On the scattering theory of the classical hyperbolic $C_n$ Sutherland model

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Abstract

In this paper, we study the scattering theory of the classical hyperbolic Sutherland model associated with the $C_n$ root system. We prove that for any values of the coupling constants the scattering map has a factorized form. As a byproduct of our analysis, we propose a Lax matrix for the rational $C_n$ Ruijsenaars–Schneider–van Diejen model with two independent coupling constants, thereby setting the stage to establish the duality between the hyperbolic $C_n$ Sutherland and the rational $C_n$ Ruijsenaars–Schneider–van Diejen models.

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1. Introduction

In the study of interacting many-particle systems, it is hard to overestimate the importance of scattering theory. At the same time, it is notoriously difficult to obtain rigorous results in this subject. It is a very fortunate situation that for certain integrable systems, defined on the real line, the scattering theory has been completely understood. In particular, we have full control over the scattering theory of the Toda systems, the Calogero–Moser–Sutherland models, and the Ruijsenaars–Schneider models associated with the $A_n$ root system (see e.g. [1–5]). Their characteristic feature is that the scattering map has a factorized form. Though these integrable many-particle systems have natural generalizations to other root systems as well (see e.g. [6]), the scattering theory of the non-$A_n$-type models is far less developed than that of the $A_n$-type systems.

In this paper, we undertake the task to understand the scattering behavior of the classical hyperbolic $C_n$ Sutherland model. (For background information on the $C_n$-type Sutherland systems see e.g. [6–9].) Recall that the phase space of this model is the cotangent bundle of
the Weyl chamber $\mathcal{C} := \{ q = (q_1, \ldots, q_n) \in \mathbb{R}^n | q_1 > \cdots > q_n > 0 \}$ and the dynamics is governed by the Hamiltonian

$$H_{C_n}(q, p) = \frac{1}{2} \sum_{c=1}^{n} p_c^2 + \sum_{1 \leq a < b \leq n} \left( \frac{g^2}{\sinh^2(q_a - q_b)} + \frac{g_2^2}{\sinh^2(q_a + q_b)} \right) + \sum_{c=1}^{n} \frac{g_2^2}{\sinh^2(2q_c)},$$

(1)

where $g$ and $g_2$ are arbitrary non-zero real numbers, the so-called coupling parameters. By the repulsive nature of the interaction, we expect that the particles move asymptotically freely for very large positive and negative values of time $t$; thus, it makes sense to study the scattering map that relates the asymptotic phases and momenta of the past and the future. Using only elementary algebraic techniques, the main goal of the paper is to show that the scattering map of the $C_n$-type model also has a factorized form, i.e., the classical phase shifts are entirely determined by the two-particle processes and by the one-particle scatterings on the external field. The precise statement is given in theorem 3. Though this result does meet our expectations, to our knowledge, its rigorous proof has not appeared in the literature before.

To understand the scattering properties of the hyperbolic $C_n$ Sutherland model, we closely follow Ruijsenaars’ seminal work [4] on the $A_n$ system. One of the upshots of his approach is that it reveals a natural action-angle duality between the hyperbolic Sutherland and the rational Ruijsenaars–Schneider models. Surprisingly, for root systems other than $A_n$, relatively less is known about the duality between the Sutherland and the Ruijsenaars–Schneider–van Diejen (RSvD) models. Even the Lax representation of the generic RSvD dynamics is missing, except for some very special one-parameter family of $C_n$ and $BC_n$ models obtained by the natural $\mathbb{Z}_2$-folding of the original $A_{2n-1}$ and $A_{2n}$ systems [10]. Only partial results [11] are known for the $D_n$ root system, too. However, as a byproduct of our scattering theoretic analysis, we obtain a natural candidate for the Lax matrix of the two-parameter family of rational $C_n$ RSvD models. The Lax matrix presented in lemma 2 generalizes the known one-parameter family of rational Lax matrices obtained by the folding procedure and it generates the rational $C_n$ RSvD Hamiltonian [12] with two independent coupling parameters.

The paper is organized as follows. To keep the presentation self-contained, in section 2, we collect the necessary background material and fix the notational conventions. In section 3, we present our results on the scattering theory of the hyperbolic $C_n$ Sutherland model. After the discussion part in section 4, we finish the paper with an appendix on some useful facts from linear algebra.

### 2. Preliminaries

In this section, we gather some basic facts about the hyperbolic $C_n$ Sutherland model. We start with a short review on some group theoretic material related to the non-compact Lie group $U(n, n)$; then, we discuss the Lax representation of the Sutherland dynamics. For details the reader may consult [6–9, 13, 14].

#### 2.1. Group theoretic background

Take an arbitrary $n \in \mathbb{N} = \{ 1, 2, \ldots \}$ and let $N := 2n$. With the aid of the unitary matrix

$$C := \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \in U(N)$$

(2)

we define the non-compact real reductive matrix Lie group

$$U(n, n) := \{ y \in GL(N, \mathbb{C}) | y^* C y = C \}.$$  

(3)
The fixed-point set of the Cartan involution $\Theta(y) := (y^{-1})^*$ is the maximal compact subgroup

$$U(n, n)_+ := \{ U \in U(n, n) | \text{U is unitary} \} \cong U(n) \times U(n). \quad (4)$$

Meanwhile, the submanifold $U(n, n)_- := \{ y \in U(n, n) | \Theta(y) = y^{-1} \}$ consists of the Hermitian elements of $U(n, n)$. Note that $C(2)$ is central inside $U(n, n)_+$.

The Lie algebra of $U(n, n)$ has the form $\mathfrak{u}(n, n) = [X \in \mathfrak{gl}(N, \mathbb{C}) | X^* C + CX = 0]$. The natural trace-pairing $\langle X, Y \rangle := \text{tr}(XY)$ provides an invariant, non-degenerate, bilinear form on $\mathfrak{u}(n, n)$. The Lie algebra involution $\theta(X) = -X^*$ corresponding to $\Theta$ induces the orthogonal $\mathbb{Z}_2$-gradation

$$\mathfrak{u}(n, n) = \mathfrak{u}(n, n)_+ \oplus \mathfrak{u}(n, n)_- \quad (5)$$

with the eigenspaces $\mathfrak{u}(n, n)_\pm := \ker(\theta \mp \text{Id})$. The bilinear form $\langle , \rangle$ is negative definite on the subalgebra $\mathfrak{u}(n, n)_+$, and positive definite on the complementary subspace $\mathfrak{u}(n, n)_-$. Note that $\mathfrak{u}(n, n)_+$ (resp. $\mathfrak{u}(n, n)_-$) consists of the anti-Hermitian (resp. Hermitian) elements of $\mathfrak{u}(n, n)$.

Now with any real $n$-tuple $q = (q_1, \ldots, q_n) \in \mathbb{R}^n$ we associate the diagonal matrices

$$q := \text{diag}(q_1, \ldots, q_n) \in \mathfrak{gl}(n, \mathbb{R}) \quad \text{and} \quad Q := \text{diag}(q, -q) \in \mathfrak{gl}(N, \mathbb{R}). \quad (6)$$

The subset $\mathfrak{a} := \{ q = \text{diag}(q, -q) | q \in \mathbb{R}^n \}$ is a maximal Abelian subspace in $\mathfrak{u}(n, n)$. Its centralizer inside $U(n, n)_+$ is the Abelian group

$$M := \{ \text{diag}(e^{iX}, e^{-iX}) | X \in \mathbb{R}^n \} \leq U(n, n)_+ \quad (7)$$

with the Lie algebra $\mathfrak{m} := \{ \text{diag}(iX, iX) | X \in \mathbb{R}^n \} \leq \mathfrak{u}(n, n)_+$. Note that both $\mathfrak{a}$ and $\mathfrak{m}$ are realized by diagonal matrices. Let $\mathfrak{a}^\perp$ (resp. $\mathfrak{m}^\perp$) denotes the subspace of the off-diagonal elements of $\mathfrak{u}(n, n)_-$ (resp. $\mathfrak{u}(n, n)_+$); then, we can write $\mathfrak{u}(n, n)_- = \mathfrak{a} \oplus \mathfrak{a}^\perp$ and $\mathfrak{u}(n, n)_+ = \mathfrak{m} \oplus \mathfrak{m}^\perp$.

The subspace $\mathfrak{m}^\perp \oplus \mathfrak{a}^\perp$ formed by the off-diagonal elements of $\mathfrak{u}(n, n)$ is invariant under the linear operator $\text{ad}_Q$, for any $Q \in \mathfrak{a}$. Therefore, the restricted operator

$$\widetilde{\text{ad}}_Q := \text{ad}_Q|_{\mathfrak{m}^\perp \oplus \mathfrak{a}^\perp} \quad (8)$$

is well defined, with the spectrum

$$\sigma(\widetilde{\text{ad}}_Q) = \{ q_a - q_b, \pm(q_a + q_b), \pm 2q_c | a, b, c \in \mathbb{N}_n, a \neq b \}, \quad (9)$$

where $\mathbb{N}_n := \{1, \ldots, n\} \subset \mathbb{N}$. The regular part of $\mathfrak{a}$ is defined by the subset

$$\mathfrak{a}_{\text{reg}} := \{ Q \in \mathfrak{a} | \widetilde{\text{ad}}_Q \text{ is invertible} \} \subset \mathfrak{a}. \quad (10)$$

Since $\mathfrak{a} \setminus \mathfrak{a}_{\text{reg}}$ is a union of finitely many hyperplanes, $\mathfrak{a}_{\text{reg}}$ is open and dense in $\mathfrak{a}$. The subset

$$c := \{ Q = \text{diag}(q, -q) | q = (q_1, \ldots, q_n) \in \mathbb{R}^n, q_1 > \cdots > q_n > 0 \} \quad (11)$$

is a connected component of $\mathfrak{a}_{\text{reg}}$, i.e. it is an open Weyl chamber. Note that the configuration space of the hyperbolic $C_n$ Sutherland model (1) can be identified with $c$. In the following, we will frequently use the identification $c \cong \{ q = (q_1, \ldots, q_n) \in \mathbb{R}^n | q_1 > \cdots > q_n > 0 \}$.

As is known, the elements of $\mathfrak{u}(n, n)_-$ can be ‘diagonalized’ by conjugation with elements from $U(n, n)_+$. More precisely, let $c^+$ denote the closure of $c$; then, the map

$$c^+ \times U(n, n)_+ \ni (Q, U) \mapsto U QU^{-1} \in \mathfrak{u}(n, n)_- \quad (12)$$

is well defined and onto. Moreover, the regular part of $\mathfrak{u}(n, n)_-$,

$$\mathfrak{u}(n, n)_-|_{\text{reg}} := \{ U QU^{-1} | Q \in c, \ U \in U(n, n)_+ \} \subset \mathfrak{u}(n, n)_-. \quad (13)$$

is open and dense subset inside $\mathfrak{u}(n, n)_-$, admitting the smooth bijective parametrization

$$c \times U(n, n)_+|_{\text{reg}} \ni (Q, U M) \mapsto U QU^{-1} \in \mathfrak{u}(n, n)_-|_{\text{reg}}. \quad (14)$$

That is, the above diffeomorphism provides the identification $(\mathfrak{u}(n, n)_-|_{\text{reg}} \cong c \times U(n, n)_+/M)$.
2.2. Lax representation of the Sutherland dynamics

Let $E \in \mathbb{C}^N$ denote the column vector with the components $E_a = 1$, $E_{a+d} = -1$ ($a \in \mathbb{N}_n$), and define

$$
\xi := ig(EE^* - I_N) + i(g - g_2)C \in \mathbb{M}_1 \subset u(n, n),
$$

where $g$ and $g_2$ are arbitrary non-zero real parameters. Utilizing the Riesz–Dunford functional calculus, with any $(q, p) \in T^*_T \cong T \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$ we associate the $N \times N$ matrix

$$
\mathcal{L}(q, p) := P - \coth(\tilde{a}d_q)\xi \in u(n, n),
$$

where $Q$ and $P$ denote the $N \times N$ diagonal matrices (6) associated with variables $q$ and $p$, respectively. Note that the matrix entries of $\mathcal{L}$ have the form

$$
\mathcal{L}_{a,b} = -\mathcal{L}_{na,a+b} = -ig \coth(qa_+ - qb), \quad \mathcal{L}_{a,n+b} = -\mathcal{L}_{na,a+b} = ig \coth(qa_+ + qb),
$$

$$
\mathcal{L}_{c,n+c} = -\mathcal{L}_{nc,c+c} = ig_2 \coth(2qc), \quad \mathcal{L}_{c,c} = -\mathcal{L}_{nc,n+c} = p_c.
$$

for any $a, b, c \in \mathbb{N}_n$, $a \neq b$. Next, consider the $m$-valued function $\Phi := ig \text{diag}(\varphi_1, \ldots, \varphi_m, \varphi_1, \ldots, \varphi_m)$ with

$$
\varphi_c(q, p) = -g \sum_{\ell \in \mathbb{N}_n\setminus\{c\}} (\sinh(qc_+ - qa)^2 + \sinh(qc_+ + qa)^2) - g_2 \sinh(2qc)^{-2},
$$

and define the $N \times N$ matrix

$$
\mathcal{B}(q, p) := \Phi(q, p) + \sinh(\tilde{a}d_q)^{-2}\xi \in u(n, n).\tag{20}
$$

For its matrix entries, we clearly have

$$
B_{a,b} = B_{na,a+b} = ig \sinh^{-2}(qa_+ - qb), \quad B_{a,n+b} = B_{na,a+b} = -ig \sinh^{-2}(qa_+ + qb),
$$

$$
B_{c,n+c} = B_{nc,c+c} = -ig_2 \sinh^{-2}(2qc), \quad B_{c,c} = B_{nc,n+c} = i\varphi_c,
$$

where $a, b, c \in \mathbb{N}_n$, $a \neq b$. As is known, the matrix-valued functions $\mathcal{L}$ and $\mathcal{B}$ provide a Lax pair for the hyperbolic $C_n$ Sutherland model. More precisely, along a smooth regular curve $q(t) \in \mathbb{c}$ ($t \in \mathbb{R}$), with $p(t) := \dot{q}(t)$, the Lax equation $\dot{\mathcal{L}} = [\mathcal{L}, \mathcal{B}]$ is satisfied if and only if $q(t)$ is a solution of the hyperbolic $C_n$ Sutherland dynamics.

An obvious consequence of the Lax representation of the dynamics is that the solution curves can be realized as projections of certain geodesics on the Riemannian manifold $U(n, n)$. Indeed, take an arbitrary solution $q(t) \in \mathbb{c}$ of the Sutherland dynamics and set $p(t) := \dot{q}(t)$. The differential equation $u(t)^{-1}\dot{u}(t) = \mathcal{B}(q(t), p(t))$ has a unique smooth solution $u(t) \in U(n, n)$ ($t \in \mathbb{R}$) with an initial condition, say, $u(0) = 1_N$. For any $t \in \mathbb{R}$, we define the positive definite matrix

$$
y(t) := u(t)e^{2\mathcal{L}(t)u(t)^{-1}}u(n, n)\tag{23}.
$$

Clearly, $y(t)$ is a smooth function of $t$ satisfying the equation $y^{-1}\dot{y} + \dot{y}y^{-1} = 4u\mathcal{L}u^{-1}$. It follows that $y(t)$ is a solution of the geodesic equation on $U(n, n)$, i.e.

$$
\frac{d}{dt} \left(\frac{y^{-1}\dot{y} + \dot{y}y^{-1}}{4}\right) = u(\mathcal{L} - [\mathcal{L}, \mathcal{B}])u^{-1} = 0,
$$

with $y(0) = e^{2\mathcal{L}(0)}$ and $y(0)^{-1}\dot{y}(0) + \dot{y}(0)y(0)^{-1} = 4\mathcal{L}(q(0), p(0))$. However, by introducing

$$
\mathcal{L}(q, p) := \cosh(\tilde{a}d_q)^{-1}\mathcal{L}(q, p) = P - \sinh(\tilde{a}d_q)^{-1}\xi \in u(n, n),\tag{25}
$$

the unique solution of the geodesic equation with the above initial conditions is the curve

$$
y(t) = e^{\mathcal{L}(0)}e^{2\mathcal{L}(q(t), p(t))}e^{\mathcal{L}(0)}.
$$


Comparing (23) and (26), we see that \( Q(t) \), and so the trajectory \( q(t) \), can be recovered by diagonalizing the matrix flow (26). In particular, we have the spectral identification

\[
\{ e^{2\tilde{q}(t)}, \ldots, e^{2q(t)}, e^{-2q(t)}, \ldots, e^{-2\tilde{q}(t)} \} = \sigma( e^{2Q(0)} e^{2LQ(0), p(0)}),
\]

whence the temporal asymptotics of the trajectory \( q(t) \) can be understood by analyzing the temporal asymptotics of the eigenvalues of the matrix flow (26). Though it is a natural matrix analytic question, to our knowledge the first reference containing the solution of this problem is Ruijsenaars’ paper [4].

3. Temporal asymptotics

In this section, we work out the temporal asymptotics of the hyperbolic \( C_\sigma \) Sutherland dynamics. Our main guide is Ruijsenaars’ result on the temporal asymptotics of the eigenvalues of exponential matrix flows (26). As dictated by theorem A2 in [4], the plan is to find the matrix entries of \( e^{2Q(0)} \) in an orthonormal basis, in which \( L(q(0), p(0)) \) is diagonal with decreasing diagonal entries. Having control over the matrix entries of \( e^{2Q(0)} \) in this new basis, simply by computing the quotients of the consecutive leading principal minors, we can determine the temporal asymptotics of the eigenvalues of the matrix flow.

In the rest of the paper, we simply write \( L \) and \( Q \) in place of \( L(q(0), p(0)) \) and \( Q(0) \). Since the initial conditions \( q(0) \) and \( p(0) = q(0) \) can be arbitrary, we think of \( L \) and \( Q \) as the matrix-valued smooth functions over the phase space \( T^*\mathbb{C} \equiv \mathbb{C} \times \mathbb{R}^n \).

3.1. Analyzing the spectrum of \( L \)

Recall that it is a crucial assumption of theorem A2 in [4] that the spectrum of matrix \( L \) is simple. To examine the spectrum of \( L \), we set up an equation for \( L \in u(n, n)_- \) and \( A := e^{2Q} \in U(n, n)_- \) as follows. By applying the linear operator \( \sinh(\text{ad}_Q) \) on \( L \) (25), we obtain

\[
\sinh(\text{ad}_Q)L = -\xi,
\]

which entails \( L e^{2Q} - e^{2Q} L = 2 e^{2Q} \xi e^{2Q} \). Now, recalling (15), we can write

\[
2igA + LA - AL = 2ig(e^{Q} E)(e^{Q} E)^* + 2i(g - g_2)C.
\]

Note that this equation is the complete analog of Ruijsenaars’ commutation relation (equation (2.4) in [4]) that he analyzed to discover the remarkable action-angle duality between the hyperbolic \( A_1 \) Sutherland and the rational \( A_2 \) Ruijsenaars–Schneider models. If \( g_2 = g \), the right-hand side of (29) is a matrix of rank 1; therefore, the analysis of the equation is relatively straightforward. Note that the special case \( g_2 = g \) corresponds to the \( \mathbb{Z}_2 \)-folding of the \( A_{2n-1} \) model. However, when \( g_2 \neq g \), the innocent looking term \( 2i(g - g_2)C \) complicates the analysis considerably.

To proceed further, we diagonalize \( L \in u(n, n)_- \). As we saw in (12), we can write

\[
L = U L U^{-1} = U \tilde{L} U^*,
\]

with some \( \tilde{L} \in \mathfrak{c}^- \) and \( U \in U(n, n)_+ \). Observe that \( \tilde{L} \) is unique, having the form \( \tilde{L} = \text{diag}(\lambda, -\lambda) \) with some \( \lambda_1 \geq \cdots \geq \lambda_n \geq 0 \), but the choice of \( U \) is not unique. However, at this point all we need is the existence of the pair \( (\tilde{L}, U) \); the consequences of the non-uniqueness will be discussed at the end of this subsection. Now from (29) we conclude that \( \tilde{L} \) and \( \tilde{A} := U^{-1} A U \in U(n, n)_- \) satisfy the equation

\[
2ig\tilde{A} + \tilde{L} \tilde{A} - \tilde{A} \tilde{L} = 2ig(U^* e^{Q} E)(U^* e^{Q} E)^* + 2i(g - g_2)C.
\]
Computationwise, it is very fortunate that the matrix $\tilde{A}^{-1} \in U(n, n)$ obeys a similar equation. Indeed, by conjugating the above equation with $C$, we obtain

$$2ig\tilde{A}^{-1} - \tilde{L}\tilde{A}^{-1} + \tilde{A}^{-1}\tilde{L} = 2ig(CU^* e^O E)(CU^* e^O E)^* + 2i(g - g^2)C.$$ (32)

By introducing the purely imaginary numbers

$$x_c = -x_{nc} := (2ig)^{-1}c \in i\mathbb{R},$$ (33)

the column vector

$$F := U^* e^O E \in \mathbb{C}^N,$$ (34)

and the real parameter

$$\varepsilon := 1 - g^2g^{-1} \in \mathbb{R},$$ (35)

for the matrix entries of $\tilde{A}$ and $\tilde{A}^{-1}$, we obtain

$$\tilde{A}_{k,l} = \frac{F_k T_l + \varepsilon C_{k,l}}{1 + x_k - x_l}, \quad (\tilde{A}^{-1})_{k,l} = \frac{(CF)_k (CF)_l + \varepsilon C_{k,l}}{1 - x_k + x_l}.$$ (36)

for any $k, l \in \mathbb{N}_N$. Note that the relations

$$\sum_{j=1}^{N} \frac{F_k T_j + \varepsilon C_{k,j}}{1 + x_k - x_j} = \frac{(CF)_k (CF)_l + \varepsilon C_{k,l}}{1 - x_k + x_l} = \delta_{k,l}.$$ (37)

obviously follow from (36).

Having equipped with the above formulae, we are able to analyze the spectrum of $L$ and the properties of the column vector $F$. For convenience, we introduce the notations

$$f_c := F_c, \quad h_c := F_{nc}, \quad z_c := f_c h_c \quad (c \in \mathbb{N}_n).$$ (38)

**Lemma 1.** *If the non-zero coupling parameters $g$ and $g^2$ satisfy $g^2 \neq 2g$, then the components of the column vector $F$ are non-zero and $L$ is a regular element of $u(n, n)$.***

**Proof.** We only show that the components of $F$ are non-zero. Proving by contraposition, suppose that $z_c = 0$ for some $c \in \mathbb{N}_n$. With $k = l = c$, from (37), we obtain the quadratic relation $\varepsilon^2 = (1 + 2x_c)^2$, i.e. $\varepsilon = \pm(1 + 2x_c)$. By comparing the real parts we obtain $\varepsilon = \pm 1$, which contradicts our assumption on the coupling parameters. Along the same line, by appropriately specializing the indices in equation (37), the regularity of $L$ also follows. □

**Remark 1.** Henceforth, we assume that the parameters $g$ and $g^2$ satisfy the additional technical condition $g^2 \neq 2g$. Note, however, that it does not restrict the values of the physically relevant positive coupling constants $g^2$ and $g^2$, since the pairs $(g, g^2)$ and $(g, -g^2)$ generate the same couplings in the model (1). In principle, without loss of generality, we could have imposed the condition $gg^2 < 0$ at the outset, thereby automatically excluding the case $g^2 = 2g$.

To conclude this subsection we wish to point out that the construction of $z_c$ (38) results in a well-defined smooth function on the phase space $T^* c$. As we saw in (14), by the regularity of $L$, the non-uniqueness of the diagonalizing matrix $U \in U(n, n)$, defined in (30) is controlled entirely by the centralizer subgroup $M$ (7). Namely, the only freedom in the choice of $U$ can be characterized by the transformations

$$U \mapsto U \text{diag}(e^{i\chi}, e^{i\chi}),$$ (39)
generated by some $\chi \in \mathbb{R}^n$. Now observe that the components $f_c$ and $h_c$ (38) of the column vector $F$ (34) transform as

$$f_c \mapsto e^{-i\chi} f_c \quad \text{and} \quad h_c \mapsto e^{-i\chi} h_c.$$  

(40)

Hence, $z_c = f_c h_c$ is independent of the choice of the representative $U$. It means that with each $L = L(q, p)$ we can associate the non-zero complex numbers $z_c = z_c(q, p)$ ($c \in \mathbb{N}_n$) in a unique and well-defined manner. To show that their dependence on the phase space variables $(q, p)$ is smooth, we note that by choosing appropriate smooth local sections of the (smooth)

fiber bundle

$$c \times U(n, n)_s \to c \times (U(n, n)_s/M) \cong (u(n, n)_s)_{\text{reg}},$$

(41)

we can work with representatives $U \in U(n, n)_s$ depending smoothly on the phase space variables in a small neighborhood of any given $(q, p) \in T^*c$. Thus, $z_c$ is smooth around any $(q, p)$, proving its smoothness over the whole phase space.

3.2. The structure of $\tilde{A}$

In this subsection, we proceed with a detailed analysis on the structure of matrix $\tilde{A}$. To this end, we make use of Jacobi’s theorem in linear algebra, i.e. we exploit some non-trivial relations between certain minors of matrices $\tilde{A}$ and $\tilde{B}$:

$$\tilde{B}(1 \cdots c \cdots n) = -\tilde{A}(n+1 \cdots n+c \cdots 2n).$$

(43)

Let $R$ and $S$ denote the $n \times n$ submatrices corresponding to the above minors on the left and on the right, respectively. By introducing the Cauchy-type $n \times n$ matrix $\Psi$ with the entries

$$\Psi_{a,b} := \frac{\overline{R}_{a,b}}{1 + x_a - x_b} \text{ if } b \neq c, \quad \text{and} \quad \Psi_{a,c} := \frac{\overline{R}_{a,c}}{1 + x_a - x_{n+c}}.$$  

(44)

from (36), we see that

$$R = \Psi + \varepsilon (1 + 2x_c)^{-1} e_{c,c}.$$  

(45)

Meanwhile, the entries of $S$ can be identified as $S_{a,b} = \overline{R}_{a,b} (a, b \in \mathbb{N}_n)$. Therefore, equation (43) can be cast into the particularly simple form

$$\det(R) + \det(\overline{R}) = 0.$$  

(46)
Since $R$ is a rank-1 perturbation of $\Psi$, the determinant formula (A.6) can be expressed as
\[
\det(R) = \det(\Psi) + \varepsilon(1 + 2x_c)^{-1}C_{c,c},
\] (47)
where $C_{c,c}$ is the cofactor of $\Psi$ associated with the entry $\Psi_{c,c}$. Since $\Psi$ is of Cauchy type, we obtain
\[
\det(\Psi) = (1 + 2x_c)^{-1}D_c\omega_c z_c \quad \text{and} \quad C_{c,c} = D_c.
\] (48)
Plugging these formulae into (46), for $z_c$, we obtain the linear relation
\[
(1 - 2x_c)\omega_c z_c + (1 + 2x_c)\overline{\omega}_c z_c + 2\varepsilon = 0.
\] (49)
Note that this single equation does not determine uniquely the complex quantity $z_c$.
To obtain an independent relation for $z_c$, we turn to Jacobi’s theorem, again. Namely, we can write
\[
\begin{pmatrix}
1 & \cdots & n & n + c \\
1 & \cdots & n & n + c
\end{pmatrix}
= \begin{pmatrix}
n + 1 & \cdots & n + c & 2n \\
n + 1 & \cdots & n + c & 2n
\end{pmatrix},
\] (50)
where the symbol $n + c$ means that the indicated row (and column) is deleted in the minor on the right-hand side. That is, on the left-hand side we have a principal minor of $\hat{B}$ of size $n + 1$, and the principal minor of $\hat{A}$ on the right-hand side has the size $n - 1$. Let $X$ and $Y$ denote the submatrices corresponding to these minors, respectively; then, we have $\det(X) = \det(Y)$. Since $Y$ is of Cauchy type, the relation $\det(Y) = D_c$ immediately follows. On the other hand, the computation of $\det(X)$ requires a longer preparation. To this end, we introduce the Cauchy-type $(n + 1) \times (n + 1)$ matrix $\Phi$ with the entries
\[
\Phi_{a,b} := \frac{\overline{h}_a h_b}{1 + x_a - x_b}, \quad \Phi_{a,n+1} := \frac{\overline{h}_a f_c}{1 + x_a - x_{a+c}},
\]
\[
\Phi_{n+1,b} := \frac{\overline{f}_b h_b}{1 + x_{n+c} - x_p}, \quad \Phi_{n+1,n+1} := |f_c|^2,
\] (51)
where $a, b \in \mathbb{N}_n$. Recalling (36), we see that
\[
X = \Phi + \varepsilon(1 + 2x_c)^{-1}e_{c,n+1} + \varepsilon(1 - 2x_c)^{-1}e_{n+1,c},
\] (52)
i.e. $X$ is a rank-2 perturbation of $\Phi$. Therefore, the determinant formula (A.8) yields
\[
\det(X) = \det(\Phi) + \varepsilon \left( C_{c,n+1} + \frac{C_{c,c+1}}{1 - 2x_c} \right) + \varepsilon^2 \frac{|C_{c,n+1}|^2 - C_{c,c}C_{n+1,n+1}}{(1 - 4x_c^2) \det(\Phi)},
\] (53)
where the $C_{i,j}$’s now denote the cofactors of $\Phi$. Using the special Cauchy-type form of $\Phi$, we obtain
\[
\det(\Phi) = -4x_c^2(1 - 4x_c^2)^{-1}D_c |\omega_c z_c|^2, \quad C_{c,n+1} = -(1 - 2x_c)^{-1}D_c \overline{\omega}_c z_c,
\] (54)
 altogether with the relations
\[
C_{c,c} = D_c |f_c|^2 \prod_{a=1}^n \frac{(x_c + x_a)(-x_c - x_a)}{(1 + x_c + x_a)(1 - x_c - x_a)},
\] (55)
\[
C_{n+1,n+1} = D_c |h_b|^2 \prod_{a=1}^n \frac{(x_c - x_a)(-x_c + x_a)}{(1 + x_c - x_a)(1 - x_c + x_a)}.
\] (56)

It immediately follows that the determinant of $X$ has the form
\[
\det(X) = -(1 - 4x_c^2)^{-1} D_c \left(4x_c^2|o_c z_c|^2 + \epsilon (o_c z_c + \overline{o_c} \overline{z}_c) + \epsilon^2 \right).
\] (57)

Finally, by putting these formulae together, we end up with the quadratic equation
\[
4x_c^2|o_c z_c|^2 + \epsilon (o_c z_c + \overline{o_c} \overline{z}_c) + \epsilon^2 + 1 - 4x_c^2 = 0.
\] (58)

Next, by solving equations (49) and (58) for $z_c$, we find the following two solutions:
\[
z_c = \pm \left(1 + \frac{1 \pm \epsilon}{2x_c} \right) \prod_{a=1}^n \left(1 + \frac{1}{x_c - x_a} \right) \left(1 + \frac{1}{x_c + x_a} \right).
\] (59)

In order to select the right one, we proceed as follows. In the phase space region where the particles are far from each other, i.e. $q_1 \gg \cdots \gg q_n \gg 0$, moving with high relative momenta, i.e. $p_1 \gg \cdots \gg p_n \gg 0$, the matrix $L$ (25) is almost diagonal; therefore, the diagonalizing matrix $U$ (30) is also nearly diagonal. Recalling equations (34) and (38), we see that in the given phase space region the value of $z_c$ is very close to $-1$. Since $z_c$ is a smooth function over the connected phase space $T^* c$, by invoking a standard continuity argument, we conclude that
\[
z_c = - \left(1 + \frac{i g^2}{\lambda_c} \right) \prod_{a=1}^n \left(1 + \frac{2i g}{\lambda_c - \lambda_a} \right) \left(1 + \frac{2i g}{\lambda_c + \lambda_a} \right).
\] (60)

Having determined $z_c$, we can find the form of the components $f_c$ and $h_c$ (38), too. As we discussed at the end of the previous subsection, by the non-uniqueness of $U$ (30), the non-zero complex quantities $f_c$ and $h_c$ are determined only up to a common phase factor (40). Therefore, purely for convenience, we may and shall assume that $f_c > 0$ for any $c \in \mathbb{N}_n$. So, we can write
\[
f_c = e^{\theta_c} |z_c|^\frac{i}{2} \quad \text{and} \quad h_c = e^{-\theta_c} z_c |z_c|^{-\frac{i}{2}}
\] (61)
with some $\theta_c \in \mathbb{R}$. Combining this parametrization with (36), we obtain the following description of the matrix $\tilde{A}$.

**Lemma 2.** With the aid of the $\lambda$-dependent functions $z_c$ (60), the matrix entries of $\tilde{A}$ take the form
\[
\tilde{A}_{a,b} = e^{\theta_a \theta_b} |z_a z_b|^\frac{i}{2} \frac{2i g}{2i g + \lambda_a - \lambda_b}, \quad \tilde{A}_{a+a,b+b} = e^{-\theta_a - \theta_b} |z_a z_b|^\frac{i}{2} \frac{2i g}{2i g - \lambda_a + \lambda_b},
\] (62)
\[
\tilde{A}_{a+b,b+a} = \tilde{A}_{a+b,a} = e^{\theta_a - \theta_b} |z_a z_b|^\frac{i}{2} \frac{2i g}{2i g + \lambda_a + \lambda_b} + \frac{i(g-g)}{i g + \lambda_a} \delta_{a,b},
\] (63)
where $a, b \in \mathbb{N}_n$.

**Remark 2.** Due to the presence of the $g^2$-dependent second term on the right-hand side of equation (63), the matrix $\tilde{A}$ can be seen as a $C_n$-type non-trivial deformation of the usual Cauchy matrices. Expanding the relation $\det(\tilde{A}) = 1$, the resulting Cauchy-type determinant formula might be of interest in other branches of mathematics and physics as well.
3.3. Asymptotic phases and momenta

Now let \( q(t) = (q_1(t), \ldots, q_n(t)) \in \mathbb{C} \) be an arbitrary solution of the hyperbolic \( C_n \) Sutherland dynamics; then, by (27) we can write

\[
\{ e^{2q_1(t)}, \ldots, e^{2q_n(t)}, e^{-2q_1(t)}, \ldots, e^{-2q_n(t)} \} = \sigma(\hat{A} e^{2\hat{L}}),
\]

where both \( \hat{L} \) and \( \hat{A} \) are computed at time \( t = 0 \). Since the diagonal entries of \( \hat{L} = \text{diag}(\lambda_1, -\lambda_1) \) are not in decreasing order, we conjugate it by the unitary \( N \times N \) matrix \( W := \text{diag}(1, R) \), where \( R \) is the unitary \( n \times n \) matrix with entries \( (R_{a,b})_{a,b} := \delta_{a+b,n+1} \) \((a, b \in \mathbb{N}_n)\). By setting

\[
\hat{L} := WLW^{-1} \quad \text{and} \quad \hat{A} := W\hat{A}W^{-1},
\]

we see that the diagonal entries of \( \hat{L} = \text{diag}(\lambda_1, \ldots, \lambda_n, -\lambda_n, \ldots, -\lambda_1) \) are decreasing, and for any \( a, b \in \mathbb{N}_n \), we have

\[
\hat{A}_{a,b} = \hat{A}_{a,b} = f_a(1 + x_a - x_b)^{-1} L_b.
\]

At this point, Ruijsenaars’ theorem [4] on the temporal asymptotics of exponential matrix flows is directly applicable. Recalling (64), for any \( c \in \mathbb{N}_n \), we obtain the asymptotic relation

\[
e^{2q_c(t)} \sim m_c e^{2\lambda_c} = e^{\ln(m_c)\cdot 2\lambda_c} \quad (t \to \infty),
\]

where \( m_c \)'s stand for the quotients of the consecutive leading principal minors of \( \hat{A} \). Since the \( c \)th leading principal minor of \( \hat{A} \) has the form

\[
P_c := \prod_{d=1}^{c} |f_d|^2 \prod_{1 \leq a < b \leq c} (1 - (x_a - x_b)^{-2})^{-1},
\]

we obtain

\[
m_c = \frac{P_c}{P_{c-1}} = |f_c|^2 \prod_{a=1}^{c-1} (1 - (x_a - x_c)^{-2})^{-1}.
\]

Therefore, for \( t \to \infty \), we have \( q_c(t) \sim q_c^+ + tp_c^+ \) with asymptotic phases and momenta

\[
q_c^+ = \frac{1}{2} \ln(m_c) = \frac{1}{2} \ln(|f_c|^2) = \frac{1}{2} \sum_{a=1}^{c-1} \ln(1 - (x_a - x_c)^{-2}) \quad \text{and} \quad p_c^+ = \lambda_c.
\]

By conjugating both \( \hat{L} \) and \( \hat{A} \) with the unitary \( N \times N \) matrix \( R_N \), we see that the diagonal entries of \( R_N \hat{L}R_N^{-1} \) are in strictly increasing order, so the \( t \to -\infty \) asymptotics can be handled similarly to the \( t \to \infty \) case. It turns out that for \( t \to -\infty \), we have \( q_c(t) \sim q_c^- + tp_c^- \), where

\[
q_c^- = \frac{1}{2} \ln(|h_c|^2) = \frac{1}{2} \sum_{a=1}^{c-1} \ln(1 - (x_c - x_a)^{-2}) \quad \text{and} \quad p_c^- = -\lambda_c.
\]

Now we are in a position to formulate the main result of the paper. Indeed, recalling the form of \( z_c \) (60), the comparison of (70) and (71) immediately leads to the precise relationships between the asymptotic phases and momenta.

**Theorem 3.** Take an arbitrary solution \( q(t) \in \mathbb{C} \) (\( t \in \mathbb{R} \)) of the hyperbolic \( C_n \) Sutherland dynamics. For \( |t| \to \infty \), the particles move asymptotically freely, i.e. for any \( c \in \mathbb{N}_n \) we have the asymptotics

\[
q_c(t) \sim q_c^\pm + tp_c^\pm \quad (t \to \pm \infty).
\]

The asymptotic momenta satisfy the relations

\[
p_c^+ = -p_c^- \quad \text{and} \quad p_1^+ > \cdots > p_n^+ > 0,
\]
and for the asymptotic phases we have

\[ q_i^+ = -q_i^- - \sum_{c=1}^{n-1} \delta(p_i^- - p_c^-, g) + \sum_{a=c+1}^{n} \delta(p_i^- - p_a^-, g) + \sum_{a=1}^{n} \delta(p_i^- + p_a^-, g) + \delta(2p_c^-, g) \]

with the 2-particle phase-shift function \( \delta(p, \mu) = 2^{-1} \ln(1 + 4\mu^2p^2) \).

4. Discussion

In this paper, we examined the scattering properties of the hyperbolic \( C_n \) Sutherland model. Under certain technical conditions, the classical repulsive particle systems are Liouville integrable, having only scattering states. Indeed, in general, the asymptotic momenta provide sufficiently many independent first integrals in involution. However, the asymptotic momenta of the hyperbolic \( C_n \) Sutherland model satisfy also the peculiar algebraic conditions

\[ p_i^+ + c = -p_i^- - \sum_{a=1}^{n} a \neq c \delta(p_i^- + p_a^-, g) + \delta(2p_c^-, g^2) \]

(74)

Following Ruijsenaars’ terminology \[5\], this distinguishing feature gives grounds for calling the \( C_n \)-type model a pure soliton system. As in the \( A_n \)-type models, we expect that this stronger notion of integrability is responsible for the factorized form of the scattering map.

We find it interesting that the application of elementary linear algebraic techniques succeeds in revealing the scattering behavior of the \( C_n \) model. Since the hyperbolic \( BC_n \) Sutherland model with three independent coupling constants is also closely tied with the matrix Lie group \( U(n, n) \) (see \[9\]), we believe that the algebraic machinery presented in this paper can be extended to understand the scattering properties of the \( BC_n \) model, too. Nevertheless, a complete classification of the pure soliton systems associated with the \( BC_n \) root system appears to be a more challenging analytic problem. For further motivation, we mention that these finite-dimensional many-particle systems are closely connected with integrable field theories. For example, it is well known that the family of \( A_n \)-type Ruijsenaars–Schneider models describe the soliton solutions of the sine-Gordon equation (see e.g. \[15, 16\]). Following \[17\], it is also natural to speculate on the mathematically adequate description of the relationship between the scattering theory of the \( BC_n \)-type particle systems and the soliton dynamics of the boundary sine-Gordon models, at both the classical and the quantum levels. We wish to come back to these issues in later publications.

To explore the scattering properties of the \( C_n \) model, we adapted Ruijsenaars’ approach \[4\] to the \( C_n \) root system. Thus, it is quite natural that the matrix \( \tilde{A} \) appearing in lemma 2 can be interpreted as the Lax matrix of the rational \( C_n \) RSvD model. Recall that matrix \( \tilde{A} \) depends on \( 2n \) real parameters \( \lambda = (\lambda_1, \ldots, \lambda_n) \) and \( \theta = (\theta_1, \ldots, \theta_n) \) with \( \lambda_1 > \cdots > \lambda_n > 0 \). Regarding the parameters \( \lambda_c \) and \( \theta_c \) as canonically conjugated positions and momenta, the natural Hamiltonian associated with the matrix \( \tilde{A} \) takes the form

\[ H(\lambda, \theta) = \frac{1}{4} \text{tr}(\tilde{A} + \tilde{A}^{-1}) = \sum_{c=1}^{n} \cosh(2\theta_c) \left( 1 + \frac{g^2}{\lambda_c^2} \right)^{1/2} \prod_{a=1}^{n} \left( 1 + \frac{4g^2}{(\lambda_c - \lambda_a)^2} \right)^{1/2} \left( 1 + \frac{4g^2}{(\lambda_c + \lambda_a)^2} \right)^{1/2} \]  

(75)

Now let us observe that, up to an irrelevant additive constant, this function can be identified with the rational limit of van Diejen’s Hamiltonian (see equation (6) in \[12\]) with two independent coupling parameters. To prove the canonicity of \( \lambda_c \) and \( \theta_c \) and to show the duality between the hyperbolic \( C_n \) Sutherland and the rational \( C_n \) RSvD models, one could
imitate Ruijsenaars’ original work [4] on the systems of type $A_n$. However, for the $A_n$-type models it has been shown recently [18] that the symplectic reduction framework provides the most efficient way to establish the duality relation. For completeness, we now briefly outline the reduction picture underlying the duality of the $C_n$ models. The compact Lie group $U(n, n)_+$ naturally acts on the manifold $U(n, n)_-$ by conjugations. The lift of this action to the cotangent bundle $T^*U(n, n)_-$ is Hamiltonian, admitting an equivariant momentum map $J : T^*U(n, n)_- \to u(n, n)_+ \cong u(n, n)_+$. In the reduction picture, our functional equation (29) corresponds to the momentum map constraint $J = \xi$ with the special choice $\xi$ (15). Solving this constraint in a gauge in which $A$ is diagonal, the outcome of the reduction procedure is the usual phase space of the Sutherland model (see e.g. [6–8]), and the Sutherland dynamics (1) is induced by the Hamiltonian $H = \text{tr}(L^2)/4$. Note, however, that in section 3 we solved the momentum map constraint in a gauge in which $L$ is diagonal. As a result, the reduced phase space can be parametrized by the $\lambda_i$’s and the $\theta_i$’s, and the dynamics of our interest (75) is induced by the Hamiltonian $H = \text{tr}(A + A^{-1})/4$. Therefore, by using two different gauges, one in which $A$ is diagonal, and one in which $L$ is diagonal, we obtain two different realizations of the reduced phase space $T^*U(n, n)_-//\xi U(n, n)_+$, whence a natural symplectomorphism between the phase spaces of the hyperbolic $C_n$ Sutherland and the rational $C_n$ RSvD models comes for free. The details, together with the generalization to the $BC_n$ systems, will be published elsewhere.

Note that the action-angle duality between the Sutherland and the RSvD models has important consequences from the perspective of scattering theory, too. Based on the duality relation, the scattering theory of the rational $BC_n$ RSvD models could also be understood. Moreover, as in the $A_n$-type models, the duality could greatly simplify the verification of the symplecticity of the Møller wave transformations both for the Sutherland and the RSvD models; thereby, the understanding of their scattering theory would be complete.

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Appendix

In this appendix, we summarize some linear algebraic facts used throughout the paper. The proofs and further details can be found e.g. in [19].

For an $n \times n$ matrix $X$, let

$$X = \begin{pmatrix} k_1 & k_2 & \cdots & k_p \\ l_1 & l_2 & \cdots & l_p \end{pmatrix}$$

(A.1)

denote the minor determinant of the $p \times p$ submatrix of $X$ lying on the intersection of rows $k_1, k_2, \ldots, k_p$ with columns $l_1, l_2, \ldots, l_p$. Jacobi’s theorem claims that there are simple relationships between the minors of $X$ and the minors of its inverse, as described below.

**Theorem A1.** Let $X$ be an invertible $n \times n$ matrix, $Y := (X^{-1})^T$, and choose a permutation

$$\sigma = \begin{pmatrix} k_1 & k_2 & \cdots & k_n \\ l_1 & l_2 & \cdots & l_n \end{pmatrix} \in S_n$$

(A.2)
of the pairwise distinct indices \(k_1, k_2, \ldots, k_n \in \mathbb{N}_n\). Then, for any \(p \in \{0, 1, \ldots, n\}\), we have

\[
Y \begin{pmatrix} k_1 & k_2 & \cdots & k_p \\ l_1 & l_2 & \cdots & l_p \end{pmatrix} = \frac{\text{sgn}(\sigma)}{\det(X)} X \begin{pmatrix} k_{p+1} & k_{p+2} & \cdots & k_n \\ l_{p+1} & l_{p+2} & \cdots & l_n \end{pmatrix},
\]

(A.3)

where \(\text{sgn}(\sigma)\) denotes the sign of permutation \(\sigma\).

In this paper, we frequently encounter Cauchy matrices and their perturbations. Recall that for the determinants of the Cauchy matrices, we have

\[
\det \left( \frac{1}{1 + \xi_k - \eta_l} \right) = \prod_{k < l} (\xi_k - \xi_l)(\eta_l - \eta_k) \prod_{k, l} (1 + \xi_k - \eta_l),
\]

(A.4)

where \(\xi_k\)’s and \(\eta_l\)’s are arbitrary complex numbers. The key formula that allows us to compute effectively the determinants of perturbed matrices is given in the following.

**Theorem A2.** Let \(X \in \mathbb{C}^{n \times n}\) be an invertible matrix and \(V, W \in \mathbb{C}^{n \times k}\) be arbitrary matrices; then, we have

\[
\det(X + VW^*) = \det(X) \det(I_k + W^*X^{-1}V).
\]

(A.5)

In particular, if \(X\) is perturbed by a multiple of the elementary matrix \(e_{a,b}\), then we can write

\[
\det(X + \alpha e_{a,b}) = \det(X) + \alpha C_{a,b},
\]

(A.6)

where \(C_{a,b}\) is the cofactor of \(X\) associated with the entry \(X_{a,b}\), i.e. it is \((-1)^{a+b}\) times the \((n-1) \times (n-1)\) minor obtained by deleting the \(a\)th row and the \(b\)th column of \(X\). For analog rank-2 perturbations, we have

\[
\det(X + \alpha e_{a,b} + \beta e_{c,d}) = \det(X) + \alpha C_{a,b} + \beta C_{c,d} + \alpha \beta (C_{a,b}C_{c,d} - C_{a,d}C_{c,b}) \det(X)^{-1}.
\]

(A.7)

Finally, if \(X\) is an invertible Hermitian matrix, the above formula simplifies to

\[
\det(X + \alpha e_{a,b} + \alpha^* e_{b,a}) = \det(X) + \alpha C_{a,b} + \alpha^* C_{b,a} + |\alpha|^2 (|C_{a,b}|^2 - C_{a,d}C_{b,c}) \det(X)^{-1}.
\]

(A.8)

Since any quadratic submatrix of a Cauchy matrix is Cauchy again, the above determinant formulae offer a relatively painless way to compute the determinants of perturbed Cauchy matrices.

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