Slow Transient Processes in the Second Sound Resonator

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The Hydrodynamics of Superfluid Turbulence (HST) describes the flows (or counterflows) of HeII in the presence of a chaotic set of vortex filaments. The HST equations govern both a slow variation of the hydrodynamic variables due to dissipation related to the vortex tangle and fast processes of the first and second sound propagation. This circumstance prevents effective numerical simulations of the problems of unsteady heat transfer in HeII. By virtue of a pertinent multi-scale perturbation analysis we show how one can eliminate the fast processes to derive the evolution equation for the slow processes only. We then demonstrate that the long-term evolution of a transient heat load of moderate intensity obeys the nonlinear heat conductivity equation. The second example of the methods developed is investigation of unsteady processes in the second sound resonator. The latter is frequently used for study of nonstationary behavior of vortex tangle, just by monitoring of the quality factor behavior. This procedure however is wrong when characteristic times of processes are comparable (or smaller) than the time constant of resonator. We show how to extract the correct information on the vortex line density (VLD) dynamics with use of procedure we developed.

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1. INTRODUCTION AND SCIENTIFIC BACKGROUND

It is appreciated that in most cases the flow (or counterflow) of HeII occurs in the presence of a chaotic set of vortex filaments, the so called superfluid turbulence. To describe various problems both for applications and for experiments it is necessary to use equations of HST. These equations unify classical equations of hydrodynamics of HeII and the famous Vinen equation describing the dynamics of the vortex tangle. The HST was widely used to describe various hydrodynamics as well as thermal phenomena in
HeII in presence of vortex tangle (see the review paper 2).

The set of HST equations is extremely cumbersome, therefore, there is no wonder that to achieve quantitative results one is bound to turn to numerical methods. Yet, numerical simulation of nonstationary flows of HeII faces one serious obstacle. The point here is that slow variation of hydrodynamic variables is accompanied by the fast processes related to propagation of the first and second sounds. If one is interested in the slow evolution, particular details of sound propagation become completely irrelevant, yet requiring rather extensive numerical resources.

Thus, it seems attractive to try to get rid of the fast modes by a pure analytical procedure. In the present paper we realize effective separation of the slow from the fast modes using multi-time asymptotic perturbation techniques (See, for example, book by Nayfeh 3). That is quite universal powerful method, its application, however, depends on concrete statement of the problem. In the present work we consider two particular examples. In the first example we follow how the initially hyperbolic-type equations of HST (describing wave type of heat transfer) lead to purely parabolic-type nonlinear heat transfer equations. Another example of the method is the study of unsteady processes in the second sound resonator.

2. THE NONLINEAR HEAT CONDUCTIVITY EQUATION

Let us consider the problem of one-dimensional heat exchange in the tube filled by HeII. We assume that a heater is switched instantly on and the thermal front starts out propagating into undisturbed bulk of helium. Initially, of course, the propagation of the heat should be realized by the second sound mechanism. But gradually, due to quantum vorticity the second sound attenuates and degenerates. The mechanism of the heat exchange should change. In order to ascertain both this new mechanism and the correspondent law we apply multi-time asymptotic perturbation techniques.

The set of equations of HST for dimensionless velocity of the normal component \( V'_n \), dimensionless temperature \( T' \) and dimensionless square root of the vortex line density (VLD) \( G = \sqrt{L/L_{\infty}} \) (for details and notations see the review article of Nemirovskii and Fiszdon 2) is reduced to the form:

\[
\frac{\partial V'_n}{\partial t'} + \frac{\partial T'}{\partial x'} = -Sh G^2 V'_n, \quad \frac{\partial T'}{\partial t'} + \frac{\partial V'_n}{\partial x'} = 0, \quad (1)
\]

\[
\frac{\partial G}{\partial t'} = \frac{\alpha}{A(T) \rho_s \rho_n} \left( \frac{\alpha}{\beta} \right) \frac{Sh}{2} (G^2 V'_n - G^3), \quad (2)
\]
with the following dimensionless variables:

\[ t = \frac{L}{c_2} t', \quad x = D x', \quad V_n = V_{n0} V'_n, \quad T = \frac{\sigma V_{n0}}{\sigma_T c_2} T', \quad L = L_\infty G^2. \quad (3) \]

It is easy to see that the dynamics of the hydrodynamic variables is specified by the dimensionless criterion the Strouhal number

\[ Sh = \frac{D}{c_2 \tau_d}, \quad (D\text{-length of channel, } \tau_d \text{ decrement of attenuation of counterflow due to vortices}). \]

The Strouhal number is defined as the ratio of the counterflow decrement to inverse time that takes the heat pulse to cross the channel. In the case of heat load of moderate intensity, of order of a few \( W/cm^2 \), the quantity \( Sh \) is of order of \( 10^{2} - 10^{5} \), i.e. much greater than unity.

For times of order \( L/(c_2 Sh) \) the heat pulse propagates according to wave-like equation. Our goal now is to clarify what will happen at later times. For that purpose we have to eliminate the fast processes. According general ideas of multi-scale asymptotic perturbation theory (see e.g. Nayfeh\(^3\)) we look for a solution to the set of the HST equations \(^1\)-\(^2\) in the form of an asymptotic series. Following this method, we introduce different time scales.

\[ t_0 = t'; \quad t_1' = \epsilon t'; \quad t_2' = \epsilon^2 t' \quad \ldots \ldots \text{ , where } \epsilon = 1/Sh << 1. \]

We look for a solution to the set of the HST equations \(^1\)-\(^2\) in the form of an asymptotic series \( V'_n = V'_0(x', t'_0, t'_1, t'_2) + \epsilon V'_1(x', t'_0, t'_1, t'_2) + \epsilon^2 V'_2(x', t'_0, t'_1, t'_2) + \ldots \ldots \text{ and similarly to } T' \text{ and } G. \)

The simple rule takes place:

\[ \frac{\partial}{\partial t'} = \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2}. \quad (4) \]

The next step in study of the slow evolution of the heat pulse consists in substituting the multi-time scale series into equations of HST \(^1\)-\(^2\). Gathering terms of the same order of magnitude with respect to \( \epsilon \) we come up with a chain of equations leading to divergent (secular) solutions. Cancelling step by step these secularities we then obtain a hierarchy of equations of different orders in parameter \( \epsilon \), governing different stages of the evolution. Omitting details of computations we concentrate on the first order in \( \epsilon \). In the first order, equations for normal velocity and temperature variations read

\[ \frac{\partial V'_{03}}{\partial t'_1} = \frac{\partial^2 V'_0}{\partial x'^2}, \quad \frac{\partial T'}{\partial t'_1} + \frac{\partial V'_0}{\partial x'} = 0, \]

Excluding quantity \( V'_0 \) from the set of equations written above we obtain

\[ \frac{\partial T'}{\partial t'_1} = \epsilon^{-1/3} \frac{\partial}{\partial x'} \left( \frac{\partial T'}{\partial x'} \right)^{1/3}. \quad (5) \]

Relation \(^5\) coincides with the widely used nonlinear heat-conductivity equation derived by Dresner\(^4\) by a pure phenomenological model. We stress, that unlike the Dresner derivation, our work shows that this relation is just the long-time asymptotic limit of the full set of equations of HST.
Fig. 1. (a) Excess attenuation of the resonance waves measured by author of [5], (b) The same quantity computed with use of asymptotic perturbation techniques. *We have to stress that this behavior is just the transient response of the resonator, and change of amplitude does not follow (directly) the evolution of VLD.*

### 3. MEASUREMENTS OF THE VORTEX LINE DENSITY IN THE SECOND SOUND RESONATOR

The second problem which we study with use of multi-scale asymptotic perturbation techniques is scrutinizing a very popular experimental method: measuring vortex line density with a second sound resonator. Although that method is standard and used in many works, we, for the sake of definiteness, will consider the recent experiments made by Hilton and Van Sciver [5]. They performed direct measurements of quantum turbulence induced by second sound shock pulses in a wide channel. In Fig. 1a, there is depicted the excess attenuation of second sound in the resonator placed across the channel, along which the intense thermal pulse is propagating. It was assumed that thermal pulse created the quantum turbulence which lead to the excess attenuation of second sound. Assuming further that the the excess attenuation $\alpha_v(t)$ is connected with the amplitude of resonance wave $A(t)$ by the well-known Hall-Vinen relation $\alpha_v(t) = \alpha_v(0)(A(0)/A(t) - 1)$, and expressing $\alpha_v(t)$ via the VLD $L(t)$ the authors drew a conclusion about the VLD evolution, coinciding (up to factor) with the curve depicted in Fig. 1a.

It is easy to see, however, that this evolution looks to be quite unreasonable. Indeed, according to current view on VLD dynamics, the quantity $L(t)$ should grow only in presence of relative velocity. But the pulse duration was $0.75\text{ ms}$, whereas $L(t)$ increases during $5 \div 7\text{ ms}$. The origin of that discrepancy is obvious. The Hall-Vinen relation based on the quality factor is valid only for very slowly changing external conditions (duration of thermal pulse) in comparison with the time constant of the resonator. Here
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the condition was inverted and the Hall-Vinen relation was inapplicable; i.e. the amplitude $A(t)$ evolved independently from the quality factor evolution. In other words the curve depicted in Fig.1. demonstrates reaction of the resonator rather than the true behavior of VLD $L(t)$.

Thus the direct procedure of measuring of VLD based on the quality factor is not correct and the question appears whether it is possible at all to extract information about VLD $L(t)$ evolution. The answer is positive and corresponding analysis is based on the general topic of our work i.e. on the fact that excess attenuation of the second sound due to quantum turbulence changes slower than duration of main thermal pulse. Then we are in position to apply the multi-scale asymptotic perturbation techniques.

Let us describe very briefly the corresponding computations. Taking the spatial Fourier transform to Eqs. (1) we arrive at the following equation for variations of the temperature $T'$ in the standing second sound wave

$$\frac{d^2 T'}{dt^2} + p(\xi) \frac{dT'}{dt} + \omega^2 T' = \omega^2 T_0 e^{i\omega t}$$

(6)

Here $T_0'$ is the pumping amplitude, $\omega$ is the resonance frequency of standing wave, $p = \alpha_b + \alpha_v(\xi)$ is the attenuation coefficient consisting of background part $\alpha_b$ due to background vorticity and variable part $\alpha_v(\xi)$ due to passing thermal pulse. The latter depends on the slow time $\xi = \epsilon t$, where $\epsilon$ is small parameter, characterizing, say, attenuation coefficient/frequency.

The corresponding analysis demonstrates that the solution is (as usual) a combination of general and particular solution. The general solution is just natural oscillations, attenuating proportional to $\exp[-(1/2) \int p(\xi) d\xi]$. As for particular solution, we seek it in a form:

$$T_{part}' = \eta(t, \epsilon) e^{i\omega t}, \quad \frac{d\eta}{dt} = [G(\xi, \epsilon) - i\omega] \eta + H(\xi, \epsilon)$$

(7)

Substituting (7) into (6) and comparing coefficients near $\eta(t, \epsilon)e^{i\omega t}$ and $e^{i\omega t}$ we obtain relations for $G$ and $H$. Representing, further, quantities $G$ and $H$ in form of series in $\epsilon$, and equating coefficient near the same powers of $\epsilon$ we obtain recurrent equations for successive determination of both $G(\xi)$ and $H(\xi)$. Restricting ourseves to the first order we obtain the following equation to determine the complex amplitude of particular solution, which is just time dependent amplitude in a resonator measured in experiment $5$

$$\frac{d\eta}{dt} = -\frac{\alpha_b + \alpha_v(\xi)}{2} \eta + (1 - \frac{(\alpha_b + \alpha_v(\xi))}{4\omega}) \frac{T_0 \omega}{2i}$$

(8)

It is seen that amplitude of second sound wave in resonator is equal to the pumping amplitude multiplied by the quality factor only in limit of very
long time. Generally that amplitude obeys a more complicated law. Let us discuss how to extract the information concerning true behavior of vortex line density. Here we have performed the following procedure. We supposed that the Vinen equation for evolution of VLD can be indeed extrapolated to very high value of applied counterflow velocity. Then we varied coefficients $\alpha$ and $\beta$ of that equation in order to select best values to fit the curve depicted in curve of Fig.1a. In Fig.1b. is a curve which corresponds to values $\alpha \approx 0.13 \alpha_{Vi}$ and $\beta \approx 0.16 \beta_{Vi}$, where $\alpha_{Vi}$ and $\beta_{Vi}$ are values offered by Vinen. Evolution of the VLD corresponding to these values is depicted in Fig.2.

4. CONCLUSION

By use of multi-scale perturbation analysis it has been shown how the wave-like behavior of the heat pulse described by initially hyperbolic equations gives way to the parabolic-type non-linear heat conductivity equation. By the similar method we study unsteady processes in the second sound resonator.

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