A refinement of the Browder–Göhde–Kirk fixed point theorem and some applications

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Abstract. The following generalization of the Browder–Göhde–Kirk fixed point theorem is proved: if $C$ is a nonempty bounded closed and convex subset of a uniformly convex normed space $X$ and $T$ is a self-mapping of $C$ such that $\|Tx - Ty\| \leq \beta (\|x - y\|)$ for all $x, y \in C, x \neq y$, where a function $\beta : (0, \infty) \to [0, \infty)$ is such that $\lim_{t \to 0^+} \frac{\beta(t)}{t} = 1$, then $T$ has a fixed point. Two modifications of this theorem as well as some accompanying results on Lipschitz-type mappings are given. An application in the theory of $L^p$-solutions of an iterative functional equation, and some refinements of the Radamacher theorem are proposed.

Mathematics Subject Classification. Primary 47H10, 47H09.

Keywords. Generalization of nonexpansive operator, Lipschitz operator, fixed-point theorem.

1. Introduction

Basing on an observation that every continuous map of a convex set satisfying a restricted Lipschitz condition must be Lipschitz continuous (see [12]), we present some generalizations of Browder–Göhde–Kirk fixed point (Browder [1], Göhde [7], Kirk [9], see also [4,6,15,16]), and propose their applications, including an extension of the classical Radamacher theorem.

Section 2 contains the auxiliary results characterizing the Lipschitz continuous functions with the aid of some weaker conditions.

In Sect. 3, we present two fixed point theorems. Let $X$ be a uniformly convex Banach space, $C \subset X$ a nonempty bounded convex closed set, and $T$ a selfmapping of $C$. Theorem 1 says that $T$ has a fixed point, if for some function $\beta : (0, \infty) \to [0, \infty)$ satisfying the conditions

$$\limsup_{t \to 0^+} \frac{\beta(t)}{t} < +\infty, \quad \liminf_{t \to 0^+} \frac{\beta(t)}{t} = 1,$$

we have

$$\|Tx - Ty\| \leq \beta (\|x - y\|), \quad x, y \in C, \ x \neq y.$$
The second result, Theorem 2, says that the last inequality can be significantly weakened, namely, \( T \) has a fixed point, if \( T \) is continuous and, for a certain function \( \beta : (0, \infty) \to [0, \infty) \) and a zero sequence \( (t_n) \) of positive numbers such that

\[
\lim_{n \to \infty} \frac{\beta(t_n)}{t_n} = 1,
\]

we have, for all \( n \in \mathbb{N} \) and for all \( x, y \in C \),

\[
\|x - y\| = t_n \implies \|T(x) - T(y)\| \leq \beta(\|x - y\|).
\]

In Sect. 4, we use Theorem 1 to get a result on the existence and uniqueness of \( L^p \)-solutions \( (1 < p < +\infty) \) of the iterative functional equation

\[
\varphi(x) = h(x, \varphi[f(x)]).
\]

In Sect. 5, we give some refinements of the classical Radamacher theorem on the differentiability of the Lipschitz mappings.

### 2. Some auxiliary results on Lipschitz continuity

We begin with the following

**Lemma 1.** Let \( X, Y \) be normed spaces, \( C \subset X \) a convex set, \( T : C \to Y \) a mapping, and \( \beta : (0, \infty) \to [0, \infty) \) a real function such that

\[
\|Tx - Ty\| \leq \beta(\|x - y\|), \quad x, y \in C, \quad x \neq y.
\]

If

\[
\limsup_{t \to 0^+} \frac{\beta(t)}{t} < +\infty,
\]

then

\[
\|Tx - Ty\| \leq L \|x - y\|, \quad x, y \in C,
\]

where

\[
L := \liminf_{t \to 0^+} \frac{\beta(t)}{t}.
\]

**Proof.** Note that conditions (2) and (1) imply that \( T \) is continuous. Indeed, from (2) there are some real positive \( M \) and \( \delta \) such that \( \beta(t) \leq Mt \) for all \( t \in (0, \delta) \), and (1) implies that \( \|Tx - Ty\| \leq M \|x - y\| \) for all \( x, y \in C \) such that \( \|x - y\| < \delta \).

Take arbitrary \( x, y \in C, \ x \neq y \). By (3), for every \( \varepsilon > 0 \) there is a \( t_\varepsilon > 0 \) and a unique \( n = n_\varepsilon \in \mathbb{N}_0 \) such that

\[
\frac{\beta(t_\varepsilon)}{t_\varepsilon} \leq L + \varepsilon,
\]

\[
0 \leq \|x - y\| - n(\varepsilon)t_\varepsilon < t_\varepsilon,
\]

and

\[
\lim_{\varepsilon \to 0} t_\varepsilon = 0.
\]
Put
\[ z_k = x + kt\varepsilon \frac{y-x}{\|y-x\|}, \quad k = 0, 1, \ldots, n(\varepsilon). \]

Then, by the convexity of \( C \),
\[ z_k \in C, \quad k = 0, 1, \ldots, n(\varepsilon); \]
clearly
\[ \|z_k - z_{k+1}\| = t\varepsilon, \quad k = 0, 1, \ldots, n(\varepsilon); \quad (7) \]
and, by (5),
\[ \|z_{n(\varepsilon)} - y\| = (\|y-x\| - nt\varepsilon) < t\varepsilon. \quad (8) \]
Hence, applying in turn: the triangle inequality, condition (1), some obvious identities, (4), (7) and (5), we get
\[
\|Tx - Ty\| = \left\| \sum_{k=0}^{n(\varepsilon)-1} (Tz_k - Tz_{k+1}) + (Tz_{n(\varepsilon)} - Ty) \right\|
\leq \sum_{k=0}^{n(\varepsilon)-1} \|Tz_k - Tz_{k+1}\| + \|Tz_{n(\varepsilon)} - Ty\|
\leq \sum_{k=0}^{n(\varepsilon)-1} \beta(\|z_k - z_{k+1}\|) + \|Tz_{n(\varepsilon)} - Ty\|
= \sum_{k=0}^{n(\varepsilon)-1} \beta(t\varepsilon) + \|Tz_{n(\varepsilon)} - Ty\|
= n(\varepsilon)\beta(t\varepsilon) + \|Tz_{n(\varepsilon)} - Ty\|
= \frac{\beta(t\varepsilon)}{t\varepsilon} (n(\varepsilon) t\varepsilon) + \|Tz_{n(\varepsilon)} - Ty\|
\leq (L + \varepsilon) n(\varepsilon) t\varepsilon + \|Tz_{n(\varepsilon)} - Ty\|
\leq (L + \varepsilon) \|x - y\| + \|Tz_{n(\varepsilon)} - Ty\|,
\]
that is
\[
\|Tx - Ty\| \leq (L + \varepsilon) \|x - y\| + \|Tz_{n(\varepsilon)} - Ty\|.
\]
Since the continuity of \( T \) and the conditions (6) and (8) imply that
\[
\lim_{\varepsilon \to 0} \|Tz_{n(\varepsilon)} - Ty\| = 0,
\]
letting \( \varepsilon \to 0 \) in the above inequality, we obtain
\[
\|Tx - Ty\| \leq L \|x - y\|,
\]
which completes the proof. \( \square \)

**Remark 1.** If \( \beta : (0, \infty) \to [0, \infty) \) is subadditive, then
\[
\limsup_{t \to 0^+} \frac{\beta(t)}{t} = \liminf_{t \to 0^+} \frac{\beta(t)}{t}.
\]
(see [8, p. 250, Theorem 7.11.1], also [11]). In this case, instead of (3) it is enough to assume that \( L < +\infty \).

**Remark 2.** The reasoning in the proof of Lemma 1 simplifies, if
\[
\lim_{t \to 0^+} \frac{\beta(t)}{t} = L.
\]
Indeed, if this condition holds, then for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that
\[
\beta(t) \leq (L + \varepsilon) t, \quad t \in (0, \delta).
\]
Take arbitrary \( x, y \in C, x \neq y \), choose \( n \in \mathbb{N} \) such that
\[
\|x - y\|_n < \delta,
\]
and put
\[
z_k := x + \frac{k}{n} (y - x), \quad k = 0, 1, \ldots, n.
\]
Of course
\[
\|z_k - z_{k+1}\| = \frac{\|x - y\|}{n}, \quad k = 0, 1, \ldots, n - 1;
\]
\[
z_0 = x, \quad z_n = y,
\]
and, by the convexity of \( C \),
\[
z_k \in C, \quad k = 0, 1, \ldots, n.
\]
Applying the triangle inequality and (1), we hence get
\[
\|Tx - Ty\| = \left\| \sum_{k=0}^{n-1} (Tz_k - Tz_{k+1}) \right\| \leq \sum_{k=0}^{n-1} \|Tz_k - Tz_{k+1}\| \leq \sum_{k=0}^{n-1} \beta(\|z_k - z_{k+1}\|)
\]
\[
= \sum_{k=0}^{n-1} \beta \left( \frac{\|x - y\|}{n} \right) \leq \sum_{k=0}^{n-1} (L + \varepsilon) \frac{\|x - y\|}{n} = (L + \varepsilon) \frac{\|x - y\|}{n},
\]
that is
\[
\|Tx - Ty\| \leq (L + \varepsilon) \frac{\|x - y\|}{n}.
\]
Since \( \varepsilon > 0 \) is chosen arbitrarily, letting \( \varepsilon \to 0 \), we conclude that \( T \) is \( L \)-Lipschitzian.

A much weaker necessary and sufficient condition for a continuous map to be Lipschitz continuous gives the following

**Lemma 2.** Let \( X \) and \( Y \) be real normed spaces and \( C \subset X \) a bounded convex set. Suppose that \( T : C \to Y \) is continuous. If there are a nonnegative real \( L \) and two positive sequences \( (t_n) \), \( (c_n) \),
\[
\lim_{n \to \infty} t_n = 0, \quad \lim_{n \to \infty} c_n = L,
\]
such that for every \( n \in \mathbb{N} \) and for all \( x, y \in C \),
\[
\|x - y\| = t_n \implies \|T(x) - T(y)\| \leq c_n t_n,
\]
then \( T \) is Lipschitz continuous, and
\[
\|Tx - Ty\| \leq L \|x - y\|, \quad x, y \in C.
\]
Proof. Take arbitrary $x, y \in C$, $x \neq y$. For every $n \in \mathbb{N}$, there is a unique $m_n \in \mathbb{N} \cup \{0\}$ such that

$$m_n t_n \leq \|x - y\| < (m_n + 1) t_n.$$ 

Put

$$z_k := x + \frac{kt_n}{\|y - x\|} (y - x), \quad k = 0, 1, \ldots, m_n.$$ 

(10)

Since

$$0 \leq \frac{m_n t_n}{\|y - x\|} \leq 1,$$

and, for each $k = 0, 1, \ldots, m_n$,

$$z_k = \left(1 - \frac{kt_n}{\|y - x\|}\right)x + \frac{kt_n}{\|y - x\|}(y - x),$$

the convexity of $C$ implies that

$$z_k \in C, \quad k = 0, 1, \ldots, m_n.$$ 

Moreover, by (10),

$$\|z_k - z_{k+1}\| = t_n, \quad k = 0, 1, \ldots, m_n - 1,$$

(11)

and, for $k = m_n$,

$$z_{m_n} - y = \left(x + \frac{kt_n}{\|y - x\|}(y - x)ight) - y = \left(\frac{kt_n}{\|y - x\|} - 1\right)(y - x),$$

we have

$$\|z_{m_n} - y\| = \|y - x\| - m_n t_n < t_n.$$ 

(12)

From (11) and (9), we get

$$\|Tz_k - Tz_{k+1}\| \leq c_n t_n, \quad k = 0, 1, \ldots, m_n - 1,$$

so, by the triangle inequality,

$$\|Tx - Ty\| = \left\| \sum_{k=0}^{m_n-1} (Tz_k - Tz_{k+1}) + (Tz_{m_n} - Ty) \right\|$$

$$\leq \sum_{k=0}^{m_n-1} \|Tz_k - Tz_{k+1}\| + \|Tz_{m_n} - Ty\|$$

$$\leq m_n c_n t_n + \|Tz_{m_n} - Ty\|$$

$$= c_n (m_n t_n) + \|Tz_{m_n} - Ty\|$$

whence, taking into account that $m_n t_n \leq \|x - y\|$, by (12), we get

$$\|Tx - Ty\| \leq c_n \|x - y\| + \|Tz_{m_n} - Ty\|.$$ 

(13)

Since, by (12), $\|z_{m_n} - y\| < t_n$, we have

$$\lim_{n \to \infty} \|z_{m_n} - y\| = 0,$$

and, in view of the assumed continuity of $T$,

$$\lim_{n \to \infty} \|Tz_{m_n} - Ty\| = 0.$$
Hence, letting $n \to \infty$ in (13), and taking into account that $\lim_{n \to \infty} c_n = L$, we conclude that

$$\|Tx - Ty\| \leq L\|x - y\|,$$

which was to be shown. \hfill \Box

3. Fixed-point theorems

Recall that a real normed vector space $(X, \|\cdot\|)$ is called \textit{uniformly convex}, if for every $\varepsilon \in (0, 2]$ there is some $\delta > 0$ such that for any two vectors $x, y \in X$ with $\|x\| = \|y\| = 1$, the condition $\|x - y\| \geq \varepsilon$ implies that $\left\| \frac{x + y}{2} \right\| \leq 1 - \delta$ (Goebel and Reich [6]; see also [13] for a generalization).

Applying Lemma 1 with $L = 1$ we obtain the following generalization of the Browder–Göhde–Kirk theorem.

**Theorem 1.** Let $X$ be a uniformly convex Banach space, $C \subset X$ a nonempty bounded convex closed set and $T$ a selfmapping of $C$. If there exists a function $\beta : (0, \infty) \to [0, \infty)$ such that

$$\|Tx - Ty\| \leq \beta(\|x - y\|), \; x, y \in C, \; x \neq y,$$

and

$$\limsup_{t \to 0^+} \frac{\beta(t)}{t} < +\infty, \quad \liminf_{t \to 0^+} \frac{\beta(t)}{t} = 1,$$

then $T$ has a fixed point in $C$.

**Proof.** Applying Lemma 1 with $L = 1$, we get

$$\|Tx - Ty\| \leq L\|x - y\|, \quad x, y \in C,$$

that is $T$ is nonexpansive, and the result follows from the original version of the Browder–Göhde–Kirk theorem. \hfill \Box

In particular, the thesis of Browder–Göhde–Kirk theorem remains true, if the nonexpansivity of the mapping $T$ is replaced for instance, by the inequality

$$\|Tx - Ty\| \leq \exp(\|x - y\|) - 1, \quad x, y \in C, \; x \neq y.$$

Lemma 2 and the Browder–Göhde–Kirk theorem yield the following:

**Proposition 1.** Let $X$ be a uniformly convex Banach space and $C \subset X$ a nonempty bounded closed and convex set. Suppose that $T : C \to C$ is continuous. If there exist two positive sequences $(t_n), (c_n)$,

$$\lim_{n \to \infty} t_n = 0, \quad \lim_{n \to \infty} c_n = 1,$$

such that for every $n \in \mathbb{N}$ and for all $x, y \in C$,

$$\|x - y\| = t_n \implies \|T(x) - T(y)\| \leq c_n t_n,$$

then $T$ has a fixed point.
This proposition improves the relevant result in [11] where the uniform continuity of $T$ is assumed.

The main result of this section reads as follows.

**Theorem 2.** Let $X$ be a uniformly convex Banach space and $C \subset X$ a nonempty bounded closed and convex set. Suppose that $T : C \to C$ is continuous. If there exist a function $\beta : (0, \infty) \to [0, \infty)$ and a sequence of positive real $(t_n)$, $\lim_{n \to \infty} t_n = 0$ satisfying the condition

$$
\lim_{n \to \infty} \frac{\beta(t_n)}{t_n} = 1,
$$

such that for every $n \in \mathbb{N}$ and for all $x, y \in C$,

$$
\| x - y \| = t_n \implies \| T(x) - T(y) \| \leq \beta(\| x - y \|),
$$

then $T$ has a fixed point.

**Proof.** Setting $c_n := \frac{\beta(t_n)}{t_n}$ we have $\lim_{n \to \infty} c_n = 1$ and for every $n \in \mathbb{N}$ and for all $x, y \in C$, if $\| x - y \| = t_n$, then

$$
\| T(x) - T(y) \| \leq c_n t_n = \beta(t_n),
$$

and the result follows from Proposition 1. \qed

### 4. An application in the theory of iterative functional equations

For a measure space $(\Omega, \Sigma, \mu)$ and a real $p > 1$, denote by $(L^p(\Omega), \| \cdot \|_p)$ the Banach space of all (equivalence classes with respect to the $\mu$-a.e. equality) of $\Sigma$-measurable functions $\varphi : \Omega \to \mathbb{R}$ such that $|\varphi|^p$ is $\mu$-integrable, and

$$
\| \varphi \|_p = \left( \int_{\Omega} |\varphi|^p d\mu \right)^{1/p}.
$$

It is well known that $(L^p(\Omega), \| \cdot \|_p)$ is a uniformly convex Banach space (Clarkson [3]).

In this section, we consider solutions $\varphi \in L^p(\Omega)$ of the iterative-type functional equation

$$
\varphi(x) = h(x, \varphi[f(x)]).
$$

We assume that $(\Omega, \Sigma, \mu)$ the given functions $f$ and $h$ satisfy the following conditions:

(i) $k \in \mathbb{N}$; $\Omega \subset \mathbb{R}^k$ is an open set; $\mu$ is the Lebesgue measure, $\mu(\Omega) = 1$; and $f : \Omega \to \Omega, f = (f_1, \ldots, f_k)$, is a locally Lipschitzian homeomorphic mapping;

(ii) $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is such that: for every $y \in \mathbb{R}$ the function $\Omega \ni x \mapsto h(x, y)$ is Lebesgue measurable, and $\mathbb{R} \ni y \mapsto h(x, y)$ is continuous for almost all $x \in \Omega$ (with respect to $\mu$);

(iii) $p \in \mathbb{R}, p > 1$, and there are $g_1, g_2 \in L^p(\Omega)$, $g_1 \leq g_2$ a.e. in $\Omega$ such that for all $x \in \Omega$ and $y \in \mathbb{R}$, the following implication holds true:
\[ g_1(f(x)) \leq y \leq g_2(f(x)) \implies g_1(x) \leq h(x, y) \leq g_2(x). \]

Applying Theorem 1, we prove the following:

**Theorem 3.** Let conditions (i)–(iii) be satisfied. Assume that there are a Lebesgue measurable function \( \alpha : \Omega \to [0, \infty) \), and a function \( \beta : [0, \infty) \to [0, \infty) \) such that

\[
|h(x, y_1)| \leq \alpha(x) \beta(|y_1 - y_2|), \quad x \in \Omega, \ y_1, y_2 \in \mathbb{R}; \quad (14)
\]

\[
[\alpha(x)]^p \leq |J_f(x)| \quad \text{a.e. in } \Omega, \quad (15)
\]

where, for \( f = (f_1, \ldots, f_k) \) and \( x = (x_1, \ldots, x_k) \), the symbol \( J_f(x) := \frac{\partial(f_1, \ldots, f_k)}{\partial(x_1, \ldots, x_k)} \) stands for the Jacobian of \( f \);

the function

\[
[0, \infty) \ni t \mapsto \left[ \beta\left(t^\frac{1}{p}\right) \right]^p \text{ is concave};
\]

and

\[
\lim_{t \to 0} \frac{\beta(t)}{t} \leq 1. \quad (17)
\]

Then there exists \( \varphi \in L^p(\Omega) \), \( g_1 \leq \varphi \leq g_2 \) a.e. in \( \Omega \), such that

\[
\varphi(x) = h(x, \varphi[f(x)]) \quad \text{a.e. for } x \in \Omega; \quad (18)
\]

moreover, if \( \beta(t_n) \neq t_n \) for a sequence of \( t_n > 0 \), \( \lim_{n \to \infty} t_n = 0 \), then such \( \varphi \) is unique, and for any \( \varphi_0 \in L^p(\Omega) \), \( g_1 \leq \varphi_0 \leq g_2 \) a.e. in \( \Omega \), the sequence \( (\varphi_n) \) defined recursively by

\[
\varphi_n(x) = h(x, \varphi_{n-1}[f(x)]) \quad \text{a.e. for } x \in \Omega; \ n \in \mathbb{R},
\]

converges to \( \varphi \) in the norm \( \| \cdot \|_p \).

**Proof.** Put

\[
C := \{ \varphi \in L^p(\Omega) : g_1 \leq \varphi \leq g_2 \text{ a.e. in } \Omega \}.
\]

It is easy to see that \( C \) is a nonempty, convex and closed subset of \( L^p(\Omega) \). Define the mapping \( T \) on \( C \) by

\[
T(\varphi)(x) := h(x, \varphi[f(x)]), \quad x \in \Omega.
\]

Take an arbitrary \( \varphi \in C \). Then, in view of Carathéodory theorem [2], conditions (ii) imply that the function \( T(\varphi) \) is Lebesgue measurable. Since \( g_1 \leq \varphi \leq g_2 \) a.e. in \( \Omega \) we have, for a.e. \( x \in \Omega \)

\[
g_1[f(x)] \leq \varphi[f(x)] \leq g_2[f(x)],
\]

whence, in view of condition (iii),

\[
g_1(x) \leq h(x, \varphi[f(x)]) \leq g_2(x)
\]

for a.e. \( x \in \Omega \), that is \( T(\varphi) \in C \), which proves that \( T \) maps \( C \) into itself.

Take arbitrary \( \varphi_1, \varphi_2 \in C \). Making use in turn of: the definition of \( T \); (14) (we use here the measurability of \( \alpha \) and \( \beta \)); (15); the theorem on change of the variables under integral (see Łojasiewicz [10]), the inclusion \( f(\Omega) \subset \Omega \); an obvious equality; the assumption that the Lebesgue measure of \( \Omega \) is 1 and
the Jensen integral inequality for the concave function (16); and the definition of the norm \(\|\cdot\|_p\), we obtain
\[
\|T(\phi_1) - T(\phi_2)\|_p = \left( \int_{\Omega} |h(x, \phi_1[f(x)]) - h(x, \phi_2[f(x)])|^p \, dx \right)^{1/p} \\
\leq \left( \int_{\Omega} |\alpha(x)\beta(|\phi_1[f(x)] - \phi_2[f(x)]|)|^p \, dx \right)^{1/p} \\
\leq \left( \int_{\Omega} |J_f(x)||\beta(|\phi_1[f(x)] - \phi_2[f(x)]|)|^p \, dx \right)^{1/p} \\
= \left( \int_{f(\Omega)} [\beta(|\phi_1(x) - \phi_2(x)|)]^p \, dx \right)^{1/p} \\
\leq \left( \int_{\Omega} [\beta(|\phi_1(x) - \phi_2(x)|)]^p \, dx \right)^{1/p} \\
= \left( \int_{\Omega} \left[ \beta \left( \left| \phi_1(x) - \phi_2(x) \right|^p \right) \right]^p \, dx \right)^{1/p} \\
\leq \beta \left( \|\phi_1 - \phi_2\|_p \right),
\]
which, taking into account (17), proves that \(T\) satisfies the conditions of Theorem 1. Since \((L^p(\Omega), \|\cdot\|_p)\) is a uniformly convex Banach space, all the assumptions of Theorem 1 are satisfied. Consequently, there is a function \(\phi \in C\) such that \(\phi = T(\phi)\).

If \(\beta(t_n) \neq t_n\) for a sequence of \(t_n > 0\) such that \(\lim_{n \to \infty} t_n = 0\), then, by the concavity of \(\beta\), it is increasing and
\[
\beta(t) < t, \quad t > 0.
\]
Since \(\lim_{n \to \infty} \beta^n(t) = 0\) for every \(t > 0\), the “moreover” result follows from Theorem 1.2 in [14].

In one-dimensional case, if \(\beta = \text{id}_{[0, \infty)}\) and \(p \geq 1\), the theory of \(L^p\)-solutions of the considered functional equation simplifies. Namely, from [14], Corollary 3.1 and Theorem 3.2, we have the following

**Remark 3.** [14] Let \(\Omega = (0, a)\) where \(0 < a \leq \infty\), and \(p \geq 1\). Assume that:

- \(f : \Omega \to \Omega\) is absolutely continuous and \(0 < f(x) < x\), \(x \in \Omega\);
- \(h : \Omega \times \mathbb{R} \to \mathbb{R}\) is such that for every \(y \in \mathbb{R}\) the function \(\Omega \ni x \mapsto h(x, y)\) is Lebesgue measurable, the function \(\mathbb{R} \ni y \mapsto h(x, y)\) is continuous for almost all \(x \in \Omega\);
- moreover, for some \(x_0 \in \Omega\) and a function \(\alpha : (0, x_0) \to [0, \infty)\); we have \(|h(x, y_1)| \leq \alpha(x) |y_1 - y_2|, \quad x \in (0, x_0), \ y_1, y_2 \in \mathbb{R}\.\)
Then
(a) if for some \( x_0 \in \Omega \) we have
\[
\alpha^p(x) \leq f'(x) \quad \text{a.e. in } (0, x_0),
\]
then there exists at most one solution \( \varphi \in L^p(\Omega) \) of (18);
(b) if for some \( x_0 \in \Omega \) and \( c \in [0, 1) \) we have
\[
\alpha^p(x) \leq cf'(x) \quad \text{a.e. in } (0, x_0),
\]
then there exists exactly one solution \( \varphi \in L^p(\Omega) \) of Eq. (18).

5. A refinement of Radamacher’s theorem

Applying Lemma 1 we obtain the following refinement of the classical Radamacher’s theorem (see, for instance [5, Theorem 3.1.6], or [10, p.161]).

**Theorem 4.** Let \( \Omega \subset \mathbb{R}^k \) be an open convex set and \( f : \Omega \to \mathbb{R}^m \) for some \( k, m \in \mathbb{N} \). If there is a function \( \beta : (0, \infty) \to [0, \infty) \) such that
\[
\lim \sup_{t \to 0^+} \frac{\beta(t)}{t} < +\infty,
\]
and
\[
\|f(x) - f(y)\| \leq \beta(\|x - y\|), \quad x, y \in \Omega, \ x \neq y,
\]
where \( \|\cdot\| \) denotes the respective Euclidean norm, then \( f \) is differentiable almost everywhere in \( \Omega \); that is, the points in \( \Omega \) at which \( f \) is not differentiable form a set of Lebesgue measure zero.

**Proof.** Assume first that \( \Omega \) is convex. By Lemma 1, we have
\[
\|f(x) - f(y)\| \leq L\|x - y\|, \quad x, y \in \Omega,
\]
where
\[
L := \lim \inf_{t \to 0^+} \frac{\beta(t)}{t} < +\infty.
\]
In view of Radamacher’s theorem, the function \( f \) is differentiable almost everywhere in \( \Omega \).

To end the proof, it is enough to note that every open set \( \Omega \subset \mathbb{R}^k \) is a countable sum of convex sets. \( \square \)

Similarly, making use of Lemma 2, we obtain the following improvement of Radamacher’s theorem.

**Theorem 5.** Let \( k, m \in \mathbb{N}, \ \Omega \subset \mathbb{R}^k \) be an open convex set and \( f : \Omega \to \mathbb{R}^m \) be continuous. If for some function \( \beta : (0, \infty) \to [0, \infty) \) there is a positive sequence \( (t_n) \) with \( \lim_{n \to \infty} t_n = 0 \), such that
\[
\lim_{n \to \infty} \frac{\beta(t_n)}{t_n} < \infty,
\]
the function \( f \) is such that for every \( n \in \mathbb{N} \) and for all \( x, y \in \Omega \),
\[
\|x - y\| = t_n \implies \|f(x) - f(y)\| \leq \beta(\|x - y\|),
\]
where \( \| \cdot \| \) denotes the Euclidean norm, then \( f \) is differentiable almost everywhere in \( \Omega \).

The results of this section show that condition (i) in Theorem 3 can be replaced by a significantly weaker one.

**Author contributions** The manuscript wrote JM.

**Declarations**

**Conflict of interest** The authors declare no competing interests.

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Accepted: September 8, 2022.