Quantum-to-classical Transition of Cosmological Perturbations for Non-vacuum Initial States

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Abstract

Transition from quantum to semiclassical behaviour and loss of quantum coherance for inhomogeneous perturbations generated from a non-vacuum initial state in the early Universe is considered in the Heisenberg and the Schrödinger representations, as well as using the Wigner function. We show explicitly that these three approaches lead to the same prediction in the limit of large squeezing (i.e. when the squeezing parameter $|r_k| \to \infty$): each two-modes quantum state $(k, -k)$ of these perturbations is equivalent to a classical perturbation that has a stochastic amplitude, obeying a non-gaussian statistics which depends on the initial state, and that belongs to the quasi-isotropic mode (i.e. it possesses a fixed phase). The Wigner function is not everywhere positive for any finite $r_k$, hence its interpretation as a classical distribution function in phase space is impossible without some coarse graining procedure. However, this does not affect the transition to semiclassical behaviour since the Wigner function becomes concentrated near a classical trajectory in phase space when $|r_k| \to \infty$ even without coarse graining. Deviations of the statistics of the perturbations in real space from a Gaussian one lie below the cosmic variance level for the $N$-particles initial states with $N = N(|k|)$ but may be observable for other initial states without statistical isotropy or with correlations between different $k$ modes. As a way to look for this effect, it is proposed to measure
the kurtosis of the angular fluctuations of the cosmic microwave background temperature.

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1 Introduction

The undeniable success of the theory of the generation of cosmological perturbations from quantum vacuum fluctuations during an inflationary era in the early Universe was its ability to correctly predict in advance three main properties of the recently discovered (non-dipole) angular fluctuations $\Delta T(\theta, \phi)$ of the cosmic microwave background (CMB) temperature. These predictions are: a) the multipole power spectrum of the fluctuations that is approximately flat for low multipoles ($C_l \propto \frac{1}{(l+1)}$ for $2 \leq l \leq 30$); b) the right location and amplitude of the first acoustic (or “Doppler”) peak of $\Delta T$ at $l = 200 - 250$; c) the Gaussian statistics of the fluctuations (see, e.g., the 4-year COBE results for the spectrum \cite{1} and statistics \cite{2} of the fluctuations with $l \leq 20$, and the results of the Saskatoon-95 experiment for $l = 80 - 350$ \cite{3} and the CAT experiment for $l = 400 - 600$ \cite{4}). Of course, this does not exclude alternative theories of the early Universe (e.g., topological defects) provided they are able to produce approximately the same observational predictions. However, it is clear that we can and even must seriously investigate quantum-gravitational effects in the early Universe because of the actual observability of their consequences at present time.

At this point, one of the most important and interesting questions of this theory arises. The initial perturbations generated during an inflationary stage were quantum, relativistic and gravitational since their dimensionless amplitude contains all three fundamental constants: the Planck constant $\hbar$, the light velocity $c$ and the Newtonian gravitational constant $G$ (below we will take $c = \hbar = 1$ unless otherwise stated). The fluctuations $\Delta T$ measured today, on the other hand, are certainly classical (though stochastic). Thus, some mechanism describing the quantum-to-classical transition of the perturbations is required. What was explicitly or implicitly used in the pioneering papers where the spectrum of tensor perturbations (gravitational waves) \cite{5} and scalar (adiabatic) perturbations \cite{6} generated from vacuum fluctuations was first correctly derived, was a kind of equivalence between the strongly squeezed vacuum fluctuations and classical perturbations with a stochastic Gaussian amplitude and a fixed temporal phase in the Heisenberg representation (see \cite{7} for a detailed discussion and explanation of this equivalence). The same applies to the Schrödinger representation, see \cite{8} where the quantum cosmological approach based on the Wheeler-de Witt equation was used, it reduces to the Schrödinger representation for each mode after separation of variables and introduction of the quasi-classical time. Finally, this equivalence can be seen as well using the Wigner function formalism \cite{7, 9}. An essential property of this approach is that this type of transition from a quantum to a classical description of the perturbations, including the complete loss of quantum coherence, does not require any actual consideration of very small interactions between a given mode of perturbations and other degrees of freedom (“environment”) in the Universe. Though these interactions do of course exist and do lead to the transformation of an initial pure quantum state into a mixed state, their actual structure is unimportant for observable quantities as far as the exponentially
small decaying mode of perturbations may be neglected (note however that it must be taken into account if one wants to calculate the entropy of cosmological perturbations\([10]\)). This is precisely why this transition was called “decoherence without decoherence” in \([7]\). Furthermore, though the quantum coherence disappears, not all coherence is lost: indeed, the equivalent classical stochastic perturbation remains partly correlated (coherent) at the classical level in the sense that it has a fixed temporal phase corresponding to the fact that it belongs to the special quasi-isotropic mode of perturbations. Remarkably, this coherence has observable consequences: it results, in particular, in the appearance of multiple peaks (called acoustic, or Doppler peaks) in the multipole power spectrum of \(\Delta T / T\) with periodic spacing \(\Delta l = \pi \eta_0 - \eta_{\text{rec}} / \eta_0 v_s\) in the case of adiabatic perturbations. Here \(v_s\) is the sound velocity in the relativistic plasma consisting of coupled baryons and photons just before recombination, \(\eta_0\) resp. \(\eta_{\text{rec}}\) are the conformal times today resp. at recombination. A similar effect (though of significantly smaller amplitude) takes place for CMB angular temperature fluctuations produced by primordial gravitational waves background \([7]\) (\(v_s = 1\) in this case).

This approach however, especially that one based on the Wigner function formalism, was strongly criticised in \([11]\) on the ground that it might be applicable to vacuum initial conditions only and not to more general ones. Hence, we adress in this paper the problem of the quantum to classical transition of quantum cosmological perturbations in the case of non-vacuum initial states. By an ”initial state” of a given Fourier mode \(k\) of a quantum field we mean its state in the WKB regime deep inside the Hubble radius at the inflationary stage, long before the Hubble radius crossing, i.e. when \(k \gg aH\) where \(k = |k|\), \(a(t)\) is the scale factor of the Friedmann-Robertson-Walker (FRW) isotropic cosmological model, \(H \equiv \dot{a} / a\) is the Hubble parameter and the dot means a time derivative with respect to the cosmological time \(t\).

Of course, a non-vacuum initial state contradicts the whole spirit of the maximally symmetric initial state of the Universe which lies at the heart of the inflationary scenario. Thus, our “wish” would rather be to show that such an assumption is in conflict with observational data. Surprisingly, as we will see below, this turns out to be more delicate. Non-vacuum initial states are typically not self-consistent in the simplest variants of the inflationary scenario since they lead to a large energy density of inflaton field quanta which is not of the type of a cosmological term and it would contradict the assumption of inflation at time \(\eta_0\) especially if we push \(\eta_0\) to \(-\infty\). However, they could appear in more complicated, double inflationary models for perturbation modes which cross the Hubble radius three or more times during inflation if we additionally assume (without justification) that these modes immediately decohere (e.g., in the particle number basis) once they reenter inside the Hubble radius. Though the latter process is hardly physically viable, still it cannot be completely excluded. Furthermore, it is an important question of principle to study whether the possibility of a classical description of highly squeezed perturbations is generic or whether it is restricted to vacuum initial states only.
Particle creation from non-vacuum initial states was considered in many papers; in the cosmological case - beginning with the pioneering paper by Parker [12], see the recent review [13] for many references. However, the primary quantities studied in these papers were the number of created particles and the average value of the energy-momentum tensor of a quantum field on a FRW background. In contrast, we are mainly interested in the field fluctuations themselves since these are the quantities (and not the number of particles) that are directly observable in the case of cosmological perturbations. This refers both to scalar metric, or density, perturbations and to tensor metric perturbations or gravitational waves. Naturally, the resulting quantum-to-classical equivalence that, as we argued in [7], is the reason for the possibility of a classical description of the perturbations in the present Universe takes place for these quantities, too.

Thus, the results of paper [7] are generalized to the case of non-vacuum initial conditions. In order not to introduce any preferred direction in space, we will consider only reflection symmetric \((r \rightarrow -r, \ k \rightarrow -k)\) quantum states. We show that a wide class of initial states become quasiclassical in the limit of high squeezing. This is shown in section 2 using the Schrödinger representation, and in section 3 using the Heisenberg representation. In section 4 we derive conditions necessary for the semiclassical behaviour of quantum scalar (adiabatic) perturbations and gravitational waves on a FRW background when expressed directly in terms of rms values of the perturbations. The Wigner function approach to this problem is studied in section 5. The specific, but important case of a thermal initial state is considered in section 6. Finally, section 7 contains conclusions and a discussion of a possible observational confirmation or rejection of a non-vacuum initial state.

## 2 Schrödinger representation

In this section, we are going to compute the wave function of different non-vacuum states in the Schrödinger representation. Let us first review the basics about the dynamics of the system that we consider here, namely that of a real massless scalar field on a flat FRW universe. Actually, the field can be massive and all the arguments presented here will go through. The space-time metric has the form

\[
ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j, \quad i, j = 1, 2, 3.\tag{1}
\]

Let us remind how the dynamics of this system will lead to the appearance of squeezed states. We first write down the Hamiltonian \(H\) in terms of the field \(y \equiv a\varphi\) and the conformal time \(\eta = \int \frac{dt}{a(t)}\). The following result is then obtained

\[
\begin{align*}
H &= \int d^3x \mathcal{H}(y, p, \partial_i y, t) \\
&= \frac{1}{2} \int d^3k \bigg[ p(k)p^\dagger(k) + k^2 y(k)y^\dagger(k) + \frac{a'}{a} \left( y(k)p^\dagger(k) + p(k)y^\dagger(k) \right) \bigg] \tag{2}
\end{align*}
\]
where
\[ p \equiv \frac{\partial \mathcal{L}(y, y')}{\partial y'} = y' - \frac{a'}{a} y \quad (3) \]

and a prime stands for derivation with respect to conformal time. Here the following Fourier transform convention is used: \( \Phi(k) \equiv (2\pi)^{-3/2} \int \Phi(r) e^{-ikr} d^3r \) for functions as well as for operators. Due to reality of the field \( y \), we have that \( y(k) = y^*(-k) \), resp. \( y^i(-k) \) for operators. Therefore, any classical field configuration is completely specified by giving the Fourier transforms in half Fourier space. Though generally this doesn’t have to be the case when the fields are quantized, we will deal here with states which are invariant under the reflection \( k \to -k \) as mentioned above. The system can be described with the help of the complex valued field modes \( f_k(\eta) \) with \( \Re f_k \equiv f_{k1} \) and \( \Im f_k \equiv f_{k2} \), \( f_k(\eta_0) = 1/\sqrt{2k} \) (we adopt a similar notation for all quantities), and momentum modes \( g_k(\eta), g_k(\eta_0) = \sqrt{k/2} \), typically used in the Heisenberg representation

\[ y(k, \eta) \equiv f_k(\eta) a(k, \eta_0) + f_k^*(\eta) a^\dagger(-k, \eta_0) , \quad \quad F(k) = f_{k1}g_{k2} - f_{k2}g_{k1} . \quad (4) \]

The functions \( f_k \) satisfy the equation
\[ f_k'' + \left( k^2 - \frac{a''}{a} \right) f_k = 0 , \quad (7) \]

and
\[ g_k = i \left( f_k' - \frac{a'}{a} f_k \right) . \quad (8) \]

The couple of functions \( f_k, g_k \) represent only three independent real functions as they satisfy a Wronskian constraint. The dynamics of the system can also be conveniently parametrized by the following three functions: the rotation angle \( \theta_k \), the squeeze angle \( \phi_k \) and the squeeze parameter \( r_k \). When \( |r_k| \to \infty \), the system is said to be highly squeezed. In that case also \( |F_k| \to \infty \). As a result of the time evolution of our system, we have
\[ S a^\dagger(\pm k, \eta_0) S^{-1} = g_k(\eta) y(\mp k, \eta_0) - i f_k(\eta) p(\mp k, \eta_0) , \]
\[ S a(\pm k, \eta_0) S^{-1} = g_k^*(\eta) y(\pm k, \eta_0) + i f_k^*(\eta) p(\pm k, \eta_0) \quad (9) \]

where \( S \) is the unitary evolution operator from time \( \eta_0 \) to time \( \eta \). When working in the Schrödinger (complex) coordinate representation, we will often adopt the following
compact notation \( y(k, \eta_0) \equiv y_0, \ p(k, \eta_0) \equiv p_0 = -i \frac{\partial}{\partial y_0}, \ y(-k, \eta_0) \equiv y_0^*, \ p(-k, \eta_0) \equiv p_0^* = -i \frac{\partial}{\partial y_0} \) where \( y_0 \) and \( y_0^* \) are considered as independent variables.

As a result of the coupling with the gravitational field, which plays here the role of an external classical field, the state \( |0, \eta_0 \rangle_S \) which is the vacuum state of the field at some given initial time \( \eta_0, \ a(k, \eta_0) \ |0, \eta_0 \rangle = 0 \ \forall k, \) will no longer be the vacuum at later times \( \eta. \) Indeed field quanta are produced in pairs with opposite momenta, we get a two-modes squeezed state. Though the state at late time is no longer annihilated by the annihilation operators \( a(k) \), it is still annihilated by the following time-dependent operator

\[
\left\{ y(k, \eta_0) + i \gamma_k^{-1}(\eta) \ p(k, \eta_0) \right\} |0, \eta \rangle_S = 0. \tag{10}
\]

In the infinite volume case, equation (10) translates into the following functional differential equation in the Schrödinger (complex) coordinate representation

\[
\left\{ y(k, \eta_0) + \gamma_k^{-1}(\eta) \frac{\delta}{\delta y(-k, \eta_0)} \right\} |0, \eta \rangle_S = 0. \tag{11}
\]

Therefore the state at late time is still Gaussian, its wave functional in the infinite volume case (the quantum variables are the functions \( y(k) \) of the continuous variable \( k \) being given by

\[
\Psi_0[y(k, \eta_0), y(-k, \eta_0)] = N_k \exp \left( -\frac{1}{2} \int d^3k \ \gamma_k \ y(k, \eta_0) \ y(-k, \eta_0) \right) \tag{12}
\]

For our purposes however, it is enough to consider our system enclosed in a finite volume and we then get as wave function of our system an (infinite) product of functions

\[
\frac{1}{\sqrt{\pi |f_k|}} \exp \left( -\frac{|y(k, \eta_0)|^2}{2 |f_k|^2} \left\{ 1 - i2 F(k) \right\} \right) \tag{13}
\]

for each pair \( k, -k. \) The commutators and correlation functions of Fourier transform operators are then \( c \)-number functions and not distributions. The time dependence of \( \Psi \) is through \( f_k, \ F(k), \) and \( N_k. \) We note the persisting symmetry under reflections in \( k\)-space, it is present in the initial state and the evolution doesn’t spoil it.

As a first generalization of the vacuum state at time \( \eta_0. \) we consider a two-modes coherent state \( |\alpha \beta, \eta_0 \rangle_S \) satisfying

\[
\begin{align*}
\ a(k, \eta_0) \ |\alpha \beta, \eta_0 \rangle_S &= \left\{ \sqrt{\frac{k}{2}} y(k, \eta_0) + i \frac{\sqrt{2k}}{\sqrt{2k}} p(k, \eta_0) \right\} |\alpha \beta, \eta_0 \rangle_S \\
&= \alpha |\alpha \beta, \eta_0 \rangle_S \left\{ \sqrt{\frac{k}{2}} \langle y \rangle_0 + i \frac{\sqrt{2k}}{\sqrt{2k}} \langle p \rangle_0 \right\} |\alpha \beta, \eta_0 \rangle_S
\end{align*} \tag{14}
\]

6
where \( \langle y \rangle_0 \) and \( \langle y \rangle_0 \) are the average values of \( y(k, \eta_0) \) and \( p(k, \eta_0) \) at time \( \eta_0 \), and it satisfies also

\[
a(-k, \eta_0) |\alpha \beta, \eta_0 \rangle_S = \left\{ \sqrt{\frac{k}{2}}y(-k, \eta_0) + \frac{i}{\sqrt{2k}}p(-k, \eta_0) \right\} |\alpha \beta, \eta_0 \rangle_S
\]

\[
= \beta |\alpha \beta, \eta_0 \rangle_S = \left\{ \sqrt{\frac{k}{2}}\langle y \rangle_0^* + \frac{i}{\sqrt{2k}}\langle p \rangle_0 \right\} |\alpha \beta, \eta_0 \rangle_S .
\] (15)

It is clear that the time-evolved state \( |\alpha \beta, \eta \rangle_S \) satisfies the following equations at any later time \( \eta \)

\[
S a(k, \eta_0) S^{-1} |\alpha \beta, \eta \rangle_S = \alpha |\alpha \beta, \eta \rangle_S = (g_k^*(\eta) \langle y \rangle + i f_k^*(\eta) \langle p \rangle) |\alpha \beta, \eta \rangle_S
\]

\[
S a(-k, \eta_0) S^{-1} |\alpha \beta, \eta \rangle_S = \beta |\alpha \beta, \eta \rangle_S = (g_k^*(\eta) \langle y \rangle^* + i f_k^*(\eta) \langle p \rangle^*) |\alpha \beta, \eta \rangle_S.
\] (16)

where \( \langle y \rangle, \langle p \rangle \) are average values at time \( \eta \). It follows from (14,16) that \( \langle y \rangle \) is the solution of eq. (7) with the initial conditions \( y(\eta_0) = \langle y \rangle_0 \), \( p(\eta_0) = \langle p \rangle_0 \), and \( \langle p \rangle = \langle y \rangle^* - \frac{\partial}{\partial \eta} \langle y \rangle \). Eqs. (13) are easily solved in the Schrödinger representation resulting in the following wave function \( \Psi_{\alpha \beta}(y_0, y_0^*, \eta) \) of the two-modes coherent state:

\[
\Psi_{\alpha \beta}(y_0, \eta) = \langle y(k, \eta_0), y(-k, \eta_0)|\alpha \beta, \eta \rangle_S
\]

\[
= N_k \exp\left[ -\gamma k (y_0 - \langle y \rangle)(y_0 - \langle y \rangle)^* + i(y_0^* \langle p \rangle + y_0 \langle p \rangle^*) \right]
\] (17)

\[
= N_k \exp\left[ -\gamma k |y_0 - \langle y \rangle|^2 + i( y_0^* \langle p \rangle + y_0 \langle p \rangle^* ) \right].
\] (18)

We could equally well arrive at that result working in the real and imaginary coordinate representation. Indeed, we can separate the variables \( y \) and \( p \) in real and imaginary part and consider the corresponding operators in the Schrödinger representation. Defining \( S a(\pm k, \eta_0) S^{-1} \equiv a_1(\eta) \pm ia_2(\eta) \), the above equations can be written as:

\[
a_1(\eta) |\alpha \beta, \eta \rangle_S = \left\{ g_k^*(\eta) \langle y_1 \rangle + i f_k^*(\eta) \langle y_1 \rangle \right\} |\alpha \beta, \eta \rangle_S
\]

\[
a_1(\eta) \Psi_{\alpha \beta}(y_0, y_0^*, \eta) = g_k^*(\eta) y_0 + f_k^*(\eta) \frac{1}{2} \frac{\partial}{\partial y_{01}} \Psi_{\alpha \beta}(y_0, y_0^*, \eta) ,
\] (19)

\[
a_2(\eta) |\alpha \beta, \eta \rangle_S = \left\{ g_k^*(\eta) \langle y_2 \rangle + i f_k^*(\eta) \langle y_2 \rangle \right\} |\alpha \beta, \eta \rangle_S
\]

\[
a_2(\eta) \Psi_{\alpha \beta}(y_0, y_0^*, \eta) = g_k^*(\eta) y_0 + f_k^*(\eta) \frac{1}{2} \frac{\partial}{\partial y_{02}} \Psi_{\alpha \beta}(y_0, y_0^*, \eta) .
\] (20)

Solving this differential system, we find that the wave function can be written as:

\[
\Psi_{\alpha \beta}(y_0, y_0^*, \eta) = \phi(y_01, \eta)\phi(y_02, \eta),
\]

where:

\[
\phi(y_01, \eta) = \sqrt{N_k} \exp\left[ -\gamma k (y_{01} - \langle y_1 \rangle)^2 + i2 y_{01} \langle p_1 \rangle \right] ,
\] (21)

in complete accordance with (13,18).
We are further interested in states which, at some initial time \( \eta_0 \), contain some arbitrary number \( N \) of particles with momenta \( k \) and \(-k\) and we denote these states by \( |N_k, N_{-k}, \eta_0 \rangle_S \equiv |N, \eta_0 \rangle_S \):

\[
|N, \eta_0 \rangle_S = \frac{1}{N!} a^{+N}(k, \eta_0) a^{+N}(-k, \eta_0) |0, \eta_0 \rangle_S .
\]  

(22)

We deal with modes initially deep inside the Hubble radius \( k \gg a'(\eta_0)/a(\eta_0) \)) and use the WKB solution for these modes. The time-evolved state defined in \((22)\) becomes:

\[
|N, \eta \rangle_S = \frac{1}{N!} \left( S a^{+}(k, \eta_0) S^{-1} \right)^N \left( S a^{+}(-k, \eta_0) S^{-1} \right)^N |0, \eta \rangle_S .
\]

(23)

In the Schrödinger coordinate representation, the following differential equation relating the \( N \)-particles two-modes wave function to the vacuum two-modes wave function is obtained:

\[
\Psi_N(y_0, y_0^*, \eta) = \frac{1}{N!} f_k^{2N} \left( \gamma^*_k y_0 - \frac{\partial}{\partial y_0^*} \right)^N \left( \gamma^*_k y_0^* - \frac{\partial}{\partial y_0} \right)^N \Psi_0(y_0, y_0^*, \eta)
\]

(24)

with

\[
\Psi_0(y_0, y_0^*, \eta) = \left( \sqrt{\pi} |f_k|^{-1} \right) \exp(-\gamma_k y_0 y_0^*) = N_k \exp(-\gamma_k y_0 y_0^*) .
\]

(25)

The action on \( \Psi_0 \) of the operator inside the right parenthesis can be easily derived using the relation \( \gamma^*_k + \gamma_k = |f_k|^2 \). Indeed:

\[
\Psi_N = \frac{1}{N!} \left( \frac{f_k}{f_k^*} \right)^N N_k \left( \gamma^*_k y_0 - \frac{\partial}{\partial y_0^*} \right)^N \left( \gamma^*_k y_0^* - \frac{\partial}{\partial y_0} \right)^N y_0^{*N} \exp(-\gamma_k y_0 y_0^*) .
\]

(26)

We make further a binomial expansion of the first parenthesis:

\[
\Psi_N = \frac{1}{N!} \left( \frac{f_k}{f_k^*} \right)^N N_k \sum_{p=0}^{N} C_N^p \left( \gamma^*_k y_0 \right)^p \left( -\frac{\partial}{\partial y_0^*} \right)^{N-p} \left\{ y_0^{*N} \exp(-\gamma_k y_0 y_0^*) \right\} .
\]

(27)

It is now convenient to use the following trick, namely to replace \((\gamma^*_k y_0)^p\) by the equivalent expression:

\[
(\gamma^*_k y_0)^p = \exp(\gamma^*_k y_0 y_0^*) \left( -\frac{\partial}{\partial y_0^*} \right)^p \left\{ \exp(-\gamma_k y_0 y_0^*) \right\}
\]

(28)

As a result we get

\[
\Psi_N = \frac{1}{N!} \left( \frac{f_k}{f_k^*} \right)^N N_k \exp(\gamma^*_k y_0 y_0^*) \sum_{p=0}^{N} C_N^p \left( -\frac{\partial}{\partial y_0^*} \right)^p \left\{ \exp(-\gamma_k y_0 y_0^*) \right\} \times \left( -\frac{\partial}{\partial y_0^*} \right)^{N-p} \left\{ \exp(-\gamma_k y_0 y_0^*) \right\} ,
\]

(29)
where we recognize the Nth derivation of a product of functions. The Leibnitz formula gives:

\[
\Psi_N = \frac{1}{N!} \left( \frac{f_k}{f_k^*} \right)^N N_k \exp(\gamma_k y_0^* y_0) \left( -\frac{\partial}{\partial y_0^*} \right)^N \left\{ y_0^{*N} \exp(- (\gamma_k^* + \gamma_k) y_0 y_0^*) \right\}
\]

\[
= \frac{1}{N!} \left( -\frac{f_k}{f_k^*} \right)^N \Psi_0 \left[ \exp \left( \frac{|y_0|^2}{|f_k|^2} \right) \left( \frac{\partial}{\partial y_0^*} \right)^N \left\{ y_0^{*N} \exp \left( -\frac{|y_0|^2}{|f_k|^2} \right) \right\} \right].
\]

(30)

The expression inside the square brackets is just a Laguerre polynomial as can be checked using Rodrigue’s formula:

\[
L_N(w) = \frac{1}{N!} e^w \left( \frac{\partial}{\partial w} \right)^N \left\{ w^N e^{-w} \right\}.
\]

(31)

Taking \( w(y^*) = -\frac{y_0}{|f_k(y)|^2} y_0^* \), where \( y_0 \) is constant, it follows that the expression inside the square brackets is just \( L_N \left( \frac{y_0^*}{|f_k|} \right) \). The final result is therefore:

\[
\Psi_N(y_0, y_0^*, \eta) = \left( -\frac{f_k}{f_k^*} \right)^N L_N \left( \frac{|y_0|^2}{|f_k|^2} \right) \Psi_0(y_0, y_0^*, \eta).
\]

(32)

This calculation can also be done differently, in a way which takes advantage of basic results pertaining to the familiar one-dimensional harmonic oscillator. Indeed, let us separate the complex variables and their corresponding operators in real and imaginary parts, as we did already previously. The definition of the state then becomes:

\[
|N, \eta\rangle_S = \frac{1}{N!} \left( S a^\dagger(k, \eta_0) S^{-1} S a^\dagger(-k, \eta_0) S^{-1} \right)^N |0, \eta\rangle_S
\]

\[
= \frac{1}{N!} \left( a_1^{12} + a_2^{12} \right)^N |0, \eta\rangle_S
\]

\[
= \frac{1}{N!} \left( \sum_{M=0}^N C_N^M a_1^{12M} a_2^{12N-2M} \right) |0, \eta\rangle_S.
\]

(33)

In this representation, the differential equation (24) becomes:

\[
\Psi_N(y_0, y_0^*, \eta) = \frac{1}{N!} f_k^{2N} (\eta) \sum_{M=0}^N C_N^M \left( \gamma_k^* y_{01} - \frac{1}{2} \frac{\partial}{\partial y_{01}} \right)^{2M} \times
\]

\[
\left( \gamma_k^* y_{02} - \frac{1}{2} \frac{\partial}{\partial y_{02}} \right)^{2N-2M} \Psi_0(y_0, y_0^*, \eta)
\]

\[
= \frac{N_k}{N!} f_k^{2N} \sum_{M=0}^N C_N^M \left( \gamma_k^* y_{01} - \frac{1}{2} \frac{\partial}{\partial y_{01}} \right)^{2M} \left\{ e^{-\gamma_k y_{01}^2} \right\} \times
\]

\[
\left( \gamma_k^* y_{02} - \frac{1}{2} \frac{\partial}{\partial y_{02}} \right)^{2N-2M} \left\{ e^{-\gamma_k y_{02}^2} \right\}.
\]

(34)
Knowing the harmonic oscillator eigenfunctions, it is straightforward to find:

\[ \Psi_N = \frac{1}{N!4^N} \left( \frac{f_k^*}{f_k} \right)^N \sum_{M=0}^{N} C_N^M H_{2M} \left( \frac{y_{01}}{|f_k|} \right) H_{2N-2M} \left( \frac{y_{02}}{|f_k|} \right) \Psi_0 . \]  

(35)

It is now possible to show that this expression is equal to (32). Taking into account the relation between Hermite and Laguerre polynomials, we write:

\[ \Psi_N = \frac{1}{N!4^N} \left( \frac{f_k^*}{f_k} \right)^N \sum_{M=0}^{N} C_N^M (-1)^M 2^{2M} M! L_{N-M}^{(-\frac{1}{2})} \left( \frac{y_{01}^2}{|f_k|^2} \right) \times \]

\[ (-1)^{N-M} 2^{2N-2M} (N-M)! L_{N-M}^{(-\frac{1}{2})} \left( \frac{y_{02}^2}{|f_k|^2} \right) \Psi_0 \]

\[ = \frac{1}{N!} \left( -\frac{f_k^*}{f_k} \right)^N \left( \sum_{M=0}^{N} L_{N-M}^{(-\frac{1}{2})} \left( \frac{y_{01}^2}{|f_k|^2} \right) \Psi_0 \right) \Psi_0 \]

\[ = \frac{1}{N!} \left( -\frac{f_k^*}{f_k} \right)^N \left( \frac{y_{01}^2 + y_{02}^2}{|f_k|^2} \right) \Psi_0 , \]  

(36)

where 8.974.4, p.1038 of [14] has been used in the last step of the above calculation.

The two expressions (32,35) are two equivalent ways to write down the wave function in the coordinate Schrödinger representation. They represent, together with (17), the basic result of this section. We note that the wave function (32) is a function of \(|y(k)|\) and therefore, like the vacuum wave function, it is invariant under reflection \(k \rightarrow -k\). As a result, it is clear that all correlation with an odd number of points will vanish, the distribution has no skewness. On the other hand, it is of course no more a Gaussian, a fact which will be important regarding the positivity of the Wigner function. The transition to semi-classicality can now be seen in this representation. The condition for a semiclassical behaviour is that the phase of the wave function should be large for a typical value of \(|y|^2 \sim \langle |y|^2 \rangle = (2N + 1) |f_k|^2\). This results in the inequality \((2N + 1)|F_k(\eta)| \gg \hbar\) (here we restored \(\hbar\) for clarity). Note that, as one could expect, the latter condition is more easily achieved for large \(N\). The probability distribution for \(y_0\) at time \(\eta\) is given by:

\[ \rho_N(|y_0|, \eta) = |\Psi_N(y_0, y_0^*, \eta)|^2 = \frac{1}{\pi |f_k|^2} L_N^2 \left( \frac{|y_0|^2}{|f_k|^2} \right) \exp \left( -\frac{|y_0|^2}{|f_k|^2} \right) . \]

(37)

This probability distribution conserves probability along the classical trajectory \(y \propto f_k(\eta)\) when the condition \((2N + 1)|F(k)| \gg 1\) is satisfied. As we shall show below in section 4, the requirement that the latter condition should be better and better satisfied with the expansion of the Universe picks up a small subset of all possible classical trajectories \(f_k(\eta)\) in phase space (namely, the quasi-isotropic mode of perturbations).
3 Heisenberg representation

Let us perform now an independent analysis of the transition to semiclassical behaviour in the Heisenberg representation. The crucial point here is that, as shown in [7], in the limit of infinite squeezing the mode functions $f_k(\eta)$ can be made real by a time-independent phase rotation (of course, in addition, $|f_k| \to \infty$). Then the momentum mode functions $g_k(\eta)$ become purely imaginary. As a result, the field operators $y(k, \eta)$ and $p(k, \eta)$ take the following form:

$$y(k, \eta) = f_k \left( a(k, \eta_0) + a^\dagger(-k, \eta_0) \right), \quad (38)$$
$$p(k, \eta) = -i g_k \left( a(k, \eta_0) + a^\dagger(-k, \eta_0) \right) = -i \frac{g_k}{f_k} y(k, \eta). \quad (39)$$

So, in the limit $|r_k| \to \infty$, the non-commutativity of $y(k, \eta)$ and $p(k, \eta)$ (and of all other operators, too) may be neglected. Due to this reason, we can introduce equivalent classical stochastic quantities $y_k(\eta) = f_k \, e(k)$ where $e(k)$ are c-number complex stochastic time-independent variables obeying the same statistics as $y(k, \eta_0)$ at the time $\eta_0$. They have zero average and unit dispersion: $\langle e(k) \rangle = 0$, $\langle e(k) e^*(k') \rangle = \delta^{(3)}(k-k')$, though they are generally not Gaussian. The definition of the equivalence is the following: we say that a complex classical stochastic field $y(\eta)$ with a corresponding momentum $p(\eta)$ and a probability distribution in phase space $\rho(|y|, |p|)$ are equivalent to quantum operators $\hat{y}(\eta)$ and $\hat{p}(\eta)$ if they give the same average values for an arbitrary function $G(y, p)$:

$$H \langle \hat{N}|G(\hat{y}, \hat{p})G^\dagger(\hat{y}, \hat{p})|\hat{N}\rangle_H = \int \int dy_1 dy_2 dp_1 dp_2 \rho(|y|, |p|) |G(y, p)|^2 \quad (40)$$

(here we omit $k$ for brevity but we restore hats over operators).

Of course, such a condition cannot be fulfilled generally, since quantum mechanics is not equivalent to classical field theory. However, in our case of infinite squeezing, due to relation (39) between $\hat{y}$ and $\hat{p}$ (while the same relation between classical stochastic quantities $y$ and $p$ holds), eq. (40) reduces to a much simpler form:

$$H \langle \hat{N}|\hat{G}(y)\hat{G}^\dagger(y)|\hat{N}\rangle_H = \int \int dy_1 dy_2 \rho(|y|) |\hat{G}(y)|^2 \quad (41)$$

where $\hat{G}(y) = G(y, p(y))$ (since all operators are mutually commuting, we do not write any more hats over them). For the $N$-particles initial state, the probability density function $\rho(|y|)$ is given by

$$\rho(|y|) = \frac{1}{\pi |f_k|^2 L_N^2} \left( \frac{|y|^2}{|f_k|^2} \right) \exp \left( -\frac{|y|^2}{|f_k|^2} \right). \quad (42)$$

So, in the limit of infinite squeezing and after use is made of relation (39) the equivalence condition takes a form which is identical to the definition of an average value
of an operator in quantum mechanics with the probability distribution \( \rho \) equal to the modulus square of a wave function satisfying the Schrödinger equation. Thus, (41) can be immediately proved by referring to the results of the previous section. However, if we want to produce a completely independent derivation in the Heisenberg representation, we should not use the fact that \( \rho \) is determined by some solution of the Schrödinger equation but instead we have to make an actual calculation of both sides of eq. (41) and prove their equality.

Further, we omit tilda over \( G \). We compute first the l.h.s. of (41):

\[
H \langle N | G(y) G^\dagger(y) | N \rangle_H = \sum_{m,n=0}^{\infty} q_m q_n^* \ H \langle N | y^m(k, \eta) y^n(k, \eta) | N \rangle_H .
\]  

(43)

We use now the expression (39) in the high squeezing limit when \( f_k(\eta) \) can be made real by a time-independent phase rotation, and write:

\[
H \langle N | G(y) G^\dagger(y) | N \rangle_H = \sum_{m,n=0}^{\infty} q_m q_n^* \frac{f_k^{m+n}}{(N!)^2} H \langle 0 | a^N(-k, \eta_0) a^N(k, \eta_0) \times \]
\[
( a(k, \eta_0) + a^\dagger(-k, \eta_0) )^m ( a^\dagger(k, \eta_0) + a(-k, \eta_0) )^n a^1N(k, \eta_0) a^1N(-k, \eta_0) | 0 \rangle_H .
\]  

(44)

We find after a somehow tedious but straightforward calculation:

\[
H \langle N | G(y) G^\dagger(y) | N \rangle_H = \sum_{m=0}^{\infty} |q_m|^2 f_k^{2m}(\eta) \ 
\sum_{i = \text{sup}(0,m-N)}^{m} \frac{m!^2(N+i)!}{i!(m-i)!(N-m+i)!} .
\]

(45)

As for the r.h.s. of (41), we get:

\[
\int \int dy_1 dy_2 \rho(|\gamma|) |G(y)|^2
\]
\[
= \sum_{m,n=0}^{\infty} q_m q_n^* \int \int dy_1 dy_2 \frac{1}{|f_k|^2} L_N \left( \left| \frac{y}{|f_k|^2} \right|^2 \right) y^m y^n \exp \left( - \frac{|y|^2}{|f_k|^2} \right) .
\]

(46)

Replacing \( L_N \) by its explicit polynomial expression, we find:

\[
\int \int dy_1 dy_2 \rho \left( |\gamma| \right) |G(y)|^2
\]
\[
= \sum_{m=0}^{\infty} |q_m|^2 f_k^{2m} \sum_{i,j=0}^{N} \frac{(-1)^{i+j} N!^2}{i!^2 j!^2 (N-i)! (N-j)!} \int_0^{\infty} x^{m+i+j} e^{-x} dx
\]
\[
= \sum_{m=0}^{\infty} |q_m|^2 f_k^{2m} \sum_{i,j=0}^{N} \frac{(-1)^{i+j} N!^2 (m+i+j)!}{i!^2 j!^2 (N-i)! (N-j)!} .
\]

(47)

One finally checks, as we did numerically, that the sums in (43) and (47) are indeed equal:

\[
\sum_{i = \text{sup}(0,m-N)}^{m} \frac{m!^2(N+i)!}{i!(m-i)!(N-m+i)!} = \sum_{i,j=0}^{N} \frac{(-1)^{i+j} N!^2 (m+i+j)!}{i!^2 j!^2 (N-i)! (N-j)!} .
\]

(48)
This ends the proof of equality (II). We note that the calculation done above yields in particular all the correlation functions. Indeed, if we set \( G(y) = y^m \), we get:

\[
H \langle N | y^{2m} | N \rangle_H = |f_k|^{2m} \sum_{i = \sup(0, m - N)}^m \frac{m!^2(N + i)!}{i!^2(m - i)!^2(N - m + i)!}. \tag{49}
\]

We see also that all the following correlation functions do vanish:

\[
H \langle N | y^m y^n | N \rangle_H = 0 \quad \text{if} \quad m \neq n. \tag{50}
\]

Having in mind the comparison of the probability density obtained here with a Gaussian one, the two-point and four-point correlation functions are of particular importance:

\[
H \langle N | y^2 | N \rangle_H = (2N + 1)|f_k|^2, \\
H \langle N | y^4 | N \rangle_H = (6N^2 + 6N + 2)|f_k|^4. \tag{51}
\]

The above results can be directly used in order to calculate the kurtosis \( Q \) (the irreducible 4-order moment) of the probability distribution:

\[
Q \equiv \frac{\langle N, \eta | y(k, \eta) | N, \eta \rangle^4}{\langle N, \eta | y(k, \eta) | N, \eta \rangle^2} - 2 = -\frac{1}{2} \left( 1 - \frac{1}{(2N + 1)^2} \right). \tag{52}
\]

It vanishes, as should be the case, when \( N = 0 \) and tends to \(-1/2\) for \( N \to \infty \). We will come back to this point in Section 6 and in the conclusion. For comparison, we present the corresponding results for a one-mode \( N \)-particles state for which

\[
\varphi = \varphi(t)a + \varphi^*(t)a^\dagger, \quad [a, a^\dagger] = 1. \tag{53}
\]

An example of such a state is the \( k = 0 \) mode. Then

\[
\langle \varphi^2 \rangle = (2N + 1)|\varphi|^2, \\
\langle \varphi^4 \rangle = (6N^2 + 6N + 3)|\varphi|^4, \\
Q \equiv \frac{\langle \varphi^4 \rangle}{\langle \varphi^2 \rangle^2} - 3 = \frac{3}{2} \left( 1 - \frac{1}{(2N + 1)^2} \right). \tag{54}
\]

These expressions will be also useful for the calculation of the curtosis of \( y \) in the coordinate representation, see section 7.

\section{4 Conditions for semiclassical behaviour of perturbations}

Let us now analyze in more detail the conditions imposed on the mode functions \( f_k(\eta) \) of the quantum field \( y \) or on the equivalent stochastic functions \( y_k(\eta) = f_k(\eta)e(k) \) by
the requirement of high squeezing $|r_k| \to \infty$ which is crucial for the quasi-classical equivalence as we have shown in sections 2 and 3. In the long-wavelength limit $k \ll aH$, the leading term in the mode functions $f_k(\eta)$ and $g_k(\eta)$ have the form \[ f_k = C_1(k)a + C_2(k)a \int_{\infty}^{\eta} \frac{d\eta'}{a^2(\eta')} \qquad g_k = O(iC_1(k)\frac{k^2}{H}) + \frac{iC_2(k)}{a} \] (55)

where $C_1(k)$ is made real and positive by a phase rotation. We assume here that the integral converges at the lower limit that takes place for all physically interesting equations of state of matter at late times, in particular, if $a(t) \propto t^n$ with $n > 1/3$.

$C_1(k)$ and $C_2(k)$ should satisfy the normalisation condition

$$C_1\Im C_2 = -\frac{1}{2}.$$ (56)

The first term in (55) is the quasi-isotropic mode, the second is the decaying mode.

In the simplest inflationary models with one slow-rolling scalar field, a more concrete expression for $C_1(k)$ and $C_2(k)$ is obtained \[ C_1 = \frac{H_k}{\sqrt{2}k^{3/2}}, \qquad C_2 = -\frac{ik^{3/2}}{\sqrt{2}H_k} \] (57)

where $H_k$ is the value of $H(\eta)$ at the moment of the first Hubble radius crossing during the inflationary stage. However, generally we need not restrict ourselves to this case only. The first term in the expression for $g_k$ in eq.(55) has a rather complicated integro-differential structure that can be defined by solving (7) using a perturbative expansion in powers of $k^2$. The given estimate is valid for a wide class of smooth functions $a(\eta)$ including arbitrary power-law and quasi-de Sitter ones.

Now let us insert (55) into the definition of $F(k)$, eq.(6). It is argued in \[ F(k) \] that it is the condition $|F(k)| \gg 1$ that is required for the quasi-classical behaviour of the vacuum initial state. In the case of $N$-particles state, the necessary condition is weaker, namely $(2N+1)|F(k)| \gg 1$ as shown above. Assuming a power-law or a quasi-de Sitter behaviour of the scale factor $a(\eta)$ and omitting numerical coefficients of the order of unity, we obtain the following condition:

$$\left| C_1^2 \frac{k^2a}{H} + \frac{|C_2|^2}{a^3H} + C_1 \Im C_2 \right| \gg \frac{1}{2N+1}.$$ (58)

If the solution is dominated by the quasi-isotropic mode, and the term with $C_2$ may be neglected, the condition (58) takes the form

$$C_1^2 \gg \frac{H}{k^2a} \cdot \frac{1}{2N+1}.$$ (59)

In terms of metric tensor perturbations (gravitational waves) which are related to $\varphi = y/a$ by the relation

$$h_{ij} \equiv -\frac{\delta g_{ij}}{a^2} = \sqrt{32\pi G} \varphi e_{ij}$$ (60)
where \( e_{ij}(k) \) is the polarisation tensor normalised by the condition \( e_{ij}e^{ij} = 1 \), \( i, j = 1, 2, 3 \), we get:

\[
k^3 h^2_g(k) \equiv \langle k^3 h_{ij}(k)h^{ij*(k)} \rangle \gg \frac{H l_P^2}{\lambda^2} \cdot \frac{1}{2N + 1}
\]  

(61)

where \( l_P = \sqrt{G} \) is the Planck length and \( \lambda = 2\pi a k^{-1} \) is the wavelength of the perturbation.

The condition (59) is increasingly better satisfied with the expansion of the Universe. Thus, the quasi-isotropic mode becomes more and more classical at later times. For the simplest inflationary initial conditions (57), the inequalities (59) and (61) are satisfied if \( \lambda H(2N + 1) \gg 1 \), in particular, everywhere in the long-wavelength regime \( \lambda H \gg 1 \) even for \( N = 0 \).

Eqs. (59) and (61) were derived under the assumption \( \lambda H \gg 1 \). In the opposite case \( \lambda H \ll 1 \), i.e. after the last Hubble crossing, eq. (54) of paper [7] must be used instead of (55). This leads to the classicality condition:

\[
k^3 h^2_g(k) \gg \frac{l_P^2}{\lambda^2} \cdot \frac{1}{2N + 1}
\]  

(62)

which was derived in [7] for the case \( N = 0 \). On the other hand, let us consider the case where the quasi-isotropic mode is negligible with respect to the decaying mode while the latter is imaginary (|\( \Re C_2 \)| \( \ll |\Re C_2 | \) in the representation where \( \Im C_1 = 0 \)). This requires, in particular, \( C_2^2 \) to be much less than the value given by (57) at the moment of any Hubble radius crossing. Note that \( \Im C_2 \) is uniquely defined by \( C_1 \) through the normalisation condition (56) while \( \Re C_2 \) is not restricted at all. Then the classicality condition takes the form:

\[
|C_2|^2 \gg \frac{a^3 H}{2N + 1}, \quad k^3 h^2_g(k) \gg \frac{l_P^2}{\lambda^3 H} \cdot \frac{1}{2N + 1}.
\]  

(63)

These inequalities become weaker with the expansion of the Universe for all interesting cases (in particular, for \( a(t) \propto t^n \) with \( n > \frac{1}{3} \)) and may even cease to be valid. Thus, the imaginary decaying mode becomes less classical and more quantum at later times. Finally, a sufficiently large real and positive decaying mode (\( |\Re C_2| \ll |\Im C_2 | \) is always classical, since in this case the inequality (58) is already satisfied due to the third term in the left-hand side. In contrast, a negative decaying mode (\( \Re C_2 < 0, |\Re C_2| \gg |\Im C_2 | \)) has a rather peculiar behaviour: being classical at early and late times, it ceases to be classical at the moment when the first and the third term in the left-hand side of (58) almost cancel each other.

All that was said above refers either to the case of a massless minimally coupled scalar field \( \varphi \), or to tensor metric perturbations (gravitational waves). In the case of scalar (or adiabatic) metric perturbations, one should first either specify a gauge, or choose some gauge-independent quantity describing the perturbations. Let us consider the gauge-independent quantity \( \Phi \) which coincides with the Newtonian
potential in the longitudinal gauge \( g_{00} = 1 + 2\Phi \). However, one should keep in mind that semiclassical behaviour may be more pronounced in terms of some quantities and less pronounced, or even absent, for some other quantities.

Let us further restrict ourselves to the case where pressure perturbations are diagonal. This includes all cases where matter is a mixture of an arbitrary number of scalar fields with arbitrary potentials (but minimally coupled to the gravitational field) and an arbitrary number of ideal fluids. Then, as is well known, in the long-wave regime \( \lambda \gg \lambda_J \sim H^{-1}c_S \) where \( \lambda_J \) is the Jeans length and \( c_S \) is the sound velocity, a Fourier mode of the gravitational potential \( \Phi_k \) may be written as:

\[
\Phi_k = \tilde{C}_1(k) \left( 1 - \frac{H}{a} \int_{\eta_1}^{\eta} a^2 d\eta \right) - \frac{4\pi G}{k^2} \tilde{C}_2(k) \frac{H}{a} \tag{64}
\]

where \( \tilde{C}_1(k) \) can be made real and positive by a phase rotation (see, e.g., [15] for a simple derivation of a more general statement valid in the synchronous gauge). Once more, the first term in the right-hand side of (64) is the quasi-isotropic mode (which may be really called the growing mode now since the corresponding density perturbation increases with the expansion of the Universe), the second term is the decaying mode. \( \eta_1 \) should be as small as possible, but not less than the moment of the first Hubble radius crossing at the inflationary stage.

If the operator \( \Phi(k, \eta) \) is represented as:

\[
\Phi(k, \eta) = \Phi_k(\eta) a(k, \eta_0) + \Phi_\ast_k(\eta) a^\dagger(-k, \eta_0) , \tag{65}
\]

then the normalisation condition for the coefficients \( \tilde{C}_1(k) \) and \( \tilde{C}_2(k) \) of the mode functions \( \Phi_k(\eta) \) in the long-wave regime has the same form as (66):

\[
\tilde{C}_1 \Im \tilde{C}_2 = -\frac{1}{2} . \tag{66}
\]

For the case of matter in a Universe consisting of one minimally coupled scalar field \( \varphi \), this statement follows from the fact that in this case, canonical variables analogous to \( y \) and \( p \) are given by [16]:

\[
\xi \equiv a\zeta = a\delta \varphi_L + \frac{\varphi'}{H} \Phi , \quad \pi_\xi = \xi' - \frac{a'}{a} \xi \tag{67}
\]

where \( \delta \varphi_L \) is the scalar field perturbation in the longitudinal gauge. Using two exact relations between \( \zeta_k \) and \( \Phi_k \):

\[
\Phi_k = -\frac{4\pi G a^2}{k^2} \frac{\varphi^2}{H} \left( \frac{H}{\varphi} \zeta_k \right) ; \\
\zeta_k = \frac{1}{4\pi G} \frac{H}{a}\frac{\varphi}{H} \left( \frac{a}{H} \Phi_k \right) , \tag{68}
\]

\[
\end{align}
\]

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and the exact background equation $\dot{H} = -4\pi G \dot{\varphi}^2$, it is straightforward to obtain the leading terms in $\zeta_k$ and $\xi_k$ in the long-wavelength region from eq. (64):

$$
\zeta_k = \tilde{C}_1 \frac{\dot{\varphi}}{H} + \tilde{C}_2 \frac{\dot{\varphi}}{H} \int_{-\infty}^{t} \frac{H^2}{2 \phi^2} dt ;
$$
$$
\xi_k = \tilde{C}_1 \frac{\varphi'}{H} + \tilde{C}_2 \frac{\varphi'}{H} \int_{-\infty}^{t} \frac{H^2}{2 \phi^2} d\eta .
$$

(69)

Since, on the other hand, $\xi_k$ should satisfy the normalisation condition $\xi_k^* \xi_k - \xi_k^* \xi_k = -i$, we arrive to (66). Note also the exact equal-time commutation relation following from the canonical commutation relations between $\xi$ and $\pi$:

$$
[\Phi(k, \eta), \Phi^+(k, \eta)] = -i \frac{4 \pi G H^2}{k} \frac{\dot{H}}{a} .
$$

(70)

For the simplest inflationary models driven by one slow-rolling minimally coupled scalar field $\varphi$:

$$
\tilde{C}_1(k) = \frac{H_k^2}{\sqrt{2k^{3/2}|\dot{\varphi}_k|}} , \quad \tilde{C}_2(k) = -\frac{i k^{3/2} |\dot{\varphi}_k|}{\sqrt{2} H_k^2}
$$

(71)

where $H_k$ and $\varphi_k$ are the values of $H(t)$ and $\varphi(t)$ at the moment of the first Hubble radius crossing during the inflationary stage. Eq. (71) leads to the standard expressions for density perturbations generated during inflation.

Let us now turn to the case of matter consisting of one hydrodynamical component with an ideal fluid equation of state $P = P(\varepsilon)$, where $P$ is the pressure, $\varepsilon$ is the energy density. The canonical variables analogous to $y$ and $p$ in this case were first found by Lukash [17]:

$$
\bar{q} \equiv \bar{aq} = -\frac{3\bar{a} H^2}{\sqrt{8\pi G} aH} \left( \frac{a}{H} \Phi \right) , \quad \bar{a} = \frac{a}{c_S} \sqrt{1 + \frac{P}{3}} , \quad c_S^2 = \frac{dP}{d\varepsilon} ,
$$

$$
\pi_{\bar{q}} = \bar{q}' - \frac{\bar{a}'}{\bar{a}} \bar{q}
$$

(72)

(it is assumed here that $c_S \neq 0$). If the sound velocity $c_S$ is constant, then $\bar{a} \propto a$. The corresponding wave equation for phonon mode functions has the form:

$$
\ddot{q}_k + \left( k^2 c_S^2 - \frac{\bar{a}''}{\bar{a}} \right) q_k = 0 .
$$

(73)

Note also the second relation between $q_k$ and $\Phi_k$:

$$
\Phi_k = -\frac{3 \sqrt{2 \pi G}}{k^2} H \bar{a}^2 \dot{q}_k ,
$$

(74)
and two useful identities for the background metric in this case:

\[ 1 + \frac{P}{\varepsilon} = -\frac{2\dot{H}}{3H^2}, \quad 1 + c_s^2 = -\frac{\dot{H}}{3HH}. \]  

(75)

The corresponding leading long-wavelength behaviour of \( q_k \) is:

\[ q_k = \frac{3}{\sqrt{8\pi G}} \left( \tilde{C}_1 + \frac{8\pi G}{3} \tilde{C}_2 \int_{\infty}^{t} \frac{dt}{a^3} \cdot \frac{c_s^2}{1 + \frac{P}{\varepsilon}} \right). \]  

(76)

Using (72) and (74), it is easy to verify that the commutation relation (70) holds in this case too, which leads to (66) once more. Though we derived the commutation relation (70) for two particular cases (one scalar field or one ideal fluid), we believe that it has a much wider range of validity and we leave the proof of this assertion for another publication.

The condition for semiclassical behaviour of the gravitational potential is:

\[ \left| \langle \Phi(k, \eta) \Phi^\dagger(k, \eta) + c.c. \rangle \right| \gg \left| [\Phi(k, \eta), \Phi^\dagger(k, \eta)] \right|. \]  

(77)

Substituting here the long-wave asymptote (74) and assuming hydrodynamical matter with a constant sound velocity (that leads to the power-law behaviour \( a(t) \propto a^n \) with \( \frac{1}{3} \leq n \leq \frac{2}{3} \)), we get the following condition of classicality similar to (58):

\[ \left| \tilde{C}_1^2 \cdot O(1) \frac{k^2}{4\pi G} \frac{k^2 c_s^2}{aH^3} + |\tilde{C}_2|^2 (n + 1) \frac{4\pi G}{k^2} \frac{H}{a} - \tilde{C}_1 \Re \tilde{C}_2 \right| \gg \frac{1}{2N + 1}. \]  

(78)

If the growing mode is dominating, then

\[ \tilde{C}_1^2 \gg \frac{4\pi G}{k^2} \frac{aH^3}{k^2 c_s^2} \frac{1}{2N + 1}. \]  

(79)

This condition is increasingly better satisfied with the expansion of the Universe. In terms of the gravitational potential, this leads to

\[ k^3 \Phi^2(k) \equiv \langle k^3 |\Phi(k)|^2 \rangle \gg \frac{l_p^2 \lambda H^3}{c_s^2} \frac{1}{2N + 1}. \]  

(80)

On the other hand, for a purely decaying imaginary mode (\( |\Re \tilde{C}_2| \ll |\Im \tilde{C}_2| \)), we get

\[ |C_2|^2 \gg \left| \frac{k^2 a}{4\pi G} \frac{1}{H^2 2N + 1} \right|, \quad k^3 \Phi^2(k) \gg \frac{l_p^2 H}{\lambda} \frac{1}{2N + 1}. \]  

(81)

This condition weakens with the expansion of the Universe. Note that the conditions for the gravitational potential (80,81) are more restrictive than the corresponding conditions for gravitational waves (61,63) since the right-hand sides of the former
conditions are in $(\lambda H)^2 \gg 1$ times larger (however, the growing mode is always semi-classical in the long-wavelength regime for the most interesting case of perturbations produced during inflation with the initial conditions (71)). As pointed above, this peculiar feature may be a result of $\Phi$ not being the best quantity to study quantum-to-classical transition.

In conclusion, we see that the condition for a perturbation to become more and more classical at later times (i.e. with the expansion of the Universe) chooses a subclass of all solutions for perturbations, namely, the quasi-isotropic modes. We will return to this point at the end of the paper.

5 The Wigner function

The Wigner function was proposed as a candidate for the probability density function of a quantum mechanical system in phase space. According to the rules of quantum mechanics such a probability density can not exist, so one can only hope to come as close as possible to it by requiring from good candidates, like the Wigner function, to fulfill some basic properties of the probability density, but not all of them. In particular, the Wigner function is real, but not definite positive. Let us calculate the Wigner function for the system under consideration:

$$W_{NN} \equiv W(y_0, y_0^*, p_0, p_0^*) = \frac{1}{(2\pi)^2} \int dx_1 \, dx_2 \, e^{-i(p_1x_1 + p_2x_2)} \Psi_N^*(y - \frac{x}{2}, \eta) \Psi_N(y + \frac{x}{2}, \eta).$$

We will use the simplified notations $y(k, \eta_0) \equiv y = y_1 + iy_2$ and the same convention for $p(k, \eta_0)$. The quantity $\bar{y}_1$, resp. $\bar{y}_2$, is canonically conjugate to $y_1$, resp. $y_2$, and so $\bar{p}_1 = 2p_1$, $\bar{p}_2 = 2p_2$.

We consider first the Wigner function $W_{\alpha\beta}$ of the coherent initial state $\Psi_{\alpha\beta}$. Actually, it is clear from (17) that it will be similar to the Wigner function $W_{00}$ of the vacuum $|0, \eta\rangle$. The following result is obtained

$$W_{\alpha\beta} = |\Psi_{\alpha\beta}|^2 \delta \left(p_1 - \frac{F(k)}{|f_k|^2} y_1\right) \delta \left(p_2 - \frac{F(k)}{|f_k|^2} y_2\right).$$

We recognize the vacuum Wigner function $W_{00}$ with $p \rightarrow p - \langle p \rangle$ and $y \rightarrow y - \langle y \rangle$.

We now turn our attention to the Wigner function $W_{NN}$ of $N$-particles states. Our method will be to compute the generating functional of $W_{NN}$. For this purpose we first write down the generating functional of $W_{mn}$:

$$\sum_{m,n=0}^\infty W_{mn} z_m v^n = \frac{1}{(2\pi)^2} \int dx_1 \, dx_2 \, e^{-i(p_1x_1 + p_2x_2)} \times$$
\[
\sum_{m,n=0}^{\infty} L_m \left( \frac{|y - x/2|^2}{|f_k|^2} \right) L_n \left( \frac{|y + x/2|^2}{|f_k|^2} \right) z^m z^n \Psi_0^* \left( y - \frac{x}{2}, \eta \right) \Psi_0 \left( y + \frac{x}{2}, \eta \right) \quad (84)
\]

The generating functional of the Laguerre polynomials is given by:
\[
\sum_{m=0}^{\infty} L_m(x) z^m = \frac{1}{1 - z} \exp\left( -\frac{xz}{1 - z} \right) \quad (|z| < 1) \quad (85)
\]

We insert the expression for the vacuum wave function into (84), replace each sum of Laguerre polynomials by the generating function (85) and rearrange terms in the exponentials. The integration reduces to Fourier transforms of gaussians. In order to give a compact result, we will set:
\[
Y = \frac{y}{|f_k|}, \quad P = \left( \frac{2 F_k}{|f_k|} y - |f_k| \bar{p} \right) \quad (86)
\]

The result of the integration can then be written in the following way:
\[
\sum_{m,n=0}^{\infty} W_{mn} z^m v^n = \frac{1}{\pi^2} \frac{1}{1 - zv} \exp \left( -\frac{(|Y|^2 + |P|^2)(1 + zv)}{1 - zv} \right) \times
\exp \left( \frac{(-|Y|^2 + |P|^2 + 2iY_1P_1 + 2iY_2P_2) v}{1 - zv} \right) \times
\exp \left( \frac{(-|Y|^2 + |P|^2 - 2iY_1P_1 - 2iY_2P_2) z}{1 - zv} \right)
\]
\[
= \frac{1}{\pi^2} \frac{1}{1 - zv} \exp \left( -\frac{(|Y|^2 + |P|^2)(1 + zv)}{1 - zv} \right) \times
\sum_{i=0}^{\infty} \frac{(-|Y|^2 + |P|^2 + 2iY_1P_1 + 2iY_2P_2)^i v^i}{i!(1 - zv)^i} \times
\sum_{j=0}^{\infty} \frac{(-|Y|^2 + |P|^2 - 2iY_1P_1 - 2iY_2P_2)^j z^j}{j!(1 - zv)^j} \quad (87)
\]

We have calculated the finite form of the series \( \sum_{m,n=0}^{\infty} W_{mn} z^m v^n \), but what we really want in order to get \( W_{NN} \) is the finite form of the series \( \sum_{m=0}^{\infty} W_{mm} u^m \). This means that we must select terms of equal order in \( z \) and \( v \) in the double sum and replace the product \( zv \) by \( u \). Looking at the last expression (87), we see that the terms that must be kept are those with \( i = j \) in the two sums. We can perform the transformation and write the relevant generating functional:
\[
\sum_{m=0}^{\infty} W_{mm} u^m = \frac{1}{\pi^2} \frac{1}{1 - u} \exp \left( -\frac{(|Y|^2 + |P|^2)(1 + u)}{1 - u} \right) \times
\sum_{i=0}^{\infty} \frac{((|Y|^2 - |P|^2)^2 + (2Y_1P_1 + 2Y_2P_2)^2)^i u^i}{(i!)^2 (1 - u)^{2i}}
\]

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\[
\frac{1}{\pi^2} \frac{1}{1-u} \exp \left( -\frac{(|Y|^2 + |P|^2)(1+u)}{1-u} \right) \times J_0 \left( 2 \left[ -u \left( (|Y|^2 - |P|^2)^2 + (2Y_1P_1 + 2Y_2P_2)^2 \right) \right]^{1/2} \right), \tag{88}
\]

where \( J_0 \) is the Bessel function.

It is now possible to identify the generating functional of a product of Laguerre polynomials by using the identity (eq. 8.976.1 in [14]):

\[
\sum_{m=0}^{\infty} L_m(x)L_m(y)u^m = \frac{1}{1-u} \exp \left( -\frac{(x+y)u}{1-u} \right) J_0 \left( \frac{2(-xyu)^{1/2}}{1-u} \right). \tag{89}
\]

If we take:

\[
x = |Y|^2 + |P|^2 + 2Y_1P_2 - 2Y_2P_1 = |Y - iP|^2,
\]
\[
y = |Y|^2 + |P|^2 - 2Y_1P_2 + 2Y_2P_1 = |Y + iP|^2, \tag{90}
\]

we can finally write (88) as:

\[
\sum_{m=0}^{\infty} W_{mm}u^m = \frac{1}{\pi^2} e^{-(|Y|^2+|P|^2)} \sum_{m=0}^{\infty} L_m(x)L_m(y)u^m. \tag{91}
\]

This gives the final result:

\[
W_{mm} = \frac{1}{\pi^2} e^{-(|Y|^2+|P|^2)} L_m(|Y - iP|^2) L_m(|Y + iP|^2) = W_{00} L_m(|Y - iP|^2) L_m(|Y + iP|^2). \tag{92}
\]

Transition to semiclassical behaviour can be analyzed using this result. We see from the last expression that the Wigner function, again in the limit of infinite squeezing \(|r_k| \to \infty\), will be delta-concentrated around classical trajectories of the system. Indeed, \(W_{00}\), the Wigner function corresponding to the initial vacuum satisfies

\[
W_{00} \to |\Psi_0|^2 \delta(P) \tag{93}
\]

and, therefore, also

\[
W_{NN} \to |\Psi_N|^2 \delta(P) \tag{94}
\]

where \(\delta(P)\) is the \(\delta\)-distribution. Thus, we obtain one more independent derivation of the relation (39) between \(y\) and \(p\) in the limit \(|r_k| \to \infty\) which results in the reduction of (40) to (41).

An interesting point is that, in contrast to the vacuum initial state where the Wigner function is positive definite, here the Wigner function is no longer positive definite before the limiting transition \(|r_k| \to \infty\) leading to (44) is performed. Thus,
$W$ cannot be interpreted as a probability distribution in classical phase space for any finite $|r_k|$. This property actually constitutes the core of the criticism of the Wigner function approach in the papers $[11]$. However, we do not want to insist on this classical interpretation of the Wigner function before the limiting transition (why should we?), and we use the Wigner function only to show that, for high squeezing, it becomes concentrated near a bunch of classical trajectories in phase space with initial stochastic amplitude $y(k, \eta_0)$ obeying the probability density function $[37]$. For this reason, too, we deliberately use the word “concentrated” instead of “peaked” because the latter term, though often used may create a wrong impression about the positivity of the fine structure of the Wigner function around these classical trajectories. On the other hand, after the limiting transition $|r_k| \to \infty$ the Wigner function acquires the structure $[94]$ which may be used as a classical probability distribution in phase space. Moreover, this probability distribution is sufficient for all problems where oscillating quantum fine structure (“wave pattern”) around classical trajectories can be neglected.

This discussion also sheds some new light on the question as to what extent any coarse graining is necessary to obtain the transition to semiclassical behaviour of quantum cosmological perturbations. A coarse graining becomes really necessary if one intends to obtain some classical probability distribution in phase space from the Wigner function before the limiting transition $|r_k| \to \infty$. However, for quantities for which the above mentioned quantum fine structure is unimportant (and all quantities observable in foreseeable future belong to this class) all parameters resulting from the correct (i.e. physically justified) coarse graining disappear after the transition $|r_k| \to \infty$, so we are allowed to forget about coarse graining at all. As a result, for these quantities, decoherence (understood as a partial loss of quantum coherence, with the remaining part of coherence being expressible in classical terms) is reached without consideration of any actual mechanism of coarse graining and decoherence (that is why we call it “decoherence without decoherence”). On the other hand, there do exist quantities and problems for which this fine structure is important (e.g., it is essential for the calculation of the entropy of cosmological perturbations). But then one should not introduce a coarse graining simply by hand or average over some degrees of freedom of the Universe arbitrarily called as “unobservable”. All such choices have to be justified by consideration of real physical processes.

The very specific case of a coherent initial state, for which the Wigner function is given by eq. $[83]$, is still instructive since it clearly shows a smooth transition between deterministic and purely stochastic behaviour of perturbations in the semiclassical limit. Indeed, if $|\langle y(\eta) \rangle|^2 \gg \max\{\frac{1}{2k}, \ |f_k(\eta)|^2\}$, we get an effective deterministic classical behaviour (a time-dependent Bose condensate of the field $y$). Realization of this inequality at all times is possible, e.g., by choosing initial conditions $\langle y \rangle_0$ and $\langle p \rangle_0$ proportional to $f_k(\eta_0)$ and $g_k(\eta_0)$ resp. with a large coefficient of proportionality. In the opposite limit $|f_k(\eta)|^2 \gg \max\{|\langle y(\eta) \rangle|^2, \ \frac{1}{2k}\}$ characteristic for real cosmological perturbations, the average value of perturbations is negligible with respect to their
rms value. Then quantum perturbations become equivalent (in the sense defined above) to classical quantities with a stochastic amplitude, a fixed phase defined by the $\delta$ multipliers in the expression for the Wigner function (83), and a practically zero average value. Of course, generally both average and rms values may be non-zero. We see that the appearance of deterministic quantities in the semiclassical limit is a very specific phenomenon restricted to a narrow class of both special initial states and special time evolution (with not too many “particles” created). In particular, even the choice of a coherent initial state which is already classical and deterministic at the initial moment $\eta_0$ does not guarantee that it remains so at later times. A constructive counterexample is provided by the case where the initial conditions $\langle y \rangle_0$ and $\langle p \rangle_0$ correspond to the pure decaying mode in the long-wavelength regime $\lambda H \gg 1$. On the contrary, transition to effective stochastic classical behaviour is a generic effect which arises as a result of high squeezing of a very wide class of initial quantum states.

6 Thermal state

Until now we have considered N-particles initial states. Let us now consider the important case of a thermal initial state. Such a state arises for example in the cosmological scenario where the stage of “thermal inflation” takes place after relaxation of the radiation-dominated Universe to the state of thermal equilibrium if we mean by $\eta_0$ the moment when this stage of late inflation begins. In this case, the probability density in the $y(k)$-representation is given by:

$$P(y(k), y(-k)) = (1 - e^{-N_0}) \langle y| \left( \sum_{N=0}^{\infty} e^{-N/N_0} |N\rangle \langle N| \right) |y\rangle$$

$$= A(N_0) \sum_{N=0}^{\infty} e^{-N/N_0} \psi_N^*(y(k)) \psi_N(y(k)). \quad (95)$$

where $N_0$ is the ratio of the temperature to the energy field quanta for given $k$ and $A(N_0)$ is a normalization constant. So, actually $N_0$ depends on $k$ but we drop the $k$ for brevity of notation. The probability to find $N$ quanta in the initial state drops rapidly for $N > N_0$.

This probability density can be calculated using the generating functionnal of squared Laguerre polynomials, already given in (89):

$$P(|y|) = \frac{(1 - e^{-N_0})}{\pi |f_k|^2} \left\{ \sum_{N=0}^{\infty} e^{-N/N_0} L_N^2 \left( \frac{|y|^2}{|f_k|^2} \right) \right\} \exp \left( -\frac{|y|^2}{|f_k|^2} \right)$$

$$= \left\{ \exp \left( -\frac{|y|^2}{|f_k|^2} 2e^{-1/N_0} \right) J_0 \left( \frac{|y|^2}{|f_k|^2} 2ie^{-1/N_0} \right) \right\} \times \frac{1}{\pi |f_k|^2} \exp \left( -\frac{|y|^2}{|f_k|^2} \right)$$

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\[
\frac{1}{\pi |f_k|^2} \exp \left( -\frac{|y|^2}{|f_k|^2} \coth \frac{1}{2N_0} \right) \int_0^\infty \frac{x^n \exp(-x \coth \frac{1}{2N_0})}{\sinh \left| \frac{y}{|f_k|^2} \right|} dx .
\]

(96)

When \( N_0 \ll 1 \), \( i.e. \) the temperature is much smaller than the energy of the quanta for given \( k \), we see that \( P \) is almost Gaussian, since the exponential dominates the Bessel function. This is of course to be expected since then our thermal state almost reduces to the vacuum state, there are practically no thermal excitations. What can be said if \( N_0 \) grows, from what temperature on will we get a noticeable departure from a Gaussian probability density? It turns out that for any value of \( N_0 \), the probability \( P \) is a symmetric single-peak function centered around zero, at first sight quite similar to a gaussian. Therefore, the relevant discriminating parameter to be calculated in order to study the gaussianity of this distribution is the kurtosis. Fortunately, the latter can be found analytically. Since \( P \) depends only on \( |y| \), its even momenta are given by the following integrals:

\[
I_{2n} = \int_0^\infty |y|^{2n} P(|y|) 2\pi |y| dy = |f_k|^{2n} \int_0^\infty x^n \exp(-x \coth \frac{1}{2N_0}) \frac{1}{\sinh \left| \frac{y}{|f_k|^2} \right|} dx .
\]

(97)

Four different integrals of the Bessel functions are given in [14], 6.621.1, with the results expressed in terms of the hypergeometric function \( F \). Using the third one, we check the normalisation \( I_0 = 1 \). With the second one, we calculate \( I_2 \) and \( I_4 \), and finally the kurtosis \( Q \):

\[
I_2 = |f_k|^2 \coth \frac{1}{2N_0} F \left( -\frac{1}{2}, 0; 1; \cosh^{-2} \frac{1}{2N_0} \right) = |f_k|^2 \coth \frac{1}{2N_0} ,
\]

(98)

\[
I_4 = 2|f_k|^4 \coth^2 \frac{1}{2N_0} F \left( -1, -\frac{1}{2}; 1; \cosh^{-2} \frac{1}{2N_0} \right) = |f_k|^4 \frac{1 + 2 \cosh^{-2} \frac{1}{2N_0}}{\sinh^2 \frac{1}{2N_0}} ,
\]

(99)

\[
Q = I_4/(I_2)^2 - 2 = \cosh^{-2} \frac{1}{2N_0} .
\]

(100)

Looking at fig.1, we see that there is a sharp transition between a low-temperature gaussian behaviour \( (Q = 0) \) and a high-temperature non-gaussian behaviour \( (Q = 1) \). Intermediate values of the kurtosis imply that \( N_0 \) is close to one, within only one order of magnitude. Fig.2 shows how the gaussian transforms into another distribution at high temperature. Scales of cosmological interest were inside, most of them even deep inside, the Hubble radius at the onset of inflation. Their energy is therefore much higher than the temperature at that time. This corresponds to values of \( N_0 \ll 1 \) for which, as can be seen from (100) and from fig.2, the probability looks very much like a Gaussian.
7 Conclusions and discussion

We have shown here that, for a wide class of initial non-vacuum states including all \(N\)-particles, thermal and most of the coherent states, quantum inhomogeneous cosmological perturbations (both adiabatic perturbations and gravitational waves) generically acquire semiclassical behaviour in the regime when their wavelength is much larger than the Hubble radius. In the language of quantum field theory, the latter regime corresponds to the high squeezing limit \(|r_k| \to \infty\). Then an equivalent classical description of these perturbations in terms of c-number stochastic quantities becomes possible. This, according to our paradigm, constitutes the transition from quantum to effectively classical behaviour of cosmological perturbations. Thus, we generalize the results of [7] obtained for the perturbations of the vacuum initial state. The main difference between vacuum and non-vacuum initial conditions is that the statistics of the stochastic amplitude of the equivalent classical field modes is Gaussian in the former case but generally non-Gaussian in the latter case.

We proved this equivalence in three different independent ways. First, we explicitly constructed the wave function in the Schrödinger representation thereby exhibiting its WKB shape. Then, we have shown that the operators in the Heisenberg representation correspond to classical stochastic functions in that limit. Finally, we have computed the Wigner function and shown that it becomes concentrated along classical trajectories. Since we do not in general choose a unique vacuum state from an infinite number of unitary non-equivalent vacuum states at the initial moment \(\eta_0\) (the case of the inflationary scenario where a unique choice is possible was used as an illustration only), our results are valid for \(N\)-particles, thermal and coherent states constructed using any possible vacuum state. From a mathematical point of view, this ambiguity is reflected in the freedom to choose the constant parameters \(C_1(k)\) and \(C_2(k)\) characterizing the long-wavelength regime (53) which satisfy only one normalization condition (54).

It turns out that the generalization from vacuum to non-vacuum initial conditions does not produce any new problem of principle for the quantum-to-classical transition. The only thing which is important for it is that the classicality conditions for perturbations derived in section 4 should be satisfied (in particular, it is enough to have a sufficiently large amplitude of the quasi-isotropic mode of perturbations in the long-wavelength regime). The Wigner function for \(N\)-particles or thermal initial states (but not for coherent initial states) is not positive definite for a finite \(r_k\). However, we have shown that this does not prevent the transition to the semiclassical behaviour in the limit \(|r_k| \to \infty\) expressed by eq. (94). We believe that this addresses the criticism against the transition to the semiclassical behaviour raised in [11] since the generalization of our considerations to the case where the scale factor \(a(\eta)\) is itself quantized but still in the WKB regime does not bring new principally different features.

Note also that we never use the so called “semiclassical gravity”, or one-loop ap-
proximation, $R_{ik} = 8\pi G \langle T_{ik} \rangle$ to study the evolution of quantum perturbations and the quantum-to-classical transition for them (these equations are usually understood as equations for the average value $\langle g_{ik} \rangle$ of the metric tensor). This approximation is inappropriate for the problem under consideration, not only because one-loop approximation is not exact when only a few quantum fields are important, but mainly due to the fact that these equations are grossly insufficient in the typical case arising in cosmology where average values of perturbations $\langle \delta g_{ik} \rangle$ are negligible (or even exactly zero) as compared to their rms values, and all their higher moments should be calculated (see also the discussion at the end of section 5). So, one should go beyond the one-loop, or the Gaussian, approximation.

Now we discuss the question whether this non-Gaussianity of perturbations which appears in the case of non-vacuum initial states is observable, and if it can be used to exclude such states using observational data on $\Delta T$ angular fluctuations of the CMB temperature. First, one should take into account that observable quantities in physical space are real and consist of sums of quantities of the type $(y(k, \eta)e^{i\varphi} + c.c.)$, irrespective of the fact whether $y$ is an operator or a c-number stochastic quantity. As a result, the statistics of quantities in real space is the same as for one-mode states. In particular, in the Gaussian case the probability distribution is the one-dimensional Gaussian distribution, not the two-dimensional one that prevails for the complex amplitude $y(k, \eta)$. Then the value of the curtosis $Q$ for quantities in real space in the case of an $N$-particles initial state is given by the expression (54) (e.g., $\varphi$ can be taken as $\Delta T$ itself). Previously, the skewness $S \equiv \langle(\Delta T)^3\rangle$ and topological characteristics, like the genus, of the CMB temperature fluctuations were mainly considered. However, $S = 0$ for all $N$-particles initial states. So, we propose to measure the kurtosis $Q$ since this is more interesting. For non-vacuum states which we considered, $Q$ is negative.

For a non-Gaussian distribution, it is easy to see that $Q$ lies below the cosmic variance level of purely Gaussian perturbations for all initial non-vacuum states where different $k$ modes are $\delta$-correlated and rms values of perturbations depend on $|k|$ only. This results due to the averaging over angular directions which is equivalent to summing over all $k$ modes with the same $|k|$. Such a sum will rapidly become Gaussian due to the central limit theorem. In particular, the curtosis of the sum of $n$ equally distributed modes with curtosis $Q$ is equal to $Q/n \sim 1/n$ while the cosmic variance limit for the Gaussian distribution $\sim 1/\sqrt{n}$. Still, the non-zero kurtosis can be observable for less symmetric and more correlated initial states. An extreme example is when $\Delta T(\theta, \phi)$ can take only 2 values $\pm \Delta_0$ with equal probability. Then the curtosis $Q = -2$ and it certainly lies above the cosmic variance level. On the other hand, this discussion shows that there is a wide class of initial non-vacuum states which cannot be distinguished from the vacuum initial state by looking at the statistics of observational quantities, so it is not easy to disprove them.

Finally, we have shown the crucial role played by the quasi-isotropic mode of perturbations in the quantum-to-classical transition. Namely, if this mode is present
then, as time goes on and the Universe expands, perturbations become more and more semiclassical. On the contrary, this does not take place for the decaying mode. Note that there is no quasi-isotropic mode in the case of the exactly homogeneous and isotropic FRW model (even fully quantized) or of the cosmological model of the Bianchi I type. There are known problems with the quantum-to-classical transition in FRW quantum cosmology (even for large values of the scale factor) summarized in paper [19]. We will not discuss here to what extent these problems are really dangerous for the FRW cosmology, but we would like to point out that the results of our paper suggest that the final solution to these problems will require consideration of more general, inhomogeneous cosmological models which contain the quasi-isotropic mode of perturbations (or its generalization to the fully non-linear case). Also, it is tempting to connect the arrow of time in the Universe with the quantum-to-classical transition. Then a possible relation between the arrow of time and the growth of inhomogeneities in the Universe (the transition from order to chaos) emerges.

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Figure 1: The kurtosis $Q$ as a function of $\log(N_0)$ for a squeezed thermal state specified by $N_0$ (for given $k$).
Figure 2: The probability distribution as a function of log$(N_0)$ again for a squeezed thermal state. In order to visualize the shape of the distribution and not its width, we plot $P$, for given $N_0$, as a function of $|y|/\sqrt{<y^\dagger y>}$. We see that for $N_0 \ll 1$, it is basically Gaussian.