REGULARISATION OF GAUSSIAN HYPERCONTRACTIVITY AND LOGARITHMIC SOBOLEV INEQUALITIES VIA FOKKER–PLANCK FLOW

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Abstract. We prove that the regularising property of the Fokker–Planck equation brings about improved versions of Nelson’s hypercontractivity and logarithmic Sobolev inequalities. In particular, our result on the logarithmic Sobolev inequality complements a recently obtained result by Eldan, Lehec and Shenfeld concerning a deficit estimate for inputs with small covariance. Our approach is based on a flow monotonicity scheme and our regularised inequalities are a consequence of an investigation into the interaction between two different flows. We apply our results to derive deficit estimates for the hypercontractivity inequality associated with the Hamilton–Jacobi equation, for Talagrand’s transportation cost inequality, the Poincaré inequality, and for Beckner’s inequality. We also explore the possibility of broadening the scope of our approach to more general measure spaces, as well as an alternative route based on mass transportation.

1. Introduction and main results

In their work on stability of the celebrated gaussian logarithmic Sobolev inequality (LSI for short), Ledoux, Nourdin and Peccati [17] identified that it is of interest to understand the relationship between the deficit (the difference between the two sides of the inequality) and the size of the covariance matrix of the input. Addressing this, Eldan, Lehec and Shenfeld [38] established that for probability distributions whose covariance is bounded above by the identity (in the sense of positive definite transformations), then the deficit is minimised on an appropriate gaussian distribution. Whilst it was also shown in [38] that such a result can dramatically fail for certain inputs (in fact, suitably chosen gaussian mixtures) with large covariance, one is certainly left wondering to what extent one may recover such failure by restricting to natural classes of inputs. Moreover it is natural to seek similar stability results for the hypercontractivity inequality associated with Ornstein–Uhlenbeck semigroup (the well-known equivalence of LSI and hypercontractivity may not persist when considering restricted classes of inputs and \textit{prima facie} hypercontractivity appears to be the stronger statement). In the present work, we address this by appealing to the regularising nature of the Fokker–Planck diffusion equation. Our viewpoint was
heavily influenced by work of Bennett, Carbery, Christ and Tao [17] (see also [18]) on regularisation of the Brascamp–Lieb inequality, as well as the resolution of the gaussian version of Talagrand’s conjecture due to Eldan and Lee [37] (see also [58]). An appealing feature of our approach is its robustness and we shall improve various functional inequalities, not only LSI, but also the hypercontractivity inequalities associated with the Ornstein–Uhlenbeck and Hamilton–Jacobi semigroups, as well as Talagrand’s quadratic transportation cost inequality, the Poincaré inequality, and Beckner’s inequality.

1.1. Functional inequalities and the Fokker–Planck equation. Throughout the paper we shall frequently use the notation

\[ \gamma_\beta(x) = \frac{1}{(2\pi \beta)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2\beta}}, \quad x \in \mathbb{R}^n, \]

for the centred and \( L^1 \)-normalised gaussian with variance \( \beta > 0 \). We abbreviate to \( \gamma \) in the case \( \beta = 1 \). The Ornstein–Uhlenbeck semigroup \( P_s, s > 0 \), is given by

\[ P_s f(x) = \int_{\mathbb{R}^n} f(e^{-s}x + \sqrt{1 - e^{-2s}}y) \, d\gamma(y) \]

for non-negative \( f \in L^1(\gamma) \). This \( u_s := P_s f \) solves the gaussian heat equation

\[ \partial_s u_s = \mathcal{L}_1 u_s, \quad u_0 = f, \quad (s, x) \in (0, \infty) \times \mathbb{R}^n, \]

where in general \( \mathcal{L}_\beta \) is given by

\[ \mathcal{L}_\beta \phi(x) = \beta \Delta \phi(x) - x \cdot \nabla \phi(x) \]

for \( \beta > 0 \). For reasons that will become apparent, as above we shall often write the solution of time-dependent PDE in the form \( u_s(x) \) rather than \( u(s, x) \) (and so we shall never use the notation \( u_s \) to mean \( \partial_s u \)).

Note that each \( P_s \) can be extended to a bounded operator on \( L^p(\gamma) \) for all \( p \in [1, \infty] \). Nelson’s fundamental gaussian hypercontractivity inequality is a quantification of the smoothing property of the Ornstein–Uhlenbeck semigroup \( P_s \), and states that

\[ \| P_s [f^\frac{1}{2}] \|_{L^q(\gamma)} \leq \left( \int_{\mathbb{R}^n} f \, d\gamma \right)^\frac{1}{q} \]

for all non-negative \( f \in L^1(\gamma) \), where \( 1 < p < q < \infty \) and \( s > 0 \) satisfy \( q - 1 \frac{1}{p - 1} = e^{2s} \).

We refer the reader to the survey paper [35] and the book [40] for detailed historical background and illuminating discussion of its vast connections to several branches of the mathematical sciences, including constructive quantum field theory. As observed by Gross [49], the hypercontractivity inequality implies the LSI inequality

\[ \text{Ent}_\gamma(f) \leq \frac{1}{2} I_\gamma(f), \]

\[ \text{Here we mean the Brascamp–Lieb inequality which generalises the Hölder, Loomis–Whitney and Young convolution inequalities arising in [27], rather than the concentration inequality for log-concave probability distributions from [25]. On the other hand, certain ideas from the latter paper concerning the preservation of log-concavity under certain diffusion processes will in fact play an important role in the present paper as well – see the forthcoming Lemma [23].} \]
by considering the special case \( p = 2 \) and \( s \to 0 \) in (1.2)\(^2\). Here, the entropy \( \text{Ent}_\gamma(f) \) and Fisher information \( I_\gamma(f) \) are defined by

\[
\begin{align*}
\text{Ent}_\gamma(f) &:= \int_{\mathbb{R}^n} f \log f \, d\gamma - \left( \int_{\mathbb{R}^n} f \, d\gamma \right) \log \left( \int_{\mathbb{R}^n} f \, d\gamma \right), \\
I_\gamma(f) &:= \int_{\mathbb{R}^n} \frac{\vert \nabla f \vert^2}{f} \, d\gamma.
\end{align*}
\]

The LSI inequality too enjoys a plethora of connections to a wide range of fields, including convex geometry, differential geometry, probability theory, and information theory; see, for example, [23], [6], and the survey paper [56]. Here we devote some discussion to a remarkable recent result of Gozlan [46] in which it is observed that Mahler’s conjecture, a long-standing open problem in convex geometry, is equivalent to a certain deficit estimate for LSI. More precisely, in [46] it is proved that the inverse functional Santaló inequality with parameter \( c > 0 \) is equivalent to the deficit estimate

\[
\text{Ent}_\gamma(\nu_1) + \text{Ent}_\gamma(\nu_2) - \frac{1}{2} I_\gamma(\nu_1) - \frac{1}{2} I_\gamma(\nu_2) \leq - \frac{1}{2} W_2(\nu_1, \nu_2)^2 + n \log \frac{2\pi}{c}
\]

for sufficiently nice log-concave probability measures \( \nu_1, \nu_2 \) and \( c > 0 \), where \( W_2 \) is Wasserstein distance (of order 2) and \( \nu_i \) is the moment probability measure of \( v_i \). It is clear that the inequality (1.5) gets stronger as \( c > 0 \) increases. Moreover, combining the above equivalence of Gozlan with work of Fradelizi–Meyer [44], it follows that the Mahler conjecture for general convex bodies holds in all dimensions if and only if (1.5) holds with \( c = e \) and all \( n \geq 1 \). We also note that there exists some \( c > 0 \), independent of \( n \), for which (1.5) holds true thanks to the work of Bourgain–Milman [26]. Motivated by this, we are interested in the following question: what kind of condition should we impose on \( v_i \) to ensure (1.5) for large \( c \gg e \)? As a first step towards this question, we consider the symmetric case of (1.5), namely the case of \( v_1 = v_2 = v \):

\[
\text{Ent}_\gamma(\frac{v}{\gamma}) - \frac{1}{2} I_\gamma(\frac{v}{\gamma}) \leq \frac{n}{2} \log \frac{2\pi}{c}.
\]

This inequality clearly holds true if \( c \leq 2\pi \) from the classical LSI (1.3) and hence the question concerns the case \( c \geq 2\pi \). We will see in the forthcoming Corollary 1.8 that (1.6) holds true for all inputs which are “semi-log-convex” in the sense that

\[
\nabla^2 \log v \geq - \frac{1}{\beta(c)} \text{id}
\]

for appropriate \( \beta(c) > 0 \).

Deficit estimates for LSI of a different nature to (1.5) have been the subject of a large number of recent papers, including [22, 36, 38, 41, 42, 43, 46, 50, 51, 53, 57, 64]. Among them, we are especially interested in the result by Eldan–Lehec–Shenfeld [38]. We are also interested in the stability of Nelson’s hypercontractivity inequality (1.2) and, unlike other papers on deficit estimates for LSI, we simultaneously consider the stability of both (1.2) and (1.3); as we have already mentioned, there is no obvious reason that the equivalence of these estimates should persist when restricting the class of input functions and typically one would expect (1.3) to be the stronger estimate.

\(^2\)Remarkably, Gross [49] showed one can recover (1.2) for all admissible exponents from LSI and hence the hypercontractivity inequality is indeed equivalent to LSI when considering general inputs.
Before introducing the various statements of our results, it is instructive to first recall the characterisation of extremisers for both inequalities. Carlen [30] first showed that equality holds in (1.3) if and only if
\[ f(x) = e^{a \cdot x + b}, \quad a \in \mathbb{R}^n, \quad b \in \mathbb{R}. \]
Building on this, Ledoux [55] showed that (among smooth inputs) equality in (1.2) holds if and only if
\[ f(x) = e^{a \cdot x + b}, \quad a \in \mathbb{R}^n, \quad b \in \mathbb{R}. \] In particular, \( f = 1 = \gamma \) is essentially the unique extremiser for both of these inequalities and consequently \( f = \frac{\beta}{\gamma} \) does not attain equality in (1.2) or (1.3) unless \( \beta = 1 \). Moreover, it is possible to directly compute the deficits (in a multiplicative sense for (1.2)) for such inputs: for \( \beta \neq 1 \),
\begin{align*}
\left\| P_\gamma \left[ \left( \frac{\gamma}{\beta} \right)^p \right] \right\|_{L^q(\gamma)} &= \beta^{\frac{np}{2}} \beta_s^{-\frac{np}{2}} < 1,
\end{align*}
where \( \beta_s > 0 \) is given by
\begin{align*}
\beta_s := 1 + \left( \beta - 1 \right) \frac{q}{p} e^{-2s}
\end{align*}
and
\begin{align*}
\text{Ent}_\gamma \left( \frac{\gamma a}{\gamma} \right) - \frac{1}{2} I_\gamma \left( \frac{\gamma a}{\gamma} \right) = -\frac{n}{2} \left( \log \beta - 1 + \frac{1}{\beta} \right) < 0.
\end{align*}
It was shown by Eldan–Lehec–Shenfeld [38] that the LSI can be improved if the covariance of the input function is small enough. Let us recall the covariance matrix of a probability measure \( \rho \) is given by
\begin{align*}
\text{cov} (\rho) := \left( \int_{\mathbb{R}^n} x_i x_j \, d\rho - \left( \int_{\mathbb{R}^n} x_i \, d\rho \right) \left( \int_{\mathbb{R}^n} x_j \, d\rho \right) \right)_{1 \leq i,j \leq n}
\end{align*}
and thus in particular we have \( \text{cov} (\gamma \gamma) = \beta \text{id} \). It is also straightforward from (1.9) to see that
\begin{align*}
\text{Ent}_\gamma \left( \frac{\gamma a}{\gamma} \right) - \frac{1}{2} I_\gamma \left( \frac{\gamma a}{\gamma} \right) \leq \text{Ent}_\gamma \left( \frac{\gamma a}{\gamma} \right) - \frac{1}{2} I_\gamma \left( \frac{\gamma a}{\gamma} \right)
\end{align*}
for all \( 0 < a \leq \beta \leq 1 \). It is a consequence of [38, Theorem 3] that, as long as \( \beta \leq 1 \), then (1.10) continues to hold even if \( \gamma a \) is replaced by any probability distribution whose covariance is \( \leq \beta \text{id} \).

**Theorem 1.1** ([38]). Let \( \beta \leq 1 \). Then for any non-negative \( v \) such that \( \int_{\mathbb{R}^n} v \, dx = 1 \) and \( \text{cov} (v) \leq \beta \text{id} \),
\begin{align*}
\text{Ent}_\gamma \left( \frac{v}{\gamma} \right) - \frac{1}{2} I_\gamma \left( \frac{v}{\gamma} \right) \leq \text{Ent}_\gamma \left( \frac{\gamma a}{\gamma} \right) - \frac{1}{2} I_\gamma \left( \frac{\gamma a}{\gamma} \right).
\end{align*}
To be precise, [38, Theorem 3] gives a bound on the deficit in terms of the eigenvalues of the covariance matrix of the input from which we immediately obtain the above statement; see the forthcoming Theorem 1.6 in Section 1.3. Also, Eldan–Lehec–Shenfeld gave a counterexample to the validity of (1.11) in the case of large covariance and \( \beta \gg 1 \) by mixing two gaussians. Its covariance blows up whilst the deficit tends to zero. Nevertheless, the basic inequality (1.10) remains true in the case \( 1 < \beta < a \) and hence it is reasonable to expect some improvement like (1.11) even when \( \beta > 1 \) by imposing additional structure which excludes the counterexample in [38]. One of our aims in this paper is to realise this expectation by
capitalising on the regularising nature of the Fokker–Planck equation as well as the preservation of semi-log-convexity and semi-log-concavity along the flow.

For $\beta > 0$, the Fokker–Planck equation with a diffusion speed $\beta$, or $\beta$-Fokker–Planck equation for short, is

$$\partial_t v_t = \mathcal{L}_\beta^* v_t := \beta \Delta v_t + x \cdot \nabla v_t + n v_t, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n,$$

where $\mathcal{L}_\beta^*$ stands for the dual of $\mathcal{L}_\beta$ with respect to $L^2(dx)$. As is well-known, the Fokker–Planck equation is closely related to the gaussian heat equation. In fact, $v_t$ solves (1.12) if and only if $u_t$ solves $\partial_t u_t = \mathcal{L}_\beta u_t$, under the transformation $u_t = \frac{v_t}{\gamma_\beta}$.

The relevancy of the Fokker–Planck equation to hypercontractivity and LSI can be seen by focusing on its stable solution or equilibrium state given by $v_t = \gamma_\beta$. In fact, the extremiser of hypercontractivity and LSI given by $f = \frac{1}{\gamma}$ can be regarded as the equilibrium state for the 1-Fokker–Planck equation. This heuristic explains why the \textit{heat-flow monotonicity} scheme is well-fitted to the theory of hypercontractivity and LSI, see [6, 56] for a comprehensive heat-flow treatment and [63] for the Fokker–Planck flow. Given such historical background and our observation on the deficits (1.7) and (1.9), we are naturally led to the investigation of the interaction between hypercontractivity, LSI, and $\beta$-Fokker–Planck flow for $\beta \neq 1$.

1.2. \textbf{Main results: Regularised gaussian hypercontractivity and LSI}. Let us now introduce a class of functions regularised by the Fokker–Planck equation. For $\beta > 0$ and initial data any non-negative finite measure $d\mu$, let $v_s$ be the corresponding $2\beta$-Fokker–Planck solution:

$$\partial_t v_s = \mathcal{L}_{2\beta}^* v_s, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \quad v_s(0, x) = d\mu(x).$$

Then for a fixed time $t_s := \frac{1}{\beta} \log 2 > 0$ we introduce our regularised class

$$\text{FP}(\beta) := \{v = v_s(t_s, \cdot) : v_s \text{ is non-negative solution to } (1.13)\}.$$

Our somewhat artificial choices of $2\beta$ and $t_s$ may raise eyebrows. They are natural in the sense that $\gamma_\beta$, the stationary solution of the $\beta$-Fokker–Planck equation, can be represented by $\gamma_\beta = v_s(t_s, \cdot)$ with initial data taken to be the Dirac delta measure supported at the origin. In particular, $\gamma_\beta$, the extremiser of the inequality (1.11), belongs to the class $\text{FP}(\beta)$. As other simple properties of $\text{FP}(\beta)$, we have the nesting property $\text{FP}(\beta_1) \supset \text{FP}(\beta_2)$ for $\beta_1 \leq \beta_2$. Also for $\alpha > 0$, $\gamma_\alpha$ belongs to $\text{FP}(\beta)$ if and only if $\alpha \geq \beta$. In particular, $\gamma$, which is the extremiser of (1.2) and (1.3), is excluded from the class $\text{FP}(\beta)$ in the case $\beta > 1$.

Our first result asserts that if the input is regularised by Fokker–Planck flow, then the hypercontractivity inequality can be improved.

**Theorem 1.2.** \textit{Let $\beta \geq 1$, $s > 0$, and $1 < p < q < \infty$ satisfy $\frac{q}{p} - 1 = e^{2s}$. Then for all $v \in \text{FP}(\beta)$,

$$\|P_s \left[\frac{v}{\gamma}\right]\|_{L^q(\gamma)} \leq H(s, p, \beta)\left(\int_{\mathbb{R}^n} \frac{v}{\gamma} d\gamma\right)^{\frac{1}{q}},$$

where

$$H(s, p, \beta) := \beta^{-\frac{1}{p'}} \beta_s^{-\frac{1}{q}} \in (0, 1],$$

and $\beta_s$ is given in (1.8). Moreover equality is established when $v = \gamma_\beta$.}
Similarly LSI can be improved in the same spirit.

**Theorem 1.3.** Let $\beta \geq 1$. Then for all $v \in \text{FP}(\beta)$,

\begin{align}
\text{Ent}_\gamma \left( \frac{v}{\gamma} \right) & \leq \frac{1}{2} I_\gamma \left( \frac{v}{\gamma} \right) - \frac{n}{2} D(\beta) \int_{\mathbb{R}^n} \frac{v}{\gamma} d\gamma,
\end{align}

where $D(\beta)$ is given by

\[ D(\beta) := \log \beta - 1 + \frac{1}{\beta} \geq 0. \]

Equality is established when $v = \gamma \beta$.

**Remark.** If $v \in \text{FP}(\beta)$ is normalised $\int v \, dx = 1$, then (1.15) can be read as

\[ \text{Ent}_\gamma \left( \frac{v}{\gamma} \right) - \frac{1}{2} I_\gamma \left( \frac{v}{\gamma} \right) \leq \text{Ent}_\gamma \left( \frac{\gamma \beta}{\gamma} \right) - \frac{1}{2} I_\gamma \left( \frac{\gamma \beta}{\gamma} \right) \]

which is the form of (1.11).

In fact, we shall prove a more general result than Theorems 1.2 and 1.3 which sheds a light on the prominent role of semi-log-convexity and semi-log-concavity in this context. More importantly, this viewpoint enables us to understand in a unified manner the result of Eldan–Lehec–Shenfeld in Theorem 1.1 and our own Theorem 1.3.

**Theorem 1.4.** Let $\beta > 0$. If the twice differentiable function $v : \mathbb{R}^n \to (0, \infty)$ belongs to $L^2(\gamma \beta^{-1})$ and satisfies

\begin{align}
\begin{cases}
\nabla^2 \log v \leq -\frac{1}{\beta} \text{id} & \text{if } 0 < \beta \leq 1, \\
\nabla^2 \log v \geq -\frac{1}{\beta} \text{id} & \text{if } 1 \leq \beta,
\end{cases}
\end{align}

then we have (1.14) and (1.15).

We recommend that the reader does not dwell on the condition $v \in L^2(\gamma \beta^{-1})$. This is imposed in order to justify various technicalities in the proof and it is reassuring to note that $\text{FP}(\beta) \subseteq L^2(\gamma \beta^{-1}) \subseteq L^1(dx)$. The following result is important since it helps to explain how Theorem 1.4 implies Theorems 1.2 and 1.3 as well as its connection to Theorem 1.1. Although these results may be considered well-known, we provide a proof in Section 2.

**Lemma 1.5.** Let $\beta > 0$.

1. If $v \in \text{FP}(\beta)$, then

\( \nabla^2 \log v \geq \frac{1}{\beta} \text{id}. \)

2. Let $v : \mathbb{R}^n \to (0, \infty)$ be twice differentiable, normalised by $\int_{\mathbb{R}^n} v \, dx = 1$, and satisfy the decay condition

\[ \lim_{|x| \to \infty} |x| v(x) = \lim_{|x| \to \infty} |\nabla v(x)| = 0. \]

Then

\( \nabla^2 \log v \geq -\frac{1}{\beta} \text{id} \Rightarrow \text{cov}(v) \geq \beta \text{id}. \)
\begin{equation}
\n\nabla^2 \log v \leq -\frac{1}{\beta} \text{id} \Rightarrow \text{cov}(v) \leq \beta \text{id}.
\end{equation}

In the case of $\beta \geq 1$, at least formally, Theorems 1.2 and 1.3 follows from Theorem 1.4 via (1.17); see Section 3 for a fully rigorous treatment. Moreover, in view of (1.19), Theorem 1.3 states that the deficit estimate (1.11) holds true in the case of large covariance inputs by imposing additional structure from the Fokker–Planck flow. In this sense, Theorem 1.3 complements the result of Eldan–Lehec–Shenfeld in Theorem 1.1.

In the case of $0 < \beta \leq 1$, the improved LSI in Theorem 1.4 follows from Theorem 1.1 thanks to (1.20). Nevertheless, it is not obvious if one could derive the improved hypercontractivity inequality for $0 < \beta < 1$ in Theorem 1.4 from Theorem 1.1. This appears to be because the assumption on the covariance in Theorem 1.1 is somewhat rigid with regard to running the standard argument from which one derives classical hypercontractivity from LSI\footnote{In fact, if one attempts to run the standard argument, then eventually one would need to apply Theorem 1.1 with an input form of $v = \gamma P_s \left[ f^{\frac{1}{2}} \right]^q$. For that purpose one has to ensure that the covariance of $\gamma P_s \left[ f^{\frac{1}{2}} \right]^q$ is small enough. However, it is not clear if such a non-linear flow preserves the covariance or not.}. On the other hand, our semi-log-convexity/semi-log-concavity assumption appears to be more flexible and this can be considered an advantage of our formulation. In fact, our framework is robust enough to improve other functional inequalities, the hypercontractivity inequality for Hamilton–Jacobi equation, the dual form of Talagrand’s quadratic transportation cost inequality, as well as Poincaré and Beckner inequalities; see Section 4.

Notions of semi-log-convexity and semi-log-concavity have appeared widely in several different contexts. For instance, semi-log-convexity, which is closely related to the Li–Yau gradient estimate \cite{59}, has recently featured in the work of Eldan–Lee \cite{37} and Lehec \cite{58}. These papers address a conjecture of Talagrand concerning a regularised version of the classical Markov inequality for gaussian measure $\gamma$ in which the inputs are restricted to those which arise from Ornstein–Uhlenbeck flow. Interestingly, the breakthrough work of Eldan–Lee \cite{37} noticed that the natural framework for such a regularising phenomenon is wider than simply Ornstein–Uhlenbeck flows and they established a stronger result for semi-log-convex inputs; our presentation of the main results in Theorems 1.2–1.4 somehow mimicks this line of development on Talagrand’s conjecture. The refinement of Eldan and Lee’s work by Lehec \cite{58} also holds in the more general setting of semi-log-convex inputs, and this perspective has been further developed by Gozlan–Li–Madiman–Roberto–Samson \cite{47} as the theory begins to grow beyond the gaussian measure.

Regarding semi-log-concavity, for example, we mention the work on the stability of the entropy jump inequality due to Ball–Nguyen \cite{9}, Bizeul \cite{19}, and the Prékopa–Leindler inequality due to Ball–Böröczky \cite{7, 8} and Böröczky–De \cite{25}. We also mention work of the third author \cite{64} and Indrei–Marcon \cite{51} where they also established improved LSI under certain a log-convexity and log-concavity assumption.
(the shape of the stability estimates in these papers differs from (1.11) in the current paper and are independent results).

### 1.3. Some consequences.

It is possible to use our main theorems to derive a number of consequences, including some new deficit estimates for functional inequalities which are related to hypercontractivity and LSI. Although we postpone the bulk of this to the forthcoming Section 4, here we present two consequences which are more directly related to the discussion up to this point. The first is concerned with deficit bounds for LSI which have weaker dependence on the dimension of the ambient space. For this we first recall [38, Theorem 3] in full generality (Theorem 1.4 being a special case).

**Theorem 1.6** ([38]). For any non-negative $v$ such that $\int_{\mathbb{R}^n} v \, dx = 1$,

\begin{equation}
\text{Ent}_\gamma \left( \frac{v}{\gamma} \right) \leq \frac{1}{2} I_\gamma \left( \frac{v}{\gamma} \right) - \frac{1}{2} \sum_{i=1}^{n} \left( \frac{1}{\beta_i} - 1 + \frac{1}{\beta_i} \right),
\end{equation}

where $\beta_1, \ldots, \beta_n$ are eigenvalues of cov$(v)$.

Motivated by this, we establish a stronger version of Theorem 1.4. For a positive definite symmetric matrix $B$ on $\mathbb{R}^n$, we denote

$$\gamma_B(x) := \frac{1}{(\det (2\pi B))^{\frac{n}{2}}} e^{-\frac{1}{2} \langle x, B^{-1} x \rangle}.$$}

**Corollary 1.7.** Let $B$ be positive definite symmetric matrix on $\mathbb{R}^n$ and $\beta_1, \ldots, \beta_n > 0$ be the eigenvalues of $B$.

1. If the twice differentiable function $v : \mathbb{R}^n \to (0, \infty)$ belongs to $L^2(\gamma_B^{-1})$ and satisfies

\begin{equation}
\nabla^2 \log v \geq -B^{-1}
\end{equation}

then we have

\begin{equation}
\left\| P_s \left[ \left( \frac{v}{\gamma} \right) \right] \right\|_{L^p(\gamma)} \leq \prod_{\beta_i \geq 1} \beta_i^{\frac{np}{2}} (\beta_i)^{-\frac{np}{2}} \left( \int_{\mathbb{R}^n} \frac{v}{\gamma} \, d\gamma \right)^{\frac{p}{2}}
\end{equation}

and

\begin{equation}
\text{Ent}_\gamma \left( \frac{v}{\gamma} \right) \leq \frac{1}{2} I_\gamma \left( \frac{v}{\gamma} \right) - \frac{1}{2} \sum_{i=1, \ldots, n, \beta_i \geq 1} \left( \frac{1}{\beta_i} - 1 + \frac{1}{\beta_i} \right).
\end{equation}

2. If the twice differentiable function $v : \mathbb{R}^n \to (0, \infty)$ belongs to $L^2(\gamma_B^{-1})$ and satisfies

\begin{equation}
\nabla^2 \log v \leq -B^{-1}
\end{equation}

then we have

\begin{equation}
\left\| P_s \left[ \left( \frac{v}{\gamma} \right) \right] \right\|_{L^p(\gamma)} \leq \prod_{\beta_i \leq 1} \beta_i^{\frac{np}{2}} (\beta_i)^{-\frac{np}{2}} \left( \int_{\mathbb{R}^n} \frac{v}{\gamma} \, d\gamma \right)^{\frac{p}{2}}
\end{equation}
and
\begin{equation}
\text{Ent}_\gamma \left( \frac{v}{\gamma} \right) \leq \frac{1}{2} I_\gamma \left( \frac{v}{\gamma} \right) - \frac{1}{2} \sum_{i=1,\ldots,n: \beta_i \leq 1} \left( \log \beta_i - 1 + \frac{1}{\beta_i} \right).
\end{equation}

Moreover, equalities in \((1.23)-(1.27)\) are established when \(v = \gamma B\).

We clarify that in the above statement, following \((1.8)\), we write \((\beta_i)_{\gamma} = 1 + (\beta_i - 1) \frac{2}{\beta} e^{-2s}\) for each eigenvalue \(\beta_i\).

Interestingly, although Corollary \(1.7\) is clearly a stronger result compared with Theorem \(1.4\), we will establish Corollary \(1.7\) by combining the one-dimensional result in Theorem \(1.4\) with a standard tensorisation argument. In particular, it is an appealing feature of our set-up that the semi-log-convexity/semi-log-concavity assumption is very well suited to the tensorisation argument.

Finally, as promised, let us state a direct consequence of our regularised LSI which allows us to obtain \((1.6)\) (i.e. symmetric case of the dual form of the inverse functional Santaló inequality) for sufficiently large \(c\). For this, first we remark that \(D : [1, \infty) \ni \beta \mapsto \log \beta + \frac{1}{\beta} - 1 \mapsto [0, \infty)\) is monotone increasing and hence there exists some increasing inverse function \(D^{-1} : [0, \infty) \to [1, \infty)\). In fact, this inverse function is given by using the Lambert \(W\) function defined on \([-e^{-1}, \infty)\):

\[D^{-1}(r) = -\frac{1}{W_0(-r-1)}, \quad r \geq 0.\]

Note that
\[-1 = W_0(-1) \leq W_0(-r-1) \leq \lim_{s \to \infty} W_0(-e^{-s-1}) = 0, \quad r \geq 0.\]

As a direct consequence of Theorem \(1.4\) we obtain the following.

**Corollary 1.8.** Let \(c \geq 2\pi\) and \(\beta(c) \geq 1\) be given by

\[\beta(c) = -\frac{1}{W_0(-\frac{2\pi}{cc})}.\]

Then for all \(v \in L^2(\gamma^{-1}_\beta(\cdot))\) satisfying

\[\int_{\mathbb{R}^n} v \, dx = 1, \quad \nabla^2 \log v \geq -\frac{1}{\beta(c)} \text{id},\]

we have \((1.6)\).

### 1.4. Overview of our approach and structure of the paper.

To prove the statement regarding the hypercontractivity inequality in Theorem \(1.4\) we use a flow monotonicity argument based on the Fokker–Planck equation. To be precise, we shall show that

\[t \mapsto \int P_t \left( \frac{v}{\gamma} \right) \frac{d\gamma}{\gamma} \]

is non-decreasing on \((0, \infty)\), where \(v_t\) is the evolution of the given input \(v\) along the \(\beta\)-Fokker–Planck equation. Comparing the above time-dependent functional at \(t = 0\) and \(t = \infty\) generates the desired inequality \((1.14)\).
the proof of the monotonicity of the above functional, we shall use a quantitative version of the fact that semi-log-convexity and semi-log-concavity are preserved under Fokker–Planck flows (see Lemma 2.1). The improved LSI in (1.14) will be derived from (1.15) by following the well-known differentiation argument (due to Gross [49]).

This part of the paper takes a great deal of inspiration from work by Bennett–Carbery–Christ–Tao [17] on regularised Brascamp–Lieb inequalities and [15] on the Young convolution inequality (both of which use classical heat flow associated with the standard Laplacian), as well as the approach to (1.2) taken by Aoki et al. [2] (using Ornstein–Uhlenbeck flow); we refer the reader forward to Sections 3.4–3.5 for further elaboration on this, including some discussion on our reasons for employing Fokker–Planck flow. We also remark that although the arguments of Eldan–Lehec–Shenfeld [38] are less close to those in the current paper, they also employed a flow (the Föllmer process from stochastic analysis).

In Section 2 after giving a proof of Lemma 1.5 we state and prove the aforementioned result concerning the preservation of semi-log-convexity and semi-log-concavity under Fokker–Planck flows. Finally, a technical result concerning regularity and decay of Fokker–Planck flows is given and will be used to justify various interchanges of limits in the proof of Theorem 1.4.

In Section 3 we will prove Theorems 1.2, 1.4 and Corollary 1.7. In Section 4 we will apply these results to improve the hypercontractivity inequality for the Hamilton–Jacobi equation, the dual form of Talagrand’s quadratic transportation cost inequality, and Poincaré and Beckner inequalities.

Finally, in Section 5 we will explore the possibility of obtaining deficit estimates for hypercontractivity, LSI, and Talagrand’s cost inequality on more general measure spaces. We also address whether one can replace log-convexity/log-concavity with log-subharmonicity/log-superharmonicity in our main results.

2. Preliminaries

Proof of Lemma 1.5. The assertion (1) follows from the fact that

\( \nabla^2 \log v(t, x) \geq -\frac{1}{1 - e^{-2\beta t}} \frac{1}{\beta} \text{id}, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n \)

for any \( \beta > 0 \) and any \( \beta \)-Fokker–Planck solution \( \partial_t v_t = \mathcal{L}_\beta v_t \) with any non-negative initial data; see [47] (1) on p.3 or [37] Lemma 1.3]. In fact, this shows that for \( v_* \), a solution to (1.13), and \( t_* = \frac{1}{2} \log 2 \),

\( \nabla^2 \log v_*(t_*, x) \geq -\frac{1}{1 - e^{-2\beta t_*}} \frac{1}{2\beta} \text{id} = -\frac{1}{\beta} \text{id} \).
Let us show (1.19) next. By translation invariance, without loss of generality we may assume \( \int_{\mathbb{R}^n} x_i \, dv = 0, \, 1 \leq i \leq n \), and hence
\[
(2.2) \quad \text{cov} \, (v) = \left( \int_{\mathbb{R}^n} x_i x_j \, dv \right)_{1 \leq i, j \leq n} = \int_{\mathbb{R}^n} x \otimes x \, dv.
\]

With this in mind, we integrate by parts
\[
\int_{\mathbb{R}^n} x_i x_j \, dv = \int_{\mathbb{R}^n} x_i x_j \frac{v}{\gamma_\beta} \, d\gamma_\beta = \beta \delta_{i,j} + \beta^2 \int_{\mathbb{R}^n} \partial_{ij} \left( \frac{v}{\gamma_\beta} \right) \, d\gamma_\beta.
\]
Here we used the decay assumption (1.18) and the normalisation. We next notice that
\[
\partial_{ij} f = f \partial_{ij} \log f + f \left( \partial_i \log f \right) \left( \partial_j \log f \right)
\]
from which we see that
\[
\text{cov} \, (v) = \beta \text{id} + \beta^2 \int_{\mathbb{R}^n} \nabla^2 \log \frac{v}{\gamma_\beta} \, dv
\]
whilst we clearly have non-negativity of the third term since
\[
\left\langle x, \left( \nabla \log \frac{v}{\gamma_\beta} \right) \otimes \left( \nabla \log \frac{v}{\gamma_\beta} \right) x \right\rangle = \left| x \cdot \nabla \log \frac{v}{\gamma_\beta} \right|^2 \geq 0
\]
for all \( x \in \mathbb{R}^n \). This completes the proof of (1.19).

For the proof of (1.20), we regard the assumption \( \nabla \log v \leq -\frac{1}{\beta} \text{id} \) as the curvature-dimension condition \( \text{CD} \left( \frac{1}{\beta}, \infty \right) \), in which case it follows from [6, Corollary 4.8.2, p.212] that we have the Poincaré inequality
\[
(2.3) \quad \int_{\mathbb{R}^n} |\varphi|^2 \, dv \leq \beta \int_{\mathbb{R}^n} |\nabla \varphi|^2 \, dv, \quad \text{for} \quad \int_{\mathbb{R}^n} \varphi \, dv = 0.
\]
Again, without loss of generality, we suppose \( \int_{\mathbb{R}^n} x_i \, dv = 0 \). From (2.2) we obtain
\[
\left\langle \theta, \text{cov} \, (v) \theta \right\rangle = \int_{\mathbb{R}^n} |\langle \theta, x \rangle|^2 \, dv.
\]
Since \( \int_{\mathbb{R}^n} x_i \, dv = 0 \), we may apply (2.3) with \( \varphi(x) := \langle \theta, x \rangle \) to see that
\[
\left\langle \theta, \text{cov} \, (v) \theta \right\rangle \leq \beta \int_{\mathbb{R}^n} |\nabla_x \langle \theta, x \rangle|^2 \, dv = \beta |\theta|^2
\]
and hence \( \text{cov} \, (v) \leq \beta \text{id} \).

In the proof of Theorem 1.3, we will flow input function \( v \) via \( \beta \)-Fokker–Planck equation:
\[
(2.4) \quad \partial_t v_t = \mathcal{L}_\beta^* v_t, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \quad v_0 = v.
\]
A crucial property in our argument is the fact that the Fokker–Planck flow preserves semi-log-convexity and semi-log-concavity in the following sense.
Lemma 2.1. Let $\beta > 0$ and $v_t$ be a $\beta$-Fokker–Planck solution of (2.4) with twice differentiable initial data $v : \mathbb{R}^n \to (0, \infty)$. Then
\[
\nabla^2 \log v \geq -\frac{1}{\beta} \text{id} \Rightarrow \nabla^2 \log v_t \geq -\frac{1}{\beta} \text{id}, \quad t > 0,
\]
and
\[
\nabla^2 \log v \leq -\frac{1}{\beta} \text{id} \Rightarrow \nabla^2 \log v_t \leq -\frac{1}{\beta} \text{id}, \quad t > 0.
\]

Proof. In this proof we make use of the fact that the solution to (2.4) has the explicit form
\[
v_t(x) = \frac{1}{(2\pi\beta(1 - e^{-2t}))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x - y|^2}{2\beta(1 - e^{-2t})}} v_0(y) \, dy.
\]
This may be rewritten as
\[
v_t(x) = e^{-\frac{|x|^2}{2\beta}} \int_{\mathbb{R}^n} \phi(x, w) \, d\mu(w),
\]
where
\[
\phi(x, w) := e^{-\frac{|w + e^{-t}x|^2}{2\beta}} v_0(w + e^{-t}x),
\]
\[
d\mu(w) := \frac{1}{(2\pi\beta(1 - e^{-2t}))^{\frac{n}{2}}} e^{-\frac{1}{2\beta(1 - e^{-2t})} |w|^2} \, dw.
\]
From the assumption $\nabla^2 \log v_0 \geq -\frac{1}{\beta} \text{id}$, we see that
\[
\nabla^2 \log \phi(x, w) = e^{-\frac{2t}{\beta}} \text{id} + e^{-2t} \nabla^2 \log v_0(w + e^{-t}x) \geq 0.
\]
Using the fact that log-convexity is preserved under superpositions, we obtain (2.5).

The preservation of semi-log-concavity in (2.6) is more subtle and can be found, for instance, in [28, 52]. For the sake of completeness, we give a proof of (2.6) here. We use the explicit form of the solution (2.7) to write
\[
v_t(x) = e^{-\frac{|x|^2}{2\beta}} \int_{\mathbb{R}^n} \gamma_{\beta(1 - e^{-2t})} e^{-\frac{|w + e^{-t}x|^2}{2\beta}} v_0(w + e^{-t}x) \, dw \]
\[
= e^{-\frac{|x|^2}{2\beta}} \int_{\mathbb{R}^n} \psi_x(w) \, dw, \quad \psi_x(w) := \gamma_{\beta(1 - e^{-2t})} e^{-\frac{|w + e^{-t}x|^2}{2\beta}} v_0(w + e^{-t}x) \]
\[
= e^{-\frac{|x|^2}{2\beta}} \Psi(x),
\]
where we suppress the dependence on $t$ in $\psi_x$ and $\Psi$.

It is sufficient to show the log-concavity of $\Psi$. To this end, we focus on
\[
\Psi(\lambda x_0 + (1 - \lambda)x_1) = \int_{\mathbb{R}^n} \psi_{\lambda x_0 + (1 - \lambda)x_1}(y) \, dy.
\]
We claim that
\[
\psi_{\lambda x_0 + (1 - \lambda)x_1}(\lambda y_0 + (1 - \lambda)y_1) \geq \psi_{x_0}(y_0)^\lambda \psi_{x_1}(y_1)^{1 - \lambda}.
\]
\[4\]The uniqueness of the solution is a consequence of Widder’s theorem [68].
Once we could prove (2.9), then we conclude from the Prékopa–Leindler inequality that

\[ \Psi(\lambda x_0 + (1 - \lambda)x_1) \geq \left( \int_{\mathbb{R}^n} \psi(x_0) \right)^\lambda \left( \int_{\mathbb{R}^n} \psi(x_1) \right)^{1-\lambda} = \Psi(x_0)^\lambda \Psi(x_1)^{1-\lambda} \]

which shows the desired log-convexity of \( \Psi \).

Towards (2.9), we regard \( \psi(x, y) := \psi_x(y) \). In fact, from this viewpoint, (2.9) is clearly equivalent to the log-concavity of \( \psi \) in the \((x, y)\)-variable:

\[ \psi(\lambda x_0, y_0) + (1 - \lambda)(x_1, y_1) \geq \psi(x_0, y_0)^\lambda \psi(x_1, y_1)^{1-\lambda}. \]

This can be seen by writing

\[ \psi(x, y) = \gamma_\beta(1 - e^{-2t})(y) \vartheta(y + e^{-t}x), \quad \vartheta(w) := e^{\frac{|w|^2}{2\gamma}} v_0(w) \]

and noting that our assumption (2.10) implies \( \vartheta \) is log-concave. It is also clear that \( \gamma_\beta(1 - e^{-2t}) \) is log-concave too. Hence

\[ \psi(\lambda x_0, y_0) + (1 - \lambda)(x_1, y_1) \geq \gamma_\beta(1 - e^{-2t})(y_0) \gamma_\beta(1 - e^{-2t})(y_1)^{1-\lambda} \vartheta(y_0 + e^{-t}x_0)^\lambda \vartheta(y_1 + e^{-t}x_1)^{1-\lambda} \]

\[ = \psi(x_0, y_0)^\lambda \psi(x_1, y_1)^{1-\lambda} \]

which gives the desired log-concavity of \( \psi \). \( \square \)

The following lemma, which is a consequence of Lemmas 1.5 and 2.1, will be used to justify our formal argument. We denote by \( \Gamma \) the carré du champ operator

\[ \Gamma \phi(x) := |\nabla \phi(x)|^2. \]

for differentiable functions \( \phi \) on \( \mathbb{R}^n \).

**Lemma 2.2.** Let \( p > 1, \beta > 0 \), and twice differentiable function \( v: \mathbb{R}^n \to (0, \infty) \) satisfy \( v \in L^2(\gamma_\beta^{-1}) \) and (1.10). Then for a solution \( v_t \) to (2.4), we have

\[ \left( \frac{\nabla}{\gamma} \right)^\frac{1}{p} \Delta W_j, \left( \frac{\nabla}{\gamma} \right)^\frac{1}{p} \nabla W_j, \left( \frac{\nabla}{\gamma} \right)^\frac{1}{p} \Gamma W_j, \left( \frac{\nabla}{\gamma} \right)^\frac{2}{p} \nabla W_1 \cdot \nabla W_2 \in L^1(d\gamma), \quad j = 1, 2, \]

for each \( t > 0 \), where \( W_1(x) := \frac{|x|^2}{\gamma} \) and \( W_2 := \log \frac{|x|}{\gamma} \).

**Proof.** Let us consider the case \( \beta \geq 1 \) first. We prove \( \left( \frac{\nabla}{\gamma} \right)^\frac{1}{p} \Delta W_2 \in L^1(\gamma) \) and remark that this case contains ideas which may be used for other terms. A direct computation shows that

\[ \left( \frac{\nabla}{\gamma} \right)^\frac{1}{p} \Delta W_2 = - \left( \frac{\nabla}{\gamma_{\beta}} \right)^\frac{1}{p} - 2 \left( \frac{\gamma_{\beta}}{\gamma} \right)^\frac{1}{p} - 2 \left( \frac{\gamma_{\beta}}{\gamma} \right)^\frac{1}{p} \nabla \gamma - \frac{\nabla \gamma}{\gamma_{\beta}} \nabla \gamma_{\beta} \right) + \left( \frac{\nabla}{\gamma_{\beta}} \right)^\frac{1}{p} \Delta \left( \frac{\gamma_{\beta}}{\gamma} \right)^\frac{1}{p} \nabla \gamma_{\beta} \nabla \gamma + \frac{\nabla \gamma_{\beta}}{\gamma_{\beta}} \nabla \gamma_{\beta} \nabla \gamma. \]
Hence \(|\left(\frac{v_t}{\gamma} \right)^{\frac{1}{p}} \Delta W_2|\) is bounded, up to a multiplicative constant, by
\[
\left(\frac{v_t}{\gamma} \right)^{\frac{1}{p} - 2} |\nabla \frac{v_t}{\gamma}|^2 \left(\frac{\gamma \beta}{\gamma} \right)^{\frac{1}{p}} + \left(\frac{v_t}{\gamma} \right)^{\frac{1}{p} - 1} \left(\frac{\gamma \beta}{\gamma} \right)^{\frac{1}{p}} |\nabla W_1| \right)^2
+ \left(\frac{v_t}{\gamma} \right)^{\frac{1}{p} - 1} |\Delta \frac{v_t}{\gamma}| \left(\frac{\gamma \beta}{\gamma} \right)^{\frac{1}{p}} |\nabla W_1| + \left(\frac{v_t}{\gamma} \right)^{\frac{1}{p} - 1} \left(\frac{\gamma \beta}{\gamma} \right)^{\frac{1}{p}} |\nabla W_1|.
\]

Note that \(u_t := \frac{v_t}{\gamma}\) satisfies the gaussian heat equation
\[
\partial_t u_t = \mathcal{L}_\beta u_t, \quad u_0 \in L^1(\gamma) \cap L^2(\gamma),
\]
since we assumed \(v_0 \in L^2(\gamma^{-1})\). Hence we may employ [48, (7.78)] to bound
\[
\sup_{y \in \mathbb{R}^n} \left|\frac{v_t}{\gamma} \right| (y) + |\nabla \frac{v_t}{\gamma} (y)| + |\Delta \frac{v_t}{\gamma} (y)| \leq C(t) \|u_0\|_{L^2(\gamma)},
\]
where \(C(t)\) is locally bounded in \(t\). Also we know from the assumption (1.16) and (2.1) that
\[
\nabla^2 \left(\log \frac{v_t}{\gamma} \right) \geq 0
\]
from which we can deduce that
\[
\log \frac{v_t}{\gamma} (y) \geq \log \frac{v_t}{\gamma} (0) + \nabla \log \frac{v_t}{\gamma} (0) \cdot y.
\]
This yields the upper bound of \(\left(\frac{v_t}{\gamma} \right)^{\frac{1}{p} - 1}\) and \(\left(\frac{v_t}{\gamma} \right)^{\frac{1}{p} - 2}\) since \(p > 1\) and hence we see that
\[
\left|\left(\frac{v_t}{\gamma} \right)^{\frac{1}{p}} (y) \Delta W_2 (y)\right|
\leq C(t) \left(e^{\left(\frac{1}{p} - 2\right) \nabla \log \frac{v_t}{\gamma} (0) \cdot y} + e^{\left(\frac{1}{p} - 1\right) \nabla \log \frac{v_t}{\gamma} (0) \cdot y} (1 + |\nabla W_1|) + |\nabla W_1|^2 \right) \left(\frac{\gamma \beta}{\gamma} \right)^{\frac{1}{p}}.
\]
If we recall \(W_1 (x) = |x|^2 / 2\) and notice that \(\left(\frac{2 \alpha}{\gamma} \right)^{\frac{1}{p}} \leq C_{\alpha, \beta} \frac{2 \alpha}{\gamma}\), then we conclude that
\[
\int_{\mathbb{R}^n} \left|\left(\frac{v_t}{\gamma} \right)^{\frac{1}{p}} (y) \Delta W_2 (y)\right| d\gamma (y)
\leq C(t) \int_{\mathbb{R}^n} \left(e^{\left(\frac{1}{p} - 2\right) \nabla \log \frac{v_t}{\gamma} (0) \cdot y} + |y|^2 + e^{\left(\frac{1}{p} - 1\right) \nabla \log \frac{v_t}{\gamma} (0) \cdot y} (1 + |y|) \left(\frac{\gamma \beta}{\gamma} \right)^{\frac{1}{p}}
\right) d\gamma (y) < \infty.
\]
We next turn to the case \(0 < \beta \leq 1\). From the assumption (1.16) and (2.1), we know that \(v_t\) is \(-\frac{1}{\beta}\)-log-concave for all \(t > 0\). In addition we have the lower bound of \(\beta\) regardless of the initial data. Hence,
\[
- \frac{1}{1 - e^{-2t \beta}} \frac{1}{\beta} \text{id} \leq \nabla^2 \log v_t \leq - \frac{1}{\beta} \text{id}
\]
for all \(t > 0\). This shows
\[
|W_2 (x)| \leq C_{t, \beta} |x|^2, \quad |\nabla W_2 (x)| \leq C_{t, \beta} |x|, \quad |\Delta W_2 (x)| \leq C_{t, \beta}
\]
and hence the matters are reduced to quantities of \(W_1\). However, these cases can be handled in a similar way as in the case \(\beta \geq 1\).
3. Proof of Theorem 1.4 and Corollary 1.7

3.1. Flow monotonicity: the key result. We recall our strategy for proving Theorem 1.4. First we prove the hypercontractivity inequality and then derive LSI from it. For the hypercontractivity inequality, we establish the monotonicity of a certain functional under Fokker–Planck flow. Fix any \( \beta > 0 \) and \( v \) satisfying the assumptions in Theorem 1.4 and let \( v_t \) be the \( \beta \)-Fokker–Planck solution with initial data \( v \). Then we introduce a functional by

\[
Q(t) := \int_{\mathbb{R}^n} \tilde{v}_t \, dx, \quad \tilde{v}_t := \gamma P_s \left[ (\frac{v_t}{\gamma})^\frac{1}{2} \right]^q, \quad t > 0.
\]

It will suffice to establish that \( Q(t) \) is non-decreasing on \((0, \infty)\) and in order to achieve this we seek to obtain an expression for \( \partial_t \tilde{v}_t \) involving either non-negative terms or terms which vanish upon integrating with respect to Lebesgue measure on \( \mathbb{R}^n \) (such as \( L^n_{\beta} \tilde{v}_t \)). Establishing such an expression is the main component of the proof of Theorem 1.4 and follows from an extremely careful investigation of the interaction between two flows, one is the Ornstein–Uhlenbeck flow \( P_s \) and the other is the \( \beta \)-Fokker–Planck flow \( v_t \). The key result underlying this phenomenon is the following.

**Proposition 3.1.** Let \( \beta > 0, \, s > 0, \, 1 < p < q < \infty \) satisfy \( \frac{q}{p} = e^{2s} \). Suppose the twice differentiable \( v : \mathbb{R}^n \to (0, \infty) \) is in \( L^2(\gamma^{-1}) \) and satisfies (1.16). If

\[
\partial_t v_t = L_{\beta}^* v_t, \quad v_0 = v,
\]

then we have that

\[
\left( \frac{\beta_s}{\beta} \right) \frac{1}{\beta_s q} \partial_t \tilde{v}_t - L_{\beta}^* \tilde{v}_t
\]

\[
\geq P_s h_t P_s \left[ h_t \Gamma(\log \frac{v_t}{\gamma}) \right] - \left| P_s \left[ h_t \nabla(\log \frac{v_t}{\gamma}) \right] \right|^2
\]

\[
+ \left( \frac{\beta_s}{\beta} \right) \frac{1}{\beta_s q} (I + N) + (1 + \frac{\beta_s}{\beta})(1 - \frac{1}{\beta}) R(h_t, h_t x)
\]

\[
- (1 - \frac{1}{\beta})^2 R(h_t, h_t x),
\]

where \( h_t := (\frac{v_t}{\gamma})^{\frac{1}{2}} \), and \( \beta_s > 0 \) and \( \tilde{v}_t \) are defined by (1.8) and (3.1), respectively. Here,

\[
I := q(\beta - \beta_s) P_s h_t P_s \left[ h_t \Delta \left( \frac{v_t}{\gamma} \right) \right] + \frac{q}{p} (\beta - \beta_s) P_s h_t P_s \left[ h_t \Gamma(\log \frac{v_t}{\gamma}) \right]
\]

\[
+ \frac{q}{p} (\beta - 2\beta + 1) P_s \left[ h_t x \cdot \nabla \left( \log \frac{v_t}{\gamma} \right) \right]
\]

\[
+ \frac{q}{p} (1 - \beta) P_s h_t P_s \left[ h_t (n - |x|^2) \right],
\]

\[
N := q(\beta - 1) P_s h_t x \cdot \nabla P_s h_t + (\beta - 1)(n - |x|^2) \left( P_s h_t \right)^2,
\]

and for \( f \) and \( g = (g_j)_{j=1}^n \),

\[
R(f, g) := P_s f P_s \left[ x \cdot g \right] - P_s \left[ f x \right] \cdot P_s g - (1 - e^{-2s}) \left( P_s f P_s \left[ \text{div} g \right] - P_s \left[ \nabla f \right] \cdot P_s g \right).
\]
To show Proposition 3.1, we use the following two lemmas.

**Lemma 3.2.** Under the assumptions in Proposition 3.1, we have

\[
\frac{\partial_t \tilde{v}_t - L_\beta^* \tilde{v}_t}{\tilde{v}_t^{1-\frac{s}{2}} v_t^\frac{s}{2}} = \frac{q}{pp'}(\beta_s - \beta)P_s h_t P_s [h_t \Delta \log \frac{v_t}{\gamma}] + \frac{q}{pp'} \beta_s I_1 + I + N,
\]

where \(h_t := \left(\frac{\psi_t}{\gamma}\right)^\frac{s}{p}\) and

\[
I_1 := P_s h_t P_s [h_t \Gamma \log \frac{v_t}{\gamma}] - |P_s [h_t \nabla \log \frac{v_t}{\gamma}]|^2.
\]

**Lemma 3.3.** Under the assumptions in Proposition 3.1, we have

\[
P_s h_t P_s [h_t \Gamma \log \frac{v_t}{\gamma}] - |P_s [h_t \nabla \log \frac{v_t}{\gamma}]|^2
= p(1 - \frac{1}{\beta})(1 - \frac{\beta_s}{\beta}) P_s h_t P_s [nh_t] - p(1 - \frac{\beta_s}{\beta})(1 + \frac{\beta_s}{\beta}) P_s h_t P_s [h_t \Delta \log \frac{v_t}{\gamma}]
+ (\frac{\beta_s}{\beta})^2 I_1 - (1 + \frac{\beta_s}{\beta})(1 - \frac{1}{\beta}) R(h_t, h_t x) + (1 - \frac{1}{\beta})^2 R(h_t, h_t x),
\]

where \(h_t := \left(\frac{\psi_t}{\gamma}\right)^\frac{s}{p}\) and \(I_1\) is given by (3.2).

Let us complete the proof of Proposition 3.1 by assuming the validity of Lemmas 3.2 and 3.3 for a moment.

**Proof of Proposition 3.1** With Lemma 3.3 in mind, we focus on the term

\[
p(1 - \frac{1}{\beta})(1 - \frac{\beta_s}{\beta}) P_s h_t P_s [nh_t] = pP_s h_t \frac{1}{\beta} P_s [h_t (\beta - \beta_s)(1 - \frac{1}{\beta}) n].
\]

We appeal to Lemma 2.1 and assumption (1.16) to see

\[
\begin{align*}
(1 - \frac{1}{\beta}) n &\geq \Delta \log \frac{v_t}{\gamma} & &\text{if } 0 < \beta < 1 \\
(1 - \frac{1}{\beta}) n &\leq \Delta \log \frac{v_t}{\gamma} & &\text{if } 1 < \beta,
\end{align*}
\]

for all \(t > 0\). For the sign of the coefficient, from the definition of \(\beta_s\) we know that

\[
\beta - \beta_s = \beta - 1 - (\beta - 1) \frac{q}{p} e^{-2s} = \frac{1}{p} (\beta - 1)(1 - e^{-2s})
\]

and hence

\[
\begin{align*}
\beta - \beta_s &\leq 0 & &\text{if } 0 < \beta < 1 \\
\beta - \beta_s &\geq 0 & &\text{if } 1 < \beta.
\end{align*}
\]

Overall, we see that for all \(t > 0\)

\[
(\beta - \beta_s)(1 - \frac{1}{\beta}) n \leq (\beta - \beta_s) \Delta \log \frac{v_t}{\gamma},
\]
regardless whether \( \beta \) is less than or greater than one. Therefore from Lemma 3.3 we obtain that for all \( t, \beta > 0 \),
\[
P_s h_t P_s \left[ h_t \Gamma \log \frac{v_t}{\gamma} \right] - \left| P_s \left[ h_t \nabla \log \frac{v_t}{\gamma} \right] \right|^2
\]
\[
\leq p(1 - \frac{\beta}{\beta}) P_s h_t P_s \left[ h_t \Delta \log \frac{v_t}{\gamma} \right] - p(1 - \frac{\beta}{\beta})(1 + \frac{\beta}{\beta}) P_s h_t P_s \left[ h_t \Delta \log \frac{v_t}{\gamma} \right] + \left( \frac{\beta}{\beta} \right)^2 I_1
\]
\[
- (1 + \frac{\beta}{\beta})(1 - \frac{1}{\beta}) R(h_t, h_t \nabla (\log \frac{v_t}{\gamma})) + (1 - \frac{1}{\beta})^2 R(h_t, h_t x)
\]
\[
= \frac{\beta}{\beta^2} \frac{p p!}{q} \left( \frac{q}{p p!} p(\beta - \beta) P_s h_t P_s \left[ h_t \Delta \log \frac{v_t}{\gamma} \right] + \frac{q}{p p!} \beta_s I_1 \right)
\]
\[
- (1 + \frac{\beta}{\beta})(1 - \frac{1}{\beta}) R(h_t, h_t \nabla (\log \frac{v_t}{\gamma})) + (1 - \frac{1}{\beta})^2 R(h_t, h_t x).
\]
By applying Lemma 3.2 we conclude the proof. \( \square \)

It remains to show Lemmas 3.2 and 3.3

**Proof of Lemma 3.2.** We remark that each of the terms appearing on the right-hand side (e.g. \( P_s \left[ \left( \frac{v_t}{\gamma} \right)^\frac{1}{\beta} \Delta \log \left( \frac{v_t}{\gamma} \right) \right] \)) are well-defined for each \( t > 0 \) since \( P_s \) is bounded on \( L^1(\gamma) \) and Lemma 2.2.

In view of the commutation relation \( \nabla P_s = e^{-s} P_s \nabla \), our goal is to show
\[
(3.3) \quad \frac{\partial_t \tilde{v}_t - \mathcal{L}_s^\beta \tilde{v}_t}{\tilde{v}_t^{1 - \frac{1}{\beta}}} = \frac{q}{p p!} p(\beta - \beta) P_s h_t P_s \left[ h_t \Delta \left( \log \frac{v_t}{\gamma} \right) \right]
\]
\[
+ \frac{q}{p p!} \beta_s \left( P_s h_t P_s \left[ h_t \Gamma \left( \log \frac{v_t}{\gamma} \right) \right] - (p e^s)^2 \Gamma(P_s h_t) \right) + I + N.
\]

Thanks to the definition of \( \tilde{v}_t \) and the fact that \( v_t \) is a solution to (2.4), we see that
\[
(3.4) \quad \partial_t \tilde{v}_t = q \tilde{v}_t^{1 - \frac{1}{\beta}} \partial_t \left( \tilde{v}_t^{\frac{1}{\beta}} \right) = q \frac{p}{q} \tilde{v}_t \left( \frac{v_t}{\gamma} \right)^{-\frac{1}{\beta}} P_s \left[ \left( \frac{v_t}{\gamma} \right)^{\frac{1}{\beta} - 1} \mathcal{L}_s^\beta \tilde{v}_t \right].
\]

Here we implicitly commuted the operators \( \partial_t \) and \( P_s \); that is
\[
(3.5) \quad \partial_t P_s \left[ \left( \frac{v_t}{\gamma} \right)^{\frac{1}{\beta}} \right] = P_s \left[ \partial_t \left( \frac{v_t}{\gamma} \right)^{\frac{1}{\beta}} \right].
\]

This can be checked as follows. By denoting the integral kernel of \( P_s \) with respect to \( d\gamma \) by \( p_s(x, y) \), we write
\[
P_s \left[ \left( \frac{v_t}{\gamma} \right)^{\frac{1}{\beta}} \right](x) = \int \left( \frac{v_t}{\gamma} \right)^{\frac{1}{\beta}}(y)\gamma^\beta(y)p_s(x, y) d\gamma(y)
\]
\[
= \int \left( \frac{v_t}{\gamma} \right)^{\frac{1}{\beta}}(y)\gamma^\beta(y)p_s(x, y) d\gamma(y)
\]
and investigate the bound of \( \partial_t \left( \frac{v_t}{\gamma} \right)^{\frac{1}{\beta}} = \frac{1}{p} (\frac{v_t}{\gamma})^{\frac{1}{\beta} - 1} \partial_t \left( \frac{v_t}{\gamma} \right) \). As we did in the proof of Lemma 2.2 we may appeal to (48) (7.78) to obtain that
\[
\sup_y |\partial_t \left( \frac{v_t}{\gamma} \right)(y)| \leq C(t) \frac{\left\| \frac{v_t}{\gamma} \right\|_{L^2(\gamma)}}{\left\| \gamma \right\|_{L^2(\gamma)}} = C(t)\|v\|_{L^2(\gamma^{-1})}
\]
and hence
\[ \partial_t \left( \frac{v_t}{\gamma} \right) \frac{1}{\gamma} \left( \frac{\gamma}{\gamma} \right)^{\frac{1}{\gamma}}(y) \leq \frac{1}{p} e^{(\frac{1}{\gamma} - 1) \log \left( \frac{\gamma}{\gamma} \right)(0)} C(t) \| u_0 \|_{L^2(\gamma \beta)} G(y), \]
where \( C(t) \) is locally bounded and
\[ G(x) := e^{(\frac{1}{\gamma} - 1) \nabla \left( \log \left( \frac{\gamma}{\gamma} \right)(0) \right)} \left( \frac{\gamma}{\gamma} \right)^{\frac{1}{\gamma}}(y). \]
The same argument in the proof of Lemma 2.2 shows that \( G(x) \in L^1(\gamma) \) from which we obtain that
\[ (3.6) \quad \int G(x) \, dP_{\gamma,x}(y) = P_{\gamma}[G(x)] < \infty, \]
for almost every \( x \in \mathbb{R}^n \) since \( P_{\gamma} \) is bounded on \( L^1(\gamma) \). This suffices to ensure (3.5).
We remark that this further shows the bound
\[ (3.7) \quad |\partial_t v(t, x)| \leq C(t) \gamma(x) P_{\gamma} \left[ e^{\frac{1}{\gamma} \left( 1 - \frac{1}{\gamma} \right) |\frac{1}{\gamma} - 1| \nabla v(t, x)|^2 + c(t) |x| \right], \]
where \( C(t), c(t) > 0 \) are locally bounded and depend on \( \| v \|_{L^2(\gamma \beta)} < \infty. \)
We then compute \( L_{\beta}^* v_{\gamma} \). Since \( \gamma \) satisfies \( L_{\beta}^* \gamma = 0 \), in general we have that
\[ (3.8) \quad L_{\beta}^* \left( \gamma^\frac{1}{\gamma} f \right) = \gamma^\frac{1}{\gamma} L_{\beta} f + (1 - \beta) \gamma^\frac{1}{\gamma} T_{\gamma} f \]
for \( q \in \mathbb{R} \setminus \{0\} \), where
\[ T_{\gamma} f := 2x \cdot \nabla f + (n - 1) \left( \frac{1}{q} |x|^2 \right) f. \]
Applying this with \( q = 1 \) and \( f = \frac{v_t}{\gamma} \), it follows that
\[ L_{\beta}^* v_{t} = L_{\beta}^* \left( \frac{v_t}{\gamma} \right) = \gamma L_{\beta} \left( \frac{v_t}{\gamma} \right) + (1 - \beta) \gamma T_{\gamma} \left( \frac{v_t}{\gamma} \right), \]
which further yields that
\[ (3.9) \quad \partial_t v_t = \frac{q}{p} v_t \left( \frac{v_t}{\gamma} \right)^{\frac{1}{\gamma}} P_{\gamma} \left[ \left( \frac{v_t}{\gamma} \right)^{-\frac{1}{\gamma}} L_{\beta} \left( \frac{v_t}{\gamma} \right) \right] + \frac{q}{p} (1 - \beta) v_t \left( \frac{v_t}{\gamma} \right)^{-\frac{1}{\gamma}} P_{\gamma} \left[ \left( \frac{v_t}{\gamma} \right)^{-\frac{1}{\gamma}} T_{\gamma} \left( \frac{v_t}{\gamma} \right) \right]. \]
On the other hand, in order to compute \( L_{\beta}^* v_{t} \), we appeal to the diffusion property of \( L_{\beta}^* \) which states that for \( \mu \in \mathbb{R} \setminus \{0\} \),
\[ (3.10) \quad L_{\beta}^* (f^\mu) = \mu f^{\mu - 1} L_{\beta}^* f - (\mu - 1) f^{\mu - 2} \Gamma(f), \]
and the commutation property
\[ (3.11) \quad L_{\beta} P_{\gamma} f = \beta P_{\gamma} L_{\beta} f + (\beta - 1) x \cdot \nabla (P_{\gamma} f). \]
A sequence of applications of (3.10), (3.8), and (3.11) reveals that
\[ L_{\beta,\alpha}^* v_t \]
\[ = \beta sv_t^{1-\frac{1}{\gamma}} \gamma^\frac{1}{\gamma} P_{\beta} \left[ \left( \frac{v_t}{\gamma} \right)^{\frac{1}{\gamma}} \right] + (\beta s - 1) q v_t^{1-\frac{1}{\gamma}} \gamma^\frac{1}{\gamma} x \cdot \nabla (P_{\gamma} \left[ \left( \frac{v_t}{\gamma} \right)^{\frac{1}{\gamma}} \right]) \]
\[ + (q - \beta_s) v_t^{1-\frac{1}{\gamma}} \gamma^\frac{1}{\gamma} T_{\gamma} q P_{\gamma} \left[ \left( \frac{v_t}{\gamma} \right)^{\frac{1}{\gamma}} \right] - (q - 1) n v_t + \beta_s q (q - 1) v_t^{1-\frac{1}{\gamma}} \gamma^\frac{1}{\gamma} \Gamma(v_t^{\frac{1}{\gamma}}). \]
We further apply the diffusion property of $\mathcal{L}$ to see that

\begin{equation}
\mathcal{L}^* \tilde{v}_t = \beta_s \frac{q}{p} \tilde{v}_t^{1-\frac{1}{\gamma}} \gamma_\gamma P_s \left[ \left( \frac{v_t}{\gamma} \right)^{\frac{1}{\gamma} - 1} \mathcal{L} \left( \frac{v_t}{\gamma} \right) \right] - \beta_s \frac{q}{p p'} \tilde{v}_t^{1-\frac{1}{\gamma}} \gamma_\gamma P_s \left[ \left( \frac{v_t}{\gamma} \right)^{\frac{1}{\gamma} - 2} \Gamma \left( \frac{v_t}{\gamma} \right) \right] \\
+ (\beta_s - 1) q \tilde{v}_t^{1-\frac{1}{\gamma}} \gamma_\gamma \nabla \left( P_s \left[ \left( \frac{v_t}{\gamma} \right)^{\frac{1}{\gamma}} \right] \right) \\
+ (q - \beta_s) \tilde{v}_t^{1-\frac{1}{\gamma}} \gamma_\gamma T_q P_s \left[ \left( \frac{v_t}{\gamma} \right)^{\frac{1}{\gamma}} \right] - (q - 1) n \tilde{v}_t + \beta_s q (q - 1) \tilde{v}_t^{1-\frac{2}{\gamma}} \Gamma \left( \tilde{v}_t^{\frac{1}{\gamma}} \right).
\end{equation}

We combine (3.9) and (3.12) and then rearrange terms to derive that

\begin{equation}
\frac{\partial_t \tilde{v}_t - \mathcal{L}^* \tilde{v}_t}{\tilde{v}_t^{1-\frac{1}{\gamma}} \gamma_\gamma} = \frac{q}{p} (\beta - \beta_s) P_s h_t P_s \left[ \left( \frac{v_t}{\gamma} \right)^{\frac{1}{\gamma} - 1} \Delta \left( \frac{v_t}{\gamma} \right) \right] + \frac{q}{p} (1 - \beta) P_s h_t P_s [h_t n] \\
+ (\beta_s - 1) P_s h_t \left[ \left( \frac{v_t}{\gamma} \right)^{\frac{1}{\gamma} - 2} \mathcal{L} \left( \frac{v_t}{\gamma} \right) \right] \\
+ q (\beta_s - 1) P_s h_t \nabla P_s h_t + \frac{q}{p} (\beta - 1) P_s h_t P_s [h_t [x]^2] \\
- (\beta_s - 1) P_s h_t \left[ \left( \frac{v_t}{\gamma} \right)^{\frac{1}{\gamma} - 2} \Gamma \left( \frac{v_t}{\gamma} \right) \right] - \frac{q}{p p'} \beta_s (p e)^2 \Gamma (P_s h_t).
\end{equation}

It now suffices to apply

\begin{equation}
\nabla \frac{v_t}{\gamma} = \frac{v_t}{\gamma} \nabla \left( \log \frac{v_t}{\gamma} \right), \quad \Delta \frac{v_t}{\gamma} = \frac{v_t}{\gamma} \Delta \left( \log \frac{v_t}{\gamma} \right) + \frac{v_t}{\gamma} \Gamma \left( \log \frac{v_t}{\gamma} \right)
\end{equation}

to obtain (3.3).

\[\square\]

\textbf{Proof of Lemma 3.3} Using the fact that

\begin{equation}
\nabla \left( \log \frac{v_t}{\gamma^\beta} \right) = \nabla \left( \log \frac{v_t}{\gamma} \right) + (1 - \frac{1}{\beta}) x,
\end{equation}

it is easy to see that

\begin{equation}
P_s h_t P_s [h_t \Gamma (\log \frac{v_t}{\gamma^\beta})] - P_s [h_t \nabla (\log \frac{v_t}{\gamma^\beta})] = I_1 + I_2 + I_3,
\end{equation}

where $I_1$ is defined in (3.2) and

\begin{equation}
I_2 := (1 - \frac{1}{\beta})^2 \left( P_s h_t P_s [h_t [x]^2] - P_s [h_t x] \cdot P_s [h_t x] \right),
\end{equation}

\begin{equation}
I_3 := -2 (1 - \frac{1}{\beta}) \left( P_s h_t P_s [h_t x \cdot \nabla \left( \log \frac{v_t}{\gamma^\beta} \right)] - P_s [h_t x] \cdot P_s [h_t \nabla \left( \log \frac{v_t}{\gamma^\beta} \right)] \right).
\end{equation}

For $I_2$, from the definition of $R(f, g)$ with $f = h_t$ and $g = h_t x$, we have that

\begin{equation}
I_2 = (1 - \frac{1}{\beta})^2 \left( (1 - e^{-2s}) P_s h_t P_s [h_t n] \right.
\end{equation}

\begin{equation}
+ \frac{1 - e^{-2K_s}}{p} \left( -\frac{1}{2} (1 - \frac{1}{\beta})^{-1} \right) I_3 + R(h_t, h_t x).\end{equation}
Similarly for $I_3$,

$$I_3 = -2(1 - \frac{1}{\beta}) \left( (1 - e^{-2s}) P_{h_t} P_s [h_t \Delta (\log \frac{v_t}{\gamma})] + \frac{1 - e^{-2s}}{p} \{ P_{h_t} P_s [h_t \Gamma (\log \frac{v_t}{\gamma})] - |P_s [h_t \nabla (\log \frac{v_t}{\gamma})]|^2 \} + R(h_t, h_t \nabla (\log \frac{v_t}{\gamma})) \right).$$

Hence, after a rearrangement, we arrive at

$$I_1 + I_2 + I_3 = (1 - \frac{1}{\beta})^2 (1 - e^{-2s}) P_{h_t} P_s [h_t n] + (1 - \frac{1}{\beta}) (1 - e^{-2s}) P_{h_t} P_s [h_t \Delta (\log \frac{v_t}{\gamma})]$$

$$+ ((1 - \frac{1}{\beta}) \frac{1 - e^{-2s}}{p} - 1)^2 I_1 + (1 - \frac{1}{\beta}) R(h_t, h_t \nabla (\log \frac{v_t}{\gamma})) + (1 - \frac{1}{\beta})^2 R(h_t, h_t x).$$

In view of the relation

$$\left( \frac{1 - \frac{1}{\beta}}{p} \right) (1 - e^{-2s}) = -\frac{1}{\beta} (\beta_s - \beta),$$

we can simplify to

$$I_1 + I_2 + I_3 = p(1 - \frac{1}{\beta}) (1 - \frac{1}{\beta}) P_{h_t} P_s [h_t n] - p(1 - \frac{1}{\beta}) (1 + \frac{\beta_s}{\beta}) P_{h_t} P_s [h_t \Delta (\log \frac{v_t}{\gamma})]$$

$$+ (\frac{\beta_s}{\beta})^2 I_1 - (1 + \frac{\beta_s}{\beta}) (1 - \frac{1}{\beta}) R(h_t, h_t \nabla (\log \frac{v_t}{\gamma})) + (1 - \frac{1}{\beta})^2 R(h_t, h_t x)$$

as desired. \qed

### 3.2. Proof of Theorems 1.2 and 1.4: hypercontractivity inequality.

**Proof of Theorem 1.4:** hypercontractivity inequality. Take arbitrary $v$ satisfying the assumptions in Theorem 1.4. To show (1.14) for this $v$, it suffices to show

$$\left\| P_s \left[ \frac{v}{\gamma} \right] \right\|_{L^q(\gamma)} \leq \left\| P_s \left[ \frac{\gamma v}{\gamma} \right] \right\|_{L^q(\gamma)} \left( \int_{\mathbb{R}^n} \frac{v}{\gamma} \, d\gamma \right)^{\frac{1}{p}}$$

thanks to (1.7). Recall the quantity $Q(t)$ defined in (3.1) and that our goal is to show that $Q(t)$ is monotone non-decreasing. To this end, we first note that

$$\frac{d}{dt} Q(t) = \int_{\mathbb{R}^n} \partial_t \tilde{v}(t, x) \, dx,$$
where the interchange of derivative and integral can be justified by observing (3.7) from the proof of Lemma 3.2. We further see from integration by parts that

\[ \frac{d}{dt}Q(t) = \int_{\mathbb{R}^n} \left( \partial_t \tilde{v}(t, x) - \mathcal{L}^s_{\beta} \tilde{v}(t, x) \right) dx. \]

To be precise, we implicitly used the fact that \( \tilde{v}(t, \cdot) \) satisfies (1.18) for each \( t > 0 \) in the above step. This can be justified by

\[ |x| \tilde{v}(t, x) \leq C(t)|x| \gamma(x) P_s \left[ \left( \frac{\beta}{\gamma} \right)^{\frac{1}{p}} \right] (x) q \rightarrow 0, \quad |x| \rightarrow \infty \]

and

\[ |\nabla \tilde{v}(t, x)| \leq C(t)(1 + |x|) \gamma(x) P_s \left[ \left( \frac{\beta}{\gamma} \right)^{\frac{1}{p}} \right] (x) q \rightarrow 0, \quad |x| \rightarrow \infty, \]

both of which follow from the argument in Lemma 2.2.

Thanks to the assumption on \( v \), we may apply Proposition 3.1 to see that

\[ \left( \frac{\beta}{\gamma} \right)^{\frac{1}{p}} \frac{1}{p} \frac{pp'}{q} \frac{d}{dt}Q(t) \geq \int_{\mathbb{R}^n} \tilde{v}^{1-\frac{2}{q}} \gamma^\frac{2}{q} \left( \left( \frac{\beta}{\gamma} \right)^{\frac{1}{p}} \frac{1}{p} \frac{pp'}{q} (I + N) + \left( 1 + \frac{\beta}{\gamma} \right)(1 - \frac{1}{\beta}) R(h_t, h_t \nabla ( \log \frac{v_t}{\gamma} )) - (1 - \frac{1}{\beta})^2 R(h_t, h_t x) \right) dx, \]

where we also used the Cauchy–Schwarz inequality to see that

\[ P_s h P_s [h \Gamma ( \log \frac{v_t}{\gamma})] - \left| P_s [h \nabla ( \log \frac{v_t}{\gamma})] \right|^2 \geq 0. \]

In the above expression, the term involving \( N \) vanishes. In fact, we see that

\[ \int_{\mathbb{R}^n} P_s \left[ \left( \frac{v_t}{\gamma} \right)^{\frac{1}{p}} \right] q |x|^2 d\gamma = - \int_{\mathbb{R}^n} P_s \left[ \left( \frac{v_t}{\gamma} \right)^{\frac{1}{p}} \right] q x \cdot \nabla \gamma dx \]

\[ = q \int_{\mathbb{R}^n} \left( \frac{v_t}{\gamma} \right)^{\frac{1}{p}} \nabla \left( \left( \frac{v_t}{\gamma} \right)^{\frac{1}{p}} \right) x d\gamma + \int_{\mathbb{R}^n} \frac{v_t}{\gamma} n d\gamma \]

\[ = q \int_{\mathbb{R}^n} \left( \frac{v_t}{\gamma} \right)^{-\frac{1}{p}} \nabla \left( \left( \frac{v_t}{\gamma} \right)^{\frac{1}{p}} \right) x d\gamma + \int_{\mathbb{R}^n} \frac{v_t}{\gamma} n dx \]

from which \( \int_{\mathbb{R}^n} \tilde{v}^{1-\frac{2}{q}} \gamma^\frac{2}{q} N dx = 0 \) follows. Hence we see that

(3.16) \[ \left( \frac{\beta}{\gamma} \right)^{\frac{1}{p}} \frac{1}{p} \frac{pp'}{q} \frac{d}{dt}Q(t) \geq \int_{\mathbb{R}^n} \tilde{v}^{1-\frac{2}{q}} \gamma^\frac{2}{q} \left( \left( \frac{\beta}{\gamma} \right)^{\frac{1}{p}} \frac{1}{p} \frac{pp'}{q} I + \left( 1 + \frac{\beta}{\gamma} \right)(1 - \frac{1}{\beta}) R(h_t, h_t \nabla ( \log \frac{v_t}{\gamma} )) - (1 - \frac{1}{\beta})^2 R(h_t, h_t x) \right) dx. \]
Next we appeal to a feature well-suited to the gaussian case, namely two identities

\[
\begin{align*}
\int_{\mathbb{R}^n} \nabla (P_s[(P_s h)^{q-2}P_s f]) \cdot g \, d\gamma \\
&= e^{-2s} \int_{\mathbb{R}^n} P_s[(P_s h)^{q-2}P_s \nabla f] \cdot g \, d\gamma \\
&\quad + e^{-2s(q-2)} \int_{\mathbb{R}^n} P_s[(P_s h)^{q-3}P_s fP_s \nabla h] \cdot g \, d\gamma,
\end{align*}
\]

for \( f, g = (g_j)_{j=1}^n, h \) and

\[
\begin{align*}
\int_{\mathbb{R}^n} \text{div} (P_s[(P_s h)^{q-2}P_s f]) g \, d\gamma \\
&= e^{-2s} \int_{\mathbb{R}^n} P_s[(P_s h)^{q-2}P_s \text{div} f] g \, d\gamma \\
&\quad + e^{-2s(q-2)} \int_{\mathbb{R}^n} P_s[(P_s h)^{q-3}P_s f \cdot P_s \nabla h] g \, d\gamma,
\end{align*}
\]

for \( f = (f_j)_{j=1}^n, g, h \). These two identities follow from the commutation property \( \nabla P_s = e^{-s}P_s \nabla \). As an immediate consequence from integration by parts, we obtain

\[
\begin{align*}
\int_{\mathbb{R}^n} P_s f P_s [g \cdot x] (P_s h)^{q-2} \, d\gamma \\
&= \int_{\mathbb{R}^n} P_s f P_s [\text{div} g] (P_s h)^{q-2} \, d\gamma + e^{-2s} \int_{\mathbb{R}^n} P_s \nabla f \cdot P_s g (P_s h)^{q-2} \, d\gamma \\
&\quad + e^{-2s(q-2)} \int_{\mathbb{R}^n} P_s f P_s g \cdot P_s \nabla h (P_s h)^{q-3} \, d\gamma,
\end{align*}
\]

and

\[
\begin{align*}
\int_{\mathbb{R}^n} P_s f \cdot P_s [gx] (P_s h)^{q-2} \, d\gamma \\
&= \int_{\mathbb{R}^n} P_s f \cdot P_s [\nabla g] (P_s h)^{q-2} \, d\gamma + e^{-2s} \int_{\mathbb{R}^n} P_s [\text{div} f] P_s g (P_s h)^{q-2} \, d\gamma \\
&\quad + e^{-2s(q-2)} \int_{\mathbb{R}^n} P_s f \cdot P_s \nabla h P_s g (P_s h)^{q-3} \, d\gamma.
\end{align*}
\]

These identities will be applied for appropriate inputs.

We focus on the term involving \( P_s \left[ h_t |x|^2 \right] \) which comes from \( I \) and \( R(h_t, h_t x) \). More precisely the contribution from such a term in (3.16) is

\[
\Upsilon_1(p, s, \beta) \int_{\mathbb{R}^n} P_s \left[ h_t |x|^2 \right] P_s h_t^{q-1} \, d\gamma,
\]

where

\[
\Upsilon_1(p, s, \beta) := \left( \frac{\beta}{\beta - 1} \right)^2 \frac{1}{\beta_s} \frac{1}{q} \frac{p}{p} (\beta - 1) - \left( 1 - \frac{1}{\beta} \right)^2.
\]
We apply (3.19) with \((f, g) = (h_t, h_t x)\) to this term to see that
\[
\frac{(\beta_s)^2}{\beta} \frac{1}{p} \frac{p^p'}{q} \frac{d}{dt} Q(t) \\
\geq \left( \frac{\beta_s}{\beta} \right)^2 \frac{1}{p} \frac{p^p'}{q} \bar{I} + (1 + \frac{\beta_s}{\beta})(1 - \frac{1}{\beta}) \int_{\mathbb{R}^n} R(h_t, h_t x \nabla (\log \frac{v_t}{\gamma})) P_s h_t^{q-2} d\gamma \\
- (1 - \frac{1}{\beta})^2 \int_{\mathbb{R}^n} \tilde{R}(h_t, h_t x) P_s h_t^{q-2} d\gamma,
\]
where
\[
\bar{I} := q(\beta - \beta_s) \int_{\mathbb{R}^n} P_s [h_t \Delta (\log \frac{v_t}{\gamma})] P_s h_t^{q-1} d\gamma \\
+ \frac{q}{p}(\beta - \beta_s) \int_{\mathbb{R}^n} P_s [h_t \Gamma (\log \frac{v_t}{\gamma})] P_s h_t^{q-1} d\gamma \\
+ \frac{q}{p} (\beta_s - 2\beta + 1 + \frac{\beta - 1}{p}) \int_{\mathbb{R}^n} P_s [h_t x \cdot \nabla (\log \frac{v_t}{\gamma})] P_s h_t^{q-1} d\gamma \\
+ \frac{q}{p}(\beta - 1) \frac{e^{-2s}}{p}(q - 1) \int_{\mathbb{R}^n} P_s [h_t x \nabla (\log \frac{v_t}{\gamma})] \cdot P_s [h_t x] P_s h_t^{q-2} d\gamma
\]
and
\[
\int_{\mathbb{R}^n} \tilde{R}(h_t, h_t x) P_s h_t^{q-2} d\gamma \\
:= e^{-2s} \int_{\mathbb{R}^n} P_s [\text{div} (h_t x)] P_s h_t^{q-1} d\gamma + \int_{\mathbb{R}^n} P_s [\nabla h_t] \cdot P_s [h_t x] P_s h_t^{q-2} d\gamma \\
+ e^{-2s}(q - 2) \int_{\mathbb{R}^n} P_s [h_t x] \cdot P_s [\nabla h_t] P_s h_t^{q-2} d\gamma - \int_{\mathbb{R}^n} |P_s [h_t x]|^2 P_s h_t^{q-2} d\gamma.
\]
We then apply (3.21) with \((f, g) = (h_t x, h_t x)\) to the term involving \(|P_s [h_t x]|^2\) to see that
\[
\int_{\mathbb{R}^n} \tilde{R}(h_t, h_t x) P_s h_t^{q-2} d\gamma = 0
\]
which shows that
\[
\frac{(\beta_s)^2}{\beta} \frac{1}{p} \frac{p^p'}{q} \frac{d}{dt} Q(t) \geq \left( \frac{\beta_s}{\beta} \right)^2 \frac{1}{p} \frac{p^p'}{q} \bar{I} \\
+ (1 + \frac{\beta_s}{\beta})(1 - \frac{1}{\beta}) \int_{\mathbb{R}^n} R(h_t, h_t x \nabla (\log \frac{v_t}{\gamma})) P_s h_t^{q-2} d\gamma.
\]
Next we focus on the term involving \(P_s [h_t x] \cdot P_s [h_t x \nabla \log \frac{v_t}{\gamma}]\). Such term on the right-hand side of (3.22) is
\[
\Upsilon_2(p, s, \beta) \int_{\mathbb{R}^n} P_s [h_t x \nabla (\log \frac{v_t}{\gamma})] \cdot P_s [h_t x] P_s h_t^{q-2} d\gamma,
\]
where
\[
\Upsilon_2(p, s, \beta) := \left( \frac{\beta_s}{\beta} \right)^2 \frac{1}{p} \frac{p^p'}{q} \frac{d}{dt} \frac{e^{-2s}}{p}(q - 1) - (1 + \frac{\beta_s}{\beta})(1 - \frac{1}{\beta}) = -(1 - \frac{1}{\beta}).
\]
Hence we may apply (3.20) with \((f, g) = (h_t \nabla \log \frac{v_t}{\gamma}, h_t)\) to see that

\[
(3.25) \quad (\beta_s) \frac{1}{\beta_s} \frac{pp'}{q} \frac{d}{dt} Q(t) \geq (\beta_s) \frac{1}{\beta_s} \frac{pp'}{q} T + (1 + \frac{\beta_s}{\beta} (1 - \frac{1}{\beta})) \int_{\mathbb{R}^n} R(h_t, h_t \nabla (\log \frac{v_t}{\gamma})) P_s h_t^{q-2} d\gamma,
\]

where

\[ T =: q(\beta - \beta_s) \int_{\mathbb{R}^n} P_s [h_t \Delta (\log \frac{v_t}{\gamma})] P_s h_t^{q-1} d\gamma \]

\[ + \frac{q}{p} (\beta_s - 2\beta + 1 + \frac{\beta - 1}{p}) \int_{\mathbb{R}^n} P_s [h_t x \cdot \nabla (\log \frac{v_t}{\gamma})] P_s h_t^{q-1} d\gamma \]

\[ + \frac{q}{p} (\beta - 1) e^{-2s} \frac{e^{-2s}}{p} (q - 1)(1 + e^{-2s}(q - 2)) \int_{\mathbb{R}^n} P_s [\nabla (\log \frac{v_t}{\gamma})] \cdot P_s [\nabla h_t] P_s h_t^{q-2} d\gamma \]

\[ + \frac{q}{p} (\beta - 1) e^{-2s} \frac{e^{-2s}}{p} (q - 1) e^{-2s} \int_{\mathbb{R}^n} P_s [\nabla (\log \frac{v_t}{\gamma})] P_s h_t^{q-1} d\gamma. \]

and

\[
\int_{\mathbb{R}^n} R(h_t, h_t \nabla (\log \frac{v_t}{\gamma})) P_s h_t^{q-2} d\gamma
\]

\[ := \int_{\mathbb{R}^n} P_s [h_t x \cdot \nabla (\log \frac{v_t}{\gamma})] P_s h_t^{q-1} d\gamma \]

\[ - \int_{\mathbb{R}^n} P_s [\nabla h_t] \cdot P_s [h_t \nabla (\log \frac{v_t}{\gamma})] P_s h_t^{q-2} d\gamma \]

\[ - e^{-2s} \int_{\mathbb{R}^n} P_s [\nabla (\log \frac{v_t}{\gamma})] P_s h_t^{q-1} d\gamma \]

\[ - e^{-2s} (q - 2) \int_{\mathbb{R}^n} P_s [h_t \nabla (\log \frac{v_t}{\gamma})] \cdot P_s [\nabla h_t] P_s h_t^{q-2} d\gamma \]

\[ - (1 - e^{-2s}) \int_{\mathbb{R}^n} \left( P_s h_t P_s [\nabla (\log \frac{v_t}{\gamma})] \right) \]

\[ - P_s [\nabla h_t] \cdot P_s [h_t \nabla (\log \frac{v_t}{\gamma})] P_s h_t^{q-2} d\gamma. \]

Finally we focus on the term involving \(P_s [h_t x \cdot \nabla \log \frac{v_t}{\gamma}]\). The contribution of such term on the right-hand side of (3.25) is

\[ \Upsilon_3(p, s, \beta) \int_{\mathbb{R}^n} P_s [h_t x \cdot \nabla \log \frac{v_t}{\gamma}] P_s h_t^{q-1} d\gamma, \]

where

\[
(3.26) \quad \Upsilon_3(p, s, \beta) = (p - 2 + e^{-2s} + \frac{(1 - e^{-2s})^2}{p} (1 - \frac{1}{\beta})) \frac{1}{p - 1} (1 - \frac{1}{\beta}).
\]
We may apply (3.19) with \((f, g) = (h_t, h_t \nabla \log \gamma_t)\) to this term. This, along with the fact that \((\beta - 1) \frac{1 - e^{-2t}}{\mu} = \beta - \beta_s\), allows us to conclude that
\[
\frac{d}{dt} Q(t) \geq 0.
\]
Hence we see that \(Q(t)\) is non-decreasing and, comparing \(t \to 0\) and \(t \to \infty\), we obtain
\[
\|P_s \left[ f \left( \frac{v_t}{\gamma} \right) \right] \|_{L^q(\gamma^s)} \leq \|P_s \left[ \frac{\gamma \beta_y}{\gamma} \right] \|_{L^q(\gamma^s)} \left( \int_{\mathbb{R}^n} \frac{v_t}{\gamma} \cdot \gamma \cdot \gamma \right) ^{\frac{q}{p}} ,
\]
since
\[
\lim_{t \to \infty} v_t = \left( \int_{\mathbb{R}^n} \frac{v}{\gamma} \cdot \gamma \right) \gamma \beta.
\]
This completes the proof of (3.14).

\[\square\]

Lemma 1.5 is formally enough to derive Theorem 1.2 from Theorem 1.4. The argument becomes rigorous once we address the assumption \(v \in L^2(\gamma^{-1})\).

Proof of Theorem 1.2. Take any \(v \in \text{FP}(\beta)\), that is, \(v = v_*(t_*, \cdot)\) and \(v_*\) is a solution to (1.13) with some non-negative finite measure \(\mu\). For the purpose of proving (1.14), we may suppose that \(\mu\) has compact support without loss of generality thanks to a standard approximation argument and the monotone convergence theorem. Then the explicit form of the solution
\[
v_* (t, x) = \frac{1}{4\pi \beta (1 - e^{-2t})} \int_{\mathbb{R}^n} e^{\frac{-|x - e^{-t}y|^2}{4\beta (1 - e^{-2t})}} \cdot \mu(y) \cdot \gamma \cdot \gamma \cdot \gamma \cdot \gamma \cdot \gamma \cdot \gamma \cdot \gamma \cdot \gamma
\]
yields that
\[
v(x) = v_*(t_*, x) \leq C_\beta \mu(\mathbb{R}^n) \left( 1_{|x| \leq 100r_\mu} + e^{-\frac{4r_\mu^2}{\beta}} \right) \in L^2(\gamma^{-1}) ,
\]
where \(r_\mu > 0\) is a radius of support of \(\mu\). Hence we may apply Theorem 1.5 to obtain (1.14) thanks to (1.17). \[\square\]

3.3. Proof of Theorem 1.3 and Theorem 1.4: Log-Sobolev inequality. The following lemma bridges our regularised hypercontractivity inequality to the LSI.

Lemma 3.4. Let \(q(s) = 1 + e^{2s} \in (2, \infty)\) for \(s \geq 0\). Assume that \(f \geq 0\) satisfy
\[
\|P_s \left[ f^\gamma \right] \|_{L^{q(s)}(\gamma)} \leq \psi(s) \|f\|_{L^1(\gamma)} ,
\]
for some positive function \(\psi(s) \in C^1((0, \infty))\) such that \(\psi(0) = 1\). Then for such \(f\), we have that
\[
\text{Ent}_\gamma(f) \leq \frac{1}{2} \lambda_1(f) + 2 \psi'(0) \|f\|_{L^1(\gamma)} ,
\]
where \(\lambda_1(f) = \frac{\|P_s \left[ f^\gamma \right] \|_{L^1(\gamma)}}{\psi(s)} ,\)
\(s \geq 0\).

Proof. We modify the argument of [6, Theorem 5.2.3] so that the dependence on \(\psi(s)\) can be tracked. Define
\[\Lambda(s) := \frac{\|P_s \left[ f^\gamma \right] \|_{L^{q(s)}(\gamma)}}{\psi(s)} ,\]
Note that $\Lambda(0) = \|f\|_{L^1(\gamma)}^\frac{1}{2}$ since $\psi(0) = 1$ and $q(0) = 2$, and hence the assumption (3.29) implies that $\Lambda'(0) \leq 0$. Now notice from direct calculations that

$$\Lambda'(0) = \frac{d}{ds} \left( \|P_s [f^\frac{1}{2}]\|_{q(s)} \right) |_{s=0} - \|f^\frac{1}{2}\|_2 \psi'(0).$$

In order to compute the first term, we invoke two formulae (see [6]):

$$\frac{d}{dq} \|h\|_{L^q(\gamma)} = \frac{1}{q} \left( \text{Ent}_\gamma(h^q) + \left( \int_{\mathbb{R}^n} h^q \, d\gamma \right) \log \left( \int_{\mathbb{R}^n} h^q \, d\gamma \right) \right),$$

$$\frac{d}{ds} \int_{\mathbb{R}^n} (P_s f)^q \, d\gamma = -q(q-1) \int_{\mathbb{R}^n} (P_s f)^q \left| \nabla P_s f \right|^2 \, d\gamma.$$

From the first formula, we see that

$$(3.29) \quad \frac{d}{dq} \|h\|_{L^q(\gamma)} = \frac{1}{q} \|h\|_q^{1-q} \text{Ent}_\gamma(h^q).$$

On the other hand, we see from $\frac{d}{ds} q(s) = 2e^{2s}$ (3.29) and the second formula that

$$\frac{d}{ds} \left( \|P_s [f^\frac{1}{2}]\|_{q(s)} \right) = - (q(s) - 1) \|P_s [f^\frac{1}{2}]\|_{1-q(s)} \left( \int_{\mathbb{R}^n} (P_s [f^\frac{1}{2}])(q(s) - 2) \left| \nabla P_s [f^\frac{1}{2}] \right|^2 \, d\gamma \right) + 2e^{2s} \frac{1}{q(s)^2} \|P_s [f^\frac{1}{2}]\|_{1-q(s)} \text{Ent}_\gamma(P_s [f^\frac{1}{2}])^{q(s)}.$$

This shows that

$$(3.30) \quad \frac{d}{ds} \left( \|P_s [f^\frac{1}{2}]\|_{q(s)} \right) |_{s=0} = \frac{1}{2} \|f\|_1^{-\frac{1}{2}} \left( -\frac{1}{2} \mathbb{L}_\gamma(f) + \text{Ent}_\gamma(f) \right)$$

and hence

$$\Lambda'(0) = -\|f^\frac{1}{2}\|_2^{-\frac{1}{2}} \left( \int_{\mathbb{R}^n} \left| \nabla [f^\frac{1}{2}] \right|^2 \, d\gamma \right) + 2 \frac{1}{4} \|f^\frac{1}{2}\|_2^{-1} \text{Ent}_\gamma([f^\frac{1}{2}]^2) - \|f\|_1^\frac{1}{2} \psi'(0)$$

$$= \frac{1}{2} \|f^\frac{1}{2}\|_2^{-\frac{1}{2}} \left( -\frac{1}{2} \mathbb{L}_\gamma(f) + \text{Ent}_\gamma(f) - 2 \psi'(0) \|f\|_1 \right)$$

which concludes the proof since $\Lambda'(0) \leq 0$. \qed

Proof of Theorem 1.4: LSI. Take an arbitrary $v$ satisfying the assumptions in Theorem 1.4. In view of the hypercontractivity inequality in Theorem 1.4, we apply Lemma 3.3 with

$$\psi(s) := \|P_v \left[ \left( \frac{\gamma}{\beta} \right)^\frac{1}{2} \right]\|_{L^{q(s)}(\gamma)} = \|P_v [f^\frac{1}{2}]\|_{L^{q(s)}(\gamma)},$$

where $f_* := \frac{\gamma}{\beta}$. From (3.30), we notice that

$$\psi'(0) = \frac{d}{ds} \|P_v [f^\frac{1}{2}]\|_{L^{q(s)}(\gamma)} |_{s=0} = \frac{1}{2} \|f_*\|_1^{-\frac{1}{2}} \left( -\frac{1}{2} \mathbb{L}_\gamma(f_*) + \text{Ent}_\gamma(f_*) \right) = \frac{n}{2} D(\beta).$$

Since $\|f_*\|_{L^1(\gamma)} = 1$, we conclude (1.15). \qed
3.4. Closure properties and reverse hypercontractivity. By making use of explicit formulae for the flows, it is possible to derive a somewhat stronger statement than the monotonicity of $Q(t)$ in terms of supersolutions of Fokker–Planck equations. In fact, we observe

\begin{equation}
(1 - e^{-2s})P_s \left[ h_t \Delta \left( \log \frac{v_t}{\gamma} \right) \right]
= P_s \left[ h_t x \cdot \nabla \left( \log \frac{v_t}{\gamma} \right) \right] - e^{-s} x \cdot P_s \left[ h_t \nabla \left( \log \frac{v_t}{\gamma} \right) \right] - \frac{1}{p} e^{-2s} P_s \left[ h_t \Gamma \left( \log \frac{v_t}{\gamma} \right) \right],
\end{equation}

and

\begin{equation}
\frac{(1 - e^{-2s})^2}{p} P_s \left[ h_t \Delta \left( \log \frac{v_t}{\gamma} \right) \right]
= P_s \left[ |x|^2 h_t \right] - e^{-2s} x^2 P_s h_t - (1 - e^{-2s}) P_s h_t - 2 \left( \frac{1 - e^{-2s}}{p} \right) e^{-s} x \cdot P_s \left[ h_t \nabla \left( \log \frac{v_t}{\gamma} \right) \right] - \frac{(1 - e^{-2s})^2}{p^2} P_s \left[ h_t \Gamma \left( \log \frac{v_t}{\gamma} \right) \right],
\end{equation}

for the term involving $h_t \Delta \log \frac{v_t}{\gamma}$, and also

\begin{equation}
\left| P_s \left[ x h_t \right] \right|^2
= e^{-2s} |x|^2 \left( P_s h_t \right)^2 + 2 e^{-s} \frac{1 - e^{-2s}}{p} P_s h_t x \cdot P_s \left[ h_t \nabla \left( \log \frac{v_t}{\gamma} \right) \right]
+ \left( \frac{1 - e^{-2s}}{p} \right)^2 \left| P_s \left[ h_t \nabla \left( \log \frac{v_t}{\gamma} \right) \right] \right|^2,
\end{equation}

and

\begin{equation}
P_s \left[ h_t \nabla \left( \log \frac{v_t}{\gamma} \right) \right] \cdot P_s \left[ x h_t \right]
= e^{-s} P_s h_t x \cdot P_s \left[ h_t \nabla \left( \log \frac{v_t}{\gamma} \right) \right] + \frac{1 - e^{-2s}}{p} \left| P_s \left[ h_t \nabla \left( \log \frac{v_t}{\gamma} \right) \right] \right|^2.
\end{equation}

Applying (3.31)–(3.34) with changes of variables, we can check that $I, N, R(h_t, h_t x)$, and $R(h_t, h_t \nabla(\log \frac{v_t}{\gamma}))$ all vanish identically. Combining this fact and Proposition 3.3.1 we may derive the following.

**Theorem 3.5.** Let $\beta > 0$, $s > 0$, and $1 < p < q < \infty$ satisfy $\frac{\beta - 1}{p - 1} = e^{2s}$. Suppose the twice differentiable $v : \mathbb{R}^n \to (0, \infty)$ is in $L^2(\gamma^{-1})$ and satisfies (1.16). Then

\begin{equation}
\partial_t v_t = \mathcal{L}_{\beta}^* v_t, \quad v_0 = v \Rightarrow \partial_t \tilde{v}_t \geq \mathcal{L}_{\beta}^* \tilde{v}_t,
\end{equation}

where $\beta_s > 0$ and $\tilde{v}_t$ are given by (1.38) and (3.1), respectively.

It is easy to see that Theorem 3.5 implies the monotonicity of $Q(t)$ and thus our regularised hypercontractivity inequality. Furthermore the property (3.35) can be slightly generalised in the following form

\begin{equation}
\partial_t v_t \geq \mathcal{L}_{\beta}^* v_t, \quad v_0 = v \Rightarrow \partial_t \tilde{v}_t \geq \mathcal{L}_{\beta}^* \tilde{v}_t,
\end{equation}

under appropriate smoothness and decay conditions on $v$. Observations of this nature emerged in [13][16] in the context of the sharp Young convolution inequality and Brascamp–Lieb inequalities. We follow these papers and refer to (3.36) as a
closure property (for supersolutions of Fokker–Planck equations) associated with
the operation \( v_t \mapsto \tilde{v}_t \). A closely related closure property for supersolutions of the
gaussian heat equation was investigated in [2] and, in fact, the result in [2] yields (3.36) in the special case \( \beta = 1 \). However there are critical differences distinguishing
the case \( \beta = 1 \) and the case \( \beta \neq 1 \). The most prominent and phenomenological
one is the fact that the closure property (3.35) or (3.36) for \( \beta \neq 1 \) cannot be true
unless one imposes some additional structure like (1.16). On the other hand, the
closure property for \( \beta = 1 \) holds true in general (under very mild decay condition
on \( v \)). It is also clear that the closure property for \( \beta = 1 \) is not enough to obtain
deficit estimate (1.14), and as we have seen, significant work is required to handle
the case \( \beta \neq 1 \).

In the case \( -\infty < q < p < 1 \), the hypercontractivity inequality is known to hold in reverse form. More precisely, for \( -\infty < q < p < 1 \) satisfying \( \frac{q-1}{p-1} = e^{2s} \), one has that

\[
\| P_s [f^{\beta}] \|_{L^q(\gamma)} \geq \left( \int_{\mathbb{R}^n} f \, d\gamma \right)^{\frac{1}{p}}
\]

for all positive \( f \in L^1(\gamma) \) and the constant is sharp. It is possible to improve the
constant in the spirit of Theorem 1.2. In fact, one can follow our argument for
1 < p < q to see that

\[
\frac{pp'}{q} (\partial_t \tilde{v}_t - \mathcal{L}_{\beta_s} \tilde{v}_t) \geq 0
\]
even when \( -\infty < q < p < 1 \) as long as \( \beta_s > 0 \). Notice also that \( \beta_s \geq 1 \) holds in
the following cases: (i) \( \beta \geq 1 \) and \( pq > 0 \), or (ii) \( 0 < \beta \leq 1 \) and \( pq < 0 \). As a
consequence, we obtain the following.

**Theorem 3.6.** Let \( \beta > 0 \), \( s > 0 \), and \( -\infty < q < p < 1 \) satisfy \( \frac{q-1}{p-1} = e^{2s} \),
\( p \neq 0, 1 - e^{-2s} \). Suppose that the twice differentiable function \( v : \mathbb{R}^n \to (0, \infty) \)
belongs to \( L^2(\gamma_{\beta_s}^{-1}) \) and satisfies (1.16).

(1) In the case \( p < 0 \), we suppose \( \beta \geq 1 \). Then

\[
\partial_t v_t = \mathcal{L}_{\beta_s}^* v_t, \quad v_0 = v \quad \Rightarrow \quad \partial_t \tilde{v}_t \geq \mathcal{L}_{\beta_s} \tilde{v}_t.
\]

(2) In the case \( 0 < p < 1 - e^{-2s} \), we suppose \( 0 < \beta \leq 1 \). Then

\[
\partial_t v_t = \mathcal{L}_{\beta_s}^* v_t, \quad v_0 = v \quad \Rightarrow \quad \partial_t \tilde{v}_t \geq \mathcal{L}_{\beta_s} \tilde{v}_t.
\]

(3) In the case \( 1 - e^{-2s} < p < 1 \), we suppose \( \beta \geq 1 \). Then

\[
\partial_t v_t = \mathcal{L}_{\beta_s}^* v_t, \quad v_0 = v \quad \Rightarrow \quad \partial_t \tilde{v}_t \leq \mathcal{L}_{\beta_s} \tilde{v}_t.
\]

Here \( \tilde{v}_t \) is given by (3.1) and \( \beta_s \) is given by (1.8).

By the same argument in the proof of Theorem 1.4 we obtain the following from
Theorem 3.6

**Theorem 3.7.** Let \( \beta > 0 \), \( s > 0 \), \( -\infty < q < p < 1 \) satisfy \( \frac{q-1}{p-1} = e^{2s} \) and
\( p \neq 0, 1 - e^{-2s} \). Suppose that the twice differentiable function \( v : \mathbb{R}^n \to (0, \infty) \)
belongs to \( L^2(\gamma_{\beta_s}^{-1}) \) and satisfies (1.16).
(1) In the case $pq > 0$, we suppose $\beta \geq 1$. Then

\begin{equation}
\|P_s [(\frac{\nu}{\gamma})^\frac{1}{\gamma}]\|_{L^q(\gamma)} \geq \|P_s [(\frac{\nu^\beta}{\gamma})^\frac{1}{\gamma}]\|_{L^q(\gamma)} \left( \int_{\mathbb{R}^n} \frac{\nu}{\gamma} d\gamma \right)^\frac{\beta}{p}.
\end{equation}

In particular, (3.38) holds for all $v \in \text{FP}(\beta)$.

(2) In the case $pq < 0$, we suppose $0 < \beta \leq 1$. Then (3.38) holds.

3.5. Further remarks on the proof of Theorem 1.4. The Fokker–Plank equation $\partial_t v = \mathcal{L}_\beta^* v$ has a close connection with the classical heat equation $\partial_t u = \beta \Delta u$ and this raises the obvious question of why we have opted to use Fokker–Planck flow as the basis for our main results. The short yet somewhat opaque answer is that Fokker–Planck flow seems to be significantly better suited to the semi-log-convexity/semi-log-concavity assumption (1.16) which underlies our main result in Theorem 1.4.

To explain this in more detail, for the sake of simplicity, let us consider the one-dimensional case $n = 1$. Our discussion here is based on the fact that one may write

\begin{equation}
\|P_s [f^\gamma]^\frac{1}{\gamma}\|_{L^q(\gamma)} = C \|f^\gamma\|_{L^q(dx)}
\end{equation}

for suitably chosen constants $C$ and $\lambda$ (depending on $p$ and $q$); see, for example, [12, Theorem 5]. As a result, the claim regarding the regularised hypercontractivity inequality (1.14) in Theorem 1.4 is equivalent to

\begin{equation}
\|v^\gamma \gamma \lambda\|_{L^q(dx)} \leq \|\gamma \lambda\|_{L^q(dx)}
\end{equation}

for $v$ satisfying

\begin{equation}
\begin{cases}
\nabla^2 \log v \leq -\frac{1}{\beta} \text{id} & \text{if } 0 < \beta \leq 1, \\
\nabla^2 \log v \geq -\frac{1}{\beta} \text{id} & \text{if } 1 \leq \beta
\end{cases}
\end{equation}

which are normalised by $\int_\mathbb{R} v \, dx = 1$. It is tempting to try to prove (3.39) using classical heat flow associated with the Laplacian on $\mathbb{R}$. That $v = \gamma \lambda$ is a maximiser indicates that it would be natural to evolve $v$ according to the classical heat equation

\begin{equation}
\partial_t u = \frac{\beta}{2} \partial_{xx} u, \quad u(t_0) = v
\end{equation}

since we have for any fixed $t_0 > 0$ that

\begin{equation}
(t - t_0)^{-\frac{1}{\beta/2}} u(t, (t - t_0)^{-\frac{1}{\beta/2}} y) \to \gamma \lambda(y) \quad (t \to \infty).
\end{equation}

Moreover, it follows from [16, Theorem 6] that

\begin{equation}
\tau \to (\tau + 1)^{-\frac{1}{p/2}} (\tau + 1)^{-\frac{1}{q/2}} \|u_1(t, \cdot)^{\frac{1}{p}} u_2(t, \cdot)^{\frac{1}{q}}\|_{L^q(dx)}
\end{equation}

gives rise to a non-decreasing functional if $\frac{1}{p} + \frac{1}{q} \geq 1 + \frac{1}{\beta} \frac{1}{q}$ and the $u_j$ are non-negative solutions of

\begin{equation}
\partial_t u_j = \frac{\beta_j}{2} \partial_{xx} u_j
\end{equation}

5For example, when $n = 1$ it is straightforward to verify that given a solution $v$ to $\partial_t v = \mathcal{L}_\beta^* v$ on $(0, \infty) \times \mathbb{R}$, if we define $u$ on $(0, \infty) \times \mathbb{R}$ by

\begin{equation}
u(\tau, y) = \frac{1}{(2\tau + 1)^{1/2}} \left( \frac{1}{2} \log(2\tau + 1), \frac{y}{(2\tau + 1)^{1/2}} \right)
\end{equation}

then $\partial_t u = \beta \partial_{yy} u$ holds with $u(0) = v(0)$. 
for appropriate diffusion parameters $\beta_1, \beta_2 > 0$ (we refer the reader to [15, Section 1.3] for the restrictions on the diffusion parameters). However, as far as we can tell, it does not seem clear how to make use of this result to prove (3.39) under the semi-log-convexity/semi-log-concavity assumption (3.40). The issue seems to be related to the preservation of semi-log-convexity/semi-log-concavity along the lines of Lemma 2.1 and mis-match of the $\beta_2$ diffusion parameter in (3.41) and the $\beta$ parameter in (3.40).

In the above sense, the semi-log-convexity/semi-log-concavity assumption (3.40) seems most harmonious with Fokker–Planck rather than classical heat flow. We also suspect that Fokker-Planck flow seems the more natural candidate for generalising the Ornstein–Uhlenbeck semigroup to more general Markov semigroups; see the forthcoming Section 5 for further details in this direction.

3.6. Proof of Corollary 1.7. Let us prove the statement (1). It suffices to show (1.23) since we may then obtain (1.24) via Lemma 3.4. Moreover, we may assume $B = \text{diag}(\beta_1, \ldots, \beta_n)$ since matters are rotationally invariant.

For the sake of simplicity, we discuss the case $n = 2$; the case $n \geq 3$ follows from by induction. Therefore, if we write $u := \gamma$, then our goal boils down to show

\[
\|P_s[u^\gamma]\|_{L^p(\gamma)} \leq \prod_{i=1}^{2} \beta_i^{\frac{1}{p'}} \left( \int_{\mathbb{R}^2} u \, d\gamma \right)^\frac{1}{p'}
\]

provided

\[
\nabla^2 \log u \geq \text{diag} (1 - \beta_1^{-1}, 1 - \beta_2^{-1}).
\]

For this, we employ the standard tensorising argument to reduce matters to the one-dimensional result in Theorem 1.4.

Firstly, we consider the case $\beta_1 \geq 1$ and $\beta_2 \geq 1$. Write

\[
P_s[u^\gamma](x_1, x_2) = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} u(y_1, y_2)^\gamma p_s(x_1, y_1) \, dy_1 \right) p_s(x_2, y_2) \, dy_2,
\]

where the integral kernel is given by

\[
p_s(x, y) := \frac{1}{(2\pi(1 - e^{-2s}))^\frac{n}{2}} e^{-\frac{1}{2} [e^{-s(x-y)^2} - 1 - e^{-2s}]} , \quad (x, y) \in \mathbb{R} \times \mathbb{R}.
\]

\[\text{If one is only interested in proving the weaker result in Theorem 1.2 (where } v \in \text{FP}(\beta) \text{ and } \beta \geq 1\text{), then an approach based on (3.39) and the results in [15] is possible by a careful choice of } r \text{ and the } \beta_i.\]
Then we use Minkowski’s integral inequality to see that
\[
\left\| P_s \left[ u^{\frac{1}{p}} \right] \right\|_{L^q(\gamma)}^q \\
= \int_{\mathbb{R}} \left\| \int_{\mathbb{R}} \left( \int_{\mathbb{R}} u(y_1, y_2)^{\frac{1}{p}} p_s(x_1, y_1) \, dy_1 \right) p_s(x_2, y_2) \, dy_2 \right\|_{L^q_s(\gamma)}^q \, d\gamma(x_2) \\
\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} u(y_1, y_2)^{\frac{1}{p}} p_s(x_1, y_1) \, dy_1 \right) p_s(x_2, y_2) \, dy_2 \right)^q \, d\gamma(x_2).
\]

For each fixed \( y_2 \), we know that
\[
\partial_{y_2}^2 \log u(y_1, y_2) \geq 1 - \frac{1}{\beta_1}
\]
from the assumption (3.43). Hence we may apply Theorem 1.4 on \( y \) to have
\[
\left\| \int_{\mathbb{R}} u(y_1, y_2)^{\frac{1}{p}} p_s(x_1, y_1) \, dy_1 \right\|_{L^q_s(\gamma)} \leq \beta_1^{-\frac{1}{p'}} \left( \int_{\mathbb{R}} u(x_1, x_2) \, d\gamma(x_1) \right)^{\frac{1}{p'}} \\
= \beta_1^{-\frac{1}{p'}} \left( \beta_1^{-\frac{1}{p'}} \int_{\mathbb{R}} V(x_2) \, d\gamma(x_2) \right)^{\frac{1}{p'}}
\]
which implies that
\[
\left\| P_s \left[ u^{\frac{1}{p}} \right] \right\|_{L^q(\gamma)} \leq \beta_1^{-\frac{1}{p'}} \left( \beta_1^{-\frac{1}{p'}} \int_{\mathbb{R}} V(x_2) \, d\gamma(x_2) \right)^{\frac{1}{p'}}
\]
We then notice from (3.43) that \( \partial_{x_2}^2 \log u(x_1, x_2) \geq 1 - \frac{1}{\beta_2} \) for each \( x_1 \) and hence
\[
\partial_{x_2}^2 \log V \geq 1 - \frac{1}{\beta_2}
\]
since the superposition preserves the log-convexity, see for instance the proof of Lemma 1.3 in [37]. This allows us to apply Theorem 1.4 on \( \mathbb{R} \) again to conclude
\[
\left\| P_s \left[ u^{\frac{1}{p}} \right] \right\|_{L^q(\gamma)} \leq \left( \prod_{i=1,2} \beta_i^{-\frac{1}{p'}} \right)^q \left( \int_{\mathbb{R}} V(x_2) \, d\gamma(x_2) \right)^{\frac{1}{p'}} \\
= \left( \prod_{i=1,2} \beta_i^{-\frac{1}{p'}} \right)^q \left( \int_{\mathbb{R}} u \, d\gamma \right)^{\frac{1}{p'}}
\]
as desired.

The case where either \( \beta_1 \) or \( \beta_2 \) (or both) are less than or equal to 1 may be handled in a similar manner. For example, if \( \beta_1 \leq 1 \) then rather than applying Theorem 1.4 to \( y_1 \mapsto u(y_1, y_2) \), we simply apply classical hypercontractivity with constant 1.

Statement (2) can be proved in a similar way except a little more care is needed at the point where we obtained the semi-log-convexity of \( V \). To show statement (2), we need to ensure the semi-log-concavity of \( V \). Although superposition does not always preserve the log-concavity, we can obtain the semi-log-concavity of \( V \) as in the argument in Lemma 2.1 based on the Prékopa–Leindler inequality. \( \square \)
4. Applications

In this section we capitalise on the wealth of connections that the hypercontractivity inequality and LSI enjoy with other important inequalities. By applying our main result in Theorem 1.4 along with the reverse hypercontractivity inequalities established in Theorem 3.7, we obtain certain improved versions of the hypercontractivity inequality for the Hamilton–Jacobi equation, the dual version of Talagrand’s inequality, the gaussian Poincaré inequality, and Beckner’s inequality. Also, at the end of the section we discuss improved versions of the dual form of hypercontractivity for the Ornstein–Uhlenbeck semigroup (viewed as a special case of the Brascamp–Lieb inequality). The underlying arguments we use in the proofs in this section are not novel and so our presentation will be somewhat terse.

4.1. Hypercontractivity of the Hamilton–Jacobi equation and Talagrand’s inequality. Hypercontractivity for the Hamilton–Jacobi equation

\[
\begin{align*}
\partial_\tau u + \frac{1}{2} |\nabla u|^2 &= 0, \quad (\tau, x) \in (0, \infty) \times \mathbb{R}^n, \\
u(0, \cdot) &= f, \quad x \in \mathbb{R}^n,
\end{align*}
\]

was investigated by Bobkov–Gentil–Ledoux in [20] and a close link with (1.2) (or, alternatively, (1.3)) was observed. To describe this, we introduce the so-called Hopf–Lax formula

\[
Q_\tau f(x) := \inf_{y \in \mathbb{R}^n} \left\{ f(y) + \frac{1}{2\tau} |x - y|^2 \right\}, \quad \tau > 0, \ x \in \mathbb{R}^n
\]

for any measurable and real-valued function \( f \) on \( \mathbb{R}^n \). There are a large number of references concerning the theory of the Hamilton–Jacobi equation and we refer the reader to, for example, Evans [39] for an excellent introduction to the theory. In particular, it is known that if \( f \) is Lipschitz continuous then Hopf–Lax formula \( Q_\tau f \) is also Lipschitz continuous and solves the initial value problem (4.1) in an almost everywhere sense (see [39 Theorem 6 on p. 128]).

Obtaining a suitable framework in which one has uniqueness of “weak” solutions to Hamilton–Jacobi equations is a highly non-trivial task (classical solutions do not exist in general). For the particular case (4.1) we consider here, [39 Theorem 8 on p. 134] is one possibility in which the notion of weak solution includes the condition that \( \partial_\tau u + \frac{1}{2} |\nabla u|^2 = 0 \) holds in an almost everywhere sense and \( u \) satisfies a certain “semi-convexity” condition. For more general Hamilton–Jacobi equations, a more suitable framework involves the notion of viscosity solutions and goes back to work of Crandall–Lions [34] (see also Crandall–Evans–Lions [34]). In this framework, if we assume that \( f \) is Lipschitz continuous and bounded on \( \mathbb{R}^n \), then \( Q_\tau f \) is the unique viscosity solution to (4.1) (see, for example, [39 Theorem 3 on p. 601]). Results of this nature have also been obtained for wider classes of initial data and, for later use, we note that Alvarez, Barron and Ishii handled lower semi-continuous initial data which are bounded below by a function of linear growth.

**Theorem 4.1** *(Theorem 5.2 in [1]).* Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) is lower semi-continuous and satisfies

\[
f(x) \geq -C(1 + |x|), \quad x \in \mathbb{R}^n
\]
for some $C > 0$. Then $Q_{\tau} f$ is the unique lower semi-continuous viscosity solution to (4.1).

In the case of gaussian measure, hypercontractivity of the Hamilton–Jacobi equation takes the form

(4.3) $\|e^{Q_{\tau} f}\|_{L^{s+\tau}(\gamma)} \leq \|e^{f}\|_{L^{s}(\gamma)}$

for all $f \in L^\infty(\mathbb{R}^n)$, $\tau > 0$, and $a \in \mathbb{R}$. Offering two approaches, Bobkov, Gentil and Ledoux obtained (4.3) in [20]; one is based on ideas of Gross [49] and a second approach is based on the so-called vanishing viscosity method.

Our aim here is to improve the constant in (4.3) in the spirit of Theorem 1.4, and it turns out that we are able to handle initial data which are uniformly convex in the sense that

$\nabla^2 f \geq (1 - \frac{1}{\beta}) \text{id}$

for some $\beta \geq 1$ (and thus, in particular, are uniformly bounded from below). As in [6], we adopt the vanishing viscosity method. The point is that, for each $\varepsilon > 0$, the solution of

$\partial_{\tau} u^\varepsilon_{\tau} - \varepsilon \mathcal{L} u^\varepsilon_{\tau} + \frac{1}{2} |\nabla u^\varepsilon_{\tau}|^2 = 0, \quad u^\varepsilon(0, \cdot) = f,$

which is a certain regularisation of (4.1), can be expressed as

(4.4) $u^\varepsilon_{\tau} := -2\varepsilon \log P_{\varepsilon \tau}[e^{-\frac{f}{\varepsilon}}],$

and thus one can derive information from hypercontractivity inequalities for the Ornstein–Uhlenbeck semigroup. Moreover, we expect to have

$\lim_{\varepsilon \downarrow 0} u^\varepsilon_{\tau} = Q_{\tau} f$

in some appropriate sense. Whilst there are important papers which investigate this type of problem in much more generality (see, for example, [14] and references therein), it shall suffice for us to establish

(4.5) $\lim \inf_{\varepsilon \downarrow 0} u^\varepsilon_{\tau}(x) \geq Q_{\tau} f(x)$

for each $x \in \mathbb{R}^n$, under the assumption that $f$ is lower semi-continuous and bounded from below. To see this, note that

$P_{\varepsilon \tau}[e^{-\frac{f}{\varepsilon}}](x)^{2\varepsilon} \leq C_{\varepsilon} \left( \int_{\mathbb{R}^n} e^{g^\varepsilon_{\tau}(x,y)} dy \right)^{2\varepsilon} \|e^{g^\varepsilon_{\tau}(x,y)}\|_{L^\infty(dy)}^{1-2\varepsilon}$

$\leq C_{\varepsilon} \left( \int_{\mathbb{R}^n} e^{-f(y)} dy \right)^{2\varepsilon} \|e^{g^\varepsilon_{\tau}(x,y)}\|_{L^\infty(dy)}^{1-2\varepsilon}$

It is instructive to note that hypercontractivity for the Ornstein–Uhlenbeck semigroup (1.2) can be equivalently reformulated in exponential form as $\|e^{P_{s} f}\|_{L^{2}(\gamma)} \leq \|e^{f}\|_{L^{2}(\gamma)}$ for $s \geq 0$ (see, for example, [5]), or equivalently again, as $\|e^{P_{s} f}\|_{L^{2}(\gamma)} \leq \|e^{s f}\|_{L^{2}(\gamma)}$ for $a, s \geq 0$. Here, all norms are taken with respect to the gaussian measure $\gamma$.

The latter approach was used in [33] and gave rise to the terminology viscosity solutions.

To see this, note that $w = \exp(-\frac{f}{\varepsilon})$ solves the gaussian heat equation $\partial_{\tau} w = \varepsilon \mathcal{L} w, w(0) = \exp(-\frac{f}{\varepsilon}).$
where \( C_\varepsilon = (2\pi(1 - e^{-2\varepsilon\tau}))^{-\varepsilon n} \), and
\[
g_\varepsilon^x(x, y) := -f(y) - \frac{2\varepsilon\tau}{2\tau(1 - e^{-2\varepsilon\tau})}|y - e^{-\varepsilon\tau}x|^2.
\]
We have
\[
\sup_{y \in \mathbb{R}^n} g_\varepsilon^x(x, y) \leq \sup_{y \in \mathbb{R}^n} \{-f(y) - \frac{1}{2\tau}|y - e^{-\varepsilon\tau}x|^2\} = -Q_\tau f(e^{-\varepsilon\tau}x).
\]

Since we are assuming that \( f \) is lower semi-continuous and bounded from below, by Theorem 4.1 we know that \( Q_\tau f \) is lower semi-continuous and therefore
\[
\liminf_{\varepsilon \downarrow 0} Q_\tau f(e^{-\varepsilon\tau}x) \geq Q_\tau f(x).
\]
From the above we obtain
\[
\limsup_{\varepsilon \downarrow 0} P_\varepsilon \tau \left[e^{-\varepsilon\tau}f\right](x)^{2\varepsilon} \leq e^{-Q_\tau f(x)}
\]
and hence (4.5).

Armed with the above observations, we obtain the following.

**Theorem 4.2.** Suppose \( a, \tau > 0 \) and \( \beta \geq 1 \) satisfy

\[
(4.6) \quad \beta(1 - \frac{1}{a}) < 1.
\]
Let \( \beta(a) \geq 1 \) be given by
\[
\frac{1}{\beta(a)} := 1 - a(1 - \frac{1}{\beta})
\]
For all \( f : \mathbb{R}^n \to \mathbb{R} \) such that

\[
(4.7) \quad \nabla^2 f \geq (1 - \frac{1}{\beta})\text{id}, \quad \int_{\mathbb{R}^n} e^{2af} \frac{\gamma}{\gamma^{\beta(\gamma)}} \, d\gamma < \infty,
\]
we have that

\[
(4.8) \quad \|e^{Q_\tau f}\|_{L^{a+\tau}(\gamma)} \leq \|e^{Q_\tau f[\frac{1}{\beta} \log \frac{2\varepsilon(\gamma)}{\gamma^{\beta(\gamma)}}]}\|_{L^{a+\tau}(\gamma)} \|e^f\|_{L^a(\gamma)}.
\]

In particular, (4.8) holds for all \( f = \frac{1}{a} \log \frac{\gamma}{v} \) where \( v \in \text{FP}(\beta(a)) \).

**Proof.** Let \( a, \tau > 0 \) be fixed and set \( b_\varepsilon > 0 \) so that

\[
(4.9) \quad \frac{1 + 2\varepsilon b_\varepsilon}{1 + 2\varepsilon a} = e^{2\varepsilon\tau}, \quad \varepsilon > 0.
\]
Note that we have \( \lim_{\varepsilon \downarrow 0} b_\varepsilon = a + \tau \) under this choice. Setting
\[
u_\varepsilon := -2\varepsilon \log P_\varepsilon \tau \left[e^{-\frac{\gamma}{\varepsilon}}\right]
\]
and
\[
q = -2\varepsilon b_\varepsilon, \quad p = -2\varepsilon a < 0, \quad s = \varepsilon\tau,
\]
we have \( \frac{q}{p + \tau} = e^{2s} \) thanks to (4.9), and also
\[
\|e^{u_\varepsilon^x}\|_{L^{b_\varepsilon}(\gamma)} = \left( \int_{\mathbb{R}^n} P_s [e^{-\frac{\gamma}{\varepsilon}}]^q \, d\gamma \right)^{-\frac{2s}{q}} = \left( \int_{\mathbb{R}^n} P_s [e^{(a^f)\gamma}]^q \, d\gamma \right)^{-\frac{2s}{q}}.
\]
If we regard $e^{a\varepsilon} = \frac{e^{\varepsilon}}{\gamma}$, then the assumption \((4.7)\) can be read as

$$\nabla^2 \log v = \nabla^2 a f - \text{id} \geq (a(1 - \frac{1}{\beta}) - 1) \text{id}, \quad v \in L^2(\gamma_{\beta(a)}).$$

Since $\beta(a) \geq 1$, we may apply Theorem \[5.7\] to see that

$$\|e^{a\varepsilon}\|_{L^p(\gamma)} \leq \|P\left[\left(\frac{\gamma_{\beta(a)}}{\gamma}\right)^{\frac{1}{\gamma}}\right]^{\frac{2}{\gamma}}\left(\int e^{a\varepsilon} \, d\gamma\right)^{\frac{1}{\gamma}} = \|P_{\tau} \left[\left(\frac{\gamma_{\beta(a)}}{\gamma}\right)^{\frac{1}{\gamma}}\right]^{-2\varepsilon} - 2\|e^{\varepsilon} \|_{L^p(\gamma)}\|e^f \|_{L^p(\gamma)}.$$

Formally, taking $\varepsilon \to 0$ we deduce the desired estimate \((4.8)\). To make this rigorous, we may invoke Fatou’s lemma and \(4.5\).

Hypercontractivity for the Hamilton–Jacobi equation \((4.3)\) is very closely related to the quadratic transportation cost inequality, or Talagrand’s inequality, which states that

$$(4.10) \quad \frac{1}{2} W_2(\gamma, v)^2 \leq \text{Ent}\left(\frac{v}{\gamma}\right)$$

for $v : \mathbb{R}^n \to (0, \infty)$ satisfying $\int v \, dx = 1$ and $\int |x|^2 v \, dx < \infty$. Here, $W_2(\cdot, \cdot)$ is the quadratic Wasserstein distance; see Subsection \[5.3\] for details. We are interested in the dual form of \((4.10)\) here. As observed in the work of Bobkov–Götze \[21\], \((4.10)\) is indeed equivalent to

$$(4.11) \quad \int_{\mathbb{R}^n} e^{Q_\tau f} \, d\gamma \leq e^{\int_{\mathbb{R}^n} f \, d\gamma}$$

for all bounded and continuous $f : \mathbb{R}^n \to \mathbb{R}$, and this proceeds by taking $\tau = 1$ and $a \to 0$ in \(4.3\). Given our improvement of \((4.3)\) in Theorem \[4.2\] it is not a surprise that we obtain an improvement of Talagrand’s inequality too.

**Corollary 4.3.** Let $\tau > 0$ and $\beta > 1$. Then

$$\|e^{Q_\tau f}\|_{L^p(\gamma)} \leq T(\tau, \beta) e^{\int_{\mathbb{R}^n} f \, d\gamma}$$

whenever $f : \mathbb{R}^n \to \mathbb{R}$ satisfies \((4.7)\) for all sufficiently small $0 < a \ll 1$, and where

$$T(\tau, \beta) := \left(e^{1 + \tau(1 - \frac{1}{\beta})} \right)^{1 - \frac{1}{\beta}} \in (0, 1).$$

**Proof.** By taking a limit $a \to 0$ in \((4.8)\), it suffices to show that

$$(4.12) \quad \lim_{a \to 0} \|e^{Q_\tau \left[\frac{1}{a} \log \frac{\gamma_{\beta(a)}}{\gamma}\right]}\|_{L^{n+\tau}(\gamma)} = T(\tau, \beta).$$

To this end, we first notice from direct computation that

$$Q_\tau \left[\frac{1}{a} \log \frac{\gamma_{\beta(a)}}{\gamma}\right](x) = \frac{1}{2\tau} \frac{\delta_{a,\alpha}}{\delta_{a,\alpha} + 1} |x|^2 - \frac{n}{2\tau} \log \alpha, \quad \delta_{a,\alpha} := \frac{\tau}{a} (1 - \frac{1}{\alpha})$$

holds true in general for $\tau, a > 0$ and $\alpha \geq 1$. Hence

$$\|e^{Q_\tau \left[\frac{1}{a} \log \frac{\gamma_{\beta(a)}}{\gamma}\right]}\|_{L^{n+\tau}(\gamma)} = \alpha^{\frac{n}{2\tau}} \left(\frac{\alpha}{\tau} + 1\right) \frac{1}{\delta_{a,\alpha} + 1} - \frac{a}{\tau} \left(\frac{\alpha}{\tau} + 1\right)^{\frac{n}{2\tau}}$$
as long as \((\frac{n}{\alpha} + 1) - \frac{n}{\beta} > 0\). By taking \(\alpha = \beta(a) \geq 1\), we see that \(\delta_{a,\beta} = \tau(1 - \frac{1}{\beta})\) and hence, recalling the definition of \(\beta(a)\) in (4.8),

\[
\lim_{a \to 0} \left\| e^{Q_{\tau}} \left[ \frac{1}{a} \log \frac{1}{a} \right] \right\|_{L^{1+\tau}(\gamma)} = \lim_{a \to 0} \left( 1 - a(1 - \frac{1}{\beta}) \right)^{\frac{1}{\tau}} \left( \frac{a}{\tau} + 1 \frac{1}{\tau(1 - \frac{1}{\beta})} + \frac{a}{\tau} \right)^{-\frac{n}{\tau(1 + \beta)}} = e^{-\frac{2}{\beta} \left( 1 + \tau - \frac{1}{\beta} \right)^{\frac{1}{\tau}}} = T(\tau, \beta)
\]
as claimed. \(\square\)

Given the improvement of the dual form of Talagrand’s inequality in Corollary 4.3, one might wonder if a similar improvement of the original version of Talagrand’s inequality is possible or not. We will address this question in Section 5.3 via a mass-transport approach.

4.2. Poincaré and Beckner inequalities. Next we consider improvements of the gaussian Poincaré inequality

\[(4.13) \int_{\mathbb{R}^n} f^2 \, d\gamma - \left( \int_{\mathbb{R}^n} |f| \, d\gamma \right)^2 \leq \int_{\mathbb{R}^n} |\nabla f|^2 \, d\gamma \]

for real-valued functions \(f \in L^2(\gamma)\) whose gradient belongs to \(L^2(\gamma)\).

Corollary 4.4. Let \(\beta > 0\). Then

\[(4.14) \frac{1}{2} \left( 1 + \frac{n}{2} D(\beta) \right) \int_{\mathbb{R}^n} f^2 \, d\gamma - \frac{1}{2} \left( \int_{\mathbb{R}^n} |f| \, d\gamma \right)^2 \leq \int_{\mathbb{R}^n} |\nabla f|^2 \, d\gamma \]

whenever \(\gamma f^2\) satisfies (1.16) and belongs to \(L^2(\gamma^\beta)^{-1})\).

Remark. Note that this result offers a gain over the classical inequality (4.13) for sufficiently small or sufficiently large \(\beta\). More precisely, this is possible when \(D(\beta) > \frac{2}{n}\) or equivalently, in terms of Lambert \(W\) functions, when

\[
\beta \notin \left[ -\frac{1}{W_1(e^{-1+2/m})}, -\frac{1}{W_0(e^{-1+2/m})} \right].
\]

Proof of Corollary 4.4. We will prove a more general inequality. For all \(p \in [1, 2)\) we show

\[(4.15) \|f\|_{L^2(\gamma)}^2 \log \frac{\|f\|_{L^2(\gamma)}}{\|f\|_{L^p(\gamma)}} \leq \frac{2(2 - p)}{p} \int_{\mathbb{R}^n} |\nabla f|^2 \, d\gamma - \frac{2 - p}{p} \|D_n(\beta)\| f^2 \|_{L^2(\gamma)},
\]

from which we can see that

\[
\|f\|_{L^2(\gamma)}^2 - \|f\|_{L^p(\gamma)}^2 \leq \frac{2(2 - p)}{p} \int_{\mathbb{R}^n} |\nabla f|^2 \, d\gamma - \frac{2 - p}{p} \|D_n(\beta)\| f^2 \|_{L^2(\gamma)},
\]
since \(x \log x \geq x - 1\). By choosing \(p = 1\), we clearly derive (4.14).

To prove (4.15), we appeal to the argument of [6, Proposition 5.1.8]. Let us introduce \(\phi(r) := \log (\|f\|_{L^p(\gamma)}), r \in (0, 1]\). This \(\phi\) is convex thanks to Hölder’s
inequality and hence, for \( p \in [1, 2) \), we obtain
\[
\frac{1}{2} \log \| f \|_{L^2_p(\gamma)}^2 = \phi\left( \frac{1}{p} \right) - \phi\left( \frac{1}{2} \right) \geq \left( \frac{1}{p} - \frac{1}{2} \right) \phi'\left( \frac{1}{2} \right)
\]
\[
= \frac{1}{2} - \frac{1}{p} \text{Ent}_{\gamma}(f^2)
\]
\[
\geq \frac{1}{2} - \frac{1}{p} \| f \|_{L^2_p(\gamma)}^2 \left( \frac{1}{2} I_{\gamma}(f^2) - \frac{n}{2} D(\beta) \| f \|_{L^2_p(\gamma)}^2 \right)
\]
through use of Theorem 1.4. Since \( I_{\gamma}(f^2) = 4 \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma \), we quickly obtain (4.15).

By Beckner’s inequality [13] for each \( p \in [1, 2) \),
\[
\int_{\mathbb{R}^n} f^2 d\gamma - \left( \int_{\mathbb{R}^n} |f|^p d\gamma \right)^{\frac{2}{p}} \leq \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma
\]
is a generalisation of the Poincaré inequality (take \( p = 1 \)) and recovers the LSI in (1.3) (take \( p \to 2 \)). Beckner obtained (4.16) by proving
\[
\int_{\mathbb{R}^n} f^2 d\gamma - \int_{\mathbb{R}^n} |P_s f|^2 d\gamma \leq (1 - e^{-2s}) \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma
\]
and combining this with hypercontractivity of the Ornstein–Uhlenbeck semigroup [1.2]. In the same manner, replacing (1.2) with our result in Theorem 1.4, we obtain the following.

**Corollary 4.5.** Let \( \beta > 0 \), \( 1 < p < 2 \) and \( s := -\frac{1}{2} \log (p - 1) \). Then
\[
\frac{1}{2 - p} \left[ \int_{\mathbb{R}^n} f^2 d\gamma - B(p, \beta) \left( \int_{\mathbb{R}^n} |f|^p d\gamma \right)^{\frac{2}{p}} \right] \leq \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma
\]
whenever \( \gamma f^p \) satisfies (1.16) and belongs to \( L^2(\gamma^{-1}) \). Here,
\[
B(p, \beta) := \int_{\mathbb{R}^n} |P_s \left[ (\frac{2\beta}{\gamma_1})^{\frac{1}{p}} \right]^2 \gamma |^2 d\gamma = \beta^\frac{2}{p} \left( 1 + (\beta - 1) \frac{2}{p} \right)^{-\frac{2}{p}}.
\]

**Remark.** We have \( B(p, \beta) < 1 \) whenever \( \beta > 1 \) and \( p \in (1, 2) \) and so we see an improvement over (4.10) in such cases. Also, it is easy to verify that \( \partial_p B(2, \beta) = \frac{2}{4} D(\beta) \) and thus, as one would expect, the limiting case \( p \to 2 \) in Corollary 4.5 recovers our regularised LSI in Theorem 1.4 (and thus gives a slightly different route compared to that in Section 3.3). In the limiting case \( p \to 1 \), one sees that \( B(p, \beta) \to 1 \) and hence Corollary 4.5 does not yield any improvement for the Poincaré inequality.

### 4.3. Regularised hypercontractivity in dual form.

Via duality, the hypercontractivity inequality (1.2) on \( \mathbb{R}^n \) (for simplicity) can be equivalently viewed as a special case of Brascamp–Lieb inequality
\[
\int_{\mathbb{R}^2} e^{-\pi x \cdot Q x} \prod_{j=1,2} f_j(L_j x)^{c_j} \, dx \leq H(c_1, c_2) \prod_{j=1,2} \left( \int_{\mathbb{R}} f_j \right)^{c_j}
\]
for nonnegative $f_1, f_2 \in L^1(\mathbb{R})$. Here, the positive semi-definite transformation $Q$ and exponents $c_1, c_2 \in (0, 1)$ are given by
\begin{equation}
Q := \frac{1}{2\pi(1 - e^{-2s})} \begin{pmatrix}
(1 - (1 - e^{-2s})^{\frac{1}{p}} & -e^{-s} \\
-e^{-s} & 1 - (1 - e^{-2s})^{\frac{1}{q'}}
\end{pmatrix}, \quad c_1 := \frac{1}{p}, \quad c_2 := \frac{1}{q'}.
\end{equation}
The condition $e^{2s} = \frac{q - 1}{p - 1}$ is equivalent to
\begin{equation}
c_1 + c_2 = 1 + (1 - e^{-2s})c_1c_2,
\end{equation}
and the constant $H(c_1, c_2)$ is given by
\begin{equation}
H(c_1, c_2) := (2\pi)^{1 - \frac{1}{2}(c_1 + c_2)} \sqrt{1 - e^{-2s}}.
\end{equation}
We note that in order to obtain (4.18) from (1.2), it suffices to choose
\begin{equation}
f = \left(\frac{f_1}{\gamma}\right)^{\frac{1}{\gamma}}
\end{equation}
and apply Hölder’s inequality. Similarly, the reverse hypercontractivity inequality (3.37) is equivalent to the inverse Brascamp–Lieb inequality
\begin{equation}
\int_{\mathbb{R}^N} e^{-\pi x \cdot Q x} \prod_{j=1}^2 f_j(L_j x)^{c_j} dx \geq H(c_1, c_2) \prod_{j=1, 2} \left(\int_{\mathbb{R}} f_j\right)^{c_j}
\end{equation}
for all positive $f_1, f_2 \in L^1(\mathbb{R})$. Here, we reuse the same notation as above, but we note that positive semi-definiteness of $Q$ and the positivity of $c_1, c_2$ do not necessarily persist in this case; we refer to [10, 11] for further discussion along these lines.

The equivalences alluded to above rest on the fact that we are referring to general input functions. When restricting to special classes of input functions there is no reason to expect such equivalences to continue to be valid and typically one might expect the dual version to be weaker. For instance, it would be reasonable to consider the dual statement (4.18) where both $f_1$ and $f_2$ are regularised in a similar manner. Indeed, this would fit into the framework of regularised Brascamp–Lieb inequalities established by Bennett–Carbery–Christ–Tao [17] using heat flow (see also [65] for an alternative approach based on mass transportation, and also [18] for regularised inverse Brascamp–Lieb inequalities). Obtaining (4.18) for such inputs from (1.14) simply relies on Hölder’s inequality (and, in fact, does not make use of the regularity imposed on one of the inputs). On the other hand, based on the form of the duality function in $L^p$ duality, there seems to be a much less clear path towards obtaining the reverse implication. The following result improves (4.18) and (4.22) and is obtained by making use of our regularised hypercontractivity inequalities in (1.14) and (3.38) (via Hölder and (4.21)). The improved constant which arises in this way takes the form
\begin{equation}
\mathcal{H}(c_1, c_2) := H(c_1, c_2) \left\| Pf \left(\left[\frac{\gamma \beta}{\gamma}\right]^{c_1}\right)\right\|_{L^p(\mathbb{R})^{c_2}((\gamma)}
\end{equation}
\textbf{Corollary 4.6.} Let $\beta > 0$, $s > 0$ and $c_1, c_2 \in \mathbb{R} \setminus \{0\}$ satisfy (4.20).
(1) In the case \(c_1, c_2 \in (0, 1)\), we suppose \(\beta \geq 1\). Then
\[
\int_{\mathbb{R}^2} e^{-\pi x \cdot Q x} \prod_{j=1,2} f_j(L_j x)^{c_j} \, dx \leq \mathcal{H}(c_1, c_2) \prod_{j=1,2} \left( \int_{\mathbb{R}} f_j \right)^{c_j}
\]
for any nonnegative \(f_2 \in L^1(\mathbb{R})\) and \(f_1 \in L^1(\mathbb{R})\) satisfying \((\log f_1)'' \geq -\frac{1}{\beta}\).

(2) In the case \(c_1 c_2 < 0\), we suppose \(\beta \geq 1\). Then
\[
\int_{\mathbb{R}^2} e^{-\pi x \cdot Q x} \prod_{j=1,2} f_j(L_j x)^{c_j} \, dx \geq \mathcal{H}(c_1, c_2) \prod_{j=1,2} \left( \int_{\mathbb{R}} f_j \right)^{c_j}
\]
for any positive \(f_2 \in L^1(\mathbb{R})\) and \(f_1 \in L^1(\mathbb{R})\) satisfying \((\log f_1)'' \geq -\frac{1}{\beta}\).

(3) In the case \(c_1, c_2 > 1\), we suppose \(\beta \leq 1\). Then
\[
\int_{\mathbb{R}^2} e^{-\pi x \cdot Q x} \prod_{j=1,2} f_j(L_j x)^{c_j} \, dx \geq \mathcal{H}(c_1, c_2) \prod_{j=1,2} \left( \int_{\mathbb{R}} f_j \right)^{c_j}
\]
for any positive \(f_2 \in L^1(\mathbb{R})\) and \(f_1 \in L^1(\mathbb{R})\) satisfying \((\log f_1)'' \leq -\frac{1}{\beta}\).

This result indicates that it would be reasonable to try and extend the regularised Brascamp–Lieb inequalities in [17] and [18], in which the input functions are given by solutions to heat equations at time \(t = 1\), and allow inputs which are semi-log-convex/semi-log-concave in the appropriate sense.

5. Further results

5.1. Hypercontractivity on general measure. Given our regularised hypercontractivity inequality on gaussian space, it is of course natural to pursue the case of more general measure spaces. Whilst significant parts of our proof of Theorem [1.4] readily generalise, certain parts make use of the specific nature of the gaussian measure, namely the commutation property \(\nabla P_s = e^{-s} P_s \nabla\). In this subsection we isolate exactly which parts of the argument offer up difficulties in extending to more general measures and we suggest possible ways to overcome these.

First, we provide the following set-up. Take a smooth, non-negative, and convex potential \(V\) on \(\mathbb{R}^n\) and fix the probability measure
\[
d\mu(x) = Z^{-1} e^{-V(x)} \, dx, \quad Z := \int_{\mathbb{R}^n} e^{-V(x)} \, dx,
\]
which is invariant and symmetric with respect to the operator
\[
\mathcal{L} = \mathcal{L}_V := \Delta - \nabla V \cdot \nabla.
\]
Then we denote a diffusion semigroup corresponding to \(\mathcal{L}\) by \(P_s : L^2(\mu) \to L^2(\mu)\), \(s \geq 0\), which satisfies
\[
\partial_s P_s f = \mathcal{L} P_s f, \quad \lim_{s \downarrow 0} P_s f = f
\]
for all \(f \in L^2(\mu)\). We remark that \(P_s\) can be extended to a bounded operator on \(L^p(\mu)\) for all \(1 \leq p \leq \infty\), and moreover it is a contraction. We refer the reader to
the book [4] for a comprehensive treatment. We will frequently use the notation
\[ dm_{\beta}(x) = Z_{\beta}^{-1} e^{-\frac{1}{\beta} V(x)} dx, \quad Z_{\beta} := \int_{\mathbb{R}^n} e^{-\frac{1}{\beta} V(x)} dx, \quad \mathcal{L}_{\beta} := \beta \Delta - \nabla V \cdot \nabla \]
for \( \beta > 0 \). In the gaussian case \( V(x) = \frac{1}{2} |x|^2 \), we have \( m_{\beta} = \gamma_{\beta} \) and the notation \( \mathcal{L}_{\beta} \) is consistent with notation we introduced in Section 1.

In this general setting, the hypercontractivity inequality (1.2) can be generalised as follows. Suppose \( \nabla^2 V \geq K \text{ id} \) for some \( K > 0 \) and let \( 1 < p < q < \infty \) and \( s > 0 \) satisfy \( \frac{1}{p} - \frac{1}{q} = e^{2Ks} \). Then
\[
\left\| P_s f^{\frac{1}{q}} \right\|_{L^q(m)} \leq \left( \int_{\mathbb{R}^n} f \ dm \right)^{\frac{1}{q}}
\]
for all non-negative \( f \in L^1(m) \). It is reasonable to expect to improve the constant in (5.1) via the Fokker–Planck equation
\[
\partial_t v_t = \mathcal{L}_{\beta}^* v_t := \beta \Delta v_t + \nabla \cdot (v_t \nabla V)
\]
in the spirit of (1.14). Here \( \mathcal{L}_{\beta}^* \) means the dual of \( \mathcal{L}_{\beta} \) with respect to the \( L^2(dx) \) inner product. Note that the assumption \( \nabla^2 V \geq K \text{ id} \) is indeed known to be related to the commutativity of \( P_s \) and \( \nabla \). In the case of \( V(x) = \frac{1}{2} |x|^2 \), it is straightforward from the explicit form of \( P_s f \) in (1.1) to see that
\[
\nabla(P_s f) = e^{-s}P_s [\nabla f] := e^{-s}(P_s[\partial_1 f], \ldots, P_s[\partial_n f]).
\]
For general potential \( V \), the convexity condition on \( V \) is in fact equivalent to a weak form of (5.3), the so-called gradient estimate
\[
|\nabla(P_s f)| \leq e^{-Ks}P_s [\nabla f] \]
for all smooth and compactly supported functions \( f \). We refer to [4] for details on this equivalence. In what follows, we will impose a slightly stronger assumption than the gradient estimate (5.4), namely
\[
|\nabla(P_s f)| \leq e^{-Ks} |P_s[\nabla f]|
\]
for all smooth \( f \in L^1(m) \).

Let us consider the case \( \beta \in (0, 1) \). We generalise the notion of semi-log-concavity (1.16) by assuming that the solution \( v_t \) to (5.2) satisfies
\[
\nabla^2 \log v_t \leq -\frac{1}{\beta} \nabla^2 V
\]
for all \( t \geq 0 \). Under this set-up, let us discuss to what extent our proof of Theorem 1.4 can be generalised.

Firstly, for technical reasons, we will restrict our attention to the case \( q = 2 \) and thus we consider \( 1 < p < q = 2 \) satisfying \( \frac{1}{p} - \frac{1}{q} = e^{2Ks} \) for some \( K > 0 \). Along the lines of our proof of Theorem 1.4 our goal is to show that the functional
\[
Q(t) := \int_{\mathbb{R}^n} P_t [h_t]^q \ dm
\]
is non-decreasing for \( t > 0 \), where \( h_t := (m_{\beta})^{\frac{1}{p}} \).
Observe that in the proof of Theorem 1.4 we made use of identities (3.17) and (3.18), both of which follow from (5.3) and are unique to the case \( V(x) = \frac{1}{2}|x|^2 \).

With this in mind, for general potential \( V \), here we impose an inequality version of (3.17) and (3.18) as follows. We say that the solution \( v_t \) to (5.2) satisfies gradient type estimates if (5.5) holds, the inequality

\[
(5.7) \quad \int_{\mathbb{R}^n} \nabla P_{2s} f \cdot g \, dm \leq e^{-2Ks} \int_{\mathbb{R}^n} P_{2s} \left[ \nabla f \right] \cdot g \, dm
\]

is satisfied for \( (f, g) = (h_t, h_t \nabla V), (h_t, h_t \nabla \left( \log \left( \frac{1}{m} \right) \right)) \), and the inequality

\[
(5.8) \quad \int_{\mathbb{R}^n} \left( \text{div} \, P_{2s} f \right) g \, dm \geq e^{-2Ks} \int_{\mathbb{R}^n} P_{2s} \left[ \text{div} f \right] g \, dm
\]

is satisfied for \( (f, g) = (h_t \nabla V, h_t), (h_t \nabla \left( \log \left( \frac{1}{m} \right) \right), h_t) \).

The relevancy of our gradient type estimates can be seen from the following facts. If we have (5.7) for \( (f, g) \), then

\[
(5.9) \quad \int_{\mathbb{R}^n} P_s f P_s \left[ g \cdot \nabla V \right] \, dm \leq \int_{\mathbb{R}^n} P_s f P_s \left[ \text{div} g \right] \, dm + e^{-2Ks} \int_{\mathbb{R}^n} P_s \left[ \nabla f \right] \cdot P_s g \, dm.
\]

Also, if we have (5.8) for \( (f, g) \), then

\[
(5.10) \quad \int_{\mathbb{R}^n} P_s f \cdot P_s \left[g \nabla V \right] \, dm \geq \int_{\mathbb{R}^n} P_s f \cdot P_s \left[ \nabla g \right] \, dm + e^{-2Ks} \int_{\mathbb{R}^n} P_s \left[ \text{div} f \right] P_s g \, dm.
\]

These two properties can be seen by integration by parts. In fact, since \( P_s \) is self-adjoint with respect to \( dm \), we have that

\[
\int_{\mathbb{R}^n} P_s f P_s \left[ g \cdot \nabla V \right] \, dm = \int_{\mathbb{R}^n} P_{2s} f \left( g \cdot \nabla V \right) \, dm
\]

\[
= \int_{\mathbb{R}^n} P_{2s} f g \cdot \nabla (-m) \, dx
\]

\[
= \int_{\mathbb{R}^n} P_s f \cdot P_s \left[ \text{div} g \right] \, dm + \int_{\mathbb{R}^n} \nabla (P_{2s} f) \cdot g \, dm,
\]

which shows the first claim.

We remark that the proof of Proposition 3.1 continues to work for general potentials. Hence, armed with (5.9) and (5.10), we can modify and generalise the proof of Theorem 1.4 under the assumptions above. In fact, in the proof of Theorem 1.4 we used identities (3.19) and (3.20) for the terms

\[
\Upsilon_1(p, s, \beta) \int_{\mathbb{R}^n} P_s \left[ h_t |\nabla V|^2 \right] P_s h_t^{q-1} \, dm,
\]

\[
(1 - \frac{1}{\beta})^2 \int_{\mathbb{R}^n} |P_s \left[ h_t \nabla V \right]|^2 P_s h_t^{q-2} \, dm,
\]

\[
\Upsilon_2(p, s, \beta) \int_{\mathbb{R}^n} P_s \left[ h_t \nabla \left( \log \left( \frac{1}{m} \right) \right) \right] \cdot P_s \left[ h_t \nabla V \right] P_s h_t^{q-2} \, dm,
\]

\[
\Upsilon_3(p, s, \beta) \int_{\mathbb{R}^n} P_s \left[ h_t \nabla V \cdot \nabla \log \left( \frac{1}{m} \right) \right] P_s h_t^{q-1} \, dm
\]
and other parts of the proof did not make use of the special nature of $V(x) = \frac{1}{2}|x|^2$. The signs of the coefficients $\Upsilon_j(p, s, \beta), j = 1, 2, 3$, are key to establishing the non-decreasingness of $Q$. Note that $\Upsilon_1(p, s, \beta) \leq 0$ and $(1 - \frac{1}{p})^2, \Upsilon_2(p, s, \beta) = -(1 - \frac{1}{p}) \geq 0$ under $0 < \beta < 1$. In the case $q = 2$, we have $\Upsilon_3(p, s, \beta) = 2e^{-2Ks} - 1 + \left(1 - \frac{1}{p}\right)(1 - \frac{1}{\beta})$ and hence if $s < \frac{1}{2K} \log 2$, then there exists some $\beta_* \in (0, 1)$ for which $\Upsilon_3(p, s, \beta_*) > 0$.

In summary, if the potential $V$ satisfies (5.5) for some $K > 0$, the exponents satisfy $0 < s < \frac{1}{2K} \log 2, 1 < p < q = 2, \frac{s}{p - 1} = e^{2Ks}$, and the following holds for some $\beta = \beta_* \in (0, 1)$ sufficiently close to 1, then $Q$ can be shown to be non-decreasing on $(0, \infty)$. The input $v_0 \in L^2(m_{\beta}^{-1})$ and its evolution $v_t$ under (5.2) are semi-log-concave in the sense that (5.6) holds and satisfy gradient type estimates. If also

$$\lim_{t \to \infty} v_t = \left(\int_{\mathbb{R}^n} \frac{v_0}{m} \, dm\right) m_{\beta}$$

in an appropriate sense, we obtain the regularised hypercontractivity inequality

$$\left\|P_s \left[\left(\frac{v_0}{m}\right)^{\frac{1}{p}}\right]\right\|_{L^q(m)} \leq \left\|P_s \left[\left(\frac{m_{\beta}}{m}\right)^{\frac{1}{p}}\right]\right\|_{L^q(m)} \left(\int_{\mathbb{R}^n} \frac{v_0}{m} \, dm\right)^{\frac{1}{p}}.$$

Admittedly, the conditions we impose are somewhat forbidding and, as far as we can tell, it is not clear how to identify concrete examples outside the gaussian case. In the following subsection, we change tack somewhat and more fruitfully we obtain regularised LSI outside the gaussian case via mass transportation rather than flow monotonicity.

5.2. Regularised LSI for general measures. On a general measure space, the gaussian LSI [13] can be generalised as follows. Provided $\nabla^2 V \geq K \text{id}$ for some $K > 0$, then

$$(5.11) \quad \text{Ent}_m(f) \leq \frac{1}{2K} I_m(f)$$

holds for all non-negative locally Lipschitz $f \in L^1(m)$, where

$$\text{Ent}_m(f) := \int_{\mathbb{R}^n} f \log f \, dm - \int_{\mathbb{R}^n} f \, dm \log \int_{\mathbb{R}^n} f \, dm,$$

$$I_m(f) := \int_{\mathbb{R}^n} \frac{|
abla f|^2}{f} \, dm.$$  

See, for example, [6] for further details. At this level of generality, the cases of equality for (5.11) was recently investigated by Ohta–Takatsu [62]. In particular, one can see from [62] that $f = \frac{m_{\beta}}{m}$ does not attain equality in (5.11) unless $\beta = 1$. Hence, as in the spirit of Theorem 1.4 we may expect to improve the constant in (5.11) under an appropriate log-convexity assumption. To avoid difficulties encountered in the previous subsection, here we investigate regularised LSI via a mass-transport approach. Note that the mass-transport proof of classical LSI is due to Cordero-Erausquin [32].
We begin with some preliminaries. For two given probability measures \( \mu, \nu \) on \( \mathbb{R}^n \) which are absolutely continuous with respect to the \( n \)-dimensional Lebesgue measure, Brenier’s theorem ensures the existence of a map \( T : \mathbb{R}^n \to \mathbb{R}^n \), which is called Brenier’s map from \( \mu \) to \( \nu \), such that

\[
(5.12) \quad \int_{\mathbb{R}^n} h(T(x)) \, d\mu(x) = \int_{\mathbb{R}^n} h(x) \, d\nu(x)
\]

holds for all \( h \in L^\infty(\mathbb{R}^n) \). If we slightly abuse notation and write \( d\mu(x) = \mu(x) \, dx \) and \( d\nu(x) = \nu(x) \, dx \) assuming \( \mu, \nu \) have densities with respect to Lebesgue measure, then (5.12) leads to the Monge–Ampère equation

\[
(5.13) \quad \mu(x) = \nu(T(x)) \det \nabla T(x)
\]

for \( \mu \)-a.e. \( x \in \mathbb{R}^n \). The reader is referred to [6, 66, 67] for further details.

For the sake of simplicity we discuss the one-dimensional case. We are able to prove the following regularised LSI.

**Theorem 5.1.** Let \( K > 0 \) and \( \beta \geq 1 \). Suppose \( v \) and \( V \) are symmetric on \( \mathbb{R} \) and satisfy

\[
V'' \geq K, \quad (\log v)'' \geq -\frac{K}{\beta}, \quad \int_{\mathbb{R}} v \, dx = 1, \quad \lim_{|x| \to +\infty} |V'(x)|v(x) = 0.
\]

Then we have

\[
(5.14) \quad \text{Ent}_m \left( \frac{v}{m} \right) - \frac{1}{2K} I_m \left( \frac{v}{m} \right) \leq \text{Ent}_m \left( \frac{m_\beta}{m} \right) - \frac{1}{2K} I_m \left( \frac{m_\beta}{m} \right) + \left( 1 - \frac{1}{\beta} \right) \int_{\mathbb{R}} V'' \left( \frac{v - m_\beta}{m} \right) \, dx.
\]

In particular, if we additionally assume \( V'' \leq L \), then

\[
(5.15) \quad \text{Ent}_m \left( \frac{v}{m} \right) - \frac{1}{2K} I_m \left( \frac{v}{m} \right) \leq \text{Ent}_m \left( \frac{m_\beta}{m} \right) - \frac{1}{2K} I_m \left( \frac{m_\beta}{m} \right) + \left( 1 - \frac{1}{\beta} \right) \frac{L - K}{K}.
\]

**Remark.** It is not immediately clear if (5.15) improves upon (5.11) or not. However, if one notices that

\[
V(x) \leq V(0) + \frac{1}{2K} |V'(x)|^2
\]

from \( V'' \geq K \), then one can see that

\[
\text{Ent}_m \left( \frac{m_\beta}{m} \right) - \frac{1}{2K} I_m \left( \frac{m_\beta}{m} \right) \leq -\frac{1}{2} \log \left( \frac{2\pi\beta}{L} \right) + \frac{L}{2K} \left( 1 - \frac{1}{\beta} \right) + \log Z,
\]

provided \( K \leq V'' \leq L \). In particular, this shows that \( \text{Ent}_m \left( \frac{m_\beta}{m} \right) - \frac{1}{2K} I_m \left( \frac{m_\beta}{m} \right) \to -\infty \) as \( \beta \to \infty \). Therefore there exists sufficiently large \( \beta = \beta(V) \geq 1 \) such that (5.15) improves (5.11).

**Proof of Theorem 5.1.** Let \( T : \mathbb{R} \to \mathbb{R} \) be Brenier’s map from \( v \) to \( m_\beta \) in which case (5.12) and (5.13) can be read as

\[
(5.16) \quad \int_{\mathbb{R}} h(T(x)) \, dv = \int_{\mathbb{R}} h(x) \, dm_\beta, \quad v(x) = m_\beta(T(x))T'(x).
\]
Then we have from (5.10) that

\[
\operatorname{Ent}_m\left(\frac{\nu}{m}\right) = \int_{\mathbb{R}} \left( \log \nu - \log m \right) dv \\
= \int_{\mathbb{R}} \left( \log m_\beta(T(x)) + \log T'(x) - \log m(x) \right) dv \\
= -\log Z_\beta + \log Z - \frac{1}{\beta} \int_{\mathbb{R}} V(T(x)) dv + \int_{\mathbb{R}} V(x) dv + \int_{\mathbb{R}} \log T'(x) dv.
\]

In the case of \( v = m_\beta \), we have that

\[
\operatorname{Ent}_m\left(\frac{m_\beta}{m}\right) = -\log Z_\beta + \log Z + \left(1 - \frac{1}{\beta}\right) \int_{\mathbb{R}} V(T(x)) dv.
\]

Regarding the Fisher information, from the definition

\[
I_m\left(\frac{\nu}{m}\right) = \int_{\mathbb{R}} \|v'\|^2 dv, \quad I_m\left(\frac{m_\beta}{m}\right) = \left(1 - \frac{1}{\beta}\right)^2 \int_{\mathbb{R}} |V'|^2 dm_\beta.
\]

Hence

\[
R(v) := \operatorname{Ent}_m\left(\frac{\nu}{m}\right) - \frac{1}{2\beta} I_m\left(\frac{\nu}{m}\right) - \operatorname{Ent}_m\left(\frac{m_\beta}{m}\right) + \frac{1}{2\beta} I_m\left(\frac{m_\beta}{m}\right).
\]

\[
= \int_{\mathbb{R}} \left( V(x) - V(T(x)) \right) dv + \int_{\mathbb{R}} \log T'(x) dv \\
- \frac{1}{2\beta} \int_{\mathbb{R}} \left(\log v\right)' + \left|v\right|^2 dv + \frac{1}{2\beta} (1 - \frac{1}{\beta})^2 \int_{\mathbb{R}} |V'|^2 dm_\beta.
\]

We know that \( V'' \geq K \) implies

\[
V(x) - V(T(x)) \leq V'(x)(x - T(x)) - \frac{K}{2} |T(x) - x|^2
\]

from which we obtain

\[
R(v) \leq -\int_{\mathbb{R}} (T(x) - x)V'(x) dv - \frac{K}{2} \int_{\mathbb{R}} |T(x) - x|^2 dv + \int_{\mathbb{R}} \log T'(x) dv \\
- \frac{1}{2\beta} \int_{\mathbb{R}} \left(\log v\right)' + \left|v\right|^2 dv + \frac{1}{2\beta} (1 - \frac{1}{\beta})^2 \int_{\mathbb{R}} |V'|^2 dm_\beta.
\]

We focus on

\[
-\frac{K}{2} \int_{\mathbb{R}} |T(x) - x|^2 dv - \frac{1}{2\beta} \int_{\mathbb{R}} \left(\log v\right)' + \left|v\right|^2 dv \\
= -\frac{1}{2} \int_{\mathbb{R}} \sqrt{K} |T(x) - x| + \frac{1}{\sqrt{K}} \left(\left(\log v\right)'(x) + V'(x)\right)^2 dv \\
+ \int_{\mathbb{R}} \left(\left(\log v\right)'(x) + V'(x)\right)(T(x) - x) dv.
\]

Notice simply that

\[
\int_{\mathbb{R}} \left(\log v\right)'(T(x) - x) dv = -\int_{\mathbb{R}} (T'(x) - 1) dv.
\]

This integration by parts can be justified from assumptions \( \lim_{|x| \to +\infty} |V'(x)v(x)| = 0 \), \( V'' \geq K \), and the fact that \( V' \) is symmetric. In fact, as we will see later, we have
from \( V'' \geq K \) and the symmetric assumption on \( V \) that \(|T(x)| \leq |x| \leq \frac{1}{\beta}|V'(x)|\). This yields that \( v(x)|x|, v(x)|T(x)| \to 0 \) as \(|x| \to +\infty\). Therefore we obtain
\[
(5.17) \quad R(v) \leq \int_\mathbb{R} \left( \log T'(x) - T'(x) + 1 \right) dv + \frac{1}{2K} (1 - \frac{1}{\beta})^2 \int_\mathbb{R} |V'|^2 dm_\beta
- \frac{1}{2} \int_\mathbb{R} |\sqrt{K} (T(x) - x) + \frac{1}{\sqrt{K}} ((\log v)' + V')|^2 dv.
\]
We then reorganise terms in the third term as
\[
|\sqrt{K} (T(x) - x) + \frac{1}{\sqrt{K}} ((\log v)' + V')|^2
= |\sqrt{\frac{K}{\beta}}(T(x) - x) + \frac{1}{\sqrt{\frac{K}{\beta}}}((\log v)' + \frac{V'}{\beta}) + (1 - \frac{1}{\beta})\sqrt{K}(T(x) - x) + (1 - \frac{1}{\beta})\frac{V'}{\sqrt{K}}|^2
= |\sqrt{\frac{K}{\beta}}(T(x) - x) + \frac{1}{\sqrt{\frac{K}{\beta}}}((\log v)' + \frac{V'}{\beta})|^2
+ 2(1 - \frac{1}{\beta})(\log v)'(T(x) - x + \frac{V'}{K}) + K(1 - \frac{1}{\beta^2})|T(x) - x + \frac{V'}{K}|^2.
\]
We further rearrange \(|T(x) - x + \frac{V'}{K}|^2\) as
\[
|T(x) - x + \frac{V'}{K}|^2
= |T(x) - x + \frac{V'(x)}{K} - \frac{V'(T(x))}{K}|^2 + 2(T(x) - x + \frac{V'(x)}{K} \frac{V'(T(x))}{K}) - |\frac{V'(T(x))}{K}|^2.
\]
Since \( \beta \geq 1 \), we obtain
\[
|\sqrt{\frac{K}{\beta}}(T(x) - x) + \frac{1}{\sqrt{\frac{K}{\beta}}}((\log v)' + V')|^2
\geq 2(1 - \frac{1}{\beta})(\log v)'(T(x) - x + \frac{V'}{K})
+ 2(1 - \frac{1}{\beta^2})(T(x) - x + \frac{V'}{K})V'(T(x)) - \frac{1}{K}(1 - \frac{1}{\beta^2})|V'(T(x))|^2.
\]
Now we appeal to the assumption that \( V, v \) are symmetric to see that
\[
(5.18) \quad (T(x) - x + \frac{V'}{K})V'(T(x)) \geq \frac{|V'(T(x))|^2}{K}
\]
as follows. Consider the case \( x > 0 \). First notice that if we define \( \psi(x) := -x + \frac{1}{K}V'(x) \), then
\[
\psi'(x) = -1 + \frac{1}{K}V''(x) \geq 0
\]
and hence \( \psi \) is monotone increasing. From the convexity assumptions on \( V \) and \( v \), Caffarelli’s contraction theorem [29, 51] shows that \( 0 \leq T' \leq 1 \). Also the symmetry of \( v, V \) yields that \( T(0) = 0 \). These two imply \( 0 \leq T(x) \leq x \) and hence \( \psi(x) \geq \psi(T(x)) \). In other words,
\[
(T(x) - x + \frac{1}{K}V'(x) \geq \frac{1}{K}V'(T(x)).
\]
Also the symmetry of \( V \) yields \( V'(0) = 0 \) and hence \( \psi(x) \geq \psi(0) = 0 \) which turns into \( V'(x) \geq Kx \geq 0 \). Therefore we see (5.18) when \( x \geq 0 \). In a similar way, one can show (5.18) even when \( x \leq 0 \).
We then apply (5.18) to obtain
\[
\left| \sqrt{K} (T(x) - x) + \frac{1}{\sqrt{K}} ((\log v)' + V') \right|^2
\geq 2\left( 1 - \frac{1}{\beta} \right) \left( \log v \right)'(T(x) - x + \frac{V'}{K}) + \frac{1}{K} (1 - \frac{1}{\beta^2}) |V'(T(x))|^2.
\]
Overall we derive
\[
R(v) \leq \int_{\mathbb{R}} (\log T'(x) - T'(x) + 1) \, dv + \frac{1}{2K} (1 - \frac{1}{\beta} )^2 \int_{\mathbb{R}} |V'|^2 \, d\mu_{\beta}
- \frac{1}{2} \int_{\mathbb{R}} \left( 2 - \frac{1}{\beta} \right) \left( \log v \right)'(T(x) - x + \frac{V'}{K}) + \frac{1}{K} (1 - \frac{1}{\beta^2}) |V'(T(x))|^2 \, dv
= \int_{\mathbb{R}} \left( \log T'(x) - T'(x) + 1 \right) \, dv + \frac{1}{2K} (1 - \frac{1}{\beta} )^2 \int_{\mathbb{R}} |V'|^2 \, d\mu_{\beta}
- \left( 1 - \frac{1}{\beta} \right) \int_{\mathbb{R}} v'(x) (T(x) - x + \frac{V'}{K}) \, dx - \frac{1}{2K} (1 - \frac{1}{\beta^2}) \int_{\mathbb{R}} |V'(T(x))|^2 \, dv.
\]
We finally use (5.18) to see that
\[
\frac{1}{2K} (1 - \frac{1}{\beta} )^2 \int_{\mathbb{R}} |V'|^2 \, d\mu_{\beta} - \frac{1}{2K} (1 - \frac{1}{\beta^2}) \int_{\mathbb{R}} |V'(T(x))|^2 \, dv
= - \frac{1}{K} \left( 1 - \frac{1}{\beta} \right) \int_{\mathbb{R}} V'' \, d\mu_{\beta},
\]
where we also use \( \lim_{|x| \to +\infty} |V'(x)|v(x) = 0 \) to validate the integration by parts step. From this we conclude
\[
R(v) \leq \int_{\mathbb{R}} \left( \log T'(x) - \frac{1}{\beta} T''(x) + \frac{1}{\beta} \right) \, dv + \left( 1 - \frac{1}{\beta} \right) \frac{1}{K} \int_{\mathbb{R}} V''(v - m_{\beta}) \, dx
\]
and since \( 0 \leq T'(x) \leq 1 \) we thus obtain (5.14).

5.3. Talagrand’s inequality. Recall that in Corollary 4.3 we considered an inequality dual to Talagrand’s quadratic transportation cost inequality and established an improvement for certain uniformly convex inputs. We close the paper by considering the original form of Talagrand’s inequality
\[
(5.19) \quad \frac{1}{2} W_2(\gamma, v)^2 \leq \text{Ent}_{\gamma}(\log v).
\]
which is valid for \( v \) satisfying \( \int_{\mathbb{R}} v \, dx = 1 \) and \( \int_{\mathbb{R}} |x|^2 v \, dx < \infty \), and provide an improvement if we restrict to semi-log-concave or semi-log-convex inputs.

In (5.19), the definition of the Wasserstein distance \( W_2 \) is as follows. First, take two probability measures \( \mu, \nu \) whose second moment is finite. Let \( \Pi(\mu, \nu) \) be the set of all couplings of \( \mu \) and \( \nu \), namely probability measures \( \pi \) on \( \mathbb{R}^n \times \mathbb{R}^n \) such that \( \pi(A \times \mathbb{R}^n) = \mu(A) \) and \( \pi(\mathbb{R}^n \times A) = \nu(A) \) hold for all Borel sets \( A \subset \mathbb{R}^n \). Then
\[
(5.20) \quad W_2(\mu, \nu)^2 := \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^n} |x - y|^2 \, d\pi(x, y).
\]
It will be important that \( W_2(\mu, \nu) \) can be represented by using Brenier’s map introduced in Section 5.2 when \( \mu \) and \( \nu \) are absolutely continuous with respect to the
n-dimensional Lebesgue measure. In fact, if $T$ is Brenier’s map pushing forward $\mu$ onto $\nu$, then
\begin{equation}
W_2(\mu, \nu)^2 = \int_{\mathbb{R}^n} |x - T(x)|^2 \, d\mu(x).
\end{equation}

We refer the reader to [66] for further details.

Our purpose in this subsection is to obtain improved version of (5.19) by virtue of semi-log-convexity and semi-log-concavity. We begin with the one dimensional case.

**Proposition 5.2.** Let $\beta > 0$ and suppose $v : \mathbb{R} \to (0, \infty)$ satisfies $\int_{\mathbb{R}} v \, dx = 1$, $\int_{\mathbb{R}} |x|^2 v \, dx < \infty$ and (1.16). When $\beta \geq 1$, we additionally assume $(- \log v)'' \geq 0$.

Then
\begin{equation}
\frac{1}{2} W_2(\gamma, v)^2 - \text{Ent}_{\gamma}(\frac{v}{\gamma}) \leq \frac{1}{2} W_2(\gamma, \gamma^\beta)^2 - \text{Ent}_{\gamma}(\frac{\gamma^\beta}{\gamma}).
\end{equation}

**Proof.** First of all, we remark that
\begin{equation}
\frac{1}{2} W_2(\gamma, \gamma^\beta)^2 - \text{Ent}_{\gamma}(\frac{\gamma^\beta}{\gamma}) = 1 + \frac{1}{2} \log \beta - \sqrt{\beta}
\end{equation}
and hence (5.22) becomes
\begin{equation}
\frac{1}{2} W_2(\gamma, v)^2 - \text{Ent}_{\gamma}(\frac{v}{\gamma}) \leq 1 + \frac{1}{2} \log \beta - \sqrt{\beta}.
\end{equation}

Let us consider $\beta \geq 1$ first. In this case, since $(- \log v)'' \geq 0$, considering $v(x)e^{-\frac{5}{2}x^2}$ for small enough $\varepsilon > 0$ and by an approximation, we may assume $(- \log v)'' \geq \varepsilon$.

Let $T$ be Brenier’s map from $v$ to $\gamma$. Then we have
\begin{equation*}
v(x) = \gamma(T(x))T'(x).
\end{equation*}

With this and (5.21) in mind, we see that
\begin{align*}
\text{Ent}_{\gamma}(\frac{v}{\gamma}) - \frac{1}{2} W_2(v, \gamma)^2 &= \int_{\mathbb{R}} \left( \frac{1}{2} |x|^2 \gamma - \frac{1}{2} T(x)^2 + \log T'(x) \right) \, dv - \frac{1}{2} \int_{\mathbb{R}} |T(x) - x|^2 \, dv \\
&= - \int_{\mathbb{R}} |T(x)|^2 \, dv + \int_{\mathbb{R}} \log T'(x) \, dv + \int_{\mathbb{R}} x T(x) \, dv \\
&= - \int_{\mathbb{R}} |x|^2 \, d\gamma + \int_{\mathbb{R}} \log T'(x) \, dv + \int_{\mathbb{R}} x T(x) T'(x) \gamma(T(x)) \, dx.
\end{align*}

If we notice that $T(x)T'(x)\gamma(T(x)) = (-\gamma(T(x)))'$, then integration by parts shows that
\begin{equation*}
\text{Ent}_{\gamma}(\frac{v}{\gamma}) - \frac{1}{2} W_2(v, \gamma)^2 = \int_{\mathbb{R}} \left( -1 - \log \frac{1}{T'(x)} + \frac{1}{T'(x)} \right) \, dv.
\end{equation*}

In this step we used that
\begin{equation}
\lim_{|x| \to +\infty} |x| \gamma(T(x)) = 0
\end{equation}

\begin{align*}
\text{Ent}_{\gamma}(\frac{v}{\gamma}) - \frac{1}{2} W_2(v, \gamma)^2 &= \int_{\mathbb{R}} \left( -1 - \log \frac{1}{T'(x)} + \frac{1}{T'(x)} \right) \, dv.
\end{align*}
and this can be justified as follows. Let $S$ be Brenier’s map pushing forward $\gamma$ to $v$. Then Caffarelli’s contraction theorem yields $S’ \leq \frac{1}{\sqrt{2}}$. On the other hand, since $T(S(x)) = x$ for $x \in \mathbb{R}$, we obtain $T'(S(x))S'(x) = 1$ for $x \in \mathbb{R}$. Hence, we enjoy $T’ \geq \frac{1}{\sqrt{2}}$ on $\mathbb{R}$ from which we can deduce $|T(x)| \geq |x| - c$ for $|x|$ sufficiently large. Here $c_1, c_2 > 0$ are some constants depending on $\varepsilon$. Since $|x|\gamma(T(x)) \leq |x|\gamma(c_1|x| - c_2)$ for $|x|$ sufficiently large, we can obtain (5.23).

We now appeal to the assumption (1.10) which can be read as $(\log v)'' \leq \frac{1}{\beta}$. Hence we may apply Caffarelli’s contraction theorem, to see that $T’ \leq \frac{1}{\sqrt{\beta}} \leq 1$. With this in mind, we notice that $x \mapsto -1 - \log x + x$ is monotone increasing in $x \in (1, \infty)$ from which we conclude

$$\text{Ent}_\gamma(\frac{v}{\gamma}) - \frac{1}{2}W_2^2(v, \gamma) \geq -1 - \frac{1}{2}\log \beta + \sqrt{\beta}.$$ 

Let us next consider $0 < \beta \leq 1$. In this case, the assumption (1.10) can be read $(\log v)'' \geq \frac{1}{\beta}$ and thus if $T$ is Brenier’s map from $\gamma$ to $v$, then $T’ \leq \sqrt{\beta} \leq 1$ follows from Caffarelli’s contraction theorem.

Next note that by using $\gamma(x) = v(T(x))T'(x)$
we obtain

$$\text{Ent}_\gamma(\frac{v}{\gamma}) = \int_\mathbb{R} \log v(T(x)) d\gamma - \int_\mathbb{R} \log \gamma(T(x)) d\gamma$$

$$= \int_\mathbb{R} \log \gamma d\gamma - \int_\mathbb{R} \log T'(x) d\gamma - \int_\mathbb{R} \log T(x) d\gamma$$

$$= \frac{1}{2} \int_\mathbb{R} (|T(x)|^2 - |x|^2) d\gamma - \int_\mathbb{R} \log T'(x) d\gamma$$

$$= \frac{1}{2} \int_\mathbb{R} |T(x) - x|^2 d\gamma - \int_\mathbb{R} |x|^2 d\gamma + \int_\mathbb{R} xT(x) d\gamma - \int_\mathbb{R} \log T'(x) d\gamma.$$ 

For the third term, we perform integration by parts to see that

$$\int_\mathbb{R} xT(x) d\gamma = \int_\mathbb{R} T(x)(-\gamma)' dx = \int_\mathbb{R} T'(x) d\gamma.$$ 

This step can be justified since $T’ \leq 1$ means that $|T(x)|\gamma(x) \leq C|x|\gamma(x)$ for some constant $C > 0$ and $|x|$ sufficiently large. By (5.24) we may now write

$$\text{Ent}_\gamma(\frac{v}{\gamma}) = \frac{1}{2}W_2^2(\gamma, v) - \int_\mathbb{R} (\log T'(x) - T'(x) + 1) d\gamma$$

and

$$\text{Ent}_\gamma(\frac{v}{\gamma}) \geq \frac{1}{2}W_2^2(\gamma, v) - (\log \sqrt{\beta} - \sqrt{\beta} + 1)$$

follows since $T’ \leq \sqrt{\beta}$. \hfill \Box

**Remark.** The technical condition $(-\log v)'' \geq 0$ imposed in the case $\beta \geq 1$ is only used to justify an integration by parts step in the proof and thus it is reasonable to hope that it could be removed. For instance, when $v$ is symmetric and has a gaussian concentration property, then one may ensure $|T(x)| \geq c_1|x| - c_2$ for some
constants $c_1, c_2 > 0$ and for all large enough $|x|$ from which one can show \[5.23\]; see [31, Theorem 1.1].

We then upgrade the one dimensional result to higher dimensions via the standard tensorising argument.

**Theorem 5.3.** Let $\beta_1, \ldots, \beta_n > 0$ denote the eigenvalues of the positive definite symmetric matrix $B$ on $\mathbb{R}^n$. Suppose $v : \mathbb{R}^n \to (0, \infty)$ satisfies $\int_{\mathbb{R}^n} v \, dx = 1$ and $\int_{\mathbb{R}^n} |x|^2 \, v \, dx < \infty$.

(1) If $v$ satisfies
\[
\nabla^2 \log v \geq -B^{-1}, \quad \nabla^2 \log v \leq 0
\]
then we have
\[
\frac{1}{2} W_2(\gamma, v)^2 - \text{Ent}_\gamma \left( \frac{v}{\gamma} \right) \leq \sum_{i=1}^{\beta_i \geq 1} \left( 1 + \frac{1}{2} \log \beta_i - \sqrt{\beta_i} \right).
\]

(2) If $v$ satisfies
\[
\nabla^2 \log v \leq -B^{-1}
\]
then we have
\[
\frac{1}{2} W_2(\gamma, v)^2 - \text{Ent}_\gamma \left( \frac{v}{\gamma} \right) \leq \sum_{i=1}^{\beta_i \leq 1} \left( 1 + \frac{1}{2} \log \beta_i - \sqrt{\beta_i} \right).
\]

Moreover, equalities in (5.25) and (5.27) are established when $v = \gamma_B$.

**Proof.** As in the proof of Corollary [1.7] we show only the assertion (1) with $n = 2$ and $B = \text{diag} (\beta_1, \beta_2)$. To this end, we set
\[
v^1(y_1) := \int_{\mathbb{R}} v(y_1, z) \, dz, \quad y_1 \in \mathbb{R}
\]
and
\[
v(y_2 \mid y_1) := \frac{v(y_1, y_2)}{\int_{\mathbb{R}} v(y_1, z) \, dz}, \quad (y_1, y_2) \in \mathbb{R}^2.
\]
Then $\int_{\mathbb{R}} v^1 \, dy_1 = 1$ and $\int_{\mathbb{R}} v(y_2 \mid y_1) \, dy_2 = 1$ for every $y_1 \in \mathbb{R}$ which follows from $\int_{\mathbb{R}^2} v \, dx = 1$. Furthermore, we can easily see that
\[
\int_{\mathbb{R}} |xy_1|^2 v^1(y_1) \, dy_1 < \infty, \quad \int_{\mathbb{R}} |y_2|^2 v(y_2 \mid y_1) \, dy_2 < \infty
\]
for every $y_1 \in \mathbb{R}$ since we have $\int_{\mathbb{R}^2} |x|^2 v \, dx < \infty$. Let $\pi^1 \in \Pi(\gamma, v^1)$ and $\pi(\cdot, \cdot \mid y_1) \in \Pi(\gamma, v(\cdot \mid y_1))$ be optimal couplings attaining equality in (5.20), respectively. We
consider a measure \( d\pi((x_1, x_2), (y_1, y_2)) := d\pi(x_2, y_2 \mid y_1) d\pi^1(x_1, y_1) \) on \( \mathbb{R}^2 \times \mathbb{R}^2 \), then we enjoy that \( \pi \in \Pi(\gamma \otimes \gamma) \). Hence by the definition in (5.20), we see that

\[
\frac{1}{2} W_2(\gamma \otimes \gamma, v)^2 
\leq \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y|^2 d\pi(x, y) 
= \frac{1}{2} \int_{\mathbb{R}^2} |x_1 - y_1|^2 \rho_1(x_1, y_1) + \frac{1}{2} \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} |x_2 - y_2|^2 d\pi(x_2, y_2 \mid y_1) \right) d\pi^1(x_1, y_1) 
= \frac{1}{2} W_2(\gamma, v^1)^2 + \frac{1}{2} \int_{\mathbb{R}} W_2(\gamma, v(\cdot \mid y_1))^2 dv^1(y_1).
\]

By (5.24), we have

\[
\partial^2_{y_2} \log v(y_2 \mid y_1) \geq -\frac{1}{\beta_2}, \quad \partial^2_{y_2} \log v(y_2 \mid y_1) \leq 0
\]

for every \( y_1 \in \mathbb{R} \). We also obtain

\[
\partial^2_{y_1} \log v^1 \geq -\frac{1}{\beta_1}, \quad \partial^2_{y_1} \log v^1 \leq 0
\]

from \( 0 \geq \partial^2_{y_1} \log v(y_1, y_2) \geq -\frac{1}{\beta_1} \) as in the argument in Corollary 1.7. Hence we may apply our one dimensional result in Proposition 5.2 to see that

\[
\frac{1}{2} W_2(\gamma, v^1)^2 \leq \text{Ent}_\gamma \left( \frac{v^1}{\gamma} \right) + 1 + \frac{1}{2} \log \beta_1 - \sqrt{\beta_1}, \\
\frac{1}{2} W_2(\gamma, v(\cdot \mid y_1))^2 \leq \text{Ent}_\gamma \left( \frac{v(\cdot \mid y_1)}{\gamma} \right) + 1 + \frac{1}{2} \log \beta_2 - \sqrt{\beta_2}
\]

for every \( y_1 \in \mathbb{R} \) in the case of \( \beta_1, \beta_2 \geq 1 \). If either \( \beta_1 \) or \( \beta_2 \) are < 1, then we replace the corresponding inequality above by the standard Talagrand inequality (5.19). Thus we can deduce that

\[
\frac{1}{2} W_2(\gamma, v)^2 \leq \text{Ent}_\gamma \left( \frac{v}{\gamma} \right) + \int_{\mathbb{R}} \text{Ent}_\gamma \left( \frac{v(\cdot \mid y_1)}{\gamma} \right) dv^1(y_1) - \sum_{i=1,2; \beta_i \geq 1} \left( -1 - \frac{1}{2} \log \beta_i + \sqrt{\beta_i} \right).
\]

Finally, it follows from the additivity of entropy that we have

\[
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\]

\[
\text{Ent}_\gamma \left( \frac{v}{\gamma} \right) + \int_{\mathbb{R}} \text{Ent}_\gamma \left( \frac{v(\cdot \mid y_1)}{\gamma} \right) dv^1(y_1) = \text{Ent}_\gamma \left( \frac{v}{\gamma} \right).
\]

For instance, see [67, Theorem 22.8] for details of additivity of entropy in general settings. For completeness, we give a proof of (5.28). Indeed, we enjoy

\[
\text{Ent}_\gamma \left( \frac{v}{\gamma} \right) = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} v(y_1, y_2) dy_2 \right) \log \left( \int_{\mathbb{R}} v(y_1, y_2) dy_2 \right) dy_1 \\
+ \int_{\mathbb{R}^2} v(y_1, y_2) \log \frac{1}{\gamma(y_1)} dy_1 dy_2
\]

and

\[
\int_{\mathbb{R}^2} \text{Ent}_\gamma \left( \frac{v(\cdot \mid y_1)}{\gamma} \right) dv^1(y_1) = -\int_{\mathbb{R}} \left( \int_{\mathbb{R}} v(y_1, y_2) dy_2 \right) \log \left( \int_{\mathbb{R}} v(y_1, y_2) dy_2 \right) dy_1 \\
+ \int_{\mathbb{R}^2} v(y_1, y_2) \log \frac{v(y_1, y_2)}{\gamma(y_2)} dy_1 dy_2.
\]
Summing up the two equalities above, we can deduce (5.28), and our proof is complete. □

It is interesting to compare the assertion (2) in Theorem 5.3 to a recent result by Mikulincer [60] about the deficit estimate of Talagrand’s inequality in terms of the covariance of the input.

**Theorem 5.4** (Theorem 2 in [60]). For any non-negative \( v \) such that \( \int_{\mathbb{R}^n} v \, dx = 1 \) and \( \int_{\mathbb{R}^n} x \, v \, dx = 0 \),

\[
\frac{1}{2} \| W_2(\gamma, v) \|^2 - \text{Ent}_\gamma \left( \frac{v}{\gamma} \right) \leq \sum_{i=1}^{n} \frac{2(1 - \beta_i) + (\beta_i + 1) \log \beta_i}{2(\beta_i - 1)},
\]

where \( \beta_1, \ldots, \beta_n \) are eigenvalues of \( \text{cov}(v) \).

It is appealing that our inequality (5.27) improves (5.29) since

\[
1 + \frac{1}{2} \log \beta_i - \sqrt{\beta_i} < \frac{2(1 - \beta_i) + (\beta_i + 1) \log \beta_i}{2(\beta_i - 1)}
\]

for all \( \beta_i \in (0, 1) \) and moreover, (5.27) is sharp as in the case of \( v = \gamma_B \). On the other hand, the assumption imposed on the input \( v \) in Theorem 5.4 is weaker than the one in Theorem 5.3; recall (1.20). This is of course reminiscent of the discussion just after Lemma 1.5 in relation to the LSI deficit estimates of Eldan–Lehec–Shenfeld. Therefore it seems reasonable to expect the following to be true:

\[
\frac{1}{2} \| W_2(\gamma, v) \|^2 - \text{Ent}_\gamma \left( \frac{v}{\gamma} \right) \leq \sum_{i=1}^{n} \left( 1 + \frac{1}{2} \log \beta_i - \sqrt{\beta_i} \right)
\]

for all non-negative \( v \) such that \( \int_{\mathbb{R}^n} v \, dx = 1 \) and \( \int_{\mathbb{R}^n} x \, v \, dx = 1 \), where \( \beta_1, \ldots, \beta_n \) are eigenvalues of \( \text{cov}(v) \).

**5.4. Hypercontractivity and LSI under a semi-log-subharmonic assumption.** Although Theorem 1.4 and Corollary 1.7 are stated in terms of the Hessian of \( \log v \), it is plausible that these results continue to hold under the weaker assumption whereby one replaces the Hessian with the Laplacian. For example, we can easily see that an assumption that

\[
\Delta \log v \geq -\frac{n}{\beta_0}, \quad \frac{1}{\beta_0} := \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\beta_i}
\]

is a weaker assumption than (1.22). In this subsection, we discuss our inequalities under (5.31). To this end, it is appropriate to refer to the work by Graczyk–Kemp–Loeb [45] where they investigated hypercontractivity for semi-group \( T_s f(x) := f(e^{-s}x) \) rather than \( P_s \) under a log-subharmonic assumption. In particular, by employing [45, Lemma 2.4], we may weaken the assumption (1.22) to a condition akin to (5.31). For certain technical reasons, on inputs \( v \) we impose that

\[
\int_{\mathbb{R}^n} v(x) \, dx < \infty, \quad v(x) e^{\frac{|x|^2}{2}} \leq C
\]
holds for some $C > 0$. We remark that if (5.32) is ensured then $v \in L^2(\gamma^{-1})$ (which is our technical assumption in Theorem 1.4).

**Theorem 5.5.** Let $\beta \geq 1$. If the twice differentiable function $v : \mathbb{R}^n \to (0, \infty)$ satisfies (5.32) and

$$\Delta \log v \geq -\frac{n}{\beta},$$

then we have (1.14) and (1.15).

**Proof.** The proof is essentially the same as the proof of Theorem 1.4. Following the argument and notation from that proof, we have only to ensure

(5.33) \[ \Delta \log v_t \geq -\frac{n}{\beta} \]

for all $t > 0$ under the assumption $\Delta \log v \geq -\frac{n}{\beta}$. To this end, we recall the formula (2.8). The assumption $\Delta \log v \geq -\frac{n}{\beta}$ can be read as

$$\Delta_x \log \phi(x, w) \geq 0$$

and hence $x \mapsto \phi(x, w)$ is log-subharmonic for each $w$. We also know that $\|\phi\|_{L^\infty} < \infty$ from the assumption (5.32). Hence we may employ [45, Lemma 2.4] to ensure that

$$\Delta_x \int_{\mathbb{R}^n} \log \phi(x, w) \, d\mu(w) \geq 0$$

which turns into (5.33). Once we ensure (5.33), we can run the same proof for Theorem 1.4 to conclude. \qed

It is also seems reasonable to expect the analogue of Theorem 5.5 when $0 < \beta \leq 1$ under a semi-log-superharmonic assumption. Namely, one may hope that for sufficiently well-behaved inputs $v$ satisfying

$$\Delta \log v \leq -\frac{n}{\beta},$$

we have both (1.14) and (1.15). Once one tries to apply the argument of Theorem 5.5 then one is naturally led to the following question. If some (sufficiently nice) function $f$ on $\mathbb{R}^n$ satisfies

$$\Delta \log f \leq 0,$$

does this continue to be satisfied by $P_t f$ for all $t > 0$? If true, this would constitute a complementary result to [45, Lemma 2.4] and, given our Lemma 2.1, appears to stand a chance. Interestingly, this is too optimistic and such preservation of log-superharmonicity can easily be seen to fail. In fact, if we take

$$f(x) = e^{x_1 x_2}, \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

then $\Delta \log f = 0$ whilst $\Delta \log P_t f > 0$ for all $t > 0$. Despite this, it remains possible that an analogue of Theorem 5.5 holds when $0 < \beta < 1$ under a semi-log-superharmonic assumption. We leave this as an interesting open problem.
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