Zero-Hopf bifurcation in the FitzHugh–Nagumo system

Rodrigo D. Euzébio\textsuperscript{a,b,*†}, Jaume Llibre\textsuperscript{b} and Claudio Vidal\textsuperscript{c}

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We characterize the values of the parameters for which a zero-Hopf equilibrium point takes place at the singular points, namely, \(O\) (the origin), \(P_+\), and \(P_-\) in the FitzHugh–Nagumo system.

We find two two-parameter families of the FitzHugh–Nagumo system for which the equilibrium point at the origin is a zero-Hopf equilibrium. For these two families, we prove the existence of a periodic orbit bifurcating from the zero-Hopf equilibrium point \(O\).

We prove that there exist three two-parameter families of the FitzHugh–Nagumo system for which the equilibrium point at \(P_+\) and at \(P_-\) is a zero-Hopf equilibrium point. For one of these families, we prove the existence of one, two, or three periodic orbits starting at \(P_+\) and \(P_-\).

Keywords: FitzHugh–Nagumo system; periodic orbit; averaging theory; zero-Hopf bifurcation

1. Introduction and statements of the main result

In this paper, we study the zero-Hopf equilibrium points and the zero-Hopf bifurcations of periodic orbits that take place at the equilibria in the FitzHugh–Nagumo system.

This system was introduced in articles of FitzHugh \cite{FitzHugh1961} and Nagumo, Arimoto, and Yoshizawa \cite{Nagumo1962} as one of the simplest models describing the excitation of neural membranes and the propagation of nerve impulses along an axon. In the MathSciNet, you can find several hundreds of papers published on this system, or related with it, but in them the zero-Hopf bifurcation has not been studied.

We consider the following FitzHugh–Nagumo partial differential system:

\[
\begin{align*}
    u_t &= u_{xx} - f(u) - v, \\
    v_t &= \delta(u - \gamma v),
\end{align*}
\]

where \(f(u) = u(u - 1)(u - a)\) and \(0 < a < 1/2\) are constants, and \(\delta > 0\) and \(\gamma > 0\) are parameters. A bounded solution \((u, v)(x, t)\) with \(x, t \in \mathbb{R}\) is called a traveling wave if \((u, v)(x, t) = (u, v)(\xi)\), where \(\xi = x - ct\) and \(c\) is the constant denoting the wave speed. Substituting \(u = u(\xi), \ v = v(\xi)\) into (1), we obtain the following ordinary differential system:

\[
\begin{align*}
    \dot{x} &= z, \\
    \dot{y} &= b(x - dy), \\
    \dot{z} &= x(x - 1)(x - a) + y + cz,
\end{align*}
\]

by introducing a new variable \(w = \dot{u}\), where the dot denotes the derivative with respect to \(\xi\), \(x = u, y = v, z = w, b = \varepsilon / c\) and \(d = \gamma\), see for more details \cite{Euzébio2014}.

In this paper, the ordinary differential system (2) will be called the FitzHugh–Nagumo differential system. We shall study this system depending on the parameters \((a, b, c, d) \in \mathbb{R}^4\).

Here, a zero-Hopf equilibrium is an equilibrium point of a three-dimensional autonomous differential system, which has a zero eigenvalue and a pair of purely imaginary eigenvalues.
In general, the zero-Hopf bifurcation is a two-parameter unfolding of a three-dimensional autonomous differential equation with a zero-Hopf equilibrium. The unfolding has an isolated equilibrium with a pair of purely imaginary eigenvalues and a zero eigenvalue if the two parameters take zero values, and the unfolding has different dynamics in the small neighborhood of this isolated equilibrium as the two parameters vary in a small neighborhood of the origin.

This kind of zero-Hopf bifurcation has been studied by Guckenheimer, Han, Holmes, Kuznetsov, Marsden, and Scheurle in [4–8], and they showed that some complicated invariant sets can bifurcate from the isolated zero-Hopf equilibrium doing the unfolding. In some cases, a zero-Hopf bifurcation implies a local birth of ‘chaos’, see for instance the articles of Baldomá and Seara, Broer and Vegter, Champneys and Kirk, and Scheurle and Marsden in [8–12]. Unfortunately, as far as we know for the moment does not exist a study which describes all the kind of bifurcations that the unfolding of a zero-Hopf bifurcation can produce, consequently, every system exhibiting a zero-Hopf bifurcation must be studied directly.

Because nobody has studied analytically the existence or non-existence of zero-Hopf equilibria and zero-Hopf bifurcations in the FitzHugh–Nagumo differential system, this is the objective of the present article. We must mention that the method used for studying a zero-Hopf bifurcation must be studied directly.

Theorem 1
There are two parameter families of the FitzHugh–Nagumo differential system for which the origin of coordinates is a zero-Hopf equilibrium point, both families are two-parametric. Namely

(i) for \( ad + 1 = 0, bd - c = 0 \) and \( d(1 - b^2d^3) > 0 \); and
(ii) for \( b = c = 0 \) and \( a < 0 \).

In the next proposition, we characterize when the equilibrium point

\[ P_+ = \left( \frac{1 + a}{2}, \frac{1}{2} \sqrt{(a - 1)^2 - \frac{4}{d}} \frac{1 + a}{2d}, \frac{1}{2d} \sqrt{(a - 1)^2 - \frac{4}{d}} 0 \right), \]

if \( d > 0 \) and \( d(a - 1)^2 - 4 > 0 \) of the FitzHugh–Nagumo differential system is a zero-Hopf equilibrium point.

Proposition 2
If \( d > 0 \) and \( d(a - 1)^2 - 4 > 0 \), there are three parameter families of the FitzHugh–Nagumo differential system for which the equilibrium point \( P_+ \) is a zero-Hopf equilibrium point, these families are two-parametric. Namely

(i) for \( b = c = 0 \) and \( (a - 1)^2d + (a + 1) \sqrt{d[(a - 1)^2d - 4]} - 6 < 0 \);
(ii) for \( a - 1 + 2/\sqrt{d} = 0, bd - c = 0 \) and \( 1 - b^2d^3 > 0 \); and
(iii) for \( a - 1 - 2/\sqrt{d} = 0, bd - c = 0 \) and \( 1 - b^2d^3 > 0 \).

In the next proposition, we characterize when the equilibrium point

\[ P_- = \left( \frac{1 + a}{2}, -\frac{1}{2} \sqrt{(a - 1)^2 - \frac{4}{d}} \frac{1 + a}{2d}, -\frac{1}{2d} \sqrt{(a - 1)^2 - \frac{4}{d}} 0 \right), \]

if \( d > 0 \) and \( d(a - 1)^2 - 4 > 0 \) of the FitzHugh–Nagumo differential system is a zero-Hopf equilibrium point.

Proposition 3
If \( d > 0 \) and \( d(a - 1)^2 - 4 > 0 \), there are three parameter families of FitzHugh–Nagumo differential system for which the equilibrium point \( P_- \) is a zero-Hopf equilibrium point, these families are two-parametric. Namely

(i) for \( b = c = 0 \) and \( (a - 1)^2d - (a + 1) \sqrt{d[(a - 1)^2d - 4]} - 6 < 0 \);
(ii) for \( a - 1 + 2/\sqrt{d} = 0, bd - c = 0 \) and \( 1 - b^2d^3 > 0 \); and
(iii) for \( a - 1 - 2/\sqrt{d} = 0, bd - c = 0 \) and \( 1 - b^2d^3 > 0 \).

Note that if \( d > 0 \) and \( d(a - 1)^2 - 4 = 0 \), then the points

\[ P_+ = P_- = \left( \frac{1 + a}{2}, \frac{1 + a}{2d}, 0 \right), \]

and the following result characterizes when \( P_+ = P_- \) is a zero-Hopf equilibrium.
Proposition 4
If \( d > 0 \) and \( d(a - 1)^2 - 4 = 0 \), there is one parameter family of FitzHugh–Nagumo differential system for which the equilibrium point \( P_+ = P_- \) is a zero-Hopf equilibrium point; this family is two-parametric. Namely \( bd - c = 0 \) and \( 1 - b^2d^2 > 0 \).

In the next two theorems, we study when the FitzHugh–Nagumo differential system having a zero-Hopf equilibrium point at the origin of coordinates has a zero-Hopf bifurcation producing some periodic orbit.

Theorem 5
Let \( (a, b, c) = (-1/d + \varepsilon \alpha_1, \beta_0 + \varepsilon \beta_1, \beta_0 d + \varepsilon \gamma_1) \) and assume \( d(1 - \beta_0^2d^2) > 0, \beta_0^2d^4a^2 - (1 - \beta_0^2d^2)^2 > 0, d \neq 1 \) and \( \varepsilon \neq 0 \) sufficiently small. Then, the FitzHugh–Nagumo differential system (2) has a zero-Hopf bifurcation in the equilibrium point of coordinates, and a periodic orbit is born at this equilibrium when \( \varepsilon = 0 \).

See Remark 9 for the type of stability of the periodic orbit which is born in the zero-Hopf bifurcation of Theorem 5.

Theorem 6
Let \( \omega \in (0, \infty) \) and \( (a, b, c) = (-\omega^2 + \varepsilon \alpha_1 + \varepsilon^2 \alpha_2, \beta_1 + \varepsilon^2 \beta_2, \varepsilon \gamma_1 + \varepsilon^2 \gamma_2) \) with \( \varepsilon \) a small parameter. If \( \gamma_1 \omega^2 - \beta_1 = 0, d\omega^2 - 1 = 0, \gamma_1 \neq 0, \omega \neq 1 \) and \( \alpha_1^2 \gamma_1^2 - (\gamma_2^2\omega^2 - \beta_2^2)^2 > 0 \), then the FitzHugh–Nagumo differential system (2) has a zero-Hopf bifurcation in the equilibrium point at the origin of coordinates, and a periodic orbit is born at this equilibrium when \( \varepsilon = 0 \).

Next, we study when the equilibrium point \( P_+ \) of the FitzHugh–Nagumo differential system has a zero-Hopf bifurcation producing some periodic orbit.

Theorem 7
Let \( (a, b, c) = (a_0 + \varepsilon \alpha_1 + \varepsilon^2 \alpha_2, \beta_1 + \varepsilon^2 \beta_2, \varepsilon \gamma_1 + \varepsilon^2 \gamma_2) \) and assume \( da_0 + 1 = 0, a_0 \gamma_1 + \beta_1 = 0, 2a_0^2 + 6\alpha_0 + 1 < 0, a_0 \in (-1, (\sqrt{5} - 3)/2) \), \( \varepsilon \) sufficiently small, and additional conditions on the parameters \( a_0, \alpha_1, \beta_1, \beta_2 \) and \( \gamma_2 \) (see for more details the proof of this theorem). Then, the FitzHugh–Nagumo differential system (2) has a zero-Hopf bifurcation in the equilibrium point \( P_+ \), producing either one, two, or three periodic orbits born at \( P_+ \) when \( \varepsilon = 0 \).

For the equilibrium point \( P_- \) of the FitzHugh–Nagumo differential system, we have the following result.

Theorem 8
Let \( (a, b, c) = (a_0 + \varepsilon \alpha_1 + \varepsilon^2 \alpha_2, \beta_1 + \varepsilon^2 \beta_2, \varepsilon \gamma_1 + \varepsilon^2 \gamma_2) \) and assume \( da_0 + 1 = 0, a_0 \gamma_1 + \beta_1 = 0, 2a_0^2 + 6\alpha_0 + 1 < 0, a_0 \in (-\sqrt{5} + 3)/2, -1) \), \( \varepsilon \) sufficiently small, and additional conditions on the parameters \( a_0, \alpha_1, \beta_1, \beta_2 \) and \( \gamma_2 \) (see for more details the proof of this theorem). Then, the FitzHugh–Nagumo differential system (2) has a zero-Hopf bifurcation in the equilibrium point \( P_- \), producing either one, two, or three periodic orbits born at \( P_- \) when \( \varepsilon = 0 \).

Theorems 5, 6, 7, and 8 are proved in section 2 using the averaging theory of first order or second order for computing periodic orbits; see a summary of this averaging theorems in the appendix.

We note that the kind of stability of the periodic solutions found in Theorems 5, 6, 7, and 8 can be studied using the eigenvalues of an adequate Jacobian matrix; see for more details the last paragraphs of the appendix.

As we see in Propositions 2 and 3 under the restrictions \( d > 0 \) and \( (a - 1)^2 - 4d > 0 \), there are three parameter families of FitzHugh–Nagumo differential systems for which the equilibrium points \( P_+ \) and \( P_- \) are zero-Hopf. According to Theorems 7 and 8 for the points \( P_+ \) and \( P_- \), we get zero-Hopf bifurcations only for the zero-Hopf equilibrium of the statement (ii) of Propositions 2 and 3, respectively.

The averaging method of first and second order do not provide any information if a Hopf bifurcation takes place in the zero-Hopf equilibrium of statements (ii) and (iii) of Propositions 2 and 3, or of Proposition 4.

Furthermore, analyzing the conditions for the existence of small–amplitude periodic solutions coming from the zero-Hopf bifurcations of Theorems 5, 6, 7, and 8, we observe that we can have zero-Hopf bifurcations at the origin and at \( P_+ \) simultaneously, therefore we can obtain either two, three, or four periodic orbits simultaneously bifurcating from both equilibria, one from the origin and one, two, or three from \( P_+ \). The same simultaneous zero-Hopf bifurcations can take place at the origin and at \( P_- \).

We must mention that many of the steps in the proofs of our theorems have been made with the help of an algebraic manipulator as mathematica.

Related works also with the differential system (2) are the ones in [19] and [20] where the authors investigate traveling wave solutions of the FitzHugh–Nagumo equation from the viewpoint of fast–slow dynamical systems. In the first paper, they studied the structure of the bifurcation diagram based on geometric singular perturbation analysis. In the second work, they proved the existence of homoclinic orbits and families of periodic orbits ending on them.

On the other hand, the analytical integrability of the FitzHugh–Nagumo system (2) depending on the parameters \( a, b, c, d \in \mathbb{R} \) has been studied in [21], and noise perturbation of this differential system were considered in [22].

2. Proofs

2.1. Proof of Propositions 1, 2, 3 and 4
System (2) has three equilibrium points, \( (0, 0, 0) \), \( P_+ \) and \( P_- \) if \( d > 0 \) and \( d(a - 1)^2 - 4 > 0 \), and only two \( (0, 0, 0) \) and \( P_+ = P_- \) if \( d > 0 \) and \( d(a - 1)^2 - 4 = 0 \).

The characteristic polynomial of the linear part of system (2) at the origin is

\[ p_1(\lambda) = \lambda^3 - (c - bd)\lambda^2 - (a + bcd)\lambda - b(1 + ad). \]
Because we must have one null eigenvalue, it is necessary that
\[ b(1 + ad) = 0 \iff b = 0 \quad \text{or} \quad 1 + ad = 0. \]

Now, we impose that the other two eigenvalues must be pure imaginary, namely, \( \pm i\omega \), then
\[ p(\lambda) = \lambda(\lambda^2 + \omega^2), \]
then, we must have
\[ c - bd = 0 \quad \text{and} \quad \omega^2 = -(a + bcd). \]

In the case \( b = 0 \), we obtain that
\[ a = -\omega^2, \quad b = 0, \quad c = 0. \]

Thus, we have proved item (ii) of Proposition 1. In the case \( 1 + ad = 0 \), we have \( c - bd = 0 \) and \( \omega^2 = -(a + bcd) \), consequently we have proved (i) in Proposition 1.

Now, we observe that the characteristic polynomial \( p_\pm(\lambda) \) of the linear part of system (2) at the points \( P_\pm \) is
\[
\begin{align*}
\lambda^3 - (c - bd)\lambda^2 - \frac{(a - 1)^2 d \pm (a + 1) \sqrt{d[(a - 1)^2 d - 4]} + 2bcd - 6\lambda}{2d} \\
- \frac{b}{2} \left( (a - 1)^2 d \pm (a + 1) \sqrt{d[(a - 1)^2 d - 4]} - 4 \right).
\end{align*}
\]

Again, we impose that the roots of \( p_\pm(\lambda) \) are 0 and the other two roots are pure imaginary, namely \( \pm i\omega \), so the following conditions must hold:
\[ b \left( (a - 1)^2 d \pm (a + 1) \sqrt{d[(a - 1)^2 d - 4]} - 4 \right) = 0, \quad c - bd = 0, \]
and
\[ \omega^2 = \frac{(a - 1)^2 d \pm (a + 1) \sqrt{d[(a - 1)^2 d - 4]} + 2bcd - 6}{2d}. \]

Analyzing the solutions of the previous system the proof of Propositions 2 and 3 follow.

The proof of Proposition 4 follows as the previous ones.

### 2.2. Proof of Theorem 5

If \( (a, b, c) = (-1/d + \varepsilon a, \beta_0 + \varepsilon \beta_1, \beta_0 d + \varepsilon y) \) and \( \varepsilon \) is a small parameter, then FitzHugh–Nagumo system (2) takes the form
\[
\begin{align*}
\dot{x} &= x, \\
\dot{y} &= (\beta_0 + \varepsilon \beta_1)(x - dy), \\
\dot{z} &= \frac{1}{d} \left( \beta_0 dz + dx^3 - dx^2 + dy + x^2 - x \right) + \varepsilon(\beta_1 dx - ax^2 + ax + yz).
\end{align*}
\]

The eigenvalues at the origin of system (3) are 0 and \( \pm \sqrt{(d^2 \beta_0^2 - 1)/d} \), so we need that \( d(d^2 \beta_0^2 - 1) = -\omega^2 < 0 \). This is true by assumption. So we take
\[ \beta_0 = \frac{1}{d} \sqrt{\frac{1}{d} - \omega^2} \quad \text{with} \quad \frac{1}{d} - \omega^2 > 0. \]

Next, we do the rescaling of the variables \( (x, y, z) = (\varepsilon X, \varepsilon Y, \varepsilon Z) \), then system (3) in the new variables \( (X, Y, Z) \) writes
\[
\begin{align*}
\dot{X} &= Z, \\
\dot{Y} &= \frac{1}{d} \sqrt{\frac{1}{d} - \omega^2} (X - dY) + \varepsilon \beta_1 (X - dY), \\
\dot{Z} &= \frac{1}{d} X + Y + \sqrt{\frac{1}{d} - \omega^2} Z + \varepsilon \left( \alpha X + \left( \frac{1}{d} - 1 \right) X^2 + (Y + \beta_1 d) Z \right) + \varepsilon^2 X^2 (X - \alpha).
\end{align*}
\]

In order to calculate the fundamental matrix solution, next we write the linear part at the origin of the ordinary differential system (4) when \( \varepsilon = 0 \) into its real Jordan normal form, that is, as
\[
J = \begin{pmatrix}
0 & -\omega & 0 \\
\omega & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
We verify that this change of variable

\[(X,Y,Z) = P(u,v,w),\]

(6)
can be done by the matrix

\[P = \begin{pmatrix}
-\frac{1}{\omega^2} & 0 & \frac{1}{\omega} \left(\frac{1}{d} - \omega^2\right)
\end{pmatrix}
\]

In the new variables \((u,v,w)\), system (4) writes

\[
\begin{align*}
\dot{u} &= -\omega v + \varepsilon \left[ \beta_1 d \left( \frac{1}{\omega} \sqrt{\frac{1}{d} - \omega^2} v - u \right) + \sqrt{\frac{1}{d} - \omega^2} \cdot \\
&\quad \left( \frac{\alpha}{\omega^2} \left( \frac{1}{d} - \omega^2 - u \right) + \frac{1}{\omega^2} \left( \frac{1}{d} - 1 \right) \left( u - \sqrt{\frac{1}{d} - \omega^2} w \right)^2 \\
&\quad + \frac{\gamma + \beta_1 d}{\omega} v \right) \right] + O(\varepsilon^2),
\end{align*}
\]

\[
\dot{v} = \omega u + \varepsilon \left[ (\gamma + \beta_1 d) v + \frac{1}{\omega^2} \left( \frac{1}{d} - 1 \right) \left( u - \sqrt{\frac{1}{d} - \omega^2} w \right)^2 \\
+ \frac{\alpha}{\omega} \left( \frac{1}{d} w - \omega^2 - u \right) \right] + O(\varepsilon^2),
\]

\[
\dot{w} = \varepsilon \left[ -\frac{\beta_1 d}{\omega} u + \frac{\alpha}{\omega^2} \left( \sqrt{\frac{1}{d} - \omega^2} w - u \right) + \frac{1}{\omega^2} \left( \frac{1}{d} - 1 \right) \right] .
\]

(7)

Now, we write this differential system in cylindrical coordinates \((r,\theta,w)\) defined by \(u = r \cos \theta\), \(v = r \sin \theta\), \(w = w\) and after we introduce \(\theta\) the new independent variable, and so we arrive to the system

\[
\begin{align*}
\frac{dr}{d\theta} &= \varepsilon \left[ \omega \sin \theta \left( \frac{\gamma + \beta_1 d}{\omega} r \sin \theta + \frac{\alpha}{\omega^2} \left( \sqrt{\frac{1}{d} - \omega^2} w - r \cos \theta \right) \\
&\quad + \frac{1}{\omega^2} \left( \frac{1}{d} - 1 \right) \left( \frac{1}{d} - \omega^2 W - r \cos \theta \right)^2 \right) + r \cos \theta \cdot \\
&\left( \frac{\beta_1 d}{\omega} \sqrt{\frac{1}{d} - \omega^2} r \sin \theta - \beta_1 d \cos \theta + \frac{1}{\omega^2} \left( \frac{\gamma + \beta_1 d}{\omega} \right) \right) \right] + O(\varepsilon^2) \\
&= \varepsilon F_1(\theta, r, w) + O(\varepsilon^2),
\end{align*}
\]

(8)

\[
\begin{align*}
\frac{dw}{d\theta} &= \varepsilon \left[ -\frac{\beta_1 d}{\omega} r \cos \theta + \frac{\alpha}{\omega^2} \left( \sqrt{\frac{1}{d} - \omega^2} w - r \cos \theta \right) + \frac{1}{\omega^2} \left( \frac{1}{d} - 1 \right) \left( \sqrt{\frac{1}{d} - \omega^2} w - r \cos \theta \right)^2 \right] + O(\varepsilon^2) \\
&= \varepsilon F_2(\theta, r, w) + O(\varepsilon^2).
\end{align*}
\]
Our previous system has the form of the differential equation (16) with \( t = \theta, x = (r, w) \in \Omega = (0, +\infty) \times \mathbb{R}, T = 2\pi, z = (r_0, w_0) \) and \( F(\theta, r, w) = (F_1(\theta, r, w), F_2(\theta, r, w)) \), and an easy computation shows that
\[
f(r_0, w_0) = (f_1(r_0, w_0), f_2(r_0, w_0))
\]
is given by
\[
f_1(r_0, w_0) = \frac{1}{2\pi} \int_0^{2\pi} F_1(\theta, r_0, w_0) d\theta = \frac{1}{2d^2\omega^2} \left[ d^2 \left( y_f^2 - \alpha \omega^2 \sqrt{\frac{1}{d} - \omega^2} \right) + 2w_0(d-1)(1-d\omega^2) \right],
\]
\[
f_2(r_0, w_0) = \frac{1}{2\pi} \int_0^{2\pi} F_2(\theta, r, w) d\theta = \frac{1}{2d^2\omega^2} \left[ d^2 \left( 2\omega^2 \left( \alpha \sqrt{\frac{1}{d} - \omega^2} + w_0 - r_0 \right) + d \left( r_0 - 2(\omega^2 + 1)w_0^2 \right) \right) \right].
\]
The system \( f_1(r_0, w_0) = f_2(r_0, w_0) = 0 \) has two solutions \((r^*_1, w^*_1)\) with \( r^*_1 > 0 \), namely
\[
(r^*_1, w^*_1) = \left( \frac{d\omega^2}{1-d}, \frac{d^2\omega^2[y_f^2 - \alpha \sqrt{\frac{1}{d} - \omega^2}]}{2(d-1)(d\omega^2 - 1)} \right),
\]
and the other solution is
\[
(r^*_2, w^*_2) = (-r^*_1, w^*_1),
\]
where
\[
\Gamma = \frac{1}{\omega^2 - \frac{1}{d}} \left[ y_f^2 \omega^4 + \alpha^2 \left( \omega^2 - \frac{1}{d} \right) \right].
\]
The first solution exists if \( d(1-d) > 0 \) and \( \Gamma > 0 \), and the second solution exists if \( d(1-d) < 0 \) and \( \Gamma > 0 \). We verify that, in both situations, the Jacobian (17) takes the value
\[
\frac{d}{\omega^2} \left( \frac{1}{d} - \omega^2 \right) \Gamma \neq 0.
\]
Note that \( \Gamma > 0 \) if and only if
\[
y_f^2 \omega^4 + \alpha^2 \left( \omega^2 - \frac{1}{d} \right) = \frac{d(\beta_0^2d^4u_0^2 - (1-\beta_0^2d^2)^2\gamma^2)}{2(\beta_0^2d^2 - 1)^3} < 0.
\]
This inequality holds by assumptions.

The rest of the proof of the theorem follows immediately from Theorem 11 if we show that the periodic solution corresponding to the equilibrium point \((r^*, w^*)\) provides a periodic orbit bifurcating from the origin of coordinates of the differential system (4) at \( \varepsilon = 0 \).

If \( d \neq 0,1 \), then Theorem 11 guarantees for \( \varepsilon \neq 0 \) sufficiently small the existence of a periodic orbit corresponding to the point \((r^*, w^*)\) of the form \((r(\theta, \varepsilon), w(\theta, \varepsilon))\) for system (8) such that \((r(0, \varepsilon), w(0, \varepsilon)) \rightarrow (r^*, w^*)\) when \( \varepsilon \rightarrow 0 \). So, system (7) has the periodic solution
\[
(a(\theta, \varepsilon) = r(\theta, \varepsilon) \cos \theta, v(\theta, \varepsilon) = r(\theta, \varepsilon) \sin \theta, w(\theta, \varepsilon)), \tag{9}
\]
for \( \varepsilon \) sufficiently small. Consequently, system (4) has the periodic solution \((X(\theta), Y(\theta), Z(\theta))\) obtained from relation (9) through the linear change of variables (6). Finally, for \( \varepsilon \neq 0 \) sufficiently small, system (3) has a periodic solution \((x(\theta), y(\theta), z(\theta)) = (\varepsilon X(\theta), \varepsilon Y(\theta), \varepsilon Z(\theta))\), which goes to the origin of coordinates when \( \varepsilon \rightarrow 0 \). Thus, it is a periodic solution starting at the zero-Hopf bifurcation point located at the origin of coordinates when \( \varepsilon = 0 \). This concludes the proof of Theorem 5.

Remark 9

We note that the eigenvalues of the matrix
\[
\left. \left( \frac{\partial (f_1, f_2)}{\partial (r_0, w_0)} \right) \right|_{(r_0, w_0) = (r^*, w^*)}
\]
in the previous proof will provide the type of stability of the periodic orbits that are born in the zero-Hopf bifurcation, but because their expression is huge, we do not consider them here.
2.3. Proof of Theorem 6

If \((a, b, c) = (-\omega^2 + \varepsilon \alpha_1 + \varepsilon^2 \alpha_2, \varepsilon \beta_1 + \varepsilon^2 \beta_2, \varepsilon \gamma_1 + \varepsilon^2 \gamma_2)\) with \(\varepsilon\) a small parameter, then the FitzHugh–Nagumo system takes the form

\[
\begin{align*}
\dot{x} &= z, \\
\dot{y} &= \varepsilon \beta_1 (x - dy) + \varepsilon^2 \beta_2 (x - dy), \\
\dot{z} &= x(x - 1)(x + \omega^2) + y + \varepsilon [\alpha_1 x(1 - x) + \gamma_1 z] + \\
&+ \varepsilon^2 [\gamma_2 z - \alpha_2 x(1 - x)].
\end{align*}
\] (10)

Rescaling the variables \((x, y, z) = (\varepsilon X, \varepsilon Y, \varepsilon Z)\) system (10) is equivalent to

\[
\begin{align*}
\dot{X} &= Z, \\
\dot{Y} &= \varepsilon \beta_1 (X - dY) + \varepsilon^2 \beta_2 (X - dY), \\
\dot{Z} &= Y - \omega^2 X + \varepsilon \left[ X (\alpha_1 + (\omega^2 - 1) X) + \gamma_1 Z \right] + \\
&+ \varepsilon^2 \left[ X(X^2 - \alpha_1 X + \alpha_2) + \gamma_2 Z \right] - \varepsilon^2 \alpha_2 X^2.
\end{align*}
\] (11)

Analogously to the first case, we shall write the linear part at the origin of system (11) when \(\varepsilon = 0\) into its real Jordan normal form as in (5). We do that considering the linear change of variables \((X, Y, Z) = P(u, v, w)\) where the matrix change of coordinates \(P\) is given by

\[
\begin{pmatrix}
0 & 1 & 1 \\
0 & 0 & \frac{\omega}{2} \\
1 & 0 & 1
\end{pmatrix}.
\]

System (11) in the new variables \((u, v, w)\) assumes the form

\[
\begin{align*}
\dot{u} &= -\omega v + \varepsilon \left[ \gamma_1 u + \frac{\omega}{2} (\omega v + w) (\alpha_1 \omega^2 + (\omega^2 - 1) (\omega v + w)) \right] + \\
&+ \varepsilon^2 \left[ \gamma_2 u + \frac{\omega}{2} (\omega v + w) (\alpha_1 \omega^2 + (\omega^2 - 1) (\omega v + w)) \right], \\
\dot{v} &= \omega u - \varepsilon \frac{\beta_1}{\omega} [\omega v + w - d\omega^2 w] - \varepsilon^2 \frac{\beta_2}{\omega} [\omega v + w - d\omega^2 w], \\
\dot{w} &= \varepsilon \frac{\beta_1}{\omega^2} [\omega v + w - d\omega^2 w] + \varepsilon^2 \frac{\beta_2}{\omega^2} [\omega v + w - d\omega^2 w].
\end{align*}
\]

Next, we write the system in cylindrical coordinates \((r, \theta, w)\) as \(u = r \cos \theta, v = r \sin \theta,\) and we introduce \(\theta\) as the new independent variable, so we obtain

\[
\begin{align*}
\frac{dr}{d\theta} &= \varepsilon \frac{1}{\omega^2} \left[ -\beta_1 \omega \sin \theta (w - d\omega^2 w + r_0 \sin \theta) + \omega^2 \gamma_1 r \cos^2 \theta + \\
&+ \omega \cos \theta (\omega \sin \theta + \omega \cos \theta) (\alpha_1 + w)\right] + \\
&+ \varepsilon^2 \frac{1}{\omega^2} \left[ -\beta_1 \omega \sin \theta (w - d\omega^2 w + r_0 \sin \theta) + \omega \cos \theta (\omega \sin \theta + \omega \cos \theta) (\alpha_1 + w)\right], \\
\frac{dw}{d\theta} &= \varepsilon \frac{\beta_1}{\omega^3} [w - d\omega^2 w + r_0 \sin \theta] + \varepsilon^2 \frac{1}{\omega^2} \left[ (w - d\omega^2 w + r_0 \sin \theta) \right] \\
&+ \beta_1 \sin \theta (\omega \sin \theta + \omega \cos \theta) (\alpha_2 + \frac{1}{\omega^2} (\omega \sin \theta + \\
&+ \omega \cos \theta) (w - \alpha_1 \cos^2 \omega + r_0 \sin \theta)) - \beta_2 \frac{r_0 \sin \theta} {\omega^3} \frac{r_0 \sin \theta} {\omega^3} + O(\varepsilon^3),
\end{align*}
\]

Using the notation of Theorem 11, we have that the averaging function (19) has the two components

\[
(f_1(r_0, w_0), f_2(r_0, w_0)) = \left( \frac{r_0(\gamma_1 \omega^2 - \beta_1)}{2 \omega^3}, \frac{\beta_1 w_0 (1 - d \omega^2)}{\omega^3} \right).
\]

Therefore, the solutions of system \(f_1(r_0, w_0) = f_2(r_0, w_0) = 0\) with \(\gamma_1 \omega^2 - \beta_1 \neq 0\) have \(r_0 = 0, \) so they are not good solutions because \(r_0\) must be positive. In order to apply the averaging of second order, we need that \(f_1 = 0 \) and \(f_2 = 0.\) So, we take
\[ \beta_1 = \gamma_1 \omega^2 \quad \text{and} \quad d = \frac{1}{\omega^2}. \]

Using the notation of Theorem 11 of the appendix, we obtain

\[
\begin{align*}
&g_1(r_0, w_0) = \frac{r_0}{2\omega^2} \left[ \gamma_2 \omega^4 - \omega^2 (\beta_2 + \gamma_1 (\alpha_1 + 2w_0)) + 2 \gamma_1 w_0 \right], \\
&g_2(r_0, w_0) = \frac{\gamma_1}{2\omega^2} \left[ \frac{r_0^2}{2} \omega^2 (\omega^2 - 1) + 2w_0^2 (\omega^2 - 1) + 2 \alpha_1 \omega^2 w_0 \right].
\end{align*}
\]

Here, we obtain that the system \( g_1(r, w) = g_2(r, w) = 0 \) has the solution

\[
\begin{align*}
r^* &= \frac{\omega}{\sqrt{2}|\gamma_1| (\omega^2 - 1)} \sqrt{\alpha_1^2 \gamma_1^2 - (\gamma_2 \omega^2 - \beta_2^2)^2}, \\
w^* &= -\frac{\omega^2 (\alpha_1 \gamma_1 + \beta_2 - \gamma_2 \omega^2)}{2 \gamma_1 (\omega^2 - 1)},
\end{align*}
\]

when

\[
\gamma_1 \neq 0, \quad \omega \neq 1 \quad \text{and} \quad \alpha_1^2 \gamma_1^2 - (\gamma_2 \omega^2 - \beta_2^2)^2 > 0. \tag{12}
\]

Then, the Jacobian (17) takes the value

\[
\frac{\alpha_1^2 \gamma_1^2 - (\gamma_2 \omega^2 - \beta_2^2)^2}{\omega^6} \neq 0.
\]

The rest of the proof of Theorem 6 follows as in the proof of Theorem 5.

2.4. Proof of Theorems 7 and 8

Let \((a, b, c) = (\alpha_0 + \varepsilon \alpha_1 + \varepsilon^2 \alpha_2, \varepsilon \beta_1 + \varepsilon^2 \beta_2, \varepsilon \gamma_1 + \varepsilon^2 \gamma_2), \varepsilon > 0 \) small enough, \( d > 0 \) and \( d(\alpha_0 - 1)^2 - 4 < 0 \). Because the arguments of the proof for the equilibria \( P_+ \) and \( P_- \) are very similar, we only prove Theorem 7.

First, we translate the point \( P_+ \) to the origin of coordinates and, maintaining the notation \((x, y, z)\) for the new coordinates, we have that the FitzHugh–Nagumo system (2) takes the form

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= \varepsilon (\beta_1 + \varepsilon \beta_2) (x - dy), \\
\dot{z} &= \frac{1}{2d} \left[ 2dx^2 + \alpha_0 dx^2 + dx^2 + \alpha_0^2 dx - 2 \alpha_0 dx + dx + 2dy - 6x \\
&\quad - \sqrt{dx} (1 + \alpha_0 + 3x) \sqrt{d(\alpha_0 + \varepsilon \alpha_1 + \varepsilon^2 \alpha_2 - 1)^2 - 4} \right] \\
&\quad + \varepsilon \left[ \alpha_1 (\alpha_0 - 1) x + \gamma_1 z + \alpha_1 x^2 + \frac{\alpha_1^2}{2} \frac{x}{\sqrt{d}} - \frac{\alpha_1}{\sqrt{d}} \frac{x}{x} \right] \\
&\quad + \frac{\varepsilon^2}{2} \left[ \alpha_2 (\alpha_0 - 1) x + \gamma_1 z + \alpha_1 x^2 + \frac{\alpha_2}{\sqrt{d}} \frac{x}{x} \right] \\
&\quad + \frac{\varepsilon^3}{2} \left[ \alpha_3 (\alpha_0 - 1) x + \gamma_1 z + \alpha_1 x^2 + \frac{\alpha_3}{\sqrt{d}} \frac{x}{x} \right]. 
\end{align*}
\]

The eigenvalues of the linear part of system (13) at the origin are

\[ 0, \quad \pm \frac{\sqrt{d(\alpha_0 + 1)^2 + (\alpha_0 + 1) \sqrt{d(\alpha_0 + 1)^2 - 4}}} {2d}. \]

We have that \( d(\alpha_0 + 1)^2 + (\alpha_0 + 1) \sqrt{d(\alpha_0 + 1)^2 - 4} - 6 = -2 < 0 \), this holds using the assumptions \( d = -1/\alpha_0 \) and \( \alpha_0 < 0 \). Next, we consider the change of variables \((x, y, z) \rightarrow (r, \theta, w)\), obtained firstly by the rescaling \((x, y, z) = (\varepsilon X, \varepsilon Y, \varepsilon Z)\), after doing the linear change of variables \((u, v, w)\) defined by \((X, Y, Z)^T = P(u, v, w)^T\) where

\[
P = \begin{pmatrix}
0 & 1 & \frac{2d}{\sigma} \\
0 & 0 & \sigma \\
\frac{\sigma}{2d} & 0 & 0
\end{pmatrix}.
\]
where
\[
\sigma = 6 - d(\alpha_0 - 1)^2 - (\alpha_0 + 1) \sqrt{d[\alpha_0(1 - \alpha_0)^2 - 4]},
\]
and finally passing to cylindrical coordinates \(u = r \cos \theta, v = r \sin \theta, w = w\). After introducing \(\theta\) as the new independent variable, the first-order averaging function \(f = (f_1, f_2)\) is given by
\[
f_1 = \frac{\sqrt{d_0} \left(6\gamma_1 - d((\alpha_0 - 1)^2\gamma_1 + 2\beta_1) - (\alpha_0 + 1)\gamma_1 \sqrt{d(\alpha_0 - 1)^2d - 4})\right)}{\sqrt{2} \left(6 - (\alpha_0 - 1)^2d - (\alpha_0 + 1)\sqrt{d(\alpha_0 - 1)^2d - 4})\right)^{3/2}},
\]
\[
f_2 = \frac{\beta_1d^{1/2}w_0 \left(\frac{(\alpha_0 - 1)^2d + (\alpha_0 + 1)\sqrt{d(\alpha_0 - 1)^2d - 4})}{\sqrt{2} \left(6 - (\alpha_0 - 1)^2d - (\alpha_0 + 1)\sqrt{d(\alpha_0 - 1)^2d - 4})\right)^{1/2}}\right)}{\sqrt{2} \left(6 - (\alpha_0 - 1)^2d - (\alpha_0 + 1)\sqrt{d(\alpha_0 - 1)^2d - 4})\right)^{1/2} \sqrt{(\alpha_0 - 1)^2d - 4}}.
\]
The solutions \((r^*, w^*)\) of \(f_1 = f_2 = 0\) have \(r^* = 0\), so they are not good. We must take \(f_1 = f_2 = 0\) and apply averaging of second order. The solutions of \(f_1 = f_2 = 0\) are either
\[
d = -\frac{1}{\alpha_0} \quad \text{and} \quad \gamma = -\frac{\beta}{\alpha_0} \quad \text{if} \ \alpha_0 \neq 0,
\]
or
\[
d = \frac{4}{(\alpha_0 - 1)^2} \quad \text{and} \quad \gamma = \frac{4\beta}{(\alpha_0 - 1)^2} \quad \text{if} \ \alpha_0 \neq 1.
\]
First, we study case (14). The expressions of the second-order averaging function \(g = (g_1, g_2)\) are too long, so we decide not to include them here. In order to get our result, first we determine \(r^* = r_0(w_0)\) such that \(g_1(r^*, w_0) = 0\), this solution is given by
\[
r^* = 6(\alpha_0 + 1)^2\beta w_0 \left[\sqrt{-\alpha_0^2 - 3\alpha_0 - 1} (2\alpha_0^2\gamma_2 + 18\alpha_0^3\gamma_2 + 2\alpha_0^5(\alpha_1\beta_1 + \beta_2
+ 30\gamma_2) + 9\alpha_0^2(\alpha_1\beta_1 + \beta_2 - \alpha_1\beta_1 - \alpha_0\beta_1 4\pi \beta_1 \sqrt{-\alpha_0^2 - 3\alpha_0 - 1} + \alpha_1
+ \beta_2 + \gamma_2) - \alpha_1\beta_1 - \alpha_0\beta_1 (4\pi \beta_1 \sqrt{-\alpha_0^2 - 3\alpha_0 - 1} + \alpha_1
- \pi \sqrt{-\alpha_0^2 - 3\alpha_0 - 1} \beta_1^2 + \alpha_0^2 (-\pi \beta_1^2 \sqrt{-\alpha_0^2 - 3\alpha_0 - 1} + \alpha_1\beta_1 + 22\beta_2
+ 60\gamma_2) + 2\alpha_0^3 (-2\pi \beta_1^2 \sqrt{-\alpha_0^2 - 3\alpha_0 - 1} + 2\alpha_1\beta_1 + 60\beta_2 + 9\gamma_2)
- 8\alpha_0^3 + 2\alpha_0^2 \beta_1 + 2\alpha_0 + 1) \alpha_0\beta_1 w_0 \right]^{-1}.
\]
It is not difficult to check that \(r^* = 0\) if \(\alpha_0 < -1\). Moreover, \(r^*\) is real only for \(\alpha_0 \in (-1, 1/2(\sqrt{3} - 3))\). Next, we substitute this value of \(r = r^*\) in the equation \(g_2(r^*, w) = 0\), and then we obtain a polynomial in the independent variable \(w\) of the form \(w h(w)\), where \(h(w)\) is a polynomial of degree 3 in \(w\). Because \(w = 0\) implies \(r^* = 0\), we conclude that we can have either one, two, or three solutions of the form \((r^*, w^*)\) with \(r^* > 0\). Consequently, by Theorem 11, we can have one, two, or three periodic solutions bifurcating from the equilibrium point \(P_+\).

Now, we consider the case (15). For this values of \(d\) and \(\gamma\), the differential system \((dr/d\theta, dw/d\theta)\) is not well defined. So, the solution (15) is not good for finding periodic orbits. This completes the proof of Theorem 7.

Remark 10

Here, we will exhibit examples showing that we have three, two, or one periodic orbits born at \(P_+\) when \(\varepsilon = 0\) in Theorem 7. First, considering \(\alpha_0 = -0.8, \alpha_1 = 1, \beta_1 = 1, \beta_2 = 1, \gamma_2 = -2\), we obtain three positive solutions for \(r^*\), and then in Theorem 7, we have three periodic orbits born at \(P_+\) when \(\varepsilon = 0\).

Second, considering \(\alpha_0 = -0.8, \alpha_1 = 1, \beta_1 = 1, \beta_2 = 1, \gamma_2 = 2\), we obtain two positive solutions for \(r^*\), and then in Theorem 7, we have two periodic orbits born at \(P_+\) when \(\varepsilon = 0\).

Third, considering \(\alpha_0 = -0.8, \alpha_1 = 1, \beta_1 = 1, \beta_2 = -1, \gamma_2 = 2\), we obtain one positive solution for \(r^*\), and then in Theorem 7, we have one periodic orbit born at \(P_+\) when \(\varepsilon = 0\).

If we consider \(\alpha_0 = -0.8, \alpha_1 = 1, \beta_1 = -1, \beta_2 = -1, \gamma_2 = 10\), we do not obtain positive solutions for \(r^*\), and in this case, we do not obtain periodic orbits bifurcating from \(P_+\).
Appendix: The averaging theory of first and second order

In this appendix, we recall the averaging theory of first and second order to find periodic orbits, see for more details [23] and [24].

The averaging theory is a classical and matured tool for studying the behavior of the dynamics of nonlinear smooth dynamical systems, and in particular of their periodic orbits. The method of averaging has a long history that starts with the classical works of Lagrange and Laplace, who provided an intuitive justification of the process. The first formalization of this procedure is due to Fatou [25] in 1928. Important practical and theoretical contributions in this theory were made by Krylov and Bogoliubov [26] in the 1930s and Bogoliubov [27] in 1945.

Theorem 11

Consider the differential system

\[ \dot{x}(t) = \varepsilon F(t, x) + \varepsilon^2 G(t, x) + \varepsilon^3 R(t, x, \varepsilon), \quad (16) \]

where \( F, G : \mathbb{R} \times D \to \mathbb{R}^n \), \( R : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n \) are continuous functions, \( T \)-periodic in the first variable, and \( D \) is an open subset of \( \mathbb{R}^n \). Assume that the following hypotheses (i) and (ii) hold.

(i) \( F(t, \cdot), G(t, \cdot) \in C^1(D) \) for all \( t \in \mathbb{R} \), \( F, G, R, D_x F \) and \( D_x G \) are locally Lipschitz with respect to \( x \), and \( R \) is differentiable with respect to \( \varepsilon \). We define \( f, g : D \to \mathbb{R}^n \) as

\[
\begin{align*}
    f(z) &= \frac{1}{T} \int_0^T F(s, z) \, ds, \\
    g(z) &= \frac{1}{T} \int_0^T \left( D_s F(s, z) \right) \left( \int_0^s F(t, z) \, dt + G(s, z) \right) \, ds.
\end{align*}
\]

(ii) For \( V \subset D \), an open and bounded set and for each \( \varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\} \), there exists \( p \in V \) such that \( f(p) + \varepsilon g(p) = 0 \) and

\[ \det \left( \frac{\partial(f + \varepsilon g)}{\partial z} \bigg|_{z=p} \right) \neq 0, \quad \text{(17)} \]

Then, for \( |\varepsilon| > 0 \) sufficiently small, there exists a \( T \)-periodic solution \( \varphi(t, \varepsilon) \) of system (16) such that \( \varphi(0, \varepsilon) \to p \) when \( \varepsilon \to 0 \).

If the function \( f \) is not identically zero, then the zeros of \( f + \varepsilon g \) are mainly the zeros of \( f \) for \( \varepsilon \) sufficiently small. In this case, Theorem 11 provides the so-called averaging theory of first order.

If the function \( f \) is identically zero and \( g \) is not identically zero, then the zeros of \( f + \varepsilon g \) are the zeros of \( g \). In this case, Theorem 11 provides the so-called averaging theory of second order.

In the case of the averaging theory of first order, we consider in \( D \) the averaged differential equation

\[ \dot{y} = \varepsilon f(y), \quad y(0) = x_0, \quad (18) \]

where

\[ f(y) = \frac{1}{T} \int_0^T F(t, y) \, dt. \quad (19) \]

Then Theorem 11 gives us information about the stability or instability of the limit cycle \( \varphi(t, \varepsilon) \). In fact, it is given by the stability or instability of the equilibrium point \( p \) of the averaged system (18). In fact, the singular point \( p \) has the stability behavior of the Poincaré map associated to the limit cycle \( \varphi(t, \varepsilon) \). In the case of the averaging theory of second order, that is, \( f \equiv 0 \) and \( g \) non-identically zero, we have that the stability and instability of the limit cycle \( \varphi(t, \varepsilon) \) coincide with the type of stability or instability of the equilibrium point \( p \) of the averaged system

\[ \dot{y} = \varepsilon^2 g(y), \quad y(0) = x_0, \]

that is, it is the same that the singular point \( p \) associated the Poincaré map of the limit cycle \( \varphi(t, \varepsilon) \).

For additional information on averaging theory, see the book [28].

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