Parallelized adiabatic gate teleportation

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We study adiabatic gate teleportation (AGT), a model of quantum computation recently proposed by D. Bacon and S. T. Flammia, Phys. Rev. Lett. 103, 120504 (2009), to investigate a new property of quantum computation, namely causal order manipulation. We develop parallelized adiabatic gate teleportation (PAGT) where a sequence of gate operations is performed in a single step of the adiabatic process by introducing a gate Hamiltonian implementing a single-qubit unitary gate. By parallelizing the AGT scheme, the necessary time for the adiabatic evolution implementing a sequence of gates increases quadratically, however it allows us to map causal order of gate operations to spatial order of interactions in the final Hamiltonian. Using this property, we also present controlled-PAGT scheme to manipulate the causal order of gate operations by a control-qubit. The scheme allows us to coherently perform two differently ordered unitary operations $GF$ and $FG$ depending on the state of a control-qubit by simultaneously applying the gate Hamiltonians of $F$ and $G$.

I. INTRODUCTION

The quantum circuit model is a standard model of quantum computation describing the relationship between input and output by a sequence of elementary gates. This model is widely used since it is shown to be universal, and it has a good correspondence to the logic circuit of classical computation. However at the same time, there is a restriction that elementary gates have to be performed from left to right without creating any loops in the quantum circuit in the quantum circuit. Thus only operations with definite causal order can be performed, whereas such a restriction may not be necessary in quantum mechanics as pointed out in [1].

The role of causal order in quantum mechanics has been widely studied, since the existence of time-loops is allowed by the theory of general relativity. Historically, this question was motivated by Deutsch in [2], where he considered quantum computation with closed time-like curves (CTCs). Bennet, Schumacher, and Svetlichny considered a different model of CTCs [3, 4], referred to as BSS CTCs [5]. Others also refer the same model as P-CTCs (closed timelike curves via quantum postselection) [6, 7], where a classical time-paradox can be resolved by considering quantum computation with CTCs. The computational power of CTCs was studied in [10], where a quantum circuit with Deutsch-type CTCs is shown to be able to solve a computational problem belonging to the PSPACE class in polynomial time. Quantum computation augmented with postselection is shown to have a power to to solve problems in the PP class in polynomial time [11].

Even besides CTCs, we can consider a situation with strange causality. For example, quantum communication without causal structure is formulated in [12]. In [1], an operation beyond causally ordered quantum computation called quantum switch has been investigated. Quantum switch is a super-map of which input is two different single-qubit unitary gates $F$ and $G$ and output is a two-qubit controlled-unitary that coherently performs two differently ordered unitary operations $FG$ and $GF$ depending on the state of a control-qubit. It was proven that quantum switch cannot be implemented within the quantum circuit model with a fixed causal order, if each of $F$ and $G$ is allowed to be used only once.

Are there any operations that cannot be implemented within the causally restricted quantum circuit model but still possible in quantum mechanics? In this paper, we show a positive result that by implementing the quantum computation adiabatically one can map the causal order of operations to a construction of Hamiltonian, and that the causal order can be manipulated to implement quantum switch introduced in [1]. To achieve this task, we analyze adiabatic gate teleportation (AGT), a model of quantum computation proposed by Bacon and Flammia [13]. In AGT, a single-qubit gate is applied to an unknown quantum state of an input qubit and transferred to an output qubit via an intermediate qubit by adiabatically changing interactions between qubits from a fixed initial Hamiltonian to a final Hamiltonian determined by the gate.

Based on AGT, we develop parallelized adiabatic gate teleportation (PAGT), a scheme where consecutive gate teleportations are implemented in one adiabatic shift from the initial Hamiltonian to the final Hamiltonian by introducing a parallelizable gate Hamiltonian implementing a single-qubit gate. In PAGT, causal order of gate operations are mapped to spatial order of interactions in the final Hamiltonian. All gate Hamiltonians are simultaneously applied, although the speed of adiabatic evolution should be slowed down due to decreasing energy gaps. Thus PAGT does not contribute to speed up performing a sequence of gate operations by parallelizing, but to eliminate the control of causal order of gate oper-
ations in the time domain. Using this property, we also present the controlled-PAGT scheme that performs the controlled_Unitary operations implemented by quantum switch. As a special case of controlled-PAGT, we show that a controlled_unitary gate is implementable for an unknown unitary gate if the unitary gate is guaranteed to be written in a orthogonal matrix form.

This paper is organized as follows. We present preliminaries to review the matrix representation of bipartite states, the adiabatic theorem, and adiabatic gate teleportation in Section II. In Section III, we introduce a parallelizable gate Hamiltonian and analyze its property. In Section IV and Section V, we present our main results, the PAGT scheme and the controlled-PAGT scheme, respectively. We show several examples of the controlled-PAGT scheme in Section VI. We summarize the results in Section VII.

II. PRELIMINARIES

A. Matrix Representation

We use the matrix representation of bipartite pure states \[\{1, 13\}\] for simplifying descriptions of our scheme. Consider a bipartite system represented by a Hilbert space given by \(\mathcal{H}_1 \otimes \mathcal{H}_2\), where \(\dim(\mathcal{H}_1) = \dim(\mathcal{H}_2) = d\) and let A, B, and C be \(d \times d\) matrices. By using a set of coefficients satisfying \(\sum_{i,j} C_{ij}^* C_{ij} = 1/d\), we denote a bipartite pure state in \(\mathcal{H}_1 \otimes \mathcal{H}_2\) in orthonormal bases \(\{\mid i\rangle_1\}\) for \(\mathcal{H}_1\) and \(\{\mid j\rangle_2\}\) for \(\mathcal{H}_2\) by

\[
\mid C\rangle := \frac{1}{\sqrt{d}} \sum_{i,j} C_{ij} \mid i\rangle_1 \mid j\rangle_2.
\]

The matrix \(C = \sum_{i,j} C_{ij} \mid i\rangle_1 \langle j\mid \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)\) determines a bipartite state and can be considered as a linear map from \(\mathcal{H}_2\) to \(\mathcal{H}_1\). Note that in our definition Eq. (1), the normalization factor \(1/\sqrt{d}\) has been introduced to guarantee the state represented by \(\mid C\rangle\) to be normalized for simplifying descriptions for \(\dim(\mathcal{H}_1) = \dim(\mathcal{H}_2) = d\), whereas in the original definition of the matrix representation of bipartite states presented in [14, 13], \(\mid C\rangle\) is not necessary to be normalized but applicable for more general cases of \(\dim(\mathcal{H}_1) \neq \dim(\mathcal{H}_2)\).

This representation of bipartite states has a convenient property

\[
(A \otimes B^T) \mid C\rangle = \mid ACB\rangle,
\]

where \(A = \sum_{i,j} A_{ij} \mid i\rangle_1 \langle j\mid \in \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_1)\), \(B = \sum_{i,j} B_{ij} \mid i\rangle_2 \langle j\mid \in \mathcal{L}(\mathcal{H}_2 \otimes \mathcal{H}_2)\), and \(T\) denotes the transposition of a matrix. For a maximally entangled state

\[
\mid I\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \mid ii\rangle,
\]

where \(I\) represents the identity matrix,

\[
\mid C\rangle = (C \otimes I_2) \mid I\rangle = (I_1 \otimes C^T) \mid I\rangle
\]

where \(I_1\) represents the identity matrix on \(\mathcal{H}_1\), and

\[
\mid I\rangle = U \otimes U^* \mid I\rangle
\]

where \(U\) is an arbitrary unitary operation hold. We note that the matrix representation is basis dependent since the description of a matrix depends on the choice of the bases \(\{\mid i\rangle_1\}\) and \(\{\mid j\rangle_2\}\). In the followings, we mainly consider qubit systems \((d = 2)\) and take the computational bases of qubits for \(\mathcal{H}_1\) and \(\mathcal{H}_2\).

B. Adiabatic theorem

Adiabatic quantum computation was first proposed in [16], as a model to solve the SAT problem by using adiabatic evolution. The evolution is governed by a time-dependent Hamiltonian represented by \(H(s) = (1 - s) H_{in} + s H_{fin}\), which starts from the initial Hamiltonian \(H_{in}\) and ends with the final Hamiltonian \(H_{fin}\). The parameter \(s\) is a function of time \(t\), varying from 0 to 1 with total time \(T\), i.e. \(s(0) = 0\) and \(s(T) = 1\). The input state is the ground state of \(H_{in}\), which is assumed to be easily prepared, and the solution of the problem is encoded in the ground state of \(H_{fin}\). The main assumption of this algorithm is that the adiabatic theorem is valid during evolution.

Roughly speaking, the adiabatic theorem states that ground state of time-dependent Hamiltonian remains its instantaneous ground state at later time provided that the change of the Hamiltonian is slow enough. However, necessary and sufficient conditions for this slowness are not yet fully understood. A traditional quantitative condition presented in [16–18, 18] for the adiabatic theorem is

\[
T \geq \Omega\left(\frac{\max_{s \in [0, 1]} \mid E_m(s) \rangle d H(s) \mid E_m(s) \rangle}{G^2}\right),
\]

where \(\mid E_m(s)\rangle\) is the instantaneous \(m\)-th eigen_state \((m = 0\) represents the ground state\) and \(G := \min_{s \in [0, 1]} \{\Delta(H(s))\}\) is the minimum energy gap between the ground state and the first excited state during the evolution. However, it has been pointed out that this quantitative condition fails in certain situations [18–20]. Recently, the necessity of this condition has been considered along with new sufficient
conditions \[1\] \[2\]. In [21], the three conditions given by
\[
(A) \quad \left| \frac{\langle E_n(t) | \dot{E}_n(t) \rangle}{E_n(t) - E_m(t)} \right| < 1, \quad t \in [0, T],
\]
\[
(B) \quad \int_0^T \left| \frac{\langle E_n(t) | \dot{E}_m(t) \rangle}{E_n(t) - E_m(t)} \right| dt < 1,
\]
\[
(C) \quad \int_0^T \left| \frac{\langle E_n(t) | \dot{E}_m(t) \rangle}{E_n(t) - E_m(t)} \right| \left| \langle E_m(t) | \dot{E}_l(t) \rangle \right| dt < 1,
\]
\[
m \neq n, m \neq l
\]
have been shown to be sufficient and (A) has been shown to be necessary \[21\].

These functions are hard to calculate in general, since it is difficult to compute the instantaneous energy eigenstate. In this paper, we refer to a stronger condition for the adiabatic theorem presented in \[23\] \[24\]. Consider the time-dependent Hamiltonian \( H(s) := (1-s)H_{ini} + sH_{fin} \), where \( H(s) \) has a unique ground state \( \forall s \in [0, 1] \). Then for any fixed \( \delta > 0 \), if
\[
T \geq \Omega \left( \frac{||H_{fin} - H_{ini}|| \| \dot{\delta} \|}{e^{\delta G^2 + \delta}} \right)
\]
is satisfied, the final state under the adiabatic evolution at the time \( T \) is \( \epsilon \)-close in \( l_2 \)-norm to the ground state of \( H_{fin} \). The matrix norm \( || \cdot || \) denotes the spectral norm defined by \( ||H|| := \max_{v \neq 0} |\langle v | H | v \rangle| / ||v|| \).

The above conditions are derived for a linear interpolation function \( s(t) = t/T \). The function \( s(t) \) can be chosen in a smarter way so that it varies faster when the energy gap is large and slower when the energy gap is small. When we change the speed of \( s(t) \) depending on the energy gap, the total time sufficient for the adiabatic theorem to hold is evaluated by
\[
T = \int_0^1 ds \frac{||H_{fin} - H_{ini}||}{\Delta(H(s))^2}.
\]

It was pointed out that the necessary time \( T \) becomes proportional to \( 1/G \), and not \( 1/G^2 \) in a certain case. By adjusting \( s(t) \) on each infinitesimal time interval, the total calculation time of the adiabatic implementation of Grover’s search algorithm \[17\] reduced from \( O(N) \) to \( O(N^{3/2}) \), which is the same as the standard implementation of Grover’s algorithm using the quantum circuit model.

\section{C. Adiabatic gate teleportation}

In this section, we review adiabatic gate teleportation (AGT) \[12\]. We consider a system consisting of only qubits in this paper, but one can easily generalize the scheme to general qudit systems. We use \( X, Y \) and \( Z \) to represent the Pauli operators, generators of \( SU(2) \). Subscripts are used to specify the Hilbert space of states or operators. In case it is not confusable, we often omit the identity operator and the symbol \( \otimes \) to represent the tensor product of operators. For example, when we are considering a Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \), \( X_1X_2 \) denotes
\[
X_1X_2 := X_1 \otimes X_2 \otimes I_3.
\]

Note that \( X_1X_2 \) does not represent a simple multiplication of two operators on the same Hilbert space since the different subscripts imply that two operators act on two different Hilbert spaces.

Before describing AGT, we introduce a special version of AGT, adiabatic teleportation. Consider a three qubit system \( \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \). Adiabatic teleportation aims to transfer unknown quantum state encoded in the first qubit \( |\phi\rangle \) (input state) to the third qubit (output state) using adiabatic evolution under a time-dependent Hamiltonian of interacting qubits. The second qubit acts as a mediator for transferring the state. The initial state is prepared in \( |\phi_1\rangle |I\rangle_{23} \), which is a ground state of the initial Hamiltonian
\[
H_{ini}^{AT} = -\omega (I_2I_3 + X_2X_3 - Y_2Y_3 + Z_2Z_3)
\]
\[
= -4\omega |I\rangle \langle I|_{23},
\]
where \( |I\rangle = (|00\rangle + |11\rangle)/\sqrt{2} \). Note that we only consider the case \( \omega > 0 \). This Hamiltonian is free on \( \mathcal{H}_1 \), so it has two degenerate ground states. We can choose an arbitrary input state \( |\phi_1\rangle \in \mathcal{H}_1 \), but the state of the second and the third qubits is fixed to be in \( |I\rangle_{23} \). The initial Hamiltonian given by Eq. (11) is a slightly modified from the one introduced in the original paper of AGT \[13\] given by \( H_{original} = -\omega (X_2X_3 + Z_2Z_3) \). The additional terms \( I_2I_3 \) and \( -Y_2Y_3 \) in Eq. (11) just shift the energy eigenvalues of the initial Hamiltonian by a constant, and do not change the eigenstates. However, the existence of these additional terms is crucial for parallelizing AGT.

We adiabatically change the initial Hamiltonian to the final Hamiltonian given by
\[
H_{fin}^{AT} = -\omega (I_1I_2 + X_1X_2 - Y_1Y_2 + Z_1Z_2)
\]
\[
= -4\omega |I\rangle \langle I|_{12}.
\]

To connect the initial and final Hamiltonian, we may introduce a time-dependent Hamiltonian in the form of
\[
H^{AT}(s) = (1-s)H_{ini}^{AT} + sH_{fin}^{AT}
\]
where a parameter denoted by \( s \) is a function of time \( t \) satisfying \( s(t = 0) = 0 \) and \( s(t = T) = 1 \) with total time \( T \) of the adiabatic evolution. The final Hamiltonian also has two degenerate ground states. The degenerate ground states have kept during the evolution. This degeneracy is kept during the evolution. By adiabatically changing \( s \), the final state turns out to be \( |I\rangle_{12} |\phi_3\rangle \). That is, an unknown state \( |\phi\rangle \) is transferred from \( \mathcal{H}_1 \) to \( \mathcal{H}_3 \). Thus this scheme is called to be adiabatic teleportation.
To show how this adiabatic teleportation scheme works, we first check that the energy gap is finite throughout the evolution. The energy gap between the (degenerate) ground states and the first excited state of Hamiltonian Eq. (13) is

$$\Delta(H_{AT}(s)) = 4\omega\sqrt{1 - 3s + 3s^2}.$$  (14)

It takes minimum value $2\omega$ at $s = 1/2$, so we always have a nonzero energy gap. According to the adiabatic theorem, the system remains in the ground states throughout the evolution if parameter $s$ is varied slowly enough.

We have introduced the adiabatic theorem for a Hamiltonian $H(s)$ with a unique ground state, but $H_{AT}(s)$ has two degenerate ground states throughout the evolution. It is necessary to guarantee that no transition between these two degenerate ground states happens during adiabatic evolution under $H(s)$ for an unknown state $|\phi\rangle$ to be faithfully transferred in adiabatic teleportation. To show this, we use stabilizer formalism and define logical operators $L_x := X_1X_2X_3$ and $L_z := Z_1Z_2Z_3$. The logical space spanned by the logical operators is preserved during the adiabatic scheme due to $[L_i, H(s)] = 0$ ($i = X, Z$). We first explain that these logical operators encode information of the unknown input state $|\phi\rangle$. In the initial stage $s = 0$, the ground state of $H_{ini}$ is $|\phi\rangle_1 |I\rangle_{23}$, which is stabilized by $X_2X_3$ and $Z_2Z_3$. Namely

$$X_2X_3 |I\rangle_{23} = Z_2Z_3 |I\rangle_{23} = |I\rangle_{23},$$  (15)

which means that $X_2X_3$ takes a value 1 at $s = 0$, so $L_x = X_1X_2X_3$ indeed represents $X_1$, and the same logic works for showing that $L_z$ represents $Z_1$ at $s = 0$. Thus logical operators encode information of the unknown state $|\phi\rangle$. We note that the term $|I\rangle \langle I|_{23}$ in the initial Hamiltonian consists of terms $I_2I_3, X_2X_3, Z_2Z_3$, and $-Y_2Y_3$. When considering of a ground state of these terms, it is sufficient to stabilize only $X_2X_3$ and $Z_2Z_3$, since other term, $I_2I_3 = (X_2X_3)^2$ and $-Y_2Y_3 = (X_2X_3)(Z_2Z_3)$, can be derived from these operators.

We next show that the encoded information is faithfully transferred to the qubit on $H_3$ at the end of the adiabatic evolution. In the final stage $s = 1$, Hamiltonian of the system is the final Hamiltonian $H_{fin}$. When we drag the Hamiltonian $H(s)$ slow enough so that the adiabatic theorem holds, the ground state is kept throughout the evolution and finally we get $|I\rangle_{12}$ for the ground state of $H_{fin}$. Since the state $|I\rangle_{12}$ is stabilized by $X_1X_2$ and $Z_1Z_2$, we have $L_x = [X_1X_2]X_3 = X_3$ and $L_z = [Z_1Z_2]Z_3 = Z_3$ at $s = 1$. Therefore, the information $X_1$ and $Z_1$ of the input state on $H_1$ is sent to $X_3$ and $Z_3$ on $H_3$ respectively through the evolution, which concludes the proof.

Here, we describe the relationship between adiabatic teleportation and usual quantum teleportation. To start with, we explain usual quantum teleportation scheme. FIG. 1 is the quantum circuit representation of quantum teleportation scheme. Alice wants to send her unknown state $|\phi\rangle$ to Bob (receiver). Alice shares a first qubit which is unknown state $|\phi\rangle_1$, and a second qubit which is a part of maximally entangled state $|I\rangle_{23}$. Bob shares a third qubit which is the other side of a maximally entangled state. Next, Alice measures her two qubits by Bell basis $\{|I\rangle, |X\rangle, |Y\rangle, |Z\rangle\}$ which are described by,

$$|I\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}},$$  (16)

$$|X\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}},$$  (17)

$$|Y\rangle = \frac{-i|01\rangle + i|10\rangle}{\sqrt{2}},$$  (18)

and

$$|Z\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}.$$  (19)

She get a measurement outcome $|V\rangle$ ($V$ is either $I, X, Y$ or $Z$) with probability 25% for each, then Bob’s state collapses to $V|\phi\rangle_3$. At last, Bob performs a correction $V$ on his third qubit according to Alice’s measurement outcome to recover the unknown state $|\phi\rangle$.

We explained usual quantum teleportation scheme above. However, we restrict our attention in the case when Alice measures the state $|I\rangle_{12}$, Bob gets the state $|\phi\rangle_3$ without any correction. This effect is considered as P-CTCs (closed timelike curves via quantum postselection) [3, 4], describing a time travel effect of a state $|\phi\rangle$ from the final stage (when the measurement is performed) of the first qubit to the initial stage (when the state $|I\rangle_{23}$ is prepared) of the third qubit via the second qubit mediator.

Adiabatic teleportation is adiabatically mimicking this P-CTCs effect (quantum teleportation and postselection) by simultaneously applying Hamiltonian on the whole system to drag the initial state $|\phi\rangle_1 |I\rangle_{23}$ to the final state $|I\rangle_{12} |\phi\rangle_3$. While postselection can not be implemented deterministically in quantum circuit model, adiabatic evolution can mimic some class of quantum computation with definite measurement outcome by dragging the system with sufficiently long time determined by the energy gap.

We move on to explain AGT scheme, which is obtained by slightly modifying adiabatic teleportation. AGT aims to obtain the output state $U|\phi\rangle_3$ where $U$ denotes a single-qubit gate (unitary operation) for an input state $|\phi\rangle_1$ by adiabatic evolution. The time-dependent Hamiltonian for AGT is given by the unitary conjugation form of Eq. (13) in terms of $U$ on $H_3$, namely

$$H_{AGT}^{AT}(s) := U_3H_{AT}(s)U_3^\dagger$$
$$= (1 - s)U_3H_{AT}U_3^\dagger + sH_{AT}^I. $$  (20)

The conjugation only affects the initial Hamiltonian

$$H_{ini}^{AGT} = -\omega U_3(I_2I_3 + X_2X_3 - Y_2Y_3 + Z_2Z_3)U_3^\dagger$$
$$= -4\omega U_3 |I\rangle \langle I|_{23} U_3^\dagger. $$  (21)
This modification is just a unitary conjugation changing only energy eigenstates and it keeps the energy eigenvalues. Thus the energy gap between the ground states and the first excited states is unchanged. The correspondence of states between adiabatic teleportation and AGT is \(|\phi\rangle_1 | I\rangle_{23} \leftrightarrow |\phi\rangle_1 U_3 | I\rangle_{23}\) at \(s = 0\) and \(|I\rangle_{12} |\phi\rangle_3 \leftrightarrow |I\rangle_{12} U_3 |\phi\rangle_3\) at \(s = 1\). AGT scheme adiabatically simulates the quantum circuit presented in FIG. 2(a), which is equivalent to the circuit in FIG. 2(b). Note that in FIG. 2(b), gate teleportation is valid applying \(U\) on the third qubit even before preparing an unknown state \(|\phi\rangle\) on the first qubit. Thus we can recognize that the circuit effectively achieves time travel of a state \(|\phi\rangle\).

\[ H_{U_{ij}} := -\omega U_{ij} (I_i I_j + X_i X_j - Y_i Y_j + Z_i Z_j) U_{ij}^\dagger \]
\[ = -4\omega U_{ij} |I\rangle \langle I| U_{ij}^\dagger. \]

We call this gate Hamiltonian in our paper. Here, the Hamiltonian is defined on bipartite Hilbert space \(\mathcal{H}_i \otimes \mathcal{H}_j\) and consists of two-body interaction, products of Pauli operators rotated by \(U\). This kind of Hamiltonian is sometimes referred to as “twisted Hamiltonian” in [25].

This gate Hamiltonian is closely related to the Heisenberg spin interaction by

\[ H_{I_{ij}} = -\omega [I_i I_j - Y_j (X_i X_j + Y_i Y_j + Z_i Z_j) Y_j^\dagger] \approx \omega Y_j (S_i \cdot S_j) Y_j^\dagger, \]

where \(S = (X, Y, Z)\) represents the direction of the spin and we neglect the identity term because it only shifts the energy spectrum and does not contribute to the adiabatic evolution. The \(Y\) conjugation term rotates the reference frame 180° around the \(y\)-axis. The term \(S \cdot S\) with positive coupling constant \(\omega\) corresponds to a antiferromagnetic Heisenberg spin interaction. Eq. (22) can be rewritten in the form

\[ H_{U_{ij}} \approx \omega S_i \cdot (UYSY^\dagger U^\dagger) Y_j \]

which may be useful when one considers experimental implementation.

To perform AGT, the following conditions have to be satisfied,

1. the gate Hamiltonian can be implemented.

2. the ground state of the initial Hamiltonian can be prepared.

3. Control the strength of Hamiltonian is possible, i.e.

We assume that these conditions are satisfied in the rest of paper. We also assume that \(H_I\) is a constant resource, and can be used freely.

**III. PARALLELIZABLE GATE HAMILTONIAN**

**A. Definition**

When one considers solving some problems by quantum computation, there are some approaches to get solutions. We review and clarify the difference between quantum circuit approach and adiabatic gate teleportation. Quantum circuit model tries to implement the unitary operation which solves the problem by describing a sequence of the elementary gates. It describes input-output relation, while in adiabatic quantum computation model, we have a solution as a ground state of final Hamiltonian. We try to get this solution starting from the ground state of initial Hamiltonian which is easy to prepare.

Computational concept of AGT is more similar to the quantum circuit model than adiabatic quantum computation, because it implements a “unitary gate”, rather than encodes the solution of a problem as a ground state of a final Hamiltonian.

In general form, the Hamiltonian that implements a unitary \(U\) in AGT is

\[ H_{U_{ij}} := -\omega U_{ij} (I_i I_j + X_i X_j - Y_i Y_j + Z_i Z_j) U_{ij}^\dagger \]
\[ = -4\omega U_{ij} |I\rangle \langle I| U_{ij}^\dagger. \]
B. Power of gate Hamiltonian

We introduce the intrinsic computational power of gate Hamiltonian. Actually, we can implement various unitary operations $U, U^T, U^*$ and $U^T$ by using the same gate Hamiltonian $H_U$. As we saw in the previous section, we can teleport the unitary gate $U$ to obtain output $U|\phi\rangle$ using Hamiltonian Eq. (20).

Now, consider the situation when instead of the final Hamiltonian $H_{fin}^T$, we use $H_{fin}^{trans} = H_{I_1}$, i.e. we use the Hamiltonian

$$H_{trans}^T(s) = (1 - s)H_{I_1}^T + sH_{I_1}^{trans}$$

$$= (1 - s)H_{U_{23}} + sH_{I_{13}}$$

$$= -4\omega[(1 - s)|I\rangle \langle I|_{23} + s|I\rangle \langle I|_{13}]U_{23}^{\dagger}$$

$$= -4\omega U_{23}^T[(1 - s)|I\rangle \langle I|_{23} + s|I\rangle \langle I|_{13}]U_{23}^{T\dagger}.$$

(25)

In the last line, we used a property Eq. (4). This Hamiltonian can be written in the unitary conjugation form of adiabatic teleportation from the first qubit to the second qubit (the third qubit acts as a mediator) with $U_2^T$ so we obtain $U^T_2|\phi\rangle_2$ as a output state after the adiabatic evolution. Therefore, the gate $U^T_2$ can be teleported by just changing the interaction order in the final Hamiltonian from $H_{I_1}$ to $H_{I_{13}}$, see FIG. 3.

Now, let us consider using the gate Hamiltonian $H_U$ in the final Hamiltonian. We can consider the following two AGT schemes.

First possibility is to take the following Hamiltonian,

$$H^{con}^T(s) := (1 - s)H_{I_23} + sH_{U_{23}}$$

$$= -4\omega[(1 - s)|I\rangle \langle I|_{23} + sU_{23}|I\rangle \langle I|_{12}]U_{23}^{\dagger}$$

$$= -4\omega(U_{23}^U)^T[(1 - s)|I\rangle \langle I|_{23} + s|I\rangle \langle I|_{12}]U_{23}^{T\dagger}$$

$$= (U_{23}^U)^T \cdot \cdot \cdot (U_{23}^U)^2 H^T(s)(U_{23}^U)^\dagger,$$

(26)

where we have used property Eq. (5) during the calculation. Therefore, we start with the initial state $|\phi\rangle_1 |I\rangle_{23}$ and get the final state as $U_2^U |I\rangle_{12} U^* |\phi\rangle_3$. This shows that the result of the adiabatic evolution with Hamiltonian Eq. (20) is to implement $U^*$. Note that we do not need to prepare the ground state of gate Hamiltonian $I \otimes U^T |I\rangle$ in this case.

The remaining possibility is to use the final Hamiltonian $H_{U_{23}} = -4\omega U_{1}|I\rangle \langle I|_{21} U_1^T$, i.e.

$$H^{dagger}^T(s) := (1 - s)H_{I_23} + sH_{U_{23}}$$

$$= (U_{23}^U)^T H^T(s)(U_{23}^U)^\dagger.$$

(27)

So we will get $U^T|\phi\rangle_3$ after AGT.

![FIG. 3. Circuit representations of original AGT (left) and transpose gate teleportation (right). One can implement both $U^*$ and $U^T$ by using the same gate Hamiltonian $H_U$ in the initial Hamiltonian.](Image)

This shows that $U, U^T, U^*, U^\dagger$ can be implemented using the gate Hamiltonian $H_U$. This is summarized in FIG. 5.

IV. PARALLELIZATION

A. Algorithm

To implement an arbitrary unitary operation on the system, we need to perform consecutive unitaries. In this section, we will consider a case where we perform $L$-ordered single-qubit unitary gate teleportations in a single step. This can be done by small alteration of original AGT scheme.

Consider $L$ gate Hamiltonians, $H_{U_1}, H_{U_2}, \cdots, H_{U_L}$. Our aim is to implement $U^{(L)} \cdots U^{(2)} U^{(1)}$ operation to the input state $|\phi\rangle$ in a single adiabatic evolution. We use $2L + 1$ qubit system $H_1 \otimes H_2 \otimes \cdot \cdot \cdot \otimes H_{2L+1}$. Initial and final Hamiltonian of this parallelized adiabatic gate teleportation (PAGT) are

$$H^{PAGT}_{init} := \sum_{j=1}^{L} H_{U_{2j+1}}^{(j)}$$

$$= -4\omega \sum_{j=1}^{L} U_{2j+1} |I\rangle \langle I|_{2j+1} U_{2j+1}^\dagger,$$

(28)

$$H^{PAGT}_{fin} := \sum_{j=1}^{L} H_{I_{2j+1}}^{(j)}$$

$$= -4\omega \sum_{j=1}^{L} |I\rangle \langle I|_{2j+1} U_{2j+1}^\dagger.$$

(29)

The main idea comes from just parallelizing AGT scheme to achieve consecutive time travel in a single
To see that this adiabatic scheme actually implements a $U^{(L)} \cdots U^{(2)}U^{(1)}$ gate teleportation, we start by considering an easier case where all gates are the identity, i.e. we use

$$H^{PAGT}_{ini}(s) := (1-s)H^{PAGT}_{ini} + sH^{PAGT}_{fin}. \quad (30)$$

FIG. 6. circuit representation of parallelized adiabatic gate teleportation (PAGT). PAGT implements $L$ times iteration of unitary gate teleportation in a single adiabatic shift. Here, the initial state $|\phi\rangle U^{(1)}_1 |I\rangle_{23} \cdots U^{(L)}_{2L+1} |I\rangle_{2L+1}$ is the ground state of the initial Hamiltonian Eq. (29), and the final state $|I\rangle_{12} \cdots |I\rangle_{2L-1} 2L \ U^{(L)} \cdots U^{(2)}U^{(1)} |\phi\rangle_{2L+1}$ is the ground state of the final Hamiltonian Eq. (29).

FIG. 5. Summary table for the power of gate Hamiltonian $H_U$. One can implement $U, U^T, U^*$ and $U^\dagger$ by using a gate Hamiltonian $H_U$.

We need to check that the input state $|\phi\rangle_1$ on $\mathcal{H}_1$ teleports to $\mathcal{H}_{2L+1}$ after the evolution. This fact can be easily proven using the same technique as in the original AGT scheme [13] which uses stabilizer formalism, also explained in Section II C. So we consider the stabilizers in the initial Hamiltonian Eq. (31) as $\{X_2X_3, Z_2Z_3, \cdots, X_{2L}X_{2L+1}, Z_{2L}Z_{2L+1}\}$ and the final Hamiltonian Eq. (32) as $\{X_1X_2, Z_1Z_2, \cdots, X_{2L-1}X_{2L}, Z_{2L-1}Z_{2L}\}$. Note that there are $2L + 1$ qubits but the number of generators is $2L$, so we have $2L$ degrees of freedom. We define logical operators, $L_x := X_1X_2\cdots X_{2L+1}, L_z := Z_1Z_2\cdots Z_{2L+1}$. Note that $L_k(k = x \text{ or } z)$ commutes with all stabilizer operators $XX$ and $ZZ$, i.e. $[L_k, H^{PAGT}_{ini}(s)] = 0$. So the subspace spanned by these logical operators is preserved throughout the evolution. In the initial stage, $\{X_2X_3, Z_2Z_3, \cdots, X_{2L}X_{2L+1}, Z_{2L}Z_{2L+1}\}$ is stabilized, so $L_x = X_1X_2\cdots X_{2L}X_{2L+1} = X_1$ and similarly $L_z = Z_1$. In the final phase, we see that $L_x = X_{2L+1}$ and $L_z = Z_{2L+1}$. So, the teleportation from the first qubit to the $2L + 1$-th qubit is faithfully achieved in the ideal case.

Now, we consider the general case Eq. (30). Since the identity gate Hamiltonian $H_{I_{ij}}$ has the property

$$\left(U_jU_j^\dagger\right)H_{I_{ij}}\left(U_jU_j^\dagger\right)^\dagger = H_{I_{ij}} \quad (34)$$

due to Eq. (3), we can write the total Hamiltonian in the unitary conjugation form of Eq. (33).

$$H^{PAGT}_{ini}(s) = \tilde{U} H^{PAT}_{ini}(s) \tilde{U}^\dagger \quad (35)$$

where

$$\tilde{U} := (U^{(L)}_{2L+1} \cdots U^{(L+1)}_{2L+1} \cdots U^{(1)}_{2L+1} \cdots U^{(1)}_{2L}) \cdots \left[U^{(1)}_{2L} \cdots U^{(1)}_{2L} \cdots U^{(1)}_{2L+1} \cdots U^{(1)}_{2L+1}\right]. \quad (36)$$

Combining these results, we conclude that this parallelized AGT Hamiltonian $H^{PAGT}_{ini}(s)$ teleports the initial state $|\phi\rangle_1 U^{(1)}_3 |I\rangle_{23} \cdots U^{(L)}_2 |I\rangle_{2L+1}$ to the final state $|I\rangle_{12} \cdots |I\rangle_{2L-1} 2L \ U^{(n)} \cdots U^{(2)}U^{(1)} |\phi\rangle$.

B. Energy gap and Time of Computation

We can perform $L$ consecutive unitary operations by iterating AGT scheme $L$ times, or performing these $L$ operations in a single step using PAGT. It is worth pointing out the main differences between AGT iteration and PAGT. PAGT uses $2L + 1$ qubit system, and the energy gap $\Delta(H^{PAGT}_{ini}(s))$ depends on $L$. Let us define
\[ \Delta E_L(s) \] as the energy gap of an \( L \)-gate PAGT scheme, and \( G_L := \min_{s \in [0,1]} \Delta E_L(s) \) to be the minimum energy gap. So, \( \Delta E_L(s) \) represents the energy gap of usual AGT scheme Eq. (14). We want to consider the scaling behavior of the energy gap \( \Delta E_L(s) \).

Energy spectrum does not change under unitary conjugation, so it suffices to analyze the case
\[
\Pi_{i \in \text{even}} Y_i \cdot H^{\text{PAGT}}(s) \cdot \Pi_{i \in \text{even}} Y_i^\dagger + (\text{constant identity})
\]
\[
= \omega \sum_{i \in \text{even}} (1-s) S_{i-1} \cdot S_i + s S_i \cdot S_{i+1} =: H^{s-\text{chain}}(s).
\]
(37)

Here, the constant identity term is for neglecting \( II \) term in the Hamiltonian, since it does not affect the energy gap. Therefore, \( \Delta E_L(s) \) behaves like the energy gap of an alternating bond 1-D anti-ferromagnetic spin chain model. Especially when \( s = 1/2 \) this is an open boundary 1-D spin chain model with size \( 2L + 1 \).

Before calculating the energy gap of Eq. (37), we note that there exists invariant subspaces during the evolution (see also [22]). The total spin defined by \( J_z := \frac{1}{2} \sum_{i=1}^{2L+1} Z_i \) commutes with Hamiltonian Eq. (37), namely \([J_z, S_i, S_{i+1}] = 0\), we can consider subspace \( S_k \) as

\[
S_k := \{ |s| : |J_z| = k |s| \}
\]
\[
\left( k = \frac{2L+1}{2}, \ldots, -\frac{1}{2}, \ldots, -\frac{2L+1}{2} \right). \quad (38)
\]

All \( 2L + 1 \) subspaces are invariant under the evolution, i.e. \( H^{s-\text{chain}}(s) S_k \subseteq S_k \). The two degenerate ground states are in the subspace \( S_0 \) and \( S_{-2} \). More precisely, \(|0\rangle_L \in S_0 \) and \(|1\rangle_L \in S_{-2} \). Considering \(|0\rangle_L \), it only transitions to another state inside \( S_{ \frac{1}{2} } \) during the evolution caused by \( H^{s-\text{chain}}(s) \). So we only need to consider the energy spectrum in \( S_{ \frac{1}{2} } \). Same logic is applicable to the state \(|1\rangle_L \) in \( S_{-2} \). There is symmetry in total spin, the Hamiltonian Eq. (37) remains unchanged by transforming \( Z_i \rightarrow -Z_i \) \( \forall i \in [1, 2L + 1] \). So the behavior of \( S_k \) is the same with \( S_{-k} \), thus it suffices to consider only the energy gap of ground state and first excited state in the subspace \( S_{ \frac{1}{2} } \). From hereon \( \Delta E_L(s) \) denotes the energy gap in this subspace.

We studied this energy gap numerically, by making use of TITPac ver.2 which is a library for diagonalizing quantum spin Hamiltonians developed by Nishimori [21]. We have calculated \( \Delta E_L(s) \) by setting \( \omega = 0.5 \), with \( s = 0.01, 0.02, \ldots, 1.00 \) for each \( L \) between 1 to 14. The result is displayed in FIG. 7. From these results, we see that the minimum energy gap \( G_L \) is obtained at \( s = 1/2 \). We plot \( G_L \) for each \( L \) in FIG. 8 and it can be seen that it scales as \( O(1/L) \).

Now, we will consider the total time \( T_L \) necessary for PAGT to be achieved. When \( s(t) \) is varied locally on time according to the energy gap \( \Delta E_L(s) \), the sufficient time calculated using Eq. (9) is,
\[
T_L = \int_0^1 ds \frac{|| H_{\text{fin}} - H_{\text{ini}} ||}{\Delta E_L(s)} = || H_{\text{fin}} - H_{\text{ini}} || T_c, \quad (39)
\]
\[
T_c := \int_0^1 ds \frac{1}{\Delta E_L(s)^2}. \quad (40)
\]

\( T_c \) is also calculated numerically, in FIG. 9. We have mentioned in in Section II B that \( T_c \) scales with \( O(1/G_L) \) in most cases (e.g. [17]). It seems that PAGT scheme also scales with \( O(1/L) = O(1/G_L) \). Together with the fact that (see appendix for proof)
\[
|| H_{\text{fin}} - H_{\text{ini}} || = O(L), \quad (41)
\]

we finally get \( T_L = O(L^2) \). Therefore, we conclude that we need total time \( O(L^2) \) to implement consecutive \( L \)-gate operation, while this time scales linearly by iterating the original AGT scheme or using quantum circuit model.

We note that our criteria Eq. (8), Eq. (9) are not shown to be a necessary condition for adiabatic theorem, and it may be possible to reduce this total time.

This shows that parallelizing AGT does not offer an advantage in terms of the total time for implementing \( L \) consecutive unitary gates. However, main advantage of PAGT lies in its ability to manipulate causal order by mapping time aligned causal order of gate operations to spatial order of interactions in the final Hamiltonian.

### C. Manipulating the order of operations

The important point of PAGT is that the order of the gate teleportations are determined in a single dragging of the Hamiltonian. This order is determined by the form of the final Hamiltonian only, and it is not irrelevant to the time parameter \( s(t) \). Let us consider the case \( L = 2 \).

In the above PAGT scheme, \( U(2)^U(1) \) will be applied to the target state. However, if the final Hamiltonian is changed to
\[
H^{\text{PAGT}}_{\text{fin}} := -4\omega (|1\rangle \langle 1| + |I\rangle \langle I|) \langle I |_{25}, \quad (42)
\]
rather than using Eq. (32), one can easily check that the output state after this PAGT becomes \( U(1)^U(2) \phi \).

The PAGT scheme enables us to encode information about unitary gates and their order separately, i.e. unitary gates are encoded in the initial Hamiltonian, and their order is determined by the final Hamiltonian. This model enables us to map causal order of operation into spatial interaction between qubits. This interesting property enables us to control order with control-qubit which we will explain in the next section.
FIG. 7. Energy gap $\Delta E_L(s)$ with respect to $s$. The energy gap between the ground states and first excited states in Eq. (37) are shown. We set the parameter $\omega = 0.5$ and plot $\Delta E_L(s)$ for each $s = 0.01, 0.02, \cdots, 1.00$. We change color for each $L$ (gate number) from 1 to 14.

V. CONTROLLED-PAGT

A. Quantum control of order of operations

So far we have introduced the parallelized version of AGT, which increases the total time compared to iterative AGT. However, we now show how different PAGT evolutions with the same initial Hamiltonian can be actually superposed using a control-qubit. This controlled-PAGT allows us to manipulate the causal order of gate operations by a control-qubit and implement a quantum switch (QS), first introduced in [1]. Assume a two-qubit system consisting of a control-qubit and a target qubit, described by $\mathcal{H}_C \otimes \mathcal{H}_T$. We also assume we can use two single-qubit unitary gates, $F$ and $G \in \mathcal{L}(\mathcal{H}_T)$. QS is a super operator (one may regard it as black box itself), $\mathcal{L}(\mathcal{H}_T) \otimes \mathcal{L}(\mathcal{H}_T) \mapsto \mathcal{L}(\mathcal{H}_C \otimes \mathcal{H}_T)$. It takes $F$ and $G$, and outputs the unitary operation $|0\rangle\langle 0|_C \otimes (GF)|_T + |1\rangle\langle 1|_C \otimes (FG)|_T$. More specifically, its action to a state is described by

$$QS(F, G)\left[|0\rangle_C\langle \phi_0|_T + |1\rangle_C\langle \phi_1|_T \right] = |0\rangle_C GF|\phi_0\rangle_T + |1\rangle_C FG|\phi_1\rangle_T.$$  \hspace{1cm} (43)

$|\phi_0\rangle, |\phi_1\rangle$ are the ket vectors for the target state and are not normalized. When control-qubit is in the $|0\rangle$ state, unitary operation $F$ acts first on the target qubit followed by $G$. And if the control-qubit is $|1\rangle$ state, the order of operation changes so $G$ acts first followed by $F$. This quantum control preserves coherence.

Rather than calling $F$ and $G$ as “unitary gates”, they are referred to as “oracles” due to existence of no-cloning theorem for boxes [27]. So we will consider now $F$ and $G$ as oracles rather than elementary gates. It was shown in [1] that it is impossible to implement QS using only one call of each unitary oracle $F, G$ in the quantum circuit model with definite causal structure. The same output
FIG. 8. Minimum energy gap $G_L$ with respect to $L$ (gate number). The data are obtained from FIG. 4 at $s = 1/2$.

FIG. 9. Evaluation of $T_e$ with respect to $L$ (gate number). We have calculated $T_e$ in Eq. (10) from the data in FIG. 4.

of Eq. (48) actually can be produced (see FIG. 11), but it requires the use of either unitary at least twice, which is forbidden in our setting.

Even if quantum circuit model can not implement quantum switch, the assumption of no definite causal structure itself is not prohibited by the axioms of quantum mechanics. It was also suggested that if one can somehow implement superposition of a quantum wire in quantum circuit, we can implement this operation, although the concrete physical system to achieve these kind of superposition has not been suggested yet. Implementation of QS operation in a quantum circuit is possible if one uses measurement and allows probabilistic success.

In the following, we show how the transformation can be simulated by PAGT. Here, we use the term “simulate”, because we are using a different resource, namely the gate Hamiltonian $H_F, H_G$ to implement unitary operation $F, G$.

We consider a system consisting of six qubits described by $H_C \otimes H_1 \otimes H_2 \otimes H_4 \otimes H_5$. The control-qubit is encoded in $H_C$ and the target qubit is encoded in $H_1 \otimes H_2 \otimes H_3 \otimes H_4 \otimes H_5$. We also assume the gate Hamiltonians are $H_F$ and $H_G$, corresponding to unitary gates $F$ and $G$. We prepare the input state $|0\rangle_C|\phi_0\rangle_1 + |1\rangle_C|\phi_1\rangle_1$ on the system $H_C \otimes H_1$. Initial Hamiltonian is written as

$$H_{\text{fin}}^{\text{QS}}(F, G) = H_{F_{23}} + H_{G_{45}}$$

$$= -4\omega(F_3|I\rangle \langle I|_{23} + G_5 |I\rangle \langle I|_{45})$$

Note that $H_C$ and $H_1$ is free with Eq. (44), it has four-fold degenerate ground state. The initial state is written in the form,

$$|\Psi(s = 0)\rangle := (|0\rangle_C|\phi_0\rangle_1 + |1\rangle_C|\phi_1\rangle_1)F_3|I\rangle_{23}G_5 |I\rangle_{45}$$

The main idea comes from the fact that we can manipulate the order of operations by the final Hamiltonian in PAGT. We design the final Hamiltonian to control the order of unitary gates by the control-qubit as follows,

$$H_{\text{fin}}^{\text{QS}} = |0\rangle\langle 0|_C \otimes H_{\text{fin}}^{(0)} + |1\rangle\langle 1|_C \otimes H_{\text{fin}}^{(1)}$$

$$= \frac{1}{2}(H_{\text{fin}}^{(0)} + H_{\text{fin}}^{(1)} + Z_C \otimes H_{\text{fin}}^{(0)} - Z_C \otimes H_{\text{fin}}^{(1)})$$

$$H_{\text{fin}}^{(0)} := -4\omega(|I\rangle \langle I|_{12} + |I\rangle \langle I|_{34})$$

$$H_{\text{fin}}^{(1)} := -4\omega(|I\rangle \langle I|_{14} + |I\rangle \langle I|_{25})$$

We construct the different interactions described by $H_{\text{fin}}^{(0)}$ and $H_{\text{fin}}^{(1)}$ through adiabatic process depending on the control-qubit. Our intuition comes from adiabatically simulating FIG. 10.

FIG. 10. Circuit representation of controlled PAGT. The scheme achieves quantum control of the order of operations by changing the final Hamiltonian corresponding to a control-qubit.

The total Hamiltonian is

$$H_{\text{total}}^{\text{QS}}(s) = (1-s)H_{\text{fin}}^{\text{QS}}(F, G) + sH_{\text{fin}}^{\text{QS}}$$

(49)
This Hamiltonian implements the operation implemented by QS, $|0\rangle|0\rangle \otimes GF_{F_T} + |1\rangle|1\rangle \otimes FG_{T}$, as we prove below. We can restrict our attention to the most simple case when $F = G = I$ (we write corresponding Hamiltonian $H^{QS\text{SAT}}(s)$), because implementing unitary $F, G$ can be written in the unitary conjugation form

$$H^{QS}(s) = \tilde{U}^{QS} H^{QS\text{SAT}}(s) \tilde{U}^{QS\dagger}$$

$$\tilde{U}^{QS} = |0\rangle\langle 0| \otimes (G_5) (F_3 F_4 F_5) + |1\rangle\langle 1| \otimes (F_3) (G_2^* G_3 G_5).$$

(50)

Therefore, considering this simplest case is enough to understand the general evolution.

We prove the achievability of adiabatic teleportation from $|0\rangle_C |\phi_0\rangle + |1\rangle_C |\phi_1\rangle$ to $|0\rangle_C |\phi_0\rangle + |1\rangle_C |\phi_1\rangle$. When using stabilizer formalism to show this, logical stabilizer of the target qubit can be considered as $L_z^{\text{Target}} = Z_{12345}, L_x^{\text{Target}} = X_{12345}$ which commute with $H^{QS}(s)$.

In the same way, we can define logical Z in control qubit $L_z^{\text{Control}} = Z_C$ since it commutes with $H^{QS\text{SAT}}$. However when we try to define logical X as $L_x^{\text{Control}} = X_C$, it does not commute with $H^{QS\text{SAT}}$. So we use this approach by this approximation now fails and we need to consider the dynamics explicitly.

Eq. (49) can be written,

$$H^{QS\text{SAT}}(t) = |0\rangle\langle 0| \otimes \left[ (1-s) H^{QS\text{SAT}}_{\text{init}} / C + s H^{0}_{\text{finj}} \right] + |1\rangle\langle 1| \otimes \left[ (1-s) H^{QS\text{SAT}}_{\text{init}} / C + s H^{0}_{\text{finj}} \right].$$

(51)

Note that $s$ is a function of $t$ where $H^{QS\text{SAT}}_{\text{init}} / C$ is the Hamiltonian where the control system $H_C$ is subtracted from $H^{QS\text{SAT}}$. We will define $H_{j}(t) := (1-s(t)) H^{QS\text{SAT}}_{\text{init}} / C + s(t) H_{\text{finj}} (j = 0, 1)$.

Let us write the initial state as $|\Psi\rangle = |0\rangle_C |\Phi_0\rangle_{12345} + |1\rangle_C |\Phi_1\rangle_{12345}$. Time evolution operator is written by

$$\mathcal{T} e^{-i \int_{t_0}^{t} dt H^{QS\text{SAT}}(t')} = |0\rangle\langle 0| \otimes \mathcal{T} e^{-i \int_{t_0}^{t} dt H_0(t')} + |1\rangle\langle 1| \otimes \mathcal{T} e^{-i \int_{t_0}^{t} dt H_1(t')}$$

(52)

where \( \mathcal{T} \) is a time-ordering operator, thus

$$|\Psi(t)\rangle := \mathcal{T} e^{-i \int_{t_0}^{t} dt H^{QS\text{SAT}}(t')} |\Psi\rangle$$

$$= |0\rangle_C \otimes \mathcal{T} e^{-i \int_{t_0}^{t} dt H_0(t')} |\Phi_0\rangle_{12345} + |1\rangle_C \otimes \mathcal{T} e^{-i \int_{t_0}^{t} dt H_1(t')} |\Phi_1\rangle_{12345}$$

$$= |0\rangle_C |\Phi_0(t)\rangle_{12345} + |1\rangle_C |\Phi_1(t)\rangle_{12345}.$$  

(53)

The state time evolution $|\Phi_k(t)\rangle (k = 1, 2)$ is the same when one consider previous PAGT scheme with $L = 2$. After the process, we have

$$|\Phi_0\rangle \rightarrow |I\rangle_{12} |J\rangle_{34} |\phi_0\rangle_5,$$

$$|\Phi_1\rangle \rightarrow |I\rangle_{14} |J\rangle_{25} |\phi_1\rangle_3.$$  

(54)

However, we must now consider the relative phase between these states. We want to know the condition that ensures the relative phase remains 0. To analyze it specifically, we will consider the following time-dependent Hamiltonian

$$H^{QS} = -4 \omega [(1-s_F)|\rangle \langle I\rangle_{23} + (1-s_G)|\rangle \langle I\rangle_{45} + |0\rangle\langle 0| \otimes (s_{12} |\rangle \langle J\rangle_{12} + s_{34} |\rangle \langle J\rangle_{34}) + |1\rangle\langle 1| \otimes (s_{14} |\rangle \langle J\rangle_{14} + s_{25} |\rangle \langle J\rangle_{25})].$$

(55)

Here we assume the situation where the Hamiltonian strength $s$ is not controlled simultaneously each other (this consideration comes from the experimental setting where we may not be able to synchronize the strength $s$ of each term). For simplicity we also set $\omega = 1/4$ in the following.

$$H_0(t) = -(1-s_F) |\rangle \langle I\rangle_{23} - (1-s_G) |\rangle \langle I\rangle_{45} - s_{12} |\rangle \langle I\rangle_{12} - s_{34} |\rangle \langle I\rangle_{34}.$$  

(56)

$$H_1(t) = -(1-s_F) |\rangle \langle I\rangle_{23} - (1-s_G) |\rangle \langle I\rangle_{45} - s_{14} |\rangle \langle I\rangle_{14} - s_{25} |\rangle \langle I\rangle_{25}.$$  

(57)

Relabelling the Hilbert space of $H_1$ to $2 \leftrightarrow 4$ and $3 \leftrightarrow 5$, we get

$$H_1 = -(1-s_G) |\rangle \langle I\rangle_{23} - (1-s_F) |\rangle \langle I\rangle_{45} - s_{14} |\rangle \langle I\rangle_{14} - s_{25} |\rangle \langle I\rangle_{25}.$$  

(58)

Comparing with Eq. (59), we know

$$s_F(t) = s_G(t),$$

$$s_{12}(t) = s_{14}(t),$$

$$s_{34}(t) = s_{25}(t).$$  

(59)

are the sufficient condition that the relative phase between $|\Phi_0\rangle$ and $|\Phi_1\rangle$ remains 0, i.e. to achieve controlled adiabatic teleportation, we restrict each of the strength of

1. gate Hamiltonian $H_F$ and $H_C$
2. first gate between control $|0\rangle$ and $|1\rangle$
3. second gate between control $|0\rangle$ and $|1\rangle$

are exactly the same for arbitrary time $t$ through the evolution. Especially, in the case that the strength of each Hamiltonian is controlled in the same manner $s(t)$ as in Eq. (59), it suffices the condition. This shows that controlled teleportation works with condition Eq. (59). Here, we note that one of the good property in AGT loses with this scheme. In AGT, precise time control is not required for implementing unitary gate because we can implement $U$ to an unknown state $|\psi\rangle$ only if the time parameter $s(t)$ is slow enough. However, now we need to synchronize the strength of $s(t)$ to satisfy Eq. (59) during the evolution.

From this analysis, we can see that the state after the evolution caused by general form Eq. (44) is

$$|\Psi(s = 1)\rangle = |0\rangle_C |\rangle \langle I\rangle_{01} |\rangle \langle J\rangle_{23} GF |\phi_0\rangle_4 + |1\rangle_C |\rangle \langle I\rangle_{03} |\rangle \langle J\rangle_{14} FG |\phi_1\rangle_2.$$  

(60)
Followed by C-SWAP (Controlled-swap) operations, we can confirm that the quantum switching operation is achieved

\[
\text{C-SWAP}_{13}\text{C-SWAP}_{24}\left|\Psi^{QS}(s = 1)\right\rangle = |1\rangle_0|0\rangle_1|1\rangle_4CGF|\phi_0\rangle_4 + |1\rangle_4FG|\phi_1\rangle_4. \quad (61)
\]

Note again that we are not using the oracle \(F\) and \(G\), but gate Hamiltonian \(H_F\) and \(H_G\).

If Eq. (59) is satisfied, it is easy to understand that the energy eigenvalues do not change from PAGT scheme. We denote the eigenvalues and the eigenvectors of \(H_j(s)\) as \(E_i^{(j)}\) and \(|\Psi_j^{(i)}(s)\rangle\) \((j = 0, 1, 0 \leq i \leq 2^5 - 1)\),

\[
H_j(s)|\Psi_j^{(i)}(s)\rangle = E_i^{(j)}|\Psi_j^{(i)}(s)\rangle. \quad (62)
\]

This allows us to find the eigenvalues and the eigenvectors of controlled version Eq. (59).

\[
H^{QS}(s)|j\rangle_C|\Psi_j^{(i)}(s)\rangle = E_i^{(j)}|\Psi_j^{(i)}(s)\rangle. \quad (63)
\]

Since the eigenvalues remain unchanged from PAGT scheme Eq. (62), AGT of the quantum switch is possible.

We stress that we control the final Hamiltonian which decides only the order of operations, and it does not depend on gate Hamiltonian. In quantum circuit model it is straightforward to count the number of oracle, however in adiabatic model, “number of calls” does not have well-defined meaning. Therefore, fair comparison between 2 models of resources is not straightforward.

Here, we will introduce the term “dependent resource” and “independent resource” instead of the notion “once” and “twice (or more times)” respectively for our purposes. For example, quantum circuit in FIG. 11 produces the same result of QS, however it uses \(F\) twice. We will denote them as \(F^{(1)}\) and \(F^{(2)}\). Then we say that these \(F^{(1)}\) and \(F^{(2)}\) are independent resource, because these operation act on different spacetime, i.e. they act either at different time or different qubits. Even if the resource used to produce the unitary \(F\) is the same, we call it “independent resource” if it is used at different times. We call them “independent” because we can only change \(F^{(1)}\) operation without changing \(F^{(2)}\), or let us assume that there is noise on \(F^{(1)}\) this affects only \(F^{(1)}\) and \(F^{(2)}\) does not affected.

Contrary to this model, our controlled-PAGT scheme uses the dependent resources \(H_F, H_G\) to implement \(|0\rangle_4|0\rangle_C \otimes F^{(1)}C^{(3)} + |1\rangle_4|1\rangle_C \otimes G^{(4)}F^{(2)}\) operations. Here, both \(F^{(1)}\) and \(F^{(2)}\) are derived from the same resource \(H_F\), and both \(G^{(3)}\) and \(G^{(4)}\) are derived from \(H_G\). Thus, we regard each \(H_F\) and \(H_G\) is dependent resource to implement \(F\) and \(G\) unitary operation in QS operation.

To summarize, we rephrase the result in (61) and conclude as follows. In definite causal structure, quantum switch cannot be implemented by quantum circuit model without using independent resource deterministically. However, controlled-PAGT can simulate this task only using dependent resource.

There are some papers which studies about the computational power of quantum switch. 28, 29. The definition of QS can be generalized so that \(F\) and \(G\) are quantum channels, CPTP maps. It was proven that the power of generalized QS has a power beyond quantum circuit. More precisely, one can achieve a channel discrimination task which cannot be achieved with usual standard quantum circuit 28. The resource comparison of the order controlled operation implemented by QS and standard quantum circuit was performed in 29. However, we note that there is no computational speed up if one can use the oracle many times. Because we can implement quantum switch in quantum circuit in polynomial time.

VI. APPLICATIONS OF PAGT

A. Controlled-unitary using AGT

We will consider a question of implementing controlled-unitary using our controlled-PAGT technique. Our task is to implement controlled-\(U\) gate from given unitary \(U\). This question is important because some well-known quantum algorithms require us to call a controlled-oracle. Phase estimation algorithm is one such example, which has important applications that include Shor’s factoring algorithm 30. However, it was shown by Soeda 31 that it is not possible to implement controlled-unitary if one can only use an uncontrolled oracle. In this section, we consider the implementation of a controlled-unitary and show that it is possible in certain case, where the oracle is written in orthogonal matrix form.

B. Failure cases

We consider the problem of implementing a controlled-\(U\) using an gate Hamiltonian \(H_U\). Let us consider the following Hamiltonian which is defined on the Hilbert space \(\mathcal{H}_C \otimes \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4 \otimes \mathcal{H}_5\),

\[
H^{CU}(s) := (1 - s)H^{CU}_{\text{ini}} + sH^{CU}_{\text{fin}}, \quad (64)
\]

\[
H^{CU}_{\text{ini}} := H_{123} + H_{45}, \quad (65)
\]

\[
H^{CU}_{\text{fin}} := |0\rangle_4\langle 0| \otimes H_{123} + |1\rangle_4\langle 1| \otimes H_{14}, \quad (66)
\]

![FIG. 11. Quantum circuit implementing quantum switch operation with independent unitary gates \(F^{(1)}\) and \(F^{(2)}\). Here, a combination of \(x\) represents SWAP operation. First two SWAP operations with black circle is performed when control-qubit is \(|1\rangle\), and last two SWAP operations with white circle is performed when control-qubit is \(|0\rangle\).](image-url)
We assume the input state is $|\Psi^{CU}(s = 0)\rangle = (|0\rangle_C|\phi_0\rangle_1 + |1\rangle_C|\phi_1\rangle_1) |I\rangle_{12} |U\rangle_{23} |I\rangle_{45}$. See FIG. 12.

![Diagram](image)

**FIG. 12.** Trial run for implementing a controlled-unitary using controlled-PAGT. The scheme fails to implement a controlled-unitary due to the incoherence in the ancilla qubits.

Simple analysis shows that it does not implement the controlled-U operation because after the scheme, we get

$$
|\Psi^{CU}(s = 1)\rangle = |0\rangle_C |I\rangle_{12} U_5 |I\rangle_{45} |\phi_0\rangle_3 \\
+ |1\rangle_C |I\rangle_{12} U_5 |I\rangle_{45} |\phi_1\rangle_3.
$$

(67)

The C-SWAP$_{24}$C-SWAP$_{35}$C-SWAP$_{15}|\Psi^{CU}(s = 1)\rangle = |I\rangle_{12} |0\rangle_C U_5 |I\rangle_{45} |\phi_0\rangle_3 + |1\rangle_C |I\rangle_{45} U |\phi_1\rangle_3.

(68)

So even after C-SWAP, ancilla state on $\mathcal{H}_4 \otimes \mathcal{H}_5$ has no inequivalent part. Therefore, we only look at output state, space on $\mathcal{H}_4 \otimes \mathcal{H}_3$ in Eq. (68), it becomes a mixed state. Thus this scheme fails to implement a controlled-unitary.

Previous consideration says that we need to erase the information of $I \otimes U |I\rangle$ in the $|0\rangle_C$ part. Therefore, we consider the following scheme

$$
H^{CU}_C(s) := (1 - s)H^{CU}_{ini} + sH^{CU}_{fin}, \quad (69)
$$

$$
H^{CU}_{ini} := H_{I23} + H_{U45}, \quad (70)
$$

$$
H^{CU}_{fin} := |0\rangle_C \otimes (H_{I12} + H_{I45}) + |1\rangle_C \otimes H_{I45}. \quad (71)
$$

We add the term $|0\rangle_C \otimes H_{I45}$ to shift the initial state $U_5 |I\rangle_{45}$ to the final state $|I\rangle_{45}$. However, this scheme generates another problem. When we focus on the evolution from $U_5 |I\rangle_{45}$ to $|I\rangle_{45}$ by adiabatic process, it depends on the form of $I \otimes U |I\rangle$. For example, in the case $U = X$, there is an energy crossing between ground state and first excited state during the evolution, in which case the deletion becomes impossible to perform.

Also, even $I \otimes U |I\rangle$ can be deleted, the final state become

$$
\text{C-SWAP}_{24}\text{C-SWAP}_{15} |\Psi^{CU}(s = 1)\rangle = e^{i\theta_U(s(t))} |0\rangle_C |I\rangle_{12} |I\rangle_{45} |\phi_0\rangle_3 + |1\rangle_C |I\rangle_{12} |I\rangle_{45} U |\phi_1\rangle_3.
$$

(72)

Here, extra relative phase factor $e^{i\theta_U(s(t))}$ arises in the $|0\rangle_C$ part. $\theta_U(s(t))$ is a functional which depends on the form of the function $s(t)$. In general, we do not extract the value of $\theta$. In this case, one has to average over the states, which gives the following density operator

$$
\rho^{CU}_{s = 1} = \int_0^{2\pi} d\theta(e^{i\theta}|0\rangle\langle\phi_0| + |1\rangle U |\phi_1\rangle)(e^{-i\theta}|0\rangle\langle\phi_0| + |1\rangle U^\dagger |\phi_1\rangle) = |0\rangle\langle 0| \otimes |\phi_0\rangle\langle\phi_0| + |1\rangle\langle 1| \otimes U |\phi_1\rangle\langle\phi_1| U^\dagger.
$$

(73)

So, the coherence between $|0\rangle_C$ and $|1\rangle_C$ is broken. The same density operator is obtained when one measure control-qubit on the $\{|0\rangle, |1\rangle\}$ basis, followed by operating unitary $U$ when one get the measurement result $|1\rangle$. This scheme does not implement the controlled-unitary either.

### C. Controlled-orthogonal

The above protocol does not work well. However, let us remind the power of gate Hamiltonian we explained in the section. We can implement $U$ and $U^T$ by using same gate Hamiltonian $H_U$. Now, let us assume that $U = O^T$ where $O$ is an orthogonal matrix, i.e. $O^T O = I$. We consider following Hamiltonian which is defined on the Hilbert space $\mathcal{H}_C \otimes \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4 \otimes \mathcal{H}_5$.

$$
H^{CO}(s) := (1 - s)H^{CO}_{ini} + sH^{CO}_{fin}.
$$

(74)

$$
H^{CO}_{ini} := H_{O^T_{12}} + H_{O^T_{45}}^2,
$$

(75)

$$
H^{CO}_{fin} := |0\rangle\langle 0| \otimes (H_{O^T_{11}} + H_{O^T_{24}}) + |1\rangle\langle 1| \otimes (H_{O^T_{12}} + H_{O^T_{45}}).
$$

(76)

![Diagram](image)

**FIG. 13.** Implementation of controlled-orthogonal by controlled-PAGT. controlled-O is implemented by using gate Hamiltonian $H_{O^T}$.

When control-qubit is $|0\rangle$, the gate teleportation acts $O^T O^T = I$. Therefore, the output state is

$$
|\Psi^{CO}(s = 1)\rangle = |0\rangle_C |I\rangle_{13} |I\rangle_{24} O^T |\phi_0\rangle_5 \\
+ |1\rangle_C |I\rangle_{13} |I\rangle_{24} O |\phi_1\rangle_5.
$$

(77)

The C-SWAP$_{23}$|\Psi^{CU}(s = 1)\rangle = |0\rangle_C |I\rangle_{13} |I\rangle_{24} |\phi_0\rangle_5 \\
+ |1\rangle_C |I\rangle_{13} |I\rangle_{24} O |\phi_1\rangle_5.

(78)
Indeed, controlled-orthogonal unitary operation is implemented. Therefore, we can implement controlled-$O^2$ operations from gate Hamiltonian $H_O$ by controlled-PAGT scheme.

Indeed, we can also consider stranger operations, such as the process showed in FIG. 14, implements a controlled-$U^2U^\dagger$ operation.

![FIG. 14. One example of implementation of a strange controlled unitary. We can implement controlled-$U^2U^\dagger$ operation by using gate Hamiltonian $H_U$ in controlled-PAGT scheme.](image)

In this chapter, we show an application of controlled-PAGT to implement controlled version of an unknown unitary gate. However, these applications are not using coherent control of the causal order of gate operations. In fact, the possibility of the tasks is based on the power of gate Hamiltonian $H_U$, which can implement $U, U^T, U^*$ and $U^\dagger$ operations. Therefore, this shows the difference of the power between gate Hamiltonian $H_U$ in AGT model, and unitary gate $U$ in standard quantum circuit model.

### VII. CONCLUSION

We have investigated a method to manipulate the causal order of gate operations by adiabatically implementing quantum computation. Our method is based on the adiabatic gate teleportation (AGT) scheme proposed by Bacon and Flammia. We introduce gate Hamiltonian as a resource to implement a unitary operation. We have shown that this resource can implement not only the unitary but also the transpose, complex conjugate, and adjoint of the unitary operation.

Using the gate Hamiltonians, we construct a parallelized AGT (PAGT) scheme, where consecutive gate operations are implemented in a single adiabatic shift. An important feature of the PAGT scheme is that information about the unitary gate and information about the causal order of gates are separately encoded in the initial Hamiltonian and the final Hamiltonian respectively. We can choose the total unitary gate by changing the gate Hamiltonian in the initial Hamiltonian, and we can choose the order of these unitary gates by changing the spatial interaction order of the final Hamiltonian. This property allows us to construct the controlled-PAGT scheme that uses a control-qubit to manipulate the causal order of the unitary gates.

However, PAGT has no advantage in terms of the computational time required to implement $L$ consecutive gate operations compared to AGT scheme since the energy gap narrows by parallelization. Quantitatively, total time scales with $O(L^2)$ to implement $L$ consecutive unitary gate operations in PAGT, while it scales $O(L)$ by just iterating the AGT scheme for $L$ times.

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### VIII. APPENDIX

#### A. Proof of the spectrum behavior

Here, we prove Eq. (79) by considering lower bound and upper bound of this equation. Lower bound is obtained by using the property,

$||A|| \leq \max_{v} \frac{||Av||}{||v||} \geq \frac{||Av_0||}{||v_0||}$

where $v_0$ is arbitrary chosen vector. In our case, choosing $v_0$ as $|\phi_0\rangle = |00110011\cdots00110\rangle$ shows that

$||H_{fin} - H_{ini}|| \geq \langle \phi_0 | H_{fin} - H_{ini} | \phi_0 \rangle = \omega L,$

which proves that the lower bound scales with $O(L)$. On the other hand, upper bound is derived by

$||H_{fin} - H_{ini}|| = |\omega \sum_{k=1}^{2L} (-1)^k S_k \cdot S_{k+1}|$

$\leq |\omega \sum_{k=1}^{2L} |S_k \cdot S_{k+1}| = 6\omega L$

Therefore, $\omega L \leq ||H_{fin} - H_{ini}|| \leq 6\omega L$, which proves that $||H_{fin} - H_{ini}|| = O(L)$. 


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