There is no complete numerical invariant for smooth conjugacy of circle diffeomorphisms

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Abstract

Classical results by Poincaré and Denjoy show that two orientation-preserving \( C^2 \) diffeomorphisms of the circle are topologically conjugate if and only if they have the same rotation number. We show that there is no possibility of getting such a complete numerical Borel invariant for the conjugacy relation of orientation-preserving circle diffeomorphisms by homeomorphisms with higher degree of regularity. For instance, we consider conjugacy by Hölder homeomorphisms or by \( C^k \)-diffeomorphisms with \( k \in \mathbb{Z}^+ \cup \{\infty\} \). The proof combines techniques from Descriptive Set Theory and a quantitative version of the Approximation by Conjugation method for circle diffeomorphisms.

1 Introduction

A fundamental theme in dynamics is the classification of systems up to appropriate equivalence relations. For instance, the equivalence relation of topological conjugacy preserves the qualitative behavior of topological dynamical systems. Here, two continuous maps \( T : X \to X \) and \( S : Y \to Y \) of compact metric spaces \( X, Y \) are called topological conjugate (or topologically equivalent) if there is a homeomorphism \( h : X \to Y \) such that \( S = h \circ T \circ h^{-1} \). Smale’s celebrated program proposes to classify topological or smooth dynamical systems up to topological conjugacy.

For the purpose of classification, one seeks dynamical invariants that are easy to compute and help determine whether two systems can be equivalent to each other. In the best case scenario, these invariants can be expressed as a single number. While one often can find invariants that are preserved under equivalence, even within specific classes of dynamical systems it turns out to be very hard to find complete invariants, that is, invariants that agree for two systems if and only if the systems are equivalent to each other. Indeed, having complete numerical invariants is one of the key notions of classifiability.

One well-studied numerical invariant is the rotation number \( \tau(f) \) for an orientation-preserving circle homeomorphism \( f \) (see Section 2.1 for its definition). We denote the collection of orientation-preserving homeomorphisms of the circle by \( \mathcal{H} \). Then the rotation number is an invariant for the equivalence relation on \( \mathcal{H} \) of conjugacy by an orientation-preserving homeomorphism. In the converse direction, Poincaré showed in 1885 that any \( f \in \mathcal{H} \) with an irrational rotation number

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A number \(\alpha\) is called Diophantine of class \(D(\nu)\) for \(\nu \geq 0\), if there exists \(C > 0\) such that
\[
|\alpha - \frac{p}{q}| \geq \frac{C}{q^{\nu+\delta}} \quad \text{for every } p \in \mathbb{Z} \text{ and } q \in \mathbb{N}.
\]
A number \(\alpha\) is called Diophantine if it is in \(D(\nu)\) for some \(\nu \geq 0\).

An irrational number \(\alpha\) is called Liouville if it is not Diophantine, that is, for every \(C > 0\) and every \(n \in \mathbb{N}\) there are infinitely many pairs \(p \in \mathbb{Z}, q \in \mathbb{N}\) such that
\[
0 < |\alpha - \frac{p}{q}| < \frac{C}{q^n}.
\]
Borel invariant for the conjugacy relation of circle diffeomorphisms by homeomorphisms with higher degree of regularity. To state our result, for some degree \((D)\) of regularity from Hölder to \(C^\infty\) let \(\mathcal{D}\) be the collection of orientation-preserving circle homeomorphisms with regularity \((D)\). We say that two circle diffeomorphisms \(S\) and \(T\) are \(\mathcal{D}\)-conjugate if there is a \(h \in \mathcal{D}\) such that \(S = h \circ T \circ h^{-1}\). Then we look at the equivalence relation \(E_\mathcal{D}\) on orientation-preserving \(C^\infty\) circle diffeomorphisms defined by \(S E_\mathcal{D} T\) if and only if \(S\) and \(T\) are \(\mathcal{D}\)-conjugate. For instance, our result holds for the relation of conjugacy by \(C^1\) diffeomorphisms.

**Theorem A.** Let \((D)\) be some degree of regularity from Hölder to \(C^\infty\) and \(\mathcal{D}\) be the collection of orientation-preserving circle homeomorphisms with regularity \((D)\). Then there is no complete numerical Borel invariant for \(\mathcal{D}\)-conjugacy of orientation-preserving \(C^\infty\) diffeomorphisms of the circle.

In fact, we show for every Liouville number \(\alpha\) that there is no complete numerical Borel invariant for \(\mathcal{D}\)-conjugacy of orientation-preserving \(C^\infty\) circle diffeomorphisms with rotation number \(\alpha\). A key ingredient of our proof is the concept of a *reduction* from the equivalence relation \(E_0\) to our conjugacy relations. We explain this concept and further basic tools from descriptive set theory in Section 2.3.

In general, the interplay between descriptive set theory and dynamical systems has been fruitful to establish several *anti-classification results* in dynamics which demonstrated in a rigorous way that classification is not possible. A pioneering work \cite{Fe74} by J. Feldman showed that there cannot exist a complete numerical invariant for measure-theoretic isomorphism of ergodic transformations or even of \(K\)-automorphisms. In their landmark paper \cite{FRW11}, M. Foreman, D. Rudolph and B. Weiss showed that the measure-isomorphism relation for ergodic transformations is not a Borel set when viewed as a subset of \(\mathcal{E} \times \mathcal{E}\), where \(\mathcal{E}\) denotes the collection of ergodic transformations. Informally speaking, this result says that determining isomorphism between ergodic transformations is inaccessible to countable methods that use a countable amount of information. Recently, Foreman and Weiss \cite{FW22} generalized the aforementioned anti-classification result from \(CR\) to the \(C^\infty\) category by proving that the measure-isomorphism relation among pairs of volume-preserving ergodic \(C^\infty\)-diffeomorphisms of on compact surfaces admitting a non-trivial circle action is not a Borel set with respect to the \(C^\infty\)-topology. In joint work with S. Banerjee we are even able to show this anti-classification result for real-analytic diffeomorphisms of the 2-torus \cite{BKpp}. Hence, von Neumann’s classification problem for the isomorphism relation in ergodic theory is impossible even when restricting to real-analytic diffeomorphisms of \(\mathbb{T}^2\). M. Gerber and the author obtained analogous results for the Kakutani equivalence relation \cite{GKpp}. Recently, Foreman and A. Gorodetski proved anti-classification results in the context of topological equivalence. For any compact manifold \(M\) of dimension at least 2, Foreman and Gorodetski showed that there is no Borel function from the \(C^\infty\) diffeomorphisms to the reals that is a complete invariant for topological equivalence \cite[Theorem 2]{FGpp}. Moreover, if \(\dim(M) \geq 5\), they proved that the set of topologically equivalent pairs of diffeomorphisms is not Borel \cite[Theorem 1]{FGpp}. We refer to the survey article \cite{Fopp} by Foreman for an overview of complexity results of structure and classification of dynamical systems. Problem 1 in \cite{Fopp} asks if there are complete numerical invariants for orientation-preserving diffeomorphisms of the circle up to conjugation by orientation-preserving diffeomorphisms. Our Main Theorem \ref{main} answers this question in the negative.

The conjugacy in Theorem \ref{main} could be either a singular or an absolutely continuous map. In Section 5.1 we show that both cases can be realized in the setting of our Main Theorem \ref{main}. In Section 6 we obtain another generalization of our Main Theorem \ref{main} to the setting of higher rank
actions: we show that there is no complete numerical Borel invariant for $D$-conjugacy of free $\mathbb{Z}^d$ actions by orientation-preserving $C^\infty$ diffeomorphisms of the circle.

Our Main Theorem \[ \text{A} \] does not rule out a classification along the lines of the Halmos–von Neumann classification of ergodic transformations with discrete spectrum by countable subgroups of the unit circle. To exclude such a classification, it suffices to show that the respective conjugacy relation is not reducible to an $S_\infty$-action. Hjorth devised the concept of turbulence as a mechanism for showing that an equivalence relation is not reducible to an $S_\infty$-action (see [Fopp, section 5.5] for details).

**Question.** Is the smooth conjugacy relation of smooth circle diffeomorphisms turbulent?

As mentioned before, the Borel/non-Borel distinction is an important benchmark for classification problems. Hence, it is a natural question if the smooth conjugacy relation of smooth circle diffeomorphisms is Borel or not.

**Question.** Is the smooth conjugacy relation of smooth circle diffeomorphisms complete analytic?

## 2 Preliminaries

### 2.1 Homeomorphisms of the circle

Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the unit circle with rotations $R_\alpha : S^1 \to S^1$, $R_\alpha(x) = x + \alpha \mod 1$. We denote the collection of orientation-preserving homeomorphisms of $S^1$ by $H$. For $x \in \mathbb{R}$ we write $[x] = \pi(x) \in S^1$, where $\pi : \mathbb{R} \to S^1$ is the quotient map. Hereby, we can define a lift of $f \in H$ as an increasing continuous function $F : \mathbb{R} \to \mathbb{R}$ with $f \circ \pi = \pi \circ F$, that is, $[F(x)] = f([x])$. Using this terminology we can introduce the rotation number.

**Definition 2.1.** Let $f \in H$ and $F$ be a lift of $f$. Define

$$\tau(F) = \lim_{n \to \infty} \frac{F^n(x) - x}{n}.$$

Then $\tau(f) := [\tau(F)]$ is called the rotation number of $f$.

The definition is justified since $\tau(F)$ exists for all $x \in \mathbb{R}$, is independent of $x$, and we have $[\tau(F_1)] = [\tau(F_2)]$ for all lifts $F_1, F_2$ of $f$ [HK95, Proposition 11.1.1]. Furthermore, one can show that the rotation number is an invariant for conjugacy by orientation-preserving homeomorphisms of the circle, that is, $\tau(f) = \tau(hfh^{-1})$ for all $h \in H$ [HK95, Proposition 11.1.3]. We also note that $\tau(R_\alpha) = \alpha$. Hence, the question of the rotation number being a complete numerical invariant can be restated as asking whether $\tau(f) = \alpha$ implies that $f$ is conjugate to $R_\alpha$. If $f \in H$ is topologically conjugate to an irrational rotation, then the conjugacy satisfies the following uniqueness property.

**Lemma 2.2.** Let $f \in H$ be topologically conjugate to an irrational rotation. Then the conjugating homeomorphism $h \in H$ is unique up to a rotation, that is, if $h_i \circ f = R_{\tau(f)} \circ h_i$ for $i = 1, 2$, then $h_1 \circ h_2^{-1}$ is a rotation.

In particular, there is a unique conjugating homeomorphism $h \in H$ satisfying $h(0) = 0$.

This lemma also shows that the regularity of the conjugacy to the irrational rotation is uniquely determined.
Proof. For the reader’s convenience we provide a proof of this folklore result (see also [He79, Proposition 3.3.3]). We start with the following observation.

Claim: If a circle homeomorphism \( h \) commutes with an irrational rotation \( R_\alpha \), then \( h \) is a rotation.

Proof: From \( h \circ R_\alpha = R_\alpha \circ h \) we also get \( h \circ R_\alpha^n = R_\alpha^n \circ h \) for all \( n \in \mathbb{Z} \), that is,
\[
h(x + n\alpha) = h(x) + n\alpha \quad \text{for all } x \in \mathbb{S}^1 \text{ and } n \in \mathbb{Z}.
\]
Since \( \alpha \) is irrational, the orbit \( \{ n\alpha \mid n \in \mathbb{Z} \} \) lies dense in \( \mathbb{S}^1 \). Hence, we obtain \( h(x + y) = h(x) + y \) for all \( x, y \in \mathbb{S}^1 \) using continuity of \( h \). In particular, taking \( x = 0 \) yields \( h(y) = y + h(0) \) for all \( y \in \mathbb{S}^1 \), that is, \( h \) is a rotation by \( h(0) \).

In the next step, we suppose that \( h_1^{-1} \circ R_{\tau(f)} \circ h_1 = f = h_2^{-1} \circ R_{\tau(f)} \circ h_2 \).

Hence, \( h_1 \circ h_2^{-1} \) commutes with \( R_{\tau(f)} \). Since \( \tau(f) \) is irrational by assumption of the lemma, we can apply the aforementioned Claim and conclude that \( h_1 \circ h_2^{-1} \) is a rotation.

**Lemma 2.3.** Let \( S, T \in \mathcal{H} \) be topologically conjugate to an irrational rotation \( R_\alpha \). Then there is a unique \( g \in \mathcal{H} \) with \( g \circ S \circ g^{-1} = T \) and \( g(0) = 0 \).

Proof. We consider any \( g_1, g_2 \in \mathcal{H} \) satisfying \( g_1(0) = 0 = g_2(0) \) and \( g_1 \circ S \circ g_1^{-1} = T = g_2 \circ S \circ g_2^{-1} \).

Furthermore, let \( h \in \mathcal{H} \) be a conjugacy between \( T \) and \( R_\alpha \). Then we obtain \( h \circ g_1 \circ S \circ (h \circ g_1)^{-1} = h \circ T \circ h^{-1} = R_\alpha = h \circ g_2 \circ S \circ (h \circ g_2)^{-1} \).

By Lemma 2.2 we have \( h \circ g_1 \circ (h \circ g_2)^{-1} = R_\beta \) for some \( \beta \in \mathbb{S}^1 \). Hence, \( g_1 \circ g_2^{-1} = h^{-1} \circ R_\beta \circ h \).

Then the condition \( g_1(0) = 0 = g_2(0) \) implies \( \beta = 0 \). Using \( \beta = 0 \) in equation (2.4) gives \( g_1 = g_2 \).

2.2 Diffeomorphisms of the circle

For \( k \in \mathbb{N} \cup \{ \infty, \omega \} \) let \( \mathcal{H}^k \) be the collection of orientation-preserving \( C^k \)-diffeomorphisms of \( \mathbb{S}^1 \). Here, the case \( k = 0 \) corresponds to \( \mathcal{H}^0 = \mathcal{H} \). Furthermore, for \( 0 < \beta < 1 \) we denote by \( \mathcal{H}^{k+\beta} \) the collection of orientation-preserving \( C^k \)-diffeomorphisms of \( \mathbb{S}^1 \) with \( \beta \)-Hölder continuous \( k \)-th derivative.
Definition 2.5. Let $k \in \mathbb{N}$. We define the $C^k$ norm on $C^k$ functions $f$ of $S^1$ as

$$\|f\|_k = \max_{0 \leq i \leq k} \max_{x \in S^1} |D^i f(x)|.$$ 

For $C^k$ diffeomorphisms $f, g$ we also define

$$\|\|f\|_k = \max\{\|f\|_k, \|f^{-1}\|_k\},$$

$$d_k(f, g) = \max\{\|f - g\|_k, \|f^{-1} - g^{-1}\|_k\}.$$ 

Obviously, $d_k$ describes a metric on $\text{Diff}^k(S^1)$ measuring the distance between the diffeomorphisms as well as their inverses in the $C^k$-topology. As in the case of a general compact manifold the following definition connects to it.

Definition 2.6. 1. A sequence of $C^\infty$ diffeomorphisms is called convergent in $\text{Diff}^\infty$ if it converges in $\text{Diff}^k(S^1)$ for every $k \in \mathbb{N}$.

2. On $\text{Diff}^\infty(S^1)$ we declare the following metric

$$d_\infty(f, g) = \sum_{k=1}^\infty \frac{d_k(f, g)}{2^k \cdot (1 + d_k(f, g))}.$$ 

The space $\mathcal{H}_\infty$ equipped with the metric $d_\infty$ is complete and separable. Hence, $\mathcal{H}_\infty$ is a Polish space. We refer to [He79] chapter I.2] for a thorough description of spaces of circle diffeomorphisms.

2.3 Some tools from Descriptive Set Theory

The main tool is the idea of a reduction for equivalence relations. See [Fopp] and [Ga09] for further information.

Definition 2.7. Let $X$ and $Y$ be Polish spaces and $E \subseteq X \times X$, $F \subseteq Y \times Y$ be equivalence relations.

A function $f : X \to Y$ reduces $E$ to $F$

if and only if

for all $x_1, x_2 \in X$: $x_1 Ex_2$ if and only if $f(x_1) Ff(x_2)$.

Such a function $f$ is called a Borel (respectively, continuous) reduction if $f$ is a Borel (respectively, continuous) function.

We use the notation $E \preceq_B F$ (respectively, $E \preceq_C F$).

$E$ being reducible to $F$ can be interpreted as saying that $F$ is at least as complicated as $E$.

In this work, we want to exclude the existence of complete numerical invariants. Thus, we consider the equality equivalence relation $=_Y \subseteq Y \times Y$ on a Polish space $Y$. Since for every Polish space $Y$ there is a Borel injection $q : Y \to \mathbb{R} \setminus \mathbb{Q}$, we can extend a Borel reduction $f$ of any equivalence relation $E \subseteq X \times X$ to $(Y, =_Y)$ to a Borel reduction $g \circ f$ of $(X, E)$ to $(\mathbb{R}, =_R)$. Thus, we can assume that Borel reductions to any $=_Y$ can be changed to Borel reductions to equality on the real numbers.

In the next step, we define the equivalence relation $E_0$ which is an important tool to exclude the existence of complete numerical invariants.
Definition 2.8. Let $E_0$ be the equivalence relation on $\{0,1\}^\mathbb{N}$ defined by setting
\[(a_n)_{n \in \mathbb{N}} E_0 (b_n)_{n \in \mathbb{N}} \text{ if and only if } \exists N \in \mathbb{N} \text{ such that } a_m = b_m \text{ for all } m > N.\]

The significance of equivalence relation $E_0$ for excluding complete numerical invariants comes from the so-called Harrington-Kechris-Louveau dichotomy (sometimes also referred to as Glimm-Effros dichotomy). It states that for any Borel equivalence relation $E$ on a Polish space $X$ exactly one of the following alternatives holds: either $E \preceq_B =_B \mathbb{R}$ or $E_0 \preceq_B E$.

To exclude the existence of complete numerical invariants, we use the following direction from the Harrington-Kechris-Louveau dichotomy (see [Fopp, Theorem 36] or [Ga09, Proposition 6.1.7]).

Theorem 2.9. Suppose that $E$ is an equivalence relation on an uncountable Polish space $X$ and $E_0 \not\preceq_B E$. Then $E \not\preceq_B =_B \mathbb{R}$.

Proof. For the reader’s convenience we include Feldman’s argument from [Fe74].

Since $E_0 \not\preceq_B E$, there is a Borel map $\psi : \{0,1\}^\mathbb{N} \to X$ such that for $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}} \in \{0,1\}^\mathbb{N}$,
\[\psi ((a_n)_{n \in \mathbb{N}}) E \psi ((b_n)_{n \in \mathbb{N}}) \text{ if and only if } a_n = b_n \text{ for all but at most finitely many } n. \tag{2.10}\]

We want to show that there does not exist a complete numerical Borel invariant for $E$, that is, there does not exist a Borel function $\varphi : X \to \mathbb{R}$ such that $x Ey$ if and only if $\varphi(x) = \varphi(y)$.

Suppose such a Borel function $\varphi$ exists. Then for $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}} \in \{0,1\}^\mathbb{N}$, $\varphi \circ \psi ((a_n)_{n \in \mathbb{N}}) = \varphi \circ \psi ((b_n)_{n \in \mathbb{N}})$ if and only if $a_n = b_n$ for all but at most finitely many $n$. Thus, for any Borel set $B \subseteq \mathbb{R}$, $\varphi \circ \psi^{-1} (B)$ is invariant under maps that consist of a permutation of finitely many terms in the elements of $\varphi \circ \psi^{-1} (B)$. If we endow $\{0,1\}^\mathbb{N}$ with the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli measure, then by the Hewitt-Savage zero-one law, any such set $(\varphi \circ \psi^{-1} (B))$ has measure 0 or 1. Therefore $\varphi \circ \psi$ is almost everywhere constant on $\{0,1\}^\mathbb{N}$. Hence, there are uncountably many elements of $\{0,1\}^\mathbb{N}$ that are mapped to the same element of $\mathbb{R}$. This contradicts (2.10). \hfill \Box

3 Strategy of proof

Let $\mathcal{H}_\alpha^\infty$ be the collection of orientation-preserving $C^\infty$-diffeomorphisms with rotation number $\alpha \in \mathbb{S}^1$. As we show below, our Main Theorem \[\] follows from the subsequent stronger Proposition which is proved in Section 5.

Proposition 3.1. Let $\alpha \in \mathbb{S}^1$ be a Liouville number. There is a continuous one-to-one map
\[\Psi : \{0,1\}^\mathbb{N} \to \mathcal{H}_\alpha^\infty\]
such that for any two sequences $a = (a_n)_{n \in \mathbb{N}}$ and $b = (b_n)_{n \in \mathbb{N}}$ the following properties hold:

(R1) If there is $N \in \mathbb{N}$ such that $a_n = b_n$ for every $n \geq N$, then the $C^\infty$-diffeomorphisms $\Psi(a)$ and $\Psi(b)$ are $C^\infty$-conjugate.

(R2) If there are infinitely many $n \in \mathbb{N}$ with $a_n \neq b_n$, then the $C^\infty$-diffeomorphisms $\Psi(a)$ and $\Psi(b)$ are not Hölder-conjugate.

Using the notions from descriptive set theory in Section 2, Proposition 3.1 implies the following statement.
Corollary 3.2. Let $\alpha \in S^1$ be a Liouville number and $D$ be the collection of orientation-preserving circle homeomorphisms with regularity $(D)$, where $(D)$ could be any degree of regularity from Hölder to $C^\infty$. Then there is a continuous reduction from $E_0$ to the $D$-conjugacy relation of diffeomorphisms in $H_\alpha^\infty$.

Proof. If $aE_0b$, then the $C^\infty$-diffeomorphisms $\Psi(a)$ and $\Psi(b)$ are $C^\infty$-conjugated by part (1) of Proposition 3.1. Hence, they are $D$-conjugate.

If $a$ and $b$ are not equivalent in $E_0$, then there are infinitely many $n \in \mathbb{N}$ with $a_n \neq b_n$. Thus, the $C^\infty$-diffeomorphisms $\Psi(a)$ and $\Psi(b)$ are not Hölder-conjugate by part (2) of Proposition 3.1 and, hence, they are not $D$-conjugate.

Altogether, the $C^\infty$-diffeomorphisms $\Psi(a)$ and $\Psi(b)$ are $D$-conjugate if and only if $aE_0b$. This shows that the continuous map $\Psi : \{0, 1\}^\mathbb{N} \to H_\alpha^\infty$ is a reduction from $E_0$ to the $D$-conjugacy relation of orientation-preserving $C^\infty$-diffeomorphisms of the circle.

Then we apply Theorem 2.9 to conclude the following result which in particular implies our Main Theorem A.

Theorem 3.3. Let $\alpha \in S^1$ be a Liouville number and $D$ be the collection of orientation-preserving circle homeomorphisms with regularity $(D)$, where $(D)$ could be any degree of regularity from Hölder to $C^\infty$. Then there is no complete numerical Borel invariant for the $D$-conjugacy relation of diffeomorphisms in $H_\alpha^\infty$.

4 Construction of the reduction

Let $\alpha \in S^1$ be a Liouville number. For every sequence $a = (a_n)_{n \in \mathbb{N}}$ we construct a diffeomorphism $T_a \in H_\alpha^\infty$ by an inductive construction process as the limit of a sequence $(T_a, n)_{n \in \mathbb{N}_0}$ of $C^\infty$-diffeomorphisms of the circle defined by

$$T_{a, n} = H_{a, n} \circ R_{\alpha n + 1} \circ H_{a, n}^{-1} \text{ for } n \in \mathbb{N}_0. \quad (4.1)$$

Here, we take $H_{a, 0} = \text{id}$ and for $n \in \mathbb{N}$ we successively construct rational numbers $\alpha_n = \frac{b_n}{q_n}$ as well as conjugation maps $H_{a, n} = H_{a, n-1} \circ h_{a, n}$ with a $C^\infty$-diffeomorphism $h_{a, n}$ satisfying

$$h_{a, n} \circ R_{1/q_n} = R_{1/q_n} \circ h_{a, n}. \quad (4.2)$$

In Subsection 4.1 we present the construction of our conjugation map $h_{a, n}$ which depends on entry $a_n$ in the sequence $a = (a_n)_{n \in \mathbb{N}}$. This map will allow explicit norm estimates that we use in Subsection 4.2 to prove convergence of the sequence $(T_{a, n})_{n \in \mathbb{N}}$ in the space $H_\alpha^\infty$ of orientation-preserving $C^\infty$-diffeomorphisms with the prescribed rotation number $\alpha$.

4.1 Construction of conjugation maps

We use a $C^\infty$-function $\psi : \mathbb{R} \to [0, 1]$ satisfying $\psi ((-\infty, 0]) = 0$, $\psi ([1, \infty)) = 1$ and $\psi$ is strictly monotone increasing on $[0, 1]$. For any $t \in (0, 1/4)$ we define a strictly increasing $C^\infty$ diffeomorphism
\( \hat{h}_t : [0, 1] \to [0, 1] \) as follows: 

\[
\hat{h}_t(x) = \begin{cases} 
  x & \text{if } x \in [0, t], \\
  (1 - \psi(t^{-1}(x - t))) \cdot x + \psi(t^{-1}(x - t)) \cdot (t \cdot (x - \frac{1}{2}) + 3t) & \text{if } x \in [t, 2t], \\
  (1 - \psi(t^{-2}(x - \frac{1-t+8t^2}{2}))) \cdot (t \cdot (x - \frac{1}{4}) + 3t) + \psi(t^{-2}(x - \frac{1-t+8t^2}{2})) \cdot (t^{-1} (x - \frac{1}{2}) + \frac{1}{2}) & \text{if } x \in \left[2t, \frac{1-t+8t^2}{2}\right], \\
  (1 - \psi(t^{-2}(x - \frac{1+t-10t^2}{2}))) \cdot (t^{-1} (x - \frac{1}{2}) + \frac{1}{2}) + \psi(t^{-2}(x - \frac{1+t-10t^2}{2})) \cdot (t \cdot (x - \frac{3}{4}) + 1 - 3t) & \text{if } x \in \left[\frac{1-t+10t^2}{2}, \frac{1+t-8t^2}{2}\right], \\
  t \cdot (x - \frac{3}{4}) + 1 - 3t & \text{if } x \in \left[\frac{1+t-8t^2}{2}, 1 - 2t\right], \\
  (1 - \psi(t^{-1}(x - 1 + 2t))) \cdot (t \cdot (x - \frac{3}{4}) + 1 - 3t) + \psi(t^{-1}(x - 1 + 2t))x & \text{if } x \in [1 - 2t, 1 - t], \\
  (1 - \psi(t^{-1}(x - 1 + 2t))) \cdot (t \cdot (x - \frac{3}{4}) + 1 - 3t) + \psi(t^{-1}(x - 1 + 2t))x & \text{if } x \in [1 - t, 1]. 
\end{cases}
\]

In particular, \( \hat{h}_t \) coincides with the identity in the neighborhood of the boundary. See Figure 1 for a visualisation of such a map. We use this explicit form of the map \( \hat{h}_t \) to obtain norm estimates.
as a power of $q_n$ in (4.6). We also note that $\hat{h}_t^N$ has slope $t^{-2}$ on

$$\hat{I}_t := \left[ \frac{1}{2} - \frac{t^2}{4}, \frac{1}{2} + \frac{t^2}{4} \right].$$

Since $\hat{h}_t$ is a strictly increasing $C^\infty$ function, it has a strictly increasing smooth inverse function satisfying

$$\hat{h}_t^{-1}(x) = \begin{cases} x & \text{if } x \in [0,t], \\
t^{-1} \cdot (x-3t) + \frac{t^2}{4} & \text{if } x \in \left[3t - \frac{t^2}{4}, 3t + \frac{t^2}{4} \right], \\
t \cdot (x - \frac{1}{2}) + \frac{t^2}{4} & \text{if } x \in [5t, 1-5t], \\
t^{-1} \cdot (x - 1 + 3t) + \frac{3}{4} & \text{if } x \in \left[1 - 3t - \frac{t^2}{4}, 1 - 3t + \frac{t^2}{4} \right], \\
x & \text{if } x \in [1-t, 1]. \\
\end{cases}$$

Hence, $\hat{h}_t^{-2}$ has slope $t^{-2}$ on

$$\hat{J}_t := \left[ 3t - \frac{t^2}{4} + 3t^2 - \frac{t^2}{8}, 3t - \frac{t^2}{4} + 3t^2 + \frac{t^2}{8} \right] \cup \left[ 1 - 3t + \frac{t^2}{4} - 3t^2 - \frac{t^2}{8}, 1 - 3t + \frac{t}{4} - 3t^2 + \frac{t^2}{8} \right].$$

For any $n \in \mathbb{N}$ we take

$$t_n = q_n^{1-2n}$$

and define $\hat{h}_{n} = \hat{h}_{t_n}$. Let $h_{q_n}$ be the lift of $\hat{h}_n$ by the cyclic $q_n$-fold covering map $\pi_{q_n}$ such that $\text{Fix}(h_{q_n}) \neq \emptyset$. Finally, we define

$$h_{a,n} = \begin{cases} h_{q_n} & \text{if } a_n = 0, \\
h_{q_n}^{-1} & \text{if } a_n = 1. \\
\end{cases}$$

Clearly, the commutativity condition (4.2) is satisfied. We also observe for any $r \in \mathbb{N}$ that

$$\|h_{a,n}\|_r \leq C_{n,r} \cdot q_n^{N(n,r)}$$

with some positive integer $N(n,r)$ and some constant $C_{n,r}$ that are independent of $q_n$ and $a \in \{0,1\}^\mathbb{N}$. With the aid of this estimate and a generalized chain rule we are able to prove an estimate on the norms of $H_{a,n} = H_{a,n-1} \circ h_{a,n}$.

**Lemma 4.7.** For every $k \in \mathbb{N}$ we have

$$\|H_{a,n}\|_k \leq C(n, H_{a,n-1}, k) \cdot q_n^{k \cdot N(n,k)},$$

where $C(n, H_{a,n-1}, k)$ is a constant depending solely on $n$, $H_{a,n-1}$, and $k$. Since $H_{a,n-1}$ is independent of $q_n$ in particular, the same is true for $C(n, H_{a,n-1}, k)$.

**Proof.** By the Faà di Bruno’s formula we get

$$\|D^k (H_{a,n-1} \circ h_{a,n})\| \leq C(k) \cdot \|H_{a,n-1}\|_k \cdot \|h_{a,n}\|_k^k$$

and

$$\|D^k (h_{a,n}^{-1} \circ H_{a,n-1}^{-1})\| \leq C(k) \cdot \|h_{a,n}^{-1}\|_k \cdot \|H_{a,n-1}^{-1}\|_k^k$$

with a constant $C(k)$ solely depending on $k$. Using (4.6) these estimates imply

$$\|H_{a,n}\|_k \leq C(k) \cdot \|H_{a,n-1}\|_k^k \cdot \|h_{a,n}\|_k^k \leq C(n, H_{a,n-1}, k) \cdot q_n^{k \cdot N(n,k)},$$

where $C(n, H_{a,n-1}, k)$ is a constant depending solely on $n$, $H_{a,n-1}$, and $k$. \hfill \square
4.2 Convergence of the sequence \((T_{a,n})_{n \in \mathbb{N}}\)

We start our proof of convergence of the sequence \((T_{a,n})_{n \in \mathbb{N}}\) in \(\mathcal{H}_{\alpha}^\infty\) by stating the following lemma based upon the mean value theorem and the chain rule.

**Lemma 4.8** ([FS05], Lemma 5.6). Let \(k \in \mathbb{N}_0\) and \(h\) be a \(C^\infty\)-diffeomorphism. Then we get for every \(\alpha, \beta \in \mathbb{R}\):

\[
d_k \left( h \circ R_\alpha \circ h^{-1}, h \circ R_\beta \circ h^{-1} \right) \leq C_k \cdot \|h\|_{k+1} \cdot |\alpha - \beta|,
\]

where the constant \(C_k\) depends solely on \(k\). In particular \(C_0 = 1\).

Since \(\alpha\) is a Liouville number, the proof of convergence of \((T_{a,n})_{n \in \mathbb{N}}\) in \(\mathcal{H}_{\alpha}^\infty\) is similar to the proof of the general criterion in [FS05, Lemma 5.7]. We additionally ensure that the same sequence \((\alpha_n)_{n \in \mathbb{N}}\) yields convergence for all \(a = (a_n)_{n \in \mathbb{N}} \in \{0,1\}^\mathbb{N}\). This is used in the third part of the lemma which will guarantee continuity of our reduction \(\Psi\).

**Lemma 4.9.** There is a sequence \((\alpha_n)_{n \in \mathbb{N}}\) of rational numbers \(\alpha_n = \frac{p_n}{q_n}\) with

\[
q_{n+1} \geq q_n^{2n} \quad \text{for every } n \in \mathbb{N}
\]

such that the following properties hold:

1. For every sequence \(a \in \{0,1\}^\mathbb{N}\) the sequence of diffeomorphisms \((T_{a,n})_{n \in \mathbb{N}}\) defined by \eqref{eq:4.1} converges in the \(\text{Diff}^\infty\)-topology to a smooth diffeomorphism \(T_a\).

2. Also the sequence of diffeomorphisms \(T_{a,n} = H_{a,n} \circ R_{\alpha_n} \circ H_{a,n}^{-1} \in \mathcal{H}_{\alpha}^\infty\) converges to \(T_a\) in the \(\text{Diff}^\infty\)-topology. Hence, \(T_a \in \mathcal{H}_{\alpha}^\infty\).

3. For every \(\varepsilon > 0\) there is \(N \in \mathbb{N}\) such that for all sequences \(a = (a_n)_{n \in \mathbb{N}}, b = (b_n)_{n \in \mathbb{N}} \in \{0,1\}^\mathbb{N}\) with \(a_n = b_n\) for all \(n \leq N\) we have

\[
d_\infty(T_a, T_b) < \varepsilon.
\]

**Proof.** 1. According to condition \eqref{eq:4.2} of our construction we have \(h_{a,n} \circ R_{\alpha_n} = R_{\alpha_n} \circ h_{a,n}\) and, hence,

\[
T_{a,n-1} = H_{a,n-1} \circ R_{\alpha_n} \circ H_{a,n-1}^{-1} = H_{a,n-1} \circ R_{\alpha_n} \circ h_{a,n} \circ h_{a,n}^{-1} \circ H_{a,n-1}^{-1} = H_{a,n} \circ R_{\alpha_n} \circ H_{a,n}^{-1}
\]

for every sequence \(a \in \{0,1\}^\mathbb{N}\). Applying Lemma 4.8 we obtain for every \(k, n \in \mathbb{N}\):

\[
d_k (T_{a,n}, T_{a,n-1}) = d_k \left( H_{a,n} \circ R_{\alpha_n+1} \circ T_{a,n}, H_{a,n} \circ R_{\alpha_n} \circ H_{a,n}^{-1} \right)
\]

\[
\leq C_k \cdot \|H_{a,n}\|_{k+1} \cdot |\alpha_{n+1} - \alpha_n|.
\]

We assume \(|\alpha - \alpha_n| \xrightarrow{n \to \infty} 0\) monotonically. Using the triangle inequality we obtain \(|\alpha_{n+1} - \alpha_n| \leq |\alpha_{n+1} - \alpha| + |\alpha - \alpha_n| \leq 2 \cdot |\alpha - \alpha_n|\) and therefore equation \eqref{eq:4.11} becomes:

\[
d_k (T_{a,n}, T_{a,n-1}) \leq C_k \cdot \|H_{a,n}\|_{k+1} \cdot 2 \cdot |\alpha - \alpha_n|.
\]
To estimate the norm of $H_{a,n}$ we use Lemma 4.7. Here, we note that $H_{a,n-1}$ depends on the sequence entries $a_1, \ldots, a_{n-1}$ only. Hence,

$$\hat{C}_n := \max_{a \in \{0,1\}^N} C(n, H_{a,n-1}, n + 1)$$

is well-defined, where $C(n, H_{a,n-1}, k)$ are the constants from Lemma 4.7. Then Lemma 4.7 yields

$$\|H_{a,n}\|_{n+1} \leq \hat{C}_n \cdot q_n^{(n+1)N(n,n+1)}$$

for every sequence $a \in \{0,1\}^N$.

Since $\alpha$ is a Liouville number, there are $q_n > q_{n-1}^{2(n-1)}$ and $p_n \in \mathbb{N}$ such that

$$\left| \alpha - \frac{p_n}{q_n} \right| < \min \left( |\alpha - \alpha_{n-1}|, \frac{1}{2 \cdot n^2 \cdot C_n \cdot \left( \hat{C}_n \cdot q_n^{(n+1)N(n,n+1)} \right)^{n+1}} \right).$$

In particular, this implies condition (4.10) and

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{2 \cdot n^2 \cdot C_n \cdot \|H_{a,n}\|_{n+1}^{n+1}} \tag{4.12}$$

for every sequence $a \in \{0,1\}^N$. It follows for every sequence $a \in \{0,1\}^N$ and every $k \leq n$ that

$$d_k(T_{a,n}, T_{a,n-1}) \leq d_n(T_{a,n}, T_{a,n-1}) \leq C_n \cdot \|H_{a,n}\|_{n+1}^{n+1} \cdot 2 \cdot |\alpha - \alpha_n| \leq C_n \cdot \|H_{a,n}\|_{n+1}^{n+1} \cdot 2 \cdot \frac{1}{2 \cdot n^2 \cdot C_n \cdot \|H_{a,n}\|_{n+1}^{n+1}} \leq \frac{1}{n^2}. \tag{4.13}$$

In the next step we show, that for every sequence $a \in \{0,1\}^N$ and for arbitrary $k \in \mathbb{N}$ our sequence $(T_{a,n})_{n \in \mathbb{N}}$ is a Cauchy sequence in $\text{Diff}^k(S^1)$, that is, $\lim_{n,m \to \infty} d_k(T_{a,n}, T_{a,m}) = 0$. For this purpose, we calculate

$$\lim_{n \to \infty} d_k(T_{a,n}, T_{a,m}) \leq \lim_{n \to \infty} \sum_{i=m+1}^{n} d_k(T_{a,i}, T_{a,i-1}) = \sum_{i=m+1}^{\infty} d_k(T_{a,i}, T_{a,i-1}). \tag{4.14}$$

We consider the limit process $m \to \infty$, i.e. we can assume $k \leq m$, and obtain from equations (4.13) and (4.14) that

$$\lim_{n,m \to \infty} d_k(T_{a,n}, T_{a,m}) \leq \lim_{m \to \infty} \sum_{i=m+1}^{\infty} \frac{1}{i^2} = 0.$$

Since $\text{Diff}^k(S^1)$ is complete, the sequence $(T_{a,n})_{n \in \mathbb{N}}$ converges consequently in $\text{Diff}^k(S^1)$ for every $k \in \mathbb{N}$ and for every sequence $a \in \{0,1\}^N$. Thus, the sequence converges in $\text{Diff}^\infty(S^1)$ by definition.
Proof of Proposition 3.1

2. To show \( \hat{T}_{a,n} \to T_a \) in \( \text{Diff}^\infty(S^1) \) we compute with the aid of Lemma 4.8 and equation (4.12) that for every \( n \in \mathbb{N} \) and \( k \leq n \) we have

\[
d_k \left( T_{a,n}, \hat{T}_{a,n} \right) \leq d_n \left( H_{a,n} \circ R_{\alpha_{n+1}} \circ H_{a,n}^{-1}, H_{a,n} \circ R_\alpha \circ H_{a,n}^{-1} \right)
\]
\[
\leq C_n \cdot \|H_{a,n}\|^{n+1} \cdot |\alpha_{n+1} - \alpha| \leq C_n \cdot \|H_{a,n}\|^{n+1} \cdot |\alpha_n - \alpha| \leq \frac{1}{n^2}.
\]

This yields \( \hat{T}_{a,n} \to T_a \) in \( \text{Diff}^k(S^1) \) for any \( k \in \mathbb{N} \) and \( \lim_{n \to \infty} d_\infty \left( \hat{T}_{a,n}, T_a \right) = 0 \) as asserted.

Furthermore, we recall that the rotation number \( \tau : \mathcal{H} \to \mathbb{S}^1 \) is a continuous map in the \( C^0 \)-topology [HK95, Proposition 11.1.6]. Since \( \tau(\hat{T}_{a,n}) = \alpha \) for all \( n \in \mathbb{N} \), we also obtain \( \tau(T_a) = \alpha \).

3. Since \( T_{a,n} \) converges to \( T_a \) in \( \text{Diff}^\infty \) by the first part of the lemma, there is \( N_1 \in \mathbb{N} \) such that \( d_\infty(T_{a,n}, T_a) < \frac{\varepsilon}{2} \) for all \( n \geq N_1 \). Analogously, there is \( N_2 \in \mathbb{N} \) such that \( d_\infty(T_{b,n}, T_b) < \frac{\varepsilon}{2} \) for all \( n \geq N_2 \). Let \( N := \max(N_1, N_2) \) and suppose that \( a_n = b_n \) for all \( n \leq N \). In particular, this implies \( T_{a,n} = T_{b,n} \) for all \( n \leq N \). Altogether we have

\[
d_\infty(T_a, T_b) \leq d_\infty(T_{a,n}, T_{a,n}) + d_\infty(T_{a,n}, T_{b,n}) + d_\infty(T_{b,n}, T_b)
\]
\[
= d_\infty(T_{a,n}, T_{a,n}) + d_\infty(T_{b,n}, T_b)
\]
\[
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

\( \square \)

5 Proof of Proposition 3.1

In combination with Lemma 2.2, the following result shows that the sequence \( (H_{a,n})_{n \in \mathbb{N}} \) converges to the unique conjugating homeomorphism \( H_a \) between the rotation \( R_\alpha \) and \( T_a \) with \( H_a(0) = 0 \).

Lemma 5.1. For every \( a \in \{0,1\}^\mathbb{N} \) the sequence \( (H_{a,n})_{n \in \mathbb{N}} \) converges in \( d_0 \) to a homeomorphism \( H_a \) with \( H_a(0) = 0 \).

Proof. Since \( h_{q_n} \) is constructed as a lift of the homeomorphism \( h_n \) by the \( q_n \)-fold cyclic covering map, we obtain

\[
\|h_{q_n} - \text{id}\|_0 \leq q_n^{-1} \quad \text{as well as} \quad \|h_{q_n}^{-1} - \text{id}\|_0 \leq q_n^{-1}.
\]

We use this to estimate

\[
\|H_{a,n}^{-1} - H_{a,n-1}^{-1}\|_0 = \|(h_{a,n}^{-1} - \text{id}) \circ H_{a,n-1}^{-1}\|_0 = \|h_{a,n}^{-1} - \text{id}\|_0 \leq q_n^{-1}.
\]

Then we obtain

\[
\|H_{a,n+k}^{-1} - H_{a,n-1}^{-1}\|_0 \leq \sum_{\ell=0}^k \|H_{a,n+\ell}^{-1} - H_{a,n+\ell-1}^{-1}\|_0 \leq \sum_{\ell=0}^k q_n^{-1}.
\]
Proof of Proposition 3.1

Since $\sum_{n=1}^{\infty} q_n^{-1} < \infty$ by Lemma 4.9, $(H_{a,n}^{\alpha})_{n \in \mathbb{N}}$ is a Cauchy sequence. This shows the uniform convergence of $(H_{a,n}^{\alpha})_{n \in \mathbb{N}}$ to a continuous map $H_{a}^{-1} : S^1 \to S^1$. Additionally, $H_{a}^{-1}$ is monotone as a uniform limit of homeomorphisms. Due to the second part of Lemma 4.9 the sequence of $C^\infty$-diffeomorphisms $T_{a,n} = H_{a,n} \circ R_a \circ H_{a,n}^{-1}$ converges to a diffeomorphism $T_a$ in the $C^\infty$-topology. Hereby, we obtain $H_{a}^{-1} \circ T_a = R_a \circ H_{a}^{-1}$. Altogether, we conclude that $H_{a}^{-1}$ is a homeomorphism.

In the next step, we observe that

$$\|H_{a,n} - H_{a}\|_0 = \|\text{id} - H_{a} \circ H_{a,n}^{-1}\|_0 = \|H_{a} \circ (H_{a}^{-1} - H_{a,n}^{-1})\|_0 \to 0 \text{ as } n \to \infty$$

by uniform continuity of $H_a$. Altogether, $(H_{a,n})_{n \in \mathbb{N}}$ converges in $d_0$ to the homeomorphism $H_{a}$.

For all $n \in \mathbb{N}$ we have $h_{a,n}(0) = 0$ and, hence, $H_{a,n}(0) = 0$. Thus, we also get $H_{a}(0) = 0$. 

We are now ready to conclude the proof of the main proposition.

Proof of Proposition 3.1

We define the map $\Psi$ by setting $\Psi(a) = T_a$, where $T_a \in \mathcal{H}^\infty$ is the limit of the sequence of diffeomorphisms $(T_{a,n})_{n \in \mathbb{N}}$ constructed by (4.1). This limit exists by the first part of Lemma 4.9. The second part of Lemma 4.9 and the convergence of $(H_{a,n})_{n \in \mathbb{N}}$ to a homeomorphism $H_{a}$ yield that $T_a$ has rotation number $\alpha$. Hence, $\Psi(a) = T_a \in \mathcal{H}^\infty$ for all $a \in \{0,1\}^\mathbb{N}$. Furthermore, the third part of Lemma 4.9 proves the continuity of $\Psi$.

Finally, we check that $\Psi$ satisfies properties (R1) and (R2). For this purpose, we examine the regularity of conjugacy between $T_a$ and $T_b$. Here, we note as a consequence of Lemma 2.3 and Lemma 5.1 that the sequence $(H_{b,n} \circ H_{a,n}^{-1})_{n \in \mathbb{N}}$ converges to the unique conjugating homeomorphism $G_{a,b} := H_b \circ H_{a}^{-1}$ between $T_a$ and $T_b$ with $G_{a,b}(0) = 0$.

(R1) Let $a = (a_n)_{n \in \mathbb{N}}$ and $b = (b_n)_{n \in \mathbb{N}}$ be sequences such that there is $N \in \mathbb{N}$ with $a_n = b_n$ for every $n \geq N$. Then

$$H_{b,n} \circ H_{a,n}^{-1} = H_{b,N} \circ h_{b,N+1} \circ \cdots \circ h_{b,n} \circ h_{a,n+1}^{-1} \circ \cdots \circ h_{a,N+1}^{-1} \circ H_{a,N}^{-1} = H_{b,N} H_{a,N}^{-1}$$

for every $n \geq N$. Hence,

$$G_{a,b} = H_b \circ H_{a}^{-1} = H_{b,N} \circ H_{a,N}^{-1} \in \mathcal{H}^\infty$$

is the unique conjugating homeomorphism between $T_a$ and $T_b$ satisfying $G_{a,b}(0) = 0$, that is, $T_a$ and $T_b$ are $C^\infty$-conjugate.

(R2) Let $a = (a_n)_{n \in \mathbb{N}}$ and $b = (b_n)_{n \in \mathbb{N}}$ be sequences such that there are infinitely many $n \in \mathbb{N}$ with $a_n \neq b_n$. We denote the set of these indices with $a_n \neq b_n$ by $\mathcal{N}$. Furthermore, we recall the definition of intervals $I_t$ and $J_t$ from equations (4.3) and (4.4). Using them, we introduce the following sets $K_n$ for each $n \in \mathbb{N}$:

- If $a_n = 1$, let $k_n,1 \in \mathbb{N}$ be the smallest integer and $k_n,2 \in \mathbb{N}$ be the largest integer such that

$$\left[ k_n,1 \frac{k_n,2}{q_{n+1}}, \frac{k_n,2}{q_{n+1}} \right] \subseteq \left[ \frac{1}{2q_n} - \frac{t_n}{4q_n}, \frac{1}{2q_n} + \frac{t_n}{4q_n} \right].$$

Then we define

$$K_n := \bigcup_{\ell=0}^{q_n-1} \left[ \frac{\ell}{q_n} + \frac{k_n,1}{q_{n+1}}, \frac{\ell}{q_n} + \frac{k_n,2}{q_{n+1}} \right]$$

in order to have $h_{a,n}(K_n) \subseteq \pi_{q_n}^{-1}(I_{t_n})$. 


We are now ready to show that an affine transformation of slope $\pi$ of slope $h$ have $\pi$ $\pm$ $q^i$ $\pm$ $q^i$ $\pm$ $q^i$ $\pm$ $q^i$. These choices guarantee that $h(b_n \circ h_{a,n}^{-1})|K_n$ has slope $t_n^{-2}$ for $n \in \mathcal{N}$. We are now ready to show that $G_{a,b}$ is not $d$-Hölder continuous for any $d \in (0,1)$.

For every $i \in \mathbb{N}$ and any component $\tilde{K}_i$ of $K_i$ there is a component $\tilde{K}_{i+1}$ of $K_{i+1}$ such that $h(a_i+1) \subseteq \tilde{K}_i$. This proves the existence of a component $\tilde{K}_n$ of $K_n$ such that $H_{a,n}|\tilde{K}_n$ is an affine transformation of slope $\prod_{i=1}^n t_i$ and $h(a_n \circ h_{a,n}^{-1}|K_n)$ is an affine transformation of slope $\prod_{i=1}^n t_i^{-2}$.

In the following, we denote $\tilde{K}_n = [x',y']$. In particular, we have $|y' - x'| \geq 8^{-1}t_n q_n^{-1}$. Moreover, we define $x,y \in \mathbb{S}^1$ by $x = H_{a,n}(x')$ and $y = H_{a,n}(y')$. We also introduce the notation $G_{a,b}^{(n+1)} = H_{b,n}^{-1} \circ G_{a,b} \circ H_{a,n}$. We observe that $G_{a,b}^{(n+1)}(x) = \lim_{m \to \infty} h_{b,n+1} \circ \cdots \circ h_{b,n+m} \circ h_{a,n+1}^{-1} \circ \cdots \circ h_{a,n}^{-1}$. Since the maps $h_i$ are $q_i^{-1}$-cyclic, we have $x',y' \in \text{Fix}(h_i)$ for all $i > n$. Thus, we obtain

$$|G_{a,b}(x) - G_{a,b}(y)| = \left| H_{b,n} \left( G_{a,b}^{(n+1)}(x') \right) - H_{b,n} \left( G_{a,b}^{(n+1)}(y') \right) \right| = |H_{b,n}(x') - H_{b,n}(y')|$$

$$= \prod_{i \in \mathcal{N}, i \leq n} t_i^{-2} \cdot |x - y|.$$ 

Suppose $n \in \mathcal{N}$. Then we obtain the following estimate using the properties of the domain $K_n$ as well as equation (4.5):

$$\frac{|G_{a,b}(x) - G_{a,b}(y)|}{|x - y|^2} = \prod_{i \in \mathcal{N}, i \leq n} t_i^{-2} \cdot |x - y|^{1 - \frac{1}{n}}$$

$$= \prod_{i \in \mathcal{N}, i \leq n} t_i^{-2} \cdot \left( \prod_{i=1}^n t_i \right)^{1 - \frac{1}{n}} \cdot |x' - y'|^{1 - \frac{1}{n}}$$

$$\geq \prod_{i \in \mathcal{N}, i \leq n} t_i^{-2} \cdot \left( \prod_{i=1}^n t_i \right)^{1 - \frac{1}{n}} \cdot (8^{-1}t_n q_n^{-1})^{1 - \frac{1}{n}}.$$
It is a well-known fact that every circle homeomorphism $T$ is uniquely ergodic (see [HK95, Theorem 11.2.9]). We denote the unique invariant probability measure by $\mu_T$. Hence, any $T$-invariant Borel subset is either $\mu_T$-null or $\mu_T$-conull. Together with the uniqueness of $\mu_T$, this implies that either $\mu_T$ is equivalent to $\mu$ or singular to $\mu$.

Moreover, $H^2$ shows that $H$ is ergodic with respect to Lebesgue measure $m$. If $\mu_T$ is equivalent to $\mu$, then the unique conjugating homeomorphism $H$ with $H(0) = 0$ maps any Lebesgue null set to a null set. In this case, the conjugacy is called absolutely continuous. If $\mu_T$ is singular to $\mu$, then conjugacy $H$ maps some Lebesgue null set to a conull set. In this case, the conjugacy is called singular.

In this section, we show that both cases can be realized in the setting of our Main Theorem. We start by showing that the conjugation maps as defined in Section 4.1 result in a singular conjugacy $H_a$. Here, we follow an approach similar to [Ma12, section 3] and [Ku18, Lemma 1.1].

**Lemma 5.2.** $H_a$ is singular.

**Proof.** Let $n \in \mathbb{N}$ and suppose that $a_n = 0$. Then we let $\ell_{n,1}, \ell_{n,3} \in \mathbb{N}$ be the smallest positive integers and $\ell_{n,2}, \ell_{n,4} \in \mathbb{N}$ be the largest positive integers such that

$$\left[ \frac{\ell_{n,1}}{q_{n+1}}, \frac{\ell_{n,2}}{q_{n+1}} \right] \leq \left[ \frac{2t_n}{q_n}, \frac{1 - t_n + \delta_n^2}{2q_n} \right] \quad \text{and} \quad \left[ \frac{\ell_{n,3}}{q_{n+1}}, \frac{\ell_{n,4}}{q_{n+1}} \right] \leq \left[ \frac{1 + t_n - \delta_n^2}{2q_n}, \frac{1 - 2t_n}{q_n} \right].$$

We also recall the condition $q_{i+1} \geq q_i^{2^i}$ for every $i \in \mathbb{N}$ from [4.10]. This implies

$$\prod_{i=1}^{n-1} q_i^{2i-1} \leq q_n$$

and, hence,

$$\prod_{i=1}^{n-1} t_i = \left( \prod_{i=1}^{n-1} q_i^{2i-1} \right)^{-1} \geq q_n^{-1},$$

where we used the definition of $t_i$ from equation (4.5). Altogether, we obtain

$$\frac{|G_{a,b}(x) - G_{a,b}(y)|}{|x - y|^{\frac{1}{d}}} \geq q_n.$$

Since there are infinitely many $n \in N$, we conclude that $G_{a,b}$ cannot be $d$-Hölder for any $d \in (0,1)$. This finishes the proof of property (R2).
5.1 Absolute continuous and singular invariant measures

Then we define the set

$$L_n = \bigcup_{k=0}^{q_n-1} \left[ \frac{k + \ell_{n,1}}{q_n}, \frac{k + \ell_{n,2}}{q_n+1} \right] \cup \left[ \frac{k + \ell_{n,3}}{q_n+1}, \frac{k + \ell_{n,4}}{q_n} \right].$$

By construction, $h_{a,n}|_{L_n}$ is an affine transformation with slope $t_n$. Furthermore, we introduce the set $M_n$ defined by

$$\bigcup_{k=0}^{q_n-1} \left[ \frac{k + 3t_n - 8t_n^2}{4q_n}, \frac{k + 3t_n + 8t_n^2}{4q_n} \right] \cup \left[ \frac{k + 1 - 3t_n}{q_n}, \frac{k + 1 - 3t_n}{4q_n} \right] \cup \left[ \frac{k + 1 - 3t_n}{q_n}, \frac{k + 1 - 3t_n}{4q_n} \right].$$

Due to $t_n < 1$ we have $M_n \subseteq L_n$. Additionally, we observe $h_{a,n}(L_n) \subseteq M_n$. By direct computation, we also obtain $m(L_n) \geq 1 - 6t_n$ and $m(M_n) \leq t_n$, where $m$ stands for the Lebesgue measure.

In an analogous manner, we handle the case of $a_n = 1$ and define sets $L_n$ and $M_n$ with $m(L_n) \geq 1 - 6t_n$ and $m(M_n) \leq t_n$ in such a way that $h_{a,n}|_{L_n}$ is an affine transformation with slope $t_n$ as well as $h_{a,n}(L_n) \subseteq M_n$.

Let $C_n = \cap_{i=1}^{n} L_i$ and $C = \cap_{i=1}^{\infty} L_i$. Since each component of $L_i$ is an interval with endpoints in $(q_{i+1}^{-1}\mathbb{Z})/\mathbb{Z}$, we have

$$m(C_n) \geq \prod_{i=1}^{n} (1 - 6t_i).$$

By definition of $t_n$ from equation (4.5) we obtain

$$\sum_{n=1}^{\infty} t_n = \sum_{n=1}^{\infty} q_n^{-2n} < \infty$$

and, hence, $\mu_T(H_a(C)) = m(C) > 0$.

By construction of $h_{a,j}$ with the aid of the $q_j$-fold covering map, $(q_{n+1}^{-1}\mathbb{Z})/\mathbb{Z}$ is pointwise fixed under $h_{a,j}$, $j > n$. Since any component of $L_n$ is an interval with endpoints in $(q_{i+1}^{-1}\mathbb{Z})/\mathbb{Z}$, we get $h_{a,j}(L_n) = L_n$ for $j > n$. Then we obtain for any $j > n$ that

$$H_{a,j}(L_n) = H_{a,n}(L_n) = H_{a,n-1}h_{a,n}(L_n) \subseteq H_{a,n-1}(M_n).$$

This yields $L_n \subseteq H_{a,j}^{-1}H_{a,n-1}(M_n)$, where $H_{a,0} = \text{id}$. Thus, the uniform limit $H_a^{-1}$ of $H_{a,j}$ satisfies $L_n \subseteq H_a^{-1}(H_{a,n-1}(M_n))$ for any $n \in \mathbb{N}$. Hereby, we observe

$$H_a(C_n) = H_a\left( \bigcap_{i=1}^{n} L_i \right) = \bigcap_{i=1}^{n} H_a(L_i) \subseteq \bigcap_{i=1}^{n} H_{a,i-1}(M_i).$$

In order to have $H_{a,i-1}(A) \subseteq H_{a,i-2}(M_{i-1})$ for a set $A \subseteq S^1$, this set $A$ has to satisfy $h_{a,i-1}(A) \subseteq M_{i-1}$ which implies the condition $A \subseteq L_{i-1}$. Since $m(M_i) \leq t_i$ and the slope of $h_{i-1}|_{L_{i-1}}$ is equal to $t_{i-1}$, this yields $m(H_a(C_n)) \leq \prod_{i=1}^{n-1} t_i$ which converges to 0 as $n \to \infty$. Therefore, $m(H_a(C)) = 0$.

Altogether, we conclude that $\mu_T$ is not equivalent to $m$ because $\mu_T(H_a(C)) > 0$ and $m(H_a(C)) = 0$. Hence, $H_a$ is a singular map. □
In the next step, we present some modifications to the construction of the conjugation maps \( h_{a,n} \) which will allow us to produce absolute continuous conjugacies. For this purpose, we recall that the map \( \hat{h}_t : [0, 1] \to [0, 1] \) from Section 4.1 coincides with the identity in a neighborhood of the boundary. Using the maps \( C_n : [0, \frac{1}{2^n+1}] \to [0, 1] \), \( C_n(x) = 2^{n+1} \cdot x \), we construct the orientation-preserving circle diffeomorphism \( \hat{h}_t \) as follows:

\[
\hat{h}_t(x) = \begin{cases} 
C_n^{-1} \circ \hat{h}_t \circ C_n(x) & \text{if } x \in [0, \frac{1}{2^n+1}] \\
x & \text{if } x \in \left[ \frac{1}{2^n+1}, 1 \right]
\end{cases},
\]

where we define the number \( t_n \) as in the previous section.

Let \( h_{q,n} \) be the lift of \( \hat{h}_t \) by the cyclic \( q_n \)-fold covering map \( \pi_{q_n} \) such that \( \text{Fix}(h_{q,n}) \neq \emptyset \). Finally, we define the conjugation maps \( \tilde{h}_{a,n} \) using the map \( h_{q,n} \) instead of \( h_{q,n} \). By the same reasoning as above, one can show that the sequence \( (\theta_{a,n})_{n \in \mathbb{N}} \) converges to a homeomorphism \( \theta_{a} \) and that the limit diffeomorphisms \( T_{a} = \theta_{a} \circ R_{a} \circ \theta_{a}^{-1} \) satisfy Proposition 3.1 as well.

We now prove the absolute continuity of \( H \) by the same method as in [Ma12, section 4] and [Ku18, Lemma 2.7].

**Lemma 5.3.** \( H_a \) is absolutely continuous.

**Proof.** We introduce the sets

\[
\tilde{L}_n = \left[ 2^{- (n+1)}, 1 \right] \quad \text{and} \quad \mathcal{L}_n = \pi_{q_n}^{-1} \left( \tilde{L}_n \right).
\]

According to our construction \( \tilde{h}_{a,n} \) is the identity on \( \mathcal{L}_n \). Let \( X = \bigcap_{n=1}^{\infty} \mathcal{L}_n \). Then we have

\[
m(X) \geq 1 - \sum_{n=1}^{\infty} m(S^1 \setminus \tilde{L}_n) = 1 - \sum_{n=1}^{\infty} 2^{-(n+1)} = \frac{1}{2}.
\]

Since \( \theta_{a,n} \) is the identity on the positive measure set \( X \), we have for any Borel set \( B \) that \( \mu_{T_{a}}(B \cap \mathbb{S}^1) = m(B \cap \mathbb{S}^1) \) and \( \mu_{\theta_{a}}(X) = m(X) > 0 \).

Assume that \( \mu_{\theta_{a}} \) is not equivalent to \( m \). Then \( \mu_{\theta_{a}} \) is singular to \( m \) and there is a Borel set \( B \subset \mathbb{S}^1 \) such that \( m(B) = 1 \) and \( \mu_{\theta_{a}}(B) = 0 \). But then we obtain the contradiction \( m(B \cap \mathbb{S}^1) = m(X) > 0 \) but \( \mu_{T_{a}}(B \cap \mathbb{S}^1) \leq \mu_{T_{a}}(B) = 0 \). Hence, \( H_a \) is absolutely continuous. \( \square \)

**6 Higher rank actions**

In this final section we turn to actions by \( \mathbb{Z}^d, d \geq 2 \), on the circle. We obtain a generalization of our Main Theorem A.

**Theorem B.** Let \( (D) \) be some degree of regularity from Hölder to \( C^\infty \) and \( D \) be the collection of orientation-preserving circle homeomorphisms with regularity \( (D) \). Then there is no complete numerical Borel invariant for \( D \)-conjugacy of free \( \mathbb{Z}^d \) actions by orientation-preserving \( C^\infty \) diffeomorphisms of the circle.

By our strategy of proof from Section 3, this theorem follows from the subsequent criterion.
Proposition 6.1. Let $\mathcal{A}$ be the space of free $\mathbb{Z}^d$ actions by orientation-preserving $C^\infty$ diffeomorphisms of the circle. There is a continuous one-to-one map

$$\Phi : \{0, 1\}^N \to \mathcal{A}$$

such that for any two sequences $a = (a_n)_{n \in \mathbb{N}}$ and $b = (b_n)_{n \in \mathbb{N}}$ the following properties hold:

(R1) If there is $N \in \mathbb{N}$ such that $a_n = b_n$ for every $n \geq N$, then the $C^\infty$-actions $\Phi(a)$ and $\Phi(b)$ are $C^\infty$-conjugate.

(R2) If there are infinitely many $n \in \mathbb{N}$ with $a_n \neq b_n$, then the $C^\infty$-actions $\Phi(a)$ and $\Phi(b)$ are not Hölder-conjugate.

In order to build the smooth $\mathbb{Z}^d$ actions, we construct generating diffeomorphisms by a slight modification to the constructions in Section 4. In particular, we have to arrange for commutativity of the generators and freeness of the action. For the latter one, the following number-theoretical lemma from [Ha09] will prove useful.

Lemma 6.2 ([Ha09], Theorem 2.1). Let $d \in \mathbb{Z}^+$, $\epsilon > 0$ and $(a_i, n)_{n \in \mathbb{N}}$ for $i = 1, \ldots, d$ be sequences of positive integers such that

1. $a_{1,n}$ divides $a_{1,n+1}$ and $\frac{a_{1,n+1}}{a_{1,n}} \geq 2^{(d+1)^n-1}$

2. $b_{i,n} < 2^{(d+1)^n-(\sqrt{2}+\epsilon)n}$ for $i = 1, \ldots, d$

3. $\lim_{n \to \infty} \frac{a_{i,n}b_{j,n}}{b_{i,n}a_{j,n}} = 0$ for all $i, j \in \{1, \ldots, d\}$, $i > j$

4. $a_{i,n}2^{-(d+1)^n-\epsilon n} < a_{i,n}2^{(d+1)^n-(\sqrt{2}+\epsilon)n}$ for $i = 1, \ldots, d$

hold for every sufficiently large $n \in \mathbb{Z}^+$. Then the numbers $\sum_{n=1}^{\infty} \frac{b_{1,n}}{a_{1,n}}, \ldots, \sum_{n=1}^{\infty} \frac{b_{d,n}}{a_{d,n}}$ and 1 are linearly independent over the integers.

Proof of Proposition 6.1. We construct the $d$ generators $T^{(i)}_{a_n}$, $i = 1, \ldots, d$, of the $\mathbb{Z}^d$ action $\Phi(a)$ as limits of AbC diffeomorphisms $T^{(i)}_{a_n} = H_{a_n} \circ R_{\alpha_n^{(i)}} \circ H_{a_n}^{-1}$, where the conjugation maps $H_{a_n} = H_{a_{n-1}} \circ h_{a_n}$ are constructed as in Section 4.1 and $(\alpha_n^{(i)})_{n \in \mathbb{N}}$ are sequences of rational numbers $\alpha_n^{(i)} = \frac{p_n^{(i)}}{q_n}$ with $p_n^{(i)}$ and $q_n$ relatively prime.

For a start, we let $H_{a,0} = \text{id}$, $q_1$ be a power of 2 and for each $i = 1, \ldots, d$ we let $p_n^{(i)}$ be an odd integer. In the induction step from $n-1$ to $n$ we construct the conjugation map $h_{a_n}$ as in Section 4.1 using the number $q_n$. Then we choose $l_n$ as a sufficiently large power of 2 such that

$$l_n > 4^{dn} \cdot C_n \cdot \| H_{a_n} \|_{n+1}^{n+1}$$

(6.3)

for all sequences $a \in \{0, 1\}^N$, where $C_n$ is the constant from Lemma 4.8. We proceed by defining the rational numbers

$$\alpha_{n+1}^{(i)} := \alpha_n^{(i)} + \frac{3^{(i-1)n}p_n^{(i)}l_nq_n^2 + 3^{(i-1)n}l_nq_n^2}{l_nq_n^2}$$

(6.4)
for each $i = 1, \ldots, d$. Since $l_n$ and $q_n$ are powers of 2, we note that $q_{n+1} = l_n q_n^2$ is a power of 2 and $p_n^{(i)}$ is relatively prime to $q_{n+1}$. Furthermore, we apply Lemma 4.8 to get the following estimate for each $i = 1, \ldots, d$ and every $a \in \{0,1\}^n$:

$$
{d_n}(T_{a,n}^{(i)} T_{a,n-1}^{(i)}) = {d_n}(H_{a,n} \circ R_{\alpha_{n+1}^{(i)}} \circ H_{a,n}^{-1}, H_{a,n-1} \circ R_{\alpha_n^{(i)}} \circ H_{a,n-1}^{-1})
$$

$$
= {d_n}(H_{a,n} \circ R_{\alpha_{n+1}^{(i)}} \circ H_{a,n}^{-1}, H_{a,n-1} \circ R_{\alpha_n^{(i)}} \circ h_{a,n} \circ h_{a,n}^{-1} \circ H_{a,n-1}^{-1})
$$

$$
= {d_n}(H_{a,n} \circ R_{\alpha_{n+1}^{(i)}} \circ H_{a,n}^{-1}, H_{a,n} \circ R_{\alpha_n^{(i)}} \circ H_{a,n}^{-1})
$$

$$
\leq C_n \cdot \|H_{a,n}\|_{n+1} \cdot \left| \alpha_n^{(i)} - \alpha_n^{(i)} \right|
$$

$$
< \frac{3^{(d-1)n}}{4^{dn} q_n^2},
$$

where we used equations (6.3) and (6.4) in the last step. As in the proof of Lemma 4.9, this estimate allows us to conclude convergence of the sequence $(T_{a,n}^{(i)})_{n \in \mathbb{N}}$ to a $C^\infty$-diffeomorphism $T_a^{(i)}$. By Lemma 5.1 we again obtain convergence of $(H_{a,n})_{n \in \mathbb{N}}$ to a homeomorphism $H_a$. Hence, for each $i = 1, \ldots, d$ we get $T_a^{(i)} = H_a \circ R_{\alpha_n^{(i)}} \circ H_a^{-1}$, where $\alpha_n^{(i)}$ is the limit of the sequence $(\alpha_n^{(i)})_{n \in \mathbb{N}}$. In particular, we have $T_a^{(i)} \cdot T_a^{(j)} = T_a^{(j)} \circ T_a^{(i)}$ for all $i, j \in \{1, \ldots, d\}$. Thus, the $C^\infty$-diffeomorphisms $T_a^{(1)}, \ldots, T_a^{(d)}$ generate a smooth $\mathbb{Z}^d$ action. The properties (R1) and (R2) follow from the corresponding properties in Proposition 3.4.

Finally, we apply Lemma 6.2 with $a_{i,n} = q_{n+1}$ and $b_{i,n} = 3^{(i-1)n}$ to verify that the numbers $\alpha^{(1)}, \ldots, \alpha^{(d)}$ and 1 are linearly independent over the integers. This yields that the action $\Phi(a)$ generated by $T_a^{(1)}, \ldots, T_a^{(d)}$ is a free action.

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