Entropy Production in High Energy Processes

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We calculate the entropy produced in the decoherence of a classical field configuration and compare it with the entropy of a fully thermalized state with the same energy. We find that decoherence alone accounts for a large fraction of the equilibrium entropy when each field mode is only moderately occupied. We apply this to theories of relativistic heavy ion collisions, which describe the initial state as a collection of coherent color fields. Our results suggest that decoherence may partly explain the rapid formation of a high entropy state in these collisions.

Recent measurements of anisotropic flow in Au+Au collisions at the Relativistic Heavy Ion Collider (RHIC) point to a very rapid formation of nearly equilibrated hot matter in these processes. Hydrodynamic calculations of the collective flow pattern yield estimates for the equilibration time \( \tau_{\text{eq}} \approx 1 \text{ fm/c} \). This is difficult to understand in microscopic theories of thermalization, which describes the apparent rate of entropy production.

The initial state of a relativistic heavy ion collision is characterized by a highly coherent configuration of quark and gluon fields. For the processes contributing to the formation of matter near the center of momentum of the two nuclei, coherent gluon fields at small values of the Bjorken variable \( x \) are most important. These fields are generated by the quasistatic color charges of the valence quarks of the nuclei and can be approximated as randomly oriented, quasiclassical color fields, often called a color glass condensate. During the nuclear collision, gluons are scattered out of this coherent field with a probability that is predicted to be close to unity. After their liberation, the gluons scatter off each other and radiate additional gluons until they reach an equilibrium state.

Parton cascade models describe the liberation and rescattering of gluons and other partons in a probabilistic framework based on the relativistic Boltzmann equation, starting from an incoherent ensemble of partons. On the other hand, models based on an initial state of the color glass condensate treat the equilibration process as nonlinear classical evolution of the initial random, but coherent color fields. While entropy production in statistical transport theories, such as the Boltzmann equation, is a well studied and understood phenomenon, the production of entropy by the decoherence of classical fields is less well understood. One mechanism for the production of entropy is the pair creation of particles in strong (chromo-)electric fields. Another mechanism is the dynamical chaos generated by the nonlinear field equations, where the Kolmogorov-Sinai (KS) entropy describes the apparent rate of entropy production.

Here we are not concerned with the microscopic description of the production of entropy; instead, we address the question of the relative contribution to entropy production by (a) the decoherence of an initially coherent field configuration and (b) the rescattering among incoherent particle-like field excitations, which ultimately leads to equilibrium. Since neither the initial color field configuration in a fast moving nucleus nor its dynamical evolution after the collision of two nuclei is very well known, we start with a simple case, for which the relevant calculations can be performed exactly, but which is sufficiently general to permit conclusions that can be applied to heavy ion reactions.

The idea that decoherence may play a major part in entropy production in heavy ion collisions is not new. The notion of a large nucleus acting as a “phase filter”, decomposing the quark-gluon wavefunction of a hadronic projectile into its incoherent components, was suggested over a decade ago. The principles of decoherence of hadronic wavefunctions were investigated extensively by Elze in the mid-1990s. The formulation of models for the small Bjorken-\( x \) components of hadronic wavefunctions as superpositions of classical color fields, e.g. the color glass condensate (CGC) model, now provides the theoretical basis for a more concrete treatment. In this framework, we need to explore the effects of the decoherence of quasiclassical, coherent color fields, which are present in the nuclei before the onset of a collision.

The quantum mechanical analogue of a classical field is a coherent state:  

\[
|\Psi[J]\rangle = \prod_k \exp(i\alpha_{k\lambda}a_{k\lambda}^\dagger - i\alpha_{k\lambda}^*a_{k\lambda})|0\rangle,  
\]  

where the amplitude \( \alpha_{k\lambda} \) is determined by the classical current \( J \) creating the field:  

\[
\alpha_{k\lambda} = (\hbar \omega_k V)^{-1/2}e_{k\lambda} \cdot J(k, \omega_k).  
\]  

Let us begin by considering a single mode \( k\lambda \). The coherent state can be written as a superposition of particle
number eigenstates:

\[ |\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{3} \]

Being a pure quantum state, \(|\alpha\rangle\) is described by a density matrix

\[ \rho_{mn} = \langle m|\alpha\rangle\langle\alpha|n\rangle, \tag{4} \]

which satisfies the relation \(\rho^2 = \rho\) and has no entropy: \(S = -\text{Tr} \ln \rho = 0\).

Complete decoherence of this quantum state corresponds to the total decay of all off-diagonal matrix elements of the density matrix, yielding the diagonal density matrix

\[ \rho_{\text{dec}}^{mn} = \langle n|\alpha\rangle^2 \delta_{mn} = e^{-|\alpha|^2} \frac{|\alpha|^2 n}{n!} \delta_{mn}. \tag{5} \]

The particle number in this mixed state follows the Poisson distribution, and the average number of particles is \(\bar{n} = |\alpha|^2\). The entropy content of the mixed state is given by

\[ S^{(\text{cs})}_{\text{dec}} = \sum_{n=0}^{\infty} e^{-\bar{n}} \frac{\bar{n}^n}{n!} \ln \left(e^{-\bar{n}} \frac{\bar{n}^n}{n!}\right) \]
\[ = e^{-\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} (n \ln \bar{n} - \bar{n} - \ln n!), \tag{6} \]

where the superscript “cs” indicates that the result holds for a coherent state. With the help of Stirling’s formula and the integral representation of the logarithm,

\[ \ln n = \int_0^\infty \frac{ds}{s} \left(e^{-s} - e^{-ns}\right), \tag{7} \]

the sum in (6) can be performed yielding an analytical result that is valid asymptotically for \(\bar{n} \gg 1\) (actually, the approximation is excellent already for \(\bar{n} \approx 1\)):

\[ S^{(\text{cs})}_{\text{dec}} = \frac{1}{2} \ln(2\pi\bar{n}) + 1 - \frac{1}{6\bar{n}} + \cdots. \tag{8} \]

It is not surprising that the entropy is proportional to \(\ln \sqrt{\bar{n}}\), because we have deleted all information about the relative signs of the amplitudes \(\langle \alpha|n\rangle\) by eliminating the off-diagonal elements of the density matrix. The number of significantly contributing elements is given by the width, \(\Delta n = \sqrt{\bar{n}}\), of the Poisson distribution.

Let us mention that the energy for a single quantum oscillator in equilibrium at temperature \(T\) is given by

\[ S_{\text{eq}} = \ln(\bar{n} + 1) + \bar{n} \ln \left(1 + \frac{1}{\bar{n}}\right), \tag{9} \]

where \(\bar{n} = \left(e^{\omega/T} - 1\right)^{-1}\) is the average occupation number. Asymptotically, for large \(\bar{n}\), one obtains \(S_{\text{eq}} \approx 2S_{\text{dec}}^{(\text{cs})}\), i.e. the thermal entropy becomes twice as large as the decoherence entropy. However, for small to moderate occupation numbers the ratio \(S_{\text{dec}}^{(\text{cs})}/S_{\text{eq}}\) is close to unity and remains above 0.75 up to \(\bar{n} = 10\). Figure 1 shows the decoherence and equilibrium entropies as a function of the average occupation number \(\bar{n}\). It is evident that, for not too large values of \(\bar{n}\), the decoherence process generates a large fraction of the entropy that can be created, and any subsequent equilibration process adds only a small amount of entropy to it. Since decoherence is usually a much faster process than thermal equilibration, our result implies that the fast entropy production observed in heavy ion collisions may be primarily due to decoherence of the initial state color fields.

What does this imply for the quantum field theory, where the field is a system of (infinitely) many coupled oscillators? Assume that, after decoherence, the system can be described as a collection of \(N\) particles, given by some distribution function over single-particle states, which were generated by the decoherence of \(N_{\text{cs}}\) coherent quantum states. Examples of such states are the internal wavefunctions of nucleons forming a large nucleus, or a quark with its comoving gluon cloud. Each coherent state contributes on average \(\bar{n} = N/N_{\text{cs}}\) partons. Then, after full equilibration, the thermal entropy is of the order of \(S_{\text{th}} \sim N_{\text{cs}}\bar{n} = N\), while for the decoherence entropy we get \(S_{\text{dec}} \sim N_{\text{cs}}\frac{1}{2} \ln(2\pi\bar{n})\). The ratio of the two entropies is

\[ \frac{S_{\text{dec}}}{S_{\text{th}}} \sim \frac{\ln(2\pi\bar{n})}{2\bar{n}}, \tag{10} \]

i.e. for large amplitude quantum states, which turn into many particles per coherent mode, the decoherence con-
tribution to the thermal entropy is small. On the other hand, if the individual occupation numbers are of order one, the contribution is sizable. This case applies to our problem of interest, the collision of two nuclei at high energy, as we will discuss below.

For the coherent color fields in colliding nuclei, the average number of decohering gluons per transverse area has been given by

\[ dN_{\text{dec}} = C_F \ln 2 Q_s^2 \Delta y \approx \frac{C_F \ln 2 Q_s^2 \Delta y}{\pi^2 \alpha_s}, \]

(11)

where \( Q_s \) is the so-called saturation scale, \( C_F = 4/3 \) is the quadratic Casimir operator in the fundamental representation of SU(3), and \( \Delta y \) is the rapidity interval over which the color fields retain their coherence. The characteristic transverse area, over which the color fields are coherent, is \( \pi/Q_s^2 \), and one can argue that the longitudinal coherence length is of the order of \( \Delta y \approx 1/\alpha_s \). We thus obtain an average number of decohering partons per coherence region:

\[ \bar{n} \approx \frac{C_F \ln 2}{\pi \alpha_s^2} \approx 3. \]

(12)

For this value, our arguments presented above indicate that the entropy produced in the decoherence process is about half of the equilibrium entropy. The total entropy per unit rapidity produced by decoherence in a Au+Au collision at the Relativistic Heavy Ion Colider is

\[ \frac{dS_{\text{dec}}}{dy} \approx \frac{Q_s^2 R^2}{2\Delta y} (\ln(2\pi \bar{n}) + 1) \approx \frac{1}{2} Q_s^2 R^2 \alpha_s \left[ \ln \frac{2C_F \ln 2}{\alpha_s^2} + 1 \right] \approx 1500, \]

(13)

where we used the values \( Q_s^2 \approx 2 \text{ GeV}^2 \), \( R = 7 \text{ fm} \), and \( \alpha_s \approx 0.3 \). This value accounts for about half of the entropy measured in the final hadron distribution.

Entropy production by decoherence of classical color fields was also discussed recently by A. Mueller on the basis of somewhat different arguments, who obtained a similar expression for the total generated entropy:

\[ S_{\text{dec}} \sim c_S Q_s^2 R^2, \]

(14)

where \( c_S \) is a nontrivial factor numerical of order one. It is the factor \( Q_s^2 R^2 \), which makes entropy generation by decoherence a large effect in his treatment, as well. Our results for the numerical factor differ in their dependence on \( \alpha_s \), but not in their order of magnitude. An unambiguous determination of the factor \( c_S \) will require a determination of the (de-)coherence length for the classical fields, which exist in the nuclei before the collision.

We finally discuss the case of particle production by a fast moving electric charge, such as a large nucleus. The occupation number of the various field modes forms the basis of the Weizsäcker-Williams (WW) approximation to interactions of charged particles and photons at high energies [13]. For a Coulomb charge \( Ze \) moving close to the speed of light with Lorentz factor \( \gamma \), the spectrum of equivalent photons is given by

\[ \frac{dn}{d\omega} = \frac{2Z^2 \alpha}{\pi \omega} \ln \left( \frac{\gamma}{\omega R} \right), \]

(15)

where \( R \) is the intrinsic size of the charge and \( \omega < \omega_{\text{max}} = \gamma/R \). In order to determine the occupation number for a photon energy \( \omega \), we need to know the coherence interval \( \Delta \omega \), corresponding to the inverse longitudinal length scale of the process which causes the destruction of the coherent field. Characteristically, \( \Delta \omega \) grows with \( \omega \). For our analysis we assume that \( \Delta \omega \sim \epsilon \omega \) with a constant parameter \( \epsilon \ll 1 \) and discretize the integral over \( \omega \) by setting \( \omega_j = \omega_{\text{min}} \epsilon^j \) with integer \( j = 1, \ldots, J \) and \( J = \epsilon^{-1} \ln(\omega_{\text{min}}/\omega_{\text{max}}) \). The entropy generated by decoherence of the initial field configuration is then obtained as

\[ S_{\text{dec}} \approx \sum_{j=1}^J \frac{1}{2} \ln \left[ 4Z^2 \alpha \ln \left( \frac{\epsilon \omega_{\text{max}}}{\omega_{\text{min}}} \right) \right] \]

\[ \approx \sum_{j=1}^J \frac{1}{2} \ln \left[ 4Z^2 \alpha \epsilon^2 (J - j) \right] \]

\[ \approx \frac{1}{2 \epsilon} \ln \frac{\omega_{\text{max}}}{\omega_{\text{min}}} \cdot \ln \left( 4Z^2 \alpha \epsilon^{-1} \ln \frac{\omega_{\text{max}}}{\omega_{\text{min}}} \right). \]

(16)

Thus the produced entropy depends crucially on the value of \( \epsilon \), and thus on the process leading to decoherence. In the case of coherent color fields, a more quantitative understanding of the coherence length of the fields before and after the collision is desirable.

In conclusion, we have shown that the decoherence of a quasiclassical state generates a significant amount of entropy. If the average occupation number of each coherent domain of the initial state is not much larger than one, the entropy released by the decoherence process is a sizable fraction (e.g., one-half) of the entropy attained after thermodynamic equilibration. If this reasoning is applied to relativistic heavy ion collisions in the framework of the color glass condensate model, a significant fraction of the measured entropy of the final state can be produced on the time scale of decoherence, \( \tau_{\text{dec}} \sim 1/Q_s < 0.2 \text{ fm}/c \). This may explain the rapid transition to a state that behaves approximately like an equilibrated QCD plasma. The precise value of the entropy generated by decoherence [13] depends sensitively on the coherence length of the entropy creating process.

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