RESUMMED HEAT-KERNEL FOR SURFACE CONTRIBUTIONS:
DIRICHLET SEMITRANSPARENT BOUNDARY CONDITIONS

S. A. FRANCHINO-VINAS¹,²,³

Abstract. In this article we consider resummed expressions for the heat-kernel’s trace of a Laplace operator, the latter including a potential and imposing Dirichlet semitransparent boundary conditions on a surface of codimension one in flat space. We obtain resummed expressions that correspond to the first and second order expansion of the heat-kernel in powers of the potential. We show how to apply these results to obtain the bulk and surface form factors of a scalar quantum field theory in $d = 4$ with a Yukawa coupling to a background. A characterization of the form factors in terms of pseudo-differential operators is given.

1. Introduction

The relation between spectral functions and the quantum theory of fields has been a narrow one [1, 2], specially when considering external background fields, including electromagnetic fields [3] and curved spacetimes [4–6]. In particular, the (one-loop) quantum fluctuations are usually given in terms of operators of Laplace type, whose heat-kernels determine the one-loop effective action [7].

Recall that, given a Laplace-type operator $D_0$ defined on a real manifold $M$ of dimension $m$, with or without boundary and the corresponding local boundary conditions, its heat-kernel (HK) is defined as $K(T; D_0) := e^{-T D_0}$. The theory establishes that under general conditions of smoothness of both the operator and the manifold, as $T \downarrow 0$ the trace of the HK possesses the following expansion [8–11]:

$$K(T; f; D_0) := \text{Tr} \left( f(x) K(x, y; T; D_0) \right) = \sum_{n=0} a_n(f; D_0) T^{(n-m)/2}.$$  (1.1)

The function $f$ is called smearing function; its role is to give a precise mathematical meaning to terms that otherwise would be divergent (in physical terms, it acts as a regulator). The coefficients $a_n(f; D_0)$ are called Gilkey–Seeley–DeWitt (GSDW) coefficients [1, 9, 12] and sometimes HAMIDEW, after Hadamard–Minakshisundaram–DeWitt [13]. They consist of volume and surface integrals, respectively over the bulk and the boundary of $M$, of local invariants (including the smearing function). One can build them by considering a linear combination of all the possible invariants with the appropriate dimensions; the numerical coefficients in front of each single term are universal, i.e. independent of the problem at hand. Additionally, it can be shown that the HK corresponds to the solution of the following heat equation with initial localized conditions:

$$(\partial_T + D_0) K(x, y; T; D_0) = 0, \quad K(x, y; 0; D_0) = \delta(x, y).$$  (1.2)

¹ HZDR, Bautzner Landstraße 400, 01328 Dresden, Germany.
² Departamento de Física, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, C.C. 67 (1900), La Plata, Argentina.
E-mail address: ³ s.franchino-vinas@hzdr.de
In the last decades, the techniques in the computation of the GSDW coefficients have shown several advancements. During this period, the community has recognized the benefits of the joint use of index theorems and functorial techniques, together with the consideration of special cases [14].

Another important milestone has been the obtention of partially resummed HK expansions. For example, it has been proved that, in the so-called covariant perturbation theory, one can resum all the derivatives acting on contributions to second [15–17] and third order in the curvatures [18–21] (see also rederivations and physical consequences in [22, 23]). Other studied scenarios include resummations in abelian bundles [24], symmetric spaces [25, 26] and powers of the curvature [27, 28]; see Ref. [29] for additional considerations.

In the above-mentioned developments, much more attention has been given to the case of manifolds without boundaries, being the study of HKs in manifolds with boundaries much less developed; a not exhaustive list of works which deal with the latter problem include Refs. [14, 30–32] and references therein.

Trying to bring more balance into this scenario, in the present manuscript we will show how to obtain resummed expressions when Dirichlet semitransparent conditions on a flat surface are considered. According to the best of our knowledge, this is the first time that resummations for surface contributions are studied.

The Dirichlet semitransparent boundary condition, also known as transmittal boundary condition [33], is equivalent to the introduction of a delta potential with support on a surface of codimension one [34–36]. The problem at hand has been widely studied in the realm of quantum field theories; the interested reader may consult Refs. [37–48]. Moreover, it has also been considered as a problem in a first quantization [49–57]. Recently, it has attracted much attention in connection with the related $\delta'$ problem, see Refs. [58–60] for $\delta - \delta'$ potentials and Ref. [61] for some generalizations of semitransparent boundary conditions.

To be precise, in the following we will thus be interested in an operator of Laplace type in $d$-dimensional flat Euclidean space $\mathbb{R}^d$, whose potential is given by a sufficiently smooth function $V$, summed to a Dirac delta function with support in a surface of codimension one (chosen without loss of generality as the $x_d = L$ plane):

$$\mathcal{D} : = -\partial^2 + \lambda \delta(x_d - L) - \zeta V(x).$$  \hspace{1cm} (1.3)

We will require the potential in Eq. 1.3 to be positive in order to deal only with a continuous spectrum, i.e. to avoid bound states. This can be guaranteed if one chooses a coupling $\lambda > 0$ and enforces $\zeta V(x) < 0$.

Regarding its HK, we may summarize the main results of the present article as follows:

- Proposition 2, in Sec. 2, provides an integral equation for the relevant HK, from which an expansion in powers of the potential $V$ is derived in Corollary 2.
- Theorem 1, in Sec. 3, gives a closed expression for the resummed-in-$\lambda$ HK at first order in $V$ and in $d = 1$, together with the corresponding GSDW coefficients.
- Theorem 2, Sec. 4, concerns the second order in $V$ and resummed-in-$\lambda$ contribution to the HK in $d = 1$; the corresponding GSDW coefficients are also listed.
- In Sec. 5, we apply the previous results to a scalar quantum field theory in $d$ dimensions, including a Yukawa coupling to a background field. We show how the resummed expansions may be applied to obtain the relevant form factors, which are shown to be pseudo-differential operators with symbols in $S^j$ for all $j > 0$ (in the classification of L. Hörmander [62]).
2. A perturbative expansion of the heat-kernel in powers of $V$

In the following sections, unless otherwise stated, we will refer to the case $d = 1$. We will comment on the possibility of applying our results to $d$ dimensional operators and come back to an arbitrary dimension in Sec. 5, when we will study a QFT. To fix the notation, we will call $\Sigma$ the support of the delta function; by analogy with the $d$-dimensional case, for $d = 1$ we will use the notation

$$\int_\Sigma d^0 x g(x) := \{g(x)\} [\Sigma] := g(L). \quad (2.1)$$

As stated in the introduction, we will show how to compute the HK of the operator $D$ to all order in the $\lambda$ parameter, albeit performing an expansion in powers of $\zeta$. To zeroth order in $\zeta$, the HK of the operator has already been computed in closed form in the literature, see Refs. [34, 63, 64]. Since it will prove crucial in our computations, we state this result.

**Proposition 1.** The HK for the Laplace-type operator $D_{\zeta = 0}$ is given by

$$K_\lambda(x, y; T) = K_0(x, y; T) - \lambda \frac{1}{2} \int_{\mathbb{R}^+} du e^{-\frac{u}{2}} K_0(|x - L| + |y - L| + u, 0; T), \quad (2.2)$$

where $K_0(x, y; T) := (4\pi T)^{-1/2} \exp \left(-\frac{(x - y)^2}{4T}\right)$ corresponds to the free HK in one-dimensional flat space.

**Corollary 1.** As an immediate consequence of Proposition 1, one can prove that the functional trace of $D_{\zeta = 0}$’s HK is given by (see [65])

$$\text{Tr} \left( K_\lambda(x, y; T) - K_0(x, y; T) \right) = \frac{1}{2} \int_\Sigma d^0 x \left[ e^{\frac{2\sqrt{T}}{\lambda}} \text{erfc} \left( \frac{\lambda \sqrt{T}}{2} \right) - 1 \right]. \quad (2.3)$$

The situation becomes more involved if one tries to perform the computation to higher orders in $\zeta$. One powerful technique that has been employed in the past as a way to develop asymptotic expansions of HKs has been the derivation of an integral equation for the HK, alternative to the differential equation (1.2); see for example [15, 64, 66]. Customarily, such integral equation would involve the free HK; instead, in the present case $K_\lambda$ will play its role.

**Proposition 2.** The HK associated to $D$ satisfies the integral equation

$$K(x, y; T; D) = K_\lambda(x, y; T) + \zeta \int_0^T ds \int_{\mathbb{R}} dz K_\lambda(x, z; T - s)V(z)K(z, y; s; D). \quad (2.4)$$

**Proof.** The proof follows from direct application of the operator $D$ on both sides of Eq. (2.4), analogously to the usual case. After using the corresponding heat equation for $K_\lambda$, one can interpret $V(z)K(z, y; s; D) = [\partial_s - \partial_z^2 + \lambda \delta(z - L)]K(z, y; s; D)$. Using the symmetry of $K_\lambda$ in its first two arguments after integrating by parts in $s$ and $z$, one obtains then the desired result. \hfill \Box

The benefit of introducing the integral equation (2.4) is that it is particularly well-suited to perform a perturbative expansion in the potential. Indeed, by direct application of the heat equation and subsequent integrations by parts in the intermediate propertimes, one can prove the following corollary.

**Corollary 2.** The HK of the operator $D$ can be expanded as a series in $\zeta$,

$$K(x, y; T; D) = \sum_{n=0}^{\infty} K^{(n)}(x, y; T; D)\zeta^n, \quad (2.5)$$
where the coefficients are made of iterated integrals of the potential $V$ and the heat-kernel $K_\lambda$:

$$K^{(n)}(x, y; T; \mathcal{D}) := \int_0^T ds_n \cdots \int_0^{s_2} ds_1 \int_\mathcal{R} dz_1 \cdots \int_\mathcal{R} dz_n K_\lambda(x, z_1; s_1) \times V(z_1) K_\lambda(z_1, z_2; s_2 - s_1) \cdots V(z_n) K_\lambda(z_n, y; T - s_n)$$

\[ (2.6) \]

and $K^{(0)}(x, y; T; \mathcal{D}) := K_\lambda(x, y; T)$.

### 3. Heat-kernel’s trace: first order in $V$

We will now show how to obtain a closed expression for the HK’s trace at linear order in $V$. Let us first introduce, as usual, a smearing function $f(x) \in C^\infty_0(\mathbb{R})$, i.e. an infinitely differentiable function with compact support. Recall also that the imaginary error function $\text{erfi}(\cdot)$ and the complementary error function $\text{erfc}(\cdot)$ are defined in terms of the error function $\text{erf}(\cdot)$ as

$$\text{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x dt e^{-t^2}, \quad \text{erfi}(x) := -i \text{erf}(ix), \quad \text{erfc}(x) := 1 - \text{erf}(x). \quad (3.1)$$

Additionally, to enhance the readability of expressions we will make two definitions; one takes advantage of the symmetry under exchange of two variables and the other regards the derivative operator:

$$g(x, y) + g(y, x) =: [g(x, y) + \{x \leftrightarrow y\}], \quad (3.2)$$

$$D_x := -i \partial_x. \quad (3.3)$$

**Theorem 1.** The trace of the operator $\mathcal{D}$’s HK, under the assumptions around Eq. (1.3) and at linear order in $V$, is given by

$$\text{Tr} \left( f(x) K^{(1)}(x, y; T; \mathcal{D}) \right) = \int_\mathcal{R} dx f(x) H^{(f)}_{1, M}(D_x; T) V(x)$$

$$+ \int_\Sigma d^2x H^{(f)}_{1, \Sigma}(D_y, D_z; T; \lambda) f(y) V(z) \bigg|_{y=z=x}, \quad (3.4)$$
where we have defined the kernels

\[
H^{(f)}_{1,\lambda}(k; T) = \frac{e^{-\frac{1}{2} (k^2 T)}}{2 \sqrt{k^2}},
\]

\[
H^{(f)}_{1,\Sigma}(k_1, k_1; T; \lambda) := \left[ -\lambda k_1 \left( \frac{k_1^2 + \lambda^2}{k_1^2 - k_2^2} k_2 \left( \lambda^2 + k_2^2 \right)^2 \right) - \lambda \left( k_1 + k_2 \right) \right] e^{-\frac{1}{2} (k_1 + k_2)^2 T} \text{erfi} \left( \frac{k_1 + k_2}{\sqrt{2T}} \right)
\]

\[
+ \left( \frac{\lambda^2}{k_1 k_2} \right) \text{erfc} \left( \frac{\lambda}{\sqrt{2T}} \right)
\]

\[
- \frac{\lambda^3 \sqrt{T}}{\sqrt{\pi} \left( \lambda^2 + k_1^2 \right) \left( \lambda^2 + k_2^2 \right)} + \frac{\lambda^2 e^{-\frac{1}{2} (k_1 + k_2)^2 T}}{k_1 k_2 \left( \lambda^2 + \left( k_1 + k_2 \right)^2 \right)}.
\]

(3.6)

Proof. A direct way to prove this theorem is to consider Corollary 2 at linear order in \( V \). For the bulk contribution we have performed all the necessary integrations by parts in order to remove all the derivatives acting on \( f \); this is possible because of the assumed properties of \( f \) and \( V \). Of course, a similar procedure cannot be implemented for the boundary terms.

Some comments are in order. First, the bulk contributions are independent of the delta potential. Indeed, if we expand for small coupling \( \lambda \), we get

\[
\text{Tr} \left( f(x) K^{(1)}_{\lambda, V} (x, y; T) \right) = \int_{\mathbb{R}} dx f(x) \frac{e^{\frac{1}{4} \lambda x^2}}{2 \sqrt{-\partial^2_x}} \left( \sqrt{T} \partial^2_x \right)^n V(x) + O \left( \lambda^4 \right).
\]

(3.7)

These are standard formulae in the literature. As an example, the small-propertime \((T)\) expansion in the second line is consistent with the results contained in [14] for the GSDW coefficients up to \( a_0 \); notice that, order by order in \( T \), it depends only on integer powers of \( \partial^2_x \), i.e. it is made of local contributions. Moreover, the formulae in Eq. (3.7) are compatible with the second order resummed result à la Barvinsky–Vilkovisky [15], see [14] and the discussion in Sec. 4 of the present manuscript.

Second, even if the presence of rational functions of the \( k_i \) may induce one to think that nonlocalities may be present in the small-\( T \) expansion of the surface contributions, an explicit calculation shows that all the corresponding coefficients are made of local invariants. Even if the computations are lengthy, this can be straightforwardly seen from the following corollary of Theorem 1.

**Corollary 3.** The surface contributions to the GSDW coefficients associated to the operator \( \mathcal{D} \), at linear order in \( V \), vanish for \( n < 4 \). For \( n \geq 4 \) an even number,
they read

\[ a_{n,\Sigma}^{(1)}(f; D) = \left\{ \begin{array}{l}
\frac{2^{1-n} \lambda}{\Gamma\left(\frac{n+1}{2}\right)} - \frac{(-1)^n (n-1) \lambda^n}{\left(\lambda^2 - \partial_1^2\right)\left(\lambda^2 - \partial_2^2\right)} + \frac{(\partial_1 + \partial_2)^n}{\partial_1 \partial_2 \left(\lambda^2 - (\partial_1 + \partial_2)^2\right)} \\
\lambda^n
\end{array} \right. \]

where, for \( n \geq 5 \) and odd, we have

\[ a_{n,\Sigma}^{(1)}(f; D) = \left\{ \begin{array}{l}
\frac{2^{1-n} \lambda^2}{\Gamma\left(\frac{n+1}{2}\right)} \left[ \frac{\left(\lambda^2 - \partial_1^2 + 2\partial_1 \partial_2\right) \partial^n_\Sigma}{\partial_1 \left(\lambda^2 - \partial_2^2\right)} \right] + \{\partial_1 \leftrightarrow \partial_2\} \\
\lambda^{n-1}
\end{array} \right. \]

Remark 1. Taking into account the universality of the numerical coefficients in the HK expansion, one can lift the one dimensional results in Corollary 3 to arbitrary dimension, including the possibility of introducing additional geometrical features such as a curved manifold \( M \) and a curved surface \( \Sigma \). Then:

1. The dependence on the dimension would be only through an overall factor \((4\pi)^{-m/2}\), see Ref. [66].

2. The coefficients in Eqs. (3.8) and (3.9) would correspond to integrals over a \((d-1)\)-dimensional hypersurface \( \Sigma \) with the appropriate measure. The variable \( \lambda \) may be taken as a space-dependent quantity (living on the surface \( \Sigma \)) and the derivatives in (3.8) and (3.9) become covariant derivatives normal to \( \Sigma \). The integrals over \( \mathbb{R} \) would be replaced by integrals over \( M \) with the corresponding measure.

3. The additional geometrical features would manifest themselves as the appearance of new nonvanishing invariants contributing to the GSDW coefficients, including curvatures of the spacetime, the extrinsic curvature (also called second fundamental form) of the hypersurface \( \Sigma \), covariant derivatives in the directions tangent to \( \Sigma \) acting on \( \lambda \) and \( V \), etc.

Coming back to Eq. (3.6), one can corroborate it by comparing with several previously known results. To begin with, if we consider a constant potential \( V \equiv V_0 \), then we obtain

\[ \text{Tr} \left( f(x)K^{(1)}(x, y; T; D) \right) \bigg|_{x = x_0}^{y = y_0} = \text{Tr} \left( K_\lambda(x, y; T) \right), \]

which is the correct result since in that case one would have \( K(D) = e^{\Gamma T V_0} K_\lambda \). As an additional check, we can also remove the smearing function. In such a situation the result greatly simplifies.

Corollary 4. Provided that the potential \( V \) and its derivatives of any order decay sufficiently fast at infinity, we can set the smearing function to unity in Eq. (3.4)
and obtain
\[
\text{Tr} \left( K^{(1)}(x, y; T; D) \right) = \sqrt{\frac{T}{4\pi}} \int_{\mathbb{R}} dx V(x) - \frac{\lambda T}{2} \int_{\Sigma} d^0x \frac{\sqrt{T}}{\lambda^2 + \lambda^2} \left( -\partial_x^2 + \lambda^2 \right) \left( -1 + e^{\frac{\sqrt{T}}{\lambda}} \text{erf} \left( \frac{\lambda \sqrt{T}}{2} \right) \right) \left( \frac{\lambda \sqrt{T}}{2} \right) \partial_x V(x).
\]

(3.11)

In this case, the GSDW coefficients listed in Corollary 3, nonvanishing only for \( n \geq 4 \), simplify to
\[
a_{n, \Sigma}^{(1)}(1; D) = \frac{2^{2-n}}{\Gamma \left( \frac{n-2}{2} \right)} \int_{\Sigma} d^0x \left( -\lambda^{n-1} + \lambda i^n (-\partial_x^2)^{(n-2)/2} \right) V, \text{ for } n \text{ even,}
\]
\[
a_{n, \Sigma}^{(1)}(1; D) = \left( \lambda^{n-1} + \lambda^2 \lambda^{n-1} (-\partial_x^2)^{(n-3)/2} \right) V, \text{ for } n \text{ odd.}
\]

(3.12)

Notice that, as previously, the denominator in Eq. (3.12) is formal: it is understood that one should first perform the division of the polynomials (which is exact for all \( n \)) before interpreting \( \partial_x^2 \) as a differential operator. The GSDW coefficients up to \( n = 5 \) coincide with the corresponding ones in Ref. [66], while the \( a_6 \) term agrees with the result in Ref. [67].

One further comparison can be done employing the large \( \lambda \) expansion, which reads
\[
\text{Tr} \left( K^{(1)}(x, y; T; D) \right) = \sqrt{\frac{T}{4\pi}} \int_{\mathbb{R}} dx V(x) + \int_{\Sigma} d^0x \left\{ -\frac{T e^{\frac{\sqrt{T}}{\lambda}}} \right. \]
\[
+ \left. \left[ \frac{\sqrt{T}}{\lambda^2} + \frac{T e^{\frac{\sqrt{T}}{\lambda}}} \right] \text{erf} \left( \frac{\sqrt{T} \partial_x}{2} \right) \partial_x \right\} \lambda^{-1} - \frac{T e^{\frac{\sqrt{T}}{\lambda}}} \right. \]
\[
\left. \left[ -\frac{T e^{\frac{\sqrt{T}}{\lambda}}} \right] \right) \left( \frac{\lambda \sqrt{T}}{2} \right) \partial_x V(x)
\]
\[
= \sqrt{\frac{T}{4\pi}} \int_{\mathbb{R}} dx V(x) - \frac{T}{2} \int_{\Sigma} d^0x \left[ \sum_{j=0}^{\infty} \frac{T^{j}}{4j!} \int_{\Sigma} d^0x \left( \frac{\lambda \sqrt{T}}{2} \right) \partial_x V(x) \right] + O \left( \lambda^{-1} \right).
\]

(3.13)

The first boundary term is independent of \( \lambda \) and is expected to be related to the HK of a free Laplacian with Dirichlet boundary conditions. Indeed, it is well-known that in such limit the operator \( D_{\zeta=0} \) acts as a free Laplacian on two half-spaces: they share \( \Sigma \) as boundary, on which Dirichlet boundary conditions are imposed. This is to be expected on physical grounds, since as \( \lambda \to \infty \) the layer \( \Sigma \) foreseeable becomes impenetrable. Taking these comments into account, the correctness of the coefficients multiplying the \( V(L) \) and \( \partial^2 V(L) \) factors can be respectively verified by comparing with Refs. [14] and [68]. Notice also that the coefficients of terms with an odd number of derivatives simply vanish in our case, because of a cancellation between contributions from different semi-spaces: this is a consequence of the fact that the outward normal-pointing vector has different sign on the two sides of \( \Sigma \).

4. Heat-kernel’s trace: second order in \( V \)

Increasing the order in the potential substantially increases the difficulty in the computation of the HK’s trace. To simplify the results to quadratic order in \( V \) and taking also into account that in physical applications it is customary to do so, in the following we will set the smearing function aside.
Theorem 2. The trace of the operator $\mathcal{D}$’s HK, under the assumptions around Eq. (1.3), at quadratic order in $V$ and neglecting total derivatives, is given by

$$
\text{Tr} \left( K^{(2)}(x, y; T; \mathcal{D}) \right) = \int_{\mathbb{R}} dx' V(x) H_{2, M}(D_x; T)V(x')
$$

$$
+ \int_{\Sigma} d^3x H_{2, \Sigma}(D_1, D_2; T; \lambda)V(x_1)V(x_2) \Bigg|_{x_1 = x_2},
$$

(4.1)

where the kernels are related to those present in the linear expansion:

$$
H_{2, M}(k; T) = \frac{T}{2} H_{1, M}^{(f)}(k; T), \quad H_{2, \Sigma}(k_1, k_2; T; \lambda) = \frac{T}{2} H_{1, \Sigma}^{(f)}(k_1, k_2; T; \lambda).
$$

(4.2)

Correspondingly, the GSDW coefficients read

$$
a_{n, \Sigma}^{(2)}(1; \mathcal{D}) = \frac{1}{2} a_{n-2, \Sigma}^{(1)}(V; \mathcal{D}).
$$

(4.3)

Proof. The correctness of Theorem 2 can be shown by appealing to the perturbative expansion in Eq. (2.6). At quadratic order in $V$ we notice that the only relevant variable is $s_2 - s_1$; changing variables we thus get

$$
\text{Tr} \left( K^{(2)}(x, y; T; \mathcal{D}) \right) = \int_0^T ds_2 \int_0^{s_2} ds_1 \times \int_{\mathbb{R}^2} dz_1 dz_2 V(z_1) K_{\lambda}(z_1, z_2; s_2 - s_1)V(z_2) K_{\lambda}(z_2, z_1; T - s_2 + s_1)
$$

$$
= \int_0^T ds_2 \int_0^{s_2} ds_1 \int_{\mathbb{R}^2} dz_1 dz_2 V(z_1) K_{\lambda}(z_1, z_2; s_2) V(z_2) K_{\lambda}(z_2, z_1; T - s_2 + s_1)
$$

$$
= \frac{T}{2} \int_0^T ds_- \int_{\mathbb{R}^2} dz_1 dz_2 V(z_1) K_{\lambda}(z_1, z_2; s_-) V(z_2) K_{\lambda}(z_2, z_1; T - s_-).
$$

(4.4)

After replacing $V(z_1) \to f(z_1)$, this is proportional to the linear order expression with a smearing function, i.e., proportional to $\text{Tr} (f K^{(1)}(\mathcal{D}))$. The fact that we are neglecting total derivatives is a consequence of the integration by parts that we have performed in obtaining the kernels of the linear expression, cf. the proof of Theorem 1. Finally, the relation between the GSDW coefficients follows directly from the second equality in Eq. (4.2).

Remark 2. The discussion in Remark 1 trivially extends to this order.

Theorem 2 clarifies the comment made after Eq. (3.7): it can be easily checked that $H_{2, M}$ is the flat-space version of the Barvinsky–Vilkovisky kernel for the quadratic contribution in $V$, which is proportional by a factor $T/2$ to $H_{1, M}^{(f)}$. Alternatively, one can perform a small-propertime expansion and see that the first terms (up to order $T^3$) agree with the contributions listed in [14], of course after neglecting boundary terms.

An additional corroboration can be done by noting that in the limit when both $V$ factors becomes a constant $V_0$ we get

$$
\text{Tr} \left( K^{(2)}(x, y; T; \mathcal{D}) \right) \overset{V = V_0}{=} \frac{T^2 V_0^2}{2} \text{Tr} \left( K_{\lambda}(x, y; T) \right),
$$

(4.5)

which again matches the expansion of $e^{TV_0} K_{\lambda}$, this time at quadratic order in $V_0$.

Instead, if just one of the $V$ becomes constant, we get the equivalent of Eq. (3.11):

$$
\text{Tr} \left( K^{(2)}(x, y; T; \mathcal{D}) \right) = \frac{TV_0}{2} \text{Tr} \left( K^{(1)}(x, y; T; \mathcal{D}) \right),
$$

(4.6)
As a matter of completeness, let us perform an expansion for small propertime and one for large coupling. As a result of the former we get the following surface contributions,

\[
\text{Tr} \left( \frac{d^{d}x}{\Sigma} \right)_{\text{surface}} = \int \left( -\frac{\lambda}{4\sqrt{\pi}} T^{5/2} + \frac{1}{16} \lambda^{2}T^{3} - \frac{\lambda}{24\sqrt{\pi}} \left( \lambda^{2} + 2\partial_{1}^{2} + \partial_{1}\partial_{2} \right) T^{7/2} \right. \\
\left. + \frac{\lambda^{2}}{128} \left( \lambda^{2} + 2\partial_{1}^{2} + \frac{1}{6}\partial_{1}\partial_{2} \right) T^{4} + O(T^{9/2}) \right) \left| V(x_{1})V(x_{2}) \right|_{x_{i}=x},
\]

whose first term can be compared with the result in Ref. [67]. On the other side, in the large coupling limit the surface terms are given by

\[
\text{Tr} \left( \frac{d^{d}x}{\Sigma} \right)_{\text{surface}} = \int \left( -\frac{\lambda}{4\sqrt{\pi}} T^{5/2} + \frac{1}{16} \lambda^{2}T^{3} - \frac{\lambda}{24\sqrt{\pi}} \left( \lambda^{2} + 2\partial_{1}^{2} + \partial_{1}\partial_{2} \right) T^{7/2} \right. \\
\left. + \frac{\lambda^{2}}{128} \left( \lambda^{2} + 2\partial_{1}^{2} + \frac{1}{6}\partial_{1}\partial_{2} \right) T^{4} + O(T^{9/2}) \right) \left| V(x_{1})V(x_{2}) \right|_{x_{i}=x},
\]

where \( F(\cdot) \) is Dawson’s integral, \( F(x) := e^{-x^2} \int_{0}^{\infty} e^{y^2} dy \). In the third line, taking into account the comments below Eq. (3.13), we have rendered explicit the small-propertime expansion of the Dirichlet limit; one can readily see that the coefficient accompanying \( \int_{\Sigma} d^{d}x V^{2}(x) \) agrees with the expression in Ref. [68].

5. **Surface form factors in QFT**

As a simple, albeit conceptually rich application of the preceding results, let us now consider a quantum scalar field in \( d \)-dimensional flat Euclidean space. We will couple it quadratically to the background field \( \sigma \), according to the following action:

\[
S := \frac{1}{2} \int d^{d}x \left[ (\partial\phi)^{2} + m^{2}\phi^{2} + \lambda\delta(x_{d} - L)\phi^{2} + \phi^{2}\sigma^{2} \right].
\]

For a general discussion of this type of theories with and without the boundary term see respectively Refs. [43, 69, 70] and [71, 72]. Following the terminology in the quantum field theory literature, the negative of the Laplacian will be denoted \( \Box := -\partial^{2} \). As usually, the one-loop contribution to the effective action can be written in terms of the operator of quantum fluctuations [73],

\[
A := \Box + m^{2} + \lambda\delta(x_{d} - L) + \sigma^{2},
\]
either as a function of its determinant or, introducing an integral over Schwinger’s propertime $T$, in terms of its HK:

$$\Gamma_{1\text{-loop}} = \frac{1}{2} \log \det A = -\frac{1}{2} \int_{0}^{\infty} \frac{dT}{T} \text{Tr} e^{-TA}. \tag{5.3}$$

Using the formulae for the HK developed in the previous sections, we can recast the one-loop effective action as

$$\Gamma_{1\text{-loop}} = \int_{\mathbb{R}^d} d^d x F_M^{(0)}(m) + \int_{\Sigma} d^{d-1} x F_{\Sigma}^{(0)}(m, \lambda)$$

$$+ \int_{\mathbb{R}^d} d^d x \left[ F_M^{(1)}(m) \sigma^2(x) + \sigma^2(x) F_M^{(2)}(\Box, m) \sigma^2(x) \right]$$

$$+ \int_{\Sigma} d^{d-1} x \left[ F_{\Sigma}^{(1)}(D_{\gamma\ell}, m, \lambda) \sigma^2(y) + F_{\Sigma}^{(2)}(D_{\gamma\ell}, D_{\gamma\ell}, m, \lambda) \sigma^2(z) \sigma^2(y) \right]_{z=y=x} + \mathcal{O}(\sigma^6, \partial \parallel \sigma), \tag{5.4}$$

where $F_M^{(i)}$ and $F_{\Sigma}^{(i)}$ are called form factors. In Eq. (5.4) we are neglecting powers of the field $\sigma$ higher than four, as well as all possible contributions involving its partial derivatives with respect to the directions tangent to the surface $\Sigma$. For our purposes, this will turn out to be enough.

A closed expression for the form factors in arbitrary dimension can be obtained; given that they are rather lengthy, we prefer to leave them to App. A. Instead, we will display here the corresponding formulae in $d \equiv 4$ dimensions, focusing on the terms that require to undergo a renormalization process; they are

$$F_M^{(0)}(m) = \frac{m^4}{128 \pi^2} \left[ \frac{4}{(d-4)} + 2 \log \left( \frac{m^2}{4 \pi \mu^2} \right) + 2 \gamma - 3 \right], \tag{5.5}$$

$$F_{\Sigma}^{(0)}(m, \lambda) = \frac{\lambda (6m^2 - \lambda^2)}{192 \pi^2} \left[ \frac{2}{(d-4)} + 2 \gamma - \frac{8}{3} + \log \left( \frac{m^2}{\pi \mu^2} \right) \right] + C_{\Sigma}^{(0)}(m, \lambda), \tag{5.6}$$

$$F_{\Sigma}^{(1)}(k, m, \lambda) = \frac{\lambda}{32 \pi^2} \left[ \frac{2}{(d-4)} + \gamma + \log \left( \frac{m^2}{4 \pi \mu^2} \right) \right] + C_{\Sigma}^{(1)}(k, m, \lambda), \tag{5.7}$$

$$F_M^{(1)}(m) = \frac{m^2}{32 \pi^2} \left[ \frac{2}{(d-4)} + \gamma - 1 + \log \left( \frac{m^2}{4 \pi \mu^2} \right) \right], \tag{5.8}$$

$$F_M^{(2)}(k^2, m) = \frac{1}{64 \pi^2} \left[ \frac{2}{(d-4)} + \gamma + \log \left( \frac{m^2}{4 \pi \mu^2} \right) \right] + C_M^{(2)}(k^2, m), \tag{5.9}$$

\footnote{The zeroth order in $\sigma$ can be read from Eq. (2.3).}
where $\gamma$ is Euler–Mascheroni constant. Additionally, we have used dimensional regularization\textsuperscript{2} and defined the functions

\begin{align}
C^{(0)}(m, \lambda) &= 4\pi m^{3} - \lambda m^{2} - (\lambda^{2} - 4m^{2})^{3/2} \arctanh\left(\sqrt{1 - \frac{4m^{2}}{\lambda^{2}}}\right), \\
C^{(1)}(k, m, \lambda) &= \frac{\lambda}{32\pi^{2}(k^{2} + \lambda^{2})} \left[2\lambda\sqrt{\lambda^{2} - 4m^{2}} \arccoth\left(\frac{\lambda}{\sqrt{\lambda^{2} - 4m^{2}}}\right) + \pi\lambda\sqrt{k^{2} + 4m^{2}} + 2\sqrt{k^{2}(k^{2} + 4m^{2})} \arctanh\left(\sqrt{k^{2}}\right) \right] - \frac{\lambda}{16\pi^{3}}, \\
C^{(2)}_{M}(k^{2}, m) &= \frac{1}{32\pi^{2}} \left[\left(\frac{4m^{2}}{k^{2}} + 1\right)^{1/2} \arcsinh\left(\sqrt{\frac{k^{2}}{4m^{2}}}\right) - 1\right].
\end{align}

5.1. The renormalization. The renormalization process follows now by absorbing the infinities of the theory into the dressed coupling constants. In the minimal subtraction (MS) scheme, one would just introduce counterterms to cancel the negative powers of $(d - 4)$; this is the path followed for example in Ref. [43]. Here we will instead use a more physical scheme, in spirit similar to the discussion in Refs. [72, 74–80], noting that the functions $C^{(1)}_{\Sigma}$ and $C^{(2)}_{M}$ play the role of running coupling constants.

Indeed, suppose that we measure the bulk quartic coupling of $\sigma$ to have the value $c_{2}$ at a scale $q^{2}$. Then its value at another scale $k^{2}$ will simply be given by $c_{2} + C^{(2)}_{M}(k^{2}, m) - C^{(2)}_{M}(q^{2}, m)$, showing that our assertion is true (up to an experimentally determined constant).

The situation is slightly different for $C^{(1)}_{\Sigma}$, inasmuch as the relevant scale for its running is determined by the momentum perpendicular to the plate. On the one side, this means that our renormalization breaks Lorentz invariance, which was anyway already broken by the plate configuration. On the other side, this is not a consequence of having neglected in Eq. (5.4) terms involving partial derivatives with respect to the parallel directions: for fields vanishing at infinity, their contributions will be boundary terms that simply vanish.

Therefore, we are lead to the conclusion that $C^{(1)}_{\Sigma}$ and $C^{(2)}_{M}$ do have physical meaning, differently to the case of $C^{(0)}_{\Sigma}$. The quantum field $\phi$ acts as a mediator between the boundary condition and the background $\sigma$, such that, if we try to confine $\phi$, then we are also automatically enforcing $\sigma$ to satisfy a boundary condition. The nature of the latter is encoded in the form factor $F^{(1)}_{\Sigma}$.\textsuperscript{2}

\textsuperscript{2}In this simple example, it proves convenient to introduce dimensional regularization by just modifying the HK’s leading power of the propertime to be proportional to $(4\pi T)^{-d/2}$. Alternatively, one could modify the power of $T$ in the denominator of Eq. (5.3) to be $T^{s}$ and take afterwards the limit $s \to 1$.\textsuperscript{2}
As a last comment, it is interesting to study the asymptotic expansion of the running coupling constants for large and small $k$ (or masses), to wit:

\[
C^{(1)}_{\Sigma}(k, m, \lambda) = \begin{cases} \\
\frac{\lambda}{128\pi m} \left[ \log \left( \frac{k^2}{m^2} \right) - 2 + 2\pi \lambda (k^2)^{-\frac{7}{2}} - (\lambda^2 - 2m^2) \log \left( \frac{k^2}{m^2} \right) \right] k^{-2} \\
+ \left( 2\lambda \sqrt{\lambda^2 - 4m^2} \arcoth \left( \frac{\lambda}{\sqrt{\lambda^2 - 4m^2}} \right) + 2m^2 \right) k^{-2} \\
+ O \left( (k^2)^{-3/2} \right), \quad k^2 \gg m^2, \\
\frac{\lambda^2}{128\pi m^4} \left[ 1 + \frac{2(k^2-\lambda^2)}{3\pi\lambda m} + \frac{\lambda^2 - k^2}{16m^2} \\
- \frac{k^4 + \lambda^4 - 2\lambda^2 k^2}{15\pi \lambda m^3} + O \left( m^{-4} \right) \right], \quad k^2 \ll m^2, 
\end{cases}
\]  
(5.13)

\[
C^{(2)}_{M}(k^2, m) = \begin{cases} \\
\frac{\lambda^2}{64\pi m^4} \left[ \log \left( \frac{k^2}{m^2} \right) - 2 + 2m^2 \left( \log \left( \frac{k^2}{m^2} \right) + 1 \right) \right] k^{-2} \\
+ m^4 \left( -2 \log \left( \frac{k^2}{m^2} \right) + 1 \right) k^{-4} + O \left( k^{-6} \right), \quad k^2 \gg m^2, \\
\frac{k^2}{384\pi m^5} \left[ 1 + \frac{k^4}{10m^2} - \frac{4}{10m^2} + \frac{17\xi^2}{2210m^4} + O \left( m^{-8} \right) \right], \quad k^2 \ll m^2, 
\end{cases}
\]  
(5.14)

For large masses, they display the decoupling of the quantum field, analogue to the Appelquist and Carazzone result for QED [81] and generalizing the result for a Yukawa coupling without boundaries [82]. Additionally, the coefficients in the expansion are local. Instead, for large $k^2$ we see the nonlocal character of the runnings. In particular, we observe that both couplings diverge for $k \to \infty$. This implies that for situations involving high energy processes in the $d$th direction, at the one-loop level, the $\sigma$ field will have to obey a strong boundary condition on $\Sigma$, i.e. almost Dirichlet.

5.2. Formal aspects of the form factors. We will close this section with a short digression on a more mathematical description of the form factors. The ultimate goal will be to evidence one fact that is usually left aside in the literature: in general, form factors can be formally discussed in the frame of pseudo-differential operators. To begin, let us recall the following definitions borrowed from Ref. [62].

**Definition 1** (Symbols). Let $m \in \mathbb{R}$ and $n \in \mathbb{N}$. The class of symbols $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ consists of functions\(^3\) (symbols) $a(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that, for all multi-indices $\alpha$ and $\beta$, a constant $C_{\alpha, \beta}$ exists for which

\[
|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{|\alpha|} m^{-|\alpha|}, \quad x, \xi \in \mathbb{R}^n.
\]  
(5.15)

**Definition 2** (Pseudo-differential operator). If $a \in S^m$ and $u \in \mathcal{S}$, then

\[
a(x, D)u(x) = (2\pi)^{-n} \int d\xi \, e^{ix\cdot\xi} a(x, \xi) \hat{u}(\xi)
\]  
(5.16)

defines a function $a(x, D)u \in \mathcal{S}$. One calls $a(x, D)$ a pseudo-differential operator of order $m$.

**Proposition 3.** The form factors $C^{(1)}_{\Sigma}, C^{(2)}_{M} \in \mathcal{S}^j$ for any $j > 0$, and therefore can be used to define pseudo-differential operators.

\(^3\)\(C^\infty\) denotes the space of infinitely differentiable functions.

\(^4\)\(\mathcal{S}\) is Schwartz’s space of functions whose derivatives are rapidly decreasing.
functions. This has been followed for example in Ref. \[ \ldots \] so that the image of the latter will have to be interpreted in terms of generalized in Definition $$\xi$$ see \[ \ldots \] marks of GSDW coefficients. Bearing in mind the universality property discussed in Remarks \[ \ldots \] the lemma of Riemann–Lebesgue $$L$$ is an endomorphism in the space $$L$$ in Definition \[ \ldots \] that they belong to $$C^\infty$$ is thus also proved. □

Consequently, to analyze the form factors we can employ all the machinery of the theory of pseudo-differential operators. In particular, it is instructive to analyze under what circumstances the integrals in Eq. \[ \ldots \] are convergent. It is known that if the symbol belongs to the class $$S^0$$, then the associated pseudo-differential operator is an endomorphism in the space $$L^2(\mathbb{R}^n)$$ of Lebesgue square-integrable functions, see [62, Theorem 18.1.11]. Although in the present case the log $$k^2$$ behaviour for large $$k$$ prevents us from using this result, we can envisage two alternatives.

The first one is to stick to $$L^2(\mathbb{R}^n)$$ as the domain on which the form factors act, so that the image of the latter will have to be interpreted in terms of generalized functions. This has been followed for example in Ref. [72].

The remaining option is to restrict further the domain, in order to obtain a square-integrable function after applying the form factors [71]. If we follow this possibility, we can subtract to the form factors a term log($$1 + k^2$$) with an appropriate coefficient, obtaining thus a symbol in the class $$S^0$$. The singular contribution is then given by a factor log($$1 + k^2$$). As an example, in $$n = 1$$ a sufficient, albeit not necessary set of conditions on $$\sigma^2$$ that imply $$C^{(1)}_{\Sigma} \sigma^2, C^{(2)}_{M} \sigma^2 \in L^2(\mathbb{R})$$ can be obtained using the lemma of Riemann–Lebesgue:\[ \ldots \] if $$\sigma^2 \in C^2(\mathbb{R})$$ and $$\sigma^2, (\sigma^2)', (\sigma^2)'' \in L^1(\mathbb{R})$$, then $$C^{(1)}_{\Sigma} \sigma^2, C^{(2)}_{M} \sigma^2 \in L^2(\mathbb{R})$$ as desired.

6. Conclusions and outlook

From the mathematical point of view, we have determined an infinite number of GSDW coefficients. Bearing in mind the universality property discussed in Remarks 1 and 2, these results may be an useful guide in more involved computations.

As a next step it will be interesting to consider small curvature corrections to our problem. One could think for example in curving the plates, i.e. generating an extrinsic curvature on $$\Sigma$$, or studying totally geodesic plates in a curved manifold. These problems may be analyzed either by expanding in powers of the curvature contributions or considering curved configurations in which form factors may be obtained to all order in the curvatures. In the latter case, homogeneous spaces are probable the most natural candidate.

From the physical point of view we have analyzed, as far as we know for the first time in the literature, surface form factors of a Yukawa theory and their implications. The most immediate consequence is the emergence of an energy-dependent semitransparent boundary condition satisfied by the background field.

One attractive possibility is to introduce a self-interacting term in the action \[ \ldots \]. If one adds a quartic interaction, i.e. $$\lambda \phi^4/12$$, then Eq. \[ \ldots \] acquires an additional term, which reads $$\lambda \phi^2 \varphi^2$$. The field $$\varphi$$ is the classical field [7] and it is clear that its

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$$^5 C^2(\mathbb{R})$$ is the space of functions of a real variable with continuous derivatives up to second order. $$L^1(\mathbb{R})$$ is the space of absolutely Lebesgue integrable functions of a real variable.
contribution to the effective action can be obtained by replacing \( \sigma^2 \to \sigma^2 + \lambda \varphi^2 \). Doing so, one can readily compare the divergent terms with the counterterms found in Ref. [43]. By the discussion in the present manuscript, the semitransparent boundary condition satisfied by the classical field \( \varphi \) acquires then a dependence with the energy involved in a given physical process. However, a subtle point is that the classical field is expected to be discontinuous; in such case, taking into account the discussion in Sec. 5.2, the action of the form factors on it will have to be interpreted in terms of generalized functions and a detailed analysis will be required.

Natural generalizations include the study of more general physical models and boundary conditions. These lines are currently being explored.

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Appendix A. Form factors in arbitrary dimension

The explicit expressions for the form factors appearing in the effective action of Eq. (5.4), for arbitrary Euclidean spacetime dimensions \( d \), read

\[
F_M^{(1)}(m) := 2^{-d-1} \pi^{-\frac{d}{2}} m^{d-2} \Gamma \left( 1 - \frac{d}{2} \right),
\]

\[
F_M^{(2)}(k^2, m) := -2^{2-2d} \pi^{-\frac{d}{2}} \left( 2 \, \frac{d}{2} \right) \left( k^2 + 4m^2 \right)^{\frac{d}{2} - 2} \times 2F_1 \left( \frac{1}{2}, 2 - \frac{d}{2}, \frac{3}{2} - \frac{d}{2}, k^2 + 4m^2 \right),
\]

\[
F_\Sigma^{(1)}(k, m, \lambda) := -\frac{2^{2-2d} \pi^{i-\frac{d+1}{2}}}{{(k^2 + \lambda^2)}} \left\{ \lambda \Gamma \left( 2 - \frac{d}{2} \right) \left( k^2 + 4m^2 \right)^{\frac{d}{2} - \frac{3}{2}} \right. \\
+ \frac{2k^2 \Gamma \left( 2 - \frac{d}{2} \right) \left( k^2 + 4m^2 \right)^{\frac{d}{2} - 2}}{\sqrt{\pi}} \left[ \Gamma \left( \frac{3}{2} - \frac{d}{2} \right) \right]^{\frac{d+1}{2} - \frac{d}{2}} \left( k^2 + 4m^2 \right)^{\frac{d}{2} - \frac{3}{2}} \\
+ \frac{2\lambda^{d-2} \Gamma \left( 2 - \frac{d}{2} \right) 2F_1 \left( \frac{3-d}{2}, 2 - \frac{d}{2}, \frac{5-d}{2} - 1; \frac{k^2}{4m^2} \right)}{\sqrt{\pi(d-3)}} \right\}.
\]
\[ \pi^{d/2} F_{(2)}^{(2)} (k_1, k_2, m, \lambda) = \left[ 2^{d-2} \Gamma \left( \frac{2}{2} \right) k_1^2 \left( \lambda^2 k_1 + \lambda^2 k_2 + k_1^2 - k_2^2 \right) \right] \]
\[ \times \left( \frac{k_1^2 + 4m^2}{k_1^2} \right)^{d/2} 2 \omega \left( \frac{1}{2}, 2 - d \frac{3}{2}, \frac{1}{2} \Gamma \left( \frac{2}{2} \right) k_1^2 \right) \]
\[ + \frac{4^{d-2} \pi \lambda^2 \Gamma \left( \frac{2}{2} - \frac{d}{2} \right)}{(k_1^2 - k_2^2) k_2 \left( \lambda^2 + k_1^2 \right)^2} \left( k_1^2 + 4m^2 \right)^{d/2} + \{k_1 \leftrightarrow k_2\} \]
\[ = \frac{2^{d-2} \Gamma (2 - \frac{d}{2}) (k_1 + k_2)^2 \left[ (k_1 + k_2)^2 + 4m^2 \right]^{d/2}}{k_1 k_2 \left( \lambda^2 + (k_1 + k_2)^2 \right)} \]
\[ \times \omega \left( \frac{1}{2}, 2 - d \frac{3}{2}, \frac{1}{2} \Gamma \left( \frac{2}{2} \right) (k_1 + k_2)^2 \right) \]
\[ = \frac{2^{-d-1} \lambda^2 (m^2)^{d/2} \Gamma (2 - \frac{d}{2})}{m^4 \left( \lambda^2 + k_1^2 \right) \left( \lambda^2 + k_2^2 \right)} \omega \left( \frac{4}{d} - 1, \frac{5}{d} - \frac{d}{2}, \frac{3}{2} \Gamma \left( \frac{2}{2} \right) (k_1 + k_2)^2 \right) \]
\[ \times \omega \left( \frac{1}{2}, 2 - d \frac{3}{2}, \frac{1}{2} \Gamma \left( \frac{2}{2} \right) (k_1 + k_2)^2 \right) \]
\[ = \frac{2^{d-2} \Gamma (3 - \frac{d}{2}) \lambda^{d-1}}{(5 - d) \left( \lambda^2 + k_1^2 \right) \left( \lambda^2 + k_2^2 \right)} \omega \left( \frac{4}{d} - 1, \frac{5}{d} - \frac{d}{2}, \frac{3}{2} \Gamma \left( \frac{2}{2} \right) (k_1 + k_2)^2 \right) \]
\[ \times \omega \left( \frac{1}{2}, 2 - d \frac{3}{2}, \frac{1}{2} \Gamma \left( \frac{2}{2} \right) (k_1 + k_2)^2 \right) \]
\[ + \frac{2^{d-2} \Gamma (2 - \frac{d}{2}) \lambda^{d-1}}{(3 - d) \left( \lambda^2 + k_1^2 \right) \left( \lambda^2 + k_2^2 \right)} \omega \left( \frac{4}{d} - 1, \frac{5}{d} - \frac{d}{2}, \frac{3}{2} \Gamma \left( \frac{2}{2} \right) (k_1 + k_2)^2 \right) \]
\[ \times \omega \left( \frac{1}{2}, 2 - d \frac{3}{2}, \frac{1}{2} \Gamma \left( \frac{2}{2} \right) (k_1 + k_2)^2 \right) \]
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