Generalised bottom-up holography and walking technicolour

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In extradimensional holographic approaches the flavour symmetry is gauged in the bulk, that is, treated as a local symmetry. Imposing such a local symmetry admits fewer terms coupling the (axial) vectors and (pseudo)scalars than if a global symmetry is imposed. The latter is the case in standard low-energy effective Lagrangians. Here we incorporate these additional, a priori only globally invariant terms into a holographic treatment by means of a Stückelberg completion and alternatively by means of a Legendre transformation. This work was motivated by our investigations concerning dynamical electroweak symmetry breaking by walking technicolour and we apply our findings to these theories.

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I. INTRODUCTION

Low-energy effective Lagrangians are an extensively used tool when it comes to the description of the low-energy phenomenology of strongly interacting theories and chiral symmetry breaking. They are formulated in terms of the degrees of freedom relevant at low scales. The construction principle for said Lagrangians derives from the global symmetries of the physical system. In the context of strongly interacting theories and chiral symmetry breaking these are the flavour symmetries.

The same problem setting is addressed by invoking holographic principles inspired by the AdS/CFT correspondence \cite{adscft}. This correspondence has originally been conjectured for $\mathcal{N} = 4$ supersymmetry, which is an exactly conformal theory. Hence, using related methods for non-conformal theories must be seen as extrapolation. Holographic descriptions of quantum chromodynamics (QCD) yield numbers in good agreement with experiment \cite{qcdholography} although in QCD the scale invariance is broken by quantum effects and finite quark masses. Indeed, holography is currently heavily used as nonperturbative tool in QCD \cite{qcdholography}. The reason why the holographic principle seems to be so successful could be the fact that the strong coupling constant appears to be really a constant below 1 GeV. Simulations on the lattice are also in favour of the existence of an infrared fixed point \cite{latticewalk,latticeflavour}.

We are concerned with a very general framework, but a considerable part of our motivation to investigate these low-energy sectors comes from dynamical electroweak symmetry breaking through technicolour theories \cite{technicolour}. In technicolour models the electroweak symmetry is broken by chiral symmetry breaking among fermions (techniquarks) in an additional strongly interacting sector. (In this respect it is closely related to quantum chromodynamics.) Walking, that means quasi-conformal technicolour models \cite{walking} with techniquarks in higher-dimensional representations \cite{walking} of the technicolour gauge group are consistent with currently available electroweak precision data. They are thus viable candidates for breaking the electroweak symmetry dynamically and will be tested at the Large Hadron Collider (LHC) which is about to become operational. These walking theories feature by construction a coupling constant that is constant over a large range of scales (typically, more than two orders of magnitude). For this reason they meet the criteria for applying holography in a better way than QCD does \cite{walking,walkingreview}. Extra-dimensional holographic frameworks have even been employed to construct technicolour-like models \cite{walking}. Although holography is one of the few tools for treating strongly coupled field theories and despite its success, the obtained results should clearly be taken with a grain of salt. Of late, also lattice simulations for walking technicolour theories are being carried out \cite{latticewalk,latticeflavour}.

The central issue of the present study is the circumstance that the holographic principle involves gauging the flavour symmetry in the bulk. On first sight this turns the flavour symmetry into a local symmetry. Imposing a local symmetry a priori eliminates many terms from the Lagrangian which would be admissible for a global symmetry. We here incorporate the full set of terms admitted by a global symmetry into a description involving a local symmetry by means of a Stückelberg completion, thereby extending previous work on a holographic description of (minimal) walking technicolour \cite{walking}. (In App. A we provide another equivalent way of achieving this which is based on a Legendre transformation to antisymmetric tensor fields \cite{walking}. In the context of effective Lagrangians a standard procedure is described in \cite{walking} and outlined again in App. B.) Understandably, when applied to technicolour the resulting larger number of parameters permits a richer phenomenology with, for example, an inverted mass hierarchy between vector and axial vector states in some areas of the parameter space.

The paper is organised as follows. In Sect. II we explain...
how to incorporate the additional terms allowed with respect to a global symmetry into a holographic treatment by means of a Stickelberg completion. In Sect. 11A we derive the spectrum of spin-one degrees of freedom in the hard-wall model \[2,13\], in Sect. 11B for the soft-wall model \[14\]. Relative to the standard holographic approach the number of parameters is enlarged. In order to increase the predictive power we discuss possible ways to constrain the system. First, in Sect. 11A we revisit the standard holographic approach where we only had one free parameter \[14\]. Second, in Sect. 11B we impose the first Weinberg sum rule on the level of the lowest lying resonances, as was done for the four-dimensional treatment of the corresponding effective Lagrangian in \[11\]. In Sect. 11C we apply our findings to walking technicolour theories—minimal walking technicolour and beyond. Sect. 11D summarises the results. In the appendices we discuss two alternative ways to incorporate the terms allowed by a global symmetry into a description involving a local symmetry. App. A is concerned with a Legendre transformation from spin-one to spin-two fields. App. B discusses another where the symmetry group is first doubled and subsequently broken spontaneously to the original symmetry content.

II. HOLOGRAPHY AND EFFECTIVE LAGRANGIAN

Let us consider an effective theory for a global flavour symmetry \( \mathcal{G} \) that is broken to \( \mathcal{H} \) by chiral symmetry breaking. In the case of technicolour, for techniquarks in non-(pseudo)real representations of the technicolour gauge group this would be the breaking \( SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V \), where \( N_f \) stands for the number of techniflavours. For (pseudo)real representations the unbroken symmetry group \( \mathcal{G} \) is enhanced to \( SU(2N_f)_L \). Assuming the breaking to the maximal diagonal subgroup, it will be broken to \( SO(2N_f)_L [Sp(2N_f)_L] \) for a real [pseudooreal] representation.

We arrange the left- and right-handed fermion fields \((U,D,\ldots)\) of the elementary theory according to,

\[
Q = (U_L, D_L, \ldots, -i\sigma^2 U_R, -i\sigma^2 D_R, \ldots)^T.
\]

From there we define the bilinear,

\[
M = \epsilon_{\alpha\beta}Q^\alpha \otimes Q^\beta,
\]

with contracted spin indices. It contains the scalar and pseudoscalar degrees of freedom,

\[
M = \left[ \frac{1}{2}(\sigma + i\Theta) + \sqrt{2}(i\Pi^a + \tilde{\Pi}^a)X^a \right]E.
\]

\( E \) parametrises the condensate which breaks the flavour symmetry and \( X^a \) are those generators of \( \mathcal{G} \) which do not commute with \( E \). Per definition, the generators \( S^a \) of the residual group \( \mathcal{H} \) commute with \( E \).

\( E S^a E = -S^a E \). (For an explicit realisation of the generators and the matrix \( E \) in the case of minimal walking technicolour see for example Ref. \[11\].) Further, we identify the vectorial degrees of freedom,

\[
A_{\mu}^{\alpha,j} \sim Q_\mu^\alpha \sigma_{\beta\gamma} \bar{Q}^{\beta,j} - \frac{1}{2} \delta^j_\beta Q_\mu^\beta \sigma_{\alpha\gamma} \bar{Q}^{\beta,k},
\]

which populate the entire unbroken group \( \mathcal{G} \), \( A_\mu = A_\mu^a T^a \). With this field content we construct the Lagrangian \[4\],

\[
\mathcal{L} = -\frac{1}{4g_5^2} F_{\mu\nu}^a F_{\kappa\lambda}^a g^{\kappa\lambda} - \frac{1}{2} A_\mu^{ab} A_\mu^{ab} g^{\mu\nu},
\]

leaving out, for the time being, terms independent of the spin-one fields like kinetic, mass, and self-interaction terms for \( M \). \( F_{\mu\nu}^a \) represents the field tensor of the vectors and is defined as \( F_{\mu\nu}^a T^a = F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \), where \( T^a \) stand for the generators of \( \mathcal{G} \). The mass matrix \( m^{ab} \) is symmetric and depends on \( M \). The Lagrangian is to be invariant under simultaneous global transformations,

\[
A_\mu \to U A_\mu U^\dagger,
\]

\[
M \to U M U^T,
\]

where \( U \in \mathcal{G} \). Accordingly, the mass term can contain the following addends \[2\],

\[
\frac{1}{2} A_\mu^{ab} m^{ab} A_\mu^{ab} g^{\mu\nu} = [r_1 \text{tr}(A_\mu A_\nu M M^\dagger) + (7) + r_2 \text{tr}(A_\mu M A_\nu^T M^T) + (7) + s \text{tr}(A_\mu A_\nu) \text{tr}(M M^\dagger)/(2N_f)]g^{\mu\nu}.
\]

In a holographic approach, the flavour symmetry is gauged in the bulk, that is, treated as a gauge symmetry. The action is postulated to be invariant under local inhomogeneous transformations,

\[
A_\mu^a T^a = A_\mu \to U[A_\mu - iU^\dagger(\partial_\mu U)]U^\dagger,
\]

of the vector field (and simultaneous transformations of \( M \)). The kinetic term \( FF \) for the spin-one particles is already invariant under these inhomogeneous transformation. To the contrary, they do not leave invariant the general mass term \[7\]. The only a priori invariant combination is a gauged kinematic term for the (pseudo)scalars \( \sim \text{tr}[(D_\mu M)(D_\nu M^\dagger)]g^{\mu\nu} \), with the covariant derivative \( D_\mu M := \partial_\mu M - iA_\mu M - iMA_\mu^T \). It contains only a special combination of the addends comprised in Eq. \[7\]. (See also Sect. IIII.)

For the more general term, the invariance under inhomogeneous transformations \[8\] can be restored through

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1 For the use in technicolour the electroweak gauge bosons would have to be coupled in on this level as well. We are, however, not considering them here, but they have been in detail in Refs. \[11\].

2 The global symmetries also admit a term \( \sim \text{tr}\{A_\mu[M(i\partial_\nu M^\dagger) - (i\partial_\nu M)M]\}g^{\mu\nu} \), which, however, does not contribute to the following analysis and is therefore omitted.
the introduction of non-Abelian Stückelberg degrees of freedom $\Phi = e^{-i\theta T^a}$. The mass term \((A^\mu)^a m^{ab}(A^\nu)_b g^{\mu\nu}\), \(A^\mu = A^\mu - i\Phi(\partial^\mu \Phi)^\dagger\). The field $\Phi$ transforms like $\Phi \rightarrow U \Phi$ under local transformations and consequently, $A^\mu = U A^\mu U^\dagger$. Also here a radial, that is, Higgs-like degree of freedom together with the field $\Phi$ is related to the field $N$ in Ref. \[11\]. (See also App. \[B\])

Once we proceed with the holographic analysis, we have to decide what to do with the Stückelberg degrees of freedom. In a general gauge they will obey their proper equation of motion which has to be solved simultaneously with the equations of motion for $A_\mu$ and $M$. In unitary gauge we set $\Phi \equiv 1$. In that case, we do not have to keep track of these degrees of freedom, but we cannot enforce additionally the frequently used $A_5 \equiv 0$ gauge. We shall pursue this approach in what follows. Another alternative is to translate from the language of spin-one fields $A_\mu$ to spin-two field $\vec{B}_\mu$, as demonstrated in App. \[A\]. There the Stückelberg fields are automatically absorbed in the spin-two fields.

Subsummarizing, the terms of the five-dimensional action in unitary gauge relevant for the following calculations are,

$$S := \int d^5x \sqrt{g}(\mathcal{L} + \mathcal{L}_M),$$

\((10)\)

where,

$$\mathcal{L}_M := \text{tr}[(\partial_\mu M)(\partial^\mu M)^\dagger]g^{\mu\nu} - m_5^2 \text{tr}(MM^\dagger),$$

\((11)\)

and with $\mathcal{L}$ given by Eq. \[5\]. The metric is anti de Sitter,

$$ds^2 = z^{-2}(-dz^2 + dz^2).$$

\((12)\)

The fifth coordinate, $z$, is interpreted as inverse energy scale. The coupling $g_5$ is fixed through matching to perturbative calculations,

$$g_5^2 = 12\pi^2/d_R,$$

\((13)\)

where $d_R$ stands for the dimension of the technicolour gauge group with respect to which the fermions transform.

According to the action \((10)\) the expectation value $M_0 E$, $E$-constant, of the field $M$ obeys the equation of motion,

$$(z^3 \partial_z z^{-3} \partial_z - z^{-2} m_5^2)M_0 = 0.$$

\((14)\)

The characteristic equation for the dimension $d$ of the scalar operator is obtained from the ansatz $M_0 \sim z^d$,

$$m_5^2 = d(d - 4).$$

\((15)\)

In the quasi-conformal case $d = 2$ and the general solution of \((13)\) can be written as,

$$M_0 = c_1 z^2 + c_W z^2 \ln(z/e),$$

\((16)\)

where $c_1$ and $c_W$ depend on the boundary conditions. In general, in the ultraviolet ($z = e$) $M_0$ is proportional to the techniquark mass matrix. Hence, in the chiral limit, $c_1 = 0$. $c_W$ is treated below.

In four-dimensional momentum ($q$) space the equation of motion for the transverse part $A_5^\perp$ of the spin-one field is found to be,

$$[(z \partial_z z^{-1} \partial_z + q^2)\delta^{ab} - g_5^2 z^{-2} m_5^{ab}]A_5^b = 0.$$

\((17)\)

It differs from the minimal holographic approach \[2, 13, 14, 19\] by the appearance of the mass matrix $m_5^{ab}$. Diagonalisation in flavour space leads to,

$$[(z \partial_z z^{-1} \partial_z + q^2) - g_5^2 z^{-2} M_0^2 \lambda_5^{ab}] W = 0,$$

\((18)\)

where hereinafter $W$ always stands as representative for the vector $V$ and the axial vector $A$ eigensolution, respectively. The corresponding eigenvalues of $m_5^{ab}/M_0$ are given by,

$$\lambda_5^2 = r_1 - r_2 + s,$$

\((19)\)

$$\lambda_A^2 = r_1 + r_2 + s.$$  

\((20)\)

The approximation $M_0 = c_W z^2 \ln(z/e) \approx c_W z^2 \ln(z_m/e)$ allows us to determine analytic expressions for the eigensolutions $W$. The thus introduced error is only slight: In the hard-wall model $z$ lives on the interval $[\epsilon, z_m]$. The approximation deviates most for small values of $z$, of the order of $\epsilon$, and goes to zero for large values of $z$ around $z_m$. In Eq. \((18)\) the term involving $M_0$ is negligible in the ultraviolet. Hence, where the approximation is not that accurate it does not play a role. To the contrary, in the infrared, where it counts, the approximation is accurate. With a soft wall $z$ has no sharp bound, but is still cut off gradually by the potential. As we shall argue, there is no need to specify a value for $z_m$ because different choices amount merely to redifinitions of constants. We thus write,

$$M_0 \approx C z^2 / g_5,$$

\((21)\)

which in the hard-wall model would correspond to $C \approx g_5 c_W \ln(z_m/e)$.

For determining the pion decay constant, we have to solve the differential equation for the axial part \((18)\) with $W \rightarrow A$ with $q^2 = 0$ and for the boundary conditions,

$$\partial_z P(z_m) = 0 \quad \text{and} \quad P(0) = 1.$$  

\((22)\)

$f_\pi$ is obtained from the solution by,

$$g_5^2 f_\pi^2 = -\lim_{\epsilon \rightarrow 0} \frac{i}{\epsilon} \partial_\epsilon P(\epsilon)/\epsilon,$$

\((23)\)

which for $\partial_z P(0) = 0$ turns into $-\partial_z^2 P(0)$. 

A. Hard-wall model

In the so-called hard-wall approach \[2,15\] \(z\) takes values on the interval \([\epsilon, z_m]\). \(z_m\) corresponds to the position of the infrared boundary and \(\epsilon\) to that of the ultraviolet. In the context of quasi-conformal theories which (almost) exhibit a conformal behaviour over a range of scales, the aforementioned interval can be identified with this range \[13\]. In phenomenologically viable technicolour models \(\epsilon \ll z_m\) and we can take the limit \(\epsilon \to 0\), which is smooth. Thus, the boundary conditions for the spin-one fields are given by,

\[ W(0) = 0 = \partial_z W(z_m). \] (24)

For \(W(0) = 0\) the solutions of the equations of motion \[18\] with \(M_0\) from \[21\] are given by,

\[ W \sim z^2 e^{-C_W z^2/2} M(1 - \frac{z^2}{4C_W}, 2, C_W z^2), \] (25)

where \(C_W = C_{\lambda_W}\) and \(M(\ldots)\) represents a Kummer function. The boundary condition at \(z = z_m\) implies,

\[ (2C_W^2 z_m^2 - M^2_{2W}) M(1 - \frac{M^2_{2W}}{4C_W}, 2, C_W z_m^2) + (4C_W + M^2_{2W}) M(-\frac{M^2_{2W}}{4C_W}, 2, C_W z_m^2) = 0. \] (26)

It yields the values for the masses for the vectorial and axial states, but can only be solved numerically. Imposing normalisation conditions on the solutions,

\[ \int_0^{z_m} \frac{dz}{z} W^2 = 1, \] (27)

the decay constants can be extracted according to,

\[ g_5 F_W = \partial_z^2 W(0), \] (28)

which also requires numerical calculations.

The solution for \(P\) which obeys the boundary conditions \[22\] reads,

\[ P = \cosh(C_A z^2/2) - \tanh(C_A z_m^2/2) \sinh(C_A z^2/2), \] (29)

and extracting the pion decay constant \(f_\pi\) according to Eq. \[23\] yields,

\[ g_5^2 f_\pi^2 = C_A \tanh(C_A z_m^2/2). \] (30)

B. Soft-wall model

The soft-wall model \[19\] is based on the incorporation of an additional dilaton field \(\phi\), such that the action becomes,

\[ S_s := \int d^5x \sqrt{g} e^{-\phi} (\mathcal{L} + \mathcal{L}_M). \] (31)

From the requirement that the mass spectrum show Regge behaviour, that is a linear spacing of the squared masses, it follows that the dilaton background has to behave like \(cz^2\), \(c = \text{constant}\), for large \(z\). Instead of the previous potential well, we have now to deal with a harmonic oscillator. The infrared boundary conditions is substituted by the normalisability of the solution on \(\mathbb{R}^+\).

First we have to determine the behaviour of the condensate \(M_{0,s}\) in the new setting. It satisfies the equation of motion,

\[ (z^3 e^{cz^2} \partial_z^2 - z^{-3} e^{-cz^2} \partial_z - z^{-2} M_{s}^2) M_{0,s} = 0. \] (32)

The characteristic equation \[15\] holds approximately for \(e^{z^2} \ll M_{s}^2\). Hence, by means of an identification of the ultraviolet we can once more set \(M_{s}^2 = -4\). The previous differential equation has the solution,

\[ g_5 M_{0,s} = C z^2 e^{cz^2}. \] (33)

\(M_{0,s}\) grows exponentially with \(z\). This hints towards the presence of an instability which must ultimately be cured by non-linear terms in the differential equation \[19\]. They would come from potential terms for the (pseudo)scalars not addressed here. Therefore, the above solution is appropriate for small values of \(z\) and we can set

\[ g_5 M_{0,s} \rightarrow C z^2. \] (34)

On one hand, one can interpret the last step as practical way of regularising the aforementioned instability. There are actually nonlinear terms which would modify the equation of motion precisely in such a way that Eq. \[33\] is an exact solution. On the other hand, one can see the previous step as one introducing an \(O(z^2)\) approximation. For that latter case, the effect of the thus introduced error will be assessed below.

The equations of motion for the spin-one fields read,

\[ [(z e^{cz^2} \partial_z - z^{-1} e^{-cz^2} \partial_z + q^2) - g_5^2 z^{-2} M_{s}^2 M_{s}^2] W_s = 0. \] (35)

With the substitution,

\[ W_s =: \tilde{W}_s e^{cz^2/2}, \] (36)

we find

\[ [(z \partial_z - z^{-1} \partial_z + q^2) - g_5^2 z^{-2} M_{s}^2 M_{s}^2] \tilde{W}_s = 0. \] (37)

Taking into account the ultraviolet boundary condition right away, for \(M_{0,s}\) from \[34\] this equation is solved by

\[ \tilde{W}_s \sim z^2 e^{-c_W z^2/2} M(1 - \frac{M^2_{2W}}{4c_W}, 2, c_W z^2), \] (38)

where \(c_W^2 := e^2 + C_W^2\). This function is normalisable only if the Kummer function truncates into a polynomial, which it does for \(M^2_{2W} = 4 n c_W\), \(n \in \mathbb{N}\). For \(n = 1\) the normalisation condition,

\[ \int_0^\infty \frac{dz}{z} \tilde{W}^2 = 1, \] (39)
selects the unique solution
\[ W = \sqrt{2} c_W z^2. \]  
(40)
This leads to the decay constant,
\[ g_5 F_W = \partial_z^2 W(0) = 2 \sqrt{2} c_W = \frac{M_\Lambda}{\sqrt{2}}. \]  
(41)
In order to determine the pion decay constant \( f_\pi \) we have to solve the differential equation,
\[ [ze^{cz^2} \partial_z z^{-1} e^{-cz^2} \partial_z - g_5^2 z^{-2} M_\Lambda^2 c^2] P_s = 0, \]  
(42)
which after the substitution
\[ P_s =: \tilde{P}_s e^{cz^2/2}, \]
becomes
\[ [z \partial_z z^{-1} \partial_z - g_5^2 z^{-2} M_\Lambda^2 - c^2 z^2] \tilde{P}_s = 0. \]  
(43)
With \( M_0, s \) from (34) and for the boundary conditions (22) this equation has the solution,
\[ \tilde{P}_s = e^{-c_A z^2/2}. \]  
(45)
Equivalently,
\[ P_s = e^{-(c_A - c)z^2/2}, \]  
(46)
which through the relation (26) yields,
\[ g_5^2 f_\pi = c_A - c. \]  
(47)
Together with \( c > 0 \) and \( M_a^2 = 4 c_A \) this relation leads directly to a lower bound for the mass of the lightest axial vector, \( M_a^2 \geq 4 g_5^2 f_\pi^2 \).

If we interpret Eq. (34) as \( O(cz^2) \) approximation instead of as regularisation we have to assess its range of applicability. To this end we calculate the shift \( \delta q_\pi^2 \) of the masses \( M_\pi^2 \) originating from the difference \( \delta U \) between the exact and the approximate potential. The normalised wave functions were given by
\[ \tilde{W}_s = \sqrt{2} c_W z^2 e^{-c_W z^2/2}, \]  
(48)
and the difference between the potentials,
\[ \delta U = C_W^2 e^{cz^2} (e^{cz^2} - 1). \]  
(49)
This leads to the shift,
\[ \delta q_\pi^2 = \int_0^\infty \frac{dz}{z^2} \tilde{W}_s^2 \delta U = \]
\[ = 2 C_W^2 c_W^2 [(c_W - c)^{-3} - c_W^{-3}]. \]  
(50)
It should be smaller than the actual masses \( M_\pi^2 = 4 c_W \). Hence,
\[ x_W^2 \sqrt{1 + x_W^2} ((1 + x_W^2)^{-3} - (1 + x_W^2)^{-3/2}) \ll 2, \]  
(51)
where \( x_W = C_W/c \). The left-hand side is divergent for small \( x_W \) and approaches zero for large \( x_W \). Hence, we need \( C_W \gg c \). Exploiting the axial part of this statement in combination with Eq. (41) leads to,
\[ x_A = \sqrt{(1 + y)^2 - y^2}/y, \]
(52)
where \( y := c/(2 g_5^2 f_\pi^2) \). Thus, we need large \( x_A \) and hence small \( y \) or \( c \ll g_5^2 f_\pi^2 \). Results from an iterative study of an analogous set of equations in (14) showed that for \( c = g_5^2 f_\pi^2 \) the deviation was still only ten percent. Through Eq. (47) this implies \( M_a^2/(4 g_5^2 f_\pi^2) \ll 2 \).

## III. Constraining the Parameter Space

The incorporation of the mass term (7) increases the number of parameters for our present analysis effectively by one. One could regard the coupling strength of the vectorial eigenstates to the condensate as this parameter. It was equal to zero in the non-generalised approach.

### A. Non-generalised holography

In the standard holographic approach (2, 14) the masses of the spin-one fields arise from the kinetic term of the \( M \) field, \( g^{\mu \nu} \text{tr}[(D_\mu M)(D_\nu M)^\dagger] \). It contains,
\[ g^{\mu \nu} \text{tr}[(A_\mu M + M A_\mu^T)(A_\nu M + M A_\nu^T)^\dagger], \]  
(53)
which corresponds to \( r_1 = 2 = r_2 \) and \( s = 0 \). Hence \( \lambda^2 = 0 \), that is, the vectorial degrees of freedom are not coupled to the condensate. The unconstrained analysis only depends on the ratio of \( r_1 + s \) and \( r_2 \). The overall magnitude of these parameters can be absorbed in a rescaling. Thus, the above three conditions only fix one degree of freedom.

### B. Weinberg sum rules

Another way to fix a degree of freedom is to assume that the first Weinberg sum rule (21)—which holds for asymptotically free gauge field theories—be satisfied already at the level of the lowest resonances (11):

#### 1. Soft-wall model

Let us start the discussion with the oblique \( S \) parameter (21) as calculated from the lowest resonances. (From this point onward, we will use the subscript \( a \) for the lightest axial vector resonance and \( \rho \) for the lightest vector resonance.) It can be interpreted as the zeroth sum rule,
\[ \frac{S}{4 \pi} = \frac{F_\rho^2}{M_\rho^2} - \frac{F_a^2}{M_a^2}. \]  
(54)
Due to Eq. (41) it is exactly equal to zero because,

\[ \frac{F_\pi^2}{M_\rho^2} = \frac{1}{g_\rho^2} = \frac{F_a^2}{M_a^2}. \]  

(55)

From the first Weinberg sum rule,

\[ f_\pi^2 = \frac{F_\rho^2}{M_\rho^2} - \frac{F_a^2}{M_a^2}. \]  

(56)

we find accordingly,

\[ 2 g_\rho^2 f_\pi^2 = M_\rho^2 - M_a^2, \]  

(57)

implying immediately that the vector must be heavier than the axial vector. The lightest vector mass, \( M_\rho^\text{min} = \sqrt{g_\rho f_\pi} \) is realised for the lightest axial mass, \( M_a^\text{min} = 2 g_\rho f_\pi \). According to Eq. (51) the associated minimal decay constant is given by, \( (F_\rho^\text{min})^1/2 = 18^{1/4} g_\rho^{1/2} f_\pi \). (See also Tab. I.)

In quasi-conformal theories the second Weinberg sum rule receives corrections [22],

\[ 8 \pi^2 \alpha f_\pi^4/d_R = F_\rho^2 - F_a^2, \]  

(58)

with the parameter \( \alpha > 0 \). (Otherwise, \( F_\rho^2 - F_a^2 = 0 \).) After imposing the first sum rule we find for this parameter,

\[ \frac{\alpha}{3} = 1 + \frac{M_a^2}{g_\rho^2 f_\pi^2}. \]  

(59)

2. Hard-wall model

In the hard-wall approach the expressions for the various masses and decay constants must be evaluated numerically and we present sum-rule constrained results together with the values for the soft-wall approach in the figures.

IV. WALKING TECHNICOLOUR

Ref. [10] lists viable candidates for dynamical electroweak symmetry breaking by technicolour theories. They can display a sufficiently large amount of walking and are consistent with electroweak precision data. For the present analysis they are characterised by the number of techniflavours \( N_f \) and the dimension \( d_R \) of the representation \( R \) of the gauge group under which the techniquarks transform. The prime candidate, minimal walking technicolour, features two techniquarks in the adjoint representation of \( SU(2) \). The adjoint representation is real and therefore the symmetry breaking pattern is \( SU(4) \rightarrow SO(4) \).

Primarily here, we present results for minimal walking technicolour. Different numbers of flavours or dimensions lead to rescalings of the masses and decay constants.

Relative to minimal walking technicolour the masses are multiplied by \( (3/d_R)^{1/2} \) and \( (2/N_f)^{1/2} \). For this scaling, \( N_f \) is given by the number of techniflavours gauged under the electroweak gauge group. This must be an even number to have complete doublets. It has proven to be advantageous to include more than two techniquarks to be close to conformality, while only gauging two under the electroweak to avoid the bounds from precision data. For these partially gauged [10] technicolour models, the number of gauged flavours is given under \( N_f^a \) in Tab. I. The decay constants of the spin-one mesons have a different scaling behaviour because of the additional factor of \( g_5 \) in Eqs. (25) and (41). Relative to their values in minimal walking technicolour the \( F^{1/2} \) scale like \( (3/d_R)^{1/4} \) and \( (2/N_f)^{1/2} \). In order to give a quantitative idea about this aforementioned scaling the minimal values for the masses in the soft-wall approach and the associated minimal values for the decay constants are given in the last four columns of Tab. I.

Minimal walking technicolour features an enhanced symmetry breaking pattern \( SU(4) \rightarrow SO(4) \) relative to non-(pseudo)real representations and the most basic \( SU(2)_L \times SU(2)_R \rightarrow SU(2)_V \) pattern known from two-flavour QCD. Of the models listed in Tab. I only those with techniquarks in the two-index symmetric representation of either \( SU(3) \) or \( SU(4) \), feature this simple pattern. In that case only mesonic fields are contained in the action [10]. There are exactly three pions which are absorbed to become the longitudinal degrees of freedom of the electroweak gauge bosons.

| \( N_f \) | \( d_R \) | \( N_f^a \) | \( F_\rho^\text{min} \) | \( F_a^\text{min} \) |
|---|---|---|---|---|
| 2 | F | 2 | 7 | 6 | 2.2 | 0.66 | 2.7 | 0.81 |
| 2 | F | 2 | 7 | 2 | 3.8 | 1.15 | 4.6 | 1.41 |
| 2 | G | 3 | 2 | 2 | 3.1 | 1.04 | 3.8 | 1.27 |
| 3 | F | 2 | 2 | 2 | 2.2 | 0.87 | 2.7 | 1.07 |
| 3 | G | 2 | 2 | 2 | 1.9 | 0.81 | 2.3 | 0.99 |
| 4 | F | 2 | 15 | 2 | 2.7 | 0.97 | 3.3 | 1.18 |
| 4 | S | 10 | 2 | 2 | 1.7 | 0.77 | 2.1 | 0.94 |
| 4 | A | 6 | 2 | 2 | 2.2 | 0.87 | 2.7 | 1.07 |
| 4 | G | 15 | 2 | 2 | 1.4 | 0.69 | 1.7 | 0.85 |
| 5 | F | 5 | 19 | 2 | 2.4 | 0.91 | 2.9 | 1.12 |
| 5 | A | 10 | 6 | 2 | 1.7 | 0.77 | 2.1 | 0.94 |
| 6 | F | 6 | 23 | 2 | 2.2 | 0.87 | 2.7 | 1.07 |

TABLE I: Various walking technicolour models from Tab. III in [10]. Minimal (axial) vector meson mass \( M_\rho^\text{min} \) and square root of the minimal (axial) vector decay constant \( (F_\rho^\text{min})^1/2 \) for various walking technicolour models characterised by the representation \( R \) (\( F=\text{fundamental}, G=\text{adjoint}, S=\text{two-index symmetric}, A=\text{two-index antisymmetric} \)) of the technicolour gauge group under which the techniquarks transform and the number of (gauged) techniflavours \( N_f \) \( (N_f^a) \).
The models with techniquarks in the fundamental representation of $SU(N > 2)$ have the symmetry breaking pattern $SU(N_f)_L \times SU(N_f)_R \rightarrow SU(N_f)_V$—always assuming the breaking to the minimal diagonal subgroup—which leads to a larger field content due to the larger number of flavours. The same holds for the model based on the two-index antisymmetric representation of $SU(5)$. The symmetry does not mix left and right fields. Consequently, we have only mesonic states among the (pseudo) scalars and (axial) vectors. There are, however, more than the three pions which are absorbed to become the longitudinal degrees of freedom of the electroweak gauge bosons. The decomposition into vector and axial vector eigenstates is straightforward, $2V = A_L + A_R$ and $2A = A_L - A_R$.

The adjoint representation is always real, and all corresponding models with different numbers of colours feature two flavours, including minimal walking technicolour. Here the symmetry mixes left and right fields. Therefore, fields with non-zero technibaryon number contribute to the action \[ \text{(10)} \] \[ \text{(11)} \] \[ \text{(14)} \] or \[ \text{(16)} \]. The vector mass in the soft-wall model is a monotonously increasing function of the axial vector mass $M_a$. For small axial vector masses $M_a$, the mass $M_\rho$ of the first vector state coincides in the two scenarios, hard- and soft-wall, respectively. Both masses are bounded from below. The axial vector mass is bounded from below due to its relation to the physical value of the pion decay constant $f_\pi$. In the soft wall model,

\[ M_a > 2g_5 f_\pi. \] \[ (60) \]

The results from standard holography have been discussed in great detail in Ref. \[ \text{[14]} \]. We here turn immediately to the results from the analysis constrained by the first Weinberg sum-rule. After imposing this constraint one free parameter remains. We choose it to be the mass $M_\rho$ of the lightest axial state.

The vector mass $M_\rho$ as function of the axial vector mass $M_a$ is shown in Fig. 1. For small axial vector masses $M_a$, the mass $M_\rho$ of the first vector state coincides in the two scenarios, hard- and soft-wall, respectively. Both masses are bounded from below. The axial vector mass is bounded from below due to its relation to the physical value of the pion decay constant $f_\pi$. In the soft wall model,

\[ M_a > 2g_5 f_\pi. \] \[ (60) \]

The vector mass in the soft-wall model is a monotonously rising function of the axial vector mass and, hence, the minimum is reached for the minimal axial-vector mass,

\[ M_\rho > \sqrt{6} g_5 f_\pi. \] \[ (61) \]

In the hard-wall model, one can go slightly below the bounds known analytically for the soft-wall model. In
the hard-wall approach in the low-mass region there are also two values for the vector mass for a given value of the axial vector mass. There is also a small region of values of $M_a$ for which two values of $M_a$ can be found. The two different solutions correspond to different values of the underlying parameters $C_A$ and $z_m$. In the soft-wall model towards larger masses, the vector mass slowly approaches the axial vector mass from above. This is not seen in the hard-wall model.

Fig. 2 displays the decay constants as functions of the mass of the corresponding state. In the soft-wall approach the curves of the vector and the axial vector coincide. The vectorial curve begins at higher values of the mass though. We have always $M_W^2/F_W = \sqrt{2} g_5$. At low masses the hard-wall results coincide with that of the soft wall. In that region the hard infrared bound is so far away from the ultraviolet bound that the states essentially do not feel it. The hard-wall model curve for the vector stays always very close (but slightly below) the soft-wall curve. This is indicative of the fact that the vector couples so strongly to the condensate that it also does (almost) not feel the hard wall. For increasing mass the decay constant of the axial vector state depart to larger values from the soft-wall result and consequently also the hard-wall vectorial result. It is more strongly influenced by the infrared boundary condition.

The contribution to the oblique $S$ parameter from the lowest resonances computed according to Eq. (53) for the hard-wall model is displayed in Fig. 3 as a function of the axial mass $M_a$. At low mass it starts out with small positive values < 0.02 and for increasing mass decreases towards negative values of the order $-$0.1. For the soft-wall model the oblique $S$ parameter is identical to zero. These values are consistent with the experimental values reported in [25].

A reduction of the $S$ parameter in quasi-conformal theories relative to its perturbative value is expected due to the modification of the second Weinberg sum rule [22]. A reduction to negative values, however, has not been anticipated: The parameter $\alpha$ characterising the modification is expected to be of order one which reduces $S$, but does not suffice to change its sign [22]. Here, however, the parameter $\alpha$ grows beyond order unity. (See Fig. 4.) In view of this fact in the hard-wall model, decreasing $S$ is still related to growing $\alpha$. (See Fig. 5.) In the soft-wall model the $\alpha$ parameter grows much more slowly with increasing axial mass $M_a$, but the $S$ parameter is constant and equal to zero.

V. SUMMARY

We have introduced a generalisation of the standard bottom-up holographic method [2, 13, 14, 18, 19]. The generalisation consists of admitting all addends in a mass term for the spin-one states which are invariant under a global symmetry transformation. All these terms would be present in the non-extradimensional treatment of an effective Lagrangian. In the holographic method, however, the flavour symmetry is gauged in the bulk and is thus local. A priori, a local symmetry admits fewer terms. In order to be able to incorporate the additional terms, we carry out a St"uckelberg completion, in the main body of the paper. In the appendix two alternative methods are presented; one based on a Legendre transformation to spin-two fields [17] and one based on a doubling and subsequent spontaneous breaking of the symmetry [11].

After laying out the framework, we determine the spectrum of low-lying spin-one states for the case of quasi-conformal theories in the hard- [2, 18] and the soft-wall approach [19]. In the generalised approach there is effectively one more parameter than in the standard version.

In order to increase predictive power, a way to constrain anew the parameter space is to impose the first Weinberg sum rule on the level of the lowest resonances. The first
Weinberg sum rule holds in general for asymptotically free gauge field theories. That it is (approximately) satisfied on the level of the first resonances is an assumption. We discuss the results for the thus constrained system and apply it to walking technicolour theories. Also and especially there the imposition of sensible constraints is essential: In a generic effective Lagrangian approach it is crucial to link the numerously arising parameters in the effective Lagrangian to parameters in the elementary theory. For QCD one can conveniently rely on experimental data. Data for physics beyond the standard model is much sparser. Here theoretically motivated postulates must be used.

Naturally, when the same four-dimensional effective Lagrangian is treated with or without extradimensions, the results differ in most of the larger parameter space of the non-extradimensional treatment. As a concrete example, in the non-extradimensional treatment presented in [11], one parameter/degree of freedom was eliminated instead by setting the perturbative oblique $S$ parameter in the elementary equal to the $S$ parameter in the effective theory. To the contrary, in the present holographic treatment the $S$ parameter comes as output, with a reduction relative to its perturbative value which is generally expected [22] in the presence of walking dynamics.

There are results from the holographic approach independent of the additional constraints, that is, whether standard holography, the first Weinberg sum rule or no constraint is imposed. On one hand, there is the scaling behaviour of the masses and decay constants with the dimension $d_R$ of the representation of the technicolour gauge group under which the techniquarks transform and the number $N_f$ of techniflavours gauged under the electroweak interactions. The masses scale like $d_R^{-1/2}$ and $N_f^{-1/2}$; the decay constants like $d_R^{-1/4}$ and $N_f^{-1/2}$. On the other hand, the minimal required mass of the axial vector is always linked to the physical value of the pion decay constant $f_\pi$, $M_a > 4\pi[6/(N_f d_R)]^{1/2}\Lambda_{\text{ew}}$. Further, in the soft-wall model the ratio between the squared mass of a spin-one state and its decay constant is always $M^2_{W}/F_{W} = \sqrt{2}g_{5}$.

Other features differ between the differently constrained holographic scenarios. While in standard holography the axial vector mass is always larger than the vector mass, the imposition of the first Weinberg sum rule in the extended scheme leads to an inversion of the mass hierarchy. Without any constraint, the generalised holography can accomodate both orderings of the mass. (After all, standard holography is only a special case of the generalised version.)

In walking theories, an oblique $S$ parameter which is reduced with respect to its perturbative value is expected [22], not, however, one which turns negative. If this feature is enforced it selects the soft-wall model where $S = 0$ or the hard-wall model at small axial-vector masses.

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APPENDIX A: LEGENDRE TRANSFORMATION OF THE SPIN-ONE SECTOR

In this appendix we discuss another way for promoting a global to a local symmetry. It is based on a Legendre transformation from spin-one to spin-two fields which is generally possible for non-Abelian gauge field theories. (See for example [17] and references therein.) Applying the Legendre transformation to the effective theory can be interpreted in two ways. On one hand, we can say that the low-energy theory has already been elevated to a gauge theory by introducing Stückelberg degrees of freedom (see above) which then properly allows to carry out the Legendre transformation. The practical approach, on the other hand, is to carry out the Legendre transformation directly for the effective theory, as one would do for a gauge theory. The outcome is the same and for this reason, the second interpretation can be seen as working prescription for promoting the effective theory to a gauge theory.

1. Gauge theory

As announced just above, we will use a specific Legendre transformation which is generally feasible for a gauge theory to promote an effective theory to a gauge theory. In the following we will always present the Lagrangian

![Graph](image-url)
densities for the sake of brevity, but with the implicit understanding that an action or even a partition function is constructed from them. For the sake of clarity, let us first look at a prototypical massive Yang–Mills gauge theory,

$$\mathcal{L}_1 = -\frac{1}{4g_5^2} F_{\mu\nu}^a F_{\mu\nu}^a g^{\mu\nu} + \frac{m^2}{2} A_{\mu}^a A_{\mu}^a g^{\mu\nu}. \quad (A1)$$

This Lagrangian is not gauge invariant under inhomogeneous transformations $\Phi$ of the vector field. The Lagrangian must therefore again be supplemented by non-Abelian St"uckelberg fields,

$$\mathcal{L}_2 = -\frac{1}{4g_5^2} F_{\mu\nu}^a F_{\kappa\lambda}^a g^{\mu\nu} g^{\kappa\lambda} + m^2 \text{tr}[(D_\mu \Phi)^\dagger (D_\nu \Phi)] g^{\mu\nu}. \quad (A2)$$

An economic way to incorporate an antisymmetric tensor field $B_{\mu\nu}^a$—and finally to carry out the aforesaid Legendre transformation—is to introduce a Gaussian term into the St"uckelberg fields,

$$\mathcal{L}_3 = \left[-\frac{1}{4g_5^2} F_{\mu\nu}^a F_{\kappa\lambda}^a + \frac{2}{3g_5^2} B_{\mu\nu}^a \tilde{B}_{a\kappa\lambda} g^{\mu\nu} g^{\kappa\lambda} + m^2 \text{tr}[(D_\mu \Phi)^\dagger (D_\nu \Phi)] g^{\mu\nu}, \right. \quad (A3)$$

and subsequently to shift the $\tilde{B}_{\mu\nu}^a$, field by a multiple $\frac{1}{g_5^2} F_{\mu\nu}^a$ of the field tensor. This leads to,

$$\mathcal{L}_4 = \left[\frac{g_5^2}{2} \tilde{B}^a_{\mu\nu} \tilde{B}_{a\kappa\lambda} - \frac{1}{2} \tilde{B}_{\mu\nu}^a F_{\kappa\lambda}^a g^{\mu\nu} g^{\kappa\lambda} + m^2 \text{tr}[(D_\mu \Phi)^\dagger (D_\nu \Phi)] g^{\mu\nu}. \right. \quad (A4)$$

The St"uckelberg fields can now be absorbed in the antisymmetric tensor fields: A gauge transformation of the Yang–Mills potential yields,

$$\mathcal{L}_5 = \left[\frac{g_5^2}{2} \text{tr}(\Phi \tilde{B}_{\mu\nu}^a \Phi^\dagger \tilde{B}_{a\kappa\lambda} \Phi^\dagger) - \text{tr}(\Phi \tilde{B}_{\mu\nu}^a \Phi^\dagger F_{\kappa\lambda})\right] \times$$

$$\times g^{\mu\nu} g^{\kappa\lambda} + \frac{m^2}{2} A_{\mu}^a A_{\nu}^a g^{\mu\nu}, \quad (A5)$$

where we made use of $\Phi^\dagger \Phi = 1$. In a partition function the functional integral for $\tilde{B}_{\mu\nu}^a$ runs already over all possible configurations. Integrating independently over all configurations of $\Phi$ merely corresponds to carrying out the former integral several times. Hence, the St"uckelberg fields can be omitted and the corresponding integration factors out from a partition function. (The previous Lagrangian corresponds to a non-Abelian non-linear sigma model, which when quantised is not perturbatively renormalisable. This can be circumvented by introducing a position-dependent dynamical mass, that is a Higgs degree of freedom. The lastmentioned step does not interfere with the previous gauge symmetry arguments.) Therefore, we can continue working with the unitary gauge expression,

$$\mathcal{L}_6 = \left[\frac{g_5^2}{2} \tilde{B}^a_{\mu\nu} \tilde{B}_{a\kappa\lambda} - \frac{1}{2} \tilde{B}_{\mu\nu}^a F_{\kappa\lambda}^a g^{\mu\nu} g^{\kappa\lambda} + \frac{m^2}{2} A_{\mu}^a A_{\nu}^a g^{\mu\nu} \right], \quad (A6)$$

from which we would like to eliminate the gauge potential. As the Lagrangian is of at most second order in the Yang–Mills field, the resulting Lagrangian up to fluctuation terms reads,

$$\mathcal{L}_7 = \frac{g_5^2}{2} \tilde{B}^a_{\mu\nu} \tilde{B}_{a\kappa\lambda} g^{\mu\nu} g^{\kappa\lambda} - \frac{1}{2} \left(\frac{1}{\sqrt{\g}} \partial_\kappa \sqrt{\g} B^{\kappa\rho} \right) (\mathcal{M}^{-1})_{\rho\mu} \left(\frac{1}{\sqrt{\g}} \partial_\lambda \sqrt{\g} B^{\lambda\mu} \right), \quad (A7)$$

where $\mathcal{M}^{bc} = \mathbb{B}^{bc} - m^2 \delta^{bc} g^{\mu\nu}$, the inverse of which exists, in general, for more than two space-time dimensions, and $B^{\mu\nu}_{bc} := f_{abc} B_{\mu\nu}^c$. Quantum fluctuations of the Yang–Mills potential induce a term which on the Lagrangian level is proportional to $\frac{1}{2} \ln \det \mathcal{M}$. Lorentz indices are raised by the metric $g^{\mu\nu}$. $\sqrt{\g}$ stands for the square root of the determinant of the metric. The whole section has entirely been written in terms of the two-form field $\tilde{B}_{\mu\nu}^a$, so that the expressions become outwardly independent of the number of space-time dimensions. This field is the dual of a $(d - 2)$-form field $B^a_{\mu_1...\mu_{d-2}}$.

For more details on these and related issues in the context of gauge field theories as such, the reader is referred to Refs. [17].

2. Effective theory

We now proceed to the effective theory described by the Lagrangian $\mathcal{L}$ for which we have carried out the St"uckelberg completion above. Again, as in the previous subsection, when it comes to the reformulation in terms of antisymmetric tensor fields, the St"uckelberg fields will be absorbed. Thus, repeating those steps leads to the equivalent of Eq. (A7),

$$\mathcal{L}_8 = \frac{g_5^2}{2} \tilde{B}^a_{\mu\nu} \tilde{B}_{a\kappa\lambda} g^{\mu\nu} g^{\kappa\lambda} - \frac{1}{2} \left(\frac{1}{\sqrt{\g}} \partial_\kappa \sqrt{\g} B^{\kappa\rho \mu} \right) (\mathcal{M}^{-1})_{\rho\mu} \left(\frac{1}{\sqrt{\g}} \partial_\lambda \sqrt{\g} B^{\lambda\mu} \right), \quad (A8)$$

where $m^{bc} := \mathbb{B}^{bc} - m^2 \delta^{bc} g^{\mu\nu}$. The term induced by quantum fluctuations is proportional to $\frac{1}{2} \ln \det m$.

As a first application, let us look at the effective potential in $B_{\mu\nu}^a$-field formulation. Notably, as opposed to the standard representation it is manifestly gauge invariant.

$$V_{\text{eff}} = \frac{g_5^2}{4} \tilde{B}^a_{\mu\nu} \tilde{B}_{a\kappa\lambda} g^{\mu\nu} g^{\kappa\lambda} - \frac{1}{2} \ln \det m. \quad (A9)$$

It contains contributions from quantum fluctuations encoded in the determinant term. $\mu$ is a dynamically induced scale which can be interpreted as the inverse volume of a lattice cell. It can be fixed by a renormalisation condition. The characteristic equation for the minimum of the potential reads,

$$g_5^2 \tilde{B}^a_{\mu\nu} = \mu (m^{-1})^{bc} f^{abc}. \quad (A10)$$

If $m^{bc}$ in $m^{bc}_{\mu\nu}$ dominates over $\mathbb{B}^{bc}_{\mu\nu}$, the previous saddle-point condition becomes approximately,

$$[g^{ag} - g_5^2 \mu (m^{-1}) f^{abc} (m^{-1}) f^{dec}] \tilde{B}^a_{\mu\nu} \approx 0. \quad (A11)$$
In general, this equation is solved by $\tilde{B}_{\mu\nu}^a = 0$, in which case the assumption that $m^{bc}$ dominate is trivially justified. Nontrivial solutions arise if the determinant of the expression inside square brackets vanishes. For $B_{\mu\nu}^{bc}$ dominating over $m^{bc}$, the lowest order calculation corresponds to the massless case,

$$g_5^2 \tilde{B}_{\mu\nu} \approx \mu(B^{-1})_{\mu\nu}^{bc} f^{abc}. \quad \text{(A12)}$$

This equation does evidently not admit the solution $\tilde{B}_{\mu\nu}^a = 0$. It has been studied in Euclidean space \cite{24} which requires numerical tools.

**APPENDIX B: LOCAL SYMMETRIES IN EFFECTIVE LAGRANGIANS**

We recapitulate here the recipe for introducing a local symmetry laid out in Ref. \cite{11} and combine it with the holographic treatment. The (pseudo)scalars $M$ transform globally under the representation $R$ of the group $G$. The spin-one field $A_{\mu}^a$ transform under the adjoint representation of a copy $G'$ of the original flavour group $G$, under which it transforms as singlet. The field $M$ is a singlet under the group $G'$. In order to connect the two fields in the two different groups a scalar field $N$ is introduced which transforms under the fundamental representation of $G$ and the antifundamental of $G'$. $A_{\mu}$ is promoted to a gauge field over $G'$ such that the covariant derivative of $N$ reads,

$$D_{\mu}N = \partial_{\mu}N - iN A_{\mu}. \quad \text{(B1)}$$

In order to recover the original symmetry content, $N$ is made to acquire a vacuum expectation value $\langle N\rangle$ such that the semisimple embedding group $G \times G'$ is broken to its diagonal subgroup $G_{\nu}$, which is again a copy of $G$. The corresponding condensate must be much stronger than the one associated to the chiral/electroweak symmetry breaking in the actual theory. The field $P_{\mu}$ defined through,

$$2i\text{tr}(N^\dagger N)P_{\mu} = [(D_{\mu}N)N^\dagger - ND_{\mu}N^\dagger]d_{F}, \quad \text{(B2)}$$

when evaluated on the vacuum expectation value of the field $N$ reproduces $A_{\mu}^a$, $\langle P_{\mu}^a \rangle = A_{\mu}^a$. Finally, the kinetic terms are given by,

$$\mathcal{L}_{\text{kin}} = -\frac{1}{2g_5^2} \text{tr}(F_{\mu\nu}F_{\lambda\kappa}g^{\mu\kappa}g^{\nu\lambda}) + \text{tr}[(D_{\mu}N)(D_{\nu}N)]g^{\mu\nu} + \text{tr}[(\partial_{\mu}M)(\partial_{\nu}M)]g^{\mu\nu}. \quad \text{(B3)}$$

The $P_{\mu}$ and $M$ fields are linked by replacing $A_{\mu}^a$ by $P_{\mu}^a$ in the mass term \cite{7}.

In the holographic treatment we pursued above we had first determined the vacuum expectation value for the field $M$ and then evaluated the action for the spin-one fields on this expectation value. The vacuum expectation value for $N$ can be incorporated in the same way. Hence, the present treatment amounts to evaluating the action for the spin-one fields on the expectation values of the spin-zero fields $M$ and $N$. Per definition, $P_{\mu}^a$ evaluated on the expectation value of $N$ coincides with $A_{\mu}^a$. Hence, all occurrences of $P_{\mu}^a$ can be replaced by $A_{\mu}^a$ and from this point onward we join the analysis laid out in the main body of the paper.

The kinetic term for the $N$ fields in Eq. \text{(B3)} when evaluated on the expectation value of $N$ contributes a hard mass term for the spin-one fields, $\sim \text{tr}(A_{\mu}A_{\mu})g^{\mu\nu}\text{tr}(NN^\dagger)$. As has already been said above, the important point concerning the condensate of $N$ is that it is much stronger than that of $M$. If we put the $N$ condensate proportional to the $M$ condensate the inclusion of the $N$ condensate amounts effectively only to a redefinition of the parameter $s$. As $r_1$ and $r_2$ are completely arbitrary and $s$ appears exclusively in the combination $r_1 + s$, this has no influence on the subsequent analysis. A different condensate function for $N$ can be introduced with the same right and effect as a different dilaton background.

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