Fluctuation dissipation relations for strongly correlated out-of-equilibrium circuits

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We consider strongly correlated quantum circuits where a dc drive is added on top of an initial out-of-equilibrium stationary state. Within a perturbative approach, we derive unifying OE fluctuation relations for high frequency current noise, shown to be completely determined by zero-frequency noise and dc current. We apply them to the fractional quantum Hall effect at arbitrary filling factors, driven by OE sources, without knowledge of the underlying model: we show that they provide robust methods for an unambiguous determination of the fractional charge and of key interaction parameters entering in the exploration of anyonic statistics within an anyon collider.

Out-of-equilibrium (OE) current noise is a valuable tool to explore strongly correlated mesoscopic conductors and circuits, especially in the high frequency domain, where it unveils underlying dynamics and models [1–14]. It is a major tool in electron quantum optics where it is essential for characterizing quantum states of electrons [15] or of emitted photons [16, 17]. It also unveils fascinating collective phenomena within strongly correlated conductors as fractional charges [13, 17–19] and statistics in the fractional quantum Hall effect (FQHE) [20] or charge splitting in the integer quantum Hall effect (IQHE) [6, 21–23].

The effect of strong correlations in such systems calls for quantum laws of electronic transport independent on interactions and the microscopic model of the system. At equilibrium, the fluctuation-dissipation theorem (FDT) which uses the differential conductance at zero voltage is such a robust law even in the presence of a nonlinear current at high voltages [24].

In the OE regime, fluctuation-dissipation relations (FDRs) have been long studied but mostly at zero frequency [25]. A widely used perturbative OE FDR for high-frequency noise, expressed in terms of the dc current, has been derived by Rogovin and Scalapino [2] for independent particles obeying electron-hole symmetry. Assuming also an initial thermalization, extensions of this relation to strongly correlated conductors and quantum circuits have been achieved even without requiring inversion symmetry [26], the analogous of electron/hole symmetry for interacting systems [7, 10, 13, 14].

Finally, in a general OE situation, including many terminal setup with time-dependent voltages, the universal non-perturbative FDRs [27] concern only asymmetries between the emission (positive frequency) and absorption (negative freq.) parts of the noise spectrum, expressed in terms of an OE non-linear admittance.

However, the recent developments of interferometry experiments involving OE stationary sources in the FQHE, such as the anyon collider shown on Fig. 1 used to probe the non-trivial statistics of anyons emitted by non-trivial sources [28], calls for an in-depth exploration of new FDRs for the full finite frequency noise, valid in absence of an initial thermal state.

In this Letter, we derive perturbative OE FDRs for the full finite frequency noise without assuming initial thermalization [6, 29–32]. We show that, as long as the perturbative approach remains valid, high frequency non-symmetrized noise is not fully determined by the dc current, as in the initially thermalized case [2, 13, 14], but also by its zero frequency counterpart. This new relation illustrates the power of the OE perturbative approach, since it can be applied to a variety of situations independently of any underlying microscopic model.

This is especially relevant for the FQHE: the OE FDRs derived for the photo-assisted noise [13] and for the high-frequency noise under a dc voltage [13, 14] have already provided robust methods implemented in recent experiments to determine the fractional charge [17, 19] for filling factors ν which are not simple fractions, though the generic effective models are not appropriate [33, 34]. We illustrate furthermore the interest of the OE FDRs for the anyon collider. We show that they give access to effective interaction-dependent parameters which are important for the exploration of anyonic statistics, proposed in Ref. [35] and recently implemented in Ref. [28].

Model The underlying Hamiltonian of the OE perturbative approach in the stationary regime [13, 36]

\[ \mathcal{H}(t) = \mathcal{H}_0 + e^{-i\omega_J t} \hat{A}^\dagger e^{i\omega_J t} \hat{A}, \]  

(1)

involves an unspecified Hamiltonian \( \mathcal{H}_0 \) and a perturbing operator \( \hat{A} \). The Josephson-like frequency \( \omega_J \) must enter only through \( e^{i\omega_J t} \) in \( \mathcal{H}(t) \) and is added on top of other dc drives already present in the system. For concreteness, we will focus here on charge transport, though the theory extends beyond that. We thus assume that there is a charge operator \( \hat{Q} \), conserved by \( \mathcal{H}_0 \), translated by a model-dependent charge \( e^* \) when acting upon by \( \hat{A} \). Then, Eq. (1) implies that:

\[ \partial_t \hat{Q} = i \frac{e^*}{\hbar} \left( e^{-i\omega_J t} \hat{A} - e^{i\omega_J t} \hat{A}^\dagger \right). \]  

(2)

This is true when \( \hat{A} \) contains the unitary operator \( e^{i\hat{\varphi}} \) where the phase operator \( \hat{\varphi} \) obeys \( [\hat{\varphi}, \hat{Q}] = e^* \). In many situations, \( \partial_t \hat{\varphi} \) obeys a Josephson-type relation with \( e^* \) instead of \( 2e \) [17, 19, 37]:

\[ \omega_J = \frac{e^*}{\hbar} V_{dc}, \]  

(3)
where $V_{dc}$ is the voltage bias. Quantum averages, denoted by $\langle \ldots \rangle$, are taken over a stationary OE initial density operator $\rho_0$ ($\langle \rho_0, \mathcal{H}_0 \rangle = 0$) thereby corresponding to non-thermal occupation probabilities of many-body $\mathcal{H}_0$’s eigenstates. These can for example arise from temperature and dc-voltages biases.

Let us give some examples. In tunneling junctions between two similar or different (hybrid) conductors, such as NIN or SIN junctions, $\hat{A}$ and $\hat{I}(t)$ respectively correspond to the tunneling and electrical current operators. In Josephson junctions, $\hat{I}(t)$ is either the quasiparticle ($e^* = e$) or the pair current ($e^* = 2e$). But the form in Eq. (1) goes beyond the transfer Hamiltonian approach, as $\mathcal{H}_0$ is not split into right and left terms, so that it can incorporate all relevant screened Coulomb interactions. One can also include in $\mathcal{H}_0$ and $\hat{A}$ strong coupling to a linear or a non-linear electromagnetic environment.

In the IQHE or the FQHE at arbitrary filling factors $\nu$, $\hat{A}$ corresponds to a weak spatially extended backscattering of electrons or quasiparticles with a fractional charge $e^*$ through a QPC, acting as a beam splitter, and $\hat{I}(t)$ is the backscattering current. The unperturbed Hamiltonian $\mathcal{H}_0$ may include edge reconstruction or inhomogeneous Coulomb interactions [21, 22], or even extended tunneling processes between counter-propagating edges. As those emanate from different contacts, such processes may not be sufficient to ensure their equilibration [33].

One may also consider OE quasiparticle sources, such as quantum dots acting as energy filters or biased QPCs. As will be illustrated later in the anyon collider depicted in Fig. 1, the Josephson-type relation in Eq. (3) may break down, motivating us to keep $\omega_f$ free.

**Main OE relations:** Letting $\delta \hat{I}_\mathcal{H}(t) = \hat{I}_\mathcal{H}(t) - I_{dc}(\omega_f)$, where the subscript $\mathcal{H}$ refers to the Heisenberg representation with respect to $\mathcal{H}(t)$ in Eq. (1), we focus on the current noise:

$$S(\omega_f; t) = \langle \delta \hat{I}_\mathcal{H}(0) \Delta \hat{I}_\mathcal{H}(t) \rangle.$$  \hspace{1cm} (4)

To express $S$ at second-order in $\hat{A}$, we replace $\delta \hat{I}_\mathcal{H}(t)$ by $\hat{I}_\mathcal{H}(t)$, or, in Eq. (2), $A_f(t)$ by $\bar{A}_\mathcal{H}(t) = e^{i \mathcal{H}_0 t} A e^{-i \mathcal{H}_0 t}$. We obtain these two building blocks:

$$\hbar \Delta X_\mathcal{H}(t) = \langle \hat{A}_\mathcal{H}(0) \bar{A}_\mathcal{H}(t) \rangle$$  \hspace{1cm} (5a)

$$\hbar \Delta X_\mathcal{H}(t) = \langle \bar{A}_\mathcal{H}(0) \hat{A}_\mathcal{H}(t) \rangle.$$  \hspace{1cm} (5b)

Being evaluated in the OE regime characterized by $\mathcal{H}_0$ and $\rho_0$, these are OE correlators which don’t satisfy any kind of detailed balance equations. They determine the current noise in Eq. (4) and its Fourier transform at $\omega$:

$$S(\omega_f; t)/e^{\omega_f^2} \simeq e^{-i \omega_f t} X_\mathcal{H}(-t) + e^{i \omega_f t} X_\mathcal{H}(t).$$  \hspace{1cm} (6a)

$$S(\omega_f; \omega)/e^{\omega_f^2} \simeq X_\mathcal{H}(\omega_f - \omega) + X_\mathcal{H}(\omega_f + \omega).$$  \hspace{1cm} (6b)

In particular, the zero frequency noise reads:

$$S(\omega_f; 0)/e^{\omega_f^2} \simeq X_\mathcal{H}(\omega_f) + X_\mathcal{H}(-\omega_f),$$  \hspace{1cm} (7)

and the dc average current

$$I_{dc}(\omega_f) = \langle \hat{I}_\mathcal{H}(t) \rangle = e^{*} (X_\mathcal{H}(\omega_f) - X_\mathcal{H}(-\omega_f)).$$  \hspace{1cm} (8)

can be interpreted as the difference of two transfer rates $X_\rightarrow, X_\leftarrow$ in opposite directions [13].

Then, at a finite frequency $\omega$, the rescaled noise in Eq. (6b) is a sum of these transfer rates evaluated at two effective potential drops in two opposite directions $\pm \omega_f - \omega$. A transfer of a charge $e^*$ in each direction is associated with the emission (resp. absorption) of a photon if $\omega > 0$ (resp. $\omega < 0$) by the correlated many-body eigenstates, thus the effective potential $\pm \omega_f - \omega$ decreases (resp. increases) with respect to $\pm \omega_f$.

Comparing Eq. (6b) to Eqs. (7) and (8), we derive the central result of this Letter, an OE FDR expressing the OE current noise at finite frequency in terms of OE current average and noise at zero frequency [38]:

$$2S(\omega_f; \omega) = S(\omega_f + \omega; 0) + S(\omega_f - \omega; 0) - e^{*} I_{dc}(\omega_f + \omega) + e^{*} I_{dc}(\omega_f - \omega).$$  \hspace{1cm} (9)

Note that the first and second lines on the r.h.s. yield the symmetric and anti-symmetric parts of the noise $2S^\pm(\omega_f; \omega) = S(\omega_f, \omega) \pm S(\omega_f, -\omega)$. The high-frequency behavior of $S^+$ is indeed totally determined by its dependence on the dc bias at zero frequency:

$$2S^+(\omega_f; \omega) = S^+(\omega_f + \omega; 0) + S^+(\omega_f - \omega; 0).$$  \hspace{1cm} (10)

Moreover, using the exact relation [7, 27] $S^{-}(\omega_f; \omega) = -2 \hbar \omega \Re(Y(\omega_f, \omega))$, connecting the anti-symmetric part of the noise to the OE admittance $Y(\omega_f, \omega)$ [39], Eq. (9) enables us to extend the validity of the relation

$$2 \hbar \omega \Re(Y(\omega_f, \omega)) = e^{*} (I_{dc}(\omega_f - \omega) - I_{dc}(\omega_f + \omega))$$ \hspace{1cm} (11)
beyond the hypothesis of initial thermalization adopted in Refs. [13, 14, 36]. Since the Kramers-Kronig relation also yields $\Im(Y(\omega_J;\omega))$ in terms of $I_{dc}$, the admittance $Y(\omega_J, \omega)$ is totally determined by the dc-current/voltage characteristic.

Finally, in the initially thermalized state at electronic temperature $T_\text{el} = 1/k_B\beta$, the OE FDR (9) reduces to the previously obtained [13, 14] FDR:

$$S(\omega_J;\omega)/e^{*2} = [1 + N(\omega_J + \omega)] I_{dc}(\omega_J + \omega) + N(\omega_J - \omega) I_{dc}(\omega_J - \omega)$$

(12)
in which $N(\omega) = (e^{\beta\omega} - 1)^{-1}$, thereby repositioning a long stream of model-dependent derivations of this relation [3, 6–8, 10, 16] into an unified framework. Note that Rogovin and Scalapino’s FDR [2] is recovered from (12) by considering the symmetric noise: $S^+(\omega_J;\omega) = e^{*} \sum_\pm \coth[\beta(\omega_J \pm \omega)/2] I_{dc}(\omega_J \pm \omega)$, which we have extended beyond its original context and without assuming inversion symmetry [26].

Indeed, for an initial thermal state, the dc current in Eq. (8), though not odd, has the sign of the dc bias [13]:

$$\omega_J I_{dc}(\omega_J) \geq 0.$$  

(13)

Also, two important generic features, obtained at zero and finite frequencies, follow from Eq. (12) at a very low temperature: the Poissonian statistics and the existence of a threshold for the emitted noise at $\omega > \omega_J$ [27]. We now exploit Eq. (9) to show their common origin and their breakdown for initial OE states. For this, we use properties of $X_{\rightarrow, e}^+ (\omega_J)$ in Eq. (5) derived from their spectral decomposition [13]. As $X_{\rightarrow, e}^+ (\omega_J) \geq 0$, the zero-frequency noise in Eq. (7), compared to Eq. (8), obeys:

$$S(\omega_J;0) \geq e^{*}|I_{dc}(\omega_J)|.$$  

(14)

This leads to a lower bound on the high-frequency noise in Eq. (9) ($\Theta$ is the Heaviside function):

$$2S(\omega_J;\omega) \geq \sum_\pm e^{*} \Theta(\mp I_{dc}(\omega_J \pm \omega)) |I_{dc}(\omega_J \pm \omega)|.$$  

(15)

Let’s consider first the case when the system is initially in the ground many-body eigenstate of $\hat{H}_0$. Then only one transfer rate survives ($X_{\rightarrow, e}^- (\omega_J < 0) = X_{\rightarrow, e}^+ (\omega_J > 0) = 0$), so that Eq. (13) holds, and Eqs. (6b),(8) imply that the inequality (14) reduces to an equality: zero-frequency noise is Poissonian. As a consequence, (15) is also saturated, from which one infers the threshold for the emission noise at $\omega_J > 0$: $S(\omega_J;\omega > \omega_J) = 0$. Therefore, single charge transfer processes are Poissonian and impose energy conservation underlying the threshold.

These two features are violated when considering OE initial states: the inequality in Eq. (14) is strict, leading to a super-Poissonian zero frequency noise. So is the inequality in Eq. (15), smoothing out the threshold at $\omega_J$, due to the non-vanishing emission noise above $\omega_J$: $S(\omega_J;\omega > \omega_J) > 0$. These purely OE effects persist even at vanishing temperatures. In order to distinguish them from thermal fluctuations, which also lead to strict inequalities in Eqs. (14) and (15) (see Eqs. (13), (12) with a finite $T_\text{el}$, let us deduce the OE noise at $\omega_J = 0$ from Eq. (9), assuming inversion symmetry for simplicity:

$$S(\omega_J = 0;\omega) = S(\omega_J = \omega;\omega = 0) - e^{*} I_{dc}(\omega_J = \omega)$$  

(16)

This shows that a finite emission noise $S(\omega_J = 0;\omega > 0)$ quantifies deviations both from the Poissonian regime and from initial thermalisation, for which it would vanish.

**Applications** The FRs are alternative laws, in the OE regime, to the equilibrium FDT, thus provide similarly a robust test of analytical, numerical or experimental results for OE noise. One can, inversely, test the validity of the underlying hypotheses of our perturbative approach by checking Eqs. (9) and (10) [5], whereas the signature of a breakdown of initial thermalization [40] would be a violation of Eq. (12). In strongly correlated conductors with OE initial many-body states, a key issue is to determine $\omega_f$ in term of the experimentally controlled parameters, such as dc-voltages and temperatures when $e^{*} \neq e$ or when the Josephson-type relation Eq. (3) breaks down. This can be achieved either by measuring the admittance, using Eq. (11), or by measuring the noise both at finite and zero frequency, using Eqs. (9), (10). One can infer $\omega_f$ from the coincidence of the functions of $\omega$ on both sides of these OE FDRs.

The determination of $\omega_f$ provides the voltage drop across a strongly correlated junction in presence of a temperature gradient $\Delta T$. In particular, by imposing $I_{dc}(\omega_f) = 0$, it offers a method based on current noise measurement to infer the Seebeck coefficient from $\omega_f/\Delta T$. Note that at zero bias voltage, the temperature gradient $\Delta T$ generates a thermoelectric current $I_{dc}(\omega_f = 0) \neq 0$ [13].

The determination of $\omega_f$ is an especially acute question in the FQHE context, which goes beyond that of the fractional charge $e^{*}$ using Eq. (3) if valid, as in recent experiments [17, 19]. The important point is that, at a given filling factor $\nu$, for example 2/3, the theoretical description by effective models cannot favor one among multiple competing candidates, which may even predict different values of $e^{*}$ [34]. As of now, because of Coulomb-induced non-universal effects such as edge reconstruction, there is no clear agreement between experiments [17, 19] and effective models, predicting power laws, [33]. In this context, the OE FDR can help us sort out, among the various models, the most suitable one for the experimental data.

Let us illustrate this point in an anyon collider, to show how can the determination of $\omega_f$ help us to pinpoint the best candidate model. As depicted on Fig. 1, two de-biased QPCs inject anyons with a fractional charge $e^{*}$,
characterized by number operators $\hat{N}_{1,2}$, which collide on the central QPC, denoted cQPC. Since equilibrium reservoirs are replaced by OE sources, the backscattering noise obeys the OE FDRs given by Eqs. (9) and (10), but not that given by Eq. (12) [26]. Let’s adopt for the edge states, as in Ref.[35], an effective model characterized by two free parameters $\lambda, \delta$ which need to be known to fix the model [34]. In case $\nu = 1/(2n+1)$, one has $\lambda = \delta = \nu$, but $\lambda, \delta$ may be renormalized by Coulomb interactions and edge reconstruction, whose role can be evaluated by determining experimentally $\lambda, \delta$. Importantly, $\lambda, \delta$ intervene directly in the exploration of fractional statistics, thus affect its interpretation.

In Ref.[35], $\lambda$ renormalizes $\hat{N}_1, \hat{N}_2$ in the OE part of $A$: $A \rightarrow e^{2i\pi \lambda(\hat{N}_1-\hat{N}_2)} A$. This derives from the equation of motion method for bosonic fields with boundary conditions fixed by $\hat{N}_1, \hat{N}_2$ [22], whose higher cumulants are taken into account within the so-called OE bosonisation [32]. If the QPCs are tuned at weak transmissions and low temperatures, $\hat{N}_{1,2}$ are Poissonian, so that their cumulants are proportional to the injected average currents $I_{1,2} = \langle d\hat{N}_{1,2}/dt \rangle$, inducing an effective dc drive:

$$\omega_J = \frac{2\pi}{e^*} \sin(2\pi \lambda) I_-, \quad \text{(17)}$$

where $I_- = I_1 - I_2$. Due to the strongly correlated Hall liquid in the sources, $I_{1,2}$ have a non-linear behavior on $V_1, V_2$, and so does $\omega_J$, which then violates the Josephson-type relation in Eq.(3) ($V_{dc} = V_1 - V_2$). By using the OE FDR to determine $\omega_J$, and assuming $e^*$ is already inferred from intrinsic noise of the QPCs, one can determine $\sin(2\pi \lambda)$, thus $\lambda$, from Eq.(17), as $I_{1,2}$, thus $I_-$, can be measured directly in the outgoing edges [28]. Indeed, we can show that $\lambda$ describes plasmonic propagation between the injection point and the cQPC, thus is related to the dc conductance by using the scattering approach for plasmons [21, 22]. One can infer the second parameter $\delta$ from the model-dependent expressions of the dc current and zero-frequency noise in Ref.[35]:

$$I_{dc}(\omega_J) = C' \sin(\pi \delta) \Im(\omega_+ + i\omega_J)^{2\delta-1} \quad \text{(18a)}$$

$$S(\omega_J; \omega = 0) = e^{*} C' \cos(\pi \delta) \Re(\omega_+ + i\omega_J)^{2\delta-1}. \quad \text{(18b)}$$

Here $C'$ is a prefactor, $\omega_J$ given by Eq.(17), and $\omega_+ = 2\pi \sin^2(\pi \lambda) I_+ / e^*$, with $I_+ = I_1 + I_2$. Though $\delta$ controls the power law, this is not an easy way to extract it, so we propose an alternative way. We notice first that, compared to equilibrium reservoirs, the validity domain of perturbation is extended: for high enough $\omega_+$, one can lower $\omega_J$ down to 0 by injecting equal currents $I_1 = I_2$ through tuning $V_1 \simeq V_2$. This is precisely the regime where anyonic statistics is best revealed [35]. Then using Eqs. (18a),(18b), one has:

$$S(\omega_+ = 0; \omega = 0) = e^{*} \cot(\pi \delta) \cot(\pi \lambda) I_+ \left( \frac{\partial I_{dc}}{\partial I_-} \right)_{I_- = 0},$$

proportional to the total injected current $I_+$ and the derivative of $I_{dc}$ at $\omega_J = 0$ (depending on $I_+$). The atypical ”Fano factor” cot$(\pi \delta)$ cot$(\pi \lambda)$ then provides $\delta$ once $\lambda$ is determined. Now we can express explicitly the high-frequency backscattering noise in Eq. (10), by injecting the dc expressions in Eqs. (18a),(18b). In particular, at $I_- = 0$, as inversion symmetry now holds, so does Eq. (16) with a fixed $\omega_+$:

$$S(\omega_J = 0; \omega) = -C' \Im(-\omega + i\omega_+)^{2\delta-1}. \quad \text{Conclusion}$$

In this Letter, we have derived perturbative OE FRs and FDRs showing that high-frequency noise is completely determined by zero-frequency transport. Due to OE initial states, zero-frequency noise is super-Poissonian, and washes out the threshold for the emitted spectrum above the dc drive. The OE FDRs offer experimental tests of their underlying hypothesis [5], in particular breakdown of initial thermalization. They provide a noise measurement method of the Seebeck coefficient in a strongly correlated junction. In the FQHE, the OE FRs permit to probe the fractional charges without relying on the microscopic model [13, 17, 19] nor on initial thermal equilibrium. The latter breaks down in the anyon collider used to prove anyonic statistics [28, 35]. The high-frequency backscattering noise does not obey the previously FDRs [13, 14] but the OE FDR derived in this Letter, which offers a protocol to extract a non-universal parameter that depends on the structure of the edge channels and enters anyonic statistics. This may prove useful in forthcoming investigations of anyonic statistics through finite-frequency correlations. Future perspectives include using the OE FDRs for shot noise thermometry [30, 41], as well for thermoelectricity in the anyon collider. Beyond current noise, they can be applied to the voltage noise across a phase-slip Josephson junction [42] as well as to the spin current noise in spin Hall insulators [43].

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SUPPLEMENTAL MATERIAL

The perturbative approach

The Hamiltonian within the present approach reads:

$$H(t) = H_0 + e^{-i\omega_J t} \hat{A} + e^{i\omega_J t} \hat{A}^\dagger$$

where we don’t specify $H_0$ nor the weak operator $\hat{A}$, and $\omega_J$ is a dc drive. In the case of a tunneling junction, and contrary to the transfer Hamiltonian description, we don’t decouple right and left sides of the junction, which could circumvent difficulties in the construction of the global Hilbert space [44]. We recall the minimal conditions for the validity of the theory:
1. $\hat{A}$ is a weak operator which does not depend on $\omega J$, and with respect to which second-order expansion is valid and yields non-vanishing results.

2. The initial density matrix $\rho_0$ commutes with the unperturbed Hamiltonian $H_0$:

$$[\rho_0, H_0] = 0,$$  

(20)

thus is diagonal with respect to the stationary OE many-body eigenstates of $H_0$.

3. Letting:

$$\hat{A}_{H_0}(t) = e^{iH_0 t} \hat{A} e^{-iH_0 t},$$  

(21)

one requires the following cancellation:

$$\langle \hat{A}_{H_0}(t) \rangle = \langle \hat{A}_{H_0}(t) \hat{A}_{H_0}(0) \rangle = 0,$$  

(22)

Let us discuss in more details the last condition, Eq. (22). If the diagonal elements of $\rho_0$ are determined only by the energy of $H_0$’s eigenstates, we have shown that $I_{dc}(\omega J = 0) = 0$ [13]. For Josephson junctions, supercurrent is negligible with a dissipative environment or a magnetic field. In case one has a temperature gradient, such a condition on $\rho_0$ is violated, so one has $I_{dc}(\omega J = 0) \neq 0$ even if Eq. (22) holds.

In the paper, focussing on charge transport, we have introduced, for clarity, additional though not necessary requirements, which ensure systematically Eq. (22). Let us explain why. First, we have assumed there is a phase operator $\hat{\varphi}$ such that, implicitly, $\hat{A} = e^{i\hat{\varphi}} \hat{A}$, where $\hat{A}$ is an unspecified operator commuting with $\hat{\varphi}$. Second, we assume the unperturbed Hamiltonian $H_0$ conserves the charge operator $Q$ conjugated to $\hat{\varphi}$, thus $H_0$ does not depend on $\hat{\varphi}$. A charge $e^*\hat{e}$ is introduced through the commutator: $[\hat{\varphi}, \hat{Q}] = e^*\hat{e}$. For transfer of multiple values of charges, either one process associated with one value dominates the others, otherwise the noise is a superposition of terms obeying individually the OE FRs we have shown. Generically, the same charge $e^*\hat{e}$ enters into the Josephson-type relation: $\omega J = e^*V_{dc}/\hbar$. But one could have, depending on coupling to the bias $V_{dc}$ and the OE setup, a different model-dependent charge $q$ such that $\omega J = q\hbar V_{dc}$. This happens for instance in quantum wires with reservoirs, where $e^*$ depends on interactions, but $q$ depends on the phase $\omega J$, in which case the noise still verifies the OE FDRs.

In a junction, $Q$ corresponds to a charge difference operator, which does not change in the perturbation $A$. One can show that $\langle \hat{A}_{H_0}(t) \rangle = 0$ because $< e^{i\hat{\varphi}_{H_0}(t)} > = 0$, which expresses precisely this conservation. One has also:

$$\langle \hat{A}_{H_0}(t) \hat{A}_{H_0}(0) \rangle = X_\varphi \langle \hat{A}_{H_0}(t) \hat{A}_{H_0}(0) \rangle = 0,$$

where $X_\varphi = < e^{i\hat{\varphi}_{H_0}(t)} e^{i\hat{\varphi}_{H_0}(0)} >$. We have used $X_\varphi = 0$ as $H_0$ does not depend on $\hat{\varphi}$, which is a trivial case of gauge invariance with respect to $\hat{\varphi}$ (or $U(1)$ symmetry). Indeed $X_\varphi$ vanishes because they correspond to overlaps of states with different charges. In a more general context [13], it is sufficient, to get Eq. (22), that the action $S_0$ associated with $H_0$, obeys, for any real $c$:

$$S_0(\hat{\varphi} + c) = S_0(\hat{\varphi}).$$

(23)

This is indeed a sufficient condition replacing $\varphi$ by an alternative phase operator $\varphi_1$ on which $H_0$ depends, not necessarily in a quadratic form. For instance, $\varphi_1$ can then be associated with a linear or even non-linear electromagnetic environment, included in $H_0$. Then the phase $\varphi$ above corresponds to intrinsic electronic degrees of freedom of the junction, thus $\hat{A}$ contains $e^{i\hat{\varphi}} e^{i\hat{\varphi}_1}$. If one considers the charge operator $Q_1$ conjugate to $\varphi_1$, its time derivative contains $I(t)$, in addition to $\delta H_0/\delta \varphi_1$.

### Inversion symmetry or initial thermalization

In the perturbative approach, in view of Eq. (22), one ends up with the two OE correlators in Eq. (5a):

$$\hbar^2 X \to (t) = \langle \hat{A}_{H_0}^1(t) \hat{A}_{H_0}(0) \rangle$$  

$$\hbar^2 X \leftarrow (t) = \langle \hat{A}_{H_0}(0) \hat{A}_{H_0}^1(t) \rangle,$$  

(24)

evaluated in the OE regime characterized by $H_0, \rho_0$ to which the system stays close through perturbation theory. They determine average current and noise:

$$I_{dc}(\omega_{dc}) \approx e^* \left[ X \leftarrow (\omega_{dc}) - X \to (\omega_{dc}) \right].$$

(25)

$$S(\omega J; \omega) / e^{*2} \approx X(\omega J - \omega) + X(\omega J + \omega).$$

(26)

The fact that only two independent correlators enter allows us to establish various links, such as the perturbative OE FDR:

$$2S(\omega J; \omega) = S(\omega J + \omega; 0) + S(\omega J - \omega; 0) - e^* I_{dc}(\omega J + \omega) + e^* I_{dc}(\omega J - \omega).$$

(27)

Once symmetrized with respect to $\omega$, we have obtained:

$$2S^+(\omega J; \omega) = S^+(\omega J + \omega; 0) + S^+(\omega J - \omega; 0).$$

(28)

Notice that validity of perturbation can be stated by a weak dc differential conductance $G_{dc}(\omega J) = dI_{dc}(\omega J)/dV_{dc}$ compared to a model-dependent scale.

We have not required any of two hypothesis simultaneously adopted by almost all works on finite frequency noise: initial thermalization and an odd dc current. This is hidden in the fact that, within the perturbative approach, $X \to$ and $X \leftarrow$ are two independent OE correlators. Each of these two restrictions, which we will discuss separately, amounts to impose each time a link between $X \to$ and $X \leftarrow$. 

First, particle-hole symmetry is generically the underlying reason for oddness of the current. But we define an alternative symmetry suitable for strongly correlated systems [13], by requiring, for the OE correlators in Eq. (2a), \( X \rightarrow t \rightarrow X_{-t} \). Thus their Fourier transforms, evaluated here at the dc drive, are related to a unique function \( X \):

\[
X_{-\omega}(\omega_j) = X(\omega) = X_{-\omega}(\omega_j).
\]

This can be described as inversion symmetry, as the transfer rate in one direction is obtained by reversing the sign of the dc drive. With this hypothesis, and with respect to the dc drive \( \omega_j \), the dc current in Eq.(25) is now odd: \( I_{dc}(\omega_j) = -I_{dc}(\omega_j) \) and the noise in Eq.(26) is even: \( S(\omega_j; \omega) = S(-\omega_j; \omega) \).

In this case, let’s specify Eq. (27) to \( \omega = 0 \). One has still an OE noise given by:

\[
S(\omega_j; 0; \omega) = S(\omega_j = \omega; \omega = 0) - e^S I_{dc}(\omega_j = \omega). \tag{30}
\]

One gets also the symmetrized noise with respect to frequency, \( S^\pm(\omega_j = 0; \omega) = S^\pm(\omega_j = \omega; 0) \), thus frequency and dc drive exchange their roles; thus one can infer one function from the other, depending on which one is the most easily accessible theoretically or experimentally.

Second, we consider now a link between the OE correlators in Eq.(24):

\[
X_{-\omega}(\omega_j) = e^{\beta \omega} X_{-\omega}(\omega). \tag{31}
\]

Though \( \beta \) could acquire a different meaning, this link arises from the choice of an initial thermalization at \( T_{cl} = 1/\beta \):

\[
\hat{\rho}_0 = e^{-\beta \hbar \omega_0}/T \cdot e^{-\beta \hbar \omega_0}. \tag{32}
\]

This leads us to recover the OE-FDR for the FF non-symmetrized [13, 14] noise in terms of the dc average current:

\[
S(\omega_j; \omega)/e^* = [1 + N(\omega_j + \omega)] I_{dc}(\omega_j + \omega) + N(\omega_j - \omega) \langle I_{dc}(\omega_j - \omega) \rangle, \tag{33}
\]

where \( N(\omega) = (e^{\beta \omega} - 1)^{-1} \). We have also obtained in the FQHE or in a conductor connected to an electromagnetic environment \([7, 10]\). Here we choose to give a different derivation of Eq. (12), compared to [13, 14], by exploiting directly the novel FDRs. Using Eqs. (8,31), the zero-frequency noise obeys:

\[
S(\omega_j; 0)/e^* = \coth \left( \frac{\beta \omega_j \omega_j}{2} \right) I_{dc}(\omega_j). \tag{34}
\]

Injecting it into the expression of the non-symmetrized finite-frequency noise in Eq. (27), we recover Eq. (33). One can also get directly the symmetrised noise, injecting Eq. (34) in Eq. (28): \( S^\pm(\omega_j; \omega) = e^* \sum_{\nu} \coth [\beta(\omega_j \pm \omega)/2] I_{dc}(\omega_j \pm \omega) \). This is the same form as Rogovin and Scalapino’s FDR, here extended to a much larger large family of strongly correlated systems and circuits described by Eq.(19). It holds beyond the particle-hole symmetry, thus oddness of the current, on which Rogovin and Scalapino have insisted [2].

It is only when we assume inversion symmetry, thus Eq. (29), that, using Eq. (31), we recover the detailed balance equation for a unique transfer rate \( X(\omega) = X_{-\omega}(\omega_j) = e^{\beta \omega} X(-\omega) \), as in Ref.[2]. Interestingly, even without inversion symmetry, the equilibrium noise, now given by \( S_{eq}(\omega) = S(\omega_j = 0; \omega) \), obeys:

\[
S_{eq}(-\omega) = e^{2\omega} S_{eq}(\omega), \tag{31}\]

and the microscopic model.

Let’s now discuss the anyon collider in the FQHE. Here we specify, as in [35], to two QPCs with weak backscattering amplitudes \( \Gamma_{1,2} \), to \( \nu \) a simple fraction with an effective bosonic model, thus four chiral fields intervene here. The injected anyons into the upper/down edges collide at the beam splitter, where we allow for extended and weak backscattering amplitudes \( \Gamma(x) \). The finite frequency noise of the backscattering current \( I_{dc} \) obeys the OE FDRs independently on the fractional filling factor \( \nu \) and the microscopic model.

The anyon collider

![An anyon collider setup in the FQHE](image)

*FIG. 2. An anyon collider setup in the FQHE. Here we specify, as in [35], to two QPCs with weak backscattering amplitudes \( \Gamma_{1,2} \), to \( \nu \) a simple fraction with an effective bosonic model, thus four chiral fields intervene here. The injected anyons into the upper/down edges collide at the beam splitter, where we allow for extended and weak backscattering amplitudes \( \Gamma(x) \). The finite frequency noise of the backscattering current \( I_{dc} \) obeys the OE FDRs independently on the fractional filling factor \( \nu \) and the microscopic model.*
processes with amplitude $\Gamma(x)$ (up to some prefactors):
\[
\hat{A} = \int dx \Gamma(x) e^{i \sqrt{\pi} [\phi_u(x) - \phi_d(x)]}.
\] (35)

One adds, for the two injecting QPCs (up to prefactors):
\[
\begin{align*}
\mathcal{H}_1 &= \Gamma_1 e^{i \sqrt{\pi} [\phi_1 - \phi(x_1)]} + h.c. \\
\mathcal{H}_2 &= \Gamma_2 e^{i \sqrt{\pi} [\phi_2 - \phi(x_2)]} + h.c.
\end{align*}
\] (36)

One has also to add the linear coupling terms between $\partial_x \phi_{1,2}$ to the dc voltages $V_{1,2}$. One could carry on perturbation with respect to weak $\Gamma_1, \Gamma_2, \Gamma$, but it turns out that one gets divergent results at zero temperature. This divergence was noticed in a similar geometry with a unique QPC, in Ref.[45], and we have explained its underlying mechanism in Ref.[46].

The OE bosonisation approach [32] has the advantage to take into account the QPCs in a non-perturbative way. It extends the equation of motion method with boundary conditions, we have initiated in [22, 47]. Boundary conditions are now given by the injected number operators $\hat{N}_{1,2}$. By solving the chiral equations of motion for $\phi_u, \phi_d$, one gets their translation $\phi_{u,d} \rightarrow (\hat{N}_1 - \hat{N}_2)$, thus $A \rightarrow e^{2\pi \lambda (\hat{N}_1 - \hat{N}_2)} A$. The parameter $\lambda$ describes plasmonic propagation along the upper edge, and we can relate it to the dc conductance without the QPCs, using the plasmon approach [22, 47]. Thus $\lambda = 1/m$, which is the value of the quantized dc conductance, nonetheless $\lambda$ deviates from this value by edge reconstruction.

The backscattering current and noise associated with Eq.(35), to second order with respect to the backscattering amplitude $\Gamma(x)$, obey Eq. (9), with the dc drive $\omega_J = 2\pi \sin(2\pi\lambda)(I_1 - I_2)/e^*$. Using the explicit expressions for the dc backscattering current and noise, one can deduce the finite-frequency non-symmetrized noise. In particular, at $I_1 = I_2$, thus at $\omega_J = 0$, we have $S(\omega_J = 0; \omega) = \frac{-C'\delta(\omega + i\omega_+)\gamma^{2\delta-1}}{\omega_+\gamma^2}$, where $\omega_+ = 2\pi \sin^2(\pi\lambda)(I_1 + I_2)/e^*$. If $\omega$ is high enough, we can let $\omega_+ = 0$ to get the equilibrium noise: $S_{eq}(\omega) = C'\sin(2\pi\delta)\omega^{2\delta-1}$. We notice that $S(\omega_J = 0; \omega) = S_{eq}(\omega)$ whenever $\delta = 1$, thus for a linear dc current, or when $\omega \ll 1$, so that one is close to a thermal state [35], which we can understand through the reduction of the OE contribution $\lambda(\hat{N}_1 - \hat{N}_2)$.

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To deduce the excess noise, $S(\omega_J; \omega) - S_{eq}(\omega)$, one cannot infer the equilibrium noise $S_{eq}(\omega)$ from $S(\omega_J = 0; \omega)$, as all additional dc drives must vanish too.

$Y(\omega_J, \omega)$ is the response of average current to a small ac modulation $\delta V(t) = v_{ac} e^{i\omega t}$ superimposed on $V_{dc}$. In particular, $Y(\omega_J, \omega = 0) = G_{dc}(\omega_J)$.  

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[32] The OE formalism we have developed in Ref.[6, 7] is also an OE bosonisation; it takes into account the effect of the second cumulant at a QPC, thus its high frequency noise, on the bosonic Green’s functions evaluated in any arbitrary distance from that QPC. This could be sufficient only if higher cumulants can be ignored, as in the IQHE with a plasmonic dispersion addressed in Ref. [30]