MEASURE RIGIDITY FOR LEAFWISE WEAKLY RIGID ACTIONS

GABRIEL PONCE AND RÉGIS VARÃO

Abstract. Given a Borel action $G \curvearrowright X$ over a Lebesgue space $X$ we show that if $G \curvearrowright X$ preserves an invariant system of packing regular metrics along a Borel lamination $\mathcal{F}$, then the ergodic measures preserved by the action are rigid in the sense that the system of conditional measures with respect to the partition $\mathcal{F}$ are induced by the given invariant metric system or are supported in a countable number of boundaries of balls. The argument we employ does not require any structure on $G$ other than second-countability and no hyperbolicity on the action as well. Our main result is interesting on its own, but to exemplify its strength and usefulness we show some applications in the context of cocycles over hyperbolic maps and to certain partially hyperbolic maps.

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1. Introduction

The goal of measure rigidity problems is to classify the measures, usually ergodic measures, which are invariant by a given dynamical system. It is not an easy task to make a good description of the set of ergodic invariant measures for a general dynamical system $T$ and, for that reason, one usually has to assume that the dynamics $T$ has stronger structures besides simple measurability.

The most classical example of ergodic measure rigidity is provided by rigid maps of the circle, that is, rotations on the circle. Given a rotation $R_\alpha : S^1 \to S^1$, of angle $2\pi \cdot \alpha$, it is well known that: if $\alpha$ is rational then the unique ergodic measures invariant by $R_\alpha$ are the atomic measures supported on a periodic orbit of the map; if $\alpha$ is irrational then $R_\alpha$ is uniquely ergodic and thus the Lebesgue measure of $S^1$ is the unique ergodic measure preserved by $R_\alpha$. An equally classical result is that a translation in $\mathbb{T}^n$, with translation vector $(\alpha_1, \ldots, \alpha_n)$, is uniquely ergodic if and only if $\{1, \alpha_1, \ldots, \alpha_n\}$ is a $\mathbb{Q}$-linearly independent set. The point here is that the most natural examples of
rigid transformations are accompanied by a rigid classification of the ergodic invariant measures preserved by the map.

There is also an extensive literature in which some type of hyperbolicity is assumed for the dynamics and then measure rigidity results are obtained, for instance [27, 18, 10, 19]. These papers deal with a global classification of the measure, but another way to tackle the measure rigidity is the study of its disintegration along certain dynamically relevant invariant structures. This method has been proven to be very efficient to obtain strong conclusions on the dynamics [27, 20, 15, 25, 26, 21, 32, 31, 4, 35, 34, 12].

On smooth ergodic theory for example, measure disintegration techniques have been an essential tool to obtain ergodicity and rigidity. In his seminal work D. Anosov proved [2] that the disintegration of the volume measure along the unstable (resp. stable) foliation of a volume preserving Anosov diffeomorphism is absolutely continuous with respect to the leaf measure. This result was generalized to the stable and unstable foliation of partially hyperbolic diffeomorphisms (see [6] for example). This is clearly not the general case, and an example of a foliation for which absolute continuity does not occur was given by A. Katok [22]. More specifically, Katok’s example shows a foliation by analytic curves of \((0, 1) \times \mathbb{R}/\mathbb{Z}\) such that there exists a full Lebesgue measure set which intersects each leaf in exactly one point. This phenomenon became known as Fubini’s nightmare.

Fubini’s nightmare and absolute continuity are two extremes among the possibilities that one could expect when studying the conditional measures along a foliation. The first one is equivalent as saying that the conditional measures are atomic measures. The last one implies that the conditional measures are absolutely continuous with respect to the Riemannian measure of the leaf, a property which is usually called leafwise absolute continuity or Lebesgue disintegration of measure. Although these are two extreme behaviors among, a priori, many possibilities for the disintegration of a measure, recent results have indicated that this dichotomy is more frequent than one would at first expect. In [29] D. Ruelle and A. Wilkinson proved that for certain skew product type of partially hyperbolic dynamics, if the fiberwise Lyapunov exponent is negative then the disintegration of the preserved measure along the fibers is atomic. Later A. Homburg [17] proved that some examples treated in [29] one can actually prove that the disintegration is composed by only one dirac measures. A. Avila, M. Viana and A. Wilkinson [4] proved that for \(C^1\)-volume preserving perturbations of the time-1 map of geodesic flows on negatively curved surfaces, the disintegration of the volume measure along the center foliation is either atomic or absolutely continuous and that in the latter case the perturbation should be itself the time-1 map of an Anosov flow. Also inside the class of derived from Anosov diffeomorphisms, G. Ponce, A. Tahzibi and R. Varão [25] exhibited an open class of volume preserving diffeomorphisms which have (mono) atomic disintegration along the center foliation. It is still not known any information on the disintegration of volume for the the derived from Anosov diffeomorphisms with zero center lyapunov exponents constructed in [24].

In the present paper we take a different approach and address the problem of obtaining measure rigidity results for dynamical systems which do
not necessarily have hyperbolic structures, but instead have some metric invariant structure along a certain direction which is compatible with the measurable structure. That is, we consider systems which are not necessarily globally rigid but admit at least an invariant lamination along which it is rigid.

A continuous $m$-dimensional foliation $\mathcal{F}$ of a smooth manifold $M$ by $C^r$-submanifolds is a partition of $M$ into $C^r$-submanifolds which can be locally trivialized by local charts, that is, for each $x \in M$ one can find open sets $U \subset M$, $V \subset \mathbb{R}^m$, $W \subset \mathbb{R}^{n-m}$, $n = \dim M$, and a homeomorphism $\varphi : U \rightarrow V \times W$, such that for every $c \in W$ the set $\varphi^{-1}(V \times \{c\})$, which is called a plaque of $\mathcal{F}$, is a connected component of $L \cap U$ for a certain $L \in \mathcal{F}$. Given a foliation $\mathcal{F}$ of $M$, we denote by $\mathcal{F}(x)$ the element of $\mathcal{F}$ which contains $x$ and call such elements the leaves of $\mathcal{F}$.

We say that an action $G \curvearrowright M$ preserves a foliation $\mathcal{F}$ of $M$ if $\mathcal{F}(g \cdot x) = g \cdot \mathcal{F}(x)$, for every $g \in G$. Given a $G$ action over $M$ and $\mathcal{F}$ a $G$-invariant foliation, we say that the action is bi-Lipschitz leafwise weakly rigid along $\mathcal{F}$ if there exists a measurable system of metrics over the leaves of $\mathcal{F}$, $\{d_L : L \in \mathcal{F}\}$, which is $G$-invariant in the sense that

$$d_{g \cdot L}(g \cdot x, g \cdot y) = d_L(x, y), \quad x, y \in L,$$

and such that the local charts are bi-Lipschitz when restricted to plaques.

One of the main results we show is that ergodic $G$-invariant measures of bi-Lipschitz leafwise weakly rigid actions are classified into two categories: the conditional measures can only be weak-atomic or Hausdorff measure along the leaves.

In the above context, we say that a measure $\mu$ is weak-atomic or that the conditional measures are weak-atomic if the disintegration of $\mu$ on foliated box of $\mathcal{F}$ the support of the disintegrated measures are contained inside the boundary of balls inside $\mathcal{F}(x)$. That is, in the case that $\mathcal{F}$ is one-dimensional then the boundary of a ball is in fact two point, hence a weak-atomic measure would have atomic disintegration.

**Theorem A.** Let $G$ be a second countable group acting on a smooth manifold $M$ by continuous maps and assume that the action is bi-Lipschitz leafwise weakly rigid along a $G$-invariant foliation of $\mathcal{F}$ of dimension $m$ by $C^r$-submanifolds, $r \geq 1$. If $G \curvearrowright M$ is ergodic with respect to a $G$-invariant measure $\mu$ then either:

a) $\mu$ is weak-atomic along $\mathcal{F}$ or;

b) the normalized conditional measures $\mu_x$ are just the $m$-dimensional Hausdorff measures on the leaves of $\mathcal{F}$.

In particular if the foliation is one-dimensional case a) means atomic disintegration.

Theorem A follows as a consequence of a much more general, though more technical result, in which we consider instead of continuous foliations a much weaker structure that we call a Borel lamination, and regular metric systems instead of the bi-Lipschitz condition on the charts. Notwithstanding we give the precise definitions latter (see Section 3) we state the main Theorem below.
Theorem B. Let $G$ be a group, with a second-countable topology, acting on $X$ by Borel automorphisms. Assume that $G \acts X$ is leafwise weakly rigid with respect to a $G$-invariant Borel lamination $\mathcal{F}$ of dimension $m$. Denote by $\{d_x\}$ the respective Borel metric system on $\mathcal{F}$ which is $G$-invariant. If $\{d_x\}$ is a regular system then, given any $G$-invariant ergodic invariant probability measure $\mu$, for $\mu$-almost every $x \in X$ either

a) $\mu$ is weak-atomic along $\mathcal{F}$ or;

b) $\mu_x$ is the Hausdorff measure $\lambda_x$ induced by $d_x$ on the leaf $\mathcal{F}(x)$.

In particular if the foliation is one-dimensional case a) means atomic disintegration.

In Theorem B above if $m > 2$ it is not at all clear that the case a) could be simplified. One should not expect to obtain atomic disintegration for the general setting. The work of Lindestrauss and Schmidt [20] is also a good illustration that one might add stronger hypothesis to the dynamics in order to obtain atomic disintegration.

Question 1. What are the restrictions to be imposed on the dynamics and/or in the Borel lamination in order to obtain on item a) of Theorem B atomic disintegration?

Question 2. Can one give interesting examples for which the disintegration can only be an integer Hausdorff measure? Notice that we can give examples even in Anosov dynamics where disintegrated measures with non-integer Hausdorff dimension appears. For instance, as done in [33], consider an Anosov automorphism on $T^3$ with three distinct eigenvalues (hence we may consider it as a partially hyperbolic system) and consider a perturbation which is volume preserving and such that the conjugacy $h$ (between these two Anosov systems) preserves the center direction and is Hölder continuous restricted to this center foliation, hence the measure $h_*\operatorname{Vol}$ has disintegration $h_*\lambda_x$ which has dimension greater than zero and smaller than one. The idea behind our question is that we, roughly, would like to apply recursively Theorem B (i.e. if item (a) occurs we want to drop the dimension of the conditional measures or obtain atomic disintegration).

1.1. Structure of the paper. In Section 2 we give some preliminaries on measure theory, such as the Rohklin disintegration theorem, the measurable choice theorem and the construction of the Hausdorff measure induced by a given metric in a metric space. In Section 3 we introduce the notions of Borel laminations and Borel metric systems which will play a central role in this paper. In Section 4 we show several technical lemmas, which are very important in the proof of the main theorem, concerning measurable properties of Borel laminations and Borel metric systems. We finish Section 4 with the definition of measure distortion of a Borel metric system, whose goal is to compare the conditional measures to the Hausdorff measure on leaves. In Section 5 we give the proof of Theorem B. We finish the paper with Section 6 where, to show the strength and applicability of our results in other situations in smooth dynamics, we apply Theorem B to $\text{Diff}^r(M)$-valued cocycles (where $M$ is a compact Riemannian manifold) and to partially hyperbolic diffeomorphisms with neutral center direction.
2. Preliminaries on measure theory

All along the paper $G$ will be a group with a second-countable topology and $(X, B, \mu)$ will be a Lebesgue space. A Borel action of $G$ over $X$, denoted by $G \curvearrowright X$, is a Borel function

$$a : G \times X \to X$$

satisfying

1) $a(e, x) = x$ for every $x \in X$, where $e$ is the neutral element of $G$;

2) $a(g_1 g_2, x) = a(g_1, a(g_2, x))$, for every $g_1, g_2 \in G$ and $x \in X$.

3) for each $g \in G$, $a(g, \cdot) : X \to X$ is a Borel map.

Recall that a map is called a Borel map if the inverse image of open (or closed) sets are Borel sets. If we consider a Borel action, as above, since the inverse of $a(g, \cdot)$ is $a(g^{-1}, \cdot)$, hence we get that the image of an open or closed set by a Borel action is a Borel set.

To simplify the notation, for $g \in G$ and $x \in X$ we will write $g \cdot x$, or $g(x)$, to denote $a(g, x)$, and $H \cdot Y$ to denote

$$H \cdot Y = \bigcup_{h \in H, y \in Y} h \cdot y$$

for $H \subset G, Y \subset X$.

Measurable partitions and Rohklin’s Theorem

Let $(X, \mu, B)$ be a probability space, where $X$ is a compact metric space, $\mu$ a probability measure and $B$ the Borelian $\sigma$-algebra of $X$. Given a partition $\mathcal{P}$ of $X$ by measurable sets, we construct a probability space $(\mathcal{P}, \hat{\mu}, \hat{B})$ in the following way: let $\pi : X \to \mathcal{P}$ be the canonical projection, that is, $\pi$ maps a point $x \in X$ to the partition element of $\mathcal{P}$ that contains it, denoted by $\mathcal{P}(x)$. We then set $\hat{\mu} := \pi_* \mu$ and $\hat{B} \in \hat{B}$ if and only if $\pi^{-1}(\hat{B}) \in B$.

**Definition 2.1.** Given a partition $\mathcal{P}$. A family of measures $\{\mu_P\}_{P \in \mathcal{P}}$ is called a system of conditional measures for $\mu$ along $\mathcal{P}$ if

i) for every continuous function $\phi : X \to \mathbb{R}$ the map $P \mapsto \int \phi \, d\mu_P$ is measurable;

ii) $\mu_P(P) = 1$ for $\hat{\mu}$-almost every $P \in \mathcal{P}$;

iii) for every continuous function $\phi : X \to \mathbb{R}$,

$$\int_M \phi \, d\mu = \int_{\mathcal{P}} \left( \int_P \phi \, d\mu_P \right) \, d\hat{\mu}.$$  

If $\{\mu_P\}_{P \in \mathcal{P}}$ is a system of conditional measures for $\mu$ along $\mathcal{P}$ we also say that the family $\{\mu_P\}$ disintegrates the measure $\mu$ or that it is the disintegration of $\mu$ along $\mathcal{P}$.

**Proposition 2.2.** [14, 28] Given a partition $\mathcal{P}$ of $X$, if $\{\mu_P\}$ and $\{\nu_P\}$ are systems of conditional measures for $\mu$ along $\mathcal{P}$, then $\mu_P = \nu_P$ for $\hat{\mu}$-a.e. $P \in \mathcal{P}$. That is, the disintegration of a measure $\mu$ along a partition $\mathcal{P}$ is unique if it exists.
Definition 2.3. A partition $\mathcal{P}$ is called a measurable partition (or countably generated) with respect to $\mu$ if there exist a family of measurable sets $\{A_i\}_{i \in \mathbb{N}}$ and a measurable set $F$ of full measure such that if $B \in \mathcal{P}$, then there exists a sequence $\{B_i\}$, where $B_i \in \{A_i, A_i^c\}$ such that $B \cap F = \bigcap_i B_i \cap F$.

The following classical result of Rohklin states that it is always possible to disintegrate a Borel measure $\mu$ along a measurable partition of a compact metric space.

Theorem 2.4 (Rohklin’s disintegration [28]). Let $\mathcal{P}$ be a measurable partition of a compact metric space $X$ and $\mu$ a Borel probability measure on $X$. Then there exists a disintegration of $\mu$ along $\mathcal{P}$.

Measures on metric spaces

Given a metric space $(X, d)$ we would like to endow $X$ with a measure which is compatible with the metric $d$ in the sense that the measure is a Borel measure on $X$. A classical way of constructing such measure is through Carathéodory Method II which we will briefly recall below. For more details we refer the reader to [11].

Definition 2.5. An outer measure $\mu^*$ on a metric space $(X, d)$ is called a metric outer measure if

$$\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$$

for any pair of subsets $E, F \subset X$ with $d(E, F) := \inf\{d(e, f) : e \in E, f \in F\} > 0$.

Associated to an outer measure $\mu^*$ on $X$ one has a measure $\mu$ defined on the $\sigma$-algebra $\mathcal{M}$ of $\mu^*$-measurable sets by

$$\mu(E) := \mu^*(E), \quad E \in \mathcal{M}.$$ 

The following theorem states that the measure $\mu$ is a Borel measure if, and only if, $\mu^*$ is actually a metric outer measure.

Theorem 2.6. [11, Theorem 3.8] Let $\mu^*$ be an outer measure on a metric space $(X, d)$. Then every Borel set in $X$ is $\mu^*$-measurable if and only if $\mu^*$ is a metric outer measure. In particular, a metric outer measure is a Borel measure.

Thus, in order to construct Borel measures on $X$ one has to construct metric outer measures on $X$. This can be done by Carathéodory’s Method II using premeasures defined on a family of subsets containing the empty set.

Definition 2.7. Let $X$ be a set and $\mathcal{C}$ be a family of subsets of $X$ such that $\emptyset \in \mathcal{C}$. A nonnegative function $\rho$ defined on $\mathcal{C}$ and such that $\rho(\emptyset) = 0$ is called a premeasure.

Theorem 2.8. [11, Section 3.3] Let $(X, d)$ be a metric space and $\mathcal{C}$ a family of subsets of $X$ with $\emptyset \in \mathcal{C}$. Let $\rho$ be a premeasure on $\mathcal{C}$. For each $\delta > 0$ define

$$\mathcal{C}_\delta := \{E \in \mathcal{C} : \text{diam}(E) \leq \delta\}$$
and for $A \subset X$ define

$$
\mu_{\rho}^{\delta}(A) := \inf \left\{ \sum_{i=0}^{\infty} \rho(E_i) : A \subset \bigcup_{i=0}^{\infty} E_i, \quad E_i \in \mathcal{C}_{\delta} \right\}.
$$

The limit

$$
\mu^{\rho,*}(A) := \lim_{\delta \to 0} \mu_{\rho}^{\delta}(A)
$$

exists, being possibly infinite, and $\mu^{\rho,*}$ is a metric outer measure on $X$.

We now define the $m$-dimensional Hausdorff measure on $X$ by taking $C$ to be family of open balls together with the empty set and by taking the premeasure $\rho$ to be the $m$-th power of the radius.

**Definition 2.9.** Let $m \in \mathbb{N} \setminus \{0\}$, $(X, d)$ be a metric space and $C = \{\emptyset\} \cup \{B(x, r) : x \in X, r > 0\}$. Define the premeasure $\rho_m : C \to \mathbb{R}$ by

$$
\rho_m(\emptyset) := 0 \quad \text{and} \quad \rho_m(B(x, r)) := r^m.
$$

The measure $\lambda$ obtained from Theorem 2.8, applied using the premeasure $\rho_m$, will be called the $m$-dimensional Hausdorff measure on $X$. When $m$ is implicit we will refer to this measure simply as the Hausdorff measure on $X$.

**Measurable choice**

We finish this preliminary section with a result by R. J. Aumann [3], which although comes from the Decision Theory in Economics, lies in the realm of measure theory. This result will be used in the study of some atomic case, as it has been used in [20].

**Theorem 2.10** (Measurable Choice Theorem, [3]). Let $(T, \mu)$ be a $\sigma$-finite measure space, let $S$ be a Lebesgue space, and let $G$ be a measurable subset of $T \times S$ whose projection on $T$ is all of $T$. Then there is a measurable function $h : T \to S$, such that $(t, h(t)) \in G$ for almost all $t \in T$.

3. **Borel laminations and metric systems**

From now on, $(X, \mathcal{B}, \mu)$ will always denote a non-atomic Lebesgue probability space endowed with the topology induced by the measurable isomorphism between $X$ and the unit interval. In particular, this topology is second countable and admits a separating sequence of Borel sets, that is, there exists a family of Borel sets $\{U_i\}_{i \in \mathbb{N}}$ for which given any two distinct points $x, y \in X$, there is $i \in \mathbb{N}$ such that $x \in U_i$ and $y \notin U_i$.

When studying a smooth dynamical system such as Anosov systems for example, it is very common to work with invariant foliations naturally associated with the system. We are also work with some invariant foliation, in fact we don’t need to work with such a strong structure and all we need is a weaker type of foliation-like structure, we have called it as Borel lamination. Roughly speaking, a partition $\mathcal{F}$ of a Lebesgue space $X$ is said to be a Borel lamination of dimension $m$ of $X$ if $\mathcal{F}$ is locally modeled by the vertical partition of the product space $(0, 1) \times I^m$, where $I^m$ is a copy of $m$ unit (open, closed or semi-open) intervals or circles.
Definition 3.1. We say that a partition $F = \{F(x)\}_{x \in X}$ of $X$ is a Borel lamination of dimension $m$ if there is a covering of $X$ by a separating family of Borel sets $\{U_i\}_{i \in \mathbb{N}}$ for $X$ such that for each $i \in \mathbb{N}$ there exists a map

$$\varphi_i : (0, 1) \times I^m \to U_i,$$

where $I^m$ is the product of $m$ sets of the collection $\{S^1, (0, 1), [0, 1), (0, 1]\}$, such that:

i) $\varphi_i$ is a Borel isomorphism;

ii) given any $y \in U_i$, then

$$F(y) \cap U_i = \varphi_i(\{y_0\} \times I^m)$$

for some $y_0 = y_0(y) \in (0, 1);$$

Each pair $(U_i, \varphi_i)$ is called a local chart of $F$. When $\varphi_i$ is implicit we sometimes abuse notation and refer to $U_i$ as being a local chart.

Given a Borel action $G \curvearrowright X$ and a Borel lamination $F$ of $X$, we say that the action preserves $F$, or that $F$ is $G$-invariant if

$$F(g(x)) = g(F(x)), \quad \forall g \in G.$$

As a particular case, if $F$ is an $m$-dimensional $C^0$-foliation of a Riemannian manifold $M$ by submanifolds then $F$ is a Borel lamination.

We consider in this work a system of metrics on a Borel lamination invariant by the group $G$. But this system cannot be arbitrary, and to motivate Definition 3.2 below (which shall be the structure for the metric that we demand) consider the usual distance on the Euclidean space $\mathbb{R}^m$. Given a ball of radius $r$ how many disjoint balls of radius $s$ with $s < r$ can we fit inside the ball of radius $r$? Let us make a naive calculation. The volume of the $r$ ball is $r^m$ and the volume of the $s$ ball is $s^m$, hence thinking in term of volume it seems that we should have around $\frac{r^m}{s^m}$ balls, call this number by $L(r, s)$. Besides the approximations we have done we still expect that

$$\frac{L(r, s)}{s^{-m}r^m}$$

to be limited even if we see things locally, that is $s$ going to zero. With that in mind we introduce the next concept.

Definition 3.2. We say that a metric $d$ in a space $X$ is $m$-packing regular, or simply regular when there is no ambiguity and when $m$ is implicit, if there exists a real number $r_0 > 0$ for which, given any $x \in X$ and $0 < s < r \leq r_0$, one can find $L(r, s)$ points $a_1, \ldots, a_{L(r, s)}$ and $U(r, s)$ points $b_1, \ldots, b_{U(r, s)}$ such that

$$\bigcup_{i=1}^{L(r, s)} B(a_i, s) \subset B(x, r) \subset \bigcup_{i=1}^{U(r, s)} B(b_i, s)$$

with

$$C_1 < \limsup_{s \to 0} \frac{L(r, s)}{s^{-m}r^m}, \quad \liminf_{s \to 0} \frac{U(r, s)}{s^{-m}r^m} < C_2,$$

and

$$\lambda \left( B(x, r) \setminus \bigcup_{i=1}^{L(r, s)} B(a_i, s) \right) \leq p \cdot \lambda(B(x, r))$$

where $\lambda$ denotes the Lebesgue measure.
for certain positive constants $C_1, C_2$ and $p < 1$ not depending on $x$ or $r$, where $\lambda$ denotes the $m$-dimensional Hausdorff measure on $(X, d)$ (see Definition 2.7).

When $d$ is a packing regular metric on $X$, the real number $r_0 > 0$ satisfying the above condition is called a packing constant associated to $d$.

We have used the standard euclidean metric to motivated our definition but with a very rough sketch. It turns out that the euclidean metric is in fact packing regular (Lemma 6.2). It is important to emphasize that one should not take from granted our definition, since packing balls is not a trivial task. Finding optimized ways of packing balls is related to what is known as Sphere Packing problems.

**Proposition 3.3.** If $\mathcal{F}$ is a one dimensional Borel lamination in a space $X$, then every Borel metric system $\{d_x\}$ on $\mathcal{F}$ is regular.

**Proof.** Let $x \in X$ and consider an orientation of $\mathcal{F}(x)$. Let $\phi : \mathbb{R} \times \mathcal{F}(x) \to \mathcal{F}(x)$ be the orientation preserving flow given by: for every $y \in \mathcal{F}(x)$ and $t \in \mathbb{R}$, $\phi(t, y)$ is the first point (according to the orientation considered) which satisfies

$$d_x(\phi(t, y), y) = |t|.$$  

As $(\mathcal{F}(x), d_x)$ is locally homeomorphic to $\mathbb{R}$ with the standard distance, the flow is well defined. By definition of $\varphi$, the function $\varphi(t, x) : (\mathbb{R}, | \cdot |) \to (\mathcal{F}(x), d_x)$ is an isometry. By Lemma 6.2 it follows that $d_x$ is packing regular in $\mathcal{F}(x)$. As $x$ is arbitrary, it follows that $\{d_x\}$ is regular as we wanted to show. 

We finish this section with more two definitions.

**Definition 3.4.** Let $\mathcal{F}$ be an $m$-dimensional Borel lamination of $(X, \mu)$. We say that a system of metrics $\{d_x\}$ is a Borel metric system for $\mathcal{F}$ if:

i) for $\mu$-almost every $x \in X$, given a local chart $\varphi_i : (0,1) \times I^m \to U_i$ with $x \in \varphi_i(c, I^m)$, for a certain $0 < c < 1$, the map $\varphi_i(c, \cdot) : (I^m, || \cdot ||) \to (\mathcal{F}(x) \cap U_i, d_x)$ is a homeomorphism over its image where $|| \cdot ||$ denotes the standard euclidian metric in $I^m$;

ii) given a Borel set $E \subset X$ and any $r \geq 0$, the union

$$B(E, r) := \bigcup_{x \in E} B_{d_x}(x, r)$$

is a measurable set where $B_{d_x}(x, r) \subset \mathcal{F}(x)$ denotes the $d_x$-ball centered at $x$ and with radius $r$.

We say that the Borel metric system $\{d_x\}$ is regular if it also satisfies:

iii) for $\mu$-almost every $x \in X$, $d_x$ is a packing regular metric on plaques of $\mathcal{F}(x)$.

**Definition 3.5.** We say that an action $G \curvearrowright X$ is leafwise weakly rigid with respect to a Borel lamination $\mathcal{F}$ on $(X, \mu)$ if $\mathcal{F}$ is $G$-invariant and there exists a Borel metric system $\{d_x\}$ on $\mathcal{F}$ which is $G$-invariant, that is, for each $y \in \mathcal{F}(x)$ one have

$$d_{g(x)}(g(x), g(y)) = d_x(x, y), \forall g \in G.$$
4. Fibered spaces and disintegration over a Borel lamination

Given a Borel lamination \( F \) of a non-atomic Lebesgue probability space \( X \), it is useful to look to \( F \) as fibers over a certain base space. It is not true that we can always choose a measurable set intersecting each plaque \( F(x) \) in exactly one point (the simplest example being the irrational flow on the 2-torus), so the quotient space \( X/F \) is not a good candidate for a base of a fibered space. In the light of this observation, instead of taking the quotient by the plaques we construct a fibered-type space over \( X \) by literally attaching over each \( x \in X \) the respective plaque \( F(x) \).

**Definition 4.1.** Given a space \( X \) and a family \( \mathcal{P} \) of subsets of \( X \). We can construct a natural fibered-type space over \( X \) where the fibers are given by the elements of the family \( \mathcal{P} \). More precisely, we define the space 

\[ X^\mathcal{P} = \bigcup_{x \in X} \{x\} \times \mathcal{P}(x) \subset X \times X, \]

endowed with the \( \sigma \)-algebra induced by the product \( \sigma \)-algebra on \( X \times X \).

We call \( X^\mathcal{P} \) the \((X, \mathcal{P})\)-fibered space or simply the \( \mathcal{P} \)-fibered space. Each subset \( \{x\} \times \mathcal{P}(x) \subset X^\mathcal{P} \) is called the fiber of \( x \) on \( X^\mathcal{P} \).

Proposition 4.2. If \( U_1 \) and \( U_2 \) are described by the local charts \( \varphi_{x_1} \) and \( \varphi_{x_2} \) of \( F \) respectively, then the conditional measures \( \mu_{U_1}^{x_1} \) and \( \mu_{U_2}^{x_2} \), of \( \mu \) on \( U_1 \) and \( U_2 \) respectively, coincide up to a constant on \( U_1 \cap U_2 \).

**Proof.** It follows from [13, Proposition 5.17]. □

Consider the classical volume preserving Kronecker irrational flow on the torus \( T^2 \). Let \( F \) be the Borel lamination given by the orbits of this flow, it follows that this is not a measurable partition in the sense of Definition 2.3. Hence, we cannot apply Rohklin’s disintegration Theorem 2.4 even on the apparently well-behaved Borel foliation. But we may always disintegrate locally and compare two local disintegrations by the above result. The proposition above implies that we can talk about disintegration of a Borel lamination even if it is not a measurable partition, as long as we have in mind that by a disintegration we understand that in a plaque there is a class of conditional measures which differ up to a multiplication of a constant.
4.1. Measure theoretical properties of Borel laminations. Given a Borel lamination $\mathcal{F}$ of $X$ and $\{d_x\}$ a Borel metric system over $\mathcal{F}$, we will denote by $X_1^F$ the fibered space $X_1^\mathcal{F}$ obtained by taking $\mathcal{P} = \{B_{d_x}(x, 1)\}_{x \in X}$ (see Definition 3.1). We will refer to $X_1^\mathcal{F}$ as the unitary $\mathcal{F}$-fibered space or simply as the unitary fibered space when $\mathcal{F}$ is implicit.

**Lemma 4.3.** Let $\mathcal{F}$ be a Borel lamination of $(X, \mu)$. Then, the vertical partition $\mathcal{F}^1 = \{\{x\} \times B_{d_x}(x, 1)\}_x$ of $X_1^\mathcal{F}$ is a measurable partition.

*Proof.* Let $\{U_i\} \subset X$ be the separating sequence of Borel sets as in the Definition 3.1. By the second item in Definition 3.1 the sets $B(U_i, 1)$ are measurable sets. Let $V_i := (U_i \times B(U_i, 1)) \cap X_1^\mathcal{F}$. Each $V_i$ is a measurable set. Now, it is easy to see that each fiber can be written as intersection of sets of the countable family of sets $\{V_i\}$ or its complement. \hfill $\square$

From now on we will make the following convention: given a $G$-invariant measure $\mu$ and $\mathcal{F}$ a $G$-invariant Borel lamination we will denote by $\mu_x$, $x \in X$, the conditional measure of $\mu$ along $\mathcal{F}$ such that each $\mu_x$ is normalized to give weight exactly one to $B_{d_x}(x, 1)$, that is,

$$\mu_x(B(x, 1)) = 1, \forall x \in X.$$ 

Given any $y \in \mathcal{F}(x)$ the measures $\mu_y$ and $\mu_x$ are proportional to each other by Proposition 4.2, that is, there exists a constant $\beta$ for which $\mu_y = \beta \cdot \mu_x$. In particular, evaluating this expressions on the balls $B_{d_x}(x, 1)$ and $B_{d_x}(y, 1)$ we see that this constant may be expressed as

$$\beta \cdot \mu_x(B_{d_x}(x, 1)) = \mu_y(B_{d_x}(x, 1)) \Rightarrow \beta = \mu_y(B_{d_x}(x, 1)),$$

or equivalently

$$\mu_y(B_{d_x}(y, 1)) = \beta \cdot \mu_x(B_{d_x}(y, 1)) \Rightarrow \beta = \frac{1}{\mu_x(B_{d_x}(y, 1))}.$$ 

**Proposition 4.4.** Let $\mathcal{F}$ be a Borel lamination of $(X, \mu)$. Then

$$x \mapsto \mu_x$$

is a measurable map, that is, given any measurable set $W \subset X$ the function

$$x \mapsto \mu_x(W)$$

is a measurable function.

*Proof.* Consider the unitary fibered space $X_1^\mathcal{F}$ and the measure $\tilde{\mu}$ defined by

$$\tilde{\mu}(\tilde{A}) = \int_X \mu_x(\tilde{A}_x) d\mu(x),$$

for any measurable set $\tilde{A} \subset X_1^\mathcal{F}$, where $\tilde{A}_x = \{y \in B_{d_x}(x, 1) : (x, y) \in \tilde{A}\}$. Since the vertical partition on $X_1^\mathcal{F}$ is a measurable partition by Lemma 4.3 the probability measure $\tilde{\mu}$ has a Rohklin disintegration along the leaves for which the conditional measures varies measurably on the base point. By uniqueness and by the definition of $\tilde{\mu}$ we have that the conditional measure on the plaque $\{x\} \times B_{d_x}(x, 1)$ is exactly $\mu_x$. By the properties of the Rohklin disintegration it follows that given any measurable set $W \subset X_1^\mathcal{F}$ we have that
$x \mapsto \mu_x(\widetilde{W}_x)$ is a measurable function. Given any measurable set $W \subset X$ let
\[
\widetilde{W} := \bigcup_{x \in W} \{x\} \times [W \cap B_{d_x}(x, 1)].
\]
Thus $\widetilde{W}_x = W \cap B_{d_x}(x, 1)$ and then we have that
\[x \mapsto \mu_x(W \cap B_{d_x}(x, 1)) = \mu_x(W)\]
is a measurable function on $x$ as we wanted to show. \hfill \Box

**Proposition 4.5.** Let $\mathcal{F}$ be a Borel lamination of $(X, \mu)$. Let $\{d_x\}$ be a Borel metric system over $\mathcal{F}$. For $r$ small enough the function
\[x \mapsto \mu_x(B_{d_x}(x, r))\]
is measurable.

**Proof.** Consider the $\mathcal{F}$-fibered space $X^\mathcal{F}$ with unitary fibers. For each $x \in X$ consider the local chart $\varphi_x: U_x \to V_x \subset \mathbb{R}^n$ and consider $\rho_x$ the metric in $V_x$ given by $d_x \circ \varphi_x^{-1}$. Denote $\varphi_{e_j}$ the function given by:
- if $t \geq 0$ then
  \[\varphi_{e_j}(t, y) = y + s \cdot e_j, \quad s > 0\]
such that $\rho_x(y, y + s \cdot e_j) = t$;
- if $t < 0$ then
  \[\varphi_{e_j}(t, y) = y + s \cdot e_j, \quad s < 0\]
such that $\rho_x(y, y + s \cdot e_j) = -t$.

Let $\varphi_j(r, x) := \varphi_x^{-1} \circ \varphi_{e_j}(r, \varphi_x(x))$. This is obviously measurable in $x$ when $r$ is fixed. By Lusin’s Theorem we can take a sequence of compact spaces $C_1, C_2, \ldots \subset X$ such that $\varphi_j(r, \cdot)|C_i, \varphi_j(-r, \cdot)|C_i$ are continuous for every $i \geq 1, j = 1, \ldots, n$ and $\mu(\bigcup_{i=1}^\infty C_i) = 1$. For each $i$ define:
\[
W_i := \bigcup_{z \in C_i} \{z\} \times B_{d_z}(z, r) = \bigcup_{z \in C_i} \bigcup_{0 \leq s \leq r, j=1}^n \{z\} \times \{\varphi_j(-s, z), \varphi_j(s, z)\}.
\]
This is a compact set thus Borel. Now, we know that
\[z \mapsto \tilde{\mu}_z(W_i)\]
is a measurable function where $\tilde{\mu}_z$ denotes the conditional of $\tilde{\mu}_z$ on the fiber of $z$. Thus for each $i$ we have that
\[z \mapsto \mu_z(B_{d_z}(z, r))\]
is measurable on $C_i$. Thus it is measurable on $C = \bigcup_{i=1}^\infty C_i$ which has full measure. \hfill \Box

**Corollary 4.6.** On the hypothesis of Proposition 4.5 if the disintegration of $\mu$ along $\mathcal{F}$ is not weakly-atomic, then for each typical $x \in X$ the function
\[r \mapsto \mu_x(B_{d_x}(x, r))\]
is continuous. Furthermore the function
\[(x, r) \mapsto \mu_x(B_{d_x}(x, r))\]
is jointly measurable.

Proof. Let \( x \in X \) be a \( \mu \)-typical point, hence \( \mu_x \) is a non-atomic measure on \( \mathcal{F}(x) \). First, let us prove that \( r \mapsto \mu_x(B_{d_x}(x, r)) \) is a continuous function. Let \( y_n \in \mathcal{F}(x) \) and \( \varepsilon_n \searrow \varepsilon \in (0, \infty) \), hence \( \mu_{x}(B_{d_x}(x, \varepsilon_n)) = \mu_x(B_{d_x}(x, \varepsilon)) + \mu_x(B_{d_x}(x, \varepsilon) \setminus B_{d_x}(x, \varepsilon)) \), because \( \mu_x \) is nonatomic

\[
\lim_{n \to \infty} \mu_x(B_{d_x}(x, \varepsilon_n) \setminus B_{d_x}(x, \varepsilon)) = 0.
\]

Then,

\[
\mu_x(B_{d_x}(x, \varepsilon_n)) \to \mu_x(B_{d_x}(x, \varepsilon)),
\]

showing the first part of the statement.

By Proposition 4.5, \( x \mapsto \mu_x(B_{d_x}(x, r)) \) is a measurable function, therefore the function \( (x, r) \mapsto \mu_x(B_{d_x}(x, r)) \) is a Carathéodory function, in particular it is a jointly measurable function (Lemma 4.51).

Lemma 4.7. On the hypothesis of Proposition 4.5 if the disintegration of \( \mu \) along \( \mathcal{F} \) is not weakly-atomic, for each \( r > 0 \) and \( x \in M \) the function

\[
y \mapsto \mu_y(B_{d_y}(y, r))
\]

is continuous when restricted to \( \mathcal{F}(x) \).

Proof. Let \( y_n \to y \), \( y_n \in \mathcal{F}(x) \), \( y \in \mathcal{F}(x) \). To prove that \( \mu_{y_n}(B_{d_y}(y_n, r)) \to \mu_y(B_{d_y}(y, r)) \) it is enough to show that

\[
\lim_{n \to \infty} \mu_y(B_{d_{y_n}}(y_n, r)) = \mu_y(B_{d_y}(y, r)), \quad \forall r \in \mathbb{R}
\]

since

\[
\mu_y(B_{d_y}(y, \varepsilon_k)) = \frac{\mu_y(B_{d_{y_n}}(y_n, 1))}{\mu_{y_n}(B_{d_{y_n}}(y_n, 1))} \cdot \mu_{y_n}(B_{d_{y_n}}(y_n, \varepsilon_k))
\]

\[
= \mu_y(B_{d_{y_n}}(y_n, 1)) \cdot \mu_{y_n}(B_{d_{y_n}}(y_n, \varepsilon_k)).
\]

Given any \( k \in \mathbb{N} \), since \( \mu_x \) is not weakly-atomic we have that

\[
\mu_x(\partial B_{d_y}(y, r)) = 0 \quad \text{and} \quad \mu_x(\partial B_{d_x}(y_n, r)) = 0, \quad \forall n \in \mathbb{N},
\]

where \( \partial B_{d_y} \) denotes the boundary of the set inside the leaf \( \mathcal{F}(x) \). Now, let \( B_n := B_{d_y}(y_n, r) \Delta B_{d_x}(y, r) \) where \( Y \Delta Z \) denotes the symmetric difference of the sets \( Y \) and \( Z \). Observe that, by passing to a subsequence of \( y_n \) if necessary, we have \( B_n \supset B_{n+1} \), for every \( n \geq 1 \). Thus

\[
\lim_{n \to \infty} \mu_x(B_n) = \lim_{n \to \infty} \mu_x \left( \bigcap_{j=1}^{n} B_j \right) = \mu_x \left( \bigcap_{j=1}^{\infty} B_j \right) \leq \mu_x(\partial B_{d_y}(y, r)) = 0.
\]

Therefore

\[
\lim_{n \to \infty} \mu_x(B_{d_x}(y, r) \setminus B_{d_x}(y_n, r)) = \lim_{n \to \infty} \mu_x(B_{d_x}(y_n, r) \setminus B_{d_x}(y, r)) = 0
\]

and consequently

\[
\lim_{n \to \infty} \mu_x(B_{d_x}(y_n, r)) = \mu_x(B_{d_x}(y, r)),
\]

as we wanted to show.
4.2. The measure distortion of a Borel metric system. The last concept we will introduce in this section is the concept of measure distortion with respect to a Borel metric system.

Definition 4.8. Let \((X, \mu)\) be a non-atomic Lebesgue space and \(\mathcal{F}\) be a Borel lamination of dimension \(m\) of \(X\). Let \(\{\mu_x\}\) denote the system of conditional measures along \(\mathcal{F}\) and let \(d = \{d_x\}\) be a Borel metric system over \(\mathcal{F}\). We define the upper and lower \(\mu\)-distortion of the Borel metric system \(d\) respectively by

\[
\Delta(\mu)(x) := \limsup_{\varepsilon \to 0} \frac{\mu_x(B_{d_x}(x, \varepsilon))}{\varepsilon^m}, \quad \Delta(\mu)(x) := \liminf_{\varepsilon \to 0} \frac{\mu_x(B_{d_x}(x, \varepsilon))}{\varepsilon^m},
\]

where \(\mu_x\) is taken to be the measure on the class of \([\mu_x]\) which gives weight one to \(B_{d_x}(x, 1)\). If the upper and lower distortions are equal then we just call it the \(\mu\)-distortion of the Borel metric system \(d\) and denote it by

\[
\Delta(\mu)(x) := \lim_{\varepsilon \to 0} \frac{\mu_x(B_{d_x}(x, \varepsilon))}{\varepsilon^m}.
\]

5. Proof of the main Theorems

5.1. Sketch of the proof. The proof will be made in two steps. The first, and easy case, is the weak-atomic case. The second case, the non weakly-atomic case is the one where the main ideas and technical problems appear.

The first observation is that ergodicity implies that the upper (resp. lower) \(\mu\)-distortion at \(x\) is constant almost everywhere. Then, using the \(G\)-invariance of the family \(\{B_{d_x}(x, r)\}\) and the ergodicity of the measure, we obtain some uniformity on the upper (resp. lower) \(\mu\)-distortion in the sense that, along a certain sequence \((\varepsilon_k)\), \(\varepsilon_k \to 0\), the ratios appearing in Definition 4.8 converge to the upper (resp. lower) \(\mu\)-distortion with the same rate for almost every point \(x \in X\). This is proven in Lemmas 5.1 and 5.2. Once proven this uniformity of the upper (resp. lower) distortion, we turn our attention to the set of all points \(\Pi\) (resp. \(\Pi^\infty\)) where such uniformity occurs and its topological characteristics when restricted to a plaque. To be more precise, we prove in Lemmas 5.2 and 5.3 that the set of points for which the uniform distortions occurs is a closed set in each plaque intersecting it.

We then turn our attention to the set \(D\) (resp. \(D^\infty\)) of points \(x\) for which \(\Pi\) (resp. \(\Pi^\infty\)) is dense in \(F(x)\), that is, \(\Pi \cap F(x) = F(x)\) (resp. \(\Pi^\infty \cap F(x) = F(x)\)). The set \(D\) (resp. \(D^\infty\)) is \(G\)-invariant, thus it has full or zero measure. We first show that if the distortion is unbounded then \(\mu(D^\infty) = 0\). If the distortion is finite and \(\mu(D) = 1\) then the denseness of \(\Pi\) on the plaques \(F(x), x \in D\), allows us to extend the uniform upper distortion to every point on the respective plaque. Using the uniformity at every point and the packing regularity of the metrics, we prove that the upper distortion is a constant times the \(\mu_x\) measure of the set \(B_{d_x}(x, 1)\) on the plaque \(F(x)\).

Applying the same argument for the set \(\Pi\) where the lower distortion is uniform we get to the same equality and conclude that the upper and lower distortions are equal, thus the limit converges and we actually have a well defined distortion. Using this fact we prove in Lemma 5.3 that \(\mu_x\) is the Hausdorff measure on the leaf \(F(x)\).
The last two cases are treated simultaneously. If the distortion is unbounded and \( \mu(D) = 0 \) or if the distortion is bounded and \( \mu(D) = 0 \), then almost every plaque has a finite number of open balls in it which are in the complement of the set \( \mathbb{P}^\infty \) or of the set \( \mathbb{P} \). We use this “holes” together with the measurable choice theorem to show that weak-atoms should appear, which yields an absurd.

5.2. Proof of Theorem \( B \)

First of all, assume that \( X \) is a Lebesgue space which contains atoms, that is, there is a countable subset \( Z \subset X \) such that \( \mu(\{z\}) > 0 \) for any \( z \in Z \). Since \( G \curvearrowright X \) is ergodic and \( Z \) is \( G \)-invariant we have \( \mu(Z) = 1 \) and since the weight of each atom is also a \( G \)-invariant function there exists \( k_0 \in \mathbb{N} \) such that \( Z \) has \( k_0 \) elements \( a_1, \ldots, a_{k_0} \) and \( \mu(a_i) = 1/k_0 \) for every \( 1 \leq i \leq k_0 \). Consider \( F_i := F(a_i) \). By the invariance of the cardinality of \( F_i \cap Z \) and ergodicity of \( G \curvearrowright X \), each \( F(a_i) \) has exactly the same number of atoms. Thus we fall in the second case of the statement.

We can now assume that \( (X, \mathcal{B}, \mu) \) is an atom-less Lebesgue space. We break the proof in two cases.

Non weakly-atomic case: Assume that the disintegration is not weak-atomic, that is, the conditional measure of boundaries of balls is zero.

Let \( \{\mu_x\} \) to be the disintegration of \( \mu \) along the leaves of \( F \). For the sake of completeness, we will denote the upper and lower distortion defined on definition 4.8 by \( \bar{\Delta} \) and \( \underline{\Delta} \) respectively, that is,

\[
\bar{\Delta}(x) := \limsup_{\varepsilon \to 0} \frac{\mu_x(B_{d_x}(x, \varepsilon))}{\varepsilon^m}, \quad \underline{\Delta}(x) := \liminf_{\varepsilon \to 0} \frac{\mu_x(B_{d_x}(x, \varepsilon))}{\varepsilon^m}.
\]

Recall that \( B_{d_x}(x, \varepsilon) \) is the ball inside \( F(x) \), centered in the point \( x \) and with radius \( \varepsilon \) with respect to the metric \( d_x \).

Observe that \( \mu_x(B_{d_x}(x, \varepsilon)) > 0 \) for every \( x \in \text{Supp}(\mu_x) \) (where the support here is inside \( F(x) \)). Thus, it makes sense to evaluate the quantities above. Also observe that, a priori, \( \bar{\Delta}(x) \) and \( \underline{\Delta}(x) \) are measurable functions but not necessarily bounded can be infinity. Also note that both \( \bar{\Delta}(x) \) and \( \underline{\Delta}(x) \) are \( G \)-invariant maps since

\[
g_*\mu_x = \mu_{g(x)} \quad \text{and} \quad g(B_{d_x}(x, \varepsilon)) = B_{d_g(x)}(g(x), \varepsilon), \forall g \in G
\]

since

\[
d_{g(x)}(g(x), g(y)) = d_x(x, y).
\]

By ergodicity of \( G \curvearrowright X \) it follows that both are constant almost everywhere, let us call these constants by \( \bar{\Delta} \) and \( \underline{\Delta} \) that is for almost every \( x \):

(3) \[ \bar{\Delta}(x) = \bar{\Delta}, \quad \text{and} \quad \underline{\Delta}(x) = \underline{\Delta} \]

Let \( D \) be a (full measure) set of points \( x \) for which (3) occurs.

5.3. Technical Lemmas for the case \( \underline{\Delta} = \infty \).

Lemma 5.1. If \( \bar{\Delta} = \infty \), there exists a sequence \( \varepsilon_k \to 0 \), as \( k \to +\infty \), and a full measure subset \( R^\infty \subset D \) such that

i) \( R^\infty \) is \( f \)-invariant;
ii) for every \( x \in R^\infty \) we have
\[
(4) \quad \frac{\mu_x(B_d(x, \varepsilon_k))}{\varepsilon_k^m} \geq k.
\]

**Proof.** Let \( k \in \mathbb{N}^* \) arbitrary. Since \( \Delta(x) = \overline{\Delta} \) for every \( x \in D \), define
\[
\varepsilon_k(x) := \sup \left\{ \varepsilon \leq 1 : \frac{\mu_x(B_d(x, \varepsilon))}{\varepsilon^m} \geq k \right\}, \quad x \in D.
\]

**Claim:** The function \( \varepsilon_k(x) \) is measurable for all \( k \in \mathbb{N} \).

**Proof.** Define \( w(x, \varepsilon) = \frac{\mu_x(B_d(x, \varepsilon))}{\varepsilon^m} \).

As we are assuming the disintegration to be non weakly-atomic, then by Corollary 4.6, for any typical \( x \in M \) the function \( w(x, \cdot) : (0, \infty) \to (0, \infty) \) is continuous and, for \( \varepsilon > 0 \) fixed the function \( w(\cdot, \varepsilon) : M \to (0, \infty) \) is a measurable function by Proposition 4.5. Given any \( k \in \mathbb{N}, k > 0 \), the continuity of \( w(x, \cdot) \) implies that
\[
\varepsilon_k^{-1}((0, \beta)) = \{ x : \varepsilon_k(x) \in (0, \beta) \}
= \bigcap_{\beta \leq r \leq 1} w(\cdot, r)^{-1}((0, k))
= \bigcap_{\beta \leq r \leq 1, r \in \mathbb{Q}} w(\cdot, r)^{-1}((0, k)).
\]

Therefore \( \varepsilon_k^{-1}((0, \beta)) \) is measurable, as it is a countable intersection of measurable sets, and consequently \( \varepsilon_k \) is a measurable function for every \( k \). \( \square \)

Note that \( \varepsilon_k(x) \) is \( G \)-invariant. Thus, by ergodicity, let \( R_k^\infty \) be a full measure set such that \( \varepsilon_k(x) \) is constant equal to \( \varepsilon_k \). It is easy to see that the sequence \( \varepsilon_k \) goes to 0 as \( k \) goes to infinity. Take \( \tilde{R}^\infty := \bigcap_{k=1}^\infty R_k^\infty \). Since each \( R_k^\infty \) has full measure, \( \tilde{R}^\infty \) has full measure and clearly satisfies what we want for the sequence \( \{ \varepsilon_k \}_{k=1}^\infty \). Finally, take \( R^\infty = G \cdot \tilde{R}^\infty \). The set \( R^\infty \) is \( G \)-invariant, has full measure and satisfies (i) and (ii). \( \square \)

We now set
\[
\Pi^\infty_x := \left\{ y \in F(x) : \frac{\mu_x(B_d(x, \varepsilon_k))}{\varepsilon_k^m} \geq k, \forall k \geq 1 \right\},
\]
and
\[
\Pi^\infty := \bigcup \Pi^\infty_x.
\]
Similarly we define \( \Pi^\infty_x \) and \( \Pi^\infty \) by dealing with \( \Delta \) in the place of \( \Delta \).

**Lemma 5.2.** For every \( x \in R^\infty \) the set \( \Pi^\infty_x \) (resp. \( \Pi^\infty \)) is a closed subset on the plaque \( F(x) \)

**Proof.** Let \( y_n \to y, y_n \in \Pi^\infty_x, y \in F(x) \). By Lemma 4.7 the map \( y \in F(x) \mapsto \mu_y(B_d(y, r)) \) is continuous for \( r > 0 \) fixed. Thus
\[
\lim_{n \to \infty} \mu_{y_n}(B_d(y_n, \varepsilon_k)) = \mu_y(B_d(y, \varepsilon_k)), \quad k \geq 1,
\]
which implies that for every $k \geq 1$ we have
\[
\frac{\mu_x(B_{d_x}(y, \varepsilon_{k}))}{\varepsilon_{k}^m} = \lim_{n \to \infty} \frac{\mu_{y_n}(B_{d_x}(y_n, \varepsilon_{k}))}{\varepsilon_{k}^m} \geq k,
\]
that is, $y \in \overline{\Pi}_x$ as we wanted. \qed

Lemma 5.3. If $\Sigma = \infty$, there are two disjoint $G$-invariant borel sets $A^\infty$ and $B^\infty$ such that

i) $\mu(A^\infty) = 0$, $\mu(B^\infty) = 1$;
ii) if $x \in A^\infty$, then there exists $k_0 \in \mathbb{N}$ such that
\[
\frac{\mu_x(B_{d_x}(x, \varepsilon_{k_0}))}{\varepsilon_{k_0}^m} < k_0;
\]
iii) if $x \in B^\infty$, then for $\varepsilon_k$ as in Lemma 5.1 we have
\[
\frac{\mu_x(B_{d_x}(x, \varepsilon_{k}))}{\varepsilon_{k}^m} \geq k.
\]

Proof. Let $x \in X \setminus \overline{\Pi}^\infty$. Then there exists $k_0 \geq 1$ such that
\[
\frac{\mu_x(B_{d_x}(x, \varepsilon_{k_0}))}{\varepsilon_{k_0}^m} < k_0.
\]
By the measurability of $x \mapsto \mu_x(B_{d_x}(x, \varepsilon_{k}))$ (see Proposition 4.5) and Lusin’s Theorem we can take a compact set $A_1^\infty$ where this function varies continuously. In particular, there exists an open set $A_2^\infty \subset X$ such that for every $y \in A_2^\infty \setminus A_1^\infty$ we have
\[
\frac{\mu_x(B_{d_x}(x, \varepsilon_{k_0}))}{\varepsilon_{k_0}^m} < k_0.
\]
Define $A^\infty = G \cdot (A_2^\infty \cap A_1^\infty)$. Clearly $A^\infty$ is a Borel $G$-invariant set and $A^\infty \subset X \setminus \overline{\Pi}^\infty$ by the invariance of $\mu_x$ and $d_x$. By ergodicity we conclude that $\mu(A^\infty) = 0$.

Now let us find the set $B^\infty$. For each $n \in \mathbb{N}$, consider $\tilde{B}_n$ the set of all points $x \in X$ satisfying
\[
\frac{\mu_x(B_{d_x}(x, \varepsilon_{k}))}{\varepsilon_{k}^m} > k - \frac{1}{n}.
\]
Using again Proposition 4.5 Lusin’s Theorem and the invariance of $\mu_x$ and $d_x$ by the action of $G$ similar to what we have done in the previous paragraph, we find a sequence of nested $G$-invariant Borel sets
\[
\ldots B_{n+1}^\infty \subset B_n^\infty \subset B_{n-1}^\infty \subset \ldots \subset B_1^\infty, \quad B_n^\infty \subset \tilde{B}_n^\infty,
\]
such that for every $y \in B_n^\infty$ we have
\[
\frac{\mu_y(B_{d_y}(y, \varepsilon_{k}))}{\varepsilon_{k}^m} > k - \frac{1}{n}.
\]
By Lemma 5.1 and by ergodicity of $\mu$ we have $\mu(B_n^\infty) = 1$ for every $n \geq 1$. Take
\[
B^\infty := \bigcap_{n=1}^{\infty} B_n^\infty.
\]
Then $B^\infty$ is a $G$-invariant Borel set, $\mu(B^\infty) = 1$ and clearly $A^\infty \cap B^\infty = \emptyset$.

Consider the following measurable set

$$D^\infty := \mathcal{F}(B^\infty) \setminus \mathcal{F}(A^\infty).$$

In other words

$$D^\infty = \{ x \in \mathcal{F}(A^\infty \cup B^\infty) : \Pi_x^\infty \cap \mathcal{F}(x) = \mathcal{F}(x) \},$$

that is, $D$ is the set of all points in $\mathcal{F}(A^\infty \cup B^\infty)$ whose plaque is fully inside $\Pi_x^\infty$.

5.4. Technical Lemmas for the case $\overline{\Delta} < \infty$.

**Lemma 5.4.** If $\overline{\Delta} < \infty$, there exists a sequence $\varepsilon_k \to 0$, as $k \to +\infty$, and a full measure subset $R \subset D$ such that

i) $R$ is $G$-invariant;

ii) for every $x \in R$, then

$$\left| \frac{\mu_x(B_{\varepsilon_k}(x, \varepsilon_k))}{\varepsilon_k^m} - \overline{\Delta} \right| \leq \frac{1}{k};$$

Proof. The proof is very similar to the proof of Lemma 5.1. Let $k \in \mathbb{N}^*$ arbitrary. Since $\overline{\Delta}(x) = \overline{\Delta}$ for every $x \in D$ define

$$\varepsilon_k(x) := \sup \left\{ \varepsilon : \left| \frac{\mu_x(B_{\varepsilon}(x, \varepsilon))}{\varepsilon^m} - \overline{\Delta} \right| \leq \frac{1}{k} \right\}.$$

Observe that such $\varepsilon_k$ exists because since the lim sup is $\overline{\Delta}$ we can take a sequence $\varepsilon_l \to 0$ such that the ratio given approaches $\overline{\Delta}$.

**Claim:** The function $\varepsilon_k(x)$ is measurable for all $k \in \mathbb{N}$.

**Proof.** Define

$$w(x, \varepsilon) = \frac{\mu_x(B_{\varepsilon}(x, \varepsilon))}{\varepsilon^m}.$$ 

As observed in the proof of Lemma 5.1, $r \mapsto w(x, r)$ is continuous and $x \mapsto w(x, r)$ is measurable. Given any $k \in \mathbb{N}$, $k > 0$, the continuity of $w(x, \cdot)$ implies that

$$\varepsilon_k^{-1}((0, \beta)) = \{ x : \varepsilon_k(x) \in (0, \beta) \} = \bigcap_{\beta \leq r \leq 1} w(\cdot, r)^{-1} \left( \left[ \overline{\Delta} + \frac{1}{k}, \infty \right) \right) \cup w(\cdot, r)^{-1} \left( \left[ 0, \overline{\Delta} - \frac{1}{k} \right) \right) \cup \bigcap_{\beta \leq r \leq 1, r \in \mathbb{Q}} w(\cdot, r)^{-1} \left( \left[ \overline{\Delta} + \frac{1}{k}, \infty \right) \right) \cup w(\cdot, r)^{-1} \left( \left[ 0, \overline{\Delta} - \frac{1}{k} \right) \right).$$

Therefore $\varepsilon_k^{-1}((0, \beta))$ is measurable, as it is a countable intersection of measurable sets, and consequently $\varepsilon_k$ is a measurable function for every $k$. □

As $\varepsilon_k(x)$ is $G$-invariant, by ergodicity we may take the full measure set $R_k$ where $\varepsilon_k(x)$ is constant equal to $\varepsilon_k$. The sequence $\varepsilon_k$ goes to 0 as $k$ goes to infinity, so we set $\tilde{R} := \bigcap_{k=1}^{\infty} R_k$. Since each $R_k$ has full measure, $\tilde{R}$ has full measure and clearly satisfies what we want for the sequence $\{\varepsilon_k\}_k$. The
set \( R = G \cdot \bar{R} \) is \( G \)-invariant, has full measure and satisfies (i) and (ii) as we wanted. \( \square \)

Similar to the definitions made in section 5.3 we set
\[
\Pi := \bigcup \Pi_x.
\]
where
\[
\Pi_x := \left\{ y \in \mathcal{F}(x) : \left| \frac{\mu_y(B_d(x, \epsilon_k))}{\epsilon_k^m} - \frac{\Delta}{k} \right| \leq \frac{1}{k}, \forall k \geq 1 \right\},
\]
similarly we define \( \Pi_x \) and \( \Pi \) with \( \Delta \) in the role of \( \Delta \).

**Lemma 5.5.** For every \( x \in R \) the set \( \Pi_x \) (resp. \( \Pi_x \)) is a closed subset on the plaque \( F(x) \)

**Proof.** Identical to the proof of Lemma 5.2. \( \square \)

**Lemma 5.6.** If \( \Delta < \infty \), there are two disjoint \( G \)-invariant borel sets \( A \) and \( B \) such that

i) \( \mu(A) = 0 \), \( \mu(B) = 1 \);

ii) if \( x \in A \), then there exists \( k_0 \in \mathbb{N} \) such that
\[
\left| \frac{\mu_x(B_{d_x}(x, \epsilon_{k_0}))}{\epsilon_{k_0}^m} - \frac{\Delta}{k_0} \right| > \frac{1}{k_0};
\]

iii) if \( x \in B \), then for \( \epsilon_k \) as in Lemma 5.7 we have
\[
\left| \frac{\mu_x(B_{d_x}(x, \epsilon_k))}{\epsilon_k^m} - \frac{\Delta}{k} \right| \leq \frac{1}{k}.
\]

**Proof.** Let \( x \in X \setminus \Pi \). Then there exists \( k_0 \geq 1 \) such that
\[
\left| \frac{\mu_x(B_{d_x}(x, \epsilon_{k_0}))}{\epsilon_{k_0}^m} - \frac{\Delta}{k_0} \right| > \frac{1}{k_0},
\]
By the measurability of \( x \mapsto \mu_x(B_{d_x}(x, \epsilon_k)) \) (see Proposition 4.5 and Lusin’s Theorem we can take a compact set \( A_1 \) where this function varies continuously. In particular, there exists an open set \( A_2 \subset X \) such that for every \( y \in A_2 \cap A_1 \) we have
\[
\left| \frac{\mu_x(B_{d_x}(x, \epsilon_{k_0}))}{\epsilon_{k_0}^m} - \frac{\Delta}{k_0} \right| > \frac{1}{k_0},
\]
Define \( A = G \cdot (A_2 \cap A_1) \). Clearly \( A \) is a Borel \( G \)-invariant set and \( A \subset X \setminus \Pi \) by the invariance of \( \mu_x \) and \( d_x \). By ergodicity we conclude that \( \mu(A) = 0 \).

Now let us find the set \( B \). For each \( n \in \mathbb{N} \), consider \( \bar{B}_n \) the set of all points \( x \in X \) satisfying
\[
\left| \frac{\mu_x(B_{d_x}(x, \epsilon_k))}{\epsilon_k^m} - \frac{\Delta}{k} \right| < \frac{1}{k} + \frac{1}{n}.
\]
Using again Proposition 4.5 Lusin’s Theorem and the invariance of \( \mu_x \) and \( d_x \) by the action of \( G \) similar to what we have done in the previous paragraph, we find a sequence of nested \( G \)-invariant Borel sets
\[
\ldots B_{n+1} \subset B_n \subset B_{n-1} \subset \ldots \subset B_1, \quad B_n \subset \bar{B}_n,
\]
such that for every $y \in B_n$ we have

$$\left| \frac{\mu_x(B_{d_x}(x, \varepsilon_k))}{\varepsilon_k^m} - \overline{\Delta} \right| < \frac{1}{k} + \frac{1}{n}.$$ 

By Lemma 5.4 and by ergodicity of $\mu$ we have $\mu(B_n) = 1$ for every $n \geq 1$. Take

$$B := \bigcap_{n=1}^{\infty} B_n.$$ 

Then $B$ is a $G$-invariant Borel set, $\mu(B) = 1$ and clearly $A \cap B = \emptyset$. □

Similar to the definition made in section 5.3 we set

$$D := F(B) \setminus F(A),$$

or equivalently

$$D = \{ x \in F(A \cup B) : \Pi_x \cap F(x) = F(x) \},$$

that is, $D$ is the set of all points in $F(A \cup B)$ whose plaque is fully inside $\Pi_x$.

5.5. **Case 1:** $\overline{\Delta} = \infty$ and $\mu(D^\infty) = 1$.

Let us prove that this case cannot occur. Consider a typical fiber $F(x)$, that is $x \in D^\infty$, and take any $k \geq 1$ fixed. By hypothesis, since the Borel metric system is regular, say with a packing constant $r_0 > 0$, we can take at least $i(k) := L(r_0, \varepsilon_k)$ disjoint balls of radius $\varepsilon_k$ inside $B_{d_x}(x, r_0)$. Let $b_1, b_2, ..., b_{i(k)}$ be the centers of such balls. Then, for each $1 \leq i \leq i(k)$

$$\sum_{i=1}^{i(k)} \mu_x(B_{d_x}(a_i, \varepsilon_k)) \leq \mu_x(B_{d_x}(x, r_0)) \Rightarrow \sum_{i=1}^{i(k)} \alpha_i \cdot \mu_x(B_{d_x}(a_i, \varepsilon_k)) \leq \mu_x(B_{d_x}(x, r_0))$$

where $\alpha_i := \mu_x(B_{d_x}(a_i, 1))$. By Lemma 4.7 there exists $\alpha > 0$ such that $\alpha_i \geq \alpha$ for every $i$. Thus,

$$\alpha \cdot \sum_{i=1}^{i(k)} k \cdot \varepsilon_k^m \leq \mu_x(B_{d_x}(x, r_0)) \Rightarrow \alpha \cdot k \cdot \varepsilon_k^m \leq \mu_x(B_{d_x}(x, r_0)).$$

Since $\limsup_{k \to \infty} i(k) \cdot \varepsilon_k^m$ is bounded from below the left side is going to infinity which is an absurd. Thus this case does not occurs.

5.6. **Case 2:** $\overline{\Delta} < \infty$ and $\mu(D) = 1$.

We will prove that if this case occurs then the conditional measures are just the Hausdorff measures of the leaves.

**Lemma 5.7.** There exists a constant $c > 1$ for which

$$\frac{1}{c} < \overline{\Delta}.$$ 

**Proof.** As $\overline{\Delta} < \infty$ by hypothesis, we just need to show that it is bounded away from zero. For any given $k \in \mathbb{N}^*$ and $x \in \Pi$ we have

$$\left| \frac{\mu_x(B_{d_x}(x, \varepsilon_k))}{\varepsilon_k^m} - \overline{\Delta} \right| \leq \frac{1}{k}.$$ (6)
Given \( \varepsilon > 0 \) take \( k_0 \in \mathbb{N} \) such that \( k_0^{-1} < \varepsilon \). Again by the regularity of the metric system, say with a packing constant \( r_0 > 0 \), we need at most \( s(k) = U(r_0, \varepsilon_k) \) points, say \( a_1, a_2, \ldots, a_{s(k)} \), to cover the ball \( B_{d_x}(x, r_0) \) with balls of radius \( \varepsilon_k \). Let \( \alpha_i := \mu_x/\mu_{a_i} \). Again by continuity (see Lemma 4.7) there exists \( \beta > 0 \) such that \( \alpha_i \leq \beta \) for all \( i \). Thus

\[
\mu_x(B_{d_x}(x, r_0)) \leq \sum_{i=1}^{s(k)} \mu_x(B(a_i, \varepsilon_k))
\]

\[
= \sum_{i=1}^{s(k)} \alpha_i \cdot \mu_{a_i}(B(a_i, \varepsilon_k))
\]

\[
\leq \beta \sum_{i=1}^{s(k)} \mu_{a_i}(B(a_i, \varepsilon_k))
\]

\[
\leq \beta \sum_{i=1}^{s(k)} \frac{\varepsilon_k^m}{k} + \Delta \varepsilon_k^m
\]

\[
= \beta \cdot s(k) \frac{\varepsilon_k^m}{k} + \Delta \cdot s(k) \varepsilon_k^m \cdot \Delta.
\]

Since \( \lim \inf s(k) \varepsilon_k^m \) is bounded from above we have that \( \beta \cdot s(k) \frac{\varepsilon_k^m}{k} \) goes to zero as \( k \to \infty \) and \( \beta \cdot s(k) \varepsilon_k^m \cdot \Delta \leq D_1 \cdot \Delta \) for a certain constant \( D_1 \). Therefore

\[
\Delta \geq (D_1)^{-1} \mu_x(B_{d_x}(x, r_0)).
\]

To finish the proof it is enough to take any \( c > D_1/\mu_x(B_{d_x}(x, r_0)) \).

\[
\square
\]

Next we are able to conclude that \( \mu_x \) is equivalent to the measure induced by distance \( d_x \).

**Lemma 5.8.** For \( \mu \) almost every \( x \in X \)

\[
\mu_x = \lambda_x.
\]

**Proof.** Take any typical plaque \( F(x) \). Let \( r_0 \) be a packing constant associated to \( d_x \). For \( 0 < r < r_0 \) and \( k \) large enough, to guarantee that \( \varepsilon_k < r \), we can take \( L(r, \varepsilon_k) := L_k \) points as in condition (1) of Definition 3.2 with respect to \( B_{d_x}(x, r) \), and write the disjoint union:

\[
J_k \sqcup \bigcup_{i=1}^{L_k} B_{a_i}(a_i, \varepsilon_k) = B_{d_x}(x, r)
\]

with

\[
\lambda_x(J_k) \leq p \cdot \lambda_x(B_{d_x}(x, r)),
\]

where \( p < 1 \) is the constant that comes from the packing regularity of \( d_x \) (see equation (2) in Definition 3.2). Then

\[
\lambda_x(J_k) + \sum_{i=1}^{L_k} \lambda_x(B_{d_x}(a_i, \varepsilon_k)) = \lambda_x(B_{d_x}(x, r)).
\]
As \( \lambda_x(B_{d_x}(x, r)) = r^m \) we have

\[
\sum_{i=1}^{L_k} \varepsilon_k^m \lambda_x(B_{d_x}(x, r)) \geq (1 - p) \cdot \lambda_x(B_{d_x}(x, r)) \Rightarrow
\]

\[
\left( \Delta - \frac{1}{k} \right)^{-1} \sum_{i=1}^{L_k} \mu_x(B_{d_x}(a_i, \varepsilon_k)) \geq (1 - p) \cdot \lambda_x(B_{d_x}(x, r)) \Rightarrow
\]

\[
\alpha \cdot \left( \Delta - \frac{1}{k} \right)^{-1} \mu_x(B(x, r)) \geq (1 - p) \cdot \lambda_x(B_{d_x}(x, r)).
\]

Taking \( k \to \infty \) and using \( c^{-1} \leq \Delta \) (see Lemma 5.7) we obtain

\[
(1 - p)^{-1} \cdot \alpha \cdot c \cdot \mu_x(B(x, r)) \geq \lambda_x(B_{d_x}(x, r)).
\]

As this is true for every \( r < r_0 \) and the \( \sigma \)-algebra can be generated by open balls of radius smaller than \( r_0 \) we conclude that \( \lambda_x < < \mu_x \).

Let us prove the other side. For \( 0 < r < r_0 \) and each \( k \) large enough, consider \( b_1, b_2, \ldots, b_{U(r, \varepsilon_k)} \) points satisfying condition (1) of Definition 3.2 with respect to \( B_{d_x}(x, r) \), i.e.

\[
B_{d_x}(x, r) \subset \bigcup_{i=1}^{U(r, \varepsilon_k)} B(b_i, \varepsilon_k) \quad \text{with} \quad \liminf_{k \to \infty} \frac{U(r, \varepsilon_k)}{\varepsilon_k^{-m} \cdot r^m} < C_2,
\]

for a certain constant \( C_2 \) that does not depend on \( r \). Then we have

\[
\mu_x(B_{d_x}(x, r)) \leq \sum_{i=1}^{U(r, \varepsilon_k)} \mu_x(B_{d_x}(b_i, \varepsilon_k)) \leq \sum_{i=1}^{U(r, \varepsilon_k)} \varepsilon_k^m \cdot \left( \frac{1}{k + \Delta} \right) = r^m \cdot \frac{U(r, \varepsilon_k)}{\varepsilon_k^{-m} \cdot r^m} \cdot \left( \frac{1}{k + \Delta} \right) = \lambda_x(B_{d_x}(x, r)).
\]

Taking the inferior limit over \( k \) on both sides we obtain

\[
\mu_x(B_{d_x}(x, r)) \leq C_2 \cdot \Delta \cdot \lambda_x(B_{d_x}(x, r)), \quad \forall r < r_0,
\]

in particular \( \mu_x << \lambda_x \).

Consequently both measures are equivalent and the Radon-Nikodym derivative \( d\mu_x/d\lambda_x \) exists and is given by

\[
\frac{d\mu_x}{d\lambda_x}(y) = \lim_{r \to 0} \frac{\mu_x(B_{d_x}(x, r))}{\lambda_x(B_{d_x}(x, r))}.
\]
In particular, by taking the limit along the subsequence \( \varepsilon_k, k \to \infty \), we conclude that

\[
\frac{d\mu_x}{d\lambda_x}(y) = \lim_{k \to \infty} \frac{\mu_x(B_{d_x}(x, \varepsilon_k))}{\varepsilon_k^n}, \quad y \in \mathcal{F}(x)
\]

which implies

\[
\frac{d\mu_x}{d\lambda_x}(y) = \Omega, \quad \forall y \in \mathcal{F}(x).
\]

That is,

\[
\mu_x = \Omega \cdot \lambda_x, \quad \mu - a.e. \quad x \in X.
\]

Finally, evaluating both sides on \( B_{d_x}(x, 1) \) we conclude that

\[
\Omega = 1 \quad \text{and} \quad \mu_x = \lambda_x, \quad \mu - a.e. \quad x \in X
\]

as we wanted to show.

\[\Box\]

5.7. **Case 3:** Either \( \Omega = \infty \) and \( \mu(D^\infty) = 0 \), or \( \Omega < \infty \) and \( \mu(D) = 0 \).

For the sake of simplicity we will make an uniform notation to deal with both situations at once. Define the sets \( \Theta, \Theta_x \) and \( D \) by:

\[
\Theta := \begin{cases} 
\Pi^\infty, & \text{if } \Omega = \infty \quad \text{and} \quad \mu(D^\infty) = 0 \\
\Pi, & \text{if } \Omega < \infty \quad \text{and} \quad \mu(D) = 0,
\end{cases}
\]

\[
\Theta_x := \begin{cases} 
\Pi^\infty_x, & \text{if } \Theta = \Pi^\infty \\
\Pi_x, & \text{if } \Theta = \Pi,
\end{cases}
\]

and

\[
D := \begin{cases} 
D^\infty, & \text{if } \Theta = \Pi^\infty \\
D, & \text{if } \Theta = \Pi.
\end{cases}
\]

Since \( \Theta_x \) is closed in the plaque \( \mathcal{F}(x) \) for every \( x \in D \) and \( \mu(\Theta) = 1 \), it is true that for a full measurable set \( \mathcal{D} \), if \( x \in \mathcal{D} \) then \( x \notin \Theta \) if, and only if, there is \( r > 0 \) with \( \mu_x(B_{d_x}(x, r)) = 0 \). Now consider \( \{q_1, q_2, \ldots\} \) to be an enumeration of the rationals.

For each \( i \geq 1 \) let us define the function \( S_i \) as

\[
S_i(x) = \max\{q_j : 1 \leq j \leq i \text{ and } \mu_y(B_{d_y}(y, q_j)) = 0 \text{ for some } y \in \mathcal{F}(x)\}.
\]

**Lemma 5.9.** \( S_i \) is an invariant measurable function for all \( i \in \mathbb{N} \).

**Proof.** For each \( i \in \mathbb{N} \) define the function \( Q_i : \mathcal{D} \to [0, \infty) \) by

\[
Q_i(x) = \mu_x(B_{d_x}(x, q_i)).
\]

By Proposition 4.5 the function \( Q_i(x) \) is measurable for every \( i \) and, by Lusin’s theorem we may take a compact set \( K \subset \mathcal{D} \) of positive measure such that \( Q_i | K \) is continuous for every \( i \). Now, given \( j \in \mathbb{N} \), let \( \sigma \) be a permutation of \( \{1, \ldots, j\} \) such that \( q_{\sigma(1)} < q_{\sigma(2)} < \ldots < q_{\sigma(j)} \). Let \( \{B_\rho\}_{\rho \in \mathbb{N}} \) be a countable basis for the topology in \( G \). Observe that for

\[
\mathcal{R}_\rho = \bigcup_{g \in B_\rho} g \cdot K = B_\rho \cdot K
\]

we have

\[
S_j^{-1}({\{q_{\sigma(j)}\}}) \cap \mathcal{R}_\rho = G(B_\rho \times \mathcal{F}(Q_j^{-1}({\{0\}}) \cap K)),
\]
which is a measurable set since $Q_{\sigma(n)}^{-1}(\{0\}) \cap K$ is a Borel set for every $\rho \in \mathbb{N}$. Now, 

$$S_j^{-1}(\{q_{\sigma(j)}\}) \cap \mathcal{K}_\rho = \bigcup_{g \in \mathcal{B}_\rho} g(\mathcal{F}(Q_{\sigma(j-1)}^{-1}(\{0\}) \cap K) \setminus (S_j^{-1}(\{q_{\sigma(j)}\}) \cap \mathcal{K}_\rho),$$

which is also a measurable set. Inductively we prove that $S_j^{-1}(\{q_{\sigma(i)}\}) \cap \mathcal{K}_\rho$ is measurable for all $1 \leq i \leq j$. Since, by ergodicity, the set $\mathcal{K} = \bigcup_\rho \mathcal{K}_\rho \subset \mathfrak{D}$ has full measure we conclude that $S_j(x)$ is measurable for every $j \geq 1$. □

Define

$$S(x) := \lim_{i \to \infty} S_i(x).$$

The function $S$ is measurable and $G$-invariant, thus it is almost everywhere equal to a constant $r_0$. This means that for a full measure set $Y \subset \mathfrak{D}$, for every $x \in Y$ the plaque $\mathcal{F}(x)$ has a finite number of balls of radius $r_0$ outside $\Theta_x$. Let us call these open balls as “bad” balls.

Now consider the set $\mathfrak{M}$ formed by the center of these “bad” balls of radius $r_0$. Notice that $\mathfrak{M}$ is a measurable set, since it is inside a set of zero measure and also that $f(\mathfrak{M}) = \mathfrak{M}$ by the invariance of the Borel metric system.

Let $\phi : (0,1) \times (0,1) \to U$ be a chart as in Definition 3.1 such that the set $\mathcal{F}(\mathfrak{M} \cap U)$, which is the $\mathcal{F}$ saturation of these “bad” balls inside $U$, has positive measure. Set $\Sigma := \pi_1(\phi^{-1}(\mathfrak{M} \cap U))$, where $\pi_1 : (0,1) \times (0,1) \to (0,1)$ is the projection onto the first coordinate. Now we may apply The Measurable Choice Theorem 2.10 to obtain a measurable function $\mathfrak{F} : \Sigma \to (0,1)$ such that $(x, \mathfrak{F}(x)) \in \phi^{-1}(\mathfrak{M} \cap U)$ for all $x \in \Sigma$. Again, using Lusin’s theorem, we may assume $\Sigma$ to be compact and such that $\mathfrak{F}$ is a continuous function.

Now consider the set $\mathfrak{M}_0 := \phi(\text{graph } \mathfrak{F})$, which is a Borel set since the graph of $\mathfrak{F}$ is a compact set. Notice that our construction implies that $\mathcal{F}(\mathfrak{M}_0)$ has positive measure. Now we take the $G$-invariant measurable set

$$\mathfrak{M}_1 := \bigcup_{g \in G} g \cdot \mathfrak{M}_0 = \bigcup_{\rho \in \mathbb{N}} \mathcal{B}_\rho \cdot \mathfrak{M}_0,$$

where $\{\mathcal{B}_\rho\}_{\rho \in \mathbb{N}}$ is a countable basis for the topology in $G$. By the $G$-invariance of $\mathfrak{M}_1$ we know that $\mu(\mathfrak{M}_1) = 1$. We can assume $\mathfrak{M}_1$ is Borel since we can take arbitrarily large compact sets $C_n \subset \mathfrak{M}_1$ whose measure differs from that of $\mathfrak{M}_1$ by at most $1/n$ and then take $C = \bigcup G \cdot C_n$ which is a Borel set with full measure.

Clearly the set $\mathcal{F}(\mathfrak{M}_1)$ has full measure as it contains $\mathfrak{M}_1$ and the set $\mathfrak{M}_1$ intersects every plaque in $\mathcal{F}(\mathfrak{M}_1)$ in a finite (constant) number of points. Notice that, for each $r \in \mathbb{R}_+$, the invariant set

$$\mathfrak{M}_1^r := \bigcup_{x \in \mathfrak{M}_1} B_{d_x}(x, r)$$

has zero or full measure. Let $\alpha_0$ such that $\mu(\mathfrak{M}_1^r) = 0$ if $r < \alpha_0$ and $\mu(\mathfrak{M}_1^r) = 1$ if $r \geq \alpha_0$. This implies that the boundaries of $B_{d_x}(x, \alpha_0)$ for $x \in \mathfrak{M}_1$ forms a set of weak-atoms, contradicting the hypothesis that we are in the non weakly-atomic case. □
6. Some applications in smooth dynamics

In this section we show some situations in smooth dynamics where the main theorem of this paper can be applied to obtain measure rigidity of the system along an specific invariant foliation.

In what follows we first prove that Borel metric systems which are bi-Lipschitz, in the sense that the local charts of the Borel lamination are bi-Lipschitz maps with Lipschitz constants depending only on the leaves, are also regular. Then we use this fact to obtain measure rigidity results in the context of cocycles valued in the set of diffeomorphisms of a certain manifold and in the context of partially hyperbolic maps with neutral center.

6.1. bi-Lipschitz systems and regularity.

Definition 6.1. Let $G \acts X$ be a leafwise weakly rigid action along a $G$-invariant $m$-dimensional Borel lamination $\mathcal{F}$, and let $\{d_x\}_{x \in X}$ be the respective Borel metric system associated to it. We say that the action is bi-Lipschitz leafwise weakly rigid (along $\mathcal{F}$) if the local charts $\varphi$ of $\mathcal{F}$ are bi-Lipschitz when restricted to plaques of $\mathcal{F}$ endowed with the metrics given by the Borel metric system. More precisely, the action is bi-Lipschitz leafwise weakly rigid if given a local chart, $\varphi : U \to \mathbb{R}^m \times \mathbb{R}^{n-m}$, and a plaque $L$ of $\mathcal{F}$ inside $U$, the map

$$\varphi|_L : L \to \mathbb{R}^m \times \{c\},$$

is bi-Lipschitz when we consider $L$ endowed with $d_x$, $x \in L$, and $\mathbb{R}^m$ endowed with the standard euclidian distance.

The first Lemma of this section shows that in $\mathbb{R}^n$ the standard euclidian metric is packing regular with any packing constant $r > 0$ and that the lower and upper bounds involved in the definition depend only on $n$. The estimative we provide is very simple and is clearly not sharp, however it is enough for our purposes.

Lemma 6.2. Consider $\mathbb{R}^n$ endowed with the standard euclidian metric and, for any $x \in \mathbb{R}^n$, $r, s > 0$, denote

- $L(r, s)$ the maximum number of disjoint balls of radius $s$ contained in the ball $B(x, r)$;
- $U(r, s)$ the minimum number of balls of radius $s$ necessary to cover $B(x, r)$.

Then, for any $r > 0$

$$n^{-n/2} \leq \limsup_{s \to 0} \frac{L(r, s)}{s^{-n} \cdot r^n}, \quad \liminf_{s \to 0} \frac{U(r, s)}{s^{-n} \cdot r^n} \leq 2^{3n/2}.$$ 

Moreover, $\mathbb{R}^n$ endowed with the standard euclidian metric is packing regular with any packing constant associated to the euclidian metric.

Proof. As we are in $\mathbb{R}^n$ with the euclidian metric any two balls of same radius are equivalent by translation, thus we can assume without loss of generality that $x = 0$. The ball $B(0, r)$ circumscribes a $n$-dimensional box whose sides have length $2r/\sqrt{n}$, i.e,

$$\prod_{i=1}^{n}(-r/\sqrt{n}, r/\sqrt{n}) \subset B(0, r).$$
The interval \((-r/\sqrt{n}, r/\sqrt{n})\) can be divided in \(2k\) smaller intervals of diameter \(r/k\sqrt{n}\), so that the \(n\)-dimensional box above can be written as a disjoint union of \(2^nk^n\) disjoint boxes \(I_j\) which are products of open intervals of diameter \(r/k\sqrt{n}\), that is,

\[
\prod_{j=1}^{2^nk^n} I_j \subset B(0, r), \quad I_j \text{ is a box whose sizes have length } r/k\sqrt{n}.
\]

Each \(I_j\) contains an open ball centered in a certain \(x_j \in I_j\) and radius \(r/2k\sqrt{n}\). Thus,

\[
L\left(r, \frac{r}{2k\sqrt{n}}\right) \geq 2^nk^n,
\]

which implies

\[
\limsup_{s \to 0} \frac{L(r, s)}{s^{-n} \cdot r^n} \geq \limsup_{k \to \infty} \frac{L\left(r, \frac{r}{2k\sqrt{n}}\right)}{(\frac{r}{2k\sqrt{n}})^{-n} \cdot r^n} = n^{-n/2}.
\]

Furthermore, let \(q > 0\) be the constant satisfying \(\lambda(B(0, r)) = q \cdot r^n\) for every \(r > 0\) and \(\lambda\) the Lebesgue measure on \(\mathbb{R}^n\). Then

\[
\lambda\left(\bigcup_{j=1}^{L\left(r, \frac{r}{2k\sqrt{n}}\right)} B(x_j, r/2k\sqrt{n})\right) \geq 2^nk^n \cdot \lambda(B(0, r/2k\sqrt{n})) = 2^nk^n \cdot q \cdot (r/2k\sqrt{n})^n = \frac{1}{n^{n/2}} \cdot \lambda(B(0, r)).
\]

In particular,

\[
(7) \quad \lambda\left(B(0, r) \setminus \bigcup_{j=1}^{L\left(r, \frac{r}{2k\sqrt{n}}\right)} B(x_j, r/2k\sqrt{n})\right) \leq \left(1 - \frac{1}{n^{n/2}}\right) \cdot \lambda(B(0, r)).
\]

Now let us work with \(U(r, s)\). Since

\[
B(0, r) \subset \bigcup_{i=1}^{n} (-r, r),
\]

by dividing the interval \((-r, r)\) in \(2k\) subintervals of length \(r/k\) we can write the product on the right side as a disjoint union of \((2k)^n\) smaller boxes \(W_j\) whose sizes have length \(r/k\). Each of this boxes are contained in a ball of radius \(r\sqrt{2}/k\). Thus

\[
U\left(r, r\sqrt{2}/k\right) \leq 2^nk^n,
\]

and, analogously to what we have done before, we have

\[
\liminf_{s \to 0} \frac{U(r, s)}{s^{-n} \cdot r^n} \leq \liminf_{k \to \infty} \frac{2^nk^n}{(r\sqrt{2}/k)^{-n} \cdot r^n} = 2^{3n/2}.
\]

Thus the estimative stated in the proposition is proved and together with (7) it implies that the euclidian metric in \(\mathbb{R}^n\) is packing regular with any packing constant associated to it.
Lemma 6.3. If \( d \) is an \( m \)-packing regular metric in a metric space \( X \) then any other metric in \( X \) strongly equivalent to \( d \) is also \( m \)-packing regular.

Proof. Assume that \( \rho \) is another metric in \( X \) which is strongly equivalent to \( d \), say
\[
A \cdot d(x, y) \leq \rho(x, y) \leq B \cdot d(x, y), \quad x, y \in X.
\]
Let \( r \) be a packing constant associated to \( d \) and let \( U_\rho(r, s) \) and \( L_\rho(r, s) \) to be the analogous of \( U(r, s) \) and \( L(r, s) \) for the metric \( \rho \). We know that
\[
B_\rho(x, r) \subset B(x, A^{-1} \cdot r) \quad \text{and} \quad B(x, B^{-1} \cdot s) \subset B_\rho(x, s).
\]
Thus,
\[
U_\rho(r, s) \leq U(A^{-1} \cdot r, B^{-1} \cdot s) \Rightarrow \\
\liminf_{s \to 0} \frac{U_\rho(r, s)}{s^{-n} \cdot r^n} \leq \liminf_{s \to 0} \frac{U(A^{-1} \cdot r, B^{-1} \cdot s)}{s^{-n} \cdot r^n} \\
= A^{-n} B^n \cdot \liminf_{s \to 0} \frac{U(A^{-1} \cdot r, B^{-1} \cdot s)}{(B^{-1}s)^{-n} \cdot (A^{-1})^n} \\
\leq A^{-n} B^n \cdot 2^{3n/2}.
\]
Analogously we have
\[
L_\rho(r, s) \geq L(B^{-1}r, A^{-1}s),
\]
which implies
\[
\limsup_{s \to 0} \frac{L_\rho(r, s)}{s^{-n} \cdot r^n} \geq \limsup_{s \to 0} \frac{L(B^{-1}r, A^{-1}s)}{s^{-n} \cdot r^n} \\
= A^n B^{-n} \cdot \limsup_{s \to 0} \frac{L(B^{-1}r, A^{-1}s)}{(A^{-1}s)^{-n} \cdot (B^{-1}r)^n} \\
\geq A^n B^{-n} \cdot n^{-n/2}.
\]
Also, denoting by \( \lambda \) and \( \lambda_\rho \) the \( m \)-Hausdorff measures induced respectively by \( d \) and \( \rho \), \( \Box \) implies
\[
\lambda_\rho(B(x, r)) \leq \lambda_\rho(B_\rho(x, B \cdot r)) = B^m r^m = B^m \cdot \lambda(B(x, r))
\]
and
\[
\lambda_\rho(B(x, r)) \geq \lambda_\rho(B_\rho(x, A \cdot r)) = A^m \cdot \lambda(B(x, r)), \quad \forall r > 0.
\]
In particular \( \lambda_\rho \) and \( \lambda \) are equivalent measures and the Radon-Nikodym derivative satisfies
\[
A^m \leq \lim_{r \to 0} \frac{\lambda_\rho(B(x, r))}{\lambda(B(x, r))} \leq B^m
\]
\[
\Rightarrow A^m \leq \frac{d\lambda_\rho}{d\lambda}(y) \leq B^m.
\]
Consequently, for a certain \( r_0 \) and for \( 0 < s < B^{-1}r < B^{-1}r_0 \) there exists points \( a_1, a_2, \ldots \) for which
\[
B_\rho(x, r) \supset B(x, B^{-1}r) = \bigcup_{i=1}^{L(B^{-1}r, s)} B(a_i, s) \sqcup J \supset \bigcup_{i=1}^{L(B^{-1}r, s)} B_\rho(a_i, A \cdot s) \sqcup J.
\]
where $J := B(x, B^{-1}r) \cup \bigcup_{i=1}^{L(B^{-1}r, s)} B(a_i, s)$, and then $\lambda(J) \leq p \cdot \lambda(B(x, B^{-1}r))$ for a certain constant $p < 1$. Therefore

$$\lambda_p(J) \leq B^m \cdot \lambda(J) \leq p \cdot B^m \cdot \lambda(B(x, B^{-1}r))$$

$$= p \cdot B^m \cdot B^{-m} \cdot r^m = p \cdot \lambda_p(B_p(x, r)).$$

Thus $\rho$ is regular as we wanted to show.

\[\square\]

**Corollary 6.4.** If $(M, g)$ is a compact Riemannian manifold and $d$ is the metric in $M$ induced by $g$ then $d$ is regular.

**Proof.** For each $x$ the norm induced by the inner product $g_x$ on $T_xM \sim \mathbb{R}^n$ is equivalent to the euclidian norm, say by constants $a(x), b(x)$. As $M$ is compact, $x \mapsto a(x)$ and $x \mapsto b(x)$ are bounded away from zero and bounded from above, which implies that the induced metric $d$ is regular since it is strongly equivalent to the metric induced by the family of euclidian inner products on each $T_xM$.

\[\square\]

Now we can prove Theorem A stated in the introduction. For the readers convenience we restate it below.

**Theorem A.** Let $G$ be a second countable group acting on a smooth manifold $M$ by continuous maps and assume that the action is bi-Lipschitz leaf-wise weakly rigid along a $G$-invariant foliation of $\mathcal{F}$ of dimension $m$ by $C^r$-submanifolds, $r \geq 1$. If $G \curvearrowright M$ is ergodic with respect to a $G$-invariant measure $\mu$ then either:

a) $\mu$ is weak-atomic along $\mathcal{F}$ or;

b) the normalized conditional measures $\mu_x$ are just the $m$-dimensional Hausdorff measures on the leaves of $\mathcal{F}$.

In particular if the foliation is one-dimensional case a) means atomic disintegration.

**Proof.** Let $\{d_x\}$ be the bi-Lipschitz metric system over $\mathcal{F}$ say

$$K_1 \cdot d(\varphi(a), \varphi(b)) \leq d_x(a, b) \leq K_2 \cdot d(\varphi(a), \varphi(b))$$

where $\varphi : U \to \mathbb{R}^n$ is a local chart, $x \in U$, $a, b \in \mathcal{F}(x) \cap U$ and $K_1$ and $K_2$ are constants depending only on the plaque of $\mathcal{F}(x)$ inside $U$ containing $x$. As $d$ is the euclidian metric in $\mathbb{R}^n$ then by Lemma 6.2 it is regular with any packing constant $r > 0$. By Lemma 6.3 it follows that $d_x$ is regular inside each $U$. In particular, by taking $3 \cdot r_0$ to be smaller then the Lebesgue number of the covering $\{U_i\}$, we can guarantee that $d_x$ is regular in any open ball $B_d(x, r)$ with $r \leq r_0$. Thus $d_x$ is indeed regular as we wanted. The result is now a direct consequence of the main theorem.

\[\square\]

6.2. Measure rigidity for diffeomorphism cocycles over hyperbolic maps.

The theory of cocycles over hyperbolic maps plays a huge role in dynamical systems theory. The most classical examples of cocycles which have been extensively studied are the group-valued cocycles over hyperbolic systems. More recently, a much broader class of cocycles have been studied, the class of group-valued cocycles. Two very recent results for this class of cocycles
are [5], where the authors obtain a Livsic type theorem, and [30] where the author obtains a rigidity result based on the boundedness of the periodic data of the cocycle.

**Definition 6.5.** Given a compact metric space $X$, a homeomorphism $f : X \to X$ and a function $A : X \to G$ where $G$ is a group, the $G$-valued cocycle over $f$, generated by $A$, is the map $\mathcal{A} : X \times \mathbb{Z} \to G$ defined by taking

1. $\mathcal{A}(x, 0) = \text{Id}$, for any $x \in X$;
2. $\mathcal{A}(x, n) := A(f^{n-1}(x)) \circ \ldots \circ A(x)$ for $x \in X$, $n \in \mathbb{N}$;
3. $\mathcal{A}(x, -n) := (\mathcal{A}(x, n))^{-1}$, for $x \in X$, $n \in \mathbb{N}$.

The set $\mathcal{A}_P = \{\mathcal{A}_P^k : f^k(p) = p, p \in X, k \in \mathbb{N}\}$ is called the periodic data set of the cocycle $\mathcal{A}$.

Associated to a cocycle $\mathcal{A} : X \times \mathbb{Z} \to G$ over a homeomorphism $f : X \to X$, there is a natural associated $\mathfrak{g} : X \times G \to X \times G$ given by

$$\mathfrak{g}(x, g) = (f(x), A(x) \cdot g).$$

If $G$ is a group acting on a manifold $M$ by an action $a : G \times M \to M$, then we can use the cocycle to define a dynamics on $X \times M$ by taking

$$F : X \times M \to X \times M, \quad F(x, v) := (f(x), a(A(x), v)).$$

A straightforward application of Theorem [3] and Corollary 6.4 gives the following result:

**Theorem C.** Let $\mathcal{A}$ be a $C^1$ cocycle, valued in $\text{Diff}^r(M)$, $r \geq 0$, where $M$ is a compact Riemannian manifold, over a continuous map $f : X \to X$ such that there exists a family $\{g_x\}_{x \in M}$ of continuous Riemannian metrics for which:

$$\mathcal{A}_x : (M, g_x) \to (M, g_{f(x)})$$

is an isometry for every $x \in M$. Let $F : X \times M \to X \times M$ be defined by

$$F(x, v) = (f(x), \mathcal{A}_x(v)).$$

Then, any $F$-invariant ergodic measure $\mu$ admits a disintegration $\{\mu_x\}$ over the fibers $\mathcal{F}(x) := \{x\} \times M$ where either:

a) $\mu_x$ is weak-atomic along $\mathcal{F}(x)$ or;

b) the normalized conditional measures $\mu_x$ are just the Hausdorff measures on the leaves of $\mathcal{F}(x)$ induced by the Riemannian metrics $g_x$.

In particular, if $\mu$ is a product measure $\lambda \times \nu$ then either $\nu$ is supported on a finite union of submanifolds of dimension at most $\dim(M) - 1$, or all the metric $g_x$ are the same (say $g$), the measure $\nu$ is the Hausdorff measure induced by $g$ and the cocycle is a cocycle of isometries.

Very recently V. Sadovskaya [30] proved that, for $\text{Diff}^r(M)$-valued cocycles over hyperbolic maps, boundedness of the periodic data provides the existence of a systems of Riemannian metrics $\tau_x$ on $M$ for which the cocycle acts as an isometry. More precisely she proved the following.

**Theorem 6.6 (Theorem 1.3 in [30]).** Let $(X, f)$ be a hyperbolic system, $M$ be a compact connected manifold and $\mathcal{A}$ a bounded $\text{Diff}^2(M)$-valued cocycle over $f$ that is $\beta$-Hölder continuous as a $\text{Diff}^q(M)$-valued cocycle with $q = 1 + \gamma$, 

where $\gamma > 0$.
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$0 < \gamma < 1$. If the periodic data set $\mathcal{A}_P$ is bounded in $\text{Diff}^q(M)$, then there exists a family of Riemannian metrics $\tau_x$ on $M$ such that

$$\mathcal{A}_x : (M, \tau_x) \to (M, \tau_{f(x)})$$

is an isometry for every $x \in X$. Moreover, for any $\alpha < \gamma$ each Riemannian metric $\tau_x$ is $\alpha$-Hölder continuous on $M$ and depends Hölder continuously on $x$ in $C^\alpha$ distance with exponent $\beta(\gamma - \alpha)$.

As a direct consequence of Theorems $C$ and $6.6$ we have:

**Theorem D.** Let $(X, f)$ be a hyperbolic system, $M$ be a compact connected manifold and $\mathcal{A}$ a bounded $\text{Diff}^2(M)$-valued cocycle over $f$ that is $\beta$-Hölder continuous as a $\text{Diff}^q(M)$-valued cocycle with $q = 1 + \gamma$, $0 < \gamma < 1$. Consider the dynamics $F : X \times M \to X \times M$ defined by

$$F(x, v) = (f(x), \mathcal{A}_x(v)).$$

If the periodic data set $\mathcal{A}_P$ is bounded in $\text{Diff}^q(M)$, then given any $F$-invariant ergodic measure $\mu$, the disintegration $\{\mu_x\}$ of $\mu$ over the fibers $\mathcal{F}(x) := \{x\} \times M$ is either:

a) weakly-atomic along $\mathcal{F}(x)$ or;

b) the Hausdorff measures on the leaves of $\mathcal{F}(x)$ induced by a Riemannian metrics $\tau_x$ for which $\mathcal{A}_x : (M, \tau_x) \to (M, \tau_f(x))$ is an isometry for every $x \in X$.

In particular, if $\mu$ is a product measure $\lambda \times \nu$ then either $\nu$ is supported on a finite union of submanifolds of dimension at most $\dim(M) - 1$, or all the metrics $\tau_x$ are the same (say $\tau$), the measure $\nu$ is the Hausdorff measure induced by $\tau$ and the cocycle is a cocycle of isometries.

### 6.3. Partially hyperbolic diffeomorphisms with neutral center.

A diffeomorphism $f : M \to M$ defined on a compact Riemannian manifold $M$ is said to be partially hyperbolic if there is a nontrivial splitting

$$TM = E^s \oplus E^c \oplus E^u$$

such that

$$Df(x)E^\tau(x) = E^\tau(f(x)), \quad \tau \in \{s, c, u\}$$

and a Riemannian metric for which there are continuous positive functions $\nu, \check{\nu}, \gamma, \check{\gamma}$ with

$$\nu, \check{\nu} < 1 \quad \text{and} \quad \nu < \gamma < \check{\gamma}^{-1} < \check{\nu}^{-1}$$

such that, for any unit vector $v \in T_pM$,

$$\|Df(p) \cdot v\| < \nu(p), \text{ if } v \in E^s(p)$$

$$\gamma(p) < \|Df(p) \cdot v\| < \check{\gamma}(p)^{-1}, \text{ if } v \in E^c(p)$$

$$\check{\nu}(p)^{-1} < \|Df(p) \cdot v\|, \quad \text{if } v \in E^u(p).$$

**Definition 6.7.** We say that $f$ has neutral center direction if there exists $K > 1$ such that

$$\frac{1}{K} \leq \|Df^n|E^c(x)|\| \leq K$$

for every $x \in M$ and any $n \in \mathbb{Z}$. 
By [16, Corollary 7.6] if $f$ has neutral center the bundle $E^c$ is tangent to a unique $f$-invariant foliation $\mathcal{F}^c$. Some important examples of partially hyperbolic diffeomorphisms with neutral center appeared recently in the construction of anomalous partially hyperbolic diffeomorphisms, contradicting Pujals’ conjecture, given in [8, 7] and [9]. The nomenclature of this property was introduced in [36] where the author study dynamical properties of such anomalous diffeomorphisms when the center dimension is one.

A direct application of the main theorem of this paper yields the following result.

**Theorem E.** Let $M$ be a compact Riemannian manifold and $f : M \to M$ be a $C^1$ partially hyperbolic diffeomorphism with neutral center. Then, given any $f$-invariant ergodic measure $\mu$, either $\mu$ has Lebesgue disintegration along the center foliation $\mathcal{F}^c$ or the conditional measures of $\mu$ along $\mathcal{F}^c$ are weakly-atomic.

**Proof.** Let $f : M \to M$ be a partially hyperbolic diffeomorphism which has neutral center direction. We can define a system of invariant metrics along the center direction in the following manner.

$\left(d_x\right)(a, b) := \sup\{d_c(f^n(a), f^n(b)) : n \in \mathbb{Z}, \ a, b, \in F^c(x)\}$. 

This system of metrics is easily seen to be invariant and, furthermore, it is a Borel metric system which is bi-Lipschitz as the next propositions show.

**Proposition 6.8.** $\{d_x\}$ is an $f$-invariant Borel metric system.

**Proof.** Let $S \subset M$ a Borel set, we need to prove that $\bigcup_{x \in S} B_{d_x}(x, r)$ is a measurable set. Let $S$ be inside a coordinate chart and $r$ be small enough so that $\bigcup_{x \in S} B_{d_x}(x, r)$ is still inside the same coordinate chart. We can map the picture to $\mathbb{R}^n$ by a homeomorphism, i.e., by the respective local chart. For each $x \in X$ consider the local chart 

$\varphi_x : U_x \to V_x \subset \mathbb{R}^n$

and consider $\rho_x$ the metric in $V_x$ given by $d_x \circ \varphi_x^{-1}$. Denote $\varphi_{ej}$ the function given by:

- if $t \geq 0$ then
  \[ \varphi_{ej}(t, y) = y + s \cdot e_j, \quad s > 0 \]

  such that $\rho_x(y, y + s \cdot e_j) = t$;

- if $t < 0$ then
  \[ \varphi_{ej}(t, y) = y + s \cdot e_j, \quad s < 0 \]

  such that $\rho_x(y, y + s \cdot e_j) = -t$.

Let $\varphi_j(r, x) := \varphi_{ej}^{-1} \circ \phi_{ej}(r, \varphi_x(x))$. This function is continuous inside plaques. Also it is measurable in the whole set, thus we can take a sequence of nested compact sets $K_i$ inside $S$ such that $\bigcup_i K_i = S \mod 0$ and such that $\varphi_j|K_i$ is continuous for every $i$. Since

$$\bigcup_{x \in K_i} B_{d_x}(x, r) = \bigcup_{j=1}^n \varphi_j((-r, 0) \times K_i) \cup \varphi_j((0, r) \times K_i),$$
we have that $\bigcup_{x \in K_i} B_d(x, r)$ is measurable for all $i \geq 1$, thus $\bigcup_{x \in S} B_d(x, r)$ is measurable as we wanted to show.

**Proposition 6.9.** The metric system $\{d_x\}$ is bi-Lipschitz.

**Proof.** Let $K > 1$ be such that

$$\frac{1}{K} \leq ||Df^n|E^c(x)|| \leq K,$$

for every $x \in M$ and every $n \in \mathbb{Z}$. In this case it is easy to see that

$$\frac{1}{K} d_c(a, b) \leq d_c(f^n(a), f^n(b)) \leq K \cdot d_c(a, b), \quad \forall n \in \mathbb{Z}.$$

By taking the supremum over $n \in \mathbb{Z}$ we obtain

$$\frac{1}{K} d_c(a, b) \leq d_x(f^n(a), f^n(b)) \leq K \cdot d_c(a, b),$$

as we wanted. \qed

By Lemma 6.3 it follows that $\{d_x\}$ is indeed a regular metric system. By Theorem B the results follows. \qed

An application of Theorem E is done by the first author in [23] to study the occurrence of some strong ergodic properties, such as the Bernoulli property, for certain partially hyperbolic diffeomorphisms with neutral center and, in particular, to jointly integrable perturbations of ergodic linear automorphisms of $\mathbb{T}^n$, $n \geq 4$.

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**Departamento de Matemática, Estatística e Computação Científica, IMECC-UNICAMP, Campinas-SP, Brazil.**  
E-mail address: gaponce@ime.unicamp.br

**Departamento de Matemática, Estatística e Computação Científica, IMECC-UNICAMP, Campinas-SP, Brazil.**  
E-mail address: regisvarao@ime.unicamp.br