The Interpretation of Linear Prediction by Interpolation Framework and Several following Results

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Abstract—This paper gives a general interpretation of Linear Prediction (LP) by interpolation framework different from the perspective of statistics. This interpretation is proved to be useful by several following results, such as: The mechanism of widely used least square estimation of LP coefficients can be explained more intuitively. In data modeling, LP coefficients cannot distinguish signals spanned by the same interpolation bases. Two new general LP constructive methods instead of least square estimation are presented with their upper bounds of approximation error and some properties given; one is based on DCT-1 and the other is based on difference operator. We also establish the relationship between LP and Taylor series.

Index Terms—Linear prediction; Interpolation framework; Interpolation basis; LP constructive methods; DCT-1; Difference operator.

I. INTRODUCTION

Linear prediction had long been widely used in speech analysis, geophysics and neurophysics even until 1975 [1]. Two-dimensional LP is a fundamental image model [2] contributing to many useful image processing algorithms, such as image restoration [3]. LP is also one of the most important methods of time series analysis [4]. It is very impressive to see the effect of LP’s signal representation in engineering. However, nearly all the theoretical descriptions such as [5] of LP are based on Kolmogorov [6] or Winer [7]’s work, which are based on the statistical background.

This paper tries to interpret LP by interpolation framework different from the perspective of statistics and is organized as follows: In Section II, We first give the equivalent analytical form of real signals that LP can represent and then introduce the interpolation framework. Section III explains LP’s approximation ability to arbitrary signals by least square estimation. Two general constructive methods are presented in Section IV. Finally, the conclusion is in Section V.

II. INTERPOLATION FRAMEWORK

A. Analytical form of LP represented signals

Textbooks about combinatorial mathematics or time series analysis such as [8], [9] usually discuss the homogeneous linear difference equation, in which it shows that several kinds of signals (for example, sequences of the exponential sum and trigonometric sum) are the solutions; hence the focus is solving difference equation. The results presented following are conversely deduced from those conclusions; though trivial, they are the bases of the discussions of the whole paper, by which the interpolation framework is introduced.

Definition. We denote a sequence of data points by \( f_n \) or \( f(n) \), which is also referred to as “infinite-length discrete signal” when \( n \) is infinite or “finite-length discrete signal” otherwise. Unless otherwise stated, we’ll not consider the trivial case when \( f_n \) is a constant sequence.

Lemma 1. Let

\[
    f_n = b_0 + \sum_{k=1}^{p_1} (b_k \cos(n \theta_k) + c_k \sin(n \theta_k)) \quad (1)
\]

be a sequence of any trigonometric sum, where \( b_0, b_k, c_k \) and \( \theta_k \) are real numbers and \( \theta_k \neq 0, \pi/2 \). Then \( f_n \) can be expressed in LP recurrence form:

\[
    f_n = \sum_{k=1}^{p} a_k f_{n-k}. \quad (2)
\]

The LP coefficient \( a_k \) in (2) is corresponding to the coefficient of polynomial equation

\[
    (r - 1)^q \prod_{k=1}^{p_1} ((r - r_k)(r - \bar{r}_k)) = 0, \quad (3)
\]

where \( r_k = \cos(\theta_k) + is\sin(\theta_k) \), \( \bar{r}_k \) is the complex conjugate of \( r_k \), and

\[
    q = \begin{cases} 
        1, & \text{if } b_0 \neq 0 \\
        0, & \text{if } b_0 = 0 
    \end{cases} \quad (4)
\]

The LP order \( p = 2p_1 + q \). Denote the expanded form of (3) by \( r^p + a'_1 r^{p-1} + \cdots + a'_p = 0 \), and then \( a_k = -a'_k \) for \( k = 1, 2, \cdots, p \).

Proof: This lemma is simply a converse of the characteristic polynomial theory of homogeneous linear difference equations [8]. (3) can be considered as the characteristic polynomial of difference equation (2) whose solution is

\[
    \hat{f}_n = d_0 + \sum_{k=1}^{p_1} (d_k r_k^n + d'_k \bar{r}_k^n). \quad (5)
\]

In order to make \( \hat{f}_n \) be real number, \( d_0 \) should be real and \( d'_k \) should be the complex conjugate of \( d_k \), i.e., \( d'_k = \bar{d}_k \). We just need to let \( d_k = \frac{1}{2} b_k - \frac{1}{2} c_k i \) and \( d_0 = b_0 \), then one of solutions of LP recurrence (2) is the given trigonometric sum (1).

Remark. In Lemma 1, the LP recurrence (2) is constructed from \( \cos(n \theta_k) \), \( \sin(n \theta_k) \) and 1, which can represent (1) with arbitrary weights of \( b_0, b_k \) and \( c_k \). The concrete signal...
represented by (2) with fixed weights of (1) should be further determined by assigning \( p \) initial values to the iteration. Similarly, this conclusion applies to the following cases.

**Lemma 2.** Any polynomial sequence
\[
f_n = b_0 + b_1 n + \cdots + b_k n^k,
\]
where \( k \) is the polynomial degree and \( b_i \) is real for \( i = 0, 1, \cdots, k \), can be iterated by \( \text{LP} \), which is constructed via characteristic polynomial equation \((r - 1)^{k + 1} = 0\). The \( \text{LP} \) order \( p = k + 1 \).

Proof: The proof is the case when the characteristic polynomial of a difference equation has only one root of 1 with repeated times \( k + 1 \). The solution of this kind difference equation is \( f_n = d_0 + d_1 n + \cdots + d_k n^k \). Just letting \( d_i = b_i \), one of the solutions of the constructed \( \text{LP} \) is \( f_n \).

**Lemma 3.** \( \text{LP} \) can express the sequence of exponential sum
\[
f_n = b_0 q_0^n + b_1 q_1^n + \cdots + b_{N-1} q_{N-1}^n,
\]
where \( b_i \) and \( q_i \) are real and \( q_i \neq q_j \) if \( i \neq j \). The \( \text{LP} \) coefficients are derived from characteristic polynomial equation \( \prod_{k=0}^{N-1} (r - r_k) = 0 \), where \( r_k = q_k \). The \( \text{LP} \) order \( p = N \).

Proof: This case refers to the characteristic polynomials with distinct real roots without repeating. Others are similar.

Each preceding lemma represents a general kind of signal that is very common to us, which is one of the reasons that we present them separately. The unified conclusion is:

**Lemma 4.** Let
\[
f_n = b_0 + \sum_{i=1}^{p_1} \sum_{k=0}^{e_i-1} (b_{ik} n^k \cos(n \theta_i) + c_{ik} n^k \rho_i^k \sin(n \theta_i)),
\]
where \( b_0, b_{ik}, c_{ik}, \rho_i \) and \( \theta_i \) are real numbers. Then \( f_n \) can be represented by \( \text{LP} \).

If \( \theta_i = 0 \) or \( \pi/2 \) for all \( i \), \( \text{LP} \) is constructed by characteristic polynomial equation \( \prod_{i=1}^{p_1} (r - r_i)^{e_i} = 0 \), where \( r_i = \rho_i \) and \( e_i \) is the repeated times of root \( r_i \). The \( \text{LP} \) order \( p = \sum_{i=1}^{p_1} e_i \).

If \( \theta_i \neq 0 \) and \( \pi/2 \) for all \( i \), the characteristic polynomial equation is \((r - 1)^q \prod_{i=1}^{p_1} ((r - r_i)^{e_i} (r - \bar{r}_i)^{e_i}) = 0 \), where \( r_i = \rho_i (\cos \theta_i + i \sin \theta_i) \) and \( q \) is defined as in (5). The \( \text{LP} \) order \( p = 2 \sum_{i=1}^{p_1} e_i + q \).

Proof: The proof needs a conclusion of difference equations: If the characteristic polynomial equation is \((r - r_0)^{k + 1} = 0 \), then the solution is \( f_n = (b_0 + b_1 n + \cdots + b_k n^k) r_0^n \). The left proof is similar with the above lemmas.

**Theorem 1.** The order \( p \) \( \text{LP} \) is equivalent to (8) in representing real signals.

Proof: The converse of Lemma 4 is also true by literature [9], which is that if a \( \text{LP} \) recurrence represents any real signal, the signal must be in the form of (8).

**B. Interpolation basis**

Note that (1) of Lemma 1 is related with the common trigonometric interpolation form [10] and (6) of Lemma 2 is corresponding to the Lagrange interpolation form, where \( f_n \)’s can be considered as the discrete points of a continuous function \( f(t) \) that is obtained by interpolation methods. Similarly, it’s natural to relate (8) to a more general interpolation form different from the above special cases, which can be written as
\[
f(t) = b_0 + \sum_{i=1}^{p_1} \sum_{k=0}^{e_i-1} (b_{ik} n^k \rho_i^k \cos(n \theta_i) + c_{ik} n^k \rho_i^k \sin(n \theta_i)).
\]

In the context of interpolation framework, we can borrow some related concepts or thoughts familiar with us, helping to understand \( \text{LP} \). For example, given some discrete points, the constructed formula (9) by interpolation methods is usually called “interplant”. Interplant gives a continuous function; however, in this paper, we only study some discrete points of the interplant iterated by \( \text{LP} \), so somewhat a “lower level” concept is needed:

**Definition.** In (8), the arbitrary changes of \( b_0, b_{ik} \) and \( c_{ik} \) with fixed \( n^k \rho_i^k \cos(n \theta_i) \) or \( n^k \rho_i^k \sin(n \theta_i) \) form the real solution space of a given \( \text{LP} \) recurrence. We call \( n^k \rho^n \cos(n \theta) \) or \( n^k \rho^n \sin(n \theta) \) used in the form of (8) the interpolation basis, which can be written in the set form
\[
\{n^k \rho^n \cos(n \theta), n^k \rho^n \sin(n \theta) \mid \theta, \rho \in \mathbb{R}, k \in \mathbb{Z} \text{ and } k \geq 0\},
\]

where \( k \) is a nonnegative integer, and \( \theta, \rho \) are arbitrary real numbers, and \( n \) is considered to be a constant when referred to this concept.

If \( f_n \) can be expressed in the form of (8), we say that \( f_n \) is spanned by its corresponding interpolation bases.

We avoid using the concept “basis” in function analysis [11] since its rigorous definition is not necessarily needed.

Under the above concept, \( \text{LP} \) recurrence is equivalent to a set of interpolation bases, while the weights of the interpolation bases can be obtained from several initial values of the recurrence.

Based on this definition and Theorem 1, we can describe an important property of \( \text{LP} \)’s signal representation ability:

**Theorem 2.** \( \text{LP} \) coefficients cannot distinguish signals spanned by the same interpolation bases.

Proof: A set of \( \text{LP} \) coefficients yields a \( \text{LP} \) recurrence with respect to a set of interpolation bases. The same set of \( \text{LP} \) coefficients can represent all the signals spanned by the corresponding set of interpolation bases without the ability to distinguish them.

**Remark.** \( \text{LP} \) coefficients related features such as \( \text{LPC} \) [12] and \( \text{PLP} \) [13] are successfully applied in speech analysis, where \( \text{LP} \) is considered as a basic model or a parameterization method of speech signals for pattern recognition. However, by Theorem 2, \( \text{LP} \) method may be insufficient
sometimes when different segments of the speech signal have the same interpolation bases, which may need to be improved.

C. Summary

We have proved that so large group of signals (including the trigonometric sum, polynomial, exponential sum and their “mixtures”) can be represented by LP. The local values of those signals have a very simple linear relation expressed by LP, though globally they may be very different and nonlinear. The intrinsic local property of those types of signals is the reason that LP can represent them. The case of arbitrary finite-length discrete signals will be investigated later.

III. INTERPRETING LEAST SQUARE ESTIMATION OF LP BY INTERPOLATION FRAMEWORK

We know that any finite-length discrete signal can be approximated by LP via least square estimation of LP coefficients, which is widely used in engineering [1]. The results of Section II cannot explain this phenomenon because those signals mentioned above are constrained to certain types. Under interpolation framework, we can give an interpretation here.

A. Automatic selection of interpolation bases

**Theorem 3.** For any given finite-length discrete signal $f_n$, the least square estimation of order $p$ LP is equivalent to selecting the best $p$ interpolation bases from the set of (10) to approximate $f_n$ by minimum error. The best selection is unique once the order $p$ is fixed.

**Proof:** Least square estimation provides a unique optimal solution of LP coefficients $a_k$’s as in (2). A unique solution of $a_k$’s determines a unique set of interpolation bases belonging to (10) by solving a difference equation specified by $a_k$’s. Different sets of LP coefficients correspond to different sets of interpolation bases respectively. The LP order $p$ is the degree of characteristic polynomial of difference equations, so that there will be $p$ interpolation bases.

**Remark.** Advantages of least square estimation: The most interesting and powerful effect of LP’s approximation by least square method is that it can select the best interpolation bases automatically from a wide range bases of (10) via arbitrary changes of parameters $\theta$, $p$ and $k$. The usual methods of interpolation or approximation generally fix the interpolation bases first and the performance is constrained to those fixed bases, such as Lagrange interpolation.

B. Interpretations of LP order adjusting methods

In least square estimation of LP coefficients, the LP order $p$ must be manually determined first. Various methods were developed to choose the best LP order, nearly all of which are based on the fact that increasing the order $p$ within certain range leads to less approximation error [1], [4]. The principle underlying this fact can be explained by Theorem 3.

According to Theorem 3, higher LP order results in more interpolation bases, which could improve the approximation performance if data points were not modeled well by less interpolation bases. This is similar to polynomial fitting. The simplest data structure is linear, so degree one polynomial is enough; when the data structure is nonlinear, we must increase the polynomial degree to fit that. The more “complex” the data, the more interpolation bases are needed.

IV. TWO GENERAL LP CONSTRUCTIVE METHODS

Although least square method can automatically select the interpolation bases for arbitrary finite-length discrete signals, it is done implicitly by LP coefficients. Based on the results of Section II, it’s natural to choose the interpolation bases directly to construct LP. In what follows, two new general constructive methods will be presented.

There’s theoretical significance of the two following methods. They can establish relationships between LP and some other branches of signal processing or mathematics. The “dense” property of constructed LP in the whole set of LP may lead to standard methods to study all LP, just like using Taylor series to study all kinds of smooth functions.

They also have potential engineering applications. For example, one of the results will tell us that adjusting the LP order is not the only way to improve LP’s approximation, while increasing the sampling frequency of data points is also beneficial.

A. DCT method

In (1) of Lemma 1, if the coefficient $c_k$ of $\sin(n\theta_k)$ is zero, (1) reduces to a sequence of cosine sum, which is associated with the discrete cosine transform (DCT) [14]. Because DCT can fit any finite-length discrete signal, a corresponding general LP constructive method follows.

**Theorem 4.** Any sequence of $f(n)$ for $n = 0, 1, \ldots, N-1$ can be approximated or parameterized by LP constructed through DCT-1. In constructing process, selecting the $p$ interpolation bases with corresponding $p$ largest absolute value weights gives the minimum upper bound of approximation error than other selections.

**Proof:** There are four common forms of DCT, referred to as DCT-1, DCT-2, DCT-3 and DCT-4 respectively [14]. Among them, DCT-1 is most appropriate for constructing LP because we can directly make it into the form of (1) in Lemma 1. DCT-1 theory states that for any given $N$ distinct points of $f(n)$ there exists a unique interpolation form of cosine sum such that

$$f(n) = b_0 + \sum_{k=1}^{N-1} b_k\cos(n\theta_k),$$

where $b_0$ and $b_k$’s are the DCT-1 of $f(n)$, and $\theta_k = \frac{2\pi k}{N}$. Since the number of interpolation bases of 1 and $\cos(n\theta_k)$ for all $k$ is $N = N-1+q$, where $q$ is defined as in (4), $2(N-1)+q$ LP coefficients are needed to iterate $f(n)$ according to Lemma [1]. It’s meaningless to do this when we want to parameterize $f(n)$ by much less LP coefficients. However, we can select some of the interpolation bases to approximate $f(n)$
instead of iterating it. After selecting probable interpolation bases, we can use Lemma 1 to construct $LP$.

We choose $p$ of the $N_I$ interpolation bases (may include the constant term 1) by letting $b_0 = 0$ and let

$$\hat{f}(n) = \sum_{k=1}^{p} b_{\sigma(k)} \cos(n\theta_{\sigma(k)})$$

where $\sigma(k)$’s are the indexes of selected interpolation bases and $p < N_I$. Then the mean-square approximation error is

$$E = \frac{1}{N} \sum_{n=0}^{N-1} (f(n) - \hat{f}(n))^2$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=p+1}^{N_I} (b_{\sigma'(k)} \cos(n\theta_{\sigma'(k)}))^2$$

$$\leq \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=p+1}^{N_I} \sum_{k=p+1}^{N_I} b_{\sigma'(k)}^2 = \sum_{k=p+1}^{N_I} b_{\sigma'(k)}^2$$, \hspace{0.5cm} (12)

where $\sigma'(k)$’s are the indexes of left interpolation bases after selection. \hspace{0.5cm} (12) indicates that the approximation error is bounded by the sum of squares of the weights with respect to the left interpolation bases. So choosing interpolation bases with largest absolute value weights gives the minimum upper bound of approximating error.

The sparsity of DCT-1, i.e., much of $b_k$’s and $b_0$ in (11) are zero or close to zero, has been demonstrated in the case of Markov-1 signal, by which DCT is regarded as a qualified approximation to the KLT optimal transform. \hspace{0.5cm} (13). The sparsity of DCT is also verified in engineering applications and successfully used in image and speech compression. \hspace{0.5cm} (14). It’s feasible to choose several interpolation bases with larger absolute value weights to approximate $f(n)$ by certain precision due to this sparsity.

According to (12), if we select more interpolation bases, the upper bound of error also becomes small. More interpolation bases cause higher LP order. This may explain some of the LP order adjusting methods.

Remark. “Dense” property of the constructed LP: We know that the set $\mathbb{Q}$ of all rational numbers is dense in the set $\mathbb{R}$ of all real numbers, which means that every real number can be approximated by rational numbers. It’s a good way to study the real number by relatively “simple” rational number in terms of this dense property. We can generalize this idea to the set of all LP, though not necessarily the rigorous definition.

The set of LP constructed by DCT-1 doesn’t include the signals spanned by the interpolation bases other than $\{\cos(n\theta)\}$. However, because the constructed LP can approximate arbitrary signals with certain precision, we may consider that the set of constructed LP is “dense” in the set of all LP; i.e., every LP can be approximated by constructed LP to some extent.

B. Difference operator method

The following result is associated with Lemma 2. We’ll use polynomial sequence to measure arbitrary finite-length discrete signal in terms of LP constructed by difference operator. The relationship between LP and Taylor series will be established.

**Theorem 5.** Let $f(n)$ for $n=0, 1, \ldots, N-1$ be equidistant sampled points of smooth function $f(x)$ on any closed interval. Then $f(n)$ can be approximated or parameterized by LP with any precision as $N$ increases in the form of

$$\hat{f}_p(n) = f(n-1) + \Delta f(n-1) + \Delta^2 f(n-1) + \ldots + \Delta^{p-1} f(n-1),$$

where $p$ is the LP order and $\Delta^k f(n-1)$ is the difference operator \hspace{0.5cm} (17)

$$\Delta^k f(n-1) = \sum_{i=0}^{k} (-1)^i c_{k}^i f(n-1-i),$$

where $k$ is a nonnegative integer and $c_{k}^i$ is the binomial coefficient. By simple manipulation, (13) can be written as

$$\hat{f}_p(n) = \sum_{k=1}^{p} a_k f(n-k).$$

**Proof:** The first part of the proof is totally combinatorial. Substituting (14) into (13) and putting $f(n-1-i)$ of the same i with different coefficient together, we have

$$\hat{f}_p(n) = \sum_{k=0}^{p-1} c_k^0 (-1)^0 f(n-1) + \sum_{k=1}^{p-1} c_k^1 (-1)^1 f(n-2) + \sum_{k=2}^{p-1} c_k^2 (-1)^2 f(n-3) + \ldots + \sum_{k=p-1}^{p-1} c_k^{p-1} (-1)^{p-1} f(n-p).$$

Because $\sum_{k=0}^{n} c_k^i = c_{i+1}^{n+1}$ \hspace{0.5cm} (8), (15) can be further simplified into

$$\hat{f}_p(n) = c_1^p (-1)^0 f(n-1) + c_2^p (-1)^1 f(n-2) + c_3^p (-1)^2 f(n-3) + \ldots + c_p^p (-1)^{p-1} f(n-p).$$

Noting that

$$f(n) - \hat{f}_p(n) = f(n) - \sum_{k=0}^{p-1} c_k^0 (-1)^0 f(n-1) - c_2^1 (-1)^1 f(n-2) - \ldots - c_p^p (-1)^{p-1} f(n-p)$$

$$= c_1^p (-1)^0 f(n) + c_2^p (-1)^1 f(n-1) + c_3^p (-1)^2 f(n-2) + \ldots + c_p^p (-1)^{p-1} f(n-p)$$

$$= \Delta^p f(n),$$

so

$$f(n) = \hat{f}_p(n) + \Delta^p f(n).$$

(17) is the key of the proof, which shows that the difference between $f(n)$ and the approximation of order $p$ LP constructed is the order $p$ difference of $f(n)$, i.e., $\Delta^p f(n)$. The difference operator $\Delta^k f(n)$ has good properties to polynomial sequences; that is, if $f(n)$ is derived from degree $k - 1$ polynomial, then $\Delta^k f(n)$ will be zero \hspace{0.5cm} (17). The left proof is based on (17) in two cases:

Case 1: Interpolation. Of course, any sequence of $N$ data points can be interpolated by degree $N - 1$ polynomial via Lagrange interpolation, when $\Delta^N f(n) = 0$; however, this
trivial case is not our concern. What we are interested in is using fewer LP coefficients to represent \( N \) data points. So only the case that \( f(x) \) is a degree \( p-1 \) polynomial satisfying \( p-1 < N-1 \) is taken into consideration, when \( \Delta^p f(n) = 0 \), and therefore \( f(n) \) can be iterated by order \( p \) LP. This case has been in fact mentioned in Lemma 2.

Cases 2: Approximation. If \( \Delta^p f(n) \) is nonzero for \( p-1 < N-1 \) or \( p \) is not small enough for parameterization despite being Case 1, LP can be used to approximate \( f(n) \). In this case, \( \Delta^p f(n) \) also has some good properties if \( f(x) \) is a smooth function on closed interval. The details are as follows.

The mean-square error of approximating \( f(n) \) by \( \hat{f}_p(n) \) for \( n = 0, 1, \cdots, N - 1 \) is

\[
E = \frac{1}{N} \sum_{n=0}^{N-1} (f(n) - \hat{f}_p(n))^2 = \frac{1}{N} \sum_{n=0}^{N-1} (\Delta^p f(n))^2.
\]

We see

\[
|\Delta^p f(n)| = \left| \sum_{i=0}^{p} (-1)^i c^i_p f(n-i) \right| \\
\leq \sum_{i \text{ is even}} c^i_p f_M(n) - \sum_{i \text{ is odd}} c^i_p f_M(n),
\]

where \( f_M(n) \) and \( f_m(n) \) are the maximum and minimum value of \( f(n) \) within a local neighbourhood at position \( n \), i.e., \( f(n-i) \) for \( i = 0, 1, \cdots, p \), respectively. Since \( \sum_{p=0}^{p} (-1)^i c^i_p = 0 \) [8], which can be written as \( \sum_{i} c^i_p = \sum_{i} c^i_p = \lambda \), we have

\[
|\Delta^p f(n)| \leq \lambda (f_M(n) - f_m(n)),
\]

where \( \lambda \) is a nonnegative constant determined by the order of difference operator. Letting \( \omega = \max_n \{f_M(n) - f_m(n)\} \), then

\[
|\Delta^p f(n)| \leq \lambda \omega
\]

for all \( n = 0, 1, \cdots, N - 1 \). (18) and (20) imply

\[
E \leq \lambda^2 \omega^2.
\]

Now discuss (21). Denote the closed interval between \( n \) and \( -p \) by \( I \). Note that \( \omega \) is related to the modulus of continuity of smooth function \( f(x) \) on closed interval \( I \) [18], which is

\[
\omega_c(\delta; f) = \sup |f(x_1) - f(x_2)|,
\]

where \( x_1 \in I, x_2 \in I \) and \( |x_1 - x_2| \leq \delta \). We know that \( \omega_c \to 0 \) as \( \delta \to 0 \) [18]. Therefore, as the length of interval \( I \) tends to zero (\( \delta \to 0 \) simultaneously), the limit of \( \omega_c \) is also zero. It’s obvious that \( \omega \leq \omega_c \). Thus, if the sampling frequency is sufficiently high (i.e., \( N \) is large enough), the length of \( I \) between \( n \) and \( n-p \) will be as short as possible, resulting in arbitrarily small value of \( \omega \). By (21), the approximation error \( E \) can also be arbitrarily small. So LP with fixed order can approximate \( f(n) \) by any precision as \( N \) increases.

As shown in Fig. 1, the increasing of \( N \) is corresponding to higher sampling frequency to continues function \( f(x) \), which makes the neighborhood of \( p \) data points more local.

Remark 1. Theorem 5 tells us that even if the LP order is unchanged, we can improve the approximating performance by dealing with the original data such as increasing the sampling frequency.

Remark 2. Obviously, the set of constructed LP by difference operator also has the “dense” property in the set of all LP.

Remark 3. Distinction of the constructed LP’s representation of different signals: Since difference operator is the same for all signals, how could they be classified by this constructed LP? The local nonlinearity of data points measured by LP order and the initial values of LP recurrence are the distinctive features. Signals derived from different degree polynomials have different LP order. If signals are from the same degree polynomial, the distinction is the set of initial values of LP recurrence. Otherwise, given the same approximation error and sampling frequency, the LP order or initial values will be distinct among different signals.

C. Relationship to Taylor series

Theorem 5 discussed the case when the order \( p \) of LP constructed by difference operator is fixed and the number \( N \) of data points is varying. What’s the effect if we fix \( N \) and change \( p \)? Taylor series is needed to answer this question.

Corollary 1. For fixed \( N \), the approximation error of LP constructed via difference operator to \( f(n) \) is determined by the smoothness of function \( f(x) \) measured by the polynomial degree of its Taylor series.
Proof: The trivial case is, when \( f(x) \) is a degree \( k \) polynomial, the order \( p \) \( LP \)'s approximation error \( \Delta^p f(n) \) of (17) in Theorem 5 is zero for \( p \geq k + 1 \), which is the case of Lagrange interpolation.

Otherwise, if \( f(x) \) can be approximated well by degree \( k \) polynomial in terms of Taylor series on a closed interval, \( f(n) \) will be nearly data points of degree \( k \) polynomial such that \( \Delta^{k+1} f(n) \) is close to zero. \( \Delta^{k+1} f(n) \) is exactly the order \( k + 1 \) \( LP \)'s approximating error by (17). Thus, increasing the degree of Taylor series makes the approximation error smaller to smooth function; correspondingly, increasing of \( LP \) order yields more precise approximation to \( f(n) \).

Corollary 1 has demonstrated that the Taylor series of \( f(x) \) affects the approximation error of constructed \( LP \) to \( f(n) \). Furthermore, they are also similar in the approximating form.

We can see from (13) that current value of \( f(n) \) is approximated by its nearest point \( f(n - 1) \) adding residual from the first-order difference to higher-order difference. If \( f(n) \) is a linear sequence, order 2 \( LP \) is enough to iterate it. When \( f(n) \) is from quadratic function, order 3 \( LP \) is required. If \( f(n) \) only can be approximated instead of being iterated, higher order \( LP \) yields smaller approximation error. Thus, when \( f(n) \) is sampled from more complex nonlinear function, we should increase the \( LP \) order to iterate or approximate it.

So is Taylor series. When \( f(x) \) is a polynomial, the Taylor series of \( f(x) \) is exactly itself, just like \( LP \)'s iteration of \( f(n) \). If \( f(x) \) is not a polynomial, the residual of approximation of Taylor series is determined by the local smoothness of \( f(x) \) measured by the polynomial degree; the higher the polynomial degree, the smaller the approximation error is.

We may call this constructed \( LP \) “the discrete version of Taylor series”. Fig 2(a) is the simplest case of this comparison. In Fig 2(a) is the approximation to a smooth function \( f(x) \) by its differential (first-order Taylor series) and (b) is approximating \( f(n) \) by order 2 \( LP \), which are very similar to each other.

V. CONCLUSIONS

We gave a general interpretation of \( LP \) in the interpolation framework as well as several following results that are useful both in engineering and theory. This interpolation framework can help us understand \( LP \) from a new viewpoint besides the widely known statistical background. We hope that there will be more useful or interesting results discovered underlying this framework.

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