A least-squares Galerkin approach to gradient recovery for Hamilton-Jacobi-Bellman equation with Cordes coefficients

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Abstract. We propose a conforming finite element method to approximate the strong solution of the second-order Hamilton-Jacobi-Bellman equation with Dirichlet boundary conditions and coefficients that satisfy the Cordes condition. We show the convergence of the continuum semismooth Newton method for the fully nonlinear Hamilton-Jacobi-Bellman equation. Using this linearization approach for the equation yields a recursive sequence of linear elliptic boundary value problems (BVPs) in nondivergence form. We numerically solve these BVPs using the least-squares gradient recovery method proposed by Lakkis and Mousavi [2021]. We offer an optimal-rate a priori and a posteriori error bounds for the approximation. The a posteriori error estimators are used to drive an adaptive refinement procedure. We close with computer experiments on both uniform and adaptive meshes to reconcile the theoretical findings.

Keywords: Hamilton–Jacobi–Bellman equations, Cordes coefficients, semismooth Newton linearization, least-squares approach, gradient recovery, optimal a priori error bound, a posteriori error bound, adaptive refinement

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1. Introduction

We develop a Galerkin least-squares numerical method to approximate a function \( u : \Omega \to \mathbb{R} \) that satisfies the following elliptic Dirichlet boundary value problem (BVP) associated to the Hamilton–Jacobi–Bellman (HJB) partial differential equation (PDE)

\[
\sup_{\alpha \in A} (\mathcal{L}^\alpha u - f^\alpha) = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad u|_{\partial \Omega} = r.
\]

Here and throughout \( \Omega \) is a bounded convex domain in \( \mathbb{R}^d \), \( d \in \mathbb{N} \) (typically \( d = 2, 3 \)), \( A \) is a compact metric space called the (admissible) control set, \( r \in H^{1/2}(\partial \Omega) \).

For each admissible control \( \alpha \) in \( A \), the corresponding forcing term \( f^\alpha \) is a member of \( L^2(\Omega) \) and the elliptic operator \( \mathcal{L}^\alpha \) is defined by a triple of functions \( (A^\alpha, b^\alpha, c^\alpha) : \Omega \to \mathbb{R}^{d \times d} \times \mathbb{R}^d \times \mathbb{R} \) and

\[
\mathcal{L}^\alpha v := A^\alpha : D^2 v + b^\alpha \cdot \nabla v - c^\alpha v \quad \text{for each} \quad v \in H^2(\Omega).
\]

Here, \( \nabla \phi \) and \( D^2 \phi \) respectively indicate the gradient and Hessian of a function \( \phi \), and \( M : N := \text{tr}(M^T N) \) defines the Frobenius product for two equally sized matrices \( M, N \in \mathbb{R}^{m \times n} \), \( m, n \in \mathbb{N} \). Additionally, if \( m = n \), \( \text{tr} M \) denotes \( M \)'s trace, i.e., the sum of its (possibly repeated) eigenvalues.

Excepting special situations, e.g., when the control set \( A \) is a singleton, equation (1.1) is not linear in \( u \) and both its mathematical and computational analysis must be approached as a fully nonlinear elliptic equation.

The HJB equation (1.2) was introduced in the context of dynamic programming developed by Bellman [1957, 2010] for optimal control models. The second order
equation considered here appears in optimal control problems with stochastic processes; we refer to Fleming and Soner [2006a] and the references therein as a source of information on the modeling and analysis.

Fully nonlinear PDEs, including the HJB equations, of which a special case is a reformulation of the Monge–Ampère equation [Lions 1984, Krylov 2001], play an essential role in many fields of natural and social sciences as well as technology. The practical relevance motivates further the search for practical numerical methods. The nonvariational nature of fully nonlinear PDEs forms a challenge to their numerical approximation via standard Galerkin methods, making finite differences a natural first resort; but the flexibility that Galerkin methods offer in terms of geometry approximation and powerful adaptive mesh refinement techniques makes seeking such methods worthwhile.

One of the main difficulties in seeking solutions to general fully nonlinear PDEs, is the lack of classical solutions, which necessitates often the investigation of solutions in a weak or generalized sense. Since the structure of such PDEs precludes a natural variational formulation the direct application of a weak solution in $H^1(\Omega)$ is not practical. The natural approach to weak solutions, instead, traced by Crandall and Lions [1983] for first order equations and extended to second order equations by Lions [1983a,b], relies on the concept of viscosity solution, rooted in the theory of vanishing viscosity methods in fluid dynamics initiated by Hopf [1950]. Independent developments to weak solutions for fully nonlinear equations and HJB can be traced also to Aleksandrov [1961] who underscored the importance of the maximum principle for viscosity solutions and subsequent work connecting to stochastic control by Krylov [1972, 1979]. This avenue has since led to a flurry in the theory of fully nonlinear problems, often in connection with the theory of stochastic control [Caffarelli and Cabrè, 1995, Fleming and Soner, 2006b, Krylov, 2009, 2018].

From a numerical perspective, a milestone was reached with the seminal result of Barles and Souganidis [1991], demonstrating that consistency, stability, and monotonicity of an approximating scheme guarantee the convergence of the approximate solution to the exact viscosity solution. The importance of the Barles–Souganidis theorem is vindicated by the extensive literature built upon it [Barles and Jakobsen 2005, Oberman 2006, Debrabant and Jakobsen 2013, Peng and Jensen 2017].

The finite difference, including semi-Lagrangian or carefully designed wide-stencil methods are well suited in approximating a viscosity solution, as they preserve the maximum principle and consistency [Motzkin and Wasow 1953, Kuo and Trudinger 1992, Bonnans and Zidani 2003, Froese and Oberman 2011]. Moreover, certain Galerkin methods, such as $P^1$ on meshes that satisfy the maximum principle, have been shown to converge to the viscosity solution, as demonstrated by Jensen and Smears [2013], Nochetto and Zhang [2018], Salgado and Zhang [2019].

While besides stability and consistency, monotonicity-based discretization methods are guaranteed to converge, providing this property in discretizations is not always immediately obvious and may preserve only in specific cases.

To ease the strict requirement of discretization monotonicity, Smears and Süli [2014] explored the HJB equation under the Cordes condition on the coefficients of the elliptic operators $L^\alpha$ [Cordes 1959]. They introduced and analyzed a finite element approximation that converges without the need to maintain monotonicity. In this approach, the underlying PDE is reformulated into a second-order bilinear form, i.e., a form in two variables which is nonlinear in at least one of its arguments, which they then discretize employing a discontinuous Galerkin finite element scheme.
The provided convergence analysis relies on the strong monotonicity of the binonlinear form, considered as a generalization of coercivity for nonlinear operators, but as a functional rather than in discretization.

In Gallistl and S"uli [2019], a similar convergence argument is employed, but with a distinction: the HJB equation is treated as a variational binonlinear form problem, utilizing a conforming finite element method. The fundamental analytical tool in this work is also the strong monotonicity of the binonlinear form.

There are two strategies to deal with nonlinear PDEs such as the HJB equation. In the first strategy, the nonlinear problem is discretized and the resulting nonlinear finite-dimensional system is linearized with a nonlinear solver such as Newton’s method. Many computational methods for approximating the solution of the HJB equation follow this approach [Oberman, 2006; Smears and S"uli, 2014; Gallistl and S"uli, 2019]. In this scenario, if either the discretization is monotone or, in the functional setting, the binonlinear form is strongly monotone, the error analysis of the method becomes possible. Furthermore, strong monotonicity establishes the framework for applying the Browder-Minty theorem to demonstrate the well-posedness of solutions to nonlinear equations. It is however not always possible to achieve a monotone discretization or a strongly monotone binonlinear form for a fully nonlinear problem. To the best of our knowledge, all methods that provide an approximate solution and offer a satisfactory convergence analysis for the HJB equation follow this strategy by somehow enforcing monotonicity.

In a second strategy, which we pursue here, the nonlinear PDE is first linearized, for example, by using Newton’s method in the appropriate infinite-dimensional space, into an iterative sequence of linear PDEs in nondivergence form at the continuum level. Subsequently, these linear PDEs are discretized. By following this strategy, the convergence rate of the nonlinear solver is independent of the discrete space parameters, such as the meshsize or polynomial degree in Galerkin methods. Furthermore any numerical method applicable to linear problems in nondivergence form can be extended to fully nonlinear problems, such as HJB equation we study herein. An instance of the second strategy is Lakkis and Pryer [2011, 2013], who derived a computational method for linear elliptic problems in nondivergence form used to solve the linearized iteration for a class of fully nonlinear problems in which the nonlinearity is algebraic, i.e., a nonlinearity that can be written without resorting to sup operations with infinitely many $\alpha$s (e.g., the Monge–Amp` ere).

In this paper, we adopt the strategy of first-linearize-then-discretize. In the linearization step we establish the Newton differentiability of the HJB operator from $H^2(\Omega)$ to $L_2(\Omega)$, subject to a uniform convergence of controls Assumption 3.6.

While Smears and S"uli [2014] have demonstrated this concept for the operator from $W^2_0(\Omega; T)$ to $L_l(\Omega)$, where $1 \leq l < s \leq \infty$, these spaces are mesh-dependent and encompass only finite element spaces (not $H^2(\Omega) \to L_2(\Omega)$).

The Newton differentiability of the HJB operator allows us to extend the classical Newton linearization to the HJB operator, even when it is not necessarily Fréchet differentiable. This extended method is referred to as the semismooth Newton linearization [Hinterm"uller, 2010; Ito and Kunisch, 2008]. With this approach, the solution of the nonlinear HJB equation is realized as the limit of a recursive sequence of solutions to linear problems in nondivergence form. We complete the linearization theory by demonstrating the superlinear convergence rate of this recursive method.

We also introduce the algorithmic form of the resulting recursive method, known as Howard’s algorithm or policy iteration, which, to the best of our knowledge, is presented here for the first time in the infinite dimensional setting for HJB PDEs.
At each iteration, this algorithm updates a space-dependent function, denoted as $q$ and taking values in the control set $A$. This function $q$ determines the linear operator $L^q$ as seen in (3.24). Consequently, Howard’s algorithm yields a sequence of control-seeking parts and nondivergence form PDE solvers, essentially constituting a parametrized linearization procedure.

In the context of connecting fully nonlinear and linear problems, the bridge is formed by nondivergence form linear operators. To discretize PDEs in this scenario, we employ a least-squares Galerkin gradient recovery method. This approach offers a straightforward means of dealing with linear equations in nondivergence form. It can be interpreted as a mixed finite element technique and provides a convenient framework for deriving a posteriori error estimates and corresponding adaptive methods, as demonstrated in [2021].

The least-squares approach enables us to replace any constraint required to ensure the problem is well-posed with an additional term in the quadratic (least-squares) form. This flexibility to work in a general space is a significant advantage, as constructing finite element approximations that exactly satisfy conditions like rotational-free or vanishing tangential trace can be challenging.

The rest of our article is arranged as follows: in §2, we clarify the problem assumptions and the existence theory concerning to the well-posedness of the strong solution for the HJB equation.

In §3, we introduce the concept of Newton differentiability for operators and illustrate its application to the HJB operator in the continuous setting, particularly from $H^2(\Omega)$ to $L^2(\Omega)$. This is done under the assumption of uniform convergence on the policy operators, as outlined in Assumption 3.6. We also discuss the use of the semismooth Newton method for linearizing the HJB equation. Subsequently, we present the algorithmic expression of the linearization procedure in the form of Howard’s algorithm, which involves solving a second-order elliptic equation in nondivergence form. In §4, we provide a review of the least-squares Galerkin approach, which includes gradient recovery as proposed in [2021]. This method is employed to solve linear elliptic PDEs in nondivergence form, and we also revisit the key error bounds associated with it. In §5, we return to the nonlinear HJB equation and discuss the error analysis of the approximation, which includes both the associated a priori and a posteriori error estimates. Owing to the a posteriori error bound, we design error indicators to be employed in an adaptive refinement strategy. Additionally, we outline the algorithms used during the implementation phase to approximate the solution of the HJB equation. In §6 we present two numerical tests that validate the theoretical results.

2. Set-up and notation

We now briefly review ellipticity and Cordes conditions in §2.1, reformulate the HJB-Dirichlet problem (1.1) into a homogeneous Dirichlet problem in §2.2 and recall that it, and thus the original heterogeneous version (1.1), admits a unique strong solution under $L^\infty(\Omega; C^0(A))$ data assumptions in §2.3–§2.5.

2.1. Assumptions on the data. Throughout this paper, concerning the BVP (1.1), we suppose that the coefficients $A^\alpha$ satisfy the uniform ellipticity condition

$$C_{2.1.4.1} I \leq A^\alpha \leq C_{2.1.4.2} I,$$

a.e. in $\Omega$, for each $\alpha \in A$ for some positive constants $C_{2.1.4.1}$ and $C_{2.1.4.2}$ independent of $\alpha \in A$, while the tensor-valued $A^\alpha$, vector-valued $b^\alpha$, nonnegative scalar-valued $c^\alpha$ satisfy satisfy one of the Cordes conditions as outlined by [2014]. These conditions are as follows
(a) there exists \( \lambda > 0 \) and \( \varepsilon \in (0, 1) \) such that for \( \alpha \in \mathcal{A} \),

\[
\frac{|A^\alpha|^2 + |b^\alpha|^2/2 \lambda + (c^\alpha/\lambda)^2}{(\text{tr} A^\alpha + c^\alpha/\lambda)^2} \leq \frac{1}{d + \varepsilon} \quad \text{a.e. in } \Omega,
\]

where for a tensor/matrix \( M \in \mathbb{R}^{n \times n} \), \( |M| \) signifies its Frobenius norm, \((M : M)^{1/2}\).

(b) or in the case of a homogeneously second-order \( \mathcal{L}^\alpha \), i.e., \( b^\alpha = 0 \) and \( c^\alpha = 0 \), take \( \lambda = 0 \) and replace

\[
\frac{|A^\alpha|^2}{(\text{tr} A^\alpha)^2} \leq \frac{1}{d - 1 + \varepsilon} \quad \text{a.e. in } \Omega
\]

for some \( \varepsilon \in (0, 1) \).

For more details on the Cordes condition see [Lakkis and Mousavi 2021 §2.2]. We take the admissible control (also known as policy) set \( \mathcal{A} \) to be a compact metric space with distance \( d_A \). The ensuing topology is used in defining spaces \( C^H(\mathcal{A}; X) \) with \( X = \mathbb{R}, \mathbb{R}^d \), or \( \mathbb{R}^{d \times d} \). Most often \( \mathcal{A} \) is in fact subset (e.g., a Lie group or a subspace) of the matrix algebra \( \mathbb{R}^{d \times d} \); we give examples in [6.1 and 6.2]. We denote by \( B^\alpha_1(\alpha) \) the open ball of center \( \alpha \) and radius \( \rho \geq 0 \) with \( d_A \) scale.

2.2. Homogeneous Dirichlet problem reformulation of (1.1). Since \( r \in H^{3/2}(\partial\Omega) \), it admits an extension to \( \tilde{r} \in H^2(\Omega) \) with boundary trace \( r (\tilde{r}|_{\partial\Omega} = r) \) satisfying

\[
||\tilde{r}||_{H^2(\Omega)} \leq C(r) ||r||_{H^{3/2}(\partial\Omega)}
\]

for some \( C(r) > 0 \) depending only on \( \Omega \). By setting \( v = u - \tilde{r} \), we rewrite (1.1) as the homogeneous problem of finding \( v \) such that

\[
\sup_{\alpha \in \mathcal{A}} (\mathcal{L}^\alpha v - f^\alpha + \mathcal{L}^\alpha \tilde{r}) = 0 \quad \text{in } \Omega \quad \text{and} \quad v|_{\partial\Omega} = 0.
\]

Therefore, we consider the homogeneous problem of finding \( u \) satisfying

\[
\sup_{\alpha \in \mathcal{A}} (\mathcal{L}^\alpha u - f^\alpha) = 0 \quad \text{in } \Omega \quad \text{and} \quad u|_{\partial\Omega} = 0.
\]

We shorten notation by introducing the HJB operator \( \mathcal{F} : H^2(\Omega) \cap H^1_0(\Omega) \rightarrow L^2(\Omega) \) by

\[
\mathcal{F}[v](x) = \sup_{\alpha \in \mathcal{A}} (\mathcal{L}^\alpha v(x) - f^\alpha(x)) \quad \text{for } x \in \Omega.
\]

2.3. Theorem (existence and uniqueness of a strong solution [Smears and Süli 2014]). Suppose that \( \Omega \) is a bounded convex domain in \( \mathbb{R}^d \), \( \mathcal{A} \) is a compact metric space under \( d_A \), and that \( A, b, c, f \in C^0(\Omega \times \mathcal{A}; X) \), for \( X = \text{Sym}(\mathbb{R}^d), \mathbb{R}^d, \mathbb{R}, \mathbb{R} \), satisfy (2.1) and (2.2) with \( \lambda > 0 \), or (2.3) with \( \lambda = 0 \) when \( b = 0 \) and \( c = 0 \) hold. Then there exists a unique function \( u \in H^2(\Omega) \cap H^1_0(\Omega) \) that satisfies the HJB equation (2.6) almost everywhere in \( \Omega \).

2.4. Remark (strong solution of the nonhomogeneous equation). Since (2.5) and (1.1) are equivalent, under the assumptions of Theorem 2.3 and for \( r \in H^{3/2}(\partial\Omega) \), there exists a unique strong solution \( u \in H^2(\Omega) \) to nonhomogeneous HJB equation (1.1).

2.5. Remark (\( L_\infty(\Omega) \) vs \( C^0(\Omega) \) data). The sufficient requirement on data to be \( C^0(\Omega) \) in Theorem 2.3 can be relaxed to be just \( L_\infty(\Omega) \). This allows us to consider \( A, b, c \) in \( L_\infty(\Omega; C^0(\mathcal{A}; X)) \) for \( X = \text{Sym}(\mathbb{R}^d), \mathbb{R}^d, \mathbb{R} \) respectively, while for \( X = \mathbb{R} \) we omit it in the notation. We also assume \( f \in L^2(\Omega; C^0(\mathcal{A})) \).
3. Semismooth Newton method

Due to the nonalgebraic nonlinearity of the HJB equation, applying a linearization methods like classical Newton is not trivial even when the operator $F$ defined in (2.7) has everywhere an invertible derivative: the problem is that such a derivative cannot be explicit found, while it can be realised by finding the appropriate $\alpha = q(x)$ for an approximating function $q : \Omega \to A$. In this section, we describe the semismooth Newton method for linearizing the fully nonlinear problem (1.1), which involves nonsmooth nonlinear operators. The basic tool here is the concept of $\text{Newton derivative}$ of $F$, a set-valued operator $\mathcal{D}F$, introduced by 

[118x577]Süli [2014] demonstrated the Newton differentiability of $F$ from $W^2_0(\Omega,T)$ to $L^2(\Omega)$ with $1 \leq l < s \leq \infty$, Theorem 3.9 extends this result for $F$ from $H^2(\Omega)$ to $L^2(\Omega)$ under an assumption on the control set $A$ described in (3.1). We then outline in some properties of the $\mathcal{D}F[v]$’s members that will ensure the superlinear convergence of the semismooth Newton method in §3.12. §3.13. We close in §3.14 with an algorithmic presentation of the semismooth Newton linearization known as Howard’s algorithm or policy iteration.

3.1. Policy and set-valued maps. Introduce the policy map set (also known as control map set)

\begin{equation}
Q := \{ q : \Omega \to A \mid q \text{ is measurable} \}
\end{equation}

and the set-valued $\text{state-to-policy}$ operator $\mathcal{N} : H^2(\Omega) \cap H^1_0(\Omega) \rightrightarrows \mathcal{Q}$ by

\begin{equation}
\mathcal{N}[v] := \left\{ q \in \mathcal{Q} : q(x) \in \text{Argmax}_{\alpha \in \mathcal{A}} (\mathcal{L}^\alpha v(x) - f^\alpha(x)) \text{ for a.e. } x \in \Omega \right\},
\end{equation}

where by the notation $\mathcal{M} : X \rightrightarrows Y$, we mean that $\mathcal{M}$ is a powerset-valued, $2^Y$-valued, map, meaning that for every $x \in X$, $M(x)$ is a subset of $Y$.

3.2. Lemma (state-to-policy operator is well defined and continuous). For any $v \in H^2(\Omega) \cap H^1_0(\Omega)$, the sets $\mathcal{N}[v]$ is nonempty. Moreover, if $(v_j)_{j \in \mathbb{N}}$ be a sequence such that $v_j \to v$ in $H^2(\Omega)$ and $(q_j)_{j \in \mathbb{N}}$ be a sequence in which $q_j \in \mathcal{N}[v_j]$, then

\begin{equation}
\lim_{j \to \infty} \inf_{q \in \mathcal{N}[v_j]} d_A(q_j, q) = 0 \text{ a.e. in } \Omega,
\end{equation}

with reminder that $d_A$ is a metric on $A$.

Proof. To prove that $\mathcal{N}[v]$ is nonempty, we refer to Theorem 10 of Smears and Süli [2014]. For almost every $x \in \Omega$, we prove (3.3) by contradiction. Suppose that there exist a sequence $(v_j)_{j \in \mathbb{N}}$ convergent to $v$ in $H^2(\Omega) \cap H^1_0(\Omega)$, a subsequence $(q_j)_{j \in \mathbb{N}}$, $q_j \in \mathcal{N}[v_j]$ and a real $\varepsilon > 0$ such that

\begin{equation}
d_A(q_j(x), q(x)) > \varepsilon \text{ for all } j \in \mathbb{N} \text{ and for all } q \in \mathcal{N}[v].
\end{equation}

It means that $q_j(x) \in \mathcal{A} \setminus \bigcup_{q \in \mathcal{N}[v]} B^\varepsilon_A(q(x))$ where

\begin{equation}
B^\varepsilon_A(q(x)) := \{ \alpha \in \mathcal{A} : d_A(\alpha, q(x)) < \varepsilon \}.
\end{equation}

Since $\mathcal{A} \setminus \bigcup_{q \in \mathcal{N}[v]} B^\varepsilon_A(q(x))$ is compact, there exists a subsequence, which we pass without changing notation, such that $q_j(x) \to \tilde{\alpha} \in \mathcal{A} \setminus \bigcup_{q \in \mathcal{N}[v]} B^\varepsilon_A(q(x))$. On the other hand, $q_j \in \mathcal{N}[v_j]$ implies that

\begin{equation}
(\mathcal{L}^\tilde{\alpha} v_j - f^\tilde{\alpha})(x) \geq (\mathcal{L}^\alpha v_j - f^\alpha)(x) \forall j \in \mathbb{N}.
\end{equation}
Since \( v_j \to v \) in \( H^2(\Omega) \), again we pass to a subsequence without changing notation such that \( v_j, \nabla v_j, D^2v_j \) tend to \( v, \nabla v, D^2v \) respectively pointwise almost everywhere in \( \Omega \). Taking the limit of the above inequality as \( j \to \infty \) shows that \( \bar{\alpha} \in N[v](x) \). This is a contradiction with \( \alpha \in A \setminus \bigcup_{q \in N[v]} B^\delta_q(\tilde{q}(x)) \).

3.3. Remark (approximating policy sequence selection). Arguing by contradiction we can also show that if the sequence of functions \( (v_j)_{j \in \mathbb{N}} \subset H^2(\Omega) \cap H^1_0(\Omega) \) converges to \( v \in H^2(\Omega) \), then for any \( q \in N[v] \) there exists a sequence of policies \( (q_j)_{j \in \mathbb{N}} \) such that \( q_j \in N[v_j] \) and \( q_j \) tends to \( q \) almost everywhere in \( \Omega \).

3.4. Proposition (policy-to-coefficient lower semicontinuity). Recall that \( A, b, c \) are in \( L_\infty(\Omega; C^0(\mathcal{A}; X)) \) for \( X = \text{Sym}(\mathbb{R}^d), \mathbb{R}^d, \mathbb{R} \) respectively. Under the assumptions of Lemma 3.2, we have

\[
(3.7) \quad \lim_{j \to \infty} \inf_{q \in N[v]} \left( |A^{q_j} - A^q| + |b^{q_j} - b^q| + |c^{q_j} - c^q| \right) = 0 \text{ a.e. in } \Omega.
\]

Proof. We want to prove that for almost all \( x \in \Omega \)

\[
(3.8) \quad \forall \varrho > 0 \quad \exists N \in \mathbb{N} : \forall j \geq N \quad \inf_{q \in N[v]} \left[ |A^{q_j} - A^q| + |b^{q_j} - b^q| + |c^{q_j} - c^q| \right] < \varrho.
\]

Since \( A, b, c \) are continuous on \( A \), so for almost all \( x \in \Omega \) we deduce that for any \( \varrho > 0 \) there is some \( \delta > 0 \) such that

\[
(3.9) \quad \inf_{q \in N[v]} d_{\mathcal{A}}(q_j(x), q(x)) < \delta \Rightarrow \inf_{q \in N[v]} \left[ |A^{q_j} - A^q| + |b^{q_j} - b^q| + |c^{q_j} - c^q| \right] < \varrho.
\]

Relation (3.3) guarantees that for large enough \( N \in \mathbb{N} \)

\[
(3.10) \quad \forall j \geq N \Rightarrow \inf_{q \in N[v]} d_{\mathcal{A}}(q_j(x), q(x)) < \delta.
\]

The assertion now follows from (3.9) and (3.10). \( \square \)

3.5. Assumption on convergence of policy sequence. While we do not use the results of Lemma 3.2 and Proposition 3.4 explicitly in this paper, we have presented them to establish the foundation for the following assumption. Although in Lemma 3.2 we have demonstrated that the sequence in (3.3) converges pointwise, in the following assumption, we assume that it converges uniformly. This assumption implies uniform convergence of the sequence in (3.7) as well and this uniform convergence ensures the Newton differentiability of the HJB operator, thereby motivating the the following assumption.

3.6. Assumption (Uniform convergence of policy sequences). For \( (v_j)_{j \in \mathbb{N}} \) as a sequence such that \( v_j \to v \) in \( H^2(\Omega) \) and \( (q_j)_{j \in \mathbb{N}} \) as a sequence in which \( q_j \in N[v_j] \), we suppose that

\[
(3.11) \quad \lim_{j \to \infty} \left\| \inf_{q \in N[v]} d_{\mathcal{A}}(q_j, q) \right\|_{L_\infty(\Omega)} = 0.
\]

Under this assumption, by employing a reasoning process analogous to the proof presented in Proposition 3.4, we can deduce the following stronger convergence

\[
(3.12) \quad \lim_{j \to \infty} \left\| \inf_{q \in N[v]} \left( |A^{q_j} - A^q| + |b^{q_j} - b^q| + |c^{q_j} - c^q| \right) \right\|_{L_\infty(\Omega)} = 0.
\]

In accordance with [Smears and Suli [2014] Remark 1], the Newton differentiability of the HJB operator, denoted as \( F : H^2(\Omega) \to L_s(\Omega) \) and defined in (2.7), generally cannot be guaranteed unless \( s > l \). To overcome this limitation for case \( s = l = 2 \), we make Assumption 3.6 allowing us to ensure Newton differentiability. We would
like to emphasize that Example 8.14 in [Ito and Kunisch 2008], which is as an illustration of non-Newton differentiability in the case of $s \leq l$, does not satisfy Assumption 3.6. This is primarily a result of nonzero differences between the values of the sequence policy operators and their limit, which prevent uniform convergence.

3.7. Definition of Newton differentiable operator. Following [Ito and Kunisch 2008], for $\mathcal{X}$ and $\mathcal{Z}$ Banach spaces and $\emptyset \subseteq B \subset \mathcal{X}$, a map $F : B \subset \mathcal{X} \to \mathcal{Z}$ is called Newton differentiable at $x \in B$ if and only if there exists an open neighborhood $N(x) \subset B$ and set-valued map with nonempty image $\mathcal{D}F[x] : N(x) \ni \text{Lin}(\mathcal{X} \to \mathcal{Z})$ such that

$$\lim_{\|e\|_{\mathcal{X}} \to 0} \frac{1}{\|e\|_{\mathcal{X}}} \sup_{D \in \mathcal{D}F[x+e]} \|F[x+e] - F[x] - De\|_{\mathcal{Z}} = 0,$$

(3.13)

while $\text{Lin}(\mathcal{X} \to \mathcal{Z})$ denotes the set of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Z}$. The operator set $D$ is called a Newton derivative of $F$ at $x$ and any member of $\mathcal{D}F[x]$ is a Newton derivative of $F$ at $x$. The operator $F$ is called Newton differentiable on $B$ with Newton derivative $\mathcal{D}F : B \ni \text{Lin}(\mathcal{X} \to \mathcal{Z})$ if $F$ is Newton differentiable at each $x \in B$.

The Newton derivative may not be unique, but when $\mathcal{D}F[x]$, the Fréchet derivative of $F$ at $x$, exists we have $\mathcal{D}F[x] = \{D[e]x\}$.

3.8. The Newton derivative of the HJB operator. We now introduce a suitable candidate for the Newton derivative of the operator $F$ in (2.7) at $v$ by

$$\mathcal{D}F : H^2(\Omega) \cap H_0^1(\Omega) \ni \text{Lin}(H^2(\Omega) \cap H_0^1(\Omega) \to L_2(\Omega))$$

(3.14)

$$\mathcal{D}F[v] := \{\mathcal{L}^q := (A^q : D^2v + b^q \cdot \nabla v - e^q) \in \mathcal{N}[v] : q \in H^2(\Omega) \cap H_0^1(\Omega),$$

where $\mathcal{N}[v]$ is defined by (3.2). The term $\mathcal{D}F[v]$ in (3.14) is a Newton derivative candidate for (2.7). We investigate this issue in Theorem 3.9 where we extend Smears and Suli [2014] Thm. 13 under the Assumption 3.6 to cover the case $H^2(\Omega) \cap H_0^1(\Omega)$.

3.9. Theorem (Newton differentiability of the HJB operator). Suppose that $\Omega$ is a bounded convex domain, $\mathcal{A}$ is a compact metric space, $\mathcal{A}, b, c$ are in $L_\infty(\Omega; C(\mathcal{A}; X))$ for $X = \text{Sym}(\mathbb{R}^d), \mathbb{R}^d, \mathbb{R}$ respectively and $f \in L_2(\Omega; C(\mathcal{A}))$, which satisfy Assumption 3.6. Then, the HJB operator $F$ defined by (2.7) is Newton differentiable with Newton derivative $\mathcal{D}F$, defined by (3.14), on $H^2(\Omega) \cap H_0^1(\Omega)$.

Proof. Suppose that $(e_j)_{j \in \mathbb{N}} \subset H^2(\Omega) \cap H_0^1(\Omega)$ be a sequence with $\|e_j\|_{H^2(\Omega)} \to 0$ and $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Let $q_j \in \mathcal{N}[v + e_j]$ and $q \in \mathcal{N}[v]$. We have

$$F[v + e_j] - F[v] - \mathcal{L}^q e_j = \mathcal{L}^{q_j} v - f^{q_j} - F[v] \leq 0 \text{ a.e. in } \Omega.$$  

(3.15)

On the other hand, we get

$$0 \leq F[v + e_j] - (\mathcal{L}^q(v + e_j) - f^q) = F[v + e_j] - F[v] - \mathcal{L}^q e_j$$

(3.16)

$$= F[v + e_j] - F[v] - \mathcal{L}^{q_j} e_j + \mathcal{L}^{q_j} e_j - \mathcal{L}^q e_j \text{ a.e. in } \Omega.$$  

(3.16) implies the first and (3.15) implies the second inequality of the following

$$\mathcal{L}^q e_j - \mathcal{L}^{q_j} e_j \leq F[v + e_j] - F[v] - \mathcal{L}^{q_j} e_j \leq 0 \text{ a.e. in } \Omega.$$  

(3.17)

By applying the absolute value to both sides, we get

$$|F[v + e_j] - F[v] - \mathcal{L}^{q_j} e_j| \leq |\mathcal{L}^q e_j - \mathcal{L}^{q_j} e_j|$$

(3.18)

$$= |(A^q - A^{q_j}) : D^2 e_j + (b^q - b^{q_j}) \cdot \nabla e_j - (c^q - c^{q_j}) e_j| \text{ a.e. in } \Omega.$$
There exists $C_{3.19}$ depending on the dimension $d$, such that
\begin{equation}
(3.19) \quad \| F[v + e_j] - F[v] - L^{g_j} e_j \|_{L^2(\Omega)} \\
\leq C_{3.19} \inf_{q \in N[v]} (|A^{q} - A^{q}| + |b^{q} - b^{q}| + |c^{q} - c^{q}|) \left( \| D^2 e_j \| + |\nabla e_j| + |e_j| \right) \quad \text{a.e. in } \Omega.
\end{equation}

Therefore, we have
\begin{equation}
(3.20) \quad |F[v + e_j] - F[v] - L^{g_j} e_j| \\
\leq C_{3.19} \inf_{q \in N[v]} (|A^{q} - A^{q}| + |b^{q} - b^{q}| + |c^{q} - c^{q}|) \left( \| D^2 e_j \| + |\nabla e_j| + |e_j| \right) \quad \text{a.e. in } \Omega.
\end{equation}

By taking $L^2(\Omega)$-norm of the both sides and then using the generalized Hölder inequality on the right hand side, we arrive at
\begin{equation}
(3.21) \quad \| F[v + e_j] - F[v] - L^{g_j} e_j \|_{L^2(\Omega)} \\
\leq C_{3.19} \left( \inf_{q \in N[v]} (|A^{q} - A^{q}| + |b^{q} - b^{q}| + |c^{q} - c^{q}|) \left( \| D^2 e_j \| + |\nabla e_j| + |e_j| \right) \right)_{L^2(\Omega)}.
\end{equation}

After dividing the both sides by $\| e_j \|_{L^2(\Omega)}$, the limit (3.12) implies that
\begin{equation}
(3.22) \quad \lim_{\| e_j \|_{L^2(\Omega)} \to 0} \frac{1}{\| e_j \|_{L^2(\Omega)}} \| F[v + e_j] - F[v] - L^{g_j} e_j \|_{L^2(\Omega)} = 0.
\end{equation}

The assertion now follows from that (3.22) holds for any $q_j \in N[v_j]$. \hfill \Box

3.10. **Remark (relaxing assumptions of Theorem 3.9).** We note two points relating to Theorem 3.9

- Apart from Assumption 3.6, we did not require the ellipticity nor the Cordes condition to show Newton differentiability of $F$ of (2.1) on the infinite-dimensional space. One may employ the semismooth Newton method to HJB equations under weaker conditions.
- It is easy to check that we can also show this concept for the HJB operator on general space $H^2(\Omega)$, which allows more general Dirichlet boundary conditions and possibly oblique (second) boundary conditions.

3.11. **Lemma (bounded invertibility of the Newton derivatives).** Under the assumptions of Theorem 3.9 on $\Omega$, $A, b, c$ and $f$ and assuming that $A, b, c$ satisfy (2.1) and either (2.2) with $\lambda > 0$ or (2.3) with $\lambda = 0$ when $b = 0$, $c = 0$, we have that for any $v \in H^2(\Omega) \cap H^0_0(\Omega)$ the all members $L^q \in D[F[v]]$, which $D[F[v]]$ is defined by $(3.14)$, are nonsingular and $\| L^q^{-1} \|$ are bounded.

**Proof.** For any $v \in H^2(\Omega) \cap H^0_0(\Omega)$ and $q \in N[v]$, thanks to Theorem 2.3 for a fixed control map $\alpha = q(\cdot)$, we deduce that $\{ L^q \}$ are invertible and also bijective. It follows from $A, b, c \in L_\infty(\Omega; C^0(\mathbb{A}; X))$ for $X = \text{Sym}(\mathbb{R}^d), \mathbb{R}^d, \mathbb{R}$ respectively that $\{ L^q \}$ are bounded. Therefore, Banach’s inverse mapping theorem implies that $\{ L^q^{-1} \}$ are also bounded. \hfill \Box

3.12. **Theorem (superlinear convergence of the semismooth Newton method).** Suppose that the operator $F$ is Newton differentiable with Newton derivative $D F$ in an open neighborhood $U$ of $u$, solution of $F[u] = 0$. If for any $\tilde{u} \in U$, all $D \in D[F[\tilde{u}]]$ are invertible and $\| (D)^{-1} \|$ are bounded, then the iteration
\begin{equation}
(3.23) \quad u_{n+1} = u_n - D_n^{-1} F[u_n], \quad D_n \in D[F[u_n]]
\end{equation}
converges superlinearly to $u$ provided $u_0$ is sufficiently close to $u$. For a proof see Ito and Kunisch [2008] Thm. 8.16.
3.13. Corollary (superlinear convergence of the linearized HJB equation).
Let the operator \( F \) as (2.7). Lemma 3.11 and Theorem 3.12 imply that by choosing an initial guess \( u_0 \) sufficiently close to the exact solution \( u \in H^2(\Omega) \cap H^1_0(\Omega) \) of (2.6), the solution of recursive problem (3.23) converges superlinearly to \( u \). By acting \( D_n = \mathcal{L}^{q_0} \) on the both sides of (3.24), it can be rewritten as

(3.24) \[ \mathcal{L}^{q_0} u_{n+1} = f^{q_0} \text{ a.e. in } \Omega. \]

3.14. Howard’s algorithm. We now present the recursive problem (3.24) algorithmically, known as Howard’s algorithm or policy iteration at the continuous level. Howard’s algorithm alternately computes a control and a value function to generate a sequence that converges to the solution of (2.6). The process is repeated until the convergence criterion is satisfied. This algorithm can be interpreted as a Newton extension to the equation with a nonsmooth operator (2.6). Howard’s algorithm for (2.6) with initial guess of \( u_0 \) is described as follows.

Compute the control map \( q_0 \) such that

(3.25) \[ q_0(x) \in \text{Argmax}_{\alpha \in A} (\mathcal{L}^{\alpha} u_0 - f^{\alpha})(x) \text{ for a.e. } x \in \Omega; \]

Solve the linear PDE

(3.26) \[ \mathcal{L}^{q_0} u_1 - f^{q_0} = 0 \text{ in } \Omega \text{ and } u_1|_{\partial \Omega} = 0; \]

for \( n \geq 1 \) do

Update the control map \( q_n \) such that

(3.27) \[ q_n(x) \in \text{Argmax}_{\alpha \in A} (\mathcal{L}^{\alpha} u_n - f^{\alpha})(x) \text{ for a.e. } x \in \Omega; \]

Solve the linear PDE

(3.28) \[ \mathcal{L}^{q_n} u_{n+1} - f^{q_n} = 0 \text{ in } \Omega \text{ and } u_{n+1}|_{\partial \Omega} = 0; \]

end for

To follow Howard’s algorithm, we need to solve the optimization problem (3.27) and the linear problem in nondivergence form (3.28) in each iteration. Lemma 3.12 implies that for any \( u_n \in H^2(\Omega) \cap H^1_0(\Omega), N[u_n] \) is nonempty, ensuring the existence of a solution to (3.27) in each iteration. For Howard’s algorithm to proceed, we also require the existence of a solution to (3.28) in each iteration. Since for each \( n \in \mathbb{N}, A^{q_n}, b^{q_n}, c^{q_n} \) satisfy the Cordes condition either (2.2) or (2.3), Theorem 2.3 ensures well-posedness of the strong solution \( u_n \in H^2(\Omega) \cap H^1_0(\Omega) \) to (3.28) for fixed \( \alpha = q_n(\cdot) \). Indeed, the strong solution \( u \) represents a fixed-point iteration of the algorithm.

Remark (2.4) and the Newton differentiability of the HJB operator on \( H^2(\Omega) \), as pointed out in Remark 3.10 imply that we can directly adapt Howard’s algorithm to the nonhomogeneous HJB equation (1.1). In this case, we need to replace the boundary condition \( u_1, u_{n+1}|_{\partial \Omega} = r \) in (3.26) and (3.28).

4. A least-squares Galerkin gradient recovery for linear nondivergence form elliptic PDEs

We review here the least-squares Galerkin gradient recovery method for linear equations in nondivergence form as proposed by Lakkis and Mousavi [2021] and improve it for applying to solve (3.26) and (3.28) in the context of Howard’s algorithm. Our improvement involves imposing a tangential trace constraint on the cost functional instead of the function space, which relies on a generalized Maxwell’s inequality presented in Lemma 4.3, extending its analogue in Lakkis and Mousavi [2021]. This enables us to derive a Miranda–Talenti type estimate in Theorem 4.6, which is crucial in proving Theorem 4.7 that the problem is well-posed using a...
Lax–Milgram argument. This sets the foundation for a finite element discretization
in Section 4.10 for which we derive a priori error estimates in Theorem 4.12 and a
posteriori estimates in Theorem 4.14.

4.1. Least-squares Galerkin for linear nondivergence form PDE. Consider
the second order elliptic linear equation in nondivergence form
\[ Lu := A \cdot \nabla^2 u + b \cdot \nabla u - cu = f \quad \text{in } \Omega \quad \text{and} \quad u_{|\partial \Omega} = r \]
where \( A, b, c \in L_\infty(\Omega; X) \) for \( X = \text{Sym}(\mathbb{R}^d), \mathbb{R}^d, \mathbb{R}^{\geq 0} \) respectively. These elements
satisfy the uniform ellipticity condition (2.1) and the Cordes condition either (2.2)
with \( \lambda > 0 \) or (2.3) with \( \lambda = 0 \) when \( b = 0, c = 0, \) and \( f \in L_2(\Omega), r \in H^{1/2}(\partial \Omega). \)

Well-posedness of the strong solution to (4.1) is achieved via Theorem 2.3, Re-
mark 2.4 and Remark 2.5. But dealing with such sizeable regular function space
of finding a unique pair of the form
\[ u = \psi \quad \text{and solving the minimization problem} \quad (4.8) \]
where \( \psi \) indicate the minimum and maximum of two values \( a, b \) and \( (4.7) \),
we propose to consider an alternative equivalent problem in which its solution resides
in a weaker space. To facilitate this, we introduce some notations.

We denote the outer normal to \( \Omega \) by \( n_\Omega(x) \) for almost all \( x \) on \( \partial \Omega \). The tangen-
tial trace of \( \psi \in H^1(\Omega; \mathbb{R}^d) \) is defined by
\[ t_\Omega \psi := (I - n_\Omega n_\Omega \cdot) \psi_{|\partial \Omega}. \]
We inset the function space
\[ W := \{ \psi \in H^1(\Omega; \mathbb{R}^d) \mid t_\Omega \psi = 0 \}, \]
equipped with the \( H^1 \)-norm and consider the following norm for \( H^1(\Omega) \times H^1(\Omega; \mathbb{R}^d), \)
\[ \| (\varphi, \psi) \|_{H^1(\Omega)}^2 := \| \varphi \|_{H^1(\Omega)}^2 + \| \psi \|_{H^1(\Omega)}^2 \]
for each \( (\varphi, \psi) \in H^1(\Omega) \times H^1(\Omega; \mathbb{R}^d) \).

We denote \( L_2(\Omega)/L_2(\partial \Omega) \) inner product of two scalar/vector/tensor-valued functions by
\[ \langle \varphi, \psi \rangle := \int_\Omega \varphi(x) \star \psi(x) \, dx, \quad \langle \varphi, \psi \rangle_{\partial \Omega} := \int_{\partial \Omega} \varphi(x) \star \psi(x) \, dS(x) \]
where \( \star \) stands for one of the arithmetic, Euclidean-scalar, or Frobenius inner
product in \( \mathbb{R}, \mathbb{R}^d, \) or \( \mathbb{R}^{d \times d} \) respectively. The notations \( a \wedge b, a \vee b \) respectively
indicate the minimum and maximum of two values \( a, b \).

By considering \( 0 \leq \theta \leq 1 \), we define the linear operator
\[ \mathcal{M}_\theta : H^1(\Omega) \times H^1(\Omega; \mathbb{R}^d) \to L_2(\Omega) \]
\[ (\varphi, \psi) \mapsto A : D\psi + b \cdot (\theta \psi + (1 - \theta) \nabla \varphi) - c\varphi = : \mathcal{M}_\theta(\varphi, \psi). \]

Initially, we assume that the boundary condition is zero, i.e., \( u_{|\partial \Omega} = 0 \). In this
regard, we introduce the following quadratic functional on \( H^1_0(\Omega) \times H^1(\Omega; \mathbb{R}^d) \)
\[ E_\theta(\varphi, \psi) := \| \nabla \varphi - \psi \|_{L_2(\Omega)}^2 + \| \nabla \times \psi \|_{L_2(\Omega)}^2 + \| t_\Omega \psi \|_{L_2(\Omega)}^2 + \| \mathcal{M}_\theta(\varphi, \psi) - f \|_{L_2(\Omega)}^2 \]
where \( \nabla \times \psi \) indicates curl of \( \psi \). We then deal with the convex minimization problem
of finding a unique pair of the form
\[ (u, g) = \arg\min_{(\varphi, \psi) \in H^1_0(\Omega) \times H^1(\Omega; \mathbb{R}^d)} E_\theta(\varphi, \psi). \]

The key point is that the problem of finding the strong solution \( u \) to (4.1) with
\( u_{|\partial \Omega} = 0 \) and solving the minimization problem (4.8) are equivalent and \( g = \nabla u \)
holds as a consequence of (4.8). In the rest of the paper, \( g \) will be synonymous with
\( \nabla u \). The Euler–Lagrange equation of the minimization problem \((4.8)\) includes in finding \((u, g) \in H^1_0(\Omega) \times H^1(\Omega; \mathbb{R}^d)\) such that
\[
\langle \nabla u - g, \nabla \varphi - \psi \rangle + \langle \nabla \times g, \nabla \times \psi \rangle + \langle t_\Omega g, t_\Omega \psi \rangle_{\partial \Omega} \\
+ \langle \mathcal{M}_\theta(u, g), M_\theta(\varphi, \psi) \rangle = \langle f, M_\theta(\varphi, \psi) \rangle \quad \text{for each} \quad (\varphi, \psi) \in H^1_0(\Omega) \times H^1(\Omega; \mathbb{R}^d).
\]
Consistent with \((4.9)\), we define the symmetric bilinear form
\[
a_\theta : (H^1_0(\Omega) \times H^1(\Omega; \mathbb{R}^d))^2 \rightarrow \mathbb{R}
\]
by
\[
(4.10) \quad a_\theta(\varphi, \psi; \varphi', \psi') := \langle \nabla \varphi - \psi, \nabla \varphi' - \psi' \rangle + \langle \nabla \times \psi, \nabla \times \psi' \rangle \\
+ \langle t_\Omega \psi, t_\Omega \psi' \rangle_{\partial \Omega} + \langle \mathcal{M}_\theta(\varphi, \psi), M_\theta(\varphi', \psi') \rangle.
\]
To establish the well-posedness of \((4.9)\) using the Lax–Milgram theorem, it suffices to demonstrate the coercivity and continuity of the bilinear form \(a_\theta\). We address this matter through the following results.

### 4.2. Lemma (Maxwell’s inequality).
For a convex \( \Omega \), every function \( \psi \in H^1(\Omega; \mathbb{R}^d) \) satisfies
\[
(4.11) \quad \|D\psi\|_{L^2(\Omega)}^2 \leq \|\nabla \cdot \psi\|_{L^2(\Omega)}^2 + \|\nabla \times \psi\|_{L^2(\Omega)}^2 + 2 \langle t_\Omega \psi, D[\psi \cdot \mathbf{n}_\Omega] - \partial_{n_\partial}(\psi \cdot \mathbf{n}_\Omega) \mathbf{n}_\Omega \rangle_{\partial \Omega}.
\]
See [Costabel and Dauge 1999 Lem. 2.2 & Rem. 2.4] for more details. The bound \((4.11)\) for \( \psi \in \mathcal{W} \) known as Maxwell’s inequality, which is
\[
(4.12) \quad \|D\psi\|_{L^2(\Omega)}^2 \leq \|\nabla \cdot \psi\|_{L^2(\Omega)}^2 + \|\nabla \times \psi\|_{L^2(\Omega)}^2.
\]

### 4.3. Lemma (generalized Maxwell’s inequality).
Let \( \Omega \) be a convex domain. There exists \( C_{\text{max}} > 0 \) such that for any \( \psi \in H^1(\Omega; \mathbb{R}^d) \),
\[
(4.13) \quad \|D\psi\|_{L^2(\Omega)}^2 \leq C_{\text{max}} \left( \|\nabla \cdot \psi\|_{L^2(\Omega)}^2 + \|\nabla \times \psi\|_{L^2(\Omega)}^2 + \|t_\Omega \psi\|_{L^2(\partial \Omega)}^2 \right).
\]

**Proof.** We argue by contradiction. Suppose inequality \((4.13)\) fails, then for any \( n \in \mathbb{N} \) there is \( \psi_n \in H^1(\Omega; \mathbb{R}^d) \) such that
\[
(4.14) \quad \|\nabla \cdot \psi_n\|_{L^2(\Omega)}^2 + \|\nabla \times \psi_n\|_{L^2(\Omega)}^2 + \|t_\Omega \psi_n\|_{L^2(\partial \Omega)}^2 < \frac{1}{n} \|D\psi_n\|_{L^2(\Omega)}^2.
\]
Without loss of generality, assume \( \|D\psi_n\|_{L^2(\Omega)} = 1 \) for all \( n \in \mathbb{N} \). Then \((4.14)\) implies that \( \|\nabla \cdot \psi_n\|_{L^2(\Omega)}^2, \|\nabla \times \psi_n\|_{L^2(\Omega)}^2, \|t_\Omega \psi_n\|_{L^2(\partial \Omega)}^2 \) all converge to zero as \( n \to \infty \). On the other hand, from \((4.11)\) and by applying Cauchy-Schwarz inequality, for all \( n \in \mathbb{N} \) we have
\[
(4.15) \quad \|D\psi_n\|_{L^2(\Omega)}^2 \leq \|\nabla \cdot \psi_n\|_{L^2(\Omega)}^2 + \|\nabla \times \psi_n\|_{L^2(\Omega)}^2 \\
+ 2 \langle t_\Omega \psi_n, D[\psi_n \cdot \mathbf{n}_\Omega] - \partial_{n_\partial}(\psi_n \cdot \mathbf{n}_\Omega) \mathbf{n}_\Omega \rangle_{\partial \Omega} \leq \|\nabla \cdot \psi_n\|_{L^2(\Omega)}^2 \\
+ \|\nabla \times \psi_n\|_{L^2(\Omega)}^2 + 2 \|t_\Omega \psi_n\|_{L^2(\partial \Omega)} \|D[\psi_n \cdot \mathbf{n}_\Omega] - \partial_{n_\partial}(\psi_n \cdot \mathbf{n}_\Omega) \mathbf{n}_\Omega\|_{L^2(\partial \Omega)}.
\]
By taking the limit of the both sides of the above inequality, when \( n \to \infty \), we get
\[
(4.16) \quad 1 = \lim_{n \to \infty} \|D\psi_n\|_{L^2(\Omega)}^2 \leq \lim_{n \to \infty} \left( \|\nabla \cdot \psi_n\|_{L^2(\Omega)}^2 + \|\nabla \times \psi_n\|_{L^2(\Omega)}^2 \\
+ 2 \|t_\Omega \psi_n\|_{L^2(\partial \Omega)} \|D[\psi_n \cdot \mathbf{n}_\Omega] - \partial_{n_\partial}(\psi_n \cdot \mathbf{n}_\Omega) \mathbf{n}_\Omega\|_{L^2(\partial \Omega)} \right) = 0,
\]
which it is a contradiction. \( \square \)
4.4. Remark (extending [Lakkis and Mousavi 2021]). Using the generalized Maxwell inequality \([4.13]\) instead of its special case \([4.12]\) in the proof arguments of Lemma 3.2, Theorem 3.6 and Theorem 3.7 of [Lakkis and Mousavi 2021] yields the following lemma and theorems.

4.5. Lemma (a Miranda-Talenti estimate). Let \(A\) satisfies the Cordes condition with \(\lambda = 0\) \([2.3]\), then there exists \(C_{4.13} > 0\) such that for any \(\psi \in H^1(\Omega; \mathbb{R}^d)\),

\[
\|\nabla \psi\|^2_{L^2(\Omega)} + \|t_\Omega \psi\|^2_{L^2(\partial \Omega)} + \|A : D\psi\|^2_{L^2(\Omega)} \geq C_{4.13} \|D\psi\|^2_{L^2(\Omega)}.
\]

4.6. Theorem (a modified Miranda-Talenti-estimate). Let \(\Omega\) be a bounded, open convex subset of \(\mathbb{R}^d\), \(0 < \rho < \frac{1}{C_{4.13} \sqrt{1}}\) and \(0 < \theta < 1\). Then for any \((\varphi, \psi) \in H^1_0(\Omega) \times H^1(\Omega; \mathbb{R}^d)\), we have

\[
\left(\frac{1}{C_{4.13} \sqrt{1}} - \theta^2\right) \|(\varphi, \psi)\|^2_{L^2(\Omega)} \leq \|\nabla \psi\|^2_{L^2(\Omega)} + \|t_\Omega \psi\|^2_{L^2(\partial \Omega)} + \|D_\lambda(\varphi, \psi)\|^2_{L^2(\Omega)} + \theta^2 \lambda \|\nabla \varphi - \psi\|^2_{L^2(\Omega)},
\]

where \(\|(\varphi, \psi)\|^2_{L^2(\Omega)} := \|D\psi\|^2_{L^2(\Omega)} + 2 \lambda \|	heta \psi - \varphi\|^2_{L^2(\partial \Omega)} + \lambda^2 \|\varphi\|^2_{L^2(\partial \Omega)}\) and \(D_\lambda(\varphi, \psi) := \nabla \cdot \psi - \lambda \varphi\).

4.7. Theorem (coercivity and continuity of \(a_\theta\)). Let \(\Omega \subset \mathbb{R}^d\) be a bounded convex open domain and the coefficients \(A, b, c\) satisfy \([2.1]\) and either \([2.2]\) with \(\lambda > 0\) or \([2.3]\) with \(\lambda = 0\) when \(b = 0\), \(c = 0\) holds. Then \(a_\theta\) is coercive and continuous, i.e., there exist \(C_{4.19} > 0\) such that for any \((\varphi, \psi), (\varphi', \psi') \in H^1_0(\Omega) \times H^1(\Omega; \mathbb{R}^d)\),

\[
a_\theta(\varphi, \psi; \varphi', \psi') := \|\nabla \varphi - \psi\|^2_{L^2(\Omega)} + \|\nabla \psi\|^2_{L^2(\Omega)} + \|t_\Omega \psi\|^2_{L^2(\partial \Omega)} + \|M_\theta(\varphi, \psi)\|^2_{L^2(\Omega)} \geq C_{4.19} \|(\varphi, \psi)\|^2_{H^1(\Omega)}
\]

\[
|a_\theta(\varphi, \psi; \varphi', \psi')| \leq C_{4.20} \|(\varphi, \psi)\|_{H^1(\Omega)} \|\varphi\|_{H^2(\Omega)} \|\psi\|_{H^2(\Omega)}.
\]

Proof. By using Lemma 4.5 and Theorem 4.6, the proof of coercivity is similar to the argument proof of Theorem 3.7 of [Lakkis and Mousavi 2021] and for the continuity, we refer to section 3.9 of [Lakkis and Mousavi 2021].

4.8. An equivalent problem to the equation with nonzero boundary. If the boundary condition of \([4.1]\) is nonhomogeneous, i.e., \(u|_{\partial \Omega} = r \neq 0\) for some \(r \in H^{3/2}(\partial \Omega)\), the functional \(E_\theta\) is replaced by the extended functional \(E_{\theta}\) as

\[
E_{\theta} : H^1(\Omega) \times H^1(\Omega; \mathbb{R}^d) \to \mathbb{R}
\]

\[
E_{\theta}(\varphi, \psi) := \|\nabla \varphi - \psi\|^2_{L^2(\Omega)} + \|\nabla \psi\|^2_{L^2(\Omega)} + \|M_\theta(\varphi, \psi) - f\|^2_{L^2(\Omega)} + \|\varphi - r\|^2_{H^{3/2}(\partial \Omega)} + \|t_\Omega(\psi - \nabla r)\|^2_{L^2(\partial \Omega)}
\]

and we then consider the Euler–Lagrange equation of the minimization problem

\[
(u, g) = \arg\min_{(\varphi, \psi) \in H^1(\Omega) \times H^1(\Omega; \mathbb{R}^d)} E_{\theta}(\varphi, \psi).
\]

Indeed, we find \((u, g) \in H^1(\Omega) \times H^1(\Omega; \mathbb{R}^d)\) such that,

\[
\langle \nabla u - g, \nabla \varphi \rangle + \langle \nabla \varphi, \nabla \psi \rangle + \langle M_\theta(u, g), M_\theta(\varphi, \psi) \rangle + \langle u, \varphi \rangle_{\partial \Omega} + \langle t_\Omega g, t_\Omega \psi \rangle_{\partial \Omega} = \langle f, M_\theta(\varphi, \psi) \rangle + \langle r, \varphi \rangle_{\partial \Omega} + \langle t_\Omega \nabla r, t_\Omega \psi \rangle_{\partial \Omega}
\]

for each \((\varphi, \psi) \in H^1(\Omega) \times H^1(\Omega; \mathbb{R}^d)\).
The well-posedness of the resulting problem can be demonstrated using a similar argument to the proof of Theorem 4.7. Accordingly, we define the bilinear form

\[
\tilde{a}_\theta : H^1(\Omega) \times H^1(\Omega; \mathbb{R}^d)^2 \to \mathbb{R}
\]

(4.24)

\[
\tilde{a}_\theta(\varphi, \psi; \varphi', \psi') := a_\theta(\varphi, \psi; \varphi', \psi') + \langle \varphi, \varphi' \rangle_{\partial \Omega}.
\]

4.9. Theorem (coercivity and continuity of \( \tilde{a}_\theta \)). Under the assumption of Lemma 4.7, there exist \( C_{4.25} \) and \( C_{4.26} > 0 \) such that for any \( (\varphi, \psi), (\varphi', \psi') \in H^1(\Omega) \times H^1(\Omega; \mathbb{R}^d) \)

(4.25)

\[
\tilde{a}_\theta(\varphi, \psi; \varphi', \psi') \geq C_{4.25} \| (\varphi, \psi) \|^2_{H^1(\Omega)}
\]

(4.26)

\[
|\tilde{a}_\theta(\varphi, \psi; \varphi', \psi')| \leq C_{4.26} \| (\varphi, \psi) \|_{H^1(\Omega)} \| (\varphi', \psi') \|_{H^1(\Omega)}
\]

Proof. Using Poincaré’s inequality: there exists \( C_{4.27} > 0 \) such that for any \( (\varphi, \psi) \in H^1(\Omega) \times H^1(\Omega; \mathbb{R}^d) \) we have

(4.27)

\[
\|\nabla \varphi\|_{L^2(\Omega)}^2 + \|\varphi\|_{L^2(\partial \Omega)}^2 \geq C_{4.27} \|\varphi\|_{H^1(\Omega)}^2,
\]

the argument is analogous to what is presented in the proof of Theorem 4.7. \( \square \)

4.10. Galerkin finite element discretization. Consider \( \mathcal{T} \) as a collection of shape-regular conforming simplicial partitions (also known as triangulations) of \( \Omega \) into simplices. Given \( T \in \mathcal{T} \), for each \( K \in T \), let \( h_K := \text{diam } K \) and write \( h := \max_{K \in T} h_K \). Having a curved boundary \( \partial \Omega \) prevents \( \bigcup_{K \in T} K \) of coinciding with \( \Omega \). In this case, one can approximate sections of \( \partial \Omega \), using line segments or simple curves. This approach results in simplices with curved sides, known as isoparametric elements. Consider the following Galerkin finite element spaces

(4.28)

\[
\tilde{U} := \mathbb{P}^k(T) \cap H_0^1(\Omega), \quad \tilde{G} := \mathbb{P}^k(T_\partial) \cap H^1(\Omega; \mathbb{R}^d).
\]

By applying these Galerkin finite element spaces, the discrete problem corresponding to zero boundary condition finds \( (u_{\tilde{U}}, g_{\tilde{G}}) \in \tilde{U} \times \tilde{G} \) such that

(4.29)

\[
a_\theta(u_{\tilde{U}}, g_{\tilde{G}}; \varphi, \psi) = \langle f, \mathcal{M}_\theta(\varphi, \psi) \rangle \quad \text{for each } (\varphi, \psi) \in \tilde{U} \times \tilde{G}.
\]

and the discrete problem corresponding to nonzero boundary condition finds \( (u_{\tilde{U}}, g_{\tilde{G}}) \in \tilde{U} \times \tilde{G} \) such that

(4.30)

\[
\tilde{a}_\theta(u_{\tilde{U}}, g_{\tilde{G}}; \varphi, \psi) = \langle f, \mathcal{M}_\theta(\varphi, \psi) \rangle + \langle r, \varphi \rangle_{\partial \Omega} + \langle t_\partial \nabla r, t_\partial \psi \rangle_{\partial \Omega} \quad \text{for each } (\varphi, \psi) \in \tilde{U} \times \tilde{G}.
\]

Since the coercivity is inherited by subspaces, Theorem 4.7 and Theorem 4.9 imply that both discrete problems \(4.29\) and \(4.30\) are well-posed. The solutions of both discrete problems satisfy the following error estimate theorems.

4.11. Theorem (a priori error estimate for linear equation in nondivergence form). Let \( T \) be in a collection \( \mathcal{T} \) of shape-regular conforming simplicial meshes on the polyhedral domain \( \Omega \subseteq \mathbb{R}^d \). Moreover assume that the strong solution \( u \) of \(4.7\) with \( u|_{\partial \Omega} = 0 \) satisfies \( u \in H^{\rho+2}(\Omega) \), for some real \( \rho > 0 \). Let \( (u_{\tilde{U}}, g_{\tilde{G}}) \in \tilde{U} \times \tilde{G} \) be the finite element approximation of \(4.29\). Then for some \( C_{4.11} > 0 \), independent of \( u \) and \( h \) we have

(4.31)

\[
\| (u, \nabla u) - (u_{\tilde{U}}, g_{\tilde{G}}) \|_{H^1(\Omega)} \leq C_{4.11} h^{k+\rho} \| u \|_{H^{\rho+2}(\Omega)}.
\]

Proof. We refer to Theorem 4.6 of [Lakkis and Mousavi 2021]. \( \square \)

With a same argument of the proof, we can also show the a priori error bound of the discrete problem corresponding to nonzero boundary as the following theorem.
4.12. Theorem (a priori error estimate for linear equation in nondivergence form with nonzero boundary). Suppose that the assumptions of Theorem 4.11 on the domain and simplicial meshes are held. Assume that the strong solution $u$ of (4.1) with $u_{|\partial\Omega} = r \neq 0$ for some $r \in H^{3/2}(\partial\Omega)$ satisfies $u \in H^{4+\rho}(\Omega)$, for some real $\rho > 0$. Let $(u_{\tilde{\Omega}}, g_{\tilde{\Omega}}) \in \tilde{U} \times \tilde{G}$ be the finite element approximation of (4.30). Then for some $C_{4.31} > 0$, independent of $u$ and $h$ we have
\[
\| (u, \nabla u) - (u_{\tilde{\Omega}}, g_{\tilde{\Omega}}) \|_{H^1(\Omega)} \leq C_{4.31} h^{\rho} \| u \|_{H^{4+\rho}(\Omega)}.
\]

4.13. Remark (domain with curved boundary). If $\Omega$ includes a curved boundary, the isoparametric finite element is used. Smooth or piecewise smooth boundary ($\partial\Omega$) ensures that the error bound of using isoparametric finite element similar to that of Theorem 4.11 and Theorem 4.12. This is established in Ciarlet [1978].

4.14. Theorem (a posteriori error-residual estimate for linear equation in nondivergence form). Let $u$ be the strong solution of (4.1).

(a) For zero boundary condition, let $(u_{\tilde{\Omega}}, g_{\tilde{\Omega}})$ be the unique solution of (4.29).

(i) The following a posteriori residual upper bounds holds
\[
\| (u, \nabla u) - (u_{\tilde{\Omega}}, g_{\tilde{\Omega}}) \|^2_{H^1(\Omega)} \leq C_{4.33} \tilde{E}_0(u_{\tilde{\Omega}}, g_{\tilde{\Omega}}).
\]

(ii) For any open subdomain $\omega \subseteq \Omega$ we have
\[
\| \nabla u_{\tilde{\Omega}} - g_{\tilde{\Omega}} \|^2_{L_2(\omega)} + \| \nabla \times g_{\tilde{\Omega}} \|^2_{L_2(\omega)} + \| M_\theta(u_{\tilde{\Omega}}, g_{\tilde{\Omega}}) - f \|^2_{L_2(\omega)} + \| t_\Omega g_{\tilde{\Omega}} \|^2_{L_2(\partial\omega \cap \partial\Omega)} \leq C_{4.29} \tilde{\omega} \| (u, \nabla u) - (u_{\tilde{\Omega}}, g_{\tilde{\Omega}}) \|^2_{H^1(\omega)},
\]

where $C_{4.29} \tilde{\omega}$ is the continuity constant of $u_\theta$ on $H^1_0(\omega) \times H^1(\omega; \mathbb{R}^d)$.

(b) For nonzero boundary condition, let $(u_{\tilde{\Omega}}, g_{\tilde{\Omega}})$ be the unique solution of (4.30).

(i) The following a posteriori residual upper bounds holds
\[
\| (u, \nabla u) - (u_{\tilde{\Omega}}, g_{\tilde{\Omega}}) \|^2_{H^1(\Omega)} \leq C_{4.34} \tilde{E}_0(u_{\tilde{\Omega}}, g_{\tilde{\Omega}}).
\]

(ii) For any open subdomain $\omega \subseteq \Omega$ we have
\[
\| \nabla u_{\tilde{\Omega}} - g_{\tilde{\Omega}} \|^2_{L_2(\omega)} + \| \nabla \times g_{\tilde{\Omega}} \|^2_{L_2(\omega)} + \| M_\theta(u_{\tilde{\Omega}}, g_{\tilde{\Omega}}) - f \|^2_{L_2(\omega)} + \| u_{\tilde{\Omega}} - r \|^2_{L_2(\partial\omega \cap \partial\Omega)} + \| t_\Omega (g_{\tilde{\Omega}} - \nabla r) \|^2_{L_2(\partial\omega \cap \partial\Omega)} \leq C_{4.29} \tilde{\omega} \| (u, \nabla u) - (u_{\tilde{\Omega}}, g_{\tilde{\Omega}}) \|^2_{H^1(\omega)},
\]

where $C_{4.29} \tilde{\omega}$ is the continuity constant of $\tilde{u}_\theta$ on $H^1(\omega) \times H^1(\omega; \mathbb{R}^d)$.

**Proof.** We refer to Theorem 4.3 of Lakkis and Mousavi [2021].

5. Discretization

In this section, we turn to the practical approximation of Howard’s algorithm, as clarified in (3.14) by applying the method from §4 to solve each iteration of problem (3.28). We discuss the resulting approximation error as well. Hereupon, we present an a priori (Theorem 5.8) and an a posteriori (Theorem 5.9) error analysis of the approximation. In the spirit of the a posteriori error estimator, we specify the error indicators in an adaptive mesh refinement algorithm. Afterwards, an approximation of the control problem is explained and finally, we close the section by offering Howard’s and adaptive algorithms, which are used in the implementation.
5.1. Discretization of recursive problem. Let the simplicial mesh $\mathcal{T}$ and the Galerkin finite element spaces $\mathbb{U}, \mathbb{U}_1, \mathcal{C}$ be as introduced in 4.10. For any fixed $\theta \in [0,1]$ and control map $q \in Q$ we associate a linear operator $M^\theta_0$ on $\{\phi, \psi\} \in H^1(\Omega) \times H^1(\Omega; \mathbb{R}^d)$ as follows
\begin{equation}
M^\theta_0(\phi, \psi) := A^H \psi + b^H \cdot (\theta \psi + (1 - \theta) \nabla \phi) - c^\theta \phi.
\end{equation}
Furthermore, we define the set-valued operator $\tilde{N} : H^1(\Omega) \times H^1(\Omega; \mathbb{R}^d) \ni \varphi$ by
\begin{equation}
\tilde{N}(\phi, \psi) := \left\{ q \in Q \mid q(x) \in \text{Argmax}_{\alpha \in A} [M^\theta_0(\phi, \psi) - f^\alpha]_x, \text{ for a.e. } x \in \Omega \right\},
\end{equation}
where $\theta$ is implicit in the notation. Recall the recursive problem (3.24) and let $(u_{u_{n}}, g_{n})$ be the least-squares Galerkin with gradient recovery finite element approximation of it at step $n$, approximated by setting $q_{n-1} \in \tilde{N}(u_{u_{n-1}}, g_{n-1})$. Indeed, $(u_{u_{n}}, g_{n})$ is the solution of (4.29) for $M_\theta = M^0_0$ and $f = f_{n-1}$. Let
\begin{equation}
(u_{u}, g_{E}) = \lim_{n \to \infty} (u_{u_{n}}, g_{n}).
\end{equation}
We analyze errors by examining the limiting behavior of both the exact solution and the approximation of the recursive problem (3.24). To report error bounds, we will consider an equivalent problem to (2.6), where $(u, \nabla u)$ represents its exact solution, and $(u_{u}, g_{E})$ represents its approximation. In this regard, it is appropriate to consider a mixed formulation of the HJB problem. Corresponding to the approach used to deal with the linear equation at each iteration, determining such a nonlinear form of the equation is straightforward.

5.2. Mixed formulation of HJB problem. To introduce the mixed formulation corresponding to HJB problem (2.6), we define the mixed HJB operator as following
\begin{equation}
\hat{F} : H^1(\Omega) \times H^1(\Omega; \mathbb{R}^d) \to L^2(\Omega)
\end{equation}
\begin{equation}
\hat{F}(\phi, \psi) := M^\theta_0(\phi, \psi) - f^\theta \text{ such that } q \in \tilde{N}(\phi, \psi),
\end{equation}
where similar to (5.2), $\theta$ is muted in the notation. Through adapting the proof of Theorem 3.9 to operator $\hat{F}$, we argue that for any $(\phi, \psi) \in H^1(\Omega) \times H^1(\Omega; \mathbb{R}^d)$
\begin{equation}
\mathcal{D}\hat{F}(\phi, \psi) := \left\{ M^\theta_0 := A^q : D + b^\theta \cdot (\theta + (1 - \theta) \nabla) - c^\theta \mid q \in \tilde{N}(\phi, \psi) \right\}
\end{equation}
is the Newton derivative of $\hat{F}$ at $(\phi, \psi)$. Based on the least-squares idea applied for the linearized problem, we introduce the following quadratic functional
\begin{equation}
\hat{E}_\theta : H^1(\Omega) \times H^1(\Omega; \mathbb{R}^d) \to \mathbb{R}
\end{equation}
\begin{equation}
\hat{E}_\theta(\phi, \psi) := \|
abla \phi - \psi \|^2_{L^2(\Omega)} + \|
abla \times \psi \|^2_{L^2(\Omega)} + \|t_\Omega(\psi)\|^2_{L^2(\partial \Omega)} + \|M^\theta_0(\phi, \psi) - f^\theta\|^2_{L^2(\Omega)}
\end{equation}
in which $q \in \tilde{N}(\phi, \psi)$. We then consider the minimization problem of finding $(u, g) \in H^1(\Omega) \times H^1(\Omega; \mathbb{R}^d)$ such that
\begin{equation}
(u, g) = \underset{(\phi, \psi) \in H^1(\Omega) \times H^1(\Omega; \mathbb{R}^d)}{\arg\min} \hat{E}_\theta(\phi, \psi).
\end{equation}
Obviously, $g = \nabla u$ and $u$ is the unique strong solution of (2.6). By applying Newton derivative (5.5), the Euler-Lagrange equation of the minimization problem (5.7) finds $(u, g) \in H^1(\Omega) \times H^1(\Omega; \mathbb{R}^d)$ such that satisfies the binonlinear problem
\begin{equation}
\langle \nabla u - g, \nabla \phi - \psi \rangle + \langle \nabla \times g, \nabla \times \phi \rangle + \langle t_\Omega g, t_\Omega \psi \rangle_{\partial \Omega} + \langle M^\theta_0(u, g) - f^\theta, M^\theta_0(\phi, \psi) \rangle = 0,
\end{equation}
for $q \in \tilde{N}(u, g)$ and each $(\phi, \psi) \in H^1(\Omega) \times H^1(\Omega; \mathbb{R}^d)$. 

Respectively, the discrete nonlinear problem finds \((u_0, g_0) \in U \times G\) such that
\[
\begin{align*}
(5.9) \quad \langle \nabla u_0 - g_0, \nabla \varphi - \psi \rangle + \langle \nabla \times g_0, \nabla \times \psi \rangle + \langle t_{\Omega} g_0, t_{\Omega} \psi \rangle_{\partial \Omega} & \quad + \langle M_0^1(u_0, g_0) - f, M_0^2(\varphi, \psi) \rangle = 0, \\
& \quad \text{for } q \in \tilde{N}(u_0, g_0) \text{ and each } (\varphi, \psi) \in U \times G.
\end{align*}
\]
To achieve linearity, applying the semismooth Newton linearization to \((5.9)\) yields a recursive bilinear form equation, as in \((4.29)\). Hence, the solutions of \((5.9)\) and \((5.3)\) coincide, and we use the same notation for them, as explained in the following remark. Consequently, to provide error bounds, we consider problem \((5.8)\) and its discrete version \((5.9)\).

5.3. **Remark.** When discretizing the HJB equation using the Galerkin finite element method, the steps of discretization and linearization are computationally commutative. This means that whether we apply the Galerkin method first to enter a finite-dimensional function space and then perform linearization, or if we linearize first and then use finite element discretization, the outcomes are equivalent.

5.4. **Supremum property.** We recall that for real numbers \(\{x^\alpha\}_\alpha, \{y^\alpha\}_\alpha\), we have
\[
(5.10) \quad \sup_\alpha x^\alpha - \sup_\alpha y^\alpha \leq \sup_\alpha |x^\alpha - y^\alpha|.
\]
This property is used to show the following claims.

5.5. **Lemma (weak monotonicity and Lipschitz continuity).** Let \(\Omega\) be a convex domain. There exist \(C_{\text{weak}} > 0\) such that every \((\varphi, \psi), (\varphi', \psi'), (\varphi'', \psi''), (\varphi''', \psi''') \in H_0^1(\Omega) \times H^1(\Omega; \mathbb{R}^d)\) and \(q \in \tilde{N}(\varphi, \psi), q' \in \tilde{N}(\varphi', \psi'), q'' \in \tilde{N}(\varphi'', \psi''), q''' \in \tilde{N}(\varphi''', \psi''')\) satisfy
\[
(5.11) \quad \|\nabla(\varphi - \varphi') - (\psi - \psi')\|_{L_2(\Omega)}^2 + \|\nabla \times (\psi - \psi')\|_{L_2(\Omega)}^2 + \|t_{\Omega}(\psi - \psi')\|_{L_2(\partial \Omega)}^2 + \|\langle M_0^1(\varphi, \psi) - f, (\varphi'') - f'' \rangle\|_{L_2(\Omega)}^2 \geq C_{\text{weak}} \|(\varphi, \psi) - (\varphi', \psi')\|_{H^1(\Omega)}^2.
\]
\[
(5.12) \quad \langle \nabla(\varphi - \varphi') - (\psi - \psi'), \nabla(\varphi'' - \varphi''') - (\psi'' - \psi''') \rangle + \langle \nabla \times (\psi - \psi'), \nabla(\psi'' - \psi''') \rangle + \langle t_{\Omega}(\psi - \psi'), t_{\Omega}(\psi'' - \psi''') \rangle_{\partial \Omega} + \langle (M_0^3(\varphi, \psi) - f), (M_0^3(\varphi', \psi') - f') \rangle + \langle (M_0^3(\varphi'', \psi'') - f''), (M_0^3(\varphi''', \psi''') - f''') \rangle
\]
\[
\leq C_{\text{weak}} \|(\varphi, \psi) - (\varphi', \psi')\|_{H^1(\Omega)} \|(\varphi'', \psi'') - (\varphi''', \psi''')\|_{H^1(\Omega)}.
\]
**Proof.** By tracking what is saying in the proof of Theorem \((4.7)\) and using \((5.10)\), the inequalities are achieved.

5.6. **Remark (non strong monotonicity).** We want to emphasize that \((5.11)\) differs from the strong monotonicity of the binonlinear form in Equation \((5.8)\). It can be verified that the corresponding binonlinear form is not strongly monotone.

5.7. **Theorem (quasi-optimality).** Let \((u_0, g_0)\) be the solution of the discrete problem \((5.9)\). It satisfies the error bound
\[
(5.13) \quad ||(u, \nabla u) - (u_0, g_0)||_{H^1(\Omega)} \leq C_{\text{weak}} \inf_{(\varphi, \psi) \in U \times G} ||(u, \nabla u) - (\varphi, \psi)||_{H^1(\Omega)}.
\]
Proof. Consider any arbitrary \((\varphi, \psi) \in \mathbb{U} \times \mathbb{G}\). Since for \(q \in \tilde{N}(u, \nabla u), M_0(q, u, \nabla u) - f^q = 0\), Lemma 5.5 and 5.9 imply that for \(q^* \in \tilde{N}(u, g)\) we have

\[
\text{(5.14)} \quad \|u, \nabla u - (u, g)\|_{H^1(\Omega)}^2
\]

\[
\leq \|
\nabla u - g\|_{L^2(\Omega)}^2 + \|
\nabla \times g\|_{L^2(\Omega)}^2 + \|t_{\Omega} g\|_{L^2(\Omega)}^2 + \left\| \left( M_0'\left(u, g\right) - f^q\right) \right\|_{L^2(\Omega)}^2
\]

\[
= \langle \nabla u - g, \nabla \varphi - \psi \rangle + \langle \nabla \times g, \nabla \times \psi \rangle + \langle t_{\Omega} g, t_{\Omega} \psi \rangle_{\partial \Omega}
\]

\[
+ \begin{cases} 
M_0'(u, g) - f^q, & \text{for } q \in \tilde{N}(u, g), \\
M_0'(q, u, \nabla u) - f^q, & \text{for } q \in \tilde{N}(u, g) \end{cases}
\]

\[
\leq C_{5.10} \|(u, \nabla u - (u, g))\|_{H^1(\Omega)}^2 \|(u, \nabla u - (\varphi, \psi))\|_{H^1(\Omega)}.
\]

5.8. Proposition (a priori error estimate for HJB equation). Suppose that the strong solution \(u\) of (2.6) satisfies \(u \in H^{r+2}(\Omega)\), for some real \(r > 0\). Then for some \(C_{5.10} > 0\), independent of \(u\) and \(h\) we have

\[
\text{(5.15)} \quad \|(u, \nabla u - (u, g))\|_{H^1(\Omega)} \leq C_{5.10} k^\rho \|u\|_{H^{r+2}(\Omega)}.
\]

Proof. Th. 4.4.20 and the error bound of the interpolation [Brenner and Scott, 2008] demonstrate the claim.

5.9. Proposition (error-residual a posteriori estimates for HJB equation). Let \((u, g)\) be as considered in (5.3) and \(q \in \tilde{N}(u, g)\).

(i) The following a posteriori residual upper bound holds

\[
\text{(5.16)} \quad \|(u, \nabla u - (u, g))\|_{H^1(\Omega)} \leq C_{5.11} \left( \|\nabla u - g\|_{L^2(\Omega)}^2 + \|\nabla \times g\|_{L^2(\Omega)}^2 \\
+ \|t_{\Omega} g\|_{L^2(\Omega)}^2 + \|M_0'(u, g) - f^q\|_{L^2(\Omega)}^2 \right).
\]

(ii) For any open subdomain \(\omega \subseteq \Omega\) the following a posteriori residual lower bound holds

\[
\text{(5.17)} \quad \|\nabla u - g\|_{L^2(\omega)}^2 + \|\nabla \times g\|_{L^2(\omega)}^2 \\
+ \|M_0'(u, g) - f^q\|_{L^2(\omega)}^2 \leq C_{5.12} \|(u, \nabla u - (u, g))\|_{H^1(\omega)}^2,
\]

where \(C_{5.12}\) is the constant of (5.12) for subdomain \(\omega \subseteq \Omega\).

Proof. 5.11 and the fact that for \(q \in \tilde{N}(u, \nabla u), M_0'(u, \nabla u) - f^q = 0\) imply (5.16).

One can easily check that (5.12) holds for any subdomain \(\omega \subseteq \Omega\) as well, which implies (5.17). □

Lemma 5.5 and consequently propositions 5.8 and 5.9 can be easily adapted to the problem with non-zero boundary, \(r \neq 0\). As we see in propositions 5.8 and 5.9, the error bounds of discretization for nonlinear problem are similar to the error bounds of discretization in linear case which have reported in theorems 4.11 and 4.14. We use the a posteriori residual error bounds of Proposition 5.9 as an error indicator in the adaptive scheme.
5.10. Implementation and error indicators. Practically, corresponding to the problem with zero boundary, at each step of the recursive problem, for fixed control map \( q \), we find \((u_0, g_0) \in U \times G \) such that

\[
\langle \nabla u_0 - g_0, \nabla \varphi - \psi \rangle + \langle \nabla \times g_0, \nabla \times \psi \rangle + \langle M_0^0(u_0, g_0), M_0^0(\varphi, \psi) \rangle + \langle t_0 g_0, \partial_\Omega \psi \rangle_{\partial \Omega} = \langle f^0, M_0^0(\varphi, \psi) \rangle \quad \text{for each } (\varphi, \psi) \in U \times G,
\]

and corresponding to the problem with nonzero boundary \((r \neq 0)\), due to (4.30), we find \((u_\tilde{r}, g_\tilde{r}) \in \tilde{U} \times \tilde{G} \) such that

\[
\langle \nabla u_\tilde{r} - g_\tilde{r}, \nabla \varphi - \psi \rangle + \langle \nabla \times g_\tilde{r}, \nabla \times \psi \rangle + \langle M_0^0(u_\tilde{r}, g_\tilde{r}), M_0^0(\varphi, \psi) \rangle + \langle t_\tilde{r} g_\tilde{r}, \partial_\Omega \psi \rangle_{\partial \Omega} = \langle r, \varphi \rangle_{\partial \Omega},
\]

Accordingly, for each \( K \in \mathcal{T} \), we consider the local error indicator by

\[
\eta^2(K) := \begin{cases} 
\| \nabla u_K - g_K \|_{L^2(K)}^2 + \| \nabla \times g_K \|_{L^2(K)}^2 + \| M^0_0(u_K, g_K) \|_{L^2(K)}^2 + \| t_\Omega g_K \|_{L^2(\partial K \cap \partial \Omega)}^2 & \text{if } r = 0 \\
\| \nabla u_\tilde{K} - g_\tilde{K} \|_{L^2(K)}^2 + \| \nabla \times g_\tilde{K} \|_{L^2(K)}^2 + \| M^0_0(u_\tilde{K}, g_\tilde{K}) - f^0 \|_{L^2(K)}^2 + \| t_\Omega g_\tilde{K} - \nabla r \|_{L^2(\partial K \cap \partial \Omega)}^2 & \text{if } r \neq 0,
\end{cases}
\]

and the global error indicator by

\[
\eta := \sqrt{\sum_{K \in \mathcal{T}} \eta^2(K)}.
\]

5.11. Approximation of control problem. As we see in Howard’s algorithm, in addition to approximating a linear equation in nondivergence form, we also need to solve or approximate a control problem at each step of the iteration appropriately. In dealing with control problems, we approximate them elementwise on \( \mathcal{T} \). For each \( K \in \mathcal{T} \), and given \((u, g), q(K)\) is evaluated as a member of the set

\[
\text{Argmax}_{\alpha \in \mathcal{A}} \int_K (M^0_0(u, g) - f^\alpha).
\]

This makes \( q \) belong to the finite element space of \( \mathcal{A} \)-valued piecewise constants \( \mathcal{A} := \mathcal{P}_0(\mathcal{T}, \mathcal{A}) \). We emphasize that the local control problem may not have a unique solution for each \( K \) and thus the control \( q \) is one of many possible choices. But we do guarantee that the algorithm will approximate one of its solutions.

We summarize Howard’s algorithm and the adaptive refinement algorithm in the next paragraphs.

5.12. Algorithm (Howard least-squares gradient recovery Galerkin HJB solver). The following pseudocode summarizes our code.

**Require:** data of the problem (1.1), parameter \( \theta \in [0, 1] \) to (1.6) mesh \( \mathcal{T} \) of domain \( \Omega \), \( k \) polynomial degree, initial guess \((u_0, g_0)\), tolerance \( \text{tol} \) and maximum number of iterations \( \text{maxiter} \).

**Ensure:** approximate control map and approximate solution to (1.1) either with \( \| (u_{n+1}, g_{n+1}) - (u_n, g_n) \|_{H^1(\Omega)} \lesssim \text{tol} \) or after \( \text{maxiter} \) iterations.

**procedure** HOWARD-GALS(data of the problem (1.1), \( \theta, \mathcal{T}, k, u_0, g_0, \text{tol}, \text{maxiter} \))

- build the Galerkin spaces \( U, \bar{U}, G \) as (4.28) and \( \mathcal{A} := \mathcal{P}_0(\mathcal{T}, \mathcal{A}) \)
- \((u_0, g_0) \leftarrow \text{projection of } (\bar{u}_0, \bar{g}_0) \text{ onto } U \times G \)
- \( n \leftarrow 1 \)
- \( \text{res} \leftarrow \text{tol} + 1 \)
While \( n < \text{maxiter} \) and \( \text{res} > \text{tol} \) do
  for \( K \in T \) do
    \( q_n(K) \in \text{Argmax}_{a \in \mathcal{A}} \int_K [\mathcal{M}_0^a(u_{n-1}, g_{n-1}) - f^a] \)
  end for
  solve for \( (u_n, g_n) \leftarrow (u_{\tilde{U}}, g_{\tilde{G}}) \) satisfying problem (5.19) with \( q \leftarrow q_n \)
  \( \text{res} \leftarrow \| (u_n, g_n) - (u_{n-1}, g_{n-1}) \|_{H_1(\Omega)} \)
  \( n \leftarrow n + 1 \)
end while
end procedure

5.13. Algorithm (adaptive least-squares gradient recovery Galerkin HJB solver).

Require: data of the problem (1.1), refinement fraction \( \beta \in (0, 1) \), tolerance \( \text{tol}_a \), and maximum number of iterations \( \text{maxiter}_a \).

Ensure: approximate solution to (1.1) either with \( \| (u, \nabla u) - (u_l, g_l) \|_{H_1(\Omega)} \lesssim \text{tol}_a \) or after \( \text{maxiter}_a \) iterations

1. Construct an initial admissible partition \( T_0 \)
2. \( \eta^2 \leftarrow \text{tol}_a^2 + 1 \)
3. \( l \leftarrow 0 \)
4. while \( l \leq \text{maxiter}_a \) and \( \eta^2 > \text{tol}_a^2 \) do
5.   \( (q_l, u_l, g_l) \leftarrow \text{HOWARD-GALS(data of the problem (1.1), } \theta, T, k, u_0, g_0, \text{tol}, \text{maxiter}) \)
6.   for \( K \in T_l \) do
7.     compute \( \eta(K)^2 \) via (5.20)
8.   end for
9.   estimate by computing \( \eta^2 \leftarrow \sum_{K \in T_l} \eta^2(K) \)
10. sort array \( (\eta(K)^2)_{K \in T_l} \) in decreasing order
11. mark the first \( \lceil \beta \# T_l \rceil \) elements \( K \) with the highest \( \eta(K)^2 \)
12. refine \( T_l \) ensuring split of all marked elements
13. \( l \leftarrow l + 1 \)
end while

6. Numerical experiments

In this section, to test the performance of our method, we report on two numerical experiments in a subset of \( \mathbb{R}^2 \) domains. In both experiments, as input Howard’s algorithm [5.12], we choose \( u_0 = 0, g_0 = 0 \) and \( \theta = 0.5 \). We set the stop criterion of the algorithm as either \( \| (u_{n+1}, g_{n+1}) - (u_n, g_n) \|_{H_1(\Omega)} < 10^{-7} \) (\( \text{tol} = 10^{-7} \)) or maximum 8 iterations (\( \text{maxiter} = 8 \)). All implementations were done by using the FEniCS package.

In the first experiment, we aim to test the convergence rate of the method. Thus the known solution is considered smooth enough, and the numerical results are presented on the uniform mesh. In the second experiment, we would like to test the performance of the adaptive scheme, the known solution is considered near singular. Through comparing the convergence rate via the adaptive with the uniform refinement, we observe the efficiency of the adaptive scheme. In both experiments, the results of the errors are plotted in logarithmic scale.

6.1. Test problem with nonzero boundary condition. We approximate (1.1) with \( \Omega = (-1, 1) \times (-1, 1) \) and \( \mathcal{A} := \text{SO}(2) \) (the special orthogonal group on \( \mathbb{R}^2 \))
parametrized as \( R/2\pi \mathbb{Z} = [0, 2\pi) \ni \alpha \mapsto e^{i\alpha} \) (in matrix form), with the metric
\[
(6.1) \quad d_A(\alpha, \beta) := |e^{i\alpha} - e^{i\beta}| = 2(1 - \cos(\alpha - \beta)),
\]
(6.2) \[
A^\alpha(x) := \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} 2 \cos(\alpha/2) \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \sin \alpha \cos \alpha,
\]
(6.3) \[
b^\alpha := 0, \quad c^\alpha := 2 - 0.5(\cos(2\alpha) + \sin(2\alpha)),
\]
(6.4) \[
f^\alpha := \mathcal{L}^\alpha u - (1 - \cos(2\alpha - \pi(x_1 + x_2))).
\]

where the exact solution is
\[
(6.5) \quad u(x) = \sin(\pi x_1) \sin(\pi x_2) + \sin(\pi(x_1 + x_2)).
\]

\( A^\alpha \) together with \( b^\alpha, c^\alpha \) satisfy the Cordes condition \((2.2)\) with \( \lambda = 1 \) and \( \epsilon = 0.45 \). We approximate a solution by following Howard’s algorithm \ref{howard}. Different measures of the error in the together linear(\( P^1 \)) and quadratic(\( P^2 \)) finite element spaces are reported in Fig. \ref{fig:convergence_rates}. To benchmark this test we use the experimental orders of convergence (EOC) associated with a numerical experiment with errors \( e_i \) and (uniform) meshsizes \( h_i, i = 0, \cdots, I \), which is defined by
\[
(6.6) \quad \text{EOC} := \frac{\log(e_{i+1}/e_i)}{\log(h_{i+1}/h_i)}.
\]

As we observe, even for a nonhomogeneous problem, the numerical results confirm the analysis convergence rate of Proposition \ref{convergence_rate_proposition}.

\[ \begin{array}{c}
\begin{array}{c}
\text{(a) } P^1 \text{ elements} \\
\text{(b) } P^2 \text{ elements}
\end{array}
\end{array} \]

\text{FIGURE 1. Convergence rates for Test problem 6.1.}

6.2. Test problem with singular solution. Consider \ref{example-problem} in the unit disk domain with the exact solution
\[
(6.7) \quad u(x) = \begin{cases} |x|^{5/3}(1 - |x|)^{5/2}(\sin(2\varphi(x)/3))^{5/2} & \text{if } 0 < \varphi(x) < 3\pi/2, \\ 0 & \text{otherwise,} \end{cases}
\]

the control set \( \mathcal{A} = \text{SO}(2) \supseteq R/2\pi \mathbb{Z} = [0, 2\pi) \) and
\[
\begin{align*}
A^\alpha(x) &= \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} 1 + (x_1^2 + x_2^2) \\ 0.005 \\ 1.01 - (x_1^2 + x_2^2) \end{bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}, \\
b^\alpha &= 0, \quad c^\alpha = 0 \quad f^\alpha = \mathcal{L}^\alpha u - (1 - \cos(2\alpha - \pi(x_1 + x_2))),
\end{align*}
\]
where \((|x|, \varphi(x))\) are polar coordinates centered at the origin. The near degenerate diffusion \( A^\alpha \) satisfies the Cordes condition \((2.3)\) with \( \epsilon = 0.008 \). Note that
the solution $u$ belongs to $H^s(\Omega)$, for any $s < 8/3$. To adaptive refinement, we follow Algorithm 5.13 by choosing $\beta = 0.3$, $\text{tol}_n = 10^{-7}$ and $\text{maxiter}_n = 8$ and tracking Howard’s algorithm to quadratic($P^2$) finite element space. We let a coarse quasi-uniform, unstructured mesh as a initial and the mesh generated by adaptive refinement is shown in Fig. 2(A). Since $u \in H^2(\Omega)$, we do not expect the advantage of the adaptive scheme over than the uniform refinement for $H^1(\Omega)$-norm error of $u_U$; it is shown in Fig. 2(B). While, since $g = \nabla u$ does not have such smoothness, we observe the superiority performance of the adaptive scheme rather than the uniform refinement for $H^1(\Omega)$-norm error of $g_G$ and as well as $H^1(\Omega)$-norm error of $(u_U, g_G)$ in Fig. 2(C), (D).

7. Conclusion

This study efficiently and practically approximated the strong solution of the fully nonlinear HJB equation in two steps. First, we linearized the nonlinear HJB equation using the semismooth Newton method. We demonstrated the Newton differentiability of the HJB operator from $H^2(\Omega)$ to $L^2(\Omega)$ without requiring ellipticity or the Cordes condition. This suggests that this linearization method can
be applied to approximate the strong solution of a well-posed HJB equation under less stringent conditions. Through semismooth Newton linearization, the fully nonlinear HJB problem transforms into a recursive linear problem in nondivergence form, with its strong solution converging to the strong solution of the HJB equation with a superlinear rate.

With appropriate modifications, one can extend Newton differentiability and the semismooth Newton linearization to nonconvex Isaacs equations.

In the second step of the approximation, we discretized the recursive linear problem in each iteration using a mixed finite element method, employing a least-squares approach with gradient recovery. A notable advantage of the least-squares approach is the ability to enforce constraints on the unknowns by incorporating square terms into the cost functional. However, each discretization that works for the linear equation in nondivergence form can also be applied.

Additionally, we have conducted an error analysis of the approximation, obtaining both a priori and a posteriori error bounds. The a priori error bound serves to demonstrate the convergence of the discretization, while the a posteriori error bound facilitates the application of adaptive refinement procedures.
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