Application of Structural Similitude for Scaling of a Pressure Vessel

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Abstract. Pressure vessels find their use in various fields, ranging from gas cylinders used in households for cooking, boilers for steam engines, fuselage of aircraft to solid rocket motors used in missiles and space shuttle. The design of such vessels is validated by performing tests on full scale prototypes. Mostly the testing of such vessels is cumbersome and expensive. This paper establishes the method to reduce the cost for testing the pressure vessels. The theory of similitude is studied to make the testing process easier by establishing structural similitude for a pressure vessel. Using similitude theory a scaled model of the prototype vessel is developed in such a way that when the scaled models’ responses are multiplied by a calculated scale factor, behaviour of the prototype could be predicted. By testing on the scaled down model, the cost of manufacturing is reduced. The pressure vessel considered here is representative of the pressures and materials used in high pressure applications. In this paper a 1/10th scaled model of the pressure vessel is developed using structural similitude theory. Buckingham pi-theorem technique has been used for dimensional analysis after studying parameters on which pressure vessel is designed and ANSYS software is used to validate the resulting pi-products. Complete similarity is achieved when predicted prototype results completely map on to prototype results.

Keywords: Dimensional Analysis, Pressure Vessels, Scaled Model, Structural Similitude.

1. Introduction

In all modern manufacturing processes extensive testing and experimental verification is needed to ensure reliability and safety of the finished structure. Huge constructions like dams, bridges, and transport aircrafts require full scale testing of large components called prototypes to verify theoretical predictions about systems’ behavior under various operating conditions before they are considered acceptable. This testing can be expensive and time consuming and remains an indispensable part of modern engineering. If accurate predictions about prototype behavior can be made using experimental results of scaled down easier to handle models, valuable time and money can be saved. This is the motivation behind similitude of models and this study.

A model of a prototype is a system which replicates the behavior of the prototype to a specific input. The experimental results of the model may then be extrapolated to predict the experimental response of the prototype using a relationship which exists between the parameters and variables of the model and prototype called scale factors. The accuracy of the results depends upon the material, design and analysis techniques of the model. These predicted experimental results are then compared with theoretical prototype results.

The similitude of systems which predicts the structural behavior of large components by use of models is termed as structural similitude [1]. The structural similitude theory is already being applied
in structural [2], vibration, and impact [3] problems in civil and mechanical engineering. It has also been investigated for developing scaled models of stiffened cylinders and shells for various applications and operating conditions [4-5]. In this paper, the structural similitude is developed for thin walled pressure vessels used in aerospace applications and the similarity obtained is studied.

2. Similitude Theory
A mathematical model of a system can be made using all the system variables and parameters. For similar systems these parameters are similar but not identical. The theory of similitude searches for a relationship (transformation) which maps the models parameters onto the prototypes parameters. This means that each model parameter is proportional to its corresponding prototype parameter. This proportional factor is called the scale factor. For example if vectors $X_{pr}$ and $X_{m}$ are the characteristic vectors of the prototype and model, then we can find a transformation matrix $\Lambda$ [6] such that:

$$X_{pr} = \Lambda X_{m} \quad \text{or} \quad X_{m} = \Lambda^{-1}X_{pr}$$  \hspace{1cm} (1)

In its simplest form diagonal elements of $\Lambda$ are the scale factors. This need not be the case always.

$$\lambda_{xi} = \frac{x_{i,pr}}{x_{i,m}}$$  \hspace{1cm} (2)

$$\Lambda = \begin{bmatrix} \lambda_{x1} & 0 & \cdots & 0 \\ 0 & \lambda_{x2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{xn} \end{bmatrix}$$  \hspace{1cm} (3)

There are two methods which are used to determine scale factors for scale model development of cylindrical shells and other structures: direct use of governing equations and dimensional analysis [7]. When the number of design parameters is large, dimensional analysis can be tedious so instead the systems’ governing equations along with the appropriate boundary and initial conditions are used to find the similarity conditions [8]. If the governing equations of the two systems can be mapped onto each other (are proportional) then the two systems are similar and the response of one can be used to find that of the other. Dimensional analysis method is used either when the mathematical model of the system is unavailable or the numbers of parameters involved are few. It requires that all parameters which affect the systems’ response to a particular input be known. The similarity conditions are in the form of non-dimensional constants which remain same for both prototype and model and are used to decide parameters and test conditions of the model. The technique used most frequently to find these constants is called the Buckingham pi theorem [9] which is used in this paper.

3. Pressure Vessels
The American society for mechanical engineers defines pressure vessels as a containment of solid, liquid or gaseous material under internal or external pressure, also capable of withstanding various other loadings [10]. Some examples of pressure vessels use are in aircrafts, solid rocket motors, nuclear reactor vessel, diving cylinder, distillation towers and many vessels in mining or oil refineries and petrochemical plants.

Pressure vessels carry liquids and gases at high pressure and so withstand pressure loading which creates stresses and strains. For most engineering applications the thin wall pressure vessel can be used which by definition has a ratio of $r/t \geq 10$.

For this study, a thin wall pressure vessel loaded internally is considered. It is capped with ellipsoidal heads which are made of the same material as the cylindrical portion and is of same thickness. The pressure considered is 5 MPa which makes it suitable for high pressure applications. The mean radius of the cylinder (and the caps) is ‘$R$’. The cylinder and heads have a uniform thickness and the vessel is subjected to an internal pressure and a zero external pressure. No other external forces act. The vessel walls are constructed of a single material.
3.1. Stress equation for thin walled pressure vessels
In a region of the cylinder that is far from the ends, three normal stresses \( \sigma_h, \sigma_l \) and \( \sigma_r \) may be calculated to characterize the thin shell stress state \[11\]. Therefore:

\[
\sigma_h = \frac{pR}{t} \\
\sigma_l = \frac{pR}{2t} \\
\sigma_r = -\left(\frac{1}{2} p\right)
\]

(4) \hspace{1cm} (5) \hspace{1cm} (6)

3.2. Strain equation in cylinder
Total radial displacement is caused by a combination of all three stresses and called ‘\( \Delta r \)’ here \[11\]:

\[
\Delta r = \left(\frac{pR^2}{2Et}\right) \left[ 2 - v + v \left(\frac{1}{2}\right) \right]
\]

(7)

The displacement of the cylinder in the longitudinal direction is given by \[12\]:

\[
\Delta z = \frac{pRy}{Et} \left(0.5 - v\right)
\]

(8)

3.3. Normal stresses for ellipsoidal head
Longitudinal stress in the head is:

\[
\sigma_l = \frac{pa}{2t}
\]

(9)

Since semi major axis of head will be equal to cylinder radius \( R \), ellipsoidal longitudinal stress is same as longitudinal stress of cylinder. Hoops stress is given by:

\[
\sigma_h = \frac{pa(2b^2-a^2)}{2tb^2}
\]

(10)

3.4. Strain in ellipsoidal head
The longitudinal strain in the head is small when compared to the cylinder and is hence not considered. Radial displacement for ellipsoidal head at the equator is given by:

\[
\Delta r = \frac{pR^2}{Et} \left(1 - \frac{a^2}{2b^2} - \frac{v}{2}\right)
\]

(11)

It is important to note that the net displacement of the cylinder radially will be a value between radial displacement for cylinder and head. This is because the value for displacement of head and cylinder are unequal (note equations (7) and (11)). To overcome this difference, moment and shear forces are set up at the joint of head with cylinder wall which increase the stress values. So the total longitudinal stress is sum of the stress calculated for membrane only (5) and stress developed in the joint. In reality this stress is localized at the joint region only and away from the center only membrane stress needs to be calculated. Bending hoop stresses are much smaller and can be ignored.

4. Dimensional analysis of pressure vessel

4.1. Dimensional analysis
The material used for the pressure vessel considered is Maraging 250 Steel. The material and geometrical properties of the considered pressure vessel are given in Table 1. The results show that the stress scale factors are independent of the material properties and only strain and displacement scale factors are affected. Displacement and strain scale factors change by the same order of magnitude by which the model material deviates from the prototype. However, this entire study comes under the domain of partial similarity which is beyond the scope of this paper. Due to which, the results have not
been included in this paper. The thickness is calculated using ASME code UG-23. After calculation and allowances, thickness of 6 mm is selected. The general similitude theory for the pressure vessel can be found from the Buckingham pi theorem by considering factors that affect the vessel’s structural behavior. The validation may then be carried out by checking that the model and prototype pi products are equal.

14 physical parameters as shown in Table 2 are considered for the mentioned pressure vessel. These are ‘\(\sigma_h\)’, ‘\(\sigma_l\)’, ‘\(\sigma_r\)’, ‘\(p\)’, ‘\(R\)’, ‘\(t\)’, ‘\(v\)’, ‘\(h\)’, ‘\(a\)’, ‘\(b\)’, ‘\(\Delta r\)’, ‘\(\Delta z\)’, ‘\(Q\)’ and ‘\(E\)’. All these variables can be represented by two dimensions which are force and length. As a result 12 dimensionless \(\pi\) products are left.

Table 1: Material Properties for Maraging 250

| Material Property                | Value  |
|----------------------------------|--------|
| \(\sigma_y\) (yield strength)   | 1560 MPa |
| \(\sigma_u\) (ultimate tensile strength) | 1760 MPa |
| \(E\) (Young’s modulus of elasticity) | 190 GPa  |
| \(v\) (Poisson ratio)           | 0.3    |
| \(p\) (Design pressure)         | 5 MPa  |
| \(R\) (external radius)         | 1000 mm|
| \(h\) (length)                  | 6300 mm|
| Head ratio                      | 2:1    |
| Semi-major axis (external)      | 1000 mm|
| Semi-minor axis (external)      | 500 mm |
| Straight flange                 | 40mm   |

Table 2: Physical Parameters and their Dimensions

| Dimension | Force (F) | Length (L) |
|-----------|-----------|------------|
| \(\sigma_h\) | 1         | -2         |
| \(\sigma_l\) | 1         | -2         |
| \(\sigma_r\) | 1         | -2         |
| \(p\) | 1         | -2         |
| \(R\) | 0         | 1          |
| \(b\) | 0         | 1          |
| \(\Delta r\) | 0      | 1          |
| \(t\) | 0         | 1          |
| \(v\) | 0         | 0          |
| \(E\) | 1         | -2         |
| \(h\) | 0         | 1          |
| \(a\) | 0         | 1          |
| \(\Delta z\) | 0    | 1          |
| \(Q\) | 1         | -2         |

4.2. Estimation of dimensionless groups

The physical relation may be expressed as a relation of following 12 dimensionless \(\pi\) products, i.e:

\[
G(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7, \pi_8, \pi_9, \pi_{10}, \pi_{11}, \pi_{12}) = 0
\]

\[
\pi_1 = (p,R,\sigma_h) \quad \pi_2 = (p,R,\sigma_l) \quad \pi_3 = (p,R,t) \quad \pi_4 = (p,R,v) \quad \pi_5 = (p,R,E) \quad \pi_6 = (p,R,h)
\]

\[
\pi_7 = (p,R,\sigma_r) \quad \pi_8 = (p,R,a) \quad \pi_9 = (p,R,b) \quad \pi_{10} = (p,R,\Delta r) \quad \pi_{11} = (p,R,\Delta z) \quad \pi_{12} = (p,R,Q)
\]

The method of obtaining the first \(\pi\) product is shown here. Rest of the \(\pi\) products are obtained by using a similar method.

\[
\pi_1 = p^a R^b \sigma h
\]

For

\[
F \geq a + 1 = 0
\]

\[
L \geq -2a + b - 2 = 0
\]

From equation (12) and (13) we get, \(a = -1\) and \(b = 0\)

So the first \(\pi\) product is: \(\pi_1 = p^{-1} R^0 \sigma h\). Rest of the \(\pi\) products determined are:

\[
\pi_1 = \frac{\sigma h}{p} \quad \pi_2 = \frac{\sigma l}{p} \quad \pi_3 = \frac{t}{R} \quad \pi_4 = \frac{v}{R} \quad \pi_5 = \frac{E}{p} \quad \pi_6 = \frac{h}{R}
\]

\[
\pi_7 = \frac{\sigma r}{p} \quad \pi_8 = \frac{a}{R} \quad \pi_9 = \frac{b}{R} \quad \pi_{10} = \frac{\Delta r}{R} \quad \pi_{11} = \frac{\Delta z}{R} \quad \pi_{12} = \frac{Q}{R}
\]
12 pi-products obtained can be divided into two groups, dimensional group and pressure group as shown in Table 3. The dimensional group contains pi-products that are ratio of dimensions and displacements of pressure vessel. The pressure group contains products related to modulus and stresses.

4.3. Estimation of scaling factors

As there are two repeating variables so we can fix two scale factors and find scale factors for the rest of the vessel parameters using the calculated pi products. Scale factors chosen are $\lambda_R$ and $\lambda_R$. Subscript ‘p’ is for prototype and ‘m’ for scaled model.

$\lambda_p = \frac{(p)_{pr}}{(p)_m} = 5$ (Pressure scale factor), $\lambda_R = \frac{(R)_{pr}}{(R)_m} = 10$ (Radius scale factor)

Similitude demands that pi products of prototype and model remain equal therefore;

$\lambda_{\sigma_h} = \lambda_p = 5$ (Hoop stress scale factor), $\lambda_{\sigma_l} = \lambda_p = 5$ (Longitudinal stress scale factor)

From 3rd pi term $\lambda_t = \lambda_R = 10$ (Thickness scale factor)

and 5th pi term gives $\lambda_E = \lambda_p = 5$ (Youngs modulus scale factor), $\lambda_R = \lambda_h = \lambda_a = \lambda b = \lambda Q = 10$

where the scale factor of $\lambda_h$ is for length, $\lambda a$ is for ellipsoidal head’s semi-major axis, $\lambda b$ is for shear stress. $\lambda v = 1$ that is poisson’s ratio for prototype = poisson’s ratio for model. The scale factors found are summarized in Table 4 where $\lambda p=5$ and $\lambda R=10$.

| Table 3: pi Products Groups | Table 4: Scale Factors Found by Applying Similitude Theory |
|----------------------------|-------------------------------------------------------------|
| Dimensional group          | Pressure group                                              |
| $\Pi_3$                    | $\Pi_1$                                                     |
| $\Pi_4$                    | $\Pi_2$                                                     |
| $\Pi_5$                    | $\Pi_7$                                                     |
| $\Pi_6$                    | $\Pi 12$                                                    |
| $\Pi 8$                    |                                                            |
| $\Pi 9$                    |                                                            |
| $\Pi 10$                   |                                                            |
| $\Pi 11$                   |                                                            |

Using these scale factors scaled model dimensions and scaled loads are found next given the dimensions and applied loads on the prototype. Similitude between the prototype and its scaled model exists when all the pi terms are same for both the model and prototype. The tabulation of prototype and 1/10th model of pressure vessel is shown in Table 5 and the pi terms obtained for both the model and prototype is shown in Table 6. It is evident from the Table 6 that pi-terms for both the prototype and model are identical. Thus pi-terms shown in mentioned table have been verified for complete similarity. The pi-products verified belong to the dimensional group and as both prototype and model dimensions are available, they can be verified. Only one of the pi-product in the pressure group has been verified. For the verification of remaining 6 pi-products ANSYS is used. The structural analysis is performed for both the model and prototype and then model results are used to predict prototype results.
Table 5: Prototype and Model Dimensions

| Parameter                        | Prototype | Scale factor | Model |
|----------------------------------|-----------|--------------|-------|
| Radius, R                        | 1000 mm   | 10           | 100 mm|
| Height, h                        | 6380 mm   | 10           | 638 mm|
| Thickness, t                     | 6 mm      | 10           | 0.6 mm|
| Semi major axis(external)        | 1000 mm   | 10           | 100 mm|
| Semi-minor axis(external)        | 500 mm    | 10           | 50 mm |
| Pressure, p                      | 5000 kPa  | 5            | 1000 kPa|
| Young’s modulus, E               | 190 GPa   | 5            | 38 GPa |

Table 6: Pi terms for model and prototype

| Pi-products | Expression | Numerical value |
|-------------|------------|-----------------|
| π3          | (t/R)pr    | 6/1000 = 0.006  |
|             | (t/R)m     | 0.6/1000 = 0.006|
| π4          | (v)pr      | 0.3             |
|             | (v)m       | 0.3             |
| π5          | (E/p)pr    | 190e09/5e06 = 38000 |
|             | (E/p)m     | 38e09/1e06 = 38000 |
| π6          | (h/R)pr    | 6300/1000 = 6.3  |
|             | (h/R)m     | 630/100 = 6.3    |
| π8          | (a/R)pr    | 1000/1000 = 1   |
|             | (a/R)m     | 100/100 = 1     |
| π9          | (b/R)pr    | 500/1000 = 0.5  |
|             | (b/R)m     | 50/100 = 0.5    |

5. FEA (Finite Element Analysis)

The geometry is made in PRO/E as shown in Figure 1 and then imported into ANSYS software. An axisymmetric FE model of the scaled pressure vessel is developed in the same way as the prototype. The requirement is to map the results found in ANSYS of the scaled model of the vessel onto the prototype by using the derived similarity parameters called scale factors and compare them to the ANSYS results of the actual prototype. After some preliminary treatment of the ANSYS result data, both the prototype and model data is compiled in the same M-file of MATLAB program. In each M-file the scale factors derived earlier using similitude theory (scale factor of 5 for stresses and 10 for geometrical parameters respectively) are used to map model stress and strain values onto the prototype. This process basically predicts prototype results using model results and similarity parameters. These mapped results are referred to as predicted prototype result from now on to avoid confusion.

5.1. Analysis of results

The structural parameters analyzed for prototype and model are Von Mises stress, Radial stress (σr), Longitudinal stress (σl), Hoop stress (σθ), Shear Stress, Q in rz plane (XY when taken in terms of cartesian coordinates), Radial displacement Δr (X coordinate displacement), Longitudinal displacement Δz (Y coordinate displacement), Radial strain and Longitudinal strain. The observation made from the comparison of prototype and predicted prototype results are summarized below.

The Von Mises stress has been used as failure criteria. Figure 2 shows the maximum Von Mises stress to be about 1300MPa which is less than the yield stress of Maraging 250 Steel (1560MPa). The vessel is not in a state of failure. The same method can be used for any other material using its respected properties. However, the limitation is to choose a material for the scaled model as per the scaled down properties.
Fig 1. Pressure vessel geometry

Table 7: Percentage errors in analyses results

| Parameter         | Percentage error |
|-------------------|------------------|
| Von Mises stress  | <1.4%            |
| Radial stress     | < 0.1 %          |
| Longitudinal stress| < 1 %           |
| Hoop stress       | < 1 %            |
| Shear stress      | < 0.35 %         |
| Radial displacement| < 0.5 %        |
| Longitudinal displacement| < 0.1 %    |

The predicted and actual prototype results match for all the stresses and displacements as is evident from the overall percentage errors shown in Table 7. Percentage error is insignificant and maximum values of error occur in the heads. However the error magnitude is negligible when compared to the actual values. For radial, longitudinal and hoop stress, the maximum percentage error is less than 1% for the stresses due to the change in the geometry from cylindrical to the head region. The percentage error in radial and longitudinal displacement is less than 0.5%.

During development of similitude theory, shear is taken as a parameter as it is created at the joint of pressure vessel. The graph in Figure 3 of shear stress results shows that it is almost zero throughout the cylindrical region but has notable presence in the heads which proves that it should be included as a parameter during dimensional analysis. The strains are not chosen as a parameter because it is dimensionless and its pi-products will always be 1 (i.e. model and prototype have same strain). The model strains have been plotted as it is (not multiplied by scale factor) along with the prototype. As shown in Figure 4 and Figure 5, the two map onto each other verifying that strains are not a similitude parameter.

Fig. 2: Prototype, predicted prototype and error plots for von mises stress

Fig. 3: Prototype, predicted prototype and error plots for shear stress
6. Conclusion
Using similitude theory, a 1/10th scaled down pressure vessel, which duplicates full scale vessel, has been analyzed and all pi products for the two are verified. Analysis of the results shows that the percentage error between the prototype and mapped model is less than 1.5% for stresses and 0.5% for displacements respectively. This guarantees that scale factors determined using the established similitude theory, can be used to make a scaled model which accurately predicts prototype response of the pressure vessel. Theory of similitude is a very useful tool as it predicts response without developing full scale models. It is very economical where large scale testing is required. It gives the rules that should be followed, if complete prototype results are to be predicted using a scaled model.

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