Regularity of the Bergman Projection on Forms 
and Plurisubharmonicity Conditions

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Vienna, Preprint ESI 1731 (2005)

November 2, 2005

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at
REGULARITY OF THE BERGMAN PROJECTION ON FORMS AND PLURISUBHARMONICITY CONDITIONS

A.-K. HERBIG & J.D. MCNEAL

Abstract. Let \( \Omega \subset \subset \mathbb{C}^n \) be a smoothly bounded domain. Suppose \( \Omega \) has a defining function, such that the sum of any \( q \) eigenvalues of its complex Hessian is non-negative. We show that this implies global regularity of the Bergman projection, \( B_{j-1} \), and the \( \bar{\partial} \)-Neumann operator, \( N_j \), acting on \( (0,j) \)-forms, for \( j \in \{q, \ldots, n\} \).

1. Introduction

A function \( f \in C^\infty(\Omega) \) is holomorphic on \( \Omega \), if it satisfies the Cauchy-Riemann equations: \( \bar{\partial} f = \sum_{k=1}^{n} \frac{\partial f}{\partial \overline{z}_k} \, \overline{d}z_k = 0 \) in \( \Omega \). Denote the set of holomorphic functions on \( \Omega \) by \( H(\Omega) \). The Bergman projection, \( B_0 \), is the orthogonal projection of square-integrable functions onto \( H(\Omega) \cap L^2(\Omega) \). Since the Cauchy-Riemann operator, \( \bar{\partial} \) above, extends naturally to act on higher order forms, we can as well define Bergman projections on higher order forms: let \( B_j \) be the orthogonal projection of square-integrable \( (0,j) \)-forms onto its subspace of \( \bar{\partial} \)-closed, square-integrable \( (0,j) \)-forms.

In this paper we give a condition on \( \Omega \) which implies that \( B_j f \) is smooth on \( \Omega \) whenever \( f \) is. Our result is the following:

Theorem 1.1. Let \( \Omega \subset \subset \mathbb{C}^n \) be a smoothly bounded domain. Suppose there exists a smooth defining function of \( \Omega \), such that the sum of any \( q \) eigenvalues of its complex Hessian is non-negative on \( \Omega \). Then the Bergman projection, \( B_{j-1} \), is globally regular for \( q \leq j \leq n \).

Global regularity of the Bergman projection is closely tied to the regularity of the \( \bar{\partial} \)-Neumann operator. Recall that the \( \bar{\partial} \)-Neumann operator, \( N_j \), is the operator, acting on square-integrable \( (0,j) \)-forms, which inverts a particular boundary value problem associated to the complex Laplacian. By a result of Boas and Straube [Boa-Str90], \( N_j \) is globally regular if and only if \( B_{j-1}, B_j \) and \( B_{j+1} \) are. Consequently, the hypothesis of our Theorem 1.1 implies that \( N_j, q \leq j \leq n \), are also globally regular.

1991 Mathematics Subject Classification. 32W05.
Key words and phrases. Bergman projection, global regularity.
Research of the first author was partially supported by a Rackham Fellowship.
Research of the second author was partially supported by an NSF grant.
Theorem 1.1 is an extension of an earlier result of Boas and Straube in [Boa-Str91], though only a partial one. There the authors show that $B_j$ is globally regular for all $j \in \{1, \ldots, n\}$, if $\Omega$ has a smooth defining function which is plurisubharmonic on $b\Omega$ (the boundary of the domain $\Omega$). Our theorem covers the case considered by Boas-Straube, when $q = 1$, but only under the stronger hypothesis that the defining function $r$ is plurisubharmonic on all $\Omega$. Our method of proof is quite different than the one in [Boa-Str91], though there are, naturally, some points in common. Our proof shares more similarities with one of Kohn [Koh99], where he determined how the range of Sobolev norms $\| \cdot \|_k$, where $\|B_0 f\|_k \leq C\|f\|_k$ holds, depends on the Diederich-Fornaess exponent – the (largest) exponent $1 > s > 0$ such that $-(−r)^s$ is plurisubharmonic – when $r$ itself is not plurisubharmonic.

To compare our proof with that of [Boa-Str91], consider the case in common to both results, i.e., assume that $r$ is plurisubharmonic on all of $\Omega$. The essential problem is to estimate some Sobolev norm larger than $1/2$ of $B_* f$ by the same Sobolev norm of $f$, say $\|B_* f\|_1$ by $\|f\|_1$. In both proofs, standard arguments reduce this problem to that of estimating $\|X B_* f\|_0$ by $\|f\|_1$, where $X$ is a tangential vector field to $b\Omega$ which is transverse to the complex tangent space. In order to achieve this estimate, $X$ must commute “nicely” with the $\bar{\partial}$-complex in some fashion, so that one can absorb the error terms which arise in comparing $X (B_* f)$ to $B_* (X f)$. In [Boa-Str91], the focus is on the commutator $[X, \bar{\partial}]$, and the plurisubharmonicity of $r$ is used to construct a special vector field $X$ so that this commutator has a small component in the complex normal direction to $b\Omega$. The role of plurisubharmonicity in this approach is that non-negativity of the matrix $\left(\frac{\partial^2 r}{\partial z_k \partial \bar{z}_l}\right)$ can be used to adjust any tangential, transverse field by adding tangential fields containing only barred derivatives (which are benign in the estimates considered) to it in such a way that the commutator has the desired property. In our proof, the focus is on the commutator $[\bar{\partial}^*, X]$, for $X$ the “natural” tangential vector field transverse to the complex tangent space (see the subsection I Section 4 below). Actually, we focus on a tangential field $T$, very closely related to $X$; the crucial property of $T$ is that it preserves the domain of $\bar{\partial}^*$. We point out that the passage from $X$ to $T$ is a lower-order adjustment and does not depend on the plurisubharmonicity of $r$, i.e., $T$ differs from $X$ by a 0th order operator. The matrix $\left(\frac{\partial^2 r}{\partial z_k \partial \bar{z}_l}\right)$ then appears as the matrix of coefficients in front of $T$, acting on various components of a $(0, q)$-form $\varphi$, closely connected to the form $f$, paired with a negligible form. The non-negativity of $\left(\frac{\partial^2 r}{\partial z_k \partial \bar{z}_l}\right)$ then allows the use of the Cauchy-Schwarz inequality to separate this pairing into separate, purely quadratic factors,
see (4.12) below. The factor involving the $T$ derivatives of $\varphi$ is then estimated by a small constant times the $\bar{\partial}$-Dirichlet form of $\varphi$ using the $\bar{\partial}$-Hardy inequality proved in Section 3.

Theorem 1.1 gives many examples of domains where the Bergman projection on higher-level forms is regular while the Bergman projection on functions is not. Suppose $D \subset \subset \mathbb{C}^n$ is a smoothly bounded domain, and $\rho$ a smooth defining function of $D$. Let $C$ be a lower bound for the sum of the eigenvalues of $i\partial\bar{\partial}\rho$ on $D$, and define

$$\tilde{D} = \{(z, w) \in \mathbb{C}^{n+m} \mid r(z, w) := \rho(z) + K(|w_1|^2 + \cdots + |w_m|^2) < 0\},$$

where $K \geq 0$ is chosen such that $K \geq |C|$. Then $\tilde{D}$ is a smoothly bounded domain, and the sum of any $(n+1)$ eigenvalues of $i\partial\bar{\partial}\rho$ is non-negative on the closure of $\tilde{D}$. Thus, by Theorem 1.1, the Bergman projection, $B^\tilde{D}_q$, on $(0, q)$-forms is globally regular for $n \leq q \leq n + m$. In [Bar84], Barrett constructed a smoothly bounded domain $D'$ in $\mathbb{C}^2$ for which the Bergman projection on functions fails to be regular. Inserting Barrett’s domain $D'$ for $D$ in the preceding construction, one obtains a smoothly bounded domain $\tilde{D}'$ in $\mathbb{C}^{2+m}$ such that $B^\tilde{D}_q$ is regular for $2 \leq q \leq 2 + m$. However, using similar arguments to those in [Bar84], one can show that $B^\tilde{D}_0$ fails to be regular.

The paper is structured as follows. In Section 2 we present the general setting and a brief review of the $\bar{\partial}$-Neumann problem and its relation to the Bergman projections. In Section 3 we derive the basic estimates which will be used for the proof of Theorem 1.1. In particular, we prove a Hardy-like inequality for the $\bar{\partial}$-complex, Proposition 3.4; this estimate is of independent interest and should have further application. In Section 4, we give the proof of Theorem 1.1. Although the total length of this paper exceeds that of [Boa-Str91], the analytic heart of our proof is relatively short and is labeled as such in Section 4.

We would like to thank K. Koenig and E. Straube for pointing out errors in an earlier version, and for their helpful suggestions.

2. Preliminaries

Throughout, let $\Omega \subset \subset \mathbb{C}^n$ be a smoothly bounded domain, i.e., $\Omega$ is bounded and there exists a smooth, real-valued function $r$ such that $\Omega = \{z \in \mathbb{C}^n \mid r(z) < 0\}$, and $\nabla r \neq 0$ when $r = 0$. The hypothesis of Theorem 1.1 will be abbreviated as follows.

**Definition 2.1.** We say that a smoothly bounded domain $\Omega \subset \subset \mathbb{C}^n$ satisfies condition $(H_q)$ if there exists a defining function $r$ for $\Omega$ such that the sum of any $q$ eigenvalues of $\left(\partial^2 r / \partial z_k \partial \bar{z}_q\right)$ is non-negative on $\overline{\Omega}$.
We shall write an arbitrary \((0, q)\)-form \(u\), \(0 \leq q \leq n\), as
\[
(2.2) \quad u = \sum'_{|J|=q} u_J d\bar{z}^J,
\]
where \(\sum'_{|J|=q}\) means that the sum is taken over strictly increasing multi-indices \(J\) of length \(q\). We define the coefficients \(u_I\) for arbitrary multi-indices \(I\) of length \(q\), so that the \(u_I\)'s are antisymmetric functions in \(I\).

Let \(\Lambda^{0,q}_c(\Omega)\) and \(\Lambda^{0,q}_c(\Omega)\) denote the \((0, q)\)-forms with coefficients in \(C^\infty(\Omega)\) and \(C^\infty_c(\Omega)\), respectively. For \((0, 1)\)-forms, we use the pointwise inner product \(\langle ., . \rangle\) defined by \(\langle d\bar{z}^k, d\bar{z}^l \rangle = \delta^k_l\). By linearity we extend this inner product to \((0, q)\)-forms.

We define the global \(L^2\)-inner product on \(\Omega\) by
\[
(u, v)_\Omega = \int_\Omega \langle u, v \rangle dV \quad \text{for} \quad u, v \in \Lambda^{0,q}_c(\Omega),
\]
where \(dV\) is the euclidean volume form. The \(L^2\)-norm of \(u \in \Lambda^{0,q}_c(\Omega)\) is then given by \(\|u\|_\Omega = \sqrt{(u, u)_\Omega}\), and we define the space \(L^2_{0,q}(\Omega)\) to be the completion of \(\Lambda^{0,q}_c(\Omega)\) under \(\|\cdot\|_\Omega\).

The Cauchy-Riemann operator, \(\tilde{\partial}\), acting on \(u \in \Lambda^{0,q}(\overline{\Omega})\) is defined as follows
\[
(2.3) \quad |(\tilde{\partial} u, v)| \leq C\|u\| \quad \text{holds for all} \quad u \in \text{Dom}(\tilde{\partial}).
\]
The Riesz representation theorem implies that, if \(v \in \text{Dom}(\tilde{\partial}^*)\), there exists a unique \(w \in L^2_{0,q}(\Omega)\), such that
\[
(\tilde{\partial} u, v) = (u, w)
\]
holds for all \(u \in \text{Dom}(\tilde{\partial})\); we write \(\tilde{\partial}^* v\) for \(w\). One can show, using integration by parts, that if \(v \in D^{0,q}(\Omega) := \text{Dom}(\tilde{\partial}^*) \cap \Lambda^{0,q}(\overline{\Omega})\), then \(v\) satisfies the following boundary conditions:
\[
\sum_{k=1}^n \frac{\partial r}{\partial z_k} v_{kl} = 0 \quad \text{on} \quad \partial \Omega
\]
for any strictly increasing multi-index $I$ of length $q - 1$. Here we mean by $kI$ the multi-index $\{k, I\}$. Denote by $\text{Dom}(\Box_q)$ those $(0, q)$-forms $u \in \text{Dom}(\bar{\partial}^*) \cap \text{Dom}(\bar{\partial})$ for which $\bar{\partial}u \in \text{Dom}(\bar{\partial}^*)$ and $\partial^* u \in \text{Dom}(\bar{\partial})$ holds. The operator $\Box_q = \bar{\partial} \partial^* + \partial^* \bar{\partial}$, defined for forms in $\text{Dom}(\Box_q)$, is called the complex Laplacian.

We introduce a convenient, if non-standard, piece of notation: if $f$ is a $C^2$ function

$$\tag{2.4} i\partial \bar{\partial} f(u, u) := \sum'_{|J|=q-1} \sum_{k,l=1}^{n} \frac{\partial^2 f}{\partial z_l \partial \bar{z}_k} u_{lJ} \bar{u}_{kJ}, \quad u \in \Lambda^{0,q}(\Omega).$$

When $q = 1$, (2.4) is standard notation and expresses the natural action of the $(1, 1)$-form $i\partial \bar{\partial} f$ on the pair of vectors $u$ and $\bar{u}$. For $q > 1$, the left-hand side of (2.4) does not have such a natural meaning. However, the right-hand side of (2.4) arises repeatedly in integration by parts arguments on the $\bar{\partial}$ complex, and it is useful to abbreviate this expression by the left-hand side of (2.4) for all levels of forms. For example, the basic identity for the $\bar{\partial}$-Neumann problem assumes the following form: if $u \in \mathcal{D}^{0,q}(\Omega)$,

$$\tag{2.5} \|\partial u\|^2 + \|\bar{\partial} u\|^2 = \sum'_{|I|=q} \sum_{k=1}^{n} \left| \frac{\partial u_I}{\partial z_k} \right|^2 + \int_{\partial \Omega} i\partial \bar{\partial} r(u, u).$$

We also mention the equivalence of the following two facts related to (2.4):

(i) $i\partial \bar{\partial} f(u, u) \geq C|u|^2$ for all $u \in \Lambda^{0,q}$.

(ii) The sum of any $q$ eigenvalues of the matrix $\left( \frac{\partial^2 f}{\partial z_k \partial \bar{z}_l} \right)$ is greater than or equal to $C$.

A proof of the equivalence of (i) and (ii) follows by diagonalizing the matrix $\left( \frac{\partial^2 f}{\partial z_k \partial \bar{z}_l} \right)$; see [Hör65] or [Cat86].

Suppose that for all $u \in \mathcal{D}^{0,q}(\Omega)$, $i\partial \bar{\partial} r(u, u) \geq 0$ on the boundary of $\Omega$. Starting with (2.5), one can show that

$$\tag{2.6} \|u\|^2 + \sum'_{|I|=q} \sum_{k=1}^{n} \left| \frac{\partial u_I}{\partial z_k} \right|^2 \leq C \left( \|\partial u\|^2 + \|\bar{\partial} u\|^2 \right)$$

holds for all $u \in \mathcal{D}^{0,q}(\Omega)$; here $C > 0$ does not depend on $u$. If inequality (2.6) holds, then the $\bar{\partial}$-Neumann operator exists: $N_q : L^2_{0,q}(\Omega) \rightarrow \text{Dom}(\Box_q)$ with $N_q \Box_q = Id$ on $\text{Dom}(\Box_q)$ and $\Box_q N_q = Id$. One of the equations which connects the $\bar{\partial}$-Neumann operator and the Bergman projection is Kohn’s formula, which says

$$B_{q-1} = Id - \bar{\partial}^* N_q \bar{\partial}.$$

Throughout the paper, we shall use the notation $|A| \lesssim |B|$ to mean $|A| \leq C|B|$ for some constant $C > 0$, which is independent of relevant parameters.
It will be mentioned, or clear from the context, what those parameters are. We call the often-used inequality $|AB| \leq \eta A^2 + \frac{1}{4\eta} B^2$ for $\eta > 0$ the (sc)-(lc) inequality. Finally, we denote the commutator of two operators, $L$ and $M$, as usual: $[L, M] = LM - ML$.

3. Basic Estimates

In this section, we derive a Hardy-like inequality for the $\bar{\partial}$-complex. This inequality, (3.5), is essential for our proof of Theorem 1.1. We start out with an energy identity for the $\bar{\partial}$-complex.

Proposition 3.1. Let $\Omega \subset \subset \mathbb{C}^n$ be a smoothly bounded domain, $r$ a smooth defining function of $\Omega$. Let $s \geq 0$ and set $\tau = (-r)^s$. Then

$$\|\sqrt{\tau} \bar{\partial} u\|^2 + \|\sqrt{\tau} \bar{\partial}^* u\|^2 = \sum_{|I|=q} \sum_{k=1}^n \left(\tau \bar{\partial}_k u_I, \bar{\partial}_k u_I\right)$$

holds for all $u \in D^{0,q}(\Omega)$.

Equation (3.2) was proved in [McN02] for any $\tau \in C^2(\Omega)$. Other, related identities, e.g., for $(0,1)$-forms and forms vanishing on $\partial \Omega$, have been obtained by several authors, starting with the basic work of Ohsawa and Takegoshi [Ohs-Tak87]; see [McN05] for references and an expository account of these identities. However, since we need the identity when $\tau = (-r)^s$, for ranges of $s$ for which $\tau \notin C^2(\Omega)$, we give the modification of the proof in [McN02] which yields Proposition 3.1.

Proof. Let $J$, $M$ and $N$ be multi-indices with $|J| = q - 1$ and $|M| = |N| = q$. For notational ease denote $\frac{\partial}{\partial z_k}$ by $\partial_k$. We write for $u \in D^{0,q}(\Omega)$

$$\bar{\partial} u = \sum_{M} \sum_{k=1}^n \bar{\partial}_k u_M d\bar{z}^k \wedge d\bar{z}^M$$

and $\bar{\partial}^* u = -\sum_{J} \sum_{l=1}^n \partial_l u_{lJ} d\bar{z}^J$.

Then we obtain

$$\left\|\sqrt{\tau} \bar{\partial} u\right\|^2 + \left\|\sqrt{\tau} \bar{\partial}^* u\right\|^2 = \sum_{M,N} \sum_{k,l=1}^n \sigma_{MN}^{k,l} \left(\tau \bar{\partial}_k u_M, \bar{\partial}_l u_N\right)$$

$$+ \sum_{J} \sum_{k,l=1}^n \left(\tau \bar{\partial}_k u_{k,l}, \partial_l u_{l,J}\right),$$
where \( \sigma_{iM}^{kM} \) is the sign of the permutation \( (iM) \) and equals zero whenever \( \{k\} \cup \{M\} \neq \{l\} \cup \{N\} \). Rearranging terms we obtain

\[
\| \sqrt{\tau} \partial u \|_2 + \| \sqrt{\tau} \partial^* u \|_2 = \sum_{M} \sum_{k=1}^{n} \| \sqrt{\tau} \bar{\partial}_M u_M \|^2
\]

\[
+ \sum_{J} \sum_{k,l=1}^{n} \int_{\Omega} \tau (\partial_l u_{lJ} \bar{\partial}_k \bar{u}_{k,J} - \bar{\partial}_k (\tau \partial_l u_{lJ}) \bar{u}_{k,J}),
\]

where we denote the last term on the right hand side by (I). Integration by parts gives

\[
(I) = \sum_{J} \sum_{k,l=1}^{n} \int_{\Omega} \partial_l (\tau \bar{\partial}_k u_{lJ}) \bar{u}_{k,J} - \bar{\partial}_k (\tau \partial_l u_{lJ}) \bar{u}_{k,J},
\]

here no boundary integral appears because \( \tau = (-r)^s \) is zero on the boundary of \( \Omega \). Since \( \partial_l \bar{\partial}_k = \bar{\partial}_k \partial_l \) it follows

\[
(I) = \sum_{J} \sum_{k,l=1}^{n} \int_{\Omega} \partial_l (\tau \bar{\partial}_k u_{lJ}) \bar{u}_{k,J} - \bar{\partial}_k (\tau \partial_l u_{lJ}) \bar{u}_{k,J}.
\]

We would like to integrate the first term on the right hand side of the above equation by parts, but some care has to be taken since \( \partial_l \tau \) is not defined on \( b\Omega \) for \( s \in (0, 1) \). For \( \epsilon > 0 \) small set \( \Omega_\epsilon = \{ z \in \Omega \mid -\epsilon < r(z) < 0 \} \). Then

\[
\sum_{J} \sum_{k,l=1}^{n} \int_{\Omega_\epsilon} \partial_l (\tau \bar{\partial}_k u_{lJ}) \bar{u}_{k,J} = -\sum_{J} \sum_{k,l=1}^{n} \int_{\Omega_\epsilon} \bar{\partial}_k (\partial_l (\tau \bar{u}_{k,J}) u_{lJ}
\]

\[
+ \sum_{J} \sum_{k,l=1}^{n} \int_{b\Omega_\epsilon} \partial_l (\tau u_{lJ} \bar{\partial}_k (r) \bar{u}_{k,J}) \frac{dS}{|\partial r|}.
\]

Let us denote the boundary integral by (II). Using that \( \tau = (-r)^s \) we can express (II) in the following manner

\[
(II) = \sum_{J} \sum_{k,l=1}^{n} \int_{b\Omega_\epsilon} -s(-r)^{s-1} \partial_l(r) u_{lJ} \bar{\partial}_k (r) \bar{u}_{k,J} \frac{dS}{|\partial r|}
\]

\[
= -\sum_{J} \sum_{k,l=1}^{n} \int_{b\Omega_\epsilon} s(-r)^{s-1} \left| \sum_{l=1}^{n} \partial_l(r) u_{lJ} \right|^2 \frac{dS}{|\partial r|}.
\]

Recall that \( u \in \mathcal{D}^{0,q}(\Omega) \) means that \( \sum_{l=1}^{n} \partial_l(r) u_{lJ} = 0 \) on \( b\Omega \) for all increasing multi-indices \( J \). Thus \( \sum_{l=1}^{n} \partial_l(r) u_{lJ} = O(\epsilon) \) on \( b\Omega_\epsilon \), which yields (II) =
$O(\epsilon^{s+1})$. Therefore, taking the limit as $\epsilon$ approaches 0, it follows that

\[
\sum' \sum_{J, k, l=1}^{n} \int_{\Omega} \partial_\tau (\tau_k u_l J) \bar{u}_{kJ} = - \sum' \sum_{J, k, l=1}^{n} \int_{\Omega} \partial_k (\partial_\tau \bar{u}_{kJ}) u_{lJ} \nabla \nabla \left( \tau_{k} \right) \bar{u}_{kJ} \bar{u}_{lJ} = - \int_{\Omega} i \partial \partial \tau (u, u) - \sum' \sum_{J, k, l=1}^{n} \int_{\Omega} \partial_\tau (u_{lJ} \partial_\tau \bar{u}_{kJ} \bar{u}_{lJ}),
\]

which proves our claimed equation (3.2). $\square$

In order to prove our main result of this section, inequality (3.5), we need to show that under the hypotheses of Theorem 1.1, one can choose a defining function, $r$, for $\Omega$ with a suitable lower bound on the complex Hessian of $-(-r)^s$, $s \in (0, 1)$.

**Lemma 3.3.** If $\Omega$ satisfies condition $(H_q)$, then for each $s \in (0, 1)$, there exists a smooth defining function $r$ satisfying

\[
i \partial \partial (-(-r)^s)(u, u) \succeq (-r)^s |u|^2 \text{ for all } u \in \Lambda_{0,q}(\Omega).
\]

The constant in $\succeq$ depends on $s$, but is independent of $z \in \Omega$ and $u \in \Lambda_{0,q}$.

For $q = 1$, Lemma 3.3 was proved in [Die-For77] (also see [Ran81]), though it was not stated in this form. The proof for general $q$ follows the same lines.

**Proof.** Let $\rho$ satisfy Definition 2.4 and set $r(z) := e^{-K|z|^2}\rho(z)$, for a constant $K > 0$ to be determined.

A straightforward computation gives

\[
i \partial \partial (-(-r)^s)(u, u) = s(-\rho)^{s-2}e^{-sK|z|^2} \left\{ K\rho^2 \left[ q|u|^2 - sK \sum'_{|J|=q-1} \sum_{k=1}^{n} \bar{z}_k u_{kJ} \right] \right\} - \rho \left[ i \partial \partial \rho(u, u) - 2sK \text{Re} \left( \sum_{|J|=q-1}^{n} \sum_{k, l=1}^{n} \bar{z}_k u_{kJ} \frac{\partial \rho}{\partial z_l} u_{lJ} \right) \right] + (1 - s) \sum'_{|J|=q-1} \sum_{k=1}^{n} \left| \frac{\partial \rho}{\partial z_k} u_{kJ} \right|^2
\]

Note that by (sc)-(lc) inequality we have

\[
2sK\rho \text{Re} \left( \sum_{k, l=1}^{n} z_k \bar{u}_{kJ} \frac{\partial \rho}{\partial z_l} u_{lJ} \right) \geq (s - 1) \left| \sum_{k=1}^{n} \frac{\partial \rho}{\partial z_k} u_{kJ} \right|^2 - \frac{(sK\rho)^2}{1 - s} \left| \sum_{k=1}^{n} \bar{z}_k u_{kJ} \right|^2
\]
for all increasing multi-indices $J$ with $|J| = q - 1$. Using this inequality and that $i\partial \bar{\partial} \rho \geq 0$, it follows

$$i\partial \bar{\partial} \left( (-r)^s \right)(u, u) \geq sK(-r)^s \left\{ q|u|^2 - K \frac{s}{1 - s} \sum'_{|J| = q-1} \left| \sum_{k=1}^{n} \bar{z}_k u_{k,J} \right|^2 \right\}.$$  

Since $\Omega$ is a bounded domain, there exists a constant $D > 0$ such that $|z|^2 \leq D$ for $z \in \overline{\Omega}$. We obtain

$$\sum'_{|J| = q-1} \left| \sum_{k=1}^{n} \bar{z}_k u_{k,J} \right|^2 \leq nq|z|^2|u|^2 \leq nqD|u|^2,$$

which implies that

$$i\partial \bar{\partial} \left( (-r)^s \right)(u, u) \geq sqK(-r)^s|u|^2 \left( 1 - K \frac{Dns}{1 - s} \right).$$

Choosing $K = \frac{1-s}{2Dns}$ then proves the claim with

$$i\partial \bar{\partial} \left( (-r)^s \right)(u, u) \geq \frac{(1-s)q}{4Dn} (-r)^s|u|^2.$$

We are prepared to prove the main result of this section.

**Proposition 3.4.** Suppose $\Omega$ satisfies condition $(H_q)$. Let $s \in [0, 1)$. Then there exists a smooth defining function, $r$, for $\Omega$ such that

$$\int_{\Omega} (-r)^{-2+s} \left| \left[ \partial^*, r \right] u \right|^2 \preceq \left( (-r)^s \partial u, \bar{\partial} u \right) + \left( (-r)^s \bar{\partial} u, \partial u \right)$$

holds for all $u \in \mathcal{D}^{0,q}(\Omega)$; the constant in $\preceq$ depends on $s$.

**Proof.** Let $s \in [0, 1)$ be fixed and let $r$ be given by Lemma 3.3. Write $(-r)^{-2+s} = \frac{1}{1-s} \frac{\partial}{\partial r} (-r)^{-1+s}$ and express $\frac{\partial}{\partial r}$ as a linear combination of $\frac{\partial}{\partial \bar{z}_k}$ derivatives and vector fields which are tangent to $b\Omega$. Then

$$\int_{\Omega} (-r)^{-2+s} \left| \left[ \partial^*, r \right] u \right|^2 \quad = \quad \frac{1}{1-s} \int_{\Omega} \left( \sum_{k=1}^{n} a_k \frac{\partial}{\partial \bar{z}_k} (-r)^{-1+s} + D_t (-r)^{-1+s} \right) \left| \left[ \partial^*, r \right] u \right|^2,$$
where $D_t(r) = 0$ on $b\Omega$. Thus we can write $D_t(r(z)) = O(r(z))$, and it follows that

$$
\frac{1}{1 - s} \int_{\Omega} (D_t(\bar{r}^{1-s}) |[\bar{\partial}^*, r] u|^2 = \int_{\Omega} (\bar{r}^{-2+s} D_t(r) |[\bar{\partial}^*, r] u|^2
\lesssim \int_{\Omega} (\bar{r}^{-1+s} |[\bar{\partial}^*, r] u|^2
\lesssim \epsilon \|(-\bar{r})^{-1+s} [\bar{\partial}^*, r] u\|^2 + \frac{1}{\epsilon}\|(-\bar{r})^\frac{s}{2} u\|^2,
$$

where the last line holds by (sc)-(lc) inequality and the fact that $[\bar{\partial}^*, r]$ is in $L^\infty$. Furthermore, integration by parts gives

$$
\int_{\Omega} \left( \sum_{k=1}^{n} a_k \frac{\partial}{\partial \bar{z}_k} (\bar{r}^{1-s}) \right) |[\bar{\partial}^*, r] u|^2 = \int_{\Omega} (-\bar{r}^{-1+s} \sum_{k=1}^{n} \frac{\partial}{\partial \bar{z}_k} (a_k |[\bar{\partial}^*, r] u|^2).
$$

Here the boundary integral vanishes since $[\bar{\partial}^*, r] u$ vanishes on the boundary. It follows that

$$
\int_{\Omega} (-\bar{r}^{-1+s} \sum_{k=1}^{n} \frac{\partial}{\partial \bar{z}_k} (a_k |[\bar{\partial}^*, r] u|^2)
\lesssim \int_{\Omega} (-\bar{r}^{-1+s} |[\bar{\partial}^*, r] u|^2 + \int_{\Omega} (-\bar{r}^{-1+s} \left| \sum_{k=1}^{n} \frac{\partial}{\partial \bar{z}_k} ([\bar{\partial}^*, r] u, [\bar{\partial}^*, r] u) \right|
\lesssim \epsilon \int_{\Omega} (-\bar{r}^{-2+s} |[\bar{\partial}^*, r] u|^2 + \frac{1}{\epsilon} \int_{\Omega} (-\bar{r})^s \left( |u|^2 + \sum_{|I|=q} \sum_{k=1}^{n} \left| \frac{\partial u_I}{\partial \bar{z}_k} \right|^2 \right)
$$

Collecting the above estimates and choosing $\epsilon > 0$ sufficiently small, we obtain

$$
(3.6) \quad \int_{\Omega} (-\bar{r})^{-2+s} |[\bar{\partial}^*, r] u|^2 \lesssim \|(-\bar{r})^\frac{s}{2} u\|^2 + \sum_{|I|=q} \sum_{k=1}^{n} \left| (-\bar{r})^\frac{s}{2} \frac{\partial u_I}{\partial \bar{z}_k} \right|^2.
$$

If $s = 0$, then the left hand side of (3.6) is dominated by $\|\bar{\partial} u\|^2 + \|\bar{\partial}^* u\|^2$, which yields inequality (3.5).

Now suppose $s \in (0, 1)$. We recall that Lemma 3.3 implies that $(-r)^s|u|^2 \lesssim i\bar{\partial}(-(-r)^s)(u, u)$ holds. Thus, using Proposition 3.1 with $\tau = (-r)^s$, we get

$$
\int_{\Omega} (-\bar{r})^{-2+s} |[\bar{\partial}^*, r] u|^2 \lesssim - \int_{\Omega} i\bar{\partial}(-r)^{s}(u, u) + \sum_{|I|=q} \sum_{k=1}^{n} \left| (-r)^\frac{s}{2} \frac{\partial u_I}{\partial \bar{z}_k} \right|^2
\lesssim \|(-r)^\frac{s}{2} \bar{\partial} u\|^2 + \|(-r)^\frac{s}{2} \bar{\partial}^* u\|^2 - 2\text{Re}([\bar{\partial}^*, (-r)^s] u, \bar{\partial}^* u).
$$
The last term on the right hand side can be easily controlled, in fact
\[
|2\text{Re}([\bar{\partial}^*, (-r)^s] u, \partial^* u)| \lesssim |((-r)^{\frac{s}{2}-1} [\bar{\partial}^*, r] u, (-r)^{\frac{s}{2}} \partial^* u)|
\]
\[
\lesssim \epsilon \int_{\Omega} (-r)^{-2+s} |[\bar{\partial}^*, r] u|^2 + \frac{1}{\epsilon} \|(-r)^{\frac{s}{2}} \partial^* u\|^2,
\]
by the (sc)-(lc) inequality. Choosing \(\epsilon > 0\) sufficiently small, we obtain
\[
\int_{\Omega} (-r)^{-2+s} |[\bar{\partial}^*, r] u|^2 \lesssim \|(-r)^{\frac{s}{2}} \partial^* u\|^2 + \|(-r)^{\frac{s}{2}} \partial u\|^2.
\]

\[
\square
\]

4. The Proof

We state a quantitative form of Theorem 1.1:

**Theorem 4.1.** Let \(\Omega\) be a smoothly bounded domain satisfying condition \((H_q)\). Then the Bergman projection, \(B_j\), is continuous on the Sobolev space \(H^{s}_0, j(\Omega)\), \(s > 0\), for \(j \in \{q - 1, \ldots, n - 1\}\).

**Proof.** We shall prove that \(\|B_j f\|_k \leq C_k \|f\|_k\) holds for all integer \(k > 0\); the general case follows by the usual interpolation arguments.

The proof goes via a downward induction on \(j\), the form level, as well as an upward induction on the \(k\), the order of differentiation, in the following manner. The induction basis (on the form level \(j = n - 1\)) is satisfied: since the \(\bar{\partial}\)-Neumann problem on \((0, n)\)-forms is an elliptic boundary value problem, the \(\bar{\partial}\)-Neumann operator, \(N_n\), gains two derivatives, which implies that \(\|B_{n-1} f\|_k \leq C_k \|f\|_k\) holds for all \(k \geq 0\) since \(B_{n-1} = I - \bar{\partial}^* N_n \partial\).

The induction basis (on the order of differentiation \(k = 0\)) is also satisfied: \(\|B_j f\| \leq C_0 \|f\|\) holds for all \(j \in \{0, \ldots, n - 1\}\) by definition.

In the following, we shall prove the case \(k = 1\) only, so that the main ideas are not cluttered by technicalities. We indicate at the end how to prove the induction step for \(k > 1\).

Let \(j \in \{q, \ldots, n - 1\}\) be fixed. Suppose that \(B_j\) is continuous on \(H^{1}_0, j(\Omega)\). We want to show that
\[
\|B_{j-1} f\|_1 \leq C_1 \|f\|_1
\]
holds for all \(f \in \Lambda^{0,j-1}(\Omega)\). We first assume that \(B_{j-1} f \in \Lambda^{0,j-1}(\Omega)\) and establish (4.2). At the end of the proof, we show how to pass from this apriori estimate to a true estimate.

I. Standard reduction.

Let \(r\) be a defining function for \(\Omega\) which satisfies (3.5) for some \(s = s_0 \in (0, 1)\) fixed, which will be chosen later. We can assume that \(\sum_{k=1}^{n} |r_{z_k}|^2 \neq 0\)
on a strip near $b\Omega$, i.e., on $S_1 = \{z \in \Omega \mid -\eta < r(z) < 0\}$ for some fixed $\eta > 0$. Let $\chi$ be a smooth, non-negative function which vanishes on $\Omega\backslash S_1$ and equals $(\sum_{k=1}^{n} |r_{z_k}|^2)^{-1}$ on $S_1$. We define $(1,0)$-vector fields as follows:

$$L_j = \frac{\partial}{\partial z_j} - \chi \cdot r_j \sum_{l=1}^{n} r_{z_l} \frac{\partial}{\partial z_l} \quad \text{for} \quad j \in \{1, \ldots, n\} \quad \text{and} \quad \mathcal{N} = \sum_{k=1}^{n} r_{z_k} \frac{\partial}{\partial z_k}.$$

Then $\{L_1, \ldots, L_{n-1}, \mathcal{N}\}$ is a basis of $(1,0)$ vector fields. Also, define the $(1,1)$ vector field $X = \mathcal{N} - \mathcal{N}$. Notice that $L_j$, $1 \leq j \leq n$, and $X$ are tangential. Since the complex $\overline{\eta} \geq \sum$ and equals $(\sum_{k=1}^{n} r_{z_k}|^2)^{-1}$ on $S_1$. We define $(1,0)$-vector fields as follows:

$$L_j = \frac{\partial}{\partial z_j} - \chi \cdot r_j \sum_{l=1}^{n} r_{z_l} \frac{\partial}{\partial z_l} \quad \text{for} \quad j \in \{1, \ldots, n\} \quad \text{and} \quad \mathcal{N} = \sum_{k=1}^{n} r_{z_k} \frac{\partial}{\partial z_k}.$$

Then $\{L_1, \ldots, L_{n-1}, \mathcal{N}\}$ is a basis of $(1,0)$ vector fields. Also, define the $(1,1)$ vector field $X = \mathcal{N} - \mathcal{N}$. Notice that $L_j$, $1 \leq j \leq n$, and $X$ are tangential. Since the complex $\overline{\eta} \geq \sum$ and equals $(\sum_{k=1}^{n} r_{z_k}|^2)^{-1}$ on $S_1$. We define $(1,0)$-vector fields as follows:

$$L_j = \frac{\partial}{\partial z_j} - \chi \cdot r_j \sum_{l=1}^{n} r_{z_l} \frac{\partial}{\partial z_l} \quad \text{for} \quad j \in \{1, \ldots, n\} \quad \text{and} \quad \mathcal{N} = \sum_{k=1}^{n} r_{z_k} \frac{\partial}{\partial z_k}.$$

Then $\{L_1, \ldots, L_{n-1}, \mathcal{N}\}$ is a basis of $(1,0)$ vector fields. Also, define the $(1,1)$ vector field $X = \mathcal{N} - \mathcal{N}$. Notice that $L_j$, $1 \leq j \leq n$, and $X$ are tangential. Since the complex $\overline{\eta} \geq \sum$ and equals $(\sum_{k=1}^{n} r_{z_k}|^2)^{-1}$ on $S_1$. We define $(1,0)$-vector fields as follows:

$$L_j = \frac{\partial}{\partial z_j} - \chi \cdot r_j \sum_{l=1}^{n} r_{z_l} \frac{\partial}{\partial z_l} \quad \text{for} \quad j \in \{1, \ldots, n\} \quad \text{and} \quad \mathcal{N} = \sum_{k=1}^{n} r_{z_k} \frac{\partial}{\partial z_k}.$$

Then $\{L_1, \ldots, L_{n-1}, \mathcal{N}\}$ is a basis of $(1,0)$ vector fields. Also, define the $(1,1)$ vector field $X = \mathcal{N} - \mathcal{N}$. Notice that $L_j$, $1 \leq j \leq n$, and $X$ are tangential. Since the complex $\overline{\eta} \geq \sum$ and equals $(\sum_{k=1}^{n} r_{z_k}|^2)^{-1}$ on $S_1$. We define $(1,0)$-vector fields as follows:

$$L_j = \frac{\partial}{\partial z_j} - \chi \cdot r_j \sum_{l=1}^{n} r_{z_l} \frac{\partial}{\partial z_l} \quad \text{for} \quad j \in \{1, \ldots, n\} \quad \text{and} \quad \mathcal{N} = \sum_{k=1}^{n} r_{z_k} \frac{\partial}{\partial z_k}.$$

Then $\{L_1, \ldots, L_{n-1}, \mathcal{N}\}$ is a basis of $(1,0)$ vector fields. Also, define the $(1,1)$ vector field $X = \mathcal{N} - \mathcal{N}$. Notice that $L_j$, $1 \leq j \leq n$, and $X$ are tangential. Since the complex $\overline{\eta} \geq \sum$ and equals $(\sum_{k=1}^{n} r_{z_k}|^2)^{-1}$ on $S_1$. We define $(1,0)$-vector fields as follows:

$$L_j = \frac{\partial}{\partial z_j} - \chi \cdot r_j \sum_{l=1}^{n} r_{z_l} \frac{\partial}{\partial z_l} \quad \text{for} \quad j \in \{1, \ldots, n\} \quad \text{and} \quad \mathcal{N} = \sum_{k=1}^{n} r_{z_k} \frac{\partial}{\partial z_k}.$$

Then $\{L_1, \ldots, L_{n-1}, \mathcal{N}\}$ is a basis of $(1,0)$ vector fields. Also, define the $(1,1)$ vector field $X = \mathcal{N} - \mathcal{N}$. Notice that $L_j$, $1 \leq j \leq n$, and $X$ are tangential. Since the complex $\overline{\eta} \geq \sum$ and equals $(\sum_{k=1}^{n} r_{z_k}|^2)^{-1}$ on $S_1$. We define $(1,0)$-vector fields as follows:

$$L_j = \frac{\partial}{\partial z_j} - \chi \cdot r_j \sum_{l=1}^{n} r_{z_l} \frac{\partial}{\partial z_l} \quad \text{for} \quad j \in \{1, \ldots, n\} \quad \text{and} \quad \mathcal{N} = \sum_{k=1}^{n} r_{z_k} \frac{\partial}{\partial z_k}.$$
II. Main apriori estimate (heart of the proof).

From here on, we call those terms allowable which are dominated by $\|f\|^2$. In order to show that $\|XB_{j-1}f\|^2$ is allowable, we need to introduce a new vector field, $T$, which equals $X$ at the first order level, but preserves membership in the domain of $\bar{\partial}^*$. 

**Lemma 4.5.** There exists a smooth, tangential vector field $T$ such that

1. $Tu \in D^{0,j}(\Omega)$ whenever $u \in D^{0,j}(\Omega)$, and
2. $\|(X - T)u\| \lesssim \|u\|$ for $A^{0,j}(\Omega)$.

**Proof.** Recall that we chose $\chi$ to be a smooth, non-negative function which vanishes on $\Omega \setminus S_{2\eta}$ and equals $\left(\sum_{k=1}^{n} |r_{z_k}|^2\right)^{-1}$ on $\bar{S}_{\eta}$ for some fixed $\eta > 0$. Set

$$Tu := Xu + \chi \cdot \bar{\partial} r \wedge ([X, [\bar{\partial}^*, r]]u).$$

(4.6)

Note that $T$ acts diagonally at the first order level, since $X$ does, and preserves the form level. It is straightforward to check that $Tu \in D^{0,j}(\Omega)$ whenever $u \in D^{0,j}(\Omega)$. The claimed property (ii), on $(X - T)u$, is obvious. $\square$

Thus, to show that $\|XB_{j-1}f\|^2$ is allowable, it suffices to prove the

**Claim:** $\|TB_{j-1}f\|^2$ is allowable.

We define $\varphi = N_j \bar{\partial} f$, note that then $\varphi \in D^{0,j}(\Omega)$ and $B_{j-1}f = f - \bar{\partial}^* \varphi$. In order to deal with certain error terms involving $\varphi$ arising in the proof of the Claim, we need the following lemma.

**Lemma 4.7.**

(i) $Q(\varphi, \varphi)$ is allowable,

(ii) $\|\varphi\|^2$, $\|N^\varphi\|^2$, and $\|L_k \varphi\|^2$, $1 \leq k \leq n$, are allowable,

(iii) $\|L_k \varphi\|^2 \lesssim \epsilon \|T\varphi\|^2 + \frac{1}{\epsilon} \|f\|^2$ for $1 \leq k \leq n$, $\epsilon > 0$,

(iv) $\|T\varphi\|^2 \lesssim \|f\|^2 + \|B_{j-1}f\|^2$.

**Proof of Lemma 4.7.** The proof of (i) follows directly from the definition of $\varphi$, i.e.,

$$Q(\varphi, \varphi) = (\bar{\partial} \varphi, \bar{\partial} \varphi) + (\bar{\partial}^* \varphi, \bar{\partial}^* \varphi)$$

$$= (\bar{\partial} N_j \bar{\partial} f, \bar{\partial} N_j \bar{\partial} f) + (\bar{\partial}^* N_j \bar{\partial} f, \bar{\partial}^* N_j \bar{\partial} f)$$

$$= \|f - B_{j-1}f\|^2 \lesssim \|f\|^2.$$

To prove (ii), we use inequality (2.6) and (i), that is

$$\|\varphi\|^2 + \|L_k \varphi\|^2 \lesssim Q(\varphi, \varphi) \lesssim \|f\|^2.$$

Inequality (iii) follows from the same arguments as those made directly below inequality (4.3).
For the proof of (iv) we use the Boas-Straube formula in [Boa-Str90], which expresses $\bar{\partial}^* N_j$ in terms of $B_{j-1}$, $B_j$ and $N_{t,j}$; here $N_{t,j}$ is the solution operator to the weighted $\bar{\partial}$-Neumann problem with weight $w_t(z) = \exp(-t|z|^2)$. Recall that $\varphi = N_j \bar{\partial} f$, and hence we are interested in the operator $N_j \bar{\partial}$ and not in the operator $\bar{\partial}^* N_j$. However, $\bar{\partial}^* N_j$ is the $L^2$-adjoint of $N_j \bar{\partial}$, and thus the formula for $\bar{\partial}^* N_j$ in [Boa-Str90], pg. 29, implies

$$N_j \bar{\partial} f = B_j \{ w_t N_{t,j} \bar{\partial} w_{-t} (f - B_{j-1} f) \}.$$  

The induction hypothesis says that $B_j$ is continuous on $H^1_0(\Omega)$. Thus

$$\| T \varphi \|^2 = \| T N_j \bar{\partial} f \|^2 \lesssim \| N_{t,j} \bar{\partial} (w_{-t} f - w_{-t} B_{j-1} f) \|^2.$$  

A theorem of Kohn in [Koh73] implies that $N_{t,j} \bar{\partial}$ is continuous on $H^1_{0,j-1}(\Omega)$ as long as $t > 0$ is sufficiently large. Actually, in [Koh73], Kohn assumes that $b\Omega$ is pseudoconvex, so his result is not immediately applicable under our hypotheses. However, on $(0,j)$-forms the basic $\bar{\partial}$-Neumann identity is (2.5) and the boundary integrand is non-negative by condition ($H_j$). The techniques in [Koh73] may now be applied to give the claimed estimate on $N_{t,j} \bar{\partial}$.

Thus we obtain

$$\| T \varphi \|^2 \lesssim \| f \|^2 + \| B_{j-1} f \|^2,$$

which proves (iv).

Now we are ready to show that $\| TB_{j-1} f \|^2$ is allowable. Since $B_{j-1} f = f - \bar{\partial}^* \varphi$, it follows that

$$\| TB_{j-1} f \|^2 = (T f, T B_{j-1} f) = (T \bar{\partial}^* \varphi, T B_{j-1} f)$$

$$= (T f, T B_{j-1} f) - (\varphi, [\partial, T^{*}T] B_{j-1} f) = (\varphi, T^{*}T \bar{\partial} B_{j-1} f).$$

Since $\bar{\partial} B_{j-1} f$ equals 0, it follows with the (sc)-(lc) inequality

$$\| TB_{j-1} f \|^2 \lesssim \| f \|^2 + \| (\partial, T^{*}T) \varphi, B_{j-1} f \|.$$  

Also,

$$[\partial, T^{*}T] = [\partial, T^{*}] T + [\partial, T] T^{*} - [\partial, T], T^{*},$$

where the last term above is of order 1. The adjoint of $[\partial, T^{*}]$ is $[\bar{\partial}, T^{*}]$, but the adjoint of $[\partial, T]$ is not $[\bar{\partial}, T^{*}]$, since $T^{*}$ does not preserve the domain of $\bar{\partial}^{*}$. However, since $X$ is self-adjoint, $T$ and $T^{*}$ only differ by terms of order 0, which allows us to integrate by parts with negligible error terms. That is,

$$\langle \varphi, [\partial, T^{*}] T^{*} B_{j-1} f \rangle = (T^{*} \bar{\partial}^* \varphi, T^{*} B_{j-1} f) - (T^{*} \varphi, \bar{\partial} T^{*} B_{j-1} f),$$

and

$$(T^{*} \varphi, \bar{\partial} T^{*} B_{j-1} f) \leq (T \varphi, \bar{\partial} T^{*} B_{j-1} f) + \| \varphi \| \cdot \| \bar{\partial} T^{*} B_{j-1} f \|$$

$$= (\bar{\partial} T \varphi, T^{*} B_{j-1} f) + \| \varphi \| \cdot \| [\partial, T^{*}] B_{j-1} f \|. $$
Thus, using again that $T$ and $T^*$ are equal at the first order level, we get
\[
\|TB_{j-1}f\|^2 \lesssim \left(\|\langle T, \bar{\partial}\rangle \varphi, TB_{j-1}f \rangle\| + \|f\|^2 + \|\varphi\| \cdot \|B_{j-1}f\|_1\right),
\]
where the last step follows since $B_{j-1}f$ is in the kernel of $\bar{\partial}$. Inequality (4.4), part (ii) of Lemma 4.7 and the (sc)-(lc) inequality yield
\[
(4.8) \quad \|TB_{j-1}f\|^2 \lesssim \|\langle T, \bar{\partial}\rangle \varphi \|^2 + \|f\|^2.
\]
Therefore, our claim follows by the (sc)-(lc) inequality if $\|\langle T, \bar{\partial}\rangle \varphi \|^2$ is allowable.

The term $\langle T, \bar{\partial}\rangle \varphi$ is equal to $[X, \vartheta]\varphi$ up to terms of order 0, where $\vartheta$ is the formal adjoint of $\bar{\partial}$. A straightforward computation gives
\[
[X, \vartheta] \varphi = \sum_{|J|=1}^n \sum_{k,l=1}^n \left\{ \frac{\partial^2 r}{\partial z_i \partial z_k} \frac{\partial \varphi_{lJ}}{\partial z_k} - \frac{\partial^2 r}{\partial z_l \partial z_k} \frac{\partial \varphi_{lJ}}{\partial z_k} \right\} dz^J.
\]
Note that the terms involving $\frac{\partial \varphi_{lJ}}{\partial z_k}$ are allowable by part (ii) of Lemma 4.7.

To deal with the remaining terms, notice that
\[
\frac{\partial}{\partial z_k} = L_k + \chi r_{z_k} X + \chi r_{z_k} \sum_{i=1}^n r_{z_i} \frac{\partial}{\partial z_k} \text{ for } k \in \{1, \ldots, n\}.
\]
Thus we obtain
\[
\sum' \sum_{|J|=1}^n \frac{\partial^2 r}{\partial z_i \partial z_k} \frac{\partial \varphi_{lJ}}{\partial z_k} = \sum' \sum_{|J|=1}^n \left\{ \frac{\partial^2 r}{\partial z_l \partial z_k} \left( (\chi X \varphi_{lJ}) r_{z_k} + L_k \varphi_{lJ} \right) + \chi r_{z_k} \sum_{i=1}^n r_{z_i} \frac{\partial \varphi_{lJ}}{\partial z_k} \right\}.
\]
By part (ii) and (iii) of Lemma 4.7, applied to the last two terms of the right hand side of the above equation, it follows that
\[
(4.9) \quad \|\langle T, \bar{\partial}\rangle \varphi \|^2 \lesssim \left\| \sum' \sum_{|J|=1}^n \frac{\partial^2 r}{\partial z_l \partial z_k} (T \varphi_{lJ}) r_{z_k} \right\|^2 + \text{ allowable}.
\]

**Lemma 4.10.** There exists a $(0, j)$-form $\psi$ such that
\[
(4.11) \quad \left| \sum' \sum_{|J|=1}^n \frac{\partial^2 r}{\partial z_l \partial z_k} (T \varphi_{lJ}) r_{z_k} \right| \lesssim |i\bar{\partial}r(T \varphi, \psi) + |[\bar{\partial}^*, r] T \varphi|.
\]

**Proof.** For notational ease, let us write $\phi$ for $T \varphi$ temporarily. We define
\[
\psi = \bar{\partial} \left( r \sum'_{|J|=1} dz^{j'} \right).
\]
For $j = 1$, (4.11) holds trivially, since $\psi_k = r_{z_k}$. For $j > 1$, an error term occurs when passing to $\psi$. In the following, we shall indicate what kind of algebraic manipulations of this error term lead to (4.11). For that we need to fix some notation: Recall that we write $kI$ for $\{k, I\}$, if $I$ is an
increasing multi-index and $k \notin I$. Furthermore, we shall mean by $k \cup I$ the increasing multi-index which equals $kI$ as a set. As before, $\sigma_{J}^{I}$ is the sign of the permutation $(I_{J})$ and is zero whenever $I$ and $J$ are not equal as set.

Notice first that $\overline{\psi}_{I} = \sum_{m \in I} r_{z_{m}} \sigma_{I}^{m(I\setminus m)}$ for any increasing multi-index $I$ of length $j$. Moreover, if $J$ is an increasing multi-index of length $j - 1$ and $k \notin J$, we can write

$$\overline{\psi}_{k,J} = r_{z_{k}} + \sum_{m \in J} r_{z_{m}} \sigma_{k,J}^{m(k\cup(J\setminus m))}.$$ 

Using this, a straightforward computation then gives

$$\sum_{|I|=j-1}^{'} \sum_{k,l=1}^{n} \frac{\partial^{2}r}{\partial z_{l} \partial z_{k}} \phi_{l,J} r_{z_{k}} - i \partial \bar{\partial}r(\phi, \psi)$$

$$= \sum_{k,l=1}^{n} \frac{\partial^{2}r}{\partial z_{l} \partial z_{k}} \left\{ \sum_{|I|=j-1}^{'} \sum_{k \in J} \phi_{l,J} r_{z_{k}} - \sum_{|I|=j-1}^{'} \phi_{l,J} \left[ \sum_{m \in J} r_{z_{m}} \sigma_{k,J}^{m(k\cup(J\setminus m))} \right] \right\}.$$ 

Let us consider the two terms in the parentheses on the right hand side for $k = l$ fixed. Notice that in this case the first term vanishes, so we only need to study the second term. That is

$$\sum_{|I|=j-1}^{'} \phi_{k,J} \left[ \sum_{m \in J} r_{z_{m}} \sigma_{k,J}^{m(k\cup(J\setminus m))} \right] = \sum_{|I|=j-2}^{'} \sum_{m=1}^{n} r_{z_{m}} \phi_{k(m\cup I)} \cdot \sigma_{m(k\cup I)}^{k(m\cup I)},$$

$$= \sum_{|I|=j-2}^{'} \sum_{m=1}^{n} r_{z_{m}} \phi_{m(k\cup I)} = \sum_{|I|=j-1}^{'} \sum_{k \in J} r_{z_{m}} \phi_{m,J},$$

which equals the sum over those components of $- [\bar{\partial}^{*}, r] \phi$ whose multi-indices contain $k$. With similar, though more elaborate, computations one obtains for the case where $k \neq l$ are fixed the following

$$\sum_{|I|=j-1}^{'} \phi_{l,J} r_{z_{k}} - \sum_{|I|=j-1}^{'} \phi_{l,J} \left[ \sum_{m \in J} r_{z_{m}} \sigma_{k,J}^{m(k\cup(J\setminus m))} \right] = \sum_{|I|=j-1}^{'} \sigma_{k,l}^{j(k\cup(J\setminus l))} \sum_{m=1}^{n} r_{z_{m}} \phi_{m,J}$$

$$= - \sum_{|I|=j-1}^{'} \sigma_{k,l}^{j(k\cup(J\setminus l))} \left[ \bar{\partial}^{*}, r \right] \phi_{I}\cdot$$
Thus the error term appearing when passing to $\psi$ is described by the following equation

$$
\sum'_{|J|=j-1} \sum_{k,l=1}^{n} \frac{\partial^2 r}{\partial z_k \partial \bar{z}_l} \phi_{l,J} r_{z_k} - i \bar{\partial} \bar{\partial} r(\phi, \psi)
$$

which implies the claimed estimate (4.11).

Recall that $i \bar{\partial} \bar{\partial} r(u, u) \geq 0$ holds for all $(0, j)$-forms $u$. Hence, it follows by the Cauchy-Schwarz inequality that

$$
|i \bar{\partial} \bar{\partial} r(T\varphi, \psi)| \leq (i \bar{\partial} \bar{\partial} r(T\varphi, T\varphi))^{\frac{1}{2}} \cdot (i \bar{\partial} \bar{\partial} r(\psi, \psi))^{\frac{1}{2}}.
$$

This, combined with Lemma 4.10 and (4.9), implies that

$$
\| [T, \bar{\partial}^*] \varphi \|^2 \lesssim \int_{\Omega} i \bar{\partial} \bar{\partial} r(T\varphi, T\varphi) + \| [\bar{\partial}^*, r] T\varphi \|^2 + \text{ allowable}.
$$

Recall that $Tu \in \mathcal{D}^{0,j}(\Omega)$ whenever $u \in \mathcal{D}^{0,j}(\Omega)$. Thus, we can apply (3.2) to (4.13), with $\tau = (-r)$, and obtain

$$
\int_{\Omega} i \bar{\partial} \bar{\partial} r(T\varphi, T\varphi) \lesssim \| (-r)^{\frac{j}{2}} \bar{\partial} T\varphi \|^2 + \| (-r)^{\frac{j}{2}} \bar{\partial}^* T\varphi \|^2
$$

$$
+ \| ([\bar{\partial}^*, r] T\varphi, \bar{\partial}^* T\varphi) \| + \text{ allowable}.
$$

Using the (sc)-(lc) inequality, we estimate the last term on the right hand side above as follows:

$$
\| ([\bar{\partial}^*, r] T\varphi, \bar{\partial}^* T\varphi) \| \lesssim \int_{\Omega} (-r)^{-2+\frac{j}{2}} \| [\bar{\partial}^*, r] T\varphi \|^2 + \| (-r)^{\frac{j}{2}} \bar{\partial} T\varphi \|^2
$$

Since $r$ is bounded from below, this implies that

$$
\| [T, \bar{\partial}^*] \varphi \|^2 \lesssim \int_{\Omega} (-r)^{-2+\frac{j}{2}} \| [\bar{\partial}^*, r] T\varphi \|^2 + \| (-r)^{\frac{j}{2}} \bar{\partial} T\varphi \|^2 + \| (-r)^{\frac{j}{2}} \bar{\partial}^* T\varphi \|^2 + \text{ allowable}.
$$

Recall that we chose $r$ to be a defining function of $\Omega$ which satisfies (3.5) for some fixed $s_0 \in (0, 1)$. We now choose $s_0 = \frac{1}{2}$, so that (3.5) gives

$$
\int_{\Omega} (-r)^{-2+\frac{j}{2}} \| [\bar{\partial}^*, r] T\varphi \|^2 \lesssim \| (-r)^{\frac{j}{2}} \bar{\partial} T\varphi \|^2 + \| (-r)^{\frac{j}{2}} \bar{\partial}^* T\varphi \|^2.
$$

Again, since $r$ is bounded from below, we may more simply write

$$
\| [T, \bar{\partial}^*] \varphi \|^2 \lesssim \| (-r)^{\frac{j}{2}} \bar{\partial} T\varphi \|^2 + \| (-r)^{\frac{j}{2}} \bar{\partial}^* T\varphi \|^2 + \text{ allowable}.
$$
The two terms on the right hand side of the above inequality contain a factor \((-r)^{\frac{1}{2}}\), which makes them possible to estimate. Consider \(\|(-r)^{\frac{1}{2}} DT \varphi\|^2\), where \(D\) will either be \(\partial\) or \(\partial^*\). Let \(\epsilon > 0\) be a fixed number, which will be chosen later. Note that \((-r)^{\frac{1}{2}} < \epsilon\) holds on the set \(S_{\epsilon 2} = \{ z \in \Omega \mid -\epsilon^2 < r(z) < 0 \}\). Let \(\zeta\) be a smooth, non-negative function such that \(\zeta = 1\) on \(\Omega \setminus S_{\epsilon 2}\) and \(\zeta = 0\) on \(S_{\epsilon 2}\). Then

\[
\|(-r)^{\frac{1}{2}} DT \varphi\|^2 \lesssim \|\zeta DT\varphi\|^2 + \epsilon\|(1 - \zeta) DT\varphi\|^2 =: A_1 + A_2.
\]

To show that the term \(A_1\) is allowable we commute:

\[
A_1 \lesssim \|\zeta [D, \varphi]\|^2 + \|\zeta TD\varphi\|^2 \lesssim \|\zeta [D, T]\varphi\|^2 + \|\zeta \varphi\|^2 + \|\zeta TD\varphi\|^2.
\]

Since \(\zeta\varphi\) is compactly supported in \(\Omega\), interior elliptic estimates give us

\[
\|\zeta \varphi\|^2 \lesssim Q(\zeta\varphi, \zeta\varphi) \lesssim \epsilon^{-4}Q(\varphi, \varphi) \lesssim \epsilon^{-4}\|f\|^2,
\]

where the last step follows by part (i) of Lemma 4.7. Also, \([\zeta, [D, T]]\) is of order 0. Thus, with part (ii) of Lemma 4.7, we obtain

\[
A_1 \lesssim \epsilon^{-4}\|f\|^2 + \|\zeta TD\varphi\|^2.
\]

If \(D = \partial\), then the second term on the right hand side of the last inequality equals 0. If \(D = \partial^*\), then

\[
\|\zeta T\partial^*\varphi\|^2 \lesssim \|\zeta TB_{j-1}f\|^2 + \|Tf\|^2 \lesssim \|\zeta [D, T]\varphi\|^2 + \|\zeta B_{j-1}f\|^2 + \|f\|^2.\]

Again, \(\zeta B_{j-1}f\) is supported in \(\Omega\), thus

\[
\|\zeta B_{j-1}f\|^2 \lesssim Q(\zeta B_{j-1}f, \zeta B_{j-1}f) \lesssim \epsilon^{-4}\|f\|^2.
\]

This concludes the proof of the term \(A_1\) being allowable. To estimate the term \(A_2\) we commute again:

\[
A_2 \lesssim \epsilon \|DT\varphi\|^2 \lesssim \epsilon \left(\|TD\varphi\|^2 + \|[D, T]\varphi\|^2\right) \lesssim \epsilon \left(\|TD\varphi\|^2 + \|\varphi\|_1^2\right).
\]

The term \(TD\varphi\) vanishes if \(D = \partial\), otherwise

\[
\|TD\varphi\|^2 = \|T\partial^*\varphi\|^2 \lesssim \|TB_{j-1}f\|^2 + \|Tf\|^2 \lesssim \|B_{j-1}f\|^2 + \|f\|^2.
\]

Therefore we get

\[
A_1 + A_2 \lesssim \epsilon^{-4}\|f\|_1^2 + \epsilon \left(\|\varphi\|_1^2 + \|B_{j-1}f\|_1^2\right) \lesssim \epsilon^{-4}\|f\|_1^2 + \epsilon \|B_{j-1}f\|_1^2,
\]

here the last estimate holds by part (iv) of Lemma 4.7. This implies in particular

\[
\|T [T, \partial^*] \varphi\|^2 \lesssim \epsilon^{-4}\|f\|_1^2 + \epsilon \|B_{j-1}f\|_1^2 \lesssim \epsilon^{-4}\|f\|_1^2 + \epsilon \|TB_{j-1}f\|^2,
\]

where the last line follows by inequality (4.4). Invoking inequality (4.8) we at last obtain

\[
\|TB_{j-1}f\|^2 \lesssim \epsilon^{-4}\|f\|_1^2 + \epsilon \|TB_{j-1}f\|^2.
\]
Choosing $\epsilon > 0$ sufficiently small so that we can absorb the last term into the left hand side, we obtain that $\|TB_{j-1}f\|^2$ is allowable. Thus we have shown that (4.2) holds, assuming both $f$ and $B_{j-1}f$ are in $\Lambda^{0,j-1}(\Omega)$.

III. Removing smoothness assumptions. Consider a sequence of approximating subdomains: define $\Omega_\delta = \{z \in \mathbb{C}^n \mid \rho_\delta(z) = \rho(z) + \delta|z|^2 < 0\}$ for sufficiently small $\delta > 0$. Note that $\Omega_\delta$ is smoothly bounded and strongly pseudoconvex. This implies in particular that the Bergman projection, $B_{j-1}^\delta$, on $\Omega_\delta$ preserves $\Lambda^{0,j-1}(\Omega_\delta)$. We can execute the above argument, replacing $\rho$ by $\rho_\delta$, for $B_{j-1}^\delta$ on $\Omega_\delta$ to obtain
\[
\|B_{j-1}^\delta f\|_{1,\Omega_\delta} \leq C_1 \|f\|_{1,\Omega_\delta} \quad \text{for } f \in \Lambda^{0,j-1}(\Omega_\delta),
\]
where the constant $C_1$ does not depend on $\delta$ since none of our estimates do. Thus $\|B_{j-1}^\delta f\|_{1,\Omega_\delta} \leq C_1 \|f\|_{1,\Omega}$ for $f \in \Lambda^{0,j-1}(\Omega)$ is a true estimate, which holds uniformly in $\delta$. Since $B_{j-1}^\delta f$ converges to $B_{j-1}f$ pointwise, it follows that $B_{j-1}$ is in fact in $H^k_{0,j-1}(\Omega)$ and $\|B_{j-1}f\|_{1,\Omega} \leq C_1 \|f\|_{1,\Omega}$ for $f \in \Lambda^{0,j-1}(\Omega)$. Because $\Lambda^{0,j-1}(\Omega)$ is dense in $H^1_{0,j-1}(\Omega)$ with respect to the Sobolev 1-norm, continuity of $B_{j-1}$ on $H^1_{0,j-1}(\Omega)$ follows.

IV. The induction step for $k > 1$.

Let $j \in \{q, \ldots, n-1\}$ be fixed. Suppose $B_j$ is continuous on $H^k_{0,j}(\Omega)$, and $B_{j-1}$ is continuous on $H^{k-1}_{0,j-1}(\Omega)$. In the following, we indicate how to obtain that $\|B_{j-1}f\|_k \leq C_k \|f\|_k$ holds for $f \in \Lambda^{0,j-1}(\Omega)$ assuming that $B_{j-1}f \in \Lambda^{0,j-1}(\Omega)$. With arguments similar to the ones preceding inequality (4.4), we obtain
\[
\|B_{j-1}f\|_k^2 \lesssim \|X^k B_{j-1}f\|^2_2 + \|f\|_k^2 + \|B_{j-1}f\|_{k-1}^2 \lesssim \|X^k B_{j-1}f\|^2 + \|f\|_k^2,
\]
where the last step follows by the induction hypothesis. For more details on how to derive (4.4) see also [Boa-Str91], pg. 83.

Thus one needs to show that $\|X^k B_{j-1}f\|$ is dominated by the Sobolev $k$-norm of $f$. In this context we call a term allowable if it is dominated by $\|f\|_k^2$. As before, it is sufficient to show that $\|T^k B_{j-1}f\|^2$ is allowable. Again, we define $\varphi = N_j \bar{\partial} f$, and compute
\[
\|T^k B_{j-1}f\|^2 = (T^k f, T^k B_{j-1}f) - (T^k \bar{\partial}^* \varphi, T^k B_{j-1}f) = (T^k f, T^k B_{j-1}f) - (\varphi, [\bar{\partial}, (T^k)^* T^k] B_{j-1}f).
\]
Thus (sc)-(lc) inequality gives
\[
\|T^k B_{j-1}f\|^2 \lesssim \|f\|_k^2 + \|\varphi, [\bar{\partial}, (T^k)^* T^k] B_{j-1}f\|.
\]
Using that \((T^k)^* = (T^*)^k\) a straightforward computation gives
\[
[\bar{\partial}, (T^k)^* T^k] = 3(T^*)^{k-1} [\bar{\partial}, T] T^k + 3T^{k-1} [\bar{\partial}, T^*] (T^*)^k
\]
+ terms of order \(2k - 1\).

Therefore
\[
\| T^k B_{j-1} f \|^2 \lesssim \| [T, \bar{\partial}^*] T^{k-1} \varphi, T^k B_{j-1} f \| + \| f \|_k^2 + \| \varphi \|_{k-1} \| B_{j-1} f \|_k.
\]

Since \(B_{j-1} f\) is continuous on \(H^k_{0,j-1}(\Omega)\), it follows that \(\| \varphi \|_{k-1}^2\) is allowable. Thus, using the \((sc)-(lc)\) inequality and \((4.4)\), it follows
\[
(4.8) \quad \| T^k B_{j-1} f \|^2 \lesssim \| [T, \bar{\partial}^*] T^{k-1} \varphi \|^2 + \| f \|^2.
\]

Obviously, one now needs estimates for \(\varphi\) similar to those estimates in Lemma 4.7. By analogous arguments as those which give Lemma 4.7, using the induction hypotheses, one obtains

**Lemma 4.7.**

(i) \(Q(T^{k-1} \varphi, T^{k-1} \varphi)\) is allowable,
(ii) \(\| \varphi \|_{k-1}^2, \| \bar{\partial} \varphi \|_{k-1}^2\) and \(\| L_{\varphi} \|_{k-1}^2\), \(1 \leq l \leq n\), are allowable,
(iii) \(\| L_{\varphi} \|_{k-1}^2 \leq \epsilon \| T \varphi \|_{k-1}^2 + \frac{1}{\epsilon} \| f \|_k^2\) for \(1 \leq l \leq n, \epsilon > 0,\)
(iv) \(\| T \varphi \|_{k-1}^2 \lesssim \| f \|_k^2 + \| B_{j-1} f \|_k^2\).

Now one just follows the proof for \(k = 1\) starting at \((4.8)\), with \(T^{k-1} \varphi\) in place of \(\varphi\) and using Lemma 4.7 instead of Lemma 4.7. This leads to the estimate
\[
\| T^k B_{j-1} f \|^2 \lesssim \| f \|_k^2,
\]
from which the apriori estimate \(\| B_{j-1} f \|_k \leq C_k \| f \|_k\) follows, under the assumptions that \(f, B_{j-1} f \in L^{0,j-1}(\Omega)\). Passing from these apriori estimates to actual estimates follows as before. \(\square\)

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