ON LARGE VALUES OF WEYL SUMS

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Abstract. A special case of the Menshov–Rademacher theorem implies for almost all polynomials $x_1Z + \ldots + x_dZ^d \in \mathbb{R}[Z]$ of degree $d$ for the Weyl sums satisfy the upper bound

$$\left| \sum_{n=1}^{N} \exp\left(2\pi i \left(x_1n + \ldots + x_d n^d\right)\right) \right| \leq N^{1/2+o(1)}, \quad N \to \infty.$$ 

Here we investigate the exceptional sets of coefficients $(x_1, \ldots, x_d)$ with large values of Weyl sums for infinitely many $N$, and show that in terms of the Baire categories and Hausdorff dimension they are quite massive, in particular of positive Hausdorff dimension in any fixed cube inside of $[0, 1]^d$. We also use a different technique to give similar results for sums with just one monomial $xn^d$. We apply these results to show that the set of poorly distributed modulo one polynomials is rather massive as well.

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2010 Mathematics Subject Classification. 11K38, 11L15, 28A78, 28A80.

Key words and phrases. Weyl sum, exceptional set, Vinogradov mean value theorem, rational exponential sums, Baire category, Hausdorff dimension.
1. Introduction

1.1. Motivation. Here we consider a new type of problems of metric number theory where the vectors of real numbers are classified by the size of the corresponding Weyl sums given by (1.1) below, rather than by their Diophantine approximation properties as in the classical settings, see [3, 8].

Clearly both points of view are ultimately related and operated in similar notions such as the Lebesgue measure and Hausdorff dimension. They are also both related to the question of uniformity of distribution modulo one of fractional parts of real polynomials. However, our study of sets of large Weyl sums also uses several new ideas and techniques. We believe that these ideas and concrete results on such a very powerful and versatile tool as exponential sums can find applications to other problems. In particular, in Section 1.4 below we give one of such applications and show that the set of polynomials which are poorly distributed modulo one is rather massive (in fact, our results are quantitative and thus more precise).

In problems of this kind, the case $d \geq 3$ is much harder than the case $d = 2$. The main reason is that Lemma 2.3 below, giving an exact size
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of Gauss sums, which we have for the case $d = 2$, does not in general have any analogues for $d \geq 3$, see also Remark 2.8 below.

1.2. Set-up and background. We now describe our main objects of study.

For an integer $d \geq 2$, let $T_d = (\mathbb{R}/\mathbb{Z})^d$ denote the $d$-dimensional unit torus.

For a vector $x = (x_1, \ldots, x_d) \in T_d$ and $N \in \mathbb{N}$, we consider the exponential sums

$$S_d(x; N) = \sum_{n=1}^{N} e \left( x_1 n + \ldots + x_d n^d \right),$$

which are commonly called Weyl sums, where throughout the paper we denote $e(x) = \exp(2\pi ix)$. From the Parseval identity

$$\int_{T_d} |S_d(x; N)|^2 \, dx = N$$

one immediately concludes that for any fixed $\alpha > 0$ the set of $x \in T_d$ with $|S_d(x; N)| \geq N^{\alpha}$ is of Lebesgue measure at most $N^{1-2\alpha}$, which is nontrivial when $1/2 < \alpha < 1$.

Furthermore, from the Vinogradov mean value theorem, in the currently known form

$$\int_{T_d} |S_d(x; N)|^{2s(d)} \, dx \leq N^{s(d)+o(1)} \quad \text{as } N \to \infty,$$

where $s(d) = d(d+1)/2$, due to Bourgain, Demeter and Guth [2] (for $d \geq 4$) and Wooley [22] (for $d = 3$) (see also a more general form due to Wooley [24]), one can derive a much stronger bound $N^{s(d)(1-2\alpha)+o(1)}$ when $1/2 < \alpha < 1$.

In fact, a special case of the Menshov–Rademacher theorem, see [10, p. 251], implies that for almost all $x \in T_d$ (with respect to the Lebesgue measure) we have

$$|S_d(x; N)| \leq N^{1/2} (\log N)^{3/2+o(1)}, \quad \text{as } N \to \infty.$$  

For completeness we give a proof of (1.2) in Appendix A.

Hence if for $0 < \alpha < 1$ we define the set

$$E_{\alpha,d} = \{ x \in T_d : |S_d(x; N)| \geq N^{\alpha} \text{ for infinitely many } N \in \mathbb{N} \},$$

and define

$$\vartheta_d = \inf \{ \alpha > 0 : \lambda(E_{\alpha,d}) = 0 \},$$

where we use $\lambda(A)$ to denote the Lebesgue measure of $A \subseteq T_d$, then by (1.2) we have

$$\vartheta_d \leq 1/2.$$
In fact we make:

**Conjecture 1.1.** For each integer \( d \geq 2 \) we have

\[
\vartheta_d = 1/2.
\]

Here we are mostly interested in the structure of the set of exceptional \( x \in T_d \) for which (1.2) does not hold. For convenience we call \( E_{\alpha,d} \) the exceptional set for each \( 0 < \alpha < 1 \) and \( d \in \mathbb{N} \). Thus we study the exceptional sets \( E_{\alpha,d} \) and show that they are massive enough in a sense of Baire categories and the Hausdorff dimension.

### 1.3. Main results

Recall that a subset of \( \mathbb{R}^d \) is called nowhere dense if its closure in \( \mathbb{R}^d \) has an empty interior. We now recall the following:

**Definition 1.2.** A subset of \( \mathbb{R}^d \) is of the first Baire category if it is a countable union of nowhere dense sets; otherwise it is called of the second Baire category.

For the basic properties and various applications of Baire categories we refer to [17, 19].

We now show that the complements of the sets \( E_{\alpha,d} \) are small.

**Theorem 1.3.** For each \( 0 < \alpha < 1 \) and integer \( d \geq 2 \), the subset \( T_d \setminus E_{\alpha,d} \) is of the first Baire category.

Alternatively, Theorem 1.3 is equivalent to the statement that the complement \( T_d \setminus \Xi_d \) to the set

\[
\Xi_d = \{ x \in T_d : \forall \varepsilon > 0, |S_d(x;N)| \geq N^{1-\varepsilon} \text{ for infinitely many } N \in \mathbb{N} \}
\]

is of first category. Indeed, let \( \alpha_j = 1 - 1/j \), \( j = 1, 2, \ldots \). Then

\[
T_d \setminus \Xi_d = T_d \setminus \left( \bigcap_{j=1}^{\infty} E_{\alpha_j,d} \right) = \bigcup_{j=1}^{\infty} (T_d \setminus E_{\alpha_j,d})
\]

is a countable union of first category sets, and is of first category too. Since also for any \( 0 < \alpha < 1 \) we have \( \Xi_d \subseteq E_{\alpha,d} \), we obtain the desired equivalence.

For sets of Lebesgue measure zero, it is common to use the Hausdorff dimension to describe their size; for the properties of the Hausdorff dimension and its applications we refer to [6, 15]. We recall that for \( U \subseteq \mathbb{R}^d \)

\[
\text{diam } U = \sup\{|u - v|_{L^2} : u, v \in U\}
\]

where \( ||w||_{L^2} \) is the Euclidean norm in \( \mathbb{R}^d \).
Definition 1.4. The Hausdorff dimension of a set $A \subseteq \mathbb{R}^d$ is defined as
\[
\dim A = \inf \left\{ s > 0 : \forall \varepsilon > 0, \exists \{U_i\}_{i=1}^{\infty}, \, U_i \subseteq \mathbb{R}^d, \right. \\
\left. \text{such that } A \subseteq \bigcup_{i=1}^{\infty} U_i \text{ and } \sum_{i=1}^{\infty} (\text{diam } U_i)^s < \varepsilon \right\}.
\]

We show that for $d \geq 2$ and any $0 < \alpha < 1$ the exceptional set $E_{\alpha,d}$ is everywhere rich in a sense that its intersection $E_{\alpha,d}(Q) = E_{\alpha,d} \cap Q$, with any cube $Q \subseteq T_d$, is of positive Hausdorff dimension, and give an explicit lower bound on this dimension.

We now define
\[
\kappa_d = \max_{\nu=1, \ldots, d} \min \left\{ \frac{1}{2\nu}, \frac{1}{2d - \nu} \right\}.
\]

We note that
\[
\lim_{d \to \infty} d\kappa_d = \frac{3}{4}
\]
and in fact if $3 \mid d$ then $\kappa_d = 3/4d$.

Theorem 1.5. For each $0 < \alpha < 1$ and any cube $\Omega \subseteq T_d$ we have

(i) for $d = 2$,
\[
\dim E_{\alpha,2}(\Omega) \geq \min\{3/2, 3(1 - \alpha)\};
\]
(ii) for $d \geq 3$,
\[
\dim E_{\alpha,d}(\Omega) \geq \min\{\kappa_d, 2\kappa_d(1 - \alpha)\}.
\]

Note that for $0 < \alpha \leq 1/2$ Theorem 1.5 asserts that
\[
\dim E_{\alpha,d}(\Omega) \geq \begin{cases} 3/2, & \text{for } d = 2, \\ \kappa_d, & \text{for } d \geq 3. \end{cases}
\]
However Conjecture 1.1 asserts that for any $\alpha \in (0, 1/2)$ and any integer $d \geq 2$ we have $\lambda(E_{\alpha,d}) > 0$ and hence we expect
\[
\dim E_{\alpha,d} = d,
\]
and even stronger
\[
\dim E_{\alpha,d}(\Omega) = d,
\]
for any cube $\Omega \subseteq T_d$. We remark that in fact we expect $\lambda(E_{\alpha,d}) = 1$ for any $\alpha \in (0, 1/2)$, see Conjecture 6.2 below.

Our approach to Theorem 1.5 is based on a version of the classical Jarník–Besicovitch theorem, see [6, Theorem 10.3] or [1] and on the
investigation of the distribution of large values of rational exponential sums with prime denominators. This question is of independent interest and it also gives us an opportunity to mention very interesting but perhaps not so well-known results of Knizhnerman and Sokolina [11, 12] about large and small values of rational exponential sums.

Furthermore, we also investigate the monomial sums

$$\sigma_d(x; N) = \sum_{n=1}^{N} e(xn^d)$$

to which the above technique does not apply. Similarly to the sets $E_{\alpha,d}$, for each $0 < \alpha < 1$ let

$$E_{\alpha,d} = \{ x \in [0, 1) : |\sigma_d(x; N)| \geq N^\alpha \text{ for infinitely many } N \in \mathbb{N} \}.$$

Similarly to Theorem 1.3 and Theorem 1.5, we also obtain the corresponding results for the monomial sums.

**Theorem 1.6.** For each $0 < \alpha < 1$ and each integer $d \geq 2$, the set $[0, 1) \setminus E_{\alpha,d}$ is of first Baire category.

We also show the positivity of the Hausdorff dimension of

$$E_{\alpha,d}(I) = E_{\alpha,d} \cap I$$

for any interval $I \subseteq \mathbb{T}$. In analogy to Theorem 1.5 we have the following result.

**Theorem 1.7.** For each $0 < \alpha < 1$ and any interval $I \subseteq \mathbb{T}$, we have

(i) for $d = 2$,

$$\dim E_{\alpha,2}(I) \geq \min\{1, 2(1 - \alpha)\};$$

(ii) for $d \geq 3$,

$$\dim E_{\alpha,d}(I) \geq (1 + 1/d) \min \{2/(d + 2), 1 - \alpha \}.$$

Note that for $0 < \alpha \leq 1/2$ Theorem 1.7 (i), for the case $I = \mathbb{T}$, asserts that

$$\dim E_{\alpha,2} = 1.$$

In fact only the case of $\alpha = 1/2$ is of interest as for $\alpha < 1/2$, and this is instant from the result of Fiedler, Jurkat and Körner [7, Theorem 2].

For $0 < \alpha \leq d/(d + 2)$ with $d \geq 3$ Theorem 1.7 (ii), for the case $I = \mathbb{T}$, asserts that

$$\dim E_{\alpha,d} \geq \frac{2(d + 1)}{d(d + 2)}.$$

However we conjecture that for each $0 < \alpha \leq 1/2$ and each $d \geq 2$ one has

$$\dim E_{\alpha,d} = 1.$$
and perhaps even stronger
\[ \dim \mathcal{E}_{\alpha,d}(J) = 1, \]
for any interval \( J \subseteq T \).

1.4. **Applications to uniform distribution modulo one.** A quantitative way to describe the *uniformity of distribution modulo one* is given by the *discrepancy*, see [5].

**Definition 1.8.** Let \( x_n, n \in \mathbb{N} \), be a sequence in \([0,1)\). The discrepancy of this sequence at length \( N \) is defined as
\[ D_N = \sup_{0 \leq a < b \leq 1} \left| \# \{ 1 \leq n \leq N : x_n \in (a,b) \} - (b-a)N \right|. \]

Recalling that a sequence is uniform distributed modulo one if and only if the corresponding discrepancy
\[ D_N = o(N) \quad \text{as } N \to \infty, \]
see [5, Theorem 1.6] for a proof. We note that sometimes in the literature the scaled quantity \( N^{-1}D_N \) is called the discrepancy, since our argument looks cleaner with Definition 1.8, we adopt it here.

For \( x \in T_d \) and the sequence
\[ x_1n + \ldots + x_d n^d, \quad n \in \mathbb{N}, \]
we denote by \( D_d(x; N) \) the corresponding discrepancy. Motivated by the work of Wooley [23, Theorem 1.4], the authors [4] have shown that for almost all \( x \in T_d \) with \( d \geq 2 \) one has
\[ D_d(x; N) \leq N^{1/2+o(1)} \quad \text{as } N \to \infty. \]

In view of Lemmas 2.2 and 5.1 below, Conjecture 1.1 is equivalent to the statement that the exponent 1/2 in (1.4) cannot be improved.

Thus, the bound (1.4), combined with Lemma 5.1 below, provides yet another way to obtain that
\[ S_d(x; N) \ll N^{1/2+o(1)}, \quad \text{as } N \to \infty, \]
holds for almost all \( x \in T_d \) (which is a slightly less precise version of (1.2)).

Let
\[ \mathcal{D}_{\alpha,d} = \{ x \in T_d : D_d(x; N) \geq N^\alpha \text{ for infinitely many } N \in \mathbb{N} \}. \]

**Theorem 1.9.** For each \( 0 < \alpha < 1 \) and integer \( d \geq 2 \) the subset \( T_d \setminus \mathcal{D}_{\alpha,d} \) is of the first Baire category.
Note that this is equivalent to the statement that the complement $T_d \setminus D_d$ to the set
\[
D_d = \{ x \in T_d : \forall \varepsilon > 0, \ D_d(x; N) \geq N^{1-\varepsilon} \text{ for infinitely many } N \in \mathbb{N} \}
\]
is of first Baire category.

For any cube $Q \subseteq T_d$ denote $D_{\alpha,d}(Q) = D_{\alpha,d} \cap Q$.

**Theorem 1.10.** For each $0 < \alpha < 1$ and any $Q \subseteq T_d$, we have

(i) for $d = 2$,
\[
\dim D_{\alpha,2}(Q) \geq \min\{3/2, 3(1-\alpha)\};
\]

(ii) for $d \geq 3$,
\[
\dim D_{\alpha,d}(Q) \geq \min\{\kappa_d, 2\kappa_d(1-\alpha)\}.
\]

In the case of monomials, For $x \in [0,1)$ we denote by $\Delta_d(x; N)$ the discrepancy of the sequence $xn^d$, $n \in \mathbb{N}$ and set
\[
D_{\alpha,d} = \{ x \in [0,1) : \Delta_d(x; N) \geq N^\alpha \text{ for infinitely many } N \in \mathbb{N} \}.
\]

We have the following analogues of Theorems 1.9 and 1.10

**Theorem 1.11.** For each $0 < \alpha < 1$ and integer $d \geq 2$ the subset $[0,1) \setminus D_{\alpha,d}$ is of the first Baire category.

Furthermore, we also have the following result. For an interval $I \subseteq T$ denote $D_{\alpha,d}(I) = D_{\alpha,d} \cap I$.

**Theorem 1.12.** For each $0 < \alpha < 1$ and any interval $I \subseteq T$, we have

(i) for $d = 2$,
\[
\dim D_{\alpha,2}(I) \geq \min\{1, 2(1-\alpha)\};
\]

(ii) for $d \geq 3$,
\[
\dim D_{\alpha,d}(I) \geq \min\{1 + 1/d, 2/(d+2), 1 - \alpha\}
\]

We remark that the case $d = 1$ is a special case. For the linear sequence $(nx)$ the celebrated result of Khintchine, see [5, Theorem 1.72], implies that for almost all $x \in [0,1)$ one has
\[
D_1(x; N) \leq N^{o(1)}, \quad \text{as } N \to \infty.
\]
2. Preliminaries

2.1. Notation and conventions. Throughout the paper, the notations $U = O(V)$, $U \ll V$ and $V \gg U$ are equivalent to $|U| \leq c|V|$ for some positive constant $c$, which throughout the paper may depend on the degree $d$ and occasionally on the small real positive parameters $\varepsilon$ and $\delta$.

We use $\# \mathcal{X}$ to denote the cardinality of set $\mathcal{X}$.

The letter $p$, with or without a subscript, always denotes a prime number.

We always identify $\mathbb{T}_d$ with half-open unit cube $[0, 1)^d$, in particular we naturally associate Euclidean norm $\|x\|_{L^2}$ with points $x \in \mathbb{T}_d$.

We say that some property holds for almost all $x \in \mathbb{T}_d$ if it holds for a set $\mathcal{X} \subseteq \mathbb{T}_d$ of Lebesgue measure $\lambda(\mathcal{X}) = 1$.

We always keep the subscript $d$ in notations for our main objects of interest such as $E_{\alpha, d}$, $S_d(x; N)$ and $\mathbb{T}_d$, but sometimes suppress it in auxiliary quantities.

2.2. Complete rational exponential sums and uniform distribution. We first recall the classical Weil bound, see, for example, [14, Chapter 6, Theorem 3]. For a prime $p$, let $\mathbb{F}_p$ denote the finite field of $p$ elements, which we identify with the set $\{0, \ldots, p-1\}$, and let $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$. Furthermore let $e_p(z) = e(z/p)$.

**Lemma 2.1.** Let $f \in \mathbb{F}_p[X]$ be a nonconstant polynomial of degree $\deg f \leq d$. Then we have

$$\sum_{\lambda \in \mathbb{F}_p} e_p(f(\lambda)) \ll \sqrt{p}.$$ 

Applying Lemma 2.1 and the completion technique (see [9, Section 12.2]) we derive the following bounds for incomplete sums. For $1 \leq N \leq p$ one has

$$\sum_{n=1}^N e_p(f(n)) \ll \sqrt{p} \log p.$$ 

Next, we consider discrete cubic boxes

$$(2.2) \quad \mathcal{B} = \mathcal{I}_1 \times \ldots \times \mathcal{I}_d \subseteq \mathbb{F}_p^d$$

with the side length

$$\ell(\mathcal{B}) = L,$$

where $\mathcal{I}_j = \{k_j + 1, \ldots, k_j + L\}$ is a set of $L \leq p$ consecutive integers, (reduced modulo $p$ if $k_j + L \geq p$), $j = 1, \ldots, d$. 
We formulate the following easy consequence of the Koksma–Szüsz inequality, see [5, Theorem 1.21].

**Lemma 2.2.** Let $\xi_i \in \mathbb{F}_p^d$, $1 \leq i \leq I$, be a sequence of $I$ vectors over $\mathbb{F}_p$ and let $\mathcal{B} \subseteq \mathbb{F}_p^d$ be a box. Let

$$R = \# \{ \xi_i \in \mathcal{B} : 1 \leq i \leq I \}.$$ 

Then we have

$$|R - \# \mathcal{B} p^{-d}| \ll (\log p)^d \max_{h \in \mathbb{F}_p^d \setminus \{0\}} \left| \sum_{i=1}^I e_p(\langle \xi_i, h \rangle) \right|,$$

where $\langle \xi, h \rangle$ denotes the scalar product of two vectors $\xi, h \in \mathbb{F}_p^d$.

### 2.3. Distribution of large rational exponential sums

For a vector $a = (a_1, \ldots, a_d) \in \mathbb{F}_p^d$ we consider the rational exponential sum

$$T_{d, p}(a) = S_d(a/p; p) = \sum_{n=1}^p e_p(a_1n + \ldots + a_dn^d).$$

We need some results about the density of the vectors $a \in \mathbb{F}_p^d$ for which the sums $T_{d, p}(a)$ are large.

For $d = 2$ the answer to the question is trivial due to the following property of Gaussian sums, see [9, Equation (1.55)].

**Lemma 2.3.** Let $p \geq 3$ and $a, b \in \mathbb{F}_p$ with $b \neq 0$, then

$$\left| \sum_{n=0}^{p-1} e_p(an + bn^2) \right| = \sqrt{p}.$$ 

We now investigate the case of $d \geq 3$. For this, we define

$$\omega_d = \liminf_{p \to \infty} \frac{1}{\sqrt{p}} \max_{a = (a_1, \ldots, a_d) \in \mathbb{F}_p^d} |T_{d, p}(a)|.$$ 

From the classical method of Mordell [16] we have

$$\sum_{a \in \mathbb{F}_p^d} |T_{d, p}(a)|^{2d} = d!p^{2d} + O(p^{2d-1}).$$

(2.3)

Hence, taking into account the contribution $|T_{d, p}(0)|^{2d} = p^{2d}$ from the zero vector $a \neq 0$ and estimating the contribution from $O(p^{d-1})$ vectors with $a_d = 0$ by Lemma 2.1, we obtain

$$\sum_{a = (a_1, \ldots, a_d) \in \mathbb{F}_p^d \atop a_d \neq 0} |T_{d, p}(a)|^{2d} = (d! - 1)p^{2d} + O(p^{2d-1}),$$

$$\sum_{a = (a_1, \ldots, a_d) \in \mathbb{F}_p^d \atop a_d = 0} |T_{d, p}(a)|^{2d} = (d! - 1)p^{2d} + O(p^{2d-1}),$$
which trivially implies that
\[ \omega_d \geq (d! - 1)^{1/2d}. \]

Knizhnerman and Sokolinskii [11,12] have given stronger lower bounds, asymptotically for \( d \to \infty \) and also for small values of \( d \), for example, \( \omega_3 \geq \sqrt{3}. \)

Furthermore, by [11, Theorem 1] we have

**Lemma 2.4.** For every integer \( d \geq 2 \) there are some positive constants \( c_d \) and \( \gamma_d \) such that
\[ |T_{d,p}(a)| \geq \gamma_d \sqrt{p} \]
for a set \( L_p \subseteq \mathbb{F}_p^d \) of cardinality \( \#L_p \geq c_d p^d \).

We now show that the vectors \( a \in \mathbb{F}_p^d \) for which the sums \( T_{d,p}(a) \) reach their extreme values are reasonably densely distributed. That is, we intend to show that the set \( L_p \) of Lemma 2.4 is quite dense. Before this we provide a result on the distribution of monomial curves.

**Lemma 2.5.** Let \( (a_1, \ldots, a_k) \in (\mathbb{F}_p^*)^k, \ k \geq 2. \) Then there exists a positive constant \( C \) which depends only on \( k \) such that for any box \( B \) as in (2.2) with the side length \( L \geq C p^{1-2k} \log p \) we have
\[ \# \{ \lambda \in \mathbb{F}_p^* : (a_1 \lambda, \ldots, a_k \lambda^k) \in B \} \geq 0.5 L^k p^{1-k}. \]

**Proof.** For a nonzero vector \( h = (h_1, \ldots, h_k) \in \mathbb{F}_p^k \setminus \{0\} \) the Weil bound, see Lemma 2.1, gives
\[ \sum_{\lambda \in \mathbb{F}_p^*} e_p \left( \sum_{j=1}^k \lambda^j a_j h_j \right) \ll p^{1/2}. \]
Combining this bound with Lemma 2.2, we finish the proof. \( \Box \)

Clearly we can replace a lower bound \( 0.5 L^k p^{1-k} \) of Lemma 2.5 with an asymptotic formula \( (1 + o(1))L^k p^{1-k} \) for slightly larger values of \( L \), namely, if \( L^{-1} p^{1-2k} \log p \to 0 \) as \( p \to \infty \). We also note that Lemma 2.5 still holds for the case \( k = 1 \).

**Lemma 2.6.** Fix \( d \geq 3. \) There is an constant \( C > 0 \) depending only on \( d \), such that for a box \( B \subseteq \mathbb{F}_p^d \) as in (2.2) with the side length \( L \geq C p^{1-\kappa d} \log p \), where \( \kappa_d \) is as in (1.3), and \( L_p \) as in Lemma 2.4, there is \( a \in B \cap L_p \).

**Proof.** Adjusting \( C \) if necessary, we can assume that \( p \) is large enough.

Clearly, if \( (a_1, \ldots, a_d) \in L_p \) then for any \( \lambda \in \mathbb{F}_p^* \) we also have \( (a_1 \lambda, \ldots, a_d \lambda^d) \in L_p \). Let \( k \) be an integer such that
\[ \kappa_d = \min\{1/2k, 1/(2d - k)\}. \]
By Lemma 2.4 we conclude that there exists \((a_1, \ldots, a_k) \in \mathbb{F}_p^k\) with \(a_i \neq 0\) for each \(1 \leq i \leq k\) such that
\[
\#L_p \cap \left( \{a_1, \ldots, a_k\} \times \mathbb{F}_p^{d-k} \right) \gg p^{d-k}.
\]

For convenience we denote this set by \(L^*_{p,k}\).

Let \(B = \mathbb{F}_p^d\) be a box with the side length \(\ell(B) = L\), which we decompose in a natural way as \(B = B_1 \times B_2 \subset \mathbb{F}_p^k \times \mathbb{F}_p^{d-k}\)

Note that we have \(#B_1 = L^k\). Let
\[
\Lambda_k = \{\lambda \in \mathbb{F}_p^* : (\lambda a_1, \ldots, \lambda^k a_k) \in B_1\}.
\]

Then Lemma 2.5 implies that
\[
\#\Lambda_k \geq 0.5 L^k p^{1-k}
\]
provided the condition
\[
L \geq C p^{1-1/2k} \log p
\]
is satisfied with a sufficiently large \(C\).

We now fix a vector \(h = (h_{k+1}, \ldots, h_d) \in \mathbb{F}_p^{d-k} \setminus \{0\}\) and consider the double exponential sums
\[
W(h) = \sum_{(a_1, \ldots, a_d) \in L^*_{p,k}} \left| \sum_{\lambda \in \Lambda_k} e_p \left( \sum_{j=k+1}^d h_j a_j \lambda^j \right) \right|^2.
\]

By the Cauchy-Schwarz inequality
\[
|W(h)|^2 \leq \#L^*_{p,k} \sum_{(a_1, a_2, \ldots, a_d) \in L^*_{p,k}} \left| \sum_{\lambda \in \Lambda_k} e_p \left( \sum_{j=k+1}^d h_j a_j \lambda^j \right) \right|^2
\]
\[
\leq \#L^*_{p,k} \sum_{(a_1, \ldots, a_d) \in \mathbb{F}_p^{d-k}} \left| \sum_{\lambda \in \Lambda_k} e_p \left( \sum_{j=k+1}^d h_j a_j \lambda^j \right) \right|^2.
\]

Now using that for any \(z \in \mathbb{C}\) we have \(|z|^2 = z \bar{z}\), and then changing the order of summations, we obtain
\[
|W(h)|^2 \leq \#L^*_{p,k} \sum_{\lambda, \mu \in \Lambda_k} \sum_{(a_1, a_2, \ldots, a_d) \in \mathbb{F}_p^{d-k}} e_p \left( \sum_{j=k+1}^d h_j a_j (\lambda^j - \mu^j) \right)
\]
\[
\leq \#L^*_{p,k} \sum_{\lambda, \mu \in \Lambda_k} \prod_{j=k+1}^d \sum_{a_j \in \mathbb{F}_p} e_p \left( h_j a_j (\lambda^j - \mu^j) \right).
\]

By the orthogonality of exponential functions, the last sum vanishes unless for every \(j = k+1, \ldots, d\) we have \(h_j (\lambda^j - \mu^j) = 0\). Since \(h\) is a nonzero vector of \(\mathbb{F}_p^{d-k}\), this is possible for at most \(2d \#\Lambda_k\) pairs.
(\lambda, \mu) \in \Lambda_k^2$, and in the case the inner sum is equal to $p^{d-k}$. Hence, for any nonzero vector $h \in \mathbb{F}_p^{d-k}$ we have

$$|W(h)|^2 \ll \#\mathcal{L}_{p,k}^* \#\Lambda_k p^{d-k}.$$ 

Using that $\#\mathcal{L}_{p,k}^* \gg p^{d-k}$, we now obtain

$$|W(h)| \ll \#\mathcal{L}_{p,k}^* (\#\Lambda_k)^{1/2}. \quad (2.6)$$

Let $R$ be the number of the vectors $(a_{k+1}, \ldots a_d, \lambda) \in \mathcal{L}_{p,k}^* \times \Lambda_k$ such that

$$(\lambda^k, a_{k+1}, \ldots, \lambda^d a_d) \in \mathcal{B}_2. \quad (2.7)$$

Combining the bound (2.6) with Lemma 2.2, we obtain

$$R = \#\mathcal{L}_{p,k}^* \#\Lambda_k (L/p)^{d-k} + O(\#\mathcal{L}_{p,k}^* (\#\Lambda_k)^{1/2} (\log p)^{d-k}).$$

Thus we conclude that $R > 0$ when

$$L^{d-k} (\#\Lambda_k)^{1/2} \geq C_0 p^{d-k} (\log p)^{d-k}$$

for some constant $C_0$ depending only on $d$ and $k$. By (2.4) this condition becomes

$$L^{d-k} (0.5 L^{1-k})^{1/2} \geq C_0 p^{d-k} (\log p)^{d-k},$$

and hence it is enough to request that

$$L \geq C p^{1 - 1/(2d-k)} (\log p)^{(d-k)/(d-k/2)} \quad (2.8)$$

for a sufficiently large constant $C$.

Combining the conditions (2.5) and (2.8), and recalling the definition of $\kappa_d$ in (1.3), we conclude that there exists a large enough constant $C$ such that the inequality

$$L \geq C p^{1 - \kappa_d} \log p$$

is sufficient to guarantee that for some $(a_{k+1}, \ldots a_d, \lambda) \in \mathcal{L}_{p,k}^* \times \Lambda_k$ we have (2.7). Since we always have $(a_1 \lambda, \ldots, a_k \lambda^k) \in \mathcal{B}_1$ when $\lambda \in \Lambda_k$ and so the result now follows. \hfill \Box

Corollary 2.7. Let $\mathcal{L}_p$ be defined as in Lemma 2.4. Then for any $k \in \mathbb{N}$ the set

$$\bigcup_{p \geq k \text{ is prime}} \mathcal{L}_p \subseteq \mathbb{T}_d$$

is dense in $\mathbb{T}_d$. 

Proof. Let \( B \) be a box of \( \mathbb{T}_d \) with the side length
\[
\ell(B) = 2Cp^{-\kappa_d} \log p,
\]
where \( C \) is as in Lemma 2.6. Define
\[
\mathfrak{B} = \{ a \in \mathbb{F}_p^d : a/p \in B \}.
\]
By Lemma 2.5 there exists \( b \in \mathfrak{B} \) such that
\[
|T_{d,p}(b)| \geq \gamma_d \sqrt{p}
\]
provided that \( p \) is large enough. Thus, we conclude that
\[
b/p \in \mathcal{L}_p \cap \mathfrak{B}.
\]
Since this holds for any box \( B \) of \( \mathbb{T}_d \), the result follows. \( \Box \)

Remark 2.8. For the case \( d = 2 \), Corollary 2.7 follows immediately from Lemma 2.3. However in general Lemma 2.3 does not hold for \( d \geq 3 \) and in fact \( a \in \mathbb{F}_p^d \) with vanishing sums \( T_{d,p}(a) = 0 \) are often densely distributed as well.

For instance, for \( d \geq 3 \) and a prime number \( p \) with \( \gcd(d, p-1) = 1 \), the map: \( x \rightarrow x^d \) permutes \( \mathbb{F}_p \). Hence, for any \( \lambda \in \mathbb{F}_p^* \) we have
\[
\sum_{n=0}^{p-1} e_p \left( \sum_{j=1}^{d} \binom{d}{j} \lambda^j n^j \right) = \sum_{n=0}^{p-1} e_p ((\lambda n + 1)^d - 1) = \sum_{n=0}^{p-1} e_p (n^d - 1) = \sum_{n=0}^{p-1} e_p (n) = 0.
\]
Assuming \( p > d \) we see that
\[
\binom{d}{j} \not\equiv 0 \pmod{p}, \quad j = 1, \ldots, d.
\]
By Lemma 2.5 for any box \( \mathfrak{B} \subseteq \mathbb{F}_p^d \) with the side length \( \ell(\mathfrak{B}) \geq Cp^{1-1/2d} \log p \) for some constant \( C \) there exists \( \lambda \in \mathbb{F}_p^* \) such that
\[
\begin{pmatrix} \binom{d}{1} \lambda, \ldots, \binom{d}{d} \lambda^d \end{pmatrix} \in \mathfrak{B}.
\]
Therefore we conclude that for any \( k \in \mathbb{N} \) the set
\[
\bigcup_{p \geq k} \{ a/p : a \in \mathbb{F}_p^d, T_{d,p}(a) = 0 \}
\]
is a dense subset of \( \mathbb{T}_d \).
2.4. Large Weyl sums. We are going to show that the small neighbourhood of $L_p$ still have large exponential sums. Namely let $B(x, \delta)$ denotes the cubic box centered at $x \in T_d$ with the side length
\[
\ell (B(x, \delta)) = 2\delta > 0.
\]
For each $\tau > 0$ and a prime $p$ we define
\[
L_{\tau,p} = \bigcup_{a \in L_p} B(a/p, p^{-\tau}).
\]
We also use $\gamma_d$ from Lemma 2.4.

We use the following version of summation by parts. Let $a_n$ be a sequence and for each $t \geq 1$ denote
\[
A(t) = \sum_{1 \leq n \leq t} a_n.
\]
Let $\psi : [1, N] \to \mathbb{R}$ be a differential function. Then
\[
\sum_{n=1}^{N} a_n \psi(n) = A(N)\psi(N) - \int_{1}^{N} A(t)\psi'(t)dt.
\]

Lemma 2.9. Let $x \in L_{\tau,p}$ for some $\tau > 0$ and prime $p$. There exists an absolute constant $c = c(d)$ such that if
\[
cp^{\tau/d}(\log p)^{-1} \geq N \geq p \quad \text{and} \quad p \mid N
\]
then
\[
|S_d(x; N)| \gg Np^{-1/2}.
\]

Proof. For any $x = (x_1, \ldots, x_d) \in L_{\tau,p}$ there exist $a = (a_1, \ldots, a_d) \in L_p$ such that
\[
\|(x_1, \ldots, x_d) - (a_1/p, \ldots, a_d/p)\|_{L^\infty} < p^{-\tau},
\]
where $\|z\|_{L^\infty}$ is the $L^\infty$-norm in $\mathbb{R}^d$. Let $\delta_j = x_j - a_j/p, 1 \leq j \leq d$. Applying summation by parts we obtain
\[
S_d(x; N) - S_d(a/p; N)
\]
\[
= \sum_{n=1}^{N} e_p \left( \sum_{j=1}^{d} a_j n^j \right) \left( e \left( \sum_{j=1}^{d} \delta_j n^j \right) - 1 \right)
\]
\[
= S_d(a/p; N)\psi(N) - \int_{1}^{N} A(t)\psi'(t)dt,
\]
where

\[ A(t) = \sum_{1 \leq n \leq t} e_p \left( \sum_{j=1}^{d} a_j n^j \right) \quad \text{and} \quad \psi(t) = e \left( \sum_{j=1}^{d} \delta_j t^j \right) - 1. \]

Note that for any \( u \in \mathbb{R} \) we have

\[ |e(u) - 1| \leq 2|u|. \]

Combining with \( |\delta_j| < p^{-\tau}, 1 \leq j \leq d \) we obtain

\[ |\psi(N)| \leq 2d N^d p^{-\tau}. \]

For the integral part of (2.9) we derive

\[
\int_{1}^{N} A(t) e \left( \sum_{j=1}^{d} \delta_j t^j \right) \left( 2\pi i \sum_{j=1}^{d} j \delta_j t^{j-1} \right) dt \\
\leq 2\pi p^{-\tau} \max_{1 \leq t \leq N} |A(t)| \int_{1}^{N} \sum_{j=1}^{d} j t^{j-1} dt \\
\leq 4d\pi N^d p^{-\tau} \max_{1 \leq t \leq N} |A(t)|.
\]

Thus combining with (2.10) and the definition of \( A(t) \), and using bound (2.1) on incomplete sums, we derive

\[ S_d(x; N) - S_d(a/p; N) \leq 8d\pi N^d p^{-\tau} \max_{1 \leq t \leq N} |A(t)| \\
\leq c_0 N^{d+1} p^{-\tau-1/2} \log p,
\]

where \( c_0 > 0 \) is some constant which depends on \( d \) only.

Since \( p \mid N \), using the periodicity of function \( e_p(n) \), we obtain

\[ |S_d(a/p; N)| = N p^{-1} |T_{d,p}(a)| \geq 0.5 \gamma_d N / p^{1/2}. \]

Combining (2.11) and (2.12) we obtain

\[ |S_d(x; N)| \geq 0.5 \gamma_d N p^{-1/2} - c_0 N^{d+1} p^{-\tau-1/2} \log p \geq 0.25 \gamma_d N p^{-1/2}
\]

provided

\[ N \leq c p^{\tau/d} (\log p)^{-1}, \]

for a sufficiently small constant \( c \) (depending only on \( d \)), which gives the desired result.

We formulate some notation for our using on the lower bound of the Hausdorff dimension of \( E_{\alpha,d} \).
Lemma 2.10. Let $\tau > d$. For any $\varepsilon > 0$ there exists $p_{\varepsilon, d}$ such that for any $p > p_{\varepsilon, d}$ and any cubic box $B \subseteq T_d$ with the side length $\ell(B) = p^{\kappa_d + \varepsilon}$ there exists a box $C \subseteq B$ with the side length $\ell(C) = p^{-\tau}$ and such that for $N = p \left\lfloor c p^{\tau/d-1} (\log p)^{-1/2} \right\rfloor$, where $c$ is as in Lemma 2.9, and all $x \in C$, we have

$$\left| S_d(x; N) \right| \gg N^{1-d/2\tau} (\log N)^{-d/2\tau}. $$

Proof. Let $B = B(z, \ell(B)/2)$ be the box. For the box $B(z, \ell(B)/5)$, Lemma 2.6 implies that there exists a point

$$c \in L_p \cap B(z, \ell(B)/5)$$

provided $p$ is large enough. Let $C = B(c, p^{-\tau}/2)$. The condition $\tau > d$ gives $\tau > \kappa_d - \varepsilon$, and hence $C \subseteq B$.

By the choice of $N = p \left\lfloor c p^{\tau/d-1} (\log p)^{-1} \right\rfloor$ and the condition $\tau > d$, Lemma 2.9 implies that for all $x \in C$ we have

$$\left| S_d(x; N) \right| \gg N^{1-d/2\tau} (\log N)^{-d/2\tau},$$

which gives the desired result. \qed

Definition 2.11 (($a, b, c$)-patterns). Let $a > b > c > 0$ and $a/b \in \mathbb{Z}$. Let $B$ be a box with with the side length $\ell(B) = a$. We divide the box $B$ into $(a/b)^d$ smaller boxes in a natural way. For each of these $(a/b)^d$ boxes we pick a smaller box, at an arbitrary location with the side length $c$. The resulting configuration of $(a/b)^d$ boxes with the side length $c$ is called an $(a, b, c)$-pattern.

An illustrative example of an $(a, b, c)$-pattern is given in Figure 2.1.

![Figure 2.1](image)

Figure 2.1. An $(a, b, c)$-pattern with $a/b = 3$ and $d = 2$.

We note that each $(a, b, c)$-pattern is a subset of $B$. For our applications we find $(a, b, c)$-patterns such that the Weyl sums are large inside of the $(a/b)^d$ small boxes. We show that for any box $B \subseteq T_d$ there
are \((a, b, c)\)-patterns which admit large Weyl sums. More precisely we have the following.

**Lemma 2.12.** Let \(\tau, \varepsilon\) and \(p_{\varepsilon,d}\) be the same as in Lemma 2.10. Let \(p > p_{\varepsilon,d}\) and \(B \subseteq T_d\) with the side length \(\ell(B) > 10p^{-\kappa_d+\varepsilon}\). There exists \(b\) such that \(p^{-\kappa_d+\varepsilon} \leq b \leq 2p^{-\kappa_d+\varepsilon}\) and \(\ell(B)/b \in \mathbb{Z}\). Furthermore, there exists a \((\ell(B), b, p^{-\tau})\)-pattern, which we denote by \(\Upsilon_B\), such that

\[
N = p \left[ c p^{\tau/d-1} (\log p)^{-1} \right]
\]

and all \(x \in \Upsilon_B\) we have

\[
|S_d(x; N)| \gg N^{1-d/2\tau} (\log N)^{-d/2\tau}.
\]

**Proof.** Since \(\ell(B)/b \in \mathbb{Z}\), we divide the box \(B\) into \(q = (\ell(B)/b)^d\) smaller boxes of equal sizes in a natural way. We label them by \(B_1, \ldots, B_q\) for convenience.

For each \(B_i, 1 \leq i \leq q\), Lemma 2.10 asserts that there exists a box \(C_i \subseteq B_i\) with the side length \(p^{-\tau}\), and for all \(x \in C_i\) we have the desired bound.

We finish the proof by taking \(\Upsilon_B = \bigcup_{i=1}^q C_i\). \(\square\)

### 2.5. Hausdorff dimension of a class of Cantor sets

By a repeated application of Lemma 2.12, we find large Weyl sums on a Cantor-like set. This implies a lower bound for the Hausdorff dimension of \(E_{\alpha,d}(\Omega)\). In this section we investigate a general construction of Cantor-like sets.

Now we show the construction of the Cantor sets by iterating the construction of \((a, b, c)\)-patterns.

Let

\[
\delta = (\delta_k)_{k=1}^\infty \quad \text{and} \quad \ell = (\ell_k)_{k=1}^\infty
\]

such that for each \(k = 1, 2, \ldots\), we have

\[
\delta_k > \delta_{k+1} \quad \text{and} \quad \ell_k > \ell_{k+1}.
\]

For convenience we also denote

\[
(2.13) \quad \delta_0 = \lambda(\Omega)^{1/d}
\]

the side length of \(\Omega\).

For each \(k \geq 0\) we ask that the triple \((\delta_k, \ell_{k+1}, \delta_{k+1})\) satisfies the condition on \((a, b, c)\) in Definition 2.11. In particular, we always assume that

\[
\delta_k/\ell_{k+1} \in \mathbb{Z}
\]

and we denote

\[
(2.14) \quad q_{k+1} = (\delta_k/\ell_{k+1})^d.
\]

for every \(k = 0, 1, \ldots\).
We start from the cube $Q$ and take a $(\delta_0, \ell_1, \delta_1)$-pattern inside of $Q$.

Let $\mathcal{C}_1$ be the collection of these $q_1$ boxes. More precisely let

$$\mathcal{C}_1 = \{B_i : 1 \leq i \leq q_1\}.$$ 

For each $B_i$ we take a $(\delta_1, \ell_2, \delta_2)$-pattern inside of $B_i$, and we denote these sub-boxes of $B_i$ by $B_{i,j}$ with $1 \leq j \leq q_2$. Let

$$\mathcal{C}_2 = \{B_{i,j} : 1 \leq i \leq q_1, 1 \leq j \leq q_2\}.$$ 

Figure 2.2 shows an example of this construction.

Suppose now we have $\mathcal{C}_k$ which is a collection of $\prod_{i=1}^{k} q_k$ boxes with the side length $\delta_k$. For each of these box $B$ we take a $(\delta_k, \ell_{k+1}, \delta_{k+1})$-pattern inside of the box $B$. Let $\mathcal{C}$ be the collections of these boxes, that is

$$\mathcal{C}_{k+1} = \{B_{i_1, \ldots, i_{k+1}} : 1 \leq i_1 \leq q_1, \ldots, 1 \leq i_{k+1} \leq q_{k+1}\}.$$ 

Our Cantor-like set is defined by

$$F = \bigcap_{k=1}^{\infty} F_k,$$

where

$$F_k = \bigcup_{B \in \mathcal{C}_k} B.$$ 

There are many possible outcomes by the above construction, we let $\Omega(Q; \delta, \ell)$ denote all possible patterns.

From our construction clearly we have $F_k \supseteq F_{k+1}$, and $F_k$ is a compact set, and hence $F$ is a nonempty compact set. Furthermore we obtain the lower bound of these Cantor sets by using the following mass distribution principle [6, Theorem 4.2].
Lemma 2.13. Let $\mathcal{X} \subseteq \mathbb{R}^d$ and let $\nu$ be a measure on $\mathbb{R}^d$ such that $\nu(\mathcal{X}) > 0$. If for any box $B(x, r)$ with $0 < r \leq \varepsilon_0$ for some $\varepsilon_0 > 0$ we have

$$\nu(B(x, r)) \ll r^s,$$

then the Hausdorff dimension of $\mathcal{X}$ is at least $s$.

Lemma 2.14. Let $F \in \Omega(\Omega; \delta, \ell)$ and let $q_{k+1}, \ k = 0, 1, \ldots$, are given by (2.14). Then

$$\dim F = \liminf_{k \to \infty} \log \prod_{i=1}^{k} q_i - \log \delta_k.$$

Proof. We show the upper bound of $\dim F$ first. Let

$$s > t = \liminf_{k \to \infty} \log \prod_{i=1}^{k} q_i - \log \delta_k.$$

Then there exists a sequence $k_j, \ j \in \mathbb{N},$ such that

$$\prod_{i=1}^{k_j} q_i \leq \delta_{k_j}^{-s}.$$

The construction of $F$ implies for each $j \in \mathbb{N}

$$F \subseteq \bigcup_{B \in C_{k_j}} B.$$

Thus for any $\varepsilon > 0$ we obtain

$$\sum_{B \in C_{k_j}} (\text{diam } B)^{s+\varepsilon} \ll \delta_{k_j}^{s+\varepsilon} \prod_{i=1}^{k_j} q_i \ll \delta_{k_j}^\varepsilon \to 0 \text{ as } j \to \infty.$$

The definition of Hausdorff dimension, see Definition 1.4, implies that $\dim F \leq s + \varepsilon$. By the arbitrary choices of $\varepsilon > 0$ and $s > t$ we obtain the upper bound

$$\dim F \leq t.$$

Now we turn to the lower bound of $\dim F$. We first define a measure on $F$ (natural measure). For each $k$ and any subset $\mathcal{A}$ let

$$\nu_k(\mathcal{A}) = \delta_k^d \prod_{i=1}^{k} \frac{1}{q_i} \int \mathbf{1}_{\mathcal{A} \cap F_k}(x) dx,$$

where $\mathbf{1}_V$ is the indicator function of a set $V$. Observe that for each $B \in C_k$ we have

$$\nu_k(B) = \prod_{i=1}^{k} q_i^{-1}.$$
We note that the measure $\nu_k$ weakly convergence to a measure $\nu$, see [15, Chapter 1].

Let $\varepsilon > 0$ then there exists $k_0$ such that for any $k \geq k_0$ we have
\begin{equation}
\prod_{i=1}^{k} q_i \geq \delta^{-t+\varepsilon}.
\end{equation}

Let $B(x, r) \subseteq T_d$ with $r \leq \delta_{k_0}$. Then there exists $k \geq k_0$ such that 
$\delta_{k+1} \leq r \leq \delta_k$.

Observe that
$$\nu(B(x, r)) \ll \left(\frac{r}{\ell_{k+1}}\right)^d \prod_{i=1}^{k} q_i^{-1}.$$

Applying $q_{k+1} = (\delta_k/\ell_{k+1})^d$, we obtain
$$\nu(B(x, r)) \ll \left(\frac{r}{\delta_k}\right)^d \prod_{i=1}^{k} q_i^{-1}.$$

Combining with the estimate (2.15) and the condition $\delta_{k+1} \leq r \leq \delta_k$, we have
$$\nu(B(x, r)) \ll r^d \delta_k^{-d-\varepsilon} \ll r^{t-\varepsilon}.$$

Applying the mass distribution principle given in Lemma 2.13, we have $\dim F \geq t - \varepsilon$. By the arbitrary choice of $\varepsilon > 0$ we obtain that $\dim F \geq t$, which finishes the proof. \hfill $\square$

2.6. Monomial exponential sums. We need the following elementary statement, see, for example [13, Equation (82)] for a more general statement.

**Lemma 2.15.** Let $a \in \mathbb{Z}$ with $\gcd(a, p) = 1$, then for integer $d \geq 2$
\begin{equation*}
\sum_{n=1}^{p^d} e \left( \frac{an^d}{p^d} \right) = p^{d-1}.
\end{equation*}

One can certainly adapt the arguments in the proof of Lemma 2.9 to get a lower bound on $\sigma_d(x; N)$. However we can achieve better results with the following approximate formula of Vaughan [20, Theorem 4.1].

**Lemma 2.16.** Let
$$x = \frac{a}{q} + \xi$$
with some relatively prime integers $a$ and $q \geq 1$. Then
$$\sigma_d(x; N) = \frac{1}{q} \sigma_d(a/q; q) \int_0^N e(\xi \gamma^d) \, d\gamma + O \left( q^{1/2+o(1)} (1 + |\xi|N^d)^{1/2} \right).$$
We now easily see that Lemma 2.16 implies the following result.

**Lemma 2.17.** Let \( a \in \mathbb{Z} \) and let \( p \) be a prime number such that \( \gcd(a, p) = 1 \). Let \( x \in [0, 1) \) with \( |x - a/p^d| < p^{-\tau} \) for some \( \tau > 0 \). There exists an absolute constant \( c > 0 \) such that for any \( \varepsilon > 0 \) If \( cp^{\tau/d} \geq N \geq p^{d/2+1+\varepsilon} \) then

\[
|\sigma_d(x; N)| \geq 0.5Np^{-1},
\]

provided that \( p \) is large enough.

**Proof.** Using Lemma 2.16 with \( \xi = x - a/p^d \) we see that the assumed upper bound on \( N \) implies that

\[
|\xi|N^d = |x - a/p^d|N^d < p^{-\tau}N^d \leq c^d.
\]

Hence taking \( c \) small enough we obtain

\[
\left| \int_0^N e(\xi \gamma^d) \, d\gamma \right| \geq \frac{2}{3}N.
\]

Therefore by Lemmas 2.15 and 2.16

\[
|\sigma_d(x; N)| \geq \frac{2}{3p^d}N\sigma_d(a/p^d; p^d) + O\left(p^{d/2+o(1)}\right) = \frac{2}{3}Np^{-1} + O\left(p^{d/2+o(1)}\right)
\]

Recalling the lower bound \( N \) we see that the first term dominates, which finishes the proof. \( \square \)

### 3. Proofs of abundance of large Weyl sums

#### 3.1. Proof of Theorem 1.3.

The idea is that we first show that the exponential sums \( S_d(x; N) \) are large at a dense subset of \( T_d \), and then we show the exponential sums are still large at the small neighbourhoods of these points. This implies that the subset \( E_{\alpha,d} \) has large topology for each \( 0 < \alpha < 1 \).

Let the sets \( L_{m,p} \) be as in Lemma 2.9.

For positive integers \( k \) and \( m \) we consider the sets

\[
G_{m,k} = \bigcup_{p \text{ is prime}} L_{m,p},
\]

and define

\[
G = \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} G_{m,k}.
\]

Using Lemma 2.9, with \( N = p \left[ cp^{n/d-1}(\log p)^{-1} \right] \), we conclude that for each \( 0 < \alpha < 1 \) we have

\[
G \subseteq E_{\alpha,d}.
\]

(3.1)
Let \( m, k \in \mathbb{N} \) and \( B \subseteq T_d \) be an arbitrary open cubic box. Then Corollary 2.7 implies that there exists an open cubic box \( \tilde{B} \subseteq G_{m,k} \) such that \( \tilde{B} \subseteq B \). It follows that \( T_d \setminus G_{m,k} \) is a nowhere dense subset. Furthermore since

\[
T_d \setminus G = \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} (T_d \setminus G_{m,k}),
\]

we obtain that the set \( T_d \setminus G \) is the countable union of nowhere dense sets, and hence \( T_d \setminus G \) is of first category. Together with (3.1) we complete the proof.

3.2. **Proof of Theorem 1.5.**

3.2.1. **Preamble.** We first note that our methods for the cases \( d = 2 \) and \( d \geq 3 \) are different. For the case \( d = 2 \) we use Lemma 2.3. As it is shown in Remark 2.8, in general Lemma 2.3 does not hold for \( d \geq 3 \), for this case we use the results from Section 2.4.

Throughout the proof we fix the cube \( Q \). In particular, all implied constants may depend on \( Q \).

We use \( \langle z \rangle \) to denote the distance in the \( L^\infty \)-norm between \( z \in \mathbb{R}^d \) and the closest point \( Z^d \).

3.2.2. **Case (i): \( d = 2 \).** For \( \tau > 2 \) we define

\[
W(\tau) = \{ x \in T : \langle qx \rangle < q^{1-\tau} \text{ for infinitely many } q \in \mathbb{N} \}.
\]

The classical Jarník–Besicovitch theorem, see [6, Theorem 10.3] or [1], asserts that

\[
\dim W(\tau) = 2/\tau.
\]

We note that the method in the proof of [6, Theorem 10.3] (or see the proof of Lemma 3.1) imply that

\[
(3.2) \quad \dim \{ x \in T : \langle px \rangle < p^{1-\tau} \text{ for infinitely primes } p \} = 2/\tau.
\]

For our purpose we need obtain an analogy of (3.2) for \( x \in \mathcal{Q} \subseteq T_2 \).

We introduce some notation first. For a prime number \( p \) we define

\[
\mathcal{A}_{\tau,p} = \bigcup_{1 \leq i,j \leq p-1} \{ x \in \mathcal{Q} : \| x - (i/p, j/p) \|_{L^\infty} < p^{-\tau} \},
\]

where \( \| z \|_{L^\infty} \) is the \( L^\infty \)-norm in \( \mathbb{R}^2 \), and

\[
\mathcal{G}_{\tau} = \bigcap_{k=1}^{\infty} \bigcup_{p \geq k, p \text{ is prime}} \mathcal{A}_{\tau,p}.
\]
Applying the arguments of [6, Theorem 10.3] to our setting $G_\tau$ we have the following.

**Lemma 3.1.** *Using the above notation for any $\tau > 2$ we have*

$$\dim G_\tau = \frac{3}{\tau}.$$  

**Proof.** For the upper bound first note that for each $p$ the set $A_{\tau,p}$ can be covered by at most $p^2$ boxes with the side length $2p^{-\tau}$. Since for each $k \in \mathbb{N}$

$$G_\tau \subseteq \bigcup_{\substack{p \geq k \\ p \text{ is prime}}} A_{\tau,p},$$

and for any $s > \frac{3}{\tau}$ we have

$$\sum_{\substack{p \geq k \\ p \text{ is prime}}} p^{2-\tau s} \ll k^{3-\tau s} \to 0 \text{ as } k \to \infty,$$

Definition 1.4 implies $\dim G_\tau \leq s$. By the arbitrary choice of $s > \frac{3}{\tau}$ we conclude

(3.3) $$\dim G_\tau \leq \frac{3}{\tau}.$$  

Now we turn to the lower bound. Let $p_k$ be a sequence rapidly increasing prime numbers such that

(3.4) $$p_1 \cdots p_k = p_{k+1}^{o(1)} \quad \text{as } k \to \infty.$$

For each $k$ define

$$H_k = \bigcup_{\substack{p_k \leq p \leq 2p_k \\ p \text{ is prime}}} A_{\tau,p}.$$  

An important fact is that for different primes $p_k \leq p, r \leq 2p_k$ the sets $A_{\tau,p}$ and $A_{\tau,r}$ are disjoint when $p_k$ is large enough. Indeed, this follows from the choice of $\tau > 2$ and that for $1 \leq a, b \leq p$ and $1 \leq c, d \leq r - 1$,

$$\| (a/p, b/p) - (c/r, d/p) \|_{L^\infty} \gg p_k^{-2}.$$  

Note that there are $p_k^{1+o(1)}$ prime numbers between $p_k$ and $2p_k$, and for each prime number $p_k \leq p \leq 2p_k$ the set $A_{\tau,p}$ contains $p_k^{2+o(1)}$ boxes with the side length $p_k^{-\tau}$, which due to the fact that the cube $Q$ is fixed. Thus the set $H_k$ consists of $p_k^{3+o(1)}$ boxes with the side length $p_k^{-\tau}$. We remark that the implied constant may depend on $Q$, however it is not hard to see that for a fixed cube $Q$ this constant does not affect the result. Let

$$H = \bigcap_{k=1}^{\infty} H_k.$$
We claim that
\[(3.5) \quad \dim \mathcal{H} \geq 3/\tau.\]

We show some explanation in the following. For each \(k \in \mathbb{N}\) let
\[F_k = \bigcap_{i=1}^{k} \mathcal{H}_i.\]

Note that \(\mathcal{H} = \bigcap_{k=1}^{\infty} F_k\). An important fact is that for any box of \(\mathcal{H}_i\) with the side length \(p_i^{-\tau}\) it contains
\[q_{i+1} = \left(\frac{p_i^{-\tau}}{p_{i+1}}\right)^3\]
uniformly distributed boxes of \(\mathcal{H}_{i+1}\) with the side length \(p_{i+1}^{-\tau}\). Denote \(q_1 = p_1^3\). It follows, also using (3.4), that \(F_k\) contains at least
\[\prod_{i=1}^{k} q_i = p_k^{3+o(1)}\]
boxes with the side length \(p_k^{-\tau}\).

By giving a measure on \(\mathcal{H}\) in a similar way as in the proof of Lemma 2.14, and then applying the mass distribution principle, see Lemma 2.13, we obtain
\[\dim \mathcal{H} \geq \liminf_{k \to \infty} \frac{\log \prod_{i=1}^{k} q_i}{\log p_k} = 3/\tau,\]
which proves the claim (3.5).

Observe that for each \(x \in \mathcal{H}\) there are infinitely \(p\) such that \(x \in \mathcal{A}_{r,p}\), and hence \(x \in \mathcal{G}_r\) and \(\mathcal{H} \subseteq \mathcal{G}_r\). By the monotonicity property of the Hausdorff dimension we see from (3.5) that
\[\dim \mathcal{G}_r \geq \tau/3,\]
which together with (3.3) finishes the proof. \(\square\)

To conclude the proof for the case \(d = 2\), it is sufficient to prove \(\mathcal{G}_r \subseteq \mathcal{E}_{\alpha,2} (\mathcal{Q})\) with some \(\tau\), since
\[(3.6) \quad \dim \mathcal{E}_{\alpha,2} (\mathcal{Q}) \geq \dim \mathcal{G}_r \geq 3/\tau.\]

Let \(x = (x_1, x_2) \in \mathcal{A}_{r,p}\) then there exists \((a, b)\) with \(1 \leq a, b \leq p - 1\) such that
\[\|(x_1, x_2) - (a/p, b/p)\|_{L^\infty} < p^{-\tau}.\]
Applying Lemma 2.3, exactly as in the proof of Lemma 2.9 we see that
\[ \sum_{n=1}^{N} e(x_1 n + x_2 n^2) \gg N/\sqrt{p}, \]
provided
\[ p \leq N \leq cp^{\tau/2}(\log p)^{-1} \quad \text{and} \quad p \mid N \]
for some absolute constant \( c > 0 \).

Furthermore, for any small \( \varepsilon > 0 \), if we have
\[ N/\sqrt{p} \geq N^{\alpha + \varepsilon}, \]
then we also have
\[ |S_2(x, N)| \gg N^{\alpha + \varepsilon}. \]
Note that the implied constant here does not depend on \( \varepsilon \). Clearly we can find \( N \) satisfying (3.7) and (3.8) simultaneously provided that
\[ \tau > \max\{2, 1/(1 - \alpha - \varepsilon)\} \]
and \( p \) is large enough. It follows that for each \( x \in A_{\tau, p} \) with large enough \( p \) there exists \( N = N_p \) such that
\[ |S_2(x; N)| \gg N^{\alpha + \varepsilon} \gg N^\alpha. \]
This implies that \( G_\tau \subseteq E_{\alpha, 2}(\Omega) \). Combining with (3.6) and (3.9) we obtain that
\[ \dim E_{\alpha, 2}(\Omega) \geq \min\{3/2, 3(1 - \alpha - \varepsilon)\}. \]
By the arbitrary choice of small and positive \( \varepsilon \), we finish the proof.

3.2.3. Case (ii): \( d \geq 3 \). We note that our method also works for \( d = 2 \), thus we only assume \( d \geq 2 \) in the following.

Let \( p_k \) be a sequence rapidly increase prime numbers such that
\[ p_1 \ldots p_k = p_{k+1}^{\alpha(1)}, \quad \text{as} \ k \to \infty. \]
Let \( \tau > 0 \) such that
\[ \tau > d. \]
As before, we define \( \delta_0 \) as the side length of \( \Omega \), that is, as in (2.13). For each \( k \in \mathbb{N} \) let
\[ \delta_k = p_k^{-\tau}, \]
and choosing \( p_k \) large enough, we see that we can assume that \( \delta_k < \delta_0 \).

Fix some sufficiently small \( \varepsilon > 0 \) and for each \( k \geq 0 \) let
\[ p_k^{\alpha_d + \varepsilon} \leq \ell_{k+1} \leq 2p_k^{\alpha_d + \varepsilon} \]
where $\kappa_d$ is given by (1.3), such that $\delta_k/\ell_{k+1} \in \mathbb{Z}$. For example, the choice

$$\ell_{k+1} = \delta_k/\lfloor \kappa_d \delta_k \rfloor$$

is satisfactory since we may choose $p_k$ such that $p_k^{\kappa_d - \varepsilon} \delta_k \geq 1$ for any small $\varepsilon > 0$.

Denote

$$q_{k+1} = \left( \frac{\delta_k}{\ell_{k+1}} \right)^d.$$  \hspace{1cm} (3.14)

Applying Lemma 2.14 to the sequences $\delta_k, \ell_k$ we obtain the following.

**Lemma 3.2.** In the above notation (3.12) and (3.14) and under the conditions (3.10), (3.11) and (3.13), for any $F \in \Omega(\mathcal{Q}; \delta, \ell)$, we have

$$\dim F = \frac{d\kappa_d}{\tau} - d\varepsilon/\tau.$$  

**Proof.** Recalling (3.10) and (3.13), we obtain

$$q_1 \cdots q_k = \frac{(p_1 \cdots p_k)^{d\kappa_d - d\varepsilon + o(1)}}{(p_1 \cdots p_{k-1})^{\tau d}} = p_k^{d\kappa_d - d\varepsilon + o(1)}$$

and

$$\frac{\log q_1 \cdots q_k}{\log p_k^\tau} = \frac{d\kappa_d}{\tau} - d\varepsilon/\tau + o(1).$$

Lemma 2.14 gives

$$\dim F = \liminf_{k \to \infty} \frac{\log q_1 \cdots q_k}{\log p_k^\tau} = \frac{d\kappa_d}{\tau} - d\varepsilon/\tau,$$

which finishes the proof. \hfill \Box

We are now going to show that there exists a pattern $F \in \Omega(\mathcal{Q}; \delta, \ell)$ such that $F \subseteq E_{\alpha,d}(\mathcal{Q})$ for some $\tau$ which may depend on $\alpha$ and $d$. Thus Lemma 3.2 implies that

$$\dim E_{\alpha,d}(\mathcal{Q}) \geq \dim F = \frac{d\kappa_d}{\tau} - d\varepsilon/\tau.$$  \hspace{1cm} (3.15)

Our construction is inductive.

For $\delta_0$ given by (2.13) and $\ell_1$ with

$$p_1^{-\kappa_d + \varepsilon} \leq \ell_1 \leq 2p_1^{-\kappa_d + \varepsilon}$$

(note that we request $\delta_0/\ell_1 \in \mathbb{Z}$), by Lemma 2.12 there exists a $(\delta_0, \ell_1, p_1^{-\varepsilon})$-pattern, which we denote by $F_1$, such that for

$$N = p_1 \left\lfloor cp_1^{\tau/d-1} (\log p_1)^{-1} \right\rfloor$$
and all $x \in F_1$ we have

$$|S_d(x; N)| \gg N^{1-d/2\tau}(\log N)^{-d/2\tau}.$$  

Now, suppose that we have a pattern $F_k$ which is a collection of $q_1 \ldots q_k$ boxes with the side length $\delta_k$. For each box $B$ again by Lemma 2.12 there exists a $(\delta_k, \ell_k, \delta_{k+1})$-pattern $\Upsilon_B \subseteq B$ such that for

$$N = p_{k+1} \left\lceil c \frac{\tau}{d - 1} (\log p_{k+1})^{-1} \right\rceil$$

and all $x \in \Upsilon_B$ we have

(3.16)  $$|S_d(x; N)| \gg N^{1-d/2\tau}(\log N)^{-d/2\tau}.$$  

Let

$$F_{k+1} = \{ \Upsilon_B : B \in F_k \}.$$  

For convenience we use the same notation to denote

$$F_{k+1} = \bigcup_{B \in F_k} \Upsilon_B.$$  

Let

$$F = \bigcap_{k=1}^{\infty} F_k.$$  

Then by (3.16) we conclude that

$$F \subseteq \mathcal{E}_{\alpha, d}(\Omega)$$

provided that

(3.17)  $$1 - d/2\tau > \alpha,$$

and the condition (3.11) holds.

The inequalities (3.11) and (3.17) imply that it is sufficient to take any $\tau$ such that

$$\tau > \max \left\{ d, \frac{d}{2(1-\alpha)} \right\}.$$  

Combining this with (3.15), and using that $d\varepsilon/\tau \leq \varepsilon$ we obtain

$$\dim \mathcal{E}_{\alpha, d}(\Omega) \geq \min \{ \kappa_d, 2\kappa_d(1-\alpha) \} - \varepsilon.$$  

Since this lower bound holds for any $\varepsilon > 0$, we conclude the proof of Theorem 1.5.
4. Proofs of abundance of large monomial sums

4.1. Proof of Theorem 1.6. For \(d, p \in \mathbb{N}\) and some \(\tau > 0\) we define the sets

\[
\mathcal{A}_{d,p,\tau} = \bigcup_{\substack{1 \leq a < p^d \\gcd(a, p) = 1}} \left\{ x \in \mathbb{T} : \left| x - a/p^d \right| < p^\tau \right\},
\]

and

\[
\mathcal{B}_{d,\tau} = \bigcap_{k=1}^{\infty} \bigcup_{p \geq k, \text{ } \text{ } p \text{ is prime}} \mathcal{A}_{d,p,\tau}.
\]

Let \(x \in \mathcal{A}_{d,p,\tau}\). Applying Lemma 2.17 we see that

\[
\left| \sum_{n=1}^{N} e(xn^d) \right| \geq 0.5Np^{-1},
\]

provided that

\[
cp^\tau/d \geq N \geq p^{d/2+1+\varepsilon}
\]

for some \(\varepsilon > 0\) and sufficiently large \(p\), where \(c > 0\) is an absolute constant.

Furthermore, for each \(0 < \alpha < 1\) if we have

\[
0.5Np^{-1} \geq N^\alpha,
\]

then we also have

\[
|\sigma_d(x; N)| \geq N^\alpha.
\]

By conditions (4.3) and (4.4) we conclude that for any \(\tau > 0\) such that

\[
\tau > \max\{d^2/2 + d, d/(1 - \alpha)\},
\]

there exists \(N\) such that the conditions (4.3) and (4.4) hold simultaneously.

It follows that there exists some \(N_{d,p,\tau}\) such that for any \(x \in \mathcal{A}_{d,p,\tau}\)

\[
|\sigma_d(x; N_{d,p,\tau})| \geq N_{d,p,\tau}^\alpha.
\]

Therefore if (4.5) holds then

\[
\mathcal{B}_{d,\tau} \subseteq \mathcal{E}_{\alpha,d}.
\]

For each \(k \in \mathbb{N}\) let

\[
\mathcal{G}(d, \tau, k) = \bigcup_{p \geq k} \mathcal{A}_{d,p,\tau}.
\]
Clearly for each \( d, \tau, k \) the set \( G(d, \tau, k) \) is an open and dense subset of \([0, 1)\), and hence \([0, 1) \setminus G(d, \tau, k)\) is a nowhere dense subset of \([0, 1)\). Therefore we obtain that the set 
\[
\bigcup_{k=1}^{\infty} [0, 1) \setminus G(d, \tau, k)
\]
is of first Baire category set. Now from (4.2) and (4.6) we obtain 
\[
[0, 1) \setminus E_{\alpha,d} \subseteq [0, 1) \setminus B_{d,\tau} = \bigcup_{k=1}^{\infty} [0, 1) \setminus G(d, \tau, k),
\]
and hence we finish the proof.

4.2. Proof of Theorem 1.7.

4.2.1. Preamble. We note that for the monomials the methods for the cases \( d = 2 \) and \( d \geq 3 \) are also different. For the case \( d = 2 \) we use Lemma 2.3, while for the case \( d \geq 3 \) we use Lemma 2.15.

Throughout the proof we fix the interval \( \mathcal{I} \subseteq \mathbb{T} \). In particular, all implied constants may depend on \( \mathcal{I} \).

4.2.2. Case (i): \( d = 2 \). This case follows by applying the similar arguments to the proof of Theorem 1.5 for the case \( d = 2 \).

For \( p \in \mathbb{N} \) and some \( \tau > 0 \) let 
\[
\mathcal{A}_{p,\tau} = \bigcup_{1 \leq a < p} \left\{ x \in \mathcal{I} : |x - a/p| < p^{-\tau} \right\},
\]
and 
\[
\mathcal{B}_\tau = \bigcap_{k=1}^{\infty} \bigcup_{p \geq k} \mathcal{A}_{p,\tau}.
\]

As we claimed before that the method in the proof of [6, Theorem 10.3] (or see the proof of Lemma 3.1) imply that 
\[
(4.7) \quad \dim \mathcal{B}_\tau = 2/\tau.
\]

Applying Lemma 2.3 and Lemma 2.9 we conclude that for any \( x \in \mathcal{A}_{p,\tau} \) there exists \( N_{p,\tau} \) such that 
\[
\sigma_2(x; N_{p,\tau}) \gg N^\alpha
\]
provided that 
\[
\tau > \max\{2, 1/(1 - \alpha)\}.
\]
Note that this is the same condition as (3.9) up to the small parameter \( \varepsilon \). Under this condition for the parameter \( \tau \) we conclude \( \mathcal{B}_\tau \subseteq \mathcal{E}_{\alpha,2} \mathcal{I}) \). Combining with (4.7) we obtain the desired result.
4.2.3. Case (ii): \( d \geq 3 \). We slightly modify the definition of the set \( \mathcal{A}_{d,p,\tau} \) in (4.1) by using \( \mathcal{J} \) instead of \( \mathcal{T} \), that is, we now set

\[
\mathcal{A}_{d,p,\tau} = \bigcup_{1 \leq a < p^d \atop \gcd(a,p) = 1} \{ x \in \mathcal{J} : |x - a/p^d| < p^{-\tau} \},
\]

while the set \( \mathcal{B}_{d,\tau} \) is still defined by (4.2).

By adapting the arguments of [6, Theorem 10.3] and Lemma 3.1 to the sets \( \mathcal{B}_{d,\tau} \) we have the following.

**Lemma 4.1.** Using the above notation for any \( \tau > 2d \) we have

\[
\dim \mathcal{B}_{d,\tau} = \frac{(d + 1)}{\tau}.
\]

**Proof.** Let \( s > (d + 1)/\tau \). Note that for any \( k \in \mathbb{N} \) we have

\[
\mathcal{B}_{d,\tau} \subseteq \bigcup_{p \geq k \atop p \text{ is prime}} \mathcal{A}_{d,p,\tau}.
\]

Since

\[
\sum_{p \geq k} p^d p^{-\tau s} \to 0 \quad \text{as} \quad k \to \infty,
\]

Definition 1.4 implies \( \dim \mathcal{B}_{d,\tau} \leq s \). By the arbitrary choice of \( s > (d + 1)/\tau \) we conclude that

\[
(4.8) \quad \dim \mathcal{B}_{d,\tau} \leq \frac{(d + 1)}{\tau}.
\]

Now we turn to the lower bound of \( \dim \mathcal{G}_{d,\tau} \). Let \( p_k \) be a sequence of rapidly increasing prime numbers satisfying (3.4). For each \( i \) let

\[
\mathcal{F}_k = \bigcup_{p_k \leq p \leq 2p_k \atop p \text{ is prime}} \mathcal{A}_{d,p,\tau},
\]

and

\[
\mathcal{F} = \bigcap_{k=1}^{\infty} \mathcal{F}_k.
\]

Clearly we have

\[
(4.9) \quad \mathcal{F} \subseteq \mathcal{B}_{d,\tau}
\]

Hence, it is sufficient to show that

\[
(4.10) \quad \dim \mathcal{F} \geq \frac{(d + 1)}{\tau}.
\]

Let \( p, q \) be two distinct prime numbers with \( p_k \leq p, q \leq 2p_k \), and let \( 1 \leq a < p^d \) and \( 1 \leq b < q^d \) such that \( \gcd(a, p) = \gcd(b, q) = 1 \). Then

\[
|aq^d - bp^d| \geq 1,
\]
and
\[ \left| \frac{a}{p^{d}} - \frac{b}{q^{d}} \right| \gg \frac{1}{p^{2d}}. \]

Since \( \tau > 2d \), we conclude that the sets \( A_{d,p,\tau} \) and \( A_{d,q,\tau} \) are disjoint for two distinct prime numbers \( p_{k} \leq p, q \leq 2p_{k} \) when \( p_{k} \) is large enough.

Note that there are \( p^{1+o(1)} \) prime numbers between \( p_{k} \) and \( 2p_{k} \), and for each prime number \( p_{k} \leq p \leq 2p_{k} \) the set \( A_{d,p,\tau} \) contains \( p^{d+o(1)} \) intervals with length \( 2p^{-\tau} \) (since the interval \( \mathcal{J} \) is fixed). Thus the set \( \mathcal{F}_{k} \) consists of \( p_{k}^{d+1+o(1)} \) intervals with length nearly \( p_{k}^{-\tau} \). As in the proof of Theorem 1.5, we remark that the implied constant may depend on \( \mathcal{J} \), however it is not hard to see that for a fixed interval \( \mathcal{J} \) this constant does not affect the result.

By (3.4), each interval of \( \mathcal{F}_{k} \) consists nearly \( p_{k}^{d+1+o(1)} \) intervals of \( \mathcal{F}_{k+1} \) of length \( p_{k+1}^{-\tau} \).

Applying the method in [6, Example 4.7], see also Lemma 3.1, we obtain the inequality (4.10) which together with (4.8) and (4.9) concludes the proof. \( \square \)

For each \( 0 < \alpha < 1 \) we intend to find some \( \tau > 2d \) such that
\[ \mathcal{B}_{d,\tau} \subseteq \mathcal{E}_{\alpha,d}(\mathcal{J}). \]

Hence, by the monotonicity property of the Hausdorff dimension and Lemma 4.1 we obtain
\[ \dim \mathcal{E}_{\alpha,d}(\mathcal{J}) \geq \dim \mathcal{B}_{d,\tau} = (d + 1)/\tau. \] (4.11)

Applying the arguments in the proof of Theorem 1.6, see (4.5), we obtain that for any
\[ \tau > \max\{d^{2}/2 + d, d/(1 - \alpha)\} > 2d, \]

and any \( A_{d,p,\tau} \) there exists some \( N_{d,p,\tau} \) such that for any \( x \in A_{d,p,\tau} \)
\[ |\sigma_{d}(x; N_{d,p,\tau})| \geq N_{d,p,\tau}^{\alpha}. \]

Thus the condition of Lemma 4.1 is satisfied. Combining with (4.11), we obtain
\[ \dim \mathcal{E}_{\alpha,d}(\mathcal{J}) \geq (1 + 1/d) \min \{2/(d + 2), 1 - \alpha\} \]

which finishes the proof.
5. PROOFS OF ABUNDANCE OF POORLY DISTRIBUTED POLYNOMIALS

5.1. Exponential sums and the discrepancy. For our applications we need the following *Koksma-Hlawka inequality*, see [5, Theorem 1.14] for a general statement.

**Lemma 5.1.** Using the above notation, for any \( x \in T_d \)

\[ S_d(x; N) \ll D_d(x; N). \]

Note that in particular, Lemma 5.1 implies \( \sigma_d(x; N) \ll \Delta_d(x; N) \) for \( x \in [0, 1) \).

5.2. Proof of Theorems 1.9 and 1.11. We see that Lemma 5.1 implies that for any cube \( Q \subseteq T_d \) or interval \( I \subseteq T \) and any \( \varepsilon > 0 \) one has

\[ E_{\alpha + \varepsilon, d}(Q) \subseteq D_{\alpha, d}(Q) \quad \text{and} \quad E_{\alpha + \varepsilon, d}(I) \subseteq D_{\alpha, d}(I). \]

Combining (5.1) for \( Q = T_d \) and \( I = T \) with Theorems 1.3 and 1.6 we obtain Theorems 1.9 and 1.11, respectively.

5.3. Proof of Theorems 1.10 and 1.12. Applying (5.1) and the monotonicity property of Hausdorff dimension we have

\[ \dim D_{\alpha, d}(Q) \geq \sup_{\varepsilon > 0} \dim E_{\alpha + \varepsilon, d}(Q) \]

and

\[ \dim D_{\alpha, d}(I) \geq \sup_{\varepsilon > 0} \dim E_{\alpha + \varepsilon, d}(I). \]

Combining this with Theorems 1.5 and 1.7 we obtain Theorems 1.10 and 1.12, respectively.

6. FURTHER RESULTS, OPEN PROBLEMS AND CONJECTURES

6.1. Further extensions of Theorems 1.3 and 1.5. On the other hand, the method of proof of Lemma 2.6 is quite robust and can be implies to some other families of polynomials, such as sparse polynomials

\[ a_1X^{m_1} + \ldots + a_dX^{m_d} \in \mathbb{F}_p[X]. \]

In turn, this can be used to obtain versions of Theorems 1.3 and 1.5 for exponential sum with sparse polynomials

\[ S_m(x; N) = \sum_{n=1}^N e(x_1n^{m_1} + \ldots + x_dn^{m_d}), \]

where \( m = (m_1, \ldots, m_d) \in \mathbb{Z}^d \) with \( 1 \leq m_1 < m_2 < \ldots < m_d \). More precisely, for each \( 0 < \alpha < 1 \) and \( m = (m_1, \ldots, m_d) \in \mathbb{Z}^d \), we define

\[ \mathcal{E}_{\alpha, m} = \{ x \in T_d : |S_m(x; N)| \geq N^\alpha \text{ for infinitely many } N \in \mathbb{N} \}. \]
We note that (2.3) can easily be extended to sparse polynomials

\[(6.1) \sum_{n=1}^{p} e_p(a_1 n^{m_1} + \ldots + a_d n^{m_d}) \geq d! p^{2d} + O(p^{2d-1}),\]

which in turn leads to full analogues of Lemmas 2.4, 2.5 and 2.6. Then we have the following direct generalisations of Theorems 1.3 and 1.5 which can be obtained at the cost of essentially only typographical changes in their proofs. For each \(0 < \alpha < 1\) and \(m = (m_1, \ldots, m_d) \in \mathbb{Z}^n\) with \(1 \leq m_1 < m_2 < \ldots < m_d\),

\[\text{(A) the subset } T_d \setminus E_{\alpha, m} \text{ is of the first Baire category; }\]

\[\text{(B) for any cube } \Omega \subseteq T_d \text{ we have,}\]

\[\dim E_{\alpha, m}(\Omega) \geq \min \left\{ \frac{d \kappa_d}{m_d}, \frac{2d \kappa_d (1 - \alpha)}{m_d} \right\},\]

where \(\kappa_d\) is given by (1.3). Note that we recover the bound of Theorem 1.5 (for \(d \geq 3\)) provided \(m_d = d\).

**Remark 6.1.** We note that (6.1) is only a lower bound rather than an asymptotic formula as (2.3). In fact, most likely an asymptotic form of (6.1) holds with \(m_1 \ldots m_d\) instead of \(d!\), see [21]. However this is inconsequential for our results.

We note that it is natural to try to improve Theorem 1.5 via an appropriate version of Lemma 2.16 for arbitrary polynomials. Unfortunately the only known result in this direction [20, Theorem 7.2] is not strong enough to lead to such an improvement.

### 6.2. Further questions about the structure of Weyl sums.

For \(x \in T\) we now define

\[\sigma(x) = \inf \{ s > 0 : S_d(x; N) \ll N^s \}\]

\[= \sup \{ s > 0 : |S_d(x; N)| \gg N^s \text{ for infinitely many } N \in \mathbb{N} \}\]

\[= \sup \{ s > 0 : |S_d(x; N)| \geq N^s \text{ for infinitely many } N \in \mathbb{N} \}.\]

Alternatively, we may also define

\[(6.2) \sigma(x) = \limsup_{N \to \infty} \frac{\log |S_d(x; N)|}{\log N}.\]

By the definition we have

\[E_{\alpha, d} \subseteq \{ x \in T_d : \sigma(x) \geq \alpha \}.\]

For each \(0 \leq \alpha \leq 1\) we define the level set

\[\Omega_{\alpha} = \{ x \in T_d : \sigma(x) = \alpha \}.\]
Clearly these sets $\Omega_\alpha$ form a decomposition of $T_d$. There are several natural questions about these sets. Note that Conjecture 1.1 asserts that for any $\alpha \in (1/2, 1]$ we have $\lambda(\Omega_\alpha) = 0$. We may make the following stronger conjecture.

**Conjecture 6.2.** For $\alpha \in [0, 1]$ we have

$$
\lambda(\Omega_\alpha) = \begin{cases} 
0 & \text{for } \alpha \neq 1/2, \\
1 & \text{for } \alpha = 1/2.
\end{cases}
$$

We may also use the Hausdorff dimension to measure the size of $\Omega_\alpha$.

**Question 6.3.** What is the Hausdorff dimension $\dim \Omega_\alpha$ of $\Omega_\alpha$?

Finally, one can also ask whether the function $\sigma(x)$ which is defined by (6.2) has multifractal structure. More precisely we ask the following:

**Question 6.4.** Does there exist a set $A \subseteq [0, 1]$ with $\lambda(A) > 0$ such that for any $\alpha \in A$ we have

$$
\dim \Omega_\alpha > 0?
$$

### 6.3. Further questions about the distribution of large complete rational sums and possible improvements of Theorem 1.5.

It is certainly natural to consider more general transformations

$$
(6.3) \quad f(X) \mapsto f(\lambda X + \mu), \quad (\lambda, \mu) \in \mathbb{F}_p^* \times \mathbb{F}_p,
$$

instead of just $f(X) \mapsto f(\lambda X)$ which is essentially used in the proof of Lemma 2.6. The transformation (6.3) is very similar to the transformation $f(X) \mapsto \lambda^{-d} f(\lambda X + \mu)$ used in the proof of [18, Lemma 4]. However, while in [18] the Deligne bound (see [9, Section 11.11]) is applied to the corresponding double exponential sums with polynomials in $\lambda$ and $\mu$, in the case of (6.3) these polynomials are singular, and so the Deligne bound does not apply. It is certainly interesting to find an alternative way, and thus improve Lemma 2.6, in which $\kappa_d$ can possibly be replaced with $1/d$.

Lemma 2.5 study the distribution of sets

$$
\{(\lambda a_1, \ldots, \lambda^d a_d) : \lambda \in \mathbb{F}_p^*\},
$$

where $a_j \in \mathbb{F}_p^*$ for each $j = 1, \ldots, d$. Lemma 2.5 asserts that for any box $\mathcal{B}$ of $\mathbb{F}_p^d$ with the side length $L \geq C p^{1-1/2d} \log p$ for some large constant $C$ there exists $\lambda \in \mathbb{F}_p^*$ such that

$$
(\lambda a_1, \ldots, \lambda^d a_d) \in \mathcal{B}.
$$

Note that there are totally $p - 1$ vectors

$$
(\lambda a_1, \ldots, \lambda^d a_d), \quad \lambda \in \mathbb{F}_p^*,
$$
thus the smallest $L$ in Lemma 2.5 should be

$$L \gg p^{1-1/d}.$$ 

One could ask that is this a sufficient condition.

**Question 6.5.** Let $(a_1, \ldots, a_d) \in (\mathbb{F}_p^*)^d$. Is it true that for any $\varepsilon > 0$ there exists a constant $C_\varepsilon$ such that any box $\mathcal{B}$ of $\mathbb{F}_p^d$ with the side length $L \geq C_\varepsilon p^{1-1/d+\varepsilon}$ contains a vector $(\lambda a_1, \ldots, \lambda^d a_d)$ for some $\lambda \in \mathbb{F}_p^*$?

It is also interesting to consider the special case that is the distribution of

$$\{ (\lambda, \lambda^2) : \lambda \in \mathbb{F}_p^* \}.$$ 

Note that studying the distribution of

$$\{ \lambda^2 : \lambda \in \mathbb{F}_p^* \}$$

is already an interesting and hard problem related to the distribution of quadratic nonresidues.

A possible approach to improving Theorem 1.5 is via finding an asymptotic formula or at least a lower bound for the average of $T_{d,p}(a)$ over small box $\mathcal{B}$ as in (2.2). In fact finding lower bounds for the moments

$$M_{\nu,d}(\mathcal{B}) = \sum_{\substack{a \in \mathcal{B} \\ a \neq 0}} |T_{d,p}(a)|^{2\nu}, \quad \nu = 1, \ldots, d,$$

of nontrivial sums with $a \neq 0$ is of independent interest. For $\mathcal{B} = \mathbb{F}_p^d$ one can easily extend the result of Mordell [16], that is, (2.3), to any $\nu = 1, \ldots, d$ and obtain

$$M_{\nu,d}(\mathbb{F}_p^d) = A_d(\nu)p^{d+\nu} + O \left( p^{d+\nu-1} \right),$$

where

$$A_d(\nu) = \begin{cases} d! - 1, & \text{for } \nu = d, \\ \nu!, & \text{for } \nu = 1, \ldots, d - 1, \end{cases}$$

see also [12, Equation (2)].

Using the same arguments as in the proof of Lemmas 2.5 and 2.6 with $k = d$, one can obtain an asymptotic formula

$$M_{\nu,d}(\mathcal{B}) = A_d(\nu) L^d p^\nu + O \left( p^{d+\nu-1} L^{1/2} (\log p)^{d-1} \right),$$

see Appendix B, which is nontrivial in the case of cubes with the side length $L \geq p^{2(d-1)/(2d-1)+\varepsilon}$ for any fixed $\varepsilon > 0$. However we are interested in much smaller boxes, for example of size of the side length about $L \sim p^{1/2+\varepsilon}$. In fact, a lower bound of the form $L^d p^{\nu+o(1)}$ for any fixed $\nu$ is sufficient for our applications.
6.4. **An approach to Conjecture 1.1.** Recall that Conjecture 1.1 asserts that $\vartheta_d = 1/2$ for each integer $d \geq 2$, and the bound (1.2) gives $\vartheta_d \leq 1/2$. Thus it is sufficient to prove that for any $0 < \alpha < 1/2$ one has $\lambda(\mathcal{E}_{\alpha,d}) > 0$.

For $0 < \alpha < 1/2$ and integer $d \geq 2$ we define

$$\mathcal{A}_{\alpha,d} = \{x \in T_d : |S(x; i)| \geq i^\alpha\}.$$ 

We can write

$$\mathcal{E}_{\alpha,d} = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} \mathcal{A}_{\alpha,d}.$$ 

**Lemma 6.6.** Let $0 < \alpha < 1/2$ then $\lambda(\mathcal{A}_{\alpha,d}) \gg 1/i$, and hence

$$\sum_{i=1}^{\infty} \lambda(\mathcal{A}_{\alpha,d}) = \infty.$$ 

**Proof.** Applying the trivial bound $|S(x; i)| \leq i$ we obtain

$$\int_{T_d} |S(x; i)|^2 dx = \int_{\mathcal{A}_{\alpha,d}} |S(x; i)|^2 dx + \int_{T_d \setminus \mathcal{A}_{\alpha,d}} |S(x; i)|^2 dx \leq i^2 \lambda(\mathcal{E}_i) + i^{2\alpha}.$$ 

Combining with Parseval identity

$$\int_{T_d} |S(x; i)|^2 dx = i$$

and the condition $0 < \alpha < 1/2$, we obtain the desired result. \qed

Suppose that the sets $\mathcal{A}_{\alpha,d,i}$ are pair independent with respect to the Lebesgue measure $\lambda$, i.e., for any $i \neq j$ we have

$$\lambda(\mathcal{A}_{\alpha,d,i} \cap \mathcal{A}_{\alpha,d,j}) = \lambda(\mathcal{A}_{\alpha,d,i}) \lambda(\mathcal{A}_{\alpha,d,j}),$$

then the Borel-Cantelli lemma and (6.6) implies that $\lambda(\mathcal{E}_{\alpha,d}) = 1$. Surely the pair independent assumption is not true, and an ordinary way to overcome this is by the following arguments. One first show that these sets are weak independent, that is there exists some constant $C > 0$ such that for any $i \neq j$ we have

$$\lambda(\mathcal{A}_{\alpha,d,i} \cap \mathcal{A}_{\alpha,d,j}) \leq C \lambda(\mathcal{A}_{\alpha,d,i}) \lambda(\mathcal{A}_{\alpha,d,j}),$$

then a variant of the Borel-Cantelli lemma gives

$$\lambda(\mathcal{E}_{\alpha,d}) \geq 1/C > 0.$$ 

Secondly one may use a zero-one law to pass from $\lambda(\mathcal{E}_{\alpha,d}) > 0$ to $\lambda(\mathcal{E}_{\alpha,d}) = 1$. 

Appendix A. Proof of the bound (1.2) and some extensions

By applying a very special case of the Menshov–Rademacher theorem, see [10, p. 251] for the general statement, we conclude that if for some sequence $c_n, n \in \mathbb{N}$ of complex numbers we have

\[(A.1) \quad \sum_{n=1}^{\infty} |c_n|^2 (\log n)^2 < \infty,\]

then the series

\[\sum_{n=1}^{\infty} c_n e(nx)\]

converges for almost all $x \in [0, 1)$.

For $x = (x_1, \ldots, x_d) \in T_d$ we have

\[e(x_1 n + \ldots + x_d n^d) = e(x_1 n) e(x_2 n^2 + \ldots + x_d n^d).\]

It follows that for any $(x_2, \ldots, x_d) \in T_{d-1}$ the series

\[\sum_{n=1}^{\infty} c_n e(x_2 n^2 + \ldots + x_d n^d) e(x_1 n)\]

converges for almost all $x_1 \in [0, 1)$. Together with the Fubini theorem, we obtain that the series

\[\sum_{n=1}^{\infty} c_n e(x_1 n + \ldots + x_d n^d)\]

converges for almost all $x \in T_d$.

Now we turn to the proof of (1.2). We denote

\[\log^+ k = \max\{1, \log k\},\]

and

\[\varphi_n(x) = e(x_1 n + \ldots + x_d n^d).\]

Fix any $\gamma > 3/2$, and write

\[S_d(x; N) = \sum_{n=1}^{N} n^{-1/2} (\log^+ n)^{-\gamma} \varphi_n(x) n^{1/2} (\log^+ n)^{\gamma}.\]

Then the summation by parts gives

\[S_d(x; N) = s_d(x; N) N^{1/2} (\log^+ N)^{\gamma}\]

\[\quad + \sum_{k=1}^{N-1} s_d(x; k) \left( (k^{1/2} (\log^+ k)^{\gamma} - (k + 1)^{1/2} (\log^+ (k + 1))^{\gamma} \right), \]

(A.2)
where

\[ s_d(x; k) = \sum_{n=1}^{k} n^{-1/2} (\log^+ n)^{-\gamma} \varphi_n(x). \]

Since the condition (A.1) is satisfied, for almost all \( x \in \mathbb{T}_d \) there exits some positive \( B_x \) such that for all \( k \in \mathbb{N} \) we have

(A.3) \[ |s_d(x; k)| \leq B_x. \]

Substituting (A.3) in (A.2) we easily conclude that for almost all \( x \in [0, 1) \) we have (1.2).

We note that the above arguments implies that for any \((x_2, \ldots, x_d) \in \mathbb{T}_{d-1}\) the bound

\[ \left| \sum_{n=1}^{N} e^{i(x_1n + \ldots + x_d n^d)} \right| \leq N^{1/2} (\log N)^{3/2 + o(1)} \]

holds for almost all \( x_1 \in [0, 1) \).

Furthermore, one can easily see that the above argument work in a much broader generality. For example, let \( f_1, \ldots, f_d \) be \( d \) functions such that for any \( n \in \mathbb{N} \) we have \( f_i(n) \in \mathbb{Z} \) for each \( i = 1, \ldots, d \). If one of these functions is eventually strictly monotonic, then for almost all \((x_1, \ldots, x_d) \in \mathbb{T}_d\) we have

\[ \left| \sum_{n=1}^{N} e^{i(x_1f_1(n) + \ldots + x_d f_d(n))} \right| \leq N^{1/2} (\log N)^{3/2 + o(1)}. \]

For instance, for \( 0 < t < \infty, a > 1 \) the bound

\[ \left| \sum_{n=1}^{N} e^{i(x_1 \lfloor nt \rfloor + x_2 \lfloor an \rfloor + x_3 \lfloor \log n \rfloor)} \right| \leq N^{1/2} (\log N)^{3/2 + o(1)} \]

holds for almost all \((x_1, x_2, x_3) \in \mathbb{T}_3\).

**Remark A.1.** For the case \( d = 2 \) we can obtain the bound \( N^{1/2} \log N \) for the estimate (1.2) in a different way. The Khinchine theorem, see [1, Introduction], implies that for almost all irrational \( x \in [0, 1) \) there exits some positive constant \( c(x) \) such that for all rational \( a/q \) with \( \gcd(a, q) = 1 \) we have

\[ \left| x - \frac{a}{q} \right| \geq \frac{c_x}{(q \log q)^2}. \]
On the other hand, by [9, Theorem 8.1], if \( |x - a/q| \leq 1/qN \) with \( \gcd(a, q) = 1 \) and \( 1 \leq q \leq N \) then for any \( y \in [0, 1) \) one has

\[
\sum_{n=1}^{N} e(yn + xn^2) \ll N/q^{1/2} + q^{1/2} \log q.
\]

Combining these two results, we conclude that for almost all \( x \in T_2 \) one has

\[
S_2(x; N) \ll N^{1/2} \log N.
\]

**Appendix B. Moments of rational exponential sums over small boxes**

Here we sketch a proof of (6.5). Clearly we can assume that \( 0 \notin I_1 \) (it is easy to see that by Lemma 2.1 discarding \( O(p^{d-1}) \) such sums changes the value of \( M_{\nu,d}(B) \) by \( O(L^{d-1}p^\nu) \), which can be absorbed in the error in (6.5)). In particular, we can assume that \( 0 \notin B \).

Observe that for any \( \lambda \in \mathbb{F}_p^* \) and \( b \in \mathbb{F}_p^d \) we have

\[
T_{d,p}(b) = T_{d,p}(\lambda \circ b),
\]

where

\[
\lambda \circ b = (\lambda b_1, \ldots, \lambda b_d).
\]

It follows that

\[
M_{\nu,d}(B) = \frac{1}{p-1} \sum_{\lambda \in \mathbb{F}_p^*} \sum_{b \in B} |T_{d,p}(\lambda \circ b)|^{2d}
\]

\[
= \frac{1}{p-1} \sum_{a \in \mathbb{F}_p^d} N(a) |T_{d,p}(a)|^{2\nu},
\]

where

\[
N(a) = \#\{(\lambda, b) \in \mathbb{F}_p^* \times B : \lambda \circ b = a\}.
\]

Let \( \Lambda(a) \) be the set of \( \lambda \in \mathbb{F}_p^* \) with \( a\lambda \in I_1 \) where \( B \) is as in (2.2). Hence for \( a = (a_1, \ldots, a_d) \), we have

\[
N(a) = \#\{\lambda \in \Lambda(a_1) : (\lambda^2 a_2, \ldots, \lambda^d a_d) \in I_2 \times \ldots \times I_d\}.
\]

By the orthogonality of characters, and then changing the order of summation and separating the contribution from \( h_2 = \ldots = h_d \) we
obtain
\[
N(a) = \frac{1}{p^{d-1}} \sum_{\lambda \in \Lambda(a_1)} \sum_{y_2 \in I_2} \cdots \sum_{y_d \in I_d} \left( \sum_{\lambda \in \Lambda(a_1)} \prod_{i=2}^d \sum_{y_j \in I_j} e_p \left( \sum_{j=2}^d h_j (\lambda^j a_j - y_j) \right) \right)
\]
(B.2)
\[
= \frac{\#\Lambda(a_1) L^{d-1}}{p^{d-1}} + R(a),
\]
where
\[
R(a) = \frac{1}{p^{d-1}} \sum_{\lambda \in \Lambda(a_1)} \prod_{i=2}^d \sum_{y_j \in I_j} e_p \left( \sum_{j=2}^d h_j \lambda^j a_j \right) \left| \sum_{\lambda \in \Lambda(a_1)} \sum_{y_j \in I_j} e_p \left( \sum_{j=2}^d h_j \lambda^j a_j \right) \right|.
\]
We note that \( N(a) = 0 \) if the first coordinate of \( a \) is zero. Combining (B.1) and (B.2), we obtain
\[
M_{\nu,d}(\mathfrak{B}) = \frac{L^{d-1}}{(p-1)p^{d-1}} \sum_{\mathfrak{a} \in \mathbb{F}_p^d \setminus \{0\}} \#\Lambda(a) |T_{d,p}(\mathfrak{a})|^{2\nu} + O(W),
\]
where
\[
W = \frac{1}{p-1} \sum_{\mathfrak{a} \neq 0} |R(\mathfrak{a})| |T_{d,p}(\mathfrak{a})|^{2\nu}.
\]
By Lemma 2.1 we obtain
\[
\frac{L^{d-1}}{(p-1)p^{d-1}} \sum_{\mathfrak{a} \in \mathbb{F}_p^d \setminus \{0\}} \#\Lambda(a) |T_{d,p}(\mathfrak{a})|^{2\nu} = \frac{L^d}{(p-1)p^{d-1}} \sum_{\mathfrak{a} \in \mathbb{F}_p^d \setminus \{0\}} |T_{d,p}(\mathfrak{a})|^{2\nu} = \frac{L^d}{(p-1)p^{d-1}} M_{\nu,d}(\mathbb{F}_p^d) + O \left( L^d p^{\nu-1} \right)
\]
\[
= \frac{L^d}{p^d} M_{\nu,d}(\mathbb{F}_p^d) + O \left( L^d p^{\nu-1} \right).
\]
Hence, recalling (6.4) we obtain
\[
M_{\nu,d}(\mathfrak{B}) = A_d(\nu) L^d p^{\nu} + O \left( L^d p^{\nu-1} + W \right).
\]
(B.3)
To estimate $W$ we note that by \cite[Equation (8.6)]{9} we have

$$R(a) \ll (p-1)/2 \prod_{i=2}^{d} \min \left\{ \frac{L}{p}, \frac{1}{|h_i|} \right\} \sum_{\lambda \in \Lambda(a_1)} e_p \left( \sum_{j=2}^{d} a_j h_j \lambda_j \right).$$

By Lemma 2.1 we now see that

$$W \ll p^{\nu-1} \sum_{a_1 \in \mathbb{F}_p^*} \sum_{h_2, \ldots, h_d=-(p-1)/2} (p-1)/2 \prod_{i=2}^{d} \min \left\{ \frac{L}{p}, \frac{1}{|h_i|} \right\} \sum_{\lambda \in \Lambda(a_1)} e_p \left( \sum_{j=2}^{d} a_j h_j \lambda_j \right).$$

Using the Cauchy inequality, as in the proof of Lemma 2.6, we have

$$\left( \sum_{a_2, \ldots, a_d \in \mathbb{F}_p} \left| \sum_{\lambda \in \Lambda(a_1)} e_p \left( \sum_{j=2}^{d} a_j h_j \lambda_j \right) \right| \right)^2 \ll p^{d-1} \sum_{a_2, \ldots, a_d \in \mathbb{F}_p} \left| \sum_{\lambda \in \Lambda(a_1)} e_p \left( \sum_{j=2}^{d} a_j h_j \lambda_j \right) \right|^2 \ll p^{2(d-1)} L.$$

Hence,

$$W \ll p^{\nu+d-2} L^{1/2} \sum_{a_1 \in \mathbb{F}_p^*} \sum_{h_2, \ldots, h_d=-(p-1)/2} (p-1)/2 \prod_{i=2}^{d} \min \left\{ \frac{L}{p}, \frac{1}{|h_i|} \right\}$$

$$\ll p^{\nu+d-1} L^{1/2} \sum_{h_2, \ldots, h_d=-(p-1)/2} (p-1)/2 \prod_{i=2}^{d} \min \left\{ \frac{L}{p}, \frac{1}{|h_i|} \right\} \ll p^{\nu+d-1} L^{1/2} (\log p)^{d-1},$$

which together with (B.3) yields (6.5).

**Acknowledgement**

The authors are grateful to Fernando Chamizo, Boris Kashin, Bryce Kerr, Sergei Konyagin and Trevor Wooley for helpful advice and discussions.

This work was supported in part by ARC Grant DP170100786.
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