On the number of graphs without large cliques

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Abstract.
In 1976 Erdős, Kleitman and Rothschild determined the number of graphs without a clique of size \( \ell \). In this note we extend their result to the case of forbidden cliques of increasing size. More precisely we prove that for \( \ell n \leq (\log n)^{1/2}/2 \) there are

\[
2^{(1 - 1/(\ell n - 1))n^2/2 + o(n^2)} K_{\ell n}\text{-free graphs of order } n.
\]

Our proof is based on the recent hypergraph container theorems of Saxton, Thomason and Balogh, Morris, Samotij, in combination with a theorem of Lovász and Simonovits.

1. Introduction

Let \( F \) be an arbitrary graph. A graph \( G \) is called \( F \)-free if \( G \) does not contain \( F \) as a (weak) subgraph. Let \( f_n(F) \) denote the number of (labeled) \( F \)-free graphs on \( n \) vertices. As every subgraph of an \( F \)-free graph is also \( F \)-free, we trivially have \( f_n(F) \geq 2^{ex(n,F)} \), where \( ex(n,F) \) denotes the maximum number of edges of an \( F \)-free graph on \( n \) vertices. It is well known [17, 9, 10] that

\[
ex(n,F) = \left(1 - \frac{1}{\chi(F) - 1}\right) \frac{n^2}{2} + o(n^2).
\]

Erdős, Kleitman and Rothschild [5] showed that in the case of cliques, i.e., for \( F = K_\ell \), this lower bound actually provides the correct order of magnitude. Erdős, Frankl and Rödl [7] later showed that a similar result holds for all graphs \( F \) of chromatic number \( \chi(F) \geq 3 \):

\[
f_n(F) = 2^{(1 + o(1))ex(n,F)}.
\]

Note that these results just provide the asymptotics of \( \log_2(f_n(F)) \). Extending an earlier result from [5] for triangles, Kolaitis, Prömel and Rothschild [12] determined the typical structure of \( K_\ell \)-free graphs by showing that almost all of them are \((\ell - 1)\)-colorable. Thus,

\[
f_n(K_\ell) = (1 + o(1)) \cdot \text{col}_n(\ell - 1),
\]

where \( \text{col}_n(\ell) \) denotes the number of (labeled) \( \ell \)-colorable graphs on \( n \) vertices. An asymptotic for \( \text{col}_n(\ell) \) is given in [15]. For additional results and further pointers to the literature see e.g. [1, 2, 5, 14].

All of the above results consider the case of a fixed forbidden graph \( F \). Much less is known if the size of the forbidden graph \( F \) increases with the size of the host graph \( G \). The study of such situations was started only recently by Bollobás and Nikiforov [6]. They showed that for any sequence \( (F_n) \) of graphs with \( v(F_n) = o(\log n) \) one has

\[
\log_2 f_n(F_n) = \left(1 - \frac{1}{\chi(F_n) - 1}\right) \left(\frac{n}{2}\right) + o(n^2).
\]
It is interesting to note that the proof of (3) completely avoids the use of the regularity lemma, a common tool for attacking this kind of questions. Indeed, because of the tower-type dependence of the size of an \( \epsilon \)-regular partition on the parameter \( \epsilon \) (see [11]), it seems hard to adapt the regularity-based proof of (1) to the case of forbidden subgraphs of non-constant size. Furthermore, observe that (3) is only non-trivial if the chromatic number \( \chi(F_n) \) is bounded. In particular, it does not determine \( \log_2 f_n(K_{\ell_n}) \) for an increasing sequence \( \ell_n \) of positive integers, because the term \( o(n^2) \) swallows the lower-order term \( \frac{n^2}{\ell_n - 1} \).

The aim of this paper is to provide the first non-trivial result for forbidden cliques of increasing size:

**Theorem 1.** Let \( (\ell_n)_{n \in \mathbb{N}} \) be a sequence of positive integers such that for every \( n \in \mathbb{N} \), we have \( 3 \leq \ell_n \leq (\log n)^{1/4}/2 \). Then

\[
\log_2 f_n(K_{\ell_n}) = \left(1 - \frac{1}{\ell_n - 1}\right) \left(\frac{n}{2}\right) + o\left(n^2/\ell_n\right).
\]

Our proof is based on the recent powerful hypergraph container results of Balogh, Morris, Samotij [4] and Saxton and Thomason [16].

The upper bound on \( \ell_n \) in our theorem is an artifact of our proof. We have no reason to believe that this bound is tight. In fact, it is not unconceivable that the statement from Theorem 1 holds up to the size of a maximal clique in the random graph \( G_{n,1/2} \) which is known to be \((2+o(1)) \log_2 n\). We leave this question to future research.

Note also that, similarly to the result of Erdős, Kleitman and Rothschild [8], our theorem just provides the asymptotics of the logarithm of \( f_n(K_{\ell_n}) \). However, our paper has already stimulated further research, and very recently a structural result in the spirit of Kolaitis, Prömel and Rothschild has been established by Balogh et al. [3].

2. Hypergraph Containers

In the proof of Theorem 1 we make use of the hypergraph container theorem proved independently by Saxton and Thomason [16] and Balogh, Morris and Samotij [4]. Before we state this theorem, we introduce some notation.

Let \( H \) be an \( r \)-uniform hypergraph with the average degree \( d \). Then for every \( \sigma \subseteq V(H) \), we define the co-degree

\[
d(\sigma) = |\{e \in E(H) : \sigma \subseteq e\}|.
\]

Moreover, for every \( j \in [r] \), we define the \( j \)-th maximum co-degree

\[
\Delta_j = \max \{d(\sigma) : \sigma \subseteq V(H) \text{ and } |\sigma| = j\}.
\]

Finally, for any \( p \in (0,1) \), we define the function

\[
\Delta(H,p) = 2^{\binom{r}{2} - 1} \sum_{j=2}^{r} 2^{-\binom{r - 1}{2}} \frac{\Delta_j}{dp^{j-1}}.
\]

We will use the following version of the general hypergraph container theorem.

**Theorem 2 (Saxton-Thomason [16]).** There exists a positive integer \( c \) such that the following holds for all positive integers \( r \) and \( N \). Let \( H \) be an \( r \)-uniform hypergraph of order \( N \). Let \( 0 \leq p \leq 1/(cr^2) \) and \( 0 < \varepsilon < 1 \) be such that \( \Delta(H,p) \leq \varepsilon/(cr^2) \). Then there exists a collection \( \mathcal{C} \subseteq \mathcal{P}(V(H)) \) such that

(i) every independent set in \( H \) is contained in some \( C \in \mathcal{C} \),

(ii) for all \( C \in \mathcal{C} \), we have \( e(H[H[C]]) \leq \varepsilon e(H) \), and
(iii) the number $|\mathcal{C}|$ of containers satisfies
\[ \log |\mathcal{C}| \leq c r^3 (1 + \log(1/\epsilon)) \cdot Np \log(1/p). \]

Theorem 2 is an easy consequence of Theorem 5.3 in the paper of Saxton and Thomason [16]; it is derived exactly as Corollary 2.7 (also therein), the only difference being that we are precise about the dependence on the edge size $r$. We also mention that our notation deviates slightly from that used in [16]: we write $p$ instead of $\tau$ and we use $\Delta(H, p)$ as an upper bound for the function $\delta(H, \tau)$ used by Saxton and Thomason. Finally, let us just note without further explanation that Theorem 2 is much weaker than the general container theorem, although it is sufficient for the purposes of this note (and is simpler to state and apply).

As a corollary of Theorem 2 we prove the following version tailored for a collection of $K_\ell$-free graphs, with $\ell$ being a function of $n$.

**Corollary 3.** For every constant $\delta > 0$ and sequence $(\ell_n)_{n \in \mathbb{N}}$ such that $3 \leq \ell_n \leq (\log n)^{1/4}/2$, the following holds for all large enough $n \in \mathbb{N}$: there exists a collection $\mathcal{G}$ of graphs of order $n$ such that

(i) every $K_\ell$-free graph of order $n$ is a subgraph of some $G \in \mathcal{G}$,
(ii) every $G \in \mathcal{G}$ contains at most $\delta (\ell_n)^{2}/c^{\ell_n}$ copies of $K_{\ell_n}$, and
(iii) the number $|\mathcal{G}|$ of graphs in the collection satisfies
\[ \log |\mathcal{G}| \leq \frac{\delta n^{2}}{\ell_n}. \]

**Proof.** Let us assume that $n$ is large enough and write $\ell := \ell_n$.

Let $H$ be a hypergraph defined as follows: the vertex set of $H$ is the edge set of $K_n$, and the edges of $H$ are the edge sets of subgraphs of $K_n$ isomorphic to $K_\ell$. Observe that the graph $H$ is an $(\ell/2)$-uniform hypergraph of order $\binom{n}{2}$ with $e(H) = \binom{n}{\ell/2}$. With some foresight, we would like to apply Theorem 2 with $\epsilon = \delta e^{-\ell}$ and $p = n - (\log \ell)/(2\ell^2)$

\[ (4) \]

and
\[ (5) \]

for every positive integer constant $c$.

Let us start with the values $\Delta_j$ for $H$. Consider some $\sigma \subseteq V(H)$ with $|\sigma| = j$, where $1 \leq j \leq \binom{n}{\ell/2}$. We can view $\sigma$ as a subgraph of $K_n$ with $v(\sigma) = |\bigcup\{e : e \in \sigma\}|$ vertices and $|\sigma| = j$ edges. The co-degree $d(\sigma)$ is then simply the number of ways in which we can extend this graph to a copy of $K_\ell$ in $K_n$. If $v(\sigma) > \ell$ then clearly $d(\sigma) = 0$, and otherwise
\[ d(\sigma) = \binom{n - v(\sigma)}{\ell - v(\sigma)} \leq n^{\ell - v(\sigma)}. \]

Note that $j \leq \binom{v(\sigma)}{2}$ implies that
\[ v(\sigma) \geq \frac{1 + \sqrt{1 + 8j}}{2} > \frac{1}{2} + \sqrt{2j}, \]

giving the bound
\[ \Delta_j \leq n^{\ell/2 - \sqrt{2j}}. \]
On the other hand, using that that \( \ell \leq (\log n)^{1/4}/2 \) and that \( n \) is sufficiently large, the average degree \( d \) of \( H \) is
\[
d = \left( \frac{n - 2}{\ell - 2} \right) \geq \left( \frac{n}{7} \right)^{\ell - 2} \geq n^{\ell - 1.9},
\]
so for \( 2 \leq j \leq \left( \frac{\ell}{2} \right) \), we have
\[
\frac{\Delta_j}{dp^{j-1}} \leq n^{1.4 - \sqrt{2j + (j-1)(\log \ell)/(2\ell^2)}}.
\]
Using the fact that \( \log(\ell)/\ell \leq 1/e \) holds for all \( \ell > 0 \), we have, for every \( 2 \leq j \leq \left( \frac{\ell}{2} \right) \), that
\[
\sqrt{2j} - \frac{(j - 1)(\log \ell)}{2\ell^2} \geq \sqrt{2j} - \frac{(j - 1)(\log \ell)}{2\sqrt{2} \ell} \geq \sqrt{2j} - \frac{\sqrt{j}}{2e \sqrt{2}} \geq 2 - \frac{1}{2e},
\]
whence, for sufficiently large \( n \),
\[
\frac{\Delta_j}{dp^{j-1}} \leq n^{1.4 - 2 + 1/(2e)} \leq n^{-1/4}.
\]
Then, using \( \ell \leq (\log n)^{1/4}/2 \), for large enough \( n \), we get
\[
\Delta(H, p) \leq e^{\ell \cdot \sum_{j=2}^{\left( \frac{\ell}{2} \right)} 2^{-(j-1)} \cdot n^{-1/4}} \leq e^{\ell} n^{-1/4} \leq \delta/(ce^{\ell^2}),
\]
which easily implies the desired bound \( \delta \) on \( \Delta(H, p) \). On the other hand, again using \( \ell \leq (\log n)^{1/4}/2 \), we have
\[
p = n^{-(\log \ell)/(2\ell^2)} \leq 1/(ce^{\ell^2}) \tag{7}
\]
for all large enough \( n \), so \( p \) satisfies \( \delta \). Therefore, we can apply Theorem 2 with parameters \( \varepsilon \) and \( p \).

We now turn to the construction of the family \( G \). Let \( C \) be a collection of subsets of \( V(H) \) given by Theorem 2. We show that the family of graphs
\[
\mathcal{G} = \{(\{n\}, C) : C \in \mathcal{C}\}
\]
satisfies the claim.

Suppose that \( I \) is some \( K_{\ell} \)-free graph on the vertex set \([n]\). Then its edge set \( E(I) \) is an independent set in \( H \), and thus there exists \( C \in \mathcal{C} \) such that \( E(I) \subseteq C \). Therefore there exists \( G \in \mathcal{G} \) such that \( I \) is a subgraph of \( G \), and the property (i) holds. Furthermore, since \( e(H[C]) \leq \varepsilon e(H) \) for each \( C \in \mathcal{C} \), it follows that the number of copies of \( K_{\ell} \) in each \( G \in \mathcal{G} \) is also bounded by \( \binom{n}{\ell}/\ell! \), satisfying property (ii). It remains to show that \( \log |\mathcal{C}| = o(n^2/\ell) \), which then implies property (iii).

Straightforward calculation yields that for large enough \( n \), we have
\[
\log |\mathcal{C}| \leq c \left( \frac{\ell}{2} \right)^{3(\ell/2)} (1 + \ell - \log \delta) \frac{n}{2} \log(1/p)
\leq \ell^3 e^2 (1 + \ell - \log \delta) n^2 \left( \ell^{-4e^2} \cdot \log(ce^{\ell^2}) \right)
\leq \delta n^2/\ell,
\]
where in the second line, we used \( \delta \) together with the fact that \( p \log(1/p) \) is monotonically decreasing. This finishes the proof of the corollary.

The requirement that \( \ell_n \leq (\log n)^{1/4}/2 \) cannot be significantly improved upon with the same method. Indeed, the requirement that \( \Delta(H, p) = o(1) \) implies that
\[
2^{\left( \frac{\ell}{2} \right)} \Delta_2/d = o(1),
\]
which, since \( \Delta_2/p = n^{-1+o(1)} \), implies \( \ell_n = O((\log n)^{1/4}) \). We
also note that the proof shows that, in fact, we have $\log |G| = n^2 e^{-\Omega(\ell_n^2 \log \ell_n)}$, which is much stronger than the bound $\log |G| = o(n^2/\ell_n)$ that we need for the proof of Theorem 1.

3. Proof of Theorem 1

Let us start with the easy part – proving the lower bound. Consider the $(\ell_n - 1)$-partite Turán graph. That is, let $T$ be the complete $(\ell_n - 1)$-partite graph of order $n$ whose partite sets have size either $[n/(\ell_n - 1)]$ or $[n/(\ell_n - 1)]$. Clearly, $T$ is a $K_{\ell_n}$-free graph, as is every subgraph of $T$. As there are at least $2^{e(T)} \geq 2^{(n^{\ell_n - 1})^{(n)^{\ell_n - 1}}} \geq 2^{(1 - \delta/n)(n^2) + o(n^2/\ell_n)}$

subgraphs of $T$, the lower bound on the number of copies of $K_n$ is much stronger than the bound $\log n/\ell_n$.

Now we turn to proving the upper bound. We show that for every $\delta > 0$ and large enough $n$, we have

$$\log f_n(K_{\ell_n}) \leq \left(1 - \frac{1 - \delta}{\ell_n - 1}\right) \left(\frac{n}{2}\right) + \delta n^2/\ell_n.$$ 

We use the following Theorem of Lovász and Simonovits.

**Theorem 4** (Lovász-Simonovits [13, Theorem 1]). Let $n$ and $\ell$ be positive integers. Then every graph of order $n$ with at least

$$\left(1 - \frac{1 - \delta}{\ell_n - 1}\right) \frac{n^2}{2}$$

edges contains at least $\left(\frac{n}{\ell}\right)^{\ell} \binom{n}{\ell}$ copies of $K_{\ell}$.

Using Theorem 4 together with Corollary 3 we can now finish the proof of Theorem 1 as follows. Fix some $\delta > 0$ and assume that $n$ is large enough. Write $\ell := \ell_n$ and apply Corollary 3 for $\delta := \delta/\ell$. We deduce that there exists a collection $\mathcal{G}$ of at most $2^{n^2/\ell}$ graphs of order $n$ such that each contains at most $\delta^{1/\ell} \binom{n}{\ell} / \ell^\ell$ copies of $K_{\ell}$ and every $K_{\ell}$-free graph of order $n$ is a subgraph of some $G \in \mathcal{G}$.

By Theorem 4, if a graph $G$ of order $n$ has at least

$$\left(1 - \frac{1 - \delta}{\ell_n - 1}\right) \frac{n^2}{2}$$

edges, then the number of copies of $K_{\ell}$ in $G$ is at least

$$k(\ell) := \left(\frac{n(1 - \delta)}{\ell_n - 1}\right)^{\ell} \left(\frac{(\ell - 1)/(1 - \delta)}{\ell}\right).$$

If $\ell \geq 1/\delta$, then $(\ell - 1)/(1 - \delta) \geq \ell$ and we can use the bounds $\left(\frac{n}{\ell}\right)^{\ell} \leq \binom{n}{\ell} \leq \left(\frac{en}{\ell}\right)^{\ell}$ to obtain

$$k(\ell) \geq \frac{n^{\ell}}{\ell^{\ell}} > \left(\frac{n}{\ell}\right)^{\ell}/e^\ell.$$ 

For $\ell < 1/\delta$, we use the definition of the (generalized) binomial coefficient to deduce that, in this case,

$$k(\ell) \geq \frac{n^{\ell}}{\ell^{\ell}} \prod_{i=1}^{\ell - 1} \left(1 - \frac{i(1 - \delta)}{\ell_n - 1}\right) > \delta^{1/\ell} \frac{n^{\ell}}{\ell^n}.$$

Since every $G \in \mathcal{G}$ has at most $\delta^{1/\ell} \binom{n}{\ell} / \ell^\ell$ copies of $K_{\ell}$, we deduce that every $G \in \mathcal{G}$ has fewer than

$$\left(1 - \frac{1 - \delta}{\ell_n - 1}\right) \frac{n^2}{2}$$
edges. We can now count the number of $K_\ell$-free graphs by counting the number of subgraphs of order $n$ of the graphs in $\mathcal{G}$,

$$f_n(K_\ell) \leq |\mathcal{G}| \cdot 2^{(1 - \frac{1}{2\ell - 1})n^2} = 2^{(1 - \frac{1}{2\ell - 1})n^2 + \delta n^2/\ell},$$

completing the proof of the theorem.

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