Characterization of uninorms with continuous underlying t-norm and t-conorm by their set of discontinuity points

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Abstract

Uninorms with continuous underlying t-norm and t-conorm are discussed and properties of the set of discontinuity points of such a uninorm are shown. This set is proved to be a subset of the graph of a special symmetric, surjective, non-increasing multifunction. A sufficient condition for a uninorm to have continuous underlying operations is also given. Several examples are included.

Keywords: uninorm, representable uninorm, additive generator, t-norm, t-conorm

1 Introduction

The (left-continuous) t-norms and their dual t-conorms have an indispensable role in many domains [5, 20, 21]. Each continuous t-norm (t-conorm) can be expressed as an ordinal sum of continuous Archimedean t-norms (t-conorms), while each Archimedean t-norm (t-conorm) is generated by an additive generator (see [1, 7]). Generalizations of t-norms and t-conorms
that can model bipolar behaviour are uninorms (see [4, 14, 22]). The class of uninorms is widely used both in theory [12, 18] and in applications [9, 23]. The complete characterization of uninorms with continuous underlying t-norm and t-conorm has been in the center of the interest for a long time, however, only partial results were achieved (see [3, 6, 10, 11, 13, 19]).

In [15] we have introduced ordinal sum of uninorms and in [16] we have characterized uninorms that are ordinal sums of representable uninorms. We would like to characterize all uninorms with continuous underlying functions and obtain a similar representation as in the case of t-norms and t-conorms. In this paper we will show that underlying operations of a uninorm $U$ are continuous if and only if $U$ is continuous on $[0, 1]^2 \setminus R$, where $R$ is the graph of a special symmetric, surjective, non-increasing multi-function and $U$ is in each point $(x, y) \in [0, 1]^2$ either left-continuous or right-continuous, or continuous. We will then continue and in [17] we will show that each uninorm with continuous underlying t-norm and t-conorm can be decomposed into an extended ordinal sum of representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms.

Let us now recall all necessary basic notions.

A triangular norm is a binary function $T: [0, 1]^2 \rightarrow [0, 1]$ which is commutative, associative, non-decreasing in both variables and 1 is its neutral element. Due to the associativity, $n$-ary form of any t-norm is uniquely given and thus it can be extended to an aggregation function working on $\bigcup_{n \in \mathbb{N}} [0, 1]^n$. Dual functions to t-norms are t-conorms. A triangular conorm is a binary function $C: [0, 1]^2 \rightarrow [0, 1]$ which is commutative, associative, non-decreasing in both variables and 0 is its neutral element. The duality between t-norms and t-conorms is expressed by the fact that from any t-norm $T$ we can obtain its dual t-conorm $C$ by the equation

$$C(x, y) = 1 - T(1 - x, 1 - y)$$

and vice-versa.

**Proposition 1**

*Let $t: [0, 1] \rightarrow [0, \infty]$ ($c: [0, 1] \rightarrow [0, \infty]$) be a continuous strictly decreasing (increasing)*
function such that $t(1) = 0$ ($c(0) = 0$). Then the binary operation $T: [0,1]^2 \rightarrow [0,1]$ ( 
$C: [0,1]^2 \rightarrow [0,1]$) given by

$$T(x,y) = t^{-1}(\min(t(0), t(x) + t(y)))$$

$$C(x,y) = c^{-1}(\min(c(1), c(x) + c(y)))$$

is a continuous t-norm (t-conorm). The function $t$ ($c$) is called an additive generator of $T$ ($C$).

An additive generator of a continuous t-norm $T$ (t-conorm $C$) is uniquely determined up to a positive multiplicative constant. Each continuous t-norm (t-conorm) is equal to an ordinal sum of continuous Archimedean t-norms (t-conorms). Note that a continuous t-norm (t-conorm) is Archimedean if and only if it has only trivial idempotent points 0 and 1. A continuous Archimedean t-norm $T$ (t-conorm $C$) is either strict, i.e., strictly increasing on $]0,1]^2$ (on $[0,1]^2$), or nilpotent, i.e., there exists $(x,y) \in ]0,1]^2$ such that $T(x,y) = 0$ ($C(x,y) = 1$). Moreover, each continuous Archimedean t-norm (t-conorm) has a continuous additive generator. More details on t-norms and t-conorms can be found in [1, 7].

A uninorm (introduced in [22]) is a binary function $U: [0,1]^2 \rightarrow [0,1]$ which is commutative, associative, non-decreasing in both variables and have a neutral element $e \in ]0,1[\ (\text{see also [4]}).$ If we take a uninorm in a broader sense, i.e., if for a neutral element we have $e \in [0,1]$, then the class of uninorms covers also the class of t-norms and the class of t-conorms. In order the stress that we assume a uninorm with $e \in ]0,1[\ we \ will \ call \ such \ a \ uninorm \ proper. \ For \ each \ uninorm \ the \ value \ \text{U}(1,0) \in \{0,1\} \ is \ the \ annihilator \ of \ \text{U}. \ A \ uninorm \ is \ called \ conjunctive \ (\text{disjunctive}) \ if \ \text{U}(1,0) = 0 \ (\text{U}(1,0) = 1).$ Due to the associativity we can uniquely define $n$-ary form of any uninorm for any $n \in \mathbb{N}$ and therefore in some proofs we will use ternary form instead of binary, where suitable.

For each uninorm $U$ with the neutral element $e \in [0,1]$, the restriction of $U$ to $[0,e]^2$ is a t-norm on $[0,e]^2$, i.e., a linear transformation of some t-norm $T_U$ on $[0,1]^2$ and the restriction
of \( U \) to \([e, 1]^2\) is a t-conorm on \([e, 1]^2\), i.e., a linear transformation of some t-conorm \( C_U \) on \([0, 1]^2\). Moreover, \( \min(x, y) \leq U(x, y) \leq \max(x, y) \) for all \((x, y) \in [0, e] \times [e, 1] \cup [e, 1] \times [0, e] \).

We will denote the set of all uninorms \( U \) such that \( T_U \) and \( C_U \) are continuous by \( \mathcal{U} \).

From any pair of a t-norm and a t-conorm we can construct the minimal and the maximal uninorm with the given underlying functions.

**Proposition 2**

Let \( T : [0, 1]^2 \rightarrow [0, 1] \) be a t-norm and \( C : [0, 1]^2 \rightarrow [0, 1] \) a t-conorm and assume \( e \in [0, 1] \).

Then the two functions \( U_{\text{min}}, U_{\text{max}} : [0, 1]^2 \rightarrow [0, 1] \) given by

\[
U_{\text{min}}(x, y) = \begin{cases} 
  e \cdot T\left(\frac{x}{e}, \frac{2}{e}\right) & \text{if } (x, y) \in [0, e]^2, \\
  e + (1 - e) \cdot C\left(\frac{x - e}{1 - e}, \frac{y - e}{1 - e}\right) & \text{if } (x, y) \in [e, 1]^2, \\
  \min(x, y) & \text{otherwise}
\end{cases}
\]

and

\[
U_{\text{max}}(x, y) = \begin{cases} 
  e \cdot T\left(\frac{x}{e}, \frac{2}{e}\right) & \text{if } (x, y) \in [0, e]^2, \\
  e + (1 - e) \cdot C\left(\frac{x - e}{1 - e}, \frac{y - e}{1 - e}\right) & \text{if } (x, y) \in [e, 1]^2, \\
  \max(x, y) & \text{otherwise}
\end{cases}
\]

are uninorms. We will denote the set of all uninorms of the first type by \( \mathcal{U}_{\text{min}} \) and of the second type by \( \mathcal{U}_{\text{max}} \).

**Definition 1**

A uninorm \( U : [0, 1]^2 \rightarrow [0, 1] \) is called *internal* if \( U(x, y) \in \{x, y\} \) for all \((x, y) \in [0, 1]^2\).

Moreover, \( U \) is called s-internal if it is internal and there exists a continuous and strictly decreasing function \( v_U : [0, 1] \rightarrow [0, 1] \) such that \( U(x, y) = \min(x, y) \) if \( y < v_U(x) \) and \( U(x, y) = \max(x, y) \) if \( y > v_U(x) \). Finally, \( U \) is called pseudo-internal if \( U \) is internal on \([0, e] \times [e, 1] \cup [e, 1] \times [0, e] \).

The following easy lemma was shown in [15].
Lemma 1
Let \( U : [0, 1]^2 \rightarrow [0, 1] \) be a uninorm such that \( T_U = \min \) and \( C_U = \max \). Then \( U \) is internal.

Similarly as in the case of t-norms and t-conorms we can construct uninorms using additive generators (see [4]).

Proposition 3
Let \( f : [0, 1] \rightarrow [-\infty, \infty] \), \( f(0) = -\infty \), \( f(1) = \infty \) be a continuous strictly increasing function. Then a binary function \( U : [0, 1]^2 \rightarrow [0, 1] \) given by

\[
U(x, y) = f^{-1}(f(x) + f(y)),
\]

where \( f^{-1} : [-\infty, \infty] \rightarrow [0, 1] \) is an inverse function to \( f \), is a uninorm, which will be called a representable uninorm.

Note that if we relax the monotonicity of the additive generator then the neutral element will be lost and by relaxing the condition \( f(0) = -\infty \), \( f(1) = \infty \) the associativity will be lost (if \( f(0) < 0 \) and \( f(1) > 0 \)). In [18] (see also [14]) we can find the following result.

Proposition 4
Let \( U : [0, 1]^2 \rightarrow [0, 1] \) be a uninorm continuous everywhere on the unit square except of the two points \((0, 1)\) and \((1, 0)\). Then \( U \) is representable.

Thus a uninorm \( U \) is representable if and only if it is continuous on \([0, 1]^2 \setminus \{(0, 1), (1, 0)\}\), which completely characterizes the set of representable uninorms.

An ordinal sum of uninorms was introduced in [15]. For any \( 0 \leq a \leq b < c \leq d \leq 1 \), \( v \in [b, c] \), and a uninorm \( U \) with the neutral element \( e \in [0, 1] \) we will use a transformation
\[ f : [0, 1] \rightarrow [a, b[ \cup \{ v \} \cup ]c, d] \text{ given by } \]
\[
\begin{cases}
(b - a) \cdot \frac{x}{e} + a & \text{if } x \in [0, e[ , \\
v & \text{if } x = e, \\
d - \frac{(1-x)(d-c)}{(1-e)} & \text{otherwise}.
\end{cases}
\]

(1)

Then \( f \) is linear on \([0, e]\) and on \([e, 1]\) and thus it is a piece-wise linear isomorphism of \([0, 1]\) to \(([a, b[ \cup \{ v \} \cup ]c, d])\) and a binary function \( U_{v}^{a, b, c, d} : ([a, b[ \cup \{ v \} \cup ]c, d])^2 \rightarrow ([a, b[ \cup \{ v \} \cup ]c, d]) \) given by
\[
U_{v}^{a, b, c, d}(x, y) = f(U(f^{-1}(x), f^{-1}(y)))
\]

(2)
is a uninorm on \(([a, b[ \cup \{ v \} \cup ]c, d])^2\). Note that in the case when \( a = b \) (\( c = d \)) we will transform only the part of the uninorm \( U \) which is defined on \([e, 1]\)^2 \(([0, e[^2]\)). The function \( f \) is piece-wise linear, however, more generally, we can use any increasing isomorphic transformation.

**Proposition 5**

Assume \( e \in [0, 1] \). Let \( K \) be an index set which is finite or countably infinite and let \((a_k, b_k)]_{k \in K}\) be a disjoint system of open subintervals (which can be also empty) of \([0, e]\), such that \( \bigcup_{k \in K} [a_k, b_k] = [0, e] \). Similarly, let \((c_k, d_k[)_{k \in K}\) be a disjoint system of open subintervals (which can be also empty) of \([e, 1]\), such that \( \bigcup_{k \in K} [c_k, d_k) = [e, 1] \). Let further these two systems be anti-comonotone, i.e., \( b_k \leq a_i \) if and only if \( c_k \geq d_i \) for all \( i, k \in K \). Assume a family of uninorms \((U_k)_{k \in K}\) on \([0, 1]^2\) such that if both \([a_k, b_k]\) and \([c_k, d_k]\) are non-empty then \( U_k \) is a proper uninorm, if \([a_k, b_k]\) is non-empty \( U_k \) is either a t-norm or a proper uninorm and if \([c_k, d_k]\) is non-empty then \( U_k \) is either a t-conorm or a proper uninorm, and finally if both \([a_k, b_k]\) and \([c_k, d_k]\) are empty only the value \( U_k(0, 1) \) is interesting. Denote \( B = \{ b_k \mid k \in K \} \setminus \{ a_k \mid k \in K \} \) and \( C = \{ c_k \mid k \in K \} \setminus \{ d_k \mid k \in K \} \). We define a function
n: \( B \rightarrow B \cup C \) given for all \( b_k \in B \) by

\[
n(b_k) = \begin{cases} 
  b_k & \text{if } U_k(1, 0) = 0, \\
  c_k & \text{else.}
\end{cases}
\]

Let the ordinal sum \( U^e = (\langle a_k, b_k, c_k, d_k, U_k \rangle \mid k \in K)^e \) be given by

\[
U^e(x, y) = \begin{cases} 
  y & \text{if } x = e, \\
  x & \text{if } y = e, \\
  (U_k)^{a_k, b_k, c_k, d_k}_{e_k} & \text{if } (x, y) \in ([a_k, b_k] \cup [c_k, d_k])^2, \\
  x & \text{if } y \in [b_k, c_k], x \in [a_k, d_k] \setminus [b_k, c_k], \\
  y & \text{if } x \in [b_k, c_k], y \in [a_k, d_k] \setminus [b_k, c_k], \\
  \min(x, y) & \text{if } (x, y) \in [b_k, c_k]^2 \setminus ([b_k, c_k]^2 \cup \{(b_k, c_k), (c_k, b_k)\}), \\
  \max(x, y) & \text{if } (x, y) \in [b_k, c_k]^2 \setminus ([b_k, c_k]^2 \cup \{(b_k, c_k), (c_k, b_k)\}), \\
  n(b_k) & \text{if } (x, y) = (b_k, c_k) \text{ or } (x, y) = (c_k, b_k), b_k \in B, c_k \in C, \\
  \min(x, y) & \text{if } (x, y) \in \{b_k\} \times [b_k, c_k] \cup [b_k, c_k] \times \{b_k\} \text{ and } b_k \in B, c_k \notin C, \\
  \max(x, y) & \text{if } (x, y) \in \{c_k\} \times [b_k, c_k] \cup [b_k, c_k] \times \{c_k\} \text{ and } b_k \notin B, c_k \in C,
\end{cases}
\]

where \( v_k = c_k \) (\( v_k = b_k \)) if there exists an \( i \in K \) such that \( b_k = a_i \), \( c_k = d_i \) and \( U_i \) is disjunctive (conjunctive) and \( v_k = n(b_k) \) if \( b_k \in B, c_k \in C \), \( v_k = b_k \) if \( b_k \in B, c_k \notin C \), \( v_k = c_k \) if \( b_k \notin B, c_k \in C \), and \((U_k)^{a_k, b_k, c_k, d_k}_{e_k}\) is given by the formula [2]. Then \( U^e \) is a uninorm.

**Example 1**

(i) Assume two representable uninorms \( U_1, U_2: [0, 1]^2 \rightarrow [0, 1] \) with respective neutral elements \( e_1, e_2 \), and their ordinal sum \( U^\frac{1}{2} = ((0, \frac{1}{4}, \frac{3}{4}, \frac{1}{2}, U_1), (\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, U_2))^\frac{1}{2} \). Then if
we denote $U_1^* = (U_1)_v^{0, v, v},$ where $v = \frac{1}{4}$ ($v = \frac{3}{4}$) if $U_2$ is conjunctive (disjunctive), and $U_2^* = (U_2)_\frac{1}{4}^{0, v, v, \frac{3}{4}},$ we can find the structure of $U_1^\frac{1}{2}$ on Figure 1. All points of discontinuity of $U_1^\frac{1}{2}$ except $(0, 1), (0, 1)$ correspond to the transformation of the points $(x, y) \in [0, 1]^2$ such that $U_1(x, y) = e_1.$ For simplicity, we will assume that $U_1(x, 1 - x) = e_1 = \frac{1}{2}$ for all $x \in [0, 1[.$

(ii) Assume $U_1 \in U_{\text{min}}$ and $U_2 \in U_{\text{max}}$ with respective neutral elements $e_1, e_2.$ Then $U_1$ and $U_2$ are ordinal sums of uninorms, $U_1 = ((e_1, e_1, 1, C_{U_1}), (0, e_1, 1, 1, T_{U_1}))^{e_1}$ and $U_2 = ((0, e_2, e_2, 1, T_{U_2}), (0, 0, e_2, 1, C_{U_2}))^{e_2}.$ If all underlying operations are continuous then the set of discontinuity points of $U_1$ is equal to the set $S_1 = \{e\} \times \{e, 0\} \cup \{e\} \times \{e\}$ and the set of discontinuity points of $U_2$ is equal to the set $S_2 = \{e\} \times \{e\} \times \{e\}.$ Both uninorms can be seen on Figure 2.

\[
\begin{array}{ccc}
  & U_1^* & \text{max} \\
  \text{min} & U_1^* & \text{max} \\
  U_1^* & \text{min} & U_1^*
\end{array}
\]

Figure 1: The uninorm $U_1^\frac{1}{2}$ from Example [1(i)]. The oblique lines denote the points of discontinuity of $U_1^\frac{1}{2}.$

More detailed discussion on the ordinal sum of uninorms can be found in [17].
Figure 2: The uninorm $U_1$ (left) and the uninorm $U_2$ (right) from Example 1(ii). The bold lines denote the points of discontinuity of $U_1$ and $U_2$.

2 Characterization of uninorms $U \in \mathcal{U}$ by means of special multi-functions

In this section we will show that for a uninorm $U$ we have $U \in \mathcal{U}$ if and only if $U$ is continuous on $[0, 1]^2 \setminus R$, where $R$ is the graph of a special symmetric, surjective, non-decreasing multi-function $r$ and $U$ is in each point $(x, y) \in [0, 1]^2$ either left-continuous, or right-continuous, or continuous. In the first part we will focus on the necessity part, i.e., we will show that each uninorm $U \in \mathcal{U}$ is continuous on $[0, 1]^2 \setminus R$, where $R$ is the graph of some symmetric, surjective, non-decreasing multi-function $r$. We will also show that $U \in \mathcal{U}$ implies that $U$ is in each point $(x, y) \in [0, 1]^2$ either left-continuous, or right-continuous, or continuous.

2.1 The necessity part

First we will recall a result from [16].

Lemma 2

Each uninorm $U : [0, 1]^2 \rightarrow [0, 1], U \in \mathcal{U}$, is continuous in $(e, e)$.

Next we show that for $x, y \in [0, 1]$ we have $U(x, y) = \min(x, y)$ or $U(x, y) = \max(x, y)$ if $x$ is an idempotent element of $U$. 

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Lemma 3

Let $U : [0, 1]^2 \to [0, 1]$ be a uninorm and let $U \in \mathcal{U}$. If $a \in [0, 1]$ is an idempotent point of $U$ then $U$ is internal on $\{a\} \times [0, 1]$.

PROOF: If $a = e$ the result is obvious. Suppose $a < e$ (the case when $a > e$ is analogous). Since $T_U$ is continuous we have $U(a, x) = \min(a, x)$ if $x \in [0, e]$. Suppose that there exists $y \in ]e, 1]$ such that $U(a, y) = c \in ]a, y[$. Then $U(a, c) = U(a, a, y) = U(a, y) = c$ and if $c \leq e$ then $c = U(a, c) \leq a$ what is a contradiction. Thus $y > c > e$. Then since $C_U$ is continuous there exists a $y_1$ such that $U(c, y_1) = y$. Then, however,

$$U(a, y) = U(a, c, y_1) = U(c, y_1) = y$$

what is again a contradiction. Thus $U$ is internal on $\{a\} \times [0, 1]$. □

For a given uninorm $U : [0, 1]^2 \to [0, 1]$ and each $x \in [0, 1]$ we define a function $u_x : [0, 1] \to [0, 1]$ by $u_x(z) = U(x, z)$ for $z \in [0, 1]$.

Lemma 4

Let $U : [0, 1]^2 \to [0, 1]$ be a uninorm, $U \in \mathcal{U}$, and assume $x \in [0, 1]$. The function $u_x$ is continuous if and only if one of the following conditions:

(i) $u_x(1) < e$,

(ii) $u_x(0) > e$,

(iii) $e \in \text{Ran}(u_x)$

is satisfied.

PROOF: If $e \in \text{Ran}(u_x)$ then there exists a $y \in [0, 1]$ such that $U(x, y) = e$. Since $U$ is monotone continuity of $u_x$ is equivalent with the equality $\text{Ran}(u_x) = [a, b]$ for some $a = U(0, x)$ and $b = U(1, x)$. Assume $c \in [0, 1]$, $c \notin \text{Ran}(u_x)$. Then $U(x, y, c) = c$ and for $z = U(y, c)$ we have $u_x(z) = c$, what is a contradiction. Thus $\text{Ran}(u_x) = [0, 1]$. If $u_x(1) = v < e$ (the case when $u_x(0) > e$ can be shown similarly) then due to monotonicity the
continuity of $u_x$ is equivalent with the equation $\text{Ran}(u_x) = [0, v]$. Assume $w \in [0, v]$ such that $w \notin \text{Ran}(u_x)$. Since $T_U$ is continuous there exists $q \in [0, e]$ such that $U(v, q) = w$, i.e., $U(x, 1, q) = w$ and then $u_x(U(1, q)) = w$ what is a contradiction.

Vice-versa, if $u_x$ is continuous and $u_x(0) < e < u_x(1)$ then evidently $e \in \text{Ran}(u_x)$.

\[\square\]

**Example 2**

For a representable uninorm $U$ the function $u_x$ is continuous for all $x \in [0, 1]$. If $U$ is conjunctive (disjunctive) then $u_0$ ($u_1$) is continuous and $u_1$ ($u_0$) is non-continuous in $0$ ($1$). For a uninorm $U \in \mathcal{U}_{\max}$ ($U \in \mathcal{U}_{\min}$) $u_x$ is continuous for all $x \in [e, 1]$ ($x \in [0, e]$) and $u_x$ is non-continuous in $e$ for all $x \in [0, e]$ ($x \in ]e, 1]$).

**Proposition 6**

Let $U : [0, 1]^2 \rightarrow [0, 1]$ be a uninorm, $U \in \mathcal{U}$. Then for each $x \in [0, 1]$ there is at most one point of discontinuity of $u_x$. Further, if $u_x$ is non-continuous in $y \in [0, 1]$ then $U(x, z) < e$ for all $z < y$ and $U(x, z) > e$ for all $z > y$.

**PROOF:** If $u_x$ is non-continuous then Lemma \[\square\] implies $e \notin \text{Ran}(u_x)$, $u_x(0) < e$ and $u_x(1) > e$. We will denote $f = \sup\{U(x, y) \mid y \in [0, 1], U(x, y) \leq e\}$ and $g = \inf\{U(x, y) \mid y \in [0, 1], U(x, y) \geq e\}$. Note that the inequality $u_x(0) < e$ ($u_x(1) > e$) implies that $f$ is the supremum ($g$ is the infimum) of a non-empty set. Then for each $f_1 < f$ there exists $f_2 \in [0, 1], f_1 < f_2 < f$ such that $U(x, y^f) = f_2$ for some $y^f \in [0, 1]$ and since $T_U$ is continuous there exists a $f_3$ such that $U(f_2, f_3) = f_1$. Thus $U(x, y^f, f_3) = f_1$. Similarly, for each $g_1 > g$ there exists $g_2 \in [0, 1], g_1 > g_2 > g$ such that $U(x, y^g) = g_2$ for some $y^g \in [0, 1]$ and since $C_U$ is continuous there exists a $g_3$ such that $U(g_2, g_3) = g_1$. Thus $U(x, y^g, g_3) = g_1$ and therefore $[0, 1] \setminus \text{Ran}(u_x)$ is a connected set. Since $u_x$ is monotone it has only one point of discontinuity. Also, if $u_x$ is non-continuous in $y \in [0, 1]$ then $U(x, z) < e$ for all $z < y$ and $U(x, z) > e$ for all $z > y$.

\[\square\]

**Proposition 7**

Let $U : [0, 1]^2 \rightarrow [0, 1]$ be a uninorm, $U \in \mathcal{U}$. Then for all $x \in [0, 1]$ the function $u_x$ is either
left-continuous or right-continuous.

**Proof:** Assume \( x \in [0, 1] \). From Proposition 6 we know that \( u_x \) is non-continuous in at most one point, and thus we will suppose that \( u_x \) is non-continuous in the point \( p \in [0, 1] \).

Further, from the proof of the same proposition we know that \([0, 1] \setminus \text{Ran}(u_x)\) is a connected set, i.e., an interval \( I \). Due to the monotonicity \( u_x(p) \) is a border point of the interval \( I \). Then it is evident that if \( u_x(p) = \inf I \) then \( u_x \) is left-continuous and \( u_x(p) < e \), and if \( u_x(p) = \sup I \) then \( u_x \) is right-continuous and \( u_x(p) > e \). \(\square\)

Next we will show that the points of discontinuity of \( u_x \) are non-increasing with respect to \( x \in [0, 1] \).

**Proposition 8**

Let \( U : [0, 1]^2 \to [0, 1] \) be a uninorm, \( U \in \mathcal{U} \). Then if for \( x \in [0, 1] \) the function of \( u_x \) is non-continuous in \( y \) then for all \( x_1 < x \) the function \( u_{x_1} \) is either continuous or non-continuous in \( y_1 \) such that \( y_1 \geq y \).

**Proof:** From the proof of Proposition 6 we see that if \( u_x \) is non-continuous in \( y \) then \( U(x, z) < e \) for all \( z < y \) and \( U(x, z) > e \) for all \( z > y \). Assume that \( u_{x_1} \) is non-continuous on \([0, 1], \) i.e., the previous result implies that it is non-continuous exactly in one point. Denote this point by \( y_1 \). Then the monotonicity implies \( U(x_1, z) < e \) for \( z < y \) and thus \( y_1 \geq y \). \(\square\)

**Corollary 1**

Let \( U : [0, 1]^2 \to [0, 1] \) be a uninorm, \( U \in \mathcal{U} \). If \( u_{x_1} \) is non-continuous in \( y \) and \( u_{x_2} \) is non-continuous in \( y \) for some \( x_1 < x_2 \) then \( u_x \) is non-continuous in \( y \) for all \( x \in [x_1, x_2] \).

**Proof:** Assume \( x \in [x_1, x_2] \). Since \( u_{x_1} \) is non-continuous in \( y \) we have \( U(x_1, z) > e \) for all \( z > y \) and monotonicity gives \( U(x, z) > e \) for all \( z > y \). Since \( u_{x_2} \) is non-continuous in \( y \) we have \( U(x_2, z) < e \) for all \( z < y \) and monotonicity gives \( U(x, z) < e \) for all \( z < y \). Thus \( u_x \) is non-continuous in \( y \). \(\square\)
Example 3
Assume a representable uninorm $U_1[0,1]^2 \rightarrow [0,1]$ with the neutral element $e_1$, a uninorm $U_2 \in U_{\max}$, and their ordinal sum $U^\frac{1}{2} = (\langle 0, \frac{1}{4}, \frac{3}{4}, 1, U_1 \rangle, \langle \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, U_2 \rangle)^\frac{1}{2}$. Then if we denote $U^*_1 = (U_1)^{0, \frac{1}{4}, \frac{3}{4}, 1}$ and $U^*_2 = (U_2)^{\frac{1}{2}, \frac{1}{2}, \frac{3}{4}, 1}$, we can find the structure of $U^\frac{1}{2}$ on Figure 3. Here $u^\frac{1}{4}_x$ is continuous and $u^\frac{3}{4}_y$ is continuous if $U_1$ is conjunctive (disjunctive). In all other cases $u^\frac{1}{2}_x$ is non-continuous. Further, $u^\frac{1}{4}_x$ is non-continuous in $e = \frac{1}{2}$ and $u^\frac{3}{4}_y$ is non-continuous in $\frac{1}{4}$.

![Figure 3: The uninorm $U^\frac{1}{2}$ from Example 3. The oblique and bold lines denote the points of discontinuity of $U^\frac{1}{2}$.](image)

Now we recall a result of [8, Proposition 1].

Proposition 9
Let $f(x,y)$ be a real valued function defined on an open set $G$ in the plane. Suppose that $f(x,y)$ is continuous in $x$ and $y$ separately and is monotone in $x$ for each $y$. Then $f(x,y)$ is (jointly) continuous on the set $G$.

Now we can show the following.

Proposition 10
Let $U : [0,1]^2 \rightarrow [0,1]$ be a uninorm, $U \in U$. If $U$ is non-continuous in $(x_0, y_0) \in [0,1]^2$ then either $u_{x_0}$ is non-continuous in $y_0$ or $u_{y_0}$ is non-continuous in $x_0$ or there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $u_z$ is non-continuous in $x_0$ and $u_v$ is non-continuous in $y_0$ either for all $z \in [y_0, y_0 + \varepsilon_1]$, $v \in [x_0, x_0 + \varepsilon_2]$, or for all $z \in [y_0 - \varepsilon_1, y_0]$, $v \in [x_0 - \varepsilon_2, x_0]$. 

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PROOF: Since $T_U$ and $C_U$ are continuous we have $(x_0, y_0) \in [0, e] \times [e, 1] \cup [e, 1] \times [0, e]$. We will assume $(x_0, y_0) \in [0, e] \times [e, 1]$ (the other case is analogous). From Proposition 9 it follows that if $U$ is non-continuous in $(x_0, y_0) \in [0, 1]^2$ then for all $\delta_1 > 0$ and all $\delta_2 > 0$ there exist $x \in ]x_0 - \delta_1, x_0 + \delta_1[\text{ and } y \in ]y_0 - \delta_2, y_0 + \delta_2[\text{ such that either } u_x \text{ is non-continuous in } y \text{ or } u_y \text{ is non-continuous in } x. \text{ Thus } U \text{ on } [x_0 - \delta_1, x_0 + \delta_1] \times [y_0 - \delta_2, y_0 + \delta_2] \text{ attain values smaller than } e \text{ and bigger than } e \text{ as well. Let } W \text{ be a subset of } [0, 1]^2 \text{ such that } (x, y) \in W \text{ if } U(x_1, y_1) < e \text{ for all } x_1 < x, y_1 < y \text{ and } U(x_2, y_2) > e \text{ for all } x_2 > x, y_2 > y. \text{ Then the set } [x_0 - \delta_1, x_0 + \delta_1] \times [y_0 - \delta_2, y_0 + \delta_2] \cap W \text{ is non-empty for all } \delta_1 > 0 \text{ and all } \delta_2 > 0. \text{ Thus } (x_0, y_0) \in W.

If $u_{x_0}$ is continuous in $y_0$ then there exists an $\varepsilon_1 > 0$ such that either $u_{x_0}(z) < e \text{ for all } z \in [y_0 - \varepsilon_1, y_0 + \varepsilon_1]$ or $u_{x_0}(z) > e \text{ for all } z \in [y_0 - \varepsilon_1, y_0 + \varepsilon_1]$. Similarly, if $u_{y_0}$ is continuous in $x_0$ then there exists an $\varepsilon_2 > 0$ such that either $u_{y_0}(v) < e \text{ for all } v \in [x_0 - \varepsilon_2, x_0 + \varepsilon_2]$ or $u_{y_0}(v) > e \text{ for all } v \in [x_0 - \varepsilon_2, x_0 + \varepsilon_2]$. Since we cannot have both $U(x_0, y_0) < e$ and $U(x_0, y_0) > e$ we have either $u_{y_0}(v) < e \text{ and } u_{x_0}(z) < e \text{ for all } z \in [y_0 - \varepsilon_1, y_0 + \varepsilon_1]$ and all $v \in [x_0 - \varepsilon_2, x_0 + \varepsilon_2]$, or $u_{y_0}(v) > e \text{ and } u_{x_0}(z) > e \text{ for all } z \in [y_0 - \varepsilon_1, y_0 + \varepsilon_1]$ and all $v \in [x_0 - \varepsilon_2, x_0 + \varepsilon_2]$. As these two cases are analogical we will assume $u_{y_0}(v) < e \text{ and } u_{x_0}(z) < e \text{ for all } z \in [y_0 - \varepsilon_1, y_0 + \varepsilon_1]$ and all $v \in [x_0 - \varepsilon_2, x_0 + \varepsilon_2]$. Then $U(x_0, y) < e \text{ for } y \in [y_0 - \varepsilon_1, y_0 + \varepsilon_1] \text{ and } U(x, y_0) < e \text{ for } x \in [x_0 - \varepsilon_2, x_0 + \varepsilon_2]$, however, $U(f, g) > e \text{ for all } f > x_0, g > y_0. \text{ Thus } u_z \text{ is non-continuous in } x_0 \text{ and } u_v \text{ is non-continuous in } y_0 \text{ for all } z \in [y_0, y_0 + \varepsilon_1], \text{ and } v \in \{x_0, x_0 + \varepsilon_2\}. \square$

Example 4

Assume two t-norms $T_1, T_2 : [0, 1]^2 \to [0, 1]$ and a t-conorm $C : [0, 1]^2 \to [0, 1]$, and their ordinal sum $U^4 = \langle \langle \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, T_1 \rangle, \langle \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, 1, C \rangle, \langle 0, \frac{1}{4}, 1, 1, T_2 \rangle \rangle$ (see Figure 4). If we define a
The following result shows that if \( U(a, b) = e \) then \( U \) is continuous in \((a, b)\).

**Proposition 11**

Let \( U : [0, 1]^2 \rightarrow [0, 1] \) be a uninorm, \( U \in \mathcal{U} \). Then if \( U(a, b) = e \) for some \( a, b \in [0, 1] \), \( a < e \) then \( U \) is continuous on \([0, 1]^2 \setminus ([0, a[ \cup ]b, 1])^2 \).

**PROOF:** If \( U(a, b) = e \) then \( a \) and \( b \) are not idempotent points. From Lemma 4 we know that \( u_a \) and \( u_b \) are continuous functions. Next we will show that for all \( x \in ]a, b[ \) there exists a \( u^x \in [0, 1] \) such that \( U(x, u^x) = e \). Assume \( f \in ]a, e[ \) (for \( f \in [e, b[ \) the proof is analogous). Since \( T_U \) is continuous and \( U(a, f) \leq a, U(f, e) = f \) there exists an \( a^f \in [0, e] \).
such that $U(f, a^f) = a$. Then $e = U(a, b) = U(f, a^f, b)$ and if $v^f = U(a^f, b)$ then $U(f, v^f) = e$.

Summarising, we get that all $u_x$ for $x \in [a, b]$ are continuous. Now since $a$ and $b$ are not idempotents we have $U(a, a) = p < a$, $U(b, b) = q > b$ and $U(a, a, b, b) = e$. Thus also all $u_x$ for $x \in [p, q]$ are continuous and then Proposition 9 implies the result. \[\Box\]

The following result follows easily from monotonicity, Proposition 6 and the previous proposition.

**Lemma 5**

Let $U : [0, 1]^2 \rightarrow [0, 1]$ be a uninorm, $U \in \mathcal{U}$. Assume $(x_0, y_0) \in [0, 1]^2$. If there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $u_z$ is non-continuous in $x_0$ and $u_v$ is non-continuous in $y_0$ for all $z \in ]y_0, y_0 + \varepsilon_1]$, $v \in ]x_0, x_0 + \varepsilon_2]$ (for all $z \in ]y_0 - \varepsilon_1, y_0[\), $v \in [x_0 + \varepsilon_2, x_0[\)$ then $U$ is non-continuous in $(x_0, y_0)$.

The following two results show that the set of discontinuity points of a uninorm $U \in \mathcal{U}$ from the set $[0, e] \times [e, 1] ([e, 1] \times [0, e])$ is connected.

**Proposition 12**

Let $U : [0, 1]^2 \rightarrow [0, 1]$ be a uninorm, $U \in \mathcal{U}$. Let $u_{x_1}$ be non-continuous in $y_1$ and $u_{x_2}$ be non-continuous in $y_2$ for $x_1 < x_2 \leq e$ ($e \leq x_1 < x_2$). Then for all $y \in [y_2, y_1]$ either there exists $x^* \in [x_1, x_2]$ such that $u_{x^*}$ is non-continuous in $y$ or there is an interval $[c, d]$, where $y \in [c, d] \subset [0, 1]$, and $p \in [x_1, x_2]$ such that $u_z$ is non-continuous in $p$ for all $z \in [c, d]$.

**PROOF:** If $u_{x_1}$ is non-continuous in $y_1$ and $u_{x_2}$ is non-continuous in $y_2$ for $x_1 < x_2 \leq e$ (the case when $e \leq x_1 < x_2$ is analogous) then $U(x_2, z) < e$ for all $z < y_2$ and $U(x_1, z) > e$ for all $z > y_1$ and monotonicity implies that for all $x \in [x_1, x_2]$ the function $u_x$ is non-continuous in some point $z \in [y_2, y_1]$. Assume the function $g : [x_1, x_2] \rightarrow [y_2, y_1]$ which assigns to $v \in [x_1, x_2]$ a point $w \in [y_2, y_1]$ such that $u_v$ is non-continuous in $w$. Then by Proposition 8 the function $g$ is non-increasing. If $q \in [y_2, y_1] \setminus \text{Ran}(g)$ then by monotonicity there exists a $p \in [x_1, x_2]$ such that $g(d) > q$ if $d < p$ and $g(d) < q$ if $d > p$. Further, since $g$ is monotone there exists an interval $[c, d]$, such that $q \in [c, d] \subset [y_2, y_1] \setminus \text{Ran}(g)$. Then for $z \in [c, d]$
we have $U(z, v) < e$ for all $v < p$ and $U(z, v) > e$ for all $v > p$ thus $u_z$ has a point of
discontinuity in $p$.  \hfill $\square$

Lemma 6

Let $U : [0, 1]^2 \rightarrow [0, 1]$ be a uninorm, $U \in \mathcal{U}$. Let $u_x$ be non-continuous in $y_1$ and $u_{y_2}$ be
non-continuous in $x$ for some $y_1 \neq y_2$. Then for all $y \in [y_1, y_2]$ ($y \in [y_2, y_1]$) the function $u_y$
is non-continuous in $x$.

PROOF: We will assume $y_1 < y_2$ (the case when $y_1 > y_2$ is analogous). Then $U(x, y) > e$
for all $y > y_1$ and $U(z, y) \leq U(z, y_2) < e$ for all $z < x$, $y \leq y_2$. Thus $u_y$ is non-continuous in $x$. \hfill $\square$

Lemma 7

Let $U : [0, 1]^2 \rightarrow [0, 1]$ be a uninorm, $U \in \mathcal{U}$. Assume $x < e$ ($x > e$) such that $u_x$
is continuous on $[0, 1]$ and let $u_y$ be non-continuous in $x$. Then for all $q \in [y, 1]$ ($q \in [0, y]$)
the function $u_q$ is non-continuous in $x$.

PROOF: We will assume $x < e$ (the case for $x > e$ is analogous). If $U(x, z) = e$ for some
$z \in [0, 1]$ then $x, z$ are not idempotents and Proposition 11 implies that $U$ is continuous on
$[a, b] \times [c, d]$ for some $a, b, c, d \in [0, 1]$ such that $x \in ]a, b[ \text{ and } z \in ]c, d[ \text{, i.e., by Proposition}
10 for all $y \in [0, 1]$ the function $u_y$ is continuous in $x$. Since $x < e$ by Lemma 4 we have
$u_x(1) < e$, i.e., $U(x, z) < e$ for all $z \in [0, 1]$. If $u_y$ is non-continuous in $x$ then $U(p, y) > e$
for all $p > x$ and $U(p, y) < e$ for all $p < x$. Assume any $q \in [y, 1]$. Then $U(p, q) \leq U(x, q) < e$ if
$p < x$ and $U(p, q) \geq U(p, y) > e$ if $p > x$, i.e., $u_q$ is non-continuous in $x$. \hfill $\square$

Next we define a multi-function.

Definition 2

A mapping $p : X \rightarrow \mathcal{P}(Y)$ is called a multi-function if to every $x \in X$ it assigns a subset of
$Y$, i.e., $p(x) \subseteq Y$. A multi-function $p$ is called

(i) non-increasing if for all $x_1, x_2 \in X$, $x_1 < x_2$ there is $p(x_1) \geq p(x_2)$, i.e., for all $y_1 \in p(x_1)$
and all $y_2 \in p(x_2)$ we have $y_1 \geq y_2$ and thus Card($p(x_1) \cap p(x_2)$) $\leq 1$,
(ii) symmetric if \( y \in p(x) \) if and only if \( x \in p(y) \).

The graph of a multi-function \( p \) will be denoted by \( G(p) \), i.e., \( (x, y) \in G(p) \) if and only if \( y \in p(x) \).

The following is evident.

**Lemma 8**

A symmetric multi-function \( p: [0, 1] \to \mathcal{P}([0, 1]) \) is surjective, i.e., for all \( y \in Y \) there exists an \( x \in X \) such that \( y \in p(x) \), if and only if we have \( p(x) \neq \emptyset \) for all \( x \in X \). The graph of a symmetric, surjective, non-increasing multi-function \( p: [0, 1] \to \mathcal{P}([0, 1]) \) is a connected line.

**Remark 1**

The graph of a symmetric, surjective, non-increasing multi-function can be divided into connected maximal segments which are either strictly decreasing, or the horizontal segments, or the vertical segments. Note that a horizontal segment corresponds to a closed interval \( Z \) such that there exists a \( y \in [0, 1] \) with \( r(x) = \{y\} \) for all \( x \in \text{int}(Z) \) (where \( \text{int}(Z) \) is the interval \( Z \) without border points) and a vertical segment corresponds to a closed interval \( V \) such that \( r(x) = V \), \( \text{Card}(V) > 1 \), for some \( x \in [0, 1] \). We say that segment corresponding to an interval \( S \) is strictly decreasing if \( y_1 \in r(x_1) \), \( y_2 \in r(x_2) \) for \( x_1, x_2 \in S \), \( x_1 < x_2 \) implies \( y_2 < y_1 \) and \( \text{Card}(r(x)) = 1 \) for all \( x \in \text{int}(S) \). The previous description implies that all horizontal, vertical and strictly decreasing segments correspond to closed intervals.

The previous results can be summarized in the following theorem. First, for any uninorm \( U: [0, 1]^2 \to [0, 1] \), \( U \in \mathcal{U} \) denote \( A = \inf\{x \mid U(x, 0) > 0\} \), \( B = \sup\{x \mid U(x, 1) < 1\} \) and let \( a, d \in [0, 1] \) be such that \( U(x, y) = e \) for some \( y \in [0, 1] \) if and only if \( x \in ]a, d[ \). Then either \( A = 1, B \neq 0 \), or \( A \neq 1, B = 0 \), or \( A = 1, B = 0 \). Note that if \( A = 1, B \neq 0 \), then \( U \) is non-continuous in \((B, 1)\), if \( A \neq 1, B = 0 \), then \( U \) is non-continuous in \((0, A)\), and if \( A = 1, B = 0 \) then \( U \) is non-continuous in \((0, 1)\). The third case is evident. Further we will explain just the second case as the first case is analogous. If \( U(x, 0) > 0 \) for some \( x \in [e, 1] \) then if \( U(x, 0) < e \) we have \( 0 = U(e, 0) \geq U(x, 0, 0) = U(x, 0) \) and thus either \( U(x, 0) = 0 \)
or $U(x, 0) > e$. Therefore $U$ is non-continuous in $(0, A)$. Due to Proposition [11] we then have $0 \leq B \leq a \leq e \leq d \leq A \leq 1$.

**Theorem 1**

Let $U : [0, 1]^2 \to [0, 1]$ be a uninorm, $U \in \mathcal{U}$. Then there exists a symmetric, surjective, non-increasing multi-function $r$ on $[0, 1]$ such that $U$ is continuous on $[0, 1]^2 \setminus R$, where $R = G(r)$. Note that $U$ need not to be non-continuous in all points from $R$.

**Proof:** We will define the set $R^* = \{(x, y) \in [0, 1]^2 \mid U \text{ is non-continuous in } (x, y)\}$. Then due to the commutativity of $U$ the set $R^*$ is symmetric, i.e., $(x, y) \in R^*$ if and only if $(y, x) \in R^*$. If we define a multi-function $r : [0, 1] \to \mathcal{P}([0, 1])$ by

$$r(x) = \begin{cases} 
\{1\} & \text{if } x \in ]0, B[, \\
\{0\} & \text{if } x \in ]A, 1[, \\
[0, B] & \text{if } x = 1, \\
[A, 1] & \text{if } x = 0, \\
\{y \mid U(x, y) = e\} & \text{if } x \in ]a, d[, \\
\{y \mid (x, y) \in R^*\} & \text{otherwise}
\end{cases}$$

then $r$ is a symmetric multi-function. Since $u_x$ is continuous if and only if $x \in [0, B[ \cup ]a, d[ \cup ]A, 1]$ Lemma [8] implies that $r$ is surjective. Moreover, it is evident that if $U$ is non-continuous in $(x_0, y_0)$ then $x_0 \in r(y_0)$.

We will further define the set

$$P = \{(x, y) \in [0, 1]^2 \mid u_x \text{ is continuous in } y \text{ and } u_y \text{ is continuous in } x\}.$$

Assume $x_1 < x_2$ and $y_1 \in r(x_1)$, $y_2 \in r(x_2)$. If $(x_1, y_1), (x_2, y_2) \in R \setminus P$ then Proposition [8] implies $y_1 \geq y_2$. 


Assume \((x_1, y_1) \in P \cap R\). Then Proposition \([10]\) implies that either \((x_3, y_1) \in R \setminus P\) for some \(x_3 \in [0, 1]\), \(x_1 < x_3 < x_2\), or \((x_1, y_3) \in R \setminus P\) for some \(y_3, y_3 < y_1\). Now if \((x_2, y_2) \in R \setminus P\) the case when \((x_3, y_1) \in R \setminus P\) implies by Proposition \([8]\) \(y_1 \geq y_2\). In the case when \((x_1, y_3) \in R \setminus P\) we have \(y_1 > y_3 \geq y_2\). Similarly we can show the case when \((x_2, y_2) \in P \cap R\) and \((x_1, y_1) \in R \setminus P\).

Finally assume \((x_1, y_1), (x_2, y_2) \in P\). Then Proposition \([10]\) implies that either \((x_4, y_2) \in R \setminus P\) for some \(x_4 \in [0, 1]\), \(x_3 < x_4 < x_2 (x_1 < x_4 < x_2)\), or \((x_2, y_4) \in R \setminus P\) for some \(y_4, y_4 > y_2\). Now if \((x_3, y_1) \in R \setminus P\) and \((x_4, y_2) \in R \setminus P\) we have \(y_1 \geq y_2\). If \((x_3, y_1) \in R \setminus P\) and \((x_2, y_4) \in R \setminus P\) we have \(y_1 \geq y_4 > y_2\). If \((x_1, y_3) \in R \setminus P\) and \((x_4, y_2) \in R \setminus P\) we have \(y_1 > y_3 \geq y_4 > y_2\). Thus in all cases \(y_1 \geq y_2\).

We have shown that \(r\) is non-increasing on \([B, a] \cup [d, A]\). Since \(r\) is evidently non-increasing also on \([0, B[ \cup ]a, d[ \cup ]A, 1]\) we see that \(r\) is non-increasing.

\[\square\]

Remark 2

\(U\) need not to be non-continuous in all points of \(R\). From the previous proof we see that \(U\) is continuous in all points from \(\{x\} \times [0, 1]\) for all \(x \in [0, B[ \cup ]a, d[ \cup ]A, 1]\). The symmetric non-increasing multi-function from the previous theorem need not to be unique. The differences can appear on \([a, d]\). However, if we require additionally that \(U(x, y) = e\) implies \((x, y) \in R\) for all \((x, y) \in [0, 1]^2\), such a multi-function is uniquely given and we will call such a multi-function the characterizing multi-function of a uninorm \(U\) for \(U \in \mathcal{U}\).

Example 5

Assume a representable uninorm \(U_1: [0, 1]^2 \rightarrow [0, 1]\) and a continuous \(T:\ [0, 1]^2 \rightarrow [0, 1]\) and a continuous \(C:\ [0, 1]^2 \rightarrow [0, 1]\). For \(e = \frac{1}{2}\) their ordinal sum \(U_{\frac{1}{2}} = ((\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, U_1), (0, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, T), (0, 0, \frac{3}{4}, 1, C))^{\frac{1}{2}}\) is a uninorm, \(U_{\frac{1}{2}} \in \mathcal{U}\). For simplicity we will assume that \(\frac{1}{2}\) is the neutral element of \(U_1\) and that \(U_1(x, 1 - x) = \frac{1}{2}\) for all \(x \in ]0, 1[\). On Figure\([5]\) we can see the characterizing multi-function \(r\) of \(U_{\frac{1}{2}}\) as well as its set of discontinuity.
Figure 5: The uninorm $U_1^2$ from Example 4. Left: the bold lines denote the points of discontinuity of $U_1^2$. Right: the oblique and bold lines denotes the characterizing multi-function of $U_1^2$.

Proposition 13

Let $U: [0,1]^2 \rightarrow [0,1]$ be a uninorm, $U \in U$. Then in each point $(x_0, y_0) \in [0,1]^2$ the uninorm $U$ is either left-continuous or right continuous.

**Proof:** From Proposition 7 we know that for all $x \in [0,1]$ the function $u_x$ is either left-continuous or right continuous. If $(x_0, y_0)$ is the point of continuity of $U$ the claim is trivial. Thus suppose that $(x_0, y_0)$ belongs to the graph of the characterizing multi-function $r$ of $U$. If $U(x_0, y_0) = e$ then by Proposition 11 the uninorm $U$ is continuous in $(x_0, y_0)$ and thus either $U(x_0, y_0) < e$ or $U(x_0, y_0) > e$. If $U(x_0, y_0) < e$ then for all $x \leq x_0$, $y \leq y_0$ also $U(x, y) < e$ and thus $u_x$ is left-continuous in $y$ and $u_y$ is left-continuous in $x$. Now for any $\varepsilon > 0$ there exists $\delta_1 > 0$ such that $|U(x_0 - \delta_1, y_0) - U(x_0, y_0)| < \frac{\varepsilon}{2}$. Since also $u_{x_0-\delta_1}$ is left-continuous in $y_0$ there exists $\delta_2 > 0$ such that $|U(x_0 - \delta_1, y_0 - \delta_2) - U(x_0 - \delta_1, y_0)| < \frac{\varepsilon}{2}$. The monotonicity of $U$ then implies that

$$0 \leq U(x_0, y_0) - U(x_0 - \delta_1, y_0 - \delta_2) = U(x_0, y_0) - U(x_0 - \delta_1, y_0) + U(x_0 - \delta_1, y_0) - U(x_0 - \delta_1, y_0 - \delta_2) < \varepsilon.$$
Taking $\delta = \min(\delta_1, \delta_2)$, by monotonicity of $U$ we have shown that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x \in [x_0 - \delta, x_0]$ and $y \in [y_0 - \delta, y_0]$ we have $|U(x, y) - U(x_0, y_0)| < \varepsilon$, i.e., that $U$ is left-continuous in $(x_0, y_0)$. Similarly, if $U(x_0, y_0) > e$ then $U$ is right-continuous in $(x_0, y_0)$. 

The previous proposition and the construction of the characterizing multi-function $r$ of a uninorm $U$ implies the following.

**Corollary 2**

Let $U : [0, 1]^2 \rightarrow [0, 1]$ be a uninorm, $U \in \mathcal{U}$. Then there exists a symmetric, surjective, non-increasing multi-function $r$ on $[0, 1]$ such that $U$ is continuous on $[0, 1]^2 \setminus R$, where $R = G(r)$ and if $U(x, y) = e$ then $(x, y) \in R$. Moreover, in each point $(x, y) \in [0, 1]^2$ the uninorm $U$ is either left-continuous or right-continuous.

**2.2 The sufficiency part**

In this part we will show that if for a uninorm $U$ there exists a symmetric, surjective, non-increasing multi-function $r$ on $[0, 1]$ such that $U$ is continuous on $[0, 1]^2 \setminus R$, where $R = G(r)$, and $U(x, y) = e$ implies $(x, y) \in R$, then $U \in \mathcal{U}$ if and only if in each point $(x, y) \in [0, 1]^2$ the uninorm $U$ is either left-continuous or right-continuous.

We will denote the set of all uninorms $U : [0, 1]^2 \rightarrow [0, 1]$ such that $U$ is continuous on $[0, 1]^2 \setminus R$, where $R = G(r)$ and $r$ is a symmetric, surjective, non-increasing multi-function such that $U(x, y) = e$ implies $(x, y) \in R$, by $\mathcal{UR}$. First we will show that there exists a uninorm $U \in \mathcal{UR}$ such that $U \notin \mathcal{U}$.
Example 6
Let $U : [0, 1]^2 \rightarrow [0, 1]$ be given by

$$U(x, y) = \begin{cases} 
0 & \text{if } \max(x, y) < e, \\
x & \text{if } y = e, \\
y & \text{if } x = e, \\
\max(x, y) & \text{otherwise}.
\end{cases}$$

Then Proposition 2 implies that $U \in \mathcal{U}_{\text{max}}$ is a uninorm, where the underlying t-norm is the drastic product and the underlying t-conorm is the maximum. This uninorm is non-continuous in points from $\{e\} \times [0, 1] \cup [0, 1] \times \{e\}$. Thus the corresponding multi-function is given by (see Figure 6)

$$r(x) = \begin{cases} 
[e, 1] & \text{if } x = 0, \\
e & \text{if } x \in ]0, e[, \\
[0, e] & \text{if } x = e, \\
0 & \text{otherwise}.
\end{cases}$$

Since $U(x, y) = e$ implies $x = y = e$ we see that $U$ is continuous on $[0, 1]^2 \setminus R$, where $R = G(r)$ and $r$ is a symmetric, surjective, non-increasing multi-function such that $U(x, y) = e$ implies $(x, y) \in R$. However, the drastic product t-norm is not continuous and thus $U \notin \mathcal{U}$.

Assume $U \in \mathcal{UR}$. Then we have two possibilities: either $U(x, y) = e$ implies $x = y = e$ for all $(x, y) \in [0, 1]^2$, or $U(x, y) = e$ for some $(x, y) \in [0, 1]^2$, $x \neq e$. In the second possibility, since $U(e, x) = x$ for all $x \in [0, 1]$ we see that the point $(e, e)$ is an inner point of a strictly decreasing segment of the characterizing multi-function $r$. Thus $T_U (C_U)$ is continuous in all points from $[0, e]^2 ([e, 1]^2)$ except possibly the point $(e, e)$. Then we have the following result.

Lemma 9
Let $T : [0, 1]^2 \rightarrow [0, 1]$ be a t-norm which is continuous on $[0, 1]^2 \setminus \{(1, 1)\}$. Then $T$ is
Figure 6: The uninorm $U$ from Example 6. The bold lines denote the characterizing multifunction $r$ of $U$.

**continuous on** $[0, 1]^2$.

**PROOF:** Assume that $T$ is not continuous in $(1, 1)$. Then there exist two sequences \( \{a_n\}_{n \in \mathbb{N}}, a_n \in ]0, 1[ \) and \( \{b_n\}_{n \in \mathbb{N}}, b_n \in ]0, 1[ \) such that \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = 1 \) and \( \lim_{n \to \infty} T(a_n, b_n) < 1 \).

Since \( T(a_n, b_n) \geq T(\min(a_n, b_n), \min(a_n, b_n)) \) we see that there exists a sequence \( \{c_n\}_{n \in \mathbb{N}}, c_n \in ]0, 1[ \) such that \( \lim_{n \to \infty} c_n = 1 \) and \( \lim_{n \to \infty} T(c_n, c_n) = 1 - \delta < 1 \), for some \( \delta > 0 \). Since $T$ is a t-norm we have \( T(1 - \frac{\delta}{2}, 1) = 1 - \frac{\delta}{2} \) and necessarily \( T(1 - \frac{\delta}{2}, 1 - \frac{\delta}{2}) \leq 1 - \delta \). Since $T$ is continuous on \( [0, 1]^2 \\setminus \{(1, 1)\} \) there exists an \( \varepsilon > 0 \) such that \( T(1 - \frac{\delta}{2}, 1 - \varepsilon) = 1 - \frac{2\delta}{3} \) and monotonicity of $T$ implies \( \varepsilon \leq \frac{\delta}{2} \). Thus

\[
1 - \frac{2\delta}{3} = T(1 - \frac{\delta}{2}, 1 - \varepsilon) \leq T(1 - \varepsilon, 1 - \varepsilon) \leq 1 - \delta,
\]

what is a contradiction. \( \square \)

By duality between t-norms and t-conorms we get the following.

**Lemma 10**

Let $C: [0, 1]^2 \to [0, 1]$ be a t-conorm which is continuous on $[0, 1]^2 \setminus \{(0, 0)\}$. Then $C$ is continuous on $[0, 1]^2$.

From the two previous results we see that if $U \in \mathcal{UR}$ and $U(x, y) = e$ for some $(x, y) \in [0, 1]^2$, we have

\[
1 - \frac{2\delta}{3} = T(1 - \frac{\delta}{2}, 1 - \varepsilon) \leq T(1 - \varepsilon, 1 - \varepsilon) \leq 1 - \delta,
\]

what is a contradiction. \( \square \)

By duality between t-norms and t-conorms we get the following.

**Lemma 10**

Let $C: [0, 1]^2 \to [0, 1]$ be a t-conorm which is continuous on $[0, 1]^2 \setminus \{(0, 0)\}$. Then $C$ is continuous on $[0, 1]^2$.

From the two previous results we see that if $U \in \mathcal{UR}$ and $U(x, y) = e$ for some $(x, y) \in [0, 1]^2$, we have

\[
1 - \frac{2\delta}{3} = T(1 - \frac{\delta}{2}, 1 - \varepsilon) \leq T(1 - \varepsilon, 1 - \varepsilon) \leq 1 - \delta,
\]

what is a contradiction. \( \square \)
[0, 1]^2, x \neq e \text{ then } U \in \mathcal{U}.

Further we will suppose that \( U(x, y) = e \) implies \( x = y = e \) for all \( (x, y) \in [0, 1]^2 \).

From the previous we see that if \( U \not\in \mathcal{U} \) then the intersection between \( R \) and the set \( S = \{e\} \times [0, 1] \cup [0, 1] \times \{e\} \) contains more than one point. This means that commutativity of \( U \) implies that the point \((e, e)\) is the border point between a horizontal (vertical) segment and a vertical (horizontal) segment of the multi-function \( r \). Lemmas 9 and 10 then imply that one of \( T_U \) and \( C_U \) is continuous. We will suppose that \( C_U \) is continuous (the case when \( T_U \) is continuous is analogical). Then \( T_U \) is non-continuous only in points from the set \( S \). We have the following result.

**Lemma 11**

Let \( U : [0, 1]^2 \longrightarrow [0, 1] \) be a uninorm, \( U \in \mathcal{UR}, U \not\in \mathcal{U} \), such that \( C_U \) is continuous on \([0, 1]^2\) and \( U(x, y) = e \) implies \( x = y = e \) for all \( (x, y) \in [0, 1]^2 \). Then there exists a point \( (x, y) \in [0, 1]^2 \) such that \( U \) is neither left-continuous, nor right-continuous in \( (x, y) \).

**PROOF:** Since \( C_U \) is continuous, the point \((e, e)\) is the border point between a horizontal segment and a vertical segment of the multi-function \( r \). Assume that the corresponding horizontal segment corresponds to the interval \([x_0, e]\) for some \( x_0 \in [0, e[ \). Since \( U \) is continuous on \([x_0, e] \times [e, 1] \cup e, 1] \times [x_0, e] \) and \( U(x, y) = e \) implies \( x = y = e \) we see that \( U(x, y) > e \) for all \( x \in [x_0, e], y \in ]e, 1[ \). On the other hand, the neutral element \( e \) and the monotonicity of \( U \) implies \( U(x, y) \in [x, y] \) for all \( x \in [x_0, e], y \in ]e, 1[ \). Thus for all \( x \in ]x_0, e[ \) we have \( \lim_{s \rightarrow e^+} U(x, s) = e \). Therefore on \([x_0, e[ \) the uninorm \( U \) is not right-continuous. Since \( U \not\in \mathcal{U} \) and \( T_U \) is continuous on \([0, 1]^2 \) we see that \( U \) is not left-continuous in some point \((x, e)\) for \( x \in [x_0, e] \). Now similarly as in Lemma 9 we can show that \( U \) is not left-continuous in some point \((x, e)\) for \( x \in [x_0, e[ \). Finally, the neutral element and the monotonicity of \( U \) imply that \( U \) is not left-continuous in some point \((x, e)\) for \( x \in ]x_0, e[ \). Summarising, there exists a point \((x, y) \in [0, 1]^2 \) such that \( U \) is neither left-continuous, nor right-continuous in \((x, y)\). \( \square \)

All previous results can be compiled into the following theorem.
Theorem 2

Let $U : [0, 1]^2 \rightarrow [0, 1]$ be a uninorm, $U \in \mathcal{U} \mathcal{R}$. Then $U \in \mathcal{U}$ if and only if in each point $(x, y) \in [0, 1]^2$ the uninorm $U$ is either left-continuous or right-continuous.

Corollary 3

Let $U : [0, 1]^2 \rightarrow [0, 1]$ be a uninorm. Then $U \in \mathcal{U}$ if and only if $U \in \mathcal{U} \mathcal{R}$ and in each point $(x, y) \in [0, 1]^2$ the uninorm $U$ is either left-continuous or right-continuous.

3 Conclusions

We have shown that a uninorm with continuous underlying t-norm and t-conorm is continuous on $[0, 1]^2 \setminus R$, where $R$ is the graph of some symmetric, surjective, non-increasing multi-function. On the other hand, we have shown also a sufficient condition for a uninorm to have continuous underlying operations. In the follow up paper [17] we will employ this result and using the characterizing multi-function of a uninorm we will show that each uninorm with continuous underlying t-norm and t-conorm can be decomposed into an extended ordinal sum of representable uninorms, continuous Archimedean t-norms, continuous Archimedean t-conorms and internal uninorms. Thus these two papers together offer a complete characterization of uninorms from $\mathcal{U}$, i.e., of uninorms with continuous underlying t-norm and t-conorm. The applications of these results are expected in all domains where uninorms are used.

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