The Effect of Gravitational Field on Brachistochrone Problem

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Abstract. The equation of the minimum time trajectory (the brachistochrone) does not contain the gravity acceleration explicitly. The proof of the gravitational field effect on the trajectory is revealed by solving functional for the uniform gravitational field and nonuniform gravitational field for the central gravity in this article. The Findings revealed that the effect of constant gravity acceleration is inversely proportional on the arc length of the cycloid, except at \( g = 0 \) m/s\(^2\), which means that the trajectory could not be formed without gravity acceleration at a location where a particle are not affected by the gravitational field, whereas in nonuniform gravitational field, the particle’s trajectory is not a cycloid and lies in two quadrant. The curve in first quadrant is a mirror image of the curve in fourth quadrant and vice versa. The difference trajectory between uniform and nonuniform gravitational cases is the proof of the existence of the gravitational field effect.

1. Introduction

The equation of free fall is \( y(t) = -\frac{1}{2} gt^2 \), where \( y \) is height, \( g \) is gravity acceleration, and \( t \) is time. The curve is a parabola. It could be seen that the effect of \( g \) is changing the curve. The equation could be obtained by integrating both sides of the velocity equation[1] \( v_y(t) = -gt \) without involve extreme problem associated with the minimum time.

According to the both equation above, the velocity formula of free fall object is \( v = \sqrt{2gy} \). The velocity is thus proportional only to the square root of the height[2]. It could be seen that there is no effect of the mass on its velocity. Free fall experiment based on various methods and technologies, can be found in every educational physics laboratory[3].

The mathematician Johann Bernoulli issued a mathematical challenge in the scholarly journal Acta Eruditorum (Transactions of scholars) in June 1696 inviting the mathematicians to solve this problem: Given two points \( A \) and \( B \) in a vertical plane, find the path \( AMB \) down which a movable point \( M \) must by virtue of its weight fall from \( A \) to \( B \) in the shortest possible time[4]. This problem was called brachistochrone problem.

The trajectory in brachistochrone problem is a cycloid. The equation for constant gravity acceleration are \( x = R(\theta - \sin \theta) \) and \( y = -R(1 - \cos \theta) \), where \( R \) is the radius of a rolling circle[5]. Similarly to the problem above associated with the effect of the mass on the free fall object velocity formula. In this research, the effect of gravity acceleration \( (g) \) could not be seen explicitly.

Unlike the free fall experiment, the brachistochrone experiment difficult to be conducted due to the dimensional of the earth[6]. According to the problem above, the existence of the effect of
gravitational field on the brachistochrone problem will be revealed by solving functional for uniform and nonuniform gravitational field for the central gravity.

2. The Brachistochrone in the Uniform Gravitational Field

According to the principle of conservation energy, there is no lost energy during the movement from point A to point B. The mechanical energy $E_{\text{me}}$ of a system is the sum of its kinetic energy $K = \frac{1}{2}mv^2$ and potential energy $U = mgy$. The velocity of a particle released from rest in cartesian coordinate is $v = \sqrt{2gy}$, where the initial velocity is zero. So that, the relationship between the velocity and the trajectory length of the particle is $v = \frac{ds}{dt} = \sqrt{2gy}$, where the derivative of trajectory length[8] is $ds = \sqrt{1 + y'^2} \, dx$.

The shortest time between two points from A to B is $t = \int_a^b \frac{ds}{\sqrt{2gy}}$, since the functional also belongs to the most elementary type and its integrand does not contain $x$ explicitly, the modified Euler equation[9] has a first integral $F - y'F_y' = C$. There are several inappropriate substitutions in solving brachistochrone problem for uniform gravitational field[10] such as substitution

$$y = \frac{a - b}{2} - \frac{a + b}{2} \cos \theta$$

to the differential equation

$$\frac{dy}{dx} = \left( \frac{b + y}{a - y} \right)^{\frac{1}{2}}, \quad \text{where} \quad a = y_1, \ b = (2gC^2)^{-1} - y_1 \quad \text{and} \quad y(x_1) = y_1, \ y(x_0) = 0.$$

Its solution are as follows: $x = \frac{y_1}{2} (\theta + \sin \theta)$ and $y = \frac{y_1}{2} (1 - \cos \theta)$, with its minimum time is $t_{\text{min}} = \int_0^\pi \left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{\frac{1}{2}} \, dx = -\infty$. It means that there is no the minimum time using the substitution above due to the integral is divergent. This procedure was assumed that solution of $y$ has been known, hereinafter variable $x$ can be determined[11]. The solution in parameter equation is not a single solution based on this procedure. The solution of $x$ and $y$ must be determined algebraically in order to obtain a single solution. One of the appropriate substitution to solve the problem above is $y' = \cot t$, so that

$$\sqrt{\frac{1}{2gy} \left( 1 + \cot^2 \theta \right)} = C,$$

the solution is $y = \frac{1}{4gC^2} (1 - \cos 2t)$, with its derivative

$$dy = \frac{1}{gC^2} \sin t \cos t \, dt.$$ Based on the substitution previously, then

$$\frac{1}{gC^2} \sin t \cos t \, dt = \frac{\cos t}{\sin t} \, dx.$$ So that $x = \frac{1}{4gC^2} (2t - \sin 2t) + k$. Let $2t = \theta$, then the solution becomes

$$x = \frac{1}{4gC^2} (\theta - \sin \theta) + k \quad \text{and} \quad y = \frac{1}{4gC^2} (1 - \cos \theta).$$ Since the curve through the points $A(x_1, y_1)$
and \( B(0,0) \), then \( \frac{1}{4gC^2} (1 - \cos \theta) = 0, \) where \( \frac{1}{4gC^2} \neq 0 \), therefore \( \theta = 0 \) and \( k = 0 \), so that the solution becomes \( x = \frac{1}{4gC^2} (\theta - \sin \theta) \) and \( y = \frac{1}{4gC^2} (1 - \cos \theta) \).

![Figure 1. The curves of brachistochrone with varying gravity acceleration in uniform gravitational field](image)

According to the \textbf{Error! Reference source not found.}. The curve is a cycloid which formed when the circle with the radius \( \frac{1}{4gC^2} \) rolling at positive x axis, where \( \theta = 0 \), which means that its movement started from point \( B(0,0) \) to point \( A(x_1, y_1) \). The relationship between the gravity acceleration and the height is \( \frac{1}{4gC^2} = \frac{y_1}{2} \), then the solution are as follows \( x = \frac{y_1}{2} (\theta - \sin \theta) \) and \( y = \frac{y_1}{2} (1 - \cos \theta) \), where \( 0 \leq \theta \leq \pi \) with its minimum time is

\[
 t_{\text{min}} = \pi \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{2gy}} \left[ \frac{dy}{dx} \right]^{\frac{1}{2}} dx = \pi \sqrt{\frac{y_1}{2g}}.
\]

The trajectory will always be a cycloid at any height, except at \( g = 0 \) m/s\(^2\), which means that at a location where a particle are not affected by the gravity, the trajectory can not be formed and a particle still standing at its place. Before applying the boundary condition, the equation still contain the gravity acceleration explicitly, the variable \( g \) no longer can be seen after the boundary condition had been applied. But literally, the equation still has relationship to the gravity acceleration based on the relationship between gravity acceleration and the radius of a rolling circle. The effect of constant gravity acceleration \( g \) on the trajectory can be seen by setting various values of variable \( g \). This solution was calculated by letting \( 4C^2 = 1 \) and \( 0 \leq \theta \leq \pi \). The curves revealed that the gravity acceleration is inversely proportional to the arc length of the cycloid and radius of a rolling circle which forming a cycloid.
3. The Brachistochrone of the Nonuniform Gravitational Field

The particle falls under the action of an inverse square force field in minimum time in this case. The gravitational potential energy for nonuniform gravitational field is $E_p = -G \frac{mm'}{r}$, where $G$ is gravitational constant, $m$ is the mass of earth, $m'$ is the mass of particle, and $r$ is the distance between $m$ and $m'$. The principle of conservation energy gives $v^2 = v_0^2 + 2Gm \left( \frac{1}{r} - \frac{1}{r_0} \right)$, since there is no initial velocity for a particle falling from height, then the velocity along any curve is $v = \sqrt{2Gm \left( \frac{1}{r} - \frac{1}{r_0} \right)}$, so that the minimum time is $t = \int \frac{ds}{\sqrt{2Gm \left( \frac{1}{r} - \frac{1}{r_0} \right)}}$, in polar coordinate $ds = \sqrt{r^2 + r'^2} \, d\phi$. The modified Euler equation gives

$$\frac{\sqrt{r_0 \, r}}{\sqrt{2Gm(r_0 - r)}} - r' \frac{\sqrt{r_0 \, r}}{\sqrt{2Gm(r_0 - r)}} \left( \frac{1}{2} \right) \frac{1}{\sqrt{r^2 + r'^2}} (2r') = C.$$ With some simplification

$$r'^2 = \frac{r_0 \, r^5 - 2C^2 \, G \, m \, (r_0 - r) \, r^2}{2C^2 \, G \, m(r_0 - r)}.$$ Since $r' = \frac{dr}{d\phi}$, then $\left( \frac{dr}{d\phi} \right)^2 = \frac{r_0 \, r^5 - 2C^2 \, G \, m \, (r_0 - r) \, r^2}{2C^2 \, G \, m(r_0 - r)}.$

Let $2C^2m=1$, then $\left( \frac{d\phi}{dr} \right)^2 = \frac{G(r_0 - r)}{r_0 \, r^5 - G(r_0 - r) \, r^2}$. So that $\phi = \pm \int \frac{G(r_0 - r)}{\sqrt{r_0 \, r^5 - G(r_0 - r) \, r^2}} \, dr$.

The integrand can not be expressed using elementary functions. Another possible solution is using the Taylor series approach$^{12}$. But the remainder term is very difficult to be obtained, therefore, the solution was calculated with Mathematica$^{14}$. The particle's trajectory lies in two quadrant. The curve in first quadrant is a mirror image of the curve in fourth quadrant and vice versa. According to the Error! Reference source not found., the particle's trajectory lies in two quadrant. The curve in first quadrant is a mirror image of the curve in fourth quadrant and vice versa. The solution was calculated by letting $D$ equal to zero, $r_0$ equal to $45 \times 10^6$ m, $G$ is gravitational constant equal to $6.67 \times 10^{-11} \frac{m^3}{kg \, s^2}$, and $r$ in interval $2.10^7 \, m \geq r > 0 \, m$. 

$$\phi = \pm \frac{2.723 \times 10^{-13} \, r \, (-2 \times 10^7 + 1 \times r)}{r^3} \times \frac{1 - (5 \times 10^{-8} \, r)}{r^2 (6.674 \times 10^{-11} + 3.337 \times 10^{-18} \, r + r^3)} \times \sqrt{-6.674 \times 10^{-11} + 3.337 \times 10^{-18} \, r + r^3} + D.$$
The verification of this result is difficult to be conducted experimentally due to the large gravitational field difficult to be produced in laboratory, therefore, the proof of this problem can be conducted by studying about the problem related to the electrostatic, light, and electromagnetic radiation, since the whole cases following the same inverse square law.

4. Conclusion
The results revealed that the effect of constant gravity acceleration is inversely proportional on the arc length of the cycloid, except at \( g = 0 \) m/s\(^2\), which means that the trajectory could not be formed without gravity acceleration at a location where a particle are not affected by the gravitational field.

In nonuniform gravitational field, the particle’s trajectory is not a cycloid and lies in two quadrant. The curve in first quadrant is a mirror image of the curve in fourth quadrant and vice versa. The difference trajectory between uniform and nonuniform gravitational cases is the proof of the existence of the gravitational field effect.

The trajectories is only affected by the central gravitational field of the earth. The \textit{brachistochrone} problem by considering the effect of gravitational field on more than one object can be used by spacecraft to land on the astronomical objects with minimum time using gravity assist from one of the astronomical object, therefore, the functional and its solution of this problem will be the next research to be conducted.

References
[1] G. B. Robert. Introductory physics I elementary mechanics. (Durham: Duke University Physics Department, NC 27708-0305) (2013) chapter 2. p. 72.
[2] C. Schiller. The adventure of physics. 29th ed. Munich. (2016) vol 1. pp. 74-75.
[3] I. A. Sianoudis, M. Petraki, M. Serris, and L. Prelorentzos. Free fall in vacuum: an educational lab-experiment. e-Journal of Science & Technology. 4, 1 (2009).
[4] E. Knobloch. Leibniz and the brachistochrone documenta math. (2012) pp 15–18.
[5] D. S. Shafer. The brachistochrone: historical gateway to the calculus of variations. *Materials matemàtics*. (2007) pp. 0001-14.
[6] B. Singh and R. Kumar. Brachistochrone problem in nonuniform gravity. *Indian Journal of Pure & Applied Mathematics*. 19, 6 (1988) pp. 575-585.
[7] J. Walker, D. Halliday, R. Resnick. Fundamentals of physics 10th ed S Jhonson et al. (John Wiley & Sons Inc 111 River Street Hoboken) (2014) chapter 8. p. 199.
[8] D. Varberg, J. P. Edwin, E. R. Steven. Calculus ed 9th. (Prentice Hall) (2006) chapter 5. p.296.
[9] L. Elsgolts. Differential equations and the calculus of variations 3rd ed (MIR Publishers USSR, 129820, Moscow I-110, GSP Pervy Rizhsky Pereulok, 2) (1977) chapter 6. p. 317.
[10] P. Soedojo. Asas-asas matematika, fisika, dan teknik. Gajah Mada University Press (1995) chapter 7. p. 413.
[11] M. Lutfi. Tinjauan terhadap siklold terbalik terkait masalah brachistochrone. *AKSIOMA: Jurnal Pendidikan Matematika*. 5, 1 (2016) pp. 7-14.
[12] S. G’omez-Aiza, R. W. G’omez, and V. Marquina. Simplified approach to the brachistochrone problem. *European Journal of physics*. 27, 5 (2006) pp. 1091-1096.
[13] M. Lutfi. Misconceptions in solving indefinite integrals for nonelementary functions using the Taylor series. *Jurnal Ilmiah Matematika dan Terapan*. 13, 1 (2016) pp. 96-107.
[14] Wolfram Research, Inc. Mathematica Version 11.2. Champaign, Illinois. (2017).