EPOCHS OF REGULARITY FOR WILD HÖLDERS-CONTINUOUS SOLUTIONS OF THE HYPODISSIPATIVE NAVIER-STOKES SYSTEM

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ABSTRACT. We consider the hypodissipative Navier-Stokes equations on $[0,T] \times \mathbb{T}^d$ and seek to construct non-unique, Hölder-continuous solutions with epochs of regularity (smooth almost everywhere outside a small singular set in time), using convex integration techniques. In particular, we give quantitative relationships between the power of the fractional Laplacian, the dimension of the singular set, and the regularity of the solution. In addition, we also generalize the usual vector calculus arguments to higher dimensions with Lagrangian coordinates.

1. INTRODUCTION

Fix $d \geq 3$. We consider the hypodissipative Navier-Stokes equations

\[
\begin{aligned}
\frac{\partial v}{\partial t} + (-\Delta)^{\gamma} v + \text{div} (v \otimes v) + \nabla p &= 0 \\
\text{div} v &= 0
\end{aligned}
\]  

(1.1)

on the periodic domain $\mathbb{T}^d$, where $0 < \gamma < 1$ denotes the strength of the fractional dissipation, $v : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ is the velocity field and $p : [0,T] \times \mathbb{T}^d \to \mathbb{R}$ is the pressure.

Recently, in the study of hydrodynamic turbulence, significant attention has been directed towards problems such as Onsager’s conjecture, which roughly states that the kinetic energy of an ideal fluid may fail to be conserved when the regularity is less than $\frac{1}{3}$.

The starting point for much of this work in recent years is a nonuniqueness result, using ideas from convex integration, due to De Lellis and Székelyhidi Jr [10]. A sequence of results, e.g. in [7, 9, 2, 14, 13, 3, 4, 5], and the references cited in these works, developed these ideas to tackle Onsager’s conjecture. In [13], Isett reached the conjectured threshold of $\frac{1}{3}$ — for the three-dimensional Euler equation on the torus, using Mikado flows and a delicate gluing technique. Further developments include Buckmaster–De Lellis–Székelyhidi, Jr.–Vicol [3], which forms the main basis for this work; we will refer to the strategy in [3] as the Onsager scheme. The scheme produces a weak solution that can attain any arbitrary energy profile (this is sometimes referred to as energy profile control).

After this recent progress, the main techniques of convex integration have also been used to construct various kinds of “wild” solutions (nonunique, or failing to conserve energy) for the Euler equations, the Navier-Stokes equations, as well as the fractional Navier-Stokes equations [4, 6, 11, 8]. For the Navier-Stokes equations,
the dissipation term \((-\Delta)v\) can dominate the nonlinear term \(\text{div}(v \otimes v)\), and this presents a difficult obstruction to convex integration. At present, this issue can be avoided by either using spatial intermittency (at the cost of non-uniform control on the solution) or considering the fractional Laplacian \((-\Delta)^\gamma\) instead. For an explanation of intermittency, as well as more history and references, we refer the interested readers to [5].

One direction of research has looked into the construction of wild solutions with *epochs of regularity* (that is, solutions that are smooth almost everywhere outside a temporal set of small dimension); this was carried out for the hyperdissipative Navier-Stokes equations (using intermittency) in [1], the Navier-Stokes equations (using intermittency) in [8], and then for the Euler equations (not using intermittency) in [12].

We note that this goal stands in contradiction to the desire to have energy profile control, since whenever the solution is smooth the energy cannot increase. These approaches make use of the Onsager scheme, with several refinements to the gluing approach of Isett [13], combined with estimates on the overlapping (glued) regions. Because energy correction is no longer required, the scheme is also simplified.

In this paper, we look at the case of the hypodissipative Navier-Stokes equations without using spatial intermittency, and try to determine for which values of \(\gamma\) in \((-\Delta)^\gamma\) one can construct spatially H"older-continuous solutions with epochs of regularity. In addition, we also extend the arguments involving the Biot-Savart operator and vector calculus (cf. the treatment in [3]) to higher dimensions.

We now state our main theorem.

**Theorem 1.** Fix \(d \geq 3\). Let \(V_1\) and \(V_2\) be smooth solutions to (1.1) such that \(\int_{\tau^d} (V_1 - V_2) (t) = 0\) for all \(t\).

For every positive \(\beta, \gamma\) such that \(\beta < \frac{1}{3}\) and \(\beta + 2\gamma < 1\), there exist

\[
\begin{cases}
  v \in C^0_t C^{\beta}_{x} \cap L^1_t C^{\beta}_{x} \\
  \mathcal{B} \subset [0, T] \text{ closed}
\end{cases}
\]

where

- \(v\) is a nonunique weak solution to (1.1) given initial data \(V_1(0)\).
- \(v\) agrees with \(V_1\) near \(t = 0\), and agrees with \(V_2\) near \(t = T\)
- \(\beta_1 = \left(\frac{1-\beta}{2}\right)^\top\), \(\dim_{\text{Hausdorff}}(\mathcal{B}) \leq \left(\frac{1+\beta}{2(1-\beta)}\right)^\top\)
- \(v|_{\mathbb{R}^d \times T^d}\) is smooth.

In particular, Theorem 1 implies that, with what we currently know about the Onsager scheme, the best fractional Laplacian we can handle (using only temporal intermittency) is \((-\Delta)^{\frac{1}{2}}\), which is quite a distance away from the full Navier-Stokes equation. This confirms the heuristic that without spatial intermittency, we want the dissipation term \((-\Delta)^\gamma v\) to be dominated by the nonlinear term \(\text{div}(v \otimes v)\). In addition, because \(L^\infty_t C^\beta_x\) is supercritical for the \(\gamma\)-hypodissipative Navier-Stokes equations when \(\beta + 2\gamma < 1\), we expect that this constraint is sharp.

The proof of Theorem 1 makes use of the strategy of the Onsager scheme, and in particular follows from an iterative proposition based on the local existence
theory, combined with a modification of Isett’s gluing technique to preserve the “good” temporal regions. The main difficulty is to optimize the length of the overlapping regions (where the cutoff functions meet). The iterative proposition is presented in Section 2, where it is shown to imply Theorem 1. The proof of the iterative proposition itself is deferred to Section 3, where, after a brief mollification argument, we reduce the issue to a series of technical estimates (first, a collection of estimates for the gluing construction, which we then treat in Section 4; and then a perturbation result arising from convex integration, which we treat in Section 5).

Remark 2. As usual (see, e.g. [11, 3]), any \( C^0_t C^\alpha_x \) solution with \( \alpha \in (0, \frac{1}{3}) \) is automatically a \( C^{\alpha-}_{t,x} \) solution. For any given \( \beta \in (0, \frac{1}{3}) \), we can construct a wild \( v \in C^\beta_{t,x} \).

For \( \gamma < \frac{1}{3} \) and \( \varepsilon \in (0, \frac{1}{3}) \), by interpolation, this leads to the construction of wild solutions in \( C^0_t C^{\frac{1}{3}-\varepsilon}_x \cap L^1_t C^{\frac{1}{3}+\varepsilon}_x \cap L^2_t C^\frac{1}{3}_x \), with the singular set having dimension less than 1, and to construction of wild solutions in \( C^0_t C^{\frac{1}{3}+\varepsilon}_x \cap L^1_t C^\frac{1}{3}_x \), with the dimension of the singular set bounded by \( \frac{1}{2} + \).

On the other hand, in the range \( \frac{1}{3} \leq \gamma < \frac{1}{2} \), for each \( \beta < 1 - 2\gamma \), the dimension of the singular set is bounded by

\[
\left( \frac{1 + \beta}{2(1 - \beta)} \right)^+ < \frac{1 - \gamma}{2\gamma}.
\]

Further comments and open questions. The arguments we use to prove Theorem 1 immediately lead to an analogous result for the Euler equations, since we treated \( (-\Delta)^\gamma v \) as an error term. In particular, in the proof of Theorem 1, we show nonuniqueness for \( C^0_t C^{\frac{1}{3}-}_x \cap L^2_t C^{\frac{1}{3}}_x \) solutions. In the Euler context, this can be compared to the nonuniqueness of \( L^2_t C^{\frac{1}{3}}_x \) solutions in [8, Theorem 1.10]. In [8], rather than using the Onsager scheme, the authors use spatial intermittency. As a consequence, the solution they construct is not spacetime continuous; their singular set \( B \) can have arbitrarily small Hausdorff dimension, and their scheme also works in two dimensions.

Two open questions remain. The first is to ask if we can further minimize the dimension of the singular set \( B \), as suggested in [12]. The second question of interest is to determine whether the construction can be adapted to construct solutions that obey some form of energy inequality. Both questions lead to natural problems that we hope to consider in future works.

Outline of the Paper. In Section 2, we specify our notational conventions and introduce the main iterative scheme underlying the proof of Theorem 1. The iterative step is formulated in Proposition 4, which is then used to prove Theorem 1. The proof of Proposition 4 is the subject of Section 3. The proof is reduced to two technical lemmas (a collection of gluing estimates, and a perturbation argument) which are treated in Section 4 and Section 5, respectively. A short appendix recalls several geometric preliminaries used throughout the paper.
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Figure 1.1. Regularity and dimension parameters. Given any $p \in (1, \infty)$ and $\tilde{\beta}$, we identify $L^p_x C^\tilde{\beta}_x$ as the point $\left( \frac{1}{p}, \tilde{\beta} \right)$. Assuming it is away from the rejected region, by simple geometry, we can construct a wild solution in $L^p_x - C^\tilde{\beta}_x$ or $L^p_x C^\tilde{\beta}_x$ by constructing one in $C^0_\alpha_x \cap L^1_x \left( \frac{1-\beta}{2} \right)^-$ where $\tilde{\beta} \leq \frac{1}{2p} + \left( 1 - \frac{3}{2p} \right) \beta$, and get the corresponding $\dim_{\text{Hausdorff}} (B)$. If $p = \frac{3}{2}$, we can arbitrarily choose $\beta < (1 - 2\gamma) \wedge \frac{1}{3}$.

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2. Preliminaries and the iteration scheme

We begin by establishing some notational conventions. We will write $A \lesssim_{x, \sim} B$ for $A \leq CB$, where $C$ is a positive constant depending on $x$ and not $y$. Similarly, $A \sim_{x, \sim} B$ means $A \lesssim_{x, \sim} B$ and $B \lesssim_{x, \sim} A$. We will omit the explicit dependence when it is either not essential or obvious by context.

For any real number $x$, we write $x+$ or $x^+$ to denote some $y \in (x, x + \varepsilon)$ where $\varepsilon$ is some arbitrarily small constant. Similarly we write $x-$ or $x^-$ for some $y \in (x - \varepsilon, x)$.

For any $N \in \mathbb{N}_0$ and $\alpha \in (0, 1)$, we write

$$\|f\|_N = \|f\|_{CN}, \quad [f]_N = \|\nabla^N f\|_0, \quad [f]_{N+\alpha} = \|\nabla^N f\|_{C^{0,\alpha}}.$$
and
\[ \|f\|_{N+\alpha} = \|f\|_{C^{N,\alpha}} := \|f\|_N + [f]_{N+\alpha}, \]
where \([\cdot]_{C^{\alpha,\alpha}}\) denotes the Hölder seminorm. We will often make use of the following elementary inequality,
\[ \|fg\|_r \lesssim \|f\|_0 [g]_r + [f]_r \|g\|_0, \]
which holds for any \(r > 0\).

**Definition 3.** For any \(T > 0\), \(\nu > 0\), vector field \(v\) and \((2,0)\)-tensor \(R\) on \([0,T] \times \mathbb{T}^d\), we say \((v,R)\) solves the \((\nu,\gamma,T)\)-fNSR equations (fractional Navier-Stokes-Reynolds) if there is a smooth pressure \(p\) such that
\[ \begin{cases} \partial_t v + \nu(-\Delta)^\gamma v + \text{div } v \otimes v + \nabla p = \text{div } R \\ \text{div } v = 0, \end{cases} \tag{2.1} \]
(2.1)
When \(R = 0\), we also say \(v\) solves the \((\nu,\gamma,T)\)-fNS equations.

### 2.1. Formulation of the iterative argument

As we described in the introduction, the proof of Theorem 1 is based on an iterative argument. We now outline the main setup of the iteration, and establish notation that will be used throughout the remainder of the paper. We begin by fixing \(\gamma \in (0,1)\) and \(\beta < \frac{1}{3}\) with \(\beta + 2\gamma < 1\).

For any natural number \(q \in \mathbb{N}_0\), we set
\[ \lambda_q := \left[ a^{b^q} \right] \tag{2.2} \]
\[ \delta_q := \lambda_q^{-2\beta} \tag{2.3} \]
with \(a \gg 1\), 0 < \(b - 1 \ll 1\) (to be chosen later). We remark that \(\lambda_q\) will be the frequency parameter (made an integer for phase functions), while \(\delta_q\) will be the pointwise size of the Reynolds stress.

With \(\alpha > 0\) sufficiently small, and \(\sigma > 0\) (to be chosen later), we set
\[ \epsilon_q := \lambda_q^{-\sigma} \tag{2.4} \]
\[ \tau_q := C_q \delta_q^{-\frac{1}{2}} \lambda_q^{-1-3\alpha} \tag{2.5} \]
where \(C_q \sim 1\) is an inessential constant such that \(\epsilon_{q-1} \tau_{q-1}^{-1} \in \mathbb{N}_1\) (for gluing purposes). For convenience, from this point on, we will not write out \(C_q\) explicitly. The parameter \(\tau_q\) will be the time of local existence for regular solutions, while the quantity \(\epsilon_q \tau_q\) will be the length of the overlapping region between two temporal cutoffs.

We now formulate the main inductive hypothesis on which the construction is based. Let \(T \geq 1\) and \(\nu \in (0,1]\) be arbitrary constants. For the first step of the induction, we pick any positive \(\epsilon_{-1}, \tau_{-1}\) such that \(5\epsilon_{-1} \tau_{-1} = \frac{T}{4}\).

For every \(q \in \mathbb{N}_0\), we assume that there exist \(v_q\) and \(R_q\) smooth such that,

(i) \((v_q,R_q)\) solves the \((\nu,\gamma,T)\)-fNS equations in \((2.1)\),
(ii) we have the estimates
\[ \|v_q\|_{L^\infty} \leq 1 - \delta_q^{1/2}, \tag{2.6} \]
\[ \|\nabla v_q\|_{L^\infty} \leq M \delta_q^{1/2} \lambda_q, \tag{2.7} \]
\[ \| R_q \|_{L^\infty} \leq \epsilon_q \delta_{q+1} \lambda_q^{-3\alpha}, \quad (2.8) \]

where \( M \) is a universal geometric constant (depending on \( d \)), and

(iii) letting \( B_q = \bigcup \mathcal{I}_i \mathcal{B}^{i,q} \) denote the current “bad” set consisting of disjoint closed intervals of length \( 5\epsilon_{q-1}\tau_{q-1} \), and letting

\[ \mathcal{G}_q = [0,T] \setminus \mathcal{B}_q = \bigcup \mathcal{I}_i^{\mathcal{B}^{i,q}} \]

denote the current “good” set consisting of disjoint open intervals, we have

\[ R_q |_{\mathcal{G}_q + B(0,\epsilon_{q-1}\tau_{q-1})} \equiv 0, \quad (2.9) \]

where \( \mathcal{G}_q + B(0,\epsilon_{q-1}\tau_{q-1}) \) denotes the \( \epsilon_{q-1}\tau_{q-1} \)-neighborhood of \( \mathcal{G}_q \), and within this neighborhood we have the improved bounds

\[ \| v_q \|_{N+1} \lesssim_N \delta_{q-1}^{1/2} \lambda_{q-1} \epsilon_{q-1}^{-N}, \quad \text{for all } N \geq 0. \quad (2.10) \]

We note the presence of \( \epsilon_q \) in (2.8), which serves to compensate for the sharp time cutoffs in our gluing construction.

The main iterative proposition is given by the following statement.

**Proposition 4** (Iteration for the \((\nu,\gamma,T)\)-fNSR equations). We fix

\[ 0 < b - 1 \ll_{\beta, \gamma} 1, \quad (2.11) \]

\[ 0 < \sigma < \left( \frac{(b-1)(1-\beta - 2b\beta)}{b+1} \right), \quad (2.12) \]

\[ 0 < \alpha \ll_{\sigma, b, \beta, \gamma} 1, \quad (2.13) \]

\[ a \gg_{\alpha, \sigma, b, \beta, \gamma} 1, \]

and suppose that \( v_q \) and \( R_q \) are smooth functions which satisfy the properties (i)–(iii) above. Then there exist \( v_{q+1} \) and \( R_{q+1} \) satisfying those same properties but with \( q \) replaced by \( q + 1 \). Moreover, we have

\[ \| v_q - v_{q+1} \|_0 + \lambda_{q+1}^{-1} \| v_q - v_{q+1} \|_1 \leq M \delta_{q+1}^{1/2} \quad (2.14) \]

and \( v_{q+1} = v_q \) on \( \mathcal{G}_q \times \mathbb{T}^d \), \( \mathcal{G}_q \subset \mathcal{G}_{q+1} \), \( |\mathcal{B}_{q+1}| \leq \epsilon_q |\mathcal{B}_q| \).

**Remark 5.** We crucially remark that the parameters \( b, \sigma, \alpha, \) and \( a \) only depend on \( \beta, \gamma \) and \( d \). In particular, they do not depend on \( q, T \) or \( \nu \) (as long as \( \nu \leq 1 \) and \( T \geq 1 \)).

The proof of Proposition 4 is given in Section 3 below. In the remainder of this section, we use this result to prove Theorem 1.

**Proof of Theorem 1 via Proposition 4.** Let \( \eta \) be a smooth temporal cutoff on \([0,T]\) such that \( 1_{[0,\frac{T}{2}]} \geq \eta \geq 1_{[0,\frac{3}{4}T]} \), and set

\[ v_0 = \eta V_1 + (1-\eta) V_2. \]

Since the dissipative terms are linear and \( \int_{\mathbb{T}^d} (V_1 - V_2) = 0 \), if we set

\[ R_0 = \partial_t \eta \mathcal{R} (V_1 - V_2) - \eta (1-\eta) (V_1 - V_2) \otimes (V_1 - V_2) \]

where \( \mathcal{R} \) is the antidivergence operator defined in Appendix A, then \((v_0, R_0)\) solves the \((1, \gamma, T)\)-fNSR equations from (2.1).
We now aim to apply Proposition 4. To do this, we rescale in time by a positive parameter $\zeta$, i.e.

$$v_0^\zeta(t, x) = \zeta v_0(\zeta t, x), \quad R_0^\zeta = \zeta^2 R_0(\zeta t, x)$$

Then $(v_0^\zeta, R_0^\zeta)$ solves the $(\zeta, \gamma, \zeta^{-1}T)$-fNS equations.

We now recall that we are allowed to make $\zeta$ arbitrarily small because of Remark 5. For $\zeta = \zeta(T, V_1, V_2, a, b, \alpha, \sigma, \beta)$ small enough, the conditions (2.6)-(2.8) of (ii) in the inductive hypothesis for Proposition 4 are satisfied for the case $q = 0$, and we also have

$$\zeta^{-1}T > 1 > \zeta.$$ 

In addition, (iii) is satisfied by letting $B^\zeta_q = \left[\frac{T}{3\zeta}, \frac{2T}{3\zeta}\right]$.

Repeatedly applying Proposition 4 for the $(\zeta, \gamma, \zeta^{-1}T)$-fNS equations, we get a sequence $(v^\zeta_q, R^\zeta_q, B^\zeta_q)$ such that

(a) $(v^\zeta_q)_{q \in \mathbb{N}_0}$ converges in $C_t^\alpha C_x^{\beta -}$ to some $v^\zeta$.
(b) $\|R^\zeta_q\|_{C_t^\alpha C_x^{\beta -}} \to 0$ as $q \to \infty$, and
(c) $B^\zeta_q+1 \subset B^\zeta_q$ and $v^\zeta = v^\zeta_q$ on $G^\zeta_q \times T^d$.

As a consequence of (c), $v^\zeta$ is smooth on each set $G^\zeta_q \times T^d$. Moreover, $v^\zeta$ is a weak solution of the $(\zeta, \gamma, \zeta^{-1}T)$-fNS equations.

To conclude, we note that the transformations

$$v_q(t, x) := \zeta^{-1} v^\zeta_q(\zeta^{-1} t, x)$$

$$v(t, x) := \zeta^{-1} v^\zeta(\zeta^{-1} t, x)$$

$$B_q := \zeta^{-1} B^\zeta_q$$

invert the time-rescaling. The bad set is then

$$B := \bigcap_q B_q.$$ 

Moreover, noting that the choice of $V_2$ was arbitrary, the solution $v$ is nonunique.

We now verify that $B$ has the desired Hausdorff dimension. Note that the set $B_q$ consists of $\sim \tau_q^{-1} \prod_{i=1}^{q-1} \epsilon_i$ intervals of length $\sim \epsilon_q \tau_q$. It therefore follows that

$$\dim_{\text{Hausdorff}}(B) \leq \dim_{\text{box}}(B) \leq \lim_{q \to \infty} \frac{\ln \left(\tau_q^{-1} \prod_{i=1}^{q-1} \epsilon_i\right)}{\ln \left(\epsilon_q^{-1} \tau_q^{-1}\right)}$$

$$= \lim_{q \to \infty} \frac{\ln(a)b^\beta \left(1 + 3\alpha - \beta - \frac{\sigma}{2b}\right)}{\ln(a)b^\beta \left(1 + 3\alpha + \sigma - \beta\right)}$$

$$= 1 - \frac{\sigma b}{(b-1)(1 + 3\alpha + \sigma - \beta)}.$$ 

Choosing $\alpha$ sufficiently small, and then choosing $\sigma$ sufficiently close to $\frac{(b-1)(1-\beta-2b\beta)}{b+1}$ and $b > 1$ sufficiently close to 1, we get the bound

$$\dim(B) \leq \left(\frac{1 + \beta}{2(1-\beta)}\right)^+.$$
as desired.

It remains to choose $\beta_1$ to ensure that the solution lies in $C^0_t C_x^{\beta-} \cap L^1_t C_x^{\beta_1}$. For this, note that since $|B_{q+1}| \lesssim \prod_{i=1}^q \epsilon_i$, we have

$$
\|v_{q+1} - v_q\|_{L^1_t C_x^{\beta_1}} \lesssim \delta^{1/2}_{q+1} \lambda^{\beta_1}_{q+1} \left( \prod_{i=1}^q \epsilon_i \right)
$$

The right-hand side is then summable in $q$, provided that

$$
-\beta + \beta_1 - \frac{\sigma}{b-1} < 0.
$$

We may therefore choose $\beta_1 < \beta + \frac{\sigma}{b-1} < \frac{1-b}{b+1} < \frac{1-\beta}{2}$, which completes the proof. \(\square\)

3. Proof of Proposition 4

In this section, we give the proof of the main iterative result, Proposition 4, which was used to prove Theorem 1 in the previous section. As we described in the introduction, the argument makes use of three steps – a mollification procedure, a gluing construction, and a perturbation result arising from convex integration. To simplify the exposition, we discuss each step below, and after isolating a few technical lemmas whose proofs are deferred to Section 4 and Section 5, we give the proof of Proposition 4.

We define the length scale of mollification

$$
\ell_q := \frac{\sqrt{2^{q+1}}}{\lambda^{1+\frac{2}{q}+\frac{2}{2\delta^2}}_q}. \quad (3.1)
$$

To simplify notation, we will often abbreviate $\ell_q$ as $\ell$ (unless otherwise indicated).

For technical convenience, we record several useful parameter inequalities. The first set of these are essential conversions,

$$
\epsilon^\frac{1}{2} q \tau^\frac{1}{2} \delta^\frac{1}{2} q e^{-1} \ll 1 \quad (3.2)
$$

$$
\lambda_q \ll \epsilon^{-1} q \ll \lambda^2_q \quad (3.3)
$$

$$
\delta^\frac{1}{2} q \lambda_{q-1} \epsilon^{2-2\alpha} q \ll \epsilon q \tau q \delta q+1. \quad (3.4)
$$

Indeed, the bound (3.2) comes from $\alpha > 0$, while the bound $\epsilon^{-1} q \ll \lambda_q+1$ in (3.3) follows by recalling that that $\alpha$ can be made arbitrarily small by (2.13), so that (2.12) implies $\sigma < 2 (b-1) (1-\beta)$, and thus

$$
-\beta b + \beta - 1 - \frac{\sigma}{2} + b > 0. \quad (3.5)
$$

Similarly, by neglecting $\alpha$, (3.4) comes from $(b-1)(1-\beta) > 0$, which is obvious.

In order to partition the time intervals for gluing, we also need the bound

$$
\epsilon q \tau q \ll \epsilon^{-1} q \tau q-1 \quad (3.6)
$$

which comes from the inequality $\sigma < (1-\beta)(b-1)$, a consequence of (2.12). We also have the special case

$$
\tau_0 \ll \frac{1}{15} \leq \frac{T}{15} = \epsilon_{-1} \tau_{-1}.
$$
because $T \geq 1$. This allows $a, b, \beta, \alpha$ to be independent of $T$, and we use this crucial fact in the proof of Theorem 1.

To control the dissipative term in the gluing construction, we will also find it useful to observe the bound

$$\tau_q \ell_q^{-a - 2\gamma} \lesssim 1, \quad (3.7)$$

which, since $\alpha$ is negligible by (2.13), comes from the inequality $\sigma < (1 - \beta) \left( \frac{1 - 2\gamma}{\gamma} \right) - 2b\beta$. Because of (2.11) and $\beta + 2\gamma < 1$, this is implied by (2.12).

Next, to control the stress size for the induction step, we note that

$$\epsilon_q^{-1} \delta_q^{\frac{1}{2}} \delta_q^{2} \lambda_{q-1 + 10\alpha} \lambda_q^{1 + 10\alpha} \lesssim \epsilon_q \delta_q + 2\lambda_q^{-4\alpha}, \quad (3.8)$$

which, after neglecting $\alpha$, comes from

$$-b\beta - \beta - b + 1 + \sigma \leq -b\sigma - b^2 (2\beta), \quad (3.8)$$

which is precisely (2.12).

Lastly, for the dissipative error in the final stress, we observe that

$$\delta_q^{1/2} \lambda_{q+1}^{-1/2} \gamma + 10\alpha \lesssim \epsilon_q \delta_q + 2\lambda_q^{-4\alpha}, \quad (3.9)$$

which comes from

$$-\beta - 1 + 2\gamma < b (-2\beta) - \sigma$$

which in view of (2.11) and $\beta + 2\gamma < 1$, is a consequence of (2.12).

3.1. The mollification step. With $\ell$ as defined in (3.1) and $\psi$ a smooth standard radial mollifier in space of length $\ell$, we set

$$v_\ell := \psi_\ell \ast v_q. \quad (3.10)$$

By standard mollification estimates and (2.7) we have

$$\|\nabla^N v_\ell\|_{L^\infty} \lesssim N \ell^{-N} \lesssim N \delta_q^{1/2} \lambda_q \ell^{-N+1} \quad (3.11)$$

for any $N \in \mathbb{N}_1$. Moreover, by setting

$$R_\ell := \psi_\ell \ast R_q + v_\ell \otimes v_\ell - \psi_\ell \ast (v_q \otimes v_q)$$

the pair $(v_\ell, R_\ell)$ solves the $(\nu, \gamma, T)$-NSR equations.

Moreover, by using (2.6), (2.7), (2.8), (3.1), (3.3) and the usual commutator estimate

$$\|(f * \psi_\ell) (g * \psi_\ell) - (fg) * \psi_\ell\|_{C^l} \lesssim \ell^{2-r} \|f\|_{C^r} \|g\|_{C^l}$$

for $f, g \in C^\infty (T^d)$ and $l > 0, r \geq 0$ (see, e.g. [3, Proposition A.2]), we obtain

$$\|R_\ell\|_{L^\infty} \lesssim N \ell^{-N+1} \|R_q\|_{C^0} + \ell^{2-N-\alpha} \|v_q\|_{C^1}^2 \lesssim \ell^{-N+1} \delta_q^{1/2} \lambda_q \ell^{N+1} \quad (3.12)$$

for all $N \in \mathbb{N}_0$. 
3.2. The gluing step. Recalling that $\tau_q$ was defined in (2.5), we set

$$t_j := j\tau_q.$$ 

Let $J$ be the set of indices $j$ such that

$$[t_j - 2\epsilon_q \tau_q, t_j + 3\epsilon_q \tau_q] \subset B_q.$$ 

These are the “bad” indices that will be part of $B_{q+1}$ and we have $\#(J) \sim \tau_q^{-1} \prod_{p=1}^{q-1} \epsilon_p$.

Then we define $J^* = \{ j \in J | j + 1 \in J \}$. These are the indices where we will apply the following local wellposedness result from [11].

Lemma 6 (Proposition 3.5 in [11]). Given $\alpha \in (0,1)$, $\nu \in (0,1]$, any divergence-free vector field $u_0 \in C^\infty(\mathbb{T}^d)$ and $T \lesssim \alpha ||u_0||_{1+\alpha}^{-1}$, there exists a unique solution $u$ to the $(\nu, \gamma, T)$-fNS equations on $[0,T] \times \mathbb{T}^d$ such that $u(0,\cdot) = u_0$ and

$$||u||_{N+\alpha} \lesssim ||u_0||_{N+\alpha} \quad \text{for all } N \in \mathbb{N}.$$ 

Using this lemma, for any $j \in J^*$, we define $v_j$ to be the solution of the hypodissipative Navier-Stokes equations

$$\partial_t v_j + \nu(-\Delta)^\gamma v_j + \text{div} v_j \otimes v_j + \nabla p_j = 0$$ 
$$\text{div} v_j = 0$$ 
$$v_j(t_j) = v_\ell(t_j)$$

on $[t_j, t_j+2] \times \mathbb{T}^N$. This is possible as

$$\tau_q \lesssim \frac{2\alpha}{\delta_q^{1/2} \lambda_q}$$

$$\lesssim \frac{\ell}{\delta_q^{1/2} \lambda_q}$$

$$\lesssim \frac{1}{||v_\ell(t_j)||_{1+\alpha}},$$

where we have implicitly used (3.3) and (3.11).

We then have the bounds

$$||v_j||_{L^\infty_t C^{N+\alpha}_x([t_j, t_j+2] \times \mathbb{T}^N)} \lesssim_N ||v_\ell(t_j)||_{C^{N+\alpha}_x}$$

$$\lesssim_N \delta_q^{1/2} \lambda_q \ell^{-N+1-\alpha},$$

for $N \in \mathbb{N}_1$.

Recall that $B_q = \bigcup_i I^{b,q}_i$ is closed and $G_q = [0,T] \setminus B_q = \bigcup_i I^{b,q}_i$ is open. Let $\{\chi^b_j\}_j \cup \{\chi^q_i\}$, be a partition of unity of $[0,T]$ such that

- $\text{supp} \chi^b_j \subset [t_j, t_j+1 + \epsilon_q \tau_q]$ for $j \in J^*$,
- $\chi^b_j \equiv 1$ in $[t_j + \epsilon_q \tau_q, t_j+1]$ for $j \in J^*$,
- $\text{supp} \chi^q_i \subset I^{b,q}_i + B(0, \tau_q + \epsilon_q \tau_q)$, and
- for $N \in \mathbb{N}_0$,

$$||\partial^N_t \chi^q_i||_{L^\infty} + ||\partial^N_t \chi^b_j||_{L^\infty} \lesssim_N (\epsilon_q \tau_q)^{-N}.$$  

(3.16)
Note that because of (2.9) and (3.6), we have $\mathcal{R}_q = 0$ on $\text{supp} \chi^q_i$.

We now define the glued solution
\[ v_q := \sum_i \chi^q_i v_q + \sum_{j \in J^*} \chi^q_j v_j. \quad (3.17) \]

We also define $\mathcal{B}_{q+1}$ as the union of the intervals $[t_j - 2\epsilon_q \tau_q, t_j + 3\epsilon_q \tau_q]$ which lie in $\mathcal{B}_q$.

We will show in Section 4 that there exists a smooth $R_q$ such that $(v_q, R_q)$ is a solution to (2.1). For convenient notation, we define the material derivatives
\[ D_{t, \ell} := \partial_t + v_\ell \cdot \nabla \]
\[ D_{t,q} := \partial_t + v_q \cdot \nabla \]

We will then obtain the following estimates, which will be used to prove Proposition 4.

**Proposition 7** (Gluing estimates). For any $N \in \mathbb{N}_0$, we have
\[ \|v_q - v_\ell\|_{N+\alpha} \lesssim_N \epsilon_q \tau_q \delta_{q+1} \ell^{-N-1+\alpha} \quad (3.18) \]
\[ \|v_q\|_{N+1} \lesssim_N \delta^2_q \lambda_q \ell^{-N} \quad (3.19) \]
\[ \|R_q\|_{N+\alpha} \lesssim_N \delta_{q+1} \ell^{-N+\alpha} \quad (3.20) \]
\[ \|D_{t,q} R_q\|_{N+\alpha} \lesssim_N (\epsilon_q \tau_q)^{-1} \delta_{q+1} \ell^{-N+\alpha} \quad (3.21) \]

We will prove Proposition 7 in Section 4 below.

We remark that the estimate (3.18) of Proposition 7, when combined with (3.2), implies in particular
\[ \|v_q - v_\ell\|_{\alpha} \lesssim \epsilon_q \tau_q \delta_{q+1} \ell^{-1+\alpha} \lesssim \delta^2_{q+1} \ell^\alpha. \quad (3.22) \]

We also note that, because future modifications of the solution from this point on will only happen in the temporal regions $[t_j - \epsilon_q \tau_q, t_j + 2\epsilon_q \tau_q]$ (where $j \in J$), we will later have $v_{q+1} = v_q$ and $R_q = 0$ outside those temporal regions. Furthermore, (2.9) and (2.10) will hold with $q$ changed to $q + 1$. 
3.3. Perturbation step. The third key step in the proof of Proposition 4 is a perturbation lemma arising from the convex integration framework. We state this result in the next proposition.

**Proposition 8 (Convex integration).** There is a smooth solution \((v_{q+1}, R_{q+1})\) to (2.1) which satisfies \(v_{q+1} = \tau_q\) outside the temporal regions \([t_j - \epsilon_q \tau_q, t_j + 2\epsilon_q \tau_q]\) \((j \in \mathcal{J})\), along with the estimates

\[
\|v_{q+1} - \tau_q\|_0 + \frac{1}{\lambda_{q+1}} \|v_{q+1} - \tau_q\|_1 \leq \frac{M}{2} \delta_{q+1}^\frac{5}{2}, \tag{3.23}
\]

and

\[
\|R_{q+1}\|_0 \lesssim \epsilon_{q+1} \delta_{q+2} \lambda_{q+1}^{-\frac{2a}{3}}, \tag{3.24}
\]

where \(M > 0\) is a universal geometric constant (depending on \(d\)).

The proof of this proposition will be given in Section 5 below.

3.4. Proof of the main iterative proposition. With the above tools in hand, we are now ready to prove Proposition 4, making use of Proposition 7 and Proposition 8, which are proved in Sections 4 and 5, respectively.

**Proof of Proposition 4.** We first observe that

\[
\|v_q - v_{q+1}\|_0 \leq \|v_q - v_\ell\|_0 + \|v_\ell - \tau_q\|_0 + \|\tau_q - v_{q+1}\|_0 \leq C \epsilon_q \delta_{q+1} \lambda_q^{-\frac{2a}{3}} + C \delta_{q+1}^\frac{4}{5} \epsilon^\alpha + \frac{M}{2} \delta_{q+1}^\frac{5}{2} \leq M \delta_{q+1}^\frac{5}{2},
\]

where \(C\) is shorthand for the implied constants of (3.10) and (3.22). Since

\[
\max\{\epsilon_q \delta_{q+1} \lambda_q^{-\frac{2a}{3}}, \epsilon^\alpha\} \to 0
\]
as \(a \to \infty\), the last inequality is true provided that \(a\) is chosen sufficiently large.

Similarly, for large \(a\), because of (2.7), (3.19) and (3.23), we have

\[
\|v_q - v_{q+1}\|_1 \leq (\|v_q\|_1 + \|\tau_q\|_1 + \|\tau_q - v_{q+1}\|_1) \leq M \delta_{q+1}^\frac{5}{2} \lambda_{q+1}.
\]

We have thus shown (2.14), which in turn implies (2.6) and (2.7) with \(q\) replaced by \(q + 1\). On the other hand, (3.24) yields the next iteration of (2.8) (for large enough \(a\)). Recalling that all the desired properties regarding \(B_{q+1}\) were established in Subsection 3.2, this completes the proof of the proposition. \(\square\)

4. Gluing estimates

In this section, we construct \(R_q\) and prove the gluing estimate results in Proposition 7, which played a key role in the proof of Proposition 4 in the previous section.

We recall that \(\tau_q\) was defined in (3.17). We first note that (3.19) follows immediately from (3.14) and (2.10). On the other hand, (3.20) and (3.21) hold automatically outside the overlapping temporal regions \([t_j, t_j + \epsilon_q \tau_q]\) \((j \in \mathcal{J})\), since \(\tau_q\) is an exact solution and the stress is therefore zero in this regime. We now consider what happens near the overlapping regions.
4.1. **Bad-bad interface.** Consider any index \( j \in \mathcal{J}^* \) such that \( j + 1 \in \mathcal{J}^* \). Then \( \text{supp}(\chi^b_j,\chi^b_{j+1}) \) lies in an interval of length \( \epsilon_q \tau_q \) where \( \tau_q \) satisfies

\[
\partial_t \tau_q + \nu(-\Delta)\tau_q + \text{div} \tau_q \otimes \tau_q + \nabla p_q = \text{div} \mathcal{R}_q,
\]

where

\[
\mathcal{R}_q = \partial_t \chi^b_j \mathcal{R}((v_j - v_{j+1}) - \chi^b_j (1 - \chi^b_j) (v_j - v_{j+1}) \otimes (v_j - v_{j+1}).
\]

and \( \mathcal{R} \) is as defined in Appendix A.

To treat the fractional Laplacian term, we recall the following lemma from [11].

**Lemma 9** (Theorem B.1 in [11]). For any \( \gamma, \epsilon > 0 \) and \( \beta \geq 0 \) such that \( \beta + 2\gamma + \epsilon \leq 1 \), we have

\[
\|(-\Delta)^\gamma f\|_{\beta} \lesssim \|f\|_{\beta + 2\gamma + \epsilon} \quad \forall f \in C^{\beta + 2\gamma + \epsilon}.
\]

As usual, we decompose \( v_j - v_{j+1} = (v_j - v_t) - (v_{j+1} - v_t) \). By symmetry, we only need to prove estimates for \( v_j - v_t \).

**Proposition 10.** For \( N \in \mathbb{N}_0 \) and \( t \in (t_j, t_j + 2\tau_q) \), we have

\[
\|v_j - v_t\|_{N+\alpha} \lesssim \epsilon_q \tau_q \delta_q \ell^{-N-1+\alpha} \quad (4.2)
\]

\[
\| (\partial_t + v_t \cdot \nabla + \nu(-\Delta)^\gamma) (v_j - v_t) \|_{N+\alpha} \lesssim \epsilon_q \delta_q \ell^{-N-1+\alpha} \quad (4.3)
\]

\[
\| D_{t,\ell} (v_j - v_t) \|_{N+\alpha} \lesssim \epsilon_q \delta_q \ell^{-N-1+\alpha} \quad (4.4)
\]

**Proof.** We observe that

\[
(\partial_t + v_t \cdot \nabla + \nu(-\Delta)^\gamma) (v_j - v_t) = -(v_t - v_j) \cdot \nabla v_j - \nabla (p_t - p_j) + \text{div} \mathcal{R}_t,
\]

and

\[
\nabla (p_t - p_j) = \mathcal{P}_1 (-(v_t - v_j) \cdot \nabla v_j - (v_t - v_j) \cdot \nabla v_j + \text{div} \mathcal{R}_t),
\]

where \( \mathcal{P}_1 \) is as defined in Appendix A, and (A.1) was implicitly used.

Then, as usual, (4.2) and (4.3) follow from Gronwall and modified transport estimates exactly as in [11, Proposition 5.3] (which in turn mirrors [3, Proposition 3.3]).

To derive (4.4) from (4.3), we observe that

\[
\|(-\Delta)^\gamma (v_j - v_t)\|_{N+\alpha} \lesssim \|v_j - v_t\|_{N+2\alpha+2\gamma} \lesssim \epsilon_q \tau_q \delta_q \ell^{-1-2\gamma-N} \lesssim \epsilon_q \delta_q \ell^{-N-1+\alpha} \quad (4.7)
\]

where the last inequality comes from (3.7).

We have proven (3.18) for any \( t \in (t_j, t_j + 2\tau_q) \).

Now we define the potentials \( z_j := \mathcal{B}v_j, z_t := \mathcal{B}v_t \), where \( \mathcal{B} \) is as defined in Appendix A.

**Proposition 11.** For \( N \in \mathbb{N}_0 \) and \( t \in (t_j, t_j + 2\tau_q) \):

\[
\|z_j - z_t\|_{N+\alpha} \lesssim \epsilon_q \tau_q \delta_q \ell^{-N-1+\alpha} \quad (4.8)
\]

\[
\|(\partial_t + v_t \cdot \nabla + \nu(-\Delta)^\gamma) (z_j - z_t)\|_{N+\alpha} \lesssim \epsilon_q \delta_q \ell^{-N-1+\alpha} \quad (4.9)
\]

\[
\| D_{t,\ell} (z_j - z_t) \|_{N+\alpha} \lesssim \epsilon_q \delta_q \ell^{-N-1+\alpha} \quad (4.10)
\]
Proof. First, we note that for any divergence-free vector field $X$ and 2-form $\omega$, we have

$$
X^i \partial_i \omega_{jk} = \partial^j \left( X^i \partial_i w_{jk} \right) - \partial_t \left( \partial^j X^i \omega_{jk} \right)
$$

$$(\partial^j \omega_{jk}) \partial^k X^i = \partial^j (\omega_{jk} \partial^k X^i)$$

$$
[d, \nabla_X] \omega = dx^i \wedge \partial_t \nabla_X \omega - \nabla_X (dx^i \wedge \partial_t \omega)
$$

$$= dx^i \wedge (\partial_t \omega_j) (\partial_j \omega) = \partial_j ((\partial_t X^j) (dx^i \wedge \omega)).$$

Because we only care about estimates instead of how the indices contract, we can write in schematic notation (neglecting indices and linear combinations):

$$
\nabla_X \delta \omega = \delta (\nabla_X \omega) + \text{div} (\nabla X \ast \omega)
$$

$$(\delta \omega) \cdot \nabla X = \text{div} (\nabla X \ast \omega)$$

$$[d, \nabla_X] \omega = \text{div} (\nabla X \ast \omega)$$

Define $\tilde{z} := z_t - z_j$. Then we have $d \tilde{z} = 0$ and $\delta \tilde{z} = v_\ell - v_j$. From (4.5) and the schematic identities above, we have,

$$
\delta (\partial_t \tilde{z} + \nabla_{v_\ell} \tilde{z} + \nu (-\Delta)^{\gamma} \tilde{z}) = \text{div} (\nabla v_\ell, \tilde{z} - \text{div} (p_\ell - p_j) + \text{div} R_\ell
$$

$$d (\partial_t \tilde{z} + \nabla_{v_\ell} \tilde{z} + \nu (-\Delta)^{\gamma} \tilde{z}) = \text{div} (\nabla v_\ell \ast \tilde{z}),$$

and thus

$$\partial_t \tilde{z} + \nabla_{v_\ell} \tilde{z} + \nu (-\Delta)^{\gamma} \tilde{z} = (-\Delta)^{-1} d \circ \text{div} (\nabla v_\ell, \tilde{z} + R_\ell) + (-\Delta)^{-1} \delta \circ \text{div} (\nabla v_\ell \ast \tilde{z}),$$

where $v_\ell, \tilde{z}$ could be $v_j$ or $v_\ell$ (they obey the same estimates by Lemma 6). As $(-\Delta)^{-1} d \circ \text{div}$ and $(-\Delta)^{-1} \delta \circ \text{div}$ are Calderón-Zygmund operators, we have

$$
\| (D_{t, \ell} + \nu (-\Delta)^{\gamma}) \tilde{z} \|_{N+\alpha} \leq \| \nabla v_\ell \|_{N+\alpha} + \| \nabla v_\ell \|_{N+\alpha} \| \tilde{z} \|_{N+\alpha} + \| R_\ell \|_{N+\alpha}
$$

$$\leq \epsilon^{-N+\alpha} \lambda q \delta q \| \tilde{z} \|_{N+\alpha} + \epsilon^{-N+\alpha} \lambda q \delta q \| \tilde{z} \|_{N+\alpha} + \epsilon^{-N+\alpha} \epsilon q \delta q + 1
$$

$$\leq \epsilon^{-N+\alpha} \epsilon q \delta q + 1 \| \tilde{z} \|_{N+\alpha} + \epsilon^{-N+\alpha} \epsilon q \delta q + 1$$

(4.11)

where we have used (3.13) to pass to the last line.

By the modified transport estimate in [11, Proposition 3.3], we also have

$$
\| \tilde{z} (t) \|_{\alpha} \leq \int_{t_j}^{t} \| (D_{t, \ell} + \nu (-\Delta)^{\gamma}) \tilde{z} (s) \|_{\alpha} ds
$$

(4.12)

$$\leq \epsilon^{-\alpha} \epsilon q \delta q + 1 \int_{t_j}^{t} \| \tilde{z} (s) \|_{\alpha} ds + \epsilon q \delta q + 1 \epsilon \alpha
$$

By Gronwall, we obtain (4.8) for $N = 0$. For $N \geq 1$, we observe that

$$
\| z_j - z_\ell \|_{N+\alpha} \leq \| \nabla (z_j - z_\ell) \|_{N-1+\alpha}
$$

$$= \| \nabla B (v_j - v_\ell) \|_{N-1+\alpha}
$$

$$\leq \| v_j - v_\ell \|_{N-1+\alpha},$$

where we have implicitly used the facts that $\nabla B$ is Calderón-Zygmund, and that $\| f \|_{L^\infty} \leq \| \nabla f \|_{L^\infty}$ for any mean-zero $f \in C^1 (\mathbb{T}^d)$ (Poincaré inequality). Then by (4.2), we obtain (4.8). From here, we note that (4.11) and (4.8) imply (4.9).
It remains to show (4.10). For this, we argue as in (4.7) and use (3.7) to write
\[
\|(-\Delta)^\gamma (z_j - z_t)\|_{N+\alpha} \lesssim \|z_j - z_t\|_{N+2\alpha+2\gamma}
\lesssim \epsilon_q \tau_q \delta_{q+1} \ell^{-2\gamma-N}
\lesssim \epsilon_q \delta_{q+1} \ell^{-N+\alpha},
\]
as desired. \hfill \Box

Combining (3.16), (4.8) and (4.2), as well as the boundedness of the Calderón-Zygmund operator $\mathcal{R}\delta$, we obtain
\[
\|\partial_t \chi_j R(v_j - v_{j+1})\|_{N+\alpha} = \|\partial_t \chi_j R\delta(z_j - z_{j+1})\|_{N+\alpha} \lesssim_N \delta_{q+1} \ell^{-N+\alpha},
\tag{4.13}
\]
and
\[
\|\chi_j^b (1 - \chi_j^b) (v_j - v_{j+1}) \otimes (v_j - v_{j+1})\|_{N+\alpha} \lesssim_N (\epsilon_q \tau_q \delta_{q+1} \ell^{-1+\alpha})^2 \ell^{-N},
\tag{4.14}
\]
for $N \in \mathbb{N}_0$ and $t \in (t_{j+1}, t_{j+1} + \epsilon_q \tau_q)$.

Before we proceed, we will need a usual singular-integral commutator estimate from [3] to handle the Calderón-Zygmund operator $\mathcal{R}\text{curl}$.

**Lemma 12** (Proposition D.1 in [3]). Let $\alpha \in (0, 1), N \in \mathbb{N}_0, T$ be a Calderón-Zygmund operator and $b \in C^{N+1, \alpha}$ be a divergence-free vector field on $\mathbb{T}^d$. Then for any $f \in C^{N+\alpha}(\mathbb{T}^d)$, we have
\[
\|[T, b \cdot \nabla] f\|_{N+\alpha, T} \lesssim_N \|b\|_{1+\alpha} \|f\|_{N+\alpha} + \|b\|_{N+1+\alpha} \|f\|_{\alpha}
\]
We are now able to establish the relevant estimates for $\overline{R}_q$.

**Proposition 13.** $\overline{R}_q$ in (4.1) admits the bounds
\[
\|\overline{R}_q\|_{N+\alpha} \lesssim_N \delta_{q+1} \ell^{-N+\alpha},
\tag{4.15}
\]
\[
\|\partial_t + \nabla q \cdot \nabla \overline{R}_q\|_{N+\alpha} \lesssim_N (\epsilon_q \tau_q)^{-1} \delta_{q+1} \ell^{-N+\alpha},
\tag{4.16}
\]
for $N \in \mathbb{N}_0$ and $t \in (t_{j+1}, t_{j+1} + \epsilon_q \tau_q)$.

**Proof.** We observe that (4.13) and (4.14) imply
\[
\|\overline{R}_q\|_{N+\alpha} \lesssim_N \delta_{q+1} \ell^{-N+\alpha} (1 + \epsilon_q \tau_q \delta_{q+1} \ell^{-1+\alpha})^2,
\]
and then (4.15) follows from (3.2).

On the other hand, we have
\[
\|\partial_t + \nabla q \cdot \nabla \overline{R}_q\|_{N+\alpha} \lesssim \|D_{t, \ell} \overline{R}_q\|_{N+\alpha} + \|\nabla q \cdot \overline{R}_q\|_{N+\alpha}
\]
where
\[
D_{t, \ell} \overline{R}_q = (\partial_t \chi_j^b) R\delta (z_j - z_{j+1}) + (\partial_t \chi_j^b) R\delta (z_j - z_{j+1}) + (\partial_t \chi_j^b) [v_{t} \cdot \nabla, R\delta] (z_j - z_{j+1})
\]
\[
+ \partial_t \left( (\chi_j^b)^2 - \chi_j^b \right) (v_j - v_{j+1}) \otimes (v_j - v_{j+1})
\]
\[
+ \left( (\chi_j^b)^2 - \chi_j^b \right) (D_{t, \ell} (v_j - v_{j+1}) \otimes (v_j - v_{j+1})
\]
\[
+ (v_j - v_{j+1}) \otimes D_{t, \ell} (v_j - v_{j+1}))
\]

The term involving $[v_t \cdot \nabla, R\delta]$ can be handled by Lemma 12. Then by (3.16), (4.15), Propositions 10 and 11, we conclude
\[
\| (\partial_t + \tau_q \cdot \nabla) R_q \|_{N+\alpha} \lesssim_N (\tau_q \delta_q)^{-1} \delta_{q+1}^2 \ell^{-N+\alpha} \\
+ \tau_q^{-1} \delta_{q+1} \ell^{-N+\alpha} \\
+ \tau_q \delta_q^2 \ell^{-2-N+2\alpha}
\]
which then yields (4.16) because of (3.2).

4.2. Good-bad interface. Next we consider any pair of indices $i$ and $j$ such that $\chi_i^b \neq 0$. By construction, we observe that $\text{supp}(\chi_i^b) \chi_j^b$ lies in an interval of length $\sim \epsilon_q \tau_q$, where $R_q$ is 0.

Without loss of generality (i.e., depending on whether $\chi_i^b$ or $\chi_j^b$ comes first in time), in this interval $v_q$ satisfies
\[
\partial_t v_q + \nu(-\Delta) v_q + \text{div} v_q \otimes v_q + \nabla p_q = \text{div} R_q
\]
where
\[
R_q = \partial_t \chi_i^b R(v_q - v_j) - \chi_i^b (1 - \chi_i^b)(v_q - v_j) \otimes (v_q - v_j)
\]
which is a perfect analogue of (4.1).

As before, we decompose
\[
v_q - v_j = (v_q - v_t) - (v_j - v_t)
\]
The estimates for $v_j - v_t$ are as above. Turning to $v_q - v_t$, the relevant estimates are given by the following result.

**Proposition 14.** For $N \in \mathbb{N}_0$ and $t \in G_q + B(0, \tau_q + \epsilon_q \tau_q)$:
\[
\| v_q - v_t \|_{N+\alpha} \lesssim_N \epsilon_q \tau_q \delta_{q+1} \ell^{-N-1+\alpha} 
\]
\[
\| (\partial_t + v_t \cdot \nabla + \nu(-\Delta) ) (v_q - v_t) \|_{N+\alpha} \lesssim_N \epsilon_q \delta_{q+1} \ell^{-N-1+\alpha} 
\]
\[
\| D_{t,t} v_q - v_t \|_{N+\alpha} \lesssim_N \epsilon_q \delta_{q+1} \ell^{-N-1+\alpha}
\]

**Proof.** By standard mollification estimates (cf. [7, Lemma 2.1]), we have
\[
\| v_q - v_t \|_{N+\alpha} \lesssim_N \ell_q \| v_q \|_{N+1+\alpha} \\
\lesssim_N \delta_{q-1}^2 \lambda_q^{-1} \ell_q^{-N-1+\alpha} \ll \delta_{q-1}^2 \lambda_q^{-1} \ell_q^{-N-1+\alpha} \\
\lesssim \epsilon_q \tau_q \delta_{q+1} \ell^{-N+1+\alpha}
\]
where we used (2.10) to pass to the second line, and (3.4) to pass to the last line. Thus (4.18) is proven.

Then as $R_q = 0$ on this temporal region, we have an analogue of (4.5) and (4.6), namely
\[
(\partial_t + v_t \cdot \nabla + \nu(-\Delta) ) (v_q - v_t) = - (v_t - v_q) \cdot \nabla v_q - \nabla (p_t - p_q) + \text{div} R_t
\]
and
\[
\nabla (p_t - p_q) = \mathcal{P}_1 (-(v_t - v_q) \cdot \nabla v_t - (v_t - v_q) \cdot \nabla v_q + \text{div} R_t)
\]
Thus we can estimate $\| \nabla (p_t - p_q) \|_{N+\alpha}$ and then
\[
\| (\partial_t + v_t \cdot \nabla + \nu(-\Delta) ) (v_q - v_t) \|_{N+\alpha}
\]
to obtain (4.19). We then argue as in (4.7) (replacing $v_j$ by $v_q$) to obtain (4.20). □

Note that we have fully proven (3.18).

To proceed, we define the potentials $z_q := Bv_q, z_\ell := Bv_\ell$. By observing that Proposition 14 plays the exact same role as Proposition 10, and by arguing exactly as in Proposition 11 (replacing $v_j$ with $v_q$, and $z_j$ with $z_q$) we obtain

$$\|z_q - z_\ell\|_{N+\alpha} \lesssim \epsilon_q \tau_q \delta_{q+1} \ell^{-N+\alpha} \quad (4.23)$$

$$\|\partial_t + v_\ell \cdot \nabla + \nu (-\Delta)^\gamma (z_q - z_\ell)\|_{N+\alpha} \lesssim \epsilon_q \delta_{q+1} \ell^{-N+\alpha} \quad (4.24)$$

$$\|D_{t,\ell} (z_q - z_\ell)\|_{N+\alpha} \lesssim \epsilon_q \delta_{q+1} \ell^{-N+\alpha} \quad (4.25)$$

for any $N \in \mathbb{N}_0$ and $\ell \in \mathcal{G}_q + B(0, \tau_q + \epsilon_q \tau_q)$.

Then, as with (4.13) and (4.14), we have

$$\|\partial_t \chi_i^g \mathcal{R} (v_q - v_j)\|_{N+\alpha} \lesssim \epsilon_q \delta_{q+1} \ell^{-N+\alpha} \quad (4.26)$$

$$\|\chi_j^g (1 - \chi_i^g) (v_q - v_j) \otimes (v_q - v_j)\|_{N+\alpha} \lesssim \epsilon_q \tau \delta_{q+1} \ell^{-1+\alpha} 2^N \quad (4.27)$$

for any $N \in \mathbb{N}_0$ and $t \in \text{supp}(\chi_i^g \chi_j^b)$.

We also have the analogue of Proposition 13. By making the obvious replacements ($v_j - v_{j+1}$ with $v_q - v_j$, $z_j - z_{j+1}$ with $z_q - z_j$, and $\chi_j^b$ with $\chi_i^g$), we have

$$\|\mathcal{R}_q\|_{N+\alpha} \lesssim \epsilon_q \delta_{q+1} \ell^{-N+\alpha} \quad (4.28)$$

$$\|\partial_t + \nabla \cdot \nabla \mathcal{R}_q\|_{N+\alpha} \lesssim \epsilon_q \tau \delta_{q+1} \ell^{-N+\alpha} \quad (4.29)$$

for any $N \in \mathbb{N}_0$ and $t \in \text{supp}(\chi_i^g \chi_j^b)$.

5. Perturbation estimates

In this section, we prove Proposition 8, the perturbation result which was used in the proof of Proposition 4 (in Section 3). We begin by recalling the definition of the Mikado flows from [3, Lemma 5.1], which is valid for any dimension $d \geq 3$ (see also [8, Section 4.1]).

For any compact subset $\mathcal{N} \subset \subset S_+^{d \times d}$, there is a smooth vector field $W : \mathcal{N} \times \mathbb{T}^d \to \mathbb{R}^d$ such that

$$\text{div}_\xi W(R, \xi) \otimes W(R, \xi) = 0 \quad (5.1)$$

$$\text{div}_\xi W(R, \xi) = 0 \quad (5.2)$$

$$\int_{\mathbb{T}^d} W(R, \xi) \ d\xi = 0 \quad (5.3)$$

$$\int_{\mathbb{T}^d} W(R, \xi) \otimes W(R, \xi) \ d\xi = R \quad (5.4)$$

Unless otherwise noted, we set $\mathcal{N} = \overline{B_{1/2}(\text{Id})}$.

By Fourier decomposition we have

$$W(R, \xi) = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} a_k \langle R \rangle e^{i2\pi(k, \xi)}.$$
and
\[ W(R, \xi) \otimes W(R, \xi) = R + \sum_{k \in \mathbb{Z}^d \setminus \{0\}} C_k(R) e^{i2\pi \langle k, \xi \rangle}, \]
where \(a_k(R)\) and \(C_k(R)\) are smooth in \(R\), and with derivatives rapidly decaying in \(k\). Furthermore, (5.1) and (5.2) imply
\[ k \cdot a_k(R) = 0 \quad (5.5) \]
and
\[ k^\flat \cdot C_k(R) = 0. \quad (5.6) \]

Now we recall the identity
\[ v \cdot (\alpha \wedge \beta) = (v \cdot \alpha) \wedge \beta - \alpha \wedge (v \cdot \beta) \]
for any vector field \(v\), 1-form \(\alpha\), and differential form \(\beta\). This implies
\[
\text{div} \xi \left( \frac{k \wedge a_k}{i2\pi |k|^2} e^{i2\pi \langle k, \xi \rangle} \right) = -\sharp \delta \xi \left( \frac{k^\flat \wedge a_k^\flat}{i2\pi |k|^2} e^{i2\pi \langle k, \xi \rangle} \right)
\]
\[ = \sharp \left( e^{i2\pi \langle k, \xi \rangle} i2\pi k \right) \left( \frac{k^\flat \wedge a_k^\flat}{i2\pi |k|^2} \right) = a_k e^{i2\pi \langle k, \xi \rangle} \quad (5.7) \]
where \(k \wedge a_k\) is an alternating \((2,0)\)-tensor dual to \(k^\flat \wedge a_k^\flat\). Note that we implicitly used (5.5).

To handle the transport error later and generalize the “vector calculus” to higher dimensions, we also introduce a local-time version of Lagrangian coordinates.

**Definition 15 (Lagrangian coordinates).** We define the backwards transport flow \(\Phi_i\) as the solution to
\[ (\partial_t + v \cdot \nabla) \Phi_i = 0 \]
\[ \Phi_i(t_i, \cdot) = \text{Id}_{d} \]
Then as in [3, Proposition 3.1], for any \(N \geq 2\) and \(|t - t_i| \lesssim \tau_q:\)
\[ \| \nabla \Phi_i(t) - \text{Id} \|_0 \lesssim |t - t_i| \| \nabla v_q \|_0 \lesssim \tau_q^{\frac{N}{2}} \lambda_q = \lambda_q^{-3\alpha} \ll 1 \quad (5.9) \]
\[ \| \nabla^N \Phi_i(t) \|_0 \lesssim |t - t_i| \| \nabla^N v_q \|_0 \lesssim \lambda_q^{-3\alpha} \ell^{-N+1} \quad (5.10) \]
We also define the forward characteristic flow \(X_i\) as the flow generated by \(v:\)
\[ \partial_t X_i(t, \cdot) = v_q(t, X_i(t, \cdot)) \]
\[ X_i(t_i, \cdot) = \text{Id}_{d} \]
Then \(\partial_t (\Phi_i(t, X_i(t, \cdot))) = 0.\) By defining their spacetime versions
\[ \Phi_i(t, x) := (t, \Phi_i(t, x)) \]
\[ X_i(t, x) := (t, X_i(t, x)) \]
we can conclude \(X_i = (\Phi_i)^{-1}\), and that \(X_i\) maps from the Lagrangian spacetime \((t, x)\) to the Eulerian spacetime \((t, x)\).

\[ ^1\text{The formalism is discussed in Tao's lecture notes, which can be found at https://terrytao.wordpress.com/2019/01/08/255b-notes-2-onsagers-conjecture/} \]
Let vol be the standard volume form of the torus. Then \( X_t(\mathbb{T})^* \text{vol} = \text{vol} \) (volume-preserving)\(^2\) and
\[
X_t(\mathbb{T})^* (\text{div } u) = \text{div} \left( X_t(\mathbb{T})^* u \right)
\]
for any vector field \( u \).\(^3\)

5.1. Constructing the perturbation. We now specify the key terms used to define our perturbation. Set
\[
R_i := X_i^* \left( \text{Id} - \frac{\overline{R}_q}{\delta_{q+1}} \right)
\]
where we treat \( \overline{R}_q \) as a \((2,0)\)-tensor. Indeed, we can write this more explicitly as
\[
R_i \circ \Phi_t = \nabla \Phi_t \left( \text{Id} - \frac{\overline{R}_q}{\delta_{q+1}} \right) \nabla \Phi_t^T
\]
(5.12)
Note that, for \(|t - t_i| \lesssim \tau_q\), we have
\[
R_i \circ \Phi_t \in B_{1/2}(\text{Id}),
\]
because \( \nabla \Phi_t \) is close to \( \text{Id} \) and \( \left\| \frac{\overline{R}_q}{\delta_{q+1}} \right\| \lesssim \ell^q \) by (3.20).

For each \( i \) let \( \rho_i \) be a smooth cutoff such that \( 1_{[t_i, t_i + \epsilon_q \tau_q]} \leq \rho_i \leq 1_{[t_i - \epsilon_q \tau_q, t_i + 2\epsilon_q \tau_q]} \) and satisfying the estimate
\[
\| \partial_N^\tau \rho_i \|_0 \lesssim (\epsilon_q \tau_q)^{-N} \quad \forall N \in \mathbb{N}_0
\]
We now define the perturbation
\[
u^{(o)} := \sum_i \delta_{q+1}^{1/2} \rho_i(t) \nabla \Phi_i^{-1} W(R_i \circ \Phi_t, \lambda_{q+1} \Phi_t).
\]

For \( t \in [t_i - \epsilon_q \tau_q, t_i + 2\epsilon_q \tau_q] \), in local-time Lagrangian coordinates with
\[
u^{(o)} := X^*_i w^{(o)},
\]
we have
\[
\nu^{(o)} = \delta_{q+1}^{1/2} \rho_i(t) W(R_i, \lambda_{q+1} \mathbb{T})
\]
\[
= \sum_{k \neq 0} \delta_{q+1}^{1/2} \rho_i(t) a_k(R_i) e^{i2\pi (\lambda_{q+1} k, \mathbb{T})} = \sum_{k \neq 0} b_{i,k} e^{i2\pi (\lambda_{q+1} k, \mathbb{T})}
\]
and therefore, by defining \( b_{i,k} := \Phi^*_i b_{i,k} \) (zero-extended outside \( \text{supp} \rho_i \)), we have
\[
u^{(o)} = \sum_i \sum_{k \neq 0} b_{i,k} e^{i2\pi (\lambda_{q+1} k, \Phi_i)}
\]

Now, for \( t \in [t_i - \epsilon_q \tau_q, t_i + 2\epsilon_q \tau_q] \), in local-time Lagrangian coordinates, we define the incompressibility corrector
\[
u^{(c)} := \sum_{k \neq 0} \delta_{q+1}^{1/2} \rho_i(t) \text{div}_{\mathbb{T}} \left( k \wedge a_k \left( R_i \right) \right) e^{i2\pi (\lambda_{q+1} k, \mathbb{T})} := c_{i,k} e^{i2\pi (\lambda_{q+1} k, \mathbb{T})}.
\]

\(^2\)\( \partial_t \left( x_t(\mathbb{T})^* \text{vol} \right) = X_t(\mathbb{T})^* \left( \mathcal{L}_{\tau_q}(\mathbb{T}) \text{vol} \right) = X_t(\mathbb{T})^* (\text{div } \tau_q(\mathbb{T}) \text{vol}) = 0. \) See also (A.2).
\(^3\)\( X_t(\mathbb{T})^* (\text{div } u) \text{vol} = X_t(\mathbb{T})^* (\mathcal{L}_u \text{vol}) = \mathcal{L}_{X_t(\mathbb{T})^* u} X_t(\mathbb{T})^* \text{vol} = \text{div} \left( X_t(\mathbb{T})^* u \right) \text{vol} \)
Because of (5.8) and the identity  
\[
div (fv) = \nabla f \cdot v + f \ div v,
\]
which holds for any smooth function \( f \) and vector field \( v \), we have  
\[
\sum_{k \neq 0} e^{i2\pi \langle \lambda_{q+1} k, x \rangle} \left( a_k(R_q) + \text{div} \left( \frac{k \wedge a_k(R_q)}{i2\pi \lambda_{q+1} |k|^2} e^{i2\pi \langle \lambda_{q+1} k, x \rangle} \right) \right)
\]  
which is divergence-free, since \( \text{div} \text{div} T = 0 \) for any alternating \((2,0)\)-tensor on the flat torus.  

In Eulerian coordinates, we define  
\[
\sum_{k \neq 0} c_{i,k} e^{i2\pi \langle \lambda_{q+1} k, \phi_i \rangle}
\]  
for \( t \in [t_i - \epsilon q, t_i + 2\epsilon q] \).  

Because of (5.11), the full perturbation  
\[
w_{q+1} := w^{(o)} + w^{(c)}
\]  
is divergence-free.  

With these ingredients in place, we now define  
\[
v_{q+1} := \nu_q + w_{q+1}
\]  
and observe that  
\[
\partial_t v_{q+1} + \text{div} (v_{q+1} \otimes v_{q+1}) + \nu (-\Delta)^\gamma v_{q+1} = \partial_t \nu_q + \text{div} (\nu_q \otimes \nu_q) + \nu (-\Delta)^\gamma \nu_q + \text{div} (w_{q+1} \otimes w_{q+1}) + \nu (-\Delta)^\gamma w_{q+1} + D_{t,q} w_{q+1} \cdot \nabla \nu_q + \nu (-\Delta)^\gamma w_{q+1}
\]

We can then define the final stress as  
\[
R_{q+1} := R_{\text{osc}} + R_{\text{trans}} + R_{\text{Nash}} + R_{\text{dis}}
\]
\[
R_{\text{osc}} := \rho \text{div} (R_q + w_{q+1} \otimes w_{q+1})
\]
\[
R_{\text{trans}} := \rho D_{t,q} w_{q+1}
\]
\[
R_{\text{Nash}} := \rho (w_{q+1} \cdot \nabla \nu_q)
\]
\[
R_{\text{dis}} := \nu \rho (-\Delta)^\gamma w_{q+1}
\]

5.2. Perturbation estimates. To establish Proposition 8, we now have to estimate the perturbation constructed in the previous subsection. The desired bounds are established in the following series of results.
Proposition 16. Suppose \( t \in [t_i - \epsilon_q \tau_q, t_i + 2\epsilon_q \tau_q] \) and \( N \in \mathbb{N}_0 \). Then we have the following estimates,

\[ \| \nabla \Phi_i \|_N + \left\| \nabla \Phi_i^{-1} \right\|_N \lesssim N \epsilon^{-N} \]

(5.13)

\[ \left\| \frac{R_t}{N} \circ \Phi_i \right\|_N \lesssim N \epsilon^{-N} \]

(5.14)

\[ \left\| b_{i,k} \right\|_N \lesssim N \delta_{q+1}^\frac{1}{2} \| \epsilon^{-N} \| |k|^{-2d} \]

(5.15)

\[ \left\| c_{i,k} \right\|_N \lesssim N \delta_{q+1}^\frac{1}{2} \lambda_{q+1} \| \epsilon^{-N} \| |k|^{-2d} \]

(5.16)

along with their material derivative analogues,

\[ \left\| D_{t,q} (\nabla \Phi_i) \right\|_N \lesssim N \delta_{q+1} \lambda_q \epsilon^{-N} \]

(5.17)

\[ \left\| D_{t,q} (\frac{R_t}{N} \circ \Phi_i) \right\|_N \lesssim N \left( \epsilon_q \tau_q \right)^{-1} \epsilon^{-N-\alpha} \]

(5.18)

\[ \left\| D_{t,q} b_{i,k} \right\|_N \lesssim N \left( \epsilon_q \tau_q \right)^{-1} \delta_{q+1} \| \epsilon^{-N} \| |k|^{-2d} \]

(5.19)

\[ \left\| D_{t,q} c_{i,k} \right\|_N \lesssim N \left( \epsilon_q \tau_q \right)^{-1} \delta_{q+1} \lambda_{q+1} \| \epsilon^{-N} \| |k|^{-2d} \]

(5.20)

Proof. We first observe that (5.9) and (5.10) imply \( \| \nabla \Phi_i \|_N \lesssim \epsilon^{-N} \). Then the fact that \( \nabla \Phi_i \) is close to \( \text{Id} \), and the elementary identity \( d(A^{-1}) = -A^{-1} (dA) A^{-1} \) (for any invertible matrix \( A \)) imply (5.13).

Next, we observe that (5.13) and (3.20) imply (5.14), via the bounds

\[ \left\| \frac{R_t}{N} \circ \Phi_i \right\|_N \lesssim N \| \nabla \Phi_i \|_0 \| \text{Id} - \frac{R_t}{N} \|_N + \| \nabla \Phi_i \|_N \| \nabla \Phi_i \|_0 \| \text{Id} - \frac{R_t}{N} \|_0 \lesssim \epsilon^{-N} \]

Then, because of (5.14), and the fact that the derivatives of \( a_k \) rapidly decay in \( k \), we obtain

\[ \left\| b_{i,k} \right\|_N = \left\| \frac{1}{2} \rho_i(t) \nabla \Phi_i^{-1} a_k \left( \frac{R_t}{N} \circ \Phi_i \right) \right\|_N \]

\[ \lesssim \delta_{q+1} \left( \left\| \nabla \Phi_i^{-1} \right\|_N \right) \left\| a_k \left( \frac{R_t}{N} \circ \Phi_i \right) \right\|_0 + \left\| \nabla \Phi_i^{-1} \right\|_0 \left\| a_k \left( \frac{R_t}{N} \circ \Phi_i \right) \right\|_N \]

\[ \lesssim \delta_{q+1} \| \epsilon^{-N} \| |k|^{-2d}, \]

which establishes (5.15).

Similarly we obtain (5.16) by writing,

\[ \left\| c_{i,k} \right\|_N = \left\| \frac{1}{2} \rho_i(t) \nabla \Phi_i^{-1} \text{div} \left( \frac{k \wedge a_k \left( \frac{R_t}{N} \circ \Phi_i \right) }{2 \pi \lambda_{q+1} |k|} \right) \circ \Phi_i \right\|_N \]

\[ \lesssim \delta_{q+1} \| \epsilon^{-N} \| |k|^{-1} \lambda_{q+1} \left( \left\| \nabla \Phi_i^{-1} \right\|_N \left\| \nabla \left( a_k \left( \frac{R_t}{N} \circ \Phi_i \right) \right) \right\|_0 + \left\| \nabla \Phi_i^{-1} \right\|_0 \left\| \nabla \left( a_k \left( \frac{R_t}{N} \circ \Phi_i \right) \right) \circ \Phi_i \right\|_N \right) \]

\[ \lesssim \delta_{q+1} \| \epsilon^{-N} \| |k|^{-2d} \lambda_{q+1} \| \epsilon^{-N} \| |k|^{-2d}, \]

where we have implicitly used the chain rule

\[ \nabla \left( a_k \left( \frac{R_t}{N} \circ \Phi_i \right) \right) \circ \Phi_i = \nabla \left( a_k \left( \frac{R_t}{N} \circ \Phi_i \right) \right) \left( \nabla \Phi_i \right)^{-1} \]

in passing to the last line.

We now turn to (5.17), writing

\[ \left\| D_{t,q} \nabla \Phi_i \right\|_N = \left\| \nabla_{\tau_q} \left( \nabla \Phi_i \right) + \nabla \partial_t \Phi_i \right\|_N \]

\[ = \left\| \left[ \nabla_{\tau_q}, \nabla \right] \Phi_i \right\|_N \]
schematic notation, which completes the proof of (5.19).

The right-hand side of the above is now bounded by

\[
\lesssim 22 AYNUR BULUT, MANH KHANG HUYNH, AND STAN PALASEK
\delta_q \lambda_q \ell^{-N}.
\]

Next, we note that (5.17), (5.12), (3.20) and (3.21) imply

\[
\|D_{t,q} (R_{i} \circ \Phi_i)\|_N
\]
\[
\lesssim \|D_{t,q} (\nabla \Phi_i) (\text{Id} - \frac{R_q}{\delta_{q+1}}) \nabla \Phi_i^T + \nabla \Phi_i \left(\text{Id} - \frac{R_q}{\delta_{q+1}}\right) D_{t,q} \nabla \Phi_i^T\|_N
\]
\[
+ \delta_{q+1} \|\nabla \Phi_i (D_{t,q} R_q) \nabla \Phi_i^T\|_N
\]
\[
\lesssim \delta_q^2 \lambda_q \ell^{-N} + (\epsilon_q \tau_q)^{-1} \ell^{-N+\alpha}\]
\[
\lesssim (\epsilon_q \tau_q)^{-1} \ell^{-N+\alpha},
\]

establishing (5.18), where in passing to the last inequality, we have implicitly used \(\delta_q^2 \lambda_q \ll \epsilon_q^2 \tau_q^{-1} \ell^\alpha\), which comes from \(\epsilon_q \ll 1\) (after \(\alpha\) is neglected).

Turning to (5.19), we recall the identity \(\partial_t (w \circ X_i) = (D_{t,q} w) \circ X_i\) (for any tensor \(w\)). We then use (5.18), (5.14), and (5.13) to write

\[
\|D_{t,q} b_{i,k}\|_N = \left\|\partial_x \left(\Phi_i^* b_{i,k} \circ X_i\right) \circ \Phi_i\right\|_N
\]
\[
= \left\|\partial_x \left((\nabla X_i) b_{i,k} \circ X_i\right) \circ \Phi_i\right\|_N
\]
\[
= \delta_{q+1} \left\|\partial_x ((\nabla X_i) \partial_t (\rho_i (a_k (R_i)) \circ \Phi_i) \right\|_N.
\]

The right-hand side of the above is now bounded by

\[
\lesssim \delta_q^{1/2} \left((\ell_q \tau_q)^{-1} \right) \left\|((\nabla X_i) a_k (R_i)) \circ \Phi_i\right\|_N
\]
\[
+ \delta_q^{1/2} \left\|\partial_x ((\nabla X_i) a_k (R_i)) \circ \Phi_i\right\|_N
\]
\[
\lesssim \delta_q^{1/2} \left((\ell_q \tau_q)^{-1} \right) ((\nabla X_i-a_k (R_i)) \circ \Phi_i\right\|_N
\]
\[
+ \delta_q^{1/2} \left\|((\nabla (R_q \circ X_i) a_k (R_i)) \circ \Phi_i\right\|_N
\]
\[
+ \delta_q^{1/2} \left\|((\nabla X_i) (\nabla a_k (R_i) \partial_t (R_i)) \circ \Phi_i\right\|_N
\]
\[
\lesssim (\ell_q \tau_q)^{-1} \delta_q^{1/2} \lambda_q \ell^{-N} |k|^{-2d}
\]
\[
+ \delta_q^{1/2} \left((\ell_q \tau_q)^{-1} \right) a_k (R_i \circ \Phi_i) \right\|_N
\]
\[
+ \delta_q^{1/2} \left((\ell_q \tau_q)^{-1} \right) (\nabla a_k (R_i \circ \Phi_i)) D_{t,q} (R_i \circ \Phi_i))\right\|_N.
\]

This leads to the bound

\[
(\ell_q \tau_q)^{-1} \delta_q^{1/2} \lambda_q \ell^{-N} |k|^{-2d} + \delta_q^{1/2} \lambda_q \ell^{-N} |k|^{-2d} + (\epsilon_q \tau_q)^{-1} \delta_q^{1/2} \lambda_q \ell^{-N+\alpha} |k|^{-2d}
\]
\[
\lesssim (\epsilon_q \tau_q)^{-1} \delta_q^{1/2} \lambda_q \ell^{-N} |k|^{-2d}
\]

which completes the proof of (5.19).

It remains to show (5.20). For this, we again use (5.21), and write, using schematic notation,

\[
\|D_{t,q} c_{i,k}\|_N
\]
\[
\sim \delta_q^{1/2} \lambda_q^{-1} |k|^{-1} \left\|\partial_x ((\rho_i (\ell_q \nabla X_i) a_k (R_i)) \circ \Phi_i)\right\|_N,
\]
which leads to the bound
\[
\delta_{q+1}^{1/2} \epsilon_q \tau_q \lambda_{q+1}^{-1} |k|^{-1} \left\| \nabla \Phi_t \right\|_N - 1 \ast \nabla \left( a_k \left( R_t \circ \Phi_t \right) \right) \ast \nabla \Phi_t^{-1} \left\| N \right\n + \delta_{q+1}^{1/2} \lambda_{q+1}^{-1} |k|^{-1} \left\| \nabla \tau_t \ast \nabla \Phi_t^{-1} \ast \nabla \left( a_k \left( R_t \circ \Phi_t \right) \right) \ast \nabla \Phi_t^{-1} \right\| N
\]
\[
+ \delta_{q+1}^{1/2} \lambda_{q+1}^{-1} |k|^{-1} \left\| \nabla \Phi_t \right\|_N - 1 \ast \nabla \left( a_k \left( R_t \circ \Phi_t \right) D_{t,q} \left( R_t \circ \Phi_t \right) \right) \ast \nabla \Phi_t^{-1} \left\| N \right\n.
\]
This expression is in turn bounded by
\[
(\epsilon_q \tau_q)^{-1} \delta_{q+1}^{1/2} \lambda_{q+1}^{-1} \ell^{-N-1} |k|^{-2d} + \delta_{q+1}^{1/2} \lambda_{q+1}^{-1} |\delta_q^1 \lambda_q \ell^{-N-1} |k|^{-2d}
\]
\[
\lesssim (\epsilon_q \tau_q)^{-1} \delta_{q+1}^{1/2} \lambda_{q+1}^{-1} \ell^{-N-1} |k|^{-2d},
\]
as desired. This completes the proof of the stated estimates. □

We now record a useful corollary which will imply (3.23).

**Corollary 17.** There is $M = M(d)$ (independent of $q$) such that
\[
\|w^{(c)}\|_0 + \lambda_{q+1}^{-1} \|\nabla w^{(c)}\|_0 \lesssim \delta_{q+1}^{1/2} \lambda_{q+1}^{-1} \ell^{-1}
\]
(5.22)
\[
\|w^{(c)}\|_0 + \lambda_{q+1}^{-1} \|\nabla w^{(c)}\|_0 \leq \frac{M}{4} \delta_{q+1}^{1/2}
\]
(5.23)
\[
\|w_{q+1}\|_0 + \lambda_{q+1}^{-1} \|\nabla w_{q+1}\|_0 \leq \frac{M}{2} \delta_{q+1}^{1/2}
\]
(5.24)

**Proof.** Without loss of generality, we may assume that $a$ is large enough to ensure $\|\nabla \Phi_t\|_0 \leq 2$.

Recall that $t \in [t_i - \epsilon_q \tau_q, t_i + 2 \epsilon_q \tau_q]$ so that
\[
\left\| w^{(c)} \right\|_1 = \left\| \sum_{k \neq 0} c_{i,k} e^{i2\pi(\lambda_{q+1} k, \Phi_i)} \right\|_1,
\]
and (5.22) follows immediately from (5.16) and (3.3).

From the proof of (5.15), there is $C = C(d)$ (independent of $q$) such that
\[
\|b_{i,k}\|_0 \leq C_0^{1/2} \delta_{q+1} |k|^{-2d}
\]
Then (5.23) and (5.24) follow immediately from (5.15), (3.3) and (5.22). □

5.3. **Stress error estimates.** Suppose $t \in [t_i - \epsilon_q \tau_q, t_i + 2 \epsilon_q \tau_q]$. To complete the proof of Proposition 8 it remains to prove
\[
\|R_{q+1}\|_0 \lesssim \epsilon_q \delta_{q+2} \lambda_{q+1}^{-4}. \quad (5.25)
\]

We will often need to use an important antidivergence estimate from [3], stated in the following lemma.

**Lemma 18 (Proposition C.2 in [3]).** For any $N \in N_1$, $u \in X(T^d)$, and $\phi \in C^\infty \left(T^d \to T^d \right)$ such that $\frac{1}{2} \leq |\nabla \phi| \leq 2$, we have
\[
\left\| R \left( u(x) e^{i2\pi(k, \phi)} \right) \right\|_\alpha \lesssim N |k|^{\alpha-1} \|u\|_0 + |k|^{\alpha-N} \left( \|u\|_0 \|\phi\|_{N+\alpha} + \|u\|_{N+\alpha} \right).
\]
(5.26)
Another fact we will use often is that when $N$ is chosen large enough (independent of $q$), we have
\[ \ell_q^{N+10\alpha} \lambda_{q+1}^{N-1-10\alpha} > 1 \] (5.27)
This comes from
\[-\beta b + \beta - 1 - \frac{\sigma}{2} + b \left( \frac{N-1-10\alpha}{N+10\alpha} \right) > 0\]
which is implied by (3.5) when $N = N(b, \beta, \sigma, \alpha)$ is large enough. Unless otherwise noted, we will be using this choice of $N$.

5.3.1. Nash error. By using (5.26) and Proposition 16, we have
\[
\left\| R \left( w^{(o)} \cdot \nabla \tau_q \right) \right\|_\alpha \lesssim \sum_{k \neq 0} \left\| R \left( b_{i,k} \cdot \nabla \tau_q e^{i2\pi (\lambda_{q+1} k, \Phi_i)} \right) \right\|_\alpha
\]
\[
\lesssim N \sum_{k \neq 0} |\lambda_{q+1} k|^{-\alpha-1} |k|^{-2d} \frac{1}{\delta_q^{\frac{2}{3}} \delta_q^{\frac{2}{3}}} \lambda_q
\]
\[+ |\lambda_{q+1} k|^{-\alpha-N} |k|^{-2d} \left( \frac{1}{\delta_q^{\frac{1}{3}} \delta_q^{\frac{1}{3}}} \lambda_q \ell^{-N-2\alpha} \right) \lambda_{q+1} \ell^{-1}
\]
\[\lesssim \delta_q^{\frac{2}{3}} \delta_q^{\frac{2}{3}} \lambda_q^{\alpha-1} \lambda_q \lesssim \epsilon_q + \delta_q + 2 \lambda_q^{-4\alpha}
\]
where we used (5.27) to pass to the last line, and (3.8) in the last inequality.

Similarly,
\[
\left\| R \left( w^{(c)} \cdot \nabla \tau_q \right) \right\|_\alpha \lesssim \sum_{k \neq 0} \left\| R \left( c_{i,k} \cdot \nabla \tau_q e^{i2\pi (\lambda_{q+1} k, \Phi_i)} \right) \right\|_\alpha
\]
\[
\lesssim N \sum_{k \neq 0} |\lambda_{q+1} k|^{-\alpha-1} |k|^{-2d} \frac{1}{\delta_q^{\frac{2}{3}} \delta_q^{\frac{2}{3}}} \lambda_q \left( \lambda_{q+1} \ell \right)^{-1}
\]
\[+ |\lambda_{q+1} k|^{-\alpha-N} |k|^{-2d} \left( \frac{1}{\delta_q^{\frac{1}{3}} \delta_q^{\frac{1}{3}}} \lambda_q \ell^{-N-2\alpha} \right) \left( \lambda_{q+1} \ell \right)^{-1}
\]
\[\lesssim \delta_q^{\frac{2}{3}} \delta_q^{\frac{2}{3}} \lambda_q^{\alpha-1} \lambda_q \lesssim \epsilon_q + \delta_q + 2 \lambda_q^{-4\alpha}
\]
The only difference is the term $(\lambda_{q+1} \ell)^{-1}$ which is less than 1 by (3.3). Thus we have
\[
R_{Nash} \lesssim \epsilon_q + \delta_q + 2 \lambda_q^{-4\alpha}
\]

5.3.2. Transport error. The important observation here is that $D_{t,q} \left( e^{i2\pi (\lambda_{q+1} k, \Phi_i)} \right) = 0$, which helps avoid an extra factor $\lambda_{q+1}$.

Arguing as above, we have
\[
\left\| RD_{t,q} w^{(o)} \right\|_\alpha \lesssim \sum_{k \neq 0} \left\| D_{t,q} b_{i,k} e^{i2\pi (\lambda_{q+1} k, \Phi_i)} \right\|_\alpha
\]
\[
\lesssim N \sum_{k \neq 0} |\lambda_{q+1} k|^{-\alpha-1} |k|^{-2d} \frac{1}{\delta_q^{\frac{2}{3}} \delta_q^{\frac{2}{3}}} \left( \epsilon_q \tau_q \right)^{-1}
\]
\[+ |\lambda_{q+1} k|^{-\alpha-N} |k|^{-2d} \left( \frac{1}{\delta_q^{\frac{1}{3}} \delta_q^{\frac{1}{3}}} \ell^{-N-\alpha} \right) \left( \epsilon_q \tau_q \right)^{-1}
\]
\[\lesssim \delta_q^{\frac{2}{3}} \delta_q^{\frac{2}{3}} \lambda_q^{\alpha-1} \left( \epsilon_q \tau_q \right)^{-1} = \epsilon_q^{-1} \delta_q^{\frac{2}{3}} \delta_q^{\frac{2}{3}} \lambda_q^{\alpha-1} \lambda_q^{1+3\alpha}
\]
\[\lesssim \epsilon_q + \delta_q + 2 \lambda_q^{-4\alpha}
\]
and
\[
\left\| \mathcal{R} \left[ D_{t,q} w^{(c)} \right] \right\| \leq \sum_{k \neq 0} \left\| \mathcal{R} \left( D_{t,q} e^{i2\pi (\lambda_{q+1} k, \Phi_i)} \right) \right\|
\]
\[
\lesssim N \sum_{k \neq 0} |\lambda_{q+1} k|^{\alpha-1} |k|^{-2d} \delta_{q+1}^{1/2} (\epsilon_q \tau_q)^{-1} (\lambda_{q+1} \ell)^{-1}
\]
\[
+ |\lambda_{q+1} k|^{\alpha-N} |k|^{-2d} \left( \delta_{q+1}^{1/2} \epsilon_{q+1} (\lambda_{q+1} \ell)^{-1} \right)
\]
\[
\lesssim \delta_{q+1}^{1/2} \lambda_{q+1}^{\alpha-1} (\epsilon_{q} \tau_q)^{-1} = \epsilon_{q}^{-1} \delta_{q+1}^{1/2} \lambda_{q+1}^{\alpha-1} \lambda_{q+1} \ell^{-1}
\]
\[
\lesssim \epsilon_{q+1} \delta_{q+2} \lambda_{q+1}^{-4\alpha}
\]

Thus we have \( R_{\text{trans}} \|_\alpha \lesssim \epsilon_{q+1} \delta_{q+2} \lambda_{q+1}^{-4\alpha} \).

5.3.3. Oscillation error. We observe that
\[
R_{\text{osc}} := \mathcal{R} \text{div} \left( R_{q} + w_{q+1} \otimes w_{q+1} \right)
\]
\[
= \mathcal{R} \text{div} \left( R_{q} + w^{(o)} \otimes w^{(o)} \right)
\]
\[
:= O_1
\]
\[
+ \mathcal{R} \text{div} \left( w^{(c)} \otimes w^{(o)} + w^{(o)} \otimes w^{(c)} + w^{(c)} \otimes w^{(c)} \right)
\]
\[
:= O_2
\]

Then, using Corollary 17, and the fact that \( \mathcal{R} \text{div} \) is a Calderón-Zygmund operator, we obtain
\[
\| O_2 \|_\alpha \lesssim \left\| w^{(c)} \right\|_\alpha \left\| w^{(o)} \right\|_\alpha + \left\| w^{(c)} \right\|_\alpha^2 \lesssim \delta_{q+1} (\ell \lambda_{q+1})^{-1}
\]
\[
= \epsilon_{q}^{-1} \delta_{q+1}^{1/2} \lambda_{q+1}^{\alpha-1} \lambda_{q+1} \ell^{-1} \lesssim \epsilon_{q+1} \delta_{q+2} \lambda_{q+1}^{-4\alpha}
\]

where we have once again used (3.8).

On the other hand, because \( \rho_q^2 = 1 \) on supp \( R_q \), we have
\[
O_1 = \mathcal{R} \text{div} \left( R_{q} + \delta_{q+1} \rho_i^2 \Phi_i^* \left( W(R_{q}, \lambda_{q+1}) \right) \otimes W(R_{q}, \lambda_{q+1}) \right)
\]
\[
= \mathcal{R} \text{div} \left( R_{q} + \delta_{q+1} \rho_i^2 \left( \text{Id} - \frac{R_{q}}{\delta_{q+1}} \right) \right)
\]
\[
+ \delta_{q+1} \rho_i^2 \Phi_i^* \left( \sum_{k \in \mathbb{Z} \setminus \{0\}} C_k(R_{k}) e^{i2\pi (\lambda_{q+1} k, \cdot)} \right)
\]
\[
= \sum_{k \in \mathbb{Z} \setminus \{0\}} \delta_{q+1} \rho_i^2 \mathcal{R} \text{div} \left( \Phi_i^* \left( C_k(R_{k}) \right) e^{i2\pi (\lambda_{q+1} k, \cdot)} \right)
\]
\[
+ \delta_{q+1} \rho_i^2 \mathcal{R} \left( d_x \left( e^{i2\pi (\lambda_{q+1} k, \cdot)} \right) \Phi_i^* \left( C_k(R_{k}) \right) \right)_3
\]

We note that
\[
O_3 = d_x \left( \Phi_i^* e^{i2\pi (\lambda_{q+1} k, \cdot)} \right) \Phi_i^* \left( C_k(R_{k}) \right) = \Phi_i^* \left( \left( d_x e^{i2\pi (\lambda_{q+1} k, \cdot)} \right) \cdot C_k(R_{k}) \right) = 0
\]
because of (5.6). Then because of (5.26) and (3.8):
\[
\|\Omega_1\|_\alpha \lesssim \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \|\delta_{q+1} R \left( \text{div} \Phi_i^{-1} C_k (R_{\alpha} \circ \Phi_i) \nabla \Phi_i^{-T} \right) e^{i 2 \pi (\lambda_{q+1} k, \Phi_i)} \|_\alpha \\
\lesssim \sum_{k \neq 0} |\lambda_{q+1} k|^{\alpha-1} |k|^{-2d} \delta_{q+1} |\ell|^{-1} + |\lambda_{q+1} k|^{-N} |k|^{-2d} (\delta_{q+1} \ell^{-N - 3\alpha}) \ell^{-1} \\
\lesssim \lambda_{q+1}^\alpha \delta_{q+1} \ell^{-1} = \epsilon_q^{-2} \delta_{q+1}^2 \lambda_{q+1}^{-1} \lambda_{q+1}^1 + \delta_{q+1} \lambda_{q+1}^{-4\alpha} \lesssim \epsilon_{q+1} \delta_{q+2} \lambda_{q+1}^{-4\alpha}
\]
Therefore \(\|R_{\text{osc}}\|_\alpha \lesssim \epsilon_{q+1} \delta_{q+2} \lambda_{q+1}^{-4\alpha}\).

5.3.4. Dissipative error. Without loss of generality, we may assume \(2\alpha + 2\gamma < 1\) (by choosing \(\alpha\) sufficiently small). Because \(R\) and \((-\Delta)^\gamma\) commute, and because \((-\Delta)^\gamma\) is a bounded map from \(C^{2\gamma + 2\alpha}\) to \(C^\alpha\) ([11, Theorem B.1]), we have
\[
\|R_{\text{dis}}\|_\alpha \lesssim \|R_{w_{q+1}}\|_{2\alpha + 2\gamma} \lesssim \|R_{w_{q+1}}\|_{1+1} \|R_{w_{q+1}}\|_{0}^{-2\gamma - 2\alpha}.
\]
Then because \(\nabla R\) is a Calderón-Zygmund operator, and because of Corollary 17.:
\[
\|R_{w_{q+1}}\|_{1+1} \lesssim \|\nabla R_{w_{q+1}}\|_{0} \lesssim \|\nabla R_{w_{q+1}}\|_{1} \lesssim \|w_{q+1}\|_{1} \lesssim \delta_{q+1} \lambda_{q+1}^{\alpha}
\]
Meanwhile, because of (5.26):
\[
\|R_{w_{q+1}}\|_{1+1} = \sum_{k \neq 0} \|R \left( (b_{i,k} + c_{i,k}) e^{i 2 \pi (\lambda_{q+1} k, \Phi_i)} \right) \|_\alpha \\
\lesssim \sum_{k \neq 0} |\lambda_{q+1} k|^{\alpha-1} |k|^{-2d} \delta_{q+1}^1 |\lambda_{q+1} k|^{-N} |k|^{-2d} \delta_{q+1}^{1/2} \ell^{-N - \alpha} \\
\lesssim \delta_{q+1} \lambda_{q+1}^{-1}
\]
Therefore:
\[
\|R_{\text{dis}}\|_\alpha \lesssim \delta_{q+1} \lambda_{q+1}^\alpha (\delta_{q+1}^{(2\gamma + 2\alpha) + (\alpha-1)(1-2\gamma - 2\alpha)}) \lesssim \epsilon_{q+1} \delta_{q+2} \lambda_{q+1}^{-4\alpha}
\]
because of (3.9), when \(\alpha = \alpha (\sigma, b, \beta, \gamma)\) is small enough. This completes the proof of (5.25), and therefore of Proposition 8.

APPENDIX A. Geometric preliminaries

We recall the Hodge decomposition
\[
\text{Id} = \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3
\]
where \(\mathcal{P}_1 := d \cdot (-\Delta)^{-1} \delta\) and \(\mathcal{P}_2 := \delta \cdot (-\Delta)^{-1} d\) and \(\mathcal{P}_3\) maps to harmonic forms (cf. [15, Section 5.8]). We observe that \(\mathcal{P}_1, \mathcal{P}_2\) are Calderón-Zygmund operators. We also recall that \(\delta = -\text{div}\), where \((\text{div} T)^{i_1 \ldots i_k} = \nabla_j T^{j i_1 \ldots i_k}\) for any tensor \(T\).

Due to the musical isomorphism, the Hodge projections \(\mathcal{P}_i\) are also defined on vector fields, and we also write \(\sharp \mathcal{P}_i \beta\) as \(\mathcal{P}_i\) for convenience (unless ambiguity arises).

Because the torus is flat, we have the identities
\[
\delta \beta (X \cdot \nabla Y) = \delta \beta (Y \cdot \nabla X) \\
\mathcal{P}_1 (X \cdot \nabla Y) = \mathcal{P}_1 (Y \cdot \nabla X)
\]
for any divergence-free vector fields \(X, Y\). On the torus, harmonic 1-forms (or vector fields) are precisely those which have mean zero.
Definition 19 (Time-dependent Lie derivative). For any smooth family of diffeomorphisms \((F_t)\) and differential forms \((\alpha_t)\) we have

\[ \partial_t (F_t^* \alpha_t) = F_t^* (\mathcal{L}_{X_t} \alpha_t + \partial_t \alpha_t) \]

where \((X_t)\) is a time-dependent vector field defined by \(\partial_t F_t = X_t \circ F_t\).

Lemma 20. For any diffeomorphism \(\Phi\), vector field \(u\) and differential form \(\alpha\), we recall the pullback identity:

\[ \Phi^* (\mathcal{L}_u \alpha) = \mathcal{L}_{\Phi^* u} \Phi^* \alpha \]  \hspace{1cm} (A.2)

Remark 21. The pullback of a 1-form has a different meaning from the pullback of a vector field, and we do not have \(\Phi^* \flat X = \flat \Phi^* X\) unless \(\Phi\) is an isometry.

We conclude this appendix by introducing several operators that play a key role in our analysis. In particular, we will make use of the antidivergence operator

\[ \mathcal{R} : C^\infty (\mathbb{T}^d, \mathbb{R}^d) \to C^\infty (\mathbb{T}^d, \mathcal{S}_0^{d \times d}) \]

given by

\[ (\mathcal{R} v)_{ij} = \mathcal{R} \delta_{ik} v^{k} , \]  \hspace{1cm} (A.3)

with

\[ \mathcal{R} \delta_{ik} := -\frac{d-2}{d-1} \Delta^{-2} \partial_i \partial_j \partial_k - \frac{1}{d-1} \Delta^{-1} \partial_i \delta_{jk} + \Delta^{-1} \partial_j \delta_{ik} + \Delta^{-1} \partial_k \delta_{ij}. \]

Note that \(\text{div} \mathcal{R} v = v - f_{\mathbb{T}^d} v = (1 - \mathcal{P}_3) v\) for any vector field \(v\). Moreover, using the musical isomorphism, the operator \(\mathcal{R}\) can also be defined on 1-forms, and we will often write \(\mathcal{R} \delta\) as \(\mathcal{R}\) to simplify notation.

We also define the higher-dimensional analogue of the Biot-Savart operator as \(\mathcal{B} := (-\Delta)^{-1} \flat \partial\), mapping from vector fields to 2-forms. We then have

\[ \mathcal{P}_2 \delta \mathcal{B} v = \mathcal{P}_2 v \]

which implies \(\mathcal{P}_2 \delta \mathcal{B} v = v - f_{\mathbb{T}^d} v = \mathcal{P}_2 v\) for any divergence-free vector field \(v\).

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