Estimation in autoregressive models with Markov regime

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Abstract

In this paper we derive the consistency of the penalized likelihood method for the number state of the hidden Markov chain in autoregressive models with Markov regime. Using a SAEM type algorithm to estimate the models parameters. We test the null hypothesis of hidden Markov Model against an autoregressive process with Markov regime.

Keywords: Autoregressive process, hidden Markov, switching, SAEM algorithm, penalized likelihood.

1 Introduction

This paper is devoted to estimate of autoregressive models with Markov regime. Our goals in this paper are:

• Estimate, using maximum likelihood estimation (MLE) methods, the parameters that define the functions, the transition probabilities of the hidden Markov chain and the noise variance, computed via SAEM, a stochastic version of EM algorithm [8], for a pre-fixed number states of the hidden Markov chain.

• Test the null hypothesis of HMMs against AR-RM.

• Derive the consistency of the penalized likelihood method for the number of state.

An autoregressive model with Markov regime (AR-MR) is a discrete-time process defined by:

\[ Y_n = f_{X_n}(Y_{n-1}, \theta_{X_n}) + \sigma \varepsilon_n \]  

(1.1)

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where \( \{X_n\} \) is a Markov chain with finite state space \( \{1, \ldots, m\} \). The transition probabilities denoted as \( a_{ij} = \Pr(X_n = j|X_{n-1} = i) \). The \( a_{ij} \) form an \( m \times m \) transition matrix \( A \). The functions \( \{f_1, \ldots, f_m\} \) belong to a parameterized family

\[
\{\theta_1(\cdot) + \theta_0 : \theta = (\theta_1, \theta_0) \in \Theta\}
\]

(1.2)

where \( \Theta \) a compact subset of \( \mathbb{R}^2 \), and \( \{\varepsilon_n\} \) is a sequence of independent identically distributed standard normal random variables, \( \mathcal{N}(0, 1) \). As process \( \{X_n\} \) is not observable then we are forced to work with simulations of the law of the hidden chain and to rely on observed data \( \{Y_n\} \) for any inference task.

The usage of Markov regime offers possibilities for modelling time series “subject to discrete shifts in regime-episodes across which the dynamic behavior of the series is markedly different”, as noted by Hamilton [17] who used a model AR-RM in the context of econometrics, for the analysis of the U.S. annual GNP (gross national product) series, with two regimes: contraction and expansion. Linear autoregressive process with Markov regime are also used in several electrical engineering areas including tracking of manoeuvring targets, failure detection and stochastic adaptative control (Douc et alii [10]).

An important class of AR-MR is the hidden Markov models (HMMs) for which the functions \( \{f_1, \ldots, f_m\} \) are constants \( (\theta_{1,i} = 0, \text{ for all } i \in \{1, \ldots, m\}) \). The HMMs are used in many different areas: basic and applied sciences, industry, economics, finance, images reconstruction, speech recognition, tomography, inverse problem, etc. [3], [22].

The advantage of using the SAEM algorithm is easiness of movement in different modal areas, that reduces the chance of the estimate to avoid a local maximum. The particularities of our problem allows us to do an exact simulation of the distribution of the hidden chain conditional to the observations, using Carter-Kohn algorithm [4].

For the hypothesis test of HMM against Linear AR-RM we follow the ideas of Giudici et al [16] then we obtain the usual asymptotical theory. They used likelihood-ratio test for HMMs, to establish that the standard asymptotic theory rests valid. They work with hidden graphical Gaussian models.

When the number \( m \) is unknown, the hypothesis test with likelihood ratio techniques fails to estimate \( m \) because regularity hypothesis are not satisfied. Particulary, the model is not identifiable, in the sense of Dachuna-Duflo [7] (227), so standard \( \chi^2 \) can not be applied.

In the HMM framework, we distinguish two cases according if the number state of the observed variables is finite or not. In the finite case, Finesso [12] gives \( \hat{m} \) a strong consistence penalized estimator of \( m \), assuming that \( m \) belongs to a bounded subset of the integers numbers. Liu and Narayan [21], also assume this bounded condition and postules a strongly consistent and efficient \( \hat{m} \) with the probability of underestimation decaying exponentially fast w.r.t. \( N \), while the probability of overestimation does not exceed \( cN^3 \). Gassiat and Boucheron [14] prove the strong consistence of a penalized \( \hat{m} \) without assumptions about upper bounds for \( m \), with the probability of underestimation and overestimation decaying exponentially fast. In the non-finite case, the likelihood ratio is not bounded, Gassiat and Keribin studies in [15] show divergence to infinity. As far as we know, the divergence rate rests unknown. In Gassiat [13] results over penalized
likelihood are given in order to obtain weak consistence for the estimator of the number state. We obtain strong consistence for a penalized \( \hat{m} \) in a linear AR-MR, and \( m \) in a bounded set.

The paper is organized as follows. Main assumptions are given in Section 2. In Section 3 for a fixed number state of the hidden Markov chain, an SAEM type algorithm is used to estimate the parameters and is present the method of simulation of the hidden Markov chain and their convergence properties. In the Section 4 we presents our results on the analysis of LR test. In Section 5 we derive the consistency of the penalized likelihood method for a number state problem. For sake the clarity the proof of the Lemma 1 is relegated to Appendix A. Appendix B is devoted to simulations.

2 Notation and assumptions

Let \( Y_0, \{X_n\} \) and \( \varepsilon_1 \) are mutually independent then

\[
p(y_n|x_n, \ldots, x_0, y_n-1, \ldots, y_0) = p(y_n|x_n, y_{n-1}).
\] (2.2)

Using (2.2) and from Markov property of \( \{X_n\} \) we have

\[
l_N(\psi) = \log p(y_1:N|y_0, \psi)
\]

\[
= \log \left( \sum_{x \in \{1, \ldots, m\}^N} p(y_1:N, X_1:N = x_1:N|\psi) \right)
\]

\[
= \log \left( \sum_{x_1:N \in \{1, \ldots, m\}^N} p(y_1:N|X_1:N = x_1:N, y_0, \psi) p(X_1:N = x_1:N|\psi) \right)
\]

\[
= \log \left( \sum_{x_1 = 1}^{m} \ldots \sum_{x_m = 1}^{m} \prod_{n=1}^{N} p(y_n|X_n = x_n, y_{n-1}) \prod_{n=1}^{N-1} a_{x_n, x_{n+1}} p(X_1 = x_1) \right) \] (2.3)

with

\[
p(y_n|y_{n-1}, X_n = i) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{(y_n - f_i(y_{n-1}, \theta_i))^2}{2\sigma^2} \right).
\]

For the consistence the MLE we will assume the followings conditions,
(C1) The transition probability $A$ is positive, this is, $a_{ij} \geq \delta$, for all $i, j \in \{1, \ldots, m\}$ for some $\delta > 0$.

This condition implies that there is an unique invariant distribution $\mu = (\mu_1, \ldots, \mu_m)$.

(C2) Let $\sum_{i=1}^{m} \log |\theta_{1,i}| \mu_i < 0$.

This condition, and the existence of the moments the $\varepsilon_1$, implies that the chain extended \{(Y_n, X_n)\} is a geometrically ergodic Markov chain on the state space $\mathbb{R} \times \{1, \ldots, m\}$ under $\psi_0$ (see Yao and Attali [21]).

(C3) Let $\psi \in \sup_{y_0,y_1,i} |p(y_1|y_0, i) < \infty$ and $\mathbb{E}(| \log b_-(y_1, y_0)|) < \infty$, where $b_-(y_1, y_0) := \sup_{\psi} \sum_{i=1}^{m} p(y_1|y_0, i)$.

(C4) For all $i, j \in \{1, \ldots, m\}$ and all $y, y' \in \mathbb{R}$, the functions $\psi \rightarrow a_{ij}$ and $\psi \rightarrow p(y'|y, i)$ are continuous.

(C5) The model is identifiable in the sense that $p_{\psi} = p_{\psi^*}$ implies that $\psi = \psi^*$. For this is sufficient that $\theta_i \neq \theta_j$ if $i \neq j$, up to an index permutation (Krisnamurthy and Yin [19]).

(C6) For all $i, j \in \{1, \ldots, m\}$ and $y, y' \in \mathbb{R}$, the functions $\psi \rightarrow a_{ij}$ and $\psi \rightarrow p(y'|y, i)$ are twice continuously differentiable over $O = \{\psi \in \Psi : |\psi - \psi_0| < \delta\}$.

(C7) Let us denote $\nabla$ for gradient operator and $\nabla^2$ for Hessian matrix,

(a) $\sup_{\psi \in O} \sup_{i,j} \|\nabla \log a_{ij}\| < \infty$ and $\sup_{\psi} \sup_{i,j} \|\nabla^2 \log a_{ij}\| < \infty$.

(b) $\mathbb{E} \psi_0 (\sup_{\psi \in O} \sup_{i} \|\nabla \log p(Y_1|Y_0, i)\|^2) < \infty$ and $\mathbb{E} \psi_0 (\sup_{\psi \in O} \sup_{i,j} \|\nabla^2 \log p(Y_1|Y_0, i)\|) < \infty$.

(C8) (a) For all $y, y' \in \mathbb{R}$ there exist an integrable function $h_{y,y'} : \{1, \ldots, m\} \rightarrow \mathbb{R}^+$ such that $\sup_{\psi \in O} p(y_1|y_0, i) \leq h_{y,y'}(i)$. 

(b) For all $y, y' \in \mathbb{R}$ there exists integrable functions $h^1_{y,y} : \mathbb{R} \rightarrow \mathbb{R}^+$ and $h^2_{y,y} : \mathbb{R} \rightarrow \mathbb{R}^+$ such that $\|\nabla \log p(y'|y, i)\| \leq h^1_{y,y}(y')$ and $\|\nabla^2 \log p(y'|y, i)\| \leq h^2_{y,y}(y')$ for all $\psi \in O$.

In the next proposition we collect some the results of Douc et alii [10] that attains our work.

**Proposition 1**

i) Assuming (C1)-(C4). Then

$$\lim_{N \rightarrow \infty} \sup_{\psi \in \Psi} \left| N^{-1} l(\psi) - H(\psi) \right|, \mathbb{P}_{\psi_0} - a.s$$

where $H(\psi) = \mathbb{E}_{\psi_0}(\log p(Y_0|Y_{-\infty:-1}, \psi_0))$. 

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ii) Assuming (C1)-C5). Then
\[
\lim_{N \to \infty} \hat{\psi}_N = \psi_0 \mathbb{P}_{\psi_0} - a.s,
\]

iii) Assuming (C1)-(C3) and (C6)-(C8) then,
\[
N^{1/2} \nabla^2 \psi l(\psi) \to I(\psi_0) \mathbb{P}_{\psi_0} - a.s.
\]

iv) Assuming (C1)-(C8) and that the Fisher information matrix for \{Y_n\}, \(I(\psi_0)\) is positive definite. Then
\[
N^{1/2}(\hat{\psi}_N - \psi_0) \to \mathcal{N}(0, I(\psi_0)^{-1}) \mathbb{P}_{\psi_0} - \text{weakly}.
\]

3 The estimation algorithm for fixed \(m\)

Since the likelihood estimator \(\hat{\psi}\) is a solution the equation \(\nabla \psi l(\psi) = 0\), and this equation do not has an analytic solution, then the maximization has to be performed numerically by considering \(m^N\) terms in the equation [2,3]. This restricts the model to observations with limited size and few states. For HMMs models in a finite space state Baum et alii [1] introduced a forward-backward algorithm as an early version of the EM algorithm. The EM algorithm was proposed by Dempster et alii [9] to maximize log-likelihood with missing data. It enables, with a recursive method, to change the problem of maximizing the log-likelihood into the problem of maximizing some functional of the completed the likelihood \(p(y_{1:N}, x_{1:N}|\psi)\) of the model:

\[
\prod_{n=1}^{N} \left[ \prod_{i,j=1}^{m} \mathbb{I}_{i,j}(X_n, X_{n-1}) \prod_{i=1}^{m} p(y_n|y_{n+1}, i) \mathbb{I}_j(X_n) \right],
\]

where \(\mathbb{I}_A(\cdot)\) denotes the function indicator over the set \(A\) and \(\mathbb{I}_{A \times B}(\cdot, \cdot) = \mathbb{I}_A(\cdot) \mathbb{I}_B(\cdot)\).

Let us describe the \(t+1\) step of the algorithm. Set

\[
Q(\psi, \psi^{(t)}) = \mathbb{E}(\log p(Y_{1:N}, X_{1:N}|\psi^{(t)})|Y_{1:N} = y_{1:N}, \psi).
\]

Then \(Q\) is the expectation of the log-likelihood of the complete data conditioned to the observed data and the value of the parameter computed at the step \(t\), \(\psi^{(t)}\). Then we have that \(Q(\psi, \psi^{(t)})\) equals to

\[
\sum_{n=1}^{N-1} \sum_{i,j=1}^{m} \mathbb{E}(\mathbb{I}_{i,j}(X_n, X_{n+1})|Y_{1:N} = y_{1:N}, \psi) \log(a_{ij})
\]

\[+ \sum_{n=1}^{N-1} \sum_{i=1}^{m} \mathbb{E}(\mathbb{I}_i(X_n)|Y_{1:N} = y_{1:N}, \psi) \log p(y_{n+1}|y_n, i). \quad (3.1)
\]
The EM is a two steps algorithm: the E step and the M step. In the E stage compute $Q(\psi, \psi(t))$ the expectation conditioned to the observed data and the current value of the parameter.

In the M step choose,

$$\psi(t+1) = \arg\max_{\psi \in \Psi} Q(\psi, \psi(t)).$$

The EM algorithm converges to a maximum-likelihood estimate for any initial value, when the complete data likelihood function is in the exponential family and a differentiability condition is satisfied.

In order to avoid local minima, we have used an stochastic approximation of the EM algorithm, the SAEM algorithm. Such algorithm has been developed by Celeux et alii in [2], [6] and [5], and its convergence has been proved by Delyon et alii [8].

The EM algorithm is modified in the following way: the (E) step is split into a simulation step (ES) and stochastic approximation step (EA):

**ES** Sample one realization $x_{1:N}^{(t)}$ of the missing data vector under $p(x_{1:N}|y_{1:N}, \psi^t)$.

**EA** Update the current approximation of the EM intermediate quantity according to:

$$Q_{t+1} = Q_t + \gamma_t \left( \log p(y_{1:N}, x_{1:N}^{(t)}|\psi') - Q_t \right)$$

where $(\gamma_t)$ satisfies the condition:

(RM) for all $t \in \mathbb{N}$, $\gamma_t \in [0, 1]$, $\sum_{t=1}^\infty \gamma_t = \infty$ and $\sum_{t=1}^\infty \gamma_t^2 < \infty$.

### 3.1 ES step

In this section we describe the simulating method used in the SAEM algorithm. For sampling under the conditional distribution,

$$p(x_{1:N}|y_{1:N}, \psi) = \mu_{x_1} p(y_1|y_0, x_1) \ldots a_{x_{N-1}x_N} p(y_N|y_{N-1}, x_N)/p(y_{1:N}|\psi),$$

for any $x_{1:N} = (x_1, \ldots, x_N) \in \{1, \ldots, m\}^N$, Carter and Kohn in [4] give a method that constitutes a stochastic version of the forward-backward algorithm proposed by Baum et alii [1]. This follows by observing that $p(x_{1:N}|y_{1:N}, \psi)$ can be decomposed as,

$$p(x_{1:N}|y_{1:N}, \psi) = p(x_N|y_{1:N}, \psi) \prod_{n=1}^{N-1} p(x_n|x_{n+1}, y_{1:N}, \psi).$$

Provided that $X_{n+1}$ is known, $p(X_n|X_{n+1}, y_{1:N}, \psi)$ is a discrete distribution, which suggests the following sampling strategy. For $n = 2, \ldots, N$, $i \in \{1, \ldots, m\}$, compute and store recursively the optimal filter $p(X_n|y_{1:n}, \psi)$ as

$$p(X_n = i|y_{1:n}, \psi) \propto p(y_n|y_{n-1}, X_n = i, \psi) \sum_{i=1}^m a_{ij} p(X_{n-1} = j|y_{1:n-1}).$$
Then, sample $X_N$ from $p(X_N | y_{1:N}, \psi)$ and for $n = N - 1, \ldots, n$, $X_n$ is sample from

$$p(X_n = i | X_{n+1} = x_{n+1}, y_{1:n}, \psi) = \frac{a_{ij_{n+1}} p(X_n = i | y_{1:n}, \psi)}{\sum_{l=1}^{m} a_{il} p(X_n = l | y_{1:n}, \psi)}.$$ 

As a consequence, the estimation procedure generate an ergodic Markov chain $\{x_{1:N}^{(t)}\}$ on the finite state space $\{1, \ldots, m\}^N$, so that $p(x_{1:N}^{(t)} | y_{1:N}, \psi)$ is its stationary distribution. Ergodicity follow from irreducibility and aperiodicity, by observing the positivity of the kernel, this is,

$$K(x_{1:N}^{(t)} | x_{1:N}^{(t-1)}, \psi) \propto p(x_{1:N}^{(t)} | \psi, y_{1:N}) \prod_{n=1}^{N-1} p(x_n^{(t)} | x_{n+1}^{(t)}, y_{1:N}) > 0.$$ 

In this case the standard ergodic result for finite Markov chains applies (for instance, Kemeny and Snell [18]),

$$\|K(x_{1:N}^{(t+1)} | x_{1:N}^{(t)}, \psi) - p(X_{1:N} | y_{1:N}, \psi)\| \leq C \rho^{t-1},$$

with $C = \text{card}(\{1, \ldots, m\}^N)$, $\rho = (1-2K^*_x) y K^* = \inf K(x' | x, \psi)$, for $x, x' \in \{1, \ldots, m\}^N$.

### 3.2 EA step

The (3.1) equation suggests us to substitute the step EA for approximations of Robins Monro (ver Duflo [11]), $s = (s_1^{(t+1)}, s_2^{(t+1)}, s_3^{(t+1)})$, defined by:

\begin{align*}
    s_1^{(t+1)}(i, n) &= s_1^{(t)}(i, n) + \gamma_t \left( \mathbb{I}_i(x_n) - s_1^{(t)}(i, n) \right) \quad (3.2) \\
    s_2^{(t+1)}(i) &= s_2^{(t)}(i) + \gamma_t \left( N_i(x_{1:N}) - s_2^{(t)}(i) \right) \quad (3.3) \\
    s_3^{(t+1)}(i, j) &= s_3^{(t)}(i, j) + \gamma_t \left( N_{ij}(x_{1:N}) - s_3^{(t)}(i, j) \right). \quad (3.4)
\end{align*}

where $N_i(x_{1:N}) = \sum_{n=1}^{N-1} \mathbb{I}_i(x_n)$ and $N_{ij}(x_{1:N}) = \sum_{n=1}^{N-1} \mathbb{I}_{i,j}(x_n, x_{n+1})$, are sufficient statistics for the chain of hidden Markov.

When $f_j(y, \theta_j) = \theta_j$, the maximization step is given by,

\begin{align*}
    \hat{\alpha}_{ij}^{(t+1)} &= \frac{s_3^{(t+1)}(i, j)}{s_2^{(t+1)}(i)} \\
    \hat{\theta}_i^{(t+1)} &= \frac{\sum_{n=1}^{N} s_1^{(t+1)}(i, n) y_n}{s_2^{(t+1)}(i)} \\
    \hat{\sigma}^2(t+1) &= \frac{1}{N-1} \sum_{n=1}^{N-1} s_1^{(t+1)}(i, n) (y_n - \hat{\theta}_i^{(t+1)})^2,
\end{align*}
and for \( f_j(y, \theta_j) = \theta_{1,j}y + \theta_{0,j} \) by,

\[
\hat{a}_{ij}^{(t+1)} = \frac{s_3^{(t+1)}(i, j)}{s_2^{(t+1)}(i)}
\]

\[
\hat{\theta}_{1,i}^{(t+1)} = \frac{\sum_{n=1}^{N-1} s_1^{(t+1)}(i, n)y_n y_{n-1} - \sum_{n=1}^{N-1} s_1^{(t+1)}(i, n)y_n \sum_{n=1}^{N} s_1^{(t+1)}(i, n)y_{n-1}}{\sum_{n=1}^{N-1} s_1^{(t+1)}(i, n)y_n^2 - \left( \sum_{n=1}^{N-1} s_1^{(t+1)}(i, n)y_n \right)^2}
\]

\[
\hat{\theta}_{0,i}^{(t+1)} = \sum_{n=1}^{N-1} s_1^{(t+1)}(i, n)y_n - \hat{\theta}_{1,i} \sum_{n=1}^{N} s_1^{(t+1)}(i, n)y_{n-1}
\]

\[
\hat{\sigma}^2_{(t+1)} = \frac{1}{N-1} \sum_{n=1}^{N-1} \sum_{i=1}^{m} s_1^{(t+1)}(i, n)(y_n - f_i(y_{n-1}, \hat{\theta}_i))^2
\]

We consider the observations \( y_{1:N} \) fixed, the previous expressions define, in an explicit way, in each one of the two cases of study, the application \( \hat{\psi} = \psi(s) \) between the sufficient statistics and the parameters space necessary to SAEM.

### 3.3 Convergence

The simulation procedure generates \( \{x^{(t)}_{1:N}\} \), a finite Markov chain. The hypotheses of Delyon et alii [8] that ensures the convergence of the SAEM algorithm are no more satisfied but in this case, we can be use the Theorem 1 of Kuhn and Lavielle in [20]:

**Theorem 1** If we suppose the conditions that guarantee the convergence of the EM algorithm, the condition (RM) and the following hypothesis,

**SAEM1** The function \( p(y_{1:N}|\psi) \) and the function \( \hat{\psi} = \psi(s) \) are \( l \) time differentiable.

**SAEM2** The function \( \psi \to K_\psi = K(\cdot|\cdot, \psi) \) is continuously differentiable on \( \Psi \). The transition probability \( K_\psi \) generates a geometrically ergodic chain with invariant probability \( p(x_{1:N}|y_{1:N}, \psi) \). The chain \( \{x^{(t)}_{1:N}\} \) takes values a compact subset. The function \( s \) is bounded.

Then, w.p 1, \( \lim_{t \to \infty} d(\psi^{(t)}, \mathcal{L}) = 0 \) where \( \mathcal{L} = \{\psi \in \Psi : \nabla_\psi l(\psi) = 0\} \) is the set of stationary points.

In our case the hypotheses of the theorem are verified, in fact, the hypothesis RM is satisfied choosing the sequence \( \gamma_t = 1/t \), SAEM1 is obtained because \( \varepsilon_1 \) is distributed normal and SAEM2 is consequence of the discussion made in §3.1. This guarantees the previous theorem and this give us the convergency.

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4 Hypothesis test

In this section we study the likelihood ratio test (LRT) for testing a model HMMs against a process AR-RM. We prove that the standard theory for LRT of a point null hypothesis is valid. Let \( \psi = (A, \theta_1, \theta_0, \sigma^2) \) and \( \psi_0 = (A, 0, \theta_0, \sigma^2) \), then the test we consider is that

\( H_0 : \theta_1 = 0 \)

against

\( H_1 : \theta_1 \neq 0. \)

Theorem 2 Assume that (C1)-(C8) hold. Then,

\[ 2(l(\hat{\psi}) - l(\psi_0)) \to \chi^2_m, \]

under \( P_{\psi_0}. \)

Proof: Using the Taylor expansion of \( l(\psi) \) around \( \hat{\psi} \),

\[ l(\psi_0) - l(\hat{\psi}) = (\psi_0 - \hat{\psi}) \nabla_{\psi} l(\hat{\psi}) + \frac{1}{2} (\psi_0 - \hat{\psi})^T \nabla^2_{\psi} l(\hat{\psi})(\psi_0 - \hat{\psi}) \]

where \( \tilde{\psi} = \lambda \psi_0 + (1 - \lambda) \hat{\psi}, \lambda \in (0, 1) \). Also \( \nabla_{\psi} l(\hat{\psi}) = 0. \) So

\[ -2(l(\psi_0) - l(\hat{\psi})) = -[N^{1/2}(\psi_0 - \hat{\psi})^T N^{-1} \nabla^2_{\psi} l(\hat{\psi})] N^{1/2}(\psi_0 - \hat{\psi}) \]

Now, since \( \tilde{\psi}_N \to \psi_0 \) \( P_0 - a.s. \) does \( \hat{\psi}_N \), and using Proposition 1-(iii-iv),

\[ N^{1/2}(\tilde{\psi}_N - \psi_0) \to \mathcal{N}(0, I(\psi_0)^{-1}) \quad P_{\psi_0} - weakly \]

and

\[ -N^{-1/2} \nabla^2_{\psi} l(\hat{\psi}) \to I(\psi_0) \quad P_{\psi_0} - a.s. \]

So the proof is complete.

The theorem says that we can employ the LRT test rejects \( H_0 \) if:

\[ -2(l(\psi_0) - l(\hat{\psi})) \geq \chi^2_{m, \alpha} \]

where \( \chi^2_{m, \alpha} \) is the \( \alpha \)-quantile of the \( \chi^2_m \) distribution.

5 Penalized estimation of the number state

In this section we presents a penalized likelihood method for selecting the number state \( m \) of the hidden Markov chain \( \{X_n\} \). For each value of \( m \geq 1 \), we consider the sets \( \Psi_m \) and \( M = \bigcup_{m \geq 1} \Psi_m \), the collection of all the different models. For a fixed \( m \), we have
seen in Section 3 that it is possible to estimate the unknown parameters for the model. Hence, it is now possible evaluate the log-likelihood chosen model \( l(\hat{\psi}_m) \).

As we assumed identifiability \((C5)\), we have that true number state, \( m_0 \) is minimal, that is, there does not exist a parameter \( \psi_m \in \Psi_m \) with \( m < m_0 \) such that \( \psi_m \) and \( \psi_{m_0} \) induce an identical law for \( \{Y_n\}_{n \geq 0} \). We said that \( \hat{m}_N \) over-estimate the number state \( m_0 \) if \( \hat{m}_N > m_0 \) and under-estimate the number state if \( \hat{m}_N < m_0 \).

The penalized maximum likelihood (PML) is defined as:

\[
C(N, m) = -\log p(y_{1:N}|y_0, x_1\hat{\psi}(N)) + \text{pen}(N, m),
\]

where \( \hat{\psi}(N) \) is the maximum likelihood of \( \psi \in \Psi_m \) based on \( N \) observations and \( \text{pen}(N, m) \) is a positive and increasing function of \( m \). A number state estimation procedure is defined as follows:

\[
\hat{m}(N) = \min \{ \arg\min_{m \geq 1} C(N, m) \}.
\]

In the following theorem we prove that the estimator PML over-estimate the number state \( m_0 \).

**Theorem 3** Assume \((C1)-(C5)\) and that \( \lim_{N \to \infty} \text{pen}(N, m) = 0 \) for all \( m \) then

\[
\liminf_{N \to \infty} \hat{m}(N) \geq m_0, \ \mathbb{P}_{\psi_0} - a.s.
\]

**Proof:** From Proposition II(i) we have:

\[
l(\psi_0) - l(\psi) \to H(\psi_0) - H(\psi),
\]

\( \psi \in \Psi_m \), and \( H(\psi) - H(\psi_0) = \mathbb{E}_{\psi_0} \left( \log \frac{p_{\psi_0}(y_0|y_{-\infty}^{-1})}{p_{\psi}(y_0|y_{-\infty}^{-1})} \right) = D(\psi_0, \psi). \)

Therefore for \( m < m_0 \):

\[
\inf_{\psi \in \Psi_m} [l(\psi_0) - l(\psi)] \to l(\psi_0) - l(\hat{\psi}) \to \inf_{\psi \in \Psi_m} D(\psi_0, \psi) > 0,
\]

\( D(\psi_0, \psi) > 0 \) since \( m_0 \) in minimal. We have:

\[
\lim_{N \to \infty} l(\hat{\psi}_{m_0}) - l(\hat{\psi}_m) = D(\psi_0, \psi) > 0.
\]

By the definition of \( C(N, m) \) and by assumption \( \lim_N \text{pen}(N, m) = 0 \),

\[
\lim_{N \to \infty} C(N, m) - C(N, m_0) = D(\psi_0, \psi) > 0,
\]

for any \( m < m_0 \). On the other hand \( C(N, \hat{m}(N)) - C(N, m_0) \leq 0 \), by the definition of \( \hat{m}(N) \) and we conclude that

\[
\liminf_{N \to \infty} \hat{m}(N) \geq m_0, \ \mathbb{P}_{\psi_0} - a.s.
\]
In the following we prove that the estimator PML under-estimate the number state.

Let us define the distribution,

\[ Q_m(y_{1:N}|x_1) = \mathbb{E}_{p(\psi)}(p(y_{1:N}|y_0, x_1, \psi)), \]

where \( p(\psi) \) is a priori distribution on \( \Psi_m \). In the following we will write the model in its vectorial form,

\[ y = Z \theta + \varepsilon, \tag{5.1} \]

where \( \varepsilon = (\sigma \varepsilon_1, \ldots, \sigma \varepsilon_N) \), \( y = y_{1:N}^t \), in the case AR-MR \( \theta = ((\theta_{0,1}, \theta_{1,1}), \ldots, (\theta_{0,m}, \theta_{1,m}))^t \),

\[ Z = \begin{pmatrix}
(1, y_0) \mathbb{I}_1(x_1) & \cdots & (1, y_0) \mathbb{I}_m(x_1) \\
\vdots & \ddots & \vdots \\
(1, y_{N-1}) \mathbb{I}_1(x_N) & \cdots & (1, y_{N-1}) \mathbb{I}_m(x_N)
\end{pmatrix}, \]

while in the case HMMs \( \theta = (\theta_1, \ldots, \theta_m)^t \)

\[ Z = \begin{pmatrix}
\mathbb{I}_1(x_1) & \cdots & \mathbb{I}_m(x_1) \\
\vdots & \ddots & \vdots \\
\mathbb{I}_1(x_N) & \cdots & \mathbb{I}_m(x_N)
\end{pmatrix}. \]

Given \( x_1, y_0 \), the likelihood function for the model (5.1) is,

\[ p(y|x_1, y_0, \psi) = \sum_{x_{2:N} \in \{1, \ldots, m\}^{N-1}} p(y, X_{2:N} = x_{2:N}|x_1, \psi) \]
\[ = \sum_{x_{2:N} \in \{1, \ldots, m\}^{N-1}} p(y|X_{2:N} = x_{2:N}, x_1, \psi)p(X_{2:N} = x_{2:N}|x_1, \psi) \tag{5.2} \]

with,

\[ p(y|X_{2:N} = x_{2:n}, x_1, \psi) = \mathcal{N}(y - Z \theta|0, \sigma^2 I_N) \]
\[ p(X_{2:N} = x_{2:N}|x_1, \psi) = a_{x_1x_2} \cdots a_{x_{N-1}x_N}. \]

Suppose the following structure of dependence for the components \( \psi \),

\[ p(\psi) = \prod_{i \in E} p(A_i) \frac{p(\theta|\sigma^2)p(\sigma^2)}{\text{det}(\Sigma)^{-1/2}}. \]

and suppose the following densities that are priors conjugated for likelihood function (5.2):

1. \( \theta \sim \mathcal{N}(\theta|0, \sigma^2 \Sigma) = (2\pi \sigma^2)^{-m/2} \text{det}(\Sigma)^{-1/2} \exp \left( -\frac{1}{2\sigma^2} \theta^\top \Sigma^{-1} \theta \right) \)
2. For $\sigma^2$ is proposed an inverted gamma $\mathcal{IG}$,

$$\sigma^2 \sim \mathcal{IG}(v_0/2, u_0/2) = \frac{u_0^{v_0/2}}{2^{v_0/2} \Gamma(v_0/2)} \left(\sigma^2\right)^{-(v_0/2+1)} \exp \left(-\frac{u_0}{2\sigma^2}\right),$$

$$\Gamma(u) = \int_0^\infty s^u e^{-s} ds.$$

3. $A_i \sim \mathcal{D}(e)$. $\mathcal{D}$ denotes a Dirichlet density with parameter vector $e = (1/2, \ldots, 1/2)$,

$$\mathcal{D}(e) = \frac{\Gamma(m/2)}{\Gamma(1/2)^m} \prod_{j=1}^m a_{ij}^{-1/2}.$$

The following Lemma gives a bound of the likelihood function normalized by $Q_m$.

**Lemma 1** The prior distribution $p(\psi)$ satisfies for all $m$ and all $y \in \mathbb{R}^N$ the following inequalities,

$$\log \frac{p(y|y_0, x_1, \psi)}{Q_m(y|x_1)} \leq \frac{m(m-1)}{2} \log(N) + c_m(N) + \log \Gamma\left(\frac{u_0}{2}\right) + \frac{\log \det(\Sigma)}{2} + \frac{(1 + v_0)}{2} \log(u_0 + y^t P y) - \frac{N}{2} - \frac{\log \det(M)}{2} - \log \Gamma\left(\frac{N}{} + \frac{v_0}{2}\right)$$

where $M^{-1} = Z^t Z + \Sigma^{-1}$, $P = I - Z M Z^t$ and for $N \geq 4$,

$$c_m(N) = -m \left(\log \frac{\Gamma(m/2)}{\Gamma(1/2)} - \frac{m(m-1)}{4N} + \frac{1}{12N}\right).$$

Lemma 1 constitutes a basic step in the proof of the following proposition,

**Proposition 2** Let $\hat{m}$ the PML number state. Then for all $m_0$, all $\psi \in \Psi_{m_0}$ and all $m > m_0$:

$$\mathbb{P}(\hat{m} > m_0) \leq \sum_{m=m_0+1}^{m_{\text{max}}} \exp(I' + \Delta \text{pen}(m_0, m)) \int_{\{y\}} (u_0 + y^t P y)^{\frac{N + v_0}{2}} Q_m(y|x_1) dy$$

where $\Delta N \text{pen}(m_1, m_2) := \text{pen}(N, m_1) - \text{pen}(N, m_2),

$$I' = \frac{m(m-1)}{2} \log(N) + c_m(N) + \log \Gamma\left(\frac{u_0}{2}\right) + \frac{\log \det(\Sigma)}{2} - \frac{N}{2} - \frac{\log \det(M)}{2} - \log \Gamma\left(\frac{N + v_0}{2}\right)$$

**Proof:** by Lemma 1,

$$\log \frac{p(y|y_0, x_1, \psi)}{Q_m(y|x_1)} \leq I.$$
with
\[ I = \frac{m(m-1)}{2} \log(N) + c_m(N) + \log \Gamma \left( \frac{u_0}{2} \right) \]
\[ + \frac{\log \det(\Sigma)}{2} + \frac{(1+u_0)}{2} \log(u_0 + y' P y) - \frac{N}{2} \frac{\log \det(M)}{2} - \log \Gamma \left( \frac{N+u_0}{2} \right), \]
also, \[ \mathbb{P}(\hat{m}(N) > m_0) = \sum_{m=m_0+1}^{m_{\max}} \mathbb{P}(\hat{m}(N) = m), \]
and therefore, \[ \mathbb{P}(\hat{m}(N) = m) \leq \mathbb{P} \left( \log p(y|y_0, x_1, \psi_0) \leq \sup_{\psi \in M} \log p(y|y_0, x_1, \psi) + \Delta \text{pen}(m_0, m) \right) \]
\[ \leq \mathbb{P} \left( \log p(y|y_0, x_1, \psi_0) \leq \log Q_m(y|x_1) + I + \Delta \text{pen}(m_0, m) \right) \]
\[ = \mathbb{E} \left( \mathbb{I} \left( \log p(y|y_0, x_1, \psi_0) \leq \log Q_m(y|x_1) + I + \Delta \text{pen}(m_0, m) \right) \right) \]
\[ = \int \mathbb{I} \left( \log p(y|y_0, x_1, \psi_0) \leq \log Q_m(y|x_1) + I + \Delta \text{pen}(m_0, m) \right) \exp \log p(y|y_0, x_1, \psi) \, dy \]
\[ \leq \int \exp(\log Q_m(y|x_1) + I + \Delta \text{pen}(m_0, m)) \, dy. \]
get:
\[ \mathbb{P}(\hat{m} > m_0) \leq \sum_{m=m_0+1}^{m_{\max}} \exp(I' + \Delta \text{pen}(m_0, m)) \int_{\{y\}} (u_0 + y' P y)^{\frac{1+u_0}{2}} Q_m(y|x_1) \, dy. \]

As a consequence of this result and the first Borel-Cantelli Lemma, the convergence of \( \hat{m} \) depends on the study of the series \( \sum_N \sum_{m=m_0+1}^{m_{\max}} \exp(I'(N,m) + \Delta \text{pen}(m_0, m)). \)

In the following theorem we find under-estimate estimator of number state \( m_0. \)

**Theorem 4** If \( \int_{\{y\}} (u_0 + y' P y)^{\frac{N+u_0}{2}} Q_m(y|x_1) \, dy < \infty \) and \( \lim_{N \to \infty} \text{pen}(N, m) - \text{pen}(N + 1, m) = 0, \) then \[ \hat{m}(N) \leq m_0 \text{ c.s } \mathbb{P}_{\psi_0}. \]

**Proof:** Let us defined \( a_N = I'(N,m) + \Delta \text{pen}(m_0, m). \) Observe that the serie \[ \sum_{m=m_0+1}^{M} \sum_{N} \exp a_N < \infty, \]
converges as consequence of the ratio criterio and this shows that \( \lim_{N \to \infty} a_{N+1} - a_N < 0. \)
In fact, \[ \lim_{N \to \infty} \frac{m(m-1)}{2} \log \left( \frac{N+1}{N} \right) + c_m(N+1) - c_m(N) = 0, \]
\[-\frac{1+\log 2}{2} < 0, \quad -\log \left( \frac{\Gamma((N+1+v_0)/2)}{\Gamma((N+v_0)/2)} \right) < 0\]

\[
\lim_{N \to \infty} \Delta_{N+1} \text{pen}(m_0, m) + \Delta_N \text{pen}(m_0, m) \\
= \lim_{N \to \infty} \text{pen}(N+1, m_0) - \text{pen}(N+1, m) - \text{pen}(N, m_0) + \text{pen}(N, m) \\
\leq \lim_{N \to \infty} \text{pen}(N, m) - \text{pen}(N + 1, m) = 0.
\]

Then we have
\[
\lim_{N \to \infty} a_{N+1} - a_N \\
= \lim_{N \to \infty} \frac{m(m-1)}{2} \log \left( \frac{N+1}{N} \right) + c_m(N+1) - c_m(N) - \frac{1+\log 2}{2} \\
- \log \left( \frac{\Gamma((N+1+v_0)/2)}{\Gamma((N+v_0)/2)} \right) + \Delta_{N+1} \text{pen}(m_0, m) - \Delta_N \text{pen}(m_0, m) < 0
\]

Thus \( \sum_N \mathbb{P}_{\psi_0}(\hat{m}(N) > m_0) < \infty \) and from the Borel-Cantelli lemma we conclude that \( \mathbb{P}_{\psi_0}(\hat{m}(N) > m_0 \ i.o) = 0 \). This is equivalent to say that \( \hat{m}(N) \leq m_0 \ c.s.-\mathbb{P}_{\psi_0} \).

One of the most common choices is \( \text{pen}(N, m) = \frac{\log(N)}{2} \dim(\Psi_m) \) (Bayesian information criteria, BIC). It is natural to use \( \dim(\Psi_m) = m(m-1) + m \dim(\Theta) + 1 \).

### A Proof of Lemma 1

The proof of this Lemma is obtained by showing the existence of constants \( C_1, C_2 \) such that:

\[
p(y|x_1, \psi) \leq C_1 Q_m(y|x_{1:N}) \tag{A.1}
\]
\[
p(x_{2:N}|x_1, \psi) \leq C_2 Q_m(x_{2:N}|x_1). \tag{A.2}
\]

This would implies that,
\[
p(y|y_0, x_1, \psi) = \sum_{x \in \{1, \ldots, m\}^N} p(y|x_{1:N}, \psi) p(x_{2:N}|x_1, \psi) \\
\leq C_1 C_2 \sum_{x \in \{1, \ldots, m\}^N} Q_m(y|x_{2:N}) Q_m(x_{2:N}|x_1) \\
= C_1 C_2 Q_m(y|x_1).
\]

and hence \( p(y|y_0, x_1, \psi) \leq C_1 C_2 Q_m(y|x_1) \).

We proceed with the evaluation of \( Q_m(x_{2:N}|x_1) \) following the proof given in the appendix of [21]. Let
\[
Q_m(x_{2:N}|x_1) = \prod_{i=1}^m \left[ \frac{\Gamma(m/2)}{\Gamma(N_i + 1/2)} \left( \prod_{i=1}^m \frac{\Gamma(N_i + 1/2)}{\Gamma(1/2)} \right) \right]
\]
and
\[
p(x_{2:N}|x_1, \psi) \leq \frac{p(x_{2:N}|x_1)}{Q_m(x_{2:N}|x_1)} \leq \prod_{i=1}^{m} \left[ \frac{\Gamma(m/2)}{\Gamma(N/2)} \left( \frac{\Gamma(N+1/2)}{\Gamma(1/2)} \right)^{N_{ij}} \right]. \tag{A.3}
\]

We have that and the right side the equation (A.3) does not exceed,
\[
\left[ \frac{\Gamma(N+m/2)\Gamma(1/2)}{\Gamma(m/2)\Gamma(N+1/2)} \right]^{m}.
\]

In Gassiat and Boucheron [14], is noted that,
\[
m \log \left[ \frac{\Gamma(N+m/2)\Gamma(1/2)}{\Gamma(m/2)\Gamma(N+1/2)} \right] \leq \frac{m(m-1)}{2} \log N + c_m(N),
\]
for \( N \geq 4 \), \( c_m(N) \) is chosen as:
\[
-m \left( \log \frac{\Gamma(m/2)}{\Gamma(1/2)} - \frac{m(m-1)}{4N} + \frac{1}{12N} \right).
\]

Then:
\[
p(x|y_0, x_{1:N}, \theta, \sigma^2) p(\theta|\sigma^2) p(\sigma^2) \leq N^{m(m-1)/2} \exp c_m(N). \tag{A.4}
\]

To evaluate \( Q(y|x_{1:N}) \) let us develop the expression,
\[
p(y|y_0, x_{1:N}, \theta, \sigma^2) p(\theta|\sigma^2) p(\sigma^2) = \mathcal{N}(y - Z\theta|0, \sigma^2 I_N) \mathcal{N}(\theta|0, \sigma^2 \Sigma) I_G(\sigma^2 v_0/2, u_0/2)
\]
\[
= (2\pi\sigma^2)^{-N/2} \exp \left( -\frac{1}{2\sigma^2} (y - Z\theta)^t (y - Z\theta) \right)
\]
\[
(2\pi\sigma^2)^{-m/2} \det(\Sigma)^{-1/2} \exp \left( -\frac{1}{2\sigma^2} \theta^t \Sigma^{-1} \theta \right)
\]
\[
\frac{u_0^{v_0/2}}{2^{v_0/2} \Gamma(v_0/2)} \left( \sigma^2 \right)^{-(v_0/2+1)} \exp \left( -\frac{u_0}{2\sigma^2} \right)
\]

The above-mentioned is equivalent to
\[
\frac{u_0^{v_0/2}(2\pi\sigma^2)^{-N/2}(2\pi\sigma^2)^{-m/2}}{2^{v_0/2} \Gamma(v_0/2)} \exp \left( \frac{(\theta - \mathbf{m})^t M^{-1}(\theta - \mathbf{m})}{2\sigma^2} \right) \left( \sigma^2 \right)^{-(v_0/2+1)} \exp \left( -\frac{u_0 + y^t P y}{2\sigma^2} \right)
\]

with \( M^{-1} = Z^t Z + \Sigma^{-1}, \mathbf{m} = MZ^t y \) and \( P = I - Z M^t Z^t \). Integrating the last expression respect to \( \theta \) and then to \( \sigma^2 \) we obtain
\[
Q(y|x_{1:N}) = \frac{u_0^{v_0/2} \det(M)^{1/2} \Gamma((N + v_0)/2)}{(\pi\sigma^2)^{N/2} \Gamma(v_0/2) \det(\Sigma)^{1/2} (u_0 + y^t P y)^{(N+v_0)/2}}, \tag{A.5}
\]

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this given,

\[
P(y_0, x_1, \psi) \leq P(y_1: N | y_0, \psi) \frac{(\pi \sigma^2)^{N/2} \Gamma(v_0/2) \det(\Sigma)^{1/2}(u_0 + yy')^{(N+v_0)/2}}{u_0^{v_0/2} \det(M)^{1/2}((N + v_0)/2)} \\
= \exp \left(-\frac{1}{2\sigma^2}(y - Z\hat{\theta})^t(y - Z\hat{\theta})\right) \frac{(\pi \hat{\sigma}^2)^{N/2} \Gamma(v_0/2) \det(\Sigma)^{1/2}(u_0 + yy')^{(N+v_0)/2}}{2^{N/2}u_0^{v_0/2} \det(M)^{1/2}((N + v_0)/2)}.
\]

with this expression and the equation (A.4) we obtain lemma 1.

\[\blacksquare\]

B Simulations

In this section we apply our results to some simulated data. We work with an HMMs and two AR-RM. We use \[pen = \frac{\log(N)}{2} \text{dim}(\Psi_m)\] (BIC). We value the likelihood function for any set of parameters \(\psi\) by computing

\[p(y_{1:N} | y_0 \psi) = \sum_{i=1}^{m} \alpha_N(i),\]

where \(\alpha_n(i) = p(y_{1:n}, X_n = i)\) can be evaluated recursively with the following formulae forward of Baum,

\[\alpha_n(j) = \sum_{i=1}^{m} \alpha_{n-1}(i)a_{ij}p(y_n | y_{n-1}, X_n = i)\]

see D. Le Nhu et alii \[23\].

B.1 HMMs

In the simulation of the HMMs we set the following parameters: \(\text{dim}(\Psi_m) = m^2 + 1\)

\(N = 500, m = 3, \sigma^2 = 1.5, \theta = (-2, 1, 4),\)

\[A = \begin{pmatrix} 0.9 & 0.05 & 0.05 \\ 0.05 & 0.9 & 0.05 \\ 0.05 & 0.05 & 0.9 \end{pmatrix},\]

the observed serie is plotted in figure 1.

The table 1 contains the values for the penalized maximum likelihood for \(m = 2, \ldots, 7\), we observe that \(\hat{m} = 3\). In this case \(\hat{\psi}\) is estimated by using the SAEM, the values are, \(\hat{\sigma}^2 = 1.49, \hat{\theta} = (-1.98, 4.09, 0.91),\)

\[\hat{A} = \begin{pmatrix} 0.8650 & 0.0274 & 0.1076 \\ 0.0404 & 0.8943 & 0.0653 \\ 0.0658 & 0.0648 & 0.8694 \end{pmatrix},\]
| \( m \) | \(-l(\psi)\) | \(pen\) | \(-l(\psi) + pen\) |
|---|---|---|---|
| 2 | 802.32 | 15.53 | 817.85 |
| 3 | 419.09 | 31.07 | 450.16 |
| 4 | 417.70 | 52.82 | 470.52 |
| 5 | 464.70 | 80.78 | 545.48 |
| 6 | 445.89 | 114.97 | 560.86 |
| 7 | 436.26 | 155.36 | 591.62 |

Table 1: The values for the PML

![Figure 1: The observed serie \( y_1, \ldots, y_{500} \) for the HMMs.](image)

In the figure 2 displayed the sequence \( \{\psi(t)\}, t = 1, \ldots, 4000 \) and we observe the convergence of the estimate.

**B.2 AR-RM**

In the first simulation of the AR-RM we set the following parameters: \( \text{dim}(\Psi_m) = m(m+1) + 1 \), \( N = 500 \), \( m = 2 \), \( \sigma^2 = 1.5 \),

\[
\theta = \begin{pmatrix} 1 \\ 0.5 \\ -0.5 \\ -1 \end{pmatrix}, \quad A = \begin{pmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{pmatrix},
\]

the observed series is plotted in figure 3.
Figure 2: Convergence of the estimate of, $\sigma^2$, $\theta$ and $A$.

| $m$ | $-l(\psi)$ | $pen$ | $-l(\psi) + pen$ |
|-----|------------|-------|------------------|
| 2   | 351.14     | 18.64 | 369.78           |
| 3   | 346.64     | 37.28 | 383.92           |
| 4   | 355.10     | 64.14 | 417.24           |
| 5   | 354.52     | 93.21 | 447.73           |
| 6   | 384.50     | 130.50| 515.00           |

Table 2: The values for the PML

The table 2 contains the values for the penalized maximum likelihood for $m = 2, \ldots, 6$, we observe that $\hat{m} = 2$. In this case $\hat{\psi}$ is estimated by using the SAEM, the values are, $\hat{\sigma}^2 = 1.42$,

$\hat{\theta} = \begin{pmatrix} 1.07 & -0.96 \\ -0.5 & 0.5 \end{pmatrix}$, $\hat{A} = \begin{pmatrix} 0.8650 & 0.1350 \\ 0.1130 & 0.8870 \end{pmatrix}$.

in the figure 4 displayed the sequence $\{\psi^{(t)}\}, t = 1, \ldots, 1000$ and we observe the convergence of the estimate.

In the second simulation of the AR-RM we set the following parameters: $N = 500$, $m = 2$, $\sigma^2 = 1.5$,

$\theta = \begin{pmatrix} 1 & -2 \\ -0.7 & 1.08 \end{pmatrix}$, $A = \begin{pmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{pmatrix}$,

the observed serie is plotted in figure 3.

In this case $m = 2$ is fixed and $\psi$ is estimated by using the SAEM, the values are, $\hat{\sigma}^2 = 1.42$,

$\hat{\theta} = \begin{pmatrix} 0.85 & -2.01 \\ -0.69 & 1.08 \end{pmatrix}$, $\hat{A} = \begin{pmatrix} 0.9093 & 0.0007 \\ 0.019 & 0.9181 \end{pmatrix}$,

in the figure 6 displayed the sequence $\{\psi^{(t)}\}, t = 1, \ldots, 1000$ and we observe the convergence of the estimate.
Figure 3: The observed serie $y_1, \ldots, y_{500}$ for the AR-MR

Figure 4: Convergence of the estimate of, $\theta_1$, $\theta_2$, $\sigma^2$, and $A$. 
Figure 5: The observed serie $y_1, \ldots, y_{500}$ for the AR-MR.

Figure 6: Convergence of the estimate of, $\theta_1$, $\theta_2$, $\sigma^2$, and $A$. 
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