ON REAL ANTI-BICANONICAL CURVES WITH ONE DOUBLE POINT ON THE 4-TH REAL HIRZEBRUCH SURFACE. II

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Abstract. A real 2-elementary K3 surfaces of type $((3,1,1),-id)$ yields a real anti-bicanonical curve $s \cup A_1'$ (disjoint union) on the 4-th real Hirzebruch surface $F_4$ where $s$ is the exceptional section of $F_4$ and the real curve $A_1'$ has one real double point. (See Section 2 below.) We give a criterion (Proposition 2.4) which determines whether the real double point is degenerate or not. One direction of the assertion of Proposition 2.4 has already been proved in Lemma 4.6 in the preceding paper [9]. In this paper we prove the inverse direction.

Contents

1. Introduction: Review of real 2-elementary K3 surfaces
1.1. Real 2-elementary K3 surfaces
1.2. Integral involutions of $L_{K3}$ of type $(S,\theta)$
1.3. Period domains
1.4. ($\mathcal{D}\mathcal{R}$)-nondegenerate marked real 2-elementary K3 surfaces
2. Real 2-elementary K3 surfaces of type $((3,1,1),-id)$

References

1. Introduction: Review of real 2-elementary K3 surfaces

1.1. Real 2-elementary K3 surfaces. In this paper we mainly discuss K3 surfaces $X$ with a non-symplectic holomorphic involution $\tau$. We often call them 2-elementary K3 surfaces $(X,\tau)$ ([5], [1], [7], [8], [9], and e.t.c.). Note that every K3 surface with a non-symplectic holomorphic involution is algebraic. Hence, it has hyperplane sections.

Definition 1.1 (Real 2-elementary K3 surface). We say that a triple $(X,\tau,\varphi)$ is a real K3 surface with non-symplectic holomorphic involution (or real 2-elementary K3 surface) if

1. $(X,\tau)$ is a K3 surface $X$ with a non-symplectic holomorphic involution $\tau$,
2. $\varphi$ is an anti-holomorphic involution on $X$, and
3. $\varphi \circ \tau = \tau \circ \varphi$.

For a 2-elementary K3 surface $(X,\tau)$, let $H^{+}_{2}(X,\mathbb{Z})$ denote the fixed part of $\tau_* : H_{2}(X,\mathbb{Z}) \to H_{2}(X,\mathbb{Z})$. It is well-known that $H_{2}(X,\mathbb{Z})$ is an even unimodular lattice of signature $(3,19)$. $H^{+}_{2}(X,\mathbb{Z})$ is a primitive hyperbolic 2-elementary sublattice of $H_{2}(X,\mathbb{Z})$. Note that $H^{+}_{2}(X,\mathbb{Z}) \subset \text{Pic}(X)$, where $\text{Pic}(X)$ denotes the Picard lattice of $X$. 

References

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Let $L_{K3}$ be an even unimodular lattice of signature $(3,19)$ and fix it. Note that the isometry class of $L_{K3}$ is unique. Let
\[ S \subset L_{K3} \]
be a primitive hyperbolic 2-elementary sublattice of $L_{K3}$.

We set $r(S) := \text{rank } S$. The non-negative integer $a(S)$ is defined by $S^*/S \cong (\mathbb{Z}/2\mathbb{Z})^{a(S)}$. We define the “parity” $\delta(S)$ of $S$ as follows.

\[ \delta(S) := \begin{cases} 
0 & \text{if } z \cdot \sigma(z) \equiv 0 \mod 2 \quad (\forall z \in L_{K3}) \\
1 & \text{otherwise},
\end{cases} \]

where $\sigma : L_{K3} \to L_{K3}$ is the unique integral involution whose fixed part is $S$.

It is known that the triplet $(r(S), a(S), \delta(S))$ determines the isometry class of the lattice $S$ [5]. Moreover, if $S$ and $S'$ are isometric primitive hyperbolic 2-elementary sublattices of the K3 lattice $L_{K3}$, then there exists an ambient automorphism $f$ of $L_{K3}$ such that $f(S') = S$ [1, 2].

We fix a half cone $V^+(S)$ of the cone
\[ V(S) := \{ x \in S \otimes \mathbb{R} \mid x^2 > 0 \}. \]

Moreover, we fix a fundamental subdivision
\[ \Delta(S) = \Delta(S)_+ \cup -\Delta(S)_+ \]
of all elements with square $-2$ in $S$.

This is equivalent to fixing a fundamental (closed) chamber (see [7])
\[ M \subset V^+(S) \]
for the group $W(-2)(S)$ generated by reflections in all elements with square $(-2)$ in $S$.

Note that $M$ and $\Delta(S)_+$ define each other by the condition $M \cdot \Delta(S)_+ \geq 0$.

Let $\theta$ be an integral involution of $S$.

**Definition 1.2.** We say that $(X, \tau, \varphi)$ is a real 2-elementary K3 surface of type $(S, \theta)$ if there exists an isometry (so-called “marking” later)
\[ \alpha : H_2(X, \mathbb{Z}) \cong L_{K3} \]
such that $\alpha(H_2^+(X, \mathbb{Z})) = S$ and the following diagram commutes:
\[
\begin{array}{ccc}
H_2^+(X, \mathbb{Z}) & \xrightarrow{\alpha} & S \\
\varphi^* \downarrow & & \downarrow \theta \\
H_2^+(X, \mathbb{Z}) & \xrightarrow{\alpha} & S.
\end{array}
\]

**Definition 1.3** (marked real 2-elementary K3 surfaces). We define that a marked real 2-elementary K3 surface of type $(S, \theta)$ is a pair
\[ ((X, \tau, \varphi), \alpha) \]
of a real 2-elementary K3 surface $(X, \tau, \varphi)$ of type $(S, \theta)$ (Definition 1.2 above) and an isometry, which is called marking,
\[ \alpha : H_2(X, \mathbb{Z}) \cong L_{K3} \]
such that
- $\alpha(H_2^+(X, \mathbb{Z})) = S$,
We call such a pair $(L, L^{-1})$ an involution of the lattice $\psi$ then we have $L$ of $\psi$.

Integral involutions of $\psi$.

1.2. Integral involutions of $\mathbb{L}_{K3}$ of type $(S, \theta)$. Let $S$ be a hyperbolic 2-elementary sublattice of $\mathbb{L}_{K3}$ and $\theta : S \to S$ be an integral involution (as above).

Definition 1.4 (Integral involution $\psi$ of $\mathbb{L}_{K3}$ of type $(S, \theta)$). Let $\psi : \mathbb{L}_{K3} \to \mathbb{L}_{K3}$ be an integral involution of the lattice $\mathbb{L}_{K3}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
S & \subseteq & \mathbb{L}_{K3} \\
\theta & \downarrow & \downarrow \psi \\
S & \subseteq & \mathbb{L}_{K3}.
\end{array}
$$

We call such a pair $(\mathbb{L}_{K3}, \psi)$ (or $\psi$ itself) an integral involution of $\mathbb{L}_{K3}$ of type $(S, \theta)$.

Let $((X, \tau, \varphi), \alpha)$ be a marked real 2-elementary K3 surface of type $(S, \theta)$ as above. If we set

$$
\psi := \alpha \circ \varphi \circ \alpha^{-1} : \mathbb{L}_{K3} \to \mathbb{L}_{K3},
$$

then we have $\psi(S) = S$, and $\psi(x) = \theta(x)$ for every $x \in S$. Hence, $(\mathbb{L}_{K3}, \psi)$ is an integral involution of $\mathbb{L}_{K3}$ of type $(S, \theta)$.

We call the integral involution $(\mathbb{L}_{K3}, \psi)$ of type $(S, \theta)$ the associated integral involution with a marked real 2-elementary K3 surface $((X, \tau, \varphi), \alpha)$ of type $(S, \theta)$ if the following diagram commutes:

$$
\begin{array}{ccc}
H_2(X, \mathbb{Z}) & \xrightarrow{\alpha} & \mathbb{L}_{K3} \\
\varphi \downarrow & & \downarrow \psi \\
H_2(X, \mathbb{Z}) & \xrightarrow{\alpha} & \mathbb{L}_{K3}.
\end{array}
$$

Definition 1.5 (the subgroup $G$). Let $\Delta(S, L)^{(\neg)}$ be the set of all elements $\delta_1$ in $S$ such that $\delta_1^2 = -4$ and there exists $\delta_2 \in S_L^\perp$ such that $(\delta_2)^2 = -4$ and $\delta = (\delta_1 + \delta_2)/2 \in L$. Let $W^{(\neg)}(S, L)$ be the subgroup of $O(S)$ generated by reflections in all elements in $\Delta(S, L)^{(\neg)}$, and $W^{(\neg)}(S, L)_M$ be the stabilizer subgroup of $M$ in $W^{(\neg)}(S, L)$. We define the subgroup $G$ to be generated by reflections $s_{\delta_1}$ in all elements $\delta_1 \in \Delta(S, L)^{(\neg)}$ which are contained either in $S_+$ or in $S_-$ and satisfy $(s_{\delta_1})_R(M) = M$, where $s_{\delta_1}$ denotes the reflection at the orthogonal hyperplane $\delta_1^{\perp}$ on $S$, $(s_{\delta_1})_R$ stands for the real extension of $s_{\delta_1}$, and we set $S_{\pm} := \{ x \in S \mid \theta(x) = \pm x \}$. Then $G$ is a subgroup of $W^{(\neg)}(S, L)_M$.

Definition 1.6 (Isometries with respect to the group $G$). Let $(\mathbb{L}_{K3}, \psi_1)$ and $(\mathbb{L}_{K3}, \psi_2)$ be two integral involutions of $\mathbb{L}_{K3}$ of type $(S, \theta)$. We define that an isometry with respect to the group $G$ from $(\mathbb{L}_{K3}, \psi_1)$ to $(\mathbb{L}_{K3}, \psi_2)$ is an isometry $f : \mathbb{L}_{K3} \to \mathbb{L}_{K3}$ such that $f(S) = S$, $f|_S \in G$, and the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{L}_{K3} & \xrightarrow{f} & \mathbb{L}_{K3} \\
\psi_1 & \downarrow & \psi_2 \\
\mathbb{L}_{K3} & \xrightarrow{f} & \mathbb{L}_{K3}.
\end{array}
$$
We say that two integral involutions \((L_{K3}, \psi_1)\) and \((L_{K3}, \psi_2)\) of type \((S, \theta)\) are isometric with respect to the group \(G\) if there exists an isometry with respect to the group \(G\) from \((L_{K3}, \psi_1)\) to \((L_{K3}, \psi_2)\). By an automorphism of an integral involution \((L_{K3}, \psi)\) of type \((S, \theta)\) with respect to the group \(G\) we mean an isometry with respect to the group \(G\) from \((L_{K3}, \psi)\) to itself. Namely, an isometry \(f : L_{K3} \to L_{K3}\) which satisfies that \(\psi \circ f = f \circ \psi\), \(f(S) = S\) and \(f|_S \in G\).

**Definition 1.7** (analytic isomorphisms with respect to \(G\)). We say that two marked real 2-elementary K3 surfaces \(((X, \tau, \varphi), \alpha)\) and \(((X', \tau', \varphi'), \alpha')\) of type \((S, \theta)\) are analytically isomorphic with respect to the group \(G\) if there exists an analytic isomorphism \(f : X \to X'\) such that \(f \circ \tau = \tau' \circ f\), \(f \circ \varphi = \varphi' \circ f\) and \(\alpha' \circ f_* \circ \alpha^{-1}|_S \in G\).

### 1.3. Period domains.

Now let us fix an integral involution \((L_{K3}, \psi)\) of type \((S, \theta)\) throughout this subsection.

We follow the formulations of period domains of marked real 2-elementary K3 surfaces (see Itenberg [3] and Nikulin-Saito [7]).

We set \(\Omega_\psi := \{\omega \in L_{K3} \otimes \mathbb{C} \mid \omega \cdot \omega = 0, \ \omega \cdot \bar{\omega} > 0, \ \omega \cdot S = 0, \ \psi_C(\omega) = \bar{\omega}\}/\mathbb{R}^\times\).

Let \(((X, \tau, \varphi), \alpha)\) be a marked real 2-elementary K3 surface of type \((S, \theta)\) satisfying
\[
\alpha \circ \varphi_* \circ \alpha^{-1} = \psi,\]
and let \(H \subset H_2(X, \mathbb{C})\) be the Poincare dual of \(H^{2,0}(X)\). The 1-dimensional subspace \(\alpha_C(H)\) of \(L_{K3} \otimes \mathbb{C}\) is regarded as an element of \(\Omega_\psi\).

**Definition 1.8.** We call \(\alpha_C(H)\) the period of a marked real 2-elementary K3 surface \(((X, \tau, \varphi), \alpha)\) of type \((S, \theta)\) satisfying \(\alpha \circ \varphi_* \circ \alpha^{-1} = \psi\).

**Definition 1.9** (Equivalence). We say that a point \([\omega] \in \Omega_\psi\) is equivalent to a point \([\omega'] \in \Omega_\psi\) if \([\omega'] = f_C([\omega])\) for an automorphism \(f \) of \((L_{K3}, \psi)\) of type \((S, \theta)\) with respect to the group \(G\).

**Lemma 1.10** ([9]). If a point \([\omega] \in \Omega_\psi\) is equivalent to \([\omega'] \in \Omega_\psi\) and \([\omega] = \psi\) is the period of some marked real 2-elementary K3 surface \(((X, \tau, \varphi), \alpha)\) of type \((S, \theta)\) satisfying \(\alpha \circ \varphi_* \circ \alpha^{-1} = \psi\), then \([\omega']\) is also the period of a marked real 2-elementary K3 surface \(((X, \tau, \varphi), \alpha')\) of type \((S, \theta)\) satisfying \(\alpha' \circ \varphi_* \circ (\alpha')^{-1} = \psi\) where \(\alpha'\) is another marking of \((X, \tau, \varphi)\).

Using the global Torelli theorem, if two periods are equivalent, then corresponding marked real 2-elementary K3 surfaces are analytically isomorphic (see Definition 1.7). The converse is also true.

The domain \(\Omega_\psi\) has two connected components which are interchanged by \(-\psi\). Since \(-\psi\) is an automorphism of \((L_{K3}, \psi)\) with respect to the group \(G\), by Lemma 1.10 it is enough to investigate the quotient space
\[
\Omega_\psi/ -\psi.
\]

Now we set
\[
L^\psi := \{x \in L_{K3} \mid \psi(x) = x\}, \quad \mathbb{L}_\psi := \{x \in L_{K3} \mid \psi(x) = -x\}.
\]

We restrict ourselves to the case
\[
S \subset \mathbb{L}_\psi, \quad \text{i.e.,} \quad \theta = -\text{id},
\]

---

1 All marked real 2-elementary K3 surfaces whose associated integral involutions are isometric to \((L_{K3}, \psi)\) with respect to \(G\) satisfy \(\alpha \circ \varphi_* \circ \alpha^{-1} = \psi\) if we change their markings appropriately (see [9]).
where “id” stands for the identity map on $S$. We set

$$\mathbb{L}_-, S := \mathbb{L}_\psi \cap S^\perp.$$  

For $[\omega] \in \Omega_\psi$ ($\omega \in \mathbb{L}_{K3} \otimes \mathbb{C}$), we consider the orthogonal decomposition $\omega = \omega_+ + i \omega_-$ ($\omega_+ \in \mathbb{L}_\psi \otimes \mathbb{R}$, $\omega_- \in \mathbb{L}_\psi \otimes \mathbb{R}$). Then we have $\omega_- \in \mathbb{L}_-, S \otimes \mathbb{R}$ and $\omega_2^2 = \omega_-^2 > 0$. Hence, both $\mathbb{L}_\psi$ and $\mathbb{L}_-, S$ are hyperbolic lattices. We set $V(\mathbb{L}_\psi) := \{ x \in \mathbb{L}_\psi \otimes \mathbb{R} \mid x^2 > 0 \}$. $V(\mathbb{L}_\psi)$ has two connected components. Let $V^+(\mathbb{L}_\psi)$ be one of those (half cone). Let $\mathcal{L}_+$ denote the set of all rays (half lines) through 0 in $V^+(\mathbb{L}_\psi)$. ($\mathcal{L}_+$ is called the hyperbolic (or Lobachevsky) space obtained from $\mathbb{L}_\psi$. ) We define the hyperbolic space $\mathcal{L}_-, S$ obtained from $\mathbb{L}_-, S$ in the same way.

Then we have the following identification:

(1.1)  
$$\Omega_\psi \cdot \psi = \mathcal{L}_+ \times \mathcal{L}_-, S \quad \text{(a direct product)}.$$

1.4. ($D\mathbb{R}$)-nondegenerate marked real 2-elementary K3 surfaces.

**Definition 1.11.** We say that an element $x(\neq 0) \in \text{Pic}(X) \otimes \mathbb{R}$ is nef (for $X$) if $x \cdot C \geq 0$ for every effective curve $C \in \text{Pic}(X)$.

**Definition 1.12 ([7]).** (i) We say that a marked 2-elementary K3 surface $((X, \tau), \alpha)$ of type $S$ is ($D$)-degenerate if there exists an element $x_0 \in \alpha^{-1}_\mathbb{R}(\mathcal{M})$ which is not nef. Namely, $x_0 \cdot C < 0$ for an effective curve $C \in \text{Pic}(X)$. This condition is equivalent (see [6], [1]) to the existence of an irreducible $(-2)$-curve on the quotient surface $Y := X/\tau$. And this condition is also equivalent to the existence of an element $\delta \in \text{Pic}(X)$ with $\delta^2 = -2$ such that $\delta = (\delta_1 + \delta_2)/2$ where $\delta_1 \in \alpha^{-1}(S)$, $\delta_2 \in \alpha^{-1}(S)_{\text{Pic}(X)}^\perp$, and $\delta_1^2 = \delta_2^2 = -4$. (ii) We say that a marked real 2-elementary K3 surface $((X, \tau, \varphi), \alpha)$ of type $(S, \theta)$ is ($D\mathbb{R}$)-degenerate if there exists a “real” element $x_0 \in \alpha^{-1}_\mathbb{R}(S_- \cap \mathcal{M})$ which is not nef, where we set $S_{\pm} := \{ x \in S \mid \theta(x) = \pm x \}$. This condition is equivalent to the existence of an element $\delta \in \text{Pic}(X)$ with $\delta^2 = -2$ such that $\delta = (\delta_1 + \delta_2)/2$ where $\delta_1 \in \alpha^{-1}(S)$, $\delta_2 \in \alpha^{-1}(S)_{\text{Pic}(X)}^\perp$ and $\delta_1^2 = \delta_2^2 = -4$, and $\delta_1$ is orthogonal to an element $x \in \alpha^{-1}_\mathbb{R}(S_- \cap \text{int}(\mathcal{M}))$, where int$(\mathcal{M})$ denote the interior part of $\mathcal{M}$, i.e., the polyhedron $\mathcal{M}$ without its faces.

Considering associated integral involutions, we have:

**Theorem 1.13 ([7]).** The natural map gives a bijective correspondence between the connected components of the period domain of ($D\mathbb{R}$)-nondegenerate marked real 2-elementary K3 surfaces of type $(S, \theta)$ and the isometry classes with respect to $G$ of integral involutions of $\mathbb{L}_{K3}$ of type $(S, \theta)$ such that the fixed part $\mathbb{L}_\psi$ of $\psi$ is hyperbolic.

2. Real 2-elementary K3 surfaces of type $((3, 1, 1), -\text{id})$

Now we fix a sublattice $S$ of the K3 lattice $\mathbb{L}_{K3}$ with the invariants

$$(r(S), a(S), \delta(S)) = (3, 1, 1).$$

We consider 2-elementary K3 surfaces of type $S \cong (3, 1, 1)$ ([3], [9]). We quote the following results from Alexeev and Nikulin [1]. See also [8] and [9].

Let $(X, \tau)$ be a 2-elementary K3 surface of type $S \cong (3, 1, 1)$. Let $A := X^\tau$ be the fixed point set (nonsingular complex curve) of $\tau$. Then we have

$$A = A_0 \cup A_1 \quad \text{(disjoint union)},$$

where $A_0$ is a nonsingular rational curve ($\cong \mathbb{P}^1$), and $A_1$ is a nonsingular curve of genus 9.
\((X, \tau)\) has a structure of an elliptic pencil \(|E + F|\), and \(\tau\) is the inversion map of the group structure of the elliptic pencil with the zero section \(A_0\).

The unique reducible fiber \(E + F\) having the following properties:

(i): \(E\) is a nonsingular rational curve \((\cong \mathbb{P}^1)\) and \(E \cdot A_0 = 1\).

(ii): \(E \cdot F = 2, \ F^2 = -2, \ F \cdot A_0 = 0, \) and \(F\) is either
a nonsingular rational curve ("type Ia case"), or
the union of two nonsingular rational curves \(F'\) and \(F''\) which are conjugate by \(\tau, \ F' \cdot F'' = 1\)
("type Ib case").

(iii): The classes \([A_0], \ [E]\) and \([F]\) generate the lattice \(H_{2+}(X, \mathbb{Z}) (\cong \mathbb{S})\). Moreover, \(A_1 \cdot E = 1, \ A_1 \cdot F = 2\). The Gram matrix of the lattice \(H_{2+}(X, \mathbb{Z})\) with respect to the basis \([E], \ [F]\) and \([A_0]\) is

\[
\begin{bmatrix}
[E] & [F] & [A_0] \\
[E] & -2 & 2 & 1 \\
[F] & 2 & -2 & 0 \\
[A_0] & 1 & 0 & -2.
\end{bmatrix}
\]

Then we have an orthogonal decomposition

\(H_{2+}(X, \mathbb{Z}) = \mathbb{Z}([A_0], \ [E] + [F]) \oplus \mathbb{Z}([F]),\)

where the subgroups \(\mathbb{Z}([A_0], \ [E] + [F])\) and \(\mathbb{Z}([F])\) are isometric to the hyperbolic plane and \(-1\) respectively.

We now consider the quotient surface \(Y := X/\tau\) (so-called "DPN surface" \([7]\)) and let \(\pi : X \to Y\) be the quotient map. We define the curves on \(Y\) as follows:

\[e := \pi(E) \quad \text{and} \quad f := \pi(F).\]

If \(F\) is the union of two nonsingular rational curves \(F'\) and \(F''\) which are conjugate by \(\tau\) and \(F' \cdot F'' = 1\), then we have \(f = \pi(F) = \pi(F' \cup F'') = \pi(F') = \pi(F'').\)

We use the same symbols \(A_0\) and \(A_1\) for their images in \(Y\) by \(\pi\). Then, the Picard group \(\text{Pic}(Y)\) of \(Y\) is generated by the classes \([e], \ [f]\) and \([A_0]\). The Gram matrix of \(\text{Pic}(Y)\) with respect to the basis \([e], \ [f]\) and \([A_0]\) is

\[
\begin{bmatrix}
[e] & [f] & [A_0] \\
[e] & -1 & 1 & 1 \\
[f] & 1 & -1 & 0 \\
[A_0] & 1 & 0 & -4.
\end{bmatrix}
\]

We have \(A_1 \cdot e = 1\) and \(A_1 \cdot f = 2\).

We next contract the exceptional curve \(f = \pi(F)\) to a point. Then we get a blow up

\[\text{bl} : Y \to \mathbb{F}_4,\]

where \(\mathbb{F}_4\) is the 4-th Hirzebruch surface. (See \([8]\).)

We set \(s := \text{bl}(A_0), \ A'_1 := \text{bl}(A_1), \ c := \text{bl}(e)\). Then \(s\) is the exceptional section of \(\mathbb{F}_4\) with \(s^2 = -4\), and \(c\) is a fiber of the fibration \(\mathbb{F}_4 \to s\) with \(c^2 = 0\).

We have

\[\text{bl}(A) = \text{bl}(A_0) + \text{bl}(A_1) = s + A'_1 \in |-2K_{\mathbb{F}_4}|.\]

Namely, \(\text{bl}(A)\) is an anti-bicanonical curve on \(\mathbb{F}_4\). Since \(s \cdot A'_1 = 0\), \(A'_1\) does not intersect the section \(s\). Since \(-2K_{\mathbb{F}_4} \sim 12c + 4s\), we have \(A'_1 \in |12c + 3s|\).
Remark 2.1 ([8]). For any real 2-elementary K3 surface \((X, \tau, \varphi)\) of type \((S, \theta)\) with \(S \cong (3, 1, 1)\), we have
\[
\theta = -\text{id},
\]
and
\[
G = \{\text{id}\}.
\]
It is known that all real 2-elementary K3 surfaces of type \(((3, 1, 1), -\text{id})\) are \((\mathcal{D}\mathbb{R})\)-non-degenerate.

Remark 2.2 ([8], [9]). \(F\) is a nonsingular rational curve if and only if \(A_1\) intersects \(f\) in two distinct points, and \(F\) is a union of two nonsingular rational curves if and only if \(A_1\) touches \(f\).

If \(A_1\) intersects with \(f\) at two distinct points, then they are real points or non-real conjugate points. In the former case the real double point of \(A'_1\) is a real node, and in the latter case it is a real isolated point. Anyway the real double point of \(A'_1\) is nondegenerate. If \(A_1\) touches to \(f\) in \(Y\), then the real double point is a real cusp (a degenerate double point).

Since all real 2-elementary K3 surfaces of type \(((3, 1, 1), -\text{id})\) are \((\mathcal{D}\mathbb{R})\)-non-degenerate, by Theorem 1.13, the connected components of the period domain of marked real 2-elementary K3 surfaces of type \(((3, 1, 1), -\text{id})\) and the isometry classes with respect to \(G = \{\text{id}\}\) of integral involutions of \(L_{K3}\) of type \(((3, 1, 1), -\text{id})\) such that the fixed part \(L_\psi\) of \(\psi\) is hyperbolic are in bijective correspondence.

However, as is written in Remark 8 in Subsection 2.4 in [8], it is possible that an isometry class of integral involutions of \(L_{K3}\) of type \(((3, 1, 1), -\text{id})\) corresponds to both type IIa case and type IIb case. In other words, such isometry classes with respect to \(G\) (in the sense of Theorem 1.13) cannot distinguish degenerate and nondegenerate double points of the curves \(A'_1\) on \(F_4\) (Recall Remark 2.2).

Therefore, we would like to determine whether the real double point of the curve \(A'_1\) on \(F_4\) is degenerate or not. See also Lemma 4.6 and Problem 1 in [9]. In order to do this, we define more strict markings of real 2-elementary K3 surfaces of type \(((3, 1, 1), -\text{id})\). Compare the following Definition 2.3 to the Definition 1.3 above.

Let \(E, F, A\) be generators of \(S\) with the Gram matrix
\[
\begin{pmatrix}
E & F & A \\
E & -2 & 2 & 1 \\
F & 2 & -2 & 0 \\
A & 1 & 0 & -2
\end{pmatrix}
\]
where \(M \cdot E \geq 0, M \cdot F \geq 0, \) and \(M \cdot A \geq 0.\)

And we set
\[
\mathbb{U} := \mathbb{Z}(A, E + F).
\]
\(\mathbb{U}\) is isometric to the hyperbolic plane, and
\[
S = \mathbb{U} \oplus \mathbb{Z}(F)
\]
is an orthogonal decomposition of \(S\).

Definition 2.3 (marked real 2-elementary K3 surfaces \(((X, \tau, \varphi), \alpha)\) of type \(((3, 1, 1), -\text{id})\)). We define that a marked real 2-elementary K3 surfaces \(((X, \tau, \varphi), \alpha)\) of type \(((3, 1, 1), -\text{id})\) is a pair of a real 2-elementary K3 surface \((X, \tau, \varphi)\) of type \(((3, 1, 1), -\text{id})\) (Definition 1.2) and a marking (isometry)
\[
\alpha : H_2(X, \mathbb{Z}) \cong L_{K3}
\]
such that
\[
\begin{align*}
\alpha(H_{2+}(X, \mathbb{Z})) &= S, \\
\alpha \circ \varphi_* &= \theta \circ \alpha \text{ on } H_{2+}(X, \mathbb{Z}),
\end{align*}
\]
\(\alpha^{-1}(V^+(S))\) contains a hyperplane section of \(X\),
\(\alpha^{-1}(\Delta(S)_+)\) contains only classes of effective curves of \(X\),
\(\alpha([A_0]) = \mathcal{A}, \alpha([E]) = \mathcal{E}\), and \(\alpha([F]) = \mathcal{F}\).

Note that every real 2-elementary K3 surface \((X, \tau, \varphi)\) of type \(((3,1,1), -\text{id})\) has such a marking \(\alpha\).

We give a criterion for the unique double point of a real anti-bicanonical curve \(bl(A) = s + A'_1\) on \(\mathbb{F}_4\) with one real double point on \(A'_1\) to be nondegenerate.

Let us consider and fix an integral involution
\[
(\mathbb{L}_{K3}, \psi)
\]
of type \(((3,1,1), -\text{id})\) again, and consider the period domain \((1.1)\) in Subsection 1.3
\[
\Omega_\psi/ - \psi = \mathcal{L}_+ \times \mathcal{L}_{-S}.
\]

**Proposition 2.4.** Let \([\omega]\) be a point \(3\) in \(\Omega_\psi/ - \psi\). Then, the real double point of the curve \(A'_1\) on \(\mathbb{F}_4\) (Remark [2.2]) which corresponds to \(((X, \tau, \varphi), \alpha)\) is nondegenerate if and only if there are no \(\nu\) \((\neq \pm F)\) in \(\mathbb{L}_{K3}\) satisfying:
\[
\nu \cdot \omega = 0, \quad \nu \cdot U = 0, \quad \text{and} \quad \nu^2 = -2.
\]

**Proof.** The \((\Leftarrow)\) direction has been proved in Lemma 4.6 of [9]. Here we prove the \((\Rightarrow)\) direction. Assume that the double point of the real anti-bicanonical curve \(bl(A)\) on \(\mathbb{F}_4\) is nondegenerate. Then \(F\) is irreducible ([8], [9]). Suppose that there exists a \(\nu\) \((\neq \pm F)\) in \(\mathbb{L}_{K3}\) satisfying that
\[
\nu \cdot \omega = 0, \quad \nu \cdot U = 0, \quad \text{and} \quad \nu^2 = -2.
\]
Then \(-\nu\) has the same properties. By Riemann-Roch Theorem, we have
\[
l(\alpha^{-1}(\nu)) + l(-\alpha^{-1}(\nu)) \geq \alpha^{-1}(\nu)^2/2 + 2 = 1.
\]
Hence, \(\alpha^{-1}(\nu)\) or \(-\alpha^{-1}(\nu)\) is effective. (See [1], p.23 or [2], p.311.) Hence, we may assume \(\alpha^{-1}(\nu)\) is effective. Then, \(\tau^*(\alpha^{-1}(\nu))\) is also effective. We have
\[
\tau^*(\alpha^{-1}(\nu))^2 = 28, \quad \tau^*(\alpha^{-1}(\nu)) \cdot [A_0] = 0, \quad \tau^*(\alpha^{-1}(\nu)) \cdot ([E] + [F]) = 0, \quad \text{and} \quad \alpha(\tau^*(\alpha^{-1}(\nu))) \cdot \omega = 0.
\]
This is a contradiction.

If \(\tau^*(\alpha^{-1}(\nu)) = \alpha^{-1}(\nu)\), then \(\tau^*(\alpha^{-1}(\nu)) + \alpha^{-1}(\nu)^2 = (2\alpha^{-1}(\nu))^2 = -8\). On the other hand, \((\tau^*(\alpha^{-1}(\nu)) + \alpha^{-1}(\nu))^2 = (n[F])^2 = -2n^2\). Hence, \(n = 2\). Thus we have \(\nu = \mathcal{F}\). This is a contradiction. Thus we have \(\tau^*(\alpha^{-1}(\nu)) \neq \alpha^{-1}(\nu)\).

Let \(\alpha^{-1}(\nu) = \sum_i \nu_i \nu_i\), where \(\nu_i\) are positive integers and \(\nu_i\) are irreducible effective classes.

Obviously we have \(\alpha(\nu_i) \cdot \omega = 0\) for any \(i\).

If \(\nu_i = [E]\) or \([F]\), then \(\nu_i \cdot ([E] + [F]) = 0\). And if \(\nu_i \neq [E]\) and \(\neq [F]\), then we have \(\nu_i \cdot ([E] + [F]) \geq 0\) for any \(i\). Here \(E\) is irreducible, and moreover, \(F\) is irreducible by the assumption.
Since \(\alpha^{-1}(\nu_i) \cdot ([E] + [F]) = 0\), we have \(\nu_i \cdot ([E] + [F]) = 0\) for any \(i\). Since \([A_0] \cdot ([E] + [F]) = 1\), we see \(\nu_i \neq [A_0]\) for any \(i\). Thus we also have \(\nu_i \cdot [A_0] \geq 0\). Since \(\alpha^{-1}(\nu) \cdot [A_0] = 0\), we have \(\nu_i \cdot [A_0] = 0\) for any \(i\).

\[2\] By the surjectivity of the period map of marked K3 surfaces ([2], p.339), \([\omega]\) is the period of a marked real
2-elementary K3 surface \(((X, \tau, \varphi), \alpha)\) of type \(((3,1,1), -\text{id})\) satisfying \(\alpha \circ \varphi \circ \alpha^{-1} = \psi\).
Thus, we have $\alpha(v_i) \cdot U = 0$ for any $i$. Since $U$ is of signature $(1,1)$, we have $(v_i)^2 < 0$ for any $i$ by the Hodge index theorem. For such $v_i$'s, we have $v_i^2 = -2$, and $v_i$'s are nonsingular rational ($\cong \mathbb{P}^1$).

Suppose $\alpha(v_i) = F$ for some $i$, say $i = 1$. Namely, $v_1 = [F]$. Let $v' := \alpha^{-1}(v) - m_1[F]$. Since $\tau^*(\alpha^{-1}(v)) \neq \alpha^{-1}(v)$ (see above), we have $v' \neq 0$. Thus, there exists a $v_i(\neq \pm[F])$ in $H_2(X, \mathbb{Z})$ such that $v_i \cong \mathbb{P}^1$ (irreducible), $v_i^2 = -2$, $\alpha(v_i) \cdot \omega = 0$, and $\alpha(v_i) \cdot U = 0$.

Eventually we can choose $v$ such that $\alpha^{-1}(v)$ is an irreducible class.

Since $\tau^*(\alpha^{-1}(v)) \neq \alpha^{-1}(v)$, $\tau^*(\alpha^{-1}(v))$ and $\alpha^{-1}(v)$ are represented by different irreducible curves respectively. Hence, we have $\tau^*(\alpha^{-1}(v)) \cdot \alpha^{-1}(v) \geq 0$.

Since $(\alpha(\tau^*(\alpha^{-1}(v))) + v)^2 = (-2) + (-2) + 2(\tau^*(\alpha^{-1}(v)) \cdot \alpha^{-1}(v)) = -2n^2$, we have $2 - n^2 \geq 0$. Thus, we have $n = 1$. Namely, we have

$$F = v + \alpha(\tau^*(\alpha^{-1}(v))),$$
i.e., $[F] = \alpha^{-1}(v) + \tau^*(\alpha^{-1}(v))$. Let $F'$ be an irreducible curve representing $\alpha^{-1}(v)$, and we set $F'' := \tau^*(F')$. Thus $F' + F''$ represents the class $[F]$. Hence, there exists a marked real 2-elementary K3 surface corresponding to the period $[\omega]$ which has $E + F' + F''$ as the unique reducible fiber of its elliptic fibration. Conversely, every marked real 2-elementary K3 surface corresponding to the period $[\omega]$ has $E + F' + F''$ as the unique reducible fiber of its elliptic fibration. Hence, we have $F = F' + F''$. This contradicts the assumption that $F$ is irreducible. This completes the proof of Proposition 2.4.\hfill $\square$

**Remark 2.5.** For curves of degree 6 with one double point on $\mathbb{R}P^2$, the corresponding criterion to Proposition 2.4 is written on the top of p. 281 in Itenberg’s paper [3].

As is written in [9], we now get the precise image ($\subset \mathcal{L}_+ \times \mathcal{L}_-$) of the period map on the set of all marked real 2-elementary K3 surfaces $((X, \tau, \varphi), \alpha)$ of type $((3,1,1), -id)$ for which the real double points of the curves $A_i'$ on $\mathbb{F}_4$ are nondegenerate. Thus we are able to continue the interesting arguments as in [3].

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**References**

[1] V.A. Alexeev, V.V. Nikulin, Del Pezzo and K3 Surfaces, MSJ Memoirs, 2006.
[2] W.P. Barth, K. Hulek, C.A.M. Peters, and A. Van de Ven, Compact Complex Surfaces, Springer, 2004.
[3] I. Itenberg, Curves of degree 6 with one non-degenerate double point and groups of monodromy of non-singular curves, Real Algebraic Geometry, Proceedings, Rennes 1991, Lecture Notes in Math., 1524, Springer, (1992), 267–288.
[4] V.V. Nikulin, Integral symmetric bilinear forms and some of their geometric applications, Math. USSR Izv., 14 (1980), 103–167.
[5] V.V. Nikulin, On the quotient groups of the automorphism groups of hyperbolic forms by the subgroups generated by 2-reflections, Algebraic-geometric applications, J. Soviet Math., 22 (1983), 1401–1476.
[6] V.V. Nikulin, Discrete reflection groups in Lobachevsky spaces and algebraic surfaces, Proc. Intern. Congr. Math. Vol. 1 (Berkeley, 1986) (Providence, RI), Amer. Math. Soc., (1987), 654–671.
[7] V.V. Nikulin, Sachiko Saito, Real K3 surfaces with non-symplectic involution and applications, Proc. London Math. Soc., 90 (2005), 591–654.
[8] V.V. Nikulin, Sachiko Saito, Real K3 surfaces with non-symplectic involution and applications. II, Proc. London Math. Soc., 95 (2007), 20–48.
[9] Sachiko Saito, On real anti-bicanonical curves with one double point on the 4-th real Hirzebruch surface, Journal of Singularities 11 (2015), 1–32.
