How Ramsey theory can be used to solve Harary’s problem for $K_{2,k}$

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Abstract

Harary’s conjecture $r(C_3, G) \leq 2q + 1$ for every isolated-free graph $G$ with $q$ edges was proved independently by Sidorenko and Goddard and Kleitman. In this paper instead of $C_3$ we consider $K_{2,k}$ and seek a sharp upper bound for $r(K_{2,k}, G)$ over all graphs $G$ with $q$ edges. More specifically if $q \geq 2$, we will show that $r(C_4, G) \leq kq + 1$ and that equality holds if $G \cong qK_2$ or $K_3$. Using this we will generalize this result for $r(K_{2,k}, G)$ when $k > 2$. We will also show that for every graph $G$ with $q \geq 2$ edges and with no isolated vertices, $r(C_4, G) \leq 2p + q - 2$ where $p = |V(G)|$ and that equality holds if $G \cong K_3$.

Introduction

At a meeting held at Kent State University in 1980, Harary posed the general problem of determining the relationship between $r(H, G)$ and the sizes (number of edges) of the given graphs. He conjectured that $r(K_3, G) \leq 2q + 1$ for every isolate-free graph $G$ with $q$ edges. This bound is sharp since $r(K_3, T) = 2q + 1$ for any tree $T$ with $q$ edges; also $r(K_3, qK_2) = 2q + 1$. Harary’s conjecture was subsequently proved independently by Sidorenko [11, 12] and by Goddard and Kleitman [6]. More generally, we can take $H$ to be any fixed graph and seek a sharp upper bound for $r(H, G)$ over all graphs $G$ with $q$ edges. In this paper we deal with the case $H = K_{2,k}$ (with $k \geq 2$) and show that these bounds are sharp. We determine all graphs $G$ where the bound is achieved. Prior to this
through private communication it is known that the main Theorem had been proved by Bollabás and Szemerédi, using results in extremal graph theory.

1 An upper bound to Ramsey number \( r(C_4, G) \)

**Theorem 1.** For every graph \( G \) with \(|E(G)| = q \geq 2\) edges and with no isolated vertices, \( r(C_4, G) \leq 2q + 1 \). Equality holds if \( G \cong qK_2 \) or \( K_3 \).

**Proof.** As we know that \( r(C_4, P_3) = 4 \), \( r(C_4, K_3) = 7 \), \( r(C_4, C_4) = 6 \), \( r(C_4, 3K_2) = 7 \), \( r(C_4, K_{1,3}) = 6 \), \( r(C_4, K_2 \cup P_3) = 6 \), \( r(C_4, 2P_3) = 7 \), \( r(C_4, 2K_2) = 5 \), \( r(C_4, K_{1,4}) = 7 \), \( r(C_4, K_2 \cup C_3) = 7 \), \( r(C_4, K_2 \cup K_{1,3}) = 7 \), \( r(C_4, 2K_2 \cup P_3) = 8 \), \( r(C_4, K_{1,3} + e) = 7 \), \( r(C_4, T_3) = 6 \) (here \( T \) represents the tree on 5 vertices containing exactly one vertex of degree 3), \( r(C_4, 4K_2) = 9 \) (see \([2, 3, 10]\)). Thus the result holds for \( q \in \{2, 3, 4\} \), with equality for \( q = 2, 3 \) corresponding to \( G \cong K_3 \) and \( G \cong 3K_2 \). Let \( G \) be a graph with \( q \geq 4 \) edges and no isolated vertices. If \( \Delta(G) = 1 \) then \( G \cong qK_2 \). We have \( r(C_4, qK_2) \leq 2(q - 2) + r(C_4, 2K_2) = 2q + 1 \), and \( r(C_4, qK_2) \geq 2q + 1 \) as a consequence of the two-coloring of \( E(K_{2q}) \) in which \( R \cong K_{1,2q-1} \). Thus \( r(C_4, qK_2) = 2q + 1 \) for \( q > 1 \). Without loss of generality, we can assume \( G \) is connected since \( r(C_4, G_1 \cup G_2) \leq r(C_4, G_1) + r(C_4, G_2) - 1 \).

**Case 1** If \( \Delta(G) = 2 \)

Then \( G \) a path or a cycle. Using known results, \( r(C_4, P_n) \leq r(C_4, C_n) \leq n + 2 \) (see \([5, 6, 8, 9]\)) it follows that \( r(C_4, G) \leq 2q \) for all such graphs \( G \) with \( q \geq 2 \) edges and maximum degree \( \Delta(G) = 2 \).

**Case 2** If \( \Delta(G) \geq 3 \) and \( \delta(G) > 1 \)

Given a vertex \( v \) in \( G \) of degree \( \Delta(G) \), let \( H = G \setminus v \) and let \( N_v \), denote the neighborhood of \( v \) in \( G \). By induction, then for any isolated vertex free graph \( H' \) obtained from \( G \) by removing \( q' \) edges, we get that \( r(C_4, H') \leq 2(q - q') + 1 \).

In the first scenario, suppose that \( (R, B) \) is a two-coloring of \( E(K_{2q}) \) in which there is no red \( C_4 \) and no blue copy of \( G \). We claim that \( \delta(R) \geq 2\Delta(G) - 1 \).
If $w$ is a vertex of with degree $\leq 2\Delta(G) - 2$ in $R$, we may delete this vertex and its neighborhood in $R$ and still have at least $2q - (2\Delta(G) - 1) = 2(q - \Delta(G)) + 1$ vertices. Thus, there is a blue copy of $H$ in the two colored complete graph that remains after $w$ and its neighborhood in $R$ are deleted. In this copy let $X$ denote the vertex set that plays the role of $N_v$. Since $w$ is adjacent to each vertex of $X$ in $B$ there is a blue copy of $G$, so the claim that $\delta(R) \geq 2\Delta(G) - 1$ is justified.

In the next scenario, since $\Delta(G) \geq 3$, we have $2\Delta(G) - 1 \geq \Delta(G) + 1$, so $\delta(R) \geq \Delta(G) + 1$. Delete an arbitrary vertex $w$ and exactly $\Delta(G) + 1$ of its neighbors in $R$. Let $Y$ denote the set of $\Delta(G) + 1$ neighbors chosen for deletion. Since $\Delta(G) \geq 3$, the complete graph that remains has at least $2q - (\Delta(G) + 2) \geq 2(q - \Delta(G)) + 1$ vertices, so it must contain a blue copy of $H$. As before, let $X$ be the set that plays the role of $N_v$. Consider the edges between $Y$ and $X$. Since there is no blue copy of $G$, each vertex in $Y$ is adjacent in $R$ to at least one vertex of $X$. Since $|X| = \Delta(G)$ and $|Y| = \Delta(G) + 1$, there must be a vertex $x \in X$ adjacent in $R$ to two or more vertices of $Y$.

\[
\text{deg}_R(w) \leq 2\Delta(G) - 2
\]
Then $x$, $w$ and the two appropriate vertices of $Y$ yield a red $C_4$, a contradiction.

**Case 3** If $\Delta(G) \geq 3$ and $\delta(G) = 1$

It should be also noted that in the case $\Delta(G) \geq 3$ and $\delta(G) = 1$ works though unless $G \setminus v$ has isolated vertices. However, connectivity of $G$ together with the inequality $r(C_4, T_{q+1}) \leq \max\{4; q + 2; r(C_4, K_{1,q})\} \leq 2q + 1$ (see [1, 4]) gives that $G$ is a connected graph which is not a tree.

Therefore, we are left with the case when $G$ has a vertex $v$ of degree $\Delta(G)$ such that $v$ is adjacent to $s$ pendent vertices of $G$, where $1 \leq s \leq q - 3$. Let $H$ be the graph obtained by removing the $s$ pendent vertices from $G$.

Suppose that $(R, B)$ is a two-coloring of $E(K_{2q})$ in which there is no red $C_4$ and no blue copy of $G$. We claim that that $\Delta(R) \leq s$. If $w$ is a vertex of degree $\geq s + 1$, let $Y$ consist of any $s + 1$ red neighbors of $w$. Define $X = Y \cup \{w\}$. By induction hypothesis, there is a blue copy of $H$ in the two colored complete graph that remains after the vertices of $Y$ are deleted. In this copy let $x$ denote the vertex set that plays the role of $N_v$. As before then $H$ can be extended to a blue $G$ as in order to avoid a red $C_4$, $s$ vertices of $Y$ will be forced to be adjacent to $x$ in blue. Therefore, $\Delta(R) \leq s$. Let $w$ be a vertex with degree $\Delta(R) \leq s$. Let $Y$ be a set containing $N_R(w)$ along with $2s - 1 - \Delta(R)$ other vertices distinct from $w$. Define $X = Y \cup \{w\}$.

\[
\text{deg}_B(w) = 2s - 1 - \Delta(R) \geq s - 1
\]

\[
\text{deg}_R(w) = \Delta(R) \leq s
\]

In the blue copy of $H$, $x$ play the role of $v \in G$.

**Figure 3: If $w$ is a vertex of red degree at least two**

By induction hypothesis, there is a blue copy of $H$ in the two colored complete graph that remains after the vertices of $Y$ are deleted. If $s > 1$ by the above argument, $x$ can be adjacent in blue to at most $s - 2$ vertices of $Y$ and can be adjacent in red to at most $s$ vertices in $Y$ (as $\Delta(R) \leq s$). A contradiction as $|Y| = 2s - 1$. If $s = 1$ as $(w, x)$ is blue, we get a copy of $G$, a contradiction.
Thus $r(C_4, G) \leq 2q$ for every graph $G$ with $q$ edges and with no isolated vertices other than $2K_2$, and the proof is complete.

2 An upper bound to Ramsey number $r(C_4, G)$ if $G$ is connected.

Theorem 2. For every isolated vertex free graph $G$ with $q \geq 2$ edges and $p \geq 3$ vertices, $r(C_4, G) \leq 2p + q - 2$. Equality holds if $G \cong K_3$.

Proof. It is easy to verify the theorem for $q \leq 4$ by the results of the previous section. If $\Delta(G) \leq 2$ then $G$ is a path or a cycle, and using known results, again it is easy to verify that the theorem is true in this case. So it suffices to show $r(C_4, G) \leq q + 2p - 3$, if $q \geq 5$.

Now assume $\Delta(G) \geq 3$. Given a vertex $v$ in $G$ of degree $\Delta(G)$, let $H = G \setminus v$ and let $N_v$, denote the neighborhood of $v$ in $G$. First assume that $H$ has no isolated vertices. By induction, then for any isolated vertex free graph $H'$ obtained from $G$ by removing $q'$ edges (and the $p'$ corresponding vertices), we get that $r(C_4, H') \leq (q - q') + 2(p - p') - 3$.

In particular, $r(C_4, H) \leq (q - \Delta(G)) + 2(p - 1) - 3$. Suppose that $(R, B)$ is a two-coloring of $E(K_{q+2p-3})$ in which there is no red $C_4$ and no blue copy of $G$. then using a similar argument as in the last proof we would get a contradiction for all possible cases expect the third case when $\Delta(G) \geq 3$ and $\delta(G) = 1$.

In this case let $v \in G$ represent vertex of degree 1 in $G$, and $N_v$ consist of $v_1$. Clearly, $|\delta(R)| \geq 1$. First assume that, $|\Delta(R)| = 1$. Let $u \in K_{2q}$ be a vertex with degree 1 and suppose it is adjacent to $w$ in red. Then the graph obtained by removing $u$ and $w$ from $R$ (say $K \cong K_{2(q-1)}$) by induction hypothesis will have a blue copy of $G \setminus v$ in it. As before let $X = \{x\}$ be the set that play the role of $N_v$. As $u$ is adjacent to all vertices of $K$ we will get a blue copy of $G$ in $K_{2q}$. Therefore, we may assume that $|\Delta(R)| > 1$. Let $u \in K_{2q}$ be a vertex with at least two neighbors in red say $w_1$ and $w_2$. Let $Y = \{w_1, w_2\}$. Then as before the complete graph obtained by removing $Y \cup \{u\}$ will contain a blue copy of $G \setminus v$ as illustrated in the following figure(since $H$ has $p - 1$ vertices and $q - 1$ edges).

Since $|N_v| = 1$, $|Y| = 2$ and there is no blue copy of $G$, the vertex $x$ must be adjacent in $R$ to the vertices $w_1$ and $w_2$. But then $uw_1xw_2x$ will yield a red $C_4$, a contradiction.
Thus, \( r(C_4, G) \leq q + 2p - 3 \) for every graph \( G \) with \( q \) edges and with no isolated vertices other than \( K_3 \), and the proof is complete.

### 3 An Upper Bound for the Ramsey Number \( r(K_{2,k}, G) \) where \( k \geq 2 \)

**Lemma 3.** \( r(K_{2,k}, G) \leq kq + 1 \) if \( G \) is a path, star or triangle.

**Proof.** We showed before that \( r(K_{2,2}, K_{1,q}) = r(C_4, K_{1,q}) \leq 2q + 1 \) if \( q \geq 2 \). Also [7] gives \( r(K_{2,k}, K_{1,2}) = r(K_{2,k}, P_3) \leq 2k + 1 \). And thus we can conclude the result is true for stars as \( r(K_{2,k}, K_{1,q}) \leq r(B_k, K_{1,q}) \leq \max\{2q + 1, \lfloor \frac{2}{3}(k + q - 1) \rfloor + 1\} \leq kq + 1 \) for \( k \geq 3 \) and \( q \geq 3 \). For paths and triangles the result follows directly from \( r(K_{2,k}, P_{q+1}) \leq k + q + 1 \) [7] and \( r(B_k, B_{2}) \leq 2k + 3 \) [13] respectively.

**Lemma 4.** For every isolated vertex free graph \( G \) with \( q \geq 2 \) edges, \( r(K_{2,k}, G) \leq kq + 1 \) if \( k \geq 2 \).

**Proof.** We will use induction on \( q \). The result is true for \( k = 2 \) or \( q \in \{2, 3\} \). Also without loss of generality \( G \) is connected and the result is true if \( G \) is a path, star or triangle (these follow from previous lemmas). Thus we can restrict our attention to \( G \) such that \( G \) is not a path, star or triangle satisfying \( \Delta(G) > 1 \). Using the argument in Theorem 1:case 3, we may assume that \( \delta > 1 \) by considering the two cases, \( \Delta(R) \leq s + k - 2 \) and \( \Delta(R) > s + k - 2 \). Thus, given a vertex in \( G \) of degree \( \Delta(G) \), let \( H = G \setminus v \) and let \( N_v \) denote the neighborhood of \( v \) in \( G \). Then \( H \) has \( q - \Delta \) edges. Suppose that \((R,B)\) is a two-coloring of \( E(K_{kq+1}) \) in which there is no red \( K_{2,k} \) and no blue copy of \( G \). We claim that that \( \delta(R) \geq (k - 1)\Delta + 1 \). If \( w \) is a vertex of with degree \( \leq (k - 1)\Delta \) in \( R \), we may delete this vertex and its neighborhood in \( R \) and still have at
least $kq + 1 - ((k - 1)\Delta + 1) \geq \max\{k(q - \Delta) + 1, p - 1\}$ vertices (as $G$ is not a star). Thus there is a blue copy of $H$ in the two-colored complete graph that remains after $w$ and its neighborhood in $R$ are deleted. In this copy let $X$ denote the vertex set that plays the role of $N_v$. Since $w$ is adjacent to each vertex of $X$ in $B$ there is a blue copy of $G$, so the claim that $\delta(R) \geq (k - 1)\Delta + 1$ is justified. Delete an arbitrary vertex $w$ and exactly $(k - 1)\Delta + 1$ of its neighbors in $R$. Let $Y$ denote the set of $(k - 1)\Delta + 1$ neighbors chosen for deletion. As before the complete graph that remains has at least $\max\{k(q - \Delta) + 1, p - 1\}$ vertices, so it must contain a blue copy of $H$. As before, let $X$ be the set that plays the role of $N_v$. Consider the edges between $Y$ and $X$. Since there is no blue copy of $G$, each vertex in $Y$ is adjacent in $R$ to at least one vertex of $X$. Since $|X| = \Delta$ and there must be a vertex $x \in X$ adjacent in $R$ to $k$ or more vertices of $Y$. Then $x$ and the $k$ appropriate vertices of $Y$ yield a red $K_{2,k}$, a contradiction. □

**Theorem 5.** For every graph $G$ with no isolated vertices, $r(K_{2,k}, G) \leq kq + 2$ if $k \geq 3$ and equality holds if $G \cong K_2$.

**Proof.** This result follows from the previous lemma together $r(K_{2,k}, K_2) = k + 2$. □

**References**

[1] E.T. Baskoro, Hasmawati and H. Assiyatun (2006). The Ramsey Numbers for Disjoint Unions of Trees, *Discrete Mathematics*, **306**: 3297-3301.

[2] M. Clancy, Some Small Ramsey Numbers (1977). *Journal of Graph Theory*, **1**: 89-91.

[3] V. Chvátal and F. Harary (1972). Generalized Ramsey theory for graphs, III. Small off-diagonal numbers, *Pacific Journal of Mathematics*, **41-2**:335-345.

[4] P. Erdős, R.J. Faudree, C. C. Rousseau and R. H. Schelp (1987). Ramsey theory and bipartite graph-tree Ramsey numbers, *Discrete Mathematics*, **72**: 103-112.

[5] R.J. Faudree and R.H. Schelp (1974). All Ramsey Numbers for Cycles in Graphs, *Discrete Mathematics*, **8**: 313-329.

[6] W. Goddard and J. Kleitman (1994). An upper bound of the Ramsey number $r(K_3, G)$ depending only on the size of the Graph $G$. *Discrete Mathematics*, **125**: 177-182.

[7] R. Häggkvist (1989). On the path-complete bipartite Ramsey numbers, *Discete Mathematics*, **75**: 243-245.

[8] G. Kafolyi and V. Rosta (2001). Generalized and Geometric Ramsey Numbers for Cycles, *Theoretical Computer Science*, **263**: 87-98.

[9] V. Rosta (1973). On a Ramsey Type Problem of J.A. Bondy and Erdős, I and II, *Journal of Combinatorial Theory, Series B*, **15**: 94-120.

[10] Radziszowski S.P.(2014). Small Ramsey Numbers. *Electronic Journal of Combinatorics*, (rev 14), DS1.

[11] A. F. Sidorenko (1991). An upper bound of the Ramsey number $r(K_3, G)$ depending only on the size of the Graph $G$. *Journal of Graph Theory*, **15**: 15-17.
[12] A. F. Sidorenko (1993). The Ramsey number of an N-edge graph vs. Triangle is at most \( 2N + 1 \), *Journal of Combinatorial Theory Series B*, **58**: 185-196.

[13] C. C. Rousseau and J. Sheehan (1978). On Ramsey number for books, *Journal of Graph Theory*, **2**: 77-87.