ABSTRACT

In a multiparty fair coin-flipping protocol, the parties output a common (close to) unbiased bit, even when some corrupted parties try to bias the output. Cleve [STOC 1986] has shown that in the case of dishonest majority (i.e., at least half of the parties can be corrupted), in any m-round coin-flipping protocol, the corrupted parties can bias the honest parties' common output bit by $\Omega(\frac{1}{\sqrt{m}})$. For more than two decades, the best known coin-flipping protocols against dishonest majority had bias $\Theta(\frac{1}{\sqrt{m}})$, where $\ell$ is the number of corrupted parties. This was changed by a recent breakthrough result of Moran et al. [TCC 2009], who constructed of corrupted parties. This was changed by a recent breakthrough result of Moran et al. [TCC 2009], who constructed a three-party fair coin-flipping protocol that when all parties are honest, the output is unbiased and this bias is $\Theta(\frac{1}{\sqrt{m}})$. In a subsequent work, Beimel et al. [Crypto 2010] extended this result to the multiparty case in which less than $\frac{m}{2}$ of the parties can be corrupted. Still for the case of $\frac{m}{2}$ (or more) corrupted parties, the best known protocol had bias $\Theta(\frac{1}{\sqrt{m}})$. In particular, this was the state of affairs for the natural three-party case.

We make a step towards eliminating the above gap, presenting an $m$-round, three-party coin-flipping protocol, with bias $O(\frac{m^2}{\sqrt{m}})$. Our approach (which we also apply for the two-party case) does not follow the “threshold round” paradigm used in the work of Moran et al. and Beimel et al., but rather is a variation of the majority protocol of Cleve, used to obtain the aforementioned $\Theta(\frac{1}{\sqrt{m}})$-bias protocol.

1. INTRODUCTION

In a multiparty fair coin-flipping (tossing) protocol, the parties output a common (close to) unbiased bit, even though some corrupted parties try to bias the output. More formally, such protocols should satisfy the following two properties: first, when all parties are honest (i.e., follow the prescribed protocol), they all output the same bit, and this bit is unbiased (i.e., uniform over $\{0,1\}$). Second, even when some parties are corrupted (i.e., collude and arbitrarily deviate from the protocol), the remaining parties should still output the same bit, and this bit should not be too biased (i.e., its distribution should be close to uniform over $\{0,1\}$). We emphasize that, unlike weaker variants of coin-flipping protocol known in the literature, the honest parties should output a common bit, regardless of what the corrupted parties do. In particular, they are not allowed to abort if a cheat was noticed.

When a majority of the parties are honest, efficient and completely fair coin-flipping protocols are known as a special case of secure multiparty computation with an honest majority [9]. When an honest majority is not guaranteed, however, the situation is more complex.

Negative results.

Cleve [13] showed that for any efficient two-party m-round coin-flipping protocol, there exists an efficient adversary to bias the output of the honest party by $\Theta(1/m)$. This lower bound extends to the multiparty case via a simple reduction.

Positive results.

Assuming one-way functions exist, Cleve [13] showed that a simple $m$-round majority protocol can be used to derive a
t-party coin-flipping protocol with bias $\Theta(\frac{\ell}{m})$ (against dishonest majority), where $\ell$ is the number of corrupted parties. For more than two decades, Cleve’s protocol was the best known fair coin-flipping protocol (without honest majority), under any hardness assumption, and for any number of parties. In a recent breakthrough result, Moran et al. [33] constructed an $m$-round, two-party coin-flipping protocol with optimal bias of $\Theta(\frac{\ell}{m})$. The result holds for any efficiently computable $m$, and under the assumption that oblivious transfer protocols exist. In a subsequent work, Beimel et al. [6] extended the result of [33] for the multiparty case in which less than $\frac{2}{3}$ of the parties can be corrupted. More specifically, for any $\ell < \frac{2}{3} \cdot t$, they presented an $m$-round, $t$-party protocol, with bias $\frac{\ell^2 - t}{m}$ against (up to) $t$ corrupted parties.

Still for the case of $\frac{2}{3}$ (or more) corrupted parties, the best known protocol was the $\Theta(\frac{\ell}{\sqrt{m}})$-bias majority protocol of [13]. In particular, this was the state of affairs for the natural three-party case (where two parties are corrupt).

### 1.1 Our Result

We present an almost-optimally fair, three-party coin-flipping protocol.

**Theorem 1.1** (Main Theorem, Informal). Assuming the existence of oblivious transfer protocols, then for any $m \in \text{poly}$ there exists an $m$-round, three-party coin-flipping protocol, with bias $\Theta(\frac{\ell}{\sqrt{m}})$ (against one, or two, corrupted parties).

As a building block towards constructing our three-party protocol, we present an alternative construction for two-party, almost-optimally fair coin-flipping protocols. Our approach does not follow the “threshold round” paradigm used in [33, 6], but rather is a variation of the aforementioned $\Theta(\frac{\ell}{\sqrt{m}})$-bias, coin-flipping protocol of [13].

### 1.2 Additional Related Work

Cleve and Impagliazzo [14] showed that in the fail-stop model, any two-party $m$-round coin-flipping protocol has bias $\Omega(\frac{\ell}{m})$; adversaries in this model are computationally unbounded, but they must follow the instructions of the protocol, except for being allowed to abort prematurely. Dachman-Soled et al. [15] showed that the same holds for $\sigma(m/\log m)$-round protocols in the random-oracle model — the parties have oracle access to a uniformly chosen function over $m$ bit strings. There is a vast literature concerning coin-flipping protocols with weaker security guarantees. Most notable among these are protocols that are secure with abort. According to this security definition, if a cheat is detected or if one of the parties aborts, the remaining parties are not required to output anything. This form of security is meaningful in many settings, and it is typically much easier to achieve; assuming one-way functions exist, secure-with-abort protocols of negligible bias are known to exist against any number of corrupted parties [11, 27, 34]. To a large extent, one-way functions are also necessary for such coin-flipping protocols [10, 25, 28, 31].

Coin-flipping protocols were also studied in a variety of other models. Among these are collective coin-flipping in the perfect information model. The result holds for unbounded and all communication is public [3, 8, 17, 36, 37], and protocols based on physical assumptions, such as quantum computation [1, 4, 5] and tamper-evident seals [32]. Perfectly fair coin-flipping protocols (i.e., zero bias) are a special case of protocols for fair secure function evaluation (SFE). Intuitively, the security of such protocols guarantees that when the protocol terminates, either everyone receives the (correct) output of the functionality, or no one does. While Cleve’s result yields that some functions do not have fair SFE, it was recently shown by Gordon et al. [23] that many interesting function families do have (perfectly) fair SFE.

### 1.3 Our Techniques

The following is a high-level description of the ideas underlying our three-party fair coin-flipping protocol. We start by describing the two-party protocol of Moran et al. [33], and explain why natural extensions of their approach (such as the one used in [6]) fail short when it comes to constructing three-party fair coin-flipping protocols. Next, we explain our new approach for two-party protocols, and then extend this approach to three parties.

Throughout, we assume without loss of generality that if a corrupted party aborts in a given round, it sends an abort message to all other parties at the end of this round (after seeing the messages sent by the non-aborting parties). To keep the discussion simple, we focus on security against fail-stop adversaries — the parties follow the prescribed protocol, but might abort prematurely. Achieving this level of security is the heart of the matter, since (assuming one-way functions exist) there exists a round-preserving reduction from protocols secure against fail-stop adversaries into protocols of full-fledged security [20].

1.3.1 The Two-Party Protocol of Moran et al.

For $m \in \mathbb{N}$, the $(2m)$-round, two-party protocol $(P_0, P_1)$ of Moran et al. [33] is defined as follows:

1. Uniformly sample $c \leftarrow \{0, 1\}$ and $i^* \leftarrow [m] = \{1, \ldots, m\}$.
2. For $i = 1$ to $m$, let
   
   \begin{align*}
   (a) \quad (d_0^i, d_1^i) &= \begin{cases} 
   \text{uniform sample from } \{0, 1\}^2, & i < i^* - 1 \\
   (c, c), & i = i^* \\
   \perp, & i > i^*
   \end{cases} \\
   (b) \quad c_i &= \begin{cases} 
   c, & i < i^* \\
   \perp, & i > i^*
   \end{cases}
   \end{align*}

3. Split each of the $3m$ values $d_0^0, d_1^0, \ldots, d_0^m, d_1^m, c_1, \ldots, c_m$ into two “shares,” using a 2-out-of-2 secret sharing scheme, and output the two sets of shares.

**Protocol 1.3** $(P_0, P_1)$.

Initial step: The parties securely compute the function $\text{SharesGen}$, where each party gets one set of shares.

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2 The protocol described below is a close variant of the original protocol of Moran et al. [33], which serves our presentation better.
Main loop: For $i = 1$ to $m$, do

(a) $P_0$ sends to $P_1$, its share of $d_i^0$, and $P_1$ sends to $P_0$, its share of $d_i^1$.

- $P_0$ reconstructs the value of $d_i^0$, and $P_1$ reconstructs the value of $d_i^1$.

(b) Each party sends to the other party its share of $c_i$.

- Both parties reconstruct the value of $c_i$.

Output: The parties output $c_i$, for the first $i$ for which $c_i \neq \perp$.

Abort: If $P_0$ aborts, party $P_1$ outputs the value of $d_i^1$ for the maximal $i \in [m]$ for which it has reconstructed this value. If there is no such $i$, $P_1$ outputs a uniform bit. (The case that $P_1$ aborts is analogously defined).

We start with few observations regarding the secure computation of SharesGen done in the above protocol.

- The computation of SharesGen is not fair: the parties get their parts of the output (i.e., their shares) in an arbitrary manner. Specifically, the corrupted party might prematurely abort after learning its part of the output, preventing the other party from getting its part.

- Since SharesGen is efficient (in $m$), assuming OT protocols exist, an (unfair) secure computation of SharesGen exists for any efficiently computable $m$.

- Ignoring negligible terms (due to the imperfection of secure computation using OT), the output of each party (when seen on its own) is a set of uniform strings. In particular, it contains no information about the other party’s shares, or about the values of $c$ and $c^*$. By construction, a party outputs a uniform bit if the other party aborts before the end of the secure computation phase. Hence, it makes no sense for a party to abort during this phase.

- Given the above observation, it is instructive to pretend that at the first step of the protocol, the output of a random execution of SharesGen was given to the parties by an honest dealer.

Note that in each round of the above protocol, both honest parties send their messages without waiting for the other party’s message. Hence, the above protocol is symmetric with respect to the parties’ role. However, since we assume no simultaneous channel (which would have trivialized the whole question), the corrupted party can postpone sending its message until it gets the message of the honest party, and then decide whether to send its message for this round or abort.

Security of the protocol.

At least on the intuitive level, the security proof of the above protocol is rather simple. Since the protocol is symmetric, we assume for concreteness that $P_0$ is corrupted and tries to bias the expected output of $P_0$ away from $\frac{1}{2}$. The following random variables are defined with respect to a random execution of $(P_0, P_1)$: let $V$ be the view of the corrupted $P_0$, right after sending the abort message, and let $V^-$ be the view $V$ without this abort message ($V^-$ and $V$ are set to the full view, if no abort occurred). Finally, for a view $v$, let $\text{val}(v)$ be the expected outcome of the non-aborting parties, conditioned on $v$, and assuming the non-aborting parties act honestly in the rest of the execution. It is not hard to verify that the bias obtained by $P_0$ is exactly $|\text{val}(V) - \text{val}(V^-)|$.

It is also easy to see that by aborting in round $(i, b)$, for some $i \in [m]$, party $P_0$ gains nothing (i.e., $|\text{val}(V) - \text{val}(V^-)|$), where the $(i, j)$'th round of the execution stands for the $j$'th step of the $i$'th loop in the execution. A slightly more complicated math yields that by aborting in round $(i, a)$, party $P_0$ only gains $\Theta(\frac{1}{m})$ bias. It follows that the maximal bias obtained by a fail-stop strategy for $P_0$ is $\Theta(\frac{1}{m})$.

Fairness via defense.

Let us present a different view of the protocol of [33]. Consider a variant of this protocol without the $d_i$'s. Namely, the parties reconstruct $c_1, \ldots, c_m$ one at a time, until they reach $c_i \neq \perp$. When an abort occurs, the remaining party outputs an unbiased coin if it has not yet reconstructed $c_i$ and outputs $c$ otherwise. It is easy to see that an aborting attacker can bias the output of the other party in this degenerate variant by $\frac{1}{2}$; that is, it simply waits until it reconstructs $c_i$ and then aborts for biasing the other party’s output towards $1 - c$.

The role of “defense” values $(d_i^0, d_i^1), \ldots, (d_m^0, d_m^1)$ is to prevent such an attack; if a party aborts after reconstructing $c_i$, the other party is guaranteed to output $c$ as well. The problem is, however, that the defense values themselves might cause a problem: a corrupted party might abort after reconstructing its defense value for the $i$'th round (and not only after reconstructing $c_i$). Indeed, by aborting in these rounds, a corrupted party does gain a bias, but only $\Theta(\frac{1}{m})$.

On extending Moran et al.’s protocol for the three-party case.

We next explain why the approach of Moran et al. does not seem to be useful for constructing fair, three-party coin-flipping protocols.

In a three-party fair coin-flipping protocol, one should deal with two non-simultaneous aborts: after one party aborts, the remaining two parties should interact in a two-party protocol to agree on their common coin. Since one of the remaining parties might be corrupted as well, this two-party protocol needs to be a fair coin-flipping protocol as well. Moreover, the expected outcome of the latter two-party protocol, whose shares are given before each round, should be equal (up to an additive difference of $\Theta(\frac{1}{m})$) to the value of the three-party protocol after this round — the expected outcome of the protocol given the reconstructed shares. Otherwise, an aborting party can significantly bias the output of the two other parties.

Consider the following natural extension of Moran et al.’s protocol to a three-party protocol. The value of $c_1, \ldots, c_m$ are as in the two-party protocol (now shared between the three parties). The defense values are not bits, but rather two vectors of shares for the two remaining parties (different shares for each possible pair), to enable them to interact in some fair two-party coin-flipping protocol if the third party aborts.

Assume that in the $i$'th round of the “outer” three-party
the last round determines the outcome only with probabil-
a smooth coin-flipping protocol as a building block for con-
value of the protocol at the
reconstruct the
same value.
the defense values, to be constructed by each party during
two corrupted parties might extrapolate, at the
m
jority protocol. In the first round the parties reconstruct
of
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engange in if the other (in this case, the honest) party aborts
m
c
before
1). As we argued above, the (long) defense values recon-
these restrictions on the two-party protocol defense val-
ues ruin the security of the outer three-party protocol: in
each round i, two corrupted (and thus colluding) parties can
reconstruct the whole two-party execution that they should
engage in if the other (in this case, the honest) party aborts
in this round. By checking whether the defense values of
this two-party execution are all ones (indicating that c = 1),
al zeros (indicating that c = 0), or mixed (indicating that
c
= 1), they get enough information for biasing the output
of the protocol by a constant value.

What causes the above three-party protocol to fail is that
its value in a given round might be changed by \( \frac{1}{2} \) (say from \( \frac{1}{2} \)
to 1). As we argued above, the (long) defense values recon-
structed before each round in the three-party protocol have
to contain many (i.e., \( m \)) samples drawn according to the
value of the protocol at the end of the round. It follows that
two corrupted parties might extrapolate, at the beginning
of such a round, the value of protocol when this round ends,
thus rendering the protocol insecure.

1.3.2 Our Smooth Two-Party Protocol

Given the above understanding, our first step is to con-
struct a two-party coin-flipping protocol, whose value only
changes slightly (i.e., smoothly) between consecutive rounds.
In the next section we use a derandomized version of such
a smooth coin-flipping protocol as a building block for con-
structing an (almost) optimally fair three-party protocol.
Consider the \( \Theta(\frac{\sqrt{m}}{\epsilon}) \)-bias coin-flipping protocol of Cleve [13]: in each of the \( m \) rounds, the parties recon-
struct the value of a coin \( c_i \in \{-1, 1\} \), and the final outcome is set to
\( \text{sign}(\sum_{i \in [m]} c_i) \).

Since the value of \( \sum_{i = 1}^{m} c_i \) is close to being uni-
form over \([-\sqrt{m}, \sqrt{m}]\), the value of the first coin \( c_1 \) changes the
protocol’s value by \( \Theta(\frac{\sqrt{m}}{\epsilon}) \). This sounds like a good
start towards achieving a smooth coin-flipping protocol. The
problem is, however, that with probability \( \Theta(\frac{\sqrt{m}}{\epsilon}) \), the sum
of \( c_1, \ldots, c_{m-1} \) is exactly zero. Hence, with this probability,
the final coin changes the protocol’s value by \( \frac{\epsilon}{2} \).

We overcome the above problem by using a weighted ma-
jority protocol. In the first round the parties reconstruct
\( m \)-coins (in a single shot), reconstruct (\( m - 1 \)) coins in the
second round, and so on, until in the very last round only a
single coin is reconstructed. Now the value of \( \sum_{i = 1}^{m} c_i \), (now each
coin is an integer) is close to being uniform over \([-m, m]\), and
the last round determines the outcome only with probabil-
ity \( \Theta(\frac{\sqrt{m}}{\epsilon}) \) (versus \( \Theta(\frac{\sqrt{m}}{\epsilon}) \) in the unweighted version). Other
rounds also enjoy a similar smoothness property. We em-
phasize that the resulting protocol does not guarantee small
bias (but only a \( \Theta(\frac{\sqrt{m}}{\epsilon}) \)-bias). Rather, we take advantage
of its smoothness and use it as a building block of a fair
two-party protocol (and then of a three-party protocol).

The protocol.

As in Moran et al. [33], the parties start by securely com-
puting a share generating function, and then use its outputs
to slowly reconstruct the output of the protocol.
Let \( \text{Ber}(\delta) \) be the Bernoulli distribution over \{0, 1\}, taking
the value one with probability \( \delta \) and zero otherwise, let \( c_i \),
be distribution induced by the sum of \( n \) uniform coins over
\{-1, 1\}, let \( B_n(k) := \text{Pr}_{x \sim c_n}[x \geq k] \), and finally, for \( i \in [n] \)
let \( \sum_n(i) := \sum_{j=1}^{n}(n+1-i) \).

We start by describing the share generating function and
then use it to describe the protocol.

Algorithm 1.4 (Function TwoPartySharesGen).
1. For \( z \in \{0, 1\} \), sample \( d_0^z \leftarrow \{0, 1\} \).
2. For \( i = 1 \) to \( m \),
   (a) Sample \( c_i \leftarrow c_{m+1-i} \).
   (b) For \( z \in \{0, 1\} \), sample \( d_i^z \leftarrow \text{Ber}(\delta_i) \), for \( \delta_i = B_{\sum_n(i+1)}(\sum_{j=1}^{n} c_i) \).\(^{3}\)
3. Split each of the \( 3m \) values
   \( d_0^0, d_1^0, \ldots, d_m^0, d_0^1, c_1, \ldots, c_m \) into two “shares”,
   using a 2-out-of-2 secret sharing scheme, to create
   two sets of shares: \( s^{#0} \) and \( s^{#1} \).
4. Output \((d_0^0, s^{#0}), (d_0^1, s^{#1})\).

Protocol 1.5 (\( \tau_2 = (P_0^2, P_2^2) \)).

Initial step: The parties securely compute the function TwoPartySharesGen. Let \((d_0^2, s^{#1})\) be the local output
of \( P_2^2 \).

Main loop: For \( i = 1 \) to \( m \), do
   (a) \( P_0^2 \) sends to \( P_2^2 \) its share of \( d_i^0 \), and \( P_2^2 \) sends to
      \( P_0^2 \) its share of \( d_i^0 \).
      \( P_0^2 \) reconstructs the value of \( d_i^0 \), and \( P_2^2 \) recon-
      structs the value of \( d_i^0 \).
   (b) Each party sends to the other party its share of
      \( c_i \).
      \( P_0^2 \) and \( P_2^2 \) reconstruct the value of \( c_i \).

Output: Both parties output one if \( \sum_{j=1}^{m} c_i \geq 0 \), and zero
otherwise.

Abort: If \( P_0^2 \) aborts, party \( P_2^2 \) outputs the value of \( d_i^1 \), for
the maximal \( i \in [m] \) for which it has reconstructed
this value (note that by construction such an \( i \) always
exists).

The case that \( P_2^2 \) aborts is analogously defined.

\(^{3}\)\( \delta_i \) is the probability that the protocol’s output is one, given
the value of the “coins” \( c_1, \ldots, c_i \) (and assuming no abort).
Namely, the parties interact in a weighted majority protocol, where in the \(i\)th round, they reconstruct, in an unfair manner, \((m + 1 - i)\) coins (i.e., \(c_i\)). If a party aborts, the remaining party outputs a defense value given to it by the honest dealer (implemented via the secure computation of TwoPartySharesGen). A few remarks are in place. First, we will only define the protocol for \(m \equiv 1 \mod 4\).\(^4\) Hence, \(\sum_{i=1}^{m} c_i \neq 0\), and the protocol’s output is a uniform bit when played by the honest parties. Second, if \(P^i_0\) aborts in the first round, the party \(P^i_1\) could simply output a uniform bit. We make \(P^i_0\) output \(d^i_m\), since this be useful when the two-party protocol will be later used as part of the three-party protocol. Finally, we can define the above protocol without exposing the coins \(c_i\)’s to the parties (in this case, the honest parties output \((d^m_m, d^m_2)\) as the final outcome). We expose the coins to the parties to make the analysis of the protocol easier to follow.

**Security of the protocol.**

Note that the defense value given in round \((i,a)\) (i.e., step \(a\) of the \(i\)th loop) is distributed according to the expected outcome of the protocol, conditioned on the coin to be given in round \((i,b)\). These defense values make aborting in round \((i,b)\), for any value of \(i\), harmless. So it is left to argue that aborting in round \((i,a)\), for any value of \(i\), is not too harmful. So we claim that, using the fact that the value reconstructed in round \((i,a)\) gives only a very noisy signal about the value of \(c_i\).

Since the protocol is symmetric, we assume for concreteness that the corrupted party is \(P^0_0\). Similar to the analysis of Moran et al.’s protocol sketched above, it suffices to bound the value of \(\text{val}(V) - \text{val}(V^-)\).

Assume that \(P^0_0\) aborts in round \((i,b)\). By construction, \(\text{val}(V^-) = \delta_i\). Since, the defense of \(P^i_1\) in round \((i,b)\) is sampled according to \(\text{Ber}(\delta_i)\), it is also the case that \(\text{val}(V) = \delta_i\).

Assume now that \(P^0_0\) aborts in round \((i,a)\). By construction, \(\text{val}(V^-) = \delta_{i-1}\). Note that \(V^-\) does contain some information about \(\delta_i\), i.e., a sample from \(\text{Ber}(\delta_i)\), and thus \(\text{val}(V^-)\) is typically different from \(\text{val}(V)\). Yet, since \(V^-\) contains only a sample from \(\text{Ber}(\delta_i)\), and this is a very noisy signal for the actual value of \(\delta_i\), we manage to prove the following.

\[
\frac{\text{val}(V) - \text{val}(V^-)}{\delta_{i-1}} = \left(\frac{\delta_i - \delta_{i-1}}{\delta_{i-1}}\right)^2.
\]

Equation (1) yields that \(\text{val}(V) - \text{val}(V^-) = O(\frac{1}{\delta})\), since by the “smoothness” of the protocol (i.e., the value of the game does not change drastically between consecutive rounds) it follows that \(\frac{\delta_i - \delta_{i-1}}{\delta_{i-1}} = O(\frac{1}{\sqrt{m}})\) with high probability.\(^5\)

### 1.3.3 Our Three-Party Protocol

We start by applying a generic approach, introduced by Beimel et al. \([6]\), to try and extend our fair two-party protocol into a three-party one. We then explain why this generic approach does not work in our case, and show how to modify it to get a fair three-party protocol.

In the first attempted three-party protocol, the parties interact in the following variant of the two-party protocol \(\pi_2^{m}\) described in Protocol 1.5. As the first step, the three parties (securely) compute the function \(\text{ThreePartySharesGen}\), defined next, rather than \(\text{TwoPartySharesGen}\) as in protocol \(\pi_2\). Let \(C_i\) be a \(z\)-biased coin over \(\{-1, 1\}\). Namely, \(C_i\) is the Bernoulli probability distribution over \(\{-1, 1\}\), taking the value 1 with probability \(\frac{1}{2} + \varepsilon\) and \(-1\) otherwise. Let \(C_m\) be the sum of \(n\) independent \(z\)-biased coins over \(\{-1, 1\}\) and let \(B_{\sum_i k} := \text{Pr}_{x_i \in \{-1, 1\}}[x_i \geq k]\). For \(\varepsilon \in \left[\frac{1}{2}, \frac{1}{2}\right]\), we define the function \(\text{TwoPartySharesGen}\) as the following variant of the function \(\text{TwoPartySharesGen}\) defined above: (1) the “coin” \(c_i\) is sampled according to \(C_{m+1-i, \varepsilon}\) (and not according \(C_{m+1-i}\) as in \(\text{TwoPartySharesGen}\)); (2) the initial defense values \(d^0_0\) and \(d^0_0\) are sampled according to \(\text{Ber}(B_{\sum_{i=1}^m 1, \varepsilon}(0))\) (and not \(\text{Ber}(\frac{1}{2}) = \text{Ber}(B_{\sum_{i=1}^m 1}(0))\) as in \(\text{TwoPartySharesGen}\)).

**Algorithm 1.6** (function \(\text{ThreePartySharesGen}\)).

1. For \(i = 1 \to m\),
   a. Sample \(c_i \leftarrow C_{m+1-i, \varepsilon}\).
   b. Let \(\varepsilon_i \in [-\frac{1}{2}, \frac{1}{2}]\) be the value such that \(\text{Pr}_{x_i} \in \{-1, 1\}[x_i \geq \sum_{j=1}^m c_j\varepsilon_i]\).
   c. For each pair of the three parties, generate shares for an execution of \(\pi_2\), by calling \(\text{TwoPartySharesGen}\).

2. Split the values of \(c_1, \ldots, c_m\) and the defense values into three set of shares using a 3-out-of-3 secret sharing scheme, and output the three sets.

**Protocol 1.7** \((\pi_3 = (P^0_0, P^1_1, P^2_2))\).

**Initial step:** The parties securely compute the function \(\text{ThreePartySharesGen}\), where each party gets one set of shares.

**Main loop:** For \(i = 1 \to m\), do

a. Each party sends to the other parties its share of their defense values.
   - Each pair \((P^0_0, P^2_2)\) of the parties reconstructs a pair of two sets of shares \((d^i_{0-1}, d^i_{0+1})\), \((d^i_{1-1}, d^i_{1+1})\), to serve as input for an execution of the two-party protocol if the third party aborts (i.e., \(P^2_2\) reconstructs \((d^i_{1-1}, d^i_{1+1})\), and \(P^0_0\), reconstructs \((d^i_{0-1}, d^i_{0+1})\).
   - Each party sends the other parties its share of \(c_i\).
   - All parties reconstruct the value of \(c_i\).

**Output:** The parties output one if \(\sum_{i=1}^m c_i \geq 0\), and zero otherwise.

**Aabort:**

- If \(P^0_0\) aborts, the parties \(P^1_1\) and \(P^2_2\) use the shares of \(d^i_{0-1}\), for the maximal \(i \in [m]\) that has been reconstructed, to interact in \(\pi_2^i\) (starting right after the share reconstruction phase). If no such \(i\) exists, the parties interact in the (full, unbiased) two-party protocol \(\pi_2^i\).

The case that \(P^1_1\) or \(P^2_2\) aborts is analogously defined.

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\(^4\)Defining the protocol for \(m \equiv 2 \mod 4\) works as well.

\(^5\)The \(O(\log m)\) factor in our actual result does not appear in the above informal argument.
similar to the analysis for the two-party protocol sketched above, it suffices to show that the defense values reconstructed by a pair of corrupted parties in round \((i, a)\) — the inputs for the two-party protocols — do not give too much information about the value of \(\tilde{\delta}_i\) — the expected outcome of the three-party protocol conditioned on the coins reconstructed at round \((i, b)\). Note that once two corrupted parties are given these defense values, which happens in round \((i, a)\), they can immediately reconstruct the whole two-party execution induced by them. This two-party execution effectively contains \(\Theta(m)\) independent samples from \(\text{Ber}(\tilde{\delta}_i)\); one sample is given explicitly as the final output of the execution, and the value of \(2m\) additional samples can be extrapolated from the \(2m\) defense values given to the two parties. Many such independent samples can be used to approximate the value of \(\tilde{\delta}_i\), with accuracy \(\Theta(\frac{1}{\sqrt{m}})\). It follows that in round \((i, a)\), two corrupted parties can rush and estimate the value of \(\tilde{\delta}_i\) with high accuracy, and then use it to bias the outcome of the three-party protocol by \(|\delta_i - \delta_{i-1} - \Theta(\frac{1}{\sqrt{m}})| \in \Omega(\frac{1}{\sqrt{m}})\).

Derandomizing the above protocol.

We handle the above problem by modifying the way the defense values given to the parties in the three-party protocol are sampled. We modify the share generating function TwoPartySharesGen to first draw \(\Theta(m^2)\) independent samples from \(C_i\), and then use these samples via a simple derandomization technique for drawing the \(\Theta(m)\) defense values given in the three-party protocol. This constitute a saving in comparing to the original TwoPartySharesGen which, at least for its straightforward implementation, requires \(\Theta(m^3)\) (independent) samples from \(C_i\). When called with \(\varepsilon = \varepsilon_i\), the definition of \(\varepsilon_i\) yields that the information such \(\Theta(m^2)\) samples contain about the value of \(\tilde{\delta}_i\) is roughly the same as in a constant number of independent samples from \(\text{Ber}(\tilde{\delta}_i)\).

It follows that the defense values given to the parties in round \((i, a)\) of the resulting three-party protocol, leak only a little information about the value of \(\tilde{\delta}_i\). This yields that a pair of colluding parties cannot bias the output of the three-party protocol by more than \(\Theta(\frac{1}{\sqrt{m}})\).

1.4 Open Problems

The existence of an optimally fair three-party coin-flipping protocol (without the \(\Theta(\log^2 m)\) factor) is still an interesting open question. A more fundamental question is whether there exists a fair coin-flipping protocol for any number of parties (against any number of corrupted parties). While constructing (at least, almost) optimally fair, \(m\)-round coin-flipping protocols for a constant (or even \(\log(m)\)) number of parties seems within the reach of our current technique, handling a super-logarithmic number of parties, not to mention \(\Omega(m)\), seems to require a completely new approach, an may not be possible at all.

Paper Organization

Formal definition of the security model is given in Section 2. We also state there (Section 2.3.1) a new game-based definition of fair coin-flipping protocols, which is equivalent to the standard real/ideal definition. Our results are formally stated in Section 3. Missing proofs can be found in the full version of the paper [26].

2. PRELIMINARIES

2.1 Multi-Party Protocols

The following discussion is restricted to no private input protocols (such restricted protocols suffice for our needs). A \(t\)-party protocol is defined using \(t\) Touring Machines (TMs) \(P_1, \ldots, P_t\), having the security parameter \(1^*\) as their common input. In each round, the parties broadcast and receive messages on a broadcast channel. At the end of protocol, each party outputs some binary string.

The parties communicate in a synchronous network, using only a broadcast channel: when a party broadcasts a message, all other parties see the same message. This ensures some consistency between the information the parties have. There are no private channels and all the parties see all the messages, and can identify their sender. We do not assume simultaneous broadcast. It follows that in each round, some parties might hear the messages sent by the other parties before broadcasting their messages. We assume that if a party aborts, it first broadcasts the message \text{Abort} to the other parties, and without loss of generality only does so at the end of a round in which it is supposed to send a message. A protocol is efficient, if its parties are PPTM, and the protocol’s number of rounds is a computable function of the security parameter.

This work focuses on efficient protocols, and on malicious, static PPT adversaries for such protocols. An adversary is allowed to corrupt some subset of the parties; before the beginning of the protocol, the adversary corrupts a subset of the parties that from now on may arbitrarily deviate from the protocol. Thereafter, the adversary sees the messages sent to the corrupted parties and controls their messages. We also consider the so called fail-stop adversaries. Such adversaries follow the prescribed protocol, but might abort prematurely. Finally, the honest parties follow the instructions of the protocol to its completion.

2.2 The Real vs. Ideal Paradigm

The security of multiparty computation protocols is defined using the real vs. ideal paradigm [12, 19]. In this paradigm, the real-world model, in which protocols is executed is compared to an ideal model for executing the task at hand. The latter model involves a trusted party whose functionality captures the security requirements of the task. The security of the real-world protocol is argued by showing that it “emulates” the ideal world protocol, in the following sense: for any real-life adversary A, there exists an ideal-model adversary (also known as simulator) A such that the global output of an execution of the protocol with A in the real-world model is distributed similarly to the global output of running A in the ideal model. The following discussion is restricted to random, no-input functionalities. In addition, to keep the presentation simple, we limit our attention to uniform adversaries.

The Real Model.

\(^7\)All results stated in this paper, straightforwardly extend to the non-uniform settings.
Let $\pi$ be a $t$-party protocol and let $A$ be an adversary controlling a subset $C \subseteq [t]$ of the parties. Let $\text{REAL}_{A,C}(\kappa)$ denote the output of $A$ (i.e., without loss of generality its view: its random input and the messages it received) and the outputs of the honest parties, in a random execution of $\pi$ on common input $1^\kappa$.

Recall that an adversary is fail stop, if until they abort, the parties in its control follow the prescribed protocol (in particular, they property toss their private random coins). We call an execution of $\pi$ with such a fail-stop adversary, a fail-stop execution.

The Ideal Model.

Let $f$ be a $t$-output functionality. If $f$ gets a security parameter (given in unary), as its first input, let $f_s(\cdot) = f(1^\kappa, \cdot)$. Otherwise, let $f_s = f$.

An ideal execution of $f$ with respect to an adversary $A$, controlling a subset $C \subseteq [t]$ of the "parties" and a security parameter $1^\kappa$, denoted $\text{IDEAL}_{f,A,C}(\kappa)$, is the output of the adversary $A$ and that of the trusted party, in the following experiment.

**Experiment 2.1.**

1. The trusted party sets $(y_1, \ldots, y_t) = f_s(X)$, where $X$ is a uniform element in the domain of $f_s$, and sends $(y_i)_{i \in C}$ to $A(1^\kappa)$.

2. $A(1^\kappa)$ sends the description of a subset $\mathcal{J} \subseteq C$ to the trusted party, and locally outputs some value.

3. The trusted party outputs $\{o_i\}_{i \in |\mathcal{J}| \setminus C}$, where $o_i$ is equal to $y_i$ in case $\mathcal{J} = \emptyset$, and the description of $\mathcal{J}$ otherwise.

An adversary $A$ is non-aborting, if it always sets $\mathcal{J} = \emptyset$.

2.2.1 $\delta$-Secure Computation

The following definitions adopts the notion of $\delta$-secure computation [7, 21, 39] for our restricted settings.

**Definition 2.2 ($\delta$-secure computation).** An efficient $t$-party protocol $\pi$ computes a $t$-output functionality $f$ in a $\delta$-secure manner [resp., against fail-stop adversaries], if for every $C \subseteq [t]$ and every [resp., fail-stop] $\text{PPT}$ adversary $A$ controlling the parties indexed by $C$, there exists a $\text{PPT}$ $A$ controlling the same parties, such that

$$\text{SD}(\text{REAL}_{A,C}(\kappa), \text{IDEAL}_{f,A,C}(\kappa)) \leq \delta(\kappa),$$

for large enough $\kappa$.

A protocol securely compute a functionality $f$, if it computes $f$ in a $\text{neg}(\kappa)$-secure manner.

The protocol $\pi$ computes $f$ in a simultaneous $\delta$-secure manner, if the above is achieved by a non-aborting $A$.

Note that being simultaneous $\delta$-secure is a very strong requirement, as it dictates that the cheating real adversary has no way to prevent the honest parties from getting their part of the output, and this should be achieved with no simultaneous broadcast mechanism.

2.3 Fair Coin-Flipping Protocols

**Definition 2.3 ($\delta$-fair coin-flipping).** For $t \in \mathbb{N}$ let $\text{CoinFlip}_t$ be the $t$-output functionality from $\{0, 1\}$ to $\{0, 1\}^t$, defined by $\text{CoinFlip}_t(i) = b_i$. A $t$-party protocol $\pi$ is $\delta$-fair coin-flipping protocol, if it computes $\text{CoinFlip}_t$ in a simultaneous $\delta$-secure manner.

2.3.1 Proving Fairness

The following lemma reduces the task of proving fairness of a coin-flipping protocol, against fail-stop adversaries, to proving the protocol is correct: the honest parties always output the same bit, and this bit is uniform in an all honest execution, and to proving the protocol is unbiased: a fail-stop adversary cannot bias the output of the honest parties by too much.

**Definition 2.4 (Correct coin-flipping protocols).** A protocol is a correct coin flipping, if

- When interacting with an $\text{fail-stop}$ adversary controlling a subset of the parties, the honest parties always output the same bit, and
- The common output in a random honest execution of $\pi$, is uniform over $\{0, 1\}$.

Given a partial view of a fail-stop adversary, we are interesting in the expected outcome of the parties, conditioned on this and the adversary making no further aborts.

**Definition 2.5 (View value).** Let $\pi$ be a protocol in which the honest parties always output the same bit value.

For a partial view $\nu$ of the parties in a fail-stop execution of $\pi$, let $C_\nu(\nu)$ denote the parties’ full view in an honest execution of $\pi$ conditioned on $\nu$ (i.e., all parties that do not abort in $\nu$ act honestly in $C_\nu(\nu)$). Let $\text{val}_\nu(\nu) = E_{\nu', C_\nu(\nu')}[\text{out}(\nu')]$, where $\text{out}(\nu')$ is the common output of the non-aborting parties in $\nu'$.

Finally, a protocol is unbiased, if no fail-stop adversary can bias the common output of the honest parties by too much.

**Definition 2.6 (Unbiased coin-flipping protocols).** A $t$-party, $m$-round protocol $\pi$ is $\alpha$-unbiased, if the following holds for every fail-stop adversary $A$ controlling the parties indexed by a subset $C \subseteq [t]$. Let $V$ be $A$’s view in a random execution of $\pi$ in which $A$ controls the parties indexed by $C$, and let $I_j$ be the index of the $j$’th round in which $A$ sent an abort message (set to $m$, if no such round). Let $V_0$ be the prefix of $V$ at the end of the $i$’th round, letting $V_0$ being the empty view, and let $V_j^-$ be the prefix of $V_i$ with the $i$’th round abort messages (if any) removed. Then

$$\frac{E_{\nu} \left[ \sum_{j \in |C|} \text{val}(V_{j^-}) - \text{val}(V_{j^+}) \right]}{\text{val}(\nu)} \leq \alpha,$$

where $\text{val} = \text{val}_\nu$ is according to Definition 2.5.

The following lemma is an alternative characterization of fair coin-flipping protocols (against fail-stop adversaries).

**Lemma 2.7.** Let $\pi$ be a correct, $\alpha$-unbiased coin-flipping protocol with $\alpha(\kappa) \leq \frac{1}{2} - \frac{1}{t^{\kappa p}}$, for some $p \in \mathbb{P}$. Then $\pi$ is a $(\alpha(\kappa) + \text{neg}(\kappa))$-secure coin-flipping protocol against fail-stop adversaries.

**Proof.** Omitted. □
2.4 Oblivious Transfer

Definition 2.8. The \(\binom{1}{2}\) oblivious transfer (OT for short) functionality, is the two-output functionality \(f\) over \(\{0,1\}^k\), defined by \(f(\sigma_0,\sigma_1,i) = (\sigma_0,\sigma_1,\sigma_i,i)\).

Protocols that securely compute OT, are known under several hardness assumptions (cf., [2, 16, 18, 24, 29, 30]).

2.5 \(f\)-Hybrid Model

Let \(f\) be a \(t\)-output functionality. The \(f\)-hybrid model is identical to the real model of computation discussed above, but in addition, each \(t\)-size subset of the parties involved, has access to a trusted party realizing \(f\). It is important to emphasize that the trusted party realizes \(f\) in a non-simultaneous manner: it sends a random output of \(f\) to the parties in an arbitrary order. When a party gets its part of the output, it instructs the trusted party to either continue sending the output to the other parties, or to send them the abort symbol (i.e., the trusted party “implements” \(f\) in a perfect non-simultaneous manner).

All notions given in Sections 2.2 and 2.3 naturally extend to the \(f\)-hybrid model, for any functionality \(f\). In addition, the proof of Lemma 2.7 straightforwardly extends to this model.

3. OUR RESULTS

In this section we formally state our main results. The proofs that follow the intuition given in Section 1.3, can be found in the full version of this paper [26].

3.1 Two-Party Protocols

Theorem 3.1. There exist a polynomial-time oracles-aided protocol \(\Pi^2\) and a polynomial-time computable function \(\text{TwoPartySharesGen}\) such that \(\Pi^2(1^m)\) is an \(m\)-round, two-party, \(O\left(\log^2 m\right)\)-fair coin-flipping protocol against unbounded fail-stop adversaries in the \(\text{TwoPartySharesGen}\)-hybrid model, for any \(m \equiv 1 \mod 4\).

Proof. Omitted.

Using standard means, the above theorem yields the existence of an almost optimal two-party fair coin-flipping protocol, in the real (non-hybrid) model.

Theorem 3.2. Assuming protocols for securely computing OT exist, then for any polynomially bounded, polynomial-time computable, integer function \(m\), there exists an \(m\)-round, \(O\left(\log^2 m\right)\)-fair, two-party coin-flipping protocol.

Proof. Omitted.

3.2 Three-Party Protocols

Theorem 3.3. There exist a polynomial-time oracles-aided protocol \(\Pi^3\) and a polynomial-time computable function \(\text{ThreePartySharesGen}\), such that \(\Pi^3(1^m)\) is an \(m\)-round, three-party, \(O\left(\log^3 m\right)\)-fair coin-flipping protocol against unbounded fail-stop adversaries in the \(\text{ThreePartySharesGen}\)-hybrid model, for any \(m \equiv 1 \mod 4\).

Proof. Omitted.

As in the two-party case, the above theorem yields the existence of an almost optimal three-party fair coin-flipping protocol, in the real (non-hybrid) model.

Theorem 3.4. Assuming protocols for securely computing OT exist, then for any polynomially bounded, polynomial-time computable, integer function \(m\), there exists an \(m\)-round, \(O\left(\log^3 m\right)\)-fair, three-party coin-flipping protocol.

Proof. Omitted.

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