Buckling analysis of laminated composites considering the effect of orthotropic material

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Abstract. In this work, the simplified forms of governing differential equations are employed to analyse the laminated composite plates subjected to buckling. Effect of orthotropic material via variation in the E₁/E₂ ratio on the buckling parameters is verified and corresponding results are provided. With the developments in the modern technology, composites have found a significant role in applications in aerospace industry. Under the compressive in-plane loads, the composites undergo buckling. Hence, it is crucial to observe and study the buckling characteristics of laminated composites with effect of material properties. We have obtained the results of the buckling loads of a laminated composite having a cross-ply lamination scheme of plates with simply supported and clamped boundary conditions. Effect of the orthotropic material on these parameters is studied via the present study.

1. INTRODUCTION

Composites and Laminates are currently mostly used in variety of structural, aerospace and mechanical engineering as having strength and much high stiffness. Hence these composites are favoured because of their property of being little weight. The failure in common structures including beams, columns and laminates are observed not only due to increasing stress but also from the phenomenon of stability. These laminates that are preferably used as thin laminate need to be checked against their ability to carry buckling load. Their ability needs to be investigated for application of different loads and BC’s. Because of the huge stiffness and minimum weights, composites have been receiving more attentiveness from the design considerations. Practically, it is seen that composites are mostly subjected to in-plane compressions leads to buckling when overloaded above their ability. Therefore, understanding the buckling phenomenon of composites became an important aspect in safety and reliability of the design of such composites. In regards of problems associated with the theoretical analysis for laminated composites, investigation on experimental basis has become priority to resolve for the stability characteristic of composite plates.

Putcha and Reddy et al. [1] established a mixed HOSDT for laminated plates subjected to buckling. Wankhade [2] performed geometric nonlinear analysis for skew type of plates. Wankhade and Bajoria [3-11] performed vibration as well as stability of SS piezolaminated plates by the virtue of FEM. Bendine and Wankhade [12] further considered vibration analysis of functionally graded piezo-plate based on LQR genetic. Gajbhiye and Ghugal [13] recently developed new 5th order SDT for thick
plates. Wankhade and Bajoria [14-15] further showed controlled vibration phenomenon of piezo-plates considering the effect of coupled loading. Bendine and Wankhade [16] observed shape control of piezo beams for optimal locations with various BC’s and loading. Wankhade and Bajoria [17] employed HOSDT to vibration characteristics of piezo- plates under actuating mechanism. Wankhade and Niyogi [18] provided buckling loads for composite for various thicknesses to span ratios and different mode. This present work performs stability of composites for changing the E1/E2 ratios and thickness to span ratios. Here results are presented to address the buckling phenomenon for orthotropic parameters with boundary conditions as SSSS and CCCC.

2. FORMULATION

It is observed that for the buckling; after the load reaches the primary buckling load, laminates remains stable. It can sustain much load as of its in-variably supported edges. Composite laminates can be formed with integrating together. Hence, these laminate possess greater strength capacity and higher modulus fibers, which are the main load carrying members. Due to this the main matrix keeps these fibers as a constituent. The ply can be laid in any orientation based on the strength criteria. Figure 1 gives a laminated composite plate applied with in-plane force.

Figure 1. Laminated plate with in-plane compressive loading

Displacement Function

\[ u(x, y, z) = u_0(x, y) - \frac{z}{2} \frac{\partial w}{\partial x} \left[ f(z) \right] \phi(x, y) \]

\[ v(x, y, z) = v_0(x, y) - \frac{z}{2} \frac{\partial w}{\partial y} \left[ f(z) \right] \psi(x, y) \]

\[ w(x, y, z) = w(x, y) \]  \hspace{1cm} (1)

Here, we used the function \( f(z) \) is taken as \( 1 - \frac{4z^2}{3h^2} \). Function \( f(z) \) may be taken as per the requirement of the theory considered as higher order (HOST), trigonometric (TSDT), hyperbolic (HySDT) etc.

GE for stability of a plate subjected to comp. in-plane load is derived as further. To achieve the GDE equations, we have first taken the equilibrium of forces and further the equilibrium of moments as stated:

The EE equations are shown follow:

\[ \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \]

\[ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \]  \hspace{1cm} (2)

In above \( N_x, N_y \) and \( N_{xy} \) are the internal forces in the normal and tangential directions.
Hence, the EE equation written in association with the moments is,
\[ \frac{\partial M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 W}{\partial x^2} + N_y \frac{\partial^2 W}{\partial y^2} + 2 N_{xy} \frac{\partial^2 W}{\partial x \partial y} = 0 \]  
(3)

The forces \( N_x, N_y \) and \( N_{xy} \) also moments \( M_x, M_y \) and \( M_{xy} \) acting on a composite are derived by taking the integration of the stress in every lamina across the laminate thickness. Taking the stress in terms of the displacement field, we have obtained the values of \( N_x, N_y \) and \( N_{xy}, M_x, M_y \) and \( M_{xy} \). The stress resultants are defined as,
\[ N_x = \int \sigma_x \, dz; \quad N_y = \int \sigma_y \, dz; \quad N_{xy} = \int \tau_{xy} \, dz \]  
(4)
\[ M_x = \int \sigma_x \, z \, dz; \quad M_y = \int \sigma_y \, z \, dz; \quad M_{xy} = \int \tau_{xy} \, z \, dz; \]  
(5)

Where \( \sigma_x, \sigma_y \) and \( \tau_{xy} \) are normal and shear stress.

Substituting \( \sigma_x, \sigma_y \) and \( \tau_{xy} \) for and in equations (4) and (5) and integrating we obtained for \( N \) layers the resultants, thus we can write,
\[
\begin{align*}
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} &= 
\begin{bmatrix}
A_1 & A_{12} & A_{16} \\
A_{12} & A_{22} & A_{26} \\
A_{16} & A_{26} & A_{36}
\end{bmatrix} 
\begin{bmatrix}
\epsilon_x^0 \\
\epsilon_y^0 \\
\gamma_{xy}^0
\end{bmatrix} 
+ 
\begin{bmatrix}
B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{36}
\end{bmatrix} 
\begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\end{align*}
\]  
(6)
\[
\begin{align*}
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} &= 
\begin{bmatrix}
B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{36}
\end{bmatrix} 
\begin{bmatrix}
\epsilon_x^0 \\
\epsilon_y^0 \\
\gamma_{xy}^0
\end{bmatrix} 
+ 
\begin{bmatrix}
D_{11} & D_{12} & D_{16} \\
D_{12} & D_{22} & D_{26} \\
D_{16} & D_{26} & D_{36}
\end{bmatrix} 
\begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\end{align*}
\]  
(7)

Where
\[
A_g = \sum_{i=1}^{N} \left( Q_{ij} \right) \left( Z_r - Z_{r-1} \right);
\]  
(8)
\[
B_g = \frac{1}{2} \sum_{i=1}^{N} \left( Q_{ij} \right) \left( Z_r^2 - Z_{r-1}^2 \right);
\]  
(9)
\[
D_g = \frac{1}{3} \sum_{i=1}^{N} \left( Q_{ij} \right) \left( Z_r^3 - Z_{r-1}^3 \right);
\]  
(10)

In above, \( A_{ij} \) represents the extensional stiffness, \( B_{ij} \) are coupling stiffness, and \( D_{ij} \) shows the flexural stiffness. For cross-ply laminates stress resultants are derived as below. In such type of laminate, if each lamina has the equal thickness, it is termed as a regular laminate. For such a laminate, equations simplifies to
\[
\begin{align*}
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} &= 
\begin{bmatrix}
A_1 & A_{12} & 0 \\
A_{12} & A_{22} & 0 \\
0 & 0 & A_{36}
\end{bmatrix} 
\begin{bmatrix}
\epsilon_x^0 \\
\epsilon_y^0 \\
\gamma_{xy}^0
\end{bmatrix} 
+ 
\begin{bmatrix}
0 & 0 & B_{16} \\
0 & 0 & B_{26} \\
B_{16} & B_{26} & 0
\end{bmatrix} 
\begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\end{align*}
\]  
(11)
\[
\begin{pmatrix}
M_x \\
M_y \\
M_{xy}
\end{pmatrix} = \begin{pmatrix}
0 & 0 & B_{16} \\
0 & 0 & B_{26} \\
B_{16} & B_{26} & 0
\end{pmatrix} \begin{pmatrix}
\epsilon_x^0 \\
\epsilon_y^0 \\
\gamma_{xy}^0
\end{pmatrix} + \begin{pmatrix}
D_{11} & D_{12} & 0 \\
D_{12} & D_{22} & 0 \\
0 & 0 & D_{36}
\end{pmatrix} \begin{pmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{pmatrix}
\] (12)

Let the laminate be oriented alternatively at 0° and 90° as of symmetric cross-ply or anti-symmetric cross-ply pattern. Substituting stress resultants from equations (11) and (12), after substituting for \(\epsilon_x^0, \epsilon_y^0, \gamma_{xy}^0, \kappa_x, \kappa_y\) and \(\kappa_{xy}\) in equations (2) and (3), we state the GE’s as

\[-A_{11} \frac{\partial^2 \phi}{\partial x^2} - A_{66} \frac{\partial^2 \phi}{\partial y^2} - (A_{12} + A_{66}) \frac{\partial^2 \psi}{\partial x \partial y} + B_{16} \frac{\partial^3 \psi}{\partial x \partial y^2} + (B_{12} + 2B_{66}) \frac{\partial^3 \psi}{\partial x^2 \partial y} = 0 \]

\[-L_{11} \frac{\partial^2 \phi}{\partial x^2} - L_{66} \frac{\partial^2 \psi}{\partial y^2} - (L_{12} + L_{66}) \frac{\partial^2 \psi}{\partial x \partial y} = 0 \]

\[-A_{22} \frac{\partial^2 \psi}{\partial y^2} - A_{66} \frac{\partial^2 \psi}{\partial x^2} - (A_{12} + A_{66}) \frac{\partial^2 \psi}{\partial x^2 \partial y} + B_{26} \frac{\partial^3 \psi}{\partial x \partial y^2} + (B_{12} + 2B_{66}) \frac{\partial^3 \psi}{\partial x^2 \partial y} = 0 \]

\[-L_{22} \frac{\partial^2 \psi}{\partial y^2} - L_{66} \frac{\partial^2 \psi}{\partial x^2} - (L_{12} + L_{66}) \frac{\partial^2 \phi}{\partial x \partial y} = 0 \]

\[-B_{11} \frac{\partial^3 \psi}{\partial x^2 \partial y} + (B_{12} + 2B_{66}) \left( \frac{\partial^3 \psi}{\partial x \partial y^2} + \frac{\partial^3 \phi}{\partial x^2 \partial y} - \frac{\partial^3 \psi}{\partial x^2 \partial y} \right) = 0 \]

\[-B_{22} \frac{\partial^3 \psi}{\partial y^2 \partial x} + (B_{26} + 2B_{66}) \left( \frac{\partial^3 \psi}{\partial x \partial y^2} + \frac{\partial^3 \phi}{\partial y^2 \partial x} - \frac{\partial^3 \psi}{\partial x^2 \partial y} \right) = 0 \]

\[-L_{11} \frac{\partial^3 \psi}{\partial x^2 \partial y} - L_{66} \frac{\partial^3 \psi}{\partial y^2 \partial x} + (L_{12} + L_{66}) \frac{\partial^3 \psi}{\partial x \partial y^2} + (J_{12} + 2J_{66}) \frac{\partial^3 \psi}{\partial x \partial y^2} = 0 \]

In the above Eqs., coefficients \(A_{ij}\) and \(B_{ij}\) are defined as following in terms of reduced stiffness coefficients \(\overline{Q}^{(k)}_{ij}\) of the \(k\)th layer.

\[(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \overline{Q}^{(k)}_{ij} (1, z, z^2) dz, \quad (i, j = 1, 2, 6)\]

\[L_{ij} = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \overline{Q}^{(k)}_{ij} f(z) dz\]

\[J_{ij} = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \overline{Q}^{(k)}_{ij} z f(z) dz\]
\[ P_{ij} = \sum_{k=1}^{n} \int \alpha_{ij} \phi_k \left[ f(z) \right] dz; S_{ij} = \sum_{k=1}^{n} \int \beta_{ij} \phi_k \left[ f(z) \right] dz, \quad (i, j = 1, 2) \]  

(20)

**BC conditions:**

The BC condition achieved with the present methodology is given as follow.

\[ M_x = D_{11} \frac{\partial \phi}{\partial x} + D_{12} \frac{\partial \psi}{\partial y} + D_{16} \left( \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x} \right) \]  

(21)

\[ M_y = D_{12} \frac{\partial \phi}{\partial x} + D_{22} \frac{\partial \psi}{\partial y} + D_{26} \left( \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x} \right) \]  

(22)

\[ M_{xy} = D_{16} \frac{\partial \phi}{\partial x} + D_{26} \frac{\partial \psi}{\partial y} + D_{66} \left( \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x} \right) \]  

(23)

\[ Q_x = kA_{55} \left( \phi + \frac{\partial w}{\partial x} \right) + kA_{45} \left( \psi + \frac{\partial w}{\partial y} \right) \]  

(24)

\[ Q_y = kA_{45} \left( \phi + \frac{\partial w}{\partial x} \right) + kA_{55} \left( \psi + \frac{\partial w}{\partial y} \right) \]  

(25)

The BC condition are given as:

SSSS: \( w = 0; M_n = 0; M_{ns} = 0 \)

CCCC: \( w = 0; \theta_x = 0; \theta_y = 0 \)

Strain Energy Equation :

\[ U = \frac{1}{2} \int \varepsilon^{T} Q \varepsilon \, dV + \frac{1}{2} \int \gamma^{T} Q \gamma \, dV + \int \left( \sigma_0 \right)^{T} \epsilon \, dV \]  

(26)

\[ U = \frac{1}{2} \int D_f \varepsilon^{T} \varepsilon \, dA + \frac{1}{2} \int \gamma^{T} D_f \gamma \, dA + \frac{1}{2} \int \left[ \frac{\partial w}{\partial x} \left( \frac{\partial w}{\partial x} \right) + \frac{\partial w}{\partial y} \left( \frac{\partial w}{\partial y} \right) \right] \sigma^{T} \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix} h \, dA \]  

\[ + \frac{1}{2} \int \left( \frac{\partial \theta_x}{\partial x} \left( \frac{\partial \theta_x}{\partial x} \right) + \frac{\partial \theta_x}{\partial y} \left( \frac{\partial \theta_x}{\partial y} \right) \right) \sigma^{T} \begin{bmatrix} \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_x}{\partial y} \end{bmatrix} h^{3} \, dA + \frac{1}{2} \int \left( \frac{\partial \theta_y}{\partial x} \left( \frac{\partial \theta_y}{\partial x} \right) + \frac{\partial \theta_y}{\partial y} \left( \frac{\partial \theta_y}{\partial y} \right) \right) \sigma^{T} \begin{bmatrix} \frac{\partial \theta_y}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \end{bmatrix} h^{3} \, dA \]  

\[ \sigma_0 = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \]  

(27)

The eigen value problem is solved for stability criterion as,

\[ [K - \lambda K_0] \alpha^T = 0, \]  

(28)
3. RESULTS AND DISCUSSION

_Buckling analysis of three layered (0°/90°/0°) laminated composite plate with effect of orthotropy_
This part shows the buckling analysis of three layered (0°/90°/0°) laminated composite plate with the effect of orthotropic material. Different parameters of the plate are considered in the analysis and the corresponding results are obtained. These parameters include varying E1/E2 ratios and with that of different BC conditions. The effects of these parameters are illustrated for various modes of buckling. Graphs for comparison of buckling loads of SSSS and CCCC (0°/90°/0°) laminated plate are shown as follows in figure 2-5.

Material Properties: E1/E2 =10, 20…40; G12/E2 = G13/E2 = 0.6; G23/E2 = 0.5; v_{12} = 0.25

![Figure 2. Effect of material orthotropy on buckling parameter for SSSS laminated plate for first mode with a/b=1.5](image1)

![Figure 3. Effect of material orthotropy on buckling parameter for CCCC laminated plate for first mode with a/b=1.5](image2)
4. CONCLUSION

In the present piece of work, the stability analysis of laminated composites has been performed. With varying the parameters which includes the span to thickness ratio for a range of values of $E_1/E_2$ ratio, the corresponding buckling loads are obtained. It is illustrated bases on the results that as thickness of lamina lowers, the plate buckles for smaller amount of load. Buckling load decreases as $(a/h)$ ratio of plate increases. The analysis is performed for three layered $(0^\circ/90^\circ/0^\circ)$ isotropic laminated plate under variation in the material properties. In this way, results provided from this work emphasizes on the magnified effects due to buckling with these parameters which includes BC condition, aspect ratio and span-to-thickness ratio.
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