Schramm-Loewner Equations Driven by Symmetric Stable Processes

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Abstract

We consider shape, size and regularity of the hulls $K_t$ of the chordal Schramm-Loewner evolution driven by a symmetric $\alpha$-stable process. We obtain derivative estimates, show that the domains $H \setminus K_t$ are Hölder domains, prove that $K_t$ has Hausdorff dimension 1, and show that the trace is right-continuous with left limits almost surely.

1 Introduction and Results

The Loewner differential equation (LE for short)

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z$$

(1.1)

takes as input a real-valued function $W_t$ ($t \geq 0$) and produces an increasing family of sets $(K_t)_{t \geq 0}$ such that $g_t$ is the (suitably normalized) conformal map from $\mathbb{H} \setminus K_t$ onto the upper halfplane $\mathbb{H}$. See Section 3. The Schramm Loewner Evolution $SLE_\kappa$ is the random process $K_t$ (or $g_t$) when $W_t = B_{\kappa t}$ where $B_t$ is Brownian motion. See [17] and the references therein.

The spectacular success of $SLE_\kappa$ in describing scaling limits of lattice models and in resolving numerous questions from probability and mathematical physics motivates the study of the Loewner equation driven by other stochastic processes. Roughly speaking, if the driving function is sufficiently continuous, then LE produces a continuous curve $\gamma(t) \in \mathbb{H}$ defined by $g_t(\gamma(t)) = W_t$. This so-called trace generates the hull in the sense that $K_t = \gamma[0, t]$ (if $\gamma$ is not a simple curve, one has to add the filled-in loops). If $W$ has a discontinuity at time $t$, then $\gamma$ has a discontinuity too and the trace grows a "branch". In fact, if $W$ is piecewise constant, then $K$ is a union of analytic curves (and the $n$-th of these curves is a geodesic for the hyperbolic metric in the half plane minus the previous $n - 1$ curves). Thus tree-like sets $K$ can be described by LE with discontinuous driving term. In the mathematical physics literature, the LE driven

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by the symmetric $\alpha$-stable process $S_t$ (plus Brownian motion) has first appeared in [13]. A mathematically rigorous treatment of some elementary properties is in [5].

Another motivation for studying random families of conformal maps comes from a circle of problems known in the complex analysis literature as Brennan’s conjecture, see [2] or [12]. The problem is to maximize

$$\beta_f(p) = \limsup_{r \to 1} \frac{\log \left( \int_0^{2\pi} |f'(re^{it})|^p dt \right)}{\log(1-r)}$$

over all bounded conformal maps $f$ of the unit disc. While it is conjectured that $\beta(p) := \sup_f \beta_f(p) = p^2/4$ for $-2 \leq p \leq 2$, there is no proof of either $\beta(p) \leq p^2/4$ or $\beta(p) \geq p^2/4$, for any nontrivial value of $p$. The lower bound just requires one example $f$, but there are no candidates for extreme domains. From work of Carleson, Jones, Makarov and others it is known that extremals can be found amongst domains with self-similar boundary, and that extremal boundaries can be approximated by “dendrites”. Whereas it is difficult to compute the above integral means for individual functions $f$, it could be easier to estimate the expected value

$$E \left[ \int_0^{2\pi} |f'(re^{it})|^p dt \right]$$

because in a rotationally invariant family this amounts to computing $E[|f'(r)|^p]$. The calculations in [16] showed that Brownian SLE does not produce examples close to extremal. At the 2001/02 Mittag-Leffler program “Probability and Conformal Mappings”, Nikolai Makarov and the second author attempted to find stochastic processes that produced large integral means, and recognized that it would be interesting to study LE driven by symmetric stable processes. The second author would like to thank Nick for these stimulating conversations. In 2003, Daniel Meyer (then graduate student at University of Washington, Seattle) performed computer experiments that suggested a nontrivial and perhaps even close to extremal integral means spectrum for the stable LE.

In this paper, we will consider LE driven by symmetric $\alpha$-stable process $W_t = S_t$, see Section 2 for the definition of symmetric stable processes and some of the basic properties. As $W_t$ satisfies a scaling relation different from Brownian scaling, stable LE does not exhibit scale invariance and thus it is no surprise that rescaling the hulls leads to deterministic sets. Indeed, we show in Section 3 that for $0 < \alpha < 2$, as $s \to 0$, the rescaled hulls $\frac{1}{s}K_{s^2}$ converge to the vertical line segment $[0, 2i]$ (in the Hausdorff metric) in probability. On the other hand, for all $\varepsilon > 0$,

$$\lim_{s \to 0^+} \mathbb{P} \left( \frac{1}{s}K_{s^2} \cap \{y > \varepsilon\} \neq \emptyset \right) = 0.$$

We will then consider continuity and metric properties of the hulls by analyzing the backward flow

$$\partial_t f_t(z) = -\frac{2}{f_t(z) - W_t}, \quad f_0(z) = z. \quad (1.2)$$

For each fixed $t > 0$, this random conformal map $f_t(z)$ of $\mathbb{H}$ has the same distribution as $g_t^{-1}(z - W_t) + W_t$ and thus $K_t$ has the same distribution as $\mathbb{H} \setminus f_t(\mathbb{H}) - W_t$. However as a family of maps, $\{f_t(\cdot), t \geq 0\}$ does not have the same distribution as $\{g_t^{-1}(\cdot), t \geq 0\}$ (see the discussion at the beginning of Section 3). Write

$$f_t(z) - W_t = X_t + i Y_t, \quad t \geq 0.$$

It is easy to see that $Y_t$ is increasing in $t \geq 0$. We prove in Section 4 that for $z = x + iy \in \mathbb{H}$ with $y < 1$, if $\alpha \in [1, 2)$, $Y$ reaches height 1 almost surely when $\alpha \in [1, 2)$, and $Y$ does not reach height 1 with positive probability when $\alpha \in (0, 1)$.
Below are some computer simulations for SLE driven by Cauchy stable processes, with $t = 0.1, 1, 10$ and $t = 100$ respectively.

Figure 1.1 ($\alpha = 1$ and $t = 0.1$)

Figure 1.2 ($\alpha = 1$ and $t = 1$)

Figure 1.3 ($\alpha = 1$ and $t = 10$)

Figure 1.4 ($\alpha = 1$ and $t = 100$)
As in the study of SLE in [16], a key role in understanding $K_t$ is therefore played by the derivative expectation $E[[f_t(z)]^p]$. However in contrast with Brownian motion, the infinitesimal generator of the symmetric $\alpha$-stable process $S$ on $\mathbb{R}$ is the fractional Laplacian $\Delta^{\alpha/2}$, which is not very amenable to calculations. Many nice smooth functions such as polynomials of order 2 and beyond are not in its domain. For this technical reason, we use the truncated symmetric standard $\alpha$-stable process $\hat{S}$ instead, which is the symmetric $\alpha$-stable process $S$ with jumps of size larger than 1 removed. Any $C^2$-smooth function on $\mathbb{R}$ is in the domain of the infinitesimal generator of $\hat{S}$. Note that for the symmetric $\alpha$-stable process $S$, jumps of size larger than 1 arrive according to a Poisson process. So there are only a finite number of jumps of size larger than 1 in any given time interval. For any $\kappa > 0$, information on SLE driven by $S = \{S_t, t \geq 0\}$ can be easily deduced from SLE driven by $\{\hat{S}_{\kappa t}, t \geq 0\}$ (see Lemma 5.3 below), which in turn can be recovered from SLE driven by $\{\tilde{S}_{\kappa t}, t \geq 0\}$ (see Lemma 5.3 below). Our main estimate here is Theorem 4.4 For $\kappa > 0$, let $W_t = \hat{S}_{\kappa t}$ and write

$$f_t(z) - W_t = X_t + iY_t, \quad t \geq 0.$$ 

After a time change $\gamma_u := \inf\{t \geq 0 : Y_t \geq Y_0 e^u\}$ and $\tilde{f}_u(z) := f_{\gamma_u}(z)$ we show in Section 4 that for every $0 < p < 2$ and $\delta > 0$ there is $\kappa > 0$ such that for $W_t = \hat{S}_{\kappa t}$ and every $0 < y < 1$

$$E \left[ \left| \tilde{f}_y \log y(z) \right|^p ; \gamma_u \log y < \infty \right] \leq C_p, \delta y^{-\delta}.$$

In particular, this implies trivial integral means,

$$\beta(1) = 0 \quad \text{a.s.}$$

for all $\kappa > 0$, and also for the (non-truncated) stable process.

We apply the above derivative estimates to prove in Section 5 that for every $T > 0$, the maps of the backward flow $f_t(z)$ of (1.2) driven by $W_t = \hat{S}_{\kappa t}$ with small $\kappa$ are uniformly $\gamma$-Hölder continuous on every bounded set $A \subset \mathbb{H}$ for $t \in [0, T]$ with $\gamma$ close to 1/6. The Hölder exponents are certainly not optimal (we believe that the correct exponent is 1/2 for all $\alpha \in (0, 2)$).

Nevertheless, this establishes enough regularity to prove that the box counting (and hence the Hausdorff) dimension of the hull $K_t$ (of SLE (1.1) driven either by $W_t = S_{\kappa t}$ or by $W_t = \hat{S}_{\kappa t}$ for every $\kappa > 0$) is 1 a.s. It also implies that the backward flow $f_t$ of (1.2) driven either by $W_t = S_{\kappa t}$ or by $W_t = \hat{S}_{\kappa t}$ for every $\kappa > 0$ is locally uniformly Hölder continuous in $\mathbb{H}$ a.s. In particular, this implies that for each $t > 0$, the domain $\mathbb{H} \setminus K_t$ is a Hölder domain almost surely.

Finally, as another application of the Hölder continuity of the maps of the backward flow $f_t(z)$ of (1.2), we prove that the trace is right-continuous with left limits (RCLL in abbreviation): Let $\{g_t, t \geq 0\}$ be SLE (1.1) driven either by $W_t = S_{\kappa t}$ or by $W_t = \hat{S}_{\kappa t}$. We show in Theorem 7.1 that for every $\alpha \in (0, 2)$ and $\kappa > 0$, almost surely, for each $t > 0$ the limit

$$\gamma(t) = \lim_{z \to W_t; z \in \mathbb{H}} g_t^{-1}(z)$$

exists, the function $t \mapsto \gamma(t)$ is RCLL, and $K_t = \gamma[0, t]$. This is achieved by first showing that with probability one, the maps $\{g_t^{-1}, 0 \leq t \leq T\}$ are equicontinuous on $\mathbb{H}$ for every $T > 0$.

Independently from and parallel to this paper, Qing-Yang Guan [9] has recently investigated the continuity properties of the trace of the Loewner equation driven by $W_t = B_{\kappa t} + S_{\theta t}$ for $\kappa \geq 0$, $\theta \geq 0$, and $S$ the symmetric $\alpha$-stable process with $0 < \alpha < 2$ (he informed us that the assumption $\kappa > 0$ in his manuscript is not needed). Thus the main result of [9] contains our Theorem 7.1 as the special case $\kappa = 0$. Whereas his proof of the RCLL property is an adaptation of the continuity proof from [16], we employ a different simpler method that takes advantage of the tree structure of the hulls (which works only for $\kappa \leq 4$), and is of independent interest.
2 Definition and Basic Properties of symmetric $\alpha$-stable process

A random variable $X$ is symmetric $\alpha$-stable if its characteristic function

$$E[e^{i\theta X}] = e^{-c|\theta|^\alpha}. \quad (2.1)$$

For $\alpha = 2$, this is the normal distribution. It is not hard to show (but nontrivial) that such $X$ exists if and only if $0 < \alpha \leq 2$ (see for instance [3, Section 6.5]).

Write $X \sim S(\alpha, c)$ where $\alpha$ is called the index and $c^{1/\alpha}$ is the scale. If $X_i \sim S(\alpha, c_i)$ are independent, then (2.1) immediately gives

$$X_1 + X_2 \sim S(\alpha, c_1 + c_2)$$

and

$$aX \sim S(\alpha, ca).$$

The symmetric $\alpha$-stable process $S = \{S_t, t \geq 0\}$ (or $\alpha$-stable Lévy motion) is a Lévy-process (meaning $S$ is right continuous with left limits, and has stationary independent increments) with $S_t - S_r$ are distributed according to the $\alpha$-stable law:

$$S_t - S_r \sim S(\alpha, t - r)$$

for $0 \leq r \leq t$. Notice that stable process is self-similar: for every $c > 0$,

$$\{S_{ct} - S_0; t \geq 0\} \equiv \left\{c^{1/\alpha} (S_t - S_0); t \geq 0\right\} \quad (2.2)$$

where $\equiv$ denotes equality in distribution. This is the analog of the classical Brownian scaling for Brownian motion. The transition density function can be obtained from the characteristic function by the inverse Fourier transform:

$$p(t, x, y) = p(t, x - y) = P_x (S_t \in [y, y + dy]) / dy = \frac{1}{2\pi} \int_\mathbb{R} e^{-i(x-y)\theta} e^{-t|\theta|^\alpha} \, d\theta.$$

Explicit formulas for $p$ exist only in a few special cases (for $\alpha = 2$ we have the normal distribution $p(t, x) = \exp(-x^2/2t)/\sqrt{2\pi t}$, and for $\alpha = 1$ the Cauchy distribution $p(t, x) = \frac{t}{\pi(t^2 + x^2)}$). However we have the following estimate (see, for example, [4])

$$p(t, x, y) \asymp t^{-\frac{1}{\alpha}} \wedge \frac{t}{|x-y|^{1+\alpha}} = t^{-\frac{1}{\alpha}} \left(1 \wedge \frac{t^{\frac{1}{\alpha}}}{|x-y|}\right)^{1+\alpha}, \quad t > 0, x, y \in \mathbb{R}. \quad (2.3)$$

Here for $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \lor b := \max\{a, b\}$. Therefore

$$P(|S_t - S_0| \geq x) \asymp 1 \wedge \frac{t}{|x|^\alpha}. \quad (2.4)$$

We thus see that for $\alpha < 2$, $S_t$ has infinity variance, and for $\alpha \leq 1$, $S_t$ is even does not integrable.

**Lemma 2.1.** For $t > 0$ and $x > 0$,

$$P\left(\max_{0 \leq r \leq t} S_r > x\right) \leq 2P(S_t > x) \asymp 1 \wedge \frac{t}{|x|^\alpha}.$$
Proof. Let $T = \inf \{ s : S_s > x \}$. Then $\mathbb{P}(T \leq t) = \mathbb{P}(T \leq t, S_t \leq S_T) + \mathbb{P}(T \leq t, S_t > S_T) \leq 2\mathbb{P}(T \leq t, S_t \geq S_T) \leq 2\mathbb{P}(S_t > x)$.

For a Borel measurable function $f$ on $\mathbb{R}$, we define the fractional Laplacian $\Delta^{\alpha/2}f = (-\Delta)^{\alpha/2}f$ at $x \in \mathbb{R}$ as follows:

$$\Delta^{\alpha/2}f(x) := c_\alpha \lim_{\varepsilon \to 0} \int_{\{|h| > \varepsilon\}} \frac{f(x + h) - f(x)}{|h|^{1+\alpha}} dh,$$

whenever the limit exists. It is easy to see that for every $f \in C^2_b(\mathbb{R})$ and every $\delta > 0$,

$$\Delta^{\alpha/2}f(x) = c_\alpha \int_{\mathbb{R}} \frac{f(x + h) - f(x) - f'(x)1_{\{|h| \leq \delta\}}}{|h|^{1+\alpha}} dh,$$

which is well-defined and is in fact a bounded continuous function in $x$. By Ito’s formula

$$t \mapsto f(S_t) - \int_0^t \Delta^{\alpha/2}f(S_r)dr$$

is a martingale for every $f \in C^2_b(\mathbb{R})$ (see, e.g., the proof of Proposition 4.1 in [1] for details). Let $A$ be the Feller generator (that is, the infinitesimal generator in the space $C_b(\mathbb{R})$ of bounded continuous functions equipped with the supremum norm $\| \cdot \|_\infty$) of the symmetric $\alpha$-stable process $S$. Then the above implies that for $f \in C^2_b(\mathbb{R})$,

$$Af(x) = \lim_{t \to 0} \frac{\mathbb{E}[f(x + S_t)] - f(x)}{t} = \Delta^{\alpha/2}f.$$

That is, $C^2_b(\mathbb{R}) \subset \mathcal{D}(A)$ and for $f \in C^2_b(\mathbb{R})$, $Af = \Delta^{\alpha/2}f$.

Let the domain $D \subset \mathbb{R}$ and let $f$ be defined in all of $\mathbb{R}$ and continuous in $D$. Then $f$ is harmonic in $D$ with respect to $S$ if $f$ has the mean value property

$$f(x) = \mathbb{E}_x[f(S_{\tau_{B(x,r)}})]$$

for all balls $B(x,r)$ with closure in $D$, where $\tau_{B(x,r)} := \inf \{ t \geq 0 : S_t \notin B(x,r) \}$. Then the ball can be replaced by any open $D_1$ with $\overline{D_1} \subset D$, see [5] Theorem 2.2.

The function

$$u(x) = \begin{cases} \frac{|x|^{\alpha-1}}{\log |x|} & \text{if } \alpha = 1 \\ |x|^{\alpha-1} & \text{if } \alpha \neq 1 \end{cases}$$

is harmonic in $\mathbb{R} \setminus \{0\}$ as is shown in [13] (for $\alpha \neq 1$ this follows from the harmonicity of the Kelvin transform $|x|^{\alpha-1}h(1/x)$ of the constant function $h \equiv 1$, cf. [11]).

This can be used to obtain a quick proof of the recurrence resp. transience of the stable process for $\alpha > 1$ resp. $\alpha < 1$:

From Ito’s formula, $\{u(S_t), t \in [0, T_0)\}$ is a non-negative local martingale, where $T_0 = \inf \{ t \geq 0 : S_t = 0 \}$ and by Fatou’s Lemma it is a supermartingale. For $0 \leq r < S_0 = x < R$, we therefore have

$$u(x) \geq \mathbb{E}_x[u(S_{\tau_r \wedge \tau_R})] = \mathbb{E}_x [u(S_{\tau_R})1_{\{\tau_R < \tau_{\tau_r}\}} + u(S_{\tau_r})1_{\{\tau_r < \tau_R\}}]$$

(2.7)
where $\tau_r = \inf \{ t : |S_t| < r \}$ and $\tau_R = \inf \{ t : |S_t| > R \}$. For $\alpha > 1$ we get
\[
\mathbb{P}_x(\tau_R < \tau_r) \leq \frac{u(x)}{u(R)} \to 0 \quad \text{as} \quad R \to \infty
\]
proving recurrence. Whereas for $\alpha < 1$ we have $\mathbb{P}_x(\tau_r < \tau_R) \leq \frac{u(x)}{u(r)}$ and so after letting $R \to \infty$, we get
\[
\mathbb{P}_x(\tau_r < \infty) \leq \frac{u(x)}{u(r)} < 1,
\]
giving the transience of $S$. Moreover, if we let $r \downarrow 0$ in the last formula, we have for $\alpha < 1$ and every $x \neq 0$
\[
\mathbb{P}_x(\sigma \{ 0 \} < \infty) = 0.
\]
Here $\sigma \{ 0 \} = \inf \{ t > 0 : S_t = 0 \}$. In other words, almost surely neither $S_t$ nor $S_{t-}$ will visit 0.

### 3 The Loewner equation

#### 3.1 Deterministic equation

Let $W_t$ be a real-valued function that is right continuous with left limits, RCLL for short. For each initial point $z \in \mathbb{C} \setminus \{ 0 \}$, the Loewner differential equation
\[
\frac{\partial_t g_t(z)}{g_t(z) - W_t} = \frac{2}{g_0(z)} = z
\]
has a unique solution up to a time $0 < T_z \leq \infty$ where $g_t(z) = W_{t-}$ or $g_t(z) = W_t$. More precisely, let
\[
T_z = \sup \left\{ t : \inf_{s \in [0,t]} |g_s(z) - W_s| > 0 \right\},
\]
then the initial value problem (3.1) has a unique solution on $[0, T_z)$ and if $T_z < \infty$ then either $\liminf_{t \to T_z^-} |g_t(z) - W_t| = 0$, or $\liminf_{t \to T_z^-} |g_t(z) - W_t| > 0$ and $g_{T_z}(z) = W_{T_z}$ (in this case, $W$ jumps at $t = T_z$). The subset
\[
K_t = \{ z \in \mathbb{H} : T_z \leq t \}
\]
is a compact subset of the closed upper half plane $\overline{\mathbb{H}}$ and is called the hull of LE (3.1). It is well-known that the map $z \mapsto g_t(z)$ is a conformal map (i.e. analytic and one-to-one) from $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$, with Laurent series $g_t(z) = z + \frac{4t}{z} + O\left(\frac{1}{z^2}\right)$ near $\infty$. From the uniqueness of normalized conformal maps it follows that $K_t \cap \mathbb{H} \neq \emptyset$ is strictly increasing in $t$. Writing $g_t(z) = x_t + iy_t$ and taking real- and imaginary parts in (3.1), the Loewner equation reads
\[
\begin{align*}
\partial_t x_t &= \frac{2}{(x_t - W_t)^2 + y_t^2} x_t - W_t \\
\partial_t y_t &= -\frac{2}{(x_t - W_t)^2 + y_t^2} y_t
\end{align*}
\]
It is easy to see that when $W \equiv 0$, $g_t(z) = \sqrt{z^2 + 4t}$ and $K_t = \gamma[0, t]$, where $\gamma(t) = i \sqrt{t}$. 

3
3.2 LE driven by stable processes

As \( t \to \infty \), the diameter of the hulls \( K_t \) tends to infinity. In fact,

\[
\frac{1}{C} \sqrt{t} \leq \text{diam } K_t \leq C(\sqrt{t} + \sup_{0 \leq r \leq s \leq t} |W_r - W_s|) \tag{3.3}
\]

for some universal \( C \) and all \( t \). What do the hulls of LE driven by stable process (that is, \( W \) in \ref{3.2} \) is a symmetric \( \alpha \)-stable process) look like if we scale them back down as \( t \to \infty \), or scale them up when \( t \to 0 \)? We will see that both the "conformally natural" and the "metrically natural" way of rescaling the hulls does not lead to any interesting sets: If we scale them so as to have (halfplane-) capacity one or so that the diameter is one, then the hulls converge to a vertical line segment as \( t \to 0 \) and to the empty set as \( t \to \infty \). To make this precise, let \( c > 0 \).

The solution to \ref{3.1} with

\[
\tilde{W}_t = \frac{1}{c} W_{c^2 t} \tag{3.4}
\]

is given by the function \( g_t(z) = \frac{1}{2} g_{c^2 t}(cz) \). It follows that the hulls are related by

\[
\tilde{K}_t = \frac{1}{c} K_{c^2 t}, \tag{3.5}
\]

If \( W_t \) is a Brownian motion with variance \( c \), then \( \frac{1}{c} W(c^2 t) \) has the same distribution which translates to the important and useful scaling invariance of the SLE hulls. If \( W_t \) is \( \alpha \)-stable then \( \frac{1}{c} W(c^2 t) \) is \( \alpha \)-stable too but the scale is different: From \ref{2.2} it follows that

\[
\frac{1}{c} W(c^2 t) \equiv c^{\frac{2}{\alpha} - 1} W_t. \tag{3.6}
\]

Let \( \{g_t(z), t \geq 0\} \) be the SLE driven by \( W = S \), the symmetric standard \( \alpha \)-stable process on \( \mathbb{R} \), with hulls \( \{K_t, t \geq 0\} \). For \( c > 0 \), define \( \tilde{g}_t(z) := c^{-1} g_{c^2 t}(cz) \). Then

\[
\frac{2}{g_t(z) - c^{-1} S_{c^2 t}} \quad \text{with } \tilde{g}_t(z) = z.
\]

So \( \{\tilde{g}_t(z), t \geq 0\} \) is SLE with hulls \( \{\tilde{K}_t = c^{-1} K_{c^2 t}, t \geq 0\} \), driven by symmetric \( \alpha \)-stable process \( \{c^{-1} S_{c^2 t}, t \geq 0\} \equiv \{S_{c^{2 - \alpha} t}, t \geq 0\} \) running at a different speed. We record this as a lemma for future reference.

**Lemma 3.1.** Let \( \{K_t, t \geq 0\} \) be the hulls of SLE driven by \( W = S \), the symmetric standard \( \alpha \)-stable process. Then for every \( c > 0 \), \( \{c^{-1} K_{c^2 t}, t \geq 0\} \) has the same distribution as the hulls of SLE driven by \( \{W_t = S_{c^{2 - \alpha} t}, t \geq 0\} \). Hence the geometric information on hulls of SLE driven by \( W_t = S_t \) and by \( W_t = S_{c^2 t} \) can be deduced one from the other.

From \ref{3.6}, it is not difficult to prove:

**Proposition 3.2.** Let \( 0 < \alpha < 2 \) and \( \{K_t, t \geq 0\} \) be the hulls of SLE driven by \( W = S \). As \( s \to 0 \), the rescaled hulls \( \frac{1}{s} K_{s^2} \) converge to the vertical line segment \( [0, 2t] \) (in the Hausdorff metric) in probability. On the other hand, for all \( \varepsilon > 0 \),

\[
\lim_{s \to 0} \mathbb{P} \left( \frac{1}{s} K_{s^2} \cap \{y > \varepsilon\} \neq \emptyset \right) = 0.
\]

The proof uses the following simple result for deterministic hulls:
Lemma 3.3. (a) If \( W_t \in [a, b] \) for all \( t \in [0, T] \), then \( K_T \subset [a, b] \times \mathbb{R} \).

(b) Let \( 0 < \varepsilon < 1 \) and \( r > 1 \). If \( I \subset \mathbb{R} \) is an interval of length \( \sqrt{T} \) and \( 10I \) the concentric interval of size \( 10\sqrt{T} \), and if
\[
\int_0^T 1_{\{W_t \in 10I\}} dt \leq \varepsilon T,
\]
then
\[
K_T \cap I \times [4\sqrt{T}, \infty) = \emptyset.
\]

Proof. (a) If \( z = x + iy \in \mathbb{H} \) with \( x < a \) (resp. \( > b \)), then \( \partial_t x_t(z) < 0 \) (resp. \( > 0 \)) and hence \( |g_t(z) - W_t| \) is bounded from below by \( |x - a| \).

(b) By means of Brownian scaling (3.4) and (3.5), we may assume \( T = 1 \) and \( I = [-1/2, 1/2] \).

Fix \( z_0 \in I \times [4\sqrt{\varepsilon}, \infty) \) and write \( g_t(z_0) = x_t + iy_t \). We may assume \( y_0 < 2 \), else trivially \( z_0 \notin K_1 \) (only the hull of the constant function \( W_t \equiv 0 \) reaches height 2). Let \( T_1 \leq 1 \) be maximal time such that \( x_t + iy_t \in [-2, 2] \times [2\sqrt{\varepsilon}, \infty) \) for all \( t \in [0, T_1] \). We will show \( T_1 = 1 \) and hence \( z_0 \notin K_1 \), proving the lemma. Up to \( T_1 \), from (3.2) we have
\[
|x_t - x_0| \leq \int_0^{T_1} \frac{|x_t - W_t|}{(x_t - W_t)^2 + y_t^2} dt
\]
\[
\leq \int_{\{W_t \in 10I\}} \frac{1}{2y_t^2} dt + \int_{\{W_t \notin 10I\}} 2|x_t - W_t| dt
\]
\[
\leq \frac{\varepsilon}{4\sqrt{\varepsilon}} + \frac{2}{3} < \frac{3}{2}
\]
Thus \( x_t \) does not reach the boundary of \([-2, 2]\). Similarly,
\[
|y_t - y_0| = \int_0^{T_1} \frac{y_t}{(x_t - W_t)^2 + y_t^2} dt
\]
\[
\leq \int_{\{W_t \in 10I\}} \frac{2}{y_t^2} dt + \int_{\{W_t \notin 10I\}} 2\frac{y_t}{(x_t - W_t)^2} dt
\]
\[
\leq \frac{\varepsilon}{2} + \frac{2y_0}{9} \leq \sqrt{\varepsilon} + \frac{y_0}{4} < \frac{y_0}{2}
\]
and therefore \( y_t \) does not reach \( 2\sqrt{\varepsilon} \). Hence \( T_1 = 1 \) and the lemma is proved. \( \Box \)

Proof of Proposition 3.2 From (3.6) we have that the rescaled hull has the same distribution as the time 1 hull of the map \( t \mapsto s^{-1}W_t \). By Lemma 2.1 the support of this function tends to zero in probability, and from Lemma 3.3 (a) it follows that the width of the hull tends to zero in probability. Since the halfplane-capacity it 1, the height has to converge to 2 and the hull converges to the segment \([0, 2] \) as \( s \rightarrow 0 \). By Lemma 3.3 (b), the second claim is equivalent to saying that the maximal amount of time that \( W_t \) spends in an interval \([x, x + \delta]\) tends to zero in probability as \( \delta \rightarrow 0 \). \( \Box \)

3.3 Stable LE on \( \mathbb{R} \)

When \( z \) is a non-zero real number, \( Z_t := g_t(z) \) of (3.1) is real-valued. We will call the real-valued equation
\[
\partial_t Z_t = \frac{2}{Z_t - W_t}
\]
the forward Loewner equation on \( \mathbb{R} \) driven by \( W_t \), and

\[
\partial_t Z_t = -\frac{2}{Z_t - W_t}
\]  

(3.10)

the backward Loewner equation on \( \mathbb{R} \). The latter corresponds to the backward flow \( f_t(z) \) of \( W \) with \( z \in \mathbb{R} \setminus \{0\} \). If \( W \) is the symmetric \( \alpha \)-stable process on \( \mathbb{R} \), then the generator of

\[
X_t = Z_t - W_t
\]

in the forward resp. backward equation is

\[
A_\pm = \pm \frac{2}{x} \frac{d}{dx} - (-\Delta)^{\alpha/2}.
\]

For the \((-\Delta)^{\alpha/2}\)-harmonic function \( u(x) = |x|^{\alpha-1} \) (\( \alpha \neq 1 \)) we have

\[
A_\pm u = \pm 2(\alpha - 1)|x|^{-3}.
\]

Thus \( u \) is superharmonic for \( A_+ \) if \( \alpha < 1 \) and for \( A_- \) if \( \alpha > 1 \). With the above reasoning we obtain

**Proposition 3.4.** For \( \alpha > 1 \), \( X_t \) is recurrent in the backward LE on \( \mathbb{R} \), whereas for \( \alpha < 1 \), \( X_t \) is transient in the forward LE on \( \mathbb{R} \) and almost surely, neither \( X_t \) nor \( X_{t-} \) visits \( 0 \).

Notice that in SLE driven by Brownian motion, which corresponds to the case \( \alpha = 2 \), the question of recurrence versus transience of \( X_t \) in the forward LE is rather subtle: If \( B_t \) is Brownian motion and \( W_t = \sqrt{\kappa}B_t \) with \( \kappa \leq 4 \), we have transience whereas for \( \kappa > 4 \) we have recurrence.

We will now prove a partial converse to Lemma 3.3 a) about the deterministic forward LE in \( \mathbb{H} \). If a point \( x_0 \) on the real line stays away by \( \varepsilon \) from the singularity, then the disc of radius \( \varepsilon \) at this point does not meet the singularity and therefore is disjoint from the hull. More generally, if the real part of some point stays away by \( \varepsilon \), then the \( \varepsilon \)-disc around this point is disjoint from the hull.

Let \( g_t(z) \) be the solution to the deterministic LE \( (3.1) \) with \( z \in \mathbb{H} \setminus \{0\} \) and define \( X_t^z = g_t(z) - W_t \). When \( z \in \mathbb{R} \setminus \{0\} \), \( Z_t = g_t(z) \) solves the forward LE \( (3.9) \) on \( \mathbb{R} \) discussed at the beginning of the section and \( X_t^z = Z_t - W_t \) is real-valued. When \( z \in \mathbb{H} \), \( X_t^z \) is complex valued. We will use \( B(z, r) \) to denote the ball in \( \mathbb{R}^2 = (\mathbb{C}) \) centered at \( z \) with radius \( r \).

**Lemma 3.5.** If \( |\text{Re} X_t^z| \geq \varepsilon \) for some \( z_0 \in \mathbb{H} \) and all \( 0 \leq t \leq T \), then \( |\text{Re} X_t^z| > 0 \) for all \( z \in B(z_0, \varepsilon) \cap \mathbb{H} \) and all \( 0 \leq t \leq T \). In particular, \( B(z_0, \varepsilon) \cap K_T = \emptyset \).

**Proof.** From \( (3.1) \) we have

\[
\partial_t (X_t^z - X_t^{z_0}) = \frac{2}{X_t^z} - \frac{2}{X_t^{z_0}} = \frac{2(X_t^{z_0} - X_t^z)}{X_t^z X_t^{z_0}}
\]

and so

\[
\partial_t |X_t^z - X_t^{z_0}|^2 = 2 \text{Re} \left[ \partial_t (X_t^z - X_t^{z_0})(X_t^z - X_t^{z_0}) \right] = -4 \frac{|X_t^z - X_t^{z_0}|^2}{|X_t^z X_t^{z_0}|^2} \text{Re}(X_t^z X_t^{z_0}).
\]

(3.11)

It follows that \( |X_t^z - X_t^{z_0}| \) is decreasing because \( \text{Re}(X_t^z X_t^{z_0}) > 0 \) as long as \( |\text{Re} X_t^{z_0}| \geq \varepsilon \) and \( |\text{Re}(X_t^z - X_t^{z_0})| < \varepsilon \). Since \( |\text{Re} X_t^{z_0}| \geq \varepsilon \) for every \( 0 \leq t \leq T \), we have for every \( z \in B(z_0, \varepsilon) \cap \mathbb{H} \),

\[
|X_t^z - X_t^{z_0}| \leq |z - z_0| < \varepsilon \quad \text{for every } 0 \leq t \leq T
\]
and so $|\Re X^*_t| > 0$ for every $0 \leq t \leq T$. \hfill \Box

Now suppose $x \in \mathbb{R} \setminus \{0\}$ and $g_t(x)$ is the solution to the LE (3.1) driven by a symmetric $\alpha$-stable process $W$ on $\mathbb{R}$ with $\alpha < 1$. As mentioned previously, $g_t(x)$ is the solution to the forward LE on $\mathbb{R}$. Proposition 3.3 tells us that for $X^*_t = g_t(x) - W_t$,

$$ r := \inf_{t \geq 0} \Re X^*_t = \inf_{t \geq 0} |X^*_t| > 0 \quad \text{a.s.} $$

We then have by Lemma 3.5

$$ B(x, r) \cap \bigcup_{t > 0} K_t = \emptyset \quad \text{a.s.} $$

### 4 Derivative Estimates

We would like to estimate the derivative of $h_t = g_t^{-1}$. Because $h_t$ satisfies the PDE

$$ \partial_t h_t(z) = -2 \partial_z h_t(z) / (h(t(z) - W_t) $$

rather than an ODE, it is usually easier to work with the time $t$ map $f_t$ of the backward Loewner equation (1.2). The connection is as follows: If $g_t$ is the solution to (3.1) driven by a function $W_t$ ($0 \leq t \leq T$), and if $f_t$ is the solution to (3.10) driven by $\tilde{W}_t = W_{T-t}$, then $f_T = g_T^{-1}$. But generally $f_t \neq g_t^{-1}$ for $t < T$. Because for the symmetric stable process, $s \mapsto W_{T-s} - W_T$ has the same distribution as $W_s$, it follows that for each fixed $T > 0$, the random conformal map $f_T(z)$ of $\mathbb{H}$ has the same distribution as $g_T^{-1}(z - W_T) + W_T$ (but the family of maps, $\{f_t(\cdot), t \geq 0\}$ does not have the same distribution as $\{g_t^{-1}(\cdot - W_t) + W_t, t \geq 0\}$). For the remainder of this section, we consider the time $t$ map $f_t$ of the backward Loewner equation (1.2).

Let $(X_t, Y_t) := Z_t - W_t$. Then by (3.10),

$$ d(X_t + iY_t) = \frac{-2}{X_t + iY_t} \, dt - dW_t = \frac{-2X_t + 2iY_t}{X_t^2 + Y_t^2} \, dt - dW_t. $$

Hence

$$ dX_t = \frac{2X_t}{X_t^2 + Y_t^2} \, dt - dW_t \quad \text{and} \quad dY_t = \frac{2Y_t}{X_t^2 + Y_t^2} \, dt. \quad (4.1) $$

In particular, we have $d \ln Y_t = \frac{2}{X_t^2 + Y_t^2} \, dt$ and so

$$ Y_t = Y_0 e^{\int_0^t \frac{2}{X_s^2 + Y_s^2} \, ds}. $$

We record a simple lemma for later use. Let $\phi_t(z) = \sqrt{z^2 - 4t}$ be the solution to the backward LE (1.2) driven by the constant function $W \equiv 0$.

**Lemma 4.1.** For every $Z_0 = X_0 + iY_0$ with $Y_0 \in (0, 1]$,

$$ Y_t \leq \Im \phi_t(iY_0) \leq \sqrt{1 + 4t} \quad \text{for every } t > 0. $$

**Proof.** From (4.1) we have $dY_t \leq 2dt/Y_t$ with equality if and only if $X_t \equiv 0$ and therefore $W_t \equiv 0$. Thus $d(Y_t^2) \leq 4dt$ and integration gives $Y_t^2 \leq Y_0^2 + 4t$ with equality if and only if $W_t \equiv 0$. \hfill \Box

For $u > 0$, define

$$ \gamma_u = \inf \{ t > 0 : Y_t \geq Y_0 e^u \} = \inf \left\{ t > 0 : \int_0^t \frac{2}{X_s^2 + Y_s^2} \, ds \geq u \right\}. \quad (4.2) $$
Theorem 4.2. Let $W_t = S_t$ be a standard symmetric $\alpha$-stable process on $\mathbb{R}$ (that is, $W \sim S(\alpha, 1)$). Then for every $z = x + iy \in \mathbb{H}$ and $u > 0$, $\mathbb{P}_z(\gamma_u < \infty) = 1$ when $\alpha \in [1, 2)$ and $\mathbb{P}_z(\gamma_u = \infty) > 0$ when $\alpha \in (0, 1)$.

Proof. Define $u_0 := \inf \{ u : \gamma_u = \infty \}$, and $(\tilde{X}_u, \tilde{Y}_u) := (X_{\gamma_u}, Y_{\gamma_u})$ for $u < u_0$. Clearly for $u < u_0$, $\tilde{Y}_u = Y_0 e^u$. Note that under $\mathbb{P}_z$, $(X_0, Y_0) = (x, y)$, so for $u < u_0$, $\tilde{Y}_u = ye^u$ and

$$\tilde{X}_u = X_{\gamma_u} = x - \int_0^u \frac{2X_s}{X_s^2 + Y_s^2} ds - W_{\gamma_u} = x - \int_0^u \tilde{X}_s ds - W_{\gamma_u}. \quad (4.3)$$

By [14] Theorem 3.1, there is a symmetric $\alpha$-stable process $Z$ on $\mathbb{R}$ such that

$$W_{\gamma_u} = \int_0^u \left( \frac{\tilde{X}_s^2 + y^2 e^{2u}}{2} \right)^{1/\alpha} dZ_s \quad \text{on } [0, u_0).$$

Thus $\tilde{X}$ satisfies the following SDE

$$d\tilde{X}_u = -\tilde{X}_u du - \left( \frac{\tilde{X}_u^2 + y^2 e^{2u}}{2} \right)^{1/\alpha} dZ_u \quad \text{on } [0, u_0) \quad \text{with } \tilde{X}_0 = x, \quad (4.4)$$

where $Z$ is a symmetric $\alpha$-stable process on $\mathbb{R}$. We can rewrite (4.4) as

$$d(e^u \tilde{X}_u) = -e^{(1-2/\alpha)u} \left( \frac{(e^u \tilde{X}_u)^2 + y^2 e^{4u}}{2} \right)^{1/\alpha} dZ_u.$$

By [7] Lemma 4.5 and Theorem 4.6, the above SDE for $U_t := e^t \tilde{X}_t$ has a unique weak solution. Moreover [7] Theorems 4.7 and 4.9 tell us that the solution has non-explosion if and only if $\alpha \in [1, 2)$ (see also [15] for the case of $\alpha \in (1, 2]$). It follows that SDE (4.3) has a unique weak solution $\tilde{X}$ that has infinite lifetime if and only if $\alpha \in [1, 2)$. Note that the process $(\tilde{X}, ye^u)$ extends $(\tilde{X}_u, \tilde{Y}_u)$ in law. So we have for $\alpha \in [1, 2)$, $u_0 = \infty$ a.s., in other words, for any $t > 0$, the original height process $Y$ can reach level $ye^t$ with probability 1. When $\alpha \in (0, 1)$, the proof of [7] Theorem 4.9 illustrates that $\mathbb{P}_z(\gamma_t = \infty) > 0$ for every $t > 0$. This proves the lemma. $\square$

As we mentioned in the Introduction, many smooth functions such as polynomials of order 2 and higher are not $\Delta^{\alpha/2}$-differentiable. For this reason, we need to look at truncated symmetric stable processes. Let

$$\tilde{S}_t := S_t - \sum_{0 < \tau \leq t} (S_\tau - S_{\tau-})1_{\{|S_\tau - S_{\tau-}| > 1\}}, \quad t \geq 0.$$ 

The process $\tilde{S}$ is a Lévy process with Lévy characteristic measure $c_\alpha |h|^{-1-\alpha} 1_{\{|h| \leq 1\}}$ (see [3] Theorem I.1). We call $\tilde{S}$ a truncated symmetric standard $\alpha$-stable process with jumps of size larger than 1 removed. Define for $f \in C^2(\mathbb{R})$,

$$\tilde{\Delta}^{\alpha/2} f(x) := \int_{-1}^1 (f(x + h) - f(x) - f'(x)h) c_\alpha |h|^{-1-\alpha} dh.$$

Note that by Taylor expansion, we have for $f \in C^2(\mathbb{R})$,

$$|\tilde{\Delta}^{\alpha/2} f(x)| \leq \sup_{w \in [x-1, x+1]} |f''(w)| \frac{1}{2} \int_{-1}^1 c_\alpha |h|^{-1-\alpha} dh = C_0 \sup_{w \in [x-1, x+1]} |f''(w)|. \quad (4.5)$$

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Thus there is a richer family of test functions at our disposal for $\Delta^{\alpha/2}$ than for $\Delta^{\alpha/2}$. Just as in \[20\], it follows from Ito’s formula that for every $f \in C^2(\mathbb{R})$,

$$t \mapsto f(S_t) - \int_0^t \Delta^{\alpha/2} f(S_r) \, dr$$

(4.6)

is a local martingale.

**Lemma 4.3.** Let $f(x) := (x^2 + a^2)^{p/2}$ where $a > 0$ and $p < 2$. Then there are constants $C_1, C_2 > 0$ depending on $p$ and $\alpha$ only such that

$$|\Delta^{\alpha/2} f(x)| \leq C_1 (x^2 + a^2)^{(p-\alpha)/2} + C_2.$$  

(4.7)

When $p = \alpha$, the right-hand side is to be interpreted as $\log(1/(x^2 + a^2))$.

**Proof.** The proof is similar to that of Lemma 2.9 in \[5\]. Assume first that $|x| \leq a$. Then $w := x/a \in [-1, 1]$. When $0 < a < 1/2$, we have

$$|\Delta^{\alpha/2} f(x)| = \lim_{\varepsilon \to 0} c_\alpha \int_{|\varepsilon| \leq \varepsilon |h| \leq 1} \frac{((x + h)^2 + a^2)^{p/2} - (x^2 + a^2)^{p/2}}{|h|^{1+\alpha}} \, dh \leq \lim_{\varepsilon \to 0} c_\alpha \int_{|\varepsilon| < |h| \leq 1} \frac{a^p \left(\left(\frac{x}{a} + \frac{h}{a}\right)^2 + 1\right)^{p/2} - \left(\frac{x}{a} + 1\right)^{p/2}}{|h|^{1+\alpha}} \, dh \leq \lim_{\varepsilon \to 0} c_\alpha \int_{|\varepsilon| < |h| \leq 1} \frac{a^p \left(\left(\frac{x}{a} + \frac{h}{a}\right)^2 + 1\right)^{p/2} - \left(\frac{x}{a} + 1\right)^{p/2}}{\alpha^{1+\alpha} |t|^{1+\alpha}} \, dt \leq a^{p-\alpha} \int_{|t| \leq 2} \frac{((w + t)^2 + 1)^{p/2} - (w^2 + 1)^{p/2}}{|t|^{1+\alpha}} \, dt + a^{p-\alpha} \int_{|2| < |t| \leq 1/a} \frac{((w + t)^2 + 1)^{p/2} - (w^2 + 1)^{p/2}}{|t|^{1+\alpha}} \, dt \leq c_4(x^2 + a^2)^{(p-\alpha)/2} + c_3.$$  

When $a \geq 1/2$, by the same calculation as above we have

$$|\Delta^{\alpha/2} f(x)| = \lim_{\varepsilon \to 0} c_\alpha \int_{|\varepsilon| < |t| \leq 1/a} \frac{a^p \left(\left(\frac{x}{a} + \frac{h}{a}\right)^2 + 1\right)^{p/2} - \left(\frac{x}{a} + 1\right)^{p/2}}{a^{1+\alpha} |t|^{1+\alpha}} \, dt \leq c_4(x^2 + a^2)^{(p-\alpha)/2}.$$  

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This proves (4.7) for the case of $|x| \leq a$.
Now assume $|x| > a$. Then $u := a/x \in (-1, 1)$. If $a < |x| \leq 1/2$, we have
\[
|\hat{\Delta}^{\alpha/2} f(x)| \leq \lim_{\varepsilon \to 0} c_\alpha \left| \int_{\{|x|<|t|\leq 1/|x|\}} x^2 \left( \left(1 + \frac{\alpha}{2}\right)^2 + \frac{x^2}{t^2} \right)^{p/2} \left(1 + \frac{x^2}{t^2} \right)^{p/2} \right| dh
\]
\[
= \lim_{\varepsilon \to 0} c_\alpha \int_{\{\varepsilon/|x| < |t| \leq 1/|x|\}} \frac{|x|^p \left((1 + t)^2 + u^2\right)^{p/2} - \left(1 + u^2\right)^{p/2}}{|x|^{1+\alpha}|t|^{1+\alpha}} |x| dt
\]
\[
= |x|^{p-\alpha} \lim_{\varepsilon \to 0} c_\alpha \left| \left( \int_{\{1/2 < |t| \leq 1/|x|\}} \frac{((1 + t)^2 + u^2)^{p/2} - (1 + u^2)^{p/2}}{|t|^{1+\alpha}} dt \right) + \int_{\{1/2 < |t| \leq 1/|x|\}} \frac{((1 + t)^2 + u^2)^{p/2} - (1 + u^2)^{p/2}}{|t|^{1+\alpha}} dt \right|
\]
\[
\leq c_5 |x|^{p-\alpha} + c_7
\]
\[
\leq c_8 (x^2 + a^2)^{(p-\alpha)/2} + c_7.
\]
When $|x| > \max\{1/2, a\}$, we have from above
\[
|\hat{\Delta}^{\alpha/2} f(x)| \leq \lim_{\varepsilon \to 0} c_\alpha \int_{\{\varepsilon/|x| < |t| \leq 1/|x|\}} \frac{|x|^p \left((1 + t)^2 + u^2\right)^{p/2} - (1 + u^2)^{p/2}}{|x|^{1+\alpha}|t|^{1+\alpha}} dt
\]
\[
= |x|^{p-\alpha} c_\alpha \int_{\{|t| \leq 1/|x|\}} \frac{((1 + t)^2 + u^2)^{p/2} - (1 + u^2)^{p/2}}{|t|^{1+\alpha}} dt
\]
\[
\leq c_5 |x|^{p-\alpha}
\]
\[
\leq c_8 (x^2 + a^2)^{(p-\alpha)/2}.
\]
This proves (4.7) for the case of $|x| > a$ and so the lemma is established.

For $\kappa > 0$, let $W_t := \hat{S}_{\kappa t}$. It follows from (4.6) that the infinitesimal generator of $W$ is $\kappa \hat{\Delta}^{\alpha/2}$ in the sense that for every $f \in C^2(\mathbb{R})$,
\[
t \mapsto f(W_t) - \int_0^t \kappa \hat{\Delta}^{\alpha/2} f(W_s) ds
\]
is a local martingale.

**Theorem 4.4.** Let $f_t(x)$ be the solution of the backward equation (4.2) driven by $W_t := \hat{S}_{\kappa t}$.
Define $f_u(z) = f_{\gamma_u}(z)$. Then for every $\alpha, \beta \in (0, 2)$ and $\delta > 0$, there is a constant $\kappa = \kappa(\alpha, \delta) > 0$ such that, for every $z = x + iy \in \mathbb{H}$ with $0 < y < 1$,
\[
\mathbb{E}_z \left[ |\hat{f}_u(z)|^\beta; \gamma_u < \infty \right] \leq e^{-(\beta-\delta)u} \left(x^2 + y^2\right)^{\beta/2} y^{-\beta} \quad \text{for } 0 < u \leq -\log y. \tag{4.8}
\]

**Proof.** Set
\[
\bar{F}(u, x, y) := \mathbb{E}_z \left[ |\hat{f}_u(z)|^\beta; \gamma_u < \infty \right].
\]
Note that since $\partial_t f_t(z) = \frac{-2}{f_t(z) - W_t}$, $\partial_t f'_t(z) = \frac{2f'_t(z)}{(f_t(z) - W_t)^2}$. Thus we have

$$\partial_t \log f'_t(z) = \frac{2}{(f_t(z) - W_t)^2}$$

and so

$$\log |f'_t(z)| = \text{Re} \log f'_t(z) = \int_0^t \text{Re} \left( \frac{2}{(f_s(z) - W_s)^2} \right) ds$$

Since $X_t + iY_t := f_t(z) - W_t$, it follows that

$$\log |f'_t(z)| = \int_0^t \frac{2\text{Re}((X_s - iY_s)^2)}{(X_s^2 + Y_s^2)^2} ds = \int_0^t \frac{2(X_s^2 - Y_s^2)}{(X_s^2 + Y_s^2)^2} ds$$

(4.9)

and so

$$\log |f'_u(z)| = \int_0^u \frac{2(\bar{X}_s^2 - \bar{Y}_s^2)}{(X_s^2 + Y_s^2)^2} ds.$$

(4.10)

Thus we have

$$\bar{F}(u, x, y) = \mathbb{E}_z \left[ \exp \left( \beta \log |f'_u(z)| \right); \gamma_u < \infty \right]$$

$$= \mathbb{E}_z \left[ \exp \left( \int_0^u \frac{2(X_s^2 - Y_s^2)}{X_s^2 + Y_s^2} ds; \gamma_u < \infty \right) \right].$$

(4.11)

Observe that by Ito’s formula (cf. [10]), the infinitesimal generator $\mathcal{L}$ of the process $(\bar{X}, \bar{Y})$ is given by

$$\mathcal{L} \varphi = -x \partial_x \varphi + y \partial_y \varphi + \frac{x^2 + y^2}{2} \kappa \Delta^{\alpha/2}_x \varphi,$$

in the sense that for any $\varphi \in C^2_0(\mathbb{R}^2)$, $t \mapsto \varphi(\bar{X}_t, \bar{Y}_t) - \varphi(\bar{X}_0, \bar{Y}_0) - \int_0^t \mathcal{L} \varphi(\bar{X}_s, \bar{Y}_s) ds$ is a local martingale. So formally, when $\alpha \in [1, 2)$, $\bar{F}$ should satisfy

$$\frac{\partial \bar{F}}{\partial t} = \mathcal{L} \bar{F} + \beta \frac{x^2 - y^2}{x^2 + y^2} \bar{F}$$

with $\bar{F}(0, x, y) = 1$. (4.12)

in some sense. Our approach is motivated by this observation. However (4.12) will not be used in our proof so we can avoid the delicate questions about the regularity of $\bar{F}$ and in which sense the equation (4.12) holds.

For $\lambda > 0$ and $\beta > 0$, define

$$\varphi(t, x, y) = e^{-\lambda t} (x^2 + y^2)^{\beta/2} y^{-\beta}.$$

By (4.5), for $\beta > 0$, there is a constant $C_{\beta, \alpha} > 0$ such that

$$|\Delta^{\alpha/2}_x (x^2 + y^2)^{\beta/2}| \leq C_{\beta, \alpha} (x^2 + y^2)^{\beta/2 - 1} \quad \text{for } |x| \geq 2.$$

On the other hand, by Lemma 1.3 there are constants $C_1, C_2 > 0$, depending only on $\alpha$ and $\beta$, such that

$$|\Delta^{\alpha/2}_x (x^2 + y^2)^{\beta/2}| \leq C_1 (x^2 + y^2)^{(\beta - \alpha)/2} + C_2.$$

Now take $\beta \in (0, 2)$. We have from above that

$$|\Delta^{\alpha/2}_x (x^2 + y^2)^{\beta/2}| \leq c (x^2 + y^2)^{\beta/2 - 1} \quad \text{for } |x| < 2 \text{ and } y \in (0, 1).$$
By increasing the value of $C_{\beta,\alpha} > 0$ if necessary, we have
\[
|\hat{\Delta}^{y/2}(x^2 + y^2)^{3/2}| \leq C_{\beta,\alpha}(x^2 + y^2)^{3/2-1} \quad \text{for every } x \in \mathbb{R} \text{ and } 0 < y \leq 1. \quad (4.13)
\]
Thus for any $z = (x, y)$ with $x \in \mathbb{R}$ and $0 < y \leq 1$, we have
\[
\hat{L}\varphi(t, x, y) + \frac{\beta x^2 - y^2}{x^2 + y^2} \varphi(t, x, y) \\
\leq \frac{-x^2 + y^2}{x^2 + y^2} \varphi(t, x, y) - \beta \varphi(t, x, y) + \frac{x^2 + y^2}{x^2 + y^2} C_{\beta,\alpha} \varphi(t, x, y) + \frac{x^2 - y^2}{x^2 + y^2} \varphi(t, x, y) \\
= -(\beta - C_{\beta,\alpha}) \varphi(t, x, y)
\]
So for any $\delta > 0$ we can choose $\kappa > 0$ small so that $C_{\beta,\alpha} \kappa < \delta$. Taking $\lambda = \beta - \delta$, we have
\[
\left(\hat{L} + \frac{\beta x^2 - y^2}{x^2 + y^2}\right) \varphi(t, x, y) \leq - (\beta - \delta) \varphi(t, x, y) = \frac{\partial}{\partial t} \varphi(t, x, y)
\]
for $x \in \mathbb{R}$ and $0 < y \leq 1$. Thus by Ito’s formula (cf [10]), for each fixed $0 < t \leq - \log y$, with $(\tilde{X}_u, \tilde{Y}_u) := (X_{\gamma_u}, Y_{\gamma_u})$ and $q(x, y) := \beta \frac{x^2 - y^2}{x^2 + y^2}$,
\[
M_z := \varphi(t - s, \tilde{X}_s, \tilde{Y}_s) \exp \left( \int_0^s q(\tilde{X}_u, \tilde{Y}_u) du \right) 1_{\gamma_t < \infty}
\]
is a supermartingale. It follows that $E_z M_0 \geq E_z M_t$ and so
\[
\varphi(t, x, y) \geq E_z \left[ \varphi(0, \tilde{X}_t, \tilde{Y}_t) \exp \left( \int_0^t q(\tilde{X}_u, \tilde{Y}_u) du \right) ; \gamma_t < \infty \right].
\]
Since $\varphi(0, x, y) \geq 1$ for $x \in \mathbb{R}$ and $y \in (0, 1]$ and $\gamma_t \in (0, 1]$ for every $t \leq - \log y$, we have
\[
\varphi(t, x, y) \geq E_z \left[ \exp \left( \int_0^t q(\tilde{X}_u, \tilde{Y}_u) du \right) ; \gamma_t < \infty \right] = \tilde{F}(t, x, y).
\]
This proves the theorem.

\section{Hölder continuity}

In this section we will first prove that the maps $f_t$ generated by the truncated stable process $\tilde{S}_{\kappa t}$ are Hölder continuous a.s. For small $\kappa$ we obtain explicit estimates for the exponent. We will then use Lemma [3.1] and the relation between $S$ and $\tilde{S}$ explained below, in order to obtain Hölder continuity for $f_t$ driven by $S$.

The proof for $\tilde{S}_{\kappa t}$ is similar to the analogous result for SLE$_\kappa$ with $\kappa \neq 4$, Theorem 5.2 in [10]. We begin with an estimate for the derivative $|f_t'|$ of the backward SLE $\{f_t, t \geq 0\}$ of [12] driven by $W_t = \tilde{S}_{\kappa t}$, using Theorem [4.4]

\textbf{Lemma 5.1.} Let $T > 0$. For $0 < \rho < 1$ and $\varepsilon > 0$ there is $\kappa = \kappa(\rho, \varepsilon) > 0$ such that for $z = x + iy$ with $-R < x < R$ and $0 < y < 1$, there is a constant $C > 0$ depending on $T, \alpha, \varepsilon, R$ and $\rho$ so that
\[
P \left( \max_{0 \leq t \leq T} |f_t'(z)| \geq y^{\rho-1} \right) \leq Cy^{2-6\rho-\varepsilon}.
\]
Proof. Fix $0 \leq t \leq T$, $z = x + iy$ and write $f_t(z) - W_t = X_t + iY_t$, $\tilde{f}_u(z) - W_u = \tilde{X}_u + i\tilde{Y}_u$. Notice $y = Y_0$. Recall by (4.10),

$$\partial_u \log |\tilde{f}'_u(z)| = \frac{\tilde{X}_u^2 - \tilde{Y}_u^2}{X_u^2 + Y_u^2},$$

so that

$$|f'_t(z)| = \exp \left( \int_0^1 \frac{\log |\tilde{f}'_u(z)|}{X_u^2 + Y_u^2} du \right).$$

Let

$$q(u) := \frac{\tilde{X}_u^2 - \tilde{Y}_u^2}{X_u^2 + Y_u^2}.$$

Since $|q(u)| \leq 1$, if $Y_t < y^\rho$, it follows that $|f'_t(z)| < y^\rho - 1$.

On $\{1 \geq Y_t \geq y^\rho\}$, since $q(u) \leq 1$,

$$|f'_t(z)| = \exp \left( \int_0^1 q(u) du + \int_{\log \frac{Y_t}{\gamma}}^{\log \frac{Y_t}{\rho}} q(u) du \right) \leq |\tilde{f}'_{(\rho-1)\log y}(z)| \frac{Y_t}{y^\rho} \leq |\tilde{f}'_{(\rho-1)\log y}(z)| y^{-\rho},$$

while on $\{Y_t > 1\}$ we have by Lemma 1.1

$$|f'_t(z)| = \exp \left( \int_0^1 q(u) du + \int_{\log \frac{Y_t}{\rho}}^{\log \frac{Y_t}{\gamma}} q(u) du \right) \leq |\tilde{f}'_{(\rho-1)\log y}(z)| Y_t \leq (\sqrt{1 + 4t}) |\tilde{f}'_{-\log y}(z)|.$$

It follows that, on $\left\{ \max_{0 \leq t \leq T} |f'_t(z)| \geq y^{\rho - 1} \right\} \subset \{Y_T > y^\rho\}$,

$$\max_{0 \leq t \leq T} |f'_t(z)| \leq |\tilde{f}'_{(\rho-1)\log y}(z)| y^{-\rho} \mathbb{1}_{\{\gamma(\rho-1)\log y < \gamma\}} + (\sqrt{1 + 4t}) |\tilde{f}'_{-\log y}(z)| \mathbb{1}_{\{\gamma-\log y < \gamma\}}.$$

Let $0 < \beta < 2$ and $\delta > 0$. Then it follows from the above and Theorem 4.4 that

$$\mathbb{P} \left( \max_{0 \leq t \leq T} |f'_t(z)| \geq y^{\rho - 1} \right) \leq \mathbb{E}_z \left[ \left( y^{1-\rho} \max_{0 \leq t \leq T} |f'_t(z)| \right)^\beta ; Y_T \geq y^\rho \right] \leq \mathbb{E}_z \left[ \left( y^{1-\rho} \beta y^{-\rho \delta} |\tilde{f}'_{(\rho-1)\log y}(z)|^{\beta} ; \gamma(\rho-1)\log y < \gamma \right] + (\sqrt{1 + 4t})^\beta \mathbb{E}_z \left[ \left( y^{1-\rho} \beta \tilde{f}'_{-\log y}(z)|^{\beta} ; \gamma-\log y < \gamma \right] \leq C y^{\beta - 2\rho\delta y^{1-\rho} (\beta - \delta) y^{-\beta} + C y^{\beta \rho \delta (\beta - \delta) y^{-\beta}}.$$

This establishes the lemma. \qed

The next theorem says that $z \mapsto f_t(z)$ is locally uniformly $\gamma$-Hölder continuous with any exponent $\gamma < \frac{4}{\beta}$ for $0 < \alpha < 2$ and small $\kappa$. We believe that the correct Hölder exponent is $\frac{1}{2}$ for every $0 < \alpha < 2$. Notice the uniformity in $t \in [0, T]$, which is important later.

**Theorem 5.2.** For every $\varepsilon > 0$ there is $\kappa > 0$ such that with $W_t = \tilde{S}_{\kappa t}$, for every bounded set $A \subset \mathbb{H}$ and every $T > 0$, a.s. all $f_t, 0 \leq t \leq T$, are Hölder continuous with exponent $1/6 - \varepsilon$ on $A$ when $0 < \alpha < 2$:

$$|f_t(z) - f_t(z')| \leq C |z - z'|^{\frac{1}{2} - \varepsilon}$$

for all $z, z' \in A$ with a random constant $C = C(A, \alpha, T, \varepsilon)$.  

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Proof. Let $R > 0$ and $b > 0$ be such that $A \subset [-R, R] \times (0, b]$. It suffices to show that
\[ \max_{0 \leq t \leq T} |f_t'(x + iy)| \leq C y^{-\frac{d}{2} - \varepsilon} \]
for all $-R < x < R$ and all $0 < y \leq b$. By Koebe distortion, it is enough to show this for dyadic points $z_{j,n} = (j + i)2^{-n}$, where $n \geq 0$ and $-R2^n < j \leq R2^n$. For every $\varepsilon > 0$ there is $\rho > 1/6 - \varepsilon$ such that the exponent $2 - 4\rho - \varepsilon$ in Lemma $6.3$ is strictly larger than $1$. Hence
\[ \sum_{n=0}^{\infty} \sum_{j=-R2^n}^{R2^n} \mathbb{P} \left( \max_{0 \leq t \leq T} |f_t'(z_{j,n})| > y^{-\frac{d}{2} - \varepsilon} \right) < \infty \]
and the theorem follows as Theorem 5.2 in [16].

It immediately follows that for each fixed $t$, the map $f_t(z)$ extends continuously to $\mathbb{H}$ a.s. In order to pass from SLE driven by truncated stable process $\tilde{S}_{\kappa t}$ to SLE driven $S_{\kappa t}$, let’s recall the following relation between $S_t$ and $\tilde{S}_t$. Note that symmetric $\alpha$-stable process has Lévy measure $c_\alpha |h|^{-1-\alpha} dh$. The jumps $\{ (S_{\kappa t} - S_{\kappa t-})1_{\{|S_{\kappa t-}| > 1\}}, t \geq 0 \}$ of size larger than $1$ form a Poisson point process with intensity measure $c_\alpha |h|^{-1-\alpha} 1_{\{|h| > 1\}} dt dh$. The process
\[ \tilde{S}_{\kappa t} := S_{\kappa t} - \sum_{r \leq T} (S_{\kappa r} - S_{\kappa r-})1_{\{|S_{\kappa r-}| > 1\}}, \quad t \geq 0, \]
has the same distribution as $\{ \tilde{S}_{\kappa t}, t \geq 0 \}$. Define $T_0 = 0$ and let
\[ T_k := \inf \{ t > T_{k-1} : |S_{\kappa t} - S_{\kappa t-}| > 1 \} \quad \text{for } k \geq 1, \]
be the $k$th jumping time of $S_{\kappa t}$ of size larger than $1$. Then $\{ T_k - T_{k-1}, k \geq 1 \}$ is a sequence of i.i.d. exponential random variables with parameter $\lambda \kappa$. Moreover, the processes
\[ \{ (S_{\kappa t+T_{k-1}} - S_{\kappa t})1_{\{|S_{\kappa t}| > 1\}}, t \in [0, T_k - T_{k-1}], k \geq 1 \}
are i.i.d., which are independent copies of $\tilde{S}_{\kappa t}$ killed at an independent exponential random time $T_1$. All this tells us that $S_{\kappa t}$ can be constructed as follows.

Let $T_0 = 0$ and $\{ T_k - T_{k-1}, k \geq 0 \}$ be an i.i.d. sequence of exponential random variables with parameter $\lambda \kappa$. Let $\{ \tilde{S}^k_{\kappa t}, t \geq 0 \}$ be a sequence of independent copies of $\{ \tilde{S}_{\kappa t}, t \geq 0 \}$. Let $\{ \xi_k, k \geq 1 \}$ be an i.i.d. sequence of random variables with density function proportional to $1_{|h| > 1} |h|^{-1-\alpha}$ in $\mathbb{H}$. These $\{ T_k, k \geq 1 \}$, $\{ \tilde{S}^k_{\kappa t}, t \geq 1 \}$ and $\{ \xi_k, k \geq 1 \}$ are all independent. For $t > 0$, let $n$ be the largest integer so that $T_n \leq t$. Define
\[ X_t := \sum_{k=1}^{n-1} \left( \tilde{S}^k_{\kappa (T_k - T_{k-1})} + \xi_k \right) + \tilde{S}^n_{\kappa (t - T_{n-1})}. \quad (5.1) \]
Then $\{ X_t, t \geq 0 \}$ has the same distribution as $\{ S_{\kappa t}, t \geq 0 \}$. From this, we immediately have the following.

Lemma 5.3. For $\kappa > 0$, let $\{ T_k, k \geq 1 \}$, $\{ \tilde{S}^k_{\kappa t}, t \geq 1 \}$ and $\{ \xi_k, k \geq 1 \}$ be as in the last paragraph, which are all independent, and let $X$ be defined by $(5.1)$. Let $\{ f_t^{(k)}, t \geq 0 \}$ be SLE driven by $S^k_{\kappa t}$. For $t > 0$, let $n$ be the largest integer so that $T_n \leq t$. Define
\[ f_t(z) := \left( f_{t - T_n}^{(n)} \left( \cdot - T_n \right) + X_{T_n} \right) \circ \cdots \circ \left( f_{T_2 - T_1}^{(2)} \left( \cdot - T_1 \right) + X_{T_1} \right) \circ f_{T_1}^{(1)}(z). \]
Then $\{ f_t(z), t \geq 0 \}$ has the same distribution as the SLE driven by $W_t = S_{\kappa t}$.

Because compositions of Hölder continuous maps are Hölder, from Theorem 5.2, Lemma 5.1 and Lemma 5.3 we obtain the following.

Corollary 5.4. For every $0 < \alpha < 2$, $\kappa > 0$, and $W_t = \tilde{S}_{\kappa t}$, for every bounded set $A \subset \mathbb{H}$ and every $t > 0$, a.s. $f_t$ is Hölder continuous on $A$. The same holds for $W_t = S_t$. 

6 Hausdorff dimension

We will now show that the hulls have Hausdorff dimension 1 almost surely. The situation is similar to [16], Section 8.2: Because \( f_t \) is Hölder continuous, the (box counting) dimension can be estimated by the convergence exponent of the Whitney decomposition of \( \mathbb{H} \setminus K_t \), which in turn is controlled by the growth of the derivative \( f_t' \) towards the boundary \( \mathbb{R} \) of \( \mathbb{H} \). For a Borel set \( K \subset \mathbb{R}^2 \), we use \( \dim_H K \) to denote its Hausdorff dimension.

**Theorem 6.1.** For each \( 0 < \alpha < 2 \), \( \kappa > 0 \), and \( W_t = S_{\alpha t} \) (or \( W_t = \widehat{S}_{\alpha t} \))

\[
\dim_H K_t = 1
\]

for all \( t \geq 0 \), almost surely.

Since \( K_t \) has empty interior by [8], \( K_t \cup \partial = \partial (\mathbb{H} \setminus K_t) \). Because \( q_t^{-1} \) has the same distribution as \( f_t \) (for fixed \( t \)), it thus suffices to show that the boundary of \( H_t = f_t(\mathbb{H}) \) has dimension 1 a.s. Denote by \( N(\epsilon) = N(\epsilon, A) \) the minimal number of disks of radius \( \epsilon \) needed to cover a set \( A \subset \mathbb{C} \).

The following is an analog of the upper bound for the dimension of the outer SLE boundary, Theorem 8.6 in [16].

**Theorem 6.2.** For each \( 0 < \alpha < 2 \) and \( 1 < a < 2 \), there is \( \kappa > 0 \) such that with \( W_t = \widehat{S}_{\alpha t} \), for all \( T > 0, h > 0 \) and \( R > 0 \), a.s. we have

\[
\lim_{\varepsilon \to 0} \varepsilon^a \max_{0 \leq t \leq T} N(\varepsilon, f_t([-R, R] \cap \{ y > h \})) = 0.
\]

**Proof.** As in [16], Section 8.2, consider a Whitney decomposition of \( H_t \) (that is a covering of \( H_t \) by essentially disjoint closed squares \( Q \subset H_t \) with sides parallel to the coordinate axes such that the side length \( d(Q) \) is comparable to the distance of \( Q \) from the boundary of \( H_t \), and such that \( d(Q) \) is an integer power of 2). Denote by \( W_t \) the collection of those squares \( Q \) for which \( Q \cap f_t([-R, R] \times (0, \infty)) \cap \{ y > h \} \neq \emptyset \), and let

\[
S(a) = \max_{0 \leq t \leq T} \sum_{Q \in W_t} d(Q)^a \leq \infty.
\]

Then the proof of Theorem 8.6 in [16] (the last displayed formula) shows that, for each \( 0 \leq t \leq T \),

\[
N\left(2^{-n}, f_t([-R, R] \cap \{ y > 2h \}) \right) \leq C(\omega) 2^{(n+O(\log n))a S(a)}.
\]

The factor \( C(\omega) \) comes from the Hölder norm of \( f_t \) and is random, but does not depend on \( t \) or \( n \). The theorem follows at once if we show \( S(a) < \infty \) a.s. To this end, we will show

\[
\mathbb{E}[S(a)] < \infty
\]

for \( a > 1 \), in analogy with the upper bound in Theorem 8.3 of [16]. By the Koebe distortion theorem, again writing \( z_{j,n} = (j + i)2^{-n} \), the quantity

\[
\overline{S}(a) = \max_{0 \leq t \leq T} \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} 1_{\{|\text{Im} f_t(z_{j,n})| > h\}} |f_t'(z_{j,n}) 2^{-n}|^a
\]

is comparable to \( S(a) \) (see (8.2) and Lemma 8.4 of [16] for the details), in particular \( S(a) \leq CS(a) \) for some universal \( C \). For \( 0 \leq t \leq T \), Lemma 4.1 yields

\[
1_{\{|\text{Im} f_t(z_{j,n})| > h\}} |f_t'(z_{j,n})| \leq 1_{\{|\gamma \text{log } (h/|y_{j,n}|) < \infty\}} |f_t'(z_{j,n})| \leq C_{T,h} |f_t'(z_{j,n})| 1_{\{|\gamma \text{log } (2^n h) < \infty\}}.
\]
Now Theorem 4.3 with $\beta = a$ and $\delta < a - 1$ implies
\[
\mathbb{E}[S(a)] \leq \sum_{n=0}^{\infty} 2R2^n2^{-an}2^{-an} < \infty,
\]
thus proving the theorem. \hfill \square

The next lemma says that the Hausdorff dimension of boundaries of simply connected domains does not increase under finite composition.

**Lemma 6.3.** Let $1 < a < 2$, let $f^{(j)} : \mathbb{H} \to \mathbb{H} \setminus K^{(j)}, 1 \leq j \leq n$, be conformal maps, and let $f = f^{(n)} \circ \cdots \circ f^{(1)}$. If $\dim_H K^{(j)} \leq a$ for all $j$, then $\dim_H \partial f(\mathbb{H}) \leq a$.

**Proof.** For the proof, just notice that
\[
\partial f(\mathbb{H}) = K^{(n)} \cup f^{(n)}(K^{(n-1)}) \cup \cdots \cup f^{(n)}(f^{(n-1)}(\cdots (f^{(2)}(K^{(1)}))))
\]
and that each of the sets in the union has dimension $\leq a$ because each map $f^{(j)}$ is smooth in $\mathbb{H}$. \hfill \square

**Proof of Theorem 6.1** Let $1 < a < 2$ and let $\kappa > 0$ be as in Theorem 6.2. Let $f_t$ be driven by $S_{\kappa t}$, and factor $f_t$ according to Lemma 5.3. Then by Theorem 6.2 the hulls of the factors of $f_t$ have Hausdorff dimension $\leq a$, and thus $\dim_H \partial f(\mathbb{H}) \leq a$ by Lemma 6.3. Letting $a$ tend to $1$, we see that the hulls driven by $W_t = S_{\kappa t}$ have Hausdorff dimensional at most 1. Because the boundary of the simply connected domain $\mathbb{H} \setminus K_t$ is connected, and because $K_t \cap \mathbb{H} \neq \emptyset$, we have $\dim_H K_t \geq 1$ and conclude $\dim_H K_t = 1$ for every $t > 0$. By the scaling Lemma 6.1, the hulls driven by $W_t = S_t$ have the same dimension as the hulls of $S_{\kappa t}$. Finally, the hulls of $\tilde{S}_{\kappa t}$ for an arbitrary (not necessarily small) $\kappa$ can be recovered from the hulls of $S_{\kappa t}$, and therefore have dimension $1$, by removing the jumps of $W_{\kappa t}$, similar to Lemma 5.3. \hfill \square

## 7 Trace continuity

The purpose of this section is to prove the following

**Theorem 7.1.** Fix $\alpha \in (0, 2)$ and $\kappa > 0$. Let $W_t = S_{\kappa t}$ or $W_t = \tilde{S}_{\kappa t}$. Then almost surely, for each $t > 0$ the limit
\[
\gamma(t) = \lim_{z \to W_t; z \in \mathbb{H}} g_t^{-1}(z)
\]
exists, the function $t \mapsto \gamma(t)$ is RCLL, and $K_t = \gamma[0, t]$.

From Theorem 5.2 we know that for each fixed $T$, $f_T(z)$ extends continuously to $\mathbb{H}$ a.s. Because the hulls $K_T$ have the same law as $\mathbb{H} \setminus f_T(\mathbb{H}) - W_T$, they are locally connected a.s. In general, this does not imply that the subsets $K_t \subset K_T$ for $t < T$ are locally connected too (for instance, it is possible that $K_t$ is not locally connected at some time $t_0$, but that due to "swallowing" $K_{t_0}$ is contained in the interior of $K_t$ for some $t_1 > t_0$, and that the boundary of $K_{t_1}$ is smooth). Nor does the equicontinuity of $f_t(z)$ generally imply equicontinuity of $g_t^{-1}(z)$. For instance, if $W_t = c(1 - t)$ with $c = 2\sqrt{3}$, then $f_t(z)$ is equicontinuous ($\mathbb{H} \setminus f_t(\mathbb{H})$ is a halfdisc of radius proportional to $\sqrt{7}$) whereas $g_t^{-1}$ is not ($K_t$ is an arc of a semicircle up to time 1 when $K_t$ is a semicircle). Because of the tree structure of the hulls, our situation is better:

**Proposition 7.2.** Let $W_t = S_{\kappa t}$ or $W_t = \tilde{S}_{\kappa t}$. For each $0 < \alpha < 2$ and each $T > 0$, a.s. each of the maps $g_t^{-1}, 0 \leq t \leq T$, has a continuous extension to $\mathbb{H}$ (which we again denote $g_t^{-1}$). Moreover, the maps $\{g_t^{-1}, 0 \leq t \leq T\}$, are equicontinuous on $\mathbb{H}$.
Proof of Theorem 7.3. Fix \( \alpha \in (0, 2) \) and \( T > 0 \), and let \( t \leq T \). Because \( g_t^{-1} \) has a continuous extension to \( \mathbb{H} \) by Proposition 7.2, \( \gamma(t) = \lim_{z \to -W_t ; z \in \mathbb{H}} g_t^{-1}(z) \) exists, and \( \gamma(t) = g_t^{-1}(W_t) \). The equicontinuity of \( g_t^{-1} \), together with the pointwise continuity of \( t \to g_t^{-1}(z) \), easily implies the continuity of \( (t, z) \to g_t^{-1}(z) \) on \([0, T] \times \mathbb{H} \). It follows immediately that \( \gamma \) is RCLL.

To prove \( K_t = [0, t] \), we need a variant of a theorem of Warschawski [18] about the modulus of continuity of conformal maps of the disc. Roug hly speaking, after suitable normalization the modulus only depends on the "roughness" of the boundary of the domain as measured by the size of bottlenecks. Let \( G \subset \mathbb{C} \) be a simply connected domain and \( a \in G \) be a marked point (in [18], \( a = 0 \) whereas here we will have \( p = \infty \)). A crosscut of \( G \) is a simple arc \( \{\sigma(t), 0 \leq t \leq 1\} \) that lies in \( G \) except for the endpoints \( \sigma(0), \sigma(1) \in \partial G \). Every crosscut separates \( G \) into two connected components. If \( a \notin \sigma \), denote \( G(\sigma) \) the component that does not contain \( p \) in its closure. Following Warschawski, define

\[
\eta_G(\delta) = \sup_{\text{diam } \sigma \leq \delta} \text{diam } G(\sigma).
\]

Thus \( \eta_G(\delta) \to 0 \) as \( \delta \to 0 \) if and only if \( \partial G \) is locally connected. Now assume that \( G = \mathbb{H} \setminus K \) and that \( f : \mathbb{H} \to G \) is the hydrodynamically normalized conformal map, \( f(z) = z + a/z + O(1/z^2) \) near \( \infty \). Denote

\[
\omega_f(r) = \sup \{|f(z) - f(z')| : z, z' \in \mathbb{H} \text{ with } |z - z'| \leq r\}
\]

the modulus of continuity of \( f \). The following is Theorem I of Warschawski [18], except for the different normalization. His proof carries over with only minor modifications.

Theorem 7.3. For each \( R > 0 \) and each function \( \eta(\delta) \) with \( \eta(0+) = 0 \) there is a function \( \omega(r) \) with \( \omega(0+) = 0 \) such that the following holds: If \( K \subset \{ |z| < R \} \), and if \( \eta_G(\delta) \leq \eta(\delta) \) for all \( \delta \), then

\[
\omega_f(r) \leq \omega(r)
\]

for all \( r > 0 \).

Proof of Proposition 7.2. Fix \( T > 0 \). Because \( f_T(z) \) extends continuously to \( \mathbb{H} \) a.s. by Corollary 5.3 and because the hulls \( K_T \) have the same law as \( \mathbb{H} \setminus f_T(\mathbb{H}) - W_T \), we have

\[
\eta_{K_T}(0+) = 0
\]

a.s., where \( G_T = \mathbb{H} \setminus K_T \). By Theorem 1.3 (i) in [8], we know that \( K_T \) and hence \( K_t, 0 \leq t \leq T \), does not have interior points. Hence every crosscut \( \sigma \) of \( \mathbb{H} \setminus K_t \) can be decomposed into crosscuts \( \sigma_j \) of \( \mathbb{H} \setminus K_T \) such that \( (\mathbb{H} \setminus K_T)(\sigma) \subset \bigcup_j (\mathbb{H} \setminus K_T)(\sigma_j) \). It follows that \( \eta_{K_T}(\delta) \leq \delta + 2\eta_{G_T}(\delta) \), for all \( t \leq T \). Now Proposition 7.2 follows from Theorem 7.3. \( \square \)
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