficient matrices $C^i_{\sigma}$ and exterior coefficient matrices $C^e_{\sigma}$, the SSO form an equivalence class of operators in the sense that Eq. (2) yields the same surface currents for a given external source for all coefficients, as long as neither the interior nor the exterior matrices vanish for any $\sigma$, and the sum $C^i_{\sigma} + C^e_{\sigma}$ is invertible. Consequently, the scattering Green tensor and the Casimir energy must be also independent of the choice of coefficients. The surface currents and the Casimir energy at any finite order of the MSE, however, in general do depend on the chosen coefficients, and hence does the rate of convergence of the MSE. This remarkable property provides an effective method to optimize convergence for different permittivities and even frequencies by suitable adjustment of coefficients. Among the infinitely many choices there are a few which we consider important to discuss explicitly: (C1) In general, the SSO has a leading singularity that diverges as $1/|\mathbf{u} - \mathbf{u}'|^\gamma$ with $\gamma = 3$ when the two surface positions $\mathbf{u}$, $\mathbf{u}'$ approach each other. There exists a choice of coefficients\textsuperscript{34} however, for which the singularity is reduced to a weaker divergence with exponent $\gamma = 1$, presumably accelerating convergence. The coefficients are $C^i_{\sigma} = \text{diag}(\epsilon_{\sigma}, \mu_{\sigma})$ and $C^e_{\sigma} = \text{diag}(\epsilon_0, \mu_0)$. (C2) An asymmetric, material independent choice of coefficients is $C^i_{\sigma} = \text{diag}(1, 0)$ and $C^e_{\sigma} = \text{diag}(0, 1)$ which implies that the electric (magnetic) surface currents are determined by the exterior (interior) solution only. For good conductors, we have observed fast convergence of the MSE with this choice. (C3) Finally, we note that the singular choice with $C^i_{\sigma} + C^e_{\sigma} = 0$, which we excluded, does not yield a Fredholm integral equation and hence does not permit a MSE. The latter choice has been employed in a computationally intensive boundary element method\textsuperscript{27} implemented in the open-source software.
For the results in this work we shall employ the choice (C1).

Physically, the required relation between the tangential surface fields $\mathbf{n}_\sigma \times \mathbf{E}$ and $\mathbf{n}_\sigma \times \mathbf{H}$ is in general obeyed only approximately at any finite order of the MSE, with the approximation converging to the exact relation with increasing MSE order. Indeed, at first order, $\mathbf{E}$ and $\mathbf{H}$ of the incident field are rescaled differently at each body by the chosen coefficients $C^i_\sigma, C^e_\sigma$, see Eq. (3). The coefficients hence set the initial field for the MSE iteration and they control how the exact tangential surface fields are build up successively by the MSE.

**Convergence of the expansion in the number of scatterings.** It is instructive to demonstrate convergence of the MSE for shapes for which the scattering T-matrix is known exactly. The interaction between two planar and parallel surfaces is determined by their Fresnel coefficients according to the Lifshitz formula. The majority of experiments measure forces between gold (Au) and/or doped silicon (Si) surfaces, and hence we consider these materials in the following applications of the MSE (for material parameters, see Methods). When the SSO $K_{11}$ describes the scatterings on the Si surface, expansion of the energy in Eq. (3) in this SSO yields MSE approximants to the Casimir interaction. MSE orders are labelled by $\text{MSE}_{kl}$ where $2(k + 1)$ is the number of scatterings between the surfaces (total number of $K_{12}$ and $K_{21}$ operators) and $l$ is the number of single-body scatterings on the Si surface (number of $K_{11}$ operators). Figure 2 shows the energy for eight different orders of MSE relative to the known exact energy at $T = 300\text{K}$ for surface separations between 100nm and $1\mu\text{m}$. While the lowest order MSE$_{00}$ with no single-body scattering on the Si surface yields already between 70% and 87% of the exact interaction, only 4 scatterings
Figure 2: Multiple scattering expansion of the Casimir energy between a silicon plate and a gold plate. Different orders of the MSE for the Casimir energy between a plate made of doped silicon and a plate made of gold (see Methods for material details), normalized to the known exact energy. Indices of MSE$_{kl}$ label the number of scatterings between the plates ($2(k+1)$) and within the silicon plate ($l$) (see text for details).
between the surfaces ($k = 1$) and 2 single-body scatterings on the Si surface ($l = 2$) are required for an accuracy of about 1%. This validation example demonstrates fast convergence of our MSE, with good homogeneity in separation.

A second validation example involves the scattering Green function $\Gamma$ for a dielectric cylinder which is fully specified by the scattering T-operator $\mathbb{T}$ of the cylinder. The latter is known exactly and constitutes the only exact result for a curved dielectric body which couples electric and magnetic polarizations upon scattering $^{37}$. When $\mathbb{T}$ is known, one can use the relation $^{35}$

$$\Gamma(r, r') = \int d\tilde{r} \int d\tilde{r}' G_0(r, \tilde{r}) \mathbb{T}(\tilde{r}, \tilde{r}') G_0(\tilde{r}', r')$$

to compute the components of $\Gamma$ in a partial wave expansion of $G_0$ where the integrations now extend over the volume of the cylinder. Specifically, vector cylindrical waves are a convenient choice to obtain the SSO $K_{11}$ of the cylinder and to extract from the MSE for $\Gamma$ the T-operator elements $T^{\alpha\alpha'}(m, \kappa, k_z)$ for $\alpha, \alpha' \in \{E, H\}$, the imaginary wave number $\kappa = \xi/c$, the wave vector $k_z$ along the cylinder axis and the angular quantum number $m$ (see Methods for details). The panels in Fig. 3 display interesting aspects of the convergence of the approximant for $T^{\alpha\alpha'}$ with the MSE $(1 - K_{11})^{-1} = \sum_{n=0}^{p} K_{11}^{p}$ for order $p = 3$. The contour plots show the ratio of the approximant and the exact T-operator elements for $m = 0, 1$ as a function of the dimensionless wave numbers $\kappa R, k_z R$ for a cylinder of radius $R$ and permittivities $\varepsilon_1 = 30$ and $\mu_1 = 1$. While at this low order overall convergence has reached already agreement of better than 85% with the exact result, the plots reveal a complex dependence of the convergence rate on wave numbers. Typically convergence accelerates with decreasing frequency scale $\kappa$ and increasing wave number $k_z$, with the exception of lowest $m = 0$ elements which show slow convergence around the static, long wave length limits $\kappa = k_z = 0$. This slow down can be understood from
Figure 3: Multiple scattering expansion of the scattering Green function of a dielectric cylinder. Contour plots of the ratio of the T-matrix elements of a dielectric cylinder of radius $R$ computed with the MSE to order $p = 3$ and the exact results, as a function of rescaled imaginary frequency $\kappa R$ and the rescaled wave vector $k_z R$ along the cylinder axis. The dielectric permittivities of the cylinder are $\epsilon = 30$ and $\mu = 1$. Shown are the lowest order T-matrix elements with angular quantum numbers $m = 0, 1$ for all four combinations of polarizations E and H. (For $m = 0$ the polarization couplings $T^{EH}, T^{HE}$ vanish.)
the presence of a logarithmic divergence in $\mathbb{T}$ for $m = 0$ which is a consequence of the infinite length of the cylinder. The observation of fast convergence of the MSE for $\mathbb{T}$ is important as it determines directly the Casimir-Polder interaction between a surface and a polarizable particle.

**Surface with sharp edge: interaction between a silicon wedge and a planar gold surface.**

We illustrate the power of our approach by applying the MSE to compute the Casimir interaction between a gold (Au) plate and wedge made of doped silicon (Si), both at zero temperature and at room temperature ($T = 300$K). The wedge is aligned symmetrically with the normal to the plate and its wings form an angle $\theta$ with the plate, see Fig. 4(a). To date, the only in principle applicable method to compute Casimir forces for this setup is a fully numerical boundary element discretization (SCUFF-EM). We note that the T-matrix of a wedge is only known for a perfect metal for which dimensional analysis implies that the interaction energy per length $L$ of the edge of the wedge at zero temperature is $E/L = -f(\theta)\hbar c/d^2$ at separation $d$, with a dimensionless function $f(\theta)$ which has been computed previously. Here, our system contains additional length scales such as the thermal wavelength $\lambda_T = \hbar c/k_B T \approx 7.6\, \mu$m at $T = 300$K and various length scales characterizing the optical properties of the materials Au and Si. Consequently, the interaction energy is expected to assume a more complex form that can no longer be parametrized by a single function of $\theta$ only. By considering simple plane waves tangential to the wings of the wedge, the matrix elements of the (non-diagonal) SSO $\mathbb{K}_{11}(\theta)$ of the wedge can be computed analytically for any magneto-dielectric material (see Supplementary Information). The MSE amounts to evaluate Eq. (4) by expanding the logarithm and $[\mathbb{1} - \mathbb{K}_{11}(\theta)]^{-1}$ into a series of iterated integrals of the operator $\Delta \mathbb{K}_{11}(\theta) = \mathbb{K}_{11}(\theta) - \mathbb{K}_{11}(\theta = 0)$. Specifically, we perform numerical integrations up
Figure 4: Casimir interaction between a silicon wedge and a gold plate. a Schematic of the wedge-plate configuration with edge-plate separation $d$ and angle $\theta$ between plate and the sides of the wedge. Other panels b to d show the Casimir energy of the wedge-plate configuration both at zero temperature and at room temperature ($T = 300K$) as a function of the separation $d$, obtained from the multiple scattering expansion (MSE), proximity force approximation (PFA) and pair-wise summation approximation (PWS), for three different angles $\theta = \pi/6$, $\theta = \pi/3$, and $\theta = 4\pi/9$. 
to orders MSE$_{04}$ and MSE$_{12}$ for two and four scatterings between plate and wedge, respectively. Fig. 4 shows the resulting attractive Casimir energy for the three angles $\theta = \pi/6$, $\pi/3$ and $4\pi/9$ for experimentally important separations below 1 $\mu$m. The interaction shows several notable features. First, a comparison with the PFA and another often used approximation, the pair-wise summation (PWS) estimate, reveals a breakdown of these approximations even at surface proximity, unless the wings of the wedge open to a plane ($\theta = 0$). Indeed, the approximation error increases dramatically from small to large angles, reaching nearly a factor of 5 for $\theta = 4\pi/9$ at the largest studied separation of 1 $\mu$m and $T = 300K$. This signifies the importance of sharp surface features for Casimir interactions. Second, we observe anomalously large thermal effects, compared to parallel plates. Comparison of the panels of Fig. 4 shows that the difference between the interaction at zero and room temperature increases dramatically with $\theta$. The PFA and PWS approximation are unable to describe the thermal effects for large $\theta$. Consequently, thermal effects also modify substantially the crossover between the quantum regime at short distances and the classical interaction $E_{\text{cl}} \sim k_B T L/d$ induced by only thermal fluctuations which usually dominate for separations larger than $\lambda_T$. For the Au/Si parallel plates considered before, the classical contribution accounts for 45% of the total interaction at $d = 1 \mu$m. Differently, for the wedge-plate geometry importance of thermal effects emerges already at much shorter separations, with $E_{\text{cl}}$ contributing to the total energy 73%, 79% and 88% with increasing values of the three angles $\theta$ at $d = 1 \mu$m. This is presumably due to the increasing relative importance of low frequencies, arising from stronger diffraction at sharper edges.

Finally, it is instructive to compare our findings to the previously computed energy between
perfect metal (PM) wedge and plate [23] which for $T = 0K$ would be represented in the plots of Fig. (4) by a horizontal line equal to $f(\theta)$. The modification factor $\chi$, defined as the ratio of the energies for the Au-Si and the PM configurations, is shown in Fig. 5 as a function of surface separation for both temperatures. A number of interesting observations are made. First, at zero temperature, the comparison illustrates the effect of finite conductivity and real optical response. Interestingly, the effect is strongest for the sharpest wedge (reduction to 28% of PM energy at $d = 100$nm), while for smaller angles $\theta$ the reduction is surprisingly smaller than for parallel plates. We interpret this observation by a penetration of high frequency field fluctuations into the material and a resulting effective smoothening of the edge which is the stronger the sharper the edge is. For less sharp edges however, the effect is smaller compared to parallel plates due to an increased relative importance of low frequencies at which the material becomes more reflective. Second, to study the combined effect of thermal and material effects, we also computed the PM energy at $T = 300K$, and compared it to the Au-Si energy. We observe that thermal fluctuations reduce the effect of finite conductivity, leading to a larger factor $\chi$, increasing with separation due to the rapidly increasing importance of thermal effects. We note that for parallel plates thermal effects are much weaker with the factor $\chi$ being nearly equal for the two temperatures (see Fig. 5). This comparison to an ideal system shows that the combined effect of conductivity and thermal corrections are strongly shape dependent and important for any experimental study of Casimir forces in complex geometries.

We further explore the MSE for this setup by studying in more detail the convergence properties. Fig. 6 shows the relative contributions of the different scattering orders, summed up to order
Figure 5: **Influence of material properties and thermal fluctuations on the wedge-plate interaction.** Modification factor $\chi$ which compares Casimir energy $\mathcal{E}_{\text{Au-Si}}$ of Au-Si plate-wedge to energy for perfect metal (PM) plate-wedge, both at $T = 0\,\text{K}$ and $T = 300\,\text{K}$. Thin curves with open dots represent $\chi_{0\,\text{K}} = \mathcal{E}_{\text{Au-Si},0\,\text{K}}/\mathcal{E}_{\text{PM},0\,\text{K}}$, and bold curves with filled dots show $\chi_{300\,\text{K}} = \mathcal{E}_{\text{Au-Si},300\,\text{K}}/\mathcal{E}_{\text{PM},300\,\text{K}}$. See text for details.
Figure 6: **Convergence of the multiple scattering expansion for the wedge-plate configuration.** Partial sums of the Casimir energy up to multiple scattering order $\text{MSE}_{kl}$ where the number of scatterings between the wedge and the plate is $2(k + 1)$ and within the silicon wedge $l$. For the series $\text{MSE}_{0l}$ the extrapolation for the limit $l \to \infty$ is also shown ($\text{MSE}_{0\text{acc}}$). The partial energies are normalized to the final total energy estimate. 

- **a** Fastest convergence is observed for the angle $\theta = \pi/6$ as visualized by curve collapse for orders $\text{MSE}_{0l}$ and $\text{MSE}_{1l}$, yielding exact Casimir energy with an accuracy of better than 1%.

- **b** Slower convergence for the larger angle $\theta = \pi/3$ where however the series $\text{MSE}_{0l}$ with truncation at $l = 4$ has reached the extrapolated energy ($\text{MSE}_{0\text{acc}}$) with a maximal error of about 1%. The next order $\text{MSE}_{1l}$ has converged for $l = 2$ within 1% accuracy.

- **c** Slowest convergence for the sharpest wedge with $\theta = 4\pi/9$ with the final extrapolation ($\text{MSE}_{0\text{acc}}$) between 2% and 9% different from the truncation of $\text{MSE}_{0l}$ at $l = 4$. Again, the higher order $\text{MSE}_{1l}$ has converged for $l = 2$ within 1% accuracy.
to the final energies for $T = 300\text{K}$ shown in Fig. 4. Here the index $l$ now counts the powers of $\Delta k_{11}(\theta)$ in the MSE. Therefore, for the lowest orders of the inter-body scatterings, MSE$_{k_0}$, the geometry dependence arises only through the SSOs of planar surfaces, and $K_{12}$, $K_{21}$ which depend on $\theta$. In other words, these orders involve $2(k+1)$ round-trips between plate and wedge, assuming all single-body scatterings to be specular as if $\Delta k_{11}(\theta) = 0$. Interestingly, the two round-trip term MSE$_{00}$ accounts already for 88% to 92% of the total interaction for $\theta = \pi/6$, and for more than 65% for $\theta = 4\pi/9$. Upon inclusion of non-specular reflections to order $l = 4$ for two round-trips and $l = 2$ for four round-trips, we observe fast convergence towards the exact energy. The rate of convergence depends on $\theta$, reaching practically the exact energy for angles $\theta = \pi/6$, $\pi/3$, with error an estimate of about 1% at all separations. When the wedge approaches a knife edge for $\theta \to \pi/2$, convergence at short distances slows down. This is expected due to giant diffraction effects for very sharp edges. Surprisingly, even for the large angle $\theta = 4\pi/9$, the few considered scatterings are sufficient to estimate the final energy very precisely. This is due to a particularly interesting property of the expansion in $\Delta k_{11}(\theta)$. With increasing $\theta$, where error estimates become more important, the MSE develops an alternating sign in $l$. Specifically, while for $\theta = \pi/6$ all contributions are positive, for $\theta = \pi/3$ the contributions at orders MSE$_{02}$ and MSE$_{04}$ are negative, and for $\theta = 4\pi/9$ the contributions at orders MSE$_{02}$, MSE$_{04}$ and MSE$_{11}$ are negative. As a result, non-linear series acceleration methods, such as the Shanks transformation[^39], allow a precise estimate of the series limit. The extrapolated limits MSE$_{0\text{acc}}$ of the series MSE$_{0l}$ are displayed in Fig. 6(b) and (c). For the two smaller angles $\theta$ these limits are within 1% of the highest considered order MSE$_{04}$, demonstrating fast convergence. For $\theta = 4\pi/9$, extrapolation changes the energy
at MSE by maximally 9% at shortest distance, suggesting a maximum error of a few percent for the final estimate since the Shanks transformation leads to an extremely accurate solution for alternating series.

In addition, we compare our results to numerical tests which we perform with the boundary element bases SCUFF-EM open-source software\textsuperscript{36}. Due to the required high computational power and time, we consider only the geometry with $\theta = \pi/3$ at $d = 100$ nm. With a finer mesh adapted locally to the edge of the wedge, and using the same optical parameters as for our MSE, we compute contributions from 66 Matsubara frequencies for $T = 300$ K. The summed total energy agrees well with our MSE result, with a difference of 2.4% which is consistent with our error estimate. However, when we compare the SCUFF-EM and MSE results for each Matsubara term, we notice a slight increase of the relative difference with frequency. This we interpret as an indication of an increasing discretization error in SCUFF-EM due to limited mesh resolution.

**Discussion**

We describe a multiple scattering representation of the Casimir interaction between bodies of arbitrary shape and material composition, positioned relative to each other without geometrical constraints. A major conceptional difference with previously developed approaches is that our method does not depend on the scattering amplitude of the bodies. Moreover, our semi-analytical MSE does not involve a discrete mesh representation of the geometry and requires no numerical computation and inversion of large matrices over boundary elements, a computationally expensive task.
Within our scheme the Casimir energy is expressed by a converging sequence of iterated integrals, extended over the surfaces of the involved bodies. We demonstrate the power of the MSE by studying the Casimir interaction of a dielectric wedge opposite a conducting plate. Interactions in this geometry are particularly challenging to compute, due to the presence of a sharp edge and the unbounded range of surface-to-surface separations in this open configuration. Remarkably, we demonstrate that within the MSE the interaction can be computed very accurately at all separations, by considering a fairly small number of scatterings. As a result of the accuracy of our method, we can provide precise predictions for experiments. For instance, for a silicon wedge with an opening angle of 60 degrees and an edge length of 10 \( \mu \text{m} \), when placed at a distance of 100 nm from a gold surface, will experience a normal Casimir force of a few pN, which is within the current experimental resolution \(^{15}\).

Our work implies several directions for future work. The Casimir-Polder interaction of an atom and an arbitrarily shaped dielectric surface, including sealed cavities, can be obtained from our MSE for the scattering Green tensor. Various further applications can be envisaged, such as forces in increasingly complex nano-electromechanical systems, actuation forces in systems composed of interlocking bodies, and torques between symmetry-breaking objects. More precise predictions could be made for Hamaker constants of the interaction between two dielectric particles of general shape, with applications in chemistry and biological physics. Importantly, we expect that for smaller dielectric contrast the rate of convergence of the MSE increases even further, making our method attractive for soft matter systems. We stress that our approach can be used also to study repulsive forces as the medium surrounding the bodies can be a general...
dissipative magneto-dielectric medium. Naturally, our description in terms of reflections and free propagations in between along closed paths should reproduce the geometric-optics limit of Casimir interactions \[^{40}\] and, more importantly, provide a framework to systematically compute corrections due to diffraction. In conclusion, our rapidly convergent MSE can provide a powerful tool to delve deeper into Casimir phenomena in submicrometre structures composed of various materials which cannot be understood by simple additive power laws and planar or spherical surface interactions.

**Methods**

**Multiple Scattering Expansion.** The MSE is constructed from the Green tensor for a homogeneous and isotropic medium with frequency dependent electric and magnetic permittivities $\epsilon_\sigma(\omega)$, $\mu_\sigma(\omega)$, respectively. For imaginary frequencies $\omega = i\xi$ and wave number $\kappa = \xi/c$ the components of $6 \times 6$ dimensional Green tensor are

\[
\begin{align*}
G_{\sigma,ij}^{EE}(r, r') &= -\frac{1}{\kappa} \left( \frac{1}{\epsilon} \frac{\partial^2}{\partial x_i \partial x'_j} + \mu \kappa^2 \delta_{ij} \right) g_\sigma(r - r') , \\
G_{\sigma,ij}^{HH}(r, r') &= -\frac{1}{\kappa} \left( \frac{1}{\mu} \frac{\partial^2}{\partial x'_i \partial x'_j} + \epsilon \kappa^2 \delta_{ij} \right) g_\sigma(r - r') , \\
G_{\sigma,ij}^{HE}(r, r') &= -\epsilon_{ijk} \frac{\partial}{\partial x_k} g_\sigma(r - r') , \\
G_{\sigma,ij}^{EH}(r, r') &= -\epsilon_{ijk} \frac{\partial}{\partial x'_k} g_\sigma(r - r') ,
\end{align*}
\]

where the upper indices label electric ($E$) and magnetic ($H$) components, $i, j \in \{x, y, z\}$ denote the spatial components, $\epsilon_{ijk}$ is the Levi-Civita symbol, and the scalar Green functions are

\[
g_\sigma(r - r') = \frac{e^{-\sqrt{\epsilon_\sigma \mu_\sigma} |r - r'|}}{4\pi |r - r'|} .
\]
Using these Green tensors, the EM field interior and exterior to the bodies can be expressed as volume integrals over the external source and surface integrals over the tangential surface components of the electric and magnetic field at the bodies. The surface integrals are discontinuous at the surfaces $S_\sigma$. We denote the two different limits when $r$ tends to the surface position $u \in S_\sigma$ from the exterior and the interior by $(E^{(\sigma)}_+, H^{(\sigma)}_+)(u)$ and $(E^{(\sigma)}_-, H^{(\sigma)}_-)(u)$, respectively. Then the discontinuity of the exterior field ($\sigma = 0$) is related to the surface currents by

$$
(E^{(0)}_+, H^{(0)}_+)(u) - (E^{(0)}_-, H^{(0)}_-)(u) = n_\sigma(u) \times (m_{\sigma+}, -j_{\sigma+})(u) \tag{7}
$$

and the discontinuity of the interior fields ($\sigma \geq 1$) is similarly given by the inside limit of the currents,

$$
(E^{(\sigma)}_+, H^{(\sigma)}_+)(u) - (E^{(\sigma)}_-, H^{(\sigma)}_-)(u) = n_\sigma(u) \times (m_{\sigma-}, -j_{\sigma-})(u). \tag{8}
$$

However, the integral solutions are also defined when $r$ is located on the surface $S_\sigma$, and we denote their values by $(E^{(\sigma)}_0, H^{(\sigma)}_0)(u)$. They are given by the simple average of the limits inside and outside,$^\text{E4}$

$$
(E^{(\sigma)}_0, H^{(\sigma)}_0)(u) = \frac{1}{2} \left[ (E^{(\sigma)}_+, H^{(\sigma)}_+)(u) + (E^{(\sigma)}_-, H^{(\sigma)}_-)(u) \right] \tag{9}
$$

both for interior and exterior fields. When we make use of the fact that the exterior field vanishes inside the bodies, e.g., $(E^{(0)}_-, H^{(0)}_-)(u) = 0$, and that the interior field vanishes outside the bodies, e.g., $(E^{(\sigma)}_+, H^{(\sigma)}_+)(u) = 0$, we obtain for $u \in S_\sigma$, $\sigma \geq 1$,

$$
2(E^{(0)}_0, H^{(0)}_0)(u) = n_\sigma(u) \times (m_{\sigma+}, -j_{\sigma+})(u), \tag{10}
$$

$$
2(E^{(\sigma)}_0, H^{(\sigma)}_0)(u) = -n_\sigma(u) \times (m_{\sigma-}, -j_{\sigma-})(u).
$$
Since the currents $m_{\sigma\pm}, j_{\sigma\pm}$ have no normal surface component, we find

$$
(m_{\sigma+}, -j_{\sigma+})(u) = -2n_\sigma(u) \times (E^{(0)}, H^{(0)})(u)
$$

$$
(m_{\sigma-}, -j_{\sigma-})(u) = 2n_\sigma(u) \times (E^{(\sigma)}, H^{(\sigma)})(u)
$$

as quoted in the main text (where we have not shown the index 0 as it is understood that the field is evaluated on the surface). For each $\sigma \geq 1$ we multiply the first equation by the diagonal $2 \times 2$ matrix $C^{e}_\sigma$ and the second equation by the diagonal $2 \times 2$ matrix $C^{i}_\sigma$, acting only on the $E$ and $H$ components, and then sum the two equations to obtain the Fredholm Eq. (2). The SSO of Eq. (1) follows by imposing the continuity conditions for the tangential surface fields, corresponding to $(j_\sigma, m_\sigma) \equiv (j_{\sigma+}, m_{\sigma+}) = -(j_{\sigma-}, m_{\sigma-})$, and substituting for the EM fields on the right hand side of the equations the surface integral representation for the EM field given in the main text. Further details are provided in the Supplementary Information.

For clarity, we provide the components of the SSO and of the operator $M$ for the choice (C1) for the coefficient matrices which are used in this work. They read

$$
K^{EE}_{\sigma\sigma'}(u, u') = \frac{2}{\mu_0 + \mu_\sigma} n_\sigma(u) \times [\mu_0 G^{HE}_0(u, u') - \delta_{\sigma\sigma'} \mu_\sigma G^{HE}_{\sigma}(u, u')]
$$

$$
K^{HH}_{\sigma\sigma'}(u, u') = \frac{2}{\epsilon_0 + \epsilon_\sigma} n_\sigma(u) \times [-\epsilon_0 G^{EH}_0(u, u') + \delta_{\sigma\sigma'} \epsilon_\sigma G^{EH}_{\sigma}(u, u')]
$$

$$
K^{EH}_{\sigma\sigma'}(u, u') = \frac{2}{\mu_0 + \mu_\sigma} n_\sigma(u) \times [\mu_0 G^{HH}_0(u, u') - \delta_{\sigma\sigma'} \mu_\sigma G^{HH}_{\sigma}(u, u')]
$$

$$
K^{HE}_{\sigma\sigma'}(u, u') = \frac{2}{\epsilon_0 + \epsilon_\sigma} n_\sigma(u) \times [-\epsilon_0 G^{EE}_0(u, u') + \delta_{\sigma\sigma'} \epsilon_\sigma G^{EE}_{\sigma}(u, u')]
$$

and

$$
M^{EE}_{\sigma}(u, r) = \frac{2\mu_0}{\mu_0 + \mu_\sigma} n_\sigma(u) \times G^{HE}_0(u, r), \quad M^{EH}_{\sigma}(u, r) = \frac{2\mu_0}{\mu_0 + \mu_\sigma} n_\sigma(u) \times G^{HH}_0(u, r)
$$

$$
M^{HE}_{\sigma}(u, r) = -\frac{2\epsilon_0}{\epsilon_0 + \epsilon_\sigma} n_\sigma(u) \times G^{EE}_0(u, r), \quad M^{HH}_{\sigma}(u, r) = -\frac{2\epsilon_0}{\epsilon_0 + \epsilon_\sigma} n_\sigma(u) \times G^{EH}_0(u, r).
$$

(13)
Computations of the MSE for the T-matrix of a dielectric cylinder. The T-operator of a dielectric cylinder of radius $R$ assumes a $2 \times 2$ block diagonal form in vector cylindrical waves labelled by the angular quantum number $m$ and the wave vector $k_z$ along the cylinder axis [20]. It is assumed that the cylinder has electric and magnetic permittivities $\epsilon$ and $\mu$, and the surrounding medium is vacuum ($\epsilon_0 = \mu_0 = 1$). On the imaginary frequency axis, and with $p = \sqrt{\kappa^2 + k_z^2}$, $p' = \sqrt{\epsilon \mu \kappa^2 + k_z'^2}$, the diagonal elements are given by [37]

\[
\begin{align*}
\mathbb{T}^{HH}(m, \kappa, k_z) &= -\frac{I_m(pR)}{K_m(pR)} \Delta_1 \Delta_4 + K^2, \\
\mathbb{T}^{EE}(m, \kappa, k_z) &= -\frac{I_m(pR)}{K_m(pR)} \frac{\Delta_2 \Delta_3 + K^2}{\Delta_1 \Delta_2 + K^2}, \\
\mathbb{T}^{HE}(m, \kappa, k_z) &= -\mathbb{T}^{EH}(m, \kappa, k_z) = \frac{1}{\sqrt{\epsilon \mu} \epsilon pR K_m(pR)} \frac{K}{K_m(pR)^2} \frac{1}{\Delta_1 \Delta_2 + K^2},
\end{align*}
\]

with

\[
K = \frac{mk_z}{\sqrt{\epsilon \mu \kappa R^2}} \left( \frac{1}{p'p} - \frac{1}{p^2} \right),
\]

and

\[
\begin{align*}
\Delta_1 &= \frac{I'_m(p' R)}{p' R I_m(p' R)} - \frac{1}{\epsilon} \frac{K'_m(pR)}{p R K_m(pR)}, \\
\Delta_2 &= \frac{I'_m(p' R)}{p' R I_m(p' R)} - \frac{1}{\mu} \frac{K'_m(pR)}{p R K_m(pR)}, \\
\Delta_3 &= \frac{I'_m(p' R)}{p' R I_m(p' R)} - \frac{1}{\epsilon} \frac{I'_m(pR)}{p R I_m(pR)}, \\
\Delta_4 &= \frac{I'_m(p' R)}{p' R I_m(p' R)} - \frac{1}{\mu} \frac{I'_m(pR)}{p R I_m(pR)},
\end{align*}
\]

where $I_m$ and $K_m$ are Bessel functions, and $I'_m$ and $K'_m$ their derivatives. We note that the polarization is not conserved under scattering, i.e., $\mathbb{T}^{EH}$, $\mathbb{T}^{HE} \neq 0$. The scattering Green tensor $\mathbb{G}$ of the cylinder can be expressed in terms of these matrix elements, following the conventional scattering method [20]. The comparison to the MSE can be performed by suitable projection. For instance, from the projection $\hat{r} \mathbb{G}^{EE} \hat{r}'$ on the radial directions $\hat{r}, \hat{r}'$ of $\mathbb{G}^{EE}(r, r')$, all four elements $\mathbb{T}^{EE}$, $\mathbb{T}^{HH}$,
$T^{HE}, T^{EH}$ can be extracted as they are multiplied by different combinations of $K_m(pr), K'_m(pr), K_m(pr'), K'_m(pr')$. Therefore, all components of the analytically computed MSE for $\Gamma^{EE}$ can be compared to the above T-matrix elements.

**MSE for the Casimir energy of wedge and plate.** The SSOs for the plate and the wedge can be directly obtained by integrating the tangential components of the free Green tensor of Eq. (5) over the surface of the bodies and using Eq. (12). A convenient choice of tangential vector fields to compute the SSO are simple vector plane waves. For the analytic expressions of this representation, see Supplementary Information. The expansion of the Casimir energy of Eq. (4) in powers of $\Delta K_{11}(\theta) = K_{11}(\theta) - K_{11}(\theta = 0)$ leads to iterated integrals of SSO over the wave vectors that are tangential to the surfaces. The highest considered order MSE$_{12}$ corresponds to a 7-dimensional integral. To obtain energy at different orders of MSE, integration of analytically computed SSOs is performed numerically using an adaptive algorithm for numerical integration over multi-dimensional rectangular regions. This program is carried out for each Matsubara frequency. To estimate the series over Matsubara frequencies from a finite number of terms, we use error estimation by bounding pairs $[11]$. The contributions from the zero frequency modes are computed directly from the exact expressions for the Casimir energy of the transverse magnetic (TM) mode in the perfect metal limit $[23]$. We note that this provides an exact treatment even of Au/Si bodies as they are conducting with a divergent electric permittivity at zero frequency, and the transverse electric (TE) mode does not contribute in the static limit.
Electric permittivity for silicon and gold. The optical properties of the doped silicon are described by the dielectric function

\[
\epsilon_{\text{Si}}(i\xi) = \epsilon_\infty + \frac{\epsilon_0 - \epsilon_\infty}{1 + \xi^2/\omega_{\text{UV}}^2} + \frac{\Omega_{\text{Si}}^2}{\xi(\xi + \gamma_{\text{Si}})},
\]

where the last term describes the carriers from doping. For gold the dielectric function has contributions from intraband effects (Drude model) and interband effects (Lorentz oscillators), with

\[
\epsilon_{\text{Au}}(i\xi) = 1 + \frac{\Omega_{\text{Au}}^2}{\xi(\xi + \gamma_{\text{Au}})} + \sum_{j=1}^{6} \frac{f_j}{\xi^2 + \gamma_j \xi + \Omega_j^2}.
\]

The parameters are provided in Tab. 1 in units of eV, unless dimensionless.

PFA and PWS approximations for the Casimir energy of wedge and plate. In the PFA the Casimir energy between an object opposite a plane is estimated by averaging the Lifshitz formula over the local separation \( H \) between the plane and the surface of the object,

\[
\mathcal{E}_{\text{PFA}} = \int_{\text{plane}} dr \, \mathcal{F}_{pp}(H).
\]

By a straightforward computation (see Supplementary Information), we obtain for the wedge-plate configuration

\[
\mathcal{E}_{\text{PFA}} = -\frac{k_B T L}{\tan \theta} \sum_{n=0}^{\infty} \int_0^{\infty} \frac{k \, dk}{2\pi} \sum_{\alpha=\text{TE,TM}} \text{Li}_2(e^{-2dq_n r_{\alpha}^{(1)}} r_{\alpha}^{(2)}),
\]

where \( q_n = \sqrt{\epsilon_0(i\xi_n)\mu_0(i\xi_n)\kappa_n^2 + k^2} \), \( \text{Li}_p(z) = \sum_{k=1}^{\infty} z^k / k^p \) is the polylogarithm function, and \( r_{\alpha}^{(\sigma)}(k, \kappa_n) \) denote the Fresnel reflection coefficients for polarization \( \alpha \) of a planar slab with electric and magnetic permittivities \( \epsilon_\sigma, \mu_\sigma \), placed in a medium with electric and magnetic permittivities \( \epsilon_0, \mu_0 \) respectively.
In the PWS approximation, the energy is estimated by adding up the interaction energies between the plate and the particles composing the other body. By doing so (see Supplementary Information for details), we get for the wedge-plate system

\[
\mathcal{E}_{PWS} = -\frac{k_B T L}{4\pi \tan \theta} \sum_{n=0}^{\infty} \frac{\epsilon_1(i\xi_n) - 1}{\epsilon_1(i\xi_n) + 1} \int_0^{\infty} k \, dk \, e^{-2d q_n} \left[ \left( 2 - \frac{\kappa_n^2}{q_n^2} \right) J^{(2)}_{TM} - \frac{\kappa_n^2}{q_n^2} J^{(2)}_{TE} \right].
\] (21)

**Application of SCUFF-EM open-source software.** For a selected wedge-plate geometry with \( \theta = \pi/3 \) and separation \( d = 100 \) nm, we performed numerical computations of the Casimir energy using the open-source software SCUFF-EM\textsuperscript{36}. Specifically, we use the module for geometries which are infinitely extended in one dimension. It computes the Casimir energy per unit length along the edge of the wedge. The configuration is described by a periodic geometry with periodic boundary conditions and the energy is obtained by numerical integration over the corresponding 1D Brillouin zone (BZ). A total of 66 Matsubara terms are summed to obtain the final energy estimate. The computation is based on the boundary element method. The mesh consists of 1098 panels per unit cell (length of 1 \( \mu \)m), describing both the planar Au slab (width of 5 \( \mu \)m and thickness of 1 \( \mu \)m) and the Si wedge with the wings truncated at the plate width. Panel size is reduced on both surfaces in proximity to the edge of the wedge to better resolve short scale field variations. Optical parameters for Au and Si are identical to the ones used for our MSE. A relative tolerance of \( 10^{-3} \) is used for the numerical BZ integration.

**Data availability** The data that support the findings of this study are available from the corresponding author, upon reasonable request.
References

1. Casimir, H. B. G. On the attraction between two perfectly conducting plates. *Proc. K. Ned Akad. Wet* **5**, 793 (1948).

2. Lifshitz, E. M. The theory of molecular attractive force between solids. *Sov. Phys. J. Exp Theoret. Phys* **2**, 73 (1956).

3. Lamoreaux, S. K. Demonstration of the Casimir force in the 0.6 to 6 µm range force in the 0.6 to 6 µm range. *Phys. Rev. Lett.* **78**, 5 (1997).

4. Mohideen, U. & Roy, A. Precision measurement of the casimir force from 0.1 to 0.9 µm. *Phys. Rev. Lett.* **81**, 4549 (1998).

5. Chan, H. B., Aksyuk, V. A., Kleiman, R. N., Bishop, D. J. & Capasso, F. Quantum mechanical actuation of microelectromechanical systems by the casimir force. *Science* **291**, 1941 (2001).

6. Bressi, B., Carugno, G., Onofrio, R. & Ruoso, G. Measurement of the casimir force between parallel metallic surfaces. *Phys. Rev. Lett.* **88**, 041804 (2002).

7. Decca, R. S., López, D., Fischbach, E. & Krause, D. E. Measurement of the casimir force between dissimilar metals. *Phys. Rev. Lett.* **91**, 050402 (2003).

8. Munday, J. N., Capasso, F. & Parsegian, V. A. Measured long-range repulsive casimir–lifshitz forces. *Nature* **457**, 170 (2009).
9. Sushkov, A. O., Kim, W. J., Dalvit, D. A. R. & Lamoreaux, S. K. Observation of the thermal casimir force. *Nat. Phys.* **7**, 230 (2011).

10. Tang, L. *et al.* Measurement of non-monotonic casimir forces between silicon nanostructures. *Nature Photonics* **11**, 97–101 (2017).

11. Bimonte, G., López, D. & Decca, R. S. Isoelectronic determination of the thermal casimir force. *Phys. Rev. B* **93**, 184434 (2016).

12. Derjaguin, B. V. Untersuchungen über die reibung und adhäsion, iv: Theorie des anhaftens kleiner teilchen. *Kolloid-Z.* **69**, 155 (1934).

13. Banishev, A. A., Wagner, J., Emig, T., Zandi, R. & Mohideen, U. Demonstration of angle-dependent Casimir force between corrugations. *Phys. Rev. Lett.* **110**, 250403 (2013).

14. Intravaia, F. *et al.* Strong casimir force reduction through metallic surface nanostructuring. *Nature Communications* **4**, 2515 (2013).

15. Wang, M., Tang, L., Ng, C. Y. *et al.* Strong geometry dependence of the casimir force between interpenetrated rectangular gratings. *Nature Commun* **12**, 600 (2021).

16. Rodriguez, A. W., Capasso, F. & Johnson, S. G. The casimir effect in microstructured geometries. *Nat. Photonics* **5** (2011).

17. Bimonte, G., Emig, T., Kardar, M. & Krüger, M. Nonequilibrium fluctuational quantum electrodynamics: heat radiation, heat transfer, and force. *Annu. Rev. Condens. Matter Phys.* **8**, 119–143 (2017).
18. Emig, T., Graham, N., Jaffe, R. L. & Kardar, M. Casimir forces between arbitrary compact objects. *Phys. Rev. Lett.* **99** (2007).

19. Kenneth, O. & Klich, I. Casimir forces in a t-operator approach. *Phys. Rev. B* **78**, 014103 (2008).

20. Rahi, S. J., Emig, T., Graham, N., Jaffe, R. L. & Kardar, M. Scattering theory approach to electrodynamic casimir forces. *Phys. Rev. D* **80**, 085021 (2009).

21. Genet, C., Lambrecht, A. & Reynaud, S. Casimir force and the quantum theory of lossy optical cavities. *Phys. Rev. A* **67**, 043811 (2003).

22. Lambrecht, A., Maia Neto, P. A. & Reynaud, S. The casimir effect within scattering theory. *New J. Phys* **8**, 243 (2006).

23. Maghrebi, M. F. *et al.* Analytical results on casimir forces for conductors with edges and tips. *PNAS* **108**, 6687–6871 (2011).

24. Messina, R., Noto, A., Guizal, B. & Antezza, M. Radiative heat transfer between metallic gratings using fourier modal method with adaptive spatial resolution. *Phys. Rev. B* **95**, 125404 (2017).

25. Fosco, C. D., Lombardo, F. C. & Mazzitelli, F. D. Proximity force approximation for the casimir energy as a derivative expansion. *Phys.Rev.D* **84**, 105031 (2011).

26. Bimonte, G., Emig, T. & Kardar, M. Material dependence of casimir forces: gradient expansion beyond proximity. *Appl. Phys. Lett.* **100**, 074110 (2012).
27. Reid, M. T. H., White, J. & Johnson, S. G. Fluctuating surface currents: an algorithm for efficient prediction of casimir interactions among arbitrary materials in arbitrary geometries. *Phys. Rev. A* **88**, 022514 (2013).

28. Rodriguez, A. W. *et al.* Classical and fluctuation-induced electromagnetic interactions in micron-scale systems: designer bonding, antibonding, and casimir forces. *Annalen Physik* **527**, 45 (2014).

29. Balian, R. & Duplantier, B. Geometry of the casimir effect. In Ciufolini et al. (ed.) *15th SIGRAV Conference on General Relativity and Gravitational Physics*, Institute of Physics Conference Series 176 (2004).

30. Balian, R. & Duplantier, B. Electromagnetic waves near perfect conductors. i. multiple scattering expansions. distribution of modes. *Ann. Phys (NY)* **104**, 300–335 (1977).

31. Balian, R. & Duplantier, B. Electromagnetic waves near perfect conductors. ii. casimir effect. *Ann. Phys. (NY)* **112**, 165–208 (1978).

32. Agarwal, G. S. Quantum electrodynamics in the presence of dielectrics and conductors. i. electromagnetic-field response functions and black-body fluctuations in finite geometries. *Phys. Rev. A* **11**, 230–242 (1975).

33. Harrington, R. F. *Time-Harmonic Electromagnetic Fields*. IEEE Press Series on electromagnetic wave theory (Wiley, New York, 2001).

34. Müller, C. *Foundations of the mathematical theory of electromagnetic waves* (Springer, 1969).
35. Bimonte, G. & Emig, T. Unifying theory for casimir forces: bulk and surface formulations. *Universe* 7, 225 (2021).

36. SCUFF-EM (2018). URL [https://github.com/homerreid/scuff-em/](https://github.com/homerreid/scuff-em/).

37. Noruzifar, E., Emig, T., Mohideen, U. & Zandi, R. Collective charge fluctuations and casimir interactions for quasi-one-dimensional metals. *Phys. Rev. B* 86, 115449 (2012).

38. Casimir, H. B. G. & Polder, D. The influence of retardation on the london-van der waals forces. *Phys. Rev.* 73, 360 (1948).

39. Shanks, D. Non-linear transformation of divergent and slowly convergent sequences. *J. Math. and Phys.* 34, 1 (1955).

40. Jaffe, R. L. & Scardicchio, A. Casimir effect and geometric optics. *Phys. Rev. Lett.* 92, 070402 (2004).

41. Braden, B. Calculating sums of infinite series. *The American Mathematical Monthly* 99, 649–655 (1992).
Table 1: Parameters of the dielectric functions for gold and doped silicon.

| Parameters for Si | value [eV, unless dimensionless] |
|-------------------|----------------------------------|
| \( \epsilon_\infty \) | 1.035 |
| \( \epsilon_0 \) | 11.87 |
| \( \omega_{\text{UV}} \) | 4.3442 |
| \( \Omega_\text{Si} \) | 0.1560 |
| \( \gamma_\text{Si} \) | 0.0425 |

| Parameters for Au |
|-------------------|
| \( \Omega_\text{Au} \) | 9.0 |
| \( \gamma_\text{Au} \) | 0.035 |
| \( f_1, \gamma_1, \Omega_1 \) | 7.091, 0.75, 3.05 |
| \( f_2, \gamma_2, \Omega_2 \) | 41.46, 1.85, 4.15 |
| \( f_3, \gamma_3, \Omega_3 \) | 2.7, 1.0, 5.4 |
| \( f_4, \gamma_4, \Omega_4 \) | 154.7, 7.0, 8.5 |
| \( f_5, \gamma_5, \Omega_5 \) | 44.55, 6.0, 13.5 |
| \( f_6, \gamma_6, \Omega_6 \) | 309.6, 9.0, 21.5 |