IDENTITIES ON ZAGIER’S RANK TWO EXAMPLES FOR NAHM’S CONJECTURE

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Abstract. Let \( r \geq 1 \) be a positive integer, \( A \) a real positive definite symmetric \( r \times r \) matrix, \( B \) a vector of length \( r \), and \( C \) a scalar. Nahm’s problem is to describe all such \( A, B \) and \( C \) with rational entries for which a specific \( r \)-fold \( q \)-hypergeometric series (denoted by \( f_{A,B,C}(q) \)) involving the parameters \( A, B, C \) is a modular form. When the rank \( r = 2 \), Zagier provided eleven sets of examples of \( (A, B, C) \) for which \( f_{A,B,C}(q) \) is likely to be a modular form. We present a number of Rogers-Ramanujan type identities involving double sums, which give modular form representations for Zagier’s rank two examples. Together with several known cases in the literature, we verified ten of Zagier’s examples and give conjectural identities for the remaining example.

1. Introduction

The famous Rogers-Ramanujan identities, first discovered by Rogers and later rediscovered by Ramanujan, assert that

\[
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q, q^4; q^5)_\infty},
\]

\[
\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_n} = \frac{1}{(q^2, q^3; q^5)_\infty}.
\]

Here and throughout this paper, we assume that \(|q| < 1\) for convergence and use the standard \( q \)-series notation

\[
(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1} (1-aq^k), \quad (a;q)_\infty := \prod_{k=0}^{\infty} (1-aq^k),
\]

\[
(a_1, \cdots, a_m; q)_n := (a_1;q)_n \cdots (a_m; q)_n, \quad n \in \mathbb{N} \cup \{\infty\}.
\]

Since the appearance of the identities (1.1) and (1.2), numerous works have been done to find similar identities, which were usually called as Rogers-Ramanujan type identities. One of the famous works on this topic is Slater’s list [20], which contains 130 of such identities such as [20, Eqs. (59),(60),(61)]

\[
\sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q; q)_n}{(q; q)_{2n+1}} = \frac{(q^2, q^{12}, q^{14}; q^{14})_\infty}{(q; q)_\infty}, \quad (\text{S. 59})
\]

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forms, and it has not been well understood yet. In particular, let $r$ be an integer, $A$, $B$, $C$ rational entries for which $f(A, B, C)$ is a scalar. Let us restrict our attention to the series

$$f_{A,B,C}(q) := \sum_{n=(n_1,\ldots,n_r)^T \in (\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}n^T An + n^T B + C}}{(q; q)_{n_1} \cdots (q; q)_{n_r}}.$$  

Nahm [15–17] posed the following problem: describe all such $A$, $B$ and $C$ with rational entries for which $f_{A,B,C}(q)$ is a modular form. For convenience, we shall call such $(A, B, C)$ as a modular triple, and call $A$ the matrix part, $B$ the vector part and $C$ the scalar part of it.

The Rogers-Ramanujan identities provided two examples $(A, B, C) = (2, 0, -1/60)$ and $(2, 1, 11/60)$ for Nahm’s problem. Nahm [17] made a conjecture providing a criterion on the matrix $A$ such that there exist $B$ and $C$ with $f_{A,B,C}(q)$ being modular forms. The conjecture is formulated in terms of Bloch group and a system of polynomial equations induced by $A$. See [23, p. 43] for detailed statement. The motivation of Nahm’s conjecture comes from physics and the modular forms $f_{A,B,C}(q)$ are expected to be characters of rational conformal field theories.

Zagier [25] studied Nahm’s problem and found many possible modular triples. In particular, when the rank $r = 1$, Zagier confirmed Nahm’s conjecture and proved that there are exactly seven modular triples $(A, B, C)$:

$$(1/2, 0, -1/40), \quad (1/2, 1/2, 1/40), \quad (1, 0, -1/48), \quad (1, 1/2, 1/24), \quad (1, -1/2, 1/24), \quad (2, 0, -1/60), \quad (2, 1, 11/60).$$

(1.8)

When the rank $r \geq 2$, Nahm’s problem becomes more difficult. For $r = 2$, extensive computer searches have been carried out by Terhoeven [21] and Zagier [25]. After searching over $A = \frac{1}{m}(a\ b\ c)$ with integers $a, b, c, m \leq 100$ which satisfy certain requirements, Zagier found eleven sets of possible modular triples and record them as [25, Table 2]. To be specific, for $A$ being

$$\left( \begin{array}{cc} \alpha & 1 - \alpha \\ 1 - \alpha & \alpha \end{array} \right), \quad \left( \begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array} \right), \quad \left( \begin{array}{cc} 4 & 1 \\ 1 & 1 \end{array} \right), \quad \left( \begin{array}{cc} 4 & 2 \\ 2 & 2 \end{array} \right), \quad \left( \begin{array}{cc} 2 & 1 \\ 1 & 3/2 \end{array} \right), \quad \left( \begin{array}{cc} 4/3 & 2/3 \\ 2/3 & 4/3 \end{array} \right)$$

and their inverses, Zagier found several values of $B$ and $C$ for which the function $f_{A,B,C}(q)$ is (or appears to be) modular. Zagier stated explicit identities which reveal

$$\sum_{n=0}^{\infty} q^{n^2+n} (-q; q)_n = (q^4, q^{10}, q^{14}; q^{14})_\infty, \quad (S. 60) \quad (1.6)$$

$$\sum_{n=0}^{\infty} q^{n^2} (-q; q)_n = (q^6, q^8, q^{14}; q^{14})_\infty, \quad (S. 61) \quad (1.7)$$

Here we use the label $S. n$ to denote the equation $(n)$ in Slater’s list [20]. A detailed introduction to Rogers-Ramanujan type identities can be found in Sills’ book [19].
the modularity of $f_{A,B,C}(q)$ only in the case $A = \binom{\alpha}{1-\alpha}$ and $A = \binom{1}{1}$. Namely, he proved that \cite{Zagier} Eq. (26)

$$f\left(\frac{\alpha}{1-\alpha}, \frac{1-\alpha}{\alpha}, \frac{\alpha}{2} \right) = \frac{1}{(q;q)_{\infty}} \sum_{n \in \mathbb{Z} + \nu} q^{\alpha n^2/2 - 1/24} (\forall \nu \in \mathbb{Q}).$$

(1.9)

Let $[x]$ denote the integer part of $x$. Zagier also stated that \cite[p. 45]{Zagier}

$$f\left(\binom{4}{1}, \binom{0}{1/2}, \binom{1}{1} \right) = \frac{1}{(q;q)_{\infty}} \sum_{n \equiv 1 \mod 10} (-1)^{|n/10|} q^{n^2/20 - 1/24},$$

(1.10)

$$f\left(\binom{4}{1}, \binom{2}{1/2}, \binom{1}{1} \right) = \frac{1}{(q;q)_{\infty}} \sum_{n \equiv 3 \mod 10} (-1)^{|n/10|} q^{n^2/20 - 1/24},$$

(1.11)

and he remarked that “these equations were not proved, but only verified to a higher order in the power series in $q$”.

Using an approach outlined by Zagier \cite{Zagier}, Vlasenko and Zwegers \cite{VlasenkoZwegers} found all modular triples $(A, B, C)$ for $A$ being $\binom{a - \lambda}{\lambda - a}$ with $a \in \mathbb{Q}$ and $\lambda \in \{\frac{1}{2}, 1, 2\}$. These include several examples missed in Zagier’s list \cite[Table 1]{Zagier}. For example, they find that for $A$ being $\binom{3/2}{1/2 3/2}$ or $\binom{3/4}{1/4 3/4}$, there are $B$ and $C$ such that $f_{A,B,C}(q)$ is modular. Since these two matrices do not satisfy Nahm’s criterion, they can be regarded as counterexamples to Nahm’s conjecture.

Motivated by the above works, the purpose of this paper is to provide a verification of Zagier’s list in the rank two case. We will state explicit Rogers-Ramanujan type identities involving double sums for each of Zagier’s example. The sum sides of our identities are essentially $f_{A,B,C}(q)$ with $(A, B, C)$ from Zagier’s list, and the product sides show clearly that they are indeed modular forms.

For convenience, we label the examples in Zagier’s list from 1 to 11 according to their order in \cite[Table 2]{Zagier}. We are able to prove nine of them. Namely, Examples 1–4, 6–9 and 11. In particular, for Example 4 we confirm Zagier’s formulas \cite{Zagier} and \cite{Zagier}. It should be mentioned that seven of Zagier’s examples have been discussed in the literature. See Table 1 for known cases and references which discuss them. Note that for Example 2, there are five choices for the vector part $B$. Lee’s arguments \cite{Lee} are applicable for only three of them. We will give unified proofs which apply to all these five cases. As for Example 10, Vlasenko and Zwegers \cite{VlasenkoZwegers} only provided conjectural identities, and these identities were later confirmed by Cherednik and Feigin \cite{CherednikFeigin} via the nilpotent double affine Hecke algebras. We will state new identities for Example 10 which is different from but equivalent to that of Vlasenko and Zwegers \cite{VlasenkoZwegers}. The only example which remains open is Example 5. We will state conjectural identities for it.

Besides the seven cases in Table 1 it appears that this is the first time to state explicit identities for other examples. Moreover, it seems to be the first time to give proofs for Examples 4, 7, 8 and 9 and two cases of Example 2 that are not covered by Lee’s discussion \cite{Lee}. It is worth mentioning that many of Zagier’s examples can be reduced to some known single sum Rogers-Ramanujan type identities from Slater’s list \cite{Slater}. The reduction processes vary for different examples. For most of the examples, we achieve it by summing over one of the indexes first. For Example
Example No. | Matrix A | Reference
--- | --- | ---
1 | \( \begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix} \) | proved by Zagier [25] (see (1.9))
2 | \( \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \) | partial results by Lee [14]
3 | \( \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \) | proved by Calinescu-Milas-Penn [5]
4 | \( \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix} \) | (1.10)–(1.11) found by Zagier [25] (without proof)
6 | \( \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix} \) | instances of the Andrews-Gordon identity [14]
10 | \( \begin{pmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{pmatrix} \) | conjectured by Vlasenko-Zwegers [22]
11 | \( \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix} \) | proved by Vlasenko-Zwegers [22]

Table 1. Examples discussed in the literature

8 we will use an integral method to find new expressions for the sums sides, and then either eliminate one of the summation indexes or follow the techniques in the author’s work [23].

The rest of this paper is organized as follows. In Section 2 we collect some useful identities which will play key roles in studying Zagier’s examples. In Section 3 we discuss Zagiers’ examples one by one. Since details of proofs for some known examples were omitted in the literature. For the sake of completeness, we shall include complete proofs or brief discussions for all these examples.

2. Preliminaries

In this section, we introduce some notations and identities that will be used frequently. To make our formulas more compact, sometimes we will use the symbols:

\[ J_m := (q^m; q^m)_\infty, \quad J_{a,m} := (q^a, q^{m-a}, q^m; q^m)_\infty. \]

We need Euler’s \( q \)-exponential identities [2, Corollary 2.2]

\[
\sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_\infty}, \quad \sum_{n=0}^{\infty} \frac{q^{(n)}_2}{(q; q)_n} = (-z; q)_\infty, \quad |z| < 1, \tag{2.1}
\]

and the Jacobi triple product identity [2, Theorem 2.8]

\[
(q, z, q/z; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{(n)}_2 z^n. \tag{2.2}
\]

We also recall Lebesgue’s identity [2, Corollary 2.7]:

\[
\sum_{n=0}^{\infty} q^{n(n+1)/2} \frac{(-zq; q)_n}{(q; q)_n} = (-zq^2; q^2)_\infty (-q; q)_\infty. \tag{2.3}
\]

As mentioned in the introduction, many of Zagier’s examples can be reduced to some identities in Slater’s list. Besides (1.5)–(1.7), we will also need:

\[
\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2}}{(-q; q)_n (q^4; q^4)_n} = \frac{(q^2, q^3, q^5, q^5)_\infty}{(q^2; q^2)_\infty}, \quad \text{(S. 19)} \tag{2.4}
\]

\[
\sum_{n=0}^{\infty} \frac{q^{2n(n+1)} (q^2; q^2)_{2n+1}}{(q^2; q^2)_{2n+1}} = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(-q; q^2)_{n+1} (q^4; q^4)_n} = \frac{(q, q^6, q^7, q^7)_\infty}{(q^2; q^2)_\infty}, \quad \text{(S. 31)} \tag{2.5}
\]
The identities (2.5)–(2.7) are usually referred as the Rogers-Selberg mod 7 identities. Note that (2.8) also appeared as [3 Entry 1.7.12], (2.9) also appeared in [3] Entries 1.7.11 and 4.2.15. A typo has been corrected for (2.17).

\[
\begin{align*}
\sum_{n=0}^{\infty} q^{2n^2+2n} (q; q^2)_n &= \sum_{n=0}^{\infty} q^{2(n^2+1)} = (q^2; q^2)_2n = (q^2, q^5, q^7, q^7)_\infty, \quad \text{(S. 32)} \\
\sum_{n=0}^{\infty} q^{2n^2} (q; q^2)_n &= \sum_{n=0}^{\infty} q^{2n^2} = (q^2, q^2)_n = (q^2, q^4, q^7, q^7)_\infty, \quad \text{(S. 33)} \\
\sum_{n=0}^{\infty} q^{n^2+2n} (q; q^2)_n &= \sum_{n=0}^{\infty} q^{n^2+2n} = (q^2; q^2)_n = (q^2, q^5, q^8)_\infty, \quad \text{(S. 34)} \\
\sum_{n=0}^{\infty} q^{n^2} (q; q^2)_n &= \sum_{n=0}^{\infty} q^{n^2} = (q^2, q^2)_n = (q^2, q^4, q^5, q^8)_\infty, \quad \text{(S. 36)} \\
\sum_{n=0}^{\infty} q^{2n^2} (q; q)_n &= \sum_{n=0}^{\infty} q^{2n^2} = (q^2, q^2)_n = (q^2, q^5, q^8)_\infty, \quad \text{(S. 39)} \\
\sum_{n=0}^{\infty} q^{2n^2+2n} (q; q)_{2n+1} &= \sum_{n=0}^{\infty} q^{2n^2+2n} = (q^2, q^2)_{2n+1} = (q^2, q^5, q^8)_\infty, \quad \text{(S. 38)} \\
\sum_{n=0}^{\infty} q^{2n^2} (q; q)_{n+1} &= \sum_{n=0}^{\infty} q^{2n^2} = (q^2; q^2)_{n+1} = (q^2, q^5, q^8, q^{10})_\infty, \quad \text{(S. 44)} \\
\sum_{n=0}^{\infty} q^{n^2-1} (q; q)_n &= \sum_{n=0}^{\infty} q^{n^2-1} = (q^2, q^2)_n = (q^2, q^4, q^{10})_\infty, \quad \text{(S. 46)} \\
\sum_{n=0}^{\infty} q^{2n^2} (q; q)_n &= \sum_{n=0}^{\infty} q^{2n^2} = (q^2; q^2)_n = \frac{(q^2, q^6, q^{10}, q^{10})_\infty}{(q^2, q^2)_n}, \quad \text{(S. 80)} \\
\sum_{n=0}^{\infty} q^{2n^2} (q; q)_n &= \sum_{n=0}^{\infty} q^{2n^2} = (q^2; q^2)_n = \frac{J_2 J_4^3}{J_2 J_1 J_1 J_4 J_4 J_6 J_6}, \quad \text{(S. 81)} \\
\sum_{n=0}^{\infty} q^{2n^2} (q; q)_n &= \sum_{n=0}^{\infty} q^{2n^2} = (q^2; q^2)_n = \frac{J_2 J_4^3}{J_2 J_1 J_2 J_1 J_4 J_4 J_6 J_6}, \quad \text{(S. 82)} \\
\sum_{n=0}^{\infty} q^{2n^2+2n} (q; q)_n &= \sum_{n=0}^{\infty} q^{2n^2+2n} = (q^2; q^2)_{2n+1} = \frac{J_2 J_3 J_4 J_4}{J_2 J_1 J_2 J_1 J_4 J_4 J_2 J_2 J_2}, \quad \text{(S. 97)} \\
\sum_{n=0}^{\infty} q^{2n^2} (q; q)_n &= \sum_{n=0}^{\infty} q^{2n^2} = (q^2; q^2)_n = \frac{J_2 J_3 J_4 J_4 J_4 J_4 J_4}{J_2 J_1 J_2 J_1 J_4 J_4 J_6 J_6 J_6}, \quad \text{(S. 117)} \\
\sum_{n=0}^{\infty} q^{2n^2} (q; q)_n &= \sum_{n=0}^{\infty} q^{2n^2} = (q^2; q^2)_n = \frac{J_2 J_3 J_4 J_4 J_4 J_4 J_4}{J_2 J_1 J_2 J_1 J_4 J_4 J_6 J_6 J_6}, \quad \text{(S. 118)} \\
\sum_{n=0}^{\infty} q^{2n^2} (q; q)_n &= \sum_{n=0}^{\infty} q^{2n^2} = (q^2; q^2)_n = \frac{J_2 J_3 J_4 J_4 J_4 J_4 J_4}{J_2 J_1 J_2 J_1 J_4 J_4 J_6 J_6 J_6}, \quad \text{(S. 119)} \\
\sum_{n=0}^{\infty} q^{2n^2} (q; q)_n &= \sum_{n=0}^{\infty} q^{2n^2} = (q^2; q^2)_n = \frac{J_2 J_3 J_4 J_4 J_4 J_4 J_4}{J_2 J_1 J_2 J_1 J_4 J_4 J_6 J_6 J_6}, \quad \text{(S. 120)}
\end{align*}
\]
For Example 8 we will provide two different proofs. For both proofs we will use an integral method to rewrite the sum sides as integrals of some infinite products. This method was applied by Rosengren [18] to prove some conjectural identities of Kanade and Russell [12]. Later it has been applied in several works. For example, it was used by Chern [7] and the author [23] to prove a conjecture of Andrews and Uncu [4]. It was also utilized by Mc Laughlin [13] and Cao and Wang [8] in finding some new multi-sum Rogers-Ramanujan type identities.

In particular, for the second proof of Example 8, we rely on the following result found from the book of Gasper and Rahman [10], which plays a key role in the author’s work [23]. Before stating it, we remark that the symbol “idem \((c_1; c_2, \ldots, c_C)\)” after an expression stands for the sum of the \((C - 1)\) expressions obtained from the preceding expression by interchanging \(c_1\) with each \(c_k\), \(k = 2, 3, \ldots, C\).

**Lemma 2.1.** (Cf. [10, Eq. (4.10.6)]) Suppose that

\[
P(z) := \frac{(a_1z, \ldots, a_Az, b_1/z, \ldots, b_B/z; q)_\infty}{(c_1z, \ldots, c_Cz, d_1/z, \ldots, d_D/z; q)_\infty}
\]

has only simple poles. We have

\[
\oint P(z) \frac{dz}{2\pi iz} = \frac{(b_1c_1, \ldots, b_Bc_1, a_1/c_1, \ldots, a_A/c_1; q)_\infty}{(q, d_1c_1, \ldots, d_Dc_1, c_2/c_1, \ldots, c_C/c_1; q)_\infty}
\times \sum_{n=0}^{\infty} \frac{(d_1c_1, \ldots, d_Dc_1, qc_1/a_1, \ldots, qc_1/a_A; q)_n}{(q, b_1c_1, \ldots, b_Bc_1, qc_1/c_2, \ldots, qc_1/c_C; q)_n}
\times \left(-c_1q^{(n+1)/2}\right)^{n(C-A)} \left(\frac{a_1 \cdots a_A}{c_1 \cdots c_C}\right)^n + \text{idem } (c_1; c_2, \ldots, c_C) \quad (2.21)
\]

when \(C > A\), or if \(C = A\) and

\[
\left|\frac{a_1 \cdots a_A}{c_1 \cdots c_C}\right| < 1. \quad (2.22)
\]

Here the integration is over a positively oriented contour so that the poles of

\[(c_1z, \ldots, c_Cz; q)_\infty^{-1}\]

lie outside the contour, and the origin and poles of \((d_1/z, \ldots, d_D/z; q)_\infty^{-1}\) lie inside the contour.

### 3. Explicit identities for rank two examples

Now we discuss Zagier’s examples one by one. Since the scalar part \(C\) of a modular triple \((A, B, C)\) is determined by the matrix and vector parts, we will not mention it. We stress here that the symbols \(F(u, v; q), F_i(q), G_i(q), H_i(q), R_i(q), S_i(q)\) and \(T_i(q)\) used below may have different meanings in each proof. Their precise meanings will be given in the context.

#### 3.1. Example 1

The matrix and vector parts for this example are

\[
A = \begin{pmatrix} \alpha & 1 - \alpha \\ 1 - \alpha & \alpha \end{pmatrix}, \quad B = \begin{pmatrix} \alpha \nu & -\alpha \nu \\ -\alpha \nu & \alpha \nu \end{pmatrix}, \quad \nu \in \mathbb{Q}.
\]
Zagier stated (1.9) and provided a proof for it. Utilizing (2.2), his result can be stated in the following equivalent form.

**Theorem 3.1.** We have

$$
\sum_{i,j \geq 0} q^{\frac{i^2}{2}+(1-\alpha)ij+\frac{j^2}{2}+\alpha v-\alpha j} \frac{(q; q)_{i}(q; q)_{j}}{(q; q)_{i}(q; q)_{j}} = \frac{(-q^{\frac{\alpha}{2}+\alpha v}, -q^{\frac{\alpha}{2}-\alpha v}, q^{\alpha}; q^{\alpha})_{\infty}}{(q; q)_{\infty}}.
$$

(3.1)

Here we reproduce Zagier’s proof [25, p. 46].

**Proof.** We have

$$
\text{LHS} = \sum_{i,j \geq 0} q^{\frac{(i-j)^2}{2}+\alpha v(i-j)+ij} \frac{(q; q)_{i}(q; q)_{j}}{(q; q)_{i}(q; q)_{j}} = \sum_{n=-\infty}^{\infty} q^{\frac{2n^2+\alpha v n}{2}} \sum_{i-j=n} q^{ij} \frac{(q; q)_{i}(q; q)_{j}}{(q; q)_{i}(q; q)_{j}} = \frac{1}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} q^{\frac{2n^2+\alpha v n}{2}} = \text{RHS}.
$$

Here for the last second equality we used the Durfee rectangle identity: for any fixed integer $n$,

$$
\sum_{i-j=n} q^{ij} \frac{(q; q)_{i}(q; q)_{j}}{(q; q)_{i}(q; q)_{j}} = \sum_{j=0}^{\infty} q^{j(i+n)} \frac{(q; q)_{j}(q; q)_{j+n}}{(q; q)_{j}(q; q)_{j+n}} = \frac{1}{(q; q)_{\infty}},
$$

(3.2)

and for the last equality we employed (2.2). □

Theorem 3.1 can also be regarded as a special case of the following identity found by Cao and Wang [8, Theorem 3.4]:

$$
\sum_{i,j \geq 0} u^{i-j} q^{\frac{(i^2)}{2}+(\frac{\alpha}{2}i)^2+(\frac{\alpha}{2}j)^2} \frac{(q; q)_{i}(q; q)_{j}}{(q; q)_{i}(q; q)_{j}} = \frac{(-uq^\alpha, -q/u, q^{\alpha+1}; q^{\alpha+1})_{\infty}}{(q; q)_{\infty}}.
$$

(3.3)

In fact, setting $a = \alpha - 1$ and $u = q^{\alpha v+1-\frac{\alpha}{2}}$ in (3.3), we recover Theorem 3.1. Note that the proof of the identity (3.3) given in [8] is different from the above. It does not use (3.2). Instead, it is based on the integral method and repeated use of (2.2).

Cao and Wang [8, Theorem 3.6] also proved that

$$
\sum_{i,j \geq 0} (-1)^{i+j} u^{i-j} q^{(i^2+i-j^2-j+(\alpha-1)(i-j)^2)/2} \frac{(q; q)_{i}(q; q)_{j}}{(q; q)_{i}(q; q)_{j}} = \frac{(u^{-1}q^{(\alpha-1)/2}, uq^{(\alpha+1)/2}, q^{\alpha}; q^{\alpha})_{\infty} + (uq^{(\alpha-1)/2}, u^{-1}q^{(\alpha+1)/2}, q^{\alpha}; q^{\alpha})_{\infty}}{(q; q)_{\infty}}.
$$

(3.4)

If we set $u = -q^{\alpha v+\frac{1}{2}}$, then we get a new modular triple with

$$
A = \begin{pmatrix} \alpha & 1 - \alpha \\ 1 - \alpha & \alpha \end{pmatrix}, \quad B = \begin{pmatrix} \alpha v \\ -\alpha v - 1 \end{pmatrix}, \quad \nu \in \mathbb{Q}.
$$

By the way, Theorem 3.1 in [8] also produces many cases for $f_{A,B,C}(q)$ being modular. For instance, the equations (3.3) and (3.4) in [8] means that $f_{A,B,C}(q)$ is modular with

$$
A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad B \in \left\{ \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \right\}.
$$
However, since $A$ is not positive definite, it is outside the scope of Nahm’s problem.

3.2. Example 2. The matrix and vector parts for this example are

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad B \in \left\{ \begin{pmatrix} -1 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

**Theorem 3.2.** We have

\[
\sum_{i,j \geq 0} \frac{q^{i^2 + 2ij + \frac{1}{2}j^2 - i + \frac{1}{2}j}}{(q; q)_i(q; q)_j} = 2 \frac{(q^4; q^4)_\infty}{(q; q)_\infty}, \quad (3.5)
\]

\[
\sum_{i,j \geq 0} \frac{q^{2i^2 + 2ij + j^2}}{(q^2; q^2)_i(q^2; q^2)_j} = \frac{1}{(q^4, q^6; q^8)_\infty}, \quad (3.6)
\]

\[
\sum_{i,j \geq 0} \frac{q^{i^2 + 2ij + \frac{1}{2}j^2 + j}}{(q; q)_i(q; q)_j} = \frac{(q^2; q^2)_\infty^3}{(q^4; q^4)_\infty}, \quad (3.7)
\]

\[
\sum_{i,j \geq 0} \frac{q^{i^2 + 2ij + i + \frac{1}{2}j}}{(q; q)_i(q; q)_j} = \frac{(q^4; q^4)_\infty}{(q; q)_\infty}, \quad (3.8)
\]

\[
\sum_{i,j \geq 0} \frac{q^{2i^2 + 2ij + 2i + 2j}}{(q^2; q^2)_i(q^2; q^2)_j} = \frac{1}{(q^3, q^5, q^6; q^8)_\infty}. \quad (3.9)
\]

Here for the second and fifth identity, we have replaced $q$ by $q^2$ so that the series contain only integral powers of $q$. We will do similar replacements for other examples as well.

**Proof.** We define

$$F(u, v; q) = \sum_{i,j \geq 0} \frac{q^{i^2 + ij + \frac{1}{2}j^2} u^i v^j}{(q; q)_i(q; q)_j} = \sum_{i \geq 0} q^i u^i \sum_{j \geq 0} \frac{q^{\frac{1}{2}(2j^2 - j)}(q^{i+\frac{1}{2}} v)^j}{(q; q)_j} = \sum_{i \geq 0} \frac{q^i u^i}{(q; q)_i} \cdot \frac{\left(-q^{i+1/2} v; q\right)_\infty}{
\left(-q^{1/2} v; q\right)_\infty} \sum_{i \geq 0} \frac{q^{i^2} u^i}{(q, -q^{1/2} v; q)_i}. \quad (3.10)
$$

Setting $(u, v) = (q^{-1}, q^{1/2})$, we have by (3.10) and (2.1) that

$$F(q^{-1}, q^{1/2}; q) = (-q; q)_\infty \sum_{i=0}^{\infty} \frac{q^{2i}}{(q^2; q^2)_i} = (-q; q)_\infty (-1; q^2)_\infty^2 = 2 \frac{(q^4; q^4)_\infty}{(q; q)_\infty}. \quad (3.11)
$$

This proves (3.5).

Setting $(u, v, q)$ as $(1, 1, q^2)$, we have by (3.10) and (2.1) that

$$F(1, 1; q^2) = (-q; q^2)_\infty \sum_{i=0}^{\infty} \frac{q^{2i}}{(-q, q^2; q^2)_i} = \frac{(q^2; q^4)_{\infty}}{(q; q^2)_{\infty}}. \quad (3.12)
$$

Replacing $q$ by $-q$ in (2.10) and substituting it into (3.12), we obtain (3.6).

Setting $(u, v) = (1, q^{1/2})$ and using (2.1), we have

$$F(1, q^{1/2}; q) = (-q; q)_\infty \sum_{i=0}^{\infty} \frac{q^{i^2}}{(q^2; q^2)_i} = (-q; q)_\infty (-q; q^2)_\infty = \frac{(q^2; q^4)_\infty}{(q; q^2)_\infty}. \quad (3.13)$$
This proves (3.7).

Setting \((u, v) = (q, q^\frac{1}{2})\) and using (2.11), we have

\[
F(q, q^\frac{1}{2}; q) = (-q; q)\sum_{i=0}^{\infty} \frac{q^{i^2+i}}{(q^2; q^2)_i} = (-q; q)\sum_{i=0}^{\infty} (-q^2; q^2)_i = \frac{(q^4; q^4)_\infty}{(q; q)_\infty}.
\]

This proves (3.8).

Setting \((q, u, v)\) as \((q^2, q^2; q^2)\), we have by (3.10) and (2.11) that

\[
F(q^2, q^2; q^2) = (-q^3; q^3)\sum_{i=0}^{\infty} \frac{q^{2i^2+2i}}{(q^2, -q^3; q^2)_i}
= (-q^3; q^3)\sum_{i=0}^{\infty} \frac{q^{2i^2+2i}}{((-q; q^2)_{i+1})}(q^2; q^2)_i.
\]

Replacing \(q\) by \(-q\) in (2.11) and then substituting it into (3.15), we obtain (3.9). \(\square\)

**Remark 1.** This example has been discussed by Lee [14, Example 3.5.2]. He offered a proof of (3.8) using Lebesgue’s identity (2.3). To be specific, using the \(q\)-binomial identity

\[
(-zq; q)_k = \sum_{r=0}^{k} \frac{(q; q)_k}{(q; q)_r (q; q)_{k-r}} z^r q^{r(r+1)/2},
\]

Lee [14, Eq. (3.25)] rewrote the sum side of (2.3) as

\[
\sum_{k=0}^{\infty} \frac{q^{k(k+1)/2} (-zq; q)_k}{(q; q)_k} = \sum_{i,j \geq 0} z^i q^{2+ij+\frac{i^2}{2}+i+j} \frac{q^{i+j+1}}{(q; q)_i (q; q)_j}.
\]

Setting \(z = q^{-2}, q^{-1}\) and 1 in (3.10), we obtain (3.5), (3.7) and (3.8), respectively. But one cannot get (3.6) and (3.9) by specializing the choice of \(z\).

Warnaar pointed out to us that (3.6) is a special case of the following identity [24, p. 235]:

\[
\sum_{n_1, \ldots, n_{k-1} \geq 0} \frac{q^{\frac{k}{2}(N_1^2 + \cdots + N_{k-1}^2)}}{(q; q)_{n_1} \cdots (q; q)_{n_{k-1}}} = \frac{(-q^\frac{1}{2}; q)_\infty}{(q; q)_\infty} \frac{(q^{\frac{1}{2}}, q^{\frac{1}{2}+1}, q^{k+1}, q^{k+1})_\infty}{(q^4; q^4)_\infty},
\]

where \(N_i = n_i + \cdots + n_{k-1}\). Indeed, setting \(k = 3\) in this identity we obtain (3.6). Meanwhile, using Lebesgue’s identity, we can also prove (3.8) using Lemma A.1 in [24].

### 3.3. Example 3

The matrix and vector parts for this example are

\[
A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \quad B \in \left\{ \begin{pmatrix} -3/2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.
\]

**Theorem 3.3.** We have

\[
\sum_{i,j \geq 0} \frac{q^{i^2+j^2-i-j+2j}}{(q; q)_i(q; q)_j} = 2q^{-1} \frac{(q^2; q^2)_\infty}{(q; q)_\infty (q^4; q^4)_\infty},
\]

\[
\sum_{i,j \geq 0} \frac{q^{i^2-2ij+j^2}}{(q^2; q^2)_i(q^2; q^2)_j} = \frac{(q^2; q^2)_\infty^3 (q^6; q^6)_\infty (q^8; q^8)_\infty (q^3; q^5; q^8)_\infty}{(q; q)_\infty^2 (q^4; q^4)_\infty^2}.
\]
Proof. We define
\[ F(u, v; q) := \sum_{i,j \geq 0} q^{\frac{1}{2} i^2 - ij + j^2 - \frac{1}{2} i + j} \frac{q^{ij} (q_i q_j)}{(q; q)_j}. \]  \hfill (3.23)

Summing over \( i \) first using (2.1), we obtain
\[ F(u, v; q) = \sum_{j=0}^{\infty} q^{j^2} v^j \frac{(-uq^{\frac{1}{2}j}; q)_{\infty}}{(q; q)_j}. \]  \hfill (3.24)

Setting \((u, v) = (q^{-\frac{1}{2}}, q^2)\) in (3.24), we have
\[ F(q^{-\frac{1}{2}}, q^2; q) = \sum_{j=0}^{\infty} \frac{q^{2j} (-q^{1-j}; q)_{\infty}}{(q; q)_j} = 2q^{-1} (-q; q)_{\infty} (q; q^2)_{\infty}. \]

This proves (3.18). Here the last equality follows from Lebesgue’s identity (2.3) with \( z = q \).

Setting \((u, v, q) = (1, 1, q^2)\) in (3.24), we have
\[ F(1, 1; q^2) = \sum_{j=0}^{\infty} \frac{q^{2j} (-q^{1-2j}; q^2)_{\infty}}{(q; q^2)_j} = (-q; q^2)_{\infty} \sum_{j=0}^{\infty} \frac{q^{2j} (-q; q^2)_{j}}{(q; q^2)_j} = (-q; q^2)_{\infty} \frac{(q^3, q^5, q^8, q^8)_{\infty}}{(q^2; q^2)_{\infty}}. \]  \hfill (3.25)

This proves (3.19). Here for the last equality we used (2.9).

Setting \((u, v) = (q^{-\frac{1}{2}}, q)\) in (3.24), we have
\[ F(q^{-1}, q; q) = \sum_{j=0}^{\infty} \frac{q^{2j} (-q^{j}; q)_{\infty}}{(q; q)_j} = (-1; q)_{\infty} \sum_{j=0}^{\infty} \frac{q^{j(j+1)/2} (-q; q)_{j}}{(q; q)_j} = 2(-q; q^2)_{\infty} (-q^2; q^2)_{\infty}. \]  \hfill (3.26)

This proves (3.20). Here for the last equality we used (2.3) with \( z = 1 \).

Setting \((u, v) = (q^{\frac{1}{2}}, 1)\) in (3.24), we have
\[ F(q^{\frac{1}{2}}, 1; q) = \sum_{j=0}^{\infty} \frac{q^{2j} (-q^{1-j}; q)_{\infty}}{(q; q)_j} = (-q; q)_{\infty} \sum_{j=0}^{\infty} \frac{q^{j(j+1)/2} (1; q)_{j}}{(q; q)_j}. \]
This proves \((3.21)\). Here for the last equality we used \((2.3)\) with \(z = q^{-1}\).

Setting \((u, v, q) = (1, q^2, q^2)\) in \((3.21)\), we have

\[
F(1, q^2; q^2) = \sum_{j=0}^{\infty} \frac{q^{2j^2 + 2j}}{(q^2; q^2)_j} (-q^{1-2j}; q^2)_\infty = (-q; q^2)_\infty \sum_{j=0}^{\infty} \frac{q^{j^2 + 2j} (-q; q^2)_j}{(q^2; q^2)_j}. \tag{3.28}
\]

Substituting \((2.8)\) into \((3.28)\), we obtain \((3.22)\).

\[\square\]

Remark 2. Equivalent forms of the identities in Theorem 3.3 were given by Calinescu, Milas and Penn \([5, \text{Eqs. (6.1)} \text{-- (6.5)}]\). They provided detailed proof for \((3.19)\) and pointed out that other identities follow in a similar way. The proof here is essentially the same with that used in the proof of \([5, \text{Theorem 6.1}]\).

3.4. Example 4. The matrix and vector parts for this example are

\[
A = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}, \quad B \in \left\{ \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1/2 \end{pmatrix} \right\}.
\]

Recall that Zagier discovered the formulas \((1.10)\) and \((1.11)\) but did not prove them. We state the following equivalent formulas and give a proof.

**Theorem 3.4.** We have

\[
\sum_{i,j \geq 0} \frac{q^{2i^2 + ij + \frac{1}{2}j^2 + \frac{1}{2}j}}{(q; q)_i(q; q)_j} = \frac{(q^4, q^6, q^{10}; q^{10})_\infty}{(q; q)_\infty}, \tag{3.29}
\]

\[
\sum_{i,j \geq 0} \frac{q^{2i^2 + ij + \frac{1}{2}j^2 + 2i + j}}{(q; q)_i(q; q)_j} = \frac{(q^2, q^8, q^{10}; q^{10})_\infty}{(q; q)_\infty}. \tag{3.30}
\]

**Proof.** We define

\[
F(u, v; q) := \sum_{i,j \geq 0} \frac{q^{2i^2 + ij + \frac{1}{2}j^2} u^i v^j}{(q; q)_i(q; q)_j}. \tag{3.31}
\]

Summing over \(j\) first using \((2.1)\), we deduce that

\[
F(u, v; q) = \sum_{i = 0}^{\infty} \frac{q^{2i^2} u^i}{(q; q)_i} \sum_{j = 0}^{\infty} \frac{q^{2j^2} (q^{2-1})^j q^{i + \frac{1}{2}} v^j}{(q; q)_j} = \sum_{i = 0}^{\infty} \frac{q^{2i^2} u^i}{(q; q)_i} (-q^{i + \frac{1}{2}} v; q)_\infty. \tag{3.32}
\]

Setting \((u, v) = (1, q^{\frac{1}{2}})\) in \((3.32)\), we have

\[
F(1, q^{\frac{1}{2}}; q) = (-q; q)_\infty \sum_{i = 0}^{\infty} \frac{q^{2i^2}}{(q^2; q^2)_i}. \tag{3.33}
\]

Using \((1.1)\) with \(q\) replaced by \(q^2\), we obtain \((3.29)\).

Next, setting \((u, v) = (q^2, q^{\frac{1}{2}})\) in \((3.32)\), we deduce that

\[
F(q^2, q^{\frac{1}{2}}; q) = (-q; q)_\infty \sum_{i = 0}^{\infty} \frac{q^{2i^2 + 2i}}{(q^2; q^2)_i}. \tag{3.34}
\]

Using \((1.2)\) with \(q\) replaced by \(q^2\), we obtain \((3.30)\). \[\square\]
### Example 5
The matrix and vector parts for this example are
\[
A = \begin{pmatrix}
1/3 & -1/3 \\
-1/3 & 4/3
\end{pmatrix}, \quad B \in \left\{ \begin{pmatrix}
-1/6 \\
2/3
\end{pmatrix}, \begin{pmatrix}
1/2 \\
0
\end{pmatrix} \right\}.
\]

We find the desired identities with the help of Maple. But we are not able to prove them at this stage. Hence we state these identities as the following conjecture.

#### Conjecture 3.5
We have
\[
\sum_{i,j \geq 0} \frac{q^{\frac{1}{3}i^2-ij+2j^2-\frac{1}{2}i+2j}}{(q^3;q^3)_i(q^3;q^3)_j} = 3 \frac{J_6 J_{15} J_{18,90} J_{27,90}}{J_3^2 J_{90}^2} - \frac{J_{10} J_{1,30} J_{4,30} J_{5,30} J_{6,30} J_{11,30} J_{14,30}}{J_3 J_{30}^2 J_{3,30}},
\]
(3.35)
\[
\sum_{i,j \geq 0} \frac{q^{\frac{1}{2}i^2-ij+2j^2+\frac{1}{2}i}}{(q^3;q^3)_i(q^3;q^3)_j} = \frac{J_{10} J_{2,30} J_{5,30} J_{7,30} J_{8,30} J_{12,30} J_{13,30} J_{14,30}}{J_3 J_{30}^2 J_{9,30}^2} + 3q^2 \frac{J_6 J_{15} J_{9,90} J_{36,90}}{J_3 J_{90}^2}.
\]
(3.36)

Below we discuss how this conjecture is formulated. We try to find the 3-dissections:
\[
\sum_{i,j \geq 0} \frac{q^{\frac{1}{3}i^2-ij+2j^2-\frac{1}{2}i+2j}}{(q^3;q^3)_i(q^3;q^3)_j} = F_0(q^3) + qF_1(q^3) + q^2F_2(q^3),
\]
(3.37)
\[
\sum_{i,j \geq 0} \frac{q^{\frac{1}{2}i^2-ij+2j^2+\frac{1}{2}i}}{(q^3;q^3)_i(q^3;q^3)_j} = G_0(q^3) + qG_1(q^3) + q^2G_2(q^3)
\]
(3.38)

with \( F_i(q), G_i(q) \in \mathbb{Z}[q] \ (i = 1, 2, 3) \).

It is easy to prove that
\[
\frac{1}{2}i^2 - ij + 2j^2 - \frac{1}{2}i + 2j \equiv 0, 1 \pmod{3}, \quad \frac{1}{2}i^2 - ij + 2j^2 + \frac{3}{2}i \equiv 0, 2 \pmod{3}.
\]

Hence
\[
F_2(q) = 0, \quad G_1(q) = 0.
\]
(3.39)

With the help of Maple, it appears that
\[
F_0(q) = 2 \frac{J_2 J_{15} J_{6,30} J_{9,30}}{J_2^2 J_{30}^2},
\]
(3.40)
\[
G_2(q) = 2 \frac{J_2 J_{15} J_{5,30} J_{12,30}}{J_2^2 J_{30}^2}
\]
(3.41)

and there are no single product representations for \( F_1(q) \) and \( G_0(q) \). This then motivates us to subtract the original series from suitable multiples of \( F_0(q^3) \) and \( q^2G_2(q^3) \) so that the results have simple product representations. As a consequence, we find the identities stated in the above conjecture.

To prove Conjecture 3.5 one might need to prove (3.40) and (3.41) first and then find representations for \( F_1(q) \) and \( G_0(q) \).
3.6. Example 6. The matrix and vector parts for this example are

\[ A = \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix}, \quad B \in \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}. \]

**Theorem 3.6.** We have

\[ \sum_{i,j \geq 0} \frac{q^{2i^2+j^2+2ij}}{(q; q)_i(q; q)_j} = \frac{(q^3, q^4, q^7; q^7)_\infty}{(q; q)_\infty}, \quad (3.42) \]

\[ \sum_{i,j \geq 0} \frac{q^{2i^2+j^2+2ij+i}}{(q; q)_i(q; q)_j} = \frac{(q^2, q^5, q^7; q^7)_\infty}{(q; q)_\infty}, \quad (3.43) \]

\[ \sum_{i,j \geq 0} \frac{q^{2i^2+j^2+2ij+2i+j}}{(q; q)_i(q; q)_j} = \frac{(q^6, q^7; q^7)_\infty}{(q; q)_\infty}. \quad (3.44) \]

As noted in [14, Example 3.4.3], these identities are special cases of the Andrews-Gordon identity [11], which states that for integers \( k, s \) such that \( k \geq 2 \) and \( 1 \leq s \leq k \),

\[ \sum_{n_1, \ldots, n_k \geq 0} \frac{q^{N_1^2 + \cdots + N_k^2 + N_s + \cdots + N_{k-1}}}{(q; q)_{n_1} \cdots (q; q)_{n_k}} = \frac{(q^s, q^{2k+1-s}, q^{2k+1}; q^{2k+1})_\infty}{(q; q)_\infty} \quad (3.45) \]

where if \( j \leq k-1 \), \( N_j = n_j + \cdots + n_{k-1} \) and \( N_k = 0 \). This is a generalization of the Rogers-Ramanujan identities.

If we set \( k = 3 \) and \( s = 3, 2, 1 \), we obtain \( (3.42) \), \( (3.43) \) and \( (3.44) \), respectively.

3.7. Example 7. The matrix and vector parts for this example are

\[ A = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1 \end{pmatrix}, \quad B \in \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \right\}. \]

**Theorem 3.7.** We have

\[ \sum_{i,j \geq 0} \frac{q^{2i^2+2j^2-2ij}}{(q; q)_i(q^4; q^4)_j} = \frac{J_4^3 J_{56} J_{6,28}}{J_2^2 J_8 J_{56} J_{24,56}} + 2q \frac{J_8 J_{56} J_{8,56}}{J_4^2 J_{4,56}}, \quad (3.46) \]

\[ \sum_{i,j \geq 0} \frac{q^{2i^2+2j^2-2ij+2i-2j}}{(q^4; q^4)_i(q^4; q^4)_j} = 2 \frac{J_8 J_{56} J_{24,56}}{J_4^2 J_{12,56}} + q \frac{J_4^3 J_{28} J_{10,56} J_{18,56}}{J_2^2 J_8 J_{56} J_{16,56} J_{24,56}}, \quad (3.47) \]

\[ \sum_{i,j \geq 0} \frac{q^{2i^2+2j^2-2ij+2i}}{(q; q^4)_i(q^4; q^4)_j} = \frac{J_8 J_{28} J_{56} J_{26,56}}{J_2^2 J_8 J_{56} J_{56} J_{16,56} J_{24,56}} + 2q^3 \frac{J_8 J_{56} J_{16,56}}{J_4^2 J_{20,56}}. \quad (3.48) \]

**Proof.** (1) We need to find a two dissection for the left side:

\[ \sum_{i,j \geq 0} \frac{q^{2i^2+2j^2-2ij}}{(q^4; q^4)_i(q^4; q^4)_j} = F_0(q^2) + q F_1(q^2), \quad F_0(q), F_1(q) \in \mathbb{Z}[q]. \quad (3.49) \]

Clearly, the parity of the exponent \( i^2 + 2j^2 - 2ij \) is the same with \( i \). Hence,

\[ F_0(q^2) = \sum_{i,j \geq 0} \frac{q^{(2i)^2+2j^2-2(2i)j}}{(q^4; q^4)_i(q^4; q^4)_j}. \quad (3.50) \]
Therefore,
\[
F_0(q) = \sum_{i,j \geq 0} \frac{q^{2i^2+j^2-2ij}}{(q^2; q^2)_{2i}(q^2; q^2)_{2j}} = \sum_{i=0}^{\infty} \frac{q^{2i^2}}{(q^2; q^2)_{2i}} \sum_{j=0}^{\infty} \frac{q^{j^2-j} \cdot q^{(1-2i)j}}{(q^2; q^2)_j}
= \sum_{i=0}^{\infty} \frac{q^{2i^2}}{(q^2; q^2)_{2i}}(-q^{1-2i}; q^2)_{\infty} = (-q; q^2)_{\infty} \sum_{i=0}^{\infty} \frac{q^{j^2} (q; q^2)_j}{(q^2; q^2)_j}.
\]
(3.51)

Replacing \(q\) by \(-q\) in (2.18) and substituting it into (3.51), we obtain
\[
F_0(q) = \frac{J_2^3 J_{28} J_{3,14}}{J_1^2 J_{4,28} J_{12,28}}.
\]
(3.52)

Next, we have
\[
q F_1(q^2) = \sum_{i,j \geq 0} \frac{q^{(2i+1)^2+2j^2-2(2i+1)j}}{(q^4; q^4)_{2i+1}(q^4; q^4)_{2j}}.
\]
(3.53)

Thus
\[
F_1(q) = \sum_{i,j \geq 0} \frac{q^{2i^2+2i+j^2-j-2ij}}{(q^2; q^2)_{2i+1}(q^2; q^2)_{2j}}.
\]
(3.54)

We have
\[
F_1(q^{\frac{1}{2}}) = \sum_{i,j \geq 0} \frac{q^{i^2+i} \cdot q^{(2j-j)/2-ij}}{(q; q)_{2i+1}(q; q)_j} = \sum_{i=0}^{\infty} \frac{q^{i^2+i}}{(q; q)_{2i+1}} \sum_{j=0}^{\infty} \frac{q^{(j-2j)/2} \cdot q^{-ij}}{(q; q)_j}
= \sum_{i=0}^{\infty} q^{i^2+i} (-q^{-i}; q)_{\infty} = 2(-q; q)_{\infty} \sum_{i=0}^{\infty} q^{(i^2+i)/2} (-q; q)_i.
\]
(3.55)

Substituting (2.14) into (3.55), we deduce that
\[
F_1(q^{\frac{1}{2}}) = 2 \frac{J_2 J_{2,14} J_{2,14}}{J_1^2 J_{1,14}}.
\]
(3.56)

Substituting (3.52) and (3.56) into (3.49), we obtain (3.46).

(2) Now we prove the second identity. Again, we need to make a 2-dissection:
\[
\sum_{i,j \geq 0} \frac{q^{i^2+2j^2-2ij+2i-2j}}{(q^4; q^4)_i(q^4; q^4)_j} = G_0(q^2) + q G_1(q^2), \quad G_0(q), G_1(q) \in \mathbb{Z}[q].
\]
(3.57)

Note that the parity of the exponent \(i^2 + 2j^2 - 2ij + 2i - 2j\) is the same with \(i\). We deduce that
\[
G_0(q^2) = \sum_{i,j \geq 0} \frac{q^{(2i^2+2j^2-2(2i)j+i+4i-2j)}}{(q^4; q^4)_{2i}(q^4; q^4)_{2j}} = \sum_{i,j \geq 0} \frac{q^{4i^2+2j^2-4ij+4i-2j}}{(q^4; q^4)_{2i}(q^4; q^4)_{2j}}.
\]
(3.58)

We have
\[
G_0(q^{\frac{1}{2}}) = \sum_{i,j \geq 0} \frac{q^{i^2-ij+i+(j^2-j)/2}}{(q^2; q^2)_i(q^2; q^2)_j} = \sum_{i=0}^{\infty} \frac{q^{i^2+i}}{(q; q)_{2i}} \sum_{j=0}^{\infty} \frac{q^{(j^2-j)/2} \cdot q^{-ij}}{(q; q)_j}
\]
Substituting (2.20) into (3.62), we obtain

\[ G_0(q^2) = \frac{2J_2 J_{14} J_{6,14}}{J_1^2 J_{3,14}}. \]  

Similarly,

\[ qG_1(q^2) = \sum_{i,j \geq 0} \frac{q^{2i^2 + 2j^2 - 2(2i+1)j + 2(2i+1) - 2j}}{(q^4; q^4)_{2i+1} (q^4; q^4)_j} = \sum_{i,j \geq 0} \frac{q^{4i^2 - 4ij + 2j^2 + 8i - 4j + 3}}{(q^4; q^4)_{2i+1} (q^4; q^4)_j}. \]  

Hence

\[ G_1(q) = \sum_{i,j \geq 0} \frac{q^{2i^2 + 4i + 1 + j^2 - 2j - 2ij}}{(q^2; q^2)_{2i+1} (q^2; q^2)_j} = \sum_{i=0}^{\infty} q^{2i^2 + 4i + 1} \sum_{j=0}^{\infty} q^{j^2 - j} \cdot q^{-(2i+1)j} \]  

\[ = \sum_{i=0}^{\infty} q^{2i^2 + 4i + 1} (q^{2i^2} - q^{2i}) \cdot (q^2; q^2)_{2i+1}. \]  

Substituting (2.20) into (3.62), we obtain

\[ G_1(q) = \frac{J_2^3 J_{14} J_{5,28} J_{9,28}}{J_1^2 J_{4,28} J_{8,28} J_{12,28}}. \]  

Now substituting (3.60) and (3.63) into (3.57), we obtain (3.47).

(3) It remains to prove (3.48). We make a 2-dissection:

\[ \sum_{i,j \geq 0} \frac{q^{i^2 + 2j^2 - 2ij + 2i}}{(q^4; q^4)_i (q^4; q^4)_j} = H_0(q^2) + qH_1(q^2). \]  

Note that the parity of \( i^2 + 2j^2 - 2ij + 2i \) is the same with \( i \). We have

\[ H_0(q^2) = \sum_{i,j \geq 0} \frac{q^{4i^2 + 2j^2 - 4ij + 4i}}{(q^4; q^4)_{2i} (q^4; q^4)_j}, \]  

\[ qH_1(q^2) = \sum_{i,j \geq 0} \frac{q^{4i^2 + 8i + 3 + 2j^2 - 4ij - 2j}}{(q^4; q^4)_{2i+1} (q^4; q^4)_j}. \]  

From (3.65) we have

\[ H_0(q) = \sum_{i,j \geq 0} \frac{q^{2i^2 + j^2 - 2ij + 2i}}{(q^2; q^2)_{2i} (q^2; q^2)_j} = \sum_{i=0}^{\infty} q^{2i^2 + 2i} \sum_{j=0}^{\infty} q^{j^2 - j} \cdot q^{(1-2i)j} \]  

\[ = \sum_{i=0}^{\infty} q^{2i^2 + 2i} (q^{2i^2} - q^{2i}) \cdot (q^2; q^2)_{2i}. \]  

Substituting (2.19) into (3.67), we obtain

\[ H_0(q) = \frac{J_2^3 J_{14} J_{1,28} J_{13,28}}{J_1^2 J_{4,28} J_{4,28} J_{8,28}}. \]
By (2.1) and (2.2), we have

\[ H_1(q^1) = q^{1/2} \sum_{i,j \geq 0} q^{i^2+2i-i-j+(j^2-j)/2} (q; q)_{2i+1}(q; q)_j = q^{1/2} \sum_{i=0}^{\infty} q^{i^2+2i} \sum_{j=0}^{\infty} q^{(j^2-j)/2} \cdot q^{-i} \cdot (q; q)_j \]

\[ = q^{1/2} \sum_{i=0}^{\infty} q^{i^2+2i} (-q^{-i}; q)_{\infty} = 2q^{1/2} (-q; q)_{\infty} \sum_{i=0}^{\infty} q^{(i^2+3i)/2} (-q; q)_i. \]  

(3.69)

Substituting (2.16) into (3.69), we obtain

\[ H_1(q^1) = 2q^{1/2} \frac{J_{2}J_{14}J_{4,14}}{J_{1}^{2}J_{5,14}}. \]  

(3.70)

Substituting (3.68) and (3.70) into (3.64), we obtain (3.48).

3.8. Example 8. The matrix and vector parts for this example are

\[ A = \begin{pmatrix} 3/2 & 1 \\ 1 & 2 \end{pmatrix}, \quad B \in \left\{ \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \end{pmatrix} \right\}. \]

Theorem 3.8. We have

\[ \sum_{i,j \geq 0} q^{3i^2+4j^2+4ij-2i} (q^4; q^4)_{i}(q^4; q^4)_{j} = \sum_{i,j \geq 0} q^{i^2+4ij+4j^2+4j} (q^4; q^4)_{i}(q^8; q^8)_{j} = \frac{J_{14}J_{28}J_{13,28}}{J_{1,28}J_{4,28}J_{8,28}J_{12,28}}, \]  

(3.71)

\[ \sum_{i,j \geq 0} q^{3i^2+4j^2+4ij} (q^4; q^4)_{i}(q^4; q^4)_{j} = \sum_{i,j \geq 0} q^{i^2+4ij+4j^2+4j} (q^4; q^4)_{i}(q^8; q^8)_{j} = \frac{J_{14}J_{28}J_{6,28}}{J_{3,28}J_{4,28}J_{11,28}J_{12,28}}, \]  

(3.72)

\[ \sum_{i,j \geq 0} q^{3i^2+4j^2+4ij+2i+4j} (q^4; q^4)_{i}(q^4; q^4)_{j} = \sum_{i,j \geq 0} q^{i^2+4ij+4j^2+4j+4j} (q^4; q^4)_{i}(q^8; q^8)_{j} = \frac{J_{14}J_{28}J_{10,28}}{J_{5,28}J_{8,28}J_{9,28}J_{12,28}}. \]

(3.73)

Here the second double sum expression in each identity was found by the author in the proof. This example is more technical than the others since we cannot eliminate the variables \( i \) or \( j \) from the left side in an easy way. To achieve this, we need to use an integral method mentioned in Section 2 to give the new double sum representations stated above.

Proof of Theorem 3.8. We define

\[ F(u, v; q) := \sum_{i,j \geq 0} q^{2i^2+(i+2j)^2} u^{i} v^{j} (q^4; q^4)_{i}(q^4; q^4)_{j}. \]  

(3.74)

By (2.1) and (2.2), we have

\[ F(u, v; q) = \oint \sum_{i=0}^{\infty} q^{2i^2} z^{i} u^{i} \sum_{j=0}^{\infty} z^{2j} v^{j} \sum_{k=-\infty}^{\infty} q^{k^2} z^{-k} \frac{dz}{2\pi iz} \]

\[ = \oint (-uzq^2; q^4)_{\infty} (-qz, -qz; q^2; q^2)_{\infty} \frac{dz}{(vz^2; q^4)_{\infty}}. \]  

(3.75)

Now we discuss the identities one by one.
Applying (2.1) and (2.2) to (3.76), we deduce that

$$F(q^{-2}, 1; q) = \int \frac{(-z; q^4)_{\infty}(-qz, -q/z, q^2; q^2)_{\infty}}{(z^2; q^4)_{\infty}} \frac{dz}{2\pi iz}$$

$$= \int \frac{(-z; q^4)_{\infty}(-qz, -q/z, q^2; q^2)_{\infty}}{(z^{2q^8}; q^8)_{\infty}(z^2q^4; q^8)_{\infty}} \frac{dz}{2\pi iz}$$

$$= \int \frac{(-qz, -q/z, q^2; q^2)_{\infty}}{(z; q^4)(z^2q^4; q^8)_{\infty}} \frac{dz}{2\pi iz}. \quad (3.76)$$

Applying (2.1) and (2.2) to (3.76), we deduce that

$$F(q^{-2}, 1; q) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{z^i}{(q^4; q^4)_i} \frac{z^{2j}q^{4j}}{(q^8; q^8)_j} \sum_{k=-\infty}^{\infty} q^{k^2}z^{-k} \frac{dz}{2\pi iz}$$

$$= \sum_{i,j \geq 0} \frac{q^{(i+2j)^2+4j}}{(q^4; q^4)_i(q^8; q^8)_j}. \quad (3.77)$$

Now we have

$$\sum_{i,j \geq 0} \frac{q^{(i+2j)^2+4j}}{(q^4; q^4)_i(q^8; q^8)_j} = \sum_{i \geq 0} \frac{q^{i^2}}{(q^4; q^4)_i} \sum_{j \geq 0} \frac{q^{4j^2-4j}q^{(4i+8)j}}{(q^8; q^8)_j}$$

$$= \sum_{i \geq 0} \frac{q^{i^2}(-q^{4i+8}; q^8)_{\infty}}{(q^4; q^4)_i} = R_0(q) + R_1(q). \quad (3.78)$$

Here $R_0(q)$ and $R_1(q)$ correspond to sums over even and odd values of $i$, respectively. To be precise,

$$R_0(q) = \sum_{i=0}^{\infty} \frac{q^{4i^2}(-q^{8i+8}; q^8)_{\infty}}{(q^8; q^8)_{\infty}}, \quad (3.79)$$

$$R_1(q) = \sum_{i=0}^{\infty} \frac{q^{4i^2+4i+1}(-q^{8i+12}; q^8)_{\infty}}{(q^4; q^4)_{2i+1}}. \quad (3.80)$$

We have

$$R_0(q^{1/2}) = \sum_{i=0}^{\infty} \frac{q^{i^2}(-q^{2i+2}; q^2)_{\infty}}{(q; q)_{2i}} = (-q^2; q^2)_{\infty} \sum_{i=0}^{\infty} \frac{q^{i^2}}{(q; q)_{2i}(-q^2; q^2)_i}$$

$$= (-q^2; q^2)_{\infty} \sum_{i=0}^{\infty} \frac{q^{i^2}(-q; q^2)_{i}}{(q^2; q^2)_{2i}} = \frac{J_{14}J_{3, 28}J_{8, 28}J_{11, 28}}{J_1J_{28}^3}. \quad (3.81)$$

Here for the last equality we used (2.18).

Similarly,

$$R_1(q^{1/2}) = q^{1/2} \sum_{i=0}^{\infty} \frac{q^{2i+1}(-q^{2i+3}; q^2)_{\infty}}{(q; q)_{2i+1}} = q^{1/2}(-q^2; q^2)_{\infty} \sum_{i=0}^{\infty} \frac{q^{2i+1}}{(q; q)_{2i+1}(-q^2; q^2)_{i+1}}$$

$$= q^{1/2}(-q^2; q^2)_{\infty} \sum_{i=0}^{\infty} \frac{q^{2i+1}(-q^2; q^2)_{i}}{(q^2; q^2)_{2i+1}} = q^{1/2} \frac{J_{14}J_{12, 28}J_{6, 28}J_{10, 28}}{J_1J_{28}^3}. \quad (3.82)$$

Here for the last equality we used (2.14) with $q$ replaced by $q^2$. 
Here for the last equality we used (2.15) with $q$ after simple verifications using the method in [9].

(2) Setting $(u, v) = (1, 1)$ in (3.75), we deduce that

$$F(1, 1; q) = \int \frac{(-q^2 z; q^4)_\infty (-q, -q/z, q^2; q^2)_\infty}{(z^2; q^4)_\infty} \frac{dz}{2\pi iz}$$

$$= \int \frac{(-q^2 z; q^4)_\infty (-q, -q/z, q^2; q^2)_\infty}{(q^4 z^2; z^2; q^8)_\infty} \frac{dz}{2\pi iz}$$

$$= \int \frac{(-q z, -q/z; q^2)_\infty}{(q^2 z; q^4)_\infty (z^2; q^8)_\infty} \frac{dz}{2\pi iz}. \quad (3.83)$$

Using (2.1) and (2.2) and arguing similarly as in (1), we get the first equality in (3.72).

Next, we have

$$\sum_{i,j \geq 0} q^{i^2+4ij+4j^2+2i} (q^4; q^4)_i (q^8; q^8)_j = \sum_{i \geq 0} q^{i^2+2i} \sum_{j \geq 0} q^{4j^2-4j} \cdot (q^{4i+4})_i (q^8; q^8)_j$$

$$= \sum_{i \geq 0} q^{i^2+2i} \frac{(-q^{4i+4}; q^8)_\infty}{(q^4; q^4)_i} = S_0(q) + S_1(q). \quad (3.84)$$

Here $S_0(q)$ and $S_1(q)$ correspond to sums over even and odd values of $i$, respectively. To be precise,

$$S_0(q) = \sum_{i=0}^{\infty} q^{4i^2+4i} \frac{(-q^{8i+4}; q^8)_\infty}{(q^4; q^4)_{2i}}, \quad (3.85)$$

$$S_1(q) = \sum_{i=0}^{\infty} q^{4i^2+8i+3} \frac{(-q^{8i+8}; q^8)_\infty}{(q^4; q^4)_{2i+1}}. \quad (3.86)$$

We have

$$S_0(q^{1/2}) = \sum_{i=0}^{\infty} q^{i^2+i} \frac{(-q^{2i+1}; q^2)_\infty}{(q; q)_{2i}} = (-q; q^2)_\infty \sum_{i=0}^{\infty} \frac{q^{i^2+i}}{(q; q)_{2i} (-q; q^2)_i}$$

$$= (-q; q^2)_\infty \sum_{i=0}^{\infty} \frac{q^{i^2+i} (-q^2; q^2)_i}{(q^2; q^2)_{2i}} = \frac{J_{14} J_{2.28} J_{10.28} J_{12.28}}{J_1 J_{28}^3}. \quad (3.87)$$

Here for the last equality we used (2.15) with $q$ replaced by $q^2$.

Similarly,

$$S_1(q^{1/2}) = q^{3/2} \sum_{i=0}^{\infty} q^{i^2+2i} \frac{(-q^{2i+2}; q^2)_\infty}{(q; q)_{2i+1}} = q^{3/2} (-q^2; q^2)_\infty \sum_{i=0}^{\infty} \frac{q^{i^2+2i}}{(q; q)_{2i+1} (-q^2; q^2)_i}$$

$$= q^{3/2} (-q^2; q^2)_\infty \sum_{i=0}^{\infty} \frac{q^{i^2+2i} (-q^2; q^2)_{i+1}}{(q^2; q^2)_{2i+1}} = q^{3/2} \frac{J_{14} J_{4.28} J_{5.28} J_{9.28}}{J_1 J_{28}^3}. \quad (3.88)$$

Here for the last equality we used (2.20).

Substituting (3.87) and (3.88) into (3.84), we obtain the second equality in (3.72) after simple verifications using the method in [9].
(3) Setting \((u, v) = (q^2, q^4)\) in (3.75), we have

\[
F(q^2, q^4; q) = \oint \frac{(-q^4 z; q^4)_\infty (-q z, -q^2 z, q^2; q^2)_\infty}{(q^4 z^2; q^4)_\infty} \frac{d z}{2\pi i z} = \oint \frac{(-q^4 z; q^4)_\infty (-q z, -q^2 z, q^2; q^2)_\infty}{(q^8 z^2; q^8)_\infty (q^4 z^2; q^4)_\infty} \frac{d z}{2\pi i z} = \oint \frac{(-q^4 z, -q z, q^2, q^2)_\infty}{(q^4 z^2; q^4)_\infty (q^4 z^2; q^8)_\infty} \frac{d z}{2\pi i z}.
\]

Applying (2.1) and (2.2) to (3.75), we obtain the first equality in (3.73).

Next, we have

\[
\sum_{i,j \geq 0} q^{i^2+4j^2+4i+4j} (q^4; q^4)_i (q^8; q^8)_j = \sum_{i \geq 0} \frac{q^{i^2+4i} \sum_{j \geq 0} q^{4j^2-4j} q^{4i+8j}}{(q^8; q^8)_j} = \sum_{i \geq 0} \frac{q^{i^2+4i} (-q^{4i+8}; q^8)_\infty}{(q^4; q^4)_i} = T_0(q) + T_1(q).
\]

Here \(T_0(q)\) and \(T_1(q)\) correspond to sums over even and odd values of \(i\), respectively. To be precise,

\[
T_0(q) = \sum_{i=0}^{\infty} \frac{q^{4i^2+8i} (-q^{8i+8}; q^8)_\infty}{(q^4; q^4)_{2i}},
\]

\[
T_1(q) = \sum_{i=0}^{\infty} \frac{q^{4i^2+12i+5} (-q^{8i+12}; q^8)_\infty}{(q^4; q^4)_{2i+1}}.
\]

We have

\[
T_0(q^4) = \sum_{i=0}^{\infty} \frac{q^{i^2+2i} (-q^{2i+2}; q^2)_\infty}{(q; q)_{2i}} = (-q^2; q^2)_\infty \sum_{i=0}^{\infty} \frac{q^{i^2+2i}}{(q; q)_{2i}} (-q^2; q^2)_i = \frac{J_{14} J_{1.28} J_{12.28} J_{13.28}}{J_{1} J_{3}^{4}}.
\]

Here for the last equality we used (2.19).

Similarly,

\[
T_1(q^4) = q^2 \sum_{i=0}^{\infty} \frac{q^{i^2+3i} (-q^{2i+3}; q^2)_\infty}{(q; q)_{2i+1}} = q^2 (-q; q^2)_\infty \sum_{i=0}^{\infty} \frac{q^{i^2+3i}}{(q; q)_{2i+1}} (-q^2; q^2)_{i+1} = q^2 \frac{J_{14} J_{2.28} J_{6.28} J_{8.28}}{J_{1} J_{3}^{4}}.
\]

Substituting (3.93) and (3.94) into (3.70), we obtain the second equality in (3.73) after simple verifications using the method in [9].

We now present a different proof for the product representations in Theorem 3.8 without using the second double sum expression in each identity. This is actually the first proof we found when doing this project. The computations in this proof are
a bit more complicated, but it tells us that the theorem can also be reduced to other identities in Slater’s list. Here we follow the techniques in the author’s work [23].

Second proof of the product representations in Theorem 3.8. (1) We rewrite (3.70) as

\[
F(q^{-2}, 1; q) = (q^2; q^4)_{\infty} \oint \frac{(-qz, -q^3z, -q/z, -q^3/z, q^4; q^4)_{\infty}}{(z, q^2z, -q^2z; q^4)_{\infty}} \frac{dz}{2\pi iz}. \tag{3.95}
\]

To prove the second equality in (3.71), we use (3.95). Applying Lemma 2.1 with \(q\) replaced by \(q^4\) and

\((A, B, C, D) = (2, 2, 3, 0), \quad (a_1, a_2) = (b_1,b_2) = (-q, -q^3), \quad (c_1, c_2, c_3) = (1, q^2, -q^2),\)

we deduce that

\[
F(q^{-2}, 1; q) = (q^2; q^4)_{\infty} (R_1(q) + R_2(q) + R_3(q)), \tag{3.96}
\]

where

\[
R_1(q) := \frac{(-q, -q, -q^3, -q^3; q^4)_{\infty}}{(q^2, -q^2; q^4)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}(-q^3, -q^3; q^4)_n}{(q^4, -q, -q^3, q^2, -q^2; q^4)_n}, \tag{3.97}
\]

\[
R_2(q) := \frac{(-q^3, -q^5, -q^{-1}, -q; q^4)_{\infty}}{(q^{-2}, -1; q^4)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{2n^2+4n}(-q^5, -q^3; q^4)_n}{(q^4, -q^3, -q^5, q^6, -q^4; q^4)_n}, \tag{3.98}
\]

\[
R_3(q) := \frac{(q^3, q^5, -q^{-1}, q; q^4)_{\infty}}{(-1, -q^{-2}; q^4)_{\infty}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n^2+4n} (q^5, q^3, q^4)_n}{(q^4, q^3, q^5, -q^4, -q^6; q^4)_n}, \tag{3.99}
\]

Substituting the identity (2.15) with \(q\) replaced by \(q^4\) into (3.97), we deduce that

\[
R_1(q) = \frac{J_3^4 J_8 J_{56} J_{24,56}}{J_1^4 J_{12,56}}. \tag{3.100}
\]

Substituting the identity (2.20) with \(q\) replaced by \(q^2\) (resp. \(-q^2\)) into (3.98) (resp. (3.99)), we deduce that

\[
R_2(q) = -\frac{1}{2q} \frac{J_2^4 J_4 J_{8,56} J_{10,28}}{J_1^2 J_8 J_{56}}, \tag{3.101}
\]

\[
R_3(q) = -\frac{1}{2q} \frac{J_1^2 J_4 J_{2,28} J_{6,28}}{J_2 J_{14} J_{28} J_{4,28} J_{12,28}}. \tag{3.102}
\]

Now substituting (3.100)–(3.102) into (3.96), using the method in [9], it is easy to verify that the second equality of (3.71) holds.
(2) From (3.83) we have
\[ F(1, 1; q) = (q^2; q^4)_\infty \int \frac{(-qz, -q^3z, -q/z, -q^3/z; q^4)_\infty}{(z, -z; q^2z; q^4)_\infty} \frac{dz}{2\pi iz}. \] (3.103)
Applying Lemma 2.1 with \( q \) replaced by \( q^4 \) and
\( (A, B, C, D) = (2, 2, 3, 0), \quad (a_1, a_2) = (b_1, b_2) = (-q, -q^3), \quad (c_1, c_2, c_3) = (1, -1, q^2), \)
we deduce that
\[ F(1, 1; q) = (q^2; q^4)_\infty (S_1(q) + S_2(q) + S_3(q)), \] (3.104)
where
\[ S_1(q) = \frac{(-q, -q, -q^3, -q^3; q^4)_\infty}{(-1, q^2; q^4)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2+4n}(-q^3, -q; q^4)_n}{(q^4, -q^4, q^2, -q, -q^3; q^4)_n}, \]
\[ = \frac{1}{2} \frac{(-q; q^2)_\infty^2}{(q^2, -q^4; q^4)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2+4n}(q^3; q^4)_n(q^6; q^8)_n}{(q^4, q^2, q^3, -q^2, -q^2; q^4)_n}, \] (3.105)
\[ S_2(q) = \frac{(q, q^3; q^4)_\infty}{(-1, -q^2; q^4)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n^2+4n}(q, q^3; q^4)_n}{(q^4, q^2, q^3, -q^2, q^4)_n}, \]
\[ = \frac{1}{2} \frac{(q^2; q^4)_\infty^2}{(-q^2, q^4; q^8)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n^2+4n}}{(q^4, q^4, q^8; q^8)_n}, \] (3.106)
\[ S_3(q) = \frac{(-q^3, -q^5, -q^{-1}, -q; q^4)_\infty}{(-q^2, -q^2, q^4; q^4)_\infty} \sum_{n=0}^{\infty} \frac{(-q^5, -q^3; q^4)_n q^{2n^2+6n}}{(q^4, -q^3, -q^6, -q^6, q^4)_n}, \]
\[ = -q^3 \frac{(-q; q^2)_\infty^2}{(q^4, q^8; q^8)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2+6n}(q^4; q^4)_n(q^4; q^8)_{n+1}}{(q^4, q^4, q^8; q^8)_n}. \] (3.107)
Substituting (2.19) with \( q \) replaced by \( q^2 \) (resp. \( -q^2 \)) into (3.105) (resp. (3.106)), we deduce that
\[ S_1(q) = \frac{1}{2} \frac{J_2 J_4 J_{28} J_{25,56} J_{26,56}}{J_1 J_8 J_{56} J_{56} J_{16,56}}, \] (3.108)
\[ S_2(q) = \frac{1}{2} \frac{J_2 J_4 J_{10,28} J_{10,28}}{J_2 J_4 J_{28} J_{28} J_{12,28}}. \] (3.109)
Substituting (2.16) (with \( q \) replaced by \( q^4 \)) into (3.107), we obtain
\[ S_3(q) = -q^3 \frac{J_2 J_2 J_3}{J_1 J_4 J_{8,56} J_{20,56} J_{24,56}}. \] (3.110)
Now substituting (3.108)–(3.110) into (3.104), using the method in [9], it is easy to verify that the second equality in (3.72) holds.

(3) From (3.89) we have
\[ F(q^2, q^4; q) = (q^2; q^4)_\infty \int \frac{(-qz, -q^3z, -q/z, -q^3/z; q^4)_\infty}{(q^2z, q^2z, -q^2z; q^4)_\infty} \frac{dz}{2\pi iz}. \] (3.111)
Applying Lemma 2.1 with
\( (A, B, C, D) = (2, 2, 3, 0), \quad (a_1, a_2) = (b_1, b_2) = (-q, -q^3), \quad (c_1, c_2, c_3) = (q^4, q^2, -q^2), \)
we deduce that
\[ F(q^2, q^4; q) = (q^2; q^4)_\infty (T_1(q) + T_2(q) + T_3(q)), \tag{3.112} \]
where
\[
T_1(q) = \frac{(-q^5, -q^7, -q^{-3}, -q^{-1}; q^4)_\infty}{(q^{-2}, -q^{-2}; q^4)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}(-q^7, -q^5; q^4)_n}{(q^4, -q^5, -q^7, q^6; -q^4)_n}
\]
\[
= \frac{(-q; q^2)_\infty^2}{(q^2; q^8)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q^4, q^4)_n(q^8; q^8)_n}, \tag{3.113}
\]
\[
T_2(q) = \frac{(-q^3, -q^5, -q^{-1}, -q; q^4)_\infty}{(q^2, -1; q^4)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2}(-q^5, -q^3; q^4)_n}{(q^4, -q^3, -q^5, -q^2; -q^4)_n}
\]
\[
= \frac{1}{2} q^{-1} \frac{(-q; q^2)_\infty^2}{(q^2, -q^4; q^4)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2, q^4)_n(q^8; q^8)_n}, \tag{3.114}
\]
\[
T_3(q) = \frac{(q^3, q^5, q^{-1}, q; q^4)_\infty}{(-1, -q^2; q^4)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n^2} (q^5, q^3; q^4)_n}{(q^4, q^3, q^5, -q^4, -q^2; q^4)_n}
\]
\[
= \frac{1}{2} q^{-1} \frac{(q; q^2)_\infty^2}{(-q^2, q^2; q^4)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n^2}}{(-q^2, q^4)_n(q^8; q^8)_n}. \tag{3.115}
\]
Substituting (2.14) (with \( q \) replaced by \( q^4 \)) into (3.113), we obtain
\[
T_1(q) = -\frac{J_2^2 J_6^2 J_{29}^2}{J_1^2 J_4 J_{16}, J_{56} J_{24}, J_{16}}. \tag{3.116}
\]
Substituting (2.18) with \( q \) replaced by \( q^2 \) (resp. \( -q^2 \)) into (3.114) (resp. (3.115)), we obtain
\[
T_2(q) = \frac{1}{2} q^{-1} \frac{J_2^2 J_4 J_{28} J_{68,68} J_{22,26}}{J_2^2 J_5 J_{36} J_{56} J_{24,26}}, \tag{3.117}
\]
\[
T_3(q) = -\frac{1}{2} q^{-1} \frac{J_2^2 J_2 J_{28} J_{48,28} J_{48,28}}{J_2^2 J_4 J_{28} J_{48,28}}. \tag{3.118}
\]
Substituting (3.116)–(3.118) into (3.112), using the method in [9], it is easy to verify that the second equality in (3.73) holds. \( \Box \)

3.9. Example 9. The matrix and vector parts for this example are
\[
A = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 3/4 \end{pmatrix}, \quad B \in \left\{ \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \right\}.
\]

**Theorem 3.9.** We have
\[
\sum_{i,j \geq 0} q^{4i^2+3j^2-4ij-4i+2j} \frac{(q^8; q^8)_i(q^8; q^8)_j}{(q^8; q^8)_i(q^8; q^8)_j}
= 2 \frac{J_{16} J_{48,112}}{J_8^2} + q \frac{J_8 J_6^2 J_{112} J_{8,112} J_{24,112}}{J_4 J_{4,112} J_{16,112} J_{28,112} J_{48,112} J_{52,112}}, \tag{3.119}
\]
Thus where

\[
\sum_{i,j \geq 0} \frac{q^{4i^2+3j^2-4ij}}{(q^8; q^8)_i(q^8; q^8)_j} = \frac{J_8 J_{56}^2 J_{112}^2 J_{40,112}}{J_4 J_{12,112} J_{28,112} J_{32,112} J_{44,112} J_{48,112}} + 2q^3 \frac{J_{16}^3 J_{32,112}}{J_8^2}, \tag{3.120}
\]

\[
\sum_{i,j \geq 0} q^{4i^2+3j^2-4ij+4j} = \frac{J_8 J_{56}^2 J_{112}^2 J_{8,112} J_{40,112}}{J_4 J_{16,112} J_{20,112} J_{28,112} J_{32,112} J_{36,112}} + 2q^7 \frac{J_{16} J_{16,112}}{J_8^2}. \tag{3.121}
\]

Proof. (1) We need to find the 4-dissection:

\[
\sum_{i,j \geq 0} q^{4i^2+3j^2-4ij+2j} \frac{(q^8; q^8)_i(q^8; q^8)_j}{(q^8; q^8)_i(q^8; q^8)_j} = F_0(q^4) + qF_1(q^4) + q^2F_2(q^4) + q^3F_3(q^4),
\]

where \(F_0(q), F_1(q), F_2(q), F_3(q) \in \mathbb{Z}[q]\). Note that

\[
4i^2 + 3j^2 - 4ij - 4i + 2j \equiv 1 - (j - 1)^2 \equiv \begin{cases} 0 & \text{(mod 4), } j \equiv 0 \text{ (mod 2)}, \\ 1 & \text{(mod 4), } j \equiv 1 \text{ (mod 2)}. \end{cases}
\]

Hence

\[
F_2(q) = F_3(q) = 0. \tag{3.123}
\]

We have

\[
F_0(q^4) = \sum_{i,j \geq 0} q^{4i^2+3(2j)^2-8ij-4i+4j} \frac{(q^8; q^8)_i(q^8; q^8)_j}{(q^8; q^8)_i(q^8; q^8)_j} = \sum_{i,j \geq 0} q^{4i^2-4i+12j^2-8ij+4j} \frac{(q^8; q^8)_i(q^8; q^8)_j}{(q^8; q^8)_i(q^8; q^8)_j}. \tag{3.124}
\]

Thus

\[
F_0(q) = \sum_{i,j \geq 0} q^{i^2+j^2+3j+2i-j} \frac{q^{i^2-2i+3j^2+j-2ij}}{(q^2; q^2)_i(q^2; q^2)_j}. \tag{3.125}
\]

We have

\[
F_0(q^{1/2}) = \sum_{i,j \geq 0} q^{(i^2-i) / 2 + (3j^2+j)/2 - ij} \frac{(q; q)_i(q; q)_j}{(q; q)_i(q; q)_j} = \sum_{j=0}^{\infty} q^{(3j^2+j)/2} \sum_{i=0}^{\infty} \frac{q^{i^2-i/2} \cdot (q^{-j})^i}{(q; q)_i}.
\]

\[
= \sum_{j=0}^{\infty} q^{(3j^2+j)/2} (-q^{-j}; q)_{\infty} = 2(-q; q)_{\infty} \sum_{j=0}^{\infty} q^{j^2} (-q; q)_j \frac{(q; q)_j}{(q; q)_{2j}}. \tag{3.126}
\]

Substituting (1.7) into (3.126), we deduce that

\[
F_0(q^{1/2}) = 2 \frac{J_2 J_{6,14}}{J_1^2}. \tag{3.127}
\]

Next, we have

\[
qF_1(q^4) = \sum_{i,j \geq 0} q^{4i^2-8ij+12j^2-8i+16j+5} \frac{(q^8; q^8)_i(q^8; q^8)_j}{(q^8; q^8)_i(q^8; q^8)_{j+1}}. \tag{3.128}
\]
Thus

\[ F_1(q) = \sum_{i,j \geq 0} q^{i^2 - 2ij + 3j^2 - 2i + 4j + 1} = \sum_{j=0}^{\infty} q^{3j^2 + 4j + 1} \sum_{i=0}^{\infty} q^{2i - 1} \cdot q^{-(2j+1)i} \]

\[ = \sum_{j=0}^{\infty} q^{3j^2 + 4j + 1} (q^2; q^2)_{2j+1} = (-q; q^2)_{\infty} \sum_{j=0}^{\infty} q^{2j^2 + 2j} (q; q^2)_{2j+1}. \quad (3.129) \]

Substituting (2.5) with \( q \) replaced by \(-q\) into (3.129), we deduce that

\[ F_1(q) = \frac{J_2 J_4^2 J_{28} J_{6.28}}{J_1 J_{1.128} J_{4.28} J_{7.28} J_{12.28} J_{13.28}}. \quad (3.130) \]

Substituting (3.123), (3.127) and (3.130) into (3.122), we obtain (3.119).

(2) We need to find the 4-dissection:

\[ \sum_{i,j \geq 0} q^{4i^2 + 3j^2 - 4ij} (q^8; q^8)_{2}(q^8; q^8)_{2j} = G_0(q^4) + qG_1(q^4) + q^2G_2(q^4) + q^3G_3(q^4), \quad (3.131) \]

where \( G_0(q), G_1(q), G_2(q), G_3(q) \in \mathbb{Z}[q] \). Note that

\[ 4i^2 + 3j^2 - 4ij \equiv -j^2 \equiv \begin{cases} 0 \pmod{4}, & j \equiv 0 \pmod{2}, \\ 3 \pmod{4}, & j \equiv 1 \pmod{2}. \end{cases} \quad (3.132) \]

It follows that

\[ G_1(q) = G_2(q) = 0. \quad (3.133) \]

We have

\[ G_0(q^4) = \sum_{i,j \geq 0} q^{4i^2 + 12j^2 - 8ij} (q^8; q^8)_{2}(q^8; q^8)_{2j}. \quad (3.134) \]

Thus

\[ G_0(q) = \sum_{i,j \geq 0} q^{i^2 + 3j^2 - 2ij} (q^8; q^8)_{2}(q^8; q^2)_{2j} = \sum_{j=0}^{\infty} q^{3j^2} (q^2; q^2)_{2j} \sum_{i=0}^{\infty} q^{2i - 1} \cdot q^{(1-2j)i} \]

\[ = \sum_{j=0}^{\infty} q^{3j^2} (q^2; q^2)_{2j} (q^{-2j+1}; q^2)_{\infty} = (-q; q^2)_{\infty} \sum_{j=0}^{\infty} q^{2j^2} (-q; q^2)_{2j}. \quad (3.135) \]

Substituting (2.7) (with \( q \) replaced by \(-q\)) into (3.135), we deduce that

\[ G_0(q) = \frac{J_2 J_4^2 J_{28} J_{6.28} J_{10.28}}{J_1 J_{3.28} J_{7.28} J_{8.28} J_{11.28} J_{12.28}}. \quad (3.136) \]

Next, we have

\[ q^3 G_3(q^4) = \sum_{i,j \geq 0} q^{4i^2 + 3(2j+1)^2 - 4i(2j+1)} (q^8; q^8)_{2}(q^8; q^8)_{2j+1} = \sum_{i,j \geq 0} q^{4i^2 - 8ij + 12j^2 - 4i + 12j + 3} (q^8; q^8)_{2}(q^8; q^8)_{2j+1}. \quad (3.137) \]

We have

\[ G_3(q^{\frac{1}{2}}) = \sum_{i,j \geq 0} q^{(i^2-i)/2 - ij + (3j^2+3j)/2} (q; q)_{2j+1} = \sum_{j=0}^{\infty} q^{(3j^2+3j)/2} \sum_{i=0}^{\infty} q^{(i^2-i)/2} \cdot q^{-ji} (q; q)_{2j+1} \]
Thus

\[ q \sum_{j=0}^{\infty} \frac{q^{(3j^2+3j)/2}(-q^{-j};q)_\infty}{(q;q)_{2j+1}} = 2(-q;q)_\infty \sum_{j=0}^{\infty} \frac{q^{j^2+j}(-q;q)_j}{(q;q)_{2j+1}}. \]  

(3.138)

Substituting (1.6) into (3.138), we deduce that

\[ G_3(q^2) = 2 \frac{J_2J_{4,14}}{J_1^2}. \]  

(3.139)

Substituting (3.133), (3.136) and (3.139) into (3.131), we obtain (3.120).

(3) We need to find the 4-dissection:

\[ \sum_{i,j \geq 0} q^{4i^2+3j^2-4ij+4j} (q^8; q^8)_i (q^8; q^8)_j = H_0(q^4) + qH_1(q^4) + q^2H_2(q^4) + q^3H_3(q^4), \]  

(3.140)

where \(H_0(q), H_1(q), H_2(q), H_3(q) \in \mathbb{Z}[q].\) Note that

\[ 4i^2 + 3j^2 - 4ij + 4j \equiv -j^2 \equiv \begin{cases} 0 \pmod{4}, & j \equiv 0 \pmod{2}, \\ 3 \pmod{4}, & j \equiv 1 \pmod{2}. \end{cases} \]  

(3.141)

It follows that

\[ H_1(q) = H_2(q) = 0. \]  

(3.142)

We have

\[ H_0(q^4) = \sum_{i,j \geq 0} q^{4i^2+12j^2-8ij+8j} (q^8; q^8)_i (q^8; q^8)_j. \]  

(3.143)

Thus

\[ H_0(q) = \sum_{i,j \geq 0} q^{2i^2+3j^2-2ij+2j} (q^2; q^2)_i (q^2; q^2)_j = \sum_{j=0}^{\infty} q^{3j^2+2j} (q^2; q^2)_j \sum_{i=0}^{\infty} q^{i^2-i} \cdot q^{(1-2j)i} (q^2; q^2)_i \]  

\[ = \sum_{j=0}^{\infty} q^{3j^2+2j} (q^2; q^2)_j (-q^{-2j}; q^2)_\infty = (-q; q^2)_\infty \sum_{j=0}^{\infty} q^{2j^2+2j} (-q; q^2)_j. \]  

(3.144)

Substituting (2.6) (with \(q\) replaced by \(-q\)) into (3.144), we deduce that

\[ H_0(q) = \frac{J_2J_{7,14}J_{28}J_{2,28}J_{10,28}}{J_1J_{4,28}J_{5,28}J_{7,28}J_{8,28}J_{9,28}}. \]  

(3.145)

Next, since

\[ q^3H_3(q^4) = \sum_{i,j \geq 0} q^{4i^2-8ij+12j^2-4ij+20j+7} (q^8; q^8)_i (q^8; q^8)_j, \]  

(3.146)

we have

\[ H_3(q^2) = q^2 \sum_{i,j \geq 0} q^{(i^2-i)/2-ij+(3j^2+5j)/2} (q; q)_i (q; q)_{2j+1} = q^2 \sum_{j=0}^{\infty} q^{3j^2+5j/2} (q; q)_{2j+1} \sum_{i=0}^{\infty} q^{(i^2-i)/2} \cdot q^{-ji} (q; q)_i \]  

\[ = q^2 \sum_{j=0}^{\infty} q^{(3j^2+5j)/2} (q; q)_{2j+1} (-q^{-j}; q^2)_\infty = 2q^{1/2} (-q; q^2)_\infty \sum_{j=0}^{\infty} q^{j^2+j} (-q; q)_j. \]  

(3.147)
Substituting (1.5) into (3.147), we deduce that

\[ H_3(q^2) = 2q^{1/2}J_2J_{2.14}J_1^2. \]  

(3.148)

Substituting (3.142), (3.145) and (3.148) into (3.140), we obtain (3.121). \( \Box \)

3.10. **Example 10.** The matrix and vector parts for this example are

\[ A = \begin{pmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{pmatrix}, \quad B \in \left\{ \begin{pmatrix} -2/3 \\ -1/3 \end{pmatrix}, \begin{pmatrix} -1/3 \\ -2/3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}. \]

Since the (1, 1)-entry and the (2, 2)-entry of \( A \) are the same, the first and the second vectors give the same identity.

This example has been discussed by Vlasenko and Zwegers [22, p. 633, Table 1]. They found the following conjectural identities, and a justification of them were provided by Cherednik and Feigin [6] via the nilpotent double affine Hecke algebras. Thus we state them as a theorem.

**Theorem 3.10.** We have

\[
\sum_{i,j \geq 0} q^{2i^2+2ij+j^2-2i-j} (q^3; q^3)_i(q^3; q^3)_j = \frac{1}{(q^3; q^3)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n \left( 2q^{45n^2+\frac{9}{2}n} + q^{45n^2+\frac{9}{2}n+4} - q^{45n^2+\frac{9}{2}n+13} \right)
\]

\[
= \frac{1}{J_3} \left( 2J_{18,45} + qJ_{12,45} + q^4J_{3,45} \right), \tag{3.149}
\]

\[
\sum_{i,j \geq 0} q^{2i^2+2ij+j^2} (q^3; q^3)_i(q^3; q^3)_j
\]

\[
= \frac{1}{(q^3; q^3)_\infty} \sum_{n=-\infty}^{\infty} \left( 2q^{45n^2+\frac{9}{2}n+2} + q^{45n^2+\frac{9}{2}n} - q^{45n^2+\frac{9}{2}n+3} \right)
\]

\[
= \frac{1}{J_3} \left( J_{21,45} - q^3J_{6,45} + 2q^2J_{9,45} \right). \tag{3.150}
\]

For each of the identity, the first equality is the one given by [22], while the second equality follows from the first one and (2.2). Note that Cherednik and Feigin [6, p. 1074] proved that the sum sides of these identities are modular functions, and then said that “Then one needs to compare only few terms in the \( q \)-expansions to establish their coincidence”.

Though we cannot find purely \( q \)-series proofs for this theorem (which would be very exciting), we find equivalent formulas for it.

**Theorem 3.11.** We have

\[
\sum_{i,j \geq 0} q^{2i^2+2ij+j^2-2i-j} (q^3; q^3)_i(q^3; q^3)_j = 3 \frac{J_{18,45}}{J_3} - \frac{J_5J_{1,15}}{J_{15}^2J_3}J_{4,15}, \tag{3.151}
\]

\[
\sum_{i,j \geq 0} q^{2i^2+2ij+j^2} (q^3; q^3)_i(q^3; q^3)_j = \frac{J_5J_{2,15}J_{7,15}}{J_{15}^2J_{6,15}} + 3q^2 \frac{J_{6,45}}{J_3}. \tag{3.152}
\]
The equivalence of Theorems 3.10 and 3.11 can be proved easily using the method in [9].

3.11. Example 11. The matrix and vector parts for this example are

\[ A = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}, \quad B \in \left\{ \begin{pmatrix} -1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}. \]

This example has been discussed by Vlasenko and Zwegers [22, p. 629, Theorem 3.2]. They gave modular form representations for it. We shall state their result in the following equivalent form.

**Theorem 3.12.** We have

\[
\sum_{i,j \geq 0} \frac{q^{i^2+j^2-ij}}{(q^2;q^2)_i(q^2;q^2)_j} = 2J_4J_{8,20} - \frac{J_2^2J_{10,20}J_{3,20}}{J_1J_4J_{1,20}J_{5,20}J_{6,20}J_{8,20}J_{9,20}},
\]

(3.153)

\[
\sum_{i,j \geq 0} \frac{q^{i^2+j^2-ij}}{(q^2;q^2)_i(q^2;q^2)_j} = \frac{J_2^2J_{10,20}J_{3,20}}{J_1J_4J_{1,20}J_{5,20}J_{6,20}J_{7,20}J_{9,20}} + 2qJ_4J_{4,20},
\]

(3.154)

We will not repeat the proof in [22]. But for the convenience of the reader, we sketch briefly the steps. To prove (3.153), one may sum over \( i \) using (2.1), and then split the sum into two sums corresponding to even and odd values of \( j \), respectively. Then for the sum with \( j \) even, we can use (2.13). For the sum with \( j \) odd, we can use (2.17).

Similarly, the identity (3.154) can be reduced to the identities (2.4) and (2.12).

So far we have discussed all the eleven rank two examples discovered by Zagier. Before closing this paper, we make two remarks. First, according to the notion used in [23], all of the above identities for Zagier’s examples are Rogers-Ramanujan type identities of index \((1,1)\), except that Theorem 3.2 also contains double sums of index \((1,2)\). Here we slightly modified the notion in [23] by allowing the product side to be finite sums of infinite products instead of just one single product. Second, Zagier [25] also stated many possible modular triples when the rank \( r = 3 \). In an undergoing work, we have verified all of them by proving some identities of index \((1,1,1)\). This will be discussed in detail in a separate paper.

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