A CONDITION FOR UNIQUE ERGODICITY OF QUADRATIC DIFFERENTIALS

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Abstract. We prove a geometric criterion for the horizontal foliation of a quadratic differential to have exactly one transverse invariant measure. Our theorem generalizes a similar result of Treviño for the horizontal flow on a translation surface [Tre14], as well as Masur’s criterion for unique ergodicity of the horizontal foliation [Mas92].

1. Introduction

A flat surface is a pair \((S, \Phi)\) where \(S\) is a Riemann surface and \(\Phi\) is a meromorphic quadratic differential on \(S\) with at most simple poles. The differential \(\Phi\) determines a flat Riemannian metric with conical singularities at the set \(\Sigma\) of zeroes and poles of \(\Phi\). We are most interested in the case when \((S, \Phi)\) has finite area in terms of the area form of \(\Phi\) (note that this forces any poles of \(\Phi\) to be simple). For example, if \(S\) is compact the \(\Phi\)-area will always be finite. In this case \(\Phi\) determines a Borel probability measure \(\lambda\) on \(S\), which we call Lebesgue measure. We would like to know if there are any other Borel probability measures naturally associated with \((S, \Phi)\).

Away from the zeroes or poles, we can write \(\Phi\) in local coordinates as \(\varphi(z)(dz)^2\) for some holomorphic function \(\varphi\). The kernel of \(\text{Im} \left( \sqrt{\varphi(z)} \, dz \right)\) determines a singular measured foliation of \(S\), which we call the horizontal foliation. Following [MS91], we say that a Borel probability measure \(\mu\) on \(S\) is ergodic for \(\Phi\) if for every \(\mu\)-measurable set \(E \subseteq S\) which is a union of leaves of the horizontal foliation, \(\mu(E) = 0\) or \(\mu(E) = 1\). A quadratic differential \(\Phi\) is uniquely ergodic if Lebesgue measure is the only ergodic measure.

If \(\alpha\) is an abelian differential on \(S\), then \((S, \alpha)\) is called a translation surface: away from zeroes of \(\alpha\), there exist charts for \(S\) whose transition maps are all translations. Since \(\alpha^2\) is a quadratic differential, we have a singular Riemannian metric and a Lebesgue measure on \((S, \alpha)\). Further, the kernel of \(\text{Im} \alpha\) determines a parallel unit vector field \(X\) on \(S \setminus \Sigma\). In this case the horizontal foliation of \(\alpha^2\) is given by a unit speed flow along \(X\), called the horizontal flow. This flow always preserves the Lebesgue measure. If \(\mu\) is an ergodic probability measure for the horizontal flow, then by observing that any measurable \(E\) which is a union of leaves is flow invariant, \(\mu(E) = 0\) or \(\mu(E) = 1\), justifying our definition of ergodicity for quadratic differentials.

There is a natural \(\text{SL}_2(\mathbb{R})\) action on the space of meromorphic quadratic differentials on \(S\), by pre-composing any chart in a flat atlas with the usual action of \(\text{SL}_2(\mathbb{R})\) on \(\mathbb{R}^2\).

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In particular, consider the subgroup of diagonal matrices:

\[ g_t = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix} \]

The action of \( g_t \) on \( \Phi \) determines a flow called the \textit{Teichmüller flow}. We are interested in proving a connection between the Teichmüller orbit of \( \Phi \) and the ergodic measures associated to \( \Phi \) in the compact case. This connection is based on the \textit{systole} of a flat surface, which is the length of the shortest simple closed curve on \((S, \Phi)\) which is not homotopically trivial. Our main theorem says that if the systole does not rapidly become small along the Teichmüller orbit of the surface, then the horizontal foliation is uniquely ergodic.

**Theorem 1.1.** Let \( \kappa(t) \) be the systole of the compact flat surface \((S, g_t \cdot \Phi)\). If:

(1) \[ \int_0^\infty (\kappa(t))^2 \, dt = \infty \]

then the horizontal foliation determined by \( \Phi \) is uniquely ergodic.

This theorem is based on work of Rodrigo Treviño, who proved a similar geometric criterion for unique ergodicity of the horizontal flow of an abelian differential \( \alpha \) [Tre14, Theorem 3].

**Theorem 1.2 (Treviño).** Let \( \kappa(t) \) be the systole of the compact translation surface \((S, g_t \cdot \alpha)\). If:

(2) \[ \int_0^\infty (\kappa(t))^2 \, dt = \infty \]

then the horizontal flow determined by \( \alpha \) is uniquely ergodic.

Theorem 1.1 implies Masur’s criterion for unique ergodicity, which is the case where \( \limsup_{t \to \infty} \kappa(t) > 0 \).

**Theorem 1.3 (Masur’s criterion [Mas92]).** If the horizontal foliation of a flat surface is minimal and not uniquely ergodic, then the Teichmüller orbit (of the class of that flat metric) leaves every compact set of the moduli space.

Theorem 1.1 also implies a result of Cheung and Eskin. Let \( \Delta_t \) denote the length of the smallest saddle connection on \((S, g_t \cdot \Phi)\), and let \( d(t) = -\log \Delta_t \). By an argument of Treviño [Tre14], we can choose \( \varepsilon = \frac{1}{2} \) in the following theorem.

**Theorem 1.4 (Cheung and Eskin [CE07]).** There is an \( \varepsilon > 0 \) such that if \( d(t) < \varepsilon \log t + C \) for some \( C \) and for all \( t > 0 \), then the horizontal foliation is uniquely ergodic.

Our argument for Theorem 1.1 proceeds in three steps: First, for every quadratic differential we construct a branched cover \( \tilde{p} : \tilde{S} \to S \) which "unfolds" \( \Phi \) to an abelian differential \( \alpha \). Statements about transverse measures associated to the horizontal foliation of \( \tilde{\Phi} \) can be restated in terms of the horizontal flow of \( \alpha \). Though these results are elementary, we need to develop them carefully for Treviño’s methods to apply to our situation.
We then prove Theorem 3.1 which is a geometric condition implying Lebesgue measure is ergodic for the horizontal foliation. This theorem heavily uses technology developed by Forni [For02]. Note that we cannot just move the geometry to the double cover and cite Treviño. Indeed, in many steps we will need to use the covering map to translate computations back and forth between the two surfaces we are working with. In the compact case, we can restate the theorem in terms of the systole alone, which would greatly simplify the statement. However, our theorem also guarantees ergodicity of the foliation in the non-compact case, which is of independent interest.

Finally, we show that in the compact case, the condition on the systole of a flat surface guarantees ergodicity of Lebesgue measure. Using the systole condition again, we can upgrade from ergodicity to unique ergodicity. The major difficulty is in constructing a map in the style of Veech [Vee78, Section 1] which normalizes a particular measure to the Lebesgue measure without distorting the geometry too much. We need to modify the known map in the abelian differential case so that it will work for our quadratic differentials.

In the last section, we provide a proof of our theorem when \( p : \hat{S} \to S \) is a genuine covering space using topological techniques that are not available in the general case. This second proof actually says something stronger: in this case, the horizontal flow "upstairs" is uniquely ergodic. While we do not have a counterexample, we do not think that the systole condition for the quadratic differential "downstairs" implies unique ergodicity of the horizontal flow "upstairs" in general. The essential issue seems to be that any branching in the double cover can generate new simple closed curves upstairs which are not controlled by the geometry downstairs. We do need \( p : \hat{S} \to S \) to be a covering map for this second proof to work, and so our theorem is not just a simple corollary of Treviño’s theorem except in special cases. However, unique ergodicity downstairs still places strong requirements on ergodic measures upstairs via the covering map.

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2. Definitions

Given a flat surface \((S, \Phi)\) we can construct its orientation double cover: there is a translation surface \((\hat{S}, \alpha)\) and a degree 2 branched covering map \( p : \hat{S} \to S \) so that \( p^* \Phi = \alpha^2 \). The branch points of \( p \) are the odd order zeroes or poles of \( \Phi \), and away from the branch points, \( p \) is a local isometry. \( \hat{S} \) is disconnected if and only if \( \Phi = \alpha^2 \) for some abelian differential \( \alpha \). See [MZ08, Figure 1] for an example. The translation surface \( \hat{S} \) comes with an involution, an isometry \( \iota : \hat{S} \to \hat{S} \) which exchanges \( \alpha \) and \(-\alpha\) and fixes the ramification points of \( p \). We can think of \((S, \Phi)\) as the orbifold quotient of \((\hat{S}, \alpha)\) by the involution \( \iota \).

**Definition 2.1.** We say that a Borel probability measure \( \mu \) on a flat surface \((S, \Phi)\) is an **invariant measure for the horizontal foliation** if there is an invariant Borel probability measure \( \nu \) for the horizontal flow on \((\hat{S}, \alpha)\) such that \( \mu = p_* \nu \). There is a simplex \( C \)
of invariant measures for the horizontal flow, and we say that an invariant measure is ergodic if it is an extreme point of the simplex $p_\ast C$.

The purpose of this definition is to ensure some kind of "flow invariance" for measures on $S$, even though we do not have a flow on $S$. We could also consider all transverse invariant measures. These gadgets give lengths to transversals to the horizontal foliation and are invariant under homotopy along the foliation, or holonomy.

**Proposition 2.2.** Every projection of a horizontal flow invariant measure on $\hat{S}$ determines a transverse invariant measure for the horizontal foliation on $S$, and vice versa.

**Proof.** Each projection $p_\ast \nu$ of an invariant measure $\nu$ for the horizontal flow of $\alpha$ determines a transverse invariant measure $\Upsilon$ for the horizontal foliation of $\Phi$:

\[
\Upsilon(\gamma) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} p_\ast \nu([0, \varepsilon] \cdot \gamma)
\]

where $[0, \varepsilon] \cdot \gamma$ is the set formed by flowing $\gamma$ along the foliation for all times between 0 and $\varepsilon$. As $\nu$ is flow-invariant upstairs, $\Upsilon$ will be a transverse invariant measure.

Conversely, given a transverse invariant measure $\Upsilon$ for the horizontal foliation of $\Phi$, we can construct an invariant measure $\mu$ on $S$. Indeed, if $\gamma$ is a small enough curve contained in a leaf of the vertical foliation of $\Phi$ in a neighborhood of a regular point, then we can move $\gamma$ by holonomy for all times in $[0, \varepsilon]$ for some small $\varepsilon > 0$ to create a rectangle $R$. We define $\mu(R) = \Upsilon(\gamma) \times \varepsilon$ for such rectangles $R$, which uniquely determines a finite Borel measure $\mu$ on $S$. Viewing $\hat{S}$ measurably as two copies of $S$, the measure $\nu = \frac{1}{2} (\mu + \iota_\ast \mu)$ is a finite measure on $\hat{S}$ with $p_\ast \nu = \mu$. Because the transverse measure $\Upsilon$ is invariant under holonomy, $\nu$ is an invariant measure for the horizontal flow on $\hat{S}$. 

Our definition of ergodicity reflects dynamical properties of the foliation, as the next proposition shows.

**Proposition 2.3.** The following are equivalent for an invariant measure $\mu$ for the horizontal foliation of a flat surface $(S, \Phi)$ of finite area.

1. $\mu$ is ergodic for the horizontal foliation.
2. For any $f \in L^2(\mu)$ such that $p^\ast f = f \circ p$ is invariant under the horizontal flow, $f$ is constant $\mu$-a.e.
3. For every measurable $E \subseteq S$ which is a union of leaves, $\mu(E) = 0$ or $\mu(E) = 1$.

The proof is standard.

3. **An Ergodicity Criterion**

We prove the following criterion for ergodicity of Lebesgue measure for a horizontal foliation. A slight strengthening in the next section will guarantee unique ergodicity.

**Theorem 3.1.** Let $(S, \Phi)$ be a flat surface, and $S_t = (S, \Phi_t) = (S, g_t \cdot \Phi)$ its Teichmüller flow. Suppose that for any $\eta > 0$ there exists a function $t \mapsto \varepsilon(t) > 0$, a one-parameter
family of subsets

\[ S_{\varepsilon(t),t} = \prod_{i=1}^{C_t} S^i_t \]

of \( S \) made up of \( C_t < \infty \) path connected components, each homeomorphic to a closed orientable surface with boundary, and functions \( t \to D^i_t > 0 \), for \( 1 \leq i \leq C_t \), such that for

\[ \Gamma^i_j = \{ \text{paths connecting } \partial S^i_t \text{ to } \partial S^j_t \} \]

and

\[ \delta_t = \min \sup_{i \neq j, \gamma \in \Gamma^i_j} \text{dist}_t(\gamma, \Sigma) \]

the following hold:

1. \( \lambda(S \setminus S_{\varepsilon(t),t}) < \eta \) for all \( t > 0 \) (where \( \lambda \) is Lebesgue measure).
2. \( \text{dist}_t(\partial S_{\varepsilon(t),t}, \Sigma) > \varepsilon(t) \) for all \( t > 0 \), measured with respect to the flat metric on \( (S, \Phi_t) \).
3. The diameter of each \( S^i_t \), measured with respect to the flat metric on \( (S, \Phi_t) \), is bounded by \( D^i_t \), and

\[ \int_0^\infty \left( \frac{1}{\varepsilon(t)^2} \sum_{i=1}^{C_t} D^i_t + \frac{C_t - 1}{\delta_t} \right)^{-2} \, dt = \infty \]

Also suppose the set of points whose leaves diverge and leave every compact subset of \( S \) has measure zero. Then Lebesgue measure is ergodic for the horizontal foliation of \( \Phi \).

The original ergodicity condition for the horizontal flow in [Tre14, Theorem 2] was designed for non-compact surfaces and needs to allow for \( \varepsilon(t) \to 0 \) as \( t \to \infty \). In the compact case, \( \varepsilon(t) \) can be chosen to be constant along the Teichmüller orbit (see the discussion by Equation 10), and the condition about divergent leaves is clearly true.

We would like to call attention to the different roles of the small parameters \( \eta, \varepsilon(t), \delta_t, \kappa(t) \) in further discussions. The number \( \eta \) is an arbitrary fixed positive number. Given \( \eta \), we construct \( \varepsilon(t) \) for the surface \( S_t \) by removing open sets of total measure at most \( \eta \) containing balls of radius \( \varepsilon(t) \) from the singular points of the surface. After removing these open sets, the resulting surface \( S \setminus S_{\varepsilon(t),t} \) is likely disconnected. The parameter \( \delta_t \) is a measure of how close we need to travel to a singularity if we want to travel in \( S \) between two connected components of \( S \setminus S_{\varepsilon(t),t} \). In general \( \delta_t \) is much smaller than \( \varepsilon(t) \), and reflects properties of the "thin" part of the surface. The systole \( \kappa(t) \) is related to \( \delta_t \) by a constant depending only on the stratum of the quadratic differential.

**Proof of Theorem 3.1.** Our proof follows the strategy for [Tre14, Theorem 2], with the additional complication of the orientation covering map. Fix \( (S, \Phi) \) as in the hypothesis and choose an abelian differential \( \alpha \) on \( \hat{S} \) with \( p^*\Phi = \alpha^2 \). Recall that when \( S_t \) is the Teichmüller geodesic flow of the surface by \( g_t \), the Lebesgue measure on \( S \) is preserved by \( g_t \). Hence \( L^2(S_t) = L^2(S) \) for all \( t \). Now we fix a function \( u \in L^2(S) \) such that \( p^*u \) is flow invariant. We assume that \( u \) is real valued and has zero average, and want to show that \( u \equiv 0 \) Lebesgue almost everywhere.
The computation \( u \equiv 0 \) proceeds by a series of lemmas. First, we will decompose \( u = m_t + h_t \), where \( m_t \) is a meromorphic function on \( S \). We will then show that \( m_t \) is zero, which is the longest step and where the geometry of our situation plays the largest role. Finally, we will show that \( h_t \) is zero using other properties of the geometry and dynamics of the foliation. This last step is where the additional assumptions in the noncompact case need to be used.

**Lemma 3.2.** The function \( u \) can be written as \( u = m_t + h_t \) in \( L^2(\tilde{S}_t) \), where \( m_t \) is meromorphic on \( S \) and \( h_t \) is orthogonal to \( m_t \). Further, \( p^*u = p^*m_t + \partial_t v_t \) for a differential operator \( \partial_t \) determined by \( \alpha_t \).

**Proof.** Let \( \mathcal{M} \) be the space of meromorphic functions in \( L^2(S) \). By using the Cauchy integral formula, one can show if a sequence of meromorphic functions converges in \( L^2(S) \), then the sequence converges uniformly on compact sets of \( S \setminus \Sigma \). Hence the \( L^2(S) \) limit of a sequence of meromorphic functions must be meromorphic, and so \( \mathcal{M} \) is a closed subspace of \( L^2(S) \). So there is a decomposition \( L^2(S) = \mathcal{M} \oplus \mathcal{M}^\perp \), and so \( u = m_t + h_t \) for unique \( m_t \in \mathcal{M} \) and \( h_t \in \mathcal{M}^\perp \).

Now observe that the linear map \( p^* : L^2(S_t) \rightarrow L^2(\tilde{S}_t) \) defined by \( p^* f = f \circ p \) isometrically embeds \( L^2(S) \) as a closed subspace of \( L^2(\tilde{S}) \). Indeed,

\[
\int_S |p^*f|^2 d\lambda = 2 \times \frac{1}{2} \int_S |f|^2 d(p_*\lambda)
\]

as \( p \) is a degree 2 covering map almost everywhere, we can view \( \tilde{S} \) measurably as two copies of \( S \) with half the measure. Hence \( \|p^*f\|_{L^2(\tilde{S})} = \|f\|_{L^2(S)} \). This shows that \( p^* \) is an isometry, and \( L^2(S) \) is closed in \( L^2(\tilde{S}) \) by completeness of \( L^2 \) spaces. Observe that if \( \iota \) is the involution of \( \tilde{S} \), then \( L^2(S) \) is an \( \iota^* \) invariant subspace of \( L^2(\tilde{S}) \).

On \( \tilde{S} \), away from the singularities we can find unit vector fields \( X, Y \) which are tangent to the horizontal, vertical foliations of \( \alpha^2 \) respectively. By a result of Forni, \( L^2(\tilde{S}_t) \) decomposes as the orthogonal direct sum of meromorphic functions on \( \tilde{S}_t = (\tilde{S}, g_t \cdot \alpha) \) and functions in the image of the anti-Cauchy-Riemann operator on \( \tilde{S} \) determined by \( g_t \cdot \alpha \):

\[
\partial_t = e^t X + ie^{-t} Y
\]

(see [For97 Proposition 3.2]). Hence we can write \( p^* u = \tilde{m}_t + \partial_t v_t \) in \( L^2(S) \), where the decomposition is again unique. By our previous computations we must also have \( p^*u = p^*m_t + p^*h_t \), with \( p^*m_t \) meromorphic on \( \tilde{S} \) and orthogonal to \( p^*h_t \). Hence by uniqueness of the decomposition, we must have \( \tilde{m}_t = p^*m_t \) and \( \partial_t v_t = p^*h_t \).

We assume without loss of generality that \( v_t \) is a function of zero average, since this assumption does not affect \( \partial_t v_t \). By this assumption, and because \( u \) is real valued, it follows that \( v_t \) is purely imaginary. Since \( p^*u \) is flow invariant, we have \( X(p^*u) = 0 \), so we also get the following fact about \( m_t \) from [Tre14 Lemma 2].

\[
\frac{d}{dt} \|p^* m_t\|_{L^2(S)}^2 = 4 \|\text{Im}(p^*m_t)\|_{L^2(\tilde{S})}^2
\]

\[6\]
We will use this differential equation and the decomposition of the surface $S$ to show that $\|m_t\|_{L^2(S)} = 0$.

Fix $\eta > 0$. For a fixed $t > 0$ we can decompose $m_t = R_t + iI_t$ into the real and imaginary parts. For a fixed $z \in p^{-1}(S_{\varepsilon(t), t})$, and a $\alpha_t$ disc of radius $R < \varepsilon(t)$ centered at $z$ in a small enough flat chart, we can use the Cauchy integral formula for derivatives to compute:

$$X_t(p^*I_t(z)) = \frac{1}{\pi} \int_0^{2\pi} \frac{\cos(s)}{R} p^*I_t(Re^{is}) \, ds$$

We can then integrate the above equation over a small annulus near $z$ to obtain the bound:

$$|X_t(p^*I_t(z))| \leq \frac{4}{\varepsilon(t)^2} \|p^*I_t\|_{L^2(\hat{S})}$$

The same bound holds for $Y_t(p^*I_t)$ by the same computation, which gives us the trivial bound:

$$\|\nabla_t p^*I_t\|_{L^\infty(p^{-1}(S_{\varepsilon(t), t}))} \leq \frac{8}{\varepsilon(t)^2} \|p^*I_t\|_{L^2(\hat{S})}$$

Here $\nabla_t$ is the gradient given by the flat metric $\alpha_t$. The same bound applies to $p^*R_t$ by the Cauchy-Riemann equations.

Now suppose $z \in S \setminus (S_{\varepsilon(t), t} \cup \Sigma)$. If $0 < \rho < \varepsilon(t)$ is the distance of $z$ to $\Sigma$, then similarly:

$$\|\nabla_t p^*R_t(z)\| \leq \frac{8}{\rho^2} \|p^*I_t\|_{L^2(\hat{S})}$$

The same bound holds for $p^*I_t$.

Since the branched cover $p$ is a local isometry by construction and $\|p^*f\|_{L^2(\hat{S})} = \|f\|_{L^2(S)}$, we see that these gradient bounds also apply to $I_t$ and $R_t$. We can use this fact to bound the norm of $m_t$ using the geometry of $S$.

**Lemma 3.3.** There is a constant $C > 0$ depending only on the stratum of the quadratic differential $\Phi$ so that for any points $a, b$ in $S_{\varepsilon(t), t}$ the following bound holds.

$$|R_t(a) - R_t(b)| \leq C \left( \frac{1}{\varepsilon(t)^2} \sum_{i=1}^{C_t} D_i^t + \frac{C_t - 1}{\delta_t} \right) \|I_t\|_{L^2(S)}$$

The same bound holds for $|I_t(a) - I_t(b)|$.

**Proof.** If $\Phi$ is holomorphic, let $\gamma$ be the unique flat geodesic on $S_t$ connecting $a$ and $b$ with minimal length. If $\Phi$ is meromorphic, then each of the at most simple poles of $\Phi$ lifts to a regular point of $\alpha$, so that $\alpha$ is a holomorphic abelian differential on $\hat{S}$. In this case, let $\beta$ be a flat geodesic on $\hat{S}$ which is the shortest among all geodesics connecting one of the two points of $p^{-1}(a)$ to one of the two points of $p^{-1}(b)$, which is possibly not unique. Let $\gamma = p(\beta)$, then $\gamma$ is a path from $a$ to $b$ on $S$. Either way, the curve $\gamma$ is a union of straight line paths between singularities or between an endpoint and a singularity. For existence of these minimal length paths, see [Str84, Chapter V, Section 18].

In the meromorphic case, the following bounds may need a factor of 2 coming from the covering map being degree 2: $\gamma$ might pass through the same subsurface of the
decomposition multiple times, but this will happen at most twice (as otherwise we could have chosen a shorter path $\beta$ on $\tilde{S}$). This does not change the result of the lemma.

Based on our gradient computations, we arrive at the following bound:

\begin{equation}
|R_t(z_i) - R_t(z_j)| \leq 8\|I_t\|_{L^2(S)} \int_0^1 \frac{ds}{(\text{dist}_t(\gamma(s), \Sigma))^2} \tag{6}
\end{equation}

We are interested in bounding the integral. There are three parts of the curve $\gamma$ we need to consider. First is the part of $\gamma$ in the "thick" part of the surface: let $\text{Thick}$ be the set of $s$ where $\text{dist}_t(\gamma(s), \Sigma) \geq \varepsilon(t)$. Here we do a trivial bound: the length of $\gamma$, since it was chosen to have minimal length, is less than the sum of the lengths $D_i$ of diameters of each of the subsurfaces of the thick part, and the maximum value of the function we are integrating is $(\varepsilon(t))^{-2}$. So we have the bound:

\begin{equation}
\int_{\text{Thick}} \frac{ds}{(\text{dist}_t(\gamma(s), \Sigma))^2} \leq \frac{1}{\varepsilon(t)^2} \sum_{i=1}^{C_t} D_i \tag{7}
\end{equation}

Now we want to bound the integral over the thin part. Here the geodesic might pass through singularities $\sigma \in \Sigma$, so we want to modify our curve $\gamma$ to avoid the singularities. By the definition of $\delta_t$, we know that after removing the open ball of radius $\delta_t$ about each $\sigma \in \Sigma$, the resulting surface is still path connected. Call this new surface $S_t \setminus D$. Then $\gamma \cap S_t \setminus D$ consists of at most $|\Sigma| + 1$ connected components, since we removed at most $|\Sigma|$ disjoint open sets from $S_t$. By choice of $\gamma$, this holds for the domain of the parametrized curve as well as its image.

To connect each of these components of $\gamma$, we can travel along the boundary of $S_t \setminus D$ at a distance $\delta_t$ from $\Sigma$. The worst case scenario for connecting two components of $\gamma \cap S_t \setminus D$ is travelling around the circumference of all of the open balls surrounding every singularity to reach the next component of $\gamma \cap S_t \setminus D$, which must start at a distance $\delta_t$ from a singularity. If $2\pi c > 0$ is the maximum angle around a conical singularity $\sigma$, then for each connection between components the worst case scenario is travelling a distance $|\Sigma| \times 2\pi c \delta_t$. We have to do this process at most $|\Sigma|$ times, since there are $|\Sigma| + 1$ components needing to be connected. In this way we obtain a new curve connecting $a$ and $b$, which is a union of geodesic segments and circular arcs. We will call this new curve $\tilde{\gamma}$. See Figure 1 for a schematic of how to obtain $\tilde{\gamma}$ from $\gamma$.

Let $\text{Thin}_1$ be the set of $s$ where $\tilde{\gamma}(s)$ is a distance $\delta_t$ from the singularity. Since all of the previous work is for $\text{Thin}_1$, we use the trivial bound to obtain an upper bound on the first thin part:

\begin{equation}
\int_{\text{Thin}_1} \frac{ds}{(\text{dist}_t(\gamma(s), \Sigma))^2} \leq \left( \frac{2\pi c |\Sigma|^2 \delta_t}{\delta_t^2} \right) = \left( \frac{2\pi c |\Sigma|^2}{\delta_t} \right) \tag{8}
\end{equation}

The third part of the integral is the set $\text{Thin}_2$ of $s$ so that $\delta_t < \text{dist}_t(\gamma(s), \Sigma) < \varepsilon(t)$, which is still within the thin part but not controlled by our other estimate. First, we split up the distance from the singularities with a trivial bound:

\[
\frac{1}{(\text{dist}_t(\tilde{\gamma}(s), \Sigma))^2} \leq \sum_{\sigma \in \Sigma} \frac{1}{(\text{dist}_t(\tilde{\gamma}(s), \sigma))^2}
\]
Figure 1. Left: the geodesic $\gamma$ connecting $a$ to $b$, which passes through the "thin" part determined by $\epsilon(t)$. Right: the modified curve $\tilde{\gamma}$, which stays at a distance at least $\delta_t$ from all singularities $\sigma$.

For a fixed singularity $\sigma$ and $k \geq 0$, consider the open annuli $A_k(\sigma) = \text{ann}(\sigma, 2^k\delta_t, 2^{k+1}\delta_t)$. The intersection of $\tilde{\gamma}(s)$ with $A_k(\sigma)$ is a union of line segments, whose intersection with the annulus has length less than the diameter $2 \times 2^{k+1}\delta_t$. Since the distance to the singularity is at most $2^k\delta_t$, the contribution of $A_k(\sigma)$ to the integral using the trivial bound is at most:

$$\frac{2^{k+2}\delta_t}{(2^k\delta_t)^2} \leq \frac{4}{2^k\delta_t}$$

Hence for one singularity, we can bound the integral by the sum of the previous bound over all annuli.

$$\int_{\text{Thin}_2} \frac{|ds|}{(\text{dist}_t(\tilde{\gamma}(s), \sigma))^2} \leq \sum_{k=0}^{\infty} \frac{4}{2^k\delta_t} = \frac{8}{\delta_t}$$

Summing over all singularities, we obtain the following bound on the third and final part of the integral:

(9) $$\int_{\text{Thin}_2} \left| \frac{ds}{(\text{dist}_t(\gamma(s), \Sigma))^2} \right| \leq \left( \frac{8|\Sigma|}{\delta_t} \right)$$

Combining (7), (8), and (9), we obtain the desired bound on (6).

$$|R_t(z_i) - R_t(z_j)| \leq 8\|I_t\|_{L^2(S)} \left( \frac{1}{\epsilon(t)^2} \sum_{i=1}^{C_t} D_i^i + \frac{2\pi c|\Sigma|^2}{\delta_t} + \frac{8|\Sigma|}{\delta_t} \right)$$

$$\leq C \left( \frac{1}{\epsilon(t)^2} \sum_{i=1}^{C_t} D_i^i + \frac{C_t - 1}{\delta_t} \right) \|I_t\|_{L^2(S)}$$

Since $C$ depends only on the number of singularities and the maximum cone angle of a singularity, it depends only on the stratum of $\Phi$.

\[\Box\]

Lemma 3.4. The meromorphic functions $m_t$ are zero.
Proof. Now consider \( p^* m_t = p^* R_t + i p^* I_t \). If \( \zeta_1, \zeta_2 \in p^{-1}(S_{\zeta(t), t}) \), then we observe each \( p(\zeta_k) \) is some \( z_k \in S_{\zeta(t), t} \), so that by definition of \( p^* R_t \) and Lemma 3.3:

\[
|p^* R_t(\zeta_1) - p^* R_t(\zeta_2)| \leq C \left( \frac{1}{\varepsilon(t)^2} \sum_{i=1}^{C_t} D_i^i + \frac{C_t - 1}{\delta_t} \right) \|I_t\|_{L^2(S)}
\]

The same bound holds for \( p^* I_t \). Also observe that the Lebesgue measure of \( p^{-1}(S_{\zeta(t), t}) \) is at least \( 1 - \eta \) (as Lebesgue downstairs is the pushforward by \( p \) of Lebesgue upstairs).

Observe that we must have:

\[
\liminf_{t \to \infty} \left( \frac{1}{\varepsilon(t)^2} \sum_{i=1}^{C_t} D_i^i + \frac{C_t - 1}{\delta_t} \right) \|p^* I_t\|_{L^2(\tilde{S})} = 0
\]

If not, then as in [Tre14, Proof of Theorem 2] there would be some constant \( c \) so that for all \( t \):

\[
0 < c < \left( \frac{1}{\varepsilon(t)^2} \sum_{i=1}^{C_t} D_i^i + \frac{C_t - 1}{\delta_t} \right) \|p^* I_t\|_{L^2(\tilde{S})}
\]

Rearrange, square both sides, and integrate:

\[
\infty = c^2 \int_0^\infty \left( \frac{1}{\varepsilon(t)^2} \sum_{i=1}^{C_t} D_i^i + \frac{C_t - 1}{\delta_t} \right)^{-2} dt < \int_0^\infty \|p^* I_t\|_{L^2(\tilde{S})}^2 dt \leq \|p^* u\|_{L^2(\tilde{S})} < \infty
\]

The first equality is by hypothesis and the third inequality follows from the differential equation (5) by integrating the imaginary part of \( p^* m_t \). This is a contradiction. Hence there must be arbitrarily large values of \( t \) so that:

\[
\|p^* m_t\|_{L^\infty(p^{-1}(S_{\zeta(t), t}))} < \eta
\]

Let \( \tau \) be one of these values of \( t \). Then we can compute:

\[
\|p^* m_\tau\|_{L^2(\tilde{S})}^2 = \int_{p^{-1}(S_{\zeta(\tau), \tau})} p^* m_\tau p^* u \, d\lambda + \int_{S_{\zeta(\tau), \tau}} p^* m_\tau p^* u \, d\lambda
\]

(note that because \( u \) is written as an orthogonal sum \( u = \partial_v \lambda + m_\tau \), any terms involving \( \partial_v \lambda \) vanish in the inner product, which leads to this integral). The idea of the bound is the following: the first integral is over a good set where \( p^* m \) is small, and the second integral is over a bad set of small measure. More precisely:

\[
\|p^* m_\tau\|_{L^2(\tilde{S})}^2 \leq \|p^* m_t\|_{L^\infty(p^{-1}(S_{\zeta(t), t}))} \|p^* u\|_{L^1(\tilde{S})} + \|p^* u\|_{L^\infty(\tilde{S})} \int_{S_{\zeta(\tau), \tau}} |p^* m_\tau| \, d\lambda
\]

\[
\leq \eta \|p^* u\|_{L^1(\tilde{S})} + \|p^* u\|_{L^\infty(\tilde{S})} (\lambda(p^{-1}(S \setminus S_{\tilde{\zeta}(\tau), \tau})))^{1/2} \|p^* m_\tau\|_{L^2(\tilde{S})}
\]

\[
\leq \sqrt{\eta} \left( \sqrt{\eta} + \|p^* u\|_{L^\infty(\tilde{S})} \right) \|p^* u\|_{L^2(\tilde{S})}
\]

Hence we can bound \( \|p^* m_\tau\|_{L^2(\tilde{S})} < \eta \) for arbitrarily large values of \( \tau \). Again by the differential equation (5), we see that \( p^* m_t \) and hence \( m_t \) must be zero for all \( t \).

\( \square \)
Since $m_t = 0$, it suffices to show that $\partial_t v_t \equiv 0$ to conclude $u \equiv 0$ almost everywhere. By the argument in [Tre14, Proof of Theorem 2], it suffices to show that the horizontal flow on $\hat{S}_t$ is recurrent to conclude that $\partial_t v_t = 0$.

First, observe that there are no horizontal cylinders in $(S, \Phi)$. If there was a horizontal cylinder, then it would intersect every neighborhood of a singularity nontrivially. Then at least one of $\varepsilon(t)$ and $\delta(t)$ would be forced to decay exponentially by our hypothesis on the measure of $S \setminus S_{\varepsilon(t),t}$, contradicting divergence of the integral. A horizontal cylinder in $\hat{S}$ would project to one on $S$, so there are no cylinders upstairs either.

Secondly, consider the set $B \subseteq S_t$ of points in divergent leaves, and $\hat{B} \subseteq \hat{S}_t$ of points in divergent flow trajectories. Observe that $\hat{B}$ has measure zero. Indeed, let $\{K_n\}_{n=1}^\infty$ be a compact exhaustion of $S_t$. Since every point on $S_t$ has either one or two pre-images under $p$ and $p$ is continuous, $\{p^{-1}(K_n)\}_{n=1}^\infty$ is a compact exhaustion of $\hat{S}_t$. So if a flow trajectory $L$ leaves every compact set of $\hat{S}_t$, it must leave each of the $\{p^{-1}(K_n)\}_{n=1}^\infty$. Then $p(L)$ must leave every compact set $K_n$, and so every compact set of $S_t$ since the $K_n$ form a compact exhaustion. So $\hat{B} \subseteq p^{-1}(B)$. Since Lebesgue measure on $S_t$ is the push-forward by $p$ of Lebesgue measure on $\hat{S}_t$, and $B$ has measure zero, it follows that $\hat{B}$ also has measure zero.

As there are no horizontal cylinders and the set of divergent flow orbits on $\hat{S}_t$ has measure zero, the horizontal flow on $\hat{S}_t$ is recurrent. Hence Lebesgue measure is ergodic for the horizontal foliation of $\Phi$. 

\[ \square \]

4. A Unique Ergodicity Criterion

Proof of Theorem 1.1 First we apply Theorem 3.1 Let $\varepsilon(t) = \varepsilon_0$ for some sufficiently small constant $\varepsilon_0 > 0$, and define the surface decomposition by $S_{\varepsilon(t),t} = \{z \in S : \text{dist}_t(z, \Sigma) \geq \varepsilon_0\}$. Note that the number of components of the decomposition is uniformly bounded in time by a constant which depends only on the strata of $\Phi$. By [MS91 Corollary 5.6], there is a constant $K$ so that if $D_t$ is the diameter of $S_t$ and $\kappa(t)$ is the systole, and $D_t > \sqrt{2/\pi}$, then:

\[ D_t \leq \frac{K}{\kappa(t)} \]

In addition, the quantity $\delta_t$ of Theorem 3.1 is at least $\kappa(t)/(2|\Sigma|)$. Indeed, the worst case scenario is if $\kappa(t)$ is the core circle of a cylinder passing through all singularities, evenly spaced around the curve. In this case, passing halfway between two singularities gives $\delta_t = \kappa(t)/(2|\Sigma|)$. By a computation, the above implies that the decomposition satisfies the hypothesis of Theorem 3.1 and hence the horizontal foliation given by $\Phi$ is ergodic with respect to Lebesgue measure.

To upgrade to unique ergodicity, we use an argument of Veech [Vee78, Section 1]. Suppose that there is another measure $\mu$ which is invariant for the horizontal foliation of $\Phi$. Then $\mu$ is non-atomic, since the foliation must be minimal by the systole condition, and must be mutually singular with respect to the ergodic Lebesgue measure $\Lambda_\Phi$. For any fixed $s \in (0,1)$, the measure $\mu(s) = s\lambda_\Phi + (1-s)\mu$ is invariant for the horizontal
foliation. If we can show that $\mu(s)$ is ergodic, then we have a contradiction because it is not extreme in the simplex of invariant measures.

On the double cover $(\hat{S}, \alpha)$, there is the Lebesgue measure $\lambda_\alpha$ and a measure $\hat{\mu}$ which is flow invariant such that $p_*\hat{\mu} = \mu$. We can also request that $\iota_*\hat{\mu} = \hat{\mu}$: if $\nu$ is flow invariant and $p_*\nu = \mu$, then consider the measure $\iota_*\nu$. By definition of the projection map $p$, we must have $p_*\iota_*\nu = \mu$. As the horizontal flow commutes with the involution $\iota$, $\iota_*\nu$ is also flow invariant. So $\frac{1}{2}(\nu + \iota_*\nu)$ is a flow invariant measure which projects to $\mu$ and is fixed by the involution.

Define $\hat{\mu}(s) = s\lambda_\alpha + (1-s)\hat{\mu}$, so that $p_*\hat{\mu}(s) = \mu(s)$. Then by [Tre14, Proof of Theorem 3] there exists a homeomorphism $\hat{F}_s$ from $\hat{S}$ to itself such that:

1. $\hat{F}_s$ induces a new translation structure on $\hat{S}$.
2. $\hat{F}_s$ preserves the horizontal foliation on $\hat{S}$ and fixes the singularities.
3. The new translation structure is induced by a unique abelian differential $\alpha(s)$ with the same horizontal foliation as $\alpha$.
4. $\hat{\mu}(s)$ pushes forward to the Lebesgue measure of $\alpha(s)$:

\[ (\hat{F}_s)_*\hat{\mu}(s) = \lambda_{\alpha(s)} \]

5. If $\gamma$ is a homotopically nontrivial simple closed curve or a saddle connection on $\hat{S}$, then $\hat{F}_s(\gamma)$ satisfies:

\[ \text{length}_{\alpha(s)}(\hat{F}_s(\gamma)) \geq s \cdot \text{length}_{\alpha}(\gamma) \]

Define an involution $\sigma_s$ on $(\hat{S}, \alpha(s))$ by $\sigma_s = \hat{F}_s \circ \iota \circ \hat{F}_s^{-1}$. Modding out $(\hat{S}, \alpha(s))$ by the action of $\sigma_s$ produces an orbifold $S'$ with a branched cover $p' : \hat{S} \to S'$. The map $\hat{F}_s$ descends to a homeomorphism $F_s : S \to S'$ such that $(F_s)_*\mu(s) = p'_*\lambda_{\alpha(s)}$. In fact, $\sigma_s$ is an isometry of $(\hat{S}, \alpha(s))$ sending $\alpha(s)$ to $-\alpha(s)$, and so sends leaves of the horizontal flow to leaves. To show this, we recall that $\hat{F}_s$ is defined on $\hat{S}$ away from singularities of $\alpha$ in local coordinates centered at 0 as:

\[ \hat{F}_s(x + iy) = x + \text{sign}(y) \cdot i\Upsilon_s(\ell_y) \]

where $\ell_y$ is the line segment between 0 and $y$. Here $\Upsilon_s$ is a transverse invariant measure defined by:

\[ \Upsilon_s(\gamma) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \hat{\mu}(s)([0, \varepsilon] \cdot \gamma) \]

where $[0, \varepsilon] \cdot \gamma$ denotes the horizontal flow of the curve $\gamma$ for all times between 0 and $\varepsilon$. Now, since $\hat{\mu}(s)$ was chosen to be involution invariant, we see that $\Upsilon_s(\gamma) = \Upsilon_s(\iota(\gamma))$. A computation in local coordinates then shows that $\sigma_s$ is an isometry of $(\hat{S}, \alpha(s))$.

Hence $S'$ has a Riemann surface structure, and in fact a flat surface structure from a quadratic differential $\Phi(s)$ such that $(p')^*(\Phi(s)) = \alpha(s)^2$. Observe also that the homeomorphism $F_s$ takes leaves of the horizontal foliation on $(S, \Phi)$ to leaves on $(S', \Phi(s))$ because $\hat{F}_s$ takes leaves to leaves. By our previous observations we have $(F_s)_*\mu(s) = \lambda_{\Phi(s)}$, so that $F_s$ takes our convex combination to Lebesgue measure.
Now consider the systoles on $S_t$ and $S'_t$. We claim their lengths are related by:

\[(15) \quad \kappa_{S'}(t) \geq s \kappa_S(t)\]

Indeed, let $\gamma$ be any simple closed nontrivial curve on $S_t$, which corresponds uniquely to a curve $F_s(\gamma)$ on $S'_t$. Up to removing the singularities on $S_t$ and $\hat{S}_t$, the map $p$ is a covering map, so we can lift $\gamma$ to a finite union of curves $\beta_i$ on $\hat{S}_t$. See Figure 2 for a schematic of the lifting. By (12), $\text{length}_{g_t}(\hat{F}_s(\beta_i)) \geq s \cdot \text{length}_{g_t}(\beta_i)$. By uniqueness of lifts, the union of the $\hat{F}_s(\beta_i)$ is a lift of $F_s(\gamma)$. Since the covering maps are isometries restricted to the $\beta_i$, we compute:

$$\text{length}_{g_t\cdot \Phi(s)}(F_s(\gamma)) = \sum_i \text{length}_{g_t\cdot \alpha(s)}(\hat{F}_s(\beta_i)) \geq s \sum_i \text{length}_{g_t\cdot \alpha}(\beta_i) = s \cdot \text{length}_{g_t\cdot \Phi}(\gamma)$$

which proves our claim.

By (15), Theorem 3.1 applies to $(S', \Phi(s))$:

$$\int_0^\infty (\kappa_{S'}(t))^2 \, dt \geq s^2 \int_0^\infty (\kappa_S(t))^2 \, dt = \infty$$

So the horizontal foliation of $(S', \Phi(s))$ is ergodic with respect to Lebesgue measure. But this is impossible, since by our construction Lebesgue measure $\lambda_{\Phi(s)}$ is a nontrivial convex combination of two invariant measures. Hence Lebesgue measure on $S$ must have been the only invariant measure to begin with, and so the horizontal foliation is uniquely ergodic.

\[\square\]

5. Covering Spaces

If the quadratic differential $\Phi$ has no odd order zeroes or poles, then $p : \hat{S} \to S$ is a genuine covering space. In this case there is a proof of unique ergodicity given Treviño’s criterion which is much shorter than our proof in the general case. The theorem is a consequence of a proposition about lengths of curves. The necessary covering space theory can be found in [Hat02, Chapter 1].
Proposition 5.1. Let \((M, \Phi)\) be a flat surface and \(P : \hat{M} \to M\) a finite degree covering map. For every simple closed curve \(\gamma\) on \(\hat{M}\) which is not homotopically trivial, there is a simple closed nontrivial curve \(\beta\) on \(M\) with \(\text{length}(\gamma) \geq \text{length}(\beta)\).

Proof. Observe that if the covering map has finite degree, then \(\hat{M}\) is also compact, so \((\hat{M}, P^*\Phi)\) is a flat surface. Let \(\gamma : [0, 1] \to \hat{M}\) be a piecewise smooth simple closed curve, and \(P\gamma\) its projection to \(M\). Covering space theory tells us that \(P_* : \pi_1(\hat{M}) \to \pi_1(M)\) is an injective map. Hence if \([\gamma] \neq 1 \in \pi_1(\hat{M})\), then \([P\gamma] \neq 1 \in \pi_1(M)\). Since \(P\) is a local isometry for the given flat structures, and the measurement of length is a local computation, we observe that, as parametrized curves, \(\text{length}(\gamma) = \text{length}(P\gamma)\). We can view \(\gamma\) as a map \(S^1 \to \hat{M}\), and \(\gamma\) being simple tells us that we can assume this map is injective. If \(P\gamma : S^1 \to M\) is injective, then \(\text{length}(\gamma) = \text{length}(P\gamma)\) as curves, and so we can take \(\beta = P\gamma\).

For the case when \(P\gamma\) is not injective, we find a subcurve which is injective. Since \(P\) is a degree 2 covering map and \(\gamma\) is injective, it follows that for each \(t_0 \in [0, 1]\), there is at most one other time \(t_1 \in [0, 1]\) with \(P\gamma(t_0) = P\gamma(t_1)\). So if \(P\gamma\) is not injective as a map from \(S^1\), there is some minimal \(s \in (0, 1)\) such that \(P\gamma(s) = P\gamma(r)\) for some \(0 \leq r < s < 1\). Define \(\beta\) to be the restriction of \(P\gamma\) to \([r, s]\). Minimality of \(s\) means that \(\beta\) is injective as a map of \([r, s]\) into \(M\), so the curve is simple, and closed because \(P\gamma(r) = P\gamma(s)\) by construction. Since \(\beta\) is the restriction of \(P\gamma\) we see that \(\text{length}(\gamma) = \text{length}(P\gamma) \geq \text{length}(\beta)\). Since \(\gamma\) and \(\beta\) are both injective maps, these lengths are their lengths as curves.

Finally, \(\beta\) is homotopically nontrivial. Indeed, consider the universal cover \(\tilde{M}\) of \(\hat{M}\), and let \(Q : \tilde{M} \to \hat{M}\) be the covering map. Since \(\hat{M}\) is a cover of \(M\), \(\tilde{M}\) is also the universal cover of \(M\). For any \(m \in Q^{-1}(\gamma(0))\), there is a unique lift \(\tilde{\gamma}\) of \(\gamma\) with \(\tilde{\gamma}(0) = m\). Observe that \(\tilde{\gamma}\) is also a lift of \(P\gamma\). Since we assumed that \(\gamma\) was an injective map on \(S^1\), we must have \(\tilde{\gamma} : [0, 1] \to \hat{M}\) is also injective, where \(\tilde{\gamma}(0) \neq \tilde{\gamma}(1)\) follows because \(\gamma\) is homotopically nontrivial. Since \(\tilde{\gamma}\) is injective, we see that \(\tilde{\gamma}(r) \neq \tilde{\gamma}(s)\) in the universal cover but \(P\tilde{\gamma}(r) = P\tilde{\gamma}(s)\). This proves that \(\beta\) is homotopically nontrivial.

Corollary 5.2. (1) If \((S, \alpha)\) is a translation surface satisfying Treviño’s criterion, then given a finite degree covering map \(p : R \to S\), the translation surface \((R, p^*\alpha)\) also satisfies Treviño’s criterion (and hence the horizontal flow given by \(p^*\alpha\) is uniquely ergodic).

(2) If \((S, \Phi)\) satisfies Treviño’s criterion and its orientation double cover is a covering space, then the horizontal flow on its double cover \((\hat{S}, \alpha)\) is uniquely ergodic (and hence the horizontal foliation given by \(\Phi\) is uniquely ergodic).

Proof. The proofs of both corollaries are essentially the same. For the translation surface case, the proposition shows that the systoles on the covering space \(R\) must satisfy:

\[
\int_0^\infty \kappa_R(t) \, dt \geq \int_0^\infty \kappa_S(t) \, dt = \infty
\]

which by Theorem \([1, 2]\) implies that the horizontal flow on \(R\) is uniquely ergodic.
For the flat surface case take \( R = \hat{S} \). Since the systoles given by \((\hat{S}, \alpha^2)\) are the same as the systoles given by \((\hat{S}, \alpha)\), we see by \([16]\) that the horizontal flow on \((\hat{S}, \alpha)\) is uniquely ergodic. By definition the simplex of invariant measures for \((S, \Phi)\) contains only the Lebesgue measure as an extreme point, so the horizontal foliation on \((S, \Phi)\) is uniquely ergodic.

\[\square\]

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