ON THE SELMER GROUP ATTACHED TO A MODULAR FORM
AND AN ALGEBRAIC HECKE CHARACTER

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Abstract

We construct an Euler system of generalized Heegner cycles to bound the Selmer group
associated to a modular form and an algebraic Hecke character. The main argument is based on
Kolyvagin’s method adapted by Bertolini and Darmon [2] and by Nekovář [17] while the key
object of the Euler system, the generalized Heegner cycles, were first considered by Bertolini,
Darmon and Prasanna in [4].

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1 Introduction

Kolyvagin [14, 11] constructs an Euler system based on Heegner points and uses it to bound the size of the Selmer group of certain (modular) elliptic curves $E$ defined over $\mathbb{Q}$, over imaginary quadratic fields $K$ assuming the non-vanishing of a suitable Heegner point. In particular, this implies that

$$\text{rank}(E(K)) = 1,$$

and the Tate-Shafarevich group $\text{III}(E/K)$ is finite. Bertolini and Darmon adapt Kolyvagin’s descent to Mordell-Weil groups over ring class fields [2]. More precisely, they show that for a given character $\chi$ of $\text{Gal}(K_c/K)$ where $K_c$ is the ring class field of $K$ of conductor $c$,

$$\text{rank}(E(K_c)^{\chi}) = 1$$

assuming that the projection of a suitable Heegner point is non-zero. More generally, one can associate to a modular form $f$ of even weight $2r$ and level $\Gamma_0(N)$ a $p$-adic Galois representation $T_p(f)$ [13, 22]. For a given number field $K$, there is a $p$-adic Abel-Jacobi map

$$\Phi_{f,K} : \text{CH}^r(X/K)_0 \rightarrow H^1(K, T_p(f)),$$

where

- $X$ is the Kuga-Sato variety of dimension $2r - 1$, that is, a compact desingularization of the $2r-2$-fold fibre product of the universal generalized elliptic curve over the modular curve $X_1(N)$,
- $\text{CH}^r(X/K)_0$ is the $r$-th Chow group of $X$ over $K$, that is the group of homologically trivial cycles on $X$ defined over $K$ of codimension $r$ modulo rational equivalence,
- $H^1(K, T_p(f))$ stands for the first Galois cohomology group of $\text{Gal}(\overline{K}/K)$ acting on $T_p(f)$.

Nekovář [17] adapts the method of Euler systems to modular forms of higher even weight to describe the image by the Abel-Jacobi map $\Phi_{f,K}$ of Heegner cycles on the associated Kuga-Sato varieties, hence showing that

$$\dim_{\mathbb{Q}_p}(\Phi_{f,K}(e_f \text{CH}^r(X/K)_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = 1$$

for a suitable projector $e_f$, assuming the non-vanishing of a suitable Heegner cycle. In Article [8], we combined these two approaches to study modular forms of higher even weight twisted by ring class characters $\chi$ of imaginary quadratic fields and showed that

$$\dim_{\mathbb{Q}_p}(\Phi_{f,\chi,K}(e_{f,\chi} \text{CH}^r(X/K)_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = 1$$

for a suitable projector $e_{f,\chi}$, assuming the non-vanishing of a suitable generalized Heegner cycle.

In this article, we study the Selmer group associated to a modular form of even weight $r + 2$ and
an unramified algebraic Hecke character \( \psi \) of \( K \) of infinity type \( (r,0) \). Our setting involves the generalized Heegner cycles introduced by Bertolini, Darmon and Prasanna in [4].

Our motivation stems from the Beilinson-Bloch-Kato conjecture that predicts that

\[
\dim_{\mathbb{Q}_p}(\Phi_{f, \psi, K}(e_{f, \psi} \text{CH}^r(W/K)_0) \otimes \mathbb{Z}_p \mathbb{Q}_p) = \text{ord}_{s=r+1} L(f \otimes \theta_{\psi}, s),
\]

where

\[
\theta_{\psi} = \sum_{a \in O_K} \psi(a)q^{N(a)}
\]
is the theta series associated to \( \psi \) [20], \( W \) is a Kuga-Sato like variety, and \( e_{f, \psi} \) is a suitable projector.

Let \( f \) be a normalized newform of level \( \Gamma_0(N) \) where \( N \geq 5 \) and even weight \( r+2 > 2 \). Denote by \( K = \mathbb{Q}(\sqrt{-D}) \) an imaginary quadratic field with odd discriminant satisfying the Heegner hypothesis, that is primes dividing \( N \) split in \( K \). For simplicity, we assume that

\[
|\mathcal{O}_K^\times| = 2.
\]

Let

\[
\psi : \mathbb{A}_K^\times \rightarrow \mathbb{C}^\times
\]
be an unramified algebraic Hecke character of \( K \) of infinity type \( (r,0) \). There is an elliptic curve \( A \) defined over the Hilbert class field \( K_1 \) of \( K \) with complex multiplication by \( \mathcal{O}_K \). We further assume that \( A \) is a \( \mathbb{Q} \)-curve, that is \( A \) is \( K_1 \)-isogenous to its conjugates in \( \text{Aut}(K_1) \). This is possible by the assumption on the parity of \( D \), (see [9, Section 11]). Consider a prime \( p \) not dividing \( ND\phi(N)N_AN_\psi \), where \( N_A \) is the conductor of \( A \) and \( N_\psi \) is the conductor of \( \psi \). We denote by \( V_f \) the \( f \)-isotypic part of the \( p \)-adic étale realization of the motive associated to \( f \) by Scholl [22] and Deligne [7] twisted by \( \frac{r+2}{2} \) and by \( V_\psi \) the \( p \)-adic étale realization of the motive associated to \( \psi \) twisted by \( \frac{r}{2} \). More precisely, \( V_\psi \) is the \( \psi \)-isotypic component of the first Galois cohomology group of

\[
\text{res}_{K_1/Q}(A) = \prod_{\sigma \in \text{Gal}(K_1/Q)} A^\sigma
\]
where \( A^\sigma \) is the \( \sigma \)-conjugate of \( A \), (see Section 2 for more details). Let \( \mathcal{O}_F \) be the ring of integers of

\[
F = \mathbb{Q}(a_1, a_2, \ldots, b_1, b_2, \ldots),
\]
where the \( a_i \)'s are the coefficients of \( f \) and the \( b_i \)'s are the coefficients of the theta series

\[
\theta_{\psi} = \sum_{a \in \mathcal{O}_K} \psi(a)q^{N(a)}
\]
associated to \( \psi \). Then \( V_f \) and \( V_\psi \) will be viewed (by extending scalars appropriately) as free \( \mathcal{O}_F \otimes \mathbb{Z}_p \)-modules of rank 2. We denote by

\[
V = V_f \otimes_{\mathcal{O}_F} \mathbb{Z}_p V_\psi
\]
the $p$-adic étale realization of the tensor product of $V_f$ and $V_\psi$ and let $V_{\rho}$ be its localization at a prime $\rho$ in $F$ dividing $p$. Let $\mathcal{O}_{F,\rho}$ be the localization of $\mathcal{O}_F$ at $\rho$. Then $V_{\rho}$ is a four dimensional representation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over $\mathcal{O}_{F,\rho}$ with coefficients in

$$\text{End}(\text{Res}_{K_1,\mathbb{Q}}(A)/\mathbb{Q}) = \bigoplus_{\sigma \in \text{Gal}(K_1/\mathbb{Q})} \text{Hom}(A,A^\sigma),$$

(see Section 2).

By the Heegner hypothesis, there is an ideal $N$ of $\mathcal{O}_K$ satisfying $\mathcal{O}_K/N = \mathbb{Z}/N\mathbb{Z}$. We can therefore fix $\Gamma_1(N)$ level structure on $A$ over $K_1$ which is a point of exact order $N$ defined over the ray class field $L_1$ of $K$ of conductor $N$. Consider a pair $(\phi_1,A_1)$ where $A_1$ is an elliptic curve defined over $K_1$ with level $N$ structure and $\phi_1 : A \rightarrow A_1$ is an isogeny over $\overline{K}$. We associate to it a codimension $r + 1$ cycle on $V$

$$\Upsilon_{\phi_1} = \text{Graph}(\phi_1)^r \subset (A \times A_1)^r \simeq (A_1)^r \times A^r$$

and define a generalized Heegner cycle of conductor 1

$$\Delta_{\phi_1} = e_r \Upsilon_{\phi_1},$$

where $e_r$ is an appropriate projector (1). Then $\Delta_{\phi_1}$ is defined over $L_1$. We consider the corestriction

$$P(\phi_1) = \text{cor}_{L_1,K} \Phi_{f,\psi,L_1}(\Delta_{\phi_1}) \in H^1(K,V_{\rho}/p)$$

where $\Phi_{f,\psi,L_1}$ is the $p$-adic étale Abel-Jacobi map. The Selmer group

$$S \subseteq H^1(K,V_{\rho}/p)$$

consists of the cohomology classes which localizations at a prime $v$ of $K$ lie in

$$\begin{cases} 
H^1(K_v^\ur/\mathcal{O}_v,V_{\rho}/p) \text{ for } v \text{ not dividing } NN_A N_{\psi} p \\
H^1_f(K_v,V_{\rho}/p) \text{ for } v \text{ dividing } p 
\end{cases}$$

where $K_v$ is the completion of $K$ at $v$, and $H^1_f(K_v,V_{\rho}/p) = H^1_{\text{cris}}(K_v,V_{\rho}/p)$ is the finite part of $H^1(K_v,V_{\rho}/p)$ [5]. Note that the assumptions we make will ensure that $H^1(K_v^\ur/K_v,V_{\rho}/p) = 0$ for $v$ dividing $NN_A N_{\psi}$. We denote by $Fr(v)$ the arithmetic Frobenius element generating $\text{Gal}(K_v^\ur/K_v)$, and by $I_v = \text{Gal}(\overline{K_v}/K_v^\ur)$.

Let $h$ be the class number of $K$, and let $a_1, \ldots, a_h$ be a set of representatives of the ideal classes of $\mathcal{O}_K$. For a cohomology class $c$ in $H^1(K,V_{\rho}/p)$, we define

$$e^v = \frac{1}{h} \sum_{i=1}^h \psi^{-1}(a_i) \cdot a_i \cdot c$$
where $a_i \cdot c$ is the image of $c$ by the map $H^1(K, V_{\rho}/p) \rightarrow H^1(K, V_{\rho}/p/V_{\rho}/p[a_i])$ induced by the projection map

$$V_{\rho}/p \rightarrow V_{\rho}/p/V_{\rho}/p[a_i].$$

Then $(c^w)^{\psi} = c^w$ and it is independent of the choice of representatives $a_1, \cdots, a_n$.

**Theorem 1.1.** Let $p$ be such that

$$(p, ND\phi(N)\Lambda_N) = 1.$$

Suppose that $V_{\rho}/p$ is a simple $\text{Aut}_{L,1}(V_{\rho}/p)$-module and suppose that $\text{Gal}(\overline{L}/L)$ does not act trivially on $V_{\rho}/p$. Suppose further that the eigenvalues of $\text{Fr}(v)$ acting on $V_{\rho}^k$ are not equal to $1$ modulo $p$ for $v$ dividing $NN_A\Lambda_N$. Assume that $P(\phi_1)^w \neq 0$ in $H^1(K, V_{\rho}/p)^w$. Then the $\psi$-eigenspace of the Selmer group $S^\psi$ has rank $1$ over $\mathcal{O}_{F,\rho}/p$.

To prove Theorem 1.1 we first consider the $p$-adic étale realization of the twisted motive $V$ associated to $f$ and $\psi$ in the middle étale cohomology of the associated Kuga-Sato variety. This provides us with a $p$-adic Abel-Jacobi map that lands in the Selmer group $S$. Next, we construct an Euler system of generalized Heegner cycles which were first considered by Bertolini, Darmon and Prasanna in [4]. These algebraic cycles lie in the domain of the $p$-adic Abel Jacobi map. In order to bound the rank of the $\psi$-eigenspace of the Selmer group $S^\psi$, we apply Kolyvagin’s descent using local Tate duality, the local reciprocity law, an appropriate global pairing of $S$ and Cebotarev’s density theorem.

Our development is an adaptation of Nekovář’s techniques [17] and Bertolini and Darmon’s approach [2]. The main novelty is that the algebraic Hecke character $\psi$ is of infinite type. In particular, the Galois representation associated to $V$ is four-dimensional over $\mathcal{O}_{F,\rho}/p$.

## 2 Motive associated to a modular form and a Hecke character

In this section, we describe the construction of the four dimensional $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation

$$V_{\rho} = (V_f \otimes \mathcal{O}_{F} \otimes \mathbb{Z}_p) V_{\psi},$$

where $\rho$ is a prime of $F$ dividing $p$. Denote by $Y_1(N)$ the affine modular curve over $\mathbb{Q}$ parametrising elliptic curves with level $\Gamma_1(N)$. Let $j : Y_1(N) \rightarrow X_1(N)$ be its proper compact desingularization classifying generalized elliptic curves of level $\Gamma_1(N)$. The assumption $N \geq 5$ allows for the definition of the generalized universal elliptic curve $\pi : \mathcal{E} \rightarrow X_1(N)$. Denote by $W_r$ the Kuga-Sato variety of dimension $r + 1$, that is a compact desingularization of the $r$-fold fiber product of $\mathcal{E}$ over $X_1(N)$. We let $W$ be the $2r + 1$-dimensional variety defined by

$$W = W_r \times A^r,$$

where $A$ is as in Section 1. We denote by $[\alpha]$ the element of $\text{End}_{K_1}(A) \otimes \mathbb{Z} \otimes \mathbb{Q}$ corresponding to an element $\alpha$ of $K$. Given two non-singular varieties $X$ and $Y$ over a number field, the group of correspondences $\text{Corr}^m(X, Y) = \text{CH}^{\dim(X) + m}(X \times Y)$ is the group of algebraic cycles of codimension
\[ \dim(X) + m \text{ on } X \times Y \text{ modulo rational equivalence, (see [3] Section 2) for more details). Consider the projectors} \]
\[ e^{(1)}_A = \left( \frac{\sqrt{-D} + \sqrt{-D}}{2\sqrt{-D}} \right)^r \quad \text{and} \quad e^{(2)}_A = \left( \frac{\sqrt{-D} - \sqrt{-D}}{2\sqrt{-D}} \right)^r, \]
and
\[ e_A = e^{(1)}_A \circ e^{(2)}_A \]
in \( \mathbb{Q}[\text{End}(A)]^r \). These projectors \( e^{(1)}_A, e^{(2)}_A \) and \( e_A \) belong to the group of correspondences
\[ \text{Corr}^0(A', A')_{\mathbb{Q}} = \text{Corr}^0(A', A') \otimes \mathbb{Z} \mathbb{Q} \]
from \( A' \) to itself. Let
\[ \Gamma_r = (\mathbb{Z}/N \rtimes \mu_2)^r \rtimes \Sigma_r \]
where \( \mu_2 = \{ \pm 1 \} \) and \( \Sigma_r \) is the symmetric group on \( r \) elements. Then \( \Gamma_r \) acts on \( W_r \), (see [22] Sections 1.1.0 and 1.1.1) for more details.) The projector \( e_W \) in \( \mathbb{Z} \left[ \frac{1}{2Nr!} \right][\Gamma_r] \) associated to \( \Gamma_r \), called Scholl’s projector, belongs to the group of zero correspondences \( \text{Corr}^0(W_r, W_r)_{\mathbb{Q}} \) from \( W_r \) to itself over \( \mathbb{Q} \), (see [3] Section 2.1). Let
\[ e_r = e_W e_A, \] (1)
be the projector in the group of zero correspondences \( \text{Corr}^0(W, W)_{\mathbb{Q}} \) from \( W \) to itself over \( \mathbb{Q} \). We consider the sheafs
\[ \mathcal{F} = j_* \text{Sym}^r(R^1\pi_* \mathbb{Z}_p) \quad \text{and} \quad \mathcal{F}_A = j_* \text{Sym}^r(R^1\pi_* \mathbb{Z}_p) \otimes e_A H^r_{\text{et}}(\overline{A'}, \mathbb{Z}_p). \]

**Proposition 2.1.** The étale cohomology group
\[ H^1_{\text{et}}(X_1(N) \otimes \overline{\mathbb{Q}}, \mathcal{F}_A) \]
is isomorphic to
\[ e_r H^{2r+1}_{\text{et}}(\overline{W} \otimes \overline{\mathbb{Q}}, \mathbb{Z}_p) \]
and
\[ H^1_{\text{et}}(X_1(N) \otimes \overline{\mathbb{Q}}, \mathcal{F}) \otimes e_A H^r_{\text{et}}(\overline{A'}, \mathbb{Z}_p). \]

Also, we have
\[ e_r H^i_{\text{et}}(\overline{W} \otimes \overline{\mathbb{Q}}, \mathbb{Z}_p) = 0 \]
for all \( i \neq 2r + 1 \).

**Proof.** The proof is a combination of [22] theorem 1.2.1 and [4] proposition 2.4]. Note that the proof in [22] theorem 1.2.1 involves \( \mathbb{Q}_p \) coefficients but it is still valid in our setting, (see the Remark following [17] Proposition 2.1). \( \square \)
Let $B = \Gamma_0(N)/\Gamma_1(N)$. We define
\[
\tilde{V} = e_B H^1_\text{et}(X_1(N) \otimes \mathbb{Q}, \mathbb{F}_A)(r+1)
\]
where $e_B = \frac{1}{|B|} \sum_{b \in B} b$. Given rational primes $\ell$ coprime to $NN_A N_{\psi}$, the Hecke operators $T_\ell$ provide correspondences on $X_1(N)$ [22], inducing endomorphisms of $\tilde{V}$. Letting
\[
I = \text{Ker} \{ T_\ell \mapsto a_\ell b_\ell, \ \forall \ell \nmid NN_A N_{\psi} \},
\]
we can define the $(f, \psi)$-isotypic component of $\tilde{V}$ by
\[
V = \tilde{V}/I\tilde{V}.
\]
Hence, there is a map $m : \tilde{V} \to V$ that is equivariant under the action of Hecke operators $T_\ell$, for $\ell$ not dividing $NN_A N_{\psi}$ and under the action of the Galois group Gal$(\mathbb{Q}/\mathbb{Q})$. The $f$-isotypic component of $e_B H^1_\text{et}(X_1(N) \otimes \mathbb{Q}, \mathbb{F}_A)(r+1)$ gives rise (by extending scalars appropriately) to $V_f$ and $e_A H^1_\text{et}(\mathbb{N}, \mathbb{Z}_p)(r+1)$ gives rise to $V_\psi$. They are free $\mathcal{O}_F \otimes \mathbb{Z}_p$-modules of rank 2. Hence,
\[
V_\phi = (V_f \otimes \mathcal{O}_F \otimes \mathbb{Z}_p, V_\psi)_\phi
\]
is a four dimensional representation of Gal$(\mathbb{Q}/\mathbb{Q})$ over $\mathcal{O}_F, \phi$ with coefficients in
\[
\text{End}(A/\mathbb{Q}) = \bigoplus_{\sigma \in \text{Gal}(\mathbb{Q}/\mathbb{Q})} \text{Hom}(A, A^\sigma).
\]

3 p-adic Abel-Jacobi map

We use Proposition [24] to view the $p$-adic étale realization of the twisted motive $V$ associated to $f$ and $\psi$ in the middle étale cohomology of the associated Kuga-Sato varieties. For an integer $n$ with $(n, pNN_A N_{\psi}) = 1$, let
\[
L_n = L_1 \cdot K_n
\]
be the compositum of the ring class field $K_n$ of $K$ of conductor $n$ with the ray class field $L_1$ of $K$ of conductor $N$. Consider the $p$-adic étale Abel-Jacobi map
\[
\text{CH}^{r+1}(W/L_n)_0 \to H^1(L_n, H^{2r+1}_{\text{et}}(W \otimes \mathbb{Q}, \mathbb{Z}_p(r+1)))
\]
where $\text{CH}^{r+1}(W/L_n)_0$ is the group of homologically trivial cycles of codimension $r+1$ on $W$ defined over $L_n$, modulo rational equivalence. This map factors through $e_r(\text{CH}^{r+1}(W/L_n)_0 \otimes \mathbb{Z}_p)$ as the Abel-Jacobi map commutes with correspondences on $W$. Composing the Abel-Jacobi map with the projectors $e_r$ and $e_B$ and with $m : \tilde{V} \to V$, we obtain a map
\[
\Phi_{f, \psi, L_n} : e_r(\text{CH}^{r+1}(W/L_n)_0 \otimes \mathbb{Z}_p) \to H^1(L_n, V),
\]
which is $T[\text{Gal}(L_n/\mathbb{Q})]$-equivariant.
Conjectures and motivation. Beilinson [1, Conjecture 5.9] predicts that
\[ \dim Q e_r CH^{r+1}(W/K)_0 = \ord_{s=r+1} L(f \otimes \theta_{\psi}, s). \]

Bloch and Kato [5] conjecture that
\[ \Phi_{f,\psi,K} : e_r CH^{r+1}(W/K)_0 \otimes Q_{\mathbb{Q}} \rightarrow S \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \]
is an isomorphism. As a consequence, one expects that
\[ \dim Q_p (\text{Im}(\Phi_{f,K}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = \ord_{s=r+1} L_p(f \otimes \theta_{\psi}, s). \]

In certain settings to be relaxed in his forthcoming work, Shnidman [23, Remark X.2] relates
\[ L'(f \otimes \theta_{\psi}, r+1) \] to the complex valued height \[ \langle \cdot, \cdot \rangle_B \] of \[ \Delta_{\phi_1} \] defined by Beilinson [1], that is
\[ L'(f \otimes \theta_{\psi}, r+1) = \langle \Delta_{\phi_1}, \Delta_{\phi_1} \rangle_B \]
up to an explicit constant. However, it is worth noting that the pairing \[ \langle \cdot, \cdot \rangle_B \] is not known to be non-degenerate.

Kolyvagin’s results [14] combined with those of Gross and Zagier [12] prove the Birch and Swinnerton-Dyer conjecture for analytic rank less than or equal to 1. This is the particular case of the Beilinson-Bloch-Kato conjectures where the modular form \( f \) is associated to an elliptic curve and \( \psi \) is the trivial character. Nekovář’s results [17, 18] that correspond to the setting where \( \psi \) is trivial provide further evidence towards a \( p \)-adic analog of the Beilinson-Bloch-Kato conjecture of the form
\[ \dim Q_p (\text{Im}(\Phi_{f,K}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = \ord_{s=r+1} L_p(f \otimes \theta_{\psi}, s) \]
due to Perrin-Riou [6, section 2.8], [19]. Shnidman [23] relates the order of vanishing
\[ \ord_{s=r+1} L_p(f \otimes \theta_{\psi}, s) \]
of the \( p \)-adic \( L \)-function at \( s = r + 1 \) to the height of the image by the \( p \)-adic Abel-Jacobi map of an appropriate generalized Heegner cycle of conductor 1. In this article, we prove that
\[ \dim Q_p (\text{Im}(\Phi_{f,K}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = 1. \]

4 Generalized Heegner cycles

We describe the construction of generalized Heegner cycles following Bertolini, Darmon and Prasanna [4]. Consider pairs \((\phi_i, A_i)\) where \( A_i \) is an elliptic curve with level \( N \) structure and
\[ \phi_i : A \rightarrow A_i \]
is an isogeny over \( \mathbb{K} \). Two pairs
\[ (\phi_i, A_i), (\phi_j, A_j) \]
are said to be *isomorphic* if there is a $K$-isomorphism $\alpha : A_i \rightarrow A_j$ satisfying $\alpha \circ \varphi_i = \varphi_j$. Recall that $\mathcal{N}$ is an ideal of $\mathcal{O}_K$ such that $\mathcal{O}_K / \mathcal{N} = \mathbb{Z} / N\mathbb{Z}$. Let $\text{Isog}^K(A)$ denote the isomorphism classes of pairs $(\varphi_i, A_i)$ with $\ker(\varphi_i) \cap A[\mathcal{N}]$ trivial. For $(\varphi_i, A_i)$ in $\text{Isog}^K(A)$, we associate a codimension $r + 1$ cycle on $W$

$$\Upsilon_{\varphi_i} = \text{graph}(\varphi_i) \subset (A \times A_i)' \simeq (A_i)' \times A' \subset W_r \times A'$$

and define a *generalized Heegner cycle*

$$\Delta_{\varphi_i} = e_r \Upsilon_{\varphi_i}.$$  

We view $\text{graph}(\varphi_i)$ as a subset of $A_i \times A$. Denote by $D_{A_i}$ the element

$$(\text{graph}(\varphi_i) - 0 \times A - \deg(\varphi_i)(A_i \times 0)) \text{ in } NS(A_i \times A),$$

where $NS(A_i \times A)$ is the Néron-Severi group of $A_i \times A$. Then by an adaptation of [18, Paragraph II.3.2], we have

$$\Delta_{\varphi_i} = D_{A_i}'$$

Let us assume that the index $i$ of $A_i$ indicates that $\text{End}(A_i)$, which is an order in $\mathcal{O}_K$, has conductor $i$. Then $\Delta_{\varphi_i}$ is defined over the compositum of the abelian extension $\tilde{K}$ of $K$ over which the isomorphism class of $A$ is defined, with the ring class field $K_i$ of conductor $i$. Since $\tilde{K}$ is the smallest extension of $K_1$ over which $\text{Gal}(\tilde{K} / \tilde{K})$ acts trivially on $A[\mathcal{N}]$, it is equal to the ray class field $L_1$ of $K$ of conductor $\mathcal{N}$. Therefore, $\Delta_{\varphi_i}$ is defined over

$$L_i = L_1 K_i;$$

Hence,

$$\Delta_{\varphi_i} \text{ belongs to } \text{CH}^{r+1}(W/L_i).$$

In fact, $\Delta_{\varphi_i}$ is homologically trivial on $W$ as shown in [4, proposition 2.7]. In the rest of this section, we consider elements $(\varphi_i, A_i)$ and $(\varphi_j, A_j)$ in $\text{Isog}^K(A)$.

**Lemma 4.1.** Consider the map

$$g \times I : A_i \times A \rightarrow A_j \times A,$$

where $g : A_i \rightarrow A_j$ is an isogeny of elliptic curves and $I : A \rightarrow A$ is the identity map. Then

$$(g \times I)_* D_{A_i} = \deg(g) \frac{\deg(\varphi_i)}{\deg(\varphi_j)} D_{A_j};$$

**Proof.** We denote the intersection pairing of two divisors by a dot. We have

$$(g \times I)_* D_{A_i} \cdot (g \times I)_* D_{A_i} = \deg(g)^2 D_{A_i} \cdot D_{A_i};$$
where

\[ D_{A_i} \cdot D_{A_i} = (\text{graph}(\varphi_i) - 0 \times A - \deg(\varphi_i)A_i \times 0) \cdot (\text{graph}(\varphi_i) - 0 \times A - \deg(\varphi_i)A_i \times 0) \]

\[ = \text{graph}(\varphi_i) \cdot \text{graph}(\varphi_i) + 0 \times A \cdot 0 \times A + \deg(\varphi_i)A_i \times 0 \cdot \deg(\varphi_i)A_i \times 0 \]

\[ - 2 \text{graph}(\varphi_i) \cdot 0 \times A - 2 \text{graph}(\varphi_i) \cdot \deg(\varphi_i)A_i \times 0 + 2 \deg(\varphi_i)A_i \times 0 \cdot 0 \times A \]

\[ = 0 + 0 - 2 \deg^2(\varphi_i) - 2 \deg(\varphi_i) + 2 \deg(\varphi_i) \]

\[ = -2 \deg^2(\varphi_i). \]

In the previous computation, the equality \( \text{graph}(\varphi_i) \cdot \text{graph}(\varphi_i) = 0 \) follows from the implication

\[ (x, \varphi_i(x)) = (x, \varphi_i(x) + P) \implies P = 0 \]

for a translation of \( \varphi_i(x) \) by some quantity \( P \). Hence,

\[ (g \times I)_* D_{A_i} \cdot (g \times I)_* D_{A_i} = -2 \deg(g)^2 \deg^2(\varphi_i). \]

The induced map

\[ (g \times I)_*: NS(A_i \times A) \to NS(A_j \times A) \]

respects the subgroups generated by complex multiplication cycles. Then, since

\[ (g \times I)_* D_{A_i} = k D_{A_j}, \]

where \( A_j = g(A_i) \) and \( k > 0 \), we have

\[ (g \times I)_* D_{A_i} \cdot (g \times I)_* D_{A_i} = k^2 D_{A_j} \cdot D_{A_j} = -2k^2 \deg^2(\varphi_j). \]

The equality \(-2 \deg(g)^2 \deg^2(\varphi_i) = -2k^2 \deg^2(\varphi_j)\) then implies that

\[ k = \deg(g) \frac{\deg(\varphi_i)}{\deg(\varphi_j)}, \]

and

\[ (g \times I)_* D_{A_i} = \deg(g) \frac{\deg(\varphi_i)}{\deg(\varphi_j)} D_{A_j}. \]

\[ \square \]

## 5 Euler system properties

We study certain global and local norm compatibilities of generalized Heegner cycles satisfying the properties of Euler systems. We have \( \mathcal{O}_F \otimes \mathbb{Z}_p = \bigoplus_{p \mid \mathfrak{p}} \mathcal{O}_{F, \mathfrak{p}} \), where \( \mathcal{O}_{F, \mathfrak{p}} \) is the completion of
\( \mathcal{O}_F \) at the prime \( \wp \) dividing \( p \). Recall that \( V_\wp = (V_f \otimes \mathcal{O}_F V_\wp)_{\wp} \) where \( \wp \) is a prime of \( F \) dividing \( p \).

For a Galois representation \( V \),

\[
F(V)
\]

will designate the smallest extension of \( F \) such that \( \text{Gal}(\overline{F}/F(V)) \) acts trivially on \( V \). We denote by \( \text{Frob}_v(F_1/F_2) \) the conjugacy class of the Frobenius substitution of the prime \( v \in F_2 \) in \( \text{Gal}(F_1/F_2) \) and by \( \text{Frob}_\infty(F_1/\mathbb{Q}) \) the conjugacy class of the complex conjugation \( \tau \) in \( \text{Gal}(F_1/\mathbb{Q}) \). A rational prime \( \ell \) is called a Kolyvagin prime if

\[
(\ell, NDN_{A\mathbb{F}} p) = 1 \quad \text{and} \quad a_\ell b_\ell \equiv \ell + 1 \equiv a_\ell^2 - b_\ell^2 + 4 \equiv 0 \mod p.
\]

Let

\[
L = K(\mu_p)(V_\wp/p),
\]

where \( \mu_p \) is the group of \( p \)-th roots of unity. Condition (2) is equivalent to

\[
\text{Frob}_\ell (L/\mathbb{Q}) = \text{Frob}_\infty (L/\mathbb{Q}).
\]

Indeed, it is enough to compare the characteristic polynomial of the complex conjugation \((x^2 - 1)^2 = x^4 - 2x^2 + 1 \) acting on \( V_\wp/p \) with roots \(-1, 1\), each with multiplicity 2, with the twist by \( r + 1 \) of the characteristic polynomial of the Frobenius substitution at \( \ell \) acting on \( V_\wp/p \) with roots

\[
\alpha_1,\alpha_2, \alpha_3, \alpha_4 \quad \text{and} \quad \alpha_2 \alpha_4
\]

satisfying

\[
\alpha_1 \alpha_2 = \ell', \quad \alpha_1 + \alpha_2 = b_\ell, \quad \alpha_3 \alpha_4 = \ell^{r+1}, \quad \alpha_3 + \alpha_4 = a_\ell.
\]

The characteristic polynomial of \( \text{Frob}(\ell) \) acting on \( V_\wp/p \) is

\[
(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4)
\]

\[
= (x^2 - (\alpha_1 \alpha_3 + \alpha_1 \alpha_4)x + \alpha_1^2 \alpha_3 \alpha_4)(x^2 - (\alpha_2 \alpha_3 + \alpha_2 \alpha_4)x + \alpha_2^2 \alpha_3 \alpha_4)
\]

\[
= (x^2 - \alpha_2 \alpha_4 x + \ell^{r+1} \alpha_1^2)(x^2 - \alpha_2 \alpha_1 x + \ell^{r+1} \alpha_2^2)
\]

We use the equality \((\alpha_1 + \alpha_2)^2 = \alpha_1^2 + \alpha_2^2 + 2\alpha_1 \alpha_2\) that is \( b_\ell^2 - 2\ell' = \alpha_1^2 + \alpha_2^2 \) to conclude that the latter equals

\[
x^4 - (\alpha_2 a_\ell + \alpha_4 a_\ell) x^3 + (\ell^{r+1} \alpha_2^2 + \ell^{r+1} \alpha_2^2 + \alpha_1 \alpha_2 a_\ell^2) x^2
\]

\[
- \ell^{r+1} (\alpha_1 a_\ell \alpha_2^2 + \alpha_2 a_\ell \alpha_1^2) x + \ell^{2r+2} \alpha_1^2 \alpha_2^2
\]

\[
= x^4 - a_\ell b_\ell x^3 + (\ell^{r+1} b_\ell^2 - 2\ell^{2r+1} + a_\ell^2 \ell') x^2 - \ell^{2r+1} a_\ell (\alpha_1 + \alpha_2)x + \ell^{4r+2}
\]

\[
= x^4 - a_\ell b_\ell x^3 + (\ell^{r+1} b_\ell^2 - 2\ell^{2r+1} + a_\ell^2 \ell') x^2 - \ell^{2r+1} b_\ell a_\ell x + \ell^{4r+2}.
\]

To twist this characteristic polynomial by \( \ell^{r+1} \), it is enough to map \( x \mapsto \ell^{r+1}x \). We obtain

\[
\ell^{4r+4} x^4 - a_\ell b_\ell \ell^{3r+3} x^3 + \ell^{2r+2} (\ell^{r+1} b_\ell^2 - 2\ell^{2r+1} + a_\ell^2 \ell') x^2 - \ell^{3r+2} b_\ell a_\ell x + \ell^{4r+2}
\]

\[
= \ell^{4r+4} \left(x^4 - a_\ell b_{\ell_0} \ell^{3r+3} x^3 + \frac{\ell^{r+1} b_\ell^2 - 2\ell^{2r+1} + a_\ell^2 \ell'}{\ell^{2r+2}} x^2 - \frac{b_\ell a_\ell}{\ell^{2r+2}} x + \frac{1}{\ell^2} \right).
\]
On the one hand, using the congruences
\[ a_\ell b_\ell \equiv \ell + 1 \equiv a_\ell^2 - b_\ell^2 + 4 \equiv 0 \mod p, \]
we find that the characteristic polynomial
\[ x^4 - 2x^2 + 1 \]
of the complex conjugation \( \tau \) acting on \( V_p/p \) is congruent to the characteristic polynomial of \( \text{Frob}(\ell) \) acting on \( V_p/p \). On the other hand, comparing the action of the Frobenius element \( \text{Frob}_\ell \) and the complex conjugation \( \tau \) on \( \zeta_p \), where \( \zeta_p \) is a \( p \)-th root of unity, we obtain
\[ \zeta_p^\ell = \text{Frob}(\zeta_p) = \text{Frob}_1(\zeta_p) = \zeta_p^{-1}. \]
This implies that \( \ell \equiv -1 \mod p \). As a consequence, Condition (2) is necessary to satisfy Equality (3).

Let \( n = \ell_1 \cdots \ell_k \) be a squarefree integer where \( \ell_i \) is a Kolyvagin prime for \( i = 1, \ldots, k \). The Galois group \( \text{Gal}(K_n/K_1) \) is the product over the primes \( \ell \) dividing \( n \) of the cyclic groups
\[ G_\ell = \text{Gal}(K_\ell/K_1) \]
of order \( \ell + 1 \). We denote by \( \sigma_\ell \) a generator of \( G_\ell \). The extensions \( L_\ell \) and \( K_\ell \) are disjoint over \( K_1 \). Indeed, if \( I = L_1 \cap K_\ell \) is a non-trivial extension of \( K_1 \), then \( \text{Gal}(I/K_1) = \prod_{\ell} \text{Gal}(K_{\ell_i}/K) \) for some \( \ell_i \) dividing \( n \). This implies that the primes \( \ell_i \) ramify in \( I \) and hence also in \( L_1 \), a contradiction since \( (n,N) = 1 \). Hence
\[ G_n = \text{Gal}(L_n/L_1) \cong \text{Gal}(K_n/K_1). \]

The Frobenius condition on \( \ell \) implies that it is inert in \( K \). Denote by \( \lambda \) the unique prime in \( K \) above \( \ell \). Writing \( n = n \ell m \), we have that \( \lambda \) splits completely in \( L_m \) since it is unramified in \( L_m \) and has the same image as \( \text{Frob}_\infty(L/K) = \tau^2 = Id \) by the Artin map. A prime \( \lambda_m \) of \( L_m \) above \( \lambda \) ramifies completely in \( L_n \). We denote by \( \lambda_n \) the unique prime in \( L_n \) above \( \lambda_m \). Consider the image of \( \Delta_{\phi_n} \) by the Abel-Jacobi map
\[ \Phi_{f,\psi,L_n} : c_f(CH^{r+1}(W/L_n)0 \otimes \mathbb{Z}_p) \rightarrow H^1(L_n,V). \]

Proposition 5.1. Consider \((A_n, \varphi_n) \sim (A_m, \varphi_m) \in \text{Isog}^V(A)\) where \( n = \ell m \) for an odd prime \( \ell \). Then
\[ T_\ell \Phi_{f,\psi,L_m} (\Delta_{\phi_n}) = \text{cor}_{L_n,L_m} \Phi_{f,\psi,L_n} (\Delta_{\phi_n}) = a_\ell b_\ell \Phi_{f,\psi,L_m} (\Delta_{\phi_n}). \]

Proof. By [21, corollary 11.4],
\[ T_\ell (\Delta_{\phi_n}) = \sum_{n_i} \Delta_{\phi_{n_i}}, \]
where the generalized Heegner cycles \( \Delta_{\phi_n} \) correspond to elements
\[(A_{n_i}, \varphi_{n_i}) \sim (A_m, \varphi_m)\]
in $\text{Isog}^\nu(A)$ for elliptic curves $A_n$ that are $\ell$-isogenous to $A_m$ respecting level $N$ structure. These elliptic curves $A_n$ correspond to $gA_m$ where

$$g \in \text{Gal}(L_n/L_m) \cong \text{Gal}(K_n/K_m) \cong \text{Gal}(K_1/K_1).$$

Hence

$$\sum n_i \Delta_{\phi_n} = \sum_{g \in \text{Gal}(L_n/L_m)} g \Delta_{\phi_n} = \text{cor}_{L_n/L_m}(\Delta_{\phi_n}) = a_\ell b_\ell \Delta_{\phi_n},$$

where the last equality follows from the action of $T_\ell$ on $V$. Finally, we apply the $p$-adic Abel-Jacobi map which commutes with $T_\ell$ to obtain $T_\ell \Phi_{f,\psi,L_n}(\Delta_{\phi_n}) = \text{cor}_{L_n/L_m} \Phi_{f,\psi,L_n}(\Delta_{\phi_n}). \quad \Box$

For an element $c \in H^1(F,M)$, we denote by $\text{res}_n(c) \in H^1(F_v,M)$ the image of $c$ by the restriction map $H^1(F,M) \rightarrow H^1(F_v,M)$ induced from the inclusion

$$\text{Gal}(\overline{F}/F_v) \rightarrow \text{Gal}(\overline{F}/F).$$

**Proposition 5.2.** Consider $(A_n, \phi_n) \sim (A_m, \phi_m) \in \text{Isog}^\nu(A)$ where $n = \ell m$. Then

$$\text{res}_{\lambda_m} \Phi_{f,\psi,L_n}(\Delta_{\phi_n}) = k \text{Frob}_\ell(L_n/L_m) \text{res}_{\lambda_m} \Phi_{f,\psi,L_m}(\Delta_{\phi_m})$$

for $k = \ell \frac{\deg(\phi_i)}{\deg(\phi_j)}$.

**Proof.** Since $\lambda_m$ completely ramifies in $L_n$, we have

$$\mathfrak{q}_{L_n}/\lambda_m \cong \mathfrak{q}_{L_m}/\lambda_m,$$

which is isomorphic to the finite field with $\ell^2$ elements as $\ell$ is inert in $K$. As a consequence, the reductions of the elliptic curves $A_n$ and $A_m$ at $\lambda_m$ and $\lambda_m$ are supersingular. Hence the $\ell$-isogeny from $A_n$ to $A_m$ reduces to the Frobenius morphism $\text{Frob}_\ell$. Therefore, we have $\text{Frob}_\ell(A_m) \equiv A_n \mod \lambda_n$. By Proposition 4.1, this implies

$$(\text{Frob}_\ell \times I)_* D_{\lambda_m} \equiv k D_{\lambda_n} \mod \lambda_n$$

where $k = \ell \frac{\deg(\phi_i)}{\deg(\phi_j)}$ from which the result follows. \quad \Box

**6 Kolyvagin cohomology classes**

We denote by

$$\Phi_{f,\psi,L_n}(\Delta_{\phi_n}) \in H^1(L_n,V_{\phi_n})$$

the image of $\Phi_{f,\psi,L_n}(\Delta_{\phi_n}) \in H^1(L_n,V)$ by the map $H^1(L_n,V) \rightarrow H^1(L_n,V_{\phi_n})$ induced by the projection $V \rightarrow V_{\phi_n}$. Let

$$y_{\phi_n} = \text{red}(\Phi_{f,\psi,L_n}(\Delta_{\phi_n}) \in H^1(L_n,V_{\phi_n}/p)$$
be the image of $\Phi_{f,\psi,L_n}(\Delta_{\phi_n}) \in H^1(L_n, V_{\psi})$ by the map $H^1(L_n, V_{\psi}) \to H^1(L_n, V_{\psi}/p)$ induced by the projection $V_{\psi} \to V_{\psi}/p$. We use certain operators \(^{[4]}\) defined by Kolyvagin in order to lift the cohomology classes $\gamma_{\phi_n} \in H^1(L_n, V_{\psi}/p)$ to Kolyvagin cohomology classes $P(\phi_n) \in H^1(K, V_{\psi}/p)$, for appropriate $n$.

**Lemma 6.1.** For all $n$,

$$H^0(L_n, V_{\psi}/p) = H^0(L_1, V_{\psi}/p) = 0$$

and $\text{Gal}(L_n(V_{\psi}/p)/L_n) \cong \text{Gal}(L_1(V_{\psi}/p)/L_1)$

**Proof.** The extensions $L_n/L_1$ and $L_1(V_{\psi}/p)/L_1$ are unramified outside primes dividing $n$ and $N_\psi N_p$. Therefore, $L_n \cap L_1(V_{\psi}/p)$ is unramified over $L_1$ and is hence contained in $L_1$, the maximal unramified extension of $K$ of conductor $\mathcal{N}$. Hence,

$$H^0(L_n, V_{\psi}/p) = H^0(L_1, V_{\psi}/p).$$

The result follows from the assumption that $V_{\psi}/p$ is a simple $\text{Aut}_{L_1}(V_{\psi}/p)$-module and $\text{Gal}(L_1/L_1)$ does not act trivially on $V_{\psi}/p$. \(\square\)

**Proposition 6.2.** The restriction map

$$\text{res}_{L_1, L_n} : H^1(L_1, V_{\psi}/p) \to H^1(L_n, V_{\psi}/p)^{G_n}$$

is an isomorphism.

**Proof.** Recall that $G_n = \text{Gal}(L_n/L_1)$ and let $G = \text{Gal}(\overline{\mathbb{Q}}/L_1)$. The result follows from the inflation-restriction sequence

$$0 \to H^1(L_n/L_1, (V_{\psi}/p)^G) \xrightarrow{\text{inf}} H^1(L_1, V_{\psi}/p) \xrightarrow{\text{res}} H^1(L_n, V_{\psi}/p)^{G_n} \to H^2(L_n/L_1, (V_{\psi}/p)^G),$$

since $H^0(L_n, V_{\psi}/p) = 0$ by Lemma 6.1 \(\square\)

Let

$$\text{Tr}_\ell = \sum_{i=0}^{\ell} \sigma_i^\ell, \quad D_\ell = \sum_{i=1}^{\ell} i \sigma_i^\ell \quad \text{in } \mathbb{Z}[G_\ell]. \quad (4)$$

They are related by

$$(\sigma_\ell - 1)D_\ell = \ell + 1 - \text{Tr}_\ell. \quad (5)$$

Define

$$D_n = \prod_{\ell \mid \mathfrak{p}} D_\ell \in \mathbb{Z}[G_n].$$

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Proposition 6.3.

\[ D_n y_{\varphi_n} \in H^1(L_n, V_{\rho}/p)^{G_n}. \]

Proof. It is enough to show that for all \( \ell \) dividing \( n \),

\[ (\sigma_\ell - 1)D_n y_{\varphi_n} = 0. \]

We have

\[ (\sigma_\ell - 1)D_n = (\sigma_\ell - 1)D_lD_m = (\ell + 1 - \text{Tr}_\ell)D_m, \]

where the last equality follows by Relation (5). Since \( \text{res}_{L_m,L_n} \circ \text{cor}_{L_n,L_m} = \text{Tr}_\ell \),

\[ (\ell + 1 - \text{Tr}_\ell)D_m = (\ell + 1)D_m \text{red}_{L_m}(\Phi_{f,\varphi,L_m}(\Delta_{\varphi_m})). \]

As a consequence, the cohomology classes \( D_n y_{\varphi_n} \in H^1(L_n, V_{\rho}/p)^{G_n} \) can be lifted to cohomology classes \( c(\varphi_n) \in H^1(L_1, V_{\rho}/p) \) such that

\[ \text{res}_{L_1,L_n} c(\varphi_n) = D_n y_{\varphi_n}. \]

We define

\[ P(\varphi_n) = \text{cor}_{L_1,K} c(\varphi_n) \text{ in } H^1(K, V_{\rho}/p). \]

For a place \( v \) of \( K \) and a cohomology class \( c \) in \( H^1(K, V_{\rho}/p) \), we denote by \( \text{res}_v(c) \) the image of \( c \) by the map \( H^1(K, V_{\rho}/p) \to H^1(K_v, V_{\rho}/p) \) induced by the inclusion \( G_{K_v} \hookrightarrow G_K \).

Proposition 6.4. Let \( v \) be a prime of \( L_1 \).

1. If \( v \vert N_A N y_{\varphi} N \), then \( \text{res}_v(P(\varphi_n)) \) is trivial.

2. If \( v \nmid N_A N y_{\varphi} N p \), then \( \text{res}_v(P(\varphi_n)) \) lies in \( H^1(K_v^w, V_{\rho}/p) \).

Proof. 1. We denote by

\[ V_{\rho}/p^{\text{dual}} = \text{Hom}(V_{\rho}/p, \mathbb{Z}/p\mathbb{Z}(1)) \]

the local Tate dual of \( V_{\rho}/p \). The local Euler characteristic formula \([16]\) Section 1.2] yields

\[ |H^1(K_v, V_{\rho}/p)| = |H^0(K_v, V_{\rho}/p)| \times |H^2(K_v, V_{\rho}/p)|. \]

Local Tate duality then implies

\[ |H^1(K_v, V_{\rho}/p)| = |H^0(K_v, V_{\rho}/p)| \times |H^0(K_v, V_{\rho}/p^{\text{dual}})| = |H^0(K_v, V_{\rho}/p)|^2 \]
as $V_{\rho}/p$ is self-dual. The Weil conjectures and the assumption on $Fr(v)$ imply that

$$(V_{\rho}/p)^{Fr(v)} = 0$$

where $< Fr(v) > = \text{Gal}(K^{ur}_v/K_v)$ and $I = \text{Gal}(\overline{K_v}/K^{ur}_v)$ is the inertia group. Therefore,

$$H^0(K_v,V_{\rho}/p) = ((V_{\rho}/p)^{Fr(v)}) = 0.$$

2. If $v$ does not divide $N_\Lambda N_\Psi N_p$, then

$$\text{res}_{L_{1,v},L_{n,v}} c(\phi_h) = \text{res}_v D_n y_n \in H^1(\overline{L_{n,v'}}/L_{n,v}, V_{\rho}/p)$$

for $v'$ above $v$ in $L_n$. The exact sequence

$$\cdots \rightarrow H^1(L_{n,v'/L_{n,v}}, (V_{\rho}/p)^I) \rightarrow H^1(L_{n,v'}, V_{\rho}/p) \rightarrow \text{res}_{V} H^1(\overline{L_{n,v'}/L_{n,v}}, V_{\rho}/p) \rightarrow \cdots$$

allows us to view the cohomology class $\text{res}_v D_n y_n$ that belongs to $\text{Ker}(\text{res})$ as an element in

$$H^1(\overline{L_{n,v'}/L_{n,v}}, V_{\rho}/p).$$

Since $v$ is unramified in $L_n/L_1$, then $\text{res}_v c(\phi_h)$ belongs to $H^1(\overline{L_{1,v'}/L_{1,v}}, V_{\rho}/p)$.

\[\square\]

7 Global extensions by Kolyvagin classes

We consider the restriction $d$ of an element $c$ of $H^1(K,V_{\rho}/p)$ to $H^1(L,V_{\rho}/p)$. Then $d$ factors through some finite extension $\tilde{L}$ of $L$. We denote by

$$L(c) = \tilde{L}^{\ker(d)}$$

the subfield of $\tilde{L}$ fixed by $\ker(d)$. Note that $L(c)$ is an extension of $L$. In this section, we study extensions of $L$ by Kolyvagin cohomology classes $c$ and $P(\phi_\eta)$, where $P(\phi_\eta)$ will play a crucial role in the proof of Theorem 11. We recall the statement of the theorem.

Theorem 11. Let $p$ be such that

$$(p, ND\phi(N)N_\Lambda) = 1.$$ 

Suppose that $V_{\rho}/p$ is a simple $\text{Aut}_{L_1}(V_{\rho}/p)$-module and suppose that $\text{Gal}(\overline{L_1}/L_1)$ does not act trivially on $V_{\rho}/p$. Suppose further that the eigenvalues of $Fr(v)$ acting on $V_{\rho}^L$ are not equal to 1 modulo $p$ for $v$ dividing $NN_\Lambda N_\Psi$. If $P(\phi_\eta)^\Psi$ is non-zero in $H^1(K,V_{\rho}/p)^\Psi$, then the $\Psi$-eigenspace of the Selmer group $S^\Psi$ has rank 1 over $\delta_{F,\rho}/p$.

Recall that $L = K(\mu_p)(V_{\rho}/p)$. 

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Lemma 7.1. We have
\[ H^1(\text{Aut}_K(V_{\rho}/p), V_{\rho}/p) = H^1(L/K, V_{\rho}/p) = 0. \]

Proof. First note that if \( p \nmid |\text{Aut}_K(V_{\rho}/p)| \), then
\[ H^1(\text{Aut}_K(V_{\rho}/p), V_{\rho}/p) = 0. \]
If \( p \) divides \(|\text{Aut}_K(V_{\rho}/p)|\), then since \( V_{\rho}/p \) is irreducible as an \( \text{Aut}_K(V_{\rho}/p) \)-module, Dickson’s lemma [24, Theorem 6.21] implies that \( \text{Aut}_K(V_{\rho}/p) \) contains \( \text{SL}_2(F_q) \) for some \( q \). In particular, it contains \( 2I \) where \( I \) is the identity map. Sah’s lemma [15, 8.8.1] states that if \( G \) is a group, \( M \) a \( G \)-representation, and \( g \) an element of Center(\( G \)), then the map \( x \mapsto (g^{-1})x \) is the zero map on \( H^1(G,M) \). Therefore, by Sah’s lemma, the map \( x \mapsto (2I-I)x = Ix \) is the zero map on \( H^1(\text{Aut}_K(V_{\rho}/p), V_{\rho}/p) \) implying that \( H^1(\text{Aut}_K(V_{\rho}/p), V_{\rho}/p) = 0 \). As a consequence, \( H^1(K(\mu_p), V_{\rho}/p) = 0 \) since \( p \) does not divide \(|\text{Gal}(K(\mu_p)/K)| = p-1 \).

We denote the Galois group \( \text{Gal}(L/K) \) by \( G \). The restriction map
\[ r : H^1(K, V_{\rho}/p) \rightarrow H^1(L, V_{\rho}/p)^G = \text{Hom}_G(\text{Gal}(\overline{Q}/L), V_{\rho}/p) \]
has kernel
\[ \text{Ker}(r) = H^1(L/K, V_{\rho}/p) = 0 \]
by Lemma 7.1. Consider the evaluation pairing
\[ [ , ] : r(S) \times \text{Gal}(\overline{Q}/L) \rightarrow V_{\rho}/p. \] (6)
We denote by \( \text{Gal}_S(\overline{Q}/L) \) the annihilator of \( r(S) \). Let \( L^S \) be the extension of \( L \) fixed by \( \text{Gal}_S(\overline{Q}/L) \) and \( G_S \) the Galois group \( \text{Gal}(L^S/L) \).

Remark 7.2. The element \( P(\varphi_1) \) belongs to \( S \) by Proposition 6.4. Also, \( L(P(\varphi_1)) \) is a subfield of \( L^S \). Indeed, assume \( \rho \in \text{Gal}_S(\overline{Q}/L) \), then \([s, \rho] = 0 \) for all \( s \in S \). Hence, \( P(\varphi_1) \) defines a cocycle of \( S \) by
\[ \rho \mapsto \rho(P(\varphi_1)) - P(\varphi_1) = 0. \]
This implies that \( \rho \) fixes \( L(P(\varphi_1)) \), a subfield of \( L^S \).

Proposition 7.3. There is a Kolyvagin prime \( q \) such that
\[ \text{res}_\beta P(\varphi_1)^w \neq 0, \]
where \( \beta \) is a prime dividing \( q \) in \( K \).
Proof. By Ceboratév’s density theorem, there is a Kolyvagin prime \( q \) such that

\[
\text{Frob}_q(L^S / \mathbb{Q}) = \tau h, \quad h^{1+\tau} \notin \text{Gal}(L^S / L(P(\varphi_1)^\psi))
\]

for some \( h \) in \( \text{Gal}(L^S / L) \). The restriction of \( \tau h \) to \( L \) is indeed \( \tau \). Let \( \beta \) be a prime of \( L \) above \( q \). Since \( \text{Frob}_\beta(L(P(\varphi_1)^\psi) / L) = (\tau h)^2 \) does not fix \( P(\varphi_1)^\psi \), \( \beta \) is not in the kernel of the Artin map. Hence \( \beta \) does not split completely in \( L(P(\varphi_1)^\psi) \). Therefore, \( \text{res}_\beta P(\varphi_1)^\psi \neq 0 \). \( \square \)

Consider the following extensions

\[
\begin{align*}
H_0 &= H_1 H_2 \\
H_1 &= L(P(\varphi_1)) & v_0 & H_2 = L(P(\varphi_q)) \\
L &= K(\mu_p)(V_{\wp}/p)
\end{align*}
\]

where \( q \) is a Kolyvagin prime satisfying

\[
\text{res}_\beta P(\varphi_1)^\psi \neq 0
\]

for some \( \beta \) in \( L \) above \( q \) (see Proposition (7.3)). We define

\[
V_i = \text{Gal}(H_i/F) \quad \text{for } i = 0, 1, 2.
\]

An automorphism of \( L(P(\varphi_1))/L \) corresponds to the multiplication of the generators of \( L(P(\varphi_1)) \) over \( L \) by an element of \( (V_\wp)_p \simeq V_\wp/p \). Hence

\[
V_1 = \text{Gal}(L(P(\varphi_1))/L) \simeq V_\wp/p.
\]

Similarly, we have \( V_2 \simeq V_\wp/p \). We recall that \( h \) is the class number of \( K \), and \( a_1, \cdots, a_h \) are a set of representatives of the ideal classes of \( \mathcal{O}_K \). For a cohomology class \( c \) in \( H^1(K, V_\wp/p) \), we defined

\[
c^\psi = \frac{1}{h} \sum_{i=1}^{h} \psi^{-1}(a_i) a_i \cdot c,
\]

where \( a_i \cdot c \) is the image of \( c \) by the map \( H^1(K, V_\wp/p) \rightarrow H^1(K, V_\wp/p/V_\wp/p [a_i]) \) induced by the projection map

\[
V_\wp/p \rightarrow V_\wp/p/V_\wp/p [a_i].
\]
Note that $c^\psi$ is independent of the choice of representatives $a_1, \ldots, a_h$. Furthermore, $(c^\psi)^\psi = c^\psi$ lies in $H^1(K, V_{\phi}/p)^\psi$. We denote by
\[
H_1^\psi = L(P(\phi_1)^\psi), \quad H_2^\psi = L(P(\phi_q)^\psi), \quad H_2^\psi = L(P(\phi_q)^\psi),
\]
and we let
\[
V_i^\psi = \text{Gal}(H_i^\psi/H_i), \quad V_i^\psi = \text{Gal}(H_i^\psi/H_i), \quad \text{for } i = 1, 2.
\]
We will show that
\[
H_1^\psi \cap H_2^\psi = L.
\]

**Lemma 7.4.** There is an isomorphism of $\mathcal{O}_{F,\psi}/p$-modules
\[
H^1(K_{\lambda}^u/K_{\lambda}, V_{\phi}/p) \rightarrow H^1(K_{\lambda}^u, V_{\phi}/p)
\]
mapping $\text{res}_\lambda P(\phi_m)$ to $\text{res}_\lambda P(\phi_m)$. Also, $\text{res}_\lambda P(\phi_1)$ is ramified.

**Proof.** This is an adaptation of [8, Section 5] that uses the properties of the Euler system considered in Proposition 5.1 and Proposition 5.2.

Recall that $\tau$ denotes complex conjugation.

**Proposition 7.5.** The extensions $H_1^\psi$ and $H_2^\psi$ are linearly disjoint over $L$.

**Proof.** Linearly independent cocycles $c_1$ and $c_2$ of $H^1(K, V_{\phi}/p)$ over $\mathcal{O}_{F,\psi}/p$ can be viewed by Lemma (7.1) as linearly independent homomorphisms $h_1$ and $h_2$ in $\text{Hom}_{\text{Gal}(L/K)}(L, V_{\phi}/p)$ over $\mathcal{O}_{F,\psi}/p$. Linearly independent homomorphisms $h_1$ and $h_2$ induce linearly disjoint extensions $\mathcal{I}_{\text{Ker}(h_1)}$ and $\mathcal{I}_{\text{Ker}(h_2)}$ of $L$. Hence, if $H_1$ and $H_2$ were not linearly disjoint over $L$, we would have that
\[
u_1 c_1 + \nu_2 c_2 = 0, \quad \text{for some } \nu_1, \nu_2 \text{ in } (\mathcal{O}_{F,\psi}/p)^*,
\]
where
\[
c_1 = F(\phi_1)^\psi \quad \text{and} \quad c_2 = F(\phi_q)^\psi.
\]
Lemma (7.3) implies that $\text{res}_\beta P(\phi_q) = \text{res}_\beta P(\phi_1)$ is ramified for $\beta \in K$ dividing $q$ and provides an isomorphism sending $\text{res}_\beta P(\phi_q)$ to $\text{res}_\beta P(\phi_q)$. The latter implies by Proposition (7.3) that $\text{res}_\beta P(\phi_q)$ is non-zero. This is inconsistent with the equality $u_1 c_1 + u_2 c_2 = 0$ since the nontrivial cocycles $c_1, c_2$ have different ramification properties.

## 8 Complex conjugation

We study the action of complex conjugation on the image by the $p$-adic Abel-Jacobi map of generalized Heegner cycles. Recall that $D_{A_j}$ is the element
\[
(\text{graph}(\phi_j) - 0 \times A - \text{deg}(\phi_j)(A_j \times 0)) \text{ in } \text{NS}(A_j \times A).
\]
In this section, we write $D_{A_j, \phi_j}$ instead of $D_{A_j}$ alone in order to keep track of the underlying map $\phi_j$. 

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Lemma 8.1. There is an element $\sigma$ in $\text{Gal}(K_j/K)$ such that

$$\tau \Phi(\Delta_{\phi_j})_{\rho} = (-1)^{\frac{\epsilon_L}{2}} \epsilon_L N^{r/2} \text{deg}^{-r}(\sigma) \sigma \Phi(\Delta_{\phi_j})_{\rho},$$

where $\epsilon_L$ is the sign of the functional equation of $L(f,s)$.

Proof. We have that $(\tau \times I)_*(D_{A_j,\phi_j}) = D_{\tau(A_j),\Phi_j}$. Article [10] shows that $\tau A_j = W_N(\sigma A_j)$ for some $\sigma$ in $G(K_j/K)$, hence

$$(\tau \times I)_*(D_{A_j,\phi_j}) = D_{\tau(A_j),\Phi_j} = D_{W_N(\sigma A_j),W_N(\sigma \circ \phi_j)}.$$  

Consider the map

$$W \times I : W_N \times A \rightarrow W_N \times A : ((E,P),A) \rightarrow ((E/\langle P \rangle,P'),A),$$

where $P'$ is such that the Weil pairing $<P,P'>$ of $P$ with $P'$ satisfies $<P,P'> = \zeta_N$ for some choice $\zeta_N$ of an $N$-th root of unity $\zeta_N$. Note that $W$ has degree $N$. Also,

$$W,f(\tau)d_\tau d_\zeta = (-1)^{\frac{\epsilon_L}{2}} \epsilon_L N^{r/2} f(\tau)d_\tau d_\zeta.$$  

This implies as in [17] Proposition 6.2] that

$$(W \times I)_* D_{W_N(\sigma A_j),W_N(\sigma \circ \phi_j)} = (-1)^{\frac{\epsilon_L}{2}} \epsilon_L N^{r/2} D_{W_N(\sigma A_j),W_N(\sigma \circ \phi_j)},$$

while Proposition [3.1] implies that the former equals $N^{r} \frac{\text{deg}^r(\phi_j)}{\text{deg}^r(\phi_j)} D_{\sigma A_j,\sigma \circ \phi_j}$. Hence,

$$D_{W_N(\sigma A_j),W_N(\sigma \circ \phi_j)} = (-1)^{\frac{\epsilon_L}{2}} \epsilon_L k_1 N^{r/2} D_{\sigma A_j,\sigma \circ \phi_j},$$

where $k_1 = \frac{\text{deg}^r(\phi_j)}{\text{deg}^r(\phi_j)}$. Applying Proposition [4.1] to the map $(\sigma \times I)$, we obtain

$$(\sigma \times I)_* (D_{A_j,\phi_j}) = k_2 D_{\sigma A_j,\sigma \circ \phi_j},$$

where $k_2 = \frac{\text{deg}^r(\sigma)}{\text{deg}^r(\phi_j)} = \text{deg}^r(\sigma) k_1$. Hence,

$$(\tau \times I)_* (D_{A_j,\phi_j}) = D_{W_N(\sigma A_j),W_N(\sigma \circ \phi_j)} = (-1)^{\frac{\epsilon_L}{2}} \epsilon_L N^{r/2} \text{deg}^{-r}(\sigma)(\sigma \times I)_*(D_{A_j,\phi_j}).$$

Therefore

$$\tau \Phi(\Delta_{\phi_j})_{\rho} = (-1)^{\frac{\epsilon_L}{2}} \epsilon_L N^{r/2} \text{deg}^{-r}(\sigma) \sigma \Phi(\Delta_{\phi_j})_{\rho}. \qed$$
Lemma 8.2. Let $\varepsilon = (-1)^{r_2} \varepsilon_k$ and $k = \deg^{-1}(\sigma) N^{r/2}$. Then

$$\tau P(\varphi_1)^\psi = \varepsilon k \sigma P(\varphi_1)^\overline{\psi} \text{ and } \tau P(\varphi_q)^\psi = -\varepsilon k \sigma P(\varphi_q)^\overline{\psi}.$$  

Proof. For an element $c$ in $H^1(K, V_{\psi}/p)$, we have that

$$\tau \cdot c^\psi = \frac{1}{h} \sum_{i=1}^{h} \overline{\psi}^{-1}(a_i) \tau \cdot (a_i \cdot c) = \frac{1}{h} \sum_{i=1}^{h} \overline{\psi}^{-1}(a_i) \tau (a_i \cdot c)\tau^{-1} = \frac{1}{h} \sum_{i=1}^{h} \overline{\psi}^{-1}(a_i) a_i \cdot \tau c \tau^{-1} = (\tau c)^\overline{\psi}.$$  

For $c = P(\varphi_j)$ and $\sigma$ in $\text{Gal}(K_j/K)$, we have that

$$\sigma \cdot c^\psi = \frac{1}{h} \sum_{i=1}^{h} \psi^{-1}(a_i) \sigma \cdot (a_i \cdot c) = \frac{1}{h} \sum_{i=1}^{h} \psi^{-1}(a_i) \sigma (a_i \cdot c)\sigma^{-1} = \frac{1}{h} \sum_{i=1}^{h} \psi^{-1}(a_i) a_i \cdot \sigma c \sigma^{-1} = (\sigma c)^\psi.$$  

Hence

$$\tau P(\varphi_1)^\psi = (\tau P(\varphi_1))^\overline{\psi} = \varepsilon k \sigma P(\varphi_1)^\overline{\psi}.$$  

The identity $\tau D_q = -D_q \tau$ indicates that

$$\tau P(\varphi_q)^\psi = (\tau P(\varphi_q))^\overline{\psi} = -\varepsilon k \sigma P(\varphi_q)^\overline{\psi}.$$  

\[\square\]

For $(v_{11}, v_{12}, v_{21}, v_{22})$ in $V_1^\psi V_2^\psi V_1^\psi V_2^\psi$, we define

$$\tau(v_{11}, v_{12}, v_{21}, v_{22}) = (\varepsilon k \sigma \tau v_{12}, \varepsilon k \sigma \tau v_{11}, -\varepsilon k \sigma \tau v_{22}, -\varepsilon k \sigma \tau v_{21}).$$

We define

$$U_0 = \{(v_{11}, v_{12}, v_{21}, v_{22}) \mid \varepsilon k \sigma \tau v_{1i} + v_{1i}, -\varepsilon k \sigma \tau v_{2j} + v_{2j}, i, j = 1, 2 \text{ generate } V_{\psi}/p\}.$$  

Then $U_0^+$ generates $V_0^+$. We let

$$L(U_0) = \{\ell \text{ rational prime } \mid \text{Frob}_\ell(H_0/\mathbb{Q}) = [\tau u, u \in U_0]\}.$$  

9 Local to Global study

Local Tate duality. Let $K_\lambda$ be a local field with residue field $F_q$ and let $A$ be a finite group with an unramified action of $\text{Gal}(K_\lambda^s/K_\lambda)$ killed by a prime $p$. Assume $p$ divides $q - 1$ so that $\mu_p \subset K_\lambda$ and let $A' = \text{Hom}(A, \mu_p)$. We denote by $K_\lambda^p$, the maximal tamely ramified extension of $K_\lambda$, and by $H^1_{ur}(K_\lambda, *)$, the group $H^1(K_\lambda^p/K_\lambda, *)$. The natural pairing $A \times A' \rightarrow \mu_p$ yields the cup product pairing

$$H^1(K_\lambda, A) \times H^1(K_\lambda, A') \rightarrow H^2(K_\lambda, \mu_p) = \mathbb{Z}/p\mathbb{Z}$$.
which induces a perfect local Tate pairing

\[ H^1(K_{\lambda}^{ur}/K_{\lambda}, A) \times H^1(K_{\lambda}', A') / H^1(K_{\lambda}^{ur}/K_{\lambda}, A) \rightarrow \mathbb{Z}/p\mathbb{Z}. \]

Let

\[ \alpha : H^1(K_{\lambda}^{ur}/K_{\lambda}, A) \xrightarrow{\sim} A/(\phi - 1)A \]

be the evaluation map at the Frobenius element \( \phi \) where

\[ Gal(K_{\lambda}^{ur}/K_{\lambda}) = \langle \phi \rangle. \]

Then \( \alpha \) is an isomorphism. The exact sequence of Galois groups

\[ 0 \rightarrow Gal(K_{\lambda}/K_{\lambda}') \rightarrow Gal(K_{\lambda}^{ur}/K_{\lambda}') \rightarrow Gal(K_{\lambda}^{ur}/K_{\lambda}) \rightarrow 0 \]

induces the exact sequence

\[ H^1(K_{\lambda}'/K_{\lambda}^{ur}, A') \rightarrow H^1(K_{\lambda}^{ur}, A') \rightarrow H^1(K_{\lambda}'/K_{\lambda}^{ur}, A') \rightarrow 0, \]

where \( H^1(K_{\lambda}', A') = 0 \) since \( Gal(K_{\lambda}/K_{\lambda}') \) is a pro-\( q \) group. Therefore,

\[ H^1(K_{\lambda}^{ur}, A') \simeq H^1(K_{\lambda}'/K_{\lambda}^{ur}, A') \simeq \text{Hom}(\mathbb{Z}/p\mathbb{Z}(1), A') \simeq \text{Hom}(\mu_p, A'). \]

Hence we have an isomorphism

\[ H^1(K_{\lambda}^{ur}, A') \xrightarrow{\sim} \text{Hom}(\mu_p, A'). \]

The exact sequence of Galois cohomology groups

\[ 0 \rightarrow H^1(K_{\lambda}^{ur}/K_{\lambda}, A') \rightarrow H^1(K_{\lambda}, A') \rightarrow H^1(K_{\lambda}^{ur}/K_{\lambda}, A')^\phi \rightarrow 0 \]

allows us to identify \( H^1(K_{\lambda}, A')/H^1(K_{\lambda}^{ur}/K_{\lambda}, A') \) with

\[ H^1(K_{\lambda}^{ur}, A')^\phi \simeq \text{Hom}(\mu_p, A')^\phi. \]

Hence, we obtain a perfect local pairing

\[ \langle \cdot, \cdot \rangle_\lambda : H^1(K_{\lambda}^{ur}/K_{\lambda}, A) \times H^1(K_{\lambda}^{ur}, A')^\phi \rightarrow \mathbb{Z}/p\mathbb{Z}. \]

**Set up of the proof.** Given a Kolyvagin prime \( \ell \), the Frobenius condition implies that it is inert in \( K \). We denote by \( \lambda \) the prime of \( K \) lying above \( \ell \). We have a perfect local pairing

\[ \langle \cdot, \cdot, \cdot \rangle_\ell : H^1(K_{\lambda}^{ur}/K_{\lambda}, (V_{\ell}/p)^{I_{\lambda}}) \times H^1(K_{\lambda}^{ur}/V_{\ell}/p) \rightarrow \mathbb{Z}/p, \]

where \( I_{\lambda} = Gal(K_{\lambda}/K_{\lambda}') \) and \( \mathcal{O}_{F, \ell} \)-linear isomorphisms

\[ \{H^1(K_{\lambda}^{ur}/V_{\ell}/p)^{I_{\lambda}} \}_{\text{dual}} \simeq H^1(K_{\lambda}'/K_{\lambda}, (V_{\ell}/p)^{I_{\lambda}}) \simeq (V_{\ell}/p)^{I_{\lambda}}/(\phi - 1), \quad (7) \]
where $\phi$ generates $\text{Gal}(K^\text{ur}/K_{\lambda})$. We denote by
\[
\text{res}_{\lambda} : H^1(K, V_{\wp}/p) \longrightarrow H^1(K_{\lambda}, V_{\wp}/p)
\]
the restriction map from $H^1(K, V_{\wp}/p)$ to $H^1(K_{\lambda}, V_{\wp}/p)$ induced by the embedding $\text{Gal}(K_{\lambda}/K_{\lambda}) \hookrightarrow \text{Gal}(K/K)$.

Restricting the domain of $\text{res}_{\lambda}$ to the Selmer group $S$, we obtain
\[
\text{res}_{\lambda} : S \longrightarrow H^1(K^\text{ur}_{\lambda}/K_{\lambda}, (V_{\wp}/p)^{I_{\lambda}}).
\]
Taking the $\mathbb{Z}/p\mathbb{Z}$-dual of this map, we obtain a homomorphism
\[
\omega_\ell : H^1(K^\text{ur}_{\lambda}, V_{\wp}/p) \longrightarrow S^{\text{dual}}.
\]
We denote
\[
X_\ell = \omega_\ell(H^1(K^\text{ur}_{\lambda}, V_{\wp}/p))
\]
in $S^{\text{dual}}$. We aim to bound the size of $S^{\text{dual}, \psi}$ by studying the set
\[
\{X_\psi^\ell\}_{\ell \in L(U_0)}.
\]

**Proposition 9.1.** The modules $\{X_\ell\}_{\ell \in L(U_0)}$ generate $S^{\text{dual}}$.

**Proof.** Let $G = \text{Gal}(L/K)$. Consider an element $s$ of $S$. The restriction map
\[
H^1(K, V_{\wp}/p) \overset{\text{res}}{\longrightarrow} H^1(L, V_{\wp}/p)^G
\]
is injective by Lemma (7.1) as
\[
\text{Ker}(\text{res}) = H^1(L/K, V_{\wp}/p) = 0.
\]
We identify $s$ with its image by restriction in
\[
H^1(L, V_{\wp}/p)^G \subset \text{Hom}_G(L, V_{\wp}/p).
\]
We will show that
\[
\{\text{res}_\ell\}_{\ell \in L(U_0)} : S \longrightarrow \{H^1(K^\text{ur}_{\lambda}/K_{\lambda}, (V_{\wp}/p)^{I_{\lambda}})\}_{\ell \in L(U_0)}
\]
is injective. As a consequence, the induced map between the duals
\[
\{H^1(K^\text{ur}_{\lambda}, V_{\wp}/p)\}_{\ell \in L(U_0)} \longrightarrow S^{\text{dual}}
\]
is surjective. Hence, it is enough to show that $\text{res}_\ell(s) = 0$ for all $\ell \in L(U_0)$ implies $s = 0$. Consider $\overline{H}_0$, the minimal Galois extension of $\mathbb{Q}$ containing $H_0$ such that $s$ factors through $\text{Gal}(\overline{H}_0/L)$. Let
\( x \) be an element of \( \text{Gal}(\overline{H}_0/L) \) such that \( x|_{\overline{H}_0} \) belongs to \( U_0 \). By Cebotarev’s density theorem, there exists \( \ell \) in \( L(U_0) \) such that \( \text{Frob}_\ell(\overline{H}_0/\mathbb{Q}) = [\tau x] \). The hypothesis \( \text{res}_\lambda(s) = 0 \) implies that \( s(\text{Frob}_\ell(\overline{H}_0/L)) = 0 \) for \( \lambda_L \) above \( \ell \) in \( L \) since \( \text{Frob}_\ell(\overline{H}_0/L) \) is a generator of \( \text{Gal}(\overline{H}_0,\lambda_{\overline{H}_0}/L_{\lambda_L}) \), where \( \lambda_{\overline{H}_0} \) is above \( \lambda_L \) in \( \overline{H}_0 \). In fact,

\[
\text{Frob}_\ell(\overline{H}_0/L) = (\tau x)^{|D(L/\mathbb{Q})|} = (\tau x)^2 = x^\tau x = (x^+)^2,
\]

where \( |D(L/\mathbb{Q})| \) is the order of the decomposition group \( D(L/\mathbb{Q}) \), also the order of the residue extension. Therefore, \( s(x^+) = 0 \) for all \( x \in \text{Gal}(\overline{H}_0/L) \) such that \( x|_{\overline{H}_0} \) belongs to \( U_0 \). Since \( U_0^+ \) generates \( V_0^+ \), we have that \( s \) vanishes on \( \text{Gal}(\overline{H}_0/L)^+ \). Hence, \( \text{Im}(s) \) lies in \( V_0/p^{-} \), the minus eigenspace of \( V_{p^2}/p \) for the action of \( \tau \). In particular, it cannot be a proper \( G \)-submodule of \( V_{p^2}/p \).

\[ \square \]

**Proposition 9.2.** The elements

\[
\text{res}_\lambda P(\phi_1)^\psi \text{ and } \text{res}_\lambda P(\phi_2)^\psi
\]

generate \( H^1(K_{\lambda}^\psi, V_{p^2}/p)^\psi \).

**Proof.** We have

\[
H^1(K_{\lambda}^\psi, V_{p^2}/p) \simeq V_{p^2}/p(K_{\lambda})
\]

by (7). The module \( V_{p^2}/p(K_{\lambda})^\psi \) is of rank 2 over \( \mathcal{O}_{F,p}/p \), hence, so is \( H^1(K_{\lambda}^\psi, V_{p^2}/p)^\psi \). The Frobenius condition on \( \ell \) implies that

\[
g_1 = \text{res}_\lambda P(\phi_1)^\psi \text{ and } g_2 = \text{res}_\lambda P(\phi_2)^\psi
\]

are linearly independent in \( H^1(K_{\lambda}^\psi, V_{p^2}/p)^\psi \) over \( \mathcal{O}_{F,p}/p \). Indeed, if they were linearly dependent, then

\[
g_1^{(\tau x_1)^2} - g_1, \text{ and } g_2^{(\tau x_2)^2} - g_2
\]

where \( \text{Frob}_\ell(H_0/\mathbb{Q}) = \tau u = (\tau x_1, \tau x_2, \tau x_2, \tau x_2) \) would also be linearly dependent. The Frobenius condition implies that

\[
\{\text{Frob}_\ell(H_1/L) = x_1, x_2 = (\tau x_1)^2, \ i = 1, 2\}
\]

generate \( (V_{p^2}/p) \), which yields a contradiction since \( (\tau x_1)^2 \) acts on \( \text{res}_\lambda P(\phi_1)^\psi \) which generates \( H_1^1 \) by \( g_1^{(\tau x_1)^2} - g_1 \) and \( (\tau x_2)^2 \) acts on \( \text{res}_\lambda P(\phi_2)^\psi \) which generates \( H_1^1 \) by \( g_2^{(\tau x_2)^2} - g_2 \). Therefore, by Lemma (7.4), \( \text{res}_\lambda P(\phi_1)^\psi \) and \( \text{res}_\lambda P(\phi_2)^\psi \) are linearly independent in \( H^1(K_{\lambda}^\psi, V_{p^2}/p)^\psi \) over \( \mathcal{O}_{F,p}/p \).

\[ \square \]
10 Reciprocity Law and Local Triviality

In this section, we use the local reciprocity law as well as the local properties of the Kolyvagin cohomology classes \( P(\varphi_n) \) to study the modules \( X_\ell \) for \( \ell \in L(U_0) \).

**Proposition 10.1.** We have
\[
\sum_{\lambda \mid \ell \mid n} <s_{\lambda}^{\psi}, \text{res}_\lambda P(\varphi_n)^{\psi}>_{\lambda} = 0.
\]

**Proof.** The proof follows [17, proposition 11.2(2)] where both the reciprocity law and the ramification properties of \( P(\varphi_n) \) in proposition 6.4 are used.

**Proposition 10.2.** 1. The element \( \omega_\ell(\text{res}_\lambda P(\varphi^{\psi}_\ell)) \) vanishes in \( X^{\psi}_\ell \) for \( \ell \in L(U_0) \).
2. The modules \( \{X^{\psi}_\ell\}_{\ell \in L(U_0)} \) are generated over \( O_{F, p} / p \) by \( \omega_q(\text{res}_\beta P(\varphi^{\psi}_{\ell q})) \).
3. The module \( S^{\psi} \) is of rank 1 over \( O_{F, p} / p \).

**Proof.** 1. Recall that \( \lambda \) is the prime above \( \ell \) in \( K \). The image of \( \text{res}_\lambda P(\varphi^{\psi}_\ell) \) by the map \( \omega_\ell : H^1(K^{ur}_\lambda, V_{p, \ell}/p) \rightarrow X^{\psi}_\ell \) is the homomorphism
\[
S \rightarrow \mathbb{Z}/p\mathbb{Z} : s^{\psi} \mapsto <s_{\lambda}^{\psi}, P(\varphi^{\psi}_\ell)>_{\lambda}.
\]

Proposition 10.1 implies that
\[
<s_{\lambda}^{\psi}, P(\varphi^{\psi}_\ell)>_{\lambda} = 0.
\]

Hence, the image by \( \omega_\ell \) of \( \text{res}_\lambda P(\varphi^{\psi}_\ell) \), one of the two generators of \( H^1(K^{ur}_\lambda, V_{p, \ell}/p) \) by proposition 9.2 vanishes.

2. Let \( \beta \) be the prime above \( q \) in \( K \). Proposition 10.1 implies that
\[
<s_{\lambda}^{\psi}, P(\varphi^{\psi}_{\ell q})>_{\lambda} + <s_{\beta}^{\psi}, P(\varphi^{\psi}_{\ell q})>_{\beta} = 0.
\]

Hence,
\[
\omega_q(\text{res}_\lambda P(\varphi^{\psi}_{\ell q})) + \omega_q(\text{res}_\beta P(\varphi^{\psi}_{\ell q})) = 0.
\]

Therefore, \( X^{\psi}_\ell \) is generated by \( \omega_q(\text{res}_\beta P(\varphi^{\psi}_{\ell q})) \) for all \( \ell \in L(U_0) \). As a consequence,
\[
\{X^{\psi}_\ell\}_{\ell \in L(U_0)} \subseteq X^{\psi}_q,
\]
where the rank one module \( X^{\psi}_q \) is generated over \( O_{F, p} / p \) by \( \omega_q(\text{res}_\beta P(\varphi^{\psi}_{\ell q})) \) for some \( \ell_0 \) in \( L(U_0) \).

3. By proposition 9.1, the set \( \{X^{\psi}_\ell\}_{\ell \in L(U_0)} \) generates \( S^{\text{dual}, \psi} \). Furthermore, \( P(\varphi_1)^{\psi} \) belongs to \( S^{\psi} \) by Proposition 6.4 and is non-zero by the hypothesis on \( P(\varphi_1)^{\psi} \) and Proposition 8.2. Therefore,
\[
\text{rank}(S^{\psi}) = \text{rank}(S^{\text{dual}, \psi}) = 1
\]
over \( O_{F, p} / p \).

\[\square\]
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