Estimation of Velocity of a Frictionless Motion of a Truck on an Infinitely Long Straight Rail

Ogwola, Peter, Sullayman Muhammad Bello.
Department of Mathematics,
Nasarawa State University, Keffi, Nigeria
ogwolapeter10@gmail.com

Abstract
This paper is concerned with estimation of velocity of a frictionless motion of a truck on an infinitely long straight rail. For simplicity assume that the Truck is controlled only by the throttle producing an accelerative force per unit mass. A discrete dynamic model of first order difference equation is to describe the system. Kalman filtering technique is applied to the discrete dynamic model to estimate the velocity of the Truck at any particular time. A computer programme is developed to simulate the system.

Keywords: Distance, estimation, kalman filter, modelling, velocity.
Introduction

In system analysis, a fundamental problem is to provide values for the unknown states or parameters of a system given noisy measurements which are some functions of these states and parameters.

According to Greg and Gary (2006), Kalman filter is defined as a set of mathematical equations that provides an efficient computational (recursive) means to estimate the state of a process, in a way that minimizes the mean of the square error. Kalman filter was originally developed for use in spacecraft navigation but turns out to be useful for many applications.

Kalman (1960) published the discrete-time filter in a Mechanical Engineering Journal and Kalman and Bucy (1961), the continuous-time filter. In the meantime, Swerling (1959) had derived an equivalent formulation of the Kalman filter and applied it to the problem of estimating the trajectories of satellites using ground-based sensor. His results were published in an Astronomy journal the year before Kalman (1960) appeared.

Materials and Methods

Kalman filter Wikipedia

The Kalman filter model assumes the true state at time k is evolved from the state at (k-1) as stated below.

\[ X_{k+1} = \Phi X_k + BU_k + \xi_k \]

(1)

Where,

- \( \Phi \) is the state transition model which is applied to the previous state \( X_k \);
- \( B \) is the control-input model which is applied to the control vector \( U_k \);
- \( \xi_k \) is the process noise which is assumed to be drawn from a zero mean multivariate normal distribution with covariance \( Q \).

\( \xi_k \sim N(0,Q) \)

At time k an observation (or measurement) \( Y_k \) of the true state \( X_k \) is made according to

\[ Y_k = HX_k + \eta_k \]

(2)

Where \( H \) is the observation model which maps the true state space into the observed space and \( \eta_k \) is the observation noise which is assumed to be zero mean Gaussian white noise with covariance \( R, \eta_k \sim N(0,R) \)

The initial state, and the noise vectors at each step \( \{X_o, \eta_1, \ldots, \eta_k, \xi, \ldots, \xi_k \} \) are all assumed to be mutually independent.

The Kalman filter is a recursive estimator. This means that only the estimated state from the previous time step and the current measurement are needed to compute the estimate for the current state and as such no history of observations and/or estimates is required. In what follows, the notation \( \hat{X}_{k|k-1} \) which is the 1 step prediction represents the estimates of \( X_k \) at time k given observations up to and including at time k-1.

\[ \hat{X}_{k|k-1} = \Phi \hat{X}_{k-1|k-1} + BU_k \]

(3)

The covariance matrix for the one step prediction error is given by

\[ P_{k|k-1} = \Phi P_{k-1|k-1} \Phi^T + Q \]

(4)

\[ \hat{X}_{k|k} = \hat{X}_{k|k-1} + K_k (Y_k - H \hat{X}_{k|k-1}) \]

(5)

\[ P_{k|k} = (1 - K_k H) P_{k|k-1} \]

(6)

Where \( K_k \) is the Kalman (Filter) gain and given by

\[ K_k = P_{k|k-1} H^T (H P_{k|k-1} H^T + R)^{-1} \]

(7)
The Kalman filter loop

The Kalman filter loop given below summarizes what is known as the Kalman filter.

\[
\dot{\hat{x}}_{k|k} = \phi \hat{x}_{k|k-1} + K_k (y_k - H \hat{x}_{k|k-1})
\]

\[
P_{k|k} = (I - E_k H) P_{k|k-1}
\]

Project ahead

\[
\hat{x}_{k+1|k} = \phi \hat{x}_{k|k}
\]

\[
P_{k+1|k} = \phi P_{k|k} \phi^T + Q
\]

Enter prior estimate \( \hat{x}_{k|k-1} \) and its covariance \( P_{k|k-1} \)

Compute kalman gain

\[
K_k = P_{k|k-1} H^T (HP_{k|k-1} H^T + R)^{-1}
\]

Update estimate with measurement \( y_k \)

\[
\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (y_k - H \hat{x}_{k|k-1})
\]

Compute covariance matrix for updated estimate

\[
P_{k|k} = (1 - E_k H) P_{k|k-1}
\]

Modelling and identification

Suppose a Truck is being driven along a straight rail, and let its distance from initial point be \( x_1(t) \) at time \( t \). For simplicity assume that the Truck is controlled only by the throttle, producing an accelerating force per unit mass. Ignoring friction, wind resistance etc. It is required to find the velocity of the Truck at time \( t = 1, 2, 3 \) seconds.

Let \( x_1(t) = x(t) \) (position) \hspace{1cm} (8)

\[
x_2(t) = x(t) \text{ (velocity)} \hspace{1cm} (9)
\]

Application of Newtons principle of classical mechanics yields

\[
m \frac{d^2 x_1(t)}{dt^2} = u(t)
\]

For a unit mass, equation (10) becomes

\[
\frac{d^2 x_1(t)}{dt^2} = \frac{u(t)}{m} = u^*(t)
\]

From eqns. (8), (9) and (11) we have,

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= u^*(t)
\end{align*}
\]

Writing eqn. (12) in matrix form gives

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} + 
\begin{bmatrix}
0 \\
1
\end{bmatrix} u^*(t)
\]

Where

\( u(t) \) = accelerating force

\( u^*(t) \) = accelerating force per unit mass

Equation (13) can be written as
\[ X_{k+1} = \Phi X_k + BU_k + \xi_k \quad \text{(14)} \]

Where,
\[
\Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_k = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \xi_k \text{ is the }
\]
modelling noise.
\[ U_k = u^*(t) = (1 \times 1) \text{ control matrix.} \]

**The estimation problem**

The estimation problem is stated as follows:

Given,
\[
X_{k+1} = \Phi X_k + BU_k + C + \xi_k \quad \text{and} \quad Y = HX_k + \eta_k \quad \text{ } \quad (15)
\]

From the observed values of \( Y_0, Y_1, \ldots, Y_k \), find an estimate \( \hat{X}_{k|k} \) of \( X_k \) which minimizes the expected loss.

Where:
\[
\Phi = (2 \times 2) \text{ constant matrix obtained from the transition model} \]
\[ B = (2 \times 1) \text{ control input matrix which is applied to the control vector } U_k. \]
\[ Y = (1 \times 1) \text{ output vector (vector measurement at time } t_k \text{), } Y = y, \text{ it is the measured value of the distance covered at } t_k \]
\[ H = (1 \times 2) \text{ constant matrix giving the ideal connection between the measurement and the State vector at time } t_k \]
\[ X_k = (2 \times 1) \text{ process state vector at time } t_k, \text{ i.e., } X_k = X(t_k) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \text{ } x_1 \text{ and } x_3 \text{ are the estimates of distance and velocity of the truck at time } t_k \text{ respectively.} \]
\[ U_k = (1 \times 1) \text{ control vector, } U_k = u^*(t), \]
\[ \eta_k = (1 \times 1) \text{ measurement error – assumed to be white noise sequence with known co-} \]

\[ \text{Variance } R \text{ and having zero cross correlation with } \xi_k \text{ sequence.} \]
\[ \xi_k = (2 \times 1) \text{ vector-assumed to be white noise sequence with known co-variance} \]

**Recursive processing of the noisy measurement (input) data**

A computer programme is written to simulate the system.

An algorithm for the programming model as in ogwola (2017) is as follows

Given the initial values \( \hat{X}_{0|0} \) and its co-variance, \( p_{0|0} \)

(i) \( K_k = P_{k|k-1} H^T (HP_{k|k-1} H^T + R)^{-1} \)

(ii) \( \hat{X}_{k|k} = \hat{X}_{k|k-1} + K_k (Y_k - H \hat{X}_{k|k-1}) \)

(iii) \( P_{k|k} = P_{k|k-1} - K_k H P_{k|k-1} \)

(iv) \( \hat{X}_{k+1|k} = \Phi \hat{X}_{k|k} \quad \text{and} \quad p_{k+1|k} = \Phi p_{k|k} \Phi^T + Q \)

(v)

The values of \( \hat{X}_{0|0} \) and \( P_{0|0} \) used were \( \begin{pmatrix} 5 \\ 3 \end{pmatrix} \) and \( \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \) respectively.
\[ \Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = (1 \quad 0) \]

The parameter \( Q, R \) that gives optimal estimates of velocity is \( \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \) and 0.4 respectively.

The distance \( (Y_k) \) of the truck at time \( t_k \) from the initial point is given in table 1.

**RESULTS**

The values of \( \Phi, Q, R, H, \hat{X}_{0|0}, P_{0|0} \) and \( Y_k \) were used to run the computer programme from which the following results in table 2 were obtained.
Table 1: Observed (measured) values of the distance \( Y_k \) covered at time \( t_k \) of the Truck in Kilometres.

| K | \( t_k \) | \( Y_k \) (Km) |
|---|---|---|
| 0 | \( t_0 \) | 4 |
| 1 | \( t_1 \) | 7 |
| 2 | \( t_2 \) | 9 |
| 3 | \( t_3 \) | 15 |
| 4 | \( t_4 \) | 16 |
| 5 | \( t_5 \) | 20 |

Table 2: Simulation Results for the system

| K | \( t_k \) | \( X_1 \) | \( X_2 \) |
|---|---|---|---|
| 0 | \( t_0 \) | 4.57 | 3 |
| 1 | \( t_1 \) | 5.22 | 1.78 |
| 2 | \( t_2 \) | 6.17 | 2.83 |
| 3 | \( t_3 \) | 10.33 | 4.67 |
| 4 | \( t_4 \) | 11.67 | 4.33 |
| 5 | \( t_5 \) | 14.01 | 5.99 |

DISCUSSIONS

In table 2, column 2 gives the time \( t_k \) it takes for the truck to reach a particular point away from the initial point. Column 3 gives the estimates of the true value of the distance \( X_1 \) in km covered from the initial point at time \( t_k \). Column 4 gives the estimates of the velocity \( X_2 \) of the truck in km/s at any particular time.

In conclusion, a sensor inside the truck measured the distance of the truck from the initial point at any time \( t_k \). The value of the distance is then imputed into computer which estimate the true value of the distance \( X_1 \) together with the estimates of the velocity \( X_2 \) of the truck through the computer programme developed.

REFERENCES

Filtering, Hamilton Printing Company.
Greg, W. and Gary, B. (2006). An introduction to the Kalman Filter” TR 95-041 Department of Computer Science, University of North Carolina at Chapel Hill, Chapel Hill, NC 27599-3175.
Kalman, R.E. (1960). A New Approach to Linear Filtering and Prediction Problems, ASME Trans, J. Basic Eng., series D., Vol. 82, pp35-45.
Kalman, R.E. and Bucy, R.S. (1961). New Results in Linear Prediction and Filtering Theory, ASME Trans., J. Basic Eng., series D., Vol. 83, pp95-108.
Ogwola, P. (2017). Application of Kalman filtering Technique for controlling the problems of river pollution (A Case Study of Warri River), Ph.D. Thesis, Nasarawa State university, Keffi, Nigeria.
Robert, G.B. and Patrick, Y. C. H. (1992). Introduction to Random Signals and Applied Swerling, P. (1959). First Order Error Propagation in a Stage wise Differential Smoothing Procedure for Satellite Observations,” J. Astronaut Sci., Vol. 6, pp46-52.