Let $X$ be a compact Kähler manifold and let $\mathcal{A}(X)_{cl}$ denote the space of closed forms on $X$. Each choice of a Kähler form $\omega$ on $X$ defines a Hodge decomposition of $\mathcal{A}(X)_{cl}$ into the subspace of $d$-exact forms and the subspace of harmonic forms $\mathcal{H}_\omega$. How does the subspace $\mathcal{H}_\omega \subset \mathcal{A}(X)_{cl}$ depend on $\omega$? In this paper we study the infinitesimal variation of $\mathcal{H}_\omega$.

The most interesting situation to which the results apply and which was the original motivation for this work is the following: Let $X$ be a Calabi-Yau manifold and $\mathcal{K}^0$ be the set of all Ricci-flat Kähler forms. Due to Calabi and Yau, the natural projection to cohomology defines a bijection between $\mathcal{K}^0$ and the Kähler cone $\mathcal{K}_X \subset H^{1,1}(X, \mathbb{R})$ of all Kähler classes. Thus, varying the Ricci-flat Kähler structure on $X$ is equivalent to moving in the open subset $\mathcal{K}_X$ of the affine space $H^{1,1}(X, \mathbb{R})$. Assuming Mirror Symmetry, changing the Kähler class in $\mathcal{K}_X$ corresponds to deforming the complex structure of the mirror partner of $X$. The space of all complex structures on the mirror partner is, a priori, a highly non-linear object. So, one has tried to introduce linear coordinates on this space corresponding to the linear coordinates on $H^{1,1}(X, \mathbb{R})$ in the A-model [6]. However, it turns out that passing from Ricci-flat Kähler forms to Kähler classes one loses useful and interesting information. In [2] we have shown that there is a relation, though still mysterious, between the shape of $\mathcal{K}^0$, the question whether the top exterior power of harmonic $(1,1)$-forms is harmonic, and the geometry of $X$, in particular the existence of extremal curves. In this paper we will show that the infinitesimal variation of the space $\mathcal{H}_\omega$ of harmonic forms is also related to the (non)-harmonicity of the product of harmonic forms. Thm. 3.9 describes the infinitesimal variation of $\mathcal{H}_\omega$ in general terms. One immediate consequence of it is the following result, which is a special case of Cor. 3.10.

Let $X$ be a compact Kähler manifold of dimension $N$ and let $\omega$ be a Kähler form on $X$. If $\alpha$ is harmonic with respect to $\omega$, then $\alpha$ is harmonic with respect to an infinitesimal change of $\omega$ by a closed $(1,1)$-form $v$ if and only if the product $\alpha \omega^{N-2}$ is harmonic with respect to $\omega$.
As \((\mathcal{H}^{1,1})_\mathbb{R}\) is the tangent space of the set \(\mathcal{K}^0\) of all Ricci-flat Kähler forms at \(\omega\),
the formula in Thm. 3.9 can also be used to calculate the second fundamental form of \(\mathcal{K}^0\) inside \(\mathcal{A}^{1,1}(X)_\text{cl}\).
Fixing a scalar product on \(H^{1,1}(X, \mathbb{R})\) the mean curvature then associates to each Ricci-flat Kähler form a unique exact \((1, 1)\)-form.

The method to prove these results is almost entirely algebraic and of some interest in itself. Due to the fortunate interplay between the Hodge \(\ast\)-operator, needed to define the Laplacian, and the Lefschetz operator, one can make use of general results on the variation of \(\mathfrak{sl}_2\)-representations presented in Sect. 1. Let \(\oplus_{n=0}^N A^n\) be a graded algebra and let \(\rho: \mathfrak{sl}_2 \to \text{End}(\oplus_{n=0}^N A^n)\) be a given representation compatible with the grading and the algebra structure (cf. Def. 1.1). Such a representation is given by the image \(L\) and \(\Lambda\) of the standard generators of \(\mathfrak{sl}_2\), respectively. We will consider infinitesimal changes \(\rho_\varepsilon\) of \(\rho\) such that \(L\) deforms to \(L + \varepsilon v\), where \(v \in A^2\) acts by multiplication. Prop. 1.3 and 1.6 show that the Lefschetz (or primitive) decomposition associated to any \(\mathfrak{sl}_2\)-representation changes in a controlled manner when \(\rho\) is deformed to such a \(\rho_\varepsilon\).

Besides its application to the variation of the space of harmonic forms the precise formulae also shed some light on the question whether the Kähler Lie algebra of \(X\) has the Jordan-Lefschetz property. The Kähler Lie algebra of a compact Kähler manifold has recently been introduced by Looijenga and Lunts [3]. Instead of looking at just one Kähler class \(\omega \in \mathcal{K}_X\) and its associated \(\mathfrak{sl}_2\)-representation \((L_\omega, \Lambda_\omega, B)\), they study the Lie algebra \(\mathfrak{g}\) that is generated by all \(L_\omega\) and \(\Lambda_\omega\) for \(\omega \in \mathcal{K}_X\). Whereas \(L_\omega\) and \(L_{\omega'}\) clearly commute for different \(\omega, \omega' \in \mathcal{K}_X\), the operators \(\Lambda_\omega\) and \(\Lambda_{\omega'}\) in general do not. If they do commute then the Lie algebra \(\mathfrak{g}\) has the Jordan-Lefschetz property and Lie algebras of this type have been classified in [3].

For tori and hyperkähler manifolds the Jordan-Lefschetz property can be verified, but for no other class of manifolds it is known to hold. As it turns out, however, the infinitesimal Jordan-Lefschetz property always holds. More precisely, we prove (Thm. 2.2):

\begin{quote}
Let \(X\) be a compact Kähler manifold and let \(\omega\) be a Kähler class on \(X\). If \(v\) is a real \((1, 1)\)-class, then the contraction operators \(\Lambda_\omega\) and \(\Lambda_{\omega+\varepsilon v}\) associated with the Kähler class \(\omega\) and its infinitesimal deformation \(\omega + \varepsilon v\) commute, i.e. \([\Lambda_\omega, \Lambda_{\omega+\varepsilon v}] = 0\).
\end{quote}

Of course, in order to decide whether the Kähler Lie algebra has the Jordan-Lefschetz property we need to pass from first order deformations to higher order deformations, which in general will be obstructed.

As an application of the result about the infinitesimal variation of the space of harmonic forms we study various sets of special Kähler forms. Usually, a Kähler class in \(\mathcal{K}_X\) is lifted to the unique Calabi-Yau Kähler form which is distinguished by the property that \(\omega^N\) is a scalar multiple of the fixed volume form. Recently, it has turned out that other forms of the Monge-Ampère equation are interesting as well. Firstly, in [4] the B-field is chosen to be a real closed \((1, 1)\)-form \(\beta\) such that \((\omega_0 + i\beta)^N\) is a scalar multiple of the volume form.
Secondly, Leung [3] has studied a modified version of the Hermite-Einstein equation which is related to Gieseker-Maruyama stability (rather than to slope-stability). It is an equation for the powers of \((\omega_0 + tR)\), where \(t\) is a small imaginary parameter and \(R\) is the curvature.

So we propose to look at the set \(K^i\) of all Kähler forms \(\omega\) such that \(\omega^{N-i}\omega^i_0\) is a scalar multiple of the volume form. These are the coefficients of \((\omega_0 + t\omega)^N\). For \(i = 0\) one obtains the set of Calabi-Yau Kähler forms and for \(i = N - 1\) the set of \(\omega_0\)-harmonic Kähler forms. For \(0 < i \leq N - 1\) the lift of a Kähler class to such a form is still unique, but might not always exist. The different sets \(K^i\) are related to each other. Each of these sets is suspected to reflect certain geometric properties of the variety \(X\). E.g. we will see that rational curves in the twistor fibres associated to a Ricci-flat Kähler form \(\omega_0\) on a K3 surface prevent Kähler classes from being represented by \(\omega_0\)-harmonic Kähler forms (Prop. 4.4). Of course, many aspects of the interplay between the geometry and the metric structure of Ricci-flat manifolds still remain to be exploited.

The paper is organized as follows. In Sect. 1 we study deformations of compatible \(sl_2\)-representations. Deforming the Lefschetz operator \(L\) to \(L + \varepsilon v\) induces a variation of the primitive decomposition \(\oplus L^j P^{n-2j}\). This variation is measured by the natural map \(L^j P^{n-2j} \rightarrow \bigoplus_{i \neq j} L^i P^{n-2i}\), which is described in Prop. 1.5. The result relies on a formula for the primitive decomposition of \(L^j(\alpha) \in L^j P^{n-2j}\) with respect to \(L + \varepsilon v\) (cf. Prop. 1.6).

In Sect. 2 the results are applied to the standard \(sl_2\)-representation on cohomology induced by the choice of a Kähler class. In particular, the infinitesimal Jordan-Lefschetz property is proved (Thm. 2.2).

In Sect. 3 we introduce the map \(h(v, -) : H_\omega \rightarrow \text{Im}(d^*)\) that measures the change of the space of \(\omega\)-harmonic forms when \(\omega\) is deformed to \(\omega + \varepsilon v\). Thm. 3.3 provides a precise formula for \(h(v, \alpha)\) which involves the exterior derivative of the primitive components of the product \(\alpha v\). The aforementioned criterion that decides whether \(\alpha\) stays harmonic follows from this.

Sect. 4 is a sequel to [2]. We introduce the series \(K^0, K^1, \ldots, K^{N-1}\) of sets of special Kähler forms, where \(K^0\) is the set of Ricci-flat Kähler forms (denoted by \(\bar{K}_X\) in [4]) and \(K^{N-1}\) is the set of Kähler forms that are harmonic with respect to a fixed Kähler form \(\omega_0\). Prop. 4.6 shows that there is a relation between the harmonicity of \(\alpha^{N-i}\omega^i_0\) for harmonic \((1, 1)\)-forms \(\alpha\) and the linearity of \(K^i\). For the set of Ricci-flat Kähler forms \(K^0\) we complement a result of [2] by showing that there is a relation between the linearity of \(K^0\) and the harmonicity of \(\alpha^2\omega^{N-2}\) for harmonic \((1, 1)\)-forms (and not of the top exterior power).

1 Deformations of compatible \(sl_2\)-representations

Let \(A = \bigoplus_{n=0}^{2N} A^n\) be a finite dimensional graded \(\mathbb{C}\)-algebra. We assume that \(A\) is \(\mathbb{Z}/2\mathbb{Z}\)-commutative with respect to the induced \(\mathbb{Z}/2\mathbb{Z}\)-grading \(A = A^{even} \oplus A^{odd}\). A representation
\[ \rho : \mathfrak{sl}_2(\mathbb{C}) \to \text{End}_\mathbb{C}(A) \] is given by the action of the three generators

\[
L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and } B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

**Definition 1.1** — We say that \( \rho \) is compatible with the grading if \( L \) and \( \Lambda \) are homogenous of degree 2 and -2, respectively, and \( B|_{A^n} = (N - n) \cdot \text{id} \). The representation \( \rho \) is compatible with the algebra structure of \( A \) if \( L(\alpha\beta) = L(\alpha)\beta \) for all \( \alpha, \beta \in A \). The representation \( \rho \) is called compatible if it is compatible with the grading and the algebra structure.

Any representation \( \rho \) induces a primitive decomposition of \( A \). If \( \rho \) is compatible with the grading this decomposition respects the grading, i.e. \( A^n = \oplus_{j \geq 0} L^j P^{n-2j} \), where \( P^m := \ker(\Lambda : A^m \to A^{m-2}) \) is the space of primitive elements. Hence, \( A = \oplus_{n=0}^{2N} \oplus_{j \geq 0} L^j P^{n-2j} \).

Recall that \( L^j : A^n \to A^{n+2j} \) is injective for \( j \leq N - n \) and that \( P^n = 0 \) for \( n > N \). Also, \( \alpha \in A^n \) for \( n \leq N \) is primitive if and only if \( L^{N-n+1}(\alpha) = 0 \).

**Example 1.2** — Let \( V \) be a vector space of dimension 2\( N \) with a complex structure \( I \) which respects a scalar product \( \langle \cdot, \cdot \rangle \) on \( V \). The exterior algebra \( \Lambda^* V^* = \oplus_{n=0}^{2N} \Lambda^n V^* \) admits a natural \( \mathfrak{sl}_2 \)-representation for which \( L \) is given by multiplication with the Kähler form \( \omega := \langle I(\cdot), \cdot \rangle \in \Lambda^2 V^* \). This is the natural \( \mathfrak{sl}_2 \)-representation on hermitian exterior algebras.

Let us consider an infinitesimal deformation \( \rho_\varepsilon \) of \( \rho \) given by an infinitesimal deformation of the identity id \( \in \text{End}_\mathbb{C}(A) \), i.e. id + \( \varepsilon \varphi : A[\varepsilon] \to A[\varepsilon] \), where \( \varphi \in \text{End}_\mathbb{C}(A) \). More precisely, \( \rho_\varepsilon(X) = (\text{id} + \varepsilon \varphi)^{-1} \cdot \rho(X) \cdot (\text{id} + \varepsilon \varphi) = \rho(X) + \varepsilon[\rho(X), \varphi] \) for any \( X \in \mathfrak{sl}_2 \). Then the deformation of the generators of \( \mathfrak{sl}_2 \) are of the form \( L_\varepsilon := L + \varepsilon[L, \varphi], \Lambda_\varepsilon = \Lambda + \varepsilon[\Lambda, \varphi], \) and \( B_\varepsilon = B + \varepsilon[B, \varphi] \). Of course, as \( \mathfrak{sl}_2 \)-representations are rigid, the deformation \( \rho_\varepsilon \) as abstract \( \mathfrak{sl}_2 \)-representation is isomorphic to the trivial deformation of \( \rho \), but the action on the fixed vector space \( A \) in general changes.

If \( \rho \) is compatible with the grading, then \( \rho_\varepsilon \) is compatible with the grading if and only if \( \varphi \) is homogenous of degree \( 0 \). Then, \( B_\varepsilon = B \) and \( [L, \varphi] \), respectively \( [\Lambda, \varphi] \) are of degree \( 2 \) and \( -2 \), respectively. Deforming \( \rho \) this way induces a variation of the decomposition \( A = \oplus_{n=0}^{2N} \oplus_{j \geq 0} L^j P^{n-2j} \). We are interested in two aspects of this.

i) The variation of \( L^j P^{n-2j} \) as a subspace of \( A^n \) is determined by the natural linear map induced by \( \varphi \):

\[ \tilde{\varphi} : L^j P^{n-2j} \to \oplus_{i \neq j} L^j P^{n-2i} \]

ii) Any element \( L^j(\alpha) \in L^j P^{n-2j} \) admits a primitive decomposition with respect to \( \rho_\varepsilon \).
In general, as \( \varphi \) could be any homomorphism of degree 0, nothing can be said about i) and ii). But as soon as the algebra structure of \( A \) is taken into account the situation becomes more interesting.

Let us assume that \( \rho \) is a compatible representation on \( A \) and that the deformation \( \rho_\varepsilon = \rho + \varepsilon[\rho, \varphi] \) has the property that the operator \([L, \varphi]\) of degree two is given by multiplication with a vector \( v \in A^2 \), i.e. \( L_\varepsilon = L + \varepsilon v \). It is easy to see that such a deformation \( \rho_\varepsilon \), where \( \varphi \) is homogenous of degree 0, is also compatible. In order to describe the primitive decomposition of \( A \) with respect to \( \rho_\varepsilon \) one has in principle to express \( \Lambda_\varepsilon \) in terms of \( \Lambda \) and \( v \). As there is no easy way to write this down, we are going to compute the primitive decomposition directly in terms of \( L \) and \( v \).

In order to formulate the results we need the following notation.

**Definition 1.3** — For \( v \in A^2 \) and a given representation \( \rho \) one defines the linear maps \( (i \geq 0) \):
\[
Q_i^v : P^m \to P^{m-2i+2}
\]
by the primitive decomposition of the product \( v\alpha = \sum_{i \geq 0} L^i Q_i^v(\alpha) \).

In fact, for compatible representations most of these maps are trivial:

**Lemma 1.4** — Let \( \alpha \in P^m \) and let \( \alpha v = \sum_{i \geq 0} L^i(\beta_i) \) be the primitive decomposition of \( \alpha v \), i.e. \( \beta_i \in P^{m-2i+2} \). Then \( \beta_i = 0 \) for \( i \geq 3 \). Equivalently, \( Q_i^v = 0 \) for \( i \geq 3 \).

*Proof.* If \( \alpha \in P^m \), then \( L^{m+1}(\alpha) = 0 \). Hence, \( L^{m+1}(\alpha v) = L^{m+1}(\alpha)v = 0 \) and, therefore, \( L^{m+1+i}(\beta_i) = 0 \). Since \( L^j \) is injective on \( P^{m-2i+2} \) for \( j \leq N - m + 2i - 2 \), one obtains \( \beta_i = 0 \) for \( i \geq 3 \). \( \square \)

The next two propositions show that infinitesimal deformations of compatible representations of the above form, i.e. \( L_\varepsilon = L + \varepsilon v \) for some \( v \in A^2 \), are rather special with regard to i) and ii).

**Proposition 1.5** — Let \( \rho \) be a compatible \( sl_2 \)-representation and let \( \rho_\varepsilon \) be a compatible infinitesimal deformation of \( \rho \), such that \( L_\varepsilon = L + \varepsilon v \), where \( v \in A^2 \) acts by multiplication. The variation of the subspace \( L^j P^{n-2j} \subset A^n \) is measured by the natural map
\[
\varphi : L^j P^{n-2j} \to \bigoplus_{i \neq j} L^i P^{n-2i}
\]
which is given by \( \varphi(L^j(\alpha)) = (N - n + j + 1)L^{j+1}Q_v^2(\alpha) - jL^{j-1}Q_v^0(\alpha) \).

The description of \( \varphi \) is an immediate consequence of the following
Proposition 1.6 — Under the assumption of the previous proposition the primitive decomposition of $L^j(\alpha)$ for $\alpha \in P^{n-2j}$ with respect to $\rho_\varepsilon$ takes the form

$$L^j(\alpha) = \begin{cases} L^j_{\varepsilon} + (\varepsilon(N - n + j + 1)Q^2_\varepsilon(\alpha)) \\ + \sum_{i=0}^{n-2j}(\varepsilon(N - n + j + 1)Q^2_\varepsilon(\alpha) + jQ^4_\varepsilon(\alpha)) \\ - L^j_{\varepsilon}^{-1}(\varepsilon j \gamma Q^0_\varepsilon(\alpha)). \end{cases}$$

Proof. Using $L^k = L^k + k\varepsilon vL^{k-1}$ one has for any $\gamma \in A$ the following equality

$$L^j(\alpha) = \begin{cases} L^j_{\varepsilon} + \varepsilon(\gamma - \varepsilon L(\gamma)) \\ - j\varepsilon(L^j_{\varepsilon}^{-1}Q^0_\varepsilon(\alpha) + L^j_{\varepsilon}Q^4_\varepsilon(\alpha) + L^j_{\varepsilon}^{-1}Q^2_\varepsilon(\alpha)). \end{cases}$$

Since $P^m = 0$ for $m > N$, we can assume that $n - 2j \leq N$. Furthermore, $L^{N-n+2j+2} : A^{n-2j-2} \to A^{2N-n+2j+2}$ is bijective. Hence, there is a uniquely defined $\gamma \in A^{n-2j-2}$ such that $L^{N-n+2j+2}(\gamma) = (N - n + 2j + 1)L^{N-n+2j}(\alpha)$. Since $\alpha \in P^{n-2j}$, one has

$$L^{N-n+2j+3}(\gamma) = (N - n + 2j + 1)L^{N-n+2j+1}(\alpha) = 0.$$ 

Thus, $\gamma$ is primitive with respect to $\rho$ and, therefore, $\varepsilon \gamma$ is primitive with respect to $\rho_\varepsilon$. Analogously, since the $Q^j_\varepsilon(\alpha)$ are primitive with respect to $\rho$, the $\varepsilon Q^j_\varepsilon(\alpha)$ are $\rho_\varepsilon$-primitive. Moreover, $(\alpha - \varepsilon L(\gamma))$ is seen to be $\rho_\varepsilon$-primitive by the following argument:

$$L^{N-n+2j+1}_{\varepsilon}(\alpha - \varepsilon L(\gamma)) = L^{N-n+2j+1}_{\varepsilon}(\alpha) - \varepsilon L^{N-n+2j+2}_{\varepsilon}(\gamma) + (N - n + 2j + 1)\varepsilon L^{N-n+2j}(\alpha) = 0.$$ 

To conclude it suffices to show that $\gamma = (N - n + 2j + 1)Q^2_\varepsilon(\alpha)$. This follows from

$$L^{N-n+2j+2}_{\varepsilon}(\gamma) = (N - n + 2j + 1)L^{N-n+2j}_{\varepsilon}(\alpha)$$

$$= (N - n + 2j + 1)L^{N-n+2j}_{\varepsilon}(Q^0_\varepsilon(\alpha) + L^j_{\varepsilon}Q^4_\varepsilon(\alpha) + L^j_{\varepsilon}Q^2_\varepsilon(\alpha))$$

$$= (N - n + 2j + 1)L^{N-n+2j+2}_{\varepsilon}(\alpha),$$

due to $Q^j_\varepsilon(\alpha) \in P^{n-2j-2i+2}$, and the injectivity of $L^{N-n+2j+2}$ on $A^{n-2j-2}$. \hfill \square

Examples 1.7 — The two most interesting special cases of the last proposition are $j = 0$ and $n - 2j = 0$. Let $\rho$ and $\rho_\varepsilon$ be as before.

i) If $\alpha \in P^n$, then the primitive decomposition of $\alpha$ with respect $\rho_\varepsilon$ has the form

$$\alpha = L_{\varepsilon}((N - n + 1)\varepsilon Q^2_\varepsilon(\alpha)) + (\alpha - \varepsilon(N - n + 1)LQ^2_\varepsilon(\alpha)).$$

ii) If $\alpha = 1 \in A^0$ and $v \in P^2$, then the primitive decomposition of $L^j(\alpha)$ with respect to $\rho_\varepsilon$ has the form

$$L^j(1) = L^j(\alpha) = L^j_{\varepsilon}(\alpha) + L^j_{\varepsilon}^{-1}(-j\varepsilon v).$$
Corollary 1.8 — A primitive vector \( \alpha \in P^n \) stays primitive with respect to \( L_\varepsilon \) if and only if \( L^{n-n}(\alpha v) = 0 \).

Proof. The vector \( \alpha \in P^n \) is \( L_\varepsilon \)-primitive if and only if \( Q^2_\varepsilon(\alpha) = 0 \) due to the above example. Since \( L^{n-n}(\alpha v) = L^{n-n+2}Q^2_\varepsilon(\alpha) + L^{n-n+1}Q^1_\varepsilon(\alpha) + L^{n-n}Q^0_\varepsilon(\alpha) = L^{n-n+2}Q^2_\varepsilon(\alpha) \) and \( Q^2_\varepsilon(\alpha) = 0 \) if and only if \( L^{n-n+2}Q^2_\varepsilon(\alpha) = 0 \), the result follows from this. \( \square \)

2 Variation of the Lefschetz decomposition

Everything in the last section immediately applies to the Lefschetz decomposition of the cohomology of a Kähler manifold. Let \( X \) be a compact Kähler manifold and let \( [\omega] \in \mathcal{K}_X \subset H^{1,1}(X, \mathbb{R}) \) be a Kähler class. The Lefschetz operator \( L(\alpha) := [\omega] \wedge \alpha \) is part of an \( sl_2 \)-representation on \( H^*(X) \). The Lefschetz decomposition is the primitive decomposition induced by this representation \( H^*(X) = \oplus_{j\geq 0} L^j H^{n-2j}(X)_{\text{prim}} \).

The reason for the existence of an \( sl_2 \)-representation on the cohomology of a Kähler manifold are the Kähler identities. The natural \( sl_2 \)-representations on the hermitian exterior algebras \( \Lambda^* T^*_x \) for all \( x \in X \) (cf. Example 1.2) are compatible with passing to cohomology, as the Laplacian commutes with \( L \) and \( \Lambda \). Although, two \( sl_2 \)-representations on \( \Lambda^* T^*_x \) associated to two Kähler structures commute, this does not hold in general on the level of cohomology, since the two Laplacian are different and \( L \) and \( \Lambda \) for one Kähler structure do not necessarily commute with the Laplacian of the other. However, the argument immediately shows that all \( sl_2 \)-representations on the cohomology of a torus do commute, as the cohomology is isomorphic to the exterior algebra of the tangent space at some point.

Changing the Kähler class \( [\omega] \in \mathcal{K}_X \) changes this decomposition. Since an infinitesimal variation of \( [\omega] \in \mathcal{K}_X \) is of the form \( [\omega] + \varepsilon v \) for some \( v \in H^{1,1}(X, \mathbb{R})_{\text{prim}} \), we can apply Prop. L0. In particular, we have

Corollary 2.1 — Let \( \alpha \in H^{n-2j}(X)_{\text{prim}} \). Then the Lefschetz decomposition of \( L^j(\alpha) \) with respect to \( [\omega] + \varepsilon v \) involves only \( L^j_{\varepsilon+1}, L^j_{\varepsilon}, \) and \( L^j_{\varepsilon-1} \). \( \square \)

Let us next come to the infinitesimal Jordan-Lefschetz property. In [6] Looijenga and Lunts introduced the Kähler Lie algebra \( g \) of a compact Kähler manifold \( X \) as the Lie subalgebra of \( gl(H^*(X)) \) generated by \( L_\omega, \Lambda_\omega \) for all Kähler classes \( \omega \). By [6, Prop. 1.6] the Kähler Lie algebra \( g \) is semi-simple. By Prop. 2.1 of the same paper the operators \( \Lambda_\omega \) and \( \Lambda_{\omega'} \) commute for all different \( \omega, \omega' \) if and only if \( g \) has degree \(-2, 0, \) and \( 2 \) only. In this case \( (g, [\Lambda, L]) \) is called a Jordan-Lefschetz pair. Due to [6, Cor. 2.6] Jordan-Lefschetz pairs can be classified. Complex tori and compact hyperkähler manifolds satisfy the Jordan-Lefschetz property, but no other series of higher dimensional compact Kähler manifolds seems to be
known. Surprisingly, the infinitesimal Jordan-Lefschetz property always holds true. This is the content of the next theorem.

**Theorem 2.2** — Let $X$ be a compact Kähler manifold and let $\omega \in H^{1,1}(X, \mathbb{R})$ be a Kähler class. For any $v \in H^{1,1}(X, \mathbb{R})$ the operators $\Lambda_\omega$ and $\Lambda_{\omega+\varepsilon v}$ associated to $\omega$ resp. its infinitesimal deformation $\omega + \varepsilon v$ commute, i.e. $[\Lambda_\omega, \Lambda_{\omega+\varepsilon v}] = 0$.

**Proof.** The assertion is a consequence of Prop. 1.6. The calculation is straightforward, but lengthy. We only indicate the main steps.

Let us introduce the notation $\Lambda := \Lambda_\omega$ and $\Lambda_\varepsilon := \Lambda_{\omega+\varepsilon v}$. In order to prove $[\Lambda_\varepsilon, \Lambda] = 0$ it suffices to show that for any primitive class $\alpha \in H^{n-2j}$ with $L^j \alpha \neq 0$ one has $[\Lambda_\varepsilon, \Lambda](L^j \alpha) = 0$.

A simple calculation proves $\Lambda L^k \beta = k(N-m-k+1)L^{k-1} \beta$ for any primitive form $\beta$ of degree $m$. Thus $\Lambda_\varepsilon \Lambda L^j \alpha = j(N-n+j+1)\Lambda_\varepsilon L^{j-1} \alpha$. Then, the $(\omega + \varepsilon v)$-primitive decomposition of $L^{j-1} \alpha$ is given by Prop. 1.6 and the resulting expressions $\Lambda_\varepsilon L^k_\varepsilon$ with $k = j, j-1, j-2$ can be computed as before. Thus, eventually $\Lambda_\varepsilon \Lambda L^j \alpha$ can be expressed as a linear combination of $\varepsilon L^{j-1}Q^2_1(\alpha), \varepsilon L^{j-2}Q^1_1(\alpha), \varepsilon L^{j-3}Q^0_0(\alpha)$, and $L^{j-2} \alpha$. Analogously, $\Lambda \Lambda_\varepsilon L^j \alpha$ can be computed by inserting the formula for the primitive decomposition of $L^j \alpha$ with respect to $\omega + \varepsilon v$ given by Prop. 1.6. The resulting formula is again a linear combination of the same terms as before and, as it turns out, the linear combination is in both cases the same. This proves $[\Lambda_\varepsilon, \Lambda] = 0$. \(\square\)

## 3 Infinitesimal change of harmonic forms

In this section we apply the results of Sect. 1 to $sl_2$-representations on the space of forms. Although the vector space is no longer finite dimensional the results are still valid, as the representations are induced by the standard $sl_2$-representation on hermitian exterior algebras. The Lefschetz decomposition discussed in the last section only reflects this on the level of cohomology. The new ingredient on the level of linear algebra in this context is the Hodge $*$-operator.

Let $X$ be a compact Kähler manifold of dimension $N$. For any Kähler form $\omega$ we denote by $\mathcal{H}_{\omega}$ the space of all forms that are harmonic with respect to $\omega$. Thus, $\mathcal{H}_{\omega} = \oplus \mathcal{H}^p_q = \oplus \mathcal{H}_{\omega}^{p,q}$. Considered as a subspace of the space $\mathcal{A}(X)_{cl}$ of all closed forms the space $\mathcal{H}_{\omega}$ depends on $\omega$. We shall compute the first order term of this dependence. A first order deformation of the Kähler form $\omega$ is of the form $\omega_\varepsilon = \omega + \varepsilon v$, where $v$ is a closed real $(1,1)$-form.

In order to study the variation of $\mathcal{H}_{\omega}$ we have to understand the deformation of the Laplacian $\Delta$ which involves the Hodge operator $*$ associated with $\omega$. Let us write the Hodge operator $*_{\varepsilon}$ associated to $\omega_\varepsilon = \omega + \varepsilon v$ in the form $*_{\varepsilon} = * + \varepsilon T_v$, where $T_v$ is a certain linear operator. We shall first describe the deformation of $\mathcal{H}_{\omega}$ to $\mathcal{H}_{\omega_\varepsilon}$ in terms of $T_v$. Later we will use the results about the deformation of $sl_2$-representations to express this purely in terms of $v$. 

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Recall that the Hodge decomposition of a form $\alpha \in \mathcal{A}(X)$ has the form $\alpha = \mathcal{H}_\omega(\alpha) \oplus dG^* \alpha \oplus d^* Gd\alpha$, where $\mathcal{H}_\omega(\alpha)$ denotes the $\omega$-harmonic part of $\alpha$ and $G$ is the Green operator with respect to $\omega$, which commutes with $d$ and $d^*$. Analogously, one has $\alpha = \mathcal{H}_\omega(\alpha) \oplus dG_\varepsilon d^* \alpha \oplus d^* G_\varepsilon d\alpha$. The infinitesimal variation of $\mathcal{H}_\omega \subset \mathcal{A}(X)_{cl}$ induced by $\omega_\varepsilon = \omega + \varepsilon v$ is determined by the canonical homomorphism

$$\tilde{h}(v, -) : \mathcal{H}_\omega \to \mathcal{A}(X)_{cl}/\mathcal{H}_\omega = \text{Im}(d).$$

The space of harmonic forms $\mathcal{H}_\omega$ does not change when passing from $\omega$ to $\omega_\varepsilon$ if and only if this map is trivial. More specifically, one has

An $\omega$-harmonic form $\alpha \in \mathcal{H}_\omega$ is harmonic with respect to $\omega_\varepsilon$, i.e. $\alpha \in \mathcal{H}_\omega$, if and only if $\tilde{h}(v, \alpha) = 0$.

The map $\tilde{h}(v, -)$ can explicitly be described in terms of $T_v$ due to the following

**Lemma 3.1** — Let $\alpha \in \mathcal{H}_\omega$. Then the $\omega_\varepsilon$-harmonic part of $\alpha$ is given by $\mathcal{H}_{\omega_\varepsilon}(\alpha) = \alpha + \varepsilon dG \ast dT_v(\alpha)$.

**Proof.** Since $\varepsilon dG \ast dT_v(\alpha)$ is exact, it suffices to show that $\beta := \alpha + \varepsilon dG \ast dT_v(\alpha)$ is harmonic with respect to $\omega_\varepsilon$. Obviously, $\beta$ is $d$-closed. In order to show that $d^* \beta = 0$, one first notes that $\varepsilon G = \varepsilon G_\varepsilon$. Also recall that $\Delta G = \Delta = \text{id}$ and $\Delta_\varepsilon G_\varepsilon = G_\varepsilon \Delta_\varepsilon = \text{id}$ on $d$-exact forms and that $d^* = - \ast d^*$. Therefore,

$$\Delta_\varepsilon(\alpha + \varepsilon dG \ast dT_v(\alpha)) = \Delta_\varepsilon(\alpha) + \varepsilon \Delta_\varepsilon G_\varepsilon d \ast dT_v \alpha$$

$$= dd^* \alpha + \varepsilon d \ast dT_v (\alpha)$$

$$= \varepsilon d(\ast + \varepsilon T_v) d(\ast + \varepsilon T_v) (\alpha) + \varepsilon d \ast dT_v (\alpha) = 0.$$

Thus, the Hodge decomposition of $\alpha$ with respect to $\omega_\varepsilon$ takes the form $\alpha = (\alpha + \varepsilon dG \ast dT_v(\alpha)) \oplus (-\varepsilon dG \ast dT_v(\alpha))$.

**Corollary 3.2** — The map $\tilde{h}(v, -) : \mathcal{H}_\omega \to \text{Im}(d)$ maps $\alpha$ to $-dG \ast dT_v(\alpha)$

**Remark 3.3** — The Green operator commutes with $d$ and defines an automorphism of $\text{Im}(d)$. Moreover, $d : \text{Im}(d^*) \to \text{Im}(d)$ is bijective. So, without losing any information we may replace $\tilde{h}(v, -)$ by the map $h(v, -) : \mathcal{H}_\omega \to \text{Im}(d^*)$ that maps $\alpha$ to $\ast dT_v(\alpha)$. Note that $h(v, -)$ is a map of degree $(-1)$. In fact, we could also consider $dT_v(\alpha)$ without losing information, but the degree of this map depends on $\alpha$ and on $N$.  

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Next we shall compute $T_v(\alpha)$ in terms of the product $\alpha v$. Throughout, we will make use of the following formula due to A. Weil (cf. [7, 8]):

\begin{equation}
* (\omega^r \alpha) = (-1)^{(p+q)(p+q+1)/2} \cdot \frac{r! \cdot i^{p-q}}{(N-(p+q)-r)!} \cdot \omega^{N-(p+q)-r} \alpha.
\end{equation}

In fact, if both sides are correctly interpreted the formula also holds for $(p+q)+r > N$. Namely, $\omega^r \alpha = 0$, since $\alpha$ is primitive and $r > N-(p+q)$, and $\omega^{N-(p+q)-r} \alpha = 0$, as the exponent is negative.

The idea to compute $T_v(\alpha)$ is the following. Using Prop. 1.6 we can compute the primitive decomposition of any $\alpha \in H_\omega$ with respect to $\omega_\varepsilon$. Formula (1) allows one to compute $*\varepsilon$ by applying it to each of the summands in the $\omega_\varepsilon$-primitive decomposition of $\alpha$. On the other hand, $*\varepsilon (\alpha) = * (\alpha) + \varepsilon T_v(\alpha)$ can also be computed by applying (1) to $*$. This gives a formula for $T_v(\alpha)$.

**Example 3.4** — Let $\alpha$ be an $\omega$-primitive form of type $(1,1)$. By [7,8] the $\omega_\varepsilon$-primitive decomposition of $\alpha$ is given by $\alpha = \omega_\varepsilon((N-1)\varepsilon Q^2_v(\alpha)) + (\alpha - \varepsilon(N-1)\omega Q^2_v(\alpha))$. The function $f = Q^2_v(\alpha)$ is determined by $Q^2_v(\alpha)\omega^N = \alpha v \omega^{N-2}$. Thus, $f = (\alpha v \omega^{N-2})/\omega^N$. Hence, the $\omega_\varepsilon$-primitive decomposition of $\alpha$ is

\[ \alpha = \omega_\varepsilon((N-1)\varepsilon \left( \frac{\alpha v \omega^{N-2}}{\omega^N} \right)) + (\alpha - \varepsilon(N-1)\omega \left( \frac{\alpha v \omega^{N-2}}{\omega^N} \right)). \]

By (1):

\[ *\varepsilon \alpha = *\varepsilon(\omega_\varepsilon((N-1)\varepsilon \left( \frac{\alpha v \omega^{N-2}}{\omega^N} \right))) + *\varepsilon(\alpha - \varepsilon(N-1)\omega \left( \frac{\alpha v \omega^{N-2}}{\omega^N} \right)) \]

\[ = \frac{1}{(N-1)!} \varepsilon \omega_\varepsilon((N-1)\varepsilon \left( \frac{\alpha v \omega^{N-2}}{\omega^N} \right)) - \frac{1}{(N-2)!} \omega_\varepsilon^{N-2}(\alpha - \varepsilon(N-1)\omega \left( \frac{\alpha v \omega^{N-2}}{\omega^N} \right)). \]

On the other hand, $*\varepsilon \alpha = *\alpha + \varepsilon T_v(\alpha) = -\frac{1}{(N-2)!} \omega^{N-2} \alpha + \varepsilon T_v(\alpha)$. Hence,

\[ T_v(\alpha) = \frac{1}{(N-3)!} \omega^3 \alpha v + \frac{N}{(N-2)!} \left( \frac{\alpha v \omega^{N-2}}{\omega^N} \right) \omega^{N-1}. \]

For the second example let us assume that $v$ is primitive. Then for the $(1,1)$-form $\alpha = \omega$ the primitive decomposition with respect to $\omega_\varepsilon = \omega + \varepsilon v$ takes the form $\omega = (\omega + \varepsilon v) - \varepsilon v$.

Indeed, $\varepsilon v$ is $\omega_\varepsilon$-primitive, for $\varepsilon v(\omega + \varepsilon v)^{N-1} = \varepsilon v \omega^{N-1} = 0$.

Thus,

\[ *\varepsilon \omega = *\varepsilon \omega_\varepsilon - \varepsilon *\varepsilon v = \frac{1}{(N-1)!} \omega_\varepsilon^{N-1} - \frac{\varepsilon}{(N-2)!} \omega_\varepsilon^{N-2} v. \]

Using, $*\varepsilon = * + \varepsilon T_v$ this yields
\[ T_v(\omega) = \frac{(N-1)}{(N-1)!} \omega^{N-2}v + \frac{1}{(N-2)!} \omega^{N-2}v = \frac{2}{(N-2)!} \omega^{N-2}v. \]

The general formula is provided by the following

**Proposition 3.5** — Let \( \alpha \) be \( \omega \)-primitive of type \( (p-j, q-j) \) and let \( \alpha v = \beta_0 + \omega \beta_1 + \omega^2 \beta_2 \) be the \( \omega \)-primitive decomposition of \( \alpha v \). Then

\[ T_v(\alpha \omega^j) = c'_0 \beta_0 \omega^{N-n+j-1} + c_1 \beta_1 \omega^{N-n+j} + c_2 \beta_2 \omega^{N-n+j+1} + c_0 \alpha v \omega^{N-n+j-1}, \]

where \( c_0 = -\eta(N-n+j) \), \( c_1 = \eta \cdot j \), \( c_2 = \eta(N-n+3j+2) \), \( \eta := (-1)^{(n-2j-2)(n-2j-1)} \frac{j!p-q}{(N-n+2)!} \), \( n = p + q \), and \( c'_0 = c_0 \) for \( j > 0 \) and \( c'_0 = 0 \) for \( j = 0 \).

**Proof.** Using \( (\ast + \varepsilon T_v)(\alpha \omega^j) = \ast_\varepsilon(\alpha \omega^j) \) and the primitive decomposition of \( \alpha \omega^j \) with respect to \( \omega \varepsilon \) given by Prop. 1.6, the claim is proven by applying (1). By Prop. 1.6 we have the \( \omega \varepsilon \)-primitive decomposition

\[ \omega^j \alpha = (N-n+j+1)\omega^j_{\varepsilon}((\varepsilon \beta_2) + \omega^2((\alpha - \varepsilon(N-n+2j+1)\omega \beta_2 + j \beta_1)) \]

\[ -j \omega^j_{\varepsilon}((\varepsilon \beta_0)). \]

Applying \( \ast_\varepsilon \) to these three terms gives:

\[ \ast_\varepsilon \omega^j_{\varepsilon}((\varepsilon(-j \beta_0)) = c_0 \varepsilon \beta_0 \omega^{N-n+j-1} \text{ for } j > 0, \]

\[ \ast_\varepsilon \omega^j_{\varepsilon}((\alpha - \varepsilon((N-n+2j+1)\beta_2 \omega + j \beta_1)) = -\eta \omega^j_{\varepsilon}(\alpha - \varepsilon((N-n+2j+1)\beta_2 \omega + j \beta_1)) \]

and \( \ast_\varepsilon \omega^j_{\varepsilon}((\varepsilon(N-n+j+1)\beta_2)) = \eta(j+1)\omega^j_{\varepsilon}(\varepsilon \beta_2). \)

The scalars \( c_1, c_2 \) can be easily computed from this. \( \square \)

**Remark 3.6** — Note that if \( N-n+j < 0 \), then \( \alpha \omega^j = 0 \). In this case the right hand side is also interpreted as zero. For \( N-n+j = 0 \) the right hand side reduces to \( c_1 \beta_1 \).

- For \( j = 0 \) the formula yields

\[ T_v(\alpha) = (-1)^{\frac{n(n+1)}{2}} \frac{p-q}{(N-n)!} ((N-n)\alpha v \omega^{N-n-1} - (N-n+2)\beta_2 \omega^{N-n+1}). \]

- For \( p = q = 1 \) we get back the formula in the example, since \( \beta_2 = (\alpha v \omega^{N-2})/\omega^N \).

- For \( \alpha = 1 \) and \( v \) primitive we find \( T_v(\omega^j) = \frac{2j!}{(N-j-1)!} v \omega^{N-j-1} \).

Passing from \( T_v(\alpha \omega^j) \) to \( h(v, \alpha \omega^j) = \ast dT_v(\alpha \omega^j) \) actually simplifies the formula due to the following
Lemma 3.7 — Let $\alpha$ be a closed primitive form and $\alpha v = \beta_0 + \beta_1 \omega + \beta_2 \omega^2$ be the primitive decomposition of the product $\alpha v$, where $v$ is a closed form of degree two. Then the primitive decomposition of $d\beta_0$, $d\beta_1$, and $d\beta_2$ are of the form: $d\beta_1 = \delta_0 + \delta_1 \omega$, $d\beta_0 = -\delta_0 \omega$, and $d\beta_2 = -\delta_1$. In particular, $d\beta_1 = 0$ if and only if $d\beta_0 = d\beta_2 = 0$.

Proof. Let $d\beta_0 = \sum \delta_0^k \omega^k$, $d\beta_1 = \sum \delta_k \omega^k$, and $d\beta_2 = \sum \delta_k^2 \omega^k$ be the primitive decomposition of $d\beta_0$, $d\beta_1$, and $d\beta_2$, respectively. We use the following three equations to deduce the result:

1) $d(\alpha v) = 0$, ii) $(d\beta_1 + d\beta_2) \omega^{N - \deg(\alpha)} = 0$, and iii) $d\beta_2 \omega^{N - \deg(\alpha) + 2} = 0$.

Since $\alpha$ and $v$ are closed one has 1), which yields

$$\sum \delta^0_k \omega^k + \sum \delta_1 \omega^{k+1} + \sum \delta_2^k \omega^k = 0.$$ 

Hence, $\delta^0_0 = 0$, $\delta^0_1 = -\delta_0$, and $\delta^0_{k+2} + \delta_{k+1} + \delta^2_k = 0$ for all $k \geq 0$. If $\deg(\alpha) \geq N - 1$ then $\delta_0 = 0$. If $\deg(\alpha) \leq N - 1$, we use that $\beta_0$ is of degree $\deg(\alpha) + 2$ to conclude that $\beta_0 \omega^{N - \deg(\alpha) - 1} = 0$. This yields 2) and therefore

$$\sum \delta_k \omega^{N - \deg(\alpha) + k} + \sum \delta_k^2 \omega^{N - \deg(\alpha) + k+1} = 0.$$ 

Since $\delta_{k+1} + \delta_k^2$ is primitive of degree $\deg(\alpha) - 1 - 2k$ we obtain $\delta_{k+1} + \delta_k^2 = 0$ for all $k \geq 0$. This already proves $d\beta_1 = \delta_0^0 \omega = -\delta_0 \omega$. Using that $\beta_1$ is primitive of degree $\deg(\alpha)$ equation 3) gives 3), i.e. $\sum \delta_k^2 \omega^{N - \deg(\alpha) + 2 + k} = 0$. Since $\delta_k^2$ is primitive of degree $\deg(\alpha) - 1 - 2k$, one obtains $\delta_k^2 = 0$ for $k > 0$. 

Lemma 3.8 — Let $\alpha$ be a closed primitive form of pure type and let $v$ be closed of type $(1,1)$. Then $\alpha v$ is harmonic if and only if $\beta_1$ is closed. As before, $\beta_1$ is the primitive form defined by the primitive decomposition $\alpha v = \beta_0 + \beta_1 \omega + \beta_2 \omega^2$.

Proof. Since $X$ is Kähler, $\alpha v$ is harmonic if and only if $\beta_0$, $\beta_1$, and $\beta_2$ are harmonic. On the other hand, if $d\beta_1 = 0$, then also $d\beta_0 = d\beta_2 = 0$. Since the forms $\beta_i$ are primitive of pure type, this implies $d \ast \beta_i = 0$, i.e. they are harmonic. (In general, a closed primitive form of pure type is harmonic. In particular, $\alpha$ is harmonic from the very beginning.)

This leads to the final formula.

**Theorem 3.9** — Let $\alpha$ be $\omega$-primitive and $\omega$-harmonic of type $(p - j, q - j)$ with $\alpha \omega^j \neq 0$. If $\alpha v = \beta_0 + \beta_1 \omega + \beta_2 \omega^2$ is the primitive decomposition of $\alpha v$ and $d\beta_1 = \delta_0 + \delta_1 \omega$ is the primitive decomposition of $d\beta_1$, then

$$h(v, \alpha \omega^j) = \lambda_1 \ast (\delta_0 \omega^{N - n + j}) + \lambda_2 \ast (\delta_1 \omega^{N - n + j + 1}),$$

where $\lambda_1 = \eta(N - n + 2j)$ for $j > 0$ and $\lambda_1 = 0$ for $j = 0$, $\lambda_2 = -\eta(N - n + 2j + 2)$, $\eta$ as before, and $n = p + q$. 

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Proof. By definition $h(v, \alpha \omega^j) = *dT_v(\alpha \omega^j)$ and by Prop. 3.5

$$
*dT_v(\alpha \omega^j) = *(c'_0 d\beta_0 \omega^{N-n+j-1} + c_1 d\beta_1 \omega^{N-n+j} + c_2 d\beta_2 \omega^{N-n+j+1}) = (c_1 - c'_0) \delta_0 \omega^{N-n+j} + (c_1 - c_2) \delta_1 \omega^{N-n+j+1}.
$$

The calculation of $\lambda_1 := c_1 - c'_0$ and $\lambda_2 := c_1 - c_2$ is straightforward. \hfill \qed

**Corollary 3.10** — If $j = 0$ the form $\alpha$ is harmonic with respect to $\omega_\epsilon = \omega + \epsilon v$ if and only if $\alpha \omega^{N-n}$ is $\omega$-harmonic. If $j > 0$ the form $\alpha \omega^j$ is harmonic with respect to $\omega_\epsilon = \omega + \epsilon v$ if and only $\alpha v$ is $\omega$-harmonic.

**Proof.** Let first $j = 0$. Since $\lambda_1 = 0$ for $j = 0$ one has $h(v, \alpha) = 0$ if and only if $\delta_1 \omega^{N-n+1} = 0$. Since $\delta_1$ is of degree $n - 1$ the latter is equivalent to the vanishing of $d\beta_2$. Now, $\beta_2$ is closed if and only if $\beta_2$ is harmonic if and only if $\beta_2 \omega^{N-n+2}$ is harmonic, but the latter is just $\alpha \omega \omega^{N-n}$.

In general, if $\alpha \omega$ is harmonic, then $\delta_0 = \delta_1 = 0$ by lemma 3.8 and hence $h(v, \alpha \omega^j) = 0$. Conversely, if $h(v, \alpha \omega^j) = 0$, then $\delta_1 = 0$, because $\delta_1$ is primitive of degree $n - 2j - 1$. For $j > 0$ the same argument shows $\delta_0 = 0$. \hfill \qed

## 4 Non-linear Kähler cones

Let $X$ be a compact Kähler manifold of dimension $N$ with a fixed Kähler form $\omega_0$. By $\mathcal{K}_X \subset H^{1,1}(X, \mathbb{R})$ we denote the Kähler cone, i.e. the set of all Kähler classes. Then $[\omega_0] \in \mathcal{K}_X$. Clearly, $\mathcal{K}_X$ is an open convex cone. By definition, every class in $\mathcal{K}_X$ can be represented by a Kähler form, but a priori there is no canonical choice.

**Definition 4.1** — For $i = 0, \ldots, N - 1$ we denote by $\mathcal{K}^i = \mathcal{K}^i_{(X, \omega_0)}$ the connected component containing $\omega_0$ of the set of all Kähler forms $\omega$, such that $\omega^{N-i} \omega^i_0 = c \cdot \omega^N_0$ for some scalar constant $c$.

**Remark 4.2** — The set $\mathcal{K}^0 = \mathcal{K}^0_{(X, \omega_0)}$ only depends on the volume form $\omega_0^N$ and not on $\omega_0$ itself. In [2] this set was denoted by $\tilde{\mathcal{K}}_X$. Due to results of Calabi and Yau the natural projection $\mathcal{K}^0 \to \mathcal{K}_X$, $\omega \mapsto [\omega]$ is bijective. If $\omega_0$ is Ricci-flat, then $\mathcal{K}^0$ is the set of all Ricci-flat Kähler forms. 

- In general, the canonical projection $\mathcal{K}^i \to \mathcal{K}_X$ is injective. This is proved using the original argument of Calabi [1]: If $\omega, \omega' \in \mathcal{K}^i$ with $[\omega] = [\omega']$, then $\omega - \omega' = dd^c \varphi$ and $dd^c \varphi(\omega^{N-i-1} + \omega^{N-i-2} \omega' + \ldots + \omega^{N-i-1}) \omega_0^i = 0$. The positivity of $\omega$, $\omega'$, and $\omega_0$ yields $\varphi = 0$.

- In general, the projection $\mathcal{K}^i \to \mathcal{K}_X$ need not be surjective for $i > 0$ (see the example below). Presumably, taking the connected component in the definition of $\mathcal{K}^i$ is superflous. It certainly is for $i = 0$ and $i = N - 1$. 

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- The only ‘linear’ cone is $K$. Indeed, $K^{N-1}$ consists of all Kähler forms $\omega$ such that $\omega_0 = c \cdot \omega_0$. Thus, $\omega - c\omega_0$ is closed and $\omega_0$-primitive, hence $\omega_0$-harmonic. Therefore, $K^{N-1}$ is an open subset of $(H^1,0)$. In some imprecise sense the sequence $K^0, K^1, \ldots, K^{N-1}$ is a stepwise linearization of the Calabi-Yau cone $K^0$.

**Example 4.3** — Often, for a class $[\alpha] \in \partial K_X$ there exists a curve $C \subset X$ such that $\int_C [\alpha] = 0$. Assume that $K^{N-1} \to K_X$ is surjective and let $\alpha \in \partial K^{N-1} \subset H^1,0$ be representing $[\alpha]$. Since $\alpha$ is in the boundary of $K^{N-1}$ it is semi-positive definite. On the other hand, $\int_C [\alpha] = 0$. Therefore, $\alpha|_C = 0$. Conversely, if we want to find examples for which $K^{N-1} \to K^0$ is not surjective, then we have to look for $[\alpha]$ such that $\alpha|_C \neq 0$.

Let $C$ be a curve of genus two and $\varphi_1, \varphi_2 \in H^{1,0}(C)$ be an orthogonal base, i.e. $\int \varphi_1 \bar{\varphi}_2 = 0$. Consider the form $\alpha = \varphi_1 \times \varphi_2 + \varphi_1 \times \varphi_2$ on $X := C \times C$, which is harmonic with respect to $\omega \times \omega$ for any Kähler form $\omega$ on $C$. Then $\alpha$ is trivial on the fibres of the two projections and $\int_\Delta [\alpha] = 0$, where $\Delta \subset C \times C$ is the diagonal. On the other hand, we may assume that $\alpha|_\Delta = \varphi_1 \land \varphi_2 + \bar{\varphi}_1 \land \varphi_2$ is not trivial. Indeed, we may change $\varphi_1$ by a complex scalar $\lambda$ and if for all $\lambda$ one has $0 = \lambda \varphi_1 \land \varphi_2 + \bar{\lambda} \varphi_1 \land \varphi_2$ then $\varphi_1 \land \varphi_2 = 0$, but $\varphi_1$ and $\varphi_2$ vanish at different points. Therefore, $\alpha$ is not semi-positive definite. Since obviously $[\alpha] \in \partial K_X$, the form $\alpha$ itself cannot be in $\partial K^{N-1}$. Hence, $K^{N-1} \to K_X$ is not surjective.

Here is another geometric relevant example which shows that the existence of certain subvarieties of the (deformed) manifold influences the size of the set of all $\omega_0$-harmonic Kähler forms.

**Proposition 4.4** — Let $\omega_I$ be a Ricci-flat Kähler form on a K3 surface $(X, I)$, where $I$ is a fixed complex structure on $X$, and let $\mathbb{P} := \{aI + bJ + cK | a^2 + b^2 + c^2 = 1\}$ be the induced twistor family of complex structures. Let $\alpha \in K_X$ be a Kähler class on $(X, I)$ and $\omega_I + \alpha'$ be its primitive decomposition with respect to $\omega_I$. If for some complex structure $\lambda \in \mathbb{P}$ there exists a smooth rational curve $C$ in $(X, \lambda)$ such that $C.(\omega_\lambda + \alpha') < 0$, then $\alpha$ cannot be represented by an $\omega_I$-harmonic Kähler form.

**Proof.** We denote by $\alpha$ and $\alpha'$ also the harmonic representatives of $\alpha$ and $\alpha'$, respectively. Then $\alpha'$ is of pure type $(1, 1)$ for any complex structure of the form $\lambda = aI + bJ + cK$. Let us first show that $(\omega_\lambda + \alpha')^2$ is independent of $\lambda$ at any point of $X$. Indeed, $(\omega_\lambda + \alpha')^2 = \omega_\lambda^2 + \alpha'^2 = \omega_\lambda^2 + \alpha^2$, as $\alpha'$ is $\omega_\lambda$-primitive for any $\lambda$. Therefore, if the harmonic representative of $\alpha$ is in fact positive definite, then $\omega_\lambda + \alpha'$ is a positive definite form on $(X, \lambda)$ for all $\lambda \in \mathbb{P}$, as otherwise the square would have to vanish for some triple $\lambda$ at at least one point of $X$. But then a rational curve as above cannot exist. Thus, $\alpha$ cannot be represented by an $\omega_I$-harmonic Kähler form. □
In other words, the part of the Kähler cone $K_X$ of the K3 surface $(X, I)$ that can be represented by harmonic Kähler forms stays constant in the twistor family. In particular, this excludes the existence of a series of smooth rational curves $C_i$ on twistor fibres $(X, \lambda_i)$ with $\int_{C_i} \omega_{\lambda_i} \to 0$. It is an open and interesting question whether any Kähler class that stays Kähler in the twistor family can actually be represented by an harmonic Kähler form.

Let us come back to the general situation. The description of the tangent space of $K^0 = \tilde{K}_X$ in [3] can easily be adapted to describe the tangent space of $K^0$ at the point $\omega_0 \in K^i$.

**Lemma 4.5** — *At the point $\omega_0 \in K^i$ the tangent space is $T_{\omega_0} K^i = (H^{1,1}_{\omega_0})_R$, i.e. any first order deformation of $\omega_0$ within $K^i$ is given as $\omega_0 + \varepsilon v$ with $v \in (H^{1,1}_{\omega_0})_R$.*

**Proof.** Indeed, $(\omega_0 + \varepsilon v)^{N-i} \omega_0^i = \omega_0^N + (N - i)\varepsilon \omega_0^{N-1} v$. Hence, for some scalar $\lambda$ the form $v - \lambda \omega_0$ is $\omega_0$-primitive and closed. Thus, $v$ is $\omega_0$-harmonic.

Thus, the sets $K^i$ have contact at $\omega_0$ of order two. The tangent space of $K^0$ can be described at every point $\omega \in K^0$ as the space of all real $\omega$-harmonic $(1,1)$-forms. The description of $T_{\omega} K^i$ for $0 < i < N - 1$ and $\omega \neq \omega_0$ is less clear, it mixes $\omega$-harmonicity and $\omega_0$-harmonicity.

We next want to know under what circumstances the ‘non-linear’ cone $K^i$ is linear, i.e. when it is contained in its tangent space $T_{\omega_0} K^i = (H^{1,1}_{\omega_0})_R$. The following proposition generalizes [3, Prop. 2.3]

**Proposition 4.6** — *The cone $K^i$ is linear if and only if $\alpha^{N-i} \omega_0^i$ is $\omega_0$-harmonic for all $\alpha \in H^{1,1}_{\omega_0}$.*

**Proof.** If $K^i$ is linear, then for all $\alpha$ in the open set $K^i \subset (H^{1,1}_{\omega_0})_R$ the form $\alpha^{N-i} \omega_0^i$ is $\omega_0$-harmonic. But then this holds for all $\alpha \in H^{1,1}_{\omega_0}$. Conversely, if $\alpha^{N-i} \omega_0^i$ is $\omega_0$-harmonic for all $\alpha \in H^{1,1}_{\omega_0}$, then $K^i$ intersects $(H^{1,1}_{\omega_0})_R$ in an open subset. Hence, $K^i \subset (H^{1,1}_{\omega_0})_R$. □

**Corollary 4.7** — *If $K^i$ is linear, then $K^{i+1}$ is linear.*

**Proof.** If $K^i$ is linear then $\alpha^{N-i} \omega_0^i$ is $\omega_0$-harmonic for all $\alpha \in H^{1,1}_{\omega_0}$. Hence, $(\alpha + t \omega_0)^{N-i} \omega_0^i$ is $\omega_0$-harmonic for all $t$. Then, also the linear coefficient of this polynomial in $t$, which is $\alpha^{N-i-1} \omega_0^{i+1}$, is $\omega_0$-harmonic for all $\alpha \in H^{1,1}_{\omega_0}$. Thus, $K^{i+1}$ is linear. □

In particular, the linearity of $K^0$ implies the linearity of all the other sets $K^i$. In this case, $K^0 = K^1 = \ldots = K^{N-1}$. In this sense, the sequence $K^0, K^1, \ldots, K^{N-1}$ goes from the most curved set $K^0$ to the linear set $K^{N-1}$.

The following result is another criterion for the linearity of $K^0$. 

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Proposition 4.8 — The set $\mathcal{K}^0$ is linear, i.e. $\mathcal{K}^0 = \mathcal{K}^1 = \ldots = \mathcal{K}^{N-1}$, if and only if $H^{1,1}_\omega = H^{1,1}_{\omega_0}$ for all $\omega$ in a neighbourhood of $\omega_0$ in $\mathcal{K}^{N-1} \subset (H^{1,1}_{\omega_0})_\mathbb{R}$.

Proof. If $\mathcal{K}^0$ is linear, then $H^{1,1}_\omega = H^{1,1}_{\omega_0}$ for all $\omega \in \mathcal{K}^0 = \mathcal{K}^{N-1}$. Conversely, if $H^{1,1}_\omega = H^{1,1}_{\omega_0}$ for all $\omega \in (H^{1,1}_{\omega_0})_\mathbb{R}$ close to $\omega_0$, then $H^{N,N}_{\omega + \varepsilon v} = H^{N,N}_{\omega_0}$ for all $v \in T_{\omega} \mathcal{K}^{N-1} = (H^{1,1}_{\omega_0})_\mathbb{R}$. Indeed, $H^{N,N}_{\omega_0} = C(\omega + \varepsilon v)^N = C(\omega^N + (N-1)\varepsilon \omega^{N-1}v)$ and $\omega^{N-1}v$ is $\omega$-harmonic. Thus, the map $\mathcal{K}^{N-1} \to \mathbb{P}(\mathcal{A}^{N,N}(X))$, $\omega \mapsto H^{N,N}_\omega$ has vanishing differential at every point in an open neighbourhood of $\omega_0$ and is, therefore, constant. In particular, for an open subset of $(H^{1,1}_{\omega_0})_\mathbb{R}$ the top exterior power $\alpha^N$ is $\omega_0$-harmonic. Therefore, the intersection of $\mathcal{K}^0$ and $(H^{1,1}_{\omega_0})_\mathbb{R}$ contains an open subset. This implies $\mathcal{K}^0 \subset (H^{1,1}_{\omega_0})_\mathbb{R}$, i.e. $\mathcal{K}^0$ is linear. \hfill \Box

The results of Sect. 3 yield

Proposition 4.9 — The following conditions are equivalent.

i) The set $\mathcal{K}^0$ is linear.

ii) For all $\alpha \in H^{1,1}_{\omega_0}$ the form $\alpha^N$ is $\omega_0$-harmonic.

iii) For all $\omega \in \mathcal{K}^0$ and all $\alpha \in H^{1,1}_{\omega_0}$ the form $\alpha^N$ is $\omega$-harmonic.

iv) For all $\omega \in \mathcal{K}^0$ and all $\alpha \in H^{1,1}_{\omega_0}$ the form $\alpha^2 \omega^{N-2}$ is $\omega$-harmonic.

Proof. The equivalence of i) and ii) was shown in [2]. Clearly, iii) implies ii). On the other hand, if $\mathcal{K}^0$ is linear, then $H^{1,1}_\omega = H^{1,1}_{\omega_0}$ and $H^{N,N}_\omega = H^{N,N}_{\omega_0}$ for all $\omega \in \mathcal{K}^0$. Hence, i) implies iii). By Corollary 3.10 the form $\alpha^2 \omega^{N-2}$ is $\omega$-harmonic for all $\alpha \in H^{1,1}_\omega$ if and only if $H^{1,1}_\omega = H^{1,1}_{\omega + \varepsilon v}$ for all $v \in (H^{1,1}_{\omega_0})_\mathbb{R}$. Hence, iv) holds if and only if the Gauss map of the embedding $\mathcal{K}^0 \subset \mathcal{A}^{1,1}(X)_{cl}$ has everywhere vanishing differential, i.e. $T_{\omega} \mathcal{K}^0$ is constant. \hfill \Box

Note that condition iv) for $\omega = \omega_0$ is equivalent to $\mathcal{K}^{N-2}$ being linear (Prop. 4.6). Thus, $\mathcal{K}^0$ is linear if and only if $\mathcal{K}^{N-2}$ is linear for any $\omega \in \mathcal{K}^0$. For Calabi-Yau manifolds this reads as follows:

Corollary 4.10 — Let $X$ be a Calabi-Yau manifold. The set $\mathcal{K}^0$ of Ricci-flat Kähler forms is linear if and only if for any Ricci-flat Kähler form $\omega$ and any $\omega$-harmonic $(1,1)$-form $\alpha$ the product $\alpha^2 \omega^{N-2}$ is $\omega$-harmonic. In this case, any class $\alpha$ in the positive cone $\mathcal{C}_X \subset H^{1,1}(X, \mathbb{R})$ (cf. [3]) is a Kähler class. \hfill \Box

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