Zeros of polynomials with four-term recurrence and linear coefficients

Khang Tran · Andres Zumba

Received: 1 August 2019 / Accepted: 16 February 2020 / Published online: 20 July 2020
© Springer Science+Business Media, LLC, part of Springer Nature 2020

Abstract
This paper investigates the zero distribution of a sequence of polynomials \( \{P_m(z)\}_{m=0}^{\infty} \) which satisfy a four-term recurrence whose coefficients are linear polynomials in \( z \). In particular, we study necessary and sufficient conditions for the reality of the zeros of \( P_m(z) \). Under these conditions, we find an explicit interval containing these zeros, whose union forms a dense subset of this interval.

Keywords  Zero distribution · Generating function · Recurrence

Mathematics Subject Classification  Primary 30C15 · Secondary 26C10 · 11C08

1 Introduction

The understanding of zeros of polynomials defined recursively plays an important role in the study of zero distribution of polynomials. A classic recurrence is the three-term recursion which is a necessary condition for a sequence of polynomials to be orthogonal. Orthogonality, in turn, establishes the reality of the zeros of the sequence of polynomials.

The sequence of polynomials \( \{P_m(z)\}_{m=0}^{\infty} \) satisfying a four-term recurrence

\[
P_m(z) + C(z)P_{m-1}(z) + B(z)P_{m-2}(z) + A(z)P_{m-3}(z) = 0, \tag{1.1}
\]

where \( A(z), B(z), \) and \( C(z) \) are linear polynomial in \( z \), has its own important role in mathematics. In [2], the authors study a transformation of the binomial-type polynomial

1 California State University, 5245 North Backer Avenue M/S PB108, Fresno, CA 93740, USA
and show that the resulting polynomial satisfies a special form of the four-term recurrence given in (1.1). This resulting polynomial satisfies a Riemann Hypothesis. The substitution $x = 1/2 + it$ reduces the need of proving Riemann Hypothesis property of the polynomial to the need of proving certain polynomial is hyperbolic (i.e., all zeros are real). However, due to the four-term recurrence nature, the sequence of polynomials may not be orthogonal and the reality of the zeros is far from trivial. In [2], the authors find sufficient conditions for the reality of these zeros for a special case of (1.1) when $A(z)$ and $B(z)$ are constant. In this special case, the condition was later shown to be necessary in [1, Proposition 1].

The goal of the present paper is to study necessary and sufficient conditions so that all the polynomials $P_m(z)$ given in (1.1) are hyperbolic when only one coefficient $C(z)$ is constant. For the zero distribution of a special four-term recurrence where the coefficients are not linear, see [3].

With the initial condition similar to that of Chebyshev polynomials of the second kind, i.e., $P_0(z) = 1$ and $P_{-m}(z) = 0, \forall m \in \mathbb{N}$, the sequence $\{P_m(z)\}$ is generated by

$$\sum_{m=0}^{\infty} P_m(z)t^m = \frac{1}{1 + C(z)t + B(z)t^2 + A(z)t^3}. \quad (1.2)$$

We apply the notations $C(z) \equiv c$, $B(z) = b_0 + b_1z$, and $A(z) = a_0 + a_1z$, and state the main result of the paper.

**Theorem 1** Suppose the sequence $\{P_m(z)\}$ is defined as above where $ca_1b_1 \leq 0$ and $ca_1 \neq 0$. The zeros of $P_m(z)$ are real for all $m \in \mathbb{N}$ if and only if

$$1 + a + b \geq 0 \quad \text{and} \quad 9 - 27a + b \geq 0, \quad (1.3)$$

where

$$a := \frac{b_0}{c^2} - \frac{b_1a_0}{c^2a_1},$$
$$b := -\frac{b_1c}{a_1}.$$ 

Under (1.3), we can find an explicit real interval containing the zeros of $P_m(z)$ by considering the cubic polynomial

$$(8a - 2)\zeta^3 + \zeta^2(-12a + b + 5) + (6a - 2)\zeta - a$$

whose only real zero on $(-\infty, -1] \cup [1, \infty)$ is denoted by $\zeta_0$. The existence and uniqueness of such zero is justified in Sect. 2. The zeros of $P_m(z)$ lie on the interval

$$\frac{c^3}{a_1} I_{a,b} - \frac{a_0}{a_1}, \quad (1.4)$$
where

\[ I_{a, b} = \left( -\infty, \frac{\zeta_0^2}{(1 - 2\zeta_0)^3} \right). \]  

(1.5)

Moreover, if we let \( \mathcal{Z}(P_m) \) be the set of zeros of \( P_m(z) \), then \( \bigcup_{m=0}^{\infty} \mathcal{Z}(P_m) \) is dense on (1.4). In the special case \( b_1 = 0, a_1 = 1, \) and \( a_0 = 0 \), we solve

\[ \zeta_0 = \frac{2a - 1 - \sqrt{1 - 3a}}{4a - 1} \]

and obtain Case (ii) of Theorem 1 in [5].

Our approach to the proof of Theorem 1 relies on the reparametrization from \( P_m(z) \) to \( P_m(z(\theta)) \) where \( z(\theta) \) is strictly monotone. This function \( z(\theta) \) is constructed by an auxiliary function \( \zeta(\theta) \) which is defined implicitly through the bivariate function \( f(\zeta, \theta) \) (cf. (2.8)). We count the number of zeros in \( \theta \) of \( P_m(z(\theta)) \), each of which yields a distinct real zero of \( P_m(z) \) by the monotonicity of \( z(\theta) \). If the number of counted zeros is the same as the degree of \( P_m(z) \), then all the zeros of \( P_m(z) \) are real by the Fundamental Theorem of Algebra.

Our paper is organized as follows. Section 2 studies the auxiliary function \( \zeta(\theta) \) and Sect. 3 establishes the monotone property of \( z(\theta) \). With all the properties in these two sections, we prove the sufficient and necessary condition for the reality of the zeros of \( P_m(z) \) in Sects. 4 and 5, respectively.

2 Auxiliary functions

Our first step is to simplify the right-side of (1.2). We note that the substitutions \( t \to t/c \) and

\[ \frac{a_1}{c^3} \zeta + \frac{a_0}{c^3} \to z \]

reduce the right-side of (1.2) to

\[ \frac{1}{1 + t + at^2 + zt^2(t - b)} =: \frac{1}{D(t, z)}. \]

We deduce that Theorem 1 is equivalent to the following theorem.

**Theorem 2** Suppose \( b \geq 0 \). The zeros of \( H_m(z) \) generated by

\[ \sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{1 + t + at^2 + zt^2(t - b)} := \frac{1}{D(t, z)} \]  

(2.1)
are real if and only if
\[ 1 + a + b \geq 0 \quad \text{and} \quad 9 - 27a + b \geq 0. \]

Since the case \( b = 0 \) is proved in Theorem 2 of [5], we only consider \( b > 0 \) in this paper. In fact, to prove the sufficient condition for the reality of the zeros of \( H_m(z) \), for each \( b > 0 \), we can ignore certain values of \( a \) as the following lemma shows.

**Lemma 1** We fix \( b > 0 \) and let \( S \) be a dense subset of \([-1 - b, (b + 9)/27]\). If

\[ Z(H_m(z, a, b)) \subset I_{a,b} \]

for all \( a \in S \), then

\[ Z(H_m(z, a^*, b)) \subset I_{a^*,b} \]

for all \( a^* \in [-1 - b, (b + 9)/27] \).

**Proof** Let \( a^* \in [-1 - b, (b + 9)/27] \) be given. By the density of \( S \) in \([-1 - b, (b + 9)/27]\), we can find a sequence \( \{a_n\} \) in \( S \) such that \( a_n \to a^* \). For any \( z^* \notin I_{a^*,b} \), we will show that \( H_m(z^*, a^*, b) \neq 0 \). We note that the zeros of \( H_m(z, a_n, b) \) lie in the interval \( I_{a_n,b} \) whose right endpoint approaches the right endpoint of \( I_{a^*,b} \) as \( n \to \infty \).

If we let \( z^*_k \), \( 1 \leq k \leq \deg H_m(z, a_n, b) \), be the zeros of \( H_m(z, a_n, b) \) then

\[ \prod_{k=1}^{\deg H_m(z, a_n, b)} \left| z^* - z^*_k \right|, \]

where \( \gamma^{(n)} \) is the leading coefficient of \( H_m(z, a_n, b) \). Since \( \deg H_m(z, a_n) \leq \lfloor m/2 \rfloor \) by Lemma 2, using this product representation and the assumption that \( z^* \notin I_{a,b} \), we conclude that there is a fixed (independent of \( n \)) \( \delta > 0 \) so that \( |H_m(z^*, a_n, b)| > \delta \), for all large \( n \). Since \( H_m(z^*, a, b) \) is a polynomial in \( a \) for any fixed \( z^* \), we conclude that

\[ H_m(z^*, a^*, b) = \lim_{n \to \infty} H_m(z^*, a_n, b) \neq 0 \]

and the result follows. \( \square \)

As suggested in the introduction, we will count the number of real zeros of \( H_m(z) \) and compare this number to its degree. The lemma below provides an upper bound for the degree.

**Lemma 2** The degree of the polynomial \( H_m(z) \) defined by (2.1) is at most \( \lfloor m/2 \rfloor \).

**Proof** This lemma follows easily from induction applied to the recurrence

\[ H_m(z) + H_{m-1}(z) + (a - bz)H_{m-2}(z) + zH_{m-3}(z) = 0, \quad m \geq 1, \]

and the initial condition \( H_0(z) \equiv 1 \) and \( H_m(z) \equiv 0 \) for \( m < 0 \). \( \square \)
To motivate the formula for the function $z(\theta)$ mentioned in the introduction, we provide some heuristic arguments. For each $z \in \mathbb{R} \setminus \{0\}$, we let $t_0 = t_0(z)$, $t_1 = t_1(z)$, and $t_2 = t_2(z)$ be the three zeros of $D(t, z)$. If $t_0$ and $t_1$ are two distinct complex conjugates and $t_2 \in \mathbb{R}$, then we let $t_0 = \tau e^{-i\theta}$, $t_1 = \tau e^{i\theta}$, and $t_2 = \zeta \tau$ where $\zeta \in \mathbb{R}$.

From the elementary symmetric equations

$$
t_0 + t_1 + t_2 = \frac{bz - a}{z}, \quad t_0 t_1 + t_0 t_2 + t_1 t_2 = \frac{1}{z}, \quad \text{and} \quad t_0 t_1 t_2 = -\frac{1}{z},
$$

we deduce that

$$
1 + e^{2i\theta} + \zeta e^{i\theta} = \frac{bz - a}{zt_0}, \quad e^{2i\theta} + \zeta e^{i\theta} + \zeta e^{3i\theta} = \frac{1}{zt_0^2}, \quad \text{and} \quad \zeta e^{3i\theta} = -\frac{1}{zt_0^3}.
$$

We divide the first equation by $e^{i\theta}$, the second by $e^{2i\theta}$, and the third by $e^{3i\theta}$ and obtain

$$
2 \cos \theta + \zeta = \frac{bz - a}{z\tau}, \quad 1 + 2\zeta \cos \theta = \frac{1}{z\tau^2}, \quad \text{and} \quad \zeta = -\frac{1}{z\tau^3}. \quad (2.3)
$$

We solve for $z$ from the third equation

$$
z = -1/\zeta \tau^3 \quad (2.4)
$$

and substitute $z$ to the first equation

$$
\tau (2 \cos \theta + \zeta) = b + a\zeta \tau^3 \quad (2.5)
$$

and the second equation

$$
\tau = -\frac{1}{\zeta} - 2 \cos \theta. \quad (2.6)
$$

From these identities, we obtain

$$
\zeta (1 + 2\zeta \cos \theta) (2 \cos \theta + \zeta) = -b\zeta^2 + a(1 + 2\zeta \cos \theta)^3, \quad (2.7)
$$

which motivates the definition of the function

$$
f(\zeta, \theta) = \zeta (1 + 2\zeta \cos \theta) (2 \cos \theta + \zeta) + b\zeta^2 - a(1 + 2\zeta \cos \theta)^3. \quad (2.8)
$$

Converse to the construction above, we have the following lemma.

**Lemma 3** For any $\theta \in (0, \pi)$, if $\zeta$ is a zero of $f(\zeta, \theta)$ and $z$ and $\tau$ are given in (2.4) and (2.6), then $\tau e^{\pm i\theta}$ and $\zeta \tau$ are the three zeros of $D(t, z)$. 

 Springer
Proof We reverse the arguments above by combining (2.7) and (2.6) to obtain (2.5). Together with (2.4), we deduce (2.3) and (2.2) follows. □

As a polynomial in $\zeta$, its reciprocal $f^*(\zeta, \theta) := \zeta^3 f(1/\zeta, \theta)$ is

$$f^*(\zeta, \theta) = (\zeta + 2 \cos \theta)(2 \zeta \cos \theta + 1) + b\zeta - a(\zeta + 2 \cos \theta)^3,$$

which, after collecting the coefficients of $\zeta$, becomes

$$-a\zeta^3 + (2 \cos \theta - 6a \cos \theta)\zeta^2 + (1 + b + 4 \cos^2 \theta - 12a \cos^2 \theta)\zeta + 2 \cos \theta - 8a \cos^3 \theta.$$  

(2.9)

For the sufficient direction of Theorem 2, we limit the domain of $\theta$ to $(\pi/2, \pi)$. Our first goal here is to show that for any $\theta \in (\pi/2, \pi)$, $f^*(\zeta, \theta)$ has exactly one real zero on the interval $(-1, 1)$ by considering the sign of this polynomial at the endpoints.

Lemma 4 For any fixed $\theta \in (\pi/2, \pi)$, if

$$2 - 8a + 8a^2 + ab \neq 0,$$

$$b + 1 - a \neq 0,$$

$$9 - 27a + b \neq 0,$$

$$1 + a + b \neq 0,$$

then we have

$$f^*(-1, \theta)f^*(1, \theta) < 0.$$

Proof If we let $x = \cos \theta$, then $f^*(-1, \theta)f^*(1, \theta)$ is a cubic polynomial in terms of $x^2$. We let that polynomial be $g(x)$ and compute its discriminant by a computer algebra package as

$$-4096b(27a^2b + 4)(2 - 8a + 8a^2 + ab)^2 < 0,$$

from which we deduce that $g(x)$ has only one real root. Then the two inequalities

$$g(0) = -(b + 1 - a)^2 < 0,$$

$$g(1) = -(9 - 27a + b)(1 + a + b) < 0$$

imply that $g(x) < 0 \ \forall x \in (0, 1)$ and the lemma follows. □

We note that Lemma 1 allows us to focus on the values of $a$ in which the all conditions of Lemma 4 are met. In fact, when $a > 0$ we know the sign of each factor $f^*(-1, \theta)$ and $f^*(1, \theta)$ in the lemma below.

Lemma 5 If $a > 0$ and $b - 27a + 9 > 0$, then $f^*(1, \theta) > 0$ and $f^*(-1, \theta) < 0$ for all $\theta \in (\pi/2, \pi)$.  

Springer
**Proof** To show $f^*(1, \theta) > 0$, we consider two cases $0 < a \leq 1$ and $a > 1$. In the first case, the inequalities

$$a(2 \cos \theta + 1) - 1 < a - 1 < 0$$

imply

$$-f^*(1, \theta) = a - (2 \cos \theta - 6a \cos \theta) - (1 + b + 4 \cos^2 \theta - 12a \cos^2 \theta)$$

$$- (2 \cos \theta - 8a \cos^3 \theta)$$

$$= (a + 6a \cos \theta + 12a \cos^2 \theta + 8a \cos^3 \theta) - (4 \cos^2 \theta + 4 \cos \theta + 1) - b$$

$$= a(1 + 2 \cos \theta)^3 - (1 + 2 \cos \theta)^2 - b$$

$$= (2 \cos \theta + 1)^2(a(2 \cos \theta + 1) - 1) - b < 0.$$

In the later case, we have

$$-f^*(1, \theta) = a - 2 \cos \theta(2 - 3a) - (1 + b + 4 \cos^2 \theta - 12a \cos^2 \theta) + 8a \cos^3 \theta$$

$$< a - 2 \cos \theta(2 - 3a) - 1 - b - 4 \cos^2 \theta + 12a + 8a \cos^3 \theta$$

$$= -2 \cos \theta(2 - 3a) - 1 - 4 \cos^2 \theta + 8a \cos^3 \theta + (-b + 27a - 9)$$

$$+ (-14a + 9)$$

$$< 0.$$

The claim that $f^*(-1, \theta) < 0$ follows from Lemma 4. 

**Remark 1** As a consequence of Lemma 5 and the fact that the leading coefficient of $f^*(\zeta, \theta)$ is $-a$, we conclude that if $a > 0$, then $f^*(\zeta, \theta)$ has one zero on each of the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$ and consequently this polynomial has exactly one zero on $(-1, 1)$.

For the case $a < 0$, we consider the lemma below.

**Lemma 6** Assume $a < 0$, $1 + b + a > 0$, and $\theta \in (\pi/2, \pi)$. If all the zeros in $\zeta$ of $f^*(\zeta, \theta)$ are real, then exactly one of them lies in the interval $(0, 1)$, and the other two lie in $(1, \infty)$.

**Proof** By Lemma 4 and (2.9), the real zeros of $f^*(\zeta, \theta)$ are positive and at least one of which lies in $(0, 1)$. If all the zeros of $f^*(\zeta, \theta)$ are real, then so are two zeros of its derivative $df^*(\zeta, \theta)/d\zeta$

$$\frac{(2 - 6a) \cos \theta \pm \sqrt{3a \left(b - 4 \cos^2 \theta + 1\right) + 4 \cos^2 \theta}}{3a}.$$

With the note that the leading coefficient of $df^*(\zeta, \theta)/d\zeta$ is positive, we will show that these two zeros lie in the interval $(1, \infty)$ by claiming that one of the two zeros lies in this interval and $df^*(\zeta, \theta)/d\zeta > 0$ when $\zeta = 1$. The lemma will follow from
the interlacing zeros of \( f^*(\zeta, \theta) \) and its derivative. The second claim comes directly from the identity

\[
\left. \frac{df^*(\zeta, \theta)}{d\zeta} \right|_{\zeta=1} = (2\cos \theta + 1)^2(1 - 3a) + b > 0.
\]

Since the two zeros of \( df^*(\zeta, \theta)/d\zeta \) are real, we have

\[
4\cos^2 \theta + 3a + 3ab - 12a \cos^2 \theta \geq 0
\]
or equivalently

\[
\cos \theta \leq -\sqrt{\frac{-3a(1 + b)}{4(1 - 3a)}}.
\]

On the other hand, the assumption \( a < 0 \) implies that \( 1 + b - 3ab > 0 \) and consequently

\[
\sqrt{1 + b} > \sqrt{\frac{-3a}{1 - 3a}}.
\]

We multiply both sides of this inequality by \( \sqrt{-3a/4(1 - 3a)} \) and apply (2.10) to get

\[
\cos \theta \leq -\sqrt{\frac{-3a(1 + b)}{4(1 - 3a)}} < \frac{3a}{2(1 - 3a)},
\]

which gives

\[
2(1 - 3a) \cos \theta - 3a < 0.
\]

Hence

\[
2(1 - 3a) \cos \theta - 3a < \sqrt{4\cos^2 \theta + 3a + 3ab - 12a \cos^2 \theta}
\]

and consequently

\[
\frac{2(1 - 3a) \cos \theta - \sqrt{4\cos^2 \theta + 3a + 3ab - 12a \cos^2 \theta}}{3a} > 1.
\]

\[\Box\]

With all the previous lemmas at our disposal, the formal proof of Theorem 2 begins by the definition of the function \( 1/\zeta(\theta) \) as the only real zero of \( f^*(\zeta, \theta) \) on the interval \((-1, 1)\). The existence and uniqueness of this zero comes from Remark 1 and Lemma 6. By the Implicit Function Theorem \( 1/\zeta(\theta) \) is smooth on \((\pi/2, \pi)\). We next define the two functions \( \tau(\theta) \) and \( z(\theta) \) according to (2.6) and (2.4) respectively. Since \( 1/\zeta(\theta) \) is
smooth on \((\pi/2, \pi)\), so is \(\tau(\theta)\). With Lemma 7 below and (2.4), the function \(\tau(\theta)\) is also smooth on \((\pi/2, \pi)\).

**Lemma 7** For any \(\theta \in (\pi/2, \pi)\), we have \(\tau(\theta) > 0\).

**Proof** We will show that \(\tau(\theta)\) has no zero on \((\pi/2, \pi)\) and the lemma will follow from

\[
\tau(2\pi/3) = -\frac{1}{\zeta(2\pi/3)} + 1 > 0.
\]

Indeed, if \(\theta_0 \in (\pi/2, \pi)\) is a zero of \(\tau(\theta)\), then \(1/\zeta(\theta_0) \neq 0\) and \(1 + 2\zeta(\theta_0) \cos \theta_0 = 0\), a contradiction to (2.7). \(\square\)

**Lemma 8** The only zero of \(1/\zeta(\theta)\) on \((\pi/2, \pi)\) is \(\cos (\sqrt{\frac{-1}{2\sqrt{a}}})\) when \(a > 1/4\).

**Proof** By the definition of \(1/\zeta(\theta)\), we note that \(\theta\) is a zero of \(1/\zeta(\theta)\) if and only if the free coefficient of \(f^*(\zeta, \theta)\)

\[
2 \cos \theta (1 - 4a \cos^2 \theta) = 0.
\] (2.11)

unless \(f^*(\zeta, \theta)\) is a constant 0 polynomial under (2.11). However, this case does not occur since the coefficient of \(\zeta\) of \(f^*(\zeta, \theta)\) is

\[
1 + b + 4 \cos^2 \theta - 12a \cos^2 \theta,
\]

which is nonzero when \(1 - 4a \cos^2 \theta = 0\) because

\[
b + \frac{1}{a} - 2 > b + \frac{27}{9 + b} - 2 = \frac{b^2 + 7b + 9}{9 + b} > 0.
\]

\(\square\)

**Lemma 9** If \(1/4 < a < b/27 + 1/3\) and \(-1 < \cos \theta < -1/2\sqrt{a}\), then \(\zeta(\theta) < 0\).

**Proof** From Lemma 8, \(\zeta(\theta)\) is continuous on \((\cos^{-1}(-1/(2\sqrt{a})), \pi)\) and does not change its sign on this interval. Thus it suffices to consider the sign of \(\zeta(\theta)\) at a single point. We consider the two cases below.

In the case \(a \geq 1/3\), we let \(\theta \to \pi\) and observe from (2.7) that \(\zeta(\theta)\) approaches \(\zeta_0\) where

\[
0 = (-2 + 8a)\zeta_0^3 + (-12a + b + 5)\zeta_0^2 + (-2 + 6a)\zeta_0 - a
= (-2 + 8a)\zeta_0^3 + (-27a + b + 9)\zeta_0^2 + (-2 + 8a)\zeta_0
+ (12a - 4)\zeta_0^2 + 3a\zeta_0^2 - 2a\zeta_0 - a.
\] (2.12)

If by contradiction that \(\zeta_0 > 0\), then \(\zeta_0 \geq 1\) by Lemma 4 and consequently

\[
3a\zeta_0^2 - 2a\zeta_0 - a = 2a\zeta_0(\zeta_0 - 1) + a(\zeta_0^2 - 1) \geq 0.
\]
Under the assumption that \( a \geq 1/3 \), we have \(-2 + 8a > 0\) and all other the coefficients of (2.12) are non-negative, which is a contradiction.

Similarly, in the case \( 1/4 < a < 1/3 \), identity (2.7) with \( \theta \to \pi \) yields
\[
0 > -b \xi_0^2 = (1 - 2 \xi_0)(\xi_0(-2 + \xi_0) - a(1 - 2 \xi_0)^2) = (1 - 2 \xi_0)(\xi_0^2(1 - 4a) + 2 \xi_0(-1 + 2a) - a).
\]

With the same arguments in the previous case, we conclude that \( \xi_0 < 0 \). \( \square \)

**Lemma 10** If \( 9 - 27a + b \geq 0 \), then \( \zeta(\theta) \to +\infty \) as \( \cos \theta \to 0^- \).

**Proof** As \( \cos \theta \to 0^- \), the reciprocal of \( f(\zeta, \theta) \) as a polynomial in \( \zeta \) approaches
\[
(b + 1)\zeta - a\zeta^3,
\]
which has a simple zero at 0. Thus exactly one of the zero in \( \zeta \) of \( f(\zeta, \theta) \) approaches \( +\infty \). Since the sum of the three zeros of \( f(\zeta, \theta) \) is
\[
\frac{1 + b + 4 \cos^2 \theta - 12a \cos^2 \theta}{8a \cos^3 \theta - 2 \cos \theta} \to +\infty,
\]
as \( \cos \theta \to 0^- \), we conclude that \( \zeta(\theta) \to +\infty \). \( \square \)

In the case \( a > 1/4 \), from Lemmas 8 and 10, the continuity of \( \zeta(\theta) \) on
\[
\left( \frac{\pi}{2}, \cos^{-1} \left( -\frac{1}{2\sqrt{a}} \right) \right),
\]
and the inequality \( |\xi(\theta)| > 1 \), we deduce that \( \zeta(\theta) \to +\infty \) as \( \cos \theta \to -1/2\sqrt{a}^+ \).

### 3 The monotonicity of \( z(\theta) \)

The goal of this section is to show that \( z(\theta) \) is strictly increasing on \( (\pi/2, \pi) \). We recall from Lemma 3 that the three zeros in \( t \) of the polynomial \( 1 + t + at^2 + z(\theta)t^2(t - b) \) are \( t_0 = \tau(\theta)e^{-i\theta}, t_1 = \tau(\theta)e^{i\theta} \), and \( t_2 = \zeta(\theta)\tau(\theta) \). Consequently
\[
z = -\frac{1 + t_0 + at_0^2}{t_0^2(t_0 - b)}.
\]

If we let \( 1 + t_0 + at_0^2 = a(t_0 - \tau_1)(t_0 - \tau_2) \), then the logarithmic derivatives of both sides and the identity
\[
dt_0 = d\tau e^{-i\theta} - i\tau e^{-i\theta} d\theta = t_0 \left( \frac{d\tau}{\tau} - i d\theta \right)
\]
give
\[
\frac{\mathrm{d}z}{z} = h(t_0) \left( \frac{\mathrm{d}\tau}{\tau} - i\mathrm{d}\theta \right),
\]
where
\[
h(t_0) := \frac{t_0}{t_0 - \tau_1} + \frac{t_0}{t_0 - \tau_2} - \frac{t_0}{t_0 - b} - 2.
\]
Since \(\mathrm{d}z/z \in \mathbb{R}\), the imaginary and the real parts of (3.2) give
\[
\text{Im } h(t_0) \frac{\mathrm{d}\tau}{\tau} = \text{Re } h(t_0) \mathrm{d}\theta
\]
and
\[
\frac{\mathrm{d}z}{z} = \text{Re } h(t_0) \frac{\mathrm{d}\tau}{\tau} + \text{Im } (h(t_0)) \mathrm{d}\theta.
\]
We multiply both sides of the second equation by \(\text{Im } h(t_0)\) and apply the first equation to obtain
\[
\text{Im } h(t_0) \frac{\mathrm{d}z}{\mathrm{d}\theta} = z |h(t_0)|^2,
\]
where
\[
\text{Im } h(t_0) = \text{Im } \left( \frac{-\tau_1 t_0}{|t_0 - \tau_1|^2} + \frac{-\tau_2 t_0}{|t_0 - \tau_2|^2} + \frac{bt_0}{|t_0 - b|^2} \right).
\]

**Lemma 11** If \(a < 0\), then the function \(z(\theta)\) is negative and strictly increasing on \((\pi/2, \pi)\).

**Proof** We first note that (2.7) has only positive solutions in \(\xi\) by Lemma 6 and consequently \(z(\theta)\) is negative by (2.4) and Lemma 7. Since \(a < 0\), we have \(\tau_1, \tau_2 \in \mathbb{R}\). With the identities \(\tau_1 + \tau_2 = -1/a\) and \(\tau_1 \tau_2 = 1/a\), we obtain
\[
\tau_1 |t_0 - \tau_2|^2 + \tau_2 |t_0 - \tau_1|^2
\]
\[
= \tau_1 \left( \tau^2 - 2\tau_2 \tau \cos \theta + \tau_2^2 \right) + \tau_2 \left( \tau^2 - 2\tau_1 \tau \cos \theta + \tau_1^2 \right)
\]
\[
= -\frac{1}{a} \tau^2 - \frac{4}{a} \tau \cos \theta - \frac{1}{a^2}
\]
\[
= \frac{1}{a^2} \left( -a \tau^2 - 4a \tau \cos \theta - 1 \right)
\]
\[
= \frac{1}{a^2} \left( a \tau (-1/\xi - 2 \cos \theta) - 4a \tau \cos \theta - 1 \right)
\]
\[
= \frac{1}{a^2} \left( a \tau/\xi - 2a \tau \cos \theta - 1 \right) < 0,
\]
and the lemma follows from (3.3). \(\square\)
Lemma 12 If $0 \leq a \leq 1/4$, then the function $z(\theta)$ is negative and strictly increasing on $(\pi/2, \pi)$.

Proof From $0 \leq a \leq 1/4$, we conclude that $\tau_1$ and $\tau_2$ are negative and thus

$$\text{Im } h(t_0) = \frac{\tau_1 \tau \sin \theta}{|t_0 - \tau_1|^2} + \frac{\tau_2 \tau \sin \theta}{|t_0 - \tau_2|^2} - \frac{b \tau \sin \theta}{|t_0 - b|^2} < 0.$$ 

Also (2.4) and Lemmas 8 and 10 imply that $z(\theta)$ is negative on $(\pi/2, \pi)$. The lemma follows from (3.3). $\square$

We now consider the case $a > 1/4$ in which $\tau_1, \tau_2 \notin \mathbb{R}$. If we write $\tau_1 = x + iy$ and $\tau_2 = x - iy$, then

$$\text{Im } \left( -\tau_1 t_0 |t_0 - \tau_2|^2 - \tau_2 t_0 |t_0 - \tau_1|^2 \right) = 2 \tau \sin \theta \left( x \tau^2 \cos^2 \theta - 2 \tau \cos \theta (x^2 + y^2) + x (x^2 + y^2 + \tau^2 \sin^2 \theta) \right)$$

$$= 2 \tau \sin \theta \left( x \tau^2 - \frac{2}{a} \tau \cos \theta + \frac{x}{a} \right)$$

$$= 2 \tau \sin \theta \left( -\frac{\tau^2}{2a} - \frac{2}{a} \tau \cos \theta - \frac{1}{2a^2} \right)$$

$$= 2 \tau \sin \theta \left( -\frac{1}{2a} (-1/\zeta - 2 \cos \theta)^2 - \frac{2}{a} (-1/\zeta - 2 \cos \theta) \cos \theta - \frac{1}{2a^2} \right)$$

$$= 2 \tau \sin \theta \left( -\frac{1}{2a \zeta^2} + \frac{2}{a} \cos^2 \theta - \frac{1}{2a^2} \right)$$

$$= \frac{\tau \sin \theta}{a^2 \zeta^2} \left( -a + \zeta^2 (4a \cos^2 \theta - 1) \right). \quad (3.5)$$

If $-1/2\sqrt{a} < \cos \theta < 0$, then $\text{Im } h(t_0) < 0$. Consequently Lemmas 8 and 10 and (3.3) imply that $z(\theta)$ is negative and strictly increasing on $(\pi/2, \cos^{-1}(-1/2\sqrt{a}))$.

For the remainder of this section, we will show $z(\theta)$ is strictly increasing when $a > 1/4$ and

$$-1 < \cos \theta < -\frac{1}{2\sqrt{a}}. \quad (3.6)$$

From Lemma 9, it suffices to show $\text{Im } h(t_0) > 0$. We first show that (3.5) is positive or equivalently

$$\zeta < \frac{-a}{\sqrt{4a^2 \cos^2 \theta - a}}. \quad (3.7)$$
Since $\zeta < -1$, this claim is trivial if
\[-1 < \frac{-a}{\sqrt{4a^2 \cos^2 \theta} - a}.\]
To prove (3.7) for the remaining case, we will show that the polynomial $f(\zeta, \theta)$ has no zero in $\zeta$ on the interval
\[- \frac{a}{\sqrt{4a^2 \cos^2 \theta} - a}, -1\]
by showing that this polynomial has one zero on each of the intervals
\[- \infty, - \frac{a}{\sqrt{4a^2 \cos^2 \theta} - a}], (-1, 0), \text{ and } (0, \infty). \quad (3.8)

We check the sign of $f(\zeta, \theta)$ at each of the endpoint of these intervals and apply the Intermediate Value Theorem. We first note that
\[f(0, \theta) = -a < 0.\]
Since the leading coefficient of $f(\zeta, \theta)$ satisfies
\[2 \cos \theta (1 - 4a \cos^2 \theta) > 0\]
by (3.6), we conclude $\lim_{\zeta \to -\infty} f(\zeta, \theta) = -\infty$ and $\lim_{\zeta \to +\infty} f(\zeta, \theta) = +\infty$. From Lemma 5, we obtain
\[f(-1, \theta) = -f^*(-1, \theta) > 0.\]

Lemma 13 Whenever $a > 1/4$, $b - 27a + 9 > 0$, and (3.6), we have
\[f\left(- \frac{a}{\sqrt{4a^2 \cos^2 \theta} - a}, \theta\right) > 0.\]

Proof The Cauchy inequality gives
\[26a + 1/a - 7 > 0.\]
We expand $f(\zeta, \theta)$ when $\zeta = -a/\sqrt{4a^2 \cos^2 \theta} - a$ and collect the terms according to $\zeta$
\[
\begin{align*}
\frac{4a^2 \cos^2 \theta - a}{a^2} f\left(- \frac{a}{\sqrt{4a^2 \cos^2 \theta} - a}, \theta\right) &= -12a \cos^2 \theta + b + 4 \cos^2 \theta + 1 + \frac{a}{\sqrt{4a^2 \cos^2 \theta} - a} 2 \cos \theta (-1 + 4a \cos^2 \theta) \\
&- \frac{\sqrt{4a^2 \cos^2 \theta} - a}{a} 2 \cos \theta (1 - 3a) - a
\end{align*}
\]
\[\begin{align*}
&= -12 a \cos^2 \theta + b + 4 \cos^2 \theta + 1 + 2 \cos \theta \sqrt{4a^2 \cos^2 \theta} - a \\
&\quad - 2 \cos \theta \sqrt{4a^2 \cos^2 \theta - a} \left( \frac{1}{a} - 3 \right) - a \\
&= -12 a \cos^2 \theta + b + 4 \cos^2 \theta + 1 + 2 \cos \theta \sqrt{4a^2 \cos^2 \theta} - a \left( 4 - \frac{1}{a} \right) - a \\
&> -12 a \cos^2 \theta + b + \frac{1}{4a} + 1 + 8 \cos \theta \sqrt{4a^2 \cos^2 \theta} - a \\
&\quad - 2 \cos \theta \frac{\sqrt{4a^2 \cos^2 \theta} - a}{a} - a \\
&\quad > -12 a \cos^2 \theta + b + \frac{1}{a} + 1 + 8 \cos \theta \sqrt{4a^2 \cos^2 \theta} - a \\
&\quad = 4a \cos^2 \theta + b + \frac{1}{a} + 1 - a \\
&\quad > 4a \frac{1}{4a} + b + \frac{1}{a} + 1 - a \\
&\quad = (b - 27a + 9) + (26a - 7 + 1/a) > 0.
\end{align*}\]

By the Intermediate Value Theorem, \( f(\zeta, \theta) \) has a zero on each of the interval in (3.8) and consequently it has no zero on 

\[\left[-\frac{a}{\sqrt{4a^2 \cos^2 \theta} - a}, -1\right].\]

Having proved that (3.5) is positive, we now show that the same conclusion holds for \( \text{Im} (h(t_0)) \). We multiply both sides of (3.4) by \( a^2 |t_0 - \tau_1|^2 |t_0 - \tau_2|^2 \) and obtain

\[\frac{a^2 |t_0 - \tau_1|^2 |t_0 - \tau_2|^2}{\tau \sin \theta} \text{Im} (h(t_0)) = -a + \frac{\zeta^2 (4a \cos^2 \theta - 1)}{\xi^2} - b \tau^2.\]

With (2.4) and (2.6), the right-side becomes

\[\frac{-a + \zeta^2 (4a \cos^2 \theta - 1)}{\xi^2} - \frac{b}{(1 + 2 \xi \cos \theta)^2}\]

or

\[\frac{(-\zeta^2 + a(2 \xi \cos \theta - 1)(2 \xi \cos \theta + 1))(1 + 2 \xi \cos \theta)^2 - b \xi^2}{\xi^2 (1 + 2 \xi \cos \theta)^2}.
\]

Using (2.7), we replace \(-b \xi^2\) by

\[\zeta (1 + 2 \xi \cos \theta)(2 \cos \theta + \zeta) - a(1 + 2 \xi \cos \theta)^3,
\]
cancel the factor $1 + 2\xi \cos \theta$, and collect the terms in the numerator by $\xi$ and it remains to show that

$$G(\xi) := -2a + 2\xi \cos \theta(1 - 3a) + \xi^3(2 \cos \theta)(4a \cos^2 \theta - 1) > 0. \quad (3.9)$$

In the first case when $a \leq 1/3$, (3.7) implies that

$$\xi^2 \cos^2 \theta > \frac{\xi^2 + a}{4a} > \frac{1 + a}{4a} \geq 1$$

or equivalently $\xi \cos \theta > 1$. With this inequality, (3.9) follows directly from

$$G(\xi) = 2\xi \cos \theta(1 - 3a) + 2(\xi^2(\xi \cos \theta)(4a \cos^2 \theta - 1) - a) > 0.$$  

On the other hand if $a > 1/3$, then we use (2.7) to solve for $-\xi^3(2 \cos \theta - 8a \cos^3 \theta)$ and reduce $G(\xi)$ to a quadratic polynomial in $\xi$

$$-3a + 4\xi \cos \theta(1 - 3a) + \xi^2 \left(-12a \cos^2 \theta + 4 \cos^2 \theta + b + 1\right),$$

which is at least

$$-3a + 4\xi \cos \theta(1 - 3a) + \xi^2(-12a \cos^2 \theta + 4 \cos^2 \theta + 27a - 8) \quad (3.10)$$

by (1.3). As a quadratic polynomial in $\xi$, the value of (3.10) at $-1$ is

$$4(3a - 1)(2 + \cos \theta - \cos^2 \theta) > 0$$

and its derivative is

$$4 \cos \theta(1 - 3a) + 2(27a - 8 + 4 \cos^2 \theta - 12a \cos^2 \theta)\xi$$

$$= 6a\xi + 4(1 - 3a) \left(\cos \theta - 2\xi + 2\xi \left(\cos^2 \theta - 1\right)\right) < 0$$

when $\xi < -1$. Thus (3.10) is positive for $\xi < -1$ and so is $G(\xi)$.

Having proved that $z(\theta)$ is strictly increasing on $(\pi/2, \pi)$, we conclude this section with the following lemma.

**Lemma 14**  The function $z(\theta)$ maps $(\pi/2, \pi)$ onto $I_{a,b}$.

**Proof** We will show that the limits of $z(\theta)$ when $\theta$ approaches $\pi/2$ and $\pi$ give the two endpoints of the interval $I_{a,b}$. Lemma 10 and (2.6) imply that $\lim_{\theta \to \pi/2} \tau(\theta) = 0$. Thus from (3.1) and the fact that $z(\theta)$ is monotone increasing, we conclude

$$\lim_{\theta \to \pi/2} z(\theta) = -\infty.$$
On the other hand, (2.8) implies that \( \lim_{\theta \to \pi} \zeta(\theta) = \zeta_0 \) which is the unique zero of
\[
(8a - 2)\zeta^3 + \zeta^2(-12a + b + 5) + (6a - 2)\zeta - a
\]
on \((-\infty, -1] \cup [1, \infty)\). The limit
\[
\lim_{\theta \to \pi} z(\theta) = \frac{\zeta_0^2}{(1 - 2\zeta_0)^3}
\]
follows from (2.4) and (2.6).

\[\square\]

4 The zeros of \( H_m(z) \)

We recall that for each \( \theta \in (\pi/2, \pi) \), the functions \( \tau(\theta) \) and \( z(\theta) \) are defined as in (2.6) and (2.4). We note that the three zeros \( t_0, t_1, t_2 = \tau(\theta)e^{\pm i\theta} \) and \( t_2 = \zeta(\theta)\tau(\theta) \) of
\[
1 + t + at^2 + zt^2(t - b)
\]
are distinct since they have different arguments. The Cauchy’s integral formula gives
\[
H_m(z) = \frac{1}{2\pi i} \oint_{|t| = \epsilon} \frac{dt}{(1 + t + at^2 + zt^2(t - b))t^{m+1}}.
\]
Since
\[
\lim_{R \to \infty} \oint_{|t| = R} \frac{dt}{(1 + t + at^2 + zt^2(t - b))t^{m+1}} = 0,
\]
we compute the residue of the integrand each distinct zero of \( (1 + t + at^2 + zt^2(t - b))t^{m+1} \) and obtain
\[
-zH_m(z) = \frac{1}{(t_0 - t_1)(t_0 - t_2)t_0^{m+1}} + \frac{1}{(t_1 - t_0)(t_1 - t_2)t_1^{m+1}} + \frac{1}{(t_2 - t_0)(t_2 - t_1)t_2^{m+1}}.
\]
The reduction of the right-side to (4.2) is the same as that in [5], which is provided below for completeness. From the expression above, we deduce that \( z \) is a nonzero root of \( H_m(z) \) if and only if
\[
\frac{1}{(t_0 - t_1)(t_0 - t_2)t_0^{m+1}} + \frac{1}{(t_1 - t_0)(t_1 - t_2)t_1^{m+1}} + \frac{1}{(t_2 - t_0)(t_2 - t_1)t_2^{m+1}} = 0.
\]
(4.1)

After multiplying the left-side of (4.1) by \( t_0^{m+3} \), and letting \( \zeta = t_2/(t_0e^{i\theta}) \) the left-side becomes

\[\square\]
\[
\frac{1}{(1 - e^{2i\theta})(1 - \zeta e^{i\theta})} + \frac{1}{(e^{2i\theta} - 1)(e^{2i\theta} - \zeta e^{i\theta})(e^{2i\theta})^{m+1}}
\]
\[
+ \frac{1}{(\zeta e^{i\theta} - 1)(\zeta e^{i\theta} - e^{2i\theta})(\zeta e^{i\theta})^{m+1}}
\]
or equivalently
\[
\frac{1}{e^{2i\theta}(-2i \sin \theta)(e^{-i\theta} - \zeta)} + \frac{1}{(2i \sin \theta)(e^{i\theta} - \zeta)(e^{2i\theta})^{m+2}}
\]
\[
+ \frac{1}{(\zeta - e^{-i\theta})(\zeta - e^{i\theta})(\zeta)^{m+1}(e^{i\theta})^{m+3}}.
\]
We multiply this expression by \((\zeta - e^{-i\theta})(\zeta - e^{i\theta})e^{i(m+3)\theta}\) and set the summation equal to zero to arrive at
\[
0 = (\zeta - e^{i\theta})e^{i(m+1)\theta} + \frac{e^{-i\theta} - \zeta}{2i \sin \theta} + \frac{1}{\zeta^{m+1}}
\]
\[
= (\zeta - e^{i\theta})e^{i(m+1)\theta} - (\zeta - e^{-i\theta})e^{-i(m+1)\theta}
\]
\[
+ \frac{2i \sin \theta}{\zeta^{m+1}}
\]
\[
= \frac{\zeta(e^{i(m+1)\theta} - e^{-i(m+1)\theta}) + e^{-i(m+2)\theta} - e^{i(m+2)\theta}}{2i \sin \theta}
\]
\[
= \frac{2i \zeta \sin ((m + 1)\theta) - 2i \sin ((m + 2)\theta) \cos \theta - 2i \cos ((m + 1)\theta) \sin \theta + 1}{\zeta^{m+1}}.
\]
We define the function \(g_m(\theta)\) on \((\pi/2, \pi)\) as in (4.2). By Lemma 8, \(g_m(\theta)\) has a vertical asymptote at \(\cos^{-1}(-1/(2\sqrt{a}))\) if \(a > 1/4\).

Lemma 15 Suppose \(1/4 < a\) and \(m \geq 6\). Let \(J_h \subset (\pi/2, \pi)\) be the interval
\[
\begin{cases}
\left(\frac{h-1}{m+1}\pi, \frac{h}{m+1}\pi\right) & \text{if } \lfloor(m+1)/2\rfloor + 2 \leq h \leq m + 1 \\
\left(\frac{\pi}{2}, \frac{h}{m+1}\pi\right) & \text{if } h = \lfloor(m+1)/2\rfloor + 1.
\end{cases}
\]
If
\[
\cos^{-1}\left(-\frac{1}{2\sqrt{a}}\right) \in J_h,
\]
then \(g(\theta)\) has at least two zeros in \(J_h\) whenever \(\lfloor(m+1)/2\rfloor + 2 \leq h \leq m\), and at least one zero whenever \(h = m + 1\) or \(h = \lfloor(m+1)/2\rfloor + 1\).
The vertical asymptote of \( g_m(\theta) \) at \( \cos^{-1}(-1/2\sqrt{a}) \) divides the interval \( J_h \) in (4.3) into two subintervals. We will show that each subinterval contains at least one zero of \( g_m(\theta) \) if \( \lfloor(m+1)/2\rfloor + 2 \leq h \leq m \). In the case \( h = m + 1 \), the subinterval on the left of the asymptote contains at least one zero of \( g_m(\theta) \). On the other hand, if \( h = \lfloor(m+1)/2\rfloor + 1 \), then the subinterval on the right contains at least one zero of \( g_m(\theta) \). We analyze these two subintervals in the two cases below.

We consider the first case when \( \theta \in J_h \) and \( \theta < \cos^{-1}(-1/2\sqrt{a}) \). From (4.2) and the inequality \( |\zeta(\theta)| > 1 \), we see that the sign of \( g_m(\theta) \) at the left-end point of \( J_h \), for \( \lfloor(m+1)/2\rfloor + 2 \leq h \leq m + 1 \), is \((-1)^{h-1}\). We now show that the sign of \( g_m(\theta) \) is \((-1)^{h-1}\) when \( \theta \rightarrow \cos^{-1}(-1/2\sqrt{a}) \). From Lemmas 8 and 10, we observe that \( \zeta(\theta) \rightarrow +\infty \) as \( \theta \rightarrow \cos^{-1}(-1/2\sqrt{a}) \). Since \( \theta \in J_h \), the sign of \( \sin((m+1)\theta) \) is \((-1)^{h-1}\) and consequently the sign of \( g_m(\theta) \) is \((-1)^{h-1}\) when \( \theta \rightarrow \cos^{-1}(-1/2\sqrt{a}) \) by (4.2). By the Intermediate Value Theorem, we obtain at least one zero of \( g_m(\theta) \) in this case.

Next we consider the case when \( \theta \in J_h \) and \( \theta > \cos^{-1}(-1/2\sqrt{a}) \). In this case the sign of \( g_m(\theta) \) at the right-end point of \( J_h \), for \( \lfloor(m+1)/2\rfloor + 1 \leq h \leq m \), is \((-1)^{h-1}\). Since \( \zeta(\theta) \rightarrow -\infty \) as \( \theta \rightarrow \cos^{-1}(-1/2\sqrt{a}) \) by Lemma 9 and the sign of \( \sin((m+1)\theta) \) is \((-1)^{h-1}\), the sign of \( g_m(\theta) \) is \((-1)^{h}\) as \( \theta \rightarrow \cos^{-1}(-1/2\sqrt{a}) \) and we obtain at least one zero of \( g_m(\theta) \) by the Intermediate Value Theorem.

We note that Lemma 1 allows us to ignore the case when an endpoint of \( J_h \) coincides with \( \cos^{-1}(-1/(2\sqrt{a})) \).

**Lemma 16** If \( a < 1/4 \), then the sign of \( g_m(\pi^-) \) is \((-1)^m\).

**Proof** As \( \theta \rightarrow \pi^- \), the leading coefficient of \( f(\xi, \theta) \) approaches \(-2 + 8a < 0\) and \( f(1, \theta) \) approaches \( 1 + a + b \geq 0 \). Thus, \( f(\xi, \theta) \) has a solution on \((1, \infty)\) when \( \theta \) is close to \( \pi \) and consequently \( \zeta(\theta) > 1 \) by the definition of \( \zeta(\theta) \) in Sect. 2. The result follows directly from (4.2) and the fact that

\[
\lim_{\theta \to \pi^-} \frac{\sin((m+1)\theta)}{\sin(\theta)} = (m+1)(-1)^m.
\]

With all the lemmas at our disposal, we now prove the sufficient condition of Theorem 2 for the two cases \( a \leq 1/4 \) and \( a > 1/4 \). In the first case, Lemma 8 shows that the function \( \zeta(\theta) \) is continuous on \((\pi/2, \pi)\). From the formula of \( g_m(\theta) \) in (4.2) and Lemma 16, this function changes its sign at the endpoints of \( J_h \), \( \lfloor(m+1)/2\rfloor + 2 \leq h \leq m + 1 \), in (4.3) and thus it has at least

\[
m - \lfloor(m+1)/2\rfloor = \lfloor m/2 \rfloor
\]

zeros on \((\pi/2, \pi)\). Each such zero gives us a real zero of \( H_m(z) \) by the monotone map \( z(\theta) \) and the reality of the zeros of \( H_m(z) \) follows from Lemma 2 and the Fundamental Theorem of Algebra. On the other hand, if \( a > 1/4 \), then we obtain at least \( \lfloor m/2 \rfloor - 1 \) of \( g_m(\theta) \) on the intervals \( J_h \), \( \lfloor(m+1)/2\rfloor + 2 \leq h \leq m \) by the same argument. By

\[
\text{(c) Springer}
\]
Lemma 15, the interval $J_h$ containing the vertical asymptote $\cos^{-1}\left(-1/2\sqrt{a}\right)$ gives us another zero of $g_m(\theta)$ and we conclude all the zeros of $H_m(z)$ are real. For the density of these zeros, the Intermediate Value Theorem shows that $\bigcup_{m=0}^{\infty} \mathcal{Z}(g_m(\theta))$ is dense on $(\pi/2, \pi)$. From Lemma 14, we conclude that $\bigcup_{m=0}^{\infty} \mathcal{Z}(H_m)$ is dense on $I_{a,b}$ since the map $z(\theta)$ is continuous.

5 The necessary condition for the reality of the zeros of $H_m(z)$

The initial setup to prove the necessary condition is similar to that in [5]. For completeness, we quickly review this setup and then focus on the key differences starting from Lemma 17. We recall some definitions (from [4]) related to the root distribution of a sequence of functions

$$f_m(z) = \sum_{k=1}^{n} \alpha_k(z) \beta_k(z)^m,$$

where $\alpha_k(z)$ and $\beta_k(z)$ are analytic in a domain $D$. We say that an index $k$ is dominant at $z$ if $|\beta_k(z)| \geq |\beta_l(z)|$ for all $l$ ($1 \leq l \leq n$). Let

$$D_k = \{z \in D : k \text{ is dominant at } z\}.$$

Let $\lim \inf \mathcal{Z}(f_m)$ be the set of all $z \in D$ such that every neighborhood $U$ of $z$ has a non-empty intersection with all but finitely many of the sets $\mathcal{Z}(f_m)$. Let $\lim \sup \mathcal{Z}(f_m)$ be the set of all $z \in D$ such that every neighborhood $U$ of $z$ has a non-empty intersection with infinitely many of the sets $\mathcal{Z}(f_m)$. The necessary condition for the reality of zeros of $H_m(z)$ relies on following theorem from Sokal ([4, Theorem 1.5]).

**Theorem 3** Let $D$ be a domain in $\mathbb{C}$, and let $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n$ ($n \geq 2$) be analytic functions on $D$, none of which is identically zero. Let us further assume a ‘no-degenerate-dominance’ condition: there do not exist indices $k \neq k'$ such that $\beta_k \equiv \omega \beta_{k'}$ for some constant $\omega$ with $|\omega| = 1$ and such that $D_k (= D_{k'})$ has nonempty interior. For each integer $m \geq 0$, define $f_m$ by

$$f_m(z) = \sum_{k=1}^{n} \alpha_k(z) \beta_k(z)^m.$$

Then $\lim \inf \mathcal{Z}(f_m) = \lim \sup \mathcal{Z}(f_m)$, and a point $z$ lies in this set if and only if either

(i) there is a unique dominant index $k$ at $z$, and $\alpha_k(z) = 0$, or
(ii) there are two or more dominant indices at $z$.

Using (4.1), we apply Theorem 3 with

$$\alpha_k(z) = \frac{1}{t_k} \prod_{i \neq k} \frac{1}{(t_i - t_k)} \quad \text{and} \quad \beta_k(z) = \frac{1}{t_k}.$$
and deduce that \( z \in \liminf \mathcal{Z}(H_m) = \limsup \mathcal{Z}(H_m) \) if and only if the two smallest (in modulus) zeros of \( P(t) + z Q(t) \) have the same modulus. Thus if we can find \( z \notin \mathbb{R} \) with this property then for large \( m \), not all the zeros of \( H_m(z) \) are real by the definition of \( \liminf \mathcal{Z}(H_m) \). The following lemma shows it is sufficient to find a suitable \( \zeta \).

**Lemma 17** Assume \( \cos \theta \neq 0 \). If \( \frac{1}{\zeta} \) is a nonreal solution of \( f^* \) such that \( |1/\zeta| < 1 \), then \( \zeta \tau^3 \notin \mathbb{R} \).

**Proof** Since \( \text{Arg} (\zeta) = -\text{Arg} (1/\zeta) \) and \( |\zeta| \neq |1/\zeta| \), we conclude that

\[
z + 1/\zeta \notin \mathbb{R}.
\]

As a consequence, (2.6) gives

\[
\tau (2 \cos \theta + \zeta) = - \left( \frac{1}{\zeta} + 2 \cos \theta \right) (2 \cos \theta + \zeta),
\]

which is nonreal after we expand the product. The lemma follows from (2.5).

From (2.4) and Lemmas 3 and 17, it suffices to find \( \theta^* \neq \pi/2 \) such that \( f^* \) has a solution \( \zeta^* \notin \mathbb{R} \) with \( |\zeta^*| < 1 \). We will find such a \( \theta^* \) for the two cases \( a < -b - 1 \) and \( a > (b + 9)/27 \).

**Case** \( a < -b - 1 \)

From (2.9), we observe that the roots in \( \zeta \) of \( f^*(\zeta, \pi/2) \) are \( 0, \pm i \sqrt{-\frac{1}{2} + b}/a \). The inequalities \( a < -b - 1 < 0 \) imply that there is \( \theta^* \) sufficiently close to \( \pi/2 \) so that \( f^* \) has a nonreal root inside the open unit disk.

**Case** \( a > (b + 9)/27 \)

We first note that the discriminant of \( f^*(\zeta, \theta) \) as a cubic polynomial in \( \zeta \) is a polynomial in \( \cos^2 \theta =: x \), which is denoted by \( \Delta(x) \). Computer algebra shows that the discriminant of \( \Delta(x) \) in \( x \) is

\[
-65536b \left( 27 a^2 b - 9 ab + b + 1 \right)^3 \left( ab^2 + b + 1 \right) < 0
\]

and thus \( \Delta(x) \) has a unique real zero denoted by \( x' \). Since

\[
\Delta(0) = 4a(b + 1)^3 > 0, \quad (5.1)
\]

\[
\Delta(1) = -4(27a - b - 9)(ab^2 + b + 1) < 0, \quad (5.2)
\]

we have \( 0 < x' < 1 \). By the definition of \( x' \), the polynomial \( f^*(\zeta, \cos^{-1} \sqrt{x'}) \) has a multiple zero which is denoted by \( \zeta' \).
We will show later that $|\zeta'| < 1$. Assuming this inequality, we choose $\sqrt[x']{\cos \theta^*} \ll 1$. From (5.1) and (5.2), we conclude that the discriminant $f^*(\zeta, \cos^{-1} \sqrt[x']{\theta})$ is negative. Since $\zeta'$ is a multiple zero of $f^*(\zeta, \cos^{-1} \sqrt[x']{\theta})$ and $f^*(\zeta, \theta^*)$ has only one real zero, the inequality $|\zeta'| < 1$ implies that for $\cos \theta^*$ sufficiently close to $\sqrt[x']{\theta}$, $f^*(\zeta, \theta^*)$ has a non-real zero inside the open unit disk.

For the remainder of this case, we prove $|\zeta'| < 1$. We note that $\zeta'$ is the zero of the remainder of the polynomial division of the cubic polynomial $f^*(\zeta, \cos^{-1} \sqrt[x']{\theta})$ and its derivative. Since this remainder is linear in $\zeta'$, we can easily solve for $\zeta'$ by using a computer algebra package

$$
\zeta' = -\frac{\sqrt[x']{\cos \theta^*}}{\cos \theta^*} < 1.
$$

(5.3)

Next, the rational function

$$
r(x) := \frac{1 + 6a + b - 3ab + (4 - 24a)x}{3a + 3ab + (4 - 12a)x}
$$

is decreasing because its derivative

$$
\frac{dr}{dx} = -\frac{4(27a^2b - 9ab + b + 1)}{(3ab - 12ax + 3a + 4x)^2} < 0
$$

since

$$
1 + b - 9ab + 27a^2b > 1 + ab > 0,
$$

where we apply the inequality $27a > b + 9$ to the first expression. We note that the inequality above also implies that the numerator and the denominator of (5.3) cannot be both zero since

$$
\frac{1 + 6a + b - 3ab}{24a - 4} \neq \frac{3a + 3ab}{12a - 4}.
$$

We also have

$$
r\left(\frac{9a + b + 1}{4(9a - 2)}\right) = -1,
$$

$$
r\left(\frac{-6ab + 3a + b + 1}{12a}\right) = 1.
$$

(5.4)

Next, we show that $r(x)$ is continuous on

$$
\left(\frac{-6ab + 3a + b + 1}{12a}, \frac{9a + b + 1}{4(9a - 2)}\right)
$$

 Springer
by showing that the vertical asymptote of \( r(x) \) is outside this interval. Indeed, we have

\[
\frac{3a + 3ab}{12a - 4} > \frac{9a + b + 1}{4(9a - 2)} > \frac{-6ab + 3a + b + 1}{12a} \tag{5.5}
\]

since the difference of the first two terms and the last two terms are

\[
\frac{27a^2b - 9ab + b + 1}{4(3a - 1)(9a - 2)} > 0
\]

and

\[
\frac{27a^2b - 9ab + b + 1}{12a(3a - 1)} > 0,
\]

respectively. As a consequence \(|r(x)| < 1\) for all \( x \) in (5.4). From (5.3), if \( x' \) is in this interval, then \(|\xi'| < 1\).

On the other hand, if \( x' \) does not belong to this interval, then the inequalities \( \Delta(0) > 0 \) and

\[
\Delta\left(\frac{9a + b + 1}{4(9a - 2)}\right) = -\frac{(27a - b - 9) (27a^2b - 9ab + b + 1)^2}{(9a - 2)^3} < 0,
\]

and the Intermediate Value Theorem imply that

\[
0 < x' < \frac{-6ab + 3a + b + 1}{12a} < \frac{9a + b + 1}{4(9a - 2)} \tag{5.6}
\]

and

\[
\Delta\left(\frac{-6ab + 3a + b + 1}{12a}\right) = \frac{(2ab - a + b + 1) (27a^2b - 9ab + b + 1)^2}{27a^3} < 0.
\]

We note that the first inequality implies \( r(x') > 1 \) and the second inequality implies \( b < 1/2 \). From (5.3) and (5.6), to prove \( |\xi'| < 1 \), it suffices to show

\[
\sqrt{9a - 2} < \frac{9a - 2}{\sqrt{9a + b + 1}}.
\]

By the monotonicity and continuity of \( r(x) \) given in (5.5), this inequality is equivalent to

\[
x' > r^{-1}\left(2 \sqrt{\frac{9a - 2}{9a + b + 1}}\right)
\]
where, with a computer algebra, the right-side is

$$-3ab + (-6ab - 6a) \sqrt{\frac{9a-2}{9a+b+1}} + 6a + b + 1$$

$$4(6a - 2) \sqrt{\frac{9a-2}{9a+b+1}} + 4(1 - 6a).$$

By the Intermediate Value Theorem applied to $\Delta(x)$, it remains to prove at the value $x$ above, $\Delta(x) > 0$. By a computer algebra, such value of $\Delta(x)$ is the product of

$$-4(27a^2b - 9ab + b + 1)^2$$

and

$$\frac{(18a^2b + 72a^2 + 2ab - 16a - 2b - 2) \left( \sqrt{\frac{9a-2}{9a+b+1}} - 1 \right) - 3 + 12a - 3b + 16ab + ab^2}{(9a + b + 1) \left( (6a - 2) \sqrt{\frac{9a-2}{9a+b+1}} - 6a + 1 \right)^3}.$$

The denominator of the expression above is negative since

$$(6a - 2) \sqrt{\frac{9a-2}{9a+b+1}} - 6a + 1 < (6a - 2) - 6a + 1 < 0.$$

To show the numerator is positive we need to show

$$\left( 1 - \frac{-3 + 12a - 3b + 16ab + ab^2}{18a^2b + 72a^2 + 2ab - 16a - 2b - 2} \right)^2 - \frac{9a-2}{9a+b+1} < 0.$$

Note that the left-side is the product of $-27a + b + 9 < 0$ and

$$\frac{108a^3b^2 + 432a^3b + a^2b^4 + 20a^2b^3 + 64a^2b^2 - 36a^2b - 2ab^3 - 14ab^2 - 4ab + 8a + b^2 + 2b + 1}{4(9a + b + 1) (9a^2b + 36a^2b + ab - 8a - b - 1)^2}.$$

We apply the inequalities $a \geq 1/3$ and $0 \leq b < 1/2$ to conclude that the four differences $432a^3b - 36a^2b, 20a^2b^3 - 2ab^3, 64a^2b^2 - 14ab^2, 8a - 4ab$ are non-negative. Thus the numerator of the expression above is positive and we complete this case.

We end the paper by proposing the following problem.

**Problem 1** Characterize all linear polynomials $A(z), B(z),$ and $C(z)$ such that the zeros of $P_m(z)$ defined in (1.1) are real for all $m$.

**References**

1. Borcea, J., Bøgvad, R., Shapiro, B.: On rational approximation of algebraic functions. Adv. Math. 204(2), 448–480 (2006)
2. Egecioglu, O., Redmond, T., Ryavec, C.: From a polynomial Riemann hypothesis to alternating sign matrices. Electron. J. Comb. 8(1), 51 (2001)
3. Goh, W., He, M., Ricci, P.E.: On the universal zero attractor of the Tribonacci-related polynomials. Calcolo 46(2), 95–129 (2009)
4. Sokal, A.: Chromatic roots are dense in the whole complex plane. Comb. Probab. Comput. 13(2), 221–261 (2004)

5. Tran, K., Zumba, A.: Zeros of polynomials with four-term recurrence. Involve J. Math. 11(3), 501–518 (2018)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.