Integrability and Scheme Independence of Even-Dimensional Quantum Geometry Effective Action

KEN-JI HAMADA

Institute of Particle and Nuclear Studies, High Energy Accelerator Research Organization (KEK), Tsukuba, Ibaraki 305-0801, Japan

Abstract

We investigate how the integrability conditions for conformal anomalies constrain the form of the effective action in even-dimensional quantum geometry. We show that the effective action of four-dimensional quantum geometry (4DQG) satisfying integrability has a manifestly diffeomorphism invariant and regularization scheme-independent form. We then generalize the arguments to six dimensions and propose a model of 6DQG. A hypothesized form of the 6DQG effective action is given.

1E-mail address : hamada@post.kek.jp
1 Introduction

Since quantum geometry (QG) is defined by functional integrations over the metric fields, diffeomorphism invariance in QG can equivalently be described as an invariance under any change of the background metric. This background-metric independence includes an invariance under a conformal change of the background metric. Thus, in even-dimensional QG well-defined on the background metric [1]–[16], conformal anomalies [17]–[29] play an important role. Therefore, to preserve diffeomorphism invariance we must formulate an even-dimensional QG while considering that conformal anomalies always exist [1]–[16].

Background-metric independence in two dimensions implies that QG can be described as a conformal field theory [2, 3]. This idea can be generalized to an arbitrary numbers of even dimensions [11, 13, 14, 15, 16]. However, as recently studied [13, 14], this generalization is not simple, because the traceless mode becomes dynamical in higher dimensions, so that higher-dimensional QG can no longer be described as a free theory. Furthermore, it has been found that the integrability condition of the conformal anomaly [21, 22] introduces a strong constraint on even-dimensional QG [7, 8, 14].

In this paper we further consider how the integrability condition of the conformal anomaly affects even-dimensional QG. We also settle the problem of the regularization scheme dependence and show that the effective action has a manifestly diffeomorphism invariant and regularization scheme-independent form.

This paper is organized as follows. In the next section we present the fundamental idea of how to preserve diffeomorphism invariance in even-dimensional QG and review how such an idea is realized in exactly solvable 2DQG [2, 3, 4]. In $D \geq 4$ dimensions, the integrability condition of the conformal anomaly not only restricts interactions of matter fields to conformally invariant ones, but also reduces the number of indefinite coefficients in the gravity sector [14]. How the integrability condition affects 4DQG is rediscussed in section 3. We then show that the effective action can be written in a diffeomorphism invariant and scheme independent form. A generalization to six dimensions [15, 16] is discussed in section 4. We show that Duff’s scheme [19] is also useful to tame the trivial anomalies in six dimensions [23, 27]. We then propose a model of 6DQG that is based on the arguments concerning integrability made in the 4DQG case. Many indefinite
coefficients that result from the existence of many curvature invariants are fixed by enforcing the integrability, and a hypothesized scheme-independent form of 6DQG effective action is given. Section 5 is devoted to conclusions and discussion.

We use the curvature convention in which \( R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} \) and \( R^\lambda_{\mu\sigma\nu} = \partial_\sigma \Gamma^\lambda_{\mu\nu} - \cdots \).

## 2 Conditions of Diffeomorphism Invariance

In this section we briefly explain how to realize diffeomorphism invariance in even-dimensional QG.

QG is defined by functional integration over the metric field as

\[
Z = \int \frac{[g^{-1} dg][dX]_g}{\text{vol(diff.)}} \exp[-I(X, g)] , \tag{2.1}
\]

where \( I \) is an invariant action and \( X \) is a matter field. In this paper we consider a conformal scalar without self-interactions, for example. The measure of the metric field is defined by the invariant norm

\[
<d g, d g>_g = \int d^D x \sqrt{g} g^{\mu\nu} g_{\lambda\sigma} (dg_\mu \partial_\nu dg_\sigma + udg_\mu d g_\nu) , \tag{2.2}
\]

where \( D = 2n \) and \( u > -1/D \). This measure can be orthogonally decomposed into the conformal mode and the traceless mode as

\[
<d \phi, d \phi>_g = \int d^D x \sqrt{g} (d \phi)^2 , \tag{2.3}
\]

\[
<d h, d h>_g = \int d^D x \sqrt{g} tr(e^{-h} d e^h)^2 . \tag{2.4}
\]

Here, the metric is decomposed as \( g_{\mu\nu} = e^{2\phi} \hat{g}_{\mu\nu} \) and \( \hat{g}_{\mu\nu} = (\hat{g} e^h)_{\mu\nu} \), where \( tr(h) = 0 \) \[3\] [3] [4].

This definition is manifestly diffeomorphism invariant/background-metric independent. However, it is not well-defined because the measures of the metric fields defined by (2.3) and (2.4) have a metric dependence, represented by \( \sqrt{g} \), in the measures themself, so that we must integrate this dependence when we quantize the conformal mode \( \phi \).
Instead, we consider measures defined on the background metric as

\[
\langle d\phi, d\phi \rangle_{\tilde{g}} = \int d^D x \sqrt{\tilde{g}} (d\phi)^2 ,
\]

\[
\langle dh, dh \rangle_{\tilde{g}} = \int d^D x \sqrt{\tilde{g}} \text{tr}(e^{-h} de^h)^2 .
\]

This replacement, however, violates diffeomorphism invariance. In fact, these norms conformally change under a general coordinate transformation generated by \(\delta g_{\mu\nu} = g_{\mu\lambda} \nabla_\nu \xi^\lambda + g_{\nu\lambda} \nabla_\mu \xi^\lambda\), which can be decomposed as

\[
\delta \phi = \frac{1}{D} \tilde{\nabla}_\lambda \xi^\lambda + \xi^\lambda \partial_\lambda \phi ,
\]

\[
\delta \bar{g}_{\mu\nu} = \bar{g}_{\mu\lambda} \nabla_\nu \xi^\lambda + \bar{g}_{\nu\lambda} \nabla_\mu \xi^\lambda - \frac{2}{D} \bar{g}_{\mu\nu} \tilde{\nabla}_\lambda \xi^\lambda ,
\]

where the relation \(\tilde{\nabla}_\lambda \xi^\lambda = \bar{\nabla}_\lambda \xi^\lambda\) is used. Therefore, these measures produce conformal anomalies \([20]\) under the general coordinate transformation.

As a lesson from 2DQG \([3, 4, 5]\), in order to preserve diffeomorphism invariance, we must add an action \(S\) as

\[
Z = \int \frac{[d\phi]_{\tilde{g}} [e^{-h} de^h]_{\bar{g}} [dX]_{\bar{g}}}{\text{vol(diff.)}} \exp[-S(\phi, \bar{g}) - I(X, g)] ,
\]

where the measures of the metric fields are now defined by (2.5) and (2.6).

Let us now briefly see how background-metric independence constrains the theory provided by (2.8). Background-metric independence for the traceless mode represents the condition that \(\bar{g}\) and \(h\) always appear in the combination \(\bar{g} = \tilde{g} e^h\) in (2.8) \([13]\). This condition guarantees, at most, that the effective action has an invariant form on the metric \(\tilde{g}\).

Background-metric independence for the conformal mode requires that \(S\) satisfies the Wess-Zumino condition \([30]\), defined by

\[
S(\phi, \tilde{g}) = S(\omega, \bar{g}) + S(\phi - \omega, e^{2\omega} \bar{g}) .
\]

Such an action is obtained by integrating the conformal anomaly within the interval \([0, \phi]\). Hence it satisfies the initial condition \(S(0, \tilde{g}) = 0\) and has a local form. In this paper we call this local action the Wess-Zumino action, because condition (2.9) is essential in the arguments concerning diffeomorphism invariance. In two dimensions it is usually called the Liouville action \([4]\).
well-known non-local forms of the integrated conformal anomaly are called
Polyakov action [1] and Riegert action [7] in two and four dimensions, re-
spectively. Why we distinguish between the local and the non-local actions
becomes clear below.

Although the Wess-Zumino condition fixes the form of $S$, some overall
coefficients remain to be determined. These coefficients should be de-
termined from the requirement of diffeomorphism invariance in a self-consistent
manner. The process used to determine them is as follows.

Under the general coordinate transformation, $\delta I = 0$, while the Wess-
Zumino action is not invariant and produces a conformal anomaly. This
property results from condition (2.9). Diffeomorphism invariance is now re-
alized dynamically in such a manner that $\delta S$ cancels conformal anomalies
calculated with loop effects of the combined theory, $I = S + I$. In other
words, we consider the regularized 1PI effective action $\Gamma$ of the combined
theory $I$ and require $\delta \Gamma = 0$ to determine $S$. This means that the tree ac-
tion $I$ is not manifestly invariant, but by including loop effects the effective
action becomes an invariant form on the metric $g$.

Here, it is worth commenting on the difference in the Wess-Zumino action
defined by (2.9) and the non-local Polyakov/Riegert action. The former pro-
duces conformal anomalies under a general coordinate transformation, while
the non-local Polyakov/Riegert action, which appears in the effective action
due to loop effects, is generally defined by the condition that it produces
conformal anomalies under a conformal change.

As an exercise, let us first discuss 2DQG coupled $N$ conformal scalars.
The tree action in the conformal gauge is given by [3, 4, 5]

\[ I = \frac{b}{4\pi} \int d^2 x \sqrt{\bar{g}} (\bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \bar{R} \phi) + I_{GF+FP} + I_M(X, \bar{g}), \]  

(2.10)

where $I_M$ is the invariant action of the $N$ free scalars. The gauge-fixing term
and the Faddeev-Popov (FP) ghost action are given by [3]

\[ I_{GF+FP} = \frac{1}{4\pi} \int d^2 x \sqrt{\bar{g}} \left( -i B_{\mu\nu} (\bar{g}^{\mu\nu} - \bar{g}^{\mu\nu}) + 2 \bar{g}^{\mu\nu} b_{\mu\lambda} \nabla_\nu c^\lambda \right), \]  

(2.11)

where the reparametrization ghost $c^\mu$ is a contravariant vector. $B_{\mu\nu}$ and the
anti-ghost $b_{\mu\nu}$ are covariant symmetric traceless tensors. The coefficient $b$ is
uniquely determined by diffeomorphism invariance.
Consider the effective action of 2DQG, which has the form
\[ \Gamma = \mathcal{I}(\phi, X, \bar{g}) + W(\bar{g}) \, , \] (2.12)
where \( W \) is a loop effect that depends only on \( \bar{g} \) because the measure is now defined on \( \bar{g} \). The condition of diffeomorphism invariance, \( \delta \Gamma = 0 \), is now given by
\[ -\frac{b}{4\pi} \int d^2 x \sqrt{\bar{g}} \omega \bar{R} + \delta_{\omega} W(\bar{g}) = 0 \, , \] (2.13)
where \( \delta_{\omega} \bar{g}_{\mu\nu} = 2\omega \bar{g}_{\mu\nu} \) and \( \omega = -\frac{1}{2} \tilde{\nabla} \lambda \xi^\lambda \). Here, \( \delta W = \delta_{\omega} W \) because \( W \) does not depend on the conformal mode \( \phi \). The second term on the l.h.s is just the conformal anomaly of the theory \( \mathcal{I} \).

From one-loop calculations using the tree action \( \mathcal{I} \), we obtain the well-known non-local Polyakov action \[ W(\bar{g}) = \frac{N - 25}{96\pi} \int d^2 x \sqrt{\bar{g}} \bar{R} \frac{1}{\Box} \bar{R} \, , \] (2.14)
where \( N \) comes from scalar matter fields and this becomes \( N - 26 \) through the effect of the ghosts. The change in the coefficient from \( N - 26 \) to \( N - 25 \) is due to a contribution from the conformal mode.

As mentioned above, diffeomorphism invariance determines the coefficient \( b \) uniquely as \[ b = \frac{25 - N}{6} \, . \] (2.15)

Using the relation
\[ -\frac{1}{24\pi} \int d^2 x \sqrt{\bar{g}} (\bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \bar{R} \phi) + \frac{1}{96\pi} \int d^2 x \sqrt{\bar{g}} \bar{R} \frac{1}{\Box} \bar{R} = \frac{1}{96\pi} \int d^2 x \sqrt{g} R \frac{1}{\Box} R \, , \] (2.16)
the effective action can be reexpressed in a manifestly invariant form,
\[ \Gamma = \frac{N - 25}{96\pi} \int d^2 x \sqrt{g} R \frac{1}{\Box} R + I_M(X, g) \, . \] (2.17)
Here, we have used the fact that the matter action is conformally invariant, so that \( I_M(X, \bar{g}) = I_M(X, g) \).
3 4D Quantum Geometry

Recently, we showed that there is a model of diffeomorphism invariant 4DQG [13, 14]. This model has many advantages in physics. In particular, it is renormalizable and asymptotically free. Also, it is capable of solving the cosmological constant problem dynamically without any fine-tuning, [9, 10] and it naturally describes our four dimensional universe at the low-energy region and for large \( N \). However, the unitarity problem remains unsolved. In this paper we do not discuss the unitarity problem, which is expected to be solved dynamically [32, 33, 35, 14].

3.1 Tree action

The tree action of 4DQG [13] is given by a proper combination of the Wess-Zumino action [7, 8] and the invariant action required by the integrability conditions discussed in [14] and also in the following subsection 3.3 as

\[
I = \frac{1}{(4\pi)^2} \int d^4x \sqrt{g} \left\{ \frac{1}{t^2} \bar{F} + a \bar{F} \phi + 2b \phi \bar{\Delta_4} \phi + b \left( \bar{G} - \frac{2}{3} \bar{\nabla} \bar{R} \right) \phi \\
+ \frac{1}{36} (2a + 2b + 3c) \bar{R}^2 + \mathcal{L}_{GF+FP} \right\} + I_{LE}(X, g), \tag{3.1}
\]

where \( \mathcal{L}_{GF+FP} \) contains the gauge-fixing term and the FP ghost Lagrangian defined below. The term \( I_{LE} \) represents lower derivative actions which include actions of conformally invariant matter fields, the Einstein-Hilbert action, and the cosmological constant term. The lower derivative gravitational actions are treated in the perturbation of the massive constants [10, 12, 14].

The invariants \( F \) and \( G \) are defined by

\[
F = R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2, \tag{3.2}
\]

\[
G = R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2. \tag{3.3}
\]

In four dimensions they are the square of the Weyl tensor and the Euler density, respectively. The operator \( \Delta_4 \) is the conformally covariant fourth-

\[\text{2In contrast to 2DQG, in which the classical limit is given by } N \to -\infty, \text{ the limit of positive large } N \text{ gives the correct classical limit in 4DQG [10].} \]
order operator \[7\],
\[
\Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu \nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}(\nabla^\mu R)\nabla_\mu ,
\] (3.4)
which satisfies \(\Delta_4 = e^{-4\phi}\Delta_4\) locally for a scalar.

In the above, we introduce the dimensionless coupling \(t\) only for the traceless mode as \(\bar{g}_{\mu\nu} = (\hat{g}^{th})_{\mu\nu}\) and consider the perturbation of \(t\). The kinetic term of the conformal mode comes from the Wess-Zumino action. Since the invariant \(R^2\) terms in the Wess-Zumino action and the invariant action cancel out in our model, the self-interactions of \(\phi\) appear only in the lower derivative actions in the exponential form, which can be treated exactly, order by order in \(t\) \([14]\).

The gauge-fixing term and the FP ghost action are given by \([33, 34]\):
\[
\mathcal{L}_{GF+FP} = 2iB^\mu N_{\mu\nu}\chi_\nu - \zeta B^\mu N_{\mu\nu}B_\nu - 2i\hat{c}^\mu N_{\mu\nu}\hat{\nabla}^\lambda \delta_B h_\nu^\mu ,
\] (3.5)
where \(\chi_\nu = \hat{\nabla}^\lambda h_\nu^\lambda\), and \(N_{\mu\nu}\) is a symmetric second-order operator. The BRST transformations are given by
\[
\begin{align*}
\delta_B h_\mu^\nu &= i\left\{\hat{\nabla}^\mu c_\nu + \hat{\nabla}_\nu c^\mu - \frac{1}{2}\delta_\nu^\mu \hat{\nabla}_\lambda c^\lambda + tc^\lambda \hat{\nabla}_\lambda h_\mu^\nu \\
&\quad + \frac{t}{2}h_\nu^\lambda (\hat{\nabla}_{\nu} c^\lambda - \hat{\nabla}^\lambda c_\nu) + \frac{t}{2}h_\nu^\lambda (\hat{\nabla}_\nu c_\lambda - \hat{\nabla}^\lambda c_\nu) + \cdots\right\}, \\
\delta_B \phi &= itc^\lambda \partial_\lambda \phi + \frac{i}{4}t\hat{\nabla}^\lambda c^\lambda , \\
\delta_B \bar{c}^\mu &= B^\mu , \quad \delta_B B^\mu = 0 , \\
\delta_B c^\mu &= itc^\lambda \hat{\nabla}_\lambda c^\mu .
\end{align*}
\] (3.6)

The first two of these equations are obtained by replacing \(\xi^\mu/t\) in the equation for general coordinate transformation, \(2.7\), with the contravariant vector ghost field \(ic^\mu\). The kinetic term of the ghost action then becomes \(t\) independent. This BRST transformation is nilpotent. Using this transformation, the gauge-fixing term and the FP ghost action can be written as \(\mathcal{L}_{GF+FP} = 2i\delta_B \{\bar{c}^\mu N_{\mu\nu}(\chi_\nu + \frac{2}{3}\zeta B^\nu)\}\) \([36]\).

The important property of this tree action is that it transforms under the general coordinate transformation \(2.7\) as
\[
\delta I = \frac{1}{(4\pi)^2} \int d^4x \sqrt{\bar{g}} \omega \left\{-a\left(F + \frac{2}{3}\Box \hat{R}\right) - b\bar{G} - c \Box \hat{R}\right\} ,
\] (3.7)
where
\[ \omega = -\frac{1}{4} \hat{\nabla}_\lambda \xi^\lambda . \] (3.8)

In the case of the BRST transformation, \( \xi^\mu \) is replaced by \( itc^\mu \).  

The \( \Box \tilde{R} \) terms in (3.7) depend on the regularization scheme. We use here Duff’s scheme [19] of dimensional regularization characterized by the equations
\[
\delta_\phi \int d^D x \sqrt{g} F = (D - 4) \int d^D x \sqrt{g} \phi \left( F + \frac{2}{3} \Box R \right) , \tag{3.9}
\]
\[
\delta_\phi \int d^D x \sqrt{g} G = (D - 4) \int d^D x \sqrt{g} \phi G . \tag{3.10}
\]
When we define the tree action \( I \), it is taken into account that Duff’s scheme will be used subsequently for computing loop effects of the effective action. As shown below, the scheme-dependent terms cancel out, and we obtain a scheme-independent effective action.

### 3.2 Effective action

As investigated in [14], the regularized effective action of the theory \( I \) has the following form:
\[
\Gamma = I(X, \phi, \bar{g}) + V_{NS}(\phi, \bar{g}) + W_F(\bar{g}, \mu) + W_G(\bar{g}) + W_{\Box R}(\bar{g}) . \tag{3.11}
\]
Here, the first term on the r.h.s. is the tree action. \( V_{NS}, W_F, W_G \) and \( W_{\Box R} \) come from loop diagrams. The former represents corrections to the Wess-Zumino action, and the latter three represent corrections to the traceless mode \( h \).

Let us first consider corrections to the traceless mode. Here, \( W_F \) is the part that is associated with the conformally invariant counterterm of \( \bar{F} \); it

---

3Even in 2DQG, although we can set \( \delta_B I = 0 \) if we use the flat background-metric and integrate out the \( B_{\mu\nu} \) field, the nilpotency of the BRST charge at the quantum level, after all, requires condition (2.15). Thus, the BRST invariance in even-dimensional QG is realized dynamically.

4Some errors in the form of the effective action in section 3.3 of ref. [14] are corrected in this section.
can be determined by computing two-point diagrams of the traceless mode. In Duff’s scheme, it has the following scale-dependent form:

\[ W_F(\bar{g}, \mu) = f (4\pi)^2 \int \frac{d^4x}{\sqrt{\bar{g}}} \left\{ -\frac{1}{4} \bar{C}_{\mu\nu\lambda\sigma} \log \left( \frac{\Delta_4^C}{\mu^4} \right) \bar{G}^{\mu\nu\lambda\sigma} - \frac{1}{18} \bar{R}^2 \right\} . \quad (3.12) \]

Here, the appearance of the \( \bar{R}^2 \) term is due to our use of Duff’s scheme. \( C \) is the Weyl tensor, and \( \Delta_4^C = \Box^2 + \cdots \) is an appropriate conformally covariant operator for the Weyl tensor. Although the explicit form of \( \Delta_4^C \) is unknown, it is known that there is a function \( W_F \) that satisfies the equation [18, 23, 25]

\[ \delta W_F(\bar{g}, \mu) = \delta_\omega W_F(\bar{g}, \mu) = \frac{f}{(4\pi)^2} \int \frac{d^4x}{\sqrt{\bar{g}}} \omega \left( \bar{F} + \frac{2}{3} \Box \bar{R} \right) , \quad (3.13) \]

where \( \delta_\omega \bar{g}_{\mu\nu} = 2\omega \bar{g}_{\mu\nu} \), with (3.8). Thus, \( W_F \) produces the type-B anomaly in the classification of [23].

The term \( W_G \) in (3.11) is the part that is associated with the conformally invariant counterterm of \( \bar{G} \). It is called the non-local Riegert action, which produces the type-A anomaly, or the Euler density in the classification of [23], and it has the form

\[ W_G(\bar{g}) = e \frac{1}{(4\pi)^2} \int \frac{d^4x}{\sqrt{\bar{g}}} \left\{ \frac{1}{8} \bar{G} \frac{1}{\Delta_4^G} \bar{G} - \frac{1}{18} \bar{R}^2 \right\} , \quad (3.14) \]

where

\[ \bar{G} = G - \frac{2}{3} \Box \bar{R} . \quad (3.15) \]

As stated above, \( W_G \) produces the type-A anomaly as

\[ \delta W_G(\bar{g}) = \delta_\omega W_G(\bar{g}) = e \frac{1}{(4\pi)^2} \int \frac{d^4x}{\sqrt{\bar{g}}} \omega \bar{G} . \quad (3.16) \]

The \( \bar{R}^2 \) term is needed to realize equation (3.16), which guarantees that \( W_G \) does not have any contribution to two-point diagrams of the traceless mode \( h \) in the flat background. This is consistent with the direct loop calculations of two-point diagrams of \( h \). Hence, \( W_G \) is related to \( h^3 \) vertex corrections in the flat background.

The coefficients \( f \) and \( e \) are scheme independent. They can be expanded by the renormalized coupling \( t_r \) as

\[ f = f_0 + f_1 t_r^2 + \cdots , \quad e = e_0 + e_1 t_r^2 + \cdots . \quad (3.17) \]
Here, $f_0$ and $e_0$ have already been computed using one-loop diagrams as

\[ f_0 = -\frac{N}{120} - \frac{199}{30} + \frac{1}{15} , \quad e_0 = \frac{N}{360} + \frac{87}{20} - \frac{7}{90} , \]

where the first term in each coefficient comes from $N$ conformal scalar fields \[19\]. The second and the last terms come from the traceless mode \[33\] and the conformal mode \[11\], respectively. The coefficients $f_1$ and $e_1$ are given by functions of $a$ and $b$, to which not only two-loop diagrams, but also one-loop (but order $t_r^2$) diagrams contribute \[14\].

The beta function for the coupling $t_r$ is given by $\beta = f t_r^3$. Since $f_0$ is negative, 4DQG is asymptotically free. Here, note that, although background-metric independence implies an invariance under any conformal change of the background metric, the usual $\beta$ function does not need to vanish. This is due to the fact that there exists a conformal anomaly, or the Wess-Zumino action.

The last term in (3.11) is a scheme-dependent part, defined by

\[ W_{\Box R}(\bar{g}) = -\frac{u}{12(4\pi)^2} \int d^4x \sqrt{\bar{g}} R^2 . \]

It is unknown whether this term is really necessary or not. In any case, the coefficient $u$ is at most order $t^2$, so that $u = u_1 t_r^2 + \cdots$.

As computed in \[14\], the correction $V_{NS}$ is scale-independent, and it merely changes coefficients $a$ and $b$ in the tree action into $\tilde{a} = a(1 + v_a)$ and $\tilde{b} = b(1 + v_b)$, where $v_a$ and $v_b$ are order $t_r^2$ at the one-loop level. The implications of this fact are discussed in the following subsection.

Now, the conditions for diffeomorphism invariance are given by the following equations \[14\]:

\[ \tilde{a} = f , \quad \tilde{b} = e , \quad c = u . \]

Since $f_1$ and $e_1$ are functions of $a$ and $b$, while $f_0$ and $e_0$ are constants independent of $a$ and $b$, we can solve these equations perturbatively, order by order in $t_r$. Note that the one-loop coefficients of $v_a$ and $v_b$ are related to the order $t_r^2$ coefficients, $f_1$ and $e_1$, of $W_F$ and $W_G$. This is reasonable, because the Wess-Zumino action originally comes from the measure, and thus is essentially a quantum effect. Thus, one-loop contributions given by quantizing the Wess-Zumino action are related to two-loop contributions.
Substituting the solutions of (3.20) into the regularized effective action, the $\bar{R}^2$ terms cancel out, and we obtain the scheme-independent and manifestly invariant effective action,

$$\Gamma = \frac{1}{(4\pi)^2} \int d^4x \sqrt{\bar{g}} \left\{ -\frac{f}{4} C_{\mu\nu\lambda\sigma} \log \left( \frac{\Delta_4}{\mu^4} \right) C^{\mu\nu\lambda\sigma} + \frac{e}{8} G \frac{1}{\Delta_4} G \right\} + I_{LE}(X, g) .$$

(3.21)

Here, the Weyl action $F$ is absorbed into the scale, $\mu$.

### 3.3 Two-loop integrability

Here, we summarize the conditions of diffeomorphism invariance discussed in ref. [14].

The condition that a theory can be made diffeomorphism invariant is that in the effective action, there is no action which produces a term that does not appear in the variation of the tree action $\delta I$, (3.7). Namely, diffeomorphism invariance implies that the action

$$W_{R^2}(\bar{g}, \mu) = \frac{r}{(4\pi)^2} \int d^4x \sqrt{\bar{g}} \bar{R} \log \left( \frac{\bar{\Delta}_4}{\mu^4} \right) \bar{R}$$

(3.22)

is not allowed, because this action produces $\bar{R}^2$ under a general coordinate transformation. Further, a scale-dependent action including the conformal mode $\phi$, for example

$$V_S(\phi, \bar{g}, \mu) = \frac{s}{(4\pi)^2} \int d^4x \sqrt{\bar{g}} \phi \bar{\Delta}_4 \log \left( \frac{\bar{\Delta}_4}{\mu^4} \right) \phi ,$$

(3.23)

is not allowed, because this action cannot be absorbed into the Wess-Zumino action by changing coefficients $a$ and $b$, and it produces a term that is not in $\delta I$ under a general coordinate transformation.

In general, parts of the effective actions, other than that which produces the type-B anomaly, must be independent of the scale $\mu$, as $V_{NS}$, $W_G$ and $W_{\Box R}$. The vanishing of $r$ and $s$, at least to order $t^2$, is demonstrated in a previous paper [14]. We give this demonstration in the following.

5The possibility that an invariant $R^2$ term appears in the effective action is not excluded. There is a possibility that such a term appears in $V_{NS}$ at order $t^4$. 

First, we expand $r$ and $s$ as $r = r_0 + r_1 t_2 + \cdots$ and $s = s_0 + s_1 t_2 + \cdots$. The vanishing of $r_0$ is guaranteed in our model because at this order, only conformally invariant vertices contribute to the one-loop diagrams. This is a consequence of the fact that the invariant $R^2$ terms with the coefficients $a$, $b$ and $c$ cancel out, so that self-interactions of the conformal mode $\phi$ do not appear in the tree action $\mathcal{I}$, except in the lower derivative terms, such as the cosmological constant in the exponential form. This fact also implies $s_0 = 0$, because there are no diagrams that contribute to $s_0$.

The vanishing of $s_1$ is proved directly by showing the finiteness of the self-energy diagram of $\phi$ [14]. Here, the fact that there are no interactions of $R^2$ is also essential. Note that we cannot explain this result by using conformal invariance, because conformal invariance does not forbid that there exists the counterterm of $\phi \Delta \Delta \phi$. It can be explained only by diffeomorphism invariance/background-metric independence.

The background-metric independence for the conformal mode implies that $W_{R^2}$ and $V_S$ are related in such a manner that $s = 0$ implies $r = 0$. Thus, $r_1 = 0$ is shown indirectly.

A more direct demonstration of $r_1 = 0$ is as follows. Since there are no self-interactions of $\phi$, two-loop diagrams that contribute to $f_1$, $g_1$ and $r_1$ can be derived from the conformally invariant vertices of $2b\phi \Delta \Delta \phi$ and $\frac{1}{t^2} F$, so that the contributions of two-loop diagrams to $r_1$ vanish. However, there are contributions from one loop (but order $t^2$) diagrams, which include the vertices of $a \tilde{F} \phi$, $b(\tilde{G} - \frac{2}{3} \Box \tilde{R}) \phi$ and $\frac{1}{t^2}(2a + 2b + 3c) \tilde{R}^2$. Here, because these vertices, with the exception of the first one, are non-conformally invariant, we must pay attention to such one-loop contributions.

As shown in [13, 14], the variation in the one-loop contributions to the effective action of our model is given by

$$\delta_{\omega} W^{(1)}(\hat{g}) = -2\text{Tr}(\omega e^{-\epsilon K}) ,$$

where $\epsilon$ is a cutoff. The matrix operator $K$ is defined by the kinetic term $\frac{1}{2} \Phi^t K \Phi$ on an arbitrary background-metric $\hat{g}$, where $\Phi = (\phi, h^{\mu}_{\nu}, X)$. The $t$-independent diagonal parts give the coefficients $f_0$ and $e_0$. The off-diagonal parts, as well as the $t$-dependent diagonal parts, give contributions of order $t^2$. Note that, unlike in the case of matter fields, we do not use the condition of conformal invariance for gravitational fields to derive this expression. We merely use the facts that $K$ is a fourth-order operator and there are no self-interactions of the conformal mode. If there are the invariant $R^2$ term with
the coefficients $a$, $b$ and $c$, we cannot describe $\delta_\omega W^{(1)}$ in such a simple form, because we do not introduce the coupling for the conformal mode $\phi$. That $\delta_\omega W^{(1)}$ is expressed in the simple form such as the r.h.s. of (3.24) is a general property of $2n$-th order operators in $2n$ dimensions, and such a quantity has been shown to be integrable \cite{13,14}. Thus, our model satisfies $r_1 = 0$.

In four dimensions, integrability places strong constraints on QG. It seems that there is no 4DQG other than ours that overcomes the integrability conditions. Thus, 4DQG may be fixed uniquely according to the conformal matter content.

4 6D Quantum Geometry

In this section we show that the arguments concerning integrability in 4DQG can be generalized to the six dimensional case. Since there are many curvature invariants in six dimensions, many indefinite coefficients appear in the definition of 6D action. However, we show below that many of them are fixed by the integrability.

4.1 Duff’s scheme in six dimensions

Recently, six-dimensional conformal anomalies have been studied in detail \cite{22}–\cite{29}. In this subsection we summarize the results of these studies and then show that we can also apply Duff’s scheme to the six-dimensional case.

In six dimensions there are 17 independent curvature invariants. We here use the following bases \cite{22,27}:

\begin{align}
K_1 &= R^3, & K_2 &= RR_{\mu\nu}R^{\mu\nu}, & K_3 &= RR_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}, \\
K_4 &= R_\mu^\nu R_\lambda^\lambda R_\sigma^\mu, & K_5 &= R_{\mu\nu}R_{\lambda\sigma}R^{\mu\lambda\nu\sigma}, & K_6 &= R_{\mu\nu}R_{\alpha\beta}R^{\mu\alpha\beta\gamma}, \\
K_7 &= R_{\mu\nu}^{\alpha\beta} R_{\lambda\sigma}^{\alpha\beta} R_{\lambda\sigma}^{\mu\nu}, & K_8 &= R_{\mu\nu\lambda\sigma}^{\alpha\beta} R_{\lambda\sigma}^{\mu\lambda\nu\sigma}, & K_9 &= R \Box R, \\
K_{10} &= R_{\mu\nu} \Box R^{\mu\nu}, & K_{11} &= R_{\mu\nu\lambda\sigma} \Box R^{\mu\nu\lambda\sigma}, & K_{12} &= R^{\mu\nu} \nabla_\mu \nabla_\nu R, \\
K_{13} &= (\nabla_\lambda R_{\mu\nu}) \nabla^\lambda R^{\mu\nu}, & K_{14} &= (\nabla_\lambda R_{\mu\nu}) \nabla^{\mu\lambda}, & K_{15} &= (\nabla_\lambda R_{\alpha\beta\gamma\delta}) \nabla^\lambda R^{\alpha\beta\gamma\delta}, & K_{16} &= \Box R^2, & K_{17} &= \Box^2 R. 
\end{align}

(4.1)

The results for conformal anomalies are summarized as follows. There are ten independent integrable curvature invariants \cite{27}. They provide a basis for
the conformal anomalies in six dimensions. In the classification of ref. [23],
the type-A anomaly is unique and given by the Euler density,

\[ G_6 = -K_1 + 12K_2 - 3K_3 - 16K_4 + 24K_5 + 24K_6 - 4K_7 - 8K_8 \, . \]  

(4.2)

Here, we normalize it as

\[ G_6 = -\frac{1}{8} \epsilon_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} \epsilon^{\lambda_1\sigma_1\lambda_2\sigma_2\lambda_3\sigma_3} R^{\mu_1\nu_1}_{\lambda_1\sigma_1} R^{\mu_2\nu_2}_{\lambda_2\sigma_2} R^{\mu_3\nu_3}_{\lambda_3\sigma_3} \, . \]  

(4.3)

There are three type-B anomalies. They are locally conformally invariant in six dimensions:

\[ F_1 = \frac{19}{800} K_1 - \frac{57}{160} K_2 + \frac{3}{40} K_3 + \frac{7}{16} K_4 - \frac{9}{8} K_5 - \frac{3}{4} K_6 + K_8 \, . \]  

(4.4)

\[ F_2 = \frac{9}{200} K_1 - \frac{27}{40} K_2 + \frac{3}{10} K_3 + \frac{5}{4} K_4 - \frac{3}{2} K_5 - 3K_6 + K_7 \, . \]  

(4.5)

\[ F_3 = -\frac{11}{50} K_1 + \frac{27}{10} K_2 - \frac{6}{5} K_3 - K_4 + 6K_5 + 2K_7 - 8K_8 \]
\[ + \frac{3}{5} K_9 - 6K_{10} + 6K_{11} + 3K_{13} - 6K_{14} + 3K_{15} \, . \]  

(4.6)

Here, \( F_1 \) and \( F_2 \) correspond to two independent combinations of the Weyl tensors, \( C_{\alpha\beta\gamma} C^{\mu\lambda\nu} C_{\lambda\sigma\gamma} \) and \( C_{\alpha\beta\gamma} C^{\lambda\sigma\nu} C_{\lambda\sigma\gamma} \), respectively. \( F_3 \) gives the kinetic term of the traceless mode, which is expressed, up to a total derivative term, as \( C_{\mu\alpha\beta\gamma} (\Box \delta^\mu + 4R^\mu - \frac{6}{5} R\delta^\mu) C^{\nu\alpha\beta\gamma} \).

The other six combinations are given by

\[ M_5 = 6K_6 - 3K_7 + 12K_8 + K_{10} - 7K_{11} - 11K_{13} + 12K_{14} - 4K_{15} \, . \]  

(4.7)

\[ M_6 = -\frac{1}{5} K_9 + K_{10} + \frac{2}{5} K_{12} + K_{13} \, . \]  

(4.8)

\[ M_7 = K_4 + K_5 - \frac{3}{20} K_9 + \frac{4}{5} K_{12} + K_{14} \, . \]  

(4.9)

\[ M_8 = -\frac{1}{5} K_9 + K_{11} + \frac{2}{5} K_{12} + K_{15} \, . \]  

(4.10)

\[ M_9 = K_{16} \, . \]  

(4.11)

\[ M_{10} = K_{17} \, . \]  

(4.12)

These are classified as trivial conformal anomalies.
In order to treat the trivial anomalies $M_5 \cdots M_{10}$, unambiguously, we use dimensional regularization. Consider the conformal variations of the functions $G_6$, $F_1$, $F_2$ and $F_3$ defined by the combinations listed above. In $D$ dimensions we obtain the equations

$$\delta \phi \int d^D x \sqrt{\bar{g}} G_6 = (D - 6) \int d^D x \sqrt{\bar{g}} \phi G_6$$

and

$$\delta \phi \int d^D x \sqrt{\bar{g}} F_i = (D - 6) \int d^D x \sqrt{\bar{g}} \phi \left( F_i + \sum_{n=5}^{10} z_{i,n} M_n \right) \quad (i = 1, 2, 3),$$

where

$$[z_{1,5}, z_{1,6}, z_{1,7}, z_{1,8}, z_{1,9}, z_{1,10}] = [\frac{1}{16}, \frac{71}{80}, \frac{15}{16}, \frac{13}{40}, \frac{159}{3200}, 0] \quad (4.15)$$

$$[z_{2,5}, z_{2,6}, z_{2,7}, z_{2,8}, z_{2,9}, z_{2,10}] = [-\frac{1}{4}, -\frac{1}{20}, -\frac{3}{4}, -\frac{7}{10}, -\frac{51}{800}, 0] \quad (4.16)$$

$$[z_{3,5}, z_{3,6}, z_{3,7}, z_{3,8}, z_{3,9}, z_{3,10}] = [1, \frac{1}{5}, 3, \frac{14}{5}, \frac{39}{200}, \frac{3}{5}] \quad (4.17)$$

Here, note that the r.h.s. of equation (4.14) is expanded in terms of $F_i$ itself and the trivial conformal anomalies. This equation suggests that Duff’s scheme also works well in six dimensions.

### 4.2 Tree action

Let us first look for a conformally covariant sixth-order operator in six dimensions [13]. It can be expanded in terms of the 21 independent operators, apart from the $\Box^3$ term, as

$$\Delta_6 = \Box^3 + v_1 R^{\mu \nu} \nabla_\mu \nabla_\nu \Box + v_2 R \Box^2 + v_3 (\nabla^\lambda R^{\mu \nu}) \nabla_\lambda \nabla_\mu \nabla_\nu$$

$$+ v_4 (\nabla^\lambda R) \nabla_\lambda \Box + v_5 (\nabla^\mu \nabla^\nu R) \nabla_\mu \nabla_\nu + v_6 (\Box R^{\mu \nu}) \nabla_\mu \nabla_\nu$$

$$+ v_7 (\Box R) \Box + v_8 R^{\alpha \beta} \nabla^\mu \nabla_\mu \nabla_\nu + v_9 R_{\mu \nu \lambda \sigma} R^{\mu \nu \lambda \sigma} \Box$$

$$+ v_{10} R^{\alpha \beta} R^{\mu \nu}_{\alpha \beta} \nabla_\mu \nabla_\nu + v_{11} R^{\mu \lambda} R^{\nu \lambda} \nabla_\mu \nabla_\nu + v_{12} R^{\mu \nu} R_{\mu \nu} \Box$$

$$+ v_{13} R R^{\mu \nu} \nabla_\mu \nabla_\nu + v_{14} R^2 \Box + v_{15} (\nabla^\lambda \Box R) \nabla_\lambda + v_{16} R_{\alpha \beta \gamma \mu} (\nabla^\mu R^{\alpha \beta \gamma \nu}) \nabla_\nu$$

$$+ v_{17} R_{\mu \nu \lambda \sigma} (\nabla_\mu R_{\nu \lambda \sigma}) \nabla_\nu + v_{18} R_{\mu \nu} (\nabla^\mu R^{\nu \lambda}) \nabla_\lambda + v_{19} R_{\mu \nu} (\nabla^\lambda R^{\mu \nu}) \nabla_\lambda$$

$$+ v_{20} R^{\mu \nu} (\nabla_\mu R) \nabla_\nu + v_{21} R (\nabla^\lambda R) \nabla_\lambda ,$$

(4.18)
From the requirement that \( \delta_\phi (\sqrt{g} \Delta_6 Y) = 0 \) is satisfied locally for a scalar \( Y \), the coefficients are determined as follows:

\[
\begin{align*}
v_1 &= 4, & v_2 &= -1, & v_3 &= 4, & v_4 &= 0, & v_5 &= 0, & v_6 &= 4, \\
v_7 &= -\frac{3}{5}, & v_8 &= \zeta_1, & v_9 &= \zeta_2, & v_{10} &= \zeta_1, & v_{11} &= 6 - \frac{3}{4} \zeta_1, \\
v_{12} &= -1 + \frac{1}{8} \zeta_1 - \zeta_2, & v_{13} &= -2 + \frac{1}{4} \zeta_1, & v_{14} &= \frac{9}{25} - \frac{1}{40} \zeta_1 + \frac{1}{10} \zeta_2, \\
v_{15} &= \frac{2}{5}, & v_{16} &= \zeta_1 + 4 \zeta_2, & v_{17} &= -\zeta_1, & v_{18} &= 6 + \frac{1}{4} \zeta_1, \\
v_{19} &= -2 - \frac{3}{4} \zeta_1 - 2 \zeta_2, & v_{20} &= 1 - \frac{1}{8} \zeta_1, & v_{21} &= -\frac{7}{25} + \frac{3}{40} \zeta_1 + \frac{1}{5} \zeta_2.
\end{align*}
\]

In six dimensions, \( \Delta_6 \) is not unique, as the two constants \( \zeta_1 \) and \( \zeta_2 \) are not determined by the conformal property alone. The terms with these arbitrary constants are collected, using the Weyl tensor, in the forms \( \zeta_1 \nabla^\mu (C^{\mu \alpha \beta \gamma} C^{\nu \alpha \beta \gamma} \nabla^\nu) \) and \( \zeta_2 \nabla^\lambda (C_{\alpha \beta \gamma \delta} C^{\alpha \beta \gamma \delta} \nabla^\lambda) \), respectively \([15]\).

Next, we look for a combination of \( G_6 \) and \( M_n \) that satisfies the following conformal property locally:

\[
\delta_\phi \left\{ \sqrt{g} \left( G_6 - \sum_{n=5}^{10} w_n M_n \right) \right\} = 6 \sqrt{g} \Delta_6 \phi.
\]

This equation determines the coefficients \( w_n \) uniquely for each \( \Delta_6 \) with \( \zeta_1 \) and \( \zeta_2 \) as

\[
\begin{align*}
w_5 &= 1 + \frac{1}{4} \zeta_1, & w_6 &= 11 + \frac{1}{2} \zeta_1 - 3 \zeta_2, & w_7 &= -6 - \frac{3}{4} \zeta_1, \\
w_8 &= 1 + \zeta_1 + 3 \zeta_2, & w_9 &= -\frac{21}{100} + \frac{9}{160} \zeta_1 + \frac{3}{20} \zeta_2, & w_{10} &= \frac{3}{5}.
\end{align*}
\]

Using equation (4.20), the Wess-Zumino action defined by integrating the conformal anomalies within the interval \([0, \phi]\) are expressed in the form

\[
S(\phi, \bar{g}) = \frac{1}{(4\pi)^3} \int d^6x \int_0^\phi d\phi \sqrt{g} \left\{ \sum_{i=1}^{3} (F_i + \sum_{n=5}^{10} z_{i,n} M_n) + bG_6 + \sum_{n=10}^{5} c_n M_n \right\}
\]

\[
= \frac{1}{(4\pi)^3} \int d^6x \sqrt{g} \left\{ \sum_{i=1}^{3} a_i F_i \phi + 3b \phi \Delta_6 \phi + b \left( G_6 - \sum_{n=5}^{10} w_n M_n \right) \phi \right\}
\]

\[
+ \sum_{n=5}^{10} \frac{\sum_{i=1}^{3} a_i z_{i,n} + bw_n + c_n}{(4\pi)^3} \int d^6x \left( \sqrt{g} L_n - \sqrt{\bar{g}} L_n \right).
\]

16
Here, the $L_n$ are local functions given by integrating the $M_n$ as

$$\delta \phi \int d^6x \sqrt{g} L_n = \int d^6x \sqrt{g} \delta \phi M_n$$

(4.22)

such that

$$L_5 = \frac{1}{30} K_1 - \frac{1}{4} K_2 + K_6, \quad L_6 = \frac{1}{100} K_1 - \frac{1}{20} K_2,$$

$$L_7 = \frac{37}{6000} K_1 - \frac{7}{150} K_2 + \frac{1}{75} K_3 - \frac{1}{10} K_5 - \frac{1}{15} K_6,$$

$$L_8 = \frac{1}{150} K_1 - \frac{1}{20} K_3, \quad L_9 = -\frac{1}{30} K_1, \quad L_{10} = \frac{1}{300} K_1 - \frac{1}{20} K_9.$$  

(4.23)

As discussed in the case of 4DQG, the integrability condition suggests that the sixth-order parts of the invariant action $I$ should be chosen such that the invariant $L_n$ terms cancel out in the sum $\mathcal{I} = S + I$. Hence, we obtain a 6DQG tree action analogous to that in 4DQG as

$$\mathcal{I} = \frac{1}{(4\pi)^3} \int d^6x \sqrt{g} \left\{ -\frac{3}{t^2} \left( F_1 + \alpha_1 \dot{F}_1 + \alpha_2 \dot{F}_2 \right) + \sum_{i=1}^{3} a_i \tilde{F}_i \phi + 3 \phi \tilde{\Delta}_6 \phi + b \left( \tilde{G}_6 - \sum_{n=5}^{10} w_n \tilde{M}_n \right) \phi ight\} + I_{LE}(X, g)$$

(4.24)

$$- \sum_{n=5}^{10} \left( \sum_{i=1}^{3} a_i z_{i,n} + bw_n + c_n \right) \tilde{L}_n \right\}.$$  

(4.25)

Here, we introduce the dimensionless coupling $t$, as in 4DQG. In six dimensions, two extra dimensionless constants $\alpha_1$ and $\alpha_2$, in addition to $\zeta_1$ and $\zeta_2$ in $\Delta_6$ and $w_n$, appear. These constants are not fixed by the arguments of the integrability. The constants $t$, $\alpha_1$ and $\alpha_2$ are renormalized, but $\zeta_1$ and $\zeta_2$ may not be.

Under a general coordinate transformation, this action changes according to

$$\delta \mathcal{I} = \frac{1}{(4\pi)^3} \int d^6x \sqrt{g} \omega \left\{ -\sum_{i=1}^{3} a_i \left( \dot{F}_i + \sum_{n=5}^{10} z_{i,n} \dot{M}_n \right) - \dot{b} \tilde{G}_6 - \sum_{n=5}^{10} c_n \dot{\tilde{M}}_n \right\},$$

(4.25)

where

$$\omega = -\frac{1}{6} \tilde{\nabla}_\lambda \xi^\lambda.$$  

(4.26)
4.3 Effective action

It is expected that the effective action of this model has the form

\[ \Gamma = \tilde{I}(X, \phi, \bar{g}) + W_{G_6}(\bar{g}) + \sum_{i=1}^{3} W_{F_i}(\bar{g}, \mu) + \sum_{n=5}^{10} W_{M_n}(\bar{g}), \quad (4.27) \]

where the tilde on \( \mathcal{I} \) denotes the inclusion of finite corrections to the Wess-Zumino action described by \( V_{NS} \) in the four-dimensional model. Here, \( W_{G_6} \) is the generalization of the non-local and scale-independent Polyakov-Riegert action \([24, 27]\). We find its complete form as

\[ W_{G_6}(\bar{g}) = \frac{e}{(4\pi)^3} \int d^6x \sqrt{\bar{g}} \left\{ \frac{1}{12} \bar{G}_6 \frac{1}{\Delta_6} \bar{G}_6 + \sum_{n=5}^{10} w_n \bar{L}_n \right\}, \quad (4.28) \]

where

\[ \bar{G}_6 = G_6 - \sum_{n=5}^{10} w_n M_n. \quad (4.29) \]

This produces the type-A anomaly,

\[ \delta W_{G_6}(\bar{g}) = \delta \omega W_{G_6}(\bar{g}) = \frac{e}{(4\pi)^3} \int d^6x \sqrt{\bar{g}} \omega \bar{G}_6, \quad (4.30) \]

where \( \delta \omega \bar{g}_{\mu\nu} = 2\omega \bar{g}_{\mu\nu} \), with \((4.26)\). This equation is realized for arbitrary values of \( \zeta_1 \) and \( \zeta_2 \). These constants, as well as \( e \), are determined according to matter content.

The action \( W_{F_i} \), which produces the type-B anomaly in Duff’s scheme, is defined by

\[ W_{F_i}(\bar{g}, \mu) = f_i \left( W'_{F_i}(\bar{g}, \mu) + \sum_{n=5}^{10} \frac{z_{i,n}}{(4\pi)^3} \int d^6x \sqrt{\bar{g}} \bar{L}_n \right). \quad (4.31) \]

The \( \bar{L}_n \) terms appear in Duff’s scheme. \( W'_{F_i} \) is a scale-dependent part defined through the equation

\[ \delta \omega W'_{F_i}(\bar{g}, \mu) = \frac{1}{(4\pi)^3} \int d^6x \sqrt{\bar{g}} \omega \bar{F}_i. \quad (4.32) \]

It is known that the coefficients \( e \) and \( f_i \) are independent of the regularization scheme.
The remaining action, $W_{M_n}$, is a scheme-dependent part defined by

$$W_{M_n}(\bar{g}) = \frac{u_n}{(4\pi)^3} \int d^6 x \sqrt{\bar{g}} \bar{L}_n .$$  \hspace{1cm} (4.33)

This action produces a trivial anomaly, $\bar{M}_n$. As in 4DQG, it is unknown whether this action is really necessary or not. Since the vertices of the tree action at zeroth order in $t$ is conformally invariant, the coefficients $u_n$ will be at most order $t^2$.

The conditions for diffeomorphism invariance are now given by

$$\tilde{a}_i = f_i , \quad \tilde{b} = e , \quad c_n = u_n ,$$  \hspace{1cm} (4.34)

where the tildes on $a_i$ and $b$ indicate the inclusions of corrections to the Wess-Zumino action. As in 4DQG, the scheme-dependent terms, $\bar{L}_n$, cancel out, and the final expression takes the invariant and scheme-independent form

$$\Gamma = \frac{e}{(4\pi)^3} \int d^6 x \sqrt{g} \frac{1}{12} \partial_6 \Gamma_6 \frac{1}{\Delta_6} \Gamma_6 + \sum_{i=1}^{3} f_i W_{F_i}(g, \mu) + I_{LE}(X, g) .$$  \hspace{1cm} (4.35)

The matter contributions to the coefficients $e$ and $f_i$ are computed in refs. [26, 28, 29].

5 Conclusions and Discussion

In this paper we have discussed how the integrability conditions for conformal anomalies constrain the form of the effective action of even-dimensional QG. We showed that the effective action of 4DQG satisfying such integrability conditions has a manifestly diffeomorphism invariant and scheme-independent form. We then generalized the arguments to six dimensions and proposed a model for 6DQG. The expected scheme-independent form of the effective action was presented.

Now, the role of conformal anomalies in even-dimensional QG is naturally understood in terms of background-metric independence/diffeomorphism invariance. In $D = 2n (\geq 4)$ dimensions, unlike the case for 2DQG, there is no critical matter content where the Wess-Zumino action vanishes. Thus, $2n$-dimensional QG is to be necessarily $2n$-th order, because of diffeomorphism invariance.
Background-metric independence does not require the vanishing of the usual beta functions in $D \geq 4$ dimensions, though it implies invariance under any conformal change of the background metric. This is due to the fact that there exist conformal anomalies, or the Wess-Zumino action, in even dimensions. We believe that conformal invariance in physics should be reinterpreted in terms of diffeomorphism invariance. In this case, the problem of dependence on the regularization scheme would disappear.

In odd dimensions, because there is no conformal anomaly, background-metric independence seems to require the theory to be finite. In three dimensions the Einstein-Hilbert+cosmological constant action is written in the Chern-Simons action, and its quantum theory is expected to be topological\cite{37}. However, for $D \geq 5$, it is unknown whether odd-dimensional QG exists or not. Since in odd dimensions, we cannot introduce a dimensionless coupling constant, it seems necessary to make the theory super-renormalizable.

There is another approach to QG based on dynamical triangulation in two\cite{38, 39, 40} and four dimensions\cite{11, 12, 13, 14}. It is expected that our model is obtained in the continuum limit of such a simplicial QG. In this paper we do not discuss quantum corrections of the lower-derivative gravitational actions. The anomalous dimensions of the gravitational constant and the cosmological constant are needed to compare the two methods\cite{10, 14}. A project involving detailed comparison in 4DQG between them has started\cite{43}.

Finally, we comment on dimensional regularization. Because dimensional regularization violates conformal invariance in general, it is not a suitable regularization for a theory in which conformal invariance plays an important role. Nevertheless, dimensional regularization is still useful, because this violation is quite small and it is expected to give correct results for sufficiently higher order loops\cite{44}.

There is an assertion that, when using dimensional regularization, we can regularize QG defined by (2.1) in a manifestly diffeomorphism invariant way if we take great care concerning the conformal mode dependence\cite{9}. At present, the relation between this approach and ours is unknown. Detailed analyses of this relation are important to prove renormalizability to all orders.

The beautiful relations obtained among integrable curvature invariants in $D$ dimensions seem to suggest the validity of dimensional regularization. Our model, at least up to order $t^2$, gives correct results because of the finiteness
of the self-energy diagrams of $\phi$, which implies that our model is rather insensitive to the conformal mode dependence. Whether or not the derived effective action at higher order is acceptable will be decided by the condition that it possesses a scheme-independent form and does not contain terms that violate diffeomorphism invariance, such as (3.22) and (3.23).

Acknowledgements

This work was supported in part by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture of Japan.

References

[1] A. Polyakov, Phys. Lett. **103B** (1981) 207; Mod. Phys. Lett. **A2** (1987) 893.

[2] V. Knizhnik, A. Polyakov and A. Zamolodchikov, Mod. Phys. Lett. **A3** (1988) 819.

[3] J. Distler and H. Kawai, Nucl. Phys. **B321** (1989) 509; F. David, Mod. Phys. Lett. **A3** (1988) 1651.

[4] N. Seiberg, Prog. Theor. Phys. Suppl. **102** (1990) 319; J. Polchinski, Proc. of String 1990 (Texas A&M, march 1990); Nucl. Phys. **B346** (1990) 253; K. Hamada, Phys. Lett. **B324** (1994) 141; Nucl. Phys. **B413** (1994) 278.

[5] K. Hamada, Phys. Lett. **B300** (1993) 322; K. Hamada and A. Tsuchiya, Int. J. Mod. Phys. **A8** (1993) 4897.

[6] H. Kawai and M. Ninomiya, Nucl. Phys. **B336** (1990) 115; H. Kawai, Y. Kitazawa and M. Ninomiya, Nucl. Phys. **B393** (1993) 280; Nucl. Phys. **B404** (1993) 684; T. Aida, Y. Kitazawa, H. Kawai and M. Ninomiya, Nucl. Phys. **B427** (1994) 158; T. Aida, Y. Kitazawa, J. Nishimura and A. Tsuchiya, Nucl. Phys. **B444**
(1995) 353;
H. Kawai, Y. Kitazawa and M. Ninomiya, Nucl. Phys. B467 (1996) 313;
T. Aida and Y. Kitazawa, Nucl. Phys. B491 (1997) 427.

[7] R. Riegert, Phys. Lett. 134B (1984) 56.

[8] E. Fradkin and A. Tseytlin, Phys. Lett. 134B (1984) 187.

[9] E. Tomboulis, Nucl. Phys. 329 (1990) 410.

[10] I. Antoniadis and E. Mottola, Phys. Rev. D45 (1992) 2013;
I. Antoniadis, P. Mazur and E. Mottola, Phys. Lett. B323 (1994) 284,
B394 (1997) 49.

[11] I. Antoniadis, P. Mazur and E. Mottola, Nucl. Phys. B388 (1992) 627.

[12] S. Odintsov, Z. Phys. C54 (1992) 531;
I. Antoniadis and S. Odintsov, Phys. Lett. B343 (1995) 76.

[13] K. Hamada and F. Sugino, Nucl. Phys. B553 (1999) 283.

[14] K. Hamada, Prog. Theor. Phys. 103 (2000) 1237 hep-th/9912098 hep-
th/0005063.

[15] D. Karakhanian, R. Manvelian and R. Mkrtchian, Mod. Phys. Lett. A11
(1996) 409 hep-th/9411068.
T. Arakelian, D. Karakhanian, R. Manvelian and R. Mkrtchian, Phys.
Lett. B353 (1995) 52.

[16] S. Odintsov and A. Romeo, Mod. Phys. Lett. A9 (1994) 3373 hep-
th/9410191.

[17] D. Capper and M. Duff, Nuovo Cimento 23A (1974) 173.

[18] S. Deser, M. Duff and C. Isham, Nucl. Phys. B111 (1976) 45.

[19] M. Duff, Nucl. Phys. B125 (1977) 334;
Twenty years of the Weyl anomaly, Class. Quant. Grav. 11 (1994) 1387
hep-th/9308075 and references therein.

[20] K. Fujikawa, Phys. Rev. D23 (1981) 2262; Phys. Rev. Lett. 44 (1980) 1733.
[21] L. Bonora, P. Cotta-Ramusino and C. Reina, Phys. Lett. B126 (1983) 305.

[22] L. Bonora, P. Pasti and M. Bregola, Class. Quant. Grav. 3 (1986) 635.

[23] S. Deser and A. Schwimmer, Phys. Lett. B309 (1993) 279 hep-th/9302047.

[24] D. Anselmi, Nucl. Phys. B567 (2000) 331 hep-th/9905005.
Phys. Lett. B476 (2000) hep-th/9908014.

[25] S. Deser, Phys. Lett. B479 (2000) 315 hep-th/9911129.

[26] F. Bastianelli, S. Frolov and A. Tseytlin, JHEP 0002 (2000) 013 hep-th/0001041.

[27] F. Bastianelli, G. Cuoghi and L. Nocetti, Consistency conditions and trace anomalies in six dimensions hep-th/0007222.

[28] P. Gilkey, J. Diff. Geom. 10 (1975) 601;
D. Toms, Phys. Rev. D26 (1982) 2713;
T. Parker and S. Rosenberg, J. Diff. Geom. 25 (1987) 199.

[29] S. Ichinose and N. Ikeda, J. Math. Phys. 40 (1999) 2259 hep-th/9810256;
F. Bastianelli and O. Corradini, hep-th/0010118.

[30] J. Wess and B. Zumino, Phys. Lett. 37B (1971) 95.

[31] M. Kato and K. Ogawa, Nucl. Phys. B212 (1983) 443;
K. Itoh, Nucl. Phys. B342 (1990) 449.

[32] E. Tomboulis, Phys. Lett. 70B (1977) 361; Phys. Lett. 97B (1980) 77;
B. Hasslacher and E. Mottola, Phys. Lett. 99B (1981) 221.

[33] E. Fradkin and A. Tseytlin, Nucl. Phys. B201 (1982) 469; Phys. Lett. 104B (1981) 377.

[34] A. Bartoli, J. Julve and E. Sánchez, Class. Quant. Grav. 16 (1999) 2283.
[35] M. Kaku, Nucl. Phys. B203 (1982) 285;
    D. Boulware, G. Horowitz and A. Strominger, Phys. Rev. Lett. 50 (1983) 1726;
    E. Tomboulis, Phys. Rev. Lett. 52 (1984) 1173.

[36] T. Kugo and S. Uehara, Nucl. Phys. B197 (1982) 378.

[37] E. Witten, Nucl. Phys. B311 (1988/89) 46.

[38] D. Weingarten, Phys. Lett. B90 (1980) 285, Nucl. Phys. B210 [FS6] (1982) 229;
    V. Kazakov, Phys. Lett. 150B (1985) 282;
    F. David, Nucl. Phys. B257 (1985) 45;
    J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl. Phys. B257 (1985) 433;
    D. Boulatov, V. Kazakov, I. Kostov and A. Migdal, Nucl. Phys. B275 (1986) 641.

[39] E. Brezin and V. Kazakov, Phys. Lett. B236 (1990) 144;
    M. Douglas and S. Shenker, Nucl. Phys. B335 (1990) 635;
    D. Gross and A. Migdal, Phys. Rev. Lett. 64 (1990) 127.

[40] J. Ambjørn, B. Durhuus and T. Jonsson, Quantum Geometry, (Cambridge Univ. Press, Cambridge, 1997).

[41] M. Agishtein and A. Migdal, Nucl. Phys. B385 (1992) 395;
    J. Ambjørn and J. Jurkiewicz, Phys. Lett. B278 (1992) 42;
    S. Catterall, J. Kogut and R. Renken, Phys. Lett. B328 (1994) 277.

[42] S. Bilke, Z. Burda, A. Krzywicki, B. Petersson, J. Tabaczek and G. Thorleifsson, Phys. Lett. B418 (1998) 266; Phys. Lett. B432 (1998) 279.

[43] H. Egawa, A. Fujitu, S. Horata, N. Tsuda and T. Yukawa, Nucl. Phys. B (Proc. Suppl.) 73 (1999) 795; Phase Transition of 4D Simplicial Quantum Gravity with U(1) Gauge Field, Proc. Lattice 99, Pisa (June-July 1999)/hep-lat/9908048;
    S. Horata, H. Egawa, N. Tsuda and T. Yukawa, Phase Structure of Four-dimensional Simplicial Quantum Gravity with a U(1) Gauge Field, [hep-lat/0004021].
S. Horata, H. Egawa and T. Yukawa, *Geometry of 4d Simplicial Quantum Gravity with a U(1) Gauge Field*, hep-lat/0010050.

[44] I. Drummond and S. Hathrell, Phys. Rev. **D21** (1980) 958; G. Shore, Phys. Rev. **D21** (1980) 2226; Ann. Phys. **128** (1980) 376; S. Hathrell, Ann. Phys. **139** (1982) 136; Ann. Phys. **142** (1982) 34.