Capacity Expansion in the College Admission Problem

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Abstract

The college admission problem plays a fundamental role in several real-world allocation mechanisms such as school choice and supply chain stability. The classical framework assumes that the capacity of each college is known and fixed in advance. However, increasing the quota of even a single college would improve the overall cost of the students. In this work, we study the problem of finding the college capacity expansion that achieves the best cost of the students, subject to a cardinality constraint. First, we show that this problem is NP-hard to solve, even under complete and strict preference lists. We provide an integer quadratically constrained programming formulation and study its linear reformulation. We also propose two natural heuristics: A greedy algorithm and an LP-based method. We empirically evaluate the performance of our approaches in a detailed computational study. We observe the practical superiority of the linearized model in comparison with its quadratic counterpart and we outline their computational limits. In terms of solution quality, we note that the allocation of a few extra spots can significantly impact the overall student satisfaction.
1 Introduction

The college admission process is a fundamental moment in a student’s life. Ensuring that each student is able to access the best possible education brings numerous benefits for the student and society. In a real-world scenario, a college can only accept a limited number of students; as a consequence, some students may enroll in a college they like, but not their most preferred. In fact, each student have personal preferences over colleges; likewise, colleges rank applicants. These rankings are used to guide the selection process, which is conditioned by the capacity constraints.

From a centralized point of view, third party institutions, such as the Ministry of Education or a private foundation, may plan to impact the education system by allocating additional admission positions to colleges, e.g., via extra funding or scholarships. Once an educational institution is targeted with additional positions, it may now become reachable by students who were previously rejected. The following question naturally arises: How should the centralized decision-maker optimally allocate new positions to colleges? Tackling this question is crucial to further understand the benefits and challenges of increasing the capacity of the educational system. In this work, we tackle this question from the lens of discrete optimization with the ultimate goal of improving the students’ welfare.

Gale and Shapley [13] introduce the college admission (CA) problem, or the hospitals-residents problem, and propose the deferred acceptance (DA) algorithm. The algorithm computes a student-college assignment such that any other possible pair is not preferred simultaneously by both agents to their current assignment; this is known as a stable matching. In practice, the DA mechanism has been extensively used to improve admission processes, e.g., [1, 9]. A substantial part of the literature related to the CA problem has focused on strategy-proof mechanisms, i.e., on mechanisms that output matchings that incentivize participants to reveal their true preferences. Sönmez [28] proved that colleges can manipulate the stable matching in their favor by falsely reporting a reduced capacity. Moreover, Romm [24] proved that the stable matching mechanism can still be manipulated even if the reported capacities are enforced during the admission process. For further details on stable matching mechanisms, see [20].

The design of a stable matching mechanism, when the number of participants of one side is increased, has already been investigated in the past. If colleges have capacity one, this is known as the entry comparative static in the stable marriage problem. In this setting, the two sides are traditionally called women and men. In [16, 14, 26] the authors proved that when a new woman is added to the instance, all men are matched weakly better. On the same path, Balinski and Sönmez [8] proved that the DA method is invariant with respect to students who improve their score in the ranking lists. Recently, Kominers [18] extended this result to the general CA problem. Our setting adds a layer of complexity to the comparative static approach: The allocation of new spots to expand colleges’ capacities is now a variable to optimize rather than a parameter.

Contributions. Our main contributions summarize as follows. On the theoretical side, we show that the underlying decision problem of minimizing the student’s cost for CA with capacity expansion is NP-hard. This is a far-from-trivial result, that looks somehow counter intuitive, since it holds under very basic assumptions. Relying on the observation that minimizing the students’ cost is equivalent to minimize the regret of each student, we formulate the problem as a quadratically-constrained integer program. Additionally, we propose some possible linearizations, we theoretically analyze their quality and we show computationally that the best linearized model can be solved significantly faster than the quadratically-constrained one by the mixed-integer solver Gurobi. We also introduce two natural heuristics, a greedy approach and a LP-based heuristic, and we compare them. We provide extensive computational results, which establish a reference point for future computational approaches and highlight important challenges. Finally,
we analyze the value of the capacity expansion by showing that even a small amount of extra positions improve considerably the overall and individual welfare of the students.

2 Preliminaries

An instance of the CA problem consists of a set of students \( \mathcal{S} = \{i_1, \ldots, i_s\} \), a set of colleges \( \mathcal{C} = \{j_1, \ldots, j_c\} \) and a set of edges \( \mathcal{E} \) between \( \mathcal{S} \) and \( \mathcal{C} \). A student and a college are linked by an edge in \( \mathcal{E} \) if they deem each other acceptable. In this work, we assume that every student-college pair is acceptable, i.e., \( \mathcal{E} = \mathcal{S} \times \mathcal{C} \). Each college \( j \in \mathcal{C} \) has an associated quota \( c_j \in \mathbb{Z}_+ \) which represents the maximum number of students that college \( j \) can admit. In this context, a matching \( M \) is a subset of \( \mathcal{E} \) in which each college \( j \) appears in at most \( c_j \) pairs and each student appears in at most 1 pair. We denote by \( M(i) \) and \( M(j) \), the college assigned to student \( i \) and the subset of students assigned to college \( j \), respectively.

An instance \( \Gamma \) of the CA problem is given by tuple \( \Gamma = \langle \mathcal{S}, \mathcal{C}, \succ, c \rangle \), where \( c \in \mathbb{Z}_+^C \) is the vector of capacities and \( \succ \) corresponds to the preferences that students have over colleges and vice-versa. We consider students and colleges whose preferences are expressed in the form of a linear order. We assume that the preference list of each agent (student or college) is strict, i.e., it includes no ties, and complete, i.e., each agent ranks the entire opposite side. We use the notation \( j \succ_i j' \) if student \( i \) prefers college \( j \) over college \( j' \). Similarly, if college \( j \) prefers student \( i \) over \( i' \), we write \( i \succ_j i' \). Whenever the context makes it clear, we drop the subscript in \( \succ \).

Given a matching \( M \), we say that a pair \( (i, j) \in \mathcal{E} \) is a blocking pair if the following two conditions are satisfied: (1) student \( i \) is unassigned or prefers college \( j \) over \( M(j) \); (2) college \( j \) is such that \( |M(j)| < c_j \) or prefers student \( i \) over at least one student in \( M(j) \). The matching \( M \) is said to be stable if it does not admit a blocking pair.

Gale and Shapley [13] showed that every instance of the CA problem admits a stable matching which can be found by the DA method. In particular, this algorithm can be designed to prioritize the students in the following sense: Let \( M \) and \( M' \) be two different stable matchings, we say that a student \( i \) weakly prefers \( M \) over \( M' \) if \( M(i) \succ_i M'(i) \) or \( M(i) = M'(i) \). The DA algorithm can be adapted so it computes the unique stable matching that is weakly preferred by all students, over all other possible stable matchings. Such unique stable matching is called student-oriented. In Section 4.2, we formally describe the student-oriented DA algorithm.

In this work, we focus on stable matchings that minimize the total student cost, which we define as follows: We denote by \( \text{rank}_i(j) \) the rank of college \( j \) in the list of student \( i \), i.e., the number of colleges preferred by student \( i \) to college \( j \). This means, for example, that the most preferred college has the lowest ranking. The total student cost of a matching \( M \) is defined as

\[
\sum_{(i,j) \in M} \text{rank}_i(j).
\]

A folklore result states that a stable matching \( M \) is student-oriented if, and only if, it is a stable matching of minimum cost, as defined in (1); we provide a proof in Appendix C. This result is crucial because it allows us to look for the best possible stable matching for the students by optimizing Expression (1) over the family of stable matchings, which can be polyhedrally described [27].

As we mentioned in Section 1, our goal is to improve the total student satisfaction, in terms of their cost, by increasing the college system’ capacity. Formally, for a non-negative vector \( t \in \mathbb{Z}_+^C \), we denote by \( \Gamma_t = \langle \mathcal{S}, \mathcal{C}, \succ, c + t \rangle \) an instance of the CA problem in which the capacity of every
college \( j \in \mathcal{C} \) is now \( c_j + t_j \). Observe that \( \Gamma_0 \) corresponds to the original instance \( \Gamma \) with no capacity expansion.

Formally, we define our main optimization problem as follows.

**Problem 1.** Given a budget \( B \in \mathbb{Z}_+ \) and a CA instance \( \Gamma \), the capacity expansion problem, is the following:

\[
\min_{t, M_t} \left\{ \sum_{(i,j) \in M_t} \text{rank}_i(j) : \text{where } t \in \mathbb{Z}_+^C \text{ is s.t. } \sum_{j \in \mathcal{C}} t_j \leq B \text{ and } M_t \text{ is a stable matching in instance } \Gamma_t \right\}.
\]

In other words, the optimal solution of Problem 1 corresponds to the best possible student-oriented stable matching that can be obtained among all feasible capacity expansions. We work under the assumption that the sum of the colleges’ capacities is greater or equal than the number of students, i.e., \( \sum_{j \in \mathcal{C}} c_j \geq |S| \).

Our main focus is to study Problem 1 from the lens of discrete optimization. Specifically, we model the stability condition in the context of capacity expansion via a quadratic constraint. In Section 3, we discuss the complexity of the problem and prove that its decision version is NP-complete. In Section 4.1, we give a quadratically constrained integer programming formulation and we study two McCormick linearizations. Finally, in Section 5, we provide a detailed computational study.

### 2.1 Related Work

The DA algorithm has been crucial behind a vast number of applications in the educational admission process: Daycare admission in Denmark [17], school admission in the USA [2, 3, 1], school and university admission in Hungary [9, 10], school admission in Singapore [29], university admission in China [31], Germany [12] and Spain [23].

Some of the applications above have motivated algorithms based on polyhedral approaches, such as the faculty recruitment in France and the students admission in Turkey [6, 7]. The first mathematical programming formulations of the CA problem were studied in [15, 30, 27, 25]. In particular, in Báïou and Balinski [5] provided a linear programming formulation of CA (with exponential size) whose extreme points are integral.

In the last years, variants of CA have also been modeled through mathematical programming. Kwanashie and Manlove [19] gave an integer formulation for CA when there are ties in the preference lists. Biró, Manlove and McBride [11] study the problem of assigning couples of doctors to hospitals. Ágoston, Biró and McBride [4] elaborated an integer model that incorporates upper and lower quotas.

### 3 Complexity Analysis

In this section, we analyze the complexity of the Problem 1. As we will see, when \( B = 1 \), the problem is polynomially solvable, but for general values of \( B \) the problem becomes NP-complete.

To give some insights behind the difficulty of Problem 1, we first study an intuitive approach for the case of \( B = 1 \) that does not always provide an optimal solution. In real life, a majority of the students may rank the same college first. Thus, when \( B = 1 \), a tentative approach is to assign the additional spot to the college which is preferred by the majority of the students. However, as the following example shows, this is not necessarily optimal.
Counterexample for the Majority Approach. Let \( S = \{i_1, i_2, i_3, i_4, i_5, i_6, i_7\} \) and \( C = \{j_1, j_2, j_3, j_4\} \). We assume that all colleges have the same preference list: \( i_1 \succ i_2 \succ \cdots \succ i_7 \). Colleges \( j_1, j_2 \) and \( j_3 \) have each capacity 1, and college \( j_4 \) has capacity 4. Students \( i_1 \) and \( i_2 \) rank colleges as \( j_2 \succ j_3 \succ j_4 \succ j_1 \). Students \( i_3 \) and \( i_4 \) rank colleges as \( j_3 \succ j_2 \succ j_4 \succ j_1 \). Students \( i_5, i_6 \) and \( i_7 \) rank colleges as \( j_1 \succ j_4 \succ j_2 \succ j_3 \). The student-oriented stable matching is \( M = \{(i_1, j_2), (i_2, j_3), (i_3, j_4), (i_4, j_4), (i_5, j_1), (i_6, j_4), (i_7, j_4)\} \) with cost 7. Now, consider Problem 1 with \( B = 1 \). For this instance, an intuitive solution such as allocating the extra post to the most preferred college, \( j_1 \), is sub-optimal. Indeed, if we expand the capacity \( c_{j_1} = 1 \) to \( c_{j_1} = 2 \), then student \( i_6 \) would prefer to go to college \( j_1 \), which leaves an extra spot in college \( j_4 \). This solution reduces the students’ cost by 1 unit. Instead, if we expand the capacity of \( j_2 \) to 2, then student \( i_2 \) is admitted by college \( j_2 \), leaving an empty spot in college \( j_3 \) that is filled by student \( i_3 \) and resulting in a cost reduction of 3 units.

As the previous example shows, the allocation of one extra spot is not trivial when we try to solve it by just looking at the students’ preferences. However, we can still solve this problem in polynomial time by doing an exhaustive search in combination with the DA algorithm. For this, we compute the student-oriented stable matching using the DA mechanism in the instance \( \Gamma_t \) with \( t = 1_j \) for each \( j \in C \), where \( 1_j \in \{0, 1\}^C \) is the indicator vector, i.e., the \( j \)-the component is 1 and the rest is 0. Once we obtain the cost for each \( j \in C \), we output the student-oriented stable matching of minimum cost. Since the DA algorithm’s runtime complexity is \( O(|S| \cdot |C|) \), then this exhaustive search runs in \( O(|S| \cdot |C|^2) \). It remains as an open question if this can be improved.

Next, we focus on the general case. Our main result is the following.

**Theorem 1.** The decision version of Problem 1 is NP-complete.

**Sketch proof of Theorem 1.** We restrict our analysis to the case in which colleges’ capacities are all 1. As we mentioned earlier, this problem is called the stable marriage problem. Manlove et al. [21] proves that the problem of finding the egalitarian stable matching\(^1\) when preferences lists are allowed to have ties is NP-hard (see their Theorem 7). We use their NP-hardness reduction for our problem. First, we show that their result can be used to prove that finding the minimum students’ cost stable matching when there are at most two ties in each preference list is NP-hard. Given an instance of such problem, we build an instance of Problem 1 by creating two clones with capacity zero and strict preferences for each college with ties in its preference list; for every such college with ties we have one extra post to allocate to the colleges. We organize the preferences of the students and of the colleges in order to ensure that the extra posts will be allocated to mimic the minimum students’ cost stable matching in the original problem.

\(\square\)

4 Methodologies

Next, we present two mathematical programs which can be solved with out-of-the-box optimization solvers and then the two heuristic approaches.

\(^1\)An egalitarian stable matching is a stable matching in which the cost of all the agents (both students and colleges) is minimized.
4.1 Integer Programming Formulations

First, we introduce some notation. Let

\[ P = \left\{ (x, t) \in [0, 1]^E \times [0, B]^C : \sum_{j \in C} x_{ij} \leq 1 \quad \forall i \in S, \quad \sum_{i \in S} x_{ij} \leq c_j + t_j \quad \forall j \in C, \quad \sum_{j \in C} t_j \leq B \right\} \]

be the set of fractional matchings (including non-stable) with capacity expansion. Specifically, note that the first two conditions set a bound on the capacity of each student and college, respectively, and the last condition establishes a budget for the extra spots to allocate. We denote by \( P_Z \) the integral points of \( P \), i.e.,

\[ P_Z = P \cap \left( \{0, 1\}^E \times \{0, \ldots, B\}^C \right). \]

In the following, we base our approach on the stable admission polytope discussed in Lemma 1 in [5]. Given a budget \( B \in \mathbb{Z}^+ \), Problem 1 can be modeled through the following integer quadratic program:

\[
\begin{align*}
\min_{x, t} & \quad \sum_{(i, j) \in E} \text{rank}_i(j) \cdot x_{ij} \\
\text{s.t.} & \quad (t_j + c_j) \cdot \left( 1 - \sum_{q \in S_{ij}} x_{iq} \right) \leq \sum_{p \in T_{ij}} x_{pj}, \quad \forall (i, j) \in E, \\
& \quad (x, t) \in P_Z,
\end{align*}
\]

where \( S_{ij} = \{ q \in C : \text{rank}_i(q) \leq \text{rank}_i(j) \} \) is the set of indices of colleges that student \( i \) prefers at least as college \( j \); similarly, \( T_{ij} = \{ p \in S : \text{rank}_j(p) < \text{rank}_i(i) \} \) is the set of indices of students that college \( j \) prefers (strictly) more than student \( i \). Constraint (2b) guarantees that the matching is stable. Note that its relaxed version, i.e., when \((x, t)\) is allowed to be fractional, is a non-convex quadratic constraint, which makes the problem challenging. Finally, Constraint (2c) establishes the integrality of the decision variables and the matching feasibility.

In an attempt to obtain a mathematical programming model that is solved efficiently in practice, we linearize the quadratic constraints (2b) through a McCormick envelope [22]. In our formulation, the quadratic term \( t_j \cdot \sum_{q \in S_{ij}} x_{iq} \) in (2b) can be linearized in at least two ways, namely

- **Aggregated Linearization**: For every \((i, j) \in E\), we define
  \[
  w_{ij} := t_j \cdot \sum_{q \in S_{ij}} x_{iq},
  \]

- **Non-Aggregated Linearization**: For every \((i, j) \in E\) and \( q \in S_{ij} \), we define
  \[
  w_{ijq} := t_j \cdot x_{iq}.
  \]

The mixed-integer programming formulation of the McCormick envelope for the aggregated linearization reads as
\[ \min_{x,t,w} \sum_{(i,j) \in E} \text{rank}_i(j) \cdot x_{ij} \]  

\[ \text{s.t.} \quad t_j - w_{ij} + c_j \cdot \left(1 - \sum_{q \in S_{ij}} x_{iq}\right) \leq \sum_{p \in T_{ij}} x_{pj}, \quad \forall (i,j) \in E, \]  

\[ -w_{ij} + t_j + B \cdot \sum_{q \in S_{ij}} x_{iq} \leq B, \quad \forall (i,j) \in E, \]  

\[ w_{ij} \leq t_j, \quad \forall (i,j) \in E, \]  

\[ w_{ij} \leq B \cdot \sum_{q \in S_{ij}} x_{iq}, \quad \forall (i,j) \in E, \]  

\[ (x,t) \in P_Z, \quad w_{ij} \geq 0. \]  

(3a)  

(3b)  

(3c)  

(3d)  

(3e)  

(3f)

Note that the original stability Constraints (2b) become Constraints (3b), i.e., linear. Constraints (3c), (3d), (3e) and the non-negativity of the \( w_{ij} \) form the McCormick envelope (see Appendix A). The rest of the constraints remain the same as well as the objective function.

It is well known that whenever at least one of the variables involved in the linearization is binary, the McCormick envelope is exact. Namely, the set of feasible solutions for the original problem coincides with the set of feasible solutions obtained from the McCormick linearization. For the aggregated linearization, we have that \( \sum_{q \in S_{ij}} x_{iq} \in \{0,1\} \) due to matching constraints in \( P_Z \).

**Corollary 1.** The projection of the feasible region given by Constraints (3b)-(3f) in the variables \( t \) and \( x \) coincides with the region given by Constraints (2b) and (2c).

We now discuss the mixed-integer programming formulation of the McCormick envelope for the non-aggregated linearization. Namely,

\[ \min_{x,t,w} \sum_{(i,j) \in E} \text{rank}_i(j) \cdot x_{ij} \]  

\[ \text{s.t.} \quad t_j - \sum_{q \in S_{ij}} w_{ijq} + c_j \cdot \left(1 - \sum_{q \in S_{ij}} x_{iq}\right) \leq \sum_{p \in T_{ij}} x_{pj}, \quad \forall (i,j) \in E, \]  

\[ -w_{ijq} + t_j + B \cdot x_{iq} \leq B, \quad \forall (i,j) \in E, q \in S_{ij}, \]  

\[ w_{ijq} \leq t_j, \quad \forall (i,j) \in E, q \in S_{ij}, \]  

\[ w_{ijq} \leq B \cdot x_{iq}, \quad \forall (i,j) \in E, q \in S_{ij}, \]  

\[ (x, t) \in P_Z, \quad w_{ijq} \geq 0. \]  

In this case, Constraints (4c), (4d), (4e) and the non-negativity of the \( w_{ijq} \) form the McCormick envelope. Similar to Corollary 1, since \( x_{iq} \in \{0,1\} \), we obtain the following corollary.

**Corollary 2.** The projection of the feasible region given by Constraints (4b)-(4f) in the variables \( t \) and \( x \) coincides with the region given by Constraints (2b) and (2c).

Therefore, both mixed-integer linear programming formulations yield the same set of feasible solutions.

Interestingly, the feasible region of the relaxed aggregated linearization, i.e., when \( (x,t) \in P_Z \) in (3f) is changed to \( (x,t) \in P \), is strictly contained in the feasible region of the relaxed non-aggregated linearization.
Theorem 2. The feasible region of the relaxed aggregated linearization model is contained in the feasible region of the relaxed non-aggregated linearization model.

Proof. The constraints that do not involve the linearization terms $w_{ij}$ or $w_{ijq}$ are trivially satisfied by a feasible solution in both formulations. Therefore, we will restrict our analysis to the remaining constraints.

Let $(x, t, w)$ be a feasible solution of the relaxed aggregated linearized program. It is immediate to verify that by defining $\bar{w}_{ijq} = w_{ij} \cdot x_{iq}$ for every $i \in S$, $j \in C$ and $q \in S_{ij}$, the constraints of the relaxed non-aggregated linearization are all met.

Theorem 2 implies that the optimal value of the relaxed aggregated linearized model is greater or equal to the optimal value of the relaxed non-aggregated linearized model. In Appendix C, we provide a counterexample that shows that the inverse of Theorem 2 is false, i.e., the inclusion in Theorem 2 is strict.

Since mixed-integer programming formulations are evaluated based on the quality of their continuous relaxation, the aggregated linearization dominates the non-aggregated one, thus it is expected to perform better in practice.

4.2 Heuristics

In this section, we present two natural methods: A greedy approach (Grdy) and an LP-based heuristic (LPH). These two heuristics rely on the computation of a student-oriented stable matching which can be done in polynomial time through the DA algorithm:

Algorithm 1 Student-oriented deferred acceptance algorithm.

Input: A CA instance $\Gamma = \langle S, C, \succ, c \rangle$.

Output: A student-oriented matching.

1: Each student starts by applying to her most preferred college. Colleges temporarily accept the most preferred applications and reject the less preferred applications which exceed their capacity.

2: Each student $s$ who has been rejected, proposes to her most preferred college to which she has not applied yet; if she has proposed to all colleges, then she does not apply. If the capacity of the college is not met, then her application is temporarily accepted. Otherwise, if the college prefers her application to one of a student $s'$ who was temporarily enrolled, $s$ is temporarily accepted and $s'$ is rejected. Vice-versa, if the college prefers all the students temporarily enrolled to $s$, then $s$ is rejected.

3: If all students are enrolled or have applied to all the colleges they rank, return the current matching. Otherwise, go to Step 2.

Greedy Approach. In Grdy, we explore the fact that the student cost is decreasing in $t$ and, iteratively, assign an extra spot to the college leading to the largest cost reduction. More precisely, Grdy performs $B$ iterations; in each iteration, the student cost reduction is evaluated for each possible allocation of one extra capacity to a college through the DA algorithm; then, the college leading to the highest cost reduction receives that one extra capacity. In the end of the this procedure, $B$ extra spots are allocated. Given that the students’ cost is reduced as we allocate extra capacities, the Greedy heuristic is a non-decreasing monotone function in the number extra capacities $B$. 
Table 1: Size of the formulations.

| | | \( |S| \) | | \( |C| \) | Integer Quadratic Program | Aggregated Linearization |
|---|---|---|---|---|---|---|
| rows | columns | rows | columns |
| 1000 | 5 | 1006 | 5005 | 21006 | 10005 |
| 8 | 1009 | 8008 | 33009 | 16008 |
| 10 | 1011 | 10010 | 41011 | 20010 |
| 15 | 1016 | 15015 | 61016 | 30015 |
| 2000 | 5 | 2006 | 10005 | 42006 | 20005 |
| 8 | 2009 | 16008 | 66009 | 32008 |
| 10 | 2011 | 20010 | 82011 | 40010 |
| 15 | 2016 | 30015 | 122016 | 60015 |

\( |S| \) \( |C| \) Integer Quadratic Program \( |S| \) \( |C| \) Aggregated Linearization

**LP-based Heuristic.** The CA problem without the stability constraints can be modeled as a minimum-cost flow problem whose polytope has integer vertices. Once we enrich this problem with the expansion of capacities, the integrality of the vertices is preserved. Hence, LPH starts by solving the linear program minimizing the student cost, Objective \((2a)\), restricted to the set \(P\). Then, using the DA algorithm, we compute the student-oriented stable matching obtained from the capacity expansion given by the linear program.

## 5 Computational Experiments

In this section we address the following questions: Which mathematical program solves the problem faster? Which are the computational limits of our exact models? Which heuristic performs better? How does the optimal allocation of extra spots impact the cost of the students?

**Experimental Setup.** For each combination of the following parameters, we created 30 instances: \( |S| = \{1000, 2000\} \), \( |C| = \{5, 8, 10, 15\} \), \( B = \{1, 2, 5, 10, 20, 30\} \). In particular, for each instance, we generated preference lists and capacities uniformly at random with the provision that no college has capacity zero and satisfy \( \sum_{j \in C} c_j = |S| \).

The formulations were coded in Python 3.7.3 and solved through Gurobi 9.1.2, restricted to a single CPU thread and 1 hour of time limit. The scripts were run on an Intel(R) Xeon(R) Gold 6226 CPU on 2.70GHz, running Linux 7.9.

**Experiments.** Next, we provide computational results comparing our four methodologies: two exact methods, the quadratic program (IQP) and the aggregated linearization (Agg-Lin), and two heuristics, Grdy and LPH. Moreover, we analyze the additional welfare of the students that can be achieved by expanding the colleges’ quota.

In Table 1, we summarize the initial size of our mathematical programming formulations, namely, number of constraints (rows) and variables (columns) at the beginning of the computation. It is worth noting that the size of the models does not depend on the number of available extra spots, but only on the number of students and colleges. Of course, the IQP model is more compact than the Agg-Lin model for the same set of parameters \(|S|\) and \(|C|\).

In Tables 2 and 3, we present the results of our experiments. The columns with “gap” report the average percentage gap between the best upper bound found by the two exact methods and the solution found by the heuristic; low values mean better capacity to find a solution close to the optimum. When running the mathematical programs in Gurobi, we feed the best solution found by the heuristics as a warm start. The other columns contain the following information about each mathematical program: “rt-g” is the average percentage root gap between the root value
| $|S|$ | $|C|$ | $B$ | LPH | Grdy | Integer Quadratic Program (IQP) | Aggregated linearization (Agg-Lin) |
|-----|-----|-----|-----|-----|-----------------|-----------------|
|     |     |     |     | gap | gap | rt-g | nodes | #g | %g | rt-g | nodes | #g | %g | b-g | time | rt-g | nodes | #g | %g | b-g | time |
| 1000 | 5  | 1  | 1.4 | 0.0 | 44.7 | 110.6 | 0 | 0.0 | 0.0 | 21.75 | 54.0 | 315.5 | 0 | 0.0 | 0.0 | 26.23 |
|      | 2  |    | 2.9 | 0.3 | 51.5 | 373.4 | 0 | 0.0 | 0.0 | 14.23 | 50.7 | 107.3 | 0 | 0.0 | 0.0 | 22.19 |
|      | 5  |    | 6.0 | 3.6 | 44.1 | 126.5 | 0 | 0.0 | 0.0 | 10.57 | 43.9 | 89.1 | 0 | 0.0 | 0.0 | 21.05 |
|      | 10 |    | 11.8| 11.0| 41.6 | 146.8 | 0 | 0.0 | 0.0 | 11.02 | 41.6 | 172.6 | 0 | 0.0 | 0.0 | 22.93 |
|      | 20 |    | 18.0| 28.9| 28.3 | 21.6 | 0 | 0.0 | 0.0 | 7.95  | 28.3 | 102.2 | 0 | 0.0 | 0.0 | 17.91 |
|      | 30 |    | 25.2| 59.1| 18.0 | 15.1 | 0 | 0.0 | 0.0 | 5.65  | 18.0 | 158.7 | 0 | 0.0 | 0.0 | 12.93 |
| 1000 | 8  | 1  | 1.6 | 0.0 | 66.2 | 25068.6| 4 | 2.1 | 0.0 | 1488.81 | 64.7 | 1929.9 | 0 | 0.0 | 0.0 | 181.38 |
|      | 2  |    | 3.2 | 0.8 | 61.1 | 819.8 | 0 | 0.0 | 0.0 | 60.67 | 60.6 | 1324.9 | 0 | 0.0 | 0.0 | 146.83 |
|      | 5  |    | 6.0 | 3.1 | 58.6 | 653.4 | 0 | 0.0 | 0.0 | 56.41 | 58.4 | 1185.1 | 0 | 0.0 | 0.0 | 111.91 |
|      | 10 |    | 10.6| 8.3 | 55.7 | 620.9 | 0 | 0.0 | 0.0 | 60.73 | 55.6 | 1204.2 | 0 | 0.0 | 0.0 | 108.18 |
|      | 20 |    | 18.9| 18.3| 44.6 | 347.0 | 0 | 0.0 | 0.0 | 64.27 | 44.6 | 702.1 | 0 | 0.0 | 0.0 | 63.78 |
|      | 30 |    | 31.7| 51.0| 27.7 | 77.6 | 0 | 0.0 | 0.0 | 22.52 | 22.7 | 432.7 | 0 | 0.0 | 0.0 | 33.15 |
| 1000 | 10 | 1  | 2.1 | 0.0 | 66.7 | 27688.8| 13 | 9.5 | 0.0 | 2427.38 | 65.6 | 8646.9 | 6 | 5.2 | 0.0 | 1324.37 |
|      | 2  |    | 2.8 | 0.9 | 65.8 | 1015.3 | 0 | 0.0 | 0.0 | 207.02 | 64.9 | 1825.4 | 0 | 0.0 | 0.0 | 331.98 |
|      | 5  |    | 5.8 | 3.7 | 62.7 | 956.3 | 0 | 0.0 | 0.0 | 170.84 | 62.4 | 1635.7 | 0 | 0.0 | 0.0 | 227.79 |
|      | 10 |    | 10.0| 8.2 | 58.3 | 819.7 | 0 | 0.0 | 0.0 | 308.82 | 58.1 | 1395.7 | 0 | 0.0 | 0.0 | 187.56 |
|      | 20 |    | 20.4| 19.9| 44.9 | 464.7 | 0 | 0.0 | 0.0 | 148.93 | 44.9 | 772.6 | 0 | 0.0 | 0.0 | 112.19 |
|      | 30 |    | 34.7| 51.4| 32.1 | 42.2 | 1 | 0.1 | 0.0 | 44.59 | 32.1 | 436.3 | 0 | 0.0 | 0.0 | 64.01 |
| 1000 | 15 | 1  | 2.2 | 0.0 | 71.8 | 11773.6| 28 | 29.5 | 0.0 | 3519.41 | 70.9 | 6765.4 | 27 | 30.5 | 0.0 | 3426.71 |
|      | 2  |    | 4.2 | 0.8 | 70.2 | 1699.7 | 0 | 0.0 | 0.0 | 1427.29 | 69.2 | 2352.5 | 1 | 0.5 | 0.0 | 964.12 |
|      | 5  |    | 7.8 | 3.6 | 67.4 | 2070.9 | 1 | 0.1 | 0.0 | 1499.12 | 66.7 | 3007.2 | 0 | 0.0 | 0.0 | 822.86 |
|      | 10 |    | 12.7| 8.9 | 65.6 | 1802.3 | 1 | 0.6 | 0.0 | 1442.69 | 65.3 | 3305.5 | 0 | 0.0 | 0.0 | 710.30 |
|      | 20 |    | 23.3| 20.1| 51.4 | 864.2 | 0 | 0.0 | 0.0 | 652.50 | 51.4 | 1480.8 | 0 | 0.0 | 0.0 | 349.08 |
|      | 30 |    | 31.4| 32.7| 44.3 | 562.8 | 0 | 0.0 | 0.0 | 450.67 | 44.3 | 1134.5 | 0 | 0.0 | 0.0 | 248.97 |

Table 2: Average results for each triplet $|S| = 1,000$, $|C|$ and $B$. 
| $|S|$ | $|C|$ | $B$ | LPH | Grdy | Integer Quadratic Program (IQP) | Aggregated linearization (Agg-Lin) |
|---|---|---|---|---|---|---|---|
| 2000 | 5 | 1 | 0.6 | 0.0 | 59.4 | 9444.3 | 0 | 0.0 | 0.0 | 538.29 | 57.5 | 1473.8 | 0 | 0.0 | 0.0 | 108.17 |
| 2 | 2.1 | 0.3 | 54.4 | 1691.5 | 0 | 0.0 | 0.0 | 538.29 | 57.5 | 1473.8 | 0 | 0.0 | 0.0 | 117.00 |
| 5 | 4.3 | 2.7 | 56.2 | 1762.2 | 0 | 0.0 | 0.0 | 538.29 | 57.5 | 1473.8 | 0 | 0.0 | 0.0 | 139.85 |
| 10 | 8.0 | 5.6 | 49.2 | 1370.4 | 0 | 0.0 | 0.0 | 538.29 | 57.5 | 1473.8 | 0 | 0.0 | 0.0 | 137.41 |
| 20 | 10.0 | 16.4 | 41.6 | 889.6 | 0 | 0.0 | 0.0 | 538.29 | 57.5 | 1473.8 | 0 | 0.0 | 0.0 | 137.41 |
| 30 | 11.4 | 33.4 | 25.3 | 187.7 | 0 | 0.0 | 0.0 | 538.29 | 57.5 | 1473.8 | 0 | 0.0 | 0.0 | 92.03 |
| 2000 | 8 | 1 | 1.3 | 0.0 | 67.4 | 11057.6 | 17 | 17.4 | 0.0 | 2793.14 | 66.7 | 3856.3 | 7 | 5.3 | 0.0 | 1665.35 |
| 2 | 2.5 | 0.6 | 67.0 | 17375.8 | 2 | 0.5 | 0.0 | 2793.14 | 66.7 | 3856.3 | 7 | 5.3 | 0.0 | 861.73 |
| 5 | 5.0 | 2.8 | 62.5 | 12946.5 | 4 | 1.8 | 0.0 | 2793.14 | 66.7 | 3856.3 | 7 | 5.3 | 0.0 | 1059.36 |
| 10 | 8.5 | 4.9 | 60.7 | 13625.2 | 1 | 0.3 | 0.0 | 2793.14 | 66.7 | 3856.3 | 7 | 5.3 | 0.0 | 1099.34 |
| 20 | 18.0 | 15.9 | 48.9 | 9011.9 | 2 | 0.8 | 0.0 | 2793.14 | 66.7 | 3856.3 | 7 | 5.3 | 0.0 | 666.67 |
| 30 | 17.5 | 23.3 | 44.3 | 6149.3 | 1 | 0.4 | 0.0 | 2793.14 | 66.7 | 3856.3 | 7 | 5.3 | 0.0 | 410.77 |
| 2000 | 10 | 1 | 1.0 | 0.0 | 71.0 | 1355.9 | 29 | 38.8 | 0.0 | 3528.57 | 70.7 | 3569.4 | 22 | 25.7 | 0.0 | 3140.23 |
| 2 | 2.4 | 0.6 | 68.9 | 24435.4 | 21 | 16.0 | 0.0 | 3528.57 | 70.7 | 3569.4 | 22 | 25.7 | 0.0 | 1853.09 |
| 5 | 4.2 | 2.3 | 67.4 | 18647.5 | 19 | 16.3 | 0.0 | 3528.57 | 70.7 | 3569.4 | 22 | 25.7 | 0.0 | 2270.44 |
| 10 | 7.1 | 5.8 | 60.7 | 17816.0 | 9 | 8.0 | 0.1 | 3528.57 | 70.7 | 3569.4 | 22 | 25.7 | 0.0 | 1904.39 |
| 20 | 13.9 | 10.9 | 56.9 | 9829.0 | 13 | 10.6 | 0.0 | 3528.57 | 70.7 | 3569.4 | 22 | 25.7 | 0.0 | 1640.89 |
| 30 | 19.1 | 21.8 | 46.2 | 5591.9 | 7 | 6.0 | 0.2 | 3528.57 | 70.7 | 3569.4 | 22 | 25.7 | 0.0 | 830.71 |
| 2000 | 15 | 1 | 1.3 | 0.0 | 73.3 | 5793.4 | 30 | 54.4 | 0.0 | 3600.18 | 73.2 | 1424.3 | 30 | 48.5 | 0.0 | 3601.82 |
| 2 | 2.5 | 0.6 | 72.3 | 2563.4 | 30 | 47.9 | 0.2 | 3600.18 | 73.2 | 1424.3 | 30 | 48.5 | 0.0 | 3514.34 |
| 5 | 5.1 | 2.8 | 69.1 | 1680.6 | 30 | 29.9 | 0.1 | 3600.18 | 73.2 | 1424.3 | 30 | 48.5 | 0.0 | 3347.38 |
| 10 | 8.4 | 6.2 | 63.9 | 649.9 | 26 | 22.1 | 0.4 | 3600.18 | 73.2 | 1424.3 | 30 | 48.5 | 0.0 | 2833.55 |
| 20 | 14.7 | 14.2 | 56.1 | 535.1 | 21 | 15.0 | 1.8 | 3600.18 | 73.2 | 1424.3 | 30 | 48.5 | 0.0 | 2140.14 |
| 30 | 21.1 | 25.0 | 48.6 | 506.1 | 14 | 7.0 | 1.4 | 3600.18 | 73.2 | 1424.3 | 30 | 48.5 | 0.0 | 1815.62 |

Table 3: Average results for each triplet $|S| = 2,000$, $|C|$ and $B$. 


and the best upper bound found by all methods; “nodes” is the average number of explored nodes; “#g” is the number of instances that reached the time limit of 1 hour; “%g” is the average last percentage gap found by Gurobi; “b-g” is the average gap between the upper bound found by the program and the best upper bound found by the two programs; “time” records the average time in seconds.

![Performance profile](image)

Figure 1: Performance profiles of the integer quadratic program and the aggregated linearization. Experiments on 1440 instances.

![Graphs](image)

Figure 2: The average gap (in %) between the two heuristics and the best upper bound found by the exact methods; the gap is represented as a function of $B$. The number of students is 2000 and each data-point is the average of 30 instances.

From Table 2 and 3, we can deduce the following observations: (i) Starting in $|S| \geq 1000$, $|C| = 15$ and $B \geq 1$, Agg-Lin outperforms IQP; moreover, given its non-significant under-performance in other smaller instances, we can conclude that Agg-Lin should be preferred. Indeed, the performance profile plot (Figure 1) stresses this conclusion too. (ii) The bulk of the computational difficulty lays in the number of colleges; this can be noticed by looking at the increase in the computing times and gaps for the exact approaches. (iii) Increasing $B$ diminishes the computing time.
of the exact models. In particular, we observe a drastic reduction in the computing time as we move from $B = 1$ to $B = 2$. One possible explanation of that phenomenon is that the additional extra position makes infeasible the branching in the second summation of the stability constraint.

(iv) When $B$ is relatively small, Grdy performs better than LPH, while, as $B$ increases, the latter performs better than the former. This is easier to see in Figure 2, where we provide a graphical representation of the average gap between the heuristics and the best upper bound. The four sub-figures are made for the instances with 2000 students, since for $|S| = 1000$ the trend is similar. Also, note that LPH has a more regular performance than Grdy. Note that for $B = 0$ the problem can be linearly solved by the DA algorithm.

In Figure 3, we address the following question: What is the impact on the students’ cost in expanding the capacities of the colleges? Given 1000 students and 15 colleges, we present the average ranking of the college in which each student is enrolled according to her own preferences. Observe that when $B = 30$, more than 150 students (more than 15% of the total) are now admitted in their most preferred college.

6 Conclusions

In this work, we have addressed the following question: How should a centralized institution allocate extra positions to colleges with the goal of reducing the cost of the students? We studied this question from a computational complexity standpoint and we provided exact and heuristic methodologies to solve it. We assessed the relative performance of our methods and provided computational evidence that our model can bring substantial improvements in a centralized system that looks to expand the capacity of the colleges.

The problem that we consider in this work opens the door for multiple research directions. First, the next step is to consider Problem 1 when preference lists are incomplete and with ties. Another important question is to design methodologies that are able to solve large-scale instances, e.g., in admission processes with 116 thousands students and 1100 colleges [10]. Finally, it would

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2The 30 instances taken in consideration are the same of line $S = 1000, C = 15, B = 1$ from Table 2.
be interesting to adapt our problem to a more realistic model that considers partial knowledge of the preferences lists.

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Appendix

A Background

In this section we describe the McCormick convex envelope [22] used to obtain a linear relaxation for bi-linear terms; if one of the terms is binary, the linearization provides an equivalent formulation. Consider a bi-linear term of the form $x_i \cdot x_j$ with the following bounds for the variables $x_i$ and $x_j$: $l_i \leq x_i \leq u_i$ and $l_j \leq x_j \leq u_j$. Let us define $y = x_i \cdot x_j$, $m_i = (x_i - l_i)$, $m_j = (x_j - l_j)$, $n_i = (u_i - x_i)$ and $n_j = (u_j - x_j)$. Note that $m_i \cdot m_j \geq 0$, from which we derive the under-estimator $y \geq x_i \cdot l_j + x_j \cdot l_i - l_i \cdot l_j$. Similarly, it holds that $n_i \cdot n_j \geq 0$, from which we derive the under-estimator $y \geq x_i \cdot u_j + x_j \cdot u_i - u_i \cdot u_j$. Analogously, over-estimators of $y$ can be defined. Make $o_i = (u_i - x_i)$, $o_j = (x_j - l_j)$, $p_i = (x_i - l_i)$ and $p_j = (u_j - x_j)$. From $o_i \cdot o_j \geq 0$ we obtain the over-estimator $y \leq x_i \cdot l_j + x_j \cdot u_i - u_i \cdot l_j$, and from $p_i \cdot p_j \geq 0$ we obtain the over-estimator $y \leq x_j \cdot l_i + x_i \cdot u_j - u_j \cdot l_i$. The four inequalities provided by the over and under estimators of $y$, define the McCormick convex (relaxation) envelope of $x_i \cdot x_j$.

B Proof of NP-Completeness

In this section, we focus on the proof of Theorem 1. Our reduction considers a capacity vector $c \in \{0, 1\}^\epsilon$ which corresponds to the well-known Stable Marriage (SM) problem, where colleges and students are traditionally called men and women.

We denote by $W$ the set of women and $M$ the set of men. Given a woman $i$ and a man $j$, in this appendix we denote by rank$_i(j)$, the rank of man $j$ in the list of woman $i$, as the position of man $j$ in the list of woman $i$. For example, if the preference list of man $m$ is $w \succ w' \succ w''$, the rankings of woman $w, w', w''$, is 1, 2 and 3 respectively. To ease the exposition, we avoid using the symbol $>$ when presenting a preference list, instead we simply separate agents by “,” and use the convention that the leftmost agents are the most preferred (e.g., we will represent the previous preference list as $w, w', w''$). Let us define the following decision problem that we call MIN-W SMT.

INSTANCE: An instance $\Gamma = \langle W, M, \succ \rangle$ with $c_v = 1$, for all $v \in W \cup M$ and a positive integer $K$.

QUESTION: Is there a stable matching $M$ such that the cost of the women

$$cost_w(M) := \sum_{(w,m) \in M} \text{rank}_w(m) \leq K?$$

By SMT (SMTI) we denote the stable marriage problem in which preference lists may include ties (and the preference lists may be incomplete, i.e., not all the agents on the other side of the bipartition may be ranked).

Corollary 3. Consider an instance of MIN-W SMT in which ties are only in women’s rankings, there is at most one tie per list, and each tie is of length two. MIN-W SMT is not approximable within $n^{1-\epsilon}$, for any $\epsilon > 0$, unless $P = NP$, where $n = |M|$ is the number of men in $\Gamma$.

Proof. Our proof is inspired by the Proof of Theorem 7 in [21]. Let $\epsilon > 0$ be given, and let $a = 3/\epsilon$. From Theorem 2 in [21], we know that the decision version of the MAX CARDINALITY SMTI problem when ties occur on the women’s side only, and each tie has length two is NP-complete; let us consider an instance of this problem: Let $M = \{m_1, m_2, \ldots, m_n\}$ be the set
of men and let \( \mathcal{W} = \{ w_1, w_2, \ldots, w_n \} \) be the set of women in a given instance \( \Gamma \) of the problem. We assume that the target value in \( \Gamma \) is equal to \( n \). Let \( O_i \) (resp. \( R_i \)) denote the preference list of man \( m_i \) (resp. woman \( w_i \)) for \( i = 1, \ldots, n \). We build an instance \( \Gamma' \) of MIN-W SMT as follows: Let \( \mathcal{M}^0 \cup \bigcup_{j=1}^{C} \mathcal{M}^j \) be the set of men, and let \( \mathcal{W}^0 \cup \bigcup_{j=1}^{C} \mathcal{W}^j \) be the set of women, where \( C := n^{[a]-1} \), \( \mathcal{M}^0 = \{ m^0_1, m^0_2, \ldots, m^0_n \} \), \( \mathcal{M}^j = \{ m^j_1, m^j_2, \ldots, m^j_n \} \) for \( j = 1, \ldots, C \), \( \mathcal{W}^0 = \{ w^0_1, w^0_2, \ldots, w^0_n \} \), and \( \mathcal{W}^j = \{ w^j_1, w^j_2, \ldots, w^j_n \} \) for \( j = 1, \ldots, C \). Thus \( \Gamma' \) comprises \( 2n^a \) men and \( 2n^a \) women, so that \( n := 2n^a \). For each \( i = 1, \ldots, n \) and \( j = 1, \ldots, C \), let \( O^j_i \) be the preference list obtained from \( O_i \) by replacing woman \( w_k \) in \( O_i \) by the corresponding woman \( w^j_k \), for every \( k = 1, \ldots, n \). We refer to the women in \( O^j_i \) as the proper women for \( m^j_i \). Similarly, we define \( R^j_i \) and the proper men for \( w^j_i \). Let us create the preference lists of the agents in \( \Gamma' \) as follows:

\[
\begin{align*}
& m^0_i : w^0_i \ldots & i = 1 \ldots n^a \\
& m^j_i : O^j_i, \mathcal{W}^0 \ldots & i = 1, \ldots, n, j = 1, \ldots, C \\
& w^0_i : m^0_i \ldots & i = 1 \ldots n^a \\
& w^j_i : R^j_i, \mathcal{M}^0 \ldots & i = 1, \ldots, n, j = 1, \ldots, C
\end{align*}
\]

where the dots “…” in the preference lists mean that the remaining agents on the other side of the bipartition are ranked strictly and arbitrarily. Note that the only ties in \( \Gamma' \) occur in the preference lists of women \( w^j_i \) for \( i = 1, \ldots, n \), \( j = 1, \ldots, C \); moreover, there is at most one tie per list, and each tie has length 2. Consider a maximum cardinality stable matching \( M \) in \( \Gamma \), where \( |M| = n \). We create a matching \( M' \) in \( \Gamma' \) as follows: For every \( h = 1, \ldots, n^a \), we add the pair \( (m^0_h, w^0_h) \) to \( M' \), and for each \( i = 1, \ldots, n \), we add the pair \( (m^i_j, w^j_i) \) to \( M' \) for \( j = 1, \ldots, C \), where \( (m_i, w_k) \in M \). Note that \( M' \) is stable in \( \Gamma' \), and it may be verified that

\[
\text{cost}_w (M') \leq n^a + n^a - 1 < n^a + \frac{n^{a+2}}{2}
\]

since, without loss of generality, we may choose \( n \geq 3 \). Therefore, also the women’s cost in the minimum women’s cost stable matching is less or equal than \( \frac{n^{a+2}}{2} \).

On the other side, let us assume that \( \Gamma \) does not have a stable matching of cardinality \( n \). Let \( M' \) be a stable matching in \( \Gamma' \). It holds that, for every \( j = 1, \ldots, C \), there is some \( i = 1, \ldots, n \) for which \( w^j_i \) cannot be matched in \( M' \) to one of her proper men. Nonetheless, in \( M', m^j_i \) and \( w^j_i \) must be partners, for every \( j = 1, \ldots, n^a \), therefore rank \( w^j_i (M(w^j_i)) > n^a \). Hence \( \text{cost}_w (M') > n^{2a-1} \), and the same result holds for the minimum cost stable matching for the women.

Therefore, the existence of a polynomial-time approximation algorithm for Min-W SM whose approximation ratio is as good as \( (2n^{2a-1}) / n^{a+2} = 2n^{a-3} \) would give a polynomial-time algorithm for determining whether \( \Gamma \) has a stable matching in which everybody is matched. To conclude, \( 2n^{a-3} = (2/21^{-3/a}) n^{1-3/a} > n^{1-3/a} = n^{1-\varepsilon} \), which ends the proof.

\[ \square \]

**Remark 1.** After the proof of Lemma 1 in [21] the authors prove that the problem of MAX CARDINALITY SMTI can be reduced to the case in which ties are at the end of the preference list. Since the ties of the new instance created in Lemma 1 from [21] are at most two, we can use their same reasoning to assume instead, without losing generality, that in the instance \( \Gamma \) and \( \Gamma' \) of Corollary 3 ties occur only at the head of a preference list.
We now define the decision version of Problem 1 for the specific case of $c_v = 1$ for all $v \in \mathcal{M} \cup \mathcal{W}$ that we denote $\text{SM}_{\text{ext}}$

**INSTANCE:** An instance $\Gamma = (\mathcal{W}, \mathcal{M}, \succ c)$ with $c_v = 1$ for all $v \in \mathcal{W} \cup \mathcal{M}$, budget $B \in \mathbb{Z}_+$ and a positive integer $K$.

**QUESTION:** Does there exist an allocation of $B$ extra spots to the men so that there is a women-oriented stable matching $M$ with a cost for the women less or equal than $K$, i.e., $\text{cost}_w(M) := \sum_{(w, m) \in M} \text{rank}_w(m) \leq K$?

In a weakly stable matching a pair $(w, m)$ is not a blocking pair if the woman $w$ (man $m$) is matched to a man $m'$ (woman $w'$) the she ranks at least as good as $m$.

**Lemma 1.** Let $O$ be a weakly stable matching of an instance $\Gamma$ of SMT, in which ties occur only among the preference lists of women. In each preference list there is at most one tie, which has length 2, and it is positioned at the head of the list. Let $K_n$ be its women’s cost where $n$ is the size of $\Gamma$. There is a polynomial reduction $f$ that provides a stable matching $O'$ in an instance $\Gamma'$ of $\text{SM}_{\text{ext}}$ where the parameters are $\tilde{K} = 2 \cdot K_n + n \cdot (n + 2L) \ (\text{cost})$ and $B = L \cdot n \ (\text{budget})$ for $L$ equal to the number of women with a tie at the head of their preference list in $\Gamma$.

**Proof.** In $\Gamma'$, the set of women is $\mathcal{W}$ and the set of men is $\mathcal{M}$. The preference lists of the men and the women can be assumed to be complete. The set of women is partitioned in two sets $\mathcal{W} = \mathcal{W}' \cup \mathcal{W}''$ where $\mathcal{W}'$ is the set of women with a tie of length two at the head of the preference list and $\mathcal{W}''$ is the set of women with a strict and complete preference list. We assume that $|\mathcal{M}| = |\mathcal{W}| = n$ and the cardinality of $\mathcal{W}'$ is $L$. Observe that every stable matching is complete (every woman is matched).

We create a new instance $\Gamma'$ of $\text{SM}_{\text{ext}}$, and we populate it with a copy of $\mathcal{M}$ and a copy of $\mathcal{W}'$. We also introduce a set of women $\mathcal{W}_0$ of cardinality $n$, a set of men $\mathcal{M}_0$ of cardinality $n$, and a set $\mathcal{X}$ of men of cardinality $L \cdot n$. For every woman $w_j \in \mathcal{W}'$, where $j = 1, \ldots, L$, we associate and introduce additional men and women in $\Gamma'$, these form a structure called batch:

- a set of women $\mathcal{W}_j = \{w_{j,i}\}_{i=1,\ldots,n}$
- a woman $y_j$
- for $h = 0, \ldots, n$ we introduce a set of men $\mathcal{V}_{j,h} = \{v_{s}^{j,h}\}_{s=1,\ldots,n}$, where each man has capacity 0. Let $\mathcal{V}_j = \bigcup_{h=0}^{n} \mathcal{V}_{j,h}$.

We denote as $\mathcal{B}_j$ the batch in $\Gamma'$ associated to woman $w_j$ in $\mathcal{W}'$, and we denote $\mathcal{V} := \bigcup_{j=1}^{n} \mathcal{V}_j$.

The preference lists of the men and women in $\mathcal{M}_0, \mathcal{W}_0$ are as follows:

\[
m_j^0 : w_j^0, \mathcal{W}_0 \setminus \{w_j^0\}, \ldots \quad \forall j = 1, \ldots, n
\]

\[
w_j^0 : m_j^0, \mathcal{M}_0 \setminus \{m_j^0\}, \mathcal{V}, \ldots, \mathcal{X} \quad \forall j = 1, \ldots, n
\]

where $\mathcal{W}_0$ in the preference list means that the set $\mathcal{W}_0$ is ordered strictly according to the indices of its elements, and the symbol “...” means that the remaining people on the other side of the bipartition are ranked strictly and arbitrarily.
Given a woman \( w_j \in \mathcal{W}' \), for \( j = 1, \ldots, L \), let \((m_{j_1}, m_{j_2}), m_{j_3} \ldots\) be her ranking of the men in \( \mathcal{M} \) (the parenthesis symbolize the tie at the head of the list). We provide some of the preference lists of the remaining men and women in batch \( B_j \) of the instance \( \Gamma' \):

\[
\begin{align*}
    w_{j,1} : & \, V_{j,1}, V_{j,0} \setminus \{v_{1}^{j,0}\}, m_{j_1}, \mathcal{V} \setminus \mathcal{V}_j, \ldots, \mathcal{X} \\
    w_{j,2} : & \, V_{j,2}, V_{j,0} \setminus \{v_{2}^{j,0}\}, m_{j_2}, \mathcal{V} \setminus \mathcal{V}_j, \ldots, \mathcal{X} \\
    w_{j,i} : & \, V_{j,i}, V_{j,0} \setminus \{v_{i}^{j,0}\}, m_{j_i}, \mathcal{V} \setminus \mathcal{V}_j, \ldots, \mathcal{X} \quad \forall i = 3 \ldots n \\
    y_j : & \, v_2^{j,0}, v_3^{j,0}, \ldots, v_n^{j,0}, V_{j,1}, M_0, \mathcal{V} \setminus \mathcal{V}_j, \ldots, \mathcal{X} \\
    v_{s,i} : & \, w_{j,i}, \mathcal{W}, \ldots \quad \forall s = 1, \ldots, n, \quad \forall i = 1, \ldots, n \\
    v_{s,0} : & \, W_j \setminus \{w_{j,s}\}, y_j, \mathcal{W}, \ldots \quad \forall s = 1, \ldots, n \\
\end{align*}
\]

where \( M_0 = \{m_1, \ldots, m_n\} \).

Finally, we have to present the preference list of the copy of every man \( m \in \mathcal{M} \) and every woman \( w \in \mathcal{W}' \) in the new instance \( \Gamma' \). Let \( m \in \mathcal{M} \). We modify the original preference list of \( m \) by substituting every woman \( w_j \in \mathcal{W}' \) (for \( j = 1, \ldots, L \)) with woman \( w_{j,r} \), where \( r = \text{rank}_{w_j}(m) \) is the rank of \( m \) in the list of \( w_j \). Once these substitutions are applied, \( m \) ranks \( \mathcal{W}_0 \), and then an arbitrary strict ordering of the remaining women.

Let \( w \in \mathcal{W}' \) and let \( m_{h_1}, \ldots, m_{h_n} \) be her strict and complete preference list in \( \Gamma \). The preference list of \( w \) in \( \Gamma' \) is \( m_1, m_{h_r}, m_2, m_{h_2}, \ldots, m_{n_r}, m_{h_n} \).

Recall that we set the capacity of the men in \( \mathcal{V} \) to 0. In \( \Gamma' \) we provide \( L \cdot n \) extra positions, i.e., when it is assigned to a man, extends his capacity by 1. The purpose of the set of men \( \mathcal{X} \) is to ensure that women are matched when there is no allocation of extra positions.\(^3\) Observe that the assignment to men in \( \mathcal{X} \) leads to a higher cost. Therefore, it is optimal to assign the \( L \cdot n \) extra capacities to the men in \( \mathcal{V} \).

Let \( O \) be a complete weakly stable matching in \( \Gamma \). From \( O \), we build a matching \( O' \) in \( \Gamma' \). Let \((w, m)\) be a pair in \( O \). If \( w \in \mathcal{W}' \), then we match the pair \((w, m)\) in \( \Gamma' \).

Otherwise, \( w = w_j \in \mathcal{W}' \) and \( r = \text{rank}_{w_j}(m) \). In batch \( B_j \) of \( O' \), we match the pairs \((w_{j,r}, m), (y_j, v^{j,0}_1)\), and for \( i \in \{1, \ldots, n\} \setminus \{r\} \) we match \((w_{j,i}, v^{j,i}_1)\). In order to do these matchings, we assign \( n \) extra positions to \( V_j \): One to \( v^{j,0}_1 \) and \( n - 1 \) to the set of men \( \{v^{j,i}_1 : i \in \{1, \ldots, n\} \setminus \{r\} \} \). All the extra positions \( L \cdot n \) are used to form \( O' \).

In \( O \) we can distinguish whether a woman is matched to a man she ranks first or not and we can distinguish if a woman is in \( \mathcal{W}' \) or \( \mathcal{W}' \). Let \( K'' \) be the cost of the women in \( \mathcal{W}' \), \( K' \) be the cost of the women in \( \mathcal{W}' \) that are matched to a man in their initial ties, and let \( K' \) be the cost of the women in \( \mathcal{W}' \) matched to a man they rank second or more. Clearly \( K' \) is also the number of women matched to a man they rank first, and the total cost of \( O \) is \( K'' = K'' + K' + K'' \). The cost of \( O' \) is \( 2K'' + (2 + 2n)K' + [2 \cdot K' + 2n(L - K'') + n(n - 1)] + n \). To ease the explanation of the cost, we divide the contributions to the cost of \( O' \) in terms: The first term is given by the contribution from the women in \( \mathcal{W}' \); the second term is given by the batches \( B_j \) of the women \( w_j \) that in \( O \) are matched to a man they rank second or more; the last contribution is by the women in \( \mathcal{W}' \). Note that we can rearrange the cost as \( \hat{K} := 2K'' + 2K' + 2nL + n^2 = 2K'' + n \cdot (n + 2L) \).

\(^3\)This also could have been done by adding a copy of themselves at the end of their list, which is usually referred as individual rationality. Matching with yourself means being unassigned to a college.
Now, we verify that \( O' \) is stable in \( \Gamma' \). Let \((w, m) \in O\). If \( w \in \mathcal{W}' \), then the stability of the pair \((w, m)\) is inherited from the stability of the pair in \( O \).

If \( w \in \mathcal{W}' \), then \( w = w_j \) for a certain \( j = 1, \ldots, l \); let us assume that \( m \) is ranked first by \( w_j \). In the preference list of \( w_j \), \( m \) is of the form \( m_{j,h} \) where \( i = 1, 2 \). The pairs \((w_{j,i}, m), (y_j, v_i^h)\), and \((w_{j,h}, v_1^h)\) for \( h = 1, \ldots, n \) and \( h \neq i \) are stable. First, since for \( h = 1, \ldots, n \) and \( h \neq i \), \( w_{j,h} \) ranks \( v_1^h \) first and vice-versa, the \( n - 1 \) pairs \((w_{j,h}, v_1^h)\) are stable. Second, if \( i = 2 \), then \( y_j \) ranks \( v_2^0 \) first, and \( v_2^0 \) cannot be matched to any of the women in \( \mathcal{W}_j \setminus \{w_{j,2}\} \) because of the previous point. If, instead, \( i = 1 \), then \( y_j \) ranks \( v_1^0 \) second, and \( y_j \) cannot be matched to \( v_2^0 \) because it has capacity 0; as before, \( v_1^0 \) cannot create a blocking pair with any of the women in \( \mathcal{W}_j \setminus \{w_{j,1}\} \) because they are matched to their most preferred man. Finally, also the pair \((w_{j,1}, m)\) is stable; in fact, \( w_{j,i} \) cannot be matched to any of the men in \( \mathcal{V}_{j,i}, \mathcal{V}_{j,0} \setminus \{v_i^0\} \) because they have all capacity 0, and, if \( i = 2 \), \( w_{j,2} \) cannot be matched to \( m_0 \) because he is matched to \( w_0 \) and they mutually rank each other first; also \( m \) cannot create a blocking pair, in fact, all the women \( w_{j',r} \) ranked in his preference list before \( w_{j,i} \) are matched to men of the form \( v_1' \) that they rank first. The case in which \( m \) is ranked second or more by \( w_j \) is analogous to the case in which \( m \) is of the form \( m_{j,h} \) in \( w_j \)'s preference list. Therefore, \( O' \) is a stable matching in \( \Gamma' \).

Note that we introduced \( n + (n - L) + L \cdot (n + 1 + n^2) \) men and women in \( \Gamma' \), which is \( O(n^3) \), therefore the reduction is polynomial. \( \square \)

**Lemma 2.** Let \( O \) be a weakly stable matching of an instance \( I \) of SMT (as in Lemma 1) which minimizes the cost of the women. Given the polynomial reduction \( f \) of Lemma 1, we obtain a stable matching \( O' \) that has minimum cost for the women in the instance \( \Gamma' \) of SMext.

**Proof.** First, we note that in \( \Gamma' \) it is not optimal to move one extra position from a given \( v_{s'}^j \) to an \( m \in \mathcal{M} \cup \mathcal{M}^0 \). In fact, if this is the case, we would match \( w_{j,i} \) or \( y_j \) to another man with an additional cost of \( 2n \), which is greater than any other gain obtained by extending one of the capacities of a man in \( \mathcal{V} \). Therefore, all the extra positions are assigned to men in \( \mathcal{V} \).

We observe that, in any stable matching in \( \Gamma' \) that assigns the extra positions to reduce the cost of the women, \( n - L \) women are matched to a certain \( v_{s'}^j \), and the remaining women are matched to men in \( \mathcal{M} \cup \mathcal{M}^0 \). For \( j = 1, \ldots, L \) and \( i = 1, \ldots, n \), let \( w_{j,i} \in \mathcal{W}_j \); if \( w_{j,i} \) is matched to \( v_1^i \), then its cost is 1; otherwise, if \( w_{j,i} \) is matched to \( m_{j,i} \in \mathcal{M} \), then its cost would be more than 2 and \( y_j \) is matched to \( v_1^i \), the cumulative cost of \( w_{j,i} \) and \( y_j \) is \( 2 \cdot \text{rank}_{w_j}(m_{j,i}) + 2n \). Therefore, the cost of \( O' \) would be \( 2 \cdot \text{cost}(O) + n \cdot (n + 2L) \).

In what follows, we prove that it is not possible to assign the extra positions differently among the men in \( \mathcal{V} \), and, therefore, achieve a lower cost than \( 2 \cdot \text{cost}(O) + n \cdot (n + 2L) \).

Let us analyze why we cannot move extra capacities between men in \( \mathcal{V} \) in order to reduce the women's cost of the stable matching. Since the matching \( O \) is a minimum cost stable matching for the women and each woman ranks in the first \( 3n \) positions only one man in \( \mathcal{M} \), a stable re-allocation of extra capacities within a batch (there are \( L \) batches) would provide a worst matching from the perspective of women’s cost within the batch. Therefore, the re-allocation of extra capacities within \( O' \) could only happen from a batch to another. There are four possible ways of re-assigning extra capacities from one batch to another. We assume that \( w \in \mathcal{W}_j \), \( w' \in \mathcal{W}_{j'} \); we now move one extra capacity from batch \( \mathcal{B}_{j'} \) to batch \( \mathcal{B}_j \) in the following way:

- from \( v_{h}^j \) to \( v_{1}^j \). Note that, by assumption, both women \( y_{j'} \) and \( y_j \) are matched, in \( O' \), to \( v_{h}^{j'} \) and \( v_{s}^j \) for a certain \( s \leq n \) respectively. If we move one extra position from \( v_{h}^j \) to \( v_{1}^j \), then...
weakly stable matching with women’s cost at least $K$ in $\Gamma$.

Theorem 3. Let $\text{SM}_{\text{ext}}$ be an instance of SMT and let $\Gamma$ be the instance obtained through the reduction $f$ of Lemma 1. Let $K$ be a constant, we claim that $\Gamma$ has a stable matching with women’s cost at least $2K + n \cdot (n + 2L)$.

Proof. From Corollary 3 we know that the problem of finding a weakly stable matching with minimum cost for the women in SMT is NP-hard. Let $f$ be an instance of SMT and let $\Gamma'$ be the instance of $\text{SM}_{\text{ext}}$ obtained through the reduction $f$ of Lemma 1. Let $K$ be a constant, we claim that $\Gamma$ has a stable matching with women’s cost at least $2K + n \cdot (n + 2L)$. If $O$ is a weakly stable matching with minimum cost for the women we denote by $O'$ the stable matching obtained through $f$ in $\Gamma'$.

First, let us assume there is no $O$ in $\Gamma$ with women’s cost less or equal than $K$. Let $O$ be a weakly stable matching with minimum cost for the women equal to $K' > K$. We prove that there is not a stable matching in $\Gamma'$ with a cost less or equal than $2K + n \cdot (n + 2L - 1)$. As we have seen in Lemma 2, the extra positions in $O'$ cannot be assigned differently to obtain a smaller cost for the women in $\Gamma'$. Let us recall that the cost of the women in $O'$ is $2K' + n \cdot (n + 2L) > 2K + n \cdot (n + 2L)$.

On the other side, assume there is $O$ in $\Gamma$ with women’s cost $K' \leq K$. Therefore, by creating $O'$ in $\Gamma'$, we may verify that in every batch $B_j$ associated to $w_j$ (recall that in $O$, $w_j$ is matched to $m_j$ for a given $i \leq n$)
• the cost of $y_i$ and $w_{ji}$ is $2 + 2n$ if $i \leq 2$ and $2i + 2n$ otherwise.

• the cost for all the other women in the batch is 1 (thus a total cost of $n - 1$).

In total, we achieve that $\text{cost}(O') = 2 \cdot K' + n \cdot (n + 2L) \leq 2 \cdot K + n \cdot (n + 2L)$.

To conclude the proof, we observe that given a matching $O'$, we can verify the women’s cost in polynomial time. 

C Results on the Mathematical Programming Formulations

The following result has already been remarked by Ágoston, Biró and McBride [4], we provide a proof of it.

**Theorem 4.** Let $\Gamma = (S, C, \succ, c)$ be an instance of the CA problem with strict preferences and complete lists. A stable matching $M_S$ is a student-oriented stable matching if and only if it is a stable matching with the minimum cost for the set of students $S$.

**Proof.** If: In a student-oriented stable matching, each student is assigned to the best university she could achieve in any stable matching [13]. Thus, each unassigned student is unassigned in every stable matching. Moreover, by the Rural Hospital Theorem, the same students are assigned in all stable matchings. Suppose that $M_S$ is a student-oriented stable matching, but it does not minimize the cost for $S$. Let $M$ be a stable matching with minimum cost for the students. Hence, the student cost of $M_S$ is strictly greater than the student cost of $M$: $\sum_{(i,j) \in M_S} \text{rank}_i(j) > \sum_{(i,j) \in M} \text{rank}_i(j)$. This means that there is at least one student $s'$ who has a lower cost in $M$ than in $M_S$, i.e., she prefers the stable matching $M$ to the stable matching $M_S$, which is a contradiction.

Only if: Let us show that a minimum cost stable matching for students $M_S$ is a student-oriented stable matching. As before, by the Rural Hospital Theorem, we observe that the set of students unassigned in $M_S$ is the same set of students unassigned in every stable matching. Hence, the set of assigned students in $M_S$ is the same for every stable matching. Let us suppose, again by contradiction, that $M_S$ is not a student-oriented stable matching. Let $M$ be a student-oriented stable matching. Denote by $S'$ the set of students whose assignment to universities differs in the two matchings. Note that the cost for $S \setminus S'$ is the same in both $M_S$ and $M$. Furthermore, $S'$ is the disjoint union of the following two sets of students: $S'_1$, the set of assigned students who prefer their matching in $M$, and $S'_2$, the set of assigned students who prefer their matching in $M_S$. By Gale and Shapley [13], $M$ is a stable matching in which all students are assigned the best university they could achieve in any stable matching. Therefore, the set $S'_2$ is empty. Hence, $S' = S'_1$, and by hypothesis

$$\sum_{(i,j) \in M_S} \text{rank}_i(j) < \sum_{(i,j) \in M} \text{rank}_i(j).$$

However, from the definition of $S'_1$, $\text{rank}_s(M(s)) \leq \text{rank}_s(M_S(s))$ for every $s \in S'_1$, which leads to a contradiction. 

The following is a counter-example to the inverse inclusion of the statement in Theorem 2.

**Example 1.** Let us consider the set of students $S = \{i_1, i_2, i_3, i_4, i_5, i_6\}$ and the set of colleges $C = \{j_1, j_2, j_3, j_4\}$. The rankings of the students are as follows:

- $i_1: j_3 \succ j_4 \succ j_1 \succ j_2$;
- $i_2: j_2 \succ j_1 \succ j_4 \succ j_3$;
- $i_3: j_2 \succ j_1 \succ j_4 \succ j_3$;
- $i_4: j_1 \succ j_3 \succ j_4 \succ j_2$;
- $i_5: j_2 \succ j_1 \succ j_3 \succ j_4$;
- $i_6: j_3 \succ j_4 \succ j_2 \succ j_1$.

22
i_4: j_1 \succ j_3 \succ j_2 \succ j_4;
i_5: j_3 \succ j_1 \succ j_4 \succ j_2;
i_6: j_1 \succ j_3 \succ j_4 \succ j_2.

The rankings of the colleges are as follows:

j_1: i_1 \succ i_3 \succ i_2 \succ i_5 \succ i_6 \succ i_4;
j_2: i_4 \succ i_1 \succ i_6 \succ i_5 \succ i_2 \succ i_3;
j_3: i_2 \succ i_1 \succ i_5 \succ i_6 \succ i_3 \succ i_4;
j_4: i_6 \succ i_3 \succ i_2 \succ i_4 \succ i_5 \succ i_1.

Finally, their capacities are c_{j_1} = c_{j_2} = 1 and c_{j_3} = c_{j_4} = 2. Given that we have to allocate optimally one extra position, the optimal solution for the relaxed aggregated linearization is x_{i_1j_3} = 1, x_{i_2j_2} = 1, x_{i_3j_4} = 0.16, x_{i_3j_2} = 0.66, x_{i_4j_1} = 0.5, x_{i_4j_2} = 0.16, x_{i_5j_3} = 0.16, x_{i_6j_1} = 0.83, x_{i_6j_4} = 0.33 and t = (0.16, 0.83, 0, 0) with the cost equal to 1.33.

On the other hand, the optimal solution for the relaxed non-aggregated linearization is x_{i_1j_3} = 0.83, x_{i_1j_4} = 0.08, x_{i_2j_2} = 1, x_{i_3j_1} = 0.11, x_{i_3j_2} = 0.66, x_{i_3j_4} = 0.02, x_{i_4j_4} = 0.66, x_{i_4j_3} = 0.16, x_{i_5j_3} = 1, x_{i_6j_1} = 0.52, x_{i_6j_4} = 0.30 and t = (0.30, 0.66, 0, 0.03) with cost equal to 1.03.

Experimental results suggest the following hypothesis: Under the additional constraint that for every student \( i \), \( \sum_{j \in C} x_{ij} \geq 1 \) holds (i.e., every student is assigned at least to a college), the inverse of Theorem 2 holds.