\( \lambda \phi^4 \) model and Higgs mass in standard model calculated by Gaussian effective potential approach with a new regularization-renormalization method

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Abstract

Basing on new regularization-renormalization method, the \( \lambda \phi^4 \) model used in standard model is studied both perturbatively and nonperturbatively (by Gaussian effective potential). The invariant property of two mass scales is stressed and the existence of a (Landau) pole is emphasized. Then after coupling with the SU(2) × U(1) gauge fields, the Higgs mass in standard model (SM) can be calculated as \( m_H \approx 138 \text{GeV} \). The critical temperature (\( T_c \)) for restoration of symmetry of Higgs field, the critical energy scale (\( \mu_c \), the maximum energy scale under which the lower excitation sector of the GEP is valid) and the maximum energy scale (\( \mu_{\text{max}} \), at which the symmetry of the Higgs field is restored) in the standard model are \( T_c \approx 476 \text{ GeV} \), \( \mu_c \approx 0.547 \times 10^{15} \text{ GeV} \) and \( \mu_{\text{max}} \approx 0.873 \times 10^{15} \text{ GeV} \) respectively.

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1 Introduction

Year after year, the standard model (SM) in particle physics enjoys its great success, especially after the discovery of the top quark in 1995\textsuperscript{[1,2]}. Now the careful phenomenological analysis even leads to a very impressive conclusion that the only unobserved particle mass, the Higgs mass $m_H$, is constrained within a rather narrow interval, say $130 \sim 150$ GeV by present experimental data\textsuperscript{[3,4,5]}.

On the other hand, the calculation on $m_H$ by quantum field theory (QFT) lagged behind the experimental progress. It is well known that at tree level, the ratio of $m_H^2$ to $m_W^2$ reads

$$\frac{m_H^2}{m_W^2} = \frac{4}{3} \frac{\lambda}{g^2}. \quad (1)$$

However, unlike the gauge coupling constant $g$, the value of $\lambda$ is unknown. So one has to resort to QFT beyond tree level. Then the calculation turns out to be rather difficult and confused due to the divergence, counter-term and the ambiguity between bare and physical parameters. Furthermore, a puzzle of so called “triviality” existed in $\lambda \phi^4$ model\textsuperscript{[6]} which rendered the situation more complicated. For many years, only a lower bound and/or an upper bound on $m_H$ were obtained\textsuperscript{[7]}.

Nine years ago, believing in triviality and introducing a large but fixed cut off $\Lambda$, we had attacked this problem by the Gaussian effective potential (GEP) method in QFT. We found\textsuperscript{[8,9]}

$$76 \text{ GeV} < m_H < 170 \text{ GeV} \quad (2)$$

which was still rather unreliable and unsatisfied since we had been bothered by all these difficulties mentioned above.

Now we are in a much better position to restudy the problem. Basing on a new regularization-renormalization (R-R) method first proposed by one of us, Yang \textsuperscript{[10,11]}, and further applied in Refs \textsuperscript{[12,13]}, we can get rid of all the annoying divergence, counter-term and bare parameters so that a clearcut value of $m_H \approx 138$ GeV will emerge after the input of the present accurate experimental data.

The organization of this paper is as follows: In section 2, the $\lambda \phi^4$ model used in SM will be studied both perturbatively and nonperturbatively (by GEP method). Then in section 3, the singularity (Landau pole) is stressed. the running coupling constant and renormalization group equation are also discussed. After a brief summary on $\lambda \phi^4$ model in section 4, section 5 is devoting to the calculation of Higgs mass in SM. The final section 6 contains the summary and discussions.
2 $\lambda \phi^4$ model with spontaneous symmetry breaking (SSB)

The Lagrangian of $\lambda \phi^4$ with wrong sign in mass term reads

$$L = \partial^\mu \phi \partial_\mu \phi + \frac{1}{2} \sigma \phi^2 - \frac{\lambda}{4!} \phi^4. \quad (3)$$

2.1 One loop (L=1) calculation

Besides the tree level (L=0) contribution to the effective potential (EP)

$$V_0 = -\frac{1}{2} \sigma \phi^2 + \frac{1}{4!} \lambda \phi^4, \quad (4)$$

the one-loop contribution to EP is evaluated as

$$V_1(\phi) = \frac{1}{2} \int \frac{d^4k_E}{(2\pi)^4} \ln(k_E^2 - \sigma + \frac{1}{2} \lambda \phi^2), \quad (<\phi> \rightarrow \phi). \quad (5)$$

Denoting $M^2 = -\sigma + \frac{1}{2} \lambda \phi^2$, we get $\frac{\partial^2 V_1}{\partial (M^2)^2} = \frac{1}{32\pi^2 M^2}$ and

$$V_1(\phi) = +\frac{1}{32\pi^2} \left\{ \left( \frac{1}{2} \lambda \phi^2 - \sigma \right)^2 \left[ \frac{1}{2} \ln \left( \frac{\lambda \phi^2 - 2\sigma}{2\mu_1^2} \right) - \frac{3}{4} \right] + C_2 \left( \frac{1}{2} \lambda \phi^2 - \sigma \right) + C_3 \right\} \quad (6)$$

with three arbitrary constants $\mu_1$, $C_2$, and $C_3$. To fix them, we calculate $\frac{dV_{\text{eff}}}{d\phi} = \frac{d}{d\phi}(V_0 + V_1) = 0$. As the symmetric phase $\phi_0 = 0$ is not interesting to us, we manage to fix the spontaneous symmetry breaking (SSB) phase at

$$\phi_1^2 = \frac{6\sigma}{\lambda} \quad (7)$$

as that at tree level by choosing $\mu_1^2 = C_2 = 2\sigma$. At the same time, the mass square of excitation at SSB phase reads

$$m_\sigma^2 \equiv \left. \frac{d^2 V_{\text{eff}}}{d\phi^2} \right|_{\phi = \phi_1} = 2\sigma \quad (8)$$

while the renormalized coupling constant is modified:

$$\lambda_R = \left. \frac{d^4 V_{\text{eff}}}{d\phi^4} \right|_{\phi = \phi_1} = \lambda \left( 1 + \frac{9\lambda}{32\pi^2} \right). \quad (9)$$

2.2 GEP method

In GEP method, we begin from a Gaussian wave functional (GWF) $|\Psi>^{[8,9,14]}$

$$|\Psi> = N_f \left\{ -\frac{1}{2\hbar} \int_{x,y} (\phi_x - \Phi_x) f_{xy}(\phi_y - \Phi_y) \right\} \quad (10)$$
with \( f_x = f d^3 x, \ \Phi_x = \langle \phi_x \rangle = \langle \Psi|\phi(x)\Psi \rangle \), etc and a correlation function of quantum fluctuation:

\[
f_{xy} = f(\vec{x} - \vec{y}) = \int \frac{d^3 p}{(2\pi)^3} f_p \exp \left[ i \frac{p}{\hbar} \cdot (\vec{x} - \vec{y}) \right]. \tag{11}
\]

Calculating the energy of system in this GWF, we get \( E = \langle \Psi|H|\Psi \rangle \) as a function of \( \Phi \) and \( f_p \). A variation \( \frac{\delta E}{\delta f_p} = 0 \) leads to \( (p = |\vec{p}|, \ \hbar = 1) \)

\[
f_p = \sqrt{p^2 + \mu^2}, \quad \mu^2 = -\sigma + \frac{\lambda}{2} \Phi^2 + \frac{\lambda}{4} f_{xx}^{-1}: \tag{12}
\]

where \( f_{xy}^{-1} \) is the inverse of \( f_{xy} \), \( \int f_{xy} f_{yz}^{-1} = \delta^3(\vec{x} - \vec{z}) \). The energy \( E \) is a function of \( \phi(\equiv \Phi) \) and \( \mu^2 \):

\[
E(\phi, \mu) = -\frac{1}{2} \sigma \phi^2 + \frac{\lambda}{24} \phi^4 + \frac{1}{2} I_0 - \frac{1}{2} (\sigma + \mu^2) I_1 + \frac{1}{8} \lambda \phi^2 I_1 + \frac{1}{32} \lambda I_1^2. \tag{13}
\]

with \( I_0(\mu^2) \equiv f_{xx} \equiv I_0, \ I_1(\mu^2) \equiv f_{xx}^{-1} \equiv I_1, \)

\[
I_0 = \int \frac{d^3 p}{(2\pi)^3} \sqrt{p^2 + \mu^2}, \quad I_1 = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{p^2 + \mu^2}}. \tag{14}
\]

The variational condition \( \frac{\partial E}{\partial \mu^2} = 0 \) leads again to \( \mu^2 \) equation (10) with a common factor \( I_2(\mu^2) \equiv I_2 \equiv -2 \frac{\partial I_1}{\partial \mu^2} \) ignored \( (I_2 = 0 \) leads to a trivial sector \( V_G \rightarrow V_0 \) see below). Then the GEP is defined as a function of one variable \( \phi \):

\[
V_G(\phi) \equiv E(\phi, \mu(\phi)) = -\frac{1}{2} \sigma \phi^2 + \frac{\lambda}{24} \phi^4 + \frac{1}{2} I_0 - \frac{1}{32} \lambda I_1^2 \tag{15}
\]

with

\[
\mu^2 = -\sigma + \frac{1}{2} \lambda \phi^2 + \frac{\lambda}{4} I_1, \quad \frac{d\mu^2}{d\phi} = \frac{8 \lambda \phi}{8 + \lambda I_2}. \tag{16}
\]

Note that \( I_0, \ I_1 \) and \( I_2 \) are all divergent. After handling them by our new \( R - R \) method, we obtain

\[
I_2(\mu^2) = -\frac{1}{4 \pi^2} \ln \left( \frac{\mu^2}{\mu_s^2} \right), \quad I_1(\mu^2) = \frac{1}{8 \pi^2} \mu^2 (\ln \left( \frac{\mu^2}{\mu_s^2} \right) - 1) + C_2 \tag{17}
\]

\[
I_0(\mu^2) = \frac{1}{32 \pi^2} \mu^4 (\ln \left( \frac{\mu^2}{\mu_s^2} \right) - \frac{3}{2}) + \frac{1}{2} C_2 \mu^2 + C_3 \tag{18}
\]

with \( \mu_s^2, \ C_2 \) and \( C_3 \) being three arbitrary constants.

For discussing the SSB, we calculate

\[
\frac{dV_{eff}}{d\phi} = \phi [ -\sigma + \frac{\lambda}{6} \phi^2 + \frac{\lambda}{4} I_1(\mu^2) ] \tag{19}
\]

Besides the symmetric phase located at \( \phi_0 = 0 \) the SSB phase \( \phi_1 \) can still be located at

\[
\phi_1^2 = \frac{6 \sigma}{\lambda} \tag{20}
\]
by choosing $\mu_s^2 = \mu_1^2 = \mu^2(\phi_1) = 2\sigma$, \( C_2 = \frac{1}{8\pi^2} \mu_1^2 \).

Meanwhile

\[
\frac{d^2V_{eff}}{d\phi^2} \bigg|_{\phi=\phi_1} = 2\sigma
\]

remains the same as before. However,

\[
\lambda_R \equiv \frac{d^4V_{eff}}{d\phi^4} \bigg|_{\phi_1} = \lambda \left[ 1 + \frac{9}{32} \frac{\lambda}{\pi^2} + \frac{3\lambda^2}{210\pi^4} \right]
\]

is further modified but closed at the order of $\lambda^3$ though GEP method amounts to add up the loop contributions of cactus diagram of $\lambda \phi^4$ model up to $L \to \infty$.

### 3 Singularity in GEP, running coupling constant and renormalization group equation

Being a nonperturbative approach in QFT, GEP method is essentially different from any perturbative calculation up to $L$ being fixed large number. To see this, let us concentrate on gap equation (the first one of (16)) \((x^2 = x, \frac{d^2}{d\phi^2} = y)\):

\[
y = \left( 2 + \frac{1}{16\pi^2} \right) x - \frac{1}{16\pi^2} x \ln x + \left( \frac{1}{\lambda} - \frac{1}{16\pi^2} \right). \tag{23}
\]

It is interesting to see that $y$ is a single-valued but not monotonic function of $x$. The SSB phase is located at \((x = 1, y = \frac{3}{\lambda})\) whereas the decreasing of $x$ to the left side can not reach the symmetric phase $y = 0$ at $x \to 0$ as long as $\lambda < 16\pi^2$. On the right side, increasing of $x$ will lead to a maximum of $y$, $y_{\text{max}}$, at $x_c$

\[
\frac{\mu_c^2}{\mu_1^2} \equiv x_c = \exp \left( \frac{32\pi^2}{\lambda} \right), \quad \frac{\phi_{\text{max}}^2}{2\sigma} \equiv y_{\text{max}} = \frac{1}{2\lambda}(x_c + 1). \tag{24}
\]

Further running of $x$ will arrive at the remote destination \((x = x_{\text{max}}, y = 0)\) where the symmetric phase $\phi_0 = 0$ is restored at high energy excitation ($\mu > \mu_c$) sector.

Return back to $V_G$ as a function of $\phi$, we see that $\phi_{\text{max}}$ corresponding to $x_c$ (or $\mu_c$) is a singular point of $V_G$ because $\frac{d\mu^2}{d\phi}$ is divergent at $x_c$. It divides $V_G$ into two branches (sectors). The low energy excitation sector ($\mu < \mu_c$) contains the SSB phase ($\phi = \phi_1$), whereas the high energy excitation sector ($\mu > \mu_c$) contains the symmetric phase ($\phi = 0$) with very low energy in the whole system. No other stationary state exists. So for low excitation particles, the system is staying at SSB phase and will not collapse to the symmetric phase ($\phi = 0$) of the other sector because of the barrier at $\phi_{\text{max}}$.

In GEP scheme, we define the running coupling constant (RCC) as \((\mathcal{J} \equiv \ln(\mu^2/(2\sigma)) )$

\[
\bar{\lambda}(\mu(\phi)) \equiv \frac{d^4V_G}{d\phi^4} = \mu^{-4} \left( 2^{21} \cdot 3 \lambda^3 \phi^2 \pi^8 \mu^2 + 2^{25} \lambda \pi^{10} \mu^4 + 2^{15} \cdot 3 \lambda^5 \phi^4 \pi^6 - 2^{20} \lambda^4 \phi^4 \pi^8 \right)
\]
\[-(2^{17} \cdot 3\lambda^4 \phi^2 \pi^6 \mu^2 + 2^{21} \pi^8 \mu^4 \lambda^2 - 2^{15} \lambda^5 \phi^4 \pi^6)J + (2^{11} \cdot 3\lambda^5 \phi^2 \pi^4 \mu^2 - 2^{16} \lambda^3 \pi^6 \mu^4)J^2 + 2^{13} \lambda^4 \phi^4 \mu^4 J^3 - 2^{5} \cdot 7\lambda^5 \pi^2 \mu^4 J^4 + 2\lambda^6 \mu^4 J^5) \left(2^{5} \pi^2 - \lambda J \right)^{-5}\]

(25)

and a beta function

\[\beta(\bar{\lambda}) \equiv \mu \frac{d}{d\mu} \bar{\lambda}(\mu) = \frac{1}{(2^{5} \pi^2 - \lambda J)^6 \mu^4} \left((-2^{13} \cdot 15\lambda^5 \pi^4 \mu^4 + 2^{12} \cdot 5\lambda^6 \phi^2 \pi^4 \mu^2)J^3 + \right.\]

\[+ \left[2^{12} \cdot 15\lambda^6 \phi^2 \pi^4 \mu^2 + 2^{17} \pi^6 (45\lambda^4 \mu^4 + \lambda^6 \phi^4 - 15\lambda^5 \phi^2 \mu^2)J^2 + 2^6 \cdot 15\lambda^6 \pi^2 \mu^4 J^4 + 2^{26} \cdot 15\lambda^2 \pi^{10} \mu^4 - 2^{27} \pi^{10} (5\lambda^3 \phi^2 \mu^2 - \lambda^4 \phi^4) - 2^{22} \pi^{8} (5\lambda^5 \phi^4 - 15\lambda^4 \phi^2 \mu^2) + 2^{16} \cdot 15\lambda^6 \phi^4 \pi^6 + \left(2^{22} \pi^{8} (15\lambda^4 \phi^2 \mu^2 - 2\lambda^5 \phi^4 - 30\lambda^3 \mu^4) + 2^{17} \pi^{6} (5\lambda^6 \phi^4 - 30\lambda^5 \phi^2 \mu^2)) \right) \right] J \]

(26)

with

\[\beta(\bar{\lambda})|_{\mu_1} = \frac{3\lambda^2}{(4\pi)^2} + \frac{135\lambda^4}{4(4\pi)^6}\]

(27)

which can be compare with \(\beta(\lambda) = \frac{3\lambda^2}{(4\pi)^2} - \frac{17\lambda^4}{3(4\pi)^6} + \cdots\) usually quoted as in Ref. [15].

Obviously, from Eq. (25), we see that there is a pole of five order in \(\bar{\lambda}\) at \(\mu = \mu_c\). On the other hand, we can define a RCC and beta function in one-loop calculation of EP

\[\bar{\lambda}^{(1)}(M) = \frac{d^4 V_{\text{eff}}}{d\phi^4} = \lambda + \frac{\lambda^2}{32\pi^2} \left(3 \ln \frac{M^2}{2\sigma} + 8 + 4 \frac{\sigma}{M^2} - 4 \frac{\sigma^2}{M^4} \right),\]

(28)

\[\beta^{(1)}(\bar{\lambda}) = M \frac{d\bar{\lambda}}{dM} = \left(3 - 4 \frac{\sigma}{M^2} + 8 \frac{\sigma^2}{M^4} \right) \frac{\lambda^2}{16\pi^4}\]

(29)

with

\[\beta(\bar{\lambda}^{(1)})|_{\mu_1} = \frac{3\lambda^2}{(4\pi)^2}.\]

(30)

It is clear that there is no pole in \(\bar{\lambda}^{(1)}(M)\). That is because the contribution in \(V_{\text{eff}}\) is a sum of L=1 diagram while \(V_G\) is a sum of L→∞ diagram. To re-find the pole from one loop EP, one may perform an improvement on \(\bar{\lambda}(M)\) by renormalization group equation (RGE). Modifying the right hand side of Eq. (30)

\[\bar{\mu} \frac{d}{d\bar{\mu}} \bar{\lambda}(\bar{\mu}) = \frac{3\lambda(\bar{\mu})^2}{(4\pi)^2}.\]

(31)

by \(\lambda \rightarrow \bar{\lambda}(\bar{\mu})\). Integrating the RGE (31) yields

\[\bar{\lambda}(\bar{\mu}) = \frac{\lambda_R}{1 - \frac{3}{16\pi^2} \lambda_R \ln \frac{\bar{\mu}}{\mu_1}}\]

(32)

where \(\mu_1 = \bar{\mu}|_{\phi_1}, \lambda_R = \bar{\lambda}(\mu_1)\).

Evidently, there is a simple pole, \(\bar{\mu} = \bar{\mu}_c = \mu_1 \exp \left(\frac{16\pi^2}{3\lambda_R} \right)\), the so-called Landau pole in Eq. (32). The location of singularity should be an invariant feature of \(\lambda \phi^4\) model at QFT level. So the difference between \(\mu_c\) and \(\bar{\mu}_c\) implies a relation between two running mass scales, \(\mu\) in GEP method and \(\bar{\mu}\) in L=1 RGE calculation.
4 Brief summary on $\lambda \phi^4$ model

(a) The $\lambda \phi^4$ model is well defined at classical level by Lagrangian shown at Eq. (3). However, it is not well defined at QFT level by $L$ solely before it is supplemented by three constants: $C_1 = -\ln \mu_1$, $C_2$ and $C_3$.

(b) While $C_3$ is trivial (it only affects the whole shift of EP), fixing $\mu_1$ and $C_2$ is equivalent to reconfirming two mass scales, $m_\sigma^2 = 2\sigma$ and $\phi_1^2 = \frac{6\sigma}{\lambda}$, in $\lambda \phi^4$ model at QFT level.

(c) Now we understand that the invariant meaning of parameter $\lambda$ in $L$ is not a coupling constant but the ratio of these two mass squares, $m_\sigma^2/\phi_1^2 = \lambda/3$, at any order of loop ($L$) expansion theory even at GEP ($L \rightarrow \infty$) level.

(d) The prominent difference between perturbative theory ($L=\text{finite}$) and nonperturbative theory ($L \rightarrow \infty$) like GEP method (or RGE) lies in the fact that in the latter case there is a singularity at GEP, the critical mass scale $\mu_c = \mu_1 \exp \frac{16\sigma^2}{3\lambda_R}$ (or Landau pole, $\bar{\mu}_c = \mu_1 \exp \frac{16\sigma^2}{3\lambda_R}$, in RGE) whereas in the former there is no singularity. An elementary example is the geometric series $S_n = 1 + r + \cdots + r^n$ is analytic whereas $S_n|_{n \rightarrow \infty} = \frac{1}{1-r}$ ($|r| < 1$) has a pole.

(e) Formally, when the mass of a physical particle exceeds a critical value, $\mu > \mu_c$, a phase transition is triggered. The system would collapse to symmetric phase, $\phi_0 = 0$. Safely speaking, $\mu = \mu_c$ is the upper bound in energy scale of $\lambda \phi^4$ model at QFT level with SSB.

(f) In summary, at QFT level, $\lambda \phi^4$ model with SSB is characterized by two mass scale: ($\phi_1^2 = 6\sigma/\lambda$ and $\mu_c^2 = 2\sigma$) and one singularity in $\mu^2$, $\mu_c^2 = \mu_1^2 \exp (32\pi^2/\lambda)$. It is a renormalizable, nontrivial and effective (up to critical energy $\mu_c$) theory.

5 Calculation of Higgs mass

We are now well prepared to calculate the Higgs mass $m_H$ in SM by GEP method.

As a generalization of Eq. (10), we start from a GWF:

$$|\Psi \rangle = \exp \left\{ -\frac{1}{2} \int_{xy} \{ [\xi(x) - \bar{\xi}(x)] F_{xy}(\bar{\xi}) [\xi(y) - \bar{\xi}(y)] - [W^\mu(x) F_{xy}(\bar{W}) W^\mu(y) \\
+W^\mu(x) F_{xy}(\bar{W}) W^\mu(y)] - Z^\mu(x) F_{xy}(\bar{Z}) Z^\mu(y) + A^\mu(x) F_{xy}^\mu(\bar{A}) A^\mu(y) \} \right\}, \quad (33)$$

where $\xi$ is the real Higgs field while $W^\mu$, $Z^\mu$ and $A^\mu$ are fields of $W$, $Z$ bosons and photon respectively, $\xi = <\Psi|\xi|\Psi >$ etc.. The quantum fluctuation correlation function

$$F_{xy}(\vec{B}) = C_B^3 \int \frac{d^3p}{(2\pi)^3} \sqrt{p^2 + \mu_B^2} \exp [iC_B \vec{p} \cdot (\vec{x} - \vec{y})], \quad (34)$$

($\vec{B} = \vec{W}$, $\vec{Z}$, $\vec{A}$, $\vec{\xi}$, $C_A = C_W = C_Z = \sqrt{3/2} \equiv C^{1/3}$, $C_\xi = 1$), is controlled by the mass parameter $\mu_B$ which is determined via variational procedure and is different for different
fields \((\xi \rightarrow \xi)\) again:

\[
\begin{align*}
\mu^2_\xi & = -\sigma + \frac{\lambda}{2} \left[ \xi^2 + \frac{1}{2} I_1(\mu^2_\xi) \right] + \frac{3}{4} \, g^2 C I_1(\mu^2_W) + \frac{3}{8} C (g^2 + g'^2) I_1(\mu^2_Z), \\
\mu^2_W & = \frac{1}{4} g^2 \left[ \xi^2 + \frac{1}{2} I_1(\mu^2_\xi) \right] + g^2 C I_1(\mu^2_W) + \frac{g^4 C}{g^2 + g'^2} I_1(\mu^2_Z) + \frac{g^2 g'^2 C}{g^2 + g'^2} I_1(\mu_A = 0), \\
\mu^2_Z & = \frac{1}{4} (g^2 + g'^2) \left[ \xi^2 + \frac{1}{2} I_1(\mu^2_\xi) \right] + 2 \frac{g^4 C}{g^2 + g'^2} I_1(\mu^2_W),
\end{align*}
\]

(35)

(36)

(37)

where \(g\) and \(g'\) are coupling constants in \(SU(2) \times U(1)\) gauge model. Some explanations are important:

(a) We need not introduce any counter terms related to gauge fields for ensuring the massless property of gauge bosons at symmetric phase \((\xi = 0)\) because the low energy symmetric phase is not at the same sector with the SSB phase \((\xi = \xi_1)\) under consideration as shown in pure \(\lambda \phi^4\) theory \((\lambda < 16\pi^2)\).

(b) While \(\mu^2_W|_{\xi_1} = m^2_W\) and \(\mu^2_Z|_{\xi_1} = m^2_Z\) are the observed mass square of W and Z bosons at SSB phase, the parameter

\[
\mu^2_\xi|_{\xi_1} \equiv \mu^2_1 = \frac{\lambda}{3} \xi^2_1 \neq 2\sigma
\]

(38)

is not the Higgs mass square as that at tree level. We will soon find the expression for Higgs mass after the quantum corrections are added.

(c) We set the mass parameter for photon field, \(\mu_A\), always zero, \(\mu_A = 0\), as shown in the last term of Eq. (36).

(d) After performing the same renormalization procedure as in \(\lambda \phi^4\) model, \(I_1(\mu^2)\) in Eqs. (35~37) has the form

\[
I_1(\mu^2) = \frac{1}{8\pi^2} \mu^2 \left( \ln \frac{\mu^2}{\mu^2_1} - 1 \right) + C_2.
\]

(39)

Taking \(C_2 = \frac{\mu^2_1}{8\pi^2} = I_1(0)\) further, we have \(I_1(\mu^2_1) = 0\). Basing on Eqs. (35)-(37) and using the experimental data\(^{[3,4,5]}\):

\[
\begin{align*}
\alpha^{-1} = \frac{4\pi}{g^2 \sin^2 \theta} & = 128.89, \quad \sin^2 \theta \equiv \frac{g'^2}{g^2 + g'^2} = 0.2317, \\
m_W & = 80.359 \text{ GeV}, \quad m_Z = 91.1884 \text{ GeV},
\end{align*}
\]

(40)

we manage to find the values of \(\mu_1, \lambda\) and \(\sigma\). Denoting

\[
w_1 = \frac{m^2_W}{\mu^2_1}, \quad a = \frac{m^2_Z}{m^2_W} = 1.2877
\]

(41)

and calculating \((36)|_{\xi_1} \times (g^2 + g'^2) - (37)|_{\xi_1} \times g^2\), one obtains

\[
\begin{align*}
\{ \sec^2 \theta - a - \frac{g^2 C}{8\pi^2} [a(\ln a - 1) - \sec^2 \theta + 2 \cos^2 \theta] \} w_1 - \frac{g^2 C}{4\pi^2} (\sec^2 \theta - \cos^2 \theta) & = \frac{g^2 C}{8\pi^2} (\sec^2 \theta - 2 \cos^2 \theta + a) w_1 \ln w_1
\end{align*}
\]

(42)
0.0210070w_1 - 0.0104422 = 0.0103063w_1 \ln w_1

One finds (apart from a meaningless solution \( w_1 \gg 1 \), see final discussion)

\[
w_1 = \frac{m_W^2}{\mu_1^2} = 0.3183153i

Substituting the value (43) into Eq. (37)|_{\xi_1} with (38), one finds:

\[
\lambda = 1.0139453
\]

Then it is easy to find the value of \( \sigma \) from Eq.(35)|_{\xi_1}:

\[
\frac{\sigma}{\mu_1^2} = 0.5034030
\]

which means that \( \mu_1^2 \) is not far from its value at tree level, \( 2\sigma \).

Apart from the fermion contribution to be added below, the Higgs mass square reads (see Ref. [9])

\[
\frac{d^2 V_{eff}}{d\xi^2}|_{\xi_1} = \frac{2\xi_1^2}{3} \left[ \frac{4(\lambda r + 3s) - \lambda(\lambda r + s)I_2(\mu_1^2)}{8r + (\lambda r + s)I_2(\mu_1^2)} \right] = \mu_1^2 \left( 1 + \frac{3s}{\lambda r} \right)
\]

where \( I_2(\mu_1^2) = -\frac{1}{4\pi^2} \ln \frac{\mu^2}{\mu_1^2} \).

\[
r = 2 + g^2 C I_2(\mu_W^2) - g^8 C^2 I_2(\mu_W^2) I_2(\mu_Z^2)/(g^2 + g'^2)^2
\]

\[
s = \frac{3}{8}(g^2 + g'^2)^2 \left[ \left( \frac{g^4}{(g^2 + g'^2)^2} - \frac{1}{4} \right) \frac{g^2 C I_2(\mu_W^2) I_2(\mu_Z^2)}{I_2(\mu_W^2) - \frac{C}{2} I_2(\mu_Z^2)} \right]
\]

We see that the quantum fluctuation effect of gauge fields on the Higgs mass is very small: \( \frac{3s}{\lambda r} = -0.00853833 \).

Moreover, the fermions will contribute to Higgs mass at one loop level as discussed in Ref. 12. A fermion with mass \( m_i \) contributes:

\[-\frac{G_i^2}{2\pi^2 m_i^2} \ln \frac{m_i^2}{\mu_1^2} \]

In SSB theory of SM, \( G_i = \left( \frac{m_i}{\xi_1} \right) \), so the Higgs mass \( m_H \) should be evaluated as

\[
\left( \frac{m_H}{\mu_1} \right)^2 = 1 + \frac{3s}{\lambda r} - \frac{1}{2\pi^2} \sum_{i=e,\mu,\tau} \frac{m_i^4 \ln \frac{m_i^2}{\mu_1^2}}{\mu_1^2} - \frac{3}{2\pi^2} \sum_{q=u,d,s,c,b,t} \frac{m_q^4 \ln \frac{m_q^2}{\mu_1^2}}{\mu_1^2}
\]

The extra factor 3 for quarks comes from the color freedom. Actually, only the top quark with \( m_t = 175 \text{GeV} \) makes the main contribution. Eventually, we find

\[
\left( \frac{m_H}{\mu_1} \right)^2 = 0.943251 \quad \text{i.e.,} \quad m_H = 138.331 \text{Gev} \approx 138 \text{Gev}.
\]
6 Summary and discussions

(1) The motivation of adopting GEP method is the following. The value of weak mixing (Weinberg) angle $\theta$ derived from the experiments of neutral current process

$$\sin^2 \theta = \frac{g'^2}{g^2 + g'^2} \approx 0.2317$$  \hspace{1cm} (50)

is different from the value derived from the mass ratio of $W, Z$ bosons,

$$1 - \frac{m_W^2}{m_Z^2} \approx 0.2234$$  \hspace{1cm} (51)

due to the quantum corrections to all orders in perturbation theory. However, to evaluate the discrepancy 3.7% in QFT is not easy. Then GEP method has the advantage of providing an analytically calculable scheme. As shown in Eqs. (35~37), we assume that the gauge fields undergo the same quantum fluctuation as that of Higgs field in a GWF but with different mass parameters, which are linking together. Hence the difference between (50) and (51) provides a possibility to find the value of $\lambda$ and thus the Higgs mass.

(2) It is interesting to see that $\lambda \approx 1$. Then we can find a critical temperature $T_c$ for restoration of SSB phase ($\xi_1$) to symmetric phase ($\xi = 0$), as discussed in Ref. [16] or Ref. [12]:

$$T_c = \sqrt{\frac{12}{\lambda}} m_H.$$  \hspace{1cm} (52)

Substituting the value of $\lambda$ and $m_H$ here, we find

$$T_c = 475.886 \text{GeV} \approx 476 \text{GeV}$$  \hspace{1cm} (53)

which is not far from 510 GeV as estimated in Ref. 12 by other method.

(3) In pure $\lambda \phi^4$ model, there is a critical mass scale $\mu_c(= \sqrt{2}\sigma \exp(16\pi^2/\lambda))$, beyond which the system will collapse to symmetric phase. After coupling with gauge fields, this critical value $\mu_\xi = \mu_c$ should be solved from Eqs. (35~37) together with the vanishing condition of denominator of $\frac{\partial \mu^2}{\partial \xi}$, i.e.,

$$8r + (\lambda r + s) I_2(\mu_\xi^2) = 0.$$  \hspace{1cm} (54)

Numerical calculation yields

$$\mu_c \approx 0.547 \times 10^{15} \text{Gev}.$$  \hspace{1cm} (55)

Furthermore, the maximum energy scale, $\mu_{\text{max}} = \mu_\xi|_{\xi = 0}$, as can be solved from Eqs. (5~7) with $\xi = 0$ is approximately:

$$\mu_{\text{max}} \approx 0.873 \times 10^{15} \text{Gev}.$$  \hspace{1cm} (56)
at which the symmetry of Higgs field is restored in high energy sector ($\mu_\xi > \mu_c$) whereas at $T = T_c$ symmetry restoration occurs at low excitation sector ($\mu_\xi < \mu_c$).

(4) The advantage of our new R-R method can be seen as follows. In Ref. [9], $I_2(\mu_1) \sim \frac{1}{2\pi^2} \ln \frac{\Lambda}{\mu_1}$ was logarithmically divergent whereas now $I_2(\mu_1^2) = 0$. So whole calculation becomes quite clear and well under control.

(5) As stressed in $\lambda\phi^4$ theory, the model is characterized by two mass scales ($\xi_1 = \sqrt{6\sigma/\lambda}$ and $\mu_1 = \sqrt{2\sigma}$) and one singularity $\mu_c = \sqrt{2\sigma \exp(16\pi^2/\lambda)}$. After coupling with gauge fields, both $\xi_1$ and $\mu_1$ are modified to some extent while keeping their ratio form $\xi_1/\mu_1 = \sqrt{3/\lambda}$ invariant. The critical value $\mu_c$ is strongly suppressed to $\mu_c \approx 0.547 \times 10^{15}$ Gev, which could be viewed as the upper bound of energy scale in SM with SSB.

Nonetheless, the whole model is well defined (reconfirmed) at nonperturbative QFT level.

(6) Because the experimental data are not quite fixed yet[5], for checking the sensitivity of our results to input, we have calculated the value of $m_H$ over a wide rage: $80.26\text{Gev} < m_w < 80.36\text{Gev}$ ($m_Z = 91.1884$ Gev) and $0.2316 < \sin^2 \theta < 0.2325$. The result shows that the average value is

$$< m_H > \approx 140.96\text{Gev}$$

while $m_H^{max} \approx 143.11\text{Gev}$ and $m_H^{min} \approx 124.92\text{Gev}$.

(7) Finally, we would like to compare some recent literatures on Higgs mass. The estimation by Altarelli and G. Isidori[17] or by Eillis et al[18] is not far from that of ours. On the other hand, the prediction of $m_H \sim 2\text{Tev}$ as in Ref. [19] seems too high to be considered.

We thank Profs. Su-qing Chen, Y-s Duan, T. Huang, H. C. Lee, K. Wu, H. L. Yu, X-m Zhang, Z-x Zhang, Z-y Zhu and Dr. H-q Zheng for discussion and encouragement. This work was supported in part by the National Science Foundation in China.
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