In this paper, we consider the plasmon resonance in multi-layer structures. We show that the plasmon mode is equivalent to the eigenvalue problem of a matrix, whose order is the same to the number of layers. For any number of layers, the exact characteristic polynomial is derived by a conjecture and is verified by using induction. It is shown that all the roots to the characteristic polynomial are real and exist in the span \([-1, 2]\), when the background field is uniform in \(\mathbb{R}^3\). Numerical examples are presented for finding all the plasmon modes, and it is surprisingly to find out that such multi-layer structures may induce so called surface-plasmon-resonance-like band.

**KEYWORDS**
- characteristic polynomial
- multi-layer structure
- plasmon modes
- plasmon resonance

**MSC CLASSIFICATION**
- 35J05
- 35P15

**1 | INTRODUCTION**

Recently, there have been increasingly great deal of interest in mathematical theory of surface localized resonance (SLR), due to its fundamental basis of many cutting-edge applications such as invisibility cloaking, near-field microscopy, molecular recognition, nano-lithography, and so on, see, for example, previous works [1–14] and references there in. SLR structure is usually constructed by high contrast material compared with ambient medium, or noble metallic material which may exhibit negative property at some special occasions. Plasmon resonance is the resonant oscillation of conduction electrons at the interface between negative and positive permittivity material caused by the background field.

It is well known that plasmon resonance depends highly on the material structure. Ammari et al. [4] studied the plasmon resonance for multiple well-separated nanoparticles under the physical model called Drude model, which describes the dependence of material parameter on the frequency of incident wave. Core-shell structures are widely used for analysis of cloaking due to anomalous localized resonance [1–3, 6, 8, 14]. Plasmon resonance for spherical structure with normal scale has been studied in these years [10, 15]. It is noted that the above structures are all isotropic material structures. In previous studies [16–18], plasmon resonance with anisotropic material structures (nanorod, slender-body) are studied and some sophisticated observations have been investigated. In most of the aforementioned works on plasmon resonance, the spectral of so called Nuemann–Poincaré (N-P) type operators are deeply studied, since it may cause the breaking of the invertibility of an integral system derived from related physical partial differential equation. Each eigenvalue is associated with one type of plasmon mode.
We mention that most mathematical formulation of plasmonic structures are structures with homogeneous material parameters. Fang et al. [19] studied the plasmon resonance and its heat generation effect of a four-layer structure. As far as we know, this is the first time to mathematically study the plasmon resonance in a piecewise constant material structure. However, the results in Fang et al. [19] cannot be generalized to plasmon resonance in any multi-layer structures, which is a quite challenging problem. The focus of this paper is to present a precisely understanding on plasmon resonance in any multi-layer structures. We simply use the conductivity problem for analysis breach, while more sophisticated electro-magnetic system shall be considered in forth coming works. We shall first derive the exact perturbed field in terms of a matrix which contains the material information, together with the structure information. This matrix plays the role of the integral operator in homogeneous material structure case. The explicit formula for the determinant of the matrix, which is equivalent to the characteristic polynomial of another matrix that does not depend on the material parameter, is a stumbling block in the analysis of plasmon modes. After struggling against complicated structure of the matrix, we come up with a conjecture of the exact formulation of characteristic polynomial. This formulation shows that the eigenvalues can be divided by pairs which add up to one. The conjecture is then verified by induction. Besides, the roots of the characteristic polynomial are all real and belong to the span \([-1, 2]\) for uniform background field in \(\mathbb{R}^3\). Such breaking through opens a wide way to analysis of localized resonance in multi-layer structures.

The organization of this paper is as follows. In Section 2, we present the mathematical formulation of plasmon resonance in conductivity problem with multi-layer structures. Some main results on the formulation of characteristic polynomial and its estimation of roots are exposed. We show some similar results for two-dimensional case in Section 3. In Section 4, numerical examples are presented in finding all the plasmon modes in a fixed multi-layer structure and plasmon resonance is simulated by using physical Drude model in \(\mathbb{R}^3\). Some conclusions are made in Section 5. Appendix A is devoted to the proofs of the main results in Section 2.

## 2 | MATHEMATICAL FORMULATION AND MAIN RESULTS

Consider the conductivity equation

\[
\begin{aligned}
\left\{ \begin{array}{ll}
\nabla \cdot \epsilon \nabla u = 0, & \text{in } \mathbb{R}^3 \\
u - H = O(|x|^{-1}), & |x| \to \infty,
\end{array} \right.
\]

(2.1)

where \(H\) is the background field which satisfies \(\Delta H = 0\) in \(\mathbb{R}^3\). The parameter \(\epsilon\) denotes the conductivity which is given by

\[
\epsilon(x) = \epsilon_c(x) \chi(D) + \epsilon_0 \chi(\mathbb{R}^3 \setminus D),
\]

(2.2)

where \(D\) is a multi-layer structure and \(\chi\) denotes the characteristic function. The values of \(\epsilon_c(x)\) in \(D\) are piecewise constants, depending on the number of the layers given. In Figure 1, a six-layer structure is presented. In general, define the \(N\)-layer structure by

\[
A_0 := \{r > r_1\}, \quad A_j := \{r_{j+1} < r \leq r_j\}, \quad j = 1, 2, \ldots, N - 1 \quad A_N := \{r \leq r_N\},
\]

(2.3)

where \(N \in \mathbb{N}\). Assume that

\[
\epsilon_c(x) = \epsilon_j, \quad x \in A_j, \quad j = 1, 2, \ldots, N.
\]

(2.4)

Suppose that the background field is uniformly distributed, that is, the field \(H\) can be represented by

\[
H = r \sum_{m=-1}^{1} a_{0,m} Y_1^m,
\]

(2.5)
FIGURE 1  Schematic of a six-layer structure.

where $a_{0,m}$, $m = -1, 0, 1$ are given real numbers. By the symmetric properties of the multi-layer structure, we suppose the total electric potential $u$ has the form

$$u = \begin{cases} 
 r \sum_{m=-1}^{1} a_{N,m} Y_1^m, & x \in A_N, \\
 r \sum_{m=-1}^{1} a_{j,m} Y_1^m + r^{-2} \sum_{m=-1}^{1} b_{j,m} Y_1^m, & x \in A_j, j = N - 1, N - 2, \ldots, 1 \\
 r \sum_{m=-1}^{1} a_{0,m} Y_1^m + r^{-2} \sum_{m=-1}^{1} b_{0,m} Y_1^m, & x \in A_0, 
\end{cases} \tag{2.6}$$

where the constants $a_{j,m}, b_{j,m}$ and $b_{0,m}, j = 1, 2, \ldots, N, m = -1, 0, 1$ are to be determined. In what follows, we define

$$\lambda_j = \frac{2 \varepsilon_{j-1} + \varepsilon_j}{\varepsilon_{j-1} - \varepsilon_j}, j = 1, 2, \ldots, N \tag{2.7}$$

The following theorem shows the formula of the perturbed field $u - H$ outside the multi-layer structure.

**Theorem 2.1.** Suppose $u$ is the solution to (2.1), with the parameter $\varepsilon$ given by piecewise constant values in (2.4), which are positive real numbers. Suppose that $H$ is given by (2.5). Then there holds

$$u - H = r^{-3} He^T Y_N (P_N^T)^{-1} e. \tag{2.8}$$

where $e := (1, 1, \ldots, 1)^T$, the matrix $P_N$ and $Y_N$ are given by

$$P_N := \begin{bmatrix} 
 \lambda_1 & -1 & -1 & \cdots & -1 \\
 2(r_2/r_1)^3 & \lambda_2 & -1 & \cdots & -1 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 2(r_{N-1}/r_1)^3 & 2(r_{N-1}/r_2)^3 & 2(r_{N-1}/r_3)^3 & \cdots & -1 \\
 2(r_N/r_1)^3 & 2(r_N/r_2)^3 & 2(r_N/r_3)^3 & \cdots & \lambda_N 
\end{bmatrix} \tag{2.9}$$
and

\[
Y_N := \begin{bmatrix}
    r_1^1 & 0 & 0 & \ldots & 0 \\
    0 & r_2^1 & 0 & \ldots & 0 \\
    0 & 0 & r_3^1 & \ldots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \ldots & r_p^1
\end{bmatrix}.
\]  

We mention that in general the permittivity \(\varepsilon_c\) are positive valued, then the system (2.1) is an elliptic system and the uniqueness of the solution is well established, and thus, one can derive that (see Appendix 2.6) the matrix \(P_N\) is invertible. However, in resonance modes, it contains negative values in the multi-layer structure, that is \(\varepsilon_j < 0\) holds for some \(j = 1, 2, \ldots, N\). We shall give a precise connection on plasmon resonance and the choice of parameters \(\varepsilon_c\) in the multi-layer structure. It can be readily seen that the plasmon modes occur when the matrix \(P_N\) satisfy \(|P_N| = 0\), where and in what follows \(|P_N|\) stands for the determinant of \(P_N\). One of our aims in this paper is to derive the exact formula for the determinant of \(P_N\). First, we define

\[
P_M^i := \begin{bmatrix}
    \lambda_i & -1 & -1 & \ldots & -1 \\
    2r_{i+1}^3 & \lambda_{i+1} & -1 & \ldots & -1 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    2r_{M-1}^3 & 2r_{M-1}^3 & 2r_{M-1}^3 & \ldots & -1 \\
    2r_{M}^3 & 2r_{M}^3 & 2r_{M}^3 & \ldots & \lambda_M
\end{bmatrix}.
\]  

We also set \(P_2^3 = 1\) in what follows. To simplify the notation we define \(t_j^i := (r_j/r_i^3)\), \(i, j = 1, 2, \ldots, N\). We present an elementary result as follows.

**Lemma 2.2.** Suppose \(N \geq 4\). The there holds the following recursive relation:

\[
|P_N| = (\lambda_1 + (\lambda_2 - 1)t_1^3)(\lambda_N + (\lambda_{N-1} - 1)t_N^3)|P_{N-1}^2| - (\lambda_1 + (\lambda_2 - 1)t_1^3)(\lambda_{N-1} + 1)(\lambda_{N-2} - 2)t_{N-1}^3)|P_{N-2}^3| - (\lambda_2 - 2)t_2^3(\lambda_2 + 1)(\lambda_{N-1} + 1)(\lambda_{N-2} - 2)t_{N-1}^3)|P_{N-1}^2| + (\lambda_2 - 2)t_2^3(\lambda_2 + 1)(\lambda_{N-1} + 1)(\lambda_{N-2} - 2)t_{N-1}^3|P_{N-2}^3|.
\]  

### 2.1 Eigenvalue problem

It is known that plasmon resonance is usually associated with some eigenvalue problem generated by the PDE system. We shall also explore the related eigenvalue problem for multi-layer structure. To simplify the analysis, we suppose that

\[
\varepsilon_j = \begin{cases}
    -\varepsilon^* + i\delta, & \text{if } j \text{ is odd}, \\
    \varepsilon_0, & \text{if } j \text{ is even},
\end{cases}
\]

where \(\varepsilon^*\) is a positive number to be chosen and \(\delta\) is some small parameter, which can be treated as a lossy parameter. \(i = \sqrt{-1}\). In this setup, one can readily obtain that

\[
\lambda_j = \begin{cases}
    \lambda, & \text{if } j \text{ is odd}, \\
    1 - \lambda, & \text{if } j \text{ is even},
\end{cases}
\]

where

\[
\lambda = \frac{2\varepsilon_0 - \varepsilon^* + i\delta}{\varepsilon_0 + \varepsilon^* - i\delta}.
\]

Then (2.8) can be rewritten by

\[
u - H = r^{-3}He^TY_N(\lambda I - K_N^2)^{-1}\hat{e},
\]

```
where \( \mathbf{e} := (1, -1, 1, \ldots, (-1)^{N-1})^T \) and the matrix \( K_N \) is given by

\[
K_N = \begin{bmatrix}
0 & 1 & -1 & \cdots & (-1)^N \\
-2(r_2/r_1)^3 & 1 & -1 & \cdots & (-1)^N \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-2(r_{N-1}/r_1)^3 & 2(r_{N-2}/r_1)^3 & -2(r_{N-1}/r_3)^3 & \cdots & (-1)^N \\
-2(r_N/r_1)^3 & 2(r_{N-2}/r_3)^3 & -2(r_N/r_3)^3 & \cdots & (1 + (-1)^N)/2
\end{bmatrix}.
\] (2.16)

The plasmon resonance mode is parallel to the eigenvalue problem of the matrix \( K_N \). In fact, similar to Deng et al. [16], we can define the plasmon resonance as follows.

**Definition 2.3.** Consider the system (2.1) associated with the \( N \)-layer structure \( D \), where the material configuration is described in (2.13). Then plasmon resonance occurs if the following condition is fulfilled:

\[
\lim_{\delta \to 0} \|\nabla (u - H)\|_{L^2(\mathbb{R}^2 \setminus B)} = \infty.
\]

According to Definition 2.3, one can show (see Fang et al. [19], for example) that the plasmon modes are the configurations which make the parameter \( \lambda \) the eigenvalue of the matrix \( K_N \) as \( \delta \to 0 \). Thus, in what follows, we shall focus on the eigenvalues of \( K_N \), or the determinant of \( P_N \). To this end, we shall consider the dependence of the determinant \( |P^i_m| \), \( i \leq m \), on \( \lambda \). First by direct computations one has

\[
|P^m_{m+1}(\lambda)| = -(\lambda^2 - \lambda) + 2t^m_{m+1},
\]

\[
|P^{-m}_{m+1}(\lambda)| = -(\lambda^2 - \lambda) + (2t^m_{2m+1} + 2t^{2m}_{2m+1} - 2t^{2m-1}_{2m+1}) \lambda,
\]

\[
|P^m_{2m+2}(\lambda)| = \lambda^3 - 2\lambda^2 + (2t^m_{2m+2} - 2t^{2m}_{2m+2} - 2t^{2m+1}_{2m+2} + 1) \lambda + 2t^m_{2m+2} + 2t^{2m+1}_{2m+2} - 2t^{2m+2}_{2m+2} \]

\[
= -((1 - \lambda)^2 - (1 - \lambda))(2t^m_{2m+2} + 2t^{2m+1}_{2m+2} - 2t^{2m+2}_{2m+2}) \big(1 - \lambda\big), \quad m \geq 1.
\] (2.17)

With the recursive formula (2.12) and (2.17), one then has

\[
|P_5(\lambda)| = \lambda(1 - t_1^2)(1 - t_2^2)(1 - t_3^2)(\lambda(1 - \lambda) + 2t_1^2) - \lambda(1 - t_1^2)(\lambda + 1)(\lambda - 2)t_2^4(1 - \lambda)
\]

\[
\quad - (\lambda + 1)t_2^2(\lambda - 2)(1 - t_3^2)(\lambda + 1)t_4^3(\lambda - 1)(\lambda - 2)t_4^5
\]

\[
= (\lambda^2 - \lambda)^2 + 2(\lambda^2 - \lambda) \sum_{(i,j) \in C_4^{m+1}} (-1)^{i+j} t_i^j + 4t_2^4 t_4^5,
\] (2.18)

where \( C_n^m \) denote the set of all combinations of \( m \) out \( n \), \( m \leq n \). For one combination \((i_1, i_2, \ldots, i_m) \in C_n^m\), \((i + 1) \leq i_1, i_2, \ldots, i_m \leq (n + i)\), we are arranging the order in an ascending way, that is, \( i_1 < i_2 < \ldots < i_m \). Then, in a similar manner, one can show that

\[
|P_6(\lambda)| = \lambda \left( (\lambda^2 - \lambda)^2 + 2(\lambda^2 - \lambda) \sum_{(i,j) \in C_4^{m+2}} (-1)^{i+j} t_i^j + 4 \sum_{(i,j,k) \in C_4^{6}} \tau_{i,j,k} t_i^j t_k^k \right),
\] (2.19)

where \( \tau_{(i,j,k)} := (-1)^{i+j+k+1} \). Similarly,

\[
|P_6(\lambda)| = -(\lambda^2 - \lambda)^3 - 2(\lambda^2 - \lambda)^2 \sum_{(i,j) \in C_4^{m+2}} (-1)^{i+j} t_i^j - 4(\lambda^2 - \lambda) \sum_{(i,j,k) \in C_4^{6}} \tau_{(i,j,k)} t_i^j t_k^k + 8t_2^4 t_4^5 t_6^5.
\] (2.20)

For the sake of simplicity, in what follows, we denote by \( \mathbf{i}^k \) the multi-index \((i_1, i_2, \ldots, i_k)\) and \( r_k \) its sign given by

\[
r_k := (-1)^{\sum_{i=1}^{k} i}.
\] (2.21)

By combining the above results, we have following result:
Theorem 2.4. Define

\[ q(\lambda) := \lambda^2 - \lambda. \]

Then for \( N \in \mathbb{N} \), there hold that

\[
|P_N(\lambda)| = \begin{cases} 
(-1)^L \left( \sum_{k=0}^{L} 2^k q(\lambda)^{L-k} \sum_{l=1}^{\lfloor N/2 \rfloor} \prod_{i=1}^{k} l_{i_2}^{l_{i_2}-1} \right), & N = 2L, \\
\lambda(-1)^L \left( \sum_{k=0}^{L} 2^k q(\lambda)^{L-k} \sum_{l=1}^{\lfloor N/2 \rfloor} \prod_{i=1}^{k} l_{i_2}^{l_{i_2}-1} \right), & N = 2L + 1. 
\end{cases} 
\] (2.22)

We want to mention that if the number of layers, \( N \), is odd then there exists one root, that is, \( \lambda = 0 \) to the determinant \( P_N(\lambda) \). This equals to the plasmon mode that \( \varepsilon^* = 2\varepsilon_0 \). The other roots are contained in a quadratic polynomial, whose constant terms are the roots of a polynomial of order \( \lfloor N/2 \rfloor \), respectively. Here, we denote by \( \lfloor t \rfloor \) the integer part of \( t \in \mathbb{R} \).

In other words, we have the following result:

Lemma 2.5. Let \( P_N(\lambda) \) be defined in (2.22), then the roots to \(|P_N(\lambda)|\) are the solutions to

\[ \lambda^2 - \lambda - q = 0, \]

where \( q \) are all the roots to the polynomial

\[
|P_N(\lambda)| = \begin{cases} 
(-1)^L \left( \sum_{k=0}^{L} 2^k q(\lambda)^{L-k} \sum_{l=1}^{\lfloor N/2 \rfloor} \prod_{i=1}^{k} l_{i_2}^{l_{i_2}-1} \right), & N = 2L, \\
\lambda(-1)^L \left( \sum_{k=0}^{L} 2^k q(\lambda)^{L-k} \sum_{l=1}^{\lfloor N/2 \rfloor} \prod_{i=1}^{k} l_{i_2}^{l_{i_2}-1} \right), & N = 2L + 1. 
\end{cases} 
\]

Proof. The proof is straight forward by results from Theorem 2.4. \( \square \)

We shall analyze all the roots to the polynomial \(|P_N(\lambda)|\). Note that (2.22) can also be written by the following compact form:

\[
|P_N(\lambda)| = (-1)^{\lfloor N/2 \rfloor} \lambda^{\lfloor N/2 \rfloor-\lfloor N/2 \rfloor} \left( \sum_{k=0}^{\lfloor N/2 \rfloor} 2^k q(\lambda)^{\lfloor N/2 \rfloor-k} \sum_{l=1}^{\lfloor N/2 \rfloor} \prod_{i=1}^{k} l_{i_2}^{l_{i_2}-1} \right). 
\] (2.23)

As mentioned before, to find the plasmon modes, it is essential to find the roots of the polynomial

\[
f_N(q) := \sum_{k=0}^{\lfloor N/2 \rfloor} 2^k q^{\lfloor N/2 \rfloor-k} \sum_{l=1}^{\lfloor N/2 \rfloor} \prod_{i=1}^{k} l_{i_2}^{l_{i_2}-1}. \] (2.24)

We have the following elementary result on the roots of \( f(q) \):

Theorem 2.6. Suppose \( f_N(q) \) is defined in (2.24). Then it exists \( \lfloor N/2 \rfloor \) real values of roots to \( f_N(q) \). Besides, let \( q^* \) be the root to \( f_N(q) \), then

\[ q^* \in [-1/4, 2]. \] (2.25)
With Theorem 2.6, one can readily show that the roots of $|P_N(\lambda)|$ are all real values. In fact, for any real solution $q^*$ to (2.24), one can derive two roots of $|P_N(\lambda)|$, more specifically,

$$
\lambda = \frac{1 \pm \sqrt{1 + 4q^*}}{2}.
$$

(2.26)

By using (3.13), one can estimate that

$$
\lambda \in [-1, 2].
$$

2.2 | Extreme case

In this part, we shall consider that the radius of the layers are extreme large. We mention that the Earth’s structure can be treated in this case. To mathematically describe this case, we set $r_i = R + c_i$, where $R \gg 1$ and $c_i$ are regular constants, $i = 1, 2, \ldots, N$. One then has

$$
t_i^j = (r_j/r_i)^3 = 1 + \mathcal{O}(1/R).
$$

Define

$$
g_{N,k} := \sum_{i_k \in C_N^{\pm}} r_{i_k},
$$

(2.27)

by straight forward computations, one then has

$$
g_{N,1} = \sum_{i_k \in C_N^{\pm}} r_{i_1} = (-1)^N - 1/2, \quad g_{N,2} = \sum_{i_k \in C_N^{\pm}} r_{i_2} = -[N/2],
$$

and

$$
g_{2N+1,2N+1} = \sum_{i_k \in C_N^{\pm}} r_{i_{2N+1}} = (-1)^{N+1}, \quad g_{N,\lfloor N/2 \rfloor} = \sum_{i_k \in C_N^{\pm}} r_{i_{\lfloor N/2 \rfloor}} = (-1)^{\lfloor N/2 \rfloor}.
$$

Besides, one can also observe that

$$
g_{2N,2k-1} = \sum_{i_{k-1} \in C_N^{\pm}} r_{i_{2k-1}} = 0, \quad k = 1, 2, \ldots, N.
$$

(2.28)

We have the following recursive formula:

**Lemma 2.7.** Suppose that $g_{N,k}$ is given by (2.27), then there holds

$$
g_{2N+1,2k} = g_{2N,2k}, \quad k = 1, 2, \ldots, N,
$$

(2.29)

and

$$
g_{2N+2,2k} = -g_{2N+1,2k-2} + g_{2N+1,2k}, \quad k = 2, 3, \ldots, N,
g_{2N+1,2k-1} = -g_{2N-1,2k-3} + g_{2N-1,2k-1}, \quad k = 2, 3, \ldots, N.
$$

(2.30)

**Proof.** First by using (2.28) we have

$$
g_{2N+1,2k} = -g_{2N,2k-1} + g_{2N,2k} = g_{2N,2k},
$$

which verifies (2.29). Next we compute

$$
g_{2N+1,2k-1} = -g_{2N,2k-2} + g_{2N,2k-1}
= -g_{2N-1,2k-3} + g_{2N-1,2k-1} + g_{2N-1,2k-1}
= -g_{2N-1,2k-3} + g_{2N-1,2k-1}.
$$
which verifies the second equation in (2.30). To prove the first equation in (2.30), we shall make use of induction. It can be simply verified that the first equation in (2.30) holds for $N \leq 4$. Suppose it holds for $N \leq N_0 - 1$, $N_0 \geq 5$. By combing the above results, one finally has

$$g_{2N_0+2,2k} = g_{2N_0+1,2k-1} + g_{2N_0+1,2k}$$

$$= -g_{2N_0-1,2k-3} + g_{2N_0-1,2k-1} + g_{2N_0+1,2k}$$

$$= g_{2N_0-2,2k-4} - g_{2N_0-2,2k-2} + g_{2N_0+1,2k}$$

$$= g_{2N_0-1,2k-4} - g_{2N_0-1,2k-2} + g_{2N_0+1,2k}$$

$$= -g_{2N_0,2k-2} + g_{2N_0+1,2k} = -g_{2N_0,2k-2} + g_{2N_0+1,2k},$$

which completes the proof. □

By using Lemma 2.7, the associated polynomial $f_N(q)$ is then given by

$$f_N(q) = \sum_{k=0}^{[N/2]} 2^k q^{[N/2]-k} g_{N,2k} + \mathcal{O}(1/R), \quad (2.31)$$

where $g_{N,2k}$ can be calculated recursively by using (2.30). By using elementary combination theory, one can show that

$$g_{N,2k} = (-1)^k C_{[N/2]}^k, \quad (2.32)$$

where $C_{[N/2]}^k$ denote the number of combinations for $k$ out of $[N/2]$. (2.31) then becomes

$$f_N(q) = \sum_{k=0}^{[N/2]} (-1)^k 2^k q^{[N/2]-k} C_{[N/2]}^k + \mathcal{O}(1/R) = (q - 2)^{[N/2]} + \mathcal{O}(1/R). \quad (2.33)$$

We can thus summarize our result as follows:

**Proposition 2.8.** Let $f_N(q)$ is defined by (2.24). Suppose $r_i = R + c_i$, $i = 1, 2, \ldots, N$, then there holds

$$\lim_{R \to \infty} f_N(q) = (q - 2)^{[N/2]}, \quad (2.34)$$

The formula (2.34), together with (2.23), indicates that the possible plasmon modes for such set of multi-layer structure are only $\lambda = -1, 2$ if the number of layers $N$ is even and $\lambda = 0, -1, 2$ if $N$ is odd.

### 3 TWO-DIMENSIONAL CASE

For the sake of completeness, we shall present the related results for two-dimensional case. Consider again the conductivity problem (2.1) but in two-dimensional space $\mathbb{R}^2$.

#### 3.1 Perturbation filed

Let us keep the notations for the multi-layer structure $D$. Suppose that the background harmonic field $H$ admits the following expansion:

$$H = \sum_{n=0}^{\infty} \bar{a}_{0,n} r^n \cos n\theta, \quad (3.1)$$
where \( r(\cos \theta, \sin \theta) \) is the polar coordinate for \( \mathbf{x} := (x, y) \). We mention that the choice of the background field \( H \) is much more general than that in three-dimensional case. Then the solution \( u \) to (2.1) can be represented by

\[
\begin{align*}
\sum_{n=0}^{\infty} \tilde{a}_{N,n} r^n \cos n\theta, & \quad \mathbf{x} \in A_N, \\
\sum_{n=-\infty}^{0} \tilde{a}_{j,n} r^n \cos n\theta, & \quad \mathbf{x} \in A_j, j = N - 1, N - 2, \ldots, 1 \\
\sum_{n=-\infty}^{\infty} \tilde{a}_{0,n} r^n \cos n\theta, & \quad \mathbf{x} \in A_0
\end{align*}
\]

(3.2)

By using transmission conditions across each layer \( A_j, j = 1, 2, \ldots, N \), one can then derive that

\[
\begin{align*}
\tilde{a}_{j,n} r^n_j + \tilde{a}_{j,-n} r^{-n}_j = \tilde{a}_{j-1,n} r^n_j + \tilde{a}_{j-1,-n} r^{-n}_j, \\
\varepsilon_n \left( \tilde{a}_{j,n} r^n_j - \tilde{a}_{j,-n} r^{-n}_j \right) = \varepsilon_{j-1} n \left( \tilde{a}_{j-1,n} r^n_j - \tilde{a}_{j-1,-n} r^{-n}_j \right)
\end{align*}
\]

(3.3)

for \( n = 0, 1, \ldots, \infty \). By following a similar strategy, one thus has the following result:

**Theorem 3.1.** Suppose \( u \) is the solution to

\[
\begin{align*}
\nabla \cdot \varepsilon \nabla u & = 0, \quad \text{in } \mathbb{R}^2 \\
u - H & = \mathcal{O}(|x|^{-1}), \quad |x| \to \infty,
\end{align*}
\]

(3.4)

where \( H \) is given in (3.1), with the parameter \( \varepsilon \) given by piecewise constant values in (2.4), which are positive real numbers. Then there holds

\[
u - H = \mathbf{e}^T \sum_{n=1}^{\infty} r^{-n} \cos n\theta \tilde{\Upsilon}_N^{(n)} \left( \tilde{P}_N^{(n)} \right)^{-1} \mathbf{e},
\]

(3.5)

where \( \mathbf{e} := (1, 1, \ldots, 1)^T \), the matrix \( \tilde{P}_N^{(n)} \) and \( \tilde{\Upsilon}_N^{(n)} \) are given by

\[
\tilde{P}_N^{(n)} := \begin{bmatrix}
\tilde{\lambda}_1 (r_2/r_1)^{2n} & (r_3/r_1)^{2n} & \cdots & (r_N/r_1)^{2n} \\
-1 & \tilde{\lambda}_2 (r_3/r_2)^{2n} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \cdots & (r_N/r_{N-1})^{2n} \\
-1 & -1 & -1 & \cdots & \tilde{\lambda}_N
\end{bmatrix}
\]

(3.6)

and

\[
\tilde{\Upsilon}_N^{(n)} := \begin{bmatrix}
r_1^{2n} & 0 & 0 & \cdots & 0 \\
0 & r_2^{2n} & 0 & \cdots & 0 \\
0 & 0 & r_3^{2n} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & r_N^{2n}
\end{bmatrix}
\]

(3.7)

In (3.6), the notations \( \tilde{\lambda}_j, j = 1, 2, \ldots, N \) are defined by

\[
\tilde{\lambda}_j = \frac{\varepsilon_{j-1} + \varepsilon_j}{\varepsilon_{j-1} - \varepsilon_j}, \quad j = 1, 2, \ldots, N.
\]

(3.8)
3.2 Plasmon modes

The perturbation formula (3.5) exhibits very rich separable resonator information. For example, to analyze the monopole resonator plasmon modes, one only needs to consider the term for \( n = 1 \) in (3.5), and dipole resonator for \( n = 2 \), and so on. In this part, we shall introduce a similar structure design as for three-dimensional case, that is, we suppose that

\[
\epsilon_i = \begin{cases} 
\epsilon_1, & i \text{ is odd,} \\
\epsilon_0, & i \text{ is even,} 
\end{cases}
\]

(3.9)

where \( \epsilon_1 \) is some negative material parameter. In such setup, the matrix \( \tilde{P}_N^{(n)} \) in (3.6) is then reduced to

\[
\tilde{P}_N^{(n)} := \begin{bmatrix}
\tilde{\lambda}_1 (r_2/r_1)^{2n} (r_3/r_1)^{2n} \ldots (r_N/r_1)^{2n} \\
-1 -\tilde{\lambda}_1 (r_3/r_2)^{2n} \ldots (r_N/r_2)^{2n} \\
\vdots \vdots \vdots \vdots \vdots \\
-1 -1 -1 \ldots (r_N/r_{N-1})^{2n} \\
-1 -1 -1 \ldots (-1)^{N-1} \tilde{\lambda}_1 
\end{bmatrix}
\]

(3.10)

All the possible plasmon modes for such setup are contained in the solution to

\[
|\tilde{P}_N^{(n)}| = 0,
\]

(3.11)

for any \( n \in \mathbb{N} \), which is equivalent to

\[
\tilde{\lambda}_1^{-N-2[N/2]} \sum_{k=0}^{[N/2]} \sum_{i_k \in C_N^{[k]}} \prod_{l=1}^{k} \left( \frac{r_{2l-1}}{r_{2l}} \right)^2 = 0.
\]

(3.12)

By following a similar arguments as in the proof of Theorem 2.6, one can show that

**Theorem 3.2.** There exists \( N \) real values of roots to (3.11). Besides, let \( \tilde{\lambda}^* \) be one root, then

\[
\tilde{\lambda}^* \in [-1, 1].
\]

(3.13)

4 NUMERICAL EXAMPLES

In this section, we shall show some numerical examples to expose our theoretical results. Let us focus on three-dimensional case. By the theoretical analysis, to find the plasmon modes for any multi-layer structure, one only needs to find the roots of characteristic polynomial according to Theorem 2.4. Moreover, to show the plasmon resonance phenomena, one only needs to observe the energy outside the structure, determined by the term

\[
r^{-3} \textbf{e}^T \mathcal{Y}_N (\lambda I - K_N^{-1})^{-1} \textbf{e}
\]

according (2.15) and Definition 2.3. We shall investigate both scenarios numerically.

4.1 Roots to characteristic polynomials

First, we consider that intervals between each two adjacent layers are equidistance. Let \( \epsilon_0 = 1 \). For \( N \)-layer structure, set

\[
r_i = N - i + 1, \quad i = 1, 2, \ldots N.
\]

(4.1)

Table 1 shows the roots to the characteristic polynomial with \( N = 19 \). In Figure 2, we plot the values of the polynomial \( f_N(q) \) in the span [1.48, 2.0] (approximately the span between the third eigenvalue and last eigenvalue), it is interesting that the values of the characteristic polynomial in the span are all very small (of the order \( 10^{-5} \)). This is surprisingly useful in real applications that this span can be used for surface-plasmon-resonance-like (SPR-like) band.
TABLE 1  Roots to the characteristic polynomial with layers that are chosen by (4.1).

| $N = 19$ | $q$ | $\lambda$ | $\epsilon_i$ |
|----------|-----|-----------|-------------|
| 1.9794  | 1.9664 | 1.9457 |
| 0.9931  | 1.9888 | 0.9888 |
| 0.0023  | 0.9931 | 0.9888 |
| 2.0000  | 0.9931 | 0.9888 |
| 0.0101  | 0.9931 | 0.9888 |
| 4.863   | 1.0066 | 0.3299 |
| 1.7867  | 1.7707 | 1.7408 |
| 0.9271  | 1.9215 | 0.9215 |
| 0.0249  | 0.9271 | 0.9215 |
| 0.0269  | 0.9271 | 0.9215 |
| 0.0306  | 0.9271 | 0.9215 |
| 0.0326  | 0.9271 | 0.9215 |
| 0.0310  | 0.9271 | 0.9215 |
| 0.0479  | 0.9271 | 0.9215 |
| 0.0674  | 0.9271 | 0.9215 |
| 0.1046  | 0.9271 | 0.9215 |

Next, we consider the radius of layers are decreasing with the same scale $s$, that is,

$$r_{i+1} = sr_i, \ i = 1, 2, \ldots \ N - 1.$$  \hspace{1cm} (4.2)

Set $r_1 = 1$ and $s = 0.8$. Table 2 exhibits all the roots. Similarly, one can also find out that the values in the span [1.22, 1.79] (again approximately the span between the third eigenvalue and last eigenvalue, see Figure 3) are all very small. Besides, it is worth mentioning that in both set up of structures, the roots $q$ are all positive values. Similar results can be found in Figures 4 and 5 for $N = 11$ and $N = 16$, respectively.

FIGURE 2  Values of $f_N(q)$ in the span [1.48, 2.0], $N = 19$. [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 2  Roots to the characteristic polynomial with layers that are chosen by (4.2).

| $N = 19$ | $q$ | $\lambda$ | $\epsilon_i$ |
|----------|-----|-----------|-------------|
| 1.7867  | 1.7707 | 1.7408 |
| 0.9271  | 1.9215 | 0.9215 |
| 0.0249  | 0.9271 | 0.9215 |
| 0.0269  | 0.9271 | 0.9215 |
| 0.0306  | 0.9271 | 0.9215 |
| 0.0326  | 0.9271 | 0.9215 |
| 0.0310  | 0.9271 | 0.9215 |
| 0.0479  | 0.9271 | 0.9215 |
| 0.0674  | 0.9271 | 0.9215 |
| 0.1046  | 0.9271 | 0.9215 |
| 0.1815  | 0.9271 | 0.9215 |
| 0.3390  | 0.9271 | 0.9215 |
| 0.2982  | 0.9271 | 0.9215 |

Set $r_1 = 1$ and $s = 0.8$. Table 2 exhibits all the roots. Similarly, one can also find out that the values in the span [1.22, 1.79] (again approximately the span between the third eigenvalue and last eigenvalue, see Figure 3) are all very small. Besides, it is worth mentioning that in both set up of structures, the roots $q$ are all positive values. Similar results can be found in Figures 4 and 5 for $N = 11$ and $N = 16$, respectively.
4.2 | Plasmon resonance illustration

We introduce the Drude model for modeling the parameter $\varepsilon_c$ with the angular frequency $\omega$ (see, e.g., Sarid and Challener [20]). In (2.13), $\varepsilon_i$ are replaced by

$$
\varepsilon_i = \varepsilon' \left(1 - \frac{\omega_p^2}{\omega(\omega + i\tau)}\right), \quad i \text{ is odd},
$$

(4.3)

where we suppose that (cf. Ammari et al. [4])

$$
\tau = 10^{14} \text{s}^{-1}; \quad \varepsilon' = 9 \cdot 10^{-12} \text{Fm}^{-1}; \quad \varepsilon_0 = (1.33)^2 \varepsilon'; \quad \omega_p = 2 \cdot 10^{15} \text{s}^{-1}.
$$

We define the polarization tensor here by

$$
\mathbf{M} := r_1^{-3} Y_N(\lambda I - K_N)^{-1}.
$$

(4.4)
The functionality of $r_{13}^{-3}$ appearing in (4.4) is to reduce the scale of the structure. In Figure 6, we show the norm of polarization tensor defined in (4.4) with the multi-layer structure designed by (4.1), where $N = 17$. It can be seen that the peaks of the norm of polarization tensor are in accordance with the plasmon modes in the setup of physical Drude model. The norm of polarization tensor decays as the frequency $\omega$ is getting small. This is due to the existence of lossy parameter $\tau$.

5 | CONCLUDING REMARKS

We considered plasmon resonance in multi-layer structures. In order to better exhibit the main idea and theoretical results, we use the simplest conductivity problem associated with uniformly distributed background field for derivation of the matrix $P_N(\lambda)$, together with its characteristic polynomial. We mention that the idea can be extended to more general electro-magnetic system with some technique adjustments. We shall derive some similar results regarding the plasmon modes in multi-layer structures for more complicated systems in forthcoming works. Localized resonance of multi-layer structures for elastic problem is another interesting and challenging problem. It is also worthwhile mentioning that the theoretical results can greatly help to find all the plasmon modes for any given multi-layer structures. More numerical observations and justifications will be implemented, especially the searching for SPR-like band will be verified in more different set up of multi-layer structures. Finally, we also want to mention that multi-layer structures have also been
used to generate generalized polarization tensors (GPTs) vanishing structures for enhancement of near cloaking, see, for example, previous works [21–23]. The authors there designed sophisticated multi-layer structures in order to make the lower orders of GPTs vanishing. However, the existence of the vanishing structures is remain open. The design of multi-layer structures together with the theoretical results in this paper might be a possible way to prove the existence of GPTs vanishing structures.

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CONFLICT OF INTEREST STATEMENT

The authors declared that they have no conflicts of interest to this work.

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APPENDIX A: PROOF OF MAIN RESULTS

A.1 | Proof of Theorem 3.1

The transmission conditions on the interface \{r = r_j\}, \(j = 1, 2, \ldots, N\), imply that

\[
\begin{align*}
  a_j, m r_j + b_j, m r_j^{-2} &= a_{j-1, m} r_j + b_{j-1, m} r_j^{-2}, \\
  \varepsilon_j \left(a_j, m - 2b_j, m r_j^{-3}\right) &= \varepsilon_{j-1} \left(a_{j-1, m} - 2b_{j-1, m} r_j^{-3}\right)
\end{align*}
\]  

(A1)

where we set \(b_{N, m} = 0\), \(m = -1, 0, 1\). By using (2.14) and some proper arrangements to the equations (A1), one obtains that

\[
\lambda_1(a_{1, m} - a_{0, m}) + 2 \sum_{j=2}^{N} (a_{j, m} - a_{j-1, m}) \left(\frac{r_j}{r_1}\right)^3 = a_{0, m},
\]

\[
- \sum_{j=1}^{l-1} (a_{j, m} - a_{j-1, m}) + \lambda_l(a_{l, m} - a_{l-1, m}) + 2 \sum_{j=l+1}^{N} (a_{j, m} - a_{j-1, m}) \left(\frac{r_j}{r_1}\right)^3 = a_{0, m},
\]

\[
- \sum_{j=1}^{N-1} (a_{j, m} - a_{j-1, m}) + \lambda_N(a_{N, m} - a_{N-1, m}) = a_{0, m},
\]

for \(m = -1, 0, 1\). In the case that the material parameters \(\varepsilon_j\), \(j = 1, 2, \ldots, N\) are all positive, the matrix \(P_N\) is invertible and then there holds

\[
a_m = a_{0, m} \left(\Xi \left(P_N^T\right)^{-1} e + e\right).
\]

(A2)

where \(a_m := (a_{1, m}, a_{2, m}, \ldots, a_{N, m})^T\) and

\[
\Xi = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 1
\end{bmatrix}.
\]

(A3)

Furthermore, by the first equation in (A1), there holds

\[
b_m = a_{0, m} \Xi^T Y_N \left(P_N^T\right)^{-1} e
\]

(A4)

where \(b_m := (b_{0, m}, b_{1, m}, \ldots, b_{N-1, m})^T\). The proof is complete by extracting the first element in \(b_m\).

A.2 | Proof of Lemma 2.2

Denote by \(E_{i,j}\) the elementary matrix which is transform of identity matrix via adding the \(j\)-th line of identity matrix to the \(i\)-th line. Then by using some elementary translation, we compute
Proof of Theorem 2.4

We shall make use of induction. It is obvious that

\[ \det(N) = \begin{vmatrix} \lambda_1 + (\lambda_2 - 1)t_2^1 - \lambda_2 - 1 & 0 & 0 \\ (2 - \lambda_2)t_2^1 & \lambda_2 & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 2t_{2N-1}^2 & \cdots & \lambda_{N-1} & -\lambda_{N-1} - 1 \\ 0 & 0 & \cdots & (2 - \lambda_{N-1})t_{N-1}^2 & \lambda_N + (\lambda_{N-1} - 1)t_{N-1}^2 \end{vmatrix} \]

\[ (A5) \]

where we only changed the first line and column, together with the last line and column. By using the notation (2.11) and the expansion theorem for determinant, one thus has

\[ |P_N| = \left(\lambda_1 + (\lambda_2 - 1)t_2^1\right) \left(\lambda_N + (\lambda_{N-1} - 1)t_{N-1}^2\right) \left|P_{N-1}^2\right| 
- \left(\lambda_1 + (\lambda_2 - 1)t_2^1\right) \left(\lambda_N + (\lambda_{N-1} - 1)t_{N-1}^2\right) \left|P_{N-2}^2\right| 
- (\lambda_2 - 2)t_2^1(\lambda_2 + 1) \left(\lambda_N + (\lambda_{N-1} - 1)t_{N-1}^2\right) \left|P_{N-1}^3\right| 
+ (\lambda_2 - 2)t_2^1(\lambda_2 + 1) \left(\lambda_N + (\lambda_{N-1} - 1)t_{N-1}^2\right) \left|P_{N-2}^3\right| \]

\[ (A6) \]

holds for all \( N \geq 4, N \in \mathbb{N} \).

**A.3 | Proof of Theorem 2.4**

We shall make use of induction. It is obvious that \(|P_N| = \lambda\) which is contained in (2.22), by using (2.17), one has the assertion for \( N = 2 \) and \( N = 3 \). Now we suppose that (2.22) holds for all \( N \leq N_0 \), \( N_0 \geq 4 \), we show that it also holds for \( N = N_0 + 1 \). Note that \( q(1 - \lambda) = q(\lambda) \), in what follows, we shall write \( q \) instead of \( q(\lambda) \) for simplicity.

Case i) \( N_0 \) is even. By making use of (2.12) and (2.22) for all \( N \leq N_0 \), one has

\[ |P_N(\lambda)| = \lambda^2 \left(1 - t_2^1\right) \left(1 - t_{N-1}^2\right) \left|P_{N-1}^2(1 - \lambda)\right| + \lambda \left(1 - t_1^1\right) \left(2 - \lambda\right)(\lambda + 1)t_{N-1}^2 \left|P_{N-2}^2(1 - \lambda)\right| 
+ t_2^1(2 - \lambda)(\lambda + 1) \left(1 - t_{N-1}^2\right) \left|P_{N-1}(\lambda)\right| + (2 - \lambda)^2(\lambda + 1)^2t_{N-1}^2 \left|P_{N-2}(\lambda)\right| \]

\[ (A7) \]

where we use the notations

\[ d_1 = \lambda^2 \left(1 - t_2^1\right) \left(1 - t_{N-1}^2\right)(-1)^{N_0/2-1}(1 - \lambda), \]

\[ d_2 = \lambda \left(1 - t_1^1\right) \left(2 - \lambda\right)(\lambda + 1)t_{N-1}^2(-1)^{N_0/2-1}, \]

\[ d_3 = t_2^1(2 - \lambda)(\lambda + 1) \lambda \left(1 - t_{N-1}^2\right)(-1)^{N_0/2-1}, \]

\[ d_4 = (2 - \lambda)^2(\lambda + 1)^2t_{N-1}^2(-1)^{N_0/2}. \]

and

\[ D_j = \left(\sum_{k=0}^{N_0/2-1-\lfloor j/4 \rfloor} 2^k q^{N_0/2-k-\lfloor j/4 \rfloor} \sum_{l_{2i} \in [N_0/2-\lfloor j/4 \rfloor]} \tau_{l_{2i}} \prod_{l=1}^{k} l_{2i}^{t_{2i}}} \right). \]
By setting

$$a_k := \sum_{i_{l_1} \in C_{N-3}}^k \tau_{i_{l_1}} \prod_{l=1}^{i_{l_1}} \ell_{i_{l_1}}^{-1}, \quad b_k := \sum_{i_{l_1} \in C_{N-3}}^k \tau_{i_{l_1}} \prod_{l=1}^{i_{l_1}} \ell_{i_{l_1}},$$

and some straightforward computations one has

$$c_k := \sum_{i_{l_1} \in C_{N-3}}^k \tau_{i_{l_1}} \prod_{l=1}^{i_{l_1}} \ell_{i_{l_1}}^{-1}, \quad d_k := \sum_{i_{l_1} \in C_{N-3}}^k \tau_{i_{l_1}} \prod_{l=1}^{i_{l_1}} \ell_{i_{l_1}}^{-1}.$$

and some straightforward computations one has

$$|P_N(\lambda)| = (-1)^{N/2} k \left( q (1 - t_2^1) (1 - t_N^{-1}) \left( \sum_{k=0}^{N/2} 2^k q^{N/2 - k - 1} a_k \right) + (q - 2) (1 - t_2^1) t_N^{-1} \left( \sum_{k=0}^{N/2} 2^k q^{N/2 - k - 1} b_k \right) + (q - 2) t_2^1 (1 - t_N^{-1}) \left( \sum_{k=0}^{N/2} 2^k q^{N/2 - k - 1} c_k \right) + (q - 2)^2 t_2^1 t_N^{-1} \left( \sum_{k=0}^{N/2} 2^k q^{N/2 - k - 2} d_k \right) \right) = : (-1)^{N/2} k \left( \sum_{k=0}^{N/2} 2^k q^{N/2 - k} g_k \right),$$

where it can be seen that

$$g_0 = (1 + t_2^1 t_N^{-1} - t_2^1 - t_N^{-1}) + (1 - t_2^1) t_N^{-1} + t_2^1 (1 - t_N^{-1}) + t_2^1 t_N^{-1} = 1,$$

(A8)

and

$$g_1 = (1 + t_2^1 t_N^{-1} - t_2^1 - t_N^{-1}) \left( \sum_{i_{l_1} \in C_{N-3}}^k \tau_{i_{l_1}} \ell_{i_{l_1}}^1 \right) + (1 - t_2^1) t_N^{-1} \left( \sum_{i_{l_1} \in C_{N-3}}^k \tau_{i_{l_1}} \ell_{i_{l_1}}^1 \right) + t_2^1 (1 - t_N^{-1}) \left( \sum_{i_{l_1} \in C_{N-3}}^k \tau_{i_{l_1}} \ell_{i_{l_1}}^1 \right) + t_2^1 t_N^{-1} \left( \sum_{i_{l_1} \in C_{N-3}}^k \tau_{i_{l_1}} \ell_{i_{l_1}}^1 \right) - (1 - t_2^1) t_N^{-1} - t_2^1 (1 - t_N^{-1}) - 2 t_2^1 t_N^{-1}$$

(A9)

$$= \sum_{i_{l_1} \in C_{N-3}} \tau_{i_{l_1}} \ell_{i_{l_1}}^1,$$

by using the relations

$$a_1 = b_1 + \sum_{j=2}^{N-2} (-1)^{j+N-1} t_N^{-1}, \quad a_1 = c_1 + \sum_{j=3}^{N-1} (-1)^{2+j} t_j^1,$$

$$b_1 = d_1 + \sum_{j=2}^{N-2} (-1)^{2+j} t_j^2, \quad c_1 = d_1 + \sum_{j=3}^{N-1} (-1)^{j+N-1} t_N^{-1}.$$
One can also find out that

\[
g_{N_{0}/2} = - (1 - t_{2}^{N_{0}/2}) b_{N_{0}/2 - 1} - t_{2}^{N_{0}/2} (1 - t_{N_{0}/2}^{N_{0}/2 - 1}) c_{N_{0}/2 - 1} + t_{2}^{N_{0}/2 - 1} d_{N_{0}/2 - 2}
\]

\[
= - (1 - t_{2}^{N_{0}/2}) t_{N_{0}/2}^{N_{0}/2 - 1} \left( \sum_{i_{0}/2-1 \in \mathbb{N}} \tau_{i_{0}/2-1} K_{i_{0}/2-1} \prod_{l=1}^{N_{0}/2-1} l_{ij}^{l_{ij}-1} \right)
\]

\[
- t_{2}^{N_{0}/2} (1 - t_{N_{0}/2}^{N_{0}/2 - 1}) \left( \sum_{i_{0}/2-1 \in \mathbb{N}} \tau_{i_{0}/2-1} K_{i_{0}/2-1} \prod_{l=1}^{N_{0}/2-1} l_{ij}^{l_{ij}-1} \right)
\]

\[
+ t_{2}^{N_{0}/2} t_{N_{0}/2}^{N_{0}/2 - 1} \left( \sum_{i_{0}/2-1 \in \mathbb{N}} \tau_{i_{0}/2-1} K_{i_{0}/2-1} \prod_{l=1}^{N_{0}/2-1} l_{ij}^{l_{ij}-1} \right)
\]

\[
= \sum_{i_{0}/2 \in \mathbb{N}} \tau_{i_{0}/2} K_{i_{0}/2} \prod_{l=1}^{N_{0}/2} l_{ij}^{l_{ij}-1}.
\]

Furthermore, one can readily derive that

\[
g_{k} = (1 + t_{2}^{N_{0}/2} - t_{2}^{N_{0}/2 - 1} - t_{2}^{N_{0}/2 - 1}) a_{k} + (1 - t_{2}^{N_{0}/2}) t_{N_{0}/2}^{N_{0}/2 - 1} (b_{k} - b_{k-1}) + t_{2}^{N_{0}/2} (1 - t_{N_{0}/2}^{N_{0}/2 - 1}) (c_{k} - c_{k-1}) + t_{2}^{N_{0}/2} (b_{k} - 2b_{k-1} + b_{k-2})
\]

for all \( k = 2, 3, \ldots, N_{0}/2 - 1 \). Thus, there holds

\[
g_{k} = \left( 1 + t_{2}^{N_{0}/2} - t_{2}^{N_{0}/2 - 1} \right) (a_{k} - c_{k} - b_{k})
\]

\[
+ \left( 1 - t_{2}^{N_{0}/2} \right) (b_{k} - b_{k-1}) + \left( 1 - t_{N_{0}/2}^{N_{0}/2 - 1} \right) (c_{k} - b_{k}) + b_{k}
\]

\[
+ \left( t_{2}^{N_{0}/2 - 1} - t_{N_{0}/2}^{N_{0}/2 - 1} \right) (b_{k-1} - b_{k-1}) + \left( t_{2}^{N_{0}/2 - 1} - t_{N_{0}/2}^{N_{0}/2 - 1} \right) (c_{k-1} - b_{k-1})
\]

\[
- \left( t_{2}^{N_{0}/2} + t_{N_{0}/2}^{N_{0}/2 - 1} \right) b_{k-1} + t_{2}^{N_{0}/2 - 1} b_{k-2},
\]

for \( k = 2, 3, \ldots, N_{0}/2 - 1 \). By straightforward computations, one has

\[
b_{k} - b_{k} = \sum_{i_{0}/2 \in \mathbb{N}} \tau_{i_{0}/2} I_{i_{0}/2}^{i_{0}/2 - 1} \prod_{l=1}^{k-1} l_{ij}^{l_{ij}-1}, \quad c_{k} - b_{k} = \sum_{i_{0}/2 \in \mathbb{N}} \tau_{i_{0}/2} I_{i_{0}/2}^{i_{0}/2 - 1} \prod_{l=1}^{k} l_{ij}^{l_{ij}-1},
\]

and

\[
a_{k} - c_{k} - b_{k} + b_{k} = \sum_{i_{0}/2 \in \mathbb{N}} \tau_{i_{0}/2} I_{i_{0}/2}^{i_{0}/2 - 1} \prod_{l=1}^{k-2} l_{ij}^{l_{ij}-1}.
\]
By substituting (A11) and (A12) into (A10) and simply computations, one thus obtains

\[
g_k = b_k + \sum_{i_{k-1} \in C_{N-4}} (-1)^{N+1} r_{i_{k-2}} t_{i_1}^{k-2} \prod_{l=1}^{k-2} t_{i_{j_l}}^{L_{i_{j_l}+1}} N + \sum_{i_{k-1} \in C_{N-4}} (-1)^{N+1} r_{i_{k-2}} t_{i_1}^{k-1} \prod_{l=1}^{k-1} t_{i_{j_l}}^{L_{i_{j_l}+1}} \sum_{i_{k-1} \in C_{N-4}} (-1)^{N+2} r_{i_{k-2}} t_{i_1}^{k-2} \prod_{l=1}^{k-2} t_{i_{j_l}}^{L_{i_{j_l}+1}} N
\]

\[+ \sum_{i_{k-1} \in C_{N-4}} (-1)^{N} r_{i_{k-2}} t_{i_1}^{k-1} \prod_{l=1}^{k-1} t_{i_{j_l}}^{L_{i_{j_l}+1}} N - \sum_{i_{k-1} \in C_{N-4}} (-1)^{N} r_{i_{k-2}} t_{i_1}^{k-1} \prod_{l=1}^{k-1} t_{i_{j_l}}^{L_{i_{j_l}+1}} N
\]

\[+ \sum_{i_{k-1} \in C_{N-4}} (-1)^{N} r_{i_{k-2}} t_{i_1}^{k-1} \prod_{l=1}^{k-1} t_{i_{j_l}}^{L_{i_{j_l}+1}} N + \sum_{i_{k-1} \in C_{N-4}} (-1)^{N} r_{i_{k-2}} t_{i_1}^{k-1} \prod_{l=1}^{k-1} t_{i_{j_l}}^{L_{i_{j_l}+1}} N \]

\[= \sum_{i_{k-1} \in C_{N-4}} r_{i_{k-2}} t_{i_1}^{k-1} \prod_{l=1}^{k-1} t_{i_{j_l}}^{L_{i_{j_l}+1}}, \]

which completes the proof for the case that \(N_0\) is even.

Case ii) \(N_0\) is odd. The proof is similar to the case that \(N_0\) is even. We shall only sketch the main ingredients. First, it can be derived that

\[
|P_N(\lambda)| = (-1)^{N/2} \left( q \left( 1 - t_2^1 \right) \left( 1 - t_N^{N-1} \right) \left( \sum_{k=0}^{N/2-1} 2^k q^{N/2-k-1} a_k \right) \right.
\]

\[+ q(q-2) \left( 1 - t_2^1 \right) t_N^{N-1} \left( \sum_{k=0}^{N/2-2} 2^k q^{N/2-k-2} b_k \right) \]

\[+ q(q-2) t_2^1 \left( 1 - t_N^{N-1} \right) \left( \sum_{k=0}^{N/2-2} 2^k q^{N/2-k-2} c_k \right) \]

\[+ (q-2)^2 t_2^1 t_N^{N-1} \left( \sum_{k=0}^{N/2-2} 2^k q^{N/2-k-2} d_k \right) \]

\[= (-1)^{N/2} \left( \sum_{k=0}^{N/2} 2^k q^{N/2-k} h_k \right). \]

Similarly, one can verify that \(h_0 = 1\) and

\[h_1 = \sum_{i \in C_N^2} r_i t_2^1, \quad h_{N/2} = t_2^1 t_4^3 \cdots t_N^{N-1}. \]
One can readily find out that $h_k, k = 2, 3, \ldots, N/2 - 1$ have the same form with (A10); thus, there holds

$$h_k = \sum_{i \in \mathbb{Z}_N^2} \tau_{i} \prod_{j=1}^{k} i_{ij-1}, \quad k = 2, 3, \ldots, N/2 - 1,$$

which completes the proof.

### A.4 Proof of Theorem 2.6

We assume on contrary that

$$q^* \not\in [-1/4, 2], \text{ or } \Re q^* \neq 0.$$  

Note that $\lambda^2 - \lambda = q^*$, it is equivalent that

$$\lambda \not\in [-1, 2], \text{ or } \Re \lambda \neq 0.$$  

Since $\lambda$ satisfies $|P_N(\lambda)| = 0$. There exists non-trial solution to the following linear equations

$$P_N(\lambda)^T y = 0, \quad \text{(A14)}$$

where $y \in \mathbb{R}^N$ and $y \neq 0$. Next, for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, define

$$u = \begin{cases} 
  a_N x_1, & x \in A_N, \\
  a_j x_1 + b_j x_{|x|^{-1}}, & x \in A_j, \; j = N - 1, N - 2, \ldots, 1 \\
  b_0 x_{|x|^{-1}}, & x \in A_0,
\end{cases} \quad \text{(A15)}$$

where $a = (a_1, a_2, \ldots, a_N)$ and $b = (b_0, b_1, \ldots, b_N-1)$ are determined by

$$a = \Xi y, \quad b = \Xi^T \Upsilon_N y,$$

where $\Upsilon_N$ is defined in (2.10). It can then be verified that $u$ is the solution to

$$\begin{cases} 
  \nabla \cdot \epsilon \nabla u = 0, \quad \text{in } \mathbb{R}^3 \\
  u = O(|x|^{-1}), \quad |x| \to \infty,
\end{cases} \quad \text{(A16)}$$

with

$$\epsilon(x) = \epsilon_c(x) \chi(D) + \epsilon_0 \chi(\mathbb{R}^3 \setminus \overline{D}), \quad \text{(A17)}$$

and the parameter $\epsilon_c$ is given by

$$\epsilon_c = \begin{cases} 
  \frac{j-2}{j+1} \epsilon_0, & \text{in } A_j, \; j \text{ is odd}, \\
  \epsilon_0, & \text{in } A_j, \; j \text{ is even}.
\end{cases} \quad \text{(A18)}$$

Note that $u$ is not identically zero in $\mathbb{R}^3$. In fact, if $u \equiv 0$, then there holds that

$$a + Y_N^{-1} M b = 0, \quad b_0 = 0, \quad \text{(A19)}$$
where

\[
M = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}.
\] (A20)

Furthermore, by transmission conditions on the boundary of each layer, there holds,

\[(I - M)\mathbf{b} = \mathbf{Y}_N(I - M^T)\mathbf{a}.\] (A21)

By combining (A19) and (A21), one thus obtains that

\[(\mathbf{Y}_N - M\mathbf{Y}_N M^T)\mathbf{a} = \mathbf{0},\] (A22)

which indicates that \(\mathbf{a} = \mathbf{0}\). This is impossible since \(\mathbf{a} = \mathbf{E}\mathbf{y}\) and \(\mathbf{y} \neq \mathbf{0}\). Thus, \(u\) is not identically zero in \(\mathbb{R}^3\).

On the other hand, one can use integral by parts to compute

\[
\int_{\mathbb{R}^3} \varepsilon |\nabla u|^2 = \sum_{j=0}^{N} \int_{A_j} \varepsilon |\nabla u|^2 = 0.\] (A23)

But by using (A18) there also holds

\[
0 = \int_{\mathbb{R}^3} \varepsilon |\nabla u|^2 = \varepsilon_0 \sum_{j=0}^{[N/2]} \int_{A_{2j}} |\nabla u|^2 + \frac{\hat{\lambda} - 2}{\lambda + 1} \sum_{j=0}^{[(N+1)/2]} \int_{A_{2j-1}} |\nabla u|^2.\] (A24)

Case i. \(\lambda \not\in [-1, 2]\). It is obvious that

\[
\frac{\hat{\lambda} - 2}{\lambda + 1} > 0.
\]

Thus, one immediately has \(\nabla u = 0\) in \(\mathbb{R}^3\) and thus \(u = C\) in \(\mathbb{R}^3\); by the decay behavior of \(u\) at infinity, one thus show that \(u \equiv 0\) in \(\mathbb{R}^3\).

Case ii. \(\Im \lambda \neq 0\). Then by using (A24), one has

\[
\Im \left( \frac{\hat{\lambda} - 2}{\lambda + 1} \right) \sum_{j=0}^{[N+1]/2} \int_{A_{2j-1}} |\nabla u|^2 = 0.\] (A25)

Thus, \(u = C\) in \(A_{2j-1}, j = 1, 2, \ldots, [(N + 1)/2]\). By using integral by parts, one can readily show that \(u = 0\) in \(A_0\), and thus, \(C = 0\). Thus, \(u \equiv 0\) in \(\mathbb{R}^3\).

We have shown that \(u \equiv 0\) either \(\lambda \not\in [-1, 2]\) or \(\Im \lambda \neq 0\), which is in contradiction with our assumption. Thus, (3.13) holds.