Abstract

In this article we associate to every lattice ideal $I_{L,\rho} \subset K[x_1, \ldots, x_m]$ a cone $\sigma$ and a graph $G_\sigma$ with vertices the minimal generators of the Stanley-Reisner ideal of $\sigma$. To every polynomial $F$ we assign a subgraph $G_\sigma(F)$ of the graph $G_\sigma$. Every expression of the radical of $I_{L,\rho}$ as a radical of an ideal generated by some polynomials $F_1, \ldots, F_s$ gives a spanning subgraph of $G_\sigma$, the $\bigcup_{i=1}^s G_\sigma(F_i)$. This result provides a lower bound for the minimal number of generators of $I_{L,\rho}$ and therefore improves the generalized Krull’s principal ideal theorem for lattice ideals. But mainly it provides lower bounds for the binomial arithmetical rank and the $A$-homogeneous arithmetical rank of a lattice ideal. Finally we show, by a family of examples, that the bounds given are sharp.

1 Introduction

Lattice ideals arise naturally in problems from diverse areas of mathematics, including toric geometry, integer programming, dynamical systems, computer algebra, graph theory, hypergeometric differential equations, mirror symmetry and computational statistics, see [6], [13], [15], [18]. A fundamental problem in the theory of lattice ideals is to determine minimal generators of the lattice ideal $I_L$ or of the lattice ideal $I_L$ up to radical. The main theorem of this article provides a lower bound for the minimal number of generators of a lattice ideal, but also it provides lower bounds for the binomial arithmetical rank and the $A$-homogeneous arithmetical rank of a lattice ideal. The lower bounds depend only on the geometry of the cone associated to the lattice ideal.

Let $K$ be an algebraically closed field of characteristic $p \geq 0$. A lattice is a finitely generated free abelian group. A partial character $(L, \rho)$ on $\mathbb{Z}^n$ is a homomorphism $\rho$ from a sublattice $L$ of $\mathbb{Z}^n$ to the multiplicative group $K^* = K - \{0\}$. Given a partial character $(L, \rho)$ on $\mathbb{Z}^n$, we define the ideal

$$I_{L,\rho} := (\{x^\alpha + \rho(\alpha)x^\alpha | \alpha = \alpha_+ - \alpha_- \in L\}) \subset K[x_1, \ldots, x_m]$$

called lattice ideal. Where $\alpha_+, \alpha_- \in \mathbb{N}^n$ and $\alpha_- \in \mathbb{N}^n$ denote the positive and negative part of $\alpha$, respectively, and $x^\beta = x_1^{b_1} \cdots x_m^{b_m}$ for $\beta = (b_1, \ldots, b_m) \in \mathbb{N}^n$. Lattice ideals are binomial ideals. The theory of binomial ideals were developed by Eisenbud and Sturmfels in [6]. If $L$ is a sublattice of $\mathbb{Z}^n$, then the saturation of $L$ is the lattice

$$\text{Sat}(L) := \{\alpha \in \mathbb{Z}^n | d\alpha \in L \text{ for some } d \in \mathbb{Z}^*\}.$$
We say that the lattice $L$ is saturated if $L = \text{Sat}(L)$. The lattice ideal $I_{L,\rho}$ is prime if and only if $L$ is saturated. A prime lattice ideal is called a toric ideal, while the set of zeroes in $K^n$ is an affine toric variety in the sense of [15], since we do not require normality.

Throughout this paper we assume that $L$ is a non-zero positive sublattice of $\mathbb{Z}^n$, that is $L \cap \mathbb{N}^n = \{0\}$. This means that the lattice ideal $I_{L,\rho}$ is homogeneous with respect to some positive grading.

The group $\mathbb{Z}^n/\text{Sat}(L)$ is free abelian, therefore is isomorphic to $\mathbb{Z}^n$, where $n = m - \text{rank}(L)$. Let $\psi$ be the above isomorphism, $e_1, \ldots, e_n$ the unit vectors of $\mathbb{Z}^n$ and $\psi(e_i + \text{Sat}(L)) = a_i \in \mathbb{Z}^n$ for $1 \leq i \leq m$.

Let $A = \{a_i | 1 \leq i \leq m\}$, we associate to the lattice ideal $I_{L,\rho}$ the rational polyhedral cone

$$\sigma = \text{pos}_\mathbb{Q}(A) := \{l_1 a_1 + \cdots + l_m a_m | l_i \in \mathbb{Q} \text{ and } l_i \geq 0\}.$$ 

A cone $\sigma$ is strongly convex if $\sigma \cap -\sigma = \{0\}$. The condition that the lattice $L$ is positive, is equivalent with the condition that the cone $\sigma$ is strongly convex.

We grade $K[x_1, \ldots, x_m]$ by setting $\text{deg}_A(x_i) = a_i$ for $i = 1, \ldots, m$. We define the $A$-degree of the monomial $x^u$ to be

$$\text{deg}_A(x^u) := u_1 a_1 + \cdots + u_m a_m \in \mathbb{N}A,$$

where $\mathbb{N}A$ is the semigroup generated by $A$. The lattice ideal $I_{L,\rho}$ is $A$-homogeneous as well as all lattice ideals with the same saturation. The binomial arithmetical rank of a binomial ideal $I$ (written $\text{bar}(I)$) is the smallest integer $s$ for which there exist binomials $f_1, \ldots, f_s$ in $I$ such that $\text{rad}(I) = \text{rad}(f_1, \ldots, f_s)$. Hence the binomial arithmetical rank is an upper bound for the arithmetical rank of a binomial ideal (written $\text{ara}(I)$), which is the smallest integer $s$ for which there exists $f_1, \ldots, f_s$ in $I$ such that $\text{rad}(I) = \text{rad}(f_1, \ldots, f_s)$. Especially, when $I$ is $A$-homogeneous and all the polynomials $f_1, \ldots, f_s$ are $A$-homogeneous, the smallest integer $s$ is called $A$-homogeneous arithmetical rank of $I$, denoted by $\text{ara}_A(I)$.

From the definitions, the generalized Krull’s principal ideal theorem and the graded version of Nakayama’s Lemma we deduce the following inequality for a lattice ideal $I_{L,\rho}$:

$$h(I_{L,\rho}) \leq \text{ara}(I_{L,\rho}) \leq \text{ara}_A(I_{L,\rho}) \leq \text{bar}(I_{L,\rho}) \leq \mu(I_{L,\rho}).$$

Here $h(I)$ denotes the height and $\mu(I)$ denotes the minimal number of generators of an ideal $I$. When $h(I) = \text{ara}(I)$ the ideal $I$ is called a set-theoretic complete intersection and when $h(I) = \mu(I)$ it is called a complete intersection. In several cases the lower bound $h(I_{L,\rho})$ given by the generalized Krull’s principal ideal theorem can be improved by using local cohomological methods, see [3], [9].

The computation of the numbers $\text{ara}(I_{L,\rho}), \text{ara}_A(I_{L,\rho}), \text{bar}(I_{L,\rho})$ is usually an extremely difficult problem and remains open even for some very simple lattice ideals, like the ideal of the Macaulay curve $(t^4, t^3u, tu^3, u^4)$ in the three dimensional projective space, see [4]. In the case that we can compute good generating sets for the ideal, sharp lower bounds for these numbers may help us to determine the exact value of them, see section 5. The numbers $\text{ara}(I_{L,\rho}), \text{bar}(I_{L,\rho})$ and $\text{ara}_A(I_{L,\rho})$, in the cases that were known up to this work, were either identical or very close to each other, see for example [1], [2], [7], [12], [17]. Also, there was no known example of a lattice ideal $I_{L,\rho}$ with the property $\text{ara}(I_{L,\rho}) \neq \text{ara}_A(I_{L,\rho})$. In this work, by providing good lower bounds for $\text{ara}_A(I_{L,\rho})$ and $\text{bar}(I_{L,\rho})$ and using the result
of Eisenbud, Evans and Storch, see [5] and [14], that \( ara(I_{L,\rho}) \) is bounded above by the dimension \( m \) of the space \( K^m \), we show that there can be very large differences between these numbers. For example, using the results of section 5 and putting \( n = 10 \) we have an example of a lattice ideal for which the height is equal to 80, the \( ara(I_{L,\rho}) \) is smaller than 90 by Eisenbud, Evans and Storch, while the \( ara_A(I_{L,\rho}) \) is exactly 1740 and \( bar(I_{L,\rho}) \) is exactly 1860.

In section 2 we recall some basic facts about lattice ideals, which are necessary for the formulation and proof of the main Theorem 4.1.

In section 3 we introduce a graph \( G_\sigma \) with vertices the minimal generators of the Stanley-Reisner ideal of the cone \( \sigma \) associated to the lattice ideal.

In section 4 we state and prove the main theorem of the article, Theorem 4.1, which provides lower bounds for the \( A \)-homogeneous arithmetical rank, the binomial arithmetical rank and the minimal number of generators of a lattice ideal.

In section 5 we compute these bounds for a special class of lattice ideals. In this case, we show that the lower bounds given by Theorem 4.5 cannot be improved, by computing the exact value of the \( A \)-homogeneous arithmetical rank and the binomial arithmetical rank for certain lattice ideals.

2 Basics on Lattice ideals

Let \( L \) be a nonzero positive sublattice of \( \mathbb{Z}^m \) and \( (L, \rho) \) be a partial character on \( \mathbb{Z}^m \).

Definition 2.1 If \( p \) is a prime number, we define \( \text{Sat}_p(L) \) and \( \text{Sat}'_p(L) \) to be the largest sublattices of \( \text{Sat}(L) \) containing \( L \) such that \( \text{Sat}_p(L)/L \) has order a power of \( p \) and \( \text{Sat}'_p(L)/L \) has order relatively prime to \( p \). If \( p = 0 \), we define \( \text{Sat}_p(L) = L \) and \( \text{Sat}'_p(L) = \text{Sat}(L) \).

Theorem 2.2 [6] Let \( (L, \rho) \) be a partial character on \( \mathbb{Z}^m \). Write \( g \) for the order of \( \text{Sat}'_p(L)/L \). There are \( g \) distinct characters \( \rho_1, \ldots, \rho_g \) of \( \text{Sat}'_p(L) \) extending \( \rho \) and for each \( j \) a unique character \( \rho'_j \) of \( \text{Sat}(L) \) extending \( \rho_j \). There is a unique partial character \( \rho' \) of \( \text{Sat}_p(L) \) extending \( \rho \). The radical, associated primes and minimal primary decomposition of \( I_{L,\rho} \) are:

\[
\text{rad}(I_{L,\rho}) = I_{\text{Sat}_p(L),\rho'},
\]
\[
\text{Ass}(K[x_1, \ldots, x_m]/I_{L,\rho}) = \{I_{\text{Sat}(L),\rho'_j} | j = 1, \ldots, g\}
\]

and

\[
I_{L,\rho} = \bigcap_{j=1}^g I_{\text{Sat}'_p(L),\rho'_j}
\]

where \( I_{\text{Sat}'_p(L),\rho'_j} \) is \( I_{\text{Sat}(L),\rho'_j} \)-primary. In particular, if \( p = 0 \), then \( I_{L,\rho} \) is a radical ideal. The associated primes \( I_{\text{Sat}(L),\rho'_j} \) of \( I_{L,\rho} \) are all minimal and have the same codimension \( \text{rank}(L) \).
We decompose the affine space $K^m$ into $2^m$ coordinate cells,

$$(K^*)^E := \{(q_1, \ldots, q_m) \in K^m | q_i \neq 0 \text{ for } i \in E, q_i = 0 \text{ for } i \notin E\},$$

where $E$ runs over all subsets of $\{1, \ldots, m\}$. We denote by $K[E] := K[\{x_i | i \in E\}]$. Let $P = (x_1, \ldots, x_m)$ be a point of $K^m$ then

$$P_E := (\delta_1^E x_1, \delta_2^E x_2, \ldots, \delta_m^E x_m) \in K^m,$$

where $\delta_i^E = 1$ if $i \in E$ and $\delta_i^E = 0$ if $i \notin E$. Note that if $P \in (K^*)^{\{1,\ldots, m\}}$ then $P_E \in (K^*)^E$.

A face $F$ of $\sigma$ is any set of the form

$$F = \sigma \cap \{x \in \mathbb{R}^n : cx = 0\},$$

where $c \in \mathbb{R}^n$ and $cx \geq 0$ for all points $x \in \sigma$. Faces of dimension one are called extreme rays. If the number of the extreme rays of a cone coincides with the dimension (i.e. the extreme rays are linearly independent), then the cone is called simplex cone.

Let $S$ be a subset of the cone $\sigma$, then $E_S := \{i \in \{1, \ldots, m\} | a_i \in S\}$. To simplify the notation we denote the point $P_{E_S}$ by $P_S$ and the cell $(K^*)^{E_S}$ by $(K^*)^S$. The $n$-dimensional algebraic torus $(K^*)^n$ acts on the affine $m$-space $K^m$ via

$$(x_1, \ldots, x_m) \to (x_1 t^{a_1}, \ldots, x_m t^{a_m}).$$

Let $j \in \{1, \ldots, g\}$. The affine toric variety $X_{A,j} := V(I_{\text{Sat}(L), \rho'})$ is the Zariski-closure of the $(K^*)^n$-orbit of a point $P_{j} = (c_{j_1}, c_{j_2}, \ldots, c_{j_m})$, where all $c_{j_i}$ are different from zero. Note that the ideal $I_{\text{Sat}(L), \rho'}$ is the kernel of the $K$-algebra homomorphism

$$\phi_j : K[x_1, \ldots, x_m] \to K[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}]$$

given by

$$\phi_j(x_i) = c_{j_i} t^{a_i} \quad \text{for all } i = 1, \ldots, m.$$

The $(K^*)^n$-orbits on the affine toric variety $X_{A,j}$ are in order-preserving bijection with the faces of the cone $\sigma$, see [8], [10], [11], for every $j$. Note that our cone $\sigma$ is the dual of the cone that is used to define the toric variety in the above references.

Actually the orbit corresponding to the face $F$ is the orbit of the point $(P_{j})_F$ and the toric variety is the disjoint union of the orbits of the points $(P_{j})_F$, for all the faces $F \in \sigma$, i.e.

$$X_{A,j} = \cup_{F \in \sigma} O((P_{j})_F).$$

Each orbit $O((P_{j})_F)$ corresponds to the relative interior of the face $F$. The orbit $O((P_{j})_F)$ is in the cell $(K^*)^F$ and there are no points of the toric varieties $X_{A,j}$ that are in the cells $(K^*)^E$, where $E$ is not in the form $E_F$ for a face $F$ of $\sigma$.

From the theorem 2.2 we have $V(I_{L, \rho}) = \cup_{j=1}^g X_{A,j}$. Therefore $V(I_{L, \rho})$ has points only on the cells in the form $(K^*)^F$ for some face $F$ of the cone $\sigma$. 

4
3 Stanley-Reisner rings

Given a set \( Y \subset Z^n \) the set of all nonnegative linear combinations \( x = l_1y_1 + \cdots + l_sy_s \), where \( y_1, \ldots, y_s \in Y \), \( l_1, \ldots, l_s \in \mathbb{Q} \) is called the positive hull of \( Y \), \( \text{pos}_\mathbb{Q}(Y) \).

Let \( \sigma \subset \mathbb{Q}^n \) be a strongly convex rational polyhedral cone and let \( R_\sigma = \{ r_1, \ldots, r_t \} \) a set of integer vectors, one for each extreme ray of \( \sigma \), therefore \( \sigma = \text{pos}_\mathbb{Q}(r_1, \ldots, r_t) \). The vectors \( r_i \) are called extreme vectors of \( \sigma \). We consider the polynomial ring \( K[Y_1, \ldots, Y_t] \) by taking one variable \( Y_i \) for each vector \( r_i \). Let \( M = Y_{i_1}^{n_1} \cdots Y_{i_l}^{n_l} \) be a monomial, we shall denote by \( \text{pos}_\mathbb{Q}(M) \) the positive hull of the vectors \( r_{i_1}, \ldots, r_{i_l} \).

The boundary of \( M \) is defined to be
\[
\partial(M) = \partial(r_{i_1}, \ldots, r_{i_l}) := \text{pos}_\mathbb{Q}(r_{i_1}, \ldots, r_{i_l}) - \text{relint}(r_{i_1}, \ldots, r_{i_l}),
\]
which is the union of all proper faces of the cone \( \text{pos}_\mathbb{Q}(r_{i_1}, \ldots, r_{i_l}) \).

By \( F(M) \) we denote the minimal face of \( \sigma \) that contains \( \{ r_{i_1}, \ldots, r_{i_l} \} \), i.e.
\[
F(M) = \cap_{\{ r_{i_1}, \ldots, r_{i_l} \} \subset F},
\]
since any intersection of faces of \( \sigma \) is a face of \( \sigma \).

The Stanley-Reisner ring of \( \sigma \) is the \( K \)-algebra
\[
K[\sigma] = K[Y_1, \ldots, Y_t]/I_\sigma,
\]
where \( I_\sigma \) is the Stanley-Reisner ideal generated by all square-free monomials \( M = Y_{i_1}Y_{i_2} \cdots Y_{i_l} \) such that \( \text{pos}_\mathbb{Q}(M) \) is not a face of \( \sigma \).

The ideal \( I_\sigma \) is a monomial ideal, so there is a unique set \( \{ M_1, \ldots, M_q \} \) of minimal square-free monomial generators of \( I_\sigma \).

**Definition 3.1** We associate to the cone \( \sigma \) a graph \( G_\sigma \) with vertices the set \( \{ M_1, \ldots, M_q \} \) of minimal monomial generators of \( I_\sigma \). There is an edge between the vertices \( M_i \) and \( M_j \) if \( \text{relint}_\mathbb{Q}(M_i) \cap \text{relint}_\mathbb{Q}(M_j) \neq \{0\} \).

**Remark 3.2** \( G_\sigma = \emptyset \) if and only if \( \sigma \) is simplex cone.

The next Theorem gives an equivalent condition for a square-free monomial to be minimal generator of \( I_\sigma \).

**Theorem 3.3** The monomial \( M \) is a minimal generator of \( I_\sigma \) iff

i) for every proper divisor \( N \) of \( M \), \( \text{pos}_\mathbb{Q}(N) \) is a face of \( \sigma \)

ii) the positive hull \( \text{pos}_\mathbb{Q}(M) \) is a proper subset of \( F(M) \)

iii) \( \text{pos}_\mathbb{Q}(M) \) is a simplex cone and every proper face of \( \text{pos}_\mathbb{Q}(M) \) is a face of \( \sigma \).
Proof. Suppose that $M = Y_{i_1}Y_{i_2} \cdots Y_{i_t}$ is a minimal generator of $I_\sigma$.

i) Assuming that $pos_\mathcal{Q}(N)$ is not a face of $\sigma$ we have $N \not\in I_\sigma$ from the definition of Stanley-Reisner ideal. But this contradicts the fact that $M$ is a minimal generator of $I_\sigma$.

ii) The positive hull of $M$ is not a face of $\sigma$, while $F(M)$ is a face of $\sigma$. Thus $pos_\mathcal{Q}(M) \neq F(M)$ and certainly $pos_\mathcal{Q}(M) \subset F(M)$.

iii) Assume that $r_{i_1}, r_{i_2}, \ldots, r_{i_t}$ are not linearly independent and consider a linear relation $d_{i_1}r_{i_1} + d_{i_2}r_{i_2} + \cdots + d_{i_t}r_{i_t} = 0$ between them, with at least one $d_{i_j} \neq 0$. Then, since $\sigma$ is strongly convex, there will be positive and negative coefficients $d_{i_j}$ in the previous relation. Let $P$ be the subset of $\{i_1, \ldots, i_t\}$ consisting of all indices $i_j$, such that the corresponding $d_{i_j}$ is positive. Then $P$ is not empty and proper. Therefore $N = \Pi_{i \in P} Y_i$ is a proper divisor of $M$ which means that $pos_\mathcal{Q}(N)$ is a face $F$ of $\sigma$. Let $c_F$ be a vector defining the face $F$.

Considering the dot product of $c_F$ and $d_{i_1}r_{i_1} + d_{i_2}r_{i_2} + \cdots + d_{i_t}r_{i_t}$ we have a contradiction, namely a negative number equal to zero. Therefore $pos_\mathcal{Q}(M)$ is a simplex cone.

Let $F$ be a proper face of $pos_\mathcal{Q}(M) = pos_\mathcal{Q}(r_{i_1}, \ldots, r_{i_t})$. Then $F = pos_\mathcal{Q}(r_{j_1}, \ldots, r_{j_k})$, where $\{r_{j_1}, \ldots, r_{j_k}\}$ is a proper subset of $\{r_{i_1}, \ldots, r_{i_t}\}$. Then $N = Y_{j_1} \cdots Y_{j_k}$ is a proper divisor of $M$, therefore $F$ is a face of $\sigma$.

Suppose that i), ii) and iii) are true. Then ii) gives us that $M$ is a generator of the Stanley-Reisner ideal, while i) ensure that $M$ is minimal.

The following lemma will be useful in the proof of the theorem 4.1.

Lemma 3.4 The monomial $x^u \in K[E_F]$ iff $deg_A(x^u) \in F$.

Proof. Obviously, $x^u$ belongs to $K[E_F]$ implies that $deg_A(x^u)$ is in $F$. Suppose that

$$deg_A(x^u) = u_1a_1 + \cdots + u_ma_m \in F.$$ 

Then

$$0 = c_F(\sum_{i=1}^m u_ia_i) = \sum_{i=1}^m u_ic_Fa_i,$$

where $c_F$ is any vector that defines the face $F$. All the terms $c_Fa_i$ are non-negative and every $u_i \geq 0$, therefore we have that $u_i = 0$ whenever $c_Fa_i$ is positive. Thus $x^u \in K[E_F]$.

4 Radical of a Lattice ideal

We consider a lattice ideal $I_{L,\rho} \subset K[x_1, \ldots, x_m]$ and the strongly convex rational polyhedral cone $\sigma = pos_\mathcal{Q}(A) \subset \mathbb{Q}^d$ corresponding to $I_{L,\rho}$. Let $I_\sigma \subset K[Y_1, \ldots, Y_t]$ be the Stanley-Reisner ideal of the cone $\sigma$, where $t$ is the number of extreme rays of the cone $\sigma$. Let $N = x_{i_1}^{n_1} \cdots x_{i_s}^{n_s}$ be a monomial in $K[x_1, \ldots, x_m]$. Set $A_N := \{a_{i_1}, \ldots, a_{i_s}\}$, we define the cone of $N$ to be

$$cone(N) := \cap_{A_N \subset pos_\mathcal{Q}(r_{j_1}, \ldots, r_{j_l})}pos_\mathcal{Q}(r_{j_1}, \ldots, r_{j_l}) \subset \sigma.$$
Note that \( \text{pos}_Q(A_N) \subseteq \text{cone}(N) \). Also, the \( \text{cone}(N) \) is not necessarily in the form

\[
\text{pos}_Q(r_{j_1}, \ldots, r_{j_i})
\]

for some extreme vectors \( r_{j_1}, \ldots, r_{j_i} \) of \( \sigma \). But in the case that every one of \( a_{i_1}, \ldots, a_{i_s} \) belongs to some extreme ray of \( \sigma \), we have that \( \text{cone}(N) = \text{pos}_Q(a_{i_1}, \ldots, a_{i_s}) \).

Let \( F \) be a polynomial in \( K[x_1, \ldots, x_m] \), we associate to \( F \) the induced subgraph \( G_\sigma(F) \) of \( G_\sigma \) with vertices those \( M_i \) with the property that there exist a monomial \( N \) in \( F \) such that \( \text{cone}(N) = \text{pos}_Q(M_i) \). The induced subgraph of a graph \( G \) by certain vertices \( V \) is the subgraph of \( G \) with these vertices and edges those edges of \( G \) that have both vertices in \( V \). A subgraph \( H \) of a graph \( G \) is called a spanning subgraph if \( V(H) = V(G) \), where \( V(G) \) denotes the set of vertices of a graph \( G \).

**Theorem 4.1** Every expression of \( \text{rad}(I_{L,\rho}) = \text{rad}(F_1, \ldots, F_s) \) gives a spanning subgraph of \( G_\sigma \), the \( \cup_{i=1}^g G_\sigma(F_i) \).

**Proof.** Suppose that \( \text{rad}(I_{L,\rho}) = \text{rad}(F_1, \ldots, F_s) \) and let \( M = Y_{i_1} \cdots Y_{i_k} \neq 0 \) be a minimal generator of the Stanley-Reisner ideal of \( \sigma \). We will prove that there exists a monomial \( N = x_{i_1} \cdots x_{i_k} \) in some \( F_i \) such that \( \text{cone}(N) = \text{pos}_Q(M) \).

Let us consider the point \( (P_j)_{\partial(M)} \), for any \( j \in \{1, \ldots, g\} \). We divide the proof into three steps:

1. We claim that \( (P_j)_{\partial(M)} \) is not a point of \( V(I_{L,\rho}) \). Recall that \( (P_j)_{\partial(M)} \) belongs to the cell \( (K^*)^{\partial(M)} \). But since every point of \( V(I_{L,\rho}) \) belongs to a cell \( (K^*)^F \) for some face \( F \) of \( \sigma \), it is enough to prove that \( E_{\partial(M)} \) is not in the form \( E_F \) for a face \( F \) of \( \sigma \).

Note that \( M \neq 0 \) is a minimal generator of the Stanley-Reisner ideal of the cone \( \sigma \) and therefore \( \text{dim}(\text{pos}_Q(M)) \geq 2 \). Also \( \sigma = \text{pos}_Q(A) \) which implies that for every extreme vector \( r_k \) of \( \sigma \) there exist \( i_k \in \{1, \ldots, m\} \) such that \( a_{i_k} = \lambda r_k \), for some \( \lambda \in Q \). Then \( r_k \in \text{pos}_Q(M) \) iff \( r_k \in \partial(M) \) iff \( a_{i_k} \in \partial(M) \) iff \( i_k \in E_{\partial(M)} \). Also \( i_k \in E_F \) iff \( a_{i_k} \in F \) iff \( r_k \in F \). Therefore \( \text{pos}_Q(M) = F \), since every face of \( \sigma \) is generated by extreme vectors. But this contradicts the fact that \( M \) is a generator of the Stanley-Reisner ideal and the claim is proved.

Therefore \( (P_j)_{\partial(M)} \) cannot be a zero of all the \( F_i \). Thus there exists at least one \( i \) such that \( F_i((P_j)_{\partial(M)}) \neq 0 \). Let \( N \) be a monomial in \( F_i \) such that \( N((P_j)_{\partial(M)}) \neq 0 \).

We have, from the definition of \( (P_j)_{\partial(M)} \) and the fact \( N((P_j)_{\partial(M)}) \neq 0 \), that \( A_N \subset \partial(M) \subset \text{pos}_Q(M) \). Therefore \( \text{cone}(N) \subset \text{pos}_Q(M) \).

The last condition implies that \( \text{deg}_A(N) \in \text{pos}_Q(M) \), that means either \( \text{deg}_A(N) \in \text{relint}(\text{pos}_Q(M)) \) or \( \text{deg}_A(N) \in \partial(M) \).

2. We claim that always we can find a monomial \( N \) in \( F_i \) such that \( \text{deg}_A(N) \in \text{relint}(\text{pos}_Q(M)) \).

Suppose that \( \text{deg}_A(N) \in \partial(M) \). But \( M \) is a minimal generator of the Stanley-Reisner ideal and therefore, from theorem 3.3, we have that \( \text{deg}_A(N) \) belongs to a face \( F \) of the cone \( \sigma \).
such that $F \subset \partial(M)$.

The polynomial $F_i$ belongs to the lattice ideal $I_{L,\rho}$, which is $A$-homogeneous and therefore has a decomposition $F_i = F_{i1} + \cdots + F_{is}$ into $A$-homogeneous components. By lemma 3.4, $\deg A(N)$ belongs to a face $F$ implies that the $A$-component, $F_{ij}$, of $N$ belongs to $K[E_F]$, since all monomials in $F_{ij}$ have the same $A$-degree. Note that $F \subset \partial(M)$ therefore $(P_j)_{\partial(M)}F = (P_j)_F$. Thus, since $F_{ij}$ involves variables belonging only to the face $F$, we have $F_{ij}(P_j)_{\partial(M)} = F_{ij}(P_j)F = F_{ij}(P_j)$ which is zero because $P_j \in V(I_{L,\rho})$.

But then $F_{ij}(P_j)_{\partial(M)} = 0$ and $F_i(P_j)_{\partial(M)} \neq 0$, so there exist a different monomial $N'$ in a different $A$-homogeneous component of $F_i$ such that $N'(P_j)_{\partial(M)} \neq 0$. This cannot be repeated indefinitely, since $F_i$ has finitely many $A$-homogeneous components. So we conclude that there must be an $N$ in $F_i$ such that $\deg A(N) \in \relint(\pos M)$ and $N((P_j)_{\partial(M)}) \neq 0$.

3rd step. For a set $S \subset \sigma$ we define $R_S$ to be the set of extreme vectors of $\sigma$ that belong to $S$. We will show that a monomial $N$ with the property $\deg A(N) \in \relint(\pos M)$ and $N((P_j)_{\partial(M)}) \neq 0$ satisfies $\cone(N) = \pos M$. Let $a_i \in A_N$, then $N((P_j)_{\partial(M)}) \neq 0$ implies that $a_i \in \partial(M)$. By theorem 3.3 we conclude that $a_i \in F$ for some face of $\sigma$. Therefore $F(a_i) \subset F \subset \partial(M)$, where $F(a_i)$ denotes the smallest face that contains $a_i$. We have that

$$R_F(a_i) \subset R_F \subset R_{\partial(M)} = R_{\pos M}.$$ 

Now we claim that if $a_i \in \pos M$, for some $R \subset \sigma$, then $R_{F(a_i)} \subset R$. Let $a_i = \sum_{r_i \in R} l_i r_i$, with $l_i \geq 0$. Multiplying by $c_{F(a_i)}$ a vector that defines the face $F(a_i)$, we have that $l_j = 0$ whenever $r_j \notin F(a_i)$. So in fact $a_i = \sum_{r_i \in R_{F(a_i)}} l_i r_i$. Note also that $\pos M(R_{F(a_i)} \cap R)$ is a face of $\sigma$ by theorem 3.3, since $R_{F(a_i)} \cap R$ is a proper subset of $R_{\pos M}$. We conclude that

$$F(a_i) \subset \pos M(R_{F(a_i)} \cap R) \subset \pos M(R_{F(a_i)}) = F(a_i).$$

Therefore $R_{F(a_i)} \cap R = R_{F(a_i)}$ which implies the claim $R_{F(a_i)} \subset R$.

To prove that $\cone(N) = \pos M$ is enough to prove that $\cup_{a_i \in A_N} R_{F(a_i)} = R_{\pos M}$. We have just proved that $\cup R_{F(a_i)} \subset R_{\pos M}$. If they are not equal then $\cup R_{F(a_i)} \subset F$, for some face $F$ of $\sigma$, since $M$ is a minimal generator of the Stanley-Reisner ideal. But if $\cup R_{F(a_i)} \subset F$ then $\deg A(N) \subset F$. Which is a contradiction, since $\deg A(N) \in \relint(\pos M)$. Therefore we have proved that for every minimal generator $M$ of the Stanley-Reisner ideal of $\sigma$ there exists at least one monomial $N$ in some $F_i$ such that $\cone(N) = \pos M$ and even more, $\deg A(N) \in \relint(\pos M)$ and $A_N \subset \partial(M)$.

**Theorem 4.2** Let $F \in K[x_1, \ldots, x_m]$ be an $A$-homogeneous polynomial, then the graph $G_\sigma(F)$ is complete.

**Proof.** Suppose that $G_\sigma(F)$ is not empty and that $M_i$, $M_j$ are two vertices of $G_\sigma(F)$. Let $N_i$ and $N_j$ be the corresponding monomials in $F$ with $\deg A(N_i) \in \relint(\pos M_i)$ and $\deg A(N_j) \in \relint(\pos M_j)$, see the proof of Theorem 4.1. Using the fact that $F$ is $A$-homogeneous we get $\deg A(N_i) = \deg A(N_j)$. Thus $\relint(\pos M_i) \cap \relint(\pos M_j) \neq \{0\}$ and
therefore, from the definition of $G_\sigma$, there is an edge between them. It follows that the subgraph $G_\sigma(F)$ is complete, since for any two vertices $M_i$ and $M_j$ of $G_\sigma(F)$ there is an edge between them.

Combining Theorems 4.1 and 4.2 we have the following corollary:

**Corollary 4.3** Every expression of $\text{rad}(I_{L,\rho}) = \text{rad}(F_1, \ldots, F_s)$, where each $F_i$ is $A$-homogeneous polynomial, gives a subgraph of $G_\sigma$ which is spanning and is a union of complete subgraphs.

Note that binomials belonging to $I_{L,\rho}$ are always $A$-homogeneous and therefore we have the following corollary:

**Corollary 4.4** Every expression of $\text{rad}(I_{L,\rho}) = \text{rad}(B_1, \ldots, B_s)$, where each $B_i$ is binomial, gives a subgraph of $G_\sigma$ which is spanning and each binomial contributes two vertices and an edge joining them or just one vertex or nothing.

For a graph $G$ we denote by $c_G$ the smallest number $s$ of complete subgraphs $G_i$ of $G$, such that the subgraph $\bigcup_{i=1}^s G_i$ of $G$ is spanning. While by $b_G$ we denote the smallest number $s$ of subgraphs $B_i$ of $G$, consisting of two vertices and an edge or just a vertex, such that the subgraph $\bigcup_{i=1}^s B_i$ is spanning. Then Corollaries 4.3 and 4.4 imply that:

**Theorem 4.5** For a lattice ideal $I_{L,\rho}$ with associated cone $\sigma$ we have $c_{G_\sigma} \leq \text{ara}_A(I_{L,\rho})$ and $b_{G_\sigma} \leq \text{bar}(I_{L,\rho})$.

Note that $b_{G_\sigma} \geq q/2$, where we recall that $q$ is the minimal number of generators of $I_\sigma$, and $c_{G_\sigma}$ is greater than or equal to the number of connected components of the graph $G_\sigma$.

Also note that the above bounds depend only on the graph $G_\sigma$, i.e. lattice ideals with associated cones rationally affine equivalent have exactly the same bound. Two cones are called *rationally affine equivalent* if there is a rational affine transformation mapping the first cone to the second bijectively.

**Corollary 4.6** Every expression of $I_{L,\rho} = (B_1, \ldots, B_s)$, where each $B_i$ is binomial, gives a subgraph of $G_\sigma$ which is spanning and each binomial contributes two vertices and an edge joining them or just one vertex or nothing. In particular $\max\{b_{G_\sigma}, h(I_{L,\rho})\} \leq \mu(I_{L,\rho})$.

The above Corollary gives a lower bound for the minimal number of generators of $I_{L,\rho}$ which improves the generalized Krull’s principal ideal theorem, see also remark 5.6.
The lower bounds are sharp

The aim of this last section is to explicitly compute the bounds for the $A$-homogeneous arithmetical rank and the binomial arithmetical rank, obtained from Theorem 4.5, for a special class of lattice ideals and show that the lower bounds given are sharp. This will be done by computing the exact values of the above numbers and proving that they are identical with the corresponding bounds, for a certain class of lattice ideals.

We consider the set of vectors $A_n = \{2e_i + e_j : 1 \leq i, j \leq n, i \neq j\}$, where $n \geq 2$ and $e_i, 1 \leq i \leq n$, is the canonical basis of $K^n$. The toric ideal $I_{A_n}$ of $A_n$, see [15], is the kernel of the $K$-algebra homomorphism $\phi : K[[x_{ij} | 1 \leq i, j \leq n, i \neq j]] \to K[t_1, \ldots, t_n]$ given by

$$\phi(x_{ij}) = t_i^2t_j.$$

Let $I_{L, \rho}$ be any lattice ideal with associated cone $\sigma = pos_{Q}(A_n)$ or rationally affine equivalent to the cone $pos_{Q}(A_n)$.

We define the following vectors in $Q^n$, with coordinates:

$$(c_T)_s = \begin{cases} 0, & \text{for } s \in T \\ 1, & \text{otherwise}, \end{cases}$$

$$(c_{i,T})_s = \begin{cases} -1, & \text{for } s = i \\ 2, & \text{for } s \in T \\ 3, & \text{otherwise}, \end{cases}$$

where $1 \leq s \leq n$, $T$ is a subset of $\{1, \ldots, n\}$ and $1 \leq i \leq n$, $i \notin T$.

Note that the $pos_Q(2e_i + e_j)$ is an extreme ray of the cone $pos_Q(A_n) \subset Q^n$ with defining vector $c_{i,j}$. Therefore the cone $pos_Q(A_n)$ has $n(n-1)$ extreme rays.

We consider the Stanley-Reisner ideal $I_{pos_Q}(A_n) \subset K[[Y_{ij} | 1 \leq i, j \leq n, i \neq j]]$. For the graph $G_{pos_Q}(A_n)$ we have the following result:

**Proposition 5.1** There are $9 \binom{n}{3} + 12 \binom{n}{4}$ vertices, $15 \binom{n}{3} + 18 \binom{n}{4}$ edges and $\binom{n}{3} + \binom{n}{4}$ connected components in the graph $G_{pos_Q}(A_n)$.

**Proof.** We claim that the minimal generators of $I_{pos_Q}(A_n)$ are the $3 \binom{n}{3}$ quadratic monomials in the form $Y_{ij}Y_{kl}$, the $6 \binom{n}{3}$ monomials in the form $Y_{ij}Y_{ki}$ and the $12 \binom{n}{4}$ monomials in the form $Y_{ij}Y_{kl}$, where $i, j, k, l \in \{1, \ldots, n\}$. Here we adopt the convention that $\binom{n}{k} = 0$ for $k > n$.

The relation $(2e_i + e_j) + (2e_k + e_j) = (2e_j + e_i) + (2e_k + e_i)$ shows that $pos_Q(Y_{ij}Y_{kj})$
cannot be a face of the cone \( pos\varrho(A_n) \). In the contrary case, taking the dot product with its defining vector in the two parts of the equality we get zero equal to a positive number, which is a contradiction. Thus \( Y_{ij}Y_{kj} \) is a generator of \( I_{pos\varrho(A_n)} \). Similarly, the relations \( 2(2e_i + e_j) + (2e_k + e_i) = 2(2e_i + e_k) + (2e_i + e_j) \) and \( (2e_i + e_j) + (2e_k + e_j) = (2e_i + e_k) + (2e_k + e_j) \) show that \( Y_{ij}Y_{ki} \) and \( Y_{ij}Y_{kl} \) are generators of \( I_{pos\varrho(A_n)} \). They are minimal, since there is no linear monomial in \( I_{pos\varrho(A_n)} \). Next we show that there is no other minimal generator of the Stanley-Reisner ideal. The only square free monomials of degree greater than or equal to two that are not divided by the previous quadratic minimal generators are in the form \( M_{i,T} = \prod_{j \in T} Y_{ij} \) for some \( T \subset \{1, \ldots , i-1, i+1, \ldots , n\} \) or \( M_{\{i,j\}} = Y_{ij}Y_{ji} \). But \( pos\varrho(M_{i,T}) \) and \( pos\varrho(M_{\{i,j\}}) \) are faces whose defining vectors are \( c_{i,T} \) and \( c_{\{i,j\}} \).

We define the index of a \( Y_{ij} \) to be the set \( \{i,j\} \) and the index of a monomial \( M \in K[\{Y_{ij}\} | 1 \leq i, j \leq n, i \neq j, M \in \text{index}(M) \} \) to be the union of the indices of the variables in \( M \).

Let \( M \) and \( N \) be minimal generators of the Stanley-Reisner ideal of \( I_{A_n} \) then there is an edge between \( M \) and \( N \) iff \( \text{relint}(M) \cap \text{relint}(N) \neq \{0\} \). Every vector in \( \text{relint}(M) \) can be written as a positive linear combination of the vectors \( e_i \), where \( i \in \text{index}(M) \). Since the vectors \( \{e_i\} \) are linearly independent, we conclude that \( \text{index}(M) = \text{index}(N) \).

Therefore two minimal generators can be vertices of a connected component of the graph \( G_{pos\varrho(A_n)} \) if they have the same index. The index of a minimal generator can be a set with three elements \( \{i,j,k\} \) or four elements \( \{i,j,k,l\} \). By explicitly computing the edges among the 9 vertices with index \( \{i,j,k\} \) we get that all of them are in the same connected component which has 15 edges and looks like the FIGURE 1. Similarly, by explicitly computing the edges among the 12 vertices with index \( \{i,j,k,l\} \) we get that all of them are in the same connected component which has 18 edges and looks like the FIGURE 2.

\[ \text{FIGURE 1} \]

\[ \text{FIGURE 2} \]

Therefore we conclude that the graph \( G_{pos\varrho(A_n)} \) has \( \binom{n}{3} \) connected components like the
one in FIGURE 1, with 9 vertices and 15 edges each, and \( \binom{n}{4} \) connected components like the one in FIGURE 2, with 12 vertices and 18 edges each.

**Corollary 5.2** Let \( L \) be a lattice with associated cone rationally affine equivalent to \( \text{pos}_Q(A_n) \), then for the ideal \( I_{L,\rho} \) we have that

\[
\bar{b}(I_{L,\rho}) \geq 5 \binom{n}{3} + 6 \binom{n}{4} \quad \text{and} \\
\text{ara}_A(I_{L,\rho}) \geq 4 \binom{n}{3} + 6 \binom{n}{4}.
\]

\[
\text{FIGURE 3}
\]

Proof. Recall that \( b_G \) is the smallest number \( s \) of subgraphs \( B_i \) of \( G \), consisting of two vertices and an edge or just a vertex, such that the graph \( \bigcup_{i=1}^{s} B_i \) is spanning. For the \( \binom{n}{3} \) connected components of \( G_{\text{pos}_Q(A_n)} \), like the one in FIGURE 1, this number is five as it can be seen in FIGURE 3. While for the \( \binom{n}{4} \) connected components of \( G_{\text{pos}_Q(A_n)} \), like the one in FIGURE 2, this number is six as it can be seen in FIGURE 4. Thus \( b_{G_{\text{pos}_Q(A_n)}} = 5 \binom{n}{3} + 6 \binom{n}{4} \).

Recall also that \( c_G \) is the smallest number \( s \) of complete subgraphs \( G_i \) of \( G \), such that the graph \( \bigcup_{i=1}^{s} G_i \) is spanning. Note also that graphs like those in FIGURE 1 have only one complete subgraph with 3 vertices and those in FIGURE 2 have only complete subgraphs with two or one vertices. Consequently, for the \( \binom{n}{3} \) connected components of \( G_{\text{pos}_Q(A_n)} \), like the one in FIGURE 1, the number \( c_G \) is four as it can be seen in FIGURE 3. While for the \( \binom{n}{4} \) connected components, like the one in FIGURE 2, this number is six as it can be seen in FIGURE 4. Therefore \( c_{G_{\text{pos}_Q(A_n)}} = 4 \binom{n}{3} + 6 \binom{n}{4} \).
The proof follows from Theorem 4.5.

Next we will prove that the lower bounds computed in Corollary 5.2 are sharp by computing the exact value of the binomial arithmetical rank and the A-homogeneous arithmetical rank for the toric ideal $I_{A_n}$.

**Proposition 5.3** The ideal $I_{A_n}$ is generated up to radical by the $5\binom{n}{3} + 6\binom{n}{4}$ binomials $x_{ij}x_{k} - x_{j}x_{ik}, x_{ij}^2x_{ki} - x_{i}^2x_{ji}, x_{ij}x_{kl} - x_{d}x_{kj}$, where $i, j, k, l \in \{1, \ldots, n\}$. Therefore

$$\text{bar}(I_{A_n}) = 5\binom{n}{3} + 6\binom{n}{4}.$$  

**Proof.** Let $J$ be the ideal in $K[\{x_{ij}|1 \leq i, j \leq n, i \neq j\}]$ generated by the $5\binom{n}{3} + 6\binom{n}{4}$ binomials $x_{ij}x_{kl} - x_{il}x_{kj}, x_{ij}x_{k} - x_{i}x_{ik}, x_{ij}^2x_{ki} - x_{i}^2x_{ji}, x_{ij}x_{kl} - x_{d}x_{kj}$, where $i, j, k, l \in \{1, \ldots, n\}$. We will use Hilbert’s Nullstellensatz to prove the theorem. Obviously $J \subseteq I_{A_n}$ and therefore $V(I_{A_n}) \subseteq V(J)$. Note that the toric variety $V(I_{A_n})$ is the Zariski-closure of the point $P = (1, \ldots, 1) \in K^{n(n-1)}$ under the toric action induced by the set of vectors $A_n$.

Let $a \in K^{n(n-1)}$ be a point in $V(J)$ with $a_{ij} \neq 0$, for some fixed indices $i, j$. There are two cases:

a) $a_{ii} = 0$. Then, using the binomials $x_{ij}^2x_{ki} - x_{i}^2x_{ji}$ and $x_{ji}x_{ki} - x_{ij}x_{kj}$, we get that $a_{ki} = 0$ and $a_{kj} = 0$ for every index $k$ different from $i, j$.

In addition, using the binomials $x_{ij}x_{kl} - x_{il}x_{kj}$ and $x_{ji}x_{kl} - x_{i}^2x_{ji}$, we have that $a_{kl} = 0$ and $a_{jk} = 0$ for every indices $k, l$ different from $i, j$.

Let $T = \{k|a_{ik} \neq 0\}$. Note that $T$ is not empty, because $j \in T$. Let $F_{i,T} = \text{pos}_{Q}\{2e_{i} + e_{k}/k \in T\}$, then $F_{i,T}$ is a face of $\sigma$ whose defining vector is $c_{i,T}$. Setting $t_{i} = 1$, $t_{k} = a_{ik}$, for every $k \in T$, and $t_{l} = 0$, for every $l \notin T$, we obtain that $a$ is in the orbit of the point $P_{F_{i,T}}$. Thus $a$ belongs to $V(I_{A_n})$.

b) $a_{ij} \neq 0$. Let $T = \{k|a_{ik} \neq 0\} \cup \{i\}$. Note that $j \in T$. Let $k \in T$ then, from the definition, $a_{ik} = 0$. Using the binomial $x_{ij}^2x_{ki} - x_{i}^2x_{ji}$ we obtain that $a_{ki} = 0$. Then, from the binomial $x_{ij}x_{k} - x_{i}x_{ik}$ we have that $a_{kj} = 0$. Finally, from the binomial $x_{ij}x_{kl} - x_{j}x_{ik}$, we conclude that $a_{jk} = 0$.

Let $\{k, l\} \subset T$ and $\{k, l\} \cap \{i, j\} = \emptyset$, then, using the binomial $x_{ij}x_{kl} - x_{il}x_{kj}$, we take $a_{kl} = 0$.

Assume that $k \notin T$, then, from the definition, $a_{ik} = 0$. The binomial $x_{ij}^2x_{ki} - x_{i}^2x_{ji}$ gives $a_{kj} = 0$, while the binomial $x_{ij}x_{k} - x_{i}x_{ik}$ gives $a_{k} = 0$. From the binomial $x_{ij}^2x_{kl} - x_{i}^2x_{ji}$ we conclude that $a_{jk} = 0$. Also $a_{k} = 0$ for every index $l$, because of the binomial $x_{ij}x_{kl} - x_{i}x_{ik}$. The binomial $x_{ij}x_{lk} - x_{i}x_{lj}$ give us that $a_{lk} = 0$ for every index $l$.

Therefore $a_{pq} \neq 0$ if $\{p, q\} \subset T$, while $a_{pq} = 0$ if $\{p, q\} \notin T$. Let $F_{T} = \text{pos}_{Q}(2e_{p} + e_{q}/\{p, q\} \subset T)$, then $F_{T}$ is a face of $\sigma$ whose defining vector is $c_{T}$.

We will prove that the point $a$ is in the orbit of the point $P_{F_{T}}$. Let $i, j \in T$ and $\omega$ be any cubic root of $a_{ij}a_{ji}$. Setting $t_{i} = a_{ij}\omega^{-1}$ and $t_{j} = a_{ji}\omega^{-1}$ we have $a_{ij} = t_{i}^{2}t_{j}$ and $a_{ji} = t_{j}^{2}t_{i}$.

For any $k \in T$ put $t_{k} = a_{jk}t_{j}^{-2}$, then of course $a_{jk} = t_{j}^{2}t_{k}$. 

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Using the binomials $x_{ji}^2x_{kj} - x_{jk}^2x_{ij}$, $x_{ij}x_{kj} - x_{jk}x_{ik}$, $x_{ij}^2x_{ki} - x_{ik}^2x_{ji}$ we conclude step by step that $a_{kj} = t_k^2t_j$, $a_{ik} = t_k^2t_i$ and $a_{ki} = t_k^2t_i$. Then for any two $k,l$ in $T$, from the binomial $x_{ij}x_{kl} - x_{il}x_{kj}$, we have that $a_{kl} = t_k^2t_l$. Put $t_l = 0$ for all $l \notin T$. Then the point $a$ is in the orbit of the point $P_T$, so it is a point of $V(I_{A_n})$.

The second part of the proposition now follows from corollary 5.2.

**Remark 5.4** We can choose the binomials $x_{ij}^2x_{jk} - x_{ji}^2$ instead of $x_{ij}^2x_{ki} - x_{ik}^2x_{ji}$ to generate the radical of $I_{A_n}$. In addition, from the proof of the above theorem we can see that the faces of the cone $\sigma$ are in the form $F_iT$ or $F_T$, for all the possible choices of $i$ and $T$.

**Proposition 5.5** The $A_n$-homogeneous arithmetical rank of $I_{A_n}$ is equal to $4 \left( \begin{array}{c} n \\ 3 \end{array} \right) + 6 \left( \begin{array}{c} n \\ 4 \end{array} \right)$.

**Proof.** The ideal $I_{A_n}$ is generated up to radical by the $A_n$-homogeneous polynomials $x_{ij}^2x_{ki} - x_{jk}^2x_{ij}$, $x_{ij}^3x_{kj} - x_{jk}^3x_{ij}$, $x_{ij}^3x_{ki} - x_{ik}^3x_{ij}$, $x_{ij}^2x_{kl} - x_{il}x_{kj}$, where $i,j,k,l \in \{1, \ldots, n\}$. The proof follows from proposition 5.3 and the observation that $(x_{ij}x_{kj} - x_{jk}x_{ik})^5$ belongs to the ideal generated by the previous $A_n$-homogeneous polynomials. Note that the $3 \left( \begin{array}{c} n \\ 3 \end{array} \right)$ binomials $x_{ij}^2x_{ki} - x_{ij}^2x_{ji}$ correspond to the $3 \left( \begin{array}{c} n \\ 3 \end{array} \right)$ complete subgraphs of $G_{pos\mathcal{Q}}(A_n)$ with two vertices like those in FIGURE 3. The $n \left( \begin{array}{c} n \\ 3 \end{array} \right)$ $A_n$-homogeneous polynomials $x_{ij}^3x_{kj} - x_{jk}^3x_{ij}$, $x_{ij}^3x_{ki} - x_{ik}^3x_{ij}$, $x_{ij}^2x_{kl} - x_{il}x_{kj}$ correspond to the $n \left( \begin{array}{c} n \\ 3 \end{array} \right)$ complete subgraphs of $G_{pos\mathcal{Q}}(A_n)$ with three vertices like the one in FIGURE 3. The $6 \left( \begin{array}{c} n \\ 4 \end{array} \right)$ binomials $x_{ij}^2x_{kl} - x_{il}x_{kj}$ correspond to the $6 \left( \begin{array}{c} n \\ 4 \end{array} \right)$ complete subgraphs of $G_{pos\mathcal{Q}}(A_n)$ with two vertices like those in FIGURE 4.

**Remark 5.6** The bounds given in corollary 5.2 are also bounds for the minimal number of generators of a lattice ideal $I_{L,\rho}$ with associated cone rationally affine equivalent to $pos\mathcal{Q}(A_n)$. In particular for any such ideal the minimal generators are greater than or equal to $5 \left( \begin{array}{c} n \\ 3 \end{array} \right) + 6 \left( \begin{array}{c} n \\ 4 \end{array} \right)$. This implies that for any such ideal, for $n \geq 3$, it is impossible to be complete intersection, since $h(I_{L,\rho}) = n(n-2)$.

**Remark 5.7** While theorem 4.1 give lower bounds for $ara_A(I_{L,\rho})$, $bar(I_{L,\rho})$ and $\mu(I_{L,\rho})$ it does not provide a lower bound for $ara(I_{L,\rho})$. Nevertheless theorem 4.1 provides certain information on the form and size of the polynomials $F_1, \ldots, F_s$ such that $rad(I_{L,\rho}) = rad(F_1, \ldots, F_s)$. We know that for every vertex we need at least one monomial in at least one of the $F_1, \ldots, F_s$, corresponding to the vertex. In particular for the ideals $I_{L,\rho}$ studied
in this section we know that \( n(n-2) \leq \text{ara}(I_{L,\rho}) \leq n(n-1) \), by the Krull’s principal ideal theorem and the results of Eisenbud, Evans and Storch \([5], [14]\). From theorem 4.1 we know that in these \( s = \text{ara}(I_{L,\rho}) \) polynomials there must be at least \( 9 \binom{n}{3} + 12 \binom{n}{4} \) monomials, in at least \( 4 \binom{n}{3} + 6 \binom{n}{4} \) \( A_n \)-homogeneous components. For example for \( n = 10 \) we know that we need a number of polynomials between 80 to 90 to generate the radical of, say, \( I_{A_{10}} \). Those polynomials should have totally at least 3600 monomials, so on the average at least 40 to 45, and therefore even for small \( n \)’s the polynomials involved are huge. It will be an interesting problem to compute these polynomials even for \( n = 10 \).

Note that in all the cases, that we know explicitly the polynomials which define a lattice ideal up to radical, the polynomials involved are all \( A \)-homogeneous, see \([1], [12], [17]\). The results of this paper show that \( A \)-homogeneous polynomials are not always enough to define a lattice ideal up to radical. Therefore we have to understand better the topic of non \( A \)-homogeneous set theoretic intersections for lattice ideals. Also these results give a different perspective relative to the famous Macaulay curve \((t^4, t^3u, tu^3, u^4)\) in the three dimensional projective space, for which we know that it is not \( A \)-homogeneous set-theoretic complete intersection, see \([16]\).

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOANNINA, IOANNINA 45110 (GREECE)

UNIVERSITÉ DE GRENOBLE I, INSTITUT FOURIER, UMR 5582, B.P.74, 38402 SAINT-MARTIN D’HÈRES CEDEX, AND IUFM DE LYON, 5 RUE ANSELME, 69317 LYON CEDEX (FRANCE)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOANNINA, IOANNINA 45110 (GREECE)