Dual correspondence between classical spin models and quantum CSS states

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The correspondence between classical spin models and quantum states has attracted much attention in recent years. However, it remains an open problem as to which specific spin model a given (well-known) quantum state maps to. Here, we provide such an explicit correspondence for an important class of quantum states where a duality relation is proved between classical spin models and quantum Calderbank-Shor-Steane (CSS) states. In particular, we employ graph-theoretic methods to prove that the partition function of a classical spin model on a hypergraph \(H\) is equal to the inner product of a product state with a quantum CSS state on a dual hypergraph \(\widetilde{H}\). We next use this dual correspondence to prove that the critical behavior of the classical system corresponds to a relative stability of the corresponding CSS state to bit-flip (or phase-flip) noise, thus called critical stability. We finally conjecture that such critical stability is related to the topological order in quantum CSS states, thus providing a possible practical characterization of such states.

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I. INTRODUCTION

Quantum entangled states play an important role in quantum computation and quantum communication. Of particular interest are error correcting codes such as CSS states [1–4] which are used for protecting information against decoherence. Because of their high entanglement properties, they have also found many applications in quantum information protocols such as encryption [5] and measurement-based quantum computation (MBQC) [6, 7]. Furthermore, an important set of such states called topological CSS states, including Kitaev’s toric code states (TC) [8] and color code states (CC) [9], have found important applications as robust memory for topological quantum computation. Such a robustness against local perturbations is a result of topological order which is a new phase of matter that is not described by symmetry breaking theory [10–12].

On the other hand, statistical physics is a well-established field with many interdisciplinary applications in recent decades, in particular with regards to criticality and phase transitions [13–15]. Accordingly, connections between quantum information theory and classical statistical mechanics has attracted much attention recently [16–21], which has led to cross-fertilization. Specifically in an important paper [22], it was shown that a partition function of a classical spin model can be written as an inner product of an entangled state and a product state. Such a mapping has found many applications in both quantum information theory and statistical mechanics [23–28]. For example, it is shown that computational power of quantum states for measurement-based quantum computation is related to computational complexity of the corresponding classical spin models. Using such a relation, it is shown that some specific classical spin models, corresponding to quantum stabilizer states which are universal resources for MBQC, are complete [29, 30] in a sense that their partition function is equivalent to partition functions of any other classical spin models. Furthermore, a similar correspondence between some classical spin models with specific topological code states has been exploited in order to study the power of a few topological states as a source for MBQC [31, 32].

Although the above applications of the correspondence between the partition function of classical spin models and quantum entangled states has only been considered for a few specific models, one might expect that a wider class of classical spin models could have specific quantum state mapping. Accordingly, finding an explicit correspondence between specific spin models and various quantum states is an important and challenging task. In particular, one might like to know what are the classical spin models corresponding to well-known quantum states. To this end, one needs to find a common tool which can be employed as a mapping mechanism for both classical spin models as well as quantum entangled states. In this paper, we use hypergraphs as such a tool to find an explicit mapping to establish a duality correspondence between the partition function of classical spin models and quantum CSS states. In particular, we show that the classical spin models map to a hypergraph which corresponds to a quantum CSS state on the corresponding dual hypergraph. Our results are very general and can in principle be applied to a wide class of models, as in spatial dimensions higher than the typical 1D or 2D systems. Furthermore, in order to show that hypergraph is a useful and practical tool and provide some specific examples, we show the details of such mapping for TC on arbitrary graphs as well as CC on D-colexes.

In addition to the possibility of the above-mentioned applications, one might also think of other possible insights. For example, we show that one can use the well-known properties of the classical systems in order to obtain important new information about the corresponding quantum CSS states. In particular, we use the
non-analytic properties of classical partition functions at the critical point of their second order phase transitions \(T = T_c\) in order to draw conclusions about the corresponding CSS states, which become relatively stable at the particular value of noise, precisely related to \(T_c\). We therefore call this new concept critical stability. This type of stability is distinctly different from the more common concept of robustness and the accuracy threshold of CSS codes associated with classical spin glasses which have been previously studied \([33-38]\). We further conjecture that critical stability as an intrinsic physical property can be used to ascertain topological order in quantum CSS states.

This paper consists of the following sections: in Sec.(II) we provide a basic review on hypergraphs which we use to establish our duality mapping. In Sec.(III) we establish our duality mapping where we prove that the partition function of a classical spin model on a hypergraph corresponds to CSS state on a dual hypergraph. We next provide some specific examples of such duality for the well known TC as well as the CC in Sec.(IV). Next as an interesting application of the duality, we prove that the critical phase transition of the classical spin model corresponds to the stability of the CSS state at a specific noise probability, thus referred to as critical stability. We end in Sec.(V) by providing some concluding remarks.

II. BASICS OF HYPERGRAPHS

A hypergraph \(H\) is an extension of an ordinary graph where each edge of the hypergraph can involve arbitrary number of vertices. In this way, each hypergraph is characterized by two sets of vertices \(V = \{v_1, v_2, ..., v_K\}\) and (hyper)edges \(E = \{e_1, e_2, ..., e_N\}\) and is denoted by \(H = (V, E)\). As an example in Fig.1(a), a hypergraph of four vertices \(v_1, v_2, v_3, v_4\) has been shown where edge \(e_1 = \{v_1\}\), edge \(e_2 = \{v_2, v_3\}\), and \(e_3 = \{v_1, v_2, v_4\}\), which is denoted by a closed curve. Degree of each edge \(e_m\) is called cardinality and is equal to the number of vertices that are involved by \(e_m\), denoted by \(|e_m|\). In Fig.1(a), degree of edges \(e_1, e_2, e_3\) are equal to 1, 2, and 3, respectively. The dual of a hypergraph \(H(V, E)\) is a hypergraph \(\hat{H}(\hat{V}, \hat{E})\) is a hypergraph \(\hat{H}(\hat{V}, \hat{E})\) where \(\hat{V} = \{\hat{v}_1, \hat{v}_2, ..., \hat{v}_N\}\) and \(\hat{E} = \{\hat{e}_1, \hat{e}_2, ..., \hat{e}_K\}\) where \(\hat{e}_i = \{\hat{v}_m|v_i \in e_m \text{ in } H\}\), i.e. duality interchanges vertices and edges \([39]\). For example, in Fig.1(b), we show the dual hypergraph of part (a).

Consider a hypergraph with \(|V| = K\) vertices and \(|E| = N\) edges. Related to each edge \(e_m\), we consider a binary vector, which is called edge vector, with \(K\) components which are denoted by \(a_m^i\) and \(j = \{1, 2, 3, ..., K\}\) where \(a_m^i = 1\) if \(v_j \in e_m\) and \(a_m^i = 0\) otherwise. For example, for hypergraph in Fig.1(a), the corresponding binary vectors to three edges are \(e_1 = (1, 0, 0, 0)\), \(e_2 = (0, 1, 1, 0)\) and \(e_3 = (1, 1, 0, 1)\) . In this way, we will have \(N\) binary vectors corresponding to \(N\) edges of the hypergraph. Furthermore, an edge vector is called dependent if it can be written as a superposition of other edge vectors of the hypergraph. The set of all independent edges is an independent set which is denoted by \(I\), where \(|I| \leq |V|\). We can also define an orthogonality relation between edges where two edges \(e_m\) and \(e_n\) are called orthogonal if and only if their corresponding binary vectors are orthogonal, i.e. \(e_m.e_n = 0\) where the symbol “.” refers to the inner product, in binary representation. Consider a hypergraph \(H = (V, E)\) with an independent set of edges, \(I\), orthogonal hypergraph of \(H\) is a hypergraph \(H^* = (V^*, E^*)\) that has the same vertices as \(H\), \(V^* = V\), but has \(K - |I|\) distinct and independent edges that are orthogonal to all edges of \(H\), see Fig.1(c).

One can show that, for any hypergraph \(H\), an orthogonal hypergraph \(H^*\) always exists. To this end, consider a hypergraph \(H = (V, E, I)\) where \(I\) is the independent subset of the hyperedges. If \(|V| = K\) and \(|I| = M\), it is clear that the number of independent edges is smaller than the number of vertices, i.e. \(M < K\). We denote the independent hyperedges by \(e_1, e_2, ..., e_M\) with a binary vector representation for each one of them.

We want to find a new hypergraph \(H^*\) that is orthogonal to \(H\). To this end, we need to find all hyperedges denoted by \(e^*\) which are orthogonal to all \(e_i\). The binary vector of \(e^*\) has \(K\) components denoted by \(a_1^*, a_2^*, ..., a_K^*\). Clearly, \(e^*\) will be orthogonal to all edges of the \(H\) if and only if, for \(i = 1, 2, ..., M\), we have \(e^*.e_i = 0\). Such a relation for all \(i\) is equivalent to \(M\) independent equations on \(K\) binary variables \(a_1^*, a_2^*, ..., a_K^*\). The number of independent solutions for such a set of equations is equal to \(K - M\). Since each independent solution is equal to a independent hyperedge of \(H^*\), we will find an orthogonal hypergraph \(H^*\) for each original hypergraph \(H\) with a distinct independent set of hyperedges.

FIG. 1: (Color online) (a) A simple hypergraph where we use purple (dark) color for hyperedges and (b) Dual hypergraph where we use yellow (light) color for vertices and purple (dark) color for hyperedges and (c) orthogonal hypergraph of part (a).
III. DUALITY MAPPING BETWEEN CLASSICAL SPIN MODELS AND QUANTUM CSS STATES

One can use hypergraphs in order to define the most general form of spin Hamiltonian:

\[ H = \sum_i J_i s_i + \sum_{i,j} J_{ij} s_i s_j + \sum_{i,j,k} J_{ijk} s_i s_j s_k + \ldots (1) \]

where \( s_i = \{\pm 1\} \), \( J_i \) refers to local magnetic field, \( J_{ij} \) and \( J_{ijk} \) refers to two-body and three-body coupling constants, and \( +... \) refers to other many-body interactions. Such form of Hamiltonian can also include d-level spins as is shown in [23]. To map the spin model to a hypergraph \( H(V,E) \), spins are represented by the vertices, and corresponding to each interaction term, we define an edge which includes all spins belonging to the interaction term. Therefore, we can re-write the above Hamiltonian in a compact form:

\[ H = \sum_{m|e_m \in E} J_m \prod_{i|v_i \in e_m} s_i. \quad (2) \]

We next map a quantum CSS state to a hypergraph, see also [40]. To this end, we use an idea which has recently been used for some quantum entangled states in [41, 42]. A quantum CSS state on \( K \) qubits is a stabilizer state that is stabilized by \( X \)-type and \( Z \)-type operators belonging to Pauli group on \( K \) qubits. To encode the structure of quantum CSS states, consider a hypergraph \( H = (V,E,I) \), with \( |V| = K \), \( |E| = N \) and \( |I| = M \). We insert \( K \) qubits in all vertices of the hypergraph. Then we define an \( X \)-type operator corresponding to each independent edge of \( H \), and also define a \( Z \)-type operator corresponding to each edge of the \( H^* = (V,E^*) \). We denote such operators by \( A_m \) and \( B_n \), respectively, where \( e_m \in I \) and \( e_n \in E \), and are given by:

\[ A_m = \prod_{i|v_i \in e_m} X_i, \quad B_n = \prod_{i|v_i \in e_n^*} Z_i. \quad (3) \]

Clearly, when two edges are orthogonal to each other it is necessary that the number of shared vertices of those edges is an even number, therefore each \( X \)-type operator commutes with other \( Z \)-type operators, i.e. \( [A_m, B_n] = 0 \). In this way, corresponding to \( H \), we will have \( M \), \( X \)-type stabilizers and \( K - M \), \( Z \)-type stabilizers which are generators of a quantum CSS state in the following form:

\[ |CSS_H\rangle = \frac{1}{2^K} \prod_{m|e_m \in I} (1 + A_m) |0\rangle^{\otimes K} \quad (4) \]

where \( |0\rangle \) is the positive eigenstate of \( Z \) and \( \frac{1}{2^K} \) is the normalization factor. Since \( A_m(1 + A_m) = (1 + A_m) \) and \( [A_m, B_n] = 0 \), one can easily check that the above state is stabilized by \( A_m \) and \( B_n \).

Furthermore, it is also possible to represent Eq.\((4)\) with \( Z \)-type operators \( B_n \) in the following form:

\[ |CSS_H\rangle = \frac{1}{2^K} \prod_{m|e_m \in I} (1 + B_n)|+\rangle^{\otimes K} \quad (5) \]

where \(|+\rangle \) is the positive eigenstate of \( X \). Therefore, we can construct a CSS state corresponding to a hypergraph with a distinct independent set.

We are now ready to present our main result. Consider a classical spin model on a hypergraph \( H = (V,E) \) given by Eq.\((2)\). The partition function is given by:

\[ Z = \sum_{\{s_i\}} \exp(-\beta \sum_{m|e_m \in E} J_m \prod_{i|v_i \in e_m} s_i) \quad (6) \]

where \( \beta \) is the reciprocal temperature with the Boltzmann constant \( k_B \) sets equal to one. We can perform a simple change of variable. To this end, corresponding to each edge \( e_m \), we define a new spin variable \( S_m \) which is related to all the spins \( s_i \) belonging to the edge \( e_m \) in the form of \( S_m = \prod_{i|v_i \in e_m} s_i \), and replace these new edge variables in the above relation. It is clear that these new variables are not necessarily independent variables.

To better understand this change of variable, consider a spin model on a hypergraph as shown in Fig.\(2(a)\), with four vertices \( v_1, v_2, v_3, v_4 \) and four edges \( e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}, e_3 = \{v_3\} \) and \( e_4 = \{v_1, v_3, v_4\} \). The classical spin model corresponding to this hypergraph will be given by: \( H = -J_{(1)} s_1 s_2 - J_{(2)} s_2 s_3 - J_{(3)} s_3 s_4 - J_{(4)} s_1 s_3 s_4 \) where the variable \( s_i \) corresponds to the vertex \( v_i \) and \( J_m \) refers to coupling constant of the edge \( e_m \). We now replace the original spin variables \( s_i \) by the edge variables \( S_m \): \( S_{(1)} = s_1 s_2, S_{(2)} = s_2 s_3, S_{(3)} = s_3 \) and \( S_{(4)} = s_1 s_3 s_4 \). By such a definition for edge variables, it is clear that there is a constraint given by \( S_{(1)} S_{(2)} S_{(3)} S_{(4)} = 1 \). In Fig.\(2(b)\), we have shown this constraint by a closed curve around the four new spin variables corresponding to edges \( e_m \). It is necessary to apply this constraint in the expression for the partition function. We thus use a Kronecker (Leopold Kroncker) delta as \( \delta(S_{(1)} S_{(2)} S_{(3)} S_{(4)}, 1) \) and the partition function can be written as: \( Z = \sum_{\{S_m\}} \delta(S_{(1)} S_{(2)} S_{(3)} S_{(4)}, 1) \times \exp[\beta(J_{(1)} S_{(1)} + J_{(2)} S_{(2)} + J_{(3)} S_{(3)} + J_{(4)} S_{(4)})] \). It is clear that the above example can be extended to any spin model and we can denote the set of all edges \( e_m \) belonging to a constraint \( C \) where such a constraint can be written as \( \prod_{m|e_m \in C} S_m = 1 \). However, it is very important to find a simple interpretation for all constraints in a general way. To this end, we define a new hypergraph \( H_C \), whose vertices are the variables \( S_m \). We consider each constraint \( C \) as an edge of hypergraph \( H_C \). Since each vertex of \( H_C \) is related to an edge of \( H \), we denote those vertices by \( v_m \) in analogy to vertices of the dual hypergraph \( \tilde{H} \). Therefore, \( H_C \) is a hypergraph in the dual space with a set of vertices corresponding to spin variables \( S_m \), which we denote by \( \tilde{v}_m \), and a set of edges corresponding to constraints \( C \). A simple lemma can now be proven in order to interpret \( H_C \).
Lemma: $H_C$ is equal to the orthogonal hypergraph of dual hypergraph $\tilde{H}$, i.e., $H_C = \tilde{H}^*$. Proof: We prove this lemma in two steps. In the first step, since $S_m = \prod_{|v_i|e_m \in C} s_i$, it is clear that each constraint in the form of $\prod_{|e_m| \in C} S_m = 1$ will hold true if and only if each vertex of $H$ is a member of an even number of edges belonging to constraint $C$. Now consider the above conclusion in a dual space where vertices of the hypergraph $H$ are edges of $\tilde{H}$ which are denoted by $\tilde{e}_i$, and constraints $C$ are denoted by $\tilde{e}_C$, the edges of $H_C$. Therefore, in a dual space, $|\tilde{e}_i \cap \tilde{e}_C| = 1$ is an even number. In the second step, consider binary vectors corresponding to edges $\tilde{e}_i$ and $\tilde{e}_C$ in the dual space. Since $|\tilde{e}_i \cap \tilde{e}_C|$ is an even number, it is simple to show that $\tilde{e}_i \cap \tilde{e}_C = 0$. Therefore, the binary vectors corresponding to constraints in $H_C$ are orthogonal to binary vectors corresponding to edges of the dual hypergraph $\tilde{H}$. In other words, $H_C$ is equal to the orthogonal hypergraph of the dual hypergraph $\tilde{H}$, and thus, the lemma is proved. As an example, in Fig.2(c), we show dual hypergraph of Fig.2(a). It is simple to check that $H_C$ is orthogonal to $\tilde{H}$.

To recap, $H_C = \tilde{H}^*$ and each constraint $C$ on spin variables $S_m$ ($\prod_{|e_m| \in C} S_m = 1$) can be written as $\prod_{|\tilde{e}_m| \in \tilde{e}^*} S_m = 1$ where $\tilde{e}_m$ and $\tilde{e}^*$ refer to a vertex and edge of $\tilde{H}^*$, respectively. We are ready to come back to Eq.(6) for the partition function of classical spin models which now finds the form:

$$Z = \sum_{\{S_m\}} e^{(-\beta \sum_{m|\tilde{e}_m \in \tilde{V}} J_m S_m)} \prod_{\tilde{e}^* \in \tilde{E}^*} \delta(\prod_{m|\tilde{e}_m \in \tilde{V}} S_m, 1),$$

(7)

where $\tilde{V}$ and $\tilde{E}^*$ are sets of vertices and edges of $\tilde{H}^*$, respectively. In the next step, we show that the above form of the partition function can be written in a quantum language. To this end, we use a simple identity for the Kronecker delta in the form of $\delta(\prod_{m|\tilde{e}_m \in \tilde{V}} S_m, 1) = (1 + \prod_{m|\tilde{e}_m \in \tilde{V}} S_m)$ and rewrite the partition function as

$$Z = \sum_{\{S_m\}} e^{(-\beta \sum_{m|\tilde{e}_m \in \tilde{V}} J_m S_m)} \prod_{\tilde{e}^* \in \tilde{E}^*} \left(1 + \prod_{m|\tilde{e}_m \in \tilde{V}} S_m\right).$$

(8)

Since each spin variable has a value of 1 or -1, it is simple to show that a summation over $\sum_{\{S_m\}} F(S_m)$, where $F$ is an arbitrary function, can be written as $\sum_{\{S_m\}} F(S_m) = 2(|+|F(Z)|+|)$. Therefore,

$$Z = 2^N N^O \langle |e^{-\beta \sum_{m|\tilde{e}_m \in \tilde{V}} J_m S_m} \rangle + \prod_{\tilde{e}^* \in \tilde{E}^*} \left(1 + \prod_{m|\tilde{e}_m \in \tilde{V}} Z_m\right).$$

(9)

where $|+\rangle$ refers to positive eigenstate of the $X$ operator and $N$ is the number of vertices of the hypergraph $\tilde{H}^*$. If we consider the number of independent edges of the $\tilde{H}$ to be equal to $M$, we can write the above as:

$$Z = 2^M \langle \alpha|Q \rangle$$

(10)

where $|\alpha\rangle$ is a product state given by $\prod_{m|\tilde{e}_m \in \tilde{V}} \exp(-\beta J_m Z_m)|+\rangle^\otimes N$ and $|Q\rangle = 2^{m|\tilde{e}_m \in \tilde{V}} \langle 1 + \prod_{m|\tilde{e}_m \in \tilde{V}} Z_m|+\rangle^\otimes N$ is an unnormalized quantum state. Comparing with Eq.(4, 5), we conclude that $|Q\rangle$ is equal to a quantum CSS state on dual hypergraph $\tilde{H}$ up to a normalization factor:

$$|Q\rangle = \prod_{\tilde{e}^* \in \tilde{E}^*} (1 + \prod_{m|\tilde{e}_m \in \tilde{V}} Z_m)|+\rangle^\otimes N$$

$$= 2^{N-M} \prod_{\tilde{e} \in \tilde{I}} (1 + \prod_{m|\tilde{e}_m \in \tilde{V}} X_m)|0\rangle^\otimes N = 2^{N-M} |CSS_{\tilde{H}}\rangle,$$

(11)

where $\tilde{I}$ is the independent set of $\tilde{H}$. We have therefore proven that the partition function of a classical spin model on $\tilde{H}$ corresponds to a quantum CSS state on $\tilde{H}$ in the following form:

$$Z_{\tilde{H}} = 2^{N+M} \langle \alpha|CSS_{\tilde{H}} \rangle.$$

(12)

IV. EXAMPLES

Using the duality relation Eq.(12), one can find specific classical spin models corresponding to well-known quantum CSS states. In this section, we show this by considering two important classes of quantum CSS states, i.e., the TC and the CC.
A. Kitaev’s toric code state and Ising model on
arbitrary graphs

First, we consider the duality mapping for TC on an arbitrary graph which is a CSS quantum state with topological order [8]. We show that TC on an arbitrary graph (lattice) corresponds to the Ising model on the same graph (lattice). To this end, consider an arbitrary graph $G$ where qubits live on the edges of $G$. Corresponding to any vertices of the graph one defines $X$-type operators in the following form:

$$A_v = \prod_{i \in v} X_i$$

where $i \in v$ refers to qubits that live on the edges of vertex $v$. For TC the $Z$-type operators that commute with $A_v$ can easily be found, where corresponding to each plaquette of the graph, an operator $B_p = \prod_{i \in \partial p} Z_i$ is defined, where $i \in \partial p$ refers to qubits belonging to the boundary of the plaquette $p$ for a square lattice, see Fig.3(a). In this way, the TC is a stabilizer state of both $X$ and $Z$-type operators in the following form:

$$|K_G\rangle = \prod_v (1 + A_v)|0\rangle^\otimes N.$$  \hfill (14)

Although the above state has been defined on a graph, it is simple to present a hypergraph representation for it. To this end, we consider all qubits of the TC as vertices of a hypergraph. Then, corresponding to each vertex of graph $G$, we define a hyperedge of the hypergraph $H$ that involves all qubits connecting to that vertex, see Fig.3(b). In this way, the TC is a stabilizer state of both $X$ and $Z$-type operators in the following form:

$$A_e = \prod_{i \in e} X_i$$

where $e$ refers to a hyperedge of the hypergraph $H$.

According to our duality mapping, a TC on a hypergraph $H$ is related to a spin model on the dual hypergraph $\tilde{H}$. In order to find the dual hypergraph relating to the TC on a graph $G$, we insert spin variables on vertices of the graph $G$ corresponding to each edge of the hypergraph $H$, see Fig.3(c). This way, the spin variables are considered as vertices of the $H$. Furthermore, each vertex of $H$ is equal to a hyperedge of $\tilde{H}$ and since each vertex of $H$ is a member of two neighboring hyperedges of $H$, each hyperedge of $\tilde{H}$ should involve two spin variables, see Fig.3(c). Therefore, the hypergraph $\tilde{H}$ is an ordinary graph that is exactly the same as the original graph $G$ and the corresponding spin models will be an Ising model on the graph $G$. In Fig.4, we show another example of a TC on a 3D square lattice where the same argument as above holds. Accordingly, the following relation holds between the partition function of Ising models on an arbitrary graph $G$ and the TC state $|K_G\rangle$:

$$Z_{Ising,G} = \langle \alpha |K_G\rangle.$$  \hfill (15)

B. Color code state on D-colexes and spin model on D-dimensional simplicial lattice

Another set of quantum CSS states with topological order are CC. They can be defined on $(D + 1)$-valent lattices with $(D + 1)$-colorable edges in D-dimensional manifold which is technically called D-colexes [45]. We next use the duality mapping to show that inner product of a product state with a CC is equal to the partition function of a spin model on a simplicial lattice.

In order to show the above result, we present the main idea of CC on two simple easy-to-imagine lattices, and then generalize to higher dimensions. In two dimensions, a 2-colex is a trivalent lattice where the edges can be colored by three different colors such that any two neighboring edges do not have the same color. In Fig.5(a), we show an example of such a structure. Corresponding
to each plaquette of a 2D trivalent lattice, we define two $X$-type and $Z$-type operators in the following form:

$$
B_p = \prod_{i \in p} Z_i, \quad A_p = \prod_{i \in p} X_i
$$

(16)

where $i \in p$ refers to all vertices belonging to a plaquette $p$. The quantum CSS state corresponding to the above operators will have the following form:

$$
|CC_2\rangle = \prod_p (1 + A_p)|0\rangle^\otimes N.
$$

(17)

It is simple to give an equivalent representation of the above state on a hypergraph. To this end, we should consider each plaquette of the lattice as a hyperedge of a hypergraph which involves all vertices belonging to that plaquette, see Fig.5(b). By such a definition it is simple to find the dual of this hypergraph. As in Fig.5(c), since each vertex of the $H$ is a member of three hyperedges, the dual hypergraph will be a triangular lattice. In this way and by the duality mapping, we conclude that the partition function of a spin model on triangular lattice with three-body interactions is equal to inner product of a product state with a CC on the original trivalent lattice.

Similar to the above argument for 2-colexes, it is simple to find the duality mapping for 3-colexes. In Fig.6, we show a 3-colexes where vertices are four-valent and all edges are colored by four different colors. A CC on such a 3-colex is defined by $X$-type and $Z$-type operators in the following form:

$$
A_c = \prod_{i \in c} X_i, \quad B_f = \prod_{i \in f} Z_i
$$

(18)

where $c$ refers to each cell of the lattice and $f$ refers to each face of the lattice. Accordingly, the quantum CSS state corresponding to these operators is in the following form:

$$
|CC_3\rangle = \prod_c (1 + A_c)|0\rangle^\otimes N.
$$

(19)

The hypergraph representation of the above state can easily be derived by relating cells of the 3-colex to hyperedges of a hypergraph $H$. Since each vertex of the $H$ is a member of four hyperedges of the $H$ (four cells of the 3-colex), the dual hypergraph will be a tetrahedron lattice with four-colorable vertices, see Fig.7. Therefore using our duality mapping, we have shown that the partition function of a spin model on a tetrahedron lattice with four-body interactions is related to a CC on the original 3-colex.

The extension of the above idea to CC in higher dimensions is straightforward. It is well-known that the dual of a $D$-colex is a $D$-simplicial lattice with $(D+1)$-colorable vertices on a closed $D$-manifold [46]. We can therefore conclude that the partition function of a spin model on a $D$-simplicial lattice with $(D+1)$-body interactions is related to a CC on a $D$-colex in the following form:

$$
Z_{D-simplicial} = \langle \alpha|CC_{D-complex}\rangle.
$$

(20)
V. CRITICAL STABILITY: A CASE STUDY OF DUALITY MAPPING

The above mapping between the partition function of a classical spin model and a quantum CSS state can provide a powerful tool in order to find how certain well-known properties on one side of the equation would have effects on the other side. This cross-fertilization could have important consequences. One clear candidate that can be considered is the non-analytic property of the classical partition function on the classical side. What physics does it correspond to on the quantum side? We next show that the corresponding CSS state of the critical spin model has a relative stability to noise, i.e. critical stability.

In order to define such a concept, suppose that a probabilistic bit-flip noise is applied to each qubit of the quantum CSS state with probability \( p \). Such a noise can lead to different patterns of errors that are denoted by \( \mathcal{E} \). In other words, each error \( \mathcal{E} \) is a product of Pauli operators \( X \) on various qubits of the CSS state. Suppose that, for a specific error \( \mathcal{E} \), the number of qubits that are affected by Pauli operators \( X \) is equal to \( l \). It is clear that the probability of such an error will be equal to \( W_\mathcal{E} = p^l(1-p)^{N-l} \) where \( N \) is the number of qubits. In this way, one can check that \( W_\mathcal{E}(p) \) is a normalized probability where \( \sum_\mathcal{E} W_\mathcal{E}(p) = \sum_{l=0}^{N} \binom{N}{l} p^l(1-p)^{N-l} = 1 \).

On the other hand, there is a probability that pattern of qubits which are affected by the noise is equal to a stabilizer of CSS state where the CSS state remains in the stabilizer space. We denote such a probability by \( W_S(p) \) which is defined in the following form:

\[
W_S(p) = \sum_{\mathcal{E} \in S} W_\mathcal{E}(p) \tag{21}
\]

where \( S \) denotes set of stabilizers of the CSS states and \( \mathcal{E} \in S \) refers to each error pattern \( \mathcal{E} \) which is equal to one of members of the \( S \). Since the stabilizers of the CSS state do not change the CSS state, we call the above quantity stability probability. Finally, we define the value of this quantity as a measure of stability of a CSS state.

Next, we show that the inner product on the right hand of the duality relation (12) is related to the stability probability \( W_S(p) \). To this end, let us represent the product state \( |\alpha\rangle \) in the following form:

\[
\frac{1}{2^N} \prod_{i|v_i \in V} (e^{\beta J}1 + e^{-\beta J}X_i)|0\rangle^{\otimes N}, \tag{22}
\]

which can be rewritten as:

\[
\frac{1}{2^N [p(1-p)]^{\frac{N}{2}}} \prod_{i|v_i \in V} ((1-p)1 + pX_i)|0\rangle^{\otimes N} \tag{23}
\]

where \( N \) is the number of qubits and \( p \in [0, \frac{1}{2}] \) is given by \( \frac{p}{1-p} = e^{-2\beta J} \). Consequently, we can write Eq.(12) as:

\[
Z = \frac{1}{[p(1-p)]^{\frac{N}{2}}} W(p) \tag{24}
\]

where \( W(p) = 2^M N^\oplus \langle 0 | \prod_{i|v_i \in V} ((1-p)1 + pX_i) |CSS \rangle \). Now, we expand the operator \( \prod_{i|v_i \in V} ((1-p)1 + pX_i) \) in this relation where it will be equal to a superposition of all errors where the factor of each error term \( \mathcal{E} \) will be equal to \( W_\mathcal{E}(p) = p^l(1-p)^{N-l} \). On the other hand, the CSS state in the relation for \( W(p) \) is a superposition of all \( X \)-type stabilizers of CSS state. Therefore, the inner product is equal to summation of \( W_\mathcal{E}(p) \) on all errors that lead to stabilizers of the CSS state which is equal to \( W_S(p) \), or stability probability of the CSS state i.e. \( W(p) = W_S(p) \).

Accordingly, the duality correspondence now finds a new form where stability probability of a CSS state is related to partition function of the corresponding classical spin model:

\[
W_S(p) = [p(1-p)]^\frac{N}{2} Z. \tag{25}
\]

The right hand side goes to zero as \( N \) diverges, leading to \( W_S(p) = 0 \) for finite \( Z \). This seems reasonable, because it is nearly impossible that bit-flip noise does not lead to an error in the quantum CSS state [47]. However, one would have to reconsider the above, if the classical partition function \( Z \) also diverges, which could happen at the critical point of a phase transition.

Under typical situations, \( Z = e^{-\beta F} \) where \( F \) is the Helmholtz free energy. However, near the critical point, fluctuations become dominant, and one can consider a fluctuation correction to \( Z = \int_0^\infty \Omega(E)e^{-\beta F}dE \), with \( \Omega(E) \) being the density of states. This can easily be achieved by an expansion about the mean energy of the system [43], which in leading term is given by

\[
Z = e^{-\beta F} \sqrt{2\pi kT^2c_v} \tag{26}
\]

where \( c_v \) is the heat capacity of the classical spin model. The heat capacity diverges as \( (T - T_{cr})^{-\alpha} \) near the critical phase transition in the thermodynamic limit, with \( \alpha \) being a critical exponent. Therefore, \( Z \sim (T - T_{cr})^{-\frac{\alpha}{2}} \).

The divergence of the classical partition function has important ramification for stability of the corresponding CSS state. Clearly, associated with the critical temperature \( T_{cr} \), there is a critical probability given by \( \frac{p_{cr}}{1-p_{cr}} = e^{-2\beta_{cr} J} \), at which the value of \( W_S(p) \) significantly increases indicating a relative stability to bit-flip noise at this particular value. We therefore call this new concept critical stability of the CSS state, defined as the nonzero value of \( W_S(p) \) due to critical behavior of the partition function at a particular noise value \( p_{cr} \). We emphasize that the actual value of \( W_S(p) \) does not need to be large. The system becomes relatively stable at the particular value of \( p = p_{cr} \) since, as \( N \) diverges, it is exactly zero everywhere except at \( p_{cr} \). We should
emphasize that since $W_\mathcal{E}(p)$ is a normalized probability function, it will be clear that $W_S(p) = \sum_{\mathcal{E} \in S} W_\mathcal{E}(p)$ is always a finite number smaller than 1, since $S$ is a subspace of the set of all error patterns of $\mathcal{E}$. Specifically, we emphasize that even at the critical point where partition function $Z$ diverges, $W(p)$ remains a finite number smaller than 1. Furthermore, one can show that critical stability also exists in the case of phase-flip noise similar to that of bit-flip noise, see the Appendix.

We would like to emphasize that our concept of critical stability is very different from the more common concept of robustness in error correcting threshold for the CSS states which have previously been considered in the literature. Indeed, since a CSS state belongs to error correcting codes, it can be protected form noise by an active error correcting protocol. Specifically, one can find errors caused by noise by measuring stabilizers of CSS state. Then it is simple to correct errors by applying suitable operators [3]. Here, our definition of stability is completely different and is related to intrinsic stability/robustness of CSS state against bit-flip noise. Moreover, the usual robustness is a threshold below which the system can be actively stabilized where critical stability occurs only at one particular value $p_{cr}$. Furthermore, we note that the concept of critical stability is relative in a sense that it is magnified at $p_{cr}$ in relation to other noise probabilities. The concept of relative stability provides an additional level of robustness to what one needs in well-known error-correcting protocols.

The critical behavior of classical spin models is a well-known phenomenon. However, the concept of critical stability is a new and interesting property of the quantum CSS state which should carefully be considered. To this end, we note that the above concept of critical stability indicates a natural or intrinsic stability to external noise. On the other hand, a class of CSS states known as topological CSS states are known to exhibit stability to other forms of external perturbations. One might wonder if such stabilities might be related. Fortunately, the mapping we have provided along with specific examples discussed in Sec.(IV) can help to shed some light on such a possible relation. We next examine, for some specific cases, whether such a relation exists.

As was shown in the Sec.(IV), each TC on an arbitrary graph corresponds to the Ising model on the same graph. For example, a TC on a 1D lattice (GHZ state) maps to a one-dimensional Ising model which does not exhibit a phase transition, i.e. the TC on a 1D lattice is not stable to noise, $p_{cr} = 0$. This is consistent with the well-known result that there is no topological order in 1D models. However, TC in higher dimensions have topological order and according to our mapping, they correspond to the Ising model in higher dimensions which is well-known to exhibit critical behavior. We also considered classical spin models corresponding to CC in different dimensions in the previous section. Each CC on a D-colex (color complex) is mapped to a classical spin model on a D-simplicial lattices that is the dual of the original D-colex. Although such classical spin models are usually very abstract, they have a specific symmetry because of their colorability property. Therefore, one expects a phase transition via a symmetry-breaking mechanism. For example, a 2D case with three-body interactions on a triangular lattice with a $Z_2 \times Z_2$ symmetry has been studied and critical behavior has been identified [44], again confirming such a relation. Another example is provided by the fact that cluster states correspond to the Ising model in a magnetic field [22] which does not exhibit a critical phase transition, again consistent such a relation, since cluster states do not have a topological phase. We therefore conjecture that topological CSS states will exhibit critical stability. In fact, we have not been able to find any counter-example to our conjecture. If true, it can provide a practical characterization of topological quantum CSS states. We again note that the connection between critical stability and the existence of topological order seems plausible from a physical point of view since they both imply stability to external perturbations.

VI. CONCLUSIONS

In this work we provided a duality relation between quantum CSS states and classical partition function of spin models (Eq.12). Such duality relation was proved by graph-theoretic methods and relied heavily on the concept of dual hypergraphs. We next provided two concrete examples of such a mapping for the well-known toric code and color code states in various dimensions in (Eq.15, Eq.20). Using such correspondence, we introduced the concept of critical stability (Eq.25) where it was shown that certain CSS states can exhibit a natural relative stability to noise at a particular value of noise probability. Furthermore, we conjectured that this intrinsic property of CSS states is related to their topological order. Our general results can open new avenues for further characterization of quantum CSS states. The generality of the duality correspondence allows one to look for quantum CSS states corresponding to well-known classical spin models, or vice versa. In this way, one might be able to find new complete models or consider classical simulatability of quantum CSS states for MBQC, just to mention a few possibilities. On the other hand, our conjecture that critical stability corresponds to topological CSS states provides a natural characterization of topological states to external noise. It will be interesting to see if other CSS, as well as non-CSS, topological states possess critical stability, thus providing means to characterize topological order in general quantum states.

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[1] A. R. Calderbank and P. W. Shor, “Good quantum error-correcting codes exist,” Phys. Rev. A, 54, 1098-1105 (1996).
[2] A. M. Steane, “Error correcting codes in quantum theory,” Phys. Rev. Lett. 77, 793-797 (1996).
[3] D. Gottesman, “Stabilizer Codes and Quantum Error Correction,” arXiv:quant-ph/9705052 (1997).
[4] D. Nigg, M. Muller, E. A. Martinez, P. Schindler, M. Henrich, T. Monz, M. A. Martin-Delgado, R. Blatt, “Quantum computations on a topologically encoded qubit,” Science 345,6194 (2014): 302-305
[5] M. Hillery, V. Bužek, A. Berthiaume, “Quantum secret sharing,” Phys. Rev. A 59, 1829 (1999).
[6] R. Raussendorf, H. J. Briegel, “A one-way quantum computer,” Phys. Rev. Lett. 86, 5188 (2001).
[7] H. J. Briegel, D. E. Browne, W. Dür, R. Raussendorf, and M. Van den Nest, “Measurement-based quantum computation,” Nat. Phys. 5, 19 (2009).
[8] A. Y. Kitaev, “Fault-tolerant quantum computation by anyons,” Ann. Phys. (N.Y.) 303, 2 (2003)
[9] H. Bombin, M. A. Martin-Delgado, “Topological quantum computation,” Phys. Rev. Lett. 97, 180501 (2006).
[10] X. G. Wen, Q. Niu, “Ground-state degeneracy of the fractional quantum Hall states in the presence of a random potential and on high-genus Riemann surfaces,” Phys. Rev. B 41, 9377 (1990).
[11] B. J. Brown, D. Loss, J. K. Pachos, C. N. Solf, and J. R. Wootton, “Quantum memories at finite temperature,” Rev. Mod. Phys. 88, 045005 (2016).
[12] B. J. Brown, N. H. Nickerson, and D. E. Browne, “Fault-tolerant error correction with the gauge color code,” Nat. Commun. 7, 12302 (2016).
[13] M. Mézard, G. Parisi, and M. A. Virasoro, Spin glass theory and beyond (World scientific, Teaneck, NJ, USA, 1987).
[14] M. Schulz, Statistical Physics and Economics: Concepts, Tools and Applications (Springer, 2003).
[15] D. Felice, C. Cafaro, S. Mancini, “Information geometric methods for complexity,” Chaos 28, 032101 (2018).
[16] J. Geraci, D. A. Lidar, “On the exact evaluation of certain instances of the Potts partition function by quantum computers,” Commun. Math. Phys. 279, 735 (2008).
[17] D. A. Lidar, O. Biham, “Simulating Ising spin glasses on a quantum computer,” Phys. Rev. E 56, 3661 (1997).
[18] J. Geraci, D. A. Lidar, “Classical Ising model test for quantum circuits,” New J. Phys. 12, 75026 (2010).
[19] R. D. Somma, C. D. Batista, G. Ortiz, “Quantum approach to classical statistical mechanics,” Phys. Rev. Lett. 99, 030603 (2007).
[20] F. Verstraete, M. Wolf, D. Perez-Garcia, and J. Cirac, “Criticality, the area law, and the computational power of projected entangled pair states,” Phys. Rev. Lett. 96 (2006).
[21] A. Montakhab, A. Asadian, “Multiparticle entanglement and quantum phase transitions in the one-, two-, and three-dimensional transverse-field Ising model”, Phys. Rev. A 82, 062313 (2010).
[22] M. Van den Nest, W. Dür, H. J. Briegel, “Classical spin models and the quantum-stabilizer formalism,” Phys. Rev. Lett. 98, 117207 (2007).
[23] G. De las Cuevas, W. Dür, H. J. Briegel, and M. A. Martin-Delgado, “Unifying all classical spin models in a Lattice Gauge Theory,” Phys. Rev. Lett. 102, 230502 (2009).
[24] G. De las Cuevas, W. Dr, M. Van den Nest and M. A. Martin-Delgado, “Quantum algorithms for classical lattice models,” New J. Phys. 13:093021 (2011).
[25] Y. Xu, G. De las Cuevas, W. Dür, H. J. Briegel, M. A. Martin-Delgado, “The U (1) lattice gauge theory universally connects all classical models with continuous variables, including background gravity,” J. Stat. Mech. 1102:P02013 (2011).
[26] V. Karimipour, M. H. Zarei, “Completeness of classical 4\(^4\) theory on two-dimensional lattices,” Phys. Rev. A 85, 032316 (2012).
[27] Mohammad Hossein Zarei, Yahya Khalili, “Systematic study of the completeness of two-dimensional classical 4 theory,” Int. J. Quantum Inform. 15, 1750051 (2017).
[28] G. De las Cuevas, T. S. Cubitt, “Simple universal models capture all classical spin physics,” Science 351,6278 : 1180-1183 (2016).
[29] M. Van den Nest, W. Dr, H. J. Briegel, “Completeness of the classical 2D Ising model and universal quantum computation,” Phys. Rev. Lett. 100, 110501 (2008).
[30] V. Karimipour, M. H. Zarei, “Algorithmic proof for the completeness of the two-dimensional Ising model,” Phys. Rev. A 86, 052303 (2012).
[31] S. Bravyi, R. Raussendorf, “Measurement-based quantum computation with the toric code states,” Phys. Rev. A 76, 022304 (2007).
[32] H. Bombin, M. A. Martin-Delgado, “Statistical mechanical models and topological color codes,” Phys. Rev. A 77, 042322 (2008).
[33] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, “Topological quantum memory,” J. Math. Phys. 43, 4452 (2002).
[34] H. G. Katzgraber, H. Bombin, M. A. Martin-Delgado, “Error threshold for color codes and random three-body Ising models,” Phys. Rev. Lett. 103, 090501 (2009).
[35] R. S. Andrist, H. G. Katzgraber, H. Bombin, and M. A. Martin-Delgado, “Error tolerance of topological codes with independent bit-flip and measurement errors,” Phys. Rev. A 94, 012318 (2016).
[36] R. S. Andrist, J. R. Wootton, and H. G. Katzgraber, “Error thresholds for Abelian quantum double models: Increasing the bit-flip stability of topological quantum memory,” Phys. Rev. A 91, 042331 (2015).
[37] R. S. Andrist, H. Bombin, H. G. Katzgraber, and M. A. Martin-Delgado, “Optimal error correction in topological subsystem codes,” Phys. Rev. A 85, 050302(R) (2012).
[38] H. Bombin, R. S. Andrist, M. Ohzeki, H. G. Katzgraber, and M. A. Martin-Delgado, “Strong Resilience of Topological Codes to Depolarization,” Phys. Rev. X 2, 021004 (2012).
[39] C. Berge, Graphs and Hypergraphs, Amsterdam, The Netherlands: North-Holland, (1973).
[40] Mohammad Hossein Zarei, “strong-weak coupling duality between two perturbed quantum many-body systems: CSS codes and Ising-like systems ,” Phys. Rev. B 96, 165146 (2017).
[41] R. Qu, J. Wang, Z. Li, Y. Bao, “Encoding Hypergraphs into Quantum States,” Phys. Rev. A 87, 022311 (2013).
A. APPENDIX: Critical stability against phase-flip noise

In analogy to critical stability of CSS states against bit-flip noise, we show that there also exists a critical stability against a phase-flip noise. To this end, suppose that a Pauli operator $Z$ is applied to each qubit of a CSS state with probability $p$. Such a noise leads to an error as a product of $Z$ operators on various qubits of the CSS state. We denote the probability of such an error by $V_Z(p)$. In analogy to the bit-flip noise, the above function is also a normalized probability function. Now, we consider Eq. (12) in a new form. To this end, we rewrite the product state $|\alpha\rangle$ in the following form:

$$|\alpha\rangle = \frac{1}{2^N} \prod_{i \in V} (e^{\beta J} 1 + e^{-\beta J} X_i) |0\rangle^{\otimes N}$$

$$= \prod_{i \in V} (|\cosh(\beta J) 1 + \sinh(\beta J) Z_i\rangle + \rangle)^{\otimes N}. \quad (27)$$

Therefore, by a change of variable in the form of $1 - 2p = e^{-2\beta J}$ where $p \in [0, \frac{1}{2}]$, $|\alpha\rangle$ will find the following form:

$$|\alpha\rangle = \frac{1}{(1-2p)^{\frac{N}{2}}} \prod_{i \in V} ((1-p) 1 + pZ_i) |+\rangle^{\otimes N}. \quad (28)$$

The operator $\prod_{i \in V} ((1-p) 1 + pZ_i)$ in the above relation is equal to superposition of all $Z$-type errors with corresponding probabilities $V_Z(p)$. On the other hand, the CSS state can also be written in terms of $Z$-type stabilizers in the form of $\prod_{m \in E} (1 + \prod_{m \in E \cdot Z_m}) |+\rangle^{\otimes N}$. Here, the operator $\prod_{m \in E \cdot Z_m}$ is equal to superposition of all $Z$-type stabilizers of the CSS state. In this way, the inner product of $|\alpha\rangle$ and the CSS state can be interpreted as total probability that $Z$-type errors lead to stabilizers of the CSS state. We denote such a probability by $V(p)$ which will be in the following form:

$$V(p) = \frac{(1 - 2p)^{\frac{N}{2}}}{2^M} Z \quad (29)$$

Therefore, in the same way as bit-flip noise, the stability probability due to phase-flip noise will be equal to a product of quickly decaying function and the partition function of the corresponding classical spin model, which again leads to critical stability at the particular value of $p_{cr}$ corresponding to the critical temperature of the spin model.

It is interesting to note that the critical probability for bit-flip noise ($p_{cr}^b$) and phase-flip noise ($p_{cr}^f$) are different since they are given by $\frac{p_{cr}^b}{1-p_{cr}^b} = e^{-2\beta_{cr}^b J}$ and $1 - 2p_{cr}^f = e^{-2\beta_{cr}^f J}$. In fact,

$$p_{cr}^f = \frac{1}{2} - \frac{p_{cr}^b / 2}{(1 - p_{cr}^b)}, \quad (30)$$

which indicates the complimentary nature of critical stability due to two different noises. That is, if a CSS state exhibits critical stability at a high value of noise probability for bit-flip noise it will exhibit such a stability for low values of probability for phase-flip noise. We emphasize that such critical probability is not a threshold, and stability occurs at one particular value of probability and nowhere else. However, one might wonder if relative stability might occur at the same value for both types of noises. Clearly, this will happen at $p_{cr}^b = p_{cr}^f = 1 - \sqrt{2}/2 \approx 0.293$. From a more physical point of view, one would expect this to occur when there is a symmetry between $X$-type and $Z$-type stabilizers of the given CSS state. This is the case with the TC on a square lattice, for example. In such a model, if we interchange the $X$-type stabilizers with $Z$-type stabilizers, the model maps to itself. We therefore expect that the TC on a square lattice exhibits critical stability at the same value for both bit-flip and phase-flip noise. Such an expectation is indeed satisfied by our classical-quantum correspondence as the TC on a square lattice corresponds to the Ising model on a square lattice which in fact exhibits a critical transition at $\tanh J = e^{-2\beta J}$ [43]. It is easy to check that this condition is satisfied by Eq. (30) above. We also expect that the same result would hold for other symmetric CSS states such as the CC on a hexagonal lattice.

[42] M. Rossi, M. Huber, D. Bru, C. Macchiavello, “Quantum Hypergraph States,” New J. Phys. 15, 113022 (2013).
[43] Pathria, R. K. Statistical Mechanics, International Series in Natural Philosophy Volume 45, Pergamon Press, Oxford, UK, 1986.
[44] R. J. Baxter, F. Y. Wu, “Exact solution of an Ising model with three-spin interactions on a triangular lattice,” Phys. Rev. Lett. 31, 1294 (1973).
[45] H. Bombin, M. A. Martin-Delgado, “Exact topological quantum order in D= 3 and beyond: Branyons and brane-net condensates,” Phys. Rev. B 75, 075103 (2007).
[46] H. Bombin, “Gauge color codes: optimal transversal gates and gauge fixing in topological stabilizer codes,” New Journal of Physics 17.8 : 083002 (2015).
[47] C. Cafaro, S. Mancini, “Quantum stabilizer codes for correlated and asymmetric depolarizing errors,” Phys. Rev. A 82, 012306 (2010).