Multiparticle random walks
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ABSTRACT

An overview is presented of recent work on some statistical problems on multiparticle random walks. We consider a Euclidean, deterministic fractal or disordered lattice and $N \gg 1$ independent random walkers initially ($t = 0$) placed onto the same site of the substrate. Three classes of problems are considered: (i) the evaluation of the average number ($S_N(t)$) of distinct sites visited (territory explored) up to time $t$ by the $N$ random walkers, (ii) the statistical description of the first passage time $t_{j,N}$ to a given distance of the first $j$ random walkers (order statistics of exit times), and (iii) the statistical description of the time $t_{j,N}$ elapsed until the first $j$ random walkers are trapped when a Euclidean lattice is randomly occupied by a concentration $c$ of traps (order statistics of the trapping problem). Although these problems are very different in nature, their solutions share the same form of a series in $\ln^{-m}(N)\ln^{m}\ln(N)$ (with $n \geq 1$ and $0 \leq m \leq n$) for $N \gg 1$. These corrective terms contribute substantially to the statistical quantities even for relatively large values of $N$.

I. INTRODUCTION

It is now a commonplace in the history of physics to cite the works of the botanist Robert Brown (circa 1827) on the stochastic movement of pollen grains suspended in water as the starting point in our understanding of the microscopic basis of diffusion. In the “annus mirabilis” of 1905, Einstein gave a successful theoretical explanation of the phenomenon in terms of an atomic theory which was later brilliantly confirmed by the experiments of Perrin and served to remove the reluctance toward the atomic hypothesis that scientists as eminent as Ostwald and Mach still entertained [1]. It was in this context that the theory of a single random walker emerged. In the simplest random walk model a particle occupies a site of a lattice (regular, fractal or disordered) and performs a jump to a randomly selected nearest neighbour of that site every time step [2]. Random jumps are an effective way of simulating the net fluctuating force experienced by the Brownian particle suspended in the liquid. This discrete definition of the random walk has gained widespread acceptance since the availability of computers because, as are all lattice and automata models, it is especially suitable for computer simulation. Random walks are also a way of describing fluctuations in the diffusion process that are completely smoothed out when we take the continuous limit represented by diffusion equations. Random walks have also been a very useful tool in the study of transport in such disordered media as fractured and porous rocks, silica aerogels and percolation clusters [3], substituting more phenomenological approaches [4], and have also been used as topological models of polymers (the so-called self-avoiding random walks [5]).

The single random walker statistical problems have been the subject of intense research since the beginning of past century and constitute now, in many areas, an almost closed discipline extensively treated in general references [2]. However, the generalization of these problems to the case of $N \geq 1$ interacting [6] or independent [7, 8] random walkers have only really started to be considered in detail in the last decade. It may seem strange to the layman in random walk theory that there are any reasons for such a late study of the multiparticle case, especially in the absence of random walker interactions, because in other fundamental physical models composed of independent particles, such as the ideal gas, the quantities of interest are obtained as simple averages over the mechanical properties of single particles. However, there are some quantities defined in a multiparticle random walk that can not be analyzed in terms of the single walker theory even when the walkers are independent! Examples are the number of distinct sites visited by a set of $N$ independent random walkers, $S_N(t)$, or the arrival time of the $j$-th particle of a set of $N$ independent random walkers at a given border, $t_{j,N}$, neither of which yield to that simple approach. Fortunately, these problems are tractable in the limit of large $N$ by resorting to asymptotic [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21] or Tauberian and Abelian techniques [2, 5, 7, 8]. The complexity of these problems is a consequence of the superposition of the trails of the random walkers in the case of the territory problem and the competition between random walkers in the case of the order statistics. Hence every random walker has an influence on the result which is indirectly correlated with the rest of the random walkers’ influences in spite of the absence of direct interactions between them.

Interest in multiparticle diffusion problems has been rekindled lately by the development of experimental techniques allowing the observation of events caused by single particles of an ensemble [22, 23]. These are powerful techniques aimed at the study of local conditions (mechanical response, viscoelasticity) inside such complex structures as fibrous polymers, the intracellular medium, etc., which determine the behaviour of molecular motors and the rate of biochemical reactions. These new research tools for
the analysis of soft disordered media also demand a better understanding of the statistical problems associated with mesoscopic samples of Brownian particles.

II. STATEMENT OF THE PROBLEMS AND DEFINITIONS

A. Substrates

We will consider three different classes of substrates the following sections: (i) $d$-dimensional Euclidean lattices ($d = 1, 2, 3$), (ii) deterministic fractals (in particular the two-dimensional Sierpinski gasket) and (iii) stochastic fractals (in particular, the two-dimensional incipient percolation aggregate embedded in the square lattice).

Euclidean lattices are reminiscent of crystalline structures and are the most customarily studied media, that the solution of our problems in this case is fundamental. Fractal lattices are good models of disorder substrates and have been used extensively in the analysis of diffusion and transport properties in these media. In contrast with classical diffusion, the mean-square displacement of a random walker in fractal lattices is given by $\langle r^2 \rangle \sim 2D_r^2$, where $D_r > 2$ is the anomalous diffusion exponent ($D_r = 2$ for classical diffusion) and $D$ is the diffusion coefficient. This slowing down of the transport has also been observed in disordered media, which consequently supports the use of fractals as models of real disordered materials.

Deterministic fractals are constructed by iteration of an unvarying rule starting with a seed and successively applying the same iteration. These kinds of fractal are only mathematical idealizations but they have the obvious advantage of being suitable for the use of exact renormalization techniques in the calculation of many statistical quantities. Unfortunately, stochastic fractals do not allow the application of these techniques because they are the result of some random process. However, they are also closer to real disordered media which share their statistical-fractal features and are the most customarily studied media, so the ones to be considered in this review are described below.

B. The number of distinct sites visited. Territory explored

A set of $N$ random walkers are placed on the same site of a lattice at $t = 0$. As time runs the random walkers move independently, jumping randomly from the site they occupy to any of its nearest-neighbours and imprinting every site they visit. We are concerned with those sites visited by any of the random walkers. Successive visits are not relevant to register a site as visited. The number $S_N(t)$ of the imprinted sites, that is, the number of distinct sites visited at time $t$ is the territory explored by the diffusing random walkers. This is the magnitude whose statistical distribution we are interested in.

The case $N = 1$ was posed at the beginning of the 1950s by Dvoretzky and Erdős and has been thoroughly studied since then. The problem was taken up again by Larralde et al. who systematically treated the multiparticle ($N > 1$) version. In these pioneering works the general features of the solution in the limit $N \gg 1$ were unveiled, and they found three time regimes...
as follows

\[ \langle S_N(t) \rangle \sim \begin{cases} 
\text{const} \times t^d, & t \ll t_x \\
\text{const} \times t^{d/2} \ln^{d/2}(x), & t_x \ll t \ll t'_x \\
N \langle S_1(t) \rangle, & t'_x \ll t 
\end{cases} \tag{1} \]

where \( x = N \) for \( d = 1 \), \( x = N/\ln t \) for \( d = 2 \) and \( x = N/\sqrt{t} \) for \( d = 3 \). The behaviour of the territory covered by a single random walker, \( \langle S_1(t) \rangle \), is also well known: \( \langle S_1(t) \rangle \sim \text{const} \times t^{1/2} \) for \( d = 1 \), \( \langle S_1(t) \rangle \sim \text{const} \times t/\ln t \) for \( d = 2 \), and \( \langle S_1(t) \rangle \sim \text{const} \times t \) for \( d = 3 \). The existence of these three regimes is easily explained:

a. **Regime I.** In this case there are so many particles at every site that all the nearest neighbours of the already visited sites are reached at the next step. It is clear that the territory covered by the random walkers grows as the volume of a hypersphere of radius \( t \), \( \langle S_N \rangle \sim \text{const} \times t^d \). The crossover time from regime I to regime II, \( t_x \), is simply derived if we take into account that regime I must break when the number of particles on the outer zone of the territory visited is of order 1. For very short times the number of particles on the outer visited sites decreases as \( N/z^d \), where \( z \) is the coordination number of the lattice, and, thus, the overlapping regime will break approximately when \( N/z^d \simeq 1 \) or, equivalently, when \( t_x = \mathcal{O}(\ln N) \).

b. **Regime II.** This is the most interesting regime. The simple regime I has broken and the random walkers move diffusively so that the radius of the territory explored grows as \( t^{1/2} \). Then the territory explored \( \langle S_N(t) \rangle \) is given, essentially, by a volume that grows as \( t^{d/2} \) modified by a factor that depends on \( N \). The explored region is divided into an inner hyperspherical core and a corona of dendritic nature characterized by filaments created by the random walkers wandering in the outer regions (Fig. 2). The overlapping of the trails of the random walkers make the problem non-trivial. In Refs. 12, 17, asymptotic techniques were used to obtain the prefactor and corrective terms of the main term of \( \langle S_N(t) \rangle \).

c. **Regime III.** Particles are very far from each other and their trails (almost) never overlap so that \( \langle S_N \rangle \sim N \langle S_1 \rangle \). The territory explored by the set of \( N \) random walkers is the sum of the territories explored by \( N \) single random walkers. The crossover time from regime II to regime III is \( t'_x = \mathcal{O}(e^N) \) for \( d = 2 \) and \( t'_x = \mathcal{O}(N^2) \) for \( d = 3 \). This regime never appears if the spectral dimension \( d_s = 2d_f/d_w \) of the substrate is \( d_s < 2 \). In particular, regime III is never reached in one-dimensional lattices.

The territory problem was also studied for fractal lattices with \( d_s < 2 \) by Larraid et al. 20, and Havlin et al. 21 who proposed the expression \( \langle S_N \rangle = \text{const} \times t^{d_s/2} / (\ln N)^{d_s/d_w} \) for the regime II with \( N \gg 1 \), where \( u = d_w/(d_w - 1) \). More recently, Dräger and Klafter 19 have also analyzed this problem using scaling arguments finding that \( \langle S_N \rangle = \mathcal{O} [t^{d_s/2}/(\ln N)^{d_s/d_w}] \) for \( t_x \ll t \), where \( v = d_w/(d_w - 1) \) and \( d_w = d_w/d_{\text{min}} \) is the chemical-diffusion exponent. 20, 21, 22. Of course, the two predic-

![FIG. 2: A snapshot of the set of sites visited by \( N = 1000 \) random walkers on the two-dimensional lattice. The visited sites are in white, the unvisited ones are in black and the internal gray points are the random walkers. The outer white circle is centered on the starting point of the random walkers and its radius is the maximum distance from that point reached by any walker at the time the snapshot was taken. The internal black circle is concentric with the former but its radius is the distance between the origin and the nearest unvisited site.](image)
of random walkers on Euclidean lattices were obtained by Lindenberg et al. 30 and Weiss et al. 10. Asymptotic results (N ∝ 1) for finitely ramified fractals were obtained much later by one of us 12 using renormalization techniques developed by van den Broeck 24. This kind of fractal has the property that, by cutting a finite number of bonds, a portion of the lattice with certain number of generations becomes isolated from the rest of the lattice. Curiously, rigorous asymptotic expansions for Euclidean lattices of arbitrary dimension were only given later by Yuste et al 18.

D. Order statistics of the trapping problem.

The “trapping” problem has for decades been one of most widely studied the areas of random walk theory 2 3. In its simpler version we have a Euclidean lattice where a population of traps occupies the sites with probability c. A single random walker starts moving from an empty site until it is absorbed by one of the traps. The statistical quantity of interest is the survival probability Φ1(t) that the random walker is not trapped by time t. This problem has its origin in Smoluchowski’s theory of coagulation of colloidal particles 3 2 31 and has been applied to many systems in physics and chemistry such as trapping of mobile defects in crystals with point sinks 32 33 34, the kinetics of luminescent organic materials 34, anchoring of polymers by chemically active sites 35 and atomic diffusion in glasslike materials 36, among others. The generalization of the trapping problem to N independent random walkers was only considered very recently by Krapivsky and Redner 10, who studied a predator-prey problem in which a static prey or “lamb” is captured by one of a set of N diffusing predators or “pride or lions” in one dimension. In their problem the N predators are placed initially at a given distance from the prey. Later, the case was studied of stochastic fractal lattices 12 24 37, the solution of the diffusion equation for Euclidean media 18 38 and computer simulations 17 39. The resulting expression in the short-time (large-distance) limit is given by

\[ h_B(z, t) \sim A \xi^{-\mu v} e^{-\xi z} \left\{ 1 + h_1 \xi^{-\nu} + \ldots \right\}, \]  

with \( \xi = z/(z^2)^{1/2} = z/(\sqrt{2D t}/d_{\text{w}}) \gg 1 \) and \( v = d_{\text{w}}^z/(d_{\text{w}}^v - 1) \). The parameters \( d_{\text{w}}, A, \mu, c \) and \( h_1 \) are listed in Table I for several lattices. It is known that the mortality function of a single random walker starting at a distance z from a lattice trap site is given by an equation formally identical to (2) in the case of Euclidean lattices 18 38 for \( \xi \gg 1 \). This form also encompasses the cases of deterministic and stochastic fractal lattices, as simulation results have shown 17. Hence, in general, we will write

\[ h_P(z, t) \sim A \xi^{-\mu v} e^{-\xi z} \left\{ 1 + h_1 \xi^{-\nu} + \ldots \right\}, \]  

for \( \xi \gg 1 \), and where \( A, \mu, c, \) and \( h_1 \) is a set of parameters characteristic of the lattice. In Table I the values of these parameters are listed for several lattices. In the absence of exact theoretical expressions for \( h_P(z, t) \), these parameters are obtained from comparison with simulation results. Figure 6 shows an example. In this figure the theoretical short-time behaviour of \( h_P(z, t) \) given by

III. SURVIVAL PROBABILITY: ABSORBING TRAPS AND BOUNDARIES

The calculation of the average territory covered by \( N \gg 1 \) random walkers and the order statistics of exit and trapping times requires the previous knowledge of the survival functions, or their complementaries, the mortality functions for short times. Suppose we have a circular boundary of radius r in a lattice and a single random walker starting at the center of this circle at \( t = 0 \). The probability that this diffusing particle has reached the distance z during the time interval \((0, t)\) is called the mortality function. We will denote it as \( h_B(z, t) \) where the subscript B indicates that this quantity refers to the boundary and z denotes any distance defined on the substrate. In stochastic fractal lattices it is common to define the chemical distance \( \ell \) as the shortest path measured along the lattice bonds 7. Between this chemical distance and the Euclidean distance there exists a scaling relation, \( \ell \sim \text{const} \times r^d_{\text{min}} \), with \( d_{\text{min}} = 1 \) for Euclidean and deterministic fractals (Sierpinski gasket, Given-Mandelbrot curve, etc.) and \( d_{\text{min}} > 1 \) for stochastic fractals (for example, \( d_{\text{min}} = 1.14(2) \) for the two-dimensional incipient percolation aggregate). Thus, in the case of stochastic fractal lattices there are at least two independent ways of defining a “spherical” boundary as the set of sites with either constant \( \ell \) (chemical boundary) or constant r (Euclidean boundary) from the origin. The anomalous diffusion coefficient \( d_{\text{w}}^z \) appears in Einsteins relation, \( \langle z^2 \rangle \sim 2Dt^{1/d_{\text{w}}} \), for the average square distance traveled by the random walker by time t, so that \( d_{\text{w}}^z = d_{\text{w}} \) for \( z = r \) and \( d_{\text{w}}^z = d_{\text{w}}^c \) for \( z = \ell \). The complementary of \( h_B(r, t) \) is the survival probability in the case of an absorbing boundary: \( h_B(r, t) = 1 - h_B(r, t) \). Similarly, we will define the mortality function of a single random walker starting from the origin site of a lattice with a trap located on site \( r \) as the probability that site \( r \) has been visited by that random walker in the time interval \((0, t)\). We will denote it as \( h_P(r, t) \) in order to distinguish it from \( h_B(r, t) \).

These functions have been studied through the application of various techniques: renormalization in the case of finitely ramified fractals 12 24 37, the solution of the diffusion equation for Euclidean media 18 38 and computer simulations 17 39. The resulting expression in the short-time (large-distance) limit is given by
TABLE I: Parameters appearing in the asymptotic expression of the mortality function $h_B(z, t)$ of a random walker starting at the origin with a boundary of trapping sites at a given distance, Eq. (2) for four substrates: the symbol dSC refers to the $d$-dimensional simple cubic lattice, Sd to the $d$-dimensional Sierpinsky lattice, GM to the Given–Mandelbrot curve ($d_w = d$ for these cases) and A2E (A2C) to the two-dimensional incipient percolation aggregate with Euclidean (chemical) boundary. These parameters are analytical for Euclidean lattices [18, 38], numerical for the deterministic fractal lattices [12, 40] and the output of a numerical fit to simulation results for the percolation aggregate [39]. In this case a 2000 aggregates average and a time interval [0, 1000] were used.

| Case  | $d_w$ | 2D | $A$ | $\mu$ | $c$ | $d$ |
|-------|-------|----|-----|-------|-----|-----|
| dSC   | 2     | 1  | $\frac{2(d/2)^{d/2-1}}{\Gamma(d/2)}$ | $1 - d/2$ | $d/2$ | $d - 3$ |
| S2    | $\ln 5/\ln 2$ | 1.05 | 2.46 | $1/2$ | 0.981 | -0.56 |
| S3    | $\ln 6/\ln 2$ | –   | 3.36 | $1/2$ | 1.31  | -0.46 |
| GM    | $\ln 22/\ln 3$ | –   | 2.5  | $1/2$ | 1.10  | -0.6 |
| A2E   | 2.8   | 1.4 | 1.6  | -1.8  | 2.1   | –   |
| A2C   | 2.4   | 1.2 | 1.1  | -0.4  | 1.1   | –   |

FIG. 3: Plot of $\ln(-\ln h_P(\ell, t))$ versus $\ln \xi$ averaged over 2000 two-dimensional incipient percolation clusters. The trap was always placed at a site a distance $\ell = 80$ from the origin. The line represents the function of Eq. (3) with the parameters listed in Table I with the parameters listed in Table I.

Eq. (3) is compared to simulation results for the two-dimensional incipient percolation aggregate. One sees that the values of $d_w$, $A$, $\mu$, $c$ listed in Table I lead to good agreement between Eq. (3) and the simulation results.

Finally, it is interesting to note that, for disordered media, the territory explored and the order statistics of the exit times are better described in terms of the chemical distance [14, 17, 39] than in terms of the Euclidean distance. The reason for the advantage of the chemical distance description is to be found in the breadth of the distribution of the mortality function at fixed time $t$ and Euclidean distance $r$ in comparison with the corresponding distribution for fixed chemical distance $\ell$. Figure 4 shows the histogram for the values of $h_P(z, t)$ for the values $r = 30$ and $\ell = 80$ at $t = 1000$ plotted from the results for 2000 percolation clusters. The same Euclidean distance in a disordered lattice corresponds to very different chemical paths depending on the holes that may block this path between the origin and the destination sites. Hence, the mortality function should exhibit large deviations from one realization of the fractal lattice to another.

FIG. 4: Plot of the histogram $N(\ln h_P)$ versus $|\ln h_P|$ for the two-dimensional incipient percolation aggregate for fixed $r$ and $t$ (solid line) and for fixed $\ell$ and $t$ (dashed line). The values are $r = 30$, $\ell = 80$, and $t = 1000$. 
TABLE II: The same as Table I but for the mortality function of a single random walker starting at a distance \(z\) from a trapping site, \(h_P(z, t)\). Results for the two-dimensional incipient percolation aggregate with a trap at a given Euclidean distance from the origin are not shown because the broadness of the distribution does not allow any reasonable fit for the limited number of aggregates used in our simulations (See Fig. 4). The parameter \(p\) is \([2(2D\pi)^{3/2}]^{1/2}p(0, 1)\), where \(p(0, 1) \approx 1.516386\). The chemical dimension \(d_c\) and the volume \(V_0\) of a hypersphere of chemical radius \(\ell = 1\), are also listed.

| Case | \(d_c\) | \(d_\ell\) | \(V_0\) | \(\hat{A}\) | \(\hat{\mu}\) | \(\hat{c}\) | \(h_1\) |
|------|---------|-------------|---------|-----------|-------------|---------|--------|
| 1SC  | 2       | 1           | 2       | \(\sqrt{2/\pi}\) | 1/2         | 1/2     | -1     |
| 2SC  | 2       | 2           | \(\pi\) | 1/\(\ln t\) | 1           | 1       | -1     |
| 3SC  | 2       | 3           | 3\(\pi/2\) | 1/\(\hat{\mu}\sqrt{\ell}\) | 1           | 3/2     | -1/3   |
| S2   | \(\ln 5/\ln 2\) | \(\ln 3/\ln 2\) | 3       | 0.61       | 1/2         | 0.98    | -0.56  |
| A2C  | 2.4     | 1.65        | 1.1     | 1.00       | 0.8         | 1.05    | -      |

IV. TERRITORY EXPLORED

A. Asymptotic expressions

In this section we give the main and two first corrective terms of the asymptotic expression (a power series in \(\ln^{-1} N\) of the territory explored \(\langle S_N(t)\rangle\) by \(N \gg 1\) random walkers, thus going beyond the leading terms (without prefactor) discussed in Sec. 11. In the previous section we have defined the survival probability \(\Gamma_P(r, t)\) as the probability that a site \(r\) has not been visited by a single random walker by time \(t\). The average number of distinct sites visited by \(N\) independent random walkers, \(\langle S_N(t)\rangle\), is then simply related to \(\Gamma_P(r, t)\) as follows:

\[
\langle S_N \rangle = \left( \sum \left( 1 - [\Gamma_P(r, t)]^N \right) \right) ,
\]

where the sum runs over all the sites of the lattice. One must notice that there are two averages implicit in Eq. 4: (a) an average over all exploration experiments performed on the same lattice, represented by the sum \(\sum \left( 1 - [\Gamma_P(r, t)]^N \right)\); and (b) a second average \(\langle \ldots \rangle\) over all possible stochastic lattices compatible with the generation rules. Of course, in the case of deterministic lattices (Euclidean, Sierpinski gaskets, etc.) only the first average is necessary. As the histogram of the survival probability \(\Gamma_P(r, t)\) is very narrow in stochastic fractal lattices when the trap is placed at fixed chemical distance \(z = \ell\) (see Fig. 4), it is not difficult to see that Eq. 4 can be approximated by

\[
\langle S_N \rangle = \sum_{m=0}^{\infty} \left( 1 - [\Gamma_P(\ell_m, t)]^N \right) \langle n(m) \rangle ,
\]

where \(\langle n(m) \rangle\) is the average number of sites separated from the origin by a chemical distance in the range \(\ell_m = m\Delta t, \ell_m + \Delta t, m = 0, 1, 2, \ldots\) Evidently, Eq. 5 is exact when applied to Euclidean lattices and deterministic fractals and, for the sake of generality, we will take it as the starting point of the subsequent derivations for all classes of substrates.

First, we replace Eq. 5 by its continuum approximation

\[
\langle S_N \rangle = \int_0^\infty \left( 1 - [\Gamma_P(\ell, t)]^N \right) d\ell \ V_0 \ell^{d_\ell-1}d\ell ,
\]

where \(d_\ell\) is the chemical dimension of the fractal lattice and \(dV(\ell) = V_0 \ell^{d_\ell-1}d\ell\) is the average number of fractal sites placed at a chemical distance between \(\ell\) and \(\ell + d\ell\) (values of \(V_0\) for several lattices are given in Table I). The asymptotic evaluation technique for \(\langle S_N \rangle\) is inspired in the behaviour of \(1 - [\Gamma_P(\ell, t)]^N\) for a fixed time \(t\). This function is plotted in Fig. 4 for several values of \(N\) in the case of the two-dimensional incipient percolation aggregate. We observe that it approaches a unit step function \(\Theta(\ell - \ell_\times)\) when \(N \to \infty, \ell_\times\) being a value that depends on \(N\). For large \(N\), \(\Gamma_P(\ell, t)\) is only non-negligible when \(\Gamma_P(\ell, t)\) is very close to 1. This occurs in the limit \(\xi = \ell/L(t) \gg 1\) \((L(t) \sim \sqrt{2D t^{1/d_\omega}}\) is the root-mean square chemical distance traveled by a single random walker by time \(t)\). On the other hand, \(1 - [\Gamma_P(\ell, t)]^N\) approaches rapidly the value 1 as \(\xi\) decreases (large times and short-distance limit). Therefore, it is clear that \(1 - [\Gamma_P(\ell, t)]^N \approx \Theta(\ell - \ell_\times)\) is a reasonable approximation which improves as \(N\) increases. The threshold chemical distance \(\ell_\times\) can be reasonably defined as the distance at which \(1 - [\Gamma_P(\ell, t)]^N\) takes the value 1/2, and from Eq. 3 we find that \(1/2 \approx N A_\times^{\mu\nu} \exp(-c\xi_\times)\) and the following approximation \(\xi_\times \approx \ln N - \hat{\mu} \ln N + \ln A_\times^{\mu\nu} + 2A\) for \(\xi_\times\) ensues. We then get \(\hat{c}\xi_\times \approx \ln N - \hat{\mu} \ln N + \ln A_\times^{\mu\nu} + \ln 2\) or

\[
\ell_\times \approx (2D)^{1/2} t^{1/d_\omega} \left( \frac{\ln N}{c} \right)^{1/v} \times \left( 1 + \frac{1 - \hat{\mu} \ln N + \ln A_\times^{\mu\nu} + \ln 2}{\ln N} \right).
\]

The integration in Eq. 6, evaluated using the Heaviside step function approximation for the quantity inside the braces, trivially yields \(\langle S_N(t)\rangle \approx V_0 \ell_\times^{d_\ell}\), and after inserting the expression for \(\ell_\times\) in Eq. 7 we finally arrive at
the following approximation for $\langle S_N \rangle$:

$$\langle S_N \rangle \approx V_0 (2D)^{d_e/2} t_{d_e/d_w} \left( \frac{\ln N}{c} \right)^{d_e/v} \times \left\{ 1 + \frac{d_e}{v} \ln 2 + \ln \hat{A} \hat{\mu} - \hat{\mu} \ln \ln N \right\}. \quad (8)$$

The elementary approach leading to Eq. (8) is not exact, but the dominant behaviour and the form of the first corrective term found (except for the ln 2 in the numerator) coincide with the prediction of a systematic and rigorous improvement of the approach discussed above. This rigorous analysis for $\langle S_N \rangle$ yields \cite{14,17}

$$\langle S_N \rangle \sim \hat{S}_N(t) \left[ 1 - \frac{d_e}{v} \sum_{n=1}^{\infty} \sum_{m=0}^{n} s^{(n)}_m (\ln N)^m \frac{(\ln N)^m}{(\ln N)^n} \right]. \quad (9)$$

with

$$\hat{S}_N(t) = V_0 (2D)^{d_e/2} t_{d_e/d_w} \left( \frac{\ln N}{c} \right)^{d_e/v}. \quad (10)$$

and

$$s_0^{(1)} = -\omega \quad (11)$$

$$s_1^{(1)} = \hat{\mu} \quad (12)$$

$$s_0^{(2)} = -(\beta - 1) \left( \frac{\pi^2}{12} + \frac{\omega^2}{2} \right) - (\hat{c}\hat{h}_1 - \hat{\mu}\omega) \quad (13)$$

$$s_1^{(2)} = -\hat{\mu}^2 + (\beta - 1)\hat{\mu}\omega \quad (14)$$

$$s_2^{(2)} = -\frac{1}{2} (\beta - 1)\hat{\mu}^2. \quad (15)$$

Here $\omega = \gamma + \ln \hat{A}\hat{\mu}$, $\gamma \approx 0.577215$ is the Euler constant, and $\beta = d_e/v = d_e/(d_w^v - 1)/d_w^v$. Inserting (11) and the definition of $\omega$ in Eq. (13) we get an expression for $\langle S_N(t) \rangle$ that almost coincides with that obtained before in Eq. (8) by a much simpler analysis. The only difference between the two expressions is that, in the first-order correction, the term ln 2 plays in Eq. (8) the role of the Euler constant $\gamma$ in Eq. (9).

In Fig. (6) the first and the second order approximations for $\langle S_N(t) \rangle$ are compared with Monte Carlo simulation results for $N = 2^3, 2^4, \ldots, 2^{14}$ at $t = 200$ in the simple cubic Euclidean lattice with $d = 1, 2$ and 3. An excellent agreement was found for $N \gtrsim 1000$. The uncertainty in the values of the parameters $\hat{A}, \hat{\mu}, \hat{c}$ and $\hat{h}_1$ appearing in the general expression of the point mortality function for fractal media in Eq. (13) does not allow a
FIG. 7: Plot of $S_N(t)/t^{d_A/d_w}$ versus $\ln N$ in the two-dimensional incipient percolation aggregate for $N = 2^0, 2^1, \ldots, 2^{13}$. The circles [triangles] are the simulation results for $t = 1000$ [$t = 500$] averaging over 2000 aggregate realizations. The dashed line is the zeroth-order theoretical prediction with $c = 1.05$ and $v = 1.7$ and the solid line is the first-order approximation with $c = 1.05$, $v = 1.7$, $\mu = 0.8$ and $A = 1$.

clear comparison in this case. However, using the tentative values listed in Table 11 for the two-dimensional percolation aggregate, good agreement has been found with Monte Carlo simulation results [17] as shown in Fig. 7.

A remarkable fact associated with the asymptotic series expression [12] for $\langle S_N \rangle$ is the large value of the corrective terms even for very large number of particles $N$. We have, for example, $\ln N/\ln N \sim 0.075$ for one mol of random walkers $N \sim 10^{23}$. Hence, these corrective terms must be included in order to make a sensible estimate of the territory expected to be covered by any number of random walkers in simulations or experiments. As discussed in Sec. III there are three time regimes in the territory problem and Eq. 9 is derived only for regime II. Once the trails of the random walkers no longer overlap (regime III), the series in Eq. 9 fails to converge, and the convergence condition $s_m^{(n)}(\ln N)^m / (\ln N)^n \ll 1$ determines the span of the regime II. We know that in the two-dimensional square lattice the parameter $\hat{A} = 1/\ln t$ (see Table 11) and, consequently, the first corrective term in the series is comparable with the main term if $|\ln \hat{A}| \sim \ln N$. This condition implies a crossover time $\tau_x \sim e^{N}$ from regime II to regime III. Similarly, the crossover time $\tau_x \sim N^2$ is found for the three-dimensional simple cubic lattice. For the one-dimensional lattice and any fractal lattice with spectral dimension $d_s < 1$ the parameter $\hat{A}$ is time independent and regime III is never reached as we have already discussed in Sect. III. Thus a unified and purely analytical criteria for the transition between the two regimes II and III emerges naturally from the series in Eq. 9.

B. Geometry of the territory explored

The asymptotic series in Eq. 9 can be used to unveil the geometric properties of the set of visited sites on Euclidean lattices [17]. In Fig. 2 one discerns a compact circular core and a dendritic ring composed of the trails of those random walkers that have travelled further than the rest. We define $R_0(N, t)$ as the average distance between the origin and the nearest unvisited site at time $t$ (the radius of the inner compact core) and $R_+(N, t)$ as the average maximum distance reached by any of the $N$ random walkers by time $t$. Assuming compact exploration in the sense of de Gennes [42], a reasonable estimate of $R_0(N, t)$ is given by $\langle |S_N(t)|/V_0 \rangle^{1/d}$. Qualitative arguments also yield an approximation for $R_+(N, t)$ on the $d$-dimensional Euclidean lattice [17]. In one dimension, the territory covered by $N$ independent random walkers is obviously a segment stretching from the maximum span on the left side of the origin to the maximum span on the right side and, consequently, we have $R_+(N, t) = \langle S_N(t) \rangle^{d=1}/2 \approx [4Dt \ln N]^{1/2}$. The $d$-dimensional walk of a random walker over the simple square or cubic lattice is now decomposed into $d$ orthogonal one-dimensional random walks and we get $R_+(N, t) \approx [4D(t/d) \ln N]^{1/2}$ because, on average, the random walker travels along each direction only the $d$-th part of the time. Taking into account that on Euclidean lattices $d_\ell = d$, $d_w = v = 2$ and $c = d/2$ as shown in Table 11 we can write the main term of Eq. 9 as $\bar{S}_N \approx V_0 [R_+(N, t)]^d$. Therefore, the thickness of the dendritic layer $R_+(N, t) - R_0(N, t) \approx \langle |S_N|/V_0 \rangle^{1/d} - \bar{S}_N/V_0 [R_+(N, t)]^{1/d}$ is given by the corrective terms in Eq. 9 as follows: $R_+(N, t) - R_0(N, t) \approx -(\ln \hat{A}(t)/\ln N)R_+(N, t)$. This means that the dendritic layer’s thickness is a fraction of the compact core radius that grows with time as $\ln \ln t$ for $d = 2$ and $\ln \sqrt{t}$ for $d = 3$ (see Table 11 for $\hat{A}$). In two dimensions this ratio grows so slowly that the set of distinct sites visited scales as $\sqrt{t}$, and is almost statistically self-similar in regime II as Fig. 8 shows. For the crossover times $\tau_x = O(e^N)$ ($d = 2$) and $\tau_x = O(N^2)$ ($d = 3$), the dendritic ring outruns the inner compact core and we enter into regime III.

V. EXIT TIMES

Now we turn to the order statistics problem of evaluating the quantities $\langle t_{j, N} \rangle$, $j = 1, 2, \ldots$, i.e., the average time (exit time) taken by the $j$-th random walker of a set of $N$ to arrive at a given border containing the origin site, and its moments. Specific techniques have
been developed for the calculation of these quantities [12, 13, 18, 39, 43]. They will be discussed below, but first it is interesting to explore the connection between the first passage time and the territory problems. We have found that for Euclidean media and also for media with spectral dimension \(d_s = 2d_f/d_w < 2\), the random walkers perform an almost compact exploration of the lattice in the sense of de Gennes [42]. This means the following: If the maximum distance \(\ell\) from the origin site is reached by any of the \(N\) random walkers at time \(t\), all the sites at a distance smaller than \(\ell\) have already been visited. The territory explored by the \(N\) random walkers is roughly a hypersphere of radius \(\ell\), and we can write:

\[
\langle S_N(t_1,N) \rangle \approx V_0 e^{\ell t}.
\]

(16)

Taking into account Eqs. (9), (10) and (11) one easily finds that Eq. (16) implies

\[
t_{1,N} \approx \left( \frac{\ell}{\sqrt{2D}} \right)^{\alpha+1} \left( \frac{e}{\ln N} \right)^{\alpha} \times \left\{\frac{1 + \mu \ln \ln N - \gamma - \ln \lambda_0}{\ln N} + \ldots\right\}
\]

(17)

where \(\alpha = d_w - 1\) and \(\lambda_0 = Ae^{\mu\ell}\). It is remarkable that this equation, derived following simple arguments, is exact to first order, i.e., all terms in Eq. (17) are exact. A rigorous approach leads to the result [12, 18, 39]:

\[
\langle t_{1,N}^m \rangle = \left[ \frac{\ell}{\sqrt{2D}} \right]^{m(\alpha+1)} \left[ \frac{e}{\ln(\lambda_0 N)} \right]^m \left\{1 + \frac{\mu \ln \ln \lambda_0 N - \gamma}{\ln \lambda_0 N} + \frac{m\alpha}{2\ln^2(\lambda_0 N)} \times \left[ (1 + m\alpha) \left( \frac{\pi^2}{6} + \gamma^2 \right) + 2\mu\gamma - 2h_1c \right. \right.
\]

\[
-2\mu (1 + m\alpha) \ln \lambda_0 N + (1 + m\alpha)\mu^2 \ln^2 \ln \lambda_0 N \right) + O\left( \frac{\ln^3 \ln \lambda_0 N}{\ln^3 \lambda_0 N} \right) \}.
\]

(18)

If we write the series in Eq. (18) in terms of \(\ln N\) instead of \(\ln \lambda_0 N\) we find that it coincides, to first order, with the approximation in Eq. (17). The moments of \(\langle t_{1,N}^m \rangle\) for \(j > 1\) have also been calculated [12, 18, 39] and the result is

\[
\langle t_{1,N}^m \rangle = \langle t_{1,N}^m \rangle^c \times \left\{ \frac{\mu e^{\mu \ell}}{\ln \lambda_0 N} \sum_{n=1}^{j-1} \frac{\Delta_n}{n} \right\}
\]

(19)

where \(j = 2, 3, \ldots\) and

\[
\Delta_n = 1 + \frac{m\alpha + 1}{\ln \lambda_0 N} \left[ (-1)^n \frac{S_n(2)}{(n - 1)!} + \mu \ln \ln \lambda_0 N \right.
\]

\[
- \mu \frac{\mu^2}{\ln^2 \lambda_0 N} \right) + O\left( \frac{\ln^2 \ln \lambda_0 N}{\ln^2 \lambda_0 N} \right).
\]

(20)

The quantities \(S_n(i)\) are the Stirling numbers of the first kind [43]. The expressions [13] and [19] are also valid for Euclidean boundaries with minor and obvious changes (\(d_w \rightarrow d_f, \ell \rightarrow r, \ldots\)). In Euclidean lattices and deterministic fractals there is no difference between using chemical or Euclidean distances. In Fig. 8 exact scaled results for the \(N\) dependence of \(\langle t_{1,N}^m \rangle^{1/m}\) are plotted against the predictions of Eq. (18) for the one-dimensional lattice, the two-dimensional square lattice and the three-dimensional simple cubic lattice. Agreement is also very good for disordered media such as the two-dimensional incipient percolation aggregate, as shown in Fig. 9.

VI. TRAPPING TIMES

A. Lifetime of the first trapped particle

The “trapping” problem is a fundamental problem of random walk theory with a long tradition [2, 3]. In this problem a lattice is randomly filled with static traps placed at certain sites, and a particle performs a random
walk on the lattice until it arrives at a site occupied by a trap, when it is absorbed. Most works have analyzed the survival probability of the random walker, \( \Phi(t) \), defined as the probability of the random walker not being absorbed in the time interval \([0, t]\). This problem has also been generalized to \( N > 1 \) random walkers, and the survival probability of the set of \( N \) random walkers has been defined analogously. In this spirit, we define the survival probability \( \Phi_N(t) \) that no particle of the initial set of \( N \) diffusing particles has been trapped by time \( t \). This survival function is given by

\[
\Phi_N(t) = \langle (1 - c)^{S_N(t)} \rangle ,
\]

where \( c \) is the concentration of traps and the angle brackets denote an average over all realizations of the \( N \) particle random walk on the lattice. Usually, \( \Phi_N(t) \) has been estimated using the Rosenstock approximation \([2, 3, 20, 21, 31]\) in which the average of the exponential is identified with the exponential of the average:

\[
\Phi_N(t) = e^{-\lambda(S_N(t))} ,
\]

with \( \lambda = -\ln(1 - c) \). Higher order terms are calculated in the extended Rosenstock approximation (or truncated cumulant expansion) first proposed by Zumofen and Blumen \([45]\). The first-order Rosenstock approximation is

\[
\Phi_N^{(1)} = \exp \left[ -\lambda \langle S_N \rangle \left( 1 + \frac{\lambda \text{Var}(S_N)}{2 \langle S_N \rangle} \right) \right] .
\]

This means that the error made by using the zeroth-order Rosenstock approximation is \( \mathcal{O}(\lambda^2 \text{Var}(S_N)) \). Consequently, this approximation performs well if the condition \( \lambda^2 \text{Var}(S_N) \ll 1 \) is satisfied. In Ref. \([21]\), it was shown that \( \text{Var}(S_N) = \mathcal{O}(t^d \ln N)^{d-2} \) on the simple Euclidean \( d \)-dimensional lattice, and we have that the zeroth-order Rosenstock approximation works well when \( \lambda^2 t^d \ln N)^{d-2} \ll 1 \). Hence, the approximation becomes poorer as \( N \) increases for \( d = 3 \), and also for long times where eventually, the Donsker-Varadhan regime settles in \([40]\). \( \Phi_N(t) \sim \exp(-t^d/(d+2)) \). If we now define \( h_{1,N}(t) = -\Phi_N(t)/dt \) as the probability that the
first absorbed particle of the initial set of \( N \) disappears during the time interval \( (t, t + dt] \), it is clear that, using Eq. (22), the average time (lifetime) at which the first particle is trapped is given by

\[
\langle t_{1,N}^m \rangle = \int_0^\infty t^m h_{1,N}(t) \, dt \\
\simeq m \int_0^\infty t^{m-1} \exp[-\lambda \langle S_N \rangle] \, dt .
\]

Inserting the main asymptotic term of \( \langle S_N \rangle \) given by Eq. (10) into Eq. (3) for the number of distinct sites visited into Eq. (24) a zeroth-order approximation for the \( r \)th moment of the first trapping time, \( t_{1,N} \) was found in Ref. 21 as follows:

\[
\langle t_{1,N}^m \rangle \simeq \frac{\Gamma(1 + 2m/d)}{(\lambda V_0)^{2m/d}} \frac{d^m}{(4D \ln N)^m} .
\]

No further terms in the expansion of \( \langle S_N \rangle \) were considered because, in the two- and three-dimensional Euclidean lattice case, they depend on time, and the analytical integration in Eq. (24) cannot be explicitly performed. In Fig. 11 numerical simulation results for \( \langle t_{1,N} \rangle \) on the two-dimensional square lattice with \( c = 4 \times 10^{-4} \) are compared with the predictions of Eq. (24) and those of Eq. (23) obtained by numerical integration using the first and second order corrective terms of \( \langle S_N \rangle \) in Eq. (9).

B. Lifetime of the \( j \)th trapped particle

Until now we have discussed the statistics of the absorption of the first particle, but the problem of the order statistics have also been solved for \( 1 < j \ll N \) [21]. This has been possible for independent random walkers. If we define \( \Psi(t) \) as the survival probability of a single random walker in a given trapping configuration, then the distribution for the absorption of \( j \) particles is given by

\[
\Psi_{j,N}(t) = \frac{\binom{N}{j} (1 - \Psi)^j \Psi^{N-j}}{\frac{\binom{N}{j}}{\binom{N}{j} \sum_{m=0}^{j} (-1)^m \binom{j}{m} \Psi^{N-j+m}}}. \tag{26}
\]

An average over all different trap configurations yields \( \Phi_N(t) = \langle \Psi_N(t) \rangle \), the survival probability of the \( N \) random walkers and \( \Phi_{j,N}(t) = \langle \Psi_{j,N}(t) \rangle \), the probability that exactly \( j \) random walkers from the set of \( N \) have been trapped in the time interval \( [0, t] \). From Eq. (26) we get

\[
\Phi_{j,N}(t) = (-1)^j \binom{N}{j} \Delta^j \Phi_N(t) \tag{27}
\]

\[
\simeq (-1)^j \binom{N}{j} \frac{d^j}{dN^j} \Phi_N(t) , \tag{28}
\]

using the backward difference formula for the \( j \)th derivative and the approximation \( \Delta^j \simeq d^j / dN^j \) when \( j \ll N \). In Fig. 12 the \( j \)th survival probability \( \Phi_{j,N}(t) \) obtained from Eq. (27) using the Rosenstock approximation for \( \Phi_N(t) \) is compared with simulation results for the two-dimensional lattice with a concentration of traps \( c = 4 \times 10^{-4} \).

The probability that the \( j \)th absorbed particle of the initial set of \( N \) disappears during the time interval \( (t, t + dt] \), \( h_{j,N}(t) \), satisfies the recurrence relation:

\[
h_{j+1,N}(t) = h_{j,N}(t) - \frac{d}{dt} \Phi_{j,N}(t) ,
\]

with \( h_{0,N}(t) \). The \( m \)th moment of the time at which the \( j \)th particle is trapped is given by

\[
\langle t_{j,N}^m \rangle = \int_0^\infty t^m h_{j,N}(t) dt , \tag{30}
\]

and, taking into account Eqs. (27) and (28), we finally arrive at a recurrence relation for these moments

\[
\langle t_{j,N}^m \rangle = \langle t_{j,N}^m \rangle + (-1)^j \binom{N}{j} \Delta^j \langle t_{1,N}^m \rangle . \tag{31}
\]

For large \( N \) and small \( j \), this relation yields

\[
\langle t_{j,N}^m \rangle \simeq \langle t_{j,N}^m \rangle + m \frac{d^m \Gamma(1 + 2m/d)}{(\lambda V_0)^{2m/d}} \frac{(\ln N)^{-1-m}}{4D^m/j} . \tag{32}
\]
where the difference operator $\Delta^j$ has been approximated by $d^j/dN^j$ and $\langle t_{j,N}^m \rangle$ was given by Eq. (22).

It is remarkable that the main asymptotic term of the ratio $\sigma_{j,N}/\langle t_{j,N} \rangle$ between the variance $\sigma_{j,N} = \sqrt{\langle t_{j,N}^2 \rangle - \langle t_{j,N} \rangle^2}$ and $\langle t_{j,N} \rangle$ depends only on the dimension of the lattice as is easily shown from Eq. (32):

$$\frac{\sigma_{j,N}}{\langle t_{j,N} \rangle} \simeq \frac{\Gamma(1+4/d) - \Gamma^2(1+2/d)}{\Gamma(1+2/d)} \frac{1}{2}$$

(33)

In Fig. 12 this ratio is plotted for $d = 1, 2$ and 3 for several values of $j$ and $N$ and compared with simulation results. Good agreement is found for $N \gtrsim 1000$.

C. One-dimensional case: rigorous results

For the one-dimensional lattice one can obtain rigorous order-statistics asymptotic expressions for the trapping times (lifetimes) from the knowledge of the order statistics of the diffusion process in the presence of two fixed traps [21], without resorting to the Rosenstock approximation. We only quote here the results up to first-order corrective terms for large $N$ (second-order terms were calculated in [21]):

$$\langle t_{j,N}^m \rangle = \frac{\Gamma(1+2m)}{(2m)!} \frac{\tau_{j,N}(m)}{(4D\ln\kappa N)^m}$$

(34)

with $\tau_{j,N}(m) = \tau_{1,N}(m) + \delta_{j,N}(m)$,

$$\tau_{1,N}(m) = 1 + \frac{m}{2} \ln \ln \kappa N - \gamma + O\left(\frac{\ln^2 \ln \kappa N}{\ln^2 \kappa N}\right),$$

(35)

and $\kappa = 1/\sqrt{\pi}$.

VII. SUMMARY AND OPEN PROBLEMS

In this review we have discussed some recent advances in the field of multiparticle independent random walks. The random walk is one of the simplest non-equilibrium models in statistical physics, and has played an important role in the study of transport throughout the past century [1, 2, 3]. Problems concerning a single random walker have been the subject of thorough study, and it is today a textbook discipline [2]. However, multiparticle random walk problems have been posed only very recently [4, 5, 6, 7] and we are just starting to understand what happens in this case. We have here dealt exclusively with independent random walkers all starting from the same site on Euclidean, deterministic fractal and disordered lattices. Three problems have been discussed:

(a) The territory problem: the estimate of the average number $\langle S_N(t) \rangle$ of distinct sites visited at time $t$ by $N$ random walkers starting from the origin at $t = 0$.

(b) Order statistics of exit times from a (hyper)spherical region: the evaluation of the moments of
the time \(t_{j,N}\) taken by the \(j\)-th, \(j = 1, 2, \ldots\), random walker out of \(N\) to arrive at a closed boundary at a given distance from the origin.

(c) Order statistics of trapping times: the estimate of the moments of the elapsed time \(t_{j,N}\) until the first \(j\) particles from a set of \(N\) are trapped in a disordered configuration of trapping sites.

These problems can not be solved by simple generalizations of the solution of the \(N = 1\) case, and specific techniques have had to be developed for the multiparticle problems even for independent random walkers. It is remarkable that these techniques lead to solutions of the aforementioned problems in terms of asymptotic series for large \(N\) which share the same mathematical form \([12, 15, 17, 18, 20, 21, 39]\). The corrective terms of these series decay logarithmically in the number \(N\) of random walkers, and are consequently, very important even when \(N\) is very large. This is reflected in the large magnitude of the fluctuations on the diffusion front (see Sec. [IV.D]).

Despite the success in the calculation of the territory explored (c.f. Sec. [IV]), the order statistics of the exit times (c.f. Sec. [V] and the order statistics of the trapping time (c.f. Sec. [VI]), there are still many open problems. For example, there exists the problem of the evaluation of the distribution of the territory explored or, equivalently, the moments \(\langle S^m_N(t)\rangle\), \(m = 2, 3, \cdots\). Only the moments of the number of distinct sites visited in a given direction of a one-dimensional lattice up to time \(t\) have been calculated rigorously \([20]\). There are good reasons to believe that these moments may be expressed by series in \(\ln N\) of the same form as those in Eq. [9], but this is only a conjecture for general media \([21]\). The development of a procedure for the calculation of these moments in Euclidean and fractal lattices is a still a challenge. The work discussed in this review could also be extended in other directions. For example, the extension of our work to the case in which the traps are randomly distributed on a fractal (deterministic or stochastic) is an interesting open problem. Another front opened recently is the study of unknotting in granular chains on a vibrating plate \([17]\).

The time corresponding to the unknotting of the first knot in a chain with \(N\) knots corresponds to the first passage time of a random walker of a set of \(N\) random walkers interacting through an excluded volume pair potential. This is a good motivation, among many others, to investigate the influence of interactions in the quantities we have considered in this review, namely, \(\langle S_N(t)\rangle\), \(\langle t_{j,N}^m\rangle\) and \(\langle t_{j,N}^m\rangle\).

Acknowledgments

This work has been supported by the Ministerio de Ciencia y Tecnología (Spain) through Grant No. BFM2001-0718.
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