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GLOBAL WELL-POSEDNESS OF 2D HYPERBOLIC PERTURBATION OF THE NAVIER-STOKES SYSTEM IN A THIN STRIP

NACER AARACH

Abstract. In this paper, we study a hyperbolic version of the Navier-Stokes equations, obtained by using the approximation by relaxation of the Euler system, evolving in a thin strip domain. The formal limit of these equations is a hyperbolic Prandtl type equation, our goal is to prove the existence and uniqueness of a global solution to these equations for analytic initial data in the tangential variable, under a uniform smallness assumption. Then we justify the limit from the anisotropic hyperbolic Navier-Stokes system to the hydrostatic hyperbolic Navier-Stokes system with small analytic data.

1. INTRODUCTION

In fluid mechanics, the Navier-Stokes equations are nonlinear partial differential equations that describe the motion of Newtonian fluids, this equation have been a tremendous topic of research since their introduction in the 30s. In our paper, we considered a hyperbolic perturbation of the incompressible Navier Stokes equations in $\mathbb{R} \times (0, \epsilon)$. We studied this system on a thin striped domain and provided it with no-slip boundary conditions. We denote $S^\epsilon = \{(x, y) \in \mathbb{R}^2 : 0 < y < \epsilon\}$ where $\epsilon$ is the width of the strip. Our system is of the following form:

$$\begin{cases}
\tau \partial^2_t U^{(\tau, \epsilon)} + \partial_t U^{(\tau, \epsilon)} + U^{(\tau, \epsilon)} \cdot \nabla U^{(\tau, \epsilon)} \\
- \epsilon^2 \Delta U^{(\tau, \epsilon)} + \nabla P^{(\tau, \epsilon)} = 0,
\end{cases} \text{ in } S \times [0, \infty[ \tag{1.1}$$

where

$$U^{(\tau, \epsilon)}(t, x, y) = \begin{pmatrix} U_1^{(\tau, \epsilon)}(t, x, y), U_2^{(\tau, \epsilon)}(t, x, y) \end{pmatrix}$$

denotes the velocity of the fluid and $P^{(\tau, \epsilon)}(t, x, y)$ the scalar pressure function, which guarantees the divergence-free property of the velocity field $U^{(\tau, \epsilon)}$. The system (1.1) is complemented by the no-slip boundary condition

$$U^{(\tau, \epsilon)}_t |_{y=0} = 0 \text{ and } U^{(\tau, \epsilon)}_t |_{y=\epsilon} = 0 \text{ in } S \times [0, \infty[.$$

Here, in the equation of the velocity, the Laplacian is $\Delta = \partial_x^2 + \partial_y^2$.

The hyperbolic Navier-Stokes equations in $\mathbb{R}^2$ space, presented here has various justifications. The system (1.1) derived by the Cattaneo approximation in 1949 for the study of the heat equation (see [8,9] ) and others (Chester, Vernotte, etc.) proposed the following hyperbolic model.

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\begin{equation}
\frac{1}{c^2} \partial_t^2 \theta + \frac{1}{\beta} \partial_t \theta - \Delta \theta = 0.
\end{equation}

This equation is called the **Telegraph equation**. It has a finite propagation speed and is compatible with the principle of relativity as well as the second law of thermodynamics, therefore it constitutes a satisfactory physical model. Later, in [2,33,34,36], the case of two(or three)-dimensional space. The dissipative hyperbolic Navier-Stokes equation (1.1) is obtained after relaxation of Euler’s equations and a change of scale variables.

This perturbation, seen as a relaxation of Euler’s equations, was considered by Brenier, Natalini, and Puel in [2], they introduced a hyperbolic system of equations, based on a relaxation approximation of the incompressible Navier-Stokes equations, following the scheme described by Jin and Xin in [3].

\begin{equation}
\begin{aligned}
&\partial_t U^\epsilon + \text{div} V^\epsilon = \nabla \cdot Q^\epsilon \\
&\partial_t V^\epsilon + \frac{\tau^2}{\tau} \nabla U^\epsilon = -\frac{1}{\tau} (V^\epsilon - U^\epsilon \otimes U^\epsilon) \\
&\text{div} U^\epsilon = 0 \\
&(U^\epsilon, V^\epsilon)_{t=0} = (U_0^\epsilon, V_0^\epsilon)
\end{aligned}
\end{equation}

In their work they proved global existence and uniqueness for the perturbed Navier-Stokes equation with initial data in \( H^2(T^2)^2 \times H^1(T^2)^2 \), where \( T^2 \) is the periodic square \( \mathbb{R}^2/\mathbb{Z}^2 \). Moreover, they proved the convergence of the solution to perturbed Navier-Stokes towards a smooth solution to Navier-Stokes.

Later this equation has been considered by Paicu and Raugel in [33,34]. In their work, they proved also a global existence and uniqueness result with significantly improved regularity for the initial data, when \( \tau \) is small enough. In fact, they only need the regularity in \( H^1(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2)^2 \). Also Hachicha in [36], she get a global result of existence and uniqueness of the perturbed Navier-Stokes in two and three space dimensions and under suitable smallness assumptions on the initial data in the space \( H^{2+\delta} \cap H^{2-1+\delta}(\mathbb{R}^n)^n \) where \( n = 2, 3 \). Moreover, for all positive time \( T \), she proved the convergence to perturbed Navier-Stokes towards solutions to the Navier-Stokes system (NS) with initial data in \( H^{2-1+s}(\mathbb{R}^n)^n, s > 0 \).

We finally mention a recently result obtained by O. Coulaud, I. Hachicha and G. Raugel in [35]. They considered a hyperbolic quasi-linear version of the Navier-Stokes equation in \( \mathbb{R}^2 \) and proved the existence and uniqueness of solutions to these equations, and exhibit smallness assumptions on the data, under which the solutions are global in time in the 2D case.

The way these authors introduce their system of equations is to take advantage of methods that are usually devoted to the study of numerical schemes, and which can be applied to every conservation law. In this article, we take the problem from another point of view. In order to describe hydro-dynamical flows on the earth, in geophysics, it is usually assumed that vertical motion is much smaller than horizontal motion and that
the fluid layer depth is small compared to the radius of the sphere, thus, they are a good approximation of global atmospheric flow and oceanic flow. The thin-striped domain in the system (1.1) is considered to take into account this anisotropy between horizontal and vertical directions. Under this assumption, it is believed that the dynamics of fluids on large scale tend towards a geostrophic balance (see [24], [25] or [39]).

The purpose of this paper is to show the existence and uniqueness of solutions to (1.1) in the striped domain \( \mathbb{R} \times (0, 1) \), For some analytically small initial data in the tangential variable. To simplify our system we eliminate the \( \tau \)-dependency. To that extend, we perform the re-scaling

\[
U^{(\tau, \epsilon)}(t, X) = \tau^\alpha U^{\epsilon}(\tau^\beta t, X/\sqrt{\tau}), \quad P^{(\tau, \epsilon)}(t, X) = \tau^\alpha P^{\epsilon}(\tau^\beta t, X/\sqrt{\tau}). \quad (1.3)
\]

We replace in system (1.1), we find that \( \alpha = -\frac{1}{2} \), \( \beta = -1 \) and \( \alpha' = -1 \), then our re-scaling 1.3 have the following form

\[
U^{(\tau, \epsilon)}(t, X) = \frac{1}{\sqrt{\tau}} U^{\epsilon}(\frac{t}{\tau}, \frac{X}{\sqrt{\tau}}), \quad P^{(\tau, \epsilon)}(t, X) = \frac{1}{\tau} P^{\epsilon}(\frac{t}{\tau}, \frac{X}{\sqrt{\tau}}). \quad (1.4)
\]

This scaling transforms the \( \tau \)-dependent equations (1.1) into the following system of equations with initial data which depend on \( \tau \):

\[
\begin{cases}
\partial_t^2 U^{\epsilon} + \partial_t U^{\epsilon} + U^{\epsilon} \cdot \nabla U^{\epsilon} - \epsilon^2 \Delta U^{\epsilon} + \nabla P^{\epsilon} = 0, \quad &\text{in } S \times ]0, \infty[ \\
\text{div } U^{(\tau, \epsilon)} = 0, \quad &\text{in } S \times ]0, \infty[ \\
U^{\epsilon}_{/t=0} = \sqrt{\tau} U^{\tau \epsilon}_0 \left( \sqrt{\tau} X \right) = U^{\epsilon}_0, \quad &\text{in } S \\
\partial_t U^{\epsilon}_{/t=0} = \tau^{\frac{1}{2}} U^{\tau \epsilon}_1 \left( \sqrt{\tau} X \right) = U^{\epsilon}_1, \quad &\text{in } S \\
U^{\epsilon}/y=0 = U^{\epsilon}/y=1 = 0, \quad &\text{in } S \times ]0, \infty[.
\end{cases} \quad (1.5)
\]

In a formal way, as in [31] and [43], taking into account this anisotropy, we also consider the initial data of the following form,

\[
U^{\epsilon}_{/t=0} = U^{\epsilon}_0 = \left( u_0 \left( x, \frac{y}{\epsilon} \right), \epsilon v_0 \left( x, \frac{y}{\epsilon} \right) \right),
\]

and

\[
U^{\epsilon}_{/t=1} = U^{\epsilon}_1 = \left( u_1 \left( x, \frac{y}{\epsilon} \right), \epsilon v_1 \left( x, \frac{y}{\epsilon} \right) \right).
\]

In this paper, we look for solutions in the form

\[
\begin{cases}
U^{\epsilon}(t, x, y) = \left( u^{\epsilon} \left( t, x, \frac{y}{\epsilon} \right), \epsilon v^{\epsilon} \left( t, x, \frac{y}{\epsilon} \right) \right) \\
P^{\epsilon}(t, x, y) = p^{\epsilon} \left( t, x, \frac{y}{3\epsilon} \right).
\end{cases} \quad (1.6)
\]
Let $\mathcal{S} := \{(x, z) \in \mathbb{R}^2 : 0 < y < 1\}$, we can rewrite the system (1.5) as follows
\[
\begin{cases}
\frac{\partial}{\partial t} u^\epsilon + \partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon + v^\epsilon \partial_y u^\epsilon - \epsilon^2 \partial_x^2 u^\epsilon - \partial_y^2 u^\epsilon + \partial_x p^\epsilon = 0, & \text{in } \mathcal{S} \times [0, \infty[ \\
\partial_x u^\epsilon + \partial_y v^\epsilon = 0, & \text{in } \mathcal{S} \times [0, \infty[ \\
(u^\epsilon, v^\epsilon) |_{t=0} = (u_0, v_0) \quad \text{and} \quad \partial_t (u^\epsilon, v^\epsilon) |_{t=0} = (u_1, v_1), & \text{in } \mathcal{S} \\
\end{cases}
\] (1.7)

Formally taking $\epsilon \to 0$ in the system (1.7), we obtain the following perturbation hydrostatic Navier-Stokes equations,
\[
\begin{cases}
\frac{\partial}{\partial t} U + \partial_t U + u \partial_x U + v \partial_y U - \partial_x^2 U + \partial_y p = 0, & \text{in } \mathcal{S} \times [0, \infty[ \\
\partial_y p = 0, & \text{in } \mathcal{S} \times [0, \infty[ \\
\partial_x U + \partial_y V = 0, & \text{in } \mathcal{S} \times [0, \infty[ \\
U |_{t=0} = U_0, & \quad \text{in } \mathcal{S} \\
\partial_t U |_{t=0} = U_1, & \quad \text{in } \mathcal{S},
\end{cases}
\] (1.8)

where the velocity $U = (u, v)$ satisfy the Dirichlet no-slip boundary condition
\[
(u, v) |_{y=0} = (u, v) |_{y=1} = 0.
\] (1.9)

Now let us state our main results.

The first result obtained in this paper is the global well-posedness of the system (1.8) with small analytic data in the tangential variable. The global well-posedness and the global analyticity of the solutions to the classical 2-D perturbed hydrostatic Navier-Stokes system are well-known (see [33] for instance). We remark that a similar global result seems open for the Prandtl equation, where only a lower bound of the lifespan to the solution was obtained (see [29], [12]).

**Theorem 1.1.** Let $a > 0$ and $s \in \left[\frac{1}{2}, 1\right]$. There exists a constant $c_0 > 0$ sufficiently small, such that, for any data $(u_0, u_1)$ verifying the compatibility condition $\int_0^1 u_0 dy = 0$, we have the smallness condition
\[
\|e^{a|D_x|}(u_0 + u_1)\|_{H^s} + \|e^{a|D_x|} \partial_y u_0\|_{H^s} + \|e^{a|D_x|} u_1\|_{H^s} \leq c_0 a,
\] (1.10)

then the system (1.8) has a unique global solution $u$ satisfying the estimate
\[
\begin{align*}
\left(\frac{1}{2}\|e^{Rt}(u + \partial_t U)\|_{L^\infty_t(\mathcal{H}^s)} + \|e^{Rt} \partial_y u_0\|_{L^\infty_t(\mathcal{H}^s)}\right) + \|e^{Rt}(\partial_t U)\|_{L^2_t(\mathcal{H}^s)} + \|e^{Rt} \partial_y u_0\|_{L^2_t(\mathcal{H}^s)} \\
+ \frac{1}{2}\|e^{Rt}(\partial_t U)\|_{L^\infty_t(\mathcal{H}^s)} \leq C\|e^{a|D_x|} \partial_y u_0\|_{H^s} + C\|e^{a|D_x|}(u_0 + u_1)\|_{H^s} + C\|e^{a|D_x|} u_1\|_{H^s},
\end{align*}
\] (1.11)

where $u_0$ is given by (2.3), and $\mathcal{R}$ is a constant determined by Poincaré inequality for the strip $\mathcal{S}$ (see (3.8)), and the functional spaces will be presented in Section 2.

The second result is the global well-posedness of the perturbed Navier-Stokes system (1.7) with small analytic data in the tangential variable.

Theorem 1.2. Let \( a > 0 \) and \( s \in ]\frac{1}{2}, 1[ \). We assume that our initial that satisfy the following smallness condition
\[
\| e^{a|D_x|} \partial_y (u_0, \epsilon v_0) \|_{\mathcal{H}^s} + \| e^{a|D_x|} \partial_y (u_0, \epsilon v_0) \|_{\mathcal{H}^s} + \| e^{a|D_x|} (u_0 + u_1, \epsilon (v_0 + v_1)) \|_{\mathcal{H}^s} + \| e^{a|D_x|} (u_1, \epsilon v_1) \|_{\mathcal{H}^s} \leq c_1 a,
\] (1.12)
for some \( c_1 \) sufficiently small. Then the system (1.7) has a unique global solution \((u, v)\), so that
\[
\left( \frac{1}{2} \right) \left[ \| e^{R t} (u + \partial_t u, \epsilon (v + \partial_t v)) \|_{L_t^\infty (\mathcal{H}^s)} + \| e^{R t} \partial_y (u, \epsilon v) \|_{L_t^\infty (\mathcal{H}^s)} + \| e^{R t} \partial_y (u, \epsilon v) \|_{L_t^\infty (\mathcal{H}^s)} + \| e^{R t} \partial_x (u, \epsilon v) \|_{L_t^\infty (\mathcal{H}^s)} \right]
+ \frac{1}{2} \left[ \| e^{R t} (\partial_t u, \epsilon \partial_t v) \|_{L_t^\infty (\mathcal{H}^s)} + \| e^{R t} (\partial_t u, \epsilon \partial_t v) \|_{L_t^\infty (\mathcal{H}^s)} + \| e^{R t} \partial_y (u, \epsilon v) \|_{L_t^\infty (\mathcal{H}^s)} \right]
+ \| e^{R t} \partial_x (u, \epsilon v) \|_{L_t^\infty (\mathcal{H}^s)} \leq C \left( \| e^{a|D_x|} \partial_y (u_0, \epsilon v_0) \|_{\mathcal{H}^s} + \| e^{a|D_x|} \partial_x (u_0, \epsilon v_0) \|_{\mathcal{H}^s} \right)
+ \| e^{a|D_x|} (u_1, \epsilon v_1) \|_{\mathcal{H}^s} + \| e^{a|D_x|} (u_0 + u_1, \epsilon (v_0 + v_1)) \|_{\mathcal{H}^s} ,
\] (1.13)
where \((u_\theta, v_\theta)\) is given by (4.1).

The main idea to prove the above two theorems is to control the new unknown \( u_\phi \) defined by (2.3), where \( u \) is the horizontal velocity and \( u_\phi \) is a weighted function of \( u \) in the dual Fourier variable with an exponential function of \((a - \lambda (t)) \| \xi \|\). By the classical Cauchy-Kovalevskaya theorem, one expects the radius of the analytically of the solutions decay in time and so the exponent, which corresponds to the width of the analytical strip, is allowed to vary with time. Using energy estimates on the equation satisfied by \( u_\phi \) and the control of the quantity which describes ” the loss of the analytical radius ”, we shall show that the analytical strip persists global in time. Consequently, our result is a global Cauchy-Kovalevskaya type theorem.

The third result concern the study of the divergence from the scaled anisotropic perturbed Navier-Stokes system (1.7) to the limit system (1.8), so in this theorem, we proved that the convergence is globally in time.

Theorem 1.3. Let \( a > 0 \) and \( s \in ]\frac{1}{2}, 1[ \), and \((u_0, v_0)\) satisfying (1.12). Let \( u_0 \) satisfy \( e^{a|D_x|} (u_0, u_1) \in (\mathcal{H}^s \cap \mathcal{H}^{s+3})^2 \), \( e^{a|D_x|} \partial_y (u_0, u_1) \in (\mathcal{H}^{s+1})^2 \), and there holds the compatibility condition \( \int_0^1 u_0 dy = 0 \) and
\[
\| e^{a|D_x|} (u_0 + u_1) \|_{\mathcal{H}^s} + \| e^{a|D_x|} \partial_y u_0 \|_{\mathcal{H}^s} + \| e^{a|D_x|} u_1 \|_{\mathcal{H}^s} \leq \frac{c_2 a}{2} + \| e^{a|D_x|} (u_0 + u_1) \|_{\mathcal{H}^{s+1}} + \| e^{a|D_x|} \partial_y u_0 \|_{\mathcal{H}^{s+1}} + \| e^{a|D_x|} u_1 \|_{\mathcal{H}^{s+1}},
\] (1.14)
for some \( c_2 \) sufficiently small, then we have
\[
\left( \frac{1}{2} \right) \left[ \| R^1 + \partial_t R^1, \epsilon (R^2 + \partial_t R^2) \|_{L_t^\infty (\mathcal{H}^s)} + \| \partial_y (R^1, \epsilon R^2) \|_{L_t^\infty (\mathcal{H}^s)} + \| \partial_x (R^1, \epsilon R^2) \|_{L_t^\infty (\mathcal{H}^s)} \right]
+ \frac{1}{2} \left[ \| \partial_t R^1, \epsilon \partial_t R^2 \|_{L_t^\infty (\mathcal{H}^s)} + \| \partial_t R^1, \epsilon \partial_t R^2 \|_{L_t^\infty (\mathcal{H}^s)} \right]
+ \| \partial_y (R^1, \epsilon R^2) \|_{L_t^\infty (\mathcal{H}^s)} + \| \partial_y (R^1, \epsilon R^2) \|_{L_t^\infty (\mathcal{H}^s)}
\]
and $v$, the convergence of the system (1.7) to the system (1.8) on a fixed time interval $[0, t]$ given by (6.6).

We remark that without the smallness conditions (1.12) and (1.14), we can not prove the convergence of the system (1.7) to the system (1.8) on a fixed time interval $[0, t]$ for $t < T^*$, where $T^*$ is the lifetime of the solution of the hydrostatic perturbed Navier-Stokes equation with the large initial data $u_0$.

**Organisation of the paper:** Our paper will be divided into several sections as follows. In section 2, we present some basic notions of the Littlewood-Paley Theory and some technical lemmas. In section 3, we prove the global wellposedness of the system (1.8) for small data in the analytic framework. Section 4 is devoted to the study of the system (1.7) and the proof of Theorem 1.2. In section 5 we present some proposition states the propagation for any $H^s$ regularity. In Section 6, we prove the convergence of the system (1.7) towards the system (1.8) when $\epsilon \to 0$. Finally, in the appendix, we give the proofs of some technical estimates.

We end this introduction by the notations that will be used in all that follows. For $f \lesssim g$, we mean that there is a uniform constant $C$, which may be different from line to line, such that $f \leq Cg$. We denote by $\langle f, g \rangle_{L^2}$ the inner product of $f$ and $g$ in $L^2(S)$. Finally, we denote by $(d_q)_{q \in \mathbb{Z}}$ (resp. $d_q(t) \big/_{q \in \mathbb{Z}}$) to be a generic element of $\ell^1(\mathbb{Z})$ so that $\sum_{q \in \mathbb{Z}} d_q = 1$ (resp. $\sum_{q \in \mathbb{Z}} \hat{d}_q(t) = 1$).

**2. Littlewood-Paley Theory and Some technical lemmas**

To introduce the result of this paper, we will recall some elements of the Littlewood-Paley theory and also introduce the function space and technique using for the proof of our result. So we define the dyadic operator in the horizontal variable, (of $x$ variable) and for all $q \in \mathbb{Z}$, we recall from [5] that

$$
\begin{align*}
\Delta^h_q a(x, y) &= \mathcal{F}^{-1}_h (\varphi(2^{-q} |\xi|) \hat{a}(\xi, y)), \\
S^h_q a(x, y) &= \mathcal{F}^{-1}_h (\psi(2^{-q} |\xi|) \hat{a}(\xi, y)).
\end{align*}
$$

where $\varphi$ and $\psi$ are a smooth function such that

$$
\text{supp } \varphi \subset \{ z \in \mathbb{R} / \frac{3}{4} \leq |z| \leq \frac{8}{3} \} \text{ and } \forall \epsilon > 0, \quad \sum_{q \in \mathbb{Z}} \varphi(2^{-q} z) = 1,
$$
supp $\psi \subset \{ z \in \mathbb{R} / |z| \leq \frac{4}{3} \}$ and $\psi(z) + \sum_{q \geq 0} \varphi(2^{-q}z) = 1.$

and

$$\forall q, q' \in \mathbb{N}, |q - q'| \geq 2, \quad \text{supp } \varphi(2^{-q} \cdot) \cap \text{supp } \varphi(2^{-q'} \cdot) = \emptyset.$$ 

And in all that follows, $\mathcal{F}a$ and $\hat{a}$ always denote the partial Fourier transform of the distribution $a$ with respect to the horizontal variable (of $x$ variable), that is, $\hat{a}(\xi, y) = \mathcal{F}_{x \to \xi}(a)(\xi, y).$ We refer to [5] and [6] for a more detailed construction of the dyadic decomposition. Combined the definition of the dyadic operator to

$$\forall z \in \mathbb{R}, \quad \psi(z) + \sum_{j \in \mathbb{N}} \varphi(2^{-j}z) = 1,$$  \tag{2.1}

implies that all tempered distributions can be decomposed with respect to the horizontal frequencies as

$$u = \sum_{q \in \mathbb{Z}} \Delta_q^u.$$ 

We now introduce the function spaces used throughout the paper. As in [31], we define the anisotropic Sobolev-type spaces $\mathcal{H}^s$, $s \in \mathbb{R}$ as follows.

**Definition 2.1.** Let $s \in \mathbb{R}$ and $\mathcal{S} = \mathbb{R} \times ]0, 1[.$ For any $u \in \mathcal{S}'(\mathcal{S})$, i.e., $u$ belongs to $\mathcal{S}'(\mathcal{S})$ and $\lim_{q \to -\infty} \| S_h^q u \|_{L^\infty} = 0$, we set

$$\| u \|_{\mathcal{H}^s} \overset{def}{=} \left( \sum_{q \in \mathbb{Z}} 2^{2qs} \| \Delta_q^h u \|_{L^2}^2 \right)^{\frac{1}{2}}.$$ 

(i) For $s \leq \frac{1}{2}$, we define

$$\mathcal{H}^s(\mathcal{S}) \overset{def}{=} \{ u \in \mathcal{S}'(\mathcal{S}) : \| u \|_{\mathcal{H}^s} < +\infty \}.$$ 

(ii) For $s \in ]k - \frac{1}{2}, k + \frac{1}{2}[$, with $k \in \mathbb{N}^*$, we define $\mathcal{H}^s(\mathcal{S})$ as the subset of distributions $u$ in $\mathcal{S}'(\mathcal{S})$ such that $\partial^k u \in \mathcal{H}^{s-k}(\mathcal{S}).$

For a better use of the smoothing effect given by the diffusion terms, we will work in the following Chemin-Lerner type spaces and also the time-weighted Chemin-Lerner type spaces.

**Definition 2.2.** Let $p \in [1, +\infty]$ and $\mathcal{T} \in ]0, +\infty[.$ Then, the space $\hat{L}^p_T(\mathcal{H}^s(\mathcal{S}))$ is the closure of $C([0, T]; \mathcal{S}(\mathcal{S}))$ under the norm

$$\| u \|_{\hat{L}^p_T(\mathcal{H}^s(\mathcal{S}))} \overset{def}{=} \left( \sum_{q \in \mathbb{Z}} 2^{2qs} \left( \int_0^T \| \Delta_q^h u(t) \|_{L^2}^p \ dt \right)^{\frac{2}{p}} \right)^{\frac{1}{2}},$$

with the usual change if $p = +\infty.$
Definition 2.3. Let \( p \in [1, +\infty] \) and let \( f \in L^1_{loc}(\mathbb{R}_+) \) be a non negative function. Then, the space \( \tilde{L}_t^p(f)(\mathcal{H}^s(S)) \) is the closure of \( C([0, T]; S(S)) \) under the norm

\[
\|u\|_{\tilde{L}_t^p(f)(\mathcal{H}^s(S))} \overset{\text{def}}{=} \left( \sum_{q \in \mathbb{Z}} 2^{2qs} \left( \int_0^t f(t') \|\Delta_q^h u(t')\|_{L^p_t L^2_x}^p \right) \right)^{\frac{1}{p}}.
\]

The following Bernstein lemma gives important properties of a distribution \( u \) when its Fourier transform is well localized. We refer the reader to [11] for the proof of this lemma.

Lemma 2.1. Let \( k \in \mathbb{N}, d \in \mathbb{N}^* \) and \( r_1, r_2 \in \mathbb{R} \) satisfy \( 0 < r_1 < r_2 \). There exists a constant \( C > 0 \) such that, for any \( a, b \in \mathbb{R}, 1 \leq a \leq b \leq +\infty \), for any \( \lambda > 0 \) and for any \( u \in L^a(\mathbb{R}^d) \), we have

\[
supp (\hat{u}) \subset \{ \xi \in \mathbb{R}^d \mid |\xi| \leq r_1 \lambda \} \implies \sup_{|\alpha| = k} \|\partial^\alpha u\|_{L^b} \leq C^k \lambda^{k+d\left(\frac{1}{a} - \frac{1}{b}\right)} \|u\|_{L^a},
\]

and

\[
supp (\hat{u}) \subset \{ \xi \in \mathbb{R}^d \mid r_1 \lambda \leq |\xi| \leq r_2 \lambda \} \implies C^{-k} \lambda^k \|u\|_{L^a} \leq \sup_{|\alpha| = k} \|\partial^\alpha u\|_{L^b} \leq C^k \lambda^k \|u\|_{L^a}.
\]

Finally to deal with the estimate concerning the product of two distribution, we shall frequently use the Bony’s decomposition (see [6] ) in the horizontal variable \((\text{ of the } x \text{ variable }) \) that for \( f, g \) two tempered distribution :

\[
f g = T^h_f g + T^h_g f + R^h(f, g),
\]

where

\[
T^h_f g = \sum_q S^h_{q-1} f \Delta^h_q g, \quad T^h_g f = \sum_q S^h_{q-1} g \Delta^h_q f
\]

and the rest term satisfied

\[
R^h(f, g) = \sum_q \Delta^h_q f \Delta^h_q g \quad \text{with} \quad \Delta^h_q f = \sum_{|q-q'| \leq 1} \Delta^h_{q'} f.
\]

Our main difficulty relies on finding a way to estimate the nonlinear terms, which allows exploiting the smoothing effect given by the above function spaces. Using the method introduced by Chemin in [13] (see also [15], [30] or [31]), for any \( f \in L^2(S) \), we define the following auxiliary function, which allows to control the analyticity of \( f \) in the horizontal variable \( x \),

\[
\begin{cases}
  u_\phi(t, x, y) = e^{\phi(t,D_x)} u(t, x, y) \overset{\text{def}}{=} \mathcal{F}^{-1}_h(e^{\phi(t,\xi)} \hat{u}(t, \xi, y)) \\
  \phi(t, \xi) = (a - \lambda \theta(t))|\xi|.
\end{cases}
\]

where the quantity \( \theta(t) \), which describes the evolution of the analytic band of \( f \), satisfies

\[
\forall t > 0, \dot{\theta}(t) \geq 0 \quad \text{and} \quad \theta(0) = 0.
\]
In what follows, we shall always assume that 
\[ T^* \overset{\text{def}}{=} \sup \left\{ t > 0 : \|u_\phi\|_{H^*} \leq \frac{1}{4C^2} \text{ and } \theta(t) \leq \frac{a}{\lambda} \right\}. \tag{2.5} \]

By virtue of (2.3) for any \( t < T^* \), there holds the following convex inequality
\[ \phi(t, \xi) \leq \phi(t, \xi - \eta) + \phi(t, \eta) \quad \forall \xi, \eta \in \mathbb{R}. \tag{2.6} \]

Before starting the obtained result, we need the following lemma to characterize the product \((fg)\phi\), indeed this product will be useful in all the rest of the paper.

**Lemma 2.2.** Let \( f \in L^2_x, g \in L^2_x \), we define \( f^+ = F^{-1}_x(|F_x(f)|) \) then, we have
\[ |(\widehat{fg})\phi(\xi)| \leq \widehat{f^+ g^+}(\xi) \quad \text{and} \quad \|f^+\|_{L^2_\xi} = \|f\|_{L^2_\xi} \]

**Proof.** Let as consider \( f, \) and \( g \) two functions in \( L^2_\xi \), we have
\[ |(\widehat{fg})\phi(\xi)| = e^{\phi(\xi)}|\widehat{f}(\xi) * \widehat{g}(\eta)| \leq e^{\phi(\xi)} \int |\widehat{f}(\xi - \eta)||\widehat{g}(\eta)|d\eta, \]

By virtue of the definition of the function \( \phi \) we have \( e^{\phi(\xi)} > 0 \) and \( e^{\phi(\xi)} \leq e^{\phi(\xi-\eta)}e^{\phi(\eta)}, \) thus
\[ |(\widehat{fg})\phi(\xi)| \leq \int e^{\phi(\xi-\eta)}|\widehat{f}(\xi - \eta)||e^{\phi(\eta)}||\widehat{g}(\eta)||d\eta \leq \int |\widehat{f}_\phi(\xi - \eta)||\widehat{g}_\phi(\eta)||d\eta \leq |\widehat{f}_\phi| * |\widehat{g}_\phi|(\xi) = \widehat{f^+}_\phi * \widehat{g^+}_\phi = \widehat{f^+ g^+}_\phi(\xi) \]

The second point of the lemma is trivial. \( \square \)

**Corollary 2.1.** For any \( f \) and \( g \) in \( L^2_\xi \), we have
\[ |(Tfg)\phi| \leq (Tf^+g^+)\phi \quad \text{and} \quad |R(f,g)\phi| \leq R(f^+g^+)\phi. \]

We next present the weighted energy estimate for the linear heat equations

**Lemma 2.3.** Let \( f \) and \( g \) two smooth enough functions on \( \mathbb{R} \times (0, 1) \), satisfy the Dirichlet boundary condition, then we have
\[ \frac{d}{dt} \left\langle \Delta^h_q f_\phi, \Delta^h_q g_\phi \right\rangle_{L^2} = \left\langle \Delta^h_q (\partial_t f)_\phi, \Delta^h_q g_\phi \right\rangle + \left\langle \Delta^h_q f_\phi, \Delta^h_q (\partial_q g)_\phi \right\rangle - 2\lambda \hat{\theta}(t) \left\langle \Delta^h_q |D_x|^\frac{1}{2} f_\phi, \Delta^h_q |D_x|^\frac{1}{2} g_\phi \right\rangle, \tag{2.7} \]

In particular if \( f = g \), we obtain that
\[ \frac{1}{2} \frac{d}{dt} \left\| \Delta^h_q f_\phi \right\|^2_{L^2} = \left\langle \Delta^h_q (\partial_t f)_\phi, \Delta^h_q f_\phi \right\rangle_{L^2} - \lambda \hat{\theta}(t) \left\| \Delta^h_q |D_x|^\frac{1}{2} f_\phi \right\|^2_{L^2}, \tag{2.8} \]
The last estimate (2.11) is trivial.

\[ \langle \Delta_q^h \big( \partial_t^2 f \big) \phi, \Delta_q^h(\partial_t f) \phi \rangle_{L^2} = \frac{1}{2} \frac{d}{dt} \| \Delta_q^h(\partial_t f) \phi \|^2_{L^2} + \lambda \dot{\theta}(t)\| \Delta_q^h D_x \frac{1}{2}(\partial_t f) \phi \|^2_{L^2}, \quad (2.9) \]

\[ \langle \Delta_q^h \big( -\partial_t^2 f \big) \phi, \Delta_q^h(\partial_t f) \phi \rangle_{L^2} = \frac{1}{2} \frac{d}{dt} \| \Delta_q^h \partial_t f \phi \|^2_{L^2} + \lambda \dot{\theta}(t)\| \Delta_q^h |D_x|^{\frac{1}{2}} \partial_t f \phi \|^2_{L^2}, \quad (2.10) \]

\[ \langle \Delta_q^h \big( (\partial_t f) \phi \big), \Delta_q^h(\partial_t f) \phi \rangle_{L^2} = \| \Delta_q^h(\partial_t f) \phi \|^2_{L^2}. \quad (2.11) \]

**Proof.** To prove the first assertion, we apply the rules of the derivation of a product, we obtain

\[ \frac{d}{dt} \langle \Delta_q^h f \phi, \Delta_q^h g \phi \rangle_{L^2} = \langle \Delta_q^h \partial_t f \phi, \Delta_q^h g \phi \rangle_{L^2} + \langle \Delta_q^h f \phi, \Delta_q^h \partial_t g \phi \rangle_{L^2} \]

\[ = \langle \Delta_q^h \partial_t f \phi, \Delta_q^h g \phi \rangle_{L^2} - \lambda \dot{\theta}(t) \langle \Delta_q^h |D_x| f \phi, \Delta_q^h g \phi \rangle_{L^2} \]

\[ + \langle \Delta_q^h f \phi, \Delta_q^h (\partial_t g) \phi \rangle_{L^2} - \lambda \dot{\theta}(t) \langle \Delta_q^h f \phi, \Delta_q^h |D_x| g \phi \rangle_{L^2} \]

\[ = \langle \Delta_q^h \partial_t f \phi, \Delta_q^h g \phi \rangle_{L^2} + \langle \Delta_q^h f \phi, \Delta_q^h (\partial_t g) \phi \rangle_{L^2} - 2\lambda \dot{\theta}(t) \langle \Delta_q^h |D_x|^{\frac{1}{2}} f \phi, \Delta_q^h |D_x|^{\frac{1}{2}} g \phi \rangle_{L^2}. \]

By using the rules of the derivation of a product and Parseval equality, we find

\[ \langle \Delta_q^h \big( (\partial_t^2 f) \phi \big), \Delta_q^h(\partial_t f) \phi \rangle_{L^2} \]

\[ = \int \Delta_q^h \big( e^{\theta(t)|D_x|} \partial_t^2 f \big) \Delta_q^h \big( e^{\theta(t)|D_x|} \partial_t f \big) dx 
\]

\[ = \int \Delta_q^h \big( e^{\theta(t)|D_x|} \partial_t^2 f \big) \Delta_q^h \big( e^{\theta(t)|D_x|} \partial_t f \big) d\xi_h 
\]

\[ = \frac{1}{2} \frac{d}{dt} \int |\Delta_q^h \big( e^{\theta(t)|D_x|} \partial_t f \big) |^2 d\xi_h 
\]

\[ + \frac{1}{2} \int 2\lambda \dot{\theta}(t) \varphi(2^{-\eta}\xi)|\xi_h| e^{\theta(t)|\xi_h|} \partial_t \tilde{f}(\xi) \varphi(2^{-\eta}\xi) e^{\theta(t)|\xi_h|} \partial_t \tilde{h}(\xi) d\xi_h 
\]

\[ = \frac{1}{2} \frac{d}{dt} \| \Delta_q^h (\partial_t^2 f) \phi \|^2_{L^2} + \lambda \dot{\theta}(t) \int |\Delta_q^h |D_x|^{\frac{1}{2}} \partial_t f \phi |^2 d\xi_h \]

\[ = \frac{1}{2} \frac{d}{dt} \| \Delta_q^h (\partial_t^2 f) \phi \|^2_{L^2} + \lambda \dot{\theta}(t)\| \Delta_q^h |D_x|^{\frac{1}{2}} (\partial_t f) \phi \|^2_{L^2}. \]

By using integration by parts, we can find the estimate (2.10)

\[ \langle \Delta_q^h \big( -\partial_t^2 f \big) \phi, \Delta_q^h(\partial_t f) \phi \rangle_{L^2} = \langle \Delta_q^h \big( \partial_t f \phi \big), \Delta_q^h(\partial_t f \phi) \rangle_{L^2} \]

\[ = \frac{1}{2} \frac{d}{dt} \| \Delta_q^h \partial_t f \phi \|^2_{L^2} + \lambda \dot{\theta}(t)\| \Delta_q^h |D_x|^{\frac{1}{2}} \partial_t f \phi \|^2_{L^2}. \]

The last estimate (2.11) is trivial. \( \square \)
The proofs of our main result rely on the following lemmas

**Lemma 2.4.** Let \(s \in [\frac{1}{2}, 1], T > 0\) and \(A, B\) and \(C\) be smooth enough functions on \(\mathbb{R} \times (0, 1)\), let \(\phi\) be defined as in (2.3) with \(\hat{\theta}(t) = \|\partial_y A_\phi(t)\|_{\mathcal{H}^s}\). There exist \(C \geq 1\) such that, for any \(t > 0\), \(\phi(t, \xi) > 0\) and for any \(B \in \tilde{L}^2_{L,\hat{\theta}(t)}(\mathcal{H}^{s+\frac{1}{2}})\) and \(C \in \tilde{L}^2_{L,\hat{\theta}(t)}(\mathcal{H}^{s+\frac{1}{2}})\), we have

\[
\sum_{q \in \mathbb{Z}} 2^{2qs} \int_0^t \|e^{Rt'} \Delta^h_q(A\partial_x B)\phi, e^{Rt'} \Delta^h_q C\phi\|_{L^2} \, dt' \lesssim C \|e^{Rt} B\phi\|_{L^2_{L,\hat{\theta}(t)}(\mathcal{H}^{s+\frac{1}{2}})} \|e^{Rt} C\phi\|_{L^2_{L,\hat{\theta}(t)}(\mathcal{H}^{s+\frac{1}{2}})}.
\]  

(2.12)

**Proof.** As in [5], using Bony’s homogeneous decomposition into para-products \(A\partial_x B\) in the horizontal variable and remainders as in Definition of a tempered distribution, we can write

\[A\partial_x B = T^h_{A}\partial_x B + T^h_{\partial_x B} A + R^h(A, \partial_x B)\]

where,

\[T^h_{A}\partial_x B = \sum_{q} S^h_{q-1} A \Delta^h_q \partial_x B \quad \text{and} \quad R^h(A, \partial_x B) = \sum_{|q'| \leq 1} \Delta^h_q A \Delta^h_{q'} \partial_x B.
\]

We have the following bound of I

\[
\int_0^t \|e^{Rt'} \Delta^h_q(A\partial_x B)\phi, e^{Rt'} \Delta^h_q C\phi\|_{L^2} \, dt' \leq M_{1,q} + M_{2,q} + M_{3,q},
\]

where

\[
M_{1,q} = \int_0^t \|e^{Rt'} \Delta^h_q(T^h_{A}\partial_x B)\phi, e^{Rt'} \Delta^h_q C\phi\|_{L^2} \, dt',
\]

\[
M_{2,q} = \int_0^t \|e^{Rt'} \Delta^h_q(T^h_{\partial_x B} A)\phi, e^{Rt'} \Delta^h_q C\phi\|_{L^2} \, dt',
\]

\[
M_{3,q} = \int_0^t \|e^{Rt'} \Delta^h_q(R^h(A, \partial_x B))\phi, e^{Rt'} \Delta^h_q C\phi\|_{L^2} \, dt'.
\]

We start by getting the estimate of the first term \(M_{1,q}\), for that we need to use the support properties given in [6], Proposition 2.10 and the definition of \(T^h_{A}\partial_x B\), and also the Lemma 2.2, we infer

\[
M_{1,q} \leq \sum_{|q' - q| \leq 4} \int_0^t e^{2Rt'} \|S^h_{q-1} A_\phi(t')\|_{L^\infty} \|\Delta^h_q \partial_x B_\phi(t')\|_{L^2} \|\Delta^h_q C_\phi(t')\|_{L^2} dt'.
\]  

(2.13)

While it follow from the Poincaré inequality on the interval \(\{0 < y < 1\}\), we have the inclusion \(\hat{H}^1_y \hookrightarrow L^\infty_y\) and

\[
\|\Delta^h_q A_\phi(t')\|_{L^\infty} \lesssim 2^\frac{q}{2} \|\Delta^h_q A_\phi(t')\|_{L^2_y(L^\infty)} \lesssim 2^\frac{q}{2} \|\Delta^h_q \partial_y A_\phi(t')\|_{L^2} \lesssim d_q(A_\phi) \|\partial_y A_\phi(t')\|_{\mathcal{H}^s},
\]

(2.14)
with \( s > \frac{1}{2} \). Here and in all that follows, we always denote \((d_q(t))_{q \in \mathbb{Z}}\) to be a generic element of \(l^1(\mathbb{Z})\) \(\sum d_q(t) = 1\). Then,
\[
\|S_{q-1}^h A_\phi(t')\|_{L^\infty} \lesssim \|\partial_y A_\phi(t')\|_{H^s},
\]
then, we replace this result in our estimate (2.13), and combining with Hölder inequality, imply that
\[
M_{1,q} \lesssim \sum_{|q-q'| \leq 4} \int_0^t \|\partial_y A_\phi(t')\|_{H^s} e^{Rt'} \|\Delta_q^h \partial_x B_\phi(t')\|_{L^2} e^{Rt'} \|\Delta_q^h C_\phi(t')\|_{L^2} dt'.
\]
Similarly, by Lemma 2.2 and considering the support properties to the Fourier transform given in \([6], \text{Proposition 2.10}\) of the terms in \(T_{\partial_x B}^h A\), we obtain
\[
M_{1,q} \lesssim 2^{-2qs} d_q^2 \|e^{Rt} B_\phi\|_{L^2_{t,\phi}(H^{s+\frac{1}{2}})} \|e^{Rt} C_\phi\|_{L^2_{t,\phi}(H^{s+\frac{1}{2}})},
\]
where
\[
d_q^2 = d_q \left( \sum_{|q-q'| \leq 4} d_{q'} 2^{(q-q')(s-\frac{1}{2})} \right)
\]
if we multiply (2.15) by \(2^{2qs}\) and summing with respect to \(q \in \mathbb{Z}\), we get
\[
\sum_{q \in \mathbb{Z}} 2^{2qs} M_{1,q} \lesssim C \|e^{Rt} B_\phi\|_{L^2_{t,\phi}(H^{s+\frac{1}{2}})} \|e^{Rt} C_\phi\|_{L^2_{t,\phi}(H^{s+\frac{1}{2}})}. \tag{2.16}
\]
Similarly, by Lemma 2.2 and considering the support properties to the Fourier transform given in \([6], \text{Proposition 2.10}\) of the terms in \(T_{\partial_x B}^h A\), we obtain
\[
M_{2,q}(t) \leq \int_0^t \left| \left< e^{Rt'} \Delta_q^h (T_{\partial_x B}^h A)_\phi, e^{Rt'} \Delta_q^h C_\phi \right> \right|_{L^2} \ dt'.
\]
\[
\leq \sum_{|q-q'| \leq 4} \int_0^t e^{2Rt'} \|S_{q-1}^h \partial_x B_\phi\|_{L^\infty_{t}(L^2_{\phi})} \|\Delta_q^h A_\phi\|_{L^2_{t}(L^\infty_{\phi})} \|\Delta_q^h C_\phi\|_{L^2} dt'.
\]
As in (2.14), we can write
\[
\|\Delta^h A\|_{L^2_t(L^\infty_x)} \lesssim \|\Delta^h \partial_y A\|_{L^2} \lesssim 2^{-\frac{q}{2}} \|\partial_y A\|_{\mathcal{H}^s}.
\]
Since \(\frac{1}{2} < s < 1\), we have
\[
M_{2,q} \leq \sum_{|q'-q| \leq 4} \int_0^t e^{2Rt'} \|S^h_{q-1} \partial_x B\|_{L^\infty_t(L^2_x)} \|\Delta^h A\|_{L^2_t(L^\infty_x)} \|\Delta^h C\|_{L^2} dt'
\]
\[
\leq \sum_{|q'-q| \leq 4} \int_0^t 2^{-\frac{q'}{2}} d_q(t) e^{Rt'} \|S^h_{q-1} \partial_x B\|_{L^\infty_t(L^2_x)} \|\partial_y A\|_{\mathcal{H}^s} e^{Rt'} \|\Delta^h C\|_{L^2} dt'
\]
\[
\leq \sum_{|q'-q| \leq 4} 2^{-\frac{q'}{2}} \left( \int_0^t 2^{\frac{q'}{2}} \sum_{t \leq t' \leq t} 2^t 2^{\frac{q'}{2}} \|\Delta^h B\|_{L^2} \|\partial_y A\|_{\mathcal{H}^s} \|\Delta^h C\|_{L^2} dt' \right) \frac{1}{2}
\]
\[
\times \left( \int_0^t e^{2Rt'} \|\partial_y A\|_{\mathcal{H}^s} \|\Delta^h C\|_{L^2} dt' \right) \frac{1}{2}.
\]
Yet we observe from Definition 2, and \(s < 1\) we have
\[
\left( \int_0^t 2^{\frac{q'}{2}} \sum_{t \leq t' \leq t} 2^t 2^{\frac{q'}{2}} \|\Delta^h B\|_{L^2} \|\partial_y A\|_{\mathcal{H}^s} \|\Delta^h C\|_{L^2} dt' \right) \frac{1}{2}
\]
\[
\lesssim \sum_{t \leq t' \leq t} 2^{\frac{q'}{2}} \left( \int_0^t \|\partial_y A\|_{\mathcal{H}^s} e^{2Rt'} \|\Delta^h B\|_{L^2} dt' \right)^{\frac{1}{2}}
\]
\[
\lesssim 2^q (1-s) d_q \|e^{Rt} B\|_{L^2_{t,\mathcal{H}^s}}.
\]
So that it comes out
\[
M_{2,q} \leq d_q 2^{-2q} \|e^{Rt} B\|_{L^2_{t,\mathcal{H}^s}} \|e^{Rt} C\|_{L^2_{t,\mathcal{H}^s}},
\]
where
\[
d_q = d_q \left( \sum_{|q'-q| \leq 4} d_q 2^{q'-q} (s-\frac{1}{2}) \right)
\]
is a suitable sequence of positive constants. Summing with respect to \(q \in \mathbb{Z}\), and using Fubini’s theorem, we get
\[
\sum_{q \in \mathbb{Z}} 2^{2qs} M_{2,q} \lesssim C \|e^{Rt} B\|_{L^2_{t,\mathcal{H}^s}} \|e^{Rt} C\|_{L^2_{t,\mathcal{H}^s}},
\]
(2.17)
where we recall that \(\dot{\theta}(t) \simeq \|\partial_y A\|_{\mathcal{H}^s} \).
To end this proof, it remains to estimate $M_{3,q}$ (is the rest term). Using the support properties given in [6, Proposition 2.10], the definition of $R^h(A, \partial_x B)$ and Bernstein lemma 2.1, we can write

$$M_{3,q} = \int_0^t \left| \left\langle e^{Rt} \Delta_q^h (R^h(A, \partial_x B)) \phi, e^{Rt'} \Delta_q^h C \phi \right\rangle \right|_{L^2} dt'$$

$$\leq 2^q \sum_{q' \geq q-3} \int_0^t \int_0^t e^{2Rt'} \| \Delta_q^h A \phi \|_{L^2(L^{\infty})} \| \Delta_q^h \partial_x B \phi \|_{L^2} \| \Delta_q^h C \phi \|_{L^2} dt'$$

$$\leq 2^q \sum_{q' \geq q-3} \int_0^t \int_0^t e^{2Rt'} 2^q (1-\frac{1}{2}) \| \partial_y A \phi \| \| \Delta_q^h B \phi \|_{L^2} \| \Delta_q^h C \phi \|_{L^2} dt'$$

$$\leq 2^q \sum_{q' \geq q-3} \int_0^t e^{2Rt'} 2^q \| \Delta_q^h B \phi \|_{L^2} \| \Delta_q^h C \phi \|_{L^2} dt'.$$

Since $\frac{1}{2} < s < 1$, we have

$$M_{3,q} \leq 2^q \sum_{q' \geq q-3} \int_0^t 2^q \| \partial_y A \phi \| \| \Delta_q^h e^{Rt'} B \phi \| \| \Delta_q^h e^{Rt'} C \phi \|_{L^2} dt'$$

$$\leq 2^q \sum_{q' \geq q-3} \int_0^t 2^q d_{q'}(B) 2^{-q'(s+\frac{1}{2})} \| e^{Rt'} B \phi \|_{H^{s+\frac{1}{2}}} \| \partial_y A \phi \| \| e^{Rt'} C \phi \|_{H^{s+\frac{1}{2}}} dt'$$

$$\leq d_q 2^{-2q} \int_0^t \| e^{Rt} B \phi \|_{H^{s+\frac{1}{2}}} \| \partial_y A \phi \| \| e^{Rt} C \phi \|_{H^{s+\frac{1}{2}}} \left( \sum_{q' \geq q-3} d_{q'} 2^{q-q' \cdot s} \right) dt'$$

$$\leq d_q 2^{-2q} \| e^{Rt} B \phi \|_{L^2_{t,\theta(t)}(H^{s+\frac{1}{2}})} \| e^{Rt} C \phi \|_{L^2_{t,\theta(t)}(H^{s+\frac{1}{2}})} dt'$$

where

$$d_q^2 = d_q \left( \sum_{q' \geq q-3} d_{q'} 2^{q-q' \cdot s} \right)$$

is a suitable sequence of positive constants. Summing with respect to $q \in \mathbb{Z}$, and using Fubini’s theorem, we finally obtain

$$\sum_{q \in \mathbb{Z}} 2^{2qs} M_{3,q} \lesssim \| e^{Rt} B \phi \|_{L^2_{t,\theta(t)}(H^{s+\frac{1}{2}})} \| e^{Rt} C \phi \|_{L^2_{t,\theta(t)}(H^{s+\frac{1}{2}})}.$$ 

(2.18)

Lemma 2.4 is then proved by summing Estimates (2.16), (2.17) and (2.18). □

**Lemma 2.5.** Let $A, B$ and $C$ be a smooth function on $[0, T] \times \mathbb{R} \times (0, 1)$, and $s \in ]\frac{1}{2}, 1[$, $T > 0$ and $\phi$ be defined as in (2.3), with $\theta(t) = \| \partial_y A \phi(t) \| \| H^s \cdot \$. There exist $C \geq 1$ such that, for any $t > 0$, $\phi(t, \xi) > 0$ and for any $B, C \in L^2_{t,\theta(t)}(H^{s+\frac{1}{2}})$, we have

$$\sum_{q \in \mathbb{Z}} 2^{2qs} \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h \left( \int_0^t \partial_x B v A \phi, e^{Rt'} \Delta_q^h C \phi \right) \right\rangle \right|_{L^2} dt'$$

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\[
\lesssim C\|e^{Rt}B\phi\|_{L^2_t(L^{\infty_y}(\mathcal{H}^{\frac{1}{2}}))}\|e^{Rt}C\phi\|_{L^2_t(L^{\infty_y}(\mathcal{H}^{\frac{1}{2}}))}.
\] (2.19)

**Proof.** As in [5], using Bony’s homogeneous decomposition into para-products \(\int_0^y \partial_y Bdy' \partial_y A\) in the horizontal variable and remainders as in Definition of a tempered distribution, we can write

\[
\partial_y A \int_0^y \partial_x Bdy' = T_{\partial_y A}^h \int_0^y \partial_x Bdy' + T_{\int_0^y \partial_x Bdy'}^h \partial_y A + R^h(\partial_y A, \int_0^y \partial_x Bdy')
\]

where,

\[
T_{\partial_y A}^h \int_0^y \partial_x Bdy' = \sum_{q \in \mathbb{Z}} S_{q-1}^h \partial_y A \Delta^h \int_0^y \partial_x Bdy'
\]

\[
R^h(\partial_y A, \int_0^y \partial_x Bdy') = \sum_{|q'-q| \leq 1} \Delta^h \partial_y A \Delta^h_{q'} \int_0^y \partial_x Bdy'.
\]

We replace, we obtain the following bound of \(\int_0^t \left| \left\langle e^{Rt'} \Delta_q^h(\partial_y A, \int_0^y \partial_x Bdy') \phi, e^{Rt'} \Delta_q^h C\phi \right\rangle \right| dt'\)

\[
\int_0^t \left| \left\langle e^{Rt'} \Delta_q^h(\partial_y A, \int_0^y \partial_x Bdy') \phi, e^{Rt'} \Delta_q^h C\phi \right\rangle \right| dt' \leq N_{1,q} + N_{2,q} + N_{3,q},
\]

where

\[
N_{1,q} = \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h(T_{\partial_y A}^h \int_0^y \partial_x Bdy') \phi, e^{Rt'} \Delta_q^h C\phi \right\rangle \right| dt'
\]

\[
N_{2,q} = \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h(T_{\int_0^y \partial_x Bdy'}^h \partial_y A) \phi, e^{Rt'} \Delta_q^h C\phi \right\rangle \right| dt'
\]

\[
N_{3,q} = \int_0^t \left| \left\langle e^{Rt'} \Delta_q^h(R^h(\partial_y A, \int_0^y \partial_x Bdy')) \phi, e^{Rt'} \Delta_q^h C\phi \right\rangle \right| dt'.
\]

We start by getting the estimate of the first term \(N_{1,q}\), for that we need to use the support properties given in [6], Proposition 2.10 and the definition of \(T_{\partial_y A}^h \int_0^y \partial_x Bdy'\), and again thanks to the Lemma 2.2 we infer

\[
N_{1,q} \leq \sum_{|q'-q| \leq 4} \int_0^t e^{2Rt'} \| S_{q'-1}^h \partial_y A\phi(t') \|_{L^2_y(L^\infty_x)} \| \Delta^h \int_0^y \partial_x B\phi(t') dy' \|_{L^2_y(L^\infty_x)} \| \Delta_q^h C\phi(t') \|_{L^2} dt'.
\] (2.20)

While it follow from the Poincaré inequality on the interval \(\{0 < y < 1\}\), we have the inclusion \(H^1 \hookrightarrow L^\infty_y\) and

\[
\| \Delta_q^h \partial_y A\phi(t') \|_{L^2_y(L^\infty_x)} \lesssim 2^q \| \Delta_q^h \partial_y A\phi(t') \|_{L^2}\lesssim d_q(A\phi) \| \partial_y A\phi(t') \|_{H^s},
\] (2.21)

Here and in all that follows, we always denote \((d_q(t))_{q \in \mathbb{Z}}\) to be a generic element of \(\ell^1(\mathbb{Z}) \sum d_q(t) = 1\). Then,\[
\| S_{q'-1}^h A\phi(t') \|_{L^\infty} \lesssim \| \partial_y A\phi(t') \|_{H^s},
\]
and

$$\| \Delta_{\phi}^{h} \int_{0}^{y} \partial_{x} B_{\phi}(t')dy' \|_{L_{t}^{\infty}(L_{x}^{2})} \lesssim 2^{q'} \| \Delta_{\phi}^{h} B_{\phi} \|_{L^{2}}$$

As a result, it come out

$$N_{1,q} \leq \sum_{|q-q'|\leq 4} 2^{q'} \int_{0}^{t} e^{2R \tau'} \| \partial_{y} A_{\phi}(t') \|_{L^{\infty}} \| \Delta_{\phi}^{h} B_{\phi} \|_{L^{2}} \| \Delta_{\phi}^{h} C_{\phi}(t') \|_{L^{2}} dt'$$

$$\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \int_{0}^{t} e^{2R \tau'} \| \Delta_{\phi}^{h} B_{\phi} \|_{L^{2}} \| \Delta_{\phi}^{h} C_{\phi}(t') \|_{L^{2}} dt'$$

$$\lesssim \sum_{|q-q'|\leq 4} 2^{q'} \left( \int_{0}^{t} e^{2R \tau'} \| \partial_{y} A_{\phi}(t') \|_{L^{\infty}} \right) \left( \int_{0}^{t} e^{2R \tau'} \| \Delta_{\phi}^{h} C_{\phi} \|_{L^{2}}^{2} dt' \right)^{\frac{1}{2}} \times \left( \int_{0}^{t} e^{2R \tau'} \| \Delta_{\phi}^{h} C_{\phi} \|_{L^{2}}^{2} dt' \right)^{\frac{1}{2}}$$

we note that $$\dot{\theta}(t) \simeq \| \partial_{y} A_{\phi}(t') \|_{H^{s}}$$, using the definition (2.3), we achieve

$$N_{1,q} \lesssim 2^{-2qs} d_{q}^{2} \| e^{Rt} B_{\phi} \|_{L_{t}^{2}(H^{s+\frac{1}{2}})} \| e^{Rt} C_{\phi} \|_{L_{t}^{2}(H^{s+\frac{1}{2}})};$$  (2.22)

where

$$d_{q}^{2} = d_{q} \left( \sum_{|q-q'|\leq 4} d_{q} 2^{(q-q')(s-\frac{1}{2})} \right)$$

if we multiply (2.22) by $$2^{2qs}$$ and summing with respect to $$q \in \mathbb{Z}$$, we get

$$\sum_{q \in \mathbb{Z}} 2^{2qs} N_{1,q} \lesssim C \| e^{Rt} B_{\phi} \|_{L_{t}^{2}(H^{s+\frac{1}{2}})} \| e^{Rt} C_{\phi} \|_{L_{t}^{2}(H^{s+\frac{1}{2}})}.$$  (2.23)

Along the same way for $$\frac{1}{2} < s < 1$$, we obtain

$$N_{2,q}(t) \leq \int_{0}^{t} \left( \int_{0}^{y} e^{R \tau'} \Delta_{\phi}^{h} \left( T_{j}^{h} \partial_{x} B_{\phi} \partial_{y} A_{\phi} \right) e^{R \tau'} \Delta_{\phi}^{h} C_{\phi} \right) dt'$$

$$\lesssim \sum_{|q-q'|\leq 4} \int_{0}^{t} e^{2R \tau'} \| S_{q-1}^{h} \int_{0}^{y} \partial_{x} B_{\phi} dy' \|_{L_{t}^{\infty}(L_{x}^{2})} \| \Delta_{\phi}^{h} \partial_{y} A_{\phi} \|_{L^{2}} \| \Delta_{\phi}^{h} C_{\phi} \|_{L^{2}} dt'$$

$$\lesssim \sum_{|q-q'|\leq 4} d_{q} \left( \int_{0}^{t} e^{2R \tau'} \| S_{q-1}^{h} \int_{0}^{y} \partial_{x} B_{\phi} \|_{L_{t}^{\infty}(L_{x}^{2})} \dot{\theta}(t') dt' \right)^{\frac{1}{2}} \times \left( \int_{0}^{t} e^{2R \tau'} \| \Delta_{\phi}^{h} C_{\phi} \|_{L^{2}}^{2} dt' \right)^{\frac{1}{2}}$$

Yet we observe from Definition 2, and $$s < 1$$ we have

$$\left( \int_{0}^{t} e^{2R \tau'} \| S_{q-1}^{h} \int_{0}^{y} \partial_{x} B_{\phi} \|_{L_{t}^{\infty}(L_{x}^{2})} \dot{\theta}(t') dt' \right)^{\frac{1}{2}} \lesssim 2^{q'(1-s)} \| e^{Rt} B_{\phi} \|_{L_{t}^{2}(H^{s+\frac{1}{2}})}.$$  (2.24)

So that it comes out

$$N_{2,q} \lesssim d_{q}^{2} 2^{-2qs} \| e^{Rt} B_{\phi} \|_{L_{t}^{2}(H^{s+\frac{1}{2}})} \| e^{Rt} C_{\phi} \|_{L_{t}^{2}(H^{s+\frac{1}{2}})}.$$
Bernstein lemma 2.1, we can write properties given in \([6], \text{Proposition 2.10}\), the definition of \(R\) Since 1

Fubini’s theorem, we get is a suitable sequence of positive constants. Summing with respect to \(q\)

\[ Fubini’s \ theorem, \ we \ finally \ obtain \]  

is a suitable sequence of positive constants. Summing with respect to \(N\), we recall that \(\dot{\theta}(t) \simeq \|\partial_{y}A_{\phi}\|_{\mathcal{H}^{s}}\).

To end this proof, it remains to estimate \(N_{3,q}\) (is the rest term). Using the support properties given in \([6], \text{Proposition 2.10}\), the definition of \(R^{h}(\partial_{y}A, \int_{0}^{t} \partial_{x}Bdy')\) and Bernstein lemma 2.1, we can write

\[
N_{3,q} = \int_{0}^{t} \left( e^{R^{t}} \Delta_{q}^{h}(R^{h}(\partial_{y}A, \int_{0}^{t} \partial_{x}Bdy'))_{\phi}, e^{R^{t}} \Delta_{q}^{h}C_{\phi} \right)_{L^{2}} dt' 
\]

\[
\leq 2^{\frac{d}{2}} \sum_{q' \geq q-3} \int_{0}^{t} e^{2R^{t}} \|\Delta_{q}^{h}\partial_{y}A_{\phi}\|_{L^{2}(\mathcal{H}^{s}(\mathcal{H}^{s})}) \|\Delta_{q}^{h}B_{\phi}\|_{L^{2}} \|\Delta_{q}^{h}C_{\phi}\|_{L^{2}} dt' 
\]

\[
\leq 2^{\frac{d}{2}} \sum_{q' \geq q-3} \int_{0}^{t} e^{2R^{t}} 2^{q'(1-\frac{1}{2})} \|\partial_{y}A_{\phi}\|_{\mathcal{H}^{s}} \|\Delta_{q}^{h}B_{\phi}\|_{L^{2}} \|\Delta_{q}^{h}C_{\phi}\|_{L^{2}} dt' 
\]

\[
\leq 2^{\frac{d}{2}} \sum_{q' \geq q-3} \int_{0}^{t} e^{2R^{t}} 2^{\frac{d}{2}} \|\partial_{y}A_{\phi}\|_{\mathcal{H}^{s}} \|\Delta_{q}^{h}B_{\phi}\|_{L^{2}} \|\Delta_{q}^{h}C_{\phi}\|_{L^{2}} dt'. 
\]

Since \(\frac{1}{2} < s < 1\), we have

\[
N_{3,q} \leq 2^{\frac{d}{2}} \sum_{q' \geq q-3} \int_{0}^{t} 2^{\frac{d}{2}} \|\partial_{y}A_{\phi}\|_{\mathcal{H}^{s}} \|\Delta_{q}^{h}e^{R^{t}}B_{\phi}\|_{L^{2}} \|\Delta_{q}^{h}e^{R^{t}}C_{\phi}\|_{L^{2}} dt' 
\]

\[
\leq 2^{\frac{d}{2}} \sum_{q' \geq q-3} \int_{0}^{t} 2^{\frac{d}{2}} d_{q}(B_{\phi}) 2^{-q'(s+\frac{1}{2})} \|e^{R^{t}}B_{\phi}\|_{\mathcal{H}^{s+\frac{1}{2}}} \|\partial_{y}A_{\phi}\|_{\mathcal{H}^{s}} d_{q}2^{-q(s+\frac{1}{2})} \|e^{R^{t}}C_{\phi}\|_{\mathcal{H}^{s+\frac{1}{2}}} dt' 
\]

\[
\leq d_{q}2^{-2q} \int_{0}^{t} \|e^{R^{t}}B_{\phi}\|_{\mathcal{H}^{s+\frac{1}{2}}} \|\partial_{y}A_{\phi}\|_{\mathcal{H}^{s}} \|e^{R^{t}}C_{\phi}\|_{\mathcal{H}^{s+\frac{1}{2}}} \left( \sum_{q' \geq q-3} d_{q}2^{q-q')s} \right) dt' 
\]

\[
\leq d_{q}2^{-2q} \|e^{R^{t}}B_{\phi}\|_{L^{2}(\mathcal{H}^{s+\frac{1}{2}})} \|e^{R^{t}}C_{\phi}\|_{L^{2}(\mathcal{H}^{s+\frac{1}{2}})} \]  

where

\[
d_{q}^{2} = d_{q} \left( \sum_{q' \geq k-3} d_{q'2^{q'(q-q')s}} \right) 
\]

is a suitable sequence of positive constants. Summing with respect to \(q \in \mathbb{Z}\), and using Fubini’s theorem, we finally obtain

\[
\sum_{q \in \mathbb{Z}} 2^{2q}N_{3,q} \lesssim \|e^{R^{t}}B_{\phi}\|_{L^{2}(\mathcal{H}^{s+\frac{1}{2}})} \|e^{R^{t}}C_{\phi}\|_{L^{2}(\mathcal{H}^{s+\frac{1}{2}})} \]  

(2.26)
Lemma 2.5 is then proved by summing Estimates (2.23), (2.25) and (2.26).

**Lemma 2.6.** For any $s \in [\frac{1}{2}, 1]$ and $t \leq T^*$, and $\phi$ be defined as in (2.3), with
\[
\dot{\eta}(t) = \|\partial_y A_{\phi}(t)\|_{H^s} + \epsilon \|\partial_y B_{\phi}(t)\|_{H^s}.
\]

Then, there exists $C \geq 1$ such that, for any $t > 0$, $\phi(t, \xi) > 0$ and for any $A \in L^2_{t,x}(H^{s+\frac{1}{2}})$ that satisfies $B(t,x,y) = -\int_0^t \partial_x A(t,x,s) ds$ and $\partial_x A = -\partial_y B$, we have
\[
\epsilon^2 \sum_{q \in \mathbb{Z}} 2^{2qs} \int_0^t \left| \left\langle e^{R_t t} \Delta_q^h(B \partial_y B)_{\phi}, e^{R_t t} \Delta_q^h C_{\phi} \right\rangle \right| dt' \leq C \|e^{R_t t}(A_{\phi}, C_{\phi})\|_{L^2_{t,x}(H^{s+\frac{1}{2}})}^2.
\]

**Proof.** As in [5], using Bony’s homogeneous decomposition into para-products in the horizontal variable and remainders as in Definition of a tempered distribution, we can write
\[
B \partial_y B = T^h \partial_y B dy' + T^h \partial_y B + R^h(\partial_y B, B)
\]

We replace, we obtain the following bound of $\int_0^t \left| \left\langle e^{R_t t} \Delta_q^h(B \partial_y B)_{\phi}, e^{R_t t} \Delta_q^h C_{\phi} \right\rangle \right| dt' \leq L_{1,q} + L_{2,q} + L_{3,q},$

where
\[
L_{1,q} = \int_0^t \left| \left\langle e^{R_t t} \Delta_q^h(T^h \partial_y B)_{\phi}, e^{R_t t} \Delta_q^h C_{\phi} \right\rangle \right| dt'
\]
\[
L_{2,q} = \int_0^t \left| \left\langle e^{R_t t} \Delta_q^h(T^h \partial_y B)_{\phi}, e^{R_t t} \Delta_q^h C_{\phi} \right\rangle \right| dt'
\]
\[
L_{3,q} = \int_0^t \left| \left\langle e^{R_t t} \Delta_q^h(R^h(\partial_y B, \partial_y B))_{\phi}, e^{R_t t} \Delta_q^h C_{\phi} \right\rangle \right| dt'.
\]

We start by getting the estimate of the first term $L_{1,q}$. Due to $\partial_y B = -\partial_x A$, one has
\[
\epsilon^2 L_{1,q} \lesssim \epsilon^2 \sum_{|q' - q| \leq 4} \int_0^t e^{2R_t t} \left\| S^h_{q' - 1} B_{\phi}(t) \right\|_{L^\infty} \left\| \Delta_q^h \partial_y B_{\phi}(t) \right\|_{L^2} \left\| \Delta_q^h C_{\phi}(t) \right\|_{L^2}
\]
\[
\lesssim \epsilon^2 \sum_{|q' - q| \leq 4} \int_0^t e^{2R_t t} \left\| S^h_{q' - 1} B_{\phi}(t) \right\|_{L^\infty} \left\| \Delta_q^h \partial_x A_{\phi}(t) \right\|_{L^2} \left\| \Delta_q^h C_{\phi}(t) \right\|_{L^2}
\]
\[
\lesssim \epsilon \sum_{|q' - q| \leq 4} \int_0^t e^{2R_t t} \left\| S^h_{q' - 1} B_{\phi}(t) \right\|_{L^\infty} \left\| \partial_x A_{\phi}(t) \right\|_{H^s} \left\| \Delta_q^h C_{\phi}(t) \right\|_{L^2}
\]
\[
\lesssim \epsilon \sum_{|q' - q| \leq 4} 2^{-\frac{q}{2}} \left( \int_0^t \left\| S^h_{q' - 1} e^{R_t t} B_{\phi}(t') \right\|_{L^\infty} \left\| \partial_x A_{\phi}(t') \right\|_{H^s} dt' \right)^{\frac{1}{2}}
\]
we get, by a similar derivation of (2.24), that
\[
\left( \int_0^t \| S_{q'-q}^h e^{RT} B_{\phi}(t') \|_{L^\infty}^2 \| \partial_x A_{\phi}(t') \|_{H^s} dt' \right)^{\frac{1}{2}} \leq d_q 2^{q'(1-s)} \| e^{RT} A_{\phi} \|_{L^2_{t,q}(H^{s+\frac{3}{2}})}.
\]
Hence we deduce from the definition 2 that
\[
L_{1,q} \lesssim d_q 2^{-2s} \| e^{RT} A_{\phi} \|_{L^2_{t,q}(H^{s+\frac{3}{2}})} \| e^{RT} e^{C_{\phi}} \|_{L^2_{t,q}(H^{s+\frac{3}{2}})}
\]  

(2.27)

Along the same way, we have
\[
L_{2,q}(t) \leq \int_0^t \left| \left( e^{RT} \Delta_q^h (T^h_{\partial_t B} B_{\phi}) \right) \right|_{L^2} dt' \lesssim \sum_{|q'-q| \leq 4} \int_0^t e^{2RT'} \| S_{q'-q}^h \partial_x A_{\phi} \|_{L^\infty} \| \Delta_q^h B_{\phi} \|_{L^2} \| \Delta_q^h C_{\phi} \|_{L^2} dt' \lesssim \sum_{|q'-q| \leq 4} 2^{q'} \int_0^t e^{2RT'} \| \partial_y A_{\phi} \|_{H^s} \| \Delta_q^h B_{\phi} \|_{L^2} \| \Delta_q^h C_{\phi} \|_{L^2} dt' \lesssim \sum_{|q'-q| \leq 4} 2^{q'} \left( \int_0^t \| \partial_y A_{\phi} \|_{H^s} \| e^{RT'} \Delta_q^h C_{\phi} \|_{L^2}^2 dt' \right)^{\frac{1}{2}} \times \left( \int_0^t \| \partial_y A_{\phi} \|_{H^s} \| e^{RT'} \Delta_q^h C_{\phi} \|_{L^2}^2 dt' \right)^{\frac{1}{2}}.
\]

Then thanks to Definition 2, we arrive at
\[
e^2 L_{2,q} \lesssim d_q 2^{-2s} \| e^{RT} B_{\phi} \|_{L^2_{t,q}(H^{s+\frac{3}{2}})} \| e^{RT} e^{C_{\phi}} \|_{L^2_{t,q}(H^{s+\frac{3}{2}})}
\]  

(2.28)

To end this proof, it remains to estimate \( L_{3,q} \) (is the rest term). Due to \( \partial_y B = -\partial_x A \), we get, by applying lemma 2.1 that
\[
L_{3,q} \lesssim 2^{\frac{s}{2}} \sum_{q' \geq q-3} \int_0^t e^{2RT'} \| \Delta_q^h \partial_x A_{\phi} \|_{L^2_{t,q}(H^{s+\frac{3}{2}})} \| \Delta_q^h B_{\phi} \|_{L^2} \| \Delta_q^h C_{\phi} \|_{L^2} dt' \lesssim 2^{\frac{s}{2}} \sum_{|q'-q| \leq 4} 2^{\frac{q'}{2}} \int_0^t e^{2RT'} \| \partial_y A_{\phi} \|_{H^s} \| \Delta_q^h B_{\phi} \|_{L^2} \| \Delta_q^h C_{\phi} \|_{L^2} dt' \lesssim 2^{\frac{s}{2}} \sum_{|q'-q| \leq 4} 2^{\frac{q'}{2}} \left( \int_0^t \| \partial_y A_{\phi} \|_{H^s} \| e^{RT'} \Delta_q^h B_{\phi} \|_{L^2}^2 dt' \right)^{\frac{1}{2}} \times \left( \int_0^t \| \partial_y A_{\phi} \|_{H^s} \| e^{RT'} \Delta_q^h C_{\phi} \|_{L^2}^2 dt' \right)^{\frac{1}{2}}.
\]

which together with Definition 2 and \( \frac{1}{2} < s < 1 \) ensures that
Lemma 2.6 is then proved by summing Estimates (2.27), (2.28) and (2.29).

3. Global existence of the perturbed hydrostatic system (1.8)

The goal of this section is to prove the global well-posedness of the limit system of the Perturbed Navier-Stokes equation, we remark that the local smooth solution of the limit system follows a standard parabolic regularization method similar to the Perturbation NS system, First, we remark that the Dirichlet boundary condition

\[(u, v)_{y=0} = (u, v)_{y=1} = 0,
\]

and the incompressible condition \(\partial_x u + \partial_y v = 0\) imply that :

\[v(t, x, y) = \int_0^y \partial_y v(t, x, s)ds - \int_0^u \partial_x u(t, x, s)ds \quad (3.1)\]

Due to the compatibility condition \(\partial_x \int_0^1 u_0dy = 0\), we deduce from \(\partial_x u + \partial_y v = 0\) that

\[\partial_x \int_0^1 u(t, x, y) dy = -\int_0^1 \partial_y v(t, x, y) dy = v(t, x, 1) - v(t, x, 0) = 0, \quad (3.2)\]

which together with the fact: \(u(t, x, y) \to 0\) as \(|x| \to \infty\), ensure that

\[\int_0^1 u(t, x, y)dy = 0. \quad (3.3)\]

Then by integrating the equations \(\partial_t^2 u + \partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u + \partial_x p = 0\) and, for \(y \in [0, 1]\) and using the fact that \(\partial_y p = 0\), we obtain

\[\partial_x p = \partial_y u(t, x, 1) - \partial_y u(t, x, 0) - \frac{1}{2} \partial_x \int_0^1 (u)^2(t, x, y)dy \quad (3.4)\]

In view of the system (1.8), we can transform it like a equation of order one in time, so if we define \(V = (u, \partial_t u)\), Then \(V\) satisfy the following equation

\[
\begin{cases}
\partial_t V + A(D)V = -\left(\begin{array}{c}
0 \\
\partial_x u + v\partial_y u + \partial_x p
\end{array}\right) \\
\partial_y p = 0 \\
\partial_x u + \partial_y v = 0 \\
(u, v)_{y=0} = (u, v)_{y=1} = 0
\end{cases}
\quad (3.5)
\]

where

\[V = \left(\begin{array}{c}
u \\
\partial_t u
\end{array}\right) \quad \text{and} \quad A(D) = \left(\begin{array}{cc}
0 & -1 \\
-\partial_y^2 & 1
\end{array}\right)\]
Then in view of (2.3) we observe that $V_{\phi}$ verifies

$$
\begin{align*}
\begin{cases}
\partial_t V_{\phi} + \lambda \hat{\theta}(t) |D_x| V_{\phi} + A(D) V_{\phi} = -\left( (u \partial_x u)_{\phi} + (v \partial_y u)_{\phi} + \partial_x p_{\phi} \right) \\
\partial_y p_{\phi} = 0 \\
\partial_x u_{\phi} + \partial_y v_{\phi} = 0 \\
(u_{\phi}, v_{\phi})/_{y=0} = (u_{\phi}, v_{\phi})/_{y=1} = 0
\end{cases}
(3.6)
\end{align*}
$$

Where $|D_x|$ denotes the Fourier multiplier with symbol $|\xi|$.

The main idea of this technique consists in the fact that if we differentiate, with respect to the time variable a function of the type $e^{\phi(t,D_x)} u(t,x,y)$, we obtain an additional “good term” which plays the smoothing role. More precisely, we have

$$
\frac{d}{dt} \left( e^{\phi(t,D_x)} V(t,x,y) \right) = -\hat{\theta}(t) |D_x| e^{\phi(t,D_x)} V(t,x,y) + e^{\phi(t,D_x)} \partial_t V(t,x,y),
$$

where $-\hat{\theta}(t) |D_x| e^{\phi(t,D_x)} u(t,x,y)$ gives a smoothing effect if $\hat{\theta}(t) \geq 0$. This smoothing effect allows to obtain our global existence and stability results in the analytic framework.

**Proof of global well-posedness of system (1.8).** By applying the dyadic operator in the horizontal variable $\Delta_q$ to (3.6) and taking the $L^2$ inner product of the resulting equation with $\Delta_q^h(L_{\phi})$ we obtain

$$
\langle \Delta_q^h \partial_t V_{\phi}, \Delta_q^h V_{\phi} \rangle_{L^2} + \lambda \hat{\theta}(t) \langle \Delta_q^h |D_x| V_{\phi}, \Delta_q^h V_{\phi} \rangle_{L^2} + \langle \Delta_q^h A(D) V_{\phi}, \Delta_q^h V_{\phi} \rangle_{L^2} = -\langle \Delta_q^h (u \partial_x u + v \partial_y u)_{\phi}, \Delta_q^h (\partial_t u)_{\phi} \rangle_{L^2} - \langle \Delta_q^h \partial_x p_{\phi}, \Delta_q^h (\partial t u)_{\phi} \rangle_{L^2}.
(3.7)
$$

In what follows, we shall use the technical lemmas in Section 2, to handle term by term in the estimate (3.7).

By applying the result of the lemma 2.3, we find that

$$
\begin{align*}
\langle \Delta_q^h \partial_t V_{\phi}, \Delta_q^h V_{\phi} \rangle_{L^2} &+ \lambda \hat{\theta}(t) \langle \Delta_q^h |D_x| V_{\phi}, \Delta_q^h V_{\phi} \rangle_{L^2} + \langle \Delta_q^h A(D) V_{\phi}, \Delta_q^h V_{\phi} \rangle_{L^2} = \frac{1}{2} \frac{d}{dt} \left( \| \Delta_q^h u_{\phi} \|^2_{L^2} \right) \\
&+ \| \Delta_q^h (\partial_t u)_{\phi} \|^2_{L^2} + \lambda \hat{\theta}(t) \| \Delta_q^h |D_x| u_{\phi} \|^2_{L^2} + \lambda \hat{\theta}(t) \| \Delta_q^h |D_x| u_{\phi} \|^2_{L^2} - \frac{1}{2} \frac{d}{dt} \| \Delta_q^h u_{\phi} \|^2_{L^2} \\
&- \lambda \hat{\theta}(t) \| \Delta_q^h |D_x| u_{\phi} \|^2_{L^2} + \frac{1}{2} \frac{d}{dt} \| \Delta_q^h \partial_y u_{\phi} \|^2_{L^2} + \lambda \hat{\theta}(t) \| \Delta_q^h |D_x| \| \partial_y u_{\phi} \|^2_{L^2} + \| \Delta_q^h (\partial_t u)_{\phi} \|^2_{L^2} \\
&= \frac{1}{2} \frac{d}{dt} \left( \| \Delta_q^h (\partial_t u)_{\phi} \|^2_{L^2} + \| \Delta_q^h \partial_y u_{\phi} \|^2_{L^2} + \| \Delta_q^h (\partial_t u)_{\phi} \|^2_{L^2} \right) + \| \Delta_q^h (\partial_t u)_{\phi} \|^2_{L^2} \\
&+ \lambda \hat{\theta}(t) \| \Delta_q^h |D_x| \| \partial_t u_{\phi} \|^2_{L^2} + \lambda \hat{\theta}(t) \| \Delta_q^h |D_x| \| \partial_t u_{\phi} \|^2_{L^2}.
\end{align*}
$$

Now, for the pressure term, using the Dirichlet boundary condition $(u,v)|_{y=0} = (u,v)|_{y=1} = 0$, and the incompressibility condition $\partial_x u + \partial_y v = 0$ and the relation $\partial_y p = 0$, we can perform integration by parts, we get

$$
D = \left| \langle \Delta_q^h \partial_x p_{\phi}, \Delta_q^h (\partial_t u)_{\phi} \rangle_{L^2} \right| = \left| \langle \Delta_q^h p_{\phi}, \Delta_q^h (\partial_t \partial_x u)_{\phi} \rangle \right| \\
= \left| \langle \Delta_q^h p_{\phi}, \Delta_q^h (\partial_t \partial_y v)_{\phi} \rangle \right| = \left| \langle \Delta_q^h \partial_y p_{\phi}, \Delta_q^h (\partial_t v)_{\phi} \rangle \right| = 0.
$$
Thus,

\[ D = \left| \langle \Delta^h_p \partial_x p_\phi, \Delta^h_q |D_x| \partial_u \phi \rangle \right|_{L^2} = 0. \]

While due to \( u/y = 0 = u/y = 1 = 0 \), by applying Poincaré inequality, we have

\[ \mathcal{R} \| \Delta^h u \phi \|_{L^2}^2 \leq \| \Delta^h \partial_y u \phi \|_{L^2}^2. \quad (3.8) \]

Then, using Lemma 2.1 and multiplying (1.6) by \( e^{2Rt} \), we achieve

\[
\frac{1}{2} \frac{d}{dt} \left( \| e^{Rt} \Delta^h_q (\partial_t u) \|_{L^2}^2 + \| e^{Rt} \Delta^h_q \partial_y u \|_{L^2}^2 \right) + \| e^{Rt} \Delta^h_q (\partial_t u) \|_{L^2}^2 \\
+ \lambda \dot{\theta}(t) \| e^{Rt} \Delta^h_q |D_x|^j (\partial_t u) \|_{L^2}^2 + \lambda \dot{\theta}(t) \| e^{Rt} \Delta^h_q |D_x|^j \partial_y u \|_{L^2}^2 \\
\leq \left| \langle \Delta^h_q (u \partial_x u) \phi, e^{2Rt} \Delta^h_q (\partial_t u) \phi \rangle \right|_{L^2} + \left| \langle \Delta^h_q (v \partial_y u) \phi, e^{2Rt} \Delta^h_q (\partial_t u) \phi \rangle \right|_{L^2}. \quad (3.9)
\]

Next, we note that

\[
\begin{align*}
J_1^q(t') &= \left| \langle \Delta^h_q (u \partial_x u) \phi, e^{2Rt'} \Delta^h_q (\partial_t u) \phi \rangle \right|_{L^2} \\
&\lesssim C 2^{-2q} \delta^2 \hat{\theta}(t) \| e^{Rt} \partial_y u \phi \|_{H^{s - 1/2}} \| e^{Rt}(\partial_t u) \phi \|_{H^{s + 1/2}}, \quad (3.10)
\end{align*}
\]

where \( \dot{\theta}(t) = \| \partial_y u \phi \|_{H^s} \) with \( s > \frac{1}{2} \).

In view of the lemma 2.5, we replace \( C_\phi \) by (\( \partial_t u \)\), \( A = B = u \), then we conclude the following estimate of \( J_2^q \)

\[
\begin{align*}
J_2^q(t') &= \left| \langle \Delta^h_q (v \partial_y u) \phi, e^{2Rt'} \Delta^h_q (\partial_t u) \phi \rangle \right|_{L^2} \\
&\lesssim C 2^{-2q} \delta^2 \hat{\theta}(t) \| e^{Rt} \partial_y u \phi \|_{H^{s - 1/2}} \| e^{Rt}(\partial_t u) \phi \|_{H^{s + 1/2}},
\end{align*}
\]

where \( \dot{\theta}(t) = \| \partial_y u \phi \|_{H^s} \) with \( s > \frac{1}{2} \).

Then we deduce from (3.9), that

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left( \| e^{Rt} \Delta^h_q (\partial_t u) \|_{L^2}^2 + \| e^{Rt} \Delta^h_q \partial_y u \|_{L^2}^2 \right) + \| e^{Rt} \Delta^h_q (\partial_t u) \|_{L^2}^2 \\
+ \lambda \dot{\theta}(t) \| e^{Rt} \Delta^h_q |D_x|^j (\partial_t u) \|_{L^2}^2 + \lambda \dot{\theta}(t) \| e^{Rt} \Delta^h_q |D_x|^j \partial_y u \|_{L^2}^2 \\
&\leq 2C 2^{-2q} \delta^2 \hat{\theta}(t) \| e^{Rt} \partial_y u \phi \|_{H^{s - 1/2}} \| e^{Rt}(\partial_t u) \phi \|_{H^{s + 1/2}} \quad (3.12)
\end{align*}
\]

Now we still have to get some information of the norm \( \| \partial_y u \phi \|_{H^s} \), for that we need to apply the dyadic operator \( \Delta^h_q \) to the equation

\[ e^{\dot{\theta}(t)|D_x|}(\partial_t^2 u + \partial_t u + u \partial_x u + v \partial_y u - \partial^2_y u + \partial_x p) = 0, \quad (3.13) \]
and then, we take the $L^2$ inner product of the resulting equation (3.13) with $\Delta_q^h u_\phi$, we obtain

\[
\langle \Delta_q^h(\partial_t^2 u_\phi), \Delta_q^h u_\phi \rangle_{L^2} + \langle \Delta_q^h(\partial_t u), \Delta_q^h u_\phi \rangle_{L^2} - \langle \Delta_q^h \partial_y^2 u_\phi, \Delta_q^h u_\phi \rangle_{L^2} \\
= -\langle \Delta_q^h(u \partial_x u + v \partial_y u), \Delta_q^h u_\phi \rangle_{L^2} - \langle \Delta_q^h \partial_x p_\phi, \Delta_q^h u_\phi \rangle_{L^2}.
\]

(3.14)

In what follows, we shall use again the technical lemmas in Section 2, to handle term by term in the estimate (3.14). We start by the term

\[
I_1 = \langle \Delta_q^h(\partial_t^2 u_\phi), \Delta_q^h u_\phi \rangle_{L^2} \quad \text{and} \quad I_2 = \langle \Delta_q^h(\partial_t u), \Delta_q^h u_\phi \rangle_{L^2}
\]

so by using integration by parts, we find

\[
I_1 = \frac{d}{dt} \int \Delta_q^h(\partial_t u_\phi)\Delta_q^h u_\phi dx - \int \Delta_q^h(\partial_t u_\phi)\Delta_q^h(\partial_t u_\phi) dx \\
+ 2\lambda \hat{\theta}(t) \int \Delta_q^h|D_x|(|\partial_t u_\phi|\Delta_q^h u_\phi) dx,
\]

and

\[
I_2 = \frac{1}{2} \frac{d}{dt} \|\Delta_q^h u_\phi\|_{L^2}^2 + \lambda \hat{\theta}(t) \|\Delta_q^h|D_x|\|_{L^2}^2.
\]

Whereas due to the boundary condition, and by integrating by part, we achieve

\[
\langle \Delta_q^h(-\partial_y^2 u_\phi), \Delta_q^h u_\phi \rangle_{L^2} = \langle \Delta_q^h \partial_y u_\phi, \Delta_q^h \partial_y u_\phi \rangle_{L^2} = \|\Delta_q^h \partial_y u_\phi\|_{L^2}^2.
\]

Now, by using the Dirichlet boundary condition $(u, v)|_{y=0} = (u, v)|_{y=1} = 0$, and the incompressibility condition $\partial_x u + \partial_y v = 0$ and the relation $\partial_y p_\phi = 0$, we can find by integrating by parts the estimate of the pressure

\[
\left|\langle \Delta_q^h \partial_x p_\phi, \Delta_q^h u_\phi \rangle_{L^2} \right| = \left|\langle \Delta_q^h p_\phi, \Delta_q^h \partial_x u_\phi \rangle \right| \\
= \left|\langle \Delta_q^h p_\phi, \Delta_q^h \partial_y v_\phi \rangle \right| = \left|\langle \Delta_q^h \partial_y p_\phi, \Delta_q^h v_\phi \rangle \right| = 0.
\]

Then by using the Lemma 2.1 and by multiplying (3.14) by $e^{2R_1}$, and then integrating the resulting inequality over time, we achieve

\[
\frac{d}{dt} \int e^{2R_1} \Delta_q^h(\partial_t u_\phi)\Delta_q^h u_\phi dx - \int e^{2R_1} \Delta_q^h(\partial_t u_\phi)\Delta_q^h(\partial_t u_\phi) dx + 2\lambda \hat{\theta}(t) \int e^{2R_1} \Delta_q^h|D_x|(|\partial_t u_\phi|\Delta_q^h u_\phi) dx \\
+ \frac{1}{2} \frac{d}{dt} \|e^{R_1} \Delta_q^h u_\phi\|_{L^2}^2 + \lambda \hat{\theta}(t) \|e^{R_1} \Delta_q^h|D_x|\|_{L^2}^2 + \|e^{R_1} \Delta_q^h \partial_y u_\phi\|_{L^2}^2
\]

\[
= -\langle \Delta_q^h(u \partial_x u_\phi, e^{2R_1} \Delta_q^h u_\phi) \rangle_{L^2} - \langle \Delta_q^h(v \partial_y u_\phi, e^{2R_1} \Delta_q^h u_\phi) \rangle_{L^2}.
\]

(3.15)

Next, we note that

\[
\begin{cases}
L_1^q = \left|\langle \Delta_q^h(u \partial_x u_\phi, e^{2R_1'} \Delta_q^h u_\phi) \rangle \right|_{L^2} \\
L_2^q = \left|\langle \Delta_q^h(v \partial_y u_\phi, e^{2R_1'} \Delta_q^h u_\phi) \rangle \right|_{L^2} \\
L_3^q = 2\lambda \hat{\theta}(t) \int e^{2R_1} \Delta_q^h|D_x|(|\partial_t u_\phi|\Delta_q^h u_\phi) dx
\end{cases}
\]
In view of the lemma 2.4-2.5, we can deduce that

\[
L_1^q = \left| \Delta_q^h(u \partial_x u, e^{2Rt} \Delta_q^h u) \right|_{L^2} \lesssim C 2^{-2q} d_q^2 \hat{\theta}(t) \| e^{Rt} u \|_{\mathcal{H}^{q+\frac{1}{2}}} \| e^{Rt} u \|_{\mathcal{H}^{q+\frac{1}{2}}}
\]  

(3.16)

and

\[
L_2^q = \left| \Delta_q^h(v \partial_x u, e^{2Rt} \Delta_q^h u) \right|_{L^2} \lesssim C 2^{-2q} d_q^2 \hat{\theta}(t) \| e^{Rt} u \|_{\mathcal{H}^{q+\frac{1}{2}}} \| e^{Rt} u \|_{\mathcal{H}^{q+\frac{1}{2}}}
\]  

(3.17)

Then we still have to estimate \( L_3^q \), therefore by cutting the derivative \( |D_x| \) into two half derivatives and the Poincaré inequality, we achieve the

\[
L_3^q = \left| 2\lambda \hat{\theta}(t) \int e^{2Rt} \Delta_q^h |D_x| (\partial_t u) \Delta_q^h u \, dx \right|
\]

\[
\lesssim 2\lambda \hat{\theta}(t) \int \left| e^{Rt} \Delta_q^h |D_x| \left( \frac{1}{2} (\partial_t u) \right) \left| e^{Rt} \Delta_q^h |D_x| \left( \frac{1}{2} u \right) \right| \, dx
\]

\[
\lesssim 2^{-2q} d_q^2 2\lambda \hat{\theta}(t) \| e^{Rt}(\partial_t u) \|_{\mathcal{H}^{q+\frac{1}{2}}} \| e^{Rt} \partial_x u \|_{\mathcal{H}^{q+\frac{1}{2}}}
\]  

(3.18)

We multiply (3.12) by 2 and we sum it with (3.15), we obtain

\[
\frac{1}{2} \frac{d}{dt} \left( \| e^{Rt} \Delta_q^h (\partial_t u) \|_{L^2}^2 + 2 \int e^{2Rt} \Delta_q^h (\partial_t u) \Delta_q^h u \, dx + \| e^{Rt} \Delta_q^h u \|_{L^2}^2 \right) - \| e^{Rt} \Delta_q^h (\partial_t u) \|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \| e^{Rt} \Delta_q^h (\partial_t u) \|_{L^2}^2 + \lambda \hat{\theta}(t) \| e^{Rt} \Delta_q^h |D_x| \left( \frac{1}{2} (\partial_t u) \right) \|_{L^2}^2 + \| e^{Rt} \Delta_q^h (\partial_t u) \|_{L^2}^2 + \| e^{Rt} \Delta_q^h (\partial_t u) \|_{L^2}^2 + \frac{d}{dt} \| e^{Rt} \Delta_q^h (\partial_t u) \|_{L^2}^2
\]

\[
\lesssim 2^{-2q} d_q^2 (2\lambda + 4C) \hat{\theta}(t) \| e^{Rt}(\partial_t u) \|_{\mathcal{H}^{q+\frac{1}{2}}} \| e^{Rt} \partial_x u \|_{\mathcal{H}^{q+\frac{1}{2}}} + 2C 2^{-2q} d_q^2 \hat{\theta}(t) \| e^{Rt} u \|_{\mathcal{H}^{q+\frac{1}{2}}}
\]

(3.19)

Multiplying (3.19) by \( 2^{2q} \) for \( s \in [\frac{1}{2}, 1] \) and then integrating over time, and summing with respect to \( q \in \mathbb{Z} \), we find that for \( t < T^* \)

\[
\left( \frac{1}{2} \| e^{Rt}(u + \partial_t u) \|_{L^\infty(\mathcal{H}^s)}^2 + \| e^{Rt} \partial_x u \|_{L^\infty(\mathcal{H}^s)}^2 \right) + \frac{1}{2} \| e^{Rt}(\partial_t u) \|_{L^\infty(\mathcal{H}^s)}^2 + \lambda \| e^{Rt} \partial_x u \|_{L^\infty(\mathcal{H}^{s+\frac{1}{2}})}^2 + 2\lambda \| e^{Rt}(\partial_t u) \|_{L^\infty(\mathcal{H}^{s+\frac{1}{2}})}^2 + 2 \lambda \| e^{Rt} \partial_x u \|_{L^\infty(\mathcal{H}^{s+\frac{1}{2}})}^2 \lesssim C \| e^{Rt} \partial_x u \|_{\mathcal{H}^{s+\frac{1}{2}}}^2 + C \| e^{Rt} (u + u) \|_{\mathcal{H}^{s+\frac{1}{2}}}^2 + C \| e^{Rt} (u + u) \|_{\mathcal{H}^{s+\frac{1}{2}}}^2
\]

\[
+ (2\lambda + 4C) \| e^{Rt} \partial_x u \|_{L^\infty(\mathcal{H}^{s+\frac{1}{2}})}^2 \| e^{Rt}(\partial_t u) \|_{L^\infty(\mathcal{H}^{s+\frac{1}{2}})}^2 + 2C \| e^{Rt} u \|_{L^\infty(\mathcal{H}^{s+\frac{1}{2}})}^2
\]

(3.20)

Taking \( \lambda = 2C \) and \( 2\lambda = \lambda + 2C \) in the above inequality leads to
Then, for any $0 < t < T$, bounded function $\Theta$ on $\mathbb{R}_+^1$, the width of the analyticity band $\Theta$ is defined by

$$\frac{1}{2}\|e^{Rt}(u + \partial_t u)\|_{\tilde{L}_t^\infty(\mathcal{H}_t^\star)} + \|e^{Rt}\partial_y u\|_{\tilde{L}_t^\infty(\mathcal{H}_t^\star)} + \|e^{Rt}(\partial_t u)_\phi\|_{\tilde{L}_t^2(\mathcal{H}_t^\star)} + \|e^{Rt}\partial_y u\|_{\tilde{L}_t^2(\mathcal{H}_t^\star)} \leq C\|e^{|D_x|}\partial_y u_0\|_{\mathcal{H}_t^\star} + C\|e^{|D_x|}(u_0 + u_1)\|_{\mathcal{H}_t^\star} + C\|e^{|D_x|}u_1\|_{\mathcal{H}_t^\star}, \quad \text{for } t \leq T^\star. \quad (3.21)$$

We recall that we already defined $\dot{\theta}(t) = \|\partial_y u_{\phi}(t)\|_{\mathcal{H}_t^\star}$ with $\frac{1}{2} < s < 1$ and $\theta(0) = 0$. Then, for any $0 < t < T^\star$, Inequality (3.21) yields

$$\theta(t) = \int_0^t \|\partial_y u_{\phi}(t')\|_{\mathcal{H}_t^\star} dt' \leq \int_0^t e^{-\mathcal{R}t'}\|e^{\mathcal{R}t'}\partial_y u_{\phi}(t')\|_{\mathcal{H}_t^\star} dt' \leq \left(\int_0^t e^{-2\mathcal{R}t'} dt'\right)^{\frac{1}{2}} \left(\int_0^t \|e^{\mathcal{R}t'}\partial_y u_{\phi}(t')\|_{\mathcal{H}_t^\star}^2 dt'\right)^{\frac{1}{2}} \leq C \|e^{\mathcal{R}t}\partial_y u\|_{\tilde{L}_t^2(\mathcal{H}_t^\star)} \leq C \left(\|e^{|D_x|}\partial_y u_0\|_{\mathcal{H}_t^\star} + \|e^{|D_x|}(u_0 + u_1)\|_{\mathcal{H}_t^\star} + \|e^{|D_x|}u_1\|_{\mathcal{H}_t^\star}\right) \leq \frac{a}{2\lambda}.$$ 

A continuity argument implies that $T^\star = +\infty$ and we have (3.21) is valid for any $t \in \mathbb{R}_+$.

4. Global well-posedness of the system (1.7)

The goal of this section is to prove the Theorem and to establish the global well-posedness of the system (1.7) with small analytic data. As in Section 2, for any locally bounded function $\Theta$ on $\mathbb{R}_+ \times \mathbb{R}$ and any $u \in L^2(S)$, we define the analyticity in the horizontal variable $x$ by means of the following auxiliary function

$$u_{\phi}(t, x, y) = \mathcal{F}^{-1}_{\xi \rightarrow x}(e^{\Theta(t, \xi)}\hat{u}(t, \xi, y)). \quad (4.1)$$

The width of the analyticity band $\Theta$ is defined by

$$\Theta(t, \xi) = (a - \lambda \tau(t))|\xi|,$$

where $\lambda > 0$ with be precised later and $\tau(t)$ will be chosen in such a way that $\Theta(t, \xi) > 0$, for any $(t, \xi) \in \mathbb{R}_+ \times \mathbb{R}$ and $\dot{\Theta}(t) = \Theta'(t) = -\lambda \dot{\tau}(t) \leq 0$. In our paper, we will choose

$$\dot{\tau}(t) = \|\partial_y u_{\phi}(t)\|_{\mathcal{H}_t^\star} + \epsilon\|\partial_y v_{\phi}(t)\|_{\mathcal{H}_t^\star} \quad \text{with} \quad \tau(0) = 0. \quad (4.2)$$

In what follows, for the sake of the simplicity, we will neglect the script $\epsilon$ and write $(u_{\phi}, v_{\phi})$ instead of $(u_{\phi}^\epsilon, v_{\phi}^\epsilon)$. In view of the system (1.7), we can transform it like a equation of order one in time, so if we define $U = (u, \partial_t u)$ and $V = (v, \partial_t v)$, Then $U$ and
V satisfy the following equation

$$\begin{cases}
\partial_t U + A_\epsilon(D)U = -\left( u\partial_x u + v\partial_y u + \partial_x p \right) \\
\epsilon^2 \left( \partial_t V + B_\epsilon(D)V \right) = -\left( \epsilon^2(u\partial_x v + v\partial_y v) + \partial_y p \right) \\
\partial_x u + \partial_y v = 0 \\
(u, v)_{y=0} = (u, v)_{y=1} = 0
\end{cases} \tag{4.3}$$

where

$$U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix} \text{ and } A_\epsilon(D) = \begin{pmatrix} 0 & -1 \\ -\epsilon^2\partial_x^2 - \partial_y^2 & 1 \end{pmatrix}$$

and

$$V = \begin{pmatrix} v \\ \partial_t v \end{pmatrix} \text{ and } B_\epsilon(D) = \begin{pmatrix} 0 & -1 \\ -\epsilon^2\partial_x^2 - \partial_y^2 & 1 \end{pmatrix}$$

Then in view of (2.3) we observe that \((U, V)_\Theta\) verifies

$$\begin{cases}
\partial_t U_\Theta + \lambda \hat{\tau}(t)|D_x|U_\Theta + A_\epsilon(D)U_\Theta = -\left( u\partial_x u_\Theta + (v\partial_y u_\Theta + \partial_x p_\Theta) \right) \\
\epsilon^2 \left( \partial_t V_\Theta + \lambda \hat{\tau}(t)|D_x|V_\Theta + B_\epsilon(D)V_\Theta \right) = -\left( \epsilon^2(u\partial_x v_\Theta + v\partial_y v_\Theta + \partial_y p_\Theta) \right) \\
\partial_x u_\Theta + \partial_y v_\Theta = 0 \\
(u_\Theta, v_\Theta)_{y=0} = (u_\Theta, v_\Theta)_{y=1} = 0
\end{cases} \tag{4.4}$$

Where \(|D_x|\) denote the Fourier multiplier of the symbol \(|\xi|\). In what follows, we recall that we use "C" to denote a generic positive constant which can change from line to line.

By applying the dyadic operator in the horizontal variable \(\Delta^h_q\) to (4.4) and taking the \(L^2\) inner product of the resulting equation with \(\Delta^h_q U_\Theta\) and \(\Delta^h_q V_\Theta\) we obtain

$$\langle \Delta^h_q \partial_t U_\Theta, \Delta^h_q U_\Theta \rangle_{L^2} + \lambda \hat{\tau}(t) \langle \Delta^h_q |D_x|U_\Theta, \Delta^h_q U_\Theta \rangle_{L^2} + \langle \Delta^h_q A_\epsilon(D)U_\Theta, \Delta^h_q U_\Theta \rangle_{L^2}$$

$$= -\langle \Delta^h_q (u\partial_x u_\Theta + v\partial_y u_\Theta), \Delta^h_q (\partial_t u_\Theta) \rangle_{L^2} - \langle \Delta^h_q \partial_x p_\Theta, \Delta^h_q (\partial_t u_\Theta) \rangle_{L^2} \tag{4.5}$$

and

$$\langle \Delta^h_q \partial_t V_\Theta, \Delta^h_q V_\Theta \rangle_{L^2} + \lambda \hat{\tau}(t) \langle \Delta^h_q |D_x|V_\Theta, \Delta^h_q V_\Theta \rangle_{L^2} + \langle \Delta^h_q B_\epsilon(D)V_\Theta, \Delta^h_q V_\Theta \rangle_{L^2}$$

$$= -\epsilon^2 \langle \Delta^h_q (u\partial_x v_\Theta + v\partial_y v_\Theta), \Delta^h_q (\partial_t v_\Theta) \rangle_{L^2} - \epsilon^2 \langle \Delta^h_q \partial_y p_\Theta, \Delta^h_q (\partial_t v_\Theta) \rangle_{L^2}. \tag{4.6}$$

By using the Dirichlet boundary condition \((u, v)|_{y=0} = (u, v)|_{y=1} = 0\), and the incompressibility condition \(\partial_x u + \partial_y v = 0\) and the relation, we can perform integration by parts, we get

$$\left| \langle \Delta^h_q \nabla p_\Theta, \Delta^h_q (\partial_t u, \partial_t v)_\Theta \rangle_{L^2} \right| = 0.$$
Then by using Lemma 2.1 and by multiplying (4.5) and (4.6) by $e^{2Rt}$, we achieve

$$
\frac{1}{2} \frac{d}{dt} \left( \| e^{Rt} \Delta_q^h (\partial_t u) e \|_{L^2}^2 + \| e^{Rt} \Delta_q^h (\partial_y u) e \|_{L^2}^2 + \| e^{Rt} \Delta_q^h (\epsilon (\partial_t v)) e \|_{L^2}^2 + \| e^{Rt} \Delta_q^h (\epsilon \partial_y v) e \|_{L^2}^2 \right)
$$

$$
+ e^2 \| e^{Rt} \Delta_q^h (\partial_x v) e \|_{L^2}^2 + e^2 \| e^{Rt} \Delta_q^h (\partial_y v) e \|_{L^2}^2 \right)
$$

$$
+ \lambda \tau (t') \| e^{Rt} \Delta_q^h |D_x|^{\frac{1}{2}} (\partial_t u) e \|^2_{L^2} + \lambda \tau (t') \| e^{Rt} \Delta_q^h |D_x|^{\frac{1}{2}} (\partial_y v) e \|^2_{L^2} + e^2 \lambda \tau (t') \| e^{Rt} \Delta_q^h |D_x|^{\frac{1}{2}} (\partial_x v) e \|^2_{L^2}
$$

Then we deduce from the lemma 2.4.2.6, that

$$
\frac{1}{2} \frac{d}{dt} \left( \| e^{Rt} \Delta_q^h (\partial_t u) e \|_{L^2}^2 + \| e^{Rt} \Delta_q^h (\partial_y u) e \|_{L^2}^2 + \| e^{Rt} \Delta_q^h (\epsilon (\partial_t v)) e \|_{L^2}^2 + \| e^{Rt} \Delta_q^h (\epsilon \partial_y v) e \|_{L^2}^2 \right)
$$

$$
+ \| e^{Rt} \Delta_q^h (\partial_x v) e \|_{L^2}^2 + \| e^{Rt} \Delta_q^h (\partial_y v) e \|_{L^2}^2 \right)
$$

Now we still have to get some information of the norm $\| \partial_x u \|_{H^s}$ and $\| \partial_x u \|_{H^s}$, for that we need to apply the dyadic operator $\Delta_q^h$ to the equation

$$
e^{\Theta(t,|D_x|)} \left( \partial_t^2 u + \partial_t u + u \partial_x u + v \partial_y u - \partial_x^2 u + \partial_x p \right) = 0
$$

and then, we take the $L^2$ inner product of the resulting equation (4.9) with $\Delta_q^h u_\Theta$ and $\Delta_q^h v_\Theta$, we obtain

$$
\langle \Delta_q^h (\partial_t^2 u) e, \Delta_q^h u_\Theta \rangle_{L^2} + \langle \Delta_q^h (\partial_t u) e, \Delta_q^h u_\Theta \rangle_{L^2}
$$

$$
- \langle \Delta_q^h \partial_x^2 u e, \Delta_q^h u_\Theta \rangle_{L^2} - \langle \Delta_q^h \partial_x u e, \Delta_q^h u_\Theta \rangle_{L^2}
$$

$$
= - \langle \Delta_q^h (u \partial_x u + v \partial_y u) e, \Delta_q^h u_\Theta \rangle_{L^2} - \langle \Delta_q^h \partial_x p e, \Delta_q^h u_\Theta \rangle_{L^2},
$$

and

$$
\langle \Delta_q^h (\epsilon \partial_t^2 v) e, \Delta_q^h v_\Theta \rangle_{L^2} + \langle \Delta_q^h (\epsilon \partial_t u) e, \Delta_q^h v_\Theta \rangle_{L^2}
$$

$$
- \langle \Delta_q^h \epsilon \partial_x^2 v e, \Delta_q^h v_\Theta \rangle_{L^2} - \langle \Delta_q^h \epsilon \partial_x u e, \Delta_q^h v_\Theta \rangle_{L^2}
$$

$$
= - \langle \Delta_q^h (u \partial_x v + v \partial_y v) e, \Delta_q^h v_\Theta \rangle_{L^2} - \langle \Delta_q^h \partial_y p e, \Delta_q^h v_\Theta \rangle_{L^2},
$$

In what follows, we shall use again the technical lemmas in Section 2, to handle term by term in the estimate (4.10) and (4.11). We start by the complicate term $I_1 = \ldots$
\[ \langle \Delta^h_q (\partial_t^2 u)_{\theta}, \Delta^h_q u_{\theta} \rangle_{L^2} \text{ and } I_2 = \langle \Delta^h_q (\epsilon \partial_t^2 v)_{\theta}, \Delta^h_q \epsilon v_{\theta} \rangle_{L^2}, \] so by using integration by parts, we find

\[
I_1 = \frac{d}{dt} \int \Delta^h_q (\partial_t u)_{\theta} \Delta^h_q u_{\theta} dx - \int \Delta^h_q (\partial_t u)_{\theta} \Delta^h_q (\partial_t u)_{\theta} dx + 2\lambda \dot{t}(t) \int \Delta^h_q |D_x| (\partial_t u)_{\theta} \Delta^h_q u_{\theta} dx
\]

\[
I_2 = \frac{d}{dt} \int \Delta^h_q (\epsilon \partial_t v)_{\theta} \Delta^h_q \epsilon v_{\theta} dx - \int \Delta^h_q (\epsilon \partial_t v)_{\theta} \Delta^h_q (\epsilon \partial_t v)_{\theta} dx + 2\lambda \dot{t}(t) \int \Delta^h_q |D_x| (\epsilon \partial_t v)_{\theta} \Delta^h_q \epsilon v_{\theta} dx
\]

Whereas due to the boundary condition, and by integrating by part, we achieve

\[
\langle \Delta^h_q - (\partial_t^2 u_{\theta} + \epsilon^2 \partial_x^2 u_{\theta}), \Delta^h_q u_{\theta} \rangle_{L^2} = ||\Delta^h_q \partial_y u_{\theta}||^2_{L^2} + \epsilon^2 ||\Delta^h_q \partial_x u_{\theta}||^2_{L^2}
\]

\[
\langle \Delta^h_q (-\epsilon \partial_t^2 v_{\theta} - \epsilon^3 \partial_x^2 v_{\theta}), \epsilon \Delta^h_q v_{\theta} \rangle_{L^2} = ||\Delta^h_q \epsilon \partial_y v_{\theta}||^2_{L^2} + \epsilon^2 ||\Delta^h_q \epsilon \partial_x v_{\theta}||^2_{L^2}.
\]

Now, by using the Dirichlet boundary condition \((u, v)|_{y=0} = (u, v)|_{y=1} = 0\), and the incompressibility condition \(\partial_x u + \partial_y v = 0\) and the relation \(\partial_y u_{\theta} = 0\), we can find by integrating by parts the estimate of the pressure

\[
\left| \langle \Delta^h_q \nabla p_{\theta}, \Delta^h_q (u, v)_{\theta} \rangle_{L^2} \right| = \left| \langle \Delta^h_q p_{\theta}, \Delta^h_q \text{ div } (u, v)_{\theta} \rangle \right| = 0.
\]

Then by using the Lemma 2.1 and by multiplying (4.10) and (4.11) by \(e^{2Rt}\), and then integrating the resulting inequality over time, we achieve

\[
\frac{d}{dt} \int e^{2Rt} \Delta^h_q (\partial_t u)_{\theta} \Delta^h_q u_{\theta} dx - \int e^{2Rt} \Delta^h_q (\partial_t u)_{\theta} \Delta^h_q (\partial_t u)_{\theta} dx + 2\lambda \dot{t}(t) \int e^{2Rt} \Delta^h_q |D_x| (\partial_t u)_{\theta} \Delta^h_q u_{\theta} dx
\]

\[
\frac{d}{dt} \int e^{2Rt} \Delta^h_q (\partial_t v)_{\theta} \Delta^h_q v_{\theta} dx - \int e^{2Rt} \Delta^h_q (\partial_t v)_{\theta} \Delta^h_q (\partial_t v)_{\theta} dx + 2\lambda \dot{t}(t) \int e^{2Rt} \Delta^h_q |D_x| (\partial_t v)_{\theta} \Delta^h_q v_{\theta} dx
\]

\[
+ \frac{1}{2} \frac{d}{dt} \|e^{Rt} \Delta^h_q u_{\theta}\|^2_{L^2} + \frac{1}{2} \frac{d}{dt} \|e^{Rt} \Delta^h_q v_{\theta}\|^2_{L^2} + \lambda \dot{t}(t) \|e^{Rt} \Delta^h q |D_x| \frac{1}{2} (u_{\theta}, v_{\theta})\|^2_{L^2} + \|e^{Rt} \Delta^h q \partial_y u_{\theta}\|^2_{L^2} +
\]

\[
\|e^{Rt} \Delta^h q \partial_y v_{\theta}\|^2_{L^2} + \epsilon^2 \|e^{Rt} \Delta^h q \partial_x u_{\theta}\|^2_{L^2} + \epsilon^2 \|e^{Rt} \Delta^h q \partial_x v_{\theta}\|^2_{L^2}\] (4.12)

\[
= - \langle \Delta^h_q (u \partial_x u + v \partial_y u)_{\theta}, e^{2Rt} \Delta^h_q u_{\theta} \rangle_{L^2} - \epsilon^2 \langle \Delta^h_q (u \partial_x v + v \partial_y v)_{\theta}, e^{2Rt} \Delta^h_q v_{\theta} \rangle_{L^2}.
\]

In view, of the lemma 2.4-2.5, and by summing 2 \times (4.8) with (4.12), we obtain

\[
\frac{d}{dt} \left( \|e^{Rt} \Delta^h_q (\partial_t u)_{\theta}\|^2_{L^2} \right) + \int e^{2Rt} \Delta^h_q (\partial_t u)_{\theta} \Delta^h_q u_{\theta} dx + \frac{1}{2} \|e^{Rt} \Delta^h q |D_x| \frac{1}{2} (u_{\theta}, v_{\theta})\|^2_{L^2} + \|e^{Rt} \Delta^h q \partial_y u_{\theta}\|^2_{L^2}
\]

\[
\frac{d}{dt} \left( \|e^{Rt} \Delta^h_q (\partial_t v)_{\theta}\|^2_{L^2} \right) + \int e^{2Rt} \Delta^h_q (\partial_t v)_{\theta} \Delta^h_q v_{\theta} dx + \frac{1}{2} \|e^{Rt} \Delta^h q |D_x| \frac{1}{2} (u_{\theta}, v_{\theta})\|^2_{L^2} + \|e^{Rt} \Delta^h q \partial_y v_{\theta}\|^2_{L^2}
\]

\[
+ \lambda \dot{t}(t) \|e^{Rt} \Delta^h q |D_x| \frac{1}{2} (u_{\theta}, v_{\theta})\|^2_{L^2} + \|e^{Rt} \Delta^h q \partial_y u_{\theta}\|^2_{L^2} + \epsilon^2 \|e^{Rt} \Delta^h q \partial_x u_{\theta}\|^2_{L^2} + \epsilon^2 \|e^{Rt} \Delta^h q \partial_x v_{\theta}\|^2_{L^2}\] (4.13)

\[
+ \frac{d}{dt} \|e^{Rt} \Delta^h q \partial_y (u, v)_{\theta}\|^2_{L^2} + \epsilon^2 \frac{d}{dt} \|e^{Rt} \Delta^h q \partial_y (u, v)_{\theta}\|^2_{L^2} + 2\lambda \dot{t}(t) \|e^{Rt} \Delta^h q |D_x| \frac{1}{2} (\partial_y u, \epsilon \partial_y v)_{\theta}\|^2_{L^2} + \lambda \dot{t}(t) \|e^{Rt} \Delta^h q |D_x| \frac{1}{2} (\partial_y u, \epsilon \partial_y v)_{\theta}\|^2_{L^2}.
\]
\begin{align*}
\lesssim 4C^{-2q_0}a^2\tau(t) \left( \|e^{Rt}\partial_\psi u_\psi\|_{\mathcal{H}^{q+\frac{1}{2}}} + \|e^{Rt}\partial_\psi u_\psi\|_{\mathcal{H}^{q+\frac{1}{2}}} + \|e^{Rt}\partial_\psi u_\psi\|_{\mathcal{H}^{q+\frac{1}{2}}}
+ 2\lambda^{-2q_0}a^2\tau(t) \|e^{Rt}\partial_\psi u_\psi\|_{\mathcal{H}^{q+\frac{1}{2}}} + 2C^{-2q_0}a^2\tau(t) \|e^{Rt}(u_\psi, v_\psi)\|_{\mathcal{H}^{q+\frac{1}{2}}}
\right)
+ 2\lambda^{-2q_0}a^2\tau(t) \|e^{Rt}\partial_\psi u_\psi\|_{\mathcal{H}^{q+\frac{1}{2}}} + 2C^{-2q_0}a^2\tau(t) \|e^{Rt}(u_\psi, v_\psi)\|_{\mathcal{H}^{q+\frac{1}{2}}}.
\end{align*}

Multiplying (4.13) by $2^q s$ for $s \in [\frac{1}{2}, 1]$ and then integrating over time, and summing with respect to $g \in \mathbb{Z}$, we find that for $t < T^*$

\begin{align*}
\left(\frac{1}{2}\|e^{Rt}(u+\partial_t u, e(v+\partial_t v))\|_{\mathcal{L}^2(H^s)}^2 + \|e^{Rt}\partial_\psi u_\psi\|_{\mathcal{L}^2(H^s)}^2, \|e^{Rt}\partial_\psi u_\psi\|_{\mathcal{L}^2(H^s)}^2 &\right)
+ \frac{1}{2}\|e^{Rt}(\partial_t u_\psi, \partial_t v_\psi)\|_{\mathcal{L}^2(H^s)}^2 + \|e^{Rt}\partial_\psi u_\psi\|_{\mathcal{L}^2(H^s)}^2 + 2\lambda \|e^{Rt}(u_\psi, v_\psi)\|_{\mathcal{L}^2(H^s)}^2
\end{align*}

\begin{align*}
+ 2\lambda \|e^{Rt}(u_\psi, v_\psi)\|_{\mathcal{L}^2(H^s)}^2 &\leq C\|e^{[D_1]}(u_0, v_0)\|_{\mathcal{H}^{s+\frac{1}{2}}}^2 + C\|e^{[D_1]}(u_0, v_0)\|_{\mathcal{H}^{s+\frac{1}{2}}}^2
+ C\|e^{[D_1]}(u_1, v_1)\|_{\mathcal{H}^{s+\frac{1}{2}}}^2 + C\|e^{[D_1]}(u_0, v_0)\|_{\mathcal{H}^{s+\frac{1}{2}}}^2
+ (2\lambda + 4C)\|e^{Rt}(u_\psi, v_\psi)\|_{\mathcal{L}^2(H^s)}^2 \|e^{Rt}(\partial_t u_\psi)\|_{\mathcal{L}^2(H^s)}^2 + 2C\|e^{Rt}(u_\psi, v_\psi)\|_{\mathcal{L}^2(H^s)}^2.
\end{align*}

Taking $\lambda = 2C$ and $2\lambda = 2C$ in the above inequality leads to

\begin{align*}
\left(\|e^{Rt}(u+\partial_t u, e(v+\partial_t v))\|_{\mathcal{L}^2(H^s)}^2 + \|e^{Rt}\partial_\psi u_\psi\|_{\mathcal{L}^2(H^s)}^2, \|e^{Rt}\partial_\psi u_\psi\|_{\mathcal{L}^2(H^s)}^2 &\right)
+ \frac{1}{2}\|e^{Rt}(\partial_t u_\psi, \partial_t v_\psi)\|_{\mathcal{L}^2(H^s)}^2 + \|e^{Rt}\partial_\psi u_\psi\|_{\mathcal{L}^2(H^s)}^2 + 2\lambda \|e^{Rt}(u_\psi, v_\psi)\|_{\mathcal{L}^2(H^s)}^2
\end{align*}

\begin{align*}
+ 2\lambda \|e^{Rt}(u_\psi, v_\psi)\|_{\mathcal{L}^2(H^s)}^2 &\leq C\|e^{[D_1]}(u_0, v_0)\|_{\mathcal{H}^{s+\frac{1}{2}}}^2 + C\|e^{[D_1]}(u_0, v_0)\|_{\mathcal{H}^{s+\frac{1}{2}}}^2
+ C\|e^{[D_1]}(u_1, v_1)\|_{\mathcal{H}^{s+\frac{1}{2}}}^2 + C\|e^{[D_1]}(u_0, v_0)\|_{\mathcal{H}^{s+\frac{1}{2}}}^2
+ (2\lambda + 4C)\|e^{Rt}(u_\psi, v_\psi)\|_{\mathcal{L}^2(H^s)}^2 \|e^{Rt}(\partial_t u_\psi)\|_{\mathcal{L}^2(H^s)}^2 + 2C\|e^{Rt}(u_\psi, v_\psi)\|_{\mathcal{L}^2(H^s)}^2.
\end{align*}

We recall that we already defined $\tilde{\tau}(t) = \|\partial_t u_\psi(t)\|_{\mathcal{H}^s} + \|\partial_t v_\psi(t)\|_{\mathcal{H}^s}$ with $\tau(0) = 0$. Then, for any $0 < t < T^*$, Inequality (3.21) yields

\begin{align*}
\tau(t) &= \int_0^t \|\partial_t u_\psi(t')\|_{\mathcal{H}^s} + \|\partial_t v_\psi(t')\|_{\mathcal{H}^s} dt'
\leq \int_0^t e^{-\lambda t'} \|e^{Rt'}(u_\psi, v_\psi)\|_{\mathcal{L}^2(H^s)} dt'
\leq \left( \int_0^t e^{-\lambda t'} dt' \right)^{\frac{1}{2}} \left( \int_0^t \|e^{Rt'}(u_\psi, v_\psi)\|_{\mathcal{L}^2(H^s)}^2 dt' \right)^{\frac{1}{2}}
\leq C\|e^{[D_1]}(u_0, v_0)\|_{\mathcal{H}^{s+\frac{1}{2}}}^2 + 2\|e^{[D_1]}(u_0, v_0)\|_{\mathcal{H}^{s+\frac{1}{2}}}^2 + 2\|e^{[D_1]}(u_1, v_1)\|_{\mathcal{H}^{s+\frac{1}{2}}}^2 + \frac{a}{\lambda}_2.
\end{align*}
A continuity argument implies that $T^* = +\infty$ and we have (3.21) is valid for any $t \in \mathbb{R}_+$. 

5. Propagation of the regularity and of the vorticity of the hyperbolic Parndtl equation (1.8)

In this section, we present first a proposition states the propagation for any $\mathcal{H}^s$ regularity on the solution of the perturbed hyperbolic Navier-Stokes equations (1.8). The second proposition allows us to control two derivatives in the normal direction $\partial_\nu$ in any $\mathcal{H}^s$, despite the difficulties raised by the boundary conditions. Those propositions will be useful in the last section when we prove the global convergence of the Theorem 1.3.

Proposition 5.1. We assume that the condition (1.10) is satisfied, then for any $s > \frac{1}{2}$ and $\frac{1}{2} < \delta < 1$, there exist a small constant $C$, such that for

$$
\lambda = C(1 + \|e^{\alpha[D_x]}\partial_t u_0\|_{\mathcal{H}^{s+1}} + \|e^{\alpha[D_x]}(u_0 + u_1)\|_{\mathcal{H}^{s+1}} + \|e^{\alpha[D_x]}u_1\|_{\mathcal{H}^{s+1}}),
$$

we have

$$
\left( \frac{1}{2} \|e^{RT}(u + \partial_t u_0)\|_{L^\infty_t(\mathcal{H}^s)} + \frac{1}{2} \|e^{RT}(\partial_t u_0)\|_{L^\infty_t(\mathcal{H}^s)} + \|e^{RT}(\partial_t u_0)\|_{L^\infty_t(\mathcal{H}^s)} + \|e^{RT}(\partial_t u_0)\|_{L^\infty_t(\mathcal{H}^s)} \right)
$$

$$
\leq C\|e^{\alpha[D_x]}(\partial_t u_0)\|_{\mathcal{H}^s} + C\|e^{\alpha[D_x]}(u_0 + u_1)\|_{\mathcal{H}^s} + C\|e^{\alpha[D_x]}u_1\|_{\mathcal{H}^s}, \quad \text{for } t < T^*. \quad (5.1)
$$

Proof of the proposition 5.1. We start by the first estimate of the temperature equation, we deduce from the lemma 2.4 that for any $s > \frac{1}{2}$ we have

$$
\int_0^t \langle \Delta_q^h(u\partial_t u_0, \Delta_q^h(\partial_t u_0)) \rangle_{L^2} dt' \leq Cd_q^2 2^{-2q^s} \|u_0\|_{L^2_{t,\delta(t)}(\mathcal{H}^{s+\frac{1}{2}})} \|\partial_t u\|_{L^2_{t,\delta(t)}(\mathcal{H}^{s+\frac{1}{2}})}. \quad (5.2)
$$

In view of the proof the lemma 2.4, we need only to proof that

$$
\int_0^t \langle \Delta_q^h(T_{\partial_\nu u} u_0, \Delta_q^h(\partial_t u_0)) \rangle_{L^2} dt' \leq Cd_q^2 2^{-2q^s} \|u_0\|_{L^2_{t,\delta(t)}(\mathcal{H}^{s+\frac{1}{2}})} \|\partial_t u\|_{L^2_{t,\delta(t)}(\mathcal{H}^{s+\frac{1}{2}})}.
$$

We infer

$$
\int_0^t \langle \Delta_q^h(T_{\partial_\nu u} u_0, \Delta_q^h(\partial_t u_0)) \rangle_{L^2} dt' \leq \sum_{|q'-q|\leq 4} \int_0^t \|S^h_{q'-1} \partial_\nu u_\phi(t')\|_{L^\infty} \|\Delta_q^h u_\phi(t')\|_{H^s} \|\Delta_q^h(\partial_t u_\phi(t'))\|_{L^2} dt'
$$

$$
\leq \sum_{|q'-q|\leq 4} 2q' \int_0^t \|\partial_\nu u_\phi(t')\|_{H^s} \|\Delta_q^h u_\phi(t')\|_{L^2} \|\Delta_q^h(\partial_t u_\phi(t'))\|_{L^2} dt'
$$

$$
\leq \sum_{|q'-q|\leq 4} 2q' \left( \int_0^t \|\partial_\nu u_\phi(t')\|_{H^s} \|\Delta_q^h u_\phi(t')\|_{L^2} dt' \right)^\frac{1}{2} \times \left( \int_0^t \|\partial_\nu u_\phi(t')\|_{H^s} \|\Delta_q^h(\partial_t u_\phi(t'))\|_{L^2} dt' \right)^\frac{1}{2},
$$

30
where \( \frac{1}{2} < \delta < 1 \), which leads to (5.2).

While it follows also from the proof of the lemma 2.4 that

\[
\int_0^t |(\Delta_q^h(T^h_{\theta}v, u + R^h(\nu, \partial_y u))_\phi, \Delta_q^h(\partial_t u)_\phi)_L^2| dt' \leq d_q^2 2^{-2q\delta} \| u_\phi \|_{L^2_{\theta, \delta}(\mathcal{H}^{q+\frac{1}{2}})} \| (\partial_t u)_\phi \|_{L^2_{\theta, \delta}(\mathcal{H}^{q+\frac{1}{2}})} ;
\]

so we have yet to determine the estimate of \( \int_0^t |(\Delta_q^h(T^h_{\nu}T^h_{\phi}, \Delta_q^h T^h_{\phi})_L^2| dt' \), we have

\[
\| \Delta_q^h v_\phi(t) \|_{L^\infty} \lesssim d_q(t) 2^\frac{q}{2} \| u_\phi(t) \|_{\mathcal{H}_{q+1}} \| \partial_y u_\phi(t) \|_{\mathcal{H}_{q+1}} ;
\]

so that

\[
\| S^h_{\theta-1} v_\phi(t') \|_{L^\infty} \lesssim 2^\frac{q}{2} \| u_\phi(t) \|_{\mathcal{H}_{q+1}} \| \partial_y u_\phi(t) \|_{\mathcal{H}_{q+1}} ;
\]

which implies that

\[
\int_0^t |(\Delta_q^h(T^h_{\nu} \partial_y u)_\phi, \Delta_q^h(\partial_t u)_\phi)_L^2| dt' \leq \sum_{|\nu - q| \leq 4} \int_0^t \| S^h_{\theta-1} v_\phi(t') \|_{L^\infty} |(\Delta_q^h \partial_y u_\phi(t'), \Delta_q^h(\partial_t u)_\phi(t'))_L^2| dt'
\]

\[
\leq \sum_{|\nu - q| \leq 4} 2^\frac{q}{2} \| u_\phi \|_{L^\infty(\mathcal{H}^{q+\frac{1}{2}})} \| \Delta_q^h \partial_y u_\phi \|_{L^2(\mathcal{H})} \left( \int_0^t \| \partial_y u_\phi(t') \|_{\mathcal{H}^{q+1}} \| \Delta_q^h(\partial_t u)_\phi(t') \|_{L^2} dt' \right)^\frac{1}{2}
\]

\[
\leq d_q^2 2^{-2q\delta} \| u_\phi \|_{L^\infty(\mathcal{H}^{q+\frac{1}{2}})} \| \partial_y u_\phi \|_{L^2(\mathcal{H})} \| (\partial_t u)_\phi \|_{L^2_{\theta, \delta}(\mathcal{H}^{q+\frac{1}{2}})} ;
\]

By summing all the terms we obtain

\[
\int_0^t |(\Delta_q^h(\nu \partial_y u)_\phi, \Delta_q^h(\partial_t u)_\phi)_L^2| dt' \leq d_q^2 2^{-2q\delta} \| (\partial_t u)_\phi \|_{L^2_{\theta, \delta}(\mathcal{H}^{q+\frac{1}{2}})} \times \left( \| u_\phi \|_{L^2_{\theta, \delta}(\mathcal{H}^{q+\frac{1}{2}})} + \| u_\phi \|_{L^2(\mathcal{H}^{q+\frac{1}{2}})} \| \partial_y u_\phi \|_{L^2(\mathcal{H})} \right) .
\]

(5.3)

Along the same way we can obtain

\[
\int_0^t \| (\Delta_q^h(\nu \partial_y u)_\phi, \Delta_q^h(\partial_t u)_\phi) \|_{L^2} \leq C d_q^2 2^{-2q\delta} \| u_\phi \|_{L^2_{\theta, \delta}(\mathcal{H}^{q+\frac{1}{2}})}^2
\]

(5.4)

and

\[
\int_0^t \| (\Delta_q^h(\nu \partial_y u)_\phi, \Delta_q^h(\partial_t u)_\phi) \|_{L^2} \leq d_q^2 2^{-2q\delta} \| u_\phi \|_{L^2_{\theta, \delta}(\mathcal{H}^{q+\frac{1}{2}})} \times \left( \| u_\phi \|_{L^2_{\theta, \delta}(\mathcal{H}^{q+\frac{1}{2}})} + \| u_\phi \|_{L^2(\mathcal{H}^{q+\frac{1}{2}})} \| \partial_y u_\phi \|_{L^2(\mathcal{H})} \right) .
\]

(5.5)

By virtue of (5.2), (5.3) (5.4) and (5.5), we deduce from (3.9) and (3.15) that

\[
\frac{d}{dt} \left( \| e^{R_t} \Delta_q^h(\partial_t u)_\phi \|_{L^2}^2 + e^{2R_t} \Delta_q^h(\partial_t u)_\phi \Delta_q^h u_\phi dx + \frac{1}{2} \| e^{R_t} \Delta_q^h u_\phi \|_{L^2}^2 \right) - \| e^{R_t} \Delta_q^h(\partial_t u)_\phi \|_{L^2}^2
\]

\[
+ \lambda \hat{\theta}(t) e^{R_t} \Delta_q^h D_x \frac{1}{2} u_\phi \|_{L^2}^2 + || e^{R_t} \Delta_q^h \partial_y u_\phi \|_{L^2}^2 + \frac{d}{dt} \| e^{R_t} \Delta_q^h \partial_y u_\phi \|_{L^2}^2
\]

(5.6)
+ 2\|e^{Rt} \Delta_q (\partial_t u)_\phi\|_{L^2}^2 + 2\lambda \hat{\theta}(t') \|e^{Rt} \Delta_q^h D_x |\frac{1}{2} (\partial_t u)_\phi\|_{L^2}^2 + 2\lambda \hat{\theta}(t') \|e^{Rt} \Delta_q^h D_x \frac{1}{\lambda} \partial_y u_\phi\|_{L^2}^2 \\
\leq 2^{-2q} \hat{d}_q^2 (\lambda + 2C) \hat{\theta}(t) \|e^{Rt} ((\partial_t u)_\phi, \partial_y u_\phi)\|_{N^{q+\frac{1}{2}}}^2 + 2C^2 2^{-2q} \hat{d}_q^2 \hat{\theta}(t) \|e^{Rt} u_\phi\|_{N^{q+\frac{1}{2}}}^2 \\
+ Cd^2 q^{-2q} \left( \|u_\phi\|_{N^{q+\frac{1}{2}}} \|\partial_y u_\phi\|_{N^{q+\frac{1}{2}}} \|\partial_y u_\phi\|_{N^*} + \|((\partial_t u)_\phi)\|_{N^{q+\frac{1}{2}}} \|\partial_y u_\phi\|_{N^{q+\frac{1}{2}}} \|\partial_y u_\phi\|_{N^*} \right) \\

Multiplying (5.6) by 2^{2q} s$ for $s > \frac{1}{2}$ and $\frac{1}{2} < \delta < 1$ and then integrating over time, and summing with respect to $q \in \mathbb{Z}$, we find that for $t < T^*$

\[
\left( \frac{1}{2} \|e^{Rt} (u + \partial_t u)_\phi\|_{L^2_{t, \theta}(N^{q+\frac{1}{2}})}^2 + \frac{1}{2} \|e^{Rt} (\partial_t u)_\phi\|_{L^2_{t, \theta}(N^q)}^2 + \|e^{Rt} \partial_y u_\phi\|_{L^2_{t, \theta}(N^q)}^2 \right) + \|e^{Rt} (\partial_t u)_\phi\|_{L^2_{t, \theta}(N^q)}^2 \\
+ \lambda \|e^{Rt} u_\phi\|_{L^2_{t, \theta}(N^{q+\frac{1}{2}})}^2 + \|e^{Rt} \partial_y u_\phi\|_{L^2_{t, \theta}(N^q)}^2 + 2\lambda \|e^{Rt} (\partial_t u)_\phi\|_{L^2_{t, \theta}(N^{q+\frac{1}{2}})}^2 + 2\lambda \|e^{Rt} \partial_y u_\phi\|_{L^2_{t, \theta}(N^{q+\frac{1}{2}})}^2 \\
\leq C \|\partial_t u_\phi\|_{L^2_{t, \theta}(N^{q+\frac{1}{2}})}^2 + \|\partial_y u_\phi\|_{L^2_{t, \theta}(N^q)}^2 + C \|s|D_s| (u_0 + u_1)\|_{N^q}^2 + C \|s|D_s| u_1\|_{N^q}^2 \\
+ (\lambda + 2\lambda) \|e^{Rt} u_\phi\|_{L^2_{t, \theta}(N^{q+\frac{1}{2}})}^2 + \|e^{Rt} \partial_y u_\phi\|_{L^2_{t, \theta}(N^q)}^2 + (\lambda + 2\lambda) \|e^{Rt} u_\phi\|_{L^2_{t, \theta}(N^{q+\frac{1}{2}})}^2 + \|e^{Rt} \partial_y u_\phi\|_{L^2_{t, \theta}(N^q)}^2. \tag{5.7}
\]

Applying Young's inequality yields

\[
C \|u_\phi\|_{L^2_{t, \theta}(N^{q+\frac{1}{2}})}^2 + \|e^{Rt} \partial_y u_\phi\|_{L^2_{t, \theta}(N^q)}^2 + \|e^{Rt} u_\phi\|_{L^2_{t, \theta}(N^{q+\frac{1}{2}})}^2 \\
\leq C \|u_\phi\|_{L^2_{t, \theta}(N^{q+\frac{1}{2}})}^2 + \|e^{Rt} \partial_y u_\phi\|_{L^2_{t, \theta}(N^q)}^2 + \|e^{Rt} u_\phi\|_{L^2_{t, \theta}(N^{q+\frac{1}{2}})}^2 + \frac{1}{2} \|e^{Rt} \partial_y u_\phi\|_{L^2_{t, \theta}(N^q)}^2. \tag{5.7}
\]

Then we achieve

\[
\left( \frac{1}{2} \|e^{Rt} (u + \partial_t u)_\phi\|_{L^2_{t, \theta}(N^{q+\frac{1}{2}})}^2 + \frac{1}{2} \|e^{Rt} (\partial_t u)_\phi\|_{L^2_{t, \theta}(N^q)}^2 + \|e^{Rt} \partial_y u_\phi\|_{L^2_{t, \theta}(N^q)}^2 \right) + \|e^{Rt} (\partial_t u)_\phi\|_{L^2_{t, \theta}(N^q)}^2 \\
+ \lambda \|e^{Rt} u_\phi\|_{L^2_{t, \theta}(N^{q+\frac{1}{2}})}^2 + \|e^{Rt} \partial_y u_\phi\|_{L^2_{t, \theta}(N^q)}^2 + 2\lambda \|e^{Rt} (\partial_t u)_\phi\|_{L^2_{t, \theta}(N^{q+\frac{1}{2}})}^2 + 2\lambda \|e^{Rt} \partial_y u_\phi\|_{L^2_{t, \theta}(N^{q+\frac{1}{2}})}^2 \\
\leq C \|s|D_s| \partial_y u_0\|_{N^q}^2 + C \|s|D_s| (u_0 + u_1)\|_{N^q}^2 + C \|s|D_s| u_1\|_{N^q}^2 \\
+ C(2 + \|\partial_y u_\phi\|_{L^2_{t, \theta}(N^{q+\frac{1}{2}})}^2) \|e^{Rt} (u, \partial_t u)_\phi\|_{L^2_{t, \theta}(N^{q+\frac{1}{2}})}^2 + \|e^{Rt} (u, \partial_t u)_\phi\|_{L^2_{t, \theta}(N^{q+\frac{1}{2}})}^2. \tag{5.8}
\]

Therefore if we take

\[
\lambda \geq C \left( 2 + \|\partial_y u_\phi\|_{L^2_{t, \theta}(N^{q+\frac{1}{2}})}^2 \right), \tag{5.9}
\]

we obtain

\[
\left( \frac{1}{2} \|e^{Rt} (u + \partial_t u)_\phi\|_{L^2_{t, \theta}(N^q)}^2 + \|e^{Rt} \partial_y u_\phi\|_{L^2_{t, \theta}(N^q)}^2 + \frac{1}{2} \|e^{Rt} (\partial_t u)_\phi\|_{L^2_{t, \theta}(N^q)}^2 \right) + \|e^{Rt} (\partial_t u)_\phi\|_{L^2_{t, \theta}(N^q)}^2 \\
+ \|e^{Rt} \partial_y u_\phi\|_{L^2_{t, \theta}(N^q)}^2 \leq C \left( \|s|D_s| \partial_y u_0\|_{N^q}^2 + \|s|D_s| (u_0 + u_1)\|_{N^q}^2 + \|s|D_s| u_1\|_{N^q}^2 \right) \tag{5.9}
\]
which in particular implies that under the condition (5.9), there hold
\[ \| \partial_y u \phi \|_{L^\infty(H^t)} \leq C \| e^{a|D_x|} \partial_y u_0 \|_{H^{t+1}} + C \| e^{a|D_x|} (u_0 + u_1) \|_{H^{t+1}} + C \| e^{a|D_x|} u_1 \|_{H^{t+1}}. \]

Then by taking \( \lambda = C(2 + \| e^{a|D_x|} \partial_y u_0 \|_{H^{t+1}} + \| e^{a|D_x|} (u_0 + u_1) \|_{H^{t+1}} + \| e^{a|D_x|} u_1 \|_{H^{t+1}}) \), where \( \frac{1}{2} < \delta < 1 \). Therefore the condition of the proposition is satisfied and then the proposition is proved.

**Proposition 5.2.** We assume that the condition (1.10) is satisfied, then for any \( s > 0 \), there exist a small constant \( C \), such that for
\[
\lambda = C(1 + \| e^{a|D_x|} \partial_y u_0 \|_{H^{t+1}} + \| e^{a|D_x|} (u_0 + u_1) \|_{H^{t+1}} + \| e^{a|D_x|} u_1 \|_{H^{t+1}}),
\]
we have
\[
\left( \frac{1}{2} \| e^{Rt} \partial_y (u + \partial_t u) \|_{L^\infty(C^s)} + \frac{1}{2} \| e^{Rt} \partial_y (\partial_t u) \|_{L^\infty(C^s)} + \| e^{Rt} \partial_y^2 u_0 \|_{L^\infty(C^s)} \right)
+ \| e^{Rt} \partial_y (u + \partial_t u) \|_{L^2(C^s)} + \| e^{Rt} \partial_y^2 u_0 \|_{L^2(C^s)} \leq C \| e^{a|D_x|} \partial_y u_0 \|_{C^1} + C \| e^{a|D_x|} \partial_y u_1 \|_{C^1}, \quad \text{for } t < T^*. \tag{5.10}
\]

**Proof.** of the proposition 5.2. We start our proof by applying the partial derivative on \( y (\partial_y) \) to (1.8), we obtain
\[
\partial_t \partial_y u + \partial_t \partial_y u + \partial_y (u \partial_x u) + \partial_y (v \partial_y u) = \partial_y (u \partial_x u) + \partial_y (v \partial_y u) - \partial_y^2 u + \partial_x \partial_y p = 0.
\]
Due to the free divergence condition of \( U \) (it mean that \( \partial_x u + \partial_y v = 0 \)), we have
\[
\partial_y (u \partial_x u) + \partial_y (v \partial_y u) = \partial_y (u \partial_x u) + \partial_y (u \partial_x \partial_y u) + \partial_y (v \partial_y u) + v \partial_y^2 u
= \partial_y (u \partial_x u) + \partial_y (u \partial_x \partial_y u) + \partial_x u \partial_y u + v \partial_y^2 u
= \partial_x \partial_y u + v \partial_y^2 u,
\]
so our equation becomes
\[
\partial_t^2 w + \partial_t w + u \partial_x w + v \partial_y w - \partial_y^2 w + \partial_x q = 0 \tag{5.11}
\]
where \( w = \partial_y u \) and \( q = \partial_y p \). From which, using that \( -\partial_y w + \partial_x q \) is vanishing on the boundary, we get, by using a similar derivation of (3.9) and (3.15), it mean that we do the scalar product of our equation with \( \Delta^h_q (w + 2 \partial_t w) \), so that
\[
\frac{d}{dt} \left( \| e^{Rt} \Delta^h_q (\partial_t w) \|_{L^2}^2 + \| e^{Rt} \Delta^h_q \partial_y w_\phi \|_{L^2}^2 \right) + 2 \| e^{Rt} \Delta^h_q (\partial_t w) \|_{L^2}^2
+ 2 \lambda \hat{\theta}(t') \| e^{Rt} \Delta^h_q |D_x|^2 (\partial_t w) \|_{L^2}^2 + 2 \lambda \hat{\theta}(t') \| e^{Rt} \Delta^h_q |D_x|^2 \partial_y w_\phi \|_{L^2}^2
\leq C \left( \| \Delta^h_q (u \partial_x w) \|_{L^2} + \| \Delta^h_q (v \partial_y w) \|_{L^2} + \| \Delta^h_q (u \partial_x \partial_y w) \|_{L^2} \right)
+ \int \Delta^h_q \partial_x p_\phi \cdot e^{2Rt} \Delta^h_q (\partial_t w(t, x, 1) - \partial_t w(t, x, 0)) \phi \, dx, \tag{5.12}
\]
and
\[
\frac{d}{dt} \int e^{2Rt} \Delta^h_q (\partial_t w) \Delta^h_q \partial_y w_\phi \, dx = - \int e^{2Rt} \Delta^h_q (\partial_t w) \Delta^h_q (\partial_t w) \phi \, dx + 2 \lambda \hat{\theta}(t) \int e^{2Rt} \Delta^h_q |D_x| (\partial_t w) \Delta^h_q w_\phi \, dx
\]
\[ + \frac{1}{2} \frac{d}{dt} \|e^{Rt} \Delta_q^h w_\phi\|_{L^2}^2 + \lambda \hat{\theta}(t) \|e^{Rt} \Delta_q^h |D_x|^\frac{1}{2} w_\phi\|_{L^2}^2 + \|e^{Rt} \Delta_q^h \partial_y w_\phi\|_{L^2}^2 \]

(5.13)

\[ = - \left\langle \Delta_q^h (u \partial_x w_\phi), e^{2Rt} \Delta_q^h w_\phi \right\rangle_{L^2} - \left\langle \Delta_q^h (v \partial_y w_\phi), e^{2Rt} \Delta_q^h w_\phi \right\rangle_{L^2} + \int_R \Delta_q^h \partial_x p_\phi \cdot e^{2Rt} \Delta_q^h (w(t, x, 1) - w(t, x, 0))_\phi dx. \]

Now, we start to estimate the pressure term, for which we denote

\[ K_q = \int_R \Delta_q^h \partial_x p_\phi \cdot \Delta_q^h (\partial_t w(t, x, 1) - \partial_t w(t, x, 0))_\phi dx. \]

In view of (3.4) and the lemma 2.3, we write

\[ K_q = \int_R \Delta_q^h \partial_x p_\phi \cdot e^{2Rt} \Delta_q^h (\partial_t w(t, x, 1) - \partial_t w(t, x, 0))_\phi dx \]

\[ = \int_R \Delta_q^h ((w_\phi(t, x, 1) - w_\phi(t, x, 0)) \partial_t w(t, x, 1) - \partial_t w(t, x, 0))_\phi dx dt \]

\[ - \frac{1}{2} \int_R \Delta_q^h (\int_0^1 (u \partial_x w(t, x, 1) - \partial_t w(t, x, 0))_\phi dx \]

\[ = \frac{1}{2} \frac{d}{dt} \int_R \Delta_q^h ((w(t, x, 1) - w(t, x, 0))^2_\phi dx + \lambda \hat{\theta}(t) \int_R \Delta_q^h |D_x| ((w(t, x, 1) - w(t, x, 0))^2_\phi dx \]

\[ - \frac{1}{2} \int_R \Delta_q^h (\int_0^1 (u \partial_x w(t, x, 1) - \partial_t w(t, x, 0))_\phi dx \]

If we use the lemma 2.3 to the following quantity \( \int_R \Delta_q^h (\int_0^1 (u \partial_x w(t, x, 1) - w(t, x, 0))_\phi dx \), we can obtain that

\[ \frac{1}{2} \int_R \Delta_q^h e^\phi \partial_x (\int_0^1 (u \partial_x w(t, x, 1) - \partial_t w(t, x, 0))_\phi dx \]

\[ = \frac{1}{2} \partial_t \int_R \Delta_q^h e^\phi \partial_x (\int_0^1 (u \partial_x w(t, x, 1) - w(t, x, 0))_\phi dx \]

\[ - \int_R \Delta_q^h e^\phi \partial_x (\int_0^1 (u \partial_x w(t, x, 1) - w(t, x, 0))_\phi dx \]

\[ + \frac{2}{2} \lambda \hat{\theta}(t) \int_R \Delta_q^h e^\phi |D_x| \partial_x (\int_0^1 (u \partial_x w(t, x, 1) - w(t, x, 0))_\phi dx \]

So we multiply our equation by \( e^{2Rt} \), we achieve that

\[ |e^{2Rt} K_q(t)| \leq \frac{1}{2} \frac{d}{dt} \|\Delta_q^h e^{Rt} w_\phi\|^2_{L^2(L_h^2)} + \lambda \hat{\theta}(t) \|\Delta_q^h e^{Rt} |D_x|^\frac{1}{2} w_\phi\|^2_{L^2(L_h^2)} \]

\[ + \frac{1}{2} \partial_t \int_R e^{2Rt} \Delta_q^h e^\phi \partial_x (\int_0^1 (u \partial_x w(t, x, 1) - w(t, x, 0))_\phi dx \]

\[ + \int_R e^{2Rt} \Delta_q^h e^\phi \partial_x (\int_0^1 (u \partial_x w(t, x, 1) - w(t, x, 0))_\phi dx \]

\[ \leq 34 \]
By applying Bony decomposition, we have for any $s > \frac{1}{2}$ and $\frac{1}{2} < \delta < 1$

$$|e^{2Rt} K_q(t)| \leq \frac{1}{2} \frac{d}{dt}\left(\|\Delta^h_{\varphi} e^{Rt} w_{\varphi}\|^2_{L^p(L^p)} + \|\Delta^h_{\varphi} e^{Rt} \partial_x (u_{\varphi})\|^2_{L^p(L^p)}\right) + \left(\frac{\lambda(t)}{2}\|\Delta^h_{\varphi} e^{Rt} |D_x|^2 w_{\varphi}\|^2_{L^p(L^p)} + \frac{\lambda(t)}{2}\|\Delta^h_{\varphi} e^{Rt} |D_x|^2 \partial_x (u_{\varphi})\|^2_{L^p(L^p)}\right) + C\left(\|\Delta^h_{\varphi} e^{Rt} w_{\varphi}\|^2_{L^p(L^p)} + \|\Delta^h_{\varphi} e^{Rt} (u \partial_t u_{\varphi})\|^2_{L^p(L^p)} + \frac{\lambda(t)}{2}\|\Delta^h_{\varphi} e^{Rt} |D_x|^2 w_{\varphi}\|^2_{L^p(L^p)}\right)$$

By applying Bony decomposition, we have for any $s > \frac{1}{2}$ and $\frac{1}{2} < \delta < 1$

$$\|e^{Rt} \Delta^h_\varphi \partial_x (u_{\varphi})\|_{L^p(L^p)} \leq \|e^{Rt} \Delta^h_\varphi \partial_x (u_{\varphi})\|_{L^2} \leq d_q 2^{-2g} \|u_{\varphi}\|_{L^\infty} \|e^{Rt} u_{\varphi}\|_{H^\delta} \leq d_q 2^{-2g} \|u_{\varphi}\|_{H^\delta} \|e^{Rt} u_{\varphi}\|_{H^\delta+1}$$

and

$$\|e^{Rt} \Delta^h_\varphi \partial_x (u \partial_t u_{\varphi})\|_{L^p(L^p)} \leq \|e^{Rt} \Delta^h_\varphi \partial_x (u \partial_t u_{\varphi})\|_{L^2} \leq d_q 2^{-2g} \|u_{\varphi}\|_{L^\infty} \|e^{Rt} \partial_t u_{\varphi}\|_{H^\delta+1} + \|\partial_t u_{\varphi}\|_{L^\infty} \|e^{Rt} u_{\varphi}\|_{H^\delta+1} \leq d_q 2^{-2g} \|u_{\varphi}\|_{H^\delta} \|e^{Rt} \partial_t u_{\varphi}\|_{H^\delta+1} + \|\partial_t u_{\varphi}\|_{H^\delta} \|e^{Rt} u_{\varphi}\|_{H^\delta+1}$$

While notice that

$$\int_0^1 \Delta^h_\varphi \partial_y u_{\varphi}(t, x, y) dy = 0,$$

then for any fixed $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, there exist $Y^0_\varphi(t, x)$ so that $\Delta^h_\varphi \partial_y u_{\varphi}(t, x, Y^0_\varphi(t, x)) = 0$. So we have

$$(\Delta^h_\varphi \partial_y u_{\varphi}(t, x, y))^2 \leq \|\Delta^h_\varphi \partial_y u_{\varphi}\|_{L^2} \|\Delta^h_\varphi \partial_y^2 u_{\varphi}\|_{L^2},$$

which implies that

$$\|\Delta^h_\varphi \partial_y u_{\varphi}(t, x, y)\|_{L^p(L^p)} \leq \|\Delta^h_\varphi \partial_y u_{\varphi}\|_{L^2} \|\Delta^h_\varphi \partial_y^2 u_{\varphi}\|_{L^2}.$$
Along the same way we obtain
\[
\int_0^t \left| \left\langle \Delta^h_q (u \partial_x w), e^{Rt} \Delta^h_q (\partial_t w) \right\rangle \right| dt' \leq C d^2 2^{-2qs} \left( \frac{1}{4} \int_0^t \| e^{Rt} w \|_{L^2_q(\mathcal{H})}^2 + \| u^2 \|_{L^2_q(\mathcal{H})}^2 + \| e^{Rt} u \|_{L^\infty_q(\mathcal{H})}^2 + \| e^{Rt} \partial_x w \|_{L^2_q(\mathcal{H})}^2 \right).
\] (5.15)

It follows from the proof of Lemma 2.4, for any \( s > \frac{1}{2} \) and \( \frac{1}{2} < \delta < 1 \)
\[
\int_0^t \left| \left\langle \Delta^h_q (T^h_v \partial_x w + R^h (u, \partial_x w)), e^{Rt'} \Delta^h_q (\partial_t w) \right\rangle \right| dt' \leq C d^2 2^{-2qs} \| e^{Rt} w \|_{L^2_{t, \delta}(\mathcal{H}^{s+\frac{1}{2}})}^2 \| e^{Rt} (\partial_t w) \|_{L^2_{t, \delta}(\mathcal{H}^{s+\frac{1}{2}})}^2.
\]

While we deduce from the lemma 2.1 and the definition 2.3
\[
\int_0^t \left| \left\langle \Delta^h_q (T^h_v \partial_x w), \Delta^h_q (\partial_t w) \right\rangle \right| dt' \leq \sum_{|q'-q| \leq 4} \int_0^t \left| \left\langle S^h_{q'-1} \partial_x w (t'), \Delta^h_q u \|_{L^\infty_q(L^2)} \right\rangle \left\langle \Delta^h_q u (t'), \| \Delta^h_q \partial_t w \|_{L^2_q(L^2)} \right\rangle dt'
\]
\[
\leq \sum_{|q'-q| \leq 4} 2'' \int_0^t \| w (t') \|_{L^q} \| \Delta^h_q u \|_{L^2_q(L^2)} \| \Delta^h_q \partial_t w \|_{L^2_q(L^2)} \| \Delta^h_q \partial_t w \|_{L^2_q(L^2)} dl'
\]
\[
\leq \sum_{|q'-q| \leq 4} 2'' \left( \int_0^t \| w (t') \|_{L^q} \| \Delta^h_q w (t') \|_{L^2_q(L^2)} \right)^{\frac{1}{2}}
\]
\[
\times \left( \int_0^t \| w (t') \|_{L^q} \| \Delta^h_q \partial_t w (t') \|_{L^2_q(L^2)} \right)^{\frac{1}{2}}
\]
\[
\leq C d^2 2^{-2qs} \| e^{Rt} w \|_{L^2_{t, \delta}(\mathcal{H}^{s+\frac{1}{2}})}^2 \| e^{Rt} (\partial_t w) \|_{L^2_{t, \delta}(\mathcal{H}^{s+\frac{1}{2}})}^2.
\]

Then we conclude for any \( s > \frac{1}{2} \)
\[
\int_0^t \left| \left\langle \Delta^h_q (u \partial_x w), e^{Rt'} \Delta^h_q (\partial_t w) \right\rangle \right| dt' \leq C d^2 2^{-2qs} \| e^{Rt} w \|_{L^2_{t, \delta}(\mathcal{H}^{s+\frac{1}{2}})}^2 \| e^{Rt} (\partial_t w) \|_{L^2_{t, \delta}(\mathcal{H}^{s+\frac{1}{2}})}^2.
\] (5.16)

In the same way we have
\[
\int_0^t \left| \left\langle \Delta^h_q (u \partial_x w), e^{Rt'} \Delta^h_q w \right\rangle \right| dt' \leq C d^2 2^{-2qs} \| e^{Rt} w \|_{L^2_{t, \delta}(\mathcal{H}^{s+\frac{1}{2}})}^2.
\] (5.17)

On the other hand, we deduce from the lemma 2.1 that for any \( s > \frac{1}{2} \)
\[
\int_0^t \left| \left\langle \Delta^h_q (T^h_v \partial_y w), \Delta^h_q (\partial_t w) \right\rangle \right| dt' \leq C d^2 2^{-2qs} \| e^{Rt} w \|_{L^2_{t, \delta}(\mathcal{H}^{s+\frac{1}{2}})}^2.
\]
\[
\sum_{|q'-q|\leq 4} \int_0^t \| S_{q'-1}^h v_{\phi}(t') \|_{L^\infty} \| \Delta_q^h \partial_y w_{\phi}(t') \|_{L^2} \| \Delta_q^h (\partial_t w)_{\phi}(t') \|_{L^2} dt' \\
\lesssim \sum_{|q'-q|\leq 4} 2^{q'} \| u_{\phi} \|_{L^\infty_t(H^{q+1})} \| \Delta_q^h \partial_y w_{\phi} \|_{L^2_t(H^q)} \left( \int_0^t \| \partial_y u_{\phi}(t') \|_{H^q} \| \Delta_q^h (\partial_t w)_{\phi}(t') \|_{L^2} dt' \right)^{\frac{1}{2}} \\
\lesssim d_q^2 2^{-2q} \| u_{\phi} \|_{L^\infty_t(H^{q+1})} \| \partial_y w_{\phi} \|_{L^2_t(H^q)} \| (\partial_t w)_{\phi} \|_{L^2_t(H^{q+1})} \right). \]

In the same way we obtain that
\[
\int_0^t \left\langle \Delta_q^h R^h(v, \partial_y w), \Delta_q^h (\partial_t w)_{\phi} \right\rangle_{L^2} dt' \\
\lesssim d_q^2 2^{-2q} \| u_{\phi} \|_{L^\infty_t(H^{q+1})} \| \partial_y w_{\phi} \|_{L^2_t(H^q)} \| (\partial_t w)_{\phi} \|_{L^2_t(H^{q+1})} \right). \]

Finally, we use the lemma 2.4, we find
\[
\int_0^t \left\langle \Delta_q^h (T_{\partial_y w}^h v), \Delta_q^h (\partial_t w)_{\phi} \right\rangle_{L^2} dt' \\
\lesssim \sum_{|q'-q|\leq 4} \int_0^t \| S_{q'-1}^h \partial_y w_{\phi}(t') \|_{L^\infty_t(H^q)} \| \Delta_q^h v_{\phi}(t') \|_{L^2_t(H^q)} \| \Delta_q^h (\partial_t w)_{\phi}(t') \|_{L^2} dt' \\
\lesssim \sum_{|q'-q|\leq 4} \int_0^t \| \partial_y w_{\phi} \|_{H^q} \| \Delta_q^h \partial_y u_{\phi} \|_{L^2} \| \Delta_q^h (\partial_t w)_{\phi} \|_{L^2} dt' \\
\lesssim C d_q^2 2^{-2q} \| \partial_y w_{\phi} \|_{L^2_t(H^q)} \| \Delta_q^h \partial_y u_{\phi} \|_{L^\infty_t(H^{q+1})} \| (\partial_t w)_{\phi} \|_{L^2_t(H^q)} \right). \]

Then by summing we deduce that
\[
\int_0^t \left\langle \Delta_q^h (v \partial_y w), \Delta_q^h (\partial_t w)_{\phi} \right\rangle_{L^2} dt' \lesssim C d_q^2 2^{-2q} \| \partial_y w_{\phi} \|_{L^2_t(H^q)} \| \Delta_q^h \partial_y u_{\phi} \|_{L^\infty_t(H^{q+1})} \| (\partial_t w)_{\phi} \|_{L^2_t(H^q)} \right) + C d_q^2 2^{-2q} \| u_{\phi} \|_{L^\infty_t(H^{q+1})} \| \partial_y w_{\phi} \|_{L^2_t(H^q)} \| (\partial_t w)_{\phi} \|_{L^2_t(H^{q+1})} \right. \]
\]

(5.18)
Along the same way we can found that
\[
\int_0^t \left\langle \Delta_q^h (T_{\partial_y w}^h v + R^h(v, \partial_y w))_{\phi}, \Delta_q^h w_{\phi} \right\rangle_{L^2} dt' \\
\lesssim d_q^2 2^{-2q} \| u_{\phi} \|_{L^\infty_t(H^{q+1})} \| \partial_y w_{\phi} \|_{L^2_t(H^q)} \| w_{\phi} \|_{L^2_t(H^{q+1})} \right). \]

Then we still have to estimate $T_{\partial_y w}^h v$, so by integration by part we can obtain
\[
\int_0^t \left\langle \Delta_q^h (T_{\partial_y w}^h v), \Delta_q^h w_{\phi} \right\rangle_{L^2} dt' \leq \int_0^t \left\langle \Delta_q^h (T_{\partial_y w}^h v), \Delta_q^h w_{\phi} \right\rangle_{L^2} dt' \\
+ \int_0^t \left\langle \Delta_q^h (T_{\partial_y w}^h v), \Delta_q^h \partial_y w_{\phi} \right\rangle_{L^2} dt'."
Due to the free divergence \( \partial_x u + \partial_y v = 0 \) we deduce

\[
\int_0^t \left| \langle \Delta_q^h(T^h_w v)_\phi, \Delta_q^h w_\phi \rangle_{L^2} \right| dt' \leq \sum_{|q'-q| \leq 1} \int_0^t \| S_{q'q}^h w_\phi(t') \|_{L^\infty(L^2)} \| \Delta_q^h \partial_x u_\phi(t') \|_{L^2(L^\infty)} \| \Delta_q^h w_\phi(t') \|_{L^2} dt'
\leq \sum_{|q'-q| \leq 1} 2q' \| \partial_y u_\phi \|_{L^2} \| \Delta_q^h w_\phi \|_{L^2} \| \Delta_q^h w_\phi(t') \|_{L^2} dt'
\leq C d_q^2 2^{-2qs} \| w_\phi \|_{L^2_{t,\phi}(H^s)}^2.
\]

While we observe that

\[
\int_0^t \left| \langle \Delta_q^h(T^h_w v)_\phi, \Delta_q^h \partial_y w_\phi \rangle_{L^2} \right| dt' \leq \sum_{|q'-q| \leq 1} \int_0^t \| S_{q'q}^h w_\phi(t') \|_{L^\infty(L^2)} \| \Delta_q^h \partial_x u_\phi(t') \|_{L^2(L^\infty)} \| \Delta_q^h \partial_y w_\phi(t') \|_{L^2} dt'
\leq \sum_{|q'-q| \leq 1} \int_0^t \| w_\phi \|_{L^\infty} \| \Delta_q^h \partial_x u_\phi \|_{L^2} \| \Delta_q^h \partial_y w_\phi \|_{L^2} dt'
\leq C d_q^2 2^{-2qs} \| w_\phi \|_{L^2_{t,\phi}(H^s)} \| \partial_y w_\phi \|_{L^2_{t,\phi}(H^s)}.
\]

By summarizing the above estimates, we obtain

\[
\int_0^t \left| \langle \Delta_q^h(v \partial_y w)_\phi, \Delta_q^h w_\phi \rangle_{L^2} \right| dt' \leq C d_q^2 2^{-2qs} \| u_\phi \|_{L^2_{t,\phi}(H^{s+\frac{1}{2}})} \| \partial_y w_\phi \|_{L^2_{t,\phi}(H^s)} \| w_\phi \|_{L^2_{t,\phi}(H^{s+\frac{1}{2}})} \tag{5.19}
\]

By inserting the resulting estimates (5.14)-(5.19) in (5.12) + (5.13) and then repeating the last step of the proof of Proposition 5.1, we achieve

\[
\frac{d}{dt} \left( \| e^{RT} \Delta_q^h(\partial_t w)_\phi \|_{L^2}^2 + e^{RT} \Delta_q^h(\partial_t w)_\phi \Delta_q^h w_\phi dx + \frac{1}{2} \| e^{RT} \Delta_q^h w_\phi \|_{L^2}^2 \right) - \| e^{RT} \Delta_q^h(\partial_t w)_\phi \|_{L^2}^2
+ \lambda \hat{\theta}(t) \| e^{RT} \Delta_q^h D_x \frac{1}{2} w_\phi \|_{L^2}^2 + \| e^{RT} \Delta_q^h \partial_y w_\phi \|_{L^2}^2 + \frac{d}{dt} \| e^{RT} \Delta_q^h \partial_y w_\phi \|_{L^2}^2 \tag{5.20}
\]

\[
+ 2 \| e^{RT} \Delta_q^h(\partial_t w)_\phi \|_{L^2}^2 + 2 \lambda \hat{\theta}(t') \| e^{RT} \Delta_q^h D_x \frac{1}{2} (\partial_t w)_\phi \|_{L^2}^2 + 2 \lambda \hat{\theta}(t') \| e^{RT} \Delta_q^h D_x \frac{1}{2} \partial_y w_\phi \|_{L^2}^2
\leq 2 \| e^{RT} \Delta_q^h(\partial_t w)_\phi \|_{L^2}^2 + 2 \lambda \| e^{RT} \Delta_q^h(\partial_t w)_\phi \|_{L^2}^2 + 2 \lambda \| e^{RT} \Delta_q^h D_x \frac{1}{2} \partial_y w_\phi \|_{L^2}^2
\leq 2^{-2qs} q'^2 (\lambda + 2C) \| e^{RT} ((\partial_t w)_\phi, \partial_y w_\phi) \|_{H^{s+\frac{1}{2}}}^2 + 2C 2^{-2qs} q'^2 \| e^{RT} \partial_t w_\phi \|_{H^{s+\frac{1}{2}}}^2
+ C d_q^2 2^{-2qs} \| w_\phi \|_{H^{s+\frac{1}{2}}} \| \partial_y w_\phi \|_{H^s} + \| (\partial_t w)_\phi \|_{H^{s+\frac{1}{2}}} \| \partial_y u_\phi \|_{H^{s+1}} + \| \partial_y w_\phi \|_{H^s} \]

\[
+ \| u_\phi \|_{H^{s+\frac{1}{2}}} e^{RT} \partial_t u_\phi \|_{H^{s+1}} + \| \partial_t u_\phi \|_{H^{s+\frac{1}{2}}} e^{RT} u_\phi \|_{H^{s+1}} + \| u_\phi \|_{H^{s+\frac{1}{2}}} e^{RT} u_\phi \|_{H^{s+2}} \tag{5.21}
\]

Multiplying (5.20) by \( 2^{2qs} \) for \( s > \frac{1}{2} \) and \( \frac{1}{2} < \delta < 1 \), and then integrating over time, and summing with respect to \( q \in \mathbb{Z} \), we find that for \( t < T^* \)
\[
\left(\frac{1}{2}\|e^{Rt}(w+\partial_t w)\|_{L^2_t(\mathcal{H}^*)}^2 + \frac{1}{2}\|e^{Rt}(\partial_t w)\|_{L^2_t(\mathcal{H}^*)}^2 + \|e^{Rt}\partial_y w\|_{L^\infty_t(\mathcal{H}^*)}^2 \right) + \|e^{Rt}(\partial_t w)\|_{L^2_t(\mathcal{H}^*)}^2 \\
+ \lambda\|e^{Rt}w\|_{L^2_t(\mathcal{H}^*)}^2 + \|e^{Rt}\partial_y w\|_{L^2_t(\mathcal{H}^*)}^2 + 2\lambda\|e^{Rt}(\partial_t w)\|_{L^2_t(\mathcal{H}^*)}^2 + 2\|e^{Rt}\partial_y w\|_{L^2_t(\mathcal{H}^*)}^2 \\
\leq C\|e^{|D_x|}\partial_y w_0\|_{\mathcal{H}^*}^2 + C\|e^{|D_x|}(w_0 + w_1)\|_{\mathcal{H}^*}^2 + C\|e^{|D_x|}w_1\|_{\mathcal{H}^*}^2 \\
+ 2C\|e^{Rt}w\|_{L^2_t(\mathcal{H}^*)}^2 + (\lambda + 2C)\|e^{Rt}w\|_{L^2_t(\mathcal{H}^*)}^2 + \|e^{Rt}(\partial_t w)\|_{L^2_t(\mathcal{H}^*)}^2 \\
+ C\|w_0\|_{L^2_t(\mathcal{H}^*)}^2 \|\partial_y w_0\|_{L^2_t(\mathcal{H}^*)}^2 + 2\|\partial_y w_0\|_{L^2_t(\mathcal{H}^*)}^2 + 2\|\partial_y u_0\|_{L^2_t(\mathcal{H}^*)}^2 + 2\|\partial_y u_0\|_{L^2_t(\mathcal{H}^*)}^2 + 2\|\partial_y u_0\|_{L^2_t(\mathcal{H}^*)}^2.
\]

Applying Young's inequality yields
\[
C\|\partial_y u_0\|_{L^\infty_t(\mathcal{H}^*)}^2 \|e^{Rt}w\|_{L^2_t(\mathcal{H}^*)}^2 \|e^{Rt}w\|_{L^2_t(\mathcal{H}^*)}^2 \\
\leq C\|\partial_y u_0\|_{L^\infty_t(\mathcal{H}^*)}^2 \|e^{Rt}w\|_{L^2_t(\mathcal{H}^*)}^2 + \frac{1}{2}\|e^{Rt}w\|_{L^2_t(\mathcal{H}^*)}^2.
\]

Then we achieve
\[
\left(\frac{1}{2}\|e^{Rt}(w+\partial_t w)\|_{L^2_t(\mathcal{H}^*)}^2 + \frac{1}{2}\|e^{Rt}(\partial_t w)\|_{L^2_t(\mathcal{H}^*)}^2 + \|e^{Rt}\partial_y w\|_{L^2_t(\mathcal{H}^*)}^2 \right) + \|e^{Rt}(\partial_t w)\|_{L^2_t(\mathcal{H}^*)}^2 \\
+ \lambda\|e^{Rt}w\|_{L^2_t(\mathcal{H}^*)}^2 + \|e^{Rt}\partial_y w\|_{L^2_t(\mathcal{H}^*)}^2 + 2\lambda\|e^{Rt}(\partial_t w)\|_{L^2_t(\mathcal{H}^*)}^2 + 2\|e^{Rt}\partial_y w\|_{L^2_t(\mathcal{H}^*)}^2 \\
\leq C\|e^{|D_x|}\partial_y w_0\|_{\mathcal{H}^*}^2 + C\|e^{|D_x|}(w_0 + w_1)\|_{\mathcal{H}^*}^2 + C\|e^{|D_x|}w_1\|_{\mathcal{H}^*}^2 \\
+ C(2 + \|\partial_y u_0\|_{L^\infty_t(\mathcal{H}^*)})\|e^{Rt}w\|_{L^2_t(\mathcal{H}^*)}^2 \\
+ (\lambda + C(2 + \|\partial_y u_0\|_{L^\infty_t(\mathcal{H}^*)})\|e^{Rt}(w, \partial_t w)\|_{L^2_t(\mathcal{H}^*)}^2 \\
+ \|\partial_y w_0\|_{L^2(\mathcal{H}^*)}^2 \|e^{Rt}u_0\|_{L^2(\mathcal{H}^*)}^2 + C\|e^{Rt}(\partial_t u)\|_{L^2(\mathcal{H}^*)}^2 + C\|e^{Rt}u_0\|_{L^2(\mathcal{H}^*)}^2.
\]

Therefore if we take
\[
\lambda \geq C(2 + \|\partial_y u_0\|_{L^\infty_t(\mathcal{H}^*)}),
\]

and using the fact that \(\|\partial_y w\|_{L^2_t(\mathcal{H}^*)} \leq C\), we obtain
\[
\left(\frac{1}{2}\|e^{Rt}(w+\partial_t w)\|_{L^2_t(\mathcal{H}^*)}^2 + \frac{1}{2}\|e^{Rt}(\partial_t w)\|_{L^2_t(\mathcal{H}^*)}^2 + \|e^{Rt}\partial_y w_0\|_{L^\infty_t(\mathcal{H}^*)}^2 \right) + \|e^{Rt}(\partial_t w)\|_{L^2_t(\mathcal{H}^*)}^2 \\
+ \|e^{Rt}\partial_y w_0\|_{L^2_t(\mathcal{H}^*)}^2 \leq C\left(\|e^{|D_x|}\partial_y u_0\|_{\mathcal{H}^*} + \|e^{|D_x|}\partial_y (u_0 + u_1)\|_{\mathcal{H}^*} + \|e^{|D_x|}u_1\|_{\mathcal{H}^*} \\
+ \|e^{|D_x|}u_0\|_{\mathcal{H}^*} + \|e^{|D_x|}u_0\|_{\mathcal{H}^*} + \|e^{|D_x|}u_1\|_{\mathcal{H}^*} \right)
Then, taking $\lambda = C(2 + \|e^{\alpha|D_x|} \partial_y u_0\|_{H^t} + \|e^{\alpha|D_x|}(u_0 + u_1)\|_{H^{t+1}} + \|e^{\alpha|D_x|} u_1\|_{H^{t+1}})$, therefore the condition of the proposition is satisfied and then the proposition is proved.

As a matter of fact, it remains to present the estimate of $\|\Delta_q^h(\partial_t^2 u)\|_{L^2}$, this estimate will serve us in the proof of the last theorem 1.3. Indeed by applying $\Delta_q^h$ to (3.13) and take the $L^2$ inner product of resulting equation with $\Delta_q^h(\partial_t^2 u)$. That yields

$$\|\Delta_q^h(\partial_t^2 u)\|_{L^2} = \langle \Delta_q^h \partial_y^2 u, \Delta_q^h(\partial_t^2 u) \rangle_{L^2} - \langle \Delta_q^h(\partial_t^2 u), \Delta_q^h(\partial_t^2 u) \rangle_{L^2} \leq I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \left| \langle \Delta_q^h \partial_y^2 u, \Delta_q^h(\partial_t^2 u) \rangle_{L^2} \right|$$

$$I_2 = \left| \langle \Delta_q^h(\partial_t^2 u), \Delta_q^h(\partial_t^2 u) \rangle_{L^2} \right|$$

$$I_3 = \left| \langle \partial_q^h(v \partial_q u), \Delta_q^h(\partial_t^2 u) \rangle_{L^2} \right|$$

$$I_4 = \left| \langle \partial_q^h \partial_x p, \Delta_q^h(\partial_t^2 u) \rangle_{L^2} \right|.$$

The fact that $(\partial_t^i u) = \partial_t u + \lambda \theta(t)|D_x|u_\phi$ implies

$$\langle \Delta_q^h(\partial_t^2 u), \Delta_q^h(\partial_t^2 u) \rangle_{L^2} = \frac{1}{2} \frac{d}{dt} \|\Delta_q^h(\partial_t^2 u)\|_{L^2}^2 + \lambda \theta(t)2^\omega \|\Delta_q^h(\partial_t^2 u)\|_{L^2}^2,$$

from which, we deduce that

$$\|\Delta_q^h(\partial_t^2 u)\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|\Delta_q^h(\partial_t^2 u)\|_{L^2}^2 \leq I_1 + I_2 + I_3 + I_4.$$

Since $\partial_x u + \partial_y v = 0$, using (3.1) and integrations by parts, we find

$$I_4 = \left| \langle \partial_q^h \partial_x p, \Delta_q^h(\partial_t^2 u) \rangle_{L^2} \right| = 0.$$

For $I_1$, $I_2$ and $I_3$ we have

$$I_2 = \left| \langle \Delta_q^h \partial_y^2 u, \Delta_q^h(\partial_t^2 u) \rangle_{L^2} \right| \leq C \|\Delta_q^h(\partial_t^2 u)\|_{L^2}^2 + \frac{1}{10} \|\Delta_q^h(\partial_t^2 u)\|_{L^2}^2,$$

$$I_2 = \left| \langle \Delta_q^h(\partial_t^2 u), \Delta_q^h(\partial_t^2 u) \rangle_{L^2} \right| \leq C \|\Delta_q^h(\partial_t^2 u)\|_{L^2}^2 + \frac{1}{10} \|\Delta_q^h(\partial_t^2 u)\|_{L^2}^2,$$

Then, we deduce that

$$\|\Delta_q^h(\partial_t^2 u)\|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} \|\Delta_q^h(\partial_t^2 u)\|_{L^2}^2 \leq C \left( \|\Delta_q^h(\partial_t^2 u)\|_{L^2}^2 + \|\Delta_q^h(v \partial_q u)\|_{L^2}^2 + \|\Delta_q^h \partial_x p\|_{L^2}^2 \right).$$

Multiplying the result by $e^{2\lambda t}$ and integrating over $[0, t]$, we get

$$\|e^{\lambda t} \Delta_q^h(\partial_t^2 u)\|_{L^2(L^2)}^2 + \frac{1}{2} \|e^{\lambda t} \Delta_q^h(\partial_t^2 u)\|_{L^2(L^2)}^2 \leq C \left( \|\Delta_q^h(e^{\alpha|D_x|} u_1\|_{L^2}^2 + ||e^{\lambda t} \Delta_q^h(\partial_t^2 u)\|_{L^2(L^2)}^2 + \|e^{\lambda t} \Delta_q^h(v \partial_q u)\|_{L^2(L^2)}^2 + \|e^{\lambda t} \Delta_q^h \partial_x p\|_{L^2(L^2)}^2 \right).$$

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Multiplying the above inequality by $2^{q(\delta+1)}$ with $\frac{1}{2} < \delta < 1$, then taking the square root of the resulting estimate, and finally summing up the obtained equations with respect to $q \in \mathbb{Z}$, we obtain

$$\|e^{Rt}(\partial_t^2 u)\|_{L^2_t(H^{\delta+1})} + \frac{1}{2} \|e^{Rt}(\partial_t u)\|_{L^2_t(H^{\delta+1})} \leq C \left( \|e^{a|Dz|}u_1\|_{H^{\delta+1}} + \|e^{Rt}(u\partial_x u)\|_{L^2_t(H^{\delta+1})} + \|e^{Rt}(v\partial_y u)\|_{L^2_t(H^{\delta+1})} \right).$$  (5.25)

Next, it follows from the law of product in anisotropic Sobolev spaces and Poincaré inequality that

$$\|e^{Rt}(u\partial_x u)\|_{L^2_t(H^{\delta+1})} \leq C \|u_0\|_{L^\infty_t(H^{\delta+2})} \|e^{Rt}\partial_y u_0\|_{L^2_t(H^{\delta+2})};$$

$$\|e^{Rt}(v\partial_y u)\|_{L^2_t(H^{\delta+1})} \leq C \|u_0\|_{L^\infty_t(H^{\delta+2})} \|e^{Rt}\partial_y u_0\|_{L^2_t(H^{\delta+1})} + \|u_0\|_{L^\infty_t(H^{\delta+2})} \|e^{Rt}\partial_y u_0\|_{L^2_t(H^{\delta+1})}.$$  

Inserting the above estimates into (5.25) and then using the smallness condition $\|u_0\|_{H^{\delta}} \leq \frac{1}{4e^{\delta}}$ and propositions 5.1 and 5.2, we finally obtain

$$\|e^{Rt}(\partial_t u)\|_{L^2_t(H^{\delta+1})} + \|e^{Rt}(\partial_t u)\|_{L^2_t(H^{\delta+1})} \leq C \left( \|e^{a|Dz|}u_1\|_{H^{\delta+1}} + \|e^{a|Dz|}\partial_y u_0\|_{H^{\delta+2}} + \|e^{a|Dz|}(u_0 + u_1)\|_{H^{\delta+2}} + \|e^{a|Dz|}\partial_y u_0\|_{H^{\delta+1}} + \|e^{a|Dz|}\partial_y (u_0 + u_1)\|_{H^{\delta+1}} \right).$$

6. **The convergence to the perturbed hydrostatic Navier-Stokes equations**

In this section, we justify the limit from the scaled perturbed anisotropic Navier-Stokes system to the perturbed hydrostatic Navier-Stokes system in a 2-D striped domain. As in the sections 3 and 4, the main idea will be to obtain a control of the difference between the two solutions in analytic spaces, by using energy estimates with exponential weights in the Fourier variable. As previously, the exponent of the exponential weight is depending on time but shall take into account now the "loss of the analyticity" for both solutions, of the re-scaled perturbed Navier-Stokes system and respectively of the perturbed hydrostatic Navier-Stokes equations. To this end, we introduce

$$\begin{cases}
R^{1,\epsilon} = u^\epsilon - u, \\
R^{2,\epsilon} = v^\epsilon - v, \\
q^\epsilon = p^\epsilon - p.
\end{cases}$$  (6.1)

Then, systems (1.7) and (1.8) imply that $(R^{1,\epsilon}, R^{2,\epsilon}, q^\epsilon)$ verifies

$$\begin{cases}
\partial_t R^{1,\epsilon} + \partial_t R^{1,\epsilon} - e^2 \partial_y^2 R^{1,\epsilon} - \partial_y^2 R^{1,\epsilon} + \partial_x q^\epsilon = F^{1,\epsilon} \text{ in } \mathbb{S} \times [0, \infty], \\
e^2 (\partial_x^2 R^{2,\epsilon} + \partial_y R^{2,\epsilon} - e^2 \partial_x^2 R^{2,\epsilon} - \partial_y^2 R^{2,\epsilon}) + \partial_y q^\epsilon = F^{2,\epsilon}, \\
\partial_x R^{1,\epsilon} + \partial_y R^{2,\epsilon} = 0 \\
(R^{1,\epsilon}, R^{2,\epsilon}) \big|_{t=0} = (u_0 - u_0, v_0 - v_0) , \\
\partial_t (R^{1,\epsilon}, R^{2,\epsilon}) \big|_{t=0} = (u_1' - u_1, v_1' - v_1) , \\
(R^{1,\epsilon}, R^{2,\epsilon}) \big|_{y=0} = (R^{1,\epsilon}, R^{2,\epsilon}) \big|_{y=1} = 0.
\end{cases}$$  (6.2)
where the remaining terms $F^{i,\epsilon}$, with $i = 1, 2$, are determined by
\[
\begin{align*}
F^{1,\epsilon} &= \epsilon^2 \partial_x^2 u - (u^{\prime} \partial_x u^{\prime} - u \partial_x u) - (v^{\prime} \partial_y u^{\prime} - v \partial_y u), \\
F^{2,\epsilon} &= -\epsilon^2 \left( \partial_x^2 v + \partial_t v - \epsilon^2 \partial_x^2 v - \partial_y^2 v + u^{\prime} \partial_x v^{\prime} + v^{\prime} \partial_y v^{\prime} \right).
\end{align*}
\]  
(6.3)

As $(R^{1,\epsilon}, R^{2,\epsilon})$ satisfies the boundary condition and also the free divergence, therefore these two conditions allows us to write
\[
R^{2,\epsilon}(t, x, y) = \int_0^y \partial_s R^{2,\epsilon}(t, x, s) ds = - \int_0^y \partial_s R^{1,\epsilon}(t, x, s) ds
\]  
(6.4)

If we replace $y$ by 1 in (6.4), we deduce from the incompressibility condition $\partial_x R^{1,\epsilon} + \partial_y R^{2,\epsilon} = 0$ that
\[
\partial_x \int_0^1 R^{1,\epsilon}(t, x, y) dy = - \int_0^1 \partial_y R^{2,\epsilon}(t, x, y) dy = R^{2,\epsilon}(t, x, 1) - R^{2,\epsilon}(t, x, 0) = 0.
\]

In what follows, for simplicity, we shall neglect the subscript $\epsilon$ in $(R^{1,\epsilon}, R^{2,\epsilon}, q^{\epsilon})$. In view of the system (6.2), we can transform it like a equation of order one in time, so if we define $G = (R^1, \partial_t R^1)$ and $H = (R^2, \partial_t R^2)$, Then $G$ and $H$ satisfy the following equation
\[
\begin{align*}
\partial_t G + A_1(D)G &= \left( F^1 - \partial_x q \right) \\
\epsilon^2 \left( \partial_t H + B_2(D)H \right) &= \left( F^2 - \partial_y q \right) \\
\partial_x R^1 + \partial_y R^2 &= 0 \\
(R^1, R^2)_{/y=0} &= (R^1, R^2)_{/y=1} = 0
\end{align*}
\]  
(6.5)

where
\[
G = \begin{pmatrix} R^1 \\ \partial_t R^1 \end{pmatrix} \quad \text{and} \quad A_1(D) = \begin{pmatrix} 0 & -1 \\ -\epsilon^2 \partial_x^2 - \partial_y^2 & 1 \end{pmatrix}
\]

and
\[
H = \begin{pmatrix} R^2 \\ \partial_t R^2 \end{pmatrix} \quad \text{and} \quad B_2(D) = \begin{pmatrix} 0 & -1 \\ -\epsilon^2 \partial_x^2 - \partial_y^2 & 1 \end{pmatrix}.
\]

In view of (2.3), we define for any suitable function $f$

\[
f_{\varphi}(t, x, y) = F_{\xi \rightarrow x}^{-1} \left( e^{\varphi(t, \xi)} \tilde{f}(t, \xi, y) \right) \quad \text{where} \quad \varphi(t, \xi) = (a - \mu \eta(t)) |\xi|,
\]  
(6.6)

where $\mu \geq \lambda$ will be determined later, and $\eta(t)$ is given by
\[
\eta(t) = \int_0^t \left( \| (\partial_y u_{\phi'}, \epsilon \partial_x u_{\phi'}) \|_{\mathcal{H}^s} + \| \partial_y u_{\phi}(t') \|_{\mathcal{H}^t} \right) dt'.
\]

We can observe that, if we take $c_0$ and $c_1$ small enough in Theorems 1.1 and 1.2 then $\varphi(t) \geq 0$ and
\[
0 \leq \varphi(t, \xi) \leq \min \left( \phi(t, \xi), \Theta(t, \xi) \right).
\]
Then in view of (6.6), we observe that \((G, H)_\varphi\) verifies

\[
\begin{align*}
&\partial_t G_\varphi + \mu \dot{\eta}(t)|D_x|G_\varphi + A_\epsilon(D)G_\varphi = \left( F^1_\varphi + \partial_x q_\varphi \right) \\
&\partial^2_t H_\varphi + \mu \dot{\eta}(t)|D_x|H_\varphi + B_\epsilon(D)H_\varphi = - \left( F^2_\varphi + \partial_y q_\varphi \right)
\end{align*}
\]

(6.7)

\[
\begin{align*}
&\partial_x R^1_\varphi + \partial_y R^2_\varphi = 0 \\
&(R^1_\varphi, R^2_\varphi)/y=0 = (R^1_\varphi, R^2_\varphi)/y=1 = 0
\end{align*}
\]

Where \(|D_x|\) denote the Fourier multiplier of the symbol \(|\xi|\). In what follows, we recall that we use “C” to denote a generic positive constant which can change from line to line. Thanks to the theorems 1.1 and 1.2, the propositions 5.1 and 5.2, we deduce for \(\frac{1}{2} < s < 1\) that

\[
\|u_\varphi^{\diamond}\|_{L^\infty(\mathbb{R}^+;H^s)} + \|(u + \partial_t u)_{\phi}\|_{L^\infty(\mathbb{R}^+;H^{s+1})} + \|\partial^2_y u_{\phi}\|_{L^2(\mathbb{R}^+;H^{s+1})} + \|\partial_t u_{\phi}\|_{L^2(\mathbb{R}^+;H^{s+1})} \leq M,
\]

(6.8)

where \(u_\varphi^{\diamond}\) and \(u_{\phi}\) are respectively determined by (4.1) and (2.3) and \(M \geq 1\) is a constant independent to \(\epsilon\).

\textit{Proof of the theorem 1.3} We apply the dyadic operator in the horizontal variable \(\Delta^h_q\) to (6.7) and taking the \(L^2\) inner product of the resulting equation with \(\Delta^h_q G_\varphi\) and \(\Delta^h_q H_\varphi\) we obtain

\[
\langle \Delta^h_q \partial_t G_\varphi, \Delta^h_q G_\varphi \rangle_{L^2} + \mu \dot{\eta}(t) \langle \Delta^h_q |D_x| G_\varphi, \Delta^h_q G_\varphi \rangle_{L^2} + \langle \Delta^h_q A_\epsilon(D) G_\varphi, \Delta^h_q G_\varphi \rangle_{L^2} = \langle \Delta^h_q (F^1)_{\varphi}, \Delta^h_q (\partial_t R^1)_{\varphi} \rangle_{L^2} - \langle \Delta^h_q \partial_x q_\varphi, \Delta^h_q (\partial_t R^1)_{\varphi} \rangle_{L^2}
\]

(6.9)

and

\[
\langle \Delta^h_q \partial_t H_\varphi, \Delta^h_q H_\varphi \rangle_{L^2} + \mu \dot{\eta}(t) \langle \Delta^h_q |D_x| H_\varphi, \Delta^h_q H_\varphi \rangle_{L^2} + \langle \Delta^h_q B_\epsilon(D) H_\varphi, \Delta^h_q H_\varphi \rangle_{L^2} = - \langle \Delta^h_q (F^2)_{\varphi}, \Delta^h_q (\partial_t R^1)_{\varphi} \rangle_{L^2} - \langle \Delta^h_q \partial_y q_\varphi, \Delta^h_q (\partial_t R^1)_{\varphi} \rangle_{L^2}.
\]

(6.10)

Due to the free divergence condition, we have

\[
\left| \langle \Delta^h_q \nabla q_\varphi, \Delta^h_q (\partial_t R^1, \partial_t R^2)_{\varphi} \rangle_{L^2} \right| = 0.
\]

Then by using Lemma 2.1, we achieve

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} &\left( \|\Delta^h_q (\partial_t R^1)_{\varphi}\|_{L^2}^2 + \|\Delta^h_q \partial_x R^1_{\varphi}\|_{L^2}^2 + \|\Delta^h_q \epsilon (\partial_t R^2)_{\varphi}\|_{L^2}^2 + \|\Delta^h_q \epsilon \partial_y R^2_{\varphi}\|_{L^2}^2 \right) \\
&+ \epsilon^2 \|\Delta^h_q \partial_x R^2_{\varphi}\|_{L^2}^2 + \epsilon^2 \|\Delta^h_q \partial_y R^1_{\varphi}\|_{L^2}^2 \right) + \|\Delta^h_q (\partial_t R^1)_{\varphi}\|_{L^2}^2 + \|\Delta^h_q (\partial_t R^2)_{\varphi}\|_{L^2}^2 \\
&+ \mu \dot{\eta}(t') \|\Delta^h_q |D_x|^{\frac{1}{2}} (\partial_t R^1)_{\varphi}\|_{L^2}^2 + \epsilon^2 \mu \dot{\eta}(t') \|\Delta^h_q |D_x|^{\frac{1}{2}} (\partial_t R^2)_{\varphi}\|_{L^2}^2 + \epsilon^2 \mu \dot{\eta}(t') \|\Delta^h_q |D_x|^{\frac{1}{2}} \partial_x R^2_{\varphi}\|_{L^2}^2 \\
&+ \mu \dot{\eta}(t') \|\Delta^h_q |D_x|^{\frac{1}{2}} \partial_y R^1_{\varphi}\|_{L^2}^2 + \mu \dot{\eta}(t') \|\Delta^h_q |D_x|^{\frac{1}{2}} \partial_y R^2_{\varphi}\|_{L^2}^2 + \epsilon^4 \mu \dot{\eta}(t') \|\Delta^h_q |D_x|^{\frac{1}{2}} \partial_y R^1_{\varphi}\|_{L^2}^2 \\
&\leq \left| \langle \Delta^h_q (F^1)_{\varphi}, \Delta^h_q (\partial_t R^1)_{\varphi} \rangle_{L^2} \right| + \left| \langle \Delta^h_q (F^2)_{\varphi}, \Delta^h_q (\partial_t R^2)_{\varphi} \rangle_{L^2} \right|.
\end{align*}
\]

(6.11)
Now we still have to take the inner product in $L^2$ with $\Delta^h_q R^1_\varphi$ and $\Delta^h_q R^2_\varphi$ to the equation
\[
\begin{align*}
&\phi(t,|Dx|)(\partial_t^2 R^1 + \partial_t R^1 - \partial_x^2 R^1 - \epsilon^2 \partial_x^2 R^1 + \partial_x p - F^1) = 0 \\
&\phi(t,|Dx|)\left(\epsilon^2 (\partial_t^2 R^2 + \partial_t R^2 - \partial_x^2 R^2 - \epsilon^2 \partial_x R^2) + \partial_x p - F^2\right) = 0,
\end{align*}
\]
we obtain
\[
\begin{align*}
&\langle \Delta^h_q (\partial_t^2 R^1)_\varphi, \Delta^h_q R^1_\varphi \rangle_{L^2} + \langle \Delta^h_q (\partial_t R^1)_\varphi, \Delta^h_q R^1_\varphi \rangle_{L^2} - \langle \Delta^h_q \partial_x^2 R^1_\varphi, \Delta^h_q R^1_\varphi \rangle_{L^2} \\
&- \epsilon^2 \langle \Delta^h_q \partial_x^2 R^1_\varphi, \Delta^h_q R^1_\varphi \rangle_{L^2} = (\langle \Delta^h_q (F^1)_\varphi, \Delta^h_q R^1_\varphi \rangle_{L^2} - \langle \Delta^h_q \partial_x q_\varphi, \Delta^h_q R^1_\varphi \rangle_{L^2} \rangle
\end{align*}
\]
and
\[
\begin{align*}
&\langle \Delta^h_q (\epsilon \partial_t^2 R^2)_\varphi, \Delta^h_q \epsilon R^2_\varphi \rangle_{L^2} + \langle \Delta^h_q (\epsilon \partial_t R^2)_\varphi, \Delta^h_q \epsilon R^2_\varphi \rangle_{L^2} - \langle \Delta^h_q \epsilon \partial_x^2 R^2_\varphi, \Delta^h_q \epsilon R^2_\varphi \rangle_{L^2} \\
&- \epsilon^2 \langle \Delta^h_q \epsilon \partial_x^2 R^2_\varphi, \Delta^h_q \epsilon R^2_\varphi \rangle_{L^2} = (\langle F^2, \Delta^h_q \epsilon R^2_\varphi \rangle_{L^2} - \langle \Delta^h_q \partial_x q_\varphi, \Delta^h_q \epsilon R^2_\varphi \rangle_{L^2} \rangle
\end{align*}
\]
In what follows, we shall use again the technical lemmas in Section 2, to handle term by term in the estimate (6.13) and (6.14). We start by the complicate term $I_1 = \langle \Delta^h_q (\partial_t^2 R^1)_\varphi, \Delta^h_q R^1_\varphi \rangle_{L^2}$ and $I_2 = \langle \Delta^h_q (\epsilon \partial_t^2 R^2)_\varphi, \Delta^h_q \epsilon R^2_\varphi \rangle_{L^2}$, so by using integration by parts, we find
\[
\begin{align*}
I_1 &= \frac{d}{dt} \int \Delta^h_q (\partial_t R^1)_\varphi \Delta^h_q R^1_\varphi dx - \int \Delta^h_q (\partial_t R^1)_\varphi \Delta^h_q (\partial_t R^1)_\varphi dx \\
&\quad + 2\mu \hat{\eta}(t) \int \Delta^h_q |Dx| (\partial_t R^1)_\varphi \Delta^h_q R^1_\varphi dx \\
I_2 &= \frac{d}{dt} \int \Delta^h_q (\epsilon \partial_t R^2)_\varphi \Delta^h_q \epsilon R^2_\varphi dx - \int \Delta^h_q (\epsilon \partial_t R^2)_\varphi \Delta^h_q (\epsilon \partial_t R^2)_\varphi dx \\
&\quad + 2\mu \hat{\eta}(t) \int \Delta^h_q |Dx| (\epsilon \partial_t R^2)_\varphi \Delta^h_q \epsilon R^2_\varphi dx
\end{align*}
\]
Then by using the Lemma 2.1, we achieve
\[
\begin{align*}
&\frac{d}{dt} \int \Delta^h_q (\partial_t R^1)_\varphi \Delta^h_q R^1_\varphi dx - \int \Delta^h_q (\partial_t R^1)_\varphi \Delta^h_q (\partial_t R^1)_\varphi dx \\
&\quad + 2\mu \hat{\eta}(t) \int \Delta^h_q |Dx| (\partial_t R^1)_\varphi \Delta^h_q R^1_\varphi dx + \frac{d}{dt} \int \Delta^h_q (\partial_t R^1)_\varphi \Delta^h_q \epsilon R^2_\varphi dx \\
&\quad - \int \Delta^h_q (\partial_t R^1)_\varphi \Delta^h_q (\partial_t R^2)_\varphi dx + 2\mu \hat{\eta}(t) \int \Delta^h_q |Dx| (\partial_t R^2)_\varphi \Delta^h_q \epsilon R^2_\varphi dx \\
&\quad + \frac{1}{2} \frac{d}{dt} \|\Delta^h_q R^1_\varphi\|^2_{L^2} + \frac{1}{2} \frac{d}{dt} \|\Delta^h_q \epsilon R^2_\varphi\|^2_{L^2} + \mu \hat{\eta}(t) \|\Delta^h_q |Dx| (\partial_t R^1)_\varphi\|^2_{L^2} \\
&\quad + \|\Delta^h_q \partial_x R^1_\varphi\|^2_{L^2} + \|\Delta^h_q \partial_x \epsilon R^2_\varphi\|^2_{L^2} + \epsilon^2 \|\Delta^h_q \partial_x R^1_\varphi\|^2_{L^2} + \epsilon^2 \|\Delta^h_q \partial_x \epsilon R^2_\varphi\|^2_{L^2} \\
&\quad = \langle \Delta^h_q (F^1)_\varphi, \Delta^h_q R^1_\varphi \rangle_{L^2} + \langle \Delta^h_q (F^2)_\varphi, \Delta^h_q \epsilon R^2_\varphi \rangle_{L^2} \quad (6.15)
\end{align*}
\]
Then by summing $2 \times (6.11)$ with (6.15) and we multiply the resulting by $2^{2qs}$ for $s \in \frac{1}{2}^\mathbb{N}, 1$ and then integrating over time, and summing with respect to $q \in \mathbb{Z}$, we find that for $t < T^*$
\begin{align*}
&\left(\frac{1}{2}\|(R^1+\partial_t R^1, \epsilon(R^2+\partial_t R^2))\|_{L_r^\infty(H^r)}^2+\frac{1}{2}\|\partial_t R^1, \epsilon R^2\|_{L_r^\infty(H^r)}^2\right)
&+\epsilon^2\|\partial_x(R^1, \epsilon R^2)\|_{L_r^2(H^r)}^2
+\mu\|\partial_t R^1, \epsilon R^2\|_{L_r^2(H^r)}^2 + \mu_\|\partial_t R^1, \epsilon R^2\|_{L_r^2(H^\frac{r}{2})}^2
&+\|\partial_y(R^1, \epsilon R^2)\|_{\tilde{L}_r^2(H^r)}^2 + 2\mu\|\partial_t R^1, \epsilon R^2\|_{\tilde{L}_r^2(H^\frac{r}{2})}^2
&+ 2\mu\|\partial_y(R^1, \epsilon R^2)\|_{\tilde{L}_r^2(H^\frac{r}{2})}^2
\leq C\epsilon^q D|\partial_y(u_0-u_0, \epsilon_0(v_0-v_0))|_{H^r}^2 + C\epsilon^2 |\partial_y(u_0-u_0, \epsilon_0(v_0-v_0))|_{H^r}^2
\end{align*}

Now we claim that
\begin{align*}
&\int_0^t \left|\left\langle \Delta_q h(F^1, \Delta_q h(R^1)\right\rangle_{L_2^r}\right| dt' + \int_0^t \left|\left\langle \Delta_q h(F^1, \Delta_q h(R^1)\right\rangle_{L_2^r}\right| dt'
\end{align*}

the proof of these estimates will be presented later in the Appendix A.

By virtue of (6.8), (6.17) and (6.18), we infer
\begin{align*}
&\sum_{i=1}^2 \int_0^t \left|\left\langle \Delta_q h(F^1, \Delta_q h(R^1)\right\rangle_{L_2^r}\right| dt' + \int_0^t \left|\left\langle \Delta_q h(F^1, \Delta_q h(R^1)\right\rangle_{L_2^r}\right| dt'
\end{align*}
from which and (6.16), we deduce that

\[
\left( \frac{1}{2} \|(R^1 + \partial_t R^1, \epsilon(R^2 + \partial_t R^2))_\varphi\|_{L^2_{\text{loc}}(H^{\frac{1}{2}})} + \|\partial_t(R^1, \epsilon(R^2))_\varphi\|_{L^2_{\text{loc}}(H^{1})} + \|\partial_y(R^1, \epsilon(R^2))_\varphi\|_{L^2_{\text{loc}}(H^{1})} \right)
\]

\[
+ \frac{\epsilon}{2} \|(R^2, \epsilon \partial_t R^2)_\varphi\|_{L^2_{\text{loc}}(H^{1})} + M \frac{\epsilon}{2} \|\partial_y R^2\|_{L^2_{\text{loc}}(H^{1})} + \epsilon \eta \|\partial_y R^2\|_{L^2_{\text{loc}}(H^{1})} \left( \frac{1}{2} \|(R^1, \partial_t R^1)_\varphi\|_{L^2_{\text{loc}}(H^{1})} + \|\partial_t R^2\|_{L^2_{\text{loc}}(H^{1})} + \|\partial_y R^2\|_{L^2_{\text{loc}}(H^{1})} \right).
\]

Applying Young's inequality gives rise to

\[
\left( \frac{1}{2} \|(R^1 + \partial_t R^1, \epsilon(R^2 + \partial_t R^2))_\varphi\|_{L^2_{\text{loc}}(H^{\frac{1}{2}})} + \|\partial_t R^1, \epsilon(R^2))_\varphi\|_{L^2_{\text{loc}}(H^{1})} + \|\partial_y R^1, \epsilon(R^2))_\varphi\|_{L^2_{\text{loc}}(H^{1})} \right)
\]

\[
+ \frac{\epsilon}{2} \|(R^2, \epsilon \partial_t R^2)_\varphi\|_{L^2_{\text{loc}}(H^{1})} + \epsilon \eta \|\partial_y R^2\|_{L^2_{\text{loc}}(H^{1})} \left( \frac{1}{2} \|(R^1, \partial_t R^1)_\varphi\|_{L^2_{\text{loc}}(H^{1})} + \|\partial_t R^2\|_{L^2_{\text{loc}}(H^{1})} + \|\partial_y R^2\|_{L^2_{\text{loc}}(H^{1})} \right).
\]

Taking \(\sqrt{\mu} = CM\) lead to (3.1), this completes the proof of the theorem 1.3.

**APPENDIX A. PROOF OF ESTIMATES (6.17) AND (6.18)**

**A.1. Proof of estimate (6.17).** We first observe that

\[
F^1_\varphi = (\epsilon^2 \partial_x^2 u - (u' \partial_x R^1 + R^1 \partial_x u) - (\epsilon' \partial_y R^1 + R^2 \partial_y u))_\varphi,
\]

\[
+ \frac{\epsilon}{2} \|(R^1 + \partial_t R^1, \epsilon(R^2 + \partial_t R^2))_\varphi\|_{L^2_{\text{loc}}(H^{\frac{1}{2}})} + \|\partial_t(R^1, \epsilon(R^2))_\varphi\|_{L^2_{\text{loc}}(H^{1})} + \|\partial_y(R^1, \epsilon(R^2))_\varphi\|_{L^2_{\text{loc}}(H^{1})} \left( \frac{1}{2} \|(R^1, \partial_t R^1)_\varphi\|_{L^2_{\text{loc}}(H^{1})} + \|\partial_t(R^2, \epsilon(R^2))_\varphi\|_{L^2_{\text{loc}}(H^{1})} + \|\partial_y(R^2, \epsilon(R^2))_\varphi\|_{L^2_{\text{loc}}(H^{1})} \right).
\]
so, we define
\[ G_1 = \int_0^t \left| \langle \Delta_q^h F_{\varphi}^1, \Delta_q^h (\partial_t R^1) \varphi \rangle \right| dt'. \]

\[ = \int_0^t \left| \langle \Delta_q^h (\epsilon^2 \partial_x^2 u - (u^t \partial_x R^1 + R^1 \partial_x u) - (v^t \partial_y R^1 + R^2 \partial_y u)) \varphi, \Delta_q^h (\partial_t R^1) \varphi \rangle \right| dt'. \]

\[ \leq I_1^q + I_2^q + I_3^q, \]

where

\[ I_1^q = \int_0^t \left| \langle \Delta_q^h (\epsilon^2 \partial_x^2 u) \varphi, \Delta_q^h (\partial_t R^1) \varphi \rangle \right| dt'. \]

\[ I_2^q = \int_0^t \left| \langle \Delta_q^h (u^t \partial_x R^1 + R^1 \partial_x u) \varphi, \Delta_q^h (\partial_t R^1) \varphi \rangle \right| dt'. \]

\[ I_3^q = \int_0^t \left| \langle \Delta_q^h (v^t \partial_y R^1 + R^2 \partial_y u) \varphi, \Delta_q^h (\partial_t R^1) \varphi \rangle \right| dt'. \]

We first observe that
\[ I_1^q \leq C d_q^2 2^{-q} \epsilon^2 \left\| u \varphi \right\|_{L^2(H^{s+2})} \left\| (\partial_t R^1) \varphi \right\|_{L^2(H^s)}. \tag{A.1} \]

For \( I_2 \), we write
\[ I_2^q = \int_0^t \left| \langle \Delta_q^h (u^t \partial_x R^1 + R^1 \partial_x u) \varphi, \Delta_q^h (\partial_t R^1) \varphi \rangle \right| dt' \leq I_{21} + I_{22}, \]

where

\[ I_{21}^q = \int_0^t \left| \langle \Delta_q^h (u^t \partial_x R^1) \varphi, \Delta_q^h (\partial_t R^1) \varphi \rangle \right| dt'. \]

\[ I_{22}^q = \int_0^t \left| \langle \Delta_q^h (R^1 \partial_x u) \varphi, \Delta_q^h (\partial_t R^1) \varphi \rangle \right| dt'. \]

Lemma (2.4) implies
\[ I_{21}^q \leq C d_q^2 2^{-q} \left\| R^1 \varphi \right\|_{L^2(H^{s+\frac{1}{2}})} \left\| (\partial_t R^1) \varphi \right\|_{L^2(H^{s+\frac{1}{2}})}, \tag{A.2} \]

For \( I_{22} \), using Bony’s decomposition for the horizontal variable, we write
\[ R^1 \partial_x u = T_{\partial_x u}^h R^1 + T_{R^1 \partial_x u}^h + R^h(R^1, \partial_x u), \]

and then, we have the following bound
\[ I_{22}^q = \int_0^t \left| \langle \Delta_q^h (T_{\partial_x u}^h R^1) \varphi, \Delta_q^h (\partial_t R^1) \varphi \rangle \right| dt' \leq I_{22,1}^q + I_{22,2}^q + I_{22,3}^q \]

with

\[ I_{22,1}^q = \int_0^t \left| \langle \Delta_q^h (T_{\partial_x u}^h R^1) \varphi, \Delta_q^h (\partial_t R^1) \varphi \rangle \right| dt'. \]

\[ I_{22,2}^q = \int_0^t \left| \langle \Delta_q^h (T_{R^1 \partial_x u}^h) \varphi, \Delta_q^h (\partial_t R^1) \varphi \rangle \right| dt'. \]

\[ I_{22,3}^q = \int_0^t \left| \langle \Delta_q^h (R^h(R^1, \partial_x u)) \varphi, \Delta_q^h (\partial_t R^1) \varphi \rangle \right| dt'. \]
\[ I_{22,3}^q = \int_0^t \left| \langle \Delta_q^h (R^h (R^1, \partial_x u)_\varphi, \Delta_q^h (\partial_t R^1)_\varphi \rangle \right| dt'. \]

Using the support properties given in [6], Proposition 2.10 and the definition of \( T_{R_t}^h \partial_x u \), we have

\[ I_{22,2}^q \leq \sum_{|q-q'| \leq 4} \int_0^t \left| \langle S_{q'-1}^h R_{q'}^1 \rangle \| \Delta_q^h \partial_x u \rangle \| L^\infty(L^2) \| \Delta_q^h (\partial_t R^1)_\varphi \| L^2 \right| \]

\[ \leq \sum_{|q-q'| \leq 4} \int_0^t 2^{q'} \left| \sum_{q} S_{q'-1}^h R_{q'}^1 \right| \| \Delta_q^h \partial_x u \rangle \| L^2 \| d_q (u_\varphi) \| u_\varphi \|_{\mathcal{H}^{q+1}} \| \partial_y u_\varphi \|_{\mathcal{H}^{q+1}} \| \Delta_q^h (\partial_t R^1)_\varphi \| L^2 \]

\[ \leq \sum_{|q-q'| \leq 4} \int_0^t d_q (u_\varphi) 2^{q'} 2^{-q's} \| \partial_y R_{q'}^1 \| \mathcal{H}^s \| u_\varphi \|_{\mathcal{H}^{q+1}} \| \partial_y u_\varphi \|_{\mathcal{H}^{q+1}} \| \Delta_q^h (\partial_t R^1)_\varphi \|_{\mathcal{H}^{q+1}} \]

\[ \leq 2^{q'} d_q (u_\varphi) \| u_\varphi \|_{\mathcal{H}^{q+1}} \| \partial_y R_{q'}^1 \| \mathcal{H}^s \| (\partial_t R^1)_\varphi \| L^2_{\mathcal{H}^{q+1}} \]

Now, we recall that

\[ \| \Delta_q^h \partial_x u \|_{\mathcal{H}^\infty} \leq \sum_{l \leq q-2} 2^{\frac{q}{2}} \| \Delta_{q-l}^h u \|_{L^2} \| \Delta^h \partial_y u \|_{L^2} \leq 2^q \| \partial_y u \|_{\mathcal{H}^s}, \]

so we can deduce

\[ I_{22,2}^q = \int_0^t \left| \langle \Delta_q^h (T_{R_t}^h R^1)_\varphi, \Delta_q^h (\partial_t R^1)_\varphi \rangle \right| \leq \sum_{|q-q'| \leq 4} \int_0^t | S_{q'-1}^h \partial_x u \|_{\mathcal{H}^\infty} \| \Delta_q^h R_{q'}^1 \| L^2 \| \Delta_q^h (\partial_t R^1)_\varphi \| L^2 | \]

\[ \leq \sum_{|q-q'| \leq 4} \int_0^t 2^{q'} \left| \partial_y u \|_{\mathcal{H}^s} \| \Delta_q^h R_{q'}^1 \| L^2 \| \Delta_q^h (\partial_t R^1)_\varphi \| L^2 \right| \]

\[ \leq \sum_{|q-q'| \leq 4} 2^{q'} \left( \int_0^t \| \partial_y u \|_{\mathcal{H}^s} \| \Delta_q^h R_{q'}^1 \|_{L^2} dt' \right)^\frac{1}{2} \left( \int_0^t \| \partial_y u \|_{\mathcal{H}^s} \| \Delta_q^h (\partial_t R^1)_\varphi \|_{L^2} dt' \right)^\frac{1}{2} \]

Using the definition of \( \dot{y}(t) \) and Definition 2.3 we have

\[ \left( \int_0^t \| \partial_y u \|_{\mathcal{H}^s} \| \Delta_q^h R_{q'}^1 \|_{L^2} dt' \right)^\frac{1}{2} \leq 2^{q(s+\frac{1}{2})} d_q \| R_{q'}^1 \|_{L^2_{\mathcal{H}^{q+\frac{1}{2}}}} \]

Then,

\[ I_{22,1}^q \leq 2^{q(s+\frac{1}{2})} d_q \| R_{q'}^1 \|_{L^2_{\mathcal{H}^{q+\frac{1}{2}}}} \| (\partial_t R^1)_\varphi \|_{L^2_{\mathcal{H}^{q+\frac{1}{2}}}} \]
where
\[ d_q^2 = d_q \left( \sum_{|q-q'| \leq 4} d_{q'} \right) \]

In a similar way, we have
\[
I_{22,3}^q = \int_0^t \left| \langle \Delta_q \delta_h(R^h(R^1, \partial_x u)\varphi), \Delta_q(\partial_t R^1)\varphi \rangle \right| dt'
\]
\[ \lesssim 2^q \sum_{q' \geq q-3} \int_0^t \| \Delta_{q'}^h R_{q'}^1 \|_{L^2} \| \Delta_{q'}^h \partial_x u_{q'} \|_{L^2_t(L^\infty)} \| \Delta_{q'}^h(\partial_t R^1)\varphi \|_{L^2_t(L^2)} dt' \]
\[ \lesssim 2^q \sum_{q' \geq q-3} \int_0^t 2^{q'} \| \Delta_{q'}^h R_{q'}^1 \|_{L^2} \| \partial_q u_{q'} \|_{H^s} \| \Delta_{q'}^h(\partial_t R^1)\varphi \|_{L^2} dt' \]
\[ \lesssim 2^q \sum_{q' \geq q-3} 2^{q'} \left( \int_0^t \| \partial_q u_{q'} \|_{H^s} \| \Delta_{q'}^h R_{q'}^1 \|_{L^2_t(L^2)} dt' \right) \left( \int_0^t \| \partial_q u_{q'} \|_{H^s} \| \Delta_{q'}^h(\partial_t R^1)\varphi \|_{L^2_t(L^2)} dt' \right) \]
\[ \lesssim 2^q \| R^1 \|_{L^2_t(L^2(\mathcal{H}^{s+\frac{1}{2}} \cup \mathcal{H}^{s+\frac{1}{2}})} \] (A.3)

Then we conclude that
\[ I_{22} \lesssim \begin{cases} C \cdot 2^{-2s} \| R^1 \|_{L^2_t(L^2(\mathcal{H}^{s+\frac{1}{2}} \cup \mathcal{H}^{s+\frac{1}{2}})} + \| u \|_{L^2_t(L^2(\mathcal{H}^{s+1}))} \| \partial_q R^1 \|_{L^2_t(L^2(\mathcal{H}^{s+\frac{1}{2}}))} \end{cases} \]

For the term \( I_3^q \), we write
\[ I_3^q = \int_0^t \left| \langle v \partial_q R^1 + R^2 \partial_q u \rangle, \Delta_q(\partial_t R^1)\varphi \right|_{L^2} dt' \leq I_{31}^q + I_{32}^q \] (A.4)

where
\[ I_{31}^q = \int_0^t \left| \langle \Delta_q^h v \partial_q R^1, \varphi \rangle \right|_{L^2} dt' \]
\[ I_{32}^q = \int_0^t \left| \langle \Delta_q^h R^2 \partial_q u, \varphi \rangle \right|_{L^2} dt' \]

Since
\[ v \partial_q R^1 = (R^2 + v) \partial_q R^1 = R^2 \partial_q R^1 + v \partial_q R^1, \]
we get
\[ I_{31}^q \leq I_{31,1}^q + I_{31,2}^q, \]
with
\[ I_{31,1}^q = \int_0^t \left| \langle \Delta_q^h (R^2 \partial_q R^1), \varphi \rangle \right|_{L^2} dt' \]
\[ I_{31,2}^q = \int_0^t \left| \langle \Delta_q^h (v \partial_q R^1), \varphi \rangle \right|_{L^2} dt' \]

Lemma (2.5) implies
\[ I_{31,1}^q \lesssim C \cdot 2^{-2s} \| R^1 \|_{L^2_t(L^2(\mathcal{H}^{s+\frac{1}{2}} \cup \mathcal{H}^{s+\frac{1}{2}})} \] (A.5)
For the term $I_{31.2}^h$, we apply Bony’s decomposition with respect to the horizontal variable

$$v \partial_y R^1 = T_v^h \partial_y R^1 + T_{\partial_y}^h v + R^h(v, \partial_y R^1).$$

Using (3.1), we have

$$||S^h_{q'} v_{\varphi}||_{L^\infty} = ||S^h_{q'-1} \int_0^t \partial_s u_{\varphi}(t, x, s) ds||_{L^\infty} \lesssim \sum_{l \leq q'-2} 2^{2l} \left| \Delta^h \partial_y u_{\varphi} \right|_{L^2} \left| \Delta^h \partial_y u_{\varphi} \right|_{L^2} \lesssim 2^{2q'} \left| u_{\varphi} \right|_{H^{q+1}} \left| \partial_y u_{\varphi} \right|_{H^{q}},$$

from which, we infer

$$\int_0^t \left| \langle \Delta^h_q (T_v^h \partial_y R^1), \Delta^h_q (\partial R^1) \rangle \right|_{L^2} dt' \lesssim \sum_{|q'-q| \leq 4} \int_0^t \left| S^h_{q'-1} v_{\varphi} \right|_{L^\infty} \left| \Delta^h_q \partial_y R^1 \right|_{L^2} \left| \Delta^h_q (\partial R^1) \right|_{L^2} dt' \lesssim \sum_{|q'-q| \leq 4} \int_0^t 2^{2q'} \left| u_{\varphi} \right|_{H^{q+1}} \left| \partial_y u_{\varphi} \right|_{H^{q}} \left| \Delta^h_q \partial_y R^1 \right|_{L^2} \left| \Delta^h_q (\partial R^1) \right|_{L^2} dt' \lesssim d^2 2^{-2q^2} \left| u_{\varphi} \right|_{L^\infty(H^{q+1})} \left| \partial_y R^1 \right|_{L^2(H^{q})} \left| (\partial R^1) \right|_{L^2_{t, H^{q}(H^{q+1})}},$$

where $\{d_q\}$ forms a suitable sequence.

In the same way, we have

$$\left| \Delta^h_q v_{\varphi}(t, x, y) \right|_{L^2(L^\infty)} \lesssim \left| u_{\varphi} \right|_{H^{q+1}} \left| \partial_y u_{\varphi} \right|_{H^{q}},$$

from which, we infer

$$\int_0^t \left| \langle \Delta^h_q (T_{\partial_y}^h v), \Delta^h_q (\partial R^1) \rangle \right|_{L^2} dt' \lesssim \sum_{|q'-q| \leq 4} \int_0^t \left| S^h_{q'-1} \partial_y R^1 \right|_{L^\infty} \left| \Delta^h_q v_{\varphi} \right|_{L^2} \left| \Delta^h_q (\partial R^1) \right|_{L^2} dt' \lesssim \sum_{|q'-q| \leq 4} \int_0^t \left| u_{\varphi} \right|_{H^{q+1}} \left| S^h_{q'-1} \partial_y R^1 \right|_{L^\infty} \left| \partial_y u_{\varphi} \right|_{H^{q}} \left| \Delta^h_q (\partial R^1) \right|_{L^2} dt' \lesssim d^2 2^{-2q^2} \left| u_{\varphi} \right|_{L^\infty(H^{q+1})} \left| \partial_y R^1 \right|_{L^2(H^{q})} \left| (\partial R^1) \right|_{L^2_{t, H^{q}(H^{q+1})}},$$

where $\{d_q\}$ forms a suitable sequence.

Finally, we have

$$\int_0^t \left| \langle \Delta^h_q (R^h(v, \partial_y R^1)), \Delta^h_q (\partial R^1) \rangle \right|_{L^2} dt' \lesssim 2^{2q} \sum_{q \geq q-3} \int_0^t \left| \Delta^h_q v_{\varphi} \right|_{L^2(L^\infty)} \left| \Delta^h_q \partial_y R^1 \right|_{L^2} \left| \Delta^h_q (\partial R^1) \right|_{L^2} dt'$$
\[
\lesssim 2^q \sum_{q' \geq q-3} \int_0^t \|u_{\varphi}\|_{1+q}^{\frac{1}{2}} \|\tilde{\partial}_y^h \partial_y R^1_{\varphi}\|_{L^q} \|\tilde{\partial}_y u_{\varphi}\|_{1+q}^{\frac{1}{2}} \|\Delta_q^h (\partial_t R^1)_{\varphi}\|_{L^2} dt'
\]
\[
\leq d_t^q 2^{-2q} \|u_{\varphi}\|_{L^\infty(\mathcal{H}^{q+1})}^{\frac{1}{2}} \|\tilde{\partial}_y R^1_{\varphi}\|_{L^2(\mathcal{H})} \|\partial_t R^1_{\varphi}\|_{L^2(t, \mathcal{H}^{q+\frac{1}{2}})}^{\frac{1}{2}}.
\]
Then we obtain the following estimates,
\[
I_{32,1}^q \lesssim C 2^{-2q} d_t^q \|u_{\varphi}\|_{L^\infty(\mathcal{H}^{q+1})}^{\frac{1}{2}} \|\tilde{\partial}_y R^1_{\varphi}\|_{L^2(\mathcal{H})} \|\partial_t R^1_{\varphi}\|_{L^2(t, \mathcal{H}^{q+\frac{1}{2}})}^{\frac{1}{2}}.
\] (A.7)

We now estimate the term \(I_{32}^q\) in (A.4). Bony’s decomposition for the horizontal variable implies
\[
I_{32}^q = \int_0^t \left| \left< \Delta^h_q (T^h_{R^2} \partial_y u + T^h_{\partial_y u} R^2 + R^h (R^2, \partial_y u)) \varphi, \Delta^h_q (\partial_t R^1) \varphi \right>_{L^2} \right| dt' \leq I_{32,1}^q + I_{32,2}^q + I_{32,3}^q,
\]
where
\[
I_{32,1}^q = \int_0^t \left| \left< \Delta^h_q (T^h_{R^2} \partial_y u) \varphi, \Delta^h_q (\partial_t R^1) \varphi \right>_{L^2} \right| dt',
\]
\[
I_{32,2}^q = \int_0^t \left| \left< \Delta^h_q (T^h_{\partial_y u} R^2) \varphi, \Delta^h_q (\partial_t R^1) \varphi \right>_{L^2} \right| dt',
\]
\[
I_{32,3}^q = \int_0^t \left| \left< \Delta^h_q (R^h (R^2, \partial_y u)) \varphi, \Delta^h_q (\partial_t R^1) \varphi \right>_{L^2} \right| dt'.
\]

We first observe that
\[
I_{32,1}^q \lesssim \sum_{\|q' - q\| \leq 4} \int_0^t \|S^h_{q'-1} R^2_{\varphi}\|_{L^\infty} \|\Delta^h_{q'} \partial_y u_{\varphi}\|_{L^q} \|\Delta^h_q (\partial_t R^1)_{\varphi}\|_{L^2} dt',
\]
\[
\lesssim \sum_{\|q' - q\| \leq 4} \int_0^t 2^{-q'} \|S^h_{q'-1} R^2_{\varphi}\|_{L^\infty} \|\partial_y u_{\varphi}\|_{\mathcal{H}^q} \|\Delta^h_q (\partial_t R^1)_{\varphi}\|_{L^2} dt'.
\]

Due to the fact that \(R^2(t, x, y) = - \int_0^y \partial_x R^1(t, x, s) ds\), we deduce
\[
\|S^h_{q'-1} R^2_{\varphi}\|_{L^\infty} \lesssim \int_0^y \|S^h_{q'-1} \partial_x R^1_{\varphi}(t, x, s)\|_{L^\infty} ds
\]
\[
\lesssim 2^{\frac{3q'}{2}} \|S^h_{q'-1} R^1_{\varphi}\|_{L^2},
\]
and then from which, we have
\[
I_{32,1}^q \lesssim \sum_{\|q' - q\| \leq 4} \int_0^t 2^{-q'} \|S^h_{q'-1} R^2_{\varphi}\|_{L^\infty} \|\partial_y u_{\varphi}\|_{\mathcal{H}^q} \|\Delta^h_q (R^1)_{\varphi}\|_{L^2} dt'
\]
\[
\lesssim \sum_{\|q' - q\| \leq 4} \int_0^t 2^{-q'} 2^{\frac{3q'}{2}} \|S^h_{q'-1} R^1_{\varphi}\|_{L^2} \|\partial_y u_{\varphi}\|_{\mathcal{H}^q} \|\Delta^h_q (R^1)_{\varphi}\|_{L^2} dt'
\]
\[
\lesssim \sum_{\|q' - q\| \leq 4} 2^{q'} \left( \int_0^t \|\partial_y u_{\varphi}\|_{\mathcal{H}^q} \|S^h_{q'-1} R^1_{\varphi}\|_{L^2}^2 dt' \right)^{\frac{1}{2}} \left( \int_0^t \|\partial_y u_{\varphi}\|_{\mathcal{H}^q} \|\Delta^h_q (R^1)_{\varphi}\|_{L^2}^2 dt' \right)^{\frac{1}{2}},
\]
and
Taking into account the definition of $\dot{\eta}(t)$ and Definition 2.3 we obtain
\[
\left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{H}^s} \|\Delta_{q}^h(\partial_t R^1_\varphi)\|_{L^2} dt'\right)^{\frac{1}{2}} \lesssim 2^{-q(s+\frac{1}{2})} d_q \|\partial_t R^1_\varphi\|_{L^2_{t,n}(\mathcal{H}^{s+\frac{1}{2}})}.
\]

Then,
\[
I_{32,1}^q \lesssim 2^{-2qs} d^2_q \|R^1_\varphi\|_{L^2_{t,n}(\mathcal{H}^{s+\frac{1}{2}})} \|\partial_t R^1_\varphi\|_{L^2_{t,n}(\mathcal{H}^{s+\frac{1}{2}})} \tag{A.8}
\]

Now, for $I_{32,2}^q$, we have
\[
I_{32,2}^q \lesssim \sum_{|q'-q| \leq 1} \int_0^t \|\Delta_{q}^h \partial_y u_\varphi\|_{L^\infty(L^2)} \|\Delta_{q}^h R^1_\varphi\|_{L^2_{t,n}(L^\infty)} \|\Delta_{q}^h(\partial_t R^1_\varphi)\|_{L^2} dt'
\]
\[
\lesssim \sum_{|q'-q| \leq 1} \int_0^t 2^q \|\partial_y u_\varphi\|_{\mathcal{H}^s} 2^q \|\Delta_{q}^h R^1_\varphi\|_{L^2} \|\Delta_{q}^h(\partial_t R^1_\varphi)\|_{L^2} dt'
\]
\[
\lesssim \sum_{|q'-q| \leq 1} 2^{q'} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{H}^s} \|\Delta_{q}^h R^1_\varphi\|_{L^2} dt'\right)^{\frac{1}{2}} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{H}^s} \|\Delta_{q}^h(\partial_t R^1_\varphi)\|_{L^2} dt'\right)^{\frac{1}{2}}
\]
\[
\lesssim d^2_q 2^{-2qs} \|R^1_\varphi\|_{L^2_{t,n}(\mathcal{H}^{s+\frac{1}{2}})} \|\partial_t R^1_\varphi\|_{L^2_{t,n}(\mathcal{H}^{s+\frac{1}{2}})}.
\]

We end by estimate $I_{32,3}$, in the same way, we have
\[
I_{32,3}^q \lesssim 2^{q} \sum_{q \geq q-3} \int_0^t \|\Delta_{q}^h R^2_\varphi\|_{L^2_{t,n}(L^\infty)} \|\Delta_{q}^h \partial_y u_\varphi\|_{L^2} \|\Delta_{q}^h(\partial_t R^1_\varphi)\|_{L^2} dt'
\]
\[
\lesssim \sum_{q \geq q-3} 2^{q} \sum_{q \geq q-3} \int_0^t 2^q \|\Delta_{q}^h R^2_\varphi\|_{L^2} \|\partial_y u_\varphi\|_{\mathcal{H}^s} \|\Delta_{q}^h(\partial_t R^1_\varphi)\|_{L^2} dt'
\]
\[
\lesssim \sum_{q \geq q-3} 2^{q} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{H}^s} \|\Delta_{q}^h R^2_\varphi\|_{L^2} dt'\right)^{\frac{1}{2}} \left(\int_0^t \|\partial_y u_\varphi\|_{\mathcal{H}^s} \|\Delta_{q}^h(\partial_t R^1_\varphi)\|_{L^2} dt'\right)^{\frac{1}{2}}
\]
\[
\lesssim d^2_q 2^{-2qs} \|R^2_\varphi\|_{L^2_{t,n}(\mathcal{H}^{s+\frac{1}{2}})} \|\partial_t R^1_\varphi\|_{L^2_{t,n}(\mathcal{H}^{s+\frac{1}{2}})}.
\]

Summing all the resulting estimate, we can achieve
\[
I_{32}^q \lesssim C d^2_q 2^{-2qs} \|R^1_\varphi\|_{L^2_{t,n}(\mathcal{H}^{s+\frac{1}{2}})} \|\partial_t R^1_\varphi\|_{L^2_{t,n}(\mathcal{H}^{s+\frac{1}{2}})}. \tag{A.9}
\]

**Remark A.1.** For the proof when we have $R^1$ and not $\partial_t R^1$, it also the same we need just to replace in all the proof with $R^1$ instead of $\partial_t R^1$.

By summing up (A.1)-(A.9), we conclude the proof of (6.17).

### A.2. Proof of estimate (6.18)

We first deduce from $\partial_x u + \partial_y v = 0$, that
\[
\epsilon^2 \int_0^t \left|\langle \Delta_{q}^h(\partial_t v_\varphi), \Delta_{q}^h(\partial_t R^2_\varphi) \rangle_{L^2} \right| dt' \lesssim C d^2_q 2^{-2qs} \epsilon^2 \|\partial_t u_\varphi\|_{L^2_{t,n}(\mathcal{H}^{s+\frac{1}{2}})} \|\partial_t R^2_\varphi\|_{L^2_{t,n}(\mathcal{H}^{s+\frac{1}{2}})}
\]
\[
\epsilon^2 \int_0^t \left|\langle \Delta_{q}^h(\partial_y v_\varphi), \Delta_{q}^h(\partial_t R^2_\varphi) \rangle_{L^2} \right| dt' \lesssim C d^2_q 2^{-2qs} \epsilon^2 \|\partial_y u_\varphi\|_{L^2_{t,n}(\mathcal{H}^{s+\frac{1}{2}})} \|\partial_t R^2_\varphi\|_{L^2_{t,n}(\mathcal{H}^{s+\frac{1}{2}})}.
\]
\[
\epsilon^2 \int_0^t \left| \langle \Delta_q^h (\partial_y^2 v), \Delta_q^h (\partial_t R^2) \rangle \right|_{L^2} \, dt' \leq C d_q^2 2^{-2qs} \epsilon^2 \| \partial_y^2 u_q \|_{L^2_q(H^{s+1})} \| (\partial_t R^2) \|_{L^2_q(H^s)},
\]
\[
\epsilon^4 \int_0^t \left| \langle \Delta_q^h (\partial_x^2 v), \Delta_q^h (\partial_t R^2) \rangle \right|_{L^2} \, dt' \leq C d_q^2 2^{-2qs} \epsilon^4 \| \partial_y u_q \|_{L^2_q(H^{s+1})} \| (\partial_t R^2) \|_{L^2_q(H^s)}.
\]

(A.10)

Then now we still have to control

\[
J_q^4 = \epsilon^2 \int_0^t \left| \langle \Delta_q^h (u' \partial_x v'), \Delta_q^h (\partial_t R^2) \rangle \right|_{L^2} \, dt'
\]

and

\[
J_q^5 = \epsilon^2 \int_0^t \left| \langle \Delta_q^h (v' \partial_y v'), \Delta_q^h (\partial_t R^2) \rangle \right|_{L^2} \, dt'.
\]

We start first by \( J_q^4 \), we have

\[
J_q^4 \leq \epsilon^2 \left( J_{41}^q + J_{42}^q \right),
\]

where

\[
J_{41}^q = \int_0^t \left| \langle \Delta_q^h (u' \partial_x R^2), \Delta_q^h (\partial_t R^2) \rangle \right|_{L^2} \, dt'
\]

\[
J_{42}^q = \int_0^t \left| \langle \Delta_q^h (u' \partial_x v), \Delta_q^h (\partial_t R^2) \rangle \right|_{L^2} \, dt'.
\]

It follows from Lemma 2.4 that

\[
J_q^5 = \int_0^t \left| \langle \Delta_q^h (u' \partial_x v), \Delta_q^h (\partial_t R^2) \rangle \right|_{L^2} \, dt' \leq C d_q^2 2^{-2qs} \| R^2 \|_{L^2_q(H^{s+1})} \| (\partial_t R^2) \|_{L^2_q(H^{s+1/2})}.
\]

(A.11)

For the second term, Bony’s decomposition for the horizontal variable gives

\[
J_q^5 = \int_0^t \left| \langle \Delta_q^h (u' \partial_x v), \Delta_q^h (\partial_t R^2) \rangle \right|_{L^2} \, dt' \leq J_{421}^q + J_{422}^q + J_{423}^q,
\]

with

\[
J_{421}^q = \int_0^t \left| \langle \Delta_q^h (T_{u'} \partial_x v), \Delta_q^h (\partial_t R^2) \rangle \right|_{L^2} \, dt'
\]

\[
J_{422}^q = \int_0^t \left| \langle \Delta_q^h (T_{u'} u'), \Delta_q^h (\partial_t R^2) \rangle \right|_{L^2} \, dt'
\]

\[
J_{423}^q = \int_0^t \left| \langle \Delta_q^h (R^h (u', \partial_x v)), \Delta_q^h (\partial_t R^2) \rangle \right|_{L^2} \, dt'.
\]

Due to

\[
\| S_{\nu - 1} u' \|_{L^\infty} \lesssim \| u' \|_{L^\infty} \| \partial_y u' \|_{L^\infty}.
\]

and the relation (3.1), we have

\[
J_{421}^q = \int_0^t \left| \langle \Delta_q^h (T_{u'} \partial_x v), \Delta_q^h (\partial_t R^2) \rangle \right|_{L^2} \, dt'
\]
\[
\sum_{|\nu'| \leq 4} \int_0^t \left| \langle S_{\nu'}^h, \partial_x v_\varphi \rangle \right| \| \Delta_h^\nu (\partial_t R^2) \varphi \|_{L^2} dt'
\]
\[
\sum_{|\nu'| \leq 4} \int_0^t \left| u_\varphi \right| \| \Delta_h^\nu (\partial_t R^2) \varphi \|_{L^2} dt'
\]
\[
\sum_{|\nu'| \leq 4} \int_0^t u_\varphi \| \Delta_h^\nu (\partial_t R^2) \varphi \|_{L^2} dt'
\]
\[
\lesssim d_q^2 2^{-2q} \| u_\varphi \|_{L^2_t(\mathcal{H}^+)} \| \partial_y u_\varphi \|_{L^2_t(\mathcal{H}^+)} (\partial_t R^2) \varphi \|_{L^2_t(\mathcal{H}^+)}.
\]

While again thanks to (3.1), we find
\[
\| S_{\nu'}^h \partial_x v_\varphi \|_{L^\infty} \lesssim \int_0^t \| S_{\nu'}^h \partial_x (u_\varphi(t, x, s)) \|_{L^\infty} ds \lesssim 2^q \| \partial_y u_\varphi \|_{\mathcal{H}^+},
\]
which leads to
\[
J_{422}^q = \int_0^t \left| \langle \Delta_h^\nu (T_{\partial_x v}^h u_\varphi), \Delta_h^\nu (\partial_t R^2) \varphi \rangle \right| \frac{1}{L^2_t(\mathcal{H}^+)} \| \partial_y u_\varphi \|_{L^2_t(\mathcal{H}^+)} (\partial_t R^2) \varphi \|_{L^2_t(\mathcal{H}^+)}.
\]

Along the same way, we obtain
\[
J_{423}^q = \int_0^t \left| \langle \Delta_h^\nu (R^h (\partial_x v, u_\varphi)), \Delta_h^\nu (\partial_t R^2) \varphi \rangle \right| \frac{1}{L^2_t(\mathcal{H}^+)} \| \partial_y u_\varphi \|_{L^2_t(\mathcal{H}^+)} (\partial_t R^2) \varphi \|_{L^2_t(\mathcal{H}^+)}.
\]

This gives rise to
\[
J_{42}^q \lesssim C 2^{-2q} d_q^2 \| u_\varphi \|_{L^2_t(\mathcal{H}^+)} \| \partial_y u_\varphi \|_{L^2_t(\mathcal{H}^+)} (\partial_t R^2) \varphi \|_{L^2_t(\mathcal{H}^+)}.
\]

Now for $J_{42}^2$, we first note that
\[
v_\varphi \partial_y u_\varphi = v \partial_y w^2 + w^2 \partial_y w^2 + v \partial_y v + w^2 \partial_y v.
\]

Lemma 2.4 yields
\[
\epsilon^2 \int_0^t \left| \langle \Delta_h^\nu (w^2 \partial_y w^2), \Delta_h^\nu (\partial_t R^2) \varphi \rangle \right| \frac{1}{L^2_t(\mathcal{H}^+)} \| \partial_y u_\varphi \|_{L^2_t(\mathcal{H}^+)} (\partial_t R^2) \varphi \|_{L^2_t(\mathcal{H}^+)}.
\]

From (A.7), we have
\[
\int_0^t \left| \langle \Delta_h^\nu (v \partial_y R^2), \Delta_h^\nu (\partial_t R^2) \varphi \rangle \right| \frac{1}{L^2_t(\mathcal{H}^+)} \| \partial_y u_\varphi \|_{L^2_t(\mathcal{H}^+)} (\partial_t R^2) \varphi \|_{L^2_t(\mathcal{H}^+)}.
\]

As for (A.3), we obtain
\[
\int_0^t \left| \langle \Delta_h^\nu (R^2 \partial_x u), \Delta_h^\nu (\partial_t R^2) \varphi \rangle \right| \frac{1}{L^2_t(\mathcal{H}^+)} \| \partial_y u_\varphi \|_{L^2_t(\mathcal{H}^+)} (\partial_t R^2) \varphi \|_{L^2_t(\mathcal{H}^+)}.
\]
Then, we deduce from the proof of (A.7) that
\[
\int_{0}^{t} \left| \langle \Delta_y^h (v \partial_y v), \Delta_y^h u_{\varphi} \rangle \right|_{L^2_x} \, dt' \\
\lesssim d_q^2 2^{-2q} \| u_{\varphi} \|_{L^\infty_t (H^{s+1})} \| \partial_y v_{\varphi} \|_{L^2_t (H^s)} \| (\partial_t R^2)_{\varphi} \|_{L^2_{t, \eta(t)} (H^{s+\frac{1}{2}})} \\
\lesssim Cd_q^2 2^{-2q} \| u_{\varphi} \|_{L^\infty_t (H^{s+1})} \| \partial_y u_{\varphi} \|_{L^2_t (H^{s+1})} \| (\partial_t R^2)_{\varphi} \|_{L^2_{t, \eta(t)} (H^{s+\frac{1}{2}})}.
\]
As a result, it comes out
\[
J_q \lesssim Cd_q^2 2^{-2q} \left( \| (R^1_{\varphi}, \varepsilon (\partial_t R^2)_{\varphi}) \|_{L^2_{t, \eta(t)} (H^{s+\frac{1}{2}})}^2 \\
+ \varepsilon^2 \| u_{\varphi} \|_{L^\infty_t (H^{s+1})} \| \partial_y R^2_{\varphi} \|_{L^2_t (H^s)} \| \partial_y u_{\varphi} \|_{L^2_t (H^{s+1})} \| (\partial_t R^2)_{\varphi} \|_{L^2_{t, \eta(t)} (H^{s+\frac{1}{2}})} \right).
\] (A.13)

**Remark A.2.** For the proof when we have \( R^2 \) and not \( \partial_t R^2 \), it also the same we need just to replace in all the proof with \( R^2 \) instead of \( \partial_t R^2 \).

By summing up (A.10)-(A.13), we conclude the proof of (6.18)

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