Cylindrically Symmetric-Static 
Brans-Dicke-Maxwell Solutions

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Abstract

We present static cylindrically symmetric electrovac solutions in the framework of the Brans-Dicke theory and show that our solution yields some of the well-known solutions for special values of the parameters of the resulting metric functions.

KEY WORDS: Brans-Dicke Theory, Electrovac solutions, Cylindrical symmetry

1 Introduction

Among various versions of the scalar-tensor theories of gravity the Brans-Dicke theory can be considered as the prominent one. Although its original motivations are rooted in the attempts to incorporate the Machian view of inertia into the general theory of relativity, it has been applied to issues apart from its original intent, ranging from inflation schemes in cosmology to quantum gravity [1]. The inclusion of scalar fields into general relativity is also favored, for example, by the low energy limit of string theories in the guise of a dilaton field. Similarly, scalar fields also appear as by-products of dimensional reduction procedure in Kaluza-Klein theories. Thus, it is worth to investigate the implications of such scalar-tensor theories for various models of space-time in comparison to general theory of relativity.

In this paper, we will obtain static cylindrically symmetric electro-vacuum solutions in the framework of the Brans-Dicke scalar tensor theory [2] and compare the solutions with corresponding Einstein-Maxwell solutions. Cylindrically symmetric, static Einstein-Maxwell electro-vacuum solutions are well known [3]. Bonnor presented general solutions with axial or longitudinal magnetic fields [4] and Raychaudhuri presented a solution with radial electric field [5]. These solutions have been also discussed by L. Witten by using Rainich conditions in

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an “already unified theory” \[6,7\]. All these solutions yield the same static vacuum solution—the Levi-Civita solution—when the parameters appearing in the metric related with electric or magnetic fields vanish \[8\]. All the solutions are singular at the axis of symmetry except the so-called Bonnor-Melvin magnetic universe \[9\] having a uniform magnetic field along the \(z\) axis. When magnetic field vanishes this solution reduces to Minkowski space-time. To the best of our knowledge, the only solution of this form in the framework of Brans-Dicke theory is given in the work of Banerjee \[10\], where only a solution with a radial electric field is discussed. In the present work, we will consider other possible field configurations of a non-null Maxwell field, where there is an axial or an angular magnetic field. An analysis similar to our work has been made in cylindrical symmetric model space-time for a dilaton scalar field coupled nonlinearly to Electromagnetic field tensor \[11\]. However, for the solutions therein, unlike our solutions, when scalar field vanishes the electromagnetic field vanishes as well, hence there is no limit such that the scalar field vanishes and the solution reduces to Einstein-Maxwell solutions.

The paper is organized as follows. In the next section we present our notation and the field equations. In the third section we present the solutions corresponding to a longitudinal, axial magnetic field and a radial electric field separately.

### 2 Field Equations

The field equations for the scalar-tensor theory of Brans-Dicke minimally coupled to Maxwell field can be obtained from the Lagrangian density written in terms of exterior forms as

\[
\mathcal{L}[\phi, g, A] = \frac{1}{2} \phi \Omega_{\mu\nu} \wedge * (\theta^\mu \wedge \theta^\nu) - \frac{\omega}{2} d\phi \wedge * d\phi - \frac{1}{2} F \wedge * F, \tag{1}
\]

where \(\theta^\mu (\mu = 0, 1, 2, 3)\) are the set of coframe basis one forms. As in the standard general theory of relativity, \(\Omega_{\mu\nu}\) is curvature 2-form derived from the torsion-free metric compatible connection one form \(\omega_{\mu\nu}\). \(*\) is the linear Hodge dual operator through which matter fields couple to the metric in the coframe formulation. \(\phi\) is the Brans-Dicke scalar field which replaces the Newtonian coupling constant \(1/G\). In this investigation the metric is coupled only to the Faraday 2-form \(F\) derived from the 4-potential \(A\). The numerical factor \(\omega\) contained in the dynamical term for \(\phi\) is the so-called Brans-Dicke parameter. The notation and convention in what follows is adopted from \[12\].

The metric equations are obtained from the Lagrangian density \[11\] by a suitable variational procedure with respect to the coframe fields \(\theta^\alpha\). This imposes the condition that the metric compatible connection one form \(\omega_{\mu\nu} = -\omega_{\nu\mu}\) to be torsion-free, for example, by introducing appropriate Lagrange multiplier 2-forms. The field equations for \(\phi\) and \(F = dA\) also follow from the variation of the \[11\] with respect to \(\phi\) and \(A\). The variation \(\delta \mathcal{L} / \delta \theta^\alpha = 0\) yields the metric
\[ \phi \ast G^\alpha = D \ast (d\phi \wedge \theta^\alpha) + \frac{\omega}{\phi} \ast T^\alpha[\phi] + \ast T^\alpha[F] \]  \hspace{1cm} (2)

where \( G^\alpha = R^\alpha - 1/2\theta^\alpha R \) is the Einstein one form, and

\[ \ast T^\alpha[\phi] = \frac{\omega}{2\phi} \{ i_\alpha (d\phi) \ast d\phi + (d\phi) \wedge i_\alpha \ast d\phi \} \]  \hspace{1cm} (3)

\[ \ast T^\alpha[F] = 1/2 \{ i_\alpha F \wedge \ast F - F \wedge i_\alpha \ast F \} \]  \hspace{1cm} (4)

are the Hodge duals of the Energy-momentum 1-forms \( T^\alpha = T^\alpha_\beta \theta^\beta \) of the fields \( \phi \) and \( F \) respectively. \( i_\alpha = i_{\epsilon_\alpha} \) is the contraction operator \( (i_\alpha \theta^\beta = \delta^\beta_\alpha) \) and \( D \) is the covariant exterior derivative acting on tensor valued forms. Unlike the Maxwell field, which only couples to the metric components, the scalar field \( \phi \) couples to the derivatives of the metric field so that one has the term

\[ D \ast (d\phi \wedge \theta^\alpha) = d \ast (d\phi \wedge \theta^\alpha) + \omega^\alpha_\beta \wedge \ast (d\phi \wedge \theta^\beta) \]  \hspace{1cm} (5)

on the right-hand-side of (2). This term is essential for the the conservation of matter, \( D \ast T^\alpha[F] = 0 \), assuming that the second Bianchi identity holds [2]. The trace of (2) together with \( \delta L/\delta \phi = 0 \) yields the equation for \( \phi \)

\[ (2\omega + 3) d \ast d\phi = \ast T = 0, \]  \hspace{1cm} (6)

since the trace of the energy momentum tensor of the Maxwell field vanishes. Added to these, one has the Maxwell’s equations

\[ d \ast F = 0, \quad dF = 0 \]  \hspace{1cm} (7)

which hold outside the sources. We solve the field equations by starting with the cylindrically symmetric static ansatzs of the canonical form [3]

\[ g = \eta_{\mu\nu} \theta^\mu \otimes \theta^\nu, \quad \eta^{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-, +, +, +), \]  \hspace{1cm} (8)

with the coframe fields are given by

\[ \theta^0 = e^{K-U} dt, \quad \theta^1 = e^{K-U} dr, \quad \theta^2 = e^{U} dz, \quad \theta^3 = e^{-U} W d\phi. \]  \hspace{1cm} (9)

The cylindrically symmetric line element has three commuting Killing fields so that one has three ignorable coordinates and all the metric functions \( K, U, W \) depend on a single coordinate which is denoted by \( r \). This also dictates that the Brans-Dicke scalar \( \phi \) and the components of the 2-form \( F \) should also be the functions of this coordinate only. For a static spacetime \( dt \otimes d\phi \) term of the metric is zero. The coordinates \( t, r, z, \varphi \) are the time-like, the radial, the axial and the angular coordinates of the cylindrical symmetric static metric with ranges \(-\infty < t, z, < \infty, 0 \leq r \leq \infty \) and \( 0 < \varphi \leq 2\pi \) respectively.

It is also possible to consider an alternative form of the static cylindrical metric by applying a complex substitution \( z \rightarrow iz \) which yields the coframe forms as:

\[ \tilde{\theta}^0 = e^{U} dt, \quad \tilde{\theta}^1 = e^{K-U} dr, \quad \tilde{\theta}^2 = e^{K-U} dz, \quad \tilde{\theta}^3 = e^{-U} W d\phi. \]  \hspace{1cm} (10)
This form of the metric is useful when we consider a radial electric field.

With the ansatzs for the metric field, the field equations become

\[-\frac{W''}{W'} + K' \frac{W'}{W} - U'^2 = \frac{\omega}{2} \left( \frac{\phi'}{\phi} \right)^2 - \left\{ (K' - U') \frac{\phi'}{\phi} \right\} + \left\{ \frac{\phi''}{\phi} + \frac{W'}{W} \frac{\phi'}{\phi} \right\} + \frac{1}{\phi} T_{00}[F] e^{2(K-U)}, \]

(11)

\[K' \frac{W'}{W} - U'^2 = \frac{\omega}{2} \left( \frac{\phi'}{\phi} \right)^2 - \left\{ (K' - U') + \frac{W'}{W} \right\} \frac{\phi'}{\phi} + \frac{1}{\phi} T_{11}[F] e^{2(K-U)}, \]

(12)

\[\frac{W''}{W'} - 2U'' - 2U' \frac{W'}{W} + K'' + U'^2 = -\frac{\omega}{2\phi^2} \phi'^2 - \frac{1}{\phi} \left\{ \phi'' + \phi' \frac{W'}{W} \right\} \]

+ \frac{1}{\phi} T_{22}[F] e^{2(K-U)}, \]

(13)

\[K'' + U'^2 = -\frac{\omega}{2} \left( \frac{\phi'}{\phi} \right)^2 - U' \frac{\phi'}{\phi} - \frac{\phi''}{\phi} + \frac{1}{\phi} T_{33}[F] e^{2(K-U)} \]

(14)

\[\phi'' + \phi' \frac{W'}{W} = 0, \]

(15)

For the metric ansatz, the only change in the field equations above is that one needs the exchange \(T_{00} \leftrightarrow T_{22}\). Since we will consider various configurations of the field \(F\), explicit form of the Maxwells’s equations will be considered below.

### 3 Solutions

The isometries of the above metric ansatzs restrict the components of the Faraday tensor that couples to the metric. In effect, the corresponding four currents \(J\) have vanishing wedge products with the Killing one-forms of the Killing vector fields \(\partial_\phi\) and \(\partial_z\). That is, one has \(dz \wedge d\phi \wedge J = 0\). In the light of these observations, one can consider \(F \sim \theta^1 \wedge \theta^2\) or \(F \sim \theta^1 \wedge \theta^3\). In addition, corresponding to a charge distribution along the symmetry axis, \(F\) assumes the form \(F \sim \theta^0 \wedge \theta^1\). However, although linear combinations of these fields also satisfy the homogeneous Maxwell’s equations, not all of such linear combinations are compatible with the metric ansatzs to begin with. For all the cases, the components of Faraday 2-forms are functions of the radial coordinate.

#### 3.1 A Magnetic field along the \(z\) direction.

First, we consider a magnetic field along \(z\)-direction, which is due to current sources circulating about the \(z\)-axis. The corresponding closed Faraday 2-form is of the form \(F = f(r) \theta^1 \wedge \theta^2\). For this configuration of the fields, the metric ansatz is considered. Maxwell equation \(d \ast F = 0\) is satisfied for \(f(r) = \alpha \frac{e^{2U-K}}{W}\)
with $\alpha$ a constant. The corresponding energy momentum tensor field has the components

$$ T_{00}[F] = T_{11}[F] = T_{22}[F] = -T_{33}[F] = \frac{1}{2} f^2. \quad (16) $$

The structure of the coupled ordinary differential equations for all the fields can be reduced to an equation for a single metric function from which all the others can be computed. Explicitly, from (15), (11)-(12) and (12)+(14) one finds

$$ \phi' = \frac{a}{W}, \quad W' = \frac{b}{\phi}, \quad W\phi = r, \quad K' = \frac{q}{W \phi}. \quad (17) $$

(a, b, q are integration constants) and in addition, from (13)+(14) it follows that

$$ \frac{\phi''}{\phi} + \frac{W''}{W} - 2U'' - 2U' W' + 2U'^2 + 2K'' + \frac{w\phi'^2}{\phi^{2}} = 0. \quad (18) $$

(17), (18) reduce to a differential equation, for example, for the metric function $U$. Therefore, one can determine $U$ and integrating back one can find all the field functions as

$$ U = \frac{1}{2} (k - 1 + p) \ln r - \ln (1 + c^2 r^p), \quad (19) $$

$$ W = W_0 r^k, \quad \phi = W_0^{-1} r^{1-k}, \quad K = q \ln r, \quad (20) $$

where the integration constants $p, q, k$ and the Brans-Dicke parameter $\omega$ are related by

$$ p^2 = 1 + 4q + (2 - 3k)k + 2\omega[(2 - k)k - 1]. \quad (21) $$

These yield the metric as

$$ g = (1 + c^2 r^p)^2 \left\{ r^{2q+1-p-k} (dr \otimes dr - dt \otimes dt) + r^{1-p+k} W_0^2 d\phi \otimes d\phi \right\} $$

$$ + \frac{r^{p+k-1}}{(1 + c^2 r^p)^2} dz \otimes dz. \quad (22) $$

In order to investigate the solutions for some specific values of parameters appearing in the field functions that are relevant to other models, it is preferable to redefine these parameters as

$$ d = \frac{1}{2} (p + k - 1), \quad q = d (d - (k - 1) + \frac{1}{2} w(k - 1)^2 + k(k - 1). $$

These redefinitions bring the metric (22) into the form

$$ g = (1 + c^2 r^{2d+1-k})^2 \left\{ r^{2d(d-k)+(\omega(k-1)+2k)(k-1)} (dr \otimes dr - dt \otimes dt) $$

$$ + W_0^2 r^{2(k-d)} d\phi \otimes d\phi \right\} + \left\{ \frac{r^d}{1 + c^2 r^{2d+1-k}} \right\}^2 dz \otimes dz, \quad (23) $$

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whereas the magnetic field strength $f$ becomes

$$f = \pm \sqrt{2c(2d + 1 - k)} \frac{r^{d(k+1-d)-k^2-\omega/2(k-1)^2}}{(1+c^2r^{1+2d-k})^2}. \quad (24)$$

In the the form given in (23), it is the easy to probe the limiting cases of the metric. For $k = 1$ this solution reduces to corresponding Einstein-Maxwell solution [3, 4]. For $c = 0$ the magnetic field vanishes and one obtains the static Brans-Dicke-Maxwell vacuum generalization of the Levi-Civita vacu um solution [18]:

$$g = r^{2d(d-k)+\omega(k-1)+2k)(k-1)}(dr \otimes dr - dt \otimes dt) + r^{2d}dz \otimes dz + W_0^2r^{2(k-d)}d\phi \otimes d\phi, \quad (25)$$

$$\phi = r^{1-k}. \quad (26)$$

Furthermore, for $c = 0$ and $k = 1$, one obtains the vacuum Levi-Civita solution of the form [9]:

$$g = r^{2d(d-1)}(-dt \otimes dt + dr \otimes dr) + r^{2d}dz \otimes dz + W_0^2r^{-2(d-1)}d\phi \otimes d\phi. \quad (27)$$

The solution (23)-(24) have five constant parameters $d, W_0, k, \omega, c$. The first two parameters are the parameters of the Levi-Civita solution which we will discuss below. The next two parameters are related with the nontrivial coupling of Brans-Dicke scalar to the metric. The last parameter represents Maxwell fields. Compared to Newtonian gravity generated by a line source which contains a single parameter related to linear mass density, in general theory of relativity, there appear two parameters. For this metric, the parameter $d$ is related to the energy density of the source of the metric and the second parameter $W_0$ is related to the global topology (conicity) of space-time [14]. In order to understand the meaning and the behavior of these parameters, several interior cylinders [14] [15] and thin shells [16] have been constructed as sources of this vacuum solution. For $d = 0$ this metric describes a straight cosmic string [17]. Thus, this solution can be considered to describe a magnetic line source or a magnetic string in Brans-Dicke-Maxwell theory.

### 3.2 A Magnetic Field Along the $\phi$-Direction.

For a magnetic field lines circulating along the $\phi$-directions, one has $F = f^{1} \wedge \theta^{3}$. This magnetic field is apparently due to the 4-current form along the $z$-axis. This solution can be found either by solving the field equations or by exchanging the coordinates $\phi$ and $z$. The Maxwell equations are satisfied for $f = \beta e^{-K}$ with $\beta$ being a constant. Skipping the details of straightforward computations, solutions to the field equations assume the form

$$g = (1 + e^{2d(k+1-k-2d)}\frac{r^{d(k+1-d)-k^2-\omega/2(k-1)^2}}{(1+c^{2}r^{1+2d-k})^2}}(dr \otimes dr - dt \otimes dt)$$
\[ + r^{2d} dz \otimes dz \right] + \left\{ \frac{W_0 r^{k-d}}{1 + c^2 r^{k-2d}} \right\}^2 d\phi \otimes d\phi, \]  
\[ \phi = W_0^{-1} r^{1-k}, \]  
\[ f = \pm c \sqrt{2} (2d - k - 1) \frac{r^{-(d-k+1)-(k+w/2(k-1))(k-1)}}{(1 + c^2 r^{k-2d})^2}. \]  

As in the previous case, for the same values of the parameters, one has the same special cases for this solution as well. Also for \( d = 0 \), the solution reduces to the Brans-Dicke version of the Bonnor-Melvin universe [9] described in terms of the metric

\[ g = (1 + c^2 r^{1+k})^2 \left\{ \frac{1}{1 + c^2 r^{1+k}} \right\}^2 d\phi \otimes d\phi. \]

For \( k = 1 \), this reduces to Einstein-Maxwell solution with the metric given by

\[ g = (1 + c^2 r^2)^2 \left\{ - dt \otimes dt + d\rho \otimes d\rho + dz \otimes dz \right\} + \left\{ \frac{W_0 r^k}{1 + c^2 r^2} \right\}^2 d\phi \otimes d\phi. \]

Note that the Bonnor-Melvin solution is not singular at the axis but the corresponding Brans-Dicke version is singular. The singularity vanishes when \( k = 1 \), however for this value of \( k \), the Brans-Dicke theory reduces to Einstein’s theory.

### 3.3 A Radial Electric Field

The solution is easily found when one considers the metric [10]. In this case, one starts with the closed Faraday 2-form \( F = f(r) \theta^0 \wedge \theta^1 \). Maxwell’s equation \( d \ast F = 0 \) yields \( f(r) = c^{2(1-k)/W} \) (up to a constant). Going through similar computations as in the preceding cases, one concludes that the metric, scalar field and \( F_{01} \) depend on the radial coordinate as

\[ g = - \left\{ \frac{r^{2\sigma}}{1 - c^2 r^{4\sigma+1-k}} \right\}^2 dt \otimes dt + (1 - c^2 r^{4\sigma+1-k})^2 \times \]
\[ \left\{ r^{4\sigma(2\sigma-k)+(w(1-k)-2k)(1-k)} (dr \otimes dr + dz \otimes dz) + W_0^2 r^{2(1-k)} d\phi \otimes d\phi \right\}. \]

\[ \phi = \frac{r^{1-k}}{W_0}, \]

\[ f = \pm c \sqrt{2} (4\sigma + 1 - k) \frac{r^{-2\sigma(2\sigma-1)-w/(2(k-1)^2)}}{(1 - c^2 r^{4\sigma+1-k})^2}. \]

This solution can also be found by replacing \( t \rightarrow iz, z \rightarrow it, c \rightarrow ic, d \rightarrow 2\sigma \) from the solution corresponding to a magnetic field in \( z \) direction. The solution
is singular at $r = c^{-2/(4\sigma+1-k)}$ and this could be taken as the boundary of the source. This solution reduces to Brans-Dicke vacuum solution when $c = 0$, the Einstein-Maxwell solution with radial electric field when $k = 1$. If $k = 1$ and $c = 0$, then the metric reduces to the conventional form of the static Levi-Civita solution:

$$g = -r^{4\sigma}dt \otimes dt + r^{4\sigma(2\sigma-1)}(dr \otimes dr + dz \otimes dz) + W_0^2 r^{2(1-\sigma)} d\phi \otimes d\phi. \quad (36)$$

For $\sigma = 0$, the Levi-Civita solution reduces to the Minkowski metric and for Einstein-Maxwell solution, the electric field also vanishes, as one expects on physical grounds. However, for the Brans-Dicke-Maxwell vacuum solution, when $\sigma = 0$, the electric field does not vanish. Hence, the solution

$$g = -\frac{dt \otimes dt}{(1 - c^2 r^{1-k})^2} + (1 - c^2 r^{1-k})^2 \times \left\{ r^{-(2k + \omega(1-k))(1-k)}(dr \otimes dr + dz \otimes dz) + W_0^2 r^{2k} d\phi \otimes d\phi \right\}, \quad (37)$$

$$\phi = W_0^{-1} r^{1-k}, \quad (38)$$

$$f = \pm \sqrt{2} c(1-k) r^{-\omega/2(1-k)^2+k(2-k)} \frac{r^{-\omega/2(1-k)^2+k(2-k)}}{(1 - c^2 r^{1-k})^2}, \quad (39)$$

is only present for Brans-Dicke theory and the electric field vanishes for $k = 1$. Similar argument holds for the solution with a magnetic field in $z$-direction.

Finally, we note that it is possible to start with $F = f(r) \theta^0 \wedge \theta^1 + h(r) \theta^2 \wedge \theta^3$, then one ends up with the same metric and the scalar function, for which the diagonal energy momentum form corresponding to (39) is shared between a magnetic and an electric field.

4 Discussion

This work presents exact solutions of the Brans-Dicke-Maxwell theory in static cylindrical symmetry. The solutions reduce to corresponding Einstein-Maxwell solutions when a constant set to a specific value ($k = 1$), to the Brans-Dicke vacuum solution when a constant related with field strength vanishes ($c = 0$), or to the general relativistic vacuum Levi-Civita solution when both the constants are set to these specific values. Unlike the Einstein-Maxwell solutions, there is no solution which is everywhere regular in the Brans-Dicke-Maxwell case.

It is interesting to note that, for all the cases considered above, although the scalar $\phi$ does not depend on the Brans-Dicke parameter $\omega$, the field strengths $F$ do have $\omega$ dependence. Therefore, the general relativistic limit of the above theory can be recovered by simply setting the metric parameter $k = 1$, (which entails $1/\phi = W_0$) rather than by inspecting the field equations in the $\omega \rightarrow \infty$ limit of the Brans-Dicke parameter. Understanding the significance and implications of the parameter $\omega$ and its physically plausible numerical values in the model spacetimes presented above, needs further study. For example,
the investigation of the motion of charged test particles in spacetime models constructed above in Brans-Dicke theory.

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