ASYMPTOTIC EXPANSION OF OSCILLATORY INTEGRALS WITH SINGULAR PHASES

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Abstract. The purpose of this article is to describe the singularities of one-dimensional oscillatory integrals, whose phases have a certain singularity, in the form of an asymptotic expansion. In the case of the Laplace integral, an analogous result is also given.

1. Introduction

In the investigation of various issues in mathematics, oscillatory integrals of the form

\[ I(t) = \int_{-\infty}^{\infty} e^{itf(x)} \varphi(x) \, dx \quad \text{for } t \in \mathbb{R} \]

appear and their properties often play important roles in it. Here \( f \) and \( \varphi \) are real-valued \( C^\infty \) functions defined on an open interval in \( \mathbb{R} \) containing the origin and the support of \( \varphi \) is contained in this interval. The functions \( f \) and \( \varphi \) are called the phase and the amplitude respectively.

It is easy to see that the integral in (1.1) can be regarded as a \( C^\infty \) function of \( t \) on \( \mathbb{R} \). Moreover, the behavior of this function for large \( t \) is well understood (cf. [1], [5], [11]). If there is \( k \in \mathbb{N} \) with \( k \geq 2 \) such that \( f'(0) = \cdots = f^{(k-1)}(0) = 0 \) and \( f^{(k)}(0) \neq 0 \), then for any positive integer \( N \),

\[ I(t) = \sum_{n=1}^{N-1} c_n t^{-n/k} + O(t^{-N/k}) \quad \text{as } t \to +\infty, \tag{1.2} \]

where \( c_n \) are constants depending on \( f \) and \( \varphi \), and the exact value of the first coefficient \( c_1 \) can be explicitly given. When \( f \) does not have a critical point, its behavior is obvious in the sense: \( I(t) = O(t^{-N}) \) as \( t \to \infty \) for any positive integer \( N \). More generally, in the case where \( k \) is a positive real number, the asymptotic expansion of \( I(t) \) can be expressed in a similar fashion to (1.2) (see [8]).

Corresponding to the above regular phase case, the case where \( f \) has singularities seems to be not well thought. In this article, we investigate the properties of integrals

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of the form
\begin{equation}
I_\alpha(t) = \int_0^\infty \exp \left( it \frac{1}{x^\alpha} \right) \varphi(x) \, dx \quad \text{for } t \in \mathbb{R},
\end{equation}
where \( \alpha \) is a positive real number and \( \varphi \) is the same as that in (1.1). There have been many interesting numerical analyses of the integral (1.3) \([2], [3], [6], [7], \text{etc.}\), while we will investigate this integral from the viewpoint of the asymptotic expansion.

From the convergence of the integral, the integral (1.3) can be regarded as a function of \( t \) defined on \( \mathbb{R} \), but the smoothness of this function is not obvious. The convergence of the derivatives of the integrand in (1.3) only implies that \( I_\alpha(t) \) is a \( C^{\lceil 1/\alpha \rceil - 1} \) function on \( \mathbb{R} \). For example, \( I_1(t) \) is continuous but it is not differentiable if \( \varphi(0) > 0 \) and \( \varphi \geq 0 \) on \( \mathbb{R} \). It is interesting to consider what kind of singularities \( I_\alpha(t) \) has. The following theorem exactly shows the singular part of \( I_\alpha(t) \).

**Theorem 1.1.** (i) If \( \alpha > 0 \) is a rational number, then for any positive integer \( N \),
\begin{equation}
I_\alpha(t) = \sum_{n=1}^N A_n t^{n/\alpha} + \sum_{n=1}^N B_n t^{n/\alpha} \log |t| + \psi_N(t) \quad \text{for } t \in \mathbb{R}^+, \tag{1.4}
\end{equation}
where \( \psi_N(t) \) is a \( C^{\lceil N + 1/\alpha \rceil - 1} \) function on \( \mathbb{R}^+ \) and
\begin{align*}
A_n &= \frac{e^{-\pi i n \alpha} \varphi^{(n-1)}(0)}{\alpha (n-1)!} \Gamma(-n/\alpha) \quad \text{if } n/\alpha \not\in \mathbb{N}, \quad A_n = 0 \quad \text{if } n/\alpha \in \mathbb{N}, \\
B_n &= \frac{-1 \varphi^{(n-1)}(0)}{\alpha (n-1)! (n/\alpha)!} \frac{t^{n/\alpha}}{\alpha} \quad \text{if } n/\alpha \in \mathbb{N}, \quad B_n = 0 \quad \text{if } n/\alpha \not\in \mathbb{N},
\end{align*}
for \( n \in \mathbb{N} \), where \( \Gamma \) means the Gamma function.

(ii) If \( \alpha > 0 \) is not a rational number, then for any positive integer \( N \),
\begin{equation}
I_\alpha(t) = \sum_{n=1}^N A_n t^{n/\alpha} + \phi_N(t) \quad \text{for } t \in \mathbb{R}^+, \tag{1.6}
\end{equation}
where \( \phi_N(t) \) is a \( C^{\lceil (N+1)/\alpha \rceil - 1} \) function on \( \mathbb{R} \) and \( A_n \) are as in (1.5).

**Remark 1.2.** (1) If the branch of \( t^{n/\alpha} \) for \( t < 0 \) is chosen as \( t^{n/\alpha} = e^{i \pi / \alpha} |t|^{n/\alpha} \), then the equation in (1.6) holds for all \( t \in \mathbb{R} \), where \( \phi_N(t) \) is a \( C^{\lceil (N+1)/\alpha \rceil - 1} \) function on \( \mathbb{R} \).

(2) We can obtain a similar result to (1.4) in the case where \( t \leq 0 \). If \( \alpha > 0 \) is a rational number, then for any positive integer \( N \),
\begin{equation}
I_\alpha(t) = \sum_{n=1}^N A_n t^{n/\alpha} + \sum_{n=1}^N B_n t^{n/\alpha} \log |t| + \phi_N(t) \quad \text{for } t \in \mathbb{R}^-, \tag{1.7}
\end{equation}
where \( B_n \) are as in (1.5) and \( \phi_N(t) \) is a \( C^{\lceil (N+1)/\alpha \rceil - 1} \) function on \( \mathbb{R}^- \).
(3) When $\alpha$ is rational, $\alpha$ is uniquely denoted by $\alpha = p/q$, where $p, q \in \mathbb{N}$ with $(p, q) = 1$. In this case, the second term in (1.4) can be rewritten as

\begin{equation}
\sum_{n=1}^{N} B_n t^n \log t = \sum_{n=1}^{N} \tilde{B}_n t^n \log t \quad \text{for } t \in \mathbb{R}_+,
\end{equation}

with

\[ \tilde{B}_n = \frac{-q \varphi'(pn)(0)}{p (pn - 1)! (qn)!} \text{ for } n \in \mathbb{N}. \]

It follows from the above theorem that $I_\alpha$ is smooth away from the origin.

**Corollary 1.3.** $I_\alpha(t)$ is a $C^\infty$ function on $\mathbb{R} \setminus \{0\}$.

The above property seems to be not so obvious, because large order derivatives of the integrand in (1.3) with respect to $t$ are not absolutely integrable.

Let us focus on the behavior of $I_\alpha(t)$ near the origin. From the above theorem, we can give an asymptotic expansion of $I_\alpha(t)$ at the origin in the case where $\alpha$ is rational:

\begin{equation}
I_\alpha(t) = \sum_{n=1}^{N} A_n t^{n/\alpha} + \sum_{n=1}^{N} B_n t^n \log t + \sum_{n=0}^{[N+1/\alpha]-2} C_n t^n + O(t^{[N+1/\alpha]-1}) \quad \text{as } t \to +0,
\end{equation}

for any positive integer $N$ with $N \geq \alpha$, where $A_n, B_n$ are the same as those in (1.5) and $C_n$ are constants depending on $\varphi, \alpha$ and, in particular,

\begin{equation}
C_n = i^n \int_0^{\infty} \frac{\varphi(x)}{x^{n+\alpha}} dx \quad \text{for } n = 0, \ldots, \left\lfloor \frac{1}{\alpha} \right\rfloor - 1,
\end{equation}

which can be directly obtained by the Taylor formula. When $\varphi(0) > 0$ and $\varphi \geq 0$ on $\mathbb{R}$, the improper integral in (1.9) is convergent if and only if $n \in \mathbb{Z}_+$ satisfies $n \leq \left\lfloor \frac{1}{\alpha} \right\rfloor - 1$. When $\alpha$ is not rational, $I_\alpha(t)$ admits the following asymptotic expansion at the origin.

\begin{equation}
I_\alpha(t) = \sum_{n=1}^{N} A_n t^{n/\alpha} + \sum_{n=0}^{[N+1/\alpha]-2} C_n t^n + O(t^{[N+1/\alpha]-1}) \quad \text{as } t \to +0,
\end{equation}

for any positive integer $N$ with $N \geq \alpha$, where $A_n, C_n$ are as in (1.5), (1.9).

In particular, the limit of $I_\alpha(t)$ as $t \to +0$ is $C_0 = \int_0^{\infty} \varphi(x) dx$ as in (1.9). In more detail, we can explicitly see the decay as $t \to +0$ of the function $\tilde{I}_\alpha(t) := I_\alpha(t) - C_0$, which is classified as in the three cases as follows.

**Corollary 1.4.**

(i) If $0 < \alpha < 1$, then

\[ \lim_{t \to 0} \frac{\tilde{I}_\alpha(t)}{t} = i \int_0^{\infty} \frac{\varphi(x)}{x^\alpha} dx. \]

(ii) If $\alpha = 1$, then

\[ \lim_{t \to +0} \frac{\tilde{I}_\alpha(t)}{t \log t} = -i \varphi(0). \]
If $1 < \alpha$, then
\[
\lim_{t \to +0} \frac{\tilde{I}_\alpha(t)}{t^{1/\alpha}} = \frac{1}{\alpha} \varphi(0) e^{-\frac{1}{2\alpha^2}\pi^2} \Gamma\left(-\frac{1}{\alpha}\right).
\]

Remark 1.5. The behavior of $I_\alpha(t)$ for large $t$ is obvious in the sense:
\[(1.11) \quad I_\alpha(t) = O(t^{-N}) \quad \text{as} \quad t \to \infty,
\]
for any positive integer $N$. This can be easily shown as follows. By changing the integral variable as $y = x^{-\alpha}$, $I_\alpha(t)$ can be expressed as
\[
I_\alpha(t) = \frac{1}{\alpha} \int_0^\infty e^{it\varphi(y^{-1/\alpha})} y^{-1/\alpha-1} dy.
\]
Repeated integrations of parts imply
\[
I_\alpha(t) = \left(-\frac{1}{it}\right)^N \int_0^\infty e^{it\varphi_N(y^{-1/\alpha})} y^{-1/\alpha-N-1} dy,
\]
for any positive integer $N$, where $\varphi_N$ is a $C^\infty$ function on $\mathbb{R}$ whose support is contained in that of $\varphi$. The equation (1.11) follows from the above equation.

Remark 1.6. Let us consider integrals of the form:
\[(1.12) \quad \hat{I}(t) = \int_0^\infty e^{it/f(x)} \varphi(x) dx \quad \text{for} \quad t \in \mathbb{R},
\]
where $f, \varphi$ are as in (1.1) and, moreover, $f$ satisfies $f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0$ and $f^{(k)}(0) \neq 0$ with $k \geq 1$. If the support of $\varphi$ is sufficiently small, then $\hat{I}(t)$ essentially satisfies the property (i) in Theorem 1.1 with $\alpha = k$. (The coefficients $A_j, B_j$ are slightly different from those in (1.5).) This can be easily shown by using the implicit function theorem.

Notation and symbols.
- We denote $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{Z}_+ := \{n \in \mathbb{Z} : n \geq 0\}$.
- When $n \leq a < n + 1$ where $n \in \mathbb{Z}$, set $[a] = n$. When $n - 1 < a \leq n$ where $n \in \mathbb{Z}$, set $[a] = n$.
- For a $C^n$ function $f$, the $k$-th derivative of $f$ is denoted by $f^{(k)}$ for $k = 1, \ldots, n$.
- Let $f(t), g(t)$ be functions defined on an interval $I \subset \mathbb{R}$. We write $f(t) \equiv g(t) \mod C^\infty(I)$ to express that $f(t) - g(t)$ is a $C^\infty$ function on $I$.

2. Some Fourier Transforms

To prove Theorem 1.1, we prepare some auxiliary lemmas concerning the Fourier transform of some functions. Let $p$ be a positive real number and let $\chi : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function satisfying that
\[(2.1) \quad \chi(x) = \begin{cases} 1 & \text{if} \quad x \geq M, \\ 0 & \text{if} \quad x \leq L, \end{cases}
\]
where $L, M$ are positive constants with $L < M$. For $t \in \mathbb{R}$, let $F_p(t)$ be the integral defined by

\begin{equation}
F_p(t) = \int_0^\infty e^{itx}x^{-p-1}\chi(x)dx.
\end{equation}

Note that this integral is the Fourier transform of the function $x \mapsto x^{-p-1}\chi(x)$ and that the integral in (2.2) absolutely converges.

The following lemma plays crucial roles in the computation below.

**Lemma 2.1.** If $0 < \alpha < 1$, then

\begin{equation}
\int_0^\infty e^{iy}y^{-\alpha-1}dy = e^{\frac{\alpha}{2}\pi i}\Gamma(\alpha).
\end{equation}

The above integral in the lemma may be considered as a generalized Fresnel integral and its value can be explicitly computed by using an elementary method of complex analysis. Indeed, its complete proof has been given in [8], [9], [10], [12], etc. Since the proof itself is essential to the analysis of this paper, we will recall its proof in Section 4.

Let us consider the property of $F_p(t)$ near the origin, which depends on whether $p$ is an integer or not.

**Lemma 2.2.** If $p > -1$ is not an integer, then $F_p(t) \equiv \tilde{A}_p t^p \mod C^\infty(\mathbb{R})$, where $\tilde{A}_p = e^{-\frac{p}{2}\pi i}\Gamma(-p)$. Here, the branch of $t^p$ for $t < 0$ is chosen as $t^p = e^{\pi pi}|t|^p$.

**Proof.** First, we consider the case where $p \in (-1, 0)$. By using $\chi(x)$ in (2.1), $F_p(t)$ is divided into two parts as follow.

\begin{equation}
F_p(t) = \int_0^\infty e^{itx}x^{-p-1}\chi(x)dx
\end{equation}

\begin{equation}
= \int_0^\infty e^{itx}x^{-p-1}dx - \int_0^\infty e^{itx}x^{-p-1}(1 - \chi(x))dx \quad \text{for } t \in \mathbb{R}.
\end{equation}

Since the support of $(1 - \chi(x))$ is compact, the second integral in (2.4) is a $C^\infty$ function of $t$ on $\mathbb{R}$. When $t > 0$, by the change of the integral variable, the first integral in (2.4) can be written as

\begin{equation}
\int_0^\infty e^{itx}x^{-p-1}dx = \int_0^\infty e^{i y^{-p-1}y}dy = t^p e^{\frac{p}{2}\pi i}\Gamma(-p)
\end{equation}

by using Lemma 2.1. When $t < 0$, by choosing the branch of $t^p$ as in the lemma,

\begin{equation}
\int_0^\infty e^{itx}x^{-p-1}dx = |t|^p \int_0^\infty e^{-iy}y^{-p-1}dy = |t|^p e^{\frac{p}{2}\pi i}\Gamma(-p) = t^p e^{\frac{p}{2}\pi i}\Gamma(-p).
\end{equation}

Therefore, the equation can be obtained in the lemma for $p \in (-1, 0)$.
Next, let us consider the case where \( p \in \mathbb{R}_+ \setminus \mathbb{Z} \) with \( p > 1 \). Since \( \chi(x) = 0 \) for \( x \leq L \), an integration by parts implies
\[
F_p(t) = \int_0^\infty e^{itx}x^{-p-1}\chi(x)dx = \frac{-1}{p} \left[ x^{-p}e^{itx}\chi(x) \right]_0^\infty + \frac{1}{p} \int_0^\infty x^{-p}(e^{itx}\chi(x))'dx
\]
\[
= \frac{i}{p} \int_0^\infty e^{itx}x^{-p}\chi(x)dx + \frac{1}{p} \int_0^\infty e^{itx}x^{-p}\chi'(x)dx \quad \text{for } t \in \mathbb{R}.
\]
(2.7)

Note that the second integral in (2.7) is a \( C^\infty \) function of \( t \) on \( \mathbb{R} \). Since there uniquely exist \( m \in \mathbb{N} \) and \( q \in (-1, 0) \) such that \( p = q + m \), the repeated process as the above implies
\[
F_p(t) \equiv \frac{(it)^m}{p(p-1)\cdots(q+1)} \int_0^\infty e^{itx}x^{-q-1}\chi(x)dx \mod C^\infty(\mathbb{R}).
\]
(2.8)

Since \( q \in (-1, 0) \), we can apply (2.5) to (2.8) and, as a result, we see that \( F_p(t) \equiv \tilde{A}_pt^m \mod C^\infty(\mathbb{R}) \), where
\[
\tilde{A}_p = \frac{i^m}{p(p-1)\cdots(q+1)} \tilde{A}_q = \frac{(-i)^m}{(-p)(-p+1)\cdots(-q+1)} e^{-\frac{2\pi i}{p}} \Gamma(-q).
\]

Using the equality \( \Gamma(\alpha + 1) = \alpha \Gamma(\alpha) \) for \( \alpha \in \mathbb{R} \setminus \mathbb{Z} \), we can obtain the equation in the lemma for \( p \in \mathbb{R}_+ \setminus \mathbb{Z} \).

On the other hand, when \( p \) is an integer, a logarithmic function appears in the singularity of \( F_p(t) \).

**Lemma 2.3.** If \( p \) is a nonnegative integer, then \( F_p(t) \equiv \tilde{B}_pt^p \log t \mod C^\infty(\mathbb{R}_+) \), where \( \tilde{B}_p = -i^p/p! \). (In particular, \( F_0(t) \equiv -\log t \mod C^\infty(\mathbb{R}_+) \).)

**Proof.** It is sufficient to show the lemma in the case of \( p = 0 \). Indeed, we can easily deal with the general case in a similar argument to that in Lemma 2.2.

The integral \( F_0(t) \) can be expressed as follows.
\[
F_0(t) = \int_L^{1/t} e^{itx}x^{-1}\chi(x)dx + \int_{1/t}^\infty e^{itx}x^{-1}\chi(x)dx
\]
\[
= \int_L^{1/t} x^{-1}\chi(x)dx + \int_L^{1/t} (e^{itx} - 1)x^{-1}\chi(x)dx + \int_{1/t}^\infty e^{itx}x^{-1}\chi(x)dx
\]
\[
=: G(t) + H(t) + K(t) \quad \text{for } t \in \mathbb{R}_+.
\]

First, let us consider the integral \( G(t) \). By integration by parts,
\[
G(t) = -\log t \cdot \chi(1/t) - \int_L^{1/t} \log x \cdot \chi'(x)dx
\]
\[
\equiv -\log t \mod C^\infty(\mathbb{R}_+)
\]
Second, let us show that $H(t)$ is a $C^\infty$ function on $\mathbb{R}$. Let

$$D := \{(x, t) \in \mathbb{R}^2 : tx < 1, \ x > 0, \ t \geq 0\}.$$  

The two $C^\infty$ functions $g, h : D \to \mathbb{C}$ can be defined by the convergent series as follows:

$$g(x, t) = \sum_{n=1}^{\infty} \frac{t^n}{n!} x^{n-1}, \quad h(x, t) = \sum_{n=1}^{\infty} \frac{t^n}{nn!} x^n.$$  

Noticing that $g(x, t) = (e^{ixt} - 1)x^{-1}$ and $\frac{\partial}{\partial x}h(x, t) = g(x, t)$ on $D$, we have

$$H(t) = \int_{1}^{\infty} g(x, t)\chi(x)dx = h(1/t, t) = \int_{1}^{\infty} h(x, t)\chi'(x)dx \quad \text{for} \ t \in \mathbb{R}_+,$$

by integration by parts. It is easy to see that $h(1/t, t)$ is a constant and that the last integral in (2.9) is a $C^\infty$ function of $t$ on $\mathbb{R}_+$.

Third, let us consider the integral $K(t)$. By changing the integral variable, we have

$$K(t) = \int_{1/t}^{\infty} e^{ixt} x^{-1} \chi(x)dx = \int_{1}^{\infty} e^{iy} y^{-1/\alpha} \chi(y/t)dy$$

$$= \int_{1}^{\infty} e^{iy} y^{-1/\alpha}dy - \int_{1}^{\infty} e^{iy} y^{-1/\alpha} (1 - \chi(y/t))dy \quad \text{for} \ t \in \mathbb{R}_+.$$  

The first integral in (2.10) is a constant defined by a convergent improper integral. It is easy to see that the second integral in (2.10) is a $C^\infty$ function of $t$ on $\mathbb{R}_+$.

Putting together the above results, we can see that $F_0(t) + \log t$ is a $C^\infty$ function on $\mathbb{R}_+$.

Remark 2.4. For $t \in \mathbb{R}_-$, the same result in Lemma 2.3 can be obtained. Indeed, if $p$ is a nonnegative integer, then $F_p(t) \equiv \tilde{B}_p t^p \log |t| \mod C^\infty(\mathbb{R}_-)$, where $\tilde{B}_p$ is as in the lemma. (In particular, $F_0(t) \equiv - \log |t| \mod C^\infty(\mathbb{R}_-)$. ) However, we do not know whether $F_p(t) - \tilde{B}_p t^p \log |t|$ is a $C^\infty$ function on $\mathbb{R}$ or not.

\[\square\]

3. Proof of Theorem 1.1

Let $R$ be a positive number such that the support of $\varphi$ is contained in $[-R, R]$.

By changing the integral variable, we have

$$I_\alpha(t) = \frac{1}{\alpha} \int_{0}^{\infty} e^{iy} \varphi(y^{-1/\alpha}) y^{-1/\alpha - 1} dy \quad \text{for} \ t \in \mathbb{R}.$$  

By using $\chi$ in (2.1) with $1/R^\alpha < L$, we divide the above integral as follows.

$$I_\alpha(t) = J(t) + E(t),$$

where

$$J(t) = \int_{0}^{\infty} e^{iy} \varphi(y^{-1/\alpha}) y^{-1/\alpha - 1} dy \quad \text{for} \ t \in \mathbb{R}_+,$$

$$E(t) = \int_{\infty}^{R^\alpha} e^{iy} \varphi(y^{-1/\alpha}) y^{-1/\alpha - 1} dy \quad \text{for} \ t \in \mathbb{R}_-.$$
with

\[
J(t) = \frac{1}{\alpha} \int_0^\infty e^{ity} \varphi(y^{-1/\alpha}) y^{-1/\alpha - 1} \chi(y) dy,
\]

\[
E(t) = \frac{1}{\alpha} \int_0^\infty e^{ity} \varphi(y^{-1/\alpha}) y^{-1/\alpha - 1} (1 - \chi(y)) dy.
\]

Since it is easy to see that \(E(t)\) is a \(C^\infty\) function on \(\mathbb{R}\), it suffices to consider the property of \(J(t)\).

The Taylor formula implies that for any \(N \in \mathbb{N}\),

\[
\varphi(x) = \sum_{n=0}^{N-1} a_n x^n + x^N \rho_N(x)
\]

where \(a_n := \varphi^{(n)}(0)/n!\) for \(n \in \mathbb{Z}_+\) and \(\rho_N(x)\) is a \(C^\infty\) function on \(\mathbb{R}\).

Substituting (3.2) into (3.1), we have

\[
J(t) = \sum_{n=0}^{N-1} \frac{a_n}{\alpha} F_{\gamma_n}(t) + R_N(t),
\]

where \(F_{\gamma_n}(t)\) is as in (2.2) with \(\gamma_n := (n + 1)/\alpha\) for \(n \in \mathbb{Z}_+\) and

\[
R_N(t) = \frac{1}{\alpha} \int_0^\infty e^{ity} y^{-\frac{N+1}{\alpha} - 1} \rho_N(y^{-1/\alpha}) \chi(y) dy.
\]

Since the function \(y \mapsto \rho_N(y^{-1/\alpha}) \chi(y)\) is bounded on \(\mathbb{R}\), it is easy to see that \(R_N(t)\) is a \(C^{\lceil \frac{N+1}{\alpha} \rceil - 1}\) function on \(\mathbb{R}\). We will consider the first term in (3.3).

(i) First, let us consider the case where \(\alpha\) is a rational number.

By applying Lemmas 2.2 and 2.3, we see that the first term in (3.3) can be expressed as

\[
\sum_{n=0}^N \frac{a_n}{\alpha} F_{\gamma_n}(t) \equiv P_N(t) + Q_N(t) \mod C^\infty(\mathbb{R}_+),
\]

with

\[
P_N(t) = \sum_{n \in S_N} \frac{a_n}{\alpha} \tilde{A}_{\gamma_n} t^{\gamma_n}, \quad Q_N(t) = \sum_{j \in T_N} \frac{a_n}{\alpha} \tilde{B}_{\gamma_n} t^{\gamma_n} \log t,
\]

where \(S_N := \{ n \in \mathbb{Z}_+ : \gamma_n \notin \mathbb{Z}, n \leq N - 1 \}\) and \(T_N := \{ n \in \mathbb{Z}_+ : \gamma_n \in \mathbb{Z}, n \leq N - 1 \}\).

Putting (3.3), (3.4) and the above regularities of \(R_N(t)\) together, we can easily show (i) in the theorem.

(ii) Since the case where \(\alpha\) is not rational can be more easily dealt with and the equation (1.6) in the theorem can be obtained in a similar fashion to the case of (i), the proof will be left to the readers.
4. Generalized Fresnel integrals

In this section, we compute the exact values of the improper integrals

\[ \int_0^\infty e^{\pm ix^{1/\alpha}} \, dx = \lim_{R \to \infty} \int_0^R e^{\pm ix^{1/\alpha}} \, dx, \]

where \(0 < \alpha < 1\), which gives a proof of Lemma 2.1. When \(\alpha = 1/2\), the above integrals can be explicitly computed as

\[ \int_0^\infty e^{\pm ix^2} \, dx = \frac{\sqrt{\pi}}{2\sqrt{2}} (1 \pm i) = \frac{\sqrt{\pi}}{2} e^{\pm \frac{\pi}{4} i} \]

by using an elementary method in complex analysis. These equations imply

\[ \int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{\pi}}{2\sqrt{2}}. \]

The above two integrals are called the Fresnel integrals. The integral (4.1) can also be explicitly computed in a similar fashion to the case of the Fresnel integrals (see [8], [9], [10], [12], etc.). Since this computation is essential to our analysis, we will give an exact proof of the following proposition.

**Proposition 4.1.**

\[ \int_0^\infty e^{\pm ix^{1/\alpha}} \, dx = \alpha \int_0^\infty e^{\pm iy^{\alpha-1}}dy = e^{\pm \frac{\alpha}{2}\pi i} \Gamma(\alpha + 1), \]

where \(0 < \alpha < 1\).

**Proof.** We remark that the first equality can be seen by exchanging the integral variable. In this proof, we deal with only the case of sign "+".

First, we prepare the four integral curves as follows.

\begin{align*}
C_1 &: z = x \quad (\varepsilon \leq x \leq R); \\
C_2 &: z = Re^{i\theta} \quad (0 \leq \theta \leq \pi/2); \\
C_3 &: z = iy \quad (\varepsilon \leq y \leq R); \\
C_4 &: z = \varepsilon e^{i\theta} \quad (0 \leq \theta \leq \pi/2),
\end{align*}

where \(\varepsilon, R\) are positive numbers with \(\varepsilon < R\) and each curve admits a direction, which can be determined as in the figure below. The anti-clockwise oriented closed curve \(\sum_{j=1}^4 C_j\) is denoted by \(C\) and the bounded domain surrounded by \(C\) is denoted by \(D\).
Let \( f(z) = e^{iz}z^{\alpha-1} \). Since \( f(z) \) is holomorphic near \( D \), the Cauchy integral theorem implies

\[
\int_C f(z)\,dz = \sum_{j=1}^{4} \int_{C_j} f(z)\,dz = 0.
\]

The integral with respect to \( C_2 \) can be estimated as

\[
\left| \int_{C_2} f(z)\,dz \right| \leq R^\alpha \int_0^\frac{\pi}{2} e^{-R\sin\theta} \,d\theta \leq R^\alpha \int_0^\frac{\pi}{2} e^{-\frac{2R}{\pi}\theta} \,d\theta = \frac{\pi}{2} R^{\alpha-1}(1 - e^{-R})
\]

by using Jordan’s inequality. Note that the above last term tends to 0 as \( R \to \infty \).

In order to estimate the integral with respect to \( C_4 \), we define \( \psi(x) = (1 - e^{-x})/x \) for \( x \neq 0 \) and \( = 1 \) for \( x = 0 \), which is a \( C^\infty \) function on \( \mathbb{R} \). By using \( \psi \), we have

\[
\left| \int_{C_4} f(z)\,dz \right| \leq \epsilon^\alpha \int_0^\frac{\pi}{2} e^{-\frac{2\epsilon}{\pi}\theta} \,d\theta = \frac{\pi}{2} \epsilon^{\alpha-1}(1 - e^{-\epsilon}) = \frac{\pi}{2} \epsilon^\alpha \psi(\epsilon).
\]

Note that the above last term tends to 0 as \( \epsilon \to 0 \).

On the other hand, the integrals with respect to \( C_1, C_3 \) can be expressed as

\[
\int_{C_1} f(z)\,dz = \int_{\varepsilon}^R e^{ix}x^{\alpha-1}\,dx;
\]

\[
\int_{C_3} f(z)\,dz = -e^{\frac{\pi}{2}i} \int_{\varepsilon}^R e^{-x}x^{\alpha-1}\,dx.
\]

Applying (4.4), (4.5), (4.6) to (4.3) and considering the limits \( R \to \infty, \epsilon \to 0 \), we have

\[
\int_0^\infty e^{ix}x^{\alpha-1}\,dx = e^{\frac{\pi}{2}i} \int_0^\infty e^{-x}x^{\alpha-1}\,dx = e^{\frac{\pi}{2}i}\Gamma(\alpha)
\]

\[
= \frac{1}{\alpha} e^{\frac{\pi}{2}i}\Gamma(\alpha + 1).
\]

\[\square\]
Remark 4.2. The above method is not available in the case where $\alpha \geq 1$. The interesting papers [8], [9], [10] deal with this general case and show that the equalities in Proposition 4.1 hold in some sense.

5. In the case of the Laplace integral

Let us consider the Laplace integral

\[ L(t) = \int_{-\infty}^{\infty} e^{-tx} \varphi(x) dx \quad \text{for } t \in \mathbb{R}, \]

where $f$ and $\varphi$ are the same as those in (1.1) and, moreover, $f$ has a minimum at the origin. If there is $k \in \mathbb{N}$ with $k \geq 2$ such that $f'(0) = \cdots = f^{(k-1)}(0) = 0$ and $f^{(k)}(0) \neq 0$, then for any positive integer $N$,

\[ L(t) = \sum_{n=1}^{N-1} \hat{c}_n t^{-n/k} + O(t^{-N/k}) \quad \text{as } t \to +\infty, \]

where $\hat{c}_n$ are constants depending on $f$ and $\varphi$, and the exact value of the first coefficient $\hat{c}_1$ can be explicitly given. The result in (5.2) can be easily shown in a similar fashion to that in the case of oscillatory integrals.

In this section, we will consider the properties of integrals of the form

\[ L_\alpha(t) = \int_{0}^{\infty} \exp \left( -t \frac{1}{x^\alpha} \right) \varphi(x) dx \quad \text{for } t \in \mathbb{R}, \]

where $\alpha$ is a positive real number and $\varphi$ is as in (5.1). In the case of the above integral, we will give a result analogous to Theorem 1.1.

**Theorem 5.1.** (i) If $\alpha > 0$ is a rational number, then for any positive integer $N$,

\[ L_\alpha(t) = \sum_{n=1}^{N} \hat{A}_n t^{n/\alpha} + \hat{B}_n t^{n/\alpha} \log t + \hat{\psi}_N(t) \quad \text{for } t \in \mathbb{R}_+, \]

where $\hat{\psi}_N(t)$ is a $C^{\lceil \frac{N+1}{\alpha} \rceil - 1}$ function on $\mathbb{R}_+$ and

\[ \hat{A}_n = \frac{\varphi^{(n-1)}(0)}{\alpha (n-1)!} \Gamma(-n/\alpha) \text{ if } n/\alpha \notin \mathbb{N}, \quad \hat{A}_n = 0 \text{ if } n/\alpha \in \mathbb{N}, \]

\[ \hat{B}_n = -\frac{1}{\alpha} \frac{\varphi^{(n-1)}(0)}{(n-1)! (n/\alpha)!} \frac{1}{(n/\alpha)!} \text{ if } n/\alpha \in \mathbb{N}, \quad \hat{B}_n = 0 \text{ if } n/\alpha \notin \mathbb{N}, \]

for $n \in \mathbb{N}$.

(ii) If $\alpha > 0$ is not a rational number, then for any positive integer $N$,

\[ L_\alpha(t) = \sum_{n=1}^{N} \hat{A}_n t^{n/\alpha} + \hat{\phi}_N(t) \quad \text{for } t \in \mathbb{R}_+, \]

where $\hat{\phi}_N(t)$ is a $C^{\lceil \frac{N+1}{\alpha} \rceil - 1}$ function on $\mathbb{R}$ and $\hat{A}_n$ are as in (5.5).
Since the proof of the above theorem can be easily given by the same way as that of Theorem 1.1, it is omitted.

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