A CHARACTERIZATION OF 1-RECTIFIABLE DOUBLING MEASURES WITH CONNECTED SUPPORTS
A CHARACTERIZATION OF 1-RECTIFIABLE DOUBLING MEASURES WITH CONNECTED SUPPORTS

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Garnett, Killip, and Schul have exhibited a doubling measure $\mu$ with support equal to $\mathbb{R}^d$ that is 1-rectifiable, meaning there are countably many curves $\Gamma_i$ of finite length for which $\mu(\mathbb{R}^d \setminus \bigcup \Gamma_i) = 0$. In this note, we characterize when a doubling measure $\mu$ with support equal to a connected metric space $X$ has a 1-rectifiable subset of positive measure and show this set coincides up to a set of $\mu$-measure zero with the set of $x \in X$ for which $\liminf_{r \to 0} \frac{\mu(B_X(x, r))}{r} > 0$.

1. Introduction

Recall that a Borel measure $\mu$ on a metric space $X$ is doubling if there is $C_\mu > 0$ so that
\[ \mu(B_X(x, 2r)) \leq C_\mu \mu(B_X(x, r)) \quad \text{for all } x \in X \text{ and } r > 0. \] (1-1)

Garnett, Killip, and Schul [Garnett et al. 2010] exhibit a doubling measure $\mu$ with support equal to $\mathbb{R}^n$, $n > 1$, that is 1-rectifiable in the sense that there are countably many curves $\Gamma_i$ of finite length such that $\mu(\mathbb{R}^n \setminus \bigcup \Gamma_i) = 0$. This is surprising given that such measures give zero measure to smooth or bi-Lipschitz curves in $\mathbb{R}^d$. To see this, note that, for such a curve $\Gamma$ and for each $x \in \Gamma$, there are $r_x, \delta_x > 0$ so that for all $r \in (0, r_x)$ there is $B_{\mathbb{R}^d}(y_{x, r}, \delta_x r) \subseteq B_{\mathbb{R}^n}(x, r_x) \setminus \Gamma$, so by the Lebesgue differentiation theorem, $\mu(\Gamma) = 0$. If $\Gamma$ is just Lipschitz and not bi-Lipschitz, however, we only know this property holds for every point in $\Gamma$ outside a set of zero length. The aforementioned result shows that Lipschitz curves of finite length can in some sense be coiled up tightly enough that this zero-length set accumulates on a set of positive doubling measure.

The notion of rectifiability of a measure that we are using is not universal. In [Azzam et al. 2015], a measure $\mu$ in Euclidean space being $d$-rectifiable means $\mu \ll \mathcal{H}^d$ and $\supp \mu$ is $d$-rectifiable. In our setting, however, we don’t require absolute continuity of our measures. To avoid ambiguity, we fix our definition below, which is the convention used in [Federer 1969, §3.2.14].

**Definition 1.1.** If $\mu$ is a Borel measure on a metric space $X$, $d$ is an integer, and $E \subseteq X$ a Borel set, we say $E$ is $(\mu, d)$-rectifiable if $\mu(E \setminus \bigcup_{i=1}^\infty \Gamma_i) = 0$ where $\Gamma_i = f_i(E_i)$, $E_i \subseteq \mathbb{R}^d$, and $f_i : E_i \to X$ is Lipschitz. We say $\mu$ is $d$-rectifiable if $\supp \mu$ is $(\mu, d)$-rectifiable.

A set $E \subseteq \mathbb{R}^n$ of positive and finite $\mathcal{H}^d$-measure is $d$-rectifiable if it is $(\mathcal{H}^d, d)$-rectifiable (see [Mattila 1995, Definition 15.3] and the few paragraphs preceding it). This is also equivalent to being covered up...
to set of $H^d$-measure zero by Lipschitz graphs [Mattila 1995, Lemma 15.4]. The example from [Garnett et al. 2010], however, shows that being almost covered by Lipschitz graphs versus Lipschitz images are not equivalent definitions for rectifiability of a measure.

Since this example was published, it has been an open question to classify which doubling measures on $\mathbb{R}^d$ are rectifiable. Very recently, Badger and Schul have given a complete description. First, for a general Radon measure in $\mathbb{R}^d$ and $A$ compact with $\mu(A) > 0$, define

$$\beta_2^{(1)}(\mu, A)^2 = \inf_L \int_A \left( \frac{\text{dist}(x, L)}{\text{diam } A} \right)^2 \frac{d\mu(x)}{\mu(A)}$$

where the infimum is taken over all lines $L \subseteq \mathbb{R}^d$.

**Theorem 1.2** [Badger and Schul 2015b, Corollary 1.12]. If $\mu$ is a Radon measure on $\mathbb{R}^d$ such that $\lim_{r \to 0} \beta_2^{(1)}(\mu, B_{\mathbb{R}^d}(x, r)) > 0$ for $\mu$-almost every $x \in \mathbb{R}^d$, then $\mu$ is 1-rectifiable if and only if

$$\sum_{x \in Q, L(\mu) \leq 1} \frac{\text{diam } Q}{\mu(Q)} < \infty \quad \mu\text{-a.e.} \quad (1-2)$$

where the sum is over half-open dyadic cubes $Q$.

It is not hard to show that, if $\mu$ is a doubling measure with supp $\mu = \mathbb{R}^d$, $d \geq 2$, then there is $c > 0$ depending on the doubling constant such that $\beta_2^{(1)}(\mu, B) \geq c > 0$ for any ball $B \subseteq \mathbb{R}^d$, so the above theorem characterizes all 1-rectifiable doubling measures with support equal to all of $\mathbb{R}^d$.

In this short note, we take a different approach and provide a complete classification of 1-rectifiable doubling measures not just with support equal to $\mathbb{R}^d$ but with support equal to any topologically connected metric space. It turns out that the rectifiable part of such a measure coincides up to a set of $\mu$-measure zero with the set of points where the lower 1-density is positive, where for $s > 0$ we define the lower $s$-density as

$$D_s(\mu, x) := \liminf_{r \to 0} \frac{\mu(B_X(x, r))}{r^s}.$$ 

**Theorem 1.3** (main theorem). Let $\mu$ be a doubling measure whose support is a topologically connected metric space $X$, and let $E \subseteq X$ be compact. Then $E$ is $(\mu, 1)$-rectifiable if and only if $D_1(\mu, x) > 0$ for $\mu$-a.e. $x \in E$.

Note that there are no other topological or geometric restrictions on $X$: the support of $\mu$ may have topological dimension two (like $\mathbb{R}^2$ for example), yet if $D_1(\mu, x) > 0$ $\mu$-a.e., then $\mu$ is supported on a countable union of Lipschitz images of $\mathbb{R}$. Also observe that the condition $D_1(\mu, x) > 0$ is a weaker condition than (1-2). An interesting corollary of the main theorem and Theorem 1.2 is the following.

**Corollary 1.4.** If $\mu$ is a doubling measure in $\mathbb{R}^d$ with connected support such that

$$\liminf_{r \to 0} \beta_2^{(1)}(\mu, B_{\mathbb{R}^d}(x, r)) > 0$$

and $D_1(\mu, x) > 0$ $\mu$-a.e., then (1-2) holds.
2. Proof of the main theorem: sufficiency

When dealing with any metric space $X$, we will let $B_X(x, r)$ denote the set of points in $X$ of distance less than $r > 0$ from $x$. If $B = B_X(x, r)$ and $M > 0$, we will denote $MB = B_X(x, Mr)$. For a Borel set $A \subseteq X$, we define the (spherical) 1-Hausdorff measure as

$$\mathcal{H}^1_\delta(A) = \inf\left\{ \sum_{i=1}^\infty 2r_i : A \subseteq \bigcup_{i=1}^\infty B_X(x_i, r_i), \quad x_i \in A, \quad r_i \in (0, \delta) \right\}$$

and $\mathcal{H}^1(A) = \inf_{\delta>0} \mathcal{H}^1_\delta(A)$.

For $A, B \subseteq X$, we set

$$\text{dist}(A, B) = \inf\{|x - y| : x \in A, \ y \in B\}$$

and, for $x \in X$, $\text{dist}(x, A) = \text{dist}([x], A)$.

**Remark 2.1.** By the Kuratowski embedding theorem, if $X$ is separable (which happens, for example, if $X = \text{supp} \mu$ for a locally finite measure $\mu$), $X$ is isometrically embeddable into $C(X)$, where $C(X)$ is the Banach space of bounded continuous functions on $X$ equipped with the supremum norm $|f| = \sup_{x \in X} |f(x)|$. Thus, we can assume without loss of generality that $X$ is the subset of a complete Banach space, and we will abuse notation by calling this space $C(X)$ as well so that $X \subseteq C(X)$.

The forward direction of the main theorem is proven for general measures in Euclidean space by Badger and Schul [2015a, Lemma 2.7], who in fact prove a higher-dimensional version. Below we provide a proof that works for metric spaces in the one-dimensional case.

**Proposition 2.2.** Let $\mu$ be a finite measure with $X := \text{supp} \mu$ a metric space, and suppose $\mu$ is 1-rectifiable. Then $D^1(\mu, x) > 0$ for $\mu$-a.e. $x \in \text{supp} \mu$.

**Proof.** Let

$$F = \{x \in \text{supp} \mu : D^1(\mu, x) = 0\},$$

and let $\varepsilon, \delta > 0$. Since $\mu$ is rectifiable, there are Lipschitz functions $f_i : A_i \to X$, where $A_i \subseteq [0, 1]$ are compact Borel sets of positive measure and $i = 1, \ldots, N$, so that

$$\mu\left( E \setminus \bigcup_{i=1}^N f_i(A_i) \right) < \delta.$$

We can extend each $f_i$ affinely on the intervals in the complement of $A_i$ to a Lipschitz function $f_i : [0, 1] \to C(X)$. Let $d = \min_{i=1,\ldots,N} \text{diam} f_i([0, 1])$ so that $r \in (0, d)$ and $x \in G := \bigcup_{i=1}^N f_i([0, 1])$ implies $\mathcal{H}^1(B_{C(X)}(x, r)) \geq r$ (simply because now the images of the $f_i$ are connected).

For each $x \in F \cap G$, there is $r_x \in (0, d/5)$ so that $\mu(B_X(x, 5r_x)) < \varepsilon r_x$. By the Vitali covering theorem [Heinonen 2001, Lemma 1.2], there are countably many disjoint balls $B_i = B_X(x_i, r_i)$ with centers in $F$ so that $\bigcup_i 5B_i \supseteq F$. Thus,

$$\mu(F \cap G) \leq \sum_i \mu(5B_i) \leq \varepsilon \sum_i r_i \leq \varepsilon \sum_i \mathcal{H}^1(B_{C(X)}(x_i, r_i)) \cap G) \leq \varepsilon \mathcal{H}^1(G).$$
Thus,

\[ \mu(F) < \delta + \varepsilon \mathcal{H}^1(G). \]

Keeping \( \delta \) (and hence \( G \)) fixed and sending \( \varepsilon \to 0 \), we get \( \mu(F) < \delta \) for all \( \delta > 0 \) and thus \( \mu(F) = 0. \)

3. Proof of the main theorem: necessity

What remains is to prove the reverse direction of the main theorem, which we summarize in the next lemma.

**Lemma 3.1.** Let \( \mu \) be a doubling measure with constant \( C_\mu > 0 \) and support \( X \), a topologically connected metric space. Then \( \{ x \in X : D^1(\mu, x) > 0 \} \) is \((\mu, 1)\)-rectifiable.

To prove Lemma 3.1, it suffices to show the following lemma.

**Lemma 3.2.** Let \( \mu \) be a doubling measure and support \( X \) a topologically connected complete metric space. If \( E \subseteq X \) is a compact set for which \( E \subseteq B_X(\xi_0, r_0/2) \) for some \( \xi_0 \in X \), \( r_0 > 0 \), and

\[ \mu(B_X(x, r)) \geq 2r \quad \text{for all} \quad x \in E \quad \text{and} \quad r \in (0, r_0), \]

(3-1)

then \( E = f(A) \) for some \( A \subseteq \mathbb{R} \) and Lipschitz function \( f : A \to X \).

**Proof of Lemma 3.1 using Lemma 3.2.** First, note that, if we define \( \bar{\mu}(A) = \mu(A \cap X) \), then \( \bar{\mu} \) is a doubling measure on \( \bar{X} \), where the closure is in \( C(X) \) (recall Remark 2.1). Moreover, the closure \( \bar{X} \) is still topologically connected but now is a complete metric space since \( C(X) \) is complete. Thus, for proving Lemma 3.1, we can assume without loss of generality that \( X \) is complete.

Let \( F := \{ x \in X : D^1(\mu, x) > 0 \} \). For \( j, k \in \mathbb{N} \), let

\[ F_{j,k} = \{ x \in F : \mu(B_X(x, r)) \geq r/j \ \text{for} \ 0 < r < k^{-1} \}. \]

Then \( F = \bigcup_{j,k \in \mathbb{N}} F_{j,k} \). Furthermore, we can write \( F_{j,k} \) as a countable union of sets \( \{ F_{j,k,\ell} \}_{\ell \in \mathbb{N}} \) with diameters less than \( 1/(3k) \). It suffices then to show that each one of these sets is \( 1 \)-rectifiable. Fix \( j, k, \ell \in \mathbb{N} \). Then the measure \( j\mu \) and the set \( F_{j,k,\ell} \) satisfy the conditions for Lemma 3.2 with \( r_0 = k^{-1} \) except that \( F_{j,k,\ell} \) is not necessarily compact. However, \( \bar{F}_{j,k,\ell} \) is a closed set still satisfying these conditions, it is totally bounded since \( \mu \) is doubling, and since \( X \) is complete, the Heine–Borel theorem implies \( \bar{F}_{j,k,\ell} \) is compact. Thus, we can apply Lemma 3.2 to get that \( \bar{F}_{j,k,\ell} \) is rectifiable. Since \( F = \bigcup_{j,k,\ell} F_{j,k,\ell} \), we now have that \( F \) is also rectifiable.

The rest of the paper is devoted to proving Lemma 3.2, so fix \( \mu, E, \xi_0, \) and \( r_0 \) as in the lemma.

**Proof of Lemma 3.2.** We will require the notion of dyadic cubes on a metric space. This theorem was originally developed by David [1988] and Christ [1990], but the current formulation we take from Hytönen and Martikainen [2012].

**Theorem 3.3.** Let \( X \) be a metric space equipped with a doubling measure \( \mu \). Let \( X_n \) be a nested sequence of maximal \( \rho^n \)-nets for \( X \) where \( \rho < 1/1000 \), and let \( c_0 = 1/500 \). For each \( n \in \mathbb{Z} \), there is a collection \( \mathcal{D}_n \) of “cubes”, which are Borel subsets of \( X \) such that:
Then there is $C$ will be connected. We now proceed with the details.

and a segment connecting the endpoints, thus giving a polygonal curve connecting $x$ (unit vector corresponding to the $|x|$ disconnections. We don’t need the endpoints of these bridges to be in $\mathbb{R}$ since these are the cubes where $E$ has large holes and thus potentially has big gaps or disconnections. We don’t need the endpoints of these bridges to be in $E$, but their union plus the set $E$ will be connected. We now proceed with the details.

Let $\tilde{X} = \bigcup X_n$, and equip $C(\tilde{X}) \oplus \mathbb{R}^{\tilde{X} \times \tilde{X}}$ (where $\mathbb{R}^{\tilde{X} \times \tilde{X}} = \bigoplus_{a \in \tilde{X} \times \tilde{X}} \mathbb{R}$; see [Munkres 1975, p. 112–117] for the notation) with norm $|a \oplus b| = \max(|a|, |b|)$, where the norm on $\mathbb{R}^{\tilde{X} \times \tilde{X}}$ is the $\ell^2$ norm.

For $x, y \in \tilde{X}$, let $[x, y]$ denote the straight line segment between them in $C(\tilde{X}) \oplus \mathbb{R}^{\tilde{X} \times \tilde{X}}$, $e_{(x,y)}$ is the unit vector corresponding to the $(x, y)$ coordinate in $\mathbb{R}^{\tilde{X} \times \tilde{X}}$, and define

$$[x, y]^* := [x, (x, |x-y|e_{(x,y)})] \cup [y, (y, |x-y|e_{(x,y)})] \cup [(x, |x-y|e_{(x,y)}), (y, |x-y|e_{(x,y)})] \subseteq C(\tilde{X}) \oplus \mathbb{R}^{\tilde{X} \times \tilde{X}}.$$

The set $[x, y]^*$ is two segments going straight up from $x$ and $y$, respectively, in the $e_{(x,y)}$ direction and a segment connecting the endpoints, thus giving a polygonal curve connecting $x$ to $y$ that hops out of

(1) For every $n$, $X = \bigcup_{\Delta \in \mathcal{D}_n} \Delta$.

(2) If $\Delta, \Delta' \in \mathcal{D} = \bigcup_{\Delta \in \mathcal{D}_n} \Delta \cap \Delta' \neq \emptyset$, then $\Delta \subseteq \Delta'$ or $\Delta' \subseteq \Delta$.

(3) For $\Delta \in \mathcal{D}$, let $n(\Delta)$ be the unique integer so that $\Delta \in \mathcal{D}_n$ and set $\ell(\Delta) = 5\rho^n(\Delta)$. Then there is $\zeta_\Delta \in X_n$ so that

$$B_X(\zeta_\Delta, c_0\ell(\Delta)) \subseteq \Delta \subseteq B_X(\zeta_\Delta, \ell(\Delta))$$

and

$$X_n = \{\zeta_\Delta : \Delta \in \mathcal{D}_n\}.$$

It is not necessary for there to exist a doubling measure but just that the metric space is geometrically doubling. Moreover, Hytönen and Martikainen [2012] use sequences of sets $X_n$ slightly more general than maximal nets.

Let $X_n$ be a nested sequence of maximal $\rho^n$-nets for $X$ where $\rho < 1/1000$ and $\mathcal{D}$ the resulting cubes from Theorem 3.3. By picking our net points $X_n$ appropriately, we may assume that $E \subseteq \Delta_0 \in \mathcal{D}$.

Lemma 3.4 [Azzam 2014, §3]. Let $\mu$ be a $C_\mu$-doubling measure and $\mathcal{D}$ the cubes from Theorem 3.3 for $X = \text{supp} \, \mu$ with admissible constants $c_0$ and $\rho$. Let $E \subseteq \Delta_0 \in \mathcal{D}$ be a Borel set, $M > 1$, and $\delta > 0$, and set

$$\mathcal{D} = \{\Delta \subseteq \Delta_0 : \Delta \cap E \neq \emptyset, \text{ there exists } \xi \in B_X(\zeta_\Delta, M\ell(\Delta)) \text{ such that dist}(\xi, E) \geq \delta\ell(\Delta)\}.$$

Then there is $C_1 = C_1(M, \delta, C_\mu) > 0$ so that, for all $\Delta' \subseteq \Delta_0$,

$$\sum_{\substack{\Delta \subseteq \Delta' \\Delta \in \mathcal{D}}} \mu(\Delta) \leq C_1\mu(\Delta').$$

The theorem is stated in [Azzam 2014] in slightly more generality. For the reader’s convenience, we provide a shorter proof in the Appendix.

Let $M, \delta > 0$, to be decided later, and let $\mathcal{D}$ be the set from Lemma 3.4 applied to our set $E$. Our goal now is to construct a metric space $Y$ containing $X$, then a curve $\Gamma' \subseteq Y$ that contains $E$ as a subset, and then show it has finite length. We will do this by adding bridges through $Y$ between net points around cubes in $\mathcal{D}$ since these are the cubes where $E$ has large holes and thus potentially has big gaps or disconnections. We don’t need the endpoints of these bridges to be in $E$, but their union plus the set $E$ will be connected. We now proceed with the details.
of $C(X)$. Let

$$Y = X \cup \bigcup_{x, y \in \tilde{X}} [x, y]^*,$$

and define a metric on $Y$ (also denoted by $|\cdot|$) by setting

$$|x - y| = \inf \sum_{i=1}^{N} |x_i - x_{i+1}|$$

where $x_1 = x$, $x_{N+1} = y$, and for each $i$, $\{x_i, x_{i+1}\} \subseteq X$ or $\{x_i, x_{i+1}\} \subseteq [x', y']^*$ for some $x', y' \in \tilde{X}$. It is easy to check that the resulting metric space $Y$ is separable and $X$ is a metric subspace in $Y$. Moreover, the following lemma is immediate from the definition of $Y$.

**Lemma 3.5.** Let $F \subseteq X$ be compact and $x, y \in \tilde{X}$. Then

$$\text{dist}([x, y]^*, F) = \text{dist}([x, y], F).$$

We will let

$$B_\Delta := B_Y(\xi_\Delta, \ell(\Delta)) \supseteq B_X(\xi_\Delta, \ell(\Delta)).$$

For $\Delta \in \mathcal{D}_n$, let

$$\Gamma_\Delta = \bigcup\{[x, y]^* \subseteq C(X) \oplus \mathbb{R}\tilde{X} \times \tilde{X} : x, y \in X_{n+n_0} \cap MB_\Delta\}$$

where $n_0$ is an integer we will pick later. Note that $\Gamma_\Delta$ is connected and contains $\xi_\Delta$.

Now define

$$\Gamma = E \cup \bigcup_{\Delta \in \mathcal{D}_n} \Gamma_\Delta.$$

**Lemma 3.6.**

$$\mathcal{H}^1(\Gamma) < \infty.$$

**Proof.** We first claim that

$$\mathcal{H}^1(E) \leq 10\mu(E). \quad (3-3)$$

Indeed, let $0 < \delta < r_0$. Take any countable collection of balls centered on $E$ of radii less than $\delta$ that cover $E$. Since $\mu$ is doubling, we can use the Vitali covering theorem [Heinonen 2001, Theorem 1.2] to find a countable subcollection of disjoint balls $B_i$ with radii $r_i < \delta$ centered on $E$ so that $E \subseteq \bigcup 5B_i$. Then

$$\mathcal{H}^1_{\delta}(E) \leq \sum_{i} 10r_i \leq 10 \sum \mu(B_i) \leq 10\mu(\{x \in X : \text{dist}(x, E) < \delta\}).$$

Since $\bigcap_{\delta > 0}\{x \in X : \text{dist}(x, E) < \delta\} = E$, sending $\delta \to 0$, we obtain $\mathcal{H}^1(E) \leq 10\mu(E)$, which proves the claim.
With this estimate in hand, we have

$$\mathcal{H}^1(\Gamma) \leq \mathcal{H}^1(E) + \sum_{\Delta \in \mathcal{P}} \mathcal{H}^1(\Gamma_{\Delta}) \leq 10\mu(E) + C \sum_{\Delta \in \mathcal{P}} \ell(\Delta) \leq 10\mu(E) + C \sum_{\Delta \in \mathcal{P}} \mu(\Delta) \leq 10\mu(E) + C \mu(\Delta_0) < \infty$$

where $C$ here stands for various constants that depend only on $\delta$, $M$, $n_0$, $\rho$, and the doubling constant $C_\mu$. \qed

**Lemma 3.7.** $\Gamma$ is compact.

**Proof.** To see this, let $x_n \in \Gamma$ be any sequence. If $x_n \in \Gamma_{\Delta}$ infinitely many times for some $\Delta \in \mathcal{P}$ or is in $E$ infinitely many times, then since each of these sets are compact, we can find a convergent subsequence with a limit in $\Gamma$. Otherwise, $x_n$ visits infinitely many $\Gamma_{\Delta_j}$ where each $\Delta_j \in \mathcal{P}$ is distinct. Then $\ell(\Delta_j) \to 0$, and since $\Delta \cap E \neq \emptyset$ for all $\Delta \in \mathcal{P}$, $\dist(x_n, E) \to 0$. Pick $x'_n \in E \cap \Delta_j$. Since $E$ is compact, there is a subsequence $x'_{n_k}$ converging to a point in $E$, and $x_{n_k}$ will have the same limit. We have thus shown that any sequence in $\Gamma$ has a convergent subsequence, which implies $\Gamma$ is compact. \qed

**Lemma 3.8.** A compact connected metric space $X$ of finite length can be parametrized by a Lipschitz image of an interval in $\mathbb{R}$; that is, $X = f([0, 1])$ where $f : [0, 1] \to X$ is Lipschitz.

A proof of this fact for Hilbert spaces is given in [Schul 2007, Corollary 3.7], but the same proof works in our setting, so we omit it. Hence, to show that $\Gamma$ (and hence $E$) is rectifiable, all that remains to show is that $\Gamma$ is connected.

**Lemma 3.9.** The set $\Gamma$ is connected.

**Proof.** Suppose for the sake of a contradiction that there exist two open and disjoint sets $A$ and $B$ that cover $\Gamma$, and set $\Gamma_A = \Gamma \cap A$ and $\Gamma_B = \Gamma \cap B$. Suppose without loss of generality that $\Gamma_{\Delta_0} \subseteq \Gamma_A$, which we may do since $\Gamma_{\Delta_0}$ is connected. We sort the proof into a series of steps.

(a) $\Gamma_B \subseteq 2B_{\Delta_0}$. To see this, suppose instead that there is $z \in \Gamma_B \setminus 2B_{\Delta_0}$. Then $z \in [x, y]^\ast \subseteq \Gamma_{\Delta}$ for some $\Delta \in \mathcal{P}$. Moreover, $\dist(z, \{x, y\}) \leq 2|x - y| \leq 4M\ell(\Delta)$ since $x, y \in MB_{\Delta}$. Since $\zeta_{\Delta} \in \Delta \subseteq \Delta_0$ and $x \in MB_{\Delta}$, we get

$$\ell(\Delta_0) \leq \dist(z, B_{\Delta_0}) \leq |z - x| + \dist(x, B_{\Delta_0}) \leq 4M\ell(\Delta) + M\ell(\Delta) = 5M\ell(\Delta).$$

For $n_0$ large enough so that $5M\rho^{n_0} < 1$, this implies $\zeta_{\Delta} \in X_{n+n_0} \cap MB_{\Delta_0}$ and so $\Gamma_{\Delta} \cap \Gamma_{\Delta_0} \neq \emptyset$. Hence, $\Gamma \subseteq \Gamma_A$ since $\Gamma_{\Delta}$ is connected, contradicting that $z \in \Gamma_B$. This proves the claim.

(b) The open sets $A' = A \cup (4B_{\Delta_0})^c$ and $B' = B \cap 2B_{\Delta_0}$ are disjoint and cover $\Gamma$. First, observe that

$$A' \cap B' = (A \cap B \cap 2B_{\Delta_0}) \cup ((4B_{\Delta_0})^c \cap B \cap 2B_{\Delta_0}) \subseteq (A \cap B) \cup ((4B_{\Delta_0})^c \cap 2B_{\Delta_0}) = \emptyset.$$
Moreover, by part (a),
\[ \Gamma \cap (A' \cup B') \supseteq \Gamma_A \cup (\Gamma_B \cap 2B_{\Delta_0}) = \Gamma_A \cup \Gamma_B = \Gamma, \]
which completes the proof of this step.

(c) Set \( \Gamma_{A'} = \Gamma \cap A' \) and \( \Gamma_{B'} = \Gamma \cap B' \). These sets are disjoint by part (b), and hence, they are compact since \( \Gamma \) was compact. We define new open sets
\[ A'' = (4B_{\Delta_0})^c \cup \bigcup_{\xi \in \Gamma_{A'}} B_Y(\xi, \text{dist}(\xi, \Gamma_{B'})/2) \]
and
\[ B'' = \bigcup_{\xi \in \Gamma_{B'}} B_Y(\xi, \text{dist}(\xi, \Gamma_{A'})/2). \]

We claim these sets are disjoint. Suppose there is \( z \in A'' \cap B'' \). Then \( z \in B_Y(\xi, \text{dist}(\xi, \Gamma_{A'})/2) \) for some \( \xi \in \Gamma_{B'} \). If we also have \( z \in B_Y(\xi', \text{dist}(\xi', \Gamma_{B'})/2) \) for some \( \xi' \in \Gamma_{A'} \), then
\[
\max\{\text{dist}(\xi, \Gamma_{B'}), \text{dist}(\xi', \Gamma_{A'})\} \leq |\xi - \xi'| \leq |\xi - z| + |z - \xi| < \frac{\text{dist}(\xi, \Gamma_{B'})}{2} + \frac{\text{dist}(\xi', \Gamma_{A'})}{2},
\]
which is a contradiction, so we must have \( z \in (4B_{\Delta_0})^c \). Since \( \xi \in \Gamma_{B'} \), we know \( \xi \in 2B_{\Delta_0} \) by part (a), and \( \xi_{\Delta_0} \in \Gamma_{\Delta_0} \subseteq \Gamma_{A'} \) implies \( \text{dist}(\xi, \Gamma_{A'}) \leq 2\ell(\Delta_0) \). Hence,
\[ B_Y(\xi, \text{dist}(\xi, \Gamma_{A'})/2) \subseteq B_Y(\xi, 3\ell(\Delta_0)) \subseteq B_Y(\xi_{\Delta_0}, 3\ell(\Delta_0)) = 3B_{\Delta_0}, \]
which proves the claim.

(d) Note that \( X \setminus (A'' \cup B'') \) is nonempty since \( X \) is connected and \( A'' \) and \( B'' \) are disjoint open sets. Moreover, \( X \setminus (A'' \cup B'') \subseteq 4B_{\Delta_0} \) and hence a bounded set; since \( X \) is a doubling metric space, \( X \setminus (A'' \cup B'') \) is in fact totally bounded and thus compact by the Heine–Borel theorem. This implies we can find a point
\[ z \in X \setminus (A'' \cup B'') \subseteq 4B_{\Delta_0} \]
of maximal distance from the compact set \( \Gamma \).

(e) Let \( \xi \in E \) be the closest point to \( z \) and \( \Delta \) the smallest cube containing \( \xi \) so that \( z \in 5B_{\Delta} \); since \( z \in 4B_{\Delta_0} \subseteq 5B_{\Delta_0} \), this is well defined. We claim \( \Delta \in \mathcal{P} \). If \( \Delta_1 \) denotes the child of \( \Delta \) that contains \( \xi \), then \( z \notin 5B_{\Delta_1} \), and so
\[
\text{dist}(z, E) = |\xi - z| > |z - \xi_{\Delta_1}| - |\xi_{\Delta_1} - \xi| \geq 5\ell(\Delta_1) - \ell(\Delta_1) = 4\rho\ell(\Delta).
\]
Thus, for \( M > 10 \), \( B_X(z, 4\rho\ell(\Delta)) \subseteq MB_{\Delta} \setminus E \), so if \( \delta < 4\rho \), then \( \Delta \in \mathcal{P} \), which proves the claim.

(f) Since \( \Delta \in \mathcal{P} \), \( X_{n(\Delta)+\rho_0} \) is a maximal \( \rho^{n(\Delta)+\rho_0} \)-net,
\[
\rho^{n(\Delta)+\rho_0} < \rho^{n_0}\ell(\Delta) < \ell(\Delta),
\]
and \( z \in 5B_{\Delta} \), we can find
\[
\zeta \in X_{n(\Delta)+n_0} \cap B_X(z, \rho^{n(\Delta)+n_0}) \quad (3-5)
\subseteq X_{n(\Delta)+n_0} \cap B_X(\zeta, 6\ell(\Delta) + \rho^{n(\Delta)+n_0})
\subseteq X_{n(\Delta)+n_0} \cap B_X(\zeta, 6\ell(\Delta)) \subseteq \Gamma_{\Delta},
\quad (3-6)
\]
where the last containment follows if we assume \( M > 6 \).

Since \( \Gamma_{\Delta} \) is connected and \( A' \) and \( B' \) are disjoint open sets, we may without loss of generality suppose \( \Gamma_{A'} \supseteq \Gamma_{\Delta} \) and let \( \zeta' \in \Gamma_{B'} \) be the closest point to \( \zeta \). Then
\[
|z - \zeta| \geq |z - \zeta'|/2 = \text{dist}(\zeta, \Gamma_{B'})/2 \quad (3-7)
\]
since otherwise would imply \( z \in B_Y(\zeta, \text{dist}(\zeta, \Gamma_{B'})/2) \subseteq \Lambda'' \), contradicting that \( z \in X \setminus (\Lambda'' \cup B'') \).

We may assume \( \zeta' \in \Gamma_{\Delta'} \) for some \( \Delta' \in \mathcal{P} \), and we assume \( \Delta' \) is the largest such cube for which this happens. Note that this implies \( \Gamma_{\Delta'} \subseteq \Gamma_{B'} \) since \( \zeta' \in \Gamma_{B'} \cap \Gamma_{\Delta'} \) and \( \Gamma_{\Delta'} \) is connected. By Lemma 3.5 with \( F = \{\zeta\} \), we can assume \( \zeta' \in X \), and so \( \zeta' \in X_{n(\Delta') + n_0} \cap MB_{\Delta'} \).

(g) We claim that \( n(\Delta) + 1 \leq n(\Delta') \leq n(\Delta) + 2 \). Note that, since
\[
5\rho^{n(\Delta)+n_0} \leq \ell(\Delta)\rho^{n_0} \leq \rho \ell(\Delta) < \ell(\Delta),
\quad (3-8)
\]
we have
\[
|\zeta' - \zeta_{\Delta}| \leq |\zeta' - \zeta| + |\zeta - \zeta_{\Delta}| \leq 2|z - \zeta| + 6\ell(\Delta) \leq 2\rho^{n(\Delta)+n_0} + 6\ell(\Delta) \leq 8\ell(\Delta). \quad (3-9)
\]
Thus, for \( M > 8 \), we must have \( n(\Delta') > n(\Delta) \); otherwise, since \( \zeta' \in \Delta \subseteq B_{\Delta} \), we would have
\[
\zeta' \in X_{n(\Delta')+n_0} \cap 8B_{\Delta} \subseteq X_{n(\Delta)+n_0} \cap MB_{\Delta} \subseteq \Gamma_{\Delta}
\]
so that \( \Gamma_{\Delta} \cap \Gamma_{\Delta'} \neq \emptyset \), which implies \( \Gamma_{A'} \cap \Gamma_{B'} \neq \emptyset \), a contradiction. Thus, \( \ell(\Delta') < \ell(\Delta) \), which proves the first inequality in the claim.

Note this implies \( \ell(\Delta') \leq \rho \ell(\Delta) \). Let \( \xi' \in \Delta' \cap E \) (which exists since \( \Delta' \in \mathcal{P} \)). Since \( \zeta' \in MB_{\Delta'} \),
\[
4\rho \ell(\Delta) \overset{(3-4)}{\leq} \text{dist}(z, E) \leq |\xi' - z| \leq |\xi' - \zeta_{\Delta'}| + |\zeta_{\Delta'} - \zeta'| + |\zeta' - \zeta| + |\zeta - z| \\
\overset{(3-7)}{\leq} \ell(\Delta') + M\ell(\Delta') + 2|z - \zeta'| + |\zeta - z| \leq (M + 1)\ell(\Delta') + 3\rho^{n(\Delta)+n_0} \\
\overset{(3-8)}{\leq} (M + 1)\ell(\Delta') + \rho \ell(\Delta)
\]
and so
\[
\frac{3\rho}{M + 1} \ell(\Delta) \leq \ell(\Delta').
\]
Thus, \( \rho < 3/(M + 1) \) implies \( \rho^2 \ell(\Delta) \leq \ell(\Delta') \), and so \( n(\Delta') \leq n(\Delta) + 2 \), which finishes the claim.

(h) Now we'll show that \( \Gamma_{\Delta} \cap \Gamma_{\Delta'} \neq \emptyset \). Observe that
\[
|\zeta_{\Delta} - \zeta_{\Delta'}| \leq |\zeta_{\Delta} - \zeta'| + |\zeta' - \zeta_{\Delta'}| \overset{(3-9)}{\leq} 8\ell(\Delta) + M\ell(\Delta') \leq (8 + M\rho)\ell(\Delta) < M\ell(\Delta)
\quad (3-10)
\]
if $\rho^{-1} > M > 9$. Since $n(\Delta') \leq n(\Delta) + 2$, we have that $\zeta_{\Delta'} \in X_{n(\Delta) + n_0} \cap MB_{\Delta}$ for $n_0 \geq 2$ and so $\zeta_{\Delta'} \in \Gamma_{\Delta}$. But $\zeta_{\Delta'} \in X_{n(\Delta) + n_0} \cap MB_{\Delta'} \subseteq \Gamma_{\Delta'}$; thus, $\Gamma_{\Delta} \cap \Gamma_{\Delta'} \neq \emptyset$, which proves the claim.

This gives us a grand contradiction since $\Gamma_{\Delta} \subseteq \Gamma_{\Delta'}$ and $\Gamma_{\Delta'} \subseteq \Gamma_{B'}$, and we assumed these sets to be disjoint. \hfill $\square$

Combining Lemmas 3.6, 3.7, 3.8, and 3.9, we have now shown that $E$ is contained in the Lipschitz image of an interval in $\mathbb{R}$. This completes the proof of Lemma 3.2. \hfill $\square$  

**Appendix: Proof of Lemma 3.4**

For $\Delta \in \mathcal{D}$, define $B_{\Delta} = B_X(\zeta_{\Delta}, \ell(\Delta))$. For $\Delta \in \mathcal{D}$, let $\xi_{\Delta} \in MB_{\Delta}$ be such that $\text{dist}(\xi_{\Delta}, E) \geq \delta \ell(\Delta)$. Let $\mathcal{M}$ be the collection of maximal cubes for which $2B_{\Delta} \subseteq E^c$ and $\tilde{\Delta} \in \mathcal{M}$ be the largest cube containing $\xi_{\Delta}$. Then if $\tilde{\Delta}^1$ denotes the parent cube of $\tilde{\Delta}$, $2B_{\tilde{\Delta}^1} \cap E \neq \emptyset$, and so

$$\delta \ell(\Delta) \leq \text{dist}(\xi_{\Delta}, E) \leq \text{diam } 2B_{\tilde{\Delta}^1} \leq 4\ell(\tilde{\Delta}^1) = \frac{4}{\rho} \ell(\tilde{\Delta}). \quad (A-1)$$

Moreover,

$$\ell(\tilde{\Delta}) \leq \frac{2M}{c_0} \ell(\Delta), \quad (A-2)$$

for otherwise $\tilde{\Delta} \supseteq c_0B_{\Delta} \supseteq MB_{\Delta} \supseteq \Delta$, and since $\Delta \cap E \neq \emptyset$, this means $2B_{\tilde{\Delta}} \cap E \neq \emptyset$, contradicting our definition of $\tilde{\Delta}$.

Let $N_{\Delta}$ be such that

$$2^{N_{\Delta}}c_0 \ell(\tilde{\Delta}) > 2M \ell(\Delta) > 2^{N_{\Delta} - 1}c_0 \ell(\tilde{\Delta}). \quad (A-3)$$

Then $2^{N_{\Delta}}c_0B_{\tilde{\Delta}} \supseteq MB_{\Delta}$, and $2^{N_{\Delta}} < \frac{4M\ell(\Delta)}{c_0 \ell(\tilde{\Delta})}$, so

$$N_{\Delta} < \log_2 \left( \frac{4M\ell(\Delta)}{c_0 \ell(\tilde{\Delta})} \right). \quad (A-4)$$

Thus,

$$\frac{\mu(\tilde{\Delta})}{\mu(\Delta)} \geq \frac{\mu(c_0B_{\Delta})}{\mu(\Delta)} \geq \frac{\mu(2^{N_{\Delta}}c_0B_{\Delta})}{\mu(\Delta)} \geq \frac{\mu(MB_{\Delta})}{\mu(\Delta)} \geq \log_2 \frac{\ell(\tilde{\Delta})}{\ell(\Delta)} \geq \log_2 \frac{c_{\mu}(4)}{c_{\mu}(4M)} =: a. \quad (A-5)$$

Since $\mu$ is doubling and $\Delta$ and $\Delta'$ are always of comparable sizes by (A-1) and (A-2), there is $b$ depending on $M$, $\delta$, $\rho$, $c_0$, and $C_{\mu}$ such that at most $b$ many cubes $\Delta \in \mathcal{M}$ with $\tilde{\Delta} = \Delta'$ for some fixed $\Delta'$. Hence, for $\Delta' \subseteq \Delta_0$ with $\Delta \cap E \neq \emptyset$,

$$\sum_{\Delta \in \mathcal{D}} \mu(\Delta) \leq \sum_{\Delta \in \mathcal{D}} a\mu(\tilde{\Delta}) = \sum_{\Delta' \in \mathcal{M}} \sum_{\Delta \subseteq MB_{\Delta}} a\mu(\tilde{\Delta}) \leq \sum_{\Delta' \in \mathcal{M}} \sum_{\Delta \subseteq MB_{\Delta}} a\mu(\Delta') \leq ab\mu(MB_{\Delta_0}) \leq abC_{\mu}^{\log_2 M/c_0 + 1} \mu(c_0B_{\Delta_0}) \leq abC_{\mu}^{\log_2 M/c_0 + 1} \mu(\Delta_0).$$
This finishes the proof of Lemma 3.4.

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