Black Saturn with dipole ring

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Abstract

We present a new stationary, asymptotically flat solution of 5D Einstein-Maxwell gravity describing a Saturn-like black object: a rotating black hole surrounded by a rotating dipole black ring. The solution is generated by combining the vacuum black Saturn solution and the vacuum black ring solution with appropriately chosen parameters. Some basic properties of the solution are analyzed and the basic quantities are calculated.

1 Introduction

In recent years the higher dimensional gravity is attracting much interest. Apart from the fact that the higher dimensional gravity is interesting in its own right, the increasing amount of works devoted to the study of the higher dimensional spacetimes is inspired from the string theory and the brane-world scenario with large extra dimensions. The gravity in higher dimensions exhibits much richer dynamics and spectrum of solutions than in four dimensions. One of the most reliable routes for better understanding of higher dimensional gravity and the related topics are the exact solutions. For example, recently discovered exact black rings solutions \footnote{The black ring solution of \cite{1} is with one rotational parameter. Black rings with two independent rotational parameters have been constructed recently in \cite{2}.} with unusual horizon topology \cite{1}, demonstrated explicitly that the 5D Einstein gravity exhibits unexpected features completely absent in four dimensions. It was shown in \cite{1} that both the black hole and the black ring can carry the same conserved charges, the mass and a single angular momentum, and therefore there is no uniqueness theorem in five dimensions. Moreover, the black rings can also carry nonconserved charges which can be varied continuously without altering the conserved charges. This fact leads to continuous (classical) non-uniqueness \cite{3}.

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Among of the most fascinating solutions of the 5D vacuum Einstein gravity discovered so far are those describing many black objects in equilibrium [4], [5] (see also [6] and [7] in the case of 5D supergravity). In particular we will be interested in the so-called black Saturn solution describing a rotating black hole surrounded by a rotating black ring. The natural question is whether a similar solution exists in the more general case of non-vacuum 5D gravity, especially in 5D Einstein-Maxwell gravity. The aim of this paper is to present a new stationary, asymptotically flat solution of 5D Einstein-Maxwell gravity describing a Saturn-like black object: a rotating black hole surrounded by a rotating dipole black ring. The solution is generated by combining the vacuum black Saturn solution and the vacuum black ring solution with appropriately chosen parameters.

2 Black Saturn with dipole ring

A method for generating exact stationary solutions of 5D Einstein-Maxwell gravity was developed in [8]. This method allows us to construct 5D Einstein-Maxwell solutions by combining two stationary solutions of the 5D vacuum Einstein equations. Through this method, it was shown that the dipole black ring can be constructed via the scheme

\[
\text{Rotating Black Ring} \oplus \text{Rotating Black Ring} \rightarrow \text{Rotating Dipole Black Ring}
\]

(1)

where the black ring solutions are with appropriately chosen parameters. It is natural to expect that the generating scheme

\[
\text{Rotating Black Ring} \oplus \text{Black Saturn} \rightarrow \text{Black Saturn with Dipole Black Ring}
\]

(2)

will produce a black Saturn solution with a dipole black ring when the parameters of the solutions are appropriately chosen.

We present below the solution generated via the scheme [2]. The spacetime geometry and the electromagnetic field are given by

\[
ds^2 = -\frac{H_2}{H_1 W} \left[ dt + \left( \frac{\omega \psi}{H_2} + q \right) d\psi \right]^2 + k^2 \frac{H_1 P}{W} Y^3 \left( d\rho^2 + dz^2 \right) + \frac{H_1 G_\psi d\psi^2 + W^2 G_\phi d\phi^2}{H_2 W},
\]

\[
F_{x\phi} = \sqrt{3} W^2 \frac{G_\phi}{\rho} \partial_\rho \mathcal{V},
\]

\[
F_{\rho\phi} = -\sqrt{3} W^2 \frac{G_\phi}{\rho} \partial_\phi \mathcal{V}.
\]

Here \( k \) and \( q \) are constants and the functions \( W, Y, \mathcal{V} \) are given by

\[
W = \frac{\bar{\mu}_1 \left[ \mu_5 (\rho^2 + \bar{\mu}_1 \mu_3)^2 (\rho^2 + \bar{\mu}_1 \mu_4)^2 - B_1^2 \mu_3 \mu_4 (\bar{\mu}_1 - \mu_5)^2 \rho^4 \right]}{\mu_4 \left[ \mu_5 (\rho^2 + \bar{\mu}_1 \mu_3)^2 (\rho^2 + \bar{\mu}_1 \mu_4)^2 + B_1^2 \bar{\mu}_1 \mu_3 \mu_4 (\bar{\mu}_1 - \mu_5)^2 \rho^2 \right]},
\]

\[
Y = \frac{(\rho^2 + \bar{\mu}_1 \mu_5) \left[ \mu_5 (\rho^2 + \bar{\mu}_1 \mu_3)^2 (\rho^2 + \bar{\mu}_1 \mu_4)^2 - B_1^2 \mu_3 \mu_4 (\bar{\mu}_1 - \mu_5)^2 \rho^4 \right]}{\mu_5 (\rho^2 + \bar{\mu}_1 \mu_3) (\rho^2 + \bar{\mu}_1 \mu_4) (\rho^2 + \bar{\mu}_1^2) (\rho^2 + \mu_4 \mu_5) (\rho^2 + \mu_3 \mu_4)},
\]
\[ V = B_1 \frac{\mu_3 \mu_4 \mu_5 (\tilde{\mu}_1 - \mu_5) (\rho^2 + \tilde{\mu}_1^2)(\rho^2 + \mu_1 \mu_3)(\rho^2 + \tilde{\mu}_1 \mu_4)}{\tilde{\mu}_1 [\mu_5 (\rho^2 + \tilde{\mu}_1^2)(\rho^2 + \tilde{\mu}_1 \mu_4)^2 - B_1^2 \mu_3 \mu_4 (\tilde{\mu}_1 - \mu_5)^2 \rho^4]} . \] (6)

The functions \( P, G_\psi, G_\phi, \omega_\psi, H_1 \) and \( H_2 \) are the metric functions of the vacuum black Saturn solution \(^4[4] \) and they are given by the following expressions

\[ P = (\mu_3 \mu_4 + \rho^2)(\mu_1 \mu_5 + \rho^2)(\mu_4 \mu_5 + \rho^2) , \] (7)

\[ G_\psi = \frac{\mu_3 \mu_5}{\mu_4} , \] (8)

\[ G_\phi = \frac{\rho^2 \mu_4}{\mu_3 \mu_5} , \] (9)

\[ \omega_\psi = 2 c_1 R_1 \sqrt{M_0 M_1} - c_2 R_2 \sqrt{M_0 M_2 + c_1^2 c_2 R_2 \sqrt{M_1 M_4 - c_1 c_2^2 R_1 \sqrt{M_2 M_4}}} \] \[ F \sqrt{G_\phi} \] \( , \) (10)

\[ H_1 = F^{-1} \left[ M_0 + c_1^2 M_1 + c_2^2 M_2 + c_1 c_2 M_3 + c_1^2 c_2^2 M_4 \right] , \] (11)

\[ H_2 = F^{-1} \frac{\mu_3}{\mu_4} \left[ M_0 \frac{\mu_1}{\mu_2} - c_1^2 M_1 \frac{\rho^2}{\mu_1 \mu_2} - c_2^2 M_2 \frac{\mu_1 \mu_2}{\rho^2} + c_1 c_2 M_3 + c_1^2 c_2^2 M_4 \frac{\mu_2}{\mu_1} \right] , \] (12)

where

\[ M_0 = \mu_2 \mu_5^2 (\mu_1 - \mu_3)^2 (\mu_2 - \mu_4)^2 (\rho^2 + \mu_1 \mu_3)^2 (\rho^2 + \mu_4 \mu_5)^2 (\rho^2 + \mu_2 \mu_3)^2 , \] (13)

\[ M_1 = \mu_1^2 \mu_2 \mu_3 \mu_4 \mu_5 \rho^2 (\mu_1 - \mu_2)^2 (\mu_2 - \mu_4)^2 (\mu_1 - \mu_3)^2 (\rho^2 + \mu_2 \mu_3)^2 , \] (14)

\[ M_2 = \mu_2 \mu_3 \mu_4 \mu_5 \rho^2 (\mu_1 - \mu_2)^2 (\mu_1 - \mu_3)^2 (\rho^2 + \mu_1 \mu_4)^2 (\rho^2 + \mu_2 \mu_5)^2 , \] (15)

\[ M_3 = \frac{2 \mu_1 \mu_2 \mu_3 \mu_4 \mu_5 (\mu_1 - \mu_3)(\mu_1 - \mu_5)(\mu_2 - \mu_4)(\rho^2 + \mu_4^2)(\rho^2 + \mu_2^2)}{\times (\rho^2 + \mu_1 \mu_4)(\rho^2 + \mu_2 \mu_3)(\rho^2 + \mu_2 \mu_5)} , \] (16)

\[ M_4 = \mu_2 \mu_3 \mu_4 \mu_5^2 (\mu_1 - \mu_5)^2 (\rho^2 + \mu_1 \mu_2)^2 (\rho^2 + \mu_2 \mu_5)^2 , \] (17)
and
\[
F = \mu_1 \mu_5 (\mu_1 - \mu_3)^2 (\mu_2 - \mu_4)^2 (\rho^2 + \mu_1 \mu_3)(\rho^2 + \mu_2 \mu_3)(\rho^2 + \mu_1 \mu_4)
\times (\rho^2 + \mu_2 \mu_4)(\rho^2 + \mu_2 \mu_5)(\rho^2 + \mu_3 \mu_5) \prod_{i=1}^{5} (\rho^2 + \mu_i^2).
\] (18)

Finally we have
\[
\mu_i = R_i - (z - a_i), \quad i = 1, 2, 3, 4, 5
\] (19)
where
\[
R_i = \sqrt{\rho^2 + (z - a_i)^2}
\] (20)
and
\[
\tilde{\mu}_1 = \sqrt{\rho^2 + (z - b_1)^2} - (z - b_1).
\] (21)

In all the above expressions \(a_i, b_1, c_1, c_2\) and \(B_1\) are (real) constants. The presented solution possesses three Killing vectors, namely \(\xi = \partial/\partial t, \zeta_\psi = \partial/\partial \psi\) and \(\zeta_\phi = \partial/\partial \phi\).

Since \((c_1, c_2) \rightarrow (-c_1, -c_2)\) and \(B_1 \rightarrow -B_1\) change the direction of rotation and the sign of the electromagnetic field we can restrict ourselves to the case \(c_1 \geq 0\) and \(B_1 \geq 0\). Let us note some limiting cases of the presented solution. For \(B_1 = 0\) and \(b_1 = a_4\) we obtain the black Saturn solution. Setting further \(c_1 = 0\) and \(a_1 = a_5 = a_4\) we obtain the \(\psi\)-spinning Myers-Perry black hole. Taking instead \((B_1 \neq 0)\) \(c_2 = 0\) and \(a_2 = a_3\) we obtain the \(S^1\) rotating dipole black ring. These limits will be discussed in more detail in section 3.10.

3 Analysis of the solution

In order to analyze the solution we shall use the canonical procedure [9] closely following the analysis of [4].

As we have already mentioned, the generated solution describes regular black object when the solution parameters are chosen appropriately. Here we will consider the following ordering of the parameters \(a_i\) and \(b_1\), namely
\[
a_1 \leq a_5 \leq a_4 \leq b_1 < a_3 \leq a_2.
\] (22)

The other bounds on the dipole parameter \(b_1\) different form (22) lead to solutions which are singular and are not of physical interest.

In order to insure the positivity and regularity of the functions \(W, Y, V\) we must impose the following constraint
\[
B_1^2 = 2 \frac{(a_3 - b_1)(b_1 - a_4)}{(b_1 - a_5)}.
\] (23)
Further, in order to remove the singularity of $H_2$ we must impose the constraint

$$c_1^2 = 2 \frac{(a_3 - a_1)(a_4 - a_1)}{(a_5 - a_1)}$$ (24)

just as for the black Saturn [4].

Since the solution is invariant under shifts in $z$-direction, its description in terms of $a_i$ and $b_1$ is redundant and it is convenient to introduce new parameters. Following [4] we shall introduce one dimensionful parameter $L^2$ and four dimensionless parameters $\kappa_i (i = 1, 2, 3)$ and $\eta$ defined as follows

$$L^2 = a_2 - a_1,$$
$$\kappa_i = \frac{a_{i+2} - a_1}{L^2},$$
$$\eta = \frac{b_1 - a_1}{L^2}.$$ (25)

As a consequence of (22) we have

$$0 \leq \kappa_3 \leq \kappa_2 \leq \eta < \kappa_1 \leq 1.$$ (26)

In terms of the new parameters the conditions (23) and (24) take the form

$$B_1^2 = 2L^2 \frac{(\kappa_1 - \eta)(\eta - \kappa_2)}{(\eta - \kappa_3)},$$ (27)
$$c_1^2 = 2L^2 \frac{\kappa_1 \kappa_2}{\kappa_3}.$$ (28)

It is also convenient to introduce the dimensionless coordinate $\bar{z}$ and dimensionless parameter $\bar{c}_2$ given by [4]

$$z = L^2 \bar{z} + a_1,$$ (29)
$$\bar{c}_2 = \frac{c_2}{c_1 (1 - \kappa_2)}.$$ (30)

### 3.1 Asymptotic behaviour of the solution

In order to study the asymptotic behaviour of the solution we shall follow the standard way and we shall introduce the asymptotic coordinates $r$ and $\theta$ defined by

$$\rho = \frac{1}{2} r^2 \sin 2\theta, \quad z = \frac{1}{2} r^2 \cos 2\theta.$$ (31)

Then in the asymptotic limit $r \to \infty$ we find
\[ W \approx 1 + 2 \frac{L^2}{r^2} (\eta - \kappa_2), \]
\[ Y \approx 1 + 2 \frac{L^2}{r^2} (\eta - \kappa_2) \sin^2 \theta, \]
\[ \frac{H_1}{H_2} \approx 1 + 2 \frac{L^2}{r^2} \left\{ \frac{\kappa_3 (1 - \kappa_1 + \kappa_2) - 2 \kappa_2 \kappa_3 (\kappa_1 - \kappa_2) \bar{c}_2 + \kappa_2 [\kappa_1 - \kappa_2 \kappa_3 (1 + \kappa_1 - \kappa_2)] \bar{c}_2^2}{\kappa_3 (1 + \kappa_2 \bar{c}_2)^2} \right\}, \]
\[ k^2 H_1 P \approx \frac{1}{r^2} k^2 (1 + \kappa_2 \bar{c}_2)^2, \]
\[ \frac{\omega_\psi}{H_2} \approx -L \sqrt{\frac{2 \kappa_1 \kappa_2}{\kappa_3} \left| 1 + \kappa_2 \bar{c}_2 \right|} + \frac{4 L^3}{r^2} \frac{\sin^2 \theta}{\kappa_3 (1 + \kappa_2 \bar{c}_2)^2} \sqrt{\frac{\kappa_2}{2 \kappa_1 \kappa_3}} \left\{ \kappa_3^2 - \kappa_3 [(\kappa_1 - \kappa_2) (1 - \kappa_1 + \kappa_3) + \kappa_2 (1 - \kappa_3)] \bar{c}_2^2 + \kappa_2 \kappa_3 [(\kappa_1 - \kappa_2) (\kappa_1 - \kappa_3 + \kappa_1 (1 + \kappa_1 - \kappa_2 - \kappa_3)] \bar{c}_2^2 - \kappa_1 \kappa_2 \kappa_3 [(2 + \kappa_1 - \kappa_2 - \kappa_3) \bar{c}_2^2], \right\} \]
\[ G_\phi \approx r^2 \cos^2 \theta, \quad G_\psi \approx r^2 \sin^2 \theta. \]

The asymptotic behaviour of the function \( V \) is
\[ V \approx 2 L^3 \sqrt{2 (\kappa_1 - \eta) (\eta - \kappa_2) (\eta - \kappa_3)} \frac{\sin^2 \theta}{r^2}. \]

Since we are interested in asymptotically flat solutions for which \( g_{t\psi} \to \sim \sin^2 \theta / r^2 \) and \( g_{\rho \rho} \to 1 / r^2 \) we chose the constants \( q \) and \( k \) to be
\[ q = L \sqrt{\frac{2 \kappa_1 \kappa_2}{\kappa_3} \frac{\bar{c}_2}{1 + \kappa_2 \bar{c}_2}}, \]
\[ k = \left| 1 + \kappa_2 \bar{c}_2 \right|^{-1}, \]

for \( \kappa_2 \bar{c}_2 \neq -1 \). The case \( \kappa_2 \bar{c}_2 = -1 \) is singular and will not be considered here. With this choice for \( q \) and \( k \) the asymptotic metric takes the form
\[ ds^2 \approx -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\psi^2 + r^2 \cos^2 \theta d\phi^2. \]

In what follows we shall show that the periodicities of the angles \( \psi \) and \( \phi \) can be chosen to be canonical ones
\[ \Delta \psi = \Delta \phi = 2\pi \]

and as a consequence we find that the spacetime is asymptotically flat.
3.2 Rod structure and balance conditions

As we have already mentioned the ordering (22) and the condition (23) insure the positivity and the regularity of the functions $W$ and $Y$. Therefore the rod structure [9] of the solution presented here is the same as the rod structure of the black Saturn. With condition (24) imposed the rod structure is the following. We have

* Semi-infinite rod $\bar{z} \in [-\infty, \kappa_3]$ and finite rod $\bar{z} \in [\kappa_2, \kappa_1]$ in direction $(0, 1, 0)$ which are sources of the $\phi\phi$-part of the metric

* The source of the $\psi\psi$-part of the metric is the semi-infinite rod $\bar{z} \in [1, \infty]$ in direction $(0, 0, 1)$.

* Two finite rods $\bar{z} \in [\kappa_3, \kappa_2]$ and $\bar{z} \in [\kappa_1, 1]$ in directions $(1, 0, \Omega_{DBR}^{\psi})$ and $(1, 0, \Omega_{BH}^{\psi})$. They correspond to the location the dipole black ring horizon and the location of the black hole, respectively. $\Omega_{DBR}^{\psi}$ and $\Omega_{BH}^{\psi}$ are the angular velocities of the horizons.

In order to cure the conical singularities at the location of a rod the coordinates $\psi$ and $\phi$ should have periods

$$\Delta \psi = 2\pi \lim_{\rho \to 0} \sqrt{\rho^2 g_{\rho\rho}/g_{\psi\psi}}, \quad \Delta \phi = 2\pi \lim_{\rho \to 0} \sqrt{\rho^2 g_{\rho\rho}/g_{\phi\phi}}. \quad (43)$$

Let us first consider the rods $\bar{z} \in [-\infty, \kappa_3]$ and $\bar{z} \in [1, \infty]$. Then the regularity conditions (23) fix the periods of $\phi$ and $\psi$ to be $\Delta \phi = 2\pi$ and $\Delta \psi = 2\pi$. These periods insure asymptotic flatness of the metric as it was discussed in (42).

Concerning the regularity on the finite rod $\bar{z} \in [\kappa_2, \kappa_1]$ we obtain

$$\Delta \phi = 2\pi \frac{(\kappa_1 - \eta)^{3/2}}{1 + \kappa_2 \bar{c}_2 \sqrt{\kappa_1(1 - \kappa_2)(1 - \kappa_3)(\kappa_1 - \kappa_2)(\kappa_1 - \kappa_3)}}. \quad (44)$$

In order to find the equilibrium (the balancing) condition for the black Saturn with dipole black ring we have to impose that the r.h.s. of (44) be equal to $2\pi$. Solving then for $\bar{c}_2$ we find

$$\bar{c}_2 = \frac{1}{\kappa_2} \left[ \varepsilon \frac{(\kappa_1 - \eta)^{3/2}}{\sqrt{\kappa_1(1 - \kappa_2)(1 - \kappa_3)(\kappa_1 - \kappa_2)(\kappa_1 - \kappa_3)}} - 1 \right] \quad (45)$$

where $\varepsilon = 1$ for $\bar{c}_2 > -\kappa_2^{-1}$ and $\varepsilon = -1$ for $\bar{c}_2 < -\kappa_2^{-1}$. As in the case of black Saturn, we have two separate sectors of the balance condition for the black Saturn with a dipole black ring.

Let us also note that the electromagnetic field is regular on the rods considered here.

Let us note again that we keep imposing the conditions (23) and (24).
3.3 Black hole horizon

The black hole horizon is located at $\rho = 0$ for $\bar{z} \in [\kappa_1, 1]$. The induced metric on the spacial cross section of the horizon is given by

$$ds^2_{BH} = \frac{2L^2(\bar{z} - \kappa_1)(\bar{z} - \kappa_3)}{(\bar{z} - \kappa_2)}w^2(\bar{z})d\phi^2 + L^2s_{BH}^2 \frac{g(\bar{z})}{w(\bar{z})}(1 - \bar{z})dw^2 + \frac{L^2 (\bar{z} - \kappa_2) d\bar{z}^2}{(1 - \bar{z})(\bar{z} - \kappa_1)(\bar{z} - \kappa_3)g(\bar{z})w(\bar{z})}. \quad (46)$$

Here the constant $s_{BH}$ and the function $g(\bar{z})$ are formally the same as for the black Saturn solution [4] and are given by

$$s_{BH} = \frac{\kappa_3(1 - \kappa_1) + \kappa_1\kappa_2(1 - \kappa_2)(1 - \kappa_3)\bar{c}_2^2}{\kappa_3 \sqrt{(1 - \kappa_1)(1 - \kappa_2)(1 - \kappa_3)[1 + \kappa_2\bar{c}_2]^2}}, \quad (47)$$

$$g(\bar{z}) = 2\kappa_1\kappa_3(1 - \kappa_1)(1 - \kappa_2)(1 - \kappa_3)(\bar{z} - \kappa_2)$$

$$\times \left[1 + \kappa_2\bar{c}_2^2\right] \left[(1 - \kappa_1)^2\kappa_3 \left[\kappa_1(\bar{z} - \kappa_2) - \kappa_3(\kappa_1 - \kappa_2(1 - \bar{z})^2 - \kappa_1\kappa_2(2 - \bar{z}))\right]\right]$$

$$+ 2\kappa_1\kappa_2\kappa_3(1 - \kappa_1)(1 - \kappa_2)(1 - \kappa_3)(1 - \bar{z})(\bar{z} - \kappa_1)\bar{c}_2$$

$$+ \kappa_1^2\kappa_2(1 - \kappa_2)^2(1 - \kappa_3)^2 \bar{z}(\bar{z} - \kappa_1)\bar{c}_2^{2} - 1. \quad (48)$$

The function $w(\bar{z})$ is defined by

$$w(\bar{z}) = \frac{\bar{z} - \kappa_2}{\bar{z} - \eta}. \quad (49)$$

The functions $g(\bar{z})$ and $w(\bar{z})$ are positive for $\kappa_1 \leq \bar{z} \leq 1$ and $s_{BH} > 0$.

The topology of the horizon is that of $S^3$. Metrically, however, the horizon is distorted $S^3$. This can be seen explicitly by computing the scalar curvature of the horizon which is not constant contrary to the case of the round $S^3$. The distortion is caused by the rotation of the black hole itself and the gravitational attraction of the dipole black ring.

The area of the spacial cross section black hole horizon can be found by straightforward calculation and the result is

$$A_{BH} = 4\pi^2L^3 \sqrt{\frac{2(1 - \kappa_1)^3}{(1 - \kappa_2)(1 - \kappa_3)}} \frac{\kappa_3(1 - \kappa_1) + \kappa_1\kappa_2(1 - \kappa_2)(1 - \kappa_3)\bar{c}_2^2}{\kappa_3(1 - \kappa_1)(1 + \kappa_2\bar{c}_2)^2}. \quad (50)$$

The other horizon quantities which are of interest are the angular velocity and the temperature and they are given by

$^3$Let us note however that the balance condition (45) is different i.e. the parameter $\bar{c}_2$ is different.
\[ \Omega_{BH} = \frac{1}{L} \sqrt{\frac{\kappa_2\kappa_3}{2\kappa_1} \frac{\kappa_3(1 - \kappa_1) - \kappa_1(1 - \kappa_2)(1 - \kappa_3)c_2}{\kappa_3(1 - \kappa_1) + \kappa_1\kappa_2(1 - \kappa_2)(1 - \kappa_3)c_2^2} (1 + \kappa_2\bar{c}_2)}, \]  
(51)

\[ T_{BH} = \frac{1}{2\pi L} \sqrt{\frac{(1 - \kappa_2)(1 - \kappa_3)}{2(1 - \kappa_1)} \frac{\kappa_3(1 - \kappa_1)(1 + \kappa_2\bar{c}_2)^2}{\kappa_3(1 - \kappa_1) + \kappa_1\kappa_2(1 - \kappa_2)(1 - \kappa_3)c_2^2}}. \]  
(52)

Let us note that although the horizon area, temperature and the angular velocity of the black hole in the black Saturn with dipole ring look at first sight the same as the corresponding quantities for the black hole in the black Saturn solution, they are in fact different since the balance condition (45) (i.e. the parameter \( \bar{c}_2 \)) is different.

The investigation of the behaviour of the electromagnetic field shows that it is regular on the black hole horizon.

### 3.4 Dipole black ring horizon

The dipole black ring horizon is located at \( \rho = 0 \) for \( \bar{z} \in [\kappa_3, \kappa_2] \). The induced metric on the spacial cross section of the horizon is

\[ ds_{DBR}^2 = \frac{2L^2(\kappa_2 - \bar{z})(\bar{z} - \kappa_3)}{(\kappa_1 - \bar{z})} w^2(\bar{z}) d\phi^2 + L^2 s_{DBR}^2 f(\bar{z}) (\kappa_1 - \bar{z}) d\psi^2 \]

\[ + \frac{L^2 y_{DBR}^2 d\bar{z}^2}{(\kappa_2 - \bar{z})(\bar{z} - \kappa_3)f(\bar{z})w(\bar{z})}. \]

Here the constant \( s_{DBR} \) and the function \( f(\bar{z}) \) are formally the same as for the black Saturn solution \[4\] and are given by

\[ s_{DBR} = \sqrt{\frac{\kappa_2(\kappa_2 - \kappa_3)}{\kappa_1(1 - \kappa_3)(1 - \kappa_3)} \frac{\kappa_3 - \kappa_3(\kappa_1 - \kappa_2)\bar{c}_2 + \kappa_1\kappa_2(1 - \kappa_3)c_2^2}{\kappa_3(1 + \kappa_2\bar{c}_2)^2}}, \]
(54)

\[ f(\bar{z}) = 2\kappa_1\kappa_3(\kappa_1 - \kappa_2)(1 - \kappa_3)(1 - \bar{z}) \]
\[ \times [1 + \kappa_2\bar{c}_2] \frac{2}{(\kappa_2 - \kappa_3)^{-1}} \left[ \kappa_3 \left[ \kappa_2(\kappa_1 - \bar{z}) + \kappa_3 \left( \kappa_2(1 - \kappa_1(2 - \bar{z})) - \kappa_1(1 - \bar{z})^2 \right) \right] \right. \]
\[ + 2\kappa_1\kappa_2\kappa_3(1 - \kappa_3)(1 - \bar{z}) (\kappa_2 - \bar{z})\bar{c}_2 \]
\[ + \kappa_1\kappa_2^2(1 - \kappa_3)^2 \bar{z}(\kappa_2 - \bar{z})c_2^2 \left. \right]^{-1}. \]
(55)

The function \( w(\bar{z}) \) and the constant \( y_{DBR} \) are defined by

\[ w(\bar{z}) = \frac{(\eta - \bar{z})(\kappa_1 - \bar{z})}{(\kappa_1 - \bar{z})(\kappa_2 - \bar{z}) + \kappa_1\kappa_2(\eta - \kappa_2)(\bar{z} - \kappa_3)}, \]
(56)
The functions $f(\bar{z})$ and $w(\bar{z})$ are positive for $\kappa_3 \leq \bar{z} \leq \kappa_2$ and $s_{DBR} > 0$, $y_{DBR} > 0$. The topology of the horizon is $S^2 \times S^1$ where $S^1$ is parameterized by the angular coordinate $\psi$ and has radius depending on $\bar{z}$. The two-sphere is parameterized by the coordinates $(\bar{z}, \phi)$ and is metrically distorted. The horizon area is found by straightforward integration and the result is

$$A_{DBR} = 4\pi L^3 \left[ \frac{2\kappa_2(\eta - \kappa_3)^3}{\kappa_1(1 - \kappa_3)} \frac{\kappa_3 - \kappa_3(1 - \kappa_3)\bar{c}_2 + \kappa_1\kappa_3(1 + \kappa_3)\bar{c}_2^2}{\kappa_3(1 + \kappa_3)\bar{c}_2} \right].$$

(58)

The angular velocity and the temperature of the horizon are given by

$$\Omega_{DBR}^\psi = \frac{1}{L} \sqrt{\frac{\kappa_1\kappa_3}{2\kappa_2}} \frac{\kappa_3 - \kappa_2(1 - \kappa_3)\bar{c}_2}{\kappa_3 - \kappa_3(1 - \kappa_2)\bar{c}_2 + \kappa_1\kappa_3(1 - \kappa_3)\bar{c}_2^2}(1 + \kappa_2\bar{c}_2),$$

(59)

$$T_{DBR} = \frac{1}{2\pi L} \left[ \frac{\kappa_1(1 - \kappa_3)(\kappa_1 - \kappa_3)(\kappa_2 - \kappa_3)^2}{2\kappa_2(\eta - \kappa_3)^3} \frac{\kappa_3(1 + \kappa_2\bar{c}_2)^2}{\kappa_3 - \kappa_3(1 - \kappa_2)\bar{c}_2 + \kappa_1\kappa_3(1 - \kappa_3)\bar{c}_2^2} \right].$$

(60)

The analysis shows that the electromagnetic field is regular on the black ring horizon.

### 3.5 ADM mass and angular momentum of the black Saturn with dipole black ring

The ADM mass and angular momentum can be found from the asymptotic form of the metric (32)-(37) and the result is

$$M_{ADM} = \frac{3\pi}{4} L^2 (\eta - \kappa_2)$$

$$+ \frac{3\pi}{4} L^2 \left\{ \frac{\kappa_3(1 - \kappa_1 + \kappa_2) - 2\kappa_2\kappa_3(\kappa_1 - \kappa_2)\bar{c}_2 + \kappa_2 [\kappa_1 - \kappa_2\kappa_3(1 + \kappa_1 - \kappa_2)]\bar{c}_2^2}{\kappa_3(1 + \kappa_2\bar{c}_2)^2} \right\}.$$  

(61)

$$J_{ADM} = \frac{\pi L^3}{\kappa_3(1 + \kappa_2\bar{c}_2)^3} \sqrt{\frac{\kappa_2}{2\kappa_1\kappa_3}} \left\{ \kappa_1^2 - \kappa_3 \left[ (\kappa_1 - \kappa_2)(1 - \kappa_1 + \kappa_3) + \kappa_2(1 - \kappa_3) \right] \bar{c}_2 + \kappa_2 \kappa_3 \left[ (\kappa_1 - \kappa_2)(\kappa_1 - \kappa_3) + \kappa_1(1 + \kappa_1 - \kappa_2 - \kappa_3) \right] \bar{c}_2^2 ight\}$$

$$- \kappa_1 \kappa_2 \left[ (\kappa_1 - \kappa_2\kappa_3)(2 + \kappa_1 - \kappa_2 - \kappa_3) \right] \bar{c}_2^3.$$  

(62)

Let us note that the ADM mass is positive as a consequence of the ordering (26).
3.6 Komar masses and angular momenta

The definition of the Komar mass and angular momentum is well known, namely

\[ M_{\text{Komar}} = \frac{3}{32\pi} \int_{\partial\Sigma} \star d\xi, \quad J_{\text{Komar}} = \frac{1}{16\pi} \int_{\partial\Sigma} \star d\tilde{\zeta}_\psi \] (63)

where \( \xi \) and \( \tilde{\zeta}_\psi \) are 1-forms dual to the timelike Killing vector \( \xi \) and the spacelike Killing vector \( \zeta_\psi \). Here \( \partial\Sigma \) is a boundary of any spacelike hypersurface \( \Sigma \). From a physical point of view the Komar integrals measure the mass and angular momentum contained in \( \partial\Sigma \). When \( \partial\Sigma \) is a three-sphere at infinity the Komar integrals coincide with the ADM mass and angular momentum of an asymptotically flat spacetime. When dealing with multi-black objects configurations the Komar integrals evaluated on the horizon cross sections are of special interest since they give the intrinsic mass and angular momenta of the black objects. These intrinsic quantities for the black objects in our solution are the following

\[ M_{\text{BH Komar}}^{\text{DBR}} = \frac{3\pi L^3}{4} \frac{\kappa_2(1 - \kappa_1) + \kappa_1\kappa_2(1 - \kappa_3)(1 - \kappa_2)\bar{c}_2^2}{\kappa_3(1 + \kappa_2\bar{c}_2)^2}, \] (64)

\[ M_{\text{Komar}}^{\text{DBR}} = \frac{3\pi L^2}{4} \frac{\kappa_2[1 - (1 - \kappa_2)\bar{c}_2][\kappa_3 - \kappa_3(\kappa_1 - \kappa_2)\bar{c}_2 + \kappa_1\kappa_2(1 - \kappa_3)\bar{c}_2^2]}{\kappa_3(1 + \kappa_2\bar{c}_2)^2}, \] (65)

\[ J_{\text{Komar}}^{\text{BH}} = -\pi L^3 \bar{c}_2 \sqrt{\frac{\kappa_1\kappa_2 \kappa_3(1 - \kappa_1) + \kappa_1\kappa_2(1 - \kappa_2)(1 - \kappa_3)\bar{c}_2^2}{2\kappa_3(1 + \kappa_2\bar{c}_2)^2}} \] (66)

\[ J_{\text{Komar}}^{\text{DBR}} = \pi L^3 \sqrt{\frac{\kappa_2}{2\kappa_1\kappa_3}} \times \frac{[\kappa_3 - \kappa_2(\kappa_1 - \kappa_3)\bar{c}_2 + \kappa_1\kappa_2(1 - \kappa_2)\bar{c}_2^2][\kappa_3 - \kappa_3(\kappa_1 - \kappa_2)\bar{c}_2 + \kappa_1\kappa_2(1 - \kappa_3)\bar{c}_2^2]}{\kappa_3(1 + \kappa_2\bar{c}_2)^3}. \] (67)

Usually one thinks that the Komar mass of every black object in a selfgravitating configuration should be positive. However, when the objects are under strong gravitational interaction the Komar masses can become negative for such tightly bound configurations. Numerical solutions of black holes with negative Komar mass but with positive ADM mass were presented in [10], [11] for the 5D Einstein-Maxwell gravity with a Chern-Simons term. For our solution the situation is the same as for the vacuum black Saturn solution [4]. The balance condition (45) with \( \varepsilon = 1 \) leads to the inequality \( -\kappa_2^{-1} < \bar{c}_2 < (1 - \kappa_2)^{-1} \) which guarantees that \( M_{\text{Komar}}^{\text{BH}} > 0 \) and \( M_{\text{Komar}}^{\text{DBR}} > 0 \). The balance condition (45) with \( \varepsilon = -1 \) leads to the inequality \( \bar{c}_2 < -\kappa_2^{-1} \) and as a consequence we have \( M_{\text{Komar}}^{\text{BH}} < 0 \) and \( M_{\text{Komar}}^{\text{DBR}} > 0 \).
3.7 Dipole charge and potential

The dipole charge is defined as

$$Q = \frac{1}{4\pi} \oint_{S^2} F$$  \hspace{1cm} (68)

where $S^2$ is a sphere on the black ring horizon. For our solution we find

$$Q = -L \sqrt{\frac{3(\eta - \kappa_2)(\eta - \kappa_3)}{2(\kappa_1 - \eta)}}. \hspace{1cm} (69)$$

Finding explicitly the potential $B$ ($H = dB$) of the dual form $H = *F$ seems to be formidable task at least in the canonical coordinates $(\rho, z)$. That is why the direct computation of the dipole potential $\Phi$ is not possible. In order to find $\Phi$ we shall proceed in the following way. First it is clear that $\Phi$ is proportional to $\sqrt{3}B_1$ i.e.

$$\Phi = \Gamma \sqrt{3}B_1$$  \hspace{1cm} (70)

where the dimensionless constant $\Gamma$ is a function of the solution parameters. What is important is that this constant does not depend on the parameters $a_2$ and $c_2$. Therefore $\Gamma$ is the same for the dipole black ring solution which is obtained in the limit $c_2 = 0$ and $a_2 = a_3$. Hence we find $\Gamma = -\frac{\pi}{2}$ and therefore we have

$$\Phi = -\frac{\sqrt{3}\pi}{2} B_1 = -\pi L \sqrt{\frac{3(\kappa_1 - \eta)(\eta - \kappa_2)}{2(\eta - \kappa_3)}}. \hspace{1cm} (71)$$

3.8 Smarr relations

Straightforward calculations show that the following Smarr-like relations are satisfied

$$M_{Komar}^{BH} = \frac{3}{2} \left[ T_{BH} A_{BH} \frac{1}{4} + J_{Komar}^{BH} \Omega_{BH} \right], \hspace{1cm} (72)$$

$$M_{Komar}^{DBR} = \frac{3}{2} \left[ T_{DBR} A_{DBR} \frac{1}{4} + J_{Komar}^{DBR} \Omega_{DBR} \right], \hspace{1cm} (73)$$

$$M_{ADM} = M_{Komar}^{BH} + M_{Komar}^{DBR} + \frac{1}{2} Q \Phi. \hspace{1cm} (74)$$

The term $\frac{1}{2} Q \Phi$ can be interpreted as the energy of the electromagnetic “hair” of the dipole black ring.
3.9 Ergosurfaces and closed timelike curves

As a consequence of the way the solution was generated and the positiveness of $W$ the existence/noexistence of ergosurfaces and closed timelike curves depends only on the seed vacuum black Saturn solution i.e. the dipole solution inherits the existence/nonexistence of ergosurfaces and closed timelike curves from the vacuum black Saturn solution. Our preliminary investigations show that there are ergosurfaces for the black hole and the black ring. However, the solution is too involved in the canonical coordinates and the explicit description is not possible. Concerning the possible existence of closed timelike curves we should say that no sign for their existence is seen. The same conclusion is also reached in [4]. The problem however remains open.

3.10 Limits of the solution

The limits of the balance black Saturn with dipole black ring are more or less clear from the way the solution was generated. By removing the black hole from the configuration we obtain the dipole black ring [3],[8]. The formal procedure is as follows. First we set the angular momentum of the black hole to zero by taking $\bar{c}_2 = 0$. Then the black hole is removed by setting $\kappa_1 = 1$.

Removing the dipole black ring from the configuration we obtain a Myers-Perry black hole with one angular momentum. This is achieved as follows. First we set $\eta = \kappa_2$ which eliminates the electromagnetic field from the black ring. Then we take the limit $\kappa_3 \rightarrow \kappa_2$ and finally set $\kappa_2 = 0$.

It is tempting to consider limit in which the black ring is removed but the electromagnetic field is preserved i.e. to repeat the above procedure without setting $\eta = \kappa_2$ expecting that the result is a dipole black hole. Unfortunately the described limit is singular as the analysis shows. One can show that there is no regular limit of a merger of the black hole and the dipole black ring.

3.11 Non-uniqueness

The balanced solution depends on four dimensionless parameters $(\kappa_1, \kappa_2, \kappa_3, \eta)$ and one dimensionful parameter $L$. Two of the parameters can be fixed by fixing the mass and angular momentum of the configuration. Therefore the solution exhibits 3-fold continuous non-uniqueness.

4 Discussions

In this paper we presented a new asymptotically flat solution of 5D Einstein-Maxwell gravity describing a Saturn-like black configuration: a rotating black hole surrounded by a rotating dipole black ring. The solution was generated by combining the vacuum black Saturn solution and the vacuum black ring solution with appropriately chosen parameters along the lines of the method developed in [8]. Some basic properties of the solution were analyzed and the basic quantities were calculated. It is interesting to see in detail how the presence of the dipole charge affects the physics of the black Saturn.
This however is very difficult and requires numerical methods since the dipole black Saturn depends on many parameters in nontrivial manner. The physical properties of the dipole black Saturn are currently under numerical investigation and the results will be presented elsewhere. Preliminary results show that, more or less, many of the physical properties of the dipole solution are similar to those of the vacuum black Saturn solution which are thoroughly enough discussed in [4]. In particular, the dipole black Saturn exhibits effects like rotational frame-dragging and countering frame-dragging. Respectively, the differences between the dipole black Saturn and the vacuum black Saturn are inherited from the differences of the dipole black ring and the vacuum black ring [3]. In particular, the dipole charge increases the self-interaction of the ring and larger angular momentum is needed to balance the ring in comparison with the vacuum black Saturn solution. Another important point is that the dipole black Saturn is expected to be stable near the extremal non-BPS limit [12].

Let us finish with prospects for future investigations. The question whether there is a more general dipole Saturn solution than the one presented here remains open. If such a solution exists, most probably it has to be generated via the scheme

\[
\text{Black Saturn} \oplus \text{Black Saturn} \rightarrow \text{New Dipole Black Saturn} \quad (75)
\]

provided that the potential singularities can be removed by an appropriate choice of the solution parameters. Since the formal operation denoted by \( \oplus \) is not commutative it is also interesting to consider the generating scheme

\[
\text{Black Saturn} \oplus \text{Rotating Black Ring} \rightarrow \text{New Dipole Black Rings} \quad (76)
\]

which will generate new dipole black ring solution provided the potential singularities can be removed.

Our solution can be extended to a solution of the 5D Einstein-Maxwell-dilaton gravity with an arbitrary dilaton coupling parameter via the solution generating methods developed in [13], [14]. Moreover these dipole solutions can be uplifted to supergravity solutions.

Finally the methods of [8], [13] and [14] can be applied to the vacuum solution of [5] describing two (or more) rotating back rings at equilibrium in order generate configurations with two (or more) dipole black rings.

Acknowledgements

The author would like to thank the Alexander von Humboldt Foundation for a stipend, and the Institut für Theoretische Physik Göttingen for its kind hospitality. The partial support by the Bulgarian National Science Fund under Grant MUF04/05 (MU 408) and VUF-201/06 is also acknowledged.

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