Abstract

The purpose of this paper is to study optimal control of McKean-Vlasov (mean-field) stochastic differential equations with jumps (McKean-Vlasov jump diffusions, for short). To this end, we first prove a Fokker-Planck equation for the law of the solution of such equations. Then we study the situation when the law is absolutely continuous with respect to Lebesgue measure. In that case the Fokker-Planck equation reduces to a deterministic integro-differential equation (PIDE) for the Radon-Nikodym derivative of the law. Combining this equation with the original state equation, we obtain a Markovian system for the state and its law. Furthermore, we apply this to formulate an Hamilton-Jacobi-Bellman (HJB) equation for the optimal control of McKean-Vlasov stochastic differential equations with jumps. Finally we apply these results to solve explicitly the following problems:

- Linear-quadratic optimal control of stochastic McKean-Vlasov jump diffusions.
- Optimal consumption from a cash flow modelled as a stochastic McKean-Vlasov differential equation with jumps.

Keywords: Jump diffusion; McKean-Vlasov differential equation; Fokker-Planck equation; optimal control; HJB equation.
1 Introduction

A McKean-Vlasov equation is a stochastic differential equation (SDE) where the coefficients depend on both the state of the solution and its probability law. It was first studied by H. McKean in 1966 [18].

The first study of optimal control of such systems was done by Andersson & Djehiche [3], who introduced a stochastic maximum principle approach and solved a mean-variance portfolio selection problem.

Buckdahn et al [8] prove that the expected value of a function of the solution of a mean-field stochastic differential equation at the terminal time satisfies a non-local PDE of mean-field type. They are however, not applying this to optimal control.

Bensoussan et al [6] work directly on a the deterministic PDE of Fokker-Planck type, which they assume is satisfied by the law of a solution of a corresponding mean-field stochastic differential equation. Then they study optimal control and Nash equilibria of games for such deterministic systems, by means of an HJB equation. We refer also to Laurière and Pirroneau [15, 16], where models with constant volatility have been considered.

The paper which seems to be closest to our paper is Pham & Wei [21]. They derive a dynamic programming principle for stochastic mean-field systems, and use this to prove that the value function is a viscosity solution of an associated HJB equation. This is applied to solve a linear-quadratic mean-field control problem.

For existence and uniqueness of the Fokker-Planck equation for McKean-Vlasov SDE, we refer to the monograph of Bogachev et al [7] and to Barbu and Röckner [4, 5].

Our paper differs from the above papers in several ways: We include jumps in the system. As far as we know none of the related papers in the literature are dealing with jump models. Our methods are different. First we use Fourier transform of measures to derive a general Fokker-Planck equation in the sense of distributions for the law of the solution of a McKean-Vlasov equation. Then we consider the situation when the law of the state is absolutely continuous with respect to Lebesgue measure, and we obtain a combined Markovian stochastic differential equation for the state and its law process. Then we use this to prove an HJB equation for optimal control of McKean-Vlasov jump diffusions. Our method allows us to obtain explicit solutions of some optimal control problems for McKean-Vlasov jump diffusions.

The paper is organized as follows: In Section 2 we recall some preliminaries that will be used throughout this work. In Section 3 we prove a Fokker-Planck equation for the law of the solution of a McKean-Vlasov jump diffusion and in Section 4 we
present the Fokker-Planck equation in the absolutely continuous case. Section 5 gives a solution formula of the Fokker-Planck equation. In Section 6 we study the optimal control of McKean-Vlasov jump diffusions by means of a Hamilton-Jacobi-Bellman (HJB) equation. Finally, in Section 7 we illustrate our theory by solving explicitly a linear-quadratic control problem for McKean-Vlasov jump diffusions and a problem of optimal consumption from a cash flow modelled by a McKean-Vlasov jump diffusion.

2 Preliminaries

We now recall some basic concepts and background results:

2.1 Radon measures

A Radon measure on \( \mathbb{R}^d \) is a Borel measure which is finite on compact sets, outer regular on all Borel sets and inner regular on all open sets. In particular, all Borel probability measures on \( \mathbb{R}^d \) are Radon measures.

In the following we let \( \mathcal{M} \) be the set of Radon measures and we let \( C_0(\mathbb{R}^d) \) be the uniform closure of the space \( C_c(\mathbb{R}^d) \) of continuous functions with compact support. Then \( \mathcal{M} \) is the dual of \( C_0(\mathbb{R}^d) \). See Chapter 7 in Folland [10] for more information.

If \( \mu \in \mathcal{M} \) is a finite measure, we define

\[
\hat{\mu}(y) := F[\mu](y) := \int_{\mathbb{R}^d} e^{-ixy} \mu(dx); \quad y \in \mathbb{R}^d
\]

(2.1)

to be the Fourier transform of \( \mu \) at \( y \).

In particular, if \( \mu(dx) \) is absolutely continuous with respect to Lebesgue measure \( dx \) with Radon-Nikodym-derivative \( m(x) = \frac{\mu(dx)}{dx} \), so that \( \mu(dx) = m(x)dx \) with \( m \in L^1(\mathbb{R}^d) \), we define the Fourier transform of \( m \) at \( y \), denoted by \( \hat{m}(y) \) or \( F[m](y) \), by

\[
F[m](y) = \hat{m}(y) = \int_{\mathbb{R}^d} e^{-ixy} m(x)dx; \quad y \in \mathbb{R}^d.
\]

(2.2)
2.2 Schwartz space of tempered distributions

Let $S = S(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing smooth $C^\infty(\mathbb{R}^d)$ real functions on $\mathbb{R}^d$. The space $S = S(\mathbb{R}^d)$ is a Fréchet space with respect to the family of seminorms:

$$
\|f\|_{k,\alpha} := \sup_{x \in \mathbb{R}^d} \{(1 + |x|^k)|\partial^\alpha f(x)|\},
$$

where $k = 0, 1, ..., \alpha = (\alpha_1, ..., \alpha_d)$ is a multi-index with $\alpha_j = 0, 1, ... (j = 1, ..., d)$ and

$$
\partial^\alpha f := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} f
$$

for $|\alpha| = \alpha_1 + ... + \alpha_d$.

Let $S' = S'(\mathbb{R}^d)$ be its dual, called the space of tempered distributions. Let $\mathcal{B}$ denote the family of all Borel subsets of $S'(\mathbb{R}^d)$ equipped with the weak* topology. If $\Phi \in S'$ and $f \in S$ we let

$$
\langle \Phi, f \rangle
$$

(2.3)

denote the action of $\Phi$ on $f$. For example, if $\Phi = m$ is a measure on $\mathbb{R}^d$ then

$$
\langle \Phi, f \rangle = \int_{\mathbb{R}^d} f(x) dm(x),
$$

and, in particular, if this measure $m$ is concentrated on $x_0 \in \mathbb{R}^d$, then

$$
\langle \Phi, f \rangle = f(x_0)
$$

is evaluation of $f$ at $x_0 \in \mathbb{R}^d$.

Other examples include

$$
\langle \Phi, f \rangle = f'(x_1),
$$

i.e. $\Phi$ takes the derivative of $f$ at a point $x_1$.

Or, more generally,

$$
\langle \Phi, f \rangle = f^{(k)}(x_k),
$$

i.e. $\Phi$ takes the $k$'th derivative at the point $x_k$, or linear combinations of the above.

If $\Phi \in S'$ we define its Fourier transform $\hat{\Phi} \in S'$ by the identity

$$
\langle \hat{\Phi}, f \rangle = \langle \Phi, \hat{f} \rangle; \quad f \in S.
$$

(2.4)
The partial derivative with respect to $x_k$ of a tempered distribution $\Phi$ is defined by

$$\langle \frac{\partial}{\partial x_k} \Phi, f \rangle = -\langle \Phi, \frac{\partial}{\partial x_k} f \rangle; \quad \phi \in S^d.$$

More generally,

$$\langle \partial^\alpha \Phi, f \rangle = (-1)^{|\alpha|}\langle \Phi, \partial^\alpha f \rangle; \quad \phi \in S^d.$$

We refer to Chapter 8 in Folland [10] for more information.

3 A Fokker-Planck equation for McKean-Vlasov jump diffusions

Let $X(t) = X_t \in \mathbb{R}^d$ be a mean-field stochastic differential equation with jumps, from now on called a **McKean-Vlasov jump diffusion**, of the form

$$dX(t) = \alpha(t, X(t), \mu_t)dt + \beta(t, X(t), \mu_t)dB(t) + \int_{\mathbb{R}^d} \gamma(t, X(t^-), \mu_t, \zeta)\tilde{N}(dt, d\zeta),$$

$$X(0) = x \in \mathbb{R}^d,$$

(3.1)

where $\mu_t = \mathcal{L}(X(t))$ is the law of $X(t)$, and $B(t), \tilde{N}$ are, respectively, a Brownian motion and a compensated Poisson random measure on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. For convenience we assume that the Lévy measure of $N$, denoted by $\nu$, satisfies the condition $\int_{\mathbb{R}^d} \zeta^2 \nu(d\zeta) < \infty$, which means that $N$ does not have many big jumps (but $N$ may still have infinite total variation near 0). Note that $\mu_t = \mathcal{L}(X(t)) \in \mathcal{M}$ for all $t$.

For simplicity we assume throughout this paper that all the coefficients $\alpha(t, x, \mu), \beta(t, x, \mu) : [0, T] \times \mathbb{R}^d \times \mathcal{M} \to \mathbb{R}$ and $\gamma(t, x, \mu, \zeta) : [0, T] \times \mathbb{R}^d \times \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ are bounded processes and $\mathbb{F}$-predictable for all $x, \mu, \zeta$. We also assume that $\alpha, \beta, \gamma$ are continuous with respect to $t$ and $x$ for all $\mu, \zeta$.

For simplicity, in the following we consider only the 1-dimensional case, i.e. $d = 1$. to $t$ and $x$ for all $\mu, \zeta$.

In this section we will prove a Fokker-Planck equation for the law of the McKean-Vlasov jump diffusion. First we explain some notation: For fixed $t, \mu, \zeta$ we write for simplicity $\gamma = \gamma(t, x, \mu, \zeta)$. Note the map

$$g \mapsto \int_{\mathbb{R}} g(x + \gamma)\mu_t(dx)$$
is a bounded linear map on $C_0(\mathbb{R})$. Therefore there is a unique positive measure, denoted by $\mu_t^{(\gamma)}$ such that

$$\langle \mu_t^{(\gamma)}, g \rangle := \int_{\mathbb{R}} g(x) \mu_t^{(\gamma)}(dx) = \int_{\mathbb{R}} g(x + \gamma) \mu_t(dx), \text{ for all } g \in C_0(\mathbb{R}).$$

(3.2)

We call $\mu_t^{(\gamma)}$ the $\gamma$-shift of $\mu_t$.

Note that $\mu_t^{(\gamma)}$ is absolutely continuous with respect to $\mu_t$.

The purpose of this section is to prove the following:

**Theorem 3.1 (Fokker-Planck equation (I))** Assume that the map $t \mapsto \mu_t$ is continuous in $S'$. Then $t \mapsto \mu_t$ is differentiable in $S'$ and

$$\frac{d}{dt} \mu_t = A^* \mu_t,$$

(3.3)

where $A^*$ is the integro-differential operator

$$A^* \mu_t = -D(\alpha(t, \cdot) \mu_t) + \frac{1}{2} D^2(\beta^2(t, \cdot) \mu_t) + \int_{\mathbb{R}} \left\{ \mu_t^{(\gamma)} - \mu_t + D(\gamma(t, \cdot) \mu_t) \right\} \nu(d\zeta),$$

(3.4)

and $D$ denotes differentiation with respect to $x$ in the sense of distributions.

**Proof.** Choose $\varphi \in C^2(\mathbb{R})$ with bounded derivatives. Then by the Itô formula for jump diffusions (see e.g. [20]), we have

$$\mathbb{E}\left[ \varphi(X_{t+h}) - \varphi(X_t) \right] = \mathbb{E}\left[ \int_t^{t+h} A \varphi(X_s) \, ds \right],$$

where

$$A \varphi(X_s) = \alpha(s, X_s, \mu_s) \varphi'(X_s) + \frac{1}{2} \beta^2(s, X_s, \mu_s) \varphi''(X_s)$$

$$+ \int_{\mathbb{R}} \left\{ \varphi(X_s + \gamma(s, X_s, \mu_s, \zeta)) - \varphi(X_s) - \varphi'(X_s) \gamma(s, X_s, \mu_s, \zeta) \right\} \nu(d\zeta),$$

where $\nu(\cdot)$ is the Lévy measure of $N(\cdot, \cdot)$. \qed

In particular, choosing, with $i = \sqrt{-1}$,

$$\varphi(x) = \varphi_y(x) = e^{-iyx}, \quad y, x \in \mathbb{R},$$

6
we get

\[ A\varphi_y (X_s) = \left( -iy\alpha (s, X_s, \mu_s) - \frac{1}{2} y^2 \beta^2 (s, X_s, \mu_s) + \int_{\mathbb{R}} \{ \exp (-iy\gamma (s, X_s, \mu_s, \zeta)) - 1 + iy\gamma (s, X_s, \mu_s, \zeta) \} \nu (d\zeta) \right) e^{-iyX_s}. \quad (3.5) \]

In general we have

\[ \mathbb{E} \left[ g (X_s) e^{-iyX_s} \right] = \int_{\mathbb{R}} g (x) e^{-iyx} \mu_s (dx) = F \left[ g (\cdot) \mu_s \right] (y). \]

Therefore

\[ \mathbb{E} [e^{-iy(s,X_s,\zeta)}e^{-iyX_s}] = \int_{\mathbb{R}} e^{-iy(s,x,\zeta)}e^{-iyx} \mu_s (dx) = \int_{\mathbb{R}} e^{-iy(x+\gamma(s,x,\zeta))} \mu (dx) \]
\[ = \int_{\mathbb{R}} e^{-iyx} \mu^{(\gamma)} (dx) = F \left[ \mu^{(\gamma)} \right] (y), \quad (3.6) \]

where \( \mu^{(\gamma)} (\cdot) \) is the \( \gamma \)-shift of \( \mu_s \). Recall that if \( w \in S' \) with \( \left( \frac{d}{dx} \right)^n w (t, x) =: D^n w (t, x) \), then we have, in the sense of distributions,

\[ F \left[ D^n w (t, \cdot) \right] (y) = (iy)^n F \left[ w (t, \cdot) \right] (y). \]

Therefore

\[ -iyF [\alpha (s, \cdot) \mu_s] (y) = F [-D (\alpha (s, \cdot) \mu_s)] (y) \quad (3.7) \]
\[ -\frac{1}{2} y^2 F [\beta^2 (s, \cdot) \mu_s] (y) = F [\frac{1}{2} D^2 (\beta^2 (s, \cdot) \mu_s)] (y). \quad (3.8) \]

Applying this and (3.6) to (3.5) we get

\[ \mathbb{E} [A\varphi_y (X_s)] = -iyF [\alpha (s, \cdot) \mu_s] (y) - \frac{1}{2} y^2 F [\beta^2 (s, \cdot) \mu_s] (y) \]
\[ + F \left[ \int_{\mathbb{R}} \{ \mu_s^{(\gamma)} - \mu_s + D (\gamma (s, \cdot, \zeta) \mu_s) \} \nu (d\zeta) \right] (y). \quad (3.9) \]

Hence

\[ \hat{\mu}_{t+h} (y) - \hat{\mu}_t (y) = \int_{\mathbb{R}} e^{-iyx} \mu_{t+h} (dx) - \int_{\mathbb{R}} e^{-iyx} \mu_t (dx) \]
\[ = \mathbb{E} [\varphi_y (X_{t+h}) - \varphi_y (X_t)] = \mathbb{E} \left[ \int_t^{t+h} A\varphi_y (X_s) ds \right] = \int_t^{t+h} L_s (y) ds, \quad (3.10) \]
where
\[
L_s(y) = \mathbb{E}[A\varphi_y(X_s)] = -iyF[\alpha(s, \cdot)\mu_s](y) - \frac{1}{2}y^2F[\beta^2(s, \cdot)\mu_s](y) + F\left[\int_\mathbb{R}\{\mu_s^{(\gamma)} - \mu_s + D(\gamma(s, \cdot, \zeta)\mu_s)\}\nu(d\zeta)\right](y).
\] (3.11)

Since \( s \mapsto L_s \) is continuous, we conclude from (3.10) that
\[
\frac{d}{dt}\widehat{\mu}_t(y) = L_t(y); \quad y \in \mathbb{R}.
\] (3.12)

Combining (3.7), (3.8) with (3.10) we get
\[
L_s(y) = F[A^*\mu_s](y),
\] (3.13)
where \( A^* \) is the integro-differential operator
\[
A^*\mu_s = -D(\alpha(s, \cdot)\mu_s) + \frac{1}{2}D^2(\beta^2(s, \cdot)\mu_s) + \int_\mathbb{R}\{\mu_s^{(\gamma)} - \mu_s + D(\gamma(s, \cdot)\mu_s)\}\nu(d\zeta).
\] (3.14)

Note that \( A^*\mu_s \) exists in \( S' \).

Combining (3.12) with (3.13) we get
\[
\frac{d}{dt}F[\mu_t](y) = F[A^*\mu_t](y).
\] (3.15)

Hence
\[
F[\mu_t](y) = F[\mu_0](y) + \int_0^t F[A^*\mu_s](y)ds = F[\mu_0] + \int_0^t (A^*\mu_s)ds(y).
\] (3.16)

Since the Fourier transform of a distribution determines the distribution uniquely, we deduce that
\[
\mu_t = \mu_0 + \int_0^t A^*\mu_sds \text{ in } S'.
\] (3.17)

Therefore
\[
\frac{d}{dt}\mu_t = A^*\mu_t \text{ in } S',
\] (3.18)
as claimed. \( \square \)
4 The Fokker-Planck equation in the absolutely continuous case

From now on we assume that $\mu_t(dx) < dx$ and put

$$m(t,x) = \frac{\mu_t(dx)}{dx},$$

so that $\mu_t(dx) = m(t,x) dx$.

**Definition 4.1** Let $K(dx)$ be the measure on $\mathbb{R}$ defined by

$$K(dx) = (1 + |x|^2)dx,$$

(4.1)

and define

$$L^1_K(\mathbb{R}) = \{ f : \mathbb{R} \mapsto \mathbb{R}; ||f||_K := \int_{\mathbb{R}} |f(x)| K(dx) < \infty \}.$$

Note that since $\mathbb{E}[|X(t)|^p] < \infty$ for $p = 1, 2$, we have $m(t, \cdot) \in L^1_K(\mathbb{R})$ for all $t$.

We assume that $(t,x) \mapsto m(t,x) \in C^{1,2}([0,T] \times \mathbb{R})$.

**Remark 4.2** In this absolutely continuous setting, and if we consider probability measures only, the space $L^1_K(\mathbb{R})$ coincides with the space $P_2(\mathbb{R})$, which is endowed with the Wasserstein metric. See e.g. Cardaliaguet [11] and Lions [14]. See also Agram [1], who applies this to an optimal control problem for McKean-Vlasov equations with anticipating law.

With a slight change of notation, we can write the McKean-Vlasov jump equation (3.1) as a stochastic differential equation in the $\mathbb{R}$-valued process $X(t) = X(t,\omega); (t,\omega) \in [0,T] \times \Omega$ as follows:

$$dX(t) = \alpha(t,X(t),m(t,\cdot))dt + \beta(t,X(t),m(t,\cdot))dB(t)$$

$$+ \int_{\mathbb{R}} \gamma(t,X(t^-),m(t,\cdot),\zeta) \tilde{N}(dt,d\zeta); \quad t > 0,$$

$$X(0) = x \in \mathbb{R},$$

where the coefficients $\alpha(t,x,m), \beta(t,x,m) : [0,T] \times \mathbb{R} \times L^1_K(\mathbb{R}) \to \mathbb{R}$ and $\gamma(t,x,m,\zeta) : [0,T] \times \mathbb{R} \times L^1_K(\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ are bounded predictable and in $C^{1,2,1}([0,T] \times \mathbb{R} \times L^1_K(\mathbb{R}))$ for all $\zeta$.

In the previous section we proved that

$$\frac{d}{dt} \mu_t = A^* \mu_t$$

in $S'(\mathbb{R}^2)$.

(4.3)
The means that
\[ \langle \frac{d}{dt} \mu_t, \varphi \rangle = \langle A^* \mu_t, \varphi \rangle \text{ for all } \varphi \in \mathcal{S}(\mathbb{R}^2). \quad (4.4) \]
By the definition of derivatives of tempered distributions, this is equivalent to
\[ \langle \mu_t, -\frac{d}{dt} \varphi \rangle = \langle \mu_t, A \varphi \rangle, \quad (4.5) \]
i.e.
\[ \int_{\mathbb{R}} -\frac{d}{dt} \varphi(t,x)m(t,x)dx = \int_{\mathbb{R}} A \varphi(t,x)m(t,x)dx, \text{ for all } \varphi \in \mathcal{S}(\mathbb{R}^2) \quad (4.6) \]
Integration by parts then gives
\[ \int_{\mathbb{R}} \varphi(t,x) \frac{d}{dt} m(t,x)dx = \int_{\mathbb{R}} \varphi(t,x)A^* m(t,x)dx, \text{ for all } \varphi \in \mathcal{S}(\mathbb{R}^2). \quad (4.7) \]
Since this holds for all \( \varphi \in \mathcal{S}(\mathbb{R}^2) \), we deduce that
\[ \frac{d}{dt} m(t,x) = A^* m(t,x) \text{ for all } t,x. \quad (4.8) \]
In this case the \( A^* \)-operator gets the form
\[ A^* m(t,x) = -(\alpha(t,x,m(t,x)))' + \frac{1}{2} (\beta^2(t,x,m(t,x)))'' \]
\[ + \int_{\mathbb{R}} \{ m^{(\gamma)}(t,x) - m(t,x) + (\gamma(t,x,m(t,x)))' \} \nu(d\zeta); t \geq 0. \quad (4.9) \]
Here \((\cdot)'\), \((\cdot)''\) denotes differentiation with respect to \( x \) and \( m^{(\gamma)} \) is the Radon Nikodym derivative of \( \mu^{(\gamma)} \) with respect to Lebesgue measure, i.e.,
\[ m^{(\gamma)}(t,x) = \frac{\mu^{(\gamma)}(dx)}{dx}. \quad (4.10) \]
We have proved the following:

**Theorem 4.3** *(Fokker-Planck equation (II))* Assume that
\( (t,x) \mapsto m(t,x) \in C^{1,2} ([0,T] \times \mathbb{R}) \) with first and second order derivatives in \( L^1 (\mathbb{R}) \) for all \( t > 0 \).
Then \( t \mapsto m(t,x) \) is differentiable and
\[ \frac{d}{dt} m(t,x) = A^* m(t,x); \quad \text{for all } t,x. \quad (4.11) \]
5 A solution formula of the Fokker-Planck equation

In this section we will deduce an explicit stochastic representation of the solution of the Fokker-Planck equation. We first make the following observation:

**Lemma 5.1** Assume that

\[ \gamma = \gamma(s, x, m, \zeta) = \gamma(s, m, \zeta) \text{ does not depend on } x. \]  

(5.1)

Then

\[ m^{(\gamma)}(t, x) = m(t, x - \gamma); \text{ for all } t, x. \]  

(5.2)

**Proof.** By (3.2) and a change of variable,

\[ \int_{\mathbb{R}} g(x) m^{(\gamma)}(t, x) dx = \int_{\mathbb{R}} g(x + \gamma) m(t, x) dx = \int_{\mathbb{R}} g(x) m(t, x - \gamma) dx \]  

(5.3)

for all \( g \in C_0(\mathbb{R}). \)

□

Then by (5.2) we see that (4.11) can be written

\[ \frac{\partial}{\partial t} m(t, x) = a(t, x, m)m(t, x) + b(t, x, m)m'(t, x) + \frac{1}{2} c^2(t, x, m)m''(t, x) \]

\[ + \int_{\mathbb{R}} \{ m(t, x - \gamma) - m(t, x) + \gamma m'(t, x) \} \nu(d\zeta); \quad t \geq 0. \]  

(5.4)

where

\[ a(t, x, m) = -\alpha'(t, x, m) + \frac{1}{2}(\beta^2(t, x, m))'', \]

\[ b(t, x, m) = -\alpha(t, x, m) + (\beta^2(t, x, m))', \]

\[ c(t, x, m) = \beta(t, x, m). \]

**Theorem 5.2** (Solution of the Fokker-Planck equation) Assume that (5.1) holds and that \( a(t, x, m) \) is bounded. Let \( X(t) \) be the solution of (4.2) and assume that \( Z(t) \) is the solution of the SDE

\[ dZ(t) = b(t, X(t), Z(t)) dt + c(t, X(t), Z(t)) dB(t) - \int_{\mathbb{R}} \gamma(t, Z(t^-), \zeta) \tilde{N}(dt, d\zeta); \]

\[ Z(0) = x \in \mathbb{R}. \]  

(5.5)

Then the solution of equation (5.4), with given initial value \( m(0, x) = m_0(x) \), can be written

\[ m(t, x) = \mathbb{E}^x \left[ \exp \int_0^t a(s, X_s, Z_s) ds m_0(Z(t)) \right]. \]  

(5.6)
Proof. This follows from the Feynman-Kac formula for jump diffusions, which can be proved by using the Dynkin formula for jump diffusions, following the same procedure as in, e.g. Section 8.2 in [19]. See also Kharroubi and Pham [12], Kromer et al [13] and Zhu et al [24]. □

6 Optimal control and an HJB equation for McKean-Vlasov jump diffusions

We now combine the results obtained in the previous sections to derive an HJB equation for optimal control of McKean-Vlasov jump diffusions.

Before we proceed we introduce and explain some notation:
If \( \psi \in C^1(L^1_K(\mathbb{R})) \) then \( \nabla_m \psi \) is the Fréchet derivative of \( \psi \) with respect to \( m \in L^1_K(\mathbb{R}) \). Therefore it is a bounded linear functional on \( L^1_K(\mathbb{R}) \). By the Riesz representation theorem \( \nabla_m \psi \) can be represented by a function \( \Psi \in L^\infty_K(\mathbb{R}) \), in the sense that the action of \( \nabla_m \psi \) on a function \( h \in L^1_K(\mathbb{R}) \) can be written
\[
\langle \nabla_m \psi, h \rangle = \int_{\mathbb{R}} \Psi(x)h(x)K(dx); \quad h \in L^1_K(\mathbb{R}). \tag{6.1}
\]

Note that since \( \mathbb{E}[X^2(t)] < \infty \), we have that \( \int_{\mathbb{R}} x^2 \mu_t(dx) < \infty \) and therefore \( m_t(\cdot) \in L^1_K(\mathbb{R}) \) if \( m_t = \frac{d}{dt} \mu_t(dx) \).

Consider a controlled version of the system (4.2), in which we have introduced a control process \( u = \{u(t), t \in [0,T]\} \), i.e.
\[
dX(t) = dX(u)(t) = \alpha(t,X(t),m(t,\cdot),u(t))dt + \beta(t,X(t),m(t,\cdot),u(t))dB(t) \tag{6.2}
+ \int_{\mathbb{R}} \gamma(t,X(t^-),m(t,\cdot),u(t),\zeta)\tilde{N}(dt,d\zeta); \quad t > 0,
\]
\[
X(0) = x \in \mathbb{R},
\]
with coefficients \( \alpha(t,x,m,u), \beta(t,x,m,u) : [0,T] \times \mathbb{R} \times L^1(\mathbb{R}) \times U \rightarrow \mathbb{R} \) and \( \gamma(t,x,m,u,\zeta) : [0,T] \times \mathbb{R} \times L^1(\mathbb{R}) \times U \times \mathbb{R} \rightarrow \mathbb{R} \).

We say that \( u \) is admissible if \( u \) is Markovian, i.e. \( u \) has the form \( u(t) = u_0(t,X(t),m(t,\cdot)) \) for some function \( u_0 : \mathbb{R}^3 \mapsto \mathbb{R} \) (see below), and there is a unique solution \( X^{(u)} \) satisfying
\[
\mathbb{E}[|X^{(u)}(t)|^p] < \infty \text{ for } p = 1, 2, \text{ for all } t. \tag{6.3}
\]
The set of admissible controls is denoted by $A$. As is customary, for simplicity (and a slight abuse) of notation we do not distinguish in notation between $u_0$ and $u$ in the following.

As in Section 4 we assume that, for each $t \in [0, T]$, the law $\mu_t = \mu^{(u)}_t$ of the corresponding $X(t) = X^{(u)}(t)$ is absolutely continuous with respect to Lebesgue measure, and we put

$$m(t, x) = \frac{\mu_t(dx)}{dx}. \quad (6.4)$$

From Theorem 4.3, we have the following Fokker-Planck equation for $m$:

$$dm(t, x) = A^* u m(t, x) dt; \quad t > 0 \quad (6.5)$$

$$m(0, x) = m_0(x); \text{ a given initial probability density on } \mathbb{R},$$

where the integro-differential operator $A^* u$ is associated to the controlled process $u$, as follows:

$$A^* u m(t, x) = -(\alpha(t, x, m, u)m(t, x))' + \frac{1}{2}(\beta^2(t, x, m, u)m(t, x))''$$

$$+ \int_\mathbb{R} \{m^{(\gamma)}(t, x, m) - m(t, x) + (\gamma(t, x, m, u, \zeta)m(t, x))'\} \nu(d\zeta); \quad t \geq 0. \quad (6.6)$$

Combining (6.2) and (6.5) we can write the dynamics of the 3-dimensional $[0, T] \times \mathbb{R} \times L^1(\mathbb{R})$ - valued process $Y(t) = (Y_0(t), Y_1(t), Y_2(t)) = (s + t, X(t), m(t, \cdot))$ as follows:

$$dY(t) = \begin{bmatrix} dY_0(t) \\ dY_1(t) \\ dY_2(t) \end{bmatrix} = \begin{bmatrix} dt \\ dX(t) \\ dm(t, \cdot) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \alpha(Y(t), u(t)) \\ A^* Y_2(t) \end{bmatrix} dt + \begin{bmatrix} \beta(Y(t), u(t)) \\ 0 \\ 0 \end{bmatrix} dB(t)$$

$$+ \int_\mathbb{R} \begin{bmatrix} \gamma(Y(t^-), u(t), \zeta) \\ 0 \\ 0 \end{bmatrix} \tilde{N}(dt, d\zeta), \quad (6.7)$$

where we have used the shorthand notation

$$\alpha(Y(t), u(t)) = \alpha(Y_0(t), Y_1(t), Y_2(t), u(t))$$

$$\beta(Y(t), u(t)) = \beta(Y_0(t), Y_1(t), Y_2(t), u(t))$$

$$\gamma(Y(t^-), u(t), \zeta) = \gamma(Y_0(t^-), Y_1(t^-), Y_2(t^-), u(t), \zeta).$$
We now introduce the performance functional:

\[ J_u(s, x, m) = J(y) = \mathbb{E}^y \left[ \int_0^T f(s + t, X(t), m(t, \cdot), u(t)) dt + g(X(T), m(T, \cdot)) \right], \tag{6.8} \]

where \( y = (y_0, y_1, y_2) \). Thus time starts at \( y_0 = s \), \( X(t) \) starts at \( x = y_1 \) and \( m(t) \) starts at \( m = y_2 \). Then we define the value function:

\[ \Phi(y) = \Phi(s, x, m) = \sup_{u \in \mathcal{A}} J_u(s, x, m), \tag{6.9} \]

where \( \mathcal{A} \) denotes the set of admissible controls.

We now give sufficient conditions under which a smooth function \( \varphi \) of the HJB equation coincides with the value function \( \Phi \), i.e. a verification theorem:

**Theorem 6.1 (An HJB equation for optimal control of McKean-Vlasov jump diffusions)** Assume that \( \gamma(t, x, m, u, \zeta) = \gamma(t, m, u, \zeta) \) does not depend on \( x \).

For \( v \in \mathbb{R} \) define \( G_v \) to be the following integro-differential operator, which is the generator of \( Y(t) = (s + t, X(t), m_t(\cdot)) \) given the control value \( v \):

\[
G_v \varphi(s, x, m) = \frac{\partial \varphi}{\partial s} + \alpha(s, x, m, v) \frac{\partial \varphi}{\partial x} + \langle \nabla_m \varphi, A^v(m) \rangle + \frac{1}{2} \beta^2(s, x, m, v) \frac{\partial^2 \varphi}{\partial x^2} \\
+ \int_{\mathbb{R}} \{ \varphi(s, x + \gamma, m) - \varphi(s, x, m) - \gamma \frac{\partial}{\partial x} \varphi(s, x, m) \} \nu(d\zeta);
\]

\( \varphi \in C^{1,2,1}([0, T] \times \mathbb{R} \times L^1_K(\mathbb{R})) \), \tag{6.10}

where \( \gamma = \gamma(s, m, u, \zeta) \). Suppose there exists a function \( \tilde{\varphi}(s, x, m) \in C^{1,2,1}([0, T] \times \mathbb{R} \times L^1_K(\mathbb{R})) \) and a Markov control \( \tilde{u} = \tilde{u}(y) \in \mathcal{A} \) such that, for all \( y \),

\[
\sup_{v \in \mathbb{R}} \left\{ f(y, v) + G_v \tilde{\varphi}(y) \right\} = f(y, \tilde{u}(y)) + G_{\tilde{u}(y)} \tilde{\varphi}(y) = 0 \tag{6.11}
\]

and

\[
\tilde{\varphi}(T, x, m) = g(x, m). \tag{6.12}
\]

Then \( \tilde{u} \) is an optimal control and \( \tilde{\varphi} = \Phi \).

**Proof.** Choose \( \varphi \in C^{1,2,1}([0, T] \times \mathbb{R} \times L^1(\mathbb{R})) \) and let \( u = u(y) \in \mathcal{A} \) be a Markov control for the Markov process \( Y(t) = (s + t, X(t), m(t, \cdot)) \). Then by the Dynkin
formula we have
\[ \mathbb{E}^y[\varphi(Y(T))] = \varphi(s, x, m) + \mathbb{E}^y\left[\int_0^T G_u\varphi(Y(t))dt\right] \] (6.13)

Suppose \( \varphi \) satisfies the conditions
\[ f(y, v) + G_v\varphi(y) \leq 0 \text{ for all } y = (s, x, m) \text{ and all } v \] and
\[ \varphi(T, x, m) \geq g(x, m) \text{ for all } x, m. \] (6.15)

Then by (6.13) we get
\[ \mathbb{E}^y[\varphi(Y(T))] \leq \varphi(s, x, m) - \mathbb{E}^y\left[\int_0^T f(Y(t), u(t))dt\right] \] (6.16)
or
\[ \varphi(s, x, m) \geq \mathbb{E}^y\left[\int_0^T f(s + t, X(t), m(t, \cdot), u(t))dt + \varphi(T, X(T), m(T, \cdot))\right] \]
\[ \geq \mathbb{E}^y\left[\int_0^T f(s + t, X(t), m(t, \cdot), u(t))dt + g(X(T), m(T, \cdot))\right] \]
\[ = J_u(s, x, m). \] (6.17)

Since this holds for all \( u \in \mathcal{A} \) we get
\[ \varphi(s, x, m) \geq \sup_{u \in \mathcal{A}} J_u(s, x, m) = \Phi(s, x, m). \] (6.18)

Now assume that \( \varphi = \hat{\varphi} \) and \( u = \hat{u} \in \mathcal{A} \) satisfy (6.11) and (6.12). Then (6.14) and (6.15) both hold with equality and hence we get equality also in (6.17). We therefore obtain the following string of inequalities
\[ \Phi(s, x, m) \leq \hat{\varphi}(s, x, m) = J_{\hat{u}}(s, x, m) \leq \sup_{u \in \mathcal{A}} J_u(s, x, m) = \Phi(s, x, m). \] (6.19)

Since the first term and the last term is the same, we have equality everywhere in this string. Hence \( \hat{\varphi} = \Phi \) and \( \hat{u} \) is optimal.

\[ \square \]

7  Examples

In this section we illustrate our results by solving explicitly some optimal control problems for McKean-Vlasov jump diffusions. Related problems (for the Fokker-Planck equation only) have been considered in Bensoussan et al. [6]
7.1 A linear-quadratic mean-field control problem

Mean-field linear quadratic control problems have been studied by Yong et al [23], Carmona et al [9]. Here we use the HJB equation to find an explicit solution.

Suppose the system is given by

\[ dX(t) = u(t) dt + \sigma \mathbb{E}[X(t)] dB(t) + \int_{\mathbb{R}} \gamma_0(\zeta) \mathbb{E}[X(t)] \tilde{N}(dt, d\zeta); \quad t > 0, \quad \text{(7.1)} \]

\[ X(0) = x \in \mathbb{R}, \quad \text{(7.2)} \]

with performance functional

\[ J(u) = \mathbb{E} \left[ -\frac{1}{2} \int_{0}^{T} u^2(t) dt - \frac{1}{2} X^2(T) \right]; \quad u \in \mathcal{A}, \]

where \( \sigma > 0 \) and \( \gamma_0(\zeta) \) are a given constant and a given function, respectively.

In this case we have, setting \( \gamma_0(\zeta) \mathbb{E}[X(t)] = \gamma \) for simplicity of notation,

\[ Ah(x) = uh'(x) + \frac{1}{2} \sigma^2 \mathbb{E}[X(t)]^2 h''(x) + \int_{\mathbb{R}} \{ h(x + \gamma) - h(x) - \gamma h'(x) \} \nu(d\zeta), \quad \text{and} \]

\[ A^* h(x) = -uh'(x) + \frac{1}{2} \sigma^2 \mathbb{E}[X(t)]^2 h''(x) + \int_{\mathbb{R}} \{ h(x - \gamma) - h(x) + \gamma h'(x) \} \nu(d\zeta), \quad h \in C^2(\mathbb{R}). \]

If we define \( q(x) = x \), then \( \mathbb{E}[X(t)] = \langle m(t, .), q \rangle = \int_{\mathbb{R}} xm(t, x) dx \), so we can write \( \gamma = \gamma_0(\zeta) \langle m, q \rangle \) and

\[ Ah(x) = uh'(x) + \frac{1}{2} \sigma^2 \langle m, q \rangle^2 h''(x) + \int_{\mathbb{R}} \{ h(x + \gamma) - h(x) - \gamma h'(x) \} \nu(d\zeta), \quad \text{and} \]

\[ A^* h(x) = -uh'(x) + \frac{1}{2} \sigma^2 \langle m, q \rangle^2 h''(x) + \int_{\mathbb{R}} \{ h(x - \gamma) - h(x) + \gamma h'(x) \} \nu(d\zeta), \quad h \in C^2(\mathbb{R}). \]

Note that \( \langle m, q' \rangle = 1, \langle m, q'' \rangle = 0 \) with this \( q \).

The HJB equation becomes

\[
\sup_{v \in \mathbb{R}} \left\{ -\frac{1}{2} v^2 + \frac{\partial \varphi}{\partial s} + v \frac{\partial \varphi}{\partial x} + \frac{1}{2} \sigma^2 \langle m, q \rangle^2 \frac{\partial^2 \varphi}{\partial x^2} + \langle \nabla_m \varphi, A^* m \rangle 
+ \int_{\mathbb{R}} \{ \varphi(s, x + \gamma, m)) - \varphi(s, x, m) - \gamma \frac{\partial \varphi}{\partial x}(s, x, m) \} \nu(d\zeta) \right\} = 0 \quad \text{(7.3)}
\]
As a candidate for the value function we try a function of the form

\[ \varphi(s, x, m) = \kappa_1(s) x^2 + \kappa_2(s) x \langle m, q \rangle + \kappa_3(s) \langle m, q \rangle^2 \]

\[ = \kappa_1(s) x^2 + \kappa_2(s) x \psi_1(m) + \kappa_3(s) \psi_2(m), \quad (7.4) \]

where \( \kappa_1, \kappa_2, \kappa_3 \) are deterministic, differentiable functions and we have defined

\[ \psi_1(m) = \langle m, q \rangle \quad \text{and} \quad \psi_2(m) = \langle m, q \rangle^2; \quad m \in L^1_K(\mathbb{R}). \quad (7.5) \]

Then, since \( \psi_1 \) is linear,

\[ \langle \nabla_m \psi_1, A^* m \rangle = \langle \psi_1, A^* m \rangle = \langle A^* m, q \rangle = u \langle m, q \rangle = u, \quad (7.6) \]

and, by the chain rule,

\[ \langle \nabla_m \psi_2, A^* m \rangle = 2 \langle m, q \rangle \langle \nabla_m \psi_1, A^* m \rangle = 2 \langle m, q \rangle \langle m, Aq \rangle = 2u \langle m, q \rangle. \quad (7.7) \]

Therefore

\[ \langle \nabla_m \varphi, A^* m \rangle = u [\kappa_2(s) x + 2\kappa_3(s) \langle m, q \rangle]. \quad (7.8) \]

Hence the HJB (6.11) becomes, with \( \kappa'_i = \kappa'_i(s) = \frac{d}{ds} \kappa_i(s) \),

\[ \max_v \left\{ -\frac{1}{2} v^2 + \left[ \kappa'_1 x^2 + \kappa'_2 x \langle m, q \rangle + \kappa'_3 \langle m, q \rangle^2 \right] + v \left[ 2\kappa_1 x + \kappa_2 \langle m, q \rangle \right] \right. \\

\[ + \frac{1}{2} \sigma^2 \langle m, q \rangle^2 2\kappa_1 + v \left[ x\kappa_2 + 2\kappa_3 \langle m, q \rangle \right] + \kappa_1 \int_{\mathbb{R}} \gamma^2(\zeta) \nu(d\zeta) \right\} = 0. \]

Maximising with respect to \( u \) gives the first order condition

\[ -v + [2\kappa_1 x + \kappa_2 \langle m, q \rangle + 2\kappa_3 \langle m, q \rangle] = 0, \]

or

\[ v = \hat{u} = (2\kappa_1 + \kappa_2) x + (\kappa_2 + 2\kappa_3) \langle m, q \rangle. \quad (7.9) \]

It remains to verify that with this value of \( u = \hat{u} \), we get (7.14) (without the sup) satisfied:
Substituting (7.10) into (7.14) we get
\[-\frac{1}{2} \left[ (2\kappa_1 + \kappa_2)^2 x^2 + 2 (2\kappa_1 + \kappa_2) (\kappa_2 + 2\kappa_3) x \langle m, q \rangle + (\kappa_2 + 2\kappa_3)^2 \langle m, q \rangle^2 \right]
+ \left[ \kappa'_1 x^2 + \kappa'_2 x \langle m, q \rangle + \kappa'_3 \langle m, q \rangle^2 \right]
+ \left[ (2\kappa_1 + \kappa_2) x + (\kappa_2 + 2\kappa_3) \langle m, q \rangle \right] \left[ 2\kappa_1 x + \kappa_2 \langle m, q \rangle \right]
+ \frac{1}{2} \sigma^2 \langle m, q \rangle^2 2\kappa_1 + \kappa_1 \int_{\mathbb{R}} \gamma^2 \nu(d\zeta)
+ \left[ (2\kappa_1 + \kappa_2) x + (\kappa_2 + 2\kappa_3) \langle m, q \rangle \right] \left[ \kappa_2 x + 2\kappa_3 \langle m, q \rangle \right]
= x^2 \left[ -\frac{1}{2} (2\kappa_1 + \kappa_2)^2 + \kappa'_1 + (2\kappa_1 + \kappa_2) 2\kappa_1 + (2\kappa_1 + \kappa_2) \kappa_2 \right]
+ x \langle m, q \rangle \left[ - (2\kappa_1 + \kappa_2) (\kappa_2 + 2\kappa_3) + \kappa'_2 + (\kappa_2 + 2\kappa_3) 2\kappa_1 + (\kappa_2 + 2\kappa_3) \kappa_2 \right]
+ \langle m, q \rangle^2 \left[ - \frac{1}{2} (\kappa_2 + 2\kappa_3)^2 + \kappa'_3 + \kappa_2 (\kappa_2 + 2\kappa_3) \right]
+ \kappa_1 (\sigma^2 + \int_{\mathbb{R}} \gamma^2 \nu(d\zeta)) + 2\kappa_3 (\kappa_2 + \kappa_3) = 0.

This holds for all \( x, \langle m, q \rangle \) if and only if the following system of differential equations (Riccati equations) are satisfied
\[
\begin{align*}
\kappa'_1 + \frac{1}{2} (2\kappa_1 + \kappa_2)^2 &= 0; \quad \kappa_1 (T) = -\frac{1}{2}, \quad (7.10) \\
\kappa'_2 + 2 (2\kappa_1 + \kappa_2) (\kappa_2 + 2\kappa_3) &= 0; \quad \kappa_2 (T) = 0, \quad (7.11) \\
\kappa'_3 + \frac{1}{2} (\kappa_2 + 2\kappa_3)^2 + \kappa_1 (\sigma^2 + \int_{\mathbb{R}} \gamma^2 \nu(d\zeta)) &= 0; \quad \kappa_3 (T) = 0. \quad (7.12)
\end{align*}
\]

By general theory of matrix Riccati equations (see e.g. Theorem 37 in [22]) there is a unique solution \( \hat{\kappa}_1 (s), \hat{\kappa}_2 (s), \hat{\kappa}_3 (s) \) satisfying (7.10), (7.11), (7.12). Hence we get the following conclusion:

**Theorem 7.1** The optimal control \( \hat{u} (t) \) of the problem to maximize
\[
J (u) := \mathbb{E} \left[ -\frac{1}{2} \int_{0}^{T} u^2 (t) \, dt - \frac{1}{2} X^2 (T) \right]; \quad u \in \mathcal{A},
\]
is given in feedback form as follows:
\[
\hat{u} (Y(t)) = \left( 2\hat{\kappa}_1 (t) + \hat{\kappa}_2 (t) \right) X (t) + \left( \hat{\kappa}_2 (t) + 2\hat{\kappa}_3 (t) \right) \mathbb{E} [X (t)]. \quad (7.13)
\]
The value function is
\[ \Phi(s, x, m) = \hat{\varphi}(s, x, m) = \hat{\kappa}_1(s) x^2 + \hat{\kappa}_2(s) x \langle m, q \rangle + \hat{\kappa}_3(s) \langle m, q \rangle^2. \]

### 7.2 An optimal consumption/harvesting problem

The following problem may for example be considered as an optimal fish population harvesting problem, or as a problem of optimal consumption from a cash flow.

Consider the following controlled mean-field SDE:
\[
\begin{cases}
    dX(t) = \left( \rho(t) - c(t) \right) E[X(t)] dt + \sigma_0(t) E[X(t)] dB(t) \\
    \quad \quad + \int \gamma_0(t, \zeta) E[X(t)] N(dt, d\zeta); \quad t > 0,
\end{cases}
\]
where \( \rho(t), \sigma_0(t) \) and \( \gamma_0(t, \zeta) \) are bounded deterministic functions. The consumption rate \( c = c(t, \omega); (t, \omega) \in [0, T] \times \Omega \) is defined to be an \( \mathbb{F} \)–predictable positive process; it is admissible if \( \mathbb{E}[\int_0^T c(t)^2 dt] < \infty \).

We want to maximize the total expected utility form the consumption, expressed by the performance functional
\[
J(c) = \mathbb{E} \left[ \int_0^T \ln \left( c(t) E[X(t)] \right) dt + \theta \mathbb{E} [X(T)] \right]; \quad c \in \mathcal{A},
\]
where \( \theta > 0 \) is a (deterministic) constant.

To put this into our framework we write, with \( m_t = m(t, \cdot), \)
\[ E[X(t)] = \langle m_t, q \rangle \text{ where } q(x) = x. \]

Then the generator \( A \) of \( m_t \) can be written as
\[
Ah(x) = \left( \rho(t) - c(t) \right) \langle m_t, q \rangle h'(x) + \frac{1}{2} \sigma_0^2(t) \langle m_t, q \rangle^2 h''(x)
\]
\[ + \int \{h(x + \gamma_0 \langle m_t, q \rangle) - h(x) - \gamma_0 \langle m_t, q \rangle h'(x)\} \nu(d\zeta); \quad h \in C^2(\mathbb{R}), \]
and the adjoint operator is
\[
A^*h(x) = -\left( \rho(t) - c(t) \right) \langle m_t, q \rangle h'(x) + \frac{1}{2} \sigma_0^2(t) \langle m_t, q \rangle^2 h''(x)
\]
\[ + \int \{h(x - \gamma_0 \langle m_t, q \rangle) - h(x) + \gamma_0 \langle m_t, q \rangle h'(x)\} \nu(d\zeta); \quad h \in C^2(\mathbb{R}). \]
Setting $\gamma_0(t, \zeta)\mathbb{E}[X(t)] = \gamma$ for simplicity of notation, the corresponding HJB equation for the value function $\varphi$ becomes

$$\sup_{c>0} \{ \ln (c/m, q) + \frac{\partial \varphi}{\partial t} + (\rho(t) - c) (m, q) \frac{\partial \varphi}{\partial x} + \frac{1}{2} \sigma_0^2(t) (m_t, q) \frac{\partial^2 \varphi}{\partial x^2} \} + \langle \nabla_m \varphi, A^* m \rangle = 0; \quad t < T,$$

(7.18)

$$+ \int_{\mathbb{R}} \{ \varphi(s, x + \gamma, m) - \varphi(s, x, m) - \gamma \frac{\partial \varphi}{\partial x}(s, x, m) \} \nu(d\zeta) = 0; \quad t < T,$$

(7.19)

with terminal value

$$\varphi(T, x, m) = \theta \langle m_T, q \rangle.$$

(7.20)

Let us guess that the value function is of the form

$$\varphi(s, x, m) = \kappa_0(s) + \kappa_1(s) \ln \langle m, q \rangle,$$

(7.21)

for some $C^1$ deterministic functions $\kappa_0(s), \kappa_1(s)$.

Define

$$\psi(m) = \langle m, q \rangle.$$

(7.22)

By the chain rule we have, since $\psi$ is linear,

$$\nabla_m \varphi = \kappa_1(s) \frac{1}{\langle m, q \rangle} \nabla_m \psi = \kappa_1(s) \frac{1}{\langle m, q \rangle} \psi.$$

(7.23)

Hence

$$\langle \nabla_m \varphi, A^* m \rangle = \kappa_1(s) \frac{1}{\langle m, q \rangle} \langle \nabla_m \psi, A^* m \rangle = \kappa_1(s) \frac{1}{\langle m, q \rangle} \langle \psi, A^* m \rangle$$

(7.24)

$$= \kappa_1(s) \frac{1}{\langle m, q \rangle} \langle A^* m, q \rangle = \kappa_1(s) \frac{1}{\langle m, q \rangle} \langle m, Aq \rangle$$

$$= \kappa_1(s) (\rho - c) \left[ \langle m, q' \rangle + \langle m, \int_{\mathbb{R}} \{ (x - \gamma_0(m, q)) - q + \gamma_0(m, q) q' \} \nu(d\zeta) \rangle \right]$$

$$= \kappa_1(s) (\rho(s) - c).$$

Here we have used that

$$\langle m, q' \rangle = \langle m, 1 \rangle = \int_{\mathbb{R}} m(t, y) dy = 1 \quad \text{and, since } q \text{ is linear,}$$

$$\int_{\mathbb{R}} \{ \langle m, q(x - \gamma) \rangle - \langle m, q \rangle + \gamma \langle m, q' \rangle \} \nu(d\zeta) = \int_{\mathbb{R}} \{ \langle m, q \rangle - \langle m, \gamma \rangle - \langle m, q \rangle + \gamma \} \nu(d\zeta) = 0.$$
Therefore the HJB equation now takes the form
\[
\sup_{c > 0} \{ \ln c + \ln \langle m, q \rangle + \kappa_0'(s) + \kappa_1'(s) \ln \langle m, q \rangle + \kappa_1(s) \langle \rho(t) - c \rangle \} = 0. \tag{7.25}
\]
The optimising value of \(c\) is the solution of the equation
\[
\frac{1}{c} - \kappa_1(s) = 0,
\]
i.e
\[
c(s) = \hat{c}(s) = \frac{1}{\kappa_1(s)}. \tag{7.26}
\]
Substituting this into (7.25) we get
\[
- \ln \kappa_1(s) + \ln \langle m, q \rangle + \kappa_0'(s) + \kappa_1'(s) \ln \langle m, q \rangle + \rho(s) \kappa_1(s) - 1 = 0. \tag{7.27}
\]
From (7.21) and (7.22) we get the terminal values
\[
\kappa_0(T) = 0, \quad \kappa_1(T) = \theta. \tag{7.28}
\]
Hence, if we choose \(\kappa_1\) such that
\[
\kappa_1'(s) = 1, \text{ i.e.} \quad \kappa_1(s) = \theta + T - s, \tag{7.29}
\]
and let \(\kappa_0(s)\) be the given by
\[
\begin{aligned}
\kappa_0'(s) &= 1 + \ln \kappa_1(s) - \rho(s) \kappa_1(s); s < T, \\
\kappa_0(T) &= 0,
\end{aligned} \tag{7.30}
\]
we see that (7.21) holds. We have proved the following:

**Theorem 7.2**  The function
\[
\varphi(s, x, m) = \kappa_0(s) + \kappa_1(s) \ln \langle m, q \rangle,
\]
with \(\kappa_0, \kappa_1\) defined by (7.29), (7.30), satisfies all the conditions of our HJB theorem and therefore
\[
\Phi(s, x, m) = \varphi(s, x, m)
\]
is the value function, and
\[
\hat{c}(s) = \frac{1}{\kappa_1(s)}
\]
is the optimal control.
8 Summary

- In this paper we introduce a *McKean-Vlasov jump diffusions*, which are solutions of mean-field stochastic differential equations with jumps.

- Using Fourier transforms of Radon measures we prove that the law process of the solution of a McKean-Vlasov jump diffusion satisfies (in the sense of distributions) a Fokker-Planck equation.

- In the case when the law of the solution is absolutely continuous with respect to Lebesgue measure, we represent the solution of the McKean-Vlasov jump diffusion as the solution of a classical 3-dimensional Markovian system. This allows us to formulate an HJB equation for the optimal control of such systems.

- Finally we illustrate our results by solving explicitly some optimal control problems for McKean-Vlasov jump diffusions.

References

[1] Agram, N. (2019). Stochastic optimal control of McKean-Vlasov equations with anticipating law. Afrika Matematika, 30(5), 879-901.

[2] Agram, N., Hu, Y., & Øksendal, B. (2018). Mean-field backward stochastic differential equations and applications. arXiv preprint arXiv:1801.03349.

[3] Andersson, D. & Djehiche, B. (2011). A maximum principle for SDEs of mean-field type. Appl. Math. Optim. 63: 341 - 356.

[4] Barbu, V., & Röckner, M. (2021). Solutions for nonlinear Fokker-Planck equations with measures as initial data and McKean-Vlasov equations. Journal of Functional Analysis, 280(7), 108926.

[5] Barbu, V., & Röckner, M. (2021). Uniqueness for nonlinear Fokker-Planck equations and weak uniqueness for McKean-Vlasov SDEs. Stochastics and Partial Differential Equations: Analysis and Computations, 9(3), 702-713.

[6] Bensoussan, A., Huang, T., & Laurière, M. (2018). Mean field control and mean field game models with several populations. arXiv preprint arXiv:1810.00783.

[7] Bogachev, V. I., Krylov, N. V., Röckner, M., & Shaposhnikov, S. V. (2015). Fokker-Planck-Kolmogorov Equations (Vol. 207). American Mathematical Soc.
[8] Buckdahn, R., Li, J., Peng, S., & Rainer, C. (2017). Mean-field stochastic differential equations and associated PDEs. The Annals of Probability, 45(2), 824-878.

[9] Carmona, R., Delarue, F., & Lachapelle, A. (2013). Control of McKean-Vlasov dynamics versus mean field games. Mathematics and Financial Economics, 7(2), 131-166.

[10] Folland, G.B. (1984). Real Analysis. Modern Techniques and Their Applications. Wiley.

[11] Cardaliaguet, P. (2010). Notes on mean field games (p. 120). Technical report.

[12] Kharroubi, I., & Pham, H. (2015). Feynman-Kac representation for Hamilton-Jacobi-Bellman IPDE. The Annals of Probability, 43(4), 1823-1865.

[13] Kromer, E., Overbeck, L., & Röder, J. A. L. (2015). Feynman-Kac for functional jump diffusions with an application to Credit Value Adjustment. Statistics & Probability Letters, 105, 120-129.

[14] Lasry, J-M. & Lions, P.-L. (2007) Mean-field games. Japan J. Math. 2, 229-260.

[15] Laurière, M. & Pirroneau, O. (2014). Dynamic programming for mean-field type control. Comptes Rendus Mathematique, 352(9), 707-713.

[16] Laurière, M. & Pirroneau, O. (2016). Dynamic programming for mean-field type control. Journal of Optimization Theory and Applications, 169(3), 902-924.

[17] Li, X., Sun, J. & Yong, J. (2016). Mean-field stochastic linear quadratic optimal control problems: closed-loop solvability. Probability, Uncertainty and Quantitative Risk, 1(1), 1-24.

[18] McKean Jr, H. P. (1966). A class of Markov processes associated with nonlinear parabolic equations. Proceedings of the National Academy of Sciences of the United States of America, 56(6), 1907.

[19] Øksendal, B. (2013). Stochastic Differential Equations. 6th edition. Springer.

[20] Øksendal, B. & Sulem, A. (2019). Applied Stochastic Control of Jump diffusions. 3rd edition. Springer.

[21] Pham, H., & Wei, X. (2017). Dynamic programming for optimal control of stochastic McKean–Vlasov dynamics. SIAM Journal on Control and Optimization, 55(2), 1069-1101.
[22] Sontag, E. (1998). Mathematical Control Theory: Deterministic Finite Dimensional Systems, 2nd Edition Texts in Applied Mathematics, Volume 6, Second Edition, New York: Springer.

[23] Yong, J. (2013). Linear-quadratic optimal control problems for mean-field stochastic differential equations. SIAM journal on Control and Optimization, 51(4), 2809-2838.

[24] Zhu, C., Yin, G., & Baran, N. A. (2015). Feynman-Kac formulas for regime-switching jump diffusions and their applications. Stochastics An International Journal of Probability and Stochastic Processes, 87(6), 1000-1032.