Mellin-type Functional Integrals with Applications

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Abstract

Conventional functional/path integrals used in physics are most often defined as the infinite-dimensional analog of Fourier transform. Likewise, the infinite-dimensional analog of Mellin transform also defines a class of functional integrals. The associated functional integrals are useful tools for probing non-commutative function spaces in general and $C^*$-algebras in particular. Functional Mellin transforms can be used to define the functional analogs of resolvents, complex powers, traces, logarithms, and determinants. Several aspects of these objects are explored and applied to various constructs in mathematical physics. As specific examples, we point out connections between functional complex powers and scattering amplitudes, construct a Mellin-based QFT generating functional, and define a parameter-dependent entropy that formally justifies the replica trick.

Keywords: Functional integration, topological groups, crossed products.
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1 Introduction

1.1 Motivation

Functional integration had its beginnings in the study of stochastic processes — particularly the Wiener process — and was therefore deeply rooted in probability theory [1]–[3]. Later, functional integrals in the form of Feynman path integrals [4] were found to be useful tools in quantum theory and partial differential equations, and so they have been extensively developed and utilized in the mathematical physics literature for decades. [5]–[9]

One consequence of this heritage is that functional integration methods in mathematical physics borrow heavily from probability constructs and are mostly confined to transformations or expansions/perturbations around quadratic-type functional integrals. These archetypical functional integrals are distinguished by the fact that the characteristic function of a Gaussian probability distribution is again a Gaussian, and the probability analogy is used to carry this notion over to the context of quantum physics. This allows functional integrals to be interpreted as a functional Fourier transform between dual Banach spaces, and this forms the basis of the vast majority of applications in physics.
But, while the probability analogy can be inspirational, it can also be restrictive. On one hand, probability theory is a useful complement to intuition, and it is easy to imagine that functional integrals based on probability distributions other than Gaussian would be useful in mathematical physics. On the other hand, expressing probability distributions through their characteristic functions can lead to an emphasis on Fourier transform (which expresses Pontryagin duality between locally compact abelian topological groups) as a guiding principle, and this encourages customary expansions around Gaussian backgrounds in the restricted setting of abelian topological groups.

However Fourier transform, although useful, is not the only game in town. It is not hard to see that the functional analog of the Mellin transform could be a useful tool. Like the functional Fourier transform, functional Mellin encodes a duality — but an algebraic rather than group duality. And for functional Mellin transforms, the probability analogy remains a profitable guide; except this time in the context of non-commutative Banach algebras. Significantly, functional Mellin provides means to represent and probe Banach algebras in the setting of non-abelian topological groups which lies beyond the power of Fourier. The purpose of this paper is to construct and develop this tool with an eye towards physics applications.

1.2 Outline of paper

The functional integral framework we will use to construct functional Mellin transforms is based on topological groups, so we start with a brief exposition of some pertinent results concerning locally compact topological groups and their associated integral operators on Banach algebras. Our proposed definition of functional integrals is then briefly reviewed, but we refer to [10] for details.

The remainder of the paper concentrates on elaborating the functional analog of the finite-dimensional Mellin transform. We use the functional Mellin transform to define functional analogs of resolvent, trace, log and determinant. In the functional context, Mellin is not just a simple transformation of Fourier, so developing and investigating the the infinite dimensional analog of Mellin transform is worthwhile.

With certain restrictions, functional Mellin represents a C*-algebra; and therefore it is an effective tool for quantum physics. Notably, one can construct and multiply operators with complex powers. We explore some properties of such operators in several examples. In particular, in §5 we examine a certain fractional-power operator whose trace is given by a sum of beta functions that reproduces four-point, tree-level, tachyon string scattering amplitudes up to normalization and momentum conservation. Moreover, the algebraic product of two such operators, which is realized through a convolution integral, yields closed tachyon tree-level amplitudes up to normalization and momentum conservation; thereby echoing the KLT relation[11]. This is not to claim that functional Mellin has any direct connection with string theory or contains any stringy physics. Rather, it supports our primary thesis; that functional Mellin is well suited to represent the C*-algebras of interesting quantum theories.

To further support this claim, §6 compares Fourier and Mellin representations of bosonic and fermionic n-point generating functions in QFT. Our treatment is not comprehensive as it does not address renormalization and gauge symmetry issues. Nevertheless, the Mellin
representations of functional trace, log, and determinant are profitable in this context. In particular, we show the functional log justifies the replica trick, and we observe connections between functional Mellin and scattering processes in QFT and string theory. Further, in appendix we prove a theorem generalizing the relation \( \text{exp tr} M = \det \text{exp} M \). Roughly stated, it says the Mellin transform and exponential map commute under appropriate conditions.

There are further reasons — beyond representing \( C^* \)-algebras — to expect that functional Mellin transforms will be useful in physics and applied mathematics. To give just a few: Crossed products, which are useful tools for \( C^* \)-algebraic quantization \[12\]–\[14\], are closely related to functional Mellin transforms. (Appendix D looks closer at this relationship.) The Mellin transform features prominently in the world-line formalism \[14\]–\[12\] of perturbative QFT. Properties of the Mellin transform allow efficient analytic treatment of harmonic integrals, asymptotic analysis of harmonic sums, and Fuchsian type partial differential equations \[15\]–\[18\]. Finally, functional integrals based on the gamma probability distribution show up in the study of constrained function spaces \[46\]–\[19\], and these are particular classes of functional Mellin transforms.

A note of caution: This exposition follows a physics not a mathematics analysis. Technical mathematical issues are not addressed: details of operator theory, complex analysis, and existence/uniqueness are mostly ignored. The goal is to develop tools for physics applications to be mathematically scrutinized later if they prove their worth. Our ultimate purpose is to use the framework to construct and represent \( C^* \)-algebras of interesting quantum systems starting from topological groups and functional Mellin.

## 2 Functional integration scheme

We briefly recall the functional integral scheme proposed in \[10\].

Our functional integrals are based on the data \((G, \mathfrak{B}, G_\Lambda)\) where \(G\) is a Hausdorff topological group, \(\mathfrak{B}\) is a Banach space that may have additional associative algebraic structure, and \(G_\Lambda := \{G_\lambda, \lambda \in \Lambda\}\) is a countable family of locally compact topological groups indexed by surjective homomorphisms \(\lambda : G \to G_\lambda\).

**Definition 2.1** A Hausdorff topological group \(G\) is a group endowed with a topology such that; (i) multiplication \(G \times G \to G\) by \((g, h) \mapsto gh\) and inversion \(G \to G\) by \(g \mapsto g^{-1}\) are continuous maps, and (ii) \(\{e\}\) is closed.

Topological groups come equipped with one-parameter subgroups.

**Definition 2.2** (\[20\] ch. 5]) A one-parameter subgroup \(\phi : \mathbb{R} \to G\) of a topological group is the unique extension of a continuous homomorphism \(f \in \text{Hom}_C(I \subseteq \mathbb{R}, G)\) such that \(f(t + s) = f(t)f(s)\) and \(f(0) = e \in G\). Let \(\mathfrak{L}(G)\) denote the set of all one-parameter subgroups \(\text{Hom}_C(\mathbb{R}, G)\) endowed with the uniform convergence topology on compact sets in \(\mathbb{R}\). The exponential function is defined by

\[
\exp_G : \mathfrak{L}(G) \to G
\]

\[
\phi \mapsto \exp_G \phi = \phi(1).
\] (2.1)
Locally compact topological groups come equipped with a crucial structure:

**Definition 2.3** $G$ is locally compact if every $g \in G$ has a neighborhood basis\(^1\) comprised of compact sets.

**Theorem 2.4** If $G$ is locally compact, then there exists a unique (up to positive scalar multiplication) Haar measure. If $G$ is compact, then it is unimodular.

Thanks to this theorem, the set $G_\Lambda$ gives us access to Banach-valued integration.

**Proposition 2.5** ([21, prop. B.34]) Let $G_\Lambda$ be a locally compact topological group, $\mu$ its associated Haar measure, and $\mathfrak{B}$ a Banach space possibly with an algebraic structure. Then the set of integrable functions $L^1(G_\Lambda, \mathfrak{B})$ consisting of equivalence classes of measurable functions equal almost everywhere with norm $\|f\|_1 := \int_{G_\Lambda} \|f(g_\Lambda)\|d\mu(g_\Lambda) \leq \|f\|_\infty \mu(\text{supp } f) < \infty$, is a Banach space. Moreover, $f \mapsto \int_{G_\Lambda} f(g_\Lambda)d\mu(g_\Lambda)$ is a linear map such that

$$\| \int_{G_\Lambda} f(g_\Lambda)d\mu(g_\Lambda) \| \leq \|f\|_\infty \mu(\text{supp } f) \quad (2.2)$$

for all $f \in L^1(G_\Lambda, \mathfrak{B})$,

$$\varphi \left( \int_{G_\Lambda} f(g_\Lambda)d\mu(g_\Lambda) \right) = \int_{G_\Lambda} \varphi(f(g_\Lambda))d\mu(g_\Lambda) \quad (2.3)$$

for all $\varphi \in \mathfrak{B}'$, and

$$L_B \left( \int_{G_\Lambda} f(g_\Lambda)d\mu(g_\Lambda) \right) = \int_{G_\Lambda} L_B(f(g_\Lambda))d\mu(g_\Lambda) \quad (2.4)$$

for bounded linear maps $L_B : \mathfrak{B}_1 \to \mathfrak{B}_2$. Moreover, Fubini’s theorem holds for all equivalence classes $f \in L^1(G_1 \times G_2, \mathfrak{B})$.

**Corollary 2.6** ([21, lemma. 1.92]) Let $\mathfrak{B}^*$ be a $*$-algebra and $\pi : \mathfrak{B}^* \to L_B(\mathcal{H})$ a representation with $L_B(\mathcal{H})$ the algebra of bounded linear operators on Hilbert space $\mathcal{H}$. Then

$$\left\langle \pi \left( \int_{G_\Lambda} f(g_\Lambda)d\mu(g_\Lambda) \right)v \mid w \rightangle = \int_{G_\Lambda} \left\langle \pi(f(g_\Lambda))v \mid w \right\rangle d\mu(g_\Lambda), \quad (2.5)$$

$$\left( \int_{G_\Lambda} f(g_\Lambda)d\mu(g_\Lambda) \right)^* = \int_{G_\Lambda} f(g_\Lambda)^*d\mu(g_\Lambda), \quad (2.6)$$

and

$$a \int_{G_\Lambda} f(g_\Lambda)d\mu(g_\Lambda)b = \int_{G_\Lambda} af(g_\Lambda)b d\mu(g_\Lambda) \quad (2.7)$$

where $v, w \in \mathcal{H}$ and $a, b \in M(\mathfrak{B}^*)$ with $M(\mathfrak{B}^*)$ the multiplier algebra of $\mathfrak{B}^*$.

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\(^1\)A neighborhood basis at $g \in G$ is a family $\mathcal{N}$ of neighborhoods such that given any neighborhood $U$ of $g$ there exists an $N \in \mathcal{N}$ such that $N \subset U$.  

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It is well known (see e.g. [21, appx. B]) that $L^1(G_\lambda, \mathfrak{B}^*)$ is a Banach $*$-algebra when equipped with the $\| \cdot \|_1$ norm, the convolution
\[
f_1 \ast f_2(g_{\lambda}) := \int_{G_\lambda} f_1(h_\lambda)f_2(h_\lambda^{-1}g_\lambda)d\mu(h_\lambda),
\] (2.8)
and the involution
\[
f^*(g_{\lambda}) := f(g_\lambda)^{-1}\Delta(g_\lambda^{-1})
\] (2.9)
where $\Delta$ is the modular function on $G_\lambda$.

The data $(G, \mathfrak{B}, G_\Lambda)$ together with its associated Banach-valued integration motivates our definition of functional integral.

**Definition 2.7** Let $\mathbf{F}(G)$ represent a space of functionals $F : G \to \mathfrak{B}$, and denote the restriction of $F$ to $G_\lambda$ by $f := F|_{G_\lambda}$. Let $\nu$ be a left Haar measure on $G_\lambda$.

A family of integral operators $\text{int}_\Lambda : \mathbf{F}(G) \to \mathfrak{B}$ is defined by
\[
\text{int}_\lambda(F) \equiv \int_G F(g)\mathcal{D}_{\lambda}g := \int_{G_\lambda} f(g_\lambda) \, d\nu(g_\lambda)
\] (2.10)
such that $f \in L^1(G_\lambda, \mathfrak{B})$ for all $\lambda \in \Lambda$. We say $F$ is integrable with respect to the integrator family $\mathcal{D}_{\lambda}g$, and $\mathbf{F}(G) \subseteq \mathbf{F}(G)$ is the space of integrable functionals (with respect to $\Lambda$).

In addition, if $\mathfrak{B}$ is an algebra, define the functional $*$-convolution and $\ast$-convolution by
\[
(F_1 \ast F_2)_{\lambda}(g) := \int_G F_1(\hat{g})F_2(\hat{g}^{-1}g)\mathcal{D}_{\lambda}\hat{g}
\] (2.11)
and
\[
(F_1 \ast F_2)_{\lambda}(g) := \int_G F_1(\hat{g}\hat{g})F_2(\hat{g}\hat{g})\mathcal{D}_{\lambda}\hat{g}
\] (2.12)
for each $\lambda \in \Lambda$.

**Proposition 2.8** $\mathbf{F}(G)$ equipped with the $*$-convolution is a Banach algebra when completed with respect to the norm $\|F\|_F := \sup_\lambda \|\text{int}_\lambda(F)\|$.

**Proof**: For any given $\lambda$, the integral operator is linear and bounded according to
\[
\|\text{int}_\lambda(F)\| \leq \int_{G_\lambda} \|f(g_\lambda)\|d\nu(g_\lambda) = \|f\|_{1,\lambda} < \infty .
\] (2.13)
Linearity is obvious. To see it is bounded, use Cauchy-Schwarz along with Proposition 2.5.

Since $\|f\|_{1,\lambda}$ is a norm on $L^1(G_\lambda, \mathfrak{B})$, it follows that $\|F\|_F := \sup_\lambda \|\text{int}_\lambda F\|$ is a norm on $\mathbf{F}(G)$. Being $\mathbf{F}(G)$ a normed space, its completion (which will be denoted by the same
symbol) is a Banach space. The \(*\)-convolution then implies

\[
\int_\lambda (F_1 \ast F_2) = \int_G (F_1 \ast F_2)(g) \mathcal{D}_\lambda g
\]

\[
= \int_{G_\lambda \times G_\lambda} f_1(\bar{g}_\lambda) f_2(g_\lambda) \, d\nu(\bar{g}_\lambda, g_\lambda)
\]

\[
= \int_{G_\lambda \times G_\lambda} f_1(\bar{g}_\lambda) f_2(g_\lambda) \, d\nu(\bar{g}_\lambda, g_\lambda)
\]

\[
= \int_{G_\lambda} \int_{G_\lambda} f_1(\bar{g}_\lambda) f_2(g_\lambda) \, d\nu(\bar{g}_\lambda) d\nu(g_\lambda)
\]

\[
= \int_{G_\lambda} \int_{G_\lambda} f_1(\bar{g}_\lambda) f_2(g_\lambda) \, d\nu(\bar{g}_\lambda) d\nu(g_\lambda)
\]

\[
= \int_\lambda (F_1) \int_\lambda (F_2) \quad (2.14)
\]

where the third line follows from left-invariance of the Haar measure and the fourth line follows from Fubini. A similar computation (using left-invariance and Fubini) establishes associativity \((F_1 \ast F_2) \ast F_3 = F_1 \ast (F_2 \ast F_3)\). Finally, given that \(\mathcal{B}\) is Banach, eq. (2.14) implies \(\|F_1 \ast F_2\|_F \leq \|F_1\|_F \|F_2\|_F\). □

If \(\mathcal{B}\) has an involutive structure, then \(\mathbf{F}(G)\) inherits this structure:

**Proposition 2.9** ([10, prop. 2.21]) Suppose \(\mathcal{B}\) is a Banach \(*\)-algebra. Then the integral operator \(\text{int}_\lambda\) is a \(*\)-homomorphism, and \(\mathbf{F}(G)\) is a Banach \(*\)-algebra when endowed with a suitable topology and involution given by \(\mathbf{F}^*(g) := \mathbf{F}(g^{-1})^* \Delta(g^{-1})\) and completed with respect to the norm \(\|\cdot\|_F\).

**Corollary 2.10** If \(\mathcal{B}\) is a \(C^*\)-algebra, then \(\mathbf{F}(G)\) is \(C^*\)-algebra when completed w.r.t. the norm \(\|\cdot\|_F\).

**Proof**: First,

\[
\text{int}_\lambda (F^*) = \int_G F^*(g) \mathcal{D}_\lambda g
\]

\[
= \int_{G_\lambda} f^*(g_\lambda) \, d\nu(g_\lambda)
\]

\[
= \int_{G_\lambda} f(g_{\lambda}^{-1})^* \Delta(g_{\lambda}^{-1}) \, d\nu(g_\lambda)
\]

\[
= \int_{G_\lambda} f(g_{\lambda})^* \, d\nu(g_\lambda)
\]

\[
= \left( \int_{G_\lambda} f(g_{\lambda}) \, d\nu(g_\lambda) \right)^*
\]

\[
= \text{int}_\lambda (F)^* \quad (2.15)
\]

where the fourth line follows by virtue of the Haar measure. Together with (2.14), this shows that the integral operators are \(*\)-homomorphisms. In particular, \(\text{Id}^* = \text{Id}\).
It remains to verify the \(\ast\)-algebra axioms. The \(\ast\)-operation is continuous for a suitable choice of topology, and linearity is obvious. Next,

\[
(F^\ast)^\ast(g) := F^\ast(g^{-1})^\ast \Delta(g^{-1}) = (F(g)^\ast)^\ast \Delta(g) \Delta(g^{-1}) = F(g)
\]  
(2.16)

and

\[
(F_1^\ast F_2^\ast)_\lambda(g) := \int_{G_\lambda} f_1^\ast(\tilde{g}_\lambda) f_2^\ast(\tilde{g}_\lambda^{-1} g_\lambda) \, d\nu(\tilde{g}_\lambda)
= \int_{G_\lambda} (f_2(\tilde{g}_\lambda^{-1} g_\lambda) \Delta(\tilde{g}_\lambda^{-1} g_\lambda) f_1(\tilde{g}_\lambda^{-1})^\ast) \, d\nu(\tilde{g}_\lambda)
= \left( \int_{G_\lambda} f_2(\tilde{g}_\lambda^{-1} g_\lambda) f_1(\tilde{g}_\lambda^{-1}) \Delta(\tilde{g}_\lambda^{-1}) \, d\nu(\tilde{g}_\lambda) \right)^\ast
= ((F_2 \ast F_1)_\lambda(g^{-1}))^\ast \Delta(g^{-1})
= (F_2 \ast F_1)_\lambda^\ast(g)
\]  
(2.17)

where we used the definition of involution, left-invariance of the Haar measure, and the fact that the modular function \(\Delta\) is a homomorphism. For the norm, \(\mathcal{B}\) a \(\ast\)-algebra and (2.15) imply \(\|\int_{\lambda}(F^\ast)\| = \|\int_{\lambda}(F)^\ast\| = \|\int_{\lambda}(F)\|\) which implies \(\|F^\ast\|_F = \|F\|_F\). Conclude that \(F(G)\) is a \(\ast\)-algebra.

Lastly, if \(\mathcal{B}\) is a \(C^\ast\)-algebra, the corollary follows from (2.14) and (2.15) since now

\[
\|\int_{\lambda}(F \ast F^\ast)\| = \|\int_{\lambda}(F)\int_{\lambda}(F)^\ast\| = \|\int_{\lambda}(F)\| \|\int_{\lambda}(F)^\ast\| = \|\int_{\lambda}(F)\|^2
\]  
(2.18)

implies \(\|F \ast F^\ast\|_F = \|F\|_F^2\). \(\Box\)

3 \ Functional Mellin transform

3.1 Motivation

One might question the utility of the functional integration scheme: After all, it eventually just boils down to a space of functions on a countable family of locally compact groups that inherits the structure of some target Banach space. But, as stressed in [10], the accompanying organization and structure carries motivational value.

To further reveal its value, consider integrals of the type \(\int_{G_\lambda} \pi(f(g_\lambda)) U(g_\lambda) \, d\nu(g_\lambda)\) where \(\mathcal{B}^\ast\) is a \(C^\ast\)-algebra, \(\pi : \mathcal{B}^\ast \to L_B(\mathcal{H})\) is a non-degenerate representation, and \(U : G_\lambda \to U(\mathcal{H})\) is a unitary representation furnished by some Hilbert space \(\mathcal{H}\). This integral represents a Fourier transform if \(G_\lambda\) is abelian, but generically \(G_\lambda\) will be non-abelian. Unfortunately, as it stands the integral is not well-defined because \(\pi(f(g_\lambda)) U(g_\lambda)\) is not a continuous function in general. However, if the multiplier algebra of \(\mathcal{B}^\ast\) (viewed as a Hilbert \(\mathcal{B}^\ast\)-module) is equipped with the strict topology, continuity is restored since then \(\pi(f(g_\lambda)) U(g_\lambda)\) is continuous for \(f(g_\lambda) \in M_s(\mathcal{B}^\ast)\) where \(M_s(\mathcal{B}^\ast)\) denotes \(M(\mathcal{B}^\ast)\) endowed with the strict topology ([21 §1.5]). Adopting the strict topology and restricting to \(f(g_\lambda) \in M_s(\mathcal{B}^\ast)\) maintains Corollary 2.6 in the form;
Proposition 3.1 ([21, lemma 1.101]) For $f \in C_C(G, M_s(\mathcal{B}^*))$ (i.e. $f : G \to M_s(\mathcal{B}^*)$ is continuous and compactly supported), and $\bar{\pi} : M(\mathcal{B}^*) \to L_B(H)$ a non-degenerate representation, there exists a linear map $f \mapsto \int_{G} f(g) \, d\nu(g)$ from $C_C(G, M_s(\mathcal{B}^*))$ to $M(\mathcal{B}^*)$ such that

$$\langle \bar{\pi} \left( \int_{G} f(g) \, d\nu(g) \right) v | w \rangle = \int_{G} \langle \bar{\pi}(f(g)) v | w \rangle \, d\nu(g),$$

(3.1) and

$$\bar{\mathcal{I}} \left( \int_{G} f(g) \, d\nu(g) \right) = \int_{G} \bar{\mathcal{I}}(f(g)) \, d\nu(g)$$

(3.2)

for $\bar{\mathcal{I}} : M(\mathcal{B}^*_1) \to M(\mathcal{B}^*_2)$ a non-degenerate homomorphism.

Under these conditions, the integral $\int_{G} \bar{\pi}(f(g)) U(g) \, d\nu(g)$ embodies the crossed-product approach to algebraic quantization [14]:

Proposition 3.2 ([21, ch. 2.3])

$$\bar{\pi} \rtimes U(f) := \int_{G} \bar{\pi}(f(g)) U(g) \, d\nu(g)$$

(3.3)

defines a $\ast$-representation of $C_C(G, M_s(\mathcal{B}^*))$ on $H$.

As our ultimate goal is to apply functional integration in the context of quantum physics, this connection to crossed products is suggestive. However, to our knowledge, virtually all crossed-product quantizations are built from some ‘dynamical system’ (see appx. D). That is, they begin with a classical system, form a commutative algebra of functions, and then build a non-commutative $C^*$-algebra via the crossed product. This is the time-honored method of classical→quantum quantization.

But the functional integral context inspires a different approach. We have seen that certain algebras have units that comprise a topological group.[10] This suggests beginning with an abstract algebra $\mathfrak{A}$ whose group of units is isomorphic to the topological group $G$ that partially defines a functional integral. In particular, for quantum physics we should start with an abstract non-commutative $C^*$-algebra. Then it is enough to specify $G \to G_\Lambda$ and its relevant representations in order to construct a concrete realization of $\mathfrak{A}$ expressed through the algebra $\mathcal{B}^*$. This understanding together with the crossed-product construction viewed from the functional integral perspective motivates the introduction of functional Mellin transforms.

### 3.2 The definition

In order to define the functional Mellin transform, specialize the functional integral data to $(G^C, \mathcal{C}^*, G_\Lambda^C)$ where $\mathcal{C}^*$ is a unital $C^*$-algebra and $G^C$ a complex topological Lie group isomorphic to the group (or a subgroup) of units $A_\mathfrak{A}$ of a complex CCIA $\mathfrak{A}$.

**Definition 3.3** A complete continuous inverse algebra (CCIA) is a Mackey complete, unital topological algebra $\mathfrak{A}$ whose group of units (invertible elements) $A_\mathfrak{A}$ is open and group inversion is continuous.
Definition 3.4 ([22 § 1.3]) A locally convex space $S_\diamond$ is Mackey complete iff the integral $\int_a^b \gamma(t) \, dt$ exists for any smooth curve $\gamma : (\alpha, \beta) \to S_\diamond$ with $\alpha < a < b < \beta$.

Having a CCIA is enough to construct a functional calculus on $A$, and one can construct complex-analytic exponential and logarithm maps;

Definition 3.5 ([22 defs. 4.7, 5.1]) Suppose $(z1 - a) \in A$. Let $B_1(1)$ be the unit ball about the identity element $1 \in A$. The exponential $\exp_a : A \to A$ and logarithm $\log_a : B_1(1) \to A$ are defined by

$$\exp_a a := \frac{1}{2\pi i} \int_\Gamma e^{z_1 - a} \, dz \quad \forall a \in A,$$

$$\log_a : a := \frac{1}{2\pi i} \int_\Gamma \log z_1 - a \, dz \quad \forall a \in B_1(1) \quad (3.4)$$

for $\Gamma$ a partially smooth contour in $\mathbb{C}$ enclosing the spectrum $\sigma(a)$.

Since $\exp_a$ and $\log_a$ are complex-analytic, these contour integral representations yield the usual series

$$\exp_a a = \sum_{n=0}^{\infty} \frac{1}{n!} a^n \quad \forall a \in A,$$

$$\log_a(1 + a) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} a^n \quad \forall a \in B_1(1). \quad (3.5)$$

Then, as an open subset of $A$, the units $A_\alpha$ inherit a manifold structure, and the complex-analytic structure of a CCIA is enough to endow $A_\alpha$ with a Lie group structure.

Definition 3.6 A topological group $G$ is a Lie group if there exists a neighborhood $U$ of $\{e\}$ such that, for every subgroup $H$, if $H \subseteq U$ then $H = \{e\}$.

Theorem 3.7 ([22 prop. 3.2, prop. 3.4, § 4]) Let $A_\alpha$ be the set of units of a complex CCIA $A$. Then group inversion $\text{inv} : A_\alpha \to A_\alpha$ is complex-analytic and, hence, $A_\alpha$ is a complex-analytic Lie group with exponential map $\exp_{A_\alpha} = \exp_A|_{A_\alpha} : T_e(A_\alpha) \to A_\alpha$ by $a \mapsto \exp_a(ta) = \phi_a(t)$ such that $d\phi_a(d/dt) = a \in T_e(A_\alpha)$ and $t \in \mathbb{R}$. If $A$ is real CCIA, then inversion is real-analytic and $A_\alpha$ is a real-analytic Lie group.

Moreover, since $A$ is Mackey complete, $A_\alpha$ becomes a BCH-Lie group:

Definition 3.8 ([22 def. 5.5]) A BCH-Lie group $G_{[\cdot, \cdot]}$ is a complex-analytic Lie group such that: i) there exists an open 0-neighborhood $U \subset T_e(G_{[\cdot, \cdot]})$ with $V := \exp_{G_{[\cdot, \cdot]}}(U)$ open in $G_{[\cdot, \cdot]}$ such that the map $\varphi := \exp_{G_{[\cdot, \cdot]}}|_U : U \to V$ is a diffeomorphism; and ii) there exists a $(0, 0)$-neighborhood $W \subseteq U \times U$ such that $\exp_{G_{[\cdot, \cdot]}} a \cdot \exp_{G_{[\cdot, \cdot]}} b \subseteq V$ and $\varphi^{-1}(\varphi(a)\varphi(b))$ is the BCH series for all $a, b \in W$. 

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Proposition 3.9 ([22, th. 5.6]) If $\mathfrak{A}$ is CCIA, then the group of units $A_{\mathfrak{A}}$ is a BCH-Lie group.

Given that CCIA $\mathfrak{A}$ yields a complex-analytic Lie group $A_\mathfrak{A}$ and $G$ is isomorphic to $A_\mathfrak{A}$ by assumption, we can construct the complex Lie group $G^C$. Recall the definition of the exponential function on a topological Lie group $G$: it associates a $g^1 \in G$ with a one-parameter subgroup $\phi_\theta \in \mathfrak{L}(G)$ by $\exp_G \phi_\theta = \phi_\theta(1) =: g^1$ where $\theta = d\phi_\theta(d/dt)$. Then, by the definition of one-parameter subgroups, $\phi_\theta(t) = \exp_G(t\theta) =: g^t$ with $t \in \mathbb{R}$, and the short-hand notation $g^t$ can be formally interpreted as the $t$-th power of $g$ in the sense that $g^t = \exp_G(t \log_G(g^1))$. Evidently the $t$-th power is characterized by two fiducial points $\phi_\theta(0) = e$ and $\phi_\theta(1) = g^1 = g$.

To extend one-parameter subgroups to complex $G^C$ follow [23 pg. 23]. Let $\mathfrak{L}(C)$ denote the Lie algebra of $C$ and consider the morphism $\gamma : \mathfrak{L}(C) \to \mathfrak{L}(G)$ by $z \mapsto z\theta$. The real analytic group $C_{\mathfrak{A}}$ underlying $C$ is simply connected so there is a unique, real analytic homomorphism $\tilde{\phi}_\theta : C \to G^C$ defined by $\tilde{\phi}_\theta(z) := (\exp_G \circ \gamma)(z) = \exp_G(z\theta)$. Now, since $d\tilde{\phi}_\theta = \gamma'$ is a morphism of complex Lie algebras, $\tilde{\phi}_\theta$ is in fact complex analytic and $d\tilde{\phi}_\theta(1) = g$. Hence $\tilde{\phi}_\theta$ is a one-parameter subgroup of $G^C$. Denote the set of complex one-parameter subgroups by $\mathfrak{L}(G^C) := \text{Hom}_C(C, G^C)$. This allows the definition of the exponential map $\exp_{G^C} : \mathfrak{L}(G^C) \to G^C$ by $\tilde{\phi}_\theta \mapsto \exp_{G^C} \tilde{\phi}_\theta = \tilde{\phi}_\theta(1)$. Note that $\tilde{\phi}_\theta(0) = e$ and $\tilde{\phi}_\theta(1) = g^1 = g$. Formally interpret $g^t = \exp_{G^C}(z \log_G(g^1))$ as a complex power of $g$.

**Proposition 3.10** ([20 prop. 5.40] ; [23 th. 1.15]) The exponential $\exp_{G^C} : \mathfrak{L}(G^C) \to G^C$ is a complex analytic map such that $\exp_{G^C} = \exp_G$.

**Proof:** First, $\mathfrak{L}(G)$ is homeomorphic to $\mathfrak{G}$; which can be identified with the Lie algebra of $G^C$ over $\mathbb{R}$ so $\exp_{G^C} \theta = \tilde{\phi}_\theta(1) = \exp_G \theta$ for all $\theta \in \mathfrak{G}$. Therefore $\exp_{G^C} = \exp_G$. Now let $U \subset G^C$ be an open connected neighborhood of $0 \in G^C$ where $\exp_G$ is invertible. The inverse $\exp_{G^C}^{-1}$ provides a local chart of $V := \exp_G(U) \subset G^C$ at the identity, and $\exp_G$ commutes with the almost complex structures on $U$ and $V$. Hence, $\exp_{G^C}$ is complex analytic in $U$. To extend the analyticity to all of $G^C$, pick any $\theta \in \mathfrak{G}$ and note that $n^{-1} \theta \in U$ for some $n \in \mathbb{N}$. Then $\exp_{G^C}(n^{-1} \theta) \in V$ is complex analytic. Therefore $(\exp_{G^C}(n^{-1} \theta))^n = \exp_G \theta$ implies $\exp_{G^C}$ (and hence $\exp_{G^C}$) is complex analytic at any $\theta \in \mathfrak{G} \simeq \mathfrak{L}(G^C)$. □

We are ready to define the functional Mellin transform given data $(G^C, \mathfrak{C}^*, G_X^C)$ where $G$ is isomorphic to the group of units of a complex CCIA.

**Definition 3.11** Given a topological group $G^C$, let $\rho : G^C \to \mathfrak{C}^*$ be a strictly-continuous injective representation, and let $\pi : \mathfrak{C}^* \to L_B(H)$ be a representation. Consider the space of integrable, equivariant functionals $\overline{F}(G^C) \subseteq F(G^C)$ where $F \in \text{Mor}_G(G^C, M_s(\mathfrak{C}^*))$ is equivariant under right-translations by $G^C$ according to $F(g) = F(h)\rho(h)$. Let $\mathfrak{C}^*$ be a unital $\mathfrak{C}^*$-algebra whose involution induces an involution on $\overline{F}(G^C)$ given by $F^*(gg^\alpha) := \rho(g^{-\alpha})F(g^{-1})^\Delta(g^{-1})^2$. This prescription is for left-invariant Haar measures. For right-invariant Haar measures impose equivariance under left-translations.
with $\alpha \in \mathbb{C}$ and $\Delta(g^{-1})$ defined by restriction to $G^C_{\lambda}$. Then the functional Mellin transform $M_\lambda : \mathcal{F}(G^C) \to \mathfrak{C}^*$ is defined by

$$M_\lambda [F; \alpha] := \int_{G^C} F(g g^\alpha) \, d\lambda g = \int_{G^C} F(g) \rho(g^\alpha) \, d\lambda g$$

(3.6) such that $g^\alpha := \exp_G(\alpha \log_G g)$, and $\pi(F(g g^\alpha)) = \pi(F(g) \rho(g^\alpha)) \in L_B(H)$ where the space of bounded linear operators $L_B(H)$ is given the strict topology. Denote the space of Mellin integrable functionals by $\mathcal{F}_S(G^C)$.\footnote{Unless specified otherwise, we take the standard branch for $\log_G$ when $\pi(\rho(g)) \in L_B(H)$.} \footnote{Identification of the tangent space $T_v G^C$ with $\mathfrak{L}(G^C)$, the Mellin integral can be explicitly formulated on $\mathfrak{L}(G^C)$:

$$\int_{G^C} f(g_\lambda) \rho(g_\lambda^\alpha) \, d\nu(g_\lambda) = \int_{\mathfrak{L}(G^C)} f(\exp_{G^C} g) \rho(\exp_{G^C}(\alpha g)) \det g \exp_{G^C} g \, dg.$$ (3.8) This is a multiple integral depending on the dimension of $G^C$. The notation $\rho(g_\lambda^\alpha)$ is understood as $\rho(g_\lambda^\alpha) = \exp_{G^C}(\alpha \rho'(g)) = \exp_{G^C} \left( \alpha \sum_{i=1}^n c_i \rho'(e_i) \right) =: \exp_{G^C} \left( \sum_{i=1}^n \alpha_i \rho'(e_i) \right)$ (3.9) and $\alpha$ is a multi-index relative to the maximal commuting subalgebra of $\mathfrak{C}^*$.

Since we don’t have a definition of Mellin transform for generic locally compact topological groups, we have first defined the functional Mellin transform. Then, according to Definition 2.7, a sufficient condition for the functional Mellin transform to exist is for $f(g) \rho(g^\alpha)$ to be integrable precisely when $\alpha \in \mathbb{S}$. This extends the usual definition of Mellin transform to the case of Banach-valued integrals over locally compact topological groups:

**Definition 3.12** Given a locally compact topological group $G^C_{\lambda}$, let $\rho : G^C_{\lambda} \to \mathfrak{C}^*$ be a strictly-continuous injective representation, and consider equivariant functions $f \in \text{Mor}_{C}(G^C_{\lambda}, \mathfrak{C}^*)$ by $g^{1+\alpha} \mapsto f(g) \rho(g^\alpha)$ such that $f \in C_{C}(G^C_{\lambda}, \mathfrak{C}^*)$ for all $\alpha \in \mathbb{S} \subset \mathbb{C}$. Then the functional $F$ is Mellin integrable if

$$|M_\lambda [F; \alpha]| \leq \int_{G^C_{\lambda}} |f(g_\lambda) \rho(g_\lambda^\alpha)| \, d\nu(g_\lambda) < \infty, \quad \alpha \in \mathbb{S}. \quad (3.7)$$

We say the Mellin transform $M_\lambda [F; \alpha]$ exists in the fundamental region $\mathbb{S}$ (which may be empty). To emphasize the fundamental region depends on $\lambda$, we sometimes write $\mathbb{S}_\lambda$.

Identifying the tangent space $T_v G^C$ with $\mathfrak{L}(G^C)$, the Mellin integral can be explicitly formulated on $\mathfrak{L}(G^C)$:

$$\int_{G^C} f(g_\lambda) \rho(g_\lambda^\alpha) \, d\nu(g_\lambda) = \int_{\mathfrak{L}(G^C)} f(\exp_{G^C} g) \rho(\exp_{G^C}(\alpha g)) \det g \exp_{G^C} g \, dg.$$ (3.8)

This is a multiple integral depending on the dimension of $G^C_{\lambda}$. The notation $\rho(g_\lambda^\alpha)$ is understood as $\rho(g_\lambda^\alpha) = \exp_{G^C}(\alpha \rho'(g)) = \exp_{G^C} \left( \alpha \sum_{i=1}^n c_i \rho'(e_i) \right) =: \exp_{G^C} \left( \sum_{i=1}^n \alpha_i \rho'(e_i) \right)$ (3.9) and $\alpha$ is a multi-index relative to the maximal commuting subalgebra of $\mathfrak{C}^*$.

\footnote{Emphasize that $M_\lambda [F; \alpha]$ depends on the chosen representation $\rho$, the topological localization $\lambda$, and the normalization of the Haar measure.}

\footnote{The class of functional Mellin transforms defined here includes the integrated form of a covariant representation of a dynamical system\footnote{Proposition 2.5 and Definition 2.7 were stated for $f \in L^1(G_\lambda, \mathfrak{B})$. In order to quote \cite{21} precisely and thereby avoid introducing technical difficulties, we restrict here to $f \in C_C(G_\lambda, \mathfrak{B}^*)$ where $C_C(G_\lambda, \mathfrak{B}^*)$ denotes the set of continuous, compactly-supported functions $f : G_\lambda \to \mathfrak{B}^*$ and $\mathfrak{B}^*$ is a $C^*$-algebra. But we point out that $C_C(G_\lambda, \mathfrak{B}^*)$ is dense in $L^1(G_\lambda, \mathfrak{B}^*)$ since $G_\lambda$ is locally compact.\cite{21}} as a special case. To relate the integrated form to functional Mellin transforms, require $\pi \circ \rho$ to be a strongly continuous unitary representation $U : G^C_{\lambda} \to L_B(H)$. Then $\pi(f(g_\lambda h_\lambda)) = \pi(f(g_\lambda) \rho(h_\lambda)) = \pi(f(g_\lambda)) U(h_\lambda)$ and the integrated form $\pi \times U(f)$ is equivalent to $\pi(M_\lambda [F; 1])$; although our definitions of $\ast$-convolution and involution are different. See appendix E for further discussion.}
Roughly speaking, the functional Mellin transform is a family of integrals represented by the right-hand side of (3.8) which can be viewed as a generalized two-sided Laplace/Fourier transform provided \( \alpha \in \mathbb{S} \).

Stating the definitions is relatively easy: the hard work involves determining \( \mathbb{S} \) given \( f, \rho, \) and \( \lambda \). For example, let \( \mathcal{C}^* = \mathbb{C} \) and \( \lambda : G^\mathbb{C} \to \mathbb{R}_+ \) the strictly positive reals so \( \rho \) acts by multiplication. Choose the standard normalization for the Haar measure \( \nu(g) = \log g \) on \( \mathbb{R}_+ \). Then \( \mathcal{M}_\lambda [F; \alpha] \) reduces to the standard Mellin transform (see appx. A):

\[
\mathcal{M}_{\mathbb{R}_+, H} [F; \alpha] = \int_0^\infty f(x)x^{\alpha} \frac{dx}{x} = \int_0^\infty f(x)x^{\alpha-1} \, dx, \quad \alpha \in \langle a, b \rangle_H \equiv \mathbb{S}_H \quad (3.10)
\]

where the subscript \( H \) indicates the normalized Haar measure. Alternatively, the integral can be transferred to the Lie algebra of \( \mathbb{R}_+ \) and computed via (3.8):

\[
\mathcal{M}_{\mathbb{R}_+, H} [F; \alpha] = \int_{-\infty}^\infty f(e^t)e^{\alpha t} \, d \log(e^t) = \int_{-\infty}^\infty f(e^t)e^{\alpha t} \, dt, \quad \alpha \in \langle a, b \rangle_H \equiv \mathbb{S}_H . \quad (3.11)
\]

For this elementary case, this is just a Fourier transform of \( g := (f \circ \exp) \) if \( \alpha \) is imaginary. In particular, if \( f(x) = e^{-A \cdot x} \) with \( A \in \mathbb{C}_+ \cong \mathbb{R}_+ \times i\mathbb{R} \) both integrals yield

\[
\mathcal{M}_{\mathbb{R}_+, H} [E^{-A}; \alpha] = \Gamma(\alpha) A^{-\alpha}, \quad \alpha \in (0, \infty)_H . \quad (3.12)
\]

When \( A \in \mathbb{C}_- \) the integrals do not converge and so \( \mathbb{S}_H = \emptyset \) in this case.

**Example 3.13** For a less trivial but well-worn example, consider the Mellin transform of the heat kernel of a free particle on \( \mathbb{R}^n \). The degrees of freedom associated with a free particle are encoded by a continuous map \( x : \mathbb{R}_+ \to \mathbb{R}^n \) which dictates we choose \( \lambda : G^\mathbb{C} \to \mathbb{R}_+ \) for position-to-position boundary conditions. In this context, \( \mathcal{H} = L^2(\mathbb{R}^n), \mathcal{C}^* = L_E(L^2(\mathbb{R}^n)) \), and the observable \( F = E^{-\Delta} \) for position-to-position boundary conditions gives rise to the heat kernel \( \langle x'_a | F(g) | x_a \rangle \equiv e^{-S_{x_a, x'_a}(g)} \) with the ‘effective action’ functional \( S_{x_a, x'_a} : G^\mathbb{C} \to \mathbb{R}_+ \) given by

\[
S_{x_a, x'_a}(g) = \pi |x'_a - x_a|^2 / g + \frac{n}{2} \log g . \quad (3.13)
\]

With the choice of the usual Haar normalization, \( \mathcal{M}_{\mathbb{R}_+, H} [E^{-S_{x_a, x'_a}}; 1] \) is the elementary kernel of the Laplacian \( \Delta \) on \( \mathbb{R}^n \) associated with position-to-position boundary conditions. Explicitly, the position-to-position elementary kernel of the Laplacian on \( \mathbb{R}^n \) is given by

\[
K(x'_a, x_a) = \langle x'_a | \mathcal{M}_{\mathbb{R}_+, H} [E^{-\Delta}; 1] | x_a \rangle
= \mathcal{M}_{\mathbb{R}_+, H} [\langle x'_a | E^{-\Delta} | x_a \rangle; 1]
= \mathcal{M}_{\mathbb{R}_+, H} [E^{-S_{x_a, x'_a}}; 1]
= \int_{\mathbb{R}_+} \exp \left\{ -\frac{\pi |x'_a - x_a|^2}{g} \right\} g^{1-n/2} \, d \log g
= \begin{cases}
-2 \log(\pi |x'_a - x_a|) & n = 2 \\
\pi^{1-n/2} \Gamma(n/2 - 1) |x'_a - x_a|^{2-n} & n > 2
\end{cases} . \quad (3.14)
\]
3.3 Algebraic properties

The functional Mellin transform inherits important properties from \( \text{int}_\Lambda \) that follow from equivariance, Proposition 2.5, and Definition 2.7. First note that if \( \alpha = 0 \in S \) then functional Mellin reduces \( \mathcal{M}_\lambda \to \text{int}_\Lambda \), so we will not consider this case any longer.

It’s easy to show that \((F_1 \ast F_2)(gg^\alpha) = (F_1 \ast F_2)(g)\rho(g^\alpha)\) so \(\tilde{\mathcal{F}}(G^C)\) is a subalgebra of \(F(G^C)\). Define a norm on \(\mathcal{F}_S(G^C)\) by \(\|F\|_S := \sup_{\alpha, \lambda} \|\mathcal{M}_\lambda[F; \alpha]\|\) with \(\alpha \in S_\lambda\). Complete \(\mathcal{F}_S(G^C)\) with respect to \(\|\cdot\|_S\) (or some other suitably defined norm). Then \(\mathcal{F}_S(G^C)\) is Banach because \(\mathcal{C}^*\) is Banach.

**Lemma 3.14** If \(F \in \mathcal{F}_S(G^C)\), then

\[
\mathcal{M}_\lambda^S[F; \alpha] := \pi(\mathcal{M}_\lambda[F; \alpha]) = \mathcal{M}_\lambda[\pi \circ F; \alpha].
\]  

(3.15)

**Proof:** By definition, \(F \in \mathcal{F}_S(G^C)\) implies \(f\) is Mellin integrable for some \(S_\lambda\). So

\[
\pi(\mathcal{M}_\lambda[F; \alpha]) = \pi \left( \int_{G^C} F(gg^\alpha) D\lambda g \right)
\]

\[
= \pi \left( \int_{G^C} f(g \lambda g^\alpha) \, d\nu(g \lambda) \right), \quad \alpha \in S_\lambda
\]

\[
= \int_{G^C} \pi(f(g \lambda g^\alpha)) \, d\nu(g \lambda), \quad \alpha \in S_\lambda
\]  

(3.16)

and the third line follows from Proposition 2.5. \(\Box\)

Crucially, under certain conditions \(\mathcal{M}_\lambda\) is a \(*\)-homomorphism:

**Lemma 3.15** If \(\mathcal{C}^*\) is commutative, then

\[
\mathcal{M}_\lambda[(F_1 \ast F_2); \alpha] = \mathcal{M}_\lambda[F_1; \alpha] \mathcal{M}_\lambda[F_2; \alpha]
\]  

(3.17)

and

\[
\mathcal{M}_\lambda[(F_1 \star F_2); \alpha] = \mathcal{M}_\lambda[F_1; \alpha] \mathcal{M}_\lambda[F_2; 1 - \alpha].
\]  

(3.18)

**Proof:** For \(\mathcal{C}^*\) commutative, \([\rho(g), \rho(h)] = 0\) for all \(g, h \in G^C\) and \([\rho'(g), \rho'(h)] = 0\) for all \(g, h \in \mathcal{G}^C\). Use Proposition 3.9 to write \(\exp_G g \exp_G h = \exp_G (g + h + C([g, h]))\) where \(C([g, h])\) represents the commutator terms in the BCH series. Then

\[
\rho((gh)^\alpha) = \rho((\exp_G g \exp_G h)^\alpha)
\]

\[
= e^{\alpha \rho'(g + h + C([g, h]))}
\]

\[
= e^{\alpha \rho'(g)} e^{\alpha \rho'(h)} = \rho(h^\alpha) \rho(g^\alpha).\]

(3.19)

We used (3.9) in line two. Line three follows from \([\rho(g), \rho(h)] = 0\) and \(C([g, h]) = -C([-g, -h])\) implies \(\rho'(C([g, h])) = 0\) and then using \([\rho'(g), \rho'(h)] = 0\).
Now put \( g \to \tilde{g}g \) in the functional Mellin transform of the \(*\)-convolution. The integral is invariant under this transformation by virtue of the left-invariant Haar measure on \( G^C_\lambda; \)

\[
\mathcal{M}_\lambda [(F_1 \ast F_2); \alpha] = \int_{G^C \times G^C} F_1(\tilde{g}) F_2(g(\tilde{g}g)^\alpha) D_\lambda \tilde{g} D_\lambda g \\
= \int_{G^C \times G^C} f_1(\tilde{g}_\lambda) \rho(\tilde{g}_\lambda^\alpha) f_2(g(\tilde{g}g)^\alpha) \, d\nu(\tilde{g}_\lambda) d\nu(g_\lambda) \\
= \int_{G^C \times G^C} F_1(\tilde{g}g) F_2(gg)^\alpha) \, D_\lambda \tilde{g} D_\lambda g \\
= \mathcal{M}_\lambda [F_1; \alpha] \mathcal{M}_\lambda [F_2; \alpha]. \tag{3.20}
\]

The second line uses commutativity of \( \mathcal{C}^* \), and the last equality uses Fubini which follows from Proposition \[2.5] and Definition \[2.7]. The \(*\)-convolution proof follows similarly. \( \square \)

**Lemma 3.16** If \( \mathcal{C}^* \) is non-commutative, but \( G^C \) is abelian, \( \rho \) is unitary, and \( \alpha_\mathbb{R} = \alpha^* \), i.e. \( \alpha_\mathbb{R} \in \mathbb{R} \cap \mathbb{S} \), then

\[
\mathcal{M}_\lambda [(F_1 \ast F_2); \alpha_\mathbb{R}] = \mathcal{M}_\lambda [F_1; \alpha_\mathbb{R}] \mathcal{M}_\lambda [F_2; \alpha^*_\mathbb{R}] \tag{3.21}
\]

and

\[
\mathcal{M}_\lambda [(F_1 \ast F_2); \alpha_\mathbb{R}] = \mathcal{M}_\lambda [F_1; \alpha_\mathbb{R}] \mathcal{M}_\lambda [F_2; 1 - \alpha_\mathbb{R}] . \tag{3.22}
\]

**Proof:** If \( G^C_\lambda \) is abelian,

\[
\rho((gh)^\alpha) = \rho(e^{\alpha (\log g + \log h)}) = \rho(e^{\alpha \log h}e^{\alpha \log g}) = \rho(h^\alpha)\rho(g^\alpha). \tag{3.23}
\]

Since \( \rho \) is unitary, \( \rho'(g) = \rho'(\log g) \) is anti-Hermitian and therefore \( \rho(g^{-\alpha_\mathbb{R}})^* = e^{-\alpha_\mathbb{R} \rho'(g)^*} = e^{\alpha_\mathbb{R} \rho'(g)} = \rho(g^{\alpha_\mathbb{R}}) \). Hence,

\[
\int_{G^C \times G^C} F_1(\tilde{g})(F_2^*)^*(g(\tilde{g}g)^\alpha) D_\lambda \tilde{g} D_\lambda g \\
= \int_{G^C \times G^C} f_1(\tilde{g}_\lambda) \rho(\tilde{g}_\lambda^\alpha) \rho(g_\lambda^{-\alpha})^*(f_2^*)^*(g_{\lambda}^{-1})^* \Delta(g_{\lambda}^{-1}) \, d\nu(\tilde{g}_\lambda) d\nu(g_\lambda) \\
= \int_{G^C \times G^C} F_1(\tilde{g}g)^*(gg)^\alpha \, D_\lambda \tilde{g} D_\lambda g \\
= \int_{G^C \times G^C} F_1(\tilde{g}g)^*(gg)^\alpha \, D_\lambda \tilde{g} D_\lambda g. \tag{3.24}
\]

The first equality follows from \( \rho(\tilde{g}^{-\alpha_\mathbb{R}})^* = \rho(\tilde{g}^{\alpha_\mathbb{R}}) \) and the definition of involution. If instead \( \rho \) is assumed real, \( \alpha_\mathbb{R} \) gets replaced by \( \alpha_\mathbb{R} \in i\mathbb{R} \cap \mathbb{S} \). If \( \rho(g) \in Z(\mathcal{C}^*) \) (e.g. \( \rho(g) = \det g \)) then there is no restriction on \( \alpha \in \mathbb{S} \). The proof for the \(*\)-convolution follows similarly. \( \square \)

For complex \( \alpha \) we don’t get representations, but with \( \rho \) unitary we have (for abelian \( G^C \))

\[
\mathcal{M}_\lambda [(F_1 \ast F_2); \alpha] = \mathcal{M}_\lambda [F_1; \alpha^*] \mathcal{M}_\lambda [F_2; \alpha], \tag{3.25}
\]
and for $\rho$ real
\[ \mathcal{M}_\lambda [(F_1 \ast F_2); \alpha] = \mathcal{M}_\lambda [F_1; -\alpha^*] \mathcal{M}_\lambda [F_2; \alpha] . \] (3.26)

Finally, for the most general case of non-commutative $\mathcal{C}^*$ and non-abelian $G^C$, we must restrict to $\alpha = 1$ for unitary $\rho$ (or $\pm i$ for real $\rho$) to get an algebra representation:

**Lemma 3.17** If $\mathcal{C}^*$ is non-commutative and $G^C$ is non-abelian, but $\rho$ is unitary, then
\[ \mathcal{M}_\lambda [(F_1 \ast F_2); 1] = \mathcal{M}_\lambda [F_1; 1] \mathcal{M}_\lambda [F_2; 1] \] (3.27)
and
\[ \mathcal{M}_\lambda [(F_1 \ast F_2); 1] = \mathcal{M}_\lambda [F_1; 1] \mathcal{M}_\lambda [F_2; 0] . \] (3.28)

**Proof:** Use $\rho((gh)^{-1})^* = \rho(gh) = \rho(g)\rho(h) = \rho(g)\rho(h^{-1})^*$ in the previous argument. \(\square\)

**Corollary 3.18** If $\alpha = 1/2$ and $\mathcal{C}^*$ is commutative or $G^C$ is abelian with $\rho$ unitary,
\[ \mathcal{M}_\lambda [(F \ast F^*); 1/2] = \mathcal{M}_\lambda [F; 1/2]^2 = \mathcal{M}_\lambda [(F \ast F^*) ; 1/2] . \] (3.29)

**Proposition 3.19** With $\alpha \in \mathbb{S}$ suitably restricted according to the previous lemmas, $\mathcal{M}_\lambda$ is a $*$-homomorphism.

**Proof:** Given the preceding lemmas, we only need to show
\[ \mathcal{M}^*_\lambda [F; \alpha] := (\mathcal{M}_\lambda [F; \alpha])^* = \left( \int_{G^C} (F(\gamma g^\alpha)) \mathcal{D}_\lambda \gamma \right)^* \]
\[ = \int_{G^C} (f(\gamma g^\alpha))^* \mathcal{D} \gamma \]
\[ = \int_{G^C} \rho(\gamma^\alpha)^*(f(\gamma))^* \mathcal{D} \gamma \]
\[ = \int_{G^C} \rho(\gamma^{-\alpha})^* f(\gamma^{-1})^* \Delta(\gamma^{-1}) \mathcal{D} \gamma \]
\[ = \int_{G^C} F^*(\gamma g^\alpha) \mathcal{D} \gamma \]
\[ = \mathcal{M}_\lambda [F^*; \alpha] \] (3.30)
where we used $f^*(\gamma g^\alpha) = \rho(g^{-\alpha})^* f(\gamma^{-1})^* \Delta(\gamma^{-1}) = \rho(g^{-\alpha})^* f^*(\gamma)$. In particular $\text{Id}^* = \text{Id}$ if $\mathcal{B}$ is a $*$-algebra. \(\square\)

**Theorem 3.20** Let $S_R$ denote the fundamental region with $\alpha \in \mathbb{S}$ sufficiently restricted to render $\mathcal{M}_\lambda$ a $*$-homomorphism. Then $\mathcal{F}_{S_R}(G^C)$ is a Banach $C^*$-algebra — when endowed with an involution defined by $F^*(g^{1+\alpha}) := F(g^{-1-\alpha})^* \Delta(g^{-1})$ and a suitable topology.
Proof: Linearity and \((F^*)^* = F\) are obvious. Next,

\[
(F_1^* F_2^*) \lambda (g^{1+\alpha}) := \int_{G^\lambda} f_1^* (\tilde{g}^\lambda) f_2^* (\tilde{g}^{-1}_\lambda g^{1+\alpha}_\lambda) \, d\nu(\tilde{g}^\lambda)
\]

\[
= \int_{G^\lambda} (f_2 (g^{-1}_\lambda \tilde{g}^\lambda) \Delta (g^{1-1}_\lambda \tilde{g}^\lambda) f_1 (\tilde{g}^{-1}_\lambda) \Delta (\tilde{g}^{-1}_\lambda))^* \, d\nu(\tilde{g}^\lambda)
\]

\[
= \left( \int_{G^\lambda} f_2 (g^{-1}_\lambda \tilde{g}^\lambda) f_1 (\tilde{g}^{-1}_\lambda) \, d\nu(\tilde{g}^\lambda) \right)^* \Delta (g^{1-1}_\lambda)
\]

\[
= ((F_2 * F_1) \lambda (g^{-1-\alpha}))^* \Delta (g^{-1})
\]

\[
= (F_2 * F_1) \lambda (g^{1+\alpha})
\]

(3.31)

using left-invariance of the Haar measure to put \(\tilde{g}^{\lambda} \rightarrow g^{1+\alpha}_\lambda \tilde{g}^{\lambda}\) in the fourth line. This gives \(\mathcal{M}_\lambda [(F_1^* F_2^*) ; \alpha] = \mathcal{M}_\lambda [(F_2 * F_1)^* ; \alpha]\). Lastly, since \(\mathfrak{C}^*\) is a \(C^*\)-algebra in any case, it follows that \(\|F\|_S = \|F^*\|_S\) and the lemmas imply \(\|F * F^*\|_S = \|F\|^2_S\). □

It is convenient to denote by \(\Pi^{(\alpha)} : \pi(M_\lambda)\) the \(\pi\)-representation of functional Mellin and \(\Pi^{(\alpha)}(F_{S\mathbb{R}}(G^\mathbb{C}))\) its corresponding image in \(L_B(\mathcal{H})\) under the various conditions that render \(\mathcal{M}_\lambda\) a \(*\)-homomorphism.

**Corollary 3.21** \(\Pi^{(\alpha)}_\lambda\) is a \(*\)-representation on Hilbert \(\mathcal{H}\).

For example, if \(\mathfrak{C}^*\) is non-commutative and unital, then \(\Pi^{(\alpha_R)}_\lambda\) denotes a functional Mellin \(*\)-representation for an abelian group and unitary \(\rho\). Similarly, \(\Pi^{(1)}_\lambda\) is a \(*\)-representation for non-commutative \(\mathfrak{C}^*\) and non-abelian \(G^\mathbb{C}\). In fact, as already mentioned, \(\Pi^{(1)}(F_{S\mathbb{R}}(G^\mathbb{C}))\) is closely related to a crossed product[21].

These inherited properties of the functional Mellin transform at least partially explain the utility of the resulting integrals in probing the local structure of topological linear Lie groups \(G^\mathbb{C}\) and \(*\)-algebras \(F_S(G^\mathbb{C})\). More importantly, (and this is the main theme of this paper) since it is not merely a restatement of Fourier transform it provides an independent tool to investigate these spaces. The following sections are exemplary.

### 4 Mellin functional tools

Before extracting useful tools from the functional Mellin transform, it is useful to gain some experience and insight by analyzing its reduction to finite-dimensional integrals under various conditions. Appendix [B contains several examples. They suggest how to define Mellin functional counterparts of resolvents, traces, logarithms, and determinants. For the most part, these are familiar objects and many have been constructed and extensively analyzed using a variety of approaches in the literature — in particular, resolvents, complex powers of operators, and zeta functions. Our purpose here is to establish them in the functional integral context and to show consistency.

Warning: From now on, to further clean up notation no distinction will be made between \(g, g_\lambda\), and \(\rho(g_\lambda)\) when it will not cause confusion. Also, instead of detailing integrable
conditions, we will generally assume (by restricting the domain of integration if necessary) continuous, Mellin integrable functions from the beginning. And as mentioned in the introduction, we will ignore technical operator theory issues in order to concentrate on tool building.

4.1 Functional resolvent

Definition 4.1 Consider \( A, Z \in \text{Mor}_C(G^C, \mathcal{C}^*) \) with \( Z \in \text{Z(F}_S(G^C)) \) a central element and \( E^{-(A-Z)} \in \text{F}_S(G^C) \). Assume \( (A - Z)(g) \) with fixed \( A \) but variable \( Z \) is invertible. Define the functional resolvent of \( A \) by

\[
R_\lambda(Z, A; \alpha) := ((A - Z))^{\alpha} := \mathcal{M}_\lambda \left[ E^{-(A-Z)}; \alpha \right].
\]

(4.1)

When \( \lambda \) localizes onto \( \mathbb{R}_+ \) and \( (A - Z)(g) = (A - Z)g \) with \( A, Z \in L_B(\mathcal{H}) \) and \( g \in \mathbb{R}_+ \), it is not hard to see this definition reduces to the standard Laplace transform definition of the resolvent for \( \alpha = 1 \).

Example 4.2 Consider an invertible \( (a - z) \in G^C \) and the associated real one-parameter subgroup \( \phi_{a-z}(\mathbb{R}) \subseteq G^C \) where \( \exp_{G^C}(a - z) = (a - z) \). For \( g \in \phi_{a-z}(\mathbb{R}) \), define the functional \( E^{-(A-Z)}(g) := e^{-(A-Z)(g)} := e^{-\rho(a-z)\rho(g)} \) such that \( \rho(a-z) = (A - zId) \in \mathcal{C}^* \) with \( z \in \mathbb{C} \). The functional resolvent for \( z \notin \sigma(A) \) is

\[
R_\lambda(Z, A; \alpha) = \int_{\phi_{a-z}(\mathbb{R})} e^{-(A-zId)\rho(g)} \rho(g^{\alpha}) D_\lambda g = (A - zId)^{-\alpha}, \quad \alpha \in (0, \infty).
\]

(4.2)

To see this note, for \( z \notin \sigma(A) \) a regular value, \( (A - zId) \) is invertible and commutes with all \( \rho(g) \), and so it can be extracted from the integral using the invariance of the Haar measure. After extraction, the remaining integral is a normalization we absorb into the measure; explicitly, we require \( \int_{\phi_{a-z}(\mathbb{R})} e^{-\rho(g)} \rho(g^{\alpha}) D_\lambda g := Id \in \mathcal{C}^* \) for all \( \lambda \in \Lambda \). On the other hand, if \( z \in \sigma(A) \), then \( (A - zId)^{-\alpha} \) can no longer be extracted and the associated integral formally corresponds to a fractional derivative of an imaginary delta functional according to \([10]\) \(^6\).

Emphasize \([12]\) is to be understood as a functional operator identity. That is, \( R_\lambda(Z, A; \alpha) \) represents a family of operators in \( \mathcal{C}^* \) that can be queried for measurable quantities. Let us exhibit two specific instantiations. To be concrete, let \( \mathcal{C}_0^* \equiv L_B(\mathcal{H}) \) with \( \mathcal{H} \) separable.

Assume \( (A - zId)^{-\alpha} \in L_B(\mathcal{H}) \) is diagonalizable and choose the ‘topological localization’ \( \lambda_f : \phi_{a-z}(\mathbb{R}) \rightarrow F \) where \( F \equiv (C^\infty)^d \) with \( C^\infty = \mathbb{C} \setminus \{0\} \) when \( d = \text{dim}(\mathcal{H}) < \infty \). When \( \mathcal{H} \) is infinite-dimensional, \( F \equiv (C^\infty)^C \) where \( (C^\infty)^C \) denotes the abelian group underlying the space of continuous, measurable functions \( g_{\lambda_f} : \mathbb{R} \rightarrow C^\infty \cong \mathbb{R}_+ \times S^1 \) which must be localized further.

\(^6\)When \( (A - Z) \) is linear on \( G^C \) it is useful to generalize for any \( \mathcal{C}^* \) the delta functional defined in \([10]\) and to formally write (under appropriate conditions on \( A \)) \( R_\lambda(Z, A; \alpha) = \text{Pv}(A - Z)^{-\alpha} + i\delta^{(\alpha-1)}(A - Z)_\lambda \). Consequently, at \( \alpha = 1 \), \( \text{Pv}(A - Z)_\lambda^{-1} \) and \( \delta(A - Z)_\lambda \) correspond to the resolvent set and spectrum respectively. The appearance of distributions here motivates extending the theory of Mellin transforms of distributions to the functional context.
say by expected values \( \langle i | \rho(g) | j \rangle \) with \( |i\rangle, |j\rangle \in \mathcal{H} \). Since \( \rho(g) \) commutes with \((A - z \text{Id})\), it is also diagonalizable. In a diagonal basis then, (4.2) localizes to

\[
R_{F,N}(Z, A; \alpha) = \int e^{-(A - z \text{Id})\rho(g)} \rho(g^\alpha) \, d\nu(g_N) = (A - z \text{Id})^{-\alpha}, \quad \alpha \in \{0, 1\}
\]  

where \( \nu(g_N) = (\log r \times \theta)/N \) with \( g = re^{i\theta} \) and \( \theta \in [-\pi/2, \pi/2] \) or \( \theta \in [\pi/2, 3\pi/2] \) depending on whether \( \sigma(A - z \text{Id}) \in \mathbb{C}_+ \) or \( \sigma(A - z \text{Id}) \in \mathbb{C}_- \) respectively. The normalizations are \( N = (\pi \Gamma(\alpha))^d \) and \( N = \int_{\mathbb{C}^d} \langle i | e^{-g^\alpha} | j \rangle \theta \log \theta = \pi \Gamma(\alpha) \) in the finite-dimensional and infinite-dimensional case respectively.

Now suppose \( A \) is self-adjoint. Since \( \phi_{a-3}(\mathbb{R}) \) is a real one-dimensional abelian topological group, the previous paragraph suggests a finer localization \( \lambda_{\mathbb{R}_+} : \phi_{a-3}(\mathbb{R}) \to (\mathbb{R}_+ \cup \mathbb{R}_-)^d \)

where \( \mathbb{R}_+ \cup \mathbb{R}_- = \mathbb{R}_+ \times \{1, -1\} \). To maintain unit normalization, choose the Haar measure \( \nu(g) = \nu(g)/\Gamma(\alpha) = \log g/\Gamma(\alpha) \). In the diagonal basis, the matrix elements of the functional resolvent for self-adjoint \( A \) reduce to

\[
\langle i | R_{\mathbb{R}_+, g}(Z, A; \alpha) | j \rangle = \frac{1}{\Gamma(\alpha)} \left( \int_0^\infty - \int_{-\infty}^0 \right) \langle i | j \rangle e^{-(A - z)g_j^\alpha} \, d\log g_j = (A - z \text{Id})_{ij}^{-\alpha} \delta_{ij} \tag{4.4}
\]

where we choose a suitable log branch and only one of the integrals will contribute depending on whether \( \sigma(A - z \text{Id}) \subset \mathbb{C}_+ \) or \( \sigma(A - z \text{Id}) \subset \mathbb{C}_- \). This verifies the operator identity \( R_{\mathbb{R}_+, g}(Z, A; \alpha) = (A - z \text{Id})^{-\alpha} \). Evidently, for both \( \lambda_F \) and \( \lambda_{\mathbb{R}_+} \) we could just as well have localized after extracting \( (A - z \text{Id})^{-\alpha} \) from \( R_\lambda(Z, A; \alpha) \).

Definition (4.1) is a “good” definition in the sense that it agrees with the standard resolvent when \((A - z \text{Id}) \in L_B(\mathcal{H}) \) and \( \alpha = 1 \) according to (4.4) and the spectral theorem. Moreover,

\[
\frac{1}{\Gamma(n)} \frac{d^{n-1}}{dz^{n-1}} (A - z \text{Id})^{-\alpha} = \frac{1}{\Gamma(n)} \int_0^{\pm \infty} e^{-(A - z_i)^\alpha} g^n \, d\log g = (A - z \text{Id})_{ii}^{-\alpha} \tag{4.5}
\]

so the objects agree at all allowed integer values of \( \alpha \). Intuitively, the functional resolvent represents a “complex derivative” of the resolvent, i.e.

\[
\frac{1}{\Gamma(n)} \frac{d^{n-1}}{dz^{n-1}} (A - z \text{Id})^{-\alpha} \sim (A - z \text{Id})^{-\alpha}.
\]

Matrix elements of these objects can be had by appealing to Proposition (3.1) and then using either of the concrete integrals in (4.3) or (4.4) (or any other explicit localization).

This example is a significant simplification because the exponent is linear in \( g \) and the localizations are a rather drastic reduction from a one-dimensional abelian subgroup down to \( F \) and \( \mathbb{R}_+ \). A still-seemingly tractable case is for \( \mathbb{C}_+ \) any locally compact topological abelian group (which is finite dimensional) and \((A - Z)(g)\) linear in \( g \) for which the Mellin distributions of appendix A.3 become relevant. In the generic case, it is much harder (if not impossible) to exhibit a closed form for \( R_\lambda(Z, A; \alpha) \), and its interpretation is not obvious.

Specializing the functional resolvent defines a functional inverse power:

\[
7\text{Observe that } \mathbb{R}/\{1, -1\} \cong \mathbb{R}_+ \text{ so } \mathbb{R}^\times \text{ is a double-cover of } \mathbb{R}_+. \text{ We take } \lambda_{\mathbb{R}_+} : \phi_{a-3}(\mathbb{R}) \to \mathbb{R}_+ \cup \mathbb{R}_- \text{ instead of } \lambda_{\mathbb{R}} : \phi_{a-3}(\mathbb{R}) \to \mathbb{R}^\times \text{ because, in a physics context where } A \text{ is skew-adjoint, we want to interpret the one-parameter subgroup as the forward and reverse time-evolution operator, i.e. as the union of two evolution semi-groups. The minus sign for the } \mathbb{R}_- \text{ integral comes from the relative phase of } \pi \text{ radians between the arguments of elements in } \phi_{a-3}(\mathbb{R}_+) \text{ and } \phi_{a-3}(\mathbb{R}_-). \text{ Intuitively, } g_+ = \log g_+ \text{ and } g_- = \log g_- \text{ “point” in opposite directions which encodes evolution-time reversal. Essentially, this is just the previous } F \text{ case with measurable functions } g : \mathbb{C} \to \mathbb{R}_+ \times S^1 \text{ restricted to } g : \mathbb{C} \to \mathbb{R}_+ \times \{1, -1\}.
\]
Definition 4.3  The functional inverse power of $A \in \text{Mor}_C(G^C, C^*)$ such that $E^{-A} \in F_{\mathcal{C}}(G^C)$ is defined by
\[
A^{-\alpha}_\lambda := (A)^{-\alpha}_\lambda = R_\lambda(0, A; \alpha) = \mathcal{M}_\lambda \left[ E^{-A}; \alpha \right].
\] (4.6)

For example, if $C^* = L_B(L^2(\mathbb{R}^n))$ then $\Delta^{-1}_\infty = \mathcal{M}_\lambda \left[ E^{-\Delta}; 1 \right]$ is the inverse of the Laplacian on $\mathbb{R}^n$. Example 3.13 is the archetype inverse power of the Laplacian on $\mathbb{R}^n$ localized by position-to-position boundary conditions, i.e. $\langle x_a' | R_H(0, \Delta; 1) | x_a \rangle = \langle x_{a'} | \Delta^{-1}_H | x_a \rangle = K(x_{a'}, x_a)$ is the elementary kernel/propagator. To construct the elementary kernel of the Laplacian on a Riemannian manifold $M^n$, one has to appropriately parametrize the space of paths on $M^n$ by the space of paths on some $G^C$ and then localize according to given boundary conditions (see e.g. [10, application 3.2]).

Example 4.4  Consider the same set-up from Example 4.2 with localization $\lambda_f$; except now with $\mathfrak{z} = 0$ and some invertible $B \in C^*$. Then, for example if $\sigma(A) \subset \mathbb{C}_+$ and $\sigma(B) \subset \mathbb{C}_+$,
\[
\mathcal{M}_\lambda \left[ E^{-B} \ast E^{-A}; \alpha \right] = \int_{\phi_a(\mathbb{R})} \int_{\phi_a(\mathbb{R})} e^{-B\tilde{g}} e^{-A\tilde{g}^{-1}g} \tilde{g}^\alpha \mathcal{D}_{\lambda\tilde{g}} \mathcal{D}_g
\]
\[
\xrightarrow{\lambda_f} \frac{1}{N^2} \int f \ e^{-B\tilde{g}} (\tilde{g}^{-\alpha})^* A^{-\alpha} d \log \tilde{g}
\]
\[
= \ (BA)^{-\alpha}_N, \quad \alpha \in \mathbb{R}_{N,N}, \quad (4.7)
\]

where we used Fubini, involution $F(g(\tilde{g})^\alpha) = \rho((\tilde{g}g)^{-\alpha})^* F^* (g^{-1})^* \Delta(g^{-1})$, and the fact that $A$ commutes with both $g$ and $\tilde{g}$ in the first line. Since $\rho(g)$ commutes with $A^{-\alpha}$ in this case, by Lemma 3.10 we get $(BA)^{-\alpha}_N = \mathcal{M}_{f,N} \left[ E^{-B}; \alpha \right] \mathcal{M}_{f,N} \left[ E^{-A}; \alpha \right] = B^{-\alpha}_N A^{-\alpha}_N = B^{-\alpha} A^{-\alpha}_N$.

Complex inverse powers are typically only valid for $\Re(\alpha) > 0$. Moreover, given some $A \in C^*$, the functional form $(A)^{-\alpha}_\lambda =: A^{-\alpha}_\lambda$ will not resemble $A^{-\alpha}$ except in some limited cases. This means we can’t profitably define a positive power simply by $\mathcal{M}_\lambda \left[ E^{-(A^{-1})}; \alpha \right]$ when $A$ is invertible, because $((A^{-1})^\lambda)^{-\alpha} =: (A^{-1})^{-\alpha}_\lambda \neq A^\alpha$ in general. In particular, this is clearly the case for differential operators. We will come back to positive powers later, but for now we continue to build tools with inverse powers.

Definition 4.1 and the characterization of delta functionals in [10], suggest that the functional inverse complex power of $A$ implicitly includes derivatives of Dirac delta functionals if $0 \in \sigma(A)$. On the other hand, if $A$ has $0 \notin \sigma(A)$, then we can define a functional zeta.

Definition 4.5  If $A^{-\alpha}_\lambda$ is trace class with $0 \notin \sigma(A)$, then the functional zeta is defined by
\[
\zeta_A(\alpha) := \text{tr} \left( A^{-\alpha}_\lambda \right) = \text{tr} \mathcal{M}_\lambda \left[ E^{-A}; \alpha \right].
\] (4.8)

Of course this trace is not always a well-defined object for all $\alpha$ in the fundamental region of $A^{-\alpha}_\lambda$. Rather, the fundamental region containing $\alpha$ is dictated by $\lambda$ and is often restricted. An effective strategy to quantify the restriction is to move the trace inside the integral. Presumably the trace of the integrand will be well-defined for certain choices of $\lambda$ and the properties of the functional Mellin transform will allow the domain of $\alpha$ to be determined. This strategy leads us to the next subsection.

\footnote{The fundamental strip of the product is the intersection of the fundamental strips of the two factors. Also, we used $O^{-\alpha} = e^{-\alpha O'}(\varphi)$ for $O \in C^*$ and $\varphi \in \mathfrak{g}^C$.}
4.2 Functional trace, logarithm, and determinant

In this subsection we define and explore the functional Mellin analogs of trace, logarithm, and determinant. When \( \alpha = 1 \), such functional objects are ubiquitous in physics. We will see that generalizing to \( \alpha \in \mathbb{S} \) is quite productive.

**Definition 4.6** Let \( A \in \text{Mor}_C(G^C, \mathbb{C}^*) \) with \( E^{-A} \in F_S(G^C) \) for some \( \lambda \)-dependent fundamental region \( \alpha \in S_\lambda \), and let \( A \in \mathbb{C}^* \) such that \( 0 \notin \sigma(A) \). The functional trace \( \text{Tr} A \in \text{Mor}_C(G^C, \mathbb{C}) \) is defined by

\[
\text{Tr} A_{\lambda}^{-\alpha} := (\text{Tr} A)_{\lambda}^{-\alpha} := \mathcal{M}_\lambda [\text{Tr} E^{-A}; \alpha] := \int_{G^C} \text{tr} (e^{-A(g)} g^\alpha) \, D_\lambda g .
\] (4.9)

Remark that, as a consequence of Proposition 2.5, the interchange of the ordinary trace and functional integral is valid only for \( E^{-A} \in F_S(G^C) \) and appropriate \( \lambda \). Then according to the definition,

\[
\text{tr} A_{\lambda}^{-\alpha} = \text{tr} \mathcal{M}_\lambda [E^{-A}; \alpha] \\
= \text{tr} \left( \int_{G^C} e^{-A(g)} g^\alpha \, D_\lambda g \right) , \quad \alpha \in S_\lambda \\
= \int_{G^C} \text{tr} (e^{-A(g)} g^\alpha) \, D_\lambda g , \quad \alpha \in \tilde{S}_\lambda \subseteq S_\lambda \\
= (\text{Tr} A)_{\lambda}^{-\alpha} \\
= \text{Tr} A_{\lambda}^{-\alpha} .
\] (4.10)

Evidently, the ordinary trace \( \text{tr} \) and functional trace \( \text{Tr} \) possess the same functional form. But the fundamental region of the functional trace depends on the chosen normalization, and taking the ordinary trace inside the integral often requires a restriction on the fundamental region of \( \mathcal{M}_\lambda [\text{Tr} E^{-A}; \alpha] \) as in the third line of (4.10). The point is we can turn the calculation around and (with appropriate normalization/regularization) give meaning to the object \( \mathcal{M}_\lambda [\text{Tr} E^{-A}; \alpha] \) through the ordinary trace and an adjustment to \( S_\lambda \).

In particular, the functional zeta can be represented as

\[
\zeta_{\lambda}(\alpha) \equiv \mathcal{M}_\lambda [\text{Tr} E^{-A}; \alpha] .
\] (4.11)

but only for appropriate \( \lambda \) and \( \alpha \). For example,

**Example 4.7** Consider a counting functional \( N \) such that \( N(g) = N g \). Let \( \mathcal{H} \) be a separable Hilbert space. Suppose \( N \in \mathbb{C}^* = L_B(\mathcal{H}) \) is self-adjoint with \( \sigma(N) = \mathbb{Z}_+ \), and let \( \{ |i \rangle, \varepsilon_i \} \) with \( i \in \{1, \ldots, \infty\} \) denote the set of orthonormal eigenvectors and associated eigenvalues of \( N \). Choose \( \lambda_{\pm} : G^C \to \mathbb{R}_+ \cup \mathbb{R}_- \) and \( \rho \) such that \( g \mapsto g \text{Id} \). The Riemann zeta function associated with \( N \) can be defined by

\[
\zeta_{N_{\Gamma}}(\alpha) = \text{tr} \int_{\mathbb{R}_+ \cup \mathbb{R}_-} e^{-Ng \rho(g^\alpha)} \, d\nu(g_{\Gamma}) .
\] (4.12)
In the orthonormal basis this is
\[\zeta_{N_\Gamma}(\alpha) = \sum_{i=1}^{\infty} \int_{\mathbb{R}_+} e^{-\epsilon_i g + \alpha \log g} \langle i | i \rangle \, d\nu(g_\Gamma), \quad \alpha \in (0, \infty)\]
\[= \int_{\mathbb{R}_+} \sum_{i=1}^{\infty} e^{-\epsilon_i g + \alpha \log g} \, d\nu(g_\Gamma), \quad \alpha \in (1, \infty)_\Gamma\]
\[= \int_{0}^{\infty} \frac{1}{e^g - 1} g^\alpha \, d\nu(g_\Gamma), \quad \alpha \in (1, \infty)_\Gamma\]
\[= \mathcal{M}_{\mathbb{R}_+ \Gamma} [\text{Tr} E^{-N}; \alpha] = \zeta(\alpha), \quad \alpha \in (1, \infty)_\Gamma\] (4.13)
where \(\nu(g_\Gamma) := \nu(g)/\Gamma(\alpha) = \log g/\Gamma(\alpha)\). Note the integral in the second line is valid for \(\alpha \in (0, \infty)\) and the integral over \(\mathbb{R}_-\) does not converge for any \(\alpha\). So exchanging summation and integration in the third line comes with the price of restricting the fundamental strip.

If instead the localization is \(\lambda_C : \mathbb{C} \to \mathbb{C}\) a smooth contour in \(\mathbb{C}^\times \equiv \mathbb{C} \setminus \{0\}\) with the branch cut for \(\log\) along the positive real axis, this has the well-known representation
\[\zeta_{N_{\Gamma_C}}(\alpha) = \int_{\mathbb{C}} \frac{1}{e^g - 1} g^\alpha \, d\nu(g_{\Gamma_C}), \quad \alpha \in (0, \infty)_{\Gamma_C \setminus \{1\}}\]
\[= \mathcal{M}_{\mathbb{C}, \Gamma_C} [\text{Tr} E^{-N}; \alpha] = \zeta(\alpha), \quad \alpha \in (0, \infty)_{\Gamma_C \setminus \{1\}}\] (4.14)
where \(d\nu(g_{\Gamma_C}) := \frac{\pi \csc(\pi \alpha)}{2\pi i(\alpha)} \frac{dg}{g}\) and \(\mathbb{C}\) starts at \(+\infty\) just below the real axis, passes around the origin clockwise, and then continues back to \(+\infty\) above the real axis.\(^9\) This is an explicit illustration that \(\mathbb{S}\) depends on both the normalization and the group. One can also define the trace of a winding/degree functional \(W\) given by \(W(g) = Wg\) where \(\sigma(W) = \mathbb{Z} \setminus \{0\}\):
\[\zeta_{W_\Gamma}(\alpha) := \int_{\mathbb{R}_+ \cup \mathbb{R}_-} \text{tr} \left( e^{-W g} \rho(g^\alpha) \right) \, d\nu(g_\Gamma), \quad \alpha \in (1, \infty)_\Gamma\]
\[= \mathcal{M}_{\mathbb{R}_+ \Gamma} [\text{Tr} E^{-W}; \alpha] = \zeta(\alpha), \quad \alpha \in (1, \infty)_\Gamma\] (4.15)
Here each integral contributes depending on the sign of \(\sigma(W)\).

**Example 4.8** Again for \(N \in L_B(\mathcal{H})\) but now with \(N(g) = N(g \pm i\pi)\), the Dirichlet eta function associated with \(N\) can be represented by
\[\eta_{N_\Gamma}(\alpha) = \int_{\mathbb{R}_+} \sum_{i=1}^{\infty} e^{-\epsilon_i (g \pm i\pi) + \alpha \log g} \, d\nu(g_\Gamma)\]
\[= \int_{0}^{\infty} \frac{-1}{e^g + 1} g^\alpha \, d\nu(g_\Gamma), \quad \alpha \in (0, \infty)_\Gamma\]
\[= \mathcal{M}_{\mathbb{R}_+ \Gamma} [\text{Tr} E^{-N}; \alpha] = \eta(\alpha), \quad \alpha \in (0, \infty)_\Gamma\] (4.16)
where \(\nu(g_\Gamma) := \log g/\Gamma(\alpha)\). The second equality uses the fact that \(\sum \int e^{-\epsilon_i g^\alpha} \, d\nu(g_\Gamma)\) converges for \(\alpha \in (0, \infty)\).
\(^9\)This assumes \(\arg g = \pm \pi\) below(respectively above) the real axis. Of course \(\pi \csc(\pi \alpha)/\Gamma(\alpha)\) can be analytically continued to the left-half plane thus obtaining a meromorphic representation of Riemann zeta.
Continuing the strategy of defining functional analogs, define the functional logarithm:

**Definition 4.9** Let $A \in \text{Mor}_C(G^C, C^*)$ be invertible and suppose that $E^{-A} \in F_{S}(G^C)$ for some fundamental region $\alpha \in S_{\lambda}$. The functional logarithm $\text{Log} A$ is defined by

$$
\text{Log} A^{-\alpha} := (\text{Log} A)^{-\alpha} := \frac{d}{d\alpha} M_{\lambda} [E^{-A}; \alpha] = \int_{G^C} e^{-\lambda(g)} g^\alpha \log g \, D\lambda g
$$

where $g^\alpha \, D\lambda g = \frac{d}{d\alpha} (g^\alpha \, d\nu(g))$ verifies $\int_{G^C} e^{-g} g^\alpha \, D\lambda g = 0$ for the chosen normalization. Note that we allow $0 \in \sigma(A^{-1})$ here if the $\alpha \to 0^+$ limit exists.

For this to be well-defined requires a suitable definition of the ordinary logarithm $\log A^{-1}$ of operators $A^{-1} \in C^*$. Choosing the standard series or the spectral representation of (3.4) is convenient and typical, but their convergence is limited to $\|A^{-1} - Id\| < 1$ relative to the norm on $C^*$. Perhaps a better choice for invertible $(A - Id) \in C^{m \times m}$ is to use the Cayley transform $C(A) := (A + Id)(A - Id)^{-1}$. In this case,

$$
\log A^{-1} = 2 \sum_{m=3}^{\infty} \frac{(-1)^{m+1}}{m} C(A^{-1})^m
$$

converges if $\sigma(A^{-1}) \subseteq \mathbb{C}_+$. [25]

Notably, since functional Log is a derivative, it commutes with functional Tr if $\alpha \in \tilde{S}_{\lambda}$;

$$
(\text{Log Tr} A)^{-\alpha} := \frac{d}{d\alpha} M_{\lambda} [\text{Tr} E^{-A}; \alpha] = \zeta_{\lambda}^A(\alpha),
$$

and

$$
(\text{Tr Log} A)^{-\alpha} := \text{Tr} M_{\lambda} [E^{-A}; \alpha] = \widehat{M}_{\lambda} [\text{Tr} E^{-A}; \alpha] = \zeta_{\lambda}^A(\alpha).
$$

But, $(\text{Log Tr} A)^{-\alpha(1)} \neq \log (\text{Tr} A)^{-\alpha(1)}$; indicating that functional Log and ordinary log are applied very differently. However, they can be equivalent:

**Example 4.10** Return to Example [4.2] but now with $\zeta = 0$ and $\sigma(A) \subseteq \mathbb{C}_+$. Then calculate

$$
\begin{align*}
\text{Log} A^{-1} &= \lim_{\alpha \to 0^+} \int_{\Phi_{\alpha}(R)} e^{-A g} g^\alpha \log g \, D\lambda g \\
&= \lim_{\alpha \to 0^+} \int_{\Phi_{\alpha}(R)} e^{-g(A^{-1} g)^\alpha} \log (A^{-1} g) \, D\lambda g \\
&\xrightarrow{\lambda} \lim_{\alpha \to 0^+} (A^{-1})^\alpha \int_{R} e^{-g} g^\alpha \, d\nu(g_N) \\
&= \lim_{\alpha \to 0^+} (A^{-1})^\alpha \int_{R} e^{-g} g^\alpha (\log g + \log A^{-1} - \psi(\alpha)) \, d\nu(g_N) \\
&= \log A^{-1}
\end{align*}
$$

(4.21)
where the third line follows because $A^{-1}$ and $\rho(g)$ commute and the $\mathbb{R}_+$ integral does not converge for any $\alpha$. (See [3.9] and appx. B for the definition and motivation for $d\tilde{v}(g)$.) Note that $\log A = (A^{-1})^\alpha \log A^{-1}$. Evidently, functional $\log$ behaves like an ordinary logarithm only for $\lim \alpha \to 0^+$; e.g. formally $(\log (E^{-A}))^{-1} = (E^{-\log A})^{-1} = A_{\lambda}$.

The final component of our triad is the functional determinant of an inverse power.

**Definition 4.11** Let $A \in \text{Mor}_C(G^C, \mathfrak{C}^*)$ with $A(g) \in \mathfrak{C}^*$ trace class and $0 \notin \sigma(A(g))$. Suppose $E^{-\text{Tr}A} \in \text{F}_{\mathbb{S}}(G^C)$ for $\alpha \in \mathbb{S}_\lambda$. The functional determinant $\det A \in \text{Mor}_C(G^C, \mathbb{C})$ is

$$
\det A^{-\alpha} := (\det A)^{-\alpha} := \mathcal{M}_\lambda [\det E^{-A}; \alpha] := \int_{G^C} \det (e^{-A(g)} g^\alpha) \, D\lambda g
$$

$$
= \int_{G^C} e^{-\text{tr}(A(g))} \det g^\alpha \, D\lambda g
$$

(4.22)

where $\det \rho(g^\alpha) = \det e^{\alpha \rho'(g)} = e^{\text{tr}(\alpha \rho'(g))} = (e^{\text{tr} \rho'(g)})^\alpha = (\det \rho(g))^\alpha$ if $\arg(e^{\text{tr} \rho'(g)}) = 0$.

In contrast to functional $\log$ and $\text{Tr}$, functional $\log$ and $\det$ do not generally commute;

$$
(\log \det A)^{(\alpha+1)} := \frac{d}{d\alpha} \mathcal{M}_\lambda [\det E^{-A}; \alpha] \neq \hat{\mathcal{M}}_\lambda [\det E^{-A}; \alpha] =: (\det \log A)^{-(\alpha+1)}
$$

(4.23)

since $\frac{d}{d\alpha} \det \rho(g^\alpha) \neq \det(\rho(g^\alpha) \log \rho(g))$ unless $\rho$ is one-dimensional.

As with the functional trace and logarithm, the determinant can be ‘taken outside the integral’ only for special choices of $\lambda$ for which the trace in the integrand is well-defined given a particular representation. It is important to realize that, at the functional level,

**Proposition 4.12** If $A_1, A_2 \in \text{F}_{\mathbb{S}}(G^C)$ such that $A_1(g)$ and $A_2(g)$ are trace class, then

$$
(\det(A_1 \ast A_2))^{-\alpha} = (\det A_1 \ast \det A_2)^{-\alpha} = \det A_1^{-\alpha} \det A_2^{-\alpha}.
$$

(4.24)

Consequently, the functional determinant possesses the multiplicative property if the pertinent functional determinants have overlapping critical strips.

**Proof:**

$$
\mathcal{M}_\lambda [\det(E^{-A_1} \ast E^{-A_2}); \alpha] = \int_{G^C} \int_{G^C} \det (e^{-A_1(\tilde{g})} e^{-A_2(\tilde{g}^{-1} g)} g^\alpha) \, D\lambda \tilde{g} \, D\lambda g
$$

$$
= \int_{G^C} \int_{G^C} e^{-\text{tr}(A_1(\tilde{g}))} e^{-\text{tr}(A_2(\tilde{g}^{-1} g))} \det g^\alpha \, D\lambda \tilde{g} \, D\lambda g
$$

$$
= \int_{G^C} \int_{G^C} \det (e^{-A_1(\tilde{g})} e^{-A_2(g)} ) \det [(\tilde{g}g)^\alpha] \, D\lambda \tilde{g} \, D\lambda g
$$

$$
= \det A_1^{-\alpha} \det A_2^{-\alpha}.
$$

(4.25)

In line three we used the fact that BCH implies $e^{\alpha \text{tr}(\log(gh))} = e^{\alpha \text{tr}(\rho'(g)+\rho'(h))}$. □
However, at the function level precipitated by a specific $\lambda$, the determinant may not have the multiplicative property with respect to $\alpha$. For starters, if $A$ is not in the multiplier algebra $M_\rho(\mathcal{C}^*)$, the functional determinant $\text{Det} A_\lambda^{-\alpha}$ doesn’t have the same form as $\det A^{-\alpha}$. But even if $A_1(g)$ and $A_2(g)$ are linear in $g$, there may not exist a consistent choice of $\lambda$ that renders both $A_{1\lambda}$ and $A_{2\lambda}$ simultaneously analytic at a common value of $\alpha$; even if their convolution is analytic there. Moreover, if such a $\lambda$ does exist, the localization implicit in $\lambda$ that achieves the reduction $\text{Det} \rightarrow \text{det}$ may introduce a “multiplicative anomaly”\textsuperscript{10}.

**Proposition 4.13** Suppose $A(g) = \rho(ag) =: A\rho(g)$ where $a \in G^C$ and $A\rho(g)$ is trace class. Then,

$$\text{Det} A_\lambda^{-\alpha} = \mathcal{N}_\lambda(\alpha) \left| \det A^{-1}\right|^\alpha e^{i\varphi_\lambda(\alpha)}, \quad \alpha \in \mathbb{S}_\lambda$$

(4.26)

where $\mathcal{N}_\lambda(\alpha)$ is a $\lambda$-dependent normalization and $\varphi_\lambda(\alpha) = 2\alpha(\arg(e^{\text{tr}(\log A^{-1})}) + n\pi)$.

**Proof:** Using the invariance of the Haar measure with $g \rightarrow A^{-1}g$ yields

$$\text{Det} A_\lambda^{-\alpha} = \mathcal{M}_\lambda \left[ \text{Det} E^{-A}; \alpha \right] = \int_{G^C_\lambda} e^{-\text{tr}(g)} \det[(A^{-1}g)^\alpha] \, d\nu(g_\lambda)$$

$$= \int_{G^C_\lambda} e^{-\text{tr}(g)} (\det A^{-1})^\alpha e^{i\alpha \arg(\det(A^{-1}))} \det g^\alpha \, d\nu(g_\lambda)$$

$$= \left| \det A^{-1}\right|^\alpha e^{2i\varphi_\lambda(\alpha)} \int_{G^C_\lambda} e^{-\text{tr}(g)} \det g^\alpha \, d\nu(g_\lambda)$$

$$=: \left| \det A^{-1}\right|^\alpha e^{2i\varphi_\lambda(\alpha)} \mathcal{N}_\lambda(\alpha), \quad \alpha \in \mathbb{S}_\lambda$$

(4.27)

where $\varphi_\lambda(\alpha) = \alpha(\arg(e^{\text{tr}(\log A^{-1})}) + n\pi)$. \qed

The normalization $\mathcal{N}_\lambda(\alpha) := \det Id_\lambda^{-\alpha}$ requires scrutiny. The definition of functional determinant assumes $A(g)$ is trace class. However, $A(g) = \rho(ag)$ will not be trace class for generic $G^C$. If it is not, then we can try to regulate $\mathcal{N}_\lambda(\alpha)$ with a positive-definite invertible fixed element $R \in G^C_\lambda$ such that $Rg$ is trace class and $e^{-\text{tr}(Rg)} \in \mathcal{F}_\mathbb{S}(G^C)$. Let $\mathcal{H}$ furnish a representation of $\pi(\mathcal{C}^*)$. Pick a basis in $\mathcal{H}$ for which $R$ is diagonal. Then

$$\mathcal{N}_\lambda(R; \alpha) := \int_{G^C_\lambda} e^{-\text{tr}(Rg)} \det g^\alpha \, d\nu(g_\lambda) = \prod_{i=1}^d r_i^{-\alpha}, \quad \alpha \in \mathbb{S}_\lambda.$$  

(4.28)

where $\dim(\mathcal{H}) = d$. Even if $d = \infty$, this product potentially can be rendered finite and well-defined if a suitable regulator $R$ exists.

Alternatively, we can simply choose the normalization/regularization associated with the choice of $\lambda$ to set $\mathcal{N}_\lambda(R; \alpha) = 1$ — which amounts to formally dividing out this factor

\textsuperscript{10}For example, if one chooses zeta function regularization to effect the reduction $\text{Det} \rightarrow \text{det}$, it is well-known that a non-vanishing Wodzicki residue at $\alpha = 0$ leads to a multiplicative anomaly.

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from the functional determinant (similarly to what is done with $\Gamma(\alpha)$). The corresponding regularized functional determinant of an operator $O$ on $H$ can then be defined by

$$\det_R O^{-\alpha} := \frac{\det \text{Ad}(R) O^{-\alpha}}{\mathcal{N}_R(R; \alpha)}$$

$$= \frac{1}{\mathcal{N}_R(R; \alpha)} \int_{G^c_\lambda} e^{-\text{tr}(ROR^{-1} g)} \det g^\alpha \, d\nu(g_\lambda)$$

$$= \frac{1}{\mathcal{N}_R(R; \alpha)} \left( \det(R^\alpha) \det(O^{-\alpha}) \right) \int_{G^c_\lambda} e^{-\text{tr}(Rg)} \det g^\alpha \, d\nu(g_\lambda)$$

$$= e^{i\varphi_{O,R}(\alpha)} \left| \frac{\det(O^{-1})}{\det(R^{-1})} \right|^\alpha$$

$$= \left( \det_R O^{-1} \right)^\alpha. \quad (4.29)$$

This is common practice: One knows how $R$ acts on $H$ and then defines the regularized determinant of $A$ by $\det_R A := \det A / \det R$.

So in particular, if $\arg \text{tr}(\log(RA^{-1})) = 0 \mod 2\pi$ and $\alpha = 1 \in \mathbb{S}_\lambda$, then

$$\det_R A^{-1} \det_R A^{-1} = \det_R A^{-2} = (\det A^{-1} / \det R^{-1})^2 \quad (4.30)$$

and the regularized determinant enjoys the usual multiplicative property. However, if instead $\arg(e^{\text{tr}(\log(RA^{-1}))}) \neq 0 \mod 2\pi$, a multiplicative anomaly obtains.

Yet another perspective on the determinant comes from Lemma 3.14 along with the fact that $\det : G^c_\lambda \to \mathbb{C}^\times$ is a (possibly projective) representation. Suppose $A$ is self-adjoint and bounded. Then choose the localization $\lambda_{R\pm} : G^c_\lambda \to \mathbb{R}_+ \cup \mathbb{R}_-$ to get

$$\mathcal{M}_{R\pm} [\det E^{-A}; \alpha] = \det \left( \mathcal{M}_{R\pm} [E^{-A}; \alpha] \right) = \det A^{\alpha}_{R\pm, \Gamma} = \det A^{-\alpha}, \quad \alpha \in \tilde{\mathbb{S}}_\lambda \quad (4.31)$$

This sidesteps the potentially thorny issue of $\mathcal{N}_\lambda(\alpha)$ — at least for self-adjoint $A$.

## 5 More Mellin functional tools

Conspicuously absent from the development so far have been functional operators $A^\alpha_\lambda$ with $\alpha \in \mathbb{C}_+$ — which we will refer to as positive powers. This section remedies the deficiency.

### 5.1 Positive powers

Recall the notion that $(A - z)^{-\alpha} \sim d^{\alpha-1}(A - z)^{-1}/dz^{\alpha-1}$. We want to view an inverse power in the same way. That is, formally $A^{-\alpha} \sim d^{\alpha-1}(A - z)^{-1}/dz^{\alpha-1}|_{z=0} = (A^{-1})^{(\alpha-1)'}$. From this perspective, there should be associated functionals in $\mathbf{F}_S(G^c)$ that roughly correspond to complex derivatives of the resolvent at the point $z = 0$. Positive powers are based on these objects.

**Definition 5.1** Suppose $A \in \text{Mor}_c(G^c, \mathbb{C}^\times)$ is invertible and $E^{-A-Z} \in \mathbf{F}_S(G^c)$. Let $\beta$ be a point in the fundamental strip of the functional resolvent of $A$. Define the functional
The functional positive power of $A$ can be defined by (cf. \cite{31,32,33})

$$
A_{\lambda,\beta}^\alpha := (A)_{\lambda,\beta}^{\alpha} := M_{\lambda} [(Id - A^{-1}(g))^{\beta} (A(g) - Id)^{\beta} = (Id - A^{-1}(g))^{-\beta}]
$$

$$
:= \int_{C^\infty} (Id - A^{-1}(g))^{-\beta} g^\alpha D_{\lambda} g , \quad \alpha \in S_{\lambda,\beta} .
$$

(5.1)

Notice the $\beta$ dependence of $A_{\lambda,\beta}^\alpha$. It exerts its influence by restricting the fundamental strip of $\alpha$. It is not surprising the fundamental strip of the functional resolvent fixes the range of validity of the positive power of $A$. After all, it fixes the inverse power as well.

**Example 5.2** Return to the context of Example 4.2. As before, $A^{-1}(g) = A^{-1} \rho(g)$, and to simplify matters assume $A^{-1} \in L_B(H)$ is positive-definite. Here we choose the localization $\lambda_{\mathbb{R}}: \phi_a(\mathbb{R}) \rightarrow \mathbb{R}^\times$ since the integrand derives from the resolvent which characterizes the spectrum of $a$ and we assume invertible $A$. We calculate

$$
A_{\lambda,\beta}^\alpha = \int_{\phi_a(\mathbb{R})} (Id - A^{-1} \rho(g))^{-\beta} \rho(g)^{\alpha} D_{\lambda} g , \quad \alpha \in S_{\beta}
$$

$$
= A^\alpha \int_{\phi_a(\mathbb{R})} (Id - \rho(g))^{-\beta} \rho(g)^{\alpha} D_{\lambda} g , \quad \alpha \in S_{\beta}
$$

$$
\langle i | \lambda_{\mathbb{R}} | j \rangle
A_{ij}^{\alpha} \delta_{ij} \int_{\mathbb{R}^\times} (1 - g)^{-\beta} g^\alpha d\nu(g_H) , \quad \alpha \in S_{H,\beta}
$$

$$
= A_{ij}^{\alpha} \delta_{ij} \left( \int_{-\infty}^{0} + \int_{0}^{1} + \int_{1}^{\infty} \right) (1 - g)^{-\beta} g^\alpha d\log g , \quad \alpha \in S_{H,\beta}
$$

(5.2)

where $A^\alpha = e^{\alpha \log A}$ and the second line follows by the measure-invariant left translation $\rho(g) \rightarrow \rho(a g)$ and from $A \rho(g) = \rho(g) A$. The third and fourth lines follow after localization and taking matrix elements in a diagonal basis.

Consider first the case $\Re(\beta) \geq 1$ and save the analysis of $0 < \Re(\beta) < 1$ for the next subsection. Since we assume $A^{-1}$ positive-definite, only the first integral contributes because the next two do not converge for any $\alpha$ when $\Re(\beta) \geq 1$. (Conversely, if $A$ is negative definite only the integral over $\mathbb{R}_+$ contributes.) Change integration variable $g \rightarrow -g$ and use the integral relation (\cite{34}, 3.194(3.))

$$
\int_{0}^{\infty} (1 + \xi x)^{-\nu} x^{\mu-1} \, dx = \xi^{-\mu} B(\mu, \nu - \mu) \quad |\arg \xi| < \pi ; \quad 0 < \Re(\mu) < \Re(\nu)
$$

(5.3)

to find

$$
A_{\mathbb{R}^+,H,\beta}^\alpha = (-1)^{\alpha-1} \frac{\Gamma(\alpha) \Gamma(\beta - \alpha)}{\Gamma(\beta)} \cdot A^\alpha = (-1)^{\alpha-1} B(\alpha, \beta - \alpha) \cdot A^\alpha \quad \alpha \in (0, \Re(\beta))
$$

(5.4)

where $B(\cdot, \cdot)$ is the beta function and we put $\alpha = \mu - 1$ and $\beta = \nu$. Evidently positive powers are more interesting than negative powers to the extent that $B(\alpha, \beta - \alpha)$ is more interesting than $\Gamma(\alpha)$. One can instead elect to use Haar measure $\nu(g_B) := \log g / [(-1)^{\alpha-1} B(\alpha, \beta - \alpha)]$ to get the presentation

$$
A_{\mathbb{R}^+,B,\beta}^\alpha = A^\alpha .
$$

(5.5)
Alternatively, if $A^{-1}$ is in the unit ball of $1 \in \mathbb{C}^*$, one can localize by $\phi_\alpha(\mathbb{R}) \to \mathbb{C}$ where $\mathcal{C}$ is a smooth contour enclosing $\sigma(A^{-1})$ without crossing any branch cut. For $g \to g^{-1}$ and choosing Haar measure $\nu(\mathbb{C}_B) := \frac{\pi \csc(\pi \alpha)}{2\pi i B(\alpha, \beta - \alpha)}$, the holomorphic functional calculus yields (strictly for $\beta \in \mathbb{Z}_+$ but formally for $\beta \in \mathbb{S}$)

$$A_{\mathcal{C}, \mathbb{C}_B, \beta}^\alpha = \int_{\mathcal{C}} (\rho(g) - A^{-1})^{-\beta} \rho(g)^{\beta - \alpha} \, d\nu(\mathbb{C}_B) = A^\alpha, \quad \alpha \in \langle 0, \Re(\beta) \rangle . \quad (5.6)$$

Given the definition of positive powers, we can define the positive-power analogs of functional trace, log, and determinant for $\alpha \in \langle 0, \Re(\beta) \rangle$:

**Definition 5.3** Let $A \in \text{Mor}_\mathcal{C}(G^c, \mathbb{C}^*)$ be invertible such that $(\text{Id} - A^{-1})_{(\beta)} \in F_S(G^c)$ is trace class. The positive-power functional trace, log, and determinant are defined by

$$\text{Tr} A_{\lambda, \beta}^\alpha := (\text{Tr} A)^\alpha_{\lambda, \beta} := \mathcal{M}_\lambda \left[ \text{Tr} (\text{Id} - A^{-1})_{(\beta)}; \alpha \right] \quad = \int_{G^c} \text{tr} \left[ (\text{Id} - A^{-1}(g))^{-\beta} g^\alpha \right] \mathcal{D}_\lambda g , \quad (5.7)$$

$$\text{Log} A_{\lambda, \beta}^{\alpha + 1} := (\text{Log} A)^{\alpha + 1}_{\lambda, \beta} := \frac{d}{d\alpha} \mathcal{M}_\lambda \left[ (\text{Id} - A^{-1})_{(\beta)}; \alpha \right] \quad =: \hat{\mathcal{M}}_\lambda \left[ (\text{Id} - A^{-1})_{(\beta)}; \alpha \right] , \quad (5.8)$$

$$\text{Det} A_{\lambda, \beta}^\alpha := (\text{Det} A)^\alpha_{\lambda, \beta} := \mathcal{M}_\lambda \left[ \text{Det} (\text{Id} - A^{-1})_{(\beta)}; \alpha \right] \quad = \int_{G^c} \det \left[ (\text{Id} - A^{-1}(g))^{-\beta} g^\alpha \right] \mathcal{D}_\lambda g \quad = \int_{G^c} \det \left[ (\text{Id} - A^{-1}(g))^{-\beta} \right] \det g^\alpha \mathcal{D}_\lambda g . \quad (5.9)$$

As in the negative power case, convergence of the integrals and the $\alpha$-limit are particularly sensitive to the chosen normalization. Recall $\Gamma(\alpha)$ played a crucial role in normalizing negative powers, and $B(\alpha, \beta - \alpha)$ will do the same for positive powers. It is a useful exercise to compare against previous examples. For instance, one finds $\text{Tr} M_A^{-\alpha}_{\mathbb{C}_B, \beta} = \zeta(-\alpha)$ where $\alpha_R \in (0, \Re(\beta))$ (use $M^{-1}$ since the trace of $M$ obviously diverges) and $\text{Log} A_{\mathbb{C}_B, \beta} = \text{Log} A$.

### 5.2 Some loose ends

Return to Example 5.2 for the case $0 < \Re(\beta) < 1$ and restrict to real $\alpha, \beta$. Perform transformations $g \to \frac{g^{-1}}{g}$ and $g \to \frac{\bar{g}^{-1}}{g^{-1}}$ (which are $SL(2, \mathbb{Z})$) in the second and third integrals respectively. Again using \textcolor{red}{key3}, all three integrals contribute to give

$$A_{\mathbb{R}^k, H, \beta}^\alpha = [A_1^\alpha B(\alpha, \beta - \alpha) + A_2^\alpha B(\alpha, 1 - \beta) + A_3^\alpha B(1 - \beta, \beta - \alpha)] \quad (5.10)$$
where \(0 < \alpha, \beta < 1\) and we have defined \(A^\alpha_k := A^\alpha e^{i\pi \phi_k}\) with \(\phi_1 = \alpha - 1, \phi_2 = 0,\) and \(\phi_3 = -\beta\). Introduce real \(a, b > 0\) and restrict to \(\alpha \equiv a\) and \(\beta \equiv 1 - b\) such that \(\beta - \alpha = c\) with \(c\) a constant. Notice that \(a + b + c = 1\). Then

\[
A^\alpha_{\mathbb{R}^+, H, 1-b} = A^\alpha_a B(a, c) + A^\alpha_b B(a, b) + A^\alpha_c B(b, c) .
\]  

(5.11)

Of course there is nothing preventing the \(\alpha, \beta\) assignments with \(a \leftrightarrow b\). Including this contribution gives

\[
A(a, b) := A^\alpha_{\mathbb{R}^+, H, 1-b} + A^\beta_{\mathbb{R}^+, H, 1-a} = \left[ e^{i\pi \phi_1} B(a, c) + e^{i\pi \phi_2} B(a, b) + e^{i\pi \phi_3} B(b, c) \right] (A^\alpha + A^\beta) .
\]  

(5.12)

**Example 5.4** Let \(\Psi_{i,j}\) with \(i \neq j\) be asymptotic states in some Hilbert space \(\mathcal{H}\). Restrict to \(i, j \in \{1, 2, 3, 4\}\) and consider the trivial functional \(\text{Id}^{-1}(g) \equiv \rho(g)\) so that \(A = \text{Id}\). We claim that \(\langle \Psi_{4,3} | \text{Id}(a,b) | \Psi_{2,1}\rangle\) mimics the **kinematics** of tree-level, open string scattering in the sense that it reproduces the four-point tachyon string scattering amplitude on the unit disk up to normalization and momentum conservation; with vertex interchange taken into account.

To see this, represent the disk by the upper-half \(C\)-plane and fix tachyon vertex operators \(\mathcal{V}_{k_1}(g_1)\) at \(-\infty\), \(\mathcal{V}_{k_2}(g_2)\) at \(0\), and \(\mathcal{V}_{k_4}(g_4)\) at \(1\). The vertex ordering \(1234\) and its (24) permutation \((1432)\) associated with \(a \leftrightarrow b\) corresponds to the second term in (5.12). Then the (23) permutation of this ordering given by (1234) along with its (34) permutation associated with \(a \leftrightarrow c\) corresponds to the first term, and the (34) permutation of \((1234)\) along with its (23) permutation associated with \(b \leftrightarrow c\) corresponds to the third term in (5.12). The relative phases account for the non-cyclic permutations of the vertex orderings. Explicitly, \(\langle \Psi_{4,3} | e^{i\pi \phi_1} (\text{Id}^a + \text{Id}^b) | \Psi_{2,1}\rangle = 2\langle \Psi_{4,2} | \Psi_{3,1}\rangle\) and \(\langle \Psi_{4,3} | e^{i\pi \phi_3} (\text{Id}^a + \text{Id}^b) | \Psi_{2,1}\rangle = 2\langle \Psi_{3,4} | \Psi_{2,1}\rangle\) which can be seen by following the minus signs accrued when transforming \(g_3\) to the corresponding integral domains. We get

\[
\frac{1}{2} \langle \Psi_{4,3} | \text{Id}(a,b) | \Psi_{2,1}\rangle = \langle \Psi_{4,3} | B(a,b) | \Psi_{2,1}\rangle + \langle \Psi_{4,2} | B(a,c) | \Psi_{3,1}\rangle + \langle \Psi_{3,4} | B(b,c) | \Psi_{2,1}\rangle .
\]  

(5.13)

This example is not surprising since the integral we are solving is essentially the string scattering amplitude integral without the absolute values. Still, it is curious that positive powers, with \(\alpha, \beta \in (0, 1)\) and \(\beta - \alpha\) fixed, appear to model scattering channels with crossing symmetry. It gets more curious.

**Example 5.5** In the same vein as Example 4.4 here we calculate the product of positive powers \(M_{\mathbb{R}^+, H^2} \left[ (\text{Id} - B^{-1})(\beta) \ast (\text{Id} - A^{-1})(\gamma) \right] ; \alpha \right] = B^\alpha_{\mathbb{R}^+, H, \beta, \lambda, \gamma} A^\alpha_{\mathbb{R}^+, H, \gamma}\) where

\[
B^\alpha_{\lambda, \beta} A^\alpha_{\lambda, \gamma} = \int_{\phi_\mathbb{R}(\mathbb{R})} \int_{\phi_\mathbb{R}(\mathbb{R})} (\text{Id} - B^{-1} \tilde{g})^{-\beta} (\text{Id} - A^{-1} \tilde{g}^{-1} \tilde{g})^{-\gamma} g^\alpha D_\alpha \tilde{g}
\]  

(5.14)

and the fundamental strip depends on \(\lambda\) and the choice of \(\beta, \gamma\). To simplify matters, assume \([B, A] = 0\). Then, by now familiar manipulations bring the right-hand side into the form

\[
\int_{\mathbb{R}^+} B^\alpha A^\alpha \left[ \int_{\mathbb{R}^+} (1 - \tilde{g})^{-\beta} \tilde{g}^\alpha (1 - \tilde{g}^{-1})^{-\gamma} g^\alpha d \log \tilde{g} \right] d \log g , \quad \alpha \in S_{H, \beta, \gamma} .
\]  

(5.15)
Unlike the previous example, the double integral here is very different from the corresponding string integral. It is most easily evaluated by expressing the inverse operators as Mellin transforms

\[
\int_{(\mathbb{R}_+\cup\mathbb{R}_-)^2}\left[\int_{(\mathbb{R}_+\cup\mathbb{R}_-)^2}\frac{e^{-(1d-\bar{g})t}e^{-(1d-g)s}}{\Gamma(\beta)\Gamma(\gamma)}t^\beta\bar{g}^\alpha s^\gamma g^\alpha\,d\log t\,d\log \bar{g}\right]\,d\log s\,d\log g.
\]

Assuming \(S_{H,\beta,\gamma}\) is not empty, the integrals converge for some \(\alpha \in S_{H,\beta,\gamma}\) and the integrands are continuous functions on their various domains, so we can interchange integration order. Integrating over \(\bar{g}, g\) (in that order)\(^{11}\) gives four integrals:

\[
I_1 = \frac{\Gamma(\alpha)^2}{\Gamma(\beta)\Gamma(\gamma)}\int_{(\mathbb{R}_+\cup\mathbb{R}_-)^2} e^{-Id(s+t)t^\beta-1(-t)^{-\alpha}s^\gamma-1(-s)^{-\alpha}}\,dt\,ds, \quad 0 < \alpha, s < 0
\]

\[
I_2 = (-1)^{\alpha-1}\frac{\Gamma(\alpha)^2}{\Gamma(\beta)\Gamma(\gamma)}\int_{(\mathbb{R}_+\cup\mathbb{R}_-)^2} e^{-Id(s+t)t^\beta-1(-t)^{-\alpha}s^\gamma-1(-s)^{-\alpha}}\,dt\,ds, \quad 0 < \alpha, s < 0
\]

\[
I_3 = (-1)^{\alpha-1}\frac{\Gamma(\alpha)^2}{\Gamma(\beta)\Gamma(\gamma)}\int_{(\mathbb{R}_+\cup\mathbb{R}_-)^2} e^{-Id(s+t)t^\beta-1(-t)^{-\alpha}s^\gamma-1(-s)^{-\alpha}}\,dt\,ds, \quad 0 < \alpha, s > 0
\]

\[
I_4 = (-1)^{2\alpha}\frac{\Gamma(\alpha)^2}{\Gamma(\beta)\Gamma(\gamma)}\int_{(\mathbb{R}_+\cup\mathbb{R}_-)^2} e^{-Id(s+t)t^\beta-1(-t)^{-\alpha}s^\gamma-1(-s)^{-\alpha}}\,dt\,ds, \quad 0 < \alpha, s > 0.
\]

\(I_1, I_2\) don’t converge for either \(t \in \mathbb{R}_+\) or \(t \in \mathbb{R}_-,\) and \(I_3, I_4\) don’t converge for \(t \in \mathbb{R}_-\). Finally, \(I_3 + I_4\) integrated over \(\mathbb{R}_+^2\) yields

\[
((-1)^{2\alpha} - 1)\frac{\Gamma(\alpha)\Gamma(\beta - \alpha)}{\Gamma(\beta)}\frac{\Gamma(\alpha)\Gamma(\gamma - \alpha)}{\Gamma(\gamma)}, \quad 0 < \alpha < \Re(\beta) < 1
\]

(5.17)

and we get

\[
B_{\alpha,\beta,\gamma}^2 = \frac{2\pi i e^{i\alpha}}{\pi \csc(\pi \alpha)} B(\alpha, \beta - \alpha) B(\alpha, \gamma - \alpha) B^\alpha A^\alpha.
\]

(5.18)

Observe the right-hand side is symmetric under \(\beta \leftrightarrow \gamma\). This was to be expected since we assumed \([B, A] = 0\) so the integration order of \(\bar{g}, g\) could be reversed due to Fubini. We can conclude then that \(0 < \alpha < \Re(\beta), \Re(\gamma) < 1\).

Now suppose \(A = Id\) and \(B = (e^{-i\pi Id}) =: \tilde{Id}\). Returning to \(a + b + c = 1\) and putting \(\alpha = b, \beta = 1 - a = b + c,\) and \(\gamma = 1 - c = a + b\) in (5.18) yields

\[
\tilde{Id}_{R^*\times R^*\times H}^bId_{R^*\times R^*\times H}^c = 2i \sin(\pi bc) B(b, c) B(b, a) Id = 2i B(a, b, c) Id
\]

(5.19)

whose trace is recognized as the tachyon four-point scattering of closed strings on the unit sphere up to normalization and momentum conservation. Note that we need \(\arg(\det A)\) and \(\arg(\det B)\) to be \(\pi\) out of phase to cancel the \(e^{i\pi \alpha}\) term in (5.18) to get this result.

\(^{11}\)Reversing the order gives the same integrals but with the \(<>\) conditions on \(t\) instead of \(s\).
We learn that the algebraic product of positive-power operators with $0 < \alpha, \beta < 1$ formally parallels “gravity = (gauge)$^2$”. Explicitly, since functional Mellin is a representation in this case,

$$
\Pi^{(b)}_{\lambda} \left( (\mathrm{Id} - O^{-1})_{(a+b)} \ast (\mathrm{Id} - O^{-1})_{(c+b)} \right) = \Pi^{(b)}_{\lambda} \left( (\mathrm{Id} - O^{-1})_{(a+b)} \right) \cdot \Pi^{(b)}_{\lambda} \left( (\mathrm{Id} - O^{-1})_{(c+b)} \right)
$$

(5.20)

where $a + b + c = 1$.

Similarly, for the product of positive-power and negative-power operators, using

$$
\Pi^{(b)}_{\lambda} \left( (\mathrm{Id} - O^{-1})_{(a+b)} \right) \cdot \Pi^{(b)}_{\lambda} \left( (\mathrm{Id} - O^{-1})_{(c+b)} \right) = \Pi^{(b)}_{\lambda} \left( (\mathrm{Id} - O^{-1})_{(a+b)} \right) \ast \Pi^{(b)}_{\lambda} \left( (\mathrm{Id} - O^{-1})_{(c+b)} \right)
$$

(5.21)

and again expressing the inverse operator as a Mellin transform yields

$$
\mathcal{M}_{\mathbb{R}_+ \times \mathbb{CR}^2} \left[ (\mathrm{Id} - B^{-1})_{(\beta)} \ast E^{-\lambda}; \alpha \right] = B^{a \beta}_{\mathbb{R}_+ \times \mathbb{CR} \times \mathbb{R}^2} A^{-\alpha}_{\mathbb{R}_+ \times \mathbb{CR} \times \mathbb{R}^2} = (-1)^{a-1} B(\alpha, \beta - \alpha) \Gamma(\alpha) B^\alpha A^{-\alpha}.
$$

(5.22)

So $I^{a \beta}_{\mathbb{R}_+ \times \mathbb{CR} \times \mathbb{R}^2} Id^{b \beta}_{\mathbb{R}_+ \times \mathbb{CR} \times \mathbb{R}^2} = (-1)^{b-1} B(a, b) \Gamma(b) Id$, and referring back to Example 4.4 gives the product of two negative-power identity operators $I^{a \beta}_{\mathbb{R}_+ \times \mathbb{CR} \times \mathbb{R}^2} I^{b \beta}_{\mathbb{R}_+ \times \mathbb{CR} \times \mathbb{R}^2} = \Gamma(b)^2 Id$. Along with (5.19), these three functional products highlight the fundamentally different algebraic underpinnings of operators in $\mathbb{C}^*$ with “positive” versus “negative” powers.

Example 5.6 Evidently, a positive power operator $A^{a \lambda}_{\lambda}$ probes an associated resolvent through functional Mellin. However, so far we have restricted to a one-parameter real subgroup $\phi_a(\mathbb{R})$ and localized to a very simple abelian group $\mathbb{R}^\times$ identified as the domain of the spectrum of operator $A$ relative to states located at the end-points of evolution-time intervals in $\mathbb{R}$. It is easy to imagine less drastic simplifications that presumably probe more properties of the resolvent by accessing more of the group structure.

As an obvious example, relax the restriction to $\mathbb{R}$ for the domain of the one-parameter subgroup. Then $\phi_a(\mathbb{C})$ generates a complex one-parameter subgroup, and the spectrum of $A$ relative to states located at marked points on the boundaries of 1-dimensional compact manifolds in $\mathbb{C}$ motivates localization to $\mathbb{C}^\times$. The countable family $G^\mathbb{C}_\lambda = \{ G^\mathbb{C}_{\lambda}, \lambda \in \Lambda \}$ that partially characterizes the functional integral will then represent a sum of integrals over all Riemann surfaces with boundary relevant to the operator $A$.

The resemblance to string scattering amplitudes for fractional positive powers is intriguing: But of course there is no string physics coming from functional Mellin. Nevertheless, the mathematical nature of $\mathbf{F}_\mathbb{S}(\mathbb{C}^\times)$ (being a $\mathbb{C}^\times$-algebra) appears to capture at least some algebraic structure of perturbative string theory; which is not too surprising since Mellin-type integrals feature in scattering amplitude and $n$-point correlation calculations in a variety of theories—especially CFTs. This suggests the resemblance of fractional positive powers to scattering amplitudes is likely more than just coincidence. We return to this idea in the next section in the context of QFT.
6 Functional Mellin in QFT

We aim to show that functional Mellin can represent some relevant objects in QFT normally constructed from functional Fourier. This can be seen already at a scopic level: Functional Fourier in QFT is defined on vector spaces of complex-valued fields and their topological duals, so their underlying groups are abelian under point-wise addition. Essentially, this means that functional Fourier in QFT derives from the Lie algebra associated with functional Mellin in the special case of an abelian group and \( \alpha = 1 \).

6.1 Generating functional

Here we construct the QFT generating functional first by functional Fourier methods and then by functional Mellin methods for comparison.

Let us review the context of the functional Fourier transform. Begin with the topological vector space \( \mathcal{P}_a\mathbb{C}^m \) of piece-wise continuous, pointed paths \( z : (T, t_a) \to (\mathbb{C}^m, z_a) \). The involution and complex structure on \( \mathbb{C}^m \) induce an involution and complex structure on \( \mathcal{P}_a\mathbb{C}^m \). Use this to complexify \( (\mathcal{P}_a\mathbb{C}^m)^\mathbb{C} \cong \mathcal{P}_a\mathbb{C}^m + i\mathcal{P}_a\mathbb{C}^m \). Let \( Z_a \cong X_a \oplus iY_a \) denote the underlying complex abelian group (under point-wise addition) of \( \mathcal{P}_a\mathbb{C}^m \), and denote its dual by \( Z_a^\ast \). Given is a nondegenerate linear operator \( G : Z_a^\ast \to Z_a \) and its inverse \( D : Z_a \to Z_a^\ast \) on an appropriate domain with a set of zero modes \( \{ z : Dz = 0 \} \). These define a quadratic form \( Q_B : Z_a \times Z_a \to \mathbb{C} \) by \( (z_1, z_2) \mapsto -\frac{1}{2}(D + D^\dagger)z_1, z_2) =: -\frac{1}{2}(Qz_1, z_2) \) rendered self-adjoint by a suitable boundary term \( B : Z_a \times Z_a \to \mathbb{C} \). We will assume \( Q_B \) is positive-definite. The complex structure \( J \) on \( Z_a \) allows the \( \mathbb{Z}_2 \)-graded decomposition \( Z_a = Z_a^+ \oplus Z_a^- \) relative to the inner product \( (z_1|z_2)_Z := \text{Id}_B(z_1, z_2) \), and it determines associated maps \( J : Z_a \to Z_a' \) and \( J' : Z_a' \to Z_a \) by \( -\frac{1}{2}(Jz_1, z_2) := (Jz_1|z_2)_Z \) and \( J'J = -\text{Id}_{Z_a} \). A subspace \( X_a \subset Z_a \) is real relative to the complex structure if \( X_a \cap JX_a = \{ 0 \} \) and \( Z_a = X_a \oplus JX_a \). Extend \( Q_B \) to \( Z_a \times Z_a' \) by duality and decomposes as

\[
Z_a \times Z_a' = \bigoplus_{\pm} \left[ (Z_a^\pm \times Z_a'^\mp) \oplus (Z_a^\mp \times Z_a'^\pm) \right] =: W_a^\pm \oplus W_a^\mp =: W_a. \tag{6.1}
\]

Note that \( W_a^\pm \) and \( W_a^\mp \) are even and odd respectively under topological duality.

Functional Fourier of the Gaussian functional induced by \( Q_B \) on \( W_a \) is defined to be

\[
\int_{W_a} e^{2\pi i (w^{e\prime}, (w^{e\prime} - w^e)) - (\pi/s)Q_B(w^{e\prime} - w^e)} D^\lambda w^e := e^{-(\pi/s)B(w^e)} \text{Det}_\lambda (sw_B) \frac{1}{2} e^{-\pi sW_B(w^{e\prime})} \tag{6.2}
\]

where \( W_B(w_1^{e\prime}, w_2^{e\prime}) := -2\langle Ww_1^{e\prime}, w_2^{e\prime} \rangle \) and \( W \) is inverse to \( Q \). At \( w^{e\prime} = 0 \) this decomposes

\[
\int_{W_a} e^{-(\pi/s)Q_B(w^{e\prime} - w^e)} D^\lambda w^e = \sum_{\{w^e\}} \int_{W_0^e} e^{-(\pi/s)Q_B(w^e)} D^\lambda w^e = \sum_{\{w^e\}} \text{Det}_\lambda (sw_B) \frac{1}{2} e^{-(\pi/s)B(w^e)} \tag{6.3}
\]

where \( W_0^e \) is a Banach space of pointed maps \( w^e : (T, t_a) \to (\mathbb{C}^{2m}, 0) \).

\[\text{Explicitly, } Q_B \text{ maps } (z_1 \times z_1', z_2 \times z_2') \mapsto -\frac{1}{2}(Qz_1 \times z_1', z_2 \times Q'z_2') \text{ where } (z_1', Q'z_2') := (Qz_1, z_2).\]

\[\text{Since } W_0^e \text{ is Banach, its integrator } D^\lambda w^e \text{ is translation invariant. Together with } w^{e\prime} = \bar{w}^e + Gw^{e\prime} \text{ for all } w^{e\prime} \text{ in the dual space of } W_a^e \setminus \text{Ker}(D), \text{ this allows the decomposition expressed by the first equality.}\]
Being an abelian group, \( W_a^e \) doesn’t support scalar multiplication of \( w^e \in W_a^e \), so direct field renormalization as practised in QFT is not available. However, the factor \( s \in \mathbb{C}_+ \) scales the quadratic form \( Q_B \) and hence, indirectly, the argument \( w^e - \bar{w}^e \). Further, adjusting any parameters in \( Q_B \) is tantamount to defining a new quadratic form \( \bar{Q}_B \). Otherwise said; the scale \( s \) and defining quadratic form \( Q_B \) are part of the specification of \( D_{\Lambda}w^e \). The same can be said for a general action functional \( S_B = Q_B + V \). Off hand, it appears \( G_{\Lambda} \) can accommodate the renormalization program of QFT, but we will not pursue the details here.

Transcribing to QFT we have: I) \( W_a^e \) corresponds to the underlying additive abelian group \( \phi \) of the Banach space of bosonic fields (see [10, application A.4] for details). Because of the \( \mathbb{Z}_2 \) structure on \( W_a^e \), elements \( w^e \in W_a^e \) correspond to 2-tuples of independent degrees of freedom \( \phi \ni (\phi^*, \phi) : \mathbb{R}^{3,1} \to \mathbb{C}^{2m} \) where \( \phi \sim z^+ \times z'^+ \) and \( \phi^* \sim z^- \times z'^- \). II) Elements \( w^{\text{eff}} \in W_a^{e'} \) correspond to 2-tuples of external sources\(^{14} \) with \( (J_{\phi^*}, J_{\phi}) \sim 2\pi w^{\text{eff}} \). III) \( D \) is a linear differential operator. IV) \( \lambda_{\text{vac}} : \phi \to \mathbb{C}^{2m} \) with \( \bar{w}^e = 0 \) and we take \( B(w^e) = 0 \) for the specific instance of vacuum-to-vacuum transitions. V) Lastly, \( s \to \pi i h \). This yields (dropping the subscript on \( Q_B \) to reflect vacuum-to-vacuum boundary conditions)

\[
\langle 0|0 \rangle_{\text{bos}} := \int_{\phi} e^{\frac{i}{\hbar} Q(\phi^*, \phi)} D_{\lambda_{\text{vac}}} (\phi^*, \phi) = \text{Det}_{\lambda_{\text{vac}}} (ihW)^{1/2} := \det (ihQ)^{-1/2} \quad (6.4)
\]

where the form \( Q(\phi^*, \phi) = -\frac{1}{2} \langle Q(\phi^*, \phi) \rangle \), and \( W \) is the inverse of \( Q \). For \( w^{\text{eff}} \neq 0 \), we get

\[
Z(J) := \langle 0|0 \rangle_{J_{\psi}} = \int_{\phi} e^{i(J_{\phi^*}, \phi^*) + i(J_{\phi}, \phi) + (1/\hbar) Q(\phi^*, \phi)} D_{\lambda_{\text{vac}}} (\phi^*, \phi) = \det (ihQ)^{-1/2} e^{(ih/2)W(J_{\phi^*}, J_{\phi})} \quad (6.5)
\]

which must be supplemented with interaction terms in the form \( S_B = Q_B + V \) and then perturbative methods must be applied to yield the generating functional employed by QFT for bosonic fields. Remark that \( \text{det}Q = \text{det}Q^2_{\mathbb{R}} \) with \( Q_{\mathbb{R}} := Q|_{X_{\mathbb{R}}} \) where \( X_{\mathbb{R}} = X_a \oplus JX_a \).

For fermionic fields \( \psi \ni (\bar{\psi}, \psi) \colon \mathbb{R}^{3,1} \to \mathbb{C}^{2m} \), construct functional Fourier for a skew-Gaussian functional induced by a pre-symplectic form \( \Omega_B \) on \( Z_a \) (see [10, § 4.2]). Transcription to QFT parallels the bosonic case where now we take \( (\bar{\psi}, \psi) \sim w^{\text{ferm}} = (z^- \times z'^+, z^+ \times z'^-) \) and \( (J_{\bar{\psi}}, J_{\psi}) \sim 2\pi w^{\text{eff}} \). This gives

\[
\langle 0|0 \rangle_{\text{ferm}} := \int_{\psi} e^{\frac{i}{\hbar} \Omega(\bar{\psi}, \psi)} D_{\lambda_{\text{vac}}} (\bar{\psi}, \psi) = \text{Pf}_{\lambda_{\text{vac}}} (hM)^{-1} := \text{pf}(h\Omega) \quad (6.6)
\]

where the form \( \Omega(\bar{\psi}, \psi) := -\frac{1}{2} \langle \Omega(\bar{\psi}, \psi) \rangle \) and \( M = \Omega^{-1} \), and the generating functional

\[
Z(J) := \langle 0|0 \rangle_{J_{\bar{\psi}}} = \int_{\psi} e^{i(J_{\bar{\psi}}, \bar{\psi}) + i(J_{\psi}, \psi) + (1/\hbar) \Omega(\bar{\psi}, \psi)} D_{\lambda_{\text{vac}}} (\bar{\psi}, \psi) = \text{pf}(h\Omega) e^{(ih/2)M(J_{\bar{\psi}}, J_{\psi})} \quad (6.7)
\]

where \( M(J_{\bar{\psi}}, J_{\psi}) = -\frac{1}{2} \langle J_{\bar{\psi}}, \Omega^{-1} J_{\psi} \rangle \). Note that \( \bar{\psi} \) and \( \psi \) are contained in orthogonal Lagrangian subspaces determined by \( \Omega \), and in this sense they are (dynamical) conjugate degrees of freedom. The interpretation, therefore, is that \( D_{\lambda_{\text{vac}}} (\bar{\psi}, \psi) \) quantifies quantum dynamics while \( D_{\lambda_{\text{vac}}} (\phi^*, \phi) \) quantifies quantum correlations.

\(^{14}\)Not to be confused with the complex-induced map \( J \).

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Now we construct the generating functional using functional Mellin. In the context of QFT, restrict to abelian $G^C$ and take $\rho(g) \in L_B(H)$ and $Q(g) = -\frac{i}{\hbar}\rho(g)\dagger Q\rho(g) \in L_B(H)$ where $Q$ is positive-definite and $\rho$ is unitary.

As we did for Fourier, extend $Q$ to $G^C \times \tilde{G}^C$ where $\tilde{G}^C$ is the Pontryagin dual and decompose into even and odd parts according to the complex grading

$$G^C \times \tilde{G}^C = \bigoplus_{\pm} \left[ \left( G^\pm \times \tilde{G}^\pm \right) \oplus \left( G^\pm \times \tilde{G}^{\mp} \right) \right] =: G^e \oplus G^o . \quad (6.8)$$

Identify $F_S(G^e)$ with bosonic observables. This gives

$$Q_{\lambda}^{-\alpha_R} \equiv \int_{G^e} e^{-S(g)} g^{\alpha_R} D_\lambda g , \quad \alpha_R \in \mathbb{S} . \quad (6.9)$$

By analogy with the Fourier case, sources could be introduced via the Pontryagin dual group of $G^e$ but this is not particularly useful or necessary as will be evident shortly.

To include interactions, consider the functional $S(g) = Q(g) + V(g)$. Expand $e^{-V(g)}$ as a formal power series and write $e^{-S(g)} = \sum_{m=0}^{\infty} a_m g^m e^{-Q(g)}$ where $a_m$ are real constants multiplying $g^m$. Hence, the integral for a non-quadratic action functional is

$$S_{\lambda}^{-\alpha_R} \equiv \int_{G^e} e^{-S(g)} g^{\alpha_R} D_\lambda g = \int_{G^e} \sum_{m=0}^{\infty} a_m e^{-Q(g)} g^{\alpha_R+m} D_\lambda g \quad (6.10)$$

which gives rise to a perturbative loop expansion up to order $M$ given by

$$S^{(M)}_{\lambda}^{-\alpha_R} := \sum_{m=0}^{M} a_m \int_{G^e} e^{-Q(g)} g^{\alpha_R+m} D_\lambda g . \quad (6.11)$$

Using the same QFT identifications as in the preceding Fourier construction yields an $\alpha_R$-dependent, bosonic generating functional defined by

$$Z_{\lambda_{\text{vac}}}^{(M)}(\alpha_R) := \langle 0 | Q_{\lambda}^{-\alpha_R} | 0 \rangle_{\text{bos}} = \int_{G^e} \langle 0 | e^{-Q(g)} g^{\alpha_R} | 0 \rangle D_\lambda g =: \int_{\Phi} e^{-Q(\Phi)} \Phi^{\alpha_R} D_{\lambda_{\text{vac}}} \Phi \quad (6.12)$$

with a perturbative generating functional up to order $M$ for a non-quadratic action functional $S(g) = Q(g) + V(g)$ given by

$$Z^{(M)}_{\lambda_{\text{vac}}} (\alpha_R) := \sum_{m=0}^{M} a_m \int_{\Phi} e^{-Q(\Phi)} \Phi^{\alpha_R+m} D_{\lambda_{\text{vac}}} \Phi . \quad (6.13)$$

In equation (6.12) we identified $G^e$ with the underlying additive abelian group of a Banach space of complex scalar bosonic fields $\Phi \in \Phi$ relative to the vacuum $|0\rangle \in H$ such that $\langle 0 | e^{-S(g)} g^{\alpha_R} | 0 \rangle = e^{-S(\Phi)} \Phi^{\alpha_R}$. According to the Fourier construction, the vacuum expectation corresponds to the determinant representation $\text{det} : L_B(H) \to \mathbb{C}$ so we have normalized $D_{\lambda_{\text{vac}}} \Phi$ by $Z_{\lambda_{\text{vac}}}(0) = \text{det}(i\hbar Q)^{-1/2}$. Note that here $D_{\lambda_{\text{vac}}} \Phi$ is translation invariant as befits an integrator over an additive abelian group.
To make contact with vacuum n-point functions in QFT: First, restrict to $\alpha_{\mathbb{R}} \in \mathbb{N}_+$. Observe that $Z^{(M)}_{\lambda}(\alpha_{\mathbb{R}}) = 0$ for $\alpha_{\mathbb{R}} + m$ an odd number since $Q$ is quadratic and $D_{\lambda}\Phi$ is translation invariant. Also, $\Phi^{\alpha_{\mathbb{R}}} := e^{\alpha_{\mathbb{R}}(\log\Phi^*+\log\Phi)/2} = \frac{1}{2}(\Phi^*\Phi + \Phi\Phi^*)^{\alpha_{\mathbb{R}}/2}$ follows from BCH since the group commutator $[\Phi^*,\Phi]$ is a c-number and $\log\Phi$ is Hermitian.\(^{15}\) Next, localize by $\lambda_{\text{vac}} : \Phi \to \mathbb{C}$ according to

$$[\Phi^*,\Phi]_{\lambda_{\text{vac}}} [\Phi^*,\Phi]_\Phi(p,p') = [\Phi^*(p),\Phi(p')] \quad p,p' \in \mathbb{R}^{3,1}. \quad (6.14)$$

This tantamount to $\Phi^2 \mapsto e^{(\log\Phi^*(p)+\log\Phi(p'))} = \frac{1}{2}(\Phi^*(p)\Phi(p') + \Phi(p)\Phi^*(p')) =: \Phi(p,p')$ and by iteration $\Phi^{2n} \mapsto \Phi(p_1,p'_1)\Phi(p_2,p'_2) \cdots \Phi(p_n,p'_n)$ up to different pairings of $p_i,p'_j$. Momentum n-point correlation functions for general QFT operators $\mathcal{O}(\Phi)$ obtain from $\mathcal{M}_\lambda[O^S;\alpha]$ with $O^S(g) \equiv O(\Phi)e^{-S(\Phi)}$.

For the fermionic generating functional, follow the Fourier construction. Replace Hermitian Q with skew-Hermitian $\Omega$ and $G^e$ with $G^o$. Identify $G^o$ with the underlying additive abelian group of a Banach space of complex scalar fermionic fields $\Psi \in \Psi$. This leads to the fermionic generating functional

$$Z_{\lambda}(\alpha_{\mathbb{R}}) := \langle 0| i\Omega^{\alpha_{\mathbb{R}}}|0 \rangle_{\text{term}} = \int_{\Psi} e^{-i\Omega(\Psi)} \Psi^{\alpha_{\mathbb{R}}} D_{\lambda_{\text{vac}}} \Psi \quad (6.15)$$

where $D_{\lambda_{\text{vac}}} \Psi$ is a translation-invariant skew-Gaussian integrator\(^{10}\) that transforms obversely to $D_{\lambda_{\text{vac}}} \Phi$ and is normalized so that $Z_{\lambda_{\text{vac}}}(0) = \text{pf}(h\Omega)$. Interactions are included in parallel with the bosonic case.

Crucially, the group commutator on $G^e \oplus G^o$ is graded because the product on $G^o$ is defined by $\Omega(g_1,g_2) = -\Omega(g_2,g_1)$. This implies $\overline{\Psi}\Psi = -\frac{1}{2}\langle \Omega \overline{\Psi},J\Psi \rangle = \frac{1}{2}\langle \Omega \Psi,J\overline{\Psi} \rangle = -\Psi\overline{\Psi}$ since $\overline{\Psi} \sim J\Psi$ and $[\Omega,J] = 0$. Hence by BCH, $\Psi^2 = e^{(\log\overline{\Psi}+\log\Psi)} = \frac{1}{2}(\overline{\Psi}\Psi - \Psi\overline{\Psi})$ because the group anti-commutator $\{\overline{\Psi},\Psi\}_\Psi$ is a c-number. Our chosen topological localization then gives $\Psi(p,p') = \frac{1}{2}(\overline{\Psi}(p)\Psi(p') - \Psi(p)\overline{\Psi}(p'))$ and

$$\{\overline{\Psi},\Psi\}_\Psi \overset{\lambda_{\text{vac}}}{\mapsto} \{\overline{\Psi},\Psi\}_\Phi(p,p') = \{\overline{\Psi}(p),\Psi(p')\}. \quad (6.16)$$

Some comments are in order: The functional Mellin objects for non-quadratic action functionals presented in this subsection should be considered preliminary since we did not adequately deal with renormalization and gauge symmetry. Although the perturbative generating functionals coming from functional Mellin are essentially the usual Fourier ones (albeit with $\alpha_{\mathbb{R}}$ in place of $\delta/\delta J$), the functional Mellin representations of resolvent, trace, log, and determinant may prove fruitful. It is satisfying to see the commuting/anti-commuting products of fields coming from the graded group commutator induced by the quadratic forms $Q$ and $\Omega$. In this regard, the bosonic and fermionic cases can be combined on $G^e \oplus G^o$ to access supersymmetric QFT generating functionals from a functional Mellin perspective — as opposed to the functional Fourier perspective of [10]. Finally, functional Mellin offers possible alternatives to the usual loop expansion displayed in (6.11). To wit, one can use the

\(^{15}\)Since $[\Phi^*,\Phi]_\Phi$ is a c-number and $\log\Phi$ is Hermitian, then $[\log\Phi^*,\log\Phi] = 0$. The multiplication here is defined by $\Phi^*\Phi = Q(\Phi) = Q(\Phi)^* = Q(\Phi^*)$. 34
functional Laguerre expansion developed in appendix E to expand $e^{-V(g)} = \sum_n a_n^{(s)} A_n^{(s)}(g)$ where

$$a_n^{(s)} = \frac{1}{c(s)_{m,n}} \int g e^{-V(g)} L_m^{(s)}(g) g^{s+1} D\lambda g$$

(6.17)

and

$$c_{m,n}^{(s)} = \delta_{m,n} \left( \frac{n + s}{n} \right) Id.$$  

(6.18)

Or one can try to formally express $e^{-V(g)}$ as an inverse functional Mellin

$$e^{-V(g)} \sim \int_{c+i\infty}^{c-i\infty} M\lambda[O^{-V}; \alpha] d\alpha$$

(6.19)

for some to-be-determined observable $O^{-V}$. The Laguerre expansion appears to yield a regrouping (relative to the loop expansion) of $s$-dependent perturbative terms that may prove favorable in some cases, and the inverse functional Mellin is potentially useful if the order of $\int D\lambda g$ and $\int d\alpha$ can be exchanged. We leave fleshing out these alternatives and investigating their feasibility and efficacy to future work.

### 6.2 Mellin scattering

Mellin-type integrals are ubiquitous in QFT and string theory scattering amplitudes. For example, Schwinger’s trick leads to the parametric representation of Feynman diagrams [41], which motivates the world-line formalism [42], which in turn can be seen as the infinite-tension limit of string scattering [43]. In this subsection we want to explore how these representations fit in to the functional Mellin formalism.

#### 6.2.1 point-to-point Green’s functions

Here we calculate Green’s functions for the Klein-Gordon operator of massive states in $L^2(\mathbb{R}^{1,3}, \mathcal{W}) \equiv \mathcal{H}$ where $\mathcal{W}$ furnishes the spin $0, 1/2, 1$ representations of $\mathbb{R}^{1,3} \rtimes SL(2, \mathbb{C})$. Of course the Green’s functions can be alternatively realized as correlation functions via functional integrals over second quantized fields according to QFT in the usual way, but it has long been recognized that simple point-particle path integrals can also do the job—even with background gauge fields included.[44]

The first step is to calculate the free scalar elementary kernel for evolution from the state $|\psi_{x_a}\rangle$ at evolution-time $t_a$ to the state $|\psi_{x_b}\rangle$ at evolution-time $t_b$.[45] The equation to solve is $(\Box + V(x)) K(x_a, x_b) = \delta(x_a - x_b)$ where $V(x) = m^2$ in our case, $x_b, x_a \in \mathbb{R}^{1,3}$, and the position-to-position *Dirichlet* elementary kernel (a.k.a. Green’s function) is

$$K(x_b, x_a) := \langle \psi_{x_b} | K_T | \psi_{x_a} \rangle$$

$$= \langle \psi_{x_b} | M_T[\mathcal{E}^{-(\Box + m^2)}; 1] | \psi_{x_a} \rangle$$

$$= M_T[\langle \psi_{x_b} | \mathcal{E}^{-(\Box + m^2)} | \psi_{x_a} \rangle; 1].$$

(6.20)

---

[16] We follow [45][46] which provide a QM path integral realization of elementary kernels/Green’s functions of linear, second order partial differential equations.
Localizing according to \( E^{-\Box (m^2)}(g) \xrightarrow{i\mathbb{R}_+} e^{-\Box (m^2)\Delta t} \) with unitary \( \rho(g) \equiv \Delta t := t_b - t_a \in i\mathbb{R}_+ \) gives

\[
\langle \psi_{x_b} | e^{-\Box (m^2)\Delta t} | \psi_{x_a} \rangle = \int_{\mathbb{C}^4} \int_{\mathbb{C}^4} \psi_{x_b}(z_b)(z_b; \mu') e^{-\Box (m^2)\Delta t} | z_a^* \rangle \langle z_a^* | \mu \rangle \psi_{x_a}(z_a^*) \, dz_b \, dz_a
\]

\[
\text{real submanifold } \int_{\mathbb{R}^{1,3}} \int_{\mathbb{R}^{1,3}} \delta(x_b, \Re(z_b)) K_{\Delta t}(z_b, z_a^*) \delta(x_a, \Re(z_a)) \, dz_b \, dz_a
\]

(6.21)

with \( U_{\Delta t} := e^{-(\Box + m^2)\Delta t} \) the evolution operator, \( z_b := z(t_b) \) and \( z_a^* := z^*(t_a) \) in \( \mathbb{C}^4 \), and fixed-point states \( \psi_{x_b}(z_b) = \delta_{\epsilon}(x_b, \Re(z_b)) \) and \( \psi_{x_a}(z_a^*) = \delta_{\epsilon}(x_a, \Re(z_a)) \). Here, \( \delta_{\epsilon} \) denotes a regularized delta distribution. The chosen regularization must respect covariance with respect to \( \mathbb{R}^{1,3} \times SL(2, \mathbb{C}) \), and the limit \( \epsilon \to 0 \) is only taken after integration.

The kernel \( K_{\Delta t}(z_b, z_a^*) \) derives from a parametrized Gaussian functional integral\(^{10}\) associated with pointed maps \( Z_a \ni \zeta : (T, t_a) \to (\mathbb{C}^4, 0) \) where \( T \equiv [t_a, t_b] \subset i\mathbb{R}, \)

\[
K_{\Delta t}(z_b, z_a^*) := \int_{\mathcal{Z}} \delta(z(t_b, \zeta) - z_b) e^{-S(z(t_b, \zeta))} \, D\zeta
\]

(6.22)

with \( z(t, \zeta) = z_a^* \cdot \Sigma(t - t_a, \zeta) \) where \( \Sigma(t - t_a, \zeta) \) is a global transformation on \( \mathbb{R}^{1,3} \) such that \( z(t_a, \zeta) = z_a^* \cdot \Sigma(0, \zeta) = z_a^* \), and the Poincaré invariant and (gauge fixed) time-reparametrization invariant action is

\[
S(z(t_b, \zeta) = Q(z(t_b, \zeta) + \int_{t_a}^{t_b} m^2 \, dt = \int_{t_a}^{t_b} z''(t, \zeta) \left( \eta_{\mu\nu} \frac{d^2}{dt^2} \right)^2 z''(t, \zeta) \, dt + \int_{t_a}^{t_b} m^2 \, dt \). \tag{6.23}
\]

Explicitly, the parametrization \( z(t, \zeta) = z_{cr}(t) + \zeta(t) \) is a variation about the critical path \( z_{cr}(t) \) with fixed end-points \( z_a^*, z_b \) given by \( z_{cr}(t) = z_a^* + \left( \frac{z_b - z_a^*}{t_b - t_a} \right)(t - t_a) \). Substituting into the functional integral yields the well-known heat kernel

\[
\langle z_b | e^{-\Box (m^2)\Delta t} | z_a \rangle = e^{-\frac{(z_b - z_a)^2}{4m^2\Delta t} - m^2\Delta t} \int_{\mathcal{Z}} \delta(\zeta(t_b)) e^{-f^b_{\epsilon}(\zeta(t))} \, D\zeta
\]

\[
e^{-\frac{(z_b - z_a)^2}{4m^2\Delta t} - m^2\Delta t}(2\Delta t)^{-2} \tag{6.24}
\]

where we used

\[
\delta(\zeta(t_b)) = \int_{\mathbb{R}^{1,3}} e^{-i(u', \zeta(t_b))} \, du' = \int_{\mathbb{R}^{1,3}} e^{-i(u' \delta_{\epsilon}(t_b, \zeta))} \, du'
\]

(6.25)

and skipped a few trivial steps that explicate \( \delta_{\epsilon} \) and calculate the function integral.\(^{18}\)

---

\(^{17}\) Since \( \Box + m^2 \) is self-adjoint, \( \rho(g) \) must be imaginary. To ensure a unitary representation \( \rho \), we need skew-Hermitian \( \log(i|\Delta t|) \in i\mathbb{R} \) which implies \( \Delta t' = -i|\Delta t|^{-1} = (\Delta t)^{-1} \) so that \( \rho(g)^{-1} = \rho(g)^{-1} \).

\(^{18}\) The skipped steps include the localization \( \lambda : Z_a \to \mathbb{R}^{1,3} \) by \( \zeta \mapsto \zeta(t_b) =: u \) for \( u \in \mathbb{R}^{1,3} \). By duality then, \( \zeta' \mapsto u' \delta_{\epsilon} \) which renders the covariance \( W(u \delta_{\epsilon}(t_a)) = |u'|^2\Delta t \). Remark that, following localization, \( K(z_b, z_a^*) \) should be regarded as a distribution; in which case the action functional must localize to an honest function and this will typically involve regularization/gauge fixing and renormalization in more complicated examples. Also, for the regularized delta distributions in \( \int_{\mathbb{R}^{1,3}} \) we used \( \delta_{\epsilon}(x) = (\epsilon)^{-1/2} e^{-\pi x^2/\epsilon} \).
Similarly, by Fourier transform
\[
\langle p_b | e^{-(\Box + m^2)\tau} p_a \rangle = \delta(p_b - p_a) e^{-\tau(p_b^2 - m^2)} \tag{6.26}
\]
where \( p_b, p_a \in \mathbb{C}^4 \) and \( \tau := \Delta t \) is interpreted as an evolution-time interval. This gives the (complex) momentum-to-momentum realization (recall Example 4.2)
\[
\langle p_b | (\Box + m^2)^{-1} p_a \rangle = \int_{i\mathbb{R}_+ \cup i\mathbb{R}_-} \langle p_b | e^{-(\Box + m^2)\tau} p_a \rangle \tau \, d\log(\tau)
\]
\[
= \left( \int_0^{1+i\infty} + \int_0^{-1+i\infty} \right) e^{-\mathcal{S}(p_a,t_a;p_b,t_b)\tau} \, d\log(\tau)
\]
\[
= \left( \int_0^{1+i\infty} + \int_0^{-1+i\infty} \right) e^{-\tau(p_b^2 - m^2)} \, d\log(\tau)
\]
\[
= \left( \int_0^{1+i\infty} + \int_0^{-1+i\infty} \right) e^{-\mathcal{S}(p_a,t_a;p_b,t_b)\tau} \, d\log(\tau)
\]
\[
\int_{i\mathbb{R}_+ \cup i\mathbb{R}_-} \langle p_b | e^{-(\Box + m^2)\tau} p_a \rangle \tau \, d\log(\tau)
\]
\[
\int_{i\mathbb{R}_+ \cup i\mathbb{R}_-} \langle p_b | e^{-\tau(p_b^2 - m^2)} \rangle \, d\log(\tau)
\]
\[
\int_{i\mathbb{R}_+ \cup i\mathbb{R}_-} \langle p_b | e^{-\tau(p_b^2 - m^2)} \rangle \, d\log(\tau)
\]
\[
\int_{i\mathbb{R}_+ \cup i\mathbb{R}_-} \langle p_b | e^{-\tau(p_b^2 - m^2)} \rangle \, d\log(\tau)
\]
with \( p_a = p_b \). A functional inverse power such as \((\Box + m^2)^{-1}\) in the momentum representation comprises a principal value for \( \Re(p_b^2) > m^2 \) plus a delta function for \( p_b^2 = m^2 \). (see 19, def. 5.5) and footnote 6 in § 4.1. Consequently
\[
K(p_b, p_a) = \langle p_b | (\Box + m^2)^{-1} p_a \rangle = \frac{1}{p_b^2 - m^2} \pm i\delta(p_b^2 - m^2) \quad \text{for } i\mathbb{R}_\pm . \tag{6.28}
\]
Choosing the opposite sign of the delta function exchanges \( i\mathbb{R}_+ \leftrightarrow i\mathbb{R}_- \).

Notice the integral-convergence conditions \( \Im(p_b^2 - m^2) \gtrless 0 \) for \( i\mathbb{R}_\pm \) naturally incorporate the Feynman prescription for the Fourier transformed propagator. Moreover, unitarity of the representation \( \rho \) in the Mellin functional integral implies the above position-to-position as well as the momentum-to-momentum \( i\mathbb{R}_\pm \) kernels are invariant under \( \tau \to \tau^{-1} \) as one can confirm by direct calculation. This is a reflection of unitarity and the multiplicative-group nature of the evolution-time interval. Because \( \tau \) is interpreted as an evolution-time interval here, invariance under \( \tau \to \tau^{-1} \) appears to indicate any UV/IR dichotomy is not fundamental — at least for quadratic operators. Meanwhile, evolution-time reversal (which means \( \tau = i|\Delta t| \to -i(|\Delta t| - \Delta t) = \tau^* \)) exchanges \( i\mathbb{R}_+ \leftrightarrow i\mathbb{R}_- ^{19} \) Recalling Example 4.2, this motivates the study of system evolution parametrized by complex one-parameter subgroups with localization \( \lambda_{\mathbb{C}^\times} : \phi_{\alpha}(\mathbb{C}) \to \mathbb{C}_\times \).

Remark 6.1 As a consistency check we calculate the inverse off-shell momentum-to-momentum propagator when \( p_a = p_b \) and \( \Re(p_b^2) > m^2 \)
\[
K^{-\alpha}(p_b, p_a) = \langle p_b | M_\star [\mathcal{E}_{\overline{\mathbb{C}^4}}^{1/2}; p_a] \rangle \]
\[
= \frac{1}{\Gamma(\alpha)} \int_{i\mathbb{R}_+ \cup i\mathbb{R}_-} e^{\frac{1}{p_b^2 - m^2} \tau} \tau^\alpha \, d\log(\tau)
\]
\[
= \left( \frac{1}{p_b^2 - m^2} \right)^{-\alpha} \quad \text{for } \Re(\alpha) > 0 . \tag{6.29}
\]

\[ ^{19} \text{If energy decrease is associated with the parameter flow along } i\mathbb{R}_+ \text{ away from the origin, then under evolution-time reversal the same holds for parameter flow along } i\mathbb{R}_- \text{ which is also away from the origin. Both cases correspond to increasing entropy which implies } \tau \text{ characterizes the “arrow of time”.} \]
In particular, for $\alpha = 1$, this yields $K^{-1}(p_b,p_a) = (p_b^2 - m^2)$ which verifies the identity
\[
\int K^{-1}(p',p'')K(p'',p)\,dp'' = \delta(p',p)\text{Id for } p_b^2 \neq m^2.
\]
We have repeatedly pointed out that generically $O^{-\alpha} \neq (O^{-1})^\alpha$ (the present example notwithstanding). The latter can be represented as
\[
(O^{-1})^\alpha_H := M_H [(\text{Id} - O)(\beta); \alpha] := \int_{G_C} (\text{Id} - O(\tau))^{-\alpha} \beta \, \mathcal{D}_H \, \tau, \quad \alpha < \beta \in S_\lambda. \quad (6.30)
\]
In the case at hand this gives (for $\Re(p_b^2) > m^2$)
\[
(K^{-1})^\alpha_{H,\beta}(p_b,p_a) = \int_{\mathbb{R}^\times} \left(1 - \frac{1}{p_b^2 - m^2}\tau\right)^{-\beta} \tau^\alpha \, d\log(\tau) = B(\alpha, \beta - \alpha)(p_b^2 - m^2)^\alpha, \quad 0 < \Re(\alpha) < \Re(\beta) \quad (6.31)
\]
for $B(\cdot, \cdot)$ the beta function. For $\beta = 2$ (recall $\alpha = 1$) we get $(K^{-1})_{H,2} = (p_b^2 - m^2)$ as expected\(^{20}\).

To go beyond the spin-0 case, consider the operator
\[
\mathcal{M}_\Gamma[\mathbb{E}^{-(\square + m^2)}; \alpha] = \int_{\phi_a(\mathbb{R})} e^{-(\square + m^2)(g)} \rho(g^\alpha) \, d\nu(g). \quad (6.32)
\]
Recall it depends on the representation $\rho$, and the one-parameter subgroup $\phi_a(\mathbb{R})$ characterizes evolution of a quantum system. If we want to talk about particle transitions, we need $\rho$ to be a unitary irreducible representation, and we need for it to act trivially on state vectors corresponding to elementary particles since the notion of ‘elementary’ implies invariance under evolution.

Accordingly, we now localize by $\lambda : \phi_a(\mathbb{R}) \to A_a$ where $A_a$ is abelian. This induces a unitary irreducible representation $g_\lambda \equiv a \to \rho(a)$ with $a \in A_a$. In light of this, for each relevant representation $r$ let us take for the Hamiltonian operator (in the momentum representation)
\[
H^{(r)}(a) = (p^2 - m^2)\rho^{(r)}(a) = (p^2 - m^2) \tau P^{(r)} / |P^{(r)} P^{(r)\dagger}| \quad (6.33)
\]
where $\tau P^{(r)} \in L(H), \tau \in i\mathbb{R}_+ \cup i\mathbb{R}_-$, and $P^{(r)}$ is a projection onto an irreducible subspace of $H$ furnished by $\mathcal{W}$. To verify the unitarity of representations $\rho^{(r)}$, notice that $\log(i|\tau|) \in i\mathbb{R}$ implies $\tau^\dagger = \tau^{-1} = -i|\tau|^{-1}$ so that
\[
\rho^{(r)}(a)^\dagger = (\tau P^{(r)} / |P^{(r)} P^{(r)\dagger}|)^\dagger = -i \frac{|\tau|^{-1}}{|P^{(r)} P^{(r)\dagger}|} P^{(r)\dagger} \quad (6.34)
\]
and therefore $\rho^{(r)}(a) \rho^{(r)}(a)^\dagger = \text{Id}$. (Recall that $\tau^\dagger \neq \tau^*$.)

\(^{20}\)Different choices of $\beta > 1$ just contribute an overall constant that can be absorbed into the normalization of $K^{-1}$. 38
The relevant objects to analyze are the total Hamiltonian

\[ H(a) := \bigoplus_r H^{(r)}(a) = \bigoplus_r (p^2 - m^2) \frac{\tau P^{(r)}}{|P^{(r)} P^{(r)^\dagger}|} \]  

(6.35)

and its elementary kernel

\[ (K(p_b, p_a))_{\mu' \mu} = \bigoplus_r \left( K^{(r)}(p_b, p_a) \right)_{\mu' \mu}. \]  

(6.36)

For higher spin state vectors labelled by \((j', j)\), the work to find the projection operators has already been done many times over and we just state the \((1/2, 0) \oplus (0, 1/2)\) and \((1/2, 1/2)\) results. For Dirac spinors we take

\[ \tau P^{(1/2)}_{\parallel} = \tau \frac{\tau}{p^2 - m^2} (\phi \pm m). \]  

(6.37)

which gives

\[ (K^{(1/2)}(p_b, p_a))_{\alpha \dot{\alpha}} = \left( \frac{\tau}{p^2 - m^2} \right)_{\alpha \dot{\alpha}} \delta(p_b - m) \delta(p_a). \]  

(6.38)

For vector bosons, the operator \(\Box + m^2\) corresponds to the unitary gauge so we choose \(P^{(1)}_{\parallel} = (\eta - (p \cdot p)/m^2)\) which yields

\[ (K^{(1)}(p_b, p_a))_{\mu \nu} = \left( \eta_{\mu \nu} - \frac{p_{\mu} p_{\nu}}{m^2} \right) \delta(p_b, p_a). \]  

(6.39)

In both calculations we used the Haar measure to re-scale \(\frac{\tau P^{(r)}}{|P^{(r)} P^{(r)^\dagger}|} \rightarrow \tau\), and we used the operator identity

\[ e^{-(p^2 - m^2)\tau} P^{(r)} = e^{-(p^2 - m^2)\tau} \tau P^{(r)} \]  

(6.40)

which can be seen by expanding the exponential and using \(P^{(r)2} = P^{(r)}\).

### 6.2.2 Green’s functions on graphs

We have seen that the point-to-point elementary kernel \(\langle b | K^{\alpha} | a \rangle\) associated with an observable \(S \in F_{S\mathbb{R}}(G^\mathbb{C})\) for the spin 0, 1/2, 1 representations of Poincaré are the Mellin transform \(\mathcal{M}_1[\langle b | E^{-S} | a \rangle; \alpha]\) for the topological localization \(\lambda : \phi_a(\mathbb{R}) \rightarrow A_a\). The generalization to many points and many observables that define a graph is well-known and leads to Feynman
diagrams. From the functional Mellin perspective, the generalization amounts to taking the Mellin transform of a Gaussian functional integral associated with the graph.

To illustrate, restrict to the scalar case. Let matrix $\varepsilon$ with components $[\varepsilon]_{i,j}$ be the incidence matrix of a connected graph $G$ with $V$ vertices and $I$ internal lines. In the momentum representation, the appropriate quadratic form is given by (compare (6.27))

$$Q_G(p, \tau) = p^T \cdot L(\tau)^{-1} p := p^T \cdot (\varepsilon D(\tau) \varepsilon^T)^{-1} p$$  \hspace{1cm} (6.41)

where $p := (p_{v_1}, \ldots, p_{v_n}, \ldots, p_{v_N})$ with each component $p \ni p_{v_i} : (\mathbb{T}, t_\alpha) \to \mathbb{R}^{1,3}$ being a sum of external momenta incident at vertex $v_i$, the collection of evolution-time intervals $\tau := (\tau_1, \ldots, \tau_1, \ldots, \tau_I)$ $\in i\mathbb{R}_+$, and $D(\tau)$ is the diagonal matrix

$$D(\tau) = \begin{pmatrix} 1/\tau_1 & \cdots & \cdots & \cdots & 1/\tau_I \\ \end{pmatrix}.$$  \hspace{1cm} (6.42)

Note that $\varepsilon \varepsilon^T$ is the graph Laplacian, and $p^T \cdot L(\tau)^{-1} p$ is the many-point generalization of $\tau p_i^2$ in (6.27) for the point-to-point case.

Upon localizing by $\lambda : \phi_\alpha(\mathbb{R}) \to i\mathbb{R}_+$, the Gaussian functional integral for free scalar propagation on $G$ with $N$ incoming (conserved) momenta evaluates to (see e.g. [10] § 3.1]

$$Z_{p, \omega_B}(p')|_{p'=0} = e^{-\langle s \omega_B(p) \rangle} \operatorname{Det}_{\lambda}(s \omega_B)^{1/2}$$  \hspace{1cm} (6.43)

where the observable $W_G(p', \tilde{p}') := -2\langle W p', \tilde{p}' \rangle$ is inverse to $Q_G(p, \tilde{p}) := -1/2\langle Q p, \tilde{p} \rangle$ according to $QW := \operatorname{Id}_{Z_{p, \omega_B}}$ and $WQ := \operatorname{Id}_{Z_{p, \omega_B}}$. Since $Q_G$ is quadratic, one can show the boundary term for each vertex $v_i$ is just the classical action $B(p_{v_i}) = S_G(p_{v_i})$ which depends parametrically on $\tau_i$. The right-hand side of (6.43) can be written in terms of the first and second Symanzik polynomials $U(\tau)$ and $F(p, \tau)$ in the usual way. [11] This yields the parametric representation of Feynman amplitudes\[21\]

$$I_G(p_1, \ldots, p_n; \alpha) = \int_{i\mathbb{R}_+} e^{-\langle F(p, \tau)/U(\tau) - tr(M^T D(\tau)^{-1} M)) \rangle U(\tau)^{(1+3)/2} \tau^\alpha} d\tau$$

$$\simeq M_\Gamma[E^{-\langle Q_G + M^2 \rangle}; \alpha]$$  \hspace{1cm} (6.44)

where $U(\tau) = \det(L(\tau)) \prod \alpha_i, F(p, \tau)/U(\tau) = Q_G(p, \tau), M$ is an $I \times I$ diagonal mass matrix, and $\alpha = (\alpha_1, \ldots, \alpha_I)$ with $\alpha_i \in \mathbb{S}$ is a multi-index\[22\]. Explicitly, the measure is $d\tau = \prod_{i=1}^I d\tau_i / \Gamma(\alpha_i)$. The right-hand side can be viewed as a Mellin functional integral over the multiplicative group of positive-definite diagonal matrices whose integrand is the exponentiated effective action functional of a graph $G$ evaluated at a classical point. Not surprisingly, the individual Mellin integrals yield Gamma function terms (that depend on $\alpha$) with their concomitant pole structure. Hence, dimensional/analytic regularization can be performed by suitable subtraction of the pole terms.

\[21\text{Recall (6.39): Since } G^C \text{ is abelian and } \mathfrak{c}^* \text{ is commutative in this case, } \rho(g^\alpha) = \prod_i \rho(g_{1i}^{\alpha_i}).\]
Instead of viewing the parametric representation as an alternative to the more basic space-time representation of Feynman diagrams, perhaps it should be viewed the other way around. That is, perhaps the Feynman diagram picture is a particular model of some operator $\mathcal{M}_\Gamma[E^{-S}, \alpha] \in \Pi^{(\alpha)}(\mathbf{F}_{S_R}(G^C))$ localized to an abelian group and represented as an evolution process of massive particles in $\mathbb{R}^{1,3}$. The former springs from QFT and the latter from the world-line formalism. Both are model-probes of an observable $E^{-S} \in \mathbf{F}_{S_R}(G^C)$.

In the previous subsection, we calculated the momentum-to-momentum realization of the negative-power operator $\Box + m^2$.$^{-1}$'. Although not true in general, in an eigenbasis of the operator we get the same result for the positive power of the inverse operator:

$$\langle p_b | (\Box + m^2)^{-\alpha}_{H, \beta} p_a \rangle = \int_{\mathbb{R}^x} (1 - (p_b^2 - m^2)\tau)^{-\beta} \tau^\alpha d \log(\tau)$$

$$= e^{i\pi(\alpha - 1)} B(\alpha, \beta - \alpha)(p_b^2 - m^2)^{-\alpha}, \quad \Re(\beta) \geq 1 \quad (6.45)$$

for $\Re(p_b^2) > m^2$. Putting $\alpha = 1$ and $\beta = 2$ gives a massive-particle-Green’s-function interpretation of a positive-power operator $A^\alpha_{H, \beta} = (\Box + m^2)^{-\alpha}_{H, \beta} \in \Pi^{(\alpha)}(\mathbf{F}_{S_R}(G^C))$. The same can be done for $L(\tau)^{-1}$.

### 6.2.3 positive-power scattering

Interestingly, in §5.2, we saw that positive-power operators, of the same form as (6.45) but with $0 < \Re(\alpha) < \Re(\beta) < 1$, exhibit a structural similarity to 4-point, tree level, tachyon string scattering. There is more overlap with string scattering.

For example, the 4-point amplitudes for three tachyons and one higher-spin massive string contain the usual Beta function factor, string-state terms, and an integral of the schematic form [52 pg. 4]

$$F^{(K)}_D(a : b_1, \ldots, b_K : c : x_1, \ldots, x_K)$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 (1 - t)^{c - a - 1}(1 - x_1 t)^{-b_1} \cdots (1 - x_K t)^{-b_K} t^{a - 1} dt. \quad (6.46)$$

This has the same form as a positive-power operator when $(Id - Ag)^{-\beta} = \Pi_i (1 - A_{ij}g_j)^{-\beta_i}$ and the normalized Haar measure is $\nu(g) = \log g/\log (1/a, c - a)$. Another source of overlap comes from the “stringy canonical forms” of [48];

$$\mathcal{I}_\{p_J\}(X, \{c\}) = (\alpha')^d \int_{\mathbb{R}^d} \prod_j p_J(x)^{-\alpha'_{i_j}} \prod_{i=1}^d x_i^{\alpha'_{X_i}} d \log x_i \quad (6.47)$$

where $p_J(x)$ are Laurent polynomials. Hiding in this integral are $n$-point open string scattering amplitudes if $\prod_j p_J(x)^{-\alpha'_{i_j}} = \prod_{1 \leq i < j \leq n - 2} p_{i,j}(x)^{\alpha'_{s_{i,j}}}$ with $n \geq 4$ and three of the $x_i$ are fixed so that $d = n - 3$. [48 pg. 3] Conjecturally, given a non-abelian group $G^C$ of rank $n - 3$, these string amplitudes could represent a localization of functional Mellin onto a Borel subgroup $B < G^C$. Presumably, the $(n - 3)(n - 2)/2$ degree polynomials $\prod_{1 \leq i < j \leq n - 2} p_{i,j}(x)^{\alpha'_{s_{i,j}}}$ would come from an operator $A \in B$ in the adjoint representation such that $(Id - Ag)^{-\beta} = \prod_{i < j}(1 - A_{ij}g_j)^{-\beta_i}$, where $g \in \mathbb{R}^{n-3}$ is a maximal torus in $B$. 

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If positive-power operators underlie scattering processes as these examples suggest, this raises a question: Why does the parameter range $1 \leq \alpha < \Re(\beta)$ correspond to particle scattering while $0 < \Re(\alpha) < \Re(\beta) < 1$ corresponds to string scattering? Of course there are many more questions that need to be asked and answered to give credence to the notion that positive-power operators govern scattering processes.

### 6.3 Density operators, entropy, and the replica trick

Since functional Mellin deals in complex groups, quantum system evolution is more naturally parametrized by complex one-parameter subgroups with localization $\lambda_{\mathbb{C}^\times} : \phi_a(\mathbb{C}) \to \mathbb{C}^\times$. We have already considered free-particle evolution by localizing to $i\mathbb{R}_+ \cup i\mathbb{R}_-$. To see what the orthogonal direction of the complex parameter characterizes, fix the evolution-time interval $i\Delta t$. This yields two copies of the real projective line $\mathbb{P}^1(\mathbb{R})$ which we will interpret as representing a fixed-energy state relative to forward and reverse evolution-time.

Take the generator of the one-parameter subgroup to be $a \equiv H$ for some Hamiltonian $H$ such that $H(g) = H g \in L_B(\mathcal{H})$. As $\mathbb{P}^1(\mathbb{R})$ is homeomorphic to $S^1$, the negative power is

$$H^{-\alpha}_{\lambda_{\mathbb{S}^1}} = \mathcal{M}_\lambda[H^{-\alpha}; \alpha] \mathcal{L}_{\lambda_{\mathbb{S}^1}} \int_{S^1} e^{-H(g)} g^\alpha d \log g = H^{-\alpha} \quad \alpha \in \mathbb{R} \cap \mathbb{S} \quad (6.48)$$

where $H = \rho(H)$. Also associated with $H$ is an observable $\text{Tr} E^{-H}$ and its associated zeta function

$$\text{Tr} H^{-\alpha}_{\lambda_{\mathbb{S}^1}} = \mathcal{M}_{\lambda_{\mathbb{S}^1}}[\text{Tr} E^{-H}, \alpha] = \int_{S^1} \text{tr} \left(e^{-Hg} g^\alpha\right) d \log g = \int_{S^1} \text{tr}(e^{-Hg}) g^\alpha d \log g = \zeta_H(\alpha). \quad (6.49)$$

where $\text{tr}(e^{-Hg})$ is the partition function of the scaled operator $H g$.

It is convenient to put $g \to g^{-1}$ in the integrals and define

$$\varrho^{-\alpha}_{\lambda_{\mathbb{S}^1}} := \frac{\int_{S^1} e^{-H(g^{-1})} g^{-\alpha} d \log g}{\text{Tr} H^{-1}_{\lambda_{\mathbb{S}^1}}} = \frac{H^{-\alpha}}{\zeta_H(1)} . \quad (6.50)$$

In an $H$-eigensystem $\{|i\rangle, \varepsilon_i\}$ with dimensionless $\varepsilon_i := E_i/k_B T_0$ for $E_i = \langle i|H|i\rangle$ and $T_0$ some reference temperature, we have $\langle i|\varrho^{-\alpha}_{\lambda_{\mathbb{S}^1}}|i\rangle = \varepsilon_i^{-\alpha}/\sum_i \varepsilon_i^{-1}$. Moreover, the integration variable can be associated with (dimensionless) energy/temperature and $\text{Tr} \varrho^{-\alpha}_{\lambda_{\mathbb{S}^1}} = 1$. Therefore we regard $\varrho^{-1}_{\lambda_{\mathbb{S}^1}}$ as an inverse-density operator with von-Neumann entropy

$$\text{tr} \log \varrho^{-1}_{\lambda_{\mathbb{S}^1}} = \text{tr} \frac{d}{d\alpha} \varrho^{-\alpha}_{\lambda_{\mathbb{S}^1}} \bigg|_{\alpha=1} = \frac{\mathcal{M}_{\lambda_{\mathbb{S}^1}}[\text{Tr} E^{-H}; 1]}{\zeta_H(1)} = \frac{\zeta_H'(1)}{\zeta_H(1)} = \frac{d}{d\alpha} \log \zeta_H(\alpha) \bigg|_{\alpha=1} = \text{tr}(\varrho^{-1} \log \varrho^{-1}) . \quad (6.51)$$

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Mellin affords a work-around: from Definition 5.3
\[ \rho_{\alpha}^* = \frac{\mathcal{M}_{\mathcal{C}_B}((\text{Id} - H^{-1})(\beta); \alpha)}{\mathcal{M}_{\mathcal{C}_B}[\text{Tr} ((\text{Id} - H^{-1})(\beta)); 1]} = \frac{\int_{\mathcal{C}} (g - H^{-1})^{-\beta} g^{\beta - \alpha} d\nu(g_{\mathcal{C}_B})}{\text{Tr} H_{\mathcal{C}_B}^1} = \frac{H^\alpha}{\text{tr} H}. \] (6.52)
Here the integration variable is simply a parameter along a contour \( \mathcal{C} \in \mathbb{C}^\times \) enclosing the spectrum \( \sigma(H^{-1}) \). From Definition 5.3 with \( \alpha = 1 \) and \( \beta = 2 \), its von-Neumann entropy is
\[ \text{tr} \log \rho_{\alpha}^{1+1} = \text{tr} \frac{d}{d\alpha} \rho_{\alpha}^* \bigg|_{\alpha=1} = \frac{\text{tr} \mathcal{M}_{\mathcal{C}_B}((\text{Id} - H^{-1})(\beta); 1)}{\text{tr} H_{\mathcal{C}_B}^1} = \frac{\mathcal{M}_{\mathcal{C}_B}[\text{Tr} ((\text{Id} - H^{-1})(\beta)); 1]}{\mathcal{M}_{\mathcal{C}_B}[\text{Tr} ((\text{Id} - H^{-1})(\beta)); 1]} = \frac{d}{d\alpha} \log \text{tr} H \bigg|_{\alpha=1} = \text{tr} (\rho \log \rho). \] (6.53)

Clearly \( \rho^{-1} \) (which is the Mellin transform of the partition function) and \( \rho \) (which is the Mellin transform of the resolvent) give the same von-Neumann entropy for invertible \( H \) since they derive from normalized eigenvalues of \( H \).

To obtain the standard representation of the density operator \( \hat{\rho} \) in terms of the partition function, put \( H \to E^{-H} \) and \( \alpha \to T_0/T \) with positive power
\[ (E^{-H})_{\alpha}^\mathcal{C}_B := \mathcal{M}_{\mathcal{C}_B}((\text{Id} - E^H)(\beta); \alpha) = (e^{-H})^\alpha \quad \alpha < \beta \in \mathbb{R} \cap S. \] (6.54)
Use this to define the \( \alpha \)-dependent partition function
\[ \mathcal{Z}_{\mathcal{C}_B}(\alpha) := \mathcal{M}_{\mathcal{C}_B}[\text{Tr} ((\text{Id} - E^H)(\beta); \alpha] = \text{tr} (e^{-H})^\alpha, \] (6.55)
density operator
\[ \hat{\rho}(\alpha) := (e^{-H})^\alpha / \mathcal{Z}_{\mathcal{C}_B}(\alpha), \] (6.56)
and entropy
\[ S(\alpha) := d \log \mathcal{Z}_{\mathcal{C}_B}(\alpha)/d\alpha. \] (6.57)
This assigns a nice physical interpretation to the parameter \( \alpha \).

**Remark 6.2** The replica trick is a means to calculate \( \log \mathcal{Z} \) and/or \( \text{tr} (\hat{\rho} \log \hat{\rho}) \). It relies on replacing \( \hat{\rho}^n \) for \( n \in \mathbb{Z}_{\geq 0} \) with \( \hat{\rho}^r \) for \( r \in \mathbb{R}_{\geq 0} \) which is hard to rigorously justify. Functional Mellin affords a work-around: from Definition 5.3. \( \hat{\mathcal{Z}}_{\mathcal{C}_B}(0) := d\mathcal{Z}(\alpha)/d\alpha|_{\alpha=0^+} = \log \mathcal{Z} \) if the limit \( \alpha \to 0^+ \) exists, and \( S(1) = d \log \mathcal{Z}_{\mathcal{C}_B}(\alpha)/d\alpha|_{\alpha=1} = \text{tr} (\hat{\rho} \log \hat{\rho}) = \text{tr}(d\hat{\rho}(\alpha)/d\alpha|_{\alpha=1}). \)

Putting this and the previous subsection together suggests complex one-parameter subgroups in the context of functional Mellin may be a useful tool for combining Lorentzian and Euclidean quantum field theory. We leave that for future work.

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22There is a close parallel between the von-Neumann entropy and \( \log \zeta(\alpha) \) where \( \zeta(\alpha) \) is the Riemann zeta function. The log of Riemann zeta acts like a spectral measure on \( (\mathbb{Z}/x\mathbb{Z})^\times \) when counting prime powers up to some cut-off integer \( x \), and it can be generalized to prime \( k \)-tuples. ([10] App. 4.8]) It would be interesting to apply the essentially functional Mellin techniques developed in [10] to count eigenvalues and \( k \)-tuple/\( k \)-correlated eigenvalues of \( H \) using the “spectral measure” \( \log \zeta_H(\alpha) \).
7 Conclusion

Fourier transform has been a central theme in the discipline of functional integrals since their inception. As is well known, it represents duality between locally compact abelian groups. Our contention is that the Mellin transform in the functional context is not just a reformulation of Fourier: It represents duality between Banach $*$-algebras. As such, we assert functional Mellin is a useful addition to the toolbox of mathematical physics. In particular, we used it to construct functional analogs of resolvent, trace, log, and determinant and presented several applications and examples. Hopefully the applications and examples have demonstrated the practical and conceptual utility of functional Mellin.

More importantly, at a broad level, functional Mellin is pertinent to non-commutative $C^*$-algebras characterizing quantum systems. In fact, given some relevant representations of a topological group $G^C$, functional Mellin defines a $C^*$-algebra for which the Mellin integrator acts as a $*$-homomorphism to the algebra of bounded linear operators on the Hilbert spaces carrying representations of $G^C$. This means that, armed with functional Mellin and a starting topological group, we can construct and represent a non-commutative $C^*$-algebra — without having to somehow deform a commutative algebra. Consequently, if one is fortunate enough to know $G^C$ and can construct relevant representations that characterize a physical system, then one can model its quantum properties without first passing through the classical realm. We exploit this idea in subsequent work.

A Mellin transforms

A.1 Basics

Most of the following basic properties, which include some non-standard aspects, can be found in [15]-[18].

**Definition A.1** Let $f : (0, \infty) \to \mathbb{C}$ be a function such that $f \in L^1(\mathbb{R}_+)$ with limits

$$\lim_{x \to 0^+} f(x) \to O(x^{-a}) \quad \text{and} \quad \lim_{x \to \infty} f(x) \to O(x^{-b})$$

for $a, b \in \mathbb{R}$. Then the Mellin transform $\tilde{f}(\alpha)$ with $\alpha \in (a, b) := (a, b) \times i\mathbb{R} \subset \mathbb{C}$ is defined by

$$\tilde{f}(\alpha) := \mathcal{M}[f(x); \alpha] := \int_0^\infty f(x) x^{\alpha-1} dx.$$  \hspace{1cm} (A.1)

The fundamental strip $(a, b) \subset \mathbb{C}$ indicates the domain of convergence. Since if

$$f(x)|_{x \to 0^+} = O(x^{-a}) \quad \text{and} \quad f(x)|_{x \to \infty} = O(x^{-b})$$ \hspace{1cm} (A.2)

then $\tilde{f}(\alpha)$ exists in $(a, b)$ where it is holomorphic and absolutely convergent. More precisely,

$$|\tilde{f}(\alpha)| \leq \int_0^1 |f(x)| x^{\alpha a-1} dx + \int_1^\infty |f(x)| x^{\alpha a-1} dx$$

$$\leq M_1 \int_0^1 x^{\alpha a-1-a} dx + M_2 \int_1^\infty x^{\alpha a-1-b} dx$$ \hspace{1cm} (A.3)
for some finite constants $M_1, M_2$.

From the definition follows some important properties (for suitable fundamental strips):

\[ c^{-\alpha} \tilde{f}(\alpha) = \mathcal{M}[f(cx); \alpha] \quad c > 0 \]
\[ \tilde{f}(\alpha + d) = \mathcal{M}[x^d f(x); \alpha] \quad d > 0 \]
\[ \frac{1}{|z|} \tilde{f}(\alpha) = \mathcal{M}[f(x'); \alpha] \quad r \in \mathbb{R} - \{0\}, \, \alpha \in \langle ra, rb \rangle \]
\[ \frac{d^n}{d\alpha^n} \tilde{f}(\alpha) = \mathcal{M}[(\log x)^n f(x); \alpha] \quad n \in \mathbb{N} \]
\[ -\alpha \tilde{f}(\alpha) = \mathcal{M} \left[ \left( \frac{d}{dx} \right) f(x); \alpha \right] \]
\[ -\tilde{f}(\alpha - 1) = \mathcal{M} \left[ \frac{d}{dx} f(x); \alpha \right] \]
\[ -\frac{1}{\alpha} \tilde{f}(\alpha + 1) = \mathcal{M} \left[ \int_0^x f(x') \, dx'; \alpha \right] \] (A.4)

The last three relations can be extended by iteration:

\[ (-1)^n \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} \tilde{f}(\alpha) = \mathcal{M} \left[ x^n \frac{d^n}{dx^n} f(x); \alpha \right] \] (A.5)

for $n \in \mathbb{N}$ and $x^{\alpha + n - m} f^{(n-m)}(x)|_0^\infty = 0 \forall m \in \{1, \ldots, n\}$,

\[ (-1)^n \frac{\Gamma(\alpha)}{\Gamma(\alpha - n)} \tilde{f}(\alpha - n) = \mathcal{M} \left[ f^{(n)}(x); \alpha \right] \] (A.6)

for $n \in \mathbb{N}$ and $x^{\alpha - n - 1 + m} f^{(n-m)}(x)|_0^\infty = 0 \forall m \in \{1, \ldots, n\}$, and

\[ (-1)^n \frac{\Gamma(\alpha)}{\Gamma(\alpha + n)} \tilde{f}(\alpha + n) = \mathcal{M} \left[ \left( \int_0^x f(x') \, dx' \right)^n; \alpha \right] \] (A.7)

where $\left( \int_0^x f(x') \, dx' \right)^n$ defines an iterated integral

\[ \left( \int_0^x f(x') \, dx' \right)^n := \int_0^x \cdots \int_0^x f(x_n) \cdots f(x_1) \, dx_1 \cdots dx_n . \] (A.8)

The last two relations show that (for functions with appropriate asymptotic conditions) the Mellin transforms of derivatives and integrals are symmetrical under $n \to -n$. Indeed, this is the basis of the definition of fractional derivatives. This suggests an application to pseudo-differential symbols of the type $A(x, d/dx) = \sum_{i=-\infty}^n a_i(x) d^i/dx^i$.

The Mellin transform is directly related to the Fourier and (two-sided) Laplace transforms by

\[ \mathcal{M}[f(x); \alpha] = \mathcal{F}[f(e^x); -i\alpha] = \mathcal{L}[f(e^{-x}); \alpha] . \] (A.9)

From these relationships, the inverse Mellin transform can be deduced;

\[ f(x) \overset{a.e.}{=} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-\alpha} \tilde{f}(\alpha) \, d\alpha \] (A.10)
where \( c \in (a, b) \) (provided \( \tilde{f}(\alpha) \) is integrable along the path). The almost everywhere (a.e.) designation can be dropped if \( f(x) \) is continuous. Moreover, if \( f(x) \) is of bounded variation about \( x_0 \), then

\[
\frac{f(x_0^+) + f(x_0^-)}{2} = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} x^{-\alpha} \tilde{f}(\alpha) \, d\alpha .
\]  

(A.11)

Using the inversion formula, the Parseval relation for the Mellin transform follows from

\[
\int_0^\infty g(x) h(x) x^{\alpha-1} \, dx = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{g}(\alpha') \tilde{h}(\alpha - \alpha') \, d\alpha',
\]  

(A.12)

assuming the necessary conditions on \( g(x) \) and \( h(x) \) to allow for the interchange of integration order. In particular,

\[
\mathcal{M} [g(x)h(x); 1] = \int_0^\infty g(x) h(x) \, dx = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{g}(\alpha') \tilde{h}(1 - \alpha') \, d\alpha'.
\]  

(A.13)

Using the inversion formula, the Parseval relation for the Mellin transform follows from

\[
\int_0^\infty g(x) h(x) x^{\alpha-1} \, dx = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{g}(\alpha') \tilde{h}(\alpha - \alpha') \, d\alpha',
\]  

(A.12)

assuming the necessary conditions on \( g(x) \) and \( h(x) \) to allow for the interchange of integration order. In particular,

\[
\mathcal{M} [g(x)h(x); 1] = \int_0^\infty g(x) h(x) \, dx = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{g}(\alpha') \tilde{h}(1 - \alpha') \, d\alpha'.
\]  

(A.13)

\[
\mathcal{M} [g(x) * h(x); \alpha] := \int_0^\infty \int_0^\infty g(x') h \left( \frac{x}{x'} \right) x^{\alpha' - 1} \, dx' \, dx = \tilde{g}(\alpha) \tilde{h}(\alpha),
\]  

(A.14)

and

\[
\mathcal{M} [g(x) * h(x); \alpha] := \int_0^\infty \int_0^\infty g(x x') h(x') x^{\alpha' - 1} \, dx' \, dx = \tilde{g}(\alpha) \tilde{h}(1 - \alpha).
\]  

(A.15)

A.2 Expansions

Definition A.2 The singular expansion of a meromorphic function \( f(b) \) with a finite set \( P \) of poles is defined to be the sum of its Laurent expansions to order \( O(b^0) \) about each pole, i.e.

\[
f(b) \asymp \sum_{\varepsilon \in P} \text{Laur}[f(b), \varepsilon; O(b^0)].
\]  

(A.16)

Theorem A.3 ([15] th. 3) Let \( f(x) \) have Mellin transform \( \tilde{f}(\alpha) \) in \( (a, b) \). Assume

\[
f(x) \big|_{x \to 0^+} = \sum_{\varepsilon, k} c_{\varepsilon, k} x^{\varepsilon} (\log x)^k + O(x^M)
\]  

(A.17)

where \(-M < -\varepsilon \leq a\) and \( k \in \mathbb{N} \). Then \( \tilde{f}(\alpha) \) can be continued to a meromorphic function in \( (-M, b) \), and it has the singular expansion

\[
\tilde{f}(\alpha) \asymp \sum_{\varepsilon, k} c_{\varepsilon, k} \frac{(-1)^k k!}{(\alpha + \varepsilon)^{k+1}}.
\]  

(A.18)

Likewise, if

\[
f(x) \big|_{x \to \infty} = \sum_{\varepsilon, k} c_{\varepsilon, k} x^{-\varepsilon} (\log x)^k + O(x^{-M})
\]  

(A.19)
where \( b \leq \varepsilon < M \) and \( k \in \mathbb{N} \), then \( \tilde{f}(\alpha) \) can be continued to a meromorphic function in \( (a, M) \), and it has the singular expansion

\[
\tilde{f}(\alpha) \asymp \sum_{\varepsilon, k} c_{\varepsilon, k} \frac{(-1)^{k+1}k!}{(\alpha - \varepsilon)^{k+1}}. \tag{A.20}
\]

Conversely, it can be shown that for \( \tilde{f}(\alpha) \) meromorphic in \( (-M, b) \) (respectively \( (a, M) \)) whose poles lie to the right (respectively left) of the fundamental strip, then

\[
f(x)|_{x \to 0^+} = \sum_{\varepsilon_k \in \mathcal{P}} \text{Res} \left[ \tilde{f}(\alpha)x^{\alpha}, \alpha = \varepsilon_k \right] + O(x^{M})
\]

\[
f(x)|_{x \to \infty} = -\sum_{\varepsilon_k \in \mathcal{P}} \text{Res} \left[ \tilde{f}(\alpha)x^{-\alpha}, \alpha = \varepsilon_k \right] + O(x^{-M}) \tag{A.21}
\]

if \( f(x) \) is at least twice differentiable. More precisely,

**Theorem A.4** ([15 th. 4]) **Let** \( f(x) \) **have Mellin transform** \( \tilde{f}(\alpha) \) **in** \( (a, b) \). Assume \( \tilde{f}(\alpha) \) is meromorphic in \( (-M, b) \) **such that**

\[
\tilde{f}(\alpha)|_{|\alpha| \to \infty} = O(|\alpha|^{-r}), \quad r > 1
\]

and

\[
\tilde{f}(\alpha) \asymp \sum_{k, \varepsilon} \frac{c_{k, \varepsilon}}{(\alpha - \varepsilon)^{k+1}}. \tag{A.23}
\]

Then

\[
f(x)|_{x \to 0^+} = \sum_{k, \varepsilon} \frac{(-1)^k k!}{k} c_{k, \varepsilon} x^{-\varepsilon} (\log x)^k + O(x^{M}). \tag{A.24}
\]

Likewise, if \( \tilde{f}(\alpha) \) is meromorphic in \( (a, M) \), then

\[
f(x)|_{x \to \infty} = \sum_{k, \varepsilon} \frac{(-1)^{k+1} k!}{k} c_{k, \varepsilon} x^{-\varepsilon} (\log x)^k + O(x^{-M}). \tag{A.25}
\]

### A.3 Mellin distributions

The relationship between Mellin and Fourier transforms allows the development of Mellin distributions. Following [16, 18];

**Definition A.5** ([16 §7] Let \( f_I : \mathbb{R}_+^n := \{ y \in \mathbb{R}^n : 0 < y < \infty \} \to \mathbb{C} \) be a function with support \( I := \{ x \in \mathbb{R}_+^n : 0 < x \leq x_0 \text{ for some } x_0 \in \mathbb{R}_+^n \} \). Take \( f_I \in L^1(\mathbb{R}_+^n) \) with limits \( \lim_{x \to 0^+} f_I(x) = O(x^{-a}) \) and \( \lim_{x \to \infty} f_I(x) = O(x^{-b}) \) for \( a, b \in \mathbb{R}^n \). Then the Mellin transform \( \tilde{f}(\alpha) \) with \( \alpha \in (a, b) := (a, b) \times i\mathbb{R}^n \subset \mathbb{C}^n \) is defined by (the analytic extension of)

\[
\tilde{f}(\alpha) := \mathcal{M}[f_I(x); \alpha] := \int_{\mathbb{R}_+^n} f_I(x)x^{-a-1}dx. \tag{A.26}
\]

The notation \((a, b)\) denotes a poly-interval \( \{ y \in \mathbb{R}^n : a < y < b \} \) and \( x^\alpha := x_1^{a_1} \cdots x_n^{a_n} \).

\(^{23}\)The substitution \( \alpha \to -\alpha \) in the exponent of \( x \) conforms with reference [16].
Now, the close relationship with the Fourier transform motivates the definition

**Definition A.6** [16, §5] Let \( v \in \mathbb{R}^n \) and define

\[
M_v(I) := \{ \psi \in C^\infty(I) : \sup_{x \in I} |x^{v+1} (x \partial_x)^\lambda \psi(x)| < \infty \}
\] (A.27)

where \( \lambda \in \mathbb{N}_0^n \) and \( \mathbb{N}_0^n \) is the set of non-negative multi-indices. Endow \( M_v(I) \) with the topology defined by the sequence of seminorms

\[
\varrho_{v,\lambda}(\psi) = \sup_{x \in I} |x^{v+1} (x \partial_x)^\lambda \psi(x)|.
\] (A.28)

Then \( M_w(I) \) for \( w \in \mathbb{R}_\infty^n := (\mathbb{R} \cup \{ \infty \})^n \) is defined to be the inductive limit of \( M_v(I) \), i.e. \( M_w(I) = \lim_{v \to w} M_v(I) \). The dual space \( M'_w(I) \) is comprised of Mellin distributions and the total space of Mellin distributions is

\[
M'(I) = \bigcup_{w \in \mathbb{R}_\infty^n} M'_w(I).
\] (A.29)

Finally, the Mellin transform of a distribution \( T \in M'_w(I) \) is defined by

\[
\tilde{T}(\alpha) := M[T; \alpha] := \langle T, x^{-\alpha-1} \rangle \hspace{1em} \Re(\alpha) < w.
\] (A.30)

Note the topological inclusions

\[
D(I) \subset M_v(I) \subset M'_w(I) \subset D'(I),
\] (A.31)

and \( \tilde{T}(\alpha) \) is well-defined on the set

\[
\Omega_T := \bigcup_{\{v: T \in M'_v(I)\}} [\Re(\alpha) < v].
\] (A.32)

**Theorem A.7** [16, §8, th. 1] \( M'(I) \) coincides with the space of distribution on \( \mathbb{R}^n \) supported on the closure of \( I \) and restricted to \( \mathbb{R}_+^n \).

Some of the important properties of the 1-dimensional Mellin transform have their analogues for distributions: The Mellin transform \( \tilde{T}(\alpha) \) is holomorphic on \( \Omega_T \) and

\[
\frac{\partial}{\partial \alpha_i} \tilde{T}(\alpha) = \langle T, x^{-\alpha-1}(- \log x_i) \rangle
\]

\[
\tilde{T}(\alpha - \beta) = M[\chi^\beta T; \alpha] \hspace{1em} \Re(\alpha) - \Re(\beta) < w
\]

\[
\alpha^\gamma \tilde{T}(\alpha) = M[(x \partial_x)^\gamma T; \alpha] \hspace{1em} \gamma \in \mathbb{N}_0^n, \ Re(\alpha) < w
\]

\[
(\alpha^\gamma + 1) \tilde{T}(\alpha + \gamma) = M[(\partial_x)^\gamma T; \alpha] \hspace{1em} |\gamma| = 1, \ Re(\alpha) < w - \gamma.
\] (A.33)
B Exponential exercises

The exponential function plays a prominent role in ordinary Mellin transforms, so we want to develop and characterize the functional counterpart by looking at some specific cases of reduction to finite dimensional groups.

Let \( E := \exp F \mathbb{S}_\mathcal{G} \mathbb{C} \) stand for the exponential on \( \mathbb{F}_\mathcal{S}(\mathbb{G}_\mathbb{C}) \) defined with the product given by the \(*\)-convolution. Suppose \( \mathcal{E}^* \equiv \mathbb{C}, \lambda : \mathbb{G}_\mathbb{C} \to \mathbb{R}_+, \) and \( \Lambda(g) = Ag \) where \( A \in \mathbb{C}_+ := \mathbb{R}_+ \times i\mathbb{R} \). For the standard Haar measure on \( \mathbb{R}_+ \) this is just the usual exponential Mellin transform

\[
\mathcal{M}_{\mathbb{R}_+, H}[E^{-\Lambda}; \alpha] := \int_{\mathbb{R}_+} e^{-Ag} g^{\alpha-1} \, dg = \frac{\Gamma(\alpha)}{A^\alpha}, \quad \alpha \in (0, \infty)_H .
\] (B.1)

In particular,

\[
\mathcal{M}_{\mathbb{R}_+, H}[E^{-\text{Id}}; \alpha] = \Gamma(\alpha), \quad \alpha \in (0, \infty)_H .
\] (B.2)

As a quick exercise, use Lemma 3.15 with \( F_1(\tilde{g}g) = e^{-\tilde{g}g} \) and \( F_2(\tilde{g}) \rho(\tilde{g}) = e^{-\tilde{g} \tilde{g}} \) to deduce

\[
\Gamma(\alpha)\Gamma(1-\alpha) = \mathcal{M}_{\mathbb{R}_+, H}[\int_{\mathbb{R}_+} e^{-\tilde{g} \tilde{g}} e^{-\tilde{g}g} d\log \tilde{g}; \alpha] = \int_0^\infty t^{\alpha-1} \frac{dt}{1+t} = \pi \csc(\pi\alpha), \quad \alpha \in (0, 1)_H .
\] (B.3)

Notice the reduction in the fundamental strip \( \mathcal{S}_\lambda \). Simple manipulations yield the standard results \( \pi \alpha \csc(\pi\alpha) = \Gamma(1+\alpha)\Gamma(1-\alpha) \) and \( \Gamma(1+\alpha)/\Gamma(\alpha-1) = \alpha(\alpha-1) \).

However, the functional Mellin transform provides a mechanism to regularize; and with a suitable choice of \( \lambda \),

\[
\mathcal{M}_{\mathbb{R}_+, \Gamma}[E^{-\Lambda}; \alpha] := \int_{\mathbb{R}_+} e^{-Ag} g^\alpha \nu(g_T) = \frac{1}{A^\alpha}, \quad \alpha \in (0, \infty)_\Gamma
\] (B.4)

for \( \nu(g_T) := \log g/\Gamma(\alpha) = \nu(g)/\Gamma(\alpha) \) where \( \nu(g) \) is the normalized Haar measure on \( \mathbb{R}_+ \). To extend the fundamental strip to the left of the imaginary axis, one can use

\[
\mathcal{M}_{\mathbb{R}_+, \Gamma^p}[E^{-\Lambda}; \alpha] := \frac{\Gamma(\alpha)}{\Gamma(\alpha+p)} \int_{\mathbb{R}_+} (Ag)^p e^{-Ag} g^\alpha \nu(g_T) = \left( -1 \right)^p \frac{\Gamma(\alpha)}{\Gamma(\alpha+p)} \mathcal{M}_{\mathbb{R}_+, H}[g^p d^p E^{-\Lambda}; \alpha], \quad \alpha \in (-p, \infty)_{\Gamma^p}, \quad p \geq 0 .
\] (B.5)

There are other ways to extend the fundamental strip to the left of the imaginary axis (besides analytic continuation). For example, defining \( \varphi^{-Ag} := e^{-Ag} - e^{-g} \) yields

\[
\mathcal{M}_{\mathbb{R}_+, H}[E^{-\Lambda}; \alpha] := \int_{\mathbb{R}_+} \varphi^{-Ag} g^\alpha \nu(g_H) = \Gamma(\alpha) \left( \frac{1}{A^\alpha} - 1 \right), \quad \alpha \in (-1, \infty)_H .
\] (B.6)
For \( \lim \alpha \to 0^+ \) this gives \( \int_{\mathbb{R}_+} g^{-Ag} \, d\nu(g_R) = -\log A \), and therefore (in this case)

\[
\mathcal{M}_{\mathbb{R}_+, H} \left[ E^{-A}; \alpha \right] \bigg|_{\alpha \to 0^+} = \frac{d}{d\alpha} \mathcal{M}_\Gamma \left[ E^{-A}; \alpha \right] \bigg|_{\alpha \to 0^+}
\]  

which suggests the definition

\[
\widetilde{\mathcal{M}}_{\lambda} [F; \alpha] \bigg|_{\alpha \to 0^+} = \frac{d}{d\alpha} \mathcal{M}_\lambda [F; \alpha] \bigg|_{\alpha \to 0^+} =: \int_{G^C} F(g) g^\alpha \log g \, D_\lambda g \bigg|_{\alpha \to 0^+} =: \int_{G^C} F(g) g^\alpha \widehat{D}_\lambda g \bigg|_{\alpha \to 0^+}
\]

if the limit exists. In particular, for \( \lambda : G^C \to C^C_\lambda \),

\[
g^\alpha \widehat{D}_\lambda g \sim \frac{d}{d\alpha} g^\alpha d\nu(g_R) =: g^\alpha d\widehat{\nu}(g_R) .
\]

For example, choosing \( \nu(g_R) \) yields \( d\widehat{\nu}(g_R) = (\log g - \psi(\alpha)) \, d\nu(g_R) \) where \( \psi(\alpha) = \Gamma'(\alpha)/\Gamma(\alpha) \). This motivates Definition 4.9 for functional Log.

Moving on to the non-abelian case, suppose \( \lambda : G^C \to GL(n, \mathbb{C})_+ := SL(n, \mathbb{C}) \times \mathbb{R}_+ \) and \( C^* \equiv \mathbb{C} \). Define the functional \( E^{-\text{Tr} A} : G^C \to \mathbb{C} \) by \( g \mapsto e^{-\text{tr}(A^{-1}g)} \) with \( A \in GL(n, \mathbb{C})_+ \), and take \( \rho : G^C \to \mathbb{C}^*_+ \) by \( g \mapsto \det g \). Then

\[
\mathcal{M}_{GL(n, \mathbb{C})_+, \Gamma_n} \left[ E^{-\text{Tr} A}; \alpha \right] = \int_{GL(n, \mathbb{C})_+} e^{-\text{tr}(A g)} \det g^\alpha d\nu(g_{\Gamma_n}) , \; \alpha \in \mathbb{S}_{\Gamma_n}
\]

\[
= \int_{GL(n, \mathbb{C})_+} e^{-\text{tr}(A g)} (\det g)^\alpha e^{i\varphi(\alpha)} d\nu(g_{\Gamma_n}) , \; \alpha \in \mathbb{S}_{\Gamma_n}
\]

\[
= \int_{GL(n, \mathbb{C})_+} e^{-\text{tr}(g)} (\det A^{-1} \det g)^\alpha e^{i\varphi(\alpha)} d\nu(g_{\Gamma_n}) , \; \alpha \in \mathbb{S}_{\Gamma_n}
\]

\[
= \det A^{-\alpha} , \; \alpha \in \mathbb{S}_{\Gamma_n}
\]

where \( \varphi(\alpha) \) is a phase, \( \nu(g_{\Gamma_n}) := \nu(g)/\Gamma_n(\alpha) \) with \( \nu(g) \) the Haar measure on \( GL(n, \mathbb{C})_+ \), and \( \Gamma_n(\alpha) \) a complex multi-variate gamma function defined by

\[
\Gamma_n(\alpha) := \mathcal{M}_{GL(n, \mathbb{C})_+, H} \left[ E^{-\text{Tr} \text{Id}}; \alpha \right] := \int_{GL(n, \mathbb{C})_+} e^{-\text{tr}(g)} \det g^\alpha d\nu(g) , \; \alpha \in \mathbb{S}_{H} .
\]

In particular, if \( \alpha = 1 \in \mathbb{S}_{\Gamma_n} \), then

\[
\mathcal{M}_{GL(n, \mathbb{C})_+, \Gamma_n} \left[ E^{-\text{Tr} A}; 1 \right] = \int_{GL(n, \mathbb{C})_+} e^{-\text{tr}(A g)} \det g \, d\nu(g_{\Gamma_n}) = (\det A)^{-1}
\]
Remark that $\Gamma_n(\alpha)$ is not a well-defined object unless one restricts to a compact subgroup of $GL(n, \mathbb{C})_+$. Otherwise, the price of extracting $\det A^{-\alpha}$ from the integral comes with the price of regularizing this possibly singular normalization.

Generalizing further, suppose $\lambda : G^C \to GL(n, \mathbb{C})_+$ but now $\mathfrak{C}^* \equiv L_B(\mathbb{C}^n)$ the space of bounded linear maps on $\mathbb{C}^n$ and $E^{-\lambda} : G^C \to L_B(\mathbb{C}^n)$ by $g \mapsto e^{-a \cdot g}$ with $a \in GL(n, \mathbb{C})_+$ and $\rho(a) = A \in L_B(\mathbb{C}^n)$. The Haar normalized functional Mellin transform yields

$$
\mathcal{M}_{GL(n, \mathbb{C})+} \left[ E^{-\lambda}; \alpha \right] := \int_{GL(n, \mathbb{C})_+} e^{-A \rho(g)} \rho(g^\alpha) \, d\nu(g), \quad \alpha \in S_H
$$

$$
= \int_{GL(n, \mathbb{C})_+} e^{-\rho(g)} \rho((A^{-1} g)^\alpha) \, d\nu(g), \quad \alpha \in S_H
$$

$$
=: A_{H}^{-\alpha}, \quad \alpha \in S_H
$$

which defines the element $A_{H}^{-\alpha} \in M_s(\mathfrak{C}^*)$ for $\alpha \in S_H$.

Unless $a$ is in the center of $GL(n, \mathbb{C})_+$ or we restrict to a subgroup of $GL(n, \mathbb{C})_+$, this can’t be reduced further without explicit computation, i.e. $A_{H}^{-\alpha} \neq (A)^{-\alpha}$ in general. However, various restrictions allow for various degrees of simplification. For example, if $A$ is self-adjoint and $G^C$ is restricted to a one-parameter subgroup generated by $\log a \in \mathfrak{g}l(n, \mathbb{C})$, then more can be done. So let us take the subgroup $\phi_{\log a}(\mathbb{R}) \leq G^C$. Then, since $a \in \phi_{\log a}(\mathbb{R})$,

$$
\mathcal{M}_{\lambda \phi} \left[ E^{-\lambda}; \alpha \right] = \int_{\phi_{\log a}(\mathbb{R})} e^{-A \rho(g)} \rho(g^\alpha) \, D\lambda g, \quad \alpha \in S_T
$$

$$
= \int_{\phi_{\log a}(\mathbb{R})} e^{-\rho(g)} \rho((a^{-1} g)^\alpha) \, D\lambda g, \quad \alpha \in S_T
$$

$$
= A^{-\alpha} \int_{\phi_{\log a}(\mathbb{R})} e^{-\rho(g)} \rho(g^\alpha) \, D\lambda g, \quad \alpha \in S_T
$$

$$
=: A^{-\alpha} \mathcal{N}_T(\alpha), \quad \alpha \in \langle 0, \infty \rangle_T
$$

The second line follows from the left-invariance of the Haar measure, and the third line from the fact that $a$ and $g$ commute and $\rho$ is a representation. Most of the time we will absorb the normalization $\mathcal{N}_T(\alpha)$ into the measure.

## C Commuting Mellin and the exponential

The fundamental relationship between exponentials, determinants, and traces in finite dimensions, i.e. $\exp \tr M = \det \exp M$, also characterizes the functional analogs. In the functional context, consider $F = -\Log A$. Then formally, for $F \in \mathbb{F}_S(G^C)$ and suitable $\lambda$,

$$
\exp \tr A^{-1} \sim \exp \int_{G^C} e^{-A(g)} \rho(g) \, D\lambda g \sim \int_{G^C} e^{-\tr(A(g))} \det g \, D\lambda g = \det A^{-1}. 
$$

This important relation represents a deep connection between Poisson processes and functional Mellin transforms/gamma integrators (as indicated for example in [10]). It is a particular case of the following theorem:
Theorem C.1 Let $E^{-\Lambda} \in \text{Mor}_{C}(\mathcal{G}^{C}, \mathcal{C}^{*})$ by $g \mapsto e^{-(e^{-\Lambda(g)})}$ and $E^{-\text{Tr } E^{-\Lambda}} \in \text{Mor}_{C}(\mathcal{G}^{C}, \mathcal{C}^{*})$ by $g \mapsto e^{-\text{tr } (e^{-\Lambda(g)})}$ with $e^{-\Lambda(g)} := \sum_{n} \frac{(-\Lambda(g))^{n}}{n!}$ trace class. If $\text{Tr } E^{-\Lambda}$ and $E^{-\text{Tr } E^{-\Lambda}}$ are Mellin integrable for a common domain $\alpha \in \mathcal{S}_{\lambda}$, then

$$e^{-\text{tr } A^{-\alpha}_{\lambda}} = \det (E^{-\Lambda})^{-\alpha}, \quad \alpha \in \mathcal{S}_{\lambda}.$$  

Proof: First, for suitable functionals $F$, an immediate consequence of the definitions and the relationship between $(\exp \text{tr})$ and $(\det \exp)$ in $\mathcal{C}^{*}$ is

$$\mathcal{M}_{\lambda} \left[ \det E^{-F}; \alpha \right] = \int_{\mathcal{C}^{*}} \det (e^{-F(g)}g^{\alpha}) \, Dg$$
$$= \int_{\mathcal{C}^{*}} e^{-\text{tr } F(g)} \det g^{\alpha} \, Dg$$
$$= \mathcal{M}_{\lambda} \left[ E^{-\text{Tr } F}; \alpha \right]$$ (C.3)

where the second line follows as soon as $F(g)$ is trace class.

Lemma C.2 Assume $\text{Tr } \tilde{F} \in \text{Mor}_{C}(\mathcal{G}^{C}, \mathcal{C})$ with $\tilde{F}(g)$ analytic. Suppose the functional Mellin transforms of $\text{Tr } \tilde{F}$ and $E^{-\text{Tr } \tilde{F}}$ exist for common $\alpha \in \mathcal{S}_{\lambda}$ for a given $\lambda$. Then

$$e^{-\mathcal{M}_{\lambda}[\text{Tr } \tilde{F}; \alpha]} = \mathcal{M}_{\lambda} \left[ E^{-\text{Tr } \tilde{F}}; \alpha \right], \quad \alpha \in \mathcal{S}_{\lambda}.$$  

Proof: For $\alpha \in \mathcal{S}_{\lambda}$, the Mellin transform of $\text{Tr } \tilde{F}$ exists by assumption so the integral of $\text{Tr } \tilde{F}$ is holomorphic and converges absolutely. Hence, $e^{-\mathcal{M}_{\lambda}[\text{Tr } \tilde{F}; \alpha]}$ represents an absolutely convergent series for $\alpha \in \mathcal{S}_{\lambda}$. Also, recall $E^{-\text{Tr } \tilde{F}} = \sum_{n} \frac{(-1)^{n}}{n!} (\text{Tr } \tilde{F})^{n}$ where $(\cdot)^{n}$ denotes the $n$-fold $*$-product in $\mathcal{F}_{SR}(\mathcal{G}^{C})$. Therefore,

$$e^{-\mathcal{M}_{\lambda}[\text{Tr } \tilde{F}; \alpha]} = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} \mathcal{M}_{\lambda} \left[ \text{Tr } \tilde{F}; \alpha \right]^{n} = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!} \mathcal{M}_{\lambda} \left[ (\text{Tr } \tilde{F})^{n}; \alpha \right]$$
$$= \lim_{N \to \infty} \mathcal{M}_{\lambda} \left[ \sum_{n=0}^{N} \frac{(-1)^{n}}{n!} (\text{Tr } \tilde{F})^{n}; \alpha \right]$$
$$= \mathcal{M}_{\lambda} \left[ E^{-\text{Tr } \tilde{F}}; \alpha \right].$$  

Moving the power of $n$ into the Mellin transform in the first line follows from induction using Lemma 3.15. The second line is obvious from the linearity of functional Mellin. The last line follows from analytic $e^{-\text{tr } F(g)}$ and the assumption that $E^{-\text{Tr } \tilde{F}}$ is Mellin integrable. Together these imply that $\lim_{N \to \infty} \sum_{n=0}^{N} \frac{(-1)^{n}}{n!} (\text{tr } \tilde{F}(g))^{n} \to e^{-\text{tr } F(g)}$ point-wise and $|\sum_{n=0}^{N} \frac{(-1)^{n}}{n!} (\text{tr } \tilde{F}(g))^{n}| \leq e^{-\text{R}(\text{tr } F(g))}$ such that $e^{-\text{R}(\text{tr } F(g))}$ is integrable. Hence, by Lebesgue

24One must prescribe $(A(g))^{n}$. Three examples are $(A(g))^{n} = A^n \rho(g)^n$ and $(A(g))^{n} = \int_{0}^{\rho(g)} A^n(s) \, ds$ and $(A(g))^{n} = \int_{0}^{\rho(g)} \int_{0}^{\rho(g)} \int_{0}^{\rho(g)} A^n(s_1)A^n(s_2) \cdots A^n(s_n) \, ds_1 \, ds_2 \cdots \, ds_n$ where $A^n \in \mathcal{C}^{*}$.

25Of course the nature of $\mathcal{G}^{C}$ and $\mathcal{C}^{*}$ may severely restrict $\alpha \in \mathcal{S}_{\lambda}$. For example, $\alpha = 1$ for non-abelian $\mathcal{G}^{C}$ and non-commutative $\mathcal{C}^{*}$.
Dominated Convergence, \( \lim_{N \to \infty} \| \sum_{n=0}^{N} \frac{(-1)^n}{n!} (\text{tr} \tilde{F}(g))^n - e^{-\text{tr} \tilde{F}(g)} \|_{L^1} \rightarrow 0 \) for all \( \lambda \in \Lambda \).

Therefore, \( \lim_{N \to \infty} \| \sum_{n=0}^{N} \frac{(-1)^n}{n!} (\text{tr} \tilde{F})^n - e^{-\text{tr} \tilde{F}} \|_{F} \rightarrow 0 \). We stress the lemma holds only for \( \alpha \in S_{\lambda} \) and it can certainly happen that \( S_{\lambda} = \emptyset \). \( \square \)

To finish the proof, put \( \tilde{F} \equiv E^{-A} \) in the lemma to get;

\[
\mathcal{M}_{\lambda} \left[ E^{-\text{tr} E^{-A}}; \alpha \right] = \int_{G^C} e^{-\text{tr} \left( e^{-A(g)} \right)} \det g^\alpha \mathcal{D} g = \left( \Det \left( E^{-A} \right) \right)^{-\alpha}_\lambda = \det \left( E^{-A} \right)^{-\alpha}_\lambda \quad (\text{C.6})
\]

and

\[
\mathcal{M}_{\lambda} \left[ \text{tr} E^{-A}; \alpha \right] = \int_{G^C} \text{tr} \left( e^{-A(g)} g^\alpha \right) \mathcal{D} g = \left( \text{tr} A \right)^{-\alpha}_\lambda = \text{tr} A^{-\alpha}. \quad (\text{C.7})
\]

\( \square \)

**Corollary C.3** Under the conditions of Lemma [C.2], replace \( \text{tr} \tilde{F} \) with \( V \in \text{Mor}_C(G^C, \mathcal{C}^*) \) where now \( \mathcal{C}^* = L_H(H) \) is the (commutative) algebra of Hermitian linear operators on some Hilbert space. Then Lemma 3.13 and Theorem 3.20 imply

\[
\mathcal{M}_{\lambda} [E^{-V}; \alpha] = e^{-\mathcal{M}_\lambda[V, \alpha]}, \quad \alpha \in S_{\lambda}. \quad (\text{C.8})
\]

In particular, if \( A(g) = Ag \) is self-adjoint, the corollary implies

\[
\left( E^{-A} \right)^{-\alpha}_\lambda = \mathcal{M}_{\lambda}[E^{-E^{-A}}; \alpha] = e^{-\mathcal{M}_\lambda[E^{-E^{-A}}; \alpha]} = e^{-A^{-\alpha}} \quad (\text{C.9})
\]

and then Proposition 4.13 leads to \( e^{-\text{tr} A^{-\alpha}} = \det e^{-A^{-\alpha}} \) when \( \alpha \in S_{\lambda} \).

**Example C.4** Continuing with our prevailing example; consider the one-parameter sub-group \( \phi_a(\mathbb{R}) \), localization \( \lambda_{\mathbb{R}^+} : \phi_a(\mathbb{R}) \rightarrow \mathbb{R}^\cup \mathbb{R}^- \), and \( A(g) = Ag \) with \( A \in \mathcal{C}^* \) self-adjoint. Let \( M_{\lambda} = -A_{\lambda}^{-1} \) and choose a suitable normalization so that \( M_{\lambda} \rightarrow M = \Lambda^{-1} \). Then

\[
e^{-c_{M^{-1}}(1)} \left( \lambda_{\mathbb{R}^+} \right) \rightarrow e^{-\text{tr} \Lambda} \left( \lambda_{\mathbb{R}^+} \right) \rightarrow e^{-\mathcal{M}_\lambda[\text{tr} E^{-\Lambda}]; 1} = \det e^{-\mathcal{M}_\lambda \left( \lambda_{\mathbb{R}^+} \right) \rightarrow \det e^{-M}. \quad (\text{C.10})
\]

Similarly, take \( \log M_{\lambda}^{-1} = A_{\lambda}^{-1} \), choose a suitable normalization so that \( M_{\lambda} \rightarrow M \) with \( M \) positive definite, and use \( (4.20) \) to get (if the limit exists)

\[
e^{-c_{M^{-1}}(0)} \left( \lambda_{\mathbb{R}^+} \right) \rightarrow e^{-\mathcal{M}_\lambda[\text{tr} E^{-\Lambda}]; 0} = \det e^{-\log M_{\lambda}^{-1} \left( \lambda_{\mathbb{R}^+} \right) \rightarrow \det M}. \quad (\text{C.11})
\]

Evidently the theorem reproduces the standard expressions for this special case [29] [30]. Moreover, the **functional** relation \( e^{-\mathcal{M}_\lambda[\text{tr} E^{-\Lambda}]; 0} = \det e^{-\log M_{\lambda}^{-1}} \) and the subsequent ‘localized’ **function** relation \( \exp \text{tr}(\log M) = \det M \) expressed in (C.11) are consistent with the conclusion in Example 6.3 that the log of a Mellin transform can be represented as the ‘topological localization’ of a functional Log.

We emphasize that Theorem [C.1] and the above properties should be understood as a family of statements at the functional level that may be explicitly realized only for appropriate choices of \( \lambda \) leading to non-empty \( S_{\lambda} \).
D Relation to crossed products

The ingredients necessary to define crossed products of C*-algebras\cite{21} are: i) a “dynamical system” \((A, G, \varepsilon)\) where \(A\) is a C*-algebra, \(G\) is a locally compact group, and \(\varepsilon : G \to Aut(A)\) is a continuous homomorphism; ii) some Hilbert space \(\mathcal{H}\); iii) an algebra representation \(\varpi : A \to L_B(\mathcal{H})\); and iv) a unitary, group representation \(U : G \to U(\mathcal{H})\). The two representations are required to satisfy the “covariance condition”

\[
\varpi(\varepsilon_g(a)) = U_g \varpi(a) U_g^*, \quad g \in G, \ a \in A.
\] (D.1)

With these objects, a \(\ast\)-representation on \(\mathcal{H}\) of \(C_c(G, A)\) (continuous compact morphisms \(f : G \to A\)) is supplied by the integral

\[
\varpi \rtimes U(f) := \int_G \varpi(f(g)) U_g \, d\mu(g)
\] (D.2)

where \(f \in C_c(G, A)\) and \(\mu\) is a Haar measure on \(G\).

A product and involution are introduced on \(C_c(G, A)\) according to

\[
(f_1 * f_2)(g) := \int_G f_1(\tilde{g}) \varepsilon_{\tilde{g}}(f_2(\tilde{g}^{-1} g)) \, d\mu(\tilde{g})
\] (D.3)

and

\[
f^*(g) := \Delta(g^{-1}) \varepsilon_g(f(g^{-1})^*)
\] (D.4)

where \(\Delta\) is the modular function on \(G\). Completion of \(C_c(G, A)\) with respect to the norm \(\|f\| := \sup \|\varpi \rtimes U(f)\|\) is a C*-algebra called the crossed product denoted by \(A \rtimes_\varepsilon G\).

The crucial property of this construction is a one-to-one correspondence between non-degenerate covariant representations associated with \((\varpi, U)\) and non-degenerate representations of \(A \rtimes_\varepsilon G\) which preserves direct sums, irreducibility, and equivalence. So the C*-algebra \(A \rtimes_\varepsilon G\) can be used to model the C*-algebra encoded in the system \((A, G, \varepsilon)\) endowed with a covariant representation \((\varpi, U)\). We recognize the covariant condition as an algebra automorphism by a group element; which, in particular, for the evolution operator in quantum mechanics becomes the integrated Heisenberg equation.

Let’s compare with functional Mellin. Suppose \(\lambda : G^\mathbb{C} \to G^\lambda\). Identify \(\pi \circ \rho \equiv U\) (with suitable restrictions if necessary) and choose \(D_\lambda g \equiv d\mu(g)\) with \(g \in G^\lambda\), then

\[
\pi(\mathcal{M}_\lambda[F, 1]) = \int_{G^\lambda} \pi(f(g)) U(g) \, d\mu(g)
\] (D.5)

where \(\pi : \mathfrak{c}^* \to L_B(\mathcal{H})\). As soon as \(\pi(f(g))\) is Mellin integrable w.r.t. \(G^\lambda\), this integral and the integral in (D.2) represent the same object in \(L_B(\mathcal{H})\) iff \(\varpi \circ f \equiv \pi \circ f\). Keep in mind that the nature of \(f \in C_C(G, A)\) versus \(f \in C_C(G^\lambda, \mathfrak{c}^*)\) is quite dependent on the nature of \(A\) versus \(\mathfrak{c}^*\): If they are both simultaneously commutative or non-commutative, then \(f\) and \(f\) at least have the chance of representing the same object if \(A\) and \(\mathfrak{c}^*\) are isomorphic. Otherwise, they are distinctly different. Mathematically, we can always choose \(A \equiv \mathfrak{c}^*\) and \(\varpi \equiv \pi\). In this case, the difference between crossed products and functional Mellin is that
A \rtimes_{\varepsilon} G is the $C^*$-algebra of $f \in C_C(G,A)$ satisfying the covariance condition \cite{D.1} while $F_S(G^C)$ is the $C^*$-algebra of equivariant $f \in C_C(G_\lambda,A)$.

For application to quantum physics, the pivotal point in this difference comes down to $\varepsilon : G \to \text{Aut}(A)$ and dynamics. Suppose $A \equiv \mathcal{C}^*$ is commutative. By Gelfand duality, there is some topological space $X$ such that $A \equiv C_0(X)$ (the algebra of complex valued continuous morphisms on $X$ vanishing at infinity). Non-trivial $\varepsilon$ reflects a basic assumption about the dynamical system; that $G$ acts on $X$ and this is accounted for by $\varepsilon_h(f(g))(x) = f(g)(h^{-1} \cdot x)$ for $x \in X$. But then the covariance condition is required to encode dynamics through the adjoint action on $L_B(H)$. Insofar as crossed-product quantization (virtually always) starts with a classical “dynamical system” $(C_0(X),G,\varepsilon)$ with covariant representation $(\varpi,U)$, the crossed product $A \rtimes_{\varepsilon} G$ realizes a concrete quantization of the commutative algebra $C_0(X)$.

On the other hand, for functional Mellin the group is already contained in $\mathfrak{A}$ by construction, and so it acts by inner automorphisms which automatically incorporates the covariance condition. Moreover, if $\mathcal{C}^* \equiv A$ is assumed to act on some $X$, then by equivariance $f(gh)(x) = f(g)(\rho(h)(x) = f(g)(h^{-1}\cdot x)$. However, the involution and product that are defined for functional Mellin do not depend on $\varepsilon$ — unlike $A \rtimes_{\varepsilon} G$. Evidently, even though $F_S(G^C)$ is a $C^*$-algebra it is not isomorphic to $A \rtimes_{\varepsilon} G$ in general, and it would be difficult to attach a classical interpretation to the functions $f \in C_C(G_\lambda,A)$ in relation to some dynamical system.

Now suppose $A \equiv \mathcal{C}^*$ is non-commutative and $G$ is its group of units for some dynamical system. Then $G$ acts on $A$ by inner automorphisms which means the covariance condition is automatic and $\varepsilon$ is unneeded. Setting $\varepsilon \equiv Id$ brings the product and involution of crossed products $A \rtimes_{Id} G$ into agreement with functional Mellin for $\alpha = 1$. But then, the only way (it seems) to save the non-commutative $C^*$-algebra structure of $A \rtimes_{Id} G$ is to insist that the morphisms $f$ be equivariant. In this situation $F_S(G^C) \cong A \rtimes_{Id} G$ and representations furnished by $\pi(M_\lambda[F,1])$ are in one-to-one relation to $A \rtimes_{Id} G$ and therefore in one-to-one relation to the system $(A,G,Id)$ with covariant representation $(\varpi \equiv \pi,U)$ and equivariant $f$.

But note this dynamical system in not classical — until expectations are taken. Whereas the previous paragraph described the quantization process classical $\to$ quantum, this paragraph describes quantum $\to$ classical.

But for quantum $\to$ classical how can we know anything about the functions $f \equiv f$ without $C_0(X)$? Happily, spectral theory allows to represent $(\pi \circ f)(g)$ in terms of an operator valued function $\hat{f}(\rho(g))$ and Mellin integrators supply the resolvent of $\rho(g)$. If we don’t venture outside of $A$ to find evolution operators and we use functional calculus to represent $\pi(f(g)) \equiv \hat{f}(\rho(g))$, then functional Mellin and a choice of $G^C$ fully determine a quantum system. That is, once we settle on $G^C$ and find relevant representations and their furnishing Hilbert spaces, functional Mellin defines a $C^*$-algebra $F_S(G^C)$ that contains quantum dynamics, and Mellin integrators furnish representations of this algebra in $L_B(H)$.

To further highlight the similarity between crossed products and functional Mellin, extend the integrated form of $(\varpi,U)$ to $G^C$ according to

$$\varpi \rtimes U^{(\alpha)}(f) := \int_{G^C} \varpi(f(g))U_{g^\alpha} \, d\mu(g). \quad (D.6)$$
Likewise, extend the involution \( f^*(g^{1+\alpha}) := \Delta(g^{-1})\varepsilon_{g^\alpha}(f(g^{-(1+\alpha)})^*) \) and define the \( * \)-product

\[
(f_1 * f_2)(g^\alpha) := \int_{G_c} f_1(\tilde{g})\varepsilon_{\tilde{g}}(f_2(\tilde{g}^{-1}g^\alpha))\,d\mu(\tilde{g}). \tag{D.7}
\]

Then we claim \( A \rtimes \varepsilon G^c \) (after completion w.r.t. to a suitable norm) is a \( C^* \)-algebra and \( \varpi \rtimes U^{(\alpha)} \) is a \( * \)-homomorphism because:

- \( (f_1 * f_2) * f_3 = f_1(f_2 * f_3) \tag{D.8} \)
  follows immediately from the star product using the invariance of the Haar measure.

- \( (f^*(g^{1+\alpha}))^* = (\Delta(g^{-1})\varepsilon_{g^\alpha}(f(g^{-(1+\alpha)}))^*) = f(g^{1+\alpha}) \tag{D.9} \)
  where the first equality follows from the covariance conditions and the second from the invariance of the Haar measure which implies \( f(g^{(1+\alpha)}) = \Delta(g^{-1})\varepsilon_{g^\alpha}(f(g^{-(1+\alpha)})) \).

\[
\|f^*\|_\alpha := \int_{G_c} \|f^*(g^{1+\alpha})\|\,d\mu(g)
= \int_{G_c} \|\Delta(g^{-1})\varepsilon_{g^\alpha}(f(g^{-(1+\alpha)})^*)\|\,d\mu(g)
= \int_{G_c} \|\Delta(g^{-1})^* (\varepsilon_{g^\alpha}(f(g^{-(1+\alpha)}))^*)^*\|\,d\mu(g)
= \int_{G_c} \|\Delta(g^{-1})\varepsilon_{g^\alpha}(f(g^{-(1+\alpha)}))\|\,d\mu(g)
= \int_{G_c} \|f(g^{-(1+\alpha)})\|\,d\mu(g)
= \|f\|_\alpha. \tag{D.10}
\]

\[
(f_1^* * f_2^*) (g^{1+\alpha}) = \int_{G_c} f_1^*(\tilde{g})\varepsilon_{\tilde{g}}(f_2^*(\tilde{g}^{-1}g^{1+\alpha}))\,d\mu(\tilde{g})
= \int_{G_c} \Delta(\tilde{g}^{-1})\varepsilon_{\tilde{g}}(f_1(\tilde{g}^{-1})^*)\varepsilon_{\tilde{g}}(\Delta(g^{-1})\varepsilon_{\tilde{g}}(f_2(\tilde{g}^{-1}g^{1+\alpha})^*))\,d\mu(\tilde{g})
= \Delta(g^{-1})\int_{G_c} \varepsilon_{\tilde{g}}(f_1(\tilde{g}^{-1})^*)\varepsilon_{\tilde{g}}(f_2(\tilde{g}^{-1}g^{1+\alpha})^*)\,d\mu(\tilde{g})
= \Delta(g^{-1})\int_{G_c} \varepsilon_{\tilde{g}g^{1+\alpha}}(f_1(\tilde{g}^{-1}g^{-(1+\alpha)})^*)\varepsilon_{g^{1+\alpha}}(f_2(\tilde{g}^*)^*)\,d\mu(\tilde{g})
= \Delta(g^{-1})\varepsilon_{g^{1+\alpha}}\int_{G_c} (f_2(\tilde{g})f_1(\tilde{g}^{-1}g^{-(1+\alpha)}))^*\,d\mu(\tilde{g})
= \Delta(g^{-1})\varepsilon_{g^{1+\alpha}} (f_2 * f_1(\varepsilon_{g^{1+\alpha}})^* = (f_2 * f_1)^*(g^{1+\alpha}) \tag{D.11}
\]

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\[ \varpi \times U^{(\alpha)}(f)^* = \int_{G^c} (\varpi(f(g))U_{g^\alpha})^* \, d\mu(g) \]
\[ = \int_{G^c} U_{g^{-\alpha}} \varpi(f(g)^*) \, d\mu(g) \]
\[ = \int_{G^c} U_{g^\alpha} \varpi(f(g^{-1})^*) \Delta(g^{-1}) \, d\mu(g) \]
\[ = \int_{G^c} \varpi(\varepsilon_g(f(g^{-1})^*) \Delta(g^{-1})) \; U_{g^\alpha} \, d\mu(g) \]
\[ = \int_{G^c} \varpi(f^*(g^{-1})) \; U_{g^\alpha} \, d\mu(g) \]
\[ = \varpi \times U^{(\alpha)}(f^*) \quad (D.12) \]

\[ \varpi \times U^{(\alpha)}(f_1 \ast f_2) = \int_{G^c \times G^c} \varpi(f_1(g))\varepsilon_g(f_2(g^{-1}\tilde{g})) \; U_{\tilde{g}} \; d\mu(g, \tilde{g}) \]
\[ = \int_{G^c \times G^c} \varpi(f_1(g))U_g \varpi(f_2(g^{-1}\tilde{g})) \; U_{g^{-1}\tilde{g}} \; d\mu(g, \tilde{g}) \]
\[ = \int_{G^c \times G^c} \varpi(f_1(g))U_g \varpi(f_2(\tilde{g})) \; U_{\tilde{g}} \; d\mu(g, \tilde{g}) \]
\[ = \varpi \times U^{(\alpha)}(f_1) \cdot \varpi \times U^{(\alpha)}(f_2) \quad (D.13) \]

where the last equality follows from Fubini.

E  Some comments on \(*\) v.s. \(\ast\)

Restrict to abelian \(G^c\) and recall lemmas 3.15 and 3.16. Allowing for complex \(\alpha\) gives
\[ \mathcal{M}_\lambda [(F_1 \ast F_2) ; \alpha] = \mathcal{M}_\lambda [F_1 ; \alpha^\ast] \mathcal{M}_\lambda [F_2 ; \alpha] \quad (E.1) \]
\[ \mathcal{M}_\lambda [(F_1 \ast F_2) ; \alpha] = \mathcal{M}_\lambda [F_1 ; \alpha] \mathcal{M}_\lambda [F_2^\ast ; 1 - \alpha^\ast] . \quad (E.2) \]

Accordingly, for the two algebras distinguished by their \(*\) v.s. \(\ast\) product, at \(\alpha = 1/2\) we get norm equality \(\|\mathcal{M}_\lambda [(\Psi \ast \Psi) ; 1/2]\| = \|\mathcal{M}_\lambda [(\Psi \ast \Psi) ; 1/2]\|\) and simultaneous representations for both commutative and non-commutative \(C^*\). But generically we get neither norm equality nor simultaneous representations.

However, if there exists a class of holomorphic \(\psi_\lambda(\alpha) := \mathcal{M}_\lambda [\Psi ; \alpha] \in C^*\) that satisfy \(\psi_\lambda(\alpha)^\ast = \psi_\lambda(\alpha^\ast)\) and \(\psi_\lambda(\alpha)\psi_\lambda(\alpha)^\ast = \psi_\lambda(\alpha)^\ast \psi_\lambda(\alpha)\). And if \(\psi_\lambda(\alpha)\) further satisfies a functional equation of the form \(\psi_\lambda(\alpha) = \varepsilon_\lambda(\alpha)\psi_\lambda(1 - \alpha)\) with \(\varepsilon_\lambda(\alpha) \in \mathbb{C}\) such that \(|\varepsilon_\lambda(\alpha)| = 1\), then (for this class) the \(\ast\)-convolution and \(\ast\)-convolution yield norm equivalence and simultaneous representations up to a phase for all \(\alpha \in \mathbb{S}\)!
Lemma E.1 Suppose $G^C$ is abelian and there exist holomorphic $\psi_\alpha(\alpha) := \mathcal{M}_\lambda[\Psi; \alpha] \in \mathcal{C}^*$ where; i) $\psi_\alpha(\alpha^*)^* = \psi_\alpha(\alpha^*)$, ii) $\psi_\lambda(\alpha)\psi_\lambda(\alpha)^* = \psi_\lambda(\alpha^*)^*\psi_\lambda(\alpha)$, and iii) $\psi_\lambda(\alpha) = \varepsilon_\lambda(\alpha)\psi_\lambda(1-\alpha)$ with $\varepsilon_\lambda(\alpha) \in \mathbb{C}$ and $|\varepsilon_\lambda(\alpha)| = 1$. Then

$$\|\mathcal{M}_\lambda[(\Psi \ast \Psi); \alpha]\| = \|\mathcal{M}_\lambda[(\Psi \ast \Psi); \alpha]\| = \|\psi_\lambda(\alpha)^2\| \quad \forall \alpha \in \mathbb{S}.$$  \hspace{2cm} (E.3)

To gain some insight into this observation, recall the nature of the $\ast$ and $\star$ convolutions in the context of standard Mellin transforms: The former is integrated along $T$ of continuous pointed maps $t$ and the latter along $\tau$ to functional Mellin for $\tau$. It results from considering the space $\varepsilon$ with $\varepsilon_\lambda(\alpha) \in \mathbb{C}$ and $\varepsilon_\lambda(\alpha^*)^*\varepsilon_\lambda(\alpha)$ is the multiplicative group of complex numbers. For a suitable topology, $\varepsilon$ is an abelian topological group under point-wise multiplication. One can attach a counting interpretation to functional Mellin for $\tau(t)$ along the $\mathbb{R}_+$ direction and an evolution interpretation along the $i\mathbb{R}_+$ directions. To explicate functional Mellin, take $\mathcal{C}^*$ and impose periodicity on $\tau$ so that $G^C \ast \mathcal{C}$ where $\mathcal{C}$ is some closed contour in $\mathbb{C}^\times$. The gamma functional integral under the localization $\lambda_C : T_0 \rightarrow \mathcal{C}$ reduces:

$$\int_{T_0} e^{i(\tau', \tau)} \mathcal{D}_\lambda(\alpha_R, \beta_R) \tau \int_\mathcal{C} e^{(it')t} t^{\alpha_R} d\nu(t_{\mathcal{C}}) \quad \alpha_R \in \mathbb{R} \cap \mathbb{S}$$ \hspace{2cm} (E.4)

where $(it' - \omega)t = (it' - \omega), \lambda_C(\tau) \in \mathbb{C}^\times = (\lambda_C(t), \tau)_{T_0} = (it' - \beta'), \tau)_{T_0} \in \mathbb{C}_+$ and $\alpha_R$ must be real according to Lemma 3.16. Under the $SL(2, \mathbb{Z})$ transformation $t \mapsto t/(1-t)$

$$\int_\mathcal{C} e^{(it'-\omega)t} t^{\alpha_R} d\nu(t_{\mathcal{C}}) \rightarrow \frac{\pi \csc(\pi \alpha_R)}{2\pi i \Gamma(\alpha_R)} \int_\mathcal{C} \frac{e^{(it'-\omega)t}}{(1-t)^{\alpha_R+1}} t^{\alpha_R-1} dt = \frac{\pi \csc(\pi \alpha_R)}{2\pi i \Gamma(\alpha_R)} \int_\mathcal{C} \sum_{n=0}^\infty t^n L_n^{(\alpha_R)}(it' - \omega) t^{\alpha_R-1} dt = \frac{\pi \csc(\pi \alpha_R)}{2\pi i \Gamma(\alpha_R)} \int_\mathcal{C} \sum_{n=0}^\infty \frac{(it')^n}{(1+it')n+\alpha_R+1} L_n^{(\alpha_R)}(t)e^{-\omega t} t^{\alpha_R-1} dt \hspace{2cm} (E.5)$$

where $L_n^{(\alpha_R)}(\cdot)$ are the generalized Laguerre polynomials on $\mathbb{C}$. In physics applications, the exponential integrand in the left integral corresponds to an evolution operator for appropriate choice of $t'$.

Not surprisingly, Laguerre polynomials are germane in this context, so define algebraic elements $A_n^{(s)} \in \mathbf{F}_{S^R}(G^C)$ where $A_n^{(s)} : G^C \rightarrow \mathcal{C}^\ast$ by $g \mapsto e^{-\rho(g)}L_n^{(s)}(\rho(g))$ with $L_n^{(s)}(\rho(g))$ being operator-valued generalized Laguerre polynomials such that (for unitary $\rho$ and $-1 < s < \mathbb{R}$)

$$\mathcal{M}_\lambda[\text{Det} (A_n^{(s)}); \alpha] = \int_{G^C} e^{-\text{tr}(g)} \text{det}(L_n^{(s)}(g)) \text{ det} g^\alpha \mathcal{D}_\lambda g := \binom{s - \alpha + n}{n}. \hspace{2cm} (E.6)$$

Parenthetically, since $\rho$ is assumed unitary, this definition satisfies

$$\mathcal{M}_\lambda[\text{Det} (A_n^{(s)}); \alpha] = \int_{G^C} e^{-\text{tr}(g)} \ast \text{det}(L_n^{(s)}(g)) \ast \text{ det} g^\alpha \mathcal{D}_\lambda g = \int_{G^C} e^{-\text{tr}(g^{-1})} \text{det}(L_n^{(s)}(g^{-1})) \text{ det} g^{-\alpha} \mathcal{D}_\lambda g. \hspace{2cm} (E.7)$$

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For $G^C \cong T_0$ under $\lambda_\mathbb{R}^+ : T_0 \to \mathbb{R}^+$ and choosing the gamma Haar measure, this reduces to

$$\mathcal{M}_{\mathbb{R}^+, \Gamma} \left[ \det (A_n^{(s)}; \alpha_{\mathbb{R}^+}) \right] = \frac{1}{\Gamma(\alpha_{\mathbb{R}^+})} \int_0^\infty e^{-t} L_n^{(s)}(t) t^{\alpha_{\mathbb{R}^+} - 1} \, dt$$

$$= \left( \frac{1}{\Gamma(\alpha_{\mathbb{R}^+})} \int_0^\infty e^{-\frac{1}{2} t} L_n^{(s)}(t) t^{-(\alpha_{\mathbb{R}^+} - 1)} \, dt \right)^*$$

$$= \left( \frac{1}{n} \lambda^{\alpha_{\mathbb{R}^+} + n} \right) \quad 0 < \alpha_{\mathbb{R}^+}. \quad (E.8)$$

This motivates to introduce a Laguerre-type functional integral defined by

$$\mathcal{L}_\lambda [F; \alpha, n, s] := \int_{G^C} F(g) \, L_n^{(s)}(g) g^\alpha \, d\lambda g := \int_{G^C} F(g) \, D\lambda l_n^{(s)}(g)$$ \quad (E.9)

where $D\lambda l_n^{(s)}(g)$ defines a family of Laguerre integrators. We can add $D\lambda l_n^{(s)}(g)$ to the list of non-Gaussian integrators introduced in [10]. Remark that Hermite polynomials (relevant for normal ordered operators) are a special class of Laguerre polynomials on $(G^C \times G^C)_\Delta$.

For suitable $F$, one can series expand $F = \sum_n a_n^{(s)} A_n^{(s)}$ with the help of Laguerre integrators. Explicitly,

$$\mathcal{M}_\lambda[F; \alpha] = \sum_{n=0}^\infty a_n^{(s)} \mathcal{L}_\lambda \left[ E^{-\Id}; \alpha, n, s \right]$$ \quad (E.10)

where

$$a_n^{(s)} = \frac{1}{c_{m,n}^{(s)}} \int_{G^C} F(g) L_m^{(s)}(g) g^{s+1} D\lambda g = \frac{1}{c_{m,n}^{(s)}} \mathcal{L}_\lambda [F; s + 1, m, s]$$ \quad (E.11)

and the constants $c_{m,n}^{(s)} = \mathcal{L}_\lambda \left[ A_m^{(s)}; s + 1, n, s \right]$. Observe that $\mathcal{L}_\lambda \left[ E^{-\Id}; \alpha, n, s \right] = \mathcal{M}_\lambda \left[ A_n^{(s)}; \alpha \right]$ and $\mathcal{L}_\lambda \left[ E^{-\Id}; \alpha, 0, s \right] = \mathcal{M}_\lambda \left[ E^{-\Id}; \alpha \right]$. For example, with the localization $\lambda_\mathbb{R}^+ : T_0 \to \mathbb{R}^+$ and $\rho(g) = g Id$ where $Id$ is the identity in $\mathfrak{e}^*$, we have $\mathcal{L}_{\mathbb{R}^+, \Gamma} \left[ E^{-\Id}; \alpha_{\mathbb{R}^+}, n, s \right] = \left( \frac{s - \alpha_{\mathbb{R}^+} + n}{n} \right) Id$ and

$$c_{m,n}^{(s)} = \mathcal{L}_{\mathbb{R}^+, \Gamma} \left[ A_m^{(s)}; s + 1, n, s \right] = \int_0^\infty e^{-g} L_m^{(s)}(g) L_n^{(s)}(g) g^{s+1} \, d\nu(g) = \delta_{m,n} \left( \frac{n + s}{n} \right) Id. \quad (E.12)$$

So calculating $\mathcal{M}_\lambda[F; \alpha]$ boils down to calculating $\mathcal{L}_\lambda \left[ F; s + 1, n, s \right]$ (for suitable $F(g)$). In particular, suppose we are given $F(g) = \sum_{m=0}^\infty F_m^{(s)} A_m^{(s)}(g)$ with $F_m^{(s)} \in \mathfrak{e}^*$. Then

$$\mathcal{L}_{\mathbb{R}^+, \Gamma} \left[ F; s + 1, n, s \right] = \sum_{m=0}^\infty F_m^{(s)} \int_0^\infty e^{-g} L_m^{(s)}(g) L_n^{(s)}(g) g^{s+1} \, d\nu(g)$$

$$= \sum_{m=0}^\infty F_m^{(s)} c_{m,n}^{(s)} \left( \frac{n + s}{n} \right) F_n^{(s)}. \quad (E.13)$$

With this diversion in mind, return to the lemma. Following [37, 38, 39], define

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**Recall example 3.13**: Evidently, to calculate the elementary kernel of the Laplacian on $\mathbb{R}^n$ one can just as well use the effective action $\tilde{S}(g) := S(g^{-1}) = \pi g |x_{a'} - x_a|^2 - n/2 \log g$. 

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**Definition E.2** For \(-1 < s \in \mathbb{R}\) and \(\Psi^{(s)} \in \mathbf{F}_{S_{\mathcal{R}}}(G^\mathbb{C})\) such that \(\Psi^{(s)}(g) = \rho(g)^{s/2}e^{-\rho(g)/2}\),

\[
\mathcal{L}_\lambda \left[\Psi^{(s)}; \alpha, n, s\right] := \int_{G^\mathbb{C}} g^{s/2}e^{-s/2} L_n^{(s)}(g) g^{\alpha} D_\lambda g . \tag{E.14}
\]

Suppose again \(\lambda_{\mathbb{R}^+} : T_0 \to \mathbb{R}_+\) and choose Haar measure \(\nu(g_{\psi}) := 2^{-(\alpha+s/2+n)} \nu(g)/\Gamma(\alpha+s/2)\). This yields a family of the \(\psi_{\lambda}(\alpha)\) of Lemma [E.1]

\[
\psi_n^{(s)}(\alpha) := \mathcal{L}_{\mathbb{R}_+, \psi} \left[\Psi^{(s)}; \alpha, n, s\right] = \frac{1}{2^{\alpha+s/2+n} \Gamma(\alpha+s/2)} \int_0^\infty t^{s/2} e^{-t/2} L_n^{(s)}(t) t^{\alpha-1} dt \quad 0 < \Re(\alpha+s/2)
\]

\[
= \frac{1}{2^{\alpha+s/2+n} \Gamma(\alpha+s/2)} \mathcal{M}_{\mathbb{R}_+, \Gamma}[\Psi^{(s)}; \alpha, s/2] := \mathcal{M}_{\mathbb{R}_+, \psi}[\Psi^{(s)}; \alpha, s/2] \tag{E.15}
\]

where we have defined \(\Psi^{(s)}(g) := e^{-\rho(g)/2} L_n^{(s)}(\rho(g)) = e^{\rho(g)/2} A_n^{(s)}(\rho(g))\).

Using the series representation of Laguerre, this can be expressed as [37, 39]

\[
\psi_n^{(s)}(\alpha) = (-2)^{-n}(s+1)^n \binom{2}{n} F_1(-n, \alpha+s/2; s+1; 2) , \tag{E.16}
\]

and thereafter analytically continued to all \(\alpha \in \mathbb{C}\). Clearly, \(\psi_n^{(s)}(\alpha)^* = \psi_n^{(s)}(\alpha)^*\) and trivially \(\psi_n^{(s)}(\alpha) \psi_n^{(s)}(\alpha)^* = \psi_n^{(s)}(\alpha)^* \psi_n^{(s)}(\alpha)\). Moreover,

**Theorem E.3**

- \(\psi_n^{(s)}(\alpha) = (-1)^n \psi_n^{(s)}(1 - \alpha)\) \(\forall \alpha \in \mathbb{C}\)
- \(\psi_n^{(s)}(\alpha) = (\alpha - 1/2) \psi_n^{(s)}(\alpha) + \frac{n(\alpha+n)}{4} \psi_{n-1}^{(s)}(\alpha)\)
- All zeros \(\alpha_0\) of \(\psi_n^{(s)}(\alpha)\) are simple and lie on the critical line \(\Re(\alpha) = 1/2\).

**proof**: The first two points follow readily from (E.16) and the identities (respectively)

\[
\binom{2}{n} F_1(a, b; c; z) = \frac{(1-z)^{-a} \binom{2}{n} F_1(a, c-b; c; z)}{\binom{2}{a} F_1(a, b; c; z)}
\]

\[
\binom{2}{n} F_1(a-1, b; c; z) = \frac{(2a-c-z(a-b)) \binom{2}{a} F_1(a, b; c; z)}{(a-c)} + \frac{a(z-1)}{(a-c)} \binom{2}{a} F_1(a+1, b; c; z) .
\]

For proof of the third point we refer to [38, th. 4] (see also [37, th. 4]). Note that, in the course of the proof, it is shown that \(\psi_n^{(s)}(1/2 + i\sigma)\) are orthogonal on \(\mathbb{R}\) with respect to the measure \(2^{s+1} \Gamma(1/2 + i\sigma + s/2) d\sigma\) where \(\sigma \in \mathbb{R}\). \(\Box\)

We have learned that there exist elements \(\Psi_n^{(s)} \in \mathbf{F}_{S_{\mathcal{R}}}(T_0)\) such that functional Mellin of both \(*\) and \(\ast\) products act by multiplication up to a phase; also

\[
\mathcal{M}_\psi \left[\left(\Psi_n^{(s)} \Psi_m^{(r)}\right) ; \alpha + s/2\right] = (-1)^n \mathcal{M}_\psi \left[\left(\Psi_m^{(s)} \Psi_n^{(r)}\right) ; \alpha + s/2\right] , \tag{E.17}
\]

and

\[
\left\|\mathcal{M}_\psi \left[\left(\Psi_n^{(s)} \Psi_n^{(s)}\right) ; \alpha + s/2\right]\right\| = \left\|\mathcal{M}_\psi \left[\left(\Psi_n^{(s)} \Psi_n^{(s)}\right) ; \alpha + s/2\right]\right\| = \left\|\psi_n^{(s)}(\alpha)^2\right\| . \tag{E.18}
\]

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Further, $\psi_n^{(s)}(\alpha_0) = 0$ iff $\alpha_0 = 1/2 + i\sigma_0$ with $\sigma_0 \in \mathbb{R}$ and $-1 < s$. Evidently, functionals of $\tau \in T_0$ degrees of freedom can be series expanded along the critical line in terms of the class of functions $\psi_n^{(s)}(\alpha) = \mathcal{M}_\psi[(\Psi_n^{(s)})]; \alpha + s/2]$. In light of this, it is curious and perhaps significant that (E.3) holds for $\psi_n^{(s)}(\alpha)$. We don’t fully understand it’s implications.

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$^{27}$However, observe that in a quantum physics setting $\text{Log } \Psi^{(s)}$ corresponds to a complex effective action, and the counting/evolution along a contour $\Gamma$ is associated with entropy/action. The fact that functional Mellin of both $\ast$ and $\ast$ products yield (projective)representations supports the idea that, in this context, $\tau(t)$ represents a complex evolution parameter along a contour in $\mathbb{C}^\times$. And the critical line $\Re(\alpha) = 1/2$ seems to indicate an isentropic process characterized by a duality or equivalence between the notions of real-valued action and information entropy. From this perspective, $\psi_n^{(s)}(\alpha)$ has the earmarks of a Loschmidt amplitude, which would suggest the isentropic process represents non-equilibrium, unitary evolution (see e.g. [10]).
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