Exact conformal field theories from mutually T-dualizable $\sigma$-models

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Exact conformal field theories (CFTs) are obtained by using the approach of Poisson-Lie (PL) T-duality in the presence of spectators. We explicitly construct some non-Abelian T-dual $\sigma$-models (here as the PL T-duality on a semi-Abelian double) on $2 + 2$-dimensional target manifolds $M \approx G \times \tilde{G}$ and $\tilde{M} \approx \tilde{G} \times \tilde{O}$, where $G$ and $\tilde{G}$ as two-dimensional real non-Abelian and Abelian Lie groups act freely on $M$ and $\tilde{M}$, respectively, while $O$ is the orbit of $G$ in $M$. The findings of our study show that the original models are equivalent to Wess-Zumino-Witten (WZW) models based on the Heisenberg ($H_4$) and $GL(2,\mathbb{R})$ Lie groups. In this way, some new T-dual backgrounds for these WZW models are obtained. For one of the duals of the $H_4$ WZW model, we show that the model is self-dual. In the case of the $GL(2,\mathbb{R})$ WZW model it is observed that the duality transformation changes the asymptotic behavior of solutions from $AdS_3 \times \mathbb{R}$ to flat space. Then, the structure and asymptotic nature of the dual spacetime of this model including the horizon and singularity are determined. We furthermore get the non-critical Bianchi type III string cosmological model with a non-vanishing field strength from T-dualizable $\sigma$-models and show that this model describes an exact CFT (equivalent to the $GL(2,\mathbb{R})$ WZW model). After that, the conformal invariance of T-dual models up to two-loop order (first order in $\alpha'$) is discussed.

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I. INTRODUCTION

The duality symmetries play an important role in string theory. On the one hand, they are specific to string theory and their study has led to important insights in understanding the spacetime geometry from the string point of view. A very important symmetry of string theory or more generally, two-dimensional sigma models, is the T-duality [1]. A study of the T-duality in string theory has led to the discovery of PL T-duality. Klimčík and Ševera in their seminal work [2] proposed a generalization of T-duality, or the so-called PL T-duality, which allows the duality to be performed on a target space without isometries. In Klimčík and Ševera’s formalism, PL T-dual sigma models are defined by PL group manifolds which constitute a Drinfeld double [3]. The classification of low-dimensional Drinfeld doubles [4, 5] has become a convenient laboratory for investigating the PL T-duality.

On the other hand, the duality symmetries in WZW models have received considerable attention because of the preservation of the conformal symmetry under the Abelian duality [6]. This duality has been investigated in the WZW models [7]. Furthermore, for the case of non-Abelian duality [8], it has been shown that the conformal symmetry is preserved when the trace of the adjoint representation of the isometry group is zero [9].

The WZW model is a well-known construction for obtaining a CFT which describes string propagation on a Lie group. For instance, the natural metric on the Lie group $SL(2,\mathbb{R})$ is precisely the three-dimensional anti-de Sitter metric. Hence, the WZW model based on Lie group $SL(2,\mathbb{R})$ can be considered as an exact CFT describing string propagation on anti-de Sitter space [10].

Up to now, only few examples of PL symmetric $\sigma$-models have been treated at the quantum level [11, 12]. Furthermore, PL symmetry in the WZW models based on the Lie supergroups have recently been studied in Refs. [13, 14]. We also refer the reader to the literature given in Ref. [15]. In Ref. [11] it has been shown that the duality relates the $SL(2,\mathbb{R})$ WZW model to a constrained $\sigma$-model defined by the $SL(2,\mathbb{R})$ group space. We have shown that the PL T-duality relates the $H_4$ WZW model to a $\sigma$-model defined on the dual Lie group $A_2 \oplus 2A_1$. We have also stressed that the dual model is conformally invariant up to two-loop order. Furthermore, we have recently shown that the PL T-duality relates the $SL(2,\mathbb{R})$ WZW model to a $\sigma$-model defined on $2 + 1$-dimensional manifold $M \approx O \times G$ in which $G$ is two-dimensional real non-Abelian Lie group $A_2$, and $O$ as a one-dimensional space is the orbit of $G$ in $M$. Accordingly, we have obtained a dual model for the $SL(2,\mathbb{R})$ WZW model yielding a new three-dimensional charged black string which is stationary and asymptotically flat.

The main purpose of this paper is to construct some new non-Abelian T-dual backgrounds for the $H_4$ and $GL(2,\mathbb{R})$ WZW models via PL T-duality approach in the presence of spectators. The original models as exact CFTs (the $H_4$ and $GL(2,\mathbb{R})$ WZW models) are constructed on $2 + 2$-dimensional target manifold $M \approx O \times G$ with $G = A_2$ and dual models on manifold $\tilde{M} \approx \tilde{G}$ with $\tilde{G} = 2A_1$, whereas, in [10] T-dual $\sigma$-models were only constructed on Lie groups in the absence of spectators. In the present work, two dual models for the $H_4$ WZW are obtained for one of which we show that...
the dual model is indeed identical to the same $H_4$ WZW model. Moreover, we get one dual model for the $GL(2,\mathbb{R})$ WZW for which the structure and asymptotic nature of the spacetime including the horizon and singularity are determined. We also obtain the non-critical Bianchi type III string cosmological model with a non-vanishing field strength from a T-dualizable $\sigma$-model on 3 + 1-dimensional target manifold $M \approx O \times G$, in which $G$ represents three-dimensional decomposable Lie group $A_2 \oplus A_1$, and then we show that this model describes an exact CFT. Finally, we discuss the conformal invariance conditions of the T-dual models up to the first order in $\alpha'$ to introduce new solutions for two-loop $B$-function equations of the $\sigma$-model with a non-vanishing field strength $H$ and the dilaton field in both cases of the absence and presence of a cosmological constant $\Lambda$.

This paper is organized as follows. In Sec. II we present a basic review of the PL T-dual $\sigma$-models construction in the presence of spectator fields. In Sec. III we get the $H_4$ and $GL(2,\mathbb{R})$ WZW models from T-dualizable $\sigma$-models constructed on 2 + 2-dimensional target manifolds $M \approx O \times G$ and $M \approx O \times \tilde{G}$. In addition, the dual backgrounds for these WZW models together with the structure and asymptotic nature of the dual spacetime of the $GL(2,\mathbb{R})$ WZW including the horizon and singularity are studied. Finally, the non-Abelian T-dualization of the non-critical Bianchi type III string cosmology solution is discussed at the end of Sec. III.

In Sec. IV we investigate the conformal invariance conditions for T-dual models up to two-loop order. Some concluding remarks are given in Sec. V.

II. CONSTRUCTION OF PL T-DUAL $\sigma$-MODELS WITH SPECTATORS

We begin this section by reviewing the construction of PL T-dual $\sigma$-models in the presence of spectator fields. First of all, for the description of PL T-duality we need to introduce the Drinfeld double group $D$ [3], which by definition has a pair of maximally isotropic subgroups $G$ and $\tilde{G}$ corresponding to the subalgebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$, respectively. The generators of $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are denoted, respectively, $T_a$ and $\tilde{T}^a$, $a = 1, \ldots, \dim G$. One says that the Lie algebras $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are compatible if the brackets

$$
[T_a, T_b] = f^c_{ab} T_c, \quad [\tilde{T}^a, \tilde{T}^b] = \tilde{f}^{ab}_{c} \tilde{T}^c, \\
[T_a, \tilde{T}^b] = f^{c\alpha}_{ab} T^c + f^{b\alpha}_{ca} \tilde{T}^c,
$$

(1)

define a Lie algebra structure on the direct sum vector space $\mathcal{D} = \mathcal{G} \oplus \tilde{\mathcal{G}}$. In this case, we say that the Lie algebra $\mathcal{D}$ is the Drinfeld double of $G$ or, equivalently, of $\tilde{G}$. Thus, the group $\mathcal{D}$ is called the Drinfeld double of $G$ (or $\tilde{G}$).

We also note that the Drinfeld double $\mathcal{D}$ is equipped with an invariant inner product $\langle \cdot, \cdot \rangle$ with the following properties

$$
\langle T_a, T_b \rangle = \delta^b_a, \quad \langle T_a, \tilde{T}^b \rangle = \delta^a_c, \quad \langle T_a, T_a \rangle = 0.
$$

(2)

In what follows we shall investigate PL T-duality transformations in the presence of spectators [2,18] of a non-linear $\sigma$-model with the following action for a bosonic string, propagating in a $d$-dimensional spacetime, with the metric $G_{\mu\nu}$, the antisymmetric tensor field $B_{\mu\nu}$ and the dilaton field $\phi$

$$
S = \frac{1}{2\pi \alpha'} \int d^d \sigma \sqrt{-h} \left[ \frac{1}{2} (h^{\alpha\beta} G_{\mu\nu} + e^{\alpha\beta} B_{\mu\nu}) \partial_\alpha x^\nu \partial_\beta x^\nu + \frac{1}{4} \alpha' \phi R^{(4)} \right].
$$

(3)

where $h_{\alpha\beta}$ is the worldsheet metric with $R^{(4)}$ the corresponding worldsheet curvature scalar and $h = \det h_{\alpha\beta}$. The indices $\alpha, \beta$ run over $(\tau, \sigma)$, and $e^{\alpha\beta}$ is an antisymmetric tensor on the worldsheet $\Sigma$. The dimensionful coupling constant $\alpha'$ turns out to be the inverse string tension. The functions $x^i : \Sigma \to \mathbb{R}$, $(\mu = 1, \ldots, \dim M)$ are obtained by the composition $x^i = X^i \circ \sigma$ of a map $x : \Sigma \to M$ and components of a coordinate map $X$ on a chart of $M$. Here, and in the following, we use the standard light-cone variables on the worldsheet, $\sigma^{\pm} = \tau \pm \sigma$.

Let us now consider a $d$-dimensional manifold $M$ and some coordinates $x^\mu = (x^i, y^\alpha)$ on it, where $x^i (i = 1, \ldots, \dim G)$ are the coordinates of Lie group $G$ acting freely from right on $M$, $y^\alpha (\alpha = 1, \ldots, d - \dim G)$ are the coordinates labeling the orbit $O$ of $G$ in the target space $M$. We note that the coordinates $y^\alpha$ do not participate in the PL T-duality transformations and are therefore called spectators [18]. Take a linear (dempotent) map $K$ from the space $T^a_y M \oplus T^\alpha_y M \oplus \mathcal{D}$ into itself. It has two eigenspaces $\mathbb{R}_\pm (y^\alpha)$ with eigenvalues $\pm 1$. They are perpendicular to each other according to the bilinear form on $T^a_y M \oplus T^\alpha_y M \oplus \mathcal{D}$. These eigenspaces may be considered as the graph of a non-degenerate linear map $E^{\pm} (y)$: $T^a_y M \oplus \tilde{G} \to T^a_y M \oplus \tilde{G}$, such that by translating this graph to the point $g \in G$ we have

$$
g^{-1} \mathbb{R}_\pm (y^\alpha) g = \text{Span} \{ X_A \pm E^{\pm}_{AB} (y^\alpha) \tilde{X}^A \},
$$

(4)

where $X_A = (T_a, \partial_\alpha)$ and $\tilde{X}^A = (T^a, dy^\alpha)$ are the basis of the spaces $T^a_y M \oplus \tilde{G}$ and $T^\alpha_y M \oplus \tilde{G}$, respectively. In order to determine the $d \times d$ matrix $E_{AB} (y, y^\alpha)$ we write the spaces $\mathbb{R}_\pm (y^\alpha)$ as follows:

$$
g^{-1} \mathbb{R}_\pm (y^\alpha) g = \text{Span} \{ g^{-1} X_A g \pm E^{\pm}_{AB} (e, y^\alpha) g^{-1} \tilde{X}^A \},
$$

(5)

in which the matrix $E^{\pm}_{AB} (e, y^\alpha)$ is defined as

$$
E^{\pm}_{AB} (e, y^\alpha) = \begin{pmatrix}
E_{0\alpha}^{\pm} (e, y^\alpha) & F_{\alpha\beta}^{\pm (1)} (e, y^\alpha) \\
F_{\alpha\beta}^{\pm (2)} (e, y^\alpha) & F_{\alpha\beta}^{\pm (y^\alpha)}
\end{pmatrix}. \quad (6)
$$

Here, submatrices $E_{0\alpha}^{\pm} (e, y^\alpha)$, $F_{\alpha\beta}^{\pm (1)} (e, y^\alpha)$ and $F_{\alpha\beta}^{\pm (2)} (e, y^\alpha)$ are functions of the variables $y^\alpha$ and $e$, where $e$ is the unit element of $G$. $F_{\alpha\beta}^{\pm (y^\alpha)}$ is also a function of $y^\alpha$ only. Here, and in the following, the minus sign stands for transpose, namely, $E_{0\alpha}^{\pm} = E_{0\alpha}^{-\pm}$, $F_{\alpha\beta}^{\pm (2)} = F_{\alpha\beta}^{-\pm}$, and $F_{\alpha\beta}^{\pm (1)} = F_{\alpha\beta}^{-\pm}$. 

Thus, $E^{\pm}_{AB} (e, y^\alpha)$ is a matrix with the property

$$
E^{\pm}_{AB} (e, y^\alpha) = E^{\pm}_{AB} (e, -y^\alpha).
$$

(7)

In what follows we shall use this property when we need to write $E^{\pm}_{AB} (e, y^\alpha)$ as a matrix with $y^\alpha$ and $-y^\alpha$.
It is convenient to define matrices \( a(g), b(g) \) and the Poisson bracket \( \Pi(g) \) in the following way
\[
\begin{align*}
g^{-1} T_s g &= a_s^b (g) T_b, \\
g^{-1} T^a g &= b^{ab} (g) T_s + (a^{-1})^a_b (g) \tilde{T}^b, \\
\Pi^{ab}(g) &= b^{ac} (g) (a^{-1})^b_c (g).
\end{align*}
\] (7)
(8)
Thus, using (43) and (44) together with (7) one gets
\[
E_{AB}^\pm (g, y^\alpha) = \left( A(g) \pm \Phi^\pm (e, y^\alpha) B(g) \right)_{A}^{-1} C
\times E_{CD}^\pm (e, y^\alpha) (A^{-1})^D_B (g),
\] (9)
where
\[
A(g) = \begin{pmatrix} a(g) & 0 \\ 0 & Id \end{pmatrix}, \quad B(g) = \begin{pmatrix} b(g) & 0 \\ 0 & 0 \end{pmatrix}.
\] (10)

We also define
\[
F_{AB}^\pm (g, y^\alpha) = A^C_A (g) E_{CD}^\pm (g, y^\alpha) A^D_B (g).
\] (11)
Considering matrix \( F_{AB}^\pm (g, y^\alpha) \) in the form
\[
F_{AB}^\pm (g, y^\alpha) = \begin{pmatrix} E_{ab}^\pm (g, y^\alpha) & \Phi_{ab}^{(1)} (g, y^\alpha) \\ \Phi_{ab}^{(2)} (g, y^\alpha) & \Phi_{ab}^{(3)} (g, y^\alpha) \end{pmatrix},
\] (12)
and then using (43), (44), (45) and (46) one can obtain the backgrounds appearing in the action of original \( \sigma \)-model. They are given in matrix notation by
\[
\begin{align*}
\mathbb{E}^\pm (g, y^\alpha) &= \left( E_{ab}^{\pm 1} (e, y^\alpha) \pm \Pi(g) \right)^{-1}, \\
\Phi^{(1)}_{ab} (g, y^\alpha) &= \mathbb{E}^\pm (g, y^\alpha) (E_{ab}^{\pm 1} (e, y^\alpha) F^{\pm 1} (e, y^\alpha)), \\
\Phi^{(2)}_{ab} (g, y^\alpha) &= \mathbb{E}^\pm (g, y^\alpha) (E_{ab}^{\pm 1} (e, y^\alpha) E^{\pm} (g, y^\alpha)), \\
\Phi (g, y^\alpha) &= \mathbb{E}^{\pm} (g, y^\alpha) - F^{\pm 2} (e, y^\alpha) \Pi(g) E^{\pm} (g, y^\alpha) \\
&\quad \times (E_{ab}^{\pm 1} (e, y^\alpha) F^{\pm 1} (e, y^\alpha)).
\end{align*}
\] (13)
(14)
(15)
(16)

Let us now introduce the elements \( V_\pm \) of subspaces \( \mathbb{R}_\pm (y^\alpha) \) as
\[
V_\pm := \partial_{\pm} y^\alpha \frac{\partial}{\partial y^\alpha} + p_\alpha^{(\mp)} dy^\alpha + \partial_{\pm} U^{-1},
\] (17)
where \( p_\alpha^{(\mp)} \in T_y^* M \) and \( l \in D \). Inserting the decomposition \( l = g \tilde{h} (g \in G, \tilde{h} \in \tilde{G}) \) into (17) we get
\[
V_\pm := R^A_\pm X_A + R_{\pm A} (B^{BA} (g) X_b + A^B_A (g) \tilde{X}^a),
\] (18)
where \( R^A_\pm \) and \( R_{\pm A} \) are the elements of the respective spaces \( T_y^* M \oplus G \) and \( T_y^* M \oplus \tilde{G} \), and are given by
\[
\begin{align*}
R^A_\pm &= (R^A_\pm, \partial_{\pm} y^\alpha) = (\partial_{\pm} g^{-1} a, \partial_{\pm} y^\alpha), \\
R_{\pm A} &= ((\partial_{\pm} \tilde{h} \tilde{h}^{-1}) a, \mp p_\alpha^{(\mp)}).
\end{align*}
\] (19)
(20)
Thus, by using the equations of motion
\[
\left< V_\pm, (X_A + E^{\pm} (g, y^\alpha) \tilde{X}^a) \right> = 0,
\] (21)
we obtain
\[
R_{\pm A} = \pm R^B_{\pm A} F^{\pm}_{ab} (g, y^\alpha) (A^{-1})^C_A (g).
\] (22)
The equation (22) can be written in terms of components. They then take the following forms
\[
\begin{align*}
\partial_+ h \tilde{h}^{-1} a &= (a^{-1})^c_a (g) \left[ R^b_+ E^{\pm}_{ab} (g, y^\alpha) \\
&\quad + \partial_+ y^\alpha \Phi^{(2)}_{ab} (g, y^\alpha) \right],
\end{align*}
\] (23)
and
\[
\begin{align*}
\partial_- h \tilde{h}^{-1} a &= -(a^{-1})^c_a (g) \left[ E^{\pm}_{ab} (g, y^\alpha) R^b_- \\
&\quad + \Phi^{(1)}_{ab} (g, y^\alpha) \partial_- y^\beta \right].
\end{align*}
\] (24)

The above results indicate that the equations (23) and (24) are nothing but the equations of motion concerning the \( \sigma \)-model described by the following action
\[
\begin{align*}
S &= \frac{1}{2} \iiint d\sigma^+ d\sigma^- \mathcal{E}^{\pm} (g, y^\alpha) R^+ R^- \\
&= \frac{1}{2} \iiint d\sigma^+ d\sigma^- \left[ E^{\pm}_{ab} (g, y^\alpha) R^+ R_- \\
&\quad + \Phi^{(1)}_{ab} (g, y^\alpha) R^+ \partial_- y^\beta + \Phi^{(2)}_{ab} (g, y^\alpha) \partial_+ y^\alpha R^- \\
&\quad + \Phi^{(3)}_{ab} (g, y^\alpha) \partial_+ y^\alpha \partial_- y^\beta \right].
\end{align*}
\] (27)

As we shall see below, one can construct another \( \sigma \)-model (denoted as usual with tilded symbols) which is said to be dual to (27) in the sense of the PL T-duality if the Lie algebras \( G \) and \( \tilde{G} \) form a pair of maximally isotropic subalgebras of the Lie algebra \( D \). In order to get the dual \( \sigma \)-model one proceeds in an analogous way so that eigenspaces \( \mathbb{R}_\pm (y^\alpha) \) are considered as
\[
\mathbb{R}_\pm (y^\alpha) = \text{Span}\{ \tilde{Y}^A \pm \tilde{E}^{\pm AB} (\tilde{e}, y^\alpha) Y_b \},
\] (28)
where \( \tilde{E}^\pm (\tilde{e}, y^\alpha) : T_y M \oplus \tilde{G} \rightarrow T_y^* M \oplus G, Y_b = (T_a, dy^\alpha) \) and \( \tilde{Y}^A = (\tilde{T}^A, \partial_a) \). With a slight abuse of the notation, comparing (5) and (28) we get the matrix form of \( \tilde{E}^\pm (\tilde{e}, y^\alpha) \) as
\[
\tilde{E}^\pm (\tilde{e}, y^\alpha) = \pm (\mathcal{A} \pm \tilde{E}^\pm (e, y^\alpha) \mathcal{B})^{-1} \tilde{E}^\pm (e, y^\alpha) \mathcal{A},
\] (29)
in which
\[
\mathcal{A} = \begin{pmatrix} 0 & 0 \\ 0 & Id \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} Id & 0 \\ 0 & 0 \end{pmatrix}.
\] (30)

---

1. Here \( Id \) means the identity matrix.
Now, using (28) and inserting the decomposition $t = \tilde{g} \tilde{h}$ into the equations (21), one can get the equations of motion for $y^a$ and $\tilde{x}^i$ corresponding to the following action

$$\tilde{S} = \frac{1}{2} \int d\sigma^+ d\sigma^- (\tilde{\phi}^{+\alpha}(\tilde{g}, y^a) \tilde{R}^\alpha_+ \tilde{R}^-_\alpha$$

$$= \frac{1}{2} \int d\sigma^+ d\sigma^- \left[ E^{+\alpha}(\tilde{g}, y^a) \tilde{R}^\alpha_+ \tilde{R}^-_\alpha + \bar{\phi}^{+\alpha}(\tilde{g}, y^a) \tilde{R}^\alpha_+ \tilde{R}^-_\alpha + \phi^{+\alpha}(\tilde{g}, y^a) \tilde{R}^\alpha_+ \tilde{R}^-_\alpha + \bar{\phi}^{+\alpha}(\tilde{g}, y^a) \tilde{R}^\alpha_+ \tilde{R}^-_\alpha \right].$$

(31)

The coupling matrices of the dual $\sigma$-model are also determined in a similar fashion [15]. Using (29) one relates them to those of the original one by

$$\tilde{E}^\pm(\tilde{g}, y^a) = \left( E^\pm(e, y^a) \pm \tilde{\Pi}(\tilde{g}) \right)^{-1},$$

(32)

$$\tilde{\phi}^{+\alpha}(\tilde{g}, y^a) = \pm \tilde{E}^\pm(\tilde{g}, y^a) F^{+\alpha}(e, y^a),$$

(33)

$$\tilde{\phi}^{-\alpha}(\tilde{g}, y^a) = \mp F^{-\alpha}(e, y^a) \tilde{E}^\pm(\tilde{g}, y^a),$$

(34)

$$\tilde{\phi}(\tilde{g}, y^a) = F(y^a) - F^{+\alpha}(e, y^a) \times \tilde{E}^+(\tilde{g}, y^a)^2 F^{+\alpha}(e, y^a).$$

(35)

The actions (27) and (31) correspond to PL T-dual $\sigma$-models [2]. Notice that if the group $G(\tilde{G})$ besides having free action on $M(\tilde{M})$, acts transitively on it, then the corresponding manifold $M(\tilde{M})$ will be the same as the group $G(\tilde{G})$. In this case only the first term appears in the actions (27) and (31).

In the PL T-duality case, dilaton shifts in both models have been obtained by quantum considerations based on a regularization of a functional determinant in a path integral formulation of PL T-duality by incorporating spectator fields [13] (see, also, [20])

$$\phi = \phi_0(y^a) + \log(\det E^\pm) - \log(\det E^\pm_0),$$

(36)

$$\tilde{\phi} = \phi_0(y^a) + \log(\det \tilde{E}^\pm),$$

(37)

where $\phi_0(y^a)$ is just a function of $y^a$.

III. T-DUALIZABLE $\sigma$-MODELS ON 2 + 2-DIMENSIONAL MANIFOLDS AS EXACT CFTS

In this section, we explicitly construct two pairs of PL T-dual $\sigma$-models on 2 + 2-dimensional target manifolds $M \approx O \times G$ and $\tilde{M} \approx O \times \tilde{G}$, where $G$ and $\tilde{G}$ as two-dimensional real non-Abelian and Abelian Lie groups act freely on $M$ and $\tilde{M}$, respectively, while $O$ is the orbit of $G$ in $M$ with the spectators $y^a = \{y_1, y_2\}$. The Lie algebras of the Lie groups $G$ and $\tilde{G}$ are denoted by $A_2$ and $2A_1$, respectively. According to Sec. III having Drinfeld doubles we can construct PL T-dual $\sigma$-models on them. The four-dimensional Lie algebra of the Drinfeld double $(A_2, 2A_1)$ is given by the following non-zero commutation relations:

$$[T_1, T_2] = T_2, \quad [T_1, \tilde{T}^2] = -\tilde{T}^2, \quad [T_2, \tilde{T}^2] = \tilde{T}^1,$$

(38)

where $\{T_1, T_2\}$ and $\{\tilde{T}^1, \tilde{T}^2\}$ are the basis of $A_2$ and $2A_1$, respectively. Notice that the double $(A_2, 2A_1)$ has non-vanishing trace in the adjoint representations. In such a situation, there is usually a conformal anomaly at one-loop associated with non-Abelian T-duality [20]. In what follows, we will also discuss the conformal anomaly appeared in the string effective Lagrangians corresponding to the T-dual models.

In order to calculate the components of right invariant one-forms $R^1_\pm$ on the Lie group $A_2$ we parametrize an element of $A_2$ as

$$g = e^{x_1 T_1} e^{x_2 T_2},$$

(39)

where $x^i = \{x_1, x_2\}$ are the coordinates of the Lie group $A_2$. $R^1_\pm$’s are then derived in the following form

$$R^1_\pm = \partial_\pm x_1, \quad R^2_\pm = e^{x_1} \partial_\pm x_2.$$  

(40)

Since the dual Lie group, $2A_1$, is Abelian, by using (7), (8) and (35) it follows that the Poisson bracket $\Pi^{ab}(\tilde{g})$ on $A_2$ vanishes. Furthermore, for obtaining the Poisson bracket on the dual group $2A_1$ we first parametrize the Lie group $2A_1$ with coordinates $\tilde{x}^i = \{\tilde{x}_1, \tilde{x}_2\}$ so that its elements are defined as in (39) by replacing untitled quantities with tilded ones. Then, using (7) and (35) for tilded quantities together with (35) the Poisson bracket on $2A_1$ is derived as follows:

$$\Pi_{ab}(\tilde{g}) = \begin{pmatrix} 0 & -\tilde{x}_2 \\ \tilde{x}_2 & 0 \end{pmatrix}. $$

(41)

In addition to the right invariant one-forms, to construct the $\sigma$-models (27) and (31) on manifolds $M$ and $\tilde{M}$ we need to determine the couplings $E^{+\alpha}_0(g, y^a)$, $\phi^{+\alpha}_\beta(g, y^a)$, $\phi^{-\alpha}_0(g, y^a)$ and $\phi^{-\alpha}_\beta(g, y^a)$. By convenient choices of the background matrices $E^{+\alpha}_0(e, y^a)$, $F^{+\alpha}_\beta(e, y^a)$, $F^{-\alpha}_\beta(e, y^a)$ and $F^{-\alpha}_\beta(e, y^a)$, we will show that the original models are equivalent to the $H_4$ and $GL(2, \mathbb{R})$ WZW models. In this way, the new dual backgrounds for these WZW models are obtained.

A. The $H_4$ WZW model from T-dualizable $\sigma$-models and its dual pairs

In this subsection, we obtain two different duals of the $H_4$ WZW model. In both cases, the original $\sigma$-models (which are equivalent to the $H_4$ WZW model) are constructed on the manifold $M \approx O \times G$ with $G = A_2$ acting freely on it, however, the spectator-dependent background matrices are chosen to be different for each model.
Case (1): In this case, we take the background matrices as

\[ E_{0\alpha}^{+} = \begin{pmatrix} 0 & e^y \\ e^{-y} & 0 \end{pmatrix}, \quad F_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \]

\[ F_{\alpha\beta}^{(1)} = \begin{pmatrix} 0 & 0 \\ e^y & 0 \end{pmatrix}, \quad F_{\alpha\beta}^{(2)} = \begin{pmatrix} 0 & -e^y \\ e^{-y} & 0 \end{pmatrix}. \] (42)

As explained above, the Poisson bracket \( \Pi^{ab}(g) \) is zero. By using relations (13)-(16), one can get the required couplings which where mentioned above. Finally, using (41) and then (27), the original \( \sigma \)-model is found to be of the form

\[ S = \frac{1}{2} \int d\sigma^{+} d\sigma^{-} \left[ - \partial_{+} y_{1} \partial_{-} y_{2} - \partial_{+} y_{2} \partial_{-} y_{1} ight. \]

\[ + e^{y_{1} + y_{2}} \left( \partial_{+} x_{1} \partial_{-} x_{2} + \partial_{+} x_{2} \partial_{-} x_{1} \right) \]

\[ + e^{y_{1} + y_{2}} \left( \partial_{+} x_{2} \partial_{-} y_{1} - \partial_{+} y_{1} \partial_{-} x_{2} \right) \right]. \] (43)

By identifying action (13) with the \( \sigma \)-model of the form (3) one can read off the background matrix. Thus, the metric and antisymmetric tensor field corresponding to the action (13) can be written as

\[ ds^{2} = -2 dy_{1} dy_{2} + 2e^{x_{1} + y_{1}} dx_{1} dx_{2}, \]

\[ B = e^{x_{1} + y_{1}} dx_{2} \wedge dy_{1}. \] (44)

Before proceeding to construct the dual \( \sigma \)-model, let us discuss the conformal invariance of the model (43). The classical canonical equivalence to the \( \sigma \)-models related by PL T-duality was done by Sfetsos in (18) (see also, 21). The canonical transformations are essentially classical and the quantum equivalence of the two \( \sigma \)-models has not yet been revealed. Equivalence can hold in some special cases but it fails in most cases. In this respect, checking the equivalence by studying conformal invariance (the vanishing of the Beta-functions) is important. But, since after a classical canonical transformation, the equivalence always holds up to first order in Planck’s constant in the semiclassical expansion (corresponding to one-loop order in \( \sigma \)-model language), only the two-loop order is the first real test of quantum equivalence of the two different \( \sigma \)-models related by PL T-duality. For these reasons it is important to check the conformal invariance conditions of our models.

In the \( \sigma \)-model context, the conformal invariance is provided by the vanishing of the \( B \)-functions equations (23), which are equivalent to the equations of motion of effective action in the string frame (23). In four dimensions, the low energy string effective action is

\[ S_{\text{eff}} = \int d^{4}x \sqrt{-G} e^{-\phi} L_{\text{eff}}, \] (46)

where \( G = \text{det} G_{\mu\nu} \), and \( L_{\text{eff}} \) is given by

\[ L_{\text{eff}} = R + (\nabla \phi)^{2} - \frac{1}{3} H^{2} + 2\Lambda. \] (47)

In this expression, \( R \) is the scalar curvature of the metric \( G_{\mu\nu} \) and \( H_{\mu\nu} \), defined by \( H_{\mu\nu} = 1/2 (\partial B_{\mu\nu} + \partial_{\nu} B_{\mu} - \partial_{\mu} B_{\nu}) \) is the torsion (the field strength) of the field \( B_{\mu\nu} \). \( \Lambda \) is a cosmological constant which is vanished for critical strings. Our analysis applies also to non-critical strings, i.e. when \( \Lambda \) is different from zero.

Consistency of the string theory requires that the action (43) be defined a conformally invariant quantum field theory. The conditions for conformal invariance can be interpreted as field equations for \( G_{\mu\nu} \), \( B_{\mu\nu} \) and \( \phi \) of the string effective action (23, 24). The vanishing of the one-loop \( B \)-functions equations gives us the conformal invariance conditions of the \( \sigma \)-model (3) up to one-loop order (zeroth order in the inverse string tension \( \alpha’ \)) (26). These equations are given by

\[ B^{(0)}_{\mu\nu} : \quad R_{\mu\nu} - (H^{2})_{\mu\nu} + \nabla_{\mu} \nabla_{\nu} \phi + O(\alpha') = 0, \] (48a)

\[ B^{(0)}_{\mu\nu} : \quad -\nabla^{\lambda} H_{\lambda\mu\nu} + H_{\mu\nu} \nabla_{\lambda} \phi + O(\alpha') = 0, \] (48b)

\[ B^{(0)} : \quad \Lambda + 1/2 \nabla^{2} \phi - 1/2 (\nabla \phi)^{2} + 1/3 H^{2} + O(\alpha') = 0. \] (48c)

We have used the conventional notations \( (H^{2})_{\mu\nu} = H_{\mu\nu} H^{\nu\mu} \). One quickly finds that the only non-zero component of \( R_{\mu\nu} \) is \( R_{y_{1}y_{1}} = -1/2 \) and then \( R = 0 \). Thus, the metric is flat in the sense that its scalar curvature vanishes (4). For the antisymmetric tensor field (15) one verifies that the only non-zero component of \( H \) is \( H_{x_{1}x_{2}y_{1}} = (e^{x_{1} + y_{1}})/2 \). It then follows that \( H^{2} = 0 \) and the only non-zero component of \( (H^{2})_{\mu\nu} \) is \( (H^{2})_{y_{1}y_{1}} = -1/2 \). Inserting the above results in the vanishing of the one-loop \( B \)-functions equations (48a), (48c), the conformal invariance conditions up to one-loop order

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2 In string theory, the cosmological constant term \( \Lambda \) is related to the dimension of spacetime, \( d \), and the inverse string tension by \( \Lambda = (d-26)/3\alpha' \), whereas, here in this paper it is, in some cases, treated as a free parameter.

3 (2, 2)-signature often appears in Kleinian geometry as the neutral \((- + +)

4 The metric (43) can be considered as the plane-parallel (pp)-wave in the so-called Rosen coordinates. To this end, one can first use the coordinate transformation \( e^{x_{1}} = \theta + \phi \), \( x_{2} = -\theta + \phi \), \( y_{1} = u \), \( y_{2} = -v \) to obtain \( ds^{2} = 2du dv + 2e^{\theta} (d\theta^{2} + dv^{2}) \), then, after the change of the metric signature by a Wick rotation as \( \theta = it \), the resulting metric turns into the pp-wave one in the Rosen coordinates.
are satisfied with \( \Lambda = 0 \) and the dilaton field

\[
\phi = \sigma_0 + \sigma_1 y_1,
\]

where \( \sigma_0 \) and \( \sigma_1 \) are integration constants. In addition to the conformal invariance of the model \((43)\) up to one-loop order, we are interested in investigating the conformal invariance of the model for higher orders in \( \alpha' \). Instead of this, we show that the model \((43)\) is equivalent to an exact CFT, namely a WZW model based on a Lie group. The WZW models represent exact solutions to the string equations of motion to all orders in \( \alpha' \). One can show that under the coordinate transformation

\[
e^{x_1} = a_+, \quad x_2 = a_- \quad y_1 = n \quad y_2 = m,
\]

action \((43)\) turns into

\[
S = \frac{1}{2} \int d\sigma^+ d\sigma^- \left\{ -\partial_- n \partial_- m - \partial_+ m \partial_- n \\
+ e^\phi (\partial_- a_+ \partial_- a_- + \partial_+ a_- \partial_- a_-) \\
+ a_+ e^\phi (\partial_- a_- \partial_- n - \partial_+ n \partial_- a_-) \right\},
\]

which is nothing but the action of WZW model based on the Lie group \( H_4 \) \((10)\) (cf. Appendix A). Therefore, the action \((43)\) as an exact CFT describes string propagation on a four-dimensional manifold with \((2,2)\)-signature. We showed that the PL T-duality relates the \( H_4 \) WZW model to a \( \sigma \)-model defined on 2 + 2-dimensional manifold \( M \cong O \times G \) only when \( G \) is the Lie group \( A_2 \).

To continue, we obtain a new dual background for the \( H_4 \) WZW model. This background is obtained from a \( \sigma \)-model which is constructed on 2 + 2-dimensional manifold \( M \cong O \times G \) with two-dimensional Abelian Lie group \( G = 2A_1 \) acting freely on it. In order to construct the dual model in the form \((31)\) we need to determine the dual couplings. Making use of \((41)\) and inserting \((42)\) into equations \((12a)\)\,-\,(12c)\) they are then read off to be

\[
\tilde{\Phi}^{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\Phi}^a \tilde{\Phi}^b = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},
\]

\[
\tilde{\Phi}^{(1)\alpha} = \begin{pmatrix} e^{\phi y_1} & 0 \\ 0 & e^{\phi y_1} \end{pmatrix}, \quad \tilde{\Phi}^{(1)\alpha} \tilde{\Phi}^{(1)\beta} = \begin{pmatrix} e^{\phi y_1} & 0 \\ 0 & e^{\phi y_1} \end{pmatrix}.
\]

Putting these pieces together into \((31)\) and using the fact that the components of the right invariant one-forms on 2\( A_1 \) are \( \tilde{R}_{\pm a} = \partial_\pm \tilde{x}_a \), the action of dual \( \sigma \)-model is obtained to be

\[
\tilde{S} = \frac{1}{2} \int d\sigma^+ d\sigma^- \left\{ -\partial_- y_1 \partial_- y_2 - \partial_+ y_2 \partial_- y_1 \\
+ \frac{1}{\Delta} \left[ (e^{\phi y_1} - \tilde{x}_2) \partial_+ \tilde{x}_1 \partial_- \tilde{x}_2 + (e^{\phi y_1} + \tilde{x}_2) \partial_- \tilde{x}_1 \partial_+ \tilde{x}_2 \\
+ e^{\phi y_1} (e^{\phi y_1} - \tilde{x}_2) \partial_+ \tilde{x}_1 \partial_- y_1 + e^{\phi y_1} (e^{\phi y_1} + \tilde{x}_2) \partial_- y_1 \partial_+ \tilde{x}_1 \right] \right\},
\]

where \( \Delta = e^{2y_1} - \tilde{x}_2^2 \). Comparing the above action with the \( \sigma \)-model action of the form \((3)\), the corresponding line element and antisymmetric field \( \tilde{B} \) take the following forms

\[
\tilde{ds}^2 = -2dy_1 dy_2 + \frac{e^{\phi y_1}}{\Delta} (d\tilde{x}_1 d\tilde{x}_2 + e^{\phi y_1} d\tilde{x}_1 dy_1),
\]

\[
\tilde{B} = -\frac{\tilde{x}_2}{\Delta} (d\tilde{x}_1 \wedge d\tilde{x}_2 + e^{\phi y_1} d\tilde{x}_1 \wedge dy_1).
\]

The line element \((54)\) is ill defined at the regions \( \tilde{x}_2 = e^{\phi y_1} \) and \( \tilde{x}_2 = -e^{\phi y_1} \). We can test whether there are true singularities by calculating the scalar curvature, which is, \( \tilde{R} = 0 \). Furthermore, one gets that the only non-zero components of the Ricci tensor and Riemann tensor field are, respectively,

\[
\tilde{R}_{y_2 y_1} = \frac{e^{y_1 y_2}}{(e^{y_1} - \tilde{x}_2)^2}, \quad \tilde{R}_{y_1 y_1} = \frac{-2e^{y_1} + 6\tilde{x}_2 e^{y_1} + \tilde{x}_2^2}{2(e^{y_1} - \tilde{x}_2)^2},
\]

and

\[
\tilde{R}_{y_1 y_2 y_2 y_1} = \frac{e^{y_1 y_2}}{(e^{y_1} + \tilde{x}_2)(e^{y_1} - \tilde{x}_2)^3}, \quad \tilde{R}_{y_1 y_1 y_2 y_2} = -\frac{e^{y_1} (e^{y_1} + 6e^{y_1} \tilde{x}_2 + \tilde{x}_2^2)}{4(e^{y_1} + \tilde{x}_2)(e^{y_1} - \tilde{x}_2)^3}.
\]

Then, the other invariant characteristics of spacetime, such as \( \tilde{R}^{\mu\nu} \tilde{R}^{\mu\nu} \) and the Kretschmann scalar are found to be zero. Therefore, the singular points are not the essential singularities, that is, they can be removed by an appropriate change of coordinates. In order to investigate the conformal invariance conditions of the dual model \((53)\) we look at the vanishing of the one-loop \( \beta \)-functions equations \((18a)-(18c)\). To this end, we find that the only non-zero component of \( \tilde{H} \) corresponding to \( \beta \)-field \((53)\) is \( \tilde{H}_{y_1 y_2 y_1} = e^{y_1 y_2}/2(e^{y_1} - \tilde{x}_2)^2 \); consequently \( \tilde{H}^2 = 0 \). Hence, equations \((18a)\) and \((18c)\) are satisfied by the new dilaton field

\[
\tilde{\phi} = b_0 + b_1 y_1 - \log \left( \frac{\tilde{x}_2 - e^{\phi y_1}}{\tilde{x}_2 + e^{\phi y_1}} \right),
\]

where \( b_0 \) and \( b_1 \) are integration constants. The dilatonic contribution, equation \((48c)\), is also satisfied if the cosmological constant of the dual theory is left invariant, that is, \( \Lambda = 0 \). Thus, it seems that under the non-Abelian T-duality the cosmological constant has been restored from the dual model to the original one.

At the end of this subsection let us discuss the invariance of the string effective Lagrangians corresponding to the \( \sigma \)-models \((13)\) and \((33)\). For these models, the two Lagrangians \( \mathcal{L}_{\alpha\beta} \) and \( \mathcal{L}_{\alpha\beta} \) yield the same expression. They are both equal to zero. The equivalence of Lagrangians holds in spite of the non-vanishing traces of the structure constants corresponding to the double \((A_2, 2A_1)\). Moreover, in the case of this example one can show that the integration weights \( \sqrt{-G e^{-\phi}} \) and \( \sqrt{-G e^{-\tilde{\phi}} \right} \) are not equal. The reason behind this can be
interpreted in two ways: firstly, the dilaton obtained in \[53\] does not follow the formula \[57\]. Secondly, due to the particularity of our model, there is a possibility of absorbing the anomalous terms into dilaton shift which is the same as a diffeomorphism transformation. An analysis similar to this has been carried out in Ref. \[34\] in a strict field theory sense, regardless of the relationship between \(\sigma\)-models and string theory effective actions. Notice that the equations \[59\] and \[60\] are the only transformations which lead to a proportionality between the integration weights \(\sqrt{-G e^{-\phi}}\) and \(\sqrt{-G e^{-\tilde{\phi}}}\) \[20\].

**Case (2): The self-duality of the \(H_4\) WZW model**

The self-duality of the WZW model under PL T-duality, as well as the \(SU(N)\) WZW model, has already been discussed in \[33\]. It turns out that the dual to the WZW model is again the same WZW model. Here we shall show that the \(H_4\) WZW model is self-dual. Let us now choose the spectator-dependent background matrices as

\[
\begin{align*}
E_{ab}^{+} &= \begin{pmatrix} 0 & -y_1 \\ y_1 & 0 \end{pmatrix}, \\
F_{ab}^{(1)} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
F_{ab}^{(2)} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
F_{a\beta} &= 0.
\end{align*}
\]

(59)

Then, using the fact that \(\Pi(g) = 0\) and utilizing formulas \[14\] and \[16\], together with \[27\], the original \(\sigma\)-model is, in this case, obtained to be of the form

\[
S = \frac{1}{2} \int d\sigma^+ d\sigma^- \left[ -y_1 e^{\tau_1} (\partial_+ x_1 \partial_- x_2 - \partial_+ x_2 \partial_- x_1) + e^{\tau_2} (\partial_+ y_1 \partial_- y_2 + \partial_+ y_2 \partial_- y_1) - \partial_+ x_1 \partial_- y_2 - \partial_+ y_2 \partial_- x_1 \right].
\]

(60)

As it is seen, the action \[61\] is indeed identical to the action of \(H_4\) WZW model (cf. Appendix A).

Analogously to Case (1), the dual model is constructed on the manifold \(M \approx O \times \tilde{G}\). The Poisson bracket on \(\tilde{G}\) is given by formula \[41\], and thus by inserting \[59\] into equations \[32\]–\[34\] the dual couplings are computed to be

\[
\begin{align*}
\tilde{G}^{ab} &= \frac{1}{y_1 + \tilde{x}_2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
\tilde{F}_{a\beta}^{(1)} &= \frac{1}{y_1 + \tilde{x}_2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\tilde{F}_{a\beta}^{(2)} &= \frac{1}{y_1 + \tilde{x}_2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{align*}
\]

(61)

Finally, inserting these into formula \[41\] the dual \(\sigma\)-model is obtained to be in the following form

\[
\tilde{S} = \frac{1}{2} \int d\sigma^+ d\sigma^- \left[ \partial_+ \tilde{x}_1 \partial_- y_1 + \partial_+ y_1 \partial_- \tilde{x}_1 \\
+ \partial_+ \tilde{x}_2 \partial_- y_2 + \partial_+ y_2 \partial_- \tilde{x}_2 - \partial_+ \tilde{x}_2 \partial_- y_1 - \partial_+ y_1 \partial_- \tilde{x}_2 \right].
\]

(62)

The line element and antisymmetric tensor field corresponding to action \[62\] can be cast in the forms

\[
\begin{align*}
\tilde{d}s^2 &= \frac{2}{(y_1 + \tilde{x}_2)} (d\tilde{x}_1 \, dy_1 + d\tilde{x}_2 \, dy_2), \\
\tilde{B} &= \frac{1}{(y_1 + \tilde{x}_2)} (d\tilde{x}_1 \wedge d\tilde{x}_2 - dy_1 \wedge dy_2).
\end{align*}
\]

(63) (64)

By using \[63\] and \[64\] the conformal invariance conditions of the model \[62\] are satisfied with zero cosmological constant and dilaton field that supports the dual background is found to be

\[
\tilde{\phi} = c_0 + c_1 \log(y_1 + \tilde{x}_2),
\]

(65)

where \(c_0\) and \(c_1\) are the constants of integration. On the other hand, it is interesting to note that if we write \[66\] as \(e^{-\tilde{\phi}} = \theta_0/(y_1 + \tilde{x}_2)\), then metric \[66\] may be expressed as

\[
\tilde{G}_{\mu\nu} = e^{-\tilde{\phi}} \eta_{\mu\nu},
\]

(66)

in which \(\eta_{\mu\nu} = 2d\tilde{x}_1 \, dy_1 + 2d\tilde{x}_2 \, dy_2\). The formula \[66\] indicates a conformal transformation \(7\) between the dual metric \(\tilde{G}_{\mu\nu}\) and flat metric \(\eta_{\mu\nu}\), and \(e^{-\tilde{\phi}}\) is a smooth, non-vanishing function of the spacetime which is called a conformal factor. We note that the conformal transformations do change geometry and they are entirely different from coordinate transformations. This is crucial since conformal transformations may lead to a different physics \[34\]. If we use the coordinate transformation

\[
\tilde{x}_i = m, \quad \tilde{x}_2 = a_+, \quad y_1 = e^{-\nu} - a_+, \quad y_2 = a_+ + m,
\]

(67)

then the dual background can be cast to \[16\]

\[
\begin{align*}
\tilde{d}s^2 &= -2dn \, dm + 2e^\nu da_+ \, da_-, \\
\tilde{B} &= -a_+ e^n \, da_+ \wedge da_-, \\
\tilde{\phi} &= c_0 + c_1 n,
\end{align*}
\]

(68)

where \(c_0\) and \(c_1\) are arbitrary constants. Here we have ignored the total derivative terms that appeared in the \(\tilde{B}\)-field part. Indeed, the solution \[68\] is identical to the background of the original \(\sigma\)-model action \[61\]. Thus, we showed that the \(H_4\) WZW model does remain invariant under the non-Abelian T-duality transformation, that is, the model is self-dual.

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6 Here we have set \(\theta_0 = 1\).

7 The conformal transformations shrink or stretch the distances between the two points described by the same coordinate system \(x^\mu\) on the manifold \(M\), but they preserve the angles between vectors which lead to a conservation of the (global) causal structure of the manifold \[34\].
B. The $GL(2, \mathbb{R})$ WZW model from T-dualizable $\sigma$-models and its dual pair

We shall show that the original $\sigma$-model \cite{27} on the $2 + 2$-dimensional manifold $M \approx O \times G$ can be equalled to the $GL(2, \mathbb{R})$ WZW model. Similar to previous examples, the isometry group $G$ that is being dualized is $A_2$. The only difference is in choosing the spectator-dependent background matrices. In this regard, the non-Abelian T-dual geometry of the $GL(2, \mathbb{R})$ WZW model is determined.

1. The original $\sigma$-model as the $GL(2, \mathbb{R})$ WZW model

Here, we choose

\begin{align*}
E_0^{ab} &= \begin{pmatrix} 0 & \frac{1}{2} e^{-2y_1} & \frac{1}{2} e^{-2y_1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & F_{a\beta} &= \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}, \\
F^{(1)}_{a\beta} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & F^{(2)}_{a\beta} &= \begin{pmatrix} 0 & e^{-2y_1} \\ e^{-2y_1} & 0 \end{pmatrix},
\end{align*}

where $b$ is a non-zero real constant. Inserting relations \cite{69} into \cite{13-10} and then using \cite{40} together with \cite{27}, the original $\sigma$-model is worked out to be

\begin{align*}
S &= \frac{1}{2} \int ds^+ \cdot ds^- \left[ \partial_{+} y_1 \partial_{-} y_1 + b \partial_{+} y_2 \partial_{-} y_2 ight. \\
&\quad \left. + \frac{1}{2} e^{x_1-2y_1} \left( \partial_{+} x_1 \partial_{-} x_2 + \partial_{+} x_2 \partial_{-} x_1 \right) + e^{x_1-2y_1} \left( \partial_{+} y_1 \partial_{-} x_2 - \partial_{+} x_2 \partial_{-} y_1 \right) \right].
\end{align*}

Now, if one uses the following coordinate transformation

\begin{align*}
e^{x_1} = \theta_-, & \quad x_2 = \theta_+, & \quad y_1 = \theta_3, & \quad y_2 = \theta,
\end{align*}

then, action (70) becomes

\begin{align*}
S &= \frac{1}{2} \int ds^+ \cdot ds^- \left[ b \partial_{+} \theta \partial_{-} \theta + \partial_{+} \theta \partial_{-} \theta_3 \\
&\quad + \frac{1}{2} e^{-2y_3} \left( \partial_{+} \theta_3 \partial_{-} \theta_3 + \partial_{+} \theta_3 \partial_{-} \theta_3 \right) + \theta e^{-2y_3} \left( \partial_{+} \theta_3 \partial_{-} \theta_3 - \partial_{+} \theta_3 \partial_{-} \theta_3 \right) \right].
\end{align*}

Using the integration by parts over the fourth term of action, it is concluded that the action (72) is nothing but the action of WZW model based on the Lie group $GL(2, \mathbb{R})$ (cf. Appendix A). Hence, the original $\sigma$-model \cite{70} can be described as an exact CFT.

The line element and the antisymmetric tensor field corresponding to action (70) are, respectively, given by

\begin{align*}
ds^2 &= dy_1^2 + b dy_2^2 + e^{x_1-2y_1} dx_1 \cdot dx_2, \\
B &= -e^{x_1-2y_1} dx_2 \wedge dy_1.
\end{align*}

To better understand action (70) we diagonalize the corresponding metric. Let

\begin{align*}
e^{x_1} &= \frac{1}{l} \left( t - l \varphi \right), & \quad x_2 &= -l \left( t + l \varphi \right), \\
e^{y_1} &= \frac{1}{r}, & \quad y_2 &= z,
\end{align*}

where $l$ is constant with the dimension of length. Then, the metric (73) and the field strength corresponding to the $B$-field (74) shall, respectively, become

\begin{equation}
ds^2 = \frac{r^2}{l^2} dt^2 + r^2 d\varphi^2 + \frac{1}{r^2} dr^2 + b dz^2,
\end{equation}

\begin{equation}
H = \frac{r}{l} dt \wedge d\varphi \wedge dr.
\end{equation}

The metric (76) describes a four-dimensional Lorentz-signature spacetime if $b$ is considered positive and when $b$ is negative, the metric has $(2, 2)$-signature. One immediately finds that the scalar curvature of the metric is $\mathcal{R} = -6$, and the non-zero components of $\mathcal{R}_{\mu\nu}$ are $\mathcal{R}_{\nu\nu} = 2g_{\nu\nu}$ where $n = (t, \varphi, r)$. Since $\mathcal{R}_{\nu\nu} = 0$, this spacetime cannot be described as an $AdS_4$ space. On the other hand, the metric is a direct product of $\mathbb{R}$ associated with the coordinate $z$ and the three-dimensional metric of $(t, \varphi, r)$, which is nothing but the $AdS_3$ space. Hence, the metric (76) corresponds to $AdS_3 \times \mathbb{R}$. Furthermore, using (76) and (77) it is concluded that the only non-zero components of $(H^2)_{\mu\nu}$ are $(H^2)_{tt} = (2r^2)/l^2, (H^2)_{\varphi\varphi} = -2r^2$ and $(H^2)_{rr} = -2/l^2$. Finally, one verifies the equations (48a) and (48b) with the following dilaton field

\begin{equation}
\phi = \zeta_0 + \zeta_1 z,
\end{equation}

where $\zeta_0$ and $\zeta_1$ are integration constants. Also, computing $H^2 = -6$ the dilatonic contribution in equation (48c) is satisfied provided that $\zeta_1^2 = b(2A - 4)$. Notice that the metric (76) has no horizon and no curvature singularity. Indeed, this solution is everywhere regular including $r = 0$. Consider now two killing vectors $\partial/\partial t$ and $\partial/\partial \varphi$ corresponding to the time translational and the rotational isometries of the metric (76), respectively. The killing field $\partial/\partial t$ becomes null at $r = 0$ and it is time-like for the whole range $r > 0$, while the killing field $\partial/\partial \varphi$ is everywhere space-like except for $r = 0$. In addition, there is another killing field such as $(1/b) \partial/\partial z$ so that it is time-like for $b < 0$, and remains space-like for $b > 0$.

2. The dual $\sigma$-model

Similar to the construction of dual $\sigma$-models for $H_4$ WZW, the dual manifold is, here, assumed to be $M \approx O \times G$ in which $G = 2A_1$. So, the Poisson structure on $2A_1$ does follow the relation (11). In order to obtain the dual $\sigma$-model for $GL(2, \mathbb{R})$ WZW, we use the action (61). The dual coupling matrices can be obtained by inserting (41) and (69) into (72-75). They are then read

\begin{align*}
\tilde{E}^{ab} &= \begin{pmatrix} 0 & \frac{1}{l^2} e^{2y_1} \\ 0 & 0 \end{pmatrix}, & \tilde{F}^{ab} &= \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}, \\
\tilde{F}^{(1)a}_{\beta} &= \begin{pmatrix} 0 \\ -\frac{1}{l^2} e^{2y_1} \end{pmatrix}, & \tilde{F}^{(2)a}_{\beta} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{align*}

Finally, the dual $\sigma$-model to the $GL(2, \mathbb{R})$ WZW model is found to be
where $\bar{\Delta} = \frac{1}{4}e^{-2y_1} - \tilde{x}_2^2$. The line element and antisymmetric field corresponding to this action may be expressed as

$$\tilde{d}s^2 = dy_1^2 + b dy_2^2 + \frac{e^{-2y_1}}{\Delta} (dx_1 dx_2 - e^{-2y_1} dy_1 dy_1), \quad (81)$$

$$\tilde{B} = - \frac{x_2}{\Delta} (dx_1 \wedge dx_2 - e^{-2y_1} dy_1 \wedge dy_1). \quad (82)$$

The scalar curvature of the metric is

$$\tilde{R} = - \frac{2(11e^{-4y_1} + 28x_2 e^{-2y_1} + 12x_2^2)}{(e^{-2y_1} - 2x_2^2)^2}. \quad (83)$$

As it can be seen from formulas $(81)$ and $(82)$, the region $\tilde{x}_2 = \frac{1}{2}e^{-2y_1}$ is a true curvature singularity (in what follows we will discuss the structure and asymptotic nature of the dual spacetime including the horizon and singularity). We deduce that the only non-zero component of the field strength corresponding to the $B$-field $(82)$ is $H_{x_1 x_2 y_1} = -(2e^{-2y_1})/(e^{-2y_1} - 2\tilde{x}_2)^2$; consequently $\tilde{H}^2 = -6(e^{-2y_1} + 2\tilde{x}_2)^2/(e^{-2y_1} - 2\tilde{x}_2)^2$. Thus, equations $(83a)$ and $(83b)$ are satisfied by the new dilaton field

$$\tilde{\phi} = \lambda_0 + \lambda_1 y_2 + \log \left( \frac{2\tilde{x}_2 + e^{-2y_1}}{2\tilde{x}_2 - e^{-2y_1}} \right), \quad (84)$$

where $\lambda_0$ and $\lambda_1$ are arbitrary constants. Also, the dilatonic contribution in $(82)$ is vanished if the cosmological constant of the dual theory does satisfy in $\lambda_1^2 = b(2\Delta - 4)$.

As in the first example of subsection A, we now discuss the presence of anomalous terms breaking the proportionality between the original and dual string effective actions. The string effective Lagrangians corresponding to the $\sigma$-models $(70)$ and $(80)$ are found to be

$$\mathcal{L}_{eff} = 4\Delta - 8, \quad (85)$$

$$\tilde{\mathcal{L}}_{eff} = 4\bar{\Delta} - 8 - \frac{64\tilde{x}_2 e^{2y_1}}{(1 - 2\tilde{x}_2 e^{2y_1})^2}. \quad (86)$$

The last term of equation $(86)$ is not invariant under PL T-duality transformation and therefore the two Lagrangians $\mathcal{L}_{eff}$ and $\tilde{\mathcal{L}}_{eff}$ are not equal. This anomaly is due to the non-vanishing traces of the structure constants of the double $(A_2, 2A_1)$; furthermore, the dilaton field obtained in $(84)$ does not follow the transformation $(83)$. The dilaton field $(84)$ is well behaved for the ranges $\tilde{x}_2 < -\frac{1}{2}e^{-2y_1}$ and $\tilde{x}_2 > \frac{1}{2}e^{-2y_1}$. We also note that a dilaton field can easily be found for the range $-\frac{1}{2}e^{-2y_1} < \tilde{x}_2 < \frac{1}{2}e^{-2y_1}$ by shifting $\lambda_0$ by an imaginary constant $(\lambda_0 \rightarrow \lambda_0 + i\pi)$.

For the range $\tilde{x}_2 < -\frac{1}{2}e^{-2y_1}$ we consider $\tilde{x}_2 + \frac{1}{2}e^{-2y_1} = -e^X$. Then, we introduce the following coordinate transformation

$$\tilde{x}_1 = Y + \frac{1}{2}(W + e^w), \quad \tilde{x}_2 = -e^X (1 + \frac{e^{-w}}{2}),$$

$$y_1 = \frac{1}{2}(W - X), \quad y_2 = V. \quad (87)$$

Under this transformation, the dual background now looks as follows

$$\tilde{d}s^2 = b dV^2 + \frac{1}{4}(dW^2 + dX^2) + \frac{1}{e^w + 1} \ dX \ dY, \quad (88)$$

$$\tilde{B} = \frac{2e^w + 1}{2(e^w + 1)} \ dX \wedge dY, \quad (89)$$

$$\tilde{\phi} = \lambda_0 + \lambda_1 V + \log \left( \frac{e^w}{e^w + 1} \right). \quad (90)$$

Here we have ignored the terms concerning $\tilde{B}$-field which are contributed to the Lagrangian as the total derivatives. Notice that there is no singularity for the metric $(88)$. In fact, this was expected since the solutions $(81), (82)$ and $(84)$ are, in this case, defined only for the range $\tilde{x}_2 < -\frac{1}{2}e^{-2y_1}$, a region which is located out of the range $\tilde{x}_2 + \frac{1}{2}e^{-2y_1} < 0$. As explained above, the true singularity of the metric $(84)$ occurs at $\tilde{x}_2 = \frac{1}{2}e^{-2y_1}$, a region which is located out of the range $\tilde{x}_2 + \frac{1}{2}e^{-2y_1} < 0$. The background $(88), (89)$ can be simplified by performing a coordinate transformation. Let us now consider the transformation $e^w = 1/(r - 1)$ so that it requires that $1 < r < \infty$. In addition, we introduce the following linear transformation

$$X = -2(t + \frac{x}{\sqrt{3}}), \quad Y = (t - \frac{x}{\sqrt{3}}), \quad V = z. \quad (91)$$

By applying the above transformation to the solutions $(88), (89)$ and $(90)$ one obtains the forms of the dual spacetime metric, antisymmetric field strength and dilas-
ton field in new coordinate base \(\{t, x, r, z\}\) as

\[
d\hat{s}^2 = -(1 - \frac{2}{r}) dt^2 + (1 - \frac{2}{3r}) dx^2 + \frac{2}{\sqrt{3}} dt dx + (1 - \frac{1}{r})^{-2} \frac{dr^2}{4r^2} + b \, dz^2, \tag{92}
\]

\[
\hat{H}_{\text{D3}} = \frac{1}{\sqrt{3} \, r^2}, \tag{93}
\]

\[
\phi = \lambda_0 + \lambda_1 \, z - \log r. \tag{94}
\]

We note that this solution is valid only for the range \(\hat{x}_2 + \frac{1}{2} e^{-2y_1} < 0\) or \(1 < r < \infty\). In order to have a solution with the range \(0 < r < 1\) we have to look at the second case, where \(\hat{x}_2 - \frac{1}{2} e^{-2y_1} > 0\). In this case it is assumed that \(\hat{x}_2 - \frac{1}{2} e^{-2y_1} = e^X - e^{-2y_1}\), for which \(X + 2y_1 > 0\). Analogously, we introduce the transformation

\[
\begin{align*}
\hat{x}_1 &= Y - \frac{1}{2} (e^w - W), \\
\hat{x}_2 &= e^X (1 - \frac{e^{-w}}{2}), \\
y_i &= \frac{1}{2} (W - X), \\
y_i &= V,
\end{align*} \tag{95}
\]

in which \(W = X + 2y_1 > 0\), i.e., \(e^w > 1\). We then define the transformation \(e^w = 1/(1 - r)\) so that it requires that \(0 < r < 1\). Using these results and also utilizing the linear transformation \(91\) one concludes that the solution given by equations \(81, 82\) and \(83\) is nothing but the solution given by \(92, 94\). Thus, the obtained solutions to both the valid ranges \(\hat{x}_2 - \frac{1}{2} e^{-2y_1} < 0\) and \(\hat{x}_2 - \frac{1}{2} e^{-2y_1} > 0\) can be expressed as a solution in the form of equations \(92\), \(94\) only with \(0 < r < \infty\). Analogously, for the range \(-\frac{1}{2} e^{-2y_1} < \hat{x}_2 < \frac{1}{2} e^{-2y_1}\) one can consider \(\hat{x}_2 + \frac{1}{2} e^{-2y_1} = e^X\) to obtain the same results presented in \(92\), \(94\).

One can simply check that the solution \(92\)-\(94\) does satisfy the equations \(48a\)-\(48c\). Considering this solution for the whole spacetime, \(0 < r < \infty\), one sees that the metric components \(92\) are ill behaved at \(r = 0\) and \(r = 1\). Looking at the scalar curvature, which is \(R = 2(4r - 7)/r^2\), we find that \(r = 0\) is a curvature singularity. Notice that the singularity at \(r = 0\) corresponds to the same true singularity at the region \(\hat{x}_2 = \frac{1}{2} e^{-2y_1}\) which mentioned above. We furthermore see that \(r = 1\) is also an event horizon. The cross term appeared in the metric is constant and thus for large \(r\) one can show that the metric is asymptotically flat. For large \(r\) the metric \(92\) approaches the following asymptotic solution

\[
d\hat{s}^2 = -dt^2 + dx^2 + \frac{2}{\sqrt{3}} dt dx + \frac{dr^2}{4r^2} + b \, dz^2. \tag{96}
\]

Performing a convenient coordinate transformation, the metric \(93\) can be simply diagonalized. We also note that the sign of \(b\) changes the signature of metric. If we introduce the new coordinates \((\hat{t}, \hat{x}, \hat{r}, z)\) by the transformation

\[
\begin{align*}
r &= e^{2\hat{t}}, \\
t &= \frac{\sqrt{3}}{2} \hat{t}, \\
x &= \hat{x} - \frac{\hat{t}}{2}, \\
z &= \frac{z}{\eta},
\end{align*} \tag{97}
\]

then, \(96\) will become

\[
d\hat{s}^2 = \begin{cases}
dt^2 + dx^2 + dt^2 + dz^2 & \text{for } b = \eta^2, \\
dt^2 + dx^2 + dt^2 - dz^2 & \text{for } b = -\eta^2.
\end{cases} \tag{98}
\]

As it is seen for \(b > 0\) the metric has \((1, 3)\)-signature, while the signature is \((2, 2)\) when \(b\) is negative. Thus, we have shown that the non-Abelian T-duality transformation (here as the PL T-duality on a semi-Abelian double) changes the asymptotic behavior of solutions from \(AdS_3 \times \mathbb{R}\) to flat space.

\[\boxed{C. \text{ The non-Abelian T-duality of non-critical Bianchi type III string cosmological model (the GL(2, \mathbb{R}) WZW model) }}\]

In this subsection, we show that the non-critical Bianchi type III string cosmology solution with a nonvanishing field strength and an appropriate dilaton field can be described by the \(GL(2, \mathbb{R})\) WZW model. In fact, we shall obtain the \(GL(2, \mathbb{R})\) WZW model from a T-dualizable \(\sigma\)-model constructed on a \(3 + 1\)-dimensional manifold \(M \approx O \times G\), in which \(G\) is three-dimensional decomposable Lie group \(A_2 \oplus A_1\) acting freely on \(M\). In this case, the non-Abelian T-duality of the model is studied here. The dual Lie group \(G\) is considered to be three-dimensional Abelian Lie group \(3A_1\). We note that the Lie algebra \(A_2 \oplus A_1\) is isomorphic to the Lie algebra of Bianchi type III. Hence, six-dimensional Lie algebra of the Drinfeld double \((A_2 \oplus A_1, 3A_1)\) is defined by the following commutation relations:

\[
\begin{align*}
[T_1, T_2] &= T_3, \\
[T_3, T_1] &= T_2, \\
[T_1, T_2] &= -T^1, \\
[T_2, T^1] &= T_3, \\
[T_3, T_1] &= 0.
\end{align*} \tag{99}
\]

Taking a convenient element of the Lie group \(A_2 \oplus A_1\) such as \(g = e^{(\ln x_1)T_1} e^{z T_2} e^{x_3 T_3}\) we immediately find that \(R_{\alpha\beta} = \partial_\alpha x_1 / x_1, R_{\alpha} = x_1 \partial_\alpha x_1, R_\alpha = \partial_\alpha x_1\). In order to study the non-Abelian T-duality of non-critical Bianchi type III string cosmological model, we consider the orbit \(O\) as a one-dimensional space with time coordinate \(y^0 = \{t\}\). Now, one can choose the spectator-dependent background matrices as

\[
\begin{align*}
E^{a\beta}_{ab} &= \begin{pmatrix}
0 & -a_0^2 e^{-2t} & 0 \\
0 & 0 & 0 \\
a_0^2 e^{-2t} & 0 & 0
\end{pmatrix},
F^{\alpha\beta}_{ab} &= \begin{pmatrix}
0 \\
0 \\
a_0^2 e^{-2t}
\end{pmatrix},
\end{align*} \tag{100}
\]

for some constants \(a_0, b\), and then use \(27\) to obtain the following background

\[
\begin{align*}
ds^2 &= -a_0^2 dt^2 - a_0^2 e^{-2t} dx_1 \, dx_2 + b \, dx_3^2, \\
B &= a_0^2 x_1 \, e^{-2t} \, dx_2 \wedge dt.
\end{align*} \tag{101}
\]

Comparing \(101\) and the general form of the string cosmology metric

\[
ds^2 = -\eta_{mn}(t) dt^2 + \sum_{a,b=1}^3 R^a_\mu R^b_\nu g_{ab}(t) \, dx^\mu \, dx^\nu, \tag{103}
\]

where

\[
\eta_{mn}(t) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

and

\[
R_{\mu
\nu} g_{ab}(t) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]
one concludes that (101) is nothing but the Bianchi type III string cosmology metric. This metric has (2, 2)-signature if $b$ is considered positive. One can easily check that the metric (101) and the field strength corresponding to the B-field (102) ($H_{1 + x_3} = a_o^2 e^{-2t}/2$) along with the dilaton field $\phi = v_o + u_i x_3$ (for some constants $v_o, v_1$) make up a solution for the vanishing of the one-loop $B$-functions equations (48a)-(48c). It is also interesting to make up a solution for the vanishing of the one-loop $\phi$-function equations (48a)-(48c). It is also interesting to note that the corresponding action to (101) and (102) is equivalent to the $GL(2, \mathbb{R})$ WZW model. This means that the obtained background can be described as an exact CFT.

The dual model is constructed on 3 + 1-dimensional manifold $\tilde{M} \approx O \times G$ with $G = 3A_1$. Finally, using (99) and (100) together with equations (32)-(35) the dual background is obtained to be

$$d\tilde{s}^2 = -\tilde{a}_o^2 dt^2 + \frac{1}{\tilde{b}} d\tilde{x}_3^2 + \frac{a_s^2 e^{-2t}}{\tilde{x}_2^2 - \frac{a_s^4}{4} e^{-4t}} (d\tilde{x}_1 d\tilde{x}_2 + a_s^2 e^{-2t} d\tilde{x}_1 dt),$$

$$\tilde{B} = \frac{\tilde{x}_2}{\tilde{x}_2^2 - \frac{a_s^4}{4} e^{-4t}} (d\tilde{x}_1 \wedge d\tilde{x}_2 + a_s^2 e^{-2t} d\tilde{x}_1 \wedge dt).$$

The dilaton field that supports the dual background is obtained in the following form

$$\tilde{\phi} = \vartheta_0 + \vartheta_1 \tilde{x}_3 - \log \left( \frac{a_o^2 + 2\tilde{x}_2 e^{2t}}{a_o^2 - 2\tilde{x}_2 e^{2t}} \right),$$

where $\vartheta_0, \vartheta_1$ are some constants.

IV. CONFORMAL INVARIANCE OF THE T-DUAL MODELS UP TO TWO-LOOP ORDER (FIRST ORDER IN $\alpha'$)

So far, we have been concerned with the conformal invariance of the T-dual models up to one-loop order (zeroth order in $\alpha'$). As mentioned in Sec. III, the conditions for conformal invariance of the $\sigma$-model with action (3) can be interpreted as field equations for $G_{\mu\nu}, B_{\mu\nu}$ and $\phi$ of the string effective action (24, 25). These equations to the first order in $\alpha'$ take the following form (26)

$$\mathcal{R}_{\mu\nu} - (H^2)_{\mu\nu} + \nabla_\mu \nabla_\nu \phi + \frac{1}{2} \alpha' \left[ \mathcal{R}_{\mu\rho\sigma\lambda} \mathcal{R}^{\rho\sigma\lambda} + 2 \mathcal{R}_{\mu\rho\sigma\nu} (H^2)_{\rho\sigma} + 2 \mathcal{R}_{\rho\sigma\mu\nu} (H^2)_{\rho\sigma} + \frac{1}{3} (\nabla_\mu H_{\rho\sigma})(\nabla_\nu H^{\rho\sigma}) \right] - \frac{1}{2} \nabla_\mu H^{\rho\sigma} H_{\nu\rho\sigma} = 0,$$

$$\nabla^\lambda H_{\lambda\mu\nu} - (\nabla^\lambda \phi) H_{\mu\nu} + \alpha' \left[ \nabla^\lambda H^{\rho\sigma} [R^\nu_{\nu\lambda\rho\sigma} - (\nabla_\lambda H_{\mu\nu}) (H^2)_{\lambda\rho\sigma} - 2 (\nabla^\lambda H^{\rho\sigma}_{\nu\lambda}) H_{\nu\lambda\rho\sigma} \right] + \mathcal{O}(\alpha'^2) = 0,$$

$$2\Lambda + \nabla^2 \phi' - (\nabla^2 \phi)^2 + \frac{2}{3} H^2 - \alpha' \left[ \frac{1}{4} \mathcal{R}_{\mu\rho\sigma\lambda} \mathcal{R}^{\mu\rho\sigma\lambda} - \frac{1}{3} (\nabla_\mu H_{\nu\rho\sigma}) (\nabla^\lambda H^{\mu\rho\sigma}) - \frac{1}{2} H^{\mu\nu} H_{\nu\rho\sigma} \right] + \mathcal{O}(\alpha'^2) = 0,$$

where $\phi' = \phi + \alpha' q H^2$ for some coefficient $q \in \mathbb{R}$, $(H^2)^{\mu\nu} = H^{\mu\sigma} H_{\rho\nu}^{\sigma}$ and $\mathcal{R}_{\mu\nu\rho\lambda}$ is the Riemann tensor field. We note that round brackets denote the symmetric part on the indicated indices whereas square brackets denote the antisymmetric part. Below using the above equations we check the conformal invariance conditions of the T-dual models up to two-loop order (first order in $\alpha'$). In fact, we introduce new solutions for two-loop $B$-function equations of the $\sigma$-model with a non-vanishing field strength $H$ and the dilaton field in both cases of the absence and presence of a cosmological constant $\Lambda$.

- As shown in subsection A of Sec. III, the background of the original $\sigma$-model (13) is given by the formulas (44) and (45) so that this model is equivalent to the $H_4$ WZW model. Therefore, it should be conformally invariant. In the case of this model, the only non-vanishing component of the Riemann tensor is $\mathcal{R}_{z_1 z_2 z_2 y_1} = -(e^{z_1 + y_1})/4$. Moreover, the only non-vanishing component of $(H^2)^{\mu\nu}$ is $(H^2)_{y_2 y_2} = -1/2$ and all components of $\nabla_\lambda H_{\mu\rho\sigma}$ vanish. Hence using these results, the field equations (107a)-(107c) are satisfied for the metric (44) and the tensor field (45) together with the dilaton field (49) and zero cosmological constant as this was expected.

- In order to investigate the conformal invariance conditions of the dual model to the $H_4$ WZW (the $\sigma$-model (53)) up to the first order in $\alpha'$, we first find that the only non-vanishing components of $(\tilde{H}^2)^{\mu\nu}$ and $(\tilde{H}^2)^{\nu\mu}$ are $(\tilde{H}^2)_{y_1 y_1} = -(e^{y_1} + \tilde{x}_2)^2/2(e^{y_1} - \tilde{x}_2)^2$ and $(\tilde{H}^2)_{y_2 y_2} = (\tilde{H}^2)_{y_2 y_2}$, respectively. Also, the only non-vanishing components of $\nabla_\lambda \tilde{H}_{\mu\rho\sigma}$ may be expressed as

$$\nabla_{\tilde{x}_2} \tilde{H}_{\tilde{x}_1 \tilde{x}_2 y_1} = \frac{2y_1}{(e^{y_1} + \tilde{x}_2)(e^{y_1} - \tilde{x}_2)^3},$$

$$\nabla_{y_1} \tilde{H}_{\tilde{x}_1 \tilde{x}_2 y_1} = -\frac{\tilde{x}_2 e^{y_1}}{(e^{y_1} + \tilde{x}_2)(e^{y_1} - \tilde{x}_2)^3}. \quad (108)$$
Using these results together with the given data for this model in subsection A of Sec. [11], one verifies the field equations (107a) and (107b) for the metric \( g_{\mu \nu} \) and the tensor field \( \sigma \) along with the dilaton field \( \tilde{\Lambda} \). The equation (107c) is also satisfied with \( \tilde{\Lambda} = 0 \).

- Under the coordinate transformation (75), the background of the original \( \sigma \)-model (70) was represented by (76) and (77). It was shown that resulting background as an exact CFT satisfies the vanishing of the one-loop \( B \)-functions equations (48a)-(48c) with the dilaton field (78). Using the expressions (76) and (77) for the background fields one may verify that the only non-vanishing components of Riemann tensor are \( \mathcal{R}_{\sigma \sigma \sigma} \) along with the dilaton field (78). The equation (107c) is also satisfied with \( \tilde{\Lambda} = 0 \).

As mentioned in the preceding section, the dual model of the \( GL(2, \mathbb{R}) \) WZW (equations (92)-(94)) does satisfy the vanishing of the one-loop \( B \)-functions equations. Unfortunately, this background does not satisfy the equations for the two-loop \( B \)-functions. One can show that for this background all equations (107a)-(107c) are satisfied except for the components of \( B^r_{\mu \nu} \), \( B^t_{\mu \nu} \), and \( B^2 \).

V. SUMMARY AND CONCLUDING REMARKS

Using the PL T-duality approach in the presence of spectators we have constructed some non-Abelian T-dualizable \( \sigma \)-models on 2 + 2-dimensional target manifolds \( M \approx O \times G \) and \( M \approx O \times \tilde{G} \), where \( G \) and \( \tilde{G} \) are two-dimensional real non-Abelian and Abelian Lie groups, respectively. We have shown that the original \( \sigma \)-models are equivalent to the \( H_4 \) and \( GL(2, \mathbb{R}) \) WZW models. In this way, we could obtain some new T-dual backgrounds for these WZW models. The most interesting feature of our results is the invariance of the \( H_4 \) WZW model under the non-Abelian T-duality. We have shown that the \( GL(2, \mathbb{R}) \) WZW model as a T-dualizable \( \sigma \)-model is equivalent to \( AdS_3 \times \mathbb{R} \) space and has no horizon and no curvature singularity, while the dual spacetime of the \( GL(2, \mathbb{R}) \) WZW model is stationary and asymptotically flat and has a single horizon and a curvature singularity. Moreover, it was shown that for the line element (70), the Killing vectors \( \partial/\partial t \) and \( (1/b) \partial/\partial z \) with \( b < 0 \) are time-like. Analogously, one can show that the dual line element (92) possesses three independent Killing vectors \( \sqrt{3}\partial/\partial t \), \( 3/2(\sqrt{3}\partial/\partial t - \partial/\partial x) \), and \((1/b) \partial/\partial z \). The first two Killing vectors become time-like for the ranges \( r > 2 \) and \( r > 4/3 \), respectively. The last Killing vector stays everywhere time-like for \( b < 0 \). Hence, the duality has involved the time-like directions. In summary, in the case of the effect of the non-Abelian T-duality (here as the PL T-duality on a semi-Abelian double) on the \( GL(2, \mathbb{R}) \) WZW model three points have been highlighted.

1. The non-Abelian T-duality transformation has changed the asymptotic behavior of solutions from \( AdS_3 \times \mathbb{R} \) to flat space.
2. This transformation has related a solution with no horizon and no curvature singularity to a solution with a single horizon and a curvature singularity.
3. The duality has involved the time-like directions.

We have also obtained the non-critical Bianchi type III string cosmological model with a non-vanishing field strength from a T-dualizable \( \sigma \)-model and have shown that this model describes an exact CFT. Most importantly, we have discussed the conformal invariance of the T-dual \( \sigma \)-models such that the duals of the \( H_4 \) WZW model are conformally invariant up to the first order in \( \alpha' \), while the conformal invariance condition for the dual spacetime of the \( GL(2, \mathbb{R}) \) WZW model has only been satisfied up to zeroth order in \( \alpha' \).

As we have shown, all our models satisfy the vanishing of the one-loop Beta-functions equations. Therefore, each pair of them consists of two canonically equivalent models. Among these models, only (55) and (60) and their dual pairs (53) and (52), respectively, satisfy the equations for two-loop \( B \)-functions.

The findings of our study showed that 2 + 2-dimensional manifold \( M \approx O \times G \) with two-dimensional real non-Abelian Lie group \( G = A_2 \) is weathly. In addition to PL symmetric backgrounds constructed out in this paper one can obtain other string and gravitational backgrounds from mutually T-dualizable \( \sigma \)-models on manifold \( M \approx O \times \tilde{G} \) with \( G = A_2 \) when the dual manifold is \( M \approx O \times \tilde{G} \), with \( \tilde{G} \) = 2\( A_1 \). In this regard, the following further developments come to mind.

- **Plane-parallel (pp)-wave**: Homogenous plane wave is generally defined by the metric of the following form (52)
  \[ ds^2 = 2udv - A_{\mu}(u) X^{\mu} X^{\nu} du^2 + dX^2, \]
  where \( dX^2 \) is the standard metric on Euclidean space \( E^d \) and \( X \in E^d \). A special case of isotropic homogenous plane wave metric can be chosen by \( A_{\mu}(u) = \lambda(u) \delta_{\mu}^\nu \). Furthermore, for special choice of \( \lambda(u) = k/u^2 \), the metric becomes (57)
  \[ ds^2 = 2udv - \frac{k}{u^2}(x^2 + y^2)du^2 + dx^2 + dy^2, \]
  where \( k \) is an arbitrary real constant. The metric (111) does satisfy the conformal invariance conditions up to the first order in \( \alpha' \), equations (107a)-(107c).
with zero field strength. The cosmological constant $\Lambda$ in this case vanishes and dilaton field is obtained to be

$$\phi = \gamma_0 + \gamma_1 u + 2 k \log u,$$

(112)

where $\gamma_0$ and $\gamma_1$ are the constants of integration. Since the field strength $H$ is zero, one can easily consider explicit expressions for the field $B$ in such a way that the terms concerning $B$-field in action of $\sigma$-model contribute to the Lagrangian as the total derivatives, which can be ignored. To obtain the non-Abelian T-dual geometry of pp-wave background in the PL T-duality approach with spectators, we first construct the original $\sigma$-model corresponding to the pp-wave metric (111). In this case, a convenient choice of the spectator-dependent matrices may be expressed as

$$F^{(1)}_{\alpha\beta} = 0, \quad F^{(2)}_{\alpha\beta} = 0.$$  

(113)

Inserting (113) into equations (113)-(116) and noting that $\Pi^{\alpha\beta}(g)$ is zero, the action of original $\sigma$-model (27) yields

$$S = \frac{1}{2} \int d\sigma^+ d\sigma^- \left[ \partial_+ y_1 \partial_- y_1 + \partial_+ y_2 \partial_- y_2 + e^x (\partial_+ x_1 \partial_- x_2 + \partial_+ x_2 \partial_- x_1) - k(y_1^2 + y_2^2) \partial_+ x_1 \partial_- x_2 \right].$$  

(114)

Carrying out the coordinates transformation $x_1 \rightarrow \ln u, x_2 \rightarrow v, y_1 \rightarrow x, y_2 \rightarrow y$ one arrives at the pp-wave metric (111) from action (114). Thus, inserting (111) and (113) into equations (32)-(35) and then using (113) one can obtain a non-Abelian T-dual $\sigma$-model to (114).

- **Gödel and Gödel-type metrics:** Among the known exact solutions of Einstein field equations gravity, the Gödel and Gödel-type metrics [38] play a special role. It was shown within the usual general relativity framework that the Gōdel metric can be considered as exact solutions in string theory for the full $O(a')$ action including both dilaton field $\phi$ and field strength $H$. Following Ref. [38], we assume that the dilaton field depends only on the $z$ coordinate, so $\phi(z) = \phi_0 + f z$ for some constants $\phi_0$, $f$. With this assumption and taking the zero field strength, $H = 0$, the field equations (107a) and (107b) are satisfied for the metric (115) in such a way that the inverse string tension $\alpha'$ has to satisfy relation $\alpha' = 4l^2 \beta$ only with $\beta = 1$ or $\beta = 3/4$. Finally, the field equation (107c) is satisfied if the following relation is held between the constants $f$, $\Lambda$ and $l$:

$$f^2 = \begin{cases} -\frac{1}{2} + 2\Lambda l^2 & \text{if } \beta = 1 \\ -\frac{3}{4} + 2\Lambda l^2 & \text{if } \beta = \frac{3}{4} . \end{cases}$$

In addition, one can check that the metric (115) for $\beta = 1/2$ (the original Gödel metric) along with the respective $B$-field and dilaton field

$$B = \frac{l^2}{2} r \, d\varphi \wedge dt,$$

$$\phi = \phi_0 + f z,$$

satisfy the field equations (107a)-(107c) provided that $\alpha' = -l^2/2$ and $f^2 = 1/2 + 2\Lambda l^2$. Now, by using the above results and by choosing the appropriate spectator-dependent background matrices we can construct a $\sigma$-model including the Gödel (Gödel-type) metric in the form (115) and the given $B$-fields. In this way, one can study the non-Abelian T-duality of the Gödel (Gödel-type) metric. We intend to address this problem in the future.

- **$AdS_4$ metric:** $AdS_4$ metric with radius $l$ and constant negative scalar curvature $\Lambda = -12/l^2$ is one of the maximal symmetric four-dimensional spacetimes. A simple form of this metric in coordinates $(z, x^+, x^-, \rho)$ is given by

$$ds^2 = \frac{l^2}{z^2} (dz^2 - dx^+ dx^- + d\rho^2).$$

(116)

Solving the field equations (107a)-(107c) for metric (116) one should be so lucky to obtain an appropriate field strength along with a dilaton field. Then he/she can study the non-Abelian T-duality of the $AdS_4$ metric.

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Appendix A: The WZW models based on the $H_4$ and $GL(2,\mathbb{R})$ Lie groups

In this appendix, we construct the WZW models based on the $H_4$ and $GL(2,\mathbb{R})$ Lie groups. To define a WZW model, in general, given a Lie algebra with generators $T_a$ and structure constants $f_{ab}^c$, one needs a non-degenerate ad-invariant symmetric bilinear form $\Omega_{ab} = < T_a , T_b >$ on Lie algebra $G$ so that it satisfies the following relation

$$f_{ab}^d \Omega_{dc} + f_{ac}^d \Omega_{db} = 0.$$  

(A1)
The WZW model based on a Lie group $G$ is defined on a Riemannian surface $\Sigma$ as a worldsheet by the following action [3]

$$I(g) = \frac{1}{2} \int d\sigma^+ d\sigma^- \Omega_{ab} L^+_a L^-_b$$

$$+ \frac{1}{12} \int_B d^3 \sigma \varepsilon^{\alpha\beta\gamma} L^\alpha_a L^\beta_b L^\gamma_c \Omega_{ad} f_{bc}^d, \quad (A2)$$

where $B$ is a three-manifold bounded by worldsheet $\Sigma$, and the components of the left invariant one-forms $L^a$'s are defined via $g^{-1} \partial_a g = L^a \gamma_a$ in which $g : \Sigma \rightarrow G$ is an element of Lie group $G$.

1. The $H_4$ WZW model

Before proceeding to construct the model, let us first introduce the oscillator Lie algebra $h_4$ of the Lie group $H_4$. The Lie algebra $h_4$ is generated by the generators $\{N, A_+, A_-, M\}$ with the following non-zero Lie brackets

$$[N, A_+] = A_+, \quad [N, A_-] = -A_-, \quad [A_-, A_+] = M. \quad (A3)$$

Using (A1) and (A3) one can simply get a non-degenerate ad-invariant bilinear form $\Omega_{ab}$ on $h_4$ as

$$\Omega_{ab} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad (A4)$$

In order to calculate the $L^a$'s on the Lie group $H_4$ we parameterize an element of $H_4$ as

$$g = e^{nM} e^{a_- A_-} e^{nN} e^{a_+ A_+}. \quad (A5)$$

Finally, the WZW action on the $H_4$ Lie group is worked out to be of the form [10]

$$I(g) = \frac{1}{2} \int d\sigma^+ d\sigma^- \left[ -\partial_+ n \partial_- m - \partial_+ m \partial_- n + e^n (\partial_+ a_+ \partial_- a_- + \partial_+ a_- \partial_- a_+) \\
+ a_+ e^n (\partial_+ a_+ \partial_- n - \partial_+ n \partial_- a_+) \right]. \quad (A6)$$

2. The $GL(2, \mathbb{R})$ WZW model

The $gl(2, \mathbb{R})$ Lie algebra is spanned by the generators $\{J_3, J_+, J_-, I\}$ which obey the following commutation rules

$$[J_3, J_+] = 2J_+, \quad [J_3, J_-] = -2J_-, \quad [J_+, J_-] = J_3, \quad [I, I] = 0. \quad (A7)$$

where $I$ is the central generator. We notice that $gl(2, \mathbb{R}) = sl(2, \mathbb{R}) \oplus u(1)$. Using (A7), a non-degenerate solution to (A1) is obtained to be of the form

$$\Omega_{ab} = \begin{pmatrix} 2a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}. \quad (A8)$$

for some non-zero constants $a$, $b$. In order to construct the WZW model based on the $GL(2, \mathbb{R})$ Lie group we parameterize the $GL(2, \mathbb{R})$ with coordinates $\{\theta_3, \theta_+, \theta_- \}$ so that its elements can be written as

$$g = e^{\theta_+ J_+} e^{\theta_3 J_3} e^{\theta_- J_-} e^{\theta I}. \quad (A9)$$

Using (A9), we then obtain

$$L^J_\pm = \partial_\mp \theta \partial_\pm \theta + \partial_\pm \theta, \quad L^J_3 = e^{-2\theta_3} \partial\_3 \theta + \partial\_3 \theta_3, \quad L^J_\pm = -\partial_\mp e^{-2\theta_3} \partial\_3 \theta - 2\partial_\pm \theta_3 + \partial_\pm \theta, \quad L^I_\pm = \partial_\pm \theta. \quad (A10)$$

Finally, the $GL(2, \mathbb{R})$ WZW action looks like

$$I(g) = \frac{1}{2} \int d\sigma^+ d\sigma^- \left[ b \partial_\pm \theta \partial_\mp \theta + 2a \partial_\pm \theta \partial_\mp \theta + e^{-2\theta_3} \partial_\mp \theta \partial_\pm \theta + \partial_\mp e^{-2\theta_3} \partial_\pm \theta \partial_\pm \theta \right]. \quad (A11)$$

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