Polynomial Inequalities in Regions Bounded by Piecewise Asymptotically Conformal Curve with Nonzero Angles in the Bergman Space

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Abstract We continue the study of estimates of algebraic polynomials in regions bounded by a piecewise asymptotically conformal curve with interior non-zero angles in the weighted Bergman space.

Keywords: Algebraic polynomials, Conformal mapping, Asymptotically conformal curve.

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1 Introduction and Main Results

Let \( G \subset \mathbb{C} \) be a finite region, with \( 0 \in G \), bounded by a Jordan curve \( L := \partial G, \Omega := \text{ext} L := \overline{\mathbb{C}} \setminus G \), where \( \overline{\mathbb{C}} := \mathbb{C} \cup \{ \infty \} \), \( \Delta := \{ w : |w| > 1 \} \) and let \( \wp_n \) denote the class of arbitrary algebraic polynomials \( P_n(z) \) of degree at most \( n \in \mathbb{N} \).

Let \( w = \Phi(z) \) be the univalent conformal mapping of \( \Omega \) onto the \( \Delta \) normalized by \( \Phi(\infty) = \infty \), \( \lim_{z \to \infty} \frac{\Phi(z)}{z} > 0 \), and \( \Psi := \Phi^{-1} \). For \( t \geq 1 \), \( z \in \mathbb{C} \), we set:

\[
L_t := \{ z : |\Phi(z)| = t \} \quad (L_1 \equiv \Omega), \quad G_t := \text{int} L_t, \quad \Omega_t := \text{ext} L_t.
\]

Let \( \{ z_j \}_{j=1}^m \) be a fixed system of distinct points on curve \( L \), located in the positive direction. For some fixed \( R_0 \), \( 1 < R_0 < \infty \), and \( z \in G_{R_0} \), consider a so-called generalized Jacobi weight function \( h(z) \) being defined as follows:

\[
h(z) := h_0(z) \prod_{j=1}^m |z - z_j|^\gamma_j, \quad z \in G_{R_0}, \tag{1.1}
\]

where \( \gamma_j > -2 \), for all \( j = 1, 2, \ldots, m \), and the function \( h_0 \) is uniformly separated from zero in \( G_{R_0} \), i.e. there exists a constant \( c_0 := c_0(G_{R_0}) > 0 \) such that, for all \( z \in G_{R_0} \)

\[
h_0(z) \geq c_0 > 0.
\]

For any \( p > 0 \) and for Jordan region \( G \), let's define:

\[
\| P_n \|_p := \| P_n \|_{A_p(h,G)} := \left( \int_G h(z) |P_n(z)|^p \, d\sigma_z \right)^{1/p} < \infty, \quad 0 < p < \infty; \tag{1.2}
\]

\[
\| P_n \|_\infty := \| P_n \|_{A_{\infty}(1,G)} := \| P_n \|_{C(\overline{G})}, \quad p = \infty,
\]

where \( \sigma_z \) is the two-dimensional Lebesgue measure.

In this work, we continue the study of the following Nikolskii-type inequality:

\[
\| P_n \|_\infty \leq c_1 \lambda_n(G,h,p) \| P_n \|_p, \tag{1.3}
\]

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where $c_1 = c_1(G, h, p) > 0$ is a constant independent of $n$ and $P_1$, and $\lambda_n(G, h, p) \to \infty$, $n \to \infty$, depending on the geometrical properties of region $G$, weight function $h$ and of $p$. The estimate of (1.3)-type for some $(G, p, h)$ was investigated in [27, pp.122-133], [17], [26, Sect.5.3], [32], [15], [2]-[8] (see, also, references therein) and others. Further, analogous of (1.3) for some regions and the weight function $h(z)$ were obtained: in [8] for $p > 1$ and for regions bounded by piecewise Dini-smooth boundary without cusps; in [11] for $p > 0$ and for regions bounded by quasiconformal curve; in [7] for $p > 1$ and for regions bounded by piecewise smooth curve without cusps; in [10] for $p > 0$ and for regions bounded by asymptotically conformal curve; in [16] for $p > 0$ and for regions bounded by piecewise smooth curves with interior (zero or nonzero) angles, in [12] for $p > 0$ and for regions bounded by piecewise asymptotically conformal curve having cusps and others.

In this work, we investigate similar problems for $z \in G$ in regions bounded by piecewise asymptotically conformal curves having interior nonzero zero angles and for weight function $h(z)$, defined in (1.1) and for $p > 0$.

Now, we begin to give some definitions and notations.

Following [24, p.97], [28], the Jordan curve (or arc) $L$ is called $K$-quasiconformal ($K \geq 1$), if there is a $K$-quasiconformal mapping $f$ of the region $D \supset L$ such that $f(L)$ is a circle (or line segment).

Let $S$ be a Jordan curve and $z = z(s), s \in [0, |S|], |S| := mes S$, denote the natural representation of $S$. Let $z_1, z_2 \in S$ be an arbitrary points and $S(z_1, z_2) \subset S$ denotes the subarc of $S$ of shorter diameter with endpoints $z_1$ and $z_2$. The curve $S$ is a quasicircle if and only if the quantity

$$ \sup_{z_1, z_2 \in L; z \in S(z_1, z_2)} \frac{|z_1 - z| + |z_2 - z|}{|z_1 - z_2|} $$

is bounded. Following to Lesley [25], the curve $S$ to be said "$c$-quasiconformal", if the quantity (1.4) bounded by positive constant $c$, independent from points $z_1, z_2$ and $z$. At the literature it is possible to find various functional definitions of the quasiconformal curves (see, for example, [29, pp.286-294], [24, p.105], [13, p.81], [30, p.107]).

The Jordan curve $S$ is called asymptotically conformal [19], [30], if

$$ \sup_{z_1, z_2 \in S; z \in S(z_1, z_2)} \frac{|z_1 - z| + |z_2 - z|}{|z_1 - z_2|} \to 1, \quad |z_1 - z_2| \to 0. $$

We will denote this class as $AC$, and will write $G \in AC$, if $L := \partial G \in AC$.

The asymptotically conformal curves occupy a special place in the problems of the geometric theory of functions of a complex variable. These curves in various problems have been studied by Anderson, Becker and Lesley [14], Dynkin [20], Pommerenke, Warschawski [31], Gutlyanskii, Ryazanov [21], [22], [23] and others. According to the geometric criteria of quasiconformality of the curves ([13, p.81], [30, p.107]), every asymptotically conformal curve is a quasicircle. Every smooth curve is asymptotically conformal but corners are not allowed. It is well known that quasicircles can be non-rectifiable (see, for example, [18], [24, p.104]). The same is true for asymptotically conformal curves.

A Jordan arc $\ell$ is called asymptotically conformal arc, when $\ell$ is a part of some asymptotically conformal curve.

Now, we define a new class of regions bounded by piecewise asymptotically conformal curve having exterior nonzero "angles" at the connecting points of boundary arcs.

Throughout this work, we will assume that $p > 0$ and the constants $c, c_0, c_1, c_2, \ldots$ are positive and constants $e_0, e_1, e_2, \ldots$ are sufficiently small positive (generally, are different in different relations), which depends on $G$ in general and, on parameters inessential for the argument, otherwise, the dependence will be explicitly stated. Also note that, for any $k \geq 0$ and $m > k$, notation $j = k, m$ denotes $j = k, k+1, \ldots, m$.

Now, let's introduce "special angles" on $L$.

**Definition 1.1.** We say that a Jordan region $G \in \text{PAC}(\nu_1, \ldots, \nu_m)$, $0 < \nu_j < 2$, $j = \overline{1, m}$, if $L := \partial G$ consists of the union of finite asymptotically conformal arcs $\{L_j\}_{j=1}^m$, connected at the points $\{z_j\}_{j=1}^m$ in $L$ such that in $z_0$-$L$ locally asymptotically conformal and for any $z_j \in L, j = \overline{1, m}$, where two arcs $L_{j-1}$ and $L_j$ meet, there exist $r_j := r_j(L, z_j) > 0$ and $\nu_j := \nu_j(L, z_j)$, $0 < r_j < 2$, such that for some $0 \leq \theta_0 < 2$ a closed maximal circular sector $S(z_j; r_j, \nu_j) := \{\zeta : |z_j - z_j| + r_j e^{i\theta_j}, \theta_0 \leq \theta < \theta_0 + \nu_j\}$ of radius $r_j$ and opening $\nu_j \pi$ lies in $G = \text{int}L$ with vertex at $z_j$. AAN
Clearly, that $PAC(\nu_1) \subset PAC(\nu_2)$, if $\nu_2 \geq \nu_1$.

**Definition 1.2.** We say that a Jordan region $G \in PAC(\nu)$, if $G \in PAC(\nu_1, \ldots, \nu_m)$, $0 < \nu_j < 2$, $j = \overline{1,m}$, where $\nu = \min(\nu_j : 0 < \nu_j < 2$, $j = \overline{1,m})$.

It is clear from Definition 1.1 (1.2), that each region $G \in PAC(\nu_1, \ldots, \nu_m)$, $0 < \nu_1, \ldots, \nu_m < 2$, $(G \in PAC(\nu))$ may have 'singularity' at the boundary points $\{z_i\}_{i=1}^m \in L$. If it does not have such 'singularity' (in this case we put $\tilde{\nu}$).

**Remark 1.1.** $(\cite{9, Theorem 1.15}, \cite{2})$ For any $G \in \nu_n$, $n \in N$, and arbitrarily small $\varepsilon > 0$, there exists $c_1 = c_1(G, p, \gamma_j) > 0$ such that

$$\|P_n\|_\infty \leq c_1(1 + 1)\frac{\gamma_j}{2-\gamma_j} + \nu \|P_n\|_p,$$

where $\tilde{\gamma} = \max \{0, \gamma_i\}$ and $\tilde{\nu} = \min \{\nu_i\}$, $i = \overline{1,m}$.

**Theorem 1.2.** Let $p > 0$. Suppose that $G \in PAC(\nu_1, \ldots, \nu_m)$ for some $0 < \nu_1, \ldots, \nu_m < 1$; $h(z)$ defined as in (1.1). Then, for any $P_n \in \nu_n$, $n \in N$, and arbitrarily small $\varepsilon > 0$, there exists $c_2 = c_2(G, p, \gamma_j) > 0$ such that

$$|P_n(z)| \leq c_2\mu_n \|P_n\|_p,$$

where

$$\mu_n := \begin{cases} n^{(2+\gamma_j)(2-\gamma_j)} + \varepsilon, & \text{if } \gamma_j > \frac{1}{2-\gamma_j} - 2 - \varepsilon, \\ (n \ln n)^{\frac{\gamma_j}{2}}, & \text{if } \gamma_j = \frac{1}{2-\gamma_j} - 2 - \varepsilon, \\ n^{\frac{1}{\gamma_j}}, & \text{if } 2 < \gamma_j < \frac{1}{2-\gamma_j} - 2 - \varepsilon. \end{cases}$$

The sharpness of the estimations (1.6) and (1.7) can be discussed by comparing them with the following result:

**Remark 1.1.** $(\cite{9, Theorem 1.15}, \cite{2})$ For any $n \in N$ there exists a polynomials $Q^*_n, T^*_n \in \nu_n$ such that for unit disk $B$ and weight function $h^*(z) = |z - 1|^2$ the following is true:

$$|Q^*_n(z)| \geq c_6n \|Q^*_n\|_{A_2(B)}, \text{ for all } z \in B;$$

$$|T^*_n(z)| \geq c_7n^2 \|T^*_n\|_{A_2(h^*, B)}.$$
According to Cauchy integral representation for the unbounded region

According to (3.1) - (3.5), we have:

The function

Then, for each

and let

Suppose that

Proof. Let us set:

and for this branch, we maintain the same designation.

For any

let us set:

The function

is analytic in \( \Omega \), continuous on \( \overline{\Omega} \), \( Q_{n,p}(\infty) = 0 \) and does not have zeros in \( \Omega \). We take an arbitrary continuous branch of the \( Q_{n,p}(z) \) and for this branch, we maintain the same designation. According to Cauchy integral representation for the unbounded region \( \Omega \), we have:

According to (3.1) - (3.5), we have:

\[
|P_n(z)|^{p/2} \leq \frac{B_m(z)\Phi^{n+1}(z)}{2\pi d(z,\Omega_{R_1})} \int_{L_{R_1}} \left| \frac{P_n(\zeta)}{B_m(\zeta)\Phi^{n+1}(\zeta)} \right|^{p/2} |d\zeta| \leq \frac{\Phi^{n+1}(z)}{\zeta - z}, \quad z \in \Omega_{R_1}.
\]
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According to Lemma 2.3, for we obtain:

\[
\left( \int_{|t|=R_1} |P_n(\zeta)|^2 \, |d\zeta| \right) \leq \int_{|t|=R_1} h(\Psi(t)) \left| P_n(\Psi(t)) \right| |\Psi'(t)|^2 \, |dt| \cdot \int_{|t|=R_1} \frac{|dt|}{h(\Psi(t)) |\Psi(t) - \Psi(w)|^2} \tag{3.8}
\]

\[
\leq \int_{|t|=R_1} h(\Psi(t)) \left| P_n(\Psi(t)) \right| |\Psi'(t)|^2 \, |dt| \cdot \int_{|t|=R_1} \frac{|dt|}{h(\Psi(t)) |\Psi(t) - \Psi(w)|^2}
\]

\[= \int_{|t|=R_1} |f_{n,p}(t)|^p \, |dt| \cdot \int_{|t|=R_1} \frac{|dt|}{h(\Psi(t)) |\Psi(t) - \Psi(w)|^2} =: A_n \cdot D_n(w),
\]

where \( f_{n,p}(t) := \frac{1}{h(\Psi(t))} P_n(\Psi(t)) |\Psi'(t)|^\frac{p-2}{2}, \quad |t| = R_1. \)

For the estimate integral \( A_n \), we divide the circle \(|t| = R_1\) into \( n \) equal parts \( \delta_n \) with \( \text{mes} \delta_n = \frac{2\pi R_1}{n} \) and by applying the mean value theorem, we get:

\[A_n := \int_{|t|=R_1} |f_{n,p}(t)|^p \, |dt| = \frac{n}{\delta_n} \sum_{k=1}^n |f_{n,p}(t_k')|^p \text{mes} \delta_k, \quad t_k' \in \delta_k.
\]

On the other hand, by applying mean value estimation

\[|f_{n,p}(t_k')|^p \leq \frac{1}{\pi \left( |t_k'|-1 \right)^2} \int_{|\xi|<|t_k'|-1} |f_{n,p}(\xi)|^p \, d\sigma_\xi,
\]

we obtain:

\[(A_n)^2 \leq \frac{n}{\delta_n} \sum_{k=1}^n \frac{\text{mes} \delta_k}{\pi \left( |t_k'|-1 \right)^2} \int_{|\xi|<|t_k'|-1} |f_{n,p}(\xi)|^p \, d\sigma_\xi, \quad t_k' \in \delta_k.
\]

By taking into account, at most two of the discs with center \( t_k' \) are intersecting, we have:

\[A_n \leq \frac{n \text{mes} \delta_1}{\left( |t_1'|-1 \right)^2} \int_{1<|\xi|<R} |f_{n,p}(\xi)|^p \, d\sigma_\xi \leq n \int_{1<|\xi|<R} |f_{n,p}(\xi)|^p \, d\sigma_\xi.
\]

According to Lemma 2.3, for \( A_n \) we get:

\[A_n \leq n \int_{G_n \setminus G} h(\zeta) |P_n(\zeta)|^p \, d\sigma_\zeta \leq n \cdot \|P_n\|^p.
\]

To estimate the integral \( D_n(w) \), denoted by \( w_j := \Phi(z_j), \varphi_j := \text{arg} \, w_j, \) for any fixed \( \rho > 1 \), we introduce:

\[
\Delta_1(\rho) := \left\{ t = re^{i\theta} : r > \rho, \, \frac{\varphi_0 + \varphi_1}{2} \leq \theta < \frac{\varphi_1 + \varphi_2}{2} \right\},
\]

\[
\Delta_2(\rho) := \left\{ t = re^{i\theta} : r > \rho, \, \frac{\varphi_1 + \varphi_2}{2} \leq \theta < \frac{\varphi_1 + \varphi_0}{2} \right\};
\]

\[
\Delta_j := \Delta_j(1), \, \Omega_j := \Psi(\Delta_j), \, \Omega_j^\rho := \Psi(\Delta_j(\rho));
\]

\[L_{\rho} := L \cap \overline{T}_{\rho}^j, \, L_{\rho} := L_{\rho} \cap \overline{T}_{\rho}^j, \quad j = 1, 2; \, L = L^1 \cup L^2, \, L_{\rho} = L_{\rho}^1 \cup L_{\rho}^2.
\]
Under these notations, from (3.8) for the $D_n(w)$, we get:

$$D_n(w) = \int_{|t|=R_1} \frac{|dt|}{\pi(\Psi(t)) |\Psi(t) - \Psi(w)|^2}$$

$$= \sum_{j=1}^{2} \int_{\Phi(L_{R_1}^j)} \frac{|dt|}{\prod_{j=1}^{2} |\Psi(t) - \Psi(w_j)|^{\gamma_j} |\Psi(t) - \Psi(w)|^2}$$

$$= \sum_{j=1}^{2} \int_{\Phi(L_{R_1}^j)} \frac{|dt|}{\prod_{j=1}^{2} |\Psi(t) - \Psi(w_j)|^{\gamma_j} |\Psi(t) - \Psi(w)|^2} = : \sum_{j=1}^{2} D_{n,j}(w),$$

since the points $\{z_j\}_{j=1}^{2} \in L$ are distinct. So, we need to evaluate the $D_{n,j}(w)$. For this, we take $z \in L_R$ and introduce the notations:

$$\Phi(L_{R_1}) = \Phi(\bigcup_{j=1}^{2} L_{R_1}^j) = \bigcup_{j=1}^{2} \Phi(L_{R_1}^j) = \bigcup_{j=1}^{2} \bigcup_{i=1}^{2} K_{i}^j(R_1),$$

where

$$K_{1}^j(R_1) := \left\{ t \in \Phi(L_{R_1}^j) : |t - w_j| < c_1 \right\}$$

$$K_{2}^j(R_1) := \Phi(L_{R_1}^j) \setminus K_{1}^j(R_1), \quad j = 1, 2.$$

Analogously,

$$\Phi(L_R) = \Phi(\bigcup_{j=1}^{2} L_{R}^j) = \bigcup_{j=1}^{2} \Phi(L_{R}^j) = \bigcup_{j=1}^{2} \bigcup_{i=1}^{2} K_{i}^j(R),$$

where

$$K_{1}^j(R) := \left\{ t \in \Phi(L_{R}^j) : |\tau - w_j| < 2c_1 \right\}$$

$$K_{2}^j(R) := \Phi(L_{R}^j) \setminus K_{1}^j(R), \quad j = 1, 2.$$

Then, after these definitions, taking arbitrary fixed $w = \Phi(z) \in \Phi(L_R)$, the quantity $D_{n,j}(w)$ can be written as follows:

$$D_{n,j}(w) = \sum_{i=1}^{2} \int_{K_{i}^j(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w_j)|^{\gamma_j} |\Psi(t) - \Psi(w)|^2} = : \sum_{i=1}^{2} D_{n,i,j}(w)$$

(3.13)

The quantity $D_{n,j}^i(w)$ we shall estimate for each $i = 1, 2$ and $j = 1, 2$ in cases separately, depending of location of the $w \in \Phi(L_R)$. Let $\varepsilon > 0$ arbitrary small fixed number.

Case 1. Let $w \in \Phi(L_{R_1}^j)$.

According to the above notations, we will make evaluations for case $w \in K_{1}^i(R)$ for each $i = 1, 2, 3$. 1.1) Let $w \in K_{1}^1(R)$. In this case, we will estimate the quantity

$$D_{n,1}(w) = \sum_{i=1}^{2} \int_{K_{i}^1(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w_j)|^{\gamma_j} |\Psi(t) - \Psi(w)|^2} = : \sum_{i=1}^{2} D_{n,1,i}(w)$$

(3.14)

for $\gamma_1 \geq 0$ and $\gamma_1 < 0$ separately.

For each $i = 1, 2$ and $j = 1, 2$ we put: $K_{i,1}^j(R_1) := \left\{ t \in \Phi(L_{R_1}^j) : |t - w_j| \geq |t - w| \right\}$, $K_{i,2}^j(R_1) := K_{i}^j(R_1) \setminus K_{i,1}^j(R_1)$. 148

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1.1.1) If $\gamma_1 \geq 0$, then

$$D_{n,1}^1(w) = \int_{K_1^1(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{2+\gamma_1} |\Psi(t) - \Psi(w)|^{2}}$$  \hspace{1cm} (3.15)$$

$$= \int_{K_1^1(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w)|^{2+\gamma_1}} + \int_{K_2^1(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{2+\gamma_1}}$$

$$= : D_{n,1}^{1,1}(w) + D_{n,1}^{1,2}(w).$$

Since $G \in PAC(\nu_1, \nu_2)$ for some $0 < \nu_1, \nu_2 < 1$, according to [25], $\psi \in Lip_\nu$ and $\Phi \in Lip_{\frac{1}{2-\nu_i}}$, $i = 1, 2$, in a some fixed neighborhood of point $z_j$. Therefore, we get:

$$D_{n,1}^{1,1}(w) \leq \int_{K_1^1(R_1)} \frac{|dt|}{|t - w|^{2+\gamma_1}(2-\nu_1)} \leq n^{(2+\gamma_1)(2-\nu_1)-1},$$  \hspace{1cm} (3.16)$$

and

$$D_{n,1}^{1,2}(w) \leq \int_{K_1^1(R_1)} \frac{|dt|}{|t - w_1|^{2+\gamma_1}(2-\nu_1)} \leq n^{(2+\gamma_1)(2-\nu_1)-1},$$  \hspace{1cm} (3.17)$$

If $\gamma_1 < 0$, then

$$D_{n,1}^1(w) = \int_{K_1^1(R_1)} \frac{|\Psi(t) - \Psi(w_1)|^{(-\gamma_1)} |dt|}{|\Psi(t) - \Psi(w)|^{2}}$$  \hspace{1cm} (3.18)$$

$$\leq \int_{K_1^1(R_1)} \frac{|dt|}{|t - w|^{2-\nu_1}} \leq \int_{K_1^1(R_1)} \frac{|dt|}{|t - w|^{2-\nu_1}}$$

$$\leq n^{2(2-\nu_1)-1}.$$  \hspace{1cm} (3.19)$$

1.1.2) If $\gamma_1 \geq 0$, then

$$D_{n,1}^2(w) = \int_{K_2^1(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{2+\gamma_1} |\Psi(t) - \Psi(w)|^{2}}$$  \hspace{1cm} (3.20)$$

$$= \int_{K_1^1(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w)|^{2+\gamma_1}} + \int_{K_2^1(R_1)} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{2+\gamma_1}}$$

$$= : D_{n,1}^{2,1}(w) + D_{n,1}^{2,2}(w).$$

and, so from Lemma 2.1 and 2.2, we get:

$$D_{n,1}^{2,1}(w) \leq \int_{K_2^1(R_1)} \frac{|dt|}{|t - w_1|^{2+\gamma_1}(2-\nu_1)} \leq n^{(2+\gamma_1)(2-\nu_1)-1},$$  \hspace{1cm} (3.21)$$

and

$$D_{n,1}^{2,2}(w) \leq 1.$$  \hspace{1cm} (3.22)$$

Therefore, from (3.19)-(3.21) for $\gamma_1 \geq 0$, we have:

$$D_{n,1}^2(w) \leq n^{(2+\gamma_1)(2-\nu_1)-1},$$  \hspace{1cm} (3.23)$$

For $\gamma_1 < 0$ from (3.14), we have:

$$D_{n,1}^2(w) = \int_{K_2^1(R_1)} \frac{|\Psi(t) - \Psi(w_1)|^{(-\gamma_1)} |dt|}{|\Psi(t) - \Psi(w)|^{2}}$$  \hspace{1cm} (3.24)$$
where \( R > \phi \).

Proof. Suppose that \( w \in K^1_2(R) \).

1.2) Let \( w \in K^1_2(R) \).

1.2.1) For any \( \gamma_1 > -2 \)

\[
D^1_{n,1}(w) = \int_{K^1_1(R)} \frac{|dt|}{|t-w|^2(1+\varepsilon)} \leq n^{1+\varepsilon}, \quad \forall \varepsilon > 0.
\]

(3.24)

and so, according to Lemmas 2.1 and 2.2, we obtain:

\[
D^1_{n,1}(w) \leq \int_{K^1_1(R)} \frac{|dt|}{|\Psi(t) - \Psi(w)|^{2+\gamma_1}} \leq 1,
\]

and

\[
D^1_{n,1}(w) \leq \int_{K^1_1(R)} \frac{|dt|}{|t-w|^{(2+\gamma_1)(2-\nu_1)}} \leq n^{(2+\gamma_1)(2-\nu_1)-1}.
\]

(3.25)

and

\[
\begin{align*}
D^2_{n,1}(w) &\leq \int_{K^1_1(R)} \frac{|dt|}{|\Psi(t) - \Psi(w)|^{2+\gamma_1}} + \int_{K^1_1(R)} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{2+\gamma_1}} \leq n^{(2+\gamma_1)(1+\varepsilon)} - 1, \\
&\leq \int_{K^1_1(R)} \frac{|dt|}{|t-w|^{(2+\gamma_1)(1+\varepsilon)}} + 1 \leq n^{(2+\gamma_1)(1+\varepsilon)} - \varepsilon, \quad \forall \varepsilon > 0.
\end{align*}
\]

Combining estimates (3.14)-(3.26), for \( w \in \Phi(L_R) \), we have:

\[
D_{n,1} \leq n^{(2+\tilde{\gamma}_1)(2-\nu_1)-1+\varepsilon}, \quad \tilde{\gamma}_1 := \max \{0; \gamma_1\}.
\]

(3.27)

Case 2. Let \( w \in \Phi(L_R^\nu) \). Analogously to the Case 1, we will obtain estimates for \( w \in K^2_2(R) \) and \( w \in K^1_2(R) \)

\[
D_{n,2}(w) \leq n^{(2+\gamma_2)(2-\nu_2)-1+\varepsilon}, \quad \tilde{\gamma}_2 := \max \{0; \gamma_2\}
\]

(3.28)

Therefore, comparing relations (3.11), (3.13), (3.27) and (3.28), we have:

\[
D_{n}(w) \leq n^{(2+\tilde{\gamma}_1)(2-\nu_1)-1} + n^{(2+\tilde{\gamma}_2)(2-\nu_2)-1},
\]

(3.29)

where \( \tilde{\gamma}_1 \) and \( \tilde{\gamma}_2 \) defined as in (3.27) and (3.28).

Now, from (3.7), (3.8), (3.9) and (3.29), for any \( z \in L_R \), we get:

\[
|P_n(z)| \leq [n^{(2+\tilde{\gamma}_1)(2-\nu_1)} + n^{(2+\tilde{\gamma}_2)(2-\nu_2)}] \|P_n\|_p
\]

Since this estimate holds for any \( z \in L_R \), then it is also true for \( z \in \overline{G} \). Therefore, we complete the proof of theorem.

3.2 Proof of Theorem 1.2

Proof. Suppose that \( G \in \text{PAC}(\nu_1, \nu_2) \) for some \( 0 < \nu_1, \nu_2 < 1 \) and \( h(z) \) is defined as in (1.1). For each \( R > 1 \), let \( w = \varphi_R(z) \) denote a univalent conformal mapping \( G_R \) onto the \( B \), normalized by \( \varphi_R(0) = 0, \varphi_R'(0) > 0 \), and let \( \{\zeta_j\}, 1 \leq j \leq m \leq n \), be a zeros of \( P_n(z) \) (if any exist) lying on \( G_R \).

Let

\[
b_{m,R}(z) := \prod_{j=1}^{m} \tilde{g}_{j,R}(z) = \prod_{j=1}^{m} \frac{\varphi_R(z) - \varphi_R(\zeta_j)}{1 - \varphi_R(\zeta_j)\varphi_R(z)},
\]

(3.30)
denote a Blaschke function with respect to zeros \( \{ \zeta_j \} \), \( 1 \leq j \leq m \leq n \), of \( P_n(z) \) \((33)\). Clearly,

\[
|b_{m,R}(z)| \equiv 1, \ z \in L_R, \ \text{and} \ |b_{m,R}(z)| < 1, \ z \in G_R. \tag{3.31}
\]

For any \( p > 0 \) and \( z \in G_R \), let us set

\[
T_{n,p}(z) := \left[ \frac{P_n(z)}{b_{m,R}(z)} \right]^{p/2}. \tag{3.32}
\]

The function \( T_{n,p}(z) \) is analytic in \( G_R \), continuous on \( \overline{G}_R \) and does not have zeros in \( G_R \). We take an arbitrary continuous branch of the \( T_{n,p}(z) \) and for this branch we maintain the same designation. Then, the Cauchy integral representation for the \( T_{n,p}(z) \) at the \( z = z_1 \) gives:

\[
T_{n,p}(z_1) = \frac{1}{2\pi i} \int_{L_R} T_{n,p}(\zeta) \frac{d\zeta}{\zeta - z_1}.
\]

Then, according to (3.31), we obtain:

\[
|P_n(z_1)|^{p/2} \leq \frac{|b_{m,R}(z_1)|^{p/2}}{2\pi} \int_{L_R} \left| \frac{P_n(\zeta)}{b_{m,R}(\zeta)} \right|^{p/2} \frac{|d\zeta|}{|\zeta - z_1|} \tag{3.33}
\]

Multiplying the numerator and the denominator of the last integrand by \( h^{1/2}(\zeta) \), replacing the variable \( w = \Phi(z) \) and applying the Hölder inequality, we obtain:

\[
\left( \int_{L_R} \left| P_n(\zeta) \right|^2 \frac{|d\zeta|}{|\zeta - z_1|} \right)^2 \leq \int_{|t|=R} h(\Psi(t)) |P_n(\Psi(t))|^p |\Psi'(t)|^2 |dt| \cdot \int_{|t|=R} \frac{|dt|}{h(\Psi(t)) |\Psi(t) - \Psi(w_1)|^2}
\]

\[
= \int_{|t|=R} |f_{n,p}(t)|^p |dt| \cdot \int_{|t|=R} \frac{|dt|}{h(\Psi(t)) |\Psi(t) - \Psi(w_1)|^2},
\]

where \( f_{n,p}(t) \) has been defined as in (3.8). Since \( R > 1 \) is arbitrary, then (3.34) holds also for \( R = R_1 := 1 + \frac{\varepsilon_1}{n}, \ 0 < \varepsilon_1 < 1 \). So, we have:

\[
\left( \int_{L_{R_1}} \left| P_n(\zeta) \right|^2 \frac{|d\zeta|}{|\zeta - z_1|} \right)^2 \leq \int_{|t|=R_1} |f_{n,p}(t)|^p |dt| \cdot \int_{|t|=R_1} \frac{|dt|}{h(\Psi(t)) |\Psi(t) - \Psi(w_1)|^2}
\]

\[
= : A_n \cdot D_n(w_1),
\]

and, \( A_n \) and \( D_n(w_j) \) have been defined as in (3.8) for \( R = R_1 \). Therefore, from (3.33) and (3.35), we have:

\[
|P_n(z_1)| \leq A_n \cdot D_n(w_1), \tag{3.36}
\]

where, according to (3.9), the estimate

\[
A_n \leq n \cdot \|P_n\|_p
\]
is satisfied. For the estimate of the quantity $D_n(w_1)$ we use the notations at the estimation of the $D_n(w)$ as in (3.11)-(3.13). Therefore, under these notations, for the $D_n(w_1)$, we get:

$$D_n(w_1) \leq \int_{\phi(L_n')} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{2+\gamma_1}} \tag{3.37}$$

$$\leq \sum_{i=1}^{2} \int_{K_i'(L_n')} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{2+\gamma_1}} = \sum_{i=1}^{2} D^i_{n,1}(w_1).$$

So, we need to evaluate the $D^i_{n,1}(w_1)$ for each $i = 1, 2$. We have:

$$D^1_{n,1}(w_1) = \int_{K_1'(L_n')} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{2+\gamma_1}} \tag{3.38}$$

$$\leq \int_{K_1'(L_n')} \frac{|dt|}{|t - w_1|^{2+\gamma_1}(2-\nu_1)} \leq \left\{ \begin{array}{ll}
n(2+\gamma_1)(2-\nu_1)-1, & \text{if } (2 + \gamma_1)(2 - \nu_1) > 1, \\
\ln n, & \text{if } (2 + \gamma_1)(2 - \nu_1) = 1, \\
1, & \text{if } (2 + \gamma_1)(2 - \nu_1) < 1,
\end{array} \right.$$ 

and

$$D^2_{n,1}(w_1) = \int_{K_2'(L_n')} \frac{|dt|}{|\Psi(t) - \Psi(w_1)|^{2+\gamma_1}} \leq \int_{K_2'(L_n')} \frac{|dt|}{|t - w_1|^{2+\gamma_1+\varepsilon}} \leq n(2+\gamma_1)(1+\varepsilon)-1. \tag{3.39}$$

Combining relations (3.37) - (3.39), we have:

$$D_n(w_1) \leq \left\{ \begin{array}{ll}
n(2+\gamma_1)(2-\nu_1)-1+\varepsilon, & \text{if } (2 + \gamma_1)(2 - \nu_1) > 1 - \varepsilon, \\
\ln n, & \text{if } (2 + \gamma_1)(2 - \nu_1) = 1 - \varepsilon, \\
1, & \text{if } (2 + \gamma_1)(2 - \nu_1) < 1 - \varepsilon,
\end{array} \right. \tag{3.40}$$

From the estimations (3.36) and (3.40), we obtain:

$$|P_n(z_1)| \leq \left\{ \begin{array}{ll}
n^{(2+\gamma_1)(2-\nu_1) \frac{1}{p}} + \varepsilon, & \text{if } (2 + \gamma_1)(2 - \nu_1) > 1 - \varepsilon, \\
(n \ln n)^{\frac{1}{p}}, & \text{if } (2 + \gamma_1)(2 - \nu_1) = 1 - \varepsilon, \\
n^{\frac{1}{p}}, & \text{if } (2 + \gamma_1)(2 - \nu_1) < 1 - \varepsilon,
\end{array} \right.$$ 

and we complete the proof of theorem. \qed

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