Thresholds of Spatially Coupled Systems via Lyapunov’s Method

Christian Schlegel and Marat Burnashev

Abstract—The threshold, or saturation phenomenon of spatially coupled systems is revisited in the light of Lyapunov’s theory of dynamical systems. It is shown that an application of Lyapunov’s direct method can be used to quantitatively describe the threshold phenomenon, prove convergence, and compute threshold values. This provides a general proof methodology for the various systems recently studied. Examples of spatially coupled systems are given and their thresholds are computed.

Index Terms—random signaling, iterative decoding, optimal joint detection

I. INTRODUCTION

In this paper we apply Lyapunov’s classic theory [1] to the case of spatially coupled information processing systems, and show that the recently proposed “potential functions” used in the proofs in [2], [3], [4] are, in fact, an example from a wide class of Lyapunov functions. Such a systematic approach to the problem provides a general tool to deal with the dynamics of spatial coupling. The required definitions and the Lyapunov theorem are described below.

Iterative signal and information processing has enjoyed a tremendous rise in popularity with the introduction of turbo coding [5], and various “statistical” analysis methods have been developed to study the performance of iterative processors, in particular the method of extrinsic information exchange (EXIT) introduced by ten Brink [6], variance transformation by Divsalar et. al. [7], and density evolution (DE), refined by Richardson and Urbanke [8].

Spatial coupling emerged in the information processing arena largely by “accident”, and in the form of low-density parity-check (LDPC) convolutional coding [9]. Researchers noted that these codes could be designed with decoding thresholds that are very close to the channel capacity. The effect of spatial coupling derives from the special structure of these codes, where a large set of random codes are linked in a controlled fashion. The performance advantage comes from “anchoring” initial symbols to known values on one (or both) side(s) of this chain of linked codes, which causes a locally smaller rate and accelerated convergence. This in turn allows the entire code to converge at signal-to-noise ratios where uniform convergence is otherwise not possible. Recently, it has been shown that spatial coupling can decrease the convergence threshold in low-density parity-check codes on binary-erasure channels all the way to the maximum-likelihood decoding threshold [10], a phenomenon known as “threshold saturation”.

This phenomenon has given rise to much research activity in attempting to use this effect to show optimal performance for certain coupled communications and coding systems [11], and to find general proof methodologies for analyzing spatially coupled systems [11], [2], [3], [4].

II. THE SYSTEM

A. Basic Dynamical Systems

We consider a discrete dynamical system, governed by the following iteration equation

$$x^{(l+1)} = f(g(x^{(l)}); \varepsilon), \quad l = 0, 1, 2, \ldots, \quad (1)$$

where $x \in \mathcal{X} = [0, 1]^d \subset \mathbb{R}^d$, $\varepsilon \in \mathcal{E} = [0, 1]$ and $f : \mathcal{X} \times \mathcal{E} \rightarrow \mathcal{X}$ is a sufficiently smooth function. Also assume that $f(z; \varepsilon)$ and $g(x; \varepsilon)$ are strictly increasing in both arguments, and that also $f(0; \varepsilon) = f(g(x); 0) = 0$.

The single-system dynamical equation corresponds to the situation where

$$x_i^{(l+1)} = f(g(x_i^{(l)}); \varepsilon), \quad i = 1, \ldots, d. \quad (2)$$

The system [2] can represent the convergence properties of an LDPC decoder, for example [12], the variance evolution of an iterative cancelation receiver [13], or similar systems described by bi-partite (Tanner) graphs, and the functions $f(\cdot)$ and $g(\cdot)$ describe the statistical behavior of key performance parameters of the two types of processing nodes in these graphs. Such equations are typically obtained by applying a density evolution analysis to the system in question. In the case where $f(\cdot)$ and $g(\cdot)$ are vector functions [11] describes the evolution of spatially coupled systems, where the joint behavior of all $d$ sub-systems needs to be studied.

In this context, one is typically interested in the largest $\varepsilon$ such that for any $x \in \mathcal{X}$ $\lim_{l \rightarrow \infty} x^{(l)} = 0$. This parameter is typically a signal-to-noise ratio [13], or a channel error rate in the case of LDPC codes [11].

With $\varepsilon \in \mathcal{E}$, and $x^{(0)} = x \in \mathcal{X}$, let

$$x^\infty(x; \varepsilon) = \lim_{l \rightarrow \infty} x^{(l)} = \lim_{l \rightarrow \infty} f(g(x^{(l)}); \varepsilon). \quad (3)$$

This limit exists for all $\varepsilon \in \mathcal{E}$ due to the monotonicity of $f$, and therefore of $x^{(l)}$ in $l$ (see [10] Lemma 15), [4] Lemma 2).

We will need the following
Definition 1: The single-system threshold is defined as
\[ \varepsilon_0^* = \sup \{ \varepsilon \in \mathcal{E} | x^\infty(1; \varepsilon) = 0 \} = \sup \{ \varepsilon \in \mathcal{E} | x^\infty(1; \varepsilon) = 0 \} \tag{4} \]

The threshold \( \varepsilon_0^* \) is the well-known threshold of iterative decoders and demodulators as discussed in the literature. It can be computed by elementary methods applied to the single-variable dynamical system (2).

In the sequel of this paper, we will focus on coupled systems of the type (1).

B. Coupled Dynamical Systems

We start with a basic (“1-dimensional”) system (1) with the state-space \( \mathcal{X} \). Assume that we have \( L \) identical independent decoders and demodulators as discussed in the literature. It can be computed by elementary methods applied to the single-variable dynamical system (2).

Now, without enlarging the space \( \mathcal{X}^L \) imagine that these \( L \) identical systems are arranged in a linear fashion from left to right, for example, and therefore there are two boundaries. We now introduce dependencies for each of the \( L \) systems on its \( w \) adjacent neighboring systems. These dependencies shall be identical, when possible, for all \( L \) systems. The only exception will be for cells close to a boundary. If some connection is not possible, it is assumed to be connected to a known value (this is the anchor value). As a result the overall system now possesses a boundary asymmetry, which will imply additional properties. As experience with spatial coupling has shown, this asymmetry, in the form of the known values starting at the boundary systems, slowly propagates with iterations into the inner systems. When the function \( f(x) \) is \( \cup \)-convex (as is usually the case) then

\[
\begin{align*}
  x_i \bigl( t+1 \bigr) &= f \left( g \left( \frac{1}{w^2} \sum_{j=0}^{w-1} \sum_{k=0}^{w-1} x_{i+j-k} \right) \right) \cdot \varepsilon \tag{5} \\
  x_i \bigl( t+1 \bigr) &= \frac{1}{w} \sum_{k=0}^{w-1} f \left( \frac{1}{w} \sum_{j=0}^{w-1} g \left( x_{i+j-k} \right) \right) \cdot \varepsilon \tag{6}
\end{align*}
\]

Both systems (5) and (6) have the same threshold and can be analyzed by similar methods. We consider the system (5) in the sequel. Note that if the function \( f(x) \) is \( \cup \)-convex (as is usually the case) then

\[
\begin{align*}
  x_i \bigl( t+1 \bigr) &= f \left( g \left( \frac{1}{w^2} \sum_{j=0}^{w-1} \sum_{k=0}^{w-1} x_{i+j-k} \right) \right) \cdot \varepsilon \\
  \leq \frac{1}{w} \sum_{k=0}^{w-1} f \left( \frac{1}{w} \sum_{j=0}^{w-1} g \left( x_{i+j-k} \right) \right) \cdot \varepsilon
\end{align*}
\]

and therefore the system (5) has convergence properties that are no worse than those of (6).

A way to investigate the system (1) was offered in (2) and developed in (3). It is based on using the following function \( U(x) : x \to \mathbb{R}^1 \), called the potential function (4).

\[
U(x) = \int_0^x g'(z) [z - f(g(z))] \, dz \tag{7}
\]

When the vector system considered is constructed from one-dimensional systems as in (3) or (6), definition (7) reduces to the one-dimensional function \( U(x) : x \to \mathbb{R}^1 \):

\[
U(x) = \int_0^x g'(z) [z - f(g(z))] \, dz. \tag{8}
\]

The motivation for using the function \( U(x) \) in (2) was based on a continuous-time approximation for the system (6), given by

\[
\frac{dx(t)}{dt} = f(g(x(t))) - x(t), \quad t > 0, \quad t \to \infty, \tag{9}
\]

and, in turn, on the close relation of an analog of the function \( U(x) \) for the system (2) to its Bethe free energy.

The main aim of the paper is to give another (more traditional) look at the problem considered based on using Lyapunov functions. We show that from that point of view the function \( U(x) \) from (7) is, in fact, an example from a wide class of Lyapunov functions for the system (1), constructed by the variable gradient method (3) Chapter 3.4.

III. LYAPUNOV FORMULATION

A. Lyapunov’s Direct Method

Essentially, Lyapunov built a theory whereby the often exceedingly complicated study of when and how dynamical systems converge is moved away from studying the behavior of individual trajectories to studying the behavior of the system in certain regions of space. This is simplified by studying Lyapunov candidate functions in these regions.

Definition 2: The solution \( x(t) = 0 \) to (1) is globally asymptotically stable if \( \lim_{t \to \infty} x(t) = 0 \) for all \( x(0) \in \mathcal{X} \).

Denote by \( \mathcal{X}_0 = \mathcal{X} \setminus \{0\} \); \( \mathcal{E}_0 = \mathcal{E} \setminus \{0\} \), and let \( \mathcal{E}_2 \subseteq \mathcal{E}_0 \) be a subset of \( \mathcal{E}_0 \).

A Lyapunov candidate function is defined in Definition 3: A continuous function \( V(x; \varepsilon) : \mathcal{X} \times \mathcal{E}_2 \to \mathbb{R}^1 \) is called a Lyapunov function for the system (1) with \( \varepsilon \in \mathcal{E}_2 \), if it satisfies the following conditions:

\[
\begin{align*}
  V(0; \varepsilon) &= 0, \quad \varepsilon \in \mathcal{E}_2; \tag{10} \\
  V(x; \varepsilon) &> 0, \quad x \in \mathcal{X}_0, \varepsilon \in \mathcal{E}_2; \tag{11} \\
  V(f(x; \varepsilon); \varepsilon) - V(x; \varepsilon) &< 0, \quad x \in \mathcal{X}, \varepsilon \in \mathcal{E}_2. \tag{12}
\end{align*}
\]

The following result, known as Lyapunov’s direct method (1892), gives sufficient conditions for global asymptotic stability of the system (1).

\footnote{This terminology stems from the fact that the integral in (4) does not depend on the curve \( C \) from 0 to \( a \) along which this integral is computed.}
**Theorem 1**: (modification of [13] Theorem 13.2). Assume that \( V(x; \varepsilon) \) is a Lyapunov function for the system (1) with \( \varepsilon \in \mathcal{E}_2 \). Then the solution \( x^{(l)} \equiv 0 \) is globally asymptotically stable.

### B. Lyapunov Function of Spatially Coupled Systems

Represent the system (1) in the form

\[
    z^{(l)} - z^{(l+1)} = q \left( z^{(l)} \right), \quad l = 0, 1, 2, \ldots, \tag{13}
\]

where \( q(z) = z - f(g(z)) \). In order to have convergence \( z^{(l)} \to z^\infty \) it is sufficient that

\[
    q \left( z^{(l)} \right) \geq 0, \quad l = 0, 1, 2, \ldots, \tag{14}
\]

is fulfilled along the trajectory \( z^{(0)}, z^{(1)}, z^{(2)}, \ldots \). In order to have convergence \( z^{(l)} \to 0 \) it is sufficient, in addition to (14), that the following condition is fulfilled:

For any \( \delta > 0 \) there exists \( \varepsilon = \varepsilon(\delta) > 0 \) such that

\[
    \left\| q \left( z^{(l)} \right) \right\| \geq \varepsilon(\delta), \quad \text{if} \quad \left\| z^{(l)} \right\| \geq \delta. \tag{15}
\]

We want to avoid dealing with the trajectory \( \{ z^{(l)} \} \) and replace condition (15) by a simpler check. For that purpose the following auxiliary result is useful. The system (13) is similar to the continuous-time system

\[
    \frac{dz(t)}{dt} = -q(z(t)), \quad t > 0, \quad t \to \infty, \tag{16}
\]

where \( q(z(t)) = z(t) - f(g(z(t))) \).

To approach this problem systematically we apply the *variable gradient* method for constructing Lyapunov functions to the system (16), [13] Chapter 3.4. It will be a Lyapunov function for the system (1) as well.

Let \( V : Z \to \mathbb{R}^1 \) be a continuously differentiable function and let

\[
    h(z) = \left( \frac{\partial V}{\partial z} \right)^\top,
\]

i.e. \( h(z) \) is the gradient of \( V(z) \). Here

\[
    \frac{\partial V}{\partial z} = \left[ \frac{\partial V}{\partial z_1}, \frac{\partial V}{\partial z_2}, \ldots, \frac{\partial V}{\partial z_n} \right] \quad \text{row-vector},
\]

\[
    h(z) \quad \text{column-vector}.
\]

The derivative of \( V(z) \) along the trajectories of (16) is given by

\[
    \frac{dV(z)}{dt} = -\frac{\partial V}{\partial z} q(z) = -h^\top(z) q(z). \tag{17}
\]

Next, construct \( h(z) \) such that \( h(z) \) is a gradient for a positive function and

\[
    \frac{dV(z)}{dt} = -h^\top(z) q(z) < 0, \quad z \in Z, \quad z \neq 0. \tag{18}
\]

Specifically, the function \( V(z) \) can be computed from the line integral

\[
    V(z) = \int_0^z h^\top(s) ds. \tag{19}
\]

Recall that the line integral of a gradient vector \( h : \mathbb{R}^n \to \mathbb{R}^n \) is path independent, and hence, integration in (19) can be taken along any path joining the origin to \( z \in Z \).

It is known [13] Proposition 3.1 that \( h(z) \) is a gradient of a real-valued function \( V : \mathbb{R}^n \to \mathbb{R}^1 \) if and only if the Jacobian matrix \( \frac{\partial h_i}{\partial z_j} \) is symmetric, i.e. iff

\[
    \frac{\partial h_i}{\partial z_j} = \frac{\partial h_i}{\partial z_j}, \quad i, j = 1, \ldots, n. \tag{20}
\]

According to the definition of Lyapunov function, choosing \( h(z) \) and arriving at \( V(z) \), it is necessary to have

\[
    V(z) > 0, \quad z \in \mathcal{Z}_0, \quad z \neq 0, \tag{21}
\]

where \( \mathcal{Z}_0 \subseteq Z \) is any open set such that \( z(t) \in \mathcal{Z}_0 \) for all \( t > 0 \). The larger the set \( \mathcal{Z}_0 \) we can find (based, perhaps, on additional information about \( z(t) \)), the less restrictive is condition (21).

Referring to (18) we look for \( h(z) \) of the form \( h(z) = B(z) q(z) \), where \( B(z) \) is an \( n \times n \)-positive-definite matrix. Then due to (18) we need \( (z \in \mathcal{Z}_0, z \neq 0) \)

\[
    h^\top(z) q(z) = q^\top(z) B^\top(z) q(z) > 0, \tag{22}
\]

and from (19) we have

\[
    V_B(z) = \int_0^z [B(s)(z - f(g(z)))]^\top ds. \tag{23}
\]

For a chosen positive-definite \( n \times n \)-matrix \( B(z) \) the function \( V_B(z) \) from (23) is a Lyapunov function for the system (1), if conditions (21)–(22) are satisfied. Then

\[
    \lim_{t \to \infty} z(t) = 0. \tag{24}
\]

We now show that the function \( V_B(z) \) is a Lyapunov function for the system (1). Represent (1) (see (13)) in the form

\[
    x^{(l+1)} - x^{(l)} = q(x^{(l)}) = x^{(l)} - f(g(x^{(l)})),
\]

where

\[
    q(x) = x - f(g(x)).
\]

Condition (22) takes the form \( (x \in \mathcal{X}_0, x \neq 0) \)

\[
    [x - f(g(x))]^\top B^\top(x) [x - f(g(x))] > 0, \tag{25}
\]

where \( \mathcal{X}_0 \subseteq \mathcal{X} \) is any set such that \( x^{(l)} \in \mathcal{X}_0 \) for all \( l \geq 0 \). In turn, (23) takes the form

\[
    V_B(z) = \int_0^z [B(s)(g(s))]^\top ds. \tag{26}
\]

Note that if we set \( B(x) = g'(x) \), then

\[
    V_B(z) = U(x), \tag{27}
\]

where \( U(x) \) is the potential function from (7).

For a chosen positive-definite \( n \times n \)-matrix \( B(x) \) the function \( V(x, B) \) from (26) is Lyapunov function for the system (1), if condition (25) is satisfied, and, moreover,

\[
    V_B(x) > 0, \quad x \in \mathcal{X}_0, \quad x \neq 0. \tag{28}
\]
We also have $V_B(0) = 0$. If both conditions (25) and (28) are satisfied, then
\[ \lim_{i \to \infty} x^{(i)} = 0. \tag{29} \]

Consider the system (6) and set $B(x) = Dg'(x)$, where $D$ is a positive-definite diagonal matrix. It was shown in [3, Theorem 1], [4, Theorem 1] that if condition (28) is satisfied, then the condition (25) is also satisfied (i.e. there exists the unique fixed point $x = 0$ along the trajectory).

We give another proof of a similar result for the system (5).

C. Convergence of Spatially Coupled Systems

Having defined the Lyapunov function of the spatially coupled system in (26), we proceed as follows: Given a certain initial condition, in our case $x_i = 0, i < 0; i > L$, represented by the anchoring of the spatially coupled system, we find the largest $\varepsilon \in E_2$, such that (10) – (12) hold.

Formally, the coupled system threshold is defined in

Definition 4: The coupled-system (5) threshold with $x_i = 0, i < 0; i > L$ is defined as
\[ \varepsilon^*_c = \sup \{ \varepsilon \in E_2 \mid x^{\infty}(1; \varepsilon) = 0 \}. \tag{30} \]

Evidently $\varepsilon^*_c \geq \varepsilon^*_e$ in general, with equality if $w = 0$, or, for example, if the $L$ identical systems are arranged in a circle such that no boundary exists.

We will also use

Definition 5: For a positive-definite matrix $B$ the coupled-system (5) threshold $\varepsilon_c(B)$ is defined as
\[ \varepsilon_c(B) = \sup \left\{ \varepsilon \in E_2 \mid \min_{x \in \mathbb{R}} V_B(x) \geq 0 \right\}. \tag{31} \]

For any positive-definite matrix $B$ we have
\[ \varepsilon^*_c(B) \leq \varepsilon^*_e. \tag{32} \]

Let $x_0 = (x_{0,-L}, \ldots, x_{0,0})$ be a fixed point of (5) and $f(g(x); \varepsilon) = \varepsilon f(g(x))$. Then $\{x_{0,i}\}$ satisfy equations ($i \in \mathbb{Z}' = \{-L, -L + 1, \ldots, 0\}$)
\[ x_{0,i} = \varepsilon f \left( g \left( \frac{1}{w^2} \sum_{k=0}^{w-1} \sum_{j=0}^{w-1} x_{0,i+j-k} \right) \right). \tag{33} \]

The following theorem represents the main result of the paper.

Theorem 2: There exists a function $w_0(f, g)$ such that for any positive-definite matrix $B$, $w \geq w_0(f, g)$, $L \geq 2w + 1$ and $\varepsilon < \varepsilon^*_e(B)$ the only fixed point of the system (5) is $x_0 = 0$.

Proof. Our proof of Theorem 2 is different from the proofs of [3, Theorem 1], [4, Theorem 1]. We have $x^{(l+1)} < x^{(l)}$ and $V_B(x^{(l+1)}) < V_B(x^{(l)})$ for all $l \geq 0$, and the sequence $\{x^{(l)}\}$ converges to a fixed point $x_0$, which is the (local) minimum of the function $V_B(x)$, but may never reach the point $x_0$. Note that if $x_0 \neq 0$ is a fixed point (i.e. $x_0 - \varepsilon f(g(x_0)) = 0$) then $V_B'(x_0) = 0$ and
\[ V_B''(x_0) = B(x_0) \left[ I_n - \varepsilon f'(g(x_0)) \right]. \]

Then it is sufficient to prove that the matrix $I_n - \varepsilon f'(g(x_0))$ has a negative eigenvalue (i.e. it is not a positive-definite matrix) and therefore $x_0$ cannot be a local minimum of the function $V_B(x)$ (all functions are continuous). We have
\[ f'(g(x_0)) = f'_y g'_{x0} x'_0 = f'_y g'_{x0} x'_0. \]

Note that if $\varepsilon$ is sufficiently small then there exists only the zero fixed point $x_0 = 0$. As $\varepsilon$ grows it reaches some $\varepsilon_1 > 0$ there appear non-zero fixed point(s) $x_0 \neq 0$. We need to show that for $\varepsilon > \varepsilon_1$ the matrix $A = \varepsilon f'(g(x_0))$ has an eigenvalue greater than 1. The matrix $A$ is non-negative (i.e. all its elements are non-negative). Therefore its spectral radius $\rho(A)$ equals its maximal eigenvalue. Moreover, if $A$ has a positive eigenvector (as in our case) then the corresponding eigenvalue is $\rho(A)$ [14, Chapter 8].

If $w = 1$ then the fixed point $x_0 = (x_{0,0}, \ldots, x_{0,0})$ and the matrix $A = \varepsilon f'(g(x_0))I_n$ is diagonal with equal diagonal elements (all that reduces to the uncoupled case). If $w > 1$ then the fixed point $x_0 = (x_{0,0}, \ldots, x_{0,n})$ consists of non-decreasing components. The $i$-th row $A_i$ of $A$ has the form
\[ A_i = \varepsilon a_i D_i, \quad a_i = f'_y(g_i y_i), \quad y_i = \frac{1}{w^2} \sum_{k=0}^{w-1} \sum_{j=0}^{w-1} x_{0,i+j-k}, \quad D_i = (D_{i,0}, \ldots, D_{i,n}), \tag{34} \]
\[ D_{i,j} = \frac{w - |i-j|}{w^2}, \quad |i-j| \leq w, \quad \sum D_{i,0} = 0, \quad |i-j| \geq w. \]

Diagonal elements of $A$ are $\{\varepsilon a_i/w, \; i = 1, \ldots, n\}$. For the matrix $D$ of rows $\{D_i\}$ and the matrix $A$ of rows $\{A_i\}$ we have

Lemma 2. For any $w \geq 1$ and $L \geq 2w + 1$ the matrix $D$ has the maximal eigenvalue $\rho(D) = 1$. The matrix $A$ has the maximal eigenvalue $\rho(A) = \varepsilon \max_i a_i$.

Therefore, if $\varepsilon > 1/\max_i a_i$ then $\rho(A) > 1$, and $I_n - \varepsilon f'(g(x_0))$ has a negative eigenvalue.

Remember that we still have the constraint (28), i.e $V_B(x) > 0, x \neq 0$, which sets the upper bound on $\varepsilon$. For $\varepsilon$, satisfying both constraints, the only fixed point of the system (5) is $x_0 = 0$.

It remains to clarify the condition $\varepsilon > 1/\max_i a_i$. We limit ourselves here to the following result.

Proposition 1: There exists a function $w_0(f, g)$ such that for any $w \geq w_0(f, g)$, $L \geq 2w + 1$ and $\varepsilon > \varepsilon^*_e$ the matrix $I_n - \varepsilon f'(g(x_0))$ has a negative eigenvalue.

From Proposition 1 the constraint (28), and Definition 5 Theorem 2 follows. □

Remark 4. It is natural to investigate the value $\sup_B \varepsilon_c(B)$, where supremum is taken over positive-definite matrices $B$. It will be done later.

IV. EXAMPLES:

A. LDPC Codes

Consider the traditional example of the $(3, 6)$-regular LDPC code ensemble defined by constant degree profile $(\lambda, \rho) =$
Then, for the binary erasure channel
\[ x^{(l+1)} = \varepsilon f(x^{(l)}), \quad l = 0, 1, 2, \ldots \]  
(35)

where
\[ f(x) = [1 - (1 - x)^5]^2, \quad 0 \leq x \leq 1. \]  
(36)

Given an initial erasure probability \( x^{(0)} \in [0, 1] \), we wish to find all \( \varepsilon \in [0, 1] \) such that \( x^{(l)} \to 0 \) as \( l \to \infty \).

Now, the single system converges for all \( \varepsilon < \varepsilon_0 \), where
\[ \varepsilon_0 = \min_{0 < x < 1} \frac{x}{f(x)} = \min_{0 < x < 1} \frac{x}{[1 - (1 - x)^5]^2} \approx 0.4294398. \]  
(37)

Indeed, the value \( \min x/f(x) \) is attained when \( f(x) - x f'(x) = 0 \), which is equivalent to
\[ 1 - \varepsilon f'(x) = 1 - 10\varepsilon(1 - x^4)[1 - (1 - x)^5], \]  
(38)

after replacing \( \varepsilon \) by \( x/f(x) \).

Formula (37) (and its natural generalization) is missing in [11], [12], although it simplifies analysis of \( \varepsilon_0 \).

The minimizing value \( x_0 = 0.26057 \) in (37) is the unique root of the equation
\[ (1 - x)^5 + 10x(1 - x)^4 - 1 = 0, \]  
(39)

and \( \varepsilon_0 \) is the single-system threshold.

We now consider the coupled case, and use the one-dimensional potential function from equation (8). For regular \((l, r)\)-LDPC-codes the function \( U(x) \) from (8) can be integrated in closed form and takes the form
\[ U(x, \varepsilon) = \frac{1}{r} \left[ \frac{1}{(1 - x)^{r-1}} - \varepsilon \right] \left[ 1 - (1 - x)^{r-1} \right]^{l}. \]  
(40)

We want to find the maximal \( \varepsilon^* = \varepsilon^*(l, r) \) such that
\[ U(x, \varepsilon^*) \geq 0, \quad x \in [0, 1], \]  
which will be the coupled-system threshold according to (21). Consider the case \( l \to \infty \) and \( l/r \to \alpha \), \( \alpha < 1 \). We now show that \( \varepsilon^*(l, r) = \alpha \).

First, we set \( \varepsilon = \alpha \) and \( l = \alpha r \). Then
\[ rU(x, \alpha) = 1 - (1 - x)^{r-1} - r(1 - x)^{r-1} - \left[ 1 - (1 - x)^{r-1} \right]^{l \alpha}, \]
\[ U'(x, \alpha) = (r - 1)(1 - x)^{r-2} \left[ x - \alpha \left[ 1 - (1 - x)^{r-1} \right]^{l \alpha - 1} \right]. \]

Since \( U(1, \alpha) = 0 \) and \( U'(x, \alpha) > 0, \quad x > \alpha \), we need to consider only \( x < \alpha \).

Small values of \( x \) can also be exclude as follows. We have
\[ x - \alpha \left[ 1 - (1 - x)^{r-1} \right]^{l \alpha - 1} \geq x - \alpha (r - 1) x^{l \alpha - 1} \geq 0, \]
\[ x \leq x_0 = \frac{1}{r - 1} \left[ \frac{1}{\alpha(r - 1)} \right]^{1/(r - 2)}, \]
where \( x_0 \geq 1/(2r) \), and \( U'(x, \alpha) \geq 0, \quad x \leq 1/(2r) \). Since \( U(0, \alpha) = 0 \), the interval that remains to be considered is \( x \leq b/r \), \( 1/2 < b < \alpha r \). Since \( (1 - x)^{r-1} \leq e^{-x} \) and \( (1 - x)^{r} \geq e^{-x(r-1)}, \) we may use the following bounds in the interval \( 1/2 < b < \alpha r \)
\[ rU(b/r, \alpha) \geq 1 - e^{-b} - be^{-b} - \left[ 1 - e^{-b/(1 - \alpha)} \right]^{l \alpha} \geq o(1) \]
for \( r \to \infty \). But now
\[ \inf_{0 \leq x \leq 1} U(x, \alpha) = o(1/r), \quad r \to \infty, \]
and we obtain

**Proposition 2:** For the ensemble of regular \((l, r)\)-LDPC codes of rate \( 1 - l/r \) with \( l/r \to \alpha \) as \( r \to \infty \), where \( \alpha < 1 \), the coupled-system threshold \( \lim_{r \to \infty} \varepsilon^*(l, r) = \alpha \).

**Proposition 2** immediately reveals the important

**Corollary 1:** The ensemble of coupled regular \((l, r)\)-LDPC codes with \( l/r \to \alpha \) as \( r \to \infty \) achieves the capacity \( 1 - \alpha \) of the binary erasure channel with erasure rate \( \alpha \).

Note: In Corollary 1 we have rederived an important result from [10] by elementary methods from Theorem 2 without the need for the concept of threshold saturation or the use of the area theorem.

### B. Multiuser Cancellation

In [13] an iterative interference cancelation system is discussed with the following 1-dimensional dynamical system equation
\[ x^{(l+1)} = \alpha g(x^{(l)}) + \sigma^2, \quad l = 0, 1, 2, \ldots \]  
(41)

where \( g(x) > 0 \) is a given bounded function and \( \sigma \geq 0 \) is a constant, the root of the normalized noise variance. We are interested in the maximum \( \alpha_0 = \alpha_0(g, \sigma) \), such that
\[ x^{(l)} \to x^{(\infty)} = x^{(\infty)}(g, \sigma) \quad \text{as} \quad r \to \infty. \]  
It is straightforward to show that \( x^{(0)}, x^{(1)}, x^{(2)}, \ldots \) is a monotonically decreasing sequence, i.e.,
\[ x^{(l+1)} = \alpha g(x^{(l)}) + \sigma^2 \leq x^{(l)}, \quad l = 0, 1, 2, \ldots \]  
(42)

In order to find stable points of the system (41) consider the equation
\[ \alpha g(x) + \sigma^2 - x = 0. \]  
(43)

The values \( x_0 = x^{(\infty)} \) and \( \alpha_0 \) defining a stable point satisfy the equations
\[ \alpha_0 g(x_0) + \sigma^2 - x_0 = 0, \]
\[ \alpha_0 g'(x_0) - 1 = 0. \]  
(44)

Therefore
\[ \alpha_0 = \frac{x_0 - \sigma^2}{g(x_0)} = \min_{x \in S} \frac{x - \sigma^2}{g(x)}, \]  
(45)

where \( S \) is the set of stationary points of the function \( (x - \sigma^2)/g(x) \), \( x > \sigma \sigma^2 \), i.e., roots of the equation
\[ g(x) - (x - \sigma^2)g'(x) = 0. \]  
(46)

In [20] spatial coupling is applied to this system, and, using the Lyapunov function [26] with [27], it is shown that the coupled system can approach the capacity of the multiple access channel.

### V. Approximations

In [2] certain approximations for behavior of the system (4) via partial differential equations were proposed. We present different approximations here.

Consider the system (5), i.e. the equation
\[ x_i^{(l+1)} = f \left( g \left( y_i^{(l)} \right) \right), \quad i \in \mathcal{L}_0 = \{-L, \ldots, L\}, \]  
(47)
where 
\[ y_i^{(l)} = \frac{1}{w^2} \sum_{m=-(w-1)}^{w-1} a(m) v_i^{(l)}_{i+m}, \]

Consider first the case \( w - 1 - L \leq i \leq 1 - w + L \). Note that 
\[ y_i^{(l)} \approx \frac{1}{w^2} \int_{-w}^{w} (w - |r|) x_i^{(l)} dr. \]  
(48)

For the fixed points \( v(x) \) of this equation we get 
\[ \int_{-(\alpha + x)}^{1} (1 - |s|) f (g(v(x + s))) ds = v(x). \]  
(53)

For the fixed points \( v(x) \) of this equation we get 
\[ \int_{-(\alpha + x)}^{1} (1 - |s|) f (g(v(x + s))) ds = v(x). \]  
(53)

Similar approximations for the right boundary can be obtained.

VI. Conclusion

References

[1] A.M. Lyapunov, The General Problem of the Stability of Motion. Kharkov, Russia: Kharkov, Mathematical Society, 1892.
[2] K. Takeuchi, T. Tanaka, and T. Kawabata, “A phenomenological study on threshold improvement via spatial coupling,” Arxiv preprint arXiv:1102.3056, 2011.
[3] Arvind Yedla, Yung-Yih Jian, Phong S. Nguyen, and Henry D. Pfister, “A Simple Proof of Threshold Saturation for Coupled Scalar Recursions,” Arxiv preprint arXiv:1204.5703, 2012.
[4] Arvind Yedla, Yung-Yih Jian, Phong S. Nguyen, and Henry D. Pfister, “A Simple Proof of Threshold Saturation for Coupled Vector Recursions,” Arxiv preprint arXiv:1208.4080v2, 2012.
[5] C. Berrou and A. Glavieux, “Near Optimum error correcting coding and decoding: turbo-codes”, IEEE Trans. Commun., vol. 44, pp. 1261-1271, Oct. 1996.
[6] S. ten Brink, “Convergence behavior of iteratively decoded parallel concatenated codes”, IEEE TCOM., vol. 49, Oct. 2001.
[7] D. Divsalar, S. Dolinar and F. Pollara, Iterative turbo decoder analysis based on density evolution, IEEE J. Select. Areas Commun., Vol. 19, No. 5, 2001.
[8] T.J. Richardson and R.U. Urbanke, “The capacity of low-density parity-check codes under message-passing decoding”, IEEE Trans. Inform. Theory, vol. 47, no. 2, pp. 599–618, Feb. 2001.
[9] A. J. Fehnström and K. S. Zigangirov, “Time-Varying Periodic Convolutional Codes with Low-Density-Parity-Check Matrix,” IEEE Trans. Inform. Theory, vol. 45, no. 5, Sept. 1999.
[10] S. Kudekar, T. Richardson, R. Urbanke, “Threshold Saturation via Spatial Coupling: Why Convolutional LDPC Ensembles Perform so well on the BEC,” ArXiv:1001.1826v2, Oct. 2010.
[11] T. J. Richardson and R. I. Urbanke, Modern Coding Theory. Cambridge, UK: Cambridge Univ. Press, 2008.
[12] C. B. Schlegel and L. C. Perez, Trellis and Turbo Coding. Wiley-IEEE Press, 2004.
[13] Wassim M. Haddad and VijaySekhar Chellaboina, Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach. Princeton Univ. Press, 2008.
[14] R. A. Horn and C. R. Johnson, Matrix Analysis. Cambridge University Press, 1985.
[15] D. Divsalar, S. Dolinar, C.R. Jones, K. Andrews, “Capacity-Approaching Protograph Codes,” IEEE JSAC, vol. 27, Aug. 09.
[16] R. G. Gallager, “Low-density parity-check codes”, IRE Trans. on Inform. Theory, pp. 21–28, Vol. 8, No. 1. January 1962.
[17] C. Schlegel, Z. Shi, and M. Burnashev, “Asymptotically optimal power allocation and code selection for iterative joint detection of coded random CDMA,” IEEE Trans. Inform. Theory, vol. 52, no. 9, September 2006, pp. 4286–4295.
[18] C. Schlegel and D. Truhachev, “Multiple Access Demodulation in the Lifted Signal Graph with Spatial Coupling IEEE Trans. Inform. Theory, Vol. 59, No. 4, pp. 2459-2470, 2012.
[19] T. Tanaka, “A statistical-mechanics approach to large-system analysis of CDMA multiuser detectors,” IEEE Trans. Inform. Theory, Vol. 48, no. 11, Nov. 2002.
[20] D. Truhachev and C. Schlegel, “Coupling data transmission for capacity-achieving multiple-access communications,” arXiv:1209.5785.