PREHOMOGENEOUS MODULES OF COMMUTATIVE LINEAR ALGEBRAIC GROUPS

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Abstract. Let $A$ be a finite dimensional commutative associative algebra with unit over an algebraically closed field of characteristic zero. The group $G(A)$ of invertible elements is open in $A$ and thus $A$ has a structure of a prehomogeneous $G(A)$-module. We show that every prehomogeneous module of a commutative linear algebraic group appears this way. In particular, the number of equivalence classes of prehomogeneous $G$-modules is finite if and only if the corank of $G$ is at most 5.

1. Introduction

Let $G$ be a linear algebraic group over an algebraically closed field $K$ of characteristic zero. A finite dimensional $G$-module $V$ is called prehomogeneous if the linear action $G \times V \to V$ is effective and has an open orbit in $V$. Prehomogeneous modules play an important role in geometry, number theory and analysis, as well as representation theory.

For connected simple algebraic groups prehomogeneous modules were classified by Vinberg [30]. A classification of irreducible prehomogeneous modules for connected reductive algebraic groups was obtained by Sato and Kimura [25] and Schpiz [27]. For more recent results on this subject, see [16] and references therein.

In this paper we study prehomogeneous modules of commutative linear algebraic groups. Denote by $\mathbb{G}_m$ the multiplicative group and by $\mathbb{G}_a$ the additive group of the ground field $K$. It is well known that any connected commutative linear algebraic group $G$ is isomorphic to $(\mathbb{G}_m)^r \times (\mathbb{G}_a)^s$ with some non-negative integers $r$ and $s$, see [15, Theorem 15.5]. We say that $r$ is the rank and $s$ is the corank of the group $G$.

It is an important problem to describe regular actions of a commutative linear algebraic group $G$ on algebraic varieties $X$ with an open orbit. If $s = 0$ then $G$ is a torus and we come to the classical theory of toric varieties, see [10, 21, 13, 9]. The case $s = 1$ is studied in [3]. It turns out that the variety $X$ in this case is toric as well, and $G$-actions on $X$ with an open orbit are determined by Demazure roots of $X$.

Another extreme $r = 0$ corresponds to embeddings of commutative unipotent (=vector) groups. This case is studied actively during last decades, see [14, 26, 1, 6, 11, 12, 5].

The aim of this paper is to study linear $G$-actions with an open orbit and to give linearizability criteria for some classes of actions of commutative groups on affine spaces. The paper is organized as follows. In Section 2 we discuss preliminary results on prehomogeneous modules of commutative linear algebraic groups. Section 3 contains basic facts on finite dimensional algebras. We recall Hassett-Tschinkel correspondence between actions of

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commutative unipotent groups on projective spaces with an open orbit and local finite dimensional algebras. Also we list all 42 local algebras of dimension up to 6. The classification is taken from [20, Section 2]. It seems to be not widely known.

In Section 4 we show that every prehomogeneous module $V$ of a commutative linear algebraic group $G$ is isomorphic to the $G(A)$-module $A$, where $A$ is a finite dimensional commutative associative algebra $A$ with unit and $G(A)$ is its group of invertible elements (Theorem 1). This result implies that the number of equivalence classes of prehomogeneous $G$-modules is finite if and only if the corank of the group $G$ is at most 5 (Corollary 1). We prove that the number of prehomogeneous modules with a finite number of orbits and the number of prehomogenous modules such that the acting group is normalized by all invertible diagonal matrices are finite for every commutative linear algebraic group $G$ (Corollaries 2 and 3). Such modules allow an explicit description in terms of the corresponding finite dimensional algebras. Proposition 4 shows that every cyclic module of a commutative linear algebraic group is obtained from a prehomogeneous module of a bigger commutative group by restriction to an action of a subgroup.

In Section 5 we deal with prehomogeneous modules in the framework of the theory of affine algebraic monoids and group embeddings. Consider an affine algebraic monoid $S$ isomorphic as a variety to an affine space. Proposition 6 claims that the monoid $S$ is the multiplicative monoid of a finite dimensional algebra if and only if the action of the group $G(S) \times G(S)$ on $S$ by left and right multiplication is linearizable. It leads to an alternative proof of Theorem 1.

In the last section we consider additive actions on toric varieties $X$, i.e., regular actions $G_a \times X \to X$ with an open orbit. Lifting such actions to the spectrum of the Cox ring of the variety $X$ we obtain actions of a commutative linear algebraic group $G$ on $A^n$ with an open orbit. We show that such actions are linearizable if and only if $X$ is a big open toric subset of a product of projective spaces (Proposition 7).

In a forthcoming paper we plan to study non-linearizable actions of commutative linear algebraic groups on $A^n$ with an open orbit or, equivalently, commutative monoid structures on affine spaces that do not correspond to finite dimensional algebras.

Important Comment. When the first version of this text has appeared, Professor Friedrich Knop pointed our attention to paper [18]. Proposition 5.1 of [18] is equivalent to our Theorem 1 and the result is proved there over an arbitrary field.

2. Preliminaries

Let $V$ be a finite dimensional vector space and $G$ a closed subgroup of the group $\text{GL}(V)$. We say that $V$ is a prehomogeneous $G$-module if the induced $G$-action on $V$ has an open orbit. A $G$-module $V$ is equivalent to a $G'$-module $V'$ if there exists an isomorphism of vector spaces $V$ and $V'$ such that the induced isomorphism of the groups $\text{GL}(V)$ and $\text{GL}(V')$ identifies the subgroups $G$ and $G'$.

Lemma 1. If $V$ is a prehomogeneous module of a commutative linear algebraic group $G$, then $\dim V = \dim G$.

Proof. By definition, the module $V$ contains an open $G$-orbit $O$. The stabilizer of a point on $O$ acts trivially on $O$ and hence on $V$. Since the group $G$ acts on $V$ effectively, the action of $G$ on $O$ is free, and $\dim V = \dim O = \dim G$. \hfill $\square$

Remark 1. For a commutative linear algebraic group $G$ there may exist a faithful $G$-module $V$ with $\dim V < \dim G$. For example, take $G = \mathbb{G}_m \times (\mathbb{G}_a)^n$ and its representation given
by \((2n \times 2n)\)-matrices of the form
\[
\begin{pmatrix}
\lambda E & A \\
0 & \lambda E
\end{pmatrix}, \text{ where } \lambda \in \mathbb{G}_m \text{ and } A \in \text{Mat}(n \times n, \mathbb{K}).
\]

**Lemma 2.** If a commutative linear algebraic group \(G\) admits a prehomogeneous \(G\)-module \(V\), then \(G\) is connected.

**Proof.** By definition, \(G\) is isomorphic as a variety to a dense open subset of the module \(V\). Hence \(G\) is irreducible or, equivalently, connected. \(\square\)

**Proposition 1.** Let \(G\) be a commutative linear algebraic group and \(V\) a prehomogeneous \(G\)-module. Then the group \(G\) coincides with its centralizer in \(\text{GL}(V)\). In particular, \(G\) is a maximal commutative subgroup of \(\text{GL}(V)\).

**Proof.** Let \(C\) be the centralizer of \(G\) in \(\text{GL}(V)\). The group \(C\) preserves an open \(G\)-orbit \(O\) in \(V\). The group of \(G\)-equivariant automorphisms of the orbit \(O\) coincides with \(G\). Hence for every \(c \in C\) there exists an element \(g \in G\) whose action on \(O\) coincides with the action of \(c\). Since \(O\) is open and dense in \(V\), we conclude that \(C = G\). \(\square\)

3. **Finite dimensional algebras and Hassett-Tschinkel correspondence**

Let \(A\) be a finite dimensional commutative associative algebra with unit over the ground field \(\mathbb{K}\). It is well known that \(A\) admits a unique decomposition \(A = A_1 \oplus \ldots \oplus A_r\) into a direct sum of local algebras \(A_i\) with maximal ideals \(\mathfrak{m}_i\); see, e.g., [7, Theorem 8.7]. Moreover, every algebra \(A_i\) decomposes as a vector space to \(\mathbb{K} \oplus \mathfrak{m}_i\), all elements in \(\mathfrak{m}_i\) are nilpotent and all elements in \(A_i \setminus \mathfrak{m}_i\) are invertible. In particular, the group of invertible elements \(G(A_i)\) equals \(\mathbb{K}^\times \oplus \mathfrak{m}_i\).

In [14], Hassett and Tschinkel established a correspondence between local algebras \(A\) of dimension \(n\) and effective actions of the commutative unipotent group \(\mathbb{G}_m^{n-1}\) on the projective space \(\mathbb{P}^{n-1}\) with an open orbit. Here the projective space is realized as the projectivization \(\mathbb{P}(A)\) and the \(\mathbb{G}_m^{n-1}\)-action comes from the action of the group \(\exp(\mathfrak{m}) = 1 + \mathfrak{m}\) on \(A\) by multiplication; see also [6, Section 1].

These results may be reformulated as follows, cf. [14, Theorem 2.14] and [6, Theorem 1.1].

**Proposition 2.** Equivalence classes of prehomogeneous modules \(V\) of the group \(\mathbb{G}_m \times \mathbb{G}_m^{n-1}\), where the torus \(\mathbb{G}_m\) acts on \(V\) by scalar multiplication, are in bijection with isomorphy classes of local algebras \(A\) of dimension \(n\). More precisely, every such prehomogeneous module is isomorphic to the \(G(A)\)-module \(A\).

**Example 1.** Let us illustrate the bijection of Proposition 2 for the local algebra \(A = \mathbb{K}[x_1, x_2]/(x_1x_2, x_1^3 - x_2^3)\). Take a basis \(\{1, x_1, x_2, x_1^2, x_2^2, x_1^3\}\) in \(A\). The exponent
\[
\exp(\alpha_1x_1 + \alpha_2x_2 + \alpha_3x_1^2 + \alpha_4x_2^2 + \alpha_5x_1^3)
\]
is equal to
\[
1 + \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_1^2 + \alpha_4x_2^2 + \alpha_5x_1^3 + \frac{1}{2}(\alpha_1^2x_1^3 + \alpha_2^2x_2^3 + 2(\alpha_1\alpha_3 + \alpha_2\alpha_4)x_1^3) + \frac{1}{6}(\alpha_1^3 + \alpha_2^3)x_1^3.
\]

Multiplying all basis vectors by this element, we obtain an explicit matrix form for the corresponding 6-dimensional prehomogeneous \((\mathbb{G}_m \times \mathbb{G}_m^{n})\)-module:
Local algebra taken from [20, Section 2].

Proposition 3. The number of isomorphy classes of local algebras of dimension $n$ is finite if and only if $n \leq 6$.

Proof. A classification of local algebras of dimension $n \leq 6$ can be extracted from [28, pages 136-150]. Explicitly it is given in [20, Section 2]. We reproduce this classification below.

For every $n \geq 7$ a continuous family of pairwise non-isomorphic local algebras of dimension $n$ is constructed in [14, Section 3]; see [14, Example 3.6] and the text preceding this example. \qed

In the following table we list all local algebras of dimension up to 6. The classification is taken from [20, Section 2].

| No. | dim $A$ | Local algebra $A$ |
|-----|---------|------------------|
| 1   | 1       | $\mathbb{K}$    |
| 2   | 2       | $\mathbb{K} [x_1, x_2] / (x_1^2)$ |
| 3   | 3       | $\mathbb{K} [x_1, x_2] / (x_1^2, x_2^2, x_1x_2)$ |
| 4   | 4       | $\mathbb{K} [x_1, x_2] / (x_1^4)$ |
| 5   | 4       | $\mathbb{K} [x_1, x_2] / (x_1^2, x_2, x_1x_2)$ |
| 6   | 4       | $\mathbb{K} [x_1, x_2] / (x_1^3, x_2^2, x_1x_2)$ |
| 7   | 4       | $\mathbb{K} [x_1, x_2] / (x_1^4)$ |
| 8   | 4       | $\mathbb{K} [x_1, x_2] / (x_1^2, x_2, x_1x_2)$ |
| 9   | 5       | $\mathbb{K} [x_1, x_2] / (x_1^4, x_2^2, x_1x_2)$ |
| 10  | 5       | $\mathbb{K} [x_1, x_2] / (x_1^2, x_2^3, x_1x_2)$ |
| 11  | 5       | $\mathbb{K} [x_1, x_2] / (x_1^4, x_2^2, x_1x_2)$ |
| 12  | 5       | $\mathbb{K} [x_1, x_2] / (x_1^2, x_2^3, x_1x_2)$ |
| 13  | 5       | $\mathbb{K} [x_1, x_2] / (x_1^2, x_2^3, x_1x_2)$ |
| 14  | 5       | $\mathbb{K} [x_1, x_2] / (x_1^2, x_2^3, x_1x_2)$ |
| 15  | 5       | $\mathbb{K} [x_1, x_2] / (x_1^2, x_2^3, x_1x_2)$ |
| 16  | 5       | $\mathbb{K} [x_1, x_2] / (x_1^2, x_2^3, x_1x_2)$ |
| 17  | 5       | $\mathbb{K} [x_1, x_2] / (x_1^2, x_2^3, x_1x_2)$ |
| 18  | 6       | $\mathbb{K} [x_1, x_2] / (x_1^3)$ |
| 19  | 6       | $\mathbb{K} [x_1, x_2] / (x_1^3, x_2^3)$ |
| 20  | 6       | $\mathbb{K} [x_1, x_2] / (x_1^3, x_2^3)$ |
| 21  | 6       | $\mathbb{K} [x_1, x_2] / (x_1^3, x_2^3)$ |
| 22  | 6       | $\mathbb{K} [x_1, x_2] / (x_1^3, x_2^3)$ |
23 6 \( \mathbb{K}[x_1, x_2]/(x_1 x_2) \)
24 6 \( \mathbb{K}[x_1, x_2]/(x_1^2 x_2, x_1 x_2^2, x_1^3 - x_2^3) \)
25 6 \( \mathbb{K}[x_1, x_2]/(x_1^2 x_2, x_1^2 x_2^2, x_1^3 x_2^3) \)
26 6 \( \mathbb{K}[x_1, x_2]/(x_1^2 x_2, x_1^3 x_2^2, x_1^4 - x_2^4) \)
27 6 \( \mathbb{K}[x_1, x_2, x_3]/(x_1^2 x_2, x_1^3 x_2^2, x_1 x_2 - x_1 x_3) \)
28 6 \( \mathbb{K}[x_1, x_2, x_3]/(x_1^2 x_2, x_1^3 x_2^2, x_1 x_2 - x_2 x_3) \)
29 6 \( \mathbb{K}[x_1, x_2, x_3]/(x_1^2 x_2, x_1^3 x_2^2, x_1 x_2 - x_2 x_3) \)
30 6 \( \mathbb{K}[x_1, x_2, x_3]/(x_1^2 x_2, x_1^3 x_2^2, x_1 x_2 - x_2 x_3) \)
31 6 \( \mathbb{K}[x_1, x_2, x_3]/(x_1^2 x_2, x_1^3 x_2^2, x_1 x_2 - x_2 x_3) \)
32 6 \( \mathbb{K}[x_1, x_2, x_3]/(x_1^2 x_2, x_1^3 x_2^2, x_1 x_2 - x_2 x_3) \)
33 6 \( \mathbb{K}[x_1, x_2, x_3]/(x_1^2 x_2, x_1^3 x_2^2, x_1 x_2 - x_2 x_3) \)
34 6 \( \mathbb{K}[x_1, x_2, x_3]/(x_1^2 x_2, x_1^3 x_2^2, x_1 x_2 - x_2 x_3) \)
35 6 \( \mathbb{K}[x_1, x_2, x_3]/(x_1^2 x_2, x_1^3 x_2^2, x_1 x_2 - x_2 x_3) \)
36 6 \( \mathbb{K}[x_1, x_2, x_3]/(x_1^2 x_2, x_1^3 x_2^2, x_1 x_2 - x_2 x_3) \)
37 6 \( \mathbb{K}[x_1, x_2, x_3]/(x_1^2 x_2, x_1^3 x_2^2, x_1 x_2 - x_2 x_3) \)
38 6 \( \mathbb{K}[x_1, x_2, x_3, x_4]/(x_1^2 x_2, x_1^3 x_2^2, x_1 x_2 - x_2 x_3) \)
39 6 \( \mathbb{K}[x_1, x_2, x_3, x_4]/(x_1^2 x_2, x_1^3 x_2^2, x_1 x_2 - x_2 x_3) \)
40 6 \( \mathbb{K}[x_1, x_2, x_3, x_4]/(x_1^2 x_2, x_1^3 x_2^2, x_1 x_2 - x_2 x_3) \)
41 6 \( \mathbb{K}[x_1, x_2, x_3, x_4]/(x_1^2 x_2, x_1^3 x_2^2, x_1 x_2 - x_2 x_3) \)
42 6 \( \mathbb{K}[x_1, x_2, x_3, x_4]/(x_1^2 x_2, x_1^3 x_2^2, x_1 x_2 - x_2 x_3) \)

4. Prehomogeneous and Cyclic Modules

In this section we show that every prehomogeneous module \( V \) of a commutative linear algebraic group \( G \) comes from a finite dimensional commutative associative algebra \( A \) with unit and discuss some corollaries of this result.

**Theorem 1.** Let \( G \) be a commutative linear algebraic group and \( V \) a prehomogeneous \( G \)-module. Then there exists a finite dimensional commutative associative algebra \( A \) with unit such that the \( G \)-module \( V \) is isomorphic to the \( G(A) \)-module \( A \). Moreover, two prehomogeneous modules are equivalent if and only if the corresponding algebras are isomorphic.

**Proof.** By Lemma 2, the group \( G \) is connected. Hence \( G \) is isomorphic to \( T \times \mathbb{G}_a^m \). Any orbit of an action of a unipotent group on an affine variety is closed, see, e.g., [22, Section 1.3]. Hence the torus \( T \) in \( G \) has positive dimension.

Denote by \( X(T) \) the lattice of characters of the torus \( T \). Consider the weight decomposition of the module \( V \) with respect to the torus \( T \):

\[
V = \bigoplus_{\chi \in X(T)} V_\chi, \quad \text{where} \quad V_\chi = \{ v \in V \mid tv = \chi(t)v \text{ for all } t \in T \}.
\]

Each subspace \( V_\chi \) is invariant under the group \( G \) and we have induced representations \( \rho_i : G \to \text{GL}(V_\chi) \) for all weights \( \chi_1, \ldots, \chi_r \) with nonzero weight subspaces. Denote the image \( \rho_i(G) \) by \( G_i \) and let \( V_i := V_{\chi_i} \). Then \( G \) is contained in \( G_1 \times \ldots \times G_r \), and thus the componentwise action of \( G_1 \times \ldots \times G_r \) on \( V \) has an open orbit. This implies that each \( G_i \) acts on \( V_i \) with an open orbit.

By Proposition 2, there exist local algebras \( A_i \) such that the \( G_i \)-modules \( V_i \) are isomorphic to the \( G(A_i) \)-modules \( A_i \) for all \( i = 1, \ldots, r \). In particular, we have \( \dim G_i = \dim V_i \). By Lemma 1, the group \( G \) coincides with \( G_1 \times \ldots \times G_r \), and thus the \( G \)-module \( V \) is isomorphic to the \( G(A) \)-module \( A \) for \( A = A_1 \oplus \ldots \oplus A_r \).
The last assertion follows from Proposition 2 and uniqueness of decomposition of an algebra $A$ into local summands. 

**Corollary 1.** Let $G$ be a commutative linear algebraic group. The number of isomorphy classes of prehomogeneous $G$-modules is finite if and only if the corank of $G$ is at most 5.

**Proof.** By Proposition 3, the number of isomorphy classes of finite dimensional algebras $A$ with fixed dimensions of local summands is finite if and only if the dimensions of local summands do not exceed 6. In our case the dimensions of local summands do not exceed the corank of $G$ plus 1, and this value is achieved when all local summands of $A$ except one are $K$.

**Remark 2.** Recall that two elements $a$ and $b$ of an algebra $A$ are associated if there exists an element $c \in G(A)$ such that $a = bc$. Theorem 1 implies that $G$-orbits in a prehomogeneous $G$-module $V$ are in bijection with association classes in the corresponding algebra $A$.

For positive integers $n$ and $r$, we denote by $p_r(n)$ the number of partitions $n = n_1 + \ldots + n_r$ with $n_1 \geq \ldots \geq n_r \geq 1$.

**Corollary 2.** Let $G$ be a commutative linear algebraic group of dimension $n$ and rank $r$. Then there exist precisely $p_r(n)$ prehomogeneous $G$-modules with a finite number of $G$-orbits. The corresponding algebras $A$ are precisely the algebras of the form $K[x]/(f(x))$, where $f(x)$ is a polynomial of degree $n$ with $r$ distinct roots.

**Proof.** The number of $G(A)$-orbits in $A$ is finite if and only if the number of $G(A_i)$-orbits in $A_i$ is finite for every local summand $A_1, \ldots, A_r$ in $A$. By [14, Proposition 3.7], the number of $G(A_i)$-orbits in $A_i$ is finite if and only if the algebra $A_i$ is isomorphic to $K[x]/(x^{n_i})$. It shows that the algebra $A$ is uniquely determined by dimensions $n_1, \ldots, n_r$ of its local summands.

**Remark 3.** A classification of irreducible prehomogeneous modules with finitely many orbits is obtained in [25, Theorem 8].

**Corollary 3.** Let $G$ be a commutative linear algebraic group of dimension $n$ and rank $r$. Then there exist precisely $p_r(n)$ prehomogeneous $G$-modules $V$ such that the group $G$ is normalized by the group of all invertible diagonal matrices on $V$. The corresponding algebras $A$ are precisely the algebras with $a^2 = 0$ for every nilpotent $a \in A$.

**Proof.** Clearly, the action of $G(A)$ on $A$ is normalized by the group of diagonal matrices if and only if it holds for every local summand of $A$. Thus it suffices to prove that in every dimension there exists a unique local algebra $A$ such that the action of $G(A)$ on $A$ is normalized by all diagonal matrices.

In [5], actions with an open orbit of the group $G_\mathfrak{m}$ on toric varieties are studied. It is shown in [5, Theorem 3.6] that any two such actions normalized by the maximal torus are isomorphic. In the case of the projective space $\mathbb{P}^m$ this unique isomorphy class of actions corresponds to the local algebra

$$K[x_1, \ldots, x_m]/(x_i x_j, 1 \leq i \leq j \leq m - 1),$$

or, equivalently, $A = K \oplus \mathfrak{m}$ with $\mathfrak{m}^2 = 0$; see [5, Example 6.1] and [6, Proposition 2.7]. This completes the proof.

Let us recall that a regular function $f$ on a $G$-module $V$ is a semi-invariant if $f(gv) = \mu(g)f(v)$ for some character $\mu$ of the group $G$ and all $g \in G$ and $v \in V$. 

This completes the proof.
Corollary 4. Let $G$ be a commutative linear algebraic group of rank $r$ and $V$ a prehomogeneous $G$-module. Then the complement of an open $G$-orbit in $V$ is a union of hyperplanes $H_1, \ldots, H_r$. Moreover, every semi-invariant on $V$ is a monomial in the linear functions defining the hyperplanes $H_1, \ldots, H_r$.

Proof. The first assertion follows from the decomposition $A_i = \mathbb{K} \oplus m_i$ and the equality $G(A_i) = A_i \setminus m_i$ for every local summand $A_i$ in $A$. For the second assertion we observe that the support of the divisor of zeroes of a semi-invariant $f$ on $V$ is contained in the complement of the open $G$-orbit, i.e., in the union of the hyperplanes $H_1, \ldots, H_r$. This observation implies the claim. \qed

The next proposition shows that finite dimensional algebras can be used to study much wider class of modules than the class of prehomogeneous modules. Let us recall that a vector $v$ in a $G$-module $V$ is cyclic if the linear span of the orbit $Gv$ coincides with $V$. A $G$-module $V$ is cyclic if it has a cyclic vector.

Proposition 4. Let $V$ be a cyclic module of a commutative linear algebraic group $G$. Then there exist a finite dimensional commutative associative algebra $A$ with unit and an injective homomorphism $G \to G(A)$ such that the $G$-modules $V$ and $A$ are isomorphic.

Proof. It is well known that every commutative linear algebraic group $G$ is isomorphic to a direct product $L \times T \times \mathbb{G}_m^n$, where $L$ is a finite abelian group and $T$ is a torus. Moreover, the action of the group $L \times T$ on $V$ is diagonalizable. Consider the weight decomposition $V = \oplus V_\nu$ of the module $V$ with respect to $L \times T$. Then every subspace $V_\nu$ is $G$-invariant and hence is a cyclic $G$-module. Enlarging the group $G$ we assume that $T$ consists of all invertible operators which act on every $V_\nu$ by scalar multiplication.

It suffices to show that each $V_\nu$ may be identified with some local algebra $A$ in such a way that the action of $G$ on $V_\nu$ is a restriction of the action of $G(A)$ on $A$ by multiplication to a closed subgroup in $G(A)$. It follows from [14, Theorem 2.14], see [6, Remark 1.2] for details. \qed

Remark 4. The condition that the $G$-module $V$ is cyclic is essential, see Remark 1.

5. AFFINE MONOIDS AND GROUP EMBEDDINGS

An affine algebraic monoid is an irreducible affine variety $S$ with an associative multiplication

$$\mu: S \times S \to S, \quad (a, b) \mapsto ab,$$

that is a morphism of algebraic varieties, and a unit element $e \in S$ such that $ea = ae = a$ for all $a \in S$. An example of an affine algebraic monoid is the multiplicative monoid of a finite dimensional associative algebra with unit.

The group of invertible elements $G(S)$ of a monoid $S$ is open in $S$. Moreover, $G(S)$ is a linear algebraic group. For a general theory of affine algebraic monoids, we refer to [30, 24, 23].

By a group embedding we mean an irreducible affine variety $X$ with an open embedding $G \hookrightarrow X$ of a linear algebraic group $G$ such that both actions by left and right multiplications of $G$ on itself can be extended to $G$-actions on $X$. In other words, the variety $X$ is a $(G \times G)$-equivariant open embedding of the homogeneous space $(G \times G)/\Delta(G)$, where $\Delta(G)$ is the diagonal in $G \times G$.

Any affine monoid $S$ defines a group embedding $G(S) \hookrightarrow S$. The converse statement is proved in [30] under the assumption that the group $G$ is reductive and in [24] for arbitrary $G$. For convenience of the reader we reproduce below the proof from [24].
Proposition 5. Let $G$ be a linear algebraic group. Then for every group embedding $G \hookrightarrow S$ there exists a structure of an affine algebraic monoid on $S$ such that the group $G$ coincides with the group of invertible elements $G(S)$.

Proof. Let us prove that the multiplication morphism $G \times G \to G$ can be extended to a morphism $S \times S \to S$. Consider the morphisms given by left and right multiplication

$$G \times S \to S \quad \text{and} \quad S \times G \to S$$

and the corresponding comorphisms

$$\mathbb{K}[S] \to \mathbb{K}[G] \otimes \mathbb{K}[S] \quad \text{and} \quad \mathbb{K}[S] \to \mathbb{K}[S] \times \mathbb{K}[G].$$

Since the morphisms $G \times S \to S$ and $S \times G \to S$ extend the multiplication $G \times G \to G$, the image of the subalgebra $\mathbb{K}[S]$ of $\mathbb{K}[G]$ is contained in the intersection

$$(\mathbb{K}[G] \otimes \mathbb{K}[S]) \cap (\mathbb{K}[S] \otimes \mathbb{K}[G]) = \mathbb{K}[S] \otimes \mathbb{K}[S].$$

This provides the desired extended morphism $S \times S \to S$. Such a morphism has the associativity property because it holds on an open dense subset $G$ in $S$. Similarly, the unit element $e \in G$ satisfies the property $es = se = s$ for all $s \in S$.

Every element of the group $G$ is invertible in $S$. Conversely, if $s \in S$ is invertible then the subvariety $sG$ is open in $S$ and thus the intersection $G \cap sG$ is non-empty. We conclude that $s$ lies in $G$ and $G(S) = G$. $\Box$

Let us recall that an action $G \times \mathbb{A}^n \to \mathbb{A}^n$ of a linear algebraic group $G$ is linearizable, if the image of $G$ in $\text{Aut}(X)$ is conjugate to a subgroup of the group $\text{GL}_n(\mathbb{K})$ of all linear transformations of $\mathbb{A}^n$.

Proposition 6. Let $S$ be an affine algebraic monoid. Assume that the variety $S$ is isomorphic to an affine space. Then $S$ is the multiplicative monoid of a finite dimensional algebra if and only if the action of the group $G(S) \times G(S)$ on $S$ by left and right multiplication is linearizable.

Proof. Assume that the group $G(S) \times G(S)$ acts linearly on the vector space $V$ identified with the variety $S$. The multiplication $V \times V \to V$ is given by the comorphism $\mathbb{K}[V] \to \mathbb{K}[V] \otimes \mathbb{K}[V]$. Since the $(G(S) \times G(S))$-action on $V$ is linear, for the restriction of the comorphism to the subspace $V^* \subseteq \mathbb{K}[V]$ of all linear functions on $V$ we have

$$V^* \to \mathbb{K}[G] \otimes V^* \quad \text{and} \quad V^* \to V^* \otimes \mathbb{K}[G].$$

So the image of $V^*$ is contained in the intersection $(\mathbb{K}[G] \otimes V^*) \cap (V^* \otimes \mathbb{K}[G]) = V^* \otimes V^*$. Hence the multiplication on $V$ is given by the linear map $V \otimes V \to V$ dual to $V^* \to V^* \otimes V^*$. This proves that the multiplication on $V$ is bilinear and thus the monoid $S$ is isomorphic to the multiplicative monoid of a finite dimensional algebra.

The converse implication is straightforward. $\Box$

Let us recall that an affine algebraic monoid $S$ is reductive if the group $G(S)$ is a reductive linear algebraic group. Since every action of a reductive group on an affine space with an open orbit is linearizable [19], we obtain the following result.

Corollary 5. If $S$ is a reductive monoid and the variety $S$ is isomorphic to an affine space, then $S$ is the multiplicative monoid of a finite dimensional algebra.

Clearly, the multiplicative monoid of a finite dimensional algebra $R$ is reductive if and only if $R$ is a semisimple $R$-module. By the Artin-Wedderburn Theorem, this is the case if and only if $R$ is a direct sum of matrix algebras $\text{Mat}(n_i \times n_i, \mathbb{K})$. 
Example 2. Let $V$ be a prehomogeneous $G$-module with trivial generic stabilizer, where $G$ is a non-commutative linear algebraic group. The inclusion of an open orbit $G \to V$ need not be a group embedding. For instance, take the group $G = \{ \begin{pmatrix} t & a \\ 0 & t^{-1} \end{pmatrix}, \ t \in \mathbb{G}_m, \ a \in \mathbb{G}_a \}$ and its tautological module $\mathbb{K}^2$. The orbit of the vector $(0, 1)$ is open in $\mathbb{K}^2$, it consists of the vectors $(a, t^{-1})$ or, equivalently, of the vectors $(x, y), y \neq 0$. The right multiplication by an element
\[
\begin{pmatrix} s & b \\ 0 & s^{-1} \end{pmatrix}^{-1}
\]
gives the vector $(sa - tb, t^{-1}s)$ or, equivalently, $(sx - by^{-1}, sy)$. Such an action can not be extended to $\mathbb{K}^2$.

6. Additive actions on toric varieties and Cox rings

Let $X$ be an irreducible algebraic variety over the ground field $\mathbb{K}$. An additive action on $X$ is a regular faithful action $\mathbb{G}_a^n \times X \to X$ with an open orbit. Let us recall that a variety $X$ is toric if $X$ is normal and there exists an action of an algebraic torus $T$ on $X$ with an open orbit. Additive actions on toric varieties are studied in [5].

If a variety $X$ admits an additive action, then every regular invertible function on $X$ is constant and the divisor class group $\text{Cl}(X)$ is a free finitely generated abelian group [4, Lemma 1]. For a toric variety $X$ these conditions imply that $X$ can be realized as a good quotient $\pi: U \xrightarrow{\pi} X$ of an open subset $U \subseteq \mathbb{A}^n$, whose complement is a collection of coordinate subspaces of codimensions at least 2 in $\mathbb{A}^n$ by a linear action of a torus $H$. Such a realization can be chosen in a canonical way. Namely, the Cox ring
\[
R(X) = \bigoplus_{[D] \in \text{Cl}(X)} H^0(X, D)
\]
of a toric variety $X$ is a polynomial ring graded by the group $\text{Cl}(X)$. The grading defines a linear action of the characteristic torus $H := \text{Spec}(\mathbb{K}[\text{Cl}(X)])$ on the total coordinate space $\mathbb{A}^n := \text{Spec}(R(X))$. A canonically defined open subset $U \subseteq \mathbb{A}^n$, whose complement is a union of some coordinate subspaces of codimensions at least 2, gives rise to the so-called characteristic space $p: U \xrightarrow{\phi} X$; we refer to [8] and [2, Chapter II] for details.

An additive action $\mathbb{G}_a^n \times X \to X$ can be lifted to an action $\mathbb{G}_a^n \times \mathbb{A}^n \to \mathbb{A}^n$ on the total coordinate space commuting with the $H$-action. This defines an action $G \times \mathbb{A}^n \to \mathbb{A}^n$ of the commutative group $G := H \times \mathbb{G}_a^n$ with an open orbit. Let us say that the action $G \times \mathbb{A}^n \to \mathbb{A}^n$ is associated with the given additive action on a toric variety $X$. 

Remark 5. For commutative monoids the statement of Corollary 5 does not hold.
We say that a toric variety $X$ is a big open subset of a toric variety $X'$ if $X$ is isomorphic to an open toric subset $W$ of the variety $X'$ such that codim$_X X' \setminus W \geq 2$.

**Proposition 7.** An action $G \times \mathbb{A}^n \to \mathbb{A}^n$ associated with an additive action on a toric variety $X$ is linearizable if and only if $X$ is a big open subset of a product of projective spaces.

**Proof.** It follows from Theorem 1 that if an action $G \times \mathbb{A}^n \to \mathbb{A}^n$ is linearizable, then in suitable coordinates we have $\mathbb{A}^n = V_1 \oplus \ldots \oplus V_r$, where $r = \text{rk}(G) = \dim H$ and every element $(t_1, \ldots, t_r) \in H$ acts on every subspace $V_i$ via scalar multiplication by $t_i$. Let $d_i := \dim V_i$. The torus $H$ acts on $\mathbb{A}^n$ linearly with characters $e_1(d_1\text{times}), \ldots, e_r(d_r\text{times})$, where $e_1, \ldots, e_r$ form a basis of the lattice of characters $\mathcal{X}(H)$.

It is easy to show (see, e.g., [2, Exercise 2.13]) that there is a unique maximal open subset $U$ in $\mathbb{A}^n$ such that there exists a good quotient $\pi: U \xrightarrow{\text{H}} X$ which is the characteristic space of $X$; namely, $U = (V_1 \setminus \{0\}) \times \ldots \times (V_r \setminus \{0\})$ and $X = \mathbb{P}(V_1) \times \ldots \times \mathbb{P}(V_r)$. Open subsets with this property are contained in $U$ and correspond to big open toric subsets of $\mathbb{P}(V_1) \times \ldots \times \mathbb{P}(V_r)$.

Conversely, consider an additive action $\mathbb{G}_a^n \times X \to X$ on a big open toric subset $X$ of $\mathbb{P}(V_1) \times \ldots \times \mathbb{P}(V_r)$. The Picard group of $X$ is freely generated by the line bundles $L_1, \ldots, L_r$ corresponding to ample generators of the Picard groups of the factors $\mathbb{P}(V_1), \ldots, \mathbb{P}(V_r)$. The space of global sections of $L_i$ is identified with the dual space $V_i^*$. By [17, Section 2.4], every line bundle $L_i$ admits a $\mathbb{G}_a^n$-linearization, and thus the lifted action of the group $G = H \times \mathbb{G}_a^n$ to the total coordinate space $\mathbb{A}^n = V_1 \oplus \ldots \oplus V_r$ of $X$ is linear. □

**Example 3.** Consider the action $\mathbb{G}_a^n \times \mathbb{A}^n \to \mathbb{A}^n$ by translations. This is an additive action on a toric variety, and the associated action coincides with the original one. Since the action is transitive, it has no fixed point and thus it is not linearizable.

**Example 4.** Let $X$ be the Hirzebruch surface $\mathbb{F}_d$. This toric variety admits an additive action normalized by the acting torus. The lifting of this action to the Cox ring extends to an action of the group $G = \mathbb{G}_m^2 \times \mathbb{G}_a^2$ on $\mathbb{A}^4$ with an open orbit. Explicitly this action is given by

$$(x_1, x_2, x_3, x_4) \mapsto (\lambda_1 x_1, \lambda_2 x_2, \lambda_1 x_3 + \lambda_1 \alpha_1 x_1, \lambda_1^d \lambda_2 x_4 + \lambda_1^d \lambda_2 \alpha_2 x_1 x_2), \quad \lambda_1, \lambda_2 \in \mathbb{G}_m, \alpha_1, \alpha_2 \in \mathbb{G}_a.$$ See [5, Example 6.4]. By Proposition 7, this action is not linearizable for $d \geq 1$. If $d = 0$ then $X \cong \mathbb{P}^1 \times \mathbb{P}^1$, and the action is linear.

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