Holographic four-point functions in Toda field theories in $\text{AdS}_2$

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Abstract

We consider Toda field theories in a classical Euclidean $\text{AdS}_2$ background. We compute the four-point functions of boundary operators in the $a_1$, $a_2$ and $b_2$ Toda field theories. They take the same form as the four-point functions of generators in the corresponding $W$-algebras. Therefore we conjecture that the boundary operators are in one-to-one correspondence with the generators in the $W$-algebras.

1 Introduction

Among two-dimensional field theories, integrable field theories distinguish themselves by the existence of an infinite number of conserved charges. In flat spacetime, two basic consequences of higher conservation laws are the absence of particle production and factorization of the higher-point S-matrix\cite{1, 2, 3, 4}. Since a Lax pair is in general hard to find, no tree-level particle production is regarded as an important sign of classical integrability, and it is a useful strategy for building classically integrable theories \cite{5, 6, 2, 7, 8}.

It is natural to wonder what one can say about integrable field theories in other two-dimensional spacetimes. The $\text{AdS}_2$ case is of particular interest since it is maximally symmetric and has applications related to the AdS/CFT correspondence\cite{9, 10, 11}. S-matrix cannot be defined in AdS spacetime due to the periodicity of particle orbits and the timelike boundary. The AdS analogs of the flat space S-matrix elements are boundary correlation functions which can be obtained as a limit of bulk correlation functions. In the weak coupling regime, they can be computed by Witten diagrams. Such observables in $\text{AdS}_2$ can be used to define a set of correlation functions in a one-dimensional conformal theory \cite{12}. Due to the existence of infinitely many conservation laws, it is intuitive to expect there should be something special about boundary correlation functions in integrable field theories.

Integrable field theories in $\text{AdS}_2$ are rare. Toda field theories with simple Lie algebras provide an important class of such examples. Toda field theories are conformal \cite{13, 14, 15} and integrable\cite{16, 17, 18, 19}. It was shown in \cite{20, 21, 22, 23} that they possess extended symmetries generated by $W$-algebras \cite{24} which are higher spin extensions of the Virasoro algebra (see \cite{25} for a review of $W$-algebras).

In this paper we focus on four-point functions of boundary operators in Toda field theories in a classical $\text{AdS}_2$ background. Unlike previous efforts devoted to the correlation
functions of the vertex operators, we are interested in the boundary operators obtained by pushing bulk scalar fields to the conformal boundary. The conformal dimensions of the boundary operators are related to the masses of the bulk fields. The potentials of Toda field theories in flat space do not have a stable minimum. When a Toda field theory is put in an arbitrary Riemann surface, the scalar fields need to couple to curvature in order to preserve integrability. Since the $AdS_2$ spacetime or its Euclidean version has a negative constant scalar curvature, the potential acquires a minimum and the scalar fields become massive.

The spectrum of the theory is determined by the set of exponents of the associated Lie algebra. We compute the tree-level four-point functions in Toda field theories with the Lie algebras $a_1$, $a_2$ and $b_2$. Adding up contributions from the exchange Witten diagrams and contact Witten diagrams, the final expressions are rather simple. Indeed, they are related to the four-point functions of generators in $\mathcal{W}$-algebra in the limit of large central charge.

This paper is organized as follows. In section 2 we briefly review Toda field theories in $AdS_2$. In section 3, we present in detail the calculation of four-point functions in the $a_1$, $a_2$ and $b_2$ Toda field theories. Conclusions are given in section 4. Some reduced $\bar{D}$-functions used in the calculation are collected in appendix A.

2 Toda field theories in $AdS_2$

The action of a Toda field theory associated with a finite-dimensional simple Lie algebras $\mathfrak{g}$ of rank $r$ in an Euclidean $AdS_2$ (hyperbolic plane) is given by

$$S_\mathfrak{g} = \int d^2 x \sqrt{g} \left( \frac{1}{2} g^{\mu \nu} \partial_\mu \phi \cdot \partial_\nu \phi + V_\mathfrak{g}(\phi) \right),$$

with

$$V_\mathfrak{g}(\phi) = \frac{1}{\beta^2} \sum_i n_i e^{\beta \alpha_i \cdot \phi} + \frac{1}{2} R Q \cdot \phi,$$

where $\alpha_i$ are simple roots of $\mathfrak{g}$ and $\phi$ is an $r$-component scalar field. We consider an $AdS_2$ with unit radius. The metric in Poincaré coordinates is

$$ds^2 = \frac{dx^2 + dz^2}{z^2}.$$

Then the scalar curvature $R = -2$. In the classical theory if $Q = 2\beta^{-1} \rho^\vee$ with the dual Weyl vector $\rho^\vee$ satisfying

$$\alpha_i \cdot \rho^\vee = 1, \quad i = 1, 2, ..., r,$$

the equations of motion are invariant under the Weyl transformation

$$g_{ab} \to e^{2\omega} g_{ab}, \quad \phi \to \phi - Q \omega.$$

Therefore the theory is classically integrable because the flat space theory has a Lax pair. The vector $Q$ needs to be modified at quantum level [14]. In this paper we mainly focus on classical aspects of the theory so we simply set $Q = 2\beta^{-1} \rho^\vee$.

The coefficients $n_i$ can be chosen arbitrarily through shifts in the scalar fields. If we require that the potential minimized at $\phi = 0$, then

$$\sum_i n_i \alpha_i = 2\rho^\vee, \quad \Rightarrow \sum_i n_i \alpha_i \cdot \alpha_j = 2, \quad j = 1, 2, ..., r.$$
The mass spectrum can be obtained by calculating the eigenvalues of the matrix \( \sum n_i \alpha^a_i \alpha^b_i \). In the standard holographic dictionary, the mass of a scalar field \( \phi_a \) can be expressed as \( m^2_a = \Delta_a (\Delta_a - 1) \) where \( \Delta_a \) is the conformal dimension of the boundary operator dual to \( \phi_a \). The spectrum of conformal dimensions coincides with the set of exponents plus one or equivalently degrees of fundamental adjoint-invariant polynomials of the corresponding Lie algebra. The sets of these values are given in Table 1.

| Lie algebra | Conformal dimensions |
|-------------|----------------------|
| \( a_n \)   | 2, 3, ..., \( n + 1 \) |
| \( b_n \)   | 2, 4, ..., \( 2n \)  |
| \( c_n \)   | 2, 4, ..., \( 2n \)  |
| \( d_n \)   | 2, 4, ..., \( 2n - 2, n \) |
| \( e_6 \)   | 2, 5, 6, 8, 9, 12    |
| \( e_7 \)   | 2, 6, 8, 12, 14, 18  |
| \( e_8 \)   | 2, 8, 12, 14, 18, 20, 24, 30 |
| \( f_4 \)   | 2, 6, 8, 12          |
| \( g_4 \)   | 2, 6                 |

Table 1: Spectrums of conformal dimensions in Toda field theories

Interestingly, the spins of generators in the \( \mathcal{W} \)-algebras are also equal to the exponents of the Lie algebras plus one, which implies that there is a one-to-one correspondence between the boundary operators and generators in the \( \mathcal{W} \)-algebras. We will check this conjecture by computing the tree-level four-point correlation functions of boundary operators in the next section.

As some concrete examples, the potentials of Toda field theories with Lie algebras of rank 1 and 2 are given by

\[
V_{a_1} = \frac{2}{\beta^2} (e^{\beta \phi} - \beta \phi - 1) = \phi^2 + \frac{1}{3} \beta \phi^3 + \frac{1}{12} \beta^2 \phi^4 + O (\beta^3),
\]

\[
V_{a_2} = \frac{2}{\beta^2} (e^{\beta \phi_1} \cosh (\sqrt{3} \beta \phi_2) - \beta \phi_1 - 1)
= (\phi_1^2 + 3\phi_2^2) + \beta \left( \frac{\phi_1^3}{3} + 3\phi_2^2 \phi_1 \right) + \beta^2 \left( \frac{\phi_1^4}{12} + \frac{3}{2} \phi_2^2 \phi_1^2 + \frac{3\phi_2^4}{4} \right) + O (\beta^3),
\]

\[
V_{b_2} = \frac{1}{5\beta^2} (4e^{\beta(\phi_1 + 3\phi_2)} + 6e^{\beta(\phi_1 - 2\phi_2)} - 10\beta \phi_1 - 10)
\]

\[
= (\phi_1^2 + 6\phi_2^2) + \beta \left( \frac{\phi_1^3}{3} + 6\phi_2^2 \phi_1 + 2\phi_2^3 \right) + \beta^2 \left( \frac{\phi_1^4}{12} + 3\phi_2^2 \phi_1^2 + 2\phi_2^3 \phi_1 + \frac{7\phi_2^4}{2} \right) + O (\beta^3),
\]

\[
V_{g_2} = \frac{1}{7\beta^2} (-14\beta \phi_1 + 9e^{\beta(\phi_1 - 5\phi_2/\sqrt{3})} + 5e^{\beta(1 + 3\sqrt{3} \phi_2)})
\]

\[
= (\phi_1^2 + 15\phi_2^2) + \beta \left( \frac{\phi_1^3}{3} + 15\phi_2^2 \phi_1 + \frac{20\phi_2^3}{\sqrt{3}} \right) + \beta^2 \left( \phi_1^4 + 3\phi_2^2 \phi_1^2 + 2\phi_2^3 \phi_1 + 80\sqrt{3} \phi_2^3 \phi_1 + 305\phi_2^4 \right) + O (\beta^3).
\]
3 Four-point functions in Toda field theories

3.1 Witten diagrams in \(AdS_2\)

We now briefly review some basic ingredients of Witten diagrams in \(AdS_2\). Witten diagrams are Feynman diagrams built from bulk-to-bulk propagators and bulk-to-boundary propagators. The bulk-to-bulk propagator of a scalar with mass \(m^2 = \Delta (\Delta - 1)\) is given by

\[
G_\Delta(x_1, z_1, x_2, z_2) = C_\Delta u^{-\Delta} F_1(\Delta, \frac{2\Delta}{2}, 2\Delta, -\frac{4}{u}),
\]

where

\[
u = \frac{(x_1 - x_2)^2 + (z_1 - z_2)^2}{z_1 z_2}, \quad C_\Delta = \frac{\Gamma(\Delta)}{2\sqrt{\pi} \Gamma(\Delta + 1/2)}.
\]

The bulk-to-boundary propagator is given by

\[
K_\Delta(x, x_0, z_0) = C_\Delta \left(\frac{z_0}{z_0 + (x - x_0)^2}\right)^\Delta.
\]

To get correlation functions for normalized operators whose two-point functions have unit coefficient, every external line has to be multiplied by a factor \(C_\Delta^{-1/2}\). For instance, an \(n\)-point contact diagram takes the form

\[
W_{\text{contact}}^{\Delta_i}(x_i) = \int x_0 z_0 \prod_{i=1}^{n} C_{\Delta_i}^{-1/2} K_{\Delta_i}(x_i, x_1, z_1).
\]

where \(\Delta_i, i = 1, ..., n\), are conformal dimensions of the external operators. The expression for \(n = 3\) can be found in [26]

\[
W_{\Delta_1\Delta_2\Delta_3}(x_1, x_2, x_3) = \prod_{i=1}^{3} C_{\Delta_i} \frac{\sqrt{\pi} \Gamma(\Delta_1 + \Delta_2 - \Delta_3) \Gamma(\frac{\Delta_1 + \Delta_2 - \Delta_3}{2}) \Gamma(\frac{\Delta_2 + \Delta_3 - \Delta_1}{2}) \Gamma(\Delta_1 + \Delta_2 + \Delta_3 - 1)}{2 \Gamma(\Delta_1) \Gamma(\Delta_2) \Gamma(\Delta_3) |x_{12}|^{\Delta_1 + \Delta_2 - \Delta_3} |x_{23}|^{\Delta_2 + \Delta_3 - \Delta_1} |x_{13}|^{\Delta_1 + \Delta_2 - \Delta_3}}.
\]

where \(x_{ij} \equiv x_i - x_j\).

We shall concentrate on tree-level four-point functions in Toda field theories. The contact Witten diagrams generated by quartic interactions can be written in terms of the \(D\)-functions [27, 28, 29] defined as

\[
D_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_i) = \int \frac{dx_0 d\bar{z}_0}{\bar{z}_0^2} \prod_{i=1}^{4} \left(\frac{\bar{z}_0}{\bar{z}_0 + (x_i - x_0)^2}\right)^{\Delta_i},
\]

It is convenient to work with the reduced \(\bar{D}\)-functions which are functions of the cross-ratio (not to confuse with the same symbol used for \(AdS_2\) coordinate)

\[
z = \frac{x_{12} x_{34}}{x_{13} x_{24}}.
\]

They are defined by extracting a kinematic factor

\[
\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_i) = \frac{\sqrt{\pi} \Gamma(\Sigma - 1/2)}{2 \prod_{i=1}^{4} \Gamma(\Delta_i)} \frac{x_{14}^{2(\Sigma - \Delta_1 - \Delta_4)} x_{34}^{2(\Sigma - \Delta_2 - \Delta_4)} D_{\Delta_1\Delta_2\Delta_3\Delta_4}(z)}{x_{13}^{2(\Sigma - \Delta_2)} x_{24}^{2(\Sigma - \Delta_3)}}.
\]
We start with the Liouville (3.2) diagram and exchange diagrams in \( \Delta = 2 \). The tree-level four-point function receives contributions from a quartic contact scalar field with

\[
\langle \bar{\varphi} x_1 \bar{\varphi} x_2 \varphi x_3 \varphi x_4 \rangle_{\text{tree}} = \beta^2 \left( 4W_{2222,2}^x + 4W_{2222,2}^y + 4W_{2222,2}^z - 2W_{2222}^{\text{contact}} \right)
\]

\[
= \frac{4\beta^2}{9\pi^2} (x_{21}^{-2} D_{2211} + x_{14}^{-2} D_{1221} + x_{24}^{-2} D_{2121} - 2D_{2222})
\]

(3.11)

where we have used (3.10) to write exchange diagrams in terms of \( D \)-functions [31].

We also need to consider exchange Witten diagrams mediated by the cubic interactions. In the special case when \( k = (\Delta_1 + \Delta_2 - \Delta_E)/2 \) is a positive integer, this diagram can be written as a sum of \( D \)-functions [31]

\[
W_{\Delta_1 \Delta_2 \Delta_3 \Delta_4 \Delta_E}(x_i) = \prod_{i=1}^4 C_{\Delta_i}^{1/2} \int \frac{dy_1 dy_2 dw_1}{w_1^2} K_{\Delta_1}(x_1, y_1, z_1) K_{\Delta_2}(x_2, y_1, w_1) \times G_{\Delta_E}(y_1, w_1, y_2, w_2) K_{\Delta_3}(x_3, y_2, w_2) K_{\Delta_4}(x_4, y_2, w_2)
\]

(3.9)

where \( \Delta_E \) is the conformal dimension of the exchange operator.

\[\Sigma = \sum_{i=1}^4 \Delta_i/2.\]

\( D \)-functions with integer indices can be evaluated recursively by using the identities in [30]. We list some of the \( \bar{\varphi} x_1 \bar{\varphi} x_2 \varphi x_3 \varphi x_4 \)\( D \)-functions [31]

\[
(\Delta_1 - l(\Delta_2 - 1) \Delta_3 \Delta_4 (x_i)) \]

(3.10)

where \( \Delta_E \) is the conformal dimension of the exchange operator. In the special case when

\[
\Delta = 2
\]

3.2 \( a_1 \) theory

We start with the Liouville (3.1 Toda) field theory. The potential is given in (2.8). The scalar field with \( m^2 = 2 \) corresponds to a boundary operator with conformal dimension \( \Delta = 2 \). The tree-level four-point function receives contributions from a quartic contact diagram and exchange diagrams in \( s \)-, \( t \)- and \( u \)-channels. We have

\[
\langle O(x_1)O(x_2)O(x_3)O(x_4) \rangle_{\text{tree}} = \beta^2 (4W_{2222,2}^x + 4W_{2222,2}^y + 4W_{2222,2}^z - 2W_{2222}^{\text{contact}})
\]

\[
= \frac{4\beta^2}{9\pi^2} (x_{21}^{-2} D_{2211} + x_{14}^{-2} D_{1221} + x_{24}^{-2} D_{2121} - 2D_{2222})
\]

(3.11)

where we have used (3.10) to write exchange diagrams in terms of \( D \)-functions. Using the explicit expressions for the \( D \)-functions given in Appendix A we get

\[
\bar{D}_{2211} + \bar{D}_{1221} + \bar{D}_{2121} - 5\bar{D}_{2222}
\]

\[
= \frac{z^3 - z + 1}{(z - 1)^2 z^2}
\]

(3.12)
and thus

$$\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle_{\text{tree}} = \frac{\beta^2}{12\pi} \left( \frac{1}{x_{12}^2 x_{23}^2 x_{34}^2 x_{14}^2} + \frac{1}{x_{13}^2 x_{23}^2 x_{24}^2 x_{14}^2} + \frac{1}{x_{12}^2 x_{24}^2 x_{34}^2 x_{13}^2} \right). \tag{3.13}$$

This looks like the connected part of a four-point function of the stress-tensor operators $T$ in the Virasoro algebra up to an overall coefficient, although it is a one-dimensional correlation function. Furthermore, the disconnected parts of both four-point functions also take the same form.

Taking into account carefully the overall coefficient, the central charge $c$ of Virasoro algebra is related to the $\beta$ by $c = 48\pi\beta^{-2}$ and $O(x)$ corresponds to $-\sqrt{2/cT(w)}$. This is consistent with tree-level three-point function computing by (3.5)

$$\langle O(x_1)O(x_2)O(x_3)\rangle_{\text{tree}} = -\frac{\beta}{\sqrt{6\pi}} \frac{1}{x_{12}^2 x_{23}^2 x_{13}^2} \sim -\frac{8}{c^3} \langle T(w_1)T(w_2)T(w_3) \rangle. \tag{3.14}$$

The central charge of Liouville theory in our convention is

$$c_{a_1} = 1 + 48\pi \left( \frac{\beta}{8\pi} + \frac{1}{\beta} \right)^2 \tag{3.15}$$

In the classical limit $c_{a_1} \to 48\pi\beta^{-2}$. One can expect that at the quantum level the central charge $c$ of Virasoro algebra is equal to the central charge of Liouville theory. If this conjecture is true, $n$-point functions beyond tree-level can be obtained from correlation functions of the stress-tensor operators $T(w_i)$.

### 3.3 $a_2$ theory

We now turn to the $a_2$ theory. The boundary operators have conformal dimensions $\Delta_1 = 2$ and $\Delta_2 = 3$, which are equal to the spins of generators in the algebra $W(2,3)$. The notation $W(2,s_1,s_2,\ldots)$ means that the algebra is generated by primary currents $W(s_i)$ of spins $s_i$ together with the Virasoro generator $T$ of spin 2.

The four-point function of four $O_1$ is the same as (3.13). The other independent four-point functions are

$$\langle O_1(x_1)O_1(x_2)O_2(x_3)O_2(x_4)\rangle_{\text{tree}} = \beta^2 \left(12 W_{2232} + 36 W_{2333} + 36 W_{2323} - 6 W_{2233} \right)$$

$$= \frac{16\beta^2}{45\pi^2} \left( 3x_{12}^{-2} D_{1133} + \frac{9}{2} x_{14}^{-2} D_{1232} + \frac{9}{2} x_{24}^{-2} D_{2132} - 6 D_{2233} \right)$$

$$= \frac{\beta^2}{4\pi x_{13}^2 x_{24}^2 x_{34}^2} \left( z^{-2} \hat{D}_{1133} + 3 \hat{D}_{1232} + 3 \hat{D}_{2132} - 7 \hat{D}_{2232} \right) \tag{3.16}$$

$$= \frac{\beta^2}{8\pi} \frac{3z^2 - 2z + 2}{x_{13}^2 x_{24}^2 x_{34}^2} (z - 1)^2 z^2$$

$$= \frac{\beta^2}{8\pi} \left( \frac{1}{x_{12}^2 x_{23}^2 x_{34}^2 x_{14}^2} + \frac{1}{x_{12}^2 x_{24}^2 x_{34}^2 x_{13}^2} + \frac{2}{x_{13}^2 x_{23}^2 x_{24}^2 x_{14}^2 x_{34}^2} \right).$$
and

\[ \langle O_2(x_1)O_2(x_2)O_2(x_3)O_2(x_4)\rangle_{\text{tree}} \]
\[ = \beta^2 (36W_{3333,2}^2 + 36W_{3333,2}^3 + 36W_{3333,2}^4 - 18W_{3333}^{\text{contact}}) \]
\[ = \frac{64\beta^2}{225\pi^2} \left( \frac{45}{8} x_{34}^4 D_{3311} + x_{14}^4 D_{1331} + x_{24}^4 D_{3131} \right) + \frac{9}{4} \left( x_{34}^{-2} D_{3322} + x_{14}^{-2} D_{2332} \right) \]
\[ + x_{24}^{-2} D_{3232} - 18 D_{3333} \]
\[ = \frac{3\beta^2}{40\pi x_{13}^6 x_{24}^6} \left( 5(\bar{D}_{3311} + \bar{D}_{1331} + \bar{D}_{3131}) + 7(D_{3322} + D_{2332} + D_{3232}) - 63 D_{3333} \right) \]
\[ = \frac{\beta^2}{x_{13}^6 x_{24}^6} \left( 80\pi (z-1)^4 z^4 \right) \]
\[ = \frac{3\beta^2}{80\pi} \left( \frac{5}{x_{13}^6 x_{24}^6 x_{14}^4} + \frac{5}{x_{13}^6 x_{24}^6 x_{13}^4} + \frac{5}{x_{13}^6 x_{24}^6 x_{12}^4} \right) \]
\[ \quad + \frac{5}{x_{13}^6 x_{24}^6 x_{14}^4} \left( 1331 - 3(10z^6 - 30z^5 + 29z^4 - 8z^3 + 39z^2 - 30z + 10) \right) \]

Comparing them to the four-point functions in the algebra \( \mathcal{W}(2,3) \) \[24]\]

\[ \langle T(w_1)T(w_2)W^{(3)}(w_3)W^{(3)}(w_4) \rangle \]
\[ = \frac{1}{w_{13}^6 w_{24}^6 w_{34}^6} \left( \frac{c^2}{6z^2} + c \left( \frac{2}{z^2} + \frac{3}{(1-z)^2} + \frac{2}{1-z} \right) \right), \quad (3.18) \]

\[ \langle W^{(3)}(w_1)W^{(3)}(w_2)W^{(3)}(w_3)W^{(3)}(w_4) \rangle \]
\[ = \frac{1}{w_{13}^6 w_{24}^6 w_{34}^6} \left( \frac{c^2}{9} \left( \frac{1}{z^6} + \frac{1}{(1-z)^6} + 1 \right) \right) \]
\[ + c \left( \frac{2}{z^4} + \frac{2}{(1-z)^2} + \frac{2}{z^3} + \frac{2}{(1-z)^3} + \frac{9}{5z^2} + \frac{9}{5(1-z)^2} + \frac{8}{5z} + \frac{8}{5(1-z)} \right) \]
\[ + \frac{16}{5z^2} \left( \frac{1}{(1-z)^2} + \frac{2}{z} + \frac{2}{1-z} \right), \quad (3.19) \]

one finds the tree-level four-point functions \[3.16\] and \[3.17\] correspond to the order \( c \) contributions to \[3.18\] and \[3.19\] respectively. The disconnected pieces of the four-point functions correspond to the order \( c^2 \) contributions. Therefore, we get the expected correspondence. This correspondence provides a useful way to obtain higher-point functions, because correlation functions of \( \mathcal{W} \)-currents and stress-tensor \( T \) can be computed by using recursion relations derived in \[24\].
3.4 \( b_2 \) theory

Finally we consider the \( b_2 \) theory. The conformal dimensions of the boundary operators are \( \Delta_1 = 2 \) and \( \Delta_2 = 4 \). The tree-level four-point functions are

\[
\langle O_1(x_1)O_2(x_2)O_2(x_3)O_2(x_4) \rangle_{\text{tree}} = \beta^2 \left( 24W_{2444,4}^2 + 144W_{2444,4}^t + 144W_{2444,4}^u - 12W_{2444}^{\text{contact}} \right) \\
= \frac{\beta^2}{6\pi x_1^4 x_2^4 x_3^4 x_4^4} \left( z^{-2}D_{1144} + 6D_{1243} + 6D_{1234} - 3D_{2244} \right) \\
= \frac{\beta^2}{3\pi x_1^4 x_2^4 x_3^4 x_4^4} (z - 1)^2 z^2,
\]

\[(3.20)\]

\[
\langle O_1(x_1)O_2(x_2)O_2(x_3)O_2(x_4) \rangle_{\text{tree}} = \beta^2 \left( 144W_{2444,4}^2 + 144W_{2444,4}^t + 144W_{2444,4}^u - 12W_{2444}^{\text{contact}} \right) \\
= \sqrt{\frac{3}{70} \beta^2 \frac{x_1^2}{x_2^4 x_3^4 x_4^4} \left( 3(z^{-2}D_{1344} + D_{1443} + D_{1434}) - \frac{11}{2}D_{2444} \right) } \\
= \sqrt{\frac{3}{70} \beta^2 \frac{x_1^2}{x_2^4 x_3^4 x_4^4} (z^2 - z + 1) } \\
= \sqrt{\frac{3}{70} \beta^2 \frac{x_1^2}{x_2^4 x_3^4 x_4^4} (z - 1)^2 z^2},
\]

\[(3.21)\]

\[
\langle O_2(x_1)O_2(x_2)O_2(x_3)O_2(x_4) \rangle_{\text{tree}} = \beta^2 \left( 144W_{2444,4}^2 + W_{2444,4}^t + W_{2444,4}^u + W_{2444,2}^t + W_{2444,2}^u - 84W_{2444}^{\text{contact}} \right) \\
= \frac{\beta^2}{\pi x_1^8 x_2^8 x_3^8 x_4^8} \left( \frac{1}{3}(D_{4411} + D_{1441} + D_{4111}) + \frac{69}{70}(D_{4422} + D_{2442} + D_{4242}) \\
+ \frac{33}{35}(D_{4433} + D_{3443} + D_{4343}) - \frac{143}{20}D_{4444} \right) \\
= \frac{\beta^2}{420\pi x_1^8 x_2^8 x_3^8 x_4^8} (z^2 - z + 1)^2 \left( 280z^6 - 840z^5 + 279z^4 + 842z^3 + 279z^2 - 840z + 280 \right),
\]

\[(3.22)\]

together with \( \langle O_1(x_1)O_1(x_2)O_1(x_3)O_1(x_4) \rangle_{\text{tree}} \) given by \[(3.13)\].

As an application of the conjectured correspondence, we now demonstrate how to compute \[(3.22)\] using recursion relations. The operator product expansion of two spin-4 currents \( W^{(4)} \) in the algebra \( \mathcal{W}(2,4) \) can be found in \[32,33,34,35,36\], and in the large central charge limit we have

\[
W^{(4)}(z)W^{(4)}(w) = \frac{c/4}{(z-w)^8} + \sum_{k=0}^{5} \frac{12(k+1)}{\Gamma(k+4)} \frac{\partial^k T(w)}{(z-w)^{6-k}} \\
+ \sum_{k=0}^{3} \frac{72\sqrt{105}(k+1)3}{\Gamma(k+8)} \frac{\partial^k W^{(4)}(w)}{(z-w)^{4-k}} + O(c^{-1})
\]

\[(3.23)\]
Following similar analysis in [24], we have the recursion relation

$$\langle W^{(4)}(w_1)W^{(4)}(w_2)\ldots W^{(4)}(w_n) \rangle$$

$$= \sum_{i=2}^{n} \frac{c/4}{(w_1-w_2)^{s}} \langle W^{(4)}(w_2)\ldots W^{(4)}(w_{i-1})W^{(4)}(w_{i+1})\ldots W^{(4)}(w_n) \rangle$$

$$+ \sum_{i=2}^{n} \sum_{k=0}^{5} \frac{12(k+1)}{\Gamma(k+4)} \frac{\partial_{w_i}^k}{(w_1-w_i)^{6-k}} \langle W^{(4)}(w_2)\ldots W^{(4)}(w_{i-1})T(w_i)W^{(4)}(w_{i+1})\ldots W^{(4)}(w_n) \rangle$$

$$+ \sum_{i=2}^{n} \sum_{k=0}^{3} \frac{72\sqrt{105}(k+1)3}{\Gamma(k+8)} \frac{\partial_{w_i}^k}{(w_1-w_i)^{14-k}} \langle W^{(4)}(w_2)\ldots W^{(4)}(w_n) \rangle + O\left(c^{n/2-2}\right).$$

(3.24)

Using the correspondence $O_1 \sim -\sqrt{2/cT}$ and $O_2 \sim -\sqrt{4/cW^{(4)}}$, we get

$$\langle O_2(x_1)O_2(x_2)O_2(x_3)O_2(x_4) \rangle_{\text{tree}}$$

$$= -5 \sum_{k=0}^{5} \frac{\beta 12(k+1)}{\sqrt{6\pi} \Gamma(k+4)} \frac{\partial_{w_2}^k}{(w_1-w_2)^{6-k}} \langle O_1(x_2)O_2(x_3)O_2(x_4) \rangle_{\text{tree}}$$

$$+ \frac{\partial_{w_3}^k}{(w_1-w_3)^{6-k}} \langle O_2(x_2)O_1(x_3)O_2(x_4) \rangle_{\text{tree}} + \frac{\partial_{w_4}^k}{(w_1-w_4)^{6-k}} \langle O_2(x_2)O_2(x_3)O_1(x_4) \rangle_{\text{tree}}$$

$$- \frac{4}{\sqrt{\pi}} \frac{36\sqrt{35} \beta (k+1)3}{\Gamma(k+8)} \frac{\partial_{w_i}^k}{(w_1-w_i)^{14-k}} \langle O_2(x_2)O_2(x_3)O_2(x_4) \rangle_{\text{tree}}.$$

(3.25)

The three-point functions can be obtained by use of (3.5)

$$\langle O_1(x_2)O_2(x_3)O_2(x_4) \rangle_{\text{tree}} = -\frac{2\beta}{\sqrt{6\pi}} \frac{1}{x_{23}^2 x_{24}^2 x_{34}^2}.$$

(3.26)

$$\langle O_2(x_2)O_2(x_3)O_2(x_4) \rangle_{\text{tree}} = -\frac{3\beta}{2\sqrt{35\pi}} \frac{1}{x_{23}^4 x_{24}^4 x_{34}^4}.$$

(3.27)

Plugging them into (3.25) one immediately obtain the expected result (3.22). It is reasonable to expect that higher-point functions can be obtained in the same way.

4 Conclusions

In this paper, we have studied the boundary correlation functions in Toda field theories in $AdS_2$. We have found a relation between boundary operators and generators in the $\mathcal{W}$-algebra by computing tree-level four-point functions in some simple Toda field theories. The conformal dimensions of boundary operators are equal to the spins of generators in the $\mathcal{W}$-algebra, and the tree-level four-point functions correspond to the connected four-point functions of the $\mathcal{W}$-currents or the stress-tensor in the large central charge. We conjecture the correspondence holds at the quantum level. If rigorously proved, this conjecture will provide useful information about higher-loop and higher-point Witten diagrams in $AdS_2$. It would be interesting to extend our study to the supersymmetric Toda field theories.
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A $\bar{D}$-functions

There are two independent cross-ratios $u$ and $v$ in higher dimensions. $\bar{D}$-functions in one dimension can be obtained from the higher dimensional ones by

$$
\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(z) = \lim_{u \to z^2} \lim_{v \to (1-z)^2} D_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v), \tag{A.1}
$$

where $D_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v)$ are $\bar{D}$-functions in general dimensions which can be computed recursively starting from

$$
\bar{D}_{1111}(u,v) = \Phi(u,v). \tag{A.2}
$$

The standard four-dimensional one-loop integral $\Phi(u,v)$ can be found in [37] and the recursion relations can be found in [30]. After taking the limit $u \to z^2$ and $v \to (1-z)^2$, we have

$$
\bar{D}_{1111} = \frac{2 \log(|z|)}{z-1} - \frac{2 \log(|1-z|)}{z}, \tag{A.3}
$$

$$
\bar{D}_{2211} = \frac{z^3 \log(|z|) + (-z^3 + 3z - 2) \log(|1-z|) - z^2 + z}{3(z-1)^2 z^3}, \tag{A.4}
$$

$$
\bar{D}_{1221} = \frac{(3-z) z^2 \log(|z|) + (z-1)^3 \log(|1-z|) + z^2 - z}{3(z-1)^2 z^2}, \tag{A.5}
$$

$$
\bar{D}_{2121} = \frac{z^2 (2z - 3) \log(|z|) - (2z + 1)(z-1)^2 \log(|1-z|) + z^2 - z}{3(z-1)^2 z^2}, \tag{A.6}
$$

$$
\bar{D}_{2222} = \frac{1}{15(z-1)^3 z^2} \left( - (z-1)^3 \left(2z^2 + z + 2\right) \log(|1-z|) - 2z^4 + 4z^3 - 4z^2 + 2z 
+ z^3 \left(2z^2 - 5z + 5\right) \log(|z|) \right), \tag{A.7}
$$

$$
\bar{D}_{1232} = \frac{1}{15(z-1)^3 z^2} \left( (2z + 3)(z-1)^4 \log(|1-z|) + 2z^4 - 4z^3 - z^2 + 3z - z^4(2z-5) \log(|z|) \right), \tag{A.8}
$$

$$
\bar{D}_{2132} = \frac{(5 - 3z) z^4 (\log(|2-2z|) - \log(2|z|)) + (3 - 5z) \log(|1-z|) + (1-z)(z(3z-8) + 3z)}{15(z-1)^4 z^2}, \tag{A.9}
$$
There are crossing relations

\[
D_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(z) = D_{\Delta_3 \Delta_2 \Delta_1 \Delta_4}(1 - z) = z^{-\Delta_1 - \Delta_2 - \Delta_3 + \Delta_4} D_{\Delta_1 \Delta_3 \Delta_2 \Delta_4}(1/z).
\]
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