Quantized Skyrmion Fields
in 2+1 Dimensions

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Abstract

A fully quantized field theory is developed for the skyrmion topological excitations of the O(3) symmetric CP$^1$-Nonlinear Sigma Model in 2+1D. The method allows for the obtainment of arbitrary correlation functions of quantum skyrmion fields. The two-point function is evaluated in three different situations: a) the pure theory; b) the case when it is coupled to fermions which are otherwise non-interacting and c) the case when an electromagnetic interaction among the fermions is introduced. The quantum skyrmion mass is explicitly obtained in each case from the large distance behavior of the two-point function and the skyrmion statistics is inferred from an analysis of the phase of this function. The ratio between the quantum and classical skyrmion masses is obtained, confirming the tendency, observed in semiclassical calculations, that quantum effects will decrease the skyrmion mass. A brief discussion of asymptotic skyrmion states, based on the short distance behavior of the two-point function, is also presented.

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1) Introduction

Skyrmions are topologically nontrivial solutions which appear in the O(3) symmetric Nonlinear Sigma Model (NSM) in 2+1D [4]. Sometimes they are called “baby skyrmions”, in order to distinguish them from their 3+1-dimensional counterparts. In the CP$^1$ version they become vortices carrying a “magnetic flux” along the spatial plane. There is a great interest in the study of the quantum properties of skyrmions because of the central role played by the NSM in some important condensed matter systems. A first example comprise the systems presenting the quantum Hall effect. A quantum field theory which neglects the spin of the electrons has been proposed for the description of this effect [2]. Later on the spin was introduced in the theory and this has been shown to be equivalent to a NSM containing, as a consequence, skyrmion configurations [3]. These skyrmions have been recently observed directly in neutron scattering experiments performed in quantum Hall systems in the $\nu = 1$ level [4].

Another important class of planar condensed matter systems in which skyrmions do play an important role are high-$T_c$ superconductors. In the absence of doping, these materials are quite well described, in the strong coupling limit, by a NSM where the nonlinear sigma field is the continuous version of the electron spin. The connection arises through the equivalence, on the continuum limit, between the 2+1-dimensional O(3) NSM and the antiferromagnetic Heisenberg model in 2D [5]. This, by its turn, provides a good description of the physics in the $CuO_2$ planes for the pure system in the strong coupling limit [6, 7]. It has been shown that in the process of doping, formation of spin textures or skyrmions accompany the introduction of holes in the $CuO_2$-planes [6, 7, 8, 9, 10, 11, 12]. These eventually lead to the destruction of the Néel ordering which is known to exist at low doping. Other very interesting effects due to skyrmions, like the modification of both the the NMR line shapes and the magnetic structure factor, the latter affecting neutron scattering cross sections, have been predicted for skyrmions introduced by doping antiferromagnetic planar systems [8].
The above examples justify the importance of having a well established and manageable quantum theory of skyrmions in the 2+1-dimensional CP\(^1\)-Nonlinear Sigma Model. A semiclassical effective lagrangian describing the skyrmion physics was derived in [8, 13]. The question of quantum effects on skyrmions was addressed in [9], where a semiclassical quantization of skyrmions was developed allowing for the obtainment of spin correlators in the presence of skyrmions, as well as the effective interaction energy of quantum skyrmions. An attempt to obtain a quantum theory of skyrmion fields was made in [14]. The skyrmion correlation functions, however, were not evaluated there. In the present work, we obtain a full quantum theory for the skyrmion field within the CP\(^1\) version of the theory, taking advantage of the fact that in this framework the skyrmions are vortices and the theory of quantum vortices developed in [15, 17] can be therefore applied. The quantum skyrmion two-point correlation function is explicitly evaluated in three different situations. Firstly in the pure CP\(^1\)-NSM. Secondly, when Dirac fermions are minimally coupled to the CP\(^1\) gauge field but their mutual direct interaction is neglected. Finally, when the electromagnetic interaction among the fermions is included. In this latter case the true, three-dimensional, electromagnetic interaction is considered, even though, the fermions are moving on a plane [18]. In all of these cases, we explicitly obtain the quantum skyrmion mass through the large distance behavior of the two-point function. This is compared to the classical mass and the tendency of quantum corrections to decrease the skyrmion mass observed semiclassically in [9], is confirmed. The skyrmion statistics is also evaluated in each case from the behavior of the phase of this function. The explicit results show that the conjecture, formulated in [14], about the condensation of skyrmions in the absence of a Chern-Simons term is actually not valid. A brief discussion about the existence of asymptotic quantum skyrmion states is presented at the conclusion.
2) The CP\(^1\)-Nonlinear Sigma Model and a Quantum Field Theory of Skyrmions

Let us consider the O(3) Nonlinear Sigma Model which describes a field \( n^a, a = 1, 2, 3 \) subject to the constraint \( n^a n^a = \rho_0^2 \). The lagrangian is simply

\[
L = \frac{1}{2} \partial_\mu n^a \partial^\mu n^a
\]  

(2.1)

and therefore the whole dynamics comes from the constraint. The great importance of this model for condensed matter systems emerges from the fact that it is the continuum limit of the O(3)-symmetric antiferromagnetic Heisenberg model in a two-dimensional square lattice \([5]\), which is described by the hamiltonian

\[
H = J \sum_{i,j} \vec{S}_i \cdot \vec{S}_j
\]  

(2.2)

with \( J > 0 \). The nonlinear sigma field \( \vec{n} \) is the continuum limit of the staggered spin corresponding to \( \vec{S} \) and \( \rho_0^2 \) satisfies \(
\frac{\hbar c}{a} = \rho_0^2 \frac{2\sqrt{2}}{\sqrt{S(S+1)}}
\), \( S \) being the spin quantum number, \( c \) the spin-wave velocity and \( a \), the lattice spacing. Using linear spin-wave theory results for the Heisenberg model, on the other hand, it is found that \( \frac{\hbar c}{a} = 1.18(2S)\sqrt{2}J \) where \( J \) is the Heisenberg antiferromagnetic coupling constant. These two relations are valid for large \( S \). Assuming the ratio of both holds for any \( S \), we establish the following relation between the coupling constants in the continuum and in the lattice, for the case \( S = 1/2 \)

\[
\rho_0^2 = 1.18\frac{\sqrt{3}}{4} J
\]  

(2.3)

It is very convenient to express the \( n^a \)-field in the so called CP\(^1\) language, in terms of a doublet of complex scalar fields \( z_i, i = 1, 2 \), which also satisfy the constraint \( |z_1|^2 + |z_2|^2 = \rho_0^2 \). The mapping between the two fields is established through

\[
n^a = \frac{1}{\rho_0} z_i^\dagger \sigma^a_{ij} z_j
\]  

(2.4)

where \( \sigma^a \) are Pauli matrices. The corresponding lagrangian in the CP\(^1\) version is

\[
L = 2 (D_\mu z_i)^\dagger (D^\mu z_i)
\]  

(2.5)
where \( D_\mu = \partial_\mu + iA_\mu \) and \( A_\mu = -\frac{i}{\rho_0^2} z_i^\dagger \partial_\mu z_i \). Notice the well known spontaneous generation of a local U(1) symmetry in this version.

It is convenient to express the \( z_i \)-fields in the polar representation

\[
z_i = \frac{\rho_i}{\sqrt{2}} e^{i\theta_i} \tag{2.6}
\]

where \( \rho_i, \theta_i \) are real fields. The CP\(^1 \) constraint becomes then

\[
\frac{1}{2} (\rho_1^2 + \rho_2^2) = \rho_0^2 \tag{2.7}
\]

Inserting (2.6) in the lagrangian (2.5), we get

\[
\mathcal{L} = \sum_{i=1}^{2} \left\{ \partial_\mu \rho_i \partial^\mu \rho_i + \rho_i^2 [A_\mu + \partial_\mu \theta_i]^2 \right\} \tag{2.8}
\]

The gauge transformation under which (2.8) is invariant is

\[
\theta_i \rightarrow \theta_i + \Lambda(x) \quad A_\mu \rightarrow A_\mu - \partial_\mu \Lambda(x) \tag{2.9}
\]

Let us introduce now an explicitly gauge invariant description through the field

\[
\chi_i = \theta_i - \frac{\partial_\alpha A_\alpha}{(-\Box)} \tag{2.10}
\]

which is invariant under (2.9). Introducing (2.10) in (2.8), we get

\[
\mathcal{L} = \sum_{i=1}^{2} \left\{ \partial_\mu \rho_i \partial^\mu \rho_i + \rho_i^2 \left[ \frac{\partial_\alpha F^{\mu\alpha}}{(-\Box)} + \partial_\mu \chi_i \right]^2 \right\} \tag{2.11}
\]

which is explicitly gauge invariant. The euclidean action corresponding to this can be rewritten, after some algebra \[15\] as

\[
S = \int d^3z \left\{ \sum_{i=1}^{2} \left[ \partial_\mu \rho_i \partial^\mu \rho_i + \rho_i^2 \partial_\mu \chi_i \partial^\mu \chi_i - \partial_\mu \left( \rho_i^2 \left[ \frac{\partial_\alpha F^{\mu\alpha}}{(-\Box)} \right] \chi_i \right) + \frac{1}{4} F_{\mu\nu} \left[ \frac{2(\rho_1^2 + \rho_2^2)}{(-\Box)} \right] F^{\mu\nu} \right\} \tag{2.12}
\]
Before proceeding, let us make some remarks about the topological aspects of the CP$^1$-Nonlinear Sigma Model. There are two nontrivial topologies associated to the mappings $S^2 \to S^2$ and $S^3 \to S^2$, which are carried on, respectively, by static and non-static configurations of the $n^a$-field which, because of the constraint, lives on an $S^2$ manifold. The corresponding topological invariants are, respectively the topological charge
\[ Q = \int d^2 x J^0 \] (2.13)
and the Hopf invariant
\[ S_H = \int d^3 x J^\mu A_\mu \] (2.14)
In the two previous expressions, $J^\mu$ is the topological current, which is given, respectively, in the Nonlinear Sigma and CP$^1$ versions as
\[ J^\mu = \frac{1}{8\pi} \epsilon^{\mu\alpha\beta} \epsilon^{abc} n^a \partial_\alpha n^b \partial_\beta n^c \] (2.15)
and
\[ J^\mu = \frac{1}{2\pi} \epsilon^{\mu\alpha\beta} \partial_\alpha A_\beta \] (2.16)

The immediate consequence of the existence of the nontrivial topology in the mapping $S^3 \to S^2$ is that, when performing functional integrations over the basic fields, we must weigh the corresponding topological sector by $e^{i\theta S_H}$. This implies that a term $\theta S_H$ must be added to the action in (2.12) [21]. In the CP$^1$ language, this is nothing but a Chern-Simons term.

As a consequence of the nontriviality of the mapping $S^2 \to S^2$, on the other hand, the theory possesses classical soliton solutions. These are called “skyrmions” and have topological charge $Q = 1$. In the Nonlinear Sigma version the classical skyrmion solution is given by [1]
\[ \vec{n}_S(\vec{x}, 0) = \rho_0 (\sin f(r) \hat{r}, \cos f(r)) \] (2.17)
with
\[ f(r) = 2 \arctan \frac{\lambda}{r} \]
where $\lambda$ is an arbitrary scale and $r$ is the radial distance in two-dimensional space. In the CP$^1$ language, the skyrmion becomes a vortex. This can be already inferred from (2.13) and (2.16) which show that in this language the topological charge is the magnetic flux of the field $A_\mu$ along the two-dimensional space. Indeed, using (2.4) and (2.17), we get

$$z^a_S(\vec{x}, 0) = \rho_0 \left( \cos f(r) \exp^{-i \frac{1}{2} \text{arg}(\vec{r})} \sin \frac{1}{2} \exp^{i \frac{1}{2} \text{arg}(\vec{r})} \right)$$  \hspace{1cm} (2.18)

and

$$\vec{A}^S_i = \frac{1}{2} \cos f(r) \partial_i \text{arg}(\vec{r}) \hspace{0.5cm} A_0 = 0$$  \hspace{1cm} (2.19)

From the explicit form of the above classical solution, we can evaluate the classical skyrmion mass (energy) which derives directly from (2.1), namely,

$$M_{cl} = \frac{1}{2} \int d^2x \vec{n}^a \cdot \vec{n}^a$$  \hspace{1cm} (2.20)

obtaining $M_{cl} = 4\pi \rho_0^2$ (see also [1, 9]).

A full quantum theory of vortices has been developed based on the idea of order-disorder duality [15, 17] and therefore we can apply it in the present case, in order to obtain a quantum field theory of skyrmions in the CP$^1$ language. For a theory containing a vector field $A_\mu$, the quantum skyrmion field is given by [13]

$$\mu(\vec{x}, t) = \exp \left\{ i 2\pi \int_{\vec{x}, L}^\infty d\xi^i \epsilon^{ij} \Pi_j(\vec{\xi}, t) \right\}$$  \hspace{1cm} (2.21)

where $\Pi^i$ is the momentum canonically conjugate to $A^i$. It follows that [13]

$$[Q, \mu(x)] = \mu(x)$$  \hspace{1cm} (2.22)

and

$$[\mu(x), A_i(y)] = \mu(x) \partial_i(y) \text{arg}(\vec{y} - \vec{x})$$  \hspace{1cm} (2.23)

These two equations show that indeed $\mu(x)$ creates a quantum vortex at $x$ and the state $|\mu>$ is an eigenstate of the topological charge $Q$ with eigenvalue equal to one. It follows from the formalism developed in [13, 17] that a path independent functional
integral describing the euclidean correlation functions of the vortex operator can be obtained by adding to the field intensity tensor \( F_{\mu\nu} \) an external field

\[
\bar{B}^{\mu\nu} = b \int_{\vec{x},L} d\xi \epsilon^{\lambda\mu\nu} \delta^3(z - \xi)
\]

where \( b / 2\pi \) is the number of units of magnetic flux (topological charge) or vorticity. For the unit skyrmion, \( b = 2\pi \). Notice that (2.24) is the magnetic field intensity (magnetic flux/ topological charge density) of a vortex with universe line along \( L \). For the CP\(^1\) theory, described by (2.12), the local (path independent) euclidean skyrmion correlation function, including the topological \( \theta \)-term is given by [15]

\[
<\mu(x)\mu^\dagger(y)> = Z_0^{-1} \int D\rho_i D\chi_i DA_\mu \exp \left\{ - \int d^3z \left[ \sum_{i=1}^2 \left[ \partial_\mu \rho_i \partial^\mu \rho_i + \rho_i^2 \partial_\mu \chi_i \partial_\mu \chi_i + \right. \right. \right.

\left. \left. \left. \partial_\mu \left( \rho_i^2 \right) \right] \left. \partial_\alpha \left( F_{\mu\alpha} + \bar{B}_{\mu\alpha} \right) \right] \left. \partial_\nu \left( F_{\nu\mu} + \bar{B}_{\nu\mu} \right) \right] \left( F^{\mu\nu} + \bar{B}^{\mu\nu} \right) \right. \left. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right.

\[
\frac{-i\theta}{4} \epsilon^{\mu\alpha\beta} \left[ \partial_\mu \left( F^{\nu\mu} + \bar{B}^{\nu\mu} \right) \right] \left( F_{\alpha\beta} + \bar{B}_{\alpha\beta} \right) \right\}
\]

(2.25)

In the above expression, \( \bar{B}^{\mu\nu} = \bar{B}^{\mu\nu}(z; x) - \bar{B}^{\mu\nu}(z; y) \), corresponding to the field operators \( \mu(x) \) and \( \mu^\dagger(x) \), respectively. Notice that we rewrote the Chern-Simons term in terms of \( F^{\mu\nu} \), up to a total derivative. The above expression for the skyrmion correlation function can be easily shown to be local (path independent), by performing the change in the functional integration variable [15]

\[
A_\mu \rightarrow A'_\mu = A_\mu + \Omega_\mu \quad ; \quad DA_\mu = DA'_\mu
\]

(2.26)

with

\[
\Omega_\mu = b \int_{S(L,L')} d^2\xi \delta^3(z - \xi)
\]

(2.27)

where \( S(L, L') \) is the surface closed by the curve \( L \) appearing in (2.24) and another arbitrary curve \( L' \). It is easy to show [15] that under (2.26),

\[
\bar{F}^{\mu\nu} \rightarrow \bar{F}^{\mu\nu} + \bar{B}^{\mu\nu}(L') - \bar{B}^{\mu\nu}(L)
\]

(2.28)

Inspection of (2.25), then, immediately shows its independence on \( L \). Arbitrary higher point correlation functions can be obtained by just inserting additional external fields \( \bar{B}_{\mu\nu} \) in (2.25).
3) The Skyrmion Correlation Function

In this section we explicitly evaluate the two-point skyrmion correlation function, Eq. (2.25). In order to perform the integration in (2.25), we are going to use the saddle-point approximation. To implement this, we introduce the constraint through a Lagrange multiplier field, thereby obtaining (2.7) as an equation of motion. Choosing a stationary solution where the moduli of the complex scalar fields $z_i$ are taken to be constants, namely, $\rho_i^2 = \rho^2_{i,0}$, amounts to selecting a given direction in the manifold of minima $\rho^2_{1,0} + \rho^2_{2,0} = 2\rho^2_0$. We are going to expand the $\rho_i$-fields around this constant solution. This approximation was used for the evaluation of vortex correlation functions in the relativistic Landau-Ginzburg theory in 2+1 and in 3+1 dimensions [13, 14]. Note that this approximation does not impose a restriction on the topological charge, which is determined by the phases of the $z_i$-fields, according to (2.18) and (2.19). Also, it goes beyond the semiclassical approximation for the skyrmion correlation function, because the quantized skyrmion operators already include full quantum effects.

In this approximation, the functional integral (2.25) becomes, after integration over $\rho$ and $\chi$,

$$<\mu(x)\mu^+(y)> = Z_0^{-1} \int DA_\mu \exp \left\{ - \int d^3z \left[ \frac{1}{4} \left( F_{\mu\nu} + \tilde{B}_{\mu\nu} \right) \left( F^{\mu\nu} + \tilde{B}^{\mu\nu} \right) + \frac{i\theta}{4} \epsilon^{\mu\alpha\beta} \left( \partial_\nu (F^{\nu\mu} + \tilde{B}^{\nu\mu}) \right) \left( \tilde{F}_{\alpha\beta} + \tilde{B}_{\alpha\beta} \right) + \xi \left( \partial_\mu A_\mu \right)^2 \right] \right\}$$  \hspace{1cm} (3.1)

where the last term is a gauge fixing which has been inserted. The above functional integral is quadratic and can be evaluated by using the euclidean propagator of the $A_\mu$ field, which is given in momentum space by

$$D^{\mu\nu}(k) = \frac{1}{[M^2 + \theta^2 k^2]} \left[ \frac{M}{k^2} \left( k^2 \delta^{\mu\nu} - k^\mu k^\nu \right) - \theta e^{\mu\lambda\nu} k_\lambda \right] + \left[ \frac{M^2 - \theta^2 k^2}{M^2 + \theta^2 k^2} \right] \frac{k^\mu k^\nu}{\xi k^4}$$  \hspace{1cm} (3.2)

where $M = 4\rho^2_0$. Inserting (2.24) in (3.1), we see that the linear term in $A_\mu$ in (3.1) can be written as

$$\int d^3z L^\nu(z; x, y) A_\nu$$  \hspace{1cm} (3.3)
where
\[ L'(z; x, y) = \tilde{B}'(z; x, y) + \tilde{C}'(z; x, y) \]
with
\[ \tilde{B}'(z; x, y) = Mb \int_{x,L} d\xi \alpha \epsilon^{\alpha\beta\nu} \partial_\beta \left[ \frac{1}{-\Box} \right] (z - \xi) \]
and
\[ \tilde{C}'(z; x, y) = i\theta b \int_{x,L} d\xi \mu (-\Box \delta^{\mu\nu} + \partial^{\mu} \partial^{\nu}) \left[ \frac{1}{-\Box} \right] (z - \xi) \]
(3.4)
Integration over \( A_\mu \) in (3.1) gives
\[ <\mu(x)\mu^\dagger(y)> = \exp \left\{ \frac{1}{2} \int d^3zd^3z' L^\mu(z; x, y)L^\nu(z'; x, y)D^{\mu\nu}(z - z') - S_L \right\} \]
(3.5)
where \( S_L \) is the line dependent renormalization factors of (3.1) which ensure the locality of \( <\mu\mu^\dagger> \):
\[ S_L = S_{1,L} + S_{2,L} = \frac{1}{4} \int d^3z \tilde{B}^{\mu\nu} \left[ \frac{M}{(-\Box)} \right] \tilde{B}^{\mu\nu} + \frac{i\theta}{4} \int d^3z \epsilon^{\mu\alpha\beta} \left[ \partial_\nu \tilde{B}^{\mu\nu} \right] \tilde{B}_{\alpha\beta} \]  
(3.6)
From the form of (3.4) we can see that only the gauge independent first two terms of (3.2) contribute to (3.5), thereby ensuring the gauge independence of the skyrmion correlation function.

There are six terms coming from the first part of the exponent in (3.3), which in an obvious notation we call \( BMB, CMC, B\theta C, BMC, B\theta B, C\theta C \). The first three ones combine in the form
\[ T_1 = -\frac{b^2}{2} \int_{x,L} d\xi \int_{x,L} d\eta \left[ -\Box \delta^{\mu\nu} + \partial^{\mu} \partial^{\nu} \right] \left[ \frac{M}{\epsilon} - \frac{|\xi - \eta|}{8\pi} \right] \]
(3.7)
where the last expression between brackets is \( \left[ \frac{M}{(-\Box)^2} \right] \equiv \mathcal{F}^{-1} \left[ \frac{M}{k^2} \right] \) and \( \epsilon \) is a regulator for the Fourier transform of \( \frac{1}{k^2} \). The last three terms combine in the form
\[ T_2 = i\frac{\theta b^2}{2} \int_{x,L} d\xi \int_{x,L} d\eta \epsilon^{\mu\nu\lambda\sigma} \partial_\lambda \left[ \frac{1}{4\pi |\xi - \eta|} \right] = i\frac{\theta b^2}{4\pi} \left[ \text{arg}(\vec{x} - \vec{y}) + \text{arg}(\vec{y} - \vec{x}) \right] \]
(3.8)
where the expression between brackets in the first part is \( \left[ \frac{1}{(-\Box)^2} \right] \equiv \mathcal{F}^{-1} \left[ \frac{1}{k^2} \right] \) and the second equality is proved in [23].

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Using (2.24) and integrating the second term in (3.7), we get

\[ T_1 = -\frac{Mb^2}{8\pi|x-y|} + \frac{1}{4} \int d^3z \tilde{B}_{\mu\nu} \left[ \frac{M}{\Box} \right] \tilde{B}^{\mu\nu} \]  

(3.9)

Inserting (2.24) in (3.6), we see that \( S_{2,L} \) cancels \( T_2 \). Both terms are actually multi-valued, as can be seen from (3.8). This is related to the statistics of skyrmions. We shall return to this point in Sect. 6. Also \( S_{1,L} \) cancels the last, line dependent term, of \( T_1 \) when all the terms are put together in (3.5). We finally conclude that only the first term of (3.9) remains in the exponent of (3.5) and

\[ <\mu(x)\mu^\dagger(y)> = \exp \left\{ -\frac{\rho_0^2 b^2}{2\pi} |x-y| \right\} \]  

(3.10)

where we have used \( M = 4\rho_0^2 \). This is our result for the skyrmion euclidean two-point function. From it we infer that the quantum skyrmion mass is \( M = \frac{4\rho_0^2}{2\pi} \). Observe that for the unit skyrmion, \( b = 2\pi \) and \( M = 2\pi\rho_0^2 \). This is to be compared with the classical skyrmion mass calculated in (2.20), namely \( M_{cl} = 4\pi\rho_0^2 \). The ratio \( M/M_{cl} = 0.5 \) confirms the fact observed semiclassically \([9]\) that quantum corrections decrease the skyrmion mass. This can be understood on general grounds by examining the expression for the classical energy given either by (2.20) or (2.2). From the latter we can infer that the classical energy must be proportional to the spin stiffness \( \rho_s = S^2J \). It is reasonable to expect that in the semiclassical approximation, it must be proportional to the renormalized spin stiffness \( \tilde{\rho}_s = Z\rho S^2J \) \([7]\). Since \( Z\rho \approx 0.7 \) \([7]\), we should expect semiclassically that \( M/M_{cl} \approx 0.7 \). Our result for the skyrmion mass, however, is based on the analysis of the large distance decay of the fully quantized skyrmion operator correlation function and includes full quantum effects, going beyond the semiclassical approach were expressions (2.20) or (2.2) are used for obtaining the skyrmion energy.
4) The Introduction of Fermions

Let us investigate in this section how the coupling of fermions influences the skyrmion properties at the quantum level. For this, let us consider the lagrangian

\[ \mathcal{L} = 2 (D_\mu z_i)^\dagger (D^\mu z_i) + i \frac{\theta}{2} \epsilon^{\mu\alpha\beta} A_\mu \partial_\alpha A_\beta + i \bar{\psi} \gamma_5 \gamma^\mu \psi A_\mu \]  

(4.1)

where \( q \) is the coupling between the fermions and the \( CP^1 \) gauge field. We can obtain an effective theory for \( A_\mu \) by integrating over the fermions. This will be done in two limits of approximation, namely, large and small fermion mass \( m \), both for small \( q \).

In this case, we can write the fermionic determinant as [24],

\[ \ln \text{Det} [i \partial + A - m] = \frac{q^2}{2} \int d^3 z \left[ \frac{A}{2} \epsilon^{\mu\alpha\beta} A_\mu \partial_\alpha A_\beta + \frac{1}{4} F_{\mu\nu} [B] F^{\mu\nu} \right] \]  

(4.2)

Let us consider the large mass limit first. In this case [24],

\[ A = \frac{q^2}{2\pi} + O \left( \frac{1}{m^2} \right) \]

and

\[ B = \frac{q^2}{16\pi m} + O \left( \frac{1}{m^3} \right) \]  

(4.3)

The result of the fermion integration can be written as the exponential of (4.2). Combining this with the result of the integration over the \( CP^1 \) fields, which can be done in the same approximation as in the previous section, we arrive at an expression for the skyrmion correlation function, which is identical to (3.1), except for the following modifications

\[ \theta \rightarrow \theta + \frac{q^2}{2\pi} \]

\[ M \rightarrow M + \frac{q^2(-\Box)}{16\pi m} \]

(4.4)

the last one being \( M \rightarrow M + \frac{q^2 k^2}{16\pi m} \) in momentum space. Substituting these modifications in (3.2), (3.4) and thereafter, we arrive at an expression like (3.7) but with the last term between brackets being now modified as

\[ \left[ \frac{M}{(-\Box)^2} \right] \rightarrow \left[ \frac{M}{(-\Box)^2} + \frac{q^2}{16\pi m(-\Box)} \right] \]  

(4.5)
The nonlocal terms cancel exactly as before. Going through the same steps as in the preceding section, after (3.7) and using the fact that
\[
\left[ \frac{1}{-\Box} \right] \equiv \mathcal{F}^{-1} \left[ \frac{1}{k^2} \right] = \frac{1}{4\pi |x - y|} \tag{4.6}
\]
we arrive at
\[
< \mu(x) \mu^\dagger(y) >_{LM} = \exp \left\{ -\rho_0 b^2 \frac{|x - y|}{2\pi} + \frac{q^2 b^2}{64\pi^2 m |x - y|} \right\} \tag{4.7}
\]
This is the quantum skyrmion correlation function in the presence of fermions with a large mass \(m\) and coupling constant \(q\). Observe that the skyrmion mass, determined by the large distance behavior of (4.7), is not modified by the presence of fermions. The short distance behavior of the two-point function, however, is completely different, being highly singular in this case.

Let us turn now to the small mass limit. In this case [24],
\[
A = \frac{q^2}{4\pi} + O(m)
\]
and
\[
B(k) = \frac{q^2}{16k} + O(m) \tag{4.8}
\]
Now, the modifications corresponding to (4.4) are
\[
\theta \to \theta + \frac{q^2}{4\pi}
\]
\[
M \to M + \frac{q^2(-\Box)^{1/2}}{16} \tag{4.9}
\]
and the expression corresponding to (4.3) is modified as
\[
\left[ \frac{M}{(-\Box)^2} \right] \to \left[ \frac{M}{(-\Box)^2} + \frac{q^2}{16(-\Box)^{3/2}} \right] \tag{4.10}
\]
Repeating the same procedure as in the last section, after (3.7) and using the fact that
\[
\left[ \frac{1}{(-\Box)^{3/2}} \right] \equiv \mathcal{F}^{-1} \left[ \frac{1}{k^3} \right] = -\frac{1}{2\pi} \ln |x - y| \tag{4.11}
\]
we get
\[
<\mu(x)\mu^\dagger(y)>_{SM} = \exp\left\{-\frac{\rho_0^2 b^2}{2\pi}|x-y| - \frac{q^2 b^2}{32\pi^2}\ln|x-y|\right\} = \exp\left\{-\frac{\rho_0^2 b^2}{4\pi}|x-y|\right\}
\]

This is the skyrmion correlation function in the small fermion mass limit (actually this order of approximation gives the zero mass limit). We see again that the large distance behavior remains unchanged and therefore the skyrmion mass is not affected by the presence of light fermions as well. A power law short distance behavior, however, is introduced.

5) The Introduction of Electromagnetic Coupling Between the Fermions

In the previous section, we analyzed the effects produced on the skyrmion correlation function by the inclusion of fermions in the theory. The interaction among the fermions themselves, however, was neglected. This does not correspond to the realistic situation in any possible application to a condensed matter system where the fermions are electrons which, of course, feel their mutual electromagnetic interaction. Therefore, let us consider in this section the case where the fermions, in spite of moving on the plane possess a real electromagnetic interaction. The 2+1-dimensional theory which describes the true electromagnetic interaction in this case is not Maxwell but a modified version of it \[18\]. Taking into account the results of \[18\], we can write the \(CP^1\) lagrangian in the presence of fermions which interact electromagnetically among themselves as

\[
\mathcal{L} = 2 \left(\bar{D}_\mu z_i\right)^\dagger\left(D^\mu z_i\right) + \frac{i}{2}\epsilon^{\mu\nu\sigma} A_\mu \partial_\nu A_\sigma + i\bar{\psi} \gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi - \bar{\psi}\gamma^\mu \psi (qA_\mu + eA_\mu) + \\
- \frac{1}{4} F_{\mu\nu} \left[\frac{1}{(-\Box)^{1/2}}\right] F^{\mu\nu} \quad (5.1)
\]

where \(e\) is the electric charge and \(A_\mu\) is the 2+1D electromagnetic field. Again, the integration over the \(CP^1\)-fields and also over the fermion fields can be made as before.
The integration over $A_{\mu}$, is then quadratic and we now arrive at an expression for the skyrmion correlation function which is identical to (3.1), except for the modifications

$$\theta \rightarrow \theta + A$$

$$M \rightarrow M + B(-\Box) + \frac{e^2}{q^2} \left[ B^2(-\Box)^{3/2} + A^2(-\Box)^{1/2} \right]$$

which is valid up to the order $e^2$. $A$ and $B$ have been displayed in the previous section and are given, respectively, in the large and small fermion mass limit by (4.3) and (4.8). Observe that both of them are of the order $O(q^2)$ and therefore the last term in the second piece of (5.2) is of the order $O(e^2q^2)$.

Inserting (5.2) in (3.2), (3.4) and thereafter, we obtain an expression identical to (3.7) except for the fact that the last term between brackets is now modified as

$$\left[ \frac{M}{(-\Box)^2} \right] \rightarrow \left[ \frac{M}{(-\Box)^2} + \frac{B}{(-\Box)} + \frac{e^2}{q^2} \left[ B^2(-\Box)^{1/2} + \frac{A^2}{(-\Box)^{3/2}} \right] \right]$$

Following the same steps as before and using (4.3), we obtain, in the large mass limit, up to the order $1/m$,

$$\langle \mu(x)\mu^\dagger(y) \rangle_{e,LM} = \exp \left\{ -\frac{\rho^2 b^2}{2\pi} |x - y| + \frac{q^2 b^2}{16\pi^2 m |x - y|} \right\}$$

In the small mass limit, we go through the same steps but use (4.8) for $A$ and $B$ in (5.2). The result is

$$\langle \mu(x)\mu^\dagger(y) \rangle_{e,SM} = \exp \left\{ -\frac{\rho^2 b^2}{2\pi} |x - y| - \frac{q^2 b^2}{32\pi^2} \left[ 1 + e^2 \left( \frac{\pi^2 + 16}{16\pi^2} \right) \right] \ln |x - y| \right\} =$$

$$= \exp \left\{ -\frac{\rho^2 b^2}{2\pi} |x - y| \right\} \frac{2^{16}}{(1 + e^2) \pi^2} \right\}$$

Observe that in the limit $e \to 0$, (5.4) and (5.5) reduce to (4.7) and (4.12), respectively.

6) The Statistics of Skyrmions

It is well known that in the presence of a Chern-Simons term, skyrmions acquire a generalized statistics determined by $\theta$ [21]. We can see this fact in the present
formulation in a quite interesting way. The statistics of a particle implies certain commutation relation of the quantized operator associated to it. When we evaluate the euclidean correlation functions of these field operators, the commutation relation manifest itself as a multivaluedness of these euclidean correlators [25, 22], each sheet of the function being associated with a certain ordering of operators. In the bosonic case, of course, the correlators are univalent. Consequently, we can easily infer the statistics of skyrmion in the different cases analyzed above, by checking the multivaluedness of the corresponding correlation functions.

As remarked at the end of Sect. 3 there is a cancellation between \( S_{2,L} \), given in (3.6) and \( T_2 \), given by (3.8) in all of the correlation functions evaluated in Sects. 3, 4, 5. The only difference in each case is the value of \( \theta \), which in the latter cases is modified according to (4.4), (4.9) or (5.2). As can be clearly seen from (3.8), however, either \( T_2 \) or \( S_{2,L} \) are defined up to a factor

\[
i\varphi = i \ n \ \left( \frac{\theta b^2}{4\pi} \right) 2\pi \tag{6.1}
\]

where \( n = 0, \pm 1, \ldots \). We are therefore led to the conclusion that the correlation functions (3.10), (4.7), (4.12), (5.4) and (5.5) are multivalued and defined up to a phase \( \varphi \), given by the above expression. The value of \( \theta \) changes according to (4.4), (4.9) or (5.2), in each case. The skyrmion statistics, in the pure CP\(^1\)-NSM is, therefore,

\[
S = \frac{\theta b^2}{4\pi} \tag{6.2}
\]

This is in agreement with the result found in [21]. When we couple the model to fermions we see from (4.4), (4.9) and (5.2) that the modification of skyrmion statistics is the same either in the presence of the electromagnetic interaction or not, giving

\[
S_{LM} = \left( \theta + \frac{q^2}{2\pi} \right) b^2 \tag{6.3}
\]

and

\[
S_{SM} = \left( \theta + \frac{q^2}{4\pi} \right) b^2 \tag{6.4}
\]

respectively in the large and small mass limits.
7) Conclusion and Remarks

In this work, the correlation functions of quantum skyrmion fields were explicitly evaluated in three different physical situations involving the 2+1-dimensional O(3) Nonlinear Sigma Model in its \( \mathbb{C}P^1 \) version. The relevant functional integrals were evaluated through a saddle-point method but full quantum effects are described by the use of skyrmion creation operators. The mass and statistics of the skyrmion quantum excitations were obtained by an analysis of the behavior of these correlation functions. Our results show that the inclusion of fermions, either in the presence of an electromagnetic interaction or not, does not change the skyrmion mass which is obtained in the pure NSM case. This is shown to be smaller than the classical one by a factor 0.5, confirming the tendency of quantum effects to reduce the mass, which was observed at semiclassical level \[9\]. The statistics of skyrmions is changed due to the presence of fermions but is not sensible to presence of an electromagnetic interaction. The short distance behavior of the skyrmion correlation functions and therefore the ultraviolet properties, on the other hand, are completely changed, both by the mere introduction of fermions and by turning on the electromagnetic interaction among them. It is a well known fact in field theory that the existence of asymptotic states associated to a certain quantum field requires a singular short distance behavior of the corresponding correlation function. Comparing (3.10) with (4.7), (4.12), (5.4) and (5.5), we conclude that only in the presence of fermions we expect asymptotic quantum skyrmion states. This may be related to the so called marginal stability of the classical skyrmion solution in the pure theory, which is associated to a scale invariance which may be broken by the introduction of \( q, e \) and \( m \). This result is rather suggestive, as several authors demonstrated that real skyrmions are introduced in the system through electron doping \[8, 9, 10, 11, 12\]. Looking at the above mentioned expressions obtained for the skyrmion correlation function, on the other hand, we see that always \( \mu = 0 \), implying that the skyrmions never condense, contrary to what was conjectured in \[14\], for \( \theta = 0 \).

There is a great potential of applicability of the present formalism in planar con-
densed matter systems which can be described by the NSM in the continuum limit. These include high temperature superconductors and systems presenting the quantum Hall effect. A preliminary application in the first case has been made in [12]. In order to describe the doping process, a constraint must be put on the fermion density. This would affect the skyrmion mass in an important way, as described in [12]. Another interesting application we can envisage for the present formalism is the calculation of magnetic form factors in the presence of skyrmions using the quantum skyrmion fields and correlation functions. This could have measurable consequences in neutron scattering experiments. The study of the recently experimentally observed skyrmions in quantum Hall systems [4], on the other hand, is now being performed using the formulation introduced in this paper. The nonrelativistic version of the CP$^1$ model, however, must be considered in this case. In conclusion, we stress that there are many possibilities of extracting interesting results from the formulation introduced here in real condensed matter systems. We are presently exploring some of them.

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