Chiral Anomaly and Ginsparg-Wilson Relation on the Noncommutative Torus

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Abstract

We evaluate chiral anomaly on the noncommutative torus with the overlap Dirac operator satisfying the Ginsparg-Wilson relation in arbitrary even dimensions. Utilizing a topological argument we show that the chiral anomaly is combined into a form of the Chern character with star products.

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§1. Introduction

Various matrix models have been proposed as a nonperturbative formulation of the superstring theory. Type IIB matrix model is one of such proposals and various properties have been investigated. In particular, a possibility of the dynamical generation of four dimensional space-time has been examined from various viewpoints. A connection to the noncommutative field theory has also been studied and several nontrivial dynamics were discussed in connection with string theory. There are still many unsolved issues such as the embedding of curved space-time, locality of the field theories on the dynamically generated space-time or the origin of four dimensional chiral fermions. Also, we need to understand how to describe global topology of space-time and gauge configurations within finite dimensional matrix models.

A possible origin of four dimensional chiral fermions using orbifold matrix models was discussed but this approach is not yet completely satisfactory since we need to restrict the degrees of freedom by hand. Another approach to define chiral fermions in four dimensions will be to mimic the Kaluza-Klein compactification with non-trivial indices. In the paper, we have proposed to use the Ginsparg-Wilson (GW) relation to define chiral structures in finite dimensional matrix models with general curved backgrounds and showed that this construction makes it possible to define a topological invariant for gauge field configurations with only finite number of degrees of freedom. Furthermore we gave an example on the fuzzy two-sphere in general background gauge fields and constructed chirality and Dirac operators satisfying the GW relation. We showed that the topological invariant coincides with the first Chern class in the commutative limit.

We can apply the same technique to classify the topological structures of the gauge field configurations on noncommutative tori. The GW relation was first introduced in finite matrix models with the background of a noncommutative torus in ref. In this paper, based on the formulation of ref. we evaluate the topological charge density for the overlap Dirac operator on the noncommutative torus and show that it becomes the star generalization of the Chern class.

In lattice gauge theory the formalism based on the GW relation has been investigated recently. The first important observation was that, in the presence of the mass defect which is introduced as a scalar background in higher $4 + x$ dimensions, a chiral fermion appears at the defect. The topological defects are a kink for $x = 1$ and a vortex for $x = 2$. So far a domain wall fermion ($x = 1$) and a vortex fermion ($x = 2$) are constructed on the lattice. From the former model a practical solution to the GW relation is obtained. This is the overlap Dirac operator. Anomaly free abelian chiral gauge theory is also constructed on
the lattice by using the GW relation. In the ordinary lattice gauge theory, chiral anomaly for the overlap Dirac operator has been studied in four dimensions and in arbitrary even dimensions. The form of the Chern character is derived by cohomological arguments at a finite lattice spacing.

In this paper, we evaluate the chiral anomaly on arbitrary even dimensional noncommutative torus for the noncommutative version of the overlap Dirac operator in the continuum limit. The chiral anomaly is evaluated as an integral of the Chern character and the only difference from the ordinary anomaly is that the product of gauge fields is replaced by the star product on noncommutative torus. Anomaly on 2-dimensional noncommutative torus is also calculated in ref.

A particular importance to consider the GW relation and topological invariants on the noncommutative torus will be that we can compare the various properties with those in the ordinary lattice gauge theories that have been intensively studied. They include the locality condition, the admissibility condition and the classification of the admissible gauge field configurations. We want to discuss them in a separate paper. Parity anomaly on noncommutative torus is also evaluated in ref.

In section 2, we give a brief review of the noncommutative torus and in section 3, the overlap Dirac operator is introduced on the noncommutative torus. Section 4 is the main part of the paper and the chiral anomaly, that is, the topological charge density is evaluated. In the appendix we give a detailed evaluation of the coefficient of the anomaly following ref.

§2. Noncommutative Torus

A noncommutative torus is one of the simplest examples of noncommutative geometries which can be realized in terms of finite matrices. More precisely, a complete basis of wave functions on noncommutative torus forms a finite dimensional matrix algebra.

We use the following specific matrix representation of the $d$-dimensional ($d$ is an even integer) noncommutative torus. First we introduce the 't Hooft matrices $U$ and $V$. They are $L \times L$-dimensional matrices defined by

\[ U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & \ddots \\ \ddots & \ddots & \ddots \\ 1 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 \\ \omega \\ \omega^2 \\ \ddots \\ \omega^{L-1} \end{pmatrix}, \quad (2.1) \]
where $\omega = e^{i\frac{2\pi}{L}}$. They are unitary and satisfy the following identities,

$$U^L = 1, \quad V^L = 1, \quad UV = e^{i\frac{2\pi}{L}}VU.$$  \hspace{1cm} (2.2)

$\exp(-i\pi nm/L)U^nV^m$ forms a complete basis of $L \times L$ matrices and any hermitian matrices can be expressed as a sum of them.

A $d$-dimensional noncommutative torus can be constructed by taking a $d/2$ tensor product of these matrices. First we define $N \times N$ matrices by

$$Z_{2i-1} = 1_L \otimes \cdots \otimes U \otimes \cdots \otimes 1_L,$$  \hspace{1cm} (2.3)

$$Z_{2i} = 1_L \otimes \cdots \otimes V \otimes \cdots \otimes 1_L$$  \hspace{1cm} (2.4)

where $i = 1, \cdots, d/2$ and $N = L^{d/2}$. On the right hand side of eqs. (2.3) (2.4), $U$ and $V$ are in the $i$-th slot respectively. They satisfy the following relation,

$$Z_\mu Z_\nu = e^{-2\pi i \Theta_{\mu\nu}} Z_\nu Z_\mu,$$  \hspace{1cm} (2.5)

where $\Theta_{\mu\nu} = \frac{L_{\text{phys}}^2}{2\pi} \Theta_{\mu\nu}$. Here $\Theta_{\mu\nu}$ is a $d \times d$ skew diagonal matrix which has the following form,

$$\Theta_{\mu\nu} = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}. \hspace{1cm} (2.6)$$

These $Z_\mu$ are building blocks of wave functions on the noncommutative torus and any $N \times N$ hermitian matrix can be expressed in terms of these matrices. For notational simplicity, we formally introduce hermitian noncommutative coordinates $\hat{x}_\mu$ as $Z_\mu = \exp(2\pi i \hat{x}_\mu / L_{\text{phys}})$. They satisfy

$$[\hat{x}_\mu, \hat{x}_\nu] = i \theta_{\mu\nu}$$  \hspace{1cm} (2.7)

where $\theta_{\mu\nu} = \frac{L_{\text{phys}}^2}{2\pi} \Theta_{\mu\nu}$. We further define a “lattice spacing” $a$ by the relation $L_{\text{phys}} = L a$.

Any $N \times N$ hermitian matrix can be expanded as

$$F = \sum_{\vec{m} \in (\mathbb{Z}^d)_L} f_{\vec{m}}(Z_1)^{m_1}(Z_2)^{m_2} \cdots (Z_d)^{m_d} \exp \left( \pi i \sum_{\mu<\nu} \Theta_{\mu\nu} m_\mu m_\nu \right)$$  \hspace{1cm} (2.8)

where $\vec{m} \in (\mathbb{Z}^d)_L$ means a summation over all integral vectors $\vec{m} \in \mathbb{Z}$ modulo $L$. With the phase $\exp \left( \pi i \sum_{\mu<\nu} \Theta_{\mu\nu} m_\mu m_\nu \right)$, hermiticity of $F$ leads to a condition that $f^*_{\vec{m}} = f_{-\vec{m} + \vec{L}}$ where $\vec{L} = (L, L, \cdots, L)$. We define noncommutative plane waves $\hat{\phi}_k$ by

$$\hat{\phi}_k = (Z_1)^{m_1}(Z_2)^{m_2} \cdots (Z_d)^{m_d} \exp \left( \pi i \sum_{\mu<\nu} \Theta_{\mu\nu} m_\mu m_\nu \right)$$  \hspace{1cm} (2.9)
where \( k_{\mu} = \frac{2\pi m_{\mu}}{L_{\text{phys}}} \) is inside the Brillouin zone \( B_L = [-\pi/a, \pi/a] \). By using \( \hat{x}_{\mu} \), \( \hat{\phi}_k \) can be simply written as \( \hat{\phi}_k = e^{ik\cdot\hat{x}} = \exp\left(2\pi i \vec{m} \cdot \vec{x}/L_{\text{phys}}\right) \) and corresponds to taking Weyl orderings. \( \hat{x}_{\mu} \) always appear in the form of \( \hat{\phi}_k \). The total number of basis wave functions is \( L^d = N^2 \). \( \hat{\phi}_k \) forms orthogonal basis on the noncommutative torus, \( \hat{\phi}_k = e^{ik \cdot \hat{x}} \), \( 1/N \text{tr} \hat{\phi}_k^\dagger \hat{\phi}_l = \delta_{k,l} \).

Following the paper, we can introduce a “delta”-function
\[
\Delta(x) = \sum_{\vec{m} \in (\mathbb{Z}^d)_L} \hat{\phi}_k e^{-2\pi i \vec{m} \cdot x/L_{\text{phys}}},
\]

(2.11)

In order to keep the periodicity in the momentum space \( \{\vec{m}\} \), the value of the coordinates \( x_{\mu} \) should be quantized as \( a \) times an integer. Furthermore, since \( m_{\mu} \) is also quantized as an integer, \( x_{\mu} \) and \( x_{\mu} + La = x_{\mu} + L_{\text{phys}} \) can be identified. Hence, \( \vec{x} \) forms a \( d \)-dimensional lattice \( \Lambda_L \) where
\[
\Lambda_L = \{(x_1, x_2, \cdots, x_d) \mid x_\mu = an_\mu, n_\mu \in (\mathbb{Z}^d)_L\}.
\]

(2.12)

These delta-functions satisfy the following identities,
\[
\text{tr} \Delta(x) = N,
\]
\[
\sum_{x \in \Lambda_L} \Delta(x) = N^2 1_N,
\]
\[
\frac{1}{N} \text{tr}[\Delta(x) \Delta(y)] = N^2 \delta_{x,y}.
\]

(2.15)

By using them, the hermitian matrix \( F \) has a coordinate representation as
\[
F = \frac{1}{N^2} \sum_{x \in \Lambda_L} \mathcal{F}(x) \Delta(x),
\]

(2.16)

\[
\mathcal{F}(x) = \frac{1}{N} \text{tr}[F \Delta(x)] = \sum_{\vec{m} \in (\mathbb{Z}^d)_L} f_{\vec{m}} \exp(2\pi i \vec{m} \cdot \vec{x}/L_{\text{phys}}).
\]

(2.17)

A product of two matrices \( F \) and \( G \) is mapped to a star product of functions
\[
\mathcal{F}(x) \star \mathcal{G}(x) \equiv \frac{1}{N} \text{tr}[FG \Delta(x)]
\]
\[
= \frac{1}{N^2} \sum_{y \in \Lambda_L} \sum_{z \in \Lambda_L} \mathcal{F}(y) \mathcal{G}(z) e^{-2i(\theta^{-1})_{\mu\nu}(x_\mu - y_\mu)(x_\nu - z_\nu)}
\]

(2.18)

and a trace over matrices is mapped to a summation over the lattice
\[
\text{tr} F = \frac{1}{N} \sum_{\vec{x} \in \Lambda_L} \mathcal{F}(\vec{x}).
\]

(2.19)
In order to define lattice derivatives, we introduce shift operators which satisfy
\[ \Gamma_{\mu} \Delta(x) \Gamma_{\mu}^\dagger = \Delta(x - a\hat{\mu}) \quad (2.20) \]
and hence
\[ \mathcal{F}(x + a\hat{\mu}) = \frac{1}{N} \text{tr} \left[ \Gamma_{\mu} F \Gamma_{\mu}^\dagger \Delta(x) \right]. \quad (2.21) \]
Here \( \hat{\mu} \) is a unit vector \((0, \cdots, 1, \cdots, 0)\) which has a non-vanishing element in the \( \mu \)-th direction. These shift operators can be constructed explicitly as
\[ \Gamma_{2i - 1} = 1_L \otimes \cdots \otimes V \otimes \cdots \otimes 1_L, \quad (2.22) \]
\[ \Gamma_{2i} = 1_L \otimes \cdots \otimes U^\dagger \otimes \cdots \otimes 1_L, \quad (2.23) \]
for \( i = 1, \cdots, \frac{d}{2} \). On the right hand side of eqs. \((2.22)(2.23)\), \( V \) and \( U^\dagger \) are in the \( i \)-th slot respectively.

§3. Ginsparg-Wilson Fermions on Noncommutative Torus

In lattice gauge theory, the overlap Dirac operator is a practical solution to the GW relation. This Dirac operator does not have species doubling for some region of parameters \( m_0 \) and \( r \), while it has a modified chiral symmetry at a finite lattice spacing.

Following the notation in ref. \(^1^1\) we first introduce a chirality operator \( \hat{\gamma} \), in addition to the ordinary chirality operator \( \gamma_{d+1} \), as
\[ \hat{\gamma} = \frac{H}{\sqrt{H^2}} \quad (3.1) \]
and the GW Dirac operator by
\[ D_{GW} = \frac{1}{a} [1 - \gamma_{d+1} \hat{\gamma}]. \quad (3.2) \]
This operator was first introduced in ref. \(^1^3\) to define chiral fermions on the noncommutative lattice. Here we can take the hermitian operator \( H \) as the same form as the ordinary overlap Dirac operator \(^1^6\) on the commutative torus : \(^\ast\)

\[ H = \gamma_{d+1} (m_0 - aD_w), \quad (3.3) \]
\[ D_w = \frac{1}{2} \left[ \gamma_{\mu} (\nabla^*_\mu + \nabla_\mu) - ar\nabla^*_\mu \nabla_\mu \right], \quad (3.4) \]

\(^\ast\) Our notations: Greek letters, \( \mu, \nu, \ldots \) run from 1 to \( d = 2n \). Repeated indices are understood to be summed over, unless noted otherwise. \( \{ \gamma_{\mu}, \gamma_\nu \} = 2\delta_{\mu\nu}, \gamma_\mu^\dagger = \gamma_\mu \) and \( \gamma_{d+1} = (-i)^n \gamma_1 \cdots \gamma_d; \gamma_{2d+1}^2 = 1 \) and \( \gamma_{d+1}^\dagger = \gamma_{d+1} \) follow from this.
where $m_0$ and $r$ are free parameters. In the absence of the gauge field, the Dirac operator is free of species doubling if $0 < \frac{m_0}{r} < 2$. $\nabla_\mu$ and $\nabla_\mu^*$ are forward and backward covariant difference operators respectively. They are operators acting on matrices defined by

$$\nabla_\mu \psi = \frac{1}{a} \left[ U_\mu \Gamma_\mu (\Gamma_\mu^\dagger)^R - 1 \right] \psi, \quad (3.5)$$

$$\nabla_\mu^* \psi = \frac{1}{a} \left[ 1 - \Gamma_\mu U_\mu^\dagger (\Gamma_\mu^\dagger)^R \right] \psi. \quad (3.6)$$

Here the superscript $R$ means that the operator acts on matrices from the right. $U_\mu$ are analogues of link variables in lattice gauge theories and $\psi$ is a fermion in the fundamental representation of the gauge group. All these variables $U_\mu, \Gamma_\mu$ and $\psi$ are $N \times N$ matrices. By using the mapping rules, explained in the previous section, from a hermitian matrix $F$ to a field on noncommutative lattice $F(x)$, $\nabla_\mu$ becomes a covariant derivative on noncommutative lattices. But here, we use matrix formulation instead of explicitly using noncommutative lattices. The gauge group is assumed to be abelian in the following merely for notational simplicity but fields on noncommutative geometry are already noncommutative and the following calculations can be straightforwardly applied to the nonabelian gauge group. Due to the property

$$\Gamma_\mu e^{ik_\mu \hat{x}_\mu} \Gamma_\mu^\dagger = e^{ik_\mu \hat{p}_\mu a}, \quad (3.7)$$

we can write a product of $\Gamma_\mu$ and $(\Gamma_\mu^\dagger)^R$ as

$$\Gamma_\mu (\Gamma_\mu^\dagger)^R = e^{i\hat{p}_\mu a}, \quad (3.8)$$

where $\hat{p}_\mu$ is defined to be an operator which picks up all the momenta of plane waves sited rightward. Note that the l.h.s gives the definition of the r.h.s, which is an operator acting on matrices. This rewriting in terms of $\hat{p}_\mu$ makes the calculation in section 4 simpler and similar to that of the ordinary lattice gauge theory.

Under the gauge transformation, $D_{GW}$ transforms covariantly since $\psi$ and $U_\mu$ transforms as follows,

$$\psi \rightarrow g\psi, \quad (3.9)$$

$$U_\mu \rightarrow g U_\mu \Gamma_\mu g^\dagger \Gamma_\mu^\dagger, \quad (3.10)$$

$$\nabla_\mu \psi \rightarrow g \nabla_\mu \psi, \quad (3.11)$$

$$\nabla_\mu^* \psi \rightarrow g \nabla_\mu^* \psi. \quad (3.12)$$

It is the most important property of the overlap Dirac operator that it satisfies the Ginsparg-Wilson relation $\gamma_{d+1} D_{GW} + D_{GW} \hat{\gamma} = 0$. Due to this relation, the fermion
action $S_F = Tr \bar{\psi} D_{GW} \psi$ is invariant under the modified chiral transformation\textsuperscript{24, 25, 11}

\[ \delta \psi = \lambda \hat{\gamma} \psi, \quad \delta \bar{\psi} = \bar{\psi} \lambda \gamma_{d+1}. \]  

(3.13)

We note here that $\lambda$ must transform covariantly as

\[ \lambda \rightarrow g \lambda g^\dagger \]  

(3.14)

under the gauge transformation.\textsuperscript{*)

The fermion integration measure, however, acquires a non-trivial jacobian under the transformation and this gives the chiral anomaly\textsuperscript{24}

\[ \delta \psi d\bar{\psi} = -2q(\lambda) \psi d\bar{\psi}, \]  

(3.15)

where

\[ q(\lambda) = \frac{1}{2} Tr \left[ \lambda^L \hat{\gamma} + \lambda^L \gamma_{d+1} \right]. \]  

(3.16)

Here $Tr$ means a trace over both operators acting on matrices and $\gamma$-matrices. In the lattice gauge theory this gives the correct chiral Ward-Takahashi identity even for finite lattice spacings, so it is especially suitable for a study of phenomena related to the axial anomaly, such as the U(1) problem\textsuperscript{27}

\section*{§4. Chiral Anomaly and Topological Charge}

We now evaluate the topological charge density $q(\lambda)$ in a weak coupling expansion. Hence we write the link variable $U_\mu$ as $U_\mu = \exp(iaA_\mu)$ and expand $q(\lambda)$ in terms of the gauge field $A_\mu$. First we expand $H$ and $H^2$ as

\[ H = H_0 + H_1 + H_2 + H_3 + \cdots, \]  

(4.1)

\[ H^2 = (H^2)_0 + (H^2)_1 + (H^2)_2 + (H^2)_3 + \cdots, \]  

(4.2)

where $H_i$ and $(H^2)_i (i = 0, 1, 2, \cdots)$ means the i-th order term of the gauge field $A_\mu$. $(H^2)_i$ can be written as

\[ (H^2)_0 = H^2_0, \]  

(4.3)

\[ (H^2)_1 = \{H_0, H_1\}. \]  

(4.4)

\textsuperscript{*) In noncommutative field theory there are two different ways to define local chiral transformations due to the ordering ambiguity of functions and the associated chiral currents have also two different forms.\textsuperscript{26} One is the invariant current and the other is the covariant current, which transforms invariantly and covariantly under the gauge transformation respectively. In this paper we only consider the covariant type since there is no local expression of Chern characters for the invariant type.
\[(H^2)_2 = \{H_0, H_2\} + H^2_1, \quad (4.5)\]
\[(H^2)_3 = \{H_0, H_3\} + \{H_1, H_2\}, \quad (4.6)\]

By noting
\[\nabla_\mu + \nabla^*_\mu = \frac{1}{a} (U_\mu \Gamma_\mu (\Gamma_\mu^\dagger)^R - \Gamma_\mu^\dagger U_\mu \Gamma_\mu^R), \quad (4.7)\]
\[\nabla_\mu^* \nabla_\mu = \frac{1}{a^2} (\nabla_\mu - \nabla^*_\mu) = \frac{1}{a^2} (U_\mu \Gamma_\mu (\Gamma_\mu^\dagger)^R + \Gamma_\mu^\dagger U_\mu \Gamma_\mu^R - 2), \quad (4.8)\]

the zero-th order term can be easily evaluated as
\[H_0 = -\gamma_{d+1} \left[ ab(\hat{p}) + i \sum_\mu \gamma_\mu \sin(\hat{p}_\mu a) \right] \equiv H_0(\hat{p}), \quad (4.9)\]
\[(H_0)^2 = (a\omega(\hat{p}))^2 \quad (4.10)\]

where
\[ab(\hat{p}) \equiv r \sum_\mu (1 - \cos(\hat{p}_\mu a)) - m_0, \quad (4.11)\]
\[a\omega(\hat{p}) \equiv \sqrt{\sum_\mu \sin^2(\hat{p}_\mu a) + (ab(\hat{p}))^2} \quad (4.12)\]

and they satisfy \(a^2(b_p^2 - b^2) = \sum_\mu \sin^2(\hat{p}_\mu a)\).

We expand the gauge field \(A_\mu\) as
\[A_\mu = \sum_k A_\mu(k) \hat{\phi}_k, \quad (4.13)\]

where \(A_\mu^*(-k) = A_\mu(k)\) since \(A_\mu^\dagger = A_\mu\). Then the first order term of \(H\) becomes
\[H_1 = \sum_k \sum_\mu A_\mu(k) \hat{\phi}_k e^{-\frac{1}{2} k_\mu a} \frac{\partial H_0}{\partial \hat{p}_\mu} \left( \hat{p} + \frac{k}{2} \right) \quad (4.14)\]

where
\[\frac{\partial H_0}{\partial \hat{p}_\mu}(\hat{p}) = -ia \gamma_{d+1} [\gamma_\mu \cos(\hat{p}_\mu a) - ri \sin(\hat{p}_\mu a)]. \quad (4.15)\]

No summation over \(\mu\) is taken here.

We now expand \((H^2)^{-\frac{1}{2}}\) for small gauge field configurations. \((H^2)^{-\frac{1}{2}}\) can be expanded as
\[\frac{1}{\sqrt{H^2}} = \int_t \frac{1}{t^2 + H^2} = \int_t P_0 \sum_{m=0}^\infty (-1)^m \left( \sum_{i=1}^\infty (H^2)_i P_0 \right)^m \]
\[= \int_t \left[ P_0 - P_0(H^2)_1 P_0 - P_0(H^2)_2 P_0^2 + P_0 \left( (H^2)_1 P_0 \right)^2 \right. \]
\[\left. - P_0(H^2)_3 P_0 + P_0 \left( (H^2)_1 P_0, (H^2)_2 P_0 \right) - P_0 \left( (H^2)_1 P_0 \right)^3 + \cdots \right] \quad (4.16)\]
where we introduced the abbreviated expressions such as $\int_t \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} dt$ and $P_0 \equiv \frac{1}{\sqrt{t^2 + (\mathcal{H}_0)^2}}$.

Then we obtain the following expression for $\hat{\gamma}$

$$
\hat{\gamma} = \frac{H}{\sqrt{H^2}} = \sum_{m=0}^{\infty} (-1)^m \int_t \left( \sum_{i=0}^{\infty} H_i \right) P_0 \left( \sum_{j=1}^{\infty} (H^2)_j P_0 \right)^m. \tag{4.17}
$$

We are now interested in terms which contain $d/2$ gauge fields. This gives the leading contribution to the chiral anomaly in $d$ dimensions. Since $(H)_i$ and $(H^2)_i$ contains at most one and two $\gamma$ matrices respectively, only the following term survives after taking a spinor trace:

$$
(-1)^{d/2} \int_t H_0 P_0 \left( (H^2)_1 P_0 \right)^{d/2}
= (-1)^{d/2} \int_t H_0 P_0 \left( \{H_0, H_1\} P_0 \right)^{d/2}
= (-1)^{d/2} \int_t H_0 P_0 \left( H_1 H_0 P_0 \right)^{d/2}. \tag{4.18}
$$

The last equality follows from the fact that the combination $H_0 P_0 H_0 = H_0^2 P_0 = P_0 H_0^2$ does not contain any $\gamma$ matrices and such a term vanishes after taking a spinor trace. Due to the same reason, all the other terms containing less or equal to the number of gauge fields $d/2$ vanish after taking a trace over spinors. Here in $d = 2n$ dimensions, the other surviving terms contain larger number of gauge fields.

Therefore, if we take the leading order term with $n$ gauge fields, the topological charge density becomes

$$
q(\lambda) = \frac{1}{2} \text{Tr} \left( \lambda \frac{H}{\sqrt{H^2}} \right) = \frac{1}{2N} \sum_p \text{Tr} \hat{\phi}_p^\dagger \lambda \frac{H}{\sqrt{H^2}} \hat{\phi}_p
= \frac{1}{2N} \sum_p \text{Tr} \hat{\phi}_p^\dagger \lambda \left[ \int_t (-1)^n H_0 P_0 \left( H_1 H_0 P_0 \right)^n \right] \hat{\phi}_p + \mathcal{O} \left( A_i^{4+1} \right)
= \frac{(-1)^n}{2N} \sum_p \sum_{k_0, k_1, \cdots, k_n} \sum_{\mu_1, \cdots, \mu_n} \int_t \text{tr} \left( \hat{\phi}_p^\dagger \hat{\phi}_{k_0} \hat{\phi}_{k_1} \cdots \hat{\phi}_{k_n} \hat{\phi}_p \right) \lambda(k_0)
\times A_{\mu_n}(k_n) \cdots A_{\mu_1}(k_1) e^{-\frac{i}{2} \sum_{i=1}^{n} (k_i)_{\mu_i a}}
\times \text{tr}_s \left[ \frac{H_0}{t^2 + H_0^2} \left( p + \sum_{j=1}^{n} k_j \right) \cdot \frac{\partial H_0}{\partial \mu_n} \left( p + \sum_{j_{n-1}=1}^{n-1} k_{j_{n-1}} + \frac{k_n}{2} \right) \cdot \frac{H_0}{t^2 + H_0^2} \left( p + \sum_{h_{n-1}=1}^{n-1} k_{h_{n-1}} \right) \right]
\times \frac{\partial H_0}{\partial \mu_{n-1}} \left( p + \sum_{j_{n-2}=1}^{n-2} k_{j_{n-2}} + \frac{k_{n-2}}{2} \right) \cdots \frac{H_0}{t^2 + H_0^2} (p + k_1 + k_2)
\times \frac{\partial H_0}{\partial \mu_2} \left( p + k_1 + \frac{k_2}{2} \right)
\times \frac{H_0}{t^2 + H_0^2} (p + k_1) \cdot \frac{\partial H_0}{\partial \mu_1} \left( p + \frac{k_1}{2} \right) \cdot \frac{H_0}{t^2 + H_0^2} (p) + \mathcal{O} \left( A_i^{4+1} \right), \tag{4.19}
$$
where the phase factor is given as

\[ \text{tr} \hat{\phi}_{k_0} \hat{\phi}_{k_n} \cdots \hat{\phi}_{k_1} = Ne^{i f(k, \theta) \delta_{k_0, -} \sum_{\lambda=1}^{n} k_\lambda}, \quad (4.20) \]

\[ f(k, \theta) = \frac{1}{2} \sum_{i=0}^{n-1} k_i^\alpha \sum_{j=i+1}^{n} k_j^\beta \theta_{\alpha \beta}. \quad (4.21) \]

Now we take a large \( L \) (continuum) limit where \( L_{\text{phys}} = L a \) is fixed. A requirement for the continuum limit is that all the external momenta \( k_i \) are much smaller than the scale \( O(L) \). Since the noncommutative phase associated with the star product is proportional to

\[ k_i^\mu k_j^\nu \theta_{\mu \nu} = \frac{a L_{\text{phys}}}{2\pi} k_i^\mu k_j^\nu \epsilon_{\mu \nu}, \quad (4.22) \]

it survives if we scale the external momenta as \( k_i \propto L^{1/2} \) simultaneously. Under this assumption, we can take a continuum limit while keeping the noncommutativity. If we rescale the length so that the external momenta are of order \( O(L^0) \), \( L_{\text{phys}} \) scales as \( O(L^{1/2}) \). In this picture, the external momenta are fixed but the size of the torus becomes very large and the geometry becomes noncommutative plane.

In the continuum limit, we expand eq.(4.19) in terms of the external momenta \( k_i^\mu \). Since each \( p_\mu \) derivatives of \( H_0 \) is of order \( O(a) \), we can take up to \( n \) more \( p_\mu \) derivatives so that eq.(4.19) survives in the continuum limit. After expanding the topological density in terms of the external momenta \( k_i \) and taking a spinor trace, all the terms containing \( H_0(p)(\partial H_0(p))^m H_0(p) \) vanish for an arbitrary odd integer \( m \). Hence, in the continuum limit, we need to take \( n \) \( p_\mu \) derivatives of all the \( H_0 \) in the numerators but the last one. Then \( q(\lambda) \) becomes

\[ q(\lambda) = \frac{(-1)^n}{2N} \sum_{p} \sum_{k_0, k_1, \cdots, k_n, \mu_1, \cdots, \mu_n} \text{tr} \left( \hat{\phi}_{k_0} \hat{\phi}_{k_n} \cdots \hat{\phi}_{k_1} \right) \lambda(k_0) \]

\[ \times A_{\mu_n}(k_n) \cdots A_{\mu_1}(k_1) \]

\[ \times \text{tr}_s \left[ (\partial_{\nu_n} H_0) (\partial_{\mu_n} H_0) (\partial_{\nu_{n-1}} H_0) (\partial_{\mu_{n-1}} H_0) \cdots (\partial_{\nu_1} H_0) (\partial_{\mu_1} H_0) H_0 \right] \]

\[ \times \left( \sum_{j_n=1}^{n} k_{j_n} \right)_{\nu_n} \left( \sum_{j_{n-1}=1}^{n-1} k_{j_{n-1}} \right)_{\nu_{n-1}} \cdots (k_1 + k_2) \epsilon_{\nu_1 \lambda_1} \int_t \left( t^2 + H_0^2 \right)^{n+1} \]

\[ + O \left( A_t^{d+1} \right). \quad (4.23) \]

If we neglect the ordering of the plane waves, this topological charge density becomes the same as the ordinary one in the lattice gauge theory. Hence it is clear now that the only difference from the commutative case is that the product of gauge fields is replaced with the noncommutative star product.
In order to evaluate the $p$ summation, we follow the procedure in the paper. First we introduce the following notations,
\[
s_\mu = \sin p_\mu a, \quad c_\mu = \cos p_\mu a. \tag{4.24}\]
By using the identities
\[
\text{tr} \gamma_{d+1} \gamma_{\mu_1} \gamma_{\nu_1} \cdots \gamma_{\mu_n} \gamma_{\nu_n} = i^n 2^n \epsilon_{\mu_1 \nu_1 \cdots \mu_n \nu_n} \tag{4.25}
\]
and
\[
\int_t \frac{1}{(t^2 + c)^{n+1}} = (2n - 1)! \frac{1}{n! 2^n} \frac{1}{c^{n + \frac{1}{2}}}, \tag{4.26}
\]
and then taking a spinor trace, we have
\[
\text{tr}_s \left[ (\partial_{\nu_n} H_0)(\partial_{\mu_n} H_0)(\partial_{\nu_{n-1}} H_0)(\partial_{\mu_{n-1}} H_0) \cdots (\partial_{\nu_1} H_0)(\partial_{\mu_1} H_0) H_0 \right] \int_t \frac{1}{(t^2 + H_0^2)^{n+1}} \\
= a^{2n} i^n \epsilon_{\nu_{n \mu_1} \cdots \nu_1 \mu_1} I(p; m_0, r) \frac{(2n - 1)!}{n!}, \tag{4.27}
\]
where
\[
I(p; m_0, r) = \left( \prod_{\mu=1}^d c_\mu \right) \left[ m_0 + r \sum_\rho (c_\rho - 1) + r \sum_\rho \frac{s_\mu^2}{c_\mu} \right] (H_0^2)^{-n - \frac{1}{2}}. \tag{4.28}
\]
In the continuum limit, the summation over the momentum $p$ can be replaced by an integral as
\[
\sum_p = (L_{\text{phys}})^d \int \frac{d^d p}{(2\pi)^d}.
\tag{4.29}
\]
We thus have
\[
q(\lambda) = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} (L_{\text{phys}})^d \sum_{k_1, \ldots, k_n} \lambda \left( -\sum_{i=1}^n k_i \right) e^{i f(k, \theta)} \times \epsilon_{\nu_{n \mu_1} \cdots \nu_1 \mu_1} [(k_n)_{\nu_n} A_{\mu_n} (k_n)] \cdots [(k_1)_{\nu_1} A_{\mu_1} (k_1)] a^d I(p; m_0, r) \frac{(-i)^n (2n - 1)!}{n!} \frac{1}{n!} \tag{4.30}
\]
\[+ O \left( A_1^{d+1} \right). \]

We emphasize again that the above eq. (4.30) is the same as that in lattice gauge theory except for the existence of the phase factor $e^{i f(k, \theta)}$. The evaluation of $I(p; m_0, r)$, which was done in ref. is written in the appendix. The result is summarized as
\[
I(m_0, r) = a^d \int \frac{d^d p}{B} I(p; m_0, r) = \sum_{n_\pi=0}^{[m_0/2r]} (-1)^{n_\pi} \frac{d}{n!} \frac{2^{d+1} \pi^{d+1}}{d!}. \tag{4.31}
\]
We note here that \( I(m_0, r) = \frac{2^{d+1} \pi^{n_1!}}{d!} \) for \( 0 < m_0/r < 2 \) and that the summation of \( x \) can be replaced in the continuum limit as
\[
a^d \sum_x = \int d^d x.
\]

Then the final expression of the chiral anomaly is written as
\[
q(\lambda) = \frac{1}{2} I(m_0, r) (L_{\text{phys}})^d \sum_{k_0, \ldots, k_n} \frac{1}{N^2} \sum_x e^{i(k_0 + \cdots + k_n) \cdot x} \lambda(k_0) e^{i f(k, \theta)}
\times \epsilon_{\nu_0 \mu_0 \cdots \nu_1 \mu_1} [(k_n)_{\nu_n} A_{\mu_n}(k_n)] \cdots [(k_1)_{\nu_1} A_{\mu_1}(k_1)] \left( -i \right)^n (2n - 1)!! \frac{1}{(2\pi)^{d+n} n!} + O \left( A_i^{d+1} \right)
= \frac{(-1)^n (2n - 1)!!}{2(2\pi)^d n!} \int (m_0, r) \epsilon_{\nu_0 \mu_0 \cdots \nu_1 \mu_1} \int d^d x \lambda(x) \star \partial_{\nu_0} A_{\mu_0}(x) \star \cdots \star \partial_{\nu_1} A_{\mu_1}(x)
+ O \left( A_i^{d+1} \right).
\]

In particular, when \( 0 < m_0/r < 2 \) with which the overlap Dirac operator does not encounter the species doubling, this reproduces the expected result.

As we discussed in eqs. (3.9)–(3.12), the action is invariant under gauge transformations and so is the topological charge if \( \lambda \) transforms covariantly as in eq. (3.14). Hence, the above \( q(\lambda) \) must be also gauge invariant if higher order terms of the gauge field are included. It will then become
\[
q(\lambda) = \frac{(-1)^n}{(4\pi)^n n!} \epsilon_{\nu_0 \mu_0 \cdots \nu_1 \mu_1} \int d^d x \lambda(x) \star F_{\nu_0 \mu_0}(x) \star \cdots \star F_{\nu_1 \mu_1}(x)
\]
for \( 0 < m_0/r < 2 \) and \( F_{\mu \nu} \) is a gauge covariant field strength in the continuum limit. This is the covariant form of the anomaly. In the case of gauge anomaly, similar calculation gives the same covariant form. A relation to the consistent form is discussed in refs. [29]

§5. Discussion

In this paper we calculated the noncommutative chiral anomaly on arbitrary even dimensional noncommutative torus with overlap Dirac operator. In the “continuum limit”, we derived the correct Chern character including the star product.

At the formal level, this looks very natural since the anomaly for the covariant chiral current in the fundamental representation is diagrammatically given by planar graphs. [23, 26] But the topologies of the gauge configuration spaces are very different between the ordinary lattice gauge theories and its reduced models (or the noncommutative lattice theories), and further investigation is necessary to understand the topological structure of gauge fields on
noncommutative torus. Namely, in the ordinary $U(1)$ gauge theories on the lattice with $N^2$ lattice points, the gauge field configurations have a topology of $U(1)^{N^2d}/U(1)^{N^2}$ while $U(N)^d/U(N)$ on the noncommutative torus. In the lattice gauge theory the admissibility condition for the gauge field $\|1-U(p)\| < \epsilon$ is imposed\(^{22}\) so that the chirality operator and the GW Dirac operator are well-defined. The topological structure of gauge fields under the admissibility condition is classified explicitly, which enables us to obtain an anomaly free abelian chiral gauge theory on the lattice\(^{17}\). It is interesting to investigate a similar condition on the noncommutative torus, with which the topological structure for the gauge fields is defined.

Another related issue is the reduction of degrees of freedom in matrix models. In papers\(^{28}\) the ordinary lattice gauge theories is embedded in matrix models by restricting the matrix degrees of freedom, and chiral anomaly and the topological structure are discussed. Their reduction is maximal. Namely, they reduced $N^2$ degrees of freedom to $N$. It is interesting and may be necessary to consider weaker conditions of reductions for constructing well-defined topological structures in matrix models.

Acknowledgements

We would like to thank H. Aoki and J. Nishimura for discussions.

Appendix A

Evaluation of $I(m_0, r)$

We now evaluate the following integral utilizing the degree of mapping following ref.\(^{19}\)

$$I(m_0, r) = a^d \int_{B} d^dp I(p; m_0, r),$$  \hspace{1cm} (A.1)

with

$$I(p; m_0, r) = \left( \prod_{\mu=1}^{d} c_\mu \right) \left\{ \sum_{\nu} s_{\nu}^2 + \left[ m_0 + r \sum_{\nu} (c_\nu - 1) \right]^2 \right\}^{-n-\frac{1}{2}} \times \left[ m_0 + r \sum_{\rho} (c_\rho - 1) + r \sum_{\rho} \frac{s_{\rho}^2}{c_\rho} \right]$$  \hspace{1cm} (A.2)

and

$$B = \left\{ k_\mu \in \mathbb{R}^d \mid -\frac{\pi}{2a} \leq k_\mu \leq \frac{3\pi}{2a} \right\}. \hspace{1cm} (A.3)$$

Here we have shifted the integration region as $k_\mu \in [-\frac{\pi}{a}, \frac{\pi}{a}] \rightarrow k_\mu \in [-\frac{\pi}{2a}, \frac{3\pi}{2a}]$ for later convenience.
We introduce a mapping from the Brillouin zone $B$ to the unit sphere $S^d$. The mapping is defined by

\begin{align*}
\theta_0 &= m_0 + r \sum_{\mu} (c_\mu - 1), \\
\theta_\mu &= s_\mu, \quad \text{for} \quad \mu = 1, \ldots, d,
\end{align*}

and

\begin{equation}
x_I = \frac{\theta_A}{\epsilon}, \quad \epsilon = \sqrt{\sum_I \theta_I^2},
\end{equation}

where $x_I$ ($I = 0, 1, \ldots, d$) is the coordinate of $\mathbb{R}^{d+1}$ in which the unit sphere $\sum_I x_I^2 = 1$ is embedded. $p_\mu \to x_I$ defines a mapping $f : T^d \to S^d$. The crucial observation is that the volume form on this sphere coincides with the integrand of $I(m_0, r)$:

\begin{equation}
\Omega = \frac{1}{d!} \epsilon A_0 \cdots A_d d x_{A_0} \wedge \cdots \wedge d x_{A_d} = \frac{1}{d!} \epsilon^{d+1} A_0 \cdots A_d d \theta_{A_0} \wedge \cdots \wedge d \theta_{A_d} = a^d I(p; m_0, r) \, dp_1 \wedge \cdots \wedge dp_d.
\end{equation}

This shows that the integral of $\Omega$ on a (sufficiently small) coordinate patch $U$ on $S^d$ is given by

\begin{equation}
\int_U \Omega = \text{sgn} \left[ I(p^j; m_0, r) \right] \int_{U^j} a^d I(p; m_0, r) \, dp_1 \wedge \cdots \wedge dp_d,
\end{equation}

where $U^j$ ($j = 1, \ldots, m$) is a component of the inverse image of $U$ under $f$, $f^{-1}(U)$:

\begin{equation}
f^{-1}(U) = U^1 \cup \cdots \cup U^m \subset T^d,
\end{equation}

and $p^j$ ($j = 1, \ldots, m$) is a certain point on $U^j$. For a sufficiently small $U$, $U^j$ are pairwise disjoint. We take preimages of a point $y \in U$ under $f$, $f^{-1}(y)$, as $p^j$. Then by summing both sides of eq. (A.8) over $j$, we have

\begin{equation}
\sum_j \int_{U^j} a^d I(p; m_0, r) \, dp_1 \wedge \cdots \wedge dp_d = (\deg f) \int_U \Omega,
\end{equation}

where the degree of the mapping $f$ is given by

\begin{equation}
\deg f = \sum_{f(p^j) = y} \text{sgn} \left[ I(p^j; m_0, r) \right].
\end{equation}

\(^*)\text{In deriving this relation, we have assumed that } U \text{ is within the range of } f, \text{ i.e., the inverse image } f^{-1}(U) \text{ is not empty. This relation itself, however, is meaningful even if } U \text{ is not within the range of } f, \text{ if one sets } \deg f = 0 \text{ for such case. As a consequence, eq. (A.12) holds even if } f : T^d \to S^d \text{ is not a surjection, i.e., not an onto-mapping.}\)
In general, the degree of the mapping \( f : T^d \to S^d \) is defined by a sum of the signature of jacobian of the coordinate transformation between \( T^d \) and \( S^d \) over preimages of a point \( y \in S^d \). An important mathematical fact is that the degree takes the same value for all coordinate patches \( U \) of \( S^d \). We thus have

\[
I(m_0, r) = \int_{T^d} d^d I(p; m_0, r) \, dp_1 \wedge \cdots \wedge dp_d = (\deg f) \int_{S^d} \Omega,
\]

where \( \int_{S^d} \Omega \) is given by the volume of the unit sphere \( S^d \):

\[
\int_{S^d} \Omega = \text{vol}(S^d) = \frac{2^{d+1} \pi^{n} n!}{d!}.
\]

We may choose any point \( y \) on \( S^d \) to evaluate the degree (A.11). We choose \( y = (1, 0, \cdots, 0) \). This requires

\[
x_{\mu}(p^j) = \frac{s_{\mu}}{\epsilon} = 0, \quad \text{for} \quad \mu = 1, \ldots, d,
\]

and

\[
x_0(p^j) = \frac{m_0 + r \sum_{\mu}(c_{\mu} - 1)}{|m_0 + r \sum_{\mu}(c_{\mu} - 1)|} = 1.
\]

Note that eq. (A.15) is equivalent to the condition \( m_0/r + \sum_{\mu}(c_{\mu} - 1) > 0 \). Now eq. (A.14) implies that \( p^j_{\mu} = 0 \) or \( \pi \) for each direction \( \mu \). We denote the number of \( \pi \)'s appearing in \( p^j \) by an integer \( n_{\pi} \geq 0 \),

\[
p^j = (\pi, \ldots, \pi, 0, \ldots, 0),
\]

irrespective of the position of \( \pi \)'s. For a given \( n_{\pi} \), the number of such \( p^j \) is \( \binom{d}{n_{\pi}} \). The second relation (A.15) on the other hand requires \( n_{\pi} < m_0/(2r) \). At those \( p^j \), we have

\[
\text{sgn} \left[ I(p^j; m_0, r) \right] = \prod_{\mu} c_{\mu} = (-1)^{n_{\pi}}.
\]

Thus eq. (A.11) gives

\[
\deg f = \sum_{n_{\pi}=0}^{[m_0/2r]} \binom{d}{n_{\pi}} (-1)^{n_{\pi}}.
\]

Combining eqs. (A.12), (A.13) and (A.18), we finally obtain

\[
I(m_0, r) = \sum_{n_{\pi}=0}^{[m_0/2r]} (-1)^{n_{\pi}} \binom{d}{n_{\pi}} \frac{2^{d+1} \pi^{n} n!}{d!}.
\]

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