Modulo Counting on Words and Trees*

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Abstract

We consider the satisfiability problem for the two-variable fragment of the first-order logic extended with modulo counting quantifiers and interpreted over finite words or trees. We prove a small-model property of this logic, which gives a technique for deciding the satisfiability problem. In the case of words this gives a new proof of $\text{ExpSpace}$ upper bound, and in the case of trees it gives a $2\cdot\text{ExpTime}$ algorithm. This algorithm is optimal: we prove a matching lower bound by a generic reduction from alternating Turing machines working in exponential space; the reduction involves a development of a new version of tiling games.

1 Introduction

Two-variable logics. Two-variable logic, $\text{FO}^2$, is one of the most prominent decidable fragments of first-order logic. It is important in computer science because of its decidability and connections with other formalisms like modal, temporal and description logics or query languages. The satisfiability problem for $\text{FO}^2$ is $\text{NExpTime}$-complete and satisfiable formulas have models of exponential size. In this paper we are interested in extensions of $\text{FO}^2$ interpreted over finite words and trees. It is known that $\text{FO}^2$ over words can express the same properties as unary temporal logic $\text{LTL}_{=}$ and $\text{FO}^2$ over trees is precisely as expressive as the navigational core of XPath, a query language for XML documents [16]. The satisfiability problem for $\text{FO}^2$ over words is shown to be $\text{NExpTime}$-complete in [9], and over trees—$\text{ExpSpace}$-complete in [4]. Recently it was shown that these complexities do not change if counting quantifiers of the form $\exists x \leq k, \exists x \geq k$ are added.

Modulo counting quantifiers. First-order logic has a limited expressive power. For example, it cannot express even such simple quantitative properties as parity. To overcome this problem, Wolfgang Thomas with his coauthors introduced in the 80s of the last century modulo counting quantifiers of the form “there exists $a \mod b$ elements $x$ such that . . . “. A survey of results in first order logic extended with modulo counting quantifiers can be found in [22]. Recent research in this area involves a study of definability of regular languages on words and its connections to algebra [22], [23]; definable languages of $(\mathbb{N}, +)$ [19], [13]; equivalences of finite structures [17]; definable tree languages [18], [5]; locality [11], [12]; extensions of Linear Temporal

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Logic \[1, 21\], complexity of the model-checking problem \[12, 0\]. Not much is known about the complexity of the satisfiability problem for this logic. The only work that we are aware of is \[15\], which proves ExpSpace-completeness of the satisfiability problem for FO\(^2\) with modulo counting quantifiers over finite words. On the other hand, a simple adaptation of automata techniques for deciding WS1S or WS2S gives a non-elementary decision procedure for the satisfiability problem of full first-order logic with modulo quantifiers over words or ranked trees.

**Our contribution.** We consider the satisfiability problem for FO\(^2\) with modulo counting quantifiers interpreted over finite words and trees. We provide an alternative to \[15\] proof of ExpSpace upper bound for the case of words. This proof is based on a small-model property of this logic and can be extended to the case of ordered and unranked trees. By ordered, we mean that the list of children of any node is ordered by the sibling relation, and by unranked, that there is no limit on the number of children. We prove that the case of such trees is 2-ExpTime-complete. For the upper bound, we again prove a small-model theorem and then give an alternating algorithm that decides the problem in exponential space. Since AExpSpace = 2-ExpTime, this gives the desired upper bound. With some obvious modifications this algorithm can be also applied to unordered or ranked trees. For the lower bound, we develop a new version of tiling games that can encode computations of alternating Turing machines working in exponential space, and can be encoded in the logic. In our encoding we do not use ordering of children, and the number of children of any node is bounded by 2, which shows that already the case of unordered and ranked trees is 2-ExpTime-hard.

The current paper is a full version of \[2\].

### 2 Preliminaries

**Tuples and modulo remainders.** By a \(k\)-tuple of numbers we mean an element of the set \(\mathbb{N}^k\). By \(\vec{0}_k\) and \(\vec{1}_k\) we will denote \(k\)-tuples consisting of, respectively, only zeros and only ones. We denote the \(i\)-th element of a \(k\)-tuple \(t\) by \(\pi_i(t)\). We will often drop the subscript \(k\) if the dimension \(k\) is clear from the context.

Consider a finite set \(X\), a number \(k \in \mathbb{N}\) and a mapping \(f\) from \(X\) to \(\mathbb{N}^k\). We say that \(f\) is zero, if for all \(x \in X\) we have \(f(x) = \vec{0}_k\). We say that \(f\) is a singleton, if there exists a unique argument \(x \in X\) such that \(f(x) = \vec{1}_k\) and for all other arguments the function \(f\) is zero.

Let \(n\) be a positive integer. By \(\mathbb{Z}_n\) we denote the set of all remainders modulo \(n\), that is the set \(\{0, 1, \ldots, n−1\}\). We denote by \(r_n\) the remainder function modulo \(n\), that is \(r_n : \mathbb{N} \rightarrow \mathbb{Z}_n\), \(r_n(x) = x \mod n\).

**Syntax and semantics.** We refer to structures with fraktur letters, and to their universes with the corresponding Roman letters. We always assume that structures have non-empty universes. We work with signatures of the form \(\tau = \tau_0 \cup \tau_{\text{nav}}\), where \(\tau_0\) is a set of unary relational symbols, and \(\tau_{\text{nav}} \subseteq \{\downarrow, \downarrow^+, \rightarrow^+, \rightarrow, \leq, \text{succ}\}\) is the set of navigational binary symbols, which will be interpreted in a special way, depending on which kind of structures, words or trees, are considered as models.

We define the two-variable fragment of first-order logic with modulo counting quantifiers \(\text{FO}^2_{\text{MOD}}\) as the set of all first-order formulas (over the signature \(\tau\)), featuring only the variables \(x\) and \(y\), extended with modulo counting quantifiers of the form \(\exists^{\leq k,l}, \exists^{= k,l} \text{ and } \exists^{\geq k,l}\), where \(l \in \mathbb{Z}_n\) is a positive integer and \(k \in \mathbb{Z}_l\). The formal semantics of such quantifiers is as follows: a formula \(\exists^{\leq k,l} y \varphi(x, y)\), where \(\varphi \in \{\leq, =, \geq\}\), is true at element \(a\) of a structure \(\mathfrak{M}\), in symbols
\( \mathcal{M}, a \models (\exists x. f(x) \varphi(x, y)) \) if and only if \( r_{1}(\{ b \in M : \varphi[a, b] \}) \propto k \). When measuring the length of formulas we will assume binary encodings of numbers \( k \) and \( l \) in superscripts of modulo quantifiers.

In \cite{15} the authors use a different notation for modulo counting quantifiers in their logic \( \text{FOMOD}^{\mathbb{Z}}[\prec] \). It seems that the main difference is that they require equality in place of our \( \succ \) comparison. The use of \( \leq \) and \( \geq \) operators makes the logic exponentially more succinct. For example the formula \( \exists x \geq 2^{n} \exists \prec 2^{n} x p(x) \) has length \( O(n) \), while an equivalent formula in \( \text{FOMOD}^{\mathbb{Z}}[\prec] \) has the form \( \sqrt{2^{2n} \exists \equiv 2^{2n} x p(x)} \) and its length is \( \Omega(2^{n} \cdot n) \).

We write \( \text{FO}_{\text{MOD}}^{2}[\tau_{\text{nav}}] \), to denote that the only binary symbols that are allowed in signatures are from \( \tau_{\text{nav}} \). We will work with two logics: \( \text{FO}_{\text{MOD}}^{2}[\leq, \text{succ}] \) and \( \text{FO}_{\text{MOD}}^{2}[\downarrow, \downarrow^{+}, \rightarrow, \rightarrow^{+}] \). The former is interpreted over finite words, where \( \leq \) denotes the linear order over word positions and \( \text{succ} \) is the successor relation. The latter logic is interpreted over finite, unranked and ordered trees. The interpretation of the symbols from \( \tau_{\text{nav}} \) is the following: \( \downarrow \) is interpreted as the child relation, \( \rightarrow \) as the right sibling relation, and \( \downarrow^{+} \) and \( \rightarrow^{+} \) as their respective transitive closures. We read \( u \downarrow^{+} w \) as “\( w \) is a child of \( u \)” and \( u \rightarrow w \) as “\( w \) is the right sibling of \( u \)”. We will also use other standard terminology like ancestor, descendant, preceding-sibling, following-sibling, etc.

In both cases of words and trees all elements of the universe can be labeled by an arbitrary number of unary predicates from \( \tau_{0} \).

**Normal form.** As usual when working with two-variable logic, it is convenient to use so-called Scott normal form \cite{20}. The main feature of such form is that nesting of quantifiers is restricted to depth two. Below we adapt this notion to the setting of \( \text{FO}_{\text{MOD}}^{2} \).

**Definition 1.** We say that a formula \( \varphi \in \text{FO}_{\text{MOD}}^{2} \) is in normal form, if

\[
\varphi = \forall x \forall y \chi(x, y) \land \bigwedge_{i=1}^{n} \forall x \exists y \chi_{i}(x, y) \land \bigwedge_{j=1}^{m} \forall x \exists \psi_{j}(x, y)
\]

where \( \succ_{i} \in \{ \leq, \geq \} \), each \( l_{j} \) is a positive integer and each \( k_{j} \in \mathbb{Z}_{l} \) and all \( \chi_{i}, \chi_{j}, \psi_{j} \) formulas are quantifier-free. We require both \( n \) and \( m \) parameters to be non-zero.

The following lemma is proved by a routine adaptation of a normal form proof from \cite{10}.

**Lemma 2.** Let \( \varphi \) be a formula from \( \text{FO}_{\text{MOD}}^{2} \) over a signature \( \tau \). There exists a polynomially computable \( \text{FO}_{\text{MOD}}^{2} \) normal form formula \( \varphi' \) over signature \( \tau' \), consisting of \( \tau \) and polynomially many additional unary symbols, such that \( \varphi \) and \( \varphi' \) are equisatisfiable.

In the next sections, when a normal form formula \( \varphi \) is considered we always assume that it is as in Definition \cite{1}. In particular we allow ourselves, without explicitly recalling the shape of \( \varphi \), to refer to its parameters \( n, m \), components \( \chi, \chi_{i}, \psi_{j} \) and numbers \( k_{j}, l_{j} \) given in modulo counting quantifiers.

Consider a conjunct \( \forall x \exists y \chi_{i}(x, y) \) from an \( \text{FO}_{\text{MOD}}^{2} \) normal form formula \( \varphi \). Let \( \mathcal{M} \) be its finite model and let \( v \in M \) be an arbitrary element from the structure. Then an element \( w \in M \) such that \( \mathcal{M} \models \chi_{i}[v, w] \) is called a \( \chi_{i} \)-witness for \( v \). Analogously, we will speak about \( \psi_{j} \)-witnesses for modulo conjuncts or simply witnesses if a formula is clear from the context.

**Order formulas.** Let us call order formulas the formulas specifying relative positions of pairs of elements in a structure with respect to the binary navigational predicates.

In the case of words, when \( \tau_{\text{nav}} = \{ \leq, \text{succ} \} \), there are five possible order formulas: \( x=y, \text{succ}(x, y), \text{succ}(y, x), x \not\succ y \land \neg \text{succ}(x, y) \land x \leq y \) and \( x \not\succ y \land \neg \text{succ}(y, x) \land y \leq x \). We denote them respectively as \( \theta_{=}, \theta_{+1}, \theta_{-1}, \theta_{<}, \theta_{>} \). Let \( \Theta_{\text{words}} \) be the set of these five formulas.
In the case of trees, when \( \tau_{\text{nav}} = \{\downarrow, \downarrow^+, \rightarrow, \rightarrow^+\} \), we use \( x \neq y \) to abbreviate the formula stating that \( x \) and \( y \) are in free position, i.e., that they are related by none of the navigational binary predicates available in the signature. There are ten possible order formulas: \( x \downarrow y \), \( y \downarrow x \), \( x \downarrow^+ y \land \neg(x \downarrow y) \), \( y \downarrow^+ x \land \neg(y \downarrow x) \), \( x \rightarrow y \), \( y \rightarrow x \), \( x \rightarrow^+ y \land \neg(x \rightarrow y) \), \( y \rightarrow^+ x \land \neg(y \rightarrow x) \), \( x \neq y \), \( x = y \). They are denoted, respectively, as: \( \theta_\downarrow \), \( \theta_\uparrow \), \( \theta_\downarrow^+ \), \( \theta_\rightarrow \), \( \theta_\rightarrow^+ \), \( \theta_\leftarrow \), \( \theta_\leftarrow^+ \), \( \theta_\uparrow \), \( \theta_\downarrow \). Let \( \Theta_{\text{trees}} \) be the set of these ten formulas. We will use symbol \( \Theta \) to denote either \( \Theta_{\text{words}} \) or \( \Theta_{\text{trees}} \) if the type of structure is not important or clear from context.

**Types and local configurations.** Following a standard terminology, an atomic 1-type over a signature \( \tau \) is a maximal satisfiable set of atoms or negated atoms involving only the variable \( x \). Usually we identify a 1-type with the conjunction of all its elements. We will denote by \( \alpha \) the set of all 1-types over \( \tau \). Note that the size of \( \alpha \) is exponential in the size of \( \tau \). For a given \( \tau \)-structure \( M \), and an element \( v \in M \) we say that \( v \) realizes a 1-type \( \alpha \), if \( \alpha \) is the unique 1-type such that \( M \models \alpha[v] \). We denote by \( \text{tp}^M(\alpha) \) the 1-type realized by \( v \).

A 1-type provides us only information about a single element of a model. Below we generalize this notion such that for each element of a structure it provides us also some information about surrounding elements and their 1-types.

**Definition 3.** An \((l_1, l_2, \ldots, l_m)\)-full type \( \pi \) over a signature \( \tau \) is a function of the type \( \pi: \Theta \to \alpha \to \{0,1\} \times \mathbb{Z}_{l_1} \times \mathbb{Z}_{l_2} \times \ldots \times \mathbb{Z}_{l_m} \) (a function that takes an order formula from \( \Theta \) and returns a function that takes a 1-type from \( \alpha \) and returns a tuple of integers). Additionally, we require that all \((l_1, l_2, \ldots, l_m)\)-full types satisfy the following conditions:

- \( \pi(\theta_{-\infty}) \) is a singleton,
- \( \pi(\theta) \) is either zero or singleton for \( \theta \in \{\theta_{-\infty}, \theta_{-1}, \theta_{-\rightarrow}, \theta_{\rightarrow}, \theta_{\leftarrow} \} \) and
- if \( \pi(\theta_{\rightarrow^+}) \) (respectively \( \pi(\theta_{\rightarrow}), \pi(\theta_{\rightarrow}), \pi(\theta_{\rightarrow}), \pi(\theta_{\rightarrow}) \)) is zero then also the function \( \pi(\theta_{\rightarrow}) \) (respectively \( \pi(\theta_{\rightarrow}), \pi(\theta_{\rightarrow}), \pi(\theta_{\rightarrow}), \pi(\theta_{\rightarrow}) \)) is zero.

In the following, if the numbers \( l_1, l_2, \ldots, l_m \) are clear from the context or not important, we will be omitting them.

Let us briefly describe the idea behind the notion of full types. Ideally, for a given element \( v \) of a structure \( M \), we would like to know the exact number of occurrences of each 1-type on each relative position. However, to keep the memory usage under control, we store only a summary of the total numbers of occurrences of each 1-type modulo each of \( l_j \) appearing in modulo quantifiers.

**Definition 4.** Let \( M \) be a \( \tau \)-structure and \( v \) an arbitrary element from \( M \). We will denote by \((l_1, l_2, \ldots, l_m)\)-\text{ftp}^M(v) the unique \((l_1, l_2, \ldots, l_m)\)-full type realized by \( v \) in \( M \), i.e., the \((l_1, l_2, \ldots, l_m)\)-full type \( \pi \) such that for all order formulas \( \theta \in \Theta \) and for all atomic 1-types \( \alpha \), the value of \( \pi(\theta)(\alpha) \) is equal to the tuple \((\text{cut}_1(W), \text{cut}_2(W), \ldots, r_{l_m}(W)) \), where \( W \) is the number of occurrences of elements of type \( \alpha \) in the relative position described by \( \theta \), namely

\[
W = |\{w \in M : M \models \theta[v, w] \land \text{tp}^M(w) = \alpha\}|
\]

and \( \text{cut}_1(W) \) is 0 if \( W \) is empty and 1 otherwise.

The following definition and lemma are basic tools used in the proofs of small-model properties \( \text{FO}^\text{MOD} \).
Definition 5 (ψ-consistency). Let ϕ be a FO^2_{MOD} formula in normal form and let α be the unique 1-type satisfying 1(αω)(α) = 1. We say that a (l_1, l_2, . . . , l_m)-full type α is ψ-consistent, if it satisfies the following conditions:

1. It does not violate the ∀ψ subformula i.e., for all order formulas θ ∈ Θ and for all 1-types β such that α(θ)(β) ≠ 0, the implication α(x) ∧ β(y) ∧ θ(x, y) ⇒ χ(x, y) holds.

2. There is a witness for each ∃ψ conjunct of ϕ. Formally, consider a number 1 ≤ i ≤ n and a subformula ∀x ∃y χ_i (x, y). We require that there exists a 1-type β and an order formula θ ∈ Θ such that α(θ)(β) ≠ 0 and the logical implication α(x) ∧ β(y) ∧ θ(x, y) ⇒ χ_i(x, y) holds.

3. Each modulo conjunct has the right number of witnesses. Consider a number 1 ≤ j ≤ m and a subformula ∀x ∃y χ_j (x, y). We require that the remainder modulo l_j of the number of witnesses encoded in the full-type satisfies the inequality χ_j ≥ k_j. Formally,

\[
\tau_j \left( \sum_{\theta \in \Theta, \beta \in \alpha} \pi_{j+1}(\alpha(x) \land \beta(y) \land \theta(x, y) = \psi_j(x, y)) \right) \geq k_j.
\]

A proof of the following lemma is a straightforward unfolding of Definitions 1 and 5 and the definition of the semantics of FO^2_{MOD}.

Lemma 6. Assume that a formula ϕ ∈ FO^2_{MOD} in normal form is interpreted over finite words or trees. Then M ⊨ ϕ if and only if every (l_1, l_2, . . . , l_m)-full type realized in M is ψ-consistent.

Let ϕ be FO^2_{MOD} formula in normal form. For proving the upper bounds in the next sections, we will estimate the size of a model by the following function. We define f(ϕ) as the function, which for a given formula returns the total number of (l_1, l_2, . . . , l_m)-full types over the signature of ϕ. To be precise, f(ϕ) is equal to (2^{l_1 l_2 . . . l_m})^{|Θ||ι|}. Note that f(ϕ) is doubly exponential in the size of the formula ϕ.

3 Finite words

In this section we focus our attention on the case of finite words. We give an alternative proof of ExpSpace upper bound for the satisfiability problem of FO^2_{MOD}[≤, succ]. Originally this result was proved in [13] by a reduction to unary temporal logic with modulo counting operators. Here we give a direct algorithm dedicated to FO^2_{MOD}[≤, succ], based on a small model property that we prove. The advantage of the method is that it allows an extension to the case of trees and other extensions (e.g., an incorporation of counting quantifiers of the form ∃, is quite obvious).

Small model property. We start by proving that satisfiable formulas in FO^2_{MOD}[≤, succ] have small models. The proof technique is similar to the pumping lemma known from the theory of finite word automata.

Lemma 7. Every normal form FO^2_{MOD}[≤, succ] formula satisfiable over finite words has a model M of size bounded by f(ϕ).

Proof. Consider a satisfiable formula ϕ ∈ FO^2_{MOD}[≤, succ] and assume that its model M is a word longer than f(ϕ). We will show that we can remove some subword from M and obtain a shorter model. By repeating this process, we will finally obtain a model of ϕ with a required size.

By the pigeonhole principle there exist two positions u, v ∈ W with equal full-types. Let M' be a word obtained from M by removing all letters from positions between u and v and
collapsing \( u \) and \( v \) into a single position. Observe that since full types of \( u \) and \( v \) are equal, for all \( j \) the remainders modulo \( l_j \) of the total number of removed 1-types on positions between \( u \) and \( v \) are equal to 0. Note also that due to presence of 0-1 part in the definition of a full type, all unique 1-types realized by the structure survive the surgery. Therefore, all full types in \( W \) remain unchanged. Since all these full types were \( \varphi \)-consistent in \( W \), they are also \( \varphi \)-consistent in \( W' \). As a consequence of Lemma 6 the word \( W' \) is indeed a model of \( \varphi \), as expected.

Algorithm. Now we are ready to present an \( \text{ExpSpace} \) algorithm for solving the satisfiability for \( \text{FO}^2_{\text{MOD}}[\leq, \text{succ}] \) interpreted over finite words. We assume that the input formula \( \varphi \) is in normal form. Since models of \( \varphi \) can have doubly-exponential length, we cannot simply guess a complete intended model. To overcome this difficulty, we guess the model on the fly, letter by letter. For each position of the guessed word we guess its full type and after checking some consistency conditions described in Definition 5, we verify if it can be linked with the previous position. This way we never have to store in memory more than two full types. Since the size of a full type is bounded exponentially in \(|\varphi|\), the whole procedure runs in \( \text{ExpSpace} \). We accept the input, if the guessing process ends after at most \( f(\varphi) \) steps.

To avoid presentational clutter in the description of our algorithm we omit the details of two simple tests \( \text{is-valid-successor} \) and \( \text{is-}\varphi\text{-consistent} \). The former takes two full types, \( \beta \) and \( \alpha \), and checks if the position of the type \( \alpha \) can indeed have a successor of type \( \beta \). It can be easily done by comparing the total number of 1-types on each relative position stored in both types. The latter test takes one full type \( \alpha \) and checks if it satisfies the conditions described in Definition 5.

Procedure 1: Satisfiability test for \( \text{FO}^2_{\text{MOD}}[\leq, \text{succ}] \)

1. MaxLength := \( f(\varphi) \)  // Maximal length of a model from Lemma 7
2. CurrentPosition := 0
3. guess a full type \( \pi \) s.t. \( \pi(\theta_\varphi) \) and \( \pi(\theta_{-1}) \) are zero  // Type of the first position
4. if not is-\varphi-consistent(\( \pi \)) then reject  // See Definition 5
5. while CurrentPosition < MaxLength do
6. if both \( \pi(\theta_{\varphi}) \) and \( \pi(\theta_{+1}) \) are zero then accept  // Type of the last position
7. guess a full type \( \beta \)  // Type of the successor
8. if not is-\varphi-consistent(\( \beta \)) then reject  // See Definition 5
9. if not is-valid-successor(\( \beta, \pi \)) then reject
10. \( \pi := \beta \)
11. CurrentPosition := CurrentPosition + 1
12. reject

The correctness of the algorithm above is guaranteed by the following lemma.

Lemma 8. Procedure 1 accepts its input \( \varphi \) if and only if \( \varphi \) is satisfiable.

Proof. Take an arbitrary satisfiable formula \( \varphi \in \text{FO}^2_{\text{MOD}}[\leq, \text{succ}] \). Lemma 7 guarantees the existence of a small model \( W \) of length not exceeding \( f(\varphi) \). At each step of the algorithm we can guess exactly the same types as in \( W \). Such types are of course \( \varphi \)-consistent and valid in the sense of \( \text{is-valid-successor} \), so the procedure accepts \( \varphi \).

For the opposite direction, assume that Procedure 1 accepts its input \( \varphi \in \text{FO}^2_{\text{MOD}}[\leq, \text{succ}] \). Then we can reconstruct the word \( W \) from the guessed full-types. Due to fact that all guessed full-types are \( \varphi \)-consistent we have the right number of witnesses to satisfy the formula. Additionally, \( \text{is-valid-successor} \) function guarantees that two consecutive positions are linked correctly. So the
guessed structure is indeed a word in which every realized type is \( \varphi \)-consistent. By Lemma \[\text{6}\] the word \( \mathcal{W} \) is a model of \( \varphi \).

4 Finite trees

In this section we prove the main result of this paper, which is the following theorem. It is a direct consequence of Theorems \[\text{12}\] and \[\text{16}\] below.

**Theorem 9.** The satisfiability problem for \( \text{FO}_2^{\downarrow, \downarrow^+, \rightarrow, \rightarrow^+} \) interpreted over finite trees is 2-ExpTime-complete.

4.1 Upper bound

We start by proving the 2-ExpTime upper bound. As in previous section, this is done by showing first a small model property and then an algorithm. The small model property is crucial in the proof of correctness of the algorithm. Actually, having defined the notion of the full type, the technique here is a rather straightforward combination of the technique from previous section and from \[\text{[4]}\]. The main difficulty here was to come up with the right notion of a full type.

4.1.1 Small model property

We demonstrate the small-model property of the logic \( \text{FO}_2^{\downarrow, \downarrow^+, \rightarrow, \rightarrow^+} \) by showing that every satisfiable formula \( \varphi \) has a tree model of depth and degree bounded by \( f(\varphi) \). This is done by first shortening \( \downarrow \)-paths and then shortening the \( \rightarrow \)-paths, as in the proof of Lemma \[\text{7}\].

**Theorem 10** (Small model theorem). Let \( \varphi \) be a normal form \( \text{FO}_2^{\downarrow, \downarrow^+, \rightarrow, \rightarrow^+} \) formula. If \( \varphi \) is satisfiable then it has a tree model in which every path has length bounded by \( f(\varphi) \) and every vertex has degree bounded by \( f(\varphi) \).

**Proof.** Let \( \mathcal{T} \) be a model of \( \varphi \). First we show how to shorten \( \downarrow \)-paths in \( \mathcal{T} \). Assume that there exists a \( \downarrow \)-path in \( \mathcal{T} \) longer than \( f(\varphi) \). Thus, by the pigeonhole principle, there are two nodes \( u \) and \( v \) on this path such that \( v \) is a descendant of \( u \) and \( \text{ftp}^\mathcal{T}_\downarrow(u) = \text{ftp}^\mathcal{T}_\downarrow(v) \). Consider the tree \( \mathcal{T}' \), obtained by removing the subtree rooted at \( u \) and replacing it by the subtree rooted at \( v \).

Observe that for all \( j \) the remainders modulo \( l_j \) of the total number of removed vertices is equal to 0. Additionally, all unique full types survive the surgery. Thus, all full types are retained from the original tree, so they are \( \varphi \)-consistent and by Lemma \[\text{6}\] the tree \( \mathcal{T}' \) is a model of \( \varphi \). By repeating this process we get a tree with all \( \downarrow \)-paths shorter than \( f(\varphi) \).

Shortening the \( \rightarrow \)-paths is done in a similar way. Assume that there exists a node \( v \) with branching degree greater than \( f(\varphi) \). Then there exist two children \( u, w \) of \( v \), with equal full types. Let \( \mathcal{T}'' \) be a tree obtained by removing all vertices between \( u \) and \( w \) (excluding \( u \) and including \( w \)) together with subtrees rooted at them. Again, the remainders modulo \( l_j \) of the total number of removed vertices are all equal to 0, and all unique types survive the surgery, so all full types are retained from \( \mathcal{T} \) and thus \( \varphi \)-consistent. Therefore the tree \( \mathcal{T}'' \) is a model of \( \varphi \). We repeat this process until we get a tree with desired branching.

4.1.2 Algorithm

In this section, we design an algorithm to check if a given \( \text{FO}_2^{\downarrow, \downarrow^+, \rightarrow, \rightarrow^+} \) formula \( \varphi \) interpreted over finite trees is satisfiable. By Lemma \[\text{2}\] we can assume that the input formula \( \varphi \) is given in normal form. Then, by Theorem \[\text{10}\] we can turn our attention to trees with degree and path length bounded by \( f(\varphi) \). Recall that \( f(\varphi) \) is doubly-exponential in \( |\varphi| \).
Let us describe the core ideas of the algorithm. It works in alternating exponential space. Since $\text{AExpSpace} = 2\text{-ExpTime}$, it can be translated to an algorithm working in $2\text{-ExpTime}$. The algorithm starts by guessing the full type of the root and then it calls a procedure that builds a tree with the given full type of the root.

The procedure called on a vertex $v$ guesses on the fly the children of $v$, starting with the leftmost child and storing in the memory at most two children at the same time. While guessing a child $v'$, it checks that its type is $\varphi$-consistent and that it can be correctly linked to $v$ and to its left sibling (if it exists). Then the procedure is called recursively on $v'$ to build the subtree rooted at $v'$, but the recursive call is done in parallel, exploiting the power of alternation. The process is continued in the same way, until the bottom of the tree is reached or the height of the constructed tree exceeds $f(\varphi)$.

There is one more point of the algorithm, namely, we have to make sure that the types of children and their descendants, guessed on the fly, are consistent with the information stored in their parent’s full type, in particular with $\theta_1$ and $\theta_{\downarrow \downarrow}$ components. To handle the first of them, we can simply compute the union of all $\theta_\alpha$ components of children, which can be done during the guessing process. Analogously, to handle the second case, we simply compute the union of all $\theta_1$ and $\theta_{\downarrow \downarrow}$ components of children’s full types.

Below we present a pseudocode of the described procedure. Similarly to the case of finite words, we omit the details of obvious methods for checking consistency. For calculating the union of full-types mentioned in the previous paragraph, we employ $\oplus$ operation defined as follows. For given $(l_1, l_2, \ldots, l_m)$-full types $\pi$ and $\beta$, the result of $\pi \oplus \beta$ is a $(l_1, l_2, \ldots, l_m)$-full type $\tau$ such that for all order formulas $\theta \in \Theta$ and all $1$-types $\alpha \in \alpha$, the following condition holds:

$$\tau(\theta)(\alpha) = (\max (\pi_0(\pi(\theta)(\alpha)), \pi_0(\beta(\theta)(\alpha))), R_1, R_2, \ldots, R_m),$$

where $R_i = r_i.(\pi_i(\pi(\theta)(\alpha))) + \pi_i(\beta(\theta)(\alpha)))$.

**Procedure 2:** Building a subtree rooted at given node

```
Input: Formula $\varphi \in \text{FO}^{\text{MOD}}_{\text{L}}[\downarrow, \downarrow, \rightarrow, \rightarrow^\downarrow]$ in normal form, full type $\pi$ of a starting node, and current level $\text{Lvl} \in \mathbb{N}$.
1 if not $\pi$-consistent($\pi$) then reject
2 if $\text{Lvl} \geq f(\varphi)$ then reject
3 if $\pi(\theta_1)$ is zero then accept
4 Guess the degree $\text{Deg} \in \{1, f(\varphi)\}$ of a node
5 Guess the full type $\beta$ of the leftmost child and check if its a valid leftmost son of $\pi$
6 $O_{\theta_{\downarrow}} := \beta(\theta_\downarrow)$ // Types of children guessed so far
7 $O_{\theta_{\uparrow \downarrow}} := \beta(\theta_1) \oplus \beta(\theta_{\downarrow \downarrow})$ // Types of descendants guessed so far
8 while $\text{Deg} > 1$ do
9 Run in parallel Procedure 2 on $(\varphi, \beta, \text{Lvl} + 1)$ // Alternation here
10 Guess a full type $\tau$ of the right brother of $\beta$ and check consistency with $\pi$
11 $O_{\theta_{\uparrow}} := O_{\theta_{\uparrow}} \oplus \pi(\theta_\uparrow), O_{\theta_{\downarrow \uparrow}} := O_{\theta_{\downarrow \uparrow}} \oplus \tau(\theta_\downarrow) \oplus \tau(\theta_{\downarrow \uparrow})$ // Updating obligations
12 $\beta := \tau$, $\text{Deg} := \text{Deg} - 1$
13 Run in parallel Procedure 2 on $(\varphi, \beta, \text{Lvl} + 1)$ // Last child
14 if $\beta(\theta_{\uparrow \downarrow})$ is not zero then reject // Not valid last node on $\rightarrow^\downarrow$-path.
15 if $\pi(\theta_1)$ and $\pi(\theta_{\downarrow \downarrow}) = O_{\theta_{\downarrow \downarrow}}$ then accept else reject
```

The following lemma guarantees the correctness of the algorithm and leads directly to the main result of this section.
Procedure 3: Satisfiability test for \( \text{FO}^2_{\text{MOD}}[↓, ↓^+, →, →^+] \)

**Input:** Formula \( ϕ ∈ \text{FO}^2_{\text{MOD}}[↓, ↓^+, →, →^+] \) in normal form.
1. **guess** a full type \( π \) s.t. \( π(θ↑) \) is zero \hspace{1cm} // Type of the root
2. Run Procedure 2 on \((ϕ, π, 1)\)

**Lemma 11.** Procedure 3 accepts its input formula \( ϕ ∈ \text{FO}^2_{\text{MOD}}[↓, ↓^+, →, →^+] \) if and only if \( ϕ \) is satisfiable over finite trees.

**Proof.** Assume that a given normal form formula \( ϕ ∈ \text{FO}^2_{\text{MOD}}[↓, ↓^+, →, →^+] \) has a tree model \( T \). Then by Theorem 10 this tree can be transformed into a tree model \( T' \) with both branching and path length bounded by \( f(ϕ) \). At each step of the algorithm we can guess exactly the same full-types as in \( T' \). Guessed types are \( ϕ \)-consistent and do not violate any of the propagation conditions given in the algorithm. Hence \( ϕ \) is accepted by the Procedure 3.

For the opposite direction assume that Procedure 3 accepts the input formula \( ϕ \). Below we construct a tree structure \( T \) and show that it is indeed a model of \( ϕ \). We start by constructing the the root and labeling it with unary symbols as required by the 1-type \( α (θ↑) \) guessed in line 1 of Procedure 3. Then we construct as many children as guessed by Procedure 2 in line 4, and recursively construct the subtrees rooted at them. At each step of the construction we take care that the types realised by the constructed nodes coincide with the types guessed by the procedure. The input type \( α \) in Procedure 3 forms a kind of obligation: we must construct a node that realises this type. Most components of this obligation are fulfilled immediately by the construction of the children, the \( θ↓ \) component is an exception. Partly it is fulfilled directly by \( θ↓ \) components of the children, partly it is passed in \( θ↓^+ \) components of the guessed types of the children as further obligations to be fulfilled in future. Line 15 guarantees local consistency between obligations of the current node and its children. Line 15 guarantees that there are no more obligations at the end of each path.

The coincidence between guessed and realised types guarantees that the realised types are \( ϕ \)-consistent, and thus by Lemma 6 the constructed tree is indeed a model of \( ϕ \).

**Theorem 12.** The satisfiability problem for \( \text{FO}^2_{\text{MOD}}[↓, ↓^+, →, →^+] \) interpreted over finite trees is in \( 2\text{-ExpTime} \).

4.2 Lower bound

In this section we prove that the satisfiability of \( \text{FO}^2_{\text{MOD}}[↓, ↓^+] \) is \( 2\text{-ExpTime} \)-hard. We exploit here the fact that \( 2\text{-ExpTime} = \text{AExpSpace} \) and provide a generic reduction from \( \text{AExpSpace} \).

This is done in two steps. First we translate computations of alternating Turing machines to winning strategies in (our version of) tiling games. These strategies are then encoded as trees and their existence is translated to the satisfiability problem in \( \text{FO}^2_{\text{MOD}}[↓, ↓^+] \).

4.2.1 Alternating Turing machines.

Let us first fix the notation concerning alternating Turing machines. By an alternating Turing machine we understand a tuple of the form \( ⟨Q, E, q₀, q_A, q_R, Σ, Γ, δ⟩ \), where \( Q \) is a finite set of states; \( E ⊆ Q \) is a set of **existential** states; \( q₀ ∈ E \) is the initial state; \( q_A, q_R ∈ Q \) are respectively accepting and rejecting states; \( Γ \) is a finite alphabet of tape symbols; \( Σ ⊆ Γ \) is an **input** alphabet; \( δ ⊆ Q × Γ × Γ × \{L, N, R\} × Q \) is the transition relation. States in \( Q \setminus E \) are called **universal**. While in general alternating machines may have multiple accepting and
rejecting states, we assume here without loss of generality, that there is exactly one accepting and exactly one rejecting state. In the following we will use, also without loss of generality, other simplifications. We assume that for each non-final (that is, non-accepting and non-rejecting) state \( q \) and for each tape symbol \( a \in \Gamma \) there are exactly two transitions of the form \( \langle q, a, \cdots, \delta \rangle \) in \( \delta \) and that these transitions are alternating between universal and existential states, so that 
\[
\delta \subseteq (E \times \Gamma \times \Gamma \times \{ L, N, R \} \times (Q \setminus E)) \cup ((Q \setminus E) \times \Gamma \times \{ L, N, R \} \times E).
\]
The machine has one tape that initially contains an input word. We are interested in machines working in exponential space: again without loss of generality we assume that given an input word of length \( n \) such a machine never uses more than \( 2^n - 1 \) tape cells, so its configuration is described by a word of length exactly \( 2^n \) in the language \( \Gamma^*Q\Gamma^+ \). A configuration of the form \( w_1qw_2 \) means that the tape of the machine contains the word \( w_1w_2 \), the machine is in state \( q \) with its head scanning the first letter of \( w_2 \). The initial configuration with an input word \( w \) is \( q_0wB^{2^n-n-1} \) where \( B \in \Gamma \) is the blank symbol. The machine moves from a configuration \( w_1qw_2 \) to its successor in accordance with the transition relation: it chooses a transition of the form \( \langle q, a, b, m, q' \rangle \) from \( \delta \) (recall that there should be exactly two such transitions if \( q \) is not final), replaces the symbol \( a \) with \( b \) on the tape, changes its state to \( q' \) and moves the head to the left (if \( m = L \)), to the right (if \( m = R \)) or leaves the head in the same position (if \( m = N \)). The input word is accepted if the initial configuration leads to acceptance, where leading to acceptance is defined recursively as follows: accepting configurations (i.e., configurations containing the accepting state) lead to acceptance; an existential configuration leads to acceptance if there exists its successor configuration that leads to acceptance; a universal configuration leads to acceptance if all its successor configurations lead to acceptance.

4.2.2 Tiling Games

Corridor tiling games, aka rectangle tiling games \( [8] \) provide a well-known technique for proving lower bounds in space complexity. Here we develop our own version of these games that is able to encode alternating Turing machines from previous section and to be encoded in \( \text{FO}_{\text{MOD}}[\downarrow, +, \downarrow] \).

By a tiling game we understand a tuple of the form \( \langle C, T_0, T_1, n, \langle t_0, \ldots, t_n \rangle, \square, L \rangle \), where \( C \) is a finite set of colors; \( T_0, T_1 \subseteq C^4 \) are two sets of tiles (these two sets are not meant to be disjoint); \( n \) is a natural number; \( \langle t_0, \ldots, t_n \rangle \) is an initial sequence of \( n+1 \) tiles; \( \square \in C \) is a special color called white; \( L \subseteq T_0 \cup T_1 \) is a set of tiles allowed in the last row.

We think of a tile \( \langle a, b, c, d \rangle \in C^4 \) as of a square consisting of four smaller squares, colored respectively with colors \( a, b, c, d \) (see Figure 1). In the following we will require that adjacent tiles have matching colors, both horizontally and vertically. Formally, we define the horizontal adjacency relation \( H = \{ \langle a, b, c, d \rangle, \langle b, e, d, f \rangle \mid a, b, c, d, e, f \in C \} \) and the vertical adjacency relation \( V = \{ \langle a, b, c, d \rangle, \langle c, d, e, f \rangle \mid a, b, c, d, e, f \in C \} \). We define a correctly tiled corridor to be a rectangle of size \( k \times 2^n \) for some \( k \in \mathbb{N} \) filled with tiles, with all horizontally adjacent tiles in \( H \) and all vertically adjacent tiles in \( V \), with first row starting with tiles \( t_0, \ldots, t_n \) and padded out with white tiles, with last row built only from tiles in \( L \), and with all edges being white. Figure 1 shows an example of correctly tiled corridor for \( n = 2 \) and initial tiles \( \langle \square, \square, \square, a \rangle, \langle \square, \square, a, b \rangle, \langle \square, \square, b, \square \rangle \).

Consider the following game. There are two players, \( \text{Prover} \) (also called the existential player) and \( \text{Spoiler} \) (also called the universal player). The task of \( \text{Prover} \) is to construct a correct tiling; the task of \( \text{Spoiler} \) is to prevent \( \text{Prover} \) from doing this. At the beginning they are given the initial row \( t_0, \ldots, t_n, (\square^4)^{2^n-n-1} \). In each move the players alternately choose one of the two sets \( T_0 \) or \( T_1 \) and build one row consisting of \( 2^n \) tiles from the chosen set. The first move is performed by \( \text{Prover} \). \( \text{Prover} \) wins if after a finite number of moves there is a correctly tiled corridor, otherwise \( \text{Spoiler} \) wins. There are two possibilities for \( \text{Spoiler} \) to win: either \( \text{Prover} \) cannot make a move.
while the last constructed row is not in \( L^* \) or the game lasts forever.

We say that a tiling game \( \langle T_0, T_1, \ldots \rangle \) is well-formed if for any play of the game and for any partial (i.e., non-complete) tiling constructed during this play, exactly one new row can be correctly constructed from tiles in \( T_0 \) and exactly one from tiles in \( T_1 \). In other words, every possible move in any play of the game is fully determined by the choice of the set of tiles, \( T_0 \) or \( T_1 \).

### 4.2.3 From Alternating Machines to Tiling Games

The first step of the lower-bound proof for \( \text{FO}^2_{\text{MOD}[\downarrow, \downarrow^*]} \) is a reduction from alternating machines to tiling games. Actually, we have defined our version of tiling games in such a way that the proof of the following theorem becomes a routine exercise.

**Theorem 13.** For all alternating Turing machines \( M \) working in exponential space and all input words \( w \) there exists a well-formed tiling game of size polynomial in the sizes of \( M \) and \( w \) such that Prover has a winning strategy in the game if and only if \( w \) is accepted by \( M \).

**Proof of Theorem 13.** Assume that \( M = \langle Q, E, q_0, q_A, q_R, \Sigma, \Gamma, \delta \rangle \) satisfies all restrictions made in Section 4.2.1. Let \( n \) be the length of \( w \) and let \( w = w_0 \ldots w_{n-1} \). We construct a game \( \langle C, T_0, T_1, n, \langle t_0, \ldots, t_n \rangle, \Gamma, \downarrow, L \rangle \), where \( C = \Gamma \cup (Q \times \Gamma) \cup \{\square\} \). The initial sequence is defined as follows: \( t_0 = \langle \square, \square, \langle q_0, w_0 \rangle \rangle \), \( t_1 = \langle \square, \square, \langle q_0, w_0 \rangle, w_1 \rangle \), \( t_i = \langle \square, \square, w_{i-1}, w_i \rangle \) for \( i \in \{2, \ldots, n-1\} \) and \( t_n = \langle \square, \square, w_n, \square \rangle \). It simply defines the upper white edge of the constructed corridor with the initial configuration of \( M \) stored in lower-right squares of tiles \( t_0, \ldots, t_{n-1} \), where both the current state \( q_0 \) of \( M \) and the scanned symbol \( w_0 \) are encoded as one color \( \langle q_0, w_0 \rangle \). The set \( L \) contains all tiles used in representations of accepting configurations of \( M \), with the lower edge white. Formally,

\[
L = (\Gamma \cup (q_A \times \Gamma) \cup \{\square\}) \times (\Gamma \cup \{\square\}) \times (\Gamma \cup \{q_A \times \Gamma\} \cup \{\square\}) \times (\Gamma \cup \{\square\}) \times (\Gamma \cup \{\square\}).
\]

Now we define the two sets \( T_0 \) and \( T_1 \). These are two sets of tiles for encoding of transitions of \( M \); we use here two sets to separate the encodings of the two transitions applicable in each non-final configuration. Both sets \( T_0 \) and \( T_1 \) contain tiles of the form \( \langle a, b, a, b \rangle \) for all \( a, b \in \Gamma \cup \{\square\} \). This allows copying the content of a tape cell of \( M \) from one configuration to its successor in positions distant from the position of the head of \( M \). Both sets \( T_0 \) and \( T_1 \) contain tiles corresponding to final configurations: all tiles from \( L \) and a similar set of tiles for rejecting configurations, with \( q_A \) replaced by \( q_R \). To finish the construction of sets \( T_0 \) and \( T_1 \), for all non-final states \( q \) and all tape symbols \( a \in \Gamma \), the set \( T_0 \) (respectively, \( T_1 \)) contains the tiles corresponding to the first (respectively, the second) transition of the form \( \langle q, a, \ldots \rangle \) in \( \delta \), defined as follows.

For all \( x, y, z \in \Gamma \cup \{\square\} \) there are three tiles corresponding to a transition of the form \( \langle q, a, b, L, q' \rangle \), namely \( \langle x, y, x, \langle q', y \rangle \rangle, \langle y, \langle q, a \rangle, \langle q', y \rangle, b \rangle \) and \( \langle \langle q, a \rangle, z, b, z \rangle \) (see Figure 2). Tiles
corresponding to a transition of the form \( \langle q, a, b, R, q' \rangle \) are \( \langle x, (q, a), x, b \rangle \), \( \langle (q, a), y, b, (q', y) \rangle \) and \( \langle y, z, (q', y), z \rangle \) for all \( x, y, z \in \Gamma \cup \{ \Box \} \). Tiles corresponding to a transition of the form \( \langle q, a, b, N, q' \rangle \) are \( \langle x, (q, a), x, (q', b) \rangle \) and \( \langle (q, a), y, (q', b), y \rangle \), for all \( x, y \in \Gamma \cup \{ \Box \} \).

This ends the definition of the game. It is easy to see that its size is polynomial in sizes of \( M \) and \( w \). Now we show that the constructed game is well-formed. This is because in any reachable configuration in any play of the game, the lower edge of the so-far constructed tiling encodes a configuration of \( M \) and thus there are precisely two (adjacent) occurrences of a color from \( Q \times \Gamma \). Consider the two tiles \( t_1 = \langle \ldots, (q, a), \ldots \rangle \) and \( t_2 = \langle \ldots, (q, a), \ldots \rangle \) with these occurrences. If \( q \) is final then the only possibility is to finish the play by completing the white lower edge, so let us assume that \( q \) is not final. Then, by assumption on \( M \), there are precisely two transitions of the form \( \langle q, a_1, \ldots, q \rangle \) in \( \delta \) and the chosen set contains tiles corresponding to only one of these transitions. Therefore it contains precisely one tile \( t'_1 \) and precisely one \( t'_2 \) such that \( \langle t'_1, t'_2 \rangle \in H \), \( \langle t_1, t'_1 \rangle \in V \) and \( \langle t_2, t'_2 \rangle \in V \), so the player has to choose these two tiles. Then the rest of the row is determined by the chosen two tiles, and again the lower edge of the constructed tiling correctly encodes a configuration of \( M \); moreover, this configuration is a successor of the configuration encoded in the previous row.

| \( x \) | \( y \) | \( y \) | \( q \) | \( a \) | \( b \) | \( z \) |
| \( x \) | \( q \) | \( a \) | \( b \) | \( b \) | \( b \) | \( z \) |
| \( x \) | \( q \) | \( a \) | \( y \) | \( y \) | \( z \) |
| \( x \) | \( q \) | \( a \) | \( y \) | \( q \) | \( b \) | \( y \) |

Figure 2: Tiles corresponding to transitions \( \langle q, a, b, L, q' \rangle \) \( \langle q, a, b, R, q' \rangle \) and \( \langle q, a, b, N, q' \rangle \)

It remains to show that the constructed game is correct. Suppose that \( M \) accepts \( w \). Then a winning strategy of the existential player is to always choose the set containing tiles corresponding to the transition that leads to acceptance. Conversely, if \( M \) does not accept \( w \), then the universal player has a winning strategy: he can always choose the set containing tiles corresponding to the transition that does not lead to acceptance.

The theorem above directly leads to the lower bound on the complexity of tiling games.

**Corollary 14.** The problem whether Prover has a winning strategy in a tiling game is 2-ExpTime-hard.

### 4.2.4 From Tiling Games to \( \text{FO}^2_{\text{MOD}}[\downarrow, \downarrow^+] \)

Here we show the second step of the lower-bound proof for \( \text{FO}^2_{\text{MOD}}[\downarrow, \downarrow^+] \), which is a reduction from tiling games to \( \text{FO}^2_{\text{MOD}}[\downarrow, \downarrow^+] \). We are going to encode strategies for the existential player as trees. Every complete path in such an encoding corresponds to a correct tiling for \( G \). The nodes on such a path, read from the root to the leaf, correspond to tiles in the tiling, read row by row from left to right. Intuitively, most nodes have just one child corresponding to the tile placed directly to the right. Nodes corresponding to tiles in the last column should have one or two children, depending on whether it is the existential or universal player’s turn. Formally the situation is a bit more complicated, because we are not able to prevent the tree from having additional branches with no meaning (for example, a node may have several children, each of which encodes the same right neighbor).

For a given number \( n \) we will be using unary predicates \( B_0, \ldots, B_{n-1} \) for counting modulo \( 2^n \); the predicate \( B_i \) will be responsible of the \( i \)-th bit of the corresponding number; \( B_0 \) is the least significant bit. This is a standard construction that can be found e.g. in [13] or [4], so we do not present the details here. In the following we will be using several macros that expand in an obvious way to formulas of length polynomial in \( n \) over predicates \( B_0, \ldots, B_{n-1} \). Some of them are listed below.
For example, the macro $Nr(x) > Nr(y)$ expands to
\[
\bigvee_{i=0}^{n} \left( B_i(x) \land \neg B_i(y) \land \bigwedge_{j=i+1}^{n} (B_j(x) \leftrightarrow B_j(y)) \right)
\]

**Theorem 15.** For all well-formed tiling games $G$ there exists a formula $\varphi \in \text{FO}_{\text{MOD}}[\downarrow, \uparrow]$ of size polynomial in the size of $G$ such that the existential player has a winning strategy in $G$ if and only if $\varphi$ is satisfiable over finite trees.

**Sketch of proof.** Let $G = \langle C, T_0, T_1, n, \{t_0, \ldots, t_n\}, \Box, L \rangle$. We are going to define the formula $\varphi$ as a conjunction of several smaller formulas responsible for different aspects of the encoding. Most of these aspects are routine; the most interesting is adjacency.

**Numbering of nodes.** We start by numbering nodes on paths in the underlying tree. These numbers encode the column numbers of the corresponding tiles. The following formulas express that the root is numbered 0, the number of any other node is the number of its father plus one modulo $2^n$, and that all rows are complete (i.e., they have $2^n$ tiles).

\[
\forall x \left( \neg (\exists y \ y \downarrow x) \Rightarrow Nr(x) = 0 \right)
\]

\[
\forall x \left( Nr(x) \neq 2^n - 1 \Rightarrow (\exists y \ x \downarrow y) \land \forall y (x \downarrow y \Rightarrow Nr(y) = Nr(x) + 1) \right)
\]

\[
\forall x \left( Nr(x) = 2^n - 1 \Rightarrow \forall y (x \downarrow y \Rightarrow Nr(y) = 0) \right)
\]

**Tiles and colors.** Let $t_0, \ldots, t_m$ be an enumeration of all tiles occurring in the game, that is in $T_0 \cup T_1 \cup \{t_0, \ldots, t_n, \Box, \Box, \Box, \Box\}$. The predicates $\text{tile}_1, \ldots, \text{tile}_m$ correspond to these tiles. For each color $c \in C$ we introduce four predicates $\pi_1^c, \pi_2^c, \pi_3^c, \pi_4^c$ that will be used to encode the four colors of a tile. The formulas below express that each node in the underlying tree corresponds to precisely one tile and is colored with the four colors of the tile. We assume here that $t_i = (c_1^i, c_2^i, c_3^i, c_4^i)$. We also introduce predicates $\text{set}T_0, \text{set}T_1$ and $\text{set}L$ corresponding to the sets $T_0, T_1$ and $L$, respectively.

\[
\forall x \left( \bigvee_{i=0}^{m} (\text{tile}_i(x) \land \bigwedge_{j \neq i} \neg \text{tile}_j(x)) \right)
\]

\[
\forall x \left( \bigwedge_{i=1}^{4} \left( \bigvee_{c \in C} (\pi_i^c(x) \land \bigwedge_{c' \neq c} \neg \pi_i^{c'}(x)) \right) \right)
\]

\[
\bigwedge_{i=0}^{m} \left( \forall x \left( \text{tile}_i(x) \leftrightarrow \pi_1^c(x) \land \pi_2^c(x) \land \pi_3^c(x) \land \pi_4^c(x) \right) \right)
\]

\[
\left( \bigvee_{t_i \in T_0} \text{tile}_i(x) \right) \land \left( \bigvee_{t_i \in T_1} \text{tile}_i(x) \right) \land \left( \bigvee_{t_i \in L} \text{tile}_i(x) \right)
\]
First and last row. The predicates First and Last are used to distinguish the first and the last row of a correct tiling. A node $x$ corresponds to a tile in the first row if there is no other tile in the same column in previous rows. The last row is described dually. The first row is built from tiles $t_0, \ldots, t_n$ and padded out with white tiles, the last row is built only from tiles in $L$. We assume here that tile white corresponds to the tile $⟨□, □, □, □⟩$.

\[\forall x (\text{First}(x) \iff \neg(\exists y \downarrow x \land \text{Nr}(x) = \text{Nr}(y)))\]
\[\forall x (\text{Last}(x) \iff \neg(\exists y x \downarrow y \land \text{Nr}(x) = \text{Nr}(y)))\]
\[\bigwedge_{i=0}^{n} \left( \forall x (\text{First}(x) \land \text{Nr}(x) = i \Rightarrow \text{tile}_i(x)) \right)\]
\[\forall x (\text{First}(x) \land \text{Nr}(x) > n \Rightarrow \text{tile}_\text{white}(x))\]
\[\forall x \left( \text{First}(x) \Rightarrow \pi_1^1(x) \land \pi_2^2(x) \right)\]
\[\forall x \left( \text{Last}(x) \Rightarrow \pi_2^3(x) \land \pi_1^4(x) \land \text{setL}(x) \right)\]

Existential and universal rows. Each element in each row is marked with predicate $E$ or $A$ depending on which player’s turn it is. Each row is marked the same (each element has the same marking as its left neighbor, if it exists). The first row is existential and then the marking alternates between existential and universal.

\[\forall x (E(x) \iff \neg A(x))\]
\[\forall x (\text{Nr}(x) \neq 0 \Rightarrow \exists y (y \downarrow x \land (E(y) \iff E(x)))\]
\[\forall x (\text{First}(x) \Rightarrow E(x))\]
\[\forall x (\text{Nr}(x) = 0 \land \neg \text{First}(x) \Rightarrow \exists y (y \downarrow x \land (E(y) \iff A(x)))\]

Universal rows have two successors. Each non-first row is marked with predicate move$^0$ or move$^1$ depending on the set of tiles ($T_0$ or $T_1$) from which it is built. Universal non-last rows have two successors, marked respectively with move$^0$ and move$^1$.

\[\forall x \left( \neg \text{First}(x) \Rightarrow (\text{move}^0(x) \iff \neg \text{move}^1(x)) \right)\]
\[\forall x (\text{Nr}(x) \neq 0 \Rightarrow \exists y (y \downarrow x \land (\text{move}^0(y) \iff \text{move}^0(x)))\]
\[\forall x (\text{move}^0(x) \Rightarrow \text{setT}_0(x)) \land \forall x (\text{move}^1(x) \Rightarrow \text{setT}_1(x))\]
\[\forall x \left( \text{Nr}(x) = 2^n - 1 \land \neg \text{Last}(x) \land A(x) \Rightarrow (\exists y (x \downarrow y \land \text{move}^0(y)) \land \exists y (x \downarrow y \land \text{move}^1(y))) \right)\]

Horizontal adjacency. This is simple. We first establish the white frame on the first and last tile in each row and then simply say that for each non-first tile in any row the left edge of
Therefore, we can say that two tiles occur in the same column (or in consecutive columns), but the number of tiles and colors.

Vertical adjacency. This is the tricky part of the encoding. The difficulty comes from the fact that we cannot number rows of the tiling and we have no means to say that two tiles occur in consecutive rows. Using predicates $B_0, \ldots, B_{n-1}$ we can number the $2^n$ columns, but the number of rows may be much higher and we cannot afford having enough predicates for numbering them. Therefore, we can say that two tiles occur in the same column (or in consecutive columns), but we cannot do the same with rows. On the other hand, when we add a new tile to a row, we have to make sure that the upper edge of the new tile matches the lower edge of the tile directly above, so we have to read the colors of the tile directly above. For non-white colors this can be done by observing that each occurrence of a color in an upper edge of a row must be matched with another occurrence of the same color in a lower edge of the previous row in the same column, therefore the number of occurrences of each color in each column should be even. Hence the color directly above the upper edge of the current row is the only non-white color that occurs an odd number of times in the current column in preceding rows. In the case of white color we have to take into consideration the upper white edge of the constructed tiling, so the color directly above is white if it occurs an even number of times in the current column in preceding rows.

Correctness of the constructed formula. Let $\varphi$ be the conjunction of all formulas mentioned above. It is not difficult to see that the size of $\varphi$ is polynomial in the size of $G$: the longest part of $\varphi$ is the encoding of tiles and colors, which is proportional in length to the sum of squared numbers of tiles and colors.

Assume that $\varphi$ has a finite model $M$. The formula $Tiles$ and $colors$ guarantees that each node in $M$ directly encodes precisely one tile. The formula $Numbering$ of nodes guarantees that nodes on each root-to-leaf path are correctly numbered modulo $2^n$, with the root numbered 0 and the leaf numbered $2^n - 1$. Thus each segment of length $2^n$ of such a path, consisting of nodes numbered from 0 to $2^n - 1$, corresponds to one row of tiles. The $Horizontal\ adjacency$ formula guarantees that (horizontally) adjacent tiles in such a row have matching colors. Similarly, $Vertical\ adjacency$ formula guarantees that tiles occurring on the same position in two consecutive rows (that is,
vertically adjacent tiles) have matching colors. The First and last row formula guarantees that the first row is initial and the last row is built from tiles in \(L\). Therefore each complete path in \(M\) encodes a correctly tiled corridor. By the Existential and universal rows formula all rows encoded in any such path are alternately marked as existential and universal, starting with an existential one. Finally, the Universal rows have two successors formula guarantees that each encoding of a universal row in \(M\) is followed by encodings of two existential rows, one build from tiles in \(T_0\) and one from tiles in \(T_1\). Thus (here we use the assumption that the game is well-formed) \(M\) covers both possible moves of the universal player. Now the strategy of the existential player is to follow the path in \(M\) corresponding to the partial tiling as it is constructed during the game. She starts in the root of \(M\). Then, every time when it is her turn, the existential player reads from \(M\) any successor row of her current position and replies with this row, moving down the tree to the position of the successor row. Every time when it is the universal player’s turn, his both possible moves are encoded as successor rows of the current position of the existential player, so she can always follow the branch chosen by the universal player. Since \(M\) is finite, each play stops after finitely many rounds with the constructed tiling corresponding to a path in \(M\). Therefore in each play we obtain a correctly tiled corridor and the existential player wins.

For the other direction, assume that the existential player has a winning strategy. We construct a model \(M\) of \(\varphi\) inductively, level by level, as follows. We start with \(2^n\) nodes, number them from 0 to \(2^n - 1\) using predicates \(B_0, \ldots, B_{n-1}\) and connect consecutive nodes with predicate \(\downarrow\). Then we label nodes numbered 0 to \(n\) with predicates \(\text{tile}_0, \ldots, \text{tile}_n\), respectively, and nodes numbered \(n + 1\) to \(2^n - 1\) with \(\{\bigcirc, \square, \square, \square\}\). Then (and later, always after adding new labels of the form \(\text{tile}_i\)) we add appropriate labels \(\pi^\downarrow_i, \text{set}_{T_0}, \text{set}_{T_1}\) and \(\text{set}_{L}\) to make the formula Tiles and colors true. Similarly, we add labels \(\downarrow^\uparrow\) to all pairs of nodes that are connected by the transitive closure of \(\downarrow\). We also label all these nodes with predicates \(\text{First}\) and \(E\). This way we obtain the encoding of the initial row in the game. Next we inductively, level by level, construct the remaining parts of \(M\).

Assume that \(M\) is constructed up to some level \(k\), with all leaves labeled \(E\), and that each path constructed so far encodes a partial tiling in some play after \(k\) rounds, with existential player’s turn. We extend \(M\) leaf by leaf. Let us consider one such leaf \(\ell\), it encodes the last tile in some row. If all tiles in this row are in the set \(L\), we label all nodes in this row with predicate Last and finish the construction. Otherwise let \(r\) be the next row given by the winning strategy of the existential player in the partial play encoded by the path from the root to \(\ell\). We take fresh \(2^n\) nodes and extend the path from root to \(\ell\) with the encoding of a row \(r\), as above, but this time labeling all new nodes with predicate \(\text{move}\) indicating the universal player’s move. We also label all new nodes with predicate \(\text{move}_0\) or \(\text{move}_1\), depending on whether \(r\) is constructed from tiles in \(T_0\) or \(T_1\), respectively. Again, if the row is final, it is marked with predicate Last, otherwise we take twice \(2^n\) fresh nodes and connect to the current leaf two segments of length \(2^n\) that encode two possible moves of the universal player. We mark each node in both segments with predicate \(E\) and thus finish the construction of level \(k + 1\) at node \(\ell\). The construction is repeated for all leaves at level \(k\). Since during the construction we follow the winning strategy, no encoded play can last forever and the construction ends after finitely many iterations. By inspection of all conjuncts one can check that the constructed tree is a model of \(\varphi\).

As a corollary of the theorem above and Corollary \[14\] we get the main theorem of this section.

**Theorem 16.** The satisfiability problem for \(\text{FO}^2_{\text{MOD}}[\downarrow, \downarrow^+]\) interpreted over finite trees is \(2\)-\(\text{ExpTime-hard.}\)
5 Conclusions and future work

We have shown that the satisfiability problem for two-variable logic extended with modulo counting quantifiers and interpreted over finite trees is 2-ExpTime-complete. The upper bound is based on the small-model property of the logic; for the lower bound we have developed a version of tiling games for AExpSpace computations.

There are several possible directions for future work. One of them is studying restrictions or extensions of the logic presented here. Natural candidates for restrictions include guarded fragment of the logic or unary alphabet restriction as in [4]; natural extensions include arbitrary uninterpreted binary symbols as in [3]. Another possibility is investigation of the (finite) satisfiability problem for FO$^2_{\text{MOD}}$ on arbitrary structures—we even do not know whether this problem is decidable. Yet another direction is to study the expressive power of the logic and to find an expressively equivalent extension of CTL.

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