Abstract
Various assumptions underlying the uniqueness theorems for black holes are discussed. Some new results are described, and various unsatisfactory features of the present theory are stressed.

1 Folklore, Conjectures
A classical result in the theory of black holes, known under the name of “no–hair Theorem”, is the following:

**Theorem 1.1** Let \((M, g)\) be a good electro–vacuum space–time with a non–empty black hole region and with a Killing vector field which is timelike in the asymptotic regions. Then \((M, g)\) is diffeomorphically isometric to a Kerr–Newman space–time or to a Majumdar–Papapetrou space–time.

This Theorem is known to be true under various definitions of “good space–time” (all of which actually imply that it cannot be a Majumdar–Papapetrou space–time), and the purpose of this paper is to discuss various problems related to the as–of–today–definition of “good space–time” needed above. Clearly, one would like to have a definition of “good space–time” as weak as possible. Moreover one would like this definition to have some degree of verifiability and “controllability”, and to be compatible with our knowledge of the structure of the theory gained by some perhaps completely different investigations.

We shall focus here on uniqueness theory of stationary electro–vacuum space–times. It is, however, worthwhile mentioning that substantial progress has been made recently in the understanding of various other models. In particular one should mention various results about uniqueness of perfect fluid models [58, 10], \(\sigma\)–models [46, 47, 48], Einstein–Yang–Mills solutions [13, 41, 70, 71], and dilatonic black holes [57], cf. also [11, 40]. Many of the questions raised here as well as some of the results presented here are also relevant to those other models.

One of the purposes of this paper is to give a careful definition of “good space–time” under which Theorem 1.1 holds, let us therefore start with the
complete basics. A couple \((M, g)\) will be called a space–time if \(M\) is a smooth, connected, Hausdorff, paracompact manifold of dimension 4 and \(g\) is a smooth non–degenerate tensor field of Lorentzian signature, say \(-+++.\) We shall also assume that there exists a smooth Killing vector field \(X\) on \(M\). To the conditions listed here we shall need to add several more conditions, each of which will be discussed below in a separate section:

1.1 Non–degenerate horizons vs. bifurcation surfaces

The uniqueness Theorem of Ruback \[67\] (cf. also Simon \[69\] [Section 4] or Masood–ul–Alam \[56\] for some simplifications of the argument, and Carter \[19\] and references therein for previous results on this problem) is oft en referred to as a uniqueness Theorem for static non–degenerate electro–vacuum black holes. This description is incorrect and misleading. Recall that given a space–time \((M, g)\) with a Killing vector field \(X\) one defines the Killing horizon \(\mathcal{N}[X]\) as the set of points on which \(X\) is null and non–vanishing. [In this definition we do not assume that \(X\) is necessarily timelike at inifinity]. By an abuse of terminology, a connected component of \(\mathcal{N}[X]\) will sometimes also be called a Killing horizon.

It is well known and in any case easily seen that there exists on \(\mathcal{N}[X]\) a function \(\kappa\), called surface gravity, which is defined by the equation

\[
\frac{1}{2} \nabla^a (X^b X_b) \bigg|_{\mathcal{N}[X]} = -\kappa X^a. \tag{1.1}
\]

\(\kappa\) is known to be constant over every connected component of \(\mathcal{N}[X]\) for electro–vacuum space–times \[7\] (cf. also \[49\] for a simple proof in the bifurcate-horizons case). A connected component \(\mathcal{N}[X]_a\) of \(\mathcal{N}[X]\) is said to be degenerate if \(\kappa|_{\mathcal{N}[X]_a} \equiv 0\). Let us mention that for Kerr–Newman metrics we have \(\kappa \neq 0\) as long as \(M^2 > Q^2 + L^2\), where \(M\) is the ADM mass of the metric, \(L\) its ADM angular momentum and \(Q\) the total charge of the electric field. On the other hand the Majumdar–Papapetrou black holes of ref. \[43\] (cf. also Appendix B) have \(\kappa \equiv 0\) throughout the Killing horizon.

Consider the Schwarzschild–Kruskal–Szekeres space–time \((M, g)\), let \(X\) be the standard Killing vector field which equals \(\partial/\partial t\) in the asymptotic regions. Recall that the Killing horizon \(\mathcal{N}[X]\) of \(X\) has four connected components, such that the set \(\mathcal{S}[X] \equiv \mathcal{N}[X] \setminus \mathcal{N}[X]\), where \(\mathcal{N}[X]\) denotes the topological closure of \(\mathcal{N}[X]\), is a smooth two–dimensional embedded submanifold of \(M\). The Killing vector \(X\) vanishes on \(\mathcal{S}[X]\). By definition, such a surface will be called a bifurcation surface of a bifurcate Killing horizon (cf. \[14\] for a justification of this terminology). Thus, given a non–identically vanishing Killing vector field \(X\), a bifurcation surface is a smooth two–dimensional compact embedded surface on which \(X\) vanishes. If such a surface \(\mathcal{S}[X]\) exists, then every connected component of \(\mathcal{N}[X]\) such that \(\mathcal{N}[X] \cap \mathcal{S}[X] \neq \emptyset\) is necessarily a non–degenerate Killing horizon \[14\] (cf. also \[43\]). It follows that the existence of a bifurcation surface \(\mathcal{S}[X]\) implies that of a non–degenerate Killing horizon, but the converse is
not true in general. A rather trivial example is obtained by removing $S[X]$ from a space–time which contains such a surface. A somewhat less trivial example is that of any vacuum space–time with a smooth compact non–degenerate Cauchy horizon with a Killing vector say, spacelike on a Cauchy surface (e.g., the Misner model for Taub–NUT space–times, or the Taub–NUT space–times themselves).

It is natural to look for conditions under which the existence of a non–degenerate Killing horizon does indeed imply that of a bifurcation surface. Some results in this direction have recently been obtained by Rácz and Wald, who have shown that there seems to be no local obstruction to the existence of a bifurcation surface, when a non–degenerate Killing horizon is present. More precisely, assuming that the non–degenerate Killing horizon has a global cross–section (i.e., a two dimensional submanifold which is intersected by every generator of the horizon precisely once), Rácz and Wald show that whenever a bifurcation surface does not exist, then one can make a local extension which contains one. Recall now that the difference between a local extension and a “real one” is the following: to obtain an extension of a space–time $(M, g)$ one constructs a space–time $(\hat{M}, \hat{g})$ and an embedding $i : M \to \hat{M}$ such that $i^*\hat{g} = g$ and $i(M) \neq \hat{M}$; for local extensions one considers a subset $\mathcal{U} \subset M$, and one constructs an extension $(\mathcal{U}, \hat{g})$ of $(\mathcal{U}, g|_\mathcal{U})$. The problem here is that sometimes there is no way of patching $(M, g)$ with $(\mathcal{U}, \hat{g})$ to obtain either a manifold (i.e., Hausdorffness of the resulting topological space might be violated) or a continuous metric. [For example, extensions where continuity of the metric and Hausdorffness cannot be simultaneously ensured can be constructed in the vacuum Einstein class using the polarized Gowdy metrics, exploiting the asymptotic behaviour of the metric near the $t = 0$ set described in [30]. Examples of local extensions in the Killing–horizon–context which cannot be turned into “real ones” have been constructed by Wald (without, however, satisfying any field equations or energy inequalities).]

We wish to emphasize, that the uniqueness theorems of [62, 68, 56] implicitly assume the existence of a compact bifurcation surface in the space–time under

---

1. Note that this example shows that Theorem 5.1 of (the otherwise excellent and in many respects fundamental) Ref. [18] is wrong.
2. In our discussion of the results of [66] we assume that the electro–vacuum equations are satisfied. This hypothesis is not made in [66], which allows for non–constant $\kappa$. It is shown in [66] that such a possibility leads to the existence of a “parallel propagated singularity” of the curvature tensor.
3. In the definition of local extension one sometimes adds some supplementary conditions on $\mathcal{U}$ and $\hat{\mathcal{U}}$ which are irrelevant for the discussion here, cf. e.g. [45, 65, 8, 33] and references therein.
4. R. Wald, private communication.
5. It is far from being obvious that in a general static space–time $(M, g)$ there will exist a hypersurface satisfying the conditions of [67, 69, 56]. By maximal globally hyperbolic we always mean maximal in the class of globally hyperbolic space–times.]
consideration. It must therefore be stressed that the existing uniqueness theory is that of stationary electro–vacuum space–times with bifurcation surfaces and not that of stationary electro–vacuum space–times with a non–degenerate horizon. A way of obtaining uniqueness results in the latter class of space–times would be to prove the following, or some variation thereof:

**Conjecture 1.2** Let \((M, g)\) be a maximal globally hyperbolic, asymptotically flat, electro–vacuum space–time with Killing vector field \(X\) which is timelike in the asymptotically flat regions. Suppose that \((M, g)\) is not the Minkowski space–time, let \(Z\) be a Killing vector field on \(M\) (perhaps, but not necessarily coinciding with \(X\)), and suppose that \(M\) contains a non–empty Killing horizon \(N[Z]\). Then for every non–degenerate connected component \(N_a\) of \(N[Z]\), the set \(S_a \equiv N_a \setminus \mathring{N_a}\) (where \(\mathring{N_a}\) is the topological closure of \(N_a\)) is a non–empty, compact, embedded, smooth submanifold of \(M\).

Let us emphasize that no uniqueness results have been established so far for space–times with a degenerate Killing horizon \((\kappa = 0)\). Thus, as of today the definition of “good” in Theorem 1.1 includes the notion of non–degeneracy. With that condition the conclusion of that Theorem can clearly be strengthened to exclude the Majumdar–Papapetrou black–holes, cf. Theorem 2.4 below.] It is customary to rule out the degenerate horizons as physically uninteresting, as their defining property is of unstable character. [Moreover, they can perhaps be discarded as physically irrelevant by thermodynamical considerations, as \(\kappa\) is related to some kind of “temperature of the black hole” \([7]\) (cf. also \([49]\) and references therein).] For the sake of mathematical completeness one would nevertheless like to have a classification of the degenerate cases as well:

**Problem 1.3** Classify all the maximal, electro–vacuum space–times \((M,g)\) satisfying the following:

1. in \(M\) there exists a Killing vector field \(X\) which is timelike in the asymptotic regions;
2. \(M\) contains a partial Cauchy surface \(\Sigma\) with asymptotically flat ends which is a complete Riemannian manifold with respect to the induced metric;
3. there are no naked singularities in \(J^+(\Sigma)\); and finally
4. \(M\) contains degenerate Killing horizons.

It seems that the only known space–times satisfying the above are those Majumdar–Papapetrou space–times \([55, 63, 43]\) which contain a finite number of black holes; the Véron solutions with an infinite number of black holes described in Appendix \(\text{B}\) are probably excluded by the condition of absence of naked

\[\text{Problem 1.3} \quad \text{Classify all the maximal, electro–vacuum space–times (M,g) satisfying the following:}\]

\[\text{1. in M there exists a Killing vector field X which is timelike in the asymptotic regions;}\]

\[\text{2. M contains a partial Cauchy surface } \Sigma \text{ with asymptotically flat ends which is a complete Riemannian manifold with respect to the induced metric;}\]

\[\text{3. there are no naked singularities in } J^+(\Sigma); \text{ and finally}\]

\[\text{4. M contains degenerate Killing horizons.}\]

It seems that the only known space–times satisfying the above are those Majumdar–Papapetrou space–times \([55, 63, 43]\) which contain a finite number of black holes; the Véron solutions with an infinite number of black holes described in Appendix \(\text{B}\) are probably excluded by the condition of absence of naked

\[\text{Problem 1.3} \quad \text{Classify all the maximal, electro–vacuum space–times (M,g) satisfying the following:}\]

\[\text{1. in M there exists a Killing vector field X which is timelike in the asymptotic regions;}\]

\[\text{2. M contains a partial Cauchy surface } \Sigma \text{ with asymptotically flat ends which is a complete Riemannian manifold with respect to the induced metric;}\]

\[\text{3. there are no naked singularities in } J^+(\Sigma); \text{ and finally}\]

\[\text{4. M contains degenerate Killing horizons.}\]

It seems that the only known space–times satisfying the above are those Majumdar–Papapetrou space–times \([55, 63, 43]\) which contain a finite number of black holes; the Véron solutions with an infinite number of black holes described in Appendix \(\text{B}\) are probably excluded by the condition of absence of naked
singularities, cf. Appendix 3. One of the difficulties which might arise here is, that there is no reason for a degenerate Killing horizon to be smooth (note that \( (1.1) \) guarantees the smoothness of the Killing horizon when \( \kappa \neq 0 \) and when the space–time metric is smooth).

It seems, moreover, that no globally hyperbolic asymptotically flat electro–vacuum space–times are known which possess a complete Cauchy surface and a degenerate Killing horizon: In the Majumdar–Papapetrou black–holes analyzed in \( [3] \) the Killing horizon is also a Cauchy horizon for the (complete) “static’ partial Cauchy surfaces \( t = \text{const} \). [Note, moreover, that those “static” partial Cauchy surfaces are not asymptotically flat in the usual sense: in addition to the asymptotically flat ends they contain ”infinite asymptotically cylindrical necks”.] On the other hand, those partial Cauchy surfaces which intersect the Killing horizon cannot probably be complete because of the singularities present.

To close this Section it should be admitted that there is not much evidence that the Majumdar–Papapetrou space–times play the role advertised in Theorem 1.1. The author bases his belief on the analysis of \( [42, 41] \) (cf. also \( [72] \)), where an argument is given that (under some yet–to–be–specified conditions) for any electro–vacuum space–time we must have \( M \geq \sqrt{Q^2 + P^2} \), where \( M \) is the ADM mass and \( Q, P \) are the global electric and magnetic charges, with the bound being saturated precisely by the Majumdar–Papapetrou space–times. The reader should, however, note that the local analysis of \( [42, 72] \) should be complemented by a global one, related to the questions raised above of existence of appropriately regular space–like surfaces, etc. To the author’s knowledge this has not been done yet.

1.2 Killing vectors vs. isometries

In general relativity there exist at least two ways for a solution to be symmetric: there might exist

1. a Killing vector field \( X \) on the space–time \((M, g)\), or there might exist

2. an action of a (non–trivial) connected Lie group \( G \) on \( M \) by isometries.

Clearly 2 implies 1, but 1 does not need to imply 2 (remove e.g. points from a space–time on which an action of \( G \) exists). In the uniqueness theory, as presented e.g. in \( [18, 21] \), one always assumes that an action of a group \( G \) on \( M \) exists. This is equivalent to the statement, that the orbits of all the (relevant) Killing vector fields are complete. When trying to classify space–times with Killing vector fields, as in Theorem 1.4, one immediately faces the question whether or not the orbits thereof are complete. It is worthwhile emphasizing that there is a constructive method of producing space–times with Killing vectors, by solving a Cauchy problem:
Theorem 1.4 Let $(\Sigma, \gamma, K, A, E)$ be initial data for electro–vacuum Einstein equations, let $(M, g)$ be any globally hyperbolic (electro–vacuum) development thereof. Suppose that there exists a vector field $\hat{X}$ defined in a neighbourhood of $\Sigma$ such that the following equations hold on $\Sigma$:

\begin{align*}
\left\{ \nabla_\mu \hat{X}_\nu + \nabla_\nu \hat{X}_\mu \right\}_\Sigma &= 0, \\
\left\{ \nabla_\alpha \left( \nabla_\mu \hat{X}_\nu + \nabla_\nu \hat{X}_\mu \right) \right\}_\Sigma &= 0, \\
\mathcal{L}_X F_{\mu\nu} \big|_\Sigma &= 0.
\end{align*}

Here $\nabla$ is the covariant derivative of $g$, $\mathcal{L}$ denotes a Lie derivative and $F_{\mu\nu}$ is the electromagnetic field tensor. Then there exists on $M$ a Killing vector field $X$ which coincides with $\hat{X}$ on $\Sigma$.

REMARKS:

1. Let us note that eqs. (1.2)–(1.3) are necessary, as they automatically hold if $\hat{X}$ is a Killing vector which moreover leaves the electromagnetic field invariant.

2. It must be emphasized that equations (1.2)–(1.3) need to hold on $\Sigma$ only. [These equations can be thought of as constraint equations for the initial data for a Killing vector field.] In other words, it is sufficient to satisfy the Killing equations on $\Sigma$ to obtain a solution of the Killing equations on any (not necessarily maximal) globally hyperbolic development thereof.

3. The vacuum equivalent of Theorem 1.4 is well known [61, 35, 28].

4. It would be of interest to obtain an equivalent of Theorem 1.4 for Einstein–Yang–Mils equations (cf. [2] for some related results).

PROOF: Let $X^\mu$ be defined as the unique solution of the problem

\begin{equation}
\Box X^\mu = -R^\mu_\nu X^\nu, \tag{1.4}
\end{equation}

\begin{align*}
X^\mu \big|_\Sigma &= \hat{X}^\mu, \\
\nabla_\alpha X^\mu \big|_\Sigma &= \nabla_\alpha \hat{X}^\mu,
\end{align*}

define

\begin{equation*}
A_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu.
\end{equation*}

From (1.4) and from the Einstein–Maxwell equations one derives the following system of equations

\begin{align*}
\Box A_{\mu\nu} &= -2 \mathcal{L}_X R_{\mu\nu} + 2 R^\lambda_{(\mu A_\nu)\lambda} + 2 R^\alpha_\mu A_{\alpha\beta} A_{\beta\nu}, \tag{1.5} \\
\nabla^\alpha \mathcal{L}_X F_{\mu\nu} &= \nabla_\alpha F_{\beta\nu} A^{\alpha\beta} + F_{\alpha\beta} \nabla^\alpha A^{\beta\nu}. \tag{1.6}
\end{align*}
Note that because of the Einstein equations the tensor field $\mathcal{L}_X R_{\mu\nu}$ can be expressed as a linear combination of $A_{\alpha\beta}$ and $\mathcal{L}_X F_{\alpha\beta}$. The initial data for (1.5)–(1.6) vanish by (1.2)–(1.3), and the vanishing of $A_{\alpha\beta}$ and of $\mathcal{L}_X F_{\mu\nu}$ follows. □

Recall now that given a Cauchy data set $(\Sigma, \gamma, K, A, E)$ for electro–vacuum Einstein equations, there exists a unique up to isometry vacuum space–time $(M, g)$, which is called the maximal globally hyperbolic vacuum development of $(\Sigma, \gamma, K)$, with an embedding $i: \Sigma \to M$ such that $i^* g = \gamma$, and such that $K$ corresponds to the extrinsic curvature of $i(\Sigma)$ in $M$. $(M, g)$ is inextendible in the class of globally hyperbolic space–times with a vacuum metric. This class of space–times is highly satisfactory to work with, as they can be characterized by their Cauchy data induced on some Cauchy surface. Moreover, the property of maximality seems to be a natural notion of completeness for globally hyperbolic space–times, and it is of interest to enquire about completeness of Killing orbits in such space–times. Before discussing that question, it seems appropriate to introduce some definitions:

Definition 1.5 We shall say that an initial data set $(\Sigma, \gamma, K, A, E)$ for electro–vacuum Einstein equations is asymptotically flat if $(\Sigma, \gamma)$ is a complete connected Riemannian manifold (without boundary), with $\Sigma$ of the form

$$\Sigma = \Sigma_{\text{int}} \bigcup_{i=1}^{I} \Sigma_i,$$

for some $I < \infty$. Here we assume that $\Sigma_{\text{int}}$ is compact, and each of the ends $\Sigma_i$ is diffeomorphic to $\mathbb{R}^3 \setminus B(R_i)$ for some $R_i > 0$, with $B(R_i)$ — coordinate ball of radius $R_i$. In each of the ends $\Sigma_i$ the fields $(g, K, A, E)$ are assumed to satisfy the following inequalities (after performing a duality rotation of the electromagnetic field, if necessary)

$$|g_{ij} - \delta_{ij}| + |r \partial_k g_{ij}| + |r K_{ij}| + |A_i| + |r \partial_i A_j| + |r E_i| \leq Cr^{-\alpha},$$

(1.8)

for some positive constant $C$ and some $\alpha > 0$, with $r = \sqrt{\sum (x^i)^2}$.

To motivate the next definition, consider a space–time with some number of asymptotically flat ends, and with a black hole region. In such a case there might exist a Killing vector field defined in, say, the domain of outer communication (cf. the next Section for a definition) of the asymptotically flat ends. It could, however, occur, that there is no Killing vector field defined on the whole space–time — an example of such a space–time has been considered by Brill in his construction of a space–time in which no asymptotically flat maximal surfaces exist. Alternatively, there might be a Killing vector field defined everywhere, however, there might be some non-asymptotically flat ends in $\Sigma$. [As an example, consider a spacelike surface in the Schwarzschild–Kruskal–Szekeres space–time in which one end is asymptotically flat, and the second
is “asymptotically hyperboloidal”. In such cases one would still like to claim that the orbits of $X$ are complete in the exterior region. To accommodate such behavior we introduce the following:

**Definition 1.6** Consider a stably causal Lorentzian manifold $(M, g)$ with an achronal spacelike surface $\Sigma$. Let $\Sigma \subset \hat{\Sigma}$ be a connected submanifold of $\hat{\Sigma}$ with smooth compact boundary $\partial \Sigma$, and let $(\gamma, K)$ be the Cauchy data induced by $g$ on $\Sigma$. Suppose finally that there exists a Killing vector field $X$ defined on $D(\Sigma)$ (here $D(\Sigma)$ denotes the domain of dependence of an achronal set $\Sigma$; we use the convention in which $D(\Sigma)$ is an open set). We shall say that $(\Sigma, \gamma, K, A, E)$ are Cauchy data for an asymptotically flat exterior region in a (non-degenerate) black-hole space-time if the following hold:

1. The closure $\bar{\Sigma} \equiv \Sigma \cup \partial \Sigma$ of $\Sigma$ is of the form \([1.7]\), with $\Sigma_{\text{int}}$ and $\Sigma_i$ satisfying the topological requirements of Definition \([1.5]\).
2. $(\Sigma, \gamma, K, A, E)$ satisfy the fall-off requirements of Definition \([1.3]\).
3. [From the Killing equations it follows that $X$ can be extended by continuity to $D(\Sigma)$.] We shall moreover require that $X$ be tangent to $\partial \Sigma$.

The above definition allows for space-times in which $\Sigma$ is a surface with boundary, the boundary in question being a bifurcation surface of a Killing horizon. The notion of non-degeneracy referred to in Definition \([1.6]\) above is related to the non-vanishing of the surface gravity of the horizon: Indeed, it follows from [66] that in situations of interest the behaviour described in Definition \([1.6]\) can only occur if the surface gravity of the horizon is constant on the horizon, and does not vanish.

The following is a straightforward generalization of the Theorem proved in [29], no details will be given:

**Theorem 1.7** Let $(M, g)$ be a smooth, electro-vacuum, maximal globally hyperbolic space-time with an achronal spacelike hypersurface $\Sigma$ and with a Killing vector $X$ (defined perhaps only on $D(\Sigma)$) such that $X$ approaches a non-zero multiple of the unit normal to $\Sigma$ as $r \to \infty$. Suppose that either

1. the data set $(\Sigma, \gamma, K, A, E)$ is asymptotically flat, or
2. $(\Sigma, \gamma, K, A, E)$ are Cauchy data for an asymptotically flat exterior region in a (non-degenerate) black-hole space-time.

Then the orbits of $X$ are complete in $D(\Sigma)$.

---

9The remaining results of [29] can be similarly generalized to the electro-vacuum case.

10Theorem 1.7 holds true for any Killing vector field, provided that asymptotic conditions somewhat stronger than those of Definition \([1.3]\) are assumed, cf. [29] for details.
Consider then a stationary black hole space–time \((M, g)\) in which an asymptotically flat Cauchy surface exists but in which the Killing orbits are not complete: Theorem 1.7 shows that \((M, g)\) can be enlarged to obtain a space–time with complete Killing orbits.

In conclusion, the results presented in this Section show that the hypothesis of completeness of Killing orbits usually made in uniqueness Theorems is unnecessary, as long as one restricts oneself to maximal globally hyperbolic space–times with well behaved Cauchy surfaces.

1.3 Asymptotic flatness, stationarity

There are at least three different ways of defining asymptotic flatness:

1. via existence of an asymptotically flat Cauchy surface, or
2. via existence of asymptotically Minkowskian coordinates, and finally
3. using conformal techniques.

More precisely, let \((M, g)\) be an electro–vacuum space–time. We shall say that a submanifold \(\Sigma\) with boundary is an asymptotically flat three–end in \(M\) if \(\Sigma\) is diffeomorphic to \(\mathbb{R}^3 \setminus B(R)\) for some \(R > 0\), where \(B(R)\) denotes a closed coordinate ball of radius \(R\), and in the local coordinates on \(\Sigma\) the fields \((g, K, A, E)\) satisfy the fall–off conditions (1.8) of Definition 1.5.

We shall say that an open submanifold \(\hat{M} \subset M\) is an asymptotically flat stationary four–end of \(M\) if \(\hat{M}\) is diffeomorphic to \(\mathbb{R} \times (\mathbb{R}^3 \setminus B(R))\), and in the local coordinates on \(\hat{M}\) the metric \(g_{\mu\nu}\), the electromagnetic potential \(A_\mu\) and the electromagnetic field \(F_{\mu\nu}\) satisfy (after performing a duality rotation of the electromagnetic field, if necessary)

\[
\left| g_{\mu\nu} - \eta_{\mu\nu} \right| + \left| r \partial_r g_{\mu\nu} \right| + \left| A_\mu \right| + \left| r F_{\mu\nu} \right| \leq Cr^{-\alpha}, \quad (1.9)
\]

\[
\partial_t g_{\mu\nu} = \partial_t F_{\mu\nu} = 0, \quad (1.10)
\]

for some constants \(C, \alpha > 0\). Here we have \(r = \sqrt{\sum (x_i)^2}\), as before.

Clearly a space–time with an asymptotically flat stationary four–end \(\hat{M}\) also contains an asymptotically flat three–end \(\Sigma\) and a Killing vector which is timelike on \(\Sigma\), but the converse needs not to be true. This is due to the fact that a timelike Killing vector field \(X\) defined on \(\Sigma\) might asymptotically approach a null (rather than timelike) Killing vector as \(r\) goes to infinity, say \(X \to r \to \infty \partial_t - \partial_\Sigma\). An explicit example of such a space–time (not satisfying any reasonable field equations or energy conditions) can be found in the Appendix A of [32]. When imposing electro–vacuum field equations such a behaviour seems

\(^{11}\)We use the (PDE motivated) convention that a submanifold \(\Sigma\) with boundary does not include its boundary \(\partial \Sigma \equiv \Sigma \setminus \Sigma\).
to be rather improbable. For the sake of completeness of understanding space–
time with Killing vectors which are timelike in the asymptotic regions it would
be of interest to prove the following:

**Conjecture 1.8** Let \((M, g)\) be an electro–vacuum space–time with an asymp-
totically flat three–end \(\Sigma\) and a Killing vector field \(X\) which is timelike on \(\Sigma\).
After performing a boost of \(\Sigma\) if necessary, \(X\) approaches a non–zero multiple
of the unit normal to \(\Sigma\) as \(r\) goes to infinity.

The following gives a plausibility argument for Conjecture 1.8: Suppose that
the Killing vector \(X\) asymptotically approaches a null vector at \(i^0\). Under these
circumstances one would expect the ADM four–momentum to be parallel to
the Killing vector \(X\), hence null. This is, however, not possible when energy
conditions are satisfied [3]. We find it likely that a proof of Conjecture 1.8 can
be given by filling in the details in this argument.

Under the conditions and conclusions of Conjecture 1.8 it is rather easy to
show that \(M\) will also contain an asymptotically flat stationary four–end \(\hat{M}\),
provided that the orbits of \(X\) through \(\Sigma\) are complete (cf. e.g. [32] [Appendix A]). In this case \(\hat{M}\) can be defined by the equation

\[
\hat{M} = \bigcup_{t \in \mathbb{R}} \phi_t(\Sigma),
\]

where \(\phi_t\) is the flow generated by the Killing vector field \(X\). This together with
the discussion of the previous Section shows the equivalence of the "3 + 1 Defi-
nition" and the "4–dim Definition" of asymptotic flatness for maximal globally
hyperbolic electro–vacuum space–times with an asymptotically timelike Killing
vector \(X\), modulo the proviso of the validity of the conclusion of Conjecture 1.8.

As far as the conformal approach is concerned, we have the following:

**Proposition 1.9** Suppose that an electro–vacuum space–time \((M, g)\) contains
an asymptotically flat stationary four–end \(\hat{M}\). Then \(M\) admits a conformal
completion satisfying the completeness requirements of [32].

**Proof:** A bootstrap of the stationary field equations in \(\hat{M}\) shows that one
can find a coordinate system and an electromagnetic gauge in which (1.10) holds
and moreover the fields satisfy\(^{13}\)

\[
|g_{\mu\nu} - \eta_{\mu\nu}| + r|\partial_i g_{\mu\nu}| + |A_\mu| + |r F_{\mu\nu}| \leq C r^{-1},
\]

(1.12)

The results of Ref. [38] and (1.12) show that \(\hat{M}\) admits a smooth conformal
completion at \(i^0\). The Appendix to [34] gives then an explicit construction of
the conformal completion at null infinity.

\(^{12}\)This result should probably follow by, e.g., a repetition of the analysis of [3].

\(^{13}\)In vacuum this observation has been independently done by D. Kennefick and N.
O’Murchadha [51].
A converse of Proposition 1.9 can be proved again under some provisos. Indeed, if the Ricci tensor falls off fast enough (in the sense of the note added in proof (3) of [3]) in the asymptotic end in question near a connected component \( \mathcal{J} \) of \( \mathcal{J} \) (and this decay probably follows from the peeling property of the electromagnetic field) then Bondi coordinates near \( \mathcal{J} \), and subsequently asymptotically Minkowskian coordinates near \( \mathcal{J} \) can be constructed. If the Killing vector does not approach an asymptotically null vector, then this construction gives an asymptotically flat stationary four–end \( \mathcal{M} \).

We wish to stress that the field equations played a significant role in the discussion above. Recall that one does not expect a general asymptotically Minkowskian space–time to admit smooth conformal completions [36, 23, 54, 1, 31]. As shown in Appendix A, the same is true for general asymptotically Minkowskian stationary space–times when one does not impose any field equations. When a stationary space–time admits a \( \mathcal{J} \) which is merely polyhomogeneous rather than smooth, i.e., when the metric has \( r^{-3} \log r \) terms in its asymptotic expansion for large \( r \), various technical difficulties arise when asymptotic flatness is defined in terms of a conformal completion and several of the results discussed in e.g. [45] require reexamination. It follows that the question of equivalence of the conformal definition of asymptotic flatness with the other ones requires a case by case analysis for each matter model. All these difficulties are, however, avoided, when using the definitions of asymptotic flatness based on existence of appropriate coordinate systems, as discussed above.

It should be noted that the question of definition of asymptotic flatness is related to that of the definition of the black–hole region. In [15, 16] one considers connected components \( \mathcal{J}^\pm_i \) of \( \mathcal{J} \) and one defines the black hole region \( \mathcal{B}_i \) as \( M \setminus J^-(\mathcal{J}^+_i) \). Similarly the white hole region \( \mathcal{W}_i \) is defined as \( M \setminus J^+(\mathcal{J}^-_i) \), and the domain of outer communication \( \langle \mathcal{J}_i \rangle \) is defined as \( J^-(\mathcal{J}^+_i) \cap J^+(\mathcal{J}^-_i) \). On the other hand, in [32] one considers an asymptotically flat stationary four–end \( \mathcal{M}_i \) and then the black hole, white hole, etc., are defined as

\[
\mathcal{B}_i = M \setminus J^-(\mathcal{M}_i), \quad \mathcal{W}_i = M \setminus J^+(\mathcal{M}_i), \\
\langle \mathcal{J}_i \rangle = \left\{ \cup_i J^-(\mathcal{M}_i) \right\} \cap \left\{ \cup_i J^+(\mathcal{M}_i) \right\}. \tag{1.13}
\]

Here the \( \mathcal{M}_i \)'s are defined as in (1.11), starting from the asymptotic three–ends \( \Sigma_i \) of \( \Sigma \). As discussed in [32], these definitions coincide with the ones based on conformal completions in vacuum; from what it said here it follows that this is also true in the electro–vacuum case modulo some provisos discussed above. The advantage of the conformal definition of black hole, etc., is that it carries over immediately to the non–stationary case [14].

---

14Note that the conformal definitions of black hole, etc., still make sense with conformal completions of poor differentiability. For such completions, however, various properties of \( \mathcal{B} \), etc., should be carefully reexamined.
2 Two uniqueness Theorems

2.1 The angular velocity of bifurcation surfaces of non–degenerate Killing horizons

Before presenting a uniqueness theorem for black holes, let us report here the following unpublished result of Wald which allows us to define the angular velocity for bifurcation surfaces of non–degenerate Killing horizons, and (in the case of non–vanishing angular velocities) a preferred (non–trivial) Killing vector \( Y \) with periodic orbits (no field equations are assumed below):

**Proposition 2.1** Consider a space–time \((M, g)\) with a Killing vector field \( X \) with complete orbits which contains an asymptotically flat stationary four–end \( \hat{M} \). Suppose moreover that there exist in \( M \) a compact, smooth, two dimensional (not necessarily connected) submanifold \( S \) with the following properties: for every connected component \( S_j \), \( j = 1, \ldots, J \) of \( S \) there exists a Killing vector \( Z_j \) with complete orbits in \( M \) which vanishes on \( S_j \). Then either \( X \) coincides with all the \( Z_j \)'s, in which case we set \( \Omega_j = 0 \) for all \( j \), or there exists on \( M \) a Killing field \( Y \) such that

1. \( Y \) commutes with \( X \),
2. \( Y \) is complete and has periodic orbits with period \( 2\pi \), and
3. for each \( j = 1, \ldots, J \) there exists \( \Omega_j \in \mathbb{R} \) such that, rescaling \( Z_j \) if necessary, we have \( Z_j = X + \Omega_j Y \).

It might be of some interest to note, that the arguments of Wald show moreover the following:

1. If any connected component of \( S \), say \( S_1 \), is not diffeomorphic to a torus or a sphere, then every Killing vector has to vanish on \( S_1 \) (this has already been observed in [38]). It follows that in such a case \((M, g)\) can have (up to proportionality) at most one Killing vector the orbits of which are complete.
2. If any connected component of \( S \), say \( S_1 \), is diffeomorphic to a sphere, then either there are at most two linearly independent Killing vectors with complete orbits in \( M \), or \( M' \) is static, spherically symmetric, and the asymptotically stationary Killing vector \( X \) vanishes on \( S \).
3. If a connected component of \( S \), say \( S_1 \), is diffeomorphic to a torus, then any Killing vector with complete orbits must have periodic orbits on \( S_1 \); moreover \((M, g_{ab})\) can have at most two linearly independent Killing vectors with complete orbits.

Proposition 2.1 can be used as a starting point for a classification of stationary space–times with bifurcation surfaces. The constant \( \Omega_i \) defined in Proposition 2.1 will be called the angular velocity of the \( i \)'th connected component of the black hole.
2.2 A uniqueness Theorem for black holes

In this Section we shall present a version of Theorem 1.1. The main steps of the proof are the Sudarsky–Wald staticity theorem [71] (cf. also [70]) and the Bunting–Masood–ul–Alam–Ruback [17, 67] uniqueness theorem for static electro–vacuum black holes. Let us start with a Definition:

Definition 2.2 (Condition C1) A quadruple $(M, g, X, \Sigma)$ will be said to satisfy the condition C1 if $(M, g)$ is a maximal globally hyperbolic electro–vacuum space–time with electromagnetic field $F$ and if moreover the following conditions are satisfied:

1. $\Sigma$ is a simply connected[15] spacelike hypersurface in $M$ satisfying the requirements of Definition 1.6.
2. $X$ is a Killing vector field defined on $D(\Sigma)$ such that $\mathcal{L}_X F = 0$. Moreover there exist constants $\alpha_i \in \mathbb{R}$ such that on every asymptotic three–end $\Sigma_i$ of $\Sigma$ we have (after performing a Lorentz “boost” of $\Sigma_i$ if necessary)

$$X \bigg|_{\Sigma_i} \to r \to \infty \alpha_i n, \quad (2.1)$$

where $n$ is the unit future directed normal to $\Sigma$. We shall normalize[16] the $\alpha_i$’s so that $\alpha_1 = 1$.

3. For every connected component $\partial \Sigma_a$ of $\partial \Sigma$ there exists a Killing vector $Z_a$ defined on $D(\Sigma)$ which vanishes on $\partial \Sigma_a$. We also require $\mathcal{L}_{Z_a} F = 0$.
4. Let the domain of outer communication $\langle \langle J(\Sigma) \rangle \rangle$ be defined by (1.14). We shall require that $D(\Sigma) \subset \langle \langle J(\Sigma) \rangle \rangle.$

(2.2)

In other words, $\Sigma$ and its domain of dependence $D(\Sigma)$ lie entirely outside the black hole and the white hole regions.

If $(M, g, X, \Sigma)$ satisfy the condition C1, then every Killing vector defined on $D(\Sigma)$ has complete orbits. We can consequently use[17] Proposition 2.1 to define the angular velocities $\Omega_a$, and to deduce the existence of a Killing vector $Y$ with periodic orbits in $D(\Sigma)$ when at least one of the $\Omega_a$’s is nonzero. We have the following preliminary result:

Proposition 2.3 Let $(M, g, X, \Sigma)$ satisfy the condition C1. Then

15 The hypothesis of simple connectedness of $\Sigma$ is used in the Theorems below to ensure the existence of a global gauge in which (2.3) holds. This hypothesis is therefore unnecessary in vacuum, or in situations in which one knows a priori (e.g., by assumption, as in [71]) that a global gauge satisfying (2.3) exists.

16 The non–vanishing of the $\alpha_i$’s for a non–trivial Killing vector field $X$ is a well known consequence of the Killing equations.

17 Strictly speaking, Proposition 2.1 has been formulated in a way which assumes the existence of Killing vectors defined globally on $M$. It can, however, be seen that its assertions hold true in situations under consideration.
1. There exists in $\mathcal{M}$ an asymptotically flat maximal hypersurface with boundary $\tilde{\Sigma}$, diffeomorphic to $\Sigma$, such that

$$\partial \tilde{\Sigma} = \partial \Sigma, \quad \mathcal{D}(\tilde{\Sigma}) = \mathcal{D}(\Sigma).$$

2. $X$ is transverse to $\tilde{\Sigma}$; in particular all the $\alpha_i$’s have the same sign and the gauge condition

$$\mathcal{L}_X A_\mu = 0$$

(2.3)

can be introduced, with the (perhaps locally defined) potentials $A_\mu$ satisfying the fall–off conditions $|f|_{1,12}$, and being continuous up–to–boundary on $\Sigma$.

3. If $X|_{\Sigma} \neq 0$, then the canonical angular momentum $J_i = J_i[Y, \tilde{\Sigma}]$ of each of the asymptotic three–ends $\tilde{\Sigma}_i$ is well–defined and finite. Here $Y$ is defined by Proposition 2.1.

Proof: Point 1 together with transversality of $X$ to $\tilde{\Sigma}$ has been proved in [32]. The existence of the (perhaps local) gauge (2.3) follows from the fact that $X$ is transverse to $\Sigma$. Note, however, that because $X$ is tangent to $\partial \Sigma$ the gauge (2.3) could become singular at $\partial \Sigma$. This is not the case, and can be seen as follows: Near a connected component of $\partial \Sigma$ one can introduce “Rindler–type” coordinates adopted to the action of the group of isometries generated by $X$, as in the proof of Lemma 4.1 of [32]. In these coordinates one can write a fairly explicit formula for a function $\lambda$ such that $A_\mu + \partial_\mu \lambda$ satisfies (2.3), and the uniform boundedness of the gauge potential in the new gauge readily follows.

To prove point 3, recall that the canonical angular momentum consists of two parts (cf. e.g. [24] or [70]), one being the standard ADM angular momentum and the second coming from the electro–magnetic field. To take care of the ADM part, note that by Proposition 1.9 in an appropriate coordinate system the fields satisfy the fall–off conditions $|f|_{1,12}$. Moreover uniqueness results for maximal surfaces show that $Y$ must be tangent to $\partial \Sigma$, which implies that

$$\mathcal{L}_X \gamma = \mathcal{L}_X K = 0.$$  

(2.4)

[Here $\gamma$ and $K$ are the induced metric and the extrinsic curvature of $\tilde{\Sigma}$.] The correctness of definition of the ADM angular momentum follows from (2.4) and from (2.3). To take care of the electro–magnetic contribution to $J_i$, let $\phi_t[Y]$ be the one parameter group of diffeomorphisms generated by $Y$. By assumption the (perhaps duality rotated) gauge bundle is trivial on each asymptotic end $\tilde{\Sigma}_i$, and we can choose $\partial \Sigma_i$ to be invariant under $\phi_t[Y]$. Let $\hat{A}$ be any gauge–potential

$^{18}$Actually $\mathcal{D}(\Sigma)$ can be foliated by such surfaces, we shall, however, not need this result.

$^{19}$We do not assume here that the $U(1)$ bundle associated to the electro–magnetic field is trivial. Eq. (2.3) should be viewed as a condition how to propagate some local trivialization of the gauge bundle on $\Sigma$ to a neighbourhood of $\Sigma$. When the gauge bundle is trivial, then (2.3) can be imposed globally because of the assumed simple–connectedness of $\Sigma$.

$^{20}$Here the Lie derivative of $A$ is defined formally as that of a vector field.
satisfying (2.3) and the fall-off conditions (1.12), define
\[ A = \frac{1}{2\pi} \int_0^{2\pi} \phi_t[Y]^* \tilde{A} dt. \] (2.5)

We have
\[ dA = \frac{1}{2\pi} \int_0^{2\pi} d\{\phi_t[Y]^* \tilde{A}\} dt = \frac{1}{2\pi} \int_0^{2\pi} \phi_t[Y]^* d\tilde{A} dt = \frac{1}{2\pi} \int_0^{2\pi} \phi_t[Y]^* F dt = F, \]
so that \( A \) is indeed a potential for \( F \). Moreover we clearly have
\[ \mathcal{L}_Y A = 0, \quad \mathcal{L}_X A = 0, \]
the latter equation holding because \( X \) and \( Y \) commute. The finiteness of the electromagnetic contribution to the canonical angular momentum follows now from eq. (27) of [71].

\[ \textbf{Theorem 2.4} \] Let \((M,g,X,\Sigma)\) satisfy the condition \( C_1 \). Assume moreover that the \( U(1) \) bundle associated to the electromagnetic field can be trivialized by performing a duality rotation and that
\[ \Omega_1 = \ldots = \Omega_I = \Omega. \] (2.6)

Here \( I \) is the number of connected components of \( \partial \Sigma \) and the \( \Omega_i \)'s are their angular velocities, as defined by Proposition 2.1. Then:

1. We necessarily have
\[ \Omega(J_1 + \cdots + J_K) \geq 0, \] (2.7)
where \( K \) is the number of asymptotic ends of \( \Sigma \), and the \( J_i \)'s are the canonical angular momenta as defined in Proposition 2.3.

2. If the equality in (2.7) is attained, then \( I = K = 1 \) and \( \langle J_S \rangle \) is isometrically diffeomorphic to a connected component of the domain of outer dependence of a (perhaps electrically and magnetically charged) Reissner–Nordström black hole.

**Proof:** Proposition 2.3 shows that the arguments of [70] or [71] apply. [The generalization of those arguments to the case in which several asymptotic ends are present, and in which \( \partial \Sigma \) has several connected components but (2.6) holds, presents no difficulties.] In particular when (2.7) is actually an equality the staticity and the vanishing of the electromagnetic field follow from [70, 71]. Point 2 follows then from Ruback’s uniqueness theorem [67].
It would be of interest to remove the condition (2.6) above, the hypothesis of simple connectedness of \( \Sigma \), as well as the condition of triviality \( U(1) \) bundle associated to the electro–magnetic field.

2.3 A uniqueness Theorem for space–times without black holes

A well known theorem of Lichnerowicz [53] asserts that a strictly stationary vacuum space–time with a hypersurface satisfying the conditions of Definition 1.5 and with one asymptotically flat three–end is necessarily flat. Here strictly stationary is defined as the requirement that the Killing vector \( X \) approaches asymptotically the unit normal to \( \Sigma \) and is timelike everywhere. In this Section we shall present an extension of this Theorem to the case where 1) many asymptotically flat ends are potentially allowed, 2) the Killing vector is not \textit{a priori} assumed to be timelike everywhere, and 3) a potentially non–vanishing electro–magnetic field is allowed. The proofs are mainly based on the results of Sudarsky and Wald, and run very much in parallel with those of the previous Section. The results in this Section are actually rather more elegant, as one avoids all the technicalities related to the bifurcation surfaces previously needed. Let us again start with a Definition:

\textbf{Definition 2.5 (Condition C2) } A quadruple \((M, g, X, \Sigma)\) will be said to satisfy the condition \( C_2 \) if \((M, g)\) is a maximal globally hyperbolic electro–vacuum space–time with electromagnetic field \( F \) and if moreover the following conditions are satisfied:

1. \( \Sigma \) is a simply connected spacelike hypersurface in \( M \) satisfying the requirements of Definition 1.5.

2. \( X \) is a Killing defined on \( D(\Sigma) \) such that \( \mathcal{L}_X F = 0 \). Moreover there exist constants \( \alpha_i \in \mathbb{R} \) such that on every asymptotic three–end \( \Sigma_i \) of \( \Sigma \) we have (after performing a Lorentz “boost” of \( \Sigma_i \) if necessary)

\[
X \mid_{\Sigma_i} \rightarrow_{r \rightarrow \infty} \alpha_i n ,
\]

where \( n \) is the unit future directed normal to \( \Sigma \). We shall normalize the \( \alpha_i \)’s so that \( \alpha_1 = 1 \).

3. Let the domain of outer communication \( \langle J(\Sigma) \rangle \) be defined by (1.14). We shall require that

\[
D(\Sigma) \subset \langle J(\Sigma) \rangle .
\]

In other words, there is no black hole or white hole in \( D(\Sigma) \).

\textsuperscript{21}It appears that the hamiltonian formalism used in [70] assumes at the outset the triviality of the gauge bundle. This formalism can, however, be replaced by \textit{e.g.} that of [50] where no such restriction is needed.
From one has, in parallel to Proposition 2.3, the following (the existence of the potentials $V_i$ below follows from (1.12) and from [70]):

**Proposition 2.6** Let $(M, g, X, \Sigma)$ satisfy the condition $C_2$. Then

1. $M$ can be foliated by a family of asymptotically flat maximal hypersurface (without boundary) $\bar{\Sigma}_r$, $r \in \mathbb{R}$, which are Cauchy surfaces for $D(\Sigma)$.

2. $X$ is transverse to $\bar{\Sigma}$; in particular all the $\alpha_i$’s have the same sign and the gauge condition

$$\mathcal{L}_X A_\mu = 0$$

(2.10)
can be introduced. In this gauge the limits

$$V_i \equiv \lim_{r \to \infty} A_0 \bigg|_{\Sigma_i}$$

exist and are angle-independent constants.

The main result of this Section is the following:

**Theorem 2.7** Let $(M, g, X, \Sigma)$ satisfy the condition $C_2$. Assume moreover that the $U(1)$ bundle associated to the electromagnetic field can be trivialized by performing a duality rotation. Then:

1. The maximal surfaces $\Sigma_\tau$ foliating $D(\Sigma)$ (cf. Proposition 2.4, Point 1) have vanishing extrinsic curvature, and the pull-back of the electromagnetic field two-form $F$ to each $\Sigma_\tau$ vanishes.

2. If moreover

$$\sum_{i=1}^{K} \alpha_i V_i Q_i = 0$$

(2.11)

then $(M, g)$ is the Minkowski space–time. Here $K$ is the number of asymptotic three–ends of $\Sigma$, $V_i$ is the electrostatic potential of the $i$’th end $\bar{\Sigma}_i$ as defined in Proposition 2.4, Point 2, $Q_i$ its total charge, and the $\alpha_i$’s are the constants of Definition 2.5.

**Remark:** (2.11) necessarily holds when we have

$$\alpha_1 V_1 = \ldots = \alpha_K V_K .$$

Indeed, under the conditions of Definition 2.5 the total charge $\sum_i Q_i$ necessarily vanishes, and (2.11) follows.

**Proof:** Proposition 2.6 allows one to apply the results of [70, 71], so that point 1 immediately follows. (2.10) and eq. (41) of [71] (appropriately generalized to the case of a finite number of asymptotic ends) show that $F \equiv 0$, so that $(D(\Sigma), g)$ is vacuum. As shown in [1] (cf. also [1]) in each of the asymptotic ends the Komar integral of the Killing vector $X$ is equal to $\alpha_i m_i$, where $m_i$ is the
ADM mass of the $i$’th end. In vacuum the divergence of the Komar integrand vanishes, so that we obtain

$$\sum_{i=1}^{K} \alpha_i m_i = 0.$$  

By Proposition 2.6, point 2, all the $\alpha_i$s have the same sign, and the result follows from the positive energy theorem. $\square$

It would be desirable to remove condition (2.11) above, the hypothesis of simple connectedness of $\Sigma$, as well as the hypothesis of the triviality of the $U(1)$ bundle associated to the electromagnetic field.

3 Folklore, Conjectures – continued

3.1 Rigidity and analyticity

The results discussed up to now allow one to obtain a reasonably satisfactory version of Theorem 1.1 for non–rotating black holes, cf. Theorem 2.4 above. To the list of problems listed in Section 1 we wish to add some further problems which arise when considering the rotating black holes. The key results concerning those are

1. Hawking’s rigidity Theorem, and
2. the Carter–Bunting–Mazur uniqueness Theorem.

Recall that Hawking [45] has proved that the isometry group of an analytic, electro–vacuum, stationary, non–static, asymptotically flat space–time with a complete Killing vector $X$ and which contains a black–hole must be at least two–dimensional (cf. [45] for a precise description of the notions used). It has already been pointed out by Carter [21] that the hypothesis of analyticity here is rather unsatisfactory: Indeed, it is well known that in regions where a Killing vector is timelike the metric must be analytic (in appropriate coordinates) [62]. However, this needs not to be true in those regions in which the Killing vector becomes null or spacelike. As the Killing vector cannot be timelike on the black–hole boundary, the hypothesis that the metric be analytic up–to–and–including the even horizon made in [45] has no justification. [Even in space–times without ergoregions, in which the “stationary” Killing vector becomes null at the event horizon, the metric needs not to be analytic up to the horizon.]

A simple example illustrating the fact, that an analytic function needs not to be analytic up to boundary is the following: Let $g$ be any smooth real valued function defined on $\partial B(1)$, where $B(1)$ denotes the closed unit disc in $\mathbb{R}^2$, and suppose that $g$ is not real analytic. Let $f : B(1) \to \mathbb{R}$ be a solution to the equation $\left\{ (\partial/\partial x)^2 + (\partial/\partial y)^2 \right\} f = 0$ such that $f|_{\partial B(1)} = g$. $f$ is real analytic on $\text{int} \, B(1)$ and clearly does not extend to an analytic function on $B(1)$.] Perhaps
the most important open problem in the uniqueness theory of black holes is therefore the following:

**Problem 3.1** Prove Hawking’s rigidity Theorem without assuming analyticity of \((M, g)\), or construct a counterexample.

### 3.2 Isometry groups in asymptotically flat space–times

Whatever the status of Hawking’s rigidity for non–analytic space–times, it is of interest to classify those stationary asymptotically flat space–times which have more than one Killing vector. Here, basing on what has been said above, one expects that the solutions will be either flat, or spherically symmetric, or axisymmetric. In other words, if there exists a Killing vector \(X\) which is timelike in the asymptotically flat ends, then there will be at least one more Killing vector field \(Y\) which

1. has an axis of symmetry (i.e., the set \(\{p : Y(p) = 0\}\) is non–empty), and
2. the orbits of which are periodic.

A Killing vector satisfying the above will be called an *axial Killing vector*. We believe that the following should be true:

**Conjecture 3.2** Let \((M, g)\) be a maximal globally hyperbolic electro–vacuum space–time with an asymptotically flat spacelike surface \(\Sigma\) satisfying the requirements of Definition 1.5 or 1.6, and with a Killing vector \(X\) which is timelike in the asymptotically flat three–ends of \(\Sigma\). Let \(G_0\) be the connected component of the identity of the group of all isometries of \((D(\Sigma), g|_{D(\Sigma)})\). Then

1. \(G_0 = \mathbb{R} \times SO(2)\), with axial generator of the \(SO(2)\) factor, or
2. \(G_0 = \mathbb{R} \times SO(3)\), with two–dimensional spheres as principal orbits of the \(SO(3)\) factor, or
3. \(G_0\) is the connected component of the identity of the Poincaré group.

[Here the \(\mathbb{R}\) action is that by time translations].

In other words, if \(Y \neq X\) is a Killing vector field on \(M\) which is not axial, then the Killing Lie algebra of \((M, g)\) is that of the Poincaré group. Some results concerning this question can be found in [6] (cf. also [5]) under, however, some supplementary conditions.
3.3 Topology of black holes

To continue with our long list of problems in the uniqueness theory of black holes, recall that the key to the Carter–Bunting–Mazur uniqueness theorem for axisymmetric black holes is Carter’s reduction of the problem to a two-dimensional harmonic map boundary value problem [18, 19]. In that construction one assumes that $(M, g)$ is asymptotically flat in the conformal sense, and that on $M$ there exist two Killing vector fields, $X$, which approaches $\partial/\partial t$ in asymptotically Minkowskian coordinates $(t, \vec{x})$ as $r \to \infty$, and $Y$, which is an axial Killing vector. One moreover assumes that the boundary of the black hole is connected and has spherical topology. Under these assumptions Carter reduces the field equations to a two-dimensional harmonic–map problem with appropriate boundary conditions [19], which has subsequently been shown to have unique solutions [19, 16, 20, 60]. In this context the establishing of the validity of Conjecture 3.2 would be rather useful, reducing the general question of classification of stationary asymptotically flat space–times with more than one Killing vector field to that of axisymmetric black–holes considered by Carter. Further improvements of the uniqueness theory of axisymmetric black holes should include

1. a justification of the black–hole–connectedness condition, and

2. a justification of the spherical topology condition.

Recall that a well–known claim of Hawking [45] asserts that a connected component of a black hole boundary must necessarily have spherical topology. The arguments used in [45] suffer from two problems:

1. As has been emphasized by G. Galloway [37], the claim in [45] that a black hole boundary cannot have toroidal topology does not seem to be sufficiently justified. Moreover

2. for degenerate black–holes one should justify the degree of differentiability of the black–hole boundary used in the proof.

The new argument of [37] eliminates the toroidal black holes at the price, however, of introducing 1) a condition on null geodesics in $(M, g)$ and 2) a “well–formedness” condition on the event horizon, cf. [37] for details. It would be of interest to perhaps justify those assumption for electro–vacuum stationary space–times.

---

22Let us mention that for spherical bifurcation surfaces the discussion of Section 2.1 guarantees the existence of an axial Killing vector field in space–time, so that under the hypotheses of Proposition 2.1 the Carter reduction process can be applied.

23It seems that the arguments of [11] and [12] can be used to eliminate toroidal black holes when analyticity of the metric is assumed.

24cf. e.g. [2] (Proposition 3.5) for a result on the “well–formedness” condition.
4 Conclusions

In this review we have attempted to present an exhaustive list of open problems of the theory of uniqueness of black holes. A possible approach to a more satisfactory theory is as follows: The establishing of Conjectures 1.2 and 1.8 would

1. lead to a considerably improved version of Theorem 2.4. Moreover

2. Proposition 2.1 together with some perhaps improved version of the topology–of–black–holes–theorem would lead to a complete classification of stationary space–times with two or more Killing vectors.

We believe that the establishing of Conjecture 1.8, and perhaps also of Conjecture 3.2, should be a not–too–difficult Corollary of the positive energy theorems. A proof of Conjecture 1.2 would considerably improve our knowledge of the structure of stationary asymptotically flat space–times. Finally, a solution to Problem 3.1 would be major progress in mathematical general relativity.

A Obstructions to smoothness of Scri for a class of stationary asymptotically Minkowskian space–times.

Let $M = \mathbb{R}^4 \setminus \{ \mathbb{R} \times B(R) \}$, where $B(R) \subset \mathbb{R}^3$ is a closed coordinate ball of radius $R$, and let $g_{\mu\nu}$ be a metric on $M$ satisfying

$$g_{\mu\nu} - \eta_{\mu\nu} - \frac{h_{\mu\nu}(\frac{r}{r})}{r} = O(r^{-1-\epsilon}) ,$$

(A.1)

$$\partial_i \ldots \partial_k \left\{ g_{\mu\nu} - \eta_{\mu\nu} - \frac{h_{\mu\nu}(\frac{r}{r})}{r} \right\} = O(r^{-1-k-\epsilon}) ,$$

(A.2)

with some $k \geq 1$, some functions $h_{\mu\nu} \in C^k(\mathbb{R}^3)$, and some $0 < \epsilon \leq 1$. Define

$$Y^\mu_{\pm} \partial_\mu = \partial_t \pm \frac{x^i}{r} \partial_i ,$$

and set

$$C^\mu_{\pm} (\frac{x}{r}) = \lim_{r \to \infty} r^2 \Gamma^\mu_{\nu\rho} Y^\nu_{\pm} Y^\rho_{\pm} ,$$

$$c^\mu_{\pm} = \frac{1}{4\pi} \int_{S^2} C^\mu_{\pm} d^2 S ,$$

$$e^\pm = \frac{1}{4\pi} \int_{S^2} \sum \frac{x^i}{r} C^i_{\pm} d^2 S ,$$

$$D^\mu_{\pm} = C^\mu_{\pm} - c^\mu_{\pm} - e^\pm Y^\mu_{\pm} .$$

Here $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols of $g_{\mu\nu}$, and $d^2 S = \sin \theta d\theta d\varphi$. We have the following:
Proposition A.1 The $D^\mu_{\pm}$’s are geometric invariants.

Proof: Consider two coordinate systems $\{x^\mu\}, \{y^\mu\}$ in which (A.1)–(A.2) hold, let us denote by $C^\mu_{\pm},\{x^{\alpha}\}$ respectively $C^\mu_{\pm},\{y^{\alpha}\}$, the quantities $C^\mu_\pm$ calculated in the coordinate system $\{x^\mu\}$, respectively $\{y^\mu\}$; similarly for $D^\mu_{\pm},\{y^{\alpha}\}$, etc.

Now it follows\(^\text{25}\) from the results in [26, 27] that there exists a Lorentz matrix $\Lambda^\mu_\nu$, a constant vector $A^\mu_\cdot$, and functions $B^\mu_\in C^{k+1}(\mathbb{R}^3)$ such that

$$y^\mu - \Lambda^\mu_\nu x^\nu - A^\mu_\log r - B^\mu_\{x^i r\} = O(r^{-\epsilon}), \quad (A.3)$$

$$\partial_{\sigma_1} \ldots \partial_{\sigma_{k+1}} \left\{ y^\mu - \Lambda^\mu_\nu x^\nu - A^\mu_\log r - B^\mu_\{x^i r\} \right\} = O(r^{-\epsilon-k-1}).$$

Suppose first that $\partial_{x^0} = \partial_{y^0}$, hence $\partial_{x^0} = \partial_{y^0}$. It follows that $\Lambda^\mu_\nu$ is a rotation matrix, and performing the inverse rotation if necessary we may without loss of generality assume that $\Lambda^\mu_\nu = \delta^\mu_\nu$. A calculation shows that

$$C^\mu_{\pm},\{y^{\alpha}\} - C^\mu_{\pm},\{x^{\alpha}\} = - \lim_{r \to \infty} 2 \frac{\partial^2 y^\mu}{\partial y^2} = A^\mu_\cdot,$$

and for $\frac{\partial}{\partial y^\mu} = \frac{\partial}{\partial y^\mu}$ the result follows.

Suppose, finally, that $\frac{\partial}{\partial x^0} \neq \frac{\partial}{\partial y^0}$. By (A.3) there exists a constant $\alpha$ such that $\alpha \frac{\partial}{\partial x^0} - \frac{\partial}{\partial y^0}$ is an asymptotically spacelike Killing vector. It then easily follows from (A.1)–(A.2) that $g_{\mu\nu}$ must be flat. Going to coordinates $\tilde{x}^\mu$ where the metric is explicitly flat we clearly have $C^\mu_{\pm},\{\tilde{x}^\alpha\} = 0$, and $D^\mu_{\pm},\{x^\alpha\} = D^\mu_{\pm},\{y^\alpha\} = 0$ follows from the calculation above.

Recall now that in [13] the existence of a system of coordinates $(\hat{u}, \hat{r}, \hat{\theta}, \hat{\phi})$ has been postulated such that, if we set

$$\hat{t} = \hat{u} + \hat{r},$$

and if we define the Cartesian coordinates $\hat{x}^i$ in terms of the spherical coordinates $(\hat{r}, \hat{\theta}, \hat{\phi})$ in the standard way, then

1. the metric in the coordinates $\hat{x}^\mu$ satisfies (A.1)–(A.2), and
2. the curves $\{x^\mu(s)\} = \{(s, s\hat{x}^i/r)\}$ are null geodesics.

Coordinates satisfying the above will be called retarded Bondi coordinates. The advanced Bondi coordinates are defined by reversing the time–orientation above.

As discussed after the proof of Proposition 1.9, such coordinates can be constructed whenever a conformal completion in lightlike directions satisfying appropriate requirements exists. Conversely, existence of Bondi coordinates implies that of conformal completions, cf. e.g. [20].

The main result of this Appendix is the following:

\(^{25}\) Due to the existence of a Killing vector the arguments of [26, 27] can be considerably simplified, cf. the remark after Corollary 1 in [26].
Theorem A.2 Consider a metric on \( M = \mathbb{R}^4 \setminus \{ \mathbb{R} \times B(R) \} \) satisfying (A.1)–(A.3) with \( k \geq 2 \).

1. Retarded Bondi coordinates exist if and only if
\[
D_+^\mu = 0.
\] (A.4)

2. Advanced Bondi coordinates exist if and only if
\[
D_-^\mu = 0.
\] (A.5)

Remarks: 1. Proposition 1.9 implies that (A.4)–(A.5) hold for electro–vacuum stationary space–times.

2. It is worthwhile mentioning that (A.4)–(A.5) provide a rather effective criterion. Consider, for example, a metric of the form
\[
g_{\mu\nu} = -\left( 1 - \frac{2m(\theta, \phi)}{r} \right) dt^2 + \left( 1 - \frac{2\tilde{m}(\theta, \phi)}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\] with some twice differentiable functions \( m(\theta, \phi) \) and \( \tilde{m}(\theta, \phi) \). One easily finds that the metric (A.3) admits Bondi coordinates only if \( m \) and \( \tilde{m} \) are constants, with \( m = \tilde{m} \).

3. If \( g_{\mu\nu} \) admits a full expansion in terms of inverse powers of \( r \) for large \( r \) and if (A.4) holds, then the transformed metric in the Bondi coordinates will also admit a full expansion.

4. If (A.4)–(A.5) do not hold, one can construct “Bondi-type” coordinates in which the metric has \( r^{-1} \ln r \) terms. If \( g_{\mu\nu} \) admits a full expansion in terms of inverse powers of \( r \) for large \( r \) but (A.4) does not hold, or if the metric admits an asymptotic expansion in terms of functions \( r^{-j} \ln^i r \), then the transformed metric in the “Bondi-type” coordinates will have an expansion in terms of \( r^{-j} \ln^i r \).

5. This result can essentially be found in [54]; the conclusions there are somewhat less definitive due to the dynamical character of the metrics considered.

Proof: We shall only prove point 1, point 2 follows by reversing time–orientation. We claim that (A.4) is necessary. Indeed, in Bondi coordinates the curves \( \{ x^\mu(s) \} = \{ (s, s\hat{x}^i/r) \} \) are null geodesics so that we have
\[
\frac{D^2 x^\mu}{ds^2} = \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{ds} \frac{dx^\rho}{ds} \sim \frac{dx^\mu}{ds} = Y^\mu_+.
\] (A.7)
(A.7) shows that (A.4) holds in Bondi’s coordinates, hence in any coordinate system by Proposition A.1. The sufficiency of (A.4) can be proved by constructing explicitly the appropriate family of null geodesics. One shows that if (A.3) holds, then the transformation leading to Bondi coordinates contains \( \log r \) terms only in the form (A.3); no details will be given. ☐
B  Majumdar–Papapetrou space–times with an infinite number of black holes

In this Appendix we shall briefly discuss a class of Majumdar–Papapetrou space–times with an infinite number of black holes considered by L. Véron. Let \( \{\vec{a}_i\}_{i=1}^{\infty} \) be an arbitrary sequence of pairwise distinct vectors in \( \mathbb{R}^3 \), and let \( m_i \) be an arbitrary sequence of non–negative numbers satisfying

\[
m \equiv \sum_{i=1}^{\infty} m_i < \infty.
\]

Consider the manifold \( M = \mathbb{R} \times (\mathbb{R}^3 \setminus \{\vec{a}_i\}) \) with a metric of the form

\[
ds^2 = -U^{-2}dt^2 + U^2(dx^2 + dy^2 + dz^2),
\]

\[
U = 1 + \sum_{i=1}^{\infty} \frac{m_i}{|\vec{x} - \vec{a}_i|}.
\]

It is easily seen, using e.g. the Harnack principle and (B.1), that \( U \) is a well defined smooth function on \( \mathbb{R}^3 \setminus \{\vec{a}_i\} \) satisfying

\[
\Delta U = 0.
\]

As has been shown in [55, 63], the metric (B.2) solves the electro–vacuum Einstein equations if we set

\[
A_\mu dx^\mu = U^{-1}dt.
\]

Let us moreover assume that there exists \( R > 0 \) such that \( \{\vec{a}_i\}_{i=1}^{\infty} \) is included in a ball of radius \( R \). By definition of \( U \) and by known properties of solutions of the Laplace equation it can be seen that we have

\[
U - 1 - \frac{m_i}{r} = O(r^{-2}),
\]

with \( m \) defined in (B.1), and with appropriately faster decay of all the derivatives of \( U - 1 - \frac{m_i}{r} \). It follows that \( M \) contains an asymptotically flat four–end \( M_1 \) in the sense of Section 1.3. The arguments of Section II of [43] show that every “point” \( \vec{x} = \vec{a}_i \) such that \( \vec{a}_i \) is not an accumulation point of \( \{\vec{a}_i\}_{i=1}^{\infty} \) corresponds to a connected component of the boundary of a black–hole when the space–time is suitably analytically extended, with a degenerate event horizon which has spacelike cross–sections of area \( 4\pi m_i^2 \). Thus, if there is an infinite number of non–vanishing coefficients \( m_i \) (which we shall henceforth assume), then the resulting maximally analytically extended space–time contains a black hole region (with respect to the asymptotic end \( M_1 \)) with an infinite number of connected components. Let us, however, note the following:

\[\text{26L. Véron, private communication.}\]
1. The metric (B.2) is “nakedly singular”, which can be seen as follows: By well known properties of solutions of Laplace equation for any point $\vec{a}_i$ which is not an accumulation point of $\{\vec{a}_i\}_{i=1}^{\infty}$ we have

$$\lim_{\vec{x} \to \vec{a}_i} F_{\mu\nu}F^{\mu\nu} = \lim_{\vec{x} \to \vec{a}_i} \frac{|\text{grad } U|^2}{U^4} = \frac{1}{m_i^2},$$

and since $m_i \to 0$ as $i$ tends to infinity the scalar $F_{\mu\nu}F^{\mu\nu}$ is unbounded on any hypersurface $t = \text{const}$. It is then easily seen that one can construct a causal curve which reaches future null infinity and on which $F_{\mu\nu}F^{\mu\nu}$ is unbounded.

2. We believe that the partial Cauchy surfaces $\Sigma_{\tau} \equiv \{t = \tau\}$ are not complete with respect to the induced metric in general. One can, however, find solutions with an infinite number of black holes for which the $\Sigma_{\tau}$’s will be complete. This is e.g. the case when the sequence $\{\vec{a}_i\}_{i=1}^{\infty}$ has $a_1$ as the only accumulation point.

It has been suggested [43][Section III] that the only Majumdar–Papapetrou space–times without naked singularities in $J^+(\Sigma)$, where $\Sigma$ is the hypersurface $\{t = 0\}$, are those with a $U$ of the form (B.3), with only a finite number of non–vanishing $m_i$’s. It would be of some interest to prove such a result.

Acknowledgements

Most of the work on this paper was done when the author was visiting the Max Planck Institut für Astrophysik in Garching; he is grateful to Jürgen Ehlers and to the members of the Garching relativity group for hospitality. Special thanks are due to Robert Wald for many discussions, and for pointing out a mistake in the previous version of this paper.

References

[1] L. Andersson, P.T. Chruściel, On “hyperboloidal” Cauchy data for vacuum Einstein equations and obstructions to smoothness of null infinity, gr-qc/9304019, Phys. Rev. Lett. 70 (1993), 2829–2832.

[2] J.M. Arms, Linearization stability of gravitational and gauge fields, Jour. Math. Phys. 20 (1979), 443–453.

[3] A. Ashtekar, G.T. Horowitz, Energy–momentum of isolated systems cannot be null, Phys. Lett. 89A (1982), 181–184.

27When analytically extended through a past event horizon, the black hole solutions of [43] possess naked singularities in $J^{-}(\Sigma)$.
[4] —, A. Magnon–Ashtekar, *On conserved quantities in general relativity*, Jour. Math. Phys. **20** (1979), 793–800.

[5] —, B. Schmidt, *Null infinity and Killing fields*, Jour. Math. Phys. **21** (1978), 862–867.

[6] —, B.C. Xanthopoulos, *Isometries compatible with asymptotic flatness at null infinity: A complete description* Jour. Math. Phys. **19** (1978), 2216–2222.

[7] J.M. Bardeen, B. Carter, S.W. Hawking, *The four laws of black hole dynamics*, Commun. Math. Phys. **31** (1973), 161–170.

[8] J.K. Beem, *Minkowski space–time is locally extendible*, Comm. Math. Phys. **72** (1980), 273–275.

[9] R. Beig, *Arnowitt–Deser–Misner energy and g_{00}* , Phys. Lett. **69A** (1978), 153–155.

[10] —, W. Simon, *On the uniqueness of static perfect fluid solutions in general relativity*, Comm. Math. Phys. **144** (1992), 373–390.

[11] P. Bizoń, *Gravitating solitons and hairy black holes*, Vienna preprint UWTh–1994–5, to appear in Acta Physica Polonica B.

[12] P. Bizon, O.T. Popp, *No hair theorem for spherical monopoles*, Class. Quantum Grav. **9** (1992), 193–205.

[13] H. Bondi, M.G.J. van der Burg, A.W.K. Metzner, *Gravitational Waves in General Relativity VII. Waves from axi–symmetric isolated systems*, Proc. Roy. Soc. London **269** (1962), 21–52.

[14] R.H. Boyer, *Geodesic Killing orbits and bifurcate Killing horizons*, Proc. Roy. Soc. London **A311** (1969), 245–252.

[15] D. Brill, *On spacetimes without maximal surfaces*, Proc. of the Third Marcel Grossman Meeting (H. Ning, ed.), North Holland, Amsterdam, 1982.

[16] G.L. Bunting, *Proof of the uniqueness conjecture for black holes*, Ph.D. Thesis, University of New England, Armidale 1987 (unpublished).

[17] —, A.K.M. Masood–ul–Alam, *Nonexistence of multiple black holes in asymptotically euclidean static vacuum space–times*, Gen. Rel. Grav. **19** (1987), 147–154.

[18] B. Carter, *Black Hole Equilibrium States*, Black Holes (C. de Witt, B. de Witt, eds.), Gordon & Breach, New York, London, Paris, 1973.
[19] —, The general theory of the mechanical, electromagnetic and thermodynamic properties of black holes, General relativity (S.W. Hawking, W. Israel, eds.), Cambridge University Press, Cambridge, 1979.

[20] —, Bunting identity and Mazur identity for non-linear elliptic systems including the black hole equilibrium problem, Commun. Math. Phys. 99 (1985), 563–591.

[21] —, Mathematical foundation of the theory of relativistic stellar and black hole configurations, Gravitation and Astrophysics (B. Carter, J.B. Hartle, eds.), Plenum, New York, 1987.

[22] Y. Choquet-Bruhat, R. Geroch, Global Aspects of the Cauchy Problem, Commun. Math. Phys. 14 (1969), 329-335.

[23] D. Christodoulou, S. Klainerman, The Global Nonlinear Stability of Minkowski Space, Princeton University Press, Princeton, 1993.

[24] P.T. Chruściel, On the invariance properties and the Hamiltonian of the unified affine electromagnetism and gravitation theories, Ann. Inst. Henri Poincaré 42 (1985), 329–340.

[25] —, On angular momentum at spatial infinity, Class. Quantum Grav. 4 (1987), L205–L210.

[26] —, On the invariant mass conjecture in general relativity, Commun. Math. Phys. 120 (1988), 233–248.

[27] —, On the structure of spatial infinity. II. Geodesically regular Ashtekar–Hansen structures, Jour. Math. Phys. 30 (1989), 2094–2100.

[28] —, On uniqueness in the large of solutions of Einstein equations (“Strong cosmic censorship”), Australian University Press, Canberra, 1991.

[29] —, On completeness of orbits of Killing vector fields, gr–qc/9304029, Class. Quantum Grav. 10 (1993), 2091–2101.

[30] —, J. Isenberg, V. Moncrief, Strong cosmic censorship in polarized Gowdy space–times, Class. Quantum Grav. 7 (1990), 1671–1680.

[31] —, M.A.H. MacCallum, D. Singleton, Gravitational waves in general relativity. XIV: Bondi expansions and the “polyhomogeneity” of Scri, ANU Research Report CMA–MR14–92 (SMS–53–92), gr–qc/9305021, submitted to Phil. Trans. Royal Soc. of London.

[32] —, R. Wald, Maximal hypersurfaces in stationary asymptotically flat space–times, Garching preprint MPA 708 (1992); gr–qc/9304009, Commun. Math. Phys., in press.
[33] C.J.S. Clarke, *Local extensions in singular space–times II*, Commun. Math. Phys. **84** (1982), 329–331.

[34] T. Damour, B. Schmidt, *Reliability of perturbation theory in general relativity*, Jour. Math. Phys. **31** (1990), 2441–2453.

[35] A. Fischer, J. Marsden, V. Moncrief, *The structure of the space of solutions of Einstein’s equations. I. One Killing field*, Ann. Inst. Henri Poincaré **28** (1980), 147–194.

[36] V. Fock, *Theorie von Raum–Zeit und Gravitation*, Akademie Verlag, Berlin, 1960.

[37] G. Galloway, *Least area tori, black holes and topological censorship*, this volume.

[38] R. Geroch, J. Hartle, *Distorted black holes*, Jour. Math. Phys. **23** (1982), 680–692.

[39] —, G. Horowitz, *Asymptotically simple does not imply asymptotically Minkowskian*, Phys. Rev. Lett. **40** (1978), 203–206.

[40] G. Gibbons, *Self–gravitating magnetic monopoles, global monopoles and black holes*, in *The physical universe: The interface between cosmology, astrophysics and particle physics*, J.D. Barrow, A.B. Henriques, M.T.V.T. Lago, M.S. Longair, eds., Springer Lecture Notes in Physics **383**, 1991.

[41] —, S.W. Hawking, G.T. Horowitz, M.J. Perry, *Positive mass theorem for black holes*, Commun. Math. Phys. **99** (1983), 285–308.

[42] —, C.W. Hull, *A Bogomolny bound for general relativity and solitons in N = 2 supergravity*, Phys. Lett. **109B** (1982), 190–193.

[43] J.B. Hartle, S.W. Hawking, *Solutions of the Einstein–Maxwell equations with many black holes*, Commun. Math. Phys. **26** (1972), 87–101.

[44] S.W. Hawking, *Black holes in general relativity*, Commun. Math. Phys. **25** (1972), 152–166.

[45] —, G.F.R. Ellis, *The Large Scale Structure of Space–time*, Cambridge University Press, Cambridge, 1973.

[46] M. Heusler, *A no–hair theorem for self–gravitating non–linear σ–models*, Jour. Math. Phys. **33** (1992), 3497–3505.

[47] —, *Staticity and uniqueness of multiple black hole solutions of σ–models*, Class. Quantum Grav. **10** (1993), 791–799.
[48] —, N. Straumann, The first law of black hole physics for a class of non linear matter models, Class. Quantum Grav. 10 (1993), 1299–1321.

[49] B. Kay, R.M. Wald, Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate Killing horizon, Phys. Rep. 207 (1991), 49–136.

[50] J. Kijowski, W. Tulczyjew, A symplectic framework for field theories, Springer Lecture Notes in Physics 107 (1979).

[51] D. Kennefick, N. Ó Murchadha, Weakly decaying asymptotically flat static and stationary solutions to the Einstein equations, University College Cork preprint, gr-qc/9311012 (1993).

[52] Y.K. Lau, R.P.A.C. Newman, The structure of null infinity for stationary simple spacetimes. Class. Quantum Grav. 10 (1993), 551–557.

[53] A. Lichnerowicz, Théories relativistes de la gravitation et de l’électromagnétisme, Masson & Cie, Paris, 1955.

[54] J. Madore, Gravitational radiation from a bounded source I, Ann. Inst. Henri Poincaré XII (1970), 285–305.

[55] S.D. Majumdar, A class of exact solutions of Einstein’s field equations, Phys. Rev. 72 (1947), 390–398.

[56] A.K.M. Masood–ul–Alam, Uniqueness proof of static charged black holes revisited, Class. Quantum Grav. 9 (1992), L53–L55.

[57] —, Uniqueness of a static charged dilaton black hole, Class. Quantum Grav., in press.

[58] —, L. Lindblom, On the spherical symmetry of static stellar models, Montana State University preprint, 1993.

[59] P.O. Mazur, Proof of uniqueness of the Kerr–Newman black hole solution, Jour. Phys. A: Math. Gen. 15 (1982), 3173–3180.

[60] —, Black hole uniqueness theorems, in General Relativity and Gravitation, Proc. of GRG 11, Stockholm, July 1986, M.A.H. MacCallum, ed., Cambridge Univ. Press, Cambridge 1987.

[61] V. Moncrief, Spacetime symmetries and linearization stability of the Einstein equations, Jour. Math. Phys. 16 (1975), 493–498.

[62] H. Müller zum Hagen, On the analyticity of stationary vacuum solutions of Einstein’s equations, Proc. Camb. Phil. Soc. 68 (1970), 199–201.
[63] A. Papapetrou, *A static solution of the equations of the gravitational field for an arbitrary charge–distribution*, Proc. Roy. Irish Acad. **A51** (1945), 191–204.

[64] R. Penrose, *Zero rest mass fields including gravitation: asymptotic behaviour*, Proc. Roy. Soc. London **A284** (1965), 252–276.

[65] I. Rácz, *Space–time extensions I*, Jour. Math. Phys. **34** (1993), 2448–2464.

[66] —, R.M. Wald, *Extensions of spacetimes with Killing horizons*, Class. Quantum Grav. **9** (1992), 2643–2656.

[67] P. Ruback, *A new uniqueness theorem for charged black holes*, Class. Quantum Grav. **5** (1988), L155–L159.

[68] W. Simon, *The multipole expansion of stationary Einstein–Maxwell fields*, Jour. Math. Phys. **25** (1984), 1035–1038.

[69] —, *Radiative Einstein–Maxwell spacetimes and “no–hair” theorems*, Class. Quantum Grav. **9** (1992), 241–256.

[70] D. Sudarsky, R.M. Wald, *Extrema of mass, stationarity and staticity, and solutions to the Einstein–Yang–Mills equations*, Phys. Rev. **D46** (1992), 1453–1474.

[71] —, —, *Mass formulas for stationary Einstein–Yang–Mills black holes and a simple proof of two staticity theorems*, Phys. Rev. **D47** (1993), R5209–R5213.

[72] K.P. Tod, *All metrics admitting super–covariantly constant spinors*, Phys. Lett. **121B** (1983), 241–244.