The multiplier algebra and BSE-functions for
certain product of Banach algebras

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Abstract

In this paper, we characterize the (left) multiplier algebra of a semidirect product algebra \( A = B \oplus I \), where \( I \) and \( B \) are closed two-sided ideal and closed subalgebra of \( A \), respectively. As an application of this result we investigate the BSE-property of this class of Banach algebras. We then for two commutative semisimple Banach algebras \( A \) and \( B \) characterize the BSE-functions on the carrier space of \( A \times_\phi B \), the \( \phi \)-Lau product of \( A \) and \( B \), in terms of those functions on carrier spaces of \( A \) and \( B \). We also prove that \( A \times_\phi B \) is a BSE-algebra if and only if both \( A \) and \( B \) are BSE.

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1 Introduction

Let \( A \) be a Banach algebra and suppose that \( I \) and \( B \) are closed two-sided ideal and closed subalgebra of \( A \), respectively. Following Bade and Dales [1, 2] we say that \( A \) is a semidirect product of \( B \) and \( I \) if \( A = B \oplus I \). Indeed, in this case the product of two elements \((b, a)\) and \((b', a')\) of \( B \oplus I \) is given by

\[
(b, a)(b', a') = (bb', aa' + ba' + ab'),
\]

and this algebra endowed with the norm \( \|(b, a)\| = \|b\| + \|a\| \) is a Banach algebra. This notion was also studied by Berndt [4] and Thomas [21, 22] and they considered under what conditions a commutative Banach algebra is the semidirect product of a subalgebra and a principal ideal. We also note that the algebra \( B \oplus I \) is a splitting extension of \( B \) by \( I \). These products not only induce new examples of Banach algebras which are interesting in its own right but also they are known as a fertile source as examples or counterexamples in functional and abstract harmonic analysis; see for example [16, 5]. An example of this type of algebras, which is of special interest, is the \( \phi \)-Lau product of two Banach algebras \( A \) and \( B \). In fact, suppose that \( \phi : B \to A \) is a contractive algebra homomorphism. Then the \( \phi \)-Lau product of \( A \) and \( B \), denoted by \( A \times_\phi B \), is defined as the \( \ell^1 \)-product space \( A \times B \) endowed with the norm \( \|(a, b)\| = \|a\| + \|b\| \) and the product

\[
(a, b)(a', b') = (aa' + \phi(b)a' + a\phi(b'), bb').
\]
for all \((a,b),(a',b') \in A \times B\). It is clear that with this norm and product, \(A \times \phi B\) is a Banach algebra. Identifying \(A\) with \(A \times \{0\}\) and \(B\) with \(\{0\} \times B\), \(A\) is a closed ideal of \(A \times \phi B\) and \(B\) is a closed subalgebra. Therefore, \(A \times \phi B\) is a semidirect product algebra of \(B\) and \(A\). This type of product was first introduced by Bhatt and Dabhi [3] for the case where \(A\) is commutative and was extended by the authors for the general case [10, 17].

In this paper, we characterize the (left) multiplier algebra of \(B \oplus I\) in terms of the individual (left) multiplier algebras of \(B\) and \(I\). We then apply this result to investigate the BSE-property of this class of Banach algebras. For two commutative semisimple Banach algebras \(A\) and \(B\) we characterize the BSE-functions on the carrier space \(\Delta(A \times \phi B)\) of \(A \times \phi B\), in terms of those functions on \(\Delta(A)\) and \(\Delta(B)\). We also show that the \(\phi\)-Lau product of two commutative semisimple Banach algebras \(A\) and \(B\) is BSE if and only if both \(A\) and \(B\) are BSE. Finally, we use these results to study the BSE-property of certain Banach algebras related to a locally compact group \(G\).

2 (left) Multipliers on semidirect product algebras

Let \(C\) be a Banach algebra and let \(X\) and \(Y\) be two Banach \(C\)-bimodules. An operator \(T : X \to Y\) is called a left \(C\)-module map if \(T(c \cdot x) = c \cdot T(x)\) for all \(c \in C\) and \(x \in X\). Right \(C\)-module and \(C\)-bimodule maps are defined similarly. We denote by \(\text{Hom}_C(X,Y)\) the space of all bounded left \(C\)-module maps from \(X\) into \(Y\). Define \(LM(C)\) to be the left multiplier algebra of \(C\). That is,

\[ LM(C) = \text{Hom}_C(C,C). \]

Suppose now that \(A\) is the semidirect product algebra \(B \oplus I\). Note that the dual \(A^\ast\) of \(A\) can be identified with \(B^\ast \times I^\ast\) in the natural way given for each \(f \in I^\ast, g \in B^\ast\) and \((b,a) \in B \oplus I\) by

\[ ((g,f),(b,a)) = g(b) + f(a). \]

We start the section with a key lemma.

**Lemma 2.1** Let \(A\) be the semidirect product algebra \(B \oplus I\). Then \(T \in LM(A)\) if and only if there exist some \(R_I \in \text{Hom}_B(I,I), S_I \in \text{Hom}_B(B,I), T_B \in LM(B)\) and \(S_B \in \text{Hom}_B(I,B)\) such that for each \(a,a' \in I\) and \(b,b' \in B\) we have

(i) \(T((b,a)) = (S_B(a) + T_B(b), R_I(a) + S_I(b)).\)

(ii) \(R_I(aa') = aR_I(a') + aS_B(a').\)

(iii) \(R_I(ab) = aS_I(b) + aT_B(b).\)

(iv) \(S_B(aa') = S_B(ab) = 0.\)

**Proof.** Suppose that \(T \in LM(A)\). Then there exist bounded linear mappings \(T_1 : A \to B\) and \(T_2 : A \to I\) such that \(T = (T_1,T_2)\). Let \(S_B(a) = T_1((a,0)), S_I(b) = T_2((0,b)), T_B(b) = T_1((0,b))\) and \(R_I(a) = T_2((a,0))\) for all \(a \in I\) and \(b \in B\). Then trivially \(S_B, S_I, T_B\) and \(R_I\) are linear mappings satisfying (i). Moreover, for every \(a,a' \in I\) and \(b,b' \in B\)

\[(1)\quad T((b,a)(b',a')) = T((bb',aa' + ba' + ab'))\]
\[ = \left( S_B(aa' + ba' + ab') + \sum_{\delta \in \Delta(B)} \Delta(\delta) T_B(b\delta), R\right) \]

and
\[
(b, a) T((b', a')) = (b, a) \left( S_B(a' + T_B(b'), R\right) + S_{I}(b'))
\]
\[
= \left( bS_B(a') + bT_B(b'), aR\right) + aS_{I}(b')
\]
\[
+ aS_{I}(a') + aR\right) + bS_{I}(b').
\]

Thus \( T \in LM(A) \) if and only if (1) and (2) coincide; that is,
\[
S_B(aa' + ba' + ab') + T_B(bb') = bS_B(a') + bT_B(b'),
\]
and
\[
R\right)(aa' + ba' + ab') + S_{I}(bb') = aR\right)(a') + aS_{I}(b') + aS_B(a')
\]
\[
+ aT_B(b') + bR\right)(a') + bS_{I}(b').
\]

Therefore \( T \in LM(A) \) if and only if the equations (3) and (4) are satisfied. Now a straightforward verification shows that if \( R\right) \in Hom_B(I, I), S_I \in Hom_B(B, I), T_B \in LM(B) \) and \( S_B \in Hom_B(I, B) \) and the equalities (ii), (iii), (iv) are satisfied, then (3) and (4) are valid. Applying (3) and (4) for suitable values of \( a, a', b, b' \) shows that \( R\right) \in Hom_B(I, I), S_I \in Hom_B(B, I), T_B \in LM(B) \) and \( S_B \in Hom_B(I, B) \) and the equalities (ii), (iii), (iv) are also satisfied, as claimed.

For a right Banach \( C \)-module \( X \) we denote by \( \langle XC \rangle \) the closed linear span of the set \( XC := \{x \cdot c : c \in C, x \in X \} \). As an immediate consequence of Lemma 2.1 we have the following result.

**Corollary 2.2** Let \( A \) be the semidirect product algebra \( B \oplus I \) such that either \( \langle IB \rangle = I \) or \( \langle I^2 \rangle = I \). Then \( T \in LM(A) \) if and only if there exist \( R\right) \in Hom_B(I, I) \cap LM(I), S_I \in Hom_B(B, I) \) and \( T_B \in LM(B) \) such that \( T\right)(ab) = aS\right)(b) + aT_B(b) \) and
\[
T((b, a)) = (T_B(b), T\right)(a) + S_{I}(b))
\]
for all \( a \in I \) and \( b \in B \).

**Remark 2.3** Let \( A \) be a Banach algebra and suppose that \( I \) and \( B \) are closed two-sided ideal and closed subalgebra of \( A \), respectively. Then for each \( \varphi \in \Delta(I) \) there is a unique \( \psi_\varphi \in \Delta(B) \cup \{0\} \) such that \( \varphi(ab) = \varphi(ba) = \varphi(a)\psi_\varphi(b) \) for all \( a \in I \) and \( b \in B \). Indeed, \( \psi_\varphi \in \Delta(B) \cup \{0\} \) is defined by \( \psi_\varphi(b) := \varphi(ba_0) \) for all \( b \in B \), where \( a_0 \in I \) is any element with \( \varphi(a_0) = 1 \); to see this, note that for each \( a \in I \) and \( b \in B \),
\[
\varphi(ab) = \varphi(ba_0) = \varphi(a)\varphi(ba_0) = \varphi(a)\psi_\varphi(b).
\]
That \( \psi_\varphi \) is unique follows trivially. We note that if \( \langle IB \rangle = I \), then \( \psi_\varphi \neq 0 \) for all \( \varphi \in \Delta(I) \).
Proposition 2.4 Let \( \mathcal{A} \) be the semidirect product algebra \( \mathcal{B} \oplus \mathcal{I} \) and let

\[
E := \{(\psi, \varphi) : \varphi \in \Delta(\mathcal{I})\} \quad \text{and} \quad F := \{(\psi, 0) : \psi \in \Delta(\mathcal{B})\}.
\]

Then \( E \) and \( F \) are disjoint and \( \Delta(\mathcal{A}) = E \cup F \).

Proof. It is clear that \( E \cap F = \emptyset \) and \( E \cup F \subseteq \Delta(\mathcal{A}) \). For the converse, suppose that \( (\psi, \varphi) \in \Delta(\mathcal{A}) \). Then, for each \( (b, a), (b', a') \in \mathcal{A} \) we have

\[
((\psi, \varphi), (b, a)(b', a')) = ((\psi, \varphi), (b, a))((\psi, \varphi), (b', a')).
\]

This implies that

\[
\psi(bb') + \varphi(ab' + ba' + aa') = \psi(b)\psi(b') + \psi(b)\varphi(a') + \varphi(a)\psi(b') + \varphi(a)\varphi(a').
\]

Taking \( b = b' = 0 \), it follows that \( \varphi(aa') = \varphi(a)\varphi(a') \) and therefore \( \varphi \in \Delta(\mathcal{I}) \cup \{0\} \). Similarly, we can see that \( \psi \in \Delta(\mathcal{B}) \cup \{0\} \). Now, if \( a = b' = 0 \), then we have \( \varphi(ba') = \psi(b)\varphi(a') \), similarly \( \varphi(a'b) = \varphi(a)\psi(b') \) for all \( a, a' \in \mathcal{I} \) and \( b, b' \in \mathcal{B} \). The equality \( \varphi = 0 \) implies that \( \psi \neq 0 \). If \( \varphi \neq 0 \), then \( \psi = \psi_\varphi \) by the above remark. \( \blacksquare \)

3 The BSE-property of semidirect product algebras

A commutative Banach algebra \( \mathcal{C} \) is called without order if for each \( c \in \mathcal{C} \), \( cc = \{0\} \) implies \( c = 0 \). For example, if \( \mathcal{C} \) is a commutative semisimple Banach algebra, then it is without order. Let \( \mathcal{C} \) be a commutative Banach algebra with carrier space \( \Delta(\mathcal{C}) \) and let \( \mathcal{C}^* \) denote the dual space of \( \mathcal{C} \). A continuous complex-valued function \( \sigma \) on \( \Delta(\mathcal{C}) \) is said to satisfy the Bochner-Schoenberg-Eberlein (BSE) inequality if there exists a constant \( C > 0 \) such that for any \( \varphi_1, ..., \varphi_n \in \Delta(\mathcal{C}) \) and \( c_1, ..., c_n \in \mathbb{C} \) the inequality

\[
\left| \sum_{j=1}^{n} c_j \sigma(\varphi_j) \right| \leq C \| \sum_{j=1}^{n} c_j \varphi_j \|_{\mathcal{C}^*}
\]

holds. Let \( C_{BSE}(\Delta(\mathcal{C})) \) denote the set of all continuous complex-valued functions on \( \Delta(\mathcal{C}) \) satisfying the BSE-inequality. The BSE-norm of \( \sigma \) denoted by \( \| \sigma \|_{BSE} \), is defined to be the infimum of all such \( C \). Takahasi and Hatori [19] showed that under this norm \( C_{BSE}(\Delta(\mathcal{C})) \) is a commutative semisimple Banach algebra. A linear operator \( T \) on \( \mathcal{C} \) is called a multiplier if it satisfies \( cT(b) = T(c)b \) for all \( b, c \in \mathcal{C} \). Suppose that \( M(\mathcal{C}) \) denotes the space of all multiplier of the commutative Banach algebra \( \mathcal{C} \) which is a unital commutative Banach algebra. Recall that for each \( T \in M(\mathcal{C}) \) there exists a unique continuous function \( \hat{T} \) on \( \Delta(\mathcal{C}) \) such that \( \hat{T}(c)(\varphi) = \hat{T}(\varphi)c(\varphi) \) for all \( c \in \mathcal{C} \) and \( \varphi \in \Delta(\mathcal{C}) \); see [15, Theorem 1.2.2]. A commutative Banach algebra \( \mathcal{C} \) without order is called a BSE-algebra (or is said to have the BSE-property) if

\[
C_{BSE}(\Delta(\mathcal{C})) = \hat{M}(\mathcal{C}).
\]
The concept of BSE-property has been first introduced and studied by Takahasi and Hatori [19] and later by several authors in various classes of commutative Banach algebras; see also [7, 8, 9, 12, 13, 14, 20].

A bounded net $(e_n)_n$ in a Banach algebra $C$ is called a $\Delta$-weak bounded approximate identity if it satisfies $\varphi(e_n) \to 1$ for all $\varphi \in \Delta(C)$. Such approximate identities were studied in [11]. It was shown in [19, Corollary 5] that a commutative Banach algebra $C$ has a $\Delta$-weak bounded approximate identity if and only if $\widehat{M(C)} \subseteq C_{BSE}(\Delta(C))$.

In the sequel, we assume that $A$ is a commutative semidirect product algebra of semisimple closed subalgebra $B$ and semisimple closed ideal $I$.

**Corollary 3.1** Let $A$ be the commutative semidirect product algebra $B \oplus I$. Then $T \in M(A)$ if and only if there exist $T_\mathcal{I} \in Hom_B(\mathcal{I}, \mathcal{I}) \cap M(\mathcal{I})$, $S_\mathcal{I} \in Hom_B(\mathcal{B}, \mathcal{I})$ and $T_B \in M(B)$ such that $T_\mathcal{I}(ab) = aS_\mathcal{I}(b) + aT_B(b)$ and

$$T((b,a)) = (T_B(b), T_\mathcal{I}(a) + S_\mathcal{I}(b))$$

for all $a \in I$ and $b \in B$.

Proof. Suppose that $T \in M(A)$. By Lemma 2.1, there exist some $R_\mathcal{I} \in Hom_B(\mathcal{I}, \mathcal{I})$, $S_\mathcal{I} \in Hom_B(\mathcal{B}, \mathcal{I})$, $T_B \in M(B)$ and $S_B \in Hom_B(\mathcal{I}, B)$ such that for each $a \in I$ and $b \in B$ we have

$$T((b,a)) = (S_B(a) + T_B(b), R_\mathcal{I}(a) + S_\mathcal{I}(b)).$$

Moreover, $S_B(ba) = 0$. Since $S_B \in Hom_B(\mathcal{I}, B)$ and $B$ is without order, it follows that $S_B = 0$ and $R_\mathcal{I} \in M(\mathcal{I})$. The converse is clear. 

**Proposition 3.2** Let $A$ be the semidirect product algebra $B \oplus I$ such that $\langle IB \rangle = I$. Then $B$ is a BSE-algebra if $A$ is so.

Proof. Suppose that $\rho \in C_{BSE}(\Delta(B))$. Define a function $\sigma$ on $\Delta(A)$ by

$$\sigma(\psi, 0) = \rho(\psi), \quad \sigma(\psi, \varphi) = \rho(\psi \varphi)$$

for all $\psi \in \Delta(B)$ and $\varphi \in \Delta(I)$. Then $\sigma$ is continuous since $\langle IB \rangle = I$ and so $\psi \varphi \neq 0$ for all $\varphi \in \Delta(I)$. Moreover, since $\rho \in C_{BSE}(\Delta(B))$, by [19, Theorem 4(i)] there is a bounded net $(b_\alpha)_\alpha$ in $B$ such that $\widehat{b_\alpha}(\psi) \to \sigma(\psi)$ for all $\psi \in \Delta(B)$. Now, if we consider $(b_\alpha, 0)_\alpha$ as a net in $A$, then

$$\widehat{(b_\alpha, 0)}(\psi, 0) \to \sigma(\psi) = \sigma(\psi, 0)$$

for all $\psi \in \Delta(B)$ and

$$\widehat{(b_\alpha, 0)}(\psi \varphi, \varphi) \to \rho(\psi \varphi, \varphi) = \sigma(\psi \varphi, \varphi)$$

for all $\varphi \in \Delta(I)$. Thus, $\sigma \in C_{BSE}(\Delta(A))$ and therefore, there is some $T \in M(A)$ such that $\widehat{T} = \sigma$. By Corollary 3.1, there exist some $T_\mathcal{I} \in Hom_B(\mathcal{I}, \mathcal{I})$, $S_\mathcal{I} \in Hom_B(\mathcal{B}, \mathcal{I})$, $T_B \in M(B)$ such that for each $a \in I$ and $b \in B$ we have

$$T((b,a)) = (T_B(b), T_\mathcal{I}(a) + S_\mathcal{I}(b))$$
On the other hand, $\hat{T}(b,0)(\psi,0) = \hat{T}(\psi,0)(\hat{b},0)(\psi,0)$ for all $\psi \in \Delta(\mathcal{B})$ and $b \in \mathcal{B}$. Thus,

$$\psi(T_B(b)) = \hat{T}(\psi,0)\psi(b).$$

It follows that $\hat{T}(\mathcal{B})(\psi) = \hat{T}(\psi,0)\hat{b}(\psi)$. Moreover, $\hat{T}_\mathcal{B}(b)(\psi) = \hat{T}_\mathcal{B}(\psi)\hat{b}(\psi)$ for all $b \in \mathcal{B}$. Therefore, $\hat{T}((\psi,0)) = \hat{T}(\psi,0)$ and consequently,

$$\rho(\psi) = \sigma(\psi,0) = \hat{T}(\psi,0) = \hat{T}_\mathcal{B}(\psi)$$

for all $\psi \in \Delta(\mathcal{B})$. This means that $\rho \in \hat{M}(\mathcal{B})$; that is, $C_{\text{BSE}}(\Delta(\mathcal{B})) \subseteq \hat{M}(\mathcal{B})$. The reverse inclusion follows from this fact that if $\mathcal{A}$ has a $\Delta$-weak bounded approximate identity, then so does $\mathcal{B}$. ■

Now, for two Banach algebras $\mathcal{A}$ and $\mathcal{B}$ we turn our attention to the BSE-property of direct sum algebra $\mathcal{A} \times_0 \mathcal{B}$. Recall that $\mathcal{A} \times_0 \mathcal{B}$ is equipped with the usual direct product multiplication and the norm defined by $\| (a, b) \| = \| a \| + \| b \|$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Also, it is clear that $\Delta(\mathcal{A} \times_0 \mathcal{B}) = E \cup F$, where

$$E = \{ (\varphi, 0) : \varphi \in \Delta(\mathcal{A}) \} \text{ and } F = \{ (0, \psi) : \psi \in \Delta(\mathcal{B}) \}.$$ 

We have the following corollary which was obtained independently in [12] through different methods. For sake of completeness and as an application of Corollary 3.1, we will give a proof of the corollary.

**Corollary 3.3** Let $\mathcal{A}$ and $\mathcal{B}$ be two commutative semisimple Banach algebras. Then $\mathcal{A} \times_0 \mathcal{B}$ is a BSE-algebra if and only if $\mathcal{A}$ and $\mathcal{B}$ are BSE-algebras.

Proof. It is easily verified that $C_0(\Delta(\mathcal{A} \times_0 \mathcal{B})) = C_0(\Delta(\mathcal{A})) \times_0 C_0(\Delta(\mathcal{B}))$. Indeed, $\sigma \in C_0(\Delta(\mathcal{A} \times_0 \mathcal{B}))$ if and only if there are $\tau \in C_0(\Delta(\mathcal{A}))$ and $\rho \in C_0(\Delta(\mathcal{B}))$ such that $\sigma = (\tau, \rho)$; that is,

$$\sigma(\varphi, 0) = \tau(\varphi) \text{ and } \sigma(0, \psi) = \rho(\psi)$$

for all $\varphi \in \Delta(\mathcal{A})$ and $\psi \in \Delta(\mathcal{B})$. Moreover,

$$C_{\text{BSE}}(\Delta(\mathcal{A} \times_0 \mathcal{B})) = C_0(\Delta(\mathcal{A} \times_0 \mathcal{B})) \cap (\mathcal{A}^{**} \times_0 \mathcal{B}^{**})|_{\Delta(\mathcal{A} \times_0 \mathcal{B})},$$

by [19, Theorem 4]. This shows that $C_{\text{BSE}}(\Delta(\mathcal{A} \times_0 \mathcal{B})) = C_{\text{BSE}}(\Delta(\mathcal{A})) \times_0 C_{\text{BSE}}(\Delta(\mathcal{B}))$. On the other hand, by Corollary 3.1 we have $M(\mathcal{A} \times_0 \mathcal{B}) = M(\mathcal{A}) \times_0 M(\mathcal{B})$. It is also easy to check that

$$M(\hat{\mathcal{A}} \times_0 \hat{\mathcal{B}}) = \hat{M}(\mathcal{A}) \times_0 \hat{M}(\mathcal{B}).$$

The rest of the proof is straightforward. ■
4 The BSE-property of $\phi$-Lau product algebras

Let $\mathcal{A}$, $\mathcal{B}$ and $\phi$ be as in the introduction. If $\phi = 0$, then the algebra $\mathcal{A} \times_{\phi} \mathcal{B}$ coincides with the direct sum algebra $\mathcal{A} \times_{0} \mathcal{B}$. We also recall from [10, Theorem 2.2] that $\Delta(\mathcal{A} \times_{\phi} \mathcal{B}) = E \cup F$, where

$$E = \{ (\varphi, \varphi \circ \phi) : \varphi \in \Delta(\mathcal{A}) \} \quad \text{and} \quad F = \{ (0, \psi) : \psi \in \Delta(\mathcal{B}) \}.$$ 

In the sequel for two commutative semisimple Banach algebras $\mathcal{A}$ ad $\mathcal{B}$ we characterize the BSE-functions on $\Delta(\mathcal{A} \times_{\phi} \mathcal{B})$ in terms of those functions on $\Delta(\mathcal{A})$ and $\Delta(\mathcal{B})$.

**Lemma 4.1** Let $\mathcal{A}$ and $\mathcal{B}$ be commutative semisimple Banach algebras and let $\phi : \mathcal{B} \to \mathcal{A}$ be a contractive algebra homomorphism with dense range. Then the following statements hold.

(i) Let $\sigma \in C_{BSE}(\Delta(\mathcal{A} \times_{\phi} \mathcal{B}))$ and define functions $\tau$ on $\Delta(\mathcal{A})$ and $\rho$ on $\Delta(\mathcal{B})$ by

$$\tau(\varphi) = \sigma(\varphi, \varphi \circ \phi) - \sigma(0, \varphi \circ \phi) \quad \text{and} \quad \rho(\psi) = \sigma(0, \psi)$$

for all $\varphi \in \Delta(\mathcal{A})$ and $\psi \in \Delta(\mathcal{B})$. Then $\tau \in C_{BSE}(\Delta(\mathcal{A}))$ and $\rho \in C_{BSE}(\Delta(\mathcal{B}))$.

(ii) Let $\tau \in C_{BSE}(\Delta(\mathcal{A}))$ and $\rho \in C_{BSE}(\Delta(\mathcal{B}))$ and define a function $\sigma$ on $\Delta(\mathcal{A} \times_{\phi} \mathcal{B})$ by

$$\sigma(\varphi, \varphi \circ \phi) = \tau(\varphi) + \rho(\varphi \circ \phi) \quad \text{and} \quad \sigma(0, \psi) = \rho(\psi)$$

for all $\varphi \in \Delta(\mathcal{A})$ and $\psi \in \Delta(\mathcal{B})$. Then $\sigma \in C_{BSE}(\Delta(\mathcal{A} \times_{\phi} \mathcal{B}))$.

Proof. (i). First note that the density of $\phi(\mathcal{B})$ in $\mathcal{A}$ implies that $\varphi \circ \phi \neq 0$ for all $\varphi \in \Delta(\mathcal{A})$, and therefore $\tau$ is well defined. It is also trivial that $\tau$ and $\rho$ are continuous. Moreover, by [19, Theorem 4(i)] there is a bounded net $(a_{\alpha}, b_{\alpha})_{\alpha}$ in $\mathcal{A} \times_{\phi} \mathcal{B}$ such that $\| (a_{\alpha}, b_{\alpha}) \| \leq \| \sigma \|_{BSE}$ and $(a_{\alpha}, b_{\alpha})(\gamma) \to \sigma(\gamma)$ for all $\gamma \in \Delta(\mathcal{A} \times_{\phi} \mathcal{B})$. Therefore, for each $\psi \in \Delta(\mathcal{B})$,

$$\hat{b}_{\alpha}(\psi) = (a_{\alpha}, b_{\alpha})(0, \psi) \to \sigma(0, \psi) = \rho(\psi)$$

and for each $\varphi \in \Delta(\mathcal{A})$ we have

$$\hat{a}_{\alpha}(\varphi) = (a_{\alpha}, b_{\alpha})(\varphi, \varphi \circ \phi) - \hat{a}_{\alpha}(\varphi \circ \phi) \to \sigma(\varphi, \varphi \circ \phi) - \sigma(0, \varphi \circ \phi) = \tau(\varphi).$$

These show that $\tau \in C_{BSE}(\Delta(\mathcal{A}))$ and $\rho \in C_{BSE}(\Delta(\mathcal{B}))$. Finally, since $\| b_{\alpha} \| + \| a_{\alpha} \| = \| (a_{\alpha}, b_{\alpha}) \| \leq \| \sigma \|_{BSE}$, it follows from [19, Remark on p. 154] that $\| \tau \|_{BSE} + \| \rho \|_{BSE} \leq \| \sigma \|_{BSE}$.

(ii). As it was mentioned above $\varphi \circ \phi \in \Delta(\mathcal{B})$ for all $\varphi \in \Delta(\mathcal{A})$ and consequently $\sigma$ is well defined. Furthermore, Since $E$ and $F$ are closed in $\Delta(\mathcal{A} \times_{\phi} \mathcal{B})$, it follows that $\sigma$ is continuous. By assumption and [19, Theorem 4(ii)] there exist bounded nets $(a_{\alpha})_{\alpha}$ and $(b_{\beta})_{\beta}$ in $\mathcal{A}$ and $\mathcal{B}$, respectively with $\| a_{\alpha} \| \leq \| \tau \|_{BSE}$ and $\| b_{\beta} \| \leq \| \rho \|_{BSE}$ such that $\hat{a}_{\alpha}(\varphi) \to \tau(\varphi)$ and $\hat{b}_{\beta}(\psi) \to \rho(\psi)$ for all $\varphi \in \Delta(\mathcal{A})$ and $\psi \in \Delta(\mathcal{B})$. If we consider the bounded net $(a_{\alpha}, b_{\beta})_{\alpha, \beta}$ in $\mathcal{A} \times_{\phi} \mathcal{B}$, then

$$(a_{\alpha}, b_{\beta})(\varphi, \varphi \circ \phi) = \hat{a}_{\alpha}(\varphi) + \hat{b}_{\beta}(\varphi \circ \phi) \to \tau(\varphi) + \rho(\varphi \circ \phi) = \sigma(\varphi, \varphi \circ \phi)$$

for all $(\varphi, \varphi \circ \phi) \in E$ and

$$(a_{\alpha}, b_{\beta})(0, \psi) = \hat{b}_{\beta}(\psi) \to \rho(\psi) = \sigma(0, \psi)$$

for all $(0, \psi) \in F$.
for all \((0, \psi) \in F\). Therefore, \((a_\alpha, b_\beta)(\gamma) \to \sigma(\gamma)\) for all \(\gamma \in \Delta(A \times_\phi B)\). Hence, \(\sigma \in C_{BSE}(\Delta(A \times_\phi B))\) and \(\|\sigma\|_{BSE} \leq \|\tau\|_{BSE} + \|\rho\|_{BSE}\). \(\Box\)

Now, assume that \(A\) and \(B\) are commutative semisimple Banach algebras and let \(\phi : B \to A\) be a contractive algebra homomorphism with dense range. Suppose that \(\Gamma = \phi^*|_{\Delta(A)} : \Delta(A) \to \Delta(B)\). Then the map \(\tilde{\phi} : C_{BSE}(\Delta(B)) \to C_{BSE}(\Delta(A))\) defined by \(\tilde{\phi}(\sigma) = \sigma \circ \Gamma\) is a contractive algebra homomorphism.

**Theorem 4.2** Let \(\phi : B \to A\) be a contractive algebra homomorphism with dense range. Then the map \(\Theta : (\tau, \rho) \to \sigma\) of Lemma 4.1 is an isometric isomorphism from
\[
C_{BSE}(\Delta(B)) \times_\phi C_{BSE}(\Delta(A))
\]
on to \(C_{BSE}(\Delta(A \times_\phi B))\).

Proof. It follows from Lemma 4.1 that \(\Theta\) is bijective and isometric. It suffices to observe that \(\Theta\) is also an algebra isomorphism. Indeed, given \(\tau_1 \in C_{BSE}(\Delta(A))\) and \(\rho_i \in C_{BSE}(\Delta(B))\), \(i = 1, 2\). Then for each \(\varphi \in \Delta(A)\),
\[
\langle \Theta((\tau_1, \rho_1)(\tau_2, \rho_2)), (\varphi, \varphi \circ \phi) \rangle = \langle \Theta(\tau_1 \tau_2 + \tilde{\phi}(\rho_1)\tau_2 + \tau_1 \tilde{\phi}(\rho_2), \rho_1 \rho_2), (\varphi, \varphi \circ \phi) \rangle = \langle \tau_1(\varphi) + \rho_1(\varphi \circ \phi), \tau_2(\varphi) + \rho_2(\varphi \circ \phi) \rangle = \langle \Theta(\tau_1, \rho_1)\Theta(\tau_2, \rho_2), (\varphi, \varphi \circ \phi) \rangle.
\]
Similarly we can show that \(\langle \Theta((\tau_1, \rho_1)(\tau_2, \rho_2)), (0, \psi) \rangle = \langle \Theta(\tau_1, \rho_1)\Theta(\tau_2, \rho_2), (0, \psi) \rangle\) for all \(\psi \in \Delta(B)\) which completes the proof. \(\Box\)

Let \(A\) and \(B\) be commutative semisimple Banach algebras and let \(\phi : B \to A\) be a contractive algebra homomorphism. Then it is easy to check that the map \(\Phi : A \times_0 B \to A \times_\phi B\) defined by
\[
\Phi(a, b) = (a - \phi(b), b) \quad ((a, b) \in A \times_0 B)
\]
is an algebra isomorphism and
\[
\|(a - \phi(b), b)\| = \|a - \phi(b)\| + \|b\| \leq \|a\| + (\|\phi\| + 1)\|b\| \leq (\|\phi\| + 1)\|(a, b)\|
\]
Moreover, Since both algebras are semisimple, it follows that \(\Phi\) is also a topological isomorphism. Therefore, there is a homeomorphism \(\widehat{\Phi}\) between \(\Delta(A \times_0 B)\) and \(\Delta(A \times_\phi B)\) given by
\[
\widehat{\Phi}(\varphi, \varphi \circ \phi) = (\varphi, 0), \quad \widehat{\Phi}(0, \psi) = (0, \psi)
\]
for all \(\varphi \in \Delta(A)\) and \(\psi \in \Delta(B)\). As an application of Corollary 3.3 we have the following result characterizing the BSE-property of \(A \times_\phi B\).

**Theorem 4.3** Let \(A\) and \(B\) be two commutative semisimple Banach algebras. Then \(A \times_\phi B\) is a BSE-algebra if and only if \(A\) and \(B\) are BSE-algebras.
Proof. By Corollary 3.3 it suffices to show that $A \times \phi B$ is a BSE-algebra if and only if $A \times 0 B$ is a BSE-algebra. In fact, suppose that $A \times \phi B$ is a BSE-algebra and let $\sigma \in C_{BSE}(\Delta(A \times B))$. Define $\tau$ on $\Delta(A \times B)$ by

$$\tau(\varphi, \varphi \circ \phi) = \sigma(\varphi, 0), \quad \tau(0, \psi) = \sigma(0, \psi)$$

for all $\varphi \in \Delta(A)$ and $\psi \in \Delta(B)$. Since $\tau = \sigma \circ \Phi^*$, it follows that $\tau \in C_{BSE}(\Delta(A \times \phi B))$, where $\Phi$ is the above isomorphism between $A \times 0 B$ and $A \times \phi B$. By assumption there is some $T \in M(A \times \phi B)$ such that $\hat{T} = \tau$. Now, consider the map $S = \Phi^{-1}T\Phi : A \times 0 B \to A \times 0 B$. Then it is trivial that $S \in M(A \times 0 B)$. We show that $\hat{S} = \sigma$. Indeed, for each $\varphi \in \Delta(A)$, $a \in A$ and $b \in B$ we have

$$\hat{S}(\varphi, 0)(a, b)(\varphi, 0) = \hat{S}(a, b)(\varphi, 0) = \langle \Phi^{-1}T\Phi(a, b), (\varphi, 0) \rangle = \langle T(a - \phi(b), b), \Phi^{-1*}(\varphi, 0) \rangle = \langle T(a - \phi(b), b), (\varphi, \varphi \circ \phi) \rangle = \hat{\tau}(\varphi, \varphi \circ \phi)\hat{a}(\varphi) = \sigma(\varphi)(a, b)(\varphi, 0).$$

Therefore, $\hat{S}(\varphi, 0) = \sigma(\varphi, 0)$. Similarly, we can show that $\hat{S}(0, \psi) = \sigma(0, \psi)$ for all $\psi \in \Delta(B)$. This shows that $C_{BSE}(\Delta(A \times B)) \subseteq \hat{M}(\hat{A} \times 0 \hat{B})$. For the reverse inclusion note that if $(u_\alpha, v_\alpha)$ is a $\Delta$-weak bounded approximate identity for $A \times \phi B$, then $\Phi^{-1}((u_\alpha, v_\alpha))$ is a $\Delta$-weak bounded approximate identity for $A \times 0 B$, which implies that $A \times 0 B$ is a BSE-algebra. Similarly, we can show that the converse is also true.

Recall that a locally compact group $G$ is called amenable if there exists a translation-invariant mean on $L^\infty(G)$. Now, let $A(G)$ be the Fourier algebra of $G$, introduced by Eymard [6]. Then $A(G)$ is always an ideal in the Fourier-Stieltjes algebra $B(G)$ and $M(A(G)) = B(G)$ when $G$ is amenable. Furthermore, the carrier space of $A(G)$ as a commutative Banach algebra is homeomorphic to the topological space $G$ where each $x \in G$ is mapped to $\varphi_x$, the point evaluation at $x$. Thus, in the following we identify $\Delta(A(G))$ with $G$.

The following result was obtained independently in a different way in [14].

**Theorem 4.4** Let $G$ be an amenable locally compact group and let $N$ be a closed normal subgroup of $G$ such that $G/N$ is compact. Suppose that $q : G \to G/N$ is the quotient map, and let

$$B(G) = [B(G/N) \circ q] \times 0 A(G).$$

Then $B(G)$ is a BSE-algebra.

Proof. Since $G/N$ is compact, it follows from [6, Corollary 2.26(3)] that the map $u \to u \circ q$ is an isometric Banach algebra isomorphism between $A(G/N) = B(G/N)$ and its image $A(G/N) \circ q$. Since both $G$ and $G/N$ are amenable groups, it follows from Corollary 3.3 and [14, Theorem
5.1] that $B(G)$ is a BSE-algebra.

The following example shows that $B(G)$ can be a BSE-algebra when $G$ is noncompact. Therefore, we do not have the dual version of the result on BSE-property of $M(G)$.

**Example 1** Let $n \geq 1$ be an integer, $p$ a prime number, $\mathbb{Q}_p$ the field of $p$-adic numbers, and $\mathbb{Z}_p$ the ring of $p$-adic integers. Then the semidirect product $G_{p,n} = \mathbb{Q}_p^n \rtimes GL(n, \mathbb{Z}_p)$ is a noncompact amenable locally compact group. Runde and Spronk [18] showed that

$$B(G_{p,n}) = [A(GL(n, \mathbb{Z}_p)) \circ q] \times_0 A(G_{p,n}).$$

Since $GL(n, \mathbb{Z}_p)$ and $G_{p,n}$ are compact and amenable groups, respectively, it follows from above theorem that $B(G_{p,n})$ is a BSE-algebra.

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