Critical relations in symmetric \(0 - \pi\) Josephson junctions

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Numerical modeling of dependences “critical current – external magnetic field” for geometrically symmetric \(0 - \pi\) Josephson junctions is performed. The calculation of critical current is reduced to non-linear eigenvalue problem. The critical curve of the contact is obtained as an envelope of the bifurcation curves of different distributions of the magnetic flux.

The structure of vortices in contact is observed explicitly and the dependence of the basic physical characteristics of these vortices on junction’s length is explored.

The comparison of numerical results and known experimental data shows good qualitative and quantitative conformity.

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I. INTRODUCTION

The opportunity of existence and interest of studying of \(0 - \pi\) Josephson contacts (further on referred as \(0-\pi\)JJs) have been shown in classical papers [1], [2] (see the short history of the problem in [3]). For last years in the specified area significant number of theoretical and experimental works are published (see for example, [4] – [10] and references therein).

In this article a numerical modeling of some experimental data [8] concerning the relations “critical current – external magnetic field” for such a junctions, is carried out.

For this purpose the transitions from Josephson to resistive regime by change the external current \(\gamma\) are mathematically interpreted as a bifurcation of one of possible static distributions of the magnetic flux \(\varphi(x)\) existing in the junction for given external magnetic field \(h_e\) (from now on all the variables are dimensionless).

Every solution of non-linear boundary value problem (BVP) for \(\varphi(x)\) generates some regular Sturm-Liouville problem (SLP). The minimal eigenvalue (EV) \(\lambda_0\) of SLP allows to decide whether the concrete solution is stable or unstable in linear approximation. Note, that such approach for the traditional model of “one-dimensional” long josephson junctions (JJJs) was proposed in [11, 12].

In such approach the critical current \(\gamma_c\) of some fixed distribution is that value of \(\gamma\), for which the minimal EV \(\lambda_0\) is vanishing at the given external magnetic field \(h_e\). In order to calculate points of bifurcation \(\gamma_c\) a non-linear eigenvalue problem is formulated, such that the current \(\gamma\) is considered as an EV under the rest given junction’s parameters including \(\lambda_0\) [13, 14]. Since for given external magnetic field the static BVP can have more than one solution, then the critical current of JJ is determined as maximal of critical currents of possible stable distributions. Thus the critical curve (CC) of the junction can be constructed as an envelope of bifurcation curves (BCs) of different distributions of the magnetic flux.

II. STATEMENT OF THE PROBLEM

Let consider \(0 - \pi\) junction of length \(2l, l < \infty\), disposed along axis \(x\), such that 0-junction (0JJ) is to the left and \(\pi\)-junction (\(\pi\)JJ) is to the right of the barrier \(x = \zeta \in (-l, l)\). The junction is geometrically symmetric if \(\zeta = 0\).

In case of overlap geometry the non-linear BVP for static distributions of magnetic flux \(\varphi(x)\) can be written as

\[-\varphi_{xx} + jC(x)\sin\varphi - \gamma = 0, \quad x \in (-l, \zeta) \cup (\zeta, l),\]
\[\varphi(\zeta - \varepsilon) = \varphi(\zeta + \varepsilon),\]
\[\varphi_x(\zeta - \varepsilon) = \varphi_x(\zeta + \varepsilon),\]
\[\varphi(\pm l) - h_e = 0,\]

where \(jC(x)\) is the amplitude of Josephson current

\[jC(x) = \begin{cases} 1, & x \in [-l, \zeta]; \\ -1, & x \in [\zeta, l]. \end{cases}\]

Possible solutions of (1) belong to a class \(C^1[-l, l]\), i.e., continuously differentiable (smooth) functions. Physically this means that the magnetic field inside the junction is continuous only, and at the sewing point \(x = \zeta\) continuity conditions (1b) and (1c) are fulfilled [4] – [7].

One can check that for \(\gamma = 0\) the common solution of equations (1a) in the corresponding intervals is expressed by the elliptic functions, see [20]. Under the given model’s parameters \(p = \{l, \zeta, h_e, \gamma\}\) the integration constants can be obtained by means of two boundary conditions (1d) and two continuity conditions (1b) and (1c) [13].

For small magnitudes of the current \(|\gamma| << 1\) the solution of (1) can be derived applying perturbation theory methods. But in all cases the solution of arising
III. STABILITY OF STATIC SOLUTIONS

Further we shall suppose continuous dependence of solutions $\varphi(x,p)$ of (1) on parameters $p \in \mathcal{P}$, $\mathcal{P} \subset \mathbb{R}^4$. By now, the solutions dependence on $p$ we shall mark only if it is necessary.

In order to analyze the stability of solutions of (1) under parameters variation we put in correspondence to each solution $\varphi(x,p)$ a regular Sturm-Liouville problem

\[
-\psi_{xx} + q(x)\psi = \lambda\psi, \quad (2a)
\]

\[
x \in (-l, \zeta) \cup (\zeta, l),
\]

\[
\psi(\zeta - \varepsilon) - \psi(\zeta + \varepsilon) = 0, \quad (2b)
\]

\[
\psi_x(\zeta - \varepsilon) - \psi_x(\zeta + \varepsilon) = 0, \quad (2c)
\]

\[
\psi_x(\pm l) = 0, \quad (2d)
\]

\[
\int_{-l}^{l} \psi^2(x) \, dx - 1 = 0, \quad (2e)
\]

where the potential $q(x) = jC(x) \cos \varphi(x)$. Here $\lambda$ is a spectral parameter and (2e) represents the norm condition.

It is easy to prove the existence of non-degenerate discrete bounded below spectrum $\{\lambda_n\}_{n=0}^\infty$ of problem (2), see \[18\]. Then the bifurcation equation of order $n$ of some solution $\varphi(x,p)$ is

\[
\lambda_n(p) = 0, \quad n = 0, 1, \ldots \quad (3)
\]

It is convenient to interpret the last equality geometrically as a surface in the parameters’ space $\mathcal{P}$. Each point on this surface is a bifurcation (critical) point of order $n$ for some solution of (1). For fixed value of two of parameters, the equation (3) describes a curve on the plane defined by other two parameters — a bifurcation curve (BC) of order $n$ of considered solution.

In this paper we mainly consider bifurcations of order 0 of solutions of (1) by change basic physical parameters — the external magnetic field $h_e$ and the external current $\gamma$. An implicit equation of BC for some solution on $(h_e, \gamma)$ plane is

\[
\lambda_0(h_e, \gamma) = 0. \quad (4)
\]

For 0JJs bifurcations of static distributions of magnetic flux under variation of the length $l$ were studied, for example, in articles \[16\], \[17\].

From mathematical point of view, the problem (1) can be considered as a collection of necessary extremum conditions for the total energy functional of the junction

\[
F[\varphi] = \int_{-l}^{l} \left[ \frac{1}{2} \varphi_x^2 + jC(x)(1 - \cos \varphi) - \gamma \varphi \right] \, dx - h_e \Delta \varphi
\]

on a set of smooth on $[-l, l]$ functions $\varphi(x)$.

Especially, the equations (1a) are Euler-Lagrange equations for $\varphi(x)$ at corresponding subintervals, and Weierstrass-Erdmann conditions yield (1d), (1l), and (1g), see \[19\]. The total magnetic flux through the junction $\Delta \varphi$ is defined traditionally

\[
\Delta \varphi = \int_{-l}^{l} \varphi_x \, dx = \varphi(l) - \varphi(-l). \quad (5)
\]

SLP (2) has to satisfy sufficient extremum conditions for $\varphi$, see \[19\]. If $\lambda_0 > 0$ for some solution $\varphi(x)$ of (1), then for the same solution the second variation of $F[\varphi]$ is positive definite and the functional (5) possesses minimum. Therefore the solution $\varphi(x)$ is stable in linear approximation.

If $\lambda_0(p) = 0$ for some $p \in \mathcal{P}$ then the solution $\varphi(x,p)$ has at $p$ a bifurcation of zero order, and the equations (2d) represents the Jacobi equations for the functional (5). For brevity we shall call solutions at any parameter’s bifurcation point as B-solutions.

In this article we apply the method for numerical construction of relations (4) proposed in \[13\], \[14\] (see also review \[15\]). At given $\lambda$ both equations (1) and (2) are considered as unique non-linear eigenvalue problem with spectral parameter $h_e$ or $\gamma$. To find eigenvalues and eigenfunctions $(\varphi(x), \psi(x))$ we use an algorithm based on continuous analogue to Newton method \[22\]. The discretization of corresponding linearized BVPs is performed by means of spline-collocation method. Such approach leads to difference scheme with block three-diagonal matrix. The scheme’s accuracy assessment, determined by Runge’s method on the sequence of uniform meshes with steps $h$, $h/2$ and $h/4$, is $O(h^4)$.

IV. DISCUSSION OF NUMERICAL RESULTS

A. Non-bifurcation static solutions

First let’s consider some solutions of (1), which exist far enough from the bifurcation point of the magnetic field $h_e$ or/and current $\gamma$, and behavior of their main physical characteristics when parameters tend to the critical values.

On Fig. 1 the main stable states (of minimal energy) of magnetic flux $\varphi(x)$ for $0$-$\pi$JJ at two values of the junction’s length $2l = 2$ and $2l = 7$ in the field $h_e = 0$ and under the current $\gamma = 0$ are demonstrated. Further, for convenience, we shall use notation $\Phi^{\pm 1}$ for mentioned
above solutions, where \( \Phi^1 \) is a fluxon and \( \Phi^{-1} \) – antifluxon.

Corresponding distributions of magnetic field (derivatives \( \phi(x) \)) are shown on Fig. 2. As in the case of ring \( \pi \)-contact, see [1, 2], the main states have finite total magnetic field (\( \Delta \Phi \approx \pm 0.147 \) at \( 2l = 2 \), and \( \Delta \Phi \approx \pm 0.468 \) for \( 2l = 7 \), respectively), finite energy (\( F[\Phi^{\pm1}] \approx -0.04 \), and \( F[\Phi^{\mp1}] \approx -0.583 \)) and contain a half-integer numbers \( N[\Phi^{\pm1}] = 0.5 \) of fluxons (magnetic flux quanta), where

\[ N[\phi] = \frac{1}{2\pi l} \int_{-l}^{l} \phi(x) \, dx, \]  

(7)

is the average magnetic flux trough the junction. At the center \( x = 0 \) we have \( \Phi^{\pm1}(0) = \pm \pi/2 \) (see Fig. 1). Moreover, the conditions of geometrical symmetry are fulfilled \( \Phi^{-1}(x) = -\Phi^1(x) \), \( \Phi_x^{-1}(x) = -\Phi_x^1(x) \), and \( \Phi_x^{\pm1}(x) = \Phi_x^{\mp1}(-x) \).

Except vortices demonstrated on Fig. 1 at \( h_c = 0 \) and \( \gamma = 0 \) there exist Meissner’s states \( M_0(x) = 0 \) and \( M_\pi(x) = \pi \) \((+2\pi k, k = 0, \pm 1, \ldots)\). Such distributions have zero total magnetic flux \( \Delta \phi \) and zero energy \( F[\phi] \), but \( N[M_0] = 0 \) and \( N[M_\pi] = 1 \). Note, that the potential of SLP induced by \( M_0 \), is \( q(x) = j_C(x) \) and \( q(x) = -j_C(x) \) for \( M_\pi \). Hence no one of Meissner’s solutions can be stable in contrast to traditional 0JJ and \( \pi JJ \), where \( q(x) = 1 \) for \( M_0 \) (stable) and \( q(x) = -1 \) for \( M_\pi \) (unstable). In particular \( \lambda_0 \approx -0.3 \) at \( 2l = 2 \) and \( \lambda_0 \approx -0.86 \) for \( 2l = 7 \). Numerical experiment confirms, that Meissner’s solutions remain unstable in all admissible range of magnitudes \( h_c \) and \( \gamma \). Thereby in case of 0-\( \pi JJ \) the traditional Meissner’s branches are missed on CCs, as it is observed in the experiment [3].

Increasing value of \( h_c \) arise new more complicated bound states of the magnetic flux in the junction — pure and mixed chains of fluxons and antifluxons. Pure chains of vortices are “composed” only of fluxons or antiflux-
ons. Mixed chains are obtained after non-linear interaction between fluxons and antifluxons. To mixed solutions of type $\Phi^m \Phi^{-m}$ correspond symmetric $\Phi^{-m} \Phi^m$, $m = 1, 2, \ldots$. On Fig. 3 chains of vortices of internal magnetic field $\varphi(x)$ for the field $h_e = 0$ and at the current $\gamma = 0$ are demonstrated: one-soliton distribution $\Phi^1$, system of three non-linearly interacting solitons $(N[\Phi^3] = 2.5)$, and also $\Phi^1 \Phi^{-1}$, pair soliton – anti- soliton, with $N[\Phi^1 \Phi^{-1}] = 0.005$: the symmetrical pair $\Phi^{-1} \Phi^1$ yields $N[\Phi^{-1} \Phi^1] \approx 0.995$. Similarly, on Fig. 4 a pure chain of fifth solitons $\Phi^5$ $(N[\Phi^5] = 4.5)$, and mixed chain $\Phi^2 \Phi^{-2}$ of two solitons and two antisolitons $(N[\Phi^2 \Phi^{-2}] = 4.002$ at $h_e = 3.6$), pinning at the inhomogeneity at the point $\zeta = 0$ are shown. For the symmetrical pair $\Phi^{-2} \Phi^2$ we have $N[\Phi^{-2} \Phi^2] \approx 2.998$.

On Fig. 5 the dependence of minimal EV of (2) on the external magnetic field $h_e$ for the junction of length $2l = 7$ at $\gamma = 0$ is represented. In accordance with 4 1 the zeroes of curves $\lambda_0(h_e, 0)$ are bifurcation points of the corresponding solutions. Each curve possesses two zeroes which correspond to the lower $h_{\text{min}}$ and upper $h_{\text{max}}$ critical magnetic field for concrete distribution $\varphi(x)$. The distance $\Delta h = h_{\text{max}} - h_{\text{min}}$ represents the domain of existence of the distribution $\varphi(x)$ on the field $h_e$ at $\gamma = 0$. In particular, for $\Phi^1$-distribution calculations yield $h_{\text{min}} \approx -1.8$ and $h_{\text{max}} \approx 2.2$, thus $\Delta h \approx 4$.

Similarly, on Fig. 6 the behavior of the minimal EV $\lambda_0$ of SLP by change the current $\gamma$ for $2l = 7$ and $h_e = 0$ is demonstrated. Each upper-down curve corresponds to two symmetrical vortex distributions with the same $\lambda_0$ and same energy $F$, but with opposite signs of number of fluxons and $\Delta \varphi$. The curve in the upper half-plane corresponds to single fluxon distributions $\Phi^1$ and $\Phi^{-1}$. Points $B_0$ ($|\gamma| \approx 0.6$) are points of bifurcation of mentioned solutions varying parameter $\gamma$. At these points arises confluence of $\Phi^1$ with Meissner’s distribution $M_0$, which is much deformed by the current. For the given junction, the distance $\Delta \gamma \approx 1.21$ between points $B_0$ defines domain of stability of distributions $\Phi^1$ via current $\gamma$ in the field $h_e = 0$.

At points $B_1$ the first EV $\lambda_1$ of SLP is vanishing. This means that there is a bifurcation of unstable distribution to another unstable one. In the experiments it is difficult to discover such transitions because of small life time [12].

Fig. 7 illustrated the deformation of the basic fluxon $\Phi^1$ by external field in $0-\pi JJ$ of length $2l = 7$ at the current $\gamma = 0$. Corresponding distributions of magnetic field $\varphi_x(x)$ in contact are given at Fig. 8. It is well observed, that the main changes are localized at the neighborhood of contact’s ends. The center $\varphi(0)$ of fluxon is conserved.

Similarly, on Fig. 9 the deformation of fluxon $\Phi^1$ by the current $\gamma$ at zero field $h_e$ is represented. Positive current $\gamma$ displaces the fluxon’s graph upwards (the maximum of corresponding soliton $\varphi_x(x)$ moves to the left), and the negative vice versa — the graph of $\varphi(x)$ moves down, and
the graph of soliton $\varphi(x)$ — to the right.

The basic numerical characteristics of the first several non-bifurcation magnetic flux vortices in contact at $2l = 7$ and $\gamma = 0$ are given in the Table I.

| Type | $h_e$ | $\lambda_0$ | $N[\varphi]$ | $\Delta \varphi/2\pi$ | $\varphi(0)/\pi$ | $Y[\varphi]/\pi$ |
|------|-------|-------------|--------------|---------------------|-----------------|------------------|
| $\Phi^{-1}$ | 0 | 0.772 | -0.5 | -0.468 | -0.5 | -0.586 |
| $\Phi^{1}$ | 0 | 0.772 | 0.5 | 0.468 | 0.5 | -0.586 |
| $\Phi^{-1}\Phi^{1}$ | 1.61 | 0.0275 | 0.885 | 1.7 | 0.664 | -1.349 |
| $\Phi^{1}\Phi^{-1}$ | 1.61 | 0.0267 | 2.115 | 1.7 | 2.336 | -1.346 |
| $\Phi^{3}$ | 2 | 0.108 | 2.5 | 2.582 | 2.5 | -1.883 |
| $\Phi^{-2}\Phi^{2}$ | 3.41 | 0.015 | 2.897 | 3.773 | 2.771 | -5.144 |
| $\Phi^{2}\Phi^{-2}$ | 3.41 | 0.012 | 4.103 | 3.773 | 4.229 | -5.140 |
| $\Phi^{5}$ | 4.6 | 0.111 | 4.5 | 5.049 | 4.5 | -9.378 |
| $\Phi^{-3}\Phi^{3}$ | 5.28 | 0.029 | 4.963 | 5.882 | 4.866 | -12.22 |
| $\Phi^{3}\Phi^{-3}$ | 5.28 | 0.023 | 6.037 | 5.882 | 6.134 | -12.217 |
| $\Phi^{7}$ | 6.4 | 0.081 | 6.5 | 7.072 | 6.5 | -18.00 |

TABLE I: Magnetic flux vortices in $0-\pi JJ$

One can see that the pure magnetic flux distributions match to half-integer values of number of fluxons $\Phi^n$.

$$N[\Phi^n] = n + \frac{1}{2},$$

where the minus sign correspond to $n > 0$ and plus — to $n < 0$, while for mixed fluxon states we have

$$N[\Phi^n\Phi^{-n}] + N[\Phi^{-n}\Phi^n] = 2n + \frac{1}{2}.$$

Let’s note that the quotients $\varphi(0)/2\pi$ fulfil the similar relationships as well.

**B. Bifurcations of static solutions**

In this article we shall mainly study bifurcations of static solutions varying external field $h_e$ and external current $\gamma$.

The locus on the plane $(h_e, \gamma)$ satisfying equation (1), we shall call a bifurcation curve. A unique BC corresponds to each solution of non-linear BVP (1).

For example, on Fig. 10 the BC of the vortex $\Phi^1$ in $0-\pi JJ$ of length $2l = 7$ at $\gamma > 0$ (solid line) and $\gamma < 0$ (dashed line) is demonstrated. The distance $\Delta h_e \approx 4$ between zeroes $h_{min} \approx -1.8$ and $h_{max} \approx 2.2$ represents the region of existence of $\Phi^1$ by change the external magnetic field $h_e$. The graphs of derivatives $\varphi_x(x)$ of solutions in a small neighborhood of bifurcation points $h_{min}$ and $h_{max}$ are shown on Fig. 10 by dotted and dash-dotted lines correspondingly.

Similarly, on Fig. 11 the BCs for the pair $\Phi^{-1}\Phi^{1}$, $\Phi^{1}\Phi^{-1}$ are demonstrated. The existence domain of dis-

FIG. 10: BCs of the main fluxon $\Phi^1$
tributions via field $\Delta h = h_{\text{max}} - h_{\text{min}}$ is separated by intermediate points $h_1$ and $h_2$, for which $\gamma_{\text{cr}} = 0$, on three subdomains.

Because of the symmetry, only the right upper quarter of CC is demonstrated. Let us note, that CC in symmetric 0-\(\pi\)JJ represents a set of branches, which sequentially correspond to pure and symmetric mixed distributions.

![Critical current as a function of length $l$](image)

**FIG. 14:** Disposition of the point $\lambda_0(0, \gamma)$ as a function of length $l$

Let us note, that for $h_\varepsilon \in (h_{\text{min}}, h_1)$ and $h_\varepsilon \in (h_2, h_{\text{max}})$ the corresponding domains of existence of distributions via current $\gamma$ are lied over/under the horizontal axes $\gamma_{\text{cr}} = 0$. Thus when $|\gamma|$ increases from zero, the mixed vortices at the mentioned domains have a current of “birth” (for given $h_e$ these are points on dotted curves) and a current of “destruction” — points on solid curves. For $h_e = h_{\text{min}}$ and $h_e = h_{\text{max}}$ the currents of “birth” and “destruction” coincide.

At the points $h_e = h_1$ and $h_e = h_2$ the critical current of distributions $\Phi^{-1}\Phi$ and $\Phi^2\Phi^{-1}$ is equal to zero.

The effect under consideration presents also in JJ with micro-inhomogeneities of resistive or shunt type [15]. Maybe, this effect can be determined experimentally using methods proposed in [21].

![Critical current for 0-\(\pi\)JJ of length $l = 1.3$](image)

**FIG. 15:** CC for 0-\(\pi\)JJ of length $l = 1.3$

All possible BCs of distributions of magnetic flux in 0-\(\pi\)JJ of length $2l = 7$, existing at widely differing $h_e$, are given on Fig. [12]. The critical curve of the junction contains all points of BCs with maximal modulo critical current at a given field $h_e$, i.e., the contact’s CC is derived as envelope of BCs of all distributions. On Fig. [13] CC of the contact mentioned above is illustrated for $h_e \leq 10$.

![Bias of intersection points of $\Phi^1$ & $\Phi^{-1}$ critical curves for different $\zeta$](image)

**FIG. 16:** Bias of intersection points of $\Phi^1$ & $\Phi^{-1}$ critical curves for different $\zeta$

A comparison of results of numerical modeling of CC (entire curve) with available experimental data [8] for short 0-\(\pi\)JJ of length $2l = 1.3$ is demonstrated on Fig. [15]. Note, that for such length the contribution of BC of mixed states on junction’s CC can be neglected.

A good qualitative correspondence between numerical and existing physical experiments is demonstrated. A plausible reason for some quantitative differences can be the existence of different in-symmetries in experimental samples, as $\zeta \neq 0$ and etc. For example, at Fig. [16] the influence of the shift $\zeta \neq 0$ on $\Phi^{\pm1}$ BCs in case $2l = 7$ is demonstrated. For $\zeta = 0$ the intersection point $C_0$ correspond to $h_{\varepsilon|\zeta=0} = 0$, while the inhomogeneity bias to the left or right ($\zeta = \mp0.5$) leads to bias of corresponding intersection points $C_{\mp0.5}$ to the right or left ($h_{\varepsilon|\zeta=\mp0.5} \approx \pm0.17$) correspondingly.

**V. CONCLUSIONS**

We have numerically modelled experimental data [8] for symmetric 0 – $\pi$ Josephson junctions.

Essentially, some stable bound states of magnetic fluxes and corresponding magnetic fields distributions are derived and demonstrated. For 0-\(\pi\)JJ of length $2l = 7$ we obtain and illustrate relations of minimal eigenvalue $\lambda_0$
on external magnetic field $h_e$ and external current $\gamma$. We illustrate numerically the important role of average magnetic field in the JJ as a measure of fluxon’s number in a vortex chains. Special BCs and CCs for some fluxons and antifluxons are represented, too.

For relatively short 0-\(\pi\)JJ a good qualitative conformity between numerical and existing physical results takes place. We propose that some quantitative differences can be caused on different in-symmetries in experimental samples.

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[1] L. N. Bulaevskii et al., Superconducting system with weak link with a current in a ground state, JETP Let. 25, 7 (1977).
[2] L. N. Bulaevskii et al., Solid State Commun. 25, 1053 (1978).
[3] C. Benjamin et al., Controllable \(\pi\) junction in a Josephson quantum-dot device with molecular spin, e-print: cond-mat/0608338.
[4] E. Goldobin, D. Koelle, and R. Kleiner, Ground state and bias current induced rearrangement of semifluxons in 0-\(\pi\) long Josephson junctions, Phys. Rev. B, 67, 224515 (2003).
[5] E. Goldobin, D. Koelle, and R. Kleiner, Ground states of one and two fractional vortices in long Josephson 0-\(\pi\) junctions, Phys. Rev. B, 70, 174519 (2004).
[6] E. Goldobin et al., Oscillatory eigenmodes and stability of one and two arbitrary fractional vortices in long Josephson 0-\(\kappa\) junctions, Phys. Rev. B, 71, 104518 (2005).
[7] M. Weides et al. High quality ferromagnetic 0 and \(\pi\) Josephson tunnel junctions, e-print: cond-mat/0604097.
[8] M. Weides et al., 0-\(\pi\) Josephson tunnel junctions with ferromagnetic barrier, e-print: cond-mat/0605656.
[9] K. Buckenmaier et al., Spectroscopy of the fractional vortex eigenfrequency in a long Josephson 0-\(\pi\) junction, e-print: cond-mat/0610043.
[10] H. Susanto, Darminto, S.A. van Gils, Static and dynamic properties of fluxon in a zig-zag 0-\(\pi\) Josephson junction, Physics Letters A, 361 (2007), pp. 270276.
[11] J. Rubinstein, Sine-Gordon equation, J. Math. Phys., 16, 96 (1970); M. B. Fogel et al., Dynamics of sine-Gordon solitons in the presence of perturbations, Phys. Rev. B 15, 3 (1977).
[12] Yu. S. Gal’pern, A. T. Filippov, Bound states of solitons in inhomogeneous Josephson junctions, Sov. Phys. JETP, 59 (1984).
[13] T. L. Boyadjiev, D. V. Pavlov, and I. V. Puzynin, Newton’s algorithm for calculation of critical parameters in the one-dimensional inhomogeneous Josephson junction, Comm. JINR Dubna, P11-88-409 (1988).
[14] T. L. Boyadjiev, D. V. Pavlov, and I. V. Puzynin, Application of the Continuous Analog of Newton Method for Computing the Bifurcation Curves in Josephson Junctions, in Proceedings of the International Conference on Numerical Methods and Applications, Sofia, August 22-27, 1988, edited by Bl. Sendov, R. Lazarov, I. Dimov.
[15] I. V. Puzynin et al., Methods of computational physics for investigation of models of complex physical systems, Particals & Nuclei 38, 1 (2007), Dubna.
[16] T. L. Boyadjiev, M. Todorov, Minimal length of Josephson junctions with stable fluxon bound states, Superconducting Science and Technology 14, (2002), pp. 1-7.
[17] E. G. Semerdjieva, T. L. Boyadjiev, and Yu. M. Shukrinov, Static vortices in long Josephson contacts of exponentially varying width, Low Temp. Phys. 30, 6 (2004).
[18] B. M. Levitan, I. S. Sargsjan, Sturm-Liouville and Dirac operators (Nauka, Moscow, 1988).
[19] I. M. Gelfand and S. V. Fomin, Calculus of Variations (Moscow, 1961).
[20] C. S. Owen, D. J. Scalapino, Vortex structure and Critical Currents in Josephson Junctions, Phys. Rev. 164, 2 (1967).
[21] A. N. Vishnik et al., Phys. of low Temp. (Russian) 14, 6 (1988).
[22] I. V. Puzynin et al., The generalized continuous analog of Newton method for numerical study of some nonlinear quantum-field models, Physics of Elementary Particles and Atomic Nuclei (JINR, Dubna) 30, 1 (1999).