CONNECTIVITY OF $h$-COMPLEXES

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Abstract. This paper verifies a conjecture of Edelman and Reiner regarding the homology of the $h$-complex of a Boolean algebra. A discrete Morse function with no low-dimensional critical cells is constructed, implying a lower bound on connectivity. This together with an Alexander duality result of Edelman and Reiner implies homology-vanishing also in high dimensions. Finally, possible generalizations to certain classes of supersolvable lattices are suggested.

1. Introduction.

If a simplicial complex $\Delta$ has a shelling in which the unique minimal faces from the shelling steps form a subcomplex of $\Delta$, then Edelman and Reiner call this subcomplex the $h$-complex of $\Delta$ with respect to this shelling. They refer to such a shelling as an $H$-shelling. Edelman and Reiner introduced and studied $h$-complexes in [3]. One motivation for $h$-complexes is that the $f$-vector of an $h$-complex is the $h$-vector of the original complex and the Euler characteristic of an $h$-complex is the Charney-Davis quantity of the original complex (cf. [13], [10], [11]).

Following [4], let $\Delta_n$ denote the $h$-complex which results from the standard shelling for the order complex of a truncated Boolean algebra $B_n - \{\hat{0}, \hat{1}\}$. Edelman and Reiner conjecture in [4] the following:

Conjecture 1.1 (Edelman-Reiner). $\tilde{H}_i(\Delta_n, \mathbb{Z})$ is nonzero if and only if $(3i+5)/2 \leq n \leq 3i+4$.

This is equivalent to saying that the reduced homology in dimension $i$ is nonzero if and only if

$$\frac{n-4}{3} \leq i \leq \frac{2n-5}{3}.$$

Our main result will be a proof of this conjecture. Afterwards we suggest possible generalizations.

Recall that a simplicial complex $\Delta$ is pure if all maximal faces are equidimensional; these maximal faces are called its facets. A pure simplicial complex is shippable if there is a total order $F_1, \ldots, F_k$ on its facets with the following property: for each $1 \leq j \leq k$, there is a unique face $\sigma_j$ contained in $F_j$ which is minimal among all faces contained in $F_j$ but not in any earlier facets. We refer to the faces $\sigma_1, \ldots, \sigma_k$ as the minimal faces of the shelling. For a shellable complex $\Delta$ of dimension $d$, the $h$-vector of $\Delta$ has coordinates $(h_1, \ldots, h_d)$, with $h_i$ counting the number of facets $F_j$ for which the minimal face $\sigma_j$ is $i$-dimensional.

Our interest will be in the Boolean algebra $B_n$, namely the partial order on subsets of $\{1, \ldots, n\}$ by inclusion. Let $\hat{B}_n$ denote the truncated Boolean algebra

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$B_n - \{0, 1\}$ consisting of all subsets except the empty set and the full set. Denote by $\Delta(P)$ the order complex of a poset $P$, i.e. the simplicial complex whose faces are the chains of comparable poset elements. It is well-known that $\Delta(\hat{B}_n)$ has a (lexicographic) shelling by labelling saturated chains with permutations in $S_n$ recording the order in which elements of $\{1, \ldots, n\}$ are successively inserted, and then ordering facets in $\Delta(\hat{B}_n)$ by the lexicographic order on the permutations labelling the saturated chains. The minimal faces in this shelling are comprised of the ranks at which the permutations have descents. It is not hard to check that these minimal faces form a subcomplex, denoted $\Delta_n$, of $\Delta(\hat{B}_n)$.

Reiner observed that the reduced homology $\tilde{H}_i(\Delta_n, \mathbb{Z})$ is nonzero for $n - 4 \leq i \leq 2n - 5$ (personal communication). A proof of his result is provided in Section 3. In light of Reiner’s observation, it will suffice to show that the homology vanishes in the remaining dimensions. We will use discrete Morse theory in Sections 4 and 5 to show this for dimensions below $\frac{2n - 5}{3}$. Then we use Theorem 4.14 (an Alexander duality result) from [4] to deduce homology vanishing for top dimensions. Theorem 4.14 of [4] is as follows (see [4] for definitions):

**Theorem 1.2** (Edelman-Reiner). Let $\omega$ be an $H$-shelling of a simplicial $d$-sphere $\Sigma$, and $\alpha$ a simplicial involution on $\Sigma$ which reverses the restriction map. Denote by $\Delta^{(h)}(\omega)$ the $h$-complex given by $\omega$. Then there is an isomorphism $\tilde{H}_i(\Delta^{(h)}(\omega), \mathbb{Z}) \to \tilde{H}_{d-1-i}(\Delta^{(h)}(\omega), \mathbb{Z})$.

Since the order complex of the truncated Boolean algebra is the first barycentric subdivision of the boundary of a simplex, it is a triangulation of a sphere. Its standard shelling is an $H$-shelling, so the above theorem applies to its $h$-complex.

We will also give a combinatorial proof that there is a dual discrete Morse function for the $h$-complex of a Boolean algebra in Section 6 yielding a second proof that its high-dimensional homology vanishes. Our motivation for this alternate proof is that it has the potential to generalize to situations where the Alexander duality result of [4] would not apply (e.g. to the poset of subspaces of a finite vector space). This dual Morse function might also be helpful for another question of [4], that of finding a combinatorial explanation for the symmetry of the Betti numbers which results from Theorem 1.2 above. Finally, Section 7 discusses possible generalizations from the Boolean algebra to other supersolvable lattices.

Before turning to the details, we quickly review the bare essentials from Forman’s discrete Morse theory (see [5]) and Chari’s combinatorial reformulation (see [3]). See [2] for more background on topological combinatorics and see [13] for background on $f$-vectors, $h$-vectors and the Charney-Davis conjecture.

**Definition 1.3.** A matching on the face poset $F(\Delta)$ of a simplicial complex $\Delta$ is acyclic if orienting matching edges upward and all other edges downward yields an acyclic directed graph. (Recall that $F(\Delta)$ is the partial order on faces by inclusion.)

Any acyclic matching on $F(\Delta)$ gives rise to a discrete Morse function on $\Delta$ whose critical cells are the faces left unmatched by the acyclic matching. The number of critical cells of various dimensions in a discrete Morse function give bounds on the Betti numbers as follows. For each $i$, $\beta_i \leq m_i$, where $m_i$ is the number of $i$-dimensional critical cells.

For simplicity, we will work exclusively with acyclic matchings rather than the corresponding discrete Morse functions. Forman proved that $\Delta$ a discrete Morse
function on a $d$-dimensional CW complex $\Delta$ with Morse numbers $m_0, m_1, \ldots, m_d$ implies that $\Delta$ is homotopy equivalent to a CW complex which has $m_i$ cells of dimension $i$ for each $i$. We will specifically use the fact that a discrete Morse function on a complex $\Delta$ with $m_i = 0$ for $i$ less than a fixed $j$ implies that the $\Delta$ is $(j-1)$-connected.

2. The $h$-complex of a truncated Boolean algebra

This section gives more detail about the standard shelling for the Boolean algebra $B_n$ in order to set up notation that we will need for the acyclic matching in later sections. The elements of $B_n$ are the subsets of $[n] := \{1, \ldots, n\}$. $B_n$ has covering relations $T < S$ for each $S = T \cup \{i\}$ and $i \in [n] \setminus T$. Label each covering relation $T < T \cup \{i\}$ by the label $i$. Each saturated chain is then labelled by the sequence of labels on its covering relations, i.e. by a permutation in $S_n$ written in one-line notation. Notice that each element of $S_n$ labels a single saturated chain, allowing us to refer to permutations and saturated chains interchangeably. Ordering these label sequences lexicographically gives a shelling order on facets of $\Delta(B_n)$.

Notice that the minimal face for a permutation $\pi$ consists of the chain supported at those ranks where $\pi$ has descents. For example, the minimal face for $\pi = 132654$ is the chain $\{1, 3\} < \{1, 3, 2, 6\} < \{1, 3, 2, 6, 5\}$ which consists of the ranks of the descents 32, 65 and 54. We can easily recover a saturated chain from its minimal face, so we also refer to minimal faces interchangeably with permutations and saturated chains. It is immediate from this description of minimal faces that the Charney-Davis quantity is the alternating sum $A_n$ of the Eulerian numbers $A_{n,k}$. The exponential generating function $\sum_{n \geq 0} A_n \frac{x^n}{n!}$ is well-known to equal $-\tanh(x)$ (see [4, p. 52]). Nonetheless, the homology of $\Delta_n$ will turn out to live in many different dimensions.

To simplify notation later, we add an initial letter $a_0 = 0$ and a final letter $a_{n+1} = n + 1$ to each permutation $a_1 \cdots a_n$. We refer to the permutation position between $a_i$ and $a_{i+1}$ as rank $i + 1$, reflecting the fact that we have adjoined $a_0$ between ranks 0 and 1. Depict the minimal shelling face

$$\{a_1, \ldots, a_{i_1}\} < \{a_1, \ldots, a_{i_1}, a_{i_2}, \ldots, a_{i_2}\} < \cdots < \{a_1, \ldots, a_{i_j}\}$$

for a permutation $\pi = a_0 a_1 \cdots a_{n+1}$ which has descents at ranks $i_1, i_1 + i_2, \ldots, i_1 + \cdots + i_{j-1}$ by an ordered collection of blocks

$$a_1, a_1, a_1, a_1, a_2, a_2, a_2, a_2, \ldots, a_{i_1}, a_{i_1}, a_{i_2}, a_{i_2}, \ldots, a_{i_j}.$$ 

By convention, order elements within each block in increasing order. Sometimes we refer to these blocks as intervals. Notice that this minimal face has dimension $j - 2$.

We will call the separators between the blocks bars. When we remove a bar and merge two consecutive blocks, we sort the two blocks so the new permutation is increasing on the merged block. When we speak of inversions between two consecutive blocks, we mean inversions in the permutation obtained by removing the separating bar without sorting the blocks.

3. Non-vanishing homology

This section shows for each integer $n$ with $\frac{3k+2}{2} \leq n \leq 3k + 4$ that the homology group $\tilde{H}_k(\Delta_n, \mathbb{Z})$ is nonzero. The approach for $2k + 3 \leq n \leq 3k + 4$ is to exhibit
a cycle which cannot be a boundary, by virtue of having a free face. Theorem 1.2 then gives us that $\tilde{H}_k(\Delta_n, \mathbb{Z}) \neq 0$ for $\left\lceil \frac{3k+5}{2} \right\rceil \leq n \leq 2k + 3$.

Definition 3.1. A free face of dimension $k$ in $\Delta_n$ is a face which is not in the boundary of any $(k+1)$-dimensional face. Thus, it is a maximal face in $\Delta_n$.

First notice that $\Delta_n$ is not pure, so there will be free faces in various dimensions. Each cycle we construct in this section will contain at least one free face, making it impossible for the cycle to be a boundary.

Example 3.2. For $k = 1, n = 3k + 4$, the minimal shelling face $13|246|57$ is a free face in $\Delta_7$. To see this, notice that any 2-face containing this edge must permute elements within one or more of the blocks $13$, $246$, $57$ in a way that maintains the descents at ranks 2 and 5, and creates one new descent. However, swapping 1 and 3 cannot avoid turning rank 2 into an ascent, and likewise the descents at ranks 2 and 5 force the labels 2, 6 and 5 into their current positions, making a 2-face containing $13|246|57$ impossible. Finally, notice that $13|246|57$ appears in the cycle $z = 13|246|57 - 13|26|457 + 3|126|457 - 3|1246|57$.

The remainder of this section generalizes this to all $n, k$ satisfying $2(k+1) + 1 \leq n \leq 3(k+1) + 1$, by giving constructions in the two extreme cases, then showing how to combine them to yield the desired range. Notice first that the cycle $z$ in Example 3.2 may be viewed as a sum over permutations of the form $(12)^{e_1}(45)^{e_2}$ that act on positions, each applied to the free face $C$. Each permutation is multiplied by its sign, to ensure that we get a cycle. That is, $z = \sum_{\pi \in \langle(12),(45)\rangle} \text{sgn}(\pi)\pi(13|246|57)$. Note that the minimal shelling face for a permutation appearing in $z$ includes either rank 1 or 2, but not both, depending on whether 13 appears in decreasing or increasing order, and likewise includes either rank 4 or 5, depending on the order of 4 and 6. Thus, the cycle is an alternating sum of $2k+1$ faces, each of dimension $k$, chosen so that each $(k-1)$-face appearing in any of these $k$-faces will occur in exactly two of them which have opposite signs. This will ensure $\partial(z) = 0$, as needed for a cycle.

More generally, for $n = 3k + 4$ we use the free face $F = 13|246|\cdots|3i + 2, 3i + 4, 3i + 6|\cdots|3k + 2, 3k + 4$ in which the $i$-th block has elements $3(i - 2) + 2, 3(i - 2) + 4, 3(i - 2) + 6$ for each $i$ strictly between 1 and $k$. A cycle $z$ is obtained by choosing the order of the last two elements of each of the first $k + 1$ blocks, i.e. for every block except the very last one. Each of these pairs of block elements determines the location of one of the descents. Thus, $z = \sum_{\pi \in \langle(12),(45),(78)\ldots,(3k+3,3k+4)\rangle} \text{sgn}(\pi)\pi(F)$ with permutations $\pi$ acting on positions.

At the other extreme, for $n = 2k + 3$, one obtains a free face $1, n|2, n-1|\ldots|(n-1)/2, (n+3)/2(n+1)/2$. A cycle again results from choosing the relative order of $i, n - i + 1$ for $1 \leq i \leq k + 1$, i.e. for pairs in each block except the last one. For $n$ satisfying $2(k+1) + 1 \leq n \leq 3(k+1) + 1$, we combine the two constructions above to
obtain a free face from the following permutation in $S_n$. For $n = 2j+3(k+1-j)+1$, begin the permutation with $1, n|2, n-1|\cdots|j-1, n-(j-1)+1$. Appended to this is the following permutation in $S_{[j,n-j+1]}$:

\[ j, j+2j+1, j+3, j+5|\cdots|n-j-4, n-j-2, n-j|n-j-1, n-j+1. \]

Again, we show this belongs to a cycle with $2^{k+1}$ faces by summing over elements of a group of size $2^{k+1}$ each multiplied by its sign. That is, for each of the first $k+1$ blocks, choose whether or not to swap the order of the last two letters. By similar reasoning to the above example, one obtains:

**Theorem 3.3** (Reiner). *For each* $2(k+1)+1 \leq n \leq 3(k+1)+1$, $\tilde{H}_k(\Delta_n) \neq 0$. *Furthermore, by Theorem 1.2* this implies $\tilde{H}_k(\Delta_n) \neq 0$ for $\lceil \frac{3k+2}{2} \rceil \leq n \leq 2k+3$.

4. A matching on $\Delta_n$

This section provides a matching on faces in $\Delta_n$ which will be shown to be acyclic in the next section. In contrast to most acyclic matchings in the literature, our matching is fairly easy to describe, but the proof of its acyclicity is much more intricate than usual.

Faces will be matched greedily based on their lowest interval which takes a certain form, described below. First we will need some notation. Denote by $I_{\text{above}}$ the interval immediately above an interval $I$ when such an interval exists, and likewise denote by $I_{\text{below}}$ the interval immediately below $I$. Let $S_C(I)$ be the size of the maximal set $S$ of consecutive intervals $J_1, \ldots, J_s$ immediately above $I$ such that (1) $|J_1| = \cdots = |J_s| = 2$, and (2) the only inversions among the blocks $I, J_1, \ldots, J_s$ are the $s$ descents separating the blocks. We call $J_1, \ldots, J_s$ the $J$-intervals or $J$-blocks of $I$. For example, in $0, 1, 2, 3, 6|5, 8|7, 9|4, 10$ the block $I = 0, 1, 2, 3, 7$ has $S_C(I) = 2$ and has $J$-blocks 5, 8 and 7, 9 but not 4, 10.

**Definition 4.1.** An interval $I$ is *matchable* if it has any of the following forms:

1. $|I| = 1$, $|I_{\text{above}}|$ is odd of size at least 3, and there is only one inversion between $I$ and $I_{\text{above}}$.
2. $|I|$ is even, $|I| \geq 4$, $I_{\text{below}}$ exists, and there are inversions between the largest element of $I_{\text{below}}$ and both of the two smallest elements of $I$.
3. $|I| \geq 4, S_C(I)$ is even, and $I$ is not also of type 2.
4. $|I| \geq 2, S_C(I)$ is odd, there is only one inversion between $I$ and $I_{\text{above}}$, and the block obtained by merging $I$ with $I_{\text{above}}$ is not matchable of type 2.

In an effort to make our proofs more readable, let us call the four types of matchable intervals above (1) 1-split, (2) 1-merged, (3) 2-merged, and (4) 2-split, respectively, reflecting the fact that a block of size 1 or 2 is split off from another block or merged with it. Notice that 0 and $n+1$ are permanently fixed in the first and last positions, so the matching may not insert bars at ranks 1 and $n+1$; the requirement for 1-merged blocks $I$ that $I_{\text{below}}$ exists will take care of this.

When we need to keep track of the fact that we are viewing $I$ as an interval in a chain $C$, then we will sometimes denote $I$ as $I(C)$. If the first matchable interval in a chain is at rank $r$, then we match it with another chain whose first matchable interval is also at rank $r$, as follows.

**Definition 4.2.** A chain $C$ with lowest rank matchable interval $I(C)$ at rank $r$ is *matched* with a chain $D$ if $D$ differs from $C$ by a single inversion and:
• $I(C)$ has type 1; $D$ is obtained from $C$ by merging $I$ with $I_{\text{above}}$.
• $I(C)$ has type 2; $D$ is obtained from $C$ by splitting $I$ of size $m$ into blocks of size $1, m - 1$ (where we list the block at higher ranks second).
• $I(C)$ has type 3; $D$ is obtained from $C$ by splitting $I$ of size $m$ into blocks of size $m - 2, 2$.
• $I(C)$ has type 4; $D$ is obtained from $C$ by merging $I$ with $I_{\text{above}}$.

To prove the above matching is well-defined, we first show that if $C$ has lowest matchable interval at rank $r$, then its partner $D$ also has lowest matchable interval at rank $r$.

**Theorem 4.3.** If the lowest matchable interval in a chain $C$ is at rank $r$, then the chain $D$ with which $C$ is matched also has no matchable intervals below rank $r$.

**Proof.** Suppose the lowest matchable interval $I$ in $C$ is 1-split. Then $|I(C)| = 1$, and $|I(D)|$ is even with size at least 4. Since $C$ and $D$ agree below rank $r$ and all intervals of size at least 4 are matchable, there cannot be any intervals of size 4 or larger in $D$ below rank $r$. Hence, $D$ has no 1-merged or 2-merged matchable intervals below rank $r$. Neither $I(C)$ nor $I(D)$ has size 2, so neither can be a $J$-interval for any lower intervals. $D$ cannot have a 2-split matchable interval $I'$ at rank $r' < r$ without $C$ also having such an interval: $C$ and $D$ agree below rank $r$, and $I'(D)$ cannot have any $J$-intervals at or above rank $r$, so $S_D(I') = S_C(I')$.

Finally, suppose $D$ had a 1-split matchable interval $I'$ at rank $r' < r$. Then $I'(C)$ would also be matchable, except perhaps for $r' = r - 1$. But then $D$ would need an interval of odd size at rank $r$, but $|I(D)|$ is even. Hence, $D$ has no matchable intervals below rank $r$. The case where $I(C)$ is 1-merged is similar with the roles of $C$ and $D$ reversed, so we omit the argument.

Now suppose $I(C)$ is 2-merged or 2-split. Once again $C$ and $D$ agree below rank $r$, and all intervals of size at least four are matchable; thus, we only need to consider the possibility that $D$ has a matchable interval $I'$ at rank $r' < r$ with $I'$ that is 1-split or 2-split. If $I'(D)$ is 1-split matchable, then as before $I'$ must occur at rank $r - 1$. Then $|I(D)| = m$ is odd with $m \geq 3$, which means $|I(C)| = m \pm 2$ is also odd. Furthermore, $|I(C)| \geq 2$ which means it will also have size at least 3, since $|I(C)|$ is odd. Furthermore, $I'(C) = I'(D)$ and $I_{\text{below}}(C) = I_{\text{below}}(D)$, so $I'(C)$ would also be 1-split matchable, a contradiction.

Now suppose $I'(D)$ is 2-split matchable. Then we would need $S_D(I')$ odd and $S_C(I')$ even, so in particular they are not equal. Since $C$ and $D$ agree below rank $r$, this means that either $I'(C)$ or $I'(D)$ must have one or more $J$-intervals at or above rank $r$. Hence, $C$ or $D$ must have a block of size 2 at rank $r$, while the other must then have a block of size 4 at rank $r$. Let us assume $|I(C)| = 4$, which means $S_C(I)$ is even. The other case is similar.

Since $S_C(I')$ is even and $C$ does not have a block of size 2 at rank $r$, both $I'(C)$ and $I'(D)$ must have an even number of $J$-blocks below rank $r$. Thus, $I'(D)$ needs an odd number of $J$-blocks above rank $r$. However, $S_D(I)$ is odd, implying $I(D)$ together with its $J$-blocks comprise an even number of prospective $J$-blocks for $I'(D)$ above rank $r$. This means that not all of the $J$-blocks for $I$ are also $J$-blocks for $I'$, so there must be at least one extra inversion among these potential $J$-blocks. In particular, either the second smallest label above rank $r$ must be smaller than the label just below rank $r$, or else the smallest label above rank $r$ must be smaller than the second smallest label below rank $r$. We can eliminate the latter possibility,
since \( I'(D) \) has at least one \( J \)-block above rank \( r \). Hence, \( I(C) \) has two labels that are smaller than the largest element of \( I_{\text{below}}(C) \), and \( |I(C)| \) is even of size at least 4. This means that \( I(C) \) is 1-merged matchable instead of 2-merged matchable, a contradiction. \hfill \Box

**Corollary 4.4.** The matching is well-defined.

*Proof.* It suffices now to check that the matching rules for 1-split and 1-merged matchable intervals are inverses to each other, and likewise for 2-merged and 2-split intervals. This is easy, and is left to the reader. \hfill \Box

**5. Acyclicity of \( \Delta_n \) matching**

Now we turn to the task of proving the matching is acyclic, and hence comes from a discrete Morse function. Unlike many acyclicity proofs in the literature, we are not aware of any function which is decreasing along directed paths, so our acyclicity proof will take another approach.

**Lemma 5.1.** If the matching had a directed cycle \( C \), then each downward step in \( C \) would eliminate a single inversion, i.e. would merge two blocks with only one inversion between them.

*Proof.* Each upward step increases permutation length (i.e. number of inversions) by exactly one, and each downward step decreases permutation length by at least one. Any cycle would have an equal number of upward and downward steps before revisiting its initial permutation, so down steps must decrease length by exactly one, in order to restore the length of the original permutation. \hfill \Box

In light of Lemma 5.1, each edge traversed in a directed cycle may be viewed as an adjacent transposition; an entire cycle would comprise a non-reduced expression for the identity permutation. It would be desirable to have a shorter, more elegant proof of acyclicity than the one below, perhaps using properties of non-reduced expressions for the identity permutation.

Before proceeding with the proof, we list a few facts it will use repeatedly:

1. By Lemma 5.1, downward steps merging two blocks are only permitted when the only inversion between the blocks is the descent separating them.
2. Since each upward step changes the lowest matchable interval from 2-merged or 1-merged to 2-split or 1-split, it must be immediately followed by a downward step which causes the lowest matchable interval to again be 2-merged or 1-merged. Otherwise the downward step could not be followed by another upward step, as would be required in a cycle.
3. There are no matching steps splitting a block of size \( m \) into smaller blocks of size \( m - 1, 1 \).

Within the proof, we refer to these facts as Observations 1, 2 and 3.

One other key ingredient will be the idea behind the 0-1 Sorting Lemma from theoretical computer science (cf. [8]), that deals with the following type of sorting procedure: an *oblivious comparison-exchange sorting procedure* is an ordered list of comparisons to be performed, where two elements are exchanged whenever they are compared and found to be out of order; this is “oblivious” in that the choice of comparisons cannot depend on the outcome of earlier comparisons.
Lemma 5.2 (0-1 Sorting Lemma). Any oblivious comparison-exchange sorting algorithm which correctly sorts lists consisting exclusively of 0’s and 1’s will correctly sort lists with arbitrary values.

The idea is that to sort numbers correctly, one must be sure for any particular value \( a \) that all numbers larger than \( a \) are sorted to above all numbers smaller than \( a \), and so for any fixed \( a \) one may treat the numbers larger than \( a \) as 1’s and those smaller than \( a \) as 0’s. In our context, we will have a particular label \( a \) and it will be quite useful to keep track of exactly which labels below it form inversions with it, and to disregard all other information about the relative order of the values below \( a \).

Remark 5.3. The proof below often speaks of rank, by which we mean rank in the original poset \( \hat{B}_n \), not in the face poset \( F(\Delta_n) \) upon which we construct a matching.

Denote by \( u_r \) the matching step which inserts a bar at rank \( r \). Denote by \( d_r \) the downward step deleting a bar from rank \( r \) by applying an adjacent transposition to replace a descent by an ascent.

Theorem 5.4. The matching on \( \Delta_n \) is acyclic.

Proof. Suppose there were a directed cycle \( C \) in the directed graph obtained from the matching on \( F(\Delta_n) \). Consider the highest rank \( t \) at which a bar is ever inserted, and let \( u_t \) be a matching step inserting such a bar \( B_t \) into a chain \( C_0 \) to obtain a partner chain \( D_1 \). Let \( u_{ik} \) be the upward step immediately preceding the first occurrence of \( d_t \) after \( u_t \). Then \( d_t \) deletes a bar at a strictly higher rank than \( i_k \) (since \( i_k \leq t \), but we are assured that \( i_k \neq t \) since there is already a bar at rank \( t \) just prior to \( u_{ik} \)).

Our proof will focus on the segment of \( C \) from just before \( u_t \) until just after \( d_t \). Let us establish some notation for the faces appearing in this segment. \( C \) must alternate between two consecutive face poset ranks \( r \) and \( r + 1 \), so denote this segment of \( C \) by \( C_0 \rightarrow D_1 \rightarrow C_1 \rightarrow \cdots \rightarrow C_{k-1} \rightarrow D_k \rightarrow C_k \). That is, denote chains at rank \( r \) by \( C_0, C_1, \ldots, C_k \) and chains at rank \( r + 1 \) by \( D_1, D_2, \ldots, D_k \); the \( j \)-th matching step in this segment, denoted \( u_{ij} \), takes \( C_j \) to \( D_j \). We chose \( k \) so that the first \( d_t \) after \( u_t \) immediately follows \( u_{ik} \), so there are \( k \) matching steps within the segment. See Figure 1.

Now we already observed that \( d_t \) deletes a bar at a strictly higher rank than where a bar was inserted by \( u_{ik} \). However, since \( u_{ik} \) is a matching step, it must have changed the lowest matchable interval \( I \) in \( C_{k-1} \) from 1-merged to 1-split or from 2-merged to 2-split. Since \( t > i_k, d_t \) cannot create a lower matchable interval than \( I(C_{k-1}) \). Thus, by Observation 2, \( d_t \) must destroy the structure which made
$I(C_{k-1})$ 1-split or 2-split matchable. We will consider two cases, depending on
whether $u_{ik}$ changes an interval $I$ either (a) from 1-merged to 1-split or (b) from 2-
merged to 2-split. Each case will lead to a contradiction, making cycles impossible.

Proof in case (a): Suppose $u_{ik}$ changed an interval $I$ from 1-merged to 1-split.
Lemma 5.5 will show $u_{ik}$ splits the block $I_{t-}$ immediately below $B_t$. Thus, the
block created immediately above $B_t$ has odd cardinality. Thus, the block created
by $u_t$ just above $B_t$ cannot have size 2, so $u_t$ must have changed an interval from
1-merged to 1-split. Therefore, $C_0$ must have a bar at rank $t - 1$. However, $C_k$
cannot have a bar at rank $t - 1$, since $u_{ik}$ split $I_{t-}$ into two blocks, with the one
at higher ranks having size at least 3. Thus, there must have been an intermediate
step $d_{t-1}$ at some point. However, Lemma 5.5 shows that steps $u_{t-1}$ are impossible
throughout the cycle, a contradiction to our ever returning to $C_0$.

Proof in case (b): Suppose alternatively that $u_{ik}$ changes a matchable interval
$I(C_{k-1})$ from 2-merged to 2-split. This is immediately followed by $d_t$ which deletes
a bar strictly above rank $i_k$, but by Observation 2, $d_t$ causes $I(D_k)$ no longer to be
2-split matchable. To do this, $d_t$ must change the parity of $S_{D_k}(I)$. Hence, $d_t$ must
delete a bar separating a $J$-block just below $B_t$ from a non-$J$-block $I_{t+}$ immediately
above $B_t$. To avoid being a $J$-block, $I_{t+}$ must either (I) have size $m > 2$ or (II)
have size $m = 2$ and have an inversion with the $J$-blocks below it other than the
descent separating $I_{t+}$ from $I_{t-}$.

Case b(I): If $m > 2$, then $u_t$ must have changed a matchable interval from
1-merged to 1-split. Notice that there is a bar at rank $t - 1$ just prior to $u_t$ but no
bar at rank $t - 1$ just after $u_{ik}$ (since there is a $J$-block immediately below rank $t$
just before $d_t$). This means we need a step $d_{t-1}$ in our cycle, but Lemma 5.6 again
such a step.

Case b(II): If $m = 2$, there must be extra inversions preventing the block $I_{t+}$
just above rank $t$ from being a $J$-block for $I(D_k)$. Lemma 5.6 will show that the
larger element $d$ in $I_{t+}$ cannot be inverted with any elements of the $J$-blocks of
$I(D_k)$, so that the extra inversion must instead involve the smaller element $a$ in
the block above $B_t$. Thus, $a$ must be smaller than the two largest labels within
$J$-blocks of $I(D_k)$, i.e. the labels just below ranks $t$ and $t - 2$ in $D_k$. Hence, $D_k$
has an inversion between the label $a$ just above rank $t$ and the label just below rank
$t - 2$. However, Proposition 5.5 will show we cannot get from the chain $C_0$ whose
lowest matchable interval is 2-merged to the situation at $D_k$ where the letter $a$
just above $B_t$ is inverted with the letter just below rank $t - 2$. This will complete our
proof.

Lemma 5.5. Let $u_t$ insert bar $B_t$ in the highest position a bar is ever inserted
within a directed cycle. If the step $u_{ik}$ immediately preceding the next $d_t$ after $u_t$
changes an interval from 1-merged to 1-split, then the block above $B_t$ has odd size.

Proof. This is because $d_t$ must cause $I$ no longer to be 1-split, by Observation 2,
and the only way to do this is for $d_t$ to delete a bar immediately above $I^{above}(D_k)$
so as to change the parity of $I^{above}(D_k)$ from odd to even; this can only be done
by merging $I^{above}(D_k)$ with an odd block immediately above it. This odd block
immediately above $B_t$ is left unchanged by all steps between $D_1$ and $D_k$, implying
that $u_t$ inserted a bar $B_t$ making the block immediately above $B_t$ odd.

Lemma 5.6. If the highest insertion $u_t$ in a directed cycle changes an interval
from 1-merged to 1-split, then steps $u_{t-1}$ are impossible in the cycle.
Proof. The step \( u_t \) split a block of even size \( n \) into blocks of size 1, \( n - 1 \). By Observation 3, we cannot have an upward step \( u_{t-1} \) while a bar is present at rank \( t \), since no matching steps split an interval of size \( m \) into smaller intervals of size \( m - 1, 1 \), where we list the higher interval second. On the other hand, when \( B_t \) is not present, then inserting \( u_{t-1} \) would create a block of size \( n - 2 \) above \( B_{t-1} \). However, \( n - 2 \) must be even of size at least 4, since \( n - 1 \) was odd of size at least 3. There are no such matching steps, so \( u_{t-1} \) is impossible.

Lemma 5.7. In case b(II), the larger element \( d \) in the block above \( B_t \) cannot be inverted with any elements of the \( J \)-blocks of \( I(D_k) \).

Proof. The point will be to show that the label \( c \) just below rank \( t \) in \( D_1 \) must still be in this position in \( D_k \). But then we know that \( u_t \) only increased the permutation length by exactly one, so that \( c < d \) since \( c \) was in the same block with \( d \) just prior to \( u_t \). Since \( c \) must be larger than all other elements of the \( J \)-blocks of \( I \), \( d \) must also be larger than all of them.

To show that \( c \) is still at the position just below \( B_t \) in \( D_k \), we will show that there could not have been a step \( u_{t-1} \) or \( d_{t-1} \) between \( D_1 \) and \( D_k \). By Observation 3 we could not have inserted a bar at rank \( t - 1 \) in this interval, because a bar was present at rank \( t \) the entire time. On the other hand, \( u_t \) must have changed a matchable interval from 2-merged to 2-split, since \( m = 2 \) and in particular is even; this implies that \( D_1 \) does not have a bar at rank \( t - 1 \) available to be deleted.

Proposition 5.8. It is impossible in case b(II) above to have a directed path from the face \( C_0 \) in which the lowest matchable interval must have been 2-merged to the face \( D_k \) where the letter \( a \) just above \( B_t \) is inverted with the letter just below rank \( t - 2 \).

Proof. Lemma 5.10 will show that \( u_t \) must have split a block of size 4 into blocks of size 2, 2. Consider the cycle element \( C_0 \) just prior to \( u_t \). Denote by \( K \) the block just below the bar \( B_{t-2} \) in \( D_1 \). (Note that \( D_k \) has a bar at rank \( t - 2 \), because \( u_t \) split a block of size 4 into blocks of size 2, 2.) For \( u_t \) to change a matchable interval from 2-merged to 2-split instead of from 1-merged to 1-split, we need the largest element of \( K \) to be smaller than \( a \). We also know that the element just above rank \( t - 2 \) in \( D_1 \) is smaller than \( a \), since \( u_t \) only increased the permutation length by one. We consider two cases, depending on whether (i) \( |K| = 1 \) or (ii) \( |K| \geq 2 \).

Case (i): \( |K| = 1 \), so \( C_0 \) has a bar at rank \( t - 3 \) as well as rank \( t - 2 \). \( D_k \) does not have a bar at rank \( t - 3 \), since \( u_{t+1} \) matches a 2-merged matchable block, and then \( d_t \) causes this block to no longer be 2-split matchable, which means all blocks between the bar inserted by \( u_{t+1} \) and \( B_t \) have size 2. Thus, there must be an intermediate step \( d_{t-3} \) prior to \( u_{t+1} \). Next we show that the cycle can never restore the situation of having bars at both ranks \( t - 3 \) and \( t - 2 \), which will give us a contradiction. First note that a bar cannot be inserted at rank \( t - 3 \) while one is present at rank \( t - 2 \). On the other hand, we cannot have a step \( u_{t-2} \) while a bar is present at rank \( t - 3 \), by the following reasoning: such a step would change a matchable interval from 1-merged to 1-split, so it would produce an odd block of size at least 3 immediately above rank \( t - 2 \); this is impossible both when a bar is present at rank \( t \) and when there is no bar at rank \( t \), since then the next lowest bar is at rank \( t + 2 \). Thus, (i) is impossible since we cannot return to having bars at both ranks \( t - 3 \) and \( t - 2 \).

Case (ii): \( |K| \geq 2 \) and all elements of \( K \) must be smaller than the label \( a \) appearing just above \( B_t \). In the spirit of the 0-1 Sorting Lemma, we now denote
numbers below rank $t$ as 1’s and 0’s depending on whether they are larger or smaller than this fixed value $a$. Regardless of the actual values, any 1 with a 0 above it must be a descent, while any 0 with a 1 above it is an ascent. Immediately after $u_i$, we need there to be 1’s just below ranks $t$ and $t - 2$. However, in $D_1$ we know that the block between ranks $t - 2$ and rank $t$ consists of one 1 and one 0, while the block $K$ just below this contains only 0’s. Hence, the 1 just below rank $t - 2$ in $D_k$ must have moved upward from below the $K$ block. Finally, Lemma 5.9 will use the idea of the 0-1 Sorting Lemma to show that this is impossible, again precluding a cycle.

Lemma 5.9. It is impossible in Case (ii) of Proposition 5.8 for a directed path to proceed from the face $D_1$ to the face $D_k$. That is, we cannot shift a label which is larger than a upward from below the $K$ block to just below rank $t - 2$.

Proof. In $D_1$, the highest 1 below rank $t - 2$ must be below rank $t - 4$, because it must be strictly below the block $K$ in order for $I(C_0)$ to avoid being 1-merged, and

$$
\begin{array}{c}
\text{d} \\
\text{a} \\
\text{B}_t \\
\text{1} \\
\text{0} \\
\text{B}_{t-2} \\
\text{0} \\
\text{0} \\
\text{K of size at least 2} \\
\text{\vdots} \\
\text{\vdots} \\
\text{\vdots} \\
\text{\vdots} \\
\text{1} \\
\text{Highest 1 below K} \\
\text{\vdots}
\end{array}
$$

Figure 2. 1’s and 0’s below $a$

Furthermore, the only 1 in the interval between ranks $t - 2$ and $t$ in $D_1$ is the one just below rank $t$ that never moves. Thus, we must eventually move a 1 upward from below rank $t - 4$ to just below rank $t - 2$. Just before moving a 1 upward to just below rank $t - 3$, we must have a bar at rank $t - 3$, since otherwise the step $d_{t-4}$ would eliminate more than one inversion, since the 1 will be larger than all the 0’s in the block above it. However, there is no bar $B_{t-3}$ in $D_1$, since $|K| \geq 2$. Thus, we need a step $u_{t-3}$, and this can only happen when there is no bar at rank $t - 2$, by Observation 3. Once we have a bar at rank $t - 3$ with a 1 immediately below it, there will henceforth be a 1 at this position until there is a step $d_{t-3}$, since bars cannot be inserted at rank $t - 4$ while a bar is present at rank $t - 3$. However, we
cannot have \( d_{t-3} \) until after \( u_{t-2} \), since again \( d_{t-3} \) would otherwise eliminate more than one inversion. Finally, it is not possible to have an upward step \( u_{t-2} \) with a bar present at rank \( t-3 \), again by a parity argument: \( u_{t-2} \) would need to change a matchable interval from 1-merged to 1-split, meaning we would need a block of odd size at least 3 immediately above \( B_{t-2} \), which is not possible since there is a bar at rank \( t \). Thus, a directed cycle cannot get from the situation just after \( u_t \) to the situation needed just prior to \( d_t \), contradicting there being a cycle. \( \square \)

**Lemma 5.10.** If \( u_{i_k} \) and \( u_t \) both change matchable intervals from 2-merged to 2-split, then \( u_t \) specifically must split a block of size 4 into blocks of size 2, 2.

**Proof.** Let \( m \) be the size of the lowest matchable interval \( I_0 \) in \( C_0 \). The idea of this lemma is that if \( m-2 \geq 2 \), then the second largest element \( b \) in the block below the bar inserted by \( u_t \) is smaller than both labels \( c, e \) in the block above this same bar. We cannot insert a bar at rank \( t-1 \) while a bar is present at rank \( t \), and the label \( b \) will not move until we insert a bar at rank \( t-2 \). Until such a bar is inserted, all labels below \( b \) in the block containing \( b \) must be smaller than both \( c \) and \( e \), because they are smaller than \( b \).

We must eventually insert a bar at rank \( t-2 \), since such a bar is present just after \( u_{i_k} \). However, we cannot have a matching step inserting such a bar, under our \( m-2 \geq 2 \) assumption, since such a step inserting a bar into a block \( I \) would require \( S_C(I) \) to be even, and we can show that \( S_C(I) \) must be odd, as follows. We have that \( S_C(I_0) \) was even, and we check next that \( S_C(I) = S_C(I_0) + 1 \).

An interval above rank \( t+2 \) is a J-interval for \( I \) if and only if it is a J-interval for \( I_0 \), since in either case the only allowable inversion between such intervals and elements in \( I \) or \( I_0 \) is a single inversion with \( e \). Thus, \( S_C(I) = S_C(I_0) + 1 \), so \( S_C(I) \) cannot be even, a contradiction to \( m-2 \) being larger than 2. \( \square \)

6. **Vanishing homology and a dual Morse function**

**Theorem 6.1.** The \( h \)-complex \( \Delta_n \) has a discrete Morse function with \( m_i = 0 \) for \( 3i + 4 < n \), so \( \Delta_n \) is \( \lfloor \frac{n+2}{3} \rfloor \)-connected.

**Proof.** Theorem 5.1 proves that our matching is acyclic, and hence gives rise to a discrete Morse function whose critical cells are the unmatched face poset elements. Since any interval of size at least four is matchable, critical cells must have block sizes \( i_1, i_2, \ldots, i_j \leq 3 \) for \( i_1 + \cdots + i_j = n + 2 \) (recalling that we adjoined \( a_0 \) and \( a_{n+1} \), increasing permutation lengths to \( n + 2 \) letters). Hence, \( 3j \geq n + 2 \) for any unmatched face of dimension \( j-2 \), so \( m_j = 0 \) for \( 3j + 4 < n \), as desired. \( \square \)

Now we apply the Alexander duality of [4] result to deduce that there is also no reduced homology in the necessary top dimensions. Alternatively, this may be verified by dualizing our matching construction, as follows.

**Theorem 6.2.** \( \Delta_n \) has a discrete Morse function with no critical cells of dimension \( i \) for \( i > (2n-5)/3 \), so \( \tilde{H}_i(\Delta_n) = 0 \) for \( i > (2n-5)/3 \).

**Proof.** We reverse the roles of ascents and descents in the original matching. That is, break any permutation into maximal blocks of decreasing labels, and put bars at the locations of all the ascents in the permutation. Thus, bars are at the ranks which are absent in the associated minimal shelling face. We may use the same matching construction as before, but with respect to this new choice of bars and
blocks for each permutation. Since matching steps inserting a bar will now eliminate exactly one inversion, more specifically a descent, there is no problem with having the bars at the missing ranks rather than at the ranks present in a face. Now all the arguments of the previous sections go through unchanged. In conclusion, there are no critical cells with four or more consecutive decreasing labels, implying there are no critical cells above dimension \((2n - 5)/3\).

\[\square\]

**Question 6.3.** Is there a nice description of the permutations giving rise to critical cells? Do some nice subset of these index a homology basis? Can we further collapse to this basis by gradient path reversal?

### 7. Possible Generalizations

Peter McNamara recently showed in [9] that supersolvability for a lattice of rank \(n\) is equivalent to it having an EL-labeling in which each edge is labelled by an integer in \(\{1, \ldots, n\}\) in such a way that each saturated chain is labelled with a permutation in \(S_n\). He calls such an EL-labeling an \(S_n\) EL-labeling. Richard Stanley previously provided an \(S_n\) EL-labeling for every supersolvable lattice in [12]. It is shown in [4] that labellings known as SL-labelings (originally introduced in [1]) give \(h\)-shelling, and that supersolvable lattices have SL-labellings, namely their \(S_n\) EL-labelings.

**Question 7.1.** If \(\Delta\) is the \(h\)-complex of a supersolvable lattice of rank \(n\) whose Möbius function is nonzero on every interval, then is \(\Delta\) at least \(\lfloor \frac{2n-5}{3} \rfloor\)-connected?

It seems plausible that \(S_n\) EL-labelings might enable one to generalize the discrete Morse function of previous sections to other supersolvable lattices. The above Möbius function requirement ensures that every interval has at least one decreasing chain. This seems essential to a matching in which all chains which include blocks of size 4 or larger are indeed matched.

**Remark 7.2.** The lattice of subspaces of a finite-dimensional vector space over a finite field is probably easier than the general question of any supersolvable lattice with nowhere-zero Möbius function. Another specific candidate would be the intersection lattice of any supersolvable arrangement.

The following lemma from [7] seems likely to be helpful, in conjunction with a filtration by partially ordering Boolean algebras (e.g. apartments in the poset of subspaces of a finite vector space).

**Lemma 7.3 (Cluster Lemma).** Let \(\Delta\) be a regular CW complex which decomposes into collections \(\Delta_\sigma\) of cells indexed by the elements \(\sigma\) in a partial order \(P\) with unique minimal element \(\emptyset = \Delta_\emptyset\). Furthermore, assume that this decomposition is as follows:

1. \(\Delta\) decomposes into the disjoint union \(\bigcup_{\sigma \in P} \Delta_\sigma\), that is, each cell belongs to exactly one \(\Delta_\sigma\).
2. For each \(\sigma \in P\), \(\bigcup_{\tau \leq \sigma} \Delta_\tau\) is a subcomplex of \(\Delta\).

For each \(\sigma \in P\), let \(M_\sigma\) be an acyclic matching on the subposet \(F(\Delta|\Delta_\sigma)\) of \(F(\Delta)\) consisting of the cells in \(\Delta_\sigma\). Then \(\bigcup_{\sigma \in P} M_\sigma\) is an acyclic matching on \(F(\Delta)\).

Topologically, the order complex of a supersolvable lattice with nowhere-zero Möbius function will consist of overlapping spheres, specifically overlapping type A Coxeter complexes. We refer readers to [6] for a potentially useful way of viewing
those chains in a Boolean algebra that do not belong to any earlier Boolean algebra as an intersection of half-spaces restricted to a sphere.

**Remark 7.4.** The Alexander duality result of [4] will not apply to most supersolvable lattices with nowhere-zero Möbius function, since these will not in general be spheres. However, there could still be a dual discrete Morse function, similar to Theorem 6.2.

**Question 7.5.** Is there a more general lower bound on connectivity for $h$-complexes of SL-shellable posets whose Möbius function is nonzero on every interval?

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