A Dynamic Game Model of Collective Choice: Stochastic Dynamics and Closed Loop Solutions

Rabih Salhab, Roland P. Malhamé and Jerome Le Ny

Abstract

We consider within the framework of the Mean Field Games theory a dynamic discrete choice model with social interactions, where a large number of agents/players are choosing between two alternatives while influenced by the group’s behavior. We introduce the “Min-LQG” optimal control problem, a modified Linear Quadratic Gaussian (LQG) optimal control problem that includes a minimum term in its final cost to capture the discrete choice phenomenon. We give an explicit solution of the Min-LQG problem and show that at each instant, the dynamic discrete choice model can be interpreted as a static discrete choice model where the cost of choosing one of the alternative includes an additional term that increases with the risk of being driven to the other alternative by the Wiener process. Finally, the mean field equations are given. The fixed point problem will be studied in a next version of this article.

Index Terms

Mean Field Games, Stochastic Optimal Control, Discrete Choice Models.

I. INTRODUCTION

Discrete choice problems arise in situations where an individual makes a choice among a set of alternatives, such as a mode of transportation [1], entry and withdrawal from the labor market, or a residential location [2]. In [3], McFadden laid the theoretical foundation of discrete choice models, which is based on utility theory. According to these models, each alternative is associated

This work was supported by NSERC under Grants 6820-2011 and 435905-13. The authors are with the department of Electrical Engineering, Polytechnique Montreal and with GERAD, Montreal, QC H3T-1J4, Canada {rabih.salhab, roland.malhame, jerome.le-ny}@polymtl.ca.
with a utility function, which is the sum of two terms. The first is a deterministic function that depends on the attributes of the person making the choice and the related alternative, while the second is a random variable reflecting the unobservable factors that contribute to the individual’s choice. Under a utility-maximizing behavior assumption, the individual’s choice is described by a set of “choice probabilities”, i.e., the probabilities of choosing each of the alternatives.

In some situations, the individuals’ choices are considerably influenced by the social behavior. For example, in schools, teenagers’ decisions to smoke are affected by some personal factors, as well as their peers’ behavior. To analyze this phenomenon, Brock and Durlauf introduced within the framework of static noncooperative game theory a discrete choice model with social interactions [4], where a large number of players are choosing between two alternatives while being influenced by the average of the choices. The authors analyze the model using an approach similar to the Mean Field Games (MFG) methodology, and inspired by statistical mechanics.

Recently, we studied a dynamic discrete choice model with social interactions in [5]–[7]. In these articles, we introduce within the framework of the MFG theory a dynamic game involving a large number of players choosing between multiple destination points. The agents’ dynamics are deterministic with random initial conditions. We show via the MFG approach that multiple approximate (epsilon) Nash equilibria may exist, each characterized by a vector describing the way the population splits between the alternatives. The strategies developed in these papers are open loop decentralized policies, in the sense that to make its choice of trajectory, an agent needs to know only its initial state and the initial distribution of the population.

In this paper, we consider the fully stochastic case where a large number of players moving according to a set of controlled diffusion processes should reach before a time $T$ one of two predefined destination points. They must do so while influenced along the path by the average of the population and developing as little effort as possible. This formulation generalizes the classical static discrete choice models. In fact, while in the classical models an agent makes its choice once, in our model, it updates its choice continuously based on the observations and occurring events. For example, along the path to choose a mode of transportation, a person is exposed to unexpected events that can affect its decision, such as weather changes, car accidents, etc.

The main contributions of the paper are as follows:

i. We introduce and solve a modified version of the standard Linear Quadratic Gaussian (LQG)
optimal control problem, that we call the “Min-LQG” optimal control problem, where
the final cost is replaced by a minimum of two distances to capture the discrete choice
phenomenon.

ii. We show that at each instant, the dynamic discrete choice model (Min-LQG problem) can
be interpreted as a static discrete choice model where the cost of choosing one of the
alternatives includes an additional term that increases with the risk of being driven to the
other alternative by the Wiener process.

iii. In contrast to [5]–[7], we consider in this paper controlled diffusion processes and develop
MFG-based decentralized closed-loop policies.

The MFG methodology that we follow in this paper was originally developed in a series of
papers by Huang et al. [8]–[10], and independently by Lions and Lasry [11]–[13]. It is a powerful
technique to analyze dynamic games involving a large number of players interacting through
the mass effect of the group. It starts by considering the limiting case (continuum of players)
which can be described by two coupled partial differential equations (PDE), a Hamilton-Jacobi-
Bellman (HJB) equation propagating backwards and a Fokker-Planck (FP) equation propagating
forwards. The former characterizes the players’ best response to the mass effect, while the
latter propagates the distribution of the players (mass effect) under their best responses to the
mass effect. Candidate sustainable macroscopic behaviors (mass effect), if they exist, are then
computed by a fixed point argument. The corresponding best responses, when applied to the
finite population, constitute approximate Nash equilibria (ε-Nash equilibria) [9], [10].

Definition 1: Consider $N$ agents, a set of strategy profiles $S = S_1 \times \ldots \times S_N$ and, for each
agent $k$, a cost function $J_k(u_1, \ldots, u_N), \forall (u_1, \ldots, u_N) \in S$. A strategy profile $(u_1^*, \ldots, u_N^*) \in S$
is called an $\varepsilon$-Nash equilibrium with respect to the costs $J_k$, if there exists an $\epsilon > 0$ such that,
for any fixed $1 \leq i \leq N$, for all $u_i \in S_i$, we have

$$J_i(u_i, u_{-i}^*) \geq J_i(u_i^*, u_{-i}^*) - \epsilon.$$

The mathematical model is presented in Section II. In Section III we introduce and solve
the Min-LQG optimal tracking problem. We give an explicit form of the generic agent’s best
response (Min-LQG optimal control law) and discuss its relation to the classical static discrete
choice problems. In section IV we introduce the Mean Field Games equations. The existence
of a sustainable mass behavior will be studied in a next version of this article.
II. MATHEMATICAL MODEL

We present in this section the dynamic discrete choice model with social interactions. We consider a dynamic noncooperative game involving a large number $N$ of players with the following dynamics:

$$dx_i(t) = (ax_i(t) + bu_i(t)) \, dt + \sigma dw_i(t), \quad 0 \leq i \leq N,$$

where $a \in \mathbb{R}$, $b \in \mathbb{R}\setminus\{0\}$, $\sigma > 0$, and $\{w_i, 1 \leq i \leq N\}$ are $N$ independent standard Wiener processes on some probability space $(\Omega, \mathcal{F}, P)$. We assume that the initial conditions $\{x_i(0), 1 \leq i \leq N\}$ are independent and identically distributed (i.i.d.) and also independent of $\{w_i, 1 \leq i \leq N\}$. Moreover, $\mathbb{E}x_i(0)^2 < \infty$. We denote by $p_0$ the probability density function of $x_i(0)$. The scalar $x_i(t)$ is the state of player $i$ at time $t$ and $u_i(t)$ its control input. Each player is associated with the following individual cost functional:

$$J_i(x_i(0), u_i(.)) = \mathbb{E} \left[ \int_0^T \left\{ \frac{q}{2} (x_i - \bar{x})^2 + \frac{r}{2} u_i^2 \right\} \, dt + \frac{M}{2} \min_{j=1,2} (x_i(T) - p_j)^2 |x_i(0)| \right],$$

for $i = 1, \ldots, N$, where $r, T > 0$ and $q, M \geq 0$. Along the path, the running cost forces the players to remain close to the average of the population $\bar{x}(t) = 1/N \sum_{i=1}^N x_i(t)$ and to develop as little effort as possible. At time $T$, a player $i$ should also be close to one of the destination points $p_1 \in \mathbb{R}$ or $p_2 \in \mathbb{R}$, otherwise it is strongly penalized by the final cost.

We define the set of admissible control laws as follows:

$$U = \left\{ u(.) \in \mathbb{R}^{[0,T] \times \Omega} | u(.) \text{ is progressively measurable and} \right\}$$

$$\mathbb{E} \int_0^T |u(s,w)|^m \, ds < \infty, \text{ for all } m \in \mathbb{N} \right\}.$$

We define the set of admissible Markov policies:

$$\mathcal{L} = \left\{ u(.) \in \mathbb{R}^{[0,T] \times \mathbb{R}} | \exists K_1 > 0, |u(t,x)| \leq K_1 (1 + |x|), \forall (t,x) \in [0,T] \times \mathbb{R}, \text{ and } \forall r > 0, \right\}$$

$$\forall T' \in (0,T), \exists K_2(r) > 0, |u(t,x) - u(t,y)| \leq K_2(r) |x - y|, \forall (x,y) \in [-r,r]^2, \forall t \in [0,T'] \right\}.$$

If $u \in \mathcal{L}$, then the stochastic differential equation (SDE) \((1)\), with $u_i$ equal to $u(t,x_i)$, has a unique strong solution. Moreover, $u(t,x_i(t,w)) \in U \ [14]$. 

April 28, 2016 DRAFT
III. THE “MIN-LQG” OPTIMAL TRACKING PROBLEM 
AND THE GENERIC AGENT’S BEST RESPONSE

Following the MFG approach, we start by assuming a continuum of agents for which one can
ascribe an assumed given deterministic macroscopic behavior $\bar{x}$. In order to compute its best
response to $\bar{x}$, a generic agent solves the following optimal control problem that we call the
“Min-LQG” optimal tracking problem:

$$
\inf_{u \in U} J(x(0), u(\cdot), \bar{x}) = \inf_{u \in U} \mathbb{E} \left[ \int_0^T \left\{ \frac{q}{2} (x - \bar{x})^2 + \frac{r}{2} u^2 \right\} dt + \frac{M}{2} \min_{j=1,2} (x(T) - p_j)^2 \right] x(0)
$$

s.t. $dx(t) = (ax(t) + bu(t)) dt + \sigma dw(t)$.

To solve explicitly the Min-LQG optimal tracking problem, we start by considering the situation
where $a = q = 0$. In this case, one can derive an explicit solution using a generalized Hopf-Cole
transformation. We then show that this solution holds in the general case.

A. Case $a = q = 0$

When $a$ and $q$ are assumed to be zero, the optimal cost-to-go function $V(t, x)$ corresponding
to (5) satisfies the following HJB equation [14]:

$$
-\frac{\partial V}{\partial t} = -\frac{b^2}{2r} \left( \frac{\partial V}{\partial x} \right)^2 + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2}
$$

$$
V(T, \cdot) = \frac{M}{2} \min_{j=1,2} (\cdot - p_j)^2.
$$

To find an explicit solution, we transform (6) into a Heat Equation using the following generalized
Hopf-Cole transformation [15] Chapter 4-Section 4.4]:

$$
\psi(t, x) = \exp \left( -\frac{b^2}{\sigma^2 r} V(t, x) \right).
$$

The function $\psi(t, x)$ satisfies the following backward Heat Equation:

$$
\frac{\partial \psi}{\partial t} = -\frac{\sigma^2}{2} \frac{\partial^2 \psi}{\partial x^2}
$$

$$
\psi(T, \cdot) = \exp \left( -\frac{Mb^2}{2\sigma^2 r} \min_{j=1,2} (\cdot - p_j)^2 \right).
$$
Lemma 1: The Heat Equation (8) admits a unique solution

\[ \psi(t,x) = \exp \left( -\frac{b^2}{\sigma^2 r} V_1(t,x) \right) P \left( \sigma w(T-t) \leq \frac{(c-x)r + Mb^2(T-t)d}{\sqrt{r^2 + Mb^2r(T-t)}} \right) 
+ \exp \left( -\frac{b^2}{\sigma^2 r} V_2(t,x) \right) P \left( \sigma w(T-t) \geq \frac{(c-x)r - Mb^2(T-t)d}{\sqrt{r^2 + Mb^2r(T-t)}} \right), \]  

(9)

where \( c = (p_1 + p_2)/2, d = (p_2 - p_1)/2, \) and for \( j = 1, 2, \)

\[ V_j(t,x) = \frac{1}{2} \Pi(t)x^2 + \beta_j(t)x + \delta_j(t). \]

\[ \Pi(t) = \frac{Mr}{r + Mb^2(T-t)} \]

\[ \beta_j(t) = -\frac{Mrp_j}{r + Mb^2(T-t)} \]

\[ \delta_j(t) = \frac{Mrp_j^2}{2(r + Mb^2(T-t))} + \frac{\sigma^2 r}{2b^2} \left( \log \left( r + Mb^2(T-t) \right) - \log r \right). \]

Proof: The unique solution of (8) is [15]:

\[ \psi(t,x) = \frac{1}{\sigma \sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} \exp \left( \frac{(x - y)^2}{2\sigma^2(T-t)} \right) \psi(T,y)dy \]

\[ = \frac{1}{\sigma \sqrt{2\pi(T-t)}} \left\{ \int_{-\infty}^{c} \exp \left( \frac{(x - y)^2}{2\sigma^2(T-t)} - \frac{Mb^2}{2\sigma^2 r}(y - p_1)^2 \right) dy \right. \]

\[ + \int_{c}^{\infty} \exp \left( \frac{(x - y)^2}{2\sigma^2(T-t)} - \frac{Mb^2}{2\sigma^2 r}(y - p_2)^2 \right) dy \}. \]

(10)

By expanding the integrands in (10) and completing the squares, we get for \( j = 1, 2, \)

\[ \frac{(x - y)^2}{2\sigma^2(T-t)} + \frac{Mb^2}{2\sigma^2 r}(y - p_j)^2 = \frac{b^2}{2\sigma^2 r} \Pi(t)x^2 + \frac{b^2}{\sigma^2 r} \beta_j(t)x + \frac{Mb^2p_j^2}{2\sigma^2(r + Mb^2(T-t))} \]

\[ + \frac{r + Mb^2(T-t)}{2\sigma^2 r(T-t)} \left( y - \frac{mr + Mb^2p_j(T-t)}{r + Mb^2(T-t)} \right)^2. \]

Now, by implementing these new expressions of the integrands in (10), and by making a change of variable \( z_j = \sqrt{\frac{r + Mb^2(T-t)}{r}} \left( y - \frac{mr + Mb^2p_j(T-t)}{r + Mb^2(T-t)} \right), \) for \( j = 1, 2, \) we get (9).

In the following, we characterize the probabilities in (9). It can be shown that the functions \( V_j(t,x), \) for \( j = 1, 2, \) are the optimal cost-to-go of the following standard LQG optimal control problems:

\[ \inf_{u^{(j)}(\cdot)} J^{(j)}(x^{(j)}(0), u^{(j)}(.)) = \inf_{u^{(j)}(\cdot)} \mathbb{E} \left[ \int_{0}^{T} \frac{r}{2} (u^{(j)})^2 dt + \frac{M}{2} (x^{(j)}(T) - p_j)^2 \bigg| x^{(j)}(0) \right] \]

\[ \text{s.t. } dx^{(j)}(t) = bu^{(j)}(t)dt + \sigma dw^{(j)}(t). \]
We recall that the optimal control law of (11) has the following form:
\[ u_s^{(j)}(t) = -\frac{b}{r} \left( \Pi(t)x^{(j)} + \beta_j(t) \right). \]  

(12)

Note that if \( M = 0 \), then the optimal control laws \( u_s^{(1)}(t) = u_s^{(2)}(t) = 0 \). Hence, the corresponding optimal states at time \( t \), \( x_s^{(j)}(t) \) for \( j = 1, 2 \), are equal to to \( x^{(j)}(0) + \sigma w^{(j)}(t) \), for \( j = 1, 2 \). Therefore,
\[
P \left( \sigma w(T - t) \leq \frac{(c - x)r + Mb^2(T - t)d}{\sqrt{r^2 + Mb^2(T - t)}} \right) = P \left( x_s^{(1)}(T) \leq c \bigg| x_s^{(1)}(t) = x \right)
\]
\[
P \left( \sigma w(T - t) \geq \frac{(c - x)r - Mb^2(T - t)d}{\sqrt{r^2 + Mb^2(T - t)}} \right) = P \left( x_s^{(2)}(T) \geq c \bigg| x_s^{(2)}(t) = x \right).
\]

To generalize this result for any \( M \geq 0 \), we define the optimal states corresponding to (12) as follows:
\[
dx_s^{(j)}(s) = u_s^{(j)}(s)ds + \sigma dw^{(j)}(s), \quad s \in [0, T],
\]

(13)

where \( x^{(j)}(0) = x(0) \), for \( j = 1, 2 \). The process \( x_s^{(1)} \) satisfies a linear SDE. After calculation, conditioning on \( \{ x_s^{(1)}(t) = x \} \), \( x_s^{(1)}(T) \) is distributed according to the normal distribution [16, page 354]:
\[
\mathcal{N} \left( \frac{x^r + Mb^2(T - t)p_1}{r + Mb^2(T - t)}, \frac{\sigma^2 r(T - t)}{r + Mb^2(T - t)} \right).
\]

Thus,
\[
P \left( \sigma w(T - t) \leq \frac{(c - x)r + Mb^2(T - t)d}{\sqrt{r^2 + Mb^2(T - t)}} \right) = P \left( x_s^{(1)}(T) \leq c \bigg| x_s^{(1)}(t) = x \right).
\]

Similarly, one can show that
\[
P \left( \sigma w(T - t) \geq \frac{(c - x)r - Mb^2(T - t)d}{\sqrt{r^2 + Mb^2(T - t)}} \right) = P \left( x_s^{(2)}(T) \geq c \bigg| x_s^{(2)}(t) = x \right).
\]

We summarize the above discussion in the following Theorem.

**Theorem 2:** The HJB equation (6) has a unique solution
\[
V(t, x) = -\frac{\sigma^2}{b^2} \log \left[ \exp \left( -\frac{b^2}{\sigma^2} V_1(t, x) \right) P \left( x_s^{(1)}(T) \leq c \bigg| x_s^{(1)}(t) = x \right) + \exp \left( -\frac{b^2}{\sigma^2} V_2(t, x) \right) P \left( x_s^{(2)}(T) \geq c \bigg| x_s^{(2)}(t) = x \right) \right],
\]

(14)

where \( V_j \), for \( j = 1, 2 \), are the optimal cost-to-go functions of the standard LQG optimal control problems (11), and \( x_s^{(j)} \), for \( j = 1, 2 \), are the corresponding optimal states.
B. General Case

We now solve the general case, i.e. \( q \geq 0 \) and \( a \in \mathbb{R} \). The optimal cost-to-go function of (5) satisfies the following HJB equation:

\[
\frac{\partial V}{\partial t} = ax \frac{\partial V}{\partial x} - \frac{b^2}{2r} \left( \frac{\partial V}{\partial x} \right)^2 + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2} + q(x - \bar{x})
\]

\[V(T, \cdot) = \frac{M^2}{2} \min_{j=1,2} (\cdot - p_j)^2.\]  

(15)

We define the following standard LQG optimal tracking problems:

\[
\inf_{u^{(j)} \in U} J^{(j)}(x^{(j)}(0), u^{(j)}(\cdot), \bar{x})
\]

s.t. \( dx^{(j)}(t) = (ax^{(j)}(t) + bu^{(j)}(t)) dt + \sigma dw^{(j)}(t) \)

for \( j = 1, 2 \), where

\[
J^{(j)}(x^{(j)}(0), u^{(j)}(\cdot), \bar{x}) = \mathbb{E} \left[ \int_0^T \left\{ \frac{q}{2} (x^{(j)} - \bar{x})^2 + \frac{r}{2} (u^{(j)})^2 \right\} dt + \frac{M}{2} (x^{(j)}(T) - p_j)^2 \bigg| x^{(j)}(0) \right].
\]

Problem (16) is the optimal control problem that solves a generic agent when \( p_j \) is the only available alternative. We recall the optimal cost-to-go \( V_j \), optimal control law \( u^{(j)}_* \) and optimal state \( x^{(j)}_* \) of (16) [17]:

\[
V_j(t, x) = \frac{1}{2} \Pi(t)x^2 + \beta_j(t)x + \delta_j(t)
\]

\[
u^{(j)}_*(t, x) = \Pi(t)x + \beta_j(t)
\]

\[
dx^{(j)}_*(t) = (ax^{(j)}_*(t) + bu^{(j)}_*(t, x^{(j)}_*(t))) dt + \sigma dw^{(j)}(t),
\]

where \( \Pi, \beta_j \) and \( \delta_j \) are the unique solutions of

\[
\dot{\Pi}(t) = \frac{b^2}{r} \Pi^2(t) - 2a\Pi(t) - q \quad \Pi(T) = M
\]

\[
\dot{\beta}_j(t) = -\left(a - \frac{b^2}{r} \Pi(t)\right) \beta_j(t) + q\bar{x} \quad \beta_j(T) = -Mp_j
\]

\[
\dot{\delta}_j(t) = \frac{b^2}{2r} \beta_j^2(t) - \frac{\sigma^2}{2} \Pi(t) - \frac{q}{2} \bar{x}^2 \quad \delta_j(T) = \frac{M}{2} p_j^2.
\]

(18)
We define the following conditional probabilities:

\[ g_1(t, x) = P \left( x_1^*(T) \leq c \bigg| x_1^*(t) = x \right) \]

\[ = \frac{1}{\sqrt{2\pi\sigma_t^2}} \int_{-\infty}^{c} \exp \left( -\frac{\left( y - \alpha(T, t)x + \frac{b_x^2}{r} \int_t^T \alpha(T, \tau) \beta_1(\tau) d\tau \right)^2}{2\sigma_t^2} \right) dy \]

\[ g_2(t, x) = P \left( x_2^*(T) \geq c \bigg| x_2^*(t) = x \right) \]

\[ = \frac{1}{\sqrt{2\pi\sigma_t^2}} \int_{c}^{\infty} \exp \left( -\frac{\left( y - \alpha(T, t)x + \frac{b_x^2}{r} \int_t^T \alpha(T, \tau) \beta_2(\tau) d\tau \right)^2}{2\sigma_t^2} \right) dy \]

\[ \alpha(s, t) = \exp \left( \int_t^s \left( a - \frac{b_x^2}{r} \Pi(\tau) \right) d\tau \right) \]

\[ \sigma_t = \sigma \sqrt{\int_t^T \alpha^2(T, \tau) d\tau}. \]

We now state the main result of this paper.

**Theorem 3:** The HJB equation (15) has a unique solution:

\[ V(t, x) = -\frac{\sigma_t^2}{b_x^2} \log \left( \exp \left( -\frac{b_x^2}{\sigma_t^2} V_1(t, x) \right) P \left( x_1^*(T) \leq c \bigg| x_1^*(t) = x \right) \right) + \exp \left( -\frac{b_x^2}{\sigma_t^2} V_2(t, x) \right) P \left( x_2^*(T) \geq c \bigg| x_2^*(t) = x \right), \]

where \( V_j \) and \( x_j^* \), for \( j = 1, 2 \), are defined in (17).

**Proof:** The functions \( \psi^{(j)}(t, x) = \exp \left( -\frac{b_x^2}{\sigma_t^2} V^{(j)}(t, x) \right) \), for \( j = 1, 2 \), satisfy the following linear PDEs:

\[ -\frac{\partial \psi^{(j)}}{\partial t} + ax \frac{\partial \psi^{(j)}}{\partial x} + \frac{\sigma_t^2}{2} \frac{\partial^2 \psi^{(j)}}{\partial x^2} - \frac{qb_x^2}{2\sigma_t^2} (x - \bar{x}) \psi^{(j)} = \left( \begin{array}{c} \frac{\partial g_1}{\partial t} + \left( ax - \frac{b_x^2}{r} \Pi x - \frac{b_x^2}{r} \beta_1 \right) \frac{\partial g_1}{\partial x} + \frac{\sigma_t^2}{2} \frac{\partial^2 g_1}{\partial x^2} \end{array} \right) \psi^{(1)} \]

\[ + \left( \begin{array}{c} \frac{\partial g_2}{\partial t} + \left( ax - \frac{b_x^2}{r} \Pi x - \frac{b_x^2}{r} \beta_2 \right) \frac{\partial g_2}{\partial x} + \frac{\sigma_t^2}{2} \frac{\partial^2 g_2}{\partial x^2} \end{array} \right) \psi^{(2)}. \]

We define \( \psi(t, x) = \psi^{(1)}(t, x) g_1(t, x) + \psi^{(2)}(t, x) g_2(t, x) \).

Noting (21) and the identities \( \frac{\partial \psi^{(j)}}{\partial x} = -\frac{b_x^2}{\sigma_t^2} (\Pi x + \beta_j) \psi^{(j)} \), for \( j = 1, 2 \), one can show that

\[ \frac{\partial \psi}{\partial t} + ax \frac{\partial \psi}{\partial x} + \frac{\sigma_t^2}{2} \frac{\partial^2 \psi}{\partial x^2} - \frac{qb_x^2}{2\sigma_t^2} (x - \bar{x}) \psi = \left( \begin{array}{c} \frac{\partial g_1}{\partial t} + \left( ax - \frac{b_x^2}{r} \Pi x - \frac{b_x^2}{r} \beta_1 \right) \frac{\partial g_1}{\partial x} + \frac{\sigma_t^2}{2} \frac{\partial^2 g_1}{\partial x^2} \end{array} \right) \psi^{(1)} \]

\[ + \left( \begin{array}{c} \frac{\partial g_2}{\partial t} + \left( ax - \frac{b_x^2}{r} \Pi x - \frac{b_x^2}{r} \beta_2 \right) \frac{\partial g_2}{\partial x} + \frac{\sigma_t^2}{2} \frac{\partial^2 g_2}{\partial x^2} \end{array} \right) \psi^{(2)}. \]
The optimal state \( x^j_r \) of (16) satisfies the SDE in (17). Therefore, by Kolmogorov Backward equation [16],
\[
\frac{\partial g_j}{\partial t} + \left( ax - \frac{b^2}{r}\Pi x - \frac{b^2}{r}\beta_j \right) \frac{\partial g_j}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 g_j}{\partial x^2} = 0,
\]
for \( j = 1, 2 \). Hence,
\[
\frac{\partial \psi}{\partial t} + ax \frac{\partial \psi}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 \psi}{\partial x^2} - \frac{qb^2}{2\sigma^2 r} (x - \bar{x}) \psi = 0
\]
(22) and \( V(t, x) \) satisfies (15). The uniqueness of the solution follows from the uniqueness of solutions to the uniform parabolic PDE (22) [15].

Having solved the HJB equation related to the Min-LQG optimal control problem (5), we now prove the existence of a unique optimal control law. We define the following function:
\[
u^*_r(t, x) = \frac{b}{r} \frac{\partial V}{\partial x} = \frac{\sigma^2}{b} \sum_{j=1}^{2} \exp \left( -\frac{b^2}{\sigma^2 r} V_j(t, x) \right) \left( -\frac{b^2}{\sigma^2 r} (\Pi x + \beta_j) g_j(t, x) + \frac{\partial g_j}{\partial x} \right) \sum_{j=1}^{2} \exp \left( -\frac{b^2}{\sigma^2 r} V_j(t, x) \right) g_j(t, x), \quad t \in [0, T)
\]
\[u^*_r(T, x) = 0.\]
(23)

**Theorem 4:** The following statements hold:

i. The function \( u^*_r \) defined in (23) has on \([0, T) \times \mathbb{R}\) the following form:
\[
u^*_r(t, x) = \frac{\exp \left( -\frac{b^2}{\sigma^2 r} V_1(t, x) \right) g_1(t, x) + \exp \left( -\frac{b^2}{\sigma^2 r} V_2(t, x) \right) g_2(t, x)}{\exp \left( -\frac{b^2}{\sigma^2 r} V_1(t, x) \right) g_1(t, x) + \exp \left( -\frac{b^2}{\sigma^2 r} V_2(t, x) \right) g_2(t, x)} u^{(1)}_r(t, x)
\]
\[\quad + \frac{\exp \left( -\frac{b^2}{\sigma^2 r} V_2(t, x) \right) g_2(t, x)}{\exp \left( -\frac{b^2}{\sigma^2 r} V_1(t, x) \right) g_1(t, x) + \exp \left( -\frac{b^2}{\sigma^2 r} V_2(t, x) \right) g_2(t, x)} u^{(2)}_r(t, x), \]
(24)

ii. \( u^*_r \) is an admissible Markov policy.

iii. \( u^*_r(t, x^*_r(t, w)) \) is the unique optimal control law of (5), where \( x^*_r(t, w) \) is the unique strong solution of the SDE in (5) with \( u \) equal to \( u^*_r(t, x) \).

**Proof:** We start by proving the first point. The functions \( \beta_j \) and \( \delta_j \) defined in (18) have the following explicit forms:
\[
\beta_j(t) = -M \alpha(T, t) p_j + q \int_T^t \alpha(\tau, t) \bar{x}(\tau) d\tau
\]
\[
\delta_j(t) = M \frac{b^2}{2r} \int_T^t \alpha^2(T, \tau) d\tau p_j^2 + \frac{qb^2}{2r} \int_T^t \int_T^\eta \alpha(\tau, \eta) \alpha(\sigma, \eta) \bar{x}(\tau) \bar{x}(\sigma) d\tau d\sigma d\eta
\]
\[- M \frac{qb^2}{r} \int_T^t \int_T^\eta \alpha(T, \sigma) \alpha(\tau, \sigma) \bar{x}(\tau) d\tau d\sigma p_j - \frac{\sigma^2}{2} \int_T^t \Pi(\tau) d\tau - \frac{q}{2} \int_T^t \bar{x^2}(\tau) d\tau,
\]
where $\alpha$ is defined in (19). We have
\[
\frac{\partial g_1}{\partial x}(t, x) = -\alpha(T, t) \exp \left( -\frac{(c - \alpha(T, t)x + \frac{\nu^2}{r} \int_t^T \alpha(T, \tau) \beta_1(\tau) d\tau)^2}{2\sigma^2 \int_t^T \alpha^2(T, \tau) d\tau} \right)
\]
\[
\frac{\partial g_2}{\partial x}(t, x) = \alpha(T, t) \exp \left( -\frac{(c - \alpha(T, t)x + \frac{\nu^2}{r} \int_t^T \alpha(T, \tau) \beta_2(\tau) d\tau)^2}{2\sigma^2 \int_t^T \alpha^2(T, \tau) d\tau} \right).
\]
By replacing the expressions of $\beta_j$ and $\delta_j$ in the expressions of $\frac{\partial g_j}{\partial x}$ and $\exp \left( -\frac{b^2}{\sigma^2 r} V_j \right)$, for $j = 1, 2$, one can show that
\[
\exp \left( -\frac{b^2}{\sigma^2 r} V(t, x) \right) \frac{\partial g_1}{\partial x}(t, x) = -\exp \left( -\frac{b^2}{\sigma^2 r} V_2(t, x) \right) \frac{\partial g_2}{\partial x}(t, x).
\]
Therefore, $u_*$ defined in (23) is equal to (24). We now prove the second point. In view of (24), the function $\frac{\partial u_*}{\partial x}$ is continuous on $[0, T) \times \mathbb{R}$. Therefore, the local Lipschitz condition holds.
Moreover, for all $(t, x) \in [0, T] \times \mathbb{R}$, we have
\[
|u_*(t, x)| \leq |u_*^{(1)}(t, x)| + |u_*^{(2)}(t, x)| \leq \frac{b}{r} (2|\Pi|\infty|x| + |\beta_1|\infty + |\beta_2|\infty).
\]
Hence, the linear growth condition is satisfied and $u_*$ is an admissible Markov policy. As a result, sufficient conditions are reunited for the SDE defined in (5) and controlled by $u_*(t, x)$ to have unique strong solution. Finally, we prove the third point. We have for Lebesgue $\times$ $P$-a.e $(t, w) \in [0, T] \times \Omega$,
\[
u_x(t, x_x(t, w)) = -\frac{b}{r} \frac{\partial V}{\partial x}(t, x_x(t, w))
= \arg\min_{u \in \mathbb{R}} \left\{ (ax_x(t, w) + bu) \frac{\partial V}{\partial x}(t, x_x(t, w)) + \frac{q}{2} (x_x(t, w) - \bar{x}(t))^2 + \frac{r}{2} u^2 \right\}.
\]
In view of $|V_j(t, x)| \leq \frac{b}{2} |\Pi|\infty|x|^2 + |\beta_j|\infty|x| + |\delta_j|\infty$, for all $(t, w) \in [0, T] \times \mathbb{R}$, one can show that for all $(t, x) \in [0, T] \times \mathbb{R}$,
\[
|V(t, x)| \leq K (1 + |x|^2),
\]
for some $K > 0$. Moreover, $V \in C^{1,2}([0, T] \times \mathbb{R})$ satisfies the HJB equation (15). Hence, by a verification Theorem, for example [14, Theorem 4.3.1], $u_*(t, x_x(t, w))$ is an optimal control of (5). The uniqueness of the optimal policy follows from the uniqueness of the solution of (15) and the convexity with respect to $u$ of the Hamiltonian $H(x, p, u, t) = (ax + bu)p + \frac{q}{2} (x - \bar{x}(t))^2 + \frac{r}{2} u^2$. 

C. Relation to Static Discrete Choice Models

In this paragraph, we discuss the relation between the Min-LQG optimal control problem and the static discrete choice models. We start by recalling some facts about the static models. In the standard binary discrete choice models, a generic person chooses between two alternatives 1 and 2. The cost that pays this person when choosing an alternative \( j \) is defined as follows:

\[
v_j = k(j) + \epsilon,
\]

where \( k(j) \) is a deterministic function that depends on the personal attributes and alternative \( j \), while \( \epsilon \) is a random variable. When \( \epsilon \) is distributed according to the extreme value distribution, then the probability that a generic person chooses an alternative \( j \) is [3]:

\[
P_j = \frac{\exp(-k(j))}{\exp(-k(1)) + \exp(-k(2))}.
\]

The Min-LQG optimal control law (24) can be written as follows:

\[
u^*(t,x) = \frac{\exp\left(-\frac{b^2}{\sigma^2} V_1(t,x)\right)}{\exp\left(-\frac{b^2}{\sigma^2} V_1(t,x)\right) + \exp\left(-\frac{b^2}{\sigma^2} V_2(t,x)\right)} u_1^{(1)}(t,x) \\
+ \frac{\exp\left(-\frac{b^2}{\sigma^2} V_2(t,x)\right)}{\exp\left(-\frac{b^2}{\sigma^2} V_1(t,x)\right) + \exp\left(-\frac{b^2}{\sigma^2} V_2(t,x)\right)} u_2^{(2)}(t,x),
\]

where

\[
\hat{V}_j(t,x) = V_j(t,x) - \frac{\sigma^2 r}{b^2} \log (g_j(t,x)), \quad j = 1, 2.
\]

\( V_j(t,x) \) is the expected cost that pays a generic agent if \( p_j \) is the only available alternative. In this case, \( u_j^{(j)}(t,x) \) is the optimal policy. Now in the presence of two alternatives, the optimal policy at time \( t \) is given by (25), which can be interpreted as a mixed strategy of two pure strategies \( u_1^{(1)}(t,x) \) (picking alternative \( p_1 \)) and \( u_2^{(2)}(t,x) \) (picking alternative \( p_2 \)). Within this framework, a generic agent at time \( t \) chooses the alternative \( p_j \) with probability

\[
P_{p_j} = \frac{\exp\left(-\frac{b^2}{\sigma^2} \hat{V}_j(t,x)\right)}{\exp\left(-\frac{b^2}{\sigma^2} \hat{V}_j(t,x)\right) + \exp\left(-\frac{b^2}{\sigma^2} \hat{V}_j(t,x)\right)}.
\]

Thus, at each time \( t \in [0, T] \), the dynamic discrete choice problem can be viewed as static discrete choice problem, where the cost of choosing alternative \( p_j \) has an additional term

\[-\frac{\sigma^2 r}{b^2} \log (g_j(t,x)) = -\frac{\sigma^2 r}{b^2} \log P\left(x_j^{(j)}(T) \geq c \big| x_j^{(j)}(t) = x\right).
\]

This additional cost increases with
the risk (probability) of being driven by the Wiener process to the other alternative when applying
the pure strategy $u^{(j)}_*(t, x)$ corresponding to $p_j$.

IV. MEAN FIELD EQUATIONS

In Section III we assume a given macroscopic behavior $\bar{x}$ and compute the generic agent’s
best response to it, which is given by (24). In the following, we write $u_*(t, x, \bar{x})$ instead of
$u_*(t, x)$ to emphasize the dependence on $\bar{x}$. We now seek a sustainable macroscopic behavior
$\bar{x}$, in the sense that it is replicated by the mean of the agents under their best responses to it.
The generic agent’s optimal state satisfies the following SDE:

$$dx_*(t) = (ax_*(t) + bu_*(t, x_*(t), \bar{x})) dt + \sigma dw(t).$$ (27)

We denote by $p(t, x)$ the probability density function of $x_*$. Hence, a sustainable macroscopic
behavior (if it exists) should satisfy the following system of equations:

$$\frac{\partial}{\partial t} p(t, x) = -\frac{\partial}{\partial x} ((ax + bu_*(t, x, \bar{x})) p(t, x)) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} p(t, x),$$

$$\bar{x}(t) = \int_\mathbb{R} yp(t, y) dy,$$

where $u_*(t, x, \bar{x})$ is defined in (24). The first equation is the Fokker-Planck equation correspond-
ing to the SDE (27) and $p(t, x)$ is the probability density function (assumed to exist) of a generic
agent as a function of time, while the second expresses the fact that the population mean $\bar{x}$ is
the mean of $x_*$. We analyze in a forthcoming version of this article the existence of a fixed point
path $\bar{x}$ satisfying (28).

V. CONCLUSION

We study within the framework of the MFG theory a dynamic discrete choice model with
social interactions. We introduce the Min-LQG optimal control problem and give an explicit form
of the generic agent’s best response (Min-LQG optimal control law). The Min-LQG problem
can be interpreted at each time $t$ as a static discrete choice model where the cost of choosing
one of the alternatives has an additional term that increases with the risk of being driven by the
Brownian motion to the other alternative. Finally, we give the Mean Field equations. The fixed
point problem will be analyzed in a next version of this article.
REFERENCES

[1] F. Koppelman and V. Sathi, “Incorporating variance and covariance heterogeneity in the generalized nested logit model: an application to modeling long distance travel choice behavior,” Transportation Research, vol. 39, pp. 825–853, 2005.

[2] C. Bhat and J. Guo, “A mixed spatially correlated logit model: formulation and application to residential choice modeling,” Transportation Research, vol. 38, pp. 147–168, 2004.

[3] D. McFadden, “Conditional logit analysis of qualitative choice behavior,” 1974.

[4] W. Brock and S. Durlauf, “Discrete choice with social interactions,” Review of Economic Studies, pp. 147–168, 2001.

[5] R. Salhab, R. P. Malhamé, and J. Le Ny, “Consensus and disagreement in collective homing problems: A mean field games formulation,” in Proceedings of the 53rd IEEE Conference on Decision and Control, Dec 2014, pp. 916–921.

[6] ——, “A dynamic game model of collective choice in multi-agent systems,” in Proceedings of the 54th IEEE Conference on Decision and Control, dec 2015.

[7] R. Salhab, R. P. Malhamé, and J. Le Ny, “A dynamic game model of collective choice in multi-agent systems,” arXiv preprint arXiv:1506.09210, 2015 (Submitted For Publication).

[8] M. Huang, P. E. Caines, and R. P. Malhamé, “Individual and mass behaviour in large population stochastic wireless power control problems: centralized and Nash equilibrium solutions,” in Proceedings of the 42nd IEEE Conference on Decision and Control, Maui, Hawaii, 2003, pp. 98–103.

[9] ——, “Large-population cost-coupled LQG problems with nonuniform agents: Individual-mass behavior and decentralized epsilon-Nash equilibria,” IEEE Transactions on Automatic Control, vol. 52, no. 9, pp. 1560–1571, 2007.

[10] M. Huang, R. P. Malhamé, and P. E. Caines, “Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle,” Communications in Information & Systems, vol. 6, no. 3, pp. 221–252, 2006.

[11] J. M. Lasry and P. L. Lions, “Jeux à champ moyen. I–le cas stationnaire,” Comptes Rendus Mathématique, vol. 343, no. 9, pp. 619–625, 2006.

[12] ——, “Jeux à champ moyen. II–horizon fini et contrôle optimal,” Comptes Rendus Mathématique, vol. 343, no. 10, pp. 679–684, 2006.

[13] ——, “Mean field games,” Japanese Journal of Mathematics, vol. 2, pp. 229–260, 2007.

[14] W. H. Fleming and H. M. Soner, Controlled Markov processes and viscosity solutions. Springer Science & Business Media, 2006, vol. 25.

[15] L. Evans, Partial Differential Equations, ser. Graduate studies in mathematics. American Mathematical Society, 1998.

[16] I. Karatzas and S. Shreve, Brownian motion and stochastic calculus. Springer Science & Business Media, 2012, vol. 113.

[17] B. D. Anderson and J. B. Moore, Optimal control: linear quadratic methods. Dover Publications, 2007.