Textures with two traceless submatrices of the neutrino mass matrix

H. A. Alhendi\textsuperscript{1}, E. I. Lashin\textsuperscript{1,2,3} and A. A. Mudlej\textsuperscript{1}

\textsuperscript{1} Department of physics and Astronomy, College of Science, King Saud University, Riyadh, Saudi Arabia
\textsuperscript{2} The Abdus Salam ICTP, P.O. Box 586, 34100 Trieste, Italy
\textsuperscript{3} Department of Physics, Faculty of Science, Ain Shams University, Cairo, Egypt

Emails: alhendi@ksu.edu.sa, elashin@ictp.it and lashin@ksu.edu.sa

February 27, 2008

Abstract

We propose a new texture for the light neutrino mass matrix. The proposal is based upon imposing zero-trace condition on the two by two sub-matrices of the complex symmetric Majorana mass matrix in the flavor basis where the charged lepton mass matrix is diagonal. Restricting the mass matrix to have two traceless sub-matrices may be found sufficient to describe the current data. Eight out of fifteen independent possible cases are found to be compatible with current data. Numerical and some approximate analytical results are presented.

PACS numbers: 14.60.Pq; 11.30.Hv; 14.60.St

1 Introduction

The observed phenomenon of neutrino oscillations [1]–[5], solar and atmospheric, provides a compelling evidence that neutrinos are massive and lepton flavors are mixed. These facts are in contrast to the Standard Model of Particle Physics, especially the Electro-weak interaction which is based on $SU(2)_L \times U(1)_Y$ gauge, where the neutrinos are massless. We will assume that the neutrinos are of Majorana type neutrinos, as favored by some theoretical considerations[6], whence the mass matrix $M$ is symmetric. In the frame work of three lepton families, the mass spectrum and flavor mixing are fully described by twelve real parameters: three charged lepton masses ($m_e, m_\mu, m_\tau$), three neutrino masses ($m_1, m_2, m_3$), three flavor mixing angles ($\theta_x, \theta_y, \theta_z$), one Dirac-type CP-violating phase ($\delta$) and two Majorana-type CP-violating phases ($\rho$ and $\sigma$).

In the flavor basis where the charged lepton mass matrix is diagonal, the mass term for Majorana neutrinos in terms of gauge eigen states, in the case of three flavors, has the form

$$\mathcal{L}_{\text{mass}} = -\frac{1}{2} \nu_L^T C^{-1} M^I \nu_L + h.c,$$

where $C$ is the charge conjugation matrix. The complex symmetric Majorana neutrino mass matrix $M$ can be diagonalized by unitary transformation that links the gauge and mass eigen states as:

$$\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix} =
\begin{pmatrix}
V_{e1} & V_{e2} & V_{e3} \\
V_{\mu1} & V_{\mu2} & V_{\mu3} \\
V_{\tau1} & V_{\tau2} & V_{\tau3}
\end{pmatrix}
\begin{pmatrix}
\nu'_1 \\
\nu'_2 \\
\nu'_3
\end{pmatrix}. $$

(2)
The mass term can be written in terms of mass eigenstates as
\[
\mathcal{L}_{\text{mass}} = -\frac{1}{2} \nu_L^T C^{-1} D \nu_L' + h.c.,
\] (3)
where
\[
V^T M^1 V = D
\] (4)
and \(D = \text{diag}(m_1, m_2, m_3)\) with \(m_i\) real positive numbers. The lepton flavor mixing matrix \(V\) contains six real parameters, three of them are mixing angles while the rest are three CP-violating phases. Following the parameterization in [7], the matrix \(V\) can be expressed as a product of the Dirac-type flavor mixing matrix \(U\) (consisting of three mixing angles and one CP-violating phase) and a diagonal matrix \(P\) (consisting of two nontrivial Majorana phases): \(V = U P\). Then we may rewrite \(M\) in equation (4) as
\[
M = U \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix} U^T
\] (5)
where two Majorana-type CP-violating phases are included into the complex neutrino mass eigenvalues \(\lambda_i\), and the relation \(|\lambda_i| = m_i\) holds. Without loss of generality, we take
\[
\lambda_1 = m_1 e^{2i\rho}, \quad \lambda_2 = m_2 e^{2i\sigma}, \quad \lambda_3 = m_3
\] (6)
The matrix \(U\) is the Dirac-type flavor mixing matrix [7]
\[
U = \begin{pmatrix}
c_x c_z & s_x c_z & s_z \\
-c_x s_y s_z - s_x c_y e^{-i\delta} & s_x s_y s_z + c_x c_y e^{-i\delta} & s_y c_z \\
-c_x c_y s_z + s_x s_y e^{-i\delta} & -s_x c_y s_z - c_x s_y e^{-i\delta} & c_y c_z
\end{pmatrix}
\] (7)
where we used the convention \(s_\alpha = \sin \alpha, c_\alpha = \cos \alpha\).

For the parameterization followed in the present work, the mixing angles \((\theta_x, \theta_y, \theta_z)\) are directly related to the angles of solar, atmospheric and CHOOZ reactor oscillations [7]:
\[
\theta_x \approx \theta_{\text{sol}}, \quad \theta_y \approx \theta_{\text{atm}}, \quad \theta_z \approx \theta_{\text{chz}},
\] (8)
and
\[
\Delta m^2_{\text{sol}} = |m^2_2 - m^2_1|, \quad \Delta m^2_{\text{atm}} = |m^2_3 - m^2_1|.
\] (9)

Beta decay, neutrinoless double-beta decays and precision cosmology are sensitive to the absolute neutrino mass scale. The dependence can be respectively characterized through two non-oscillation parameters and mass sum parameter as follows: the effective electron neutrino mass term
\[
M_\beta = \sqrt{|U_{e1}|^2 m_1^2 + |U_{e2}|^2 m_2^2 + |U_{e3}|^2 m_3^2},
\] (10)
the effective mass term of neutrinoless double beta decay
\[
M_{\beta\beta} = m_3 \left| \frac{m_1}{m_3} U_{e1} e^{2i\rho} + \frac{m_2}{m_3} U_{e2} e^{2i\sigma} + U_{e3} \right|,
\] (11)
and the sum mass parameter
\[
\Sigma = m_1 + m_2 + m_3.
\] (12)
A recent global analysis of neutrino oscillation data\cite{8}, at the confidence level of 95%, gives the best estimates of the oscillation parameters as

\[
\begin{align*}
\Delta m^2_{\text{atm}} &= (2.4^{+0.5}_{-0.6}) \times 10^{-3} \text{eV}^2, \\
\Delta m^2_{\text{sol}} &= (7.92 \pm 0.7) \times 10^{-5} \text{eV}^2, \\
\sin^2 \theta_{\text{sol}} &= 0.314 \pm 0.057, \\
\sin^2 \theta_{\text{atm}} &= 0.44 \pm 0.18, \\
\sin^2 \theta_{\text{chz}} &= 0.9 \pm 2.3 \times 10^{-2},
\end{align*}
\] (13)

where $\theta_{\text{atm}}$ and $\theta_{\text{sol}}$ are the angles relevant for atmospheric and solar neutrino oscillation respectively. While, $\theta_{\text{chz}}$ is the angle relevant to CHOOZ reactor experiment for neutrino oscillation\cite{5}. A useful parameter $R_\nu$ can be defined as:

\[
R_\nu = \frac{\Delta m^2_{\text{sol}}}{\Delta m^2_{\text{atm}}},
\] (14)

which has possible values ranging, at the level of confidence 95%, as

\[
0.025 \leq R_\nu \leq 0.049,
\] (15)

this constraint constitutes a very tight criteria for the model to be phenomenologically acceptable.

Whereas the three masses of charged leptons ($m_e, m_\mu, m_\tau$) have precisely been measured\cite{9}, we have, concerning the absolute neutrino mass scale, only experimental bounds for $M_\beta$, $M_{\beta\beta}$ and $\Sigma$ as follow (see \cite{8} and ref. therein)

\[
\begin{align*}
M_\beta &< 1.8 \text{ eV} \\
M_{\beta\beta} &= 0.58^{+0.22}_{-0.16} \text{ eV} \\
\Sigma &< 1.4 \text{ eV}
\end{align*}
\] (16)

The lower bound for $M_{\beta\beta}$ disappears in case of absence of the neutrinoless double-beta decay. At present, there is no available precise information on any of the CP violating phases.

The present available data on neutrinos, even those in the foreseeable future, can not fully determine all the parameters in the neutrino mass matrix. A challenging theoretical task is to find out a Majorana mass matrix of the light neutrino consistent with the current data as summarized in equation (13). Several attempts have been made to obtain phenomenologically acceptable patterns of the neutrino mass matrix $M$, such as texture zeros \cite{10}–\cite{11}, Zero sum condition \cite{12} and determinant zero requirement \cite{13} for the mass matrix. There are also many proposal for the mass matrix based on some symmetry group as in \cite{14}.

In this work we impose the condition that the trace of two possible $2 \times 2$ submatrices is zero. The traceless condition can be considered as a non trivial generalization of the zero-textures, since that, a zero-element can be viewed as a zero-trace of a $1 \times 1$ sub-matrix. Taking these submatrices in pairs we obtain 15 independent possibilities. Out of these, we find just 8 of them to be phenomenologically acceptable. However there are other five cases which can be considered to be marginally accepted. The numerical and some analytical approximate results are reported for all these thirteen cases.

The plan of the paper is as follows: in section 2, necessary formulas for the calculations are introduced beside a classification of the resulting mass patterns. Sections 3, 4 and 5 are respectively devoted to the resulting three possible mass patterns. For each pattern, we present the textures of $M$ with two independent vanishing traces and compute the expressions of the two neutrino mass ratios and the Majorana phases and other relevant parameters. Numerical and some approximate analytical results are presented. Consistency of models with experimental results are discussed. We end up by conclusions and discussions in section 5.

## 2 Fifteen Possible texture with two Traceless submatrices

As $M$ is $3 \times 3$ symmetric complex matrix, it totally has six independent complex entries. If we impose the condition that the trace of two possible $2 \times 2$ submatrices is zero, then we have in total 6 independent submatrices
with zero trace. When these independent submatrices are taken into pairs, we get 15 possibilities that can be written as

\[
M_{rs} + M_{ij} = 0 \\
M_{\alpha\beta} + M_{nm} = 0.
\] (17)

where each subscript runs over \(e(1), \mu(2)\) and \(\tau(3)\), but \((rs) \neq (ij)\) and \((\alpha\beta) \neq (nm)\).

Using equation (5) we then obtain the following constraint relations:

\[
\sum_{l=1}^{3} (U_{rl} U_{sl} + U_{il} U_{jl}) \lambda_l = 0 \\
\sum_{l=1}^{3} (U_{\alpha l} U_{\beta l} + U_{nl} U_{ml}) \lambda_l = 0
\] (18)

The solutions of equation (18) can be written as:

\[
\begin{align*}
\lambda_1 &= a_3 b_2 - a_2 b_3 \\
\lambda_3 &= b_1 a_2 - a_1 b_2 \\
\lambda_2 &= a_1 b_3 - a_3 b_1 \\
\lambda_3 &= b_1 a_2 - a_1 b_2
\end{align*}
\] (19)

where

\[
\begin{align*}
a_l &= U_{rl} U_{sl} + U_{il} U_{jl} \\
b_l &= U_{\alpha l} U_{\beta l} + U_{nl} U_{ml}
\end{align*}
\] (20)

One can observe that the left-hand sides of equation (19) contain Majorana-type CP-violating phases, while the right-hand sides contain the Dirac-type CP-violating phase. Therefore two Majorana phases must depend upon the Dirac-type CP-violating phase. This dependence results simply from the texture of traceless submatrices that we have taken.

Comparing equation (19) with equation (6), we get the two neutrino mass ratios:

\[
\begin{align*}
\frac{m_1}{m_3} &= \frac{a_3 b_2 - a_2 b_3}{b_1 a_2 - a_1 b_2} \\
\frac{m_2}{m_3} &= \frac{a_1 b_3 - a_3 b_1}{b_1 a_2 - a_1 b_2}
\end{align*}
\] (21)

and the two Majorana phases:

\[
\begin{align*}
\rho &= \frac{1}{2} \arg \left[ \frac{a_3 b_2 - a_2 b_3}{b_1 a_2 - a_1 b_2} \right] \\
\sigma &= \frac{1}{2} \arg \left[ \frac{a_1 b_3 - a_3 b_1}{b_1 a_2 - a_1 b_2} \right]
\end{align*}
\] (22)

With the inputs of three flavor mixing angles and the Dirac-type CP-violating phase, we would be able to predict the relative magnitude of three neutrino masses, the values of two Majorana phases and \(R_{\nu}\). The absolute neutrino mass scale can then be predicted by matching, for example, the value of \(\Delta m_{\text{sol}}^2\). The other remaining parameters \(M_\beta, M_{\beta\beta}\) and \(\Sigma\) can also be predicted. This predictability allows us to examine whether the chosen texture of \(M\) with two traceless submatrices is empirically acceptable or not. The input values of \(\theta_x\) and \(\theta_y\) should be consistent with the bounds given by equation (13), typical ones can be taken as \(\theta_x = 34^0\) and \(\theta_y = 42^0\). The Dirac-type
Figure 1: $R_\nu$ as a function of $\delta$, whereas $\theta_x = 34^0$, $\theta_y = 42^0$ and $\theta_z = 5^0$ for the model $D_1$. For illustrative purposes the vertical range is restricted to the interval from zero to one. The sharp singling out of $\delta = 93^0$ and $\delta = 268^0$ as consistent values with the relation $0.025 \leq R_\nu \leq 0.049$ is clear.

CP-violating phase is not constrained. The strategy followed to obtain a good choice for $\delta$ is to plot the parameter $R_\nu$ as a function of $\delta$ while maintaining $\theta_x \approx 34^0$, $\theta_y = 42^0$ and $\theta_z \approx 5^0$. The constraint in equation (15) turns out to be generically very selective for the appropriate choice of $\delta$, as shown in Fig. 1 for a particular model $D_1$ (according to the nomenclature explained later).

The resulting mass patterns turn out to be classified according to the following three classes:

- **Degenerate case** which is characterized by $m_1 \sim m_2 \sim m_3$ and is denoted by $D$.
- **Normal hierarchy** which is characterized by $m_1 \sim m_2 < m_3$ and is denoted by $N$.
- **Inverted hierarchy** which is characterized by $m_1 \sim m_2 > m_3$ and is denoted by $I$.

In all our subsequent discussion we follow this order and nomenclature.

To work out the explicit expressions of $\lambda_1/\lambda_3$ and $\lambda_2/\lambda_3$ in each case, we adopt the parameterization given in equation (7) for the Dirac-type flavor mixing matrix, from which we then obtain the analytical results for $m_1/m_3$, $m_2/m_3$, $\rho$, $\sigma$, $R_\nu$, $M_{\beta}$ and $M_{\beta\beta}$. However the analytical results as well as the approximate one (for small $s_z$) are too lengthy to be displayed here, and thus we quote only the numerical results based on the exact formulae in equation (19). We present, for each model, the expressions of $a$’s and $b$’s coefficients and the analytic approximate results for $R_\nu$. However for the sake of presentation, the approximate analytical results of all parameters are presented for just one case for each pattern class.

### 3 Degenerate models

**Pattern $D_1$:** $M_{ee} + M_{\mu\mu} = 0$, $M_{e\mu} + M_{\mu\tau} = 0$. In this pattern the required quantities $a$’s and $b$’s as given by equation (20) are

\[
\begin{align*}
a_1 &= c^2_x c^2_z + (-c_x s_y s_z - s_x c_y e^{-i\delta})^2, \\
a_2 &= s^2_x c^2_z + (-s_x s_y s_z + c_x c_y e^{-i\delta})^2, \\
a_3 &= s^2_z + s^2_y c^2_z, \\
b_1 &= c_x c_z(-c_x s_y s_z - s_x c_y e^{-i\delta}) + (-c_x s_y s_z - s_x c_y e^{-i\delta})(-c_x c_y s_z + s_x s_y e^{-i\delta}), \\
b_2 &= s_x c_z(-s_x s_y s_z + c_x c_y e^{-i\delta}) + (-s_x s_y s_z - s_x c_y e^{-i\delta})(-c_x c_y s_z - s_x s_y e^{-i\delta}), \\
b_3 &= s_x s_y c_z + s_y c^2_z s_x,
\end{align*}
\]

which are sufficient to calculate all other quantities. The corresponding expression for $R_\nu$, expanded at the leading power of $s_z$, is

\[
R_\nu \approx \left| -s_x s_y^3 \right| \left( \frac{2c_x c_y s^2_z s_y^2 - c_x c_y s^2_x + s^2_x s^2_y - 2s^3_x s_y + s_z s_y}{2c_x c_y s^2_z s_y^2 - c_x c_y s^2_x + s^2_x s^2_y - 2s^3_x s_y + s_z s_y} \right) + O(s_z).
\]
In this pattern, to match the experimental results, the required values are \((\theta_x = 34^0, \theta_y = 42^0, \delta = 92.755^0, \theta_z = 5^0)\). For these inputs we obtain \(m_1/m_3 = 1.050052527, m_2/m_3 = 1.048504453, \rho = 87.72^0, \sigma = 95^0\) and \(R_\nu = 0.033\). The mass \(m_3\) fitted from the observed \(\Delta m^2_{\text{sol}}\) is \(m_3 = 0.156 \text{ eV}\). Then the derived values for the other remaining parameters are \(\Delta m^2_{\text{atm}} = 2.5 \times 10^{-3} \text{ eV}^2\), \(M_\beta = 0.164 \text{ eV}\), \(M_\beta = 0.160 \text{ eV}\) and \(\Sigma = 0.48 \text{ eV}\). There is no tuning required for the mixing angles \(\theta_x, \theta_y\), to assure their consistency with the relation \(0.025 < R_\nu < 0.05\) as is shown in Fig. 2(a). In this pattern the numerically estimated mass matrix \(M\) is

\[
M = m_3 \begin{pmatrix}
-1.0268 + 0.000489 i & 0.027333 - 0.00049 i & 0.21400 + 0.00037 i \\
0.027333 - 0.00049 i & 1.0268 - 0.000513 i & -0.02737 + 0.000517 i \\
0.21400 + 0.00037 i & -0.02737 + 0.000517 i & 0.99942 - 0.000507 i \\
\end{pmatrix}
\]

(25)

**Pattern D\(_2\):** \(M_{ee} + M_{\tau \tau} = 0, M_{ee} + M_{\mu \mu} = 0\). In this pattern the required quantities \(a\)'s and \(b\)'s as given by equation (20) are

\[
a_1 = c_x^2 c_z^2 + (-c_x c_y s_z + s_x s_y e^{-i\delta})^2, \quad a_2 = s_x^2 c_z^2 + (-s_x c_y s_z - c_y e^{-i\delta})^2, \quad a_3 = s_x^2 + c_x^2 c_y, \\
b_1 = c_x^2 c_z^2 + (-c_x c_y s_z - s_x s_y e^{-i\delta})^2, \quad b_2 = s_x^2 c_z^2 + (-s_x s_y s_z + c_y e^{-i\delta})^2, \quad b_3 = s_x^2 + s_y^2 c_z^2,
\]

(26)

which are sufficient to calculate all other quantities.

Using \(s_x\) as a small parameter, we expand in terms of its powers and keep only leading terms. The analytical approximate formula for the mass ratio is:

\[
\frac{m_1}{m_3} \approx \frac{m_2}{m_3} \approx c_x^{-1} \sqrt{1 - 4 s_x^2 c_x^2 s_y^2} + O(s_x) 
\]

(27)

The corresponding expressions for \(\rho, \sigma\) are

\[
\rho \approx -\frac{1}{2} \tan^{-1} \left( \frac{2 s_y^2 c_x s_z}{1 + 2 s_x^2 s_y^2} \right) + O(s_z), \quad \sigma \approx \frac{1}{2} \tan^{-1} \left( \frac{2 c_x^2 c_y s_z}{1 - 2 c_x^2 s_y^2} \right) + O(s_z),
\]

(28)

while the corresponding expressions for \(R_\nu, M_\beta\) and \(M_\beta\) are

\[
R_\nu \approx \frac{2 c_y s_y s_x}{c_x c_\delta (1 + 2 c_x^2 c_y^2 - 2 c_y^2 - c_z^2)} \left| s_z + O(s_x^2) \right|
\]

(29)

and

\[
M_\beta = m_3 + O(s_z), \quad M_\beta = m_3 \sqrt{\frac{1 - 4 s_x^2 c_x^2 c_y^2}{1 - 4 c_x^2 c_y^2}} + O(s_z).
\]

(30)

In this pattern, to match the experimental results, the required values are \((\theta_x = 34^0, \theta_y = 44.65^0, \delta = 90^0, \theta_z = 5^0)\). For these inputs we obtain \(m_1/m_3 = 1.015274786, m_2/m_3 = 1.014764820, \rho = 89.99^0, \sigma = 89.86^0\) and \(R_\nu = 0.035\). The mass \(m_3\) fitted from the observed \(\Delta m^2_{\text{sol}}\) is \(m_3 = 0.28 \text{ eV}\). Then the derived values for the other remaining parameters are \(\Delta m^2_{\text{atm}} = 2.4 \times 10^{-3} \text{ eV}^2\), \(M_\beta = 0.28 \text{ eV}\), \(M_\beta = 0.28 \text{ eV}\) and \(\Sigma = 0.84 \text{ eV}\). The angle \(\theta_y\) is highly constrained around 45\(^0\) for a reasonable choice of \(\theta_x\) as can be seen from Fig. 2 (b). In this pattern the numerically estimated mass matrix \(M\) is

\[
M = m_3 \begin{pmatrix}
-0.99982 + 0.0017117 i & 0.12446 - 0.00027 i & 0.12299 + 0.00004 i \\
0.12446 - 0.00027 i & 0.99981 - 0.001714 i & -0.01515 + 0.0017267 i \\
0.12299 + 0.00004 i & -0.01515 + 0.0017267 i & 0.99982 - 0.001714 i \\
\end{pmatrix}
\]

(31)
In this pattern, to match the experimental results, the required values are \((\theta_x = 34^0, \theta_y = 42^0, \delta = 272.9^0, \theta_z = 5^0)\). For these inputs we obtain \(m_1/m_3 = 1.008141341, m_2/m_3 = 1.007871889, \rho = 177.55^0, \sigma = 5.32^0\) and \(R_\nu = 0.034\). The mass \(m_3\) fitted from the observed \(\Delta m_{\text{sol}}^2\) is \(m_3 = 0.38\) eV. Then the derived values for the other remaining parameters are \(\Delta m_{\text{atm}}^2 = 2.4 \times 10^{-3} \text{ eV}^2\), \(M_\beta = 0.38\) eV, \(M_{\beta\beta} = 0.38\) eV and \(\Sigma = 1.15\) eV. In this pattern the numerically estimated mass matrix \(M\) is

\[
M = m_3 \begin{pmatrix}
1.0001 - 0.000939 i & -0.093575 - 0.00001 i & 0.084241 + 0.00011 i \\
-0.093575 - 0.00001 i & 0.09358 + 0.00006 i & 0.99567 - 0.00001 i \\
0.084241 + 0.00011 i & 0.99567 - 0.00001 i & 0.09355 - 0.00001 i
\end{pmatrix}
\]

\[
(34)
\]

4 Normal hierarchy models

**Pattern N1:** \(\nu_{e} + \nu_{\mu} = 0, M_{ee} + M_{\mu\mu} = 0\). In this pattern the required quantities \(a\)'s and \(b\)'s as given by equation (20) are

\[
a_1 = c_x c_z (-c_x s_y s_z - s_x c_y e^{-i \delta}) + (-c_x s_y s_z - s_x c_y e^{-i \delta}) (-c_x s_y s_z + s_x c_y e^{-i \delta}) \\
a_2 = s_x c_z (-s_x s_y s_z + c_x c_y e^{-i \delta}) + (-s_x s_y s_z + c_x c_y e^{-i \delta}) (-s_x s_y s_z - c_x c_y e^{-i \delta}) \\
a_3 = s_x s_y c_z + s_y c_y e^{-i \delta} \\
b_1 = c_x^2 c_z^2 + (-c_x s_y s_z - s_x c_y e^{-i \delta}) (-c_x s_y s_z + s_x c_y e^{-i \delta})
\]

\[
(35)
\]

\[R_\nu \approx \frac{2 c_\delta c_\zeta c_\zeta s_x - 4 s_y^2 s_x^2 + 2 s_y^2 + 2 s_y^2 - 1}{-2 c_\delta c_\zeta c_\zeta s_x^2 + 2 s_y^2 s_x^2 - s_x^4} + O(s_x).
\]

\[
(33)
\]
\[ b_2 = s_z^2 c_z^2 + (-s_x s_y s_z + c_x c_y) e^{-i \delta} (-s_x c_y s_z - c_x s_y e^{-i \delta}) \]
\[ b_3 = s_z^2 + s_y c_z^2 c_y, \]  
which are sufficient to calculate all other quantities.

Using \( s_z \) as a small parameter, we expand in terms of its powers and keep only the leading terms. The analytical approximate formulae for the mass ratios are:

\[
\frac{m_1}{m_3} \approx s_y s_x \sqrt{\frac{2 s_x c_x c_y - c_y^2 - s_x^2 s_y^2}{N_m}} + O(s_z), \quad \frac{m_2}{m_3} \approx c_x s_y \sqrt{\frac{-2 c_y s_y c_x s_x + s_x^2 s_y^2 - 1}{N_m}} + O(s_z)  
\]

where

\[
N_m = 4 c_\delta s_x^3 c_x s_y s_y^2 - 2 s_y^2 c_y s_x c_x c_\delta - 4 c_\delta s_y s_x^3 c_x + 2 c_\delta s_y s_x c_x c_\delta + 4 c_y s_y s_x^2 s_y^2 - 4 c_y s_y s_x^4 s_y - 4 c_y s_y s_x^2 s_y^2 + 2 c_y s_y s_x^2 s_y^2 - s_x^4 s_y^2 + s_x^4 s_y^2 + 3 s_x^2 s_y^2 - s_y^2.  
\]

The corresponding expression for \( \rho \) and \( \sigma \) are

\[
\rho \approx \frac{1}{2} \tan^{-1} \left( \frac{N_{\rho1}}{N_{\rho2}} \right) + O(s_z), \quad \sigma \approx \frac{1}{2} \tan^{-1} \left( \frac{N_{\sigma1}}{N_{\sigma2}} \right) + O(s_z),  
\]

where

\[
N_{\rho1} = s_x s_\delta (-4 c_\delta s_x^3 c_x s_y + c_x^2 c_y s_y + 4 c_x^3 c_y^3 s_y + c_x - c_x^3 + 2 c_\delta s_y s_x - 4 c_\delta s_y s_x c_x^2 + 2 c_\delta s_y s_x c_y^2), \\
N_{\rho2} = (4 s_\delta^2 c_\delta - 4 c_\delta)(s_x c_x - s_y s_x^2) + (1 - 2 s_\delta^2)(s_y s_x^2 - s_y s_x^2 + s_y^3 s_x^4) + c_x s_x^3 c_\delta + c_\delta c_y c_x s_y - c_y s_x^2, \\
N_{\sigma1} = -s_\delta (4 c_\delta c_x^2 c_y^2 s_y s_x + 2 c_x c_y^2 s_y s_x - 2 c_x^2 c_y^2 c_\delta - 2 c_\delta c_y c_x s_y + 4 c_x c_y^3 s_y - c_x^2 s_y + s_x^2 s_x^2 c_y - c_y s_y s_x), \\
N_{\sigma2} = c_\delta s_x + c_x c_y s_x^2 + (1 - 2 s_\delta^2)(s_y c_x s_x^2 + c_x s_y s_x^2 - c_x s_y) - c_\delta s_x^3 + (-4 s_\delta^2 c_\delta + 3 c_\delta) s_y s_x c_y + (4 s_\delta^2 c_\delta - 2 c_\delta) s_y s_x c_y,  
\]

while the corresponding expressions for \( R_\nu \), \( M_{\beta\beta} \) and \( M_\beta \) are

\[
R_\nu \approx \left| \frac{R_1}{R_2} \right| + O(s_z)  
\]

where

\[
R_1 = -2 s_x c_x c_y c_\delta s_y^2 + 2 s_y^2 s_x^2 - s_y^2 \\
R_2 = 2 s_x^3 c_x c_y c_\delta s_y - 4 s_x^3 c_x c_y + 2 s_x c_x c_\delta s_y + 4 s_y c_y s_x^2 s_x^2 - 2 s_y c_y s_x^4 - 4 s_y c_y s_x^2 s_x^2 + 2 s_y c_y s_x^2 - 3 s_y s_x^4 + s_x^2 + 2 s_y^2 s_x^2 - s_x^2.  
\]

\[
M_{\beta\beta} \approx m_3 \frac{s_y s_x s_y}{4} \sqrt{\frac{1}{M_1}} + O(s_z),  
\]

where

\[
M_1 = s_y^4 s_x^2 + 3 s_y^4 s_x^2 - s_y^2 - 3 s_x^2 s_y^2 - s_y^4 s_x^2 + (4 c_\delta^2 - 2) c_y s_y s_x^4 + (2 - 4 c_\delta^2) c_y s_y s_x^2 + s_x^2 + s_y^2 - 4 c_\delta s_x^2 c_y s_x^2 + 4 c_\delta s_y s_x^2 c_x - 2 c_\delta s_y s_x c_x + 2 s_y^2 c_y s_x c_\delta,  
\]

and

\[
M_\beta \approx m_3 c_x \sqrt{\frac{s_y^2 s_x^2 - 2 s_x^2 s_y}{M_2}} + O(s_z)  
\]
where

\[
M_2 = 2c_3s_y s_x c_z + s_x^4 + (2 - 4c_3^2)c_y s_y s_x^3 + 3s_x^2 s_y^2 - 2s_x^2 c_y s_x c_z + (4c_3^2 - 2)c_y s_y s_x^2 \\
+ 4c_3 s_x^3 c_x s_y s_x^3 - 4s_x s_y s_x^3 c_z - s_x^2 - s_y^2 - s_y^4 c_x^2 - 3s_x^4 s_y + s_y^4 s_x
\]

(45)

In this pattern, the required input values in order to match the experimental results are \((\theta_x = 33^0, \theta_y = 42^0, \delta = 138.1^0, \theta_z = 5^0)\). For these inputs we obtain \(m_1/m_3 = 0.6563621852, m_2/m_3 = 0.6416046720, \rho = 79.56^0, \sigma = 148.87^0\) and \(R_\nu = 0.033\). The mass \(m_3\) fitted from the observed \(\Delta m^2_{\text{sol}}\) is \(m_3 = 0.064\) eV. Then the derived values for the other remaining parameters are \(\Delta m^2_{\text{atm}} = 2.4 \times 10^{-3}\) eV\(^2\), \(M_\beta = 0.042\) eV, \(M_{\beta\beta} = 0.021\) eV and \(\Sigma = 0.15\) eV. As it is evident from Fig. 3 (a), the available parameter space permits the choice of \(\theta_x\) and \(\theta_y\) in the acceptable range without tuning. In this pattern the numerically estimated mass matrix \(M\) is

\[
M = m_3 \begin{pmatrix}
-0.33257 - 0.00386 i & -0.33256 - 0.003873 i & 0.45632 + 0.003944 i \\
-0.33256 - 0.003873 i & 0.67411 - 0.003791 i & 0.33257 + 0.003870 i \\
0.45632 + 0.003944 i & 0.33257 + 0.003870 i & 0.64681 + 0.003947 i
\end{pmatrix}
\]

(46)

**Pattern N\(_2\):** \(M_{\mu\mu} + M_{\mu\tau} = 0, M_{\mu\tau} + M_{\tau\tau} = 0\). In this pattern the required quantities \(a\)'s and \(b\)'s as given by equation (20) are

\[
a_1 = (-c_x s_y s_z - s_x c_y e^{-i\delta})c_x c_z + (-c_x s_y s_z - s_x c_y e^{-i\delta})(-c_x s_y s_z + s_x s_y e^{-i\delta}), \\
a_2 = s_x c_z(-s_x s_y s_z + c_x c_y e^{-i\delta}) + (s_x s_y s_z + c_x c_y e^{-i\delta})(-s_x s_y s_z + c_x c_y e^{-i\delta}), \\
a_3 = s_y c_z(s_x + c_y c_z), \\
b_1 = (-s_x s_y s_z - s_x c_y e^{-i\delta})c_x c_z + (-c_x s_y s_z + s_x s_y e^{-i\delta})^2, \\
b_2 = (-s_x s_y s_z + c_x c_y e^{-i\delta})s_x c_z + (-s_x c_y s_z - c_x s_y e^{-i\delta})^2, \\
b_3 = s_y c_z(s_x + c_y c_z).
\]

(47)

which are sufficient to calculate all other quantities. The corresponding expression for \(R_\nu\), expanded at the leading power of \(s_z\), is

\[
R_\nu \approx 4c_3 s_x^3 c_y s_y^6 c_3 - 2c_3 s_x^3 c_y s_y^6 c_3 - 4s_x^7 c_x s_x^3 c_3 - 6s_x^3 c_x s_x^3 c_3 - 2s_x^2 c_x s_x^3 c_3 - 4s_y^6 c_y s_y^3 c_3 + 2s_y^3 c_y s_y^3 c_3 - 2s_y^4 s_y^3 c_3 + s_y^4 s_y^3 c_3 + O(s_z).
\]

(48)
where

\[ D_{N^2} = -4c_x s_x^5 c_y s_y^4 c_y + 2c_x s_x^5 c_y s_y^4 c_y + 4s_x^7 c_z s_y^3 c_y - 6s_y^5 c_x s_y^3 c_y + 2s_y^4 c_x s_y^3 c_y \]
\[ + 6s_y^5 c_y s_x^6 - 4s_y^6 s_x^6 + 7s_y^6 s_x^6 - s_y^6 s_x^6 + 8s_y^6 s_x^6 + 4s_y^4 s_x^4 + s_y^4 s_x^2. \]  

(49)

In this pattern, matching the experimental results, we find that the required input values are \( \theta_x = 41.5^0, \theta_y = 47^0, \delta = 195^0, \theta_z = 5^0 \). For these inputs we obtain \( m_1/m_3 = 0.767492621, m_2/m_3 = 0.7584041167, \rho = 96.62^0, \sigma = 8.4^0 \) and \( R_v = 0.033 \). The mass \( m_3 \) fitted from the observed \( \Delta m^2_{\text{Sol}} \) is \( m_3 = 0.076 eV \). Then the derived values for the other remaining parameters are \( \Delta m^2_{\text{atm}} = 2.3 \times 10^{-3} eV^2, M_\beta = 0.058 eV, M_{\beta\beta} = 0.007 eV \) and \( \Sigma = 0.19 eV \). In this model \( \theta_x \) turns out to be a little bit out of the allowed range. In this pattern the numerically estimated mass matrix \( M \) is

\[ M = m_3 \begin{pmatrix} -0.092 - 0.0024 i & -0.4444 + 0.000018 i & 0.617 + 0.00029 i \\ -0.444 + 0.00018 i & 0.639 + 0.00014 i & 0.444 - 0.00018 i \\ 0.617 + 0.00029 i & 0.444 - 0.000018 i & 0.444 - 0.000018 i \end{pmatrix} \]  

(50)

5 Inverted hierarchy models

**Pattern I\textsubscript{1}:** \( M_{ee} + M_{\mu\mu} = 0, M_{ee} + M_{\tau\tau} = 0 \). The \( a \)'s and \( b \)'s quantities can be inferred from the corresponding ones of cases \( D_1 \) and \( N_1 \). The corresponding expression for \( R_v \), expanded at the leading power of \( s_z \), is

\[ R_v \approx \left| \frac{2c_y s_y^3 - 4c_y s_y^3 s_x^2 - s_y^4 + 2s_y^2 s_x^2}{D_{I_1}} \right| + O(s_z), \]  

(51)

where

\[ D_{I_1} = -1 + (4 + 4c_x^2)c_y s_y^3 s_x^4 - 8c_y s_y^3 s_x^2 + 4s_y^2 + 2s_y^2 + (-3 + 4c_x^2)s_y^3 s_x^4 + (-4c_x^2 + 8)s_y^3 s_x^2 - 2c_y s_y \]
\[ - 4s_x^4 + 8c_y s_y^2 s_x^2 + (-8 + 4c_y^2)s_x^4 s_y^2 + (-2 - 4c_y^2)c_y s_y^3 s_x^2 \]  

(52)

In this pattern, to match the experimental results, the required input values are \( \theta_x = 43^0, \theta_y = 44^0, \delta = 26^0, \theta_z = 5^0 \). For these inputs we obtain \( m_1/m_3 = 4.230294841, m_2/m_3 = 4.298049784, \rho = 103.57^0, \sigma = 15.37^0 \) and \( R_v = 0.033 \). The mass \( m_3 \) fitted from the observed \( \Delta m^2_{\text{Sol}} \) is \( m_3 = 0.012 eV \). Then the derived values for the other remaining parameters are \( \Delta m^2_{\text{atm}} = 2.3 \times 10^{-3} eV^2, M_\beta = 0.05 eV, M_{\beta\beta} = 0.003 eV \) and \( \Sigma = 0.11 eV \). In this pattern the numerically estimated mass matrix \( M \) is

\[ M = m_3 \begin{pmatrix} -0.2982 + 0.0109 i & 3.1198 + 0.17560 i & -2.8549 - 0.17093 i \\ 3.1198 + 0.17560 i & 0.29818 - 0.01085 i & 0.2982 - 0.01090 i \\ -2.8549 - 0.17093 i & 0.2982 - 0.01090 i & 1.0592 + 0.03131 i \end{pmatrix} \]  

(53)

**Pattern I\textsubscript{2}:** \( M_{ee} + M_{\mu\mu} = 0, M_{ee} + M_{\tau\tau} = 0 \). The \( a \)'s and \( b \)'s quantities can be inferred from the corresponding ones of cases \( D_1 \) and \( N_2 \). The corresponding expression for \( R_v \), expanded at the leading power of \( s_z \), is

\[ R_v \approx \left| \frac{-6s_x c_x c_y c_5 s_y^4 + 4s_x c_x c_y c_5 s_y^4 + 6s_x c_y s_y^4 - 4s_x^2 s_y^2 - 3s_y^4 + 2s_y^2}{D_{I_2}} \right| + O(s_z), \]  

(54)

where

\[ D_{I_2} = -1 + (4c_x^2 - 10)s_x^2 s_y^2 + 6s_x^3 c_x c_y c_5 s_y^4 - 8s_x^3 c_x c_y c_5 s_y^2 + (-4c_x^2 + 2)s_y^4 s_x^2 \]
\[ + (4c_x^2 - 5)s_y^4 s_x^2 + (-4c_x^2 + 12)s_x^2 s_y^2 - 4s_x^4 + 2s_y^4 + 4s_x^2 + 2s_x c_x c_y c_5 s_y^2 \]  

(55)

10
In this pattern, to match the experimental results, the required values are \((\theta_x = 34^0, \theta_y = 43^0, \delta = 118^0, \theta_z = 5^0)\). For these inputs we obtain \(m_1/m_3 = 2.062421918, m_2/m_3 = 2.099403819, \rho = 77.05^0, \sigma = 132.41^0\) and \(R_v = 0.034\). The mass \(m_3\) fitted from the observed \(\Delta m_{\text{sol}}^2\) is \(m_3 = 0.026\) eV. Then the derived values for the other remaining parameters are \(\Delta m_{\text{atm}}^2 = 2.2 \times 10^{-3} \text{ eV}^2\), \(M_\beta = 0.054\) eV, \(M_{\beta\beta} = 0.034\) eV and \(\Sigma = 0.135\) eV. In this pattern the numerically estimated mass matrix \(M\) is

\[
M = m_3 \begin{pmatrix}
-1.3165 - 0.03164 i & -1.0155 - 0.02210 i & 1.2241 + 0.02439 i \\
-1.0155 - 0.02210 i & 1.3165 + 0.03163 i & -0.17371 - 0.02687 i \\
1.2241 + 0.02439 i & -0.17371 - 0.02687 i & 1.0155 + 0.02213 i
\end{pmatrix}
\]

(Pattern I3: \(M_{\mu\mu} + M_{\mu\tau} = 0, M_{\mu\tau} + M_{\tau\tau} = 0\). The \(a\)'s and \(b\)'s quantities can be inferred from the corresponding ones of cases D_{2} and N_{1}. The corresponding expression for \(R_v\), expanded at the leading power of \(s_z\), is

\[
R_v \approx \left| \frac{s_y c_s s_x c_x + 2 s_x^2 s_y^2 - s_y^2}{-s_y c_s s_x c_x - s_x^2 s_y^2} \right| + O(s_z),
\]

(Pattern I4: \(M_{\mu\mu} + M_{\tau\tau} = 0, M_{ee} + M_{\tau\tau} = 0\). In this pattern the required quantities \(a\)'s and \(b\)'s as given by equation (20) are

\[
a_1 = c_x s_z^2 + s_x e^{-i2\delta}, \quad a_2 = s_x^2 s_z^2 + c_x e^{-i2\delta}, \quad a_3 = c_x^2,
\]

where the coefficients \(b\)'s have the values as given in equation (35).

Using \(s_z\) as a small parameter, expanding in terms of its power and keeping only leading terms, we have the analytical approximate formulae for the mass ratios:

\[
\begin{align*}
\frac{m_1}{m_3} &\approx c_{2x}^{-1} \sqrt{1 - 2c_x^2 - 4c_x^2 s_y c_y + 4c_x^4 s_y c_y + 4c_x^2 c_y^2 - 4c_x^4 c_y^2 - 8c_x^2 s_y c_y c_x^2 + c_x^4 + 8c_x^4 c_y^2 s_y} + O(s_z) \\
\frac{m_2}{m_3} &\approx c_{2x}^{-1} \sqrt{4c_y^2 - 4c_x^2 s_y c_y - 4c_y^4 + 4c_x^2 s_y c_y - 8c_x^2 c_y^2 + 8c_x^2 c_y^2 + 4c_x^4 c_y^4 - 4c_x^4 c_y^2 - 8c_x^2 s_y c_y c_x^2 + c_x^4 + 8c_x^4 c_y^2 s_y} + O(s_z)
\end{align*}
\]

The corresponding expression for \(\rho\) and \(\sigma\) are

\[
\rho \approx \frac{1}{2} \tan^{-1} \left( \frac{s_x^2 s_{25}}{s_x^2 c_{26} - s_{2y} c_x^2} \right) + O(s_z), \quad \sigma \approx \frac{1}{2} \tan^{-1} \left( \frac{-c_x^2 s_{25}}{s_{2y} s_x^2 - c_x e^{25}} \right) + O(s_z),
\]

While the corresponding expression for \(R_v\), \(M_{\beta\beta}\) and \(M_{\beta\tau}\) are

\[
R_v \approx \left| \frac{1 + 4s_x^4 - 8s_x^4 s_y^2 - 4s_y^4 + 8s_x^2 s_y^2 - 2s_y^2}{(-8s_x^2 + 4) s_y c_y s_x^2 - 2s_x^4 + 3s_y^4 + (-4 + 8c_x^2) s_y c_y s_x^2 + 4s_x^4 s_y^2 - 4s_x^2 s_y^2} \right| + O(s_z),
\]

11
\[ M_{\beta\beta} \approx m_3 s_{2y} + O(z), \quad M_{\beta} \approx m_3 c_{2z} \sqrt{M_3}, \quad (63) \]

where
\[ M_3 = 8c_{3z}^2 s_y c_y - 8c_x^2 s_y c_y c_3^2 - 12c_{2x}^2 c_y c_3 + 4c_y^2 s_y c_y - 4c_x^2 c_y c_3 + 12c_{3z}^2 c_y c_3^2 - 12c_{2z}^2 c_y c_3 + c_x - 4c_y^2 - 4c_y^4 (64) \]

In this pattern, no need for tuning to match the experimental results as it is clear from the parameter space in Fig. 3(b). The required values are \((\theta_x = 34^0, \theta_y = 42^0, \delta = 31^0, \theta_z = 5^0)\). For these inputs we obtain \(m_1/m_3 = 1.601974769, m_2/m_3 = 1.585960893, \rho = 76.95^0, \sigma = 134.67^0\) and \(R_{\nu} = 0.034\). The mass \(m_3\) fitted from the observed \(\Delta m^2_{soli}\) is \(m_3 = 0.039\) eV. Then the derived values for the other remaining parameters are \(\Delta m^2_{\text{atm}} = 2.4 \times 10^{-3}\) eV\(^2\), \(M_{\beta} = 0.063\) eV, \(M_{\beta\beta} = 0.039\) eV and \(\Sigma = 0.165\) eV. In this pattern the numerically estimated mass matrix \(M\) is
\[ M = m_3 \begin{pmatrix} -0.97926 - 0.01129 i & 0.12874 - 0.92428 i & 0.11713 + 0.83357 i \\ 0.12874 - 0.92428 i & -0.10446 + 0.10832 i & 0.97931 + 0.01130 i \\ 0.11713 + 0.83357 i & 0.97931 + 0.01130 i & 0.10442 - 0.10834 i \end{pmatrix} \quad (65) \]

**Pattern I5:** \(M_{ee} + M_{\tau\tau} = 0, M_{ec} + M_{\mu\tau} = 0\). The \(a\)'s quantities can be inferred from the corresponding ones of cases \(D_2\), while \(b\)'s are given as
\[ b_1 = c_x c_z (c_x c_z - c_x c_y s_z + s_x s_y e^{-i\delta}), \quad b_2 = -s_x c_z (-s_x c_z + s_x c_y s_z + c_x s_y e^{-i\delta}), \quad b_3 = s_z (s_z + c_y c_z). \quad (66) \]

The corresponding expression for \(R_{\nu}\), expanded at the leading power of \(s_z\), is
\[ R_{\nu} \approx \frac{-c_y^3 (c_y + 2c_y c_x^2 + 2s_y - 4s_y c_x)}{D_{15}} + O(s_z), \quad (67) \]

where
\[ D_{15} = -1 + 4c_x^2 + 2c_y^2 + 4c_y^2 c_x^2 - 4c_y^2 c_x^2 c_3 - 4c_y^2 c_x^2 c_5 + 4c_y^2 c_x^2 c_3 s_y + 2c_y^2 s_y - 6c_y^2 c_x^2 s_y - 8c_y s_y c_x^4 - 8c_y^2 c_x^2 \]
\[-c_y^4 - 2c_y s_y - 3c_y^4 c_x - 4c_x^4 - 4c_y^4 c_x c_3 - 4c_y^4 c_x c_3 - 4c_y^4 c_x c_3 - 4c_y^4 c_x c_3 - 4c_y^4 c_x c_3 - 4c_y^4 c_x c_3 + 8c_y^4 c_x^2 + 4c_y^4 c_x^2 + 4c_y^4 c_x^2 \quad (68) \]

In this pattern, there is no tuning for the angles \(\theta_x\) to match the experimental results. The required values are \((\theta_x = 44.73^0, \theta_y = 42^0, \delta = 90^0, \theta_z = 5^0)\). For these inputs we obtain \(m_1/m_3 = 4.137879853, m_2/m_3 = 4.076054427, \rho = 51.24^0, \sigma = 136.24^0\) and \(R_{\nu} = 0.033\). The mass \(m_3\) fitted from the observed \(\Delta m^2_{soli}\) is \(m_3 = 0.012\) eV. Then the derived values for the other remaining parameters are \(\Delta m^2_{\text{atm}} = 2.5 \times 10^{-3}\) eV\(^2\), \(M_{\beta} = 0.051\) eV, \(M_{\beta\beta} = 0.004\) eV and \(\Sigma = 0.115\) eV. In this pattern the numerically estimated mass matrix \(M\) is
\[ M = m_3 \begin{pmatrix} -0.35325 + 0.0220 i & -2.9235 - 0.39769 i & 2.7916 + 0.35549 i \\ -2.9235 - 0.39769 i & 0.98990 + 0.0765 i & 0.35326 - 0.0220 i \\ 2.7916 + 0.35549 i & 0.35326 - 0.0220 i & 0.35325 - 0.0220 i \end{pmatrix} \quad (69) \]

**Pattern I6:** \(M_{ee} + M_{\mu\mu} = 0, M_{ec} + M_{\mu\tau} = 0\). The \(a\)'s and \(b\)'s quantities can be inferred from the corresponding ones of cases \(D_2\) and \(D_3\). The corresponding expression for \(R_{\nu}\), expanded at the leading power of \(s_z\), is
\[ R_{\nu} \approx \frac{s_y^4 - 2s_y s_y^2 s_y^2 - s_y^4 + 2s_y^2 s_y^2}{s_y^4 s_y^4 s_y^4 s_y^4 - 2s_y^4 s_y^4 s_y^4 s_y^4 - s_y^4 s_y^4 s_y^4 s_y^4 + s_y^4 s_y^4 s_y^4 s_y^4 - s_y^4 - s_y^4 s_y^4 s_y^4 s_y^4} + O(s_z), \quad (70) \]

In this pattern, no tuning is needed to match the experimental results. The required values are \((\theta_x = 42^0, \theta_y = 42^0, \delta = 170^0, \theta_z = 5^0)\). For these inputs we obtain \(m_1/m_3 = 8.699398197, m_2/m_3 = 8.840977186, \rho = 174.85^0, \sigma = 83.86^0\) and \(R_{\nu} = 0.032\). The mass \(m_3\) fitted from the observed \(\Delta m^2_{soli}\) is \(m_3 = 0.006\) eV. Then the derived
values for the other remaining parameters are \( \Delta m_{\text{atm}}^2 \approx 2.4 \times 10^{-3} \text{ eV}^2 \), \( M_\beta = 0.049 \text{ eV} \), \( M_{\beta\beta} = 0.005 \text{ eV} \) and \( \Sigma = 0.105 \text{ eV} \). In this pattern the numerically estimated mass matrix \( M \) is

\[
M = m_3 \begin{pmatrix}
0.8601 - 0.01705 i & 6.4626 - 0.14461 i & -5.8026 + 0.13222 i \\
6.4626 - 0.14461 i & -0.8601 + 0.01705 i & 0.9140 + 0.00166 i \\
-5.8026 + 0.13222 i & 0.9140 + 0.00166 i & 0.8602 - 0.01705 i
\end{pmatrix}
\]

(71)

**Pattern I7**: \( M_{ee} + M_{\tau\tau} = 0 \), \( M_{\mu\mu} + M_{e\tau} = 0 \). The \( a \)’s and \( b \)’s quantities can be inferred from the corresponding ones of cases \( D_2 \) and \( D_3 \). The corresponding expression for \( R_\nu \), expanded at the leading power of \( s_z \), is

\[
R_\nu \approx \left| \frac{c_y^4 (2 c_s s_x c_z - 1 + 2 c_x^2)}{D_{17}} \right| + O(s_z),
\]

(72)

where

\[
D_{17} = 1 + 4 c_x^2 c_y^2 c_z^2 - 4 c_y^2 c_x^4 c_z^2 - 4 c_y^2 + 4 c_y^4 c_x^2 c_z^2 + 8 c_y^2 c_x^2 c_z^2 - 4 c_y^4 s_x c_z c_y - 4 c_y^2 c_x^2 c_z^2 + 4 c_x^4
\]

\[
-12 c_s c_x^4 c_y c_z - 2 c_y^2 + 8 c_s c_x^2 c_y c_z + 3 c_y^4 c_x^2 - 4 c_y^4 c_x^2 c_z - 8 c_y^4 c_x^2 c_z^2 - 2 c_y^4 c_x c_z c_x + 2 c_y^2 c_z^2 c_x^2 c_y
\]

(73)

In this pattern, no tuning for the angles \( \theta_x \) to match the experimental results. The required values are \( \theta_x = 31^0, \theta_y = 44^0, \delta = 147^0, \theta_z = 5^0 \). For these inputs we obtain \( m_1/m_3 = 1.475002097, m_2/m_3 = 1.462135796, \rho = 81.84^0, \sigma = 154.74^0 \) and \( R_\nu = 0.033 \). The mass \( m_3 \) fitted from the observed \( \Delta m_{\text{sol}}^2 \) is \( m_3 = 0.046 \text{ eV} \). Then the derived values for the other remaining parameters are \( \Delta m_{\text{atm}}^2 = 2.5 \times 10^{-3} \text{ eV}^2 \), \( M_\beta = 0.067 \text{ eV} \), \( M_{\beta\beta} = 0.036 \text{ eV} \) and \( \Sigma = 0.180 \text{ eV} \). In this pattern the numerically estimated mass matrix \( M \) is

\[
M = m_3 \begin{pmatrix}
-0.77989 + 0.00498 i & -0.77991 + 0.00497 i & 0.96962 - 0.00542 i \\
-0.77991 + 0.00497 i & 0.98867 - 0.00644 i & 0.10578 + 0.00582 i \\
0.96962 - 0.00542 i & 0.10578 + 0.00582 i & 0.77989 - 0.00498 i
\end{pmatrix}
\]

(74)

**Pattern I8**: \( M_{\mu\mu} + M_{\tau\tau} = 0, M_{ee} + M_{e\tau} = 0 \). The \( a \)’s and \( b \)’s parameters can be inferred from the corresponding ones of cases \( D_2 \) and \( D_3 \). The corresponding expression for \( R_\nu \), expanded at the leading power of \( s_z \), is

\[
R_\nu \approx \left| \frac{4 s_y^4 - 8 s_y^4 s_x^2 - 4 s_y^2 + 8 s_x^2 s_y^2}{4 s_y^4 s_x^4 - 8 s_y^4 s_x^4 s_x^2 + 8 s_x^2 s_y^4 + 4 s_x^2 s_y^2 - 8 s_x^2 s_y^2 + 4 s_x^2 s_y^2 - 8 s_x^2 s_y^2} \right| + O(s_z).
\]

(75)

In this pattern, no need for tuning for the angles \( \theta_x \) to match the experimental results. The acceptable values are \( \theta_x = 42.9^0, \theta_y = 42^0, \delta = 33^0, \theta_z = 5^0 \). For these inputs we obtain \( m_1/m_3 = 10.88384938, m_2/m_3 = 11.06236809, \rho = 18.5^0, \sigma = 110.84^0 \) and \( R_\nu = 0.032 \). The mass \( m_3 \) fitted from the observed \( \Delta m_{\text{sol}}^2 \) is \( m_3 = 0.004 \text{ eV} \). Then the derived values for the other remaining parameters are \( \Delta m_{\text{atm}}^2 = 2.4 \times 10^{-3} \text{ eV}^2 \), \( M_\beta = 0.049 \text{ eV} \), \( M_{\beta\beta} = 0.004 \text{ eV} \) and \( \Sigma = 0.103 \text{ eV} \). In this pattern the numerically estimated mass matrix \( M \) is

\[
M = m_3 \begin{pmatrix}
0.8378 + 0.1050 i & -8.0358 - 0.90178 i & 7.2547 + 0.79963 i \\
-8.0358 - 0.90178 i & 0.8375 + 0.1049 i & 1.0923 + 0.0118 i \\
7.2547 + 0.79963 i & 1.0923 + 0.0118 i & -0.8376 - 0.1048 i
\end{pmatrix}
\]

(76)

The remaining two models of \( M_{11} + M_{33} = 0 \), \( M_{12} + M_{23} = 0 \) and \( M_{11} + M_{24} = 0 \), \( M_{21} + M_{33} = 0 \) fail to be consistent with the experimental data for any reasonable choice of the two mixing angle \( \theta_x \) and \( \theta_y \).
| Model | Trace conditions | Status | $\theta_x$ | $\theta_y$ | $\theta_z$ | $\delta$ | $R_\nu$ | $\frac{m_1}{m_2}$ | $\frac{m_2}{m_3}$ | $\rho$ | $\sigma$ | $m_3$ | $M_3$ | $M_{33}$ | $\Sigma$ | $\Delta m^2_{\text{atm}}$ |
|-------|-----------------|--------|------------|------------|------------|-------|-------|----------------|----------------|-------|-------|-------|-------|-------|-------|----------------|
| D\textsubscript{1} | (11, 22), (12, 23) | allowed | 34 | 42 | 5 | 92.755 | 0.033 | 1.050 | 1.049 | 87.72 | 95 | 0.156 | 0.164 | 0.160 | 0.484 | 0.0025 |
| D\textsubscript{2} | (11, 33), (11, 22) | allowed | 34 | 44.65 | 5 | 90 | 0.035 | 1.0152 | 1.0147 | 89.99 | 89.86 | 0.276 | 0.281 | 0.277 | 0.84 | 0.0024 |
| D\textsubscript{3} | (22, 33), (21, 33) | allowed | 34 | 42 | 5 | 272.9 | 0.034 | 1.0081 | 1.0079 | 177.55 | 5.32 | 0.38 | 0.38 | 0.38 | 1.15 | 0.0024 |
| N\textsubscript{1} | (12, 23), (11, 23) | allowed | 33 | 42 | 5 | 138.1 | 0.033 | 0.656 | 0.642 | 79.56 | 148.87 | 0.064 | 0.042 | 0.021 | 0.148 | 0.0024 |
| N\textsubscript{2} | (12, 23), (12, 33) | disallowed | 41.5 | 47 | 5 | 195 | 0.033 | 0.767 | 0.758 | 96.62 | 8.4 | 0.076 | 0.058 | 0.007 | 0.191 | 0.0023 |
| I\textsubscript{1} | (11, 22), (11, 23) | disallowed | 43 | 44 | 5 | 26 | 0.033 | 4.230 | 4.298 | 103.57 | 15.37 | 0.012 | 0.05 | 0.003 | 0.111 | 0.0023 |
| I\textsubscript{2} | (11, 22), (21, 33) | allowed | 34 | 43 | 5 | 118 | 0.034 | 2.0602 | 2.0904 | 77.05 | 132.41 | 0.026 | 0.054 | 0.034 | 0.135 | 0.0022 |
| I\textsubscript{3} | (12, 23), (22, 33) | allowed | 34 | 42 | 5 | 304 | 0.032 | 1.677 | 1.694 | 166.78 | 47.34 | 0.037 | 0.062 | 0.036 | 0.160 | 0.0024 |
| I\textsubscript{4} | (22, 33), (11, 23) | allowed | 34 | 42 | 5 | 31 | 0.034 | 1.6019 | 1.5859 | 76.95 | 134.67 | 0.039 | 0.063 | 0.039 | 0.165 | 0.0024 |
| I\textsubscript{5} | (11, 33), (23, 11) | disallowed | 44.73 | 42 | 5 | 90 | 0.033 | 4.137 | 4.076 | 51.24 | 136.24 | 0.012 | 0.051 | 0.004 | 0.115 | 0.0025 |
| I\textsubscript{6} | (11, 22), (22, 33) | disallowed | 42 | 42 | 5 | 170 | 0.032 | 8.699 | 8.841 | 174.85 | 83.86 | 0.006 | 0.050 | 0.005 | 0.105 | 0.0024 |
| I\textsubscript{7} | (11, 33), (21, 33) | allowed | 31 | 44 | 5 | 147 | 0.033 | 1.475 | 1.462 | 81.84 | 154.74 | 0.046 | 0.067 | 0.036 | 0.180 | 0.0025 |
| I\textsubscript{8} | (11, 33), (22, 33) | disallowed | 42.9 | 42 | 5 | 33 | 0.032 | 10.88 | 11.062 | 18.5 | 110.84 | 0.004 | 0.049 | 0.004 | 0.103 | 0.0024 |

Table 1: The 13 patterns for the two-vanishing traces. The trace corresponding to the index $(ab, ij)$ is $M_{ab} + M_{ij} = 0$. All the angles are measured in degrees, masses in eV and $\Delta m^2_{\text{atm}}$ in eV$^2$. 

\[ D, M, \Sigma, \Delta \]
6 Conclusion and discussion

In this work several patterns of Majorana neutrino mass matrix, consistent with the present available observed data, are derived. The models are based on textures possessing two $2 \times 2$ sub-matrices with vanishing trace. The new proposed texture can be considered as a non-trivial generalization of the zero-texture as explained in the introduction.

In our work we have thirteen possible acceptable patterns for Majorana mass matrix out of fifteen ones. The resulting models fall into three distinct classes namely, degenerate case ($D_1 \cdots D_3$), normal hierarchy case ($N_1$ and $N_2$), and inverted hierarchy case ($I_1 \cdots I_8$). The numerical results of our study is summarized in Table (1) and Table (2) for a quick reference.

Our numerical study reveals that there are eight models ($D_1, D_2, D_3, N_1, I_2, I_3, I_4$ and $I_7$), for which the mixing angles ($\theta_x, \theta_y$ and $\theta_z$) can be adjusted to fall into the acceptable range given by equation (13). In the remaining five models ($N_2, I_1, I_5, I_6$ and $I_8$), $\theta_z$ falls in the range $41.5^0 \leq \theta_z \leq 45^0$, which is out of the acceptable range. These kinds of models can be considered as empirically ruled out. An avenue for curing these models could be provided by a small perturbation over the adopted textures. In our subsequent discussion we only focus on the successful models namely ($D_1, D_2, D_3, N_1, I_2, I_3, I_4$ and $I_7$).

The numerical study points out that Dirac-type phase $\delta^1$ tends to be around $\pi$ or $\frac{\pi}{2}$ in the degenerate case, while in other cases no general trend could be observed. Concerning a possible relation between Dirac and Majorana phases that could be revealed by numerical study, we find the two phases $\rho$ and $\delta$ almost satisfying the relation $\rho \approx \frac{\pi}{2}$ in the normal hierarchy case, while for the degenerate case there is the relation $\rho \approx \delta$ or $\rho \approx -\frac{\pi}{2}$, and for the inverted hierarchy case $\rho \approx \frac{\pi}{2}$ is obeyed except for $I_4$ where $\rho \approx 2\delta$ is satisfied.

Another possible relation between $\sigma$ and $\delta$ could be easily recognized. In the normal hierarchy and degenerate case, the relation $\sigma \approx \delta$ or $\sigma \approx -\frac{\pi}{2}$ is satisfied. The inverted hierarchy cases have no specific general relation which is obeyed.

The non oscillation parameters $M_\beta, M_{\beta\beta}$ and $\Sigma$ are consistent with the bounds given in equation (16). In all the successful models, $M_\beta, M_{\beta\beta}$ and $m_3$ have the same order of magnitude. The mass sum parameter is always constrained to be $\Sigma \leq 1.15$ eV which is safe with the cosmological bound in equation (16).

All successful models are found to be still consistent with experimental data in the limit of vanishing $\theta_z$ while keeping $\theta_x \approx 34^0$ and $\theta_y \approx 42^0$ constants. The same thing still holds, when $\theta_z$ is stretched to its upper bound ($10^0$). In these limits, little changes take place for the other parameters.

Regarding the hierarchical structure of the mass matrices, as it is evident from table, 2, all successful models have clear hierarchical structure except the mass matrix of model $N_1$ whose elements have all the same order of magnitude. These hierarchical properties are restricted to the real parts, but for the imaginary part they are always very small in comparison with the real ones with the exception of model $I_4$.

Final remark, related to when we restrict the study to the parameter space $(\theta_x \approx 34^0, \theta_y \approx 42^0$ and $\theta_z \approx 5^0)$, while varying $\delta$ under the condition $0.025 \leq R_\nu \leq 0.049$, we find that all successful models turn out to be tightly constrained in order to have a quasi degenerate spectrum, $(m_1 \sim m_2)$, for $m_1$ and $m_2$ and no strong hierarchy between $m_1 \sim m_2$ and $m_3$ can occur. This can be considered as a general prediction for these class of models.

Acknowledgement

One of the authors, E. I. Lashin would like to thank both of A. Smirnov and S. Petcov for useful discussions. Part of this work was done within the associate scheme of ICTP.

References

[1] Y. Fukuda et al., Phys. Lett. B 436,33 (1998) ; Phys. Rev. Lett. 81, 1562 (1998) .
For a review, see: C. K. Jung, C. McGrew, T. Kajita, and T. Mann, Annu. Rev. Nucl. Part. Sci.51, 451 (2001) .
[2] SNO Collaboration, Q.R. Ahmad et al., Phys. Rev. Lett. 89, 011301 (2002) ; Phys. Rev. Lett. 89, 011302 (2002) .
[3] KamLAND collaboration, K. Eguchi et al., Phys. Rev. Lett. 90, 021802 (2003) .
[4] K2K Collaboration, M. H. Ahn et al., Phys. Rev. Lett. 90, 041801 (2003) .

1It is numerically observed that if there is an acceptable value for $\delta$ say $\delta_1$, then there is another one $\delta_2$ such that $\delta_1 + \delta_2 = 2\pi$
[5] CHOOZ Collaboration, M. Apollonio et al., *Phys. Lett.* B 420, 397 (1998); Palo Verde Collaboration, F. Boehm et al., *Phys. Rev. Lett.* 84, 3764 (2000).

[6] For a review, see, for example, B. Kayser, *hep-ph/0211134*.

[7] Z. Z. Xing, *Int. J. Mod. Phys.* A 19, 1 (2004).

[8] G. L. Fogli et al., *Prog. Part. Nucl. Phys.* 57, 742 (2006).

[9] Particle Data Group, K. Hagiwara et al., *Phys. Rev.* D 66, 010001 (2002).

[10] P. H. Frampton, S. L. Glashow and D. Marfatia, *Phys. Lett.* B 536, 79 (2002).

[11] Z. Z. Xing, *Phys. Lett.* B 530, 159 (2002); *ibid.* 569, 30 (2003).

[12] Xiao-Gang He and A. Zee, *Phys. Rev.* D 68, 037302 (2003).

[13] G.C. Branco, R. Felipe, F. Joaquim and T. Yanagida, *Phys. Lett.* B 562, 265 (2003).

[14] Ernest Ma, *Mod. Phys. Lett.* A 22, 101 (2007); *Mod. Phys. Lett.* A 21, 2931 (2006); *Phys. Rev.* D 73, 057304 (2006); *Phys. Lett.* B 583, 157 (2004).
| Model | Trace conditions | $M$                                                                 |
|-------|------------------|--------------------------------------------------------------------|
| $D_1$ | (11, 22), (12, 23) | $m_3$ $\begin{bmatrix} -1.0268 + 0.000489 i & 0.027333 - 0.00049 i & 0.21400 + 0.00037 i \\ 0.027333 - 0.00049 i & 1.0268 - 0.000513 i & -0.02737 + 0.000517 i \\ 0.21400 + 0.00037 i & -0.02737 + 0.000517 i & 0.99942 - 0.000507 i \end{bmatrix}$ |
| $D_2$ | (11, 33), (11, 22) | $m_3$ $\begin{bmatrix} -0.99982 + 0.0017117 i & 0.12446 - 0.00027 i & 0.12299 + 0.00004 i \\ 0.12446 - 0.00027 i & 0.99981 - 0.001714 i & -0.01515 + 0.0017267 i \\ 0.12299 + 0.00004 i & -0.01515 + 0.0017267 i & 0.99982 - 0.001714 i \end{bmatrix}$ |
| $D_3$ | (22, 33), (21, 33) | $m_3$ $\begin{bmatrix} 1.0001 - 0.000939 i & -0.093575 - 0.00001 i & 0.084241 + 0.00011 i \\ -0.093575 - 0.00001 i & 0.99958 - 0.000006 i & 0.99957 - 0.000001 i \\ 0.084241 + 0.00011 i & 0.99957 - 0.000001 i & 0.99955 - 0.000001 i \end{bmatrix}$ |
| $N_1$ | (12, 23), (11, 23) | $m_3$ $\begin{bmatrix} -0.33257 - 0.00386 i & -0.33256 - 0.003873 i & 0.45632 + 0.003944 i \\ -0.33256 - 0.003873 i & 0.67411 - 0.003791 i & 0.33257 + 0.003870 i \\ 0.45632 + 0.003944 i & 0.33257 + 0.003870 i & 0.64681 - 0.003947 i \end{bmatrix}$ |
| $N_2$ | (12, 23), (12, 33) | $m_3$ $\begin{bmatrix} -0.092 - 0.0024 i & -0.4444 + 0.000018 i & 0.617 + 0.00029 i \\ -0.444 + 0.000018 i & 0.639 + 0.000014 i & 0.444 - 0.000018 i \\ 0.617 + 0.00029 i & 0.444 - 0.000018 i & 0.444 - 0.000018 i \end{bmatrix}$ |
| $I_1$ | (11, 22), (11, 23) | $m_3$ $\begin{bmatrix} 3.1198 + 0.17560 i & 0.29818 - 0.01085 i & 0.2982 - 0.01090 i \\ -0.2982 + 0.01090 i & 3.1198 + 0.17560 i & -2.8549 - 0.17903 i \\ 0.2982 - 0.01090 i & -2.8549 - 0.17903 i & 1.0592 + 0.03131 i \end{bmatrix}$ |
| $I_2$ | (11, 22), (21, 33) | $m_3$ $\begin{bmatrix} -1.3165 - 0.03164 i & -1.0155 - 0.02210 i & 1.2241 + 0.02439 i \\ -1.0155 - 0.02210 i & 1.3165 + 0.03163 i & -0.17371 - 0.02687 i \\ 1.2241 + 0.02439 i & -0.17371 - 0.02687 i & 1.0155 + 0.02213 i \end{bmatrix}$ |
| $I_3$ | (12, 23), (22, 33) | $m_3$ $\begin{bmatrix} 0.98880 + 0.01487 i & -1.0068 - 0.00011 i & 0.90785 - 0.00166 i \\ -1.0068 - 0.00011 i & 0.01346 - 0.00008 i & 1.0068 + 0.00010 i \\ 0.90785 - 0.00166 i & 1.0068 + 0.00010 i & -0.01346 + 0.00010 i \end{bmatrix}$ |
| $I_4$ | (22, 33), (11, 23) | $m_3$ $\begin{bmatrix} 0.12874 - 0.92428 i & 0.10446 + 0.10832 i & 0.97931 + 0.01130 i \\ 0.12874 - 0.92428 i & -0.10446 + 0.10832 i & 0.97931 + 0.01130 i \\ 0.11713 + 0.83357 i & 0.97931 + 0.01130 i & 0.10442 - 0.10834 i \end{bmatrix}$ |
| $I_5$ | (11, 33), (23, 11) | $m_3$ $\begin{bmatrix} -0.35325 + 0.02200 i & -2.9235 - 0.39769 i & 2.7916 + 0.35549 i \\ -2.9235 - 0.39769 i & 0.98990 + 0.0765 i & 0.35326 - 0.02209 i \\ 2.7916 + 0.35549 i & 0.35326 - 0.02209 i & 0.35326 - 0.02209 i \end{bmatrix}$ |
| $I_6$ | (11, 22), (22, 33) | $m_3$ $\begin{bmatrix} 0.8601 - 0.01705 i & 6.4626 + 0.14461 i & -5.8026 + 0.13222 i \\ 6.4626 - 0.14461 i & -8.601 + 0.01705 i & 0.9140 + 0.00166 i \\ -5.8026 + 0.13222 i & 0.9140 + 0.00166 i & 8.602 - 0.17050 i \end{bmatrix}$ |
| $I_7$ | (11, 33), (21, 33) | $m_3$ $\begin{bmatrix} -0.77989 + 0.00498 i & -0.77991 + 0.00497 i & 0.96962 - 0.00542 i \\ -0.77991 + 0.00497 i & 0.98867 - 0.00664 i & 0.10578 + 0.00582 i \\ 0.96962 - 0.00542 i & 0.10578 + 0.00582 i & 0.77989 - 0.00498 i \end{bmatrix}$ |
| $I_8$ | (11, 33), (22, 33) | $m_3$ $\begin{bmatrix} 0.8378 + 0.1050 i & -8.0358 - 0.90178 i & 7.2547 + 0.79963 i \\ -8.0358 - 0.90178 i & 0.8375 + 0.1049 i & 1.0923 + 0.0118 i \\ 7.2547 + 0.79963 i & 1.0923 + 0.0118 i & -0.8376 - 0.1048 i \end{bmatrix}$ |

Table 2: The mass estimates for the 13 acceptable patterns. The trace corresponding to the index $(ab, ij)$ is $M_{ab} + M_{ij} = 0.$