Singularity structures and impacts on parameter estimation in finite mixtures of distributions

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Abstract

Singularities of a statistical model are the elements of the model’s parameter space which make the corresponding Fisher information matrix degenerate. These are the points for which estimation techniques such as the maximum likelihood estimator and standard Bayesian procedures do not admit the root-$n$ parametric rate of convergence. We propose a general framework for the identification of singularity structures of the parameter space of finite mixtures, and study the impacts of the singularity structures on minimax lower bounds and rates of convergence for the maximum likelihood estimator over a compact parameter space. Our study makes explicit the deep links between model singularities, parameter estimation convergence rates and minimax lower bounds, and the algebraic geometry of the parameter space for mixtures of continuous distributions. The theory is applied to establish concrete convergence rates of parameter estimation for finite mixture of skew-normal distributions. This rich and increasingly popular mixture model is shown to exhibit a remarkably complex range of asymptotic behaviors which have not been hitherto reported in the literature.

1 Introduction

In the standard asymptotic theory of parametric estimation, a customary regularity assumption is the non-singularity of the Fisher information matrix defined by the statistical model (see, for example, [36] (pg. 124); or [51], Sec. 5.5). This condition leads to the cherished root-$n$ consistency, and in many cases the asymptotic normality of parameter estimates. When the non-singularity condition fails to hold, that is, when the true parameters represent a singular point in the statistical model, very little is known about the asymptotic behavior of their estimates.

The singularity situation might have been brushed aside as idiosyncratic by some parametric statistical modelers in the past. As complex and high-dimensional models are increasingly embraced by statisticians and practitioners alike, singularities are no longer a rarity — they start to take a highly visible place in modern statistics. For example, the many zeros present in a high-dimensional linear regression problem represent a type of singularities of the underlying model, points corresponding to rank-deficient Fisher information matrices [26]. In another example, the zero skewness in the family of skewed distributions represents a singular point [15]. In both examples, singularity points are quite easy to spot out — it is the impacts of their presence on improved parameter estimation procedures and the asymptotic properties such procedures entail that are nontrivial matters occupying the best efforts of many researchers in the past decade. The textbooks by [8, 26], for instance, address such issues for high-dimensional regression problems, while the recent papers by [37, 24, 25] investigate statistical inference in the skewed families for distribution. By contrast, with finite mixture models — a popular

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and a rich class of modeling tools for density estimation and heterogeneity inference \cite{42, 40, 41, 31} and a subject of this paper, the singularity phenomenon is not quite well understood, to the best our knowledge, except for specific instances.

One of the simplest instances is the singularity of Fisher information matrix in an (overfitted) finite mixture that includes a homogeneous distribution, a setting studied by \cite{33, 34}. \cite{34} analyzed a test of heterogeneity based on finite mixtures, addressing the challenge arising from the aforementioned singularity. Recent works on the related topic include \cite{17, 18, 12, 11, 21, 32}. \cite{45} investigated likelihood-based parameter estimation in a somewhat general parametric modeling framework, subject to the constraint that the Fisher information matrix is one rank deficient. For overfitted finite mixtures, \cite{14} showed that under a condition of strong identifiability, there are estimators which achieve the generic convergence rate $n^{-1/4}$ for parameter estimation. Recent works also established generic behaviors of estimation under somewhat broader settings of overfitted finite mixture models with both maximum likelihood estimation and Bayesian estimation \cite{46, 43, 29}. Under sufficiently strong identifiability conditions for kernel densities, a sharp local minimax lower bound of parameter estimation in over-fitted finite mixture models were recently obtained \cite{27}.

The family of mixture models is far too rich to submit an uniform kind of behavior of parameter estimation, due to a weak identifiability phenomenon induced by underlying singularities that are much more pervasive than previously thought. In fact, it was shown recently that even classical models such as the location-scale Gaussian mixtures, and the shape-rate Gamma mixtures, do not admit such a generic rate of convergence for an estimation method such as MLE or Bayesian estimation with a non-informative prior \cite{28}. For instance, singularities arise in the finite mixtures of Gamma distributions, even when the number of mixing components is known — this phenomenon results in an extremely slow convergence behavior for the model parameters lying in the vicinity of singular points, even though such parameters are (perfectly) identifiable. Finite mixtures of Gaussian distributions, though identifiable, exhibit both minimax lower bounds and maximum likelihood estimation rates that are directly linked to the solvability of a system of real polynomial equations, rates which deteriorate quickly with the increasing number of extra mixing components. The results obtained for such specific instances contain considerable insights about parameter estimation in finite mixture models, but they only touch upon the surface of a general and complex phenomenon. Indeed, as we shall see in this paper there is a rich spectrum of asymptotic behavior in which regular (non-singular) mixtures, strongly identifiable mixtures, and weakly identifiable mixture models (such as the one studied by \cite{28}) occupy but a small spot.

**Main results** The goal of this paper is to present a general and theoretical framework for analyzing parameter estimation behavior in finite mixture models. We address directly the situations where the non-singularity condition of the Fisher information matrix may not hold. Our approach is to take on a systematic investigation of the singularity structure of a compact and multi-dimensional parameter space of mixture models, and then study the impacts of the presence of singularities on parameter estimation. There is a remarkable heterogeneity of the mixture model parameter space that we can shed some light on: it will be shown that different parts of the parameter space may admit different convergence rates, by several standard estimation methods. Parameters of differenter types may possess different estimation rates, e.g., location vs scale of the same mixture component. Even parameters of the same type may carry distinct rates of estimation, such shape parameters associated with different mixture components.

To obtain such a fine-grained picture of the parameter space, several fundamental concepts will be introduced. In particular, the natural-valued *singularity level* will be useful in describing the convergence behavior of the (discrete) mixing measure that arises in the mixture model. Specifically, a
mixture density of the form \( p_G(x) = \int f(x|\eta) dG(\eta) \), where \( f \) denotes a kernel density, corresponds to mixing measure \( G \) on a suitable parameter space. If \( G = \sum_{i=1}^{k} p_i \delta_{\eta_i} \), then it is often denoted that \( p_G(x) = \sum_{i=1}^{k} p_i f(x|\eta_i) \). The singularity level for a mixing measure \( G \) describes in a precise manner the variation of the mixture likelihood \( p_G(x) \) with respect to changes in mixing measure \( G \). Now, Fisher information singularities simply correspond to points in the parameter space which identify a mixing measure whose singularity level is non-zero. Within the set of Fisher information singularities the parameter space can be partitioned into disjoint subsets determined by different singularity levels.

Given an i.i.d. \( n \)-sample from a (true) mixture density \( p_{G_0} \), where \( G_0 \) admits a singularity level \( r \). This will imply, under some mild conditions on \( f \), that a standard estimation method such as maximum likelihood estimation and Bayesian estimation with a non-informative prior carries the rate of convergence \( n^{-1/2(r+1)} \), which is also a minimax lower bound (up to a logarithmic factor). Here, the convergence rate is expressed in terms of a suitable Wasserstein metric on the space of mixing measures. Thus, singularity level 0 results in root-\( n \) convergence rate for mixing measure estimation. Fisher singularity corresponds to singularity level 1 or greater than 1, resulting in convergence rates \( n^{-1/4}, n^{-1/6}, n^{-1/8} \) or so on.

Convergence in Wasserstein metric on mixing measures is easily translated into convergence of the supporting atoms \( [43] \). But each atom of the mixing measure may be composed of different types (e.g., location, scale, shape). To anticipate the heterogeneity of parameters of different types, we introduce vector-valued singularity index, which extends the notion of natural-valued singularity level described earlier. The singularity index describes the variation of the mixture likelihood with respect to changes of individual parameter of each type. A singularity index \( \kappa \) corresponds to singularity level \( (||\kappa||_\infty - 1) \) for the mixing measure, but it tells us much more: the convergence rate for estimating the \( j \)-th component of the atoms \( \eta \) via MLE or Bayesian method will be \( n^{-1/2\kappa_j} \). One can go further to capture “complete heterogeneity”: via singularity matrix, it can be shown that each parameter may allow a possibly different convergence rate depending on the parameter’s values. The complete picture of the distribution of singularity structure, however, can be extremely complex to derive. Remarkably, there are examples of finite mixtures for which the compact parameter space can be partitioned into disjoint subsets whose singularity level or elements of singularity index and singularity matrix range from 0 to 1 to 2, . . . , up to infinity. As a result, if we were to vary the true parameter values, we would encounter a phenomenon akin to that of “phase transition” on the statistical efficiency of parameter estimation occurring within the same model class.

**Techniques** A major component of our general framework is a procedure for characterizing subsets of points having the same singularity structure, via common singularity level and so on. It will be shown that these points are in fact a subset of a real affine variety. A real affine variety is a set of solutions to a system of real polynomial equations. The polynomial equations can be derived explicitly by the kernel density functions that define a given mixture distribution. The study of the solutions of polynomial equations is a central subject of algebraic geometry \([48],[16]\). The connections between statistical models and algebraic geometry have been studied for discrete Markov random fields \([20]\), as well as finite mixtures of categorical data \([1]\). For finite mixtures of continuous distributions, the link to algebraic geometry is distilled from a new source of algebraic structure, in addition to the presence of mixing measures: it is traced to the partial differential equations satisfied by the mixture model’s kernel density function. For Gaussian mixtures, it is the relation captured by Eq. \( \text{(3)} \) for the Gaussian kernel. The partial differential equations can be nonlinear, with coefficients given by rational functions defined in terms of model parameters. It is this relation that is primarily responsible for the complexity of the singularity structure. A quintessential example of such a relation is given by Eq. \( \text{(2)} \) for the skew-normal kernel densities.
Although our method for the analysis of singularity structure and the asymptotic theory for parameter estimation can be used to re-derive old and existing results such as those of [14, 28], a substantial outcome is establish fresh new results on mixture models for which no asymptotic theory have hitherto been achieved. This leads us to a story of finite mixtures of skew-normal distributions. The skew-normal distribution was originally proposed in [4, 6, 5]. The skew-normal generalizes normal (Gaussian) distribution, which is enhanced by the capability of handling asymmetric (skewed) data distributions. Due to its more realistic modeling capability for multi-modality and asymmetric components, skew-normal mixtures are increasingly adopted in recent years for model based inference of heterogeneity by many researchers [39, 2, 3, 38, 47, 22, 35, 44, 10, 54]. Due to its usefulness, a thorough understanding of the asymptotic behavior of parameter estimation for skew-normal mixtures is also of interest in its own right.

The singularity structure of the skew-normal mixtures is perhaps one of the more complex among the parametric mixture models that we have typically encountered in the literature. By comparison, strongly identifiable models admit the singularity level 0 (and singularity indices of all ones) for all parameter values residing in a compact space, which results in the $n^{-1/2}$ convergence rates of model parameters. Most mixture models whose kernel density function has only one type of parameter, such as location mixtures or scale mixtures, are in this category. Location-scale Gaussian mixtures are a step up in the complexity, in that all their parameter values carry the same singularity structure, which depends only on the number of extra mixing components. Yet this is not the picture of skew-normal mixtures, which exhibits the kind of complete heterogeneity described earlier. We will be able to identify subsets with singularity level (also, index/matrix) that vary all the way up to infinity. Even in the setting of mixtures with known number of mixing components, the singularity structure is remarkably complex. Thus, the results for skew-normal mixtures present an useful illustration for the full power of the general theory for finite mixtures of continuous distributions.

The source of complexity of skew-normal mixtures is the structure of the skew-normal kernel density. The evidence for the latter was already made clear by [15, 37, 24, 25], whose works provided a thorough picture of the singularities for the class of skew-normal densities, and their impacts on the non-standard rates of convergence of MLE. Not only can we recover the results of [24, 25] in terms of rates of convergence, because they correspond to a trivial “mixture” that has exactly one skew-normal component, an entirely new set of results are established for mixtures of two or more components. It is in this setting that new types of singularities arise out of the interactions between distinct skew-normal components. These interactions define the subset of singular points of a given structure that can be characterized by a system of real polynomial equations. This algebraic geometric characterisation allows us to establish either the precise singularity structure or an upper bound for the mixture model’s entire parameter space.

It is also worth mentioning, finally, as we move through increasingly sophisticated concepts of singularity structure (level, index, matrix), we also introduce increasingly structured notions of transportation distance for the space of underlying mixing measures. These distances, which behave asymptotically as semi-polynomials of the parameter perturbations, prove to be the right tool for linking up the information culled from a data sample (via its likelihood function) and the algebraic structure of the parameter space of inferential interest.

The plan for the remainder of our paper is as follows. Section 2 lays out the notations and relevant concepts such as parameter spaces and the underlying geometries. Section 3 presents the general framework of analysis of singularity structure and the impact on convergence rates of parameter estimation for singular points of a given singularity structure. Section 4 illustrates the theory on the finite mixture of skew-normal distributions, by giving concrete minimax bounds and MLE convergence rates for this class of models for the first time. We conclude with a discussion in Section 5. Additional concepts and results and full proofs are given in the Appendices.
Notation We utilize several familiar notions of distance for mixture densities, with respect to Lebesgue measure $\mu$. They are total variation distance $V(p_G, p_{G_0}) = \frac{1}{2} \int |p_G(x) - p_{G_0}(x)|d\mu(x)$ and Hellinger distance $h^2(p_G, p_{G_0}) = \frac{1}{2} \int \left( \sqrt{p_G(x)} - \sqrt{p_{G_0}(x)} \right)^2 d\mu(x)$.

Additionally, for any $\kappa_1, \kappa_2 \in \mathbb{R}^d$, we denote $\kappa_1 \preceq \kappa_2$ iff all the components of $\kappa_1$ are less than or equal to the corresponding components of $\kappa_2$. Furthermore, $\kappa_1 \prec \kappa_2$ iff $\kappa_1 \preceq \kappa_2$ and $\kappa_1 \neq \kappa_2$. Additionally, the expression “$\gtrsim$” will be used to denote the inequality up to a constant multiple where the value of the constant is fixed within our setting. We write $a_n \sim b_n$ if both $a_n \gtrsim b_n$ and $a_n \lesssim b_n$ hold. Finally, for any $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to $x$.

2 Preliminaries

A finite mixture of continuous distributions admits density of the form $p_G(x) = \int f(x|\eta)dG(\eta)$ with respect to Lebesgue measure on an Euclidean space for $x$, where $f(x|\eta)$ denotes a probability density kernel, $\eta$ is a multi-dimensional parameter taking values in a subset of an Euclidean space $\Theta$, $G$ denotes a discrete mixing distribution on $\Theta$. The number of support points of $G$ represents the number of mixing components in mixture model. Suppose that $G = \sum_{i=1}^k p_i \delta_{\eta_i}$, then $p_G(x) = \sum_{i=1}^k p_i f(x|\eta_i)$.

2.1 Parameter spaces and geometries

There are different kinds of parameter space and geometries that they carry which are relevant to the analysis of mixture models. We proceed to describe them in the following.

Natural parameter space The customarily defined parameter space of the $k$-mixture of distributions is that of the mixing component parameters $\eta_i$, and mixing probabilities $p_i$. Throughout this paper, it is assumed that $\eta_i \in \Theta$, which is a compact subset of $\mathbb{R}^d$ for some $d \geq 1$, for $i = 1, \ldots, k$. The mixing probability vector $p = (p_1, \ldots, p_k) \in \Delta^{k-1}$, the $(k-1)$-probability simplex. For the remainder of the paper, we also use $\Omega$ to denote the compact subset of the Euclidean space for parameters $(p, \eta)$.

Example 2.1. The skew-normal density kernel on the real line has three parameters $\eta = (\theta, \sigma, m) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$, namely, the location, scale and skewness (shape) parameters. It is given by, for $x \in \mathbb{R}$,

$$f(x|\theta, \sigma, m) := \frac{2}{\sigma} f \left( \frac{x - \theta}{\sigma} \right) \Phi(m(x - \theta)/\sigma),$$

where $f(x)$ is the standard normal density and $\Phi(x) = \int f(t)1(t \leq x) dt$. This generalizes the Gaussian density kernel, which corresponds to fixing $m = 0$. The parameter space for the $k$-mixture of skew нормals is therefore a subset of an Euclidean space for the mixing probabilities $p_i$ and mixing component parameters $\eta_i = (\theta_i, \sigma_i, m_i) \in \mathbb{R}^3$. For each $i = 1, \ldots, k$, $\theta_i, \sigma_i, m_i$ are restricted to reside in compact subsets $\Theta_1 \subset \mathbb{R}, \Theta_2 \subset \mathbb{R}_+, \Theta_3 \subset \mathbb{R}$ respectively, i.e., $\Theta = \Theta_1 \times \Theta_2 \times \Theta_3$.

Semialgebraic sets The singularity structure of the parameter space submits to a different geometry. It will be described in terms of the zero sets (sets of solutions) of systems of real polynomial equations. The zero set of a system of real polynomial equations is called a (real) affine variety $[16]$. In fact, the sets which describe the singularity structure of finite mixtures are not affine varieties per se. We will see that they are the intersection between real affine varieties – the real-valued solutions of a finite collection of equations of the form $P(p, \eta) = 0$, and the set of parameter values satisfying $Q(p, \eta) > 0$, for some real polynomials $P$ and $Q$. The intersection of these sets is also referred to as semialgebraic sets.
**Example 2.2.** Continuing on the example of skew-normal mixtures, we will see that first two types of singularities that arise from the mixture of skew-normals are solutions of the following polynomial equations

(i) Type A: \( P_1(\eta) = \prod_{j=1}^{k} m_j. \)

(ii) Type B: \( P_2(\eta) = \prod_{1 \leq i \neq j \leq k} \left\{ (\theta_i - \theta_j)^2 + \left[ \sigma_i^2 (1 + m_i^2) - \sigma_j^2 (1 + m_j^2) \right]^2 \right\}. \)

These are just two among many more polynomials and types of singularities that we will be able to enumerate in the sequel. We quickly note that Type A refers to the one (and only) kind of singularity intrinsic to the skew-normal kernel: \( P_1 = 0 \) if either one of the \( m_j = 0 \) — one of the skew-normal components is actually normal (symmetric). This type of singularity has received in-depth treatments by a number of authors [15, 37, 24, 25]. One the other hand, Type B is intrinsic to a mixture model, as it describes the relation of parameters of distinct mixing components \( i \) and \( j \).

**Space of mixing measures and transportation distance** As described in the Introduction, a goal of this work is to turn the knowledge about the nature of singularities of parameter space \( \Omega \) into that of statistical efficiency of parameter estimation procedures. For this purpose, the convergence of parameters in a mixture model is most naturally analyzed in terms of the convergence in the space of mixing measures endowed by transportation distance (Wasserstein distance) metrics [43]. This is because the role played by parameters \( p, \eta \) in the mixture model is via mixing measure \( G \). It is mixing measure \( G \) that determines the mixture density \( p_G \) according to which the data are drawn from. Since the map \( (p, \eta) \mapsto G(p, \eta) = G = \sum p_i \delta_{\eta_i} \) is many-to-one, we shall treat a pair of parameter vectors \( (p, \eta) = (p_1, \ldots, p_k; \eta_1, \ldots, \eta_k) \) and \( (p', \eta') = (p'_1, \ldots, p'_{k'}; \eta'_1, \ldots, \eta'_{k'}) \) to be equivalent if the corresponding mixing measures are equal, \( G(p, \eta) = G(p', \eta') \). For ease of notation we often omit the arguments when the context is clear for \( G = G(p, \eta) \) and \( G' = G(p', \eta') \).

For \( r \geq 1 \), the Wasserstein distance of order \( r \) between \( G \) and \( G' \) takes the form (cf. [52]),

\[
W_r(G, G') = \left( \inf_{i,j} \sum_{i,j} q_{ij} \|\eta_i - \eta'_j\|_r^r \right)^{1/r},
\]

where \( \| : \|_r \) is the \( \ell_r \) norm endowed by the natural parameter space, the infimum is taken over all couplings \( q \) between \( p \) and \( p' \), i.e., \( q = (q_{ij})_{ij} \in [0, 1]^{k \times k'} \) such that \( \sum_{i=1}^{k} q_{ij} = p_j \) and \( \sum_{j=1}^{k'} q_{ij} = p'_i \) for any \( i = 1, \ldots, k \) and \( j = 1, \ldots, k' \). For the specific example of skew-normal mixtures, if \( \eta = (\theta, v, m) \) and \( \eta' = (\theta', v', m') \), then \( \|\eta - \eta'\|_r^r := |\theta - \theta'|^r + |v - v'|^r + |m - m'|^r \).

Suppose that a sequence of probability measures \( G_n = \sum_i \rho_i \delta_{\eta_i} \) tending to \( G_0 \) under \( W_r \) metric at a vanishing rate \( \omega_n = o(1) \). If all \( G_n \) have the same number of atoms \( k_n = k_0 \) as that of \( G_0 \), then the set of atoms of \( G_n \) converge to the \( k_0 \) atoms of \( G_0 \), up to a permutation of the atoms, at the same rate \( \omega_n \) under \( \| : \|_r \). If \( G_n \) have the varying \( k_n \in [k_0, k] \) number of atoms, where \( k \) is a fixed upper bound, then a subsequence of \( G_n \) can be constructed so that each atom of \( G_0 \) is a limit point of a certain subset of atoms of \( G_n \) — the convergence to each such limit also happens at rate \( \omega_n \). Some atoms of \( G_n \) may have limit points that are not among \( G_0 \)'s atoms — the total mass associated with those “redundant” atoms of \( G_n \) must vanish at the generally faster rate \( \omega_n^r \), since \( r \geq 1 \).

### 2.2 Inhomogeneity and generalized transportation distance

Although the standard Wasserstein metrics \( W_r \) proved to be a convenient tool for the analysis of the space of mixing measures [43, 29, 28], they are inadequate in describing the inhomogeneous behavior
of individual parameters present in the model. As we have seen in the previous paragraph, the convergence rate of mixing measures \( G_n \) under Wasserstein metric \( W_\kappa \) induces the same rate of convergence for the atoms of \( G_n \), denoted by \( \eta \). By way of example, suppose that \( \eta \) is in fact made up of three parameters \( \eta = (\theta, v, m) \), as illustrated in the case of skew-normal mixtures, this implies that same upper bound \( \omega_n \) holds for all individual components \( \theta, v \) and \( m \) of the model parameter. Thus this fails to demonstrate the situations in which different parameter components may in fact exhibit distinct convergence behavior. For instance, we will see that in normal and skew-normal mixtures, scale parameters may converge more slowly than location parameters.

To derive inhomogeneous convergence rates of different model parameters, we introduce a general version of the optimal transport distance, which can be formulated as follows.

**Definition 2.1.** For any \( \kappa = (\kappa_1, \ldots, \kappa_d) \in \mathbb{N}^d \), let

\[
d_\kappa(\theta_1, \theta_2) := \left( \sum_{i=1}^{d} |\theta_1^{(i)} - \theta_2^{(i)}|^{\kappa_i} \right)^{1/\|\kappa\|_\infty}
\]

where \( \theta_i = (\theta_i^{(1)}, \ldots, \theta_i^{(d)}) \in \mathbb{R}^d \), as \( i = 1, 2, \) and \( \|\kappa\|_\infty = \max_{1 \leq i \leq d} \{ \kappa_i \} \). The generalized transportation distance with respect to \( \kappa \) is given by

\[
\tilde{W}_\kappa(G, G') := \left( \inf_{q_{ij}} \sum_{i,j} q_{ij} d_\kappa^{\|\kappa\|_\infty}(\eta_i, \eta'_j) \right)^{1/\|\kappa\|_\infty}
\]

where the infimum is taken over all couplings \( q \) between \( p \) and \( p' \).

For instance, in skew-normal mixtures, if \( \eta = (\theta, v, m), \eta' = (\theta', v', m') \), and \( \kappa = (2, 1, 1) \), then \( d_\kappa(\eta, \eta') = (|\theta - \theta'|^2 + |v - v'| + |m - m'|)^{1/2} \). For general \( \kappa, d_\kappa \) is a semi-metric — it satisfies “weak” triangle inequality, i.e., it only satisfies the triangle inequality up to some positive constant less than 1, except when all \( \kappa_i \) are identical. This implies that \( \tilde{W}_\kappa(G, G_0) \) is a semi-metric that only satisfies weak triangle inequality (a proof of this fact is given by Lemma 6.1 in Appendix F). An easy relation between the generalized transportation distance and the standard Wasserstein distance can be obtained.

**Lemma 2.1.** For any \( \kappa \in \mathbb{N}^d \) such that \( \|\kappa\|_\infty = r \geq 1 \), we have

\[
\tilde{W}_\kappa(G, G') \gtrsim W_r(G, G'),
\]

where equality holds when \( \kappa_i = r \) for all \( 1 \leq i \leq d \).

The role of \( \kappa \) in the definition of generalized transportation distance is to capture the inhomogeneous behaviors of different parameters. In particular, assume that a sequence of probability measures \( G_n \in \mathcal{O}_k \) tending to \( G_0 \) under \( \tilde{W}_\kappa \) metric at a rate \( \omega_n = o(1) \) for some \( \kappa \in \mathbb{N}^d \). If all \( G_n \) have the same number of atoms \( k_n = k_0 \) as that of \( G_0 \), then the set of atoms of \( G_n \) converges to the \( k_0 \) atoms of \( G_0 \), up to a permutation of the atoms, at the same rate \( \omega_n \) under \( d_\kappa \) metric. Therefore, the \( i \)-th component of each atom of \( G_n \) will converge to the \( i \)-th component of corresponding atom of \( G_0 \) at rate \( (\omega_n)^{\|\kappa\|_\infty/\kappa_i} \) for any \( 1 \leq i \leq d \). Similar implication also holds for the case \( G_n \) having its number of components varying from \( k_0 \) to \( k \).

**Complete inhomogeneity of parameter space** As we have discussed above, different types of parameter (location, scale, skewness) may exhibit inhomogeneous convergence behavior. It is a tribute to the complexity of finite mixture models that it is not uncommon that different parameters of the same
type within the same model may also exhibit such inhomogeneity as well. To properly handle such situations, one needs a notion of “blocked generalized transportation distance”. Due to space constraint, the detailed formulation of this notion as well as convergence rate analysis of parameter estimation under this semi-metric are deferred to Appendix C. Our analysis with singularity structure of finite mixture models in the main text will focus on using Wasserstein distance or generalized transportation distance.

2.3 Estimation settings: e- and o-mixtures

The impact of singularities on parameter estimation behavior is dependent on whether the mixture model is fitted with a known number of mixing components, or if only an upper bound on the number of mixing components is known. The former model fitting setting is called “e-mixtures” for short, while the latter “o-mixtures” (“e” for exact-fitted and “o” for over-fitted).

Specifically, given an i.i.d. $n$-sample $X_1, X_2, \ldots, X_n$ according to the mixture density $p_{G_0}(x) = \int f(x|\eta)G_0(d\eta)$, where $G_0 = G(p^0, \eta^0) = \sum_{i=1}^{k_0} p^0_i \delta_{\eta^0_i}$ is unknown mixing measure with exactly $k_0$ distinct support points. We are interested in fitting a mixture of $k$ mixing components, where $k \geq k_0$, using the $n$-sample $X_1, \ldots, X_n$. In the e-mixture setting, $k = k_0$ is known, so an estimate $G_n$ (such as the maximum likelihood estimate) is selected from ambient space $\mathcal{E}_{k_0}$, the set of probability measures $G = G(p, \eta)$ with exactly $k_0$ support points, where $(p, \eta) \in \Omega$. In the o-mixture setting, $G_n$ is selected from ambient space $\mathcal{E}_k$, the set of probability measures $G = G(p, \eta)$ with at most $k$ support points, where $(p, \eta) \in \Omega$.

Throughout this paper, several conditions on the kernel density $f(x|\eta)$ are assumed to hold. Firstly, the collection of kernel densities $f$ as $\eta$ varies is linearly independent. It follows that the mixture model is identifiable, i.e., $p_G(x) = p_{G_0}(x)$ for almost all $x$ entails $G = G_0$. Secondly, we say $f(x|\eta)$ satisfies a uniform Lipschitz condition of order $r$, for some $r \geq 1$, if $f$ as a function of $\eta$ is differentiable up to order $r$, and that the partial derivatives with respect to $\eta$, namely $\partial^{[\kappa]} f / \partial \eta^{[\kappa]}$, for any $\kappa = (\kappa_1, \ldots, \kappa_d) \in \mathbb{N}^d$ such that $|\kappa| := \kappa_1 + \ldots + \kappa_d = r$ satisfy the following: for any $\gamma \in \mathbb{R}^d$,

$$
\sum_{|\kappa| = r} \left| \frac{\partial^{[\kappa]} f(x|\eta_1)}{\partial \eta^{[\kappa]}} (x|\eta_1) - \frac{\partial^{[\kappa]} f(x|\eta_2)}{\partial \eta^{[\kappa]}} (x|\eta_2) \right) \gamma^\kappa \leq C \|\eta_1 - \eta_2\|_r \|\gamma\|_r
$$

for some positive constants $\delta$ and $C$ independent of $x$ and $\eta_1, \eta_2 \in \Theta$. It is simple to verify that most kernel densities used in mixture modeling, including the skew-normal kernel, satisfy the uniform Lipschitz property over compact domain $\Theta$, for any finite $r \geq 1$.

3 Singularity structure in finite mixture models

3.1 Beyond Fisher information

Given a mixture model

$$
\left\{ p_G(x) \bigg| G = G(p, \eta) = \sum_{i=1}^{k} p_i \delta_{\eta_i}, (p, \eta) \in \Omega \right\}
$$

from some given finite $k$ and $f$ a given kernel density (e.g., skew-normal), let $l_G$ denote the score vector — column vector made of the partial derivatives of the log-likelihood function $\log p_G(x)$ with respect to each of the model parameters represented by $(p, \eta)$. The Fisher information matrix is then given by $I(G) = \mathbb{E}(l_G l_G^\top)$, where the expectation is taken with respect to $p_G$. We say that the parameter
vector \((p, \eta)\) (and the corresponding mixing measure \(G\)) is a singular point in the parameter space (resp., ambient space of mixing measures), if \(I(G)\) is degenerate. Otherwise, \((p, \eta)\) (resp., \(G\)) is a non-singular point.

According to the standard asymptotic theory, if the true mixing measure \(G_0\) is non-singular, and the number of mixing components \(k_0 = k\) (that is, we are in the e-mixture setting), then basic estimators such as the MLE or Bayesian estimator yield the optimal root-\(n\) rate of convergence. Although the standard theory remains silent when \(I(G_0)\) is degenerate, it is clear that the root-\(n\) rate may no longer hold. Moreover, there may be a richer range of behaviors for parameter estimation, requiring us to look into the deep structure of the zeros of \(I(G_0)\). This will be our story for both settings of e-mixtures and o-mixtures. In fact, the (determinant of the) Fisher information matrix \(I(G_0)\) is no longer sufficient in assessing parameter estimation behaviors.

Example 3.1. To illustrate in the context of skew-normal mixtures, where parameter \(\eta = (\theta, v, m)\), observe that the mixture density structure allows the following characterization: \(I(G)\) is degenerate if and only if the collection of partial derivatives

\[
\left\{ \frac{\partial p_G(x)}{\partial p_j}, \frac{\partial p_G(x)}{\partial \eta_j} \right\} = \left\{ \frac{\partial p_G(x)}{\partial p_j}, \frac{\partial p_G(x)}{\partial \theta_j}, \frac{\partial p_G(x)}{\partial v_j}, \frac{\partial p_G(x)}{\partial m_j} \right\} \quad j = 1, \ldots, k
\]

as functions of \(x\) are not linearly independent. This is equivalent to, for some coefficients \((\alpha_{ij})\), \(i = 1, \ldots, 4\) and \(j = 1, \ldots, k\), not all of which are zeros, there holds

\[
\sum_{j=1}^{k} \alpha_{1j} f(x|\eta_j) + \alpha_{2j} \frac{\partial f}{\partial \theta}(x|\eta_j) + \alpha_{3j} \frac{\partial f}{\partial v}(x|\eta_j) + \alpha_{4j} \frac{\partial f}{\partial m}(x|\eta_j) = 0,
\]

for almost all \(x \in \mathbb{R}\). Lemma 4.1 later shows that the (Fisher information matrix’s) singular points are the zeros of some polynomial equations.

We have seen that for the e-mixtures \(G\) is non-singular if the collection of density kernel functions \(f(x|\eta)\) and their first partial derivatives with respect to each model parameter are linearly independent. This condition is also known as the first-order identifiability. For o-mixtures, the relevant notion is the second-order identifiability [14, 43, 29]: the condition that the collection of kernel density functions \(f(x|\eta)\), their first and second partial derivatives, are linearly independent. This condition fails to hold for skew-normal kernel densities, whose first and second partial derivatives are linked by the following nonlinear partial differential equations:

\[
\begin{align*}
\frac{\partial^2 f}{\partial \theta^2} - 2 \frac{\partial f}{\partial v} + m^3 + m \frac{\partial f}{\partial m} &= 0, \\
2m \frac{\partial f}{\partial m} + (m^2 + 1) \frac{\partial^2 f}{\partial m^2} + 2vm \frac{\partial^2 f}{\partial v \partial m} &= 0.
\end{align*}
\]

The proof of these identities can be found in Lemma 6.2 in Appendix F. Note that if \(m = 0\), the skew-normal kernel becomes normal kernel, which admits a (simpler) linear relationship:

\[
\frac{\partial^2 f}{\partial \theta^2} = 2 \frac{\partial f}{\partial v}.
\]

This relation, also noted previously by [11, 32], plays a fundamental role in the analysis of finite mixtures of location-scale normal distributions [28]. Unlike Gaussian kernel’s, the nonlinear relations expressed by above PDEs for skew-normal density kernel underscore the exceptional complexity of general mixture models — the inhomogeneity of the parameter space. Analyzing this requires the development of a general theory that we now embark on.
3.2 Likelihood in Wasserstein neighborhood

Instead of dwelling on the Fisher information matrix, we employ a direct approach by studying the behavior of the likelihood function \( p_G(x) \) as \( G \) varies in a Wasserstein neighborhood of a mixing measure \( G_0 = \sum_{i=1}^{k_0} p_i^0 \delta_{\eta_i^0} \).

Fix \( r \geq 1 \), and consider an arbitrary sequence of \( G_n \in \mathcal{O}_k \), such that \( W_r(G_n, G_0) \to 0 \). Let \( k_n \leq k \) be the number of distinct support points of \( G_n \). There exists a subsequence of \( G_n \) for which \( k_n \) is constant in \( n \) and each supporting atom \( \eta_i^0 \) as \( i \in \{1, \ldots, k_0\} \) of \( G_0 \) is the limit point of exactly \( s_i \) atoms of \( G_n \). Additionally, there may be also a subset of atoms of \( G_n \) whose limits are not among the atoms of \( G_0 \). Without loss of generality, we assume that there are \( \bar{l} \geq 0 \) such limit points. By relabelling its atoms, we can express \( G_n \) as

\[
G_n = \sum_{i=1}^{k_0+\bar{l}} \sum_{j=1}^{s_i} p_{ij}^n \delta_{\eta_i^j},
\]

where \( \eta_{ij}^n \to \eta_i^0 \) for all \( i = 1, \ldots, k_0 + \bar{l}, j = 1, \ldots, s_i \). Additionally, \( \sum_{i=1}^{k_0+\bar{l}} s_i = k_n \). Thus, from here on we replace the sequence of \( G_n \) by this subsequence. To simplify the notation, \( n \) will be dropped from the superscript when the context is clear. In addition, we use the notation \( \Delta \eta_{ij} := \eta_{ij} - \eta_i^0 \) for \( i = 1, \ldots, k_0 + \bar{l}, j = 1, \ldots, s_i \). Also, \( p_i := \sum_j p_{ij} \), and \( \Delta p_i := p_i - p_i^0 \), for \( i = 1, \ldots, k_0 + \bar{l} \) where \( p_i^0 = 0 \) as \( k_0 + 1 \leq i \leq k_0 + \bar{l} \). For the setting of \( \epsilon \)-mixtures, the sequence of elements \( G_n \) is restricted to \( \mathcal{E}_{k_0} \subset \mathcal{O}_k \), so \( k_n = k_0 \) for all \( n \). It follows that \( s_i^0 = 1 \) for all \( i = 1, \ldots, k_0 \) and \( \bar{l} = 0 \), so the notation is simplified further: let \( \Delta \eta_i := \Delta \eta_{i1} = \eta_i - \eta_i^0, \Delta p_i := \Delta p_i = p_i - p_i^0 \) for all \( i = 1, \ldots, k_0 \).

The following lemma relates Wasserstein metric to a semipolynomial of degree \( r \) (a semipolynomial is a polynomial of a collection of variables and/or the absolute value of some of the variables).

**Lemma 3.1.** Fix \( r \geq 1 \). For any element \( G \) represented by Eq. (4), define

\[
D_r(G, G_0) := \sum_{i=1}^{k_0+\bar{l}} \sum_{j=1}^{s_i} p_{ij} \| \Delta \eta_{ij} \|^r_r + \sum_{i=1}^{k_0+\bar{l}} |\Delta p_i|.
\]

Then \( W_r(G, G_0) \sim D_r(G, G_0) \), as \( W_r(G, G_0) \downarrow 0 \).

To investigate the behavior of likelihood function \( p_G(x) \) as \( G \) tends to \( G_0 \) in Wasserstein distance \( W_r \), by representation (4),

\[
p_G(x) - p_{G_0}(x) = \sum_{i=1}^{k_0+\bar{l}} \sum_{j=1}^{s_i} p_{ij} (f(x|\eta_{ij}) - f(x|\eta_i^0)) + \sum_{i=1}^{k_0+\bar{l}} \Delta p_i f(x|\eta_i^0).
\]

By Taylor expansion up to order \( r \),

\[
p_G(x) - p_{G_0}(x) = \sum_{i=1}^{k_0+\bar{l}} \sum_{j=1}^{s_i} p_{ij} \sum_{|\kappa|=1}^r \frac{\partial^{|\kappa|} f}{\partial \eta^{|\kappa|}} (x|\eta_i^0) + \sum_{i=1}^{k_0+\bar{l}} \Delta p_i f(x|\eta_i^0) + R_r(x),
\]

where \( R_r(x) \) is the Taylor remainder. Moreover, it can be verified that \( \sup_x |R_r(x)/W_r^r(G, G_0)| \to 0 \) since \( f \) is uniform Lipschitz up to order \( r \). We arrive at the following formulae, which measures the
speed of change of the likelihood function as \( G \) varies in the Wasserstein neighborhood of \( G_0 \):

\[
\frac{p_G(x) - p_{G_0}(x)}{W_r^p(G, G_0)} = \sum_{|\kappa|=1}^{r} \sum_{i=1}^{k_0+i} \sum_{j=1}^{s_i} \left( p_{ij} \frac{(\Delta \eta_{ij})^\kappa}{\kappa!} \right) \frac{\partial |\kappa| f}{\partial \eta^\kappa}(x|\eta_i^0) + \sum_{i=1}^{k_0+i} \Delta p_i \frac{f(x|\eta_i^0)}{W_r^p(G, G_0)} + o(1).
\]

(7)

The right hand side of Eq. (7) is a linear combination of the partial derivatives of \( f \) evaluated at \( G_0 \). It is crucial to note, by Lemma 3.2, each coefficient of this linear representation is asymptotically equivalent to the ratio of two semipolynomials.

Equation (7) highlights the distinct roles of model parameters and the kernel density function in the formation of a mixture model’s likelihood. The former appears only in the coefficients, while the latter provides the partial derivatives which appear as if they are basis functions for the linear combination. We wrote “as if”, because the partial derivatives of kernel \( f \) may not be linearly independent functions (recall the examples in Section 3.1). When a partial derivative of \( f \) can be represented as a linear combination of other partial derivatives, it can be eliminated from the expression in the right hand side. This reduction process may be repeatedly applied until all partial derivatives that remain are linearly independent functions. This motivates the following concept:

**Definition 3.1.** The following representation is called \( r \)-minimal form of the mixture likelihood for a sequence of mixing measures \( G \) tending to \( G_0 \) in \( W_r \) metric:

\[
\frac{p_G(x) - p_{G_0}(x)}{W_r^p(G, G_0)} = \sum_{l=1}^{T_r} \left( \frac{\xi^{(r)}_l(G)}{W_r^p(G, G_0)} \right) H^{(r)}_l(x) + o(1),
\]

(8)

which holds for almost all \( x \), with the index \( l \) ranging from 1 to a finite \( T_r \), if

1. \( H^{(r)}_l(x) \) for all \( l \) are linearly independent functions of \( x \), and
2. coefficients \( \xi^{(r)}_l(G) \) are polynomials of the components of \( \Delta \eta_{ij} \) and \( \Delta p_i \).

It is sufficient, but not necessary, to select functions \( H^{(r)}_l \) from the collection of partial derivatives \( \frac{\partial |\kappa| f}{\partial \eta^\kappa} \) evaluated at particular atoms \( \eta_i^0 \) of \( G_0 \), where \( |\kappa| \leq r \), by adopting the elimination technique. The precise formulation of \( \xi^{(r)}_l(G) \) and \( H^{(r)}_l(x) \) will be determined explicitly by the specific \( G_0 \). The \( r \)-minimal form for each \( G_0 \) is not unique, but they play a fundamental role in the notion of singularity level of a mixing measure relative to an ambient space that we now define.

**Definition 3.2.** Fix \( r \geq 1 \) and let \( \mathcal{G} \) be a class of discrete probability measures which has a bounded number of support points in \( \Theta \). We say \( G_0 \) is \( r \)-singular relative to \( \mathcal{G} \), if \( G_0 \) admits a \( r \)-minimal form given by Eq. (8), according to which there exists a sequence of \( G \in \mathcal{G} \) tending to \( G_0 \) under \( W_r \) such that

\[
\xi^{(r)}_l(G)/W_r^p(G, G_0) \to 0 \text{ for all } l = 1, \ldots, T_r.
\]

We now verify that the \( r \)-singularity notion is well-defined, in that it does not depend on any specific choice of the \( r \)-minimal form. This invariant property is confirmed by part (a) of the following lemma. Part (b) establishes a crucial monotonic property.

**Lemma 3.2.** (a) (Invariance) The existence of the sequence of \( G \) in the statement of Definition 3.2 holds for all \( r \)-minimal forms once it holds for at least one \( r \)-minimal form.

(b) (Monotonicity) If \( G_0 \) is \( r \)-singular for some \( r > 1 \), then \( G_0 \) is \((r-1)\)-singular.

The monotonicity of \( r \)-singularity naturally leads to the notion of singularity level of a mixing measure \( G_0 \) (and the corresponding parameters) relative to an ambient space \( \mathcal{G} \).
Definition 3.3. The singularity level of $G_0$ relative to a given class $\mathcal{G}$, denoted by $\ell(G_0|\mathcal{G})$, is

0, if $G_0$ is not $r$-singular for any $r \geq 1$;

$\infty$, if $G_0$ is $r$-singular for all $r \geq 1$;

otherwise, the largest natural number $r \geq 1$ for which $G_0$ is $r$-singular.

The role of the ambient space $\mathcal{G}$ is critical in determining the singularity level of $G_0 \in \mathcal{G}$. Clearly, by definition if $G_0 \in \mathcal{G} \subset \mathcal{G}'$, $r$-singularity relative to $\mathcal{G}$ entails $r$-singularity relative to $\mathcal{G}'$. This means $\ell(G_0|\mathcal{G}) \leq \ell(G_0|\mathcal{G}')$.

Let us look at the following examples.

- Take $\mathcal{G} = \mathcal{E}_{k_0}$, i.e., the setting of e-mixtures. If the collection of $\{\partial^k f / \partial \eta^k(x|\eta_j)|j = 1, \ldots, k_0; |\kappa| \leq 1\}$ evaluated at $G_0$ is linearly independent, then $G_0$ is not 1-singular relative to $\mathcal{E}_{k_0}$. It follows that $\ell(G_0|\mathcal{G}) = 0$.

- Take $\mathcal{G} = \mathcal{O}_k$ for any $k > k_0$, i.e., the setting of o-mixtures. It is not difficult to check that $G_0$ is always 1-singular relative to $\mathcal{O}_k$ for any $k > k_0$. Thus, $\ell(G_0|\mathcal{G}) \geq 1$. Now, if the collection of $\{\partial^k f / \partial \eta^k(x|\eta_j)|j = 1, \ldots, k_0; |\kappa| \leq 2\}$ evaluated at $G_0$ is linearly independent, then it can be observed that $G_0$ is not 2-singular relative to $\mathcal{O}_k$. Thus, $\ell(G_0|\mathcal{G}) = 1$.

The conditions described in the two examples above are in fact referred to as strong identifiability conditions studied by [4], [3], [29]. The notion of singularity level generalizes such identifiability conditions, by allowing us to consider situations where such conditions fail to hold. This is the situation where $\ell(G_0|\mathcal{G}) = 2, 3, \ldots, \infty$. The significance of this concept can be appreciated by the following theorem.

Theorem 3.1. Let $\mathcal{G}$ be a class of probability measures on $\Theta$ that have a bounded number of support points, and fix $G_0 \in \mathcal{G}$. Suppose that $\ell(G_0|\mathcal{G}) = r$, for some $0 \leq r \leq \infty$.

\begin{enumerate}[(i)]  
  \item If $r < \infty$, then $\inf_{G \in \mathcal{G}} \left\| \frac{p_G - p_{G_0}}{W_r^s(G, G_0)} \right\|_\infty > 0$ for any $s \geq r + 1$.  
  \item If $r < \infty$, then $\inf_{G \in \mathcal{G}} \frac{V(p_G, p_{G_0})}{W_r^s(G, G_0)} > 0$ for any $s \geq r + 1$.
\end{enumerate}

The following theorem establishes that the bounds obtained above are tight under some condition.

Theorem 3.2. Consider the same setting as that of Theorem 3.1.

\begin{enumerate}[(i)]  
  \item If $1 \leq r < \infty$ and in addition,
    \begin{enumerate}[(a)]  
      \item $f$ is $(r+1)$-order differentiable with respect to $\eta$ and for some constant $c_0 > 0$,
        \[ \sup_{\|\eta - \eta'\| \leq c_0} \int_{x \in \mathcal{X}} \left( \frac{\partial^s f}{\partial \eta^s(x|\eta)} \right)^2 / f(x|\eta') dx < \infty \]  
        for any index $\alpha$ such that $|\alpha| = s$ and $s \in \{1, \ldots, r\}$.
      \item There is a sequence of $G \in \mathcal{G}$ tending to $G_0$ in Wasserstein metric $W_r$ and the coefficients of the $r$-minimal form $\xi^{(r)}(G)$ satisfy $\xi^{(r)}(G)/W_r(G, G_0) \to 0$ for all real number $s \in [1, r + 1]$ and $l = 1, \ldots, T_r$. Additionally, all the masses $p_{ij}$ in the representation [4] of $G$ are bounded away from 0.
    \end{enumerate}
\end{enumerate}
Then, for any real number \( s \in [1, r + 1) \),

\[
\liminf_{G \in \mathcal{G}, W_1(G, G_0) \to 0} \frac{h(p_G, p_{G_0})}{W_1^s(G, G_0)} = \liminf_{G \in \mathcal{G}, W_s(G, G_0) \to 0} \frac{V(p_G, p_{G_0})}{W^s_s(G, G_0)} = 0.
\]

(ii) If \( \ell(G_0|\mathcal{G}) = \infty \) and the conditions (a), (b) in part (ii) hold for any \( l \in \mathbb{N} \) (here, the parameter \( r \) in these conditions is replaced by \( l \)), then the conclusion of part (i) holds for any \( s \geq 1 \).

We make a few remarks.

• Part (i) and part (ii) of Theorem 3.1 show how the singularity level of \( G_0 \) relative to ambient space \( \mathcal{G} \) may be used to translate the convergence of mixture densities (under the sup-norm and the total variation distance) into the convergence of mixing measures under a suitable Wasserstein metric \( W_s \). Part (i) of Theorem 3.2 shows a sufficient condition under which the power \( r + 1 \) in the bounds from Theorem 3.1 cannot be improved.

• In part (i) of Theorem 3.2 the condition regarding the integrand of the partial derivative of \( f \) (cf. Eq. (31)) can be easily checked to be satisfied by many kernels, such as Gaussian kernel, Gamma kernel, Student t’s kernel, and skew-normal kernel.

• Condition (b) regarding the sequence of \( G \) appears somewhat opaque in general, but it will be illustrated in specific examples for skew-normal mixtures in the sequel. It is sufficient, but not necessary, for verifying the \( r \)-singularity of \( G_0 \) to construct the sequence of \( G \) so that \( \xi_l^{(r)}(G) = 0 \) for all \( 1 \leq l \leq T_r \), provided such a sequence exists. This requires finding an appropriate parameterization of a sequence of \( G \) tending toward \( G_0 \) that satisfy a number of polynomial equations defined in terms of the parameter perturbations.

We are ready to state the impact of the singularity level of mixing measure \( G_0 \) relative to an ambient space \( \mathcal{G} \) on the rate of convergence for an estimate of \( G_0 \), where \( \mathcal{G} = \mathcal{E}_{k_0} \) in e-mixtures, and \( \mathcal{G} = \mathcal{O}_k \) in o-mixtures. Let \( \mathcal{G} \) be structured into a sieve of subsets defined by the maximum singularity level relative to \( \mathcal{G} \).

\[
\mathcal{G} = \bigcup_{r=1}^{\infty} \mathcal{G}_r, \quad \mathcal{G}_r := \left\{ G \in \mathcal{G} \mid \ell(G|\mathcal{G}) \leq r \right\}, \quad r = 0, 1, \ldots, \infty.
\]

The first part of the following theorem gives a minimax lower bound for the estimation of the mixing measure \( G_0 \), given that the singularity level of \( G_0 \) is known up to a constant \( r \geq 1 \). The second part gives a quick result on the convergence rate of a point estimate such as the MLE.

**Theorem 3.3.**

(a) Fix \( r \geq 1 \). Assume that for any \( G_0 \in \mathcal{G}_r \), the conclusion of part (i) of Theorem 3.2 holds for \( \mathcal{G}_r \) (i.e., \( \mathcal{G} \) is replaced by \( \mathcal{G}_r \) in that theorem). Then, for any real number \( s \in [1, r + 1) \) there holds

\[
\inf_{\mathcal{G}_r \in \mathcal{G}_r, G_0 \in \mathcal{G}_r} \sup_{p_{G_0}} E_{p_{G_0}} W_s(\widehat{G}_n, G_0) \gtrsim n^{-1/2s}.
\]

Here, the infimum is taken over all sequences of estimates \( \widehat{G}_n \in \mathcal{G}_r \) and \( E_{p_{G_0}} \) denotes the expectation taken with respect to product measure with mixture density \( p_{G_0}^{G_0} \).
Proof. Part (a) of this theorem is a consequence of the conclusion of Theorem 3.2 part (iii). The proof of this fact is rather standard, and similar to that of Theorem 1.1. of [28] and is omitted. Part (b) follows immediately from part (ii) of Theorem 3.1 as we have \( h(p_{\hat{G}_n}, p_{G_0}) \geq V(p_{\hat{G}_n}, p_{G_0}) \geq W_{r+1}(\hat{G}_n, G_0) \). \( \square \)

We conclude this section with some comments. It is well-known that many density estimation methods, such as MLE and Bayesian estimation applied to a compact parameter space for parametric mixture models, guarantee a root-\( n \) rate (up to a logarithmic term) of convergence under Hellinger distance metric on the density functions (cf. [50, 23, 19]). It follows that, as far as we are concerned, the remaining work in establishing the convergence behavior of mixing measure estimation (as opposed to density estimation) lies in the calculation of the singularity levels, i.e., the identification of sets \( G_r \). For skew-normal mixtures, such calculations will be carried out in Section 4 (with further details given in Appendices E and F). For the settings of \( G_0 \) where we are able to obtain the exact singularity levels, we can also construct the sequence of \( G \) required by part (i) of Theorem 3.2. Whenever the exact singularity level is obtained, we automatically obtain a (local) minimax lower bound and a matching upper bound for MLE convergence rate under a Wasserstein distance metric, thanks to the above theorem. In some cases, however, the singularity level of \( G_0 \) may be not determined exactly, but only an upper bound given. In such cases, only an upper bound to the convergence rate of the MLE can be obtained, while minimax lower bounds may be unknown.

### 3.3 Inhomogeneity of parameter space

Singularity level is an useful concept for deriving rates of convergence for the mixing measure under a suitable Wasserstein metric, which in turn entails upper bound on the rate of convergence for individual model parameters (which make up the atoms of the mixing measure). However, different parameters may actually admit different convergence behaviors. To study this phenomenon of inhomogeneity, a more elaborate notion is required to describe the singularity structure of the parameter space. In this section we shall introduce such a notion, which we call singularity index, along with generalized transportation distance for the mixing measures already given by Def. 2.1.

As in the previous section, we adopt the strategy of investigating the behavior of the likelihood function \( p_G(x) \) as \( G \) varies in a generalized transportation neighborhood of \( G_0 \). In particular, for any fixed \( \kappa \in \mathbb{N}^d \), consider a sequence of \( G_n \in \mathcal{O}_\kappa \) such that \( \tilde{W}_\kappa(G_n, G_0) \to 0 \). Let \( G_n \) be represented as in (4). To avoid notational cluttering, we again drop \( n \) from the superscript when the context is clear. The following lemma relates generalized transportation distance to a semipolynomial of order \( \|\kappa\|_\infty \).

**Lemma 3.3.** Fix \( \kappa \in \mathbb{N}^d \). For any element \( G \) represented by Eq. (4), define

\[
D_\kappa(G, G_0) := \frac{k_0 + l}{\sum_{i=1}^{k_0+l} \sum_{j=1}^{k_0} p_{ij} d_{ij}^{\|\kappa\|_\infty} (\eta_{ij}, \eta_{0i})} + \sum_{i=1}^{k_0+l} |\Delta p_i|.
\]

Then \( \tilde{W}_\kappa^{\|\kappa\|_\infty} (G, G_0) \asymp D_\kappa(G, G_0) \), as \( \tilde{W}_\kappa(G, G_0) \downarrow 0 \).
Denote \( r := \|\kappa\|_\infty \) (this notational choice is deliberate, as we will see shortly). By virtue of the argument from Section 3.2, using Taylor expansion up to the \( r \)-order we have

\[
\frac{p_G(x) - p_{G_0}(x)}{\bar{W}_\kappa^{\|\kappa\|_\infty} (G, G_0)} = \sum_{|\alpha| = 1}^{r} \sum_{i=1}^{k_0+l} \sum_{j=1}^{s_i} \left( \frac{p_{ij}(\Delta \eta_{ij})^\alpha / \alpha!}{\bar{W}_\kappa^{\|\kappa\|_\infty} (G, G_0)} \right) \frac{\partial |\alpha| f(x|\eta_i^0)}{\partial \eta^\alpha} +
\]

\[
\sum_{i=1}^{k_0+l} \frac{\Delta p_i}{W^{\|\kappa\|_\infty} (G, G_0)} f(x|\eta_i^0) + \frac{R_r(x)}{W^{\|\kappa\|_\infty} (G, G_0)},
\]

where \( R_r(x) \) is the Taylor remainder. By Lemma 2.1 we readily verify that

\[
\sup_x |R_r(x)/\bar{W}_\kappa^{\|\kappa\|_\infty} (G, G_0)| \lesssim \sup_x |R_r(x)/W_r^{\|\kappa\|_\infty} (G, G_0)| \to 0
\]
as long as \( f \) is uniform Lipschitz up to order \( r \). Thus,

\[
\frac{p_G(x) - p_{G_0}(x)}{\bar{W}_\kappa^{\|\kappa\|_\infty} (G, G_0)} = \sum_{|\alpha| = 1}^{r} \sum_{i=1}^{k_0+l} \sum_{j=1}^{s_i} \left( \frac{p_{ij}(\Delta \eta_{ij})^\alpha / \alpha!}{\bar{W}_\kappa^{\|\kappa\|_\infty} (G, G_0)} \right) \frac{\partial |\alpha| f(x|\eta_i^0)}{\partial \eta^\alpha}
\]

\[
+ \sum_{i=1}^{k_0+l} \frac{\Delta p_i}{W^{\|\kappa\|_\infty} (G, G_0)} f(x|\eta_i^0) + o(1).
\]

We are ready to define the following concept.

**Definition 3.4.** For any \( \kappa \in \mathbb{N}^d \), the following representation is called \( \kappa \)-minimal form of the mixture likelihood for a sequence of mixing measures \( G \) tending to \( G_0 \) in \( \bar{W}_\kappa \) distance:

\[
\frac{p_G(x) - p_{G_0}(x)}{\bar{W}_\kappa^{\|\kappa\|_\infty} (G, G_0)} = \sum_{l=1}^{T_\kappa} \left( \frac{\xi_l^{(\kappa)}(G)}{\bar{W}_\kappa^{\|\kappa\|_\infty} (G, G_0)} \right) H_l^{(\kappa)}(x) + o(1),
\]

which holds for almost all \( x \), with the index \( l \) ranging from 1 to a finite \( T_\kappa \), if

1. \( H_l^{(\kappa)}(x) \) for all \( l \) are linearly independent functions of \( x \), and
2. coefficients \( \xi_l^{(\kappa)}(G) \) are polynomials of the components of \( \Delta \eta_{ij} \), and \( \Delta p_i, p_{ij} \).

It is clear that \( \kappa \)-minimal form is a general version of \( r \)-minimal form when \( \kappa = (r, \ldots, r) \). The procedure for constructing \( \kappa \)-minimal forms is similar that of \( r \)-minimal forms, and will be given in Section 3.5 where we specifically search for a subset of linearly independent partial derivatives up to the order \( \|\kappa\|_\infty \). The multi-index \( \kappa \)-form provides the basis for the notion of multi-index singularity that we now define.

**Definition 3.5.** Let \( G \) be a class of discrete probability measures which has a bounded number of support points in \( \Theta \). For any \( \kappa \in \mathbb{N}^d \), we say that \( G_0 \) is \( \kappa \)-singular relative to \( G \), if \( G_0 \) admits a \( \kappa \)-minimal form given by Eq. (10), according to which there exists a sequence of \( G \in G \) tending to \( G_0 \) under \( W_\kappa \) distance such that

\[
\xi_l^{(\kappa)}(G)/\bar{W}_\kappa^{\|\kappa\|_\infty} (G, G_0) \to 0 \text{ for all } l = 1, \ldots, T_\kappa.
\]

Like \( r \)-singularity, the notion of \( \kappa \)-singularity possesses a crucial monotonic property in terms of partial order with vector:
Lemma 3.4. (a) (Invariance) The existence of the sequence of \( G \) in the statement of Definition 3.3 holds for all \( \kappa \)-minimal forms once it holds for at least one \( \kappa \)-minimal form.

(b) (Monotonicity) If \( G_0 \) is \( \kappa \)-singular for some \( \kappa \in \mathbb{N}^d \), then \( G_0 \) is \( \kappa' \)-singular for any \( \kappa' \leq \kappa \).

Let \( \mathbb{N} := \mathbb{N} \cup \{\infty\} \). The monotonicity of \( \kappa \)-singularity naturally leads to the following notion of singularity index of a mixing measure \( G_0 \) (and the corresponding parameters) relative to an ambient space \( \mathcal{G} \):

Definition 3.6. For any \( \kappa \in \mathbb{N}^d \), we say \( \kappa \) is a singularity index of \( G_0 \) relative to a given class \( \mathcal{G} \) if and only if \( G_0 \) is \( \kappa' \)-singular relative to \( \mathcal{G} \) for any \( \kappa' < \kappa \), and there is no \( \kappa' \geq \kappa \) such that \( G_0 \) remains \( \kappa' \)-singular relative to \( \mathcal{G} \). Define the singularity index set

\[
\mathcal{L}(G_0|\mathcal{G}) := \left\{ \kappa \in \mathbb{N}^d : \kappa \text{ is a singularity index of } G_0 \text{ relative to } \mathcal{G} \right\}.
\]

This definition suggests that the singularity index set may not be always a singleton in general. The following proposition clarifies the relation between singularity level \( \ell(G_0|\mathcal{G}) \) and singularity index set \( \mathcal{L}(G_0|\mathcal{G}) \).

Proposition 3.1. Assume that \( \ell(G_0|\mathcal{G}) = r \) for some \( r \geq 0 \). Then,

(i) If \( r = 0 \), then \( \mathcal{L}(G_0|\mathcal{G}) = \{(1, \ldots, 1)\} \).

(ii) If \( r = \infty \), then \( \mathcal{L}(G_0|\mathcal{G}) = \{ (\infty, \ldots, \infty) \} \).

(iii) If \( 1 \leq r < \infty \), then there exists \( \kappa \in \mathcal{L}(G_0|\mathcal{G}) \) such that \( \kappa \leq (r + 1, \ldots, r + 1) \) and at least one component of \( \kappa \) is \( r + 1 \).

(iv) If \( r \geq 1 \), and \( G_0 \) is not \( \infty \)-singular relative to \( \mathcal{G} \) for some \( \infty \in \mathbb{N}^d \), then there exists \( \kappa \in \mathcal{L}(G_0|\mathcal{G}) \) such that \( \kappa \leq \infty \).

(v) If some finite \( \kappa \in \mathcal{L}(G_0|\mathcal{G}) \), then \( \ell(G_0|\mathcal{G}) \leq \|\kappa\|_\infty - 1 \). Moreover, if \( \kappa \) is unique, then \( \ell(G_0|\mathcal{G}) = \|\kappa\|_\infty - 1 \).

This proposition establishes that when the singularity index set of a mixing measure \( G_0 \) relative to \( \mathcal{G} \) is a singleton, one can determine the corresponding singularity level of \( G_0 \) immediately. We will give several examples of ambient space \( \mathcal{G} \) and kernel \( f \) under which this situation holds (cf. the examples in Section 3.4). The role of the singularity index is in determining minimax lower bound and convergence rate of estimation for individual parameters that make up the mixing measure \( G_0 \). Briefly speaking, provided that \( \kappa \) is a singularity index of \( G_0 \), then any density estimation method, such as MLE or Bayesian estimation, that guarantees, say, a root-\( n \) rate of convergence toward density \( p_{G_0} \) under Hellinger metric will lead to the convergence rate \( n^{-1/2}\|\kappa\|_\infty \) of estimating \( G_0 \) under generalized transportation metric \( \tilde{W}_{\kappa} \), which is also minimax under additional conditions on kernel density \( f \). The implication of such results is that the \( i \)-th component of each atom of \( G_0 \) can be estimated with rate \( n^{-1/2\kappa_i} \) where \( \kappa_i \) is the \( i \)-th component of \( \kappa \). This theory will be presented in Appendix B.

**Complete inhomogeneity** Although the singularity index captures the inhomogeneous convergence behavior of different components of an atom of \( G_0 \), i.e., parameters of different types such as location, scale and skewness, it is possible that every parameter in a mixture model admits a different convergence rate, including those of the same type but associating with different mixture components. We call this phenomenon ”complete inhomogeneity”. To characterize this, we shall introduce blocked generalized transportation distance by replacing uniform semi-metric \( d_{\kappa} \) in the formulation of generalized
transportation distance as possibly different semi-metrics \( d_K \), with respect to the \( i \)-th atom \( \eta_i^0 \) of \( G_0 \) where \( K_i \in \mathbb{N}^d \) for all \( 1 \leq i \leq k_0 \). The best possible convergence rates of \( \eta_i^0 \), therefore, will be determined by the optimal choices of \( K_i \) for any \( i \). To quantify these optimal choices, we define a new notion of singularity matrix in terms of a matrix \( K \) which treat all \( K_i \) as its rows. With this concept in place, we can establish rates of convergence for estimating \( G_0 \), its atoms, as well as components of these atoms based on specific values of the singularity matrix. Due to space constraint, the detailed formulation and discussion of singularity matrix are deferred to Appendix C.

### 3.4 Revisiting known results on finite mixtures

In this section, singularity structures of parameter space will be examined to shed some light on previously known or recent results on the parameter estimation behavior of several classes of finite mixtures.

**O-mixtures with second-order identifiable kernels**  As being studied by [14, 43, 29, 46], the second order identifiability condition of kernel density \( f \) simply means that the collection of \( \{ \partial^k f / \partial \eta^k (x | \eta) \} | j = 1, \ldots, k_0; | \eta | \leq 2 \) evaluated at \( G_0 \) is linearly independent.

**Proposition 3.2.** Assume that \( f \) is second-order identifiable and admits uniform Lipschitz condition up to the second order. Then, \( \ell(G_0|O_k) = 1 \) and \( L(G_0|O_k) = \{ (2, \ldots, 2) \} \).

By Proposition 3.2 and Theorem 3.3, the convergence rate of estimating \( G_0 \) under o-mixtures of second order identifiable kernel \( f \) is \( n^{-1/4} \). Moreover, by Theorem 7.2 in Appendix B, the components of each atom of \( G_0 \) also admit uniform convergence rate \( n^{-1/4} \).

**Univariate Gaussian o-mixtures**  Location-scale Gaussian mixtures are among the most popular mixture models in statistics. For simplicity, consider univariate Gaussian o-mixtures and let \( G_0 \in \mathcal{E}_{k_0} \subset \mathcal{O}_{k_0,c_0} \) for some \( k > k_0 \) and small constant \( c_0 > 0 \). That is, the ambient space \( \mathcal{O}_{k_0,c_0} \subset \mathcal{O}_k \) contains only (discrete) probability measures whose point masses are bounded from below by \( c_0 \). Let \( \{ f(x|\theta, v = \sigma^2) \} \) be the family of univariate location-scale Gaussian distributions. Recall the partial differential equation, Eq. (3), satisfied by Gaussian kernels [11, 32, 28]. Following [28], denote by \( \tau(k-k_0) \) the minimal value of \( r > 0 \) such that the following system of polynomial equations

\[
\sum_{j=1}^{k-k_0+1} \sum_{n_1+n_2=\alpha \atop n_1,n_2 \geq 0} c_j^2 b_j^{n_1} b_j^{n_2} = 0 \quad \text{for each } \alpha = 1, \ldots, s
\]  

(11)

does not have any solution for the unknowns \( (a_j, b_j, c_j)^{k-k_0+1} \) such that all of \( c_j \)'s are non-zeros, and at least one of the \( a_j \)'s is non-zero. By means of the argument from Proposition 2.2 in [28], we can quickly verify that the singularity level of \( G_0 \) is \( \ell(G_0|\mathcal{O}_{k_0,c_0}) = r(k-k_0) - 1 \). It leads to the convergence rate \( n^{-1/2r(k-k_0)} \) of estimating mixing measure \( G_0 \) when we overfit Gaussian mixture models by \( k \) components, as established by [28]. However, we can say more: it turns out that the location parameters and the scale parameters in the Gaussian mixtures admit different rates of convergence. This is due to examining the singularity index of \( G_0 \).

**Proposition 3.3.** For any \( G_0 \in \mathcal{E}_{k_0} \cap \mathcal{O}_{k_0,c_0} \), we obtain

\[
L(G_0|\mathcal{O}_{k_0,c_0}) = \begin{cases} 
\{ (\tau(k-k_0), \frac{\tau(k-k_0)}{2} ) \}, & \text{if } \tau(k-k_0) \text{ is an even number} \\
\{ (\tau(k-k_0), \frac{\tau(k-k_0)+1}{2} ) \}, & \text{if } \tau(k-k_0) \text{ is an odd number}.
\end{cases}
\]  

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The result of Proposition 3.3 indicates that under Gaussian o-mixtures the best possible convergence rate of location parameter is \( n^{-1/2} \Gamma(k-k_0) \) while that of scale parameter is \( n^{-1/2} \Gamma(k-k_0) \) when \( k \) is even or is \( n^{-1/2} \Gamma(k-k_0)+1 \) when \( k \) is odd. These convergence rates are sharp, thanks to part (a) of Theorem 7.2 in Appendix B. Thus, in an overfitted Gaussian mixture, the more overfitted the model is, the slower the estimation rate. Moreover, the scale parameter is generally more efficient to estimate than the location parameter.

**Gamma mixtures** The Gamma family of densities takes the form \( f(x|a, b) := \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-bx) \) for \( x > 0 \), and 0 otherwise, where \( a, b \) are positive shape and rate parameters, respectively. The Gamma kernel admits the following partial differential equation:

\[
\frac{∂f}{∂b}(x|a, b) = \frac{a}{b} f(x|a, b) - \frac{a}{b} f(x|a+1, b).
\]

As demonstrated by [28], this identity leads to two disjoint categories of the parameter values of \( G_0 \), which are respectively called “generic cases” and “pathological cases”. In particular, denote \( G_0 = \sum_{i=1}^{k_0} \pi_i \delta(a_i^0, b_i^0) \) where \( k_0 \geq 2 \). Assume that \( a_i^0 \geq 1 \) for all \( 1 \leq i \leq k_0 \). Now, we define

(A1) Generic cases: \( \left\{ |a_i^0 - a_j^0|, |b_i^0 - b_j^0| \right\} \neq \{1, 0\} \) for all \( 1 \leq i, j \leq k_0 \).

(A2) Pathological cases: \( \left\{ |a_i^0 - a_j^0|, |b_i^0 - b_j^0| \right\} = \{1, 0\} \) for some \( 1 \leq i, j \leq k_0 \).

For the Gamma o-mixtures setting, following [28] we also define the constrained set of \( O_k \)

\[
O_{k,c_0} = \left\{ G = \sum_{i=1}^{k'} \pi_i \delta(a_i, b_i) \left| k' \leq k \text{ and } |a_i - a_j| \not\in [1 - c_0, 1 + c_0] \cup [2 - c_0, 2 + c_0] \forall (i, j) \right. \}
\]

where \( c_0 > 0 \). The singularity structure of \( G_0 \) is clarified by the following result.

**Proposition 3.4.** Fix any \( G_0 \in E_{k_0} \).

(a) For generic cases specified by (A1), there holds

\( a1 \) For e-mixtures: \( ℓ(G_0|E_{k_0}) = 0 \), \( ℓ(G_0|E_{k_0}) = \{(1, 1)\} \).

\( a2 \) For o-mixtures: \( ℓ(G_0|O_{k,c_0}) = 1 \), \( ℓ(G_0|O_{k,c_0}) = \{(2, 2)\} \).

(b) For pathological cases specified by (A2), there holds

\( b1 \) For e-mixtures: \( ℓ(G_0|E_{k_0}) = \infty \), \( ℓ(G_0|E_{k_0}) = \{(\infty, \infty)\} \).

\( b2 \) For o-mixtures: \( ℓ(G_0|O_{k,c_0}) = \infty \), \( ℓ(G_0|O_{k,c_0}) = \{(\infty, \infty)\} \).

Part (a) of Proposition 3.4 entails the generic \( n^{-1/2} \) and \( n^{-1/4} \) rates of parameter estimation in the e-mixture and o-mixture settings, respectively. If true parameter values belong to pathological cases, however, part (b) of the proposition implies that polynomial rates of parameter estimation are not possible, due to infinite singularity level.

Although non-trivial convergence behavior obtained in previous work can also recovered via our notion of singularity levels and indices, with the exception of location-scale Gaussian mixtures, none of these examples exhibit the complex inhomogeneity of the parameter space. To demonstrate the full spectrum of complexity of finite mixture models, we will apply the theory on finite mixtures of skew-normal distributions starting in Section 4.
3.5 Construction of \(r\)-minimal and \(\kappa\)-minimal forms

We have seen that minimax rates of parameter estimation for finite mixtures can be read off from the singularity structures, via notions of singularity levels and indices, of the parameter space. For the remainder of Section 3 we shall present a general procedure or calculating singularity level/index for a given mixing measure.

To this end, one needs to first construct \(r\)-minimal forms and \(\kappa\)-minimal forms. Since the latter can be constructed in much the same manner as the former, we will focus our presentation on \(r\)-minimal forms. A simple way of constructing an \(r\)-minimal form is to select a subset of partial derivatives of \(f\) taken up to order \(r\) such that all these functions are linearly independent. A simple procedure is to start from the smallest order \(r = 1\) and then move up to \(r = 2, 3, \ldots\) and so on. For each \(r\), assume that we have obtained a linearly independent subset of partial derivatives up to order \(r - 1\). Now, going over the ordered list of \(r\)-th partial derivatives: \(\{\partial^{|\kappa|}f/\partial \eta^{|\kappa|} | \kappa \in \mathbb{N}^d, |\kappa| = r\}\). For each \(\kappa\) such that \(|\kappa| = r\), if the partial derivative of \(f\) of order \(\kappa\) can be expressed as a linear combination of other partial derivatives among those already selected, then this one is eliminated. The process goes on until we exhaust the list of the partial derivatives.

**Example 3.2.** Continuing from Example 3.1 suppose that \(G_0\) satisfies Eq. (1). According to the proof of Lemma 4.1 we can choose \(\alpha_{4k} \neq 0\), so the partial derivative may be eliminated via the reduction:

\[
\frac{\partial f(x|\eta^0)}{\partial m} = -\sum_{j=1}^{k} \frac{\alpha_{1j}}{\alpha_{4k}} f(x|\eta_j^0) + \frac{\alpha_{2j}}{\alpha_{4k}} \frac{\partial f(x|\eta_j^0)}{\partial \theta} + \frac{\alpha_{3j}}{\alpha_{4k}} \frac{\partial f(x|\eta_j^0)}{\partial v} - \sum_{j=1}^{k-1} \frac{\alpha_{4j}}{\alpha_{4k}} \frac{\partial f(x|\eta_j^0)}{\partial m}
\]

Note that this elimination step is valid only for a subset of \(G_0 = G(p^0, \eta^0)\) for which Eq. (1) holds. That is, only if \(P_1(\eta^0) = 0\) or \(P_2(\eta^0) = 0\).

**Example 3.3.** If \(f(x|\eta) = f(x|\theta, v, m)\) where \(m = 0\), the skew-normal kernel becomes the Gaussian kernel. Due to (3), all partial derivatives with respect to both \(\theta\) and \(v\) can be eliminated via the following reduction: for any \(\kappa_1, \kappa_2 \in \mathbb{N}\), for any \(j = 1, \ldots, k_0\),

\[
\frac{\partial^{\kappa_1+\kappa_2} f(x|\eta_j^0)}{\partial \theta^{\kappa_1}v^{\kappa_2}} = \frac{1}{2^{\kappa_2}} \frac{\partial^{\kappa_1+2\kappa_2} f(x|\eta_j^0)}{\partial \theta^{\kappa_1+2\kappa_2}}
\]

This elimination is valid for all parameter values \((p^0, \eta^0)\), and \(r\)-minimal forms for all orders.

**Example 3.4.** For the skew-normal kernel density \(f(x|\eta) = f(x|\theta, v, m)\), Eq. (2) yields the following reductions: for any \(j = 1, \ldots, k_0\), any \(\eta = (\theta, v, m) = \eta_j^0 = (\theta_{j}^0, v_{j}^0, m_{j}^0)\) such that \(m \neq 0\)

\[
\frac{\partial^2 f}{\partial \theta^2} = 2\frac{\partial f}{\partial v} - \frac{m^3 + m}{v} \frac{\partial f}{\partial m}.
\]
\[
\frac{\partial^2 f}{\partial v \partial m} = -\frac{1}{v} \frac{\partial f}{\partial m} - \frac{m^2 + 1}{2vm} \frac{\partial^2 f}{\partial m^2}.
\]

Differentiating results in a ripple effect on subsequent eliminations at higher orders. For examples, partial derivatives up to the third order of \(f\) evaluated at \(\eta = \eta_j^0 = (\theta_{j}^0, v_{j}^0, m_{j}^0)\) for any \(j = 1, \ldots, k_0\).
where \( m^0 \neq 0 \) can be expressed as follows:

\[
\frac{\partial^3 f}{\partial \xi \partial \eta \partial \mu} = 2 \frac{\partial^2 f}{\partial \eta \partial \mu} - \frac{m^3 + m}{\mu} \frac{\partial^2 f}{\partial \xi \partial \mu}, \\
\frac{\partial^3 f}{\partial \eta^2 \partial \mu} = 2 \frac{\partial^2 f}{\partial \eta \partial \mu} + \frac{m^3 + m}{\mu} \frac{\partial f}{\partial \mu}, \\
\frac{\partial^3 f}{\partial \xi \partial \eta \partial \mu} = 2 \frac{\partial^2 f}{\partial \xi \partial \mu} - \frac{m^3 + m}{\mu} \frac{\partial f}{\partial \mu}, \\
\frac{\partial^3 f}{\partial \eta \partial^2 \partial \mu} = 2 \frac{\partial^2 f}{\partial \eta \partial \mu} - \frac{3m^2 + 1}{\mu} \frac{\partial f}{\partial \mu} - \frac{m^3 + m}{\mu^2} \frac{\partial f}{\partial \mu}, \\
\frac{\partial^3 f}{\partial \xi \partial^2 \partial \mu} = \frac{2m^3 + 1}{\mu} \frac{\partial^3 f}{\partial \xi \partial \mu} - \frac{3m^2 + 1}{\mu} \frac{\partial f}{\partial \mu}, \\
\frac{\partial^3 f}{\partial \eta^3 \partial \mu} = \frac{2m + 1}{\mu} \frac{\partial^3 f}{\partial \eta \partial \mu} - \frac{2m^2}{\mu} \frac{\partial f}{\partial \mu}, \\
\frac{\partial^3 f}{\partial \xi^3 \partial \mu} = \frac{4m^2 \mu^2}{\mu^2 - 2} \frac{\partial^3 f}{\partial \xi \partial \mu} + \frac{2m^3 + 1}{\mu^2} \frac{\partial f}{\partial \mu} + \frac{2m + 1}{\mu^2} \frac{\partial f}{\partial \mu}, \\
\frac{\partial^3 f}{\partial \xi \partial \eta^2 \partial \mu} = \frac{4m^2 \mu^2}{\mu^2 - 2} \frac{\partial^3 f}{\partial \xi \partial \mu} - \frac{2m^2}{\mu^2 - 2} \frac{\partial f}{\partial \mu}.
\]

(14)

All three examples above demonstrate how the dependence among partial derivatives of kernel density \( f \), among different orders \( \kappa \), and among those evaluated at different component \( i \), has a deep impact on the representation of \( r \)-minimal forms.

In general, the \( r \)-minimal form \([3] \) may be expressed somewhat more explicitly as follows

\[
\frac{p_G(x) - p_{\text{G}_0}(x)}{W^r(G, G_0)} = \sum_{(i, \kappa) \in I, \kappa} \xi_{i, \kappa}^{(r)}(G) W^r_i(G_0, G) H^{(r)}_{i, \kappa}(x|G_0) + \sum_{i=1}^{k_0} \xi_{i}^{(r)}(G) W^r_i(G_0, G) f(x|\rho^0_i) + o(1).
\]

where \( I \subset \{1, \ldots, k_0\} \) and \( \kappa \subset \mathbb{N}^d \) of elements \( \kappa \) such that \( |\kappa| \leq r \). It is emphasized that the sets \( I \) and \( \kappa \) are specific to a particular \( r \)-minimal form under consideration. \( H^{(r)}_{i, \kappa} \) are a collection of linearly independent partial derivatives of \( f \) that are also independent of all functions \( f(x|\rho^0_i) ). H^{(r)}_{i, \kappa} \) are taken from the collection of partial derivatives with order at most \( r \). Moreover, \( \xi_{i, \kappa}^{(r)} \) and \( \xi_{i}^{(r)} \) take the following polynomial forms:

\[
\xi_{i, \kappa}^{(r)}(G) = \sum_{j=1}^{s_i} p_{ij}(\Delta \eta_{ij})^{\kappa_i} + \sum_{i', \kappa', \kappa'} \beta_{i, \kappa, i', \kappa'}(G_0) \sum_{j=1}^{s_{i', \kappa', \kappa'}} p_{ij}(\Delta \eta_{ij})^{\kappa'_{i, \kappa'}}, \quad (15) \\
\xi_{i}^{(r)}(G) = \Delta \rho_i + \sum_{i', \kappa', \kappa'} \gamma_{i, i', \kappa'}(G_0) \sum_{j=1}^{s_{i', \kappa', \kappa'}} p_{ij}(\Delta \eta_{ij})^{\kappa'_{i, \kappa'}}.
\]

(16)

In the right hand side of each of the last two equations, \( i' \) is taken from a subset of \( \{1, \ldots, k_0\} \) and \( \kappa' \) is from a subset of \( \mathbb{N}^d \) such that \( |\kappa| \leq |\kappa'| \leq r \). The actual detail of these subsets depend on the specific elimination scheme leading to the \( r \)-minimal form. Likewise, the non-zero coefficients \( \beta_{i, \kappa, i', \kappa'}(G_0) \) and \( \gamma_{i, i', \kappa'}(G_0) \) arise from the specific elimination scheme. We include argument \( G_0 \) in these coefficients to highlight the fact that they may be dependent on the atoms of \( G_0 \) (cf. Example 3.2 and 3.4).

By the definition of \( r \)-singularity for any \( r \geq 1 \), \( G_0 \) is \( r \)-singular relative to \( \mathcal{G} \) if there exists a sequence of \( G \) tending to \( G_0 \) in the ambient space \( \mathcal{G} \) such that the sequences of semipolynomial fractions, namely, \( \xi_{i, \kappa}^{(r)}(G)/W^r_i(G, G_0) \) and \( \xi_{i}^{(r)}(G)/W^r_i(G, G_0) \) (whose numerators are given by Eq. (15) and Eq. (16)), must vanish. As a consequence, the question of \( r \)-singularity for a given element \( G_0 \) is determined by the limiting behavior of a finite collection of infinite sequences of semipolynomial fractions.
3.6 Polynomial limits of $r$-minimal and $\kappa$-minimal form coefficients

The limiting behavior of semipolynomial fractions described above is independent of a particular choice of the $r$-minimal form, in a sense that we now explain. In part (a) of Lemma 3.4, we established an invariance property of the $r$-singularity, which does not depend on a specific form of the $r$-minimal form. Let us restrict the basis functions to be members of the collection of all partial derivatives of $f$ up to order $r$. In the proof of part (b) of Lemma 3.4 it was shown that the coefficients $\xi_l^{(r)}(G)$ have to be elements of a set of polynomials of $\Delta \eta_{ij}$, $\Delta p_i$, and $p_{ij}$, which are closed under linear combinations of its elements. Let us denote this set by $\mathcal{P}(G, G_0)$, which is invariant with respect to any specific choice of the basis functions (from the collection of partial derivatives) for the $r$-minimal form. Moreover, $G_0$ is $r$-singular if and only if a sequence of $G$ tending to $G_0$ in $W_r$ can be constructed such that for any element $\xi_l^{(r)}(G) \in \mathcal{P}(G, G_0)$, we have $\xi_l^{(r)}(G)/W_r^*(G, G_0) \to 0$. Equivalently,

$$\xi_l^{(r)}(G)/D_r(G, G_0) \to 0 \text{ for all } \xi_l^{(r)}(G) \in \mathcal{P}(G, G_0).$$

Extracting the limits of a single multivariate semipolynomial fraction (a.k.a. rational semipolynomial functions) is quite challenging in general, due to the interaction among multiple variables involved. Analyzing the limits of not one but a collection of multivariate rational semipolynomials is considerably more difficult. To obtain meaningful and concrete results, we need to consider specific systems of multivariate rational semipolynomials that arise from the $r$-minimal form.

In the remainder of this paper we will proceed to do just that. By working with specific choices of kernel density $f$, it will be shown that under the compactness of the parameter spaces, one can extract a subset of limits from the system of rational semipolynomials $\xi_l^{(r)}(G)/D_r(G, G_0)$. These limits are expressed as a system of polynomials admitting non-trivial solutions. For a given $r \geq 1$, if the extracted system of polynomial limits does not contain admissible solutions, then it means that there does not exist any sequence of mixing measures $G$ for which a valid $r$-minimal form can be constructed, because (17) is not fulfilled. This would entail the upper bound $\ell(G_0|\mathcal{G}) < r$. On the other hand, if the extracted system of polynomial limits does contain at least one admissible solution, this is a hint that the $r$-singularity level of $G_0$ relative to the ambient space $G$ might hold. Whether this is actually the case or not requires an explicit construction of a sequence of $G \in \mathcal{G}$ (often building upon the admissible solutions of the polynomial limits) and then the verification that condition (17) indeed holds. For the verification purpose, it is sufficient (and simpler) to work with a specific choice of $r$-minimal form, as Definition 3.2 allows.

Due to the asymptotic equivalence between generalized transportation distance and a semipolynomial in Lemma 3.3, the studies of $\kappa$-singularity also reduce to an investigation of limiting behaviors of semipolynomial fractions as those in the case of $r$-singularity described earlier. However, as $D_\kappa(G, G_0)$ is an inhomogeneous semipolynomial, the limiting behaviors of ratios $\xi_l^{(\kappa)}(G)/W_\kappa^*(G, G_0)$ is generally more challenging to investigate than those of ratios from $r$-minimal form in Definition 3.2. This will be seen via examples in the sequel.

The foregoing description, along with the presentation in the previous subsection on the construction of $r$-minimal and $\kappa$-minimal forms, provides the outline of a general procedure which links the determination of the singularity structure of parameter space to the solvability of a system of polynomial limits. This procedure will be illustrated carefully in Section 4 for the remarkable world of mixtures of skew-normal distributions.
4 O-mixtures of skew-normal distributions

In this section, we study parameter estimation behavior for skew-normal mixtures. Our motivation is two-fold: First, as discussed in the Introduction, skew-normal mixture models are widely embraced in applications despite little or no known theoretical results, so understanding their theoretical properties is of interest in their own right. Second, skew-normal mixtures appear to be an ideal illustration for the general theory and tools developed in the previous section, which helps to shed some light on the remarkably complex structure of a mixture distribution’s parameter space.

On the flip side, our presentation will be necessarily (and unfortunately) quite technical. To alleviate the technicality, we focus the presentation in this section on singularity structures in the overfitted setting subject to certain restrictions on probability mass and other parameters. This case is quite interesting because it illustrates the full power of the general method of analysis that was described in Section 3 in a concrete fashion, yielding a general result while revealing sufficiently complex structures.

For readers interested in the finer details of skew-normal mixtures, in Appendix E we summarize the singularity structure of $G_0$ relative to the ambient space $E_{k_0}$ (that is, e-mixture setting), for which a more complete picture of the singularity structure is achieved, with full details laid out in Appendix F. For o-mixtures, further results can be found in a technical report [30]. Such elaborate picture might be appreciated by practitioners and experts of the skew-normals, but they can be safely skipped by the rest of the audience.

**Lemma 4.1.** For skew-normal density kernel $f(x|\eta)$, the collection of $\{\partial^\kappa f / \partial \eta^\kappa(x|\eta_j)\}_{j=1, \ldots, k_0}$, where $0 \leq |\kappa| \leq 1$ is not linearly independent if and only if $\eta = (\eta_1, \ldots, \eta_k)$ are the zeros of either polynomial $P_1$ or $P_2$, which are defined as follows:

**Type A:** $P_1(\eta) = \prod_{j=1}^{k_0} m_j$.

**Type B:** $P_2(\eta) = \prod_{1 \leq i \neq j \leq k_0} \left\{ (\theta_i - \theta_j)^2 + \left[ \sigma_i^2 (1 + m_i^2) - \sigma_j^2 (1 + m_j^2) \right]^2 \right\}$.

This lemma leads us to consider

$$S_0 = \left\{ G = G(p, \eta) \left| (p, \eta) \in \Omega, P_1(\eta) \neq 0, P_2(\eta) \neq 0 \right. \right\}. \quad (18)$$

In o-mixtures, we will see that $\ell(G_0|\mathcal{O}_{k,c_0})$ and $\mathcal{L}(G_0|\mathcal{O}_{k,c_0})$ may grow with $k - k_0$, the number of extra mixing components. The main exercise is to arrive at suitable $r$-minimal and $\kappa$-minimal forms, for which the behavior of its coefficients can be analyzed. Section 3.5 describes a general strategy for the construction of $r$-minimal form (or equivalently $(r, r, r)$-minimal form) based on the partial derivatives of the density kernel $f$ with respect to the parameters $\eta = (\theta, v, m)$ up to order $r$. This is also a strategy that we would like to utilize for $\kappa$-minimal forms for any $\kappa \in \mathbb{N}^3$.

For skew-normal kernel density $f$, the following lemma provides an explicit form for reducing a partial derivative of $f$ to other partial derivatives of lower orders.

**Lemma 4.2.** For any $r \geq 1$, denote

$$A_1^r = \{ (\alpha_1, \alpha_2, \alpha_3) : \alpha_1 \leq 1, \alpha_3 = 0, \text{ and } |\alpha| \leq r \},$$

$$A_2^r = \{ (\alpha_1, \alpha_2, \alpha_3) : \alpha_1 \leq 1, \alpha_2 = 0, \alpha_3 \geq 1, \text{ and } |\alpha| \leq r \},$$

$$F_r = A_1^r \cup A_2^r.$$
Let \( f(x|\eta) = f(x|\theta, v, m) \) denote the skew-normal kernel. Then, for any \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3 \) and \( m \neq 0 \), there holds
\[
\frac{\partial^{[\alpha]} f}{\partial \theta^{\alpha_1} \partial v^{\alpha_2} \partial m^{\alpha_3}} = \sum P_{\alpha_1, \alpha_2, \alpha_3}^{\kappa_1, \kappa_2, \kappa_3}(m) \frac{\partial^{[\kappa]} f}{\partial \theta^{\kappa_1} \partial v^{\kappa_2} \partial m^{\kappa_3}},
\]
where \( \kappa \) in the above sum satisfies \( \kappa \in \mathcal{F}_{[\kappa]} \) and \( \kappa_1 + 2\kappa_2 + 2\kappa_3 \leq \alpha_1 + 2\alpha_2 + 2\alpha_3 \). Additionally, \( P_{\alpha_1, \alpha_2, \alpha_3}^{\kappa_1, \kappa_2, \kappa_3}(m) \), \( H_{\alpha_1, \alpha_2, \alpha_3}^{\kappa_1, \kappa_2, \kappa_3}(m) \), and \( Q_{\alpha_1, \alpha_2, \alpha_3}^{\kappa_1, \kappa_2, \kappa_3}(v) \) are polynomials in terms of \( m \) and \( v \), respectively.

Next, we show that the partial derivatives on the RHS of the above identity are in fact linearly independent, under additional assumptions on \( G_0 \).

**Lemma 4.3.** Recall the notation from Lemma 4.2. If \( G_0 \in \mathcal{S}_0 \), then for any \( r \geq 1 \), the collection of partial derivatives of the skew-normal density kernel \( f(x|\eta) \), namely
\[
\left\{ \frac{\partial^{[\kappa]} f(x|\eta)}{\partial \theta^{\kappa_1} \partial v^{\kappa_2} \partial m^{\kappa_3}} \mid \kappa = (\kappa_1, \kappa_2, \kappa_3) \in \mathcal{F}_r, \eta = \eta_0^1, \ldots, \eta_0^r \right\}
\]
are linearly independent.

Figure 1 gives an illustration of Lemma 4.3 when \( r = 3 \). Armed with the foregoing lemmas we can easily obtain a suitable minimal form for the mixture densities of skew-normals.

### 4.1 Illustrations via special cases

To illustrate our techniques and results, consider a special case in which \( G_0 \) has exactly one atom, and \( k = k_0 + 1 = 2 \). The general result is presented in Section 4.2. As a warm-up exercise, we verify that

**Claim: \( G_0 \) is 1-singular and (1,1,1)-singular.** Indeed, \( G_0 \in \mathcal{S}_0 \) implies that all first order derivatives of \( f \) are linearly independent. Hence, from Eq. (8) we obtain (1,1,1)-minimal form:
\[
\frac{p_{G}(x) - p_{G_0}(x)}{W_2(G, G_0)} \asymp \frac{1}{W_2(G, G_0)} \left( \Delta p_{1, f}(x|\eta_1^0) + \sum_{i=1}^2 p_{1i} \Delta \theta_{1i} \frac{\partial f(x|\eta_1^0)}{\partial \theta} \right) + \sum_{i=1}^2 p_{1i} \Delta v_{1i} \frac{\partial f(x|\eta_1^0)}{\partial v} + \sum_{i=1}^2 p_{1i} \Delta m_{1i} \frac{\partial f(x|\eta_1^0)}{\partial m} + o(1).
\]

Since \( k = 2 \) and \( k_0 = 1 \), \( \Delta p_{1, f}(x|\eta_1^0) = 0 \), \( \sum_{i=1}^2 p_{1i} \Delta v_{1i} = 0 \), \( \sum_{i=1}^2 p_{1i} \Delta m_{1i} = 0 \), so that all of the coefficients in (19) are 0. Hence, \( G_0 \) is 1-singular and (1, 1, 1)-singular relative to \( \mathcal{O}_{2, c_0} \). In light of Prop. 3.2, it is non-trivial to show that

**Claim: \( G_0 \) is 2-singular and (2,2,2)-singular.** Indeed, using the method of elimination described in Example 3.4 we obtain the following 2-minimal and (2,2,2)-minimal form:
\[
\frac{1}{W_2^2(G, G_0)} \left( \sum_{\kappa \in \mathcal{F}_2} \xi_{\kappa_1, \kappa_2, \kappa_3}^{(2)} \frac{\partial^{[\kappa]} f}{\partial \theta^{\kappa_1} \partial v^{\kappa_2} \partial m^{\kappa_3}}(x|\eta_1^0) \right) + o(1),
\]
where \( \xi^{(2)}_{2} \equiv \xi^{(2)}_{2,\kappa,\kappa} \) are given by

\[
\xi^{(2)}_{1,0} = \sum_{i=1}^{2} p_{1i} \Delta \theta_{1i}, \quad \xi^{(2)}_{0,1} = \sum_{i=1}^{2} p_{1i} \Delta v_{1i} + \sum_{i=1}^{2} p_{1i} (\Delta \theta_{1i})^2,
\]

\[
\xi^{(2)}_{0,0,1} = -\frac{(m_{01}^0 + m_{02}^0)}{2v_1^0} \sum_{i=1}^{2} p_{1i} (\Delta \theta_{1i})^2 - \frac{1}{v_1^0} \sum_{i=1}^{2} p_{1i} \Delta v_{1i} \Delta m_{1i} + \sum_{i=1}^{2} p_{1i} \Delta m_{1i},
\]

\[
\xi^{(2)}_{0,2,0} = \sum_{i=1}^{2} p_{1i} (\Delta v_{1i})^2, \quad \xi^{(2)}_{0,0,2} = -\frac{(m_{01}^0 + 1)}{2v_1^0 m_1^0} \sum_{i=1}^{2} p_{1i} \Delta v_{1i} \Delta m_{1i} + \sum_{i=1}^{2} p_{1i} (\Delta m_{1i})^2,
\]

\[
\xi^{(2)}_{1,1,0} = \sum_{i=1}^{2} p_{1i} \Delta \theta_{1i} \Delta v_{1i}, \quad \xi^{(2)}_{1,0,1} = \sum_{i=1}^{2} p_{1i} \Delta \theta_{1i} \Delta m_{1i},
\]

where we note in particular that the formulas for \( \xi^{(2)}_{0,1,0}, \xi^{(2)}_{0,0,1} \) and \( \xi^{(2)}_{0,0,2} \) are the results of the elimination step via Eqs. (12) and (13).

It remains to construct a sequence of \( G \) tending to \( G_0 \) so that \( \xi^{(2)}_{2}/W_2^2(G,G_0) \) vanish for all \( \kappa = (\kappa_1, \kappa_2, \kappa_3) \in \mathcal{F}_2 \). Define

\[
\overline{M} = \max \left\{ \left| \Delta \theta_{11} \right|, \left| \Delta \theta_{12} \right|, \left| \Delta v_{11} \right|^{1/2}, \left| \Delta v_{12} \right|^{1/2}, \left| \Delta m_{11} \right|^{1/2}, \left| \Delta m_{12} \right|^{1/2} \right\}.
\]

Then, \( \xi^{(2)}_{2,\kappa,\kappa} = O(\overline{M}^{\kappa_1+2\kappa_2+2\kappa_3}) \). Moreover, it follows from Lemma 3.1 that \( W_2^2(G,G_0) \approx \overline{M}^2 \). The sequence of \( G \) and the corresponding \( \Delta \theta_{1i}, \Delta v_{1i}, \) and \( \Delta m_{1i} \), will be chosen so that \( W_2^2(G,G_0) \approx \overline{M}^2 \), and for all \( \kappa \in \mathcal{F}_2, \xi^{(2)}_{2,\kappa,\kappa} \) vanish for all \( \kappa = (\kappa_1, \kappa_2, \kappa_3) \). For any \( \kappa \in \mathcal{F}_2 \) such that \( \kappa_1 + 2\kappa_2 + 2\kappa_3 \geq 3 \), we have \( \xi^{(2)}_{2,\kappa,\kappa} = O(\overline{M}^s) \) where \( s \geq 3 \), so \( \xi^{(2)}_{2,\kappa,\kappa} \). We thus only need to consider the coefficients for which \( \kappa_1 + 2\kappa_2 + 2\kappa_3 \leq 2 \) and \( \kappa_1 \leq 1 \). They include \( \xi^{(2)}_{0,1,0}/W_2^2(G,G_0) \), \( \xi^{(2)}_{0,0,1}/W_2^2(G,G_0) \), and \( \xi^{(2)}_{0,0,1}/W_2^2(G,G_0) \). We will see shortly that if these terms vanish, we must be able to extract a solvable system of polynomial equations.

Let \( \Delta \theta_{1i}/\overline{M} \to a_i, \Delta v_{1i}/\overline{M} \to b_i, \Delta m_{1i}/\overline{M} \to c_i, p_{1i} \to d_i^2 \) for all \( 1 \leq i \leq 2 \) (such limits always exist for some subsequence of \( G \), in which case we use to replace \( G \)). For \( \xi^{(2)}_{0,1,0}/W_2^2(G,G_0) \) to hold, by dividing its numerator and denominator by \( \overline{M} \), it is necessary that

\[
d_1^2 a_1 + d_2^2 a_2 = 0.
\]

For \( \xi^{(2)}_{0,0,1}/W_2^2(G,G_0) \) to hold, by dividing its numerator and denominator by \( \overline{M}^2 \), the following holds

\[
d_1^2 a_1^2 + d_2^2 a_2^2 + d_1^2 b_1 + d_2^2 b_2 = 0.
\]

Finally, for \( \xi^{(2)}_{0,0,1}/W_2^2(G,G_0) \) to hold, by dividing its numerator and denominator by \( \overline{M}^2 \) and noting that

\[
\left( \sum_{i=1}^{2} p_{1i} \Delta v_{1i} \Delta m_{1i} \right)/W_2^2(G,G_0) = O(\overline{M}^2) \to 0,
\]

we obtain

\[
-(m_0^0)^3 + m_1^0 (d_1^2 a_1^2 + d_2^2 a_2^2) + d_1^2 c_1 + d_2^2 c_2 = 0.
\]

Thus, we have obtained a system of polynomial equations (21), (22), and (23). Since \( p_{11} + p_{12} = p_1^0 = 1 \) and \( p_{1i} \geq c_0 \) for all \( 1 \leq i \leq 2 \), we have \( d_1^2 + d_2^2 = 1 \) and \( d_i^2 \) are bounded away from 0 for \( 1 \leq i \leq 2 \). Additionally, at least one among \( a_1, a_2, b_1, b_2, c_1, c_2 \) is non-zero.
Figure 1: Illustration of the elimination steps from a complete collection of derivatives of $f$ up to the order 3 to a reduced system of linearly independent partial derivatives, cf. Lemma 4.3. The circled partial derivatives are eliminated from the partial derivatives present in the 3-minimal form. $A \rightarrow B$ means that $B$ is included in the representation of the minimal form when $A$ is eliminated.

One non-trivial solution to the above system of polynomial equations is $d_1 = d_2$, $a_1 = -a_2$, $b_1 = b_2 = -a_2^2$, $c_1 = c_2 = [(m_0^1)^3 + m_0^1]a_2^2/(2v_0^1)$. Given this solution, we can now select a sequence of $G$ by letting $p_{11} = p_{12} = 1/2$, $\Delta \theta_{11} = -\Delta \theta_{12}$, $\Delta v_{11} = \Delta v_{12} = -(\Delta \theta_{11})^2$, and $\Delta m_{11} = \Delta m_{12} = (\Delta \theta_{11})^2[(m_0^1)^3 + m_0^1]/(2v_0^1)$. It is simple to verify that $W_2^2(G, G_0) \simeq M^2$ and $\zeta^{(2)}_{\kappa_1, \kappa_2, \kappa_3}/W_2^2(G, G_0) \rightarrow 0$ for $\kappa_1 + 2\kappa_2 + 2\kappa_3 \leq 2$ and $\kappa_1 \leq 1$, i.e., all coefficients of the 2-minimal form vanish to 0. Hence, $G_0$ is 2-singular and (2,2,2)-singular relative to $O_{2,c_0}$.

**Claim: $G_0$ is 3-singular and (3,3,3)-singular.** The proof for this is similar to the argument that $G_0$ is 2-singular and (2,2,2)-singular. In particular, a 3-minimal and (3,3,3)-minimal form can be obtained by applying the reductions (14), which eliminate all third order partial derivatives in terms of lower order ones that are in fact linearly independent by the condition that $G_0 \in S_0$. As in the foregoing paragraphs, as long as $\Delta \theta_{1i}$, $\Delta v_{1i}$, and $\Delta m_{1i}$ are chosen such that $W_3^3(G, G_0) \times M^3$, we can obtain a system of polynomials that turn out to share the same solution as the one described. This leads to the same choice of sequence for $G$ according to which all coefficients of the 3-minimal form vanish to 0. Thus, $G_0$ is 3-singular and (3,3,3)-singular relatively to $O_{2,c_0}$. In order to establish the singularity level and singularity index of $G_0$, we will show that

**Claim: $G_0$ is not (4,2,2)-singular.** This claim immediately entails, thanks to Lemma 3.2 and Lemma 3.4, that $G_0$ is not 4-singular and (4,4,4)-singular relative to $O_{k,c_0}$, and so by definition $\ell(G_0|O_{2,c_0}) = 3$. Indeed, following the same approach as above, we obtain a (4,2,2)-minimal form and their rational
semipolynomial coefficients, from which we extract the following system of real polynomial limits:

\[
d^2a_1 + d^2a_2 = 0, \\
d^2a_1^2 + d^2a_2^2 + d^2b_1 + d^2b_2 = 0, \\
-\frac{(m_1^0)^3 + m_1^0}{2v_1^0}(d_1^2a_1^2 + d_2^2a_2^2) + d_1^2c_1 + d_2^2c_2 = 0, \\
\frac{1}{3}(d_1^2a_1^3 + d_2^2a_2^3) + d_1^2a_1b_1 + d_2^2a_2b_2 = 0, \\
-\frac{(m_1^0)^3 + m_1^0}{6v_1^0}(d_1^2a_1^3 + d_2^2a_2^3) + d_1^2a_1c_1 + d_2^2a_2c_2 = 0, \\
\frac{1}{6}(d_1^2a_1^4 + d_2^2a_2^4) + d_1^2a_1b_1 + d_2^2a_2b_2 + \frac{1}{2}(d_1^2b_1^2 + d_2^2b_2^2) = 0, \\
\frac{(m_1^0)^3 + m_1^0}{12v_1^0}(d_1^2a_1^4 + d_2^2a_2^4) - \frac{(m_1^0)^3 + m_1^0}{v_1^0}(d_1^2a_1c_1 + d_2^2a_2c_2) - \\
\frac{(m_1^0)^2 + 1}{v_1^0m_1^0}(d_1^2b_1c_1 + d_2^2b_2c_2) + d_1^2c_1 + d_2^2c_2 = 0, \\
\tag{24}
\]

such that at least one among \(a_1, a_2, b_1, b_2, c_1, c_2\) is non-zero and \(d_1, d_2 \neq 0\).

At the first glance, the behavior of this system appears dependent on the specific value of \(v_1^0, m_1^0\). However, if we remove the third, fifth and eighth equations, we obtain a system of real polynomials that does not depend on the specific value of \(G_0\). In fact, it can be verified that this system does not admit any non-trivial real solution, using a standard tool (Groebner bases method) from computational algebra [16]. Thus, there does not exist any sequence of \(G \in \mathcal{O}_{2,c_0}\) according to which all coefficients of the 4-minimal form vanish. This implies that \(G_0\) is not \((4,2,2)\)-singular relative to \(\mathcal{O}_{2,c_0}\). We proceed to show

**Claim:** \(\mathcal{L}(G_0|\mathcal{O}_{2,c_0}) = \{(4,2,2)\}\). It suffices to verify that \(G_0\) is \((3,r,r)\)-singular, \((r,1,r)\)-singular, and \((r,r,1)\)-singular for any \(r \geq 1\). As \(G_0\) is \((3,3,3)\)-singular relative to \(\mathcal{O}_{2,c_0}\), it is sufficient to validate the previous claims when \(r \geq 4\). Select the same sequence of \(G\) as in the foregoing argument, i.e., \(p_{11} = p_{12} = 1/2, \Delta \theta_{11} = -\Delta \theta_{12}, \Delta v_{11} = \Delta v_{12} = -\Delta \theta_{11}^2, \text{ and } \Delta m_{11} = \Delta m_{12} = (\Delta \theta_{11})^2[(m_1^0)^3 + m_1^0]/(2v_1^0)\). If \(\kappa = (3, r, r)\), then \(\|\kappa\|_\infty = r \geq 4\). It is simple to verify that

\[
W_\kappa^{\|\kappa\|_\infty}(G_0, G_0) = W_\kappa^r(G_0, G_0) \asymp |\Delta \theta_{11}|^3.
\]

Similarly, if \(\kappa \in \{(r,1,r), (r,r,1)\}\) and \(r \geq 4\), we have

\[
W_\kappa^{\|\kappa\|_\infty}(G_0, G_0) = W_\kappa^r(G_0, G_0) \asymp |\Delta \theta_{11}|^2 \gg |\Delta \theta_{11}|^3.
\]

Hence, as long as \(\kappa \in U_r := \{(3,r,r), (r,1,r), (r,r,1)\}\) and \(r \geq 4\), we have that \(W_\kappa^{\|\kappa\|_\infty}(G_0, G_0) \gtrsim |\Delta \theta_{11}|^3\). Due to the choices of \(\Delta \theta_{11}, \Delta v_{11}, \text{ and } \Delta m_{11}\), it is easy to check that

\[
\sum_{i=1}^2 p_{1i}(\Delta \theta_{1i})^{\alpha_1}(\Delta v_{1i})^{\alpha_2}(\Delta m_{1i})^{\alpha_3} = O(|\Delta \theta_{11}|^{\alpha_1+2\alpha_2+2\alpha_3})
\]

for any \(\alpha = (\alpha_1, \alpha_2, \alpha_3)\). Invoking the previous bounds, the following limit holds

\[
\frac{|\sum_{i=1}^2 p_{1i}(\Delta \theta_{1i})^{\alpha_1}(\Delta v_{1i})^{\alpha_2}(\Delta m_{1i})^{\alpha_3}|}{W_\kappa^{\|\kappa\|_\infty}(G_0, G_0)} \lesssim \frac{|\Delta \theta_{11}|^{\alpha_1+2\alpha_2+2\alpha_3}}{|\Delta \theta_{11}|^3} \to 0
\]

(25)
for any $\alpha_1 + 2\alpha_2 + 2\alpha_3 \geq 4$ and $\kappa \in \mathcal{U}_r$ where $r \geq 4$.

Now, for any $\kappa \in \mathcal{U}_r$ and $r \geq 4$, we obtain the following $\kappa$-minimal form:

$$\frac{1}{\hat{W}_\kappa|\kappa|\infty(G, G_0)} \left( \sum_{\tau \in F_r} \xi^{(r)}_{\tau_1, \tau_2, \tau_3} \frac{\partial^{|\tau|} f}{\partial \theta_1^{\tau_1} \partial \theta_2^{\tau_2} \partial \eta_3^{\tau_3}}(x|\eta^{(0)}_1) \right) + o(1).$$

Similar to the formulations of $\xi^{(2)}_{\xi, \tau_2, \tau_3}$ in (20), $\xi^{(r)}_{\xi, \tau_2, \tau_3}$ will only consist of monomials of the following form $\sum_{i=1}^2 p_{1i}(\Delta \theta_{1i})^{\alpha_1}(\Delta v_{1i})^{\alpha_2}(\Delta m_{1i})^{\alpha_3}$ where $\alpha_1 + 2\alpha_2 + 2\alpha_3 \geq \tau_1 + 2\tau_2 + 2\tau_3$. Note that, the constraint with $\alpha = (\alpha_1, \alpha_2, \alpha_2)$ stems from the representation in Lemma 4.2. By Eq. (25) and the structure of $\xi^{(r)}_{\xi, \tau_2, \tau_3}$, for any $\tau \in F_r$ such that $\tau_1 + 2\tau_2 + 2\tau_3 \geq 4$, we have $\xi^{(r)}_{\xi, \tau_2, \tau_3}|\hat{W}_\kappa|\kappa|\infty(G, G_0) \to 0$. On the other hand, for any $\tau \in F_r$ such that $\tau_1 + 2\tau_2 + 2\tau_3 \leq 3$, denote $\tilde{\xi}^{(r)}_{\xi, \tau_2, \tau_3}$ a coefficient by leaving out all the monomials $\sum_{i=1}^2 p_{1i}(\Delta \theta_{1i})^{\alpha_1}(\Delta v_{1i})^{\alpha_2}(\Delta m_{1i})^{\alpha_3}$ with $\alpha_1 + 2\alpha_2 + 2\alpha_3 \geq 4$ from the original coefficient $\xi^{(r)}_{\xi, \tau_2, \tau_3}$. Invoking again (25), the following holds

$$\xi^{(r)}_{\xi, \tau_2, \tau_3}|\hat{W}_\kappa|\kappa|\infty(G, G_0) - \tilde{\xi}^{(r)}_{\xi, \tau_2, \tau_3}|\hat{W}_\kappa|\kappa|\infty(G, G_0) \to 0,$$

where, by direct computations, the formulations of $\tilde{\xi}^{(r)}_{\xi, \tau_2, \tau_3}$ when $\tau \in F_r$ and $\tau_1 + 2\tau_2 + 2\tau_3 \leq 3$ are given as follows

$$\tilde{\xi}^{(r)}_{1,0,0} = \sum_{i=1}^2 p_{1i} \Delta \theta_{1i}, \quad \tilde{\xi}^{(r)}_{0,1,0} = \sum_{i=1}^2 p_{1i} \Delta v_{1i} + \sum_{i=1}^2 p_{1i} (\Delta \theta_{1i})^2,$$

$$\tilde{\xi}^{(r)}_{0,0,1} = \frac{(m_1^0)^3 + m_1^1}{2} \sum_{i=1}^2 p_{1i} (\Delta \theta_{1i})^2 + \sum_{i=1}^2 p_{1i} \Delta m_{1i},$$

$$\tilde{\xi}^{(r)}_{1,1,0} = \sum_{i=1}^2 p_{1i} \Delta \theta_{1i} \Delta v_{1i}, \quad \tilde{\xi}^{(r)}_{1,0,1} = \sum_{i=1}^2 p_{1i} \Delta \theta_{1i} \Delta m_{1i}.$$
4.2 A general theorem for skew-normal o-mixtures

In this section we shall present results on singularity structures of \( G_0 \) for the general case \( k > k_0 \). To do so, we define the system of the limiting polynomials that characterizes both the singularity level and singularity index of \( G_0 \). Recall the notation introduced by the statement of Lemma 4.2 where \( P_{\alpha_1,\alpha_2,\alpha_3}(m) \), \( H_{\alpha_1,\alpha_2,\alpha_3}(m) \), and \( Q_{\alpha_1,\alpha_2,\alpha_3}(v) \) are polynomials in terms of \( m, m, v \), respectively, that arise in the decomposition of partial derivatives of the skew-normal kernel function.

For given \( r \geq 1 \), for each \( i = 1, \ldots, k_0 \), the system of limiting polynomial is given by the following equations of real unknowns \( (a_j, b_j, c_j, d_j)_{j=1}^{k-k_0+1} \):

\[
\sum_{j=1}^{k-k_0+1} \sum_{\alpha} \frac{P_{\alpha_1,\alpha_2,\alpha_3}(m_0)}{H_{\alpha_1,\alpha_2,\alpha_3}(m_0)} Q_{\alpha_1,\alpha_2,\alpha_3}(v_i^0) = 0 \quad \beta \in \mathcal{F}_r \cap \{ \beta_1 + 2\beta_2 + 2\beta_3 \leq r \} \tag{26}
\]

where the range of \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3 \) in the above sum satisfies \( \alpha_1 + 2\alpha_2 + 2\alpha_3 = \beta_1 + 2\beta_2 + 2\beta_3 \).

Note that the above system of polynomial equations is the general version of the systems of polynomial equations described in the examples of Section 4.1. There are \( 2r-1 \) equations in the above system of \( 4(k-k_0+1) \) unknowns. A solution of (26) is considered non-trivial if all of \( d_j \) are non-zeros while at least one among \( a_1, \ldots, a_i, b_1, \ldots, b_i, c_1, \ldots, c_i \) is non-zero. We say that system (26) is unsolvable if it does not have any non-trivial (or admissible) solution. Note that increasing \( r \) makes the system more constrained. The main result of this section is the following.

**Theorem 4.1.** For each \( i = 1, \ldots, k_0 \), let \( \rho(v_i^0, m_i^0, k-k_0) \) be the minimum \( r \) for which system of polynomial equations (26) does not admit non-trivial solutions. Let \( G_0 \in \mathcal{S}_0 \) and define

\[
R(G_0, k) = \max_{1 \leq i \leq k_0} \rho(v_i^0, m_i^0, k-k_0). \tag{27}
\]

(i) Then, \( \ell(G_0|\mathcal{O}_{k,c_0}) \leq R(G_0, k) - 1 \).

(ii) Moreover, there exists \( \kappa \in \mathcal{L}(G_0|\mathcal{O}_{k,c_0}) \) such that

\[
\kappa \leq \begin{cases} 
R(G_0, k) & \text{if } R(G_0, k) \text{ is an even number} \\
R(G_0, k) + \frac{1}{2} & \text{if } R(G_0, k) \text{ is an odd number}
\end{cases}
\]

**Remark** We make the following comments regarding the results of Theorem 4.1.

(i) If \( k - k_0 = 1 \), we can obtain \( R(G_0, k) = 4 \) from the examples given in Section 4.1 (although in the examples we only worked out the case that \( k_0 = 1 \), for general \( k_0 \geq 1 \) the techniques are the same). Since \((4,2,2)\) is the unique singularity index of \( G_0 \), the bounds with singularity level and singularity index are tight.

(ii) Since the index component for the location parameter dominates that of shape and scale parameters \((4 > 2)\), estimating shape-scale parameters may be more efficient than estimating location parameter in skew-normal o-mixtures.

(iii) In order to determine \( R(G_0, k) \), we need to find the value of \( \rho(v_i^0, m_i^0, k-k_0) \) for all \( 1 \leq i \leq k_0 \). One may ask whether the value of \( \rho(v_i^0, m_i^0, k-k_0) \) depends on the specific values of \( v_i^0, m_i^0 \). The structure of \( \rho(v_i^0, m_i^0, k-k_0) \) will be looked at in more detail in the next subsection.
4.3 Properties of the system of limiting polynomial equations

The goal of this subsection is the present additional results on the structure of function $\rho(v, m, k-k_0)$, which is a fundamental quantity in Theorem 4.1 (Here, $v_i^0, m_i^0$ are replaced by $v, m$). It is difficult to obtain explicit values for $\rho(v, m, k-k_0)$ in general. Nonetheless, we can obtain a nontrivial upper bound for $\rho$. Now, let $\mathcal{X}_1 := \{(v, m) \in \Theta_2 \times \Theta_3 : m \neq 0\}$. Recall that $\rho(v, m, l)$, where $l = k-k_0 \geq 1$, is the minimum value according to which system (26) does not admit non-trivial real-solution.

**Proposition 4.1.** Let $\overline{\tau}(l)$ be defined as in (11). For all $l = 1, 2, \ldots$, there holds

$$\sup_{(v, m) \in \mathcal{X}_1} \rho(v, m, l) \leq \overline{\tau}(l).$$

**Remarks** (i) The proof of this proposition is given in Appendix F, which proceeds by verifying that system (11) forms a subset of equations that defines system (26). Combining with the statement of Theorem 4.1, we immediately obtain, provided that $G_0 \in \mathcal{S}_0$,

$$\ell(G_0|\mathcal{O}_{k,c_0}) \leq \overline{\tau}(l) - 1.$$

Moreover, there exists $\kappa \in \mathcal{L}(G_0|\mathcal{O}_{k,c_0})$ such that

$$\kappa \leq \begin{cases} (\overline{\tau}(l), \frac{\overline{\tau}(l)}{2}, \frac{\overline{\tau}(l)}{2}), & \text{if } \overline{\tau}(l) \text{ is an even number} \\ (\overline{\tau}(l), \frac{\overline{\tau}(l) + 1}{2}, \frac{\overline{\tau}(l) + 1}{2}), & \text{if } \overline{\tau}(l) \text{ is an odd number}. \end{cases}$$

Thus we are able to relate the singularity structure of a mixing measure in a location-scale Gaussian mixture model to that of the same mixing measure in a skew-normal mixture model.

(ii) Combining the above remark with the results established by Theorem 3.3 Proposition 3.3 and Theorem 7.2 leads us to conclude the following interesting results: in terms of mixing measure, it is statistically more efficient to estimate mixing measure of skew-normal o-mixtures than to estimate mixing measure of Gaussian o-mixtures that carry the same number of extra mixing components. Regarding individual parameters, estimating location and scale parameters of skew-normal o-mixtures is more efficient than estimating location and scale parameters of Gaussian o-mixtures.

**Dependence of $\rho$ on $(v, m)$** To understand the role of parameter value $(v, m)$ on singularity levels and indices, we shall construct a partition of the parameter space for $(v, m)$ based on the value of function $\rho$. For each $l, r \geq 1$, define an “inverse” function

$$\rho_l^{-1}(r) = \{(v, m) \in \mathcal{X}_1 : \rho(v, m, l) = r\}.$$ 

Additionally, take

$$\underline{\rho}(l) = \min \{ r : \rho_l^{-1}(r) \neq \emptyset \}, \overline{\rho}(l) = \max \{ r : \rho_l^{-1}(r) \neq \emptyset \}.$$ 

It follows from Proposition 4.1 that $\overline{\rho}(l) \leq \overline{\tau}(l)$. In addition, $\rho_l^{-1}(r)$ are mutually disjoint for different values of $r$. So, for each fixed amount of overfitting $l \geq 1$, the parameter space may be partitioned by

$$\mathcal{X}_1 = \bigcup_{r=\underline{\rho}(l)}^{\overline{\rho}(l)} \rho_l^{-1}(r).$$
Proposition 4.2. For each \( l \geq 1, r \geq 1, \rho_l^{-1}(r) \) is a semialgebraic set.

Proof. For each \( r \geq 1 \), let \( \mathcal{A}_r \) be the collection of all \((v, m) \in \Xi_1\) such that the system of polynomial equations (26) contains admissible solutions. Furthermore, \( \mathbb{B}_r \) denotes the collection of all solutions \((v, m, \{a_i\}_{i=1}^l, \{b_i\}_{i=1}^l, \{c_i\}_{i=1}^l, \{d_i\}_{i=1}^l)\) of the system of polynomial equations (26), i.e., we treat \( v, m \) as two additional unknowns of the system. Since \( \mathcal{P}_{\alpha_1, \alpha_2, \alpha_3}(m), H_{\alpha_1, \alpha_2, \alpha_3}(m), \) and \( Q_{\alpha_1, \alpha_2, \alpha_3}(v) \) are polynomial functions of \( m, m \) and \( v \), resp., for all \( \alpha, \beta \), by definition \( \mathbb{B}_r \) is a semialgebraic set for all \( r \geq 1 \). By Tarski-Seidenberg theorem [7], since \( \mathcal{A}_r \) is the projection of \( \mathbb{B}_r \) from dimension \((4l + 2)\) to dimension 2, \( \mathcal{A}_r \) is a semialgebraic set for all \( r \geq 1 \). It follows that \( \mathcal{A}_r \subset \mathbb{B}_r \subset \mathcal{A}_{r-1} \) for all \( r \geq 1 \). Since \( \rho_l^{-1}(r) = \mathcal{A}_r \cap \mathcal{A}_{r-1} \) for all \( r \geq 1 \), the conclusion of the proposition follows.

The following result gives us some exact values of \( \rho(l) \) and \( \overline{\rho}(l) \) in specific cases.

Proposition 4.3. \( \begin{align*} & (a) \text{ If } l = k - k_0 = 1, \text{ then } \rho(l) = \overline{\rho}(l) = 4. \\ & (b) \text{ If } l = k - k_0 = 2, \text{ then } \rho(l) = 5 \text{ and } \overline{\rho}(l) = 6. \text{ Thus, } \Xi_1 \text{ is partitioned into two subsets, both of which are non-empty because } \{(1, -2), (1, 2)\} \subset \rho_1^{-1}(5), \text{ and } (1, 10) \in \rho_1^{-1}(6). \end{align*} \)

From the definition of \( R(G_0, k) \), we can write

\[ R(G_0, k) = \max \left\{ r \mid \text{there is } i = 1, \ldots, k_0 \text{ such that } (v_i^0, m_i^0) \in \rho_{k-k_0}^{-1}(r) \right\}. \]

According to the Proposition 4.3 if \( k - k_0 = 1 \), we have \( R(G_0, k) = 4 \) (see also our earlier remark). If \( k - k_0 = 2 \), we may have either \( R(G_0, k) = 5 \) or 6, depending on the value of parameters \((v, m)\) that provide the support for \( G_0 \).

We end this section by noting that we have just provided specific examples in which \( R(G_0, k) - 1 \) may vary with the actual parameter values that define \( G_0 \). Although this provides upper bounds of the singularity level and singularity index, we have not actually proved that the singularity level and singularity index of \( G_0 \) may generally vary with its parameter values. We will be able to do so when we work with the e-mixture setting. The analysis of singularity structure of parameter space under that setting is laid out in Appendices E and F for the interested.

5 Discussion and concluding remarks

Understanding the behavior of parameter estimates of mixture models is useful because the mixing parameters represent explicitly the heterogeneity of the underlying data population that mixture models are most suitable for. In this paper, a general theory for the identification of singularity structure arising from finite mixture models is proposed. It is shown that the singularity structures of the model’s parameter space directly determine minimax lower bounds and maximum likelihood estimation convergence rates, under conditions on the compactness of the parameter space.

The systematic identification of singularity structures and the implications on parameter estimation is a crucial step toward the development of more efficient model-based inference procedures. It is our view that such procedures must account for the presence of singular points residing in the parameter space of the model. As a matter of fact, there are quite a few examples of such efforts applied to specific statistical models, even if the picture of the singularity structures associating with those models might not have been discussed explicitly. This raises a question of whether or not it is possible to extend and generalize such techniques in order to address the presence of singularities in a direct fashion. We give several examples:
(1) For overfitted mixture models, methods based on likelihood-based penalization techniques were shown to be quite effective (e.g., [49, 13]). Our work shows that parameter values residing in the vicinity of regions of high singularity levels should be hard to estimate efficiently. Can a penalization technique be generalized to regularize the estimates toward subsets containing singularity points of lower levels?

(2) Suitable choices of Bayesian prior have been proposed to induce favorable posterior contraction behavior for overfitted finite mixtures [46]. Can we develop an appropriate prior for the mixture model parameters, given our knowledge of singular points residing in the parameter space?

(3) Reparametrization is an effective technique that can be employed to combat singularities present in the class of skewed distributions [25]. It would be interesting to study if such reparameterization technique can be systematically developed for mixture models as well.

Finally, we also expect that the theory of singularity structures carries important consequences on the computational complexity of parameter estimation procedures, including both optimization and sampling based methods. The inhomogeneous nature of the singularity structures reveals a complex picture of the likelihood function: regions in parameter space that carry low singularity levels/indices may observe a relatively high curvature of the likelihood surface, while high singularity levels imply a “flatter” likelihood surface along a certain subspace of the parameters. Such a subspace is manifested by our construction of sequences of mixing measures that attest to the condition of \( r \)-singularity or \( \kappa \)-singularity in general. It is of interest to exploit the explicit knowledge of singularity structures obtained for a given mixture model class, so as to improve upon the computational efficiency of the optimization and sampling procedures that operate on the model’s parameter space.

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6 Appendix A: Proofs of key results

This Appendix contains the proofs of key results in the paper.

6.1 Proof of statements in Section 3

PROOF OF LEMMA 3.2 (a) The existence of the sequence of $G$ described in the definition of a $r$-minimal form implies for that sequence, $(pG(x) - pG_0(x))/W_r^r(G, G_0) \to 0$ holds for almost all $x$. Now take any $r$-minimal form $(\xi)$ given by the same sequence of $G$. Let $C(G) = \max_{i=1}^{T_r} \frac{\xi_i^r(G)}{W_r^r(G, G_0)}$. We will show that $\liminf C(G) = 0$, which concludes the proof. Suppose that this is not the case, so we have $\liminf C(G) > 0$. It follows that

$$
\sum_{l=1}^{T_r} \left( \frac{\xi_i^r(G)}{C(G)W_r^r(G, G_0)} \right) H_i^{(r)}(x) \to 0.
$$

Moreover, all the coefficients in the above display are bounded from above by 1, one of which is in fact 1. There exists a subsequence of $G$ by which these coefficients have limits, one of which is 1. This is a contradiction due to the linear independence of functions $H_i^{(r)}(x)$.

(b) Let $G$ be an element in the sequence that admits a $r$-minimal form such that $\xi_i^r(G)/W_r^r(G_0, G) \to 0$ for all $l = 1, \ldots, T_r$. It suffices to assume that the basis functions $H_i^{(r)}$ are selected from the collection of partial derivatives of $f$. We will show that the same sequence of $G$ and the elimination procedure for the $r$-minimal form can be used to construct a $r-1$-minimal form by which

$$
\xi_i^{(r-1)}(G)/W_r^{r-1}(G_0, G) \to 0
$$

for all $l = 1, \ldots, T_r-1$. There are two possibilities to consider.

First, suppose that each of the $r$-th partial derivatives of density kernel $f$ (i.e., $\partial^\kappa f/\partial \eta^\kappa$, where $|\kappa| = r$) is not in the linear span of the collection of partial derivatives of $f$ at order $r-1$ or less. Then, for each $l = 1, \ldots, T_r-1$, $\xi_i^{(r-1)}(G) = \xi_i^{(r)}(G)$ for some $l' \in [1, T_r]$. Since $W_r^{r-1}(G, G_0) \subseteq W_r^r(G, G_0)$, due to the fact that the support points of $G$ and $G_0$ are in a bounded set, we have that

$$
\xi_i^{(r-1)}(G)/W_r^{r-1}(G_0, G) \leq \xi_i^{(r)}(G)/W_r^r(G_0, G)
$$

which vanishes by the hypothesis.

Second, suppose that some of the $r$-th partial derivatives, say, $\partial^{|\kappa|} f / \partial \eta^{|\kappa|}$ where $|\kappa| = r$, can be eliminated because they can be represented by a linear combination of a subset of other partial derivatives $H_i^{(r-1)}$ (in addition to possibly a subset of other partial derivatives $H_i^{(r)}$) with corresponding finite coefficients $\alpha_{\kappa,i,l}$. It follows that for each $l = 1, \ldots, T_r-1$, the coefficient $\xi_i^{(r-1)}(G)$ that defines the $r-1$-minimal form is transformed into a coefficient in the $r$-minimal form by

$$
\xi_i^{(r)}(G) := \xi_i^{(r-1)}(G) + \sum_{\kappa:|\kappa|=r} \sum_{i=1}^{k_0} \alpha_{\kappa,i,l} \sum_{j=1}^{s_l} p_{ij} (\Delta \eta_{ij})^{\kappa}/\kappa!.
$$

Since $\xi_i^{(r)}(G)/W_r^r(G, G_0)$ tends to 0, so does $\xi_i^{(r-1)}(G)/W_r^{r-1}(G, G_0)$. By Lemma 3.1 for each $\kappa$ such that $|\kappa| = r$, $\sum_{i=1}^{k_0} \sum_{j=1}^{s_l} p_{ij} (\Delta \eta_{ij})^{\kappa}/\kappa! = o(D_r(G_0, G)) = o(W_r^{r-1}(G, G_0))$. Combining with Eq. (28) it follows that $\xi_i^{(r-1)}(G)/W_r^{r-1}(G, G_0)$ tends to 0, for each $l = 1, \ldots, T_r-1$. This completes the proof.
PROOF OF THEOREM \[\text{(i)}\] It suffices to prove the first inequality for \(s = r + 1\). Firstly, we will demonstrate that

\[
\liminf_{G \in \mathcal{G}: W_s(G, G_0) \to 0} \left\| p_G - p_{G_0} \right\|_{\infty}/W^s_s(G, G_0) > 0.
\]

If this is not true, then there exists a sequence of \(G\) such that \(W_s(G, G_0) \to 0\), and for almost all \(x\), \((p_G(x) - p_{G_0}(x))/W^s_s(G, G_0) \to 0\). Take any \(s\)-minimal form for this ratio, we have

\[
\frac{p_G(x) - p_{G_0}(x)}{W^s_s(G, G_0)} = \sum_{t=1}^{T_s} \left( \frac{\xi^{(s)}_t(G)}{W^s_s(G, G_0)} \right) H^{(s)}_t(x) + o(1) \to 0.
\]

For each \(G\) in the sequence, let \(C(G) = \max_l \frac{\xi^{(s)}_l(G)}{W^s_s(G, G_0)}\). If \(\liminf C(G) = 0\), then this means \(G_0\) is \(s\)-singular, so \(\ell(G|G) \geq s\). This violates the given assumption. So we have \(\liminf C(G) > 0\). It follows that

\[
\sum_{t=1}^{T_s} \left( \frac{\xi^{(s)}_t(G)}{C(G)W^s_s(G, G_0)} \right) H^{(s)}_t(x) \to 0.
\]

Moreover, all coefficients in the above display are bounded from above by 1, one of which is in fact 1. There exists a subsequence of \(G\) by which these coefficients have a limit, one of which is 1. This is also a contradiction due to the linear independence of functions \(H^{(s)}_t\).

Therefore, we can find a positive number \(\epsilon_0\) such that \(\left\| p_G - p_{G_0} \right\|_{\infty} \geq W^s_s(G, G_0)\) as soon as \(W_s(G, G_0) \leq \epsilon_0\). Now, to obtain the conclusion of part (i), it suffices to demonstrate that

\[
\inf_{G \in \mathcal{G}: W_s(G, G_0) > \epsilon_0} \left\| p_G - p_{G_0} \right\|_{\infty}/W^s_s(G, G_0) > 0.
\]

If this is not the case, there is a sequence \(G'\) such that \(W_s(G', G_0) > \epsilon_0\) and \(\left\| p_{G'} - p_{G_0} \right\|_{\infty}/W^s_s(G', G_0) \to 0\). Since \(\Theta\) is compact and \(G\) contains only probability measures with bounded number of support points in \(\Theta\), we can find \(G^* \in \mathcal{G}\) such that \(W_s(G', G^*) \to 0\) and \(W_s(G^*, G_0) \geq \epsilon_0\). As \(W_s(G', G_0) \to W_s(G^*, G_0) > 0\), we have \(\left\| p_{G'} - p_{G_0} \right\|_{\infty} \to 0\). Now, due to the first order uniform Lipschitz condition of \(f\), we obtain \(p_{G'}(x) \to p_{G^*}(x)\) for all \(x \in \mathcal{X}\). Thus, \(p_{G^*}(x) = p_{G_0}(x)\) for almost all \(x \in \mathcal{X}\), which entails that \(G^* = G_0\), a contradiction. This completes the proof.

(ii) Turning to the second inequality, we also firstly demonstrate that

\[
\liminf_{G \in \mathcal{G}: W_s(G, G_0) \to 0} \frac{V(p_G, p_{G_0})}{W^s_s(G, G_0)} > 0.
\]

If it is not true, then we have a sequence of \(G\) such that \(W_s(G, G_0) \to 0\) and \(V(p_G, p_{G_0})/W^s_s(G, G_0) \to 0\). By Fatou’s lemma

\[
0 = \liminf_{G \in \mathcal{G}: W_s(G, G_0) \to 0} \frac{V(p_G, p_{G_0})}{C(G)W^s_s(G, G_0)} \geq \int \liminf_{G} \left| \frac{\xi^{(s)}_l(G)}{C(G)W^s_s(G, G_0)} H^{(s)}_l(x) \right| dx.
\]

The integrand must be zero for almost all \(x\), leading to a contradiction as before. Hence, to obtain the conclusion of part (ii), we only need to show that

\[
\inf_{G \in \mathcal{G}: W_s(G, G_0) > \epsilon_0} \frac{V(p_G, p_{G_0})}{W^s_s(G, G_0)} > 0.
\]

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where \( \epsilon_0 > 0 \) such that \( V(p_G, p_{G_0}) \geq W^*_s(G, G_0) \) for any \( W_s(G, G_0) \leq \epsilon_0 \). If it is not true, then using the same argument as that of part (i), there is a sequence of \( G' \) such that \( W_s(G', G^*) \to 0 \), \( V(p_{G'}, p_{G_0}) \to 0 \), while \( W_s(G^*, G_0) \geq \epsilon_0 \) and \( p_{G'}(x) \to p_{G^*}(x) \) for all \( x \in X \). By Fatou’s lemma,

\[
0 = \lim \inf V(p_{G'}, p_{G_0}) \geq \int \lim \inf |p_{G'}(x) - p_{G_0}(x)| dx = V(p_{G^*}, p_{G_0}),
\]

which leads to \( G^* = G_0 \), a contradiction. The proof is concluded.

### 6.2 Proofs for section 4

#### PROOF OF THEOREM 4.1

The reader is recommended to go over the special cases given earlier in Section 4.1 before embarking on this proof. Our strategy is clear: First, we obtain a valid \( \kappa \)-minimal form for \( G_0 \), cf. Eq. (10). This requires a method for obtaining linearly independent basis functions \( H_i(x) \) out of the partial derivatives of kernel density \( f \). Second, we obtain the polynomial limits of collection of coefficients of the \( \kappa \)-minimal form. Third, we obtain bounds on \( r \) according to which this system of limiting polynomials does not admit non-trivial real solutions. This yields upper bounds on the singularity level and singularity index of \( G_0 \).

**Step 1: Construction of \( \kappa \)-minimal form** Let \( \kappa = (\kappa_1, \kappa_2, \kappa_3) \in \mathbb{N}^3 \) be an index vector, and put \( r = \| \kappa \|_\infty \). By Lemma 4.2 and Lemma 4.3 a \( \kappa \)-th minimal form for \( G_0 \) can be obtained as

\[
\frac{p_G(x) - p_{G_0}(x)}{W^*_\kappa(G, G_0)} = \frac{A_1(x) + B_1(x)}{W^*_\kappa(G, G_0)},
\]

where \( A_1(x) \) and \( B_1(x) \) are given as follows

\[
A_1(x) = \sum_{i=1}^{k_0} \sum_{\beta \in \mathcal{F}_r} \left( \sum_{j=1}^{s_i} \sum_{\alpha} \frac{P_{\alpha_j,\beta_2,\beta_3}^{\beta_1,\beta_2,\beta_3}(m_0)}{H_{\alpha_j,\beta_2,\beta_3}(m_0)} \frac{p_{ij}((\Delta \theta_{ij})^{\alpha_1} (\Delta \nu_{ij})^{\alpha_2} (\Delta m_{ij})^{\alpha_3})}{\alpha_1! \alpha_2! \alpha_3!} \right) \times \\
\frac{\partial^{[\beta]} f}{\partial \theta_{\alpha_1} \partial \nu_{\alpha_2} \partial m_{\alpha_3}}(x|\theta_i^0, \nu_i^0, m_i^0),
\]

\[
B_1(x) = \sum_{i=1}^{k_0} \Delta p_{i,} f(x|\theta_i^0, \nu_i^0, m_i^0).
\]

In the above expression, due to condition \( G_0 \in \mathcal{O}_{\kappa, \epsilon_0} \), the number of redundant limit points in the \( \kappa \)-minimal form is \( \bar{i} = 0 \), so \( i \) in the above sum runs from 1 to \( k_0 \). It is also important to note that \( \alpha \) in the above sum satisfies \( |\alpha| \leq r \) and \( \alpha_1 + 2 \alpha_2 + 2 \alpha_3 \geq \beta_1 + 2 \beta_2 + 2 \beta_3 \).

Suppose that there exists a sequence of \( G \) tending to \( G_0 \) under \( W^*_\kappa \) such that all the coefficients of \( A_1(x)/W^*_\kappa(G, G_0) \) and \( B_1(x)/W^*_\kappa(G, G_0) \) vanish, so that \( G_0 \) is \( \kappa \)-singular relative to \( \mathcal{O}_{\kappa, \epsilon_0} \). Then for all \( 1 \leq i \leq k_0 \), we obtain that \( \Delta p_i/W^*_\kappa(G, G_0) \to 0 \) and

\[
E_{\beta_1, \beta_2, \beta_3}(\theta_i^0, \nu_i^0, m_i^0) := \sum_{j=1}^{s_i} \sum_{\alpha} \frac{P_{\alpha_j,\beta_2,\beta_3}^{\beta_1,\beta_2,\beta_3}(m_0)}{H_{\alpha_j,\beta_2,\beta_3}(m_0)} \frac{p_{ij}((\Delta \theta_{ij})^{\alpha_1} (\Delta \nu_{ij})^{\alpha_2} (\Delta m_{ij})^{\alpha_3})}{\alpha_1! \alpha_2! \alpha_3!} \to 0,
\]

for all \( \beta \in \mathcal{F}_r \).
By Lemma 3.3, \( \tilde{W}_r^i(G, G_0) \preceq D_\kappa(G_0, G) \). So, \( \sum_{i=1}^{k_0} |\Delta p_{ij}| / D_\kappa(G_0, G) \to 0 \). It follows that

\[
\left\{ \sum_{i=1}^{k_0} \sum_{j=1}^{s_i} p_{ij} (|\Delta \theta_{ij}|^{\kappa_1} + |\Delta \nu_{ij}|^{\kappa_2} + |\Delta m_{ij}|^{\kappa_3}) \right\} / D_r(G_0, G) \to 1.
\]

This means there exists some index \( i^* \in \{1, \ldots, k_0\} \) such that

\[
\sum_{j=1}^{s_{i^*}} p_{i^*j} (|\Delta \theta_{i^*j}|^{\kappa_1} + |\Delta \nu_{i^*j}|^{\kappa_2} + |\Delta m_{i^*j}|^{\kappa_3}) / D_r(G_0, G) \to 0.
\]

By multiplying the inverse of the above term with \( E_{\beta_1, \beta_2, \beta_3}(\theta_0^i, \nu_0^i, m_0^i) \) as \( \beta \in F_r \) and using the fact that \( \tilde{W}_r^i(G, G_0) \preceq D_\kappa(G_0, G) \)

\[
F_{\beta_1, \beta_2, \beta_3}(\theta_0^i, \nu_0^i, m_0^i) := \sum_{j=1}^{s_{i^*}} \sum_{\alpha} \frac{p_{\alpha_1, \alpha_2, 3}(m_{1}^i)}{H_{\alpha_1, \alpha_2, 3}^i(G_0)} \frac{p_{i^*j} (\Delta \theta_{i^*j})^{\alpha_1} (\Delta \nu_{i^*j})^{\alpha_2} (\Delta m_{i^*j})^{\alpha_3}}{\alpha_1! \alpha_2! \alpha_3!} \to 0
\]

where \( \alpha \) in the above sum satisfies \( |\alpha| \leq r \) and \( \alpha_1 + 2\alpha_2 + 3\alpha_3 \geq \beta_1 + 2\beta_2 + 2\beta_3 \).

**Step 2: Greedy extraction of polynomial limits** We proceed to extract polynomial limit of coefficient \( F_{\beta_1, \beta_2, \beta_3}(\theta_0^i, \nu_0^i, m_0^i) \) for each admissible index vector \( \beta \). Let

\[
\overline{M}_g = \max \left\{ |\Delta \theta_{i^*j}|^{\kappa_1/r}, \ldots, |\Delta \theta_{i^*s_{i^*}}|^{\kappa_1/r}, |\Delta \nu_{i^*j}|^{\kappa_2/r}, \ldots, |\Delta \nu_{i^*s_{i^*}}|^{\kappa_2/r}, |\Delta m_{i^*j}|^{\kappa_3/r}, \ldots, |\Delta m_{i^*s_{i^*}}|^{\kappa_3/r} \right\}.
\]

Then the denominator of coefficient \( F_{\beta_1, \beta_2, \beta_3} \) satisfies \( \sum_{j=1}^{s_{i^*}} p_{i^*j} (|\Delta \theta_{i^*j}|^{\kappa_1} + |\Delta \nu_{i^*j}|^{\kappa_2} + |\Delta m_{i^*j}|^{\kappa_3}) \preceq \overline{M}_g^r \), while the numerator consists of a finite number of monomials of power indices \( \alpha = (\alpha_1, 2\alpha_2, 3\alpha_3) \), each of which is asymptotically bounded from above by \( M_g^{r(\alpha_1/\kappa_1 + \alpha_2/\kappa_2 + \alpha_3/\kappa_3)} \). Accordingly, the only meaningful monomials in the numerator of \( F_{\beta_1, \beta_2, \beta_3} \) are those associated with \( \alpha \) such that

\[
\alpha_1/\kappa_1 + \alpha_2/\kappa_2 + \alpha_3/\kappa_3 \leq 1.
\]

Coupling this with the constraints that \( |\alpha| \leq r \) and \( \alpha_1 + 2\alpha_2 + 3\alpha_3 \geq \beta_1 + 2\beta_2 + 2\beta_3 \), we only need to consider asymptotically dominating monomials, i.e., those with minimal index vector \( \alpha \). They are the indices that satisfy \( \alpha_1 + 2\alpha_2 + 3\alpha_3 = \beta_1 + 2\beta_2 + 2\beta_3 \) (which would also entail that \( |\alpha| \leq \beta_1 + 2\beta_2 + 2\beta_3 \leq r \), due to the definition of index set \( F_r \)). In general, for a given index vector \( \beta \), \( \alpha = \beta \) is clearly one such minimal index, even though there may be others. Now for a fixed \( r \geq 1 \), we wish to select a minimal index \( \kappa \), subject to \( \|\kappa\|_{\infty} = r \), for which the “greedy” choice \( \alpha = \beta \) always satisfies constraint (29). The clear answer is to select \( \kappa = (r, r/2, r/2) \) if \( r \) is not even, \( \kappa = (r, r+1/2, (r+1)/2) \) if \( r \) is selected instead.

From this point on, let \( \kappa = (r, r/2, r/2) \). Denote the limits for the relevant subsequences, which exist due to the boundedness: \( \Delta \theta_{i^*j} / \overline{M}_g \to a_j, \Delta \nu_{i^*j} / \overline{M}_g \to b_j, \) and \( \Delta m_{i^*j} / \overline{M}_g \to c_j, \) and \( p_{i^*j} \to d_j^2 \).
for each \( j = 1, \ldots, s_i \). Here, at least one element of \((a_j, b_j, c_j)_{j=1}^{s_i}\) equals to -1 or 1. For any \( \beta = (\beta_1, \beta_2, \beta_3) \) such that \( \beta \in \mathcal{F} \) and \( \beta_1 + 2\beta_2 + 2\beta_3 \leq r \), by dividing the numerator and denominator of \( F_{\beta_1, \beta_2, \beta_3}(\theta_i^0, v_i^0, m_i^0) \) by \( \mathcal{M}_g^{\theta_1^0 + 2\theta_2^0 + 2\theta_3^0} \) (i.e., the lowest order of \( \mathcal{M}_g \) in the numerator of \( F_{\beta_1, \beta_2, \beta_3}(\theta_i^0, v_i^0, m_i^0) \)), we obtain the following system of equations

\[
\sum_{j=1}^{s_i^*} \sum_{\alpha} P_{\alpha_1, \alpha_2, \alpha_3}(m_i^0) \frac{a_j^{\alpha_1} b_j^{\alpha_2} c_j^{\alpha_3}}{a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3}} = 0, \tag{30}
\]

where the range of \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) in the above sum satisfies \( \alpha_1 + 2\alpha_2 + 2\alpha_3 = \beta_1 + 2\beta_2 + 2\beta_3 \).

The above system of polynomial equations is the general version of system of polynomial equations (24) that we considered in Section 4.1. Now, one of the elements of \( \mathcal{O} \) is clear that \( W_\kappa(G_0) \) admits a non-trivial solution, because the latter has a larger number of unknowns (i.e., thus less constrained). This cannot be the case if \( G_0 \) is \( \kappa \)-singular relative to \( \mathcal{O} \), where \( \kappa = (r, r/2, r/2) \), then the system (30) must admit a non-trivial solution for unknowns \((a_j, b_j, c_j)_{j=1}^{s_i^*}\).

In other words, \( G_0 \) is not \( \kappa \)-singular relative to \( \mathcal{O} \) as long as the system (30) does not admit any non-trivial solution.

**Step 3: Upper bound for singularity level and indices**

There are two distinct features of system of polynomial equations (30). First, \( s_i^* \) varies in \( \{1, 2, \ldots, k_0\} \) as \( G \in \mathcal{O}_{k_0} \) tends to \( G_0 \). Second, the value of \( s_i^* \) of the subsequence of \( G \) is subject to the constraint that \( s_i^* \leq k - k_0 + 1 \). (This constraint arises due to number of distinct atoms of \( G \), \( \sum_{j=1}^{k_0} s_j \leq k' \leq k \) and all \( s_j \geq 1 \) for all \( 1 \leq j \leq k_0 \).

It follows from these two observations that the system (30) admits a non-trivial solution only if the system (26) also admits a non-trivial solution, because the latter has a larger number of unknowns (i.e., thus less constrained). This cannot be the case if \( r \geq R(G_0, k) \), by the definition given in Eq. (27).

Thus, \( G_0 \) is not \((R(G_0, k), R(G_0, k)/2, R(G_0, k)/2)\)-singular relative to \( \mathcal{O} \). The conclusion of the theorem is immediate by invoking Proposition 3.1 part (iv) and (v).

### 6.3 Auxiliary lemmas

**Lemma 6.1.** For any \( \kappa = (\kappa_1, \ldots, \kappa_d) \in \mathbb{N}^d \), if all the components of \( \kappa \) are not identical then \( W_\kappa(G, G_0) \) is a semi-metric satisfying weak triangle inequality.

**Proof.** The proof of this lemma follows the same argument from Chapter 6 in [52]. In particular, it is clear that \( W_\kappa(G, G_0) = W_\kappa(G_0, G) \). Additionally, we can verify that \( d_\kappa \) satisfies weak triangle inequality. In particular, we can demonstrate the following simple bound

\[
d_\kappa(\theta_1, \theta_2) + d_\kappa(\theta_2, \theta_3) \geq C d_\kappa(\theta_1, \theta_3)
\]

where \( C = \min_{1 \leq i \leq d} \frac{1}{2^\kappa_i} \). The proof for this inequality is straightforward from the application of Cauchy-Schwarz’s inequality. However, \( d_\kappa \) does not satisfy the standard triangle inequality as \( \kappa_i \) are not all identical.
Now, from the definition of $\tilde{W}_\kappa$ we have the following equivalent representation

$$\tilde{W}_\kappa(G, G_0) = \inf \left\{ \left[ Ed_{\kappa}^{\|\kappa\|_\infty}(X, Y) \right]^{1/\|\kappa\|_\infty}, \text{law}(X) = G, \text{law}(Y) = G_0 \right\}.$$  

For any finitely discrete probability measures $G_1, G_2, G_3$, let $(X_1, X_2)$ represent the optimal coupling for $\tilde{W}_\kappa(G_1, G_2)$ and $(Z_2, Z_3)$ be optimal coupling for $\tilde{W}_\kappa(G_2, G_3)$. According to Gluing Lemma (cf. page 11 in [52]), there exists random variables $(X'_1, X_2, X'_3)$ such that law$(X'_1, X_2) = \text{law}(X_1, X_2)$ and $\text{law}(X'_2, X'_3) = \text{law}(Z_2, Z_3)$. Therefore, we obtain

$$\tilde{W}_\kappa(G_1, G_3) = \left[ Ed_{\kappa}^{\|\kappa\|_\infty}(X'_1, X'_3) \right]^{1/\|\kappa\|_\infty} \leq \frac{1}{C} \left\{ \left[ Ed_{\kappa}^{\|\kappa\|_\infty}(X'_1, X'_2) \right]^{1/\|\kappa\|_\infty} + \left[ Ed_{\kappa}^{\|\kappa\|_\infty}(X'_2, X'_3) \right]^{1/\|\kappa\|_\infty} \right\} = \frac{1}{C} \left[ \tilde{W}_\kappa(G_1, G_2) + \tilde{W}_\kappa(G_2, G_3) \right].$$  

As a consequence, generalized transportation distance $\tilde{W}$ satisfies 'weak' triangle inequality. To demonstrate that $\tilde{W}$ does not satisfy the standard triangle inequality, we choose $G_1 = \delta_{\theta_1}, G_2 = \delta_{\theta_2}, \text{and } G_3 = \delta_{\theta_3}$. As there exist $\theta_1, \theta_2, \text{and } \theta_3$ such that $d_\kappa$ does not satisfy the standard triangle inequality, it directly implies that these $\tilde{W}$ does not satisfy standard triangle inequality with these choices of $G_1, G_2, \text{and } G_3$. We achieve the conclusion of the lemma.

Lemma 6.2. Let $\{f(x|\theta, \sigma, m), \theta \in \Theta_1, \sigma \in \Theta_2, m \in \Theta_3\}$ be a class of skew normal distribution. Denote $v := \sigma^2$, then

$$\begin{align*}
\frac{\partial^2 f}{\partial \theta^2}(x|\theta, \sigma, m) - 2 \frac{\partial f}{\partial v}(x|\theta, \sigma, m) + \frac{m^3}{v} + m \frac{\partial f}{\partial m}(x|\theta, \sigma, m) &= 0, \\
2m \frac{\partial f}{\partial m}(x|\theta, \sigma, m) + (m^2 + 1) \frac{\partial^2 f}{\partial m^2}(x|\theta, \sigma, m) + 2vm \frac{\partial^2 f}{\partial v \partial m}(x|\theta, \sigma, m) &= 0.
\end{align*}$$

Proof. Direct calculation yields

$$\begin{align*}
\frac{\partial^2 f}{\partial \theta^2}(x|\theta, \sigma, m) &= \left\{ \left( -\frac{2}{\sqrt{2\pi} \sigma^3} + \frac{2(x-\theta)^2}{\sqrt{2\pi} \sigma^5} \right) \Phi \left( \frac{m(x-\theta)}{\sigma} \right) - \frac{2m(m^2 + 2(x-\theta))}{\sqrt{2\pi} \sigma^4} f \left( \frac{m(x-\theta)}{\sigma} \right) \right\} \exp \left( -\frac{(x-\theta)^2}{2\sigma^2} \right), \\
\frac{\partial f}{\partial v}(x|\theta, \sigma, m) &= \left\{ \left(-\frac{1}{\sqrt{2\pi} \sigma^3} + \frac{(x-\theta)^2}{\sqrt{2\pi} \sigma^5} \right) \Phi \left( \frac{m(x-\theta)}{\sigma} \right) - \frac{m(x-\theta)}{\sqrt{2\pi} \sigma^4} f \left( \frac{m(x-\theta)}{\sigma} \right) \right\} \exp \left( -\frac{(x-\theta)^2}{2\sigma^2} \right), \\
\frac{\partial f}{\partial m}(x|\theta, \sigma, m) &= \frac{2(x-\theta)}{\sqrt{2\pi} \sigma^2} f \left( \frac{m(x-\theta)}{\sigma} \right) \exp \left( -\frac{(x-\theta)^2}{2\sigma^2} \right), \\
\frac{\partial^2 f}{\partial m^2}(x|\theta, \sigma, m) &= \frac{-2m(x-\theta)^3}{\sqrt{2\pi} \sigma^4} f \left( \frac{m(x-\theta)}{\sigma} \right) \exp \left( -\frac{(x-\theta)^2}{2\sigma^2} \right), \\
\frac{\partial^2 f}{\partial v \partial m}(x|\theta, \sigma, m) &= \left( -\frac{2(x-\theta)}{\sqrt{2\pi} \sigma^4} + \frac{(m^2 + 1)(x-\theta)^3}{\sqrt{2\pi} \sigma^6} \right) \exp \left( -\frac{(x-\theta)^2}{2\sigma^2} \right).
\end{align*}$$

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From these equations, we can easily verify the conclusion of our lemma.

7 Appendix B: Impacts of singularity index on parameter estimation

In this appendix we present minimax lower bounds and convergence rates for parameter estimation that arise from mixing measure’s singularity index. These results underscore the inhomogeneity of the parameter space that make up the mixing measure’s atoms, while generalizing the theorems established in Section 3.2. We start with the following result that lowers bound distances between mixing densities in terms of generalized transportation metric of corresponding mixing measures based on given singularity index.

Theorem 7.1. Let $\mathcal{G}$ be a class of probability measures on $\Theta$ that have a bounded number of support points, and fix $G_0 \in \mathcal{G}$. Suppose that $\kappa \in \mathcal{L}(G_0|\mathcal{G})$, let $r = \|\kappa\|_{\infty}$. There hold:

(i) If $r < \infty$, then
$$\inf_{G \in \mathcal{G}} \frac{\|p_G - p_{G_0}\|_{\infty}}{W_{\kappa'}(G, G_0)} > 0 \text{ for any } \kappa' \succeq \kappa.$$

(ii) If $r < \infty$, then
$$\inf_{G \in \mathcal{G}} \frac{V(p_G, p_{G_0})}{W_{\kappa'}(G, G_0)} > 0 \text{ for any } \kappa' \succeq \kappa.$$

(iii) If $1 \leq r < \infty$ and in addition,

(a) $f$ is $(r+1)$-order differentiable with respect to $\eta$ and for some constant $c_0 > 0$,
$$\sup_{|\eta - \eta'| \leq c_0} \int_{x \in \mathcal{X}} \left( \frac{\partial^{r+1} f}{\partial \eta^\alpha}(x|\eta) \right)^2 f(x|\eta') dx < \infty \quad (31)$$

for any $|\alpha| = r + 1$.

(b) For any $\kappa' \in \mathbb{R}^d$ such that $(1, \ldots, 1) \preceq \kappa' \prec \kappa$, there is a sequence of $G \in \mathcal{G}$ tending to $G_0$ in generalized transportation distance $W_{\kappa''}$ where $\kappa''_i = [\kappa'_i]$ for all $1 \leq i \leq d$ and the coefficients of the $\kappa''$-minimal form $\xi^{(\kappa'')}(G)$ satisfy $\xi^{(\kappa'')}(G)/W_{\kappa''}(G, G_0) \to 0$ for all $l = 1, \ldots, T_{\kappa''}$. Additionally, all the masses $p_{ij}$ in the representation (4) of $G$ are bounded away from 0.

Then, for any $\kappa' \in \mathbb{R}^d$ such that $(1, \ldots, 1) \preceq \kappa' \prec \kappa$,
$$\liminf_{G \in \mathcal{G} : W_{\kappa'}(G, G_0) \to 0} \frac{h(p_G, p_{G_0})}{W_{\kappa''}(G, G_0)} = 0.$$

(iv) If $r = \infty$ and the condition (a) in part (iii) holds for any $l \in \mathbb{N}$ (here, the parameter $r$ in these conditions is replaced by $l$) while the condition (b) in part (iii) holds, then the conclusion of part (iii) holds for any $\kappa' \in \mathbb{R}^d$ such that $(1, \ldots, 1) \preceq \kappa' \prec \kappa$.

The proof of Theorem 7.1 is similar to that of Theorem 3.1 and Theorem 3.2; it is omitted for the brevity of the paper. Now, denote $\mathcal{M}(\mathcal{G}) = \{ \kappa \in \mathbb{N}^d : \exists G \in \mathcal{G} \text{ such that } \kappa \in \mathcal{L}(G|\mathcal{G}) \}$. Instead of partitioning via singularity levels, we shall use finer structures via singularity indices. Let $\mathcal{G}$ be structured into a sieve of subsets defined by the elements of $\mathcal{M}(\mathcal{G})$ as follows.
\[G = \bigcup_{\kappa \in \mathcal{M}(G)} G_{\kappa}, \text{ where } G_{\kappa} := \left\{ G \in \mathcal{G} \mid \exists \kappa' \in \mathcal{L}(G) \text{ such that } \kappa' \preceq \kappa \right\}, \text{ as } \kappa \in \mathcal{M}(G).\]

The following theorem is the counterpart of Theorem 3.3.

**Theorem 7.2.** (a) Fix \(\kappa \in \mathcal{M}(G)\). Assume that for any \(G_0 \in G_{\kappa}\), the conditions of part (iii) of Theorem 7.1 hold for \(G_{\kappa}\) (i.e., \(G\) is replaced by \(G_{\kappa}\) in that theorem) and \(\kappa'\) satisfies condition (b) of part (iii) of that theorem as long as \(\kappa' \in \mathcal{L}(G_0)\). Then, for any \(\kappa' \in \mathbb{R}^d\) such that \((1, \ldots, 1) \preceq \kappa' \prec \kappa\) there holds

\[
\inf_{\hat{G}_n \in G_{\kappa}} \sup_{G_0 \in G_{\kappa}} E_{p_{G_0}} \tilde{W}_{\kappa'}(\hat{G}_n, G_0) \gtrsim n^{-1/2}||\kappa'||_{\infty}.
\]

Here, the infimum is taken over all sequences of estimates \(\hat{G}_n \in G_{\kappa}\) and \(E_{p_{G_0}}\) denotes the expectation taken with respect to product measure with mixture density \(p_{G_0}\).

(b) Let \(G_0 \in G_{\kappa}\) for some fixed \(\kappa \in \mathcal{M}(G)\). Let \(\hat{G}_n \in G_{\kappa}\) be a point estimate for \(G_0\), which is obtained from an \(n\)-sample of i.i.d. observations drawn from \(p_{G_0}\). As long as \(h(p_{\hat{G}_n}, p_{G_0}) = O_P(n^{-1/2})\), we have

\[
\tilde{W}_{\kappa}(\hat{G}_n, G_0) = O_P(n^{-1/2}||\kappa'||_{\infty}).
\]

We have some remarks regarding Theorem 7.2: (i) If we define \(\kappa_{(\min)}^i = \min_{\kappa \in \mathcal{L}(G_0)\mathcal{G}} \kappa_i\) for any \(1 \leq i \leq d\), then the result of part (b) implies that the best possible convergence rate of estimating \(i\)-th component of atoms of \(G_0\), in a local minimax sense, is \(n^{-1/2}\kappa_{(\min)}^i\) as \(1 \leq i \leq d\); (ii) The result of part (b) implies that if \(G_0\) is not \(\kappa'\)-singular relative to \(G\) for some \(\kappa' \in \mathbb{N}^d\), then we obtain

\[
\tilde{W}_{\kappa'}(\hat{G}_n, G_0) = O_P(n^{-1/2}||\kappa'||_{\infty})
\]

where \(\hat{G}_n\) is a sequence of estimates specified in part (b) of Theorem 7.2.
8 Appendix C: Complete inhomogeneity via singularity matrices

In this appendix, we provide a new notion, namely, singularity matrix, which characterizes the inhomogeneity of convergence behavior of different atoms of a fixed mixing measure under a finite mixture model. To simplify the presentation, we will only consider the e-mixtures setting, i.e., the number of mixture components is known. Let \( G_0 = G_0(p^0, \eta^0) \) be a true mixing measure with weights \( p^0 \) and atoms \( \eta^0 \) such that it has \( k_0 \) components. To capture distinct convergence rates of atoms of \( G_0 \), we introduce a blocked version of transportation distance with respect to \( G_0 \) as follows. Recall that for any \( \kappa = (\kappa_1, \ldots, \kappa_d) \in \mathbb{N}^d \), we defined

\[
d_{\kappa}(\theta_1, \theta_2) : = \left( \sum_{i=1}^{d} |\theta_1^{(i)} - \theta_2^{(i)}|^{\kappa_i} \right)^{1/\|\kappa\|_{\infty}}
\]

for any \( \theta_i = (\theta_i^{(1)}, \ldots, \theta_i^{(d)}) \in \mathbb{R}^d \) as \( 1 \leq i \leq 2 \).

**Definition 8.1.** Given any matrix \( K \in \mathbb{N}^{k_0 \times d} \) and mixing measure \( G_0(p^0, \eta^0) \) with \( k_0 \) components, the blocked transportation distance with respect to matrix \( K \) and mixing measure \( G_0 \) is given by

\[
\widehat{W}_K(G(p, \eta), G_0(p^0, \eta^0)) := \left( \inf \left\{ \sum_{i,j} q_{ij} d_{\|K\|_{\infty}}(\eta_i, \eta_j^0) \right\} \right)^{1/\|K\|_{\infty}}
\]

for any probability measure \( G = G(p, \eta) \) with weights \( p \) and atoms \( \eta \) such that it has exactly \( k_0 \) components. The infimum in the above formulation is taken over all couplings \( q \) between \( p \) and \( p^0 \), while \( K_j \) is the \( j \)-th row of matrix \( K \) for any \( 1 \leq j \leq k_0 \) and \( \|K\|_{\infty} := \max_{1 \leq i \leq k_0} \{ \|K_i\|_{\infty} \} \).

Henceforth in this appendix, for any matrix \( K \) we denote \( K_i \) as its \( i \)-th row and \( K_{ij} \) as its element in the \( i \)-th row and \( j \)-th column. Additionally, for any two matrices \( K, K' \in \mathbb{N}^{k_0 \times d} \), we denote \( K \preceq K' \) if \( K_i \preceq K'_i \) for all \( 1 \leq i \leq k_0 \). In general, the blocked transportation distance is not a metric as it is not symmetric except when all the columns of \( K \) are identical. Furthermore, it also does not satisfy the standard triangle inequality except when all the elements of \( K \) are identical. We have the following result relating blocked transportation distance to generalized transportation distance and standard Wasserstein distance.

**Proposition 8.1.** Let \( K \in \mathbb{N}^{k_0 \times d} \) be any matrix. Then, we have

\[
\widehat{W}_K(G, G_0) \geq W_K(G, G_0) \geq W_{\|\kappa\|_{\infty}}(G, G_0)
\]

for any \( \kappa \) such that \( K_i \preceq \kappa \) for all \( 1 \leq i \leq d \) and \( \|K\|_{\infty} = \|\kappa\|_{\infty} \). The first inequality holds when \( K_i \) are all equal to \( \kappa \). The second inequality holds when all the elements of \( K \) are equal to \( \|\kappa\|_{\infty} \).

The matrix \( K \) utilized in the definition of blocked transportation distance helps to capture fine-grained differences in convergence behavior toward individual components of each distinct atom of the true mixing measure \( G_0 \). In particular, assume that a sequence of probability measures \( G_n \in \mathcal{E}_{k_0} \) tending to \( G_0 \) under \( \widehat{W}_K \) distance at a rate \( \omega_n = o(1) \) for some matrix \( K \in \mathbb{N}^{k_0 \times d} \). Similar to the interpretation with generalized transportation metric, the \( i \)-th atom of \( G_0 \) receives a converging sequence from an atom of \( G_n \), to be also labeled \( i \), at the rate \( (\omega_n)^{\|K_i\|_{\infty}/\|K_i\|_{\infty}} \) under \( d_{K_i} \) semi-metric. Hence, the \( j \)-th component of \( i \)-th atom of \( G_n \) converges to the \( j \)-th component of corresponding atom of \( G_0 \) at rate \( (\omega_n)^{\|K_{ij}\|_{\infty}/\|K_{ij}\|_{\infty}} \) for any \( 1 \leq i \leq k_0 \) and \( 1 \leq j \leq d \).

Similar to approaches to the singularity level and singularity index, we are also interested in analyzing the behavior of likelihood function \( p_G \) as \( G \in \mathcal{E}_{k_0} \) varies in a neighborhood of \( G_0 \) according to
the blocked transportation distance. To avoid the ambiguity in our argument, we will repeat the key steps from establishing singularity level and singularity index in developing the key notions. In particular, for any fixed matrix $K \in \mathbb{N}^{k_0 \times d}$, we consider a sequence $G_n \in \mathcal{E}_{k_0}$ such that $\hat{W}(G_n, G_0) \to 0$. We can argue that up to a permutation of atoms’ labels, $G_n$ can be represented as $[\mathbf{T}]$ where $\mathbf{T} = 0$ and $s_i = 1$ for all $1 \leq i \leq k_0$, i.e., $G_n = \sum_{i=1}^{k_0} p_i^0 \cdot \delta_{\eta_i^0}$. To avoid notational cluttering, we also drop $n$ from the superscript when the context is clear. Now, denote $\|K\|_{\infty} = r$. By carrying Taylor expansion up to $r$-th order, we achieve that

$$
p_G(x) - p_{G_0}(x) = \frac{1}{\hat{W}_K^r(G, G_0)} = \sum_{|\alpha|=1}^{r} \sum_{i=1}^{k_0} \left( \frac{p_i(\Delta \eta_i)^{\alpha}}{W_r(G, G_0)} \right) \frac{\partial |\alpha| f(x | \eta_i^0)}{\partial \eta^\alpha_i} + \sum_{i=1}^{k_0} \frac{\Delta p_i}{W_r(G, G_0)} f(x | \eta_i^0) + o(1),$$

as long as $f$ is uniform Lipschitz up to order $r$. The above representation motivates the following definition of matrix minimal form

**Definition 8.2.** For any $K \in \mathbb{N}^{k_0 \times d}$, the following representation is called $K$-minimal form of the mixture likelihood for a sequence of mixing measures $G$ tending to $G_0$ in $\hat{W}_K$ distance:

$$
p_G(x) - p_{G_0}(x) = \sum_{i=1}^{T_K} \left( \frac{\xi_i^{(K)}(G)}{\hat{W}_K^{\infty}(G, G_0)} \right) H_i^{(K)}(x) + o(1),$$

which holds for all $x$, with the index $l$ ranging from $1$ to a finite $T_K$, if

1. $H_i^{(K)}(x)$ for all $l$ are linearly independent functions of $x$, and
2. coefficients $\xi_i^{(K)}(G)$ are polynomials of the components of $\Delta \eta_{ij}$, and $\Delta p_i, p_{ij}$.

It is clear that $K$-minimal form is a general version of $\kappa$-minimal form for any $\kappa \in \mathbb{N}^d$ when $K_i = \kappa$ for all $1 \leq i \leq k_0$. Similar to $\kappa$-minimal form in Section 3.3, matrix $K$-minimal form leads to our notion of matrix singularity, which we now define.

**Definition 8.3.** For any $K \in \mathbb{N}^{k_0 \times d}$, we say that $G_0$ is $K$-singular relative to $\mathcal{E}_{k_0}$, if $G_0$ admits a $K$-minimal form given by Eq. (32), according to which there exists a sequence of $G \in \mathcal{E}_{k_0}$ tending to $G_0$ under $\hat{W}_K$ distance such that

$$
\xi_i^{(K)}(G)/\hat{W}_K^{\infty}(G, G_0) \to 0 \text{ for all } l = 1, \ldots, T_K.
$$

We note that the limiting behavior of ratios (coefficients) $\xi_i^{(K)}(G)/\hat{W}_K^{\infty}(G, G_0)$ is generally more challenging to investigate than those of coefficients of $\kappa$-minimal form in Definition 3.5 and $r$-minimal form in Definition 3.2 due to the complex nature of blocked transportation distance $\hat{W}_K^{\infty}(G, G_0)$. In particular, we can demonstrate that as $\hat{W}_K^{\infty}(G, G_0) \to 0$, $\hat{W}_K^{\infty}(G, G_0) \sim D_K(G, G_0)$ where

$$
D_K(G, G_0) := \sum_{i=1}^{k_0} p_i d_{K_i}^{\infty}(\eta_i, \eta_i^0) + \sum_{i=1}^{k_0} |\Delta p_i|.
$$

As $\hat{W}_K^{\infty}(G, G_0)$ is asymptotically equivalent to a rather complicated inhomogeneous semipolynomial form, the vanishing of ratios $\xi_i^{(K)}(G)/\hat{W}_K^{\infty}(G, G_0)$ will be difficult to fathom if the values of vector $K_i$ are very different. Similar to singularity levels and singularity indices, the matrix singularity notion also possesses a crucial monotonic property in terms of a partial order of matrices:
Lemma 8.1. (a) (Invariance) The existence of the sequence of $G$ in the statement of Definition 8.3 holds for all $K$-minimal forms once it holds for at least one $K$-minimal form.

(b) (Monotonicity) If $G_0$ is $K$-singular to $E_{k_0}$ for some $K \in \mathbb{N}^{k_0 \times d}$, then $G_0$ is $K'$-singular for any $K' \preceq K$.

The monotonicity of $K$-singularity leads to the following notion of singularity matrix of a mixing measure $G_0$ relative to an ambient space $E_{k_0}$.

Definition 8.4. For any $K \in \mathbb{N}^{k_0 \times d}$, we say $K$ is a singularity matrix of $G_0$ relative to a given class $E_{k_0}$ if and only if $G_0$ is $K'^{-}$-singular relative to $E_{k_0}$ for any $K' \preceq K$, and there is no $K' \succeq K$ such that $G_0$ remains $K'^{+}$-singular relative to $E_{k_0}$.

Denote $\mathcal{M}(G_0|E_{k_0}) = \{ K \in \mathbb{N}^{k_0 \times d} : K$ is singularity matrix of $G_0 \}$, i.e., the set of all singularity matrices of $G_0$ relative to $E_{k_0}$. The significance of singularity matrix notion can be summarized by the following results.

Theorem 8.1. Fix $G_0 \in E_{k_0}$. Take a $K \in \mathcal{M}(G_0|E_{k_0})$, let $r = \|K\|_{\infty}$.

(i) If $r < \infty$, then $\inf_{G \in \mathbb{P}} \frac{\|pG - p_{G_0}\|_{\infty}}{\hat{W}_{K'}[\|K\|_{\infty}]} > 0$ for any $K' \succeq K$.

(ii) If $r < \infty$, then $\inf_{G \in \mathbb{P}} \frac{V(pG, p_{G_0})}{\hat{W}_{K'}[\|K\|_{\infty}]} > 0$ for any $K' \succeq K$.

(iii) Let $\hat{G}_n \in E_{k_0}$ be a point estimate for $G_0$, which is obtained from an $n$-sample of i.i.d. observations drawn from $p_{G_0}$. As long as $h(p_{\hat{G}_n}, p_{G_0}) = O_P(n^{-1/2})$ and $r < \infty$, we obtain

$$\hat{W}_{K'}(\hat{G}_n, G_0) = O_P(n^{-1/2}[K']_{\infty}).$$

for any $K' \succeq K$.

Let $K_{ij}^{(\min)} = \min_{K \in \mathcal{M}(G_0|E_{k_0})} K_{ij}$ for any $1 \leq i \leq k_0$ and $1 \leq j \leq d$. Part (iii) provides a guarantee for the estimation rate of the $j$-th component of $i$-th atom of $G_0$ to be at most $n^{-1/2}K_{ij}^{(\min)}$.

8.1 Examples of singularity matrices with e-mixtures

In this section, we provide several examples of singularity matrices of e-mixtures models that we have studied thus far in the paper, including e-mixtures of first order identifiable kernels, Gamma e-mixtures, and skew-normal e-mixtures. The proofs of these results are quite similar to those for singularity levels and indices, and will be omitted for the brevity of the paper.

E-mixtures of first order identifiable kernels As studied by [29], the first order identifiability of kernel density $f$ means that the collection of $\{\partial^k f/\partial \eta^k(x|\eta_j) | j = 1, \ldots, k_0; |k| \leq 1 \}$ evaluated at $G_0$ are linearly independent. It is not difficult to establish the following result regarding singularity matrices of $G_0$ under first order identifiability condition of $f$.

Proposition 8.2. Assume that $f$ is first order identifiable and admits uniform Lipschitz condition up to the first order. Then, $K = 1_{k_0 \times d}$ is the unique element of $\mathcal{M}(G_0|E_{k_0})$ where $1_{k_0 \times d}$ is the matrix with all elements to be 1.
Proposition 8.3. For any $G_0 \in \mathcal{E}_{k_0}$, we obtain

(a) Generic cases: $K = 1_{k_0 \times 2}$ is the unique singularity matrix of $\mathcal{M}(G_0|\mathcal{E}_{k_0})$.

(b) Pathological cases: let $A = \{ i : \exists j \text{ such that } \{ |a_i^0 - a_j^0|, |b_i^0 - b_j^0| \} = \{ 1, 0 \} \}$. Let $K \in \mathbb{N}^{k_0 \times 2}$ such that $K_i = (\infty, \infty)$ when $i \in A$ and $K_i = (\infty, \infty)$ when $i \in A^c$. Then, matrix $K$ is the unique element of $\mathcal{M}(G_0|\mathcal{E}_{k_0})$.

We wish to emphasize that the non-polynomial convergence rates of atoms in $A$ under pathological cases are just the upper bounds. It is possible that the actual convergence rates of these atoms in $A$ may be better. We leave this question for future exploration.

Proposition 8.4. If $G_0 \in \mathcal{S}_0$, then $K = 1_{k_0 \times 3}$ is the unique element of $\mathcal{M}(G_0|\mathcal{E}_{k_0})$.

When $G_0 \not\in \mathcal{S}_0$, we further consider subsets of the complement of $\mathcal{S}_0$. In particular, we give results for two subsets of the complement: $G_0 \in \mathcal{S}_1$ or $G_0 \in \mathcal{S}_2$, which is defined in Section 10. Briefly speaking, in that definition, for each index $i = 1, \ldots, k_0$, $I_i$ collects all the atoms homologous to the $i$-th atom of $G_0$. The following result establishes singularity matrix of $G_0 \in \mathcal{S}_1$.

Proposition 8.5. Given $G_0 \in \mathcal{S}_1$. Denote $A = \{ i : I_i \text{ has more than one elements} \}$. Let $K \in \mathbb{N}^{k_0 \times 3}$ be such that $K_i = (1, 1, 2)$ for all $i \in A$ and $K_i = (1, 1, 1)$ for all $i \in A^c$. Then, matrix $K$ is the unique element of $\mathcal{M}(G_0|\mathcal{E}_{k_0})$.

As a consequence of Proposition 8.5, the atoms of $G_0 \in \mathcal{S}_1$ can be divided into two blocks according to their convergence rates. For those without any homologous structure, their convergence rates are $n^{-1/2}$ (up to a log factor); however, for those with homologous structure with conformant property, their convergence rates of location, scale parameters admit $n^{-3/2}$ convergence rate while shape parameters admit $n^{-1/4}$ rate. Under the setting of $G_0 \in \mathcal{S}_2$, the singularity matrix becomes somewhat more complicated as being demonstrated by the following result.

Proposition 8.6. Given $G_0 \in \mathcal{S}_2$. Denote $A = \{ i : I_i \text{ has more than one elements} \}$ and $B = \{ j : m_j^0 = 0 \}$. Let $K \in \mathbb{N}^{k_0 \times 3}$ be such that $K_i = (1, 1, 2)$ for all $i \in A$, $K_i = (3, 2, 3)$ for all $i \in B$, and $K_i = (1, 1, 1)$ for all $i \in (A \cup B)^c$. Then, matrix $K$ is the unique element of $\mathcal{M}(G_0|\mathcal{E}_{k_0})$.

The above result indicates that the atoms of $G_0 \in \mathcal{S}_3$ can be divided into three blocks according to their convergence rates. For the “Gaussian atoms” in $G_0$, i.e., those indexed by set $B$, the convergence rates of location and shape parameters of these atoms are $n^{-1/6}$ while those of scale parameters of these atoms are $n^{-1/4}$. For other atoms of $G_0$, we arrive at the same regime of convergence as those in Proposition 8.5.
9 Appendix D: Proofs for Sections 3 and 4

For completeness we collect the remaining proofs of the statements described in the main text.

9.1 Proofs for Section 3

PROOF OF THEOREM 3.2. Since the proofs for part (i) and (ii) are similar, we only provide the proof for part (i). The proof of this part is the generalization of that of part (c) in Theorem 3.2 in [29]. By means of Taylor expansion up to \( r \)-th order, we have

\[
h^2(p_G, p_{G_0}) < \int_\mathcal{X} \left( \frac{(p_G(x) - p_{G_0}(x))^2}{p_{G_0}(x)} \right) dx = \int_\mathcal{X} \frac{\left( \sum_{l = 1}^{T_r} \xi_i^{(r)}(G)H_i^{(r)}(x) + R_r(x) \right)^2}{p_{G_0}(x)} dx
\]

where the last inequality is due to Cauchy-Schwarz’s inequality. Here, \( R_r(x) \) has the following form

\[
R_r(x) = \sum_{i=1}^{k_0} \sum_{j=1}^{s_i} \sum_{|\alpha|=r+1} \frac{r+1}{\alpha!} (\Delta \eta_{ij})^\alpha \int_0^1 (1-t)^r \frac{\partial^{r+1}f}{\partial \eta_i^\alpha}(x|\eta_i^0 + t\Delta \eta_{ij}) dt.
\]

Due to condition (a), the following result holds

\[
\int_\mathcal{X} \left( H_i^{(r)}(x) \right)^2 p_{G_0}(x) dx < \infty
\]

for any \( 1 \leq l \leq T_r \). Combining the above result with the assumption that \( \xi_i^{(r)}/W_1^s(G, G_0) \to 0 \) in (b) for all \( s \in [1, r+1] \) and \( l = 1, \ldots, T_r \), we achieve that

\[
\int_\mathcal{X} \left( \frac{\xi_i^{(r)}(G)H_i^{(r)}(x)}{W_1^{2s}(G, G_0)p_{G_0}(x)} \right) dx \to 0
\]

(33)

for all \( 1 \leq l \leq T_r \). Additionally, as \( p_{G_0}(x) > p_i^0 f(x|\eta_i^0) \) for all \( 1 \leq i \leq k_0 \), for any \( s < r + 1 \), we have

\[
\frac{h^2(p_G, p_{G_0})}{W_1^{2s}(G, G_0)} \leq \int_\mathcal{X} R_r^2(x) \frac{1}{W_1^{2s}(G, G_0)p_{G_0}(x)} dx
\]

\[
\leq \sum_{i=1}^{k_0} \int_\mathcal{X} \left( \frac{\sum_{j=1}^{s_i} \sum_{|\alpha|=r+1} \frac{r+1}{\alpha!} (\Delta \eta_{ij})^\alpha \int_0^1 (1-t)^r \frac{\partial^{r+1}f}{\partial \eta_i^\alpha}(x|\eta_i^0 + t\Delta \eta_{ij}) dt)^2}{W_1^{2s}(G, G_0)p_i^0 f(x|\eta_i^0)} \right) dx
\]

\[
\leq \sum_{i=1}^{k_0} \int_\mathcal{X} \left( \frac{\sum_{j=1}^{s_i} \sum_{|\alpha|=r+1} \frac{r+1}{\alpha!} (\Delta \eta_{ij})^\alpha \int_0^1 (1-t)^r \frac{\partial^{r+1}f}{\partial \eta_i^\alpha}(x|\eta_i^0 + t\Delta \eta_{ij}) dt)^2}{W_1^{2s}(G, G_0)p_i^0 f(x|\eta_i^0)} \right) dx,
\]
where the last inequality is due to Cauchy-Schwarz’s inequality. Now, for any \( s < r + 1 \), by utilizing Lemma 3.1 and the assumption that \( p_{ij} \) are bounded away from 0 in condition (b), we obtain that
\[
\frac{|(\Delta \eta_{ij})^\alpha|}{D_1(G, G_0)} \leq \frac{|(\Delta \eta_{ij})^\alpha|}{D_1(G_0, G)} \leq |\Delta \eta_{ij}|^s \rightarrow 0, \tag{34}
\]
for any \( |\alpha| = r + 1 \). According to the hypothesis, as \( \Delta \eta_{ij} < c_0 \), we have
\[
\int_X \left( \int_0^1 (1 - t)^r \frac{\partial^{\alpha+1} f}{\partial \eta_0^\alpha}(x|\eta_i^0 + t \Delta \eta_{ij}) dt \right)^2 \, dx < \int_X \left( \frac{\partial^{\alpha+1} f}{\partial \eta_0^\alpha}(x|\eta_i^0) \right)^2 \, dx < \infty. \tag{35}
\]
By combining (33), (34), and (35), we achieve \( h(p_G, p_{G_0}) / W_1^s(G, G_0) \to 0 \), which yields the conclusion of this part.

**PROOF OF LEMMA 3.4**  The proof idea of Lemma 3.4 is a generalization of that of Lemma 3.2.

(a) The existence of the sequence of \( G \) described in the definition of a \( \kappa \)-minimal form implies for that sequence, \( (p_G(x) - p_{G_0}(x)) / W_1 \| \kappa \|_{\infty} (G, G_0) \to 0 \) holds for almost all \( x \). Now take any \( \kappa \)-minimal form \( \{10\} \) given by the same sequence. Let \( C(G) = \max_{l=1}^{T_x} \frac{\xi_l^\kappa(G)}{W_1 \| \kappa \|_{\infty} (G_0, G)} \). We will show that \( \lim \inf C(G) = 0 \), which concludes the proof. Suppose this is not the case, so we have \( \lim \inf C(G) > 0 \). It follows that
\[
\sum_{l=1}^{T_x} \frac{\xi_l^\kappa(G)}{C(G) W_1 \| \kappa \|_{\infty} (G_0, G)} H_l^\kappa(x) \to 0.
\]
Moreover, all the coefficients in the above display are bounded from above by 1, one of which is in fact 1. There exists a subsequence of \( G \) by which these coefficients have limits, one of which is 1. This is a contradiction due to the linear independence of functions \( H_l^\kappa(\cdot) \).

(b) It suffices to establish the conclusion of this part when \( \| \kappa \|_{\infty} = 1 \leq \| \kappa' \|_{\infty} \leq \| \kappa \|_{\infty} \). Let \( G \) be an element in the sequence that admits a \( \kappa \)-minimal form such that \( \xi_l^\kappa(G) / W_1 \| \kappa \|_{\infty} (G_0, G) \to 0 \) for all \( l = 1, \ldots, T_x \). It suffices to assume that the basis functions \( H_l^\kappa \) are selected from the collection of partial derivatives of \( f \). We will show that the same sequence of \( G \) and the elimination procedure for the \( \kappa \)-minimal form can be used to construct a \( \kappa' \)-minimal form by which
\[
\xi_l^\kappa(G) / W_1 \| \kappa \|_{\infty} (G_0, G) \to 0
\]
for all \( l = 1, \ldots, T_{x'} \). When \( \| \kappa' \|_{\infty} = \| \kappa \|_{\infty} \), as \( \kappa' \leq \kappa \) and the support points of \( G \) and \( G_0 \) are in a bounded set, it is straightforward that \( \hat{W}_1 \| \kappa' \|_{\infty} (G_0, G) \leq \hat{W}_1 \| \kappa \|_{\infty} (G_0, G) \). Therefore, by choosing \( T_{x'} = T_{x'} \) and \( \xi_l^\kappa(G) = \xi_l^\kappa(G) \) for any \( 1 \leq l \leq T_{x'} \), we obtain
\[
\xi_l^\kappa(G) / \hat{W}_1 \| \kappa \|_{\infty} (G_0, G) \leq \xi_l^\kappa(G) / \hat{W}_1 \| \kappa \|_{\infty} (G_0, G) \to 0
\]
for any \( 1 \leq l \leq T_{x'} \). This results in a valid \( \kappa' \)-form. It remains to consider the case \( \| \kappa' \|_{\infty} = \| \kappa \|_{\infty} - 1 \). There are two possibilities.

First, suppose that each of the \( \| \kappa \|_{\infty} \)-th partial derivatives of density kernel \( f \) (i.e., \( \partial^{\alpha} f / \partial \eta_0^\alpha \), where \( |\alpha| = \| \kappa \|_{\infty} \)) is not in the linear span of the collection of partial derivatives of \( f \) at order \( \| \kappa \|_{\infty} - 1 \) or less. Then, for each \( l = 1, \ldots, T_{x'} \), \( \xi_l^{\kappa'}(G) = \xi_{l'}^{\kappa}(G) \) for some \( l' \in [1, T_x] \). Since \( \hat{W}_1 \| \kappa' \|_{\infty} (G, G_0) \geq \hat{W}_1 \| \kappa \|_{\infty} (G, G_0) \), we have that
\[
\xi_l^{\kappa'}(G) / \hat{W}_1 \| \kappa' \|_{\infty} (G, G_0) \lesssim \xi_l^{\kappa}(G) / \hat{W}_1 \| \kappa \|_{\infty} (G, G_0)
\]
which vanishes by the hypothesis.

Second, suppose that some of the \( \| \kappa \|_\infty \)-th partial derivatives, say, \( \partial^{[\beta]} f / \partial \eta^{[\beta]} \) where \( |\beta| = \| \kappa \|_\infty \), can be eliminated because they can be represented by a linear combination of a subset of other partial derivatives \( H_{ij}^{(\kappa)} \) (in addition to possibly a subset of other partial derivatives \( H_{ij}^{(\kappa')} \)) with corresponding finite coefficients \( \alpha_{\beta,i,t} \). It follows that for each \( l = 1, \ldots, T_{\kappa'} \), the coefficient \( \xi_{ij}^{(\kappa')} (G) \) that defines the \( \kappa' \)-minimal form is transformed into a coefficient in the \( \kappa \)-minimal form by

\[
\xi_{ij}^{(\kappa)} (G) := \xi_{ij}^{(\kappa')} (G) + \sum_{\beta, i, t = |\kappa|} \sum_{i=1}^{k_0 + i} \sum_{j=1}^{s_i} \alpha_{\beta,i,t} p_{ij} (\Delta \eta_{ij})^{[\beta]} / \beta !.
\]

Since \( \xi_{ij}^{(\kappa)} (G) / W_{\kappa} \|\kappa\|_\infty (G, G_0) \) tends to 0, so does \( \xi_{ij}^{(\kappa')} (G) / W_{\kappa'} \|\kappa'\|_\infty (G, G_0) \). By Lemma 3.3 for each \( \beta \) such that \( |\beta| = \| \kappa \|_\infty = r \), \( \sum_{i=1}^{k_0 + i} \sum_{j=1}^{s_i} p_{ij} (\Delta \eta_{ij})^{[\beta]} / \beta ! = o(W_{r-1}(G_0, G)) = o(D_{\kappa'}(G_0, G)) = o(W_{\kappa'} \|\kappa'\|_\infty (G_0, G)) \). It follows that \( \xi_{ij}^{(\kappa')} (G) / W_{\kappa'} \|\kappa'\|_\infty (G, G_0) \) tends to 0, for each \( l = 1, \ldots, T_{\kappa'} \). This completes the proof.

PROOF OF PROPOSITION 3.1 Part (i) and (ii) are immediate from the definition of \( \ell (G_0 | G) \) and \( \mathcal{L}(G_0 | G) \).

To prove part (iii), note that there are two possibilities. First, \( G_0 \) is \( \kappa \)-singular relative to \( G \) for any \( \kappa < (r + 1, \ldots, r + 1) \). Thus, \( (r + 1, \ldots, r + 1) \in \mathcal{L}(G_0 | G) \), we are done. Second, there exists \( \kappa < (r + 1, \ldots, r + 1) \) such that \( G_0 \) is not \( \kappa \)-singular relative to \( G \). Denote \( A = \{ \kappa' < \kappa : \kappa < \kappa_{G_0} \ and \ G_0 \ is \ not \ \kappa' \text{-singular} \} \). It is clear that \( |A| \) is finite. Since \( \ell (G_0 | G) = r \), \( G_0 \) is \( (r, \ldots, r) \)-singular relative to \( G \), which also implies that it is \( \kappa' \)-singular for any \( \kappa' < (r, \ldots, r) \) according to part (b) of Lemma 3.4. Therefore, at least one component of \( \kappa \) is \( r + 1 \). If \( G_0 \) is \( \kappa' \)-singular relative to \( G \) for all \( \kappa < \kappa \), then \( \kappa \in \mathcal{L}(G_0 | G) \), which concludes the proof. If there exists \( \kappa^{(1)} < \kappa \) such that \( G_0 \) is not \( \kappa^{(1)} \)-singular relative to \( G \), then at least one component of \( \kappa^{(1)} \) is \( r + 1 \) by the fact that \( \ell (G_0 | G) = r \). If \( G_0 \) is \( \kappa' \)-singular relative to \( G \) for any \( \kappa < \kappa^{(1)} \), then \( \kappa' \in \mathcal{L}(G_0 | G) \). If the previous assumption does not hold, then we also achieve \( \kappa^{(2)} \) such that \( G_0 \) is not \( \kappa^{(2)} \)-singular relative to \( G \). By repeating the same argument, we eventually will have an index \( 1 \leq s \leq |A| \) such that \( G_0 \) is not \( \kappa^{(s)} \)-singular while it is \( \kappa' \)-singular for any \( \kappa' < \kappa^{(s)} \), which implies that \( \kappa^{(s)} \in \mathcal{L}(G_0 | G) \). As \( \kappa^{(s)} < \kappa < (r + 1, \ldots, r + 1) \), we achieve the conclusion of part (iii).

For part (iv), \( r \geq 1 \) implies that \( G_0 \) is \( r \)-singular relative to \( G \). Moreover, \( G_0 \) is \( \kappa \)-singular relative to \( G \) for all \( \kappa < (r, \ldots, r) \). Since \( G_0 \) is not \( \overline{\kappa} \)-singular by the hypothesis, we must have \( (1, \ldots, 1) < \overline{\kappa} \). Due to the boundedness of \( \overline{\kappa} \), the existence of non-empty \( \mathcal{L}(G_0 | G) \) follows.

For part (v), since \( \kappa \leq \kappa' := (\| \kappa \|_\infty, \ldots, \| \kappa \|_\infty) \), by definition \( G_0 \) is not \( \kappa' \)-singular relative to \( G \). It follows that \( \ell (G_0 | G) < \| \kappa \|_\infty \). If \( \kappa \) is unique, then the conclusion is immediate from part (iii).

PROOF OF PROPOSITION 3.2 The assumption of second order identifiability entails that \( G_0 \) is neither 2-singular nor \((2, \ldots, 2)\)-singular relative to \( O_k \) (cf. proof of Theorem 3.2 of [28]). As a consequence of Proposition 3.1 it suffices to demonstrate that \( G_0 \) is \( \kappa \)-singular relative to \( O_k \) for any \( \kappa \) such that all of its components are \( r \) except for one component to be 1 as \( r \geq 1 \). Without loss of generality, let \( \kappa = (1, r, \ldots, r) \). For any \( \eta \in \mathbb{R}^d \), let \( \eta^{(i)} \) denote the \( i \)-th component of \( \eta \) for any \( 1 \leq i \leq d \). To simplify our proof argument, we firstly consider the basic case of \( r = 2 \). Now, construct sequence of \( G \to G_0 \) such that \( G \) always has \( k_0 + 1 \) support points. Specifically using the representation \( G = \sum_{i=1}^{k_0} \sum_{j=1}^{s_i} p_{ij} \delta_{\eta_{ij}} \) where \( s_1 = 2 \) and \( s_i = 1 \) for all \( 2 \leq i \leq k_0 \). Additionally, \( \eta_{11} \) and \( \eta_{12} \) are chosen such that \( \Delta \eta_{11} = -\Delta \eta_{12} \) and \( (\Delta \eta_{11})^2 / \Delta \eta_{11}^{(1)} \to 0 \) for all \( 2 \leq i \leq d \). For
other atoms of $G$, we choose $\eta_{ij} = \eta_{i}^0$ for all $2 \leq i \leq k_0$. As for the mass of $G$'s atoms, we choose $p_{11} = p_{12} = p_{0}^1/2$ and $p_{ij} = p_{0}^0$ for all $2 \leq i \leq k_0$. From this choice of $G$, we can verify that $\hat{W}_G^r(G, G_0) \succ p_0^1(|\Delta \eta_{11}^{(1)}| + \sum_{i=2}^d (|\Delta \eta_{11}^{(1)}|)^2) \succ |\Delta \eta_{11}^{(1)}|$. By carrying out Taylor expansion of the likelihood function up to the second order, we obtain a $\kappa$-minimal form for the sequence $G$,

$$
\frac{p_G(x) - p_{G_0}(x)}{W^r_G(G, G_0)} \propto \sum_{i=1}^2 \sum_{|\alpha| \leq 2} \left(\Delta \eta_{11}^{(1)}\alpha \frac{\partial f^{(1)}|\alpha|}{\partial \eta^{\alpha}}(x|\eta_1^0)\right)
\frac{|\Delta \eta_{11}^{(1)}|}{|\Delta \eta_{11}^{(1)}|}
$$

(36)

thanks to the second-order identifiability condition on $f$. Since the $\kappa$-minimal form coefficients all vanish due to our choice of $\eta_{11}$ and $\eta_{12}$, we conclude that $G_0$ is $\kappa$-singular relative to $O_k$ when $\kappa = (1, r, \ldots, r)$ and $r = 2$.

Now, for general value of $r \geq 3$ and $\kappa = (1, r, \ldots, r)$, with the choice of $\eta_{11}$ and $\eta_{12}$ such that $\Delta \eta_{11} = -\Delta \eta_{12}$ and $(\Delta \eta_{11}^{(1)})^{r}/|\Delta \eta_{11}^{(1)}| \to 0$ for any $2 \leq i \leq d$ we can verify that $\hat{W}_G^r(G, G_0) \succ |\Delta \eta_{11}^{(1)}|$ and

$$
\left(\frac{p_{11}(\Delta \eta_{11}^{(1)})^\alpha + p_{12}(\Delta \eta_{12}^{(1)})^\alpha}{\hat{W}_G^r(G, G_0)}\right) \to 0
$$

(37)

for any $|\alpha| \geq 3$. By means of Taylor expansion up to the $r$ order, a $\kappa$-minimal form for the sequence $G$ is as follows

$$
\frac{p_G(x) - p_{G_0}(x)}{W^r_G(G, G_0)} \propto \sum_{i=1}^2 \sum_{|\alpha| \leq r} \left(\Delta \eta_{11}^{(1)}\alpha \frac{\partial f^{(1)}|\alpha|}{\partial \eta^{\alpha}}(x|\eta_1^0)\right)
\frac{|\Delta \eta_{11}^{(1)}|}{|\Delta \eta_{11}^{(1)}|}
\sum_{i=1}^2 \sum_{|\alpha| \leq 2} \left(\Delta \eta_{11}^{(1)}\alpha \frac{\partial f^{(1)}|\alpha|}{\partial \eta^{\alpha}}(x|\eta_1^0)\right)
\frac{|\Delta \eta_{11}^{(1)}|}{|\Delta \eta_{11}^{(1)}|}
\to 0.
$$

Here, the second asymptotic result is due to results in [27] while the last limit is due to the choice of $G$. Therefore, $G_0$ is $\kappa$-singular relative to $O_k$ when $\kappa = (1, r, \ldots, r)$ for any $r \geq 1$. It follows that $(2, \ldots, 2)$ is the unique singularity index of $G_0$ relative to $O_k$, i.e., $L(G_0|O_k) = \{(2, \ldots, 2)\}$.

**PROOF OF PROPOSITION 3.3** Here, we only provide the proof of this proposition for the case $\tau(k - k_0)$ is an even number as the argument for the case $\tau(k - k_0)$ is an odd number is similar. We follow the argument from the proof of Proposition 2.2 in [28]. Denote $v = \sigma^2$ and $\tau = \tau(k - k_0)$. For any $\|k\|_\infty = r$ and $r \geq 1$, let $G \in O_{k, c_0} \to G_0$ under $\hat{W}_G$ distance. According to Step 1 to Step 3 in the proof of Proposition 2.2 in [28] we have the $\kappa$-minimal form for the sequence $G$ as

$$
\frac{p_G(x) - p_{G_0}(x)}{\hat{W}_G^r(G, G_0)} \propto \frac{A_1(x) + B_1(x)}{\hat{W}_G^r(G, G_0)}
$$

where $A_1(x) = \sum_{i=1}^{k_0} (p_{i}^0 - p_{0}^0) f(x|\theta_i^0, v_i^0)$ and $B_1(x) = \sum_{i=1}^{k_0} \sum_{j=1}^{s_i} \sum_{n_{1, n_2}} (\Delta \theta_{ij}^{n_1} (\Delta v_{ij}^{n_2}) \frac{\partial f^{(1)}(\theta_i^0)}{\partial \theta^{n_1}}(x|\theta_i^0), v_i^0)$. The natural indices $n_1, n_2$ in the sum satisfy $n_1 + 2n_2 = \alpha$ and $n_1 + n_2 \leq r$. To obtain the conclusion of the proposition, we divide the proof argument into the following key steps.
**Step 1** We will show $G_0$ is not $(\tau, \tau/2)$-singular relative to $O_{k,c0}$. In fact, by choosing $\kappa = (\tau, \tau/2)$ and assume that all the coefficients of $A_1(x)/\tilde{W}_k^r(G, G_0)$ and $B_1(x)/\tilde{W}_k^r(G, G_0)$ go to 0, with the same argument as Step 8 of the proof of Proposition 2.2 in [28] we eventually reach to the following system of limits

$$
E_{\alpha} = \frac{\sum_{j=1}^{s_1} p_{1j}^{n} \sum_{n_1+n_2=r} (\Delta \theta_{1j})^{n_1} (\Delta v_{1j})^{n_2}}{\sum_{j=1}^{s_1} p_{1j}^{n} (|\Delta \theta_{1j}|^r + |\Delta v_{1j}|^{r/2})} \rightarrow 0
$$

Denote $\overline{p} = \max_{1 \leq j \leq s_1} \{p_{1j}\}$ and $\overline{M} = \max \left\{ |\Delta \theta_{11}|, \ldots, |\Delta \theta_{1s_1}|, |\Delta v_{11}|^{1/2}, \ldots, |\Delta v_{1s_1}|^{1/2} \right\}$. Let $\Delta \theta_{1j}/\overline{M} \rightarrow a_j$, $\Delta v_{1j}/\overline{M} \rightarrow b_j$, $p_{1j}/\overline{p} \rightarrow c_j^2 > 0$ for all $j = 1, \ldots, s_1$. By dividing both the numerator and denominator of $E_{\alpha}$ by $\overline{M}^s$, we quickly achieve the system of polynomial equations (11). From the definition of $\tau$, this system does not admit any non-trivial solution, which is a contradiction. As a consequence, $G_0$ is not $(\tau, \tau/2)$-singular relative to $O_{k,c0}$.

**Step 2** We will show that $G_0$ is $(l, \tau/2 - 1)$-singular and $(\tau - 1, l)$-singular relative to $O_{k,c0}$ for any $l \geq \tau$. Indeed, the sequence of $G$ provided in the proof of Proposition 2.2 in [28] is sufficient to verify these results. In particular, let $G$ be constructed as

$$
\theta_{1j} = \theta_{1j}^0 + \frac{a_j^*}{n}, \quad v_{1j} = v_{1j}^0 + \frac{2b_j^*}{n^2}, \quad p_{1j} = \frac{p_{1j}^0}{\sum_{j=1}^{s_1} (c_j^*)^2}, \quad \text{for all } j = 1, \ldots, k - k_0 + 1,
$$

and $\theta_{1} = \theta_{1j}^0$, $v_{1} = v_{1j}^0$, $p_{1} = p_{1j}^0$ for all $i = 2, \ldots, k_0$ where $(c_j^*, a_j^*, b_j^*)_{j=1}^{k-k_0+1}$ is a non-trivial solution of the system of equations (11) with $r = \tau - 1$. With these choices of $G$, we can easily verify that $\tilde{W}_k^r(G, G_0) \asymp (1/n)^{\tau-2}$ when $\kappa = (l, \tau/2 - 1)$ or $\tilde{W}_k^r(G, G_0) \asymp (1/n)^{\tau-1}$ when $\kappa = (\tau - 1, l)$ for any $l \geq \tau$. As $B_1(x) = O(n^{-\tau})$ (see Step 5 in the proof of Proposition 2.2 in [28]) and $A_1(x) = 0$, it implies that all the coefficients of $A_1(x)/\tilde{W}_k^r(G, G_0)$ and $B_1(x)/\tilde{W}_k^r(G, G_0)$ go to 0 when $\kappa = (l, \tau/2 - 1)$ or $\kappa = (\tau - 1, l)$. Hence, $G_0$ is $(l, \tau/2 - 1)$-singular and $(\tau - 1, l)$-singular relative to $O_{k,c0}$ for any $l \geq \tau$.

In summary, the results of Step 1 and Step 2 demonstrate that $(\tau, \tau/2)$ is the unique singularity index of $G_0$ relative to $O_{k,c0}$, which concludes the proof.

### 9.2 Proofs for Section 4

**PROOF OF LEMMA 4.1** For any $k_0 \geq 1$ and $k_0$ different pairs $\eta_1 = (\theta_1, \sigma_1, m_1), \ldots, \eta_{k_0} = (\theta_{k_0}, \sigma_{k_0}, m_{k_0})$, let $\alpha_{ij} \in \mathbb{R}$ for $i = 1, \ldots, 4$, $j = 1, \ldots, k_0$ such that for almost all $x \in \mathbb{R}$

$$
\sum_{j=1}^{k_0} \alpha_{ij} f(x|\eta_j) + \alpha_{2j} \frac{\partial f}{\partial \theta}(x|\eta_j) + \alpha_{3j} \frac{\partial f}{\partial \sigma}(x|\eta_j) \alpha_{4j} \frac{\partial f}{\partial m}(x|\eta_j) = 0.
$$

We can rewrite the above equation as

$$
\sum_{j=1}^{k_0} \left\{ [\beta_{1j} + \beta_{2j}(x - \theta_j) + \beta_{3j}(x - \theta_j)^2] \Phi \left( \frac{m_j(x - \theta_j)}{\sigma_j} \right) \exp \left( -\frac{(x - \theta_j)^2}{2\sigma_j^2} \right) + (\gamma_{1j} + \gamma_{2j}(x - \theta_j)) f \left( \frac{m_j(x - \theta_j)}{\sigma_j} \right) \exp \left( -\frac{(x - \theta_j)^2}{2\sigma_j^2} \right) \right\} = 0,
$$

(38)
If we choose $\beta_j = \frac{2 \alpha_{1j}}{\sqrt{2\pi} \sigma_j} - \frac{\alpha_{3j}}{\sqrt{2\pi} \sigma_j^3}$, $\beta_{2j} = \frac{2 \alpha_{2j}}{\sqrt{2\pi} \sigma_j^3}$, $\beta_{3j} = \frac{\alpha_{3j}}{\sqrt{2\pi} \sigma_j^5}$, $\gamma_{1j} = \frac{-2 \alpha_{2j} \sigma_{2j}^2}{\sqrt{2\pi} \sigma_j}$, and $\gamma_{2j} = -\frac{\alpha_{3j} \sigma_{3j}^2}{\sqrt{2\pi} \sigma_j^3} + \frac{2 \alpha_{4j}}{\sqrt{2\pi} \sigma_j^3}$ for all $j = 1, \ldots, k_0$.

"Only if" direction: Assume by contrary that the conclusion does not hold, i.e., both type A and type B conditions do not hold. Denote $\sigma_{j+k_0} = \frac{\sigma_j^2}{1 + m_j^2}$ for all $1 \leq j \leq k_0$. For the simplicity of the argument, we assume that $\sigma_i$ are pairwise different and $\frac{\sigma_i^2}{1 + m_i^2} \not\in \left\{ \frac{\sigma_j^2}{1 + m_j^2} : 1 \leq j \leq k_0 \right\}$ for all $1 \leq i \leq k_0$. The argument for the other cases is similar. Now, $\sigma_j$ are pairwise different as $1 \leq j \leq 2k_0$. The equation (38) can be rewritten as

$$
\sum_{j=1}^{2k_0} \left\{ \beta_{1j} + \beta_{2j}(x - \theta_j) + \beta_{3j}(x - \theta_j)^2 \right\} \Phi \left( \frac{m_j(x - \theta_j)}{\sigma_j} \right) \exp \left( -\frac{(x - \theta_j)^2}{2\sigma_j^2} \right) = 0, \quad (39)
$$

where $m_j = 0$, $\theta_{j+k_0} = \theta_j$, $\beta_{1(j+k_0)} = \frac{2 \gamma_{1j}}{\sqrt{2\pi}}$, $\beta_{2(j+k_0)} = \frac{2 \gamma_{2j}}{\sqrt{2\pi}}$, $\beta_{3j} = 0$ as $k_0 + 1 \leq j \leq 2k_0$. Denote $\bar{t} = \arg \max_{1 \leq i \leq 2k_0} \{ \sigma_i \}$. Multiply both sides of (39) with $\exp \left( \frac{(x - \theta_{\bar{t}})^2}{2\sigma_{\bar{t}}^2} \right) / \Phi \left( \frac{m_{\bar{t}}(x - \theta_{\bar{t}})}{\sigma_{\bar{t}}} \right)$ and let $x \to +\infty$ if $m_{\bar{t}} \geq 0$ or let $x \to -\infty$ if $m_{\bar{t}} < 0$ on both sides of the new equation, we obtain $\beta_{1\bar{t}} + \beta_{2\bar{t}}(x - \theta_{\bar{t}}) + \beta_{3\bar{t}}(x - \theta_{\bar{t}})^2 \to 0$. It implies that $\beta_{1\bar{t}} = \beta_{2\bar{t}} = \beta_{3\bar{t}} = 0$. Repeatedly apply the same argument to the remaining $\sigma_i$ until we obtain $\beta_{1i} = \beta_{2i} = \beta_{3i} = 0$ for all $1 \leq i \leq 2k_0$. It is equivalent to $\alpha_{1i} = \alpha_{2i} = \alpha_{3i} = \alpha_{4i} = 0$ for all $1 \leq i \leq k_0$, which is a contradiction.

"If" direction: There are two possible scenarios.

Type A singularity There exists some $m_j = 0$ as $1 \leq j \leq k_0$. In this case, we assume that $m_1 = 0$. If we choose $\alpha_{1j} = \alpha_{2j} = \alpha_{3j} = \alpha_{4j} = 0$ for all $2 \leq j \leq k_0$, then equation (38) can be rewritten as

$$
\beta_{11} + \frac{\gamma_{11}}{\sqrt{2\pi}} + \frac{\beta_{21}}{\sqrt{2\pi}}(x - \theta_1) + \frac{\beta_{31}}{2}(x - \theta_1)^2 = 0.
$$

By choosing $\alpha_{31} = 0$, $\alpha_{11} = \frac{\alpha_{21} \sigma_{21}}{\sqrt{2\pi} \sigma_1}$, $\alpha_{21} = -\frac{\alpha_{41} \sigma_1}{\sqrt{2\pi}}$, the above equation always equal to 0. Since $\alpha_{11}, \alpha_{21}, \alpha_{41}$ are not necessarily zero, the first-order identifiability (i.e., linear independence condition) is violated.

Type B singularity There exists indices $1 \leq i \neq j \leq k_0$ such that $\left( \frac{\sigma_i^2}{1 + m_i^2}, \theta_i \right) = \left( \frac{\sigma_j^2}{1 + m_j^2}, \theta_j \right)$. Without loss of generality, we assume that $i = 1, j = 2$. If we choose $\alpha_{1j} = \alpha_{2j} = \alpha_{3j} = \alpha_{4j} = 0$ for all $3 \leq j \leq k_0$, then equation in (38) can be rewritten as

$$
\sum_{j=1}^{2} \left\{ \beta_{1j} + \beta_{2j}(x - \theta_j) + \beta_{3j}(x - \theta_j)^2 \right\} \Phi \left( \frac{m_j(x - \theta_j)}{\sigma_j} \right) \exp \left( -\frac{(x - \theta_j)^2}{2\sigma_j^2} \right) + \frac{1}{\sqrt{2\pi}} \left( \sum_{j=1}^{2} \gamma_{1j} + \sum_{j=1}^{2} \gamma_{2j}(x - \theta_1) \right) \exp \left( -\frac{(m_j^2 + 1)(x - \theta_1)^2}{2\sigma_1^2} \right) = 0.
$$
Now, we choose $\alpha_{1j} = \alpha_{2j} = \alpha_{3j} = 0$ for all $1 \leq j \leq 2$, $\frac{\alpha_{11}}{\sigma_1^2} + \frac{\alpha_{42}}{\sigma_2^2} = 0$ then the above equation always hold. Since $\alpha_{41}$ and $\alpha_{42}$ need not be zero, the first-order identifiability condition is violated.

This concludes the proof.

**PROOF OF LEMMA 3.2** The proof proceeds via induction on $\alpha_1 + 2\alpha_2 + 2\alpha_3$. To ease the presentation, we denote throughout this proof the following notation

$$U_\alpha = \{ \kappa \in \mathcal{F}_{[\alpha]} : \kappa_1 + 2\kappa_2 + 2\kappa_3 \leq \alpha_1 + 2\alpha_2 + 2\alpha_3 \}.$$  

As $\alpha_1 + 2\alpha_2 + 2\alpha_3 \leq 2$, we can easily check the conclusion of the lemma. Assume that the conclusion holds for any $\alpha_1 + 2\alpha_2 + 2\alpha_3 \leq k - 1$. We shall demonstrate that it also holds for $\alpha_1 + 2\alpha_2 + 2\alpha_3 = k$.

Indeed, there are two settings:

**Case 1: $\alpha_1 = k$** Under this setting, $\alpha_2 = \alpha_3 = 0$. From the induction hypothesis,

$$\frac{\partial^{[\kappa]} f}{\partial \theta^{\alpha_1} \partial v^{\alpha_2} \partial m^{\alpha_3}} = \frac{\partial}{\partial \theta} \left( \frac{\partial^{[\kappa]-1} f}{\partial \theta^{\alpha_1-1} \partial v^{\alpha_2} \partial m^{\alpha_3}} \right)$$  

$$= \frac{\partial}{\partial \theta} \left( \sum_{\kappa \in U_{(\alpha_1-1,\alpha_2,\alpha_3)}} \frac{P^{\kappa_1,\kappa_2,\kappa_3}_{0,\alpha_1-1,\alpha_2,\alpha_3}(m)Q^{\kappa_1,\kappa_2,\kappa_3}_{0,\alpha_1-1,\alpha_2,\alpha_3}(v)}{\partial \theta^{\kappa_1} \partial v^{\kappa_2} \partial m^{\kappa_3}} \right)$$  

$$= \sum_{\kappa \in U_{(\alpha_1-1,\alpha_2,\alpha_3)}} \frac{P^{\kappa_1,\kappa_2,\kappa_3}_{0,\alpha_1-1,\alpha_2,\alpha_3}(m)Q^{\kappa_1,\kappa_2,\kappa_3}_{0,\alpha_1-1,\alpha_2,\alpha_3}(v)}{\partial \theta^{\kappa_1} \partial v^{\kappa_2} \partial m^{\kappa_3}} \frac{\partial^{[\kappa]+1} f}{\partial \theta^{\kappa_1+1} \partial v^{\kappa_2} \partial m^{\kappa_3}}$$  

$$+ \sum_{\kappa \in U_{(\alpha_1-1,\alpha_2,\alpha_3)}} \frac{P^{\kappa_1,\kappa_2,\kappa_3}_{0,\alpha_1-1,\alpha_2,\alpha_3}(m)Q^{\kappa_1,\kappa_2,\kappa_3}_{0,\alpha_1-1,\alpha_2,\alpha_3}(v)}{\partial \theta^{\kappa_1} \partial v^{\kappa_2} \partial m^{\kappa_3}} \frac{\partial^{[\kappa]+1} f}{\partial \theta^{\kappa_1+1} \partial v^{\kappa_2} \partial m^{\kappa_3}}  \tag{40}$$

where the second equality is due to the application of the hypothesis for $\alpha_1 - 1 + 2\alpha_2 + 2\alpha_3 = k - 1$. For any $\kappa \in U_{(\alpha_1-1,\alpha_2,\alpha_3)}$ such that $\kappa_1 = 1$,

$$\frac{\partial^{[\kappa]+1} f}{\partial \theta^{\kappa_1+1} \partial v^{\kappa_2} \partial m^{\kappa_3}} = \frac{\partial^{[\kappa]-1} f}{\partial \theta^{\kappa_2} \partial m^{\kappa_3}} \left( \frac{2 \partial f}{\partial v} - \frac{m^3 + m}{v} \frac{\partial f}{\partial m} \right)$$  

$$= 2 \frac{\partial^{[\kappa]} f}{\partial \theta v^{\kappa_2+1} m^{\kappa_3}} - \frac{\partial^{[\kappa]-1} f}{\partial \theta^{\kappa_2} \partial m^{\kappa_3}} \left( \frac{m^3 + m}{v} \frac{\partial f}{\partial m} \right).  \tag{41}$$

From the inductive hypothesis, since $\kappa_1 + 2\kappa_2 + 2\kappa_3 = 2\kappa_2 + 2\kappa_3 + 1 \leq k - 1$,

$$\frac{\partial^{[\kappa]} f}{\partial \theta v^{\kappa_2+1} m^{\kappa_3}} = \sum_{\kappa' \in U_{(0,\kappa_2+1,\kappa_3)}} \frac{P^{\kappa_1,\kappa_2,\kappa_3}_{0,\kappa_2+1,\kappa_3}(m)Q^{\kappa_1,\kappa_2,\kappa_3}_{0,\kappa_2+1,\kappa_3}(v)}{\partial \theta^{\kappa_1} \partial v^{\kappa_2} \partial m^{\kappa_3}} \frac{\partial^{[\kappa']} f}{\partial \theta^{\kappa_2} \partial v^{\kappa_2} \partial m^{\kappa_3}}.  \tag{42}$$

In addition,

$$\frac{\partial^{[\kappa]-1} f}{\partial \theta^{\kappa_2} \partial m^{\kappa_3}} \left( \frac{m^3 + m}{v} \frac{\partial f}{\partial m} \right) = \sum_{\beta \leq \kappa} A_{\beta_1,\beta_2}(m) \frac{\partial^{[\beta]} f}{\partial \theta^{\beta_1} \partial v^{\beta_2} \partial m^{\beta_2}}.  \tag{43}$$

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Since $2\beta_1 + 2\beta_2 \leq 2\kappa_2 + 2\kappa_3 \leq k - 1$, from the hypothesis,

$$\frac{\partial^{\mid\beta\mid} f}{\partial v^{\beta_1} \partial m^{\beta_2}} = \sum_{\kappa' \in U(\beta_1, \beta_2)} \frac{P_{0,\beta_1,\beta_2}(m)}{H_{0,\beta_1,\beta_2}(m)Q_{0,\beta_1,\beta_2}(v)} \frac{\partial^{\mid\kappa'\mid} f}{\partial \theta^{\kappa_1} \partial v^{\kappa_2} \partial m^{\kappa_3}}.$$  (44)

Combining equations (40), (41), (42), (43), and (44), we arrive at the conclusion of the lemma.

**Case 2: $\alpha_1 \leq k - 1$** Under this setting, assume without loss of generality that $\alpha_2 \geq 1$.

$$\frac{\partial^{\mid\alpha\mid} f}{\partial \theta^{\alpha_1} \partial v^{\alpha_2} \partial m^{\alpha_3}} = \frac{\partial}{\partial v} \left( \frac{\partial^{\mid\alpha\mid - 1} f}{\partial \theta^{\alpha_1} \partial v^{\alpha_2} \partial m^{\alpha_3}} \right) = \frac{\partial}{\partial v} \left( \sum_{\kappa \in U(\alpha_1, \alpha_2, \alpha_3)} \frac{P_{\kappa_1,\kappa_2,\kappa_3}(m)}{H_{\kappa_1,\kappa_2,\kappa_3}(m)Q_{\kappa_1,\kappa_2,\kappa_3}(v)} \frac{\partial^{\mid\kappa\mid} f}{\partial \theta^{\kappa_1} \partial v^{\kappa_2} \partial m^{\kappa_3}} \right) = \sum_{\kappa \in U(\alpha_1, \alpha_2, \alpha_3)} \frac{\partial}{\partial v} \left( \frac{P_{\kappa_1,\kappa_2,\kappa_3}(m)}{H_{\kappa_1,\kappa_2,\kappa_3}(m)Q_{\kappa_1,\kappa_2,\kappa_3}(v)} \frac{\partial^{\mid\kappa\mid} f}{\partial \theta^{\kappa_1} \partial v^{\kappa_2} \partial m^{\kappa_3}} \right) + \sum_{\kappa \in U(\alpha_1, \alpha_2, \alpha_3)} \frac{\partial}{\partial v} \left( \frac{P_{\kappa_1,\kappa_2,\kappa_3}(m)}{H_{\kappa_1,\kappa_2,\kappa_3}(m)Q_{\kappa_1,\kappa_2,\kappa_3}(v)} \frac{\partial^{\mid\kappa\mid + 1} f}{\partial \theta^{\kappa_1} \partial v^{\kappa_2} \partial m^{\kappa_3}} \right).$$  (45)

Denote $A := \sum_{\kappa \in U(\alpha_1, \alpha_2, \alpha_3)} \frac{P_{\kappa_1,\kappa_2,\kappa_3}(m)}{H_{\kappa_1,\kappa_2,\kappa_3}(m)Q_{\kappa_1,\kappa_2,\kappa_3}(v)} \frac{\partial^{\mid\kappa\mid + 1} f}{\partial \theta^{\kappa_1} \partial v^{\kappa_2} \partial m^{\kappa_3}}$, we further have that

$$A = \sum_{\kappa \in U(\alpha_1, \alpha_2, \alpha_3); \kappa_3 = 0} \frac{P_{\kappa_1,\kappa_2,\kappa_3}(m)}{H_{\kappa_1,\kappa_2,\kappa_3}(m)Q_{\kappa_1,\kappa_2,\kappa_3}(v)} \frac{\partial^{\mid\kappa\mid + 1} f}{\partial \theta^{\kappa_1} \partial v^{\kappa_2} \partial m^{\kappa_3}} + \sum_{\kappa \in U(\alpha_1, \alpha_2, \alpha_3); \kappa_2 = 0, \kappa_3 \geq 1} \frac{P_{\kappa_1,\kappa_2,\kappa_3}(m)}{H_{\kappa_1,\kappa_2,\kappa_3}(m)Q_{\kappa_1,\kappa_2,\kappa_3}(v)} \frac{\partial^{\mid\kappa\mid + 1} f}{\partial \theta^{\kappa_1} \partial v^{\kappa_2} \partial m^{\kappa_3}}.$$  (46)

Since $m \neq 0$, for any $\kappa \in U(\alpha_1, \alpha_2, \alpha_3)$ such that $\kappa_2 = 0$ and $\kappa_3 \geq 1$, we have

$$\frac{\partial^{\mid\kappa\mid + 1} f}{\partial \theta^{\kappa_1} \partial v^{\kappa_2 + 1} \partial m^{\kappa_3}} = \frac{\partial^{\mid\kappa\mid - 1} f}{\partial \theta^{\kappa_1} \partial v^{\kappa_2} \partial m^{\kappa_3 - 1}} \left( -\frac{1}{v} \frac{\partial f}{\partial m} - \frac{m^2 + 1}{2mv} \frac{\partial^2 f}{\partial m^2} \right) = -\frac{1}{v} \frac{\partial^{\mid\kappa\mid - 1} f}{\partial \theta^{\kappa_1} \partial v^{\kappa_2} \partial m^{\kappa_3 - 1}} - \frac{\partial^{\mid\kappa\mid - 1} f}{\partial \theta^{\kappa_1} \partial v^{\kappa_2} \partial m^{\kappa_3 - 1}} \left( \frac{m^2 + 1}{2mv} \frac{\partial^2 f}{\partial m^2} \right).$$  (47)

Since $|\kappa| = \kappa_1 + \kappa_3 \leq k - 1$ and $|\kappa_1| \leq 1$, we have $(\kappa_1, 0, \kappa_3) \in F_k$. Additionally, we can represent

$$\frac{\partial^{\mid\kappa\mid - 1} f}{\partial \theta^{\kappa_1} \partial v^{\kappa_2} \partial m^{\kappa_3 - 1}} \left( \frac{m^2 + 1}{2mv} \frac{\partial^2 f}{\partial m^2} \right) = \sum_{1 \leq \tau \leq \kappa_3 + 1} A_\tau(m) \frac{\partial^{\mid\kappa\mid + \tau} f}{\partial \theta^{\kappa_1} \partial v^{\kappa_2} \partial m^{\kappa_3}} B_\tau(m) C_\tau(v) \theta^{\kappa_1} \partial v^{\kappa_2} \partial m^{\kappa_3},$$

where $A_\tau(m), B_\tau(m), C_\tau(v)$ are some polynomials of $m$ and $v$. Since $\kappa_1 + \tau \leq \kappa_1 + \kappa_3 + 1 \leq k$ and $\kappa_1 \leq 1$, we have $(\kappa_1, 0, \tau) \in F_k$. Combining these results with equations (45), (46), and (47), we achieve the conclusion of the lemma.
PROOF OF LEMMA 4.3  The proof of this lemma proceeds by induction on $r$. If $r = 1$,\[
\left\{ \frac{\partial^{[\alpha]} f}{\partial^{\alpha_1} v^{\alpha_2} m^{\alpha_3}} : (\alpha_1, \alpha_2, \alpha_3) \in \mathcal{F}_r \right\} = \left\{ \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial m} \right\},
\]
which are linearly independent with respect to $G_0 \in S_0$ due to the conclusion of Lemma 4.1. Assume that the conclusion of the lemma holds up to $r$. We will demonstrate that it continues to hold for $r + 1$. In fact,\[
\left\{ \frac{\partial^{[\alpha]} f}{\partial^{\alpha_1} v^{\alpha_2} m^{\alpha_3}} : (\alpha_1, \alpha_2, \alpha_3) \in \mathcal{F}_{r+1} \right\} = \left\{ \frac{\partial^{[\alpha]} f}{\partial^{\alpha_1} v^{\alpha_2} m^{\alpha_3}} : (\alpha_1, \alpha_2, \alpha_3) \in \mathcal{F}_r \right\} \cup \left\{ \frac{\partial^{r+1} f}{\partial \theta \partial v}, \frac{\partial^{r+1} f}{\partial \theta \partial m^r}, \frac{\partial^{r+1} f}{\partial m^{r+1}} \right\}.
\]
(48)
Assume that there are coefficients $\beta^{(i)}_{\alpha_1,\alpha_2,\alpha_3}$ where $1 \leq i \leq k_0$ and $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{F}_{r+1}$ such that for all $x$\[
\sum_{i=1}^{k_0} \sum_{(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{F}_{r+1}} \beta^{(i)}_{\alpha_1, \alpha_2, \alpha_3} \frac{\partial^{[\alpha]} f}{\partial^{\alpha_1} v^{\alpha_2} m^{\alpha_3}}(x|\eta_i^0) = 0.
\]
Using the fact from (48), we rewrite the above equation as\[
\sum_{i=1}^{k_0} \sum_{(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{F}_r} \beta^{(i)}_{\alpha_1, \alpha_2, \alpha_3} \frac{\partial^{[\alpha]} f}{\partial^{\alpha_1} v^{\alpha_2} m^{\alpha_3}}(x|\eta_i^0) + \beta^{(i)}_{1,r,0} \frac{\partial^{r+1} f}{\partial \theta \partial v^r}(x|\eta_i^0) + \beta^{(i)}_{0,r+1,0} \frac{\partial^{r+1} f}{\partial \theta \partial m^r}(x|\eta_i^0) + \beta^{(i)}_{0,0,r+1} \frac{\partial^{r+1} f}{\partial m^{r+1}}(x|\eta_i^0) = 0.
\]
(49)
Equation (49) can be rewritten as\[
\sum_{i=1}^{k_0} \sum_{j=1}^{2r+3} \gamma^{(r+1)}_{j,i} (x - \theta_i^0)^{j-1} f \left( \frac{x - \theta_i^0}{\sigma_i^0} \right) \Phi \left( \frac{m_i^0 (x - \theta_i^0)}{\sigma_i^0} \right) + \sum_{i=1}^{k_0} \sum_{j=1}^{2r+2} \tau^{(r+1)}_{j,i} (x - \theta_i^0)^{j-1} \exp \left( -\frac{(m_i^0)^2 + 1}{2v_i^0} (x - \theta_i^0)^2 \right) = 0,
\]
where $\gamma^{(r+1)}_{j,i}$ are a combination of $\beta^{(i)}_{\alpha_1,\alpha_2,\alpha_3}$ when $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{F}_{r+1}$ and $\alpha_3 = 0$. Additionally, $\tau^{(r+1)}_{j,i}$ are a combination of $\beta^{(i)}_{\alpha_1,\alpha_2,\alpha_3}$ when $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{F}_{r+1}$. Due to the fact that there are no type A or type B singularities in $\left\{ \eta^0_1, \ldots, \eta^0_{k_0} \right\}$, by using the same argument as that of the proof of Lemma 4.1 we obtain that $\gamma^{(r+1)}_{j,i} = 0$ for all $1 \leq i \leq k_0$, $1 \leq j \leq 2r + 3$ and $\gamma^{(r+1)}_{j,i} = 0$ for all $1 \leq i \leq k_0$, $1 \leq j \leq 2r + 2$. It can be checked that $\gamma^{(r+1)}_{2r+3,i} = 0$ implies $\beta^{(i)}_{0,r+1,0} = 0$ while $\gamma^{(r+1)}_{2r+2,i} = 0$ implies $\beta^{(i)}_{1,r,0} = 0$ for all $1 \leq i \leq k_0$. Similarly, $\gamma^{(r+1)}_{2r+2,i} = 0$ implies $\beta^{(i)}_{0,0,r+1} = 0$ while $\gamma^{(r+1)}_{2r+1,i} = 0$ implies $\beta^{(i)}_{1,0,r} = 0$ for all $1 \leq i \leq k_0$. As a consequence, Eq. (49) is reduced to\[
\sum_{i=1}^{k_0} \sum_{(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{F}_r} \beta^{(i)}_{\alpha_1, \alpha_2, \alpha_3} \frac{\partial^{[\alpha]} f}{\partial^{\alpha_1} v^{\alpha_2} m^{\alpha_3}}(x|\eta_i^0) = 0.
\]
(50)
According to the hypothesis with $r$, we obtain that $\beta^{(i)}_{\alpha_1, \alpha_2, \alpha_3} = 0$ for all $1 \leq i \leq k_0, (\alpha_1, \alpha_2, \alpha_3) \in \mathcal{F}_r$. This concludes our proof.
PROOF OF PROPOSITION 4.1 From the formation of system of polynomial equations (26), if we choose \(\beta_3 = 0\) (i.e., we only reduce to derivatives with respect to the location and scale parameter), then we have:

\[ P_{\alpha_1, \alpha_2, \alpha_3}^r(m) / H_{\alpha_1, \alpha_2, \alpha_3}(m) Q_{\alpha_1, \alpha_2, \alpha_3}(v) = 2^{2v} \text{ when } \alpha_3 = 0 \text{ and } P_{\alpha_1, \alpha_2, \alpha_3}^r(m) / H_{\alpha_1, \alpha_2, \alpha_3}(m) Q_{\alpha_1, \alpha_2, \alpha_3}(v) = 0 \text{ as } \alpha_3 \geq 1 \text{ for any } v, m \text{ and } \alpha_1 + 2\alpha_2 + 2\alpha_3 = \beta_1 + 2\beta_2 + 2\beta_3. \]

This shows that the system of polynomial equations (26) contains the following system of equations:

\[
\sum_{j=1}^{l} \sum_{\alpha_1 + 2\alpha_2 = \beta_1 + 2\beta_2} 2^{\alpha_2} a_j^{\alpha_1} b_j^{\alpha_2} / \alpha_1! \alpha_2! = 0, \tag{51}
\]

where \(\beta_1 + 2\beta_2 \leq r \) and \(\beta_1 \leq 1\). This is precisely the system of polynomial equations (11) if we replace \(d_j\) by \(x_j\), \(a_j\) by \(y_j\), \(2b_j\) by \(z_j\), \(\alpha_1, \alpha_2\) by \(n_1, n_2\). Now, if we choose \(r > 7(l)\), the system of polynomial equations (51) has only trivial solution \(a_j = b_j = 0\) for all \(1 \leq j \leq l\). Substitute these results back to system of polynomial equations (26), we also obtain \(c_j = 0\) for all \(1 \leq j \leq l\), which is a contradiction. This completes our proof.

PROOF OF PROPOSITION 4.3 The proof of part (a) is straightforward from the discussion in Section 4.1. For the proof for part (b), we will present an explicit form for the system of polynomial equations to illustrate the variability of \(\underline{\rho}(l)\) and \(\overline{\rho}(l)\) based on the values of \((m, v)\).

(b) As \(l = 2\) and \(r = 6\), the system of polynomial equations (26) can be rewritten as:

\[
\begin{align*}
\sum_{i=1}^{3} d_i^2 a_i &= 0, \quad \sum_{i=1}^{3} d_i^2 a_i^2 + d_i b_i &= 0, \quad \sum_{i=1}^{3} -(m^3 + m)d_i^2 a_i^2 + 2vd_i^2 c_i = 0, \\
\sum_{i=1}^{3} d_i^2 a_i^3 + d_i a_i b_i &= 0, \quad \sum_{i=1}^{3} -(m^3 + m)d_i^2 a_i^3 + 6vd_i^2 c_i = 0, \\
\sum_{i=1}^{3} (m^3 + m)^2/12v^2 d_i^2 a_i^4 - (m^3 + m)/v d_i^2 a_i^2 c_i - m^2/v m d_i^2 b_i c_i + d_i^2 c_i^2 &= 0, \\
\sum_{i=1}^{3} 1/6 d_i^2 a_i^4 + d_i^2 a_i^2 b_i + 1/2 d_i^2 b_i^2 &= 0, \quad \sum_{i=1}^{3} 1/30 d_i^2 a_i^5 + 1/3 d_i^2 a_i^3 b_i + 1/2 d_i^2 a_i^2 b_i^2 &= 0, \\
\sum_{i=1}^{3} (m^3 + m)^2/120v^2 d_i^2 a_i^5 - (m^3 + m)/6v d_i^2 a_i^3 c_i - m^2/v m d_i^2 a_i c_i b_i + 1/2 d_i^2 a_i c_i^2 &= 0, \\
\sum_{i=1}^{3} 1/12 d_i^2 a_i^6 + 1/12 d_i^2 a_i^4 b_i + 1/2 d_i^2 a_i^2 b_i^2 + 1/6 d_i^2 b_i^3 &= 0, \\
\sum_{i=1}^{3} (m^3 + m)^3/720v^3 d_i^2 a_i^6 + (m^3 + m)^2/24v^2 d_i^2 a_i^4 c_i + m^2 + m/4v d_i^2 a_i^2 c_i^2 + (m^2 + 1)/8v^2 m^2 d_i^2 b_i^2 c_i - m^2 + 1/4mv d_i^2 b_i c_i^2 + 1/6 d_i^2 c_i^3 &= 0. \tag{52}
\end{align*}
\]

When \(r = 4\), the system of polynomial equations (26) contains the first 7 equations in the system of polynomial equations (52). Now, \(m\) and \(v\) are considered as two additional variables in the above system of polynomial equations. Hence, there are 13 variables with only 7 equations. If we choose \(d_1 = d_2 = d_3\) and take the lexicographical ordering \(a_1 > a_2 > a_3 > b_1 > b_2 > b_3 > c_1 > c_2 > c_3 > m > v\), the Grobner bases (cf. [9]) of the above system of polynomial equations will return a non-trivial solution (due to the complexity of the roots, we will not present them here). As a consequence, \(\underline{\rho}(l) \geq 5\) under the case \(l = 2\).
For \( l = 2 \) and \( r = 5 \), the system of polynomial equations (26) retains the first 9 equations in system (52). It can be checked that if we choose \( m = \pm 2, v = 1 \), then the system of polynomial equations when \( r = 5 \) does not have any non-trivial solution (note that, we also use the same lexicographical order as that being used in the case \( r = 4 \)). So, \( \rho(l) = 5 \). However, we can check that the value of \( m = \frac{1}{10} \) (close to 0 in general) and \( v = 1 \) will lead the system of polynomial equations (52) to not having any non-trivial solution. Thus, \( \overline{\rho}(l) = 6 \). This concludes the proof or part (b) of the proposition.
Appendix E: Theory for skew-normal e-mixtures – a summary

E-mixtures are the setting in which the number of mixing components is known $k = k_0$. In this appendix, we provide a summary of singularity level and singularity index of mixing measure $G_0$ relative to the ambient space $\mathcal{E}_{k_0}$, where $k_0$ is the number of supporting atoms for $G_0$.

Recall from the previous sections the definition of $S_0$, the subset $S_0 \subset \mathcal{E}_{k_0}$ of measure $G_0 = G_0(p^0, \eta^0)$ such that $(p^0, \eta^0)$ satisfy $P_1(\eta^0)P_2(\eta^0) \neq 0$. $P_1$ and $P_2$ are polynomials given in the statement of Lemma 4.1. It is simple to verify that for any $G_0 \in S_0$, as a consequence of this lemma, the Fisher information matrix $I(G_0)$ is non-singular. It follows that

**Theorem 10.1.** If $G_0 \in S_0$, then $\ell(G_0|\mathcal{E}_{k_0}) = 0$ and $\mathcal{L}(G_0|\mathcal{E}_{k_0}) = \{(1, 1, 1)\}$.

We turn our attention to the singularity structure of set $\mathcal{E}_{k_0} \setminus S_0$. For any $G_0 \in \mathcal{E}_{k_0} \setminus S_0$, the parameters of $G_0$ satisfy $P_1(\eta^0)P_2(\eta^0) = 0$. Accordingly, for each pair of $(i, j) = 1, \ldots, k_0$ the two components indexed by $i$ and $j$ are said to be homologous if

$$(\theta_i^0 - \theta_j^0)^2 + [v_i^0(1 + (m_i^0)^2) - v_j^0(1 + (m_j^0)^2)]^2 = 0.$$ 

Moreover, for each $1 \leq i \leq k_0$, let $I_i$ denote the set of all components homologous to (component) $i$. By definition, it is clear that if $i$ and $j$ are homologous, $I_i \equiv I_j$. Therefore, these homologous sets form equivalence classes. From here on, when we say a homologous set $I$, we implicitly mean that it is the representation of the equivalent classes.

Now, the homologous set consists of the indices of skew-normal components that share the same location and a rescaled version of the scale parameter. A non-empty homologous set $I$ is said to be conformant if for any $i \neq j \in I$, $m_i^0m_j^0 > 0$. A non-empty homologous set $I$ is said to be nonconformant if we can find two indices $i, j \in I$ such that $m_i^0m_j^0 < 0$. Additionally, $G_0$ is said to be conformant if all the homologous sets are conformant or nonconformant (NC) if at least one homologous set is nonconformant. Now, we define a partition of $\mathcal{E}_{k_0} \setminus S_0$ as follows $\mathcal{E}_{k_0} = S_0 \cup S_1 \cup S_2 \cup S_3$, where

$$S_1 = \left\{ G = G(p, \eta) \in \mathcal{E}_{k_0} | P_1(\eta) \neq 0, P_2(\eta) = 0, G \text{ is conformant} \right\}$$

$$S_2 = \left\{ G = G(p, \eta) \in \mathcal{E}_{k_0} | P_1(\eta) = 0, P_2(\eta) = 0 \text{ then } G \text{ is conformant} \right\}$$

$$S_3 = \left\{ G = G(p, \eta) \in \mathcal{E}_{k_0} | P_2(\eta) = 0 \text{ and } G \text{ is nonconformant} \right\}.$$

Figure 2 summarizes singularity levels of elements residing in $\mathcal{E}_{k_0}$, except for $S_3$.

10.1 Singularity structure of $G_0 \in S_1 \cup S_2$

The main results of this subsection are the following two theorems.

**Theorem 10.2.** If $G_0 \in S_1$, then $\ell(G_0|\mathcal{E}_{k_0}) = 1$ and $\mathcal{L}(G_0|\mathcal{E}_{k_0}) = \{(1, 1, 2)\}$.

The results of Theorem 10.2 imply that the convergence rate of estimating mixing measure $G_0$ is $n^{-1/2}$ but individual parameters of $G_0$ admit different rates: it is $n^{-1/2}$ for location and scale parameters and $n^{-1/4}$ for skewness parameter. Therefore, it is generally more efficient to estimate location and scale parameter than estimating skewness parameter under the setting $G_0 \in S_1$.

**Theorem 10.3.** If $G_0 \in S_2$, then $\ell(G_0|\mathcal{E}_{k_0}) = 2$ and $\mathcal{L}(G_0|\mathcal{E}_{k_0}) = \{(3, 2, 3)\}$.

Unlike the results from Theorem 10.2 under the setting $G_0 \in S_2$, the convergence rate of estimating mixing measure $G_0$ is $n^{-1/4}$. However, the convergence rate of location and skewness parameter is $n^{-1/6}$ while that of scale parameter is $n^{-1/4}$. The illustration of proof idea of the above theorems will be first given shortly, and while a complete proof is presented in Appendix F.
Figure 2: The singularity level of $G_0$ relative to $E_{k_0}$ is determined by partition based on zeros of polynomials $P_1, P_2$ into subsets $S_0, S_1, S_2, S_3$. Here, "NC" stands for nonconformant.

Figure 3: The level of singularity structure of $G_0 \in S_3$ when $P_1(\eta^0) = 0$. Here, "NC" stands for nonconformant. The term $\pi(G_0)$ is defined in (70).
Figure 4: The level of singularity structure of \( G_0 \in S_3 \) when \( P_1(\eta^0) \neq 0 \). Here, "NC" stands for nonconformant. The term \( \pi(G_0) \) is defined in (70).

10.2 Singularity structure of \( G_0 \in S_3 \): a summary

The singularity level and singularity index of \( S_3 \) are much more complex than those of previous settings of \( G_0 \). \( S_3 \) does not admit an uniform level of singularity structure for all its elements — it needs to be partitioned into many subsets via intersections with additional semialgebraic sets of the parameter space. In addition, we can establish the existence of subsets that correspond to the infinite singularity level and singularity index. In most cases when the singularity level and singularity index are finite, we may be able to provide some bounds rather than giving an exact value. As in o-mixtures setting in Section 4.2, the unifying theme of such bounds is their connection to the solvability of a system of real polynomial equations.

If \( G_0 = G_0(p^0, \eta^0) \in S_3 \), then its corresponding parameters satisfy \( P_2(\eta^0) = 0 \), i.e., there is at least one homologous set of \( G_0 \). Moreover, at least one such homologous set is nonconformant. For any \( G_0 \in S_3 \), let \( I_1, \ldots, I_t \) be all nonconformant homologous sets of \( G_0 \). The singularity structures of \( S_3 \) arise from the zeros of the following polynomials:

- Type C(1): \( P_3(p^0, \eta^0) := \prod_{1 \leq i \neq j \leq k_0} \left( \sum_{s \in I_i \cap |S| \geq 2} p_j^0 \prod_{l \neq j} m_l^0 \right) \).

- Type C(2): \( P_4(p^0, \eta^0) := \prod_{1 \leq i \neq j \leq k_0} \left[ u_{ij}^2 + (m_i^0 \sigma_j^0 + m_j^0 \sigma_i^0)^2 + (p_i^0 \sigma_j^0 - p_j^0 \sigma_i^0)^2 \right] \), where \( u_{ij}^2 = (\theta_i^0 - \theta_j^0)^2 + \left( \nu_i^0(1 + (m_i^0)^2) - \nu_j^0(1 + (m_j^0)^2) \right)^2 \).

Type C singularities, including both C(1) and C(2), are distinguished from Type A and Type B singularities by the fact that the Type C polynomials are defined by not only component parameters \( \eta^0 \), but also mixing probability parameters \( p^0 \). Note that C(1) singularity implies that there is some homologous set \( I_i \) of \( G_0 \) such that \( \prod_{s \in I_i \cap |S| \geq 2} \left( \sum_{j \in S} p_j^0 \prod_{l \neq j} m_l^0 \right) = 0 \). A homologous set of \( G_0 \) having the above property is said to contain type C(1) singularity locally. Similarly, type C(2) singularity implies that there is some pair \( 1 \leq i \neq j \leq k_0 \) such that \( u_{ij}^2 + (m_i^0 \sigma_j^0 + m_j^0 \sigma_i^0)^2 + (p_i^0 \sigma_j^0 - p_j^0 \sigma_i^0)^2 = 0 \). A homologous set of \( G_0 \) having this pair is said to contain type C(2) singularity locally. It can be easily checked that a homologous set containing type C(2) singularity must also contain type C(1) singularity.
since \( P_4(p^0, \eta^0) = 0 \) entails \( P_3(p^0, \eta^0) = 0 \). Now, we define the following partition of \( S_3 \) according to the definition of type \( C(1) \) and \( C(2) \) singularity: 
\[
S_{31} = \{ G = G(p, \eta) \in S_3 \mid P_3(p, \eta) \neq 0 \}
\]
\[
S_{32} = \{ G = G(p, \eta) \in S_3 \mid P_3(p, \eta) = 0, P_4(p, \eta) \neq 0 \}
\]
\[
S_{33} = \{ G = G(p, \eta) \in S_3 \mid P_3(p, \eta) = 0, P_4(p, \eta) = 0 \}.
\]

Due to the highly technical nature of our analysis of the singularity structure of \( S_3 \), we defer the detailed analysis to Section 11.3 in Appendix F. Here, we only provide a summary of such results. We can demonstrate that (1, 1, \( \ell \)) \( \ell(G_0|\mathcal{E}_{k_0}) + 1 \) is the unique singularity index of \( G_0 \in S_3 \) relative to \( \mathcal{E}_{k_0} \) when \( P_1(\eta^0) \neq 0 \), i.e., there are no Gaussian components in \( G_0 \). Additionally, when \( P_1(\eta^0) = 0 \), i.e., there are some Gaussian components in \( G_0 \), then we have (3, 2, \( \max \{ 2, \ell(G_0|\mathcal{E}_{k_0}) \} + 1 \)) as the unique singularity index of \( G_0 \). Therefore, studying singularity index of \( G_0 \in S_3 \) is equivalent to studying singularity level of \( G_0 \in S_3 \).

Figure 3 and 4 provide illustrations of singularity levels of \( G_0 \in S_3 \). Specifically, when \( G_0 \in S_{31} \), it is shown that \( \ell(G_0|\mathcal{E}_{k_0}) \leq \max \{ 2, \bar{\pi}(G_0) \} \), where \( \bar{\pi}(G_0) \) is defined by a system of polynomial equations that we obtain via a method of greedy extraction of polynomial limits, see Section 11.3.1. In some specific cases, the precise singularity level of \( G_0 \in S_{31} \) will be given. If \( G_0 \in S_{32} \), we need a more sophisticated method of extraction for polynomial limits; our technique is illustrated on a specific example of \( G_0 \) in Section 11.3.2. Finally, if \( G_0 \in S_{33} \), it is shown that \( \ell(G_0|\mathcal{E}_{k_0}) = \infty \) in Section 11.3.3.

### 11 Appendix F: Theory of skew-normal e-mixtures – with proofs

This Appendix contains a self-contained and detailed treatment of singularity structure of e-mixtures of skew-normal distributions for which a brief summary of the results was given earlier in Appendix E. This Appendix should be skipped at the first reading.

#### 11.1 Singularity structure of \( S_1 \)

In the following, we shall present the proof of Theorem 10.2 for a simple setting of \( G_0 \in S_1 \), which illustrates the complete proofs, and also helps to explain why the partition of according to \( S_1 \), i.e., the notion of conformant, arises in the first place. The simplified setting is that all components of \( G_0 \) are homologous to one another. By definition all components of \( G_0 \) are non-Gaussian (because \( P_1(\eta^0) \neq 0 \)). Thus, we have \( \theta_1^0 = \ldots = \theta_{k_0}^0 \) and \( v_1^0 \frac{1}{1 + (m_1^0)^2} = \ldots = v_{k_0}^0 \frac{1}{1 + (m_{k_0}^0)^2} \). Additionally, \( m_i^0 \neq 0 \) for all \( 1 \leq i \leq k_0 \). Since \( G_0 \) is conformant, \( m_i^0 \) share the same sign for all \( 1 \leq i \leq k_0 \). Without loss of generality, we assume \( m_i^0 > 0 \). To demonstrate that (1, 1, 2) is the unique singularity index of \( G_0 \), we need show that \( G_0 \) is 1-singular and (1,1,1)-singular, but not (1,1,2)-singular.

**Claim:** \( G_0 \) is 1-singular and (1,1,1)-singular

Given constraints on the parameters of \( G_0 \), it is simple to arrive at the following 1-minimal and (1,1,1)-minimal form (cf. Eq. (53)):

\[
\frac{1}{W_1(G, G_0)} \left\{ \sum_{i=1}^{k_0} \left[ \beta_1^{(1)} + \beta_2^{(1)}(x - \theta_1^0) + \beta_3^{(1)}(x - \theta_1^0)^2 \right] f \left( \frac{x - \theta_1^0}{\sigma_1^0} \right) \Phi \left( \frac{m_1^0(x - \theta_1^0)}{\sigma_1^0} \right) \right. \\
+ \left. \left[ \gamma_1^{(1)} + \gamma_2^{(1)}(x - \theta_1^0) \right] \exp \left( -\frac{m_1^0}{2\sigma_1^0} (x - \theta_1^0)^2 \right) \right\} + o(1),
\]
where coefficients $\beta_{1i}^{(1)}, \beta_{2i}^{(1)}, \beta_{3i}^{(1)}, \gamma_{1}^{(1)}, \gamma_{2}^{(1)}$ are the polynomials of $\Delta \theta_j, \Delta v_j, \Delta m_j,$ and $\Delta p_l$:

\[
\beta_{1i}^{(1)} = \frac{2\Delta p_i}{\sigma_i^0} - \frac{p_i \Delta v_i}{(\sigma_i^0)^3}, \quad \beta_{2i}^{(1)} = \frac{2p_i \Delta \theta_i}{(\sigma_i^0)^3}, \quad \beta_{3i}^{(1)} = \frac{p_i \Delta v_i}{(\sigma_i^0)^5},
\]

\[
\gamma_{1}^{(1)} = \sum_{j=1}^{k_0} -\frac{p_j m_j^0 \Delta \theta_j}{\pi(\sigma_j^0)^2}, \quad \gamma_{2}^{(1)} = \sum_{j=1}^{k_0} -\frac{p_j m_j^0 \Delta v_j}{2\pi(\sigma_j^0)^4} + \frac{p_j \Delta m_j}{\pi(\sigma_j^0)^2}.
\]

Note that, the conditions $m_i^0 \neq 0$ for all $1 \leq i \leq k_0$ allow us to have that $f\left(\frac{x - \theta_i^0}{\sigma_i^0}\right) \Phi\left(\frac{m_i^0(x - \theta_i^0)}{\sigma_i^0}\right)$ and $\exp\left(-\frac{(m_i^0)^2 + 1}{2v_i^0}(x - \theta_i^0)^2\right)$ are linearly independent. It is clear that if a sequence of $G$ (represented by Eq. (4)) is chosen such that $\Delta \theta_i = \Delta v_i = \Delta p_i = 0$ for all $1 \leq i \leq k_0$, and $\sum_{i=1}^{k_0} p_i \Delta m_i / v_i^0 = 0$, then we obtain $\beta_{1i}^{(1)} / W_1(G, G_0) = \beta_{2i}^{(1)} / W_1(G, G_0) = \beta_{3i}^{(1)} / W_1(G, G_0) = \gamma_{1}^{(1)} / W_1(G, G_0) = \gamma_{2}^{(1)} / W_1(G, G_0) = 0$. Hence, $G_0$ is 1-singular and (1,1,1)-singular relative to $E_{k_0}$.

**Claim: $G_0$ is not (1,1,2)-singular** Indeed, suppose that this is not true. Let $\kappa = (1, 1, 2)$. Then from Definition 3.5 for any sequence of $G$ that tends to $G_0$ under $\tilde{W}_\kappa$, all coefficients of the $\kappa$-minimal form must vanish. A $\kappa$-minimal form is given as follows:

\[
\frac{1}{\tilde{W}_\kappa^2(G, G_0)}\left[\sum_{i=1}^{k_0} \left(\sum_{j=1}^{5} \beta_{ji}^{(2)}(x - \theta_i^0)^{j-1}\right) f\left(\frac{x - \theta_i^0}{\sigma_i^0}\right) \Phi\left(\frac{m_i^0(x - \theta_i^0)}{\sigma_i^0}\right) + \left(4 \sum_{j=1}^{k_0} \gamma_{j}^{(2)}(x - \theta_i^0)^{j-1}\right) \exp\left(-\frac{(m_i^0)^2 + 1}{2v_i^0}(x - \theta_i^0)^2\right)\right] + o(1), \tag{54}
\]

where $\beta_{ji}^{(2)}, \gamma_{j}^{(2)}$ are polynomials of $\Delta \theta_i, \Delta v_i, \Delta m_i,$ and $\Delta p_l$ for $l = 1, \ldots, k_0$:

\[
\beta_{1i}^{(2)} = \frac{2\Delta p_i}{\sigma_i^0} - \frac{p_i \Delta v_i}{(\sigma_i^0)^3} + \frac{p_i(\Delta \theta_i)^2}{4(\sigma_i^0)^5}, \quad \beta_{2i}^{(2)} = \frac{2p_i \Delta \theta_i}{(\sigma_i^0)^3}, \quad \beta_{3i}^{(2)} = \frac{p_i \Delta v_i}{(\sigma_i^0)^5},
\]

\[
\gamma_{1}^{(2)} = \sum_{j=1}^{k_0} -\frac{p_j m_j^0 \Delta \theta_j}{\pi(\sigma_j^0)^2} - \frac{2p_j m_j^0 \Delta \theta_j \Delta m_j}{\pi(\sigma_j^0)^4}, \quad \gamma_{2}^{(2)} = \sum_{j=1}^{k_0} -\frac{p_j m_j^0 \Delta v_j}{\pi(\sigma_j^0)^2} + \frac{2p_j m_j^0 \Delta v_j \Delta m_j}{\pi(\sigma_j^0)^4}.
\]

Now, $\beta_{ji}^{(2)} / \tilde{W}_\kappa^2(G, G_0) \to 0$ leads to $\Delta p_i / \tilde{W}_\kappa^2(G, G_0), \Delta \theta_i / \tilde{W}_\kappa^2(G, G_0), \Delta v_i / \tilde{W}_\kappa^2(G, G_0) \to 0$ for all $1 \leq i \leq k_0$ (The rigorous argument for that result is in Step 1.1 of the full proof of this theorem in
Appendix F). Combining with Lemma 3.3 we obtain

$$\sum_{i=1}^{k_0} p_i |\Delta m_i|^2 / \tilde{W}_2^2(G, G_0) \neq 0. \quad (55)$$

Additionally, the vanishing of coefficients $\gamma_j(2)/\tilde{W}_2^2(G, G_0)$ for $1 \leq j \leq 4$ entails

$$\left( \sum_{i=1}^{k_0} p_i \Delta m_i / v_i^0 \right) / \tilde{W}_2^2(G, G_0) \to 0,$$

$$\left( \sum_{i=1}^{k_0} p_i m_i^0 (\Delta m_i)^2 / (v_i^0)^2 \right) / \tilde{W}_2^2(G, G_0) \to 0. \quad (56)$$

Combining (55) and (56), it follows that

$$\left( \sum_{i=1}^{k_0} p_i m_i^0 (\Delta m_i)^2 / (v_i^0)^2 \right) / \sum_{i=1}^{k_0} p_i |\Delta m_i|^2 \to 0,$$

which is a contradiction due to $m_i^0 > 0$ for all $1 \leq i \leq k_0$. Hence, $G_0$ is not (1,1,2)-singular relative to $E_{k_0}$. We conclude that $\ell(G_0 | E_{k_0}) = 1$ and $\mathcal{L}(G_0 | E_{k_0}) = \{(1, 1, 2)\}$.

11.2 Singularity structure of $S_2$

To illustrate the singularity level and singularity index of $G_0 \in S_2$, we consider a simple setting of $G_0 \in S_2$ in which $m_i^0, m_j^0, \ldots, m_{k_0}^0 = 0$, leaving out the possible setting of conformant homologous sets and generic components.

Claim: $G_0$ is 2-singular and (2,2)-singular To establish this, we look at 2-minimal and (2,2)-minimal form for $(p_G(x) - p_{G_0}(x))/W_2^2(G, G_0)$, which is asymptotically equal to

$$\frac{1}{W_2^2(G, G_0)} \left[ \sum_{i=1}^{k_0} \left( \sum_{j=1}^{5} \zeta_{ij}^{(2)} (x - \theta_i^0)^{j-1} \right) f \left( \frac{x - \theta_i^0}{\sigma_i^0} \right) \right], \quad (57)$$

where $\zeta_{ij}^{(2)}$ are the polynomials in terms of $\Delta \theta_j$, $\Delta v_j$, $\Delta m_j$, and $\Delta p_j$ as $1 \leq i, j \leq k_0$ and $1 \leq l \leq 5$. To make all the coefficients vanish, it suffices to have $(\Delta v_i)^2 / W_2^2(G, G_0) \to 0$ and

$$[- \frac{p_i \Delta v_i}{2(\sigma_i^0)^3} - \frac{p_i (\Delta \theta_i)^2}{2(\sigma_i^0)^3} + \frac{3p_i (\Delta v_i)^2}{8(\sigma_i^0)^5} - \frac{2p_i \Delta \theta_i \Delta m_i}{\sqrt{2\pi} (\sigma_i^0)^2} + \frac{\Delta p_i}{\sigma_i^0}] / W_2^2(G, G_0) \to 0,$$

$$\left[ \frac{\Delta \theta_i}{(\sigma_i^0)^3} + \frac{2 \Delta m_i}{\sqrt{2\pi} (\sigma_i^0)^2} - \frac{3 \Delta \theta_i \Delta v_i}{2(\sigma_i^0)^5} - \frac{2 \Delta v_i \Delta m_i}{\sqrt{2\pi} (\sigma_i^0)^4} \right] / W_2^2(G, G_0) \to 0,$$

$$\left[ \frac{\Delta v_i}{2(\sigma_i^0)^5} + \frac{\Delta \theta_i \Delta v_i}{2(\sigma_i^0)^5} + \frac{\Delta v_i \Delta m_i}{\sqrt{2\pi} (\sigma_i^0)^4} \right] / W_2^2(G, G_0) \to 0,$$

$$\left[ \frac{\Delta \theta_i \Delta v_i}{2(\sigma_i^0)^5} + \frac{\Delta v_i \Delta m_i}{\sqrt{2\pi} (\sigma_i^0)^4} \right] / W_2^2(G, G_0) \to 0. \quad (58)$$

This can be achieved by choosing a sequence of $G \to G_0$ in $W_2$ such that $\Delta \theta_i = \Delta v_i = \Delta m_i = \Delta p_i = 0$ for all $2 \leq i \leq k_0$; only for component 1 do we set $\Delta \theta_1 = -2 \Delta m_1 \sigma_1^0 / \sqrt{2\pi}$ and $\Delta v_1 = (\Delta \theta_1)^2 / 2$. It follows that $G_0$ is 2-singular and (2,2,2)-singular relative to $E_{k_0}$.
Claim: $G_0$ is not $(3,2,3)$-singular

It also entails that $G_0$ is not 3-singular relative to $E_{k_0}$. Now, let $\kappa = (3, 2, 3)$. The $\kappa$-minimal form of $(p_G(x) - p_{G_0}(x))/\tilde{W}^3_\kappa(G, G_0)$ is asymptotically equal to

$$\frac{1}{\tilde{W}^3_\kappa(G, G_0)} \left( \sum_{i=1}^{k_0} \left( \sum_{j=1}^{7} \zeta_{(3)}^{(i)}(x - \theta_0^i)^{j-1} \right) f \left( \frac{x - \theta_0}{\sigma_i^0} \right) \right),$$

where $\zeta_{(3)}^{(i)}$ are the polynomials in terms of $\Delta \theta_j$, $\Delta v_j$, $\Delta m_j$, and $\Delta p_j$ as $1 \leq i, j \leq k_0$ and $1 \leq l \leq 7$. Suppose that there exists a sequence $G \to G_0$ under $\tilde{W}_\kappa$ such that all the coefficients of the $\kappa$-minimal form vanish. For any $1 \leq i \leq k_0$, it follows after some calculations that

$$C_1^{(i)} := -\frac{p_i \Delta v_i}{2(\sigma_1^0)^3} - \frac{p_i (\Delta \theta_i)^2}{2(\sigma_1^0)^3} + \frac{3p_i (\Delta v_i)^2}{8(\sigma_1^0)^5} - \frac{2p_i \Delta \theta_i \Delta m_i}{\sqrt{2\pi}(\sigma_1^0)^3} + \frac{3p_i (\Delta \theta_i)^2 \Delta v_i}{4(\sigma_1^0)^5} + \frac{2p_i \Delta \theta_i \Delta v_i \Delta m_i}{\sqrt{2\pi}(\sigma_1^0)^4} + \frac{\Delta \theta_i}{\sigma_1^0} \tilde{W}^3_\kappa(G, G_0) \to 0,$$

$$C_2^{(i)} := \frac{p_i \Delta \theta_i}{(\sigma_1^0)^3} + \frac{2p_i \Delta m_i}{\sqrt{2\pi}(\sigma_1^0)^3} - \frac{3p_i \Delta \theta_i \Delta v_i}{2(\sigma_1^0)^5} - \frac{2p_i \Delta v_i \Delta m_i}{\sqrt{2\pi}(\sigma_1^0)^4} - \frac{p_i (\Delta \theta_i)^3}{2(\sigma_1^0)^5} - \frac{3p_i (\Delta \theta_i)^2 \Delta m_i}{\sqrt{2\pi}(\sigma_1^0)^4} + \frac{15p_i \Delta \theta_i (\Delta v_i)^2}{8(\sigma_1^0)^7} + \frac{2p_i (\Delta v_i)^2 \Delta m_i}{\sqrt{2\pi}(\sigma_1^0)^6} - \tilde{W}^3_\kappa(G, G_0) \to 0,$$

$$C_3^{(i)} := \frac{p_i \Delta v_i}{2(\sigma_1^0)^5} + \frac{p_i (\Delta \theta_i)^2}{2(\sigma_1^0)^5} - \frac{3p_i (\Delta v_i)^2}{4(\sigma_1^0)^7} + \frac{2p_i \Delta \theta_i \Delta m_i}{\sqrt{2\pi}(\sigma_1^0)^4} - \frac{3p_i (\Delta \theta_i)^2 \Delta v_i}{2(\sigma_1^0)^7} - \frac{5\Delta \theta_i \Delta v_i \Delta m_i}{\sqrt{2\pi}(\sigma_1^0)^6} - \tilde{W}^3_\kappa(G, G_0) \to 0,$$

$$C_4^{(i)} := \frac{p_i \Delta \theta_i \Delta v_i}{2(\sigma_1^0)^7} + \frac{p_i \Delta v_i \Delta m_i}{\sqrt{2\pi}(\sigma_1^0)^6} + \frac{p_i (\Delta \theta_i)^3}{6(\sigma_1^0)^7} - \frac{p_i (\Delta m_i)^3}{3\sqrt{2\pi}(\sigma_1^0)^4} + \frac{p_i (\Delta \theta_i)^2 \Delta m_i}{\sqrt{2\pi}(\sigma_1^0)^6} - \frac{5p_i \Delta \theta_i (\Delta v_i)^2}{4(\sigma_1^0)^9} - \frac{2p_i (\Delta v_i)^2 \Delta m_i}{\sqrt{2\pi}(\sigma_1^0)^8} - \tilde{W}^3_\kappa(G, G_0) \to 0,$$

$$C_5^{(i)} := -\frac{p_i (\Delta v_i)^2}{8(\sigma_1^0)^9} - \frac{5p_i (\Delta v_i)^3}{16(\sigma_1^0)^11} + \frac{p_i (\Delta \theta_i)^2 \Delta v_i}{4(\sigma_1^0)^9} + \frac{p_i \Delta \theta_i \Delta v_i \Delta m_i}{\sqrt{2\pi}(\sigma_1^0)^8} - \tilde{W}^3_\kappa(G, G_0) \to 0,$$

$$C_6^{(i)} := \frac{p_i \Delta \theta_i (\Delta v_i)^2}{8(\sigma_1^0)^11} + \frac{p_i (\Delta v_i)^2 \Delta m_i}{4\sqrt{2\pi}(\sigma_1^0)^10} - \tilde{W}^3_\kappa(G, G_0) \to 0,$$

$$C_7^{(i)} := p_i (\Delta v_i)^3/48(\sigma_1^0)^3 - \tilde{W}^3_\kappa(G, G_0) \to 0.$$  

Since the system of limits in (60) holds for any $1 \leq i \leq k_0$, to further simplify the argument without loss of generality, we consider $k_0 = 1$. Under that scenario, we can rewrite $\tilde{W}^3_\kappa(G, G_0) = p_1 (|\Delta \theta_1|^3 + |\Delta v_1|^2 + |\Delta m_1|^3)$ where $p_1 = 1$. Additionally, for the simplicity of the presentation, we denote $C_i := C_i^{(1)}$ for any $1 \leq i \leq 7$. Now, our argument is organized into the following key steps

**Step 1.1:** We will argue that $\Delta \theta_1, \Delta v_1, \Delta m_1 \neq 0$. If $\theta_1 = 0$, by combining the vanishing of $C_5$, we achieve $(\Delta v_1)^2/\tilde{W}^3_\kappa(G, G_0) \to 0$. Since $C_3 \to 0$, we further obtain that $\Delta v_1/\tilde{W}^3_\kappa(G, G_0) \to 0$. Combining the previous result with $C_4 \to 0$ eventually yields that $(\Delta m_1)^3/\tilde{W}^3_\kappa(G, G_0) \to 0$. Hence, $1 = p_1(|\Delta v_1|^2 + |\Delta m_1|^3)/\tilde{W}^3_\kappa(G, G_0) \to 0$, which is a contradiction.

If $\Delta v_1 = 0$, then $C_1 + \Delta \theta_1 C_2 \to 0$ implies that $(\Delta \theta_1)^2/\tilde{W}^3_\kappa(G, G_0) \to 0$. Combining this result with $C_4 \to 0$, we achieve $(\Delta m_1)^3/\tilde{W}^3_\kappa(G, G_0) \to 0$, which also leads to a contradiction.
If $\Delta m_1 = 0$, then $C_4 \to 0$ leads to
\[
\left[ \frac{\Delta \theta_1 \Delta v_1}{2(\sigma_1^0)^2} + \frac{(\Delta \theta_1)^3}{6(\sigma_1^0)^4} \right] / \tilde{W}_k^2(G, G_0) \to 0.
\] (61)

The combination of the above result and $C_3 \to 0$ implies that $\Delta v_1 / \tilde{W}_k^2(G, G_0) \to 0$. Combine the former results with (61), we obtain $(\Delta \theta_1)^3 / \tilde{W}_k^2(G, G_0) \to 0$, which is also a contradiction. Overall, we obtain the conclusion of this step.

**Step 1.2:** If $|\Delta v_1|^2$ is the maximum among $|\Delta \theta_1|^3$, $|\Delta v_1|^2$, and $|\Delta m_1|^3$. Since $C_5 \to 0$, it leads to $(\Delta v_1)^2 / \tilde{W}_k^2(G, G_0) \to 0$, which is a contradiction.

**Step 1.3:** If $|\Delta \theta_1|^3$ is the maximum among $|\Delta \theta_1|^3$, $|\Delta v_1|^2$, and $|\Delta m_1|^3$. Denote $(\Delta v_1)^2 / (\Delta \theta_1)^3 \to k_1$ and $\Delta m_1 / \Delta \theta_1 \to k_2$. Since $C_5 \to 0$ leads to $(\Delta v_1)^2 / \tilde{W}_k^2(G, G_0) \to 0$, we obtain $k_1 = 0$. As $C_2 \to 0$, we obtain
\[
\left[ -\Delta \theta_1 / (\sigma_0^0)^3 + 2\Delta m_1 / \sqrt{2\pi}(\sigma_0^0)^2 \right] / (|\Delta \theta_1| + |\Delta v_1| + |\Delta m_1|) \to 0.
\]

By dividing both the numerator and denominator of this ratio by $\Delta \theta_1$, we quickly obtain the equation $1/(\sigma_0^0)^3 + 2k_2 / \sqrt{2\pi}(\sigma_1^0)^2 = 0$, which yields the solution $k_2 = -\sqrt{\pi} / \sqrt{2\sigma_1^0}$.

Applying the result $(\Delta v_1)^2 / \tilde{W}_k^2(G, G_0) \to 0$ to $C_3 \to 0$ and $C_4 \to 0$, we have $M_1 \to 0$ and $M_2 \to 0$ where the formations of $M_1$, $M_2$ are as follows:
\[
M_1 := \left( \frac{\Delta v_1}{2(\sigma_0^0)^5} + \frac{(\Delta \theta_1)^2}{2(\sigma_0^0)^5} + \frac{2(\Delta \theta_1)(\Delta m_1)}{\sqrt{2\pi}(\sigma_0^0)^4} \right) / (|\Delta \theta_1|^3 + |\Delta v_1|^2 + |\Delta m_1|^3),
\]
\[
M_2 := \left( \frac{(\Delta \theta_1)(\Delta v_1)}{2(\sigma_0^0)^7} + \frac{(\Delta v_1)(\Delta m_1)}{\sqrt{2\pi}(\sigma_0^0)^6} + \frac{(\Delta \theta_1)^3}{6(\sigma_0^0)^7} - \frac{(\Delta m_1)^3}{3\sqrt{2\pi}(\sigma_1^0)^4} + \frac{(\Delta \theta_1)^2(\Delta m_1)}{\sqrt{2\pi}(\sigma_1^0)^6} \right) / (|\Delta \theta_1|^3 + |\Delta v_1|^2 + |\Delta m_1|^3).
\]

Now, \( \left( \frac{\Delta \theta_1}{(\sigma_0^0)^2} + \frac{2\Delta m_1}{\sqrt{2\pi}\sigma_0^0} \right) M_1 - M_2 \) yields that
\[
\left[ \frac{(\Delta m_1)^3}{3\sqrt{2\pi}} + \frac{2(\theta_1)(\Delta m_1)^2}{\pi\sigma_0^0} + \frac{2(\Delta \theta_1)(\Delta m_1)}{\sqrt{2\pi}(\sigma_0^0)^2} + \frac{(\Delta \theta_1)^3}{3(\sigma_1^0)^3} \right] / (|\Delta \theta_1|^3 + |\Delta v_1|^2 + |\Delta m_1|^3) \to 0.
\]

By dividing both the numerator and denominator of this term by $(\Delta \theta_1)^3$, we obtain the equation $k_3^3 + \frac{2k_2^3}{\pi\sigma_1^0} + \frac{2k_2}{\sqrt{2\pi}(\sigma_0^0)^2} + \frac{1}{3(\sigma_1^0)^3} = 0$. Since $k_2 = -\sqrt{\pi} / \sqrt{2\sigma_1^0}$, this equation yields $\pi / 6 - 1 / 3 = 0$, which is a contradiction. Therefore, this step cannot hold.

**Step 1.4:** If $|\Delta m_1|^3$ is the maximum among $|\Delta \theta_1|^3$, $|\Delta v_1|^2$, and $|\Delta m_1|^3$. The argument in this step is similar to that of Step 1.3. In fact, by denoting $\Delta \theta_1 / \Delta m_1 \to k_3$ and $(\Delta v_1)^2 / (\Delta m_1)^3 \to k_4$ then we also achieve $k_4 = 0$ and $k_3 = -\sqrt{2} / \sqrt{2\sigma_1^0}$ (by $C_2 \to 0$). Now by using the limits $C_3, C_4 \to 0$ as that of Step 1.3 and after some calculations, we obtain the equation $k_3^3 + \frac{2k_2^3}{\sqrt{2\pi}(\sigma_1^0)^2} + \frac{2k_2^3}{\pi\sigma_1^0} + \frac{1}{3\sqrt{2\pi}} = 0$, which is a contradiction. Therefore, this step cannot hold.
which also does not admit \( k_3 = -\frac{\sqrt{2}}{\sqrt{\pi} \sigma_1} \) as a solution — a contradiction. As a consequence, \( G_0 \) is not (3,2,3)-singular relative to \( \mathcal{E}_{k_0} \).

Since \( G_0 \) is 2-singular but not 3-singular relative to \( \mathcal{E}_{k_0} \), we obtain that \( \ell(G_0|\mathcal{E}_{k_0}) = 2 \). To demonstrate that (3,2,3) is the singularity index of \( G_0 \), we need to verify that \( G_0 \) is (3,1,3)-singular, (3,2,2)-singular, and (2,2,3)-singular relative to \( \mathcal{E}_{k_0} \). In particular, combining these results with the fact that \( G_0 \) is (2,2,2)-singular relative to \( \mathcal{E}_{k_0} \), it implies that for any \( \kappa' \prec (3,2,3) \), \( G_0 \) is \( \kappa' \)-singular relative to \( \mathcal{E}_{k_0} \).

\textbf{Claim:} \( G_0 \) is (3,1,3)-singular \quad \text{Here, we choose } k_0 = 1 \text{ and denote } \kappa_1 = (3,1,3). \text{ Similar to the argument for } G_0 \text{ is not (3,2,3)-singular, the vanishing of all coefficients of the (3,1,3)-minimal form leads to}

\begin{align*}
\left( \frac{\Delta \theta_1}{(\sigma_1^0)^3} + \frac{2\Delta m_1}{\sqrt{2\pi}(\sigma_1^0)^2} - \frac{(\Delta \theta_1)^3}{2(\sigma_1^0)^5} - \frac{3(\Delta \theta_1)^2\Delta m_1}{\sqrt{2\pi}(\sigma_1^0)^4} \right) \frac{1}{W_\kappa_1^3(G, G_0)} &\to 0, \\
\left( \frac{\Delta v_1}{2(\sigma_1^0)^3} + \frac{(\Delta \theta_1)^2}{2(\sigma_1^0)^5} + \frac{\Delta \theta_1\Delta m_1}{\sqrt{2\pi}(\sigma_1^0)^4} \right) \frac{1}{W_\kappa_1^3(G, G_0)} &\to 0, \\
\left( \frac{(\Delta \theta_1)^3}{6(\sigma_1^0)^7} - \frac{(\Delta m_1)^3}{3\sqrt{2\pi}(\sigma_1^0)^4} + \frac{(\Delta \theta_1)^2\Delta m_1}{2\pi(\sigma_1^0)^6} \right) \frac{1}{W_\kappa_1^3(G, G_0)} &\to 0
\end{align*}

where \( \frac{1}{W_\kappa_1^3(G, G_0)} = |\Delta \theta_1|^3 + |\Delta v_1| + |\Delta m_1|^3 \). \text{ By choosing } \Delta \theta_1 = -\frac{2\sigma_1^0\Delta m_1}{\sqrt{2\pi}} \text{ and } \Delta v_1 = (\Delta \theta_1)^2, \text{ then all the above limits satisfy as long as } \Delta \theta_1 \to 0. \text{ Therefore, } G_0 \text{ is (3,1,3)-singular relative to } \mathcal{E}_{k_0}.

\textbf{Claim:} \( G_0 \) is (3,2,2)-singular \quad \text{Here, we choose } k_0 = 1 \text{ and denote } \kappa_2 = (3,2,2). \text{ The vanishing of all coefficients of the (3,2,2)-minimal form leads to}

\begin{align*}
\left( \frac{\Delta \theta_1}{(\sigma_1^0)^3} + \frac{2\Delta m_1}{\sqrt{2\pi}(\sigma_1^0)^2} - \frac{3\Delta \theta_1\Delta v_1}{2(\sigma_1^0)^5} - \frac{2\Delta v_1\Delta m_1}{\sqrt{2\pi}(\sigma_1^0)^4} - \frac{(\Delta \theta_1)^3}{2(\sigma_1^0)^5} \right) \frac{1}{W_\kappa_2^3(G, G_0)} &\to 0, \\
\left( \frac{\Delta v_1}{2(\sigma_1^0)^3} + \frac{(\Delta \theta_1)^2}{2(\sigma_1^0)^5} + \frac{2\Delta \theta_1\Delta m_1}{\sqrt{2\pi}(\sigma_1^0)^4} \right) \frac{1}{W_\kappa_2^3(G, G_0)} &\to 0, \\
\left( \frac{(\Delta \theta_1\Delta v_1)}{2(\sigma_1^0)^7} + \frac{(\Delta \theta_1)^3}{\sqrt{2\pi}(\sigma_1^0)^9} + \frac{(\Delta \theta_1)^2\Delta m_1}{6(\sigma_1^0)^6} \right) \frac{1}{W_\kappa_2^3(G, G_0)} &\to 0
\end{align*}

where \( \frac{1}{W_\kappa_2^3(G, G_0)} = |\Delta \theta_1|^3 + |\Delta v_1|^2 + |\Delta m_1|^2 \). \text{ We can verify that by choosing } \Delta \theta_1 = -\frac{2\sigma_1^0\Delta m_1}{\sqrt{2\pi}} \text{ and } \Delta v_1 = (\Delta \theta_1)^2, \text{ then all the above limits satisfy as long as } \Delta \theta_1 \to 0. \text{ Therefore, } G_0 \text{ is (3,2,2)-singular relative to } \mathcal{E}_{k_0}.

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$G_0$ is (2,2,3)-singular  Here, we choose $k_0 = 1$ and denote $\kappa_3 = (2, 2, 3)$. The vanishing of all coefficients of the (2,2,3)-minimal form leads to

$$
\begin{align*}
\left( \frac{\Delta \theta_1}{(\sigma_1^0)^3} + \frac{2\Delta m_1}{\sqrt{2\pi}(\sigma_1^0)^2} - \frac{3\Delta \theta_1 \Delta v_1}{2(\sigma_1^0)^5} - \frac{2\Delta v_1 \Delta m_1}{\sqrt{2\pi}(\sigma_1^0)^4} \right) / \tilde{W}_3^3(G, G_0) & \to 0, \\
\left( \frac{\Delta \theta_1}{2(\sigma_1^0)^5} + \frac{(\Delta \theta_1)^2}{2(\sigma_1^0)^3} + \frac{2\Delta \theta_1 \Delta m_1}{\sqrt{2\pi}(\sigma_1^0)^4} \right) / \tilde{W}_3^3(G, G_0) & \to 0, \\
\left( \frac{\Delta \theta_1 \Delta v_1}{2(\sigma_1^0)^7} + \frac{v_1 \Delta m_1}{\sqrt{2\pi}(\sigma_1^0)^6} - \frac{(\Delta m_1)^3}{3\sqrt{2\pi}(\sigma_1^0)^4} \right) & / \tilde{W}_3^3(G, G_0) \\
\end{align*}
$$

where $\tilde{W}_3^3(G, G_0) = |\Delta \theta_1|^2 + |\Delta v_1|^2 + |\Delta m_1|^3$. By using $\Delta \theta_1 = -\frac{2\sigma_1^0 \Delta m_1}{\sqrt{2\pi}}$ and $\Delta v_1 = (\Delta \theta_1)^2$, we can verify that all the above limits satisfy as long as $\Delta \theta_1 \to 0$. Therefore, $G_0$ is (2,2,3)-singular relative to $E_{k_0}$.

In sum, we have shown under the setting of $G_0 \in S_2$, (3,2,3) is the singularity index of $G_0$ relative to $E_{k_0}$. To demonstrate further that (3,2,3) is the unique singularity index of $G_0$, we need to show that $G_0$ is $\kappa$-singular relative to $E_{k_0}$ where $\kappa \in \{(r, r, 2), (2, r, r), (r, 1, r)\}$ for any $r \geq 1$. As $r \leq 3$, we have verified these results hold above with the choice of $k_0 = 1$, $\Delta \theta_1 = -\frac{2\sigma_1^0 \Delta m_1}{\sqrt{2\pi}}$, and $\Delta v_1 = (\Delta \theta_1)^2$.

We now argue that these choices of $k_0$ and $G$ are also sufficient to obtain previous results for any $r \geq 4$. In fact, with the previous choices of $k_0$ and $G$, it is clear that $(\Delta \theta_1)^{\alpha_1}(\Delta v_1)^{\alpha_2}(\Delta m_1)^{\alpha_3} / \tilde{W}_k^r(G, G_0) \to 0$ for any $\kappa \in \{(r, r, 2), (2, r, r), (r, 1, r)\}$ where $r \geq 4$ and $|\alpha| \geq 4$. With these results, for any $\kappa$ from the previous set, if all coefficients of any $\kappa$-minimal form of $G$ vanish, they will eventually lead to one of three systems of limits (the denominator is changed to $\tilde{W}_k^r(G, G_0)$) that we used to demonstrate that $G_0$ is $(3,1,3)$-singular, $(3,2,2)$-singular, and $(2,2,3)$-singular relative to $E_{k_0}$ above. These systems of limits with new denominator still hold with our choices of $k_0$ and $G$.

As a consequence, (3,2,3) is the unique singularity index of $G_0$ relative to $E_{k_0}$, i.e., $L(G_0|E_{k_0}) = \{(3,2,3)\}$. Therefore, we achieve the conclusion of the theorem under the setting of $G_0 \in S_2$.

### 11.3 Singularity structure of $S_3$

To develop intuition and obtain bounds for singularity structure of $G_0 \in S_3$, we start by considering a simple case similar to the exposition of subsection 11.1 and subsection 11.2. That is, $G_0$ has only one homologous set of size $k_0$. $G_0 \in S_3$ means that $m_i^0$ do not share the same signs for all $i = 1, \ldots, k_0$. To investigate the singularity structure for $G_0$, we first obtain an $\kappa$-minimal form, for any $\|\kappa\|_\infty = r$ such that $\kappa_3 = r$ where $r \geq 2$, of $(p_G(x) - p_{G_0}(x)) / \tilde{W}_k^r(G, G_0)$ by

$$
\begin{align*}
\frac{1}{\tilde{W}_k^r(G, G_0)} & \sum_{i=1}^{k_0} \left( \sum_{j=1}^{2r+1} \beta_{ji}^{(r)} (x - \theta_i^0)^{j-1} f \left( \frac{x - \theta_i^0}{\sigma_i^0} \right) \Phi \left( \frac{m_i^0 (x - \theta_i^0)}{\sigma_i^0} \right) + \sum_{j=1}^{2r} \gamma_j^{(r)} (x - \theta_i^0)^{j-1} \exp \left( -\frac{(m_i^0)^2 + 1}{2v_i^0} (x - \theta_i^0)^2 \right) \right) + o(1),
\end{align*}
$$

(62)
where $\beta_{ji}^{(r)}$, $\gamma_{ij}^{(r)}$ are polynomials of $\Delta \theta_i$, $\Delta v_i$, $\Delta m_i$, and $\Delta p_i$ as $1 \leq i, l \leq k_0$ and $1 \leq j \leq 2r + 1$. For concrete formulas of $\beta_{ji}^{(r)}$, $\gamma_{ij}^{(r)}$, we note that for any $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ such that $|\alpha| \leq r$, there holds

$$\frac{\partial^{|\alpha|} f}{\partial \theta^{\alpha_1} \partial v^{\alpha_2} \partial m^{\alpha_3}} = \left( \sum_{i=1}^{2r+1} \frac{U_i^{\alpha_1,\alpha_2,\alpha_3}(m)}{V_i^{\alpha_1,\alpha_2,\alpha_3}(v)} (x - \theta)^{-i-1} \right) f \left( \frac{x - \theta}{\Delta x} \right) f \left( \frac{m(x - \theta)}{\sigma} \right) + \frac{1}{\sigma} \left( \sum_{i=1}^{2r+1} \frac{L_i^{\alpha_1,\alpha_2,\alpha_3}(v)}{N_i^{\alpha_1,\alpha_2,\alpha_3}(v)} (x - \theta)^{-i-1} \right) f \left( \frac{x - \theta}{\Delta x} \right) \Phi \left( \frac{m(x - \theta)}{\sigma} \right).$$

In the above display $U_i^{\alpha_1,\alpha_2,\alpha_3}(m), V_i^{\alpha_1,\alpha_2,\alpha_3}(v), N_i^{\alpha_1,\alpha_2,\alpha_3}(v)$ are polynomials in terms of $m, v$ and $L_i^{\alpha_1,\alpha_2,\alpha_3}$ are some constant numbers. As $\alpha_3 \geq 1$, we can further check that $L_i^{\alpha_1,\alpha_2,\alpha_3} = 0$ for all $1 \leq i \leq 2r$ and $\alpha_1, \alpha_2$ such that $|\alpha| \leq r$. It follows that

$$\beta_{ji}^{(r)} = \frac{2\Delta p_i}{\sigma_j} \sum_{j=1}^{1} + 1 \frac{1}{\sigma_i} \sum_{|\alpha| \leq r} \frac{L_i^{\alpha_1,\alpha_2,\alpha_3}(v_0)}{N_i^{\alpha_1,\alpha_2,\alpha_3}(v_0)} \frac{p_i(\Delta \theta_i)^{\alpha_1} (\Delta v_i)^{\alpha_2} (\Delta m_i)^{\alpha_3}}{\alpha_1! \alpha_2! \alpha_3!},$$

and

$$\gamma_{ij}^{(r)} = \sum_{i=1}^{k_0} \sum_{|\alpha| \leq r} \frac{U_i^{\alpha_1,\alpha_2,\alpha_3}(m_0)}{V_i^{\alpha_1,\alpha_2,\alpha_3}(v_0)} \frac{p_i(\Delta \theta_i)^{\alpha_1} (\Delta v_i)^{\alpha_2} (\Delta m_i)^{\alpha_3}}{\alpha_1! \alpha_2! \alpha_3!},$$

where $1 \leq i \leq k_0$ and $1 \leq j \leq 2r + 1$. Since $L_j^{\alpha_1,\alpha_2,\alpha_3} = 0$ as $\alpha_3 \geq 1$, we further obtain that

$$\beta_{ji}^{(r)} = \frac{2\Delta p_i}{\sigma_j} \sum_{j=1}^{1} + 1 \frac{1}{\sigma_i} \sum_{\alpha_1 + \alpha_2 \leq r} \frac{L_j^{\alpha_1,\alpha_2,0}(v_0)}{N_j^{\alpha_1,\alpha_2,0}(v_0)} \frac{p_i(\Delta \theta_i)^{\alpha_1} (\Delta v_i)^{\alpha_2}}{\alpha_1! \alpha_2!}.$$

Therefore, $\beta_{ji}^{(r)}$ are polynomials of $\Delta \theta_i$, $\Delta v_i$, $\Delta m_i$, while $\gamma_{ij}^{(r)}$ are polynomials of $\Delta \theta_i, \Delta v_i, \Delta m_i$, for $1 \leq i \leq k_0, 1 \leq j \leq 2r + 1$.

Suppose that there is a sequence of $G$ tending to $G_0$ in $\tilde{W}_r$ distance such that all coefficients of its $\kappa$-minimal form in $\tilde{W}_r$ vanish. It can be checked that $\beta_{ji}^{(r)} / \tilde{W}_r(G, G_0) \rightarrow 0$ for all even number $j \in [1, 2r + 1]$ entails that $\Delta \theta_i / \tilde{W}_r(G, G_0) \rightarrow 0$ for all $1 \leq i \leq k_0$. Similarly, $\beta_{ji}^{(r)} / \tilde{W}_r(G, G_0) \rightarrow 0$ for all odd $j \in [3, 2r + 1]$. It follows that, as $\beta_{ji}^{(r)} / \tilde{W}_r(G, G_0) \rightarrow 0$, we obtain $\Delta \theta_i / \tilde{W}_r(G, G_0) \rightarrow 0$. So, $\beta_{ji}^{(r)} / \tilde{W}_r(G, G_0) \rightarrow 0$, we must have $\Delta \theta_i / \tilde{W}_r(G, G_0) \rightarrow 0$, $\Delta \theta_i / \tilde{W}_r(G, G_0) \rightarrow 0$, and $\Delta v_i / \tilde{W}_r(G, G_0) \rightarrow 0$ for all $1 \leq i \leq k_0$. Note that, these results hold for any choice of $1 \leq \kappa_1, \kappa_2 \leq r$. Additionally, they also imply that

$$\frac{\sum_{i=1}^{k_0} |\Delta p_i| + \sum_{i=1}^{k_0} |\Delta \theta_i|^{\kappa_1} + |\Delta v_i|^{\kappa_2}}{\tilde{W}_r(G, G_0)} \rightarrow 0.$$

If $\Delta m_i = 0$ for all $1 \leq i \leq k_0$, then by means of Lemma \[[\text{lem3}]\] $\sum_{i=1}^{k_0} |\Delta p_i| + \sum_{i=1}^{k_0} (|\Delta \theta_i|^{\kappa_1} + |\Delta v_i|^{\kappa_2}) = D_\kappa(G_0, G) \times \tilde{W}_r(G, G_0)$, which contradicts with the above limit. Therefore, we have $\max_{1 \leq i \leq k_0} |\Delta m_i| > 0$.

Turning to $\gamma_{ij}^{(r)}$ and the fact that $\Delta p_i / \tilde{W}_r(G, G_0) \rightarrow 0, \Delta \theta_i / \tilde{W}_r(G, G_0) \rightarrow 0, \Delta v_i / \tilde{W}_r(G, G_0) \rightarrow 0$, if $\gamma_{ij}^{(r)} / \tilde{W}_r(G, G_0) \rightarrow 0$ as $1 \leq l \leq 2r$, we also have that

$$\left( \sum_{i=1}^{k_0} \sum_{\alpha_3 \leq r} \frac{U_l^{\alpha_0,0,\alpha_3}(m_0)}{V_l^{\alpha_0,0,\alpha_3}(v_0)} \frac{p_l(\Delta m_i)^{\alpha_3}}{\alpha_3!} \right) / \tilde{W}_r(G, G_0) \rightarrow 0.$$
We can verify that as $1 \leq l \leq 2r$ is odd, $U_{ji}^{0,0,\alpha_3}(m_l^0) = 0$ for all $\alpha_3 \leq r$ and $1 \leq i \leq k_0$. Additionally, as $1 \leq l \leq 2r$ is even, the above system of limits becomes

$$
\left( \sum_{i_1-i_2=l/2}^{\frac{k_0}{i_1!}} q_{i_1,i_2} \sum_{i=1}^{l/2} p_i(m_i^0)^{i_1-2i_2-1}(\Delta m_i^{i_1})^{i_1} (\sigma_i^0)^{l} \right) / \tilde{W}_k^r(G, G_0) \to 0,
$$

where $1 \leq i_1 \leq r$, $i_2 \leq (i_1-1)/2$ as $i_1$ is odds or $i_2 \leq i_1/2 - 1$ as $i_1$ is even. Here, $q_{i,j}$ are the integer coefficients that appear in the high order derivatives of $f(x|\theta, \sigma, m)$ with respect to $m$:

$$
\frac{\partial^{s+1} f}{\partial m^{s+1}} = \sum_{j=0}^{(s-1)/2} \frac{q_{s+1,j} m^{s-2j}}{\sigma^{2s+2-2j}} (x-\theta)^{2s-2j+1} f \left( \frac{x-\theta}{\sigma} \right) f \left( \frac{m(x-\theta)}{\sigma} \right)
$$

when $s$ is an odd number and

$$
\frac{\partial^{s+1} f}{\partial m^{s+1}} = \sum_{j=0}^{s/2} \frac{q_{s+1,j} m^{s-2j}}{\sigma^{2s+2-2j}} (x-\theta)^{2s-2j+1} f \left( \frac{x-\theta}{\sigma} \right) f \left( \frac{m(x-\theta)}{\sigma} \right)
$$

when $s$ is an even number. For instance, when $s = 0$, we have $q_{1,0} = 2$ and when $s = 1$, we have $q_{2,0} = -2$.

Summarizing, under that simple setting of $G_0$ in order for all the coefficients in the $\kappa$-minimal form (62) to vanish, i.e., we have $\beta^{(r)}_{ji} / \tilde{W}_k^r(G, G_0) \to 0$ and $\gamma^{(r)}_{ji} / \tilde{W}_k^r(G, G_0) \to 0$, only the third component of $\kappa$, i.e., $\kappa_3 = r$, plays a key role while the first two components $\kappa_1$ and $\kappa_2$ of $\kappa$ can be of any values from 1 to $r$. Additionally, the value $\kappa_3 = r$ is determined by the system of limits (63), i.e., that system is the important factor to determine the singularity structure of $G_0 \in S_3$. Let $r_{\text{max}}$ to be the maximum number $r$ such that system of limits (63) holds. From the definition of singularity level, it is clear that $r_{\text{max}} = \ell(G_0|E_{k_0})$. These observations under this simple setting of $G_0 \in S_3$ shed light on the following important result regarding singularity index of any $G_0 \in S_3$ relative to $E_{k_0}$ whose rigorous proof is deferred to Appendix F.

**Theorem 11.1.** Suppose that $G_0 \in S_3$.

(a) If $\ell(G_0|E_{k_0}) < \infty$ and $P_1(\eta^0) \neq 0$, i.e., there are no Gaussian components in $G_0$, then we have $L(G_0|E_{k_0}) = \{(1, 1, \ell(G_0|E_{k_0}) + 1)\}$.

(b) If $\ell(G_0|E_{k_0}) < \infty$ and $P_1(\eta^0) = 0$, i.e., there are some Gaussian components in $G_0$, then we have $L(G_0|E_{k_0}) = \{(3, 2, \max\{2, \ell(G_0|E_{k_0})\} + 1)\}$.

(c) If $\ell(G_0|E_{k_0}) = \infty$, then $L(G_0|E_{k_0}) = (\infty, \infty, \infty)$.

The above results imply that we only need to focus on studying the singularity level to understand singularity structure of $G_0 \in S_3$, i.e., we can choose $\kappa = (r, r, r)$. To illustrate the behaviors of singularity levels of $G_0 \in S_3$, we will continue exploring the structure of system of limits (63) under the simple setting of $G_0 \in S_3$ when it has only one homologous set of size $k_0$.

### 11.3.1 Singularity structure of $G_0 \in S_{31}$

Recall that $G_0$ has only one homologous set of size $k_0$. Let $\kappa = (r, r, r)$. From the above argument that, as we have $\beta^{(r)}_{ji} / \tilde{W}_k^r(G, G_0) \to 0$ when $1 \leq i, l \leq k_0$ and $1 \leq j \leq 2r + 1$, we obtain
\[ \Delta p_i / \bar{W}_\alpha^r (G, G_0) \to 0, \quad \Delta \theta_i / \bar{W}_\alpha^r (G, G_0) \to 0, \quad \Delta v_i / \bar{W}_\alpha^r (G, G_0) \to 0 \]

for all \( 1 \leq i \leq k_0 \). Combining with Lemma 3.3, it follows that

\[ \sum_{i=1}^{k_0} p_i |\Delta m_i|^r / \bar{W}_\alpha^r (G, G_0) \neq 0. \]  

(64)

Since we have \( \max_{1 \leq i \leq k_0} |\Delta m_i| > 0 \), a combination of (63) and (64) leads to

\[ \left( \sum_{i_1-i_2=l/2} q_{i_1,i_2} / i_1! \sum_{i=1}^{k_0} p_i (m_i^0)^{i_1-2i_2-1} (\Delta m_i)^{i_2} \right) / \sum_{i=1}^{k_0} p_i |\Delta m_i|^r \to 0, \]  

(65)

for any even \( l \) such that \( 1 \leq l \leq 2r \). Let \( q_i = p_i / \sigma_i^0 \), \( m_i^0 = m_i / \sigma_i^0 \), and \( \Delta t_i = \Delta m_i / \sigma_i^0 \) for all \( 1 \leq i \leq k_0 \), then the above limits can be rewritten as

\[ \left( \sum_{i_1-i_2=l/2} q_{i_1,i_2} / i_1! \sum_{i=1}^{k_0} q_i (t_i^0)^{i_1-2i_2-1} (\Delta t_i)^{i_2} \right) / \sum_{i=1}^{k_0} q_i |\Delta t_i|^r \to 0, \]  

(66)

where in the summation of the above display, \( 1 \leq i_1 \leq r, i_2 \leq (i_1-1)/2 \) as \( i_1 \) is odd, or \( i_2 \leq i_1/2 - 1 \) as \( i_1 \) is even and \( l \) is an even number ranging from 2 to \( 2r \). These are the limits of the ratio of two semipolynomial functions. The existence of these limits will be shown to entail the existence of zeros of a system of polynomial equations.

**Greedy extraction of limiting polynomials**

As explained in the main text, it is generally difficult to obtain all polynomial limits of the system of rational semipolynomial functions given by (66). However, it is possible to obtain a subset of polynomial limits via a greedy method of extraction. We shall demonstrate this technique for the specific case \( r = 3 \), and then present a general result, not unlike what we have done in subsections 4.1 and 4.2 for \( \alpha \)-mixtures. For \( r = 3 \), we only have three possible choices of \( l \) in (66), which are \( l = 2, 4 \) and \( 6 \). As \( l = 2 \), we have \( (i_1, i_2) = (1, 0) \). As \( l = 4 \), we obtain \( (i_1, i_2) \in \{(2, 0), (3, 1)\} \). Finally, as \( l = 6 \), we get \( (i_1, i_2) = (3, 0) \). Here, we can compute that \( q_{1,0} = 2, q_{2,0} = -2, q_{3,1} = -2, q_{3,0} = 2 \). Therefore, as \( r = 3 \), the system of limits (66) becomes

\[
\left( \sum_{i=1}^{k_0} q_i |\Delta t_i| \right) / \sum_{i=1}^{k_0} q_i |\Delta t_i|^3 \to 0, \\
\left( \sum_{i=1}^{k_0} q_i (t_i^0)^2 (\Delta t_i)^3 \right) / \sum_{i=1}^{k_0} q_i |\Delta t_i|^3 \to 0, \\
\left( \sum_{i=1}^{k_0} q_i (t_i^0)^3 (\Delta t_i)^3 \right) / \sum_{i=1}^{k_0} q_i |\Delta t_i|^3 \to 0. \]  

(67)

Denote \( |\Delta t_{k_0}| := \max_{1 \leq i \leq k_0} \{||\Delta t_i||| \} \). In each of the limiting expressions in the above display, we shall divide both the numerator and denominator of the left hand side by \( |\Delta t_{k_0}|^\alpha \), where \( \alpha \) is the smallest degree that appears in one of the monomials in the numerator. Since \( |\Delta t_i|/|\Delta t_{k_0}| \) is bounded, there exist a subsequence according to which \( |\Delta t_i|/|\Delta t_{k_0}| \) tends to a constant, say \( k_i \), for each \( i = 1, \ldots, k_0 \). Note that at least one of the \( k_i \) is non-zero. Moreover, we obtain the following equations in the limit

\[
\sum_{i=1}^{k_0} q_i^0 k_i = 0, \quad \sum_{i=1}^{k_0} q_i^0 (k_i)^2 = 0, \quad \sum_{i=1}^{k_0} q_i^0 (t_i^0)^2 (k_i)^3 = 0. 
\]
Since \( q_i^0 = p_i^0 / \sigma_i^0 \), \( r_i^0 = m_i^0 / \sigma_i^0 \) for all \( 1 \leq i \leq k_0 \), by rescaling \( k_i \), the above system of polynomial equations can be rewritten as

\[
\sum_{i=1}^{k_0} p_i^0 k_i = 0, \quad \sum_{i=1}^{k_0} p_i^0 m_i^0(k_i)^2 = 0, \quad \sum_{i=1}^{k_0} p_i^0 (m_i^0)^2 (k_i)^3 = 0.
\]

Now we shall apply the greedy extraction technique to the general system (66). This involves dividing both the numerator and the denominator of the left hand side in each equation of the system by \( (\Delta t_{k_0})^{l/2} \) for any \( 2 \leq l \leq 2r \) and \( l \) is even. This leads to the existence of solution for the following system of polynomial equations

\[
\sum_{i=1}^{k_0} p_i^0 (m_i^0)^{l/2-1} k_i^{l/2} = 0,
\]

where the index \( l \) is even and \( 2 \leq l \leq 2r \). In this system, at least one of \( k_i \) is non-zero.

At this point, by a contrapositive argument we immediately deduces that if system of polynomial equations (68) does not have a valid solution for the \( k_i \), one of which must be non-zero, then \( G_0 \) is not \( \kappa \)-singular relative to \( \mathcal{E}_{k_0} \). It follows that \( \ell(G_0|\mathcal{E}_{k_0}) \leq r - 1 \) and thus the singularity index \((1,1,\ell(G_0|\mathcal{E}_{k_0}) + 1)\) of \( G_0 \) relative to \( \mathcal{E}_{k_0} \) is bounded by \((1,1,r)\) according to Theorem 11.1. This connection motivates a deeper investigation into the behavior of the system of real polynomial equations (68).

**Behavior of system of limiting polynomial equations** We proceed to study the solvability of the system of polynomial equations like (68). Consider two parameter sequences \( \mathbf{a} = \{a_i\}_{i=1}^{k_0}, \mathbf{b} = \{b_i\}_{i=1}^{k_0} \) such that \( a_i > 0, b_i \neq 0 \) for all \( 1 \leq i \leq l \) and \( b_i \) are pairwise different. Additionally, there exists two indices \( 1 \leq i_1 \neq j_1 \leq l \) such that \( b_{i_1} b_{j_1} < 0 \). We can think of \( a_i \) as taking the role of \( p_i^0 \) and \( b_i \) the role of \( m_i^0 \).

Define \( \pi(k_0, \mathbf{a}, \mathbf{b}) \) to be the minimum value of \( s \geq 1 \) such that the following system of polynomial equations

\[
\sum_{i=1}^{k_0} a_i b_i^u c_i^{u+1} = 0, \text{ for } u = 0, 1, \ldots, s
\]

(69)

does not admit any non-trivial solution, by which we require that at least one of \( c_i \) is non-zero. For example, if \( s = 2 \), and \( k_0 = 2 \), the above system of polynomial equations is

\[
a_1 c_1 + a_2 c_2 = 0, \quad a_1 b_1 c_1^2 + a_2 b_2 c_2^2 = 0, \quad a_1 b_1^2 c_1^3 + a_2 b_2^2 c_2^3 = 0.
\]

In general, it is difficult to determine the exact value of \( \pi(k_0, \mathbf{a}, \mathbf{b}) \) since it depends on the specific values of parameter sequences \( \mathbf{a} \) and \( \mathbf{b} \). However, it is possible to obtain some nontrivial bounds:

**Proposition 11.1.** Let \( k_0 \geq 2 \).

(a) If for any subset \( I \) of \( \{1, 2, \ldots, k_0\} \) we have \( \sum_{i \in I} a_i \prod_{j \in I \setminus \{i\}} b_j \neq 0 \), then \( \pi(k_0, \mathbf{a}, \mathbf{b}) \leq k_0 - 1 \).

(b) If there is a subset \( I \) of \( \{1, 2, \ldots, k_0\} \) such that \( \sum_{i \in I} a_i \prod_{j \in I \setminus \{i\}} b_j = 0 \), then \( \pi(k_0, \mathbf{a}, \mathbf{b}) = \infty \).

(c) Under the same condition as that of part (a):

If \( k_0 = 2 \), then \( \pi(k_0, \mathbf{a}, \mathbf{b}) = 1 \).

If \( k_0 = 3 \), and \( \sum_{i=1}^{k_0} a_i \prod_{j \neq i, j \leq k_0} b_j > 0 \), then \( \pi(k_0, \mathbf{a}, \mathbf{b}) = 1 \). Otherwise, \( \pi(k_0, \mathbf{a}, \mathbf{b}) = 2 \).
Remarks  (i) Applying part (a) of this proposition to system \((68)\), since \(G_0 \in S_{31}\), i.e \(P_3(p^0, \eta^0) = \sum_{k_0} p^0_i \prod_{j \neq i} m^0_j \neq 0\), \(G_0\) is not \(\overline{\sigma}(k_0, \{p^0_i\}_{i=1}^{k_0}, \{m^0_i\}_{i=1}^{k_0}) + 1\)-singular relative to \(E_{k_0}\). Therefore, the singularity level of \(G_0\) is at most \(\overline{\sigma}(k_0, \{p^0_i\}_{i=1}^{k_0}, \{m^0_i\}_{i=1}^{k_0})\) and the singularity index of \(G_0\) is at most \((1, 1, \overline{\sigma}(k_0, \{p^0_i\}_{i=1}^{k_0}, \{m^0_i\}_{i=1}^{k_0}) + 1)\) according to Theorem 11.1 (ii) Part (a) provides a mild condition of parameter sequences \(a, b\) under which a nontrivial finite upper bound can be obtained. A closer investigation of the proof establishes that this bound is tight, i.e., there exists \((a, b)\) such that \(\overline{\sigma}(k_0, a, b) = k_0 - 1\) holds. This motivates the definition of \(S_{31}\). (iii) Part (b) suggests the possibility of infinite level of singularity as well as singularity index, even as \(k_0\) is fixed. We will show that this happens when \(G_0 \in S_{33}\). (iv) Part (c) suggests that the singularity levels and singularity indices of \(G_0\) may be different for different values of \((p^0, \eta^0)\) for the same \(k_0\).

General bounds for singularity level and singularity index of \(G_0 \in S_{31}\) So far, we assume that \(G_0\) has exactly one homologous set without C(1) singularity of size \(k_0\). Now, we suppose that \(G_0\) has more than one nonconformant homologous set without C(1) singularity of components, and that there are no Gaussian components (i.e., \(P_1(\eta^0) = \prod_{k_0=1}^{k_0} m^0_i \neq 0\). It can be observed that the singularity level of \(G_0\) can be bounded in terms of a number of system of polynomial equations of the same form as Eq. \((68)\), which are applied to disjoint subsets of nonconformant homologous components. The application to each subset yields a corresponding system of polynomial limits like \((66)\). If none of such systems admit non-trivial solutions, then we are absolutely certain that their corresponding systems of limiting equations cannot hold. As a consequence, we obtain that \(\ell(G_0|E_{k_0}) \leq \overline{\sigma}(G_0)\), where

\[
\overline{\sigma}(G_0) := \max_I \overline{\sigma}(|I|, \{p^0_i\}_{i \in I}, \{m^0_i\}_{i \in I}), \tag{70}
\]

where the maximum is taken over all nonconformant homologous subsets \(I\) of components of \(G_0\).

If, on the other hand, \(G_0\) has one or more Gaussian components, in addition to having some nonconformant homologous subsets, then by combining the argument presented in Section 11.2 with the foregoing argument, we deduce that the singularity level of \(G_0\) is at most \(\max\{2, \overline{\sigma}(G_0)\}\). Summarizing, combining with Theorem 11.1 we have the following theorem regarding the upper bounds of singularity levels and singularity indices of \(G_0 \in S_{31}\) whose rigorous proof is deferred to Appendix F.

**Theorem 11.2.** Suppose that \(G_0 \in S_{31}\).

(a) If \(P_1(\eta^0) \neq 0\), then \(\ell(G_0|E_{k_0}) \leq \overline{\sigma}(G_0) \leq k^* - 1 \leq k_0 - 1\) and singularity index \((1, 1, \ell(G_0|E_{k_0}) + 1) \leq (1, 1, \overline{\sigma}(G_0) + 1)\).

(b) If \(P_1(\eta^0) = 0\), then \(\ell(G_0|E_{k_0}) \leq \max\{2, \overline{\sigma}(G_0)\} \leq \max\{2, k^* - 1\} \leq \max\{2, k_0 - 1\}\) and singularity index \((3, 2, \max\{2, \ell(G_0|E_{k_0})\} + 1) \leq (3, 2, \max\{2, \overline{\sigma}(G_0)\} + 1)\).

where \(k^*\) is the maximum length among all nonconformant homologous sets without C(1) singularity of \(G_0\).

**Exact calculations in special cases** Since our proof method was to extract only an (incomplete) subset of polynomial limits, we could only speak of upper bounds of the singularity level and singularity index, not lower bounds in general. For some special cases of \(G_0 \in S_{31}\), with extra work we can determine the exact singularity level and singularity index of \(G_0\). This is based on the specific value of \(k^*\), which is defined to be the maximum length among all nonconformant homologous sets without C(1) singularity of \(G_0\) in Theorem 11.2.

**Proposition 11.2.** (Exact singularity structure) Assume that \(G_0 \in S_{31}\) and \(P_1(\eta^0) \neq 0\).
From Theorem 11.1, it is sufficient to demonstrate that \( \ell(G_0|\mathcal{E}_{k_0}) = 1 \) and \( \mathcal{L}(G_0|\mathcal{E}_{k_0}) = \{(1, 1, 2)\} \).

(b) Let \( k^* = 3 \). In addition, if all homologous sets \( I \) of \( G_0 \) such that \( |I| = k^* \) satisfy

\[
0, \text{then } \ell(G_0|\mathcal{E}_{k_0}) = 1 \text{ and } \mathcal{L}(G_0|\mathcal{E}_{k_0}) = \{(1, 1, 2)\}. \text{ Otherwise, } \ell(G_0|\mathcal{E}_{k_0}) = 2 \text{ and } \mathcal{L}(G_0|\mathcal{E}_{k_0}) = \{(1, 1, 3)\}.
\]

11.3.2 Singularity structure of \( \mathcal{S}_{32} \)

For the simplicity of the argument in this section, we go back to the simple setting of \( G_0 \), i.e., \( G_0 \) has only one homologous set of size \( k_0 \). Since \( G_0 \in \mathcal{S}_{32} \), we have \( P_\delta(p^0, \eta^0) = \sum_{i=1}^{k_0} p_i^0 \prod_{j \neq i} m_j^0 = 0 \).

This entails that \( \overline{\sigma}(k_0, \{p_i^0\}, \{m_j^0\}) = \infty \) according to part (b) of Proposition 11.1. As a result, \( \overline{\sigma}(G_0) = \infty \), i.e., the upper bound given by Theorem 11.2, that is, \( \ell(G_0|\mathcal{E}_{k_0}) \leq \overline{\sigma}(G_0) \), is no longer meaningful for \( \mathcal{S}_{32} \). This does not necessarily imply that the singularity level and singularity index for \( G_0 \in \mathcal{S}_{32} \) is infinite. It simply means that the system of polynomial equations in (68) will not lead to any contradiction for any order \( r \). In fact, these equations described by (68) are no longer sufficient to express the polynomial limits of the system (65). The issue is that our greedy extraction of polynomial limits for the system (65) treats each equation of the system separately. For instance, in system (67), a special case of system (65) when \( r = 3 \), we do not consider the interaction between two summations \( \sum_{i=1}^{k_0} q_i^0 \Delta t_i^2 \) and \( \sum_{i=1}^{k_0} \frac{1}{3} q_i^0 \Delta t_i^2 \) in the numerator of the second limit. As a result, the limiting polynomials obtained are dependent only on the lowest order monomial terms that appear in the numerator of each of the \( (r, r, r) \)-minimal form’s coefficients.

To go further with \( \mathcal{S}_{32} \), we introduce a more sophisticated technique for the polynomial limit extraction, which seeks to partially account for the interactions among different summations in the numerators of all the limits in system (65). This can be achieved by keeping not only the lowest order monomial in the numerator of the \( (r, r, r) \)-minimal form’s coefficient, but also the second lowest order monomials. As a result, we can extract a larger set of polynomial limits than (63). This would allow us to obtain a tighter bound of the singularity level and singularity index for elements of \( \mathcal{S}_{32} \). Although our extraction technique is general, the system of limiting polynomials that can be extracted is difficult to express explicitly for large values of \( k_0 \). For this reason in the following we shall illustrate this technique of polynomial limit extraction on a specific case of \( k_0 = 2 \).

**Proposition 11.3.** Assume that \( G_0 \in \mathcal{S}_{32} \) and \( G_0 \) has only one homologous set of size \( k_0 \). Then as \( k_0 = 2 \), we have \( \ell(G_0|\mathcal{E}_{k_0}) = 3 \) and \( \mathcal{L}(G_0|\mathcal{E}_{k_0}) = \{(1, 1, 4)\} \).

**Remark:** (i) The assumption that \( G_0 \) has only one homologous set is just for the convenience of the argument. The conclusion of this proposition still holds when \( G_0 \in \mathcal{S}_{32} \) has multiple homologous sets and the maximum length of homologous sets with \( C(1) \) singularity is 2. (ii) By using the same technique, we can demonstrate that \( \ell(G_0|\mathcal{E}_{k_0}) = k_0 + 1 \) and \( \mathcal{L}(G_0|\mathcal{E}_{k_0}) = \{(1, 1, k_0 + 2)\} \) when \( k_0 \leq 5 \) and \( G_0 \in \mathcal{S}_{32} \) has only one homologous set of size \( k_0 \). We conjecture that this result also holds for general \( k_0 \).

**Proof.** From Theorem 11.1 it is sufficient to demonstrate that \( \ell(G_0|\mathcal{E}_{k_0}) = 3 \). The proof proceeds in two main steps.
**Step 1:** We will demonstrate that \( G_0 \) is 3-singular and (3, 3, 3)-singular relative to \( E_{k_0} \). As \( r = 3 \), the system (65) consists of the following limiting equations, as \( q_i \to q_i^0 > 0 \) and \( \Delta t_i \to 0 \) for all \( i = 1, 2, \)

\[
\sum_{i=1}^{2} q_i \Delta t_i / \sum_{i=1}^{2} q_i |\Delta t_i|^3 \to 0,
\]

\[
\left( \sum_{i=1}^{2} q_i t_i^0 (\Delta t_i)^2 + \frac{1}{3} q_i (\Delta t_i)^3 \right) / \sum_{i=1}^{2} q_i |\Delta t_i|^3 \to 0,
\]

\[
\left( \sum_{i=1}^{2} \frac{2}{3} q_i (t_i^0)^2 (\Delta t_i)^3 \right) / \sum_{i=1}^{2} q_i |\Delta t_i|^3 \to 0,
\]

where \( q_i = p_i / \sigma_i^0, t_i^0 = \rho_i^0 / \sigma_i^0, t_i^0 = m_i^0 / \sigma_i^0, \) and \( \Delta t_i = \Delta m_i / \sigma_i^0 \) for all \( i = 1, 2 \). The condition of C(1) singularity means \( P_3(p^0, \eta^0) = 0 \). That is \( p_1^0 \rho_2^0 + p_2^0 \rho_1^0 = 0 \). So, \( q_1 t_1^0 + q_2 t_2^0 = 0 \). By choosing \( \Delta t_2 = 1/n, \Delta t_1 = \frac{1}{n} \left( -\frac{q_2}{q_1} + \frac{1}{n^2} \right) \) where \( q_1 = q_1^0 + 1/n \) and \( q_2 = -q_1 t_1^0 / t_2^0 + 1/n^2 \), we can check that all of the above limits are satisfied. Hence, \( G_0 \) is 3-singular and (3,3,3)-singular relative to \( E_{k_0} \).

**Step 2:** It remains to show that \( G_0 \) is not 4-singular and (4,4,4)-singular relative to \( E_{k_0} \), and hence, \( G_0 \)'s singularity level is 3. Let \( r = 4 \), the system (65) consists of the following limiting equations

\[
\sum_{i=1}^{2} q_i^n \Delta t_i^n / \sum_{i=1}^{2} q_i^n |\Delta t_i|^4 \to 0,
\]

\[
\left( \sum_{i=1}^{2} q_i t_i^0 (\Delta t_i)^2 + \frac{1}{3} q_i (\Delta t_i)^3 \right) / \sum_{i=1}^{2} q_i |\Delta t_i|^4 \to 0,
\]

\[
\left( \sum_{i=1}^{2} \frac{1}{3} q_i (t_i^0)^2 (\Delta t_i)^3 + \frac{1}{4} q_i t_i^0 (\Delta t_i)^4 \right) / \sum_{i=1}^{2} q_i |\Delta t_i|^4 \to 0,
\]

\[
\sum_{i=1}^{2} q_i (t_i^0)^3 (\Delta t_i)^4 / \sum_{i=1}^{2} q_i |\Delta t_i|^4 \to 0.
\]

In order to account for the second-lowest order monomials of the numerator in each of the equations, we raise the order of the denominator in each equation to the former. That is,

\[
K_1 := \sum_{i=1}^{2} q_i \Delta t_i / \sum_{i=1}^{2} q_i |\Delta t_i|^2 \to 0,
\]

\[
K_2 := \left( \sum_{i=1}^{2} q_i t_i^0 (\Delta t_i)^2 + \frac{1}{3} q_i (\Delta t_i)^3 \right) / \sum_{i=1}^{2} q_i |\Delta t_i|^3 \to 0,
\]

\[
K_3 := \left( \sum_{i=1}^{2} \frac{1}{3} q_i (t_i^0)^2 (\Delta t_i)^3 + \frac{1}{4} q_i t_i^0 (\Delta t_i)^4 \right) / \sum_{i=1}^{2} q_i |\Delta t_i|^4 \to 0,
\]

\[
K_4 := \sum_{i=1}^{2} q_i (t_i^0)^3 (\Delta t_i)^4 / \sum_{i=1}^{2} q_i |\Delta t_i|^4 \to 0.
\]

We assume without loss of generality that \( |\Delta t_2| \) is the maximum between \( |\Delta t_1| \) and \( |\Delta t_2| \). Denote \( \Delta t_1 = k_1 \Delta t_2 \) where \( k_1 \in [-1, 1] \) and \( k_1 \to k_1' \). The vanishing of \( K_1 \) yields \( q_1^0 k_1' + q_2^0 = 0 \). So, \( k_1' = -q_2^0 / q_1^0 = t_2^0 / t_1^0 \).
Divide both the numerator and denominator of $K_1$ by $(\Delta t_2)^2$, we obtain $(q_1k_1 + q_2)/\Delta t_2 \to 0$. Write $u = k_1 + q_2/q_1$, then $q_1u/\Delta t_2 \to 0$, which implies that $u/\Delta t_2 \to 0$.

Next, divide both the numerator and denominator of $K_2$ by $(\Delta t_2)^3$, we obtain

$$\left(\sum_{i=1}^{2} q_i t_0^0 (\Delta t_i)^2 + \frac{1}{3} q_i (\Delta t_i)^3 \right) / (\Delta t_2)^3 \to 0.$$  

Plug in the formula of $k_1$ and the fact that $u/\Delta t_2 \to 0$, it follows that

$$\left( q_1 t_0^0 \left( \frac{q_2}{q_1} \right)^2 + q_2 t_0^0 \right) / (\Delta t_2) \to -\frac{1}{3} (q_1^0(k_1')^3 + q_2^0).$$

Thus, we get $P_1 := (t_1^0 q_2 + t_2^0 q_1)/\Delta t_2 \to -\frac{q_1^0}{3q_2^0} (q_1^0(k_1')^3 + q_2^0)$. It is simple to verify that this limit is non-zero, otherwise we would have $q_1^0 = q_2^0$, which violates the definition that $G_0$ does not have C(2) singularity, i.e., $G_0 \in S_{32}$.

Continuing, divide both the numerator and denominator of $K_3$ by $(\Delta t_2)^4$, and with the same argument, we obtain $P_2 := (t_1^0 q_2 - t_2^0 q_1)/(t_1^0 q_2 + t_2^0 q_1) / \Delta t_2 \to -\frac{3(q_1^0)^2}{4q_2^0} (q_1^0(k_1')^4 + q_2^0)q_2^0$.

By dividing $P_2$ by $P_1$ and let it to vanish, we can extract the following polynomial in the limit:

$$4(q_1^0(k_1')^3 + q_2^0)(t_1^0 q_2 - t_2^0 q_1) = 9q_1^0(q_1^0(k_1')^4 + q_2^0).$$

By plugging in $k_1' = -q_2^0/q_1^0$ and $t_1^0 q_2 + t_2^0 q_1 = 0$, we can deduce that $q_1^0 = q_2^0$, which is a contradiction. Thus, we conclude that $G_0$ is not 4-singular and (4,4,4)-singular relative to $E_{k_0}$.

### 11.3.3 Singularity structure of $G_0 \in S_{33}$

As we can see from the proof of Proposition 11.2, the condition of without C(2) singularity plays a major role in guaranteeing that $G_0 \in S_{32}$ is not (4,4,4)-singular relative to $E_{k_0}$ when $G_0$ has only one homologous set of $k_0 = 2$. Therefore, for elements $G_0$ in $S_{33}$, we expect the singularity level and singularity index of $G_0$ may be very large. In fact, we can show that

**Theorem 11.3.** If $G_0 \in S_{33}$, then $\ell(G_0|E_{k_0}) = \infty$ and $\mathcal{L}(G_0|E_{k_0}) = \{(\infty, \infty, \infty)\}$.

**Proof.** Here, we present the proof for $k_0 = 2$. For general values of $k_0$, the proof is similar and deferred to a complete proof in Section 12. For $k_0 = 2$, the condition that $G_0 \in S_{33}$ entails $P_4(p_0, \eta_0^0) = 0$, i.e $p_1^0/\sigma_1^0 = p_2^0/\sigma_2^0$ and $m_1^0/\sigma_1^0 = -m_2^0/\sigma_2^0$. By choosing $\Delta m_1/\sigma_1^0 = -\Delta m_2/\sigma_2^0$, $p_1 = p_2 = p_1^0 = p_2^0$, we can check that

$$\sum_{i=1}^{2} p_i(m_i^0)^u(\Delta m_i)^r / (\sigma_i^0)^u + v + 1 = 0,$$

for all odd numbers $u \in [1, v]$ when $v$ is even number, or for all even numbers $u \in [0, v]$ when $v$ is odd number.

Take order $r \geq 1$ to be an arbitrary natural number and let $\kappa = (r, r, r)$. Incorporating the identity in the previous display into (62) and (63), we obtain the vanishing of all $\gamma_1^{(r)}/\hat{W}_k^r(G_1, G)$ for all $1 \leq l \leq 2r$ and $l$ is even. If we choose $\Delta \theta_i = \Delta v_i = 0$ for all $1 \leq i \leq 2$, we also have the coefficients $\beta_j^{(r)}/\hat{W}_k^r(G_1, G) = 0$ for all $1 \leq i \leq 2$ and $1 \leq j \leq 2r + 1$. Additionally, we also have $\gamma_l^{(r)}/\hat{W}_k^r(G_1, G) = 0$ for all $1 \leq l \leq 2r$ and $l$ is odd. Hence, $G_0$ is $r$-singular and $(r, r, r)$-singular relative to $E_{k_0}$ for any $r \geq 1$. As a consequence, $\ell(G_0|E_{k_0}) = \infty$. Combining with the result of Theorem 11.1 we achieve the conclusion of the theorem.
12 Remaining proofs of technical results on skew-normal e-mixtures

12.1 Proofs of statements in Appendix E

**FULL PROOF OF THEOREM 10.2** Here, we shall complete the proof of Theorem 10.2 which is the generalization of the argument in Section 10.1 for a special case of \( G_0 \). Note that, the idea of this generalization is also used to the other settings of \( G_0 \not \subset S_1 \). Now, we consider the possible existence of generic components in \( G_0 \), i.e., there are no homologous sets or symmetry components.

Let \( u_1 = 1 < u_2 < \ldots < u_{\overline{\tau}_1} \in [1, k_0 + 1] \) such that \( \left( \frac{v_i^0}{1 + (m_j^0)^2}, \theta_j^0 \right) = \left( \frac{v_i^0}{1 + (m_j^0)^2}, \theta_j^0 \right) \) and \( m_i^0 \neq 0 \) for all \( i, j \), \( l \leq u_{i+1} - 1, 1 \leq i \leq \overline{\tau}_1 - 1 \). The constraint \( m_i^0 m_j^0 > 0 \) is due to the conformant property of the homologous sets of \( G_0 \). By definition, we have \( |I_{u_i}| = u_{i+1} - u_i \) for all \( 1 \leq i \leq \overline{\tau}_1 - 1 \) where \( I_{u_i} \) denotes the set of all components homologous to component \( u_i \).

To show that \( G_0 \) is 1-singular and (1,1,1)-singular, we construct a sequence of \( G \in \mathcal{E}_{k_0} \) such that \( (p_i, \theta_i, v_i, m_i) = (\beta_i^0, \theta_i^0, v_i^0, m_i^0) \) for all \( u_2 \leq i \leq k_0 \), i.e., all the components of \( G \) and \( G_0 \) are identical from index \( u_2 \) up to \( k_0 \). Hence, in the construction of the components from index \( u_1 \) to \( u_2 - 1 \) of \( G \) we consider only the homologous set \( I_{u_1} \) of \( G_0 \). Utilizing the argument from the special case proof of Theorem 10.2 in Section 10.1, the construction of the sequence of \( G \) is specified by \( \Delta \theta_i = \Delta v_i = \Delta \eta_i = 0 \) and \( \sum_{i=1}^{u_2-1} p_i \Delta m_i / v_i^0 = 0 \). Thus \( G_0 \) is 1-singular and (1,1,1)-singular. It remains to demonstrate that \( G_0 \not \subset S_1 \) is not (1,1,2)-singular relative to \( \mathcal{E}_{k_0} \).

Indeed, let \( \kappa = (1,1,2) \) and consider any sequence \( G \in \mathcal{E}_{k_0} \rightarrow G_0 \) under \( \widetilde{W}_\kappa \) distance. Since \( \widetilde{W}_\kappa^2(G, G_0) \asymp D_\kappa(G_0, G) \) (cf. Lemma 3.3), we have the \( \kappa \)-minimal form for the sequence \( G \) as

\[
\frac{p_G(x) - p_{G_0}(x)}{\tilde{W}_\kappa^2(G, G_0)} \asymp \frac{A_1(x) + A_2(x)}{D_\kappa(G_0, G)},
\]

where \( A_1(x)/D_\kappa(G_0, G) \) and \( A_2(x)/D_\kappa(G_0, G) \) are linear combinations of the elements of the forms \( \frac{\partial |\alpha| f}{\partial \theta_1^{\alpha_1} \theta_2^{\alpha_2} m_3^{\alpha_3}}(x | \eta_1^0) \) for any \( 1 \leq i \leq k_0 \) and \( 0 \leq |\alpha| \leq 2 \). In \( A_1(x)/D_\kappa(G_0, G) \), the indices of the components range from 1 to \( s_{\overline{\tau}_1} - 1 \). In \( A_2(x)/D_\kappa(G_0, G) \), the indices of the components range from \( u_1 \) to \( k_0 \). It is convenient to think of the term \( A_1(x)/D_\kappa(G_0, G) \) as the linear combination of homologous components, and \( A_2(x)/D_\kappa(G_0, G) \) as the linear combination of generic components, i.e., no Gaussian nor homologous components.

Regarding \( A_2(x)/D_\kappa(G_0, G) \), since we have the system of partial differential equations in \( \mathcal{F}_2 \), the collection of functions in \( \left\{ \frac{\partial |\alpha| f}{\partial \theta_1^{\alpha_1} \theta_2^{\alpha_2} m_3^{\alpha_3}}(x | \eta_1^0) : |\alpha| \leq 2, 1 \leq i \leq k_0 \right\} \) are not linearly independent. Employing the same strategy described in Section 4, we obtain a reduced system of linearly independent partial derivatives in Lemma 4.3. This is the set \( \left\{ \frac{\partial |\alpha| f}{\partial \theta_1^{\alpha_1} \theta_2^{\alpha_2} m_3^{\alpha_3}}(x | \eta_1^0) : \alpha \in \mathcal{F}_2, 1 \leq i \leq k_0 \right\} \). Let \( \lambda_{1,2,3}^{(2)}(\eta_1^0)/D_\kappa(G_0, G) \) be the coefficients of the terms \( \frac{\partial |\alpha| f}{\partial \theta_1^{\alpha_1} \theta_2^{\alpha_2} m_3^{\alpha_3}}(x | \eta_1^0) \) for any \( s_{\overline{\tau}_1} \leq i \leq k_0 \) and \( \alpha \in \mathcal{F}_2 \). The formulae for \( \lambda_{1,2,3}^{(2)} \) will be given later in Case 2.

Regarding \( A_1(x)/D_\kappa(G_0, G) \), by exploiting the fact that \( \left( \frac{v_j^0}{1 + (m_j^0)^2}, \theta_j^0 \right) = \left( \frac{v_j^0}{1 + (m_j^0)^2}, \theta_j^0 \right) \) for
all $u_i \leq j, l \leq u_{i+1} - 1$, $1 \leq i \leq \tilde{t}_1 - 1$, the term $A_1(x) / D_\kappa(G_0, G)$ can be written as

$$\frac{A_1(x)}{D_\kappa(G_0, G)} = \frac{1}{D_\kappa(G_0, G)} \left( \sum_{l=1}^{\tilde{t}_1 - 1} \sum_{i=u_l}^{u_{l+1}-1} \left[ \sum_{j=1}^{5} \beta_{jil}^{(2)}(x - \theta_{u_l}^0)^{j-1} \left( \frac{x - \theta_{u_l}^0}{\sigma_i^0} \right) \Phi \left( \frac{m_i^0(x - \theta_{u_l}^0)}{\sigma_i^0} \right) \right] + \sum_{j=1}^{4} \gamma_{jil}^{(2)}(x - \theta_{u_l}^0)^{j-1} \exp \left( - \frac{m_0^u(x - \theta_{u_l}^0)^2}{2\sigma_{u_l}^0} \right) \right),$$

where $f(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})$. (This form is a general version of Eq. 54 in Section 10.1) when $\tilde{t}_1 = 2, u_1 = 1, u_2 = k_0 + 1$. The detailed formulas of $\beta_{jil}^{(2)}$ and $\gamma_{jil}^{(2)}$ for $1 \leq l \leq \tilde{t}_1 - 1, u_l \leq i \leq u_{l+1} - 1, \text{ and } 1 \leq j \leq 5$ are thus similar to that of (54). Here, we rewrite their general formulas for the transparency of subsequent arguments:

$$\begin{align*}
\beta_{2il}^{(2)} &= \frac{2\Delta p_i}{\sigma_i^0} - \frac{p_i \Delta v_i}{(\sigma_i^0)^3} - \frac{p_i (\Delta \theta_i)^2}{(\sigma_i^0)^3} + \frac{3p_i (\Delta v_i)^2}{4(\sigma_i^0)^5}, \\
\beta_{2il}^{(2)} &= \frac{2p_i \Delta \theta_i}{(\sigma_i^0)^3} - \frac{6p_i \Delta \theta_i \Delta v_i}{(\sigma_i^0)^5}, \\
\gamma_{2il}^{(2)} &= \frac{5p_j m_j^0 (\Delta v_j)^2}{8\sigma_i^0 6} = \frac{p_j m_j^0 (\Delta v_j)^2}{(\sigma_i^0)^4} + \frac{p_j \Delta m_j}{2(\sigma_i^0)^4} + \frac{p_j \Delta m_j}{(\sigma_i^0)^6}.
\end{align*}$$

where $1 \leq l \leq \tilde{t}_1 - 1$ and $u_l \leq i \leq u_{l+1} - 1$. Now, suppose that all the coefficients of $A_1(x) / D_\kappa(G_0, G)$ and $A_2(x) / D_\kappa(G_0, G)$ go to 0. It implies that $\gamma_{jil}^{(2)} / D_\kappa(G_0, G)$ ($1 \leq j \leq 4, 1 \leq l \leq \tilde{t}_1 - 1$), $\beta_{jil}^{(2)} / D_\kappa(G_0, G)$ ($1 \leq j \leq 5, u_l \leq i \leq u_{l+1} - 1, 1 \leq l \leq \tilde{t}_1 - 1$), and $\lambda_{jil}^{(2)} / D_\kappa(G_0, G)$ (for all $|\alpha| \leq 2$) go to 0. From the formation of $D_\kappa(G_0, G)$, we can find at least one index $1 \leq i^* \leq k_0$ such that $(|\Delta p_i^*| + p_i^*(|\Delta \theta_i^*| + |\Delta v_i^*| + |\Delta m_i^*|^2)) / D_\kappa(G_0, G) \neq 0$. Let

$$\tau(p_i^*, \theta_i^*, v_i^*, m_i^*) = |\Delta p_i^*| + p_i^*(|\Delta \theta_i^*| + |\Delta v_i^*| + |\Delta m_i^*|^2).$$

Now, there are two possible cases for $i^*$:
Case 1 \( u_1 \leq i^* \leq u_{11} - 1. \) Without loss of generality, we assume that \( u_1 \leq i^* \leq u_2 - 1. \) Denote
\[
d(p_{i^*}, \theta_{i^*}, v_{i^*}, m_{i^*}) = \sum_{j=u_1}^{u_2-1} |\Delta p_j| + p_j(|\Delta \theta_j| + |\Delta v_j| + |\Delta m_j|^2).
\]
Since \( \tau(p_{i^*}, \theta_{i^*}, v_{i^*}, m_{i^*})/D_\kappa(G_0, G) \neq 0, \) we have
\[
d(p_{i^*}, \theta_{i^*}, v_{i^*}, m_{i^*})/D_\kappa(G_0, G) \neq 0.
\]
Therefore, for \( 1 \leq j \leq 5 \) and \( u_1 \leq i \leq u_2 - 1, \) \( D_j := \frac{\alpha_{ji1}}{d(p_{i^*}, \theta_{i^*}, v_{i^*}, m_{i^*})} \rightarrow 0. \) Now, our argument for this case is organized further into two steps:

**Step 1.1** From the vanishes of \( D_2 \) and \( D_3, \) we obtain
\[
p_i \Delta \theta_i/d(p_{i^*}, \theta_{i^*}, v_{i^*}, m_{i^*}) \to 0, \text{ and } p_i \Delta v_i/d(p_{i^*}, \theta_{i^*}, v_{i^*}, m_{i^*}) \to 0
\]
for all \( u_1 \leq i \leq u_2 - 1. \) Combining this result with \( D_1 \to 0, \) we achieve for all \( u_1 \leq i \leq u_2 - 1 \) that
\[
\Delta p_i/d(p_{i^*}, \theta_{i^*}, v_{i^*}, m_{i^*}) \to 0.
\]
These results eventually show that
\[
U := \left( \sum_{j=u_1}^{u_2-1} p_j(\Delta m_j)^2 \right)/d(p_{i^*}, \theta_{i^*}, v_{i^*}, m_{i^*}) \neq 0.
\]

**Step 1.2** Since \( p_i \Delta \theta_i/d(p_{i^*}, \theta_{i^*}, v_{i^*}, m_{i^*}) \to 0 \) and \( p_i \Delta v_i/d(p_{i^*}, \theta_{i^*}, v_{i^*}, m_{i^*}) \to 0, \) by using the result that \( \gamma_{41}^{(2)}/d(p_{i^*}, \theta_{i^*}, v_{i^*}, m_{i^*}) \to 0, \) we have
\[
V := \left[ \sum_{j=u_1}^{u_2-1} \frac{p_j m_j^0(\Delta m_j)^2}{(\sigma_j^0)^4} \right]/d(p_{i^*}, \theta_{i^*}, v_{i^*}, m_{i^*}) \to 0.
\]
As \( U \neq 0, \) we obtain
\[
V/U = \left[ \sum_{j=u_1}^{u_2-1} \frac{p_j m_j^0(\Delta m_j)^2}{(\sigma_j^0)^4} \right]/ \sum_{j=u_1}^{u_2-1} p_j(\Delta m_j)^2 \to 0.
\] (71)
Since \( m_i^0 m_j^0 > 0 \) for all \( u_1 \leq i, j \leq u_2 - 1, \) without loss of generality we assume that \( m_j^0 > 0 \) for all \( s_1 \leq j \leq s_2 - 1. \) However, it implies that
\[
\left[ \sum_{j=u_1}^{u_2-1} \frac{p_j m_j^0(\Delta m_j)^2}{(\sigma_j^0)^4} \right]/ \sum_{j=u_1}^{u_2-1} p_j(\Delta m_j)^2 \geq m_{\min} \sum_{j=u_1}^{u_2-1} p_j(\Delta m_j)^2 / \sum_{j=u_1}^{u_2-1} p_j(\Delta m_j)^2,
\] (72)
where \( m_{\min} := \min_{u_1 \leq j \leq u_2 - 1} \left\{ \frac{m_j^0}{(\sigma_j^0)^4} \right\}. \) Combining with (71), \( m_{\min} = 0 \) — a contradiction. In sum, Case 1 cannot happen.
Case 2 $u_{i_1}^* \leq i^* \leq k_0$. We can write down the formation of $A_2(x)/D_{k_0}(G_0, G)$ as follows

$$\frac{A_2(x)}{D_{k_0}(G_0, G)} = \frac{1}{D_{k_0}(G_0, G)} \left( \sum_{i=\alpha \in F_2} \sum_{i=\alpha}^{k_0} \lambda^{(2)}_{\alpha, \alpha_2, \alpha_3}(\eta_i^0) \frac{\partial f}{\partial \theta_{\alpha_1, \alpha_2, \alpha_3}}(x|\eta_i^0) \right),$$

where $\lambda^{(2)}_{\alpha, \alpha_2, \alpha_3}(\eta_i^0)$ are given by

$$\lambda^{(2)}_{0,0,0}(\eta_i^0) = \Delta \lambda_i, \quad \lambda^{(2)}_{1,0,0}(\eta_i^0) = \Delta \lambda_i \eta_i, \quad \lambda^{(2)}_{0,1,0}(\eta_i^0) = \Delta \lambda_i x^2 \frac{1}{i^2} \eta_i \eta_i - \Delta \lambda_i x \eta_i \eta_i - \Delta \lambda_i \eta_i \eta_i + \frac{1}{i^2} \eta_i \eta_i - \Delta \lambda_i \eta_i \eta_i,$$

and

$$\lambda^{(2)}_{0,2,0}(\eta_i^0) = \Delta (\Delta \lambda_i)^2, \quad \lambda^{(2)}_{0,0,2}(\eta_i^0) = \Delta (\Delta \lambda_i)^2 + \frac{1}{2} \Delta \lambda_i x \eta_i \eta_i + \Delta \lambda_i \eta_i \eta_i,$$

and

$$\lambda^{(2)}_{1,1,0}(\eta_i^0) = \Delta \lambda_i \eta_i \eta_i$$

From the assumption with the coefficients of $A_2(x)/D_{k_0}(G_0, G)$, we have $\lambda^{(2)}_{\alpha, \alpha_2, \alpha_3}(\eta_i^0)/D_{k_0}(G_0, G) \to 0$ for any $u_{i_1}^* \leq i \leq k_0$. From the hypothesis with $i^*$, we have $\tau(p \alpha, \theta_i, v^*, m^*)/D_{k_0}(G_0, G) \not\to 0$.

Therefore, it leads to $\lambda^{(2)}_{\alpha, \alpha_2, \alpha_3}(\eta_i^0)/\tau(p \alpha, \theta_i, v^*, m^*)$ for any $u_{i_1}^* \leq i \leq k_0$ and $\alpha \in F_2$.

Now, since $\lambda^{(2)}_{0,1,0}(\eta_i^0)/\tau(p \alpha, \theta_i, v^*, m^*) \to 0$, we obtain $\Delta \lambda_i \eta_i / \tau(p \alpha, \theta_i, v^*, m^*) \to 0$. Combining this result with $\lambda^{(2)}_{0,0,1}(\eta_i^0)/\tau(p \alpha, \theta_i, v^*, m^*) \to 0$, we have $\Delta \lambda_i \eta_i / \tau(p \alpha, \theta_i, v^*, m^*) \to 0$. Furthermore, as $\lambda^{(2)}_{0,0,1}(\eta_i^0)/\tau(p \alpha, \theta_i, v^*, m^*) \to 0$, we get $\Delta \lambda_i \eta_i / \tau(p \alpha, \theta_i, v^*, m^*) \to 0$. Hence, since $\lambda^{(2)}_{0,0,0}(\eta_i^0)/\tau(p \alpha, \theta_i, v^*, m^*) \to 0$, we ultimately obtain

$$1 = \frac{|\Delta \lambda_i | + p \alpha(|\Delta \lambda_i | + |\Delta \lambda_i | + |\Delta \lambda_i | + |\Delta \lambda_i |)}{\tau(p \alpha, \theta_i, v^*, m^*)} \to 0,$$

which is a contradiction. As a consequence, Case 2 cannot happen.

Summarizing, not all the coefficients $\gamma^{(2)}_{j, l} / D_{k_0}(G_0, G)$ (1 \leq j \leq 4, 1 \leq l \leq \bar{t}_1 - 1), $\beta^{(2)}_{j, l} / D_{k_0}(G_0, G)$ (1 \leq j \leq 5, u \leq i \leq u + 1 - 1, 1 \leq l \leq \bar{t}_1 - 1), $\lambda^{(2)}_{\alpha, \alpha_2, \alpha_3}(\eta_i^0)/D_{k_0}(G_0, G)$ (for all $\alpha \in F_2$) go to 0. From Definition 3.3, $G_0$ is not (1,1,2)-singular relative to $E_{k_0}$. This concludes our proof.

**FULL PROOF OF THEOREM 10.3** Here, we address the general setting of $G_0 \in S_2$, which accounts for the possible presence of both generic components and conformant homologous sets. The proof idea is the generalization of the argument in Section 11.2 for a special case of $G_0$. Without loss of generality, we assume that $m^0_1, m^0_2, \ldots, m^0_{\bar{t}_2} = 0$ where $1 \leq \bar{t}_2 \leq k_0$ denotes the largest index $i$ such that $m^0_i = 0$. The remaining components are either conformant homologous sets or generic components. Using the exact same constructions as that of Section 11.2 we establish easily that $G_0$ is $(r, 1, r)$-singular, $(2, r, r)$-singular, and $(r, r, 2)$-singular relative to $E_{k_0}$ for any $r \geq 1$. It remains to show that $G_0$ is not (3,2,3)-singular relative to $E_{k_0}$.

Let $\kappa = (3,2,3)$. Consider the $\kappa$-minimal form for any sequence $G \in E_{k_0} \to G_0$ under $W_\kappa$ distance. Since $W_\kappa^3(G, G_0) \propto D_{\kappa}(G_0, G)$ (cf. Lemma 3.3), we have

$$\frac{p_G(x) - p_{G_0}(x)}{W_\kappa^3(G, G_0)} \approx \frac{A'_1(x) + A'_2(x)}{D_{\kappa}(G_0, G)}.$$

where $A'_1(x)/D_{\kappa}(G_0, G)$ is the linear combination of Gaussian components, i.e., the indices of components range from 1 to $\bar{t}_2$, while $A'_2(x)/D_{\kappa}(G_0, G)$ is the linear combination of conformant homologous components and generic components.
Suppose that all the coefficients of $A'_1(x)/D_\kappa(G_0,G), A'_2(x)/D_\kappa(G_0,G)$ go to 0. Similar to the argument in the proof of Theorem 10.2 observe that there is some index $i \in [1, k_0]$ such that $(|\Delta p_2| + p_2(|\Delta \theta_1|^3 + |\Delta v_2|^2 + |\Delta m_3|^3))/D_\kappa(G_0,G) \neq 0$. There are two possible cases regarding $i$.

**Case 2.1** $i \in [1, \bar{t}_2]$. Applying a similar argument as that from Section 11.2 where we have only Gaussian components, we conclude that not all of the coefficients of $A'_1(x)/D_\kappa(G_0,G)$ vanish, which is a contradiction. Therefore, Case 2.1 cannot happen.

**Case 2.2** $i \in [\bar{t}_2 + 1, k_0]$. Define

$$D_{\kappa', new}(G_0,G) = \sum_{i=\bar{t}_2+1}^{k_0} (|\Delta p_i| + p_i(|\Delta \theta_i|^{\kappa_1} + |\Delta v_i|^{\kappa_2} + |\Delta m_i|^{\kappa_3})),$$

for any $\kappa' \in \{(1,1,2), (3,2,3)\}$. The idea of $D_{\kappa', new}(G_0,G)$ is that we truncate the value of $D_{\kappa'}(G_0,G)$ from the index 1 to $\bar{t}_2$, i.e., all the indices correspond to Gaussian components.

Let $\kappa_1 = (1,1,2)$. Since $\kappa = (3,2,3) > \kappa_1$, it is clear that $D_{\kappa, new}(G_0,G) \leq D_{\kappa_1, new}(G_0,G)$. Since $D_{\kappa, new}(G_0,G)/D_\kappa(G_0,G) \neq 0$, we have $D_{\kappa_1, new}(G_0,G)/D_{\kappa_1}(G_0,G) \neq 0$. By multiplying all the coefficients of $A'_2(x)/D_{\kappa_1}(G_0,G)$ with $D_{\kappa}(G_0,G)/D_{\kappa_1, new}(G_0,G)$, we eventually obtain all the coefficients of $A'_2(x)/D_{\kappa_1, new}(G_0,G)$ go to 0. However, by utilizing the same argument as in the proof of Theorem 10.2 we reach to the conclusion that the second order Taylor expansion is sufficient to have all the coefficients of $A'_2(x)/D_{\kappa_1, new}(G_0,G)$ not vanish. Thus, not all the coefficients of $A'_2(x)/D_{\kappa_1}(G_0,G)$ go to 0, which is a contradiction. As a consequence, Case 2.2 also cannot happen.

In sum, under no circumstance can all the coefficients of $A'_1(x)/D_\kappa(G_0,G)$ and $A'_2(x)/D_\kappa(G_0,G)$ be made to vanish. Hence, $G_0 \in S_2$ is not (3,2,3)-singular relative to $E_{k_0}$. As a consequence, (3,2,3) is the unique singularity index of $G_0$ relative to $E_{k_0}$, which concludes the proof.

### 12.2 Proofs of statements in Appendix F

**PROOF OF PROPOSITION 11.1** (a) The proof proceeds by induction on $l$. When $l = 1$, the conclusion clearly holds. Assume that that conclusion of the proposition holds for $l - 1$. We will demonstrate that it also holds for $l$. Denote $y_i = a_i c_i$ and $z_i = b_i c_i$ for all $1 \leq i \leq l + 1$. Then, we can rewrite system of polynomial equations (69) as follows:

$$\sum_{i=1}^{l+1} z^u_i y_i = 0$$

for any $0 \leq u \leq l$. If there exists some $1 \leq i_1 \leq l + 1$ such that $c_{i_1} = 0$, then we go back to the case $l - 1$, which we have already known from the hypothesis that we do not have non-trivial solution. Therefore, we assume that $c_i \neq 0$ for all $1 \leq i \leq l + 1$, which implies that $y_i \neq 0$ for all $1 \leq i \leq l + 1$. Now, the system of equations has the form of Vandermonde matrix, which is

$$\begin{bmatrix}
1 & 1 & \ldots & 1 \\
z_1 & z_2 & \ldots & z_{l+1} \\
\vdots & \vdots & \ddots & \vdots \\
z_1^s & z_2^s & \ldots & z_{l+1}^s
\end{bmatrix}.$$  

By suitable linear transformations, we can rewrite the original system of equations as the following equivalent equations

$$\prod_{j \neq i} (z_j - z_i) y_i = 0$$

for all $1 \leq i \leq l + 1$. Since $y_i \neq 0$ for all $1 \leq i \leq l + 1$, we obtain

$$\prod_{j \neq i} (z_j - z_i) = 0$$

for all $1 \leq i \leq l + 1$. As a consequence, there exists a partition $J_1, J_2, \ldots, J_s$ of $\{1,2,\ldots,l+1\}$ for some $1 \leq s \leq \lceil l/2 \rceil$ such that if $i_2, i_3 \in J_u$ for $1 \leq u \leq s$, we have $z_{i_2} = z_{i_3}$ and for any $1 \leq i \neq j \leq s$, any two elements $z_{i_4} \in J_i$, $z_{j_4} \in J_j$ are different. Choose any $j_i \in J_i$ for all $1 \leq i \leq s$. It is clear.
that the system of equations can be rewritten as \( \sum_{i=1}^{s} z_i^u \sum_{j \in J_i} y_j = 0 \) for all \( 0 \leq u \leq l + 1 \). If \( s \geq 2 \), it indicates that \( |J_i| \leq l \) for all \( 1 \leq i \leq s \). Now, if we have some \( 1 \leq i_4 \leq s \) such that \( \sum_{j \in J_{i_4}} y_j = 0 \) then we obtain \( \sum_{j \in J_{i_4}} a_j c_j = 0 \). Since \( z_{i_4} = z_{i_4} \) for any \( i_1, i_2 \in J_{i_4} \), this equation can be rewritten as 

\[
\sum_{j \in J_{i_4}} a_j \prod_{v \neq j} b_v = 0,
\]

which is a contradiction to the assumption of part (a) of the proposition. Therefore, 
\( \sum_{j \in J_i} y_j \neq 0 \) for all \( 1 \leq i \leq s \). However, by using the same argument as before, again by linear transformation, we can rewrite the new system of polynomial equations as 
\( \sum_{j \in J_i} y_j \prod_{v \neq i} (z_{ju} - z_{jv}) = 0 \) for all \( 1 \leq i \leq s \). This implies that there should be some \( 1 \leq u_1 \neq u_2 \leq s \) such that \( z_{j_{u_1}} = z_{j_{u_2}} \), which is a contradiction.

As a consequence, we have \( s = 1 \), i.e., \( |I_1| = l + 1 \). Hence, \( b_1 c_1 = b_2 c_2 = \ldots = b_{l+1} c_{l+1} \).

Combining this fact with the equation \( \sum_{i=1}^{l+1} a_i c_i = 0 \), we obtain \( \sum_{i=1}^{l+1} a_i \prod_{j \neq i} b_j = 0 \), which is a contradiction to the assumption of the proposition. This concludes the proof.

(b) We choose \( c_i = 0 \) for all \( i \notin I \subseteq \{1, \ldots, l\} \). The system of polynomial equations (69) becomes \( \sum_{i \in I} a_i b_i c_i^{u+1} = 0 \) for all \( u \geq 0 \). Notice that by choosing \( b_i c_i = b_j c_j \) for all \( i, j \in I \), we have 
\[
\sum_{i \in I} a_i b_i^u c_i^{u+1} = b_j c_j \sum_{i \in I} a_i c_i = 0 \quad \text{for some } j \in I \text{ and for all } u \geq 1 \text{ as long as } \sum_{i \in I} a_i c_i = 0.
\]

Combining all the conditions, we obtain \( \sum_{i \in I} a_i \prod_{j \neq i} b_j = 0 \), which completes the proof.

(c) The result for the case \( l = 1 \) is obvious. For the case \( l = 2 \), after replacing \( c_3 \) in terms of \( c_1, c_2 \), we obtain the following quadratic equation \( (a_1 a_3 b_1 + a_2^2 b_3) c_1^2 + a_1 a_2 b_3 c_1 + 2a_1 a_2 b_3 c_1 c_2 + (a_2 a_3 b_2 + a_2^2 b_3) c_2^2 = 0 \). Note that, \( c_1, c_2 \neq 0 \) due to the assumption of part (c). Therefore, we do not have solution of this quadratic equation when \( a_1^2 + 2a_2^2 b_3^2 < (a_1 a_3 b_1 + a_2^2 b_3)(a_2 a_3 b_2 + a_2^2 b_3) \). It is equivalent to \( \sum_{i=1}^{3} a_i \prod_{j \neq i} b_j > 0 \), which confirms our hypothesis. We are done.

**FULL PROOF OF THEOREM 11.1** Here, we only provide the proof for part (b) as the proofs for part (a) and part (c) are similar. This is a generalization of the argument in Section 11.3. Under this situation, apart from the nonconformant homologous sets, we also have in \( G_0 \) the presence of Gaussian components components and possibly some conformant homologous sets, in addition to some generic components.

Let \( u_1 = 1 < u_2 < \ldots < u_{t_3} \in [1, k_0 + 1] \) such that \( (\frac{v_j^0}{1 + (m_j^0)^2}, \theta_j^0) = (\frac{v_j^0}{1 + (m_j^0)^2}, \theta_j^0) \) for all \( u_i \leq j, l \leq u_{i+1} - 1, 1 \leq i \leq t_3 - 1 \), i.e., all the nonconformant homologous components are from index 1 to \( u_{t_3} \). The remaining components are either Gaussian ones or conformant homologous sets or generic ones. It follows that \( |I_{u_i}| = u_{i+1} - u_i \) for all \( 1 \leq i \leq t_3 - 1 \) and all \( I_{u_i} \) are nonconformant homologous sets. We divide the argument for our proof into two main steps

**Step 1** \( G_0 \) is not \( (3, 2, \max \{2, \ell(G_0) \kappa_0 \}) + 1 \)-singular relative to \( E_0 \). In fact, for any \( \kappa = (\kappa_1, \kappa_2, \kappa_3) \in \mathbb{N}^3 \) such that \( \|\kappa\|_\infty = r \) and \( \kappa_3 = r \) where \( r \geq 1 \), consider the \( \kappa \)-minimal form for any sequence \( G \in E_0 \to G_0 \) under \( \tilde{W}_\kappa \) distance. Since \( \tilde{W}_\kappa(G, G_0) \simeq D_\kappa(G_0, G) \) (cf. Lemma 3.3), we have 
\[
\frac{p_G(x) - p_{G_0}(x)}{W_\kappa^*(G, G_0)} \leq B_1(x) + B_2(x) \frac{D_\kappa(G_0, G)}{D_\kappa(G_0, G)},
\]

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where \( B_1(x)/D_(\kappa)(G_0, G) \) is the linear combination of nonconformant homologous components, i.e., the indices of components range from 1 to \( \tilde{\tau}_3 \) while \( B_2(x)/D_(\kappa)(G_0, G) \) is the linear combination of conformant homologous components, Gaussian components, and generic components.

Now, suppose that all the coefficients of \( B_1(x)/D_(\kappa)(G_0, G) \), \( B_2(x)/D_(\kappa)(G_0, G) \) go to 0. Similar to the argument employed in the proof of Theorem 10.2, there is some index \( i \in [1, k_0] \) such that \( (|\Delta p_i| + p_i(|\Delta \theta_i|\kappa^1 + |\Delta v_i|\kappa^2 + |\Delta m_i|\kappa^3))/D_(\kappa)(G_0, G) \not\rightarrow 0 \). Now, there are two possible scenarios regarding \( i \).

**Case 1.1** \( i \in [1, u_{\tilde{\tau}_3} - 1] \). Under that case, we can check that
\[
\frac{B_1(x)}{D_(\kappa)(G_0, G)} = \frac{1}{D_(\kappa)(G_0, G)} \left( \sum_{i=1}^{\tilde{\tau}_3} \sum_{i=u_i}^{u_{i+1}-1} \left[ \sum_{j=1}^{2r+1} \beta_{ij}(x - \theta_{\sigma_i^0})^{j-1} \right] f(x - \theta_{\sigma_i^0}) \Phi \left( \frac{m_i(x - \theta_{\sigma_i^0})}{\sigma_i^0} \right) \right) + \\
\left[ \sum_{j=1}^{2r} \gamma_{ij}(x - \theta_{\sigma_i^0})^{j-1} \right] \exp \left( -\frac{(m_{\sigma_i^0})^2 + 1}{2\sigma_{\sigma_i^0}^2} (x - \theta_{\sigma_i^0})^2 \right).
\]

This representation of \( B_1(x)/D_(\kappa)(G_0, G) \) is the general formulation of the equation [6.2] in Section 11.3 where \( i_3 = 2, u_1 = 1 \), and \( u_2 = k_0 + 1 \). Since \( i \in [1, u_{\tilde{\tau}_3} - 1] \), there exists some index \( l^* \in [1, \tilde{\tau}_3 - 1] \) such that \( i \in [u_{l^*}, u_{l^*+1} - 1] \). By means of the same argument as that of Section 11.3 for \( \beta_{ij}/D_(\kappa)(G_0, G) \rightarrow 0 \) and \( \gamma_{ij}/D_(\kappa)(G_0, G) \rightarrow 0 \), we can extract the following system of limits
\[
\left( \sum_{i_1 = i_2}^{u_{l^*+1} - 1} \sum_{i=u_{l^*}}^{u_{l^*+1}-1} \frac{p_i(m_i)^{i_1-2i_2-1}(\Delta m_i)^{i_1}}{(\sigma_i^0)^{l^*}} \right) \sum_{i=u_{l^*}}^{u_{l^*+1}-1} p_i |\Delta m_i|^{\kappa^3} \rightarrow 0,
\]
for any even \( l \) such that \( 1 \leq l \leq 2r \) where \( q_{l,j} \) are the integer coefficients that appear in the high order derivatives of \( f(x|\sigma, m) \) with respect to \( m \) defined as above equation [6.3] and \( 1 \leq i_1 \leq r \), \( i_2 = (i_1 - 1)/2 \) as \( i_1 \) is odd or \( i_2 = i_1/2 - 1 \) as \( i_1 \) is even. Let \( r_{\text{max}} \) to be maximum number that the above system of limits holds for any \( l^* \in [1, \tilde{\tau}_3 - 1] \). It would indicate that \( r = r_{\text{max}} \) is the best possible value that \( G_0 \) is \((\kappa_1, \kappa_2, r)\)-singular relative to \( E_{\kappa_0} \) for any \( \kappa_1, \kappa_2 \leq r \). Hence, from the definition of singularity level, we would have \( \ell(G_0|E_{\kappa_0}) = r_{\text{max}} \). If we choose \( \kappa = (3, 2, \max \{2, \ell(G_0|E_{\kappa_0})\} + 1) \), then as \( \kappa_3 = \max \{2, \ell(G_0|E_{\kappa_0})\} + 1 \geq \ell(G_0|E_{\kappa_0}) + 1 \) we achieve that \( G_0 \) is not \( \kappa \)-singular relative to \( E_{\kappa_0} \).

**Case 1.2** \( i \in [u_{\tilde{\tau}_3}, k_0]. \) Using the same argument as that in the proof of Theorem 10.3, the third order Taylor expansion is sufficient so that not all the coefficients of \( B_2(x)/D_(\kappa', G_0, G) \) go to 0 where \( \kappa' = (3, 2, 3) \) and
\[
D_(\kappa', G_0, G) = \sum_{i=u_{\tilde{\tau}_3}}^{k_0} (|\Delta p_i| + p_i(|\Delta \theta_i|\kappa^1 + |\Delta v_i|\kappa^2 + |\Delta m_i|\kappa^3)).
\]
If we choose \( \kappa = (3, 2, \max \{2, \ell(G_0|E_{\kappa_0})\} + 1) \), then we have \( \kappa' = (3, 2, 3) \not< \kappa \), which leads to \( D_(\kappa', G_0, G)/D_(\kappa)(G_0, G) \not\rightarrow 0 \). As all the coefficients of \( B_2(x)/D_(\kappa)(G_0, G) \) vanish, it leads to all the coefficients of \( B_2(x)/D_(\kappa', G_0, G) \) go to 0, which is a contradiction. Thus, Case 1.2 cannot happen.

In summary, if we choose \( \kappa = (3, 2, \max \{2, \ell(G_0|E_{\kappa_0})\} + 1) \), then not all the coefficients of \( B_1(x)/D_(\kappa)(G_0, G), B_2(x)/D_(\kappa)(G_0, G) \) go to 0. Therefore, \( G_0 \) is not \((3, 2, 3)\)-singular relative to \( E_{\kappa_0} \),
**Step 2** To demonstrate that \((3, 2, \max \{2, \ell(G_0|E_{k_0})\} + 1)\) is the unique singularity index of \(G_0\), we need to verify that \(G_0\) is \((r, r, \max \{2, \ell(G_0|E_{k_0})\})\)-singular, \((2, r, r)\)-singular, and \((r, 1, r)\)-singular relative to \(E_{k_0}\) for any \(r \geq 1\). The later two results are straightforward from the fact that there is at least one Gaussian component in \(G_0\). In particular, by choosing sequence \(G\) such that all the masses of \(G\) and \(G_0\) are identical while all the atoms of \(G\) and \(G_0\) are identical except for one Gaussian component of \(G_0\). With that Gaussian component of \(G_0\), we choose the corresponding component of \(G\) similar to that in Section **11.2** According to this choice of \(G\), we can easily check that \(G_0\) is \((2, r, r)\)-singular and \((r, 1, r)\)-singular relative to \(E_{k_0}\). Finally, to demonstrate that \(G_0\) is \((r, r, \max \{2, \ell(G_0|E_{k_0})\})\), it comes directly from our analysis in Case 1.1 with the definition of \(r_{\max}\), which guarantees the existence of \(G\) to make all of these systems of limits vanish. Therefore, we achieve the conclusion of this step.

As a consequence, \((3, 2, \max \{2, \ell(G_0|E_{k_0})\} + 1)\) is the unique singularity index of \(G_0\), i.e., \(L(G_0|E_{k_0}) = \{(3, 2, \max \{2, \ell(G_0|E_{k_0})\} + 1)\}. We achieve the conclusion of the theorem.

**FULL PROOF OF THEOREM** **11.2** Here, we only provide the proof for part (b) as the proof for part (a) is similar. This is a generalization of the argument in Section **11.3** and similar to the proof argument of that of Theorem **11.1** Under this situation, apart from the nonconformant homologous sets without \(C(1)\) singularity, we also have for \(G_0\) the presence of Gaussian components and possibly some conformant homologous sets, in addition to some generic components.

Let \(u_1 = 1 < u_2 < \ldots < u_{\gamma_3} \in [1, k_0 + 1]\) such that \((\frac{v_j^0}{1 + (m_j^0)^2}, \theta_j^0) = (\frac{v_l^0}{1 + (m_l^0)^2}, \theta_l^0)\) for all \(u_i \leq j, l \leq u_{i+1} - 1, 1 \leq i \leq \gamma_3 - 1\), i.e., all the nonconformant homologous components without type \(C(1)\) singularity are from index 1 to \(u_{\gamma_3}\). The remaining components are either Gaussian ones or conformant homologous sets or generic ones. It follows that \(|I_{u_i}| = u_{i+1} - u_i\) for all \(1 \leq i \leq \gamma_3 - 1\) and all \(I_{u_i}\) are nonconformant homologous sets without \(C(1)\) singularity.

From the result of Theorem **11.1** it is sufficient to focus on obtaining the upper bound on the singularity level of \(G_0\). Let \(\kappa = (\mathfrak{r}, \mathfrak{r}, \mathfrak{r})\) where \(\mathfrak{r} = \max \{3, \mathfrak{r}(G_0) + 1\}\). Consider the \(\mathfrak{r}\)-minimal and \(\kappa\)-th minimal form for any sequence \(G \in E_{k_0} \to G_0\) under \(\tilde{W}_\kappa\) distance. Since \(\tilde{W}_\kappa(G_0, G) \asymp D_\kappa(G_0, G)\) (cf. Lemma **3.3**), we have

\[
\frac{p_G(x) - p_{G_0}(x)}{\tilde{W}_\kappa(G_0, G)} \asymp \frac{B_1(x) + B_2(x)}{D_\kappa(G_0, G)},
\]

where \(B_1(x)/D_\kappa(G_0, G)\) is the linear combination of nonconformant homologous components, i.e., the indices of components range from 1 to \(\gamma_3\) while \(B_2(x)/D_\kappa(G_0, G)\) is the linear combination of conformant homologous components, Gaussian components, and generic components.

Now, suppose that all the coefficients of \(B_1(x)/D_\kappa(G_0, G), B_2(x)/D_\kappa(G_0, G)\) go to 0. Similar to the argument employed in the proof of Theorem **10.2** there is some index \(l \in [1, k_0]\) such that \((|\Delta p_{l}| + p_{l}(|\Delta \theta_{l}|^\mathfrak{r} + |\Delta v_{l}|^\mathfrak{r} + |\Delta m_{l}|^\mathfrak{r})/D_\kappa(G_0, G) \neq 0\). Now, there are two possible scenarios regarding \(l\).

**Case 1.1** \(l \in [1, u_{\gamma_3} - 1]\). Similar to Case 1.1 in the proof of Theorem **11.1** we can check that

\[
\frac{B_1(x)}{D_\kappa(G_0, G)} = \frac{1}{D_\kappa(G_0, G)} \left( \sum_{l=1}^{\gamma_3-1} \sum_{i=u_l}^{u_{l+1}-1} \left[ \sum_{j=1}^{2\mathfrak{r}} \beta_{ij}^l (x - \theta_{u_l}^0, \gamma_{j1}^l, \sigma_{ij}^0) \left( x - \theta_{u_l}^0 \right)^{\gamma_{j1}^l} \right] f \left( x - \theta_{u_l}^0, \sigma_{ij}^0 \right) \right) + \left[ \sum_{j=1}^{2\mathfrak{r}} \gamma_{ji}^{(\mathfrak{r})} (x - \theta_{u_l}^0, \gamma_{j1}^l, \sigma_{ij}^0) \left( x - \theta_{u_l}^0 \right)^{\gamma_{j1}^l} \left( x - \theta_{u_l}^0 \right)^{\gamma_{j1}^l} \right] \exp \left( - \frac{(m_{ij}^0)^2 + 1}{2v_{ij}^0} (x - \theta_{u_l}^0)^2 \right).
\]
Since \( l \in [1, \lfloor \frac{3}{2} \rfloor - 1 \), there exists some index \( l^* \in [1, \lfloor \frac{3}{2} \rfloor - 1 \) such that \( l^* \in [u_{l^*}, u_{l^*+1} - 1] \). By means of the same argument as that of Section 11.3 for \( \beta_{ij}^{(T)} / D_\kappa(G_0, G) \to 0 \) and \( \gamma_{ij}^{(T)} / D_\kappa(G_0, G) \to 0 \), we can extract the following system of polynomial limits:

\[
\sum_{i=\lfloor u_{l^*} \rfloor}^{\lfloor u_{l^*+1} \rfloor - 1} p_i^0 (m_i^0)^{1/2-1} (k_i)^{1/2} = 0,
\]

where at least one of \( k_i \) differs from 0. Here, \( l \) is any even number such that \( 2 \leq l \leq 2\tau \). From the formulation of \( \overline{\sigma}(G_0) \) since \( \overline{\sigma}(G_0) + 1 \geq \overline{\sigma}(\mathbb{I}_{u_1}, \{ r_i \}, \{ m_i^0 \}) + 1 \), we can guarantee that the above system of polynomial equations does not have any non-trivial solution, which is a contradiction. Therefore, Case 1.1 cannot happen.

**Case 1.2** \( l \in [u_{r_3}, k_0] \). Using the same argument as that in the proof of Theorem 11.3, the third order Taylor expansion is sufficient so that not all the coefficients of \( B_2(x) / D_{\kappa',new}(G_0, G) \) go to 0 where \( \kappa' = (3, 2, 3) \) and

\[
D_{\kappa',new}(G_0, G) = \sum_{i=\lfloor u_{r_3} \rfloor}^{k_0} (|\Delta p_i| + p_i(|\Delta \theta_i|^2 + |\Delta v_i|^2 + |\Delta m_i|^3)).
\]

Since \( \overline{\sigma} \geq 3 \), we have \( \kappa' = (3, 2, 3) < \kappa = (\overline{\sigma}, \overline{\sigma}, \overline{\sigma}) \), which leads to \( D_{\kappa',new}(G_0, G) / D_\kappa(G_0, G) \not\to 0 \). As all the coefficients of \( B_2(x) / D_\kappa(G_0, G) \) vanish, it follows that all the coefficients of \( B_2(x) / D_{\kappa',new}(G_0, G) \) go to 0, which is a contradiction. Thus, Case 1.2 cannot happen.

In sum, for any sequence of \( G \) tending to \( G_0 \) in \( \mathcal{W}_\kappa \), not all the coefficients of \( B_1(x) / D_\kappa(G_0, G) \) and \( B_2(x) / D_\kappa(G_0, G) \) go to 0. By Definition 3.5, we conclude that \( G_0 \in S_2 \) is not \( \kappa \)-singular relative to \( \mathcal{E}_{k_0} \). As a consequence, \( \ell(G_0|\mathcal{E}_{k_0}) \leq \overline{\sigma} - 1 = \max \left\{ 2, \overline{\sigma}(G_0) \right\} \), which leads to the singularity index \( 3, 2, \max \{ 2, \ell(G_0|\mathcal{E}_{k_0}) \} + 1 \leq (3, 2, \max \{ 2, \overline{\sigma}(G_0) \} + 1) \). This concludes part (b) of the theorem.

**PROOF OF PROPOSITION 11.2** Here, we utilize the same assumption on \( G_0 \) as that in the proof of Theorem 11.2, i.e., all the nonconformant homologous sets without C(1) singularity are from index 1 to \( u_{r_3} \). We also rearrange the components of \( G \) such that the first nonconformant homologous set without C(1) singularity \( I_{u_1} \) has exactly \( k^* \) elements, i.e., \( u_2 - u_1 = k^* \). As \( u_1 = 1 \), we have \( u_2 = k^* + 1 \).

(a) We will demonstrate that \( G_0 \) is 1-singular and (1,1,1)-singular relative to \( \mathcal{E}_{k_0} \). Indeed, the sequence of \( G \) is constructed as follows: \( p_i = p_i^0, \theta_i = \theta_i^0, v_i = v_i^0 \) for all \( u_2 = k^* + 1 \leq i \leq k_0 \), i.e., we match all the components of \( G \) and \( G_0 \) except the first \( k^* \) components of \( G_0 \). Now, by proceeding in the same way as described in Section 11.3 up to Eq. (66), to verify that \( G_0 \) is indeed 1-singular and (1,1,1)-singular, the choice of the first \( k^* \) components of \( G \) needs to satisfy

\[
\sum_{i=1}^{u_2-1} q_i |\Delta t_i| / \sum_{i=1}^{u_2-1} q_i |\Delta t_i| \to 0,
\]

where \( q_i = p_i / \sigma_i^0 \) and \( \Delta t_i = \Delta m_i / \sigma_i^0 \) as \( u_1 \leq i \leq u_2 - 1 \). A simple choice is to take the first \( k^* \) components of \( G \) by \( \sum_{i=1}^{u_2-1} q_i \Delta t_i = q_1 \Delta t_1 + q_2 \Delta t_2 = 0 \), which is always possible. We conclude that \( G_0 \) is 1-singular and (1,1,1)-singular relative to \( \mathcal{E}_{k_0} \). Since \( \overline{\sigma}(G_0) = 1 \) as \( k^* = 2 \), by combining with the upper bound of Theorem 11.2, we have \( \ell(G_0|\mathcal{E}_{k_0}) = 1 \). According to the result of Theorem 11.1 we achieve \( \mathcal{L}(G_0|\mathcal{E}_{k_0}) = \{(1, 1, 2)\} \).

(b) There are two cases to consider in this part
**Case 1:** All the homologous sets \( I \) of \( G_0 \) such that \(|I| = k^*\) satisfy \( \sum_{i \in I} p_i^0 \prod_{j \in I \setminus \{i\}} m_j^0 > 0 \). To demonstrate that \( G_0 \) is 1-singular and \((1,1,1)\)-singular relative to \( \mathcal{E}_{k_0} \), we utilize the same construction of \( G \) as that of part (a), i.e., \( p_i = p_i^0, \theta_i = \theta_i^0, v_i = v_i^0 \) for all \( u_2 = k^* + 1 \leq i \leq k_0 \) and \( \sum_{i = u_1}^{u_2 - 1} q_i \Delta t_i = 0 \). Next, we will show that \( G_0 \) is not 2-singular and \((2,2,2)\)-singular relative to \( \mathcal{E}_{k_0} \). Using the same argument as that of the proof of Theorem 11.2, we obtain the following system of limiting rational polynomial functions:

\[
\begin{align*}
&\sum_{i = u_1^*}^{u_1^* + 1 - 1} q_i \Delta t_i / \sum_{i = u_1^*}^{u_1^* + 1 - 1} q_i |\Delta t_i|^2 \rightarrow 0, \\
&\sum_{i = u_1^*}^{u_1^* + 1 - 1} q_i t_i^0 (\Delta t_i)^2 / \sum_{i = u_1^*}^{u_1^* + 1 - 1} q_i |\Delta t_i|^2 \rightarrow 0,
\end{align*}
\]

where \( l^* \) is some index in \([1, l_3 - 1]\) and \( q_i = p_i / \sigma_i^0, \Delta t_i = \Delta m_i / \sigma_i^0, t_i^0 = m_i^0 / \sigma_i^0 \) for all \( u_1^* \leq i \leq u_1^* + 1 \). By employing the greedy extraction technique being described in Section 11.3.1, we obtain the following system of polynomial equations:

\[
\begin{align*}
&\sum_{i = u_1^*}^{u_1^* + 1 - 1} p_i^0 c_i = 0, \\
&\sum_{i = u_1^*}^{u_1^* + 1 - 1} p_i^0 m_i^0 c_i^2 = 0,
\end{align*}
\]

where at least one of \( c_i \) differs from 0. Now, we have two possible scenarios:

**Case 1.1:** \(|I_{u_1^*}| = u_1^* + 1 - u_1^* = 2\). Then, by solving the above system of equations, we obtain \( \sum_{i \in I_{u_1^*}} p_i^0 \prod_{j \in I_{u_1^*} \setminus \{i\}} m_j^0 = 0 \), which means \( I_{u_1^*} \) is nonconformant homologous set with C(1) singularity of \( G_0 \) — a contradiction to the fact that \( G_0 \in \mathcal{S}_{31} \).

**Case 1.2:** \(|I_{u_1^*}| = u_1^* + 1 - u_1^* = k^* = 3\). Then, by solving the above system of equations, we obtain \( \sum_{i \in I_{u_1^*}} p_i^0 \prod_{j \in I_{u_1^*} \setminus \{i\}} m_j^0 < 0 \) — a contradiction to the assumption of Case 1.

Thus, \( G_0 \) is not 2-singular and \((2,2,2)\)-singular relative to \( \mathcal{E}_{k_0} \). As a consequence, \( \ell(G_0 | \mathcal{E}_{k_0}) = 1 \) and \( L(G_0 | \mathcal{E}_{k_0}) = \{(1, 1, 2)\} \) according to Theorem 11.1.

**Case 2:** There exists at least one nonconformant homologous set \( I \) of \( G_0 \) such that \(|I| = k^*\) satisfies \( \sum_{i \in I} p_i^0 \prod_{j \in I \setminus \{i\}} m_j^0 < 0 \). Without loss of generality, we assume the homologous set \( I_{u_1} \) of \( G_0 \) to have the property \( \sum_{i \in I_{u_1}} p_i^0 \prod_{j \in I_{u_1} \setminus \{i\}} m_j^0 < 0 \). We will show that \( G_0 \) is 2-singular and \((2,2,2)\)-singular relative to \( \mathcal{E}_{k_0} \). In fact, we construct the sequence of \( G \) by letting \( p_i = p_i^0, \theta_i = \theta_i^0, v_i = v_i^0 \) for all \( u_2 = k^* + 1 \leq i \leq k_0 \). In order for \( G_0 \) to be 2-singular and \((2,2,2)\)-singular, it is sufficient that

\[
\begin{align*}
&\sum_{i = u_1}^{u_2 - 1} q_i \Delta t_i / \sum_{i = u_1}^{u_2 - 1} q_i |\Delta t_i|^2 \rightarrow 0, \\
&\sum_{i = u_1}^{u_2 - 1} q_i t_i^0 (\Delta t_i)^2 / \sum_{i = u_1}^{u_2 - 1} q_i |\Delta t_i|^2 \rightarrow 0.
\end{align*}
\]
The simple solution to the above system of limits is \( u_2 - 1 \sum_{i=u_1}^{u_2-1} q_i t_i = 0 \) and \( u_2 - 1 \sum_{i=u_1}^{u_2-1} q_i t_i^2 = 0 \). One solution to these two equations is \( p_i = p_i^0 \) and \( \Delta m_i = (\sigma_i^0)^2 d_i / n \) for all \( u_1 \leq i \leq u_2 - 1 \) where \( d_1, d_2, d_3 \) satisfy

\[
\sum_{i=u_1}^{u_2-1} p_i^0 d_i = 0, \quad \sum_{i=u_1}^{u_2-1} p_i^0 m_i^0 d_i^2 = 0,
\]

which is guaranteed to have non-trivial solution as \( \sum_{i \in I} p_i^0 \prod_{j \in I \setminus \{i\}} m_j^0 < 0 \). Therefore, \( G_0 \) is 2-singular and \( (2,2,2) \)-singular relative to \( E_{k_0} \). Since \( \pi(G_0) = 2 \) as \( k^* = 3 \), combining with the upper bound of Theorem \ref{thm:main} we obtain \( \ell(G_0 | E_{k_0}) = 2 \) under Case 2. According to Theorem \ref{thm:main-1} it also indicates that \( \mathcal{L}(G_0 | E_{k_0}) = \{(1, 1, 3)\} \), which concludes our proof.

**FULL PROOF OF THEOREM \ref{thm:main-3}** Here, we shall provide the complete proof of Theorem \ref{thm:main-3}, which is also the generalization of the argument in Section \ref{sect:1.3.3}. Indeed, without loss of generality, we assume that \( (p_i^0 / \sigma_i^0, m_i^0 / \sigma_i^0) = (p_i^0 / \sigma_i^0, -m_i^0 / \sigma_i^0) \). Next, we proceed to choosing a sequence of \( G \in E_{k_0} \) as follows: \( p_i = p_i^0, \theta_i = \theta_i^0, v_i = v_i^0 \) for all \( 1 \leq i \leq k_0 \), and \( m_1 = m_1^0 + 1/n, m_2 = m_2^0 - \sigma_2^0 / n \sigma_1^0, m_i = m_i^0 \) for all \( 3 \leq i \leq k_0 \). The choice of \( m_1, m_2 \) is taken to guarantee that \( \Delta m_1 / \sigma_1^0 + \Delta m_2 / \sigma_2^0 = 0 \) as we have discussed in Section \ref{sect:1.3.3} Then, we can check that

\[
\sum_{j=1}^{\frac{2}{a}} p_j (m_j^0)^u (\Delta m_j)^v / (\sigma_j^0)^{u+v+1} = 0 \text{ for all odd numbers } u \leq v \text{ when } v \text{ is even number or for all even numbers } 0 \leq u \leq v \text{ when } v \text{ is odd number. From here, the completion of the proof follows in the same way as that of the special case previously described.}