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Generalized emptiness formation probability in the six-vertex model

F. Colomo, A. G. Pronko, and A. Sportiello

Abstract. In the six-vertex model with domain wall boundary conditions, the emptiness formation probability is the probability that a rectangular region in the top left corner of the lattice is frozen. We generalize this notion to the case where the frozen region has the shape of a generic Young diagram. We derive here a multiple integral representation for this correlation function.

1. Introduction

The six-vertex model with domain wall boundary conditions [1–3] attracts interest, in particular, for its phase separation and limit shape phenomena [4–11]. These can be studied analytically provided that appropriate correlation functions are known. In this respect, the first results concerned the probabilities of observing various specific configurations near the boundary, see [12–15].

An example of correlation function for configurations away from the boundaries is the emptiness formation probability (EFP) [16], see also [17, 18]. This is a nonlocal correlation function, describing the probability of having the first \( s \) consecutive horizontal edges along a given column, all in a given state. In the thermodynamic limit the EFP has a simple stepwise behaviour, with the jump occurring exactly in correspondence of the phase separation curve (or frozen boundary of the limit shape, or arctic curve) — a property that allowed for the determination of the analytic expression of the arctic curve [19, 20].

In view of a deeper understanding of limit shape phenomena, and to address them on wider settings, it is desirable to extend the above mentioned results to regions of the lattice with more generic shapes. Some preliminary studies in this direction have already shown the presence of two important features, namely, the occurrence of a spatial phase transition in the case of a domain of varying shape [22,23], and the fact that, even in the case of generic domains, arctic curves can be determined from the knowledge of the corresponding boundary correlation functions [24].

In the present paper we introduce a nonlocal correlation function that can provide further advances in these directions. For the six-vertex model on a square domain, it describes the probability of having an \( s \)-tuple of horizontal edges (one edge per row, for the first \( s \) rows, with corresponding column indices forming a weakly ordered sequence), all in a given state. When the horizontal edges are in
the same column, this function reduces to the EFP. We thus call it *generalized emptiness formation probability* (GEFP).

To compute the GEFP, we use the quantum inverse scattering method [25, 26]. In the derivation, we follow the method developed in [16] for calculating the EFP, see also [18]. Here we provide the result in the form of a multiple integral representation, Eq. (5.6), of which Eq. (5.17) in [16] is a particular case. The obtained representation is reminiscent of analogous multiple integral representations for correlation functions in quantum spin chains [27–31], asymmetric simple exclusion process [32–34], and stochastic six-vertex model [35].

### 2. Definition of the GEFP

In this section we recall the definition of the six-vertex model with domain wall boundary conditions, and introduce the GEFP.

We consider the six-vertex model on a square lattice formed by the intersection of \(N\) horizontal and \(N\) vertical lines (the \(N \times N\) lattice). We use the standard formulation of the model in terms of configurations of arrows pointing along the edges of the lattice, and subjected to the ice rule, namely at each lattice site (vertex) there are exactly two incoming and two outgoing arrows. The six allowed vertex configurations of arrows and the corresponding Boltzmann weights are shown in Fig. 1 (see, e.g., [36, 37], for further details). The domain wall boundary conditions mean that all arrows on the left and right boundaries are outgoing, while all arrows on the top and bottom boundaries are incoming.

The partition function is defined as follows:

\[
Z_N = \sum_{\mathcal{C}} W(\mathcal{C}), \quad W(\mathcal{C}) = a^{n_1+n_2}b^{n_3+n_4}c^{n_5+n_6}.
\]  

(2.1)

Here, \(\mathcal{C}\) is a configuration of the six-vertex model with domain wall boundary conditions, and \(n_i = n_i(\mathcal{C}), i = 1, \ldots, 6\), is the number of vertices of type \(i\) in \(\mathcal{C}\), \(\sum_i n_i = N^2\). Let us introduce the parameters

\[
\Delta = \frac{a^2 + b^2 - c^2}{2ab}, \quad t = \frac{b}{a}. \tag{2.2}
\]

The function \(Z_N/(a^{N(N-1)}c^N)\) is a polynomial in \(b^2/a^2\) and \(c^2/a^2\), and hence, a polynomial in the parameters \(\Delta\) and \(t\). Correlation functions, which can be defined as probabilities of occurrence of certain arrow configurations, are rational functions in \(\Delta\) and \(t\).

We are interested in the probability of observing some specific configuration of arrows on some given set of edges on the \(N \times N\) lattice. For each edge \(e\) of the lattice we define the characteristic function

\[
\chi_e(\mathcal{C}) = \begin{cases} 
1 & \text{if arrow on } e \text{ points left or down} \\
0 & \text{if arrow on } e \text{ points right or up.}
\end{cases} \tag{2.3}
\]

![Figure 1. The six vertices and their weights.](image)
Let us choose $s$ edges, $e_1, \ldots, e_s$, $1 \leq s \leq N$, with edge $e_j$, $j = 1, \ldots, s$, located on the $j$th horizontal line, counting from the top, and between the $r_j$th and $(r_j + 1)$th vertical lines, counting from the right. For reasons that will be apparent below, we require the $r_j$’s to form a weakly increasing sequence,

$$1 \leq r_1 \leq r_2 \leq \cdots \leq r_s \leq N.$$  \hfill (2.4)

We denote by $G_{N,s}^{(r_1, \ldots, r_s)}$ the probability of observing all arrows on the horizontal edges $e_1, \ldots, e_s$ to be pointing left,

$$G_{N,s}^{(r_1, \ldots, r_s)} = \frac{1}{Z_N} \sum_{\mathcal{C}} W(\mathcal{C}) \prod_{j=1}^s \chi_{e_j}(\mathcal{C}),$$  \hfill (2.5)

see Fig. 2b. It is clear that, setting $r_1 = \cdots = r_s = r$, the present definition reduces to that of EFP in \cite{10}. We thus call this $s$-point correlation function GEFP.

The GEFP satisfies some relations, which follow from the definition and properties of the model. Due to the ice-rule, if any $r_j \leq j$, then the probability of the configuration measured by the GEFP vanishes, and therefore

$$G_{N,s}^{(r_1, \ldots, r_s)} > 0, \quad r_j \geq j, \quad j = 1, \ldots, s.$$  \hfill (2.6)

On the other hand, if $r_s = N$, then the arrow on the edge $e_s$ may only point left, due to the domain wall boundary conditions, and

$$G_{N,s}^{(r_1, \ldots, r_s)} \bigg|_{r_s = N} = G_{N,s-1}^{(r_1, \ldots, r_{s-1})},$$  \hfill (2.7)

that is, GEFP reduces to that with $s \mapsto s - 1$.

We also emphasize that, due to the ice rule and domain wall boundary conditions, the GEFP equivalently measures the probability that the vertices at the intersection between the $j$th horizontal line and the $l_j$th vertical line, $l_j > r_j$, $j = 1, \ldots, s$, are all of type 2. In other words, it gives the probability of observing in the top left corner a frozen region with the shape of a Young diagram $\mu_s = (m_1, \ldots, m_s)$, with rows of length $m_j = N - r_j$, $j = 1, \ldots, s$, see Fig. 2b.

Finally, we note that the knowledge of the GEFP gives direct access to the partition function of the six-vertex model on a quite general class of domains on
the square lattice. Specifically, given the set of values \( r_1, \ldots, r_j, j = 1, \ldots, s \), let us consider the domain obtained by removing from the top left corner of the \( N \times N \) lattice the vertices corresponding to the Young diagram \( \mu \), see Fig. 2c. In the considered setting, the modified domain still has boundary conditions of domain wall type, with outgoing arrows on all horizontal external edges, and incoming arrows on all vertical external edges, a feature already discussed in [24], see also [23]. The partition function of the six-vertex model on the modified domain is exactly given, modulo the factor \( a^{\mu_s}/Z_N \), by the GEFP.

### 3. Quantum inverse scattering method calculations

Here we turn to the calculation of the GEFP. The method developed in [16] (see also [18]) in the case of EFP, appears to be applicable to the GEFP as well. It consists of three steps which we briefly expose below: first, evaluate the GEFP for the inhomogeneous model, using the integrability of the six-vertex model, second, take the homogeneous limit in that expression, and, third, rewrite the resulting expression as a multiple integral.

The first step is essentially based on commutation relations for operators entering the quantum monodromy matrix of the six-vertex model (the Yang-Baxter algebra). These relations make it possible to derive certain recurrence relations for the GEFP, which, together with certain initial conditions for the recurrences, can be solved.

The whole procedure of this step is applicable to the inhomogeneous model, whose weights are parameterized by two sets of spectral parameters (rapidity variables) \( \lambda := \{\lambda_1, \ldots, \lambda_N\} \) and \( \nu := \{\nu_1, \ldots, \nu_N\} \), and by the crossing parameter \( \eta \) such that the weights of the \((j, k)\)-vertex are given by

\[
a_{jk} = a(\lambda_j, \nu_k), \quad b_{jk} = b(\lambda_j, \nu_k), \quad c_{jk} = c,
\]

where

\[
a(\lambda, \nu) \equiv \sin(\lambda - \nu + \eta), \quad b(\lambda, \nu) \equiv \sin(\lambda - \nu - \eta), \quad c \equiv \sin 2\eta.
\]

The essential point of this parametrization is that the parameter \( \Delta \) (defined in (2.2)) is independent of the position of the vertex, \( \Delta = \cos 2\eta \). We also denote

\[
\varphi(\lambda, \nu) = \frac{c}{a(\lambda, \nu)b(\lambda, \nu)}, \quad d(\lambda, \nu) = \sin(\lambda - \nu), \quad e(\lambda, \nu) = \sin(\lambda - \nu + 2\eta).
\]

The partition function of the inhomogeneous six-vertex model with domain wall boundary conditions is given by the celebrated Izergin-Korepin formula [24]:

\[
Z_N(\lambda; \nu) = \frac{\prod_{j,k=1}^{N} a(\lambda_j, \nu_k)b(\lambda_j, \nu_k)}{\prod_{1 \leq j < k \leq N} d(\lambda_k, \lambda_j)d(\nu_j, \nu_k) \det\{\varphi(\lambda_j, \nu_k)\}^{(j,k=1,\ldots,N)}},
\]

Originally, the formula (3.4) was proven in [2] by showing that it satisfies certain properties, derived in [1], which completely determine the partition function. Below, we shall often omit to indicate explicitly the dependence on the sets of spectral parameters \( \lambda, \nu \), when no confusion may arise.

Formula (3.4) can also be proven by considering a recurrence relation valid for generic values of the spectral parameters, which follows from repeated application of the Yang-Baxter algebra to reduce the partition function with respect to the weights of a boundary row (or column). Relations of this kind were first proposed in [13] to compute one-point boundary correlation functions. In [16] it was further
observed that these relations can be recurrently applied \( s \) times to compute the EFP. Here our main observation is that the same method works also in case of the GEFP. Denoting

\[
\tilde{G}^{(r_1, \ldots, r_s)}_{N,s} = Z_N G^{(r_1, \ldots, r_s)}_{N,s}
\]

and applying the very same sequence of steps outlined in [16, Section 3], we obtain the following recurrence relation (see also Eq. (4.1) of that paper):

\[
\tilde{G}^{(r_1, \ldots, r_s)}_{N,s}(\lambda; \nu) = c \prod_{k=r_1+1}^{N} a(\lambda_k, \nu_1) \sum_{j=1}^{r_1} \prod_{k=1 \atop k \neq j}^{r_1} b(\lambda_k, \nu_j) e(\lambda_k, \nu_j) \prod_{l=2}^{N} a(\lambda_j, \nu_l) \times \tilde{G}^{(r_2, \ldots, r_s-1)}_{N-1,s-1}(\lambda \setminus \{\lambda_j\}; \nu \setminus \{\nu_j\}).
\]

(3.6)

Just as in [16], in the derivation of the recurrence relation (3.6) it is crucial that the parameters \( \lambda_1, \ldots, \lambda_{r_1} \) are generic, and that \( G^{(r_1, \ldots, r_s)}(\lambda; \nu) \) is totally symmetric under permutations of these parameters. The permutation symmetry is a consequence of the Yang-Baxter algebra, provided that \( r_2, \ldots, r_s \geq r_1 \). We now apply the relation \( s \) times, thus requiring the conditions that \( \lambda_1, \ldots, \lambda_{r_s} \) are generic, and \( r_1 \leq r_2 \leq \ldots \leq r_s \), that is Eq. (2.4). As a result, in the right-hand side we are left with the partition functions on the \((N-s) \times (N-s)\) lattice,

\[
\tilde{G}^{(r_1, \ldots, r_s)}_{N-s,0} = Z_{N-s},
\]

(3.7)

which is known, being given by the expression (3.4). Thus the relation (3.7) provides the initial condition for the recurrence relation (3.6).

As a consequence, relation (3.6) yields an expression for the GEFP in the form of an \( s \)-fold sum of \((N-s) \times (N-s)\) determinants. This sum can be regarded as the result of expanding an \( N \times N \) determinant with respect to \( s \) columns, that leads to the following expression:

\[
G^{(r_1, \ldots, r_s)}_{N,s} = \frac{1}{\det[\varphi(\lambda_j, \nu_k)_{j,k=1,\ldots,N}]} \prod_{j=1}^{s} \frac{\prod_{k=r_j+1}^{N} d(\nu_j, \nu_k)}{\prod_{k=1}^{r_j} a(\lambda_k, \nu_j) \prod_{k=r_j+1}^{N} b(\lambda_k, \nu_j)} \times \det \left[ \begin{array}{c} \exp\{\lambda_j \partial_{\varepsilon_k}\} \quad (k \leq s) \\ \varphi(\lambda_j, \nu_k) \quad (k > s) \\ \end{array} \right]_{j,k=1,\ldots,N} \prod_{1 \leq j < k \leq s} e(\varepsilon_j, \varepsilon_k) \prod_{j=1}^{s} \prod_{k=r_j+1}^{N} d(\lambda_k, \varepsilon_j) \Bigg|_{\varepsilon_1,\ldots,\varepsilon_s = 0}.
\]

(3.8)

The essential part in this expression is the \( N \times N \) determinant involving shift operators \( \exp\{\lambda_j \partial_{\varepsilon_k}\} \). Note also the invariance under the change: \( \exp\{\lambda_j \partial_{\varepsilon_k}\} \mapsto \exp\{\lambda_j - \lambda \partial_{\varepsilon_k}\} \) and \( \varepsilon_j \mapsto \varepsilon_j + \lambda \), where \( \lambda \) is an arbitrary parameter.

4. Homogeneous limit

The second step consists in evaluating the homogeneous limit, \( \lambda_1, \ldots, \lambda_N \to \lambda \) and \( \nu_1, \ldots, \nu_N \to \nu \), in the expression (3.8) obtained from the quantum inverse scattering method calculations.

The Boltzmann weights depend on \( \lambda - \nu \) only, thus we set \( \nu = 0 \) without loss of generality. Due to the above mentioned shift invariance, the homogeneous limit
of the expression (3.8) can be evaluated, see [16] Section 5.1,

\[ G_{r_1, \ldots, r_s}^{(\tau_1, \ldots, \tau_s)} = \frac{(-1)^s N!}{a^s b^{(N-r)s}} \frac{\prod_{j=1}^s (N-j)!}{\det \frac{\partial_i^{j}}{\omega} \frac{\partial_{i+k}^{j+k-2} \varphi}{\varphi}}_{j,k=1, \ldots, N} \times \det \left[ \begin{array}{cc} \partial_j^{k} \varphi & \text{for } k \leq s \\ \partial_j^{k+2} \varphi & \text{for } k > s \end{array} \right]_{j,k=1, \ldots, N, 1 \leq j < k \leq s} \]

\[ \times \prod_{j=1}^s \left( \frac{\sin(\varepsilon_j + \lambda + \eta) \sin(\varepsilon_j + \lambda - \eta)}{\sin(\varepsilon_j - \varepsilon_k + 2\eta)} \right) \]

\[ \times \prod_{j=1}^s \left( \frac{\sin(\varepsilon_j)^N - [\sin(\varepsilon_j - 2\eta)]^r_j}{[\sin(\varepsilon_j + \lambda - \eta)]^N} \right)_{\varepsilon_1, \ldots, \varepsilon_s = 0}, \tag{4.1} \]

where

\[ a = a(\lambda) \equiv a(\lambda, 0), \quad b = b(\lambda) \equiv b(\lambda, 0), \quad \varphi = \varphi(\lambda) \equiv \varphi(\lambda, 0) = c/ab. \tag{4.2} \]

To simplify expression (4.1), we introduce the polynomials

\[ K_n(x) = (-1)^n n! \varphi^{n+1} \frac{\det \left[ \begin{array}{cc} x_j^{k-1} \varphi & \text{for } k = 1 \\ \partial_j^{k+3} \varphi & \text{for } k > 2 \end{array} \right]_{j,k=1, \ldots, n+1}}{\det \frac{\partial_j^{k+2} \varphi}{\varphi}}_{j,k=1, \ldots, n+1} \tag{4.3} \]

In terms of these polynomials the expression (4.1) can be written as an \( s \times s \) determinant,

\[ G_{r_1, \ldots, r_s}^{(\tau_1, \ldots, \tau_s)} = (-1)^s \det \left[ [K_{N-s+j-1}(\partial_{r_i})]_{j,k=1, \ldots, s} \right] \prod_{1 \leq j < k \leq s} \frac{1}{\rho(\varepsilon_j) \rho(\varepsilon_k) \tilde{\omega}(\varepsilon_j) \tilde{\omega}(\varepsilon_k) - 1} \prod_{j=1}^s \left( \omega(\varepsilon_j)^N - [\rho(\varepsilon_j)]^N \right)_{\varepsilon_1, \ldots, \varepsilon_s = 0}, \tag{4.4} \]

where

\[ \omega(\varepsilon) = \frac{a}{b} \frac{\sin \varepsilon}{\sin(\varepsilon - 2\eta)}, \quad \rho(\varepsilon) = \frac{b}{c} \frac{\sin(\varepsilon - 2\eta)}{\sin(\varepsilon + \lambda - \eta)} = \frac{1}{\omega(\varepsilon) - 1}, \tag{4.5} \]

and the tilde stands for the transformation \( \eta \to -\eta \). Note also the relations

\[ \tilde{\omega}(\varepsilon) = \frac{t^2 \omega(\varepsilon)}{2 t \Delta \omega(\varepsilon) - 1}, \quad \tilde{\rho}(\varepsilon) = \frac{1}{1 - \tilde{\omega}(\varepsilon)}, \tag{4.6} \]

where \( \Delta \) and \( t \) are given by (2.2). Relations (4.6) imply that all functions in (4.4) are expressed rationally in terms of the function \( \omega(\varepsilon) \).

5. Multiple integral representation

The third and last step consists in rewriting the expression (4.4) as a multiple contour integral.

As in the case of EFP in [16], to express GEFP as an \( s \)-fold contour integral, we consider a particular boundary correlation function for the model on the \( N \times N \) lattice, namely \( H_N^{(r)} \), which gives the probability of observing the sole vertex of type 5 in the first row from the top, exactly at the \( r \)th site from the right. In [13] it was shown that

\[ H_N^{(r)} = K_{N-1}(\partial_{r}) \omega(\varepsilon)^N - [\rho(\varepsilon)]^N \big|_{\varepsilon = 0}. \tag{5.1} \]
Below, we will use the corresponding generating function,

\[ h_N(z) = \sum_{r=1}^{N} H_N^{(r)} z^{r-1}. \] (5.2)

The following identity plays a crucial role in the derivation of an integral representation for the GEFP: for any function \( f(z) \) regular near the origin,

\[ K_{N-1}(\partial_z f(\omega(\varepsilon))) |_{\varepsilon = 0} = \text{Res}_{z=0} \frac{(z - 1)^{N-1} h_N(z) f(z)}{z^{N}}. \] (5.3)

The proof is based on the fact that the function \( f(z) \), being regular near the origin, can be treated as a polynomial of degree \( N-1 \), since higher powers in \( z \) do not contribute to either sides of the identity (recall that \( \omega(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \)). The identity (5.3) thus reduces to a linear relation in an \( N \) dimensional vector space. For details of the proof, see [16, Section 5.3].

Before applying identity (5.3) to the determinant representation (4.4), let us introduce the multivariate functions

\[ h_{N,s}(z_1, \ldots, z_s) = \frac{\det \left[ \frac{k^{s-1}(z_j - 1)^{s-k} h_{N-k+1}(z_j)}{z_j^{s-k+1} (z_j - z_k)} \right]_{j,k=1,\ldots,s}}{\prod_{1 \leq j < k \leq s} (z_j - z_k)}. \] (5.4)

These functions are symmetric polynomials of degree \( N-1 \) in each of their variables, and satisfy the relation

\[ h_{N,s}(z_1, \ldots, z_s)|_{z_s=1} = h_{N,s-1}(z_1, \ldots, z_{s-1}). \] (5.5)

These functions are closely related to the partially inhomogeneous Izergin-Korepin partition function [16, 20].

Using now the identity (5.3) within the determinant representation (4.4), and recalling relations (4.5) and (4.6), we obtain the following multiple integral representation for the GEFP:

\[ G_{N,s}^{(r_1, \ldots, r_s)} = (-1)^s \oint \cdots \oint \prod_{j=1}^{s} \frac{[t^2 - 2\Delta t] z_j + 1]^{s-j}}{z_j^{r_j} (z_j - 1)^{s-j+1}} \times \prod_{1 \leq j < k \leq s} \frac{z_j - z_k}{z_j z_k - 2\Delta t z_j + 1} h_{N,s}(z_1, \ldots, z_s) \left(\frac{dz}{2\pi i}\right)^{s}. \] (5.6)

Here, the integrations are performed over simple counterclockwise oriented contours surrounding the origin and no other singularity of the integrand.

As a simple check of representation (5.6), we note that it satisfies relations (2.6) and (2.7). Concerning the first relation, let us consider the integrand in the limit \( z_1 \to 0, \ldots, z_s \to 0 \), performed in this order for convenience, keeping at each stage the contribution of leading order in the corresponding variable. In this limit the integrand behaves as \( \prod_{j=1}^{s} z_j^{j-r_j-1} \), and thus the integral vanishes unless \( r_j \geq j, j = 1, \ldots, s \).

Turning to relation (2.7), we observe that for \( r_s \geq N \), the integrand has no pole at infinity in \( z_s \), and thus the corresponding integration contour can be deformed to enclose the poles at \( z_s = (2\Delta t z_j - 1)/(t^2 z_j) \), \( j = 1, \ldots, s-1 \), and at \( z_s = 1 \). The contribution of each of the first \( s-1 \) poles vanishes, due to the property

\[ h_{N,s}(z_1, \ldots, z_s)|_{z_s=\frac{2\Delta t z_j - 1}{t^2 z_j}} \propto z_j, \quad z_j \to 0, \quad j = 1, \ldots, s-1, \] (5.7)
discussed in some detail in appendix A. The property (5.7) implies for the integrand of the remaining $(s - 1)$-fold integral the behaviour $z_j^{r_j - 1}$ as $z_j \to 0$, and thus ensures the vanishing of the corresponding integration. Therefore, we only need to evaluate the contribution of the simple pole at $z_s = 1$. Using the relation (5.5), we reproduce representation (5.6) with $s \to s - 1$, and hence get (2.7).

The representation (5.6) is our main result. It generalizes Eq. (5.17) in [16] for the EFP. Direct comparison shows that the two representations differ only in the simple replacement of the factor $(z_1 \ldots z_s)^r$ in the denominator of the integrand, with $(z_1^r \ldots z_s^r)$, where the $r_j$’s form a weakly increasing sequence, see (2.4). Note that, while this may seem a minor modification of the formula, it actually raises a problem concerning the symmetrization of the integrand with respect to permutations of the integration variables $z_1, \ldots, z_s$.

It is worth emphasizing that symmetrization of the integrand is necessary, for example, to perform a saddle-point analysis of the integral representation (5.6) for large $s$, to study the behaviour of the GEFP in the thermodynamic limit. The symmetrization issue can be fixed, for example, for the choice $r_j = N - s + j$, $j = 1, \ldots, s$, that gives access to the partition function of the six-vertex model on a square domain with a cut-off triangle. Another interesting example corresponds to the choice $r_1 = l$, $r_2 = \cdots = r_s = r$, providing the boundary correlation function for the model in an L-shaped domain, as defined in [22], and thus giving access to the corresponding arctic curve, using the method proposed in [24]. These special cases will be studied in detail elsewhere.

In conclusion, we have introduced the GEFP, a generalization of EFP, in the six-vertex model with domain wall boundary conditions. Our main motivation is that the GEFP is a powerful tool to study the six-vertex model on variously shaped portions of the square lattice. The GEFP can be represented, in particular, as a multiple integral, that is a particularly suitable form to address its asymptotic behaviour in the scaling limit. We believe this will bring further insights on phase separation and limit shape phenomena.

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Appendix A.

Here we prove the property (5.7). For simplicity, we consider the case $s = N$ and take $j = 1$; for generic $s$ and $j$ the same result will follow due to the total symmetry with respect to the variables $z_1, \ldots, z_N$, and to the relation (5.5). We thus need to prove that the function $h_{N,N}(z_1, \ldots, z_N)$ has a simple zero at the point

$$z_N = \frac{2t \Delta z_1 - 1}{t^2 z_1},$$

(A.1)
as $z_1 \to 0$.

In [16] it was shown the function $h_{N,N}(z_1, \ldots, z_N)$ can be expressed in terms of the partition function $Z_N(\lambda) \equiv Z_N(\lambda; \nu)|_{\nu_1=\ldots=\nu_N=0}$ as follows:

$$h_{N,N}(z_1, \ldots, z_N) = \frac{Z_N(\lambda)}{Z_N} \prod_{j=1}^{N} \left[ \frac{a}{a(\lambda_j)} \right]^{N-1}, \quad (A.2)$$

where $Z_N = Z_N(\lambda)|_{\lambda_1=\ldots=\lambda_N=\lambda}$ and

$$z_j = \frac{a(b(\lambda_j))}{b(a(\lambda_j))}, \quad j = 1, \ldots, N. \quad (A.3)$$

We recall that $a \equiv a(\lambda)$, $b \equiv b(\lambda)$, are related to the parameter $t$ used in the main text by $t = b/a$. Due to the Izergin-Korepin formula,

$$Z_N(\lambda) = \frac{\prod_{j=1}^{N}[a(\lambda_j)b(\lambda_j)]^N}{\prod_{n=0}^{N-1} n! \prod_{1 \leq j < k \leq N} d(\lambda_k, \lambda_j)} \det \left( \partial^{-1}_{\lambda_k} \varphi(\lambda_k) \right)_{j,k=1,\ldots,N}, \quad (A.4)$$

Let us now consider the relation $[A.1]$. In terms of the rapidities of the inhomogeneous partition function, it implies that $\lambda_N = \lambda_1 - 2\eta$, see $[A.3]$. Clearly, the function $Z_N(\lambda)|_{\lambda_N=\lambda_1-2\eta}$ is an entire function in $\lambda_1$ and hence the function $h_{N,N}(z_1, \ldots, z_N)|_{z_N = \frac{2(\Delta z_1 + 1)}{\Delta z_1}}$ is entire in $z_1$. Furthermore, since

$$\varphi(\pm \eta + \epsilon) = \frac{1}{\epsilon} \pm \cot 2\eta + O(\epsilon), \quad \epsilon \to 0, \quad (A.5)$$

the first and the last columns of the determinant in $[A.4]$ coincide as $\lambda_1 \to \eta$, and the function $Z_N(\lambda)|_{\lambda_N=\lambda_1-2\eta}$ has a simple zero at the point $\lambda_1 = \eta$. Equivalently,

$$h_{N,N}(z_1, \ldots, z_N)|_{z_N = \frac{2(\Delta z_1 + 1)}{\Delta z_1}} \propto z_1, \quad z_1 \to 0, \quad (A.6)$$

that is exactly the property $[5.7]$.

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