On the automorphism group of non-singular plane curves fixing the degree

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Abstract. This note is devoted, after the result of Harui [11], to solve the following questions for non-singular plane curve of degree $d$ over an algebraically closed field $K$ of zero characteristic.

(1) Given a finite non-trivial group $G$ that appears as an automorphism group of a plane non-singular curve $C$ inside $\mathbb{P}^2_K$ of a fixed degree $d \geq 4$. Is the defining equation of such $C$ unique? i.e. is there, up to $K$-equivalence, a unique homogenous polynomial $F_C(X; Y; Z)$ of degree $d$ that depends on a set of parameters, as its monomials’ coefficients with some restrictions on these parameters, such that any $G$ as above is given by the equation $F_C(X; Y; Z) = 0$ in $\mathbb{P}^2(K)$ via a certain specialization of the parameters and vice versa?

By the work of Henn [12] and Komiya-Kuribayashi [14], we conclude that the answer of this question is positive for $d = 4$. That is, all non-singular plane quartics $C$ with $Aut(C) \cong G$ satisfy a unique equation (endowed with a set of restrictions on the parameters), up to change of variables.

In Chapter 1, we show that for $d > 4$ the answer is no more positive and counter examples are provided. Moreover, if $G$ be a finite group that appears as the full automorphism group of a non-singular projective plane curve $C$ of degree $d > 4$ over $K$, then it may happen that there are non isomorphic plane homogeneous equations of degree $d$ (each of them given by a set of parameters which some restrictions in the parameters) such that their automorphism groups (when we specialize the parameters) are isomorphic to $G$. Moreover, the number of the different equations (up to $K$-isomorphism) is bounded above by the number of different conjugacy classes of the injective representations of $G$ inside $PGL_3(K)$, but this bound is usually far to be sharp. Furthermore, we prove the same results when $K$ has a positive characteristic $p$ with $p > (d - 1)(d - 2) + 1$ where $d$ is the degree of the curve.

(2) In Chapter 2, we are concerned with the following classical problem. Fix a degree $d \geq 4$, which kind of groups could appear as the full automorphism groups of non-singular projective plane curves and what are the defining equations that are associated to each such group?

The results of Harui in [11] simplify the problem to a certain list of finite groups that one should work with. These groups can be classified up to conjugation as: cyclic groups, an element of $Ext^1(-, -)$ of a cyclic group by a dihedral group, an alternating group $A_4$ (resp. $A_5$), an octahedral group $S_4$, a subgroup of $Aut(F_3)$ (the automorphism group of the Fermat curve), a subgroup of $Aut(K_4)$ (the automorphism group of the Klein curve) or a finite primitive subgroup of $PGL_3(K)$ (for the classification of such groups, we refer to Mitchell [16]).

Now, we present in this section the list of the cyclic groups $G$ that appears as an effective action of plane projective non-singular curves of degree $d$ over an algebraic closed field of zero characteristic and in particular, for any cyclic subgroup of $Aut(C)$, its order should divide one of the following numbers: $d - 1, d, d^2 - 3d + 3, (d - 1)^2, d(d - 2)$ or $d(d - 1)$. Moreover, we attach to each cyclic group an equation $C$ (unique up to $K$-isomorphism) that admits the cyclic group $G$ as a subgroup of the full automorphism group. Also, we provide an algorithm of computation of the full list once the degree is fixed together with the implementation of this algorithm in SAGE program.

Furthermore, we determine the full automorphism group and a characterization of the unique equation for the curve $C$ that admits an automorphism of order $d^2 - 3d + 3, (d - 1)^2, d(d - 2)$ or $d(d - 1)$ in $Aut(C)$.

We also consider the cases when $Aut(C)$ has a cyclic subgroup of order $\ell d$ or $\ell(d - 1)$ with $1 \leq \ell < d - 2$.

(3) In Chapter 3, we give the analogy of the results of Henn in [12] and Komiya-Kuribayashi in [14] concerning the full automorphism groups of non-singular plane curves of degree 4. Thus, we determine the automorphism groups that appear for projective non-singular, degree 5 plane curves which are non-hyperelliptic and the list of the defining equations of these curves. Similar arguments can be applied to deal with higher degrees.

It should be noted that, we present the above three chapters as three independent notes in the context. Hence, they might be read separately without any inconvenience.

The subject of this note is very classical, and we wrote them for a lack of a precise reference, as far as we know. Therefore, any further information, missing references, comments or suggestions are welcomed.

Moreover, the above results can be reformulated in the more precise language of the moduli space (which will be a forthcoming updated version of chapters 2 and 3), as follows.

Let $M_d$ be the moduli space of smooth, genus $g$ curves over an algebraically closed field $K$ of zero characteristic. Denote by $M_d(G)$ the subset of $M_d$ of curves $\delta$ such that $G$ (as a finite non-trivial group) is isomorphic to a subgroup of $Aut(\delta)$ and let $M_d(G)$ be the subset of curves $\delta$ such that $G \cong Aut(\delta)$. Now, for an integer $d \geq 4$, let $M_d^{\text{PGd}(1)}$ be the subset of $M_d$ representing smooth, genus $g$ plane curves of degree $d$ (in such case, $g = (d - 1)(d - 2)/2$) and consider the sets $M_d^{\text{PGd}(1)}(G) := M_d^{\text{PGd}(1)} \cap M_d(G)$ and $M_d^{\text{PGd}(1)}(G) := M_d(G) \cap M_d^{\text{PGd}(1)}$. We denote by $\rho(M_d^{\text{PGd}(1)}(G))$ the elements of $\delta \in M_d^{\text{PGd}(1)}(G)$ such that $G$ acts on a plane model associated to $\delta$ by $\rho_\delta \circ P_{\ell d}G^{-1}$ for certain $P$. We have, $M_d^{\text{PGd}(1)}(G) = \bigcup_{\delta \in \rho(M_d^{\text{PGd}(1)}(G))}$ where $A := \{\rho_\delta : G \rightarrow PGL_3(K)\}/\sim$ and $\rho_\delta \sim \rho_\delta'$ if and only if $\rho_\delta(G) = P \rho_\delta'(G)P^{-1}$ for some $P \in PGL_3(K)$. A similar decomposition follows for $M_d^{\text{PGd}(1)}(G)$.

Henn and Komiya-Kuribayashi proved that $M_d^{\text{PGd}(1)}(G)$ is always equal to $\rho(M_d^{\text{PGd}(1)}(G))$ and in chapter 1, we prove that this does not happen for higher degrees $d \geq 5$ in particular, we prove that the loci $M_d^{\text{PGd}(1)}(\mathbb{Z}/d - 1)$ where $d \geq 5$ is an odd integer and also the locus $M_d^{\text{PGd}(1)}(\mathbb{Z}/3)$ are not ES-Irreducible hence they are not irreducible.

In chapter 2, we give an algorithm (once $g$ is fixed) to decide the list of values that $m$ assumes so that $M_d^{\text{PGd}(1)}(\mathbb{Z}/m)$ is not empty, and how many elements are in the set $A$. Moreover, we prove that if the group $G$ has an element of order $(d - 1)^2, d(d - 1), d(d - 2)$ or $d^2 - 3d + 3$ then $M_d^{\text{PGd}(1)}(G)$ is exactly a set with a unique element therefore, it is irreducible. Also, we study the situations when $G$ has an element of order $\ell d$ or $\ell(d - 1)$. In chapter 3, we determine the exact locus where $\rho(M_d^{\text{PGd}(1)}(G))$ is not trivial, which covers the finite groups that appear for non-singular plane curves of degree 5 in $\mathbb{P}^2(K)$. 


CHAPTER 1

On the locus of smooth plane curves with a fixed automorphism group

1. Abstract

Let \( M_g \) be the moduli space of smooth, genus \( g \) curves over an algebraically closed field \( K \) of zero characteristic. Denote by \( M_g(G) \) the subset of \( M_g \) of curves \( \delta \) such that \( G \) (as a finite non-trivial group) is isomorphic to a subgroup of \( \text{Aut}(\delta) \) and let \( \tilde{M}_g(G) \) be the subset of curves \( \delta \) such that \( G \cong \text{Aut}(\delta) \) where \( \text{Aut}(\delta) \) is the full automorphism group of \( \delta \). Now, for an integer \( d \geq 4 \), let \( M^\text{Pl}_g \) be the subset of \( M_g \) representing smooth, genus \( g \) plane curves of degree \( d \) (in such case, \( g = (d-1)(d-2)/2 \)) and consider the sets \( M^\text{Pl}_g(G) := M^\text{Pl}_g \cap M_g(G) \) and \( \tilde{M}^\text{Pl}_g(G) := \tilde{M}_g(G) \cap M^\text{Pl}_g \).

In this paper, we study some aspects of the irreducibility of \( M^\text{Pl}_g(G) \) and the interrelations with the uniqueness of plane non-singular equations (depending on a set of parameters) for the elements inside \( M^\text{Pl}_g(G) \). Also, we introduce the concept of being equation strongly irreducible (ES-Irreducible) for which the locus \( M^\text{Pl}_g(G) \) is represented by a unique plane equation associated with certain parameters. Henn, in [12], and Komiya-Kuribayashi, in [14], observed that \( M^\text{Pl}_3(G) \) is ES-Irreducible and in this note we prove that this phenomena does not occur for any odd \( d > 4 \). More precisely, let \( \mathbb{Z}/m\mathbb{Z} \) be the cyclic group of order \( m \), we prove that \( M^\text{Pl}_3(\mathbb{Z}/(d-1)\mathbb{Z}) \) is not ES-Irreducible for any odd integer \( d \geq 5 \) and the number of the irreducible components of such loci is at least two. Furthermore, we conclude the previous result when \( d = 6 \) for the locus \( M^\text{Pl}_3(\mathbb{Z}/3\mathbb{Z}) \).

Lastly, we prove an analogy of these statements when \( K \) is any algebraically closed field of positive characteristic \( p \) such that \( p > (d-1)(d-2) + 1 \).

2. Introduction

Let \( K \) be an algebraically closed field of zero characteristic and fix an integer \( d \geq 4 \). We consider, up to \( K \)-isomorphism, a projective non-singular curve \( \delta \) of genus \( g = (d-1)(d-2)/2 \) and assume that \( \delta \) has a non-singular plane model, say \( C \subset \mathbb{P}^2 \) of degree \( d \) whose defining equation is given by a homogenous polynomial \( F(X; Y; Z) = 0 \). Recall that, all such plane models are \( K \)-equivalents that is, we can obtain all the others by applying a change of variables \( P \in \text{PGL}_3(K) \) on the defining equation \( F(X; Y; Z) = 0 \). Hence, where is no abuse of notations, we may replace \( \delta \) by \( C \) also, the full automorphism group of \( \delta \) as a finite subgroup of \( \text{PGL}_3(K) \) can be identified with \( \text{Aut}(C) \), the set of all elements \( \sigma \in \text{PGL}_3(K) \) that retain the defining equation of \( C \) (i.e. \( F(\sigma(X; Y; Z)) = \lambda F(X; Y; Z) \) for some \( \lambda \in K^* \)).

A classical question is to determine, up to \( K \)-equivalence, the different plane equations of non-singular plane curves over \( K \) of degree \( d \) with automorphism group isomorphic to \( \text{Aut}(C) \). Another question, is to classify the groups that appear as exact automorphism groups of algebraic curves of a certain degree \( d \). For \( d = 4 \), Henn in [12] and Komiya-Kuribayashi [14], answered the above natural questions (see also Lorenzo’s thesis [15] §2.1 and §2.2, in order to fix some minor details). It appears for \( d = 4 \) the following phenomena: consider a finite group \( G \) where \( G \cong \text{Aut}(\delta) \) for certain \( \delta \), then there is exactly a unique homogenous polynomial of degree 4 in \( X, Y \) and \( Z \), where the coefficients depend on a set of parameters (with certain algebraic restrictions on these parameters). Moreover, any specialization of the parameters in \( K \) corresponds to a unique \( \delta \in \tilde{M}^\text{Pl}_3(G) \) and any \( \delta \in \tilde{M}^\text{Pl}_3(G) \) has at least one non-singular plane model with a certain specialization of the parameters in \( K \). If this phenomena appears for some \( g \), we say that the locus \( M^\text{Pl}_g(G) \) is ES-Irreducible (see §2 for the
precise definition) which is a weaker condition than the irreducibility of this locus in the moduli space \( M_g \). In particular, it follows by Henn [12] and Komiya-Kuribayashi [14] that \( M_g^{pl}(G) \) is always ES-Irreducible.

The motivation of this work is that we did not expect \( M_g^{pl}(G) \) to be ES-Irreducible in general. In order to construct counter examples where \( M_g^{pl}(G) \) is not ES-Irreducible: we need first, a group \( G \) such that there exist at least two injective representations \( \rho_i : G \to PGL_3(K) \) with \( i = 1, 2 \) which are not conjugate (i.e. there is no transformation \( P \in PGL_3(K) \) with \( P\rho_1(G)P^{-1} = \rho_2(G) \), more details are included in §2) and for the zoo of groups that could appear for non-singular plane curves [11], we consider \( G \) a cyclic group of order \( m \). Secondly, one needs to prove the existence of non-singular plane curves with automorphism group conjugate to \( \rho_i(G) \) for each \( i = 1, 2 \).

The main result of the paper is that for any odd degree \( d(\geq 5) \), the locus \( M_g^{pl}(\mathbb{Z}/(d-1)\mathbb{Z}) \) is not ES-irreducible and it has at least two irreducible components moreover, for degree \( d = 5 \) with \( m \neq 4 \), we observed that the locus \( M_g^{pl}(\mathbb{Z}/m\mathbb{Z}) \) is ES-Irreducible whenever it is not empty. Furthermore, by [3], we know that the only group \( G \) for which \( M_g^{pl}(G) \) is not ES-Irreducible is that when \( G \cong \mathbb{Z}/4\mathbb{Z} \), and by [2] we suspect that the only case for which \( M_g(\mathbb{Z}/m\mathbb{Z}) \) is not ES-Irreducible is that when \( m \) divides \( d \) or \( d - 1 \) (this is true until degree 9 and the tables for possible cyclic groups are listed in [2]). In section §5, we give an example when \( d \) is even, concretely that \( M_g^{pl}(\mathbb{Z}/3\mathbb{Z}) \) is not ES-Irreducible. At the end (§6) of this paper we prove that the above examples of non-irreducible loci are also valid when \( K \) is an algebraically closed of positive characteristic \( p \), provided that the characteristic \( p \) is big enough once we fixed the degree \( d \).

To conclude the introduction, we relate the above locus with the classical one of the moduli space of curves and the problem of irreducibility versus the problem of being ES-Irreducible.

We denote by \( \rho(M_g^{pl}(G)) \) the set of all elements \( \delta \in M_g^{pl}(G) \) such that \( G \) acts on a plane model associated to \( \delta \) by \( P\rho(G)P^{-1} \) for certain \( P \) (because the linear systems \( s_\delta^g \) are unique, the same property follows for any plane model varying the conjugation matrix \( P \)). Therefore, \( M_g^{pl}(G) = \cup_{|\rho| \in A\rho(M_g^{pl}(G))} A := \{ \rho \mid \rho : G \to PGL_3(K) \} / \sim \)

such that \( \rho_\alpha \sim \rho_\nu \) if and only if \( \rho_\alpha(G) = P\rho_\nu(G)P^{-1} \) for some \( P \in PGL_3(K) \). A similar decomposition follows for \( M_g^{pl}(G) \).

Now, for being not ES-Irreducible, we are interested in the situations for which the locus \( \rho(M_g^{pl}(G)) \) is non-empty for at least two elements of \( A \). Also, a much deeper question is when does it happen that the locus \( \rho(M_g^{pl}(G)) \) or \( \rho(M_g^{pl}(G)) \) is irreducible? The locus \( \rho(M_g^{pl}(G)) \) is related, as a subset of the locus of genus \( g \) curves which have a Galois subcover of Galois group \( G \), with some prescribed ramification from the group \( \rho(G) \) (see Remark [13] for some examples) and by the works of Cornalba [8] and Cutaneau [8], we know that it is irreducible when \( G \) is cyclic. Therefore, \( \rho(M_g^{pl}(\mathbb{Z}/m\mathbb{Z})) \) (resp. \( \rho(M_g^{pl}(\mathbb{Z}/m\mathbb{Z})) \)) are subsets in an irreducible locus and moreover, there is a topological invariant which computes the irreducible components of \( M_g(G) \) (see [7] [2]). In particular, we can decide when \( M_g(G) \) is irreducible or not. Unfortunately, we do not have any insight for the irreducibility of the locus \( \rho(M_g^{pl}(G)) \) except possibly the computations that is given in §2 for \( M_6(\mathbb{Z}/8\mathbb{Z}) \) and the result: \( M_g^{pl}(G) \) is irreducible when \( G \) has an element of order \( (d - 1)^2 \), \( d(d - 1) \), \( d(d - 2) \) or \( d^2 - 3d + 3 \) since this locus has only one element by [2].

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3. On the locus \( M_g^{pl}(G) \) and its relation with a set of equations via certain parameters

Consider a projective non-singular curve \( \delta \) of genus \( g := \frac{(d-1)(d-2)}{2} \geq 2 \) over \( K \) with \( G \), a finite non-trivial group, inside \( Aut(\delta) \). We always assume that \( \delta \) admits a non-singular plane equation, and we consider \( \delta \) up to \( K \)-isomorphism, as a point in \( M_g^{pl}(G) \). A set of non-singular plane equations can be associated to \( \delta \) in \( P^2(K) \), let \( C : F(X; Y; Z) = 0 \) be a particular equation where \( F(X; Y; Z) \) is a homogenous polynomial of degree \( d \) and
G acts as \(\rho(G)\) (retaining invariant the equation \(C\)) for some \(\rho: G \hookrightarrow PGL_3(K)\) (a fixed representation of \(G\) inside \(PGL_3(K)\)). If \(\tilde{C}: \tilde{F}(X; Y; Z) = 0\) is any other plane equation of \(\delta\) then it is projectively equivalent to \(C\) through a change of variables \(P \in PGL_3(K)\) applied to \(\{X, Y, Z\}\) and the automorphism group \(G\). In other words, \(F(P(X; Y; Z)) = \lambda F_2(X; Y; Z)\) and \(P\rho(G)P^{-1}\) is a subgroup of \(\text{Aut}(\tilde{C})\). Now, fix \(\rho: G \hookrightarrow PGL_3(K)\) and let us denote by \(\rho(M^3_9(G))\) the subset of \(M^3_9(G)\) such that each \(\delta \in \rho(M^3_9(G))\) has a non-singular plane equation with an effective action by \(P^{-1}\rho(G)P\) on its automorphism group for certain \(P \in PGL_3(K)\) or equivalently, \(\delta\) has a plane non-singular equation with \(\rho(G)\) preserves such equation in particular, \(\rho(G)\) is a subgroup of its full automorphism group.

**Lemma 1.** Consider the set \(M^3_9(G)\) for non-trivial \(G\) and the set \(A = \{\rho: G \hookrightarrow PGL_3(K)\}/\sim\) where \(\rho_1 \sim \rho_2\) if and only if \(\exists P \in PGL_3(K)\) such that \(\rho_1(G) = P\rho_2(G)P^{-1}\). Write \(A = \{[\rho_1], \ldots, [\rho_n]\}\) then \(M_9(G)\) is the disjoint union of \(\rho_i(M_9(G))\) where \(i = 1, \ldots, n\) and \(\rho_i\) is a representative of the class \([\rho_i]\).

Fix \([\rho] \in A\) then for \(\delta \in \rho(M^3_9(G))\), we can associate infinitely many non-singular plane curves which are \(K\)-isomorphic through a change of variables \(P \in PGL_3(K)\), but we can consider only the family of such curves with \(G\) acts exactly as \(\rho(G)\) for some fixed \(\rho\) in \([\rho]\). Under this restriction, \(\delta\) can be associated with a non-empty family of non-singular plane curves such that any two models are isomorphic, through a projective transformation \(P\) satisfying \(P^{-1}\rho(G)P = \rho(G)\). Recall that, it is a necessary condition for a projective plane curve of degree \(d\) to be non-singular that the defining equation has degree \(d \geq 1\) in each variable, and by a diagonal change of variables \(P\), we can assume that the monomials with the maximal exponent have coefficients equal to \(1\). Consequently, we reduce the case to the set of \(K\)-isomorphic non-singular plane curve \(F(X; Y; Z) = 0\) associated to \(\delta\) with \(\rho(G)\) fixes the equation and each term of the core of \(F(X; Y; Z)\) (i.e the sum of all monomials with the highest exponent) is monic.

**Lemma 2.** Let \(G\) be a non-trivial finite group and consider \(\rho: G \hookrightarrow PGL_3(K)\) such that \(\rho(M^3_9(G))\) is non-empty. There exists a unique homogenous polynomial \(F(X; Y; Z) = 0\) of degree \(d\) of the above prescribed form and is endowed with certain parameters as the coefficients of the lower order terms (some restrictions should be imposed so that \(F(X; Y; Z)\) is non-singular). Moreover, every element of \(\rho(M^3_9(G))\) corresponds to a set of equations obtained by assigning certain values to the parameters in the given equation \(F(X; Y; Z) = 0\) and vice versa, that is, every concrete values of the parameters corresponds to a point of \(\rho(M^3_9(G))\).

**Proof.** Let \(\sigma \in G\) be an automorphism of maximal order \(m > 1\) and choose an element \(\rho\) in \([\rho] \in A\) such that, \(\rho(\sigma)\) is diagonal of the form \(\text{diag}(1, \zeta_m^a, \zeta_m^b)\) where \(\zeta_m\) a primitive \(m\)-th root of unity in \(K\). Following the same technique in \([3]\) or \([2]\) (for a general discussion), we can associate to the set \(\rho(M^3_9(< \sigma >))\) a non-singular plane equation \(F_{m,(a,b)}(X; Y; Z)\) with a certain set of parameters (which may have some restrictions in order to ensure the non-singularity). Now imposing that \(\rho(G)\) should fix the equation \(F_{m,(a,b)}\), we obtain some specific algebraic relations between the parameters that should satisfy in general the final equation \(F_{m,(a,b)}\) which we want to reach. The remaining part of the lemma is straightforward.

It is difficult to determine the groups \(G\) and \(\rho\) such that \(\rho(M^3_9(G))\) is non-empty for some fixed \(g\). Henn \([12]\) obtained this determination for \(g = 3\), Badr-Bars \([3]\) for \(g = 6\) and for a general implementation of any degree, we refer to \([2]\) in which we formulate an algorithm to determine the \(\rho\)'s when \(G\) is cyclic.

**Definition 3.1.** Write \(M^3_9(G)\) as \(\cup_{[\rho]} \rho(M^3_9(G))\), we define the number of the equation components of \(M^3_9(G)\) to be the number of elements \([\rho] \in A\) such that \(\rho(M^3_9(G))\) is not empty. We say that \(M^3_9(G)\) is equation irreducible if \(M^3_9(G) = \rho(M^3_9(G))\) for a certain \(\rho: G \hookrightarrow PGL_3(K)\).

Of course, if \(M^3_9(G)\) is not equation irreducible then it is not irreducible and the number of the irreducible equation components of \(M^3_9(G)\) is a lower bound for the number of irreducible components.

Here, we state some questions that appear naturally concerning the locus \(\rho(M^3_9(G))\):

**Question 3.** Is \(\rho(M^3_9(G))\) a subset of the locus of smooth projective, genus \(g\) curves with a Galois subcover of group isomorphic to \(G\) with a prescribed ramification?
We believe that the answer to this question for $K = \mathbb{C}$ (i.e. Riemann surfaces) should be always true from the work of Breuer [5].

**Question 4.** Is $\rho(M_g^{Pl}(G))$ an irreducible set when $G$ is a cyclic group?

It is to be noted that when $K = \mathbb{C}$, Cornalba [8], with $G$ cyclic of prime order, and Catanese [6], for general order, obtained that the locus of smooth projective curves of genus $g$ with a cyclic Galois subcover of group isomorphic to $G$ with a prescribed ramification is irreducible.

Concerning the irreducibility question, we prove in [2] that if $G$ has an element of large order $(d-1)^2$, $d(d-1)$, $d(d-2)$ or $d^2-3d+3$ then $\rho(M_g^{Pl}(G))$ has at most one element therefore, is irreducible.

At the end of this section, we give positive answers to the above questions for certain $\rho$ for $\rho(M_g^P(\mathbb{Z}/8))$. In this example, we obtain that $\rho(M_g^P(\mathbb{Z}/8)) = M_g^P(\mathbb{Z}/8)$ (See also Remark [14] for the explicit Galois subcover and the ramification data for the locus $\rho(M_6(\mathbb{Z}/4\mathbb{Z}))$).

**Definition 3.2.** Given $M_g^{Pl}(G)$, we define $\tilde{M}_g^{Pl}(G)$ to be the subset of the elements inside $M_g^{Pl}(G)$ whose automorphism group is isomorphic to $G$. Similarly for $\rho(\tilde{M}_g^{Pl}(G))$ and we have $\tilde{M}_g^{Pl}(G) = \bigcup_{[\rho] \in A} \rho(\tilde{M}_g^{Pl}(G))$. On the other hand, the number of strongly equation irreducible components of $M_g^{Pl}(G)$ is defined to be the number of the elements $[\rho] \in A$ such that $\rho(\tilde{M}_g^{Pl}(G))$ is not empty.

We say that $\tilde{M}_g^{Pl}(G)$ is equation strongly irreducible (or simply, ES-irreducible) if it is not empty and $M_g^{Pl}(G) = \rho(\tilde{M}_g^{Pl}(G))$ for some $\rho : G \to PGL_3(K)$.

It is a direct consequence to ask the same questions above for the locus $\rho(\tilde{M}_g^{Pl}(G))$ instead of $\rho(M_g^{Pl}(G))$. Furthermore, in this language we can formulate the main result in [12] as follows.

**Theorem 5** (Henn, Komiya-Kuribayashi). If $G$ is a non-trivial group that appears as the full automorphism group of a non-singular plane curve of degree 4, then $M_3^{Pl}(G)$ is ES-Irreducible.

**Remark 6.** Henn in [12], observed that $M_3^{Pl}(\mathbb{Z}/3)$ already has two irreducible equation components, and therefore has at least two irreducible components, but one of such components has a bigger automorphism group namely, $S_3$ the symmetry group of order 3.

### 3.1. The loci $M_6^{Pl}(\mathbb{Z}/8)$ and $M_8^{Pl}(\mathbb{Z}/8)$.

Consider in $M_6$ an element $\delta$ which has a smooth non-singular plane model with an effective action of the cyclic group of order 8 in particular, $\delta \in M_6^0(\mathbb{Z}/8\mathbb{Z})$. Following [2], [9] or the table §4 in this note, we can associate to $\delta$ an equation of the form $X^5 + Y^4Z + XZ^4 + \beta X^3Z^2 = 0$ for certain (s) $\beta$ which depend (s) on $\delta$, and there is only an element in $A$ in the case of degree 5 with cyclic group of order 8. In particular, we fix an equation for $\delta$ such that $\rho(\mathbb{Z}/8\mathbb{Z}) = \langle diag(1, \xi_8, \xi_8^4) \rangle$ where $\xi_8$ is a 8-th primitive root of unity in $K$, and because $\delta$ is non-singular then we should have $\beta \neq \pm 2$. In this locus we have $\rho(M_6^0(\mathbb{Z}/8\mathbb{Z})) = M_6^0(\mathbb{Z}/8\mathbb{Z})$.

Now, let us compute all the equations of the form $X^5 + Y^4Z + XZ^4 + \beta X^3Z^2 = 0$ that can be associated to the fixed curve $\delta$. This corresponds to equations obtained by a change of variables through a transformation $P \in PGL_3(K)$ such that $P < \langle diag(1, \xi_8, \xi_8^4) \rangle \in G^{-1} = \langle diag(1, \xi_8, \xi_8^4) \rangle$ and the new equation has a similar form $X^5 + Y^4Z + XZ^4 + \beta X^3Z^2 = 0$.

Without any loss of generality, we can suppose that $Pdiag(1, \xi_8, \xi_8^4)P^{-1} = diag(1, \xi_8, \xi_8^4)$ hence in order to have the same eigenvalues which are pairwise distinct, we may assume that $P$ is a diagonal matrix, say $P = diag(1, \lambda_2, \lambda_3)$. Therefore, we get an equation of the form: $X^5 + \lambda_2^3 \lambda_3 Y^4 Z + \lambda_2^4 X^4 Z + \beta \lambda_2^3 X^3 Z^2 = 0$. From which we must have $\lambda_2^2 \lambda_3 = \lambda_2^4 = 1$, thus $\lambda_2$ is 1 or -1. Hence, we obtain a bijection map $\varphi : M_6^0(\mathbb{Z}/8\mathbb{Z}) \to A^1(K) \setminus \{-2, 2\} / \sim$

$$\alpha \mapsto [\beta] = \{\beta, -\beta\}$$
where \( a \sim b \Leftrightarrow b = a \text{ or } a = -b \). Moreover, by the work that we did in [3], we know that \( X^5 + Y^4Z + XZ^4 + \beta X^3Z^2 = 0 \) has a bigger automorphism group than \( \mathbb{Z}/8\mathbb{Z} \) if and only if \( \beta = 0 \), therefore, we have a bijection map
\[
\tilde{\varphi} : M_p^0(\mathbb{Z}/8\mathbb{Z}) \rightarrow \mathbb{A}^1(K) \setminus \{ -2, 0, 2 \}/\sim
\]
and observe that \( 0 \in \mathbb{A}^1(K) \) is the only point which had no identification by the relation rule \( \sim \). The above sets, when \( K \) is the complex field, are irreducible.

Moreover, if we consider the Galois cyclic cover of degree 8 given by the action of the automorphism of order 8 on \( X^5 + Y^4Z + XZ^4 + \beta X^3Z^2 = 0 \), we obtain that it ramifies at the points \((0 : 1 : 0), (0 : 0 : 1)\) with ramification index 8 as well as the four points \((1 : 0 : a)\) where \( 1 + a^4 + \beta a^2 = 0 \) with ramification index 2 if \( \beta \neq \pm 2 \). That is, \( M_p^0(\mathbb{Z}/8\mathbb{Z}) \) is inside the locus of curves in \( M_8 \) which have a cyclic Galois cover of degree 8 of a genus zero curve and which ramifies at 6 points, 2 points with ramification index 8 and the other 4 points are with ramification index 4.

### 4. Preliminaries on automorphism on plane curves

We recall that, given a non-singular plane curve \( C \) of degree \( d \geq 4 \) over an algebraic closed field \( K \) of genus \( g \geq 2 \) then \( Aut(C) \) is a finite subgroup of \( PGL_3(K) \) and it satisfies one of the following situations (for more details, see Mitchell [16]):

1. fixes a point \( P \) and a line \( L \) with \( P \not\in L \) in \( \mathbb{P}^2(K) \),
2. fixes a triangle, i.e. exists 3 points \( S := \{ P_1, P_2, P_3 \} \) of \( \mathbb{P}^2(K) \), such that is fixed as a set,
3. \( Aut(C) \) is conjugate of a representation inside \( PGL_3(K) \) of one of the finite primitive group namely, the Klein group \( PSL(2,7) \), the icosahedral group \( A_5 \), the alternating group \( A_6 \), the Hessian groups \( Hess_{216}, Hess_{72} \) or \( Hess_{36} \).

For a non-zero monomial \( cX^iY^jZ^k \) we define its exponent as \( \max\{i,j,k\} \). For a homogeneous polynomial \( F \), the core of \( F \) is defined as the sum of all terms of \( F \) with the greatest exponent. Let \( C_0 \) be a smooth plane curve, a pair \((C,G)\) with \( G \leq Aut(C) \) is said to be a descendant of \( C_0 \) if \( C \) is defined by a homogeneous polynomial whose core is a defining polynomial of \( C_0 \) and \( G \) acts on \( C_0 \) under a suitable coordinate system.

**Theorem 7** (Harui). (see [IT] §2) Let \( G \) be a subgroup of \( Aut(C) \). Then \( G \) satisfies one of the following statements:

1. \( G \) fixes a point on \( C \) and then it is cyclic.
2. \( G \) fixes a point not lying on \( C \) and it satisfies a short exact sequence of the form
   \[
   1 \rightarrow N \rightarrow G \rightarrow G' \rightarrow 1,
   \]
   with \( N \) a cyclic group of order dividing \( d \) and \( G'' \) is isomorphic to a cyclic group \( C_m \) of order \( m \), a Diemedral group \( D_{2m} \), \( A_4 \), \( A_5 \) or \( S_4 \), where \( m \) is an integer \( \leq d - 1 \). Moreover, if \( G' \cong D_{2m} \), then \( m|d-2 \) or \( N \) is trivial.
3. \( G \) is conjugate (by certain \( P \in PGL_3(K) \)) to a subgroup of \( Aut(F_{d}) \) where \( F_{d} \) is the Fermat curve \( X^d + Y^d + Z^d \) and \( (G,C) \) is a descendant of \( F_{d} \). In particular, \( |G| \leq 6d^2 \).
4. \( G \) is conjugate to a subgroup of \( Aut(K_{d}) \) where \( K_{d} \) is the Klein curve curve \( X^{d-1}Y + Y^{d-1}Z + Z^{d-1}X \) and \( (G,C) \) is a descendant of \( K_{d} \). Therefore \( |G| \leq 3(d^2 - 3d + 3) \).
5. \( G \) is conjugate to a finite primitive subgroup \( PGL_3(K) \) namely, the Klein group \( PSL(2,7) \), the icosahedral group \( A_5 \), the alternating group \( A_6 \), or the Hessian groups \( Hess_{216}, Hess_{72}, Hess_{36} \).

**The Hessian group**: A representation of the Hessian group of order 216 inside \( PGL_3(K) \) is given by \( Hess_{216} = \langle S, T, U, V \rangle \) with,
\[
S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}; \quad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}; \quad V = \frac{1}{\omega - \omega^2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}; \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix};
\]
always \( \omega \) means a primitive 3rd root of unity. Also, we consider the primitive subgroups \( \text{Hess}_{36} = \langle S, T, V \rangle \) and \( \text{Hess}_{72} = \langle S, T, V, UVU^{-1} \rangle \). It should be noted that, representations of \( \text{Hess}_{216} \) inside \( \text{PGL}_3(K) \) forms a unique set up to conjugation (see Mitchell [16] page 217), and because \( \text{Hess}_{72} \) and \( \text{Hess}_{36} \) are defined geometrically as subgroups of \( \text{Hess}_{216} \) in \( \text{PGL}_3(K) \) see [10] §23,p.25 their representations inside \( \text{PGL}_3(K) \) are also unique up to conjugation.

Remark 8. In particular, for the Hessian groups \( \text{Hess}_{216} \), \( \text{Hess}_{72} \) and \( \text{Hess}_{36} \), the locus \( \tilde{M}^\text{Pl}_g(\text{Hess}_s) \) is ES-Irreducible as long as is not empty (where \( s \in \{36, 72, 216\} \)) because the set \( A \) is trivial (with the notation of §2).

Now, we may ask the following.

Question 9. Consider \( G \), a non-trivial group, where the set \( A \) is giving by one element (see the notion of \( A \) in previous section). Is it true that the topological invariant in [7] §2 is trivial for \( M_g(G) \) in order to be irreducible? Is it true that \( M^\text{Pl}_g(G) \) are irreducible?

From the above and the interplay with the notion of being ES-irreducible or not of the locus \( M^\text{Pl}_g(G) \), one also could ask the following question in group theory,

Question 10. Fix a finite non-cyclic group \( G \) of \( \text{PGL}_3(K) \). How many elements are inside the set
\[
A = \{ \rho|\rho: G \rightarrow \text{PGL}_3(K) \}/ \sim
\]
where \( p_1 \sim p_2 \) if \( p_1(G) = P^{-1}p_2(G)P \) for some \( P \in \text{PGL}_3(K) \)?

5. Cyclic groups in smooth plane curves of degree 5 and \( M^\text{Pl}_g(\mathbb{Z}/m\mathbb{Z}) \)

Note that, given a smooth plane curve \( C : F_C(X; Y; Z) = 0 \) of degree \( d \) such that \( \text{Aut}(C) \) is non-trivial, we study it up to \( K \)-isomorphism, that is, two of them are \( K \)-isomorphic if one transforms to the other by a change of variables \( P \in \text{PGL}_3(K) \).

By a change of variables, we suppose that the cyclic group of order \( m \) acting on a smooth plane curve of degree 5 is given in \( \text{PGL}_3(K) \) by a diagonal matrix \( \text{diag}(1; \xi_m; \xi_m^a) \) (where \( \xi_m \) an \( m \)-th primitive root of unity and \( a \leq b \) are positive integers) and we call this element by type \( m, (a, b) \). Following the same proof of [9] §6.5 (or see [2] for general treatment with an algorithm of computation for any \( d \)), we have the following table with the corresponding equations (depending of certain parameters) that have an effective cyclic group of certain orders. Such groups are the possible situations for which \( \rho(M^\text{Pl}_g(\mathbb{Z}/m\mathbb{Z})) \) is non-empty:

| Type: \( m, (a, b) \) | \( F(X; Y; Z) \) |
|------------------------|----------------|
| 20, (4, 5)             | \( X^4 + Y^4 + XZ^4 \) |
| 16, (1, 12)            | \( X^4 + Y^4Z + XZ^4 \) |
| 15, (1, 11)            | \( X^6 + Y^4Z + Y^3 \) |
| 13, (1, 10)            | \( X^4Y + Y^4Z + Z^4X \) |
| 10, (2, 5)             | \( X^5 + Y^5 + XZ^4 + \beta_{2,0}X^2Z^2 \) |
| 8, (1, 4)              | \( X^8 + Y^4Z + XZ^4 + \beta_{2,0}X^2Z^2 \) |
| 5, (1, 2)              | \( X^5 + Y^5 + Z^5 + \beta_{3,4}X^2Y^2Z + \beta_{3,3}XY^3Z \) |
| 5, (0, 1)              | \( Z^5 + L_{5,Z} \) |
| 4, (1, 3)              | \( X^5 + X(Z^4 + Y^4 + \beta_{2,4}Y^2Z^2) + \beta_{2,1}X^3YZ \) |
| 4, (1, 2)              | \( X^5 + X(Z^4 + Y^4) + \beta_{2,0}X^3Z^2 + \beta_{3,2}X^2Y^2Z + \beta_{2,2}Y^2Z^3 \) |
| 4, (0, 1)              | \( Z^4L_{1,Z} + L_{5,Z} \) |
| 3, (1, 2)              | \( X^5 + Y^4Z + YZ^4 + \beta_{2,1}X^3YZ + X^2(\beta_{3,0}Z^3 + \beta_{3,3}Y^3) + \beta_{3,2}XY^2Z^2 \) |
| 2, (0, 1)              | \( Z^4L_{1,Z} + Z^2L_{3,Z} + L_{5,Z} \) |

where \( L_{i,U} \) means a homogeneous polynomial of degree \( i \) which does not contain the variable \( U \) and \( \beta_{i,j}, \in K \), and remains in the table above to introduce restriction in the parameters \( \beta_{i,j} \) so that \( F(X, Y, Z) = 0 \) is non-singular (which will be omitted).
By the above table we find that $M^P_6(Z/mZ)$ is not empty, only for the values that $m$ are included in the previous list moreover, for $m \neq 4,5$, we have $M^P_6(Z/mZ) = \rho(M^P_6(Z/mZ))$ where $\rho$ is obtained such that $\rho(Z/mZ) = <\text{diag}(1, \xi_6, \xi_6^m)>$. Thus, the corresponding loci $M^P_6(Z/mZ)$ where $m \neq 4,5$ are ES-Irreducible if they are non-empty.

Now, we consider the remaining cases $M^P_6(Z/mZ)$ with $m = 4$ or $5$.

Clearly, the equations of type $5, (1, 2)$ always have a bigger automorphism group by permuting $X$ and $Z$. Therefore, there is at most one equation that define curves of degree 5 with the full automorphism group is isomorphic to $C_5$ (Observe that the number of conjugacy classes of representations of $C_5$ inside $PGL_2(K)$ is three). In particular, $M^P_6(Z/5Z) = \rho(M^P_6(Z/5Z))$ where $\rho(Z/5Z) = <\text{diag}(1, 1, \xi_5)>$ and hence $M^P_6(Z/5Z)$ is ES-Irreducible if it is non-empty.

Now, we work for the cyclic groups of order 4. The type $4, (1, 3)$ always has a bigger automorphism group by permuting $Y$ and $Z$ conveniently (after a change of variable on $Y$ to make $\alpha = 1$). In fact, this type is not irreducible (being of the form $XG$ hence is singular and will be out of our scope in this note.

Therefore, we have $M^P_6(Z/4Z) = \rho_1(M^P_6(Z/4Z)) \cup \rho_2(M^P_6(Z/4Z))$ where $\rho_1$ corresponds to type $4, (0, 1)$ and $\rho_2$ to type $4, (1, 2)$.

5.1. On type $4, (0, 1)$. Consider the non-singular plane curve defined by the equation $\tilde{C} : X^5 + Y^5 + Z^4X + \beta X^3Y^2$ where $\beta \neq 0$. This curve admits a cyclic element of order 4 namely, $\sigma := [X; Y; \xi_4Z]$ with axis $Z = 0$ and center $(0; 0; 1)$ (we say that is an homology when cyclic elements fix such geometric construction). It follows by Mitchell [10] §5, that $Aut(\tilde{C})$ should fix a point, a line or a triangle. If $Aut(\tilde{C})$ fixes a triangle and neither a line nor a point is leaved invariant then, by Harui [11] §5, $\tilde{C}$ is a descendant of the Fermat curve $F_5$ or the Klein curve $K_5$ but this is impossible because $4 \nmid |Aut(F_3)| (= 150)$ and $4 \nmid |Aut(K_5)| (= 39)$. Therefore, $Aut(\tilde{C})$ should fix a line and a point off that line. It follows by [11] [Lemma 3.7] that the point $(0; 0; 1)$ is an inner Galois point of $\tilde{C}$ (and it is unique by Yoshihara [20] §2 [Theorem 4]). Therefore, the unique inner Galois point is fixed by the full $Aut(\tilde{C})$ in particular, $Aut(\tilde{C})$ is cyclic ([13] Lemma 11.44) and automorphisms of $\tilde{C}$ are of the form $[X + \alpha_2Y; \beta_1X + \beta_2Y; \gamma_1X + \gamma_2Y + \gamma_3Z]$. From the coefficient of $YZ^4$ we must have $\alpha_2 = 0$ consequently, $\gamma_1 = \gamma_2 = 0$ (Coefficients of $Z^3X^2$ and $XZ^3Y$) then $\beta_1 = 0$ (Coefficient of $XY^4$). That is, $Aut(\tilde{C})$ is cyclic and is generated by a diagonal element $[X; \nu Y; tZ]$ which in turns implies that $\nu^5 = \nu^2 = t^4 = 1$ hence $\nu = 1$ and $t$ is a 4-th root of unity and the result follows.

Therefore, with the above argument we conclude the following result.

**Proposition 11.** The locus set $\rho(1)(M^P_6(Z/4Z))$ is non-empty.

5.2. On type $4, (1, 2)$. Consider the non-singular plane curve defined by the equation $\tilde{C} : X^5 + X(Z^4 + \alpha Y^4) + \beta Y^2Z^3$ where $\alpha \beta \neq 0$. This curve admits a cyclic group generated by the automorphism $\tau := [X; \xi_4Y; \xi_4^2Z]$. For the same reason as above (i.e $4 \nmid |Aut(K_3)|, |Aut(F_3)|$), $\tilde{C}$ is not a descendant of the Fermat curve $F_5$ or the Klein curve $K_5$. Moreover, $Aut(\tilde{C})$ is not conjugate to an icoshedral group $A_5$ (no elements of order 4), the Klein group $PSL(2, 7)$, the Hessian group $Hess_{216}$ or the alternating group $A_6$ (since by [11] [Theorem 2.3], $|Aut(\tilde{C})| \leq 150$). Moreover, we claim to prove the following.

**Claim 1.** $Aut(\tilde{C})$ is not conjugate to any of the Hessian subgroups namely, $Hess_{36}$ or $Hess_{72}$.

Because both groups contains reflections (but no four groups) then all reflections in the group will be conjugate ( [10] Theorem 11), Therefore, it suffices to consider the case $P \tau^2 P^{-1} = \lambda[Z; Y; X]$ this in turns gives $P$ of the forms

$$
\begin{pmatrix}
1 & \alpha_2 & \alpha_3 \\
\beta_1 & 0 & \beta_3 \\
1 & -\alpha_2 & -\alpha_3
\end{pmatrix}
$$

or

$$
\begin{pmatrix}
1 & \alpha_2 & \alpha_3 \\
0 & \beta_2 & 0 \\
-1 & \alpha_2 & -\alpha_3
\end{pmatrix}
$$

For both situations, by comparing the coefficients of $XY^4$ and $Y^4Z$ (resp. $L_{5,Y}$), we get $\alpha_3 = 1$ (since $[Z; Y; X] \in Aut(P\tilde{C})$) in particular, the second transformation is not an option. Furthermore, from the coefficients of $X^2Y^3$ and $Y^3Z^2$ we should have $\beta_1 = \beta_3$ a contradiction and we conclude the result.
Consequently, $\text{Aut}(\tilde{C})$ should fix a line and a point off that line. Now, because $\tau \in \text{Aut}(\tilde{C})$ is of the form $\text{diag}(1; a; b)$ such that $1, a, b$ (resp. $1, a^2, b^2$) are pairwise distinct then automorphisms of $\tilde{C}$ are of the forms $\tau_1 := [X; vY + wZ; Z + iZ]$, $\tau_2 := [vX + wY; Y; sX + tZ]$ or $\tau_3 := [vX + wY; sX + tY; Z]$ (because the fixed point only could be $(0:0:0)$ or $(1:0:0)$ or $(0:1:0)$ and the line that is left invariant is one of the reference lines).

Now, if $\tau_1 \in \text{Aut}(\tilde{C})$ then $s = 0 = w$ (Coefficient of $Y^5$ and $Z^5$) and the same conclusion if $\tau_2$ (resp. $\tau_3$) $\in \text{Aut}(\tilde{C})$ from the coefficients of $X^3Y^2$ and $Y^4Z$ (resp. $Z^3X$ and $YZ^4$). Hence, automorphisms of $\tilde{C}$ are all diagonal and moreover, if $[X; vY; sZ] \in \text{Aut}(\tilde{C})$ then we must have $v^4 = v^2s^3 = 1$ that is $\text{Aut}(\tilde{C})$ is cyclic of order 4.

Consequently, the following result follows.

**Proposition 12.** The locus set $p_1(M_6^{P_l}(\mathbb{Z}/4\mathbb{Z}))$ is non-empty.

**Corollary 13.** The locus set $M_6^{P_l}(\mathbb{Z}/m\mathbb{Z})$ is ES-Irreducible if and only if $m \neq 4$. If $m = 4$ then $M_6^{P_l}(\mathbb{Z}/m\mathbb{Z})$ has exactly two irreducible equation components and hence the number of its irreducible components is at least two.

**Remark 14.** Observe that the Galois cover of degree 4 corresponding to $p_1(M_6^{P_l}(\mathbb{Z}/4\mathbb{Z}))$:

$$C_1 := Z^4L_{1, X} + L_{5, Z} = 0 \rightarrow C_1/ < [X; Y; \xi_4Z] >$$

is ramified exactly at six points with ramification index 4. Indeed, the fixed points of $\sigma^i$ for $i = 1, 2, 3, 4$ are all the same where $\sigma = \text{diag}(1, 1, \xi_4)$ therefore, we only need to consider the ramification points of $\sigma$ in particular, the ramification index is 4. Now, by the Hurwitz formula we get $10 = 4(2g_0 - 2) + 3k$ where $g_0$ is the genus of $C_1/ < [X, Y, \xi_4Z] >$ hence we are forced to $g_0 = 0$ and $k = 6$. On the other hand, the Galois cover of degree 4 corresponding to $p_2(M_6^{P_l}(\mathbb{Z}/4\mathbb{Z}))$, gives that

$$C_2 := X^5 + X(Z^4 + Y^4) + \beta_{2, 0}X^3Z^2 + \beta_{3, 2}X^2Y^2Z + \beta_{5, 2}Y^2Z^3 = 0 \rightarrow C_2/ < [X; Y; \xi_4^2Z] >$$

is ramified at the points $(0:1:0), (0:0:1)$ with ramification index 4 and at the 4 points namely, $(1:0:a)$ where $1 + a^4 + \beta_{2, 0}a^2 = 0$ with ramification index 2 provided that $\beta_{2, 0} \neq \pm 2$. The situation with $\beta_{2, 0} = \pm 2$ is that the equation is singular or non-geometrically irreducible, which is not of our concern in this work.

**Remark 15.** Given $G$ a non-trivial finite group such that $\widetilde{M_6^{P_l}}(G)$ is non-empty, by a tedious work [3], one can show that $M_6^{P_l}(\mathbb{Z}/(d-1)\mathbb{Z})$ is ES-Irreducible except the case $G \cong \mathbb{Z}/4\mathbb{Z}$.

**Theorem 16.** Let $d \geq 5$ be an odd integer and consider $g = (d-1)(d-2)/2$ as usual. Then $M_6^{P_l}(\mathbb{Z}/(d-1)\mathbb{Z})$ is not ES-Irreducible and it has at least two irreducible components.

**Proof.** The above argument for concrete curves of Type 4, $(0, 1)$ and Type 4, $(1, 2)$ is valid for any odd degree $d \geq 5$ and the proof is quite similar. In other words, let $\tilde{C}$ and $\tilde{C}$ be the non-singular plane curves of types $d - 1, (0, 1)$ and $d - 1, (1, 2)$ defined by the equations $X^d + Y^d + Z^{d-1}X + \beta X^{d-2}Y^2 = 0$ and $X^d + X(Z^{d-1} + Y^{d-1}) + \beta Y^2Z^{d-2} = 0$ where $\beta \neq 0$. Then, $\text{Aut}(\tilde{C})$ and $\text{Aut}(\tilde{C})$ are cyclic of order $d - 1$ generated by $[X; Y; \xi_{d-1}Z]$ and $[X; \xi_{d-1}Y; \xi_{d-1}Z]$ respectively which are cyclic groups which are not conjugate, therefore belongs to two different $[\rho]^i$'s.

**On type** $d - 1, (0, 1)$: With a homology of order $d - 1 \geq 4$ inside $\text{Aut}(\tilde{C})$ we conclude that $\text{Aut}(\tilde{C})$ should fix a point, a line or a triangle (See [16]§5). Moreover, the center $(0; 0; 1)$ of this homology is an inner Galois point (Lemma 3.7) and then it is unique by Yoshihara ([20]§2 [Theorem 4]). Therefore, it should be fixed by the $\text{Aut}(\tilde{C})$ (in particular, the axis $Z = 0$ is left invariant by [16] [Theorem 4]) hence $\text{Aut}(\tilde{C})$ is cyclic ([13] Lemma 11.44) and automorphisms of $\tilde{C}$ are of the form $\text{diag}(1; v; t)$ such that $v^d = t^{d-1} = v = 1$. That is, $|\text{Aut}(\tilde{C})| = d - 1$.

**On type** $d - 1, (1, 2)$: First, we proof the following claim.
Claim 2. The full automorphism group of non-singular plane curves of type \( d-1, (1,2) \) is not conjugate to any of the primitive groups mentioned in Theorem [7] (5). Moreover, \( \tilde{C} \) is not a descendant of the Fermat curve \( F_d : X^d + Y^d + Z^d \) or Klein curve \( K_d : X^{d-1}Y + Y^{d-1}Z + Z^{d-1}X \) in particular, \( \text{Aut}(\tilde{C}) \) fixes a line and a point off this line.

We consider the case \( d \geq 7 \) (for \( d = 5 \) we refer to the above results) and the claim is straightforward. Indeed, the alternating group \( A_6 \) has no elements of order \( \geq 6 \), the Klein group \( PSL(2,7) \) which is the only simple group of order 168 has no elements of order \( \geq 8 \) (also, no elements of order 6 are present inside \([19]\)) therefore the groups \( A_6, A_5 \) and the Klein group do not appear as the full automorphism group. Moreover, elements inside the Hessian group \( \text{SmallGroup}(216,153) \) have orders 1, 2, 3, 4 or 6 then \( Hess_{216} \) (consequently, \( Hess_36 \) and \( Hess_72 \)) do not appear except possibly for \( d = 7 \). On the other hand, \( d-1 \nmid 3(d^2 - 3d + 3) \) that is \( \tilde{C} \) is not a descendant of the Klein curve \( K_d \) also, is not a descendant of the Fermat curve because \( d-1 \nmid 6d^2 \) (except possibly for \( d = 7 \)). Finally, it remains to treat the case \( d = 7 \) for the Hessian groups or being a Fermat’s descendant. In this case, the Fermat curve has automorphism group generated by the four transformations \( [\xi_1X; Y; Z], [X; \xi_1Y; Z], [X; Z; Y] \) and \( [Y; Z; X] \) consequently elements of \( \text{Aut}(F_7) \) are of the forms \( [X; \xi_1\xi_2Y; \xi_1^2Z], [\xi_1^2Z; \xi_1Y; Y], [X; \xi_1^2Z; \xi_1Y], [\xi_1^2Y; X; \xi_1^2Z], [\xi_1^4Y; Z; X] \) or \( [\xi_1^3Z; X; \xi_1^2Y] \) and one can easily verify that non of them has order 6 consequently, being a descendant of the Fermat curve \( F_d \) does not occur. Also, by the same argument as Claim 1 on Type 4, (1, 2) we exclude the Hessian groups and hence the claim is proved.

Now, the full automorphism group shall fix a point and a point which does not belong to this line thus all automorphisms of \( \tilde{C} \) satisfy one of the forms \( [X; vY + wZ; sY + tZ], [vX + wZ; Y; sX + tZ] \) or \( [vX + wZ; Y; sX + tY; Z] \) (because \( \tilde{C} \) has an automorphism \( [X; \xi_1\xi_2Y; \xi_1^2Z] \) of the form \( diag(1; v; s) \) with 1, \( v, s \) (resp. \( Y^2 \) and \( Z^3 \)) are pairwise distinct). If \( [X; vY + wZ; sY + tZ] \in \text{Aut}(\tilde{C}) \) then \( s = 0 = w \) (Coefficient of \( Y^d \) and \( Z^d \)) and the same conclusion if \( [vX + wZ; Y; sX + tZ] \) (resp. \( vX + wY; sX + tY; Z] \) \( \in \text{Aut}(\tilde{C}) \) from the coefficients of \( X^{d-2}Y^2 \) and \( Y^{d-1}Z \) (resp. \( Z^{d-2}X^2 \) and \( YZ^{d-1} \)). Hence, automorphisms of \( \tilde{C} \) are diagonal and moreover, if \( diag(1; v; s) \in \text{Aut}(\tilde{C}) \) then we must have \( v^{d-1} = s^{d-1} = v^2s^{d-2} = 1 \) that is \( v = \xi_1 \) and \( s = \xi_3^{d-1} \) such that \( d-1 \nmid 2r - r' \) hence \( \text{Aut}(\tilde{C}) \) is cyclic of order \( d-1 \).

6. On the locus \( M_{10}^{\text{red}}(\mathbb{Z}/3\mathbb{Z}) \).

By a similar argument as this one that has been done for degree 5, we obtain the following smooth plane equations of degree 6 with an effective cyclic group of order 3 in the automorphism group.

| Type: \( m, (a, b) \) | \( F(X; Y; Z) \) |
|----------------------|------------------|
| 3, (0, 1)            | \( Z^6 + Z^3L_3Z + L_6Z \) |
| 3, (1, 2)            | \( X^5Y + Y^5Z + Z^5X + \alpha Z^2X^4 + \beta X^2Y^4 + \gamma Y^2Z^4 + \delta X^3Y^2Z + \mu XY^3Z^2 + \eta X^2Y^2Z^3 \) |

6.1. On type 3, (1, 2).

Proposition 17. Consider the plane curve \( \tilde{C} \) defined by the equation

\[
X^5Y + Y^5Z + Z^5X + \mu_1Z^2X^4 + \mu_2X^2Y^4 + \mu_3Y^2Z^4 + \delta_1X^3Y^2Z + \delta_2XY^3Z^2 + \delta_3X^2YZ^3 = 0.
\]

Then, \( \text{Aut}(\tilde{C}) \) is not conjugate to any of the finite primitive groups inside \( PGL_3(K) \) namely, the Klein group \( PSL(2,7) \), the icosahedral group \( A_5 \), the alternating group \( A_6 \), the Hessian group \( Hess_{216} \) or to any of its subgroups \( Hess_{72} \) or \( Hess_{36} \).

Proof. Let \( \tau \in \text{Aut}(\tilde{C}) \) be an element of order 2 such that \( \tau \sigma \tau = \sigma^{-1} \) where \( \sigma := [X; \omega Y; \omega^2Z] \) then \( \tau \) has one of the forms \( [X; \beta Z; \gamma X] \), \( [\beta Y; \beta X; Z] \) or \( [\beta Z; Y; \beta X] \) a contradiction because non of these forms retain \( \tilde{C} \). Therefore, \( \text{Aut}(\tilde{C}) \) does not contain an \( S_3 \) as a subgroup hence is not conjugate to \( A_5 \) or \( A_6 \). Moreover, it is well known that \( PSL(2,7) \) contains an octahedral group of order 24 (but not an icosahedral group of order 60) and since all elements of order 3 in \( PSL(2,7) \) are conjugate (see \([19]\)) then by the same argument as above we conclude that \( \text{Aut}(\tilde{C}) \) is not conjugate \( PSL(2,7) \).
Now, if $\text{Aut}(\tilde{C})$ is conjugate (through a transformation $P$) to any of the Hessian groups then we can assume that $PSP^{-1} = \lambda S$ (because we did not fix the equation model for a curve having automorphism group any of the Hessian groups). In particular, $P$ has the form $[Y; \alpha Z; \beta X]$, $[Z; \alpha X; \beta Y]$ or $[X; \alpha Y; \beta Z]$ but none of these transformation satisfy $\{[X; Z; Y], [Y; X; Z] [Z; X; Y]\} \subseteq \text{Aut}(P\tilde{C})$ thus $\text{Aut}(\tilde{C})$ is not conjugate to any of the Hessian groups.

\section*{Lemma 18.} If a finite group $G$ inside $\text{PGL}_3(K)$ fixes a line and a point off that line moreover if it contains a diagonal element which is not a homology then the fixed line must be one of the reference lines or $ZH$.

\section*{Notations.} Let $\mathcal{T} := \Gamma \cup \{-3\}$ where $\Gamma$ is the set of all roots of the polynomial $g_1(v)g_2(v)g_3(v)$ such that

$g_1(v) := v \times (v^2 + v + 7)(v(v + 7) + 49) \times (v^4 - 39v^2 + 441)$

$g_2(v) := (v(v(v + 2)(v^2 + v + 4) + 3) - 1)$

$g_3(v) := (19 + (v - 3)(7 + (v - 4)v)(3 + (v - 2)v))$

Now, we can prove the main result for this section.

\section*{Theorem 19.} Consider the curve defined by the equation $\tilde{C} : X^5Y + Y^5Z + Z^5X + \delta_3 X^2YZ^3$ and assume for simplicity that $\delta_3 \notin \mathcal{T}$. Then $\text{Aut}(\tilde{C})$ is cyclic of order 3 and is generated by the transformation $\text{diag}(1; \omega; \omega^2)$.

\section*{Proof.} It follows, by Proposition 17, that $\text{Aut}(\tilde{C})$ is not conjugate to any of the finite primitive groups that have been mentioned in Theorem 7. Therefore, the automorphisms group (being a finite group in $\mathbb{P}^2(K)$) and by Mitchell in [16] either fixes a line and a point off that line or it fixes a triangle. In what follows, we treat each of these two cases.

(1) If $\text{Aut}(\tilde{C})$ fixes a point, it must be one of the reference points (by lemma 18). Consequently, $\text{Aut}(\tilde{C})$ is cyclic since all the reference points lie on $\tilde{C}$. Furthermore, elements of $\text{Aut}(\tilde{C})$ are of the forms

$\tau_1 := [X; vY + wZ; sY + tZ], \tau_2 := [vX + wZ; Y; sX + tZ]$ or $\tau_3 := [vX + wY; sX + tY; Z]$

$\tau_1$ to be in $\text{Aut}(\tilde{C})$, we must have $w = 0 = s$ (coefficients of $X^5Z$ and $XY^5$) and similarly, for $\tau_2$ (resp. $\tau_3$) through the coefficients of $Y^5X$ and $Z^6$ (resp. coefficients of $YZ^5$ and $X^5Z$).

Now, for $\tau_2$ to be in $\text{Aut}(\tilde{C})$, we must have $w = 0 = s$ (coefficients of $X^5Z$ and $XY^5$) and similarly, for $\tau_2$ (resp. $\tau_3$) through the coefficients of $Y^5X$ and $Z^6$ (resp. coefficients of $YZ^5$ and $X^5Z$).

(2) If $\text{Aut}(\tilde{C})$ fixes a triangle and there exist neither a line or a point fixed by it, then by Harui 11, $\tilde{C}$ is a descendant of the Fermat curve $F_6 : X^6 + Y^6 + Z^6$ or the Klein curve $K_6 : X^6Y + Y^5Z + Z^5X$. Hence, $\text{Aut}(\tilde{C})$ is conjugate to a subgroup of $\text{Aut}(F_6) := <[\xi_6 Y; Z], [X; \xi_6 Y; Z], [Y; X; Z], [X; Z; Y]> \text{ or to a subgroup of } \text{Aut}(K_6) := <[X; Y; Z], [X; \xi_2 Y; \xi_4 Z]>

$\bullet$ Suppose first that $\text{Aut}(\tilde{C})$ is conjugate (through $P$) to a subgroup of $\text{Aut}(F_6)$ then $PSP^{-1}$ should be one of the automorphisms $S, S^{-1}, [\xi_6 Y; \xi_6 Z; X], [\xi_6 Z; X; \xi_6 Y]$ (these are all the automorphisms of order 3 in $\text{Aut}(F_6)$ which are not homologies) and it suffices to consider the situations $PSP^{-1} = \lambda M$ where $M = S, [Y; Z; X], [Y; \xi_6 Z; X]$ or $[Y; \xi_6 Z; X]$ since any other $M$ is conjugate to one of these inside $\text{Aut}(F_6)$. Now, if $M = S$ then $P \in \text{PGL}_3(K)$ is of the form $[Y; \alpha Z; \beta X], [Z; \alpha X; \beta Y]$ or $[X; \alpha Y; \beta Z]$ and obviously, none of these transformations gives $X^6 + Y^6 + Z^6 \in P\tilde{C}$ a contradiction. On the other hand, if $M = [Y; Z; X]$ (resp. $[Y; \xi_6 Z; X]$) then $P$ has the form

$\begin{pmatrix}
\lambda & \lambda \omega \beta_2 & \lambda \omega \beta_3 \\
1 & \beta_2 & \beta_3 \\
\lambda^2 & \lambda^2 \omega \beta_2 & \lambda^2 \omega^2 \beta_3
\end{pmatrix}$

where $\lambda^3 = 1$ (resp. $\lambda^3 \xi_6 = 1$) and $\lambda^3 \xi_2^2 = 1$) moreover $X^5Z \in P\tilde{C}$ (since $\delta_3$ is not a root of $g_1(v)$). In particular, $P\tilde{C}$ should has the form

$P\tilde{C} : X^6 + Y^6 + Z^6 + k_1(X^5Z + Y^5X + aZ^5Y) + \text{other terms}$
where \( a = 1 \) (resp. \( \xi_6 \) or \( \xi_2^2 \)) and then such transformations exist only if \( \delta_3 \) is a root of \( g_2(v)g_3(v) \). Consequently, \( \tilde{C} \) is not a descendant of the Fermat sextic curve.

- Lastly, assume that \( Aut(C) \) is conjugate (through \( P \)) to a subgroup of \( Aut(K_6) \) with

\[
P\tilde{C} : X^5Y + Y^5Z + Z^5X + \text{lower terms.}
\]

Then, we must have \( PSP^{-1} = \lambda S \) for some \( \lambda \in K^* \). Indeed, elements of order 3 inside \( Aut(K_6) \) (which are not homologies) are \( S, S^{-1}, [\xi_2^a; Y; \xi_1^4aZ; X] \) and \( [\xi_1^4aZ; X; \xi_1^2Y] \) and it is enough to consider the cases \( PSP^{-1} = \lambda M \) where \( M \) could be assumed to have any of the values \( S, S^{-1}, [\xi_2^a; Y; \xi_1^4aZ; X] \) and \( [\xi_1^4aZ; X; \xi_1^2Y] \) with \( a = 0, 1, 2 \) (because any other value is conjugate to one of those inside \( Aut(K_6) \)). Now, assume that \( PSP^{-1} = \lambda[\xi_2^aY; \xi_1^4aZ; X] \) (respectively, \( [\xi_1^4aZ; X; \xi_1^2Y] \)) then \( P \) has the form

\[
\left( \begin{array}{ccc}
\lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\
1 & \beta_2 & \beta_3 \\
\lambda_2^2 & \lambda_2^2 & \lambda_3^2 
\end{array} \right) \quad \text{(respectively, } \left( \begin{array}{ccc}
\lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\
1 & \beta_2 & \beta_3 \\
\lambda_2^2 & \lambda_2^2 & \lambda_3^2 
\end{array} \right) \text{)}
\]

where \( \lambda^3 = \xi_2^a \). Since \( \delta_3 \neq 0, -3 \) then non of the above transforms \( \tilde{C} \) to a descendant of the Klein curve (because \( X^6 \) appears in \( P\tilde{C} \)). On the other hand, if \( PSP^{-1} = \lambda S \) then \( P \) fixes one of the variables and permutes the others hence the resulting core is different from \( X^5Y + Y^5Z + Z^5X \) consequently \( PSP^{-1} = \lambda S \). That is, \( P \) has one of the forms \( [Y; \alpha Z; \beta X], [Z; \alpha X; \beta Y] \) or \( [X; \alpha Y; \beta Z] \) consequently \( \tilde{C} \) is transformed to the form \( k_0(X^5Y + Y^5Z + Z^5X) + \delta_3k_1 \) (monomial).

More precisely, this monomial is one of the monomials \( X^3Y^3Z^3 \), \( X^2Y^2Z^3 \) or \( Z^2XY^3 \) consequently, \( [Z; X; Y] \notin Aut(P\tilde{C}) \). Thus, \( Aut(P\tilde{C}) \leq \tau := [X; \xi_2Y; \xi_1^4Z] \) moreover \( \tau \in Aut(P\tilde{C}) \) if and only if \( 7|\tau \) that is, \( Aut(\tilde{C}) \) has order 3.

This completes the proof. \( \square \)

6.2. On type 3, (0, 1).

**Lemma 20.** Consider the non-singular plane curve \( \tilde{C} : Z^6 + Z^3L_{3, Z} + L_{6, Z} \). Then the full automorphism group of \( \tilde{C} \) is conjugate to the Hessian group \( Hess_{216} \) or it leaves invariant a point, a line or a triangle.

**Proof.** Since \( Aut(\tilde{C}) \) contains a homology (fix a line and a point off in \( \mathbb{P}^2(K) \)) of period 3 namely, \( \sigma := [X; Y; \omega Z] \) then the result follows because \( Hess_{216} \) is the only multiplicative group that contains such homologies and does not leave invariant a point, a line or a triangle (Theorem 9 [16]). \( \square \)

Now, we can prove our main result for this section.

**Theorem 21.** The automorphisms group of the curve \( Z^6 + X^5Y + XY^5 + \alpha_3 Z^3X^3 \) such that \( \alpha_3 \neq 0 \) is cyclic and has order 3.

**Proof.** Assume that \( Aut(\tilde{C}) \) is conjugate (through a transformation \( P \)) to the Hessian group \( Hess_{216} \) then we can assume that \( P\tau P^{-1} = \lambda S \) for some \( \lambda \in K^* \). Hence, \( P = [\alpha X + \alpha Y; \beta_1 X + \beta_2 Y; Z] \) and clearly, \( \{[Z; Y; X], [X; Z; Y]\} \notin Aut(P\tilde{C}) \) a contradiction. Now, it follows by lemma 20 that \( Aut(\tilde{C}) \) should fix a point, a line or a triangle. In what we treat each case.

1. If \( Aut(\tilde{C}) \) fixes a line and a point off that line and if \( \tilde{C} \) admits a bigger non cyclic automorphism group then \( Aut(\tilde{C}) \) satisfies a short exact sequence of the form \( 1 \to C_3 \to Aut(\tilde{C}) \to G' \to 1 \) where \( G' \) is conjugate to \( C_m \) (\( m = 2, 3 \) or 4), \( D_{2m} \) (\( m = 2 \) or 4), \( A_4, S_4 \) or \( A_5 \).

If \( G' \) is conjugate to \( C_3, A_4, S_4 \) or \( A_5 \) then there exists (by Sylow’s theorem) a subgroup \( H \leq Aut(\tilde{C}) \) of order 9. In particular, \( H \) is conjugate to \( C_3 \) or \( C_3 \times C_3 \) but both cases do not occur. Indeed, if \( H = C_3 \) then \( Aut(\tilde{C}) \) has an element of order 9 a contradiction since \( 9 \nmid d - 1, d, (d - 1)^2, d(d - 2), d(d - 1), d^2 - 3d + 3 \) with \( d = 6 \) (for more details, we refer to [2]) moreover if \( H = C_3 \times C_3 \) then there exists \( \tau \in Aut(\tilde{C}) \) of order 3 such that \( \tau \sigma = \sigma \tau \) hence \( \tau = [vX + wY; sX + tY; Z] \). Comparing the coefficients of \( Z^3Y^3 \) and \( X^6 \) in \( \tau(\tilde{C}) \) we get \( w = 0 \) and \( v^6t = vt^5 = t^3 = 1 \) (thus \( \tau \in \sigma \)) a contradiction.
By a similar argument, we exclude the cases $C_4$ and $D_{2m}$ because for each $\text{SmallGroup}(6m, ID)$ there must be an element $\tau$ of order 2 or 4 that commutes with $\sigma$. Finally, if $G'$ is conjugate to $C_2$ then there exists an element $\tau$ of order 2 such that $\sigma\tau\sigma^{-1}$ and one can easily verify that such an element does not exist.

We conclude that $\text{Aut}(\tilde{C})$ should be cyclic (in particular, is commutative) hence can not be of order 3. (otherwise; there exists an element $\tau \in \text{Aut}(\tilde{C})$ of order 3 which commutes with $\sigma$ and by a previous argument such elements do not exist).

(2) If $\text{Aut}(\tilde{C})$ fixed a triangle and neither a point nor a line is fixed, then it follows by Harui [11] that $\tilde{C}$ is a descendant of the Fermat curve $F_6$ or the Klein curve $K_6$. The last case does not happen because $P\sigma P^{-1}$ should be an automorphism of $K_6$ of order 3 whose Jordan form is given as $\sigma$ (i.e a homology) but there are no such elements in $\text{Aut}(K_6)$.

Now, suppose that $\tilde{C}$ is a descendant of $F_6$ that is, $\tilde{C}$ can be transformed (through $P$) into a curve $P\tilde{C}$ with core $X^6 + Y^6 + Z^6$ then $P = [\alpha_1 X + \alpha_2 Y; \beta_1 X + \beta_2 Y; Z]$ because in $\text{Aut}(F_6)$ there are only two sets of homologies of order 3 namely; $[\omega X; Y; Z], [X; \omega Y; Z], [X; Y; \omega Z]$ and $[\omega^2 X; Y; Z], [X; \omega^2 Y; Z], [X; Y; \omega^2 Z]$ and they are not conjugate in $\text{PGL}_3(K)$. Moreover, elements of the first set are all conjugate to $[X; Y; \omega Z]$ inside $\text{Aut}(F_6)$ so it suffices to consider $P\sigma P^{-1} = \lambda \sigma$. Now, $P\tilde{C}$ has the form

$$C': \mu_0 X^6 + \mu_1 Y^6 + Z^6 + \alpha_3(\alpha_1 X + \alpha_2 Y)^3 Z^3 + \mu_2 X^5 Y + \mu_3 X^4 Y^2 + \mu_4 X^3 Y^3 + \mu_5 X^2 Y^4 + \mu_6 X Y^5$$
where $\mu_0 := \alpha_1 \beta_1 (\alpha_2^2 + \beta_2^2) = 1$ and $\mu_1 := \alpha_2 \beta_2 (\alpha_1^2 + \beta_1^2) = 1$. In particular, $(\alpha_1 \beta_1)(\alpha_2 \beta_2) \neq 0$ therefore, $[X; vZ; wY], [vZ; wY; X], [wY; vZ; X]$ and $[vZ; X; wY] \notin \text{Aut}(P\tilde{C})$ (because of the monomial $XY^2Z^3$) moreover, $[wY; X; vZ] \in \text{Aut}(P\tilde{C})$ only if $\alpha_1 = \alpha_2$ and $w = v^3 = 1$ hence $P\tilde{C} = Z^6 + \alpha_3 x_0^3(X + Y)^3 Z^3 + \alpha_1(1 + X + Y)(\beta_1 X + \beta_2 Y) \left(\alpha_1^2 (X + Y)^2 + (\beta_1 X + \beta_2 Y)^4\right)$.

Consequently, $\beta_1 = \beta_2$ (because we are assuming $[Y; X; vZ] \in \text{Aut}(P\tilde{C})$) a contradiction. Finally, if $[X, \xi_0 Y, \xi_0' Z] \in \text{Aut}(P\tilde{C})$ then $r = 0$ and $2|r'$ (since $\alpha_1 \alpha_2 \neq 0$) that is, $|\text{Aut}(P\tilde{C})| = 3$ and we are done.

As a conclusion of the results that are introduced in this section we get the following result.

**Theorem 22.** The locus $M_{10}^0(\mathbb{Z}/3\mathbb{Z})$ is not ES-Irreducible and it has at least two irreducible components.

**7. Positive characteristic**

Now, suppose that $K$ is an algebraically closed field of positive characteristic $p > 0$. Consider a non-singular plane curve $C$ in $\mathbb{P}^2(K)$ of degree $d$ and assume that the order of $\text{Aut}(C)$ is coprime with $p$, $p \nmid d(d - 1)$, $p \geq 7$ and the order of $\text{Aut}(F_d)$ and $\text{Aut}(K_d)$ are coprime with $p$ where $F_d : X^d + Y^d + Z^d = 0$ is the Fermat curve and $K_d : X^{d-1} Y + Y^{d-1} Z + Z^{d-1} X = 0$ is the Klein curve. Then, all the techniques that appeared in Harui [11], can be applied: Hurwitz bound, Arakawa and Ohakawa inequalities and so on. In particular, the arguments of all the previous sections hold.

Consider the $p$-torsion of the degree 0 Picard group of $C$, which is a finitely generated $\mathbb{Z}/(p)$-module of dimension $\gamma$ (always $\gamma \leq g$ where $g$ is the genus of $C$), we call $\gamma$ the $p$-rank of $C$.

For a point $P$ of $C$ denote by $\text{Aut}(C)_P$ the subgroup of $\text{Aut}(C)$ that fixes the place $P$.

**Lemma 23.** Assume that $\text{Aut}(C)_P$ is prime to $p$ for any point $P$ of $C$ and the $p$-rank of $C$ is trivial. Then $\text{Aut}(C)$ is prime to $p$.

**Proof.** Consider $\sigma \in \text{Aut}(C)$ of order $p$, then the extension $\mathbb{K}(\sigma)/\mathbb{K}(C)^p$ is a finite extension of degree $p$ and is unramified everywhere (because if it ramifies at a place $P$ then $\sigma$ will be an element of $\text{Aut}(C)_P$ giving a contradiction). But, if $\gamma = 0$ (i.e. the $p$-rank is trivial for $C$) then, from Deuring-Shafarevich formula [13 Theorem11.62], we obtain that $\frac{1}{\gamma} = p$ where $\gamma'$ is the $p$-rank for $C/ < \sigma >$ which is impossible. Therefore, such extensions do not exist. □
Lemma 24. Consider $C$ a plane non-singular curve of degree $d \geq 4$. If $p > (d-1)(d-2) + 1$, then $\text{Aut}(C)_p$ is coprime with $p$ for any point $P$ of the curve $C$.

Proof. By [13] Theorem 11.78 the maximal order of the $p$-subgroup of $\text{Aut}(C)_p$ is at most $\frac{4p}{(p-1)^2}g^2$. Hence, with $g = \frac{(d-1)(d-2)}{2}$ and assuming that $p > \frac{4p}{(p-1)^2}g^2$, we obtain the result.

Lemma 25. Let $C$ be a non-singular curve of genus $g \geq 2$ defined over an algebraic closed field $\mathbb{K}$ of characteristic $p > 0$. Suppose that $C$ has an unramified subcover of degree $p$, i.e. $\Phi : C \rightarrow C'$ of degree $p$. Then $C'$ has genus $\geq 2$, $g \equiv 1 \pmod{p}$ and $\gamma \equiv 1 \pmod{p}$. In particular, for the existence of such subcover, one needs to assume that $p < g$.

Proof. The Hurwitz formula for $\Phi$ gives the equality $(2g - 2) = p(2g' - 2)$ where $g'$ is the genus of $C'$. We have $g' \neq 0$ or $1$ because $g \geq 2$, therefore $g' \geq 2$ and $g - 1 \equiv 0 \pmod{p}$. Now, consider Deuring-Shafararich formula, which in such unramified extension could be read as $\gamma = \beta \equiv p(\gamma' - 1)$ where $\gamma'$ the $p$-rank of $C'$. If $\gamma = 1$ then there is nothing to prove and if $\gamma > 1$ then the congruence is clear. Finally, if $\gamma = 0$ then this situation is not possible.

Corollary 26. Let $C$ be a non-singular plane curve of degree $d$ and genus $g \geq 2$ defined over an algebraic closed field $\mathbb{K}$ of characteristic $p > 0$. Suppose that $p > (d-1)(d-2) + 1 > g$. Then the order of $\text{Aut}(C)$ is coprime with $p$.

Proof. Suppose $\sigma \in \text{Aut}(C)$ of order $p$, then $\mathbb{K}(C)/\mathbb{K}(C)^\sigma$ is a separable degree $p$ extension, and by Lemma 23 it is unramified everywhere. By Lemma 25 we find that such extensions do not exist.

And as a direct consequence of the above lemmas and because all techniques in the previous sections, from [11], are applicable when $\text{Aut}(C)$ is coprime with $p$, then we obtain:

Corollary 27. Assume $p > 13$. The automorphism groups of the curves $\tilde{C} : X^5 + Y^5 + Z^4X + \beta X^3Y^2$ and $\tilde{C} : X^5 + X(Z^4 + Y^4) + \beta Y^2Z^3$ such that $\beta \neq 0$, are cyclic of order 4. Moreover, $\tilde{C}$ is not isomorphic to $\tilde{C}$ for any choice of the parameters.

Proof. Only we need to mention that the linear $g_2$-systems for the immersion of the curve inside $\mathbb{P}^2$ are unique up to conjugation in $\text{PGL}_3(\mathbb{K})$ see [13] Lemma 11.28 (also the curves $\tilde{C}$ and $\tilde{C}$ have cyclic covers of degree 4 with different type of the cover, from Hurwitz equation, therefore they belong to different irreducible components in the moduli space of genus 6 curves).

Corollary 28. For $p > 13$ we have that the locus $M^P_6(\mathbb{Z}/4\mathbb{Z})$ of the moduli space of positive characteristic, has at least two irreducible components.

Similarly we obtain the following result from results in §4,

Corollary 29. For $p > (d-1)(d-2) + 1$ where $d \geq 5$ is an odd integer, the locus $M^P_6(\mathbb{Z}/(d-1)\mathbb{Z})$ of the moduli space over positive characteristic $p$ is not ES-Irreducible and it has at least two strongly equation components. In particular, it has at least two irreducible components.
Plane non-singular curves with large cyclic automorphism groups

1. Abstract

In this elementary note, we present a result for obtaining the full list of the cyclic groups that could appear as an effective action for projective non-singular, non-hyperelliptic plane curves $C$ of a fixed degree $d$ over an algebraic closed field of zero characteristic. In particular, for any cyclic subgroup of $\text{Aut}(C)$, its order should divide one of the following numbers: $d-1, d, d^2-3d+3, (d-1)^2, d(d-2)$ or $d(d-1)$. Moreover, we attach to each cyclic group an equation $C$ (unique up to $K$-isomorphism) that admits the cyclic group $G$ as a subgroup of the full automorphism group. Also, we provide an algorithm of computation of the full list (once the degree is fixed) together with the implementation of this algorithm in SAGE program. On the other hand, we determine the full automorphism group and a characterization of the unique equation for the curve $C$ that admits an automorphism of order $d^2-3d+3, (d-1)^2, d(d-2)$ or $d(d-1)$ in $\text{Aut}(C)$. We also consider the cases when $\text{Aut}(C)$ has a cyclic subgroup of order $\ell d$ or $\ell(d-1)$ with $1 \leq \ell < d-2$.

2. Introduction

Consider $C$, a non-singular projective plane curve of degree $d \geq 4$ over an algebraically closed field $K$ of zero characteristic. Write by $F(X, Y, Z) = 0$ the defining equation of $C$ in $\mathbb{P}^2(K)$ (in particular, the polynomial $F$ has degree $\geq d-1$ in each variable because $C$ is non-singular). Denote by $\text{Aut}(C)$, the full automorphism group of $C$ over $K$, which is the finite subgroup of $\text{PGL}_3(K)$ that preserves the defining equation $F(X, Y, Z) = 0$ of $C$.

Harui in [11] gave a list of the possible $G \leq \text{Aut}(C)$ that could appear as: cyclic groups, an element of $\text{Ext}^1(-, -)$ of a cyclic group by a dihedral group, an alternating group $A_4$ (resp. $A_5$), an octahedral group $S_4$, a subgroup of $\text{Aut}(F_4)$ (the automorphism group of the Fermat curve), a subgroup of $\text{Aut}(K_4)$ (the automorphism group of the Klein curve) or a finite primitive subgroup of $\text{PGL}_3(K)$ (for the classification of such groups, we refer to Mitchell [16]).

For a fixed degree $d$, the previous result does not give the precise list of groups that appear and and the corresponding equations with an effective action given by $G$ (up to $K$-isomorphism). In the literature, as far as we know, this is only given for $d = 4$ by Henn [12] (see also the manuscript [3] for $d = 5$). Recall that, two non-singular plane curves, with equations given by $F_1(X, Y, Z) = 0$ and $F_2(X, Y, Z)$ respectively, are $K$-isomorphic if and only if $F_1$ can be transformed through a change of variables $P \in \text{GL}_3(K)$ to $F_2$ and in particular $\text{Aut}(C_2)$ is conjugate inside $\text{PGL}_3(K)$ to $\text{Aut}(C_1)$ in the sense that $\text{Aut}(C_2) = P \text{Aut}(C_1) P^{-1}$.

Now, let $d \geq 5$ be an arbitrary but fixed, we present the first clear statements which lists the exact cyclic groups $G$ (of order $m$) associated with equations (equipped with a family of parameters) of the plane curves that have an effective action given by $G$ (up to $K$-isomorphism). In particular, we prove that the order $m$ of any cyclic group that might appear should divide one of the following integers: $d-1, d, d^2-3d+3, (d-1)^2, d(d-2)$ or $d(d-1)$ [Theorem 30]. On the other hand, we formulate the main result as an algorithm (CAGPC) together with a complete implementation in SAGE program (for more details, one may use the link in Remark 31). On the other hand, it is very classical to determine those curves with large automorphism group, see for example [18] and the interest of their equations up to $K$-isomorphism. Consequently, the second facet of this chapter is to determine (up to $K$-isomorphism) the smooth plane curves of degree $d$ which have a cyclic subgroup of large order in their automorphism group (by a large order we mean $d^2-3d+3, (d-1)^2, d(d-2)$ or $d(d-1)$). For such cases, we determine the equation of the curve $F(X; Y; Z) = 0$ (up to $K$-isomorphism) and its full automorphism group. Moreover, we solve the same question for non-singular plane curves with a cyclic...
3. Cyclic automorphism group of non-singular plane curves

Fix an integer \( d \geq 4 \) and let \( C \) be a non-singular projective plane curve of degree \( d \). Our aim here is to determine the cyclic groups that appear as subgroups inside \( Aut(C) \). We follow a similar approach of Dolgachev [9] which treat the same question for \( d = 4 \) (see [4], §2.1).

Let \( \sigma \in Aut(C) \) be of maximal order \( m \), then we can assume, without any loss of generality, that \( \sigma \) is given in its canonical Jordan form by \((x : y : z) \mapsto (\xi_m^a y : \xi_m^b z)\) where \( \xi_m \) is a primitive \( m \)-th root of unity in \( K \) and \( a, b \) integers such that \( 0 \leq a \neq b \leq m - 1 \) with \( a \leq b \) and \( \gcd(a, b) \) coprime with \( m \) if \( ab \neq 0 \) (we can reduce to \( \gcd(a, b) = 1 \) and \( \gcd(d, m) = 1 \) if \( a = 0 \).

Throughout this paper, we also use also the following notations.

- \( L_{n, \ast} \) denotes a degree \( i \) homogeneous polynomial in \( K[X, Y, Z] \) without the variable \( \ast \) where \( \ast \in \{ X, Y, Z \} \).
- \( S(i)_m := \{ j : u \leq j \leq d - 1, d - j \equiv 0 \pmod{m} \} \).
- \( S^d_{u,X} m, (a, b) := \{ i : u \leq i \leq d - u \text{ and } ai + (d - i)b = 0 \pmod{m} \} \).
- \( S^{d-1}_{u,X} m, (a, b) := \{ i : 1 \leq i \leq d - u \text{ and } ai + (d - 1 - i)b = 0 \pmod{m} \} \).
- \( S^{d-1}_{\gamma,Y} m, (a, b) := \{ i : 0 \leq i \leq j \text{ and } ai + (j - i)b = 0 \pmod{m} \} \).
- \( S^{d-1}_{\gamma,Y} m, (a, b) := \{ i : 0 \leq i \leq j \text{ and } ai + (j - i)b = 0 \pmod{m} \} \).
- \( \Gamma_m := \{ (a, b) \in \mathbb{Z}^2 : \gcd(a, b) = 1, 1 \leq a \neq b \leq m - 1 \} \).
- \( \alpha \) always is an element in \( K^* \) which by a change of variables can be assumed equal to 1.

where \( u, j, m, d, a \) and \( b \) are non-negative integers.

**Theorem 30.** Let \( C \) be a non-singular projective plane curve of degree \( d \) over an algebraically closed field \( K \) of zero characteristic. If \( G \) is a non-trivial cyclic subgroup of \( Aut(C) \) of order \( m \), then the classification of such curves up to their cyclic subgroups is given by the following. (note that, we attach to each type an equation of \( C \) which is unique up to \( K \)-isomorphism).

1. The curve \( C \) is of the form \( Z^{d-1}L_{1,Z} + \left( \sum_{j \in S(2)_m} Z^{d-j}L_{j,Z} \right) + L_d.Z \), which is type \( (0, 1, 1) \).
2. The curve \( C \) has the form \( Z^{d} + \left( \sum_{j \in S(1)_m} Z^{d-j}L_{j,Z} \right) + L_d.Z \), and \( C \) is of type \( (0, 1, 1) \).
3. All reference points lie on \( C \) which is of type \( m, (a, b) \) for some \( m \) such that \( a = (d - 1)a + b = (d - 1)b \pmod{m} \).
4. Two reference points lie on \( C \) which is of type \( m, (a, b) \) for one of the following subcases.

\(^{1}\)We warn the reader that may happen that the curve given for a certain type \( m(a, b) \) could be never geometrically irreducible for any parameter and we should discard from the list.
(4.1) First, \( m \mid d(d-2) \) and \( (a, b) \in \Gamma_m \) such that \((d-1)a + b \equiv 0 \pmod m \) and \(a + (d-1)b \equiv 0 \pmod m \). Moreover, \( C \) is given by
\[
X^d + \sum_{j=2}^{d-1} X^{d-j} \sum_{i \in S(2)^{m, (a, b)}} \beta_{ji} Y^i Z^{j-i} + (Y^{d-1} Z + \alpha Y Z^{d-1} + \sum_{i \in S_2^{d, m, (a, b)}} \beta_{di} Y^i Z^{d-i}) = 0,
\]

(4.2) The second case with \( m \mid (d-1)^2 \) and \( (a, b) \in \Gamma_m \) such that \((d-1)a + b \equiv 0 \pmod m \) and \((d-1)b \equiv 0 \pmod m \). Furthermore, the defining equation of \( C \) is
\[
X^d + \sum_{j=2}^{d-1} X^{d-j} \sum_{i \in S(2)^{m, (a, b)}} \beta_{ji} Y^i Z^{j-i} + X(\alpha Y^{d-1} + \sum_{i \in S_2^{d-1, m, (a, b)}} \beta_{di} Y^i Z^{d-1-i}) + (Y^{d-1} Z + \sum_{i \in S_2^{d, m, (a, b)}} \beta_{di} Y^i Z^{d-i}) = 0.
\]

(4.3) Lastly, \( m \mid (d-1) \) and \( (a, b) \in \Gamma_m \) such that \((d-1)b \equiv 0 \pmod m \) and \((d-1)a \equiv 0 \pmod m \). In such case \( C \) has the form
\[
X^d + \sum_{j=2}^{d-1} X^{d-j} \sum_{i \in S(2)^{m, (a, b)}} \beta_{ji} Y^i Z^{j-i} + \sum_{i \in S_2^{d, m, (a, b)}} \beta_{di} Y^i Z^{d-i} + X(\alpha Y^{d-1} + \sum_{i \in S_2^{d-1, m, (a, b)}} \beta_{di} Y^i Z^{d-1-i}),
\]

(5) One reference points lie in \( C \) that is of type \( (a, b) \) for some \( m \mid d(d-1) \) and \( (a, b) \in \Gamma_m \) such that \(da \equiv 0 \pmod m\) and \((d-1)b \equiv 0 \pmod m\). Moreover, \( C \) is defined by
\[
C : X^d + Y^d + \sum_{j=2}^{d-1} X^{d-j} \sum_{i \in S(2)^{m, (a, b)}} \beta_{ji} Y^i Z^{j-i} + \sum_{i \in S_2^{d, m, (a, b)}} \beta_{di} Y^i Z^{d-i} + X(\alpha Y^{d-1} + \sum_{i \in S_2^{d-1, m, (a, b)}} \beta_{di} Y^i Z^{d-1-i}) = 0.
\]

(6) None of the reference points lie on \( C \) and we obtain type \( m(a, b) \) for \( m \mid d \) and \( (a, b) \in \Gamma_m \) such that \(da \equiv 0 \pmod m\) and \(db \equiv 0 \pmod m\). Furthermore, we have
\[
C : X^d + Y^d + Z^d + \sum_{j=2}^{d-1} X^{d-j} \sum_{i \in S(2)^{m, (a, b)}} \beta_{ji} Y^i Z^{j-i} + \sum_{i \in S_2^{d, m, (a, b)}} \beta_{di} Y^i Z^{d-i} = 0.
\]

Here, \( \alpha, \beta_{ij}, \gamma_{ij}, \alpha_{ij} \) are in \( K \) and \( \alpha \neq 0 \).

**Remark 31.** The above result and its proof give an algorithm to list for every fix degree \( d \), all the cyclic groups that could appear with an equation (up to \( K \)-isomorphisms), for the complete algorithm and its implementation in SAGE see the link http://mat.uab.cat/~eslam/CAGPC.sage, and read the last section of this chapter for a list of cyclic groups that appears for lower degree \( d \) with their equations.

**Remark 32.** The above result is also true when \( K \) is an algebraic closed field of characteristic \( p > 0 \) and we assume from the beginning that \( m \) is always coprime with \( p \), covering the cyclic groups of order coprime with the characteristic.

**Proof.** Let \( \varphi \) be a generator of order \( m := |G| \). One can choose coordinates so that \( \varphi \) is represented by \((X; Y; Z) \mapsto (X; \xi_m^a Y; \xi_m^b Z)\) where, \( \xi_m \) is a primitive \( m \)-th root of unity in \( K \) and \(a, b\) are integers with \( 0 \leq a \neq b \leq m-1 \) (without loss of generality, one can assume that \( a \leq b \), with \( gcd(b, m) = 1 \) if \( a = 0 \) and with \( gcd(a, b) = 1 \) if \( a \neq 0 \)). Now, we have two cases, either \( a = 0 \) or \( a \neq 0 \). In the following we treat each of these cases.

**Case I:** Suppose first that \( a = 0 \), write: \( F(X; Y; Z) = \lambda Z^d + (\sum_{j=1}^{d-1} Z^{d-j} L_j, Z) + L_d, Z \).
If $\lambda = 0$, then $(d - 1)b \equiv 0 \pmod{m}$. Hence, $m|d - 1$ and we can take a generator $(a, b) = (0, 1)$. Therefore, by checking each monomial’s invariance, we obtain type $m$, $(0, 1)$ which is defined by the equation $Z^{d - 1}L_{1,Z} + \left( \sum_{j \in S(2)_m} Z^{d-j}L_{j,Z} \right) + L_{d,Z}$, and (1) arises.

If $\lambda \neq 0$, then $db \equiv 0 \pmod{m}$. From which, we have $m|d$ and $(a, b) = (0, 1)$ is a generator for each such $m$. Consequently, we get type $m$, $(0, 1)$ of the form $Z^{d} + \left( \sum_{j \in S(1)_m} Z^{d-j}L_{j,Z} \right) + L_{d,Z}$. which is precisely (2).

**Case II:** Suppose that $a \neq 0$, then, necessarily, $m > 2$. Now, Let $P_1 = (1; 0; 0), P_2 = (0; 1; 0)$ and $P_3 = (0; 0; 1)$ be the reference points, therefore, we have the following four subcases:

i.: All reference points lie in $C$, 
ii.: Two reference points lie in $C$, 
iii.: One reference point lies in $C$, 
iv.: None of the reference points lie in $C$.

We will treat each of these subcases.

- If all reference points lie on $C$, then the possibilities for the defining equation are now:

$$C : \sum_{j = 1}^{\lfloor \frac{d}{2} \rfloor} \left( X^{d-j}L_{j,X} + Y^{d-j}L_{j,Y} + Z^{d-j}L_{j,Z} \right).$$

It is obvious that $B_i$ can not appear in $L_{1,j}$, whenever $j \neq i$, where $B_1 := X$, $B_2 := Y$ and $B_3 := Z$. Hence, by the change of the variables $X, Y$ and $Z$ (for instance, $L_{1,X} \leftrightarrow Y, L_{1,Y} \leftrightarrow Z$ and $L_{1,Z} \leftrightarrow X$), we can assume that

$$C : X^{d-1}Y + Y^{d-1}Z + \alpha Z^{d-1}X + \sum_{j = 2}^{\lfloor \frac{d}{2} \rfloor} \left( X^{d-j}L_{j,X} + Y^{d-j}L_{j,Y} + Z^{d-j}L_{j,Z} \right).$$

The first three factors implies that $a = (d - 1)a + b = (d - 1)b \pmod{m}$. In particular, $m|(d - 2)^2 + (d - 1)$. Now, for each $m$, let $L_m$ be the set $L_m := \{(a, b) \in \Gamma_m : a = (d - 1)a + b = (d - 1)b \pmod{m}\}$. Then, for any $(a, b) \in L_m$, by checking each monomial invariance, we get type $m$, $(a, b)$ of the form (3). Furthermore, this type is $K$-equivalent to any type $m$, $(a', b') \in <m, (a, b)>$. So, to complete the classification for a certain $m$, it suffices to choose another $(a_0, b_0) \in L_m - (a, b)$ till $L_m = \phi$. This proves (3).

- If two reference points lie in the smooth plane curve $C$, then by re-scaling the matrix $\varphi$ and permuting the coordinates, we can assume that $(1; 0; 0) \notin C$. The equation is then

$$C : X^d + X^{d-2}L_{2,X} + X^{d-3}L_{3,X} + \ldots + X L_{d-1,X} + L_{d,X} = 0,$$

since $L_{1,X}$ is not invariant by $\varphi$ since $ab \neq 0$. Moreover, $Z^d$ and $Y^d$ are not in $L_{d,X}$, by the assumption that only $(1; 0; 0) \notin C$.

Assume first that $Y^{d-1}Z$ and $YZ^{d-1}$ are in $L_{d,X}$. Then, $(d - 1)a + b \equiv 0 \pmod{m}$ and $a + (d - 1)b \equiv 0 \pmod{m}$. In particular, $m|(d - 1)^2 - 1$ and for each $m \in \mathbb{Z}$ where $L_m$ is defined as follows

$$L_m := \{(a, b) \in \Gamma_m : (d - 1)a + b \equiv 0 \pmod{m}, a + (d - 1)b \equiv 0 \pmod{m}\}.$$ 

If $(a, b) \in L_m$, then we obtain type $m$, $(a, b)$ of the form

$$X^{d} + \left( \sum_{j = 2}^{d-1} X^{d-j} \sum_{i \in S(2)_m^{j,X,(a,b)}} \beta_{ij} Y^{i} Z^{j-i} \right) + \left( Y^{d-1}Z + \alpha Y Z^{d-1} \right) + \sum_{i \in S_2^{d,X,m,(a,b)}} \beta_{d,i} Y^{i} Z^{d-i} = 0,$$

where $\alpha \neq 0$ and (4.1) follows.

Assume second that $Y^{d-1}Z \in L_{d,X}$ and $YZ^{d-1} \notin L_{d,X}$. Then, by the non-singularity, $Z^{d-1}$ is in $L_{d-1,X}$. That is, $(d - 1)a + b \equiv 0 \pmod{m}$ and $(d - 1)b \equiv 0 \pmod{m}$. Hence, $m|(d - 1)^2$ and $(a, b) \in L_m$. 


\[(a, b) \in \Gamma : \ (d - 1) a + b \equiv 0 \ (mod \ m), \ (d - 1)b \equiv 0 \ (mod \ m)\] 
if we check monomials invariance for a certain \((a, b)\), we obtain the type \(m, (a, b)\) of the form 
\[X^d + \sum_{j=2}^{d-2} X^{d-j} \left( \sum_{i \in S^{(2)\omega, b}_m} \beta_{j_1} Y^i Z^{j-i} \right) + X(\alpha Z^{d-1} + \sum_{i \in S^{d-1, X}_m} \beta_{d-1}) Y^i Z^{d-1-i} +\]
\[+ (Y^{d-1}Z + \sum_{i \in S^d X \ m, (a,b)} \beta_{d_1} Y^i Z^{d-1-i}) = 0.\]

That proves the case 4.2.

Up to a permutation of \(Y\) and \(Z\), we can assume that \(Y^{d-1}Z\) and \(YZ^{d-1}\) are not in \(L_{d, X}\). By the non-singularity, \(Z^{d-1}\) and \(Y^{d-1}\) should be in \(L_{d-1, X}\) consequently, \((d - 1)b \equiv 0 \ (mod \ m)\) and \((d - 1)a \equiv 0 \ (mod \ m)\). Therefore, \(m|(d - 1)\) and we get type \(m, (a, b)\) of (4.3) defined by the equation 
\[X^d + \sum_{j=2}^{d-2} X^{d-j} \left( \sum_{i \in S^{(2)\omega, b}_m} \beta_{j_1} Y^i Z^{j-i} \right) + \sum_{i \in S^{d-1, X}_m} \beta_{d_1} Y^i Z^{d-1-i} +\]
\[+ X \left(Z^{d-1} + \alpha Y^{d-1} + \sum_{i \in S^d X \ m, (a,b)} \beta_{(d-1)} Y^i Z^{d-1-i}\right) = 0,\]

- If one reference points lie in the \(C\), then by normalizing the matrix \(\varphi\) and permuting the coordinates, we can assume that \((1; 0, 0), (0; 1; 0) \not\in C\). We then write 
\[C : X^d + Y^d + X^{d-2} L_{2, X} + X(d - 3) L_{3, X} + \ldots + X L_{d-1, X} + L_{d, X} = 0,\]
such that \(Z^d \not\in L_{d, X}\). Also, by the non-singularity, we have that \(Z^{d-1} \in L_{d-1, X}\), then \(da \equiv 0 \ (mod \ m)\) and \((d - 1)b \equiv 0 \ (mod \ m)\). Hence, \(m|d(d - 1)\) define \(L_m\) to be \(\{(a, b) \in \Gamma : da \equiv 0 \ (mod \ m), \ (d - 1)b \equiv 0 \ (mod \ m)\}\). Consequently, for each element in this set we obtain type \(m, (a, b)\) of the form 
\[C : X^d + Y^d + \sum_{j=2}^{d-2} (X^{d-j} \left( \sum_{i \in S^{(2)\omega, b}_m} \beta_{j_1} Y^i Z^{j-i} \right) + \sum_{i \in S^{d-1, X}_m} \beta_{d_1} Y^i Z^{d-1-i} +\]
\[+ X \left(\alpha Z^{d-1} + \sum_{i \in S^d X \ m, (a,b)} \beta_{(d-1)} Y^i Z^{d-1-i}\right) = 0,\]
which produces (5)

- If none of the reference points lie in the smooth plane curve \(C\), then 
\[C : X^d + Y^d + Z^d + \left( \sum_{j=2}^{d-1} X^{d-j} L_{j, X}\right) + L_{d, X} = 0,\]
where \(L_{1, X}\) does not appear since \(ab \neq 0\). Clearly, \(da = db = 0 \ (mod \ m)\), from which \(m|d)\) for each \(m\), and each \((a, b) \in L_m\) \(:= \{(a, b) \in \Gamma_m : da \equiv 0 \ (mod \ m), \ db \equiv 0 \ (mod \ m)\}\), we obtain type \(m, (a, b)\) of the following form 
\[C : X^d + Y^d + Z^d + \sum_{j=2}^{d-1} (X^{d-j} \left( \sum_{i \in S^{(2)\omega, b}_m} \beta_{j_1} Y^i Z^{j-i} \right) + \sum_{i \in S^{d-1, X}_m} \beta_{d_1} Y^i Z^{d-1-i} +\]
\[+ X \left(\alpha Z^{d-1} + \sum_{i \in S^d X \ m, (a,b)} \beta_{(d-1)} Y^i Z^{d-1-i}\right) = 0.\]

This completes the proof of the main result.

\[\square\]

**Corollary 33.** Let \(G\) be a cyclic subgroup of \(\text{Aut}(C)\) where \(C\) is a non-singular plane curve of degree \(d \geq 4\). Then, \(|G|\) must divide one of the following \(d - 1, d, d^2 - 3d + 3, (d - 1)^2, d(d - 2), d(d - 1)\). In particular, \(|G| \leq d(d - 1)\).
4. Characterization of curves where $\text{Aut}(C)$ has a cyclic subgroup of large order

We study next non-singular plane curves $C$ (up to $K$-isomorphism) that admits a $\sigma \in \text{Aut}(C)$ of order $d^2 - 3d + 3$, $(d - 1)^2$, $d(d - 2)$, $d(d - 1)$, or $\ell d$, with $\ell \geq 2$. In particular we are interested in determine the full automorphism group of the curve and equations of such curves.

Before a detailed study we recall the following results concerning $\text{Aut}(C)$ which will be useful. Because linear systems $g_2$ are unique, we always assume $C$ is given by a plane equation $F(X, Y, Z) = 0$ and $\text{Aut}(C)$ is a finite subgroup of $\text{PGL}_3(K)$ which fix the equation $F$. Moreover $\text{Aut}(C)$ satisfies (see Mitchell [16]) one of the following situations:

1. fixes a point $P$ and a line $L$ with $P \not\in L$ in $\text{PGL}_3(K)$,
2. fixes a triangle, i.e. exists 3 points $S := \{P_1, P_2, P_3\}$ of $\text{PGL}_3(K)$, such that is fixed as a set,
3. $\text{Aut}(C)$ is conjugate of a representation inside $\text{PGL}_3(K)$ of one of the finite primitive group namely, the Klein group $\text{PSL}(2,7)$, the icosahedral group $A_5$, the alternating group $A_6$, the Hessian groups $\text{Hess}_{216}$, $\text{Hess}_{72}$ or $\text{Hess}_{36}$.

It is classically known that if $G$ a subgroup of automorphisms of a non-singular plane curve $C$ fixes a point on $C$ then $G$ is cyclic [13] Lemma 11.44, and recently Harui in [11] [2] provided the lacked result in the literature on the type of groups that could appear in non-singular plane curves. Before introduce Harui statement we need to define descendent of a plane non-singular curve. For a non-zero monomial $cX^iY^jZ^k$, $c \in K \setminus \{0\}$, we define its exponent as $\max\{i, j, k\}$. For a homogeneous polynomial $F$, the core of $F$ is defined as the sum of all terms of $F$ with the greatest exponent. Let $C_0$ be a smooth plane curve, a pair $(C, G)$ with $G \leq \text{Aut}(C)$ is said to be a descendent of $C_0$ if $C$ is defined by a homogeneous polynomial whose core is a defining polynomial of $C_0$ and $G$ acts on $C_0$ under a suitable coordinate system.

**Theorem 34** (Harui). If $G \leq \text{Aut}(C)$ where $C$ is a non-singular plane curve of degree $d \geq 4$ then $G$ satisfies one of the following

1. $G$ fixes a point on $C$ and then cyclic.
2. $G$ fixes a point not lying on $C$ and it satisfies a short exact sequence of the form
   $$1 \to N \to \text{Aut}(C) \to G' \to 1,$$
   with $N$ a cyclic group of order dividing $d$ and $G'$ is conjugate to a cyclic group $C_m$, a Dihedral group $D_{2m}$, the alternating groups $A_4$, $A_5$ or the permutation group $S_4$, where $m$ is an integer $\leq d - 1$. Moreover, if $G' \cong D_{2m}$, then $m|(d - 2)$ or $N$ is trivial.
3. $G$ is conjugate to a subgroup of $\text{Aut}(F_d)$ where $F_d$ is the Fermat curve $X^d + Y^d + Z^d$. In particular, $|G| 6d^2$ and $(G, C)$ is a descendant of $F_d$.
4. $G$ is conjugate to a subgroup of $\text{Aut}(K_d)$ where $K_d$ is the Klein curve curve $XY^{d-1} + YZ^{d-1} + ZX^{d-1}$ and $(G, C)$ is a descendant of $K_d$.
5. $G$ is conjugate a finite primitive subgroup of $\text{PGL}_3(K)$, i.e. the Klein group $\text{PSL}(2,7)$, the icosahedral group $A_5$, the alternating group $A_6$, the Hessian groups $\text{Hess}_{216}$, $\text{Hess}_{72}$ or $\text{Hess}_{36}$ inside $\text{PGL}_3(K)$.

where

Next assume as usual $C$ a non-singular plane curve of degree $d \geq 4$ with $\sigma \in \text{Aut}(C)$ of exact order $m$ such that acts on $F(X, Y, Z) = 0$ by $(x, y, z) \mapsto (x, \xi^a_m y, \xi^b_m z)$.

4.1. Curves with a cyclic automorphism of order $d(d - 1)$.

The following result appears in Harui [11] [3].

**Proposition 35** (Harui). A non-singular projective plane curve $C$ of degree $d \geq 5$ with $\text{Aut}(C)$ cyclic group of order $d(d - 1)$ if and only if it is projectively equivalent to $X^d + Y^d + XZ^{d-1}$.

**Proposition 36.** Let $C$ be a non-singular projective plane curve of degree $d \geq 4$. Then, $\text{Aut}(C)$ contains an element of order $d(d - 1)$ if and only if $C$ is projectively equivalent to $X^d + Y^d + \alpha XZ^{d-1}$ where $\alpha \neq 0$. In particular, $\text{Aut}(C)$ is cyclic of order $d(d - 1)$ for $d \geq 5$. 

Remark 37. Recall that for $d = 4$ the automorphism group of $X^d + Y^d + \alpha XZ^{d-1}$ is $C_4 \cong \mathbb{Z}/4\mathbb{Z}$, and for $d > 4$ it is the group $C_6 \cong \mathbb{Z}/6\mathbb{Z}$, where $\mathbb{Z}/n\mathbb{Z}$ is the group of integers modulo $n$.

Claim: $S$ is the group of order $2$ generated by $\alpha$. Let $S$ be a subgroup of $\text{Aut}(C)$ of order $2$. It is clear that $S$ is cyclic.

Proof. (→) Indeed, $[X; \zeta_{d(d-1)}^j Y; \zeta_{d(d-1)}^d Z]$ is an element of $\text{Aut}(C)$ of order $d(d-1)$.

Claim: $S$ is projectively equivalent to $X^d + Y^d + \alpha XZ^{d-1}$ where $\alpha \neq 0$. The last part for which $\text{Aut}(C)$ is cyclic of order $d(d-1)$ is followed by Harui [35].

4.2. Plane curves with a cyclic automorphism of order $(d-1)^2$.

Proposition 38. The automorphism group of the non-singular projective plane curve of degree $d \geq 5$ defined by

$$C : X^d + Y^d + \alpha XZ^{d-1}$$

such that $\alpha \neq 0$ is cyclic of order $(d-1)^2$.

Remark 39. The same result holds for $d = 4$, see [12] (or also [4]).
PROOF. We have $\text{Aut}(C)$ contains a homology of order $d - 1 \geq 4$ namely, $[X; \zeta_{d-1} Y; Z]$ therefore (see Mitchell [10] §5), $\text{Aut}(C)$ should fix a point, a line or a triangle. Moreover, $(d - 1)^2 | |\text{Aut}(C)|$ because the transformation $[X; \zeta_{(d-1)^2} Y; \zeta_{(d-1)^2} Z] \in \text{Aut}(C)$ is of order $(d - 1)^2$.

(1) If $\text{Aut}(C)$ fixes a triangle and neither a point nor a line is fixed then, it follows by Harui §4, that $C$ is either a descendant of the Fermat curve $F_d$ or the Klein curve $K_d$. But, non of them contains an element of order $(d - 1)^2$ because $\text{Aut}(F_d)$ (resp. $\text{Aut}(K_d)$) has elements of order at most $2d$ (resp. $d^2 - 3d + 3 < (d - 1)^2$).

(2) Assume that $\text{Aut}(C)$ fixes a point not lying on $C$ and satisfies a short exact sequence of the form:

$$1 \rightarrow N \rightarrow \text{Aut}(C) \rightarrow G' \rightarrow 1,$$

with $N$ a cyclic group of order dividing $d$ and $G'$ is conjugate to a cyclic group $C_m$, a diedral group $D_{2m}$, $A_4$, $A_5$ or $S_4$, where $m$ is an integer $\leq d - 1$. Moreover, if $G' \cong D_{2m}$, then $m |(d - 2)$ or $N$ is trivial. Since $|N|$ and $(d - 1)^2$ are coprime, then $G'$ should contain an element of order $(d - 1)^2$ a contradiction because each of these groups have elements of order at most max $\{5, d - 1\} < (d - 1)^2$.

Consequently, $\text{Aut}(C)$ fixes a point on $C$, thus it is cyclic of order divisible by $(d - 1)^2$ and $\leq d - 1)$. Therefore, $|\text{Aut}(C)| = (d - 1)^2$.

As analogue of Proposition 38 we prove the following result.

**Proposition 40.** A non-singular projective plane curve $C$ of degree $\geq 4$ has an automorphism of order $(d - 1)^2$ if and only if it is isomorphic to $X^d + Y^{d-1}Z + aXZ^{d-1}$ such that $\alpha \neq 0$. In particular, $\text{Aut}(C)$ is cyclic of order $(d - 1)^2$ moreover, if $G$ is a non-cyclic automorphism group with $(d - 1)^2$ dividing $|G|$ then $G$ does not contain any element of this order.

**Proof.** It suffices to prove the first part because the other part follows directly by Proposition 38 and the remark 39.

$\leftarrow$ Clearly, $[X; \zeta_{(d-1)^2} Y; \zeta_{(d-1)^2} Z]$ is an automorphism that has order $(d - 1)^2$.

$\rightarrow$ One can easily check that $(d - 1)^2$ does not divide any of the integers $d - 1$, $d$, $d^2 - 3d + 3$, $d(d - 2)$, $d(d - 1)$. Therefore, by Theorem 39 $C$ is isomorphic to type $(d - 1)^2, (a, b)$ of the form

$$X^d + \sum_{j=2}^{d-2} X^{d-j} \left( \sum_{i \in S(2)^{d-1}_j, (a, b)} \beta_{ji} Y^i Z^{d-1-i} \right) + X(aZ^{d-1} + \sum_{i \in S(d-1)^2, (a, b)} \beta_{(d-1)i} Y^i Z^{d-1-i}) +$$

$$+ \left( Y^{d-1}Z + \sum_{i \in S_2^{d-1}} \beta_{di} Y^i Z^{d-1-i} \right) = 0$$

for some $(a, b) \in \Gamma_{(d-1)^2}$ such that $(d - 1)a + b \equiv 0 \mod (d - 1)^2$ and $(d - 1)b \equiv 0 \mod (d - 1)^2$. Every solution is of the form $a = (d - 1)k - k'$ and $b = (d - 1)k'$ then we can take a generator $a = 1$ and $b = (d - 1)(d - 2)$ because $[X; \zeta_{(d-1)^2} Y; \zeta_{(d-1)^2} Z]^{(d-1)k-k'} = [X; \zeta_{(d-1)^2} Y; \zeta_{(d-1)^2} Z]$. Now,

$$S(2)^{d-1}_j, (a, b) := \{ i : 0 \leq i \leq j \text{ and } i + (j - i)(d - 1)(d - 2) = 0 \mod (d - 1)^2 \}$$

$$= \{ i : 0 \leq i \leq j \text{ and } (d - 1)^2 \mid (j - i)(d - 1) \}$$

$$= \{ i : 0 \leq i \leq j \text{ and } (d - 1)^2 \mid j(d - 1) - di \}$$

$$= \phi \quad \forall j = 2, \ldots, d - 2.$$ 

because if $(d - 1)^2 | j(d - 1) - di$ then $d(j + 1)$ a contradiction since $0 < j + 1 < d$.

$$S_1^{d-1,X} m, (a, b) := \{ i : 1 \leq i \leq d - 1 \text{ and } i + (d - 1 - i)(d - 1)(d - 2) = 0 \mod (d - 1)^2 \}$$

$$= \{ i : 1 \leq i \leq d - 1 \text{ and } (d - 1)^2 \mid di \}$$

$$= \{ i : 1 \leq i \leq d - 1 \text{ and } (d - 1)^2 \mid i \}$$

$$= \phi.$$
because $0 < i < (d - 1)^2$ so $(d - 1)^2 \nmid i$.

$$S_{d}^{X} m, (a, b) := \{ i : 2 \leq i \leq d - 2 \text{ and } i + (d - i)(d - 1)(d - 2) = 0 \mod (d - 1)^2 \} \subseteq \{ i : 2 \leq i \leq d - 2 \text{ and } (d - 1)^2 \mid di - (d - 1) \} = \phi.$$  

because $0 < di - (d - 1) \leq d^2 - 3d + 1 < (d - 1)^2$. Thus, $C$ is isomorphic to $X^d + \alpha XZ^{d-1} + Y^{d-1}Z = 0$. \hfill \Box

4.3. Plane curves with an automorphism of order $d(d - 2)$.

**Proposition 41.** The full automorphism group of $C : X^d + Y^{d-1}Z + \alpha YZ^{d-1}$ of degree $d \geq 4$ such that $\alpha \neq 0$ is classified as follows.

1. If $d \neq 4, 6$, then it is the central extension

   $$< \sigma, \tau \mid \sigma^2 = \tau^{d(d-2)} = 1 \text{ and } \sigma \tau \sigma = \tau^{-(d-1)} >$$

   of $D_{2(d-2)}$ by $\mathbb{Z}_d$. In particular, $|\text{Aut}(C)| = 2d(d - 2)$.

2. If $d = 6$, then it is a central extension of $S_4$ by $\mathbb{Z}_6$. Thus, $|\text{Aut}(C)| = 144$.

3. If $d = 4$, then $C$ is isomorphic to the Fermat quartic $F_4$ thus $\text{Aut}(C) \simeq \mathbb{Z}_4^2 \times S_3$.

**Proof.** Let $\mu \in K$ such that $\mu^{d(d-2)} = \alpha$, then $C$ is projectively equivalent, through the transformation $[X: \mu Y : \mu^{-(d-1)}Z]$, to the curve $C' : X^d + Y^{d-1}Z + YZ^{d-1}$ and therefore by Harui [11, §6], $\text{Aut}(C)$ is a central extension of $D_{2(d-2)}$ by $\mathbb{Z}_d$ $(d \neq 4, 6)$, a central extension of $S_4$ by $\mathbb{Z}_d$ $(d = 6)$ or $\simeq \mathbb{Z}_4^2 \times S_3$ $(d = 4)$. To prove the remaining part, it suffices to notice that $\text{Aut}(C')$ is generated by $\sigma := [X; Z; Y]$ and $\tau := [X; \zeta_{d(d-2)}Y; \zeta_{d(d-2)}^{-1}Z]$.

As an analogue of Propositions[30] and [40], we have the following result.

**Proposition 42.** A non-singular projective plane curve $C$ of degree $d \geq 4$ admits an automorphism of order $d(d - 2)$ if and only if it is isomorphic to $X^d + Y^{d-1}Z + YZ^{d-1}$. In particular, $\text{Aut}(C)$ is a central extension of $D_{2(d-2)}$ by $\mathbb{Z}/d$ $(d \neq 4, 6)$, a central extension of $S_4$ by $\mathbb{Z}/d$ $(d = 6)$ or $\simeq (\mathbb{Z}/4)^2 \times S_3$ $(d = 4)$.

**Proof.** It suffices to prove the first part and the second part is an immediate consequence of Proposition[41]

(→) Clearly, $d(d - 2)$ does not divide any of the integers $d - 1, d, d^2 - 3d + 3, (d - 1)^2, d(d - 1)$. Therefore, by Theorem[30] $C$ is isomorphic to type $d(d - 2), (a, b)$ of the form

$$X^d + \left( \sum_{j=2}^{d-1} X^{d-j} \sum_{i \in S(2)^m, (a, b)} \beta_{j,i} Y^i Z^{j-i} \right) + (Y^{d-1}Z + \alpha YZ^{d-1} + \sum_{i \in S_2^d, (a, b)} \beta_{d,i} Y^i Z^{d-i}) = 0,$$

fore some $(a, b) \in \Gamma_{d(d-2)}$ such that $(d - 1)a + b \equiv 0 \mod d(d - 2)$ and $a + (d - 1)b \equiv 0 \mod d(d - 2)$. Every solution of the form $(k, dk' + k)$ such that $k$ and $dk' + k$ are coprime and $d - 2|k + k'$, $k + (d - 1)k'$. We can take a generator $k = 1$ and $k' = d - 3$ because $[X; \zeta_{d(d-2)}Y; \zeta_{d(d-2)}^{d(d-3)+1}Z]^k = [X; \zeta_{d(d-2)}^k Y; \zeta_{d(d-2)}^{d(d-3)+k}Z]$ (keep in mind that $d - 2|k + k'$). Therefore, we have

$$S_{d}^{X} m, (a, b) := \{ i : 0 \leq i \leq j \text{ and } i + (j - i)(d(d - 3) + 1) = 0 \mod d(d - 2) \} = \{ i : 0 \leq i \leq j \text{ and } d(d - 2) \mid j(d - 1) - di \} = \phi \ \forall j = 2, ..., d - 2.$$  

because if $d(d - 2) \mid j(d - 1) - di$ then $d \mid j$ a contradiction since $0 < j < d$.

$$S_{d}^{X} m, (a, b) := \{ i : 2 \leq i \leq d - 2 \text{ and } i + (d - i)(d(d - 3) + 1) = 0 \mod d(d - 2) \} \subseteq \{ i : 2 \leq i \leq d - 2 \text{ and } d - 2 \mid d - 1 - i \} = \phi.$$
Thus, C is isomorphic to $X^d + Y^{d-1}Z + \alpha YZ^{d-1}$ such that $\alpha \neq 0$.

4.4. Plane curves with an automorphism of order $d^2 - 3d + 3$.

**Proposition 43.** The full automorphism group of C: $X^{d-1}Y + Y^{d-1}Z + \alpha Z^{d-1}X$ of degree $d \geq 5$ such that $\alpha \neq 0$ the semidirect product of $\mathbb{Z}/3$ by $\mathbb{Z}/d^2 - 3d + 3$ given by

$$\langle \tau, \sigma \mid \tau^{d^2-3d+3} = \sigma^3 = 1 \text{ and } \tau\sigma = \sigma\tau^{-(d-1)} \rangle.$$  

In particular, $|\text{Aut}(C)| = 3(d^2 - 3d + 3)$.

**Proof.** Indeed, C can be transformed to the Klein curve $K_d$ through the transformation $[X; \mu Y; \mu^{-(d-2)}Z]$ where $\mu \in K$ is defined by the equation $\alpha = \mu^{d^2-3d+3}$. Now, it follows by Harui [11] §3 that $\text{Aut}(C)$ is a semidirect product of $\mathbb{Z}/3$ acting on $\mathbb{Z}/d^2 - 3d + 3$. To prove the last part, it suffices to notice that $\tau := [X; \zeta_{d^2-3d+3}^j Y; \zeta_{d^2-3d+3}^{-(d-2)} Z]$ and $[Z; X; Y]$ are in $\text{Aut}(K_d)$.

**Remark 44.** For $d = 4$ recall that the Klein curve $K_4$ the order of the automorphism group is 168, isomorphic to $\text{PSL}_2(\mathbb{F}_7)$ [12].

**Proposition 45.** Any non-singular projective plane curve C of degree $d \geq 4$ which has an automorphism of order $d^2 - 3d + 3$ is isomorphic to the Klein curve $K_d$: $X^{d-1}Y + Y^{d-1}Z + Z^{d-1}X$. In particular, for $d \geq 5$, $\text{Aut}(C)$ is isomorphic to

$$\langle \tau, \sigma \mid \tau^{d^2-3d+3} = \sigma^3 = 1 \text{ and } \tau\sigma = \sigma\tau^{-(d-1)} \rangle.$$  

**Proof.** Since $d^2 - 3d + 3 \nmid d - 1$, d, $d(d - 1)$, $d(d-2)$, $(d-1)^2$ for every $d \geq 5$ then, C is projectively equivalent to a plane curve of type $d^2 - 3d + 3$, $(a, b)$ of the form

$$X^{d-1}Y + Y^{d-1}Z + \alpha Z^{d-1}X +$$

$$+ \sum_{j=2}^{\left\lfloor \frac{d}{2} \right\rfloor} X^{d-j} \left( \sum_{i \in S(1)^{m,(a,b)}} \beta_{ji} Y^i Z^{j-i} \right) + Y^{d-j} \left( \sum_{i \in S(1)^{m,(a,b)}} \alpha_{ji} Y^i X^{j-i} \right) + Z^{d-j} \left( \sum_{i \in S(1)^{m,(a,b)}} \gamma_{ji} X^i Y^{j-i} \right),$$

for some $(a, b) \in \Gamma_{d^2-3d+3}$ such that $a = (d - 1)a + b = (d - 1)b (mod d^2 - 3d + 3)$. We can take a generator $a = 1$ and $b = d^2 - 4d + 5$ because every solution is of the form $(k, (d^2 - 3d + 3)k' - (d - 2)k)$ and we have

$$[X; \zeta_{d^2-3d+3}^j Y; \zeta_{d^2-3d+3}^{d^2-4d+5}\zeta^j] = [X; \zeta_{d^2-3d+3}^j Y; \zeta_{d^2-3d+3}^{d^2-4d+5+j}] = [X; \zeta_{d^2-3d+3}^j Y; \zeta_{d^2-3d+3}^{(d^2-3d+3)k' - (d-2)k}].$$

Consequently, $S(1)^{m,(a,b)} := \{ i : 0 \leq i \leq j \text{ and } (d^2 - 4d + 5) \equiv 1 (mod d^2 - 3d + 3) \}$

$$= \{ i : 0 \leq i \leq j \text{ and } (d^2 - 3d + 3) \equiv 1 (mod d^2 - 3d + 3) \}$$

$$= \phi \forall j = 2, \ldots, \left\lfloor \frac{d}{2} \right\rfloor.$$  

because $-(d^2 - 3d + 3) < (\left\lfloor \frac{d}{2} \right\rfloor - 1) \leq -j + 1 \leq (d - 2) - i(d - 1) + 1 \leq j(d - 2) + 1 \leq \frac{d(d-2)}{2} + 1 < d^2 - 3d + 3 \forall j = 2, \ldots, \left\lfloor \frac{d}{2} \right\rfloor$. Consequently, the only possibility is that $j(d - 2) - i(d - 1) + 1 = 0$ which in turns implies that $d|2j - i - 1$ a contradiction since $0 < 2j - i - 1 < d$.

$$S_{m,(a,b)}^{1,Y} := \{ i : 0 \leq i \leq j \text{ and } (d^2 - 4d + 5)i + (d - j) \equiv 1 (mod (d^2 - 3d + 3)) \}$$

$$= \{ i : 0 \leq i \leq j \text{ and } (d^2 - 3d + 3) \equiv 1 (mod (d^2 - 3d + 3)) \}$$

$$= \phi.$$  

because $-(d^2 - 3d + 3) < \frac{d^2 - 3d + 2}{2} \leq (d - j) -(d - 2)i - 1 \leq (d - 2) - 1 - d = d^2 - 3d + 3 \forall j = 2, \ldots, \left\lfloor \frac{d}{2} \right\rfloor$. Therefore, the only possibility is that $(d - j) - (d - 2)i - 1 = 0$ in particular $d - 2$ divides $j - 1$ a contradiction since $0 < j - 1 < d - 2$.

$$S_{m,(a,b)}^{1,Z} := \{ i : 0 \leq i \leq j \text{ and } i + (d - j)(d^2 - 4d + 5) \equiv 1 (mod (d^2 - 3d + 3)) \}$$

$$= \{ i : 0 \leq i \leq j \text{ and } (d^2 - 3d + 3) \equiv 1 (mod (d^2 - 3d + 3)) \}$$

$$= \phi.$$
because $0 < \frac{(d-3)}{2} + 1 \leq (d - j)(d - 2) - i + 1 \leq (d - 2)^2 + 1 < d^2 - 3d + 3$

Consequently, $C$ is isomorphic to $X^{d-1}Y + Y^{d-1}Z + \alpha Z^{d-1}X$ such that $\alpha \neq 0$. The second part of the result follows by Proposition 13. 

\[\square\]

4.5. Plane curves with automorphisms of orders $\ell d$ and $\ell(d - 1)$.

We are interested in non-singular plane curves $C$ of arbitrary but fixed degree $d \geq 5$ whose automorphism groups contain homologies $\sigma$ of period $d$ (resp. $d - 1$), recall that an homology is an automorphism that under a change of variables is of type $m, (a, b)$ with $ab = 0$. In these situations, the genus of $C/\langle \sigma \rangle$ is zero and $C$ has an unique outer (resp. inner) Galois point $P$ if $d \geq 5$ (see [11] Lemma 3.7 and [20] for the uniqueness of the outer (resp. inner) Galois point when $d \geq 5$), and we refer to [20] for the definition of inner or outer Galois point. Moreover, for a given non-singular plane curve $C$ of degree $d \geq 5$, if exists $P$ an outer (resp. inner) Galois point, because it is unique, one deduces that $P$ is fixed by the full group $Aut(C)$, otherwise $\tau(P)$ will give another outer point for $\tau \in Aut(C)$ (similarly for the inner Galois point).

4.5.1. Plane curves with automorphisms of orders $\ell(d - 1)$.

**Lemma 46.** If a non-singular plane curve $C$ of degree $d \geq 5$ has an automorphism of order $\ell(d - 1)$ $(2 \leq \ell \leq d)$ then $\ell$ should be a divisor of $d$ or $d - 1$. In particular, $d \equiv 0 \pmod{\ell}$ or $d \equiv 1 \pmod{\ell}$.

**Proof.** Clear, because $\ell(d - 1)$ does not divide $d - 1$, $d$, $d^2 - 3d + 3$, $d(d - 2)$.

**Proposition 47.** Let $d \geq 5$ where $d \equiv 0 \pmod{\ell}$ and consider the non-singular plane curve $C: X^d + Y^d + \alpha XZ^{d-1} + \sum_{2 \leq j = k \leq d-2} \beta_{jk} X^{d-j}Y^j$

Then, $Aut(C)$ is cyclic of order divisible by $\ell(d - 1)$.

**Proof.** Since $\sigma := [X; \zeta^d_{d-1}(1)Y; \zeta^d_{d-1}(1)Z] \in Aut(C)$ of order $\ell(d - 1)$ then, $C_3$ is not a descendant of the Klein curve $K_d$ (because $\ell(d - 1) \nmid 3(d^2 - 3d + 3)$) and is not a descendant of the Fermat curve $F_d$ (because $\ell(d - 1) > 2d$ for all $\ell \geq 3$ and for $\ell = 2$ we have $2(d^2 - 1) \nmid 6d^3$). Moreover, $Aut(C_3)$ contains a homology of period $d - 1 \geq 4$ namely, $[X; Y; \zeta^d_{d-1}(1)Z] = [X; \zeta_{d-1}(1)Y; \zeta_{d-1}(1)Z]$, therefore, it must fix a line and a point off that line. Now, since $[X; \zeta^d_{d-1}(1)Y; Z] \in Aut(C_3)$ is a homology of period $d - 1$ with center $(0; 0; 1)$ then by Harui [11] §3, the point $(0; 0; 1)$ is an inner Galois point of $C_3$. Moreover, it is the unique such point by Yoshihara [20] §2 [Theorem 4] hence should be fixed by $Aut(C_3)$. Consequently, $Aut(C_3)$ is cyclic of order divisible by $\ell(d - 1)$.

**Proposition 48.** Let $C$ be a non-singular plane curve of degree $d \geq 5$ where $d \equiv 0 \pmod{\ell}$. Assume that $Aut(C)$ contains an element of order $\ell(d - 1)$ with $\ell \geq 2$. Then $C$ is isomorphic to $X^d + Y^d + \alpha XZ^{d-1} + \sum_{2 \leq j = k \leq d-2} \beta_{jk} X^{d-j}Y^j$

In particular, $Aut(C)$ is cyclic.

**Proof.** Clearly, $\ell(d - 1)$ is not a divisor of $d - 1$, $d$, $d^2 - 3d + 3$, $(d - 1)^2$ or $d(d - 2)$. Therefore, $C$ is projectively equivalent to type $\ell(d - 1), ((d - 1)k, \ell k')$ of Theorem 30 (5) (where $k', (d - 1)k$ are coprime $< \ell(d - 1)$) of the form

\[
X^d + Y^d + \sum_{j=2}^{d-2} \left( X^{d-j} \sum_{i \in S(2)_{m,(a,b)}} \beta_{ji} Y^i Z^{j-i} \right) + \sum_{i \in S^d_{m,(a,b)}} \beta_{di} Y^i Z^{d-i} +
+ X(\alpha Z^{d-1} + \sum_{i \in S^d_{m,(a,b)}} \beta_{(d-1)i} Y^i Z^{d-1-i}) = 0
\]
Claim 1: $\ell(d - 1), (d - 1, \ell)$ is a generator of any type $\ell(d - 1), ((d - 1)k, \ell k')$.

Let $k \equiv m \pmod{\ell}$, then $[X; \zeta^{d-1}_{\ell(d-1)} Y; \zeta^{\ell}_{\ell(d-1)} Z]^{(k-m)(d-1)+\ell'} = [X; \zeta^{m(d-1)}_{\ell(d-1)} Y; \zeta^{\ell k'}_{\ell(d-1)} Z] = [X; \zeta^{k(d-1)}_{\ell(d-1)} Y; \zeta^{\ell k'}_{\ell(d-1)} Z].$

Now, we have

$$S_{d,X}^1 m, (a, b) := \{ i : 1 \leq i \leq d - 1 \text{ and } (d - 1)i + (d - i)\ell = 0 \pmod{\ell(d - 1)} \}$$

$$\subseteq \{ i : 1 \leq i \leq d - 1 \text{ and } (d - 1) | (i - 1) \}$$

$$= \{ 1 \}$$

But, since $(d - 1)i + (d - i)\ell = (d - 1) + (d - \ell) \neq 0 \pmod{\ell(d - 1)}$ then $S_{d,X}^1 m, (a, b) = \phi$.

$$S_{d-1,X}^1 m, (a, b) := \{ i : 1 \leq i \leq d - 1 \text{ and } (d - 1)i + (d - 1 - i)\ell = 0 \pmod{\ell(d - 1)} \}$$

$$\subseteq \{ i : 1 \leq i \leq d - 1 \text{ and } (d - 1) | i \}$$

$$= \{ d - 1 \}$$

But since $(d - 1)i + (d - 1 - i)\ell = (d - 1)2 \neq 0 \pmod{\ell(d - 1)}$ then $S_{d-1,X}^1 m, (a, b) = \phi$.

$$S(2)_{m, (a, b)} := \{ i : 0 \leq i \leq j \text{ and } (d - 1)i + (j - i)\ell = 0 \pmod{\ell(d - 1)} \}$$

$$\subseteq \{ i : 0 \leq i \leq j \text{ and } (d - 1) | (j - i) \}$$

Since $0 \leq j - i \leq d - 2 < d - 1$ then $j = i$ which in turns implies that $i = j = \ell k$ thus $S(2)_{m, (a, b)} = \phi$ if $\ell \nmid j$ and $\{ j \}$ otherwise.

A similar argument as above is done when $d \equiv 1 \pmod{\ell}$:

**Proposition 49.** Suppose that $d \geq 5$ where $d \equiv 1 \pmod{\ell}$ with $\ell \geq 2$ and let $C$ be the non-singular plane curve defined by

$$X^d + Y^{d-1} Z + \alpha XZ^{d-1} + \sum_{2 \leq j = \ell k \leq d-2} \beta_0 X^{d-j} Z^j$$

Then, $Aut(C)$ is cyclic of order divisible by $\ell(d - 1)$.

**Proof.** Since $\sigma := [X; \zeta^{(d-1)}_{\ell(d-1)} Y; \zeta^{(d-1)}_{\ell(d-1)} Z] \in Aut(C_3)$ of order $\ell(d - 1)$ then, $C_3$ is not a descendant of the Klein curve $K_d$ (because $\ell(d - 1) \nmid (3(d^2 - 3d + 3)$) and is not a descendant of the Fermat curve $F_d$ (because $\ell(d - 1) > 2d$ for all $\ell \geq 3$ and for $\ell = 2$ we have $(2d - 2) \nmid 6d^2$). Moreover, $Aut(C_3)$ contains a homology of period $d - 1 \geq 4$ namely, $[X; \zeta^{\ell}_{\ell(d-1)} Y; Z] = [X; \zeta^{\ell}_{\ell(d-1)} Y; \zeta^{(d-1)}_{\ell(d-1)} Z]^{\ell}$ therefore, it must fix a line and a point off that line. Now, since $[X; \zeta^{\ell}_{\ell(d-1)} Y; Z] \in Aut(C_3)$ is a homology of period $d - 1$ with center $(0; 1; 0)$ then by Harui [11, §3, the point $(0; 0; 1)$ is an inner Galois point of $C_3$. Moreover, it is the unique such point by Yoshihara [20] §2 [Theorem 4] hence should be fixed by $Aut(C_3)$. In particular, $Aut(C_3)$ is cyclic. 

And with a similar proof as Proposition 48 we obtain,

**Proposition 50.** Let $C$ be a non-singular plane curve of degree $d \geq 5$ where $d \equiv 1 \pmod{\ell}$ with $2 \leq \ell \leq d - 1$. Assume that $Aut(C)$ contains an element of order $\ell(d - 1)$. Then $C$ is isomorphic to

$$X^d + Y^{d-1} Z + \alpha XZ^{d-1} + \sum_{2 \leq \ell k \leq d-2} \beta_0 X^{d-j} Z^j$$

Consequently, $Aut(C)$ is cyclic.

In particular

**Corollary 51.** Let $C$ be a non-singular plane curve of degree $d \geq 5$ such that $Aut(C)$ contains an element of order $\ell(d - 1)$ with $2 \leq \ell \leq d - 1$. Then, $Aut(C)$ is cyclic and contains a homology of period $d - 1$. In particular, $C$ has an unique inner Galois point.
Remark 52. If $C$ is a non-singular plane curve of degree $d \geq 5$ such that $\text{Aut}(C)$ contains an element $\sigma$ of order $d - 1$ then if $\sigma$ is an homology, then by [11, Lemma 3.7] $C$ has an inert Galois point $P \in C$, and because it is fixed for $\text{Aut}(C)$ because it is unique the inert Galois point by [20, Theorem 4], then $P \in C$ is fixed by the full $\text{Aut}(C)$, and therefore $\text{Aut}(C)$ is a cyclic group by [13, Lemma 11.44]. In the case that $\sigma$ is not an homology (say is of type $m, (a, b)$ with $ab \neq 0$) then $\text{Aut}(C)$ is not necessarily a cyclic group and will depend of the parameters of the defining equation for the given type $m, (a, b)$.

In this subsection we proved if $C$ non-singular plane curve of degree $d \geq 5$ such that $\text{Aut}(C)$ contains an element of order $(d - 1)$ with $\ell \geq 2$, then $\text{Aut}(C)$ has always an homology of order $d - 1$.

4.5.2. Plane curves with automorphism of orders $\ell d$.

Lemma 53. If a non-singular plane curve $C$ of degree $d \geq 5$ has an automorphism of order $\ell d$ ($2 \leq \ell \leq d - 1$) then $\ell$ should be a divisor of $d - 1$ or $d - 2$. In particular, $d \equiv 1 \pmod{\ell}$ or $d \equiv 2 \pmod{\ell}$.

Proof. Obvious, since $\ell d$ does not divide $d - 1$, $d$, $d^2 - 3d + 3$, $(d - 1)^2$.

Proposition 54. Suppose that $d \geq 5$ such that $d \equiv 1 \pmod{\ell}$ for some $3 \leq \ell \leq d - 1$. Let $\tilde{C}$ be the non-singular plane curve of degree $d$ defined by

$$\tilde{C} : X^d + Y^d + \alpha XZ^{d-1} + \sum_{2 \leq j = m \leq d-2} \beta_{jm} X^{d-1} Z^j$$

where $\alpha \neq 0$. Then, $\text{Aut}(\tilde{C})$ should fix a line and a point off that line.

Proof. Since $[X; c_{\ell d} Y; c_{\ell d} Z] \in \text{Aut}(C)$ of order $\ell d$ then, $\text{Aut}(\tilde{C})$ contains a homology of period $> 4$ namely, $[X; c_{\ell d} Y; Z] = [X; c_{\ell d} Y; c_{\ell d} Z] \ell d$ therefore, it must fix a line and a point off that line or it fixes a triangle. If it fixes a triangle and neither a point nor line is leaved invariant, then $\tilde{C}$ must be a descendant of the Klein curve $K_d$ or the Fermat curve $F_d$. But, since $|\text{Aut}(K_d)| = 3(d^2 - 3d + 3)$ is not divisible by $\ell d$ for all $\ell = 2, 3, ..., d - 3$ then $\tilde{C}$ is not a descendant of $K_d$. On the other hand, $\text{Aut}(F_d)$ has elements of order at most $2d$ then $\tilde{C}$ can not be a descendant of the Fermat curve for all $\ell \geq 3$. Therefore, $\text{Aut}(C_1)$ must fix a line and a point off that line.

Remark 55. The above proposition when $\ell = 2$ with a similar proof concludes that then $\text{Aut}(\tilde{C})$ fix a line and a point off that line or $(\tilde{C}, \text{Aut}(\tilde{C}))$ is a descend of Fermat curve.

Lemma 56. Automorphisms of $\tilde{C}$ are of the form $[\alpha_1 X + \alpha_3 Z; Y; \gamma_1 X + \gamma_3 Z]$.

Proof. Since $[X; c_{\ell d}^2 Y; Z] \in \text{Aut}(\tilde{C})$ is a homology of period $d$ and center $(0; 1; 0)$ then by Harui [11 §3, the point $(0; 1; 0)$ is an outer Galois point of $\tilde{C}$. Moreover, $\tilde{C}$ is not isomorphic to the Fermat curve $F_d$ therefore by Yoshihara [20 §2 [Theorem 4'], this point is the unique outer Galois point hence should be fixed by $\text{Aut}(\tilde{C})$. In particular, automorphisms of $\tilde{C}$ are of the form $[\alpha_1 X + \alpha_3 Z; \beta_1 X + \beta_2 Y + \beta_3 Z; \gamma_1 X + \gamma_3 Z]$ which in turns implies that $\beta_1 = 0 = \beta_3$ (Coefficients of $XY^{d-1}$ and $Y^{d-1} Z$).

Proposition 57. Let $C$ be a non-singular plane curve of degree $d \geq 5$ where $d \equiv 1 \pmod{\ell}$. Assume that $\text{Aut}(C)$ contains a cyclic group of order $\ell d$. Then $C$ is isomorphic to

$$\tilde{C} : X^d + Y^d + \alpha XZ^{d-1} + \sum_{2 \leq j = m \leq d-2} \beta_{jm} X^{d-1} Z^j$$

In particular, $\text{Aut}(C)$ consists of elements of the form $[\alpha_1 X + \alpha_3 Z; Y; \gamma_1 X + \gamma_2 Z]$.

Remark 58. Unfortunately as in the previous subsections there are different groups that may appear in $\text{Aut}(C)$ depending of the parameters $\beta_{jm}$.
Proof. Clearly, $\ell d$ is not a divisor of $d - 1$, $d$, $d^2 - 3d + 3$, $(d - 1)^2$ or $d(d - 2)$. Therefore, $C$ is projectively equivalent to type $\ell d, (\ell n, m d)$ of Theorem [31] (where $\ell n m d$ are coprime $< \ell d$) of the form

$$X^d + Y^d + \sum_{j=2}^{d-2} (X^d - j) \sum_{i \in S(2)_{m,a,b}} \beta_{ji} Y^i Z^{d-i} + \sum_{i \in S_1^{d-1-X} m,a,b} \beta_{di} Y^i Z^{d-i} + X (\alpha Z^{d-1} + \sum_{i \in S_1^{d-1-X} m,a,b} \beta_{di} Y^i Z^{d-1-i}) = 0$$

Claim 1: $\ell d, (\ell, d)$ is a generator of any type $\ell d, (\ell, n, m d)$. If $n \equiv k' \pmod{\ell}$, then $[X; \zeta_{\ell d} Y; \zeta_{\ell d} Z]^{(m-k')d+n} = [X; \zeta_{\ell d} Y; \zeta_{\ell d} Z]$.

Claim 2: $S_1^{d,X} \ell d, (\ell, d) = \phi = S_1^{d-1,X} \ell d, (\ell, d)$

Indeed, we have

$$S_1^{d,X} \ell d, (\ell, d) := \{ i : 1 \leq i \leq d - 1 \text{ and } \ell i \equiv (d-i)d \equiv 0 \pmod{\ell d} \}$$

$$\subseteq \{ i : 1 \leq i \leq d - 1 \text{ and } \ell d \mid i(d - \ell) - d \}$$

$$= \phi$$

The last equality because if $i(d - \ell) - d = \ell dk$ then $d \mid i$ a contradiction.

$$S_1^{d-1,X} \ell d, (\ell, d) := \{ i : 1 \leq i \leq d - 1 \text{ and } \ell i + (d - 1 - i)d \equiv 0 \pmod{\ell d} \}$$

$$\subseteq \{ i : 1 \leq i \leq d - 1 \text{ and } \ell d \mid i(d - \ell) \}$$

$$= \phi$$

The last equality because if $i(d - \ell) = \ell dk$ then $d \mid i$ but because $(d, \ell) = 1$ thus $d \mid i$ a contradiction.

Finally, $i \in S(2)^{j,X}_{\ell d,(\ell, d)}$ then $\ell i - (j - i)d = \ell dk$ from which we get $d \mid i$ thus $i = 0$ then $j = \ell k$ hence $j$ is divisible by $\ell$.

There are also similar statements when $d = \ell k + 2 \geq 5$ of the previous results with similar techniques, we state only the result.

Proposition 59. Let $C$ be a non-singular plane curve of degree $d = \ell k + 2 \geq 5$ such that $Aut(C)$ contains a cyclic group of order $\ell d$ with $3 \leq \ell \leq d - 2$. Then $C$ is isomorphic to

$$X^d + Y^{d-1}Z + \alpha Y Z^{d-1} + \sum_{2 \leq i \leq \ell m + 1 \leq d-2} \beta_{di} Y^i Z^{d-i}$$

and $Aut(C)$ has only elements of the form $[X; \beta_2 Y + \beta_3 Z; \gamma_2 Y + \gamma_3 Z]$.

Remark 60. The above proposition with $\ell = 2$ we deduce that $Aut(C)$ fixes a point and a line off (as the previous case) or $(C, Aut(C))$ is a descendent of the Fermat curve.

In this situation where $Aut(C)$ has an element of order $\ell d$ and with [11] Lemma 3.7 we obtain,

Corollary 61. Let $C$ be a non-singular plane curve of degree $d \geq 5$. Then, there exists $G \simeq Aut(C)$ cyclic of order $\ell d$ for some $\ell \geq 1$ if and only if $Aut(C)$ contains an homology of order $d$. In particular, $C$ has a unique outer Galois point $P \notin C$ that should be fixed by $Aut(C)$.

5. Tables of types of cyclic groups

In this section, we introduce tables for lower degree of type of cyclic groups and equations that are obtained for result of \S 2. It might happen that two types $m, (a, b)$ and $m, (a', b')$ are isomorphic through a permutation of the variables or $F(X; Y; Z)$ decomposes into a product $X.G(X; Y; Z)$ then after removing such cases we get the following tables:
6. Tables of Type $m(a, b)$ for degree $d \leq 9$

The following tables are obtained by running the Sage programm concerning the first theorem in this note, see the programm in http://mat.uab.cat/~eslam/CAGPC.sagews

Table 1. Quartics

| Type: $m_i(a, b)$ | $F(X; Y; Z)$ |
|-----------------|-------------|
| 12, (3, 4)      | $X^4 + Y^4 + aXZ^3$ |
| 9, (1, 6)       | $X^4 + Y^3Z + aXZ^3$ |
| 8, (1, 5)       | $X^3 + Y^3Z + aYZ^3$ |
| 7, (1, 5)       | $X^3Y + Y^3Z + aZ^3X$ |
| 6, (2, 3)       | $X^4 + Z^4 + aXY^3 + \beta_2X^2Z^2$ |
| 4, (1, 2)       | $X^4 + Y^4 + Z^4 + \beta_2X^2Z^2 + \beta_3XY^2Z$ |
| 4, (0, 1)       | $Z^4 + L_4Z$ |
| 3, (1, 2)       | $X^4 + X(Z^4 + aY^3) + \beta_2X^2YZ + \beta_4Y^2Z^2$ |
| 3, (0, 1)       | $Z^4L_1Z + L_4Z$ |
| 2, (0, 1)       | $Z^4 + Z^2L_2Z + L_4Z$ |

Table 2. Quintics

| Type: $m_i(a, b)$ | $F(X; Y; Z)$ |
|-----------------|-------------|
| 20, (4, 5)      | $X^5 + Y^5 + aXZ^4$ |
| 16, (1, 12)     | $X^5 + Y^4Z + aXZ^4$ |
| 15, (1, 11)     | $X^5 + Y^4Z + aYZ^4$ |
| 13, (1, 10)     | $X^4Y + Y^4Z + aZ^4X$ |
| 10, (2, 5)      | $X^5 + Y^5 + aXZ^4 + \beta_2X^3Z^2$ |
| 8, (1, 4)       | $X^5 + Y^4Z + aXZ^4 + \beta_2X^3Z^2$ |
| 5, (1, 2)       | $X^5 + Y^5 + Z^5 + \beta_3X^2YZ^2 + \beta_4X^2Y^3Z$ |
| 5, (0, 1)       | $Z^5 + L_5Z$ |
| 4, (1, 2)       | $X^5 + X(Z^4 + aY^4) + \beta_2X^3Z^2 + \beta_3X^2Y^2Z + \beta_5Y^2Z^3$ |
| 4, (0, 1)       | $Z^4L_1Z + L_5Z$ |
| 3, (1, 2)       | $X^5 + Y^4Z + aYZ^4 + \beta_2X^3YZ + X^2(\beta_1Z^3 + \beta_3Y^3) + \beta_4XY^2Z^2$ |
| 2, (0, 1)       | $Z^4L_1Z + Z^2L_3Z + L_5Z$ |
### Table 3. Sextics

| Type: $m, (a, b)$ | $F(X; Y; Z)$ |
|-------------------|-------------|
| $30, (5, 6)$      | $X^6 + Y^6 + \alpha XZ^5$ |
| $25, (1, 20)$     | $X^6 + Y^5Z + \alpha XZ^5$ |
| $24, (1, 19)$     | $X^6 + Y^5Z + \alpha YZ^5$ |
| $21, (1, 17)$     | $X^5Y + Y^5Z + \alpha XZ^5$ |
| $15, (5, 3)$      | $X^6 + Y^6 + \alpha XZ^5 + \beta_{3,3}X^3Y^3$ |
| $12, (1, 7)$      | $X^6 + Y^5Z + \alpha YZ^5 + \beta_{6,3}Y^3Z^5$ |
| $10, (5, 2)$      | $X^6 + Y^6 + \alpha XZ^5 + \beta_{2,2}X^2Y^2 + \beta_{4,4}X^2Y^4$ |
| $8, (1, 3)$       | $X^6 + Y^5Z + \alpha YZ^5 + \beta_{4,2}X^2Y^2Z^2$ |
| $6, (1, 2)$       | $X^6 + Y^6 + Z^6 + \beta_{3,0}X^3Z^3 + \beta_{4,2}X^2Y^2Z^2 + \beta_{5,4}XY^4Z$ |
| $6, (1, 3)$       | $X^6 + Y^6 + Z^6 + \beta_{2,0}X^4Z^2 + \beta_{6,3}Y^3Z^3 + X^2(\beta_{4,0}Z^4 + \beta_{4,3}Y^3Z)$ |
| $6, (0, 1)$       | $Z^6 + L_{6, Z}$ |
| $5, (4, 3)$       | $X^6 + XZ^5 + \alpha XY^5 + \beta_{3,1}X^3YZ^2 + \beta_{4,3}X^2Y^3Z + \beta_{6,2}Y^2Z^4$ |
| $5, (4, 1)$       | $X^6 + XZ^5 + \alpha XY^5 + \beta_{2,1}X^4YZ + \beta_{4,2}X^2Y^2Z^2 + \beta_{6,3}Y^3Z^3$ |
| $5, (0, 1)$       | $Z^5L_{1, Z} + L_{6, Z}$ |
| $4, (1, 3)$       | $X^6 + Y^5Z + \alpha YZ^5 + \beta_{6,3}Y^3Z^5 + \beta_{2,1}X^4YZ + X^2(\beta_{4,0}Z^4 + \beta_{4,2}Y^2Z^2 + \beta_{4,4}Y^4)$ |
| $3, (0, 1)$       | $Z^6 + Z^3L_{3, Z} + L_{6, Z}$ |
| $2, (0, 1)$       | $Z^6 + Z^4L_{2, Z} + Z^2L_{4, Z} + L_{6, Z}$ |
### Table 4. degree 7

| Type: \(m,(a,b)\) | \(F(X;Y;Z)\) |
|---------------------|-----------------|
| 42, (6, 7)          | \(X^7 + Y^7 + \alpha XZ^6\) |
| 36, (1, 30)         | \(X^7 + Y^6 Z + \alpha XZ^6\) |
| 35, (1, 29)         | \(X^7 + Y^6 Z + \alpha YZ^6\) |
| 31, (1, 26)         | \(X^6 Y + Y^6 Z + \alpha XZ^6\) |
| 21, (3, 7)          | \(X^7 + Y^7 + \alpha XZ^6 + \beta_{3,0} X^4 Z^3\) |
| 18, (1, 12)         | \(X^7 + Y^6 Z + \alpha XZ^6 + \beta_{3,0} X^4 Z^3\) |
| 14, (2, 7)          | \(X^7 + Y^7 + \alpha XZ^6 + \beta_{2,0} X^5 Z^2 + \beta_{4,0} X^3 Z^4\) |
| 12, (1, 6)          | \(X^7 + Y^7 + \alpha XZ^6 + \beta_{2,0} X^5 Z^2 + \beta_{4,0} X^3 Z^4\) |
| 9, (1, 3)           | \(X^7 + Y^6 Z + \alpha XZ^6 + \beta_{3,0} X^4 Z^3 + \beta_{5,3} X^2 Y^3 Z^2\) |
| 7, (1, 2)           | \(X^7 + Y^7 + Z^7 + \beta_{3,1} X^4 Y Z^2 + \beta_{5,4} X^2 Y^4 Z + \beta_{6,2} X Y^2 Z^4\) |
| 7, (0, 1)           | \(Z^7 + L_{7,z}\) |
| 6, (5, 4)           | \(X^7 + X Z^6 + \alpha Y^6 + \beta_{3,0} X^4 Z^3 + \beta_{4,2} X^2 Y^3 Z^2 + \beta_{5,4} X^2 Y^4 Z + \beta_{7,2} Y^2 Z^5\) |
| 6, (4, 3)           | \(X^7 + X Z^6 + \alpha Y^6 + \beta_{2,0} X^5 Z^2 + \beta_{3,1} X Y^3 + \beta_{4,0} X^3 Z^4 + X^2 \beta_{5,3} Y^3 Z^2 + \beta_{7,2} Y^3 Z^4\) |
| 6, (0, 1)           | \(Z^6 L_{1,z} + L_{7,z}\) |
| 5, (1, 4)           | \(X^7 + Y^6 Z + \alpha Y Z^6 + \beta_{2,1} X^5 Y Z + \beta_{4,1} X^3 Y^2 Z^2 + \beta_{5,3} X^2 Y^3 Z^3 + X^2 (\beta_{5,0} Z^5 + \beta_{5,2} Y^5)\) |
| 4, (1, 2)           | \(X^7 + Y^6 Z + \alpha XZ^6 + \beta_{2,0} X^5 Z^2 + \beta_{3,1} X^4 Y^2 Z + \beta_{5,3} X^2 Y^2 Z^3 + \beta_{6,4} X Y^4 Z^2 + \beta_{7,2} Y^2 Z^5 + X^2 (\beta_{4,0} Z^4 + \beta_{4,1} Y^4)\) |
| 3, (1, 2)           | \(X^7 + X Z^6 + \alpha X Y^6 + \beta_{2,1} X^4 Y Z + \beta_{4,1} X^3 Y^2 Z^2 + \beta_{5,3} X Y^3 Z^3 + \beta_{7,2} Y^2 Z^5 + \beta_{7,3} Y Z^2 + X^4 \beta_{3,0} (Z^3 + \beta_{3,1} Y^3) + X^2 (\beta_{3,1} Y^4 Z + \beta_{5,4} Y^4)\) |
| 3, (0, 1)           | \(Z^6 L_{1,z} + Z^4 L_{4,z} + L_{7,z}\) |
| 2, (0, 1)           | \(Z^6 L_{1,z} + Z^4 L_{3,z} + Z^2 L_{5,z} + L_{7,z}\) |
| Type: $m, (a, b)$ | $F(X; Y; Z)$ |
|-----------------|----------------|
| 56, (7, 8)      | $X^8 + Y^8 + \alpha XZ^7$ |
| 49, (1, 42)     | $X^8 + Y^7 Z + \alpha XZ^7$ |
| 48, (1, 41)     | $X^8 + Y^7 Z + \alpha YZ^7$ |
| 43, (1, 37)     | $X^7 Y + Y^7 Z + \alpha XZ^7$ |
| 28, (7, 4)      | $X^8 + Y^8 + \alpha XZ^7 + \beta_{4,4} X^4 Y^4$ |
| 24, (1, 17)     | $X^8 + Y^7 Z + \alpha YZ^7 + \beta_{8,4} Y^4 Z^7$ |
| 16, (1, 9)      | $X^8 + Y^7 Z + \alpha YZ^7 + \beta_{8,5} Y^5 Z^7 + \beta_{8,6} Y^6 Z^7$ |
| 14, (7, 2)      | $X^8 + Y^8 + \alpha XZ^7 + \beta_{2,2} X^6 Y^2 + \beta_{4,4} X^4 Y^4 + \beta_{6,6} X^2 Y^6$ |
| 12, (1, 5)      | $X^8 + Y^7 Z + \alpha YZ^7 + \beta_{8,4} Y^4 Z^7 + \beta_{4,2} X^4 Y^4 Z^2$ |
| 8, (1, 2)       | $X^8 + Y^8 + Z^8 + \beta_{4,0} X^4 Z^4 + \beta_{5,2} X^4 Y^2 Z^3 + \beta_{6,4} X^2 Y^4 Z^2 + \beta_{7,6} XY^6 Z$ |
| 8, (1, 3)       | $X^8 + Y^8 + Z^8 + \beta_{1,2} X^4 Y^4 Z^2 + \beta_{8,4} Y^4 Z^6 + X^2 (\beta_{0,1} Y^5 Z^5 + \beta_{0,2} Y^5 Z)$ |
| 8, (1, 4)       | $X^8 + Y^8 + Z^8 + \beta_{2,0} X^6 Z^2 + \beta_{4,0} X^4 Z^4 + \beta_{5,4} X^3 Y^4 Z + \beta_{6,0} X^2 Y^6 + \beta_{7,4} X Y^4 Z^3$ |
| 7, (6, 5)       | $Z^8 + L_{8,x}$ |
| 7, (6, 1)       | $X^8 + XZ^7 + \alpha XY^7 + \beta_{4,1} X^4 Y^3 Z^3 + \beta_{5,3} X^3 Y^3 Z^2 + \beta_{6,5} X^2 Y^5 Z + \beta_{8,2} Y^2 Z^6$ |
| 7, (0, 1)       | $X^8 + XZ^7 + \alpha XY^7 + \beta_{5,1} X^5 Y^2 Z^2 + \beta_{5,3} X^3 Y^4 Z + \beta_{6,2} X^2 Y^3 Z^4 + \beta_{8,5} Y^5 Z^3 +$ |
| 6, (1, 5)       | $Z^7 L_{1,2} + L_{8,x}$ |
| 4, (0, 1)       | $Z^8 + Z^4 L_{4,2} + L_{8,x}$ |
| 3, (1, 2)       | $X^8 + Y^7 Z + \alpha YZ^7 + \beta_{8,4} Y^4 Z^7 + \beta_{2,1} X^6 YZ + \beta_{4,2} X^4 Y^2 Z^2 + \beta_{8,4} Y^4 Z^7 +$ |
|                | $X^2 (\beta_{6,0} Z^6 + \beta_{6,3} Y^3 Z^3 + \beta_{6,6} Y^6) + X (\beta_{7,2} Y^2 Z^5 + \beta_{7,5} Y^5 Z^2)$ |
| 2, (0, 1)       | $Z^8 + Z^4 L_{1,2} + Z^4 L_{4,2} + Z^4 L_{6,2} + L_{8,2}$ |
| Type: $m(a, b)$ | $F(X; Y; Z)$ |
|---------------|----------------|
| 72, (8, 9)    | $X^9 + Y^9 + \alpha XZ^8$ |
| 64, (1, 56)   | $X^9 + Y^8 Z + \alpha XZ^8$ |
| 63, (1, 55)   | $X^9 + Y^8 Z + \alpha YZ^8$ |
| 57, (1, 50)   | $X^9 Y + Y^8 Z + \alpha XZ^8$ |
| 36, (4, 9)    | $X^9 + Y^9 + \alpha XZ^8 + \beta_{4,0} X^3 Z^4$ |
| 32, (1, 24)   | $X^9 + Y^8 Z + \alpha XZ^8 + \beta_{4,0} X^5 Z^4$ |
| 24, (8, 3)    | $X^9 + Y^9 + \alpha XZ^8 + \beta_{3,3} X^6 Y^3 + \beta_{6,4} X^3 Y^6$ |
| 21, (1, 13)   | $X^9 + Y^8 Z + \alpha YZ^8 + \beta_{3,3} X^3 Y^3 Z^3$ |
| 18, (2, 9)    | $X^9 + Y^9 + \alpha XZ^8 + \beta_{2,0} X^7 Z^2 + \beta_{4,0} X^3 Z^4 + \beta_{6,0} X^3 Z^6$ |
| 16, (1, 8)    | $X^9 + Y^8 Z + \alpha XZ^8 + \beta_{2,0} X^7 Z^2 + \beta_{4,0} X^5 Z^4 + \beta_{6,0} X^3 Z^6$ |
| 12, (4, 3)    | $X^9 + Y^9 + \alpha XZ^8 + \beta_{3,3} X^6 Y^3 + \beta_{4,0} X^5 Z^4 + \beta_{6,0} X^3 Y^6 + \beta_{7,3} X^2 Y^3 Z^4$ |
| 9, (1, 2)     | $X^9 + Y^9 + Z^9 + \beta_{5,1} X^4 Y^3 Z^2 + \beta_{6,3} X^3 Y^3 Z^3 + \beta_{7,5} X^2 Y^5 Z^2 + \beta_{8,7} X Y^7 Z$ |
| 9, (1, 3)     | $X^9 + Y^9 + Z^9 + \beta_{3,3} X^4 Y^3 Z^3 + \beta_{6,0} X^3 Z^3 + \beta_{7,3} X^2 Y^3 Z^5 + \beta_{5,0} X^3 Y^6 + \beta_{6,0} X^3 Z^6$ |
| 9, (0, 1)     | $Z^9 + L_{9,9}$ |
| 8, (7, 6)     | $X^9 + XZ^8 + \alpha YZ^8 + \beta_{4,0} X^5 Z^4 + \beta_{4,0} X^3 Y^2 Z^3 + \beta_{4,0} X^3 Y^4 Z^2 + \beta_{7,3} X^2 Y^6 Z + \beta_{9,2} Y^2 Z^7$ |
| 8, (7, 4)     | $X^9 + XZ^8 + \alpha XZ^8 + \beta_{2,0} X^7 Z^2 + \beta_{4,0} X^5 Z^4 + \beta_{5,4} X^4 Y^4 Z + \beta_{5,0} X^3 Z^3 + \beta_{7,3} X^2 Y^4 Z^3 + \beta_{6,0} Y^2 Z^5$ |
| 8, (7, 2)     | $X^9 + XZ^8 + \alpha XZ^8 + \beta_{3,3} X^6 Y^2 Z + \beta_{4,0} X^5 Z^4 + \beta_{6,4} X^3 Y^4 Z^2 + \beta_{7,3} X^2 Y^3 Z^5 + \beta_{5,0} X^3 Y^6 + \beta_{9,0} Y^6 Z^3$ |
| 8, (0, 1)     | $Z^9 L_{1,9} + L_{9,9}$ |
| 7, (1, 6)     | $X^9 + Y^8 Z + \alpha YZ^8 + \beta_{2,1} X^7 Y Z + \beta_{4,2} X^5 Y^2 Z^2 + \beta_{6,3} X^3 Y^3 Z^3 + \beta_{8,4} X Y^4 Z^3 + \beta_{9,0} Z^7 + \beta_{9,7} Y^7$ |
| 6, (2, 3)     | $X^9 + Y^9 + \alpha XZ^8 + \beta_{2,0} X^7 Z^2 + \beta_{3,3} X^6 Y^3 + \beta_{4,0} X^5 Z^4 + \beta_{5,3} X^3 Y^3 Z^2 + \beta_{7,3} X^2 Y^3 Z^3 + \beta_{8,0} Z^6 + \beta_{8,0} Y^6$ |
| 4, (3, 2)     | $X^9 + XZ^8 + \alpha XZ^8 + \beta_{2,0} X^7 Z^2 + \beta_{4,0} X^5 Y^2 Z^3 + \beta_{5,2} X^4 Y^6 Z^3 + \beta_{5,4} X^4 Y^3 Z^2 + \beta_{6,2} X^2 Y^3 Z^6 + \beta_{9,0} Y^4 Z^3 + \beta_{9,0} Y^2 Z^5 + \beta_{9,0} Y^6 Z^3$ |
| 4, (0, 1)     | $Z^9 L_{1,9} + Z^{14} L_{5,9} + L_{9,9}$ |
| 3, (0, 1)     | $Z^9 + Z^6 L_{3,9} + Z^3 L_{6,9} + L_{9,9}$ |
| 2, (0, 1)     | $Z^6 L_{1,9} + Z^6 L_{3,9} + Z^4 L_{5,9} + Z^2 L_{7,9} + L_{9,9}$ |
CHAPTER 3

Automorphism group for non-singular plane curves with degree 5

1. Abstract

Henn in [12] obtains the exact list of groups that appears as automorphism group of a plane non-singular curve of degree 4 in an algebraic closed field $K$ of zero characteristic, and also give an equation for such groups. In this chapter we present the analog of Henn’s result for degree 5 non-singular plane curves. Similar arguments can be applied to deal with higher degree.

2. Cyclic subgroups for degree 5 non-singular plane curve

Fix $C$ a non-singular curve of degree 5 over an algebraic closed field of zero characteristic, and assume that $Aut(C)$ is not trivial, and write by $F(X,Y,Z) = 0$ the curve in $\mathbb{P}^2(K)$. Also assume $\sigma \in Aut(C)$ is an element of exact order $m$ in $Aut(C)$ and we may assume, by a change of variables in $K$, that $\sigma$ maps $(x : y : z) \mapsto (x : \xi_m^a y : \xi_m^b z)$ where $\xi_m$ is a primitive $m$-th root of unity in $K$ and $a, b$ naturals such that $0 \leq a \neq b \leq m - 1$ with $a \leq b$ and $gcd(a, b)$ coprime with $m$ if $ab \neq 0$, (and we can reduce to $gcd(a, b) = 1$) and $gcd(b, m) = 1$ if $a = 0$, we call type $(m, (a, b))$ such automorphism. In this note with $m \geq 2$, $C_m$ denote also the cyclic group of order $m$.

Then by a change of variables we may have one of the following situations by [2] (which follow the argument of Dolgachev did in degree 4 [9]) where $L_{i,j}$ means an homogenous degree $i$ polynomial in the variables $\{X, Y, Z\}$ such that the variable $*$ does not appears, $\alpha$ is always a non-zero element (which by a change of variables can be always assumed equal to 1), and $\beta_{i,j} \in K$:

| Type: $(m, (a, b))$ | $F(X; Y; Z)$ |
|---------------------|--------------|
| 20, (4, 5)          | $X^5 + Y^5 + \alpha XZ^4$ |
| 16, (1, 12)         | $X^5 + Y^4Z + \alpha XZ^4$ |
| 15, (1, 11)         | $X^5 + Y^4Z + \alpha YZ^4$ |
| 13, (1, 10)         | $X^4Y + Y^4Z + \alpha Z^4X$ |
| 10, (2, 5)          | $X^5 + Y^5 + \alpha XZ^4 + \beta_{2,0}X^3Z^2$ |
| 8, (1, 4)           | $X^5 + Y^4Z + \alpha XZ^4 + \beta_{2,0}X^3Z^2$ |
| 5, (1, 2)           | $X^5 + Y^5 + Z^5 + \beta_{3,1}X^2YZ^2 + \beta_{4,3}XY^3Z$ |
| 5, (0, 1)           | $Z^5 + L_{5, Z}$ |
| 4, (1, 3)           | $X^5 + X(Z^4 + \alpha Y^4 + \beta_{4,2}Y^2Z^2) + \beta_{2,1}X^3YZ$ |
| 4, (1, 2)           | $X^5 + X(Z^4 + \alpha Y^4) + \beta_{3,2}X^3Z^2 + \beta_{3,2}XY^2Z^2 + \beta_{5,2}Y^2Z^2$ |
| 4, (0, 1)           | $Z^4L_{1, Z} + L_{5, Z}$ |
| 3, (1, 2)           | $X^5 + Y^4Z + \alpha YZ^4 + \beta_{2,1}X^3YZ + X^2(\beta_{3,0}Z^3 + \beta_{3,0}Y^3) + \beta_{4,2}XY^2Z^2$ |
| 2, (0, 1)           | $Z^4L_{1, Z} + Z^2L_{3, Z} + L_{5, Z}$ |
3. General properties of the full automorphism group

Before a detailed study of the automorphisms of degree 5 we recall the following results concerning $Aut(C)$ which will be useful. Because linear systems $g_2$ are unique, we always assume $C$ is given by a plane equation $F(X,Y,Z) = 0$ and $Aut(C)$ is a finite subgroup of $PGL_3(K)$ which fix the equation $F$. Moreover $Aut(C)$ satisfies (see Mitchell [16]) one of the following situations:

1. fixes a point $P$ and a line $L$ with $P \notin L$ in $PGL_3(K)$,
2. fixes a triangle, i.e. exists 3 points $S := \{P_1, P_2, P_3\}$ of $PGL_3(K)$, such that is fixed as a set,
3. $Aut(C)$ is conjugate of a representation inside $PGL_3(K)$ of one of the finite primitive group namely, the Klein group $PSL(2,7)$, the icosahedral group $A_5$, the alternating group $A_6$, the Hessian groups $Hess_{216}$, $Hess_72$ or $Hess_{36}$.

It is classically known that if $G$ a subgroup of automorphisms of a non-singular plane curve $C$ fixes a point on $C$ then $G$ is cyclic [13, Lemma 11.44], and recently Harui in [11] §2 provided the lacked result in the literature on the type of groups that could appear in non-singular plane curves. Before introduce Harui statement we need to define descendents of a plane non-singular curve. For a non-zero monomial $cX^iY^jZ^k$, $c \in K \setminus \{0\}$, we define its exponent as $\text{max}\{i,j,k\}$. For a homogeneous polynomial $F$, the core of $F$ is defined as the sum of all terms of $F$ with the greatest exponent. Let $C_0$ be a smooth plane curve, a pair $(C,G)$ with $G \leq Aut(C)$ is said to be a descendant of $C_0$ if $C$ is defined by a homogeneous polynomial whose core is a defining polynomial of $C_0$ and $G$ acts on $C_0$ under a suitable coordinate system.

THEOREM 62 (Harui). If $G \leq Aut(C)$ where $C$ is a non-singular plane curve of degree $d \geq 4$ then $G$ satisfies one of the following

1. $G$ fixes a point on $C$ and then cyclic.
2. $G$ fixes a point not lying on $C$ and it satisfies a short exact sequence of the form
   \[ 1 \rightarrow N \rightarrow G \rightarrow G' \rightarrow 1, \]
   with $N$ a cyclic group of order dividing $d$ and $G'$ (which is a subgroup of $PGL_2(K)$) is conjugate to a cyclic group $C_m$, a Dihedral group $D_{2m}$, the alternating groups $A_4$, $A_5$ or the permutation group $S_4$, where $m$ is an integer $\leq d-1$. Moreover, if $G' \cong D_{2m}$, then $m|(d-2)$ or $N$ is trivial.
3. $G$ is conjugate to a subgroup of $Aut(F_d)$ where $F_d$ is the Fermat curve $X^d + Y^d + Z^d$. In particular, $|G| \mid 6d^2$ and $(G, C)$ is a descendant of $F_d$.
4. $G$ is conjugate to a subgroup of $Aut(K_d)$ where $K_d$ is the Klein curve $XY^{d-1} + YZ^{d-1} + ZX^{d-1}$ hence $|G| \mid 3(d^2 - 3d + 3)$ and $(G,C)$ is a descendant of $K_d$.
5. $G$ is conjugate a finite primitive subgroup of $PGL_3(K)$, i.e. the Klein group $PSL(2,7)$, the icosahedral group $A_5$, the alternating group $A_6$, the Hessian groups $Hess_{216}$, $Hess_72$ or $Hess_{36}$ inside $PGL_3(K)$.

Recall also the following statement [11, Theorem 2.3],

THEOREM 63. Given $C$ non-singular plane curve of degree $d \neq 4, 6$ we have
\[ |Aut(C)| \leq 6d^2. \]

COROLLARY 64. Given $C$ non-singular plane curve of degree 5, then $Aut(C)$ is not conjugate to the Hessian group $Hess_{216}$, the Klein group $PSL(2,7)$ or the alternating group $A_6$.

And the following from [2]

PROPOSITION 65. Given $C$ a non-singular plane curve of degree $d$ and let $m$ the order or an element of $Aut(C)$ then $m$ divides one of the following naturals: $d - 1$, $d$, $(d-1)^2$, $d(d-2)$, $d(d-1)$ or $d^2 - 3d + 3$.

Now we recall in the statement different results proved for cyclic subgroups in $Aut(C)$ obtained by Badr and Bars in [2]:
Theorem 66. Let $C$ a non-singular plane curve of degree $d$, and $\sigma \in \text{Aut}(C)$. Then,

1. if $\sigma$ has order $d(d-1)$, then $\text{Aut}(C) = \langle \sigma \rangle$ and $C$ isomorphic to $X^d + Y^d + \alpha XZ^{d-1} = 0$ with $\alpha \neq 0$.
2. if $\sigma$ has order $(d-1)^2$, then $\text{Aut}(C) = \langle \sigma \rangle$ and $C$ isomorphic to $X^d + Y^{d-1}Z + \alpha XZ^{d-1} = 0$ with $\alpha \neq 0$.
3. if $\sigma$ has order $d(d-2)$ then $C$ is isomorphic to $X^d + Y^{d-1}Z + \alpha YZ^{d-1} = 0$ with $\alpha \neq 0$ and $d \neq 4, 6$, we have $\text{Aut}(C) = \langle \sigma, \tau \rangle \sigma(\tau^2 = \sigma^{d(d-2)} = 1$, and $\tau \sigma \tau = \sigma^{-(d-1)} > 1$.
4. if $\sigma$ has order $d^2 - 3d + 3$ then $C$ is isomorphic to $K_d$ and for $d \geq 5$ we have $\text{Aut}(C) = \langle \sigma, \tau \rangle |_{\sigma^d = 1}$ and $\tau \sigma \tau = \sigma^{-(d-1)} > 1$.
5. if $\sigma$ has order $\ell(d-1)$ with $\ell \geq 2$ then $\text{Aut}(C)$ is cyclic of order $\ell(d-1)$ with $\ell$ of the same conclusion if $\sigma$ is an homology.
6. if $\sigma$ has order $\ell d$ with $\ell \geq 3$ then $\text{Aut}(C)$ fix a line and a point off that line with the point not in $C$, and $\text{Aut}(C)$ is an exterior group as in Theorem 65 (2) with $N$ of order $d$. When $\ell = 2$ may be a decendent of the Fermat curve or $\text{Aut}(C)$ is an exterior group as in Theorem 65 (2) with $N$ of order $d$.

Now, assume as usual $C$ a non-singular plane curve of degree $d = 5$ with $\sigma \in \text{Aut}(C)$ of exact order $m$ such that acts on $F(X, Y, Z) = 0$ by $(x, y, z) \mapsto (x, \xi_m y, \xi_m z)$ and we assume that $m$ is maximal.

The following result determines the the full automorphism groups of such curves with elements of large orders in $\text{Aut}(C)$.

Corollary 67. For non-singular plane curves of degree 5 over an algebraic closed field of zero characteristic we have:

1. The cyclic group $C_{20}$ appears as $\text{Aut}(C)$ inside $PGL_3(K)$ generated by the transformation $(x, y, z) \mapsto (x, \xi_{20} y, \xi_{20} z)$ up to conjugation by $P \in PGL_3(K)$, and $C$ is isomorphic (through $P$) to the plane non-singular curve $X^5 + Y^5 + \alpha XZ^4$.
2. The cyclic group $C_{16}$ appears as $\text{Aut}(C)$ inside $PGL_3(K)$ generated by the transformation $(x, y, z) \mapsto (x, \xi_{16} y, \xi_{16} z)$ up to conjugation by $P \in PGL_3(K)$, and $C$ is isomorphic (through $P$) to the plane non-singular curve $X^5 + Y^4Z + \alpha XZ^4$.
3. The group $\text{SmallGroup}(30, 1) \cong \langle \sigma, \tau \rangle \sigma(\tau^2 = \sigma^{15} = 1 > 0$ of order 30 appears as $\text{Aut}(C)$ inside $PGL_3(K)$ given by $\sigma : (x, y, z) \mapsto (x, \xi_{15} y, \xi_{15} z)$ and $\tau : (x, y, z) \mapsto (x, \mu z, \mu^{-1} y)$ up to conjugation by $P \in PGL_3(K)$, and $C$ is isomorphic (through $P$) to the curve $X^5 + Y^4Z + \alpha YZ^4 = 0$ where $\mu^3 = \alpha$.
4. The group $\text{SmallGroup}(39, 1) \cong \langle \sigma, \sigma^{13} \rangle = \langle \sigma \rangle$ is of order 39 appears as $\text{Aut}(C)$ inside $PGL_3(K)$ given by $\sigma : (x, y, z) \mapsto (x, \xi_{13} y, \xi_{13} z)$ and $\sigma : (x, y, z) \mapsto (y, z, x)$ up to conjugation by $P \in PGL_3(K)$, and $C$ is isomorphic (through $P$) to the curve $K_1 : X^3 Y + Y^4 Z + Z^2 X = 0$.
5. The cyclic group $C_8$ appears as $\text{Aut}(C)$ inside $PGL_3(K)$ given by the transformation $(x, y, z) \mapsto (x, \xi_8 y, \xi_8 z)$ up to conjugation by $P \in PGL_3(K)$, and $C$ is isomorphic (through $P$) to the plane non-singular curve $X^5 + Y^4Z + \alpha XZ^4 + \beta_20 X^3 Z^2$, with $\alpha \beta_{20} = 0$.

Proof: Straightforward the first statements because corresponds to apply theorem 66 when the curve $C$ has a cyclic automorphism of order: $d(d-1)$, $(d-1)^2$, $d(d-2)$ or $d^2 - 3d + 3$ respectively, with $d = 5$ by use the table in §2. The last statement where $\text{Aut}(C)$ has a cyclic automorphism of order $\ell(d-1)$ with $\ell = 2$, by apply Theorem 66 and table in §2, we only need to observe that if $\text{Aut}(C)$ should be bigger, then always is cyclic and should be the group of order 16, therefore the only restriction to impose is $\beta_{20} = 0$ to ensure that the curve has exact automorphism group $C_8$. □

4. Determination of the automorphism group with small cyclic subgroups

Observe that remains to study $\text{Aut}(C)$ for curves $C$ of degree $d = 5$ where its larger order for any element in $\text{Aut}(C)$ has order $2d$, or $d$ by table in §2 and the results of the previous section from Theorem 65.
Proposition 68. Suppose that $C$ of degree 5 admits $\sigma \in \text{Aut}(C)$ of order 10 as an element of highest order in $\text{Aut}(C)$, then we reduce after conjugation by certain $P \in \text{PGL}_3(K)$ that $\sigma$ acts on $X^5 + Y^5 + \alpha XZ^2 + \beta_0 X Z^2$ where $\alpha \beta_{2,0} \neq 0$ as $\sigma: (x, y, z) \mapsto (x, \xi_{10}^2 y, \xi_{10}^2 z)$ and one of the following situations happens:

1. If $\alpha = 5$ and $\beta_{2,0} = 10$ then $C$ is isomorphic to the Fermat quintic $F_3: X^5 + Y^5 + Z^5$ and $\text{Aut}(C)$ is conjugate to $\text{SmallGroup}(150, 5)$. In particular, it is generated by $\eta_1, \eta_2, \eta_3, \eta_4$ of orders $2, 3, 5, 5$ respectively such that

\[(\eta_1 \eta_2)^2 = (\eta_1 \eta_3)(\eta_3 \eta_1)^{-1} = (\eta_1 \eta_4)(\eta_4 \eta_1)^{-1} = \eta_1 \eta_2 \eta_1 (\eta_3 \eta_4)^{-3} = \eta_2 \eta_3 \eta_2 (\eta_4 \eta_3)^{-1} = 1.

2. If $(\alpha, \beta_{2,0}) \neq (5, 10)$ then $\text{Aut}(C)$ is cyclic of order 10. Moreover, in such case if $1 + \alpha + \beta_{2,0} \neq 0$ then $C$ is a descendant of the Fermat curve defined by

\[PC: X^5 + Y^5 + Z^5 + \frac{5 - 3\alpha + \beta_{2,0}}{1 + \alpha + \beta_{2,0}} (X^4 Z + X Z^4) + \frac{2(5 + \alpha - \beta_{2,0})}{1 + \alpha + \beta_{2,0}} (X^3 Z^2 + X^2 Z^3).

Proof. Recall that the condition $\alpha \neq 0$, we can change the variables with a $P$ and fix a concrete value for $\alpha$ and because an element of $\text{Aut}(C)$ where 10 divides the order should be of order 20 from previous result in last section, and because we fix here the highest order should be 10 we can assume that $\beta_{2,0} \neq 0$. Thus we reduce that $C$ has an equation (up to $K$-isomorphism) of the form $X^5 + Y^5 + \alpha XZ^4 + \beta_{2,0} X Z^4 + Z^2 = 0$ with $\alpha \beta_{2,0} \neq 0$. This curve admits a homology $\sigma^2$ of order 5 > 3 therefore $\text{Aut}(C)$ fixes a line and a point off that line or $C$ is a descendant of Fermat curve (by Theorem 66). Moreover, the center $(0; 1; 0)$ of this homology is an outer Galois point (by Lemma 3.7) and if $C$ is not isomorphic to the Fermat curve $F_3: X^5 + Y^5 + Z^5$ then it is unique (Theorem 4’ (20)) hence should be fixed by $\text{Aut}(C)$.

Assume first that $\text{Aut}(C)$ fixes a line and a point off that line and $C$ is not the Fermat quintic. In that case, $\text{Aut}(C) \subseteq \text{PGL}_3(K)$ consists of elements of the form $[\alpha_1 X + \alpha_3 Z; Y; \gamma_1 X + \gamma_2 Z]$ and $\text{Aut}(C)$ satisfies a short exact sequence $1 \rightarrow C_5 \rightarrow \text{Aut}(C) \rightarrow G' \rightarrow 1$, where $C_5$ generated by $\sigma' := [X; \xi_{12}^6 Y; Z]$ and $G'$ contains an element of order 2 from the projection of the $\text{Aut}(C)$ given by $\delta := [X; Y; \xi_{12}^6 Z]$ and hence $G'$ is conjugate to $C_2, C_4, S_3, A_4, S_4$ or $A_5$. Now, there is no group of order 30 or 60 which contains elements of order 10 and no higher orders therefore $G'$ is not conjugate to $S_3$ or $A_4$. Moreover, assume that $G'$ is conjugate to $S_4$ then $\text{Aut}(C)$ is conjugate to $\text{SmallGroup}(120, 5)$ or $\text{SmallGroup}(120, 35)$ (because they are the only groups of order 120 with elements of order 10 and no higher orders) but there are no elements $\tau \in \text{Aut}(C)$ of order 3 or 10 that commutes with $\delta$ thus $\text{Aut}(C)$ is not conjugate to any of these two groups hence $G'$ cannot be conjugate to $S_4$.

On the other hand, groups of order 20 that contain elements of order 10 and no higher orders are: $\text{SmallGroup}(20, m)$ where $m = 1, 3, 4, 5$ and since there is no element $\tau \in \text{Aut}(C)$ of order 4 such that $\sigma' \tau = \tau$ or $\sigma' \tau = \tau$ then $m = 1, 3$ moreover there is no element $\tau$ of order 2 in $\text{Aut}(C)$ which commutes with $\delta$ therefore $m = 4, 5$ that is $G'$ is not conjugate to $C_4$. Furthermore, groups of order 300 that contain elements of order 10 and no higher orders are: $\text{SmallGroup}(300, m)$ where $m = 25, 26, 27, 41, 43$. If $m = 43$ or 41 then $\text{Aut}(C)$ contains exactly 3 element of order 2 a contradiction (because $\text{Aut}(C)$ should have at least 15 such elements). On the other hand, $m = 25, 26, 27$ because there are no elements of order 2 in $\text{Aut}(C)$ such that $\tau \delta = \delta \tau$ consequently $G'$ is not conjugate to $A_5$.

Finally, we assume that $C$ is a descendant of the Fermat curve. This happens through $P \in \text{PGL}_3(K)$ such that $P \sigma P^{-1} \in \text{Aut}(F_5)$ is of order 10 therefore $P \sigma P^{-1}$ has one of the forms $[X; \xi_{10}^2 Z; \xi_{10}^5 Y], [\xi_{10} Y; X; \xi_{10}^4 Z]$ or $[\xi_{10}^2 Z; \xi_{10}^5 Y; X]$ with $5 \parallel (a + b)$. In what follows, we treat each of these cases.

If $P \sigma P^{-1} = \lambda[X; \xi_{10}^2 Z; \xi_{10}^5 Y]$ then $\lambda = \xi_{10}^2, 5a + b + 2$ and $P = [\alpha_2 Y; \beta_1 X + \beta_1 Z; \xi_{10}^{2a+2} \beta_1 X + \xi_{10}^{2a+3} \beta_1 Z]$. This transforms $C$ into $F_5'$, a descendant of the Fermat quintic, where $Y^2 Z^2$ and $Y^3 Z^2$ have coefficients 0 and $\xi_{10}^{4a-6} \beta_{2,0} \alpha_3 \beta_3^2 \neq 0$ respectively which is a contradiction (since $[X; \xi_{10}^2 Z; \xi_{10}^5 Y] \in \text{Aut}(F_5)$). Therefore, this case should be excluded.

If $P \sigma P^{-1} = \lambda[X; \xi_{10} Y; X; \xi_{10}^2 Z]$ then $\lambda = \xi_{10}^{-2b}, 5a - 2b + 2, a - 2b - 3$ and $P = [\alpha_1 X + \alpha_3 Z; \xi_{10}^{-2b} \alpha_1 X + \xi_{10}^{2-2b-5} \alpha_3 Z; \gamma_{12} Y]$. Therefore, $C$ is transformed through $P$ to a descendant of $F_5$ where the monomial $X^5$ appears with coefficient $2a_0^2 \neq 0$ and $Y^5$ does not appear a contradiction.
If \( P \sigma P^{-1} = \lambda [\zeta_{10}^6 Z; \zeta_{10}^2 Y; X] \) then \( \lambda = \zeta_{10}^{2-2a}, \) 5\(|b - 2a + b - 2a - 3 = P = [\alpha_1 X + \alpha_3 Z; Y; \zeta_{10}^{1-2a} \alpha_1 X + \zeta_{10}^{-3-2a} \alpha_3 Z] \). It suffices to consider the case \( a = 1 \) and \( b = 0 \) that is, \( P = [\alpha_1 X + \alpha_3 Z; Y; \alpha_1 X - \alpha_3 Z] \). In particular, C is without loss of generality that \( \text{Aut} \) and because 2 does not divide the degree \( C \) plane curve that the curve \( \sigma \) curve. Now for type 2(0,1) for any choose of the free parameters, obtaining the result, because for commuting with \( \text{Aut} \) where \( \zeta \) is isomorphic to cyclic element of order 4 where \( \alpha \).

Now, if \( \alpha \neq 5 \) or \( \beta = 0 \) then \( [X; \zeta_{10}^2 Z; \zeta_{10}^2 Y; X; \zeta_{10}^2 Z; X], \) \( [\zeta_{10}^2 Z; X; \zeta_{10}^2 Y] \neq \text{Aut}(P) \). Furthermore, \( [\zeta_{10}^2 Z; \zeta_{10}^2 Y; X] \) or \( [X; \zeta_{10}^2 Y; \zeta_{10}^2 Z] \in \text{Aut}(P) \) if \( b = 0 \) that is \( \text{Aut}(P) \) is cyclic of order 10.

This completes the proof.

The following lemma, is very useful to discard all the groups with a subgroup isomorphic to \( C_2 \times C_2 \) in non-singular curves of degree 5, in particular help to simplify also the results above:

**Lemma 69.** Suppose that \( C \) is a non-singular plane curve of degree 5 with \( C_2 \times C_2 \) as a subgroup of \( \text{Aut}(C) \). Then such \( C \) does not exists, in particular there is no non-singular plane curve of degree 5 with full automorphism group isomorphic to the groups: \( C_2 \times C_2, A_4, S_4 \) and \( A_5 \).

**Proof.** By Mitchel [Mi] and Harui [Ha] the group \( C_2 \times C_2 \) inside \( \text{PGL}_3(K) \) giving invariant a non-singular plane curve \( C \) of degree \( d \) should fix a point not belonging to the curve, or is a descendant of Fermat or Klein curve. Now for \( d = 5 \) could not be a descendant of Fermat of Klein curve because 4 does not divide the order of automorphism of the Fermat of Klein curve of degree 5. Therefore the subgroup \( C_2 \times C_2 \) fixes a point not in \( C \) and because 2 does not divide the degree \( d = 5 \) by Harui main theorem [Ha] we have that we can think the elements of \( C_2 \times C_2 \) in a short exact sequence:

\[ 1 \rightarrow N = 1 \rightarrow G \rightarrow G \rightarrow 1 \]

where \( G \) is isomorphic to \( C_2 \times C_2 \) but \( N \) is the subgroup that acts on \( X \), thus the subgroup \( G \) fixes the variable \( X \) in the equation of \( C \) and we can reduce by [Ha] that \( G \) only acts in the variables \( Y, Z \) as a matrix in \( \text{PGL}_2(K) \). Let now \( \sigma, \tau \in G \leq \text{GL}_2(K) \) of order two, by a transformation \( P \in \text{GL}_2(K) \) we can assume that \( \sigma = \text{diag}(1, -1) \) and \( \tau = [[a, b], [c, -a]] \neq \sigma \), (and recall that \( \sigma \tau = \tau \sigma \)) and therefore we may assume that the curve \( C \) has a model of type 2(0,1). Now all possible \( \tau \) does not retain invariant the equation of type 2(0,1) for any choose of the free parameters, obtaining the result, because for commuting with \( \sigma \) the automorphism \( \tau = \text{diag}(-1, 1) \) and therefore \( C \) has the equation: \( Z^4 L_1 Z + 2 Z_3 Z + L_5 Z \) and simultaneously \( Y^4 L_1 Y + Y^4 L_3 Y + L_5 Z \) which is impossible, or \( \tau \) is the transformation \( [X, Y, Z] \rightarrow [X, bZ, cY] \) with \( bc \neq 0 \), but we obtain that \( C \) has simultaneously also the above two equations, convenient multiplied for some constants which is not possible.

**Proposition 70.** Suppose that \( C \) of degree 5 admits \( \sigma \in \text{Aut}(C) \) of order 4 as element of higher order in \( \text{Aut}(C) \), then we reduce after conjugation by certain \( P \in \text{PGL}_3(K) \) one of the following situations:

1. \( \text{Aut}(C) \) is isomorphic to cyclic element of order 4 where \( \text{Aut}(C) = \sigma > \sigma(x, y, z) = (x, \xi_4 y, \xi_4^2 z) \) (up to conjugation by \( P \in \text{PGL}_3(K) \)) and the curve \( C \) satisfies \( F(X, Y, Z) = 0 \) (up to \( P \), i.e. up to \( K \)-isomorphism) given by \( X^5 + X(Z^4 + \alpha Y^4) + \beta_2 X^2 Z^2 + \beta_2 X^2 Y^2 Z + \beta_2 Y^2 Z^3 = 0 \), with \( \alpha \beta_2 \neq 0 \).

2. \( \text{Aut}(C) \) is isomorphic to cyclic element of order 4 where \( \text{Aut}(C) = \langle \sigma > \sigma(x, y, z) = (x, y, \xi_4 z) \) (up to conjugation by \( P \in \text{PGL}_3(K) \)) and the curve \( C \) satisfies \( F(X, Y, Z) = 0 \) (up to \( P \), i.e. up
to $K$-isomorphism) given by $Z^4Y + L_{5,Z}(X,Y) = 0$ such that $L_{5,Z}'(X,\zeta_mY') \neq \zeta_mL_{5,Z}'(X,Y')$ where $(m,r) = (8,1), (16,1)$ or $(20,4)$.

**Proof.** We consider the situations in §2 concerning types 4, $(a,b)$.

We observe first that $C$ can not be a descendant of the Fermat curve $F_5$ or the Klein curve $K_5$ because $|\text{Aut}(F_5)| = 150$ and $|\text{Aut}(K_5)| = 144$ and $4 \nmid |\text{Aut}(F_3)|$ or $|\text{Aut}(K_5)|$ and $\text{Aut}(C)$ is not conjugate to $A_5$ since there are no elements of order 4. Consequently, $\text{Aut}(C)$ is conjugate to $\text{Hess}_{36}, \text{Hess}_{72}$ or it should fix a line and a point off that line by the result of Harui. Moreover, for the last case, we need to consider the situation of a short exact sequence of the form

$$1 \to N = 1 \to \text{Aut}(C) \to G' \to 1,$$

where $G'$ should contain an element of order 4. That is, $G'$ is conjugate to a cyclic group $C_4$ or a Dihedral group $D_8$ (by use of the previous lemma $[20]$).

1. Type 4, $(1,3)$; here the automorphism group is bigger and also has a subgroup of order 32 when we impose the coefficient of $X^2YZ$ to be trivial which would give a contradiction on Harui result.

But in Type 4 $(1,3)$ the equation is $F(X,Y,Z) = X \cdot G(X,Y,Z)$ and therefore not of degree 5 plane non-singular curve (which is connected). Thus this situation can not happen.

2. Type 4, $(0,1)$; This curve admits a homology of order $d - 1$ with center $(0;0;1)$ then it follows by Harui [11] that this point is an inner Galois point of $C$ and moreover it is unique by Yoshihara [20].

Therefore, this point should be fixed by $\text{Aut}(C)$ consequently, $\text{Aut}(C)$ is cyclic. It follows by the assumption that $C$ is not conjugate to any of the above (In particular, types 8, $(1,4)$, 16, $(1,12)$ or 20, $(4,5)$) then $\text{Aut}(C)$ is cyclic of order 4. To be more precise, for the type 4, $(0,1)$ we can write it as $Z^4(aX + bY) + L_{5,Z}(X,Y) = 0$ and with $Y' := aX + bY$ we can rewrite it with the same type as $Z^4Y' + L_{5,Z}'(X,Y') = 0$. Now, it is necessary to impose the condition that $L_{5,Z}'(X,\zeta_mY') \neq \zeta_mL_{5,Z}'(X,Y')$ where $(m,r) = (8,1), (16,1)$ or $(20,4)$ (otherwise; we get a bigger automorphism group conjugate to those of types 8, $(1,4)$, 16, $(1,12)$ or 20, $(4,5)$).

3. Type 4, $(1,2)$; First from the same reason as Type 4 $(1,3)$ we need to assume that $\beta_{5,2} \neq 0$. First, we'll show that $\text{Aut}(C)$ is not conjugate to any of the Hessian subgroups $\text{Hess}_{36}$ or $\text{Hess}_{72}$. Both groups contains reflections but no four groups hence all reflections in the group will be conjugate ( [16] Theorem 11). Therefore, it suffices to consider the case $P4, (1,2)^2P^{-1} = \lambda[Z;Y;X]$ this in turns gives solutions non of them transform $\tilde{C}$ to a smooth curve $P\tilde{C}$ with $\{(X;Z;Y), (Y;Z;X), (Z;Y;X)\} \subseteq \text{Aut}(P\tilde{C})$. Indeed, $P$ has one of the forms $[a_1X + a_2Y + a_3Z; \beta_1X + \beta_3Z; a_1X - a_2Y - a_3Z]$ or $[a_1X + a_2Y + a_3Z; b_2Y - a_1X + a_2Y - a_3Z]$. For both cases, we must have $a_1 = a_3$ (coefficients of $XY^4$ and $Y^4Z$) (in particular, the second case does not occur) moreover, from the coefficients of $X^3XY$ and $Y^3Z$, we get $\gamma_1 = \gamma_2$ a contradiction. Consequently, the claim follows and $\text{Aut}(C)$ should fix a line and a point off that line.

Now, if $C$ admits a bigger non-cyclic automorphism group then it should non-commutative by Harui and contain an element of order 2 with does not commute with the cyclic element given by the type and we can reduce to a subgroup conjugate to the dihedral group $D_8$ and moreover $\text{Aut}(C)$ fixes the point $(1;0;0)$ and the line $X = 0$. In particular, automorphisms of $C$ are of the form $[X;vY + wZ; sY + tZ]$. Since there is no element $\tau \in \text{Aut}(C)$ of order 2 such that $\tau^{\sigma} = \sigma^{-1}$ then $\text{Aut}(C)$ is not conjugate to $D_8$ or $S_4$. In particular, it is cyclic of order 4. To be more precise, if $Aut(C)$ is cyclic of order $4k > 4$ then $k = 2, 4$ or 5. If $k = 5$ that is $C$ is projectively equivalent to type 20, $(4,5)$ and hence $\sigma$ is projectively equivalent to a homology of order 4 a contradiction (Indeed, $\text{Aut}(C')$ contains exactly two elements of order 4 and both are homologies) and similarly, if $k = 2$ or 4. That is, $C$ is not isomorphic to any of the above with the full automorphism group is cyclic of order $4k$. 

□

Now we deal with quintic curves with a cyclic element of order 5 as an element in $\text{Aut}(C)$ of highest order.
Proposition 71. Suppose that $C$ has an automorphism of type $5, (1,2)$ and we write $C$ as $X^5 + Y^5 + Z^5 + \beta_3 X^2 Y Z^2 + \beta_4 X Y^3 Z = 0$ such that $\beta_{3,1} \neq 0$ or $\beta_{4,3} \neq 0$. Suppose the highest order of an element in $\text{Aut}(5)$ is $5$, then $C$ is a descendant of the degree $5$ Fermat curve with $\text{Aut}(C)$ conjugate to $D_{10}$.

Proof. $\text{Aut}(C)$ is not conjugate to any of the Hessian groups $\text{Hess}_{30}$ or $\text{Hess}_{72}$ and is not conjugate to a subgroup of $\text{Aut}(K_5)$ since there are no elements of order $5$. On the other hand, always $C$ admits a bigger automorphism group isomorphic to $D_{10}$ through the transformation $[Z; Y, X]$ (in particular, $\text{Aut}(C)$ is not cyclic). Consequently, $C$ is a descendant of the Fermat quintic or $\text{Aut}(C)$ fixes a line and a point off that line or $\text{Aut}(C)$ is conjugate to $A_5$ (as a finite primitive subgroups of $PGL_3(K)$) but this last situation is not possible by previous lemma [29].

(1) $\text{Aut}(C)$ fixes a line and a point off that line

This line should be $Y = 0$ and the point is $(0; 1; 0)$ because $\sigma, \tau \in \text{Aut}(C)$ with $\sigma(x, y, z) = (x, \xi_5 y, \xi_5^2 z)$ and $\tau(x, y, z) = (z, y, x)$.

Hence elements of $\text{Aut}(C)$ are of the form $[\alpha_1 X + \alpha_3 Z; Y; \gamma_1 X + \gamma_3 Z]$. From the coefficients of $Y^3 Z^2$ and $Y^3 X^2$ we get $\alpha_1 = 0 = \gamma_3$ or $\alpha_3 = 0 = \gamma_1$. Because $\beta_{3,1} \neq 0$ or $\beta_{4,3} \neq 0$ (otherwise; $C$ is the Fermat quintic and hence admits automorphisms of order $10 > 5$ a contradiction) therefore, $\alpha_{3,1} = \gamma_{3,3} = 1$ and $\beta_{3,1} \gamma_{1,3} = 1$ or $(\alpha_{3,1} \gamma_{1,3})^2 = 1$ thus $\text{Aut}(C)$ has order $10$. In particular, $\text{Aut}(C)$ is conjugate to $D_{10}$.

(2) If $C$ is a descendant of the Fermat curve $F_5$ and neither a line nor a point is leaved invariant. Then, through a transformation $P_5, (1,2) \ P^{-1} = 5, (1,2)$. Indeed, elements of order $5$ in $\text{Aut}(F_5)$ are of the form $\lambda_5, (a, b)$ and if $P_5, (1,2) \ P^{-1} = \lambda_5, (a, b)$ then $(a, b) \in \{(1,2), (2,1), (3,4), (4,3), (1,4), (4,1)\}$ but all are conjugate in $\text{Aut}(F_5)$. Now, it is straightforward to verify that there are no more automorphisms in $\text{Aut}(F_5) \cap \text{Aut}(C)$ that is, in such case $\text{Aut}(C)$ is conjugate to $D_{10}$.

Proposition 72. Suppose that $C$ has an automorphism of type $5, (0,1)$ and $C$ has an expression as $Z^5 + L_{5,2}$. Assume that the highest order element of $\text{Aut}(C)$ is $5$, then $C$ has $\text{Aut}(C)$ cyclic of order $5$.

Proof. This curve has a homology $\sigma$ of order $d$ with center $(0;0;1)$ and axis $Z = 0$ then (by Harui) this point is an outer Galois point of $C$. Moreover, because automorphisms of $C$ has order $\leq 5$ then it is not isomorphic to the Fermat curve thus (by Yoshihara) this Galois point is unique hence should be fixed by $\text{Aut}(C)$. In particular, $\text{Aut}(C)$ fixes a line $(Z = 0)$ and a point off that line $(0;0;1)$ and therefore, elements of $\text{Aut}(C)$ have the form $[\alpha_1 X + \alpha_2 Y; \beta_1 X + \beta_2 Y; Z]$. Furthermore, $\text{Aut}(C)$ satisfies a short exact sequence

$$1 \to N \to \text{Aut}(C) \to G' \to 1,$$

with $N$ is cyclic of order dividing $5$ and $G'$ is conjugate to $C_m, D_{2m}, A_4, S_4$ or $A_5$ where $m \leq 4$ and in case of $D_{2m}$ we have $m|5$ or $N$ is trivial.

If $N$ is trivial then $G'$ should be conjugate to $A_5$ (because non of the other groups contains elements of order $5$) in particular, $D_{10}$ is a subgroup of $\text{Aut}(C)$ but one can easily verify that there exists no elements $\tau \in \text{Aut}(C)$ of order $2$ such that $\tau \sigma \tau = \sigma^{-1}$ hence $N$ can not be trivial.

If $N$ has order $5$ then for any value of $G'$ (except possibly the trivial group, $C_2$, $C_4$ or $A_4$ such that $\text{Aut}(C)$ is conjugate to $D_{10}$, SmallGroup$(20,3)$ or $A_5$) there are elements of order $> 5$ in $\text{Aut}(C)$ a contradiction. As above $D_{10}$ is not conjugate to a subgroup of $\text{Aut}(C)$ therefore $G'$ is conjugate to $C_2$ or $A_4$. On the other hand, there are no elements $\tau \in \text{Aut}(C)$ of order $4$ such that $(\tau \sigma)^2 = 1$ hence $\tau \sigma = \sigma^{-1}$ thus $G'$ is not conjugate to $C_4$. Consequently, $\text{Aut}(C)$ is cyclic of order $5$.

Remains yet the study of curves where its large automorphism element has order at most $3$. In particular, it is not conjugate to $A_5, \text{Hess}_{30}$ or $\text{Hess}_{72}$ because each of them contain elements of order $> 3$. Therefore, in such case $\text{Aut}(C)$ should fix a line and a point off that line or it is conjugate to a subgroup of $\text{Aut}(F_5)$ or $\text{Aut}(K_5)$. 

\[\Box\]
Proposition 73. Consider a curve of type 3(1, 2), and we can consider that \( C : X^5 + Y^4Z + YZ^4 + \beta_{2,1}X^3YZ + X^2(\beta_{3,0}Z^3 + \beta_{3,3}Y^3) + \beta_{4,2}XY^2Z^2 = 0 \), and assume that the highest order of an element in \( Aut(C) \) is 3. Then \( Aut(C) \) is cyclic of order 3 (if \( \beta_{3,0} \neq \beta_{3,3} \)) or conjugate to \( S_3 \) (otherwise).

Proof.
If \( C \) is a descendant of the Klein curve then \( Aut(C) \) is conjugate to a subgroup of \( Aut(K_5) \) and hence can not be of order > 3. Indeed, because \( |Aut(K_5)| = 3^2.7 \) then any subgroup of order > 3 inside \( Aut(K_5) \) which does not contain elements of order > 3 are isomorphic to \( C_3 \times C_3 \) therefore there exist \( \tau \in Aut(C) \) of order 3 that commutes with \( \sigma := [X; \xi_3 Y, \xi_3^2 Z] \). This gives \( \tau \) of the forms \( [X; \alpha Y; \mu Z], [Y; \alpha Z; \mu X] \) or \( [Z; \alpha X; \mu Y] \) but since the variable \( X \) can not be permuted with \( Y \) or \( Z \) then \( \tau = [X; \alpha Y; \mu Z] \) with \( \alpha = \omega \) (resp. \( \omega^2 \)) and \( \mu = \omega^2 \) (resp. \( \omega \)) a contradiction.

If \( C \) is a descendant of the Fermat curve then \( Aut(C) \) is cyclic of order 3 or conjugate to \( S_3 \) inside \( Aut(F_5) \). Indeed, \( |Aut(F_5)| = 2.3.5^2 \) hence any subgroup of order > 3 is conjugate to \( S_3 \) (note that \( Aut(F_5) \) contains no elements of order 6) or it contains elements of order 5 > 3. Now, if \( Aut(C) \) is conjugate to \( S_3 \) then there exists \( \tau \in Aut(C) \) of order 2 such that \( \tau 3, (1, 2) \tau = 3, (2, 1) \) which in turns implies that \( \tau \) has the form \( [X; \beta Z; \beta^{-1}Y] \). But, \( [X; \beta Z; \beta^{-1}Y] \) \( \in \) \( Aut(C) \) iff \( \beta^3 = 1 \) and \( \beta_{3,0} = \beta_{3,3} \).

If \( Aut(C) \) fix a point then this point should be one of the reference points \( P_1 := (1; 0; 0), P_2 := (0; 1; 0) \) or \( P_3 := (0; 0; 1) \) (because these are the only points which are fixed by \( 3, (1, 2) \)). If the fixed points is \( P_2 \) or \( P_3 \) then \( Aut(C) \) is cyclic of order 3 since both points belong to \( C \). If the fixed point is \( P_1 \) then the line that is leaved invariant should be \( X = 0 \) hence automorphisms of \( C \) have the form \( [X; \beta_2 Y + \beta_3 Z; \gamma_2 Y + \gamma_3 Z] \). Moreover, it follows by Harui and the assumption that there are no elements in \( Aut(C) \) of order > 3 that \( Aut(C) \) satisfies a short exact sequence of the form

\[ 1 \rightarrow N \rightarrow Aut(C) \rightarrow G' \rightarrow 1, \]

where \( N \) is trivial and \( G' \) is conjugate to \( C_3, S_3 \) or \( A_4 \). Now, let \( \tau := [X; \beta_2 Y + \beta_3 Z; \gamma_2 Y + \gamma_3 Z] \) be an element of order 2 then it has one of the following forms

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & \beta_3 \\
0 & \gamma_2 & \beta_2
\end{pmatrix}
\]

where \( \beta_2^3 + \beta_3 \gamma_2 = 1 \) and it suffices to consider the last case because non of the first three transformations retain \( C \). If \( Aut(C) \) is conjugate to \( A_4 \) then we can suppose that \( (3, (1, 2) \tau)^3 = Id \) therefore, \( \begin{pmatrix}
1 & 0 & 0 \\
0 & \omega \beta_2 & \omega \beta_3 \\
0 & \omega^2 \gamma_2 & -\omega^2 \beta_2
\end{pmatrix} \) has order 3. That is,

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \omega \beta_2 & \omega \beta_3 \\
0 & \omega^2 \gamma_2 & -\omega^2 \beta_2
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 2 \beta_2^3 + \gamma_2 \beta_3 (\omega^2 - 1) \beta_2 \beta_3 \\
0 & (1 - \omega) \gamma_2 \beta_2 & \omega \beta_2^3 + \gamma_2 \beta_3 \\
1 & 0 & 0 \\
0 & \omega^2 + (1 - \omega^2) \gamma_2 \beta_3 & (\omega^2 - 1) \beta_2 \beta_3 \\
0 & 0 & \omega + (1 - \omega) \gamma_2 \beta_3
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \omega \beta_2 & \omega \beta_3 \\
0 & \omega^2 \gamma_2 & -\omega^2 \beta_2
\end{pmatrix}
\]

From which we get \( \beta_3 = 0 \) or \( \beta_2 = \frac{\omega}{\omega^2 - 1} \). If \( \beta_3 = 0 \) then \( \beta_2 = -1 \) and \( \gamma_2 = 0 \) (yields no automorphism) and the second possibility implies that \( 1 = \gamma_2 \beta_3 = \frac{\omega \beta_3}{\omega^2} \) a contradiction. Consequently, \( Aut(C) \) can not be conjugate to \( A_4 \).

This proves the result. \( \square \)

Proposition 74. Consider a curve of type 2, (0, 1) and we can consider that \( C : Z^4L_{1,2} + Z^2L_{3,2} + L_{5,2} = 0 \). Assume that the highest order of any element of \( Aut(C) \) is two. Then the automorphism group of \( C \) is cyclic of order 2.
Proof. \( C \) is not a descendant of the Klein curve because \( 2 \nmid |\text{Aut}(K_5)| (= 63) \). Also, if \( C \) is a descendant of the Fermat curve then \( \text{Aut}(C) \) can not be conjugate to a bigger subgroup of \( \text{Aut}(F_5) \) because \( |\text{Aut}(F_5)| = 2 \cdot 3 \cdot 5^2 \) thus subgroups of order \( \geq 2 \) should contains elements of order 3 or 5 or a contradiction. Finally, if \( \text{Aut}(C) \) fixes a line and a point off that line then by Harui and our assumption that there are no automorphisms of order \( \geq 2 \) we get that \( \text{Aut}(C) \) satisfies a short exact sequence of the form

\[
1 \rightarrow N = 1 \rightarrow \text{Aut}(C) \rightarrow G' \rightarrow 1,
\]

where \( G' \) contains an element of order 2 and no higher orders thus \( G' \) should be conjugate to \( C_2 \) or \( C_2 \times C_2 \). Thus by lemma \( \text{[69]} \) we conclude.

Now in the previous results we need to ensure that there is an equation \( C \) which the highest order automorphism is of exact order \( m \) in the cases of \( m \leq 5 \).

**Lemma 75.** Take \( C \) a plane non-singular curves of degree 5 with the equation given by Type \( m, (a, b) \) with \( m \leq 5 \). Then exist an equation \( C \) of Type \( m, (a, b) \) with certain specification value at the parameters such that the element inside \( \text{Aut}(C) \) of highest order is \( m \).

Proof. The proof is only a computation that the plane equations with bigger automorphism group that the one obtained imposes restriction in the locus and does not cover all the specialization of the parameters which gives a non-singular plane curve. For example for \( C \) of Type 5(1,2), they have at least \( D_{10} \) in the automorphism group, and the equation with parameters satisfies such situation have freedom of two parameters, and the locus of curves with \( D_{10} \) as a strictly subgroup are giving only for a point, and there are specialization of such parameters given non-singular plane curve which does not corresponds the point corresponding to the Fermat of degree 5, therefore \( D_{10} \) is realized. A similar computation can be done for the other situations. The case \( m = 4 \) one can read in [1].

Therefore we can conclude as follows, when we list the exact groups that appears as \( \text{Aut}(C) \) for \( C \) non-plane singular curve of degree 5 (some of them we give in GAP notation), an equation (up to \( K \)-isomorphism) with effective action by the group (and some of them will have exact full automorphism group the predicted by the group but remain to introduce the parameter restrictions in order to be the equation geometrically irreducible and do not have a bigger automorphism group), and the third column corresponds to the presentation of the group inside \( \text{PGL}_3(K) \) that the automorphism group happens:

**Table 2.** Full Automorphism of quintics

| \( \text{Aut}(C) \) | \( F(X; Y; Z) \) | Generators |
|----------------|----------------|------------|
| (150, 5)      | \( X^5 + Y^5 + Z^5 \) | \( [\xi_2X; Y; Z], [X; \xi_5Y; Z] \) |
| (39, 1)       | \( X^4Y + Y^4Z + Z^4X \) | \( [X; \xi_{13}Y; \xi_{13}Z], [Y; Z; X] \) |
| (30, 1)       | \( X^5 + Y^4Z + \alpha Y Z^4 \) | \( [X; \xi_{15}Y; \xi_{15}Z], [X; \mu Z; \mu^{-1} Y] \) |
| \( C_{20} \)  | \( X^5 + Y^5 + \alpha X Z^4 \) | \( [X; \xi_{20}Y; \xi_{20}Z] \) |
| \( C_{16} \)  | \( X^5 + Y^4 Z + \alpha X Z^4 \) | \( [X; \xi_{16}Y; \xi_{16}Z] \) |
| \( C_{10} \)  | \( X^5 + Y^5 + \alpha X Z^4 + \beta_{2,0} X^3 Z^2 \) | \( [X; \xi_{10}Y; \xi_{10}Z] \) |

where \( \alpha \neq 0 \) which can take always 1 for a change of variables, and \( \mu \) satisfies \( \mu^3 = \alpha \).
Table 3. Full Automorphism of quintics

| Group | Equation | Automorphism Group |
|-------|----------|--------------------|
| $D_{10}$ | $X^5 + Y^5 + Z^5 + \beta_{1,1}X^2YZ^2 + \beta_{1,3}XY^3Z$ | $[X; \xi_5Y; \xi_5^2Z]$, $[Z, Y, X]$ |
| $C_8$ | $X^5 + Y^4Z + \alpha XZ^4 + \beta_{2,0}X^4Z^2$ ($\beta_{2,0} \neq 0$) | $[X; \xi_8Y; \xi_4^2Z]$ |
| $S_3$ | $X^5 + Y^4Z + YZ^4 + \beta_{2,1}X^4YZ + X^2(Z^3 + Y^3) + \beta_{4,2}XY^2Z^2$ (not above) | $[X; \xi_3Y; \xi_4^2Z]$, $[X; Z; Y]$ |
| $C_5$ | $Z^5 + L_{5, z}$ (not above) | $[X; \xi_5Z]$ |
| $C_4$ | $X^5 + X(Z^4 + \alpha Y^4) + \beta_{2,0}X^4Z^2 + \beta_{3,2}X^2Y^2Z + \beta_{5,2}Y^2Z^4$ | $[X; \xi_4Y; \xi_4^2Z]$ |
| $C_4$ | $Z^4L_{1, z} + L_{5, z}$ (not above) | $[X; Y; \xi_4Z]$ |
| $C_3$ | $X^5 + Y^4Z + \alpha YZ^4 + \beta_{2,1}X^3YZ + X^2(\beta_{3,0}Z^3 + \beta_{3,3}Y^3) + \beta_{4,2}XY^2Z^2$ ($\beta_{3,0} \neq \beta_{3,3}$) (not above) | $[X; \xi_3Y; \xi_4^2Z]$ |
| $C_2$ | $Z^4L_{1, z} + Z^4L_{3, z} + L_{5, z}$ (not above) | $[X; Y; \xi_2Z]$ |
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