MEASURE AND INTEGRATION

On barycenters of probability measures

by

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Summary. A characterization is presented of barycenters of the Radon probability measures supported on a closed convex subset of a given space. A case of particular interest is studied, where the underlying space is itself the space of finite signed Radon measures on a metric compact and where the corresponding support is the convex set of probability measures. For locally compact spaces, a simple characterization is obtained in terms of the relative interior.

1. The main goal of the present note is to characterize the barycenters of the Radon probability measures supported on a closed convex set. Let $X$ be a Fréchet space. Without loss of generality, the topology on $X$ is generated by the translation-invariant metric $\rho$ on $X$ (for details see [2]).

We denote the set of Radon probability measures on $X$ by $\mathcal{P}(X)$. The barycenter $a \in X$ of a measure $\mu \in \mathcal{P}(X)$ is, by definition,

$$(1) \quad a = \int_X x \mu(dx),$$

if the latter integral exists in the weak sense, that is,

$$(2) \quad \Lambda a = \int_X \Lambda x \mu(dx)$$

for all $\Lambda \in X^*$, where $X^*$ is the topological dual of $X$. More details on such integrals can be found in [2, Chapter 3].

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Note that if (1) exists, then
\[
(3) \quad a = \int_{\text{supp} \mu} x \mu(dx),
\]
and, by the Hahn–Banach separation theorem, \( a \in \text{co(supp} \mu) \), where \( \text{co(} \cdot \text{)} \) stands for the convex hull. From now on, we will use the bar over a set to denote its topological closure.

The following theorem gives a characterization of the barycenters of measures from \( \mathcal{P}(X) \).

**Theorem 1.** Let \( M \subset X \) be a non-empty compact convex set, and let \( a \in M \). Then the following statements are equivalent:

(i) There exists \( \mu \in \mathcal{P}(X) \) with \( \text{supp} \mu = M \) and with barycenter \( a \).

(ii) We have
\[
(4) \quad M = \overline{V_a},
\]
where \( V_a = \{ x \in M \mid \exists \alpha > 0 : -\alpha x + (1 + \alpha)a \in M \} \).

**Remark 1.** We note that the condition (4) is non-local and concerns the whole set \( M \).

**Remark 2.** We require \( M \) to be compact in order to ensure the separability of \( M \) and the existence of weak integrals (see, e.g., [2, Theorem 3.27]). If \( X \) is finite-dimensional, the theorem holds without this requirement.

**Proof of Theorem 1.** (a) First, we prove that (i) \( \Rightarrow \) (ii). Let \( c \in M \) and let \( U_\delta(c) \) be the open ball of radius \( \delta > 0 \) centered at \( c \). Because \( M \) is the support of \( \mu \), one has \( \mu(U_\delta(c)) > 0 \). Also, since \( M \) is compact, so is \( \overline{U_\delta(c)} \cap M \), and
\[
(5) \quad c_\delta = \frac{1}{\mu(U_\delta(c))} \int_{U_\delta(c)} x \mu(dx) \in M
\]
is well-defined. It is easy to show that
\[
(6) \quad \lim_{\delta \to +0} c_\delta = c
\]
in the weak topology \( \sigma(X, X^*) \). Indeed, take any \( A \in X^* \). Since \( A \) is continuous, for every \( \varepsilon > 0 \) there exists \( \delta_0 > 0 \) such that \( x \in U_{\delta_0}(c) \) implies \( |A(x) - A(c)| = |A(x - c)| < \varepsilon \). Then, it follows from the definition of the weak integral that
\[
(7) \quad |A(c_\delta - c)| \leq \frac{1}{\mu(U_\delta(c))} \int_{U_\delta(c)} |A(x - c)| \mu(dx) < \varepsilon
\]
whenever \( \delta \in (0, \delta_0) \). This means that \( c_\delta \to c \) in the weak topology as \( \delta \to +0 \).
On barycenters of probability measures

Further, for any $\delta > 0$, either $\mu(U_\delta(c)) = 1$ or $0 < \mu(U_\delta(c)) < 1$. We show that in both cases $c_\delta \in V_a$. Indeed, if $\mu(U_\delta(c)) = 1$, then $c_\delta = a$ and thus $c_\delta \in V_a$. If $0 < \mu(U_\delta(c)) < 1$, set

\[
(8) \quad \tilde{c}_\delta = \frac{1}{\mu(X \setminus U_\delta(c))} \int_{X \setminus U_\delta(c)} x \, \mu(dx) \in M.
\]

Clearly, $\alpha c_\delta + (1 - \alpha)\tilde{c}_\delta \in M$, $\alpha \in [0, 1]$, by convexity. Moreover, $a = \mu(U_\delta(c))c_\delta + (1 - \mu(U_\delta(c)))\tilde{c}_\delta$. Therefore, by a simple geometric argument and by the definition of $V_a$, it is clear that $c_\delta \in V_a$.

Since $X$ is a locally convex space and since $V_a$ is convex, the closures of $V_a$ in the weak and original topologies coincide. Consequently, by passing to the limit $\delta \to +0$, one arrives at

\[
(9) \quad c = \lim_{\delta \to +0} c_\delta \in \overline{V}_a.
\]

This concludes the proof of the claim.

(b) We prove that (ii)$\Rightarrow$(i) by constructing $\mu \in \mathcal{P}(X)$ with support $M$ and barycenter $a$.

Being a metric compact, $M$ is separable, hence there exists $M_0$ such that $\overline{M_0} = M = \overline{V}_a$. Without loss of generality, one can think that $M_0 = \{x_n\}_{n=1}^{\infty} \subset V_a$ and $\overline{x_n}_{n=1}^{\infty} = M$. By the definition of $V_a$, there exist $\{\alpha_n\}_{n=1}^{\infty}$ such that $\alpha_n > 0$ and $-\alpha_n x_n + (1 + \alpha_n) a \in M$.

Let us define the discrete measure

\[
(10) \quad \mu = \sum_{k=1}^{\infty} \frac{1}{2^n} \cdot \frac{\alpha_n \delta_{x_n} + \delta_{-\alpha_n x_n + (1+\alpha_n) a}}{1 + \alpha_n},
\]

where $\delta_x$ is the delta measure at $x$. Clearly, this is a Radon probability measure, and a simple computation shows that its barycenter is $a$. Indeed, for every $\Lambda \in X^*$ one has

\[
(11) \quad \int_X \Lambda x \, \mu(dx) = \sum_{k=1}^{\infty} \frac{1}{2^n} \cdot \frac{\alpha_n A x_n + (-\alpha_n A x_n + (1 + \alpha_n) A a)}{1 + \alpha_n} = \Lambda a.
\]

It remains to prove that $\text{supp} \, \mu = M$. First, we note that $\{x_n\}_{n=1}^{\infty} \subset \text{supp} \, \mu$. Consequently, $M = \overline{x_n}_{n=1}^{\infty} \subset \text{supp} \, \mu$, and therefore $M \subset \text{supp} \, \mu$. By the definition $\textbf{(10)}$ one also has $\text{supp} \, \mu \subset M$, which concludes the proof. \hfill $\blacksquare$

Further, we will use the following standard notation from convex analysis. For $a, b \in X$ we define the (line) segment $[a, b]$ and the open (line) segment $(a, b)$ to be

\[
(12) \quad [a, b] = \{x \in X \mid x = (1 - \lambda) a + \lambda b, \, \lambda \in [0, 1]\}, \quad (a, b) = \{x \in X \mid x = (1 - \lambda) a + \lambda b, \, \lambda \in (0, 1)\},
\]
Let us recall that the relative interior of a set $M$ is
\begin{equation}
\text{relint}(M) = \{ x \in M \mid \exists U(x) : U(x) \cap \text{aff}(M) \subset M \},
\end{equation}
where $U(x)$ is an open neighborhood of $x$ and $\text{aff}(M)$ is the affine hull of $M$. Also, we recall that the relative algebraic interior of $M$ is the set
\begin{equation}
\text{core}(M) = \{ x \in M \mid \forall y \in \text{aff}(M) \exists \alpha > 0 : [x, -\alpha y + (1+\alpha)x] \subset M \}.
\end{equation}

It is well-known that any locally compact topological vector space is finite-dimensional (see, e.g., [2]), in which case the following corollary holds.

**Corollary 1.1.** If $X$ is a locally compact space and $M \subset X$ is a non-empty closed convex set, then the set of barycenters of the Borel probability measures with support $M$ coincides with the relative interior of $M$.

**Proof.** We note that in finite-dimensional spaces any probability Borel measure is Radon. It is also well-known (see [3]) that in such spaces the relative interior and the relative algebraic interior of $M$ coincide and are non-empty.

Now, let $a \in \text{relint}(M) = \text{core}(M)$ be any point. By the definition of the relative algebraic interior, for every $y \in M \subset \text{aff}(M)$, the segment $[y, a]$ can be prolonged beyond the point $a$ within $M$. This means that $y \in V_a$, and thus $M \subset V_a$. Hence, by Theorem 1 (see also Remark 2), there exists $\mu \in \mathcal{P}(X)$ with $\text{supp} \mu = M$ and with barycenter $a$.

It remains to prove that if for some $a \in M$ one has $\overline{V_a} = M$, then $a \in \text{relint}(M)$. Notice that $V_a$ is a non-empty convex set. Since we are dealing with a finite-dimensional space, $V_a$ has a non-empty relative interior, and $\text{relint}(V_a) = \text{relint}(\overline{V_a}) = \text{relint}(M)$. Let $x \in \text{relint}(V_a) \subset V_a$. It follows from the definition of $V_a$ that there exists a segment $[x, y] \subset M$ such that $a \in (x, y)$. Since $x$ also belongs to $\text{relint}(M)$, there exists an open neighborhood $U(x)$ of $x$ such that $U(x) \cap \text{aff}(M) \subset M$.

By convexity of $M$, one obtains
\begin{equation}
(1-\lambda)(U(x) \cap \text{aff}(M)) + \lambda y \subset M, \quad \lambda \in [0, 1].
\end{equation}

It is also easy to verify directly that
\begin{equation}
(1-\lambda)(U(x) \cap \text{aff}(M)) + \lambda y = ((1-\lambda)U(x) + \lambda y) \cap \text{aff}(M), \quad \lambda \in [0, 1].
\end{equation}

Combining (15) and (16), and noticing that for $\lambda \in [0, 1)$ the set $(1-\lambda)U(x) + \lambda y$ is an open neighborhood of $(1-\lambda)x + \lambda y$, one sees that any point of $(x, y)$ belongs to $\text{relint}(M)$ by the very definition (13) of the relative interior. In particular, this means that $a \in \text{relint}(M)$.

**2.** It is tempting to think that Corollary 1.1 holds in infinite-dimensional spaces, too. Unfortunately, this is not the case even for Hilbert spaces, as the following counterexample shows.
Let $X$ be the Hilbert space of real sequences endowed with the $l^2$-scalar product, and let $M$ be the Hilbert cube, a compact convex set,

\begin{equation}
M = \prod_{k=1}^{\infty} \left[ -\frac{1}{k}, \frac{1}{k} \right].
\end{equation}

We take $a = \{a_k\}_{k=1}^{\infty} \in M$, where $a_k = \frac{1}{k+1}$. It is easy to construct a measure $\mu_k \in P(\mathbb{R})$ with $\text{supp} \, \mu_k = [-1/k, 1/k]$ such that

\begin{equation}
\frac{1}{k+1} = \int_{[-1/k, 1/k]} x \, \mu_k(dx).
\end{equation}

Having done that, consider the product of these measures restricted to $X$,

\begin{equation}
\mu = \bigotimes_{k=1}^{\infty} \mu_k \big|_X.
\end{equation}

One usually defines the product of measures on the product of spaces, having in mind the product topology. Even though the corresponding induced topology on $X$ is strictly coarser than the $l_2$-norm topology, they both generate the same Borel sigma-algebra on $X$. Thus, it is clear that $\mu$ can be seen as a Borel measure on the Hilbert space $X$. Moreover, since $X$ is a complete and separable metric space, $\mu$ is Radon.

It is clear by construction that $\text{supp} \, \mu \subset M$. We prove the other inclusion by reductio ad absurdum.

Let $b \in M$, and suppose that $\mu(U_{\varepsilon}(b)) = 0$ for some $\varepsilon > 0$, where $U_{\varepsilon}(b)$ is the ball of radius $\varepsilon$ centered at $b$. Choose $N$ such that

\begin{equation}
\sum_{n>N} \frac{4}{n^2} < \frac{\varepsilon^2}{2}.
\end{equation}

Then

\[
0 = \mu\left\{ x \in X \left| \sum_{n=1}^{\infty} (x_n - b_n)^2 < \varepsilon^2 \right\} \geq \mu\left\{ x \in M \left| \sum_{n=1}^{N} (x_n - b_n)^2 < \varepsilon^2/2 \right\} \right.
\]

\[
= \bigotimes_{k=1}^{N} \mu_k\left\{ x \in M \left| \sum_{n=1}^{N} (x_n - b_n)^2 < \varepsilon^2/2 \right\} \right. \bigotimes_{k=N+1}^{\infty} \mu_k\left\{ x \in M \left| \sum_{n=N+1}^{\infty} (x_n - b_n)^2 < \varepsilon^2/2 \right\} \right.
\]

The latter is positive, which gives a contradiction and yields $\text{supp} \, \mu = M$.

Now, we prove that $a$ is the barycenter of $\mu$. Thanks to the Riesz representation theorem, there exists $\{\lambda_k\}_{k=1}^{\infty} \in X$ such that for every $x = \{x_k\}_{k=1}^{\infty} \in X$ one has

\begin{equation}
Ax = \sum_{k=1}^{\infty} \lambda_k x_k.
\end{equation}
By the definition of the barycenter we write
\[ \int_X \Lambda x \mu(dx) = \int \sum_{k=1}^{\infty} \lambda_k x_k \mu(dx) = \sum_{k=1}^{\infty} \lambda_k \int M \mu(dx) = \sum_{k=1}^{\infty} \lambda_k a_k = \Lambda a, \]
where one can interchange the sum and the integral by dominated convergence since \( M \) is a bounded set in \( X \). This shows that \( a \) is indeed the barycenter of \( \mu \).

Next, we recall that in infinite-dimensional spaces the relative interior and relative algebraic interior do not necessarily coincide (see [3]). However, from (13) and (14) one sees that the former is a subset of the latter. Thus, it is sufficient to show that \( a \) does not belong to the relative algebraic interior of \( M \). We prove this again by contradiction.

Suppose that \( a \in \text{core}(M) \). Then the segment \([0, a]\) can be prolonged beyond \( a \) within \( M \). In other words, there exists \( \alpha > 0 \) such that \((1+\alpha)a \in M\).

The latter is equivalent to
\[ -1/k \leq (1+\alpha)a_k \leq 1/k, \quad k = 1, 2, \ldots. \]
Multiplying by \( k + 1 \) and letting \( k \to \infty \) yield
\[ -1 \leq 1 + \alpha \leq 1, \]
which contradicts \( \alpha > 0 \) and concludes the proof.

3. Now, we describe the set of barycenters of measures on the space of probability measures. Let \( K \) be a metric compact space and \( X = \mathcal{M}(K) \) the space of signed finite Radon measures on \( K \). By the Riesz–Markov theorem, \( X \) can be identified with the topological dual \( C^*(K) \) of the space \( C(K) \) of continuous functions on \( K \). We endow \( C^*(K) \) with the weak-* topology \( \sigma(C^*(K), C(K)) \). Having in mind the canonical embedding \( C(K) \hookrightarrow C^{**}(K) \), one can say that this topology is the weakest topology which makes continuous all the functionals from \( C^{**}(K) \) that correspond to elements of \( C(K) \). This topology is locally convex, as is the corresponding topology \( \tau_w \) on \( X \). The restriction of \( \tau_w \) to the convex set \( M = \mathcal{P}(K) \subset X \) of probability measures on \( K \) produces the usual topology of weak convergence on \( M \) and thus makes this set compact.

The barycenter \( \mu \in X \) of a measure \( \eta \in \mathcal{P}(X) \) is, by definition,
\[ \mu = \int_X \nu \eta(d\nu), \]
if the latter integral exists in the weak sense. That is, since \( (C^*(K))' = C(K) \), where \((\cdot)'\) is the topological dual in the weak-* topology, \( \mu \) is the barycenter of \( \eta \) if and only if for every \( f \in C(K) \),
\[ \int_K f(x) \mu(dx) = \int_X \left( \int_K f(x) \nu(dx) \right) \eta(d\nu). \]
Also, note that
\[ \mu = \int_{\text{supp} \eta} \nu(\text{d}\nu), \]
and, by the Hahn–Banach separation theorem, one has \( \mu \in \text{co}(\text{supp} \eta) \).

The following result characterizes measures from \( X \) with support \( M \).

**Theorem 2.** The set of barycenters of the measures from \( \mathcal{P}(X) \) with support \( M \) coincides with the set of the measures from \( M \) with support \( K \).

**Proof.** (a) First, we prove that the barycenter of a measure from \( \mathcal{P}(X) \) with support \( M \) is a measure from \( M \) with support \( K \).

Take any \( \eta \in \mathcal{P}(X) \) such that \( \text{supp} \eta = M \), and let \( \mu \in M \) be its barycenter. We prove that \( \text{supp} \mu \) is exactly \( K \) by contradiction.

Indeed, suppose this is not the case. Then there exists a non-zero non-negative continuous bounded function \( f \in C_b(K) \) such that
\[ \int_{K} f(x) \mu(\text{d}x) = 0. \]
Using (25) one gets
\[ \int_{M} \int_{K} f(x) \nu(\text{d}x) \eta(\text{d}\nu) = 0, \]
and since the integrand is non-negative,
\[ \int_{K} f(x) \nu(\text{d}x) = 0, \]
\( \eta \)-almost surely on \( M \).

The latter, in fact, holds for all \( \nu \in M = \mathcal{P}(K) \), due to continuity in \( \nu \) of the left-hand side of (29) with respect to the topology of weak convergence.

Consequently, by choosing \( \nu \) to be the delta measure at an arbitrary point of \( K \), one immediately obtains
\[ f(x) = 0, \quad x \in K, \]
which contradicts \( f \neq 0 \) and concludes the proof of the claim.

(b) Now, assume that \( \mu \in M \) and \( \text{supp} \mu = K \). Let
\[ A = \left\{ (a_1, a_2, \ldots) \in [0, 1]^\infty \bigg| a_j \geq 0, \sum_{j=1}^{\infty} a_j = 1 \right\} \]
be a closed subset of \([0, 1]^\infty\) endowed with the \( l_1 \)-norm. Since \( A \) is separable, there exists a Radon probability measure \( \lambda \) on \([0, 1]^\infty\) with support \( A \) (see, e.g., the proof of Theorem 1).
Let us also introduce the Radon probability measure $\lambda \otimes \mu^\infty = \lambda \otimes \bigotimes_{j=1}^{\infty} \mu_j$ on $A \times K^\infty = A \times \prod_{j=1}^{\infty} K_j$, where the $\mu_j$ are copies of $\mu$, and the $K_j$ are copies of $K$. It is easy to see that

$$\text{supp}(\lambda \otimes \mu^\infty) = A \times K^\infty.$$  

Indeed, for any open neighborhood $U(c)$ of $c = (c_a; c_1, \ldots) \in A \times K^\infty$, by the definition of the product topology, there exists an open set of the form

$$U_a(c_a) \times \prod_{j=1}^{\infty} U_j(c_j),$$

where $U_a(c_a) \subset A$ and $U_j(c_j) \subset K_j$ are open neighborhoods of $c_a$ and $c_j$, respectively, such that $U_j(c_j) \neq K_j$ only for finitely many $j \in \mathbb{N}$. Then, for large enough $N$ one has

$$\lambda \otimes \mu^\infty(U(c)) \geq \lambda(U_a(c_a)) \prod_{j=1}^{N} \mu(U_j(c_j)) > 0,$$

which proves (32).

The next step is to define the map $F : A \times K^\infty \to M$ by

$$(34) \quad F(a, x) = \sum_{j=1}^{\infty} a_j \delta_{x_j}.$$  

It is easy to show that $F$ is continuous. Indeed, let $a^{(n)} \to a^* \in A$ in $l_1$-norm, and $x^{(n)} \to x^* \in K^\infty$ in the product topology. We will prove that $F(a^{(n)}, x^{(n)})$ converges to $F(a^*, x^*)$ weakly. For every $f \in C(K)$,

$$(35) \quad \left| \int_{K} f(y) F(a^{(n)}, x^{(n)})(dy) - \int_{K} f(y) F(a^*, x^*)(dy) \right|$$

$$\leq \sum_{j=1}^{\infty} |a^{(n)}_j f(x^{(n)}_j) - a^*_j f(x^*_j)|$$

$$\leq \sup_{x \in K} |f(x)| \|a^{(n)} - a^*\|_{l_1} + \sum_{j=1}^{\infty} a^*_j |f(x^{(n)}_j) - f(x^*_j)| \to 0,$$

where the latter term tends to zero thanks to the dominated convergence theorem. This proves the continuity of $F$.

Now, let us define the measure $\eta$ to be the pushforward of $\lambda \otimes \mu^\infty$ under $F$ given by

$$(36) \quad \eta = (\lambda \otimes \mu^\infty) \circ F^{-1},$$

which is readily verified to be a Radon probability measure.
We prove that this measure is supported on $M$. Indeed, since it is known (see, e.g., [1, Ex. 8.1.6]) that
\[ F(A \times K^\infty) = M, \]
for every open neighborhood $U(\nu)$ of $\nu \in M$ there exists $(a, x) \in A \times K^\infty$ such that $F(a, x) \in U(\nu)$. Consequently, due to $F$ being continuous and due to (32), one has $\eta(U(\nu)) > 0$, and thus, since $\nu$ is arbitrary, $\text{supp} \eta = M$.

It remains to check that the barycenter of $\eta$ is $\mu$. One can write
\[
\int \int_{M \times K} f(y) \nu(dy) \eta(d\nu) = \int \int_{A \times K^\infty} f(y) F(a, x)(dy) (\lambda \otimes \mu^\infty)(da, dx)
\]
\[
= \sum_{j=1}^{\infty} \int_{A} a_j \lambda(da) \int_{K^\infty} f(x_j) \mu^\infty(dx)
\]
\[
= \sum_{j=1}^{\infty} \int_{A} a_j \lambda(da) \int_{K} f(x) \mu(dx) = \int_{K} f(x) \mu(dx),
\]
where we use the definition (36) of $\eta$, Fubini’s theorem, and the dominated convergence to interchange the sum and the integrals.

According to (25), the formula (38) means exactly that the barycenter of $\eta$ is $\mu$. This concludes the proof of the theorem. ■

As a final remark we point out that our proof relies heavily on the fact that $K$ is compact. However, barycenters are well-defined for a wider class of Radon probability measures (with finite first moments). An open question of interest is to characterize such measures as well.

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