Renormgroup symmetry for solution functionals

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The paper contains generalization of the renormgroup algorithm for boundary value problems of mathematical physics and related concept of the renormgroup symmetry, formulated earlier by authors with reference to models based on differential equations. These algorithm and symmetry are formulated now for models with non local (integral) equations. We discuss in detail and illustrate by examples applications of the generalized algorithm to models with not local terms which appear as linear functionals of the solution.

1 Introduction

The concept of RenormGroup Symmetry (RGS) appeared in mathematical physics in the beginning of 90s \cite{1,2} (see, also reviews \cite{3,4}) being borrowed from theoretical physics. In turn, the concept of the Renormalization Group (RG) for the first time has appeared in the most complex branch of last — in the Quantum Field Theory (QFT). A presence of a group structure (Lie groups of transformations) in the QFT calculation results was discovered in the beginning of 50s by Stueckelberg and Petermann \cite{5} (see, also \cite{6,7}).

This structure and exact symmetry of a solution underlying have been then used by N.N.Bogoliubov for creation of regular method for improving of an approximate solution of QFT problems — the Renormalization Group Method (=RGM) \cite{8} (see also \cite{9,10,11}). This method contains elements of the theory of continuous group transformations (theory of Sophus Lie). Improvement of approximation properties appears to be most essential in a vicinity of the solution singularity.

Turn now to the fact that in the case of problems described by complicated equations — as, e.g., in the transfer theory (integro-differential Boltzmann equation) or in the quantum field theory (an infinite chain of engaged integro-differential Dyson-Schwinger equations) — only some components of solution or their integrated characteristics obey enough simple symmetry. So, in QFT the central object in RG transformations is the so called “function of an invariant charge” $\hat{\alpha}$ (or “running coupling constant”), representing specific product of Lorentz-invariant amplitudes of propagators $d_i$, vertices $\Gamma_k$ and the expansion parameter $\alpha$

$$\hat{\alpha} = \alpha \Gamma^2 \prod_i d_i.$$  

At the same time, Schwinger-Dyson equations include only functions $d_i \Gamma_k$ and the coupling constant $\alpha$, separately, but not their product $\hat{\alpha}$. In the one-velocity plane transfer problem the RG-invariance property is related to the asymptotics of “density of particles,
moving deep into the medium” \( n_+(x) \), \( x \to \infty \) not entering the Boltzmann equation\(^1\).

The renormalization group concept was transferred to mathematical physics \([1,2]\) with the same pragmatic goal, as in QFT, in mind — “improvement” of the solution behaviour in vicinity of a singularity. Remind that proliferation of the RG method from QFT to others fields of theoretical physics (the theory of critical phenomena, physics of polymers and so on) has caused various and sometimes essentially different (in comparison with initial) forms of realization of RG ideology (see, for example, the review \([12]\)). In application to problems of mathematical physics there appeared different variants of formulations of RG method \([13]-[21]\).

For the boundary value problems (BVPs) of mathematical physics using DE, we developed (see, e.g., \([1,2]\), and also recent reviews \([1] \text{ p.232-249}, [22]\)) the RG algorithm, which is quite distinct from earlier ones. To this difference became obvious, we remind, that in a basis of the RG Method, as it was initially formulated by N.N.Bogoliubov and one of authors \([8]\) for QFT problems, lays the use of an \textit{exact} group property of a solution. One of the well-known formulations of this property is the functional equation (representing only a group composition law) for the invariant charge in QFT. In every concrete case, revealing of similar symmetry (i.e., of group property) for the solution demands a special, usually non-standard, analysis (see, for example, discussion in the papers \([23,12,13]\)), that is an algorithmic drawback of the RG technique.

Coming back to mathematical physics (MP) we note, that here we usually deal with the problems based on systems of DEs the symmetry of which can be found by a regular way with the help of Lie group analysis. In problems of MP this feature appeared as decisive in creating “RG-algorithm” which has united RG ideology of QFT with a regular way of symmetry construction for BVP solutions. Due to this algorithm also there arised the concept of “renormgroup symmetry” for solutions of BVP: these symmetries result from calculation procedure similar to that used in the modern group analysis.

At the initial stage \([1,2]\), application of RG algorithm was mainly limited to problems based on DEs though formally this algorithm can be used in any problem, for which it is possible to specify a regular way of calculation symmetries for basic equations. Hence, transition to such objects, which up to a recent time were not a subject of the group analysis, in particular, to the integral and integro-differential equations, essentially expands the area of RGS applications.

In just mentioned cases integral relations form a skeleton of a problem. They, however, can appear as some independent objects for application of RGS, constructed for solutions of DE. Frequently, of physical interest is not the solution itself in all range of change of variables and parameters, but rather its certain integral characteristic, a solution functional. This characteristic can appear, for example, in result of averaging (integrating) over one of independent variables\(^2\) or when passing to new integral representation, e.g., to Fourier representation. In this case, RG algorithm can be applied not for improving of a particular solution with the subsequent calculation of its integral characteristic, but

\(^1\)And representable as the integral \( \int_0^1 n(x, \vartheta) \, d \cos \vartheta \) of the solution of kinetic equation \( n(x, \vartheta) \).

\(^2\)see, for example, previous footnote.
directly for improvement of the solution functional for an approximate solution. These motives gave a stimulus for expanding of the RG algorithm to models with non-local (integral) equations.

The contents of the paper is structured as follows. In section 2 an introductory example to RGS algorithm in mathematical physics is presented, illustrated by a solution of a simple BVP. In section 3, generalization of RG algorithm, developed in application to BVP for DE [3] (and reviewed in [1] and [22]), is formulated for models with non-local equations. In section 4, the review of recent results received on the basis of the modified RG algorithm is presented, and also efficiency of a method in application to some already known solutions is shown. Some generalities, uniting concepts of RG symmetry, the functional self-similarity, are considered in Appendix.

2 The introductory example to the RGS algorithm

Generally, the RG can be defined as a continuous one-parameter group of specific transformations of a partial solution (or solution characteristic) of a problem, a solution that is fixed by boundary conditions. The RG transformation involves boundary condition parameters and corresponds to some change in a way of imposing this condition.

For illustration, consider transformations \( T_a \),

\[
\bar{x}^i = f^i(x, a), \quad f^i(x, a_0) = x^i, \quad i = 1, \ldots, n, \tag{1}
\]

depending on a real parameter \( a \), where \( x \in \mathbb{R}^n \). A set \( G \) of these transformations form a one-parameter local group if the functions \( f^i(x, a) \) satisfy the composition rule

\[
T_b T_a = T_{\phi(a,b)}, \quad f^i(f^i(x, a), b) = f^i(x, \phi(a, b)), \quad \phi(a, a_0) = a, \quad \phi(a_0, b) = b, \tag{2}
\]

that can be transformed to the simplest form with \( \phi(a, b) = a + b \) and with \( a_0 = 0 \) in (1).

For a given solution of some physical problem renormgroup transformations in the simplest case are defined as transformations of (1) type, i.e. as simultaneous one-parameter group transformations \( R_t \) of two variables, say \( x \) and \( g \),

\[
x \rightarrow x' = x/t, \quad g \rightarrow g' = \bar{g}(t, g), \tag{3}
\]

the first being the scaling of a coordinate \( x \) (or reference point) and the second – a more complicated functional transformation of the solution characteristic \( g \). Hence, the RG transformation corresponds to a change in the parametrization for the same solution, while the equation (2) for the function \( \bar{g} \) has the form

\[
\bar{g}(x, g) = \bar{g}(x/t, \bar{g}(t, g)), \quad \bar{g}(1, g) = g, \tag{4}
\]

and guarantees the group property \( R_{\tau t} = R_\tau R_t \) fulfillment for transformations [8]. These are just the RG functional equations and transformations for a massless QFT model with one coupling constant [8]. In that case \( x = Q^2/\mu^2 \) is the ratio of a four-momentum \( Q \)
squared to a “normalization” momentum $\mu$ squared and $g$ is a coupling constant, while $\bar{g}$ is the so-called effective coupling.

Geometrically, transformations (1) mean that any point $x \in \mathbb{R}^n$ is carried by this transformations into the point $\bar{x}$ whose locus is a continuous curve passing through $x$, known as of a path curve of the group $G$. The group property (2) means that any point of a path curve is carried by $G$ into points of the same curve. The locus of the images $T_a(x)$ is also termed the $G$-orbit of the point $x$. The correspondence between transformations (1) and (3) means that for RG transformations a curve in the $\{x, g\}$ plane that defines the solution of a physical problem is the path curve of the renormgroup $R_t$. In other words, the solution of the problem coincides with the $R_t$-orbit of a boundary manifold – the point $\{x = x_0, g = g_0\}$. Upon the RG transformations the reference (boundary) point $\{x_0, g_0\}$ is shifted to some other value $\{x_1, g_1\}$, while the solution remains unaltered, i.e. the solution curve is the invariant manifold of the group $R_t$ (like the invariant charge in QFT [11]).

Hence, the general problem of searching for the RG transformations may be reformulated as follows: the solution of the physical problem should coincide with the orbit of the renormalization group.

In mathematical physics a solution of a physical problem usually appears as a solution of some BVP. Then the corresponding RG transformations may be obtained from the symmetry group related to this BVP, provided the boundary condition is also involved in group transformations. The key point here is the fact that the corresponding symmetry group are calculated using the regular algorithms of modern group analysis, provided the basic mathematical model is formulated in terms of differential (or integro-differential) equations.

Let this model be given by a system of $k$-th order differential equations, identified with its frame,

$$F_\sigma(x, u, u^{(1)}, \ldots, u^{(k)}) = 0, \quad \sigma = 1, \ldots, s. \quad (5)$$

In the paper we use the terminology of differential algebra and notations for variables accepted in the group analysis [24]:

$$x = \{x^i\}, \quad u = \{u^\alpha\}, \quad u^{(1)} = \{u^\alpha_i\}, \quad u^{(2)} = \{u^\alpha_{ij}\}, \ldots, \quad (6)$$

where $\alpha = 1, \ldots, m; \ i, j, \ldots = 1, \ldots, n$. Variables $x$ and $u$ are referred to as independent variables and differential variables, respectively, having consecutive derivatives $u^{(1)}$, $u^{(2)}$, ... etc. Differential variables are related by a system of equations

$$u^\alpha_i = D_i (u^\alpha), \quad u^\alpha_{ij} = D_j (u^\alpha_i) = D_j D_i (u^\alpha), \ldots \quad (7)$$

via the operator of the total differentiation

$$D_i = \frac{\partial}{\partial z^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + \ldots \quad (8)$$

Locally analytical function of variables (6), for example, the function $F(x, u, u^{(1)}, \ldots, u^{(k)})$ with the highest order derivative $k$ refers to as differential function of the $k$-th order, and
a set of all such functions with any values of \(k\) forms the space of differential functions \(\mathcal{A}[x,u]\). Any function \(F \in \mathcal{A}[x,u]\) gives rise to a differential manifold \([F]\), determined by an infinite system of equations

\[
[F]: \quad F = 0, \quad D_i F = 0, \quad D_i D_j F = 0, \ldots
\]

The manifold \([F]\) is called the frame of the \(k\)-th order partial differential equation

\[
F\left(x, u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^k u}{\partial x^k}\right) = 0.
\]

According with the definition a system of \(s\)-th order differential equations is said to be invariant under a group \(G\) if the frame of the system is an invariant manifold for the extension of the group \(G\) to the \(s\)-th order derivatives [25, p.209]. When utilizing an infinitesimal group generator

\[
X = \xi^i \partial_{x^i} + \eta^\alpha \partial_{u^\alpha}, \quad \xi^i, \eta^\alpha \in \mathcal{A},
\]

with coordinates \(\xi^i, \eta^\alpha\), which are functions of group variables \(\{x^i, u^\alpha\}\), this definition leads to the invariance criterion in the following form

\[
X_{(k)} F_\sigma \bigg|_{[F_\sigma]} = 0, \quad \sigma = 1, \ldots, s,
\]

where \(X_{(k)}\) denotes \(X\) extended to all derivatives, involved in \(F_\sigma\) and the symbol \([F]\) means evaluated on the frame (9). Solving a system of linear homogeneous partial differential equations (known as the determining equations) for coordinates \(\xi^i, \eta^\alpha\) gives a set of infinitesimal operators (11) (or group generators) which correspond to the admitted vector field of the symmetry group \(G\) and form a Lie algebra \(L\).

Let the Lie group \(G\) with the generator

\[
X = \xi^i \partial_t + \xi^x \partial_x + \eta \partial_y,
\]

be defined for the system of the first order partial differential equations

\[
y_t = F(t,x,y,y_x).
\]

The typical BVP for (14) is the Cauchy problem with the boundary manifold defined by

\[
t = 0, \quad y = \psi(x).
\]

The solution of this Cauchy problem is the \(G\)-invariant solution, iff for any generator (13) the function \(\psi\) obeys the Equation [26, §29]:

\[
\eta(0,x,\psi) - \xi^x(0,x,\psi)\partial_x \psi - \xi^t(0,x,\psi) F(0,x,\psi,\partial_x \psi) = 0.
\]

The solution of the Cauchy problem (14)–(15) coincides with orbit of the group \(G\) and the boundary manifold is partially invariant manifold of the group \(G\) with the defect \(\delta = 1\).
This example gives an instructive idea of constructing generators of renormgroup sym-
metries. The milestones here are: a) considering the BVP in the extended space of group
variables that involve parameters of boundary conditions in group transformations, b)
calculating the admitted group using the infinitesimal approach, c) checking the invari-
ance condition akin to (16), with the goal to find the symmetry group with the orbit which
coincides with the BVP solution and d) utilizing the RG symmetry to find the improved
(renormalized) solution of the BVP.

The full algorithm was described in details in our previous publications [3, 4] and will
be also touched upon in the next section while here we will only give a general grasp at
the problem using a trivial example, the BVP for the Hopf equation

\[ v_z + vv_x = 0, \quad v(0, x) = \epsilon U(x), \quad (17) \]

where \( U \) is the invertible function of \( x \). Introducing \( u = \epsilon v \) we insert the boundary
amplitude directly in the input equation

\[ u_z + \epsilon uu_x = 0, \quad u(0, x) = U(x). \quad (18) \]

For small values of \( \epsilon z \ll 1 \), i.e. near the boundary, \( z \to 0 \), or for small amplitude at
the boundary, \( \epsilon \to 0 \), a perturbation theory solution to (18) has the form of a trunca-
ted power series in \( \epsilon z \),

\[ u = U - \epsilon z U_x + O((\epsilon z)^2). \quad (19) \]

It is obvious that this solution is invalid for large distances from the boundary, when
\( \epsilon z U_x \approx 1 \). The renormgroup symmetry gives the root to improving the perturb-
tation theory result and restore the correct structure of the BVP solution.

With the goal to obtain this symmetry we extend the list of variables, involved in
group transformations, adding the parameter \( \epsilon \) to the list of independent variables. Then
we calculate the admitted symmetry group \( G \), with the generator

\[ X = \xi^z \partial_z + \xi^x \partial_x + \xi^t \partial_t + \eta \partial_u, \quad (20) \]

using the classical Lie calculational algorithm (see, e.g. [25]) that employs the infinitesimal
criterion (12). The solution of determining equations gives coordinates of the generator

\[ \xi^z = \psi^1, \quad \xi^x = \epsilon u \psi^1 + \psi^2 + x(\psi^3 + \psi^4), \quad \xi^t = \epsilon \psi^4, \quad \eta = u \psi^3, \quad (21) \]

where \( \psi^i, \quad i = 2, 3, 4 \), are arbitrary functions of \( \epsilon, u, x - \epsilon uz \) and \( \psi^1 \) is an arbitrary
function of all group variables. These formulas define an infinite-dimensional Lie algebra with four
generators\(^3\)

\[ X_1 = \psi^1 (\partial_z + \epsilon u \partial_x), \quad X_2 = \psi^2 \partial_x, \quad X_3 = \psi^3 (x \partial_x + u \partial_u), \quad X_4 = \psi^4 (\epsilon \partial_t + x \partial_x). \quad (22) \]

\(^3\)In case when the amplitude \( \epsilon \) is not involved in transformations we have only three generators (see,
e.g. [24 p.222]).
Suppose we are given a particular solution of the BVP (18), \( u - W(z, x, \epsilon) = 0 \), which defines an invariant manifold of the group (20), (21). The corresponding invariance condition, evaluated on the frame (18), looks similar to (12),

\[
(W - xW_x)\psi^3 - W_x\psi^2 - (\epsilon W_x + xW_x)\psi^4 = 0. \tag{23}
\]

This equation is valid for all \( z \), hence it remains valid for \( z = 0 \), when \( W \) is replaced by \( U(x) \). In this limit, \( z \to 0 \), the condition (23) gives a relationship between \( \psi^i \), \( i = 2, 3, 4 \), (no restrictions are imposed on \( \psi^1 \)), that can be easily prolonged on \( z \neq 0 \),

\[
\psi^2 = -\chi(\psi^3 + \psi^4) + (u/U_\chi)\psi^3, \quad \chi = x - \epsilon uz, \tag{24}
\]

where the derivative \( U_\chi \) should be expressed either in terms of \( \chi \) or \( u \) in account of the boundary condition. Inserting (24) in (21) we get the group of a smaller dimension with the generators

\[
R_1 = \psi^1 (\partial_x + \epsilon u \partial_x), \quad R_2 = u \psi^3 [(\epsilon z + 1/U_\chi) \partial_x + \partial_u], \quad R_3 = \epsilon \psi^4 (zu \partial_x + \partial_\epsilon). \tag{25}
\]

The above procedure, that transforms (22) to (25) we refer to as the restriction of the group (20) on a particular solution.

The solution of the BVP defines a manifold, which appears to be invariant for any generator \( R_i \) just by a method of construction, hence (25) defines the desired RG symmetries. This means that the solution of the BVP can be constructed using any generator of the RG algebra (25), say, the generator \( R_3 \). Without loss of generality we choose \( \epsilon \psi^4 = 1 \) and obtain the following finite RG transformations (\( a \) is a group parameter)

\[
x' = x + az, \quad \epsilon' = \epsilon + a, \quad z' = z, \quad u' = u. \tag{26}
\]

Here \( z \) and \( u \) are invariants of RG transformations, while transformations of \( \epsilon \) and \( x \) are translations, which in case of \( x \) also depend upon \( z \) and \( u \). For \( \epsilon = 0 \) we have, in view of (19), \( x = H(u) \), where \( H(u) \) is defined as the function inverse to \( U(x) \). Then eliminating \( a, z \) and \( u \) from (26) and omitting primes over the variables, we get the desired solution of the BVP (18) in the non-explicit form

\[
x - \epsilon uz = H(u). \tag{27}
\]

This in fact is the improved PT solution (19), which is valid not only for small \( \epsilon \ll 1 \), provided the dependence (27) can be resolved in a unique way.

The peculiarity of the procedure of constructing RG symmetries is the multi-choice first step that depends on the way in which boundary conditions are formulated and the form in which the admitted symmetry group is calculated. For example, instead of calculating the Lie point symmetry group we may consider the Lie-Bäcklund symmetries with the canonical generator \( R = \kappa \partial_u \), where \( \kappa \) depends not only on \( z, x, \epsilon, u \), but on higher order derivatives of \( u \) as well. We may look for \( \kappa \) in the form of a power series in \( \epsilon \), and the invariance condition (23) is formulated as vanishing of \( \kappa \) at \( z \to 0 \). Depending
on the choice of the zero-order term representation we get either infinite or a truncated power series for $\kappa$, say, a linear in $\epsilon$ form,

$$ R = \kappa \partial_u , \quad \kappa = 1 - \frac{u_x}{U_x(u)} - \epsilon z u_x. $$

This RG generator (28) is equivalent to the Lie point generator $R_2$ from (25), thus giving the same result.

Another possibility of calculating RG symmetries for BVP (18) is offered by taking into account some additional differential constraints which are consistent with the boundary conditions and input equations. For example, if the boundary condition in (18) is linear in its argument, $U(x) = -x$, the differential constraint may be taken as $u_{xx} = 0$; this equality reflects the invariance of the original equation with respect to the second-order Lie-Bäcklund symmetry group. Calculating the Lie point symmetry group for the joint system of this constraint and the Hopf equation gives another way to finding RG symmetries for the BVP (18).

The above example demonstrates the milestones of the RGS method in mathematical physics. The details of the general approach will be discussed in the next section. Here we just point to the fact that in order to construct RGS we employ the symmetry group calculated in a regular way using the modern group analysis technique. This group is considered in the extended space of variables that includes the parameters and boundary data involved in group transformations. The invariance of the perturbative solution is used to find the particular RGS generator that leaves the particular BVP solution unaltered. Then the utilization of the finite group transformations restore the desired structure of the BVP solution.

3 The scheme of RG algorithm for non-local problems

A procedure of construction and use of RGS with reference to BVP for DE is described in details in our previous works, for example, in reviews [3, 4] and illustrated in the previous section. Nevertheless, for connectivity of a statement, the basic stages of the scheme are briefly depicted below with accent on those changes, which it is necessary to introduce in RG algorithm in order to make it applicable to non-local problems.

The starting point in these problems is the system of $\nu \geq 1$ integro-differential (including differential and integral) equations for functions $u = \{u^\alpha\}, \alpha = 1, \ldots, m$ of variables $x = \{x^i\}, i = 1, \ldots, n$,

$$ [E] : \quad E_\nu(u(x)) = 0 , $$

with the non-local terms depending upon integrals of these functions, and supplemented by appropriate boundary or initial conditions. We assume, that we know some approximate solution, $u^\alpha = U^\alpha$, for example, represented by a truncated power series in powers of a small parameter or in powers of a small deviation from a boundary of an area, where the solution is known.
Then, conditionally, the scheme of realization of RG algorithm can be expressed as a sequence of four steps, submitted by the scheme on the Figure 1.

(I) construction of basic manifold,
(II) calculation of the admitted symmetry group and
(III) its restriction on a particular BVP solution leading to revealing of RGS, and also
(IV) searching for an analytical solution that is adequate to this RGS.

### 3.1 Construction of RG manifold

The initial problem is the construction of RGS and appropriate transformations that also touch upon parameters of particular solutions. Therefore the purpose of a first step (I) consists of involving in group transformations in this or that way the parameters entering both in the equations of a problem, and in boundary conditions on which this partial solution depends. This purpose is achieved by construction of a special manifold $\mathcal{RM}$ which we believe is given as a system of $s$ DE of the $k$-th order and $q$ non-local relations,

$$F_\sigma(z,u,u^{(1)},\ldots,u^{(k)}) = 0, \quad \sigma = 1,\ldots,s,$$

$$F_\sigma(z,u,u^{(1)},\ldots,u^{(r)},J(u)) = 0, \quad \sigma = 1 + s,\ldots,q + s.$$
Non-local variables $J(u)$ included in (31) are introduced by integrated operations

$$J(u) = \int \mathcal{F}(u(z))dz.$$

(32)

A presence of relations (31) characterizes the basic difference of $\mathcal{R}\mathcal{M}$ for non-local problems from the case of BVP for DE, for which the manifold $\mathcal{R}\mathcal{M}$ is differential.

Let’s note that generally $\mathcal{R}\mathcal{M}$ does not coincide with a system of initial equations. Only on occasion, which will be told about below, and at the additional clauses expanding and specifying the list of variables and the parameters entering in RG transformations, it is possible to establish conformity between (29) and (30–31). However it is not possible to execute a first step of the algorithm for any boundary problems simply treating $[E]$ as $\mathcal{R}\mathcal{M}$, and the concrete form of a realization of the first step depends both on a form of the initial equations (29), and on the form in which boundary conditions are presented. Formulated earlier for models with DE the general approach to construction of $\mathcal{R}\mathcal{M}$ remains also valid for non-local problems. In this sense several specified earlier (see, for example, [4]) and illustrated in section I ways of constructing this manifold are possible. These ways do not exhaust all opportunities to construction of $\mathcal{R}\mathcal{M}$, rather they emphasize a variety of approaches to realization of a first step depending on a character of a problem under consideration. A choice of a concrete realization more often is dictated both by a form of the Eq. (29) and boundary conditions to them, and a type of the approximate solution. Such multi-variant situation is inherent only in the first step of the algorithm and is aimed on covering the widest variety of the problems, analyzed by this method. Already the following step of the scheme is carried out in frameworks of the well-developed group-theoretical methods.

### 3.2 Calculation of the transformation group

The next step (II) consists in calculation the most wide admitted symmetry group $\mathcal{G}$ for Eqs. (30)-(31). Here the essential change of RG algorithm is required in comparison with its realization for differential manifold $\mathcal{R}\mathcal{M}$. Really, in application to $\mathcal{R}\mathcal{M}$, defined only by a system of DE (30), the question was about a local group of transformations in space of differential functions $\mathcal{A}$, at which system (30) remains unchanged. At transition to manifold $\mathcal{R}\mathcal{M}$ which is set by the system of Eqs. (30)-(31), classical Lie algorithm, using the infinitesimal approach, appears inapplicable. The basic obstacle here is that $\mathcal{R}\mathcal{M}$ in this case is not determined locally in space of differential functions, therefore the basic advantage of Lie computational algorithm, namely, representation of DetEq as the over-determined system of equations is not realized here. Also in frameworks of the classical group analysis the procedure of prolongation of the group operator of point transformations on non-local variables is not defined. Probable ways to overcome these complexities while performing the second step of RG algorithm are specified below.

At modification of the RG algorithm we lean on the direct method of calculation of symmetries which was advanced in [28]–[29] and used for finding symmetries for the
Boltzmann kinetic equation, the equations of motion of viscous-elastic media and Vlasov-Maxwell equations in the kinetic theory of plasma. This method is based on a generalization of a group of symmetry, the so-called Lie-Bäcklund symmetry group (terms “higher” or “generalized” symmetry are also in use), defined by the generator of the form (31) prolonged on all higher-order derivatives,

\[ X = \xi_i \partial_{z^i} + \eta^\alpha \partial_{u^\alpha} + \zeta^\alpha \partial_{u^\alpha_i} + \zeta^\alpha_{i_1 i_2} \partial_{u^\alpha_{i_1 i_2}} + \ldots , \]

\[ \zeta_i^\alpha = D_i (\varkappa^\alpha) + \xi^j u^\alpha_{ij} , \quad \zeta_{i_1 i_2}^\alpha = D_{i_1} D_{i_2} (\varkappa^\alpha) + \xi^j u^\alpha_{j i_1 i_2} , \quad \varkappa^\alpha = \eta^\alpha - \xi^i u^\alpha_i , \] (33)

with coordinates \( \xi^i([z,u]), \eta^\alpha([z,u]), \zeta^\alpha([z,u]) \ldots \), being differential functions from the space \( \mathcal{A} \). A set of all Lie-Bäcklund operators forms an infinite-dimensional Lie algebra \( L_B \), and any operator of a form \( X^* = \xi^i D_i \) is the Lie-Bäcklund operator for any differential function \( \xi^i([z,u]) \); the set \( L^* \) of operators \( X^* \) forms an ideal in \( L_B \). This property allows to introduce a notion of equivalence of two Lie-Bäcklund operators \( X_1, X_2 \in L_B \) if \( X_1 - X_2 \in L^* \) (written as \( X_1 \sim X_2 \)). In particular, any Lie-Bäcklund operator \( X \in L_B \) is equivalent to the operator (33) with \( \xi^i = 0 \),

\[ X \sim Y = X - \xi^i D_i = \varkappa^\alpha \partial_{u^\alpha} , \quad \varkappa^\alpha \equiv \eta^\alpha - \xi^i u^\alpha_i . \] (34)

The operator \( Y \) is known as the canonical representation of \( X \), and in notation (34) we imply the prolongation of action of the operator on all higher-order derivatives according to formulas (33). It is essential, that in the group of infinitesimal transformations \( G \) with operator (34) and the parameter \( a \) only dependent variables \( u^\alpha \) are changed,

\[ u'^\alpha = u^\alpha + a \varkappa^\alpha + O(a^2) , \quad z'^i = z^i , \] (35)

while independent variables \( z^i \) remain unchanged. This property has allowed to formulate the concept of symmetry groups of IDE of form (31) as a local group of transformations \( G \) with operator (34), at which the form of the function \( F_\sigma \) remains unchanged for any value of a group parameter \( a \). Differentiation of the appropriate invariance condition, which has been written down for the function \( F_\sigma \) dependent on the transformed dependent variable \( u'^\alpha \), with respect to the group parameter \( a \) and transition to a limit \( a \to 0 \) gives determining equations (DetEq). In difference from a case of basic DE, these DetEq generally also are non-local.

With the help of the canonical operator \( Y \), the invariance criterion for Eq. (31) with respect to the admitted group can be written down in an infinitesimal form

\[ Y F_\sigma \bigg|_{[F_\sigma]} = 0 , \quad \sigma = 1 + s , \ldots , q + s , \quad \text{where} \quad Y \equiv \int dz \varkappa(z) \frac{\delta}{\delta u(z)} . \] (36)

Meaning generalization of action of the canonical group operator not only on differential functions, but also on functionals, here in definition of \( Y \) variational differentiation [29] is used. For integral functionals (32) a derivative of \( \delta J/\delta u(z) \) with respect to a function \( u \) is defined via the principal (linear) part of an increment of a functional as a limit (if it
exists) (see [31]):

\[
\frac{\delta J[u]}{\delta u(z)} = \lim_{\epsilon \to 0} \frac{J[u + \delta u_\epsilon] - J[u]}{\int_\Delta d\tau \delta u_\epsilon(\tau)}; \quad z \in (\tau_1, \tau_2).
\]

Here an infinitesimal variation \(\delta u_\epsilon(z) \geq 0\) is a continuously differentiable function given on fixed interval \(\Delta = [\tau_1, \tau_2]\) which differ from zero only in \(\epsilon\)-vicinity of a point \(z\), and the norm \(\|\delta u_\epsilon\|_{C^1} \to 0\) at \(\epsilon \to 0\).

It is checked up by direct calculation that the action of the operator \(Y\) on any differential function and its derivatives, for example \(u, u_z, \ldots\) gives a usual result: \(Y u = \xi, Y u_z = D_z(\xi)\) etc. Hence, if \(F_\sigma = 0\) is a usual DE, formulas (36) result to local DetEq, and in a case when \(F_\sigma = 0\) has a form of a system of IDE, formulas (36) can be considered as non-local DetEq, dependent both upon local and non-local variables.

Treating local and non-local variables in DetEq as independent variables allows to divide these equations into local and non-local. A procedure of a solution of local DetEq is carried out in a standard way, using Lie algorithm based on splitting of a system of overdetermined equations with respect to local variables and their derivatives. In result expressions for coordinates of the group operator are found, determining so-called group of intermediate symmetries [29] which are used further at the analysis of non-local DetEq. Procedure of the solution of non-local DetEq is carried out similarly, by substituting coordinates of group operator of intermediate symmetry found in non-local DetEq and splitting them with the help of variational differentiation. Hence, construction of symmetries for non-local equations also becomes an algorithmic procedure, which can be presented as a sequence of the following operations:

a) definition of a set of local group variables,
b) construction of DetEq on a basis of infinitesimal invariance criterion, which uses a generalization of the definition of the canonical operator,
c) separation of DetEq into local and non-local,
d) solution of local DetEq with use of the standard Lie algorithm,
e) solution non-local DetEq with the help of the operation of the variational differentiation.

These operations are generalization of the second step of algorithm on a case, when \(RM\) is the integral or the integro-differential manifold.

In summary we describe an operation of prolongation of a symmetry group on non-local variables. To execute standard (in the classical group analysis) operation of prolongation of the operator of Lie point group on the non-local variable defined, for example, by integral relationship [32], first write down this operator in the canonical form, \(Y\), and then formally prolong it on non-local variable \(J\)

\[
Y + \xi^l \partial J \equiv \xi \partial_u + \xi^l \partial_J.
\]

4 Definition of a variational derivative for arbitrary functional see in [32].
The integral relation between $\kappa$ and $\kappa^J$ is obtained by applying operator (37) to Eq. (32) which was introduced as a definition of the variable $J$. Substituting explicit expressions for the coordinate $\kappa$ of the operator $Y$ and calculating the resulting integrals we obtain the required coordinate $\kappa^J$ of the prolonged operator,

$$\kappa^J = \int \frac{\delta J(u)}{\delta u(z)} \kappa(z) \, dz \equiv \int \frac{\delta F(u(z'))}{\delta u(z)} \kappa(z) \, dz \, dz' = \int F_u \kappa(z) \, dz. \quad (38)$$

Here for brevity only one argument of a generator’s coordinate is specified, namely the argument upon which the integration is fulfilled.

The actions described in this paragraph resulting in operators of the admitted group in non-canonical (33) or in canonical (34) representation, make essence of the second step of RG algorithm.

### 3.3 Restriction a group on a solution

The group found on the second step, $\mathcal{G}$, which is determined by operators (33) and (34), is generally wider, than the renormalization group of interest. The last is related to the concrete particular solution of a boundary problem, hence with the goal to get RGS it is necessary to make the third step (III), consisting in restriction of the group $\mathcal{G}$ on a manifold, set by this particular solution. From the mathematical point of view, this procedure consists in checking vanishing conditions for a linear combination of coordinates $\kappa^j_\alpha$ of the canonical operator equivalent to (33), on some particular solution $U^\alpha(z)$ of a boundary problem

$$\left\{ \sum_j A^j \kappa^\alpha_j \equiv \sum_j A^j \left( \eta^\alpha_j - \xi^i_j u^\alpha_i \right) \right\} \bigg|_{u^\alpha = U^\alpha(z)} = 0. \quad (39)$$

The form of the condition set by relation (39), is common for any solution of the BVP, but a way of realization of the restriction procedure of a group in each separate case is different. Usually a particular approximate solution for a concrete BVP is used. On the general scheme it is specified as the dotted arrow connecting “initial object” — the approximate solution for particular BVP — with that object which arises in a result of the third step.

At calculation of combination (39) on a concrete particular solution $U^\alpha(z)$ it is transformed from a system of DE for group invariants to algebraic relationships. We shall note two consequences of the specified actions. First, the procedure of restriction results in a set of relations between various $A^j$ and thus “links” coordinates of various group operators $X_j$, admitted by $\mathcal{RM}$ (30)–(31). Second, it eliminates (in part or completely) an arbitrariness which can arise in values of coordinates $\xi^i, \eta^\alpha$ in a case of the infinite group $\mathcal{G}$.

As a rule, the procedure of restriction of the group $\mathcal{G}$ reduces its dimension. At that the general element (33) of this new group $\mathcal{RG}$ after performance of this procedure is
represented by a linear combination of new generators $R_i$ with coordinates $\hat{\xi}^i$, $\hat{\eta}^\alpha$,

$$X \Rightarrow R = \sum_j B^j R_j \, , \quad R_j = \hat{\xi}^i \partial x^i + \hat{\eta}^\alpha \partial u^\alpha \, ,$$

and arbitrary constants $B^j$.

The set of operators $R_j$, each containing the required solution of a problem in the invariant manifold, defines a group of transformations, which by analogy with RG for models with DE we also refer to as renormgroup.

The statement of the third step of the algorithm given in this section finishes the description of a procedure of construction of RGS. In the following section it is shown, how RGS are used for achievement of an ultimate goal of the RG algorithm, namely improvement of an approximate solution.

### 3.4 Construction of RG-invariant solution

Three steps described above completely define the regular algorithm of construction of RGS, but one more, the fourth (IV), final step is necessary. This step consists in use of RGS operators for finding analytical expressions for new, improved (in comparison with initial) BVP solutions.

From the mathematical point of view realization of this step consists in use of renorm-group invariance conditions which are set by a joint system of the equations (30)-(31) and vanishing conditions for a linear combination of coordinates $\hat{\zeta}_j^\beta$ of the canonical operator equivalent to (40),

$$\sum_j R^j \hat{\zeta}_j^\alpha \equiv \sum_j B^j \left( \hat{\eta}_j^\alpha - \hat{\xi}_j^i u^\alpha_i \right) = 0 \, .$$

(41)

Necessity of use $\mathcal{RM}$ while constructing BVP solution is marked on Figure 1 by the dotted arrow connecting these two objects.

It is clear, that the form of (41) is close to (39). However, contrary to a previous step, differential variables $u^\alpha_i$ in (41) are not replaced with the approximate expressions for BVP solutions $U(z)$ but are treated as usual dependent variables.

It is clear, that the form of (41) is close to (39). However, contrary to a previous step, differential variables $u^\alpha_i$ in (41) are not replaced with the approximate expressions for BVP solutions $U(z)$ but are treated as usual dependent variables.

In the most widespread case, when the renormgroup appears as a one parametric Lie point group, the invariance conditions yield first order partial differential equations. Solutions of the connected characteristic equations give group invariants (similar to invariant charges in QFT) through which the required solution of BVP is expressed. Generally for arbitrary RGS the RG invariance conditions, written down for any BVP, do not represent characteristic equations for the Lie point group operator. They can have more complex form, for example, being represented as a combination of partial DE and ordinary DE of a high order. However the common approach to construction of BVP solution as the RG invariant solution remains in force.

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5Here we should point on existing the analogy between this concept and concept of functional self-similarity [33], [34] (see also discussion in the Appendix A).
The description of the fourth step finishes the description of the regular algorithm of construction of RGS for models with IDE. It should be noted that two last steps are in the same root as for models with DE. The following section contains a number of concrete examples showing the ability of RGS algorithm.

4 Construction of RGS for integral models

4.1 The example with functionals of the solution of Hopf equation

Let’s proceed with a simple illustrative example, that we have discussed earlier at the beginning of the paper, i.e. an initial problem for Hopf equation (17). We have shown that the solution of the BVP (17) can be constructed using any generator of the RG algebra (25). Let we are interested not in the solution in all space, but only in some solution characteristic at a specific point, say, a value of its first derivative at a point \( x = 0 \), which formally can be introduced by a linear functional of \( u \),

\[
 u_x(z,0) \equiv u^0_x = -\int_{-\infty}^{+\infty} dx \delta'(x)u(t,x).
\]  

(42)

The dependence of \( u^0_x \) upon \( z \) can be easily restored using the prolongation of the linear combination of RG generators (25) on the solution functional (42). We use again the last generator from the list (25) in its simplest form with \( \epsilon \psi^4 = 1 \). Then we write down this generator in the canonical form and calculate its prolongation using formulas (37) and (38). Restricting the RG operator obtained after such prolongation to the space of group variables \( \{z,\epsilon,u^0_x\} \), we get the RG generator for the solution functional (42). For concreteness we choose \( U = -x \) and get the following generator

\[
 R_4 = \partial_t - z(u^0_x)^2 \partial_{u^0_x}.
\]  

(43)

The initial condition for the \( u^0_x \) at \( z = 0 \) is known, \( u^0_x(z = 0) = -1 \), hence the use of the invariant of this generator, \( J^0 = \epsilon z - 1/u^0_x = 1 \) restores the desired dependence \( u^0_x = -1/(1-\epsilon z) \), which is valid from the point \( z = 0 \) up to the singularity point \( z_{\text{sing}} = 1/\epsilon \). We note that this result is obtained without construction of BVP solution, using only the appropriate RGS. On the first sight the considered methodical example and the construction carried out looks clumsily enough and it is much easier to proceed from the trivial solution (27). But in more complex situations the explicit form of the solution is frequently unknown, whereas it is possible to construct RGS. In two subsequent sections of the paper, 4.2 and 4.3, we give examples that serve as an illustration of this statement.
4.2 Nonlinear optics: development of a singularity on the laser beam axis

We proceed our illustrations with analyzing the BVP for a system of two first order partial DE, which include Hopf equation as a particular case and are widely used in gas dynamics, optics and plasma physics:

\[
v_z + vv_x - \alpha I_x = 0, \quad I_z + vI_x + Iv_x = \nu I v/x.
\]

For concreteness below the terminology of nonlinear optics is used where \( \text{(44)} \) are known as nonlinear geometrical optics equations (see, for example, the review \[35\]) and are utilized to describe the evolution of a laser beam in a nonlinear medium. In this case \( I(z,x) \) stands for the beam intensity and \( v(z,x) \) is the derivative of a beam eikonal with respect to transverse coordinate, \( \alpha \) is a factor of nonlinear refraction, \( z \) and \( x \) are coordinates in a direction of a beam propagation and in a perpendicular direction, respectively; \( \nu = 1 \) and \( \nu = 0 \) for cylindrical and for plane case, respectively.

The nonlinear medium occupies a half-space \( z \geq 0 \) and boundary conditions for equations \( \text{(44)} \) are set at \( z = 0 \),

\[
v(0,x) = 0 \quad I(0,x) = I(x).
\]

The specific choice of the zero value of an eikonal derivative on the boundary corresponds to a collimated beam with the distribution of the beam intensity upon a transverse coordinate \( x \), characterized by the function \( I(x) \) with the maximal value equal to unity.

Various analytical approaches (see the papers \[35 \[36 \[37 \]) do not provide us with a universal method for finding solutions to BVP \( \text{(44)-(45)} \), suitable for any geometry of a problem and any boundary data, while the use of RG algorithm offers new opportunities (see, for example, \[38 \[4 \]). In particular, it allows to prolong the solution, known in a small vicinity of the boundary \( z \simeq 0 \) of a nonlinear medium, up to a vicinity of a solution singularity, which occurrence represents the most attracting physical effect. The singularity is formed on a beam axis hence the behaviour of \( I(x) \) and \( v(x) \) at \( x \to 0 \) is of particular interest.

The surprising thing is that this behaviour can be understood without knowledge of the complete solution via the application of RG algorithm to two functionals of the BVP solution \( \text{(41)-(45)} \), namely, to the intensity of a laser beam \( I^0(z) \equiv I(z,0) \) and to the second derivative of the eikonal \( W^0(z) \equiv v_x(z,0) \), calculated on an axis of a beam and related to this solution by formal relationships

\[
I^0(z) = \int dx \delta(x)I(z,x), \quad W^0(z) = \int dx \delta(x)v_x(z,x).
\]

Boundary conditions for these functionals with the account of \( \text{(45)} \) are given as

\[
I^0(0) = 1, \quad W^0(0) = 0.
\]
explicit form is defined by the profile of the beam intensity \( I(x) \) at \( z = 0 \). We present two examples, corresponding to cylindrical and plane ("slit") laser beams with various \( I(x) \).

For a cylindrical beam (\( \nu = 1 \)) with the parabolic intensity distribution \( I(x) = 1 - x^2 \), the RGS generator has the form

\[
R_{\text{par}} = (1 - 2\alpha z^2) \partial_z - 2\alpha z x \partial_x - 2\alpha (x - vz) \partial_v + 4\alpha I z \partial I .
\]

(48)

To determine the dependence of \( I^0 \) and \( W^0 \) on the coordinate \( z \) we prolong (48) on non-local variables (solution functionals) \( I^0 \) and \( W^0 \) that gives the following generator in the reduced space of variables \( \{z, I^0, W^0\} \),

\[
R_4 = (1 - 2\alpha z^2) \partial_z + 4\alpha I^0 z \partial I^0 - 2\alpha (1 - 2zW^0) \partial W^0 .
\]

(49)

The use of two invariants of generator (49), \( J_1 = (1 - 2\alpha z^2) I^0 \) and \( J_2 = W^0 (1 - 2\alpha z^2) + 2\alpha z \) with evident equalities \( J_1 = 1 \) and \( J_2 = 0 \), which follow at the account of boundary conditions (47), immediately gives expressions

\[
I^0 = \frac{1}{1 - 2\alpha z^2} , \quad W^0 = -\frac{2\alpha z}{1 - 2\alpha z^2} .
\]

(50)

These formulas describe spatial dependence of variables \( I^0(z) \) and \( W^0(z) \), starting from a boundary of a nonlinear medium \( z = 0 \) up to the point \( z_{\text{sing}} = 1/\sqrt{2\alpha} \), where the solution singularity occur, i.e. where the beam intensity and the eikonal derivative turns to infinity; beyond this point there is an area of rays intersection, where equations (44) can not be applied. Expressions (50) also follow from formulas, obtained earlier [39] without use of RG algorithm. However, the RGS algorithm here presents an elegant way of obtaining these formulas without calculating the complete solution to BVP.

Curves of typical dependencies of variables \( I^0(z) \) and \( W^0(z) \) upon the dimensionless coordinate \( z/z_{\text{sing}} \) at \( \alpha = 0.1 \) are given on Figure 2. The change of the parameter \( \alpha \) does not change a type of the curve for the intensity \( I^0 \), whilst values of \( W^0 \) on the right panel vary proportionally to \( \sqrt{\alpha} \). Block curves correspond to formulas (50), i.e. to parabolic distribution of intensity of a cylindrical beam at the medium boundary; dotted curves refer to plane geometry of the beam, considered below.

The procedure of prolongation of the operator on non-local variables uses a canonical form of RG generators (Lie-Bäcklund operators) and is suitable also in that case when this generator is given by a higher-order Lie-Bäcklund symmetry. Such case is realized for a plane laser beam with “soliton” profile of the intensity distribution at the boundary,
Figure 2: Dependencies of the intensity of a laser beam (at the left) and the second derivative of its eikonal (on the right) on the beam axis \( x = 0 \) at various distance \( z/z_{\text{sing}} \) from the boundary of a nonlinear medium \( z = 0 \), plotted with the use of formulas for the cylindrical \( (50) \) (block curves) and plane \( (51) \) (dotted curves) geometry.

\( I(x) = \cosh^{-2}(x) \), when the appropriate RGS generator has rather cumbersome form

\[
R^{\text{sol}} = \left\{ \frac{I}{(Iv^2_x + \alpha I^2_x)^2} \left[ \left( \frac{1}{2} (Iv^2_x - \alpha I^2_x) (v^2 + 4\alpha(1-I)) + 4\alpha vIv_x v_x \right) v_{xx} \right. \right.
\]

\[
+ \left. (2\alpha v (Iv^2_x - Iv^2_x + \alpha v_x I_x (v^2 + 4\alpha(1-I))) \left( I_{xx} - \frac{I^2_x}{2I} \right) \right) - v(1 - tv_x) - \alpha t I_x \right\} \partial v
\]

\[
+ \left\{ \frac{I}{(Iv^2_x + \alpha I^2_x)^2} \left[ \left( \frac{1}{2} (Iv^2_x - \alpha I^2_x) (v^2 + 4\alpha(1-I)) + 4\alpha vIv_x I_x \right) \left( I_{xx} - \frac{I^2_x}{2I} \right) \right. 
\]

\[
- \left. 2v (Iv^2_x - Iv^2_x + v_x I_x (v^2 + 4\alpha(1-I))) I v_{xx} + \frac{1}{4I} (Iv^2_x + \alpha I^2_x) \left[ 4\alpha I^2_x 
\right.
\]

\[
+ (I_x v - 2Iv_x x^2) \right] - I(2 - tv_x) + tv I_x \right\} \partial I. \tag{51}
\]

Prolongation of \( (51) \) on non-local variables \( (46) \) gives the more simple operator in space of functionals

\[
R_5 \equiv \mathcal{X}^0 I_0 \partial I_0 + \mathcal{X}^{W_0} \partial W_0 = \left( 4 - 5I^0 - zI_z^0 + 2(I^0 - 1) \frac{I^0 I_z^0}{(I^0 z^0)^2} \right) \partial I_0
\]

\[
+ \left( \frac{I^0}{I_0} + z \frac{I_{zz}^0}{I_0^2} \right)^2 - (I^0 - 1) \left[ \frac{I^0}{(I^0 z^0)^2} + 2 \frac{I^0}{(I^0 z^0)^3} - 2 \right] \partial W_0. \tag{52}
\]

While obtaining this formula we used the relation between derivatives of functions \( I \) and \( v_x \) with respect to spatial variables on the beam axis (at \( x = 0 \)), which follows from the initial equations,

\[
v_{xxx}(z, 0) = \frac{1}{\alpha I^0} \left[ \frac{I^0}{(I^0 z^0)^2} + 10 \left( \frac{I^0}{I^0 z^0} \right)^3 - 8 \frac{I^0}{(I^0 z^0)^2} \right],
\]

\[
v_x(z, 0) = \frac{I^0}{I^0 z^0}, \quad I_{xx}(z, 0) = \frac{1}{\alpha} \left[ 2 \left( \frac{I^0}{I^0 z^0} \right)^2 - \frac{I^0}{I^0 z^0} \right]. \tag{53}
\]
The beam intensity and its second eikonal derivative on an axis are defined from RG invariance condition (41), which is equivalent to vanishing of coordinates $\kappa_I^0$ and $\kappa_W^0$ of the operator (52). This condition gives two ODE of the second and the third order respectively. Solving at first ODE of the second order with the initial conditions (47) and the additional condition on the first derivative, $(I^0_z/\sqrt{I^0-1})_{z \to 0} = 2\sqrt{\alpha}$ which follows from the relation (53) at $z = 0$, we obtain in the implicit form the law of variation of $I^0$ and $W^0$ (compare with formulas (50) for a parabolic beam),

$$z = \frac{\sqrt{I^0-1}}{\sqrt{\alpha I^0}}, \quad W^0 = -\frac{2\alpha z I^0}{1 - 2\alpha z^2 I^0}. \quad (54)$$

These formulas are valid from the boundary of a nonlinear medium $z = 0$ up to a point where the solution singularity occurs. The coordinate of the solution singularity is found in view of the fact that the derivative of $W^0$ turns to infinity at this point, that gives $z_{sing} = 1/2\sqrt{\alpha}$, and the value of the intensity $I^0$ in this point is equal to two. The solution of the remaining ODE of the third order gives the same result. Dependencies of $I^0$ and $W^0$ upon the dimensionless coordinate $z/z_{sing}$ at $\alpha = 0.1$ are plotted on Figures 2 by dotted curves. Without prolongation of RGS on non-local variables the result (54) with the use of RG algorithm was obtained earlier in [4], though in a more complicated way.

Summarizing this paragraph, we note, that universality of a procedures of prolongation of RG generators presented either as point group operators, or Lie-Bäcklund group operators has allowed to describe from uniform positions an occurrence of a singularity of the BVP solution to (44)-(45), using for this purpose the reduced description in terms of solution functionals.

### 4.3 RGS for solution functionals of plasma kinetic equations

In the paragraph we continue with demonstrations of potentialities of RGS algorithm for nonlocal models. In contrast to the previous paragraphs, where we described the prolongation of the RGS generators on solution functionals for differential models, here we consider the case when integral relations form a skeleton of a problem. A vivid example is the model that is used in the plasma kinetic theory and define the evolution of a collisionless inhomogeneous plasma.

The macroscopic state of plasma particles is governed by distribution functions $f^\alpha$ (for each species of plasma particles with mass $m_\alpha$ and charge $e_\alpha$), that depend on time $t$, a coordinate $x$ of a particle, and its velocity $v$ (for simplicity we consider the one-dimensional plane geometry). Evolution of distribution functions is described by Vlasov kinetic equations [40],

$$f_t^\alpha + vf_x^\alpha + (e_\alpha/m_\alpha) E(t,x)f_v^\alpha = 0 \quad (55)$$

supplemented by Poisson and Maxwell equations for the electric field $E$,

$$E_x - 4\pi \sum_\alpha \int dve_\alpha f^\alpha = 0, \quad E_t + 4\pi \sum_\alpha \int dvve_\alpha f^\alpha = 0. \quad (56)$$
The joint system of equations (55) and the first equation in (56) is often referred to as Vlasov-Poisson (VP) equations. We are interested in a solution to the Cauchy problem to equations (55) with the initial conditions that correspond to the electron and ion distribution functions specified at $t = 0$

$$f^\alpha|_{t=0} = f^\alpha_0(x, v).$$  

(57)

The admitted symmetry group for (55), (56), calculated by the method prescribed in section 3.2, is given by time and space translations, Galilean boosts and two generators of dilations [30]. VP equations seem to be the simplest one-dimensional mathematical model, which is commonly used to describe the evolution of inhomogeneous plasma, e.g., the expansion of a plasma slab. Even so analytical methods fail to create the spatially symmetric solution of (55)–(56) for the distribution functions with initial zero mean velocity. Thus, with the goal to find physically reasonable solution we are forced to simplify the basic system of VP equations.

One possible way to simplify the system (55), (56) is to study dynamics of plasma expansion in quasi-neutral approximation [41, 42], suitable for a description of plasma flows with characteristic scale of density variation large as compared with Debye length for plasma particles. It means that one can neglect the field terms in Poisson and Maxwell equations (56) and consider the total charge and current densities equal to zero. Hence, particle distribution functions $f^\alpha(t, x, v)$ for the electrons ($\alpha = e$) and ions ($\alpha = 1, 2, ...$) obey kinetic equations (55) and are assumed to satisfy the non-local quasi-neutrality conditions,

$$\int dv \sum_\alpha e_\alpha f^\alpha = 0, \quad \int dv v \sum_\alpha e_\alpha f^\alpha = 0,$$

(58)

while the electric field $E$ is expressed in terms of moments of distribution functions:

$$E(t, x) = \left(\int dv v^2 \partial_x \sum_\alpha e_\alpha f^\alpha\right) \left(\int dv \sum_\alpha \frac{e_\alpha^2}{m_\alpha} f^\alpha\right)^{-1}.$$

(59)

Analytical study of such yet simplified model represents the essential difficulties, but due to application of RG algorithm it is possible not only to construct solution at various initial particle distribution functions [42] but also to find particles density and energy spectra without calculations of distribution functions for particles in an explicit form.

To construct RGS we consider a set of local (55) and non-local (58) equations as $\mathcal{R}\mathcal{M}$, in which the electric field $E(t, x)$ appears as an unknown function of the coordinate $x$ and time $t$. The Lie group of point transformations admitted by this manifold is calculated in a way similar to that used in section 3.2. Here, besides time and space translations, the Galilean boosts and three operators of dilations, there arises a new projective group generator [13]. Precisely this generator enables to construct a class of exact solutions to the initial problem that are of interest, as a linear combination of the generator of time translations and the projective group generator leaves the approximate PT solution of the initial value problem $f^\alpha = f^\alpha_0(x, v) + O(t)$ invariant at $t \to 0$, i.e. it is the RGS operator

$$R_6 = (1 + \Omega^2 t^2)\partial_t + \Omega^2 tx\partial_x + \Omega^2 (x - vt)\partial_v.$$

(60)
The generator \( 60 \) is the only which selects the spatially symmetric initial distribution functions with zero mean velocity. The value \( \Omega \) can be treated as the ratio of the ion acoustic velocity to the gradient length \( L_0 \).

Group invariants of the RG operator \( 60 \) are particle distribution functions \( f^\alpha \) and combinations \( J_3 = x/\sqrt{1 + \Omega^2 t^2} \) and \( J_4 = v^2 + \Omega^2 (x - vt)^2 \). Hence, BVP solutions, i.e. distribution functions at any time \( t \neq 0 \), are expressed with the help of these invariants in terms of initial distributions \( 57 \),

\[
f^\alpha = f_0^\alpha (I^{(\alpha)}), \quad I^{(\alpha)} = \frac{1}{2} J_4 + \frac{e_\alpha}{m_\alpha} \Phi_0 (J_3). \tag{61}\]

Here the dependence of \( \Phi_0 \) upon the invariant \( J_3 = x/\sqrt{1 + \Omega^2 t^2} \) is defined by quasi-neutral conditions \( 58 \), and the electric field \( E = -\Phi_x \) is found with the help of a potential

\[
\Phi(t, x) = \Phi_0 (J_3) (1 + \Omega^2 t^2)^{-1}. \tag{62}\]

A variety of examples, illustrating these formulas for the plasma slab consisting of different groups of hot and cold electrons and ions of various species, is found in \[43\].

Distribution functions \( 61 \) give exhaustive information on the kinetics of plasma bunch expansion. However, for practical applications rough integral characteristics, such as partial ion density, \( n^q(t,x), (q = 1, 2, \ldots) \) and ion energy spectra, \( dN_q/d\varepsilon \), might be more useful,

\[
n^q = \int_{-\infty}^{\infty} dv f^q(t,x,v), \quad \frac{dN_q}{d\varepsilon} = \frac{1}{m_q v} \int_{-\infty}^{\infty} dx \left( f^q(t,x,v) + f^q(t,x,-v) \right). \tag{63}\]

In view of the complex dependence upon the invariant \( I^{(\alpha)} \) it is not always possible to carry out direct integration of a distribution function over velocity in the analytical form, therefore here the procedure of prolongation of the operator on solution functionals described in section \textcolor{red}{3.1} comes to the aid. The density \( n^q(t,x) \) is a linear functional of \( f^q \), hence we prolong the generator \( R_6 \) on the solution functional \( 63 \) to get the following RG generator in the space of variables \( \{t,x,n^q\} \),

\[
R_7 = (1 + \Omega^2 t^2) \partial_t + \Omega^2 tx \partial_x - \Omega^2 t n^q \partial_{n^q}. \tag{64}\]

Two invariants of this generator, namely \( J_3 \) and \( J_4^q = n^q \sqrt{1 + \Omega^2 t^2} \) are related for arbitrary \( t \neq 0 \) via their initial values: \( J_3|_{t=0} = x' \), \( J_4^q|_{t=0} = N_q (x') \). Therefore, we get formulas that characterize spatial-temporal distribution of the density of ions of a given species in terms of the initial density distribution

\[
n^q = \frac{n_q}{\sqrt{1 + \Omega^2 t^2}} N_q (\chi), \quad N_q = \int_{-\infty}^{\infty} dv f_0^q, \quad \chi = \frac{x}{\sqrt{1 + \Omega^2 t^2}}. \tag{65}\]
The general form of \( dN_q/d\varepsilon \) is rather complicated but its asymptotic behavior at \( \Omega t \to \infty \) is described by the same function \( N_q \),

\[
\frac{dN_q}{d\varepsilon} \approx \sqrt{\frac{2}{m_q \varepsilon}} \Omega N_q \left( \varepsilon = \frac{m_q U^2}{2} \right), \quad 2\varepsilon/T_q \gg (\Omega t)^{-2}, \quad U = \Omega \chi. \tag{66}
\]

Relations (65) express a well-known FS property \[33\] of a solution: the product \( n^q \sqrt{1 + \Omega^2 t^2} \), being one of invariants of the RG generator, is expressed in terms of some universal function \( N_q \) of another invariant of the generator (64), and a form of this function is set by initial conditions, i.e. it is defined in terms of initial distribution functions of particles \( f_0^q \).

Such representation has a common enough nature and is a typical property of BVP solutions obtained with the help of RGS which is pertinent to name \( \Phi \)-theorem (on analogy with known \( \Pi \)-theorem). We briefly discuss this property in concluding remarks.

In order to illustrate formulas (65) we apply them to plasma slab that contains ions of several, say two, types (the index \( q = 1, 2 \) corresponds to heavy and light ions, respectively) with initial Maxwellian velocity distribution functions, and the electrons obeying a two-temperature Maxwellian distribution function with densities and temperatures of the cold and hot components \( n_c^0 \) and \( n_h^0 \) \((n_c^0 + n_h^0 = \sum_q Z_q n_q^0)\) and \( T_c \) and \( T_h \), respectively.

In this case the density distribution and, hence, the ion energy spectrum is expressed as

\[
N_q = \exp \left[ \mathcal{E} \left( \frac{Z_q T_c}{T_q} \right) - \frac{U^2}{2v_{T_q}^2} \left( 1 + \frac{Z_q m_e}{m_q} \right) \right], \quad q = 1, 2, \quad v_{T_q}^2 = \frac{T_q}{m_q}, \tag{67}
\]

where the function \( \mathcal{E} \) is defined in the implicit form

\[
n_c^0 = \sum_{q=1,2} Z_q n_q^0 \exp \left[ (1 + (Z_q T_c/T_q)) \mathcal{E} - (U^2/2v_{T_q}^2) \right. \times \left. (1 + (Z_q m_e/m_q)) \right] - n_h^0 \exp \left[ (1 - (T_c/T_h)) \mathcal{E} \right]. \tag{68}
\]

Figure 3 demonstrates the plots of \( N_q \) for the following plasma parameters: \( T_{1,2}/T_c = 0.1; T_h/T_c = 1000; n_{h0}/Z_{1}n_{10} = 5 \times 10^{-4} \). Block curves show dependence of a dimensionless “universal” density of plasma ions \( \tilde{N}_q = (n_q^0/n_{c0})N_q \), referred to the maximal density of cold electrons, upon the dimensionless “coordinate” \( \chi^2 = (J_3/L_0)^2 \). Dotted curves present the distribution of the dimensionless density of cold and hot electrons (short and long strokes respectively). These curves demonstrate the high end of the energy spectrum for light ions \((q = 2)\) with a sharp decrease, as well as the spatial separation of ion species which is of current interest in experiments on interaction of short laser pulses with thin foil targets (see, e.g., [44]). Similar results are obtained for more complex distribution functions [43] and beyond the scope of the model used for the one-dimensional expansion, for example for spherically-symmetric expansion of a plasma bunch [45].

Summarizing the paragraph 4.3 let’s note, that here a modified RG algorithm for calculation of RGS for non-local systems of the equations, and a procedure of prolongation of RG operators on solution functionals are used simultaneously.
Figure 3: The curves describing the dependence of invariants of the RGS operator (64): the “universal” density $N_q$ of plasma ions – carbon ions (curves (C)) and protons (curves (H)) – is represented as a function of a dimensionless “coordinate” $\chi^2 = (x/L_0)^2/(1+\Omega t^2)$. For illustration here by dotted curves with short and long strokes the dependencies of a dimensionless density for hot and cold electrons upon $\chi^2$ are also shown.

5 Conclusion

The realization of the program that expands opportunities of application of the RG algorithm to problems of mathematical physics, is the goal of the paper, which is obviously formulated in section 1 for non-local problems. The appropriate class covers now (besides problems on a basis of DE, discussed in section 2 in an introductory example) the models containing non-local terms, including integral and integro-differential equations.

This formulation preserves the former general scheme of construction of RG algorithm as four consecutive steps [4]. However the form of realization of these steps significantly varies, that is most brightly shown on first two stages of the algorithm (see figure 1), related to construction of non-local RG manifold and a calculation of admitted symmetry group. Here in view of absence of a regular computational algorithm (similar to Lie algorithm for DE) while performing of the second stage of algorithm various realizations are possible. As an illustration we choose and state in more detail the variant that is based on use of a canonical operator.

In the following section 4 efficiency of use of procedure of prolongation the RG operator on non-local variables is shown with the purpose of the reduced description of the solution in terms of the integrated characteristic, the solution functional. It is essential, that the knowledge of a solution in an explicit form therewith is not required.

The examples given here serve as an illustration of the program formulated in section 3. They show application of new algorithm to integro-differential systems (expansion of a plasma bunch) and a procedure of prolongation on solution functionals (Cauchy problem for Hopf equation, evolution of a laser beam in nonlinear optics). The example presented
in [4,5] is the most valuable, as here the calculation of RG symmetries for a solution of the non-local system of equations is supplemented by a procedure of prolongation of the RG operator obtained on the solution functional, the density of plasma particles, with the purpose of revealing its variation law.

At the formulation and discussion of the analytical form of results an accent is made on a role of invariants of appropriate RG operators. The manifested common regularities are considered in the Appendix.

The results of section 4 testify to universality of a method of RenormGroup Symmetries. Therefore they allow to look forward to a further expansion of a class of the problems which can be investigated with the help of RGS method, and to new objects, for which the application of RG algorithm yet is not a standard procedure.

Here we mean an infinite systems of looped integro-differential equations, similar to systems for correlation functions in statistical physics and to systems of the equations for generalized Green functions – propagators and vertex functions – in the quantum field theory.

The work was partially supported by RFBR grant No.05-01-00631, grant of Scientific School No.2339.2003.2 and ISTC project 2289.

References

[1] Kovalev V F, Pustovalov V V 1990 Theor. Math. Phys. 81 1060

[2] Shirkov D V in “Renormalization group ‘91”, (Proc. of Second Intern. Conf., Sept. 1991, Dubna, USSR) ed D V Shirkov and V B Priezzhev (WS, Singapore, 1992) p 1-10; Kovalev V F, Krivenko S V and Pustovalov V V ibid., p 300-314

[3] Kovalev V F, Pustovalov V V and Shirkov D V 1998 J. Math. Phys. 39 1170

Kovalev V F, Pustovalov V V and Shirkov D V 1997 Preprint hep-th/9706056

[4] Kovalev V F, Shirkov D V 2001 Phys. Reports 352(4-6) 219

[5] Stueckelberg E E C and Petermann A 1951 Helv. Phys. Acta 24 317;

Stueckelberg E E C and Petermann A 1953 Helv. Phys. Acta 26 499 (in French)

[6] Gell-Mann M and Low F 1954 Phys.Rev. 95 1300

[7] Bogoliubov N N and Shirkov D V 1955 Doklady AN SSSR 103 203 (in Russian)

[8] Bogoliubov N N and Shirkov D V 1955 Doklady AN SSSR 103 391 (in Russian)

[9] Bogoliubov N N and Shirkov D V 1956 Nuovo Cim. 3 845

[10] Bogoliubov N N and Shirkov D V 1956 Sov.Phys.JETP 3 57

[11] Bogoliubov N and Shirkov D 1980 Introduction to the Theory of Quantized Fields (Wiley-Interscience, New York, 1949 and 1980)
[12] Shirkov D V 1994 *Russian Math. Surveys* **49**:5 155
Shirkov D V 1996 *Preprint* hep-th/9602024

[13] Chen L Y, Goldenfeld N and Oono Y 1996 *Phys. Rev. E* **54**(1) 376-94

[14] Kunihiro T 1995 *Progr. Theor. Phys.* **94**(4) 503-14

[15] Ei S I, Fujii K, Kunihiro T 2000 *Ann. Phys.* **280**(4) 236-79

[16] Hatta Y, Kunihiro T 2002 *Ann. Phys.* **298** 24-57

[17] Pashko O, Oono Y 2000 *Intern. J. Mod. Phys. B* **14**(6) 555-561

[18] Frasca M 1998 *Phys. Rev. A* **58** 771–774

[19] de Vega H J, Salgado J F J 1997 *Phys.Rev.D* **56** 6524-6532

[20] Kunihiro T 1998 *Prog. Theor. Phys. Supplement*, **131** 459-471

[21] Bricmont J and Kupiainen A 1992 *Comm. Math. Physics* **150** 193-203;
   Bricmont J, Kupiainen A and Lin G 1994 *Comm. Pure Appl. Math.* **47** 893-922;
   Bricmont J, Kupiainen A and Xin J 1996 *J. Diff. Eqs.* **130** 9–35

[22] Kovalev V F 2002 *Acta Physica Slovaca* **52**(4) 353-62

[23] Shirkov D V 1988 *Int. J. Mod. Phys.* **A3** 1321-41

[24] *CRC Handbook of Lie Group Analysis of Differential Equations* ed Ibragimov N H
   1994 vol 1; 1995 vol 2; 1996 vol 3 (CRC Press, Boca Raton, Florida, USA)

[25] Ibragimov N H 1996 *Elementary Lie Group analysis and Ordinary Differential Equations*
   (John Wiley & Dons, Chichester-New York)

[26] Ovsyannikov L V 1982 *Group analysis of differential equations* (Acad.Press, New
   York)

[27] Ibragimov N H 1985 *Transformation groups applied to mathematical physics* (Riedel-
   Publ.,Dordrecht-Lancaster)

[28] Grigoryev Yu N, Meleshko S V 1987 *Sov.Phys.Dokl.* **32** 874-76;
   Grigoryev Yu N, Meleshko S V 1986 *Preprint Inst. Theor. and Appl. Mechanics SD
   AN SSSR* (Novosibirsk) No 18-86;
   Meleshko S V 1991 *Classification of the solutions with degenerate hodograph of the
   gas dynamics and plasticity equations. Doctoral thesis Sverdlovsk;
   Meleshko S V 1998 *Symmetry analysis and mathematical modelling* (The Interna-
   tional Institute for Symmetry Analysis and Mathematical Modelling) 45-59;
   Grigoryev Yu N, Meleshko S V 1990 *Arch. Mech.* **42** 693-701
[29] Kovalev V F, Krivenko S V, Pustovalov V V 1992 *JETP Letters* 55 253;

[30] Kovalev V F, Krivenko S V, Pustovalov V V 1993 *Differential Equations* 29 1568-78
Kovalev V F, Krivenko S V, Pustovalov V V 1993 *Differential Equations* 29 1712-21

[31] Buslaev V S 1980 *Calculus of variations* (Leningrad, Izd. LGU)

[32] Volterra V 1982 *Theory of functionals, integral and integrodifferential equations* (Moscow, Nauka) (in Russian)

[33] Shirkov D V 1984 *Theor.Math.Phys.* 60 778-82

[34] Kovalev V F, Shirkov D V 1999 *Theor.Math.Phys* 121 1315-32
Kovalev V F, Shirkov D V 2000 *Preprint* math-ph/0001056

[35] Akhmanov S A, Sukhorukov A P, Khokhlov R V 1968 *Sov.Phys.Usp.* 10 609

[36] Vlasov S N, Talanov V I 1997 *Self-focusing of waves* (Inst. Appl. Phys. RAS, N.Novgorod, 220 p) (in Russian)

[37] Bergé L 1998 *Phys.Reports* 303 259

[38] Kovalev V F and Shirkov D V 1997 *J. of Nonlin. Opt. Phys. & Materials* 6 443-54

[39] Akhmanov S A, Sukhorukov A P, Khokhlov R V 1968 *Sov.Phys.JETP* 23 1025

[40] Vlasov A A 1938 *Journ.Exp.Theor.Phys.* 8(3) 291-317 (in Russian);
Vlasov A A 1967 *Usp.Phys.Nauk* 93(3) 444-70 (in Russian)

[41] Dorozhkina D S and Semenov V E 1998 *Phys. Rev. Lett.* 81 2691

[42] Kovalev V F, Bychenkov V Yu, Tikhonchuk V T 2001 *JETP Lett.*, 74 10

[43] Kovalev V F, Bychenkov V Yu, Tikhonchuk V T 2002 *JETP* 95(2) 226-41

[44] Maksimchuk A, Flippo K, Krause H *et al.* 2004 *Plas.Phys.Rep.* 30(6) 473-95

[45] Kovalev V F and Bychenkov V Yu 2003 *Phys.Rev.Lett.* 90(18) 185004-(1-4)

[46] Rayleigh 1915 *Nature* 95 66;
Riabouchinsky D P 1915 *Nature* 95 105;
Bridgman P W 1932 *Dimensional Analysis* (New Haven: Yale University Press);
Sedov L I 1981 *Methods of similarity and dimension in mechanics* (Moscow, Nauka) (in Russian)
6 Appendix. Invariant representation of solution and \(\Phi\)-theorem

In the main text of this paper the role of invariants of RG transformation in construction of BVP solutions was repeatedly marked. Here we consider the relation of representations of BVP solution with the concept of functional self-similarity and with the well-known principles of the group analysis.

Let us remind that BVP solutions which are obtained with the use of RG algorithm are invariant solutions of RG operators. In group analysis of differential equations the explicit expression of solutions through invariants uses the well-known theorem of invariant representation of a regular (non-singular) manifold (see. [26, §18, ch.5], and [24, Vol.3, p.6]):

Let a manifold \(M \subset \mathbb{R}^N\) admit a group \(G\). Suppose \(M\) be a nonsingular manifold of a group \(G\), i.e., an infinitesimal operator of group \(G\) does not vanish identically on \(M\). Then, \(M\) can be represented by a system of equations, left-hand sides of which are invariants of group \(G\), i.e., have the form:

\[
\Phi_k(J_1(z), J_2(z), \ldots, J_{N-1}(z)) = 0, \quad k = 1, \ldots, s.
\]  

(69)

Here \(J_1(z), J_2(z), \ldots, J_{N-1}(z), \quad z \in \mathbb{R}^N\), form a basis of invariant of \(G\). Hence, equations (69) with arbitrary functions \(\Phi_k\) of \(N-1\) variables furnish the general form of non-singular invariant manifold of the group \(G\). In particular, it gives a transparent comment of the well-known \(\Pi\)-theorem [46].

Turn now to BVP solutions which are obtained with the use of RG algorithm. For RG invariant solutions there exists a more general statement as compared with \(\Pi\)-theorem, \(\Phi\)-theorem:

An invariant solution of a boundary-value problem can be represented by a system of equations of the form (69) written down in terms of functional invariants \(\phi_i\) of the problem.

These \(\phi_i\) are understood as invariants of appropriate functional transformations involving not only dependent and independent variables of the equations, but also parameters of boundary conditions, that is invariants of renormgroup transformations.

In essence, \(\Phi\)-theorem is an analogue of the theorem of invariant representation with reference to solutions of BVP having the property of the functional self-similarity. In this case, one should consider sub-manifold \(\mathcal{R}\mathcal{M}\) invariant with respect to renormgroup \(\mathcal{RG}\) as a nonsingular manifold.

In a special case, when in (69) \(s = 1\), and in the functional invariant \(\phi(y, \{a\})\), containing the required function \(y\), variables are separated, the solution can be written down in an explicit form close to the representation, which emerges from the \(\Pi\)-theorem:

\[
y = \phi_{(1)}^{-1}(\Phi(\ldots, \{a\})\); \quad \phi = \Phi(\ldots, \phi_i, \ldots), \quad i = 1, \ldots, N-1.
\]  

(70)

Here, the function \(\phi_{(1)}^{-1}\) is a reverse one to \(\phi\) with respect to its first argument. Due to this solution \(y\) appears dependent not only on the remaining functional invariants, \(\phi_i\), but
also on variables and parameters, \{a\}, entering into the invariant \(\phi\). Thus, as well as for power self-similarity, BVP solutions are not, generally, invariants of RG transformations, but are expressed through certain combinations of invariants of RGS operators.

Expressions \((61)\) for distribution functions of plasma particles in an expanding bunch serve as an example of BVP solution, being such invariants.

As the second example, we take a QFT model with two coupling constants \(g\) and \(h\). Here, invariant quantities, e.g., observed effective scattering cross-sections \(\sigma_\nu(s)\), are expressible in terms of RG invariants — two invariant coupling functions \(\bar{g}(s/\mu^2; g, h)\), \(\bar{h}(s/\mu^2; g, h)\) and of the ratio \(m^2/s\) — by relations

\[
\sigma_\nu(s) = \Sigma_\nu\left(m^2/s, \bar{g}, \bar{h}\right).
\]  

(71)

In turn, functions \(\bar{g}\) and \(\bar{h}\) should be found from system of two functional relations (see, e.g., eqs.\((48.37)\) in Ref. \([11]\))

\[
\begin{align*}
G(y/x, \bar{g}(x,y; g,h), \bar{h}(x,y; g,h)) &= G(y;g,h); \\
H(y/x, \bar{g}(x,y; g,h), \bar{h}(x,y; g,h)) &= H(y;g,h),
\end{align*}
\]  

(72)

containing two arbitrary functions, \(G\) and \(H\), of two arguments. Due to this, to find each of \(\sigma_\nu(s)\), one needs to have explicit expressions for three defining functions \(\Sigma_\nu, G\) and \(H\).

Note also that the procedure of numerical defining of the parameters \(g\) and \(h\) from boundary data (in fact, from observed quantities) involves at least two implicit relations \((71)\).

At the same time, functionals of functions which determine, according to formulae \((65)\), the density distributions of particles of expanding plasma, are not invariants of the RG operator. Another example when a BVP solution is constructed with the help of invariants of RG transformations, but is not such invariant itself, is submitted by formulae \((50)\) and \((54)\) for functionals \((46)\) in a problem of a beam refraction in a nonlinear medium. Generally, when it is impossible to express the BVP solution in an explicit form, one should use general formulae \((69)\) instead of the representation \((70)\).