Hamiltonian structure of the guiding-center Vlasov-Maxwell equations

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The Hamiltonian structure of the guiding-center Vlasov-Maxwell equations is presented in terms of a Hamiltonian functional and a guiding-center Vlasov-Maxwell bracket. The bracket, which is shown to satisfy the Jacobi identity exactly, is used to show that the guiding-center momentum and angular-momentum conservation laws can also be expressed in Hamiltonian form.

I. INTRODUCTION

The complementary Lagrangian and Hamiltonian formulations of dissipationless plasma equations have a rich history in plasma physics. In the Lagrangian formulation for the Vlasov-Maxwell equations, for example, one of several types of variational (Euler, Lagrange, or Euler-Poincaré) principles is used to derive these dissipationless plasma equations, while their exact conservation laws can be derived through the Noether method.

In the Hamiltonian formulation of dissipationless plasma equations, on the other hand, the equations for a given set of plasma fields ψ are, first, written in Hamiltonian form:

\[ \frac{\partial \psi_a}{\partial t} = j_{ab} \circ \delta H / \delta \psi_b, \]  

(1)

in terms of a Hamiltonian functional \( H[\psi] \) and the antisymmetric Poisson operator \( j^{ab}(\psi) \) acting (denoted as \( \circ \)) on the functional derivative \( \delta H / \delta \psi^b \). Next, the evolution of an arbitrary functional \( F[\psi] \) is expressed in Hamiltonian form:

\[ \frac{\partial F}{\partial t} = \left\langle \frac{\delta F}{\delta \psi^a} \right| \frac{\partial \psi^a}{\partial t} \right\rangle = \left\langle \frac{\delta F}{\delta \psi^a} \right| j^{ab} \circ \frac{\delta H}{\delta \psi^b} \right\rangle = [F, H], \]  

(2)

which is then used to construct a bracket \([, ,]\) on functionals, where the inner product \( \langle | \rangle \) involves suitable integrations over the domains of the field components.

For example, we consider the Vlasov-Maxwell equations with field components \( \psi = (f, E, B) \), which are expressed in Hamiltonian form:

\[ \frac{\partial f(x, p, t)}{\partial t} = -\left\{ f, \frac{\delta H}{\delta f} \right\} + 4\pi q \left\{ f, x \right\} \cdot \frac{\delta H}{\delta E}, \]  

(3)

\[ \frac{\partial E(x, t)}{\partial t} = 4\pi c \nabla \times \frac{\delta H}{\delta B} + 4\pi q \int_x \left\{ f, \frac{\delta H}{\delta f} \right\} \]  

(4)

\[ \frac{\partial B(x, t)}{\partial t} = -4\pi c \nabla \times \frac{\delta H}{\delta E} = j_{BE} \circ \frac{\delta H}{\delta E}, \]  

(5)

where the Vlasov function \( f \) is defined in six-dimensional particle phase space \( z = (x, p) \), while the electric and magnetic fields \( (E, B) \) are defined in three-dimensional configuration space \( x \), with the single-particle noncanonical Poisson bracket

\[ \{f, g\} = \nabla f \cdot \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \cdot \nabla g + \frac{q}{c} B \cdot \frac{\partial f}{\partial p} \times \frac{\partial g}{\partial p}, \]  

(6)

and the Hamiltonian functional

\[ H = \int_x \frac{1}{2m} f^2 + \int_x \frac{1}{\pi} \left( |E|^2 + |B|^2 \right), \]  

(7)

with \( \delta H / \delta f = |p|^2 / 2m \), \( \delta H / \delta E = E / 4\pi \), and \( \delta H / \delta B = B / 4\pi \). In Eqs. (3) and (4), summation over particle species is implied whenever an integration over the Vlasov distribution \( f \) appears, where \( \int_x \) and \( \int_z \) denote integrations over particle momentum and particle phase space, respectively, while \( \int_z \) denotes an integral over configuration space.

By expressing the evolution of an arbitrary functional \( F[f, E, B] \) on the Vlasov-Maxwell fields

\[ \frac{\partial F}{\partial t} = \int_z \frac{\partial F}{\partial t} \frac{\delta F}{\delta f} \]  

(8)

we easily arrive at the Vlasov-Maxwell bracket

\[ \left\{ F, H \right\} = \int_z \left\{ \frac{\delta F}{\delta f}, \frac{\delta H}{\delta f} \right\} \]  

(9)

where integrations by parts were performed. The proof that the Vlasov-Maxwell bracket satisfies the Jacobi identity for arbitrary functionals \( (F, G, K) \):

\[ \left[ \left[ F, G \right], K \right] + \left[ \left[ G, K \right], F \right] + \left[ \left[ K, F \right], G \right] = 0, \]  

(10)

was given in the Appendix of Ref. [3], where the Poisson bracket was broken into canonical and non-canonical parts, and Appendix B of Ref. [9], where properties of
the full Poisson bracket 15 were used. We note that the Jacobi property of the Vlasov-Maxwell bracket 10 is inherited from the Jacobi property of the Poisson bracket 9, which requires that \( \nabla \cdot \mathbf{B} = 0 \).

The purpose of the present paper is to follow a similar construction for the guiding-center Hamiltonian structure directly from the guiding-center Vlasov-Maxwell equations. This approach is in contrast to the Lie-transform construction of a Hamiltonian structure for the reduced Vlasov-Maxwell equations 10, which is derived from the guiding-center Vlasov-Maxwell bracket (9) and automatically guarantees that the reduced Vlasov-Maxwell bracket satisfies the Jacobi property. Here, while there is no guarantee that the reduced Vlasov-Maxwell bracket will satisfy the Jacobi property, its derivation is simple.

II. GUIDING-CENTER VLASOV-MAXWELL EQUATIONS

The variational formulations of the guiding-center Vlasov-Maxwell equations were presented by Pfirsch and Morrison 111 and more recently by Brizard and Tronci 12, whose works also included a derivation of exact conservation laws for energy-momentum and angular momentum through the Noether method. The guiding-center equations of motion considered here are the simplest equations derived from a variational principle 13, 14.

First, the guiding-center single-particle Lagrangian for a charged particle (of charge \( q \) and mass \( m \)) moving in a reduced phase space, with guiding-center position \( \mathbf{X} \) and guiding-center parallel momentum \( p_{||} \), is expressed as

\[
L_{gc} = \left( \frac{q}{c} \mathbf{A} + p_{||} \hat{\mathbf{b}} \right) \frac{d\mathbf{X}}{dt} - \left( q \Phi + K_{gc} \right),
\]

where the electromagnetic potentials \( \Phi, \mathbf{A} \) yield the electric field \( \mathbf{E} = -\nabla \Phi - c^{-1} \partial \mathbf{A} / \partial t \) and the magnetic field \( \mathbf{B} = \nabla \times \mathbf{A} = \hat{\mathbf{b}} \mathbf{B} \), and the guiding-center kinetic energy is \( K_{gc} = p_{||}^2 / 2m + \mu B \), where \( \mu \) denotes the guiding-center magnetic moment (which is a guiding-center invariant). The guiding-center equations of motion are derived from the guiding-center Lagrangian 111 as Euler-Lagrange equations, which are expressed as

\[
\frac{d\mathbf{X}}{dt} = \{\mathbf{X}, K_{gc}\}_{gc} + q \mathbf{E}^* \cdot \{\mathbf{X}, \mathbf{X}\}_{gc},
\]

\[
\frac{dp_{||}}{dt} = \{p_{||}, K_{gc}\}_{gc} + q \mathbf{E}^* \cdot \{\mathbf{X}, p_{||}\}_{gc},
\]

where the guiding-center Poisson bracket 14

\[
\{f, g\}_{gc} = \frac{\mathbf{B}^*}{B^*} \cdot \left( \nabla f \frac{\partial g}{\partial p_{||}} - \frac{\partial f}{\partial p_{||}} \nabla g \right)
- \frac{c \hat{\mathbf{b}}}{q B} \cdot \nabla f \times \nabla g
\]

is used without the ignorable gyromotion pair \((\mu, \theta)\), and the effective fields are

\[
\begin{align*}
\mathbf{E}^* &= \mathbf{E} - (p_{||}/q) \partial \hat{\mathbf{b}} / \partial t \\
\mathbf{B}^* &= \mathbf{B} + (p_{||}c/q) \nabla \times \hat{\mathbf{b}} \\
B_{||}^* &= \hat{\mathbf{b}} \cdot \mathbf{B}^* = B + (p_{||}c/q) \hat{\mathbf{b}} \cdot \nabla \times \hat{\mathbf{b}}
\end{align*}
\]

We note that the guiding-center Poisson bracket 14 satisfies the Jacobi property for arbitrary functions \((f, g, h)\):

\[
\{\{f, g\}_{gc}, h\}_{gc} + \{\{g, h\}_{gc}, f\}_{gc} + \{\{h, f\}_{gc}, g\}_{gc} = 0,
\]

subject to the condition

\[
\nabla \cdot \mathbf{B}^* = \nabla \cdot \mathbf{B} = 0,
\]

which is satisfied by the definition 14. We also note that the guiding-center Jacobian \( J_{gc} = 2\pi m B_{||} \) satisfies the guiding-center Liouville theorem

\[
\frac{\partial J_{gc}}{\partial t} + \nabla \cdot \left( J_{gc} \frac{d\mathbf{X}}{dt} \right) + \frac{\partial}{\partial p_{||}} \left( J_{gc} \frac{dp_{||}}{dt} \right) = 0,
\]

with the guiding-center equations of motion (12)-(13).

Next, we introduce the guiding-center Vlasov-Maxwell equations 12 for the guiding-center fields \( \Psi_{gc} = (F_{gc}, \mathbf{E}, \mathbf{B}) \):

\[
\begin{align*}
\frac{\partial F_{gc}}{\partial t} &= -\nabla \cdot \left( F_{gc} \frac{d\mathbf{X}}{dt} \right) - \frac{\partial}{\partial p_{||}} \left( F_{gc} \frac{dp_{||}}{dt} \right), \\
\frac{\partial \mathbf{E}}{\partial t} &= c \nabla \times \left( \mathbf{B} - 4\pi \mathbf{M}_{gc} \right) - 4\pi q \int_f F_{gc} \frac{d\mathbf{X}}{dt} \\
&= c \nabla \times \mathbf{H}_{gc} - 4\pi \mathbf{J}_{gc}, \\
\frac{\partial \mathbf{B}}{\partial t} &= -c \nabla \times \mathbf{E},
\end{align*}
\]

where the phase-space density \( F_{gc} \equiv F J_{gc} \) is defined in terms of the guiding-center Vlasov function \( F \) and the guiding-center Jacobian \( J_{gc} \), the guiding-center momentum integral \( \int_f \equiv \int dp_{||} \int dt \) excludes the guiding-center Jacobian \( J_{gc} \), and summation over particle species is implied whenever an integral over \( F_{gc} \) appears. In addition, the guiding-center magnetic field \( \mathbf{H}_{gc} \equiv \mathbf{B} - 4\pi \mathbf{M}_{gc} \) is defined in terms of the guiding-center magnetization

\[
\mathbf{M}_{gc} \equiv \int_f F_{gc} \left( -\mu \hat{\mathbf{b}} + \frac{q}{c} p_{||} \frac{d\mathbf{X}}{dt} \right),
\]

which is expressed in terms of the intrinsic guiding-center magnetization \(-\mu \hat{\mathbf{b}}\) and the moving guiding-center electric-dipole moment 15

\[
\left( \frac{q}{\Omega} \frac{d\mathbf{X}}{dt} \times \frac{p_{||} \hat{\mathbf{b}}}{mc} \right) \times \frac{p_{||} \hat{\mathbf{b}}}{qc} = \frac{q}{c} p_{||} \frac{d\mathbf{X}}{dt},
\]

where we introduced the symmetric dyadic tensor

\[
\mathbb{P}_{||} \equiv \frac{cp_{||}}{qB} \left( \mathbf{I} - \hat{\mathbf{b}} \hat{\mathbf{b}}^* \right).
\]
We note that, while the guiding-center magnetization \[22\] is derived from the guiding-center Lagrangian \[11]\: \( M_{gc} \equiv \int_p F_{gc} \delta L_{gc} / \delta B \), the guiding-center polarization \( P_{gc} \equiv \int_p F_{gc} \delta L_{gc} / \delta E \equiv 0 \) is absent in our model \[10].

We now express Eqs. \( 19-21 \) in Hamiltonian form \( \partial \Phi^a_{gc} / \partial t = J^{ab}_{gc} \delta H_{gc} / \delta \Phi^b_{gc} \): \[\begin{align*}
\frac{\partial F_{gc}}{\partial t} &= -B_\parallel \left\{ \frac{F_{gc}}{B_\parallel}, \frac{\delta H_{gc}}{\delta F_{gc}} \right\}_{gc} \\
&\quad - 4\pi q \delta^* \frac{\delta H_{gc}}{\delta E} \cdot B^* \left\{ X, \frac{X \cdot B}{B_\parallel} \right\}_{gc}, \quad (25)
\frac{\partial E}{\partial t} &= 4\pi c \nabla \times \left( \frac{\delta H_{gc}}{\delta B} - \frac{q}{c} \int_p F_{gc} \cdot \frac{dX}{dt} \right) \\
&\quad - 4\pi q \int_p F_{gc} \cdot \frac{dX}{dt}, \quad (26)
\frac{\partial B}{\partial t} &= -4\pi c \nabla \times \frac{\delta H_{gc}}{\delta E}, \quad (27)
\end{align*}\]

where the guiding-center velocity in Eq. \( 26 \) is \[\frac{dX}{dt} = \left\{ X, \frac{\delta H_{gc}}{\delta F_{gc}} \right\}_{gc} + 4\pi q \delta^* \frac{\delta H_{gc}}{\delta E} \cdot \{ X, X \}_{gc}. \quad (28)\]

Here, the guiding-center Hamiltonian functional is \[\begin{align*}
H_{gc} &= \int Z F_{gc} K_{gc} + \int X \frac{1}{8\pi} (|E|^2 + |B|^2) , \quad (29)
\end{align*}\]

where \( \int Z \) denotes an integration over the guiding-center phase space while \( \int X \) denotes an integral over the three-dimensional guiding-center configuration space, from which we obtain the functional derivatives \( \left( \frac{\delta H_{gc}}{\delta F_{gc}}, \frac{\delta H_{gc}}{\delta E}, \frac{\delta H_{gc}}{\delta B} \right) = \left( \frac{K_{gc}}{E_0 + B_0}, B_\parallel / 4\pi + \int_p F_{gc} \cdot \vec{b} \right), \quad (30)\)

where \( \delta K_{gc} / \delta B = \mu \vec{b} \) and we introduced the definition \[4\pi q \delta^* \frac{\delta H_{gc}}{\delta E} \equiv 4\pi q \left( \frac{\delta H_{gc}}{\delta E} + \frac{P_\parallel \cdot \nabla \times \frac{\delta H_{gc}}{\delta E}}{} \right) = qE - P_\parallel \frac{\partial \vec{b}}{\partial t} = qE^*, \quad (31)\]

after making use of Faraday’s law \[24\]. We note that the additional guiding-center Maxwell equations are \[\begin{align*}
\nabla \cdot E &= 4\pi q \int_p F_{gc} \equiv 4\pi \partial_{gc}, \quad (32)
\nabla \cdot B &= 0, \quad (33)
\end{align*}\]

where Eq. \( 32 \) is connected to Eq. \( 26 \) through the guiding-center charge conservation law \( \partial \rho_{gc} / \partial t + \nabla \cdot J_{gc} = 0 \), while Eq. \( 33 \) can be viewed as an initial condition of the Faraday equation \[21\].

### III. GUIDING-CENTER VLASOV-MAXWELL BRACKET

The guiding-center Vlasov-Maxwell bracket is now constructed from the guiding-center Vlasov-Maxwell equations \( 25-27 \) and the Hamiltonian functional \[22\]: \[\begin{align*}
\frac{\partial F_{gc}}{\partial t} &= \int Z \frac{\partial F_{gc}}{\partial t} + \int X \left( \frac{\partial E}{\partial t} \cdot \frac{\delta F_{gc}}{\delta E} + \frac{\partial B}{\partial t} \cdot \frac{\delta F_{gc}}{\delta B} \right) \\
&= \left\langle \frac{\partial F_{gc}}{\partial \Psi^a_{gc}}, \frac{\partial \Psi^a_{gc}}{\partial t} \right\rangle_{gc} = \left\langle \frac{\partial F_{gc}}{\partial \Psi^a_{gc}}, j_{gc}^b \frac{\partial H_{gc}}{\delta \Psi^b_{gc}} \right\rangle, \quad (34)
\end{align*}\]

where the guiding-center Vlasov-Maxwell bracket for two arbitrary functionals \( (F, G) \) of the fields \( \Psi = (F_{gc}, E, B) \) is defined in terms of the Poisson structure: \[\begin{align*}
[F, G]_{gc} &\equiv \left\langle \frac{\partial F_{gc}}{\partial \Psi^a_{gc}}, j_{gc}^b \frac{\partial G_{gc}}{\delta \Psi^b_{gc}} \right\rangle. \quad (35)
\end{align*}\]

Here, the antisymmetric Poisson operator \( j_{gc}^{ab}(\Psi) \) guarantees the antisymmetry property: \( [F, G]_{gc} = -[G, F]_{gc} \); and the bilinearity of Eq. \( 35 \) guarantees the Leibniz property: \( [F, G]_{gc} = [F, G]_{gc} + [G, F]_{gc} \). The Jacobi property:

\[\begin{align*}
J acc[F, G, K]_{gc} &\equiv \left[ [F, G]_{gc}, K \right]_{gc} + [G, K]_{gc}, F]_{gc} \\
&= 0, \quad (36)
\end{align*}\]

which holds for arbitrary functionals \( (F, G, K) \), involves the guiding-center Poisson operator \( j_{gc}^{ab}(\Psi) \).

From Eq. \( 35 \), we can now extract the guiding-center Vlasov-Maxwell bracket expressed in terms of two arbitrary guiding-center functionals \( (F, G) \) as

\[\begin{align*}
[F, G]_{gc} &= \int Z F_{gc} \delta F_{gc} / \delta F_{gc} + \int X \left( \frac{\delta G_{gc}}{\delta F_{gc}} \cdot \left\{ X, \frac{\delta F_{gc}}{\delta F_{gc}} \right\}_{gc} \right) \\
&\quad + (4\pi q)^2 \int Z F_{gc} \left( \delta^* \frac{\delta F_{gc}}{\delta E}, \{ X, X \}_{gc} \delta G_{gc} / \delta E \right) \\
&\quad + 4\pi c \int X \left( \frac{\delta F_{gc}}{\delta E} \cdot \nabla \times \frac{\delta G_{gc}}{\delta B} - \frac{\delta G_{gc}}{\delta E} \nabla \times \frac{\delta F_{gc}}{\delta B} \right), \quad (37)
\end{align*}\]

where \( \delta^*(\cdot) / \delta E \) is defined in Eq. \( 31 \). The guiding-center bracket \( 37 \) is analogous to Eq. \( 37 \). The Maxwell sub-bracket \( 39 \), where the Maxwell sub-bracket (last term) is identical in both cases, while the guiding-center Vlasov sub-bracket (first term) is connected by guiding-center phase-space transformation of the Vlasov sub-bracket in Eq. \( 39 \). The Interaction sub-bracket in Eq. \( 39 \), proportional to \( 4\pi q \), is transformed into the guiding-center Interaction sub-brackets in Eq. \( 37 \), where the quadratic term involving the antisymmetric dyadic Poisson bracket
\( \{X, X\}_{gc} \) represents the moving electric-dipole contribution to guiding-center magnetization, which is absent in Eq. 49. We note that, in contrast to recent work of Burby [17, 18], the guiding-center Vlasov-Maxwell bracket (47) is expressed in terms of functional derivatives involving the Vlasov-Maxwell fields \( (F_{gc}, E, B) \) since there is no guiding-center polarization in our model.

A. Guiding-center momentum conservation law

The conservation laws of energy-momentum and angular momentum for the guiding-center Vlasov-Maxwell (26-27) were recently derived by Brizard and Tronci [12]. As an application of the guiding-center Vlasov-Maxwell bracket (47), we explore the time derivative of the guiding-center Vlasov-Maxwell (vector-valued) momentum functional

\[
P_{gc} \equiv \int_X P_{gc} = \int_Z F_{gc} \rho_{\|} + \int_X E \times B / 4\pi c, \tag{38}
\]

where each component \( P_{gc} \) satisfies the functional evolution equation

\[
\frac{\partial P_{gc}}{\partial t} = \left[ P_{gc}, H_{gc} \right]_{gc}
= \int_Z F_{gc} \left\{ \frac{\delta P_{gc}}{\delta F_{gc}}, \frac{\delta H_{gc}}{\delta F_{gc}} \right\}_{gc}
- 4\pi q \int_Z F_{gc} \frac{\delta^* P_{gc}}{\delta E} \cdot \left\{ X, \frac{\delta H_{gc}}{\delta F_{gc}} \right\}_{gc}
+ 4\pi q \int_Z F_{gc} \frac{\delta^* H_{gc}}{\delta E} \cdot \left\{ X, \frac{\delta P_{gc}}{\delta F_{gc}} \right\}_{gc}
\nonumber

+ (4\pi q)^2 \int_Z F_{gc} \frac{\delta^* P_{gc}}{\delta E} \cdot \left\{ X, P_{gc} \right\}_{gc} \frac{\delta^* H_{gc}}{\delta E}
+ 4\pi c \int_X \frac{\delta H_{gc}}{\delta B} \cdot \nabla \times \frac{\delta P_{gc}}{\delta E}
- 4\pi c \int_X \frac{\delta^* P_{gc}}{\delta E} \cdot \nabla \frac{\delta H_{gc}}{\delta E}. \tag{39}
\]

Here, the functional derivatives of the guiding-center Hamiltonian functional (20) are given in Eq. (30), and the functional derivatives of the \( z \)-component of the guiding-center Vlasov-Maxwell momentum (48) are

\[
\left( \frac{\delta P_{gc}}{\delta F_{gc}}, \frac{\delta^* P_{gc}}{\delta E}, \frac{\delta^* P_{gc}}{\delta B} \right)
= \left( \begin{array}{c}
\rho_{\|} b_z \\
B \times \hat{z} \\
4\pi q \int_Z F_{gc} \hat{z} \times B
\end{array} \right),
\]

and

\[
4\pi c \frac{\delta^* P_{gc}}{\delta E} = B \times \hat{z} + \rho_{\|} \nabla \times (B \times \hat{z})
= B^* \times \hat{z} + (p_{gc} c / q) \nabla b_z. \tag{40}
\]

In Eq. (69), we now evaluate

\[
4\pi q \frac{\delta^* P_{gc}}{\delta E} \cdot \left\{ X, K_{gc} \right\}_{gc} = \hat{z} \cdot \nabla K_{gc} + \left\{ \frac{\delta P_{gc}}{\delta F_{gc}}, K_{gc} \right\}_{gc},
\]

so that

\[
4\pi q \frac{\delta^* H_{gc}}{\delta E} \cdot \left\{ X, \frac{\delta P_{gc}}{\delta F_{gc}} \right\}_{gc} + 4\pi q \frac{\delta^* P_{gc}}{\delta E} \cdot \left\{ X, X \right\}_{gc} = \hat{z} \cdot \left\{ \frac{\delta P_{gc}}{\delta F_{gc}}, K_{gc} \right\}_{gc},
\]

and, hence, we find

\[
\frac{\partial P_{gc}}{\partial t} = \int_Z F_{gc} \hat{z} \cdot \left( q E^* - \nabla K_{gc} \right) \tag{42}
\]

Next, we find

\[
4\pi c \frac{\delta^* H_{gc}}{\delta B} \cdot \nabla \times \frac{\delta P_{gc}}{\delta E} = \nabla \cdot \left( \frac{B^2}{8\pi} \hat{z} \right) + \int_Z F_{gc} \hat{z} \cdot \nabla K_{gc},
\]

where we made use of Eq. (32), so that Eq. (32) yields the guiding-center Vlasov-Maxwell momentum conservation law \( \partial P_{gc} / \partial t = [P_{gc}, H_{gc}]_{gc} = 0 \). The derivation of the angular guiding-center momentum conservation law \( \partial P_{gc} / \partial t = [P_{gc}, K_{gc}]_{gc} = 0 \), where \( P_{gc} = \int_X P_{gc} \cdot \nabla \phi / \partial \phi \) follows similar steps.

B. Guiding-center Casimir functionals

Casimir functionals \( \mathcal{C} \) satisfy the bracket property \( [\mathcal{C}, \mathcal{K}]_{gc} = 0 \), which holds for an arbitrary functional \( \mathcal{K} \). A standard example is the guiding-center entropy functional (omitting Boltzmann’s constant)

\[
S_{gc}[F_{gc}, B] \equiv - \int_Z F_{gc} \ln \left( F_{gc} / B_{\|} \right), \tag{43}
\]

for which we obtain

\[
[S_{gc}, \mathcal{K}]_{gc} = \int_Z F_{gc} \left\{ \frac{\delta S_{gc}}{\delta F_{gc}}, \frac{\delta \mathcal{K}}{\delta F_{gc}} \right\}_{gc}
+ 4\pi q \int_Z F_{gc} \frac{\delta \mathcal{K}}{\delta E} \left\{ X, \frac{\delta S_{gc}}{\delta F_{gc}} \right\}_{gc}
\nonumber

- 4\pi c \int_X \frac{\delta \mathcal{K}}{\delta B} \cdot \nabla \times \frac{\delta S_{gc}}{\delta E}. \tag{44}
\]
where $\delta S_{gc}/\delta F_{gc} = -1 - \ln(F)$, with $F \equiv F_{gc}/B_{\parallel}^*$, and

$$\frac{\delta S_{gc}}{\delta B} = \int F \hat{b} - \frac{q}{c} \frac{\partial F}{\partial p_{\parallel}} \left( B^* \frac{\partial F}{\partial p_{\parallel}} + \frac{e\hat{b}}{q} \times \nabla F \right),$$

which is obtained after using the magnetic variations

$$\begin{align*}
\delta B^* &= \delta B + \nabla \times (P_{\parallel} \cdot \delta B), \\
(c/q) \delta \hat{b} &= \delta B \cdot \partial P_{\parallel}/\partial p_{\parallel}, \\
\delta B_{\parallel} &= \delta \hat{b} \cdot B^* + \hat{b} \cdot \delta B^* \end{align*}.$$ \hspace{1cm} (45)

Hence, we find the guiding-center bracket identity

$$[S_{gc}, K]_{gc} = -\int \mathbf{J}_{ac} \left\{ \frac{\delta K}{\delta F_{gc}} \right\}_{gc} + 4\pi q \int \frac{\delta^p K}{\delta E} \left( B^* \frac{\partial F}{\partial p_{\parallel}} + \frac{e\hat{b}}{q} \times \nabla F \right) - 4\pi c \int \frac{\delta S_{gc}}{\delta B} \cdot \nabla \frac{\delta K}{\delta E} = 0,$$

where the first term on the right side vanishes since it is an exact phase-space divergence, while the last term cancels out the second term. The expression [43] for the guiding-center Vlasov-Maxwell entropy was recently mentioned by Burby and Tronci [14] and might find applications in the dissipative guiding-center bracket formulation [20–22] of the guiding-center Vlasov-Maxwell-Landau model (e.g., see Ref. [23]).

C. Jacobi property of the guiding-center Vlasov-Maxwell bracket

We now verify that the guiding-center bracket [37] satisfies the Jacobi property [36]. According to the Bracket theorem [8], the proof of the Jacobi property involves only the explicit dependence of the guiding-center Vlasov-Maxwell bracket [37] on the guiding-center fields $F_{gc}(F_{gc}, B)$, where we note that the dependence on the magnetic field $B$ enters through the guiding-center Poisson bracket [13], while the electric field $E$ is explicitly absent.

Hence, we can write the double-bracket involving three arbitrary guiding-center functionals $(F, G, K)$:

$$\left[ [F, G]_{gc}, K \right]_{gc} = \int F_{gc} \left\{ \frac{\delta^p [F, G]_{gc}}{\delta F_{gc}}, \frac{\delta K}{\delta F_{gc}} \right\}_{gc}$$

$$+ 4\pi q \int F_{gc} \frac{\delta^p K}{\delta \mathbf{E}} \left\{ \mathbf{X}, \frac{\delta^p [F, G]_{gc}}{\delta F_{gc}} \right\}_{gc}$$

$$- 4\pi c \int \frac{\delta^p [F, G]_{gc}}{\delta B} \cdot \nabla \frac{\delta K}{\delta \mathbf{E}},$$ \hspace{1cm} (46)

where the terms involving $\delta^p [F, G]_{gc}/\delta \mathbf{E}$ vanish on the basis of the Bracket theorem. Here, the functional derivative $\delta^p [F, G]_{gc}/\delta F_{gc}$ involves the explicit dependence on the guiding-center Vlasov distribution $F_{gc}$ in the Vlasov and Interaction sub-brackets, while $\delta^p [F, G]_{gc}/\delta B$ in the Maxwell sub-bracket involves the explicit dependence on the magnetic field $B$, which appears through $(\hat{b}, B^*, B_{\parallel}^*)$ in the guiding-center Poisson bracket [13] and the dyadic tensor [24], where $(\delta \hat{b}, \delta B^*, \delta B_{\parallel}^*)$ are given in Eq. [45].

The proof of the Jacobi property for the guiding-center bracket [37] involves using several identities derived from the guiding-center Poisson bracket [13] leading to an expansion in powers of $\epsilon = 4\pi q$ up to third order:

$$\mathcal{J}_{ac}[F, G, K] = \int F_{gc} \left\{ \left\{ f, g \right\}_{gc}, k \right\}_{gc} + \left\{ \left\{ g, k \right\}_{gc}, f \right\}_{gc} + \left\{ \left\{ k, f \right\}_{gc}, g \right\}_{gc}$$

$$+ \epsilon \int F_{gc} \left[ F^* \left\{ \left\{ X^i, g \right\}_{gc}, k \right\}_{gc} + \left\{ \left\{ g, k \right\}_{gc}, X^i \right\}_{gc} + \left\{ \left\{ k, X^i \right\}_{gc}, g \right\}_{gc} \right] + \odot \right]$$

$$+ \epsilon^2 \int F_{gc} \left[ F^{*g}_i G^*_j \left\{ \left\{ X^i, X^j \right\}_{gc}, k \right\}_{gc} + \left\{ \left\{ X^i, k \right\}_{gc}, X^j \right\}_{gc} + \left\{ \left\{ k, X^j \right\}_{gc}, X^i \right\}_{gc} \right] + \odot \right]$$

$$+ \epsilon^3 \int F_{gc} \left[ F^{*g}_i G^*_j K^*_l \left\{ \left\{ X^i, X^j \right\}_{gc}, X^\ell \right\}_{gc} + \left\{ \left\{ X^i, X^\ell \right\}_{gc}, X^j \right\}_{gc} + \left\{ \left\{ \ell, X^j \right\}_{gc}, X^i \right\}_{gc} \right],$$ \hspace{1cm} (47)

where $(f, g, k) \equiv (\delta F/\delta F_{gc}, \delta G/\delta F_{gc}, \delta K/\delta F_{gc})$ and $(F^*, G^*, K^*) \equiv (\delta^p F/\delta E, \delta^p G/\delta E, \delta^p K/\delta E)$, while all additional terms have cancelled out exactly (details of the proof are presented elsewhere [24]). Here, summation over the repeated indices $(i, j, \ell)$ denoting the components of the vector fields $(F^*, G^*, K^*)$ is implied, and the symbol $\odot$ denotes cyclic permutations of the functionals $(F, G, K)$. In Eq. [47], it is clear that the Jacobi property of the guiding-center Vlasov-Maxwell bracket [37] is inherited
from the Jacobi property \((\text{16})\) of the guiding-center Poisson bracket \((\text{14})\), since each term in Eq. \((\text{47})\) vanishes identically because of this latter property. Hence, the Jacobi property for the guiding-center Vlasov-Maxwell bracket \((\text{37})\) holds under the condition \((\text{17})\).}

IV. DISCUSSION

In the present paper, we derived the Hamiltonian structure of the guiding-center Vlasov-Maxwell equations introduced by Brizard and Tronci \((\text{12})\). The associated guiding-center momentum and angular-momentum conservation laws were also presented in Hamiltonian form in terms of the guiding-center Vlasov-Maxwell bracket \((\text{37})\). Since the guiding-center kinetic energy and Poisson bracket associated with the guiding-center Vlasov-Maxwell model considered here are both independent of the electric field, the effects of guiding-center polarization only appear through the moving electric-dipole contribution to the guiding-center magnetization \((\text{22})\).

Future work will consider the inclusion of guiding-center polarization into our guiding-center Vlasov-Maxwell equations, as well as applications of the guiding-center Hamiltonian structure in gauge-free gyrokinetic Vlasov-Maxwell theory \((\text{25, 26})\), where the perturbed electromagnetic fields \((\text{E}_1, \text{B}_1)\) appear explicitly in the gyrocenter Lagrangian (e.g., see Ref. \((\text{27})\)). We also plan to use the guiding-center Vlasov-Maxwell bracket \((\text{37})\) to explore extensions of Hamiltonian functional perturbation theory \((\text{28})\).

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Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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