NEARLY MINIMAX ROBUST ESTIMATOR OF THE MEAN VECTOR BY ITERATIVE SPECTRAL DIMENSION REDUCTION

BY AMIR-HOSSEIN BATENI, ARSHAK MINASYAN AND ARNAK S. DALALYAN

CEREMADE, Université Paris Dauphine - PSL
Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16
Amirhossein.bateni@ensae.fr

CREST, ENSAE, IP Paris
5 avenue Henry Le Chatelier, 91764 Palaiseau
arshak.minasyan@ensae.fr; arnak.dalalyan@ensae.fr

We study the problem of robust estimation of the mean vector of a sub-Gaussian distribution. We introduce an estimator based on spectral dimension reduction (SDR) and establish a finite sample upper bound on its error that is minimax-optimal up to a logarithmic factor. Furthermore, we prove that the breakdown point of the SDR estimator is equal to $1/2$, the highest possible value of the breakdown point. In addition, the SDR estimator is equivariant by similarity transforms and has low computational complexity. More precisely, in the case of $n$ vectors of dimension $p$—at most $\epsilon n$ out of which are adversarially corrupted—the SDR estimator has a squared error of order $(r_\Sigma/n + \epsilon^2 \log(1/\epsilon)) \log p$ and a running time of order $p^3 + np^2$. Here, $r_\Sigma \leq p$ is the effective rank of the covariance matrix of the reference distribution. Another advantage of the SDR estimator is that it does not require knowledge of the contamination rate and does not involve sample splitting. We also investigate extensions of the proposed algorithm and of the obtained results in the case of (partially) unknown covariance matrix.

1. Introduction. Robust estimation of a finite-dimensional parameter is a classical problem in statistics. The broad goal of robust estimation is to design statistical procedures that are not very sensitive to small changes in data or to small departures from the modeling assumptions. A typical example, extensively studied in the literature, and considered in the present work, is when the data set contains outliers.

The literature on robustness to outliers in parametric estimation is very rich; it would be impossible to review here all the important contributions. For an in-depth exposition of by now classical results and approaches, such as the influence function, the breakdown point and the efficiency, we refer to the books (Huber and Ronchetti, 2011; Maronna et al., 2006; Rousseeuw et al., 2011). In their vast majority, these well-established approaches treated the dimension of the parameter as a fixed and small constant. This simple setting was convenient for mathematical analysis and for computational purposes, but somewhat disconnected from many practical situations. Furthermore, it was hiding some fascinating phenomena that emerge only when the dimension is considered as a parameter that might be large, in the same way as the sample size.

More recently, (Chen et al., 2018) considered the problem of estimating the mean and the covariance matrix of a Gaussian distribution in the high-dimensional setting. The authors namely uncovered a new phenomenon: under the Huber contamination, the componentwise

MSC2020 subject classifications: Primary 62H12; secondary 62F35.
Keywords and phrases: breakdown point, minimax optimality, spectral method, robustness.
The goal of this paper is to make a step forward by designing an estimator which is not only nearly rate optimal and computationally tractable, but also has a breakdown point equal to $1/2$, which is the highest possible value of the breakdown point. To construct the estimator, termed iterative spectral dimension reduction or SDR, we combine and suitably adapt ideas from (Lai et al., 2016) and (Diakonikolas et al., 2017). The main underlying observation is that if we remove some clear outliers and restrict our attention to the subspace spanned by the eigenvectors of the sample covariance matrix corresponding to small eigenvalues, then the sample mean of the projected data points is a rate-optimal estimator. This allows us to iteratively reduce the dimension and eventually to estimate the remaining low-dimensional median is not minimax-rate optimal whereas the Tukey median is. More precisely, if a $p$-dimensional mean vector is to be estimated from $n$ independent vectors drawn from the mixture distribution $(1 - \varepsilon)N_\varepsilon(\mu, \Sigma) + \varepsilon Q$ (where $\varepsilon \in (0, 1/2)$ is the rate of contamination and $Q$ is the unknown distribution of outliers), then the mean squared error of the componentwise median is of order $p/n + p\varepsilon^2$ while that of Tukey’s median is of order $p/n + \varepsilon^2$. This extra factor $p$ in front of $\varepsilon^2$ has been proven in (Lai et al., 2016) to be present in the error of another widely used robust estimator of the mean, the geometric median (Minsker, 2015). Thus, as long as only statistical properties of the estimators are considered, Tukey’s median is thus superior to its competitors, the componentwise and the geometric medians. However, the componentwise and the geometric medians are better than the Tukey’s median in terms of the breakdown point: their breakdown point is equal to $1/2$ (Lopuhaa and Rousseeuw, 1991) whereas that of Tukey’s median is $1/3$ (Donoho and Gasko, 1992). This is one of the appealing phenomena taking place in the high dimensional setting.

Another specificity of the high dimensional setting uncovered by (Chen et al., 2018) was the lack of computational tractability of the estimators that are statistically optimal. Indeed, Tukey’s median is computationally intractable, but minimax-rate optimal, whereas the componentwise and the geometric medians are computationally tractable but statistically sub-optimal. This observation led to the development of a number of computationally tractable estimators having an error with a better dependence on dimension than that of Tukey’s median (Dalalyan and Minasyan, 2020; Diakonikolas et al., 2016, 2017, 2018; Dong et al., 2019; Lai et al., 2016). In particular, most estimators introduced in these papers allow to conciliate computational tractability (i.e., are computable in time polynomial in $n, p, 1/\varepsilon$) and statistical optimality up to logarithmic factors.

The first two plots show that SDR is as accurate as IRM for small $\varepsilon$, with SDR outperforming IRM for $\varepsilon$ close to $1/2$. IRM and SDR are naturally much more accurate than GM and CM. The last plot shows that the running time of SDR is comparable to that of GM and is much smaller than that of IRM. More details on these experiments are provided in Section 5.

Fig 1: Plots that help to visually compare four robust estimators: SDR (our estimator), geometric median (GM) given by (1), componentwise median (CM), iteratively reweighted mean (IRM) of (Dalalyan and Minasyan, 2020). The first two plots show that SDR is as accurate as IRM for small $\varepsilon$, with SDR outperforming IRM for $\varepsilon$ close to $1/2$. IRM and SDR are naturally much more accurate than GM and CM. The last plot shows that the running time of SDR is comparable to that of GM and is much smaller than that of IRM. More details on these experiments are provided in Section 5.
component of the mean by a standard robust estimator such as the componentwise median or the trimmed mean, see Algorithm 1.

The main contributions of this paper are methodological and theoretical. The SDR estimator, thoroughly defined in Section 2, is a fast and accurate method for robustly estimating the mean of a set of points. It depends on one tuning parameter, the threshold used for identifying and removing clear outliers, and on the dimension reduction regime. Our theoretical considerations provide some recommendations for their choices and our numerical experiments reported in Section 5 confirm the relevance of these choices. Importantly, the SDR estimator does not require as input the rate of contamination \( \varepsilon \) but only an upper bound on \( \varepsilon \). As for theoretical contributions of this paper, we state in Section 3 an upper bound on the error of the SDR estimator, showing that it is nearly minimax-rate optimal and has a breakdown point equal to \( \frac{1}{2} \). This is done in the general case of a sub-Gaussian distribution with heterogeneous covariance matrix contaminated by adversarial noise. In Section 4, we further investigate the error of the SDR estimator in the case where only an approximation to the covariance matrix is available.

The papers that are the closest to the present one are (Lai et al., 2016), (Diakonikolas et al., 2017) and (Dalalyan and Minasyan, 2020). The spectral dimension reduction scheme was proposed by (Lai et al., 2016) along with an initial sample splitting step ensuring the independence of the estimators over different subspaces. In the case of spherical Gaussian distribution contaminated by non-adversarial outliers, the paper states that the proposed estimator has a squared error at most of order \( \frac{p \log^2 p \log(p/\varepsilon)}{n + \varepsilon^2} \log p. \) Compared to this, our results are valid in the more general setting of sub-Gaussian distribution, with arbitrary covariance matrix and adversarial contamination. In addition, our estimator does not rely on sample splitting and, therefore, has a risk with a better dependence on \( p \). As compared to the filtering method of (Diakonikolas et al., 2017), our estimator has the advantage of being independent of \( \varepsilon \) and our error bound is valid for every covariance matrix and every confidence level. On the down side, our error bound has an extra factor \( \log p \) in front of \( \varepsilon^2 \). We believe that this factor is an artifact of the proof, but we were unable to remove it. Finally, compared to the iteratively reweighted mean (Dalalyan and Minasyan, 2020), the SDR estimator studied in the present paper has a higher breakdown point, does not require the knowledge of \( \varepsilon \) and is much faster to compute. The advantages and shortcomings of these estimators are summarized in Table 1 and Figure 1.

### Table 1

Properties of various robust estimators. Agnostic mean, iteratively reweighted mean and iterative filtering are the estimators studied in (Lai et al., 2016), (Dalalyan and Minasyan, 2020) and (Diakonikolas et al., 2017), respectively. The error rates reported for Tukey’s median, componentwise median, geometric median and the agnostic mean have been proved for non-adversarial contamination. The squared error rate is provided in the case of a covariance matrix satisfying \( \| \Sigma \|_{op} = 1 \).

| Estimator                        | Comput. tractable | Breakdown point | knowledge of \( \varepsilon \) | Squared error rate | knowledge of \( \Sigma \) or \( \Sigma \propto I \) |
|---------------------------------|-------------------|-----------------|------------------------------|-------------------|---------------------------------|
| Comp./Geom. Median             | yes               | 0.5             | no                           | \( \frac{r\Sigma}{n} + \varepsilon^2 p \) | no                         |
| Tukey’s Median                 | no                | 0.33            | no                           | \( \frac{r\Sigma}{n} + \varepsilon^2 \) | no                         |
| Agnostic Mean                  | yes               | –               | yes                          | \( \frac{(p/n) \log p}{p} \) log \( p \) | yes                        |
| Iter. Reweighted Mean          | yes               | 0.28            | yes                          | \( \frac{(r\Sigma}{n}) + \varepsilon^2 \log (1/\varepsilon) \) | yes                        |
| Iterative Filtering            | yes               | –               | yes                          | \( \frac{(p/n) \log^2 p + \varepsilon^2 \log (1/\varepsilon)}{p} \) | yes                        |
| SDR (this paper)               | yes               | 0.5             | no                           | \( \frac{r\Sigma}{n} + \varepsilon^2 \log (1/\varepsilon) \) | log \( p \)                |
Notation. For any pair of integers \( k \) and \( d \) such that \( 1 \leq k \leq d \), we denote by \( \mathcal{Y}^d_k \) the set of all \( k \)-dimensional linear subspaces \( V \) of \( \mathbb{R}^d \). For \( V \in \mathcal{Y}^d_k \), we write \( k = \dim(V) \) and denote by \( P_V \) the orthogonal projection matrix onto \( V \). \( S^{d-1} \) stands for the unit sphere in \( \mathbb{R}^d \). For a \( d \times d \) symmetric matrix \( M \), we denote by \( \lambda_1(M), \ldots, \lambda_d(M) \) its eigenvalues sorted in increasing order, and use the notation \( \lambda_{\min}(M) \leq \lambda_1(M), \lambda_{\max}(M) \geq \lambda_d(M) \), \( \|M\|_{\text{op}} \leq \max(\|\lambda_{\min}(M)\|, \|\lambda_{\max}(M)\|) \) and \( \text{Tr}(M) = (\lambda_1 + \ldots + \lambda_d)(M) \). For any integer \( n > 0 \), we set \( [n] = \{1, \ldots, n\} \). We will denote by \( \mathcal{O} \subset [n] \) the subscripts of the outliers and by \( \mathcal{I} = [n] \setminus \mathcal{O} \) the subscripts of the inliers. We also use notation \( \log_+(x) = \max\{0, \log(x)\} \).

Algorithm 1 SDR(\( X_1, \ldots, X_n; \Sigma, t \))

1. let \( p \) the dimension of \( X_1 \)
2. let \( \hat{\mu}^{GM} \) be the geometric median of \( X_1, \ldots, X_n \)
3. let \( \mathcal{S} \leftarrow \{i : \|X_i - \hat{\mu}^{GM}\| \leq t/\sqrt{p}\} \)
4. let \( \bar{X}_\mathcal{S} \) be the sample mean of the filtered sample \( \{X_i : i \in \mathcal{S}\} \)
5. let \( \hat{\Sigma}_S \) be the covariance matrix of the filtered sample \( \{X_i : i \in \mathcal{S}\} \)
6. if \( p > 1 \) then
7. let \( V \) be the span of the top \( \lfloor p/e \rfloor \) principal components of \( \hat{\Sigma}_S - \Sigma \)
8. let \( P_V \) be the orth. projection onto \( V \)
9. let \( P_{V^\perp} \) be the orth. projection onto the orth. complement of \( V \)
10. let \( \hat{\mu} \leftarrow P_{V^\perp} \bar{X}_\mathcal{S} + \text{SDR}(P_V X_1, \ldots, P_V X_n; P_V \Sigma P_V, t) \)
11. else
12. let \( \hat{\mu} \leftarrow \hat{\mu}^{GM} \)
13. end if
14. return \( \hat{\mu} \)

2. Adversarially corrupted sub-Gaussian model and spectral dimension reduction.

We assume that a set \( X_1, \ldots, X_n \) of \( n \) data points drawn from a distribution \( P_n \) is given. This set is assumed to contain at least \( n - [n\varepsilon] \) inliers, the remaining points being outliers. All the points lie in the \( p \)-dimensional Euclidean space and the inliers are independently drawn from a reference distribution, assumed to be sub-Gaussian with mean \( \mu^* \in \mathbb{R}^p \) and covariance matrix \( \Sigma \). To state the assumptions imposed on the observations in a more precise way, let us recall that the random vector \( \zeta \) is said to be sub-Gaussian with zero mean and identity covariance matrix, if \( \mathbb{E}[\zeta] = 0 \), \( \mathbb{E}[\zeta \zeta^\top] = I_p \), and for some \( s > 0 \), we have
\[
\mathbb{E}[v^\top \zeta] \leq \exp\{s\|v\|^2/2\}, \quad \forall v \in \mathbb{R}^p.
\]
The parameter \( s \) is commonly called the variance proxy and the writing \( \zeta \sim \text{SG}_p(s) \) is used.

**Definition 1.** We say that the data generating distribution \( P_n \) is an adversarially corrupted sub-Gaussian distribution with mean \( \mu^* \), covariance matrix \( \Sigma \), variance proxy \( s \) and contamination rate \( \varepsilon \), if there is a probability space on which we can define a sequence of random vectors \( (X_1,Y_1), \ldots,(X_n,Y_n) \) such that
1. \( Y_1, \ldots, Y_n \) are independent and \( \Sigma^{-1/2}(Y_i - \mu^*) \sim \text{SG}_p(s) \) for every \( i \in [n] \).
2. the cardinality of \( \mathcal{O} = \{i \in [n] : Y_i \neq X_i\} \) is at most equal to \( n\varepsilon \).
3. the distribution of \( (X_1, \ldots, X_n) \) is \( P_n \).

We write then \( P_n \in \text{SGAC}(\mu^*, \Sigma, s, \varepsilon) \). In the particular case where all \( Y_i \) are Gaussian, we will write \( P_n \in \text{GAC}(\mu^*, \Sigma, \varepsilon) \).

\(^1\text{SGAC stands for sub-Gaussian with adversarial contamination.}\)
For an overview of various kinds of contamination models we refer the interested reader to (Bateni and Dalalyan, 2020). The adversarial contamination considered throughout this work is perhaps the most general one considered in the literature as the elements of the set \( \mathcal{O} \)—called outliers—may be chosen using \( \mu^*, \Sigma, \varepsilon \) but also \( Y_1, \ldots, Y_n \) by an omniscient adversary. Note that in this setting, even the set \( \mathcal{O} \) is random and depends on \( Y_1, \ldots, Y_n \). Therefore, the inliers \( \{ \mathbf{X}_i : i \in \mathcal{I} \} \) cannot be considered as independent random variables. The problem studied in this work consists in estimating the mean \( \mu^* \) of the reference distribution from the adversarially corrupted observations \( \mathbf{X}_1, \ldots, \mathbf{X}_n \).

The estimator we analyze in this work is termed iterative spectral dimension reduction (SDR). It is closely related to the agnostic mean (Lai et al., 2016) and to iterative filtering (Diakonikolas et al., 2017) estimators. We will prove that SDR enjoys most of desired properties in the setting of robust estimation of the sub-Gaussian mean.

The parameters given as input to the iterative spectral dimension reduction algorithm are a strictly decreasing sequence of positive integers \( p_0, \ldots, p_L \) such that \( p_0 = p \) and a positive threshold \( t > 0 \). We recall that the geometric median is defined by

\[
\hat{\mu}^{GM} \in \arg \min_{\mu \in \mathbb{R}^p} \sum_{i=1}^n \| \mathbf{X}_i - \mu \|_2. \tag{1}
\]

The algorithm for computing the SDR estimator reads as follows.

1. Start by setting \( \mathbf{V}_0 = \mathbf{I}_p \).
2. For \( \ell = 0, \ldots, L - 1 \) do
   a) Define \( \hat{\mu}^{(\ell)} \in \mathbb{R}^{p_\ell} \) as the geometric median of \( \{ \mathbf{V}_\ell^T \mathbf{X}_i : i \in [n] \} \).
   b) Define the set \( \mathcal{S}^{(\ell)} = \{ i \in [n] : \| \mathbf{V}_\ell^T \mathbf{X}_i - \hat{\mu}^{(\ell)} \|_2 \leq t \sqrt{n} \} \) of filtered data points.
   c) Let \( \bar{\mathbf{X}}^{(\ell)} \) and \( \hat{\Sigma}^{(\ell)} \) be the mean vector and the covariance matrix of the filtered sample \( \{ \mathbf{X}_i : i \in \mathcal{S}^{(\ell)} \} \), that is
      \[
      \bar{\mathbf{X}}^{(\ell)} = \frac{1}{|\mathcal{S}^{(\ell)}|} \sum_{i \in \mathcal{S}^{(\ell)}} \mathbf{X}_i, \quad \hat{\Sigma}^{(\ell)} = \frac{1}{|\mathcal{S}^{(\ell)}|} \sum_{i \in \mathcal{S}^{(\ell)}} (\mathbf{X}_i - \bar{\mathbf{X}}^{(\ell)}) \otimes (\mathbf{X}_i - \bar{\mathbf{X}}^{(\ell)}).
      \]
   d) Set \( \hat{\mu}^{(\ell)} = \mathbf{V}_\ell \mathbf{U}_\ell^T \mathbf{U}_\ell \mathbf{V}_\ell^T \bar{\mathbf{X}}^{(\ell)} \), where \( \mathbf{U}_\ell \) is a \( (p_\ell - p_{\ell+1}) \times p_\ell \) orthogonal matrix the rows of which are the eigenvectors of \( \mathbf{V}_\ell^T (\hat{\Sigma}^{(\ell)} - \Sigma) \mathbf{V}_\ell \) corresponding to its \( (p_\ell - p_{\ell+1}) \) smallest eigenvalues.
   e) Set \( \mathbf{V}_{\ell+1} = \mathbf{V}_\ell (\mathbf{U}_\ell^T)^\dagger \in \mathbb{R}^{p \times p_{\ell+1}} \), where \( \mathbf{U}_\ell^\dagger \) is a \( p_{\ell+1} \times p_\ell \) orthogonal matrix orthogonal to \( \mathbf{U}_\ell \), that is \( \mathbf{U}_\ell^\dagger \mathbf{U}_\ell^T = \mathbf{0} \).
3. Define \( \hat{\mu}^{(L)} \) as the geometric median of \( \mathbf{V}_L^T \mathbf{X}_i \) for \( i = 1, \ldots, n \) and set \( \mathcal{S}^{(L)} = \{ i \in [n] : \| \mathbf{V}_L^T \mathbf{X}_i - \hat{\mu}^{(L)} \|_2 \leq t \sqrt{p_L} \} \).
4. Define \( \hat{\mu}^{(L)} = \mathbf{V}_L \mathbf{V}_L^T \bar{\mathbf{X}}^{(L)} \), the average of filtered and projected vectors.
5. Return \( \hat{\mu}^{SDR} = \hat{\mu}^{(0)} + \hat{\mu}^{(1)} + \ldots + \hat{\mu}^{(L)} \).

The steps described above can be summarised as follows. At each iteration \( \ell < L \), we start by determining a filtered subsample \( \mathcal{S}^{(\ell)} \) and a “nearly-outlier-orthogonal” subspace \( \mathcal{U}_\ell = \text{Im} (\mathbf{V}_\ell \mathbf{U}_\ell^T) \) of \( \mathbb{R}^p \) of dimension \( p_\ell - p_{\ell+1} \). We define the projection of \( \hat{\mu}^{SDR} \) onto \( \mathcal{U}_\ell \) as the sample mean of the filtered and projected subsample, and we move to the next step for determining the projection of \( \hat{\mu}^{SDR} \) onto the remaining part of the space. At the last iteration \( L \), when the dimension is well reduced, the projection of \( \hat{\mu}^{SDR} \) onto the subspace \( \mathcal{U}_L \) is defined as the average of the filtered subsample projected onto \( \mathcal{U}_L \). The subspaces \( \mathcal{U}_\ell \) are two-by-two orthogonal and span the whole space \( \mathbb{R}^p \). Each subspace is determined
from the spectral decomposition of the covariance matrix of the data points projected onto $(\mathcal{Y}_0 \oplus \cdots \oplus \mathcal{Y}_{t-1})^\perp$, after removing the points lying at an abnormally large distance from the geometric median.

2.1. Choice of the dimension reduction regime. The analysis of the error of the SDR estimator conducted in this work leads to an upper bound in which the sequence $(p_0, \ldots, p_L)$ is involved only through the expression

$$ F(p_0, \ldots, p_L) = \sum_{\ell=1}^L \frac{p_{\ell-1}}{p_{\ell}}. $$

Therefore, an appealing way of choosing this sequence is to minimize the function $F$ under the constraint that the sequence is decreasing and $p_0 = p$ and $p_L = 1$. It follows from the inequality between the arithmetic and geometric means that $F(p_0, \ldots, p_L) \geq Lp^{1/L}$.

Furthermore, the equality is achieved in the case when all the terms in the definition of $F$ are equal, i.e., when for some $c > 0$ we have $p_{\ell-1} = cp_{\ell}$ for every $\ell \in [L]$. Since $p_0 = p$ and $p_L = 1$, this yields $c = p^{1/L}$ or, equivalently, $L = \log p/\log c$. Using these relations, we find that the function $F$ is lower bounded by $Le = (c/\log c) \log p$. The last step is to find the minimum of the function $c \mapsto c/\log c$ over the interval $(1, \infty)$. One easily checks that this function has a unique minimum at $c = e$. All these considerations advocate for using the dimension reduction regime defined by

$$ p_0 = p, \quad p_\ell = \lceil p_{\ell-1}/e \rceil + 1, \ 1 \leq \ell \leq L, \quad p_L = 1, $$

where $\lceil x \rceil$ is the largest integer strictly smaller than $x$. Such a definition of $(p_\ell)$ ensures that $p_{\ell-1}/p_\ell \leq e$ and that $L \leq 2\log p$. In the rest of the paper, we assume that the sequence $(p_\ell)$ is chosen as in (2).

2.2. Choice of the threshold. The SDR procedure has one important tuning parameter: the threshold $t$ used to discard clearly outlying data points. Let us introduce the auxiliary notation

$$ \bar{r}_n = \sqrt{\xi_\varepsilon + \frac{2 \log(2/\delta)}{\sqrt{n}}} , \quad \text{and} \quad \tau = \frac{1}{4} \wedge \frac{\bar{r}_n}{\log(2/\bar{r}_n)}. $$

Note that $\bar{r}_n$ is essentially the quantile, up to a universal constant factor, of order $1 - \delta$ of the distribution of $\|\bar{Y}_n - \mu^*\|_2$ where $\bar{Y}_i$'s are independently drawn from $\mathcal{N}_p(\mu^*, \Sigma)$ with $\|\Sigma\|_{op} = 1$. Our theoretical results advocate for using the value $t = t_1 + t_2$, where

$$ t_1 = \frac{2(1 + \bar{r}_n)}{1 - 2\varepsilon^*}, \quad t_2 = 1 + \frac{\bar{r}_n}{\sqrt{\tau}} + \sqrt{2 + \log(2/\tau)}, $$

where $\varepsilon^* < 1/2$ is the largest value of the contamination rate that the algorithm may handle.

Let $\xi_1, \ldots, \xi_n$ be independent Gaussian with zero mean and covariance $\Sigma$. The expression of $t_1$ is obtained as an upper bound on the quantile of order $1 - \delta/2$ of the distribution of the random variable

$$ T_1 = \sup_{V} \frac{2}{n(1 - 2\varepsilon^*) \dim(V) \sum_{i=1}^n \|P_{V} \xi_i\|_2}, $$

We relax here the assumption that all the entries $p_\ell$ are integers.

To check this inequality, one can use the fact that $3 \leq p_{L-2} \leq p_0^2 e^2 - L + e/(e - 1)$. This implies $L \leq 2\log p$ for $p \geq 6$. For smaller values of $p$, the inequality can be checked by direct computations.
see Lemma 2 and its proof for further details. Similarly, \( t_2 \) is defined so that the event

\[
\sup_V \sum_{i=1}^n \mathbb{I} \left( \| P_v \xi_i \|_2^2 > t_2^2 \dim(V) \right) \leq n \tau
\]

has a probability at least \( 1 - \delta/2 \). The related computations are deferred to Section 7.3. Although we tried to get sharp values for these thresholds \( t_1 \) and \( t_2 \), it is certainly possible to improve these values either by better mathematical arguments or by empirical considerations. Of course, smaller values of the thresholds \( t_1 \) and \( t_2 \) satisfying aforementioned conditions lead to an SDR estimator having smaller error.

### 3. Assessing the error of the SDR estimator.

The iterative spectral dimension reduction estimator defined in previous sections has some desirable properties of a robust estimator that are easy to check. In particular, it is clearly equivariant by translation, orthogonal linear transform and global scaling. Furthermore, the breakdown point of the estimator is equal to that of the geometric median, that is to \( 1/2 \). This means that even if almost the half of data points are chosen to be infinitely large, the estimator will not “break down” in the sense of becoming infinitely large. However, the fact that the estimated value does not become infinitely large, it might be not very close to the true mean. The next theorem shows that this is not the case and that the error of the SDR estimator has a nearly rate-optimal behavior even when the contamination rate is close to \( 1/2 \). The adverb “nearly” is used here to reflect the presence of the \( \sqrt{\log p} \) factor in the error bound, which is not present in the minimax rate.

**Theorem 1.** Let \( \varepsilon^* \in (0, 1/2) \), and \( \delta \in (0, 1/2) \). Define \( \bar{r}_n \) and \( \tau \) as in (3). For every \( \varepsilon \leq \varepsilon^* \), let \( \hat{\mu}^{\text{SDR}} \) be the estimator returned by Algorithm 2 with

\[
t = 3 - 2\varepsilon^* \left( 1 + \frac{\bar{r}_n}{\sqrt{\tau}} \right) + \sqrt{2 + 2 \log \left( \frac{1}{\tau} \right)}.
\]

There exists a universal constant \( C \) such that for every \( P_n \in \text{GAC}(\mu^*, \Sigma, \varepsilon) \) with \( \varepsilon \leq \varepsilon^* \) and \( \| \Sigma \|_{op} = 1 \), the probability of the event

\[
\| \hat{\mu}^{\text{SDR}} - \mu^* \|_2 \leq \frac{C \sqrt{\log p}}{1 - 2\varepsilon^*} \left( \sqrt{\frac{r_x}{n}} + \varepsilon \sqrt{\log(2/\varepsilon)} + \sqrt{\frac{\log(1/\delta)}{n}} \right)
\]

is at least \( 1 - \delta \). Moreover, the constant \( C \) from the last display can be made explicit by replacing the effective rank \( r_x \) by the dimension \( p \) in the definition of \( \bar{r}_n \): That is, for every \( \delta \in (0, 1/5) \) the inequality

\[
\| \hat{\mu}^{\text{SDR}} - \mu^* \|_2 \leq \frac{156 \sqrt{2 \log p}}{1 - 2\varepsilon^*} \left( \sqrt{\frac{2p}{n}} + \varepsilon \sqrt{\log(2/\varepsilon)} + \sqrt{\frac{3 \log(2/\delta)}{n}} \right)
\]

holds with probability at least \( 1 - 5\delta \).

If we compare this result with its counterpart established in (Dalalyan and Minasyan, 2020) for the iteratively reweighted mean, besides the extra \( \log p \) factor, we see that the above error bound does not reduce to the error of the empirical mean when the contamination rate goes to zero. We do not know whether this is just a drawback of our proof, or it is an intrinsic

---

4Since in this theorem \( \Sigma \) is assumed to be known, we can always divide all the data points \( X_i \) by \( \| \Sigma \|_{op}^{1/2} \) to get a data set with a covariance matrix satisfying \( \| \Sigma \|_{op} = 1 \).
property of the estimator. Our numerical experiments reported later on suggest that it might be a property of the estimator.

There is another logarithmic factor, $\sqrt{\log(2/\varepsilon)}$, present in the second term of the error bounds provided by the last theorem, which does not appear in the minimax rate. There are computationally intractable robust estimators of the Gaussian mean, such as the Tukey median, that have an error bound free of this factor. However, all the known error bounds provably valid for polynomial time algorithms has this extra $\sqrt{\log(2/\varepsilon)}$ factor. Furthermore, this factor is known to be unavoidable in the case of sub-Gaussian model with adversarial contamination\(^5\), see (Lugosi and Mendelson, 2021, Section 2).

As shows the next theorem, the claims of Theorem 1 carry over the sub-Gaussian reference distributions with some slight modifications. These modifications mainly stem from the following lemma assessing the tail behavior of the singular values of a matrix having independent and sub-Gaussian columns.

**Lemma 1** (Vershynin (2012), Theorem 5.39). Let $\xi_{1:n}$ be a matrix consisting of sub-Gaussian vectors with variance proxy $s$. There is a universal constant $C_0$ such that for every $t > 0$ and for every pair of positive integers $n$ and $p$, we have

$$
\Pr(s_{\min}(\xi_{1:n}) \leq \sqrt{n} - C_0 s(\sqrt{p} + t)) \leq e^{-t^2},
$$

$$
\Pr(s_{\max}(\xi_{1:n}) \geq \sqrt{n} + C_0 s(\sqrt{p} + t)) \leq e^{-t^2}.
$$

Note that in the Gaussian case $s = 1$ and the constant $C_0$ can be chosen equal to $\sqrt{2}$. The last lemma leads to the following adaptations in the values of the thresholds used in the SDR estimator. First, we introduce auxiliary definitions

$$
\tau = \frac{1}{4} \left( \frac{r_{n,s}}{\log_{\log_2(2/r_{n,s})}} \right),
$$

with

$$
r_{n,s} = \frac{3\sqrt{s}(\sqrt{p} + 2\sqrt{\log(2/\delta)})}{\sqrt{n}}.
$$

Then, we set $t = t_1 + t_2$ with

$$
t_1 = \frac{2(1 + C_0 r_{n,s} \sqrt{s})}{1 - 2\epsilon^*}, \quad t_2 = 1 + C_0 \sqrt{s} \left( \frac{r_{n,s}}{\sqrt{\tau}} + \sqrt{2 + 2 \log(1/\tau)} \right),
$$

where $C_0$ is the same as in Lemma 1.

Now we are ready to state the theorem for the for sub-Gaussian distributions showing that the SDR estimator with the threshold depending on the variance proxy $s$ yields the same upper bound on $\ell_2$ distance between our estimator $\hat{\mu}_{SDR}$ and the true value $\mu^*$ replacing the effective rank $r_\Sigma$ with the dimension $p$.

**Theorem 2** (Sub-Gaussian version). Let $\varepsilon^* \in (0, 1/2)$, and $\delta \in (0, 1/2)$. Define $r_{n,s}$ and $\tau$ as in (4). For every $\varepsilon \leq \varepsilon^*$, let $\hat{\mu}_{SDR}$ be the estimator returned by Algorithm 2 with

$$
t = \frac{3 - 2\varepsilon^*}{1 - 2\varepsilon^*} \left( 1 + C_0 r_{n,s} \sqrt{\frac{s}{\tau}} \right) + C_0 s \sqrt{2 + 2 \log(1/\tau)},
$$

where $C$ is a universal constant. Then, there exists a constant $C_0$ depending only on the variance proxy $s$ such that for every $P_n \in \text{SGAC}(\mu^*, \Sigma, s, \varepsilon)$ with $\varepsilon \leq \varepsilon^*$ and $\|\Sigma\|_{op} = 1$,\(^5\)

\(^5\)Both sub-Gaussianity of the reference distribution and the adversarial nature of the contamination are important for getting the extra $\sqrt{\log(2/\varepsilon)}$ factor in the minimax rate.
the probability of the event

$$
\| \hat{\mu}_{\text{SDR}} - \mu^* \|_2 \leq \frac{C_\delta \sqrt{\log p}}{1 - 2\varepsilon^*} \left( \sqrt{\frac{p}{n}} + \varepsilon \sqrt{\log(2/\varepsilon)} + \sqrt{\frac{\log(1/\delta)}{n}} \right)
$$

is at least $1 - \delta$.

4. The case of unknown covariance matrix. The SDR estimator, as defined in Algorithm 1, requires the knowledge of covariance matrix $\Sigma$. In this section we consider the case where the matrix $\Sigma$ is unknown, but an approximation of the latter is available. Namely, we assume that we have access to a matrix $\tilde{\Sigma}$ and to a real number $\gamma > 0$ such that $\| \Sigma - \tilde{\Sigma}\|_{\text{op}} \leq \gamma \| \Sigma\|_{\text{op}}$. In such a situation, we can replace in the SDR estimator the true covariance matrix by its approximation $\tilde{\Sigma}$. This will necessarily require to adjust the threshold $t$ accordingly. The goal of the present section is to propose a suitable choice of $t$ and to show the impact of the approximation error $\gamma$ on the estimation accuracy.

As mentioned, the parameter $t$ used in Algorithm 1 needs to be properly tuned in order to account for the approximation error in the covariance matrix. To this end, we introduce the following auxiliary notation similar to those presented in (3):

$$
\tau_n = \sqrt{C_\gamma \tau} + \sqrt{2 \log(2/\delta)} \frac{\sqrt{n}}{\tau} \quad \text{and} \quad \tau = \frac{1}{4} \left( \frac{\tau_n}{\log(2/\tau)} \right),
$$

where $C_\gamma = (1 + \gamma)/(1 - \gamma)$. Compared to (3), the main difference here is the presence of the factor $C_\gamma$ (which is equal to one if $\gamma = 0$) and the substitution of the effective rank of $\Sigma$ by that of its approximation $\tilde{\Sigma}$. In the rest of this section, we assume that $\Sigma$ is invertible.

**Theorem 3.** Let $\varepsilon^* \in (0, 1/2)$, $\delta \in (0, 1/2)$ and define $\tilde{\tau}_n$ and $\tau$ as in (5). Assume that $\tilde{\Sigma}$ satisfies $\| \tilde{\Sigma}^{-1/2} \Sigma \tilde{\Sigma}^{-1/2} - I_p \|_{\text{op}} \leq \gamma$ for some $\gamma \in (0, 1/2]$. Let $\hat{\mu}_{\text{SDR}}$ be the output of SDR$(X_1, \ldots, X_n; \tilde{\Sigma}, \tilde{\tau}_n)$, see Algorithm 1, with $\tilde{\tau}_n = \sqrt{\tilde{\Sigma}} \tau + \sqrt{2 \log(2/\delta)} \frac{\sqrt{n}}{\tau}$.

Then, there exists a universal constant $C$ such that for every data generating distribution $P_n \in \text{GAC}(\mu^*, \Sigma, \varepsilon)$ with $\varepsilon \leq \varepsilon^*$, the probability of the event

$$
\| \hat{\mu}_{\text{SDR}} - \mu^* \|_2 \leq C \frac{\| \Sigma \|_{\text{op}}^{1/2} \sqrt{\log p}}{1 - 2\varepsilon^*} \left( \frac{\tau_n}{n} + \varepsilon \sqrt{\log(2/\varepsilon)} + \sqrt{\frac{\log(1/\delta)}{n}} \right)
$$

is at least $1 - \delta$.

On the one hand, if the value of $\gamma$ is at most of order $\sqrt{\tau} \log(1/\varepsilon) + \varepsilon \log(1/\varepsilon)$ then Theorem 3 implies that the estimation error is of the same order as in the case of known covariance matrix $\Sigma$ (Theorem 1). For instance, if the matrix $\Sigma$ is assumed to be diagonal, one can defined $\tilde{\Sigma}$ as the diagonal matrix composed of robust estimators of the variances of univariate contaminated Gaussian samples; see, for instance, Section 2 in (Comminges et al., 2021). For recent advances on robust estimation of (non-diagonal) covariance matrices by computationally tractable algorithms we refer the reader to (Cheng et al., 2019b).

On the other hand, if the value of $\gamma$ for which the condition $\| \tilde{\Sigma} - \Sigma\|_{\text{op}} \leq \gamma \| \Sigma\|_{\text{op}}$ is known to be true is of larger order than $\sqrt{\tau} \log(1/\varepsilon) + \varepsilon \log(1/\varepsilon)$, then $\sqrt{\varepsilon^*}$ dominates the
other terms appearing in the error bound (6). Moreover, if $\gamma$ is of constant order, then we get the error rate $\sqrt{\frac{\gamma}{n}} + \sqrt{\varepsilon}$, which is in line with previously known bounds for computationally tractable estimators; see for example (Lai et al., 2016, Theorem 1.1), (Diakonikolas et al., 2017, Theorem 3.2), (Dalalyan and Minasyan, 2020, Theorem 4).

5. Numerical experiments. We conducted numerical experiments on synthetic contaminated data to corroborate our theoretical results. The main goal of these experiments is to display statistical and computational features of the SDR and their dependence on various parameters. Moreover, we compared SDR to some other estimators proposed in the literature as well as to the oracle (empirical mean of the inliers). To do so, we selected componentwise median (CM), geometric median (GM) and Tukey’s median (TM) as the three classic estimators of the context, and the iteratively reweighted mean (IRM), introduced in (Dalalyan and Minasyan, 2020), as an example of optimization based method.

5.1. Implementation details. The experiments were run on a laptop with a 1.8 GHz Intel Core i7 and 8 GB of RAM. R codes of the experiments are freely available on the last author’s website. For GM and TM the R packages Gmedian\(^6\) (Cardot et al. (2013)) and TukeyRegion\(^7\) (Liu et al. (2019)) were used. IRM had been already implemented in R using Mosek\(^8\).

To optimize SDR, several choices were made. First, since geometric median is used in SDR as a rough estimator of the location, we limited it to at most 15 iterations and to stop at an accuracy of order 1. See the reference manual of Gmedian to have more details on these parameters. Second, since at the last step of the SDR one can use any estimator which is robust in low-dimensional setting, we chose to use the median of the projected data points. Finally, we adjusted the numerical constant in the threshold $t$. In all the experiments, we assumed that the true value of $\varepsilon$ is known and used $\varepsilon^* = \varepsilon$.

5.2. Experimental setup. Experiments were conducted on synthetic data sets obtained by applying a contamination scheme to $n$ i.i.d. samples drawn from $\mathcal{N}_p(0, I_p)$. The following contamination schemes were considered.

- **Contamination by uniform outliers (CUO):** the locations of $n\varepsilon$ outliers are chosen at random independently of the inliers. The outliers are independent Gaussian with identity covariance matrix and with means having coordinates independently drawn from the uniform in $[0, 3]$ distribution.
- **Gaussian mixture contamination (GMC):** the locations of $n\varepsilon$ outliers are chosen at random independently of the inliers. The outliers are independent $\mathcal{N}_p(\mu, I_p)$. In our experiments, we chose $\mu$ such that $||\mu|| = 15$.
- **Contamination by “smallest” eigenvector (CSE):** We replace the $n\varepsilon$ samples most correlated with the smallest principal eigenvector $v_p$ of the sample covariance matrix, by $n\varepsilon$ vectors all equal to $\sqrt{p} v_p$ ($v_p$ is assumed to be a unit vector). In contrast with the two previous schemes, this one is adversarial.

Each experiment was repeated 50 times for SDR, CM, GM, the oracle and 10 times for IRM and TM. The tolerance probability $\delta$ was set to 0.1 in all the experiments. In the figures, points on the curves are median values of the error or of the running time for these trials whereas

\(^6\)https://cran.r-project.org/package=Gmedian
\(^7\)https://cran.r-project.org/web/packages/TukeyRegion/index.html
\(^8\)www.mosek.com
vertical bars overlaid on the points show the spread between the first and third quartiles. Since the computation of TM is prohibitively costly and is possible only for small sample sizes and dimensions, it is excluded from most of the experiments.

5.3. Statistical accuracy. At the upper left panel of Figure 2, we illustrate the behavior of the risk when the sample size increases for four different contamination levels: \( \varepsilon \in \{0.1, 0.2, 0.3, 0.4\} \). The data are of dimension 60 and generated by the GMC scheme. The median estimation error converges respectively to the values 0.18, 0.36, 0.62 and 1.06. According to our theoretical result, the limit of the error should be proportional to \( \frac{\varepsilon \log(1/\varepsilon)}{1-2\varepsilon} \). This is confirmed by the experimental results, since the ratio between the empirical limit of the median error and \( \frac{\varepsilon \log(1/\varepsilon)}{1-2\varepsilon} \) for each \( \varepsilon \) is between 0.58 and 0.69.

At the upper right panel of Figure 2, the dependence of the error on the dimension is displayed. To better illustrate the effect of the dimension on the estimation error, we carried out our experiment on data sets of small sample size \( n = 100 \) with CUO contamination. We compared the error of GM, CM, IRM and the oracle. In this plot, we clearly observe the supremacy of SDR and IRM as compared to GM and CM, which is in line with theoretical results. An important observation is that the error of the SDR estimator is very close to those of the IRM estimator and the oracle. This suggests that the factor \( \sqrt{\log p} \) present in our theoretical results might be an artifact of the proof rather than an intrinsic property of the estimator, at least for nonadversarial contamination.

The last experiment aiming to display the behavior of the estimation error is depicted in the lower left panel of Figure 2. The examined synthetic datasets were generated by the CSE scheme with \( \varepsilon = 0.2 \). We measured the error for different values of the dimension and for sample size \( n = 10p \) proportional to the dimension. In this case, the term \( \sqrt{p/n} \) in the risk bound remains unchanged and we may perceive if the dimension virtually effects the term dependent on \( \varepsilon \) in the bound. The plot clearly confirms that the error is stable for SDR as it is for IRM and the oracle, in sharp contrast with GM and CM. The last point, of course, is not surprising since the risks of GM and CM scale as \( \varepsilon \sqrt{p} \). Once again, this plot suggests that the factor \( \sqrt{\log p} \) present in the SDR’s risk bound might be unnecessary.

5.4. Computational efficiency. We conducted another experiment in order to better understand the computational complexity of SDR. Note that the computational cost of SDR comes from two operations done at each iterations: SVD of sample covariance matrix and computation of geometric median. We see that SDR can be computed in a reasonable time even in high dimensions. For instance, for \( n = 10000 \) and \( p = 1000 \) it takes nearly 26 seconds (tested over 20 trials).

At the lower right panel of Figure 2, we plotted the running times (in seconds) of GM, CM, IRM and SDR for various dimensions. Sample size in this experiment was set to 100, contamination rate was \( \varepsilon = 0.2 \) and CSE contamination scheme was used. As expected, IRM has substantially larger running time compared to SDR, GM and CM; this is due to the semidefinite programming solver running at each iteration of IRM. The fact that SDR is faster than GM (even though GM is deployed at each iteration of SDR) is explained by our choice of computing only a rough approximation of GM within SDR (limiting to 15 iterations and a tolerance parameter set to 1).

5.5. Breakdown point. A natural measure of robustness of an estimator is its resistance to a large fraction of outliers. The goal here is to demonstrate empirically our theoretical result showing that the breakdown point of the SDR estimator is 1/2.
Fig 2: The upper left panel illustrates the convergence of SDR’s median error when the sample size tends to infinity, for various contamination rates. The limiting values are shown by gray lines. The upper right panel shows the effect of dimension on the error. We see that in the case of SDR this effect is almost the same as for IRM and the oracle. The lower left panel plots the quantities when the sample-size increases proportionally to the dimension. Once again, we see that SDR is almost as accurate as IRM and the oracle. The lower right panel plots the running times of different estimators for various dimensions. It shows the huge computational gain of the SDR estimator as compared to IRM.

In Figure 3, at the left panel, we evaluated the error of the estimators on samples of size 100 and dimension 10 generated by the CSE scheme, for various values of $\varepsilon$. We can observe that SDR preserves its robustness with large contamination rates and outperforms other estimators, excepted the oracle. More precisely, SDR and IRM have roughly the same error up to $\varepsilon = 0.28$. Starting from this value, the error of IRM starts a steep deterioration joining CM and GM.

At the right panel of Figure 3, we plotted the error as a function of the contamination rate for TM, CM, GM, IRM and SDR. Data used in this experiment were of size 100 and dimension 3, corrupted by GMC scheme. For this type of contamination, we observe that the IRM estimator remains robust even for $\varepsilon$ close to 1/2, whereas the error of TM deteriorates signif-
Fig 3: The left panel compares the robustness of various estimators by displaying the estimation error for different contamination rates under SCE scheme. SDR outperforms other estimators (even the oracle). Results displayed in the right panel are obtained by a similar experiment conducted for the GMC scheme. SDR is remarkably stable for different contamination schemes, while we see that SDR and Tukey’s median may behave poorly for $\varepsilon > 1/3$.

As a conclusion, for two contamination schemes which are challenging for iteratively reweighted mean and Tukey’s median, SDR shows very stable behavior.

6. Summary, related work and conclusion. We have proved that the multivariate mean estimator obtained by the iterative spectral dimension reduction method enjoys several appealing properties in the setting of sub-Gaussian observations subject to adversarial contamination. More precisely, in addition to being rigid transform equivariant and having breakdown point equal to $1/2$, the estimator has been shown to achieve the nearly minimax rate. Furthermore, the SDR estimator has low computational complexity, confirmed by reported numerical experiments. Indeed, its computational complexity is of the same order as that of computing the sample covariance matrix and performing a SVD on it. Presumably, at the cost of a moderate drop in accuracy, further speed-ups can be obtained by randomization (Halko et al., 2011) in the spirit of the prior work (Cheng et al., 2019a; Depersin and Lecué, 2022).

Notably, we have proved that the SDR estimator achieves the nearly optimal error rate without requiring the precise knowledge of the contamination rate. It however requires the knowledge of the covariance matrix. To alleviate this constraint, we have also established estimation guarantees in the case where an approximation of the covariance matrix is used instead of the true one. We have conducted numerical tests that show that the SDR is both fast and accurate.

Many recent works studied the problem of robust estimation in more complex high dimensional settings such linear regression or sparse mean and covariance estimation; see (Balakrishnan et al., 2017; Cheng et al., 2021; Chinot, 2020; Chinot et al., 2020; Collier and Dalalyan, 2019; Dalalyan and Thompson, 2019; Goes et al., 2020; Liu et al., 2020; Pensia et al., 2020) and the references therein. It is under current investigation whether the results of the present paper can be extended to these settings.

Another interesting avenue for future research is to find an estimator that is rate-optimal, computationally tractable, with breakdown point equal to $1/2$ and, in the same time, asymptotically optimal in the sense that its risk is of order $\sqrt{p/n}$ when $\varepsilon$ tends to zero. On a related
note, it would be interesting to push further the exploration of second-order properties of the risk started in (Minasyan, 2020). Finally, an open question is how the minimax risk blows-up when the contamination rate tends to $\varepsilon$. For the SDR estimator studied in this work, we established an upper bound of order $1/(1-\varepsilon)$. However, we have no clue whether this is optimal, and our intuition is that it is not.

### 7. Proof of Theorem 1

Before proving Theorem 1, we provide some auxiliary lemmas and propositions.

#### 7.1. Bounding the projected error of the average of filtered observations

For any $J \subset [n]$, we define $\mathbf{Z}_J$ and $\hat{\Sigma}_J^Z$ as the sample average and sample covariance matrix of the subsample $\{\mathbf{Z}_i : i \in J\}$, that is

$$\mathbf{Z}_J = \frac{1}{|J|} \sum_{i \in J} \mathbf{Z}_i, \quad \hat{\Sigma}_J^Z = \frac{1}{|J|} \sum_{i \in J} \mathbf{Z}_i \mathbf{Z}_i^\top - \mathbf{Z}_J \mathbf{Z}_J^\top.$$

The main building block of the proof is the following result.

**Proposition 1.** Let $S \subset [n]$ be an arbitrary set. We define its subsets $S_I = S \cap I$ and $S_O = S \cap O$. Let $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$ and $\mu^Z$ be arbitrary points in $\mathbb{R}^q$ with $q \geq 2$. Let $\hat{\Sigma}^Z$ be an arbitrary $q \times q$ covariance matrix and let $P_k$ be the projection matrix projecting onto the subspace spanned by the bottom $k$ eigenvectors of $\hat{\Sigma}^Z - \Sigma^Z$, for $k = 1, \ldots, q - 1$. We have

$$\|P_k(\mathbf{Z}_S - \mu^Z)\|_2 \leq \left\{ 2\omega_O \|\hat{\Sigma}^Z - \Sigma^Z\|_{op} + \frac{\omega_O^2}{\omega_I} \left( \lambda_q - \lambda_1 \right)(\Sigma^Z) + \frac{\delta_Z^2}{q-k} \right\}^{1/2} + \|P_k \xi_S^Z\|_2,$$

where $\omega_O = |S_O|/|S|$, $\omega_I = 1 - \omega_O$, $\xi_i^Z = \mathbf{Z}_i - \mu^Z$ and $\delta_Z = \inf_\mu \max_{s \in S} \|Z_i - \mu^Z\|_2$.

Furthermore, if $|S| \leq q - k$, then

$$\|P_k(\mathbf{Z}_S - \mu^Z)\|_2 \leq \left\{ 2\omega_O \|\hat{\Sigma}^Z_S - \Sigma^Z\|_{op} + \frac{\omega_O^2}{\omega_I} \left( \lambda_q - \lambda_1 \right)(\Sigma^Z) \right\}^{1/2} + \|P_k \xi_S^Z\|_2.$$

**Proof.** Since there is no risk of confusion, we remove the superscript $Z$ from $\Sigma^Z, \hat{\Sigma}^Z$ and so on. Since $\mathbf{Z}_S = \omega_I \mathbf{Z}_S + \omega_O \mathbf{Z}_S$, yields $\mathbf{Z}_S - \mathbf{Z}_S = \omega_O \left( \mathbf{Z}_S - \mathbf{Z}_S \right)$, the triangle inequality implies that

$$\|P_k(\mathbf{Z}_S - \mu)\|_2 \leq \|P_k(\mathbf{Z}_S - \mathbf{Z}_S)\|_2 + \|P_k(\mathbf{Z}_S - \mu)\|_2 \leq \omega_O \|P_k(\mathbf{Z}_S - \mathbf{Z}_S)\|_2 + \|P_k \xi_S\|_2.$$  \hspace{1cm} (7)

Moreover, one can check that

$$\hat{\Sigma}^Z_S = \omega_I \hat{\Sigma}^Z_S + \omega_O \hat{\Sigma}^Z_S + \omega_I \omega_O (\mathbf{Z}_S - \mathbf{Z}_S) \otimes^2$$

$$\geq \omega_I \hat{\Sigma}^Z_S + \omega_I \omega_O (\mathbf{Z}_S - \mathbf{Z}_S) \otimes^2.$$  \hspace{1cm} (8)

Hence, multiplying from left and right by $P_k$ and computing the largest eigenvalue of both sides, we get

$$\omega_I \omega_O \|P_k(\mathbf{Z}_S - \mathbf{Z}_S)\|_2 \leq \lambda_q \left( P_k(\hat{\Sigma}^Z - \mathbf{Z}_S) P_k^\top \right) \leq \lambda_q \left( P_k(\Sigma - \mathbf{Z}_S) P_k^\top \right) + \lambda_q (\Sigma - \hat{\Sigma}^Z_S)$$

$$\leq \lambda_q (\hat{\Sigma}^Z_S - \Sigma) + \omega_I \lambda_q (\Sigma - \hat{\Sigma}^Z_S) + \omega_O \lambda_q (\Sigma).$$  \hspace{1cm} (9)
On the other hand, using the Weyl inequality (several times) and the identity (8), we get
\[
\lambda_k(\hat{\Sigma}_S - \Sigma) \leq \omega_I \lambda_q(\hat{\Sigma}_{S_k} - \Sigma) + \omega_O \lambda_k(\hat{\Sigma}_{S_o} - \Sigma + \omega_I (\hat{Z}_{S_k} - \hat{Z}_{S_o})^{\otimes 2}) \\
\leq \omega_I \lambda_q(\hat{\Sigma}_{S_k} - \Sigma) + \omega_O \lambda_{k+1}(\hat{\Sigma}_{S_o} - \Sigma) + \omega_I \lambda_{I-1}((\hat{Z}_{S_k} - \hat{Z}_{S_o})^{\otimes 2}) \\
\leq \omega_I \lambda_q(\hat{\Sigma}_{S_k} - \Sigma) + \omega_O \lambda_{k+1}(\hat{\Sigma}_{S_o} - \Sigma - \omega_O \lambda_1(\Sigma).
\]
(10)

For the middle term of the right hand side, we can use the following upper bound
\[
\lambda_{k+1}(\hat{\Sigma}_{S_o}) \leq \frac{\lambda_{k+1}(\hat{\Sigma}_{S_o}) + \ldots + \lambda_q(\hat{\Sigma}_{S_o})}{q - k} \\
\leq \frac{\text{Tr}(\hat{\Sigma}_{S_o})}{q - k} = \frac{1}{q - k} \text{Tr}\left(\frac{1}{|S_o|} \sum_{i \in S_o} (Z_i - \hat{Z}_{S_o})^{\otimes 2}\right) \\
= \frac{1}{q - k} \inf_\mu \text{Tr}\left(\frac{1}{|S_o|} \sum_{i \in S_o} (Z_i - \mu)^{\otimes 2}\right) \\
= \inf_\mu \frac{1}{(q - k)|S_o|} \sum_{i \in S_o} \|Z_i - \mu\|^2_2 \leq \frac{\delta_k^2}{q - k}.
\]

Combining (9), (10) and the last display, we get
\[
\omega_I \omega_O \|p_k(\hat{Z}_{S_k} - \hat{Z}_{S_o})\|^2_2 \leq 2\omega_I \|\hat{\Sigma}_{S_k} - \Sigma\|_{op} + \omega_O (\lambda_q(\Sigma) - \lambda_1(\Sigma)) + \frac{\omega_O \delta_k^2}{q - k}.
\]

Dividing both sides by \(\omega_I \omega_O\), we arrive at
\[
\|p_k(\hat{Z}_{S_k} - \hat{Z}_{S_o})\|^2_2 \leq \frac{2}{\omega_O} \|\hat{\Sigma}_{S_k} - \Sigma\|_{op} + \frac{1}{\omega_I} \left(\lambda_q(\Sigma) - \lambda_1(\Sigma) + \frac{\delta_k^2}{q - k}\right).
\]

Combining this inequality with (7), we obtain the first claim of the proposition. To get the second claim, we simply remark that \(\lambda_{k+1}(\hat{\Sigma}_{S_o}) = 0\) since the rank of the matrix \(\hat{\Sigma}_{S_o}\) is less than \(|S_o| \leq |S| \leq q - k\).
\]

7.2. Bounding the error of the geometric median of projected observations. We assume in this section that \(V\) is a linear subspace of \(\mathbb{R}^p\) of dimension \(k\) and consider the geometric median \(\hat{\mu}_V^{\text{GM}}\) of the projected vectors \(P_V X_1, \ldots, P_V X_n\) that is
\[
\hat{\mu}_V^{\text{GM}} \in \arg \min_{\mu \in \mathbb{R}^k} \sum_{i=1}^n \|P_V X_i - \mu\|_2.
\]

**Lemma 2.** With probability at least \(1 - \delta\), for all linear subspaces \(V \subset \mathbb{R}^p\), we have
\[
\frac{\|\hat{\mu}_V^{\text{GM}} - P_V \mu^*\|^2_2}{\sqrt{\dim(V)}} \leq \frac{2\sqrt{\|\Sigma\|_{op}}}{1 - 2\varepsilon} \left(1 + \sqrt{\frac{\text{Tr}(\Sigma)}{n}} + \frac{\sqrt{\log(1/\delta)}}{\sqrt{n}}\right).
\]

**Proof.** It follows from (Dalalyan and Minasyan, 2020, Lemma 2) that
\[
\|\hat{\mu}_V^{\text{GM}} - P_V \mu^*\|_2 \leq \frac{2}{n(1 - 2\varepsilon)} \sum_{i=1}^n \|P_V \xi_i\|_2.
\]
Upper bounding the last term using Cauchy-Schwartz’s inequality, one obtains
\[
\|\hat{\mu}^{GM} - P_V \mu^*\|_2 \leq \frac{2}{\sqrt{n(1-2\varepsilon)}} \left( \sum_{i=1}^{n} \|P_V \xi_i\|_2^2 \right)^{1/2}.
\]

Let \(e_1, \ldots, e_k\) be any orthonormal basis of \(V\), with \(k = \dim(V)\). We have
\[
\|\hat{\mu}^{GM} - P_V \mu^*\|_2 \leq \frac{2}{\sqrt{n(1-2\varepsilon)}} \left( k \sup_{\|e\|_2=1} \sum_{i=1}^{n} |e_i^\top \xi_i|_2^2 \right)^{1/2}
\]
\[
= \frac{2}{\sqrt{n(1-2\varepsilon)}} \left( k \sup_{\|e\|_2=1} \|\xi_1, \ldots, \xi_n\|_2^2 e_2 \right)^{1/2}
\]
\[
= \frac{2\sqrt{k}}{\sqrt{n(1-2\varepsilon)}} \|\xi_1, \ldots, \xi_n\|_{\text{op}}.
\]

By Corollary 5.35 in (Vershynin, 2012), the inequality
\[
\|\xi_1, \ldots, \xi_n\|_{\text{op}} \leq \|\Sigma\|_\text{op}^{1/2} \left( \sqrt{n} + \sqrt{\text{r}_\Sigma + \sqrt{2 \log(1/\delta)}} \right)
\]
holds with probability at least \(1 - \delta\). This yields the desired inequality.

7.3. **Bounding the number of filtered out observations.** Throughout this section, we assume without loss of generality that \(\|\Sigma\|_{\text{op}} = 1\).

**Lemma 3.** Let \(\tau\) and \(\delta\) be two numbers from \((0, 1)\). Define
\[
z = 1 + \frac{\sqrt{\text{r}_\Sigma + \sqrt{2 \log(1/\delta)}}}{\sqrt{n\tau}} + \sqrt{2 + 2 \log \left( \frac{1}{\tau} \right)}.
\]
With probability at least \(1 - \delta\), we have
\[
\sup_{V} \sum_{i=1}^{n} \mathbb{I} \left( \|P_V \xi_i\|_2 > z^2 \dim(V) \right) \leq n\tau,
\]
where the supremum is over all linear subspaces \(V\) of \(\mathbb{R}^P\).

**Proof.** Let us define the random variable
\[
T_n = \sup_{V} \sum_{i=1}^{n} \mathbb{I} \left( \|P_V \xi_i\|_2 > z^2 \dim(V) \right).
\]
In what follows, we write \(d_V\) for the dimension of the subspace \(V\). We check that
\[
\mathbb{P}(T_n > n\tau) \leq \mathbb{P} \left( \exists V \subset \mathbb{R}^P, \exists J \subset [n] \text{ with } |J| = n\tau, \text{ s.t. } \min_{i \in J} \|P_V \xi_i\|_2^2 \geq z^2 \dim(V) \right)
\]
\[
\leq \sum_{J \subset [n]} \mathbb{P} \left( \sup_{V} \min_{i \in J} \|P_V \xi_i\|_2^2 / d_V \geq z^2 \right)
\]
\[
= \left( \frac{n}{n\tau} \right) \mathbb{P} \left( \sup_{V} \min_{i \in \{1, \ldots, n\tau\}} \|P_V \xi_i\|_2^2 / d_V \geq z^2 \right)
\]
We also write

\[ \sum_{i=1}^{n \tau} \| \mathbf{v}_i \|^2 \geq z^2 \]

(11)

Given a linear subspace \( V \subset \mathbb{R}^p \) of dimension \( d_V \), let \( e_1^V, \ldots, e_{d_V}^V \) be an orthonormal basis of \( V \). Using (11), we get

\[
P(T_n \geq n \tau) \leq \left( \frac{n}{n \tau} \right) P \left( \sup_{V} \frac{1}{n \tau d_V} \sum_{i=1}^{d_V} \sum_{i=1}^{n \tau} |\xi_i e_i^V|^2 \geq z^2 \right)
\]

\[
\leq \left( \frac{n}{n \tau} \right) P \left( \sup_{\|e\|=1} \frac{1}{n \tau} \sum_{i=1}^{n \tau} |\xi_i e|^2 \geq z^2 \right)
\]

\[
= \left( \frac{n}{n \tau} \right) P \left( \|\xi_1, \ldots, \xi_{n \tau}\|_\text{op} \geq z \sqrt{n \tau} \right).
\]

By (Vershynin, 2012, Corollary 5.35), the inequality

\[ \|\xi_1, \ldots, \xi_{n \tau}\|_\text{op} \leq \left( \sqrt{n \tau} + \sqrt{\mathbf{F}_x} + \sqrt{2 \log(1/\delta_0)} \right) \]

holds with probability at least \( 1 - \delta_0 \). Taking \( \delta_0 = \delta / \left( \frac{n}{n \tau} \right) \) and using the inequality \( \log \left( \frac{n}{n \tau} \right) \leq n \tau (1 + \log(1/1)) \), we can conclude that \( P(T_n \geq n \tau) \leq \delta \). This proves the lemma. \( \square \)

**Lemma 4.** Let \( \tau \in (0, 1/2) \) and \( \delta \in (0, 1/2) \) be arbitrary. Set

\[ t = \frac{3 - 2 \varepsilon^*}{1 - 2 \varepsilon^*} \left( 1 + \frac{\sqrt{\mathbf{F}_x} + \sqrt{2 \log(2/\delta)}}{\sqrt{n \tau}} \right) + \sqrt{2 + 2 \log(1/\tau)}, \]

where \( \varepsilon^* < 1/2 \). Assume that \( \{ \mathbf{X}_i \} \) are drawn from \( \text{GAC}(\mu^*, \Sigma, \varepsilon) \) model with \( \varepsilon \leq \varepsilon^* \). Then, with probability at least \( 1 - \delta \), for any linear subspace \( V \) of \( \mathbb{R}^p \), the inequalities

\[ N_V = \sum_{i \in I} \mathbb{I} \left( \frac{\| P_V \mathbf{X}_i - \hat{\mu}_V^\text{GM} \|_2}{\sqrt{\dim(V)}} \leq t \right) \geq n(1 - \varepsilon - \tau), \]

\[ \frac{\| P_V \mu^* - \hat{\mu}_V^\text{GM} \|_2}{\sqrt{\dim(V)}} \leq \frac{2}{1 - 2 \varepsilon^*} \left( 1 + \frac{\sqrt{\mathbf{F}_x} + \sqrt{2 \log(2/\delta)}}{\sqrt{n \tau}} \right), \]

where \( \hat{\mu}_V^\text{GM} \) is the geometric median of \( P_V \mathbf{X}_1, \ldots, P_V \mathbf{X}_n \).

**Proof.** To avoid unnecessary technicalities, we assume in this proof that \( n \tau \) is an integer. We also write

\[ t_1 = \frac{2}{1 - 2 \varepsilon^*} \left( 1 + \frac{\sqrt{\mathbf{F}_x} + \sqrt{2 \log(2/\delta)}}{\sqrt{n}} \right) \]

\[ t_2 = 1 + \frac{\sqrt{\mathbf{F}_x} + \sqrt{2 \log(2/\delta)}}{\sqrt{n \tau}} + \sqrt{2 + 2 \log(1/\tau)}, \]

so that \( t \geq t_1 + t_2 \). Simple algebra yields

\[ N_V = \sum_{i \in I} \mathbb{I} \left( \| P_V \mathbf{X}_i + P_V \mu^* - \hat{\mu}_V^\text{GM} \|_2 \leq t \sqrt{\dim(V)} \right) \]

\[ \geq \sum_{i=1}^{n} \mathbb{I} \left( \| P_V \mathbf{X}_i + P_V \mu^* - \hat{\mu}_V^\text{GM} \|_2 \leq t \sqrt{\dim(V)} - n \varepsilon \right) \]
\[ n - n\varepsilon - \sum_{i=1}^{n} \mathbb{I}(\parallel P_V \xi_i + P_V \mu^* - \tilde{\mu}^G \parallel_2 > t \sqrt{\dim(V)}) \]

\[ \geq n - n\varepsilon - \sum_{i=1}^{n} \mathbb{I}(\parallel P_V \xi_i \parallel_2 + \parallel P_V \mu^* - \tilde{\mu}^G \parallel_2 > t \sqrt{\dim(V)}) , \]

where in the last step, we used the triangle inequality. According to Lemma 2, on an event \( \Omega_1 \) of probability at least \( 1 - \delta/2 \), we have \( \parallel P_V \mu^* - \tilde{\mu}^G \parallel_2 \leq t_1 \sqrt{\dim(V)} \) for every \( V \). This implies that on this event,

\[ N_V \geq n - n\varepsilon - \sum_{i=1}^{n} \mathbb{I}(\parallel P_V \xi_i \parallel_2 > t_2 \sqrt{\dim(V)}) , \quad \forall V \in \mathbb{R}^p . \]

Using Lemma 3, we get that on an event \( \Omega_2 \) of probability at least \( 1 - \delta/2 \), the sum on the right hand side of the last display is less than \( n \tau \). Therefore, on the intersection of the events \( \Omega_1 \) and \( \Omega_2 \), we have \( N_V \geq n(1 - \varepsilon - \tau) \) for every linear subspace \( V \) of \( \mathbb{R}^p \).

7.4. Estimating the mean from a low-dimensional adversarial projection. In this section, we consider the following problem. We assume that for a right hand side of the last display is less than \( n \tau \).

Using once again the triangle inequality, we arrive at

\[ \text{Using Lemma 3, we get that on an event } \Omega_2 \text{ of probability at least } 1 - \delta/2, \text{ the sum on the right hand side of the last display is less than } n \tau. \text{ Therefore, on the intersection of the events } \Omega_1 \text{ and } \Omega_2, \text{ we have } N_V \geq n(1 - \varepsilon - \tau) \text{ for every linear subspace } V \text{ of } \mathbb{R}^p. \]

\[ \text{Lemma 5. For every positive threshold } t > 0, \text{ we have} \]

\[ \| \tilde{\mu}_V - \mu^*_V \|_2 \leq \frac{\| P_V \xi \|_2}{N_V} + \frac{1}{N_V} \left\| \sum_{i \in S_V \cap \Omega} P_V \xi_i \right\|_2 + \frac{n \varepsilon (t \sqrt{q} + \| \tilde{\mu}^G - \mu^*_V \|_2)}{N_V}. \]

\[ \text{Proof. For this estimator, using the triangle inequality and the fact that } X_i = \mu^* + \xi_i \text{ for every } i \in I, \text{ we have} \]

\[ \| \tilde{\mu}_V - \mu^*_V \|_2 = \frac{1}{N_V} \left\| \sum_{i \in S_V} (P_V X_i - P_V \mu^*) \right\|_2 \]

\[ \leq \frac{1}{N_V} \left\| \sum_{i \in S_V \cap \Omega} P_V \xi_i \right\|_2 + \frac{1}{N_V} \left\| \sum_{i \in S_V \cap \Omega} P_V (X_i - \mu^*) \right\|_2 \]

\[ \leq \frac{1}{N_V} \left\| \sum_{i \in S_V \cap \Omega} P_V \xi_i \right\|_2 + \frac{n \varepsilon (t \sqrt{q} + \| \tilde{\mu}^G - \mu^*_V \|_2)}{N_V}. \]

Using once again the triangle inequality, we arrive at

\[ \| \tilde{\mu}_V - \mu^*_V \|_2 \leq \frac{\| P_V \xi \|_2}{N_V} + \frac{1}{N_V} \left\| \sum_{i \in S_V \cap \Omega} P_V \xi_i \right\|_2 + \frac{n \varepsilon (t \sqrt{q} + \| \tilde{\mu}^G - \mu^*_V \|_2)}{N_V}. \]
For every positive threshold \( t > 0 \), we have
\[
\|\hat{\mu}_V - \mu_t\|^2 \leq \max_{|S| \geq m} \frac{1}{|S|} \sum_{i \in S} P_i |\xi_i| + \frac{\sqrt{m(2r_x + 3t) + m^2 \log(2ne/m)}}{n - m}. \tag{12}
\]

**7.5. Bounding stochastic errors.** Throughout this section, without loss of generality, we assume that \( \|\Sigma\|_{op} = 1 \).

**Lemma 7.** For any positive integer \( m \leq n \) and any \( t > 0 \), we have
\[
P\left( \max_{|S| \geq n-m} \frac{1}{|S|} \sum_{i \in S} P_i |\xi_i| \leq \frac{n\|\hat{\xi}_n\|^2}{n - m} + \frac{\sqrt{m(2r_x + 3t) + m^2 \log(2ne/m)}}{n - m} \right) \geq 1 - e^{-t}.
\]

**Proof.** Using the triangle inequality, one has
\[
\frac{1}{|S|} \sum_{i \in S} P_i |\xi_i| \leq \frac{1}{n - m} \sum_{i \in S} P_i |\xi_i| + \sum_{i \in S^c} P_i |\xi_i|
\]
\[
\leq \frac{1}{n - m} \sum_{i \in S} P_i |\xi_i| + \frac{1}{n - m} \sum_{i \in S^c} P_i |\xi_i|
\]
\[
\leq \frac{n\|\hat{\xi}_n\|^2}{n - m} + \frac{1}{n - m} \max_{|S| \leq n} \sum_{i \in S} |\xi_i|^2.
\]

For \( s \in [1, m] \), we choose \( t_s \) by
\[ t_s = 3s \log \left( \frac{2ne}{s} \right) + 3t, \]
so that \( t_m = 3m \log \left( \frac{2ne}{m} \right) + 3t \) and
\[ \left( \frac{ne}{s} \right)^s e^{-t_s/3} \leq 2^{-s} e^{-t}. \]

For every \( J \) of cardinality \( m \), the random variable \( \| \sum_{j \in J} \xi_j \|^2 \) has the same distribution as \( m \sum_{j=1}^p \alpha_j (\Sigma) \alpha_j^2 \), where \( \alpha_1, \ldots, \alpha_p \) are i.i.d. standard Gaussian. Hence, using the union bound, the well-known upper bound on the binomial coefficients and (Comminges and Dalalyan, 2012, Lemma 8), we have
\[
P\left( \max_{|J| \leq m} \frac{1}{|J|} \sum_{i \in J} |\xi_i| \geq m (2r_x + t_m) \right) \leq \sum_{s=1}^{m} \binom{n}{s} P\left( \left\| \sum_{i=1}^{s} \xi_i \right\|_2^2 \geq m (2r_x + t_m) \right)
\]
\[
\leq \sum_{s=1}^{m} \left( \frac{ne}{s} \right)^s P\left( \left\| \sum_{i=1}^{s} \xi_i \right\|_2 \geq s (2r_x + t_s) \right)
\]
\[
\leq \sum_{s=1}^{m} \left( \frac{ne}{s} \right)^s e^{-t_s/3} \leq e^{-t}.
\]
This entails that with probability at least $1 - e^{-t}$, we have
\[
\max_{|J| \leq m} \left\| \sum_{i \in J} \xi_i \right\|_2 \leq \sqrt{m(2r_x + tm)} \leq \sqrt{m(2r_x + 3t)} + m \sqrt{3 \log(2ne/m)}.
\]
Combining this inequality with (12), we get the claim of the lemma. \(\square\)

In the two next lemmas, given a set $S \subset \{n\}$, we look at the sample average and sample covariance matrix of the subsample $\{X_i : i \in S\}$,
\[
\bar{X}_S = \frac{1}{|S|} \sum_{i \in S} X_i, \quad \hat{\Sigma}_S = \frac{1}{|S|} \sum_{i \in S} X_i X_i^\top - \bar{X}_S \bar{X}_S^\top.
\]

**Lemma 8.** There exists a positive constant $A$ such that, for any positive integer $m \leq n$ and any $t \geq 1$, with probability at least $1 - 2e^{-t}$, the inequality
\[
\left\| \hat{\Sigma}_S - \Sigma \right\|_{op} \leq A \sqrt{m} r_x + r_x + m \log(2ne/m) \frac{2t}{n - m} + \|\bar{\xi}_S\|_2^2
\]
is satisfied for every $S \subset \{n\}$ of cardinality $\geq n - m$.

**Proof.** The triangle inequality implies
\[
\left\| \hat{\Sigma}_S - \Sigma \right\|_{op} \leq \left\| \frac{1}{|S|} \sum_{i \in S} (X_i - \mu) \otimes^2 - \Sigma \right\|_{op} + \left\| (\mu - \bar{X}_S) \otimes^2 \right\|_{op}
= \left\| \frac{1}{|S|} \sum_{i \in S} \xi_i \xi_i^\top - \Sigma \right\|_{op} + \|\bar{\xi}_S\|_2^2. \tag{13}
\]
Using again the triangle inequality, one gets
\[
\left\| \sum_{i \in S} \xi_i \xi_i^\top - \Sigma \right\|_{op} \leq \left\| \sum_{i \in S} \xi_i \xi_i^\top - \Sigma \right\|_{op}
\leq \left\| \sum_{i=1}^n (\xi_i \xi_i^\top - \Sigma) \right\|_{op} + \left\| \sum_{i \in S^c} (\xi_i \xi_i^\top - \Sigma) \right\|_{op}
\leq \left\| \sum_{i=1}^n (\xi_i \xi_i^\top - \Sigma) \right\|_{op} + \max_{|J| \leq m} \left\| \sum_{i \in J} (\xi_i \xi_i^\top - \Sigma) \right\|_{op}. \tag{14}
\]
In view of (Koltchinskii and Lounici, 2017, Theorems 4 and 5), one can show that there exists a positive universal constant $A_1$ such that for every $t \geq 1$ and every set $J$ of cardinality $s$, the inequality
\[
\left\| \sum_{j \in J} \xi_j \xi_j^\top - s \Sigma \right\|_{op} \leq A_1 (\sqrt{m} r_x + r_x + t)
\]
is satisfied with probability at least $1 - e^{-t}$. We define $t_s$ by
\[
t_s = s \log \left( \frac{2ne}{s} \right) + t,
\]
so that $t_s \leq t_m = m \log \left( \frac{2ne}{m} \right) + t$ and
\[
\left( \frac{ne}{s} \right)^s e^{-t_s} \leq 2^{-s} e^{-t}.
\]
Applying the union bound and the well-known upper bound on the binomial coefficients, this yields
\[
P\left(\max_{|J| \leq m} \left\| \sum_{j \in J} (\xi_j \xi_j^\top - \Sigma) \right\|_\infty \geq A_1(\sqrt{m} \bar{r}_x + r_x + t_m) \right) 
\leq \sum_{s=1}^{m} \left( \frac{n}{s} \right) P\left( \left\| \sum_{j=1}^{s} \xi_j \xi_j^\top \right\|_\infty \geq A_1(\sqrt{m} \bar{r}_x + r_x + t_s) \right) 
\leq \sum_{s=1}^{m} \left( \frac{n}{s} \right) e^{-ts} \leq e^{-t}.
\]

One deduces from (14) that, with probability at least \(1 - 2e^{-t}\),
\[
\left\| \frac{1}{|S|} \sum_{i \in S} \xi_i \xi_i^\top - \Sigma \right\|_\infty \leq A_1\left(\sqrt{n} + \sqrt{m}\right) \sqrt{\bar{r}_x + 2r_x + m \log(2ne/m) + 2t} \leq A \sqrt{n} \bar{r}_x + r_x + m \log(2ne/m) + 2t.
\]

Combining this with (13), one gets the claim of the lemma. \(\square\)

**Lemma 9.** For any positive integer \(m \leq n\) and any \(t > 0\), with probability at least \(1 - 4e^{-t}\), the inequality
\[
\left\| \hat{\Sigma}_S - \Sigma \right\|_\infty \leq \frac{5p + (8 \log(2ne/m) + 2)m + 7t}{n - m} + \frac{2 \sqrt{p} + \sqrt{t}}{\sqrt{n} - \sqrt{m}} + \|\bar{\xi}_S\|^2_2
\]
is satisfied for every \(S \subset [n]\) of cardinality \(\geq n - m\).

**Proof.** In this proof, without loss of generality, we assume that the matrix \(\Sigma\) is invertible. In view of (13) and (14), we have
\[
\left\| \hat{\Sigma}_{S} - \Sigma \right\|_\infty \leq \left\| \frac{1}{|S|} \sum_{i=1}^{n} (\xi_i \xi_i^\top - \Sigma) \right\|_\infty + \max_{|J| \leq m} \left\| \frac{1}{|S|} \sum_{i \in J} (\xi_i \xi_i^\top - \Sigma) \right\|_\infty + \|\bar{\xi}_S\|^2_2. \tag{15}
\]
Let us define \(\zeta_i := \Sigma^{-1/2} \xi_i\) for all \(i \in [n]\). For every set \(J\) of cardinality \(s\), it holds that
\[
\left\| \sum_{i \in J} (\xi_i \xi_i^\top - \Sigma) \right\|_\infty \leq \left\| \Sigma \right\|_\infty \left\| \sum_{i \in J} (\zeta_i \zeta_i^\top - I_p) \right\|_\infty = \max \left( \lambda_{\max} \left( \sum_{i \in J} \zeta_i \zeta_i^\top \right) - s, s - \lambda_{\min} \left( \sum_{i \in J} \zeta_i \zeta_i^\top \right) \right) = \max \left( \sigma_{\max}^2 (\zeta_J) - s, s - \sigma_{\min}^2 (\zeta_J) \right),
\]
where \(\zeta_J\) is the \(s \times n\) random matrix obtained by concatenating the vectors \(\zeta_i\) with \(i \in J\). By (Vershynin, 2012, Corollary 5.35), we know that for every \(x > 0\)
\[
\sqrt{s} - \sqrt{p} - x \leq \sigma_{\min} (\zeta_J) \leq \sigma_{\max} (\zeta_J) \leq \sqrt{s} + \sqrt{p} + x
\]
is true with probability at least \(1 - 2e^{-x^2/2}\). This yields\(^9\)
\[
\left\| \sum_{i \in J} (\xi_i \xi_i^\top - \Sigma) \right\|_\infty \leq \max \left( (\sqrt{p} + x)(2\sqrt{s} + \sqrt{p} + x), (\sqrt{p} + x)(2\sqrt{s} - \sqrt{p} - x) \right)
\]
\(^9\)We provide the argument only in the case \(\sqrt{s} \geq \sqrt{p} + x\), but the conclusion is true for every value \(s\).
\[ \leq p + x^2 + 2\sqrt{ps} + 2x\sqrt{p} + 2x\sqrt{s} \]

with probability at least \( 1 - 2e^{-x^2/2} \). By applying the same technique as in the proof of Lemma 8, we can set

\[ t_s = 2\sqrt{s\log\left(\frac{2ne}{s}\right)} + t, \]

and obtain

\[ \mathbf{P}\left( \max_{|J| \leq m} \left\| \sum_{j \in J} (\xi_j \xi_j^\top - \Sigma) \right\|_{op} \geq p + t^2_m + 2\sqrt{pm} + 2tm\sqrt{p} + 2tm\sqrt{m} \right) \leq 2e^{-t}. \]

Hence, going back to (15), we can show that the inequalities

\[
\left\| \hat{\Sigma}_S - \Sigma \right\|_{op} \leq \frac{p + t + 2\sqrt{pm} + 2\sqrt{tm}}{n - m} + \frac{p + 4t + 4m\log(2ne/m) + 2\sqrt{pm}}{n - m} \\
+ \frac{4(\sqrt{p} + \sqrt{m})\sqrt{m\log(2ne/m)} + t}{n - m} + \left\| \xi_S \right\|_2^2
\]

\[
\leq \frac{5p + 8m\log(2ne/m) + 2m + 7t}{n - m} + \frac{2(\sqrt{p} + \sqrt{t})(\sqrt{m} + \sqrt{m})}{n - m} + \left\| \xi_S \right\|_2^2
\]

hold with probability at least \( 1 - 4e^{-t} \), and this proves the lemma. \( \square \)

7.6. Putting all the pieces together. All the ingredients provided, we can now compile the complete proof of Theorem 1.

Taking \( U_L := V_L \), the algorithm detailed in (2) returns \( \hat{\mu}^{SDR} = \sum_{\ell=0}^L \hat{\mu}^{(\ell)} \) with \( \hat{\mu}^{(\ell)} \in \mathcal{W}_L = \text{Im}(V_\ell U_\ell^\top) \) for every \( \ell \in \{0, \ldots, L\} \) where the two-by-two orthogonal subspaces \( \mathcal{W}_0, \ldots, \mathcal{W}_L \) span the whole space \( \mathbb{R}^p \). This allows us to decompose the risk:

\[
\left\| \hat{\mu}^{SDR} - \mu^* \right\|^2_2 = \sum_{\ell=0}^L \left\| \hat{\mu}^{(\ell)} - P_{\mathcal{W}_\ell} \mu^* \right\|^2_2 = \sum_{\ell=0}^L \left\| P_{\mathcal{W}_\ell} (X_\ell - \mu^*) \right\|^2_2
\]

where \( P_\ell := U_\ell^\top U_\ell \) is the projection matrix projecting onto the subspace of \( \mathbb{R}^{p_\ell} \) spanned by the bottom \( p_\ell - p_{\ell+1} \) eigenvectors of \( V_\ell^\top (\hat{\Sigma}_\ell^{(\ell)} - \Sigma) V_\ell \) for \( \ell = 0, \ldots, L \) with the convention that \( p_{L+1} = 0 \).

For \( \ell \in \{0, \ldots, L - 1\} \), we intend to apply Proposition 1 to \( Z_\ell = V_\ell^\top X_i \) and \( \mu^Z = V_\ell^\top \mu^* \) in order to upper bound the error term \( \text{Err}_\ell := \left\| P_{\ell} \left( V_\ell^\top X_i - V_\ell^\top \mu^* \right) \right\|_2 \). Using the inequalities

\[
\left\| V_\ell^\top (\hat{\Sigma}_\ell^{(\ell)} - \Sigma) V \right\|_{op} \leq \left\| \hat{\Sigma}_\ell^{(\ell)} - \Sigma \right\|_{op}, \quad \lambda_{p_\ell}(V_\ell^\top \Sigma V) \leq \lambda_{p}(\Sigma), \quad \lambda_1(V_\ell^\top \Sigma V) \geq \lambda_1(\Sigma)
\]

that are true for every orthogonal matrix \( V \), and keeping in mind the definition of \( P_{\ell} \), we get

\[
\text{Err}_\ell \leq \left\{ 2\omega_0 \left\| \hat{\Sigma}_\ell^{(\ell)} - \Sigma \right\|_{op} + \frac{\omega_0^2}{1 - \omega_0} \left( \lambda_p(\Sigma) + \frac{\delta_\ell}{p_{\ell+1}} \right) \right\}^{1/2} + \left\| P_{\ell} V_\ell^\top \xi_S^{(\ell)} \right\|_2,
\]

where we have used the notation

\[
\omega_0 = \max_{\ell} \frac{|S^{(\ell)} \cap \mathcal{O}|}{|S^{(\ell)}|}, \quad \xi_i = X_i - \mu^*
\]
Combining (16), (17), inequality $p_n \in \mathcal{O}$ and, therefore, we have
\[
\frac{|S^i(\ell)|}{|S^i(\ell) \cap \mathcal{O}|} = \frac{|S^i(\ell) \cap \mathcal{O}| + |S^i(\ell) \cap \mathcal{O}|}{|S^i(\ell) \cap \mathcal{O}|} = 1 + \frac{n(1 - \varepsilon - \tau)}{n\varepsilon} + 1 = 1 - \tau
\]
and, therefore, $\omega_{\mathcal{O}} \leq \varepsilon/(1 - \tau)$. We set $\eta := \varepsilon + \tau \leq 3/4$ and apply Lemma 4 to infer that $\omega_{\mathcal{O}} \leq \omega_{\mathcal{O}}/(1 - \omega_{\mathcal{O}}) \leq \varepsilon/(1 - \eta) \leq 4\varepsilon$ is true with probability at least $1 - \delta$. Furthermore, we know that $\delta_{\ell} \leq \max_{i \in S^i(\ell)} \|\widetilde{V}_i\mathbf{X}_1 - \tilde{\mu}_i(\ell)\|_2 \leq t\sqrt{p_{\ell}}$. This yields
\[
\text{Err}_{\ell} \leq \left\{8\varepsilon\|\tilde{\Sigma}^{(\ell)} - \Sigma\|_{op} + 16\varepsilon^2 \left( (\lambda_p - \lambda_1)(\Sigma) + \frac{2p_{\ell}}{p_{\ell + 1}} \right) \right\}^{1/2} + \|P_{\delta_{\ell}}\tilde{\xi}_{S^i(\ell)}\|_2.
\]
Let us introduce the shorthand
\[
T_1 = \max_{\ell \in [L]} \|\tilde{\Sigma}^{(\ell)} - \Sigma\|_{op} + \varepsilon(\lambda_p - \lambda_1)(\Sigma).
\]
This leads to
\[
\text{Err}_{\ell} \leq \left\{8\varepsilon T_1 + \frac{16\varepsilon^2 t^2 p_{\ell}}{p_{\ell + 1}} \right\}^{1/2} + \|P_{\delta_{\ell}}\tilde{\xi}_{S^i(\ell)}\|_2. \quad (16)
\]
For the last error term, since $p_L = 1$ we have by Lemma 5
\[
\text{Err}_{L} \leq \|P_{\delta_{L}}\tilde{\xi}_{S^i(L)}\|_2 + \frac{n\varepsilon(\sqrt{p_{L}} + \|P_{\delta_{L}}\mu^* - \tilde{\mu}^{GM}_{\delta_{L}}\|_2)}{|S^i(L)|}
\]
\[
\leq \|P_{\delta_{L}}\tilde{\xi}_{S^i(L)}\|_2 + \frac{\varepsilon t + \varepsilon\|P_{\delta_{L}}\mu^* - \tilde{\mu}^{GM}_{\delta_{L}}\|_2}{1 - \eta}
\]
\[
\leq \|P_{\delta_{L}}\tilde{\xi}_{S^i(L)}\|_2 + 4\varepsilon t + 4\varepsilon\|P_{\delta_{L}}\mu^* - \tilde{\mu}^{GM}_{\delta_{L}}\|_2. \quad (17)
\]
Combining (16), (17), inequality $p_{\ell} \leq \alpha p_{\ell + 1}$, as well as the Minkowski inequality, we get
\[
\|\mu^* - \tilde{\mu}_{\text{SDR}}\|_2 \leq \left\{ \sum_{\ell = 0}^L \text{Err}_{\ell}^2 \right\}^{1/2}
\]
\[
\leq \left\{8\varepsilon LT_1 + e\varepsilon t^2 + 16\varepsilon^2(t + \|P_{\delta_{L}}\mu^* - \tilde{\mu}^{GM}_{\delta_{L}}\|_2) \right\}^{1/2} + \left\{ \sum_{\ell = 0}^L \|P_{\delta_{\ell}}\tilde{\xi}_{S^i(\ell)}\|_2^2 \right\}^{1/2}
\]
\[
\leq 2\sqrt{2\varepsilon LT_1 + 9\varepsilon t\sqrt{L} + 4\varepsilon\|P_{\delta_{L}}\mu^* - \tilde{\mu}^{GM}_{\delta_{L}}\|_2 + \frac{\|P_{\delta_{L}}\tilde{\xi}_{S^i_0}\|_2^2}{1/2} \right\}.
\]
(18)
To ease notation, let us set
\[
r_n = \frac{\left(2r_x + 3\log(2/\delta)\right)^{1/2}}{n}.
\]
In view of Lemma 7, with probability at least $1 - \delta$, we have
\[
\left\{ \sum_{\ell = 0}^L \|P_{\delta_{\ell}}\tilde{\xi}_{S^i(\ell)}\|_2^2 \right\}^{1/2} \leq \left\{ \sum_{\ell = 0}^L \left( \frac{\|P_{\delta_{\ell}}\tilde{\xi}_n\|_2}{1 - \eta} + \frac{r_n\sqrt{\eta} + \eta\sqrt{3\log(2e/\eta)}}{1 - \eta} \right)^2 \right\}^{1/2}
\]
\[
\leq \left\{ \sum_{\ell = 0}^L \left(4\|P_{\delta_{\ell}}\tilde{\xi}_n\|_2 + 4r_n\sqrt{\eta} + 10\eta\sqrt{\log(2/\eta)} \right)^2 \right\}^{1/2}
\]
\[
\leq 4\|\tilde{\xi}_n\|_2 + 4r_n\sqrt{\eta} + 10\eta\sqrt{L\log(2/\eta)}.
\]
Since the random variable $\|\xi_n\|^2_2$ has the same distribution as $\frac{1}{n} \sum_{j=1}^{p} \lambda_j(\Sigma) \gamma_j^2$, where $\gamma_1, \ldots, \gamma_p$ are i.i.d. standard Gaussian, by (Comminges and Dalalyan, 2012, Lemma 8) we have

$$\|\xi_n\|^2_2 \leq \frac{2r_x + 3 \log(2/\delta)}{n} = r_n^2$$

with probability at least $1 - \delta$. Therefore, with probability at least $1 - 2\delta$,

$$\left( \sum_{l=0}^{L} \|P_{\mathcal{Y}_l} \xi_n^{(l)}\|^2_2 \right)^{1/2} \leq 4r_n \left(1 + \sqrt{L\eta} \right) + 10\eta \sqrt{L \log(2/\eta)}.$$  \hfill (19)

Next, Lemma 4 and the fact that $\dim(\mathcal{Y}_L) = 1$ imply that with probability at least $1 - \delta$

$$\|P_{\mathcal{Y}_L} \mu^* - \hat{\mu}^{\text{GM}}_{\mathcal{Y}_L}\|_2 \leq \frac{2(1 + \sqrt{2r_n})}{1 - 2\varepsilon^*}.$$  \hfill (20)

Recall that we have chosen $t$ in such a way that

$$t \leq 3(1 + \sqrt{2r_n / \sqrt{T}}) + 1.6 \sqrt{\log(2/\tau)}.$$  \hfill (21)

Combining (18), (19), (20) and (21), we arrive at the inequality

$$\|\mu^{\text{SDR}} - \mu^*\|_2 \leq 2\sqrt{2\varepsilon L T_1} + 9ct\sqrt{L} + \frac{8\varepsilon (1 + \sqrt{2r_n})}{1 - 2\varepsilon^*} + 4r_n \left(1 + \sqrt{L\eta} \right) + 10\eta \sqrt{L \log(2/\eta)}$$

\[
\leq 2\sqrt{2\varepsilon L T_1} + \frac{27\varepsilon \sqrt{L \log(2/\eta)}}{1 - 2\varepsilon^*} + 14.4\varepsilon \sqrt{L \log(2/\tau)}
+ \frac{8\varepsilon (1 + \sqrt{2r_n})}{1 - 2\varepsilon^*} + 4r_n \left(1 + \sqrt{L\eta} \right) + 10\eta \sqrt{L \log(2/\eta)}
\]

that holds with probability at least $1 - 3\delta$. In the upper bound obtained above, only the term $T_1$ remains random. We can upper bound this term using Lemma 8. It implies that with probability at least $1 - 2\delta$, we have

$$T_1 \leq A \sqrt{n r_x} + r_x + \eta \log(2e/\eta) + 2\log(1/\delta) / n (1 - \eta) + (4r_n (1 + \sqrt{\eta}) + 10\eta \sqrt{\log(2/\eta)})^2 + \varepsilon$$

$$\leq 2A \left( \sqrt{2r_n} + r_n^2 + 4\eta \log(2/\eta) \right) + (7.5r_n + 10\eta \sqrt{\log(2/\eta)})^2 + \varepsilon.$$

Consequently,

$$\sqrt{\varepsilon T_1} \leq \left\{ 2A \varepsilon \left( \sqrt{2r_n} + r_n^2 + 4\eta \log(2/\eta) \right) \right\}^{1/2} + \left( 7.5r_n + 10\eta \sqrt{\log(2/\eta)} \right) \sqrt{\varepsilon} + \varepsilon$$

$$\leq \left\{ 2A \varepsilon \left( \sqrt{2r_n} + r_n^2 + 4\eta \log(2/\eta) \right) \right\}^{1/2} + 5.4r_n + 7.1\eta \sqrt{\log(2/\eta)} + \varepsilon$$

$$\leq \left\{ 2A \varepsilon \left( \sqrt{2r_n} + 4\eta \log(2/\eta) \right) \right\}^{1/2} + (\sqrt{A} + 5.4)r_n + 7.1\eta \sqrt{\log(2/\eta)} + \varepsilon$$

$$\leq \left( A/\sqrt{2} + 2\sqrt{2A}\varepsilon \eta \log(2/\eta) + (\sqrt{A} + 5.4)r_n + 7.1\eta \sqrt{\log(2/\eta)} + \varepsilon$$

$$\leq \varepsilon + Ar_n / \sqrt{2} + 2\eta \sqrt{2A} \log(2/\eta) + (\sqrt{A} + 5.4)r_n + 7.1\eta \sqrt{\log(2/\eta)} + \varepsilon$$

$$\leq \varepsilon + \left( A/\sqrt{2} + (\sqrt{A} + 5.4) r_n + (7.1 + 2\sqrt{2A}) \varepsilon \sqrt{\log(2/\tau)} + (9.1 + 2\sqrt{2A}) \varepsilon \sqrt{\log(2/\varepsilon)} \right).$$

These inequalities imply that there exists a universal constant $C$ such that

$$\|\hat{\mu}^{\text{SDR}} - \mu^*\|_2 \leq \frac{C (r_n + \tau \sqrt{\log(2/\tau)} + \varepsilon \sqrt{\log(2/\varepsilon)} + r_n \varepsilon / \sqrt{\tau}) \sqrt{L}}{1 - 2\varepsilon^*}.$$  \hfill (22)
We choose

$$\tau = \frac{1}{4} \sqrt{\frac{\tilde{r}_n}{\log \frac{2}{\tau}}}, \quad \text{with} \quad \tilde{r}_n = \frac{\sqrt{\tau} + \sqrt{2\log(2/\delta)}}{\sqrt{n}}.$$  

Note that \( r_n \leq \sqrt{2r_n} \) and, furthermore, \( \tau = 1/4 \) whenever \( \tilde{r}_n \geq 1/2 \). Therefore, \( r_n \varepsilon / \sqrt{\tau} \leq r_n + \varepsilon \). Inserting this value of \( \tau \) in (22) leads to

$$\|\hat{\mu}_{\text{SDR}} - \mu^*\|_2 \leq \frac{C(r_n + \varepsilon \sqrt{\log(2/\varepsilon)}) \sqrt{L}}{1 - 2\varepsilon^*},$$

where \( C \) is a universal constant, the value of which is not necessarily the same in different places where it appears. Replacing \( r_n \) by its expression, and upper bounding \( L \) by \( 2 \log p \), we arrive at

$$\|\hat{\mu}_{\text{SDR}} - \mu^*\|_2 \leq \frac{C \sqrt{\log p}}{1 - 2\varepsilon^*} \left( \frac{\sqrt{\tau} + \varepsilon \sqrt{\log(2/\varepsilon)}}{n} \right).$$

Note that this inequality holds true on an event of probability at least \( 1 - 5\delta \).

To prove the second part of the theorem, we use Lemma 9 instead of Lemma 8 in order to bound the term \( T_1 \). Moreover, in the definitions of \( r_n \) and \( r^* \), the effective rank \( r^*_\Sigma \) is replaced by the dimension \( p \). Then, with probability at least \( 1 - 2\delta \), we have

$$T_1 \leq \frac{5p + n\eta(8 \log(2\varepsilon /\eta) + 2)}{n(1 - \eta)} + 7 \log(2/\delta) + 2 \sqrt{p + \log(2/\varepsilon)} +$$

$$+ \left( 4r_n (1 + \sqrt{\eta}) + 10\eta \sqrt{\log(2/\eta)} \right)^2 + \varepsilon$$

$$\leq (10r_n^2 + 15r_n + 72\eta \log(2/\eta)) + (7.5r_n + 10\eta \sqrt{\log(2/\eta)})^2 + \varepsilon.$$

Then, repeating the same steps as for the previous case where the effective rank is used instead of dimension we arrive at the following inequality

$$\sqrt{\varepsilon T_1} \leq \left\{ \varepsilon (10r_n^2 + 15r_n + 72\eta \log(2/\eta)) \right\}^{1/2} + (7.5r_n + 10\eta \sqrt{\log(2/\eta)}) \sqrt{\varepsilon} + \varepsilon$$

$$\leq 12r_n + 16\tau \sqrt{\log(2/\tau)} + 18\varepsilon \sqrt{\log(2/\varepsilon)}.$$

Combining the obtained inequalities, plugging in the values of \( \tau \) and \( r_n \) and bounding \( L \) by \( 2 \log p \) we arrive at a final bound which reads as

$$\|\hat{\mu}_{\text{SDR}} - \mu^*\|_2 \leq \frac{156 \sqrt{2 \log p}}{1 - 2\varepsilon^*} \left( \sqrt{\frac{2p}{n}} + \varepsilon \sqrt{\log(2/\varepsilon)} + \sqrt{\frac{3 \log(2/\delta)}{n}} \right),$$

which concludes the proof.

8. Proof of Theorem 3. The proof follows the same steps as that of Theorem 1. The assumption \( \|\Sigma^{-1/2} \Sigma^{-1/2} - I_p\|_{\text{op}} \leq \gamma \) gives upper and lower bounds on the effective rank of \( \tilde{\Sigma} \), which we formulate as a separate lemma. Therefore, the choice of the threshold parameter \( \tilde{\ell}_\gamma \) stated in Theorem 3 makes Lemmas 3 and 4 applicable to this case as well. To bound the operator norm of \( \|\Sigma^{(\ell)} - \tilde{\Sigma}\|_{\text{op}} \) we make use of the assumption \( \|\Sigma^{-1/2} \Sigma^{-1/2} - I_p\|_{\text{op}} \leq \gamma \) and Lemma 8 using triangle inequality. We provide the full proof for reader’s convenience.

Before proceeding with the proof we first formulate and prove an auxiliary lemma for bounding the effective rank of \( \tilde{\Sigma} \) using that of \( \Sigma \).
**Lemma 10.** Let $\Sigma$ and $\tilde{\Sigma}$ be symmetric positive definite matrices such that $\|\Sigma^{-1/2}\tilde{\Sigma}\Sigma^{-1/2} - I_p\|_{op} \leq \gamma$. Then,

$$r_x \cdot \frac{1 - \gamma}{1 + \gamma} \leq r_x \cdot \frac{1 + \gamma}{1 - \gamma},$$

where by $r_x$ we denote the effective rank of matrix $\Sigma$, i.e. $r_x = \text{Tr}(\Sigma)/\|\Sigma\|_{op}$.

**Proof of Lemma 10.** We start with upper- and lower-bounding the operator norm of $\tilde{\Sigma}$. Using triangle inequality we have

$$\|\tilde{\Sigma}\|_{op} - \|\Sigma\|_{op} \leq \|\Sigma\|_{op} \cdot \|\Sigma^{-1/2}\tilde{\Sigma}\Sigma^{-1/2} - I_p\|_{op} \leq \gamma \|\Sigma\|_{op}.$$  

This readily yields

$$(1 - \gamma)\|\Sigma\|_{op} \leq \|\tilde{\Sigma}\|_{op} \leq (1 + \gamma)\|\Sigma\|_{op}. \tag{23}$$

Moreover, for any pair of positive definite matrices $A, B \succeq 0$ the following holds $\text{Tr}(A)\lambda_1(B) \leq \text{Tr}(AB) \leq \text{Tr}(A) \cdot \|B\|_{op}$. Hence, combining the cyclic property of trace, the trace inequality and the fact that the spectrum of the matrix $\Sigma^{-1/2}\tilde{\Sigma}\Sigma^{-1/2}$ is between $1 - \gamma$ and $1 + \gamma$, we get both the upper and the lower bounds for $\text{Tr}(\tilde{\Sigma})$. The upper bound reads as

$$\text{Tr}(\tilde{\Sigma}) = \text{Tr}(\Sigma^{1/2}\Sigma^{-1/2}\tilde{\Sigma}\Sigma^{-1/2}\Sigma^{1/2}) = \text{Tr}(\Sigma^{1/2}\tilde{\Sigma}\Sigma^{-1/2}) \leq \|\Sigma^{-1/2}\tilde{\Sigma}\Sigma^{-1/2}\|_{op} \text{Tr}(\Sigma) \leq (1 + \gamma)\text{Tr}(\Sigma). \tag{24}$$

Similarly, the lower bound can be obtained as follows

$$\text{Tr}(\tilde{\Sigma}) \geq \lambda_1(\Sigma^{1/2}\tilde{\Sigma}\Sigma^{-1/2}) \text{Tr}(\Sigma) \geq (1 - \gamma)\text{Tr}(\Sigma). \tag{25}$$

Therefore, combining (23) and (25) we get the lower bound for $r_x$, while combining (23) and (24) yields the upper bound, concluding the proof of the lemma.

We denote the effective rank of matrix $\Sigma$ by $r_x = \text{Tr}(\Sigma)/\|\Sigma\|_{op}$. This allows us to decompose the risk:

$$\|\hat{\mu}^{\text{SDR}} - \mu^*\|^2 = \sum_{\ell=0}^L \|\hat{\mu}^{(\ell)} - P_{\mathcal{U}_\ell}\mu^*\|^2 = \sum_{\ell=0}^L \|P_{\mathcal{U}_\ell}(V_\ell^T \tilde{X}_\ell - V_\ell^T \mu^*)\|^2$$

where $P_\ell := U_\ell^T U_\ell$ is the projection matrix projecting onto the subspace of $\mathbb{R}^{p_\ell}$. This bound allows us to decompose the risk:

$$\|\hat{\mu}^{\text{SDR}} - \mu^*\|^2 = \sum_{\ell=0}^L \|P_\ell(V_\ell^T \tilde{X}_\ell - V_\ell^T \mu^*)\|^2$$

Taking $U_L := V_L$, the Algorithm 1 returns $\hat{\mu}^{\text{SDR}} = \sum_{\ell=0}^L \hat{\mu}^{(\ell)}$ with $\hat{\mu}^{(\ell)} \in \mathcal{U}_\ell = \text{Im}(V_\ell U_\ell^T)$ for every $\ell \in \{0, \ldots, L\}$ where the two-by-two orthogonal subspaces $\mathcal{U}_0, \ldots, \mathcal{U}_L$ span the whole space $\mathbb{R}^p$. This allows us to decompose the risk:

$$\|\hat{\mu}^{\text{SDR}} - \mu^*\|^2 = \sum_{\ell=0}^L \|\hat{\mu}^{(\ell)} - P_{\mathcal{U}_\ell}\mu^*\|^2 = \sum_{\ell=0}^L \|P_{\mathcal{U}_\ell}(V_\ell^T X_\ell - V_\ell^T \mu^*)\|^2$$

where $P_\ell := U_\ell^T U_\ell$ is the projection matrix projecting onto the subspace of $\mathbb{R}^{p_\ell}$ spanned by the bottom $p_\ell - p_{\ell+1}$ eigenvectors of $V_\ell^T (\Sigma^{(\ell)} - \tilde{\Sigma}) V_\ell$ for $\ell = 0, \ldots, L$ with the convention that $p_{L+1} = 0$.

For $\ell \in \{0, \ldots, L - 1\}$, we intend to apply Proposition 1 to $Z_\ell = V_\ell^T X_\ell$ and $\mu^Z = V_\ell^T \mu^*$ in order to upper bound the error term $\text{Err}_\ell := \|P_{\ell}(V_\ell^T X_\ell - V_\ell^T \mu^*)\|_{op}$. Using the inequalities

$$\|V^T (\Sigma^{(\ell)} - \tilde{\Sigma}) V\|_{op} \leq \|\Sigma^{(\ell)} - \tilde{\Sigma}\|_{op}, \quad \lambda_{p_\ell}(V^T \tilde{\Sigma} V) \leq \lambda_{p_\ell}(\tilde{\Sigma}), \quad \lambda_1(V^T \tilde{\Sigma} V) \geq \lambda_1(\tilde{\Sigma})$$

that are true for every orthogonal matrix $V$, and keeping in mind the definition of $P_\ell$, we get

$$\text{Err}_\ell \leq \left\{2\omega\|\Sigma^{(\ell)} - \tilde{\Sigma}\|_{op} + \frac{\omega^2}{1 - \omega} \left(\lambda_{p_\ell} - \lambda_1(\tilde{\Sigma}) + \frac{\delta^2}{p_{\ell+1}}\right)^{1/2} + \|P_{\ell}(V_\ell^T \xi_{\ell'}^{(\ell)}\|_{2}\right\}^{1/2},$$

where $\omega = \frac{\delta}{n_\ell}$ and $\xi_{\ell'}^{(\ell)} = \sum_{\ell'=0}^{\ell} (V_\ell V_\ell^T - I_p)$.
where we have used the notation

\[ \omega_{\mathcal{O}} = \max_{\ell} \frac{|\mathcal{S}^{(\ell)} \cap \mathcal{O}|}{|\mathcal{S}^{(\ell)}|}, \quad \xi_i = X_i - \mu^* \]

and \( \delta_{\ell} = \inf_{\mu} \max_{\xi \in \mathcal{S}^{(\ell)}} \| V_\ell^T (X_i - \mu) \|_2 \). Note that when \( \mathcal{O} \) and \( (\mathcal{S}^{(\ell)})^c \) are of cardinality less than \( n\varepsilon \) and \( n(\varepsilon + \tau) \), respectively, we have \( \omega_{\mathcal{O}} \leq \varepsilon /(1 - \tau) \) and

\[ \frac{\omega_{\mathcal{O}}}{1 - \omega_{\mathcal{O}}} \leq \frac{\varepsilon}{1 - \varepsilon - \tau}. \]

Since \( C_{\gamma} r_{\varepsilon} \geq r_{\varepsilon} \) (by Lemma 10) then Lemma 4 holds for the new threshold \( \tilde{t}_{\gamma} \) as well. We set \( \eta := \varepsilon + \tau \leq 3/4 \) and apply Lemma 4 to infer that \( \omega_{\mathcal{O}} \leq \varepsilon/(1 - \eta) \leq 4\varepsilon \) is true with probability at least \( 1 - \delta \). Furthermore, we know that \( \delta_{\ell} \leq \max_{\xi \in \mathcal{S}^{(\ell)}} \| V_\ell^T X_i - \mu^{(\ell)} \|_2 \leq \tilde{t}_{\gamma} \frac{1}{p_{\ell+1}}. \) This yields

\[ \text{Err}_\ell \leq \left\{ 8\varepsilon \| \tilde{S}^{(\ell)} - \tilde{\Sigma} \|_{op} + 16\varepsilon^2 \left( \lambda_p - \lambda_1 (\tilde{\Sigma}) + \frac{\tilde{t}_{\gamma}^2 p_{\ell}}{p_{\ell+1}} \right) \right\}^{1/2} + \| P_{\mathcal{F}, \ell} \tilde{\xi} \|_2. \]

Let us introduce the shorthand \( \tilde{T}_1 = \max_{\ell \in [L]} \| \tilde{S}^{(\ell)} - \tilde{\Sigma} \|_{op} + \varepsilon (\lambda_p - \lambda_1 (\tilde{\Sigma})) \). This leads to

\[ \text{Err}_\ell \leq \left\{ 8\varepsilon \tilde{T}_1 + 16\varepsilon^2 \frac{\tilde{t}_{\gamma}^2 p_{\ell}}{p_{\ell+1}} \right\}^{1/2} + \| P_{\mathcal{F}, \ell} \tilde{\xi} \|_2. \quad (26) \]

For the last error term, since \( p_{\ell} = 1 \) we have by Lemma 5

\[ \text{Err}_{\mathcal{L}} \leq \left\| P_{\mathcal{F}, \ell} \tilde{\xi} \right\|_2^2 + \frac{n\varepsilon (\tilde{t}_{\gamma} \sqrt{p_{\ell}} + \| P_{\mathcal{F}, \ell} \mu^* - \tilde{\mu}^{GM} \|_2)}{|\mathcal{S}(\mathcal{L})|} \]

\[ \leq \left\| P_{\mathcal{F}, \ell} \tilde{\xi} \right\|_2^2 + \frac{\varepsilon \tilde{T}_1 + \varepsilon \| P_{\mathcal{F}, \ell} \mu^* - \tilde{\mu}^{GM} \|_2}{1 - \eta} \]

\[ \leq \left\| P_{\mathcal{F}, \ell} \tilde{\xi} \right\|_2^2 + 4\varepsilon \tilde{T}_1 + 4\varepsilon \| P_{\mathcal{F}, \ell} \mu^* - \tilde{\mu}^{GM} \|_2. \quad (27) \]

Combining (26), (27), inequality \( p_{\ell} \leq e p_{\ell+1} \), as well as the Minkowski inequality, we get

\[ \| \mu^* - \tilde{\mu}^{SDR} \|_2 \leq \left\{ \sum_{\ell=0}^L \text{Err}_\ell \right\}^{1/2} \]

\[ \leq \left\{ 8\varepsilon L \tilde{T}_1 + 9\varepsilon \tilde{T}_1 \sqrt{L} + 4\varepsilon \| P_{\mathcal{F}, \ell} \mu^* - \tilde{\mu}^{GM} \|_2 \right\}^{1/2} + \left\{ \sum_{\ell=0}^L \left( \| P_{\mathcal{F}, \ell} \tilde{\xi} \|_2 \right)^2 \right\}^{1/2} \]

\[ \leq 2\sqrt{2\varepsilon L \tilde{T}_1} + 9\varepsilon \tilde{T}_1 \sqrt{L} + 4\varepsilon \| P_{\mathcal{F}, \ell} \mu^* - \tilde{\mu}^{GM} \|_2 + \left\{ \sum_{\ell=0}^L \left( \| P_{\mathcal{F}, \ell} \tilde{\xi} \|_2 \right)^2 \right\}^{1/2}. \quad (28) \]

To ease notation, let us set

\[ r_n = \left( \frac{2r_{\varepsilon} + 3 \log(2/\delta)}{n} \right)^{1/2}. \]

In view of Lemma 7, with probability at least \( 1 - \delta \), we have

\[ \left\{ \sum_{\ell=0}^L \left( \| P_{\mathcal{F}, \ell} \tilde{\xi} \|_2 \right)^2 \right\}^{1/2} \leq \left\{ \sum_{\ell=0}^L \left( \frac{\| P_{\mathcal{F}, \ell} \tilde{\xi} \|_2}{1 - \eta} + r_n \sqrt{\eta + \eta \sqrt{3 \log(2e/\eta)}} \right)^2 \right\}^{1/2}. \]
implies that with probability at least $T$, that holds with probability at least $\tilde{T}$.

Recall that we have chosen $\tilde{T}$ such a way that
$$\tilde{T} \leq 3\left(1 + C_\gamma \sqrt{2r_n}/\sqrt{\gamma}\right) + 1.6\sqrt{\log(2/\delta)}.$$  
(31)

Combining (28), (29), (30) and (31), we arrive at the inequality
$$\|\hat{\mu}_{\text{DR}} - \mu^*\|_2 \leq 2\sqrt{2\epsilon L_1} + 9\epsilon_1 \sqrt{L} + 8\epsilon(1 + C_\gamma \sqrt{2r_n}/1 - 2\epsilon) + 4r_n(1 + \sqrt{L_\eta}) + 10\eta \sqrt{L \log(2/\eta)}.$$  
(32)

that holds with probability at least $1 - 3\delta$. In the upper bound obtained above, only the term $T_1$ remains random. To bound $T_1$ we first apply a triangle inequality then use Lemma 8. It implies that with probability at least $1 - 2\delta$, we have
$$\tilde{T}_1 \leq A\sqrt{8}r_n + r_n + n\eta \log(2r_n) + 2\log(1/\delta) / (n(1 - \eta)) + 4r_n(1 + \sqrt{L_\eta}) + 10\eta \sqrt{L \log(2/\eta)}^2 + (1 + \gamma)\epsilon + \gamma$$
$$\leq 2\sqrt{2\epsilon L_1} + 2\epsilon_1 \sqrt{L} + 4\epsilon(1 + \sqrt{L_\eta}) + 10\eta \sqrt{L \log(2/\eta)} + 14.4\epsilon \sqrt{L \log(2/\eta)} + 8\epsilon \log(2r_n) + 4r_n(1 + \sqrt{L_\eta}) + 10\eta \sqrt{L \log(2/\eta)} + 10\eta \sqrt{L \log(2/\eta)}^2 + (1 + \gamma)\epsilon + \gamma$$
(32)

Consequently,
$$\sqrt{\epsilon_1} \leq \left\{2A\epsilon(\sqrt{2r_n} + r_n + 4\eta \log(2r_n))\right\}^{1/2} + (7.5r_n + 10\eta \sqrt{\log(2/\eta)})^2 + (1 + \gamma)\epsilon + \epsilon.$$  
(33)

Consequently,
$$\sqrt{\epsilon_1} \leq \left\{2A\epsilon(\sqrt{2r_n} + r_n + 4\eta \log(2/\eta))\right\}^{1/2} + (7.5r_n + 10\eta \sqrt{\log(2/\eta)})^2 + (1 + \gamma)\epsilon + \epsilon + \sqrt{2\epsilon}.$$  
(33)

Consequently,
$$\sqrt{\epsilon_1} \leq \left\{2A\epsilon(\sqrt{2r_n} + r_n + 4\eta \log(2/\eta))\right\}^{1/2} + (7.5r_n + 10\eta \sqrt{\log(2/\eta)})^2 + (1 + \gamma)\epsilon + \epsilon + \sqrt{2\epsilon}.$$  
(33)
These inequalities imply that there exists a universal constant C such that
\[
\|\hat{\mu}^\text{SDR} - \mu^*\|_2 \leq \frac{C(C_{\gamma}r_n + \tau \sqrt{\log(2/\tau)} + \varepsilon \sqrt{\log(2/\varepsilon)} + r_n \varepsilon / \sqrt{\tau} + \sqrt{\varepsilon \gamma}) \sqrt{L}}{1 - 2\varepsilon^*}.
\]
(32)
Let us denote \(\log^+(x) = \max\{0, \log(x)\}\) the positive part of logarithm, then we choose
\[
\tau = \frac{1}{4} \sqrt{\frac{\tilde{r}_n}{\log^+(2/\tilde{r}_n)}}, \quad \text{with} \quad \tilde{r}_n = \sqrt{C_{\gamma}r_E + 2 \log(2/\delta)} \sqrt{n}.
\]
Note that \(r_n \leq \sqrt{2}r_n\) and, furthermore, \(\tau = 1/4\) whenever \(\tilde{r}_n \geq 1/2\). Therefore, \(r_n \varepsilon / \sqrt{\tau} \leq r_n + \varepsilon\). Inserting this value of \(\tau\) in (32) leads to
\[
\|\hat{\mu}^\text{SDR} - \mu^*\|_2 \leq \frac{C(C_{\gamma}r_n + \varepsilon \sqrt{\log(2/\varepsilon)} + \sqrt{\varepsilon \gamma}) \sqrt{L}}{1 - 2\varepsilon^*}.
\]
where \(C\) is a universal constant, the value of which is not necessarily the same in different places where it appears. Replacing \(r_n\) by its expression, upper bounding \(L\) by \(2 \log p\), and using the fact that \(C_{\gamma} \leq 3\) for \(\gamma \in (0, 1/2]\) we arrive at
\[
\|\hat{\mu}^\text{SDR} - \mu^*\|_2 \leq \frac{C \sqrt{\log p}}{1 - 2\varepsilon^*}\left(\sqrt{\frac{r_E}{n}} + \sqrt{\frac{\log(1/\delta)}{n}} + \varepsilon \sqrt{\log(2/\varepsilon)} + \sqrt{\varepsilon \gamma}\right).
\]
Note that this inequality holds true on an event of probability at least \(1 - 5\delta\).

9. Extension to Sub-Gaussian distributions. This section is devoted to the proof of Theorem 2, which is an extension of Theorem 1 to the case when the \(1 - \varepsilon\) portion of observations are sub-Gaussian. First, we formulate some technical lemmas necessary for the proof of Theorem 2 postponing the full proof to the end of the present section.

Recall that a random vector \(\zeta\) with zero mean and identity covariance matrix is sub-Gaussian with variance proxy \(s > 0\), \(\zeta \sim \text{SG}_p(s)\), if
\[
\mathbb{E}[e^{v^\top \zeta}] \leq \exp\left\{\frac{g}{2} \|v\|^2\right\}, \quad \forall v \in \mathbb{R}^p.
\]
The concentration inequality for sub-Gaussian vectors is a well-known fact (see, e.g. (Rigollet and Hütter, 2019), Theorem 1.19) that if \(\zeta \sim \text{SG}_p(s)\) then for all \(\delta \in (0, 1)\), it holds
\[
\mathbb{P}(\|\zeta\|_2 \leq 4 \sqrt{ps} + \sqrt{8s \log(1/\delta)}) \geq 1 - \delta.
\]
(33)
In the definition of SGAC(\(\mu^*, \Sigma, s, \varepsilon\)) we assume that the \(1 - \varepsilon\) portion of the data \(\{X_i\}_{i=1}^n\) are sub-Gaussian, that is \(X_i = \mu^* + \Sigma^{1/2} \xi_i\) for all \(i \in \mathcal{I}\), where the set \(\mathcal{I}\) is of cardinality at least \((1 - \varepsilon)n\). Denote \(\xi_i = \Sigma^{1/2} \xi_i\) for all \(i = 1, \ldots, n\) and assume that \(\|\Sigma\|_{op} = 1\).

First, we show that with the choice of threshold parameter the analogous to Lemma 2, Lemma 3, Lemma 4 lemmas hold true. Notice that all three lemmas are using the concentration bound for the operator norm of (sub)-Gaussian vectors. In the case of Gaussian vectors we make use of (Vershynin, 2012, Corollary 5.35), while the analogous result for the sub-Gaussian distributions is also known (Vershynin, 2012, Theorem 5.39). For the readers convenience the latter is formulated in Lemma 1.

Lemma 11. Let \(J \subset \{1, \ldots, n\}\) be a subset of cardinality \(m\). For every \(\delta \in (0, 1)\), it holds that
\[
\mathbb{P}\left(\left\|\sum_{j \in J} \xi_j\right\|_2 \leq 4 \sqrt{pm} + \sqrt{8m \log(1/\delta)}\right) \geq 1 - \delta.
\]
Without loss of generality, we assume that \( J = \{1, \ldots, m\} \). On the one hand, \( \| \Sigma \|_{\text{op}} = 1 \) implies that
\[
\left\| \sum_{i=1}^{m} \xi_i \right\|_2 \leq \left\| \sum_{i=1}^{m} \zeta_i \right\|_2.
\]
On the other hand, \( \zeta_1 + \ldots + \zeta_m \sim \text{SG}_p(m\sigma) \). Applying inequality (33) to this random variable yields the desired result.

We now state the versions of Lemma 2 and Lemma 4 that are valid in the setting of sub-Gaussian vectors. Notice that the only difference is in the choice of the threshold; thanks to Lemma 1, the threshold now includes the universal constant \( C_0 \) and the variance proxy \( s \). The proofs of these lemmas are omitted, since they are the same as in the Gaussian case presented in Section 7.3 (except for bounding the operator norm of a matrix with sub-Gaussian columns we use Lemma 1).

**Lemma 12.** With probability at least \( 1 - \delta \), for all linear subspaces \( V \subset \mathbb{R}^p \), we have
\[
\frac{\| \hat{\mu}^{GM} - P_V \mu^* \|_2}{\sqrt{\dim(V)}} \leq 2 \sqrt{\| \Sigma \|_{\text{op}}(1 + \frac{C_0 s \sqrt{p} + 2s \sqrt{\log(1/\delta)}}{\sqrt{n}})},
\]
where the constant \( C_0 \) is the same constant as in Lemma 1.

**Lemma 13.** Let \( \tau \) and \( \delta \) be two numbers from \((0, 1)\). Define
\[
z = 1 + \frac{C_0 s \sqrt{p} + 2 \log(1/\delta)}{\sqrt{n} \tau} + C_0 s \sqrt{2 + 2 \log(1/\tau)}
\]
with the same constant \( C_0 \) as in Lemma 1. Then, with probability at least \( 1 - \delta \), we have
\[
\sup_V \sum_{i=1}^{n} \mathbb{I} \left( \| P_V \xi_i \|_2^2 > z^2 \dim(V) \right) \leq n \tau,
\]
where the supremum is over all linear subspaces \( V \subset \mathbb{R}^p \).

**Lemma 14 (Koltchinskii and Lounici (2017), Theorem 9).** There is a constant \( A_3 > 0 \) depending only on the variance proxy \( \tau \) such that for every pair of integers \( n \geq 1 \) and \( p \geq 1 \), we have
\[
P\left( \| \zeta_{1:n}^{\top} - n \Sigma \|_{\text{op}} \geq A_3 \left( \sqrt{(p + t)n + p + t} \right) \right) \leq e^{-t}, \quad \forall t \geq 1.
\]

**Lemma 15.** There exists a positive constant \( A \) such that, for any positive integer \( m \leq n \) and any \( t \geq 1 \), with probability at least \( 1 - 2e^{-t} \), the inequality
\[
\| \tilde{\Sigma}_S - \Sigma \|_{\text{op}} \leq A \sqrt{np + p + m \log(2ne/m)} + 2t
\]
is satisfied for every \( S \subset [n] \) of cardinality \( \geq n - m \).

**Proof.** The proof of this lemma is similar to the proof of Lemma 8 with the only difference that instead of Theorems 4 and 5 from (Koltchinskii and Lounici, 2017) we now use Lemma 14.
LEMMA 16. For any positive integer \( m \leq n \) and any \( t > 0 \), with probability at least \( 1 - e^{-t} \), we have
\[
\max_{|S| \geq n - m} \left\| \frac{1}{|S|} \sum_{i \in S} P \xi_i \right\|_2 \leq \frac{n \| P \xi \|_2}{n - m} + \sqrt{m \bar{s}(4 \sqrt{p} + 2 \sqrt{2t}) + 2m \sqrt{2\bar{s} \log(2ne/m)}}.
\]

PROOF. The proof of this theorem is similar to that of Lemma 7, with the only exception that now we need to bound the maximum of a norm of a sum of at most \( m \) sub-Gaussian vectors, where the maximum is taken over all subsets of \([n]\) of size at most \( m \). Since, each sub-Gaussian vector has a variance proxy \( s \) then using Lemma 11 along with union bound, we have
\[
P\left( \max_{|J| \leq m} \left\| \sum_{i \in J} \xi_i \right\|_2 \geq \sqrt{m \bar{s}(4 \sqrt{p} + t_m)} \right) \leq \sum_{l=1}^m \binom{n}{l} P\left( \left\| \sum_{i=1}^l \xi_i \right\|_2 \geq \sqrt{l \bar{s}(4 \sqrt{p} + t_l)} \right) \leq \sum_{l=1}^m \left( \frac{ne}{l} \right)^l \left( \sqrt{l \bar{s}(4 \sqrt{p} + t_l)} \right) \leq \sum_{l=1}^m \left( \frac{ne}{l} \right)^l e^{-t_l/3} \leq e^{-t}.
\]
Therefore, we obtain that with probability at least \( 1 - e^{-t} \) the inequality
\[
\max_{|J| \leq m} \left\| \sum_{i \in J} \xi_i \right\|_2 \leq \sqrt{m \bar{s}(4 \sqrt{p} + 2 \sqrt{2t}) + 2m \sqrt{2\bar{s} \log(2ne/m)}}
\]
holds. Then, combining with
\[
\frac{1}{|S|} \left\| \sum_{i \in S} P \xi_i \right\|_2 \leq \frac{n \| P \xi \|_2}{n - m} + \frac{1}{n - m} \max_{|J| \leq m} \left\| \sum_{i \in J} \xi_i \right\|_2.
\]
yields the desired result.

9.1. Proof of Theorem 2. All the ingredients provided, we can now compile the complete proof of Theorem 2.

Taking \( U_L := V_L \), the algorithm detailed in (2) returns \( \hat{\mu}_{SDR} = \sum_{\ell=0}^L \hat{\mu}^{(\ell)} \) with \( \hat{\mu}^{(\ell)} \in \mathcal{H}_\ell = \text{Im}(V_\ell U_\ell^*) \) for every \( \ell \in \{0, \ldots, L\} \) where the two-by-two orthogonal subspaces \( \mathcal{H}_0, \ldots, \mathcal{H}_L \) span the whole space \( \mathbb{R}^p \). This allows us to decompose the risk:
\[
\| \hat{\mu}_{SDR} - \mu^* \|_2^2 = \sum_{\ell=0}^L \| P_{\ell^*} (X_\ell - \mu^*) \|_2^2 = \sum_{\ell=0}^L \| P_{\ell^*} (V_\ell^T X_\ell - V_\ell^T \mu^*) \|_2^2,
\]
where \( P_{\ell} := U_\ell^T U_\ell \) is the projection matrix projecting onto the subspace of \( \mathbb{R}^{p_{\ell^*}} \) spanned by the bottom \( p_{\ell} - p_{\ell+1} \) eigenvectors of \( V_\ell^T (\Sigma^{(\ell)} - \Sigma) V_\ell \) for \( \ell = 0, \ldots, L \) with the convention that \( p_{L+1} = 0 \).

For \( \ell \in \{0, \ldots, L-1\} \), we intend to apply Proposition 1 to \( Z_\ell = V_\ell^T X_\ell \) and \( \mu^Z = V_\ell^T \mu^* \) in order to upper bound the error term \( \text{Err}_\ell := \| P_{\ell^*} (V_\ell^T X_\ell - V_\ell^T \mu^*) \|_2^2 \). Using the inequalities
\[
\| V^T (\hat{\Sigma}^{(\ell)} - \Sigma) V \|_{op} \leq \| \hat{\Sigma}^{(\ell)} - \Sigma \|_{op}, \quad \lambda_{p_{\ell}} (V^T \Sigma V) \leq \lambda_p (\Sigma), \quad \lambda_1 (V^T \Sigma V) \geq \lambda_1 (\Sigma)
\]

that are true for every orthogonal matrix $V$, and keeping in mind the definition of $P_\ell$, we get
\[
\text{Err}_\ell \leq \left\{ 2\omega O \| \tilde{\Sigma}^{(\ell)} - \Sigma \|_{\text{op}} + \frac{\omega^2 O}{1 - \omega O} \left( (\lambda_p - \lambda_1)(\Sigma) + \frac{\delta^2 \ell}{p_{\ell+1}} \right) \right\}^{1/2} + \| P_\ell V^\top \xi_{S_\ell^{(\ell)}} \|_2^2,
\]
where we have used the notation
\[
\omega O = \max_{\ell} \frac{|S^{(\ell)} \cap O|}{|S^{(\ell)}|}, \quad \xi_i = X_i - \mu^*
\]
and $\delta_\ell = \inf_i \max_{\ell} \| (V^\top_i (X_i - \mu)) \|_2^2$. Note that when $O$ and $(S^{(\ell)} \cap O)$ of cardinality less than $n\varepsilon$ and $n(\varepsilon + \tau)$, respectively, we have $\omega O \leq \varepsilon/(1 - \tau)$ and $1 - \omega O \leq 1 - \varepsilon - \tau$.

We set $\eta := \varepsilon + \tau \leq 3/4$ and apply Lemma 13 to infer that $\omega O \leq \varepsilon/(1 - \eta) \leq 4\varepsilon$ is true with probability at least $1 - \delta$. Furthermore, we know that $\delta_\ell \leq \max_{i \in S^{(\ell)}} \| V^\top_i X_i - \mu^i \|_2 \leq t\sqrt{p}\tau$. This yields
\[
\text{Err}_\ell \leq \left\{ 8\varepsilon |S^{(\ell)} \cap O| + 16\varepsilon^2 \left( (\lambda_p - \lambda_1)(\Sigma) + \frac{t^2 p_{\ell+1}}{p_{\ell+1}} \right) \right\}^{1/2} + \| P_\ell \xi_{S_\ell^{(\ell)}} \|_2^2.
\]

Let us introduce the shorthand $T_1 = \max_{\ell \in [L]} |S^{(\ell)} \cap O|$. This leads to
\[
\text{Err}_\ell \leq \left\{ 8\varepsilon T_1 + 16\varepsilon^2 \frac{p_{\ell+1}}{p_{\ell+1}} \right\}^{1/2} + \| P_\ell \xi_{S_\ell^{(\ell)}} \|_2^2. \tag{34}
\]

For the last error term, since $p_L = 1$ then, by the combination of Lemma 12 and Lemma 13, we have
\[
\text{Err}_L \leq \left\| P_\ell \xi_{S_2^{(L)}} \right\|_2^2 + \frac{n\varepsilon (t\sqrt{p} + \| P_\ell \mu^* - \hat{\mu}^{GM}_\ell \|_2)}{|S^{(L)}|} \leq \left\| P_\ell \xi_{S_2^{(L)}} \right\|_2^2 + \frac{\varepsilon t + \varepsilon \| P_\ell \mu^* - \hat{\mu}^{GM}_\ell \|_2}{1 - \eta} \leq \left\| P_\ell \xi_{S_2^{(L)}} \right\|_2^2 + 4\varepsilon t + 4\varepsilon \| P_\ell \mu^* - \hat{\mu}^{GM}_\ell \|_2. \tag{35}
\]

Combining (34), (35), inequality $p_{\ell+1} \leq c p_{\ell+1}$, as well as the Minkowski inequality, we get
\[
\| \mu^* - \hat{\mu}^{SDR} \|_2 = \left\{ \sum_{\ell=0}^{L} \text{Err}_\ell^2 \right\}^{1/2} \leq \left\{ 8\varepsilon L (T_1 + 4\varepsilon t^2) + 16\varepsilon^2 (t + \| P_\ell \mu^* - \hat{\mu}^{GM}_\ell \|_2)^2 \right\}^{1/2} + \left\{ \sum_{\ell=0}^{L} \left\| P_\ell \xi_{S_\ell^{(\ell)}} \right\|_2^2 \right\}^{1/2} \leq 2\sqrt{2L T_1 + 9\varepsilon t} + 4\varepsilon \| P_\ell \mu^* - \hat{\mu}^{GM}_\ell \|_2 + \left\{ \sum_{\ell=0}^{L} \left\| P_\ell \xi_{S_\ell^{(\ell)}} \right\|_2^2 \right\}^{1/2}. \tag{36}
\]

To ease notation, let us set
\[
r_{n,\varepsilon} = 4\sqrt{\varepsilon} (\sqrt{p} + 2\sqrt{\log(2/\delta)}) / \sqrt{n}.
\]

In view of Lemma 16, with probability at least $1 - \delta$, we have
\[
\left\{ \sum_{\ell=0}^{L} \left\| P_\ell \xi_{S_\ell^{(\ell)}} \right\|_2^2 \right\}^{1/2} \leq \left\{ \sum_{\ell=0}^{L} \left( \left\| P_\ell \xi_{\mu^*} \right\|_2^2 + r_{n,\varepsilon} \sqrt{n} + 2\varepsilon \sqrt{\log(2e/\eta)(1 - \eta)} \right) \right\}^{1/2}.
imply that with probability at least

\[ \| \xi_n \|_2 \leq 4r_{n,s} \sqrt{\eta} + 10\eta \sqrt{L \log(2/\eta)}. \]

Moreover, since the random variable \( \xi_n \) is sub-Gaussian with variance proxy \( s/n \), hence by Lemma 9 we have

\[ \| \xi_n \|_2^2 \leq 16s(\sqrt{\eta} + 2\sqrt{\log(2/\delta)})^2 = r_{n,s}^2 \]

with probability at least \( 1 - \delta \). Therefore, with probability at least \( 1 - 2\delta \),

\[ \left( \sum_{\ell=0}^{L} \| \mathbb{W}_\ell \xi_{\ell}^+ \|_2 \right)^{1/2} \leq 4r_{n,s}(1 + \sqrt{\eta L}) + 10\eta \sqrt{L \log(2/\eta)}. \]

Next, the combination of Lemma 12 and Lemma 13 and the fact that \( p_L = \dim(\mathbb{W}_L) = 1 \) imply that with probability at least \( 1 - \delta \)

\[ \| \mathbb{W}_\ell \mu^* - \hat{\mu}_{\mathbb{W}_\ell}^{GM} \|_2 \leq \frac{2(1 + C_{n,s} \sqrt{\delta})}{1 - 2\delta}, \]

where \( C \) is the same universal constant as in Lemma 1. Recall that we have chosen \( t \) in such a way that

\[ t \leq \frac{3(1 + C_0r_{n,s} \sqrt{s}/4 \sqrt{T})}{1 - 2\delta} + 1.6C_0s \sqrt{\log(2/\tau)}. \]

Combining (36), (37), (38) and (39), we arrive at the inequality

\[ \| \hat{\mu}^{SDR} - \mu^* \|_2 \leq 2\sqrt{2\varepsilon L T_1} + 9\varepsilon \sqrt{L} + \frac{2\varepsilon(1 + C_{n,s} \sqrt{s}/4)}{1 - 2\varepsilon} + 4r_{n,s}(1 + \sqrt{\eta L}) + 10\eta \sqrt{L \log(2/\eta)} \]

\[ \leq 2\sqrt{2\varepsilon L T_1} + \frac{27\varepsilon \sqrt{L}(1 + C_{n,s} \sqrt{s}/4 \sqrt{T})}{1 - 2\varepsilon} + 14.4C_0s \varepsilon \sqrt{L \log(2/\tau)} \]

\[ + \frac{8\varepsilon(1 + C_{n,s} \sqrt{s}/4)}{1 - 2\varepsilon} + 4r_{n,s}(1 + \sqrt{\eta L}) + 10\eta \sqrt{L \log(2/\eta)} \]

that holds with probability at least \( 1 - 3\delta \). In the upper bound obtained above, only the term \( T_1 \) remains random. We can upper bound this term using Lemma 15. It implies that with probability at least \( 1 - 2\delta \), we have

\[ T_1 \leq A \sqrt{np} + p + n\eta \log(2e/\eta) + 2\log(1/\delta) \]

\[ \leq A_s(r_{n,s}^2 + r_{n,s}^2 + 8\eta \log(2/\eta)) + (7.5r_{n,s} + 10\eta \sqrt{\log(2/\eta)})^2 + \varepsilon, \]

where \( A_s \) is a constant depending only on the variance proxy \( s \), the value of which is not necessarily the same in further simplifications of the expression from the last display. Then, using the triangle inequality several times we arrive at the following expression

\[ \sqrt{T_1} \leq \left\{ A_s(r_{n,s}^2 + r_{n,s}^2 + 8\eta \log(2/\eta)) \right\}^{1/2} + (7.5r_{n,s} + 10\eta \sqrt{\log(2/\eta)}) \sqrt{\varepsilon} + \varepsilon \]

\[ \leq (A_s + \sqrt{A_s^2/2 + 5.4}r_{n,s} + (7.1 + 2\sqrt{2A_s})\tau \sqrt{\log(2/\tau)} + (9.1 + 2\sqrt{2A_s})\varepsilon \sqrt{\log(2/\varepsilon)}. \]

These inequalities imply that there is a universal constant \( C \) such that

\[ \| \hat{\mu}^{SDR} - \mu^* \|_2 \leq \frac{Ca(A_s r_{n,s} \sqrt{s} + \tau \sqrt{\log(2/\tau)} + \varepsilon \sqrt{\log(2/\varepsilon)} + r_{n,s} \varepsilon / \sqrt{s} \sqrt{T})}{1 - 2\delta}. \]

(40)
Let us denote \( \log_+(x) = \max\{0, \log(x)\} \) the positive part of logarithm, then we choose
\[
\tau = \frac{1}{4} \sqrt{\frac{\bar{r}_{n,s}}{\log_+(2/\bar{r}_{n,s})}}, \quad \text{with} \quad \bar{r}_{n,s} = \frac{3\sqrt{3}(\sqrt{p} + 2\sqrt{\log(2/\delta)})}{\sqrt{n}}.
\]
Note that \( r_{n,s} \leq \sqrt{5} \bar{r}_{n,s} \) and, furthermore, \( \tau = 1/4 \) whenever \( \bar{r}_{n,s} \geq 1/2 \). Therefore, \( r_{n,s} \varepsilon / \sqrt{\tau} \leq r_{n,s} + \varepsilon \). Inserting this value of \( \tau \) in (40) leads to
\[
\|\hat{\mu}^{\text{SDR}} - \mu^*\|_2 \leq C_s \left( A_s \sqrt{\frac{p}{n}} + \frac{\varepsilon}{1 - 2\varepsilon^*} \log(p/\varepsilon) \sqrt{L} \right).
\]
where \( C \) is a universal constant. Replacing \( r_{n,s} \) by its expression, and upper bounding \( L \) by \( 2\log p \), we arrive at
\[
\|\hat{\mu}^{\text{SDR}} - \mu^*\|_2 \leq \frac{C_s \sqrt{\log(p)}}{1 - 2\varepsilon^*} \left( A_s \sqrt{\frac{p}{n}} + \varepsilon \sqrt{\log(2/\varepsilon)} + A_s \sqrt{\frac{\log(1/\delta)}{n}} \right).
\]
Note that this inequality holds true on an event of probability at least \( 1 - 5\delta \).

**Funding.** The authors were supported by the grant Investissements d’Avenir (ANR-11-IDEX-0003/Labex Ecodec/ANR-11-LABX-0047) and by the FAST Advance grant. The third author was supported in part by the center Hi!PARIS.

**REFERENCES**

Balakrishnan, S., Du, S. S., Li, J., and Singh, A. (2017). Computationally efficient robust sparse estimation in high dimensions. In *COLT 2017*, pages 169–212.

Bateni, A.-H. and Dalalyan, A. S. (2020). Confidence regions and minimax rates in outlier-robust estimation on the probability simplex. *Electron. J. Statist.*, 14(2):2653–2677.

Cardot, H., Cenac, P., and Zitt, P.-A. (2013). Efficient and fast estimation of the geometric median in hilbert spaces with an averaged stochastic gradient algorithm. *Bernoulli*, 19:18–43.

Chen, M., Gao, C., and Ren, Z. (2018). Robust covariance and scatter matrix estimation under Huber’s contamination model. *Ann. Statist.*, 46(5):1932–1960.

Cheng, Y., Diakonikolas, I., and Ge, R. (2019a). High-dimensional robust mean estimation in nearly-linear time. In *Proceedings of SODA 2019*, pages 2755–2771.

Cheng, Y., Diakonikolas, I., Ge, R., and Woodruff, D. P. (2019b). Faster algorithms for high-dimensional robust covariance estimation. In Beygelzimer, A. and Hsu, D., editors, *COLT 2019*, volume 99 of *Proceedings of Machine Learning Research*, pages 727–757. PMLR.

Cheng, Y., Diakonikolas, I., Kane, D. M., Ge, R., Gupta, S., and Sohankar, M. (2021). Outlier-robust sparse estimation via non-convex optimization. *CoRR*, abs/2109.11515.

Chinot, G. (2020). Erm and rerm are optimal estimators for regression problems when malicious outliers corrupt the labels. *Electron. J. Statist.*, 14(2):3563–3605.

Chinot, G., Lécué, G., and Lerasle, M. (2020). Robust high dimensional learning for Lipschitz and convex losses. *J. Mach. Learn. Res.*, 21:Paper No. 233, 47.

Collier, O. and Dalalyan, A. S. (2019). Multidimensional linear functional estimation in sparse gaussian models and robust estimation of the mean. *Electron. J. Statist.*, 13(2):2830–2864.

Comminges, L., Collier, O., Ndaoud, M., and Tsybakov, A. B. (2021). Adaptive robust estimation in sparse vector models. *Ann. Statist.*, 49(3):1347–1377.

Comminges, L. and Dalalyan, A. S. (2012). Tight conditions for consistency of variable selection in the context of high dimensionality. *The Annals of Statistics*, 40(5):2667–2696.

Dalalyan, A. and Thompson, P. (2019). Outlier-robust estimation of a sparse linear model using \( \ell_1 \)-penalized Huber’s M-estimator. In *NeurIPS 32*, pages 13188–13198.

Dalalyan, A. S. and Minasyan, A. (2020). All-in-one robust estimator of the gaussian mean. *math.ST*, arXiv:2002.01432.

Depersin, J. and Lécué, G. (2022). Robust sub-Gaussian estimation of a mean vector in nearly linear time. *Ann. Statist.*, 50(1):511–536.
Diakonikolas, I., Kamath, G., Kane, D. M., Li, J., Moitra, A., and Stewart, A. (2016). Robust estimators in high dimensions without the computational intractability. In *IEEE 57th Annual Symposium on Foundations of Computer Science, FOCS 2016*, pages 655–664.

Diakonikolas, I., Kamath, G., Kane, D. M., Li, J., Moitra, A., and Stewart, A. (2017). Being robust (in high dimensions) can be practical. In *Proceedings of the 34th International Conference on Machine Learning, ICML 2017*, volume 70 of *Proceedings of Machine Learning Research*, pages 999–1008.

Diakonikolas, I., Kamath, G., Kane, D. M., Li, J., Moitra, A., and Stewart, A. (2018). Robustly learning a gaussian: Getting optimal error, efficiently. In *Proceedings of SODA 2018*, pages 2683–2702. SIAM.

Dong, Y., Hopkins, S. B., and Li, J. (2019). Quantum entropy scoring for fast robust mean estimation and improved outlier detection. In *NeurIPS 2019*, pages 6065–6075.

Donoho, D. L. and Gasko, M. (1992). Breakdown Properties of Location Estimates Based on Halfspace Depth and Projected Outlyingness. *The Annals of Statistics*, 20(4):1803 – 1827.

Dong, Y. S., Hopkins, S. B., and Li, J. (2019). Quantum entropy scoring for fast robust mean estimation and improved outlier detection. In *NeurIPS 2019*, pages 6065–6075.

Donoho, D. L. and Gasko, M. (1992). Breakdown Properties of Location Estimates Based on Halfspace Depth and Projected Outlyingness. *The Annals of Statistics*, 20(4):1803 – 1827.

Goes, J., Lerman, G., and Nadler, B. (2020). Robust sparse covariance estimation by thresholding Tyler’s M-estimator. *Ann. Statist.*, 48(1):86–110.

Halko, N., Martinsson, P. G., and Tropp, J. A. (2011). Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions. *SIAM Review*, 53(2):217–288.

Huber, P. and Ronchetti, E. (2011). *Robust Statistics*. Wiley Series in Probability and Statistics. Wiley.

Koltchinskii, V. and Lounici, K. (2017). Concentration inequalities and moment bounds for sample covariance operators. *Bernoulli*, 23(1):110 – 133.

Lai, K. A., Rao, A. B., and Vempala, S. S. (2016). Agnostic estimation of mean and covariance. In *IEEE 57th Annual Symposium on Foundations of Computer Science, FOCS 2016*, pages 665–674.

Liu, L., Shen, Y., Li, T., and Caramanis, C. (2020). High dimensional robust sparse regression. In Chiappa, S. and Calandra, R., editors, *AISTATS 2020*, volume 108 of *Proceedings of Machine Learning Research*, pages 411–421. PMLR.

Liu, X., Mosler, K., and Mozharovskyi, P. (2019). Fast computation of tukey trimmed regions and median in dimension $p > 2$. *Journal of Computational and Graphical Statistics*, 28:682–697.

Lopuhaa, H. P. and Rousseeuw, P. J. (1991). Breakdown points of affine equivariant estimators of multivariate location and covariance matrices. *The Annals of Statistics*, 19(1):229–248.

Lugosi, G. and Mendelson, S. (2021). Robust multivariate mean estimation: The optimality of trimmed mean. *The Annals of Statistics*, 49(1):393 – 410.

Maronna, R., Martin, D., and Yohai, V. (2006). *Robust Statistics: Theory and Methods*. Wiley Series in Probability and Statistics. Wiley.

Minasyan, A. G. (2020). Excess-risk consistency of group-hard thresholding estimator in robust estimation of Gaussian mean. *Journal of Contemporary Mathematical Analysis*, 55(3):208–212.

Minsker, S. (2015). Geometric median and robust estimation in banach spaces. *Bernoulli*, 21(4):2308–2335.

Pensia, A., Jog, V., and Loh, P.-L. (2020). Robust regression with covariate filtering: Heavy tails and adversarial contamination. preprint arXiv:2009.12976.

Rigollet, P. and Hütter, J.-C. (2019). High dimensional statistics. *Lecture notes (MIT)*.

Rousseeuw, P., Hampel, F., Ronchetti, E., and Stahel, W. (2011). *Robust Statistics: The Approach Based on Influence Functions*. Wiley Series in Probability and Statistics. Wiley.

Vershynin, R. (2012). *Introduction to the non-asymptotic analysis of random matrices*, page 210–268. Cambridge University Press.