An asymptotic approximation for the Riemann zeta function revisited

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Abstract

We revisit a representation for the Riemann zeta function \( \zeta(s) \) expressed in terms of normalised incomplete gamma functions given by the author and S. Cang in Methods Appl. Anal. 4 (1997) 449–470. Use of the uniform asymptotics of the incomplete gamma function produces an asymptotic-like expansion for \( \zeta(s) \) on the critical line \( s = 1/2 + it \) as \( t \to +\infty \). The main term involves the original Dirichlet series smoothed by a complementary error function of appropriate argument together with a series of correction terms. It is the aim here to present these correction terms in a more user-friendly format by expressing them in inverse powers of \( \omega \), where \( \omega^2 = \pi s/(2i) \), multiplied by coefficients involving trigonometric functions of argument \( \omega \).

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1. Introduction

The computation of the Riemann zeta function on the critical line \( s = 1/2 + it, t \geq 0 \) is normally carried out using the real function \( Z(t) = e^{i\vartheta(t)} \zeta(1/2 + it) \), where the phase angle

\[ \vartheta(t) = \text{arg} \Gamma(\frac{1}{4} + \frac{1}{2}it) - \frac{1}{2}t \log \pi. \]

In [3], a representation of \( Z(t) \) on the critical line was given in the form

\[ Z(t) = 2\Re e^{i\vartheta(t)} \left\{ \sum_{n=1}^{\infty} n^{-s} Q\left(\frac{1}{2}s, \pi n^2 i\right) - \pi^{s/2} e^{\pi is/4} \right\} , \]

wherein \( s = 1/2 + it, t \geq 0 \) and \( Q(a, z) = \Gamma(a, z)/\Gamma(a) \) is the normalised (upper) incomplete gamma function. The behaviour of \( Q\left(\frac{1}{2}s, \pi n^2 i\right) \) for large \( t \) changes abruptly in the neighbourhood of its transition point \( \frac{1}{2}s = \pi n^2 i \); that is, when the summation index \( n \) roughly equals the Riemann-Siegel cut-off value \( N_t \) given by

\[ N_t = \left\lfloor (t/2\pi)^{1/2} \right\rfloor , \]

where square brackets denote the integer part. Then for large \( t \), the function \( |Q\left(\frac{1}{2}s, \pi n^2 i\right)| \sim 1 \) when \( n \lesssim N_t \) and decays (algebraically) to zero when \( n \gtrsim N_t \). As a consequence, the absolutely convergent sum in (1.1) represents the original Dirichlet series “smoothed” by the incomplete gamma function.

\[ \text{The sum in (1.1) converges absolutely since } \Gamma(a, z) \sim e^{-z} z^{a-1} \text{ as } |z| \to \infty \text{ in } \arg z < 3\pi/4 \text{, and so late terms behave like } (-)^n/n^2. \]
The uniform asymptotic expansion of \( Q(a, z) \) valid for \( a \to \infty \) when \( |z| \in [0, \infty) \) in the domains \( | \arg z | < \pi \) and \( | \arg (z/a) | < 2\pi \) was employed in \cite{3} to derive an asymptotic approximation for \( Z(t) \) valid as \( t \to \infty \). The resulting expansion involved the Dirichlet sum smoothed by a complementary error function of appropriate argument together with asymptotic correction terms in inverse powers of \( \omega \), where \( \omega^2 = \pi s/(2i) \), decorated by functions \( S_k(\omega) \) involving \( \csc \omega \) and its derivatives. In \cite{2} an attempt was made to establish the asymptotic nature of this expansion; this was not successful, however, due to the use of an insufficiently precise error bound for the incomplete gamma function.

In \cite{3} the quantities \( S_k(\omega) \) were treated as computable functions. The purpose of the present paper is to revisit the expansion (1.1) with the aim of presenting the coefficients in the correction terms in a more user-friendly format. This is achieved using the expansion of the functions \( S_k(\omega) \) as polynomials in \( 1/\omega \) of degree \( k - 1 \).

2. The asymptotic approximation for \( Z(t) \)

To make this paper self-contained, we summarise in this section the main steps in the derivation of the asymptotic approximation derived from (1.1) and presented in \cite{3}. We employ the uniform asymptotic expansion of the normalised incomplete gamma function valid as \( a \to \infty \) in \( | \arg a | < \pi \) in the form \cite{4} p. 181

\[
Q(a, z) = \frac{1}{2} \erfc (\eta \sqrt{a/2}) + \frac{e^{-\frac{1}{2}a\eta^2}}{\sqrt{2\pi a}} \left\{ \sum_{k=0}^{m-1} c_k(\eta)a^{-k} + a^{-m}R_m(a, \eta) \right\}
\]  

(2.1)

for \( m = 1, 2, \ldots \), where

\[
\frac{1}{2}\eta^2 = \lambda - 1 - \log \lambda, \quad \lambda = \frac{z}{a}, \quad \mu = \lambda - 1
\]

(2.2)

and the quantity \( R_m(a, \eta) \) is the remainder term in the expansion truncated after \( m \) terms. The choice of the square root branch for \( \eta(\lambda) \) is made such that \( \eta(\lambda) \) and \( \lambda - 1 \) have the same sign when \( \lambda > 0 \); we then have \( \eta \simeq \lambda - 1 \) when \( \lambda \simeq 1 \). The coefficients \( c_k(\eta) \) are given by

\[
c_k(\eta) = \frac{(-)^k W_k(\mu)}{\mu^{2k+1}} - \frac{(-)^k (\frac{1}{2})_k}{\eta^{2k+1}}
\]

(2.3)

where \( W_k(\mu) \) is a polynomial in \( \mu \) of degree \( 2k \) and the coefficients \( \alpha_{r,k} \) are discussed in \cite{3} Appendix]; see also \cite{4}. The first few coefficients are given by

\[
\begin{align*}
\alpha_{00} &= 1 \\
\alpha_{01} &= 1, \quad \alpha_{11} = 1, \quad \alpha_{21} = \frac{17}{4} \\
\alpha_{02} &= 3, \quad \alpha_{12} = 5, \quad \alpha_{22} = \frac{25}{2}, \quad \alpha_{32} = \frac{1}{14}, \quad \alpha_{42} = \frac{5}{288} \\
\alpha_{03} &= 15, \quad \alpha_{13} = 35, \quad \alpha_{23} = \frac{105}{4}, \quad \alpha_{33} = \frac{77}{12}, \quad \alpha_{43} = \frac{49}{288}, \quad \alpha_{53} = \frac{1}{288}, \quad \alpha_{63} = -\frac{139}{51840}.
\end{align*}
\]

The modified complementary error function is introduced by

\[
\erfc (z; m) = \erfc z - \frac{e^{-z^2}}{\sqrt{\pi}} \sum_{r=0}^{m-1} \frac{(-)^r (\frac{1}{2})_r}{z^{2r+1}} \quad (z \neq 0; \quad m = 1, 2, \ldots),
\]

(2.4)

which corresponds to the deletion from \( \erfc z \) of the first \( m \) terms of its asymptotic expansion for \( |z| \to \infty \) in \( | \arg z | < \frac{1}{2} \pi \). Then the expansion (2.1) can rewritten in the form

\[
Q(a, z) = \frac{1}{2} \erfc (\eta \sqrt{a/2}; m) + \frac{e^{-\frac{1}{2}a\eta^2}}{\sqrt{2\pi a}} \left\{ \sum_{k=0}^{m-1} \frac{(-)^k W_k(\mu)}{\mu^{2k+1}}a^{-k} + a^{-m}R_m(a, \eta) \right\}.
\]

(2.5)
We now substitute the expansion (2.5) into (1.1) when we identify the parameters \( a, \lambda \equiv \lambda_n \) and \( \mu \equiv \mu_n \) (and correspondingly \( \eta \equiv \eta_n \)) with
\[
a = \frac{1}{2} s = \frac{1}{4} + \frac{1}{2}it, \quad \lambda_n = 2\pi n^2t/s, \quad \mu_n = (\pi n/\omega)^2 - 1, \quad \omega^2 = \pi s/(2i). \tag{2.6}
\]
We also make use of the well-known expansion so that
\[
\frac{1}{\Gamma\left(\frac{1}{2} s\right)} = (2\pi)^{-1/2} \left(\frac{1}{2} s\right)^{(1-s)/2} e^{s/2} \left\{ \sum_{r=0}^{m-1} \gamma_r \left(\frac{1}{2} s\right)^{-r} + \left(\frac{1}{2} s\right)^{-m} H_m(s) \right\} \quad (t \to \infty),
\]
where the \( \gamma_r \) are the Stirling coefficients with \( \gamma_1 = 1, \gamma_2 = -1/12, \gamma_2 = 1/288, \gamma_3 = 1/15360, \ldots \), and \( H_m(s) \) is a remainder term. Since
\[
e^{-\frac{1}{2} a n^2} = (-)^n n^e e^{pi s/14(27\pi e/s)^s/2},
\]
we then find that
\[
Z(t) = \Re e^{it(t)} \left\{ \sum_{n=1}^{\infty} n^{-s} \text{erfc} \left( \frac{1}{2} \eta_n \sqrt{\pi}; m\right) - \frac{e^{\pi i s/4}}{\sqrt{\pi s}} \left( \frac{2\pi e}{s}\right)^{s/2} \left\{ \sum_{r=0}^{m-1} \frac{(-)^r A_r(s)}{(\frac{1}{2} s)^r} + \frac{1}{12} A_r(s) \right\} \right\}, \tag{2.7}
\]
where \( R_m \) is the remainder that contains contributions from the remainder term in \( Q(a, z) \) and that in the expansion of \( 1/\Gamma\left(\frac{1}{2} s\right) \). We do not consider this term here; an attempt was made in [2] to bound \( R_m \) and demonstrate that (2.7) is an asymptotic expansion, but the bounds employed on the incomplete gamma function were not sharp enough to achieve this.

The coefficients \( A_r(s) \) are given by
\[
A_r(s) = (-)^r \gamma_r - 2 \sum_{n=1}^{\infty} \frac{(-)^n W_r(\mu_n)}{\mu_n^{2r+1}} = \sum_{n=-\infty}^{\infty} \frac{(-)^{n-1} W_r(\mu_n)}{\mu_n^{2r+1}}
\]
where \( W_r(\mu_0) = W_r(-1) = (-)^r \gamma_r; \) see [3] (A.1). From the definitions of \( W_k(\mu_n) \) and \( \mu_n \) in (2.5) and (2.6), we then find that
\[
A_r(s) = \sum_{k=0}^{2r} (-)^k \alpha_{r,k} \sum_{n=-\infty}^{\infty} (-)^n \left( \frac{\omega^2}{\omega^2 - (\pi n)^2} \right)^{2r+1-k}.
\]
The inner sum may be expressed in terms of the functions \( S_k(\omega) \) defined by
\[
S_k(\omega) := 2^{k-1} \sum_{n=-\infty}^{\infty} (-)^n \omega^k \left( \frac{\omega^2}{\omega^2 - (\pi n)^2} \right)^k \quad k = 1, 2, \ldots \tag{2.8}
\]
so that
\[
A_r(s) = 2^{-2r} \omega^{2r+1} B_r(\omega), \quad B_r(\omega) = \sum_{k=0}^{2r} (-)^k \alpha_{r,k} \frac{S_{2r+1-k}(\omega)}{\omega^{2k}}. \tag{2.9}
\]
After some routine algebra we finally obtain the expansion on the critical line for large \( t \) in the form

**Theorem 1.** As \( t \to +\infty \) on the critical line \( s = \frac{1}{2} + it \), we have the expansion
\[
Z(t) = \Re e^{it(t)} \left\{ \sum_{n=1}^{\infty} n^{-s} \text{erfc} \left( \frac{1}{2} \eta_n \sqrt{\pi}; m\right) - \frac{1}{\sqrt{2\pi}} \left( \frac{\pi e^{1/2}}{s}\right)^{s/2} \left( T_m(\omega) + \omega^{-2m+1} R_m \right) \right\}, \tag{2.10}
\]
where \( \omega^2 = \pi s/(2i) \) and
\[
T_m(\omega) := \sum_{r=0}^{m-1} (\pi i/4)^r B_r(\omega) \tag{2.11}
\]
with the coefficients \( B_r(\omega) \) defined in [2.4].
In [3] the functions \( S_k(\omega) \) appearing in the coefficients \( B_r(\omega) \) were treated as computable functions. In the next section we express \( B_r(\omega) \), and hence \( T_m(\omega) \), as a finite series in inverse powers of \( \omega \) with coefficients containing certain trigonometric functions.

3. The coefficients in the expansion

In this section we examine the correction term \( T_m(\omega) \) defined in (2.11) and express it as a finite series in descending powers of \( \omega \). To achieve this we first observe that the functions \( S_k(\omega) \) defined in (2.8) satisfy

\[
S_1(\omega) = \csc \omega, \quad S_{k+1}(\omega) = \frac{1}{\omega} S_k(\omega) - \frac{1}{k} \frac{d S_k(\omega)}{d \omega} \quad (k \geq 1).
\]

From this we find with the help of Mathematica that

\[
S_{k+1}(\omega) = \csc \omega \sum_{r=0}^{k} \frac{b_{r,k}}{\omega^r}, \tag{3.1}
\]

where the coefficients \( b_{r,k} \) for \( 0 \leq r, k \leq 6 \) are

\[
\begin{align*}
b_{00} &= 1; \\
b_{01} &= \cot \omega, \quad b_{11} = 1; \\
b_{02} &= \cot^2 \omega + \csc^2 \omega, \quad b_{12} = 3 \cot \omega, \quad b_{22} = 3; \\
b_{03} &= \cot^3 \omega + 5 \cot \omega \csc^2 \omega, \quad b_{13} = 6(\cot^2 \omega + \csc^2 \omega), \quad b_{23} = 15 \cot \omega, \quad b_{33} = 15; \\
b_{04} &= \cot^4 \omega + 18 \cot^2 \omega \csc^2 \omega + 5 \csc^4 \omega, \quad b_{14} = 10(\cot^3 \omega + 5 \cot \omega \csc^2 \omega), \\
b_{24} &= 45(\cot^2 \omega + \csc^2 \omega), \quad b_{34} = 105 \cot \omega, \quad b_{44} = 105; \\
b_{05} &= \cot^5 \omega + 58 \cot^3 \omega \csc^2 \omega + 61 \cot \omega \csc^4 \omega, \quad b_{15} = 15(\cot^4 \omega + 18 \cot^2 \omega \csc^2 \omega + 5 \csc^4 \omega), \\
b_{25} &= 105(\cot^3 \omega + 5 \cot \omega \csc^2 \omega), \quad b_{35} = 420(\cot^2 \omega + \csc^2 \omega), \quad b_{45} = 945 \cot \omega, \\
b_{55} &= 945; \\
b_{06} &= \cot^6 \omega + 179 \cot^4 \omega \csc^2 \omega + 479 \cot^2 \omega \csc^4 \omega + 61 \csc^6 \omega, \\
b_{16} &= 21(\cot^5 \omega + 58 \cot^3 \omega \csc^2 \omega + 61 \cot \omega \csc^4 \omega), \\
b_{26} &= 210(\cot^4 \omega + 18 \cot^2 \omega \csc^2 \omega + 5 \csc^4 \omega), \quad b_{36} = 1260(\cot^3 \omega + 5 \cot \omega \csc^2 \omega), \\
b_{46} &= 4725(\cot^2 \omega + \csc^2 \omega), \quad b_{56} = 10395 \cot \omega, \quad b_{66} = 10395.
\end{align*}
\]

Then, from (2.8) and (3.1), we find \( B_0(\omega) = \csc \omega \) and

\[
B_1(\omega) = \alpha_{10} S_3(\omega) - \frac{2}{\omega} \alpha_{11} S_2(\omega) + \frac{4}{\omega^2} \alpha_{12} S_1(\omega)
\]

\[
= \csc \omega \left( \frac{\alpha_{10}}{2} \sum_{r=0}^{2} \frac{b_{r,2}}{\omega^r} - 2 \alpha_{11} \sum_{r=0}^{1} \frac{b_{r,1}}{\omega^{r+1}} + \frac{4}{\omega^2} \alpha_{12} b_{00} \right)
= \csc \omega \sum_{k=0}^{2} \frac{d_{1,k}}{\omega^k},
\]

where

\[
\begin{align*}
d_{10} &= \frac{1}{\pi} \alpha_{10} b_{02} = \frac{1}{\pi} (\cot^2 \omega + \csc^2 \omega), \\
d_{11} &= \frac{1}{\pi} \alpha_{10} b_{12} - 2 \alpha_{11} b_{01} = -\frac{1}{2} \cot \omega, \\
d_{13} &= \frac{1}{\pi} \alpha_{10} b_{22} - 2 \alpha_{11} b_{11} + 4 \alpha_{12} b_{00} = -\frac{1}{6}.
\end{align*}
\]
In this manner we obtain after some effort

\[ B_r(\omega) = \csc \omega \sum_{k=0}^{2r} \frac{d_{r,k}}{\omega^k} \quad (r \geq 1), \]

where

\[ d_{r,k} = \sum_{p=0}^{k} \frac{(-2)^{k-p} \alpha_{r,k-m} b_{m,p}}{(p+2r-k)!} \quad (p^* = 2r-k+p+1). \]

It then follows upon reversing the order of summation that

\[ T_m(\omega) = \csc \omega \sum_{r=0}^{m-1} (\pi i/4)^r \sum_{k=0}^{2r} \frac{d_{r,k}}{\omega^k} = \csc \omega \sum_{k=0}^{2m-2} \frac{1}{\omega^k} \sum_{r=[k/2]}^{m-1} (\pi i/4)^r d_{r,k}. \]

Finally, we obtain the correction term in the following form:

**Theorem 2.** The correction term \( T_m(\omega) \) appearing in the expansion \( \mathcal{E}(t) \) has the form

\[ T_m(\omega) = \csc \omega \sum_{r=0}^{m-1} (\pi i/4)^r \sum_{k=0}^{2m-2} \frac{D_k(m)}{\omega^k}, \quad (3.2) \]

where the coefficients \( D_k(m) \equiv D_k(m; \omega) \) are given by

\[ D_k(m) = \sum_{r=[k/2]}^{m-1} (\pi i/4)^r d_{r,k}. \quad (3.3) \]

The coefficients \( D_k(\omega) \) are easily seen to satisfy

\[ D_k(m + 1) = D_k(m) + (\pi i/4)^m d_{m,k} \quad (0 \leq k \leq 2m - 4) \]

and involve the trigonometric functions \( \csc \omega \) and \( \cot \omega \). We now present the explicit representation\(^2\) of the coefficients \( D_k(m) \) for \( m = 1, 2, 3 \) and 4, where for brevity we have set \( \epsilon = \pi i/4 \).

\[ m = 1 : \quad D_0(1) = 1; \]

\[ m = 2 : \quad D_0(2) = D_0(1) + \frac{1}{2} \epsilon (\cot^2 \omega + \csc^2 \omega) \]
\[ D_1(2) = -\frac{1}{8} \epsilon \cot \omega, \quad D_2(2) = -\frac{1}{6} \epsilon; \]

\[ m = 3 : \quad D_0(3) = D_0(2) + \frac{1}{8} \epsilon^2 (\cot^4 \omega + 18 \cot^2 \omega \csc^2 \omega + 5 \csc^4 \omega) \]
\[ D_1(3) = D_1(2) + \frac{5}{12} \epsilon^2 (\cot^3 \omega + 5 \cot \omega \csc^2 \omega) \]
\[ D_2(3) = D_2(2) - \frac{1}{24} \epsilon^2 (\cot^2 \omega + \csc^2 \omega) \]
\[ D_3(3) = -\frac{1}{6} \epsilon^2 \cot \omega, \quad D_4(3) = \frac{1}{3} \epsilon^2; \]

\[ m = 4 : \quad D_0(4) = D_0(3) + \frac{1}{48} \epsilon^3 (\cot^6 \omega + 179 \cot^4 \omega \csc^2 \omega + 479 \cot^2 \omega \csc^4 \omega + 61 \csc^6 \omega) \]
\[ D_1(4) = D_1(3) - \frac{7}{48} \epsilon^3 (\cot^5 \omega + 58 \cot^3 \omega \csc^2 \omega + 61 \cot \omega \csc^4 \omega) \]
\[ D_2(4) = D_2(3) \]
\[ D_3(4) = D_3(3) + \frac{7}{72} \epsilon^3 (\cot^3 \omega + 5 \cot \omega \csc^2 \omega) \]
\[ D_4(4) = D_4(3) + \frac{49}{144} \epsilon^3 (\cot^2 \omega + \csc^2 \omega) \]
\[ D_5(4) = \frac{497}{144} \epsilon^3 \cot \omega, \quad D_6(4) = \frac{1003}{1440} \epsilon^3. \]

\(^2\)The \( O(\epsilon^3) \) contribution in \( D_2(4) \) is zero.
Table 1: Details of the computations when $t = 2600$, $N_t = 20$, $Z(t) = -0.63210232$ for different values of $m$ and truncation index $K$ in the main sum.

| $m = 2$ | Correction term | Main Sum | $|Z_{\text{approx}} - Z(t)|$ |
|---------|----------------|----------|-----------------------------|
| $K$     |                |          |                             |
| 10      | 0.17012 33165 89694 33 | -0.46197 90063 18404 31 | 2.269 $\times 10^{-08}$ |
| 20      | 0.08577 29470 58489 99 | -0.46197 90041 44898 25 | 9.502 $\times 10^{-11}$ |
| 30      | 0.08577 29470 58489 99 | -0.46197 90040 66916 27 | 1.704 $\times 10^{-11}$ |

| $m = 3$ | Correction term | Main Sum | $|Z_{\text{approx}} - Z(t)|$ |
|---------|----------------|----------|-----------------------------|
| $K$     |                |          |                             |
| 10      | 0.08577 29470 58489 99 | -0.54632 93736 04227 97 | 2.315 $\times 10^{-11}$ |
| 20      | 0.08577 29470 58489 99 | -0.54632 93735 81347 73 | 2.693 $\times 10^{-13}$ |
| 30      | 0.08577 29470 58489 99 | -0.54632 93735 81099 88 | 2.146 $\times 10^{-14}$ |

| $m = 4$ | Correction term | Main Sum | $|Z_{\text{approx}} - Z(t)|$ |
|---------|----------------|----------|-----------------------------|
| $K$     |                |          |                             |
| 10      | 0.08577 29470 58489 99 | -1.16468 98036 52714 66 | 5.792 $\times 10^{-14}$ |
| 20      | 0.08577 29470 58489 99 | -1.16468 98036 52772 40 | 1.874 $\times 10^{-16}$ |
| 30      | 0.08577 29470 58489 99 | -1.16468 98036 52772 58 | 6.799 $\times 10^{-18}$ |

4. Numerical results

In this section we describe computations using the expansion (2.10) with the correction term $T_m(\omega)$ given in (3.2) and (3.3). The terms in the main sum in (2.10) decay rapidly beyond $n \approx N_t$. In the neighbourhood of the transition point of $Q(\frac{1}{2}s, \pi n^2 i)$ given by $\lambda_n = 1$, we have

$$\eta_n \approx \frac{\pi^2}{\omega^2} \left( n^2 - \frac{\omega^2}{\pi^2} \right) \approx \frac{2\pi i}{s} \left( n^2 - \frac{t}{2\pi} \right),$$

so that the argument of the modified complementary error function in (2.10) for large $t$ when $n \approx N_t$ is

$$\frac{1}{2} \eta_n \sqrt{s} \approx \pi t^{-1/2} e^{\pi i/4} (n^2 - N_t^2) \approx 2\pi t^{-1/2} e^{\pi i/4} N_t (n - N_t).$$

From the asymptotic behaviour

$$\text{erfc}(z; m) \sim \frac{e^{-z^2}}{\sqrt{\pi}} z^{2m+1} \frac{(-)^m (\frac{1}{2})^m}{z^{2m+1}} \quad (|z| \to \infty, \quad |\arg z| < \frac{3}{4} \pi),$$

it is then found that when $n = N_t + K$, where integer $K \ll N_t$, the magnitude of the terms in the main sum is given approximately by

$$|n^{-s}\text{erfc}\left(\frac{1}{2} \eta_n \sqrt{s}; m\right)| = (4\pi K^2)^{-m-1/2} O((t/2\pi)^{-1/4}).$$

The decay of the terms is therefore controlled by $K^{-2m-1/2}$ together with a scaling factor depending weakly on $t$ like $t^{-1/4}$. Thus if $K = 10$, for example, the magnitude of the terms in the main sum is approximately of order $10^{-14}t^{-1/4}$ when $m = 4$. 
Asymptotic representation of $Z(t)$

Table 2: Details of the computations when $t = 2 \times 10^5$, $N_t = 178$, $Z(t) = -3.51142011$ for different values of $m$ and truncation index $K$ in the main sum.

| $m = 2$ | Correction term $= +0.04418$ | Main Sum | $|Z_{\text{approx}} - Z(t)|$ |
|---------|------------------------------|----------|-------------------------------|
| $K$     |                              |          |                               |
| 10      | $-3.46723$                  | $96042$  | $56646$ 04                   |
|         |                              |          | $5.980 \times 10^{-10}$      |
| 20      | $-3.46723$                  | $96036$  | $78939$ 00                   |
|         |                              |          | $2.032 \times 10^{-11}$      |
| 30      | $-3.46723$                  | $96036$  | $61437$ 02                   |
|         |                              |          | $2.815 \times 10^{-12}$      |

| $m = 3$ | Correction term $= +0.03215$ | Main Sum | $|Z_{\text{approx}} - Z(t)|$ |
|---------|------------------------------|----------|-------------------------------|
| $K$     |                              |          |                               |
| 10      | $-3.47926$                  | $93298$  | $09067$ 88                   |
|         |                              |          | $5.521 \times 10^{-12}$      |
| 20      | $-3.47926$                  | $93298$  | $03596$ 26                   |
|         |                              |          | $4.958 \times 10^{-14}$      |
| 30      | $-3.47926$                  | $93298$  | $03549$ 81                   |
|         |                              |          | $3.130 \times 10^{-15}$      |

| $m = 4$ | Correction term $= -0.03047$ | Main Sum | $|Z_{\text{approx}} - Z(t)|$ |
|---------|------------------------------|----------|-------------------------------|
| $K$     |                              |          |                               |
| 10      | $-3.54189$                  | $12817$  | $70598$ 54                   |
|         |                              |          | $1.251 \times 10^{-14}$      |
| 20      | $-3.54189$                  | $12817$  | $70611$ 01                   |
|         |                              |          | $2.921 \times 10^{-17}$      |
| 30      | $-3.54189$                  | $12817$  | $70611$ 04                   |
|         |                              |          | $8.377 \times 10^{-19}$      |

In the computations we define the truncated main sum as

$$\Re e^{\theta(t)} \sum_{k=1}^{N_t+K} n^{-s} \text{erfc} \left( \frac{1}{2} \eta_n \sqrt{s}; m \right)$$

$$= \Re e^{\theta(t)} \left\{ \sum_{k=1}^{N_t} n^{-s} \left( 2 - \text{erfc} \left( \frac{1}{2} \eta_n \sqrt{s}; m \right) \right) + \sum_{k=N_t+1}^{N_t+K} n^{-s} \text{erfc} \left( \frac{1}{2} \eta_n \sqrt{s}; m \right) \right\}, \quad (4.1)$$

where $N_t$ is the Riemann-Siegel cut-off value defined in (1.2) and we have made use of the result $\text{erfc} (-z; m) = 2 - \text{erfc} (z; m)$. The correction term is given by

$$\Re e^{\theta(t)} \sqrt{2t} \left( \frac{\pi e^{1/2}}{\omega} \right)^s T_m(\omega).$$

The difference between these two contributions then yields the value $Z_{\text{approx}}$. An example of the results is displayed in Tables 1 and 2, where the value $Z(t)$ was obtained by Mathematica using the command RiemannSiegelZ[$t$]. The values in the tables confirm that the accuracy increases as both $m$ and the truncation index $K$ increase.

### 5. Concluding remarks

We have revisited an expansion derived in [34] for $Z(t)$ on the critical line $s = \frac{1}{2} + it$ as $t \to +\infty$ in which the main sum is the original Dirichlet series smoothed by a complementary error function. The correction term in this expansion has been expressed as a series in inverse powers of $\omega$, where $\omega^2 = \pi s/(2i)$, multiplied by coefficients involving $\csc \omega$ and its derivatives. Numerical results are presented to illustrate the accuracy achievable with this expansion.

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3We note that the factor 2 in the sum over $1 \leq n \leq N_t$ in (4.1) yields the standard Riemann-Siegel sum $2 \sum_{n=1}^{N_t} n^{-1/2} \cos \theta(t - t \log n)$.
A difficulty arises in the use of the expansion (2.10) when \((t/2\pi)^{1/2}\) is close to an integer i.e., at a discontinuity in \(N_t\). This arises because

\[
\frac{\omega}{\pi} = \left(\frac{t}{2\pi}\right)^{1/2} \left(1 - \frac{i}{2t}\right)^{1/2},
\]

so that \(\csc \omega\) and \(\cot \omega\) become large, but never singular as \(\omega\) always has a small imaginary part (that decreases with increasing \(t\)). This results in the term in the main sum and correction term corresponding to \(\eta_n \approx 0\) becoming large. A means of overcoming this problem, by removing and combining these terms, is given in [3, Section 5].

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