PRESERVING COARSE PROPERTIES

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June 30, 2015

Abstract. The aim of this paper is to investigate properties preserved and co-preserved by coarsely n-to-1 functions, in particular by the quotient maps $X \to X/\sim$ induced by a finite group $G$ acting by isometries on a metric space $X$. The coarse properties we are mainly interested in are related to asymptotic dimension and its generalizations: having finite asymptotic dimension, asymptotic Property C, straight finite decomposition complexity, countable asymptotic dimension, and metric sparsification property. We provide an alternative description of asymptotic Property C and we prove that the class of spaces with straight finite decomposition complexity coincides with the class of spaces of countable asymptotic dimension.

1. Introduction

The main topic of this paper is preservation and co-preservation of coarse properties by certain classes of functions. The most important of them is the class of coarsely n-to-1 functions recently introduced by Miyata and Virk [11]. As shown in Section 5 that class is contained in the class of functions of asymptotic dimension 0 introduced in [2].

A class of functions $f : X \to Y$ preserves a coarse property $\mathcal{P}$ if $f(X)$ has $\mathcal{P}$ whenever $X$ has $\mathcal{P}$. A class of functions $f : X \to Y$ co-preserves a coarse property $\mathcal{P}$ if $X$ has $\mathcal{P}$ whenever $f(X)$ has $\mathcal{P}$.

The coarse properties we are mainly interested in are related to asymptotic dimension and its generalizations: having finite asymptotic dimension, asymptotic Property C, straight finite decomposition complexity, countable asymptotic dimension, and metric sparsification property.

Besides investigating properties being preserved and co-preserved by coarsely n-to-1 functions and providing an alternative description of asymptotic Property C, our main result of the paper is that $X$ being of straight finite decomposition complexity is actually equivalent to $X$ having countable asymptotic dimension.

2. Preliminaries

One of the main ideas in topology is approximating general topological spaces $X$ by simplicial complexes. This is done by first selecting a cover $\mathcal{U}$ of $X$ and then
constructing its nerve $N(\mathcal{U})$. Recall $N(\mathcal{U})$ has $\mathcal{U}$ as its vertices and $[U_0, \ldots, U_n]$ is an $n$-simplex of $N(\mathcal{U})$ if $\bigcap_{i=0}^n U_i \neq \emptyset$.

One of the most important characteristics of a simplicial complex $K$ is its **combinatorial dimension** $\dim(K)$, the supremum over all $n$ such that $K$ has an $n$-simplex. Therefore it makes sense to introduce the dimension of a family of subsets of a set $X$:

**Definition 2.1.** The **dimension** $\dim(\mathcal{U})$ of a family of subsets of a set $X$ is the combinatorial dimension of its nerve.

That leads to a concise explanation of the covering dimension $\dim(X)$ of a topological space: $\dim(X) \leq n$ if every open cover of $X$ can be refined by an open cover of dimension at most $n$.

Its dualization in coarse topology leads to the following definition:

**Definition 2.2.** A metric space $X$ is of asymptotic dimension $\text{asdim}(X)$ at most $n$ if every uniformly bounded cover of $X$ can be coarsened to a uniformly bounded family of dimension at most $n$.

It turns out it makes sense to look at a metric space at different scales $R \geq 0$.

**Definition 2.3.** Given a family of subsets $\mathcal{U}$ of metric space $X$ and a scale $R \geq 0$, the **dimension** $\dim_R(\mathcal{U})$ of $\mathcal{U}$ at scale $R$ (or $R$-dimension in short) is the dimension of the family $B(\mathcal{U}, R)$ of $R$-balls $B(U, R), U \in \mathcal{U}$. Here $B(U, R)$ consists of $U$ and all points $x$ in $X$ such that $d_X(x, u) < R$ for some $u \in U$. The $R$-multiplicity of a cover $\mathcal{U}$ is defined to be $\dim_R(\mathcal{U}) + 1$.

**Observation 2.4** (see [10] or [1]). $\text{asdim}(X) \leq n$ if and only if for each $R > 0$ there is a uniformly bounded cover $\mathcal{U}$ of $X$ such that $\dim_R(\mathcal{U}) \leq n$.

The easiest case of estimating $R$-dimension is in the case of unions of $R$-disjoint families:

**Observation 2.5.** If $\mathcal{U} = \bigcup_{i=0}^n U_i$ and each $U_i$ is $R$-disjoint (that means $d(x, y) \geq R$ for $x$ and $y$ belonging to different elements of $U_i$), then $\dim_R(\mathcal{U}) \leq n$.

**Proof.** It is sufficient to consider $R = 0$ as $R$-balls of elements of $U_i$ form a disjoint family for $R > 0$. Notice each $x \in X$ belongs to at most one element of $U_i$, hence it belongs to at most $(n + 1)$ elements of $U$.

**Lemma 2.6.** Suppose $f : X \to Y$ is coarse with control $E$. If $\mathcal{V}$ is an $E(d)$ disjoint collection of sets in $Y$ then $\{f^{-1}(V) \mid V \in \mathcal{V}\}$ is a $d$-disjoint collection.

**Proof.** If two points are at distance at most $d$ in $X$ then their images are at most $E(d)$ apart in $Y$ hence they cannot belong to different elements of $\mathcal{V}$.

For technical reasons it is convenient to achieve the situation of Observation 2.5.

**Lemma 2.7.** Suppose $X$ is a metric space, $n \geq 1$, and $M, R > 0$. If $\mathcal{U} = \{U_s\}_{s \in S}$ is a cover of $X$ of $R$-dimension at most $n$, then there is a cover $\mathcal{V} = \bigcup_{i=0}^n \mathcal{V}^i$ of $X$ such that each $\mathcal{V}^i$ is $\frac{R}{n+1}$-disjoint. Furthermore:
(1) every element of $\mathcal{V}$ is contained in an intersection of at most $(n+1)$-many $R$-neighborhoods of elements of $\mathcal{U}$. In particular, for each finite subset $T$ of $S$ we have $B(W_T, -R/(2n+2)) \subset \bigcap_{t \in T} B(U_t, R)$ (see the proof below for notation);
(2) if $\mathcal{U}$ is $M$-bounded then $\mathcal{V}$ is $(M+2R)$-bounded.

Proof. Define $f_s(x) = \text{dist}(x, X \setminus B(U_s, R))$. For each finite subset $T$ of $S$ define
\[ W_T = \left\{ x \in X \mid \min \{ f_t(x) \mid t \in T \} > \sup \{ f_s(x) \mid s \in S \setminus T \} \right\} = \left\{ x \in X \mid f_t(x) > f_s(x) \mid \forall t \in T, \forall s \in S \setminus T \right\}. \]
Notice $W_T = \emptyset$ if $T$ contains at least $n+2$ elements. Also, notice Fact 1: $W_T \cap W_F = \emptyset$ if both $T$ and $F$ are different but contain the same number of elements. Let us estimate the Lebesgue number of $W = \{ W_T \}_{T \subseteq S}$. Given $x \in X$ arrange all non-zero values $f_s(x)$ from the largest to the smallest. Add 0 at the end and look at gaps between those values. The largest number is at least $R$, there are at most $(n+1)$ gaps, so one of them is at least $\frac{R}{n+1}$. That implies the ball $B(x, R/(2n+2))$ is contained in one $W_T$ ($T$ consists of all $t$ to the left of the gap) hence the Lebesgue number is at least $R/(2n+2)$. Define $\mathcal{V}^i$ as
\[ \{ B(W_T, -R/(2n+2)) \} = \{ \{ x \in W_T \mid B(x, R/(2n+2)) \subset W_T \} \}, \]
where $T$ ranges through all subsets of $S$ containing exactly $i$ elements. By Fact 1 above, each $\mathcal{V}^i$ is an $\frac{R}{n+1}$-disjoint family.

For every finite subset $T$ of $S$ we have $B(W_T, -R/(2n+2)) \subset \bigcap_{t \in T} B(U_t, R)$ by the definition of $W_T$ as $f_t$ is nonzero on $W_T$ for every $t \in T$. This proves (1). Consequently, sets $W_T$ are $(M + 2R)$-bounded and so are their subsets $B(W_T, -R/(2n+2))$, elements of $\mathcal{V}$, proving (2). \hfill \Box

With the help of Lemma 2.7 we are going to show similarity of asymptotic property C to having countable asymptotic dimension (see 8.2).

Definition 2.8 (Dranishnikov [5]). A metric space $X$ has asymptotic property $C$ if for every sequence $R_1 < R_2 < \ldots$ there exists $n \in \mathbb{N}$ such that $X$ is the union of $R_i$-disjoint families $\mathcal{U}_i$, $1 \leq i \leq n$, that are uniformly bounded.

Theorem 2.9. A metric space $X$ has asymptotic property C if and only if there is a sequence of integers $n_i \geq 0$, $i \geq 1$, such that for any sequence of positive real numbers $R_i$, $i \geq 1$, there is a finite sequence $\mathcal{V}_i$, $i \leq n$, of uniformly bounded families of subsets of $X$ such that the dimension of $\mathcal{V}_i$ at scale $R_i$ is at most $n_i$ for $i \leq n$ and $X = \bigcup_{i=1}^{n} \mathcal{V}_i$.

Proof. In one direction the proof is obvious: namely, $n_i = 0$ for all $i$ works.

Suppose there is a sequence of integers $n_i \geq 0$, $i \geq 1$, such that for any sequence of positive real numbers $R_i$, $i \geq 1$, there is a finite sequence $\mathcal{V}_i$, $i \leq n$, of uniformly bounded families of subsets of $X$ such that the dimension of $\mathcal{V}_i$ at scale $R_i$ is at most $n_i$ for $i \leq n$ and $X = \bigcup_{i=1}^{n} \mathcal{V}_i$.

Given an increasing sequence $M_i$ of positive real numbers, first define $m_j$ as $\sum_{i \leq j} n_i$ and then define $R_i$ as $\sum_{j \leq i} (n_j + 1) \cdot M_{m_j+1}$. Pick a finite sequence $\mathcal{V}_i$, $i \leq n$, of uniformly bounded families of subsets of $X$ such that the dimension of $\mathcal{V}_i$ at
Definition 3.1. A function $f : X \to Y$ of metric spaces is $C$-bornologous, where $C : [0, \infty) \to [0, \infty)$, if the image of every $r$-bounded set in $X$ is $C(r)$-bounded. $f$ is bornologous if it is $C$-bornologous for some function $C : [0, \infty) \to [0, \infty)$. $f$ is $(a,b)$-Lipschitz if it is $C$-bornologous for $C(r) = ar + b$.

A function $f : X \to Y$ of metric spaces is $R$-close to $g : X \to Y$, if $d(f(x), g(x)) \leq R, \forall x \in X$. If $f$ is close to $g$ if it is $R$-close to $g$ for some $R > 0$.

A bornologous function $f : X \to Y$ of metric spaces is a coarse equivalence if there exists a bornologous function $g : Y \to X$, so that $f \circ g$ is close to the identity $id_Y$ and that $g \circ f$ is close to the identity $id_X$.

Theorem 3.2. Suppose $X$ is a metric space such that $d(x, y) \geq 1$ if $x \neq y$. If $\sim$ is an equivalence relation whose equivalence classes are uniformly bounded, then the projection $p : X \to X/\sim$ is a coarse equivalence if $X/\sim$ is equipped with the Hausdorff metric $d_H$.

Proof. Suppose each equivalence class is of diameter less than $R$. Notice

$$d(x, y) - 2R \leq d_H([x], [y]) \leq d(x, y) + 2R$$

for all $x, y \in X$. That means that $p$ is $(1, 2R)$-Lipschitz and any selection function $s : X/\sim \to X$ is $(1, 2R)$-Lipschitz as well. Since $s \circ p$ is $R$-close to the identity $id_X$ and $p \circ s$ is the identity on $X/\sim$, both $p$ and $s$ are coarse equivalences. 

4. Coarsely n-to-1 functions

Definition 4.1 (Condition $B_n$ of \[11\]). A bornologous function $f : X \to Y$ of metric spaces is coarsely n-to-1 (with control $C$) if there is a function $C : [0, \infty) \to [0, \infty)$ so that for each subset $B$ of $Y$ with $\text{diam}(B) \leq r$, $f^{-1}(B) = \bigcup A_i$ for some subsets $A_i$ of $X$ with $\text{diam}(A_i) < C(r)$ for $i = 1, \ldots, n$.

See [8] for other conditions equivalent to $f$ being coarsely n-to-1.

An example of a coarse n-to-1 map is $z \mapsto z^n$ in the complex plane. Here is a more general case:

Example 4.2. If a finite group $G$ acts on a metric space $X$ by bornologous functions, then the projection $p : X \to X/G$ is coarsely $|G|$ to 1 if $X/G$ is given the Hausdorff metric.

Proof. Change the original metric $\rho$ on $X$ to

$$d(x, y) := \sum_{g \in G} \rho(g \cdot x, g \cdot y)$$

and notice it is coarsely equivalent to $\rho$. Therefore, the Hausdorff metrics induced by both $d$ and $\rho$ on $X/G$ are coarsely equivalent. Notice $G$ acts on $X$ via isometries.
with respect to the metric \(d\). That means we can reduce our proof to the case of \(G\) acting on \(X\) by isometries.

First, notice that \(p: X \to X/G\) is 1-Lipschitz, hence bornologous. Indeed, if \(d(x, y) < r\) and \(z \in G \cdot y\), then \(z = g \cdot y\) for some \(g \in G\) and \(d(g \cdot x, g \cdot y) = d(x, y) < r\). That means \(z \in B(G \cdot x, r)\) and \(d(G \cdot x, G \cdot y) \leq d(x, y)\).

Given \(x \in X\) and given \(r > 0\) the point-inverse \(p^{-1}(B(x, r))\) is contained in \(B(G \cdot x, r)\) which is clearly the union \(\bigcup_{y \in G \cdot x} B(y, r)\) of at most \(|G|\) many sets of diameter at most \(2 \cdot r\). \(\square\)

It is easy to check that the property of being coarsely \(n\)-to-1 is a coarse property: if two maps are coarsely equivalent and one of them is coarsely \(n\)-to-1 then the other one is coarsely \(n\)-to-1 as well. Furthermore, the composition of a coarsely \(n\)-to-1 map with a coarse equivalence (from the left or from the right) is coarsely \(n\)-to-1.

**Definition 4.3.** A set \(A\) is \(R\)-connected if for every pair of points \(x, y \in A\) there exist points \(x_0 = x, x_1, \ldots, x_k = y\) in \(A\) for which \(d(x_i, x_{i+1}) \leq R\). An \(R\)-component is a maximal \(R\)-connected set.

**Lemma 4.4.** If a function \(f: X \to Y\) of metric spaces is coarsely \(n\)-to-1 with control \(C\), then for each subset \(B \subseteq Y\) with \(\text{diam}(B) \leq r\), the number of \(R\)-components of \(f^{-1}(B)\), \(R \geq C(r)\), is at most \(n\) and each \(R\)-component has diameter at most \(2n \cdot R\).

**Proof.** Given a subset \(B \subseteq Y\) with \(\text{diam}(B) \leq r\), express \(f^{-1}(B)\) as \(\bigcup_{i=1}^{n} A_i\) for some subsets \(A_i\) of \(X\) with \(\text{diam}(A_i) \leq C(r)\) for \(i = 1, \ldots, n\). Consider \(R \geq C(r)\) and pick an \(R\)-component \(A\) of \(f^{-1}(B)\). Notice \(A\) is the union of some sets among the family \(\{A_i\}_{i=1}^{n}\). Also, every two points in \(A\) can be connected by an \(R\)-chain of points. If that chain contains two points from the same set \(A_i\), then it can be shortened by eliminating all points between them. That means there is an \(R\)-chain that has at most \(2n\) points and \(\text{diam}(A) \leq 2n \cdot R\). \(\square\)

**Lemma 4.5** (Lemma 3.6 of [11]). Suppose \(f: X \to Y\) is coarsely \(n\) to \(1\) with control \(C\). Then for every cover \(\mathcal{U}\) of \(X\) and for every \(r > 0\) we have

\[
\dim_r(f(\mathcal{U})) \leq (\dim_{C(r)}(\mathcal{U}) + 1) \cdot n - 1
\]

**Proof.** Assume \(\dim_{C(r)}(\mathcal{U}) = m < \infty\). Let \(C\) be a control function of \(f\). We may assume \(C(r) \to \infty\) as \(r \to \infty\). Let’s count the number of elements of \(f(\mathcal{U})\) that intersect a given set \(B\) of diameter at most \(r\). It can be estimated from above by the sum of the numbers of elements of \(\mathcal{U}\) intersected by sets \(A_i\), where \(f^{-1}(B)\) is expressed as \(\bigcup_{i=1}^{n} A_i\) for some subsets \(A_i\) of \(X\) with \(\text{diam}(A_i) \leq C(r)\) for \(i = 1, \ldots, n\).

Each of those sets intersects at most \((m + 1)\) elements of \(\mathcal{U}\), so the total estimate for \(f^{-1}(B)\) is \(n \cdot (m + 1)\). That proves \(\dim_r(f(\mathcal{U})) \leq (\dim_{C(r)}(\mathcal{U}) + 1) \cdot n - 1\). \(\square\)

**Corollary 4.6** (Miyata-Virk [11]). Suppose \(f: X \to Y\) is a surjective function of metric spaces that is coarsely \(n\)-to-1 for some \(n \geq 1\). If \(\text{asdim}(X)\) is finite, then \(\text{asdim}(Y) \leq n \cdot (\text{asdim}(X) + 1) - 1\).

**Definition 4.7.** \(f: X \to Y\) is coarsely surjective if \(Y \subseteq B(f(X), R)\) for some \(R > 0\).
**Proposition 4.8** (Structure of coarsely n to 1 maps). Every coarsely n-to-1 map \( f : X \to Y \) factors as \( f = q \circ p \), where \( p : X \to Z \) is a coarse equivalence, \( q : Z \to Y \) is coarsely n to 1, and \( q^{-1}(y) \) has at most n points for each \( y \in Y \).

**Proof.** Given a metric \( \rho \) on \( X \) change it to the one defined by \( d(x, y) = \max(1, \rho(x, y)) \) if \( x \neq y \), and note it is coarsely equivalent to \( \rho \). Using Lemma 4.4 find \( R > 0 \) so that \( R \)-components of fibers \( f^{-1}(\{y\}) \) of \( f \) have diameter at most \( 2n \cdot R \) and there are at most \( n \) of them. Define an equivalence relation \( \sim \) on \( X \) as follows: \( x \sim z \) if and only if \( f(x) = f(z) \) and both \( x \) and \( z \) belong to the same \( R \)-component of their fiber. By Theorem 3.2 the projection \( p : X \to X/\sim \) is a coarse equivalence if \( X/\sim \) is equipped with the Hausdorff metric. Obviously, there is \( q : X/\sim \to Y \) such that \( f = q \circ p \) and each fiber of \( q \) has at most \( n \) elements. As \( p \) is a coarse equivalence, \( q \) is bornologous and coarsely n-to-1. \( \square \)

**Proposition 4.9.** Suppose \( f : X \to Y \) is coarsely n-to-1 with control \( D \) and coarse with control \( E \). Suppose \( \mathcal{U} \) is a cover of \( X \).

1. If \( \mathcal{U} \) is \( b \)-bounded and of \( D(r) \)-dimension \( m \) then \( f(\mathcal{U}) \) is \( E(b) \)-bounded of \( r \)-dimension at most \( (m + 1) \cdot n \);
2. If \( \mathcal{U} \) is \( b \)-bounded and of \( D(r) \)-dimension \( m \) then there exists \( (E(b) + r) \)-bounded cover \( \mathcal{V} = \bigcup_{i=1}^{n(m+1)} V^i \) of \( f(X) \) so that each \( V^i \) is a \( \frac{r}{(n(m+1))} \)-disjoint family.

**Proof.** (1) follows from Lemma 4.6

For (2) use (1) and Lemma 2.7 \( \square \)

5. **Asymptotic dimension of functions**

The well-known Hurewicz Theorem for maps (also known as Dimension-Lowering Theorem, see [8, Theorem 1.12.4 on p.109]) says \( \dim(X) \leq \dim(f) + \dim(Y) \) if \( f : X \to Y \) is a closed map of separable metric spaces and \( \dim(f) \) is defined as the supremum of \( \dim(f^{-1}(y)) \), \( y \in Y \). Bell and Dranishnikov [1] proved a variant of the Hurewicz Theorem for asymptotic dimension without defining the asymptotic dimension of a function. However, Theorem 1 of [1] may be restated as \( \text{asdim}(X) \leq \text{asdim}(f) + \text{asdim}(Y) \), where \( \text{asdim}(f) \) is the smallest integer \( n \) such that \( \text{asdim}(f^{-1}(B_R(y))) \leq n \) uniformly for all \( R > 0 \).

**Definition 5.1.** Given a function \( f : X \to Y \) of metric spaces, its **asymptotic dimension** \( \text{asdim}(f) \) is the supremum of \( \text{asdim}(A) \) such that \( A \subset X \) and \( \text{asdim}(f(A)) = 0 \).

In [1] there is a concept of a family \( \{X_n\} \) of subsets of \( X \) satisfying \( \text{asdim}(X_n) \leq n \) uniformly. Notice that in our language this means there is one function that serves as an \( n \)-dimensional control function for all \( X_n \).

**Corollary 5.2** (Bell-Dranishnikov [1]). Let \( f : X \to Y \) be a Lipschitz function of metric spaces. Suppose that, for every \( R > 0 \),

\[ \text{asdim}\{f^{-1}(B_R(y))\} \leq n \]

uniformly (in \( y \in Y \)). If \( X \) is geodesic, then \( \text{asdim}(X) \leq \text{asdim}(Y) + n \).

**Corollary 5.3.** If \( f \) is a coarsely n-to-1 function, then \( \text{asdim}(f) = 0 \).
Proof. Let $C$ be a control function of $f$. By [4.4] for each subset $B$ of $Y$ with \( \text{diam}(B) \leq r \) and for each $R > C(r)$, the number of $R$-components of $f^{-1}(B)$ is at most $n$ and each $R$-component has diameter at most $n \cdot R$. The last fact is sufficient to conclude $\text{asdim}(f) \leq 0$. \qed

6. Preservation of coarse properties

The last several sections will be devoted to the issue of preservation of coarse invariants by coarsely $n$-to-1 functions: MSP, finite decomposition complexity, countable asymptotic dimension, Asymptotic Property C, and Property A.

Theorem 6.1. If $f : X \to Y$ is coarsely $n$-to-1, coarse, and coarsely surjective, then $\text{asdim}X \leq \text{asdim}Y \leq (\text{asdim}X + 1)n - 1$.

Proof. The first inequality follows easily from Proposition 4.9 (3) when using the definition of asymptotic dimension in terms of $n$-many $R$-disjoint uniformly bounded families covering the space. The second inequality is the main result of [11]. \qed

Theorem 6.2. Suppose $f : X \to Y$ is coarsely $n$-to-1 and coarsely surjective. If $X$ has Asymptotic Property C, then $Y$ has Asymptotic Property C.

Proof. We may assume $f$ to be surjective as all properties in question are coarse invariants. Pick functions $C, E : [0, \infty) \to [0, \infty)$ such that $d_Y(f(x), f(y)) \leq E(d_X(x, y))$ for all $x, y \in X$ and for each subset $B$ of $Y$ with \( \text{diam}(B) \leq r \), the number of $C(r)$-components of $f^{-1}(B)$ is at most $n$ and each $C(r)$-component has diameter at most $2n \cdot C(r)$. We may assume $E(r) \to \infty$ as $r \to \infty$.

Suppose $X$ has asymptotic property C and $R_1 < R_2 < \ldots$. There exists $m$ and there are uniformly bounded families $U_i, i = 1, \ldots, m$ such that the family $U = \bigcup_{i=1}^m U_i$ is a cover of $X$ and each $U_i$ is $C(n \cdot R_{i,n})$-disjoint. By Proposition 4.9 we have $f(U_i) = \bigcup_{j=1}^n V_{i,j}$ where $V_{i,j}$ is an $R_{i,n}$-disjoint and uniformly bounded family for $j \in \{1, \ldots, n\}$. We have obtained a collection of $m \cdot n$ uniformly bounded families $V_{i,j}$ covering $f(X)$. Furthermore, $V_{i,j}$ is $R_{n,(i-1)+j}$-disjoint as it is $R_{i,n}$-disjoint and $R_{n,(i-1)+j} \leq R_{in}$. This proves the theorem. \qed

Theorem 6.3. Suppose $f : X \to Y$ is coarsely surjective of asymptotic dimension $\text{asdim}(f) = 0$. If $Y$ has Asymptotic Property C, then $X$ has Asymptotic Property C.

Proof. Pick a control function $E : [0, \infty) \to [0, \infty)$ of $f$. Suppose $Y$ has asymptotic property C. Choose $R_1 < R_2 < \ldots$. There exist $m$ and uniformly bounded families $U_i, i = 1, \ldots, m$ such that the family $U = \bigcup_{i=1}^m U_i$ is a cover of $Y$ and each $U_i$ is $E(R_i)$-disjoint. Therefore each $f^{-1}(U_i)$ is $R_i$-disjoint. Since $\text{asdim}(f) = 0$, there is $M > 0$ such that each $R_m$-component of $f^{-1}(U), U \in U_i$ is of diameter at most $M$. If we define $V_i$ as consisting of $R_m$-components of $f^{-1}(U), U \in U_i$, then each $V_i$ is uniformly bounded, $R_i$-disjoint, and $X = \bigcup_{i=1}^m V_i$. That concludes the proof of $X$ having asymptotic property C. \qed
Question 6.4. Suppose \( f : X \to Y \) is coarsely surjective of finite asymptotic dimension \( \operatorname{asdim}(f) \). If \( Y \) has Asymptotic Property C, does \( X \) have Asymptotic Property C?

Corollary 6.5. If \( f : X \to Y \) is coarsely \( n \)-to-1 and coarsely surjective, then \( X \) has Asymptotic Property C if and only if \( Y \) has Asymptotic Property C.

7. Metric Sparsification Property

Definition 7.1. (see [4]) A metric space \( X \) has MSP (Metric Sparsification Property) if for all \( R > 0 \) and for each positive \( c < 1 \) there exists \( S > 0 \) such that for all probability measures \( \mu \) on \( X \) there exists an \( R \)-disjoint family \( \{\Omega_i\}_{i \geq 1} \) of subsets of \( X \) of diameter at most \( S \) satisfying

\[
\sum_{i=1}^{\infty} \mu(\Omega_i) > c.
\]

Remark 7.2. As noted in [3] a metric space \( X \) has MSP if and only if there is \( c > 0 \) such that for all \( R > 0 \) there exists \( S > 0 \) with the property that for all probability measures \( \mu \) on \( X \) there exists an \( R \)-disjoint family \( \{\Omega_i\}_{i \geq 1} \) of subsets of \( X \) of diameter at most \( S \) satisfying

\[
\sum_{i=1}^{\infty} \mu(\Omega_i) > c.
\]

Proposition 7.3. If \( \operatorname{asdim}(X) \) is finite, then \( X \) has MSP.

Proof. Apply 2.7 to detect, for each \( R > 0 \), a uniformly bounded cover that decomposes into a union of \( (n+1) \) families, each \( R \)-disjoint. Therefore, given a probability measure \( \mu \) on \( X \), one of those families adds up to a subset \( \Omega \) of \( X \) whose measure is at least \( \frac{1}{n+1}. \)

Theorem 7.4. If \( f : X \to Y \) is coarsely \( n \)-to-1, \( f \) is coarsely surjective, and \( X \) has MSP, then \( Y \) has MSP.

Proof. We may assume \( f \) to be surjective as all properties in question are coarse invariants. Suppose \( f \) is coarsely \( n \)-to-1 with control \( D \), and bornologous with control \( E \). We have to find, for each \( R \), an \( S > 0 \) such that for any probability measure \( \mu \) on \( Y \) there is an \( R \)-disjoint family \( \Omega_j \) in \( Y \) so that

\[
\mu(\bigcup_{j=1}^{\infty} \Omega_j) > 1/(2n)
\]

and diameter of each \( \Omega_j \) is at most \( S \). As \( X \) has MSP there exists \( B > 0 \) so that for every probability measure \( \lambda \) on \( X \) there is a \( D(nR) \)-disjoint family \( \Omega'_j \) in \( X \) so that

\[
\lambda(\bigcup_{i=1}^{\infty} \Omega'_i) > 1/2
\]

and diameter of each \( \Omega'_i \) is at most \( B \). We will prove that \( S = E(B) + nR \) suffices.

Step 1: Transferring a measure to \( X \). Suppose \( \mu \) is a probability measure on \( Y \). For each \( y \in Y \) choose (by surjectivity) \( x_y \in f^{-1}(\{y\}) \). Define a probability measure \( \lambda \) on \( X \) by

\[
\lambda(A) = \mu(\{y \in Y \mid x_y \in A\}), \quad \text{for every } A \subset X.
\]

□

\[
\mu(\bigcup_{j=1}^{\infty} \Omega_j) > 1/(2n)
\]

and diameter of each \( \Omega_j \) is at most \( S \). As \( X \) has MSP there exists \( B > 0 \) so that for every probability measure \( \lambda \) on \( X \) there is a \( D(nR) \)-disjoint family \( \Omega'_j \) in \( X \) so that

\[
\lambda(\bigcup_{i=1}^{\infty} \Omega'_i) > 1/2
\]

and diameter of each \( \Omega'_i \) is at most \( B \). We will prove that \( S = E(B) + nR \) suffices.

Step 1: Transferring a measure to \( X \). Suppose \( \mu \) is a probability measure on \( Y \). For each \( y \in Y \) choose (by surjectivity) \( x_y \in f^{-1}(\{y\}) \). Define a probability measure \( \lambda \) on \( X \) by

\[
\lambda(A) = \mu(\{y \in Y \mid x_y \in A\}), \quad \text{for every } A \subset X.
\]
Note that $\lambda(A) = \mu(f(A))$, $\forall A \subset X$. Choose a $D(nR)$-disjoint family $\Omega'_i$ in $X$ so that

$$\lambda\left(\bigcup_{i=1}^{\infty} \Omega'_i\right) > 1/2$$

and diameter of each $\Omega'_i$ is at most $B$. In particular, the $D(nR)$-dimension of the collection $\{\Omega'_i\}$ is at most 1.

**Step 2:** Transferring a cover to $Y$. By Proposition 4.9 (2) (for $\mu = 1$) there exists an $(E(b) + nR)$-bounded cover $V = \bigcup_{i=1}^{n} V^i$ of $f(\bigcup_{i=1}^{\infty} \Omega'_i)$ so that each $V^i$ is an $(n \cdot r)/n$-disjoint family. As $\mu(f(\bigcup_{i=1}^{\infty} \Omega'_i)) = 1/2$, there exists $i_0$ so that $\mu(V^{i_0}) \geq 1/(2n)$. This completes the proof as $\{\Omega_j\} := V^{i_0}$ works.

**Corollary 7.5.** Suppose $X$ and $Y$ are of bounded geometry. If $f : X \to Y$ is coarsely $n$-to-1, $f$ is coarsely surjective, and $X$ has Property A, then $Y$ has Property A.

**Proof.** It was proved in [3] that Property A is equivalent to MSP for spaces of bounded geometry.

**Definition 7.6.** A bornologous function $f : X \to Y$ of metric spaces has MSP if $f^{-1}(A)$ has MSP for every subset $A$ of $Y$ of asymptotic dimension 0.

**Corollary 7.7.** If $\text{asdim}(f)$ is finite or $f$ is coarsely $n$-to-1, then $f$ has MSP.

**Proposition 7.8.** $f : X \to Y$ has MSP if and only if for every $c, R, K > 0, c < 1$ there is $S > 0$ such that for every probability measure $\mu$ on $X$ such that the diameter of $f(\text{supp}(\mu))$ is less than $K$ there is a subset $\Omega$ of $X$ whose $R$-components are $S$-bounded and $\mu(\Omega) > c$.

**Proof.** Suppose there is $c, R, K > 0, c < 1$ such that for every $n > 1$ there is a probability measure $\mu_n$ on $X$ with diameter of $f(\text{supp}(\mu_n))$ less than $K$ such that for any subset $\Omega$ of $X$ satisfying $\mu_n(\Omega) > c$ there is an $R$-component of $\Omega$ of diameter bigger than $n$.

$$\bigcup_{n>1} f(\text{supp}(\mu_n))$$

cannot be a bounded set and, by picking a subsequence of measures, we may achieve $A = \bigcup_{n>1} f(\text{supp}(\mu_n))$ being of asymptotic dimension 0. Since $f^{-1}(A)$ has MSP, there is $S > 0$ such that for any measure $\mu$ on $X$ there is a subset $\Omega$ of $X$ whose $R$-components have diameter at most $n$ and $\mu(\Omega) > c$. Pick $n > S$ and consider $\mu = \mu_n$. The set $\Omega$ picked for that measure has an $R$-component of diameter bigger than $n$, a contradiction.

**Theorem 7.9.** If $f : X \to Y$ has MSP and $Y$ has MSP, then $X$ has MSP.

**Proof.** Suppose $R_X > 0$. Pick $R_Y > 0$ such that $d_X(x, y) \leq R_X$ implies $d_Y(f(x), f(y)) \leq R_Y$. Choose $K > 0$ such that for any probability measure $\mu$ on $Y$ there is a subset $\Omega$ of measure bigger than 0.5 whose $R_Y$-components are $K$-bounded. Pick $S > 0$ with the property that for every probability measure $\mu$ on $X$ such that the diameter of $f(\text{supp}(\mu))$ is less than $K$ there is a subset $\Omega$ of $X$ whose $R$-components are $S$-bounded and $\mu(\Omega) > 0.5$. 

Given a probability measure $\mu$ on $X$ transfer it to $Y$ as follows: $\lambda(A) = \mu(f^{-1}(A)), \forall A \subset Y$. Find a subset $\Lambda$ of $Y$ satisfying $\lambda(\Lambda) > 0.5$ whose $R_Y$-components are $K$-bounded. Only countably many $\Lambda_i$ of those $R_Y$-components are of interest as the rest have measure 0. Given $i$ consider $f^{-1}(\Lambda_i)$ and find a subset $\Omega_i$ of it whose $R_X$-components are $S$-bounded and $\mu(\Omega_i) > 0.5 \cdot \mu(f^{-1}(\Lambda_i))$. Look at $\Omega = \bigcup_i \Omega_i$ and notice its $R_X$-components are $S$ bounded and $\mu(\Omega) > 0.25$.

8. Countable asymptotic dimension

Countable asymptotic dimension was introduced in [7] as a generalization of the concept of straight finite decomposition complexity introduced by Dranishnikov and Zarichnyi [6]. One of the main results of this section is that actually the two concepts are equivalent.

A partition of a set is a covering by disjoint sets. We also introduce a notation: if $U$ is a collection of subsets of $X$ and $A$ is a subset of $X$ then $A \cap U = \{A \cap U : U \in U\}$.

Definition 8.1. $X$ is of straight finite decomposition complexity [6] if for any increasing sequence of positive real numbers $R_1 < R_2 < \ldots$ there a sequence $V_i, i \leq n$, of families of subsets of $X$ such that the following conditions are satisfied:

1. $V_1 = \{X\}$,
2. each element $U \in V_i, i < n$, can be expressed as a union of at most 2 families from $V_{i+1}$ that are $R_i$-disjoint,
3. $V_n$ is uniformly bounded.

Definition 8.2. A metric space $X$ is of countable asymptotic dimension if there is a sequence of integers $n_i \geq 1, i \geq 1$, such that for any sequence of positive real numbers $R_i, i \geq 1$, there is a sequence $V_i$ of families of subsets of $X$ such that the following conditions are satisfied:

1. $V_1 = \{X\}$,
2. each element $U \in V_i$ can be expressed as a union of at most $n_i$ families from $V_{i+1}$ that are $R_i$-disjoint,
3. at least one of the families $V_i$ is uniformly bounded.

Corollary 8.3. In the definition of spaces of countable asymptotic dimension we may assume each $V_i$ to be a partition of $X$, i.e., a disjoint collection of subsets covering $X$.

Proof. Use Lemma 8.6 as an inductive step. The initial step for $i = 1$ holds as $\{X\}$ is an obvious partition of itself. □

Theorem 8.4. Metric space $X$ is of countable asymptotic dimension if and only if it is of straight finite decomposition complexity (sFDC).

Proof. One direction is fairly simple by definition. Suppose $X$ is of countable asymptotic dimension. Choose $R_1 < R_2 < \ldots$. As $X$ is of countable asymptotic dimension we obtain a sequence of families $V_i$ corresponding to $R_{n_1} < R_{n_1+n_2} < R_{n_1+n_2+n_3} < \ldots$. In particular, we may assume $V_2$ consists of $R_{n_1}$-disjoint families $W_{1i}, \ldots, W_{n_1}$ covering $X$. Let $W_i$ denote the union of all elements of $W_i$. We will construct families $U_i$ corresponding to the definition of the sFDC for $i \in \{1, \ldots, n_1\}$ inductively:

1. $U_1 = \{X\};$
2. \( U_2 = W_1 \cup \bigcup_{i=2}^{n_1} W_i \). \( X \), the only element of \( U_1 \), is a union of at most two \( R_1 \)-disjoint families in \( U_2 \):
   2a. \( W_1 \), which is \( R_{n_1} \)-disjoint by definition hence it is also \( R_1 \)-disjoint;
   2b. \( \{ \cup_{i=2}^{n_1} W_i \} \) which is \( R_1 \)-disjoint as it is a one-element collection.

3. the inductive step is the following: for \( m < n_1 \) define \( U_m = W_1 \cup \ldots \cup W_{m-1} \cup \bigcup_{i=m}^{n_1} W_i \). Every element of \( U_m \) is a union of at most two \( R_m \)-disjoint families in \( U_m \): every element of \( W_1 \cup \ldots \cup W_{m-1} \cup \bigcup_{i=m}^{n_1} W_i \) can be expressed as a union of two families:
   3a. \( W_{m-1} \), which is \( R_{n_1} \)-disjoint by definition hence it is also \( R_n \)-disjoint;
   3b. \( \{ \bigcup_{i=m}^{n_1} W_i \} \) which is \( R_n \)-disjoint as it is a one element collection.

4. \( U_{n_1} = W_1 \cup \ldots \cup W_{n_1} \). Again, every element of \( U_{n_1} \) is a union of at most two \( R_{n_1} \)-disjoint families in \( U_m \): every element of \( W_1 \cup \ldots \cup W_{n_1-2} \) appears in \( U_{n_1} \) as well and \( \bigcup_{i=n_1-2}^{n_1} W_i \) can be expressed as a union of two families:
   4a. \( W_{n_1-1} \), which is \( R_{n_1} \)-disjoint by definition;
   4b. \( W_{n_1} \), which is \( R_{n_1} \)-disjoint by definition.

We have thus obtained \( U_{n_1} = V_2 \). Proceed in the same way for every element of \( V_2 \) to obtain \( U_i \) for \( i \in \{ n_1 + 1, \ldots, n_1 + n_2 \} \) with \( U_{n_1+n_2} = V_2 \). By induction we eventually obtain a uniformly bounded family. \( \square \)

**Proposition 8.5.** If \( X \) is of countable asymptotic dimension then for every \( R \) there exists \( n \) so that \( X \) can be covered by \( n \)-many collections \( U_1, \ldots, U_n \) of subsets of \( X \), all of which are uniformly bounded and \( R \)-disjoint.

**Proof.** Suppose \( X \) is of countable asymptotic dimension and choose covers \( V_i \) corresponding to sequence \( R_i := R + i \). By the definition we eventually obtain a collection \( V_i \), which can be decomposed into a finite number of uniformly bounded \( R_i \)-disjoint families, hence they are also \( R \)-disjoint, which suffices. \( \square \)

**Lemma 8.6.** Suppose \( U_i, V_{i+1} \) and \( V_{i+2} \) are covers of space \( X \) with the following properties.

1. \( U_i \) is a partition of \( X \);
2. each element of \( U_i \) is contained in a union of at most \( n_i \) families from \( V_{i+1} \) that are \( R_i \)-disjoint;
3. each element of \( V_{i+1} \) can be expressed as a union of at most \( n_{i+1} \) families from \( V_{i+2} \) that are \( R_{i+1} \)-disjoint.

Then there exists a partition \( U_{i+1} \) of \( X \) so that:

- a: each element of \( U_i \) can be expressed as a union of at most \( n_i \) families from \( U_{i+1} \) that are \( R_i \)-disjoint;
- b: each element of \( U_{i+1} \) is contained in a union of at most \( n_{i+1} \) families from \( V_{i+2} \) that are \( R_{i+1} \)-disjoint;
- c: each element of \( U_{i+1} \) is contained in some element of \( V_{i+1} \).

**Proof.** Choose \( U \in U_i \) and let \( W_1, W_2, \ldots, W_{n_i} \) denote a collection of \( R_i \)-disjoint families from \( V_{i+1} \) so that \( \bigcup_{i=1}^{n_i} W_i \) contains \( U \). Define \( \widetilde{W}_i = \{ U \cap W \mid W \in W_i \} \) for every \( i \) and note that \( \widetilde{W}_1, \widetilde{W}_2, \ldots, \widetilde{W}_{n_i} \) is a collection of \( R_i \)-disjoint families for which \( \widetilde{W} = \bigcup_{i=1}^{n_i} \bigcup \widetilde{W}_i \) equals \( U \). The collection of subsets \( \bigcup_{i=1}^{n_i} \widetilde{W}_i \) may be well-ordered in the form \( \{ W_j \}_{j \in J} \) where \( J \) is a well-ordered set. Define a collection \( \{ U_j \}_{j \in J} \) by the rule

\[ \forall j \in J : \quad U_j = W_j \setminus \bigcup_{k<j} W_k. \]
Note that \{U_{j}\}_{j \in J} is a partition of \(U\). Undo the well-ordering by reindexing sets \{U_{j}\}_{j \in J} back into a collection of \(R\)-disjoint families \(\hat{W}_1, \hat{W}_2, \ldots, \hat{W}_n\), which together constitute a partition of \(U\); the reindexing should be exactly the inverse to previous well-ordering (i.e., if \(W \in \hat{W}_1\) was given index \(j_0 \in J\) then \(U_{j_0}\) should belong to \(\hat{W}_1\)) and \(R\)-disjointness is preserved as we have only decreased the sets. Let \(U' = \bigcup_{k \geq 1}^{n} \hat{W}_k\) denote a collection of obtained sets.

Since \(U_t\) is a partition of \(X\) and \(U_{t'}\) is a partition of \(U\) for every \(U \in U_t\), the collection \(U_{t+1} = \bigcup_{U \in U_t} U_{t'}\) of subsets is a partition of \(X\). Furthermore, each element of \(U_t\) can be expressed as a union of at most \(n_t\) families from \(U_{t+1}\) that are \(R_t\)-disjoint by construction. This proves a.

To prove b note that every element of \(U_{t+1}\) was obtained by taking an intersection of some element of \(V_{t+1}\) by some sets. In particular, every element of \(U_{t+1}\) is contained in some element of \(V_{t+1}\).

Statement b follows from c and (3).

\[\square\]

**Theorem 8.7.** Suppose \(f : X \rightarrow Y\) is coarsely \(n\)-to-1 with control \(D\), coarsely surjective and coarse with control \(E\). Then \(Y\) is of countable asymptotic dimension if and only if \(X\) is of countable asymptotic dimension.

**Proof.** Suppose \(Y\) is of countable asymptotic dimension. Choose \(R_1 < R_2 < \ldots\). According to the definition of the countable asymptotic dimension choose for \(E(R_1) < E(R_2) < \ldots\) a sequence \(V_i\) of families of subsets of \(Y\). Define \(U_t = \{f^{-1}(V) \mid V \in V_i\}\). Sequence \(U_t\) (actually its finite subsequence, see (3) below in the proof) proves \(X\) to be of countable asymptotic dimension:

1. \(U_1 = \{X\}\).
2. if \(V \in V_i\) can be expressed as a union of at most \(n_i\)-many \(E(R_i)\)-disjoint families from \(V_{i+1}\) then \(f^{-1}(V)\) can be expressed as a union of at most \(n_i\)-many \(R_i\)-disjoint families from \(U_{i+1}\) by taking the preimages and applying Lemma 2.6.
3. suppose \(V_m\) is uniformly bounded by \(b\). We will redefine \(U_{m+1}\). For every \(V \in V_m\) the preimage \(f^{-1}(V)\) can be expressed as a disjoint union of at most \(n\)-many \((nD(b) + (n - 1)R_t)\)-bounded \(R_t\)-disjoint subsets of \(X\). Let \(U_{m+1}\) consist of all such sets. The collection \(U_{m+1}\) is uniformly bounded by \(D(b)\) and every element of \(U_{m+1}\) can be expressed as a union of at most \(n\)-many elements of \(U_{m+1}\). This concludes the proof. For the sake of formal argument we may define \(U_{m+1} = U_j, \forall j \geq m + 1\) and note that the sequence of integers for \(X\) may be taken to be \((\max\{n_i, n\})_i\).

Suppose \(X\) is of countable asymptotic dimension. We may assume \(f\) to be surjective: the justification is a simple exercise. Choose \(R_1 < R_2 < \ldots\). We will define a sequence \(V_i\) of families of subsets of \(Y\) satisfying the conditions in the definition of the countable asymptotic dimension for parameters \(R_1 < R_2 < \ldots\).

According to the definition of the countable asymptotic dimension for \(X\) there exists a sequence \(\{n_i\}_{i \geq 1}\) such that for \(D(nm_1R_1) < D(nm_2R_2 + 2nm_1R_1) < \ldots\) (the pattern of increasing parameters is not yet visible; however, it will become apparent that appropriate parameters may be chosen depending on \(\{n_i\}\), \(\{R_i\}\) and \(n\)) there exists a sequence \(U_t\) of partitions of \(X\) with appropriate properties. Define \(V_1 = \{Y\}\).
Collection $\mathcal{U}_2$ is a partition of $X$ of $D(nn_1R_1)$-multiplicity at most $n_1$. By Lemma 4.5 the collection $f(\mathcal{U}_2)$ is a cover of $Y$ of $(nn_1R_1)$-multiplicity at most $nn_1$. By Lemma 4.7 there exists a cover $\mathcal{V}_2$ of $Y$ consisting of $(nn_1)$-many $R_1$-disjoint families with the following property:

every element of $\mathcal{V}_2$ is contained in the $(nn_1R_1)$-neighborhood of some element of $f(\mathcal{U}_2)$. (*)

The next step: the definition of $\mathcal{V}_3$ is actually an inductive step in the construction of sequence $\mathcal{V}_i$. For the sake of simplicity we only present it for the case $i = 3$.

Choose any $U \in \mathcal{U}_2$. Recall that $U$ may be covered by a collection of $n_2$-many $D(nn_2R_2 + 2nn_1R_1)$-disjoint families $\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_{n_2}$ (some of these families may be empty in order to get exactly $n_2$-many of them). Let $\mathcal{W} = \bigcup_{j=1}^{n_2} \mathcal{W}_j$ denote the corresponding partition of $U$. By Lemma 4.5 the collection $f(\mathcal{W})$ is a cover of $f(U)$ of $(nn_2R_2 + 2nn_1R_1)$-multiplicity at most $nn_2$. We now expand the sets by $nn_1R_1$ as suggested by (*): the collection of $(nn_1R_1)$-neighborhoods of family $\mathcal{W}$ is a cover of the $(nn_1R_1)$-neighborhood of $f(U)$ of $(nn_2R_2)$-multiplicity at most $nn_2$. By Lemma 4.7 there exists a cover $\mathcal{V}_3^U$ of the $(nn_1R_1)$-neighborhood of $f(U)$ consisting of $(nn_2)$-many $R_2$-disjoint families with the following property:

every element of $\mathcal{V}_3^U$ is contained in $(nn_2R_2 + nn_1R_1)$-neighborhood of some element of $f(\mathcal{U}_3)$. (**)

Define a collection

$$\mathcal{V}_3 = \bigcup_{U \in \mathcal{U}_2} \mathcal{V}_3^U.$$ 

Observe the following properties of obtained $\mathcal{V}_3$:

a: The collection $\mathcal{V}_3$ covers $Y$ as every $\mathcal{V}_3^U$ covers $f(U)$ for every $U \in \mathcal{U}_2$ and the collection $\{f(U)\}_{U \in \mathcal{U}_2}$ covers $Y$;

b: Every element of $\mathcal{V}_2$ is by (*) contained in $(nn_1R_1)$-neighborhood of $f(U)$ for some element $U \in \mathcal{U}_2$. By construction $\mathcal{V}_3^U$ is a cover of the $(nn_1R_1)$-neighborhood of $f(U)$ consisting of $(nn_2)$-many $R_2$-disjoint families. In particular: every element of $\mathcal{V}_2$ can be covered by a collection of $(nn_2)$-many $R_2$-disjoint families from $\mathcal{V}_3$.

Note that a. and b. are conditions required by the definition of the countable asymptotic dimension.

Proceed by induction: use the (*)-type conditions (i.e., conditions stating "every element of $\mathcal{V}_i$ is contained in the $L(\{n_i\}, \{R_i\}, n)$-neighborhood of some element of $f(\mathcal{U}_i)$." Examples are condition (*) and condition (**) for all $U_i$.) to continue the sequence $D(nn_1R_1) < D(nn_2R_2 + 2nn_1R_1) < \ldots$ of disjointness of $\mathcal{U}_i$: the sequence depends only on $\{n_i\}, \{R_i\}$ and $n$. Note that if $\mathcal{U}_i$ is uniformly bounded then the obtained $\mathcal{V}_i$ is (again, by the (*)-type condition) uniformly bounded as well, which completes the proof.

References

[1] G. Bell and A. Dranishnikov, A Hurewicz-type theorem for asymptotic dimension and applications to geometric group theory, Trans. Amer. Math. Soc. 358 (2006), no. 11, 4749–4764.

[2] N. Brodskiy, J.Dydak, M. Levin, and A.Mitra, Hurewicz Theorem for Assouad-Nagata dimension, Journal of the London Math. Soc. (2008) 77 (3): 741–756.

[3] J. Brodzki, G. A. Niblo, J. Špakula, R. Willett, and N. Wright, Uniform local amenability, J. Noncommut. Geom. 7 (2013), no. 2, 583–603.
[4] Xiaoman Chen, Romain Tessera, Xianjin Wang, and Guoliang Yu. Metric sparsification and operator norm localization. Adv. Math., 218(5):1496–1511, 2008.
[5] A. Dranishnikov, Asymptotic Topology, Russian Math. Surveys 55:6 (2000), 1085–1129.
[6] A. Dranishnikov and M. Zarichnyi, Asymptotic dimension, decomposition complexity, and Haver’s property C, arXiv:1301.3484.
[7] J. Dydak, Coarse Amenability and Discreteness, arXiv:1307.3943.
[8] J. Dydak and Ž. Virk, Inducing maps between Gromov boundaries, in preparation.
[9] R. Engelking, Theory of dimensions finite and infinite, Sigma Series in Pure Mathematics, vol. 10, Heldermann Verlag, 1995.
[10] M. Gromov, Asymptotic invariants for infinite groups, in Geometric Group Theory, vol. 2, 1–295, G. Niblo and M. Roller, eds., Cambridge University Press, 1993.
[11] T. Miyata and Ž. Virk, Dimension-Raising Maps in a Large Scale, Fundamenta Mathematicae 223 (2013), 83-97.

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