Soliton generation in optical fiber networks

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We consider the problem of soliton generation in branched optical fibers. A model based on the nonlinear Schrödinger equation on metric graphs is proposed. Number of generated solitons is computed for different branching topologies considering different initial pulse profiles. Experimental realization of the model is discussed.

I. INTRODUCTION

Optical solitons attracted much attention due to their potential applications in optoelectronics and information technologies. The idea of using optical solitons as carriers of information in high-speed communication systems was first proposed in the pioneering paper by Hasegawa and Tappert [1]. Later due to the advances made in fiber technology, it became possible to realize optical solitons experimentally in different versions (bright, dark, etc) [2]-[10]. This fact caused great interest to finding the soliton solutions of governing nonlinear wave equations, such as nonlinear Schrödinger equations with different nonlinearities. An important problem in the context of optical solitons is the problem of soliton generation in optical media. Mathematically, such problem is reduced to the initial value (Cauchy) problem for nonlinear Schrödinger equation, which allows to find soliton solution and number of generated solitons using given initial condition. For optical fibers such problem was studied in the Refs. [12]-[25]. In [12] an effective method for finding number of solitons generated in optical fibers was proposed. Later, it was extended for some other initial conditions [13]. Strict mathematical treatment of soliton generation on a half line as initial-boundary-value problem was considered presented [15]. Soliton generation in optical fibers for a dual-frequency input was studied in [16]. In [17] a theory of the generation of new spectral components in optical fibers pumped with a solitonic pulse and a weak continuous wave was proposed and the wave number matching conditions for this process was derived. In [17] characteristics of wavelength-tunable femtosecond soliton pulse generation using optical fibers in a negative dispersion region are studied experimentally and theoretically using the extended nonlinear Schrödinger equation, in which the wavelength dependence of parameters is considered. A comprehensive analysis of the generation of optical solitons in a monomode optical fibre from a superposition of soliton-like optical pulses at different frequencies in [18], where it is found that there exists a critical frequency separation above which wavelength-division multiplexing with solitons is feasible. Soliton generation and their instability are investigated in a system of two parallel-coupled fibers, with a pumped (active) nonlinear dispersive core and a lossy (passive) linear one in [24]. A theory of the generation of new spectral components in optical fibers pumped with a solitonic pulse have been studied. Bright-gap-soliton generation in finite optical lattices was discussed in [21]. Despite the fact that certain progress is made on theoretical and experimental study of soliton generation in optical fibers, all the studies are restricted by considering long, unbranched fibers. However, branched fibers are more attractive from the viewpoint of practical applications, as in many cases information-communication systems use optical fiber networks. Modeling of soliton generation and dynamics in optical fiber networks requires solving of nonlinear Schrödinger equation on metric graphs.

We note that soliton dynamics described by integrable nonlinear wave equations attracted much attention during past decade [26]-[38]. In [26] nonlinear Schrödinger equation on metric graphs is studied and condition for integrability is derived in the form of a sum rule for nonlinearity coefficients. In [27] such study is extended to Ablowitz-Ladik equation. Stationary Schrödinger equation on metric graphs and standing wave soliton in networks are studied in [28]-[30], [31]-[34]. Integrable sine-Gordon equation on metric graphs is studied in [31], [35]-[38]. Linear and nonlinear systems of PDE on metric graphs are considered in [34]-[41].

In this paper we consider the problem of soliton generation in branched optical fibers, or, optical fiber networks described in terms of the initial value problem for nonlinear Schrödinger equation on metric graphs. For different given initial conditions, we derive number of solitons generated by considering different network topologies. Unlike linear optical fibers, pulse generation and soliton dynamics in fiber networks strongly depend on the topology of latter. Depending on at which branch or vertex the initial pulse located, the number of solitons and their dynamics can be different. This fact provides powerful tool for tuning of the optical fiber network architecture and optimization of signal and information transmission. This paper is organized as follows. In the next section we briefly recall treatment of the problem of soliton generation for linear (unbranched) optical fibers. In section III we give formulation of the problem and its solution for star branched (Y-junction) optical fibers. Section IV extends the study for other network topologies, modeled by loop and tree graphs. Section V presents some concluding remarks.
II. SOLITON GENERATION IN LINEAR OPTICAL FIBERS

Let us first, following the Refs. [12, 13], recall solution of the problem for linear, i.e. unbranched optical fibers. The governing equation for the pulse generation and evolution in optical fibers is the following nonlinear Schrödinger equation

\[ i \frac{\partial \psi}{\partial t} + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + |\psi|^2 \psi = 0, \]  

where \( \psi \) is the normalized complex amplitude of the pulse envelope. The problem of soliton generation in optical fibers is reduced to the Cauchy problem for Eq. (1).

Such problem can be solved, e.g., using inverse scattering method [12, 13, 16]. In [12] it was solved for the initial conditions given by \( \psi(x,0) = -i q(x) \), with

\[
q(x) = \begin{cases} 
0, & \text{for } |x| > \frac{1}{2} a \\
 b, & \text{for } |x| \leq \frac{1}{2} a 
\end{cases} \quad b > 0. \tag{2}
\]

The evolution of the wave function upon generation of the soliton can be obtained via solving the following eigenvalue problem

\[ Au = \lambda u, \tag{3} \]

where

\[
A = \begin{pmatrix} 
i \frac{\partial}{\partial x} & \psi(x,0) \\ -\psi^* (x,0) & -i \frac{\partial}{\partial x} \end{pmatrix}. \tag{4}
\]

Each discrete eigenvalue \( \lambda = \eta + i \eta \) with \( L^2 \)-integrable eigenfunction corresponds to the generated soliton with the amplitude \( 2 \eta \) moving with the velocity \( 2 \zeta \).

It was shown in [12] that number of generated solitons is given by expression

\[ N = < \frac{1}{2} + \frac{F}{\pi} > = < \frac{1}{2} + \frac{ab}{\pi} >, \tag{5} \]

where \( F = \int_{-\infty}^{\infty} |\psi(x,0)| dx \) and \( < ... > \) denotes the integer smaller than the argument. Similar result for the number of solitons was obtained in the Ref. [12] for the initial condition given by

\[ q(x) = \beta e^{i \phi}(-a|x|), \quad \alpha, \beta > 0. \]

Later, Kivshar considered the problem of soliton generation for super Gaussian initial pulse and showed that Eq. (5) is general formula for arbitrary initial profile [13]. More detailed treatment of the problem of soliton generation in optical fibers was presented in [16]. In particular, the authors of [16] analyzed scenarios for soliton generation in an ideal fiber for an input that consists of either two in-phase or out-of-phase soliton-like optical pulses at different frequencies by considering symmetric initial input pulse given by

\[ \psi(x,0) = \text{sech}(x) [e^{i \phi t} + e^{-i \phi t}], \]

and asymmetric pulse given by

\[ \psi(x,0) = i \text{ sech}(x) [e^{i \phi t} - e^{-i \phi t}] \]

with the soliton solutions, respectively given as

\[
\psi(x, t) = \xi \eta e^{i \phi(\xi^2)} \frac{e^{i \zeta x} \cosh \left[ \eta (x + \xi t) + i \varphi \right] + e^{-i \zeta x} \cosh \left[ \eta (x - \xi t) - i \varphi \right]}{\xi^2 \cosh \eta (x + \xi t) \cosh \eta (x - \xi t) + \eta^2 \sin \xi (x + i \eta t) \sin \xi (x - i \eta t)}, \]

and

\[
\psi(x, t) = -i \xi \eta e^{i \phi(\xi^2)} \frac{e^{i \zeta x} \cosh \left[ \eta (x + \xi t) + i \varphi \right] - e^{-i \zeta x} \cosh \left[ \eta (x - \xi t) - i \varphi \right]}{\xi^2 \cosh \eta (x + \xi t) \cosh \eta (x - \xi t) + \eta^2 \cos \xi (x + i \eta t) \cos \xi (x - i \eta t)}, \tag{6}
\]

where \( \phi(t) = -i (\xi^2 - \eta^2) t + \alpha, \quad \alpha = \ln |\lambda_0|, \quad \tan \varphi = \frac{\eta}{\xi}. \]

In the next section we extend these studies to
the case of branched optical fibers, i.e. fiber networks.

![FIG. 3: Sketch for the H-graph](image)

### III. DESCRIPTION OF THE MODEL FOR Y-JUNCTION

Consider branched optical fiber having the form of the Y-junction. Such system can be considered as basic star graph with semi-infinite branches connected at the point $O$ called vertex, or branching point of the graph (see Fig. 1). The coordinates in such domain are defined as $x_1 \in (-\infty, 0]$ and $x_{2,3} \in [0, \infty)$, where $0$ corresponds to the branching point. Then the problem of generation of soliton and its propagation can be described in terms of the Cauchy problem for nonlinear Schrödinger equation on basic star graph, which is given by

$$i \frac{\partial \psi_j}{\partial t} + \frac{\partial^2 \psi_j}{\partial x^2} + \beta_j |\psi_j|^2 \psi_j = 0,$$

where $\psi_j$ is the normalized complex amplitude of the pulse envelope on $j$th bond (branch) of the graph and $q_j(x)$ is the initial profile of the amplitude. To solve this equation, one needs to impose the boundary conditions at the branching point (vertex) of the graph and determine the asymptotic of the wave function at the branch ends. One of the vertex boundary conditions can be chosen as the continuity of wave function [26]

$$\sqrt{\beta_1} \psi_1(0, t) = \sqrt{\beta_2} \psi_2(0, t) = \sqrt{\beta_3} \psi_3(0, t),$$

while, second one can be current conservation at the vertex, providing the Kirchhoff rule, which can be written as

$$\frac{1}{\sqrt{\beta_1}} \frac{d \psi_1}{dx} \Big|_{x=0} = \frac{1}{\sqrt{\beta_2}} \frac{d \psi_2}{dx} \Big|_{x=0} + \frac{1}{\sqrt{\beta_3}} \frac{d \psi_3}{dx} \Big|_{x=0}. $$

Soliton solutions of Eq. (7) were obtained in [26], where condition for integrability of the problem was derived in the form of a sum rule for nonlinearity coefficients. Here we consider the problem of soliton generation for Y-junction of the optical fiber for the initial pulse profile given as (see, Fig. 1)

$$q_1(x) = \begin{cases} 0, & x < -\frac{1}{2} a \\ b, & -\frac{1}{2} a \leq x \leq 0 \end{cases}$$

Such initial profile implies that soliton is generated around the branching point on each branch. Using the same method as that for linear optical fiber, we can compute the number of generated solitons, $N$ for such profile:

$$N = \left( \frac{1}{2} + \frac{F}{\pi} \right),$$

where

$$F = \frac{3}{2} \int_{-\infty}^{\infty} |\psi_j(x, 0)| dx = \frac{ab}{2} \left[ \frac{2}{\beta_1} + \frac{2}{\beta_2} + \frac{2}{\beta_3} \right].$$

As follows from [26], the problem given by Eqs. (7), (8) and (9) is integrable if the sum rule $\beta_1^{-1} = \beta_2^{-1} + \beta_3^{-1}$. Therefore for integrable case, when the soliton solution of the problem can be obtained, one should take into account this sum rule for finding the number of generated solitons. Difference between Eqs. (5) and (12) comes from the constant factor

$$\left( \frac{2}{\beta_1^{-1}} + \frac{2}{\beta_2^{-1}} + \frac{2}{\beta_3^{-1}} \right).$$

This allows tuning the soliton number and dynamics using different choices of the set $\beta_j, (j = 1, 2, 3)$. In addition, for simplicity, the above initial pulse profiles in Eqs. (10) and (11) are given at the vertex and have the same widths, $a$ and heights, $b$. However, in general case one can choose different widths and heights for different bonds. This also provides additional tool for tuning of the soliton number and dynamics.

Another initial pulse profile, for which the soliton number and solutions in Y-junction of fibers can be explicitly obtained is given by

$$\psi_j(x, 0) = \sqrt{\frac{2}{\beta_j}} \text{sech}(x) \left[ e^{i(2\omega x + \theta)} + e^{-i(2\omega x + \theta)} \right],$$

where $2\omega$ and $\theta$ are the frequency detuning and the phase difference between the two solitons, correspondingly. The two-soliton solution of the problem given by Eqs. (7), (8) and (9) can be written as

![FIG. 4: Initial pulse profile on the optical fiber H-shaped network](image)
\[
\psi_j(x, t) = \sqrt{\frac{2}{\beta_j}} e^{i\theta_j(x,t)} \sqrt{\frac{e^{i\xi x} \cosh[\eta(x+\xi t)+i\varphi] + e^{-i\xi x} \cosh[\eta(x-\xi t)-i\varphi]}{\xi^2 \cosh \eta(x+\xi t) \cosh \eta(x-\xi t) + \eta^2 \sin \xi (x+i\eta t) \sin \xi (x-i\eta t)}},
\]

which is valid under the constraint:
\[
\frac{1}{\beta_1} = \frac{1}{\beta_2} + \frac{1}{\beta_3},
\]

(16)

Corresponding soliton number is given as
\[
N = \frac{1}{2} + \frac{F}{\pi},
\]

(17)

where
\[
F = 2\pi \left( \frac{2}{\beta_1} + \frac{2}{\beta_2} + \frac{2}{\beta_3} \right) \text{sech} \left( \frac{\pi\omega}{2} \right) \cos \left( \frac{\theta}{2} \right)
\]

we consider the following initial conditions:
\[
\psi_j(x, 0) = -i \sqrt{\frac{2}{\beta_j}} q_j(x)
\]

where the initial pulse profiles are given by (see, Fig.4)
\[
q_{1,2}(x) = \begin{cases} 
0, & -\infty < x < -\frac{1}{2}a \\
1, & -\frac{1}{2}a \leq x \leq 0 
\end{cases}
\]

(20)

\[
q_3(x) = \begin{cases} 
1, & 0 \leq x < \frac{1}{2}a \\
0, & \frac{1}{2}a \leq x \leq L 
\end{cases}
\]

(21)

\[
q_{4,5}(x) = \begin{cases} 
1, & 0 \leq x < \frac{1}{2}a \\
0, & \frac{1}{2}a \leq x < \infty 
\end{cases}
\]

(22)

FIG. 5: Sketch for tree graph

FIG. 6: Initial pulse profile on the tree-shaped optical fiber network

IV. OTHER NETWORK TOPOLOGIES

The above treatment of the problem for soliton generation in optical fiber networks can be extended to the case of more complicated topologies. Here we demonstrate this for so-called \( H \)-graph and tree graph. For \( H \)-graph, presented in Fig.3 the coordinates are defined as \( x_{1,2} \in (\infty; 0], x_3 \in [0; L], x_{4,5} \in [0; +\infty), \) where \( L \) is the length of bond \( e_3 \), i.e. the distance between two vertices.

For NLS equation (4) and the vertex boundary conditions given by
\[
\sqrt{\beta_1} \psi_1(0, t) = \sqrt{\beta_2} \psi_2(0, t) = \sqrt{\beta_3} \psi_3(L, t),
\]

\[
\sqrt{\beta_3} \psi_3(L, t) = \sqrt{\beta_4} \psi_4(0, t) = \sqrt{\beta_5} \psi_5(0, t),
\]

(18)

\[
\frac{1}{\sqrt{\beta_1}} \frac{d\psi_1}{dx} |_{x=0} + \frac{1}{\sqrt{\beta_2}} \frac{d\psi_2}{dx} |_{x=0} = \frac{1}{\sqrt{\beta_3}} \frac{d\psi_3}{dx} |_{x=0},
\]

\[
\frac{1}{\sqrt{\beta_3}} \frac{d\psi_3}{dx} |_{x=L} = \frac{1}{\sqrt{\beta_4}} \frac{d\psi_4}{dx} |_{x=0} + \frac{1}{\sqrt{\beta_5}} \frac{d\psi_5}{dx} |_{x=0}.
\]

(19)

Then we the number of generated optical solitons in such system we have explicit expression:
\[
N = \left( \frac{1}{2} + \frac{F_1 + F_2}{\pi} \right),
\]

(23)

where
\[
F_1 = \frac{ab_1}{2} \left[ \sqrt{\frac{2}{\beta_1}} + \sqrt{\frac{2}{\beta_2}} + \sqrt{\frac{2}{\beta_3}} \right],
\]

(24)

and
\[
F_2 = \frac{ab_2}{2} \left[ \sqrt{\frac{2}{\beta_3}} + \sqrt{\frac{2}{\beta_4}} + \sqrt{\frac{2}{\beta_5}} \right].
\]

(25)
Similarly, one can find number of generated solitons for the tree graph, presented in Fig. 6. The vertex boundary conditions for such graph as given by
\[
\sqrt{\beta_1} \psi_1(0, t) = \sqrt{\beta_2} \psi_2(0, t) = \sqrt{\beta_3} \psi_3(0, t),
\]
\[
\sqrt{\beta_{1i}} \psi_{1i}(L_{1i}, t) = \sqrt{\beta_{1ij}} \psi_{1ij}(0, t),\quad i, j = 1, 2, \quad (26)
\]
and
\[
\frac{1}{\sqrt{\beta_1}} \frac{d\psi_1}{dx}|_{x=0} = \frac{1}{\sqrt{\beta_2}} \frac{d\psi_2}{dx}|_{x=0} + \frac{1}{\sqrt{\beta_3}} \frac{d\psi_3}{dx}|_{x=0},
\]
\[
\frac{1}{\sqrt{\beta_{1i}}} \frac{d\psi_{1i}}{dx}|_{x=L_{1i}} = \frac{1}{\sqrt{\beta_{11i}}} \frac{d\psi_{11i}}{dx}|_{x=0} + \frac{1}{\sqrt{\beta_{112}}} \frac{d\psi_{112}}{dx}|_{x=0},\quad i = 1, 2, (27)
\]
Furthermore, we choose the initial pulse profile at each vertex \((\psi(x, 0) = -i \sqrt{\beta_j} q_i(x))\) in the forms
\[
q_1(x) = \begin{cases} 0, & -\infty < x < -\frac{1}{2}a \\ b_1, & -\frac{1}{2}a \leq x \leq 0 \end{cases} \quad (28)
\]
\[
q_{11}(x) = \begin{cases} b_1, & 0 \leq x \leq \frac{1}{2}a \\ b_2, & \frac{1}{2}a < x < L_{11} - \frac{1}{2}a \end{cases} \quad (29)
\]
\[
q_{12}(x) = \begin{cases} b_1, & 0 \leq x \leq \frac{1}{2}a \\ b_2, & \frac{1}{2}a < x < L_{12} - \frac{1}{2}a \\ b_3, & L_{12} - \frac{1}{2}a \leq x \leq L_{12} \end{cases} \quad (30)
\]
\[
q_{11j}(x) = \begin{cases} b_{1+1}, & 0 \leq x \leq \frac{1}{2}a \\ b_{1}, & \frac{1}{2}a < x < \infty \end{cases} \quad (31)
\]
where \(i, j = 1, 2, L_{11}\) and \(L_{12}\) are lengths of bonds \(e_{11}\) and \(e_{12}\) respectively.

Then for the generated soliton number we have
\[
N = \left( \frac{1}{2} + \frac{F_1 + F_2 + F_3}{\pi} \right), \quad (32)
\]
where
\[
F_1 = \frac{ab_1}{2} \left[ \sqrt{\frac{2}{\beta_1}} + \sqrt{\frac{2}{\beta_{11}}} + \sqrt{\frac{2}{\beta_{12}}} \right] \quad (33)
\]
\[
F_2 = \frac{ab_2}{2} \left[ \sqrt{\frac{2}{\beta_{11}}} + \sqrt{\frac{2}{\beta_{11}}} + \sqrt{\frac{2}{\beta_{12}}} \right] \quad (34)
\]
\[
F_3 = \frac{ab_3}{2} \left[ \sqrt{\frac{2}{\beta_{12}}} + \sqrt{\frac{2}{\beta_{12}}} + \sqrt{\frac{2}{\beta_{12}}} \right] \quad (35)
\]
The number of parameters in Eq. (32) is much higher that in the case of star graph. This implies that tree-shaped optical fiber network provides more wider possibility for tuning the generated soliton number and their dynamics.

V. CONCLUSIONS

In this paper we studied the problem of soliton generation in optical fiber networks using a model based NLS equation on metric graphs. Initial value (Cauchy) problem for NLS equation on metric graphs is solved for different graph topologies, such as star, tree and H-graphs. For branched optical fibers one can choose the initial pulse profile in different ways (e.g., at the vertex or branch, at given vertex or branch, with different shapes at different vertices). Therefore, unlike to linear (un-branched) fibers, soliton generation for optical fiber networks have richer dynamics and tools for manipulation by solitons numbers. The above method can be applied for different network topologies, provided a network has three and more semi-infinite outgoing branches. This allows to use our model for the problem of tunable soliton generation in optical fiber networks, which is of importance for practical applications in the areas, where optical fibers are used for information (signal) transfer.

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