Low-energy conductivity of single- and double-layer graphene from the uncertainty principle

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Abstract
The minimum conductivity value and the linear dependence of conductivity on charge density near the Dirac point in single- and double-layer graphene are derived from the energy–time uncertainty principle applied to ballistic charge carriers.

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1. Introduction
One of the most intriguing properties of both single- and double-layer (bilayer) graphene (for a review see [1]) is the finite value of conductivity in the limit of vanishing density of charge carriers that does not lead to any metal–insulator transition at low temperatures [2]. This minimum conductivity value is universal: it is the same for graphene samples with different mobilities and is practically independent of temperature in a broad temperature range. Experiments confirm that the minimum conductivity of both single- and double-layer graphene is equal to $\sigma_{\text{min}} = 4e^2/h$ (or $e^2/h$ per valley per spin) [2–4]. However, most theoretical works predict that the minimum conductivity of double-layer graphene is different from that of single-layer graphene, which is supposed to attain a value $\pi$ times less than the experiments show, i.e. of only $e^2/\pi h$ per valley per spin. This value has been found using several assumptions and methods: it was obtained in ideal wide graphene strips from the calculation of the mode-dependent transmission probability for different quantization conditions of transversal momenta [5, 6] (a similar approach for double-layer graphene leads to $\sigma_{\text{min}} = e^2/2h$ per valley per spin [7]), in planar systems with low carrier densities and linear dispersion relations studied with reduced (3+1)-dimensional gauge theories [8], and in disordered degenerate semiconductors by employing a mean-field theory [9], a technique that involves disorder-averaged propagators [10] (the same technique applied to double-layer graphene yields $\sigma_{\text{min}} = (3/4)e^2/h$ per valley per spin [11]) or a superfield representation of a weakly disordered system of two-dimensional Dirac fermions with a random mass with zero average [12].

Different calculation methods sometimes lead to different results for the same system. For example, for a system of non-interacting fermions it was predicted that $\sigma_{\text{min}} = (e^2/h)(j\pi/2)$ when calculated with the Kubo formula and $\sigma_{\text{min}} = (e^2/h)(4j/\pi)$ when an alternative definition of the longitudinal conductivity is used [13]; here $j = 1$ for single-layer graphene and $j = 2$ for double-layer graphene. Moreover, in the disordered graphene case (where the density of states at the Dirac point is finite), the type of disorder is important: in single-layer graphene away from the Dirac point, a linear dependence of conductivity on charge concentration was obtained for strong scatterers, while a logarithmic dependence characterizes weak scatterers [14], the minimum conductivity being $4e^2/\pi h$ for no disorder and weak disorder that preserves one of the chiral symmetries of the ideal graphene Hamiltonian. In addition, numerical simulations of the finite-size Kubo formula showed that the linear dependence of conductivity on carrier concentration near the Dirac point occurs for screened Coulomb scatterers but not for short-range scatterers [15]. In double-layer graphene, on the other hand, charge transport calculations in the self-consistent Born approximation revealed that the minimum conductivity is $(e^2/h)(24/\pi)$ in the weak-disorder regime and $(e^2/h)(8/\pi)$ in the strong-disorder limit [16].

A method to obtain the correct minimum conductivity of single- and double-layer graphene is important for a better understanding of the physics in this unusual material. The aim of this paper is to demonstrate that the experimentally confirmed universal value of the minimum conductivity in single- and double-layer graphene can be retrieved from a Landauer-type calculation of the conductivity if, in addition, an energy–time uncertainty relation is applied to the ballistic...
charge carriers. Moreover, it is shown that near the Dirac
point the conductivity depends linearly on charge carrier
concentration for both single- and double-layer graphene,
in agreement with experiments.

2. Conductivity calculation in single-layer graphene

Because the universal nature of the minimum conductivity
value per valley per spin is similar to that of the conductance
step in a two-dimensional electron gas described by the
Schrödinger equation and subject to a transverse constriction,
we choose a Landauer-type formalism to find the conductivity
of single-layer graphene. The Dirac-like Hamiltonian for
low-energy charge carriers has, in this case, the form

\[ H = \hbar v_F \sigma \cdot k = \hbar v_F \begin{pmatrix} 0 & k_x - i k_y \\ k_x + i k_y & 0 \end{pmatrix}, \]

(1)

where \( \sigma = (\sigma_x, \sigma_y) \) consists of Pauli matrices, \( k = (k_x, k_y) \)
is the momentum/wavevector of the charge carriers and \( v_F \) is the Fermi velocity (with \( c \) the speed of light), the
dispersion relation being given by \( E = \pm |\hbar v_F k| \).

Let us assume that charge transport occurs along the
x-direction when an electric potential is applied between two
leads separated by a distance \( L \). From (1) it follows that at
the Dirac point, i.e. for energy \( E = 0 \), the components \( \psi_1, \psi_2 \) of the
spinor wavefunction satisfy the equations [6]

\[ \begin{pmatrix} \frac{1}{\hbar} \frac{\partial}{\partial x} - \frac{1}{\hbar} \frac{\partial}{\partial y} \\ \frac{1}{\hbar} \frac{\partial}{\partial x} + \frac{1}{\hbar} \frac{\partial}{\partial y} \end{pmatrix} \psi_2(x, y) = 0 \]

(2)

and have solutions of the form \( \psi_1(x, y) \propto \exp(\pm i k_x x + i k_y y), \psi_2(x, y) \propto \exp(\pm i k_{x'} x + i k_{y'} y) \), with \( k_{x'} = -i k_x, k_{y'} = i k_y \). Unlike in [6], we do not consider imaginary wavevectors,
since these are ruled out by the band structure of single-layer
graphene, which has no energy gap. The only possibility is
then \( k_y = k_{y'} = 0 \) at the Dirac point. If the charge carriers in
the electrodes have Fermi energy \( E_F \) and wavenumber \( k_F \),
their x-component wavevector in the leads being \( \pm \sqrt{k_x^2 - k_y^2} \),
their, the transmission coefficient between the leads is

\[ T = \frac{\cos^2 \phi \cos^2 \varphi}{\cos^2 \phi \cos^2 \varphi (k_F L) + (\sin \phi \sin \varphi - 1)^2 \sin^2 (k_F L)} \]

(4)

is again unity at normal incidence, i.e. for \( k_y = 0 \) (see also [17]). Numerical simulations [17] show that
the transmission coefficient is higher than 0.9 for \( \phi \) angles up
to 20° if potential barriers are taken into account, a fact that
properly accounts for charge carriers that contribute to the
conductance in real devices and that are not exactly normal
to the leads. Therefore, \( T = 1 \) is a suitable approximation
of real situations (see also the description of the device used for
conductance measurements in [4]).

In order to derive the conductivity, we follow the treatment in [18]. More precisely, in single-layer graphene
the conductance of charge carriers with energy \( E \) and density
of states per valley per spin \( D(E) = |E|/[2\pi(\hbar v_F)^2] \) [19] is given by

\[ G(E) = \frac{e^2}{h} \frac{\pi(\hbar v_F)^2}{8} \]  

(5)

Normally, at the Dirac point \( G \) should vanish since \( D(0) = 0 \).
However, even at low temperatures the energy of ballistic
electrons that contribute to the measured current is not fixed.
In fact, the measurement process of electron transmission from
one lead to another takes a time interval of the order of electron traversal time, i.e. of the order of \( L/v_F \).
Since measurement processes cannot distinguish between the
different degenerate charge carriers in single-layer graphene,
the average of the time interval between successive detections of
charge carriers is \( \bar{t} = L/4v_F \) and the uncertainty in the energy
of ballistic charge carriers can be estimated as

\[ \Delta E \equiv \hbar/\bar{t} = 4\hbar v_F/L. \]

(6)

For \( L = 200 \text{ nm}, \Delta E = 13 \text{ meV}. \) Due to the uncertainty in
energy, the measured conductance is an average of \( G \) over an energy range equal to \( \Delta E \), an average that for low
temperatures and normal incidence, i.e. for \( T(E) = 1 \) and
\( dE/dk_x = \hbar v_F \), is given by

\[ \bar{G}(E) = \frac{e^2}{h} \frac{1}{8} \int_{E - \Delta E/2}^{E + \Delta E/2} |E| dE. \]

(7)

This conductance can be regarded as the conductance per unit
width of the graphene strip. The averaging procedure that
must be performed in the case of single-layer graphene is not
encountered in quantum wires in which electrons obey the
Schrödinger equation because in the latter case the density
of states does not depend on energy.

The result of the averaging process depends on the energy
value around which it is performed. Around the Dirac point,
\( \bar{G}(0) = e^2/4h v_F = e^2/Lh \), which leads to a conductivity
value per valley per spin of

\[ \sigma(0) = \sigma_{\text{min}} = e^2/h, \]

(8)
in agreement with experimental data. If the average is taken
away from the Dirac point, around \( E > \Delta E/2 \), we obtain

\[ \bar{G}(E) = (e^2/h)(E/h v_F), \]

while for \( 0 < E < \Delta E/2 \), the result
is \( \tilde{G}(E) = (e^2/h)(E^2 + \Delta E^2/4)/\hbar v_F \Delta E \). Note that away from the Dirac point the averaging procedure does not modify the conductivity value; its effect is important only around the Dirac point.

Experiments indicate that the conductivity away from the Dirac point is proportional to the density of carriers, a fact that can be justified by our approach. More precisely, if all electrons with energy \( 0 < E < E_F \) participate at transport, the total conductance away from the Dirac point per valley per spin and per unit width of the graphene flake, \( G_{\text{tot}} \), is given at low temperatures by

\[
G_{\text{tot}} = \frac{e^2}{h} \frac{1}{v_F} \int_0^{E_F} E \, dE = \frac{e^2}{h} 2\pi \hbar v_F N, \tag{9}
\]

where \( N = \int_0^{E_F} D(E) \, dE \) is the low-temperature carrier density in single-layer graphene; the carrier density in single-layer graphene can be changed by (in fact, it is directly proportional to) a gate voltage. At high temperatures the energy dependence of the Fermi-Dirac distribution must be explicitly taken into account, but the result in (9) still holds.

The existence of the minimum conductivity value in single-layer graphene and the linear dependence of conductivity on carrier density are unique features of charge transport in graphene and are a direct consequence of both the linear dispersion relation and the chiral behavior of charge carriers.

3. Conductivity calculation in double-layer graphene

The same method of obtaining the minimum conductivity from the time–energy uncertainty principle can also be applied to double-layer graphene. In this case, the Hamiltonian is [2]

\[
H = -\frac{\hbar^2}{2m} \begin{pmatrix} 0 & (k_x + ik_y)^2 \\ (k_x - ik_y)^2 & 0 \end{pmatrix}, \tag{10}
\]

the chiral character of the charge carriers being preserved, so that the wavefunction is again a spinor. In the previous section, we have discarded imaginary wavevectors at the Dirac point since no bandgap exists in the dispersion relation of single-layer graphene. In double-layer graphene both propagating and evanescent waves are theoretically possible, a fact that led to the prediction of a vanishing transmission coefficient for normal incidence [17]. However, as expected for no-bandgap materials, the evanescent waves do not actually contribute to transport properties of ballistic electrons. This conclusion is supported by conductivity measurements, as well as by experimental data which show that p–n junctions in double-layer graphene are highly transparent [20, 21]. Then, if we assume that only propagating waves are relevant, following the same reasoning as in the previous section, we arrive at the conclusion that, again, at the Dirac point the transmission coefficient of charge carriers is \( T = 1 \) for normal incidence, i.e. for \( k_x = 0 \). Experimental data suggest that, as for single-layer graphene, the value of the transmission coefficient is close to unity in a significant angular range, an assumption that describes well the real situations.

Unlike in single-layer graphene, in bilayer structures the energy dispersion relation is parabolic: \( E = \pm \hbar^2 k_x^2/2m \), the charge carriers having a finite mass \( m = 0.05m_0 \), with \( m_0 \) the free electron mass. Due to this parabolic dispersion relation, in bilayer graphene the average transit time of charge carriers between the leads is energy dependent (or \( k_x \) dependent). More precisely, one can estimate the average time interval between successive detection of charge carriers as \( \tau = Lm/4hk_x \). The uncertainty in the energy (or wavenumber) of charge carriers can in this case be meaningfully established if we write (6) in the form

\[
(h^2k_x \Delta k_x/m)^2 \tau \cong \hbar, \tag{11}
\]

from which we obtain \( \Delta k_x = 4L/m \). The low-temperature conductance per valley per spin and per unit width should then be expressed as

\[
\tilde{G}(k_x) = \frac{e^2}{h} \frac{1}{\Delta k_x} \int_{k_x-\Delta k_x/2}^{k_x+\Delta k_x/2} D(k_x) \, dk_x = \frac{e^2}{h} \frac{1}{\Delta k_x} \int_{k_x-\Delta k_x/2}^{k_x+\Delta k_x/2} |k_x| \, dk_x, \tag{12}
\]

where \( D(k_x) = |k_x|/2\pi \) is the density of states in bilayer graphene in \( k \)-space for \( k_z = 0 \). As in the previous section, the averaging procedure is essential near the Dirac point, the conductivity per valley per spin at the Dirac point taking the minimum value

\[
\sigma(0) = \sigma_{\text{min}} = e^2/h, \tag{13}
\]

which is identical to the value for single-layer graphene. This result is in agreement with experiments. Away from the Dirac point i.e. for \( k_x > \Delta k_x/2 \), the averaging procedure does not modify the conductance \( \tilde{G}(k_x) = (e^2/h)D(k_x) \), while at intermediate values of the \( x \) component of the wavevector a parabolic dependence on \( k_x \) is expected, as in the previous section. From the form of the conductance away from the Dirac point, it follows that the conductivity is proportional to the carrier density, which can be modified by applying a gate voltage. In deriving this result we have used the chiral behavior of charge carriers, which predicts total transmission at normal incidence, and the dispersion relation in bilayer graphene, which, although similar to that in common semiconductors that obey the Schrödinger equation, differs from the latter through the absence of an energy gap.

The absence of the energy gap in the density of states leads to the manifestation of energy uncertainty in the immediate vicinity of the Dirac point, the uncertainty principle having no direct influence on conductance away from the Dirac point.

4. Discussion

The assumptions used in this paper to arrive at the minimum conductivity and at the carrier dependence of conductivity around the Dirac point in the single- and double-layer graphene cases are totally different from the ones in the literature. More precisely, we can group the attempts to arrive at the minimum conductivity value in graphene in two categories: in the first one \( \sigma_{\text{min}} \) is found by extrapolating
to the low-density limit the semiclassical Boltzmann theory for charged impurity scattering [22], and in the second approach quantum interference effects due to potential fluctuations on much larger scales than the graphene lattice spacing modify the minimum conductivity value in clean graphene [23–26]. These two approaches, as expected from their different starting points, lead to different predictions. In the first case, valid in the regime of diffusive transport through the electron–hole puddles induced by disorder, the minimum conductivity was found to decrease from \(8e^2/h\) in clean samples to the observed value of \(4e^2/h\) in dirtier samples as the disorder/the concentration of charged impurities in the substrate or near the graphene/substrate interface increases; \(\sigma_{\text{min}}\) was thus found to be non-universal, but dependent on the concentration of Coulomb long-range scatterers. On the other hand, in the second approach the weak antilocalization phenomenon leads to an increase of the conductivity as the disorder strength increases in the long-range disorder regime, when intervalley scattering can be neglected (weak localization is expected for short-range potential disorder [23]). This result has been obtained both numerically [23–25] and analytically [26]. The contradicting predictions of the two approaches mentioned above have been compared and found to agree away from, but not near to, the Dirac point (at least for weak disorder) [27]. In this paper the transport was considered to be ballistic, and the graphene lattice was considered to be perfect, i.e. no disorder was assumed. As such, a different obtained value of \(\sigma_{\text{min}}\) in comparison to these other methods is no surprise. However, our result also differs from the clean graphene limit, in which the minimum conductivity is \(e^2/2\pi h\) per valley per spin, since the energy of the ballistic charge carriers is not considered fixed in our approach. Indeed, we assume a measurement-related uncertainty principle that implies an energy averaging of the conductivity. This averaging procedure is relevant only for charge carriers that satisfy the Dirac equation, i.e. charge carriers in graphene, because only in this case the density of states of ballistic charge carriers that propagate along the direction of the applied electric field depends on energy. Moreover, the averaging procedure affects the conductivity value only around the Dirac point. Summarizing, the assumptions that we use in the derivation of the minimum conductivity value and of the carrier concentration dependence of conductivity around the Dirac point have no analogy in the literature. Therefore, a direct comparison of our theory with the predictions of other theories is not possible. Of course, in real graphene samples one must consider scattering events and, particularly near the Dirac point, the occurrence of electron–hole puddles, which have not been taken into account in our treatment. These effects can lead to values of the minimum conductivity that differ slightly from \(e^2/h\) per valley per spin. However, the results in this paper show that the observed value of the minimum conductivity in both single- and double-layer graphene can be obtained even in clean samples, where the density of states vanishes at the Dirac point, if the measurement-induced uncertainty is considered.

5. Conclusions

We have shown that the minimum conductivity in both single- and double-layer graphene takes the experimentally confirmed value in wide-enough samples such that the energy discretization due to the transverse confinement of carriers is not relevant, if the time–energy uncertainty relation is taken into account. This uncertainty relation does not influence the conductivity away from the Dirac point, and therefore is seen only for mesoscopic structures with no energy gap. We have also recovered the linear dependence of conductivity on charge density away from the Dirac point. These results are a consequence of the peculiar energy dispersion relation and the chiral behavior of charge carriers in both single- and double-layer graphene.

References

[1] Castro Neto A H, Guinea F, Peres N M R, Novoselov K S and Geim A K 2009 The electronic properties of graphene Rev. Mod. Phys. 81 109–62

[2] Geim A K and Novoselov K S 2007 The rise of graphene Nat. Mater. 6 183–91

[3] Novoselov K S, McCann E, Morozov S V, Fal’ko V I, Katsnelson M I, Zeitler U, Jiang D, Schedin F and Geim A K 2006 Unconventional quantum Hall effect and Berry’s phase of 2\(\pi\) in bilayer graphene Nat. Phys. 2 177–80

[4] Novoselov K S, Geim A K, Morozov S V, Jiang D, Katsnelson M I, Grigorieva I V, Dubonos S V and Firsov A A 2005 Two-dimensional gas of massless Dirac fermions in graphene Nature 438 197–200

[5] Twarzydo J, Trauzettel B, Titov M, Rycerz A and Beenakker C W J 2006 Sub-Poissonian shot noise in graphene Phys. Rev. Lett. 96 246 802

[6] Katsnelson M I 2006 Zitterbewegung, chirality, and minimal conductivity in graphene Eur. Phys. J. B 51 157–160

[7] Katsnelson M I 2006 Minimal conductivity in bilayer graphene Eur. Phys. J. B 52 151–3

[8] Gorbar E V, Gysyn V P, Miransky V A and Shovkovy I A 2002 Magnetic field driven metal-insulator phase transition in planar systems Phys. Rev. B 66 045108

[9] Fradkin E 1986 Critical behavior of disordered degenerate semiconductors. II. Spectrum and transport properties in mean-field theory Phys. Rev. B 33 3263–8

[10] Peres N M R, Guinea F and Castro Neto A H 2006 Electronic properties of two-dimensional carbon Phys. Rev. B 73 125411

[11] Nilsson J, Castro Neto A H, Guinea F and Peres N M R 2006 Electronic properties of graphene multilayers Phys. Rev. Lett. 97 266 801

[12] Ziegler K 1998 Delocalization of 2D Dirac fermions: the role of a broken supersymmetry Phys. Rev. Lett. 80 3113–6

[13] Cserti J 2007 Minimal longitudinal dc conductivity of perfect bilayer graphene Phys. Rev. B 75 0334057

[14] Ostrovsky P M, Gorny I V and Mirlin A D 2006 Electron transport in disordered graphene Phys. Rev. B 74 235443

[15] Nomura K and MacDonald A H 2007 Quantum transport of massless Dirac fermions Phys. Rev. Lett. 98 076602

[16] Koshino M and Ando T 2006 Transport in bilayer graphene Phys. Rev. B 73 245405

[17] Katsnelson M I, Novoselov K S and Geim A K 2006 Chiral tunneling and the Klein paradox in graphene Nat. Phys. 2 620–5

[18] Ferry D K and Goodnick S M 1997 Transport in Nanostructures (Cambridge: Cambridge University Press)

[19] Fang T, Konar A, Xing H and Jena D 2007 Carrier statistics and quantum capacitance of graphene sheets and ribbons Appl. Phys. Lett. 91 092109
[20] Oostinga J B, Heersche H B, Liu X, Morpurgo A F and Vandersypen L M K 2008 Gate-induced insulating state in bilayer graphene devices Nat. Mater. 7 151–7
[21] Gorbachev R V, Tikhonenko F V, Mayorov A S, Horsell D W and Savchenko A K 2007 Weak localization in bilayer graphene Phys. Rev. Lett. 98 176805
[22] Adam S, Hwang E H, Galitski V M and Das Sarma S 2007 A self-consistent theory for graphene transport Proc. Natl Acad. Sci. 104 18392–7
[23] Suzuura H and Ando T 2002 Crossover from symplectic to orthogonal class in a two-dimensional honeycomb lattice Phys. Rev. Lett. 89 266 603
[24] Bardarson J H, Tworzydło J, Brouwer P W and Beenakker C W J 2007 One-parameter scaling at the Dirac point in graphene Phys. Rev. Lett. 99 106801
[25] Tworzydło J, Groth C W and Beenakker C W J 2008 Finite difference method for transport properties of massless Dirac fermions Phys. Rev. B 78 235 438
[26] Schuessler A, Ostrovsky P M, Gornyi I V and Mirlin A D 2009 Analytic theory of ballistic transport in disordered graphene Phys. Rev. B 79 075 405
[27] Adam S, Brouwer P W and Das Sarma S 2009 Crossover from quantum to Boltzmann transport in graphene Phys. Rev. B 79 201 404