A THEORY OF DUALITY FOR ALGEBRAIC CURVES

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Abstract. We formulate a refined theory of $g^r_n$, using the methods of [7], and use the theory to give a geometric interpretation of the genus of an algebraic curve. Using principles of duality, we prove generalisations of Plucker’s formulae for algebraic curves. The results hold for arbitrary characteristic of the base field $L$, with some occasional exceptions when $\text{char}(L) = 2$, which we observe in the course of the paper.

1. A refined theory of $g^r_n$

The purpose of this section is to refine the general theory of $g^r_n$, in order to take into account the notion of a branch for a projective algebraic curve. We will rely heavily on results proved in [7]. We also refer the reader there for the relevant notation. Unless otherwise stated, we will assume that the characteristic of the field $L$ is zero, making the modifications for non-zero characteristic in the final section. By an algebraic curve, we always mean a projective irreducible variety of dimension 1.

Definition 1.1. Let $C \subset P^w$ be a projective algebraic curve of degree $d$ and let $\Sigma$ be a linear system of dimension $R$, contained in the space of algebraic forms of degree $e$ on $P^w$. Let $\phi_\lambda$ belong to $\Sigma$, having finite intersection with $C$. Then, if $p \in C \cap \phi_\lambda$ and $\gamma_p$ is a branch centred at $p$, we define:

\[
I_p(C, \phi_\lambda) = I_{\text{italian}}(p, C, \phi_\lambda)
\]
\[
I^\Sigma_p(C, \phi_\lambda) = I^\Sigma_{\text{italian}}(p, C, \phi_\lambda)
\]
\[
I^\Sigma_{\text{mobile}}(C, \phi_\lambda) = I^\Sigma_{\text{italian}}(p, C, \phi_\lambda)
\]
\[
I_{\gamma_p}(C, \phi_\lambda) = I_{\text{italian}}(p, \gamma_p, C, \phi_\lambda)
\]
\[
I^\Sigma_{\gamma_p}(C, \phi_\lambda) = I^\Sigma_{\text{italian}}(p, \gamma_p, C, \phi_\lambda)
\]
\[ I_{\gamma_p}(C, \phi) = I_{\Sigma . mobile}(p, \gamma_p, C, \phi) \]

where \( I_{\text{italian}} \) was defined in [7].

It follows that, as \( \lambda \) varies in \( \text{Par}_\Sigma \), we obtain a series of weighted sets:

\[ W_\lambda = \{ n_{\gamma_1^{p_1}}, \ldots, n_{\gamma_1^{p_1}}, \ldots, n_{\gamma_{n_{p_m}}} \} \]

where;

\[ \{ p_1, \ldots, p_i, \ldots, p_m \} = C \cap \phi_\lambda, \text{ for } 1 \leq i \leq m, \]

\[ \{ \gamma_{p_1}^{1}, \ldots, \gamma_{p_i}^{j(i)}, \ldots, \gamma_{p_i}^{n_i} \}, \text{ for } 1 \leq j(i) \leq n_i, \text{ consists of the branches of } C \text{ centred at } p_i \]

and

\[ I_{\gamma_{p_i}^{j(i)}}(C, \phi_\lambda) = n_{\gamma_{p_i}^{j(i)}} \]

By the branched version of the Hyperspatial Bezout Theorem, see [7], the total weight of any of these sets, which we will occasionally abbreviate by \( C \cap \phi_\lambda \), is always equal to \( de \). Let \( r \) be the least integer such that every weighted set \( W_\lambda \) is defined by a linear subsystem \( \Sigma' \subset \Sigma \) of dimension \( r \).

**Definition 1.2.** We define;

\[ \text{Series}(\Sigma) = \{ W_\lambda : \lambda \in \text{Par}_\Sigma \} \]

\[ \text{dimension}(\text{Series}(\Sigma)) = r \]

\[ \text{order}(\text{Series}(\Sigma)) = de \]

We then claim the following;

**Theorem 1.3.**

(i). \( r \leq R \), with equality iff every weighted set \( W_\lambda \) of the series is cut out by a single form of \( \Sigma \).
(ii). $r \not\leq R$ iff there exists a form $\phi_\lambda$ in $\Sigma$, containing all of $C$.

Proof. We first show the equivalence of (i) and (ii). Suppose that (i) holds and $r \not\leq R$. Then, we can find a weighted set $W$ and distinct elements $\{\lambda_1, \lambda_2\}$ of $\text{Par}_\Sigma$ such that $W = W_{\lambda_1} = W_{\lambda_2}$. Let $\{\phi_{\lambda_1}, \phi_{\lambda_2}\}$ be the corresponding algebraic forms of $\Sigma$ and consider the pencil $\Sigma_1 \subset \Sigma$ defined by these forms. We claim that:

$$W = C \cap (\mu_1 \phi_{\lambda_1} + \mu_2 \phi_{\lambda_2}), \text{ for } [\mu_1 : \mu_2] \in P^1 \text{ (1)}$$

This follows immediately from the results in [7] that the condition of multiplicity at a branch is linear and the branched version of the Hyperspatial Bezout Theorem. Now choose a point $p \in C$ not contained in $W$. Then, the condition that an algebraic form $\phi_\lambda$ passes through $p$ defines a hyperplane condition on $\text{Par}_k$, hence, intersects $\text{Par}_\Sigma_1$ in a point. Let $\phi_{\lambda_0}$ be the algebraic form in $\Sigma_1$ defined by this parameter. Then, by (1), we have that:

$$W \cup \{p\} \subseteq C \cap \phi_{\lambda_0}$$

Hence, the total multiplicity of intersection of $\phi_{\lambda_0}$ with $C$ is at least equal to $de + 1$. By the branched version of the Hyperspatial Bezout Theorem, $C$ must be contained in $\phi_{\lambda_0}$. Conversely, suppose that (i) holds and there exists a form $\phi_{\lambda_0}$ in $\Sigma$ containing all of $C$. Let $W$ be cut out by $\phi_{\lambda_1}$ and consider the pencil $\Sigma_1 \subset \Sigma$ generated by $\{\phi_{\lambda_0}, \phi_{\lambda_1}\}$. By the same argument as above, we can find $\phi_{\lambda_2}$ in $\Sigma_1$, distinct from $\phi_{\lambda_1}$, which also cuts out $W$. Hence, by (i), we must have that $r \not\leq R$. Therefore, (ii) holds.

The argument that (ii) implies (i) is similar.

We now prove that (ii) holds. Using the Hyperspatial Bezout Theorem, the condition on $\text{Par}_\Sigma$ that a form $\phi_\lambda$ contains $C$ is linear. Let $H$ be the linear subsystem of $\Sigma$, consisting of forms containing $C$ and let $h = \text{dim}(H)$. Let $K \subset \Sigma$ be a maximal linear subsystem, having finite intersection with $C$. Then $K$ has no form in common with $H$ and $\text{dim}(K) = R - h - 1$. We claim that every weighted set in $\text{Series}(\Sigma)$ is cut out by a unique form from $K$. For suppose that $W = C \cap \phi_\mu$ is such a weighted set and consider the linear system defined by $< H, \phi_\lambda >$. If $\phi_\mu$ belongs to this system and has finite intersection with $C$, then clearly $(C \cap \phi_\lambda) = (C \cap \phi_\mu)$. Using linearity of multiplicity at a branch and the Hyperspatial Bezout Theorem again (by convention, a form
containing \( C \) has infinite multiplicity at a branch), we must have that 
\[
(C \cap \phi_{\lambda}) = (C \cap \phi_{\mu}).
\]
Now consider \( K \cap <H, \phi_{\lambda}> \). We have that;
\[
\text{codim}(K \cap <H, \phi_{\lambda}>) \leq \text{codim}(K) + \text{codim}(<H, \phi_{\lambda}>) = (h + 1) + (R - (h + 1)) = R.
\]
Hence, \( \text{dim}(K \cap <H, \phi_{\lambda}>) \geq 0 \). We can, therefore, find a form \( \phi_{\mu} \) belonging to \( K \) such that \( W = (C \cap \phi_{\mu}) \). We need to show that \( \phi_{\mu} \) is the unique form in \( K \) defining \( W \). This follows by the argument given above. It follows immediately that \( r = \text{dim}(K) = R - h - 1 \). Hence, \( r \leq R \) iff \( h \geq 0 \). Therefore, (\( ii \)) is shown.

Using this theorem, we give a more refined definition of a \( g^r_n \).

**Definition 1.4.** Let \( C \subset P^w \) be a projective algebraic curve. By a \( g^r_n \) on \( C \), we mean the collection of weighted sets, without repetitions, defined by \( \text{Series}(\Sigma) \) for some linear system \( \Sigma \), such that \( r = \text{dimension}(\text{Series}(\Sigma)) \) and \( n = \text{order}(\text{Series}(\Sigma)) \). If a branch \( \gamma_p^j \) appears with multiplicity at least \( s \) in every weighted set of a \( g^r_n \), as just defined, then we allow the possibility of removing some multiplicity contribution \( s' \leq s \) from each weighted set and adjusting \( n \) to \( n' = n - s' \).

**Remarks 1.5.** The reader should observe carefully that a \( g^r_n \) is defined independently of a particular linear system. However, by the previous theorem, for any \( g^r_n \), there exists a \( g^r_{n'} \) with \( n \leq n' \) such that the following property holds. The \( g^r_{n'} \) is defined by a linear system of dimension \( r \), having finite intersection with \( C \), such that each there is a bijection between the weighted sets \( W \) in the \( g^r_{n'} \) and the \( W_{\lambda} \) in \( \text{Series}(\Sigma) \). The original \( g^r_n \) is obtained from the \( g^r_{n'} \) by removing some fixed point contribution.

We now reformulate the results of Section 2 and Section 5 in [7] for this new definition of a \( g^r_n \). In order to do this, we require the following definition;

**Definition 1.6.** Suppose that \( C \subset P^w(L) \) is a projective algebraic curve and \( C^{\text{ext}} \subset P^w(K) \) is its non-standard model. Let a \( g^r_n \) be given on \( C \), defined by a linear system \( \Sigma \) after removing some fixed point contribution. We define the extension \( g^r_{n,\text{ext}} \) of the \( g^r_n \) to the nonstandard model \( C^{\text{ext}} \) to be the collection of weighted sets, without repetitions, defined by \( \text{Series}(\Sigma) \) on \( C^{\text{ext}} \), after removing the same fixed point contribution. Note that, by definability of multiplicity at a branch, see
Theorem 6.5 of [7], if \( \gamma^j_p \) is a branch of \( C \) and;

\[
I_{\text{italian}}(p, \gamma^j_p, C, \phi_\lambda) \geq k, \quad (\lambda \in \text{Par}_\Sigma(L))
\]

then;

\[
I_{\text{italian}}(p, \gamma^j_p, C, \phi_\lambda) \geq k, \quad (\lambda \in \text{Par}_\Sigma(K))
\]

Hence, it is possible to remove the same fixed point contribution of \( \text{Series}(\Sigma) \) on \( C^{\text{ext}} \). See also the proof of Lemma 1.7.

It is a remarkable fact that, after introducing the notion of a branch, the definition is independent of the particular linear system \( \Sigma \). This is the content of the following lemma;

**Lemma 1.7.** The previous definition is independent of the particular choice of linear system \( \Sigma \) defining the \( g^r_n \).

**Proof.** We divide the proof into the following cases;

Case 1. \( \Sigma \subset \Sigma' \);

By the proof of Theorem 1.3, we can find a linear system \( \Sigma_0 \subset \Sigma \subset \Sigma' \) of dimension \( r \), having finite intersection with \( C \), such that the \( g^r_n \) is defined by removing some fixed contribution from \( \Sigma_0 \). Here, we have also used the fact that the base point contributions (at a branch) of \( \{\Sigma_0, \Sigma, \Sigma'\} \) are the same. Again, by Theorem 1.3, if \( W_\lambda \) is a weighted set defined by \( \Sigma' \) on \( C^{\text{ext}} \), then it appears as a weighted set \( V_\lambda \) defined by \( \Sigma_0 \) on \( C^{\text{ext}} \). Hence, it appears as a weighted set \( V_\lambda \) defined by \( \Sigma \) on \( C^{\text{ext}} \). By the converse argument and the remark on base point contributions, the proof is shown.

Case 2. \( \Sigma \) are \( \Sigma' \) are both linear systems of dimension \( r \), having finite intersection with \( C \), such that \( \text{degree}(\Sigma) = \text{degree}(\Sigma') = n \);

By Theorem 1.3, every weighted set \( W \) in the \( g^r_n \) is defined uniquely by weighted sets \( W_\lambda_1 \) and \( V_\lambda_2 \) in \( \text{Series}(\Sigma_1) \) and \( \text{Series}(\Sigma_2) \) respectively. Let \( (C^{\text{ms}}, \Phi^{\text{ms}}) \) be a non-singular model of \( C \). Using the method of Section 5 in [7] to avoid the technical problem of presentations of \( \Phi^{\text{ms}} \) and base point contributions, we may, without loss of generality, assume that there exist finite covers \( W_1 \subset \text{Par}_\Sigma \times C^{\text{ms}} \) and \( W_2 \subset \text{Par}_{\Sigma'} \times C^{\text{ms}} \).
such that;

\[ j_{k,\Sigma}(\lambda, p_j) \equiv Mult_{(W_1/Par_\Sigma)}(\lambda, p_j) \geq k \text{ iff } I_{italian}(p, \gamma_p j, C, \phi_\lambda) \geq k \]

\[ j_{k,\Sigma'}(\lambda', p_j) \equiv Mult_{(W_2/Par_{\Sigma'})}(\lambda', p_j) \geq k \text{ iff } I_{italian}(p, \gamma_p j, C, \psi_{\lambda'}) \geq k \]

Then consider the sentences;

\[ (\forall \lambda \in Par_\Sigma)(\exists! \lambda' \in Par_{\Sigma'}) \forall x \in C^{ns}[\bigwedge_{k=1}^{n}(j_k(\lambda, x) \leftrightarrow j_k(\lambda', x))] \]

\[ (\forall \lambda' \in Par_{\Sigma'})(\exists! \lambda \in Par_\Sigma) \forall x \in C^{ns}[\bigwedge_{k=1}^{n}(j_k(\lambda', x) \leftrightarrow j_k(\lambda, x))] \] (*)

in the language of \(< P^1(L), C_i >\), considered as a Zariski structure with predicates \(\{C_i\}\) for Zariski closed subsets defined over \(L\), (see [12]). We have, again by results of [12] or [5], that \(< P^1(L), C_i >\prec < P^1(K), C_i >\), for the nonstandard model \(P(K)\) of \(P(L)\). It follows immediately from the algebraic definition of \(j_k\) in [12], that, for any weighted set \(W_{\lambda_1}\) defined by \(Series(\Sigma)\) on \(C^{ext}\), there exists a unique weighted set \(V_{\lambda_2}\) defined by \(Series(\Sigma')\) on \(C^{ext}\) such that \(W_{\lambda_1} = V_{\lambda_2}\), and conversely. Hence, the proof is shown.

Case 3. \(\Sigma\) are \(\Sigma'\) are both linear systems of dimension \(r\), having finite intersection with \(C\);

Let \(n_1 = degree(\Sigma)\) and \(n_2 = degree(\Sigma')\). Then the original \(g_{n,ext}^r\) is obtained from \(Series(\Sigma)\), by removing a fixed point contribution of multiplicity \(n_1 - n\), and, is obtained from \(Series(\Sigma')\), by removing a fixed point contribution of multiplicity \(n_2 - n\). We now imitate the proof of Case 2, with the slight modification that, in the construction of the sentences given by (*), we make an adjustment of the multiplicity statement at the finite number of branches where a fixed point contribution has been removed. The details are left to the reader. \(\square\)

Now, using Definition 1.6, we construct a specialisation operator \(sp: g_{n,ext}^r \rightarrow g_n^r\). We first require the following simple lemma;

**Lemma 1.8.** Let \(C \subset P^w(L)\) be a projective algebraic curve and let \(C^{ext} \subset P^w(K)\) be its nonstandard model. Let \(p' \in C^{ext}\) be a non-singular point, with specialisation \(p \in C\). Then there exists a unique branch \(\gamma'_p\) such that \(p' \in \gamma'_p\).
Proof. We may assume that $p' \neq p$, otherwise $p$ would be non-singular and, by Lemma 5.4 of [7], would be the origin of a single branch $\gamma_p$. Let $(C^{ns}, \Phi)$ be a non-singular model of $C$, then $p'$ must belong to the canonical set $V[\Phi]$, hence there exists a unique $p'' \in C^{ns}$ such that $\Phi(p'') = p'$. By properties of specialisations, $p'' \in C^{ns} \cap V_{p_j}$ for some $p_j \in \Gamma_{[\Phi]}(x, p)$. Hence, by definition of a branch given in Definition 5.15 of [7], we must have that $p' \in \gamma^j_p$. The uniqueness statement follows as well.

We now make the following definition;

Definition 1.9. Let $C \subset P^w(L)$ be a projective algebraic curve and let $C^{ext} \subset P^w(K)$ be its non-standard model. Given a $g^r_n$ on $C$ with extension $g^{r,ext}_n$ on $C^{ext}$, we define the specialisation operator;

$$sp : g^{r,ext}_n \to g^r_n$$

by;

$$sp(\gamma^j_{p'}) = \gamma^j_p, \text{ for } p' \in \text{NonSing}(C^{ext}) \text{ and } \gamma^j_p \text{ as in Lemma 1.8.}$$

$$sp(\gamma^j_p) = \gamma^j_p, \text{ for } p \in \text{Sing}(C^{ext}) = \text{Sing}(C) \text{ and } \{\gamma^1_p, \ldots, \gamma^j_p, \ldots, \gamma^s_p\} \text{ enumerating the branches at } p.$$ 

$$sp(n_1\gamma^{j_1}_{p_1} + \ldots + n_r\gamma^{j_r}_{p_r}) = n_1sp(\gamma^{j_1}_{p_1}) + \ldots + n_rsp(\gamma^{j_r}_{p_r}),$$

for a linear combination of branches with $n_1 + \ldots + n_r = n$

It is also a remarkable fact that, after introducing the notion of a branch, the specialisation operator $sp$ is well defined. This is the content of the following lemma;

Lemma 1.10. Let hypotheses be as in the previous definition, then, if $W$ is a weighted set belonging to $g^{r,ext}_n$, its specialisation $sp(W)$ belongs to $g^r_n$.

Proof. We may assume that there exists a linear system $\Sigma$, having finite intersection with $C$, such that $\text{dimension}(\Sigma) = r$ and $\text{degree}(\Sigma) = n_1$, with the $g^r_n$ and $g^{r,ext}_n$ both defined by $\text{Series}(\Sigma)$, after removing some fixed point contribution $W_0$ of multiplicity $n_1 - n$. Let $W$ be a weighted set of the $g^{r,ext}_n$, then $W \cup W_0 = (C \cap \phi^r)$, for some unique $\lambda' \in Par_{\Sigma}$. 

We claim that $sp(W \cup W_0) = C \cap \phi_\lambda$, for the specialisation $\lambda \in Par_\Sigma$ of $\lambda' \,(\ast)$. As $sp(W_0) = W_0$, it then follows immediately from linearity of $sp$, that $sp(W)$ belongs to the $g_n^r$ as required. We now show $(\ast)$. Let $p \in C$ and let $\gamma_p$ be a branch centred at $p$. By $\gamma_p^{ext}$, we mean the branch at $p$, where $p$ is considered as an element of $C^{ext}$. We now claim that:

$$I_{\gamma_p}(C, \phi_\lambda) = I_{\gamma^{ext}}(C, \phi_\lambda) + \sum_{p' \in (\gamma_p \setminus p)} I_{\gamma^{ext}_{p'}} (C, \phi_\lambda) \,(\ast\ast)$$

Let $(C^{ns}, \Phi) \subset P^{w'}(L)$ be a non-singular model of $C$, such that $\gamma_p$ corresponds to $C^{ns,ext} \cap V_q$, where $q \in \Gamma_\Phi(x, p)$ and $V_q$ is defined relative to the specialisation from $P(K)$ to $P(L)$. Let $C^{ns,ext,ext} \subset P^{w'}(K')$ be a non-standard model of $C^{ns,ext}$, such that $\gamma_0^{ext}$ corresponds to $C^{ns,ext,ext} \cap V_q$, where $V_q$ is defined relative to the specialisation from $P(K')$ to $P(K)$. Then, for $p' \in (\gamma_p \setminus p)$, we can find $q' \in V_q \cap C^{ns,ext}$ such that $\gamma_p^{ext}$ corresponds to $V_q \cap C^{ns,ext,ext}$. We may choose a suitable presentation $\Phi_{\Sigma_1}$ of $\Phi$, such that $Base(\Sigma_1)$ is disjoint from $\Gamma_\Phi(x, p)$, and, therefore, disjoint from $\Gamma_\Phi(x, p')$, for $p' \in (\gamma_p \setminus p)$. Let $\{\overline{\phi_\lambda}\}$ denote the lifted family of on $C^{ns}$ from the presentation $\Phi_{\Sigma_1}$. In this case, we have, by results of [7], that:

$$I_{\gamma_p}(C, \phi_\lambda) = I_q(C^{ns}, \overline{\phi_\lambda})$$

$$I_{\gamma^{ext}}(C, \phi_\lambda) = I_q(C^{ns}, \overline{\phi_\lambda})$$

$$I_{\gamma^{ext}_{p'}}(C, \phi_\lambda) = I_q'(C^{ns}, \overline{\phi_\lambda}) \, (1)$$

By summability of specialisation, see [3];

$$I_q(C^{ns}, \overline{\phi_\lambda}) = I_q(C^{ns}, \overline{\phi_\lambda}) + \sum_{q' \in C^{ns} \cap (V_q \setminus q)} I_{q'}(C^{ns}, \overline{\phi_\lambda}) \, (2)$$

Combining (1) and (2), the result $(\ast\ast)$ follows, as required. Now, suppose that a branch $\gamma_p$ occurs with non-trivial multiplicity in $sp(C \cap \phi_\lambda)$. By Definition 1.9, the contribution must come from either $I_{\gamma^{ext}_{p'}}(C, \phi_\lambda)$ or $I_{\gamma^{ext}_{p'}}(C, \phi_\lambda)$, for some $p' \in (\gamma_p \setminus p)$. Applying $sp$ to $(\ast\ast)$, one sees that the branch $\gamma_p$ occurs with multiplicity $I_{\gamma_p}(C, \phi_\lambda)$. It follows that $sp(C \cap \phi_\lambda) = C \cap \phi_\lambda$, hence $(\ast)$ is shown. The lemma then follows.

$\Box$

We can now reformulate the results of Section 2 and Section 5 of [7] in the language of this refined theory of $g_n^r$. We first make the following
Definition 1.11. Let $C \subset P^w$ be a projective algebraic curve and let a $g^r_n$ be given on $C$. Let $W$ be a weighted set in this $g^r_n$ or its extension $g^r_{n,ext}$ and let $\gamma_p$ be a branch centred at $p$. Then we say that;

$\gamma_p$ is $s$-fold (s-plo) for $W$ if it appears with multiplicity at least $s$.

$\gamma_p$ is multiple for $W$ if it appears with multiplicity at least $2$.

$\gamma_p$ is simple for $W$ if it is not multiple.

$\gamma_p$ is counted (contato) $s$-times in $W$ if it appears with multiplicity exactly $s$.

$\gamma_p$ is a base branch of the $g^r_n$ if it appears in every weighted set.

$\gamma_p$ is $s$-fold for the $g^r_n$ if it is $s$-fold in $W$ for every weighted set $W$ of the $g^r_n$.

$\gamma_p$ is counted $s$-times for the $g^r_n$ if it is $s$-fold for the $g^r_n$ and is counted $s$-times in some weighted set $W$ of the $g^r_n$.

We then have the following;

Theorem 1.12. Local Behaviour of a $g^r_n$

Let $C$ be a projective algebraic curve and let a $g^r_n$ be given on $C$. Let $\gamma_p$ be a branch centred at $p$, such that $\gamma_p$ is counted $s$-times for the $g^r_n$. If $\gamma_p$ is counted $t$ times in a given weighted set $W$, then there exists a weighted set $W'$ in $g^r_{n,ext}$ such that $sp(W') = W$ and $sp^{-1}(t\gamma_p)$ consists of the branch $\gamma_p$ counted $s$-times and $t - s$ other distinct branches $\{\gamma_{p1}, \ldots, \gamma_{pt-s}\}$, each counted once in $W'$.

Proof. Without loss of generality, we may assume that the $g^r_n$ is defined by a linear system $\Sigma$ of dimension $r$, having finite intersection with $C$. Let $W$ be the weighted set defined by $\phi_\lambda$ in $\Sigma$. Suppose that $s = 0$, then $\gamma_p$ is not a base branch for $\Sigma$. Hence, by Lemma 5.25 of [7], we can find $\lambda' \in V_\lambda$, generic in $Par_\Sigma$, and distinct $\{p_1, \ldots, p_t\} = C^{ext} \cap \phi_{\lambda'} \cap (\gamma_p \setminus p)$ such that the intersections at these points are transverse. Let $W'$ be the weighted set defined by $\phi_{\lambda'}$ in $g^r_{n,ext}$. By the proof of (*) in Lemma 1.10, we have that $sp(W') = W$. By the construction of $sp$ in Definition 1.9,
we have that \( sp^{-1}(t_{\gamma_p}) \) consists of the distinct branches \( \{\gamma_{p_1}, \ldots, \gamma_{p_t}\} \), each counted once in \( W' \). If \( s \geq 1 \), then \( \gamma_p \) is a base branch for \( \Sigma \). By Lemma 5.27 of [7], we have that \( I_{\text{italian}}^{\Sigma,\text{mob}}(p, \gamma_p, C, \phi_{\lambda}) = t - s \). The result then follows by application of Lemma 5.28 in [7] and the argument given above.

\[ \square \]

We now note the following;

**Lemma 1.13.** Let a \( g_n^r \) be given on a projective algebraic curve \( C \). Let \( W_0 \) be any weighted set on \( C \) with total multiplicity \( n' \). Then the collection of weighted sets given by \( \{W \cup W_0\} \) for the weighted sets \( W \) in the \( g_n^r \) defines a \( g_n^{r+n'} \).

**Proof.** Let the original \( g_n^r \) be obtained from a linear system \( \Sigma \) of dimension \( r \) and degree \( n'' \), having finite intersection with \( C \), after removing some fixed point contribution \( J \) of total multiplicity \( n'' - n \). Let \( \{\phi_0, \ldots, \phi_r\} \) be a basis for \( \Sigma \) and let \( \{n_1 \gamma_{p_1}, \ldots, n_m \gamma_{p_m}\} \) be the branches appearing in \( W_0 \) with total multiplicity \( n_1 + \ldots + n_m = n' \) (†). Let \( \{H_1, \ldots, H_m\} \) be hyperplanes passing through the points \( \{p_1, \ldots, p_m\} \) and let \( G \) be the algebraic form of degree \( n' \) defined by \( H_1^{n_1} \cdot \ldots \cdot H_m^{n_m} \).

Let \( \Sigma' \) be the linear system of dimension \( r \) defined by the basis \( \{G \cdot \phi_0, \ldots, G \cdot \phi_r\} \). As we may assume that \( C \) is not contained in any hyperplane section, \( \Sigma' \) has finite intersection with \( C \). We claim that \( g_{n''}^r(\Sigma) \subset g_{n''+n'deg(C)}^r(\Sigma') \), in the sense that every weighted set \( W_\lambda \) defined by \( g_{n''+n'deg(C)}^r(\Sigma') \) is obtained from the corresponding \( V_\lambda \) in \( g_{n''}^r(\Sigma) \) by adding a fixed weighted set \( W_1 \supset W_0 \) of total multiplicity \( n'deg(C) \) (∗). The proof then follows as we can recover the original \( g_n^r \) by removing the fixed point contribution \( J \cup (W_1 \setminus W_0) \) from \( g_{n''+n'deg(C)}^r(\Sigma') \). In order to show (∗), let \( W_1 \) be the weighted set defined by \( C \cap G \). By the branched version of the Hyperspatial Bezout Theorem, see Theorem 5.13 of [7], this has total multiplicity \( n'deg(C) \).

We claim that \( W_0 \subset W_1 \) (**) Let \( \gamma_{p_j} \) be a branch appearing in (†) with multiplicity \( s \). By construction, we can factor \( G \) as \( H^s \cdot R \), where \( H \) is a hyperplane passing through \( s \). We need to show that;

\[ I_{\gamma_{p_j}}(C, H^s \cdot R) \geq s \]

or equivalently,

\[ I_{p_j}(C^{n_s}, H^s \cdot R) = I_{p_j}(C^{n_s}, H^s \cdot R) \geq s \]
for a suitable presentation $C^{ns}$ of a non-singular model of $C$, see Lemma 5.12 of [7], where we have used the “lifted” form notation there. Using the method of conic projections, see section 4 of [7], we can find a plane projective curve $C'$ birational to $C^{ns}$, such that the point $p_j$ corresponds to a non-singular point $q$ of $C'$ and;

$$I_{p_j}(C^{ns}, \overline{H} \cdot R) = I_{q}(C', \overline{H} \cdot R) = I_{q}(C', \overline{H} \cdot R)$$

The result then follows by results of the paper [6] for the intersections of plane projective curves. This shows (**). We now need to prove that, for an algebraic form $\phi_\lambda$ in $\Sigma$ and a branch $\gamma^j_p$ of $C$;

$$I_{\gamma^j_p}(C, \phi_\lambda \cdot G) = I_{\gamma^j_p}(C, \phi_\lambda) + I_{\gamma^j_p}(C, G)$$

This follows by exactly the same argument, reducing to the case of intersections between plane projective curves and using the results of [6]. The result is then shown.

Theorem 1.14. Birational Invariance of a $g^r_n$

Let $\Phi : C_1 \leftrightarrow C_2$ be a birational map between projective algebraic curves. Then, given a $g^r_n$ on $C_2$, there exists a canonically defined $g^r_n$ on $C_1$, depending only on the class $[\Phi]$ of the birational map. Conversely, given a $g^r_n$ on $C_1$, there exists a canonically defined $g^r_n$ on $C_2$, depending only on the class $[\Phi^{-1}]$ of the birational map. Moreover, these correspondences are inverse.

Proof. By Lemma 5.7 of [7], $[\Phi]$ induces a bijection;

$$[\Phi]^* : \bigcup_{O \in C_2} \gamma_O \rightarrow \bigcup_{O \in C_1} \gamma_O$$

of branches, with inverse given by $[\Phi^{-1}]^*$.

Then $[\Phi]^*$ extends naturally to a map on weighted sets of degree $n$ by the formula;

$$[\Phi]^*(n_1 \gamma^j_{p_1} + \ldots + n_r \gamma^j_{p_r}) = n_1[\Phi]^*(\gamma^j_{p_1}) + \ldots + n_r[\Phi]^*(\gamma^j_{p_r})$$

for a linear combination of branches $\{\gamma^j_{p_1}, \ldots, \gamma^j_{p_r}\}$ with $n = n_1 + \ldots + n_r$. Therefore, given a $g^r_n$ on $C_2$, we obtain a canonically defined collection $[\Phi]^*(g^r_n)$ of weighted sets on $C_1$ of degree $n$ (*). It is trivial to see that $[\Phi^{-1}]^* \circ [\Phi]^*(g^r_n)$ recovers the original $g^r_n$ on $C_2$, by the fact the map $[\Phi]^*$ on branches is invertible, with inverse given
by $[\Phi^{-1}]^\ast$. Let $C^{ns}$ be a non-singular model of $C_1$ and $C_2$ with morphisms $\Phi_1 : C^{ns} \to C_1$ and $\Phi_2 : C^{ns} \to C_2$ such that $\Phi \circ \Phi_1 = \Phi_2$ and $\Phi^{-1} \circ \Phi_2 = \Phi_1$ as birational maps (see the proof of Lemma 5.7 in [7]). We then have that $[\Phi]^\ast(g_n^r) = [\Phi_1^{-1}]^\ast \circ [\Phi_2]^\ast(g_n^r)$. It remains to prove that this collection given by $(\ast)$ defines a $g_n^r$ on $C_1$. We will prove first that $[\Phi_2]^\ast(g_n^r)$ defines a $g_n^r$ on $C^{ns}$ $(\dagger)$. Let the original $g_n^r$ on $C_2$ be defined by a linear system $\Sigma$, having finite intersection with $C_2$, such that dimension $(\Sigma) = r$ and degree $(\Sigma) = n', \text{after removing some fixed branch contribution of multiplicity } n' - n$. We may assume that $n' = n$, as if the fixed branch contribution in question is given by $W_0$ and $g_n^r \cup W_0 = g_n^{r'}$, then $[\Phi_2]^\ast(g_n^r) \cup [\Phi_2]^\ast(W_0) = [\Phi_2]^\ast(g_n^{r'})$, hence it is sufficient to prove that $[\Phi_2]^\ast(g_n^{r'})$ defines a $g_n^{r'}$. Let $W_1$ be the fixed branch contribution of the $g_n^r$ on $C_2$ and let $g_n^{r''} \subset g_n^r$ be obtained by removing this fixed branch contribution. It will be sufficient to prove that $[\Phi_2]^\ast(g_n^{r''})$ defines a $g_n^r$ on $C^{ns}$ as $[\Phi_2]^\ast(g_n^r) = [\Phi_2]^\ast(g_n^{r''}) \cup [\Phi_2]^\ast(W_1)$ and we may then use Lemma 1.13. Let $\Phi_{\Sigma_1}$ and $\Phi_{\Sigma_2}$ be presentations of the morphisms $\Phi_1$ and $\Phi_2$. We may assume that $\text{Base}(\Sigma_1)$ and $\text{Base}(\Sigma_2)$ are disjoint. Let $\{\phi_x\}$ denote the lifted family of forms on $C^{ns}$, defined by the linear system $\Sigma$ and the presentation $\Phi_{\Sigma_2}$. We claim that $[\Phi_2]^\ast(g_n^{r''})$ is defined by this system after removing its fixed branch contribution. In order to see this, we first show that for any branch $\gamma_p^j$ of $C$,

$$I_{\gamma_p^j}^{\Sigma \text{,mobile}}(C, \phi_\lambda) = I_{p_j}^{\Sigma \text{,mobile}}(C^{ns}, \overline{\phi_\lambda}) \ (\ast) \ (1)$$

where $p_j$ corresponds to $\gamma_p^j$ in the fibre $\Gamma_{[\Phi_2]}(x, p)$, see Section 5 of [7]. By Definition 2.20 and Lemma 5.23 of [7], we have that;

$$I_{p_j}^{\Sigma \text{,mobile}}(C^{ns}, \overline{\phi_\lambda}) = \text{Card}(C^{ns} \cap (\mathcal{V}_{p_j} \setminus p_j) \cap \overline{\phi_\lambda}) \text{ \ for } \lambda' \in \mathcal{V}_\lambda, \text{ generic in } \text{Par}_\Sigma$$

$$I_{\gamma_p^j}^{\Sigma \text{,mobile}}(C, \phi_\lambda) = \text{Card}(C \cap (\gamma_p^j \setminus p) \cap \phi_\lambda) \text{ \ for } \lambda' \in \mathcal{V}_\lambda, \text{ generic in } \text{Par}_\Sigma$$

As $(\gamma_p^j \setminus p)$ is in biunivocal correspondence with $(\mathcal{V}_{p_j} \setminus p_j)$ under the morphism $\Phi_2$, we obtain immediately the result $(\ast)$. Now, using Lemma 5.27 of [7], we have that, if $\gamma_p^j$ appears in a weighted set $W_\lambda$ of the $g_n^{r''}$ with multiplicity $s$, then the corresponding branch $\gamma_{p_j}$ appears in the weighted set $[\Phi_2]^\ast(W_\lambda)$ with multiplicity equal to $s = I_{p_j}^{\text{mobile}}(C^{ns}, \overline{\phi_\lambda})$. Again, using Lemma 5.27 of [7], we obtain that $[\Phi_2]^\ast(W_\lambda)$ is given by $C^{ns} \cap \overline{\phi_\lambda}$, after removing all fixed point contributions of the linear system $\Sigma$. We, therefore, obtain that $[\Phi_2]^\ast(g_n^{r''})$ is defined by $\Sigma$, after removing all fixed point contributions, as required. This proves $(\dagger)$. 
We now claim that, for the given \( g^r_n \) on \( C^{ns} \), \([\Phi^{-1}]^*(g^r_n)\) defines a \( g^r_n \) on \( C_1 \), (††). Let \( \Phi_{\Sigma_3} \) be a presentation of the morphism \( \Phi^{-1} \). If \( \phi_\lambda \) is a form belonging to the linear system \( \Sigma \) defined on \( C^{ns} \), using the presentations \( \Phi_{\Sigma_1} \) and \( \Phi_{\Sigma_3} \) of \( \Phi \) and \( \Phi^{-1} \), we obtain a lifted form \( \phi_\lambda \) on \( C_1 \) and a lifted form \( \phi_\lambda \) on \( C^{ns} \) again. We now claim that, for \( p \in C^{ns} \):

\[
I_{\phi_\lambda}^{\Sigma,\text{mobile}}(C^{ns}, \phi_\lambda) = I_{\phi_\lambda}^{\Sigma,\text{mobile}}(C^{ns}, \phi_\lambda) \tag{2}
\]

In order to see this, first observe that we can obtain the lifted system of forms \( \{\phi_\lambda\} \) directly from the linear system \( \Sigma_4 \), obtained by composing bases of the linear systems \( \Sigma_1 \) and \( \Sigma_3 \). The corresponding morphism \( \Phi_{\Sigma_4} \) defines a birational map of \( C^{ns} \) to itself, which is equivalent to the identity map \( Id \). Now the result follows immediately from Definition 2.20 and Lemma 2.16 of \([7]\), both multiplicities are witnessed inside the canonical set \( W \) of \( \Phi_{\Sigma_4} \), which, in this case, is just the domain of definition of \( \Phi_{\Sigma_4} \) on \( C^{ns} \), see Definition 1.30 of \([4]\). Now, returning to the proof of (††), we may suppose that the given \( g^r_n \) on \( C^{ns} \) is defined by the linear system \( \Sigma \), after removing all fixed point contributions. Combining (1) and (2), we have that, for a branch \( \gamma^j_p \) of \( C_1 \):

\[
I_{\phi_\lambda}^{\Sigma,\text{mobile}}(C_1, \phi_\lambda) = I_{\phi_\lambda}^{\Sigma,\text{mobile}}(C^{ns}, \phi_\lambda) = I_{\phi_\lambda}^{\Sigma,\text{mobile}}(C^{ns}, \phi_\lambda)
\]

The result now follows from the same argument as above, using Lemma 5.27 of \([4]\). This completes the theorem.

**Remarks 1.15.** Using the quoted Theorem 1.33 of \([7]\), one can use the Theorem to reduce calculations involving \( g^r_n \) on projective algebraic curves to calculations on plane projective curves. We will use this property extensively in the following sections.

\[\square\]

We finally note the following;

**Lemma 1.16.** For a given \( g^r_n \), we always have that \( r \leq n \).

**Proof.** The proof is almost identical to Lemma 2.24 of \([7]\). We leave the details to the reader.

\[\square\]
2. A Theory of Complete Linear Series on an Algebraic Curve

We now develop further the theory of \( g^n_r \) on an algebraic curve \( C \), analogously to classical results for divisors on non-singular algebraic curves. We will first assume that \( C \) is a plane projective algebraic curve, defined by some homogeneous polynomial \( F(X,Y,Z) \). Without loss of generality, we will use the coordinates \( x = X/Z \) and \( y = Y/Z \) for local calculations on the curve \( C \), defined in this system by \( f(x,y) = 0 \). Using Theorem 1.14, we will later derive general results for \( g^n_r \) on an algebraic curve from the corresponding calculations for the plane case.

We consider first the case when \( r = 1 \). By results of the previous section, a \( g^1_n \) is defined by a pencil \( \Sigma \) of algebraic curves \( \{ \phi(x,y) + \lambda \phi'(x,y) = 0 \} \lambda \in \mathbb{P}^1 \) (in affine coordinates), after removing some fixed point contribution, where, by convention, we interpret the algebraic curve \( \phi(x,y) + \infty \phi'(x,y) = 0 \) to be \( \phi'(x,y) = 0 \). We assume that the \( g^1_n \) is, in fact, cut out by this pencil. Now suppose that \( \gamma_p \) is a branch of \( C \). We may assume that \( p \) corresponds to the origin \( O \) of the affine coordinate system \( (x,y) \), (use a linear transformation and the result of Lemma 2.1) By Theorem 6.1 of [7], we can find algebraic power series \( \{ x(t), y(t) \} \), with \( x(t) = y(t) = 0 \), parametrising \( \gamma_p \). We can now substitute the power series in order to obtain a formal expression of the form:

\[
\frac{\phi(x(t),y(t))}{\phi'(x(t),y(t))} = \frac{u(t)v(t)}{v(t)u(t)} = t^{i-j}u(t)v(t)^{-1}, \quad \text{where } \{ u(t), v(t), u(t)v(t)^{-1} \} \text{ are units in } L[[t]].
\]

We then define:

\( \text{ord}_{\gamma_p}(\frac{\phi}{\phi'}) = i - j \),

\( \text{val}_{\gamma_p}(\frac{\phi}{\phi'}) = 0 \),

\( \text{if } i > j, (\frac{\phi}{\phi'} \text{ has a zero of order } i - j) \)

\( \text{ord}_{\gamma_p}(\frac{\phi}{\phi'}) = j - i \),

\( \text{val}_{\gamma_p}(\frac{\phi}{\phi'}) = \infty \),

\( \text{if } i < j, (\frac{\phi}{\phi'} \text{ has a pole of order } j - i) \)

\( \text{ord}_{\gamma_p}(\frac{\phi}{\phi'}) = \text{ord}_t(h(t) - h(0)) \),

\( \text{val}_{\gamma_p}(\frac{\phi}{\phi'}) = h(0) \),

\( \text{if } i = j \text{ and } h(t) = u(t)v(t)^{-1} \)
Observe that in all cases, \(\text{ord}_{\gamma_p}\) gives a positive integer, while \(\text{val}_{\gamma_p}\) determines an element of \(P^1\). In order to see that this construction does not depend on the particular power series representation of the branch, we require the following lemma;

**Lemma 2.1.** Let \(\{C, \gamma_p, \phi, \phi', g_n^1, \Sigma\}\) be as defined above, then;

\[
\text{ord}_{\gamma_p}(\frac{\phi}{\phi'}) = I_{\gamma_p}(C, \phi - \lambda \phi') ,
\]

if \(\gamma_p\) is not a base branch for the \(g_n^1\) and \(\frac{\phi}{\phi'}(p) = \text{val}_{\gamma_p}(\frac{\phi}{\phi'}) = \lambda\).

\[
\text{ord}_{\gamma_p}(\frac{\phi}{\phi'}) = I_{\gamma_p}^\Sigma, \text{mobile}(C, \phi - \lambda \phi'),\]

if \(\gamma_p\) is a base branch for the \(g_n^1\) and \(\lambda = \text{val}_{\gamma_p}(\frac{\phi}{\phi'})\) is unique such that, for \(\mu \neq \lambda\);

\[
I_{\gamma_p}(C, \phi - \lambda \phi') > I_{\gamma_p}(C, \phi - \mu \phi').
\]

**Proof.** Suppose that \(\gamma_p\) is not a base branch for the \(g_n^1\), then \(\frac{\phi}{\phi'}(p) = \lambda\) is well defined, if we interpret \((c/0) = \infty\) for \(c \neq 0\), and \(\phi - \lambda \phi'\) is the unique curve in the pencil passing through \(p\). It is trivial to check, using the facts that \(\phi(p) = \phi(x(0), y(0))\) and \(\phi'(p) = \phi'(x(0), y(0))\), that, in all cases, \(\text{val}_{\gamma_p}(\frac{\phi}{\phi'}) = \lambda\) as well. By Theorem 6.1 of [7], we have that;

\[
I_{\gamma_p}(C, \phi - \lambda \phi') = \text{ord}_t([\phi - \lambda \phi'](x(t), y(t))]
\]

If \(\lambda = 0\), then \(\phi(p) = 0\) and \(\phi'(p) \neq 0\), hence, by a straightforward algebraic calculation, \(\phi(x(t), y(t)) = t^i u(t)\), for some \(i \geq 1\), and \(\phi'(x(t), y(t)) = v(t)\) for \(\{u(t), v(t)\}\) units in \(L[[t]]\). Therefore, \(\text{ord}_{\gamma_p}(\frac{\phi}{\phi'}) = \text{ord}_t \phi(x(t), y(t))\) and the result follows.

If \(\lambda = \infty\), then \(\phi(p) \neq 0\) and \(\phi(p) = 0\), hence, \(\phi(x(t), y(t)) = u(t)\) and \(\phi'(x(t), y(t)) = t^j v(t)\), for some \(j \geq 1\), and \(\{u(t), v(t)\}\) units in \(L[[t]]\). Therefore, \(\text{ord}_{\gamma_p}(\frac{\phi}{\phi'}) = \text{ord}_t \phi'(x(t), y(t))\) and the result follows.

If \(\lambda \neq \{0, \infty\}\), then \(\phi(x(t), y(t)) = u(t)\) and \(\phi'(x(t), y(t)) = v(t)\) with \(\{u(t), v(t)\}\) units in \(L[[t]]\). As \(v(t)\) is a unit in \(L[[t]]\), we have that;

\[
\text{ord}_t(\frac{u(t)}{v(t)} - \frac{u(0)}{v(0)}) = \text{ord}_t(v(t)(\frac{u(t)}{v(t)} - \frac{u(0)}{v(0)}) = \text{ord}_t(u(t) - \frac{u(0)}{v(0)} v(t))
\]

Hence, by definition of \(\text{ord}_{\gamma_p}\);
\[ \text{ord}_\gamma(p) = \text{ord}_\gamma((\phi - \lambda \phi')(x(t), y(t))) \]

and the result follows.

Now suppose that \( \gamma_p \) is a base branch for the \( g_n \), then \( \phi(p) = \phi'(p) = 0 \) and we have that \( \phi(x(t), y(t)) = t^i u(t) \) and \( \phi'(x(t), y(t)) = t^j v(t) \), for some \( i, j \geq 1 \) and \( \{u(t), v(t)\} \) units in \( L[[t]] \). Again, we divide the proof into the following cases;

\( i > j \). In this case, by definition, \( \text{val}_\gamma(p) = 0 \). We compute;

\[ \text{ord}_\gamma(t^i u(t) - \lambda t^j v(t)) \]

When \( \lambda = 0 \), we obtain, by Theorem 6.1 of [7], that \( I_{\gamma_p}(C, \phi) = i \) and, for \( \lambda \neq 0 \), that \( I_{\gamma_p}(C, \phi - \lambda \phi') = j \). Using Lemma 5.27 of [7], we obtain that \( I_{\gamma_p}^{\Sigma, \text{mobile}}(C, \phi) = i - j = \text{ord}_\gamma((\phi - \lambda \phi')) \), as required.

\( i < j \). In this case, by definition, \( \text{val}_\gamma(p) = \infty \). The computation for \( \text{ord}_\gamma \) is similar, with the critical value being \( \lambda = \infty \).

\( i = j \). We compute;

\[ \text{ord}_\gamma(t^i u(t) - \lambda t^j v(t)) \]

Again, there exists a unique value of \( \lambda = \frac{u(0)}{v(0)} = \text{val}_\gamma(p) \neq \{0, \infty\} \) such that \( \text{ord}_\gamma(u(t) - \lambda v(t)) = k \geq 1 \). By the same calculation as above, we have that \( I_{\gamma_p}^{\Sigma, \text{mobile}}(C, \phi - \lambda \phi') = k \), for this critical value of \( \lambda \). By a similar algebraic calculation to the above, using the fact that \( v(t) \) is a unit, we also compute \( \text{ord}_\gamma((\phi - \lambda \phi')) = k \), hence the result follows.

We now show the following;

**Lemma 2.2.** Given any algebraic curve \( C \subset P^w \), with function field \( L(C) \), for a non-constant rational function \( f \in L(C) \) and a branch \( \gamma_p \), we can unambiguously define \( \text{ord}_{\gamma_p}(f) \) and \( \text{val}_{\gamma_p}(f) \).

**Proof.** The proof is similar to the above. We may, without loss of generality, assume that \( p \) corresponds to the origin of a coordinate system \((x_1, \ldots, x_w)\). Using Theorem 6.1 of [7], we can find algebraic power series \((x_1(t), \ldots, x_w(t))\) parametrising the branch \( \gamma_p \). By the assumption that \( f \) is non-constant, we can find a representation of \( f \) as a
Now, suppose that $\gamma$ is also an open subset of $C$ be an open subset of $C^2 \Sigma \phi$ be defined by the pencils $\sigma \phi$ and $\phi$ rational function by Lemma 2.1, to show that; val $\{g \in \mathbb{C}, \phi \mid$ of the particular representation of $\phi$ found by $\gamma$ for both the $\lambda_{\phi}$ and $\delta_{\phi}$, we also have that $\frac{\delta_{\phi}}{\phi_2}$ and $\frac{\delta_{\phi}}{\phi_4}$ are defined and equal at $p$, using the argument in Lemma 2.1, we have that val$_{\gamma_p}(\frac{\delta_{\phi}}{\phi_2}) = val_{\gamma_p}(\frac{\delta_{\phi}}{\phi_4})$. It is therefore sufficient, again by Lemma 2.1, to show that;

$$I_{\gamma_p}(C, \phi_1 - \lambda \phi_2) = I_{\gamma_p}(C, \phi_3 - \lambda \phi_4),$$

for $\frac{\delta_{\phi}}{\phi_2}(p) = \frac{\delta_{\phi}}{\phi_4}(p) = \lambda$ ($\dagger$)

Suppose that $I_{\gamma_p}(\phi_1 - \lambda \phi_2) = m$, then, by Lemma 5.25 of [7], we can find $X \in Y_{\lambda} \cap P^1$ and $\{p_1, \ldots, p_m\} = V \cap Y_{\lambda} \cap (\phi_1 - \lambda \phi_2) = 0$ witnessing this multiplicity. As $\{p_1, \ldots, p_m\}$ lie inside $V$, we also have that $\{p_1, \ldots, p_m\} \subset V \cap Y_{\lambda} \cap (\phi_3 - \lambda \phi_4) = 0$, hence $I_{\gamma_p}(C, \phi_3 - \lambda \phi_4) \geq m$. The result ($\dagger$) then follows from the converse argument.

Now, suppose that $\gamma_p$ is one of the finitely many branches of $C$, not lying inside $V$. We will just consider the case when $\gamma_p$ is a base branch for both the $g_n$ and the $g_m$ defined above, the other cases being similar. In order to prove (*) for this branch, it is sufficient, by Lemma 2.1, to show that;

$$I_{\gamma_p \sigma_1,\text{mobile}}^\Sigma(C, \phi_1 - \lambda \phi_2) = I_{\gamma_p \sigma_2,\text{mobile}}^\Sigma(C, \phi_3 - \mu \phi_4),$$

for the critical values $\{\lambda, \mu\}$

and that the critical values $\{\lambda, \mu\}$ coincide, ($\dagger\dagger$).
Using the argument to prove (†), witnessing the corresponding multiplicities in the canonical set \( V \), it follows that for any \( \nu \in P^1 \);

\[
I_{\gamma_p}^{\Sigma_1, \text{mobile}}(C, \phi_1 - \nu \phi_2) = I_{\gamma_p}^{\Sigma_2, \text{mobile}}(C, \phi_3 - \nu \phi_4), \quad (†††)
\]

If the critical values \( \{\lambda, \mu\} \) were distinct, we would have that;

\[
I_{\gamma_p}^{\Sigma_1, \text{mobile}}(C, \phi_1 - \lambda \phi_2) > I_{\gamma_p}^{\Sigma_1, \text{mobile}}(C, \phi_1 - \mu \phi_2) \quad || \quad I_{\gamma_p}^{\Sigma_2, \text{mobile}}(C, \phi_3 - \lambda \phi_4) < I_{\gamma_p}^{\Sigma_2, \text{mobile}}(C, \phi_3 - \mu \phi_4)
\]

which is clearly a contradiction. Hence, \( \lambda = \mu \) and the result (††) follows from (†††). The lemma is shown. □

**Lemma 2.3. Birational Invariance of \( \text{ord}_{\gamma_p} \) and \( \text{val}_{\gamma_p} \)**

Let \( \Phi : C_1 \leftrightarrow C_2 \) be a birational map between projective algebraic curves with corresponding isomorphisms \( \Phi^* : L(C_2) \to L(C_1) \) and \( [\Phi]^* : \bigcup_{\gamma \in C_2} \gamma_p \to \bigcup_{\phi \in C_1} \phi_q \). Then, for non-constant \( f \in L(C_2) \) and \( \gamma_p \) a branch of \( C_2 \), \( \text{ord}_{\gamma_p}(f) = \text{ord}_{[\Phi]^*\gamma_p}(\Phi^* f) \) and \( \text{val}_{\gamma_p}(f) = \text{val}_{[\Phi]^*\gamma_p}(\Phi^* f) \).

**Proof.** Let \( f \) be represented as a rational function by \( \frac{\phi}{\phi_2} \), as in Lemma 2.2, and consider the \( g_1 \) on \( C_2 \), defined by the linear system \( \Sigma = \{\phi_1 - \lambda \phi_2\}_{\lambda \in P^1} \). Let \( \Phi_{\Sigma_1} \) be a presentation of the birational map \( \Phi \). Using this presentation, we may lift the system \( \Sigma \) to a corresponding linear system \( \{\overline{\phi_1 - \lambda \phi_2}\}_{\lambda \in P^1} \). It is trivial to check that \( \Phi^* f \) is represented by the rational function \( \frac{\overline{\phi_1 - \lambda \phi_2}}{\phi_2} \). The proof of Theorem 1.14 shows that, for a branch \( \gamma_p \) of \( C_2 \);

\[
I_{\gamma_p}^{\Sigma, \text{mobile}}(C_2, \phi_1 - \lambda \phi_2) = I_{[\Phi]^*\gamma_p}^{\Sigma, \text{mobile}}(C_1, \overline{\phi_1} - \lambda \overline{\phi_2}), \quad (*)
\]

We now need to consider the following cases;

Case 1. \( \gamma_p \) and \( [\Phi]^*\gamma_p \) are not base branches for \( \Sigma \) on \( C_2 \) and \( C_1 \).

Case 2. \( \gamma_p \) is not a base branch, but \( [\Phi]^*\gamma_p \) is a base branch for \( \Sigma \) on \( C_2 \) and \( C_1 \).

Case 3. \( \gamma_p \) is a base branch and \( [\Phi]^*\gamma_p \) is a base branch for \( \Sigma \) on \( C_2 \) and \( C_1 \).
For Case 1, we have, by Lemma 2.1 and (*);

\[ \text{ord}_{\gamma_p}(\phi_1,\phi_2) = I_{\gamma_p}(C_2,\phi_1 - \lambda\phi_2) = I_{[\Phi]^*\gamma_p}(\phi_1,\phi_2) = \text{ord}_{[\Phi]^*\gamma_p}(\phi_1,\phi_2) \]

where \( \phi_1(\phi_2(q)) = \text{val}_{\gamma_p}(\phi_1,\phi_2) = \text{val}_{[\Phi]^*\gamma_p}(\phi_1,\phi_2) = \lambda \) and \([\Phi]^*\gamma_p = \gamma_q\).

For Case 3, we have, by Lemma 2.1, (*) and a similar argument to the previous lemma, to show that the critical value \( \lambda = \text{val}_{\gamma_p}(\phi_1,\phi_2) \) is also the critical value \( \text{val}_{[\Phi]^*\gamma_p}(\phi_1,\phi_2) \) for the lifted system at the corresponding branch \([\Phi]^*\gamma_p\), that;

\[ \text{ord}_{\gamma_p}(\phi_1,\phi_2) = I_{\gamma_p}(C_1,\phi_1 - \lambda\phi_2) = I_{[\Phi]^*\gamma_p}(\phi_1,\phi_2) = \text{ord}_{[\Phi]^*\gamma_p}(\phi_1,\phi_2) \]

Case 2 is similar, we leave the details to the reader.

The lemma now follows from the previous lemma, that the definitions of \( \text{ord}_{\gamma_p}(f) \), \( \text{ord}_{[\Phi]^*\gamma_p}(\Phi f) \), \( \text{val}_{\gamma_p}(f) \) and \( \text{val}_{[\Phi]^*\gamma_p}(\Phi f) \) are independent of their particular representations.

We now show;

**Lemma 2.4.** Let \( C \) be a projective algebraic curve, then, to any non-constant rational function \( f \) on \( C \), we can associate a \( g_1^n \) on \( C \), which we will denote by \((f)\), where \( n = \deg(f) \).

**Proof.** We define the weighted set \((f = \lambda)\) as follows;

\[ (f = \lambda) := \{n_{\gamma_1}, \ldots, n_{\gamma_r}\} \]

where \( \{\gamma_1, \ldots, \gamma_r\} = \{\gamma : \text{val}_\gamma(f) = \lambda\} \) and \( n_{\gamma} = \text{ord}_\gamma(f) \).

As \( \lambda \) varies over \( P^1 \), we obtain a series of weighted sets \( W_\lambda \) on \( C \). We claim that this series does in fact define a \( g_1^n \). In order to see this, let \( f \) be represented as a rational function by \( \frac{\phi}{\phi'} \). As before, we consider the pencil \( \Sigma \) of forms defined by \( (\phi - \lambda\phi')_{\lambda \in P^1} \). We claim that the series is defined by this system \( \Sigma \), after removing its fixed branch contribution, (*) In order to see this, we compare the weighted sets \((f = \lambda)\) and \( C \cap (\phi - \lambda\phi') \). For a branch \( \gamma_p \) which is not a fixed branch of the system \( \Sigma \), we have, using Lemmas 2.1 and 2.2, that;
\[ \gamma_p \in (f = \lambda) \text{ iff } \text{val}_{\gamma_p}(f) = \lambda \text{ iff } \frac{d}{df}(p) = \lambda \text{ iff } p \in C \cap (\phi - \lambda \phi') \]

In this case, by Lemmas 2.1 and 2.2, we have that;

\[ n_{\gamma_p} = \text{ord}_{\gamma_p}(f) = \text{ord}_{\gamma_p}(\frac{d}{df}) = I_{\gamma_p}(C, \phi - \lambda \phi') \]

For a branch \( \gamma_p \) which is a fixed branch of the system \( \Sigma \), we have, by Lemmas 2.1 and 2.2, that;

\[ \gamma_p \in (f = \lambda) \text{ iff } \text{val}_{\gamma_p}(\phi) = \lambda \text{ iff } p \in C \cap (\phi - \lambda \phi') \text{ and } \lambda \text{ is a critical value for the system } \Sigma \text{ at } \gamma_p. \]

In this case, by Lemmas 2.1 and 2.2, we have that;

\[ n_{\gamma_p} = \text{ord}_{\gamma_p}(f) = \text{ord}_{\gamma_p}(\frac{d}{df}) = I_{\gamma_p}^{\Sigma, \text{mobile}}(C, \phi - \lambda \phi') \] (1)

Let \( I_{\gamma_p} = \min_{\mu \in F} I_{\gamma_p}(C, \phi - \mu \phi') \) be the fixed branch contribution of \( \Sigma \) at \( \gamma_p \). Then, at the critical value \( \lambda \) for the system \( \Sigma \);

\[ I_{\gamma_p}^{\Sigma, \text{mobile}}(\phi - \lambda \phi') = I_{\gamma_p}(C, \phi - \lambda \phi') - I_{\gamma_p} \] (2)

Hence, the result (*) follows from (1), (2) and the definition of \( C \cap (\phi - \lambda \phi') \).

Finally, we show that \( n = \text{deg}(f) \). Let \( \Gamma_f \) be the correspondence determined by the rational map \( f : C \sim P^1 \). By classical arguments, \( \text{deg}(f) \) is equal to the cardinality of the generic fibre \( \Gamma_f(\lambda) \), for \( \lambda \in P^1 \). Fixing a presentation \( \frac{\phi}{\phi'} \) for \( f \), if \( U \subset \text{NonSing}(C) \) is the canonical set for this presentation, one may assume that the generic fibre \( \Gamma_f(\lambda) \) lies inside \( U \). By Lemma 2.17 of [1], one may also assume that the corresponding weighted set of the \( g^1_n \) defined by \( (f = \lambda) \) consists of \( n \) distinct branches, centred at the points of the generic fibre \( \Gamma_f(\lambda) \). Therefore, the result follows.

\[ \square \]

Remarks 2.5. By convention, for a non-zero rational function \( c \in L \setminus \{0\} \), we define \( (c = 0) \) and \( (c = \infty) \) to be the empty weighted sets. The notion of a weighted set in a \( g^1_n \), generalises the classical notion of the divisor on a non-singular curve. Using the above theorem, we can make sense of the notion of linear equivalence of weighted sets.

We make the following definition;
Definition 2.6. Linear equivalence of weighted sets

Let $C$ be an algebraic curve and let $A$ and $B$ be weighted sets on $C$ of the same total multiplicity. We define $A \equiv B$ if there exists a $g^r_n$ on $C$ such that $A$ and $B$ belong to this $g^r_n$ as weighted sets.

Theorem 2.7. Let hypotheses be as in the previous definition. If $A \equiv B$, then there exists a rational function $g$ on $C$, such that $A$ is defined by $(g = 0)$ and $B$ is defined by $(g = \infty)$, possibly after adding some fixed branch contribution.

Proof. If $r = 0$ in the definition, then we must have that $A = B$. Hence, we obtain the statement of the theorem by adding the fixed branch contribution $A$ to the empty $g_0$, defined by $(c = 0) = (c = \infty)$, for a non-constant $c \in L^*$. Otherwise, by the definition of a $g^r_n$, we may, without loss of generality, find a pencil $\Sigma$ of algebraic forms, $\{\phi - \lambda \phi'\}_{\lambda \in P^1}$, having finite intersection with $C$, such that;

$$A = C \cap (\phi - \lambda_1 \phi),$$

$$B = C \cap (\phi - \lambda_2 \phi') \quad (\lambda_1 \neq \lambda_2)$$

Let $f$ be the rational function on $C$ defined by $\frac{\phi}{g}$. If $A$ and $B$ have no branches in common (with multiplicity), (†), then the pencil $\Sigma$ can have no fixed branches and, by Lemma 2.4, we have that;

$$A = (f = \lambda_1)$$

$$B = (f = \lambda_2) \quad (\lambda_1 \neq \lambda_2)$$

Now we can find an algebraic automorphism $\alpha$ of $P^1$, taking $\lambda_1$ to 0 and $\lambda_2$ to $\infty$. We will assume that $\{\lambda_1, \lambda_2\} \neq \infty$, in which case $\alpha$ can be given, for a coordinate $z$ on $P^1$, by the Mobius transformation $\frac{z - \lambda_1}{z - \lambda_2}$. The other cases are left to the reader. Let $g$ be the rational function on $C$ defined by $\alpha \circ f$. Now, suppose that $\gamma$ is a branch of $C$, with $val_\gamma(f) = \lambda$ and $ord_\gamma(f) = m$. Then, we claim that $val_\gamma(g) = \alpha(\lambda)$ and $ord_\gamma(g) = m$, (†). If $\lambda \neq \{\lambda_2, \infty\}$, using the method before Lemma 2.1, we obtain the following power series representation of $g$ at $\gamma$;

$$\frac{(\lambda + \mu t^m + o(t^m)) - \lambda_1}{(\lambda + \mu t^m + o(t^m)) - \lambda_2} = \left[\frac{1}{(\lambda - \lambda_2)} - \frac{\mu}{(\lambda - \lambda_2)^2} t^m + o(t^m)\right] \frac{1}{(\lambda - \lambda_2) t^m + o(t^m)}$$

$$= \frac{\lambda - \lambda_1}{\lambda - \lambda_2} + t^m \left[\frac{\mu(\lambda - \lambda_2) - \mu(\lambda - \lambda_1)}{(\lambda - \lambda_2)^2} \right] + o(t^m)$$
and the claim (*) follows from the assumption that \( \lambda_1 \neq \lambda_2 \). If \( \lambda = \lambda_2 \), we obtain the following power series representation of \( g \) at \( \gamma \):

\[
\frac{(\lambda + \mu t^m + o(t^m))_1 - \lambda_1}{(\mu t^m + o(t^m))_1} = \frac{1}{t^m} \cdot [(\lambda - \lambda_1) + \mu t^m + o(t^m)] \cdot [\mu + o(1)]^{-1}
\]

which gives that \( \text{val}_\gamma(g) = \infty = \alpha(\lambda_2) \) and \( \text{ord}_\gamma(g) = m \), using the fact that \( \lambda \neq \lambda_1 \). Finally, if \( \lambda = \infty \), the Mobius transformation at \( \infty \) is given by \( \frac{1}{\frac{1}{\lambda_1}} = \frac{1}{\frac{1}{\lambda_1} - \lambda_2} \) and \( g \) may be represented at \( \gamma \) by \( \frac{\phi - \lambda_1 \phi'}{\phi - \lambda_2 \phi'} \). We then obtain the power series representation of \( g \) at \( \gamma \):

\[
\frac{(t^m u(t) - \lambda_1 t^m v(t))}{(t^m u(t) - \lambda_2 t^m v(t))} = \frac{(u(t) - \lambda_1 t^m v(t))}{(u(t) - \lambda_2 t^m v(t))} = \frac{1 - \lambda_1 t^m w(t)}{1 - \lambda_2 t^m w(t)}
\]

\[
= 1 + (\lambda_2 - \lambda_1)t^m w(t) + o(t^m), \text{ for } \{u(t), v(t), w(t)\} \text{ units in } L[[t]]
\]

which gives that \( \text{val}_\gamma(g) = 1 = \alpha(\infty) \) and \( \text{ord}_\gamma(g) = m \), using the fact that \( \lambda_1 \neq \lambda_2 \) again. This gives the claim (*). It follows that the weighted sets \( (f = \lambda) \) correspond exactly to the weighted sets \( (g = \alpha(\lambda)) \), in particular the \( g^1_n \) defined by \( (f) \) and \( (g) \), as in Lemma 2.4, is the same. With this new parametrisation of the \( g^1_n \), we then have that:

\[
A = (g = 0)
\]

\[
B = (g = \infty)
\]

Hence, the result follows, with the assumption (†). If \( A \) and \( B \) have branches in common, with multiplicity, we let \( A \cap B \) denote the weighted set consisting of these common branches (with multiplicity). Then, the same argument holds, replacing \( A \) by \( A \setminus B = A - (A \cap B) \) and \( B \) by \( B \setminus A = B - (A \cap B) \). After adding the fixed branch contribution \( (A \cap B) \) to the \( g^1_n \) defined by \( (g) \), we then obtain the result. Note that, by Lemma 1.13, this addition defines a \( g^1_{n+n'} \), where \( n' \) is the total multiplicity of \( (A \cap B) \).

\[\square\]

**Remarks 2.8.** The definition we have given of linear equivalence of weighted sets on a projective algebraic curve \( C \) generalises the modern
Modern Definition; Let $A$ and $B$ be effective divisors on a smooth projective algebraic curve $C$, then $A \equiv B$ iff $A - B = \text{div}(g)$, for some $g \in L(C)^\ast$.

See, for example, p161 of [11] for relevant definitions and notation.

We now show that our definition is the same in this case. First, observe that there exists a natural bijection between the set of effective divisors on $C$, in the sense of [11], and the collection of weighted sets on $C$, $(\ast)$. This follows immediately from the fact, given in Lemma 5.29 of [7], that, for each point $p \in C$, there exists a unique branch $\gamma_p$, centred at $p$. Secondly, observe that the notion of $\text{div}(g)$, for $g \in L(C)$, as given in [11], is the same as the notion of $\text{div}(g)$ which we give in Definition 2.9 below, (taking into account the identification $(\ast)$), $(\dagger)$. This amounts to checking that, for a point $p \in C$, with corresponding branch $\gamma_p$;

$$v_p(g) = \text{ord}_{\gamma_p}(g)$$

where $v_p(g)$ is defined in p152 of [11], and we temporarily adopt the convention that $\text{ord}_{\gamma_p}(g) = 0$ if $\text{val}_{\gamma}(g) \neq \{0, \infty\}$ and $\text{ord}_{\gamma_p}(g)$ is counted negatively if $\text{val}_{\gamma_p}(g) = \infty$. First, one can use the fact, given in Lemma 4.9 of [7], together with remarks from the final section of this paper, that there exists a birational map $\phi : C \dashrightarrow C'$, such that $C'$ is a plane projective algebraic curve, and $p$ corresponds to a non-singular point $p' \in C'$ with $\{p, p'\}$ lying inside the canonical sets associated to $\phi$. Using the calculation given below, in Lemma 2.10, for $\text{ord}_{\gamma_p}$, and the definition of $v_p$, one can assume that $v_p(g) \geq 0$ and $g \in O_{p,C}$. Let $g' \in L(C')$ denote the corresponding rational function to $g$ on $L(C)$. It is then a trivial algebraic calculation, using the fact that the local rings $O_{p,C}$ and $O_{p',C'}$ are isomorphic, to show that $v_p(g) = v_p(g')$. It also follows from Lemma 2.3 that $\text{ord}_{\gamma_p}(g) = \text{ord}_{\gamma_p}(g')$. Hence, it is sufficient to check $(\dagger)$ for the plane projective curve $C'$. We may, without loss of generality, assume that $v_p(g') \geq 1$ and that $g'$ is represented in some choice of affine coordinates $\{x, y\}$ by the polynomial $q(x, y)$. If $Q(X, Y, Z)$ denotes the projective equation of this polynomial and $p$ corresponds to the origin of this coordinate system, then;
where \( h \) is a defining equation for \( C' \) in the coordinate system \( \{x, y\} \) and \( I_p \) is the algebraic intersection multiplicity. It also follows from Lemma 2.1, that;

\[
ord_{\gamma_p}(g') = I_{\gamma_p}(C, Q)
\]

Hence, it is sufficient to check that;

\[
I_p(C, Q) = I_{\gamma_p}(C, Q)
\]

This calculation was done in the paper [6], hence (††) and therefore (†) is shown. Thirdly, it remains to check that the definitions of linear equivalence are the same. In order to see this, observe that we can write (for effective divisors or weighted sets \( A \) and \( B \));

\[
A - B = (A \setminus B) + (A \cap B) - [(B \setminus A) + (A \cap B)] = (A \setminus B) - (B \setminus A), \quad (\dagger\dagger\dagger)
\]

If \( A \equiv B \) in the sense of weighted sets (Definition 2.6), then the calculation (\dagger\dagger\dagger) (which removes the fixed branch contribution) and Theorem 2.7 shows that \( A - B = \text{div}(g) \), for some rational function \( g \in L(C) \), where, here, \( \text{div}(g) \) is as defined in Definition 2.9. By (†), it then follows that \( A \equiv B \) as effective divisors. Conversely, if \( A \equiv B \) as effective divisors, then there exists a rational function \( g \in L(C) \) such that \( A - B = \text{div}(g) \), in the sense of the modern definition given above. The above calculations (\dagger\dagger\dagger) and (†) then show that \( \text{div}(g) = (A \setminus B) - (B \setminus A) \), in the sense of Definition 2.9 below. It follows, by Lemma 2.4, that there exists a \( g^1_n \) to which \( (A \setminus B) \) and \( (B \setminus A) \) belong as weighted sets. Adding the fixed branch contribution \( (A \cap B) \) to this \( g^1_n \), we then obtain that \( A \equiv B \) in the sense of Definition 2.6, as required.

**Definition 2.9.** Let \( C \) be a projective algebraic curve and let \( f \) be a non-zero rational function on \( C \). Then we define \( \text{div}(f) \) to be the weighted set \( A - B \) where;

\[
A = (f = 0), \quad B = (f = \infty)
\]

We now require the following lemma;
Lemma 2.10. Let $C$ be a projective algebraic curve, and let $f$ and $g$ be non-zero rational functions on $C$. Then;

$$\text{div}(\frac{1}{f}) = -\text{div}(f)$$

$$\text{div}(fg) = \text{div}(f) + \text{div}(g)$$

$$\text{div}(\frac{f}{g}) = \text{div}(f) - \text{div}(g)$$

Proof. In order to prove the first claim, it is sufficient to show that, for a branch $\gamma$ of $C$;

$$\text{val}_\gamma(f) = 0 \iff \text{val}_\gamma(\frac{1}{f}) = \infty$$

$$\text{val}_\gamma(f) = \infty \iff \text{val}_\gamma(\frac{1}{f}) = 0$$

and $\text{ord}_\gamma$ is preserved in both cases. This follows trivially from the relevant power series calculation at a branch. Namely, we can represent $f$ by $\frac{\phi}{\varphi^m}$ and $\frac{1}{f}$ by $\frac{\varphi^m}{\phi}$. Substituting the branch parametrisation, we obtain that;

$$\text{val}_\gamma(f) = 0, \text{ord}_\gamma(f) = m \iff f \sim t^m u(t), \ m \geq 1, u(t) \in L[[t]] \text{ a unit.}$$

$$\text{val}_\gamma(f) = \infty, \text{ord}_\gamma(f) = m \iff \frac{1}{f} \sim t^{-m} u(t)^{-1}$$

and the calculation for $\text{val}_\gamma(f) = \infty, \text{ord}_\gamma(f) = m$ is similar.

In order to prove the second claim, we need to verify the following cases at a branch $\gamma$ of $C$;

Case 1. If $\text{val}_\gamma(f) = \text{val}_\gamma(g) \in \{0, \infty\}$, $\text{ord}_\gamma(f) = m$ and $\text{ord}_\gamma(g) = n$

then $\text{val}_\gamma(fg) \in \{0, \infty\}$ and $\text{ord}_\gamma(fg) = m + n$

Case 2. If $\text{val}_\gamma(f) \neq \text{val}_\gamma(g) \in \{0, \infty\}$, $\text{ord}_\gamma(f) = m$ and $\text{ord}_\gamma(g) = n$

then $\text{val}_\gamma(fg) \in \{0, \infty\}$ and $\text{ord}_\gamma(fg) = |m - n|$

Case 3. If exactly one of $\text{val}_\gamma(f)$ and $\text{val}_\gamma(g)$ is in $\{0, \infty\}$, with $\text{ord}_\gamma(f)$ or $\text{ord}_\gamma(g) = m$
then \( \text{val}_\gamma(fg) \in \{0, \infty\} \), with \( \text{ord}_\gamma(fg) = m \).

Case 4. If neither of \( \text{val}_\gamma(f) \) and \( \text{val}_\gamma(g) \) are in \( \{0, \infty\} \) then \( \text{val}_\gamma(fg) \) is not in \( \{0, \infty\} \).

If \( f \) is represented by \( \frac{\psi}{\phi} \) and \( g \) is represented by \( \frac{\psi'}{\phi'} \), then we can represent \( fg \) by \( \frac{\psi\psi'}{\phi\phi'} \). The proof of these cases then follow by elementary power series calculations at the branch \( \gamma \). For example, for Case 2, if \( \text{val}_\gamma(f) = 0 \) and \( \text{ord}_\gamma(f) = m \), \( \text{val}_\gamma(g) = \infty \) and \( \text{ord}_\gamma(g) = n \), then we have;

\[
f \sim t^mu(t), \ g \sim t^{-m}v(t), \ fg \sim t^n t^{-m}u(t)v(t) = t^{n-m}w(t), \text{ for } \{u(t), v(t), w(t)\} \text{ units in } L[[t]].
\]

The third claim follows from the first two claims. \( \square \)

We now claim the following;

**Theorem 2.11. Transitivity of Linear Equivalence**

Let \( C' \) be an algebraic curve. If \( A, B, C \) are weighted sets on \( C' \) of the same total multiplicity, then, if \( A \equiv B \) and \( B \equiv C \), we must have that \( A \equiv C \).

**Proof.** By Theorem 2.7, we can find rational functions \( f \) and \( g \) on \( C' \), such that;

\[
(A \setminus B) - (B \setminus A) = \text{div}(f)
\]

\[
(B \setminus C) - (C \setminus B) = \text{div}(g)
\]

By Lemma 2.9, we have that;

\[
\text{div}(fg) = (A \setminus B) - (B \setminus A) + (B \setminus C) - (C \setminus B)
\]

By drawing a Venn diagram, one easily checks that;

\[
(A \setminus B) - (B \setminus A) = (A \cap B^c \cap C^c) + (A \cap B^c \cap C) - (A^c \cap B \cap C^c) - (A^c \cap B \cap C)
\]
\[(B \setminus C) - (C \setminus B) = (A \cap B \cap C) + (A^c \cap B \cap C) - (A \cap B^c \cap C) \]

\[\| \]

\[(A \setminus C) - (C \setminus A) = (A \cap B^c \cap C^c) + (A \cap B \cap C^c) - (A^c \cap B^c \cap C) - (A^c \cap B \cap C)\]

Hence, \(\text{div}(fg) = (A \setminus C) - (C \setminus A)\). Now, given the \(g^1\) defined by the rational function \(fg\), as in Lemma 2.4, it follows that \((A \setminus C)\) and \((C \setminus A)\) belong to this \(g^1\) as weighted sets. We can now add the fixed branch contribution \(A \cap C\) to this \(g^1\), giving a \(g^1 + s\), to which \(A\) and \(C\) belong as weighted sets. Therefore, the result follows.

\[\square\]

As an immediate corollary, we have;

**Theorem 2.12.** Let \(C\) be a projective algebraic curve, then \(\equiv\) is an equivalence relation on weighted sets for \(C\) of a given multiplicity.

We also have;

**Theorem 2.13.** Linear Equivalence preserved by Addition

Let \(C'\) be a projective algebraic curve and suppose that \(\{A, B, C, D\}\) are weighted sets on \(C'\) with;

\[A \equiv B \text{ and } C \equiv D\]

then;

\[A + C \equiv B + D\]

**Proof.** By Definition 2.6, we can find a \(g^r\) containing \(C\) and \(D\) as weighted sets. If \(s\) is the total multiplicity of \(A\), then, by Lemma 1.13, we can add the weighted set \(A\) as a fixed branch contribution to this \(g^r\) and obtain a \(g^r_{n+s}\), containing \(A + C\) and \(A + D\) as weighted sets. Hence, by Definition 2.6 again, we have that;

\[A + C \equiv A + D \ (1)\]

Similarly, one shows, by adding \(D\) as a fixed branch contribution to the \(g^r_n\) containing \(A\) and \(B\) as weighted sets, that;
A + D \equiv B + D \ (2)

The result then follows immediately by combining (1), (2) and using Theorem 2.11.

We now develop further the theory of \( g_n^r \) on a projective algebraic curve \( C \). We begin with the following definition:

**Definition 2.14. Subordinate \( g_n^r \)**

Let \( \{g_n^r, g_n^s\} \) be given on \( C \) with the same order \( n \). Then we say that:

\[ g_n^r \subseteq g_n^i \]

if every weighted set in \( g_n^r \) is included in the weighted sets of the \( g_n^i \).

We now claim the following:

**Theorem 2.15. Amalgamation of \( g_n^r \)**

Let \( \{g_n^r, g_n^s\} \) be given on \( C \), having a common weighted set \( G \), then there exists \( t \) with \( r \leq t, s \leq t \) and a \( g_n^t \) such that \( g_n^r \subseteq g_n^t \) and \( g_n^s \subseteq g_n^t \).

**Proof.** Assume first that \( \{g_n^r, g_n^s\} \) have no fixed branch contribution and are defined exactly by linear systems. Then we can find algebraic forms \( \{\phi_0, \psi_0\} \) such that:

\[ G = (C \cap \phi_0 = 0) = (C \cap \psi_0 = 0) \]

and;

\( g_n^r \) is defined by \( C \cap (\epsilon_0\phi_0 + \epsilon_1\phi_1 + \ldots + \epsilon_r\phi_r = 0) \)

\( g_n^s \) is defined by \( C \cap (\eta_0\psi_0 + \eta_1\psi_1 + \ldots + \eta_s\psi_s = 0) \)

Now consider the linear system \( \Sigma \) defined by:

\[ \epsilon\phi_0\psi_0 + \psi_0(\epsilon_1\phi_1 + \ldots + \epsilon_r\phi_r) + \phi_0(\eta_1\psi_1 + \ldots + \eta_s\psi_s) = 0 \]

and let \( g_n^t \) be defined by \( \Sigma \). As \( \text{deg}(\psi_0\phi_0) = \text{deg}(\psi_0) + \text{deg}(\phi_0) \), we have that \( m = 2n \). We claim that the fixed branch contribution of \( g_{2n}^t \)
is exactly $G$, $(\ast)$. In order to see this, observe that we can write an algebraic form in $\Sigma$ as;

$$\psi_0 \phi_\xi + \phi_0 \psi_\eta$$

If $\gamma$ is a branch counted $w$-times in $G$, then, using the proof at the end of Lemma 1.13 and linearity of multiplicity at a branch, see [7];

$$I_\gamma(C, \psi_0 \phi_\xi) = I_\gamma(C, \psi_0) + I_\gamma(C, \phi_\xi) \geq w$$

$$I_\gamma(C, \phi_0 \psi_\eta) = I_\gamma(C, \phi_0) + I_\gamma(C, \psi_\eta) \geq w$$

$$I_\gamma(C, \psi_0 \phi_\xi + \phi_0 \psi_\eta) = \min\{I_\gamma(C, \psi_0 \phi_\xi), I_\gamma(C, \phi_0 \psi_\eta)\} \geq w (\dagger)$$

Hence, $\gamma$ is $w$-fold for the $g^t_{2n}$ and $G$ is contained in the fixed branch contribution of the $g^t_{2n}$. In order to obtain the exactness statement, $(\ast)$, first observe that, if $\gamma$ is a fixed branch of the $g^t_{2n}$, then, in particular, it belongs to $(C \cap \phi_0 \psi_0 = 0)$. Hence, it belongs either to $(C \cap \phi_0 = 0)$ or $(C \cap \psi_0 = 0)$. Hence, it belongs to $G$. Now, using the fact that the original $\{g^r_n, g^s_n\}$ had no fixed branch contribution, we can easily find $\phi_\xi_0$ and $\psi_\eta_0$ with $G$ disjoint from both $(C \cap \phi_\xi_0 = 0)$ and $(C \cap \psi_\eta_0 = 0)$. Then, by the same argument $(\dagger)$, we obtain, for a branch $\gamma$ of $G$;

$$I_\gamma(C, \psi_0 \phi_\xi_0 + \phi_0 \psi_\eta_0) = w$$

hence, $\gamma$ is counted $w$-times in $C \cap (\psi_0 \phi_\xi_0 + \phi_0 \psi_\eta_0 = 0)$ and, therefore, $(\ast)$ holds, as required. Now, as $G$ had total multiplicity $n$, removing this fixed branch contribution from the $g^t_{2n}$, we obtain a $g^t_n$. We then claim that $g^r_n \subseteq g^t_n$ and $g^s_n \subseteq g^t_n$, $(\ast\ast)$. By Definition 2.14, it is sufficient to check that, if $\{W_1, W_2\}$ are weighted sets appearing in $\{g^r_n, g^s_n\}$, defined by $(C \cap \phi_\xi = 0)$ and $(C \cap \psi_\eta = 0)$, then they appear in the $g^t_n$. We clearly have that both $\psi_0 \phi_\xi$ and $\phi_0 \psi_\eta$ belong to $\Sigma$ and the calculation $(\dagger)$ shows that;

$$C \cap (\psi_0 \phi_\xi = 0) = W_1 + G$$

$$C \cap (\phi_0 \psi_\eta = 0) = W_2 + G$$

Hence, the result $(\ast\ast)$ follows after removing the fixing branch contribution $G$. The fact that $r \leq t$ and $s \leq t$ then follows easily from the definition of the dimension of a $g^r_n$ and Theorem 1.3.
Now consider the case when the \( \{g_n^r, g_n^s\} \) are defined exactly by linear systems and have a fixed branch contribution. Let \( G_1 \subseteq G \) and \( G_2 \subseteq G \) be these fixed branch contributions and let \( G_3 = G_1 \cap G_2 \). We claim that the fixed branch contribution of the \( g_{2n}^t \) defined by \( \Sigma \), as given above, in this case is exactly \( G_3 + G \). The proof is similar to the above and left to the reader. Now, removing the fixed branch contribution \( G \), we obtain a series \( g_n^t \) with fixed branch contribution \( G_3 \). A similar proof to the above, left to the reader, shows that this \( g_n^t \) contains the original \( \{g_n^r, g_n^s\} \). Finally, we need to consider the case when the \( \{g_n^r, g_n^s\} \) are defined, after removing some fixed branch contribution from linear series. Let \( G_1 \) and \( G_2 \), with total multiplicity \( r_1 \) and \( r_2 \), be these fixed branch contributions and let \( \{g_n^r + r_1, g_n^s + r_2\} \) be the series obtained from adding these fixed branch contributions to \( \{g_n^r, g_n^s\} \). In this case, the linear system \( \Sigma \), as given above, defines a \( g_{2n+r_1+r_2}^t \). We claim that the weighted set \( G \cup G_1 \cup G_2 \), of total multiplicity \( n + r_1 + r_2 \), is contained in the fixed branch contribution of this series. This follows from a similar calculation, using the method above, the details are left to the reader. Removing this weighted set from the \( g_{2n+r_1+r_2}^t \), we obtain a \( g_n^t \) and a similar calculation shows that this contains the original \( \{g_n^r, g_n^s\} \), again the details are left to the reader. □

As a corollary, we have;

**Theorem 2.16.** Let a \( g_n^r \) be given on \( C \), then there exists a unique \( g_n^t \) on \( C \), with \( r \leq t \leq n \), such that;

\[
g_n^r \subseteq g_n^t
\]

and, for any \( g_n^r \) such that \( g_n^r \subseteq g_n^s \), we have that;

\[
g_n^s \subseteq g_n^t
\]

**Proof.** By Lemma 1.16, we can find \( r \leq t \leq n \) and a \( g_n^t \) on \( C \), with \( g_n^r \subseteq g_n^s \) and \( t \) maximal with this property. If \( g_n^r \subseteq g_n^s \), then \( \{g_n^r, g_n^s\} \) would contain a common weighted set. By Theorem 2.15, we could then find \( t' \leq n \) such that \( s \leq t', \ t \leq t' \) and \( g_n^s \subseteq g_n^{t'}, g_n^{t'} \subseteq g_n^{t''} \). If \( g_n^s \nsubseteq g_n^t \), then, by elementary dimension considerations, we would have that \( t < t' \leq n \) and \( g_n^r \subseteq g_n^{t'} \), contradicting maximality of \( t \). Hence, \( g_n^s \subseteq g_n^t \). The uniqueness statement also follows from a similar amalgamation argument, using Theorem 2.15. □
We can then make the following definition:

**Definition 2.17.** We call a $g_n^r$ on $C$ complete if it cannot be strictly contained in a $g_n^t$ of greater dimension. If $G$ is any weighted set on $C$ of total multiplicity $n$, then we define $|G|$ to be the unique complete $g_n^t$ to which $G$ belongs.

We then have that:

**Theorem 2.18.** Let $G$ be a weighted set on $C$, then, $G \equiv G'$ if and only if $G'$ belongs to $|G|$. In particular, $G \equiv G'$ if and only if $|G| = |G'|$.

**Proof.** The proof of the first part of the theorem is quite straightforward. By definition, if $G'$ belongs to $|G|$, then $G \equiv G'$. Conversely, if $G' \equiv G$, then, by Definition 2.6, we can find a $g_n^1$, containing the given weighted sets $G$ and $G'$. By Theorem 2.16, we can find a unique complete $g_n^t$ on $C$, with $1 \leq t \leq n$, such that $g_n^1 \subseteq g_n^t$. As $G$ belongs to this $g_n^t$ as a weighted set, it follows by Definition 2.17 that $|G| = g_n^t$. Hence, $G'$ belongs to $|G|$ as required. For the second part, if $G \equiv G'$, then, by the first part, $G'$ belongs to $|G|$. It follows immediately from Definition 2.17 and Theorem 2.16, that $|G| \subseteq |G'|$. Reversing this argument, we have that $|G'| \subseteq |G|$, hence $|G| = |G'|$ as required. Conversely, if $|G| = |G'|$, then clearly $G \equiv G'$ by Definition 2.6. □

We now make the following definition:

**Definition 2.19. Linear System of a Weighted Set**

Let $G$ be a weighted set on a projective algebraic curve $C$, then we define the Riemann-Roch space $\mathcal{L}(C, G)$ or $\mathcal{L}(G)$ to be the vector space defined as:

$$\{ g \in L(C)^* : \text{div}(g) + G \geq 0 \} \cup \{0\}$$

where $\text{div}(g)$ was defined in Definition 2.9.

**Remarks 2.20.** That $\mathcal{L}(G)$ defines a vector space follows easily from Lemma 2.10, the fact that, for non-constant rational functions $\{f, g, f + g\} \subset L(C)$ and a branch $\gamma$ of $C$, we have that;

$$\text{ord}_\gamma(f + g) \geq \min\{\text{ord}_\gamma(f), \text{ord}_\gamma(g)\}, \ (*)$$
where, for this remark only, \( \text{ord}_\gamma \) is counted negatively if \( \text{val}_\gamma \) is infinite, and an argument on constants, (**) . We now give a brief proof of (*);

We just consider the following 2 cases;

Case 1. \( \text{val}_\gamma(f) < \infty \) and \( \text{val}_\gamma(g) < \infty \)

We then have, substituting the relative parametrisations, that;

\[
f \sim c + c_1 t^m + \ldots \text{ and } g \sim d + d_1 t^n + \ldots , \text{ where } \text{ord}_\gamma(f) = m \geq 1, \text{ ord}_\gamma(g) = n \geq 1 \text{ and } \{c_1, d_1\} \subset L \text{ are non-zero}. \text{ Then;}
\]

\[
f + g \sim (c + d) + c_1 t^m + d_1 t^n + \ldots
\]

If \((f + g) - (c + d) \equiv 0\), as an algebraic power series in \( L[[t]] \), then \((f + g) = (c + d)\) as a rational function on \( C \), contradicting the assumption. Hence, we obtain that \( \text{ord}_\gamma(f + g) = \min\{\text{ord}_\gamma(f), \text{ord}_\gamma(g)\} \), if \( m \neq n \) or \( m = n \) and \( c_1 + d_1 \neq 0 \), and \( \text{ord}_\gamma(f + g) > \min\{\text{ord}_\gamma(f), \text{ord}_\gamma(g)\} \) otherwise. Hence, (*) is shown in this case.

Case 2. \( \text{val}_\gamma(f) = \text{val}_\gamma(g) = \infty \)

We then have that;

\[
f \sim c_1 t^{-m} + \ldots \text{ and } g \sim d_1 t^{-n} + \ldots , \text{ where } \text{ord}_\gamma(f) = -m \leq -1, \text{ ord}_\gamma(g) = -n \leq -1 \text{ and } \{c_1, d_1\} \subset L \text{ are non-zero}. \text{ Then;}
\]

\[
f + g \sim c_1 t^{-m} + d_1 t^{-n} + \ldots
\]

By the assumption that \( f + g \) is not a constant, if \( m = n \) and \( c_1 + d_1 = 0 \), we must have higher order terms in \( t \) in the Cauchy series for \((f + g)\), hence \( \text{ord}_\gamma(f + g) > \min\{\text{ord}_\gamma(f), \text{ord}_\gamma(g)\} \). Otherwise, we have that \( \text{ord}_\gamma(f + g) = \min\{\text{ord}_\gamma(f), \text{ord}_\gamma(g)\} \), hence (*) is shown in this case as well.

The remaining cases are left to the reader. One should also consider the case of constants, (**) . Technically, one cannot define \( \text{ord}_\gamma \) for a constant in \( L \). However, we did, by convention, define \( \text{div}(c) = 0 \), for \( c \in L^* \), in Remarks 2.5.
We now show the following;

**Lemma 2.21.** For a weighted set $G$, $\dim(\mathcal{L}(G)) = t + 1$, where $t$ is given in Definition 2.17. In particular, $\mathcal{L}(G)$ is finite dimensional.

**Proof.** Let $t$ be given by Definition 2.17. If $t = 0$, then $G = (0)$ and $\mathcal{L}(G) = L$. This follows easily from the well known fact that the only regular functions on a projective algebraic curve are the constants (see, for example, [11], p59). In this case, we then have that $\dim(\mathcal{L}(G)) = 1$, as required. Otherwise, let $t \geq 1$ be given as in Definition 2.17, with the unique complete $g^t_n$ containing $G$. After adding some fixed branch contribution $W$, we can find a linear system $\Sigma$, having finite intersection with $C$, with basis $\{\phi_0, \ldots, \phi_j, \ldots, \phi_t\}$ defining this $g^t_n$. Moreover, we may assume that $C \cap \phi_0 = G \cup W$, $(\ast)$. Let $\{f_1, \ldots, f_j, \ldots, f_t\}$ be the sequence of rational functions on $C$ defined by $f_j = \frac{\phi_j}{\phi_0}$. We claim that:

$$\text{div}(f_j) + G \geq 0, \text{ for } 1 \leq j \leq t \ (**)$$

In order to show $(**)$, it is sufficient to prove that, for a branch $\gamma$ with $\text{val}_\gamma(f_j) = \infty$, we have that $\gamma$ belong to $G$ and, moreover, that $\gamma$ is counted at least $\text{ord}_\gamma(f_j)$ times in $G$. Let $\Sigma_j$ be the pencil of forms defined by $(\phi_j - \lambda \phi_0)_{\lambda \in \mathbb{P}^1}$. By the proof of Lemma 2.4, we have that $(f_j = \infty)$ is defined by $(C \cap \phi_0)$, after removing the fixed branch contribution of this pencil. By $(\ast)$ and the fact that the fixed branch contribution of $\Sigma_j$ includes $W$, we have that $(f_j = \infty) \subseteq G$. Hence, $(**) \text{ is shown as required.}$ By Definition 2.19, we then have that $f_j$ belongs to $\mathcal{L}(G)$. We now claim that there do not exist constants $\{c_0, \ldots, c_j, \ldots, c_t\} \subset L$ such that:

$$c_0 + c_1 f_1 + \ldots + c_j f_j + \ldots + c_t f_t = 0 \ (***)$$

as rational functions on $C$. If so, we would have that:

$$c_0 \phi_0 + c_1 \phi_1 + \ldots + c_j \phi_j + \ldots + c_t \phi_t$$

vanished identically on $C$, contradicting the fact that $\Sigma$ has finite intersection with $C$. Hence, by $(***)$, $\{1, f_1, \ldots, f_t\} \subset \mathcal{L}(G)$ are linearly independent and $\dim(\mathcal{L}(G)) \geq t + 1$. Conversely, suppose that $\dim(\mathcal{L}(G)) \geq k + 1$, then we can find $\{1, f_1, \ldots, f_j, \ldots, f_k\} \subset \mathcal{L}(G)$ which are linearly independent, $(\dagger)$. By the usual method of equating denominators, we can find algebraic forms $\{\phi_0, \ldots, \phi_k\}$ of the same
degree, such that \( f_j \) is represented by \( \phi_j \), for \( 1 \leq j \leq k \). Let \( \Sigma \) be the linear system defined by this sequence of forms. By \((†)\), \( \Sigma \) has finite intersection with \( C \). Let \( W \), having total multiplicity \( n' \), be the fixed branch contribution of this system and let \( (C \cap \phi_0) = G_0 \cup W \). We claim that \( G_0 \subseteq G \), \((††)\). Suppose not, then there exists a branch \( \gamma \) with \( I_{\Sigma, \text{mobile}}(C, \phi_0) = s \), where \( \gamma \) is counted strictly less than \( s \)-times in \( G_0 \). By the definition of \( I_{\Sigma, \text{mobile}} \), we can find a form \( \phi_\lambda \) belonging to \( \Sigma \), distinct from \( \phi_0 \), witnessing this multiplicity. Consider the pencil \( \Sigma_\lambda \) defined by \( (\phi_\lambda - \mu \phi_0) \), \( \mu \in \mathbb{P} \). We then clearly have that \( I_{\Sigma_\lambda, \text{mobile}}(C, \phi_0) = s \) as well, \((†††)\). Let \( f_\lambda = \phi_\lambda \). By the proof of Lemma 2.4, we have that \((f_\lambda = \infty)\) is defined by \((C \cap \phi_0)\), after removing the fixed branch contribution of \( \Sigma_\lambda \). By \((†††)\), it follows that the branch \( \gamma \) is counted \( s \)-times in \((f_\lambda = \infty)\) and therefore \( \text{div}(f_\lambda) + G \not\subseteq 0 \). However, \( f_\lambda \) is a linear combination of \( \{1, \ldots, f_k\} \), hence \( f_\lambda \in \mathcal{L}(G) \), which is a contradiction. Hence, \((††)\) is shown. Now, consider the \( g_k \) defined by \( \Sigma \). Let \( W' \) be the weighted set \( G \setminus G_0 \) of total multiplicity \( n'' \). By Lemma 1.13, we can add the weighted set \( W' \) to the \( g_k \) and obtain a \( g_{n+n''}^k \) with fixed branch contribution \( W' \cup W \). Now, removing the fixed branch contribution \( W \) from this \( g_{n+n''}^k \), we obtain a \( g_{n+n''-n}^k \) containing \( G \) exactly as a weighted set. It follows, from Definition 2.17, that \( k \leq t \). Hence, in particular, \( \text{dim}(\mathcal{L}(G)) \) is finite and \( \text{dim}(\mathcal{L}(G)) \leq t + 1 \). Therefore, the lemma is proved.

We now extend the notion of linear equivalence to include virtual, or non-effective, weighted sets.

**Definition 2.22.** We define a generalised weighted set \( G \) on \( C \) to be a linear combination of branches:

\[
n_1 \gamma_{p_1} + \ldots + n_r \gamma_{p_r}
\]

where \( \{n_1, \ldots, n_r\} \) belong to \( \mathbb{Z} \). If \( \{n_1, \ldots, n_r\} \) belong to \( \mathbb{Z}_{\geq 0} \), we call the weighted set effective. Otherwise, we call the weighted set virtual. We define \( n = n_1 + \ldots + n_r \) to be the total multiplicity or degree of \( G \).

**Remarks 2.23.** It is an easy exercise to see that there exist well defined operations of addition and subtraction on generalised weighted sets. It is also easy to check that any generalised weighted set \( G \) may be written uniquely as \( G_1 - G_2 \), where \( \{G_1, G_2\} \) are disjoint effective weighted sets.

**Definition 2.24.** Let \( A \) and \( B \) be generalised weighted sets on \( C \) of the same total multiplicity. Let \( \{A_1, A_2\} \) and \( \{B_1, B_2\} \) be the unique
effective weighted sets, as given by the previous remark. Then we define;

\[(A_1 - A_2) \equiv (B_1 - B_2) \text{ iff } (A_1 + B_2) \equiv (B_1 + A_2)\]

and;

\[A \equiv B \text{ iff } (A_1 - A_2) \equiv (B_1 - B_2)\]

**Remarks 2.25.** Note that if \(\{A'_1, A'_2\}\) and \(\{B'_1, B'_2\}\) are any effective weighted sets such that;

\[A = A'_1 - A'_2 \text{ and } B = B'_1 - B'_2\]

then \(A \equiv B \text{ iff } A'_1 + B'_2 \equiv B'_1 + A'_2\)

The proof is just manipulation of effective weighted sets. We clearly have that;

\[A_1 + A'_2 = A'_1 + A_2 \text{ and } B_1 + B'_2 = B'_1 + B_2\ (\ast)\]

We then have;

\[A \equiv B \text{ iff } A_1 + B_2 \equiv B_1 + A_2 \text{ (Definition 2.20)}\]

\[\text{iff } A_1 + A'_2 + B_2 \equiv B_1 + A_2 + A'_2 \text{ (Theorem 2.12)}\]

\[\text{iff } A'_1 + A_2 + B_2 \equiv B_1 + A_2 + A'_2 \text{ (by \(\ast\))}\]

\[\text{iff } A'_1 + B_2 \equiv B_1 + A'_2 \text{ (Theorem 2.12)}\]

\[\text{iff } A'_1 + B_2 + B'_1 \equiv B_1 + B'_1 + A'_2 \text{ (Theorem 2.12)}\]

\[\text{iff } A'_1 + B_1 + B'_2 \equiv B_1 + B'_1 + A'_2 \text{ (by \(\ast\))}\]

\[\text{iff } A'_1 + B'_2 \equiv B'_1 + A'_2 \text{ (Theorem 2.12)}\]

We then have;

**Theorem 2.26.** Transitivity of Linear Equivalence

Let \(C'\) be an algebraic curve. If \(A, B, C\) are generalised weighted sets on \(C'\) of the same total multiplicity, then, if \(A \equiv B\) and \(B \equiv C\), we must have that \(A \equiv C\).

**Proof.** Let \(\{A_1, A_2\}\), \(\{B_1, B_2\}\) and \(\{C_1, C_2\}\) be the effective weighted sets as given by Remarks 2.23. Then, by Definition 2.24, we have that;
\[(A_1 + B_2) \equiv (B_1 + A_2)\] and \[(B_1 + C_2) \equiv (C_1 + B_2)\]

By Theorem 2.12, we have that;

\[(A_1 + B_1 + B_2 + C_2) \equiv (C_1 + B_1 + B_2 + A_2)\]

It then follows, by Definition 2.6, that there exists a \(g_{n}^1\), containing \((A_1 + B_1 + B_2 + C_2)\) and \((C_1 + B_1 + B_2 + A_2)\) as weighted sets. Clearly \((B_1 + B_2)\) is contained in the fixed branch contribution of this \(g_{n}^1\). Removing this fixed branch contribution, we obtain;

\[A_1 + C_2 \equiv C_1 + A_2\]

By Definition 2.20, we then have that \(A \equiv C\) as required. \(\square\)

It follows immediately from Theorem 2.11 and Theorem 2.22 that;

**Theorem 2.27.** Let \(C\) be a projective algebraic curve, then \(\equiv\) is an equivalence relation on generalised weighted sets for \(C\) of a given total multiplicity.

**Remarks 2.28.** Again, the definition of linear equivalence that we have given for generalised weighted sets on a smooth projective algebraic curve \(C\) is equivalent to the modern definition for divisors. More precisely, we have;

**Modern Definition:** Let \(A\) and \(B\) be divisors on a smooth projective algebraic curve \(C\), then \(A \equiv B\) iff \(A - B = \text{div}(g)\), for some \(g \in L(C)^*\).

See, for example, p161 of [11] for relevant definitions and notation. In order to show that our definition is the same, use Remarks 2.8 and the following simple argument;

\[A \equiv B\] as generalised weighted sets iff \(A_1 + B_2 \equiv B_1 + A_2\)

where \(\{A_1, A_2, B_1, B_2\}\) are the effective weighted sets given by Definition 2.24. Then;

\[A_1 + B_2 \equiv B_1 + A_2\] iff \((A_1 + B_2) - (B_1 + A_2) = \text{div}(g)\) \((g \in L(C)^*)\)
by Remarks 2.8, where \( \text{div}(g) \) is the modern definition. By a straightforward calculation, we have that;

\[
(A_1 + B_2) - (B_1 + A_2) = A - B
\]

as divisors or generalised weighted sets.

Hence, the notions of equivalence coincide.

We also have;

**Theorem 2.29. Linear Equivalence Preserved by Addition**

Let \( C' \) be a projective algebraic curve and suppose that \( \{A, B, C, D\} \) are generalised weighted sets on \( C' \) with;

\[
A \equiv B \text{ and } C \equiv D
\]

then;

\[
A + C \equiv B + D
\]

**Proof.** Let \( \{A_1, A_2\}, \{B_1, B_2\}, \{C_1, C_2\} \) and \( \{D_1, D_2\} \) be effective weighted sets as given by Remarks 2.19. Then, by Definition 2.20, we have that;

\[
A_1 + B_2 \equiv B_1 + A_2 \text{ and } C_1 + D_2 \equiv D_1 + C_2
\]

Hence, by Theorem 2.20;

\[
A_1 + B_2 + C_1 + D_2 \equiv B_1 + A_2 + D_1 + C_2 \quad (*)
\]

We clearly have that;

\[
A + C = (A_1 + C_1) - (A_2 + C_2) \text{ and } B + D = (B_1 + D_1) - (B_2 + D_2)
\]

as an identity of generalised weighted sets. Moreover, as \( (A_1 + C_1), (A_2 + C_2), (B_1 + D_1) \) and \( (B_2 + D_2) \) are all effective, we can apply Remarks 2.23 and \( (*) \) to obtain the result.

We now make the following definition;

**Definition 2.30.** Let \( G \) be a generalised weighted set on a projective algebraic curve \( C \), then we define \( |G| \) to be the collection of generalised
weighted sets $G'$ with $G' \equiv G$. We define $\text{order}(\vert G \vert)$ to be the total multiplicity (possibly negative) of any generalised weighted set in $\vert G \vert$.

**Remarks 2.31.** If $G$ is an effective weighted set, the collection defined by Definition 2.30 is not the same as the collection given by Definition 2.17, as it includes virtual weighted sets. Unless otherwise stated, we will use Definition 2.17 for effective weighted sets. This convention is in accordance with the Italian terminology, we hope that this will not cause too much confusion for the reader.

We now show that the notions of linear equivalence introduced in this section are birationally invariant;

**Theorem 2.32.** Let $\Phi : C_1 \dasharrow C_2$ be a birational map. Let $A$ and $B$ be generalised weighted sets on $C_2$, with corresponding generalised weighted sets $[\Phi]^*A$ and $[\Phi]^*B$ on $C_1$. Then $A \equiv B$, in the sense of either Definition 2.6 or 2.24, iff $[\Phi]^*A \equiv [\Phi]^*B$.

**Proof.** Suppose that $A \equiv B$ in the sense of Definition 2.6. Then, there exists a $g'_n$ on $C_2$ containing $A$ and $B$ as weighted sets. By Theorem 1.14, there exists a corresponding $g'_n$ on $C_1$, containing $[\Phi]^*A$ and $[\Phi]^*B$ as weighted sets. Hence, again by Definition 2.6, $[\Phi]^*A \equiv [\Phi]^*B$. The converse is similar, using $[\Phi^{-1}]^*$. If $A \equiv B$ in the sense of Definition 2.24, then the same argument works.

□

As a result of this theorem, we introduce the following definition;

**Definition 2.33.** Let $\Phi : C_1 \dasharrow C_2$ be a birational map. Then, given a generalised weighted set $A$ on $C_2$, we define;

$$[\Phi]^*|A| = ||[\Phi]^*A|$$

where, in the case that $A$ is effective, $|A|$ can be taken either in the sense of Definition 2.17 or Definition 2.30.

**Remarks 2.34.** The definition depends only on the complete series $|A|$, rather than its particular representative $A$. This follows immediately from Definition 2.17, Definition 2.30 and Theorem 2.32.

We finally introduce the following definition;

**Definition 2.35.** Summation of Complete Series
Let $A$ and $B$ be generalised weighted sets, defining complete series $|A|$ and $|B|$, in the sense of Definition 2.30. Then, we define the sum:

$$|A| + |B|$$

to be the complete series, in the sense of Definition 2.30, containing all generalised weighted sets of the form $A' + B'$ with $A' \in |A|$ and $B' \in |B|$. If $A$ and $B$ are effective weighted sets with $|A|, |B|$ taken in the sense of Definition 2.17, then we make the same definition for the sum in the sense of Definition 2.17.

**Remarks 2.36.** This is a good definition by Theorem 2.13 and Theorem 2.29.

**Definition 2.37.** Difference of Complete Series

Let $A$ and $B$ be generalised weighted sets, defining complete series $|A|$ and $|B|$, in the sense of Definition 2.30. Then, we define the difference:

$$|A| - |B|$$

to be the complete series, in the sense of Definition 2.30, containing all generalised weighted sets of the form $A' - B'$ with $A' \in |A|$ and $B' \in |B|$. If $A$ and $B$ are effective weighted sets with $|A|, |B|$ taken in the sense of Definition 2.17, then we can in certain cases define a difference in the sense of Definition 2.17. (This is called the residual series, the reader can look at [10] for more details)

**Remarks 2.38.** This is again a good definition, for generalised weighted sets $\{A, B\}$, it follows trivially from the previous definition and the fact that $\{A, -B\}$ are also generalised weighted sets.

3. **A geometrical definition of the genus of an algebraic curve**

The purpose of this section is to give a geometrical definition of the genus of an algebraic curve. In the case of a singular curve, this cannot be achieved using purely algebraic methods. Our treatment follows the presentation of Severi in [10].

We begin with the following lemma;
Lemma 3.1. Let $C$ be a projective algebraic curve, and suppose that a $g_n^1$ is given on $C$, with no fixed branch contribution. Then there exist a finite number of weighted sets $W_\lambda$ in the $g_n^1$, possessing multiple branches.

Proof. We may assume that the $g_n^1$ is defined by a pencil $\Sigma$, having finite intersection with $C$. Let $\theta(\lambda)$ be the statement:

$$\theta(\lambda) \equiv \forall y [(y \in \phi_\lambda \cap C) \rightarrow y \in \text{NonSing}(C) \land R.Mult_y(C, \phi_\lambda) = 1]$$

See also Lemma 2.17 of [7]. Then $\theta$ defines a constructible condition on $\text{Par}_\Sigma$ and moreover, using Lemma 2.17 of [7], we have that $\theta$ holds on an open subset $U \subset \text{Par}_\Sigma$. For $\lambda \in U$, we have that the intersection $(C \cap \phi_\lambda)$ is transverse, in the sense of Lemma 2.4 of [7], and is contained in $W$. Now, using Lemma 5.29 of [7] and the definition of $(C \cap \phi_\lambda)$, it follows that each branch of the corresponding weighted set $W_\lambda$ is counted once and lies inside $W$. Hence, if $W_\lambda$ is a weighted set, possessing multiple branches, we must have that $\lambda \in (\text{Par}_\Sigma \setminus U)$. As $\text{Par}_\Sigma$ has dimension 1, this is a finite set, hence the result follows. □

We now make the following definition;

Definition 3.2. Let $C$ be a projective algebraic curve and suppose that a $g_n^1$ is given on $C$, with no fixed branch contribution. Then we define the Jacobian of this $g_n^1$ to be the weighted set:

$$\text{Jac}(g_n^1) = \{\alpha_{\gamma_1}, \ldots, \alpha_{\gamma_j}, \ldots, \alpha_{\gamma_r}\}$$

where $\{\gamma_1, \ldots, \gamma_j, \ldots, \gamma_r\}$ consists of the finitely many branches which are multiple for some weighted set $W_{\lambda_j}$ of the $g_n^1$ and $\gamma_j$ appears with multiplicity $\alpha_{\gamma_j} + 1$ in $W_{\lambda_j}$.

Remarks 3.3. This is a good definition by Lemma 3.1 and the fact that any branch $\gamma$ can only appear in one weighted set $W_\lambda$, using the hypothesis that the $g_n^1$ has no fixed branches.

We now analyse further the Jacobian of a $g_n^1$, with no fixed branch contribution. Using Theorem 1.14 of this paper and Theorem 4.16 of [7], we will derive general results for projective algebraic curves from consideration of the case where $C$ is a plane projective curve, having at most nodes as singularities, (†). Until the end of Theorem 3.21, this assumption will be in force.
Lemma 3.4. Let the $g_n^1$, without fixed branch contribution, be given on $C$. Then there exists a rational function $h$ on $C$, defining this $g_n^1$, such that the weighted set;

$$G = (h = \infty)$$

consists of $n$ distinct branches, each counted once, lying inside $NonSing(C)$.

Proof. By Theorem 2.7, we may assume that the given $g_n^1$ is defined by $(g)$ for some rational function $g$ on $C$. Using the proof of Lemma 3.1, we may assume, that, for generic $\lambda \in P^1$, the weighted set $(g = \lambda)$ consists of $n$ distinct branches, lying inside $NonSing(C)$, each counted once. Using the proof of Theorem 2.7, we can find a Mobius transformation $\alpha$ of $P^1$, taking $\lambda$ to $\infty$, such that $h = \alpha \circ g$ also defines the given $g_n^1$ and such that $G = (h = \infty) = (g = \lambda)$. The result follows.

We now show the following, the reader should refer to [7] for the relevant notation;

Lemma 3.5. Suppose that $\deg(C) = m$, then there exists a homographic change of variables of $P^2$, such that, in this new coordinate system $(x', y')$;

(i). The line at $\infty$ cuts $C$ transversely in $m$ distinct non-singular points.

(ii). The tangent lines to $C$ parallel to the $y'$-axis all have 2-fold contact (contatto), and are based at non-singular points of $C$.

(iii). The branches of $J = Jac(g_n^1)$ and $G = (h = \infty)$ are all in finite position, with base points distinct from the points of contact in (ii).

Proof. We use the fact that a generic point of $C$ has character $(1, 1)$. The proof of this result requires duality arguments, which may be found later in the paper. It follows, by Remark 6.6 of [7], that there exist only finitely many non-ordinary branches. Hence, there exist finitely many tangent lines \{l_{r_1}, \ldots, l_{r_r}\}, based at \{p_1, \ldots, p_r\}, (possibly with repetitions), such that;
\[ I_{\gamma_j}(C, p_j, l_{\gamma_j}) \geq 3, \quad (\text{for } 1 \leq j \leq r) \]

By assumption, \( C \) has at most finitely many nodes as singularities. Let \( \{q_1, \ldots, q_s\} \) be the base points of these nodes and suppose that \( \{l_{\gamma_1 q_1}, l_{\gamma_2 q_2}, \ldots, l_{\gamma_s q_s}\} \) are the 2s tangent lines (possibly with repetitions) corresponding to these nodes. Let \( \{l_{\gamma_1 q_{1s}}, l_{\gamma_2 q_{2s}}, \ldots, l_{\gamma_s q_{ss}}\} \) are the 2s tangent lines (possibly with repetitions) corresponding to these nodes. Let \( \{l_{\gamma_1'}, \ldots, l_{\gamma_t'}\} \) define the tangent lines to each of the branches appearing in \( (\text{Jac}(g_1^n) \cup G) \). Now choose a point \( P \) not lying on \( C \) or any of the above defined tangent lines. Let \( \Sigma = \{l_P^\lambda\}_{\lambda \in \mathbb{P}^1} \) be the pencil defined by all lines passing through the point \( P \). Then \( \Sigma \) defines a \( g_1^m \) on \( C \) without fixed branches.

By the proof of Lemma 3.1, for generic \( \lambda \), \( l_P^\lambda \) intersects \( C \) transversely in \( m \) distinct branches, based at non-singular points of \( C \). Moreover, we may assume, using the fact that the pencil has no base branches, that \( (C \cap l_P^\lambda) \) is disjoint from \( \text{Jac}(g_1^n) \) and \( G \) \((\ast)\). By construction, we also have that, if \( l_{Px} \) belongs to \( \Sigma \) and defines the tangent line \( l_{\gamma_x} \) to a branch \( \gamma_x \) based at \( x \), in the sense of Definition 6.3 of \([7]\), then \( x \in \text{NonSing}(C) \), \( l_{Px} \) has 2-fold contact (contatto) with the branch \( \gamma_x \), \((\ast\ast)\), and the branch \( \gamma_x \) does not appear in \( (\text{Jac}(g_1^n) \cup G) \),\((\ast\ast\ast)\). Now choose a homography, sending the point \([0 : 1 : 0]\) and the line \( Z = 0 \), in the original coordinates \([X : Y : Z]\), to \( P \) and \( l_P^\lambda \). Let \([X' : Y' : Z']\) be the new coordinate system defined by this homography. For the affine coordinate system \((x', y')\), defined by \( X' = \frac{X}{Z} \) and \( Y' = \frac{Y}{Z} \), we have that the line at \( \infty \) has the property \((\ast)\), and, by \((\ast)\), the branches of \( \text{Jac}(g_1^n) \) and \( G \) are all in finite position. The lines parallel to the \( y' \)-axis correspond to the lines, excluding \( l_P^\lambda \), in the pencil defined by \( \Sigma \), in this new coordinate system. Hence, \((\ii)\) follows immediately from \((\ast\ast)\) and \((\iii)\) then follows from \((\ast\ast\ast)\). The lemma is proved.

We now claim the following;

**Lemma 3.6.** Let \( C \) be given in the coordinate system defined by Lemma 3.5, henceforth denoted by \((x, y)\). Then the \( g_1^m \) on \( C \), given by the lines parallel to the \( y \)-axis, and the line at \( \infty \), is defined by \((x)\), as in Lemma 2.4. Moreover, \( J' = \text{Jac}(g_1^n) \) consists exactly of the branches of contact between \( C \) and the lines parallel to the \( y \)-axis, while \( G' = (x = \infty) \) consists of \( m \) distinct branches, centred at non-singular points of \( C \).

**Proof.** In order to prove the first claim, first observe that, by its construction, the given \( g_1^m \) has no fixed branch contribution. Moreover, it is generated by the lines \( X = 0 \) and \( Z = 0 \), that is defined by the pencil of lines \((X - \lambda Z)_{\lambda \in \mathbb{P}^1}\). By Lemma 2.4, because \( x \) is represented by \((\frac{X}{Z})\),
as a rational function on $C$, we have that the series $(x)$ is defined by
$(X - \lambda Z)_{\lambda \in \mathbb{P}^1}$, after removing its fixed branch contribution. As this
series has no fixed branch contribution, the result follows. In order to
compute $J' = \text{Jac}(g_m^1)$, we need to determine;

$$\{\lambda \in \mathbb{P}^1 : C \cap (X - \lambda Z) \text{ contains a multiple branch}\}$$

This corresponds to the set;

$$\{\lambda \in \mathbb{P}^1 : I_{\gamma_p}(C, p, (X - \lambda Z)) \geq 2\}, \text{ for some } p \in C \cap (X - \lambda Z), \gamma_p \text{ a branch at } p.$$ 

As $C$ has at most nodes as singularities, the order of each branch $\gamma$
on $C$ is 1, see Definition 6.3 of [7]. Using Theorem 6.2 of [7], we then have that $I_{\gamma_p}(C, p, (X - \lambda Z)) \geq 2$ iff $(X - \lambda Z)$ is the tangent line $l_{\gamma_p}$
of $\gamma_p$. By construction of the $g_m^1$, this can only occur if $(X - \lambda Z)$ is the
tangent line to an ordinary branch, see Definition 6.3 of [7] again, centred at a non-singular point of $C$, (*) . When $\lambda = \infty$, the corresponding line in the pencil is given by the line at $\infty$, which cuts $C$ transversely, hence this possibility is excluded. Therefore, the only possible values of $\lambda$ occur for lines parallel to the $y$-axis. By (*), for such a branch $\gamma_p$
of contact, we have;

$$I_{\gamma_p}(C, p, (X - \lambda Z)) = 2$$

Now, by definition, $\text{mult}_{\gamma_p}(\text{Jac}(g_m^1)) = 2 - 1 = 1$. Hence, the re-
result follows. For the final part of the lemma, it is easy to verify that
$\{\gamma \in C : \text{val}_{\gamma}(x) = \infty\}$ correspond to the branches of intersection be-
tween $C$ and the line at $\infty$. The result then follows by $(i)$ of Lemma 3.5.

We now claim;

**Lemma 3.7.** Let $\gamma$ be a branch of $C$, in finite position, with coordi-
nates $(a, b)$, which does not belong to $J' = \text{Jac}(g_m^1)$. Then one can find
a power series representation of $\gamma$ of the form;

$$y(x) = b + c_1(x - a) + c_2(x - a)^2 + \ldots (\ast)$$
Remarks 3.8. In the representation (\(*\)), given in Lemma 3.7, we have slightly abused the terminology of Theorem 6.2 in [7]. We mean, here, that \((x, y(x))\) should parametrise the branch \(\gamma\), in the sense that, for any algebraic function \(F(x, y)\), \(F(x, y(x)) \equiv 0\) iff \(F\) vanishes on \(C\), otherwise \(F\) has finite intersection with \(C\) and;

\[
ord_{(x-a)} F(x, y(x)) = ord_{(x-a)} F(a + (x - a), b + c_1(x - a) + \ldots) = I_{\gamma(a,b)}(C, F)
\]

Proof. (Lemma 3.7)

First make the linear change of coordinates \(x' = x - a\) and \(y' = y - b\), so that, in this new coordinate system, the branch \(\gamma\) is centred at \((0, 0)\). Using Theorem 6.2 of [7], we can find algebraic power series \(\{x'(t), y'(t)\}\), parametrising the branch \(\gamma\) in the coordinate system \((x', y')\), of the form;

\[
x'(t) = a_1 t + a_2 t^2 + \ldots
\]

\[
y'(t) = b_1 t + b_2 t^2 + \ldots
\]

It is then a trivial calculation to check that;

\[
x(t) = a + a_1 t + a_2 t^2 + \ldots
\]

\[
y(t) = b + b_1 t + b_2 t^2 + \ldots
\]

parametrises the branch \(\gamma\) in the coordinate system \((x, y)\), with the terminology similar to the previous remarks.

Using the fact that the branch has order 1, which is preserved by the homographic change of coordinates, the vector \((a_1, b_1) \neq 0\). If \(a_1 = 0\), then the tangent line \(l_\gamma\) to the branch, in the coordinate system \((x', y')\), would be parallel to the \(y'\)-axis, hence the translation of \(l_\gamma\) by \((a, b)\), which is the tangent line to \(\gamma\) in the coordinate system \((x, y)\) would be parallel to the \(y\)-axis. Therefore, by Lemma 3.6, \(\gamma\) would belong to \(J' = Jac(g_m^1)\), contradicting the assumption of the lemma. Hence, we can assume that \(a_1 \neq 0\). As \(\frac{dx'}{dt}|_{t=0} \neq 0\), we can apply the inverse function theorem to the power series \(x'(t)\) and find an algebraic power series \(t(x')\), with \(\frac{dt}{dx'}|_{x'=0} \neq 0\), such that \(x'(t(x')) = x'\). Then we can write;
\[ y'(t(x')) = b_1 t(x') + b_2 t(x')^2 + \ldots \]

We claim that the sequence \((x', y'(t(x'))))\) parametrises the branch \(\gamma\) in the coordinate system \((x', y')\), \((\ast)\). We clearly have that, for any algebraic function \(F(x', y')\);

\[ F(x'(t), y'(t)) \equiv 0 \text{ iff } F(x'(t(x')), y'(t(x')))) \equiv 0 \text{ iff } F(x', y'(t(x'))) \equiv 0 \]

If \(\text{ord}_t F(x'(t), y'(t)) = m < \infty\), then we have;

\[ F(x'(t), y'(t)) = t^m u(t), \text{ for a unit } u(t) \in L[[t]]. \]

We then have that;

\[ F(x', y'(t(x'))) = t(x')^m u(t(x')), (1) \]

As \(\frac{dt}{dx}|_{x'=0} \neq 0\), we have that;

\[ t(x') = x' v(x'), \text{ for a unit } v(x') \in L[[x']], (2) \]

Combining (1) and (2) gives that;

\[ F(x', y'(t(x'))) = (x' v(x'))^m u(x' v(x')) = (x')^m v(x')^m u(x' v(x')), \]

where \(v(x')^m u(x' v(x'))\) is a unit in \(L[[x']]\).

This implies that \(\text{ord}_x F(x', y'(t(x'))) = m\) as well, hence \((\ast)\) is shown. It follows easily that the sequence \((a + x', b + y'(t(x')))\) parametrises the branch \(\gamma\) in the coordinate system \((x, y)\), with the above extension of terminology. Hence, if we let \(y(x) = b + y'(t(x-a))\), then so does the sequence \((x, y(x))\), with the convention of Remarks 3.8. The lemma is then shown.

\[ \square \]

Now let \(C\) be defined in the coordinate system \((x, y)\) by \(f = 0\) and let \(h\) be the rational function, given by Lemma 3.4, in this coordinate system. With hypotheses as in Lemma 3.7, any rational function \(\theta\) formally determines a Cauchy series \(\theta(x, y(x))\), in \((x-a)\), see the explanation at the beginning of Section 2, which we will also denote by \(\theta\).
We have that;
\[ f(x, y(x)) \equiv 0 \]
and, hence;
\[ f_x + f_y \frac{dy}{dx} = 0, \quad \frac{dy}{dx} = -\frac{f_x}{f_y} \]
\[ \frac{dh}{dx} = h_x + h_y \frac{dy}{dx} = h_x + h_y \cdot -\frac{f_x}{f_y} = \frac{h_x - h_y f_x}{f_y} \quad (*) \]

The calculation (*) should be justified carefully at the level of Cauchy series. The first part and the case when \( h \) belongs to the polynomial ring \( \mathbb{L}[x, y] \) was considered in Lemma 2.10 of [7], (**). In general, we can find \( \{h_1, h_2\} \) in \( \mathbb{L}[x, y] \) such that \( h = h_2 h_1 \). Using the result (**), and the quotient rule for differentiating rational functions, it is sufficient to check that for the formal (algebraic) power series in \((x - a)\), determined by \( h_2 \);

\[ \frac{d(1/h_2)}{dx} = \frac{dh_2}{dx} \cdot -\frac{1}{(h_2)^2} \quad (***) \]

This can be shown by a similar calculation to that in Lemma 2.10 of [7]. Namely, we can find a sequence of polynomials \( \{h_2^n\}_{m \geq 0} \) in \((x - a)\), converging to \( h_2 \) in the power series ring \( \mathbb{L}[[x - a]] \). The result (***) holds for each \( \{h_2^n\} \), hence, by general continuity results for multiplication in \( \mathbb{L}[[x - a]] \), it is sufficient to show that;

\[ \frac{d(1/h_2^n)}{dx} \rightarrow \frac{d(1/h_2)}{dx} \quad \text{and} \quad \frac{dh_2^n}{dx} \rightarrow \frac{dh_2}{dx} \]

The second part of this calculation was done in Lemma 2.10 of [7], (even in non-zero characteristic). The first part follows from the second part by representing \((1/h_2)\) as \((x - a)^{-n} u_2(x - a)\), for some \( n \geq 0 \) and a unit \( u_2(x - a) \in \mathbb{L}[[x - a]] \), and finding a sequence of units \( \{u_2^n(x - a)\}_{m \geq 1} \) in \( \mathbb{L}[[x - a]] \) such that \( u_2^n \rightarrow u_2 \) and \((1/h_2^n) = (x - a)^{-n} u_2^n \). One can then use the product rule for an algebraic power series and the function \((x - a)^{-n}\), along with continuity of addition in \( \mathbb{L}[[x - a]] \).

Using (*), we can consider \( \frac{dh}{dx} \) as a rational function on the curve \( C \) and, for a branch \( \gamma \), define \( \text{val}_\gamma(\frac{dh}{dx}) \) and \( \text{ord}_\gamma(\frac{dh}{dx}) \). For convenience, we will use the notation \( \text{ord}_\gamma(\frac{dh}{dx}) = -k \), for a positive integer \( k \), to mean
that \( val_\gamma \left( \frac{dh}{dx} \right) = \infty \) and \( ord_\gamma \left( \frac{dh}{dx} \right) = k \). We now claim the following:

**Lemma 3.9.** Let \( \gamma \) be a branch of \( C \), distinct from \( G \cup G' \cup J' \), then, for the \( g^1_n \) defined by Lemma 3.1, \( \gamma \) is counted (conto) \( s \)-times in some weighted set iff \( ord_\gamma \left( \frac{dh}{dx} \right) = s - 1 \). If \( \gamma \) belongs to \( G \), then \( ord_\gamma \left( \frac{dh}{dx} \right) = -2 \).

**Proof.** For the first part of the lemma, suppose that \( \gamma \) is counted \( s \)-times in some weighted set. Then, by Lemma 3.4, \( val_\gamma (h) < \infty \) and \( ord_\gamma (h) = s \). By Lemma 3.7 and the construction before Lemma 2.1, \( h \) determines an algebraic power series, at the branch \( \gamma \), of the form:

\[
h = \lambda + (x - a)^s \psi(x - a), \text{ with } \psi(0) \neq 0, \lambda < \infty
\]

We then have that:

\[
\frac{dh}{dx} = (x - a)^{s-1} [s \psi(x - a) + (x - a) \psi'(x - a)]
\]

At \( x = a \), the expression in brackets reduces to \( s \psi(0) \neq 0 \), hence \( ord_\gamma \left( \frac{dh}{dx} \right) = s - 1 \) as required. The converse statement is also clear by this calculation. If \( \gamma \) belongs to \( G \), then, by Lemma 3.4, \( val_\gamma (h) = \infty \) and \( ord_\gamma (h) = -1 \). Then \( h \) determines an algebraic power series at \( \gamma \) of the form:

\[
h = (x - a)^{-1} \psi(x - a), \text{ with } \psi(0) \neq 0
\]

We then have that:

\[
\frac{dh}{dx} = -(x - a)^{-2} [\psi(x - a) - (x - a) \psi'(x - a)]
\]

At \( x = a \), the expression in brackets reduces to \( \psi(0) \neq 0 \), hence \( ord_\gamma \left( \frac{dh}{dx} \right) = -2 \) as required. \( \square \)

It remains to consider the branches \( G' \cup J' \). We achieve this by the following geometric constructions:

(i). Construction for \( G' \):

We use the change of variables \( x = \frac{1}{x'} \) and \( y = \frac{y'}{x'} \). This is a homography as the map:

\[
\Theta : (x', y') \mapsto \left( \frac{1}{x'}, \frac{y'}{x'} \right)
\]
is the restriction to affine coordinates of the map;

\[ \Theta : [X' : Y' : Z'] \mapsto [Z' : Y' : X'] \]

As \( \Theta \) is a homography, the character of corresponding branches is preserved. Let \( F(x', y') \) define the equation of \( C \) in this new coordinate system. The point \( P \), given by \([0 : 1 : 0]\), relative to the coordinate system \((x, y)\), is fixed by this homography, hence the weighted set \( G'_1 \), corresponding to \( G' \), consists of branches in finite position relative to the coordinate system \((x', y')\), and is defined by \( C \cap (x' = 0) \). As the branches of \( G'_1 \) are simple, they cannot coincide with any of the branches of contact of tangents to \( C \) parallel to the \( y' \)-axis. Hence, we can apply the power series method given by Lemma 3.7. Let \( H(x', y') \) be the rational function corresponding to \( h(x, y) \) from Lemma 3.4. We claim the following;

**Lemma 3.10.** Let hypotheses be as in (i). Then, for every branch \( \gamma \) of \( G'_1 \), we have that \( \text{ord}_\gamma (\frac{dH}{dx'}) = 0 \).

**Proof.** By (i) and (iii) of Lemma 3.5, we have that \( G = (h = \infty) \) and \( G' \) are disjoint. Hence, for every branch \( \gamma \) of \( G' \), \( \text{val}_\gamma (h) < \infty \). For a given branch \( \gamma \) of \( G' \), let \( \text{val}_\gamma (h) = c \) and consider the rational function \( h - c \). We clearly have that \( \text{val}_\gamma (h - c) = 0 \). If \( \text{ord}_\gamma (h - c) \geq 2 \), then \( \gamma \) would be multiple for \( (h = c) \), hence, by Lemma 3.4, would belong to \( J = \text{Jac}(g'_n) \). This contradicts the fact, from (i) of Lemma 3.5, that \( J \) and \( G' \) are disjoint. Therefore, we must have that \( \text{ord}_\gamma (h - c) = 1 \). It follows that, for any branch \( \gamma \) of \( G'_1 \), we can find a constant \( c \) such that \( \text{ord}_\gamma (H - c) = 1 \) as well. Now, if \( \gamma \) is centred at \((a, b)\) in the coordinate system \((x', y')\), then, using the method of Lemma 3.7, \( H \) determines an algebraic power series at \( \gamma \) of the form;

\[ H(x', y'(x')) = c + c_1(x' - a) + \ldots (c_1 \neq 0) \]

It follows, from differentiating this expression, that \( \text{ord}_\gamma (\frac{dH}{dx'}) = 0 \) as required.

\( \square \)

We now claim;

**Lemma 3.11.** Let hypotheses be as in (i). Then;
\[
\frac{dH}{dx'} = \frac{dh}{dx} \cdot \frac{ds}{dx'}
\]

as an identity of Cauchy series, for corresponding branches satisfying the requirements that they are in finite position and are not branches of contact for tangent lines parallel to the \(y'\)-axis or \(y\)-axis respectively.

**Proof.** Let \(\gamma'\) and \(\gamma\) be such corresponding branches, centred at \(p' = (a, b)\) and \(p = \left(\frac{1}{a}, \frac{1}{a}\right)\), of \(F = 0\) and \(f = 0\) respectively. Let \(\gamma'\) be parametrised, as in Lemma 3.7, by the sequence \((x', y'(x'))\). Then, we claim that the sequence:

\[
(x(x'), y(x', y'(x'))) = (x(x'), y(x', y'(x'))) = \left(1, \frac{y'(x')}{x'}\right)
\]

parametrises the corresponding branch \(\gamma\), in the sense of Remarks 3.8, \((\ast)\). This follows easily from the fact that the morphism \(\Theta\), given in \((i)\), is a homography of \(P^2\), hence, in particular, one has that, for any algebraic function \(\theta\);

\[
I_{\gamma'}(p', C, \Theta^*(\theta)) = I_{\gamma}(p, C, \theta)
\]

We now claim that:

\[
(x(x'), y(x', y'(x'))) = (x(x'), y(x(x'))), \ (\ast\ast)
\]

as an identity of algebraic power series in \((x' - a)\). By \((\ast)\) and Lemma 3.7, both sequences define valid parametrisations, \((\theta_1(x - a), \theta_2(x - a))\) and \((\theta_1(x - a), \theta_3(x - a))\) in the sense of Remarks 3.8, of \(\gamma\). Moreover, we clearly have that the initial terms of both sequences are identically equal. Now, observe that \(\text{ord}_{(x' - a)}(x(x') - \frac{1}{a}) = 1\), hence we may apply the method of Lemma 3.7 (Inverse Function Theorem), in order to find an algebraic power series \(t(z) \in zL[[z]]\) such that:

\[
\theta_1(t(x' - a)) = \frac{1}{a} + (x' - a)
\]

and

\[
\left(\frac{1}{a} + (x' - a), \theta_2(t(x' - a))\right), \left(\frac{1}{a} + (x' - a), \theta_3(t(x' - a))\right)
\]

both define valid parametrisations of \(\gamma\) in the sense of Remarks 3.8, \((\ast\ast\ast)\). Now, supposing that \(\theta_2(t(x' - a)) = \theta_3(t(x' - a))\), then \((\theta_2 - \theta_3)(t(x' - a)) \equiv 0\), hence, it follows straightforwardly that \((\theta_2 - \theta_3)(x - a) \equiv 0\) and \(\theta_2(x - a) = \theta_3(x - a)\). Then \((\ast\ast)\) is shown.
We may, therefore, assume that;

\[ \theta_2(t(x' - a)) = \sum_{n=0}^{\infty} a_n(x' - a)^n, \theta_3(t(x' - a)) = \sum_{n=0}^{\infty} b_n(x' - a)^n \]

and \( a_n \neq b_n \) for some \( n \geq 1 \). Consider the rational function 
\[ \psi_n = \frac{1}{n!} \frac{d^n y}{dx^n} \] on \( C \), given by the explanation after the proof of Lemma 3.7. By (***), and Lemma 2.1, we would then have that both \( val_{\gamma}(\psi_n) = a_n \) and \( val_{\gamma}(\psi_n) = b_n \), which is clearly a contradiction. Hence, the claim (**) is shown.

Now, we have, by (**), for the corresponding branches \( \gamma' \) and \( \gamma; \)

\[ H(x', y'(x')) = h(x(x'), y(x', y'(x'))) = h(x(x'), y(x')), (\dagger) \]

Applying the chain rule for differentiating Cauchy series, (\dagger\dagger), to the composition;

\[ H : x' \mapsto x(x') \mapsto h(x(x'), y(x')) \]

and using (\dagger), the lemma follows. However, we should still justify the use of (\dagger\dagger) in the following form;

A Chain Rule for Cauchy Series.

Let \( q(x') \in L(x') \) be a rational function, such that \( q(a) \) is defined as an element of \( L \), and let \( h(x) \) define a Cauchy series in \( \text{Frac}(L[[x]]) \). Then \( H(x') = h(q(x') - q(a)) \) defines a Cauchy series in \( \text{Frac}(L[[x' - a]]) \) and;

\[ \frac{dH}{dx'} = \frac{dh}{dx} (q(x') - q(a)) \cdot \frac{dq}{dx'}, (\star) \]

as an identity of Cauchy series in \( \text{Frac}(L[[x' - a]]) \).

In order to see the first part of the claim, observe that \( q(x') - q(a) \) can be expanded as an algebraic power series in \( (x' - a)L[[x' - a]] \), hence the formal substitution of \( q(x') - q(a) \) in \( h(x) \) determines a Cauchy series in \( \text{Frac}(L[[x' - a]]) \). For the second part of the claim, choose a sequence \( \{h_m\}_{m \geq 0} \) of polynomials in \( L[x, 1/x] \) such that \( \{h_m\} \rightarrow h \) in the non-archimidean topology induced by the standard valuation on the field \( \text{Frac}(L[[x]]) \). Let \( H_m(x') = h_m(q(x') - q(a)) \) be the corresponding rational function, which also determines a Cauchy series in
the field \( \text{Frac}(L[[x' - a]]) \). By the chain rule for rational functions, (*) holds, replacing \( H \) by \( H_m \) and \( h \) by \( h_m \). Hence, the identity (*) also holds at the level of Cauchy series in \( \text{Frac}(L[[x' - a]]) \). We have that the sequence \( \{H_m\}_{m \geq 0} \to H \), in the non-archimidean topology induced by the standard valuation on the field \( \text{Frac}(L[[x' - a]]) \). This follows from the fact that, for sufficiently large \( m \);

\[
\text{ord}_{(x' - a)}(h - h_m)(q(x') - q(a)) \geq \text{ord}_x(h - h_m)
\]

By calculations shown above (even in non-zero characteristic), we have that \( \text{ord}_{(x' - a)}(h - h_m)(q(x') - q(a)) \geq \text{ord}_x(h - h_m) \)

We now combine Lemmas 3.10 and Lemmas 3.11, in order to obtain;

**Lemma 3.12.** Let \( C \) be given in the original coordinate system \((x, y)\), then, for every branch \( \gamma \) of \( G' \), we have that \( \text{ord}_\gamma (\frac{dh}{dx}) = 2 \).

**Proof.** By Lemma 3.11 and the fact that \( x(x') = \frac{1}{x'} \), we have;

\[
\frac{dH}{dx'} = -\frac{1}{x'} \frac{dh}{dx} = -x^2 \frac{dh}{dx}; \quad (*)
\]

This identity holds at the level of Cauchy series in \( L[[x' - a]] \) for corresponding branches satisfying the conditions of Lemma 3.10. However, using Lemma 3.10, we can also consider \( G = \frac{dH}{dx'} + \frac{1}{x^2} \frac{dh}{dx} \) as a rational function on the curve \( C \). By (*), for an appropriate branch \( \gamma \), satisfying the conditions of Lemma 3.10, we have that the Cauchy series expansion of \( G \) at \( \gamma \) is identically zero. By Lemma 2.1, this can only occur if \( G \) vanishes identically on \( C \). Hence, we can assume that (*) holds at the level of rational functions on \( C \) as well. By Lemma 3.10, we found that, for a branch \( \gamma \) of \( G'_1 \), \( \text{ord}_\gamma (\frac{dH}{dx'}) = 0 \). Using the fact that the branches of \( G'_1 \) are centred at \( x' = 0 \), we have, straightforwardly, that, for such a branch \( \gamma \), \( \text{ord}_\gamma (-x'^2 \frac{dH}{dx'}) = 2 \). Using Lemma 2.3, applied to the homography \( \Theta \) given in (i), we have that \( \text{ord}_\gamma (\frac{dh}{dx}) = 2 \) for any given branch of \( G'' \), as required.

\( \square \)

(ii). Construction for \( J' \);
We use the change of variables $x = y'$ and $y = x'$. This is a homography as the map;

$$\Theta : (x', y') \rightarrow (y', x')$$

is the restriction to affine coordinates of the map;

$$\Theta : [X' : Y' : Z'] \mapsto [Y' : X' : Z']$$

Again, as $\Theta$ is a homography, the character of corresponding branches is preserved. Let $F(x', y')$ denote the equation of $C$ in this new coordinate system. The point $P$, given by $[0 : 1 : 0]$, relative to the coordinate system $(x, y)$, corresponds to the point $[1 : 0 : 0]$ in the coordinate system $(x', y')$. Hence, the weighted set $J'_1$, corresponding to $J'$, consists of branches in finite position relative to the coordinate system $(x', y')$, defined by the branches of contact between $C$ and tangents parallel to the $x'$-axis. In particular, they cannot be branches of contact between $C$ and tangents parallel to the $y'$-axis. Hence, we can again apply the power series method given by Lemma 3.7. Let $H(x', y')$ be the rational function corresponding to $h(x, y)$ from Lemma 3.4. We claim the following:

**Lemma 3.13.** Let hypotheses be as in (ii). Then, for every branch $\gamma$ of $J'_1$, we have that $\text{ord}_{\gamma}(\frac{dH}{dx'}) = 0$.

**Proof.** The proof is similar to Lemma 3.10. By (iii) of Lemma 3.5, we have that $G = (h = \infty)$ and $J'$ are disjoint. Hence, for every branch $\gamma$ of $J'$, $\text{val}_{\gamma}(h) < \infty$. For a given branch $\gamma$ of $J'$, let $\text{val}_{\gamma}(h) = c$ and consider the rational function $h - c$. We then have that $\text{val}_{\gamma}(h - c) = 0$. If $\text{ord}_{\gamma}(h - c) \geq 2$, then $\gamma$ would be multiple for $(h = c)$, hence, by Lemma 3.4, would belong to $J = \text{Jac}(g^1)$, contradicting the fact, from (iii) of Lemma 3.5, that $J$ and $J'$ are disjoint. Therefore, we must have that $\text{ord}_{\gamma}(h - c) = 1$. It follows that, for any branch $\gamma$ of $J'_1$, we can find a constant $c$ such that $\text{ord}_{\gamma}(H - c) = 1$ as well. Now, if $\gamma$ is centred at $(a, b)$ in the coordinate system $(x', y')$, then, using the method of Lemma 3.7, $H$ determines an algebraic power series at $\gamma$ of the form;

$$H(x', y'(x')) = c + c_1(x' - a) + \ldots (c_1 \neq 0)$$

By differentiating this expression, we have that $\text{ord}_{\gamma}(\frac{dH}{dx'}) = 0$ as required.

$\square$
We now make an obvious extension to Lemma 3.7;

**Lemma 3.14.** Let \( \gamma \) be a branch of \( C \), in finite position, with coordinates \((a, b)\), such that the tangent line of \( \gamma \) is not parallel to the \( x \)-axis. Then one can find a power series representation of \( \gamma \) of the form;

\[
x(y) = a + c_1(y - b) + c_2(y - b)^2 + \ldots (\ast)
\]

**Proof.** The proof is the same as Lemma 3.7. \( \square \)

We can also make a similar extension to the remarks made between the proof of Lemma 3.7 and Lemma 3.9. Namely, we can define \( \frac{dx}{dy}, \frac{dh}{dy} \) (as rational functions on \( C \)), and we have that;

\[
\frac{dx}{dy} = -\frac{f_y}{f_x}, \quad \frac{dh}{dy} = \frac{h_y f_x - h_x f_y}{f_x}
\]

We now claim the following;

**Lemma 3.15.** Let \( C \) be given in the original coordinate system \((x, y)\) and let \( \gamma \) be a branch of \( C \), centred at \((a, b)\), with the property that its tangent line \( l_\gamma \) is neither parallel to the \( x \)-axis nor to the \( y \)-axis. Then the sequences \((x(y), y(x(y)))\) and \((x(y), y)\) both parametrise the branch \( \gamma \) and moreover;

\[
(x(y), y(x(y))) = (x(y), y)
\]

as an identity of sequences of algebraic power series in \( L[[y - b]] \).

**Proof.** By Lemmas 3.7 and 3.14, the sequences \((x, y(x))\) and \((x(y), y)\) both parametrise the branch \( \gamma \), as algebraic power series, in \( L[[x - a]] \) and \( L[[y - b]] \) respectively. Moreover, we claim that;

\[
\text{ord}_{(y-b)}(x(y) - a) = 1, (\ast)
\]

This follows from a close inspection of the proof of Lemma 3.7, using the fact that \( \{a_1, b_1\} \) given there are both non-zero, by the hypotheses on the branch \( \gamma \). Now, by (\ast) and the fact that \( y(x) \) defines an algebraic power series in \( L[[x - a]] \), the substitution of \( x(y) \) for \( x \) in this power series defines an algebraic power series in \( L[[y - b]] \). We need to show that the sequence \((x(y), y(x(y)))\) still parametrises \( \gamma \), (\ast\ast). The
proof is very similar to Lemma 3.7. Suppose that $F(x, y)$ is an algebraic function. Then we have that;

$$F(x, y(x)) \equiv 0 \text{ iff } F(x(y), y(x(y))) \equiv 0 \text{ iff } F \text{ vanishes on } C.$$  

If $\text{ord}_{(x-a)} F(x, y(x)) = m < \infty$, then we have;

$$F(x, y(x)) = (x - a)^m u(x, a), \text{ where } u(x, a) \text{ is a unit in } L[[x - a]].$$

We then have that;

$$F(x(y), y(x(y))) = (x(y) - a)^m u(x(y), a), \ (1)$$

By $(\ast)$, we have that;

$$(x(y) - a) = (y - b)v(y, b), \text{ where } v(y, b) \text{ is a unit in } L[[y - b]], \ (2)$$

Combining (1) and (2) gives that;

$$F(x(y), y(x(y))) = (y - b)^m v(y, b)^m u((y - b)v(y, b)), \text{ where } u(x) \text{ is a unit in } L[[x]]$$

It is easily checked that $v(y, b)^m u((y - b)v(y, b))$ defines a unit in $L[[y - b]]$. Hence;

$$\text{ord}_{(y-b)} F(x(y), y(x(y))) = m$$

as well. This proves $\ast \ast$ as required. In order to show the last part of the lemma, we use $\ast$ above and the method of Lemma 3.11, (uniqueness of parametrisation of a branch with given first term in the sequence, this was the proof of $\ast \ast$ in that Lemma). The details are left to the reader.

□

Lemma 3.16. Let $C$ be given in the original coordinate system $(x, y)$. Then;

$$\frac{dh}{dy} = \frac{dh}{dx} \cdot \frac{dx}{dy} \ (\ast)$$

where the identity can be taken either at the level of Cauchy series, for branches $\gamma$ of $C$ with the property that they are in finite position.
and their tangent line is neither parallel to the $x$-axis nor to the $y$-axis, or at the level of rational functions on $C$.

**Proof.** For a branch $\gamma$, centred at $(a, b)$, satisfying the requirements of the lemma, we consider the composition;

$$h : y \mapsto x(y) \mapsto h(x, y(x))|_{x(y)} = h(x(y), y(x(y))) = h(x(y), y) \quad (\dagger)$$

where the final identity comes from Lemma 3.15. Applying a chain rule for Cauchy series, $(\dagger\dagger)$, to $(\dagger)$, we obtain the result ($\ast$) at the level of Cauchy series. Again, we justify $(\dagger\dagger)$ in the following form, one should compare the result with the version given in Lemma 3.11;

**Another Chain Rule for Cauchy Series.**

Let $x(y)$ define an algebraic power series in $L[[y - b]]$, with constant term $x(b) = a$, and let $h(x)$ define a Cauchy series in $\text{Frac}(L[[x - a]])$. Then $g(y) = h(x(y))$ defines a Cauchy series in $\text{Frac}(L[[y - b]])$ and;

$$\frac{dg}{dy} = \frac{dh}{dx}|_{x(y)} \cdot \frac{dx}{dy} \quad (\ast\ast)$$

as an identity of Cauchy series in $\text{Frac}(L[[y - b]])$.

The first part of the claim follows easily from the obvious fact that $x(y) - a$ belongs to $(y - b)L[[y - b]]$, hence, as $h(x)$ is a Cauchy series in $\text{Frac}(L[[x - a]])$, the formal substitution of $x(y)$ in $h(x)$ (and $\frac{dh}{dx}$) defines a Cauchy series in $\text{Frac}(L[[y - b]])$. In order to show $(\ast\ast)$, let $\{x_n(y)\}_{n \geq 1}$ define a sequence of polynomials in $L[y - b]$ with the property that $\{x_n(y)\}_{n \geq 1} \to x(y)$ in the non-archimedean topology on $L[[y - b]]$ and $x_n(b) = a$. Let $g_n(y) = h(x_n(y))$ be the corresponding Cauchy series in $\text{Frac}(L[[y - b]])$. Then we have that;

$$\frac{dg_n}{dy} = \left(\frac{dh}{dx}|_{x_n(y)} \cdot \frac{dx_n}{dy}\right) \quad (\ast\ast\ast)$$

as an identity of Cauchy series in $\text{Frac}(L[[y - b]])$, by the proof of the previous version of the Chain Rule in Lemma 3.11. Now, by proofs given in the paper [7], we have that $(\frac{dx_n}{dy})_{n \geq 1} \to \frac{dx}{dy}$ as algebraic power series in $L[[y - b]]$. We now claim that $\frac{dh}{dx}|_{x_n(y)} \to \frac{dh}{dx}|_{x(y)}$ as Cauchy series in $\text{Frac}(L[[y - b]])$, $(\ast\ast\ast)$. In order to see this, observe that we can write $\frac{dh}{dx} = h_0 + h_1$, where $h_1$ belongs to $(x - a)L[[x - a]]$, and $h_0 = \frac{dh}{dx} - h_1$ is a finite sum of terms of order at most 0. We then have,
by explicit calculation, that:

\[
h_1 = \sum_{m \geq 1} a_m (x-a)^m
\]

\[
h_1(x(y)) - h_1(x_n(y)) = \sum_{m \geq 1} a_m [(x(y) - a)^m - (x_n(y) - a)^m]
\]

\[
= \sum_{m \geq 1} a_m [((x_n(y) - a) + (x - x_n)(y))^m - (x_n(y) - a)^m]
\]

\[
= \sum_{m \geq 1} a_m (x - x_n)(y)r_m(y), \text{ with } \text{ord}_{y}(y-b) r_m(y) \geq m - 1
\]

Using the definition of convergence, given any \( s \geq 1 \), we can find \( n(s) \) such that \( \text{ord}_{y}(y-b)(x_n(x_n(y))) \geq s \). The above calculation then shows that \( \text{ord}_{y}(y-b)[h_1(x_n(y)) - h_1(x_n(s)(y))] \geq s \) as well. Hence, we must have that \( \{ h_1(x_n(y)) \}_{n \geq 1} \rightarrow h_1(x(y)) \) in \( L[[y-b]] \). As \( h_0 \) is a finite sum of terms, it follows easily, by continuity of the basic operations \{+, \}, that \( \{ h_0(x_n(y)) \}_{n \geq 1} \rightarrow h_0(x(y)) \) in \( \text{Frac}(L[[y-b]]) \). Hence, the result (***) follows. A similar argument shows that \( \{ g_n(y) \}_{n \geq 1} \rightarrow g(y) \) in \( \text{Frac}(L[[y-b]]) \). Hence, using arguments in [7], we have that \( \{ \frac{d}{dy}g_n \}_{n \geq 1} \rightarrow g \) in \( \text{Frac}(L[[y-b]]) \), as well. Now, (**) follows from uniqueness of limits and continuity of multiplication for the non-archimedean topology on \( \text{Frac}(L[[y-b]]) \).

In order to complete the proof of Lemma 3.16, it remains to prove that the identity (*) may be taken at the level of rational functions on \( C \). This follows from the same argument given at the beginning of Lemma 3.12.

We now show the following;

**Lemma 3.17.** Let hypotheses be as in (ii), then;

\[
\frac{dH}{dx} = \frac{dh}{dy} = \frac{dh}{dx} \frac{dx}{dy} = -\frac{f_2}{f_2} \frac{dh}{dx}
\]

where the identities may be taken either at the level of Cauchy series, for corresponding branches satisfying the requirement that they are in finite position and are not branches of contact for tangent lines parallel to the \( x' \)-axis or \( y' \)-axis and tangent lines parallel to the \( x \)-axis or \( y \)-axis respectively, or at the level of rational functions on \( C \).

**Proof.** Let \( \gamma \) and \( \gamma' \) be such corresponding branches, centred at \( (a,b) \) and \( (b,a) \) respectively, in the coordinate systems \( (x',y') \) and \( (x,y) \). By

\[
\frac{dH}{dx} = \frac{dh}{dy} = \frac{dh}{dx} \frac{dx}{dy} = -\frac{f_2}{f_2} \frac{dh}{dx}
\]
Lemma 3.7, we can find a parametrisation \((x', y'(x'))\) of \(\gamma\) such that;

\[
\frac{dH}{dx'} = \frac{d}{dx'} H(x', y'(x')), \tag{1}
\]

as a Cauchy series in \(\text{Frac}(L[[x' - a]])\). As the morphism \(\Theta\) given in (ii) is a homography, we have that the sequence;

\[
(x(x', y'(x')), y(x', y'(x'))) = (y'(x'), x')
\]

also parametrises \(\gamma'\) in the sense of Remarks 3.8, see the corresponding proof of Lemma 3.11. Moreover, we have that;

\[
H(x', y'(x')) = h(y'(x'), x'), \tag{2}
\]

as a Cauchy series in \(\text{Frac}(L[[x' - a]])\), by definition of \(H\) and \(h\). By the hypotheses on \(\gamma'\), we can find a parametrisation of \(\gamma'\), given by Lemma 3.14, of the form \((x(y), y)\). Making the substitution \(y = x'\), gives an identical parametrisation \((x(x'), x')\) in the variable \(x'\). Using the uniqueness result for such parametrisations, see the proof of Lemma 3.11, we have that;

\[
(y'(x'), x') = (x(x'), x'), \tag{3}
\]

as sequences of algebraic power series in \(\text{Frac}(L[[x' - a]])\). We, therefore, have, combining (1), (2) and (3), that;

\[
\frac{dH}{dx'} = \frac{d}{dx'} h(y'(x'), x') = \frac{d}{dx'} h(x(x'), x') = \frac{dh}{dy} |_{y=x'}
\]

This gives the first identity at the level of Cauchy series in \(\text{Frac}(L[[x' - a]])\). The second identity, at the branch \(\gamma'\), for Cauchy series, comes from Lemma 3.16. The third identity, for Cauchy series, follows from the remarks made between Lemmas 3.14 and 3.15. In order to obtain the identities at the level of rational functions on \(C\), it is sufficient to consider the case \(\frac{dH}{dx'} = \frac{dh}{dy}\), the other identities having already been considered previously. Let \(G\) be the rational function corresponding to \(\frac{dh}{dy}\) in the coordinate system \((x', y')\). We may consider \(\frac{dH}{dx'} - G\) as a rational function on the curve \(C\), in this coordinate system. For a parametrisation \((x', y'(x'))\) of the branch \(\gamma\), we then have that;

\[
\frac{dH}{dx'}(x', y'(x')) = \frac{dH}{dx'} = \frac{dh}{dy} |_{y=x'} = \frac{dh}{dy}(y'(x'), x') = G(x', y'(x'))
\]
Hence, \((\frac{dH}{dx} - G)(x', y(x')) \equiv 0\). This implies, by arguments already shown in this paper, that \(\frac{dH}{dx} - G\) vanishes identically on \(C\), hence \(\frac{dH}{dx}' = \frac{dh}{dy}\) as rational functions on \(C\), as required.

\[\square\]

We can now show the following;

**Lemma 3.18.** Let \(C\) be given in the original coordinate system \((x, y)\), then, for each branch \(\gamma\) of \(J'\), \(ord_{\gamma}(\frac{dh}{dx}) = -1\).

*Proof.* By Lemma 3.17, we have that;

\[
\frac{dH}{dx}' = -\frac{f_y}{f_x} \frac{dh}{dx}, (*)
\]

as an identity of rational functions on \(C\). We now claim that, for a branch \(\gamma\) of \(J'\), \(ord_{\gamma}(-\frac{f_y}{f_x}) = 1\) (**). In order to see this, observe that, by \((ii)\) of Lemma 3.5, such a branch is based at a non-singular point \(q\) of \(C\). Hence, the vector \((f_x, f_y)\), evaluated at \(q\), is not identically zero. By hypotheses on \(\gamma\), that the tangent line is parallel to the \(y\)-axis, we have that \(f_y(q) = 0\), hence \(f_x(q) \neq 0\). In particular, this implies that \(0 < val_{\gamma}(f_x) < \infty\) and, therefore, that \(ord_{\gamma}(-f_x) = 0\). It also implies that \(val_{\gamma}(f_y) = 0\). In order to show (**), it is therefore sufficient to prove that \(ord_{\gamma}(f_y) = 1\). By the result of Lemma 2.1, this is equivalent to showing that;

\[I_{\gamma}(C, f_y = 0) = 1, (†)\]

We achieve this by the following method;

Let \(q = (a, b)\) and let \(l_{\gamma}\) be the tangent line to the branch \(\gamma\). By the assumption that \(\gamma\) belongs to \(J'\), \((ii)\) of Lemma 3.5 and the fact that \(\gamma\) is based at the non-singular point \(q\), we have;

\[I_{\gamma}(C, l_{\gamma}) = I_q(C, l_{\gamma}) = I_q(f = 0, l_{\gamma}) = 2\]

As was shown in the paper [6], \(I_q\) is symmetric for plane algebraic curves. Hence, we must have that;

\[I_q(l_{\gamma}, C) = I_q(l_{\gamma}, f = 0) = 2, (††)\]

as well. By the assumption that \(l_{\gamma}\) is parallel to the \(y\)-axis, we clearly have that the sequence \((a, b + t)\) parametrises this tangent line. Hence,
using Lemma 2.1 and ††, we have that;

\[ \text{ord}_t f(a, b + t) = 2 \text{ and } f(a, b + t) = t^2 u(t) \text{ for a unit } u(t) \in L[[t]] \]

Applying the Rule for differentiating algebraic power series, given in Lemma 3.7, we have that;

\[ f_y(a, b + t) = 2tu(t) + t^2 u'(t), \]  

Hence;

\[ \text{ord}_t f_y(a, b + t) = I_q(l_\gamma, f_y = 0) = I_q(f_y = 0, l_\gamma) = 1 \]

This clearly implies that \( f_y = 0 \) is non-singular at \( q \) and that the tangent line \( l_{(f_y = 0)} \) is distinct from \( l_\gamma \). Hence, the intersection between \( C \) and \( f_y = 0 \) is algebraically transverse at \( q \), therefore, as was shown in [6], we obtain (†) and then (**) follows as well.

In order to finish the proof of the lemma, we use the fact from Lemma 3.13, that, for a branch \( \gamma \) of \( J' \), \( \text{ord}_\gamma \left( \frac{dh}{dx} \right) = 0 \). Combined with (*) and (**), it follows easily that \( \text{ord}_\gamma \left( \frac{dh}{dx} \right) = -1 \) as required.

We can now summarise what we have shown in the following theorem;

**Theorem 3.19. Theorem on Differentials**

Let hypotheses and notation be as in the remarks immediately before Lemma 3.9, then;

\( \left( \frac{dh}{dx} = 0 \right) = J \cup 2G' \) and \( \left( \frac{dh}{dx} = \infty \right) = J' \cup 2G \)

In particular, one has that \( J + 2G' \equiv J' + 2G \).

**Proof.** By Lemma 3.9, a branch \( \gamma \), distinct from \( G \cup G' \cup J' \), is counted in \( \left( \frac{dh}{dx} = 0 \right) \) exactly if it appears in \( J = \text{Jac}(g_{m}^1) \), and is, moreover, also counted with the same multiplicity. By Lemma 3.12, each branch \( \gamma \) of \( G' \) is counted twice in \( \left( \frac{dh}{dx} = 0 \right) \). Again, by Lemma 3.9, each branch \( \gamma \) of \( G \) is counted twice in \( \left( \frac{dh}{dx} = \infty \right) \). Finally, by Lemma 3.18, each

\(^1\)An alternative argument is required when \( \text{char}(L) = 2 \), we leave the details to the reader
branch $\gamma$ of $J' = \text{Jac}(g^1_n)$ is counted once in $(\frac{dh}{dx} = \infty)$. Hence, the first part of the lemma is shown. As $\frac{dh}{dx}$ defines a rational function on $C$, if $s = \text{deg}(\frac{dh}{dx})$, then, by Lemma 2.4, we can associate a $g^1_s$, defined by $(\frac{dh}{dx})$, to which $J + 2G'$ and $J' + 2G$ belong as weighted sets. Hence, by the definition of linear equivalence given in Definition 2.6, we have that $J + 2G' \equiv J' + 2G$ as required.

\[ \square \]

Remarks 3.20. In the above theorem, one may assume that $G$ and $G'$ denote any 2 weighted sets of the given $g^1_n$ and $g^1_m$ respectively. In order to see this, suppose that $G_1$ and $G'_1$ are any two such weighted sets. Then we claim that:

\[ J' + 2G_1 \equiv J' + 2G \text{ and } J + 2G'_1 \equiv J + 2G' \text{ (*)} \]

In order to show (*), by Theorem 2.7 we may assume that $G - G_1 = \text{div}(f_1)$ and $G' - G'_1 = \text{div}(f_2)$ for rational functions $\{f_1, f_2\}$ on $C$. By Lemma 2.9, we then have that:

\[ 2G - 2G_1 = \text{div}(f_1^2) \text{ and } 2G' - 2G'_1 = \text{div}(f_2^2) \]

Hence, in particular, by Lemma 2.4 and Definition 2.6;

\[ 2G \equiv 2G_1 \text{ and } 2G' \equiv 2G'_1 \text{ (**)} \]

The claim (*) then follows by adding the fixed branch contributions $J'$ and $J$ to the series defining the equivalences in (**), and using Lemma 1.13. From (*), we deduce the equivalence;

\[ J' + 2G_1 \equiv J + 2G'_1 \]

immediately from Theorem 3.19 and Theorem 2.10, as required.

We now improve Theorem 3.19 as follows;

**Theorem 3.21.** Let a $g^1_{n_1}$ and $g^1_{n_2}$ be given on $C$, with no fixed branch branch contribution. Then, if $J_1 = \text{Jac}(g^1_{n_1})$, $J_2 = \text{Jac}(g^1_{n_2})$ and $\{G_1, G_2\}$ denote any 2 weighted sets appearing in $\{g^1_{n_1}, g^1_{n_2}\}$, then;

\[ J_1 + 2G_2 \equiv J_2 + 2G_1 \]
Proof. By Lemma 3.4, we can find rational functions \( \{ h_1, h_2 \} \), defining the \( \{ g_{n_1}^1, g_{n_2}^1 \} \) on \( C \), satisfying the conclusion of the lemma with respect to \( G_1^\infty = (h_1 = \infty) \) and \( G_2^\infty = (h_2 = \infty) \). An inspection of Lemma 3.5 shows that we can also obtain the conclusion there with \( J_1 \) replacing \( J \) and \( G_1^\infty \) replacing \( G \). Now one can follow through the proof up to Theorem 3.19, in order to obtain:

\[
J_1 + 2G' \equiv J' + 2G_1^\infty \quad \text{and} \quad J_2 + 2G' \equiv J' + 2G_2^\infty
\]

By Remarks 3.20, we can replace \( G_1^\infty \) by \( G_1 \) and \( G_2^\infty \) by \( G_2 \), in order to obtain:

\[
J_1 + 2G' \equiv J' + 2G_1 \quad \text{and} \quad J_2 + 2G' \equiv J' + 2G_2
\]

We then obtain, by Theorem 2.24;

\[
J_1 - 2G_1 \equiv J' - 2G' \equiv J_2 - 2G_2
\]

Hence, by Theorem 2.23;

\[
J_1 - 2G_1 \equiv J_2 - 2G_2
\]

and, therefore, the lemma follows from Theorem 2.24 again. \( \square \)

We now improve Definition 3.2.

**Definition 3.22.** Let \( C \) be a projective algebraic curve and let an arbitrary \( g_n^1 \) be given on \( C \). Let \( K \) be the fixed branch contribution of this \( g_n^1 \), with total multiplicity \( m \) and let \( g_{n-m}^1 \) be obtained by removing this fixed branch contribution. Then we define;

\[
\text{Jac}(g_n^1) = 2K + \text{Jac}(g_{n-m}^1)
\]

**Remarks 3.23.** Note that, as a consequence of the definition, if a \( g_n^1 \) is given on \( C \), then a branch \( \gamma \), which is counted \( \beta \) times for the \( g_n^1 \) and \( \beta + \alpha \) times in a particular weighted set of the \( g_n^1 \), is counted \( 2\beta + \alpha - 1 \) times in \( \text{Jac}(g_n^1) \).

We then have an improved version of Theorem 3.21;
Theorem 3.24. Let $C$ be a plane projective algebraic curve and let a $g^1_n$ be given on $C$. Then, if $G$ is any weighted set in this $g^1_n$ and $J = \text{Jac}(g^1_n)$, then the series $|J - 2G|$ is independent of the particular $g^1_n$.

Proof. Suppose that a $g^1_{n_1}$ and a $g^1_{n_2}$ are given on $C$ and let $\{K_1, K_2\}$ be their fixed branch contributions of multiplicity $\{m_1, m_2\}$ respectively. Removing these contributions, we obtain a $g^1_{n_1-m_1}$ and a $g^1_{n_2-m_2}$ on $C$ with no fixed branch contributions. Let $\{G'_1, G'_2\}$ be weighted sets of $\{g^1_{n_1}, g^1_{n_2}\}$, then we can find weighted sets $\{G'_1, G'_2\}$ of $\{g^1_{n_1-m_1}, g^1_{n_2-m_2}\}$ such that:

$$G_1 = K_1 + G'_1 \text{ and } G_2 = K_2 + G'_2 \quad (\ast)$$

Let $J'_1 = \text{Jac}(g^1_{n_1-m_1})$ and $J'_2 = \text{Jac}(g^1_{n_2-m_2})$, then we have, by Theorem 3.21:

$$J'_1 + 2G'_2 \equiv J'_2 + 2G'_1$$

Therefore, by Theorem 2.12:

$$(J'_1 + 2K_1) + 2(K_2 + G'_2) \equiv (J'_2 + 2K_2) + 2(K_1 + G'_1)$$

If $J_1 = \text{Jac}(g^1_{n_1})$ and $J_2 = \text{Jac}(g^1_{n_2})$, we then have, by Definition 3.22 and (\ast), that:

$$J_1 + 2G_2 \equiv J_2 + 2G_1$$

Hence, the result follows by Theorem 2.24 and Definition 2.25.

We now transfer this result to an arbitrary projective algebraic curve $C$, using birationality. We first require:

Lemma 3.25. Let $[\Phi] : C_1 \leftrightarrow C_2$ be a birational map. Let a $g^1_n$ be given on $C_2$, with corresponding $[\Phi]^*(g^1_n)$ on $C_1$. Then we have that:

$$[\Phi]^*([\text{Jac}(g^1_n)]) = \text{Jac}([\Phi]^*(g^1_n))$$

Proof. The result follows trivially from the fact that $[\Phi]^*$ defines a bijection on branches, see Lemma 5.7 of [7], the definition of $\text{Jac}(g^1_n)$ given in Definition 3.22 and the definition of $[\Phi]^*(g^1_n)$ given in Theorem 1.14.
We can now extend Theorem 3.24;

**Theorem 3.26.** Let $C$ be an arbitrary projective algebraic curve and let $g_n^1$ be given on $C$. Then, if $G$ is any weighted set in this $g_n^1$ and $J = \Jac(g_n^1)$, then the series $|J - 2G|$ is independent of the particular $g_n^1$.

**Proof.** By Theorem 1.33 of [7], we find a birational $\Phi : C \rightsquigarrow C'$, with $C'$ a plane projective curve. Suppose that $g_n^1$ and $g_n^2$ are given on $C$. Let $\{G_1, G_2\}$ be weighted sets of $\{g_n^1, g_n^1\}$ and let $\{J_1, J_2\}$ be the Jacobians of $\{g_n^1, g_n^1\}$. Using Lemma 3.25, we, therefore, obtain that $\{\Phi^{-1}*(G_1), \Phi^{-1}*(G_2)\}$ are weighted sets of $\{\Phi^{-1}*(g_n^1), \Phi^{-1}*(g_n^2)\}$ and $\{\Phi^{-1}*(J_1), \Phi^{-1}*(J_2)\}$ are Jacobians of $\{\Phi^{-1}*(g_n^1), \Phi^{-1}*(g_n^2)\}$. By the result of Theorem 3.24, we obtain;

$$\Phi^{-1}*(J_1) - 2\Phi^{-1}*(G_1) \equiv \Phi^{-1}*(J_2) - 2\Phi^{-1}*(G_2)$$

Applying $\Phi^*$ to this equivalence, using Theorem 2.32 and the fact that this map is linear on branches, we obtain that;

$$J_1 - 2G_1 \equiv J_2 - 2G_2$$

Hence, by Definition 2.30, $|J_1 - 2G_1| = |J_2 - 2G_2|$ as required. $\Box$

Using this result, we are able to show the following;

**Theorem 3.27.** Let $C$ be a projective algebraic curve and let $g_n^1$ be given on $C$. Then, if $\{g_n^1, g_n^1\}$ are any two series with;

$$g_n^1 \subseteq g_n^r \text{ and } g_n^1 \subseteq g_n^r$$

then;

$$\Jac(g_n^1) \equiv \Jac(g_n^1)$$

**Proof.** Let $\{G, G'\}$ be weighted sets of $\{g_n^1, g_n^1\}$ and let $\{J, J'\}$ be their Jacobians. By Theorem 3.26, we have that;

$$J - 2G \equiv J' - 2G'$$

As $\{G, G'\}$ both belong to the same $g_n^r$, we also have that $G \equiv G'$. By Theorem 2.29, we then obtain that $J \equiv J'$ as required.
As a result of this theorem, we can make the following definition;

**Definition 3.28. Jacobian Series**

Let \( C \) be a projective algebraic curve and let a \( g^r_n \) be given on \( C \), with \( r \geq 1 \), then we define \( \text{Jacob}(g^r_n) \) to be the complete series containing the Jacobians \( \text{Jac}(g^1_n) \) for any \( g^1_n \subseteq g^r_n \).

**Remarks 3.29.** If we choose any two subordinate \( g^1_n \subseteq g^r_n \), \( g^1'_n \subseteq g^r_n \) then, we have, by Theorem 2.18 and Theorem 3.27, that \( |\text{Jac}(g^1_n)| = |\text{Jac}(g^1'_n)| \). Hence, there does exist a complete linear series with the property required of the definition.

We now make the further;

**Definition 3.30.** Let \( C \) be a projective algebraic curve and let \( G \) be an effective weighted set, defining a complete series \( |G| \), with \( \dim(|G|) \geq 1 \). Then, we define;

\[ |G_j| = \text{Jacob}(|G|) \]

We then have;

**Theorem 3.31. Fundamental Theorem of Jacobian Series**

Let \( C \) be a projective algebraic curve and let \( \{A,B\} \) be effective weighted sets on \( C \), with multiplicity \( \{m,n\} \) respectively, defining complete series \( |A| \) and \( |B| \), with \( \dim(|A|) \geq 1 \) and \( \dim(|B|) \geq 1 \). Then;

\[ |(A + B)_j| = |A_j| + |2B| = |B_j| + |2A| \]

**Proof.** Consider first the complete series \( |A| + |B| = |A + B| \). By the hypotheses, we can find a \( g^1_m \subseteq |A| \), (\*). Adding \( B \) as a fixed branch contribution to this \( g^1_m \), we obtain a \( g^1_{m+n} \subseteq |A + B| \). We then have, by Definition 3.22;

\[ \text{Jac}(g^1_{m+n}) = \text{Jac}(g^1_m) + 2B \]
It then follows immediately from Remarks 3.29 and Definition 3.30 that:

\[(A + B)_j = |Jac(g^1_m) + 2B| = |Jac(g^1_m)| + |2B|\]

Again, by Remarks 3.29, (\(*\)) and Definition 3.30, we have that:

\[|Jac(g^1_m)| = |A_j|\]

Hence,

\[(A + B)_j = |A_j| + |2B|\]

as required. The remaining equality is similar.

\[\Box\]

We now have:

**Theorem 3.32. Canonical Series**

*Let \( C \) be a projective algebraic curve and let \( A \) be an effective weighted set, such that \( \text{dim}(|A|) \geq 1 \), (\(*\)). Then the series;

\[|A_j| - |2A|\]

*depends only on \( C \) and not on the choice of \( A \) with the requirement (\(*\)).*

**Proof.** Suppose that \( \{A, B\} \) are effective weighted sets satisfying (\(*\)). By Theorem 3.31, we have that:

\[|A_j| + |2B| = |B_j| + |2A|\]

Subtracting the series \(|2A| + |2B|\) from both sides, we obtain that:

\[|A_j| - |2A| = |B_j| - |2B|\]

as required.

\[\Box\]

We now make the following;
Definition 3.33. Genus of a Projective Algebraic Curve

Let $C$ be a projective algebraic curve, then we define the genus $\rho$ of $C$ by the formula;

$$2\rho - 2 = r - 2n$$

where $n = \text{order}(|A|)$ and $r = \text{order}(|A_j|)$, for any effective weighted set with the property that $\dim(|A|) \geq 1$.

Remarks 3.34. This is a good definition by Theorem 3.32.

It is clear from the definition that the genus $\rho$ is rational. We also have;

Theorem 3.35. The genus $\rho$ is a birational invariant.

Proof. Let $\Phi : C \dashrightarrow C'$ be a birational map. If $A$ is an effective weighted set on $C'$ with $\dim(|A|) \geq 1$, then $[\Phi]^*A$ is an effective weighted set on $C$ with $\dim(|[\Phi]^*A|) \geq 1$. We clearly have that $\text{order}(|A|) = \text{order}(|[\Phi]^*A|)$. We also have that $\text{Jac}(|[\Phi]^*A|) = [\Phi]^*(\text{Jac}(|A|))$, using Lemma 3.25. In particular, we then have that $\text{order}(|A_j|) = \text{order}(|([\Phi]^*A)_j|)$. The result then follows immediately from Definition 3.33. \qed

Theorem 3.36. Let $C$ be a plane projective algebraic curve of order $n$, having $d$ nodes as singularities, then;

$$\rho = \frac{(n-1)(n-2)}{2} - d$$

In particular, the genus $\rho$ of any projective algebraic curve is an integer.

Proof. Fix a generic point $P$ in the plane, and consider the pencil of lines passing through $P$. This defines a $g^1_n$ on $C$, where $n = \text{order}(C)$. Using Lemma 3.5 and the corresponding coordinate system $(x, y)$, we may assume that $\text{Jac}(g^1_n)$ consists exactly of the non-singular branches of $C$ (each counted once) in finite position, whose tangent lines are parallel to the $y$-axis. (Here, the branch formulation of a $g^1_n$ is critical, the lines in the given pencil passing through the nodes of $C$ make no contribution to $\text{Jac}(g^1_n)$ as they pass through each branch of the node transversely.) It follows that $r$ in Definition 3.33 is exactly the number of these branches. By Remarks 4.2 below, this is exactly the class of $C$, which gives the following formula relating the class and the genus;
Now, applying the Plucker formula, given in Section 4, we have;

\[ r = n(n - 1) - 2d, \quad (*) \]

Combining (†) and (*), we obtain;

\[ \rho = \frac{n(n-1) - 2d}{2} - (n - 1) = \frac{(n-1)(n-2)}{2} - d \]

as required. The remaining part of the theorem then follows immediately from Theorem 3.35 and the fact that any projective algebraic curve is birational to a plane projective curve, having at most nodes as singularities. (Note when \( C \) has no nodes, the genus gives the number of alcoves of \( n \) lines in general position).

\[ \square \]

**Theorem 3.37.** Let \( C \) be a plane projective curve of order \( n \), having \( d \) nodes as singularities, then;

\[ d \leq \frac{(n-1)(n-2)}{2} \]

In particular, \( \rho \geq 0 \), for any projective algebraic curve.

**Proof.** By (*) of Theorem 3.36, we have;

\[ d \leq \frac{n(n-1)}{2} \leq \frac{(n-1)(n+2)}{2}, \quad (***) \]

As the parameter space \( \text{Par}_{n-1} \) of plane curves, having dimension \( n - 1 \), has dimension \( \frac{(n-1)(n+2)}{2} \), by (**), we can find a plane curve \( C' \) of order \( n - 1 \), passing through the \( d \) nodes of \( C \) and through \( \frac{(n-1)(n+2)}{2} - d \) further non-singular points of \( C \). As \( C \) is irreducible, the curve \( C' \) cannot contain \( C \) as a component, therefore, by Bezout’s Theorem, must intersect \( C \) in exactly \( n(n - 1) \) points, counted with multiplicity. Using Lemma 1.11 of [9], we have the total intersection multiplicity contributed by the nodes is at least \( 2d \), and, clearly, the total intersection multiplicity contributed by the further non-singular points is at least \( \frac{(n-1)(n+2)}{2} - d \). Therefore,

\[ n(n - 1) \geq (2d + \frac{(n-1)(n+2)}{2} - d) \]

which gives that;
as required. Again, using the fact that any projective algebraic curve is birational to a plane projective curve, having at most nodes as singularities, and combining the first part of this theorem with Theorem 3.36, we obtain that $\rho \geq 0$.

We now require the following definition;

**Definition 3.38.** Let $C$ be a projective algebraic curve. We say that $C$ is rational if it is birational to a line.

We now show the following result;

**Theorem 3.39.** Let $C$ be a projective algebraic curve, then $C$ is rational if and only if its genus $\rho = 0$.

**Proof.** By Theorem 3.35, it is sufficient to show that the genus of a line is zero. As a line in the plane has no nodes and is of order 1, this follows immediately from Theorem 3.36. Conversely, suppose that the genus $\rho$ of $C$ is zero. We can assume that $C$ is a plane projective curve, having just $d$ nodes as singularities. By Theorem 3.36, we then have that:

$$d = \frac{(n-1)(n-2)}{2} \quad (1)$$

Now, consider the linear system $\Sigma$ of curves of degree $(n-1)$, passing through the $d$ nodes of $C$. As $Par_{n-1}$ has dimension $\frac{(n-1)(n+2)}{2}$, using (1), this system has dimension:

$$h \geq \frac{(n-1)(n+2)}{2} - \frac{(n-1)(n-2)}{2} = 2(n-1) \quad (2)$$

In particular, as we can assume that that $n \geq 2$, otherwise the result is proved, we have that $h \geq 2$. Now, choose a further $(2n-3)$ non-singular points on $C$. By (2), there exists a linear system $\Sigma_1$ of plane curves of degree $(n-1)$, passing through the $d$ nodes of $C$ and these further non-singular points, and this system has dimension:

$$h_1 \geq (2n-2) - (2n-3) = 1 \quad (3)$$

Now, as $C$ is irreducible, any form $F_\lambda$ in the linear system $\Sigma_1$ has finite intersection with $C$, hence, using the Branched Version of Bezout’s
Theorem, given in Theorem 5.13 of [7], given a curve $F_\lambda$ belonging to $\Sigma_1$, $C \cap F_\lambda$ defines an effective weighted set $W_\lambda$ of total multiplicity $n(n - 1)$. We, therefore, obtain a $g_{n(n-1)}^{h_1}$ on $C$. By construction, the number of base branches for this $g_{n(n-1)}^{h_1}$ is at least;

$$2d + (2n - 3) = (n - 1)(n - 2) + (2n - 3) = n(n - 1) - 1$$

We consider the following cases;

(i). One of these $n(n - 1) - 1$ base branches is 2-fold for the $g_{n(n-1)}^{h_1}$.

(ii). There exists a further base branch for the $g_{n(n-1)}^{h_1}$. (†)

Using the Branched Version of Bezout’s theorem again, we would then have that the total intersection multiplicity of intersection is contributed by the base branches. By Theorem 1.3, this implies that some form contains all of $C$, contradicting the fact that $C$ is irreducible. Hence, (†) cannot occur. It follows that each base branch is 1-fold, for the $g_{n(n-1)}^{h_1}$ and there are exactly $n(n - 1) - 1$ base branches. By Definition 1.4, we can remove this base branch contribution, in order to obtain a $g_1^{h_1}$ with no base branches. By Lemma 1.16 and (3), we then have that $h_1 = 1$. The $g_1^1$ then obtained is simple, by Definition 2.29 of [7], hence, by Lemma 2.30 of [7], defines a birational map to $P^1$. This proves the theorem.

□

Remarks 3.40. The characterisation of rational curves in terms of the vanishing of their genus may lead to applications in the field of Zariski structures, see the remarks immediately after Theorem 1.3 of [7]. As the technique we have used to define the genus of an algebraic curve is geometric rather than algebraic, one would hope to extend the definition to Zariski curves, and therefore, to find examples of non-algebraic Zariski curves which exhibit properties of lines.

4. A Plucker Formula for Plane Algebraic Curves

The purpose of this section is to give a geometric proof of an elementary Plucker formula for plane projective algebraic curves. The proof that we give depends on the method of the Italian school, however the underlying geometrical idea is due to Plucker. One can find other modern algebraic proofs, see, for example [1]. In order to state the theorem, we first require the following definition;
Definition 4.1. Class of a Plane Algebraic Curve

Let $C \subset \mathbb{P}^2$ be a plane projective algebraic curve. Then we define the class of $C$ to be the number of its tangent lines passing through a generic point $Q \in \mathbb{P}^2$.

Remarks 4.2. In order to see that this is a good definition, let $F(X, Y, Z)$ be a defining equation for $C$. For a non-singular point $p$ of $C$, the equation of the tangent line $l_p$ is given by:

$$F_X(p)U + F_Y(p)V + F_Z(p)V = 0$$

For the finitely many singular points $\{p_1, \ldots, p_j, \ldots, p_r\}$, which are the origins of branches $\{\gamma_1^{(1)}, \ldots, \gamma_j^{(1)}, \ldots, \gamma_r^{(r)}\}$, we obtain finitely many tangent lines $\{l_{\gamma_1^{(1)}}, \ldots, l_{\gamma_j^{(1)}}, \ldots, l_{\gamma_r^{(r)}}\}$. It is easily checked that the union $V$ of these finitely many tangent lines is defined over the field of definition of $C$. Hence, we may assume that $Q$ does not lie on any of these tangents. It also follows from duality arguments, see Section 5, that there exist finitely many bitangent lines $\{l_1, \ldots, l_s\}$ (Check this). Again, it is easily checked that the union $W$ of these finitely many tangent lines is defined over the field of definition of $C$. Hence, we can assume that $Q$ does not lie on any of these bitangents. Now, the condition $Cl_{\geq m}(Q)$ that there exist at least $m$ tangent lines, centred at non-singular points of $C$, passing through $Q$, is given by:

$$\exists x_1 \neq \ldots \neq x_m [\land_{1 \leq j \leq m} (\text{NS}(x_j) \land F_X(x_j)Q_0 + F_Y(x_j)Q_1 + F_Z(x_j)Q_2 = 0)]$$

The condition $Cl_{= m}(Q)$ that there exist exactly $m$ tangent lines, centred at non-singular points of $C$, passing through $Q$, is given by:

$$Cl_{\geq m}(Q) \land \neg Cl_{\geq m+1}(Q)$$

It is clear that each predicate $Cl_m$ is defined over the field of definition of $C$, hence, if it holds for some generic $Q \in \mathbb{P}^2$, it holds for any generic $Q \in \mathbb{P}^2$. Finally, observe that for generic $Q$, there can only exist finitely many tangent lines passing through $Q$. (We leave the reader to check this result.) Hence, the class does define a non-negative integer $m \geq 0$. Observe that it is possible for $m = 0$, for example if $C$ is a line. Also, observe that it is possible for $\neg Cl_m(P)$ ($m \geq 0$) to hold, if $P$ is not generic, for example if $C$ is strange. Excluding exceptional cases, one can show, using duality arguments, that a generic point of a projective algebraic curve is an ordinary simple point. In this case,
it follows that the union $W$ of the tangent lines to non-ordinary simple
points is defined over the field of definition of $C$, hence that the class
is witnessed by ordinary simple points.

We now give a statement of the elementary Plucker formula;

**Theorem 4.3.** Let $C$ be a plane projective algebraic curve of order $n$
and class $m$, with $d$ nodes. Then:

$$n + m + 2d = n^2$$

**Remarks 4.4.** We refer the reader to the papers [7] and [9] for some
relevant terminology. The reader should observe that we are using the
weaker definition of a node as the origin of exactly 2 non-singular
branches with distinct tangent directions. The original formula is usu-
ally stated with cusps, however we will not require this stronger version.

**Proof.** Theorem 4.3

Let the class of $m$ of $C$ be witnessed by a generic point $Q$ of $P^2$.
Let $\{\tau_1, \ldots, \tau_m\}$ be the finitely many tangents of $C$ passing through $Q$.
We can assume that these tangent lines are all based at non-singular
points of $C$ and do not coincide with the tangent directions of any of
the finitely many nodes of $C$. Choose a line $l$, passing through $Q$, which
does not coincide with any of these tangent lines and does not inter-
sect $C$ in any of the finitely many nodes. Let $\{X, Y, Z\}$ be a choice of
coordinates such that the line $l$ corresponds to $Z = 0$, hence defines
the line at infinity in the affine coordinate system $\{x = \frac{X}{Z}, y = \frac{Y}{Z}\}$. Let
$F(x, y) = 0$ define $C$ in this coordinate system. By construction, we
have that the tangent lines $\{\tau_1, \ldots, \tau_m\}$ are all parallel in the coordi-
nate system $\{x, y\}$. Let $(\alpha, \beta)$ be the gradient vector of each of these
lines. We define the translation $C_t$ of $C$ as follows;

$$C_t = \{(x, y) : (x - t\alpha, y - t\beta) \in C\} = \{(x, y) : F(x - t\alpha, y - t\beta) = 0\}$$

Now observe that we can find polynomials $\lambda_{i,j}(t)$, for $0 \leq (i+j) \leq n$,
such that;

$$F(x - t\alpha, y - t\beta) = F(x, y; t) = \sum_{0 \leq (i+j) \leq n} \lambda_{i,j}(t)x^iy^j \quad (*)$$

Let $\{T_0, T_1\}$ be coordinates for $P^1$ such that $t = \frac{T_0}{T_1}$. Making the
substitutions $\{t = \frac{T_0}{T_1}, x = \frac{X}{Z}, y = \frac{Y}{Z}\}$ in $(*)$, and projectivising the
resulting equation, by multiplying through by a suitable denominator, we obtain an algebraic (not necessarily linear) family of curves, parametrised by \( P^1 \). Let \( \text{Par}_t \subset P^{\frac{n(n+3)}{2}} = \text{Par}_n \) be the image of the map:

\[
\Phi : P^1 \rightarrow P^{\frac{n(n+3)}{2}}, \ T \mapsto (\lambda_{(0,0)}(T_1) : \ldots : \lambda_{(i,j)}(T_1) : \ldots)
\]

Then \( \text{Par}_t \) defines a projective algebraic curve, parametrised by \( P^1 \), whose points determine each curve in the algebraic family \( \{C_t\}_{t \in P^1} \). For a given \( t_0 \in \text{Par}_t \), we can define a tangent line \( l_{t_0} \) to \( \text{Par}_t \) by:

\[
(\lambda_{(0,0)}(t_0) + \mu \lambda'_{(0,0)}(t_0) : \ldots : \lambda_{(i,j)}(t_0) + \mu \lambda'_{(i,j)}(t_0), \ldots)
\]

**Remarks 4.5.** Observe that, in the case when \( t_0 \) is a smooth point of \( \text{Par}_t \), \( l_{t_0} \) defines the tangent line of \( \text{Par}_t \) at \( t_0 \) in the sense of Section 1 of [7], (\*). This follows easily from the fact that, for any homogeneous polynomial \( G(X_{(0,0)}, \ldots, X_{(i,j)}, \ldots) \) vanishing on \( \text{Par}_t \), we have from;

\[
G(\lambda_{(0,0)}(t), \ldots, \lambda_{(i,j)}(t), \ldots) = 0
\]

that;

\[
\frac{\partial G}{\partial X_{(0,0)}} \lambda_{(0,0)}(t) \lambda'_{(0,0)}(t) + \ldots + \frac{\partial G}{\partial X_{(i,j)}} \lambda_{(i,j)}(t) \lambda'_{(i,j)}(t) + \ldots = 0 \quad (1)
\]

and from;

\[
G(s \lambda_{(0,0)}(t_0), \ldots, s \lambda_{(i,j)}(t_0), \ldots) = 0
\]

that;

\[
\frac{\partial G}{\partial X_{(0,0)}} \lambda_{(0,0)}(t_0) s \lambda'_{(0,0)}(t_0) + \ldots + \frac{\partial G}{\partial X_{(i,j)}} \lambda_{(i,j)}(t_0) s \lambda'_{(i,j)}(t_0) + \ldots = 0 \quad (2)
\]

In order to extend these considerations to singular points, let \( U = \Phi^{-1}(\text{NonSing}(\text{Par}_t)) \) and consider the covers \( V \subset U \times P^{\frac{n(n+3)}{2}} \) and \( V^* \subset U \times P^{\frac{n(n+3)}{2}} \) defined by;

\[
V = \{(t, y) : y \in l_\Phi(t)\}
\]

\[
V^* = \{(t, \lambda) : H_\lambda \supset l_\Phi(t)\}
\]
Let $\bar{V}$ and $\bar{V}^*$ define the Zariski closure of these two covers inside $P^1 \times P^{(n+1)(n+2)}$. Let $Par_t^{ns}$ be a nonsingular model of $Par_t$ with birational map $\Phi' : Par_t^{ns} \leftrightarrow Par_t$ and let $\Phi'' = \Phi'^{-1} \circ \Phi$. Using the method of Lemma 6.13 in [7], one can show that, for $t_0 \in P^1 \setminus U$, the fibre $\bar{V}^*(t_0)$ consists of parameters for forms $H_\lambda$ containing the tangent line $l_{\gamma_j}^p$, where $\Phi(t_0) = p$ and the branch $\gamma_j^p$ corresponds to $p_j \in Par_t^{ns}$, with $\Phi''(t_0) = p_j$. Using the duality argument in Lemma 5.3 below, one then deduces that the fibre $\bar{V}(t_0)$ defines the tangent line $l_{\gamma_j}^p$. (**).

It then follows, by combining the arguments (*) and (**), that, even for a singular point $t_0$ of $Par_t$, $l_{t_0}$ defines a tangent line to the branch corresponding to $\Phi''(t_0)$.

Following the Italian terminology, we refer to the pencil of curves defined by $l_{t_0}$, for given $t_0 \in P^1$, as the curves infinitamente vicine to $C_{t_0}$, and we refer to any curve in the pencil, distinct from $C_{t_0}$, by $F(x, y; t_0 + dt_0)$.

This notation is motivated by the following fact;

Let $\Phi : P^1 \rightarrow Par_t$ be the parametrisation of $Par_t$, then, for any polynomial representation of $\Phi$ of the form;

$$\Phi : t \mapsto (\lambda_0(t) : \lambda_1(t) : \ldots)$$

$l_{t_0}$ is generated by $(\lambda_0(t_0) : \lambda_1(t_0) : \ldots)$ and $(\lambda_0'(t_0) : \lambda_1'(t_0) : \ldots)$

The proof of this fact follows easily from the above remarks or from the observation that, if $h(t)$ is a polynomial with;

$$(\lambda_0(t) : \lambda_1(t) : \ldots) = h(t)(\mu_0(t) : \mu_1(t) : \ldots)$$

then $(\lambda_0'(t_0) : \lambda_1'(t_0) : \ldots)$ corresponds to;

$h(t_0)(\mu_0(t_0) : \mu_1(t_0) : \ldots) + h'(t_0)(\mu_0(t_0) : \mu_1(t_0) : \ldots)$

and hence belongs to the tangent line generated by;

$$\{(\mu_0(t_0) : \mu_1(t_0) : \ldots), (\mu_0'(t_0) : \mu_1'(t_0) : \ldots)\}$$
We now continue the proof of Theorem 4.3;

Claim 1: Let \( t_0 = 0 \), then the algebraic curve \( C_{dt_0} \), infinitamente vicine to \( C_{t_0} = C \), passes through the finitely many points \( \{p_1, \ldots, p_m\} \) which witness the class of \( C \) and, moreover;

\[
I_{p_j}(C_{t_0}, C_{dt_0}) = 1, \text{ for } 1 \leq j \leq m
\]

In order to prove Claim 1, let \( p_j = (a_j, b_j) \) be such a point, then we obtain a parametrisation of the tangent line \( l_{p_j} \) of the form;

\[
(x(s), y(s)) = (a_j + \alpha s, b_j + \beta s)
\]

We may assume, see Remarks 4.2, that \( p_j \) is an ordinary simple point, hence that \( I_{p_j}(C_{t_0}, l_{p_j}) = 2 \). This implies that;

\[
F(x(s), y(s); t) = F(x(s) - \alpha t, y(s) - \beta t) = (s - t)^2u(s - t),
\]

\[
u(s - t) \text{ a unit in } L[[s - t]] \quad (*)
\]

Now, differentiating both sides of the equation \((*)\) with respect to \( t \), we obtain that;

\[
\frac{\partial F}{\partial t} \big|_{(x(s), y(s), t)} = -2(s - t)u(s - t) - (s - t)^2u'(s - t)
\]

\[
= -(s - t)[2u(s - t) + (s - t)u'(s - t)]
\]

Setting \( t = 0 \), we obtain that;

\[
F_{dt_0}(x(s), y(s)) = -s[2u(s) + su'(s)]
\]

Hence, \( I_{p_j}(C_{dt_0}, l_{p_j}) = 1 \). This implies, by arguments given in [6], that \( I_{p_j}(C_{t_0}, C_{dt_0}) = 1 \) as well.

Claim 2: Let \( p \) be a non-singular point of \( C \), in finite position, then \( C_{dt_0} \) passes through \( p \) iff \( p \) is one of the finitely many points \( \{p_1, \ldots, p_m\} \) witnessing the class of \( C \).

One direction of the claim follows immediately from Claim 1. Conversely, suppose that \( p = (p_1, p_2) \) lies in finite position and \( C_{dt_0}(p_1, p_2) \), \((\dagger)\). We have a parametrisation of the line \( L_p \) of the form;
\[(x(t), y(t)) = (p_1 + \alpha t, p_2 + \beta t)\]

where \(L_p\) denotes any of the tangent lines witnessing the class of \(C\), translated to \(p\). We then have that;

\[F(x(t), y(t); t) = F(p_1, p_2) = 0 \tag{**}\]

Differentiating (**) with respect to \(t\), we obtain that;

\[F_x\big|_{x(t), y(t); t} x'(t) + F_y\big|_{x(t), y(t); t} y'(t) + F_t\big|_{x(t), y(t); t} = 0\]

Setting \(t = 0\), we obtain that;

\[F_x\big|_p \alpha + F_y\big|_p \beta + F_{dt_0}\big|_p = 0, \tag{***}\]

By (†), (**) and the fact that \(p\) is non-singular, we obtain that \(L_p\) is the tangent line to \(C\) at \(p\). Hence, \(p\) must witness the class of \(C\).

Claim 3. Let \(\{q_1, \ldots, q_n\}\) be the finitely many points lying at infinity. Then \(C_{dt_0}\) passes through \(q_j\), for \(1 \leq j \leq n\), and, moreover;

\[I_{q_j}(C_{t_0}, C_{dt_0}) = 1\]

The points at infinity are given by the intersections of \(C\) with the line \(l\) passing through \(Q\). By the choice of \(l\), given at the beginning of the proof, these intersections all define non-singular points of \(C\). If \(q \in (C \cap l)\) and \(I_q(C, l) \geq 2\), then \(l\) would define the tangent line of \(C\) at \(q\). This contradicts the fact that \(l\) was chosen to avoid the finitely many tangent lines \(\{\tau_1, \ldots, \tau_m\}\) witnessing the class of \(C\). Hence, \(I_q(C, l) = 1\), (†), and the fact that there exist \(n\) distinct points at infinity then follows by an application of Bezout’s theorem, using the assumption that \(\deg(C) = n\). We now claim that, if \(q \in (C \cap l)\), then \(q\) belongs to \(\{C_t\}_{t \in \mathbb{P}^1}\), (†). In order to see this, first observe that the line \(l\) at infinity (defined by \(Z = 0\) in the choice of coordinates \(\{X, Y, Z\}\)) is fixed by the translation \(\theta_t\) along the tangent lines witnessing the class of \(C\);

\[\theta_t: (x, y) \mapsto (x - t\alpha, y - t\beta): (X: Y: Z) \mapsto (X - t\alpha Z: Y - t\alpha Z: Z)\]

The claim (†) then follows from the fact that \(C_t\) is defined as \(\text{Zero}(F \circ \theta_t)\) and the defining equation \(F\) of \(C\) vanishes on \(q\). We further claim that \(q\) belongs to \(C_{dt_0}\), for \(t_0 = 0\), (**). In order to see this, choose a system of coordinates \(\{x', y'\}\) such that \(q = (q_1, q_2)\) is in finite position.
It is a simple algebraic calculation to show that we can find a polynomial \( G(x', y'; \bar{z}) \) such that the family of curves \( \{C_t\}_{t \in \mathbb{P}^1} \) is represented by \( G(x', y'; \Phi_{\text{new}}(t)) \), for a choice of morphism \( \Phi_{\text{new}} : \mathbb{P}^1 \to \mathbb{P}^{n(n+3)/2} \). As before, let \( \text{Par}_{\text{new}}^t \) be the image of \( \text{Par}_t \) under the morphism \( \Phi_{\text{new}} \). It is a straightforward algebraic calculation to show that, if \( \theta : \mathbb{P}^2 \to \mathbb{P}^2 \) is chosen to be a homographic change of variables, then the corresponding induced morphism \( \Theta : \mathbb{P}^{n(n+3)/2} \to \mathbb{P}^{n(n+3)/2} \), on the parameter space for projective algebraic curves of degree \( n \), is also a homography. By construction, we have that \( \Theta \circ \Phi = \Phi_{\text{new}} \), \( \ast \ast \ast \), where \( \Phi \) denoted the old parametrisation of \( \text{Par}_t \). As \( \Theta \) is a homography, using the identity \( \ast \ast \ast \), and the chain rule, we obtain that, for corresponding points \( \{\Phi(t), \Phi_{\text{new}}(t)\} \) of \( \{\text{Par}_t, \text{Par}_{t_{\text{new}}}^t\} \), the tangent line \( l_{\Phi(t)} \) of \( \text{Par}_t \) is mapped by \( \Theta \) to the tangent line \( l_{\Phi_{\text{new}}(t)} \) of \( \text{Par}_{t_{\text{new}}}^t \). It therefore follows that, for given \( t_0 \in \mathbb{P}^1 \), the curves infinitamente vicine to \( C_{t_0} \) (see Remarks 4.5), can be computed by differentiating with respect to the parameter \( t \) in either of the coordinate systems \( \{\{x, y\}, \{x', y'\}\} \).

By \( \ast \), we have that:

\[
G(q_1, q_2; t) = 0, \text{ for } t \in \mathbb{P}^1
\]

Therefore, differentiating with respect to \( t \), we have that:

\[
\frac{\partial G}{\partial t}(q_1, q_2; t) = 0
\]

In particular, setting \( t = t_0 = 0 \), we obtain the claim \( \ast \ast \). We now consider the pencil of curves defined by \( \{C_{t_0}, C_{d_{t_0}}\} \), which clearly has finite intersection with \( l \), hence defines a \( g^1_n \). By \( \ast \ast \), we have that the set of intersections \( (C \cap l) \) are base points (branches) for this \( g^1_n \). By \( \dagger \) and results of [7], we have that the base branch contribution of any of these intersections is 1. In particular, it follows that, for a generic choice of \( C_{d_{t_0}} \), that \( I_{q_j}(l, C_{d_{t_0}}) = 1 \), for \( 1 \leq j \leq n \). Now applying the usual argument on tangent lines, we obtain that \( I_q(C_{t_0}, C_{d_{t_0}}) = 1 \), for \( 1 \leq j \leq n \) as well. The result of Claim 3 then follows.

Claim 4. Let \( \{p_1, \ldots, p_d\} \) be the \( d \) nodes of \( C \). Then \( C_{d_{t_0}} \) passes through \( p_j \), for \( 1 \leq j \leq d \), and, moreover;

\[
I_{p_j}(C_{t_0}, C_{d_{t_0}}) = 2
\]

Let \( p_j = (c_j, d_j) \) be such a point, and let;
be a parametrisation of any of the tangent lines \( \{ \tau_1, \ldots, \tau_m \} \), translated to \( p_j \). We have that;

\[
F(x(t), y(t); t) = F(c_j, d_j) = 0 \quad (\ast)
\]

Moreover, as \((x(t), y(t))\) defines a node of the translated curve \( C_t \), we have that

\[
F_x \big|_{(x(t), y(t); t)} = 0 \quad \text{and} \quad F_y \big|_{(x(t), y(t); t)} = 0.
\]

Hence, differentiating \((\ast)\) with respect to \( t \), we obtain that,

\[
F_t \big|_{(x(t), y(t); t)} = 0
\]

Setting \( t = 0 \), we obtain that \( F_{dt}|_{p_j} = 0 \), that is \( C_{dt_0} \) passes through \( p_j \). Hence, the first part of the claim is shown. The second part of the claim depends heavily on a geometric argument. Let \( l_{t_0} \) be the tangent line to \( Par_t \), at \( t_0 \), as given immediately before Remarks 4.5. As explained in Remarks 4.5, the curve \( C_{dt_0} \) corresponds to a point \( Q \) on \( l_{t_0} \), and we denote by \( O \), the point corresponding to \( C_{t_0} \) in \( Par_t \), hence \( l_{t_0} = l_{OQ} \). By Remarks 4.5, the line \( l_{t_0} \) defines the tangent line to the branch \( \gamma_{t_0} \), corresponding to \( t_0 \), in the parametrisation \( \Phi : P^1 \to Par_t \).

Similarly to the argument in Remarks 4.5, let;

\[
U = (\Phi^{-1}(NonSing(Par_t)) \setminus \{ t_0 \}) \subset P^1
\]

and consider the covers \( V \subset U \times P^{\frac{(n+1)(n+2)}{2}} \) and \( V^* \subset U \times P^{\frac{(n+1)(n+2)}{2}} \) defined by;

\[
V = \{ (t, y) : y \in l_{O\Phi(t)} \}
\]

\[
V^* = \{ (t, \lambda) : H_{\lambda} \supset l_{O\Phi(t)} \}
\]

Let \( \widetilde{V} \) and \( \widetilde{V}^* \) define the Zariski closure of these two covers inside \( P^1 \times P^{\frac{n(n+3)}{2}} \). Using the method of Lemma 6.14 in [7], one can show that the fibre \( \widetilde{V}^*(t_0) \) consists of parameters for forms \( H_{\lambda} \) containing the tangent line \( l_{t_0} \). Using the duality argument in Lemma 5.4 below, one then deduces that the fibre \( \widetilde{V}(t_0) \) defines the tangent line \( l_{t_0} \). The details are left to the reader. We now have that \( \widetilde{V}(t_0, Q) \) holds, hence, as \( t_0 \) is regular for the cover \( (\widetilde{V}/P^1) \), given generic \( t'_0 \in \mathcal{V}_{t_0} \cap P^1 \), we can find \( Q' \in \mathcal{V}_Q \cap P^{\frac{n(n+3)}{2}} \), such that \( \widetilde{V}(t'_0, Q') \), hence, \( Q' \) belongs to
Denote by $D_{Q'}$ the plane curve of order $n$ corresponding to the point $Q'$ in $Par_n$. We now consider the pencil of curves $P$ defined by the line $l_{O \Phi(t'_0)} \subset Par_n$. By construction, this pencil contains the plane curves $\{C_{t_0}, C_{t'_0}, D_{Q'}\}$. As $t'_0 \in Par_1$, the plane curve $C_{t'_0}$ is an infinitesimal translation of $C_{t_0}$ in the direction defined by the tangent lines $\{\tau_1, \ldots, \tau_m\}$. By the assumption at the beginning of the proof, for a given node $p_j$, this direction does not coincide with either of its two distinct tangent lines $\{l_{\gamma^1_{pj}}, l_{\gamma^2_{pj}}\}$. We are, therefore, able to apply Theorem 1.13 of [9], to conclude that $C_{t_0} \cap C_{t'_0} \cap V_{p_j}$ consists of exactly two distinct points $\{p^1_j, p^2_j\}$, situated on the branches $\{\gamma^1_{pj}, \gamma^2_{pj}\}$ of $C_{t_0}$, and moreover these intersections are transverse. By elementary facts on linear systems, $\{p^1_j, p^2_j\}$ are base points for the pencil defined by $P$, and, moreover, $C_{t_0} \cap C_{t'_0} = C_{t_0} \cap D_{Q'}$. It follows immediately that $C_{t_0} \cap D_{Q'} \cap V_{p_j}$ also consists of exactly the two points $\{p^1_j, p^2_j\}$. Again, by elementary facts on linear systems, see [7] for more details, these intersections are also transverse. We have, therefore, shown, using the non-standard definition of intersection multiplicity, given in [6] or [7], that $I_{p_j}(C_{t_0}, C_{t_{d_{t_0}}}) = 2$. Hence, Claim 4 is shown.

We now complete the proof of Theorem 4.3. Combining Claims 1, 2, 3 and 4, the total multiplicity of intersection between $C$ and $C_{d_{t_0}}$ is $n + m + 2d$. By construction, $\text{deg}(C)\text{deg}(C_{d_{t_0}}) = n.n = n^2$. Hence, Bezout’s Theorem gives that $n + m + 2d = n^2$, as required. By the results of [9], Plucker’s formula also holds for the algebraic definition of intersection multiplicity.

\[\square\]

**Remarks 4.6.** The proof that we have given follows Plucker’s original geometrical idea and the presentation of Severi in [10]. Although long, the methods used are almost entirely geometrical, and adapt easily to handle cases where the singularities of $C$ are more complicated. The reader is invited to extend the approach to these situations. One can find algebraic proofs in the literature, for example in [1]. Unfortunately, these proofs generally fail to give a precise calculation of intersection multiplicity for $C$ and some other curve $C'$, a problem that we observed in Remarks 1.12 of [9].

5. **The Transformation of Branches by Duality**

The purpose of this section is to give a general account of the theory of duality and to develop the connection with the theory of branches
A THEORY OF DUALITY FOR ALGEBRAIC CURVES

We first give a brief account of Grassmannians on $P^w$. We define;

$$G_{w,j} = \{ P_j \subset P^w \}, \ (0 \leq j \leq w)$$

where $P_j$ is a plane of dimension $j$. It follows from classical well known arguments that $G_{w,j}$ may be given the structure of a smooth algebraic variety, see for example [1] p193. (These arguments begin with the observation that $G_{w,j}$ can be identified with the set of $j + 1$-dimensional subspaces of a $w + 1$-dimensional vector space over $L$.)

The fact that $G_{w,j}$ defines a projective algebraic variety follows from the Plucker embedding;

$$\rho : G_{w,j} \to P(\wedge^{j+1}L^{w+1}) = P^{C_{j+1}^{w+1}}$$

given by sending a $j + 1$ dimensional subspace of $L^{w+1}$, with basis $\{v_1, \ldots, v_{j+1}\}$, to the multivector $v_1 \wedge \ldots \wedge v_{j+1}$. It is easily checked that the Plucker map $\rho$ defines a morphism of algebraic varieties, is injective on points and its differential $d\rho_x$ has maximal rank, for $x \in G_{w,j}$.

In the case when the underlying field $L = \mathbb{C}$, one can then use the Immersion Theorem and Chow’s Theorem to show that $\text{Image}(\rho)$ has the structure of a projective algebraic variety. In general, one can show;

The image of the Grassmannian $G_{w,j}$ under the Plucker embedding $\rho$ is defined by a linear system of quadrics.

For want of a convenience reference, we leave the reader to check that the proof given of this result in [1] holds for arbitrary characteristic.

We now observe the following duality between $G_{w,j}$ and $G_{w,w-(j+1)}$, for $0 \leq j \leq w-1$;

**Lemma 5.1.** If $P_j \subset P^w$ is a $j$-dimensional plane, then;

$$\{ \lambda \in P^{w*} : H_{\lambda} \supset P_j \} \ (1)$$

determines a $w - (j + 1)$-dimensional plane $P_{w-(j+1)}^*$ in $P^{w*}$. Conversely, if $P_{w-(j+1)}^* \subset P^{w*}$ is a $w - (j + 1)$-dimensional plane, then;

$$\{ x \in P^w : x \in \bigcap_{\lambda \in P_{w-(j+1)}^*} H_{\lambda} \} \ (2)$$
determines a $j$-dimensional plane $P_j$ in $P^w$. Moreover, these correspondences are inverse and determine a closed projective variety;

$$I \subset G_{w,j} \times G_{w,w-(j+1)}$$

Proof. The proof is quite elementary. If $P_j \subset P^w$ is a $j$-dimensional plane, then one can find an independent sequence $\{\tilde{a}_0, \ldots, \tilde{a}_j\}$ defining it, $(\ast)$, see Section 1 of [7]. The condition that a hyperplane $H_\lambda$ contains $P_j$ is then given by the conditions;

$$a_{0i}\lambda_0 + a_{1i}\lambda_1 + \ldots + a_{wi}\lambda_w = 0, \ (0 \leq i \leq j)$$

It is elementary linear algebra, using $(\ast)$, to see that these conditions determine a plane of codimension $(j + 1)$ in $P^w$. For the converse direction, if $P^*_w \subset P^w$ is a $w - (j + 1)$-dimensional plane, then one can find an independent sequence $\{\tilde{b}_0, \ldots, \tilde{b}_{w-(j+1)}\}$ defining it, $(\ast\ast)$. The condition that $x \in P^w$ is contained in the intersection of the planes defined by $P^*_w$ is then given by the conditions;

$$x_{0i}b_{0i'} + x_{1i}b_{1i'} + \ldots + x_{wi}b_{wi'} = 0, \ (0 \leq i' \leq w - (j + 1))$$

By the same elementary linear algebra argument, using $(\ast\ast)$, these conditions determine a plane of codimension $(w - j)$ in $P^w$, that is a plane of dimension $j$. The fact that these correspondences are inverse follows immediately from the relations;

$$a_{0i}b_{0i'} + a_{1i}b_{1i'} + \ldots + a_{wi}b_{wi'} = 0, \ (0 \leq i \leq j, 0 \leq i' \leq w - (j + 1)) \ (\dagger)$$

and an elementary dimension argument. The final part of the lemma follows by checking that the relations $(\dagger)$ can be defined by matrix multiplication on representatives of $G_{w,j}$ and $G_{w,w-(j+1)}$. If $\{U_I\}$ and $\{U_J\}$ define the standard open affine covers of these Grassmannians, as given on p193 of [1], then these relations clearly define closed algebraic subvarieties $I_{I,J} \subset U_I \times U_J$. Using standard patching arguments, we then obtain a closed algebraic subvariety $I \subset G_{w,j} \times G_{w,w-(j+1)}$ determining the duality correspondence.

\[\square\]

Remarks 5.2. The duality correspondence $I$ clearly induces bijective maps, in the sense of model theory;
\( * : G_{w,j} \to G_{w,w-(j+1)} \)
\( *^{-1} : G_{w,w-(j+1)} \to G_{w,j} \)

When the underlying field \( L \) has characteristic 0, it follows that they define isomorphisms in the sense of algebraic geometry. However, in non-zero characteristic, it is difficult to determine whether these maps define morphisms or are separable, therefore, whether the maps are inverse in the sense of algebraic geometry. As we will be concerned with the behaviour of algebraic curves under duality, we will be able to deal with this problem using arguments on Frobenius that we have already seen in [7]. The reader should also note that \( * \) is not canonical, even as a set theoretic map, for it depends on a particular identification of \( P^w \) and \( P^{w*} \). We will, henceforth, denote the (set theoretic) inverse \( *^{-1} \) by \( * \). This is motivated by the fact that we can identify \( P^w \) naturally with \( P^{w**} \), in which case the relation (2), defining \( *^{-1} \) in the previous lemma, becomes an instance of the relation (1).

As an application of the above, we have:

**Lemma 5.3. Tangent Variety of a Projective Algebraic Curve**

Let \( C \subset P^w \) be any projective algebraic curve and let;

\[ V \subset \text{NonSing}(C) \times P^w \text{ be } \{(x,y) : x \in \text{NonSing}(C) \land y \in l_x}\]  

Then \( V \) defines an irreducible algebraic variety and, if \( \overline{V} \subset C \times P^w \) defines its Zariski closure, then, for a singular point \( p \), which is the origin of branches \( \{\gamma_p^1, \ldots, \gamma_p^m\} \), the fibre \( \overline{V}(p) \) consists exactly of the finite union \( \bigcup_{1 \leq j \leq m} l_{\gamma_p^j} \) of the tangent lines to the \( m \) branches at \( p \).

**Proof.** The fact that \( V \) defines an irreducible algebraic variety follows easily from arguments given in Section 1 of [7]. Let \( G_{w,1} \) be the Grassmannian of lines in \( P^w \), let \( G_{w,w-2} \) be the Grassmannian of planes of codimension 2 in \( P^{w*} \) and let \( I \) be the duality correspondence between \( G_{w,1} \) and \( G_{w,w-2} \), as defined in Lemma 5.1. Without loss of generality, we can find algebraic forms \( \{G_1, \ldots, G_{w-1}\} \) defining \( \text{NonSing}(C) \), see Section 1 of [7]. For each \( G_j \), with \( 1 \leq j \leq w-1 \), the differential \( dG_j \) determines a morphism;

\[ dG_j : \text{NonSing}(C) \to P^{w*} : x \mapsto dG_j(x) = \left( \frac{\partial G_j}{\partial X_0}(x) : \ldots : \frac{\partial G_j}{\partial X_n}(x) \right) \]
We then obtain a morphism;

$$\Phi_1 : \text{NonSing}(C) \to G_{w,w-2} : x \mapsto (dG_1(x), \ldots, dG_{w-1}(x)) = P_x$$

Using the duality correspondence $I$ and the observation that, for $x \in \text{NonSing}(C)$, the tangent line $l_x$ is determined by the intersection of the hyperplanes determined by $dG_j(x)$, for $1 \leq j \leq w-1$, see Section 1 of [7], we obtain a morphism;

$$\Phi_2 : \text{NonSing}(C) \to G_{w,1} : x \mapsto l_x$$

By construction, we must clearly have the duality $I(l_x, P_x)$, whenever $x \in \text{NonSing}(C)$, (†). Let $C_1 = \text{Im}(\Phi_1)$ and $C_2 = \text{Im}(\Phi_2)$. Let $\Gamma_{\Phi_1} \subset C \times C_1$ and $\Gamma_{\Phi_2} \subset C \times C_2$ be the irreducible correspondences defined by $\text{Graph}(\Phi_1)$ and $\text{Graph}(\Phi_2)$. Let $\Gamma^*_x \subset C \times G_{w,w-2}$ be the dual correspondence to $\Gamma_{\Phi_2}$ defined by;

$$\Gamma^*_x(y, z) \equiv \exists z (\Gamma_{\Phi_2}(y, z) \land I(z, y))$$

We then have that $\Gamma^*_x$ defines a closed irreducible projective variety. By (†), $\Gamma^*_x$ agrees with $\Gamma_{\Phi_1}$ on the open subset obtained by restricting to $\text{NonSing}(C)$. Hence, we have that $\Gamma_{\Phi_1}$ defines the dual correspondence to $\Gamma_{\Phi_2}$. Let $W_{\Phi_1} \subset C \times C_1 \times P^{w*}$ and $W_{\Phi_2} \subset C \times C_2 \times P^w$ be defined by;

$$W_{\Phi_1} = \{(y, P, z) : \Gamma_{\Phi_1}(y, P) \land z \in P\}$$

$$W_{\Phi_2} = \{(y, l, z) : \Gamma_{\Phi_2}(y, l) \land z \in l\}$$

We have that $W_{\Phi_1}$ and $W_{\Phi_2}$ are closed irreducible projective varieties. Let $p_{13} : C \times C_{1,2} \times P^w \to C \times P^w$ be the projection map. Let $V^*$ be defined as in Lemma 6.13 of [7]. We have that $p_{13}(W_{\Phi_1})$ and $p_{13}(W_{\Phi_2})$ agree with $V^*$ and $V$, restricted to $\text{NonSing}(C)$, hence $p_{13}(W_{\Phi_1}) = V^*$ and $p_{13}(W_{\Phi_2}) = V$. It follows that, for a singular point $p$ of $C$, the fibre $V^*(p)$ consists of a finite number of lines determined by the planes appearing in the fibre $V^*(p)$. The result of the theorem then follows from the description of the fibre $V^*(p)$ given in Lemma 6.13 of [7].

One can also formulate the following "desingularised version" of Lemma 5.3;
**Lemma 5.4.** Let $C \subset P^w$ be any projective algebraic curve, with a choice of nonsingular model $C^{ns} \subset P^w$ and birational morphism $\Phi : C^{ns} \to C$. Then;

*Desingularised Version of Lemma 5.3:*

Let $U = \Phi^{-1}(\text{NonSing}(C)) \subset C^{ns}$ and let;

$V \subset U \times P^w$ be $\{(x, y) : x \in U \land y \in l_{\Phi(x)}\}$

Then $V$ defines an irreducible algebraic variety and if $\overline{V} \subset C^{ns} \times P^w$ defines its Zariski closure, then, for $p_j \in C^{ns}$, corresponding to a branch $\gamma_j^p$ of $C$, the fibre $\overline{V}(p_j)$ consists exactly of the tangent line $l_{\gamma_j^p}$.

**Proof.** The proof is merely a question of changing the parameter space from $C$ to $C^{ns}$ and adapting the argument of the previous lemma. The details are left to the reader.

In a similar vein, we also have;

**Lemma 5.5.** *Intuitive Construction of Tangent Lines*

Let $C \subset P^w$ be a projective algebraic curve and let $O \in C$ be a given fixed point, possibly singular. Let;

$V \subset (\text{NonSing}(C) \setminus \{O\}) \times P^w$ be $\{(x, y) : x \in \text{NonSing}(C) \land y \in l_{Ox}\}$

Then $V$ defines an irreducible algebraic variety and, if $\overline{V} \subset C \times P^w$ defines its Zariski closure, then, if $O$ is the origin of branches $\{\gamma_1^p, \ldots, \gamma_m^p\}$, the fibre $\overline{V}(O)$ consists exactly of the finite union $\bigcup_{1 \leq j \leq m} l_{\gamma_j^p}$ of the tangent lines to the $m$ branches at $O$.

**Proof.** The proof is the same as Lemma 5.3, except that we use the definition of $V^*$ given in Lemma 6.14 of [7] and the description of the fibre $\overline{V}^*(O)$ given in Lemma 6.14 of [7].

We also require the following result;
Lemma 5.6. Convergence of Intersections of Tangents

Let $C \subset P^2$ be a plane projective algebraic curve, not equal to a line, let $O$ be a nonsingular point, which is the origin of a branch $\gamma_O$ of character $(1, k-1)$, ($k \geq 2$), and suppose that char($L$) is zero or coprime to $k$. Let;

$$V \subset (NSing(C) \setminus \{O\}) \times P^2 = \{(x, y) : x \in NSing(C) \land y \in l_O \cap l_x\}$$

Then $V$ defines a generically finite cover of $NonSing(C)$. Moreover, if $\bar{V} \subset C \times P^2$ defines its Zariski closure, then the fibre $\bar{V}(O)$ consists exactly of the point $O$.

Proof. The fact that $V$ defines a generically finite cover of $NonSing(C)$ follows easily from the assumption that $C$ is not a line. The infinite fibres of the cover $V$ correspond to the, at most, finitely many points $\{p_1, \ldots, p_n\} \subset (NonSing(C) \setminus \{O\})$ such that $l_O$ defines their tangent line. Let $U = (NonSing(C) \setminus \{O, p_1, \ldots, p_n\})$ and let $V^{res}$ be the restriction of the cover $V$ to $U$. We clearly have that $\bar{V}(O) = \bar{V}^{res}(O)$ and that the fibres of $V^{res}$ consist of a unique intersection. In particular, the Zariski closure $\bar{V}^{res}$ is irreducible. Using the fact that $O$ is nonsingular and an important property of Zariski structures, given in Theorem 3.3 of [12], it follows that the fibre $\bar{V}^{res}(O)$, hence the fibre $\bar{V}(O)$ also consists of a unique point $O'$. In order to show that $O = O'$, we employ an argument using infinitesimals;

We may assume that $O$ lies at the origin $(0,0)$ of the affine coordinate system $(x,y)$ and that the tangent line $l_O$ corresponds to the line $y = 0$ in this coordinate system. Using Lemma 3.7, we can find a parametrisation of the branch $\gamma_O$, in the sense of Theorem 6.1 of [7], of the form $(t, y(t))$, where $y(t)$ is an algebraic power series. By the assumption on the tangent line $l_O$, the definition of a parametrisation in Theorem 6.1 of [7] and the assumption on the character of the branch, we must have that $ord_t(y(t)) = k$, so we can assume that $y(t) = t^k u(t)$, for the given $k \geq 2$, where $u(t) \in L[[t]]$ is a unit. Let $F(x, y)$ define $C$ in the coordinate system $(x,y)$, then we have that;

$$F(t, y(t)) = 0 \text{ and } F_x|_{(t, y(t))}.1 + F_y|_{(t, y(t))}.y'(t) = 0, \text{ (*)}$$

as a formal identity in the power series ring $L[[t]]$, see [7] for similar calculations. We now work in the nonstandard model $K = L[[\epsilon]]^{alg}$. It
follows easily from the paper [4] that we can interpret \( \epsilon \) as an infinitesimal in \( V_0 \). By construction, we clearly have that the identity (\( * \)) holds, replacing \( t \) by \( \epsilon \), (\( \ast \)). By definition of the specialisation map, given in [4], and (\( \ast \)), we then have that \((\epsilon, y(\epsilon)) \in \text{NonSing}(C) \cap V_{(0,0)}\). Using (\( \ast \)) again, we also have that the equation of the tangent line \( l_{(\epsilon,y(\epsilon))} \) in the nonstandard model \( K \) is given by;

\[
(y - y(\epsilon)) = y'(\epsilon)(x - \epsilon) \quad (\star \star \star)
\]

We now compute the intersection of \( l_{(\epsilon,y(\epsilon))} \) with \( y = 0 \). By (\( \star \star \star \)), we obtain that;

\[
l_{(0,0)} \cap l_{(\epsilon,y(\epsilon))} = (x_\epsilon, 0), \text{ where } x_\epsilon = \frac{(\epsilon y'(\epsilon) - y(\epsilon))}{y'(\epsilon)}
\]

Using the assumption on \( \text{char}(L) \), it is easy to calculate that \( \text{ord}_t(ty'(t) - y(t)) \geq k \) and \( \text{ord}_t(y'(t)) = k - 1 \). It follows that \( \text{ord}_t(x_t) \geq 1 \), considered as an element of \( L[[t]] \), hence, by the definition of the specialisation map in [4], we must have that \( x_\epsilon \in V_0 \) and \((x_\epsilon, 0) \in V_{(0,0)} = V_0 \). By construction, we have that \( \bar{V}_{\text{res}}((\epsilon, y(\epsilon)), (x_\epsilon, 0)) \), hence, by specialisation, we have that \( \bar{V}_{\text{res}}(O, O) \). It follows that the fibre \( \bar{V}_{\text{res}}(O) \) consists exactly of the point \( O \) as required. \( \square \)

We now make the following definition;

**Definition 5.7.** Let \( C \subset P^2 \) be a plane projective curve. We define a nonsingular point \( p \), which is the origin of a branch \( \gamma_p \) having character \((1, r)\), for \( r \geq 2 \), to be a flex. We define \( p \) to be an ordinary flex, if \( r = 2 \).

We now consider the duality construction applied to plane projective curves. We claim the following;

**Lemma 5.8.** Let \( C \subset P^2 \) be a plane projective curve, defined in the coordinate system \( \{X, Y, Z\} \) by the irreducible polynomial \( F(X, Y, Z) \). Then, if \( C \) is not a line, the differential;

\[
dF : \text{NonSing}(C) \to P^{2*} : x \mapsto l_x = (\frac{\partial F}{\partial X}(x) : \frac{\partial F}{\partial Y}(x) : \frac{\partial F}{\partial Z}(x))
\]

defines a morphism, with the property that \( C^* = \text{Im}(dF) \subset P^{2*} \) is also a plane projective curve. If \( \text{char}(L) \neq 2 \), then the following conditions are equivalent;
(i). *C* has finitely many flexes.

(ii). \(dF: C \xymap{\sim} C^*\) defines a birational morphism and \(C = C^{**}\).

In particular, if \(\text{char}(L) = 0\), then (ii) always holds, hence any plane projective curve over \(L\), with \(\text{char}(L) = 0\), can only have finitely many flexes, and if \(C\) has infinitely many flexes, the duality morphism \(dF\) is inseperable.

*Proof.* The fact that \(dF\) defines a morphism on \(\text{NonSing}(C)\) follows easily from the fact that the partial derivatives \(\left\{ \frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z} \right\}\) cannot all vanish at a nonsingular point of \(C\). By basic results in algebraic geometry, the image \(\text{Im}(dF) \subset P^2\) is constructible and irreducible. As \(C\) is not a line, it must have infinite distinct tangent lines, hence, \(C^* = \text{Im}(dF)\) defines a plane projective algebraic curve. We first show;

\[(i) \Rightarrow (ii)\]

As \(C\) has finitely many flexes, there exists an open subset \(U \subset \text{NonSing}(C)\) with the property that, for \(x \in U\), the corresponding branch \(\gamma_x\) has character \((1,1)\). As \(C^*\) is a projective algebraic curve, we can find a further open set \(V \subset U\) such that \(dF(V) \subset \text{NonSing}(C^*)\). If \(G(U,V,W)\) is a defining equation for \(C^*\), then, using the first part of this Lemma and Lemma 5.1, in order to identify \(P^{2**}\) with \(P^2\), we obtain a morphism;

\[dG: \text{NonSing}(C^*) \rightarrow P^2\]

We now claim that;

\[(dG \circ dF) = \text{Id}_V: V \rightarrow P^2\ (**\)

This implies immediately that \(C^{**} = C\) and \(dF: C \xymap{\sim} C^*\) is a birational map, with birational inverse \(dG: C^* \xymap{\sim} C\). We now show (**). Suppose that \(x_0 \in V\), with corresponding \(y_0 = dF(x_0) \in \text{NonSing}(C^*)\). As the branch \(\gamma_{x_0}\) has character \((1,1)\), and \(\text{char}(L) \neq 2\), we have that the result of Lemma 5.6 holds for \(x_0\), (replacing \(O\) in the Lemma). That is, if \(V_1 \subset C \times P^2\) is the closed subvariety given in Lemma 5.6, then the fibre \(V_1(x_0) = x_0\) and, if \(x \in C\), with \(x \neq x_0\), then \(V_1(x) = l_{x_0} \cap l_x\). As \(y_0 \in \text{NonSing}(C^*)\), we can apply Lemma 5.5, and obtain a closed subvariety \(V_2 \subset C^* \times P^{2*}\) such that the fibre
\[ V_2(y_0) = l_{y_0} \] and, if \( y \in C^* \), with \( y \neq y_0 \), then \( V_2(y) = l_{y_0} \). Shrinking \( V \) if necessary, we can assume that if \( x \in V \) with \( x \neq x_0 \), then \( dF(x) \neq y_0 \) and \( l_x \cap l_{x_0} \) is a point, (†). Now, define the closed relation \( S \subset V \times P^2 \times P^{2*} \) by;

\[
S(x, x', y) \equiv V_1(x, x') \land V_2(dF(x), y)
\]

By construction of \( \{V_1, V_2\} \), we have that if \( x \in V \) with \( x \neq x_0 \), then the fibre \( S(x) = \{(x', y) : x' \in l_{x_0} \cap l_x, y \in l_{y_0}dG(x)\} \). Using the assumption (†), the fibre \( S(x) \) consists of a point and a line. Moreover, the point of intersection \( l_{x_0} \cap l_x \in P^2 \) is in dual correspondence with the line \( l_{y_0}dF(x) \subset P^{2*} \), (**) Using Lemma 5.1, this follows from determining the intersection of the pencil of lines parametrised by \( l_{y_0}dF(x) = l_{dF(x_0)}dF(x) \). By construction of \( dF \), this is exactly the intersection of tangent lines \( l_{x_0} \cap l_x \).

We now use (**) and a simple limiting argument, to show that this duality must also hold for the fibre \( S(x_0) \). Let \( R_2 \subset P^2 \times P^{2*} \) be the incidence relation, given by \( R_2(u, v) \) iff \( v \in H_u \), where \( H_u \) is the linear form with coefficients given by \( u \). By (**), we have that;

\[
S(x) \subset R_2 \ (x \in V, x \neq x_0) \quad (***)
\]

Using elementary model theoretic arguments, (****) is a closed condition on \( V \). Hence, we must have that \( S(x_0) \subset R_2 \) as well, (*****).

Again, by construction of \( \{V_1, V_2\} \), we have that the fibre \( S(x_0) = \{(x_0, y) : y \in l_{y_0}\} \). By (******), this shows that \( x_0 \) must be in dual correspondence with \( l_{y_0} \). We, therefore, must have that \( dG(y_0) = x_0 \), hence (*) is shown.

\[(ii) \Rightarrow (i).\]

There are two approaches to this problem. We will show later in the section that if \( (ii) \) holds and \( [(dF)^{-1}]^* \) is the induced bijection on branches guaranteed by Lemma 5.7 of [7], then, for a given branch \( \gamma \) of \( C \) with character \( (\alpha, \beta) \), the corresponding branch \( \gamma^* \) of \( C^* \) has character \( (\beta, \alpha) \). If \( C \) had infinitely many flexes, this would imply that \( C^* \) had infinitely many singular points, which is impossible. Hence, (i) must hold. For now, we will give a more direct algebraic proof.
Suppose that (ii) holds and $C$ has infinitely many flexes. By Remarks 6.6 of [7] and using a similar argument to the above, we can assume that there exists an open $U \subset \text{NonSing}(C)$, with the property that $dF(U) \subset \text{NonSing}(C^*)$ and every $p \in U$ is a flex. For ease of notation, we abbreviate the dual morphism $dF$ by $F^{\text{dual}}$. We will show directly that if $p$ is in $U$, then:

$$dF^{\text{dual}}_{(a,b)} : T_pC \to T_{F^{\text{dual}}(p),C^*}$$

is identically zero, $(*)$.

where we have used the differential and tangent space notation, given on p170 of [2]. By standard algebraic considerations, see Section 1 of [7], this implies that the morphism $F^{\text{dual}}$ is ramified in the sense of algebraic geometry at every point of $U$. Using results of [8], see particularly Theorem 2.8, and a standard results about Zariski covers, that there can only exist finitely many ramification points, we conclude that the morphism $F^{\text{dual}}$ is inseparable. In particular, it cannot be birational as required.

In order to show $(*)$, let $\{X,Y,Z\}$ be a choice of coordinates for $P^2(L)$ and, without loss of generality, assume that $U \subset (Z \neq 0)$. Working in the coordinate system $\{x = \frac{X}{Z}, y = \frac{Y}{Z}\}$, without loss of generality, we can assume that a given flex $O$ of $U$ is located at the origin $(0,0)$ of the coordinate system $(x,y)$ and its tangent line corresponds to $y = 0$. Arguing as in Lemma 5.5, we can find a parametrisation of the branch $\gamma_O$ of the form $(t, y(t))$, where $\text{ord}_t y(t) \geq 3$. As in Lemma 5.6, we can consider $x_\epsilon = [\epsilon : y(\epsilon) : 1]$ as defining a point in $C \cap \mathcal{V}_O$, for an appropriate choice of non-standard model $K$. The coordinates of $F^{\text{dual}}(x_\epsilon)$ in the dual space $P^{2*}$ are then given by taking the cross product $\phi(\epsilon) \times \phi'(\epsilon)$, where $\phi(\epsilon) = (\epsilon, y(\epsilon), 1)$. We have:

$$\phi(\epsilon) \times \phi'(\epsilon) = (\epsilon, y(\epsilon), 1) \times (1, y'(\epsilon), 0) = (-y'(\epsilon), 1, \epsilon y'(\epsilon) - y(\epsilon))$$

so $F^{\text{dual}}(x_\epsilon) = [-y'(\epsilon) : 1 : \epsilon y'(\epsilon) - y(\epsilon)] \in C^* \cap \mathcal{V}_{F^{\text{dual}}(O)}$, see also the corresponding calculation in Lemma 5.6. Now let $\{U,V,W\}$ be projective coordinates for $P^{2*}(L)$ and let $\{u = \frac{U}{V}, w = \frac{W}{V}\}$. Let $F^{\text{dual}}_u$ and $F^{\text{dual}}_w$ be the components of $F^{\text{dual}}$ with respect to the affine coordinate system $(u,w)$. Then:

$$F^{\text{dual}}_u(\epsilon, y(\epsilon)) = -y'(\epsilon) \quad F^{\text{dual}}_v(\epsilon, y(\epsilon)) = \epsilon y'(\epsilon) - y(\epsilon)$$
Differentiating these expressions with respect to $\epsilon$, see [7] for the justification of such calculations in non-zero characteristic, we obtain;

$$\frac{\partial F^{\text{dual}}}{\partial x}|_{(\epsilon, y(\epsilon))} \cdot 1 + \frac{\partial F^{\text{dual}}}{\partial y}|_{(\epsilon, y(\epsilon))} \cdot y'(\epsilon) = -y''(\epsilon)$$

$$\frac{\partial F^{\text{dual}}}{\partial x}|_{(\epsilon, y(\epsilon))} \cdot 1 + \frac{\partial F^{\text{dual}}}{\partial y}|_{(\epsilon, y(\epsilon))} \cdot y'(\epsilon) = \epsilon y''(\epsilon) \quad (**)$$

Now setting $\epsilon = 0$ in (**) and using the fact that $\text{ord}_t y(t) \geq 3$, we obtain immediately the result (**), hence this direction of the lemma is shown.

In order to finish the result, observe that if $\text{char}(L) = 0$, one can use Lemma 5.6 directly and the argument of the first part of this Lemma ((i) $\Rightarrow$ (ii)) to show directly that the dual morphism $dF : C \leftrightarrow C^*$ is birational. If $C$ has infinitely many flexes, then one can use the argument of the second part of the Lemma ((ii) $\Rightarrow$ (i)) to show that the dual morphism $dF : C \rightarrow C^*$ is inseparable.

Remarks 5.9. For the remainder of this section, we will always assume that $\text{char}(L) \neq 2$ a given projective algebraic curve $C$ has finitely many flexes. As the duality morphism is then birational, this will allow us to use the theory of branches that we have have developed in previous papers. The exceptional case that $C$ has infinitely many flexes was studied extensively in the paper [3]. We give a brief summary of the main results;

(Corollary 2.2) Let $C$ be a non-singular projective algebraic curve of degree $n$, with infinitely many flexes. Then, if $\text{char}(L) = p \neq 2$, we have that $p | n - 1$.

(Proposition 3.7) Let $C$ be a non-singular projective algebraic curve of degree $p + 1$, with infinitely many flexes, and $\text{char}(L) = p \neq 2$, then $C$ is projectively equivalent to the plane curve with equation;

$$XY^p + YZ^p + ZX^p = 0$$

In particular, by direct calculation, the duality morphism $dF$ is purely inseparable, hence, biunivocal, and $C = C^{**}$.

(Corollary 4.3, Lemma 4.4 and Proposition 4.5)
Let $C$ be a generic non-singular projective algebraic curve of degree $dp + 1$, with $(d > 1)$, then the duality morphism $dF$ is biunivocal, but $C \neq C^{**}$.

The case of singular projective algebraic curves with infinitely many flexes is more difficult. For example, the graph of Frobenius, given by:

$$YZ^{p-1} = X^p$$

has degree $p$, infinitely many flexes and is singular. The dual curve $C^*$ is given by $X = 0$, in particular $C^{**}$ is a point, hence $C \neq C^{**}$, and the duality morphism $dF$ is purely inseparable, therefore, biunivocal.

We will return to these examples and the case when $\text{char}(L) = 2$ in the final section of this paper.

We now make the following definition:

**Definition 5.10.** Let $C \subset P^2$ be a plane projective curve. We define a multiple tangent line $l$ of $C$ to be a line which is tangent to at least 2 branches of $C$.

We now prove the following:

**Lemma 5.11.** Let $C$ be a projective algebraic curve, not equal to a line, with finitely many flexes, then every multiple tangent line of $C$ corresponds to a singularity of $C^*$, in particular, $C$ has finitely many multiple tangent lines.

**Proof.** By the hypotheses and Lemma 5.7, the duality morphism $dF$ is birational. By Lemma 5.7 of [7], the duality morphism induces a bijection $[(dF)^{-1}]^*$ between the branches of $C$ and of $C^*$. We first claim the following:

Let $\gamma$ be a branch of $C$ with tangent line $l_\gamma$, and let $O_\gamma$ be the point defined by $l_\gamma$ in $P^{2*}$, then the corresponding branch $[(dF)^{-1}]^*(\gamma)$ of $C^*$ passes through $O_\gamma$. ($\ast$)

The claim ($\ast$) is trivially true, by definition of the duality morphism, if $\gamma$ is centred at a non-singular point of $C$, ($\ast\ast$). Otherwise, we obtain the result by a straightforward limiting argument;
We use the notation and variety $V$ of Lemma 5.4. Then define $S \subset C^{ns} \times P^2 \times P^{2*}$ by:

$$S(x, y, z) \equiv V(x, y) \cap \Gamma_{[\Phi \circ dF]}(x, z)$$

where $\Phi \circ dF : C^{ns} \to C^*$ is a birational morphism. Let $R_1 \subset P^2 \times P^{2*}$ be the incidence relation defined by $R_1(u, v)$ iff $u \in H_v$, where $H_v$ is the linear form with coefficients given by $v$. If $x \in \Phi^{-1}(\text{NonSing}(C))$, then, by ($\ast$) and Lemma 5.4, we have that the fibre $S(x)$ consists of the tangent line $l_\Phi(x) \subset P^2$ and its corresponding point of the dual space $dF \circ \Phi(x) \in P^{2*}$. In particular $S(x) \subset R_1$, ($\ast \ast \ast$). As $S$ and $R_1$ are closed varieties, the relation ($\ast \ast \ast$) holds for all $x \in C^{ns}$. Now fix any branch $\gamma$ of $C$ with corresponding point $Q_\gamma \in C^{ns}$, then, by ($\ast \ast \ast$), the fibre $S(Q_\gamma)$ consists of the tangent line $l_\gamma$ and its corresponding point $O_\gamma \in P^{2*}$. From the definition of $S$ we have that $\Phi \circ dF(Q_\gamma) = O_\gamma$. Hence, the corresponding branch $[(dF)^{-1}]^*(\gamma)$ of $C^*$ must be centred at $O_\gamma$, see Lemma 5.7 of [7]. Therefore, the result ($\ast$) is shown.

Now, if $l$ is a multiple tangent line to $C$, by the previous definition, there exist at least 2 distinct branches $\{\gamma_1, \gamma_2\}$ of $C$ such that $l = l_{\gamma_1} = l_{\gamma_2}$. By the claim ($\ast$), if $O_l$ is the corresponding point of $P^{2*}$, then the corresponding branches $\{[(dF)^{-1}]^*(\gamma_1), [(dF)^{-1}]^*(\gamma_2)\}$ of $C^*$ both pass through $O_l$. By Lemma 5.4 of [7], $O_l$ must be a singular point of $C^*$. Hence, the Lemma follows immediately from the fact that a plane projective curve can only have finitely many singular points.

\[\Box\]

**Lemma 5.12.** Let $C$ be a projective algebraic curve, with finitely many flexes, then $Cl(C)$, as defined in Definition 4.1, is the same as $\text{deg}(C^*)$ and $\text{deg}(C)$ is the same as $Cl(C^*)$. In particular, if $C$ has at most nodes as singularities, then:

$$\text{deg}(C^*) = n(n - 1) - 2d$$

**Proof.** By Definition 1.12 of [7], $\text{deg}(C^*)$ is given by the number of distinct intersections $\{p_1, \ldots, p_n\}$ of $C^*$ with a generic line $l \subset P^{2*}$. Let $O_l$ be the corresponding generic point of $P^2$ and let $\{L_{p_1}, \ldots, L_{p_n}\}$ be the corresponding lines of $P^2$. We clearly have that each line $L_{p_j}$ passes through $O_l$, for $1 \leq j \leq n$. If $\{U_{[dF]}, V_{[dF]}\}$ are the canonical open subsets of $\{C, C^*\}$, with respect to the birational morphism $dF$, see Section 1 of [7], we can assume that all the intersections $\{p_1, \ldots, p_n\}$ lie inside $V_{[dF]}$. Let $\{q_1, \ldots, q_n\}$ be the corresponding non-singular points
of $C$. By the definition of the duality map $dF$, the tangent line $l_{q_j}$ is exactly $L_{p_j}$, for $1 \leq j \leq n$. Hence, by Definition 4.1, we must have that $Cl(C) \geq n$. If $Cl(C) \geq n + 1$, we could, without loss of generality, find a further $q_{n+1}$, distinct from $\{q_1, \ldots, q_n\}$, lying inside $U_{dF}$, witnessing the class of $C$, see Definition 4.1. In this case, the tangent line $l_{q_{n+1}}$ would pass through $O$, hence, again just using the definition of the duality map $dF$, the corresponding point $p_{n+1}$ of $P^{2*}$ would lie on $C^* \cap l \cap V_{dF}$. This would give at least $n + 1$ intersections of $C$ and $l$, which is a contradiction. Hence, $Cl(C) = deg(C^*) = n$ as required.

The claim that $deg(C^*) = Cl(C)$ follows from the same argument, replacing $C$ by $C^*$ and $C^*$ by $C^{**}$, and using the fact that $C = C^{**}$. Finally, the relation $deg(C^*) = n(n - 1) - 2d$ follows immediately from the Plucker formula, proved in Theorem 4.3, and the relation $Cl(C) = deg(C^*)$, which we have just shown.

We now show the following important result:

**Theorem 5.1. Transformation of Branches by Duality**

Let $C$ be a plane projective algebraic curve, with finitely many flexes, then, if $\gamma$ is a branch of $C$ with character $(\alpha, \beta)$, such that $\{\alpha, \alpha + \beta\}$ are coprime to $\text{char}(L) = p$, the corresponding branch of $C^*$ has character $(\beta, \alpha)$.

**Remarks 5.13.** One can find algebraic "proofs" of this result in the literature. These proofs use a local parametrisation of the branch and an analysis of the resulting parametrisation after applying the duality morphism. Unfortunately, such proofs fail to give the correct answer in the case when $C$ has infinitely many flexes. The explanation of this discrepancy is that in such cases, the resulting "parametrisations" fail to give parametrisations in the sense of Theorem 6.1 of [7]. One requires the fact that the duality morphism is birational in order to show this stronger claim. This suggests that a geometric proof of this result is required. The one that we give is based on Severi’s methods.

**Proof.** Let $\gamma^0$ be a given branch of $C$, with character $(\alpha, \beta)$, as in the statement of the theorem, centred at $O$, and let $l_{\gamma^0}$ be its tangent line. By (\ast) of Lemma 5.11, the corresponding branch $\gamma^{0*}$ of $C^*$ is centred at the corresponding point $O' = O_{l_{\gamma^0}}$ of the dual space $P^{2*}$. Let $l$ be a generic line through $O'$, and let $A_l$ be the corresponding point of the dual space $P^2$. As $O_{l_{\gamma^0}} \in l$, we have, by duality, that $A_l \in l_{\gamma^0}$, and, by genericity, that $A_l \neq O$. The line $l$ then parametrises the pencil
of lines passing through \( A_l \). Now pick a further generic point \( B \) belonging to \( l \) and let \( l_B \) be the corresponding line in the dual space \( P^2 \). Again, as \( B \in l \), we obtain, by duality that \( A_l \in l_B \), and, by genericity, that \( l_B \neq l_{\gamma^0} \). Now choose a parametrisation \( \Phi : P^1 \to l_B \) such that \( \Phi(0) = A_l \). We will denote the corresponding pencil of lines in \( P^2 \) by \( \{ l_t \}_{t \in P^1} \). By construction, we have that \( l_0 = l \) and that the pencil \( \{ l_t \}_{t \in P^1} \) is centred at \( B \). Now, considering \( P^2 \) as parametrising the set of lines in \( C \), where \( n = \text{deg}(C) \). Each line \( l_t \subset P^2 \), then defines a \( g_{1n} \subset g_{2n} \) on \( C \). We will denote this \( g_{1n} \) by \( g_{1n,t} \).

By a suitable choice of coordinates, we may, without loss of generality, assume that the line \( l_B \) corresponds to the line \( l_{\infty} \) in the affine coordinate system \( \{ x = \frac{X}{Z}, y = \frac{Y}{Z} \} \), the point \( \Phi(0) = A_l \) corresponds to \( [0 : 1 : 0] \), the point \( O \) corresponds to \( [0 : 0 : 1] \) and the point \( \Phi(\infty) \) corresponds to \( [1 : 0 : 0] \). By choice of \( l_B \), we can assume that the line \( l_{\infty} \) intersects \( C \) transversely in ordinary simple points \( \{ p_1, \ldots, p_n \} \). Let \( t_j = \Phi^{-1}(p_j) \) and let \( U = P^1 \setminus \{ t_1, \ldots, t_n, \infty \} \). By the choice of \( A_l \), we can assume that \( 0 \in U \). Now, for \( t \in U \), the corresponding \( g_{1n,t} \), defined above, has no fixed branch contribution. We then have that \( g_{1n,t} = (h_t) \), where \( h_t \) is the non-constant rational function on \( C \) defined by \( h_t(x, y) = x - ty \), see Lemma 2.4. In particular, we have that \( g_{1n,0} \) is defined by \( (x = 0) \). Now let \( U_{[dF]} \) be the canonical set associated to the birational morphism \( dF : C \to C^* \), and let \( V \subset U_{[dF]} \) be obtained by removing:

(i). All the finitely many flexes from \( U_{[dF]} \).

(ii). All the finitely many points on \( C \cap l_{\infty} \) from \( U_{[dF]} \).

(iii). All the finitely many non-singular points of \( U_{[dF]} \), whose tangent line is parallel to the \( y \)-axis.

Now, using the method before Lemma 3.9, we can, for \( t \neq \infty \), define the rational function \( \frac{dh_t}{dx} \). By the calculation there, if \( C \) is defined by \( f(x, y) = 0 \) in the coordinate system \( (x, y) \), we have that:

\[
\frac{dh_t}{dx} = \frac{h_x f_y - h_y f_x}{f_y} = 1 + \frac{f_x}{f_y}
\]

Then, using the notation of Lemma 2.4, for \( t \in U \setminus \{ 0 \} \), \( \frac{dh_t}{dx} = 0 \) is the same as \( (-\frac{f_y}{f_x} = t) \), considered as weighted sets on \( C \). As \( -\frac{f_y}{f_x} \) is a non-constant rational function on \( C \), then, by Lemma 2.4, we can
associate a $g^{m_1}_t = (-\frac{f_y}{f_x})$ to it.

Let $J_t = Jac(g^{m_1}_t)$, see Definition 3.2 and Definition 3.22. Then $J_t$ consists of a weighted set of branches $\{\alpha_1 \gamma^1_t, \ldots, \alpha_r \gamma^r_t\}$. We claim the following:

$$(J_t \cap V) = (g^{m_1} \cap V) \quad \text{for } (t \in U \setminus \{0\}) \quad (\dagger)$$

In order to show $(\dagger)$, let $\gamma$ be a branch of $V$. By the definition of $V$, $\gamma$ is centred at the finite position $(a, b)$, has character $(1, 1)$ and its tangent line $l_\gamma$ is not parallel to the $y$-axis. Let $(x, y(x))$ be a parametrisation of $\gamma$ in the form given by Lemma 3.7. If $\gamma$ belongs to $J_t$, then $ord_\gamma(h_t) = 2$, $val_\gamma(h_t) < \infty$ and $h_t$ determines an algebraic power series at the branch $\gamma$;

$$h_t = \lambda + (x - a)^2 \psi(x - a) \quad \text{with } \psi(0) \neq 0, \lambda < \infty$$

We then have that;

$$\frac{dh_t}{dx} = (x - a)[2\psi(x - a)] + [(x - a)\psi'(x - a)] \quad \text{(char}(L) \neq 2)$$

At $x = a$, the expression in brackets reduces to $2\psi(0) \neq 0$, hence $ord_\gamma(\frac{dh_t}{dx}) = 1$ and $val_\gamma(\frac{dh_t}{dx}) = 0$. This shows that $\gamma$ is counted once in the weighted set $\left(\frac{dh_t}{dx} = 0\right)$, and, therefore, once in the weighted set $W_t$ of the corresponding $g^{m_1}_t$ defined above. If $\gamma$ belongs to the $g^{m_1}_t$, then, $val_\gamma(\frac{dh_t}{dx}) = 0$, for some $t \in U \setminus \{0\}$. Reversing the above argument, using the fact that $val_\gamma(h_t) < \infty$, we obtain that $ord_\gamma(\frac{dh_t}{dx}) = 2$, hence $\gamma$ belongs to $J_t$ as required. Therefore, $(\dagger)$ is shown. Note that an almost identical argument to the above was carried out in Lemma 3.9.

We now consider the behaviour of the $g^{m_1}_t$ at the branch $\gamma^0$. We claim that $\gamma^0$ is counted $\beta$ times in the weighted set $W_0$ of this $g^{m_1}_m$, $(\dagger\dagger)$. Let $(x(s), y(s))$ be a parametrisation of the branch at $(0,0)$. Using the definition of a parametrisation in Theorem 6.1 of [7] and the fact that the tangent line $l_{\gamma^0}$ is given by $x = 0$, we obtain immediately that $ord_x(x(s)) = \alpha + \beta$ and $ord_y(y(s)) = \alpha$. Using the fact that $f(x(s), y(s)) = 0$, we obtain that;

$$f_x|_{(x(s), y(s))}x'(s) + f_y|_{(x(s), y(s))}y'(s) = 0$$
If either $f_x$ or $f_y$ is identically zero on $C$, then the image of the map $dF$ is contained in a line, which implies, by Lemma 5.8, that $C$ has infinitely many flexes. Hence, we must have that $x'(s) \equiv 0$ iff $y'(s) \equiv 0$. If both $x'(s) \equiv 0$ and $y'(s) \equiv 0$, we can find algebraic power series \{x_1(s), y_1(s)\} such that $x(s) = x_1(s^p)$ and $y(s) = y_1(s^p)$. This would contradict the construction of a parametrisation, as defined in Theorem 6.1 of [7], see also the method used in [8], Remarks 2.3. Hence, we have $x'(s) \neq 0$, and $y'(s) \neq 0$, and:

$$-rac{f_y}{f_x}(x(s), y(s)) = \frac{x'(s)}{y'(s)}$$

By the assumption on \{\alpha, \beta\} in the statement of the theorem, we obtain that \text{ord}_p\left(-\frac{f_y}{f_x}(x(s), y(s))\right) = (\alpha + \beta - 1) - (\alpha - 1) = \beta$. By definition of the $g_m^{1\prime}$, this implies (†††).

Now consider the $g_m^{1\prime}$ on $C^{\ast}$, given by the pencil \{l_t\}_{t \in P^1}. By Theorem 1.14, and the fact that $C$ and $C^{\ast}$ are birational, this transfers to a $g_m^{1\prime}$ on $C$. By choice of $B$, the $g_m^{1\prime}$ on $C$ and $C^{\ast}$ have no base branches. We claim that $g_m^{1\prime} = g_m^{1\prime\prime}$ (†††). Let \{V, V^{\prime}\} be canonical sets associated to the birational map $dF : C \dashrightarrow C^{\ast}$, where $V$ was defined above. Then, using Lemma 2.17 of [7], there exists an open subset $U^{\prime} \subset U \subset P^1$ such that, for $t \in U^{\prime}$, the corresponding weighted sets \{W_t, W_t^{\prime}\} of the \{g_m^{1\prime}, g_m^{1\prime\prime}\} on $C$, consist of branches(points) which are simple for \{W_t, W_t^{\prime}\} respectively, based inside $V$ or $V^{\prime}$. In this case, a point $p \in W_t^{\prime}$ corresponds to a transverse intersection between $l_t$ and $C^{\ast}$. The line $l_p'$ in the dual space $C$ is then a tangent line to a non-singular point $p'$ of $C$, having character $(1, 1)$. We, therefore, have that $l_{p'}(C, l_p) = 2$, hence $p'$ is counted once for $\text{Jac}(g_m^{1\prime\prime})$. Combining this with the result (†), we obtain that $W_t = W_t^{\prime}$, hence (†††) is shown.

We now finish the proof of the theorem. By Theorem 1.12, the fact that the $g_m^{1\prime}$ on $C$ has no fixed branches, and the result (†††), for generic $t \in V_0$, the weighted set $W_t \cap C \cap \gamma^0$ of the $g_m^{1\prime}$ consists of $\beta$ distinct branches(points), centred in $U' \cap \gamma^0$. By (†††), these correspond to $\beta$ transverse intersections $C^{\ast} \cap l_t \cap \gamma^{0\ast}$. It follows immediately that $I_{p'}(C^{\ast}, l_0) = \beta$, hence, as $l_0 = l$ was chosen to be generic through $O'$, the order of the branch is $\beta$. We now follow through the same argument, replacing $C$ by $C^{\ast}$ and $C^{\ast}$ by $C^{\ast\ast} = C$. It follows immediately that the class of the branch $\gamma^{0\ast}$ must be the order of the branch $\gamma^0$. Hence, the character of the branch $\gamma^{0\ast}$ is $(\beta, \alpha)$ as required.
Remarks 5.14. The assumption that $\{\alpha, \alpha+\beta\}$ are co-prime to $\text{char}(L) = p$ is necessary. Consider the projective algebraic curve $C$ defined by;

$$y = x^p + x^{p+2}$$

In projective coordinates, this is defined by $YZ^{p+1} = X^pZ^2 + X^{p+2}$ and the duality morphism $dF$ is given by;

$$dF : [X : Y : Z] \mapsto [2X^{p+1} : -Z^{p+1} : 2X^pZ - YZ^p]$$

$$(x, y) \mapsto (-2x^{p+1}, y - 2x)$$

The duality morphism is clearly separable, hence $C$ must have finitely many flexes. $C$ is non-singular in the affine coordinate system $(x, y)$, and the character of the unique branch $\gamma_0$ of $C$ at the origin $(0, 0)$ is $(1, p-1)$. The proof of the theorem shows that the order of the corresponding branch $\gamma_0^*$ of $C^*$ is $(p+1) \neq (p-1)$, for $\text{char}(L) \neq 2$.

One can easily construct further examples using appropriate rational parametrisations.

6. A Generalised Plucker Formula

The purpose of this section is to give a geometric proof of a generalisation of the Plucker formula of Section 4. We also discuss the question of representation of plane algebraic curves in greater detail. We first require the following definition;

Definition 6.1. We will define a plane projective algebraic curve $C$ to be normal if it is has finitely many flexes and all its branches $\gamma$ have character $(\alpha, \beta)$, with $\{\alpha, \alpha+\beta\}$ coprime to $\text{char}(L) = p$.

We first show the following;

Lemma 6.2. Let $C$ be a normal plane projective curve with $\text{class}(C) = m$, as defined in Definition 4.1, genus$(C) = \rho$, as defined in Definition 3.33 and order$(C) = n$. Then;

$$\rho = \frac{1}{2}[m + \sum_{\gamma}(\alpha(\gamma) - 1)] - (n - 1)$$

where the sum is taken over the finitely many singular branches and, for such a branch $\gamma$, $\alpha(\gamma)$ gives the order of the branch. In particular,
if $C$ is a normal projective curve, having at most nodes as singularities, we obtain the formula shown earlier:

$$\rho = \frac{m}{2} - (n - 1)$$

Proof. By Remarks 4.2 and the fact that $C$ has finitely many flexes, we can suppose that, for generic $P$, the $m$ tangent lines of $C$ passing through $P$ are based at ordinary simple points. In particularly, $P$ does not lie on any of the finitely many tangent lines belonging to the singular branches of $C$. We consider the $g^1_n$ defined by the pencil of lines passing through $P$. We have that $\text{Jac}(g^1_n)$ consists exactly of the $m$ ordinary branches witnessing the class of $C$ and the finitely many singular branches $\gamma$, each counted $\alpha(\gamma) - 1$ times. In particular;

$$\text{order}(\text{Jac}(g^1_n)) = m + \sum_\gamma (\alpha(\gamma) - 1)$$

Now we obtain the first part of the lemma from the fact that $\text{order}(g^1_n) = n$ and Definition 3.33 of the genus of $C$. If $C$ has at most nodes as singularities, then it has no singular branches. Therefore, $(\alpha(\gamma) - 1) = 0$ for any branch $\gamma$ of $C$. The second part of the lemma then follows from the previous formula.

Using duality, we have;

**Lemma 6.3.** Let $C$ be a normal plane projective algebraic curve, not equal to a line, with the invariants $\{m, n, \rho\}$ as defined above. Then;

$$\rho = \frac{1}{2}[n + \sum_\gamma (\beta(\gamma) - 1)] - (m - 1)$$

where the sum is taken over the finitely many flexes, and, for such a branch $\gamma$, $\beta(\gamma)$ gives the class of the branch.

Proof. As $C$ is normal and not equal to a line, we may apply Lemma 5.8, Lemma 5.12 and Theorem 5.1. In particular, we have that $C^*$ has finitely many flexes, and, as in the previous lemma, for generic $P$, the tangent lines of $C^*$, passing through $P$, are based at ordinary simple points. We consider the $g^1_{n'}$ on $C^*$ defined by the pencil of lines passing through $P$. We have that $n' = \text{order}(\text{class}(C^*) = m$ and $\text{class}(C^*) = \text{order}(C) = n$ by Lemma 5.12. We also have that $\text{Jac}(g^1_{m})$ consists exactly of the $n$ ordinary branches witnessing the class of $C^*$ and the finitely many singular branches, each counted $(\beta(\gamma) - 1)$ times,
by Theorem 5.1. Hence;

\[ \text{order}(\text{Jac}(g^1_m)) = n + \sum_{\gamma} (\beta(\gamma) - 1) \]

where the sum is taken over the finitely many flexes of \( C \), using the fact that \( C \) and \( C^* \) are birational, given in Lemma 5.8. We also have that \( \rho = \text{genus}(C) = \text{genus}(C^*) \) by Lemma 5.8 and Theorem 3.35. Hence, we obtain the first part of the theorem from the fact that \( \text{order}(g^1_m) = m \) and Definition 3.33 of the genus of \( C^* \).

\[ \square \]

**Theorem 6.4. Generalised Plucker Formula**

Let \( C \) be a normal plane projective algebraic curve, not equal to a line, with \( \{m,n\} \) defined as in the previous lemmas. Then;

\[ 3m - 3n = \sum_{\gamma} (\beta(\gamma) - 1) - \sum_{\gamma} (\alpha(\gamma) - 1) \]

where the sums are taken over the finitely many flexes and finitely many singular branches of \( C \) respectively. In particular, if \( C \) is a plane projective curve, having at most nodes as singularities and no flexes, then it is either a line or a smooth conic.

**Proof.** The first part of the theorem follows immediately by combining the first formulas given in Lemma 6.2 and Lemma 6.3. If \( C \) is a smooth plane projective curve, having at most nodes as singularities and no flexes, then it must be normal as all its branches \( \gamma \) have character \( (1,1) \) \((\text{char}(L) \neq 2)\). If \( C \) is not a line, then we can apply the first part of the theorem, to obtain that \( m = n \). Using the Plucker formula given in Theorem 4.3, we have that \( m = n^2 - n - 2d \), where \( d \) is the number of nodes of \( C \). Hence, \( n^2 - n - 2d = n \), which gives that \( d = \frac{n(n-2)}{2} \). If \( n \geq 3 \), this contradicts Theorem 3.37. Hence, \( n = 2 \) and \( d = 0 \). The second part of the theorem then follows.

\[ \square \]

We now apply the above formulas to the study of normal plane projective curves, having at most nodes as singularities.

**Theorem 6.5.** Let \( C \) be a normal plane projective curve, not equal to a line, having at most nodes as singularities, with the convention on summation of branches given above and \( \{m,n,\rho,d\} \) as defined in the previous lemmas. Then, we obtain the class formula;
\[3m - 3n = \sum_{\gamma}(\beta(\gamma) - 1) \quad (1)\]

and the genus formula;

\[6\rho + 3n - 6 = \sum_{\gamma}(\beta(\gamma) - 1) \quad (2)\]

and the node formula;

\[3n(n - 2) - 6d = \sum_{\gamma}(\beta(\gamma) - 1) \quad (3)\]

In particular, if \(C\) has at most ordinary flexes, and \(i\) is the number of these flexes, we obtain the class formula, referred to as Plucker III’ in [10];

\[3m = 3n + i \quad (4)\]

and the genus formula;

\[6\rho = i - 3n + 6 \quad (5)\]

and the node formula, referred to as Plucker III in [10];

\[6d = 3n(n - 2) - i \quad (6)\]

Proof. The class formula (1) follows from the Generalised Plucker formula and the fact that \((\alpha(\gamma) - 1) = 0\) for any branch \(\gamma\) of \(C\), as \(C\) has at most nodes as singularities. The genus formula (2) follows from (1) and the formula \(\rho = \frac{m}{2} - (n - 1)\), given in Lemma 6.2. The node formula (3) follows from (1) and the Plucker formula \(m = n(n - 1) - 2d\), given in Theorem 4.3. The formulas (4), (5), (6) all follow immediately from the corresponding formulas (1), (2), (3) and the fact that;

\[i = \sum_{\gamma}(\beta(\gamma) - 1)\]

as an ordinary flex has character \((1, 2)\), hence, for the corresponding branch \(\gamma\), \((\beta(\gamma) - 1) = 1.\]

\[\square\]

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