EULER-MAHONIAN DISTRIBUTIONS OF TYPE $B_n$

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Abstract. Adin, Brenti, and Roichman introduced the pairs of statistics ($\text{ndes}, \text{nmaj}$) and ($\text{fdes}, \text{fmaj}$). They showed that these pairs are equidistributed over the hyperoctahedral group $B_n$, and can be considered “Euler-Mahonian” in that they generalize the Carlitz identity. Further, they asked whether there exists a bijective proof of the equidistribution of their statistics. We give such a bijection, along with a new proof of the generalized Carlitz identity.

1. Introduction

Statistics such as inversion number, major index, and descent number have been well studied for the symmetric group $S_n$. MacMahon [7] showed that inversion number and major index are equidistributed and Carlitz [4] described the joint distribution of descent and major index—the “Euler-Mahonian” distribution—with the following theorem.

Theorem 1 ([4]). For positive integers $n$,

$$\sum_{r \geq 0} [r + 1] q^r t^r = \sum_{w \in S_n} t^{\text{des}(w)} q^{\text{maj}(w)} \prod_{i=0}^n (1 - t q^i),$$

where $[r + 1]_q = 1 + q + \cdots + q^r$.

The Coxeter group generalization of inversion number is length. (An element $w$ in a Coxeter group has length $k$ if $w = s_1 \cdots s_k$ is a minimal expression for $w$ as a product of simple reflections.) Adin and Roichman [3] generalized MacMahon’s result to the hyperoctahedral group $B_n$ in demonstrating that a new statistic, the flag major statistic, is equidistributed with length. Adin, Brenti, and Roichman [1], introduced new statistics on $B_n$, the negative descent, the negative major, and the flag descent. They proved that the pairs of statistics ($\text{fdes}, \text{fmaj}$) and ($\text{ndes}, \text{nmaj}$) are equidistributed over $B_n$. Moreover, they showed these bivariate distributions are “type $B_n$ Euler-Mahonian” in the sense that they generalize Theorem 1 as follows. (Chow and Gessel provide an alternative generalization in [5].)

Theorem 2. For positive integers $n$,

$$\sum_{r \geq 0} [r + 1]^n t^r = \sum_{w \in B_n} t^{\text{ndes}(w)} q^{\text{nmaj}(w)} \prod_{i=1}^n (1 - t^2 q^{2i}) \quad \text{[1, Theorem 3.2],}$$

$$= \sum_{w \in B_n} t^{\text{fdes}(w)} q^{\text{fmaj}(w)} \prod_{i=1}^n (1 - t^2 q^{2i}) \quad \text{[1, Theorem 4.2].}$$

One of the drawbacks of [1] is that while their proof of (2) is quite brief and elementary, their proof of (3) is rather indirect and tedious. Further, they remark (see the discussion after [1, Corollary 4.5]) that “it would be interesting to have a direct combinatorial (i.e., bijective)

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proof" that these type $B_n$ Euler-Mahonian statistics are equidistributed. The purpose of this note is to provide such a proof (Theorem 7). Along the way we give new combinatorial proofs of (2) and (3) (Theorems 3 and 5, respectively).

2. Overview

To any word $w = w_1 \cdots w_n$ whose letters are totally ordered, let $\text{Des}(w) := \{i : w_i > w_{i+1}\}$, and let $\text{des}(w) := |\text{Des}(w)|$, called the descent set and descent number, respectively. We define the major statistic as 

$$\text{maj}(w) := \sum_{i \in \text{Des}(w)} i.$$ 

Let $S_n$ denote the set of all permutations of the set $[n] := \{1, 2, \ldots, n\}$. The refined Eulerian polynomial is the generating function for the joint distribution of $(\text{des}, \text{maj})$ over $S_n$:

$$S_n(t, q) := \sum_{u \in S_n} t^{\text{des}(u)} q^{\text{maj}(u)}.$$ 

This function is the numerator of the right-hand side of (1). For $q = 1$, we have $S_n(t) = \sum_{u \in S_n} t^{\text{des}(u)}$, the classical Eulerian polynomial shifted by $t^{-1}$.

The hyperoctahedral group, $B_n$, has as its elements all signed permutations of $[n]$ such that $|w| = |w_1| \cdots |w_n|$ is a permutation in $S_n$. (We write $\bar{i}$ for $-i$.) The precise way in which the flag and negative statistics are defined for $w$ in $B_n$ depends on how we choose to put a total order on $[n] \cup [\bar{n}]$, though the distribution over $B_n$ is independent of this choice. It will be most convenient to have the lexicographic order:

$$\bar{1} < \bar{2} < \cdots < \bar{n} < 1 < 2 < \cdots < n.$$ 

We remark that [3] uses this order, whereas [1] uses the usual integer ordering.

Our approach is to provide two very explicit combinatorial proofs that the pairs of statistics in question are distributed over $B_n$ as

$$S_n(t, q) \cdot \prod_{i=1}^{n} (1 + tq^i).$$

For $q = 1$ this is the intriguing formula $(1 + t)^n S_n(t)$. (The first author in fact has a different combinatorial proof that flag descents are distributed in this fashion; see [6].)

In fact, we will demonstrate a finer result. Namely, that for each pair of statistics we have a way of assigning $2^n$ signed permutations to each unsigned permutation $u \in S_n$ so that the resulting collection has distribution

$$t^{\text{des}(u)} q^{\text{maj}(u)} \prod_{i=1}^{n} (1 + tq^i).$$

The signed permutations in this collection are thus identified with a pair $(u, J)$, $J \subset [n]$, with “weight” $t^{\text{des}(u)+|J|} q^{\text{maj}(u)+\sum_{j \in J} j}$. This weight-preserving bijection can then be composed to
obtain the desired bijection. If \( w \) corresponds to \((u, J)\) with respect to \((\text{ndes}, \text{nmaj})\) and \( v \) corresponds to \((u, J)\) with respect to \((\text{fdes}, \text{fmaj})\), then we identify \( w \) and \( v \) with one another:

\[
\begin{array}{c}
w \quad \text{neg} \quad \text{flag} \quad v \\
\end{array}
\]

This idea is formalized with Theorem 7.

The paper is organized as follows. Section 3 presents the negative statistics, Section 4 discusses the flag statistics, and Section 5 exhibits the desired bijection.

### 3. The negative statistics

As in \([1]\), define the negative descent number of \( w \in B_n \) as

\[
\text{ndes}(w) := \text{des}(w) + |\{i : w_i < 0\}|
\]

and the negative major index as

\[
\text{nmaj}(w) := \text{maj}(w) + \sum_{w_i < 0} |w_i|.
\]

For example if \( w = 57612\bar{4}\bar{3} \), \( \text{ndes}(w) = 4 + 3 = 7 \), and \( \text{nmaj}(w) = 16 + 12 = 28 \). Let

\[
B_n^{\text{neg}}(t, q) := \sum_{w \in B_n} t^{\text{ndes}(w)} q^{\text{nmaj}(w)},
\]

be the generating function for this pair of statistics.

To any \( u = u_1 \cdots u_n \) in \( S_n \) we can associate \( 2^n \) signed permutations in the following manner. Define the standardization of a signed permutation \( w \in B_n \), \( \text{st}(w) \), to be the unsigned permutation in \( S_n \) that is obtained by replacing the smallest letter of \( w \) with 1, the next smallest with 2, and so on. For example, \( \text{st}(57612\bar{4}\bar{3}) = 3764521 \).

Now given any \( u \) in \( S_n \), let

\[
B(u) = \{w \in B_n : \text{st}(w) = u\}.
\]

We have

\[
B_n = \bigcup_{u \in S_n} B(u) \quad \text{(disjoint union)}.
\]

**Remark.** This partition of \( B_n \) is employed by Adin, Brenti, and Roichman (see equation (5) of \([1]\)), though in different language. It plays a role in their proof of their Theorem 3.2 (equation (2) above) and a refined version of the same result from a different work \([2, \text{Theorem 6.7}]\). This partition will play a role in our proof as well, though the proofs are fundamentally distinct.

It is easy to see that every element of \( B(u) \) is uniquely determined by its set of negative letters. Let \( u_J \) denote the member of \( B(u) \) that is a permutation of the set

\[
\{i : i \in [n] \setminus J\} \cup \{\bar{j} : j \in J\}.
\]

For example, if \( u = 3142 \), then \( u_{\{1,3\}} = 2143 \) since \( \bar{1} \) and \( \bar{3} \) are the negative letters and \( \text{st}(2143) = 3142 \).

The following result and Theorem \([1]\) implies equation (2) of Theorem 2.
Theorem 3. We have
\[ B^{(neg)}_n(t, q) = S_n(t, q) \cdot \prod_{i=1}^{n} (1 + tq^i). \]

Theorem 3 follows immediately from the subsequent stronger claim.

Theorem 4. For any \( u \in S_n \), we have
\[ t^{\des(u)}q^{\maj(u)} \prod_{i=1}^{n} (1 + tq^i) = \sum_{w \in B(u)} t^{\ndes(w)}q^{\nmaj(w)}. \]

Proof. Let \( w = u_J \) for any \( J \subset [n] \). We want to show that
\[ t^{\ndes(w)}q^{\nmaj(w)} = t^{\des(u)}q^{\maj(u)} \prod_{j \in J} tq^j. \]

Since \( \text{st}(w) = u \), we have \( \text{Des}(w) = \text{Des}(u) \). In particular, \( \text{des}(w) = \text{des}(u) \) and \( \text{maj}(w) = \text{maj}(u) \). By definition of \( w \), we know \( \{ w_i : w_i < 0 \} = J \). Comparing with the definition of the negative statistics, this completes the proof. \( \square \)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{\( B(u) \) for \( u = 3142 \)}
\end{figure}

It is informative to observe that Theorem 4 implies the set \( B(u) \) has, up to a shift, a weight-preserving bijection with a boolean algebra given by
\[ J \leftrightarrow u_J. \]

That is, if \( w \in B(u) \) corresponds to \( J = \{ j_1, \ldots, j_k \} \), then \( \ndes(w) = \des(u) + k \) and \( \nmaj(w) = \maj(u) + (j_1 + \cdots + j_k) \). Figure 1 provides an illustration for \( u = 3142 \).
4. The Flag Statistics

From [1], our definitions for the flag statistics are as follows. For \( w \in B_n \),
\[
\text{fdes}(w) := \begin{cases} 
2 \text{des}(w) + 1 & \text{if } w_1 < 0 \\
2 \text{des}(w) & \text{if } w_1 > 0,
\end{cases}
\]
and
\[
\text{fmaj}(w) := 2 \text{maj}(w) + |\{ i : w_i < 0 \}|.
\]
So, for example, if \( w = \overline{5761243} \), then \( \text{Des}(w) = \{2, 3, 5, 6\} \), \( \text{fdes}(w) = 2 \cdot 4 + 1 = 9 \), and \( \text{fmaj}(w) = 2 \cdot 16 + 3 = 35 \). Let \( \text{wt}(w) = t^{\text{fdes}(w)}q^{\text{fmaj}(w)} \) be the weight of \( w \). The generating function is:
\[
B_n^{\text{flag}}(t, q) = \sum_{w \in B_n} \text{wt}(w) = \sum_{w \in B_n} t^{\text{fdes}(w)}q^{\text{fmaj}(w)}.
\]

Before proceeding, it will be helpful to provide another partition of \( B_n \). To any \( u = u_1 \cdots u_n \) in \( S_n \), we can assign \( 2^n \) signed permutations by independently assigning minus signs to the letters of \( u \) in all possible ways. That is, let
\[
B'_n = \bigcup_{u \in S_n} B'(u) \quad \text{(disjoint union)}.
\]
We have
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\]

Just as with Theorem 3, we have the following, which, with Theorem 1 completes the proof of Theorem 2.

**Theorem 5 ([1], Theorem 4.4).** We have
\[
B_n^{\text{flag}}(t, q) = S_n(t, q) \cdot \prod_{i=1}^{n} (1 + tq^i).
\]

**Corollary 1 ([1], Corollary 4.5).** The statistics \((\text{fdes}, \text{fmaj})\) and \((\text{ndes}, \text{nmaj})\) are equidistributed, i.e.,
\[
B_n^{\text{flag}}(t, q) = B_n^{\text{neg}}(t, q).
\]

Theorem 5 will follow from the following theorem, analogous to Theorem 3. With the combinatorial proof of Theorem 6 in hand we will be able to provide the desired bijective proof of Corollary 1.

**Theorem 6.** For any \( u \) in \( S_n \), we have:
\[
t^{\text{des}(u)}q^{\text{maj}(u)} \prod_{i=1}^{n} (1 + tq^i) = \sum_{w \in B'(u)} t^{\text{fdes}(w)}q^{\text{fmaj}(w)}.
\]

Theorem 6 will follow from Lemma 2. First, we need one more idea. Let \( \Delta_i \) be the operator that negates the first \( i \) letters of a signed permutation, i.e.,
\[
\Delta_i w = \overline{w_1 \cdots w_i} w_{i+1} \cdots w_n.
\]
Lemma 1. Fix \( u \in S_n \) and let \( w \in B'(u) \) be such that all the letters \( w_1, \ldots, w_{j+1} \) have the same sign. Then

\[
\text{wt}(\Delta_j w) = \begin{cases} 
\text{wt}(w)/tq^j & \text{if } j \in \text{Des}(u) \\
\text{wt}(w) \cdot tq^j & \text{if } j \notin \text{Des}(u).
\end{cases}
\]

Proof. Suppose everything to the left of \( w_{j+1} \) (inclusive) is positive. Then if \( j \notin \text{Des}(u) \) we have lost nothing and gained \( tq^j \), since \( w_1 \) is now negative and we have \( j \) new negative numbers. If we have \( j \in \text{Des}(u) \), then we still gain \( tq^j \), but we have lost the descent in \( j \), and with it \( t^2q^{2j} \), for a net weight change of \( t - tq - j \).

If everything to the left of \( w_{j+1} \) is negative the situation is similar. If \( j \in \text{Des}(u) \) we have gained no new descents, lost \( j \) negative numbers, and changed \( w_1 \) from negative to positive; a net weight change of \( t - tq - j \). If we have \( j \notin \text{Des}(u) \), then we gain \( t^2q^{2j} \) as \( j \) becomes a descent, but we have lost \( j \) negative numbers and \( w_1 \) has gone from negative to positive, yielding a change of \( t^2q^{2j}/tq^j = tq^j \).

This completes the proof. \( \square \)

Notice that the \( \Delta_i \) commute, so for a subset \( J = \{j_1 < \cdots < j_k\} \) of \( [n] \), there is no ambiguity in defining the composite operator \( \Delta_J = \Delta_{j_1} \cdots \Delta_{j_k} \).

Lemma 2. For any \( u \in S_n \), we have

\[
\text{wt}(\Delta_J u) = t^{\text{des}(u)} q^{\text{maj}(u)} \prod_{j \in J \setminus \text{Des}(u)} tq^j,
\]

where \( J \triangle \text{Des}(u) = (J \cup \text{Des}(u)) \setminus (J \cap \text{Des}(u)) \) is the symmetric difference of \( J \) and \( \text{Des}(u) \).

Proof. First notice that \( \text{fdes}(u) = 2 \text{des}(u) \) and \( \text{fmaj}(u) = 2 \text{maj}(u) \). Since the \( \Delta_j \) commute, we can apply \( \Delta_{j_k} \) first, \( \Delta_{j_{k-1}} \) second, and so on. This allows us to apply Lemma 1 repeatedly, giving us

\[
\text{wt}(\Delta_J u) = t^{2\text{des}(u)} q^{2\text{maj}(u)} \prod_{j \in J \setminus \text{Des}(u)} tq^j
\]

\[
= t^{\text{des}(u)} q^{\text{maj}(u)} \prod_{j \in J \triangle \text{Des}(u)} tq^j,
\]

as desired. \( \square \)

Now we can prove Theorem 6.

Proof of Theorem 6. By Lemma 2, it suffices to show that

\[
B'(u) = \{\Delta_J u : J \subset [n]\}.
\]

Because the two sets have the same cardinality, it suffices to show \( \Delta_J u \in B'(u) \) for any \( J \).

This is clear since \( |\Delta_J u| = u \). \( \square \)
5. The bijection

From Lemma 2 we notice in particular that there is a unique signed permutation with the same flag statistics as $u$ has ordinary statistics, obtained by taking $J = \text{Des}(u)$. Define

$$w_{\text{min}}(u) = \Delta_{\text{Des}(u)} u.$$

Given any signed permutation $w$ we have $\Delta_i^2 w = w$ and, if $i < j$,

$$\Delta_i \Delta_j w = \Delta_j \Delta_i w = w_1 \cdots w_i w_{i+1} \cdots w_j w_{j+1} \cdots w_n.$$

In particular the composite operator $\Delta_i \Delta_{i+1}$ simply negates letter $i+1$. Since the $\Delta_i$ commute and generate all possible sign changes, we can identify permutations in $B'(u)$ with subsets of $[n]$ by the correspondence

$$J \longleftrightarrow \Delta_J w_{\text{min}}(u) = \Delta_J \Delta_{\text{Des}(u)} u.$$

Lemma 2 can now be interpreted to show that the boolean algebra generated by subsets of $[n]$ respects the statistics $(\text{fdes}, \text{fmaj})$ on $B'(u)$. Specifically, if $w \in B'(u)$ corresponds to $J = \{j_1, \ldots, j_k\}$, then $\text{fdes}(w) = \text{des}(u) + k$ and $\text{fmaj}(w) = \text{maj}(u) + (j_1 + \cdots + j_k)$.

See Figure 2 for an example with $u = 3142$.

![Figure 2. $B'(u)$ for $u = 3142$.](image)

Implicit in here is the bijection that Adin, Brenti, and Roichman wanted.

**Theorem 7.** The bijection

$$u_J \longleftrightarrow \Delta_J \Delta_{\text{Des}(u)} u$$

is weight-preserving in that

$$\text{ndes}(u_J) = \text{fdes}(\Delta_J \Delta_{\text{Des}(u)} u)$$

$$\text{nmaj}(u_J) = \text{fmaj}(\Delta_J \Delta_{\text{Des}(u)} u).$$
Example. Let $w = 2134$ with $\text{ndes}(w) = 4$ and $\text{nmaj}(w) = 11$. We first find $u = \text{st}(w) = 1423$. Since $J = \{2, 3, 4\}$ is the set of negative numbers in $w$ and $\text{Des}(u) = \{2\}$, we take $v = \Delta_3\Delta_4(1423) = 1423$, with the desired flag descent and flag major numbers, $\text{fdes}(v) = 4$ and $\text{fmaj}(v) = 11$.

In the other direction, let $v = 4312$ with $\text{fdes}(v) = 4$ and $\text{fmaj}(v) = 8$. First we see that $u = |v| = 4312$ and $v = \Delta_2\Delta_4u$. Since $\text{Des}(u) = \{1, 2\}$, we get $J = \{1, 4\}$ and $u_J = 3214$ with $\text{ndes}(u_J) = 4$ and $\text{nmaj}(u_J) = 8$, as desired.

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