BESOV SPACES ASSOCIATED WITH NON-NEGATIVE OPERATORS ON BANACH SPACES

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Abstract. Motivated by a variety of representations of fractional powers of operators, we develop the theory of abstract Besov spaces $B^{s, A}_{q, X}$ for non-negative operators $A$ on Banach spaces $X$ with a full range of indices $s \in \mathbb{R}$ and $0 < q \leq \infty$. The approach we use is the dyadic decomposition of resolvents for non-negative operators, an analogue of the Littlewood-Paley decomposition in the construction of the classical Besov spaces. In particular, by using the reproducing formulas for fractional powers of operators and explicit quasi-norms estimates for Besov spaces we discuss the connections between the smoothness of Besov spaces associated with operators and the boundedness of fractional powers of the underlying operators.

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1. Introduction

This paper is devoted to the theory of Besov spaces from the operator-theoretic point of view and extending some recent work on Besov spaces associated with operators (see, for example, Besov spaces associated with heat kernels [17] or sectorial operators [37] or, in particular, sectorial operators of angle zero [57]). A new point in our development of the Besov space $B_{s,A}^{q,X}$ associated with an operator $A$ is that we allow the operator $A$ to be a general non-negative operator (not necessarily defined densely or with dense range) on a Banach space $X$ and allow the space $B_{s,A}^{q,X}$ to admit a full range of the smoothness index $s \in \mathbb{R}$ and size index $0 < q \leq \infty$.

The story originates from the theory of fractional powers of operators. Roughly speaking, the concept of fractional powers of operators is a continuous analogue of the discrete scale of regularity associated with operators in the theory of abstract Cauchy problems and PDEs (see [65, Chapters 3 and 5] for more information on the history and development of fractional powers of operators). In general, it is difficult to describe accurately the domains of fractional powers (i.e., the fractional domains for short) for non-negative operators on Banach spaces. To some extent, an effective substitute for the fractional domain is the so-called interpolation space, especially in the study of regularity questions (see [14] for the theory of interpolation spaces and [64, 88] for the application of interpolation spaces in the regularity theory of abstract Cauchy problems).

In a series of papers [50]-[55], H. Komatsu developed the theory of fractional powers of non-negative operators on Banach spaces. In particular, in [51] H. Komatsu revealed that the real interpolation space $(X,D(A^{\alpha}))_{\theta,q}$ between the fractional domain $(D(A^{\alpha}), \|A^{\alpha} \cdot\|)$ and the underlying space $(X, \|\|)$ can be characterized via a novel space $D_{s}^{\alpha,q}(A)$. More precisely, let $A$ be a non-negative operator defined densely on a Banach space $(X, \|\|)$. For $s > 0$ and $1 \leq q \leq \infty$, the so-called Komatsu space $D_{s}^{\alpha,q}(A)$ is given by

$$D_{s}^{\alpha,q}(A) := \left\{ x \in X : \int_{0}^{\infty} \left\| t^{s} A^{\alpha}(t + A)^{-n} x \right\|^{q} \frac{dt}{t} < \infty \right\}$$

endowed with the norm

$$\|x\|_{D_{s}^{\alpha,q}(A)} := \|x\| + \left\{ \int_{0}^{\infty} \left\| t^{s} A^{\alpha}(t + A)^{-n} x \right\|^{q} \frac{dt}{t} \right\}^{1/q}$$

(with the usual modification if $q = \infty$), where $n$ is an integer greater than $s$. It can be verified that the norm $\|\cdot\|_{D_{s}^{\alpha,q}(A)}$ is independent of integers $n > s$ in the sense of equivalent norms [51, Proposition 1.2], so that $D_{s}^{\alpha,q}(A)$ is well-defined as a Banach space (in the sense of equivalent norms). Furthermore, it also can be verified that the interpolation space $(X,D(A^{\alpha}))_{\theta,q}$ coincides with $D_{\theta,s}^{\alpha,q}(A)$ for $\alpha > 0$ and $0 < \theta < 1$ (see [51, Theorems 3.1 and 3.2]).

From the operator-theoretic point of view, the Komatsu spaces $D_{s}^{\alpha,q}(A)$ yield a pioneering style of abstract (inhomogeneous) Besov spaces. Subsequently, T. Muramatu [71] discussed the inhomogeneous Besov spaces $D_{s}^{\alpha,q}(A)$ with $0 < q \leq \infty$. 

References
and \( \varphi(s) \), a weight function on \( \mathbb{R}_+ \), for non-negative operators \( A \) on quasi-Banach spaces \( X \), where the Besov quasi-norm was essentially the same as that of the Komatsu space \( D^q_\alpha(A) \) with \( s > 0 \). An alternative improvement of the abstract inhomogeneous Besov spaces \( D^q_\alpha(A) \) with the smoothness index extended from \( s > 0 \) to \( s \in \mathbb{R} \) was given by T. Kobayashi and T. Muramatu \([19]\) while the size index is still restricted in the range of \( 1 \leq q \leq \infty \). Furthermore, abstract (homogeneous) Besov spaces \( \dot{B}^q_\alpha(A) \) with \( 1 \leq q \leq \infty \) and \( \varphi \), the weight function on \( \mathbb{R}_+ \) mentioned above, were discussed by T. Matsumoto and T. Ogawa \([67]\) for non-negative operators \( A \) on Banach spaces.

Recall that a closed linear operator on a Banach space is non-negative if and only if it is sectorial (see \([65\), Proposition 1.2.1]). Thanks to sectoriality, an alternative approach to fractional powers of non-negative operators on Banach spaces is the so-called functional calculus (see, for example, \([38\), Chapter 3\]). Using the language of functional calculus, M. Haase \([37\), Section 7\] revealed that the Komatsu spaces \( D^q_\alpha(A) \) mentioned above follow a common pattern, i.e.,

\[
D^q_\alpha(A) = \left\{ x \in X : \int_0^\infty ||t^{-s}\psi(tA)x||^q \frac{dt}{t} < \infty \right\}
\]

in the sense of equivalent norms

\[
||x||_{D^q_\alpha(A)} \simeq ||x|| + \left\{ \int_0^\infty ||t^{-s}\psi(tA)x||^q \frac{dt}{t} \right\}^{1/q}
\]

(with the usual modification if \( q = \infty \)) for \( s > 0 \) and \( 1 \leq q \leq \infty \), where \( \psi \) is a bounded holomorphic function satisfying an appropriate decay estimate on some sector relating to \( A \). In other words, the Komatsu spaces can be characterized by using the natural functional calculus for sectorial operators on Banach spaces.

In particular, applying \( \psi(z) = z^\alpha(1 + z)^{-\alpha} \) with \( 0 < s < \text{Re} \alpha \) to the functional calculus of \( A \) yields the Komatsu space \( D^q_\alpha(A) \), immediately (also, see \([65\), Definition 11.3.1\]). We refer the reader to \([38\), Chapter 6\] for more information on this topic.

More recent work due to C. Kriegler and L. Weis \([57\), Section 5\] reveals that the Komatsu spaces can also be characterized via the Mihlin functional calculus for sectorial operators with angle zero. More precisely, let \( A \) be a sectorial operator of angle zero on a Banach space \( X \) as defined in \([57\), Section 5\], where an additional assumption that \( R(A) = X \) is posed in the definition, so that \( A \) is injective whenever it is sectorial of angle zero. Then the Komatsu spaces \( D^q_\alpha(A) \) admit the so-called Littlewood-Paley decomposition, i.e.,

\[
D^q_\alpha(A) = \dot{B}^q_\alpha(A) := \left\{ x \in \dot{X}_{-N} + \dot{X}_N : \sum_{n=0}^\infty ||2^n s \varphi_n(A)x||^q < \infty \right\}
\]

in the sense of equivalent norms

\[
||x||_{D^q_\alpha(A)} \simeq ||x||_{\dot{B}^q_\alpha(A)} := \left\{ \sum_{n=0}^\infty ||2^n s \varphi_n(A)x||^q \right\}^{1/q}
\]

(with the usual modification if \( q = \infty \)) for \( s \in \mathbb{R} \) and \( 1 \leq q \leq \infty \), where \( \{ \varphi_n \} \) is an inhomogeneous partition of unity on \( \mathbb{R}_+ \) and \( \dot{X}_{\pm N} \), with \( N > |s| \), are the extrapolation spaces given by the completions of \( D(A^{\pm N}) \) with respect to \( ||A^{\pm N}|| \).

A historical account of the classical theory of Besov spaces on \( \mathbb{R}^n \) is beyond the scope of this work, and we refer the reader to, for instance, \([81\), [40\), [89\), [77\].
We shall, however, recall briefly the main techniques used to develop the theory of Besov spaces which are relevant to our work. Recall that the classical Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ are related closely to the negative Laplacian or its square root on the Euclidean space $\mathbb{R}^n$. Indeed, applying $A = -\Delta_p$, i.e., the negative Laplacian on $L^p(\mathbb{R}^n)$ with $1 < p < \infty$, to the Komatsu space $D^s_q(A)$ yields the classical Besov space $B^s_{p,q}(\mathbb{R}^n)$ for $s > 0$ and $1 < q \leq \infty$ in the sense of equivalent norms. So far, compared with the classical approaches to Besov spaces (for example, the initial method of moduli of continuity due to O. V. Besov [15], the Hardy–Littlewood maximal operator characterizations associated with the Poisson kernel and Gaussian kernel due to H. Taibleson [52], the Fourier analytic method and the real variable techniques due to J. Peetre [73] and the approach via the Calderón reproducing formulas due to H.-Q. Bui, M. Paluszynski and M. H. Taibleson [18]), the dyadic techniques due to J. Peetre [73] and the approach via the Calderón reproducing formulas due to H.-Q. Bui, M. Paluszynski and M. H. Taibleson [18], the dyadic decomposition of resolvents provides an alternative approach to the Besov spaces from the operator-theoretic point of view.

In addition to the real-variable techniques developed in the framework of the classical harmonic analysis, there is some recent work on abstract Besov spaces associated with the negative generators of strongly continuous (or, in particular, bounded analytic) semigroups. More precisely, the work connected with Peetre’s approach includes, for instance, [28, 13, 42, 68, 19], where Besov spaces associated with self-adjoint positive operators or the Schrödinger operators are discussed. The work in [17] is devoted to a class of homogeneous Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ with $-1 < s < 1$ and $1 < p, q \leq \infty$ for the negative generators $L$ of analytic semigroups with heat kernels satisfying an upper bound of Poisson type and Hölder continuity. And Besov spaces $B^s_{p,q}(L)$ with a full range of indices $s \in \mathbb{R}$ and $0 < p, q \leq \infty$ are systematically studied in [48] for the negative generators $L$ of semigroups with heat kernels satisfying an upper bound of Gauss type, Hölder continuity and a Markov property. Moreover, a quite recent work on Lipschitz spaces $\Lambda^s$ (i.e., $B^s_{\infty,\infty}$) associated with the Schrödinger operators is given in [29] by using the language of semigroups.

Our work starts from a unified representation of fractional powers of non-negative operators on Banach spaces. Let $A$ be a non-negative operator on a Banach space $X$ and let $z \in \mathbb{C}_+$. It can be verified that

$$ A^z x = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha + z)\Gamma(\beta - z)} \int_0^\infty \lambda^{\alpha - 1} A^\alpha (\lambda + A)^{-\alpha - \beta} x \frac{d\lambda}{\lambda}, \quad x \in D(A^{\alpha + \epsilon}), $$

where $\epsilon > 0$ and $\alpha, \beta \in \mathbb{C}_+$ satisfying $-\text{Re} \alpha < \text{Re} z < \text{Re} \beta$ (see Proposition 2.12 below). It is the representation (1.2) that motivates us to deal with a dyadic decomposition of the integrand and define the inhomogeneous Besov space $B^s_{q,X}$ associated with $A$ to be the completion of the subspace

$$ \left\{ x \in D(A) : \sum_{i=k}^{\infty} \left\| 2^{i(s+\alpha)} A^\beta (2^i + A)^{-\alpha - \beta} x \right\|^q < \infty \right\} \subset X $$

with respect to the quasi-norm

$$ \|x\|_{B^s_{q,X}} := \left\| (2^k + A)^{-\alpha} x \right\| + \left\{ \sum_{i=k}^{\infty} \left\| 2^{i(s+\alpha)} A^\beta (2^i + A)^{-\alpha - \beta} x \right\|^q \right\}^{1/q} $$

(with the usual modification if $q = \infty$) for $s \in \mathbb{R}$ and $0 < q \leq \infty$, where $k \in \mathbb{Z}$ and $\alpha, \beta \in \mathbb{C}_+$ satisfying $-\text{Re} \alpha < s < \text{Re} \beta$ (see Definition 3.6 below).
Indeed, the dyadic decomposition of resolvents of non-negative operators is an analogue of the standard frequency restriction in the construction of the classical Besov spaces on \( \mathbb{R}^n \) and it allows us to construct a variety of Besov spaces with a full range of the smoothness index \( s \in \mathbb{R} \) and size index \( 0 < q \leq \infty \) (see Section 3 below). Moreover, the continuous scale of indices \( \alpha \) and \( \beta \) in the representation (1.2) allows us to describe more exactly the lifting property as well as other properties of interest for fractional powers of non-negative operators on Besov spaces \( B_{q,X}^{s,A} \). Some results are novel even in the case \( 1 \leq q \leq \infty \). More precisely, thanks to the representation (1.2) with continuous scale of indices \( \alpha \) and \( \beta \) and reproducing formula (2.42) of fractional powers of operators, Theorems 5.2 and 5.7 provide equivalent quasi-norms for the lifting and interpolation of Besov spaces, which improves [51, Theorem 3.1] and [37, Corollary 7.3 (b)] in the sense of equivalent (quasi-)norms even in the case \( 1 \leq q \leq \infty \) (see Remark 5.8 below for more information). The paper is organized as follows. In Section 2 we provide the reader with a concise introduction to the concepts of non-negative operators and their fractional powers and present some estimates and representations of fractional powers of operators used in this paper. Section 3 is devoted to the construction of (inhomogeneous and homogeneous) Besov spaces associated with non-negative operators (i.e., abstract Besov spaces for short). Section 4 contains some basic properties of abstract Besov spaces, such as quasi-norm equivalence, continuous embedding and translation invariance. In Section 5 we discuss the connections between the smoothness of abstract Besov spaces and the fractional powers of the underlying operators, including the lifting property, smoothness reiteration and interpolation spaces. Finally, in Section 6 we compare our new Besov spaces with some classical ones, as well as the known Besov spaces developed by the integral transform method, or the functional calculus approach or the Littlewood-Paley decomposition technique, and we show that our new Besov spaces recover many known Besov spaces in the literature.

Notation. Throughout this paper, we shall use the following notation:

\[ \begin{align*}
\mathbb{R}^+_+ & := [0, \infty), \\
\mathbb{C}^+_+ & := \{ \alpha \in \mathbb{C} : \text{Re} \alpha > 0 \} \text{ and } \mathbb{C}^+_+ := \mathbb{C}^+_+ \cup \{0\}, \\
\Sigma_\theta & := \{ z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta \} \text{ for } \theta \in (0, \pi).
\end{align*} \]

We use the notation \( f(t) \equiv c \) to mean that \( f \) is a constant function taking the value \( c \). Also, we use the notation \( a \lesssim b \) to mean that \( a \leq Cb \) for some positive constant \( C \) independent of other relevant quantities and the notation \( a \sim b \) to mean that \( a \lesssim b \lesssim a \) for the sake of simplicity. Moreover, by \( q' \) we denote the conjugate of \( q \in (1, \infty) \), i.e., \( q' \in (1, \infty) \) satisfies that

\[ \frac{1}{q} + \frac{1}{q'} = 1. \]

For the consistency of notation, two standard classical inequalities are formulated as follows, which we shall need to get explicit quasi-norm estimates for abstract Besov spaces. One is, for \( 0 < q < \infty \),

\[ (a + b)^q \leq C_q(a^q + b^q), \quad a, b \geq 0, \]

with \( C_q = \max\{1, 2^{q-1}\} \) and the other is, for \( 0 < q \leq 1 \),

\[ \left( \sum_{i=1}^{\infty} a_i \right)^q \leq \sum_{i=1}^{\infty} a_i^q, \quad a_i \geq 0, i \in \mathbb{N}. \]
2. Fractional powers of operators

In this section we present a number of estimates and representations for the fractional powers of non-negative operators which we shall need for what follows. Some of this material is quite standard and some is relatively new. \( X \) is a complex Banach space and \( A \) is a closed linear operator on \( X \) in this section.

2.1. Non-negative operators. The class of non-negative operators in Banach spaces was introduced by A. V. Balakrishnan \cite{10} to discuss the fractional powers of operators and subsequently studied by H. Komatsu in a series of papers \cite{50}-\cite{55}. In particular, in \cite{52} H. Komatsu gave them this nomenclature.

Recall that \( A \) is said to be non-negative on \( X \) if \((-\infty, 0) \subset \rho(A)\) and

\[
M_A := \sup_{\lambda > 0} \| \lambda(A + \lambda)^{-1} \| < \infty.
\]

(2.1)

Thanks to a simple decomposition of the identity operator, i.e.,

\[
I = \lambda(A + \lambda)^{-1} + \lambda^{-1}(A + \lambda)^{-1}, \quad \lambda \in \rho(A),
\]

(2.2)

it can be seen that \( A \) is non-negative on \( X \) if and only if \((-\infty, 0) \subset \rho(A)\) and

\[
L_A := \sup_{\lambda > 0} \| A(A + \lambda)^{-1} \| < \infty.
\]

(2.3)

For convenience, we call \( M_A \) and \( L_A \) the non-negativity constants of \( A \). Moreover, thanks to (2.3), if a non-negative operator \( A \) is injective then the inverse \( A^{-1} \) of \( A \) is also non-negative with the non-negativity constants \( M_{A^{-1}} = L_A \) and \( L_{A^{-1}} = M_A \). Sometimes a non-negative operator \( A \) with \( 0 \in \rho(A) \) is also called a positive operator.

Also, recall that a family \( \{A_t\}_{t \in \Lambda} \) of non-negative operators \( A_t \) is said to be uniformly non-negative if

\[
M = M_{\{A_t\}_{t \in \Lambda}} := \sup_{t \in \Lambda} M_{A_t} < \infty,
\]

(2.4)

where \( M_{A_t} \) is the non-negativity constant of \( A_t \) given by (2.1). Clearly, a family \( \{A_t\}_{t \in \Lambda} \) of non-negative operators \( A_t \) is uniformly non-negative if and only if

\[
L = L_{\{A_t\}_{t \in \Lambda}} := \sup_{t \in \Lambda} L_{A_t} < \infty,
\]

(2.5)

where \( L_{A_t} \) is the non-negativity constant of \( A_t \) given by (2.3). Moreover, we call \( M \) and \( L \) the uniform non-negativity constants of \( \{A_t\}_{t \in \Lambda} \) for convenience.
Some uniformly non-negative spaces are listed as follows, which are simple but important in the theory of fractional powers of operators and will be used frequently in the sequel.

**Example 2.1.** Let $A$ be non-negative on $X$. The following statements hold.

(i) \( \{ (t + A)^{-1} \}_{t > 0} \) is uniformly non-negative with the uniform non-negativity constants \( M_{(t + A)^{-1}} \leq M_A + L_A \) and \( L_{(t + A)^{-1}} \leq M_A \).

(ii) \( \{ A(t + A)^{-1} \}_{t > 0} \) is uniformly non-negative with the uniform non-negativity constants \( M_{A(t + A)^{-1}} \leq M_A + L_A \) and \( L_{A(t + A)^{-1}} \leq L_A \).

(iii) \( \{ (s + A)(t + A)^{-1} \}_{s,t > 0} \) is uniformly non-negative with the uniform non-negativity constants \( M_{((s + A)(t + A)^{-1})_{s,t > 0}} \leq M_A + L_A \).

Finally, it is necessary to point out that a concept related closely to non-negative operators is the so-called sectorial operators due to T. Kato [47]. It is known nowadays that a closed linear operator on a Banach space is non-negative if and only if it is sectorial (see [65] Proposition 1.2.1). Thanks to sectoriality, it is possible to develop the theory of functional calculus for non-negative operators on Banach spaces. Sectorial operators and functional calculi of them will be discussed in Section 2.2 below. We also refer the reader to, for instance, [58, 86, 9, 36, 11, 59, 12, 33, 65 Chapter 4] and [38 Chapter 2] for more information on this topic.

### 2.2. Fractional powers of operators.

Generally speaking, there are three different approaches to fractional powers of non-negative operators on Banach spaces. One is the integral operator approach, including the complex integrals, i.e., the so-called Dunford-Riesz integrals due to sectoriality of non-negative operators (see, for example, [30, 38] and reference therein). And the third approach involves some abstract (incomplete) Cauchy problem and the fractional powers of the underlying operator can be given by the (unique) solution of the abstract (incomplete) Cauchy problem in a suitable way (see [2, 38] and [38 Chapter 1] for more recent work [39, Theorem 6.2]).

Now we recall fractional powers of non-negative operators starting from real integrals. Let $A$ be non-negative on $X$ and let $\alpha \in \mathbb{C}_+$. Initially, suppose that $A \in \mathcal{L}(X)$, one can define the fractional power $A^\alpha$ of $A$ by the Balakrishnan-Komatsu integral, i.e.,

\[
A^\alpha := \frac{\Gamma(n)}{\Gamma(\alpha) \Gamma(n - \alpha)} \int_0^\infty \lambda^n A^\alpha(\lambda + A)^{-\alpha} \frac{d\lambda}{\lambda},
\]

where Re $\alpha < n \in \mathbb{N}$. Thanks to (2.1) and (2.3), it can be seen that the integral given in the right-hand side of (2.6) absolutely converges. It can also be seen that the part given in the right-hand side of (2.6) is independent of integers $n > \text{Re } \alpha$ which is a simple consequence of the use of integration by parts. This implies that $A^\alpha$ given in (2.6) is well-defined as a bounded linear operator on $X$. By using the well-known resolvent equation (see [5 Appendix B, Proposition B.4]), one can verify the additivity of $A^\alpha$. Furthermore, by use of additivity of fractional powers one can also verify the injectivity of $A^\alpha$ whenever $A$ is injective.
The fractional powers of unbounded non-negative operators can be defined by using a standard approximation technique. More precisely, if $A$ is unbounded with $0 \in \rho(A)$, it makes sense to consider the fractional power $(A^{-1})^\alpha$ of $A^{-1}$ and, thanks to the injectivity of $(A^{-1})^\alpha$ as mentioned above, it is possible to define the fractional power $A^\alpha$ of $A$ as the inverse of $(A^{-1})^\alpha$, i.e.,

$$A^\alpha := [(A^{-1})^\alpha]^{-1}.$$ 

And if $A$ is unbounded while $0 \in \sigma(A)$, it makes sense to consider the fractional power $(A + \epsilon)^\alpha$ of $A + \epsilon$ with $\epsilon > 0$ due to the fact that $0 \in \rho(A + \epsilon)$ and define the fractional power $A^\alpha$ of $A$ by the strong limit

$$A^\alpha := s \lim_{\epsilon \to 0} (A + \epsilon)^\alpha$$

with maximal domain (see [65, Definitions 5.1.2 and 5.1.3]).

The positive powers of non-negative operators not only look like the classical powers of numbers but also work like the classical powers of numbers. More precisely, they satisfy the following two laws of exponents. One is additivity, i.e.,

$$(2.7) \quad A^\alpha A^\beta = A^{\alpha + \beta}, \quad \alpha, \beta \in \mathbb{C}_+.$$ 

The other is multiplicativity, i.e.,

$$(2.8) \quad (A^\alpha)^\beta = A^{\alpha \beta}, \quad 0 < \alpha < \pi/\omega, \beta \in \mathbb{C}_+,$$ 

where $\omega \in (0, \pi)$ is the sectoriality angle of $A$ (see [65, Theorems 5.1.11 and 5.4.3]).

The negative or imaginary powers of non-negative operators can be defined analogously. If this is the case, injectivity of non-negative operators is needed to ensure the single-valuedness of the desired fractional powers. More precisely, assume that the non-negative operator $A$ is injective. Thanks to (2.7), it is easy to verify the injectivity of $A^\alpha$ for each $\alpha \in \mathbb{C}_+$ (also see [65, Corollary 5.2.4 (ii)]), and therefore one can define the negative power $A^{-\alpha}$ of $A$ as the inverse of $A^\alpha$, i.e.,

$$A^{-\alpha} := (A^\alpha)^{-1}, \quad \alpha \in \mathbb{C}_+.$$ 

Compared with positive or negative powers, the imaginary powers of non-negative operators have a slightly more complicated configuration and are the most mysterious objects in the field of fractional powers of operators, even the domains of them have not been fully understood as yet (see Remark 2.11 below). In view of the importance of the closedness for unbounded operators on abstract spaces, one can define the imaginary powers of non-negative operators by using a closed extension of suitable operators. For example, one can consider the composition $A^{-1} A^{1+it}$ and define the imaginary power $A^it$ of $A$ by its closed extension in the following way:

$$(2.9) \quad A^it := (1 + A)^2 A^{-1} A^{1+it} (1 + A)^{-2}, \quad t \in \mathbb{R}.$$ 

See [65, Definition 7.1.2]. Moreover, one may define $A^0 := I$, the identity operator, for the sake of convenience.

Fractional (positive, negative or imaginary) powers of unbounded non-negative operators admit a general version of laws of exponents. On the one hand, in contrast to (2.7), merely the following inclusion holds:

$$(2.10) \quad A^\alpha A^\beta \subset A^{\alpha+\beta}, \quad \alpha, \beta \in \mathbb{C}.$$ 

On the other hand, analogous to (2.8), it holds that

$$(A^\alpha)^\beta = A^{\alpha \beta}, \quad -\pi/\omega < \alpha < \pi/\omega, \beta \in \mathbb{C},$$

where $\omega \in (0, \pi)$ is the sectoriality angle of $A$.
where \( \omega \in (0, \pi) \) is the sectoriality angle of \( A \). See \cite{65} Theorems 7.1.1 and 7.1.3, or \cite{38} Proposition 3.2.1).

In addition to the laws of exponents, the spectral mapping theorem also plays an important role in the theory of fractional powers of operators. The spectral mapping theorem for fractional powers of non-negative operators reads that

\[
\sigma(A^\alpha) = \{ \mu^\alpha : \mu \in \sigma(A) \}, \quad \alpha \in \mathbb{C}_+.
\]

See \cite{38} Proposition 3.1.1 (j) or \cite{65} Theorem 5.3.1. As for general spectral mapping theorems for functional calculi, we refer the reader to \cite{65} Section 4.3 (for the Hirsch functional calculus) and \cite{38} Section 2.7 (for the natural functional calculus).

Next we turn to representations of fractional powers of non-negative operators. Let \( A \) be non-negative on \( X \) and let \( \alpha \in \mathbb{C}_+ \) with \( \text{Re} \alpha < n \in \mathbb{N} \). Thanks to additivity of fractional powers of bounded non-negative operators, by using a standard approximation argument one can conclude from (2.6) that

\[
A^\alpha x = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n - \alpha)} \int_0^\infty \lambda^\alpha A^n(\lambda + A)^{-n} x \frac{d\lambda}{\lambda}, \quad x \in D(A^m),
\]

where \( m \) is the smallest integer greater than \( \text{Re} \alpha \). Also, see \cite{65} Page 59, (3.4)]. In particular, if \( A \) is injective, replacing \( A \) by \( A^{-1} \) in (2.11) yields

\[
\|A^\alpha\| \leq C_{\alpha,n} (L + KM)^n,
\]

where \( L \) and \( M \) are the uniform non-negativity constants of \( \{A_t\}_{t \in \Lambda} \) given in (2.4) and (2.5), respectively, \( K = \sup_{t \in \Lambda} \|A_t\| \) and

\[
C_{\alpha,n} = \frac{\Gamma(\text{Re} \alpha)\Gamma(n - \text{Re} \alpha)}{\Gamma(\alpha)\Gamma(n - \alpha)}
\]

In particular, applying \( A_t \equiv A \in \mathcal{L}(X) \) to (2.13) yields

\[
\|A^\alpha\| \leq C_{\alpha,n} (L_A + M_A)^n,
\]

applying \( A_t = t(t + A)^{-1} \) with \( t > 0 \) to (2.13) yields

\[
\sup_{t > 0} \|t^\alpha(t + A)^{-\alpha}\| \leq C_{\alpha,n} M_A^n (L_A + M_A + 1)^n,
\]

applying \( A_t = A(t + A)^{-1} \) with \( t > 0 \) to (2.13) yields

\[
\sup_{t > 0} \|A^\alpha(t + A)^{-\alpha}\| \leq C_{\alpha,n} L_A^n (L_A + M_A + 1)^n,
\]
and applying $A_{s,t} = (s + A)(t + A)^{-1}$ with $0 \leq s/t \leq c$ for given $c > 0$ to (2.13) yields
\begin{equation}
\sup_{0 \leq s/t \leq c} \|(s + A)^{\alpha}(t + A)^{-\alpha}\| \leq C_{\alpha,n}(L_A + M_A)^n(L_A + cM_A + 1)^n,
\end{equation}
where $M_A$ and $L_A$ are the non-negativity constants of $A$ given in (2.1) and (2.7), respectively.

Remark 2.3. Let $\beta \in \mathbb{C}$, with $\beta = \alpha$ or $\Re \alpha < \Re \beta < m \in \mathbb{N}$. For given $c > 0$, we have
\begin{equation}
\sup_{0 \leq s/t \leq c} \|(s + A)^{\alpha}(t + A)^{-\beta}\| \leq C \|(s + A)^{\alpha}(s + A)^{-\beta}\|,
\end{equation}
and we also have
\begin{equation}
\sup_{0 \leq s/t \leq c} \|(s + A)^{\alpha}(t + A)^{-\beta}\| \leq C \|(t + A)^{-(\beta - \alpha)}\|,
\end{equation}
where $C = C_{\alpha,n}(L_A + M_A)^{m}(L_A + cM_A + 1)^{m}$ with $C_{\alpha,n}$, $M_A$ and $L_A$ given by (2.13), (2.11) and (2.3), respectively.

Proof. The statements can be verified by use of (2.11) and (1.5). Indeed, from (2.11) and (1.5), it follows that
\begin{equation}
\sup_{\lambda > 0} \|(1 + \lambda)^{\alpha}A_{t}^{-1}\| \leq C_{\alpha,n}\sup_{\lambda > 0} \|(1 + \lambda)^{\alpha}A_{t}^{-1}\| \leq C_{\alpha,n}(L + KM)^n,
\end{equation}
from which (2.13) follows immediately. Note that (2.15) is a direct consequence of (2.13), while (2.16), (2.17) and (2.18) are all simple consequences of (2.13) and Example 2.1 where the uniform non-negativity constants of the corresponding families of operators are used. Moreover, (2.19) and (2.20) are two direct consequences of (2.18). The proof is complete. \qed

Remark 2.3. It is necessary to point out that sometimes we may obtain more precise estimates for a variety of families of operators by a routine calculation starting from (2.11) and (1.5), rather than the direct use of (2.13). Indeed, the estimates given in (2.15), (2.16), (2.17) and (2.18) can be improved in the following way.

(i) Analogous to (2.13), by use of (2.11) and (1.5) one can verify that
\begin{equation}
\sup_{\lambda > 0} \|(1 + \lambda)^{\alpha}A_{t}^{-1}\| \leq C_{\alpha,n}M_{A}^{n},
\end{equation}
\begin{equation}
\sup_{\lambda > 0} \|(1 + \lambda)^{\alpha}A_{t}^{-1}\| \leq C_{\alpha,n}L_{A}^{n},
\end{equation}
\begin{equation}
\sup_{0 \leq s/t \leq c} \|(s + A)^{\alpha}(t + A)^{-\alpha}\| \leq C_{\alpha,n}(L_A + \max\{c, 1\} \cdot M_A)^n,
\end{equation}
where $0 < \Re \alpha < n \in \mathbb{N}$, $C_{\alpha,n}$ is given by (2.14), and $M_A$ and $L_A$ are the non-negativity constants of $A$ given by (2.1) and (2.3), respectively. Clearly, the estimates (2.19)*, (2.17)* and (2.18)* are more precise than (2.16), (2.17) and (2.18), respectively. Moreover, (2.19)* and (2.17)* are also given in [33] Corollary 3.1.13 in the case $0 < \alpha < n \in \mathbb{N}$.

(ii) Compared with (2.15), an alternative estimate of $\|A^{\alpha}x\|$ can be given by the so-called moment inequality [65] Lemma 3.1.7, i.e.,
\begin{equation}
\|A^\alpha x\| \leq C(\alpha, n, M_A)\|A^n x\|\|x\|^{|n - \Re \alpha|/n}, \quad x \in D(A^\alpha),
\end{equation}
where $0 < \Re \alpha < n \in \mathbb{N}$, $M_A$ is the non-negativity constant of $A$ given by (2.1) and
\begin{equation}
C(\alpha, n, M_A) = \frac{\Gamma(n + 1)}{|\Gamma(\alpha)\Gamma(n - \alpha)|}\frac{M_A^{nRe\alpha}(M_A + 1)^nRe\alpha}{Re\alpha(n - Re\alpha)}.
\end{equation}
In particular, if $A \in L(X)$, from (2.21) it follows that
\[ ||A^{\alpha}|| \leq C(\alpha, n, M_A)||A||^{Re\alpha}, \]
an estimate different from (2.15). Also, see [65, Remark 5.1.2].

From this we turn to ergodicity of non-negative operators. In view of (2.11), a link between fractional powers and integral powers, it is necessary to point out two simple facts for bounded linear operators. More precisely, for $S, T \in L(X)$ satisfying commutativity, it is routine to verify that
\[ (S + T)^n = \sum_{k=0}^{n} \binom{n}{k} S^k T^{n-k}, \quad n \in \mathbb{N}, \]
and that
\[ S^n - T^n = (S - T) \sum_{k=0}^{n-1} T^k S^{n-1-k}, \quad n \in \mathbb{N}. \]

The following lemma improves slightly [65, Theorem 6.1.1] and [67, Lemma 2.3], which can be verified by using a standard density argument and some operator identities, especially (2.22) and (2.23) above.

**Lemma 2.4.** Let $A$ be a non-negative operator on $X$ and let $\alpha \in \mathbb{C}_+$. The following statements hold.

(i) $D(A) = \overline{D(A^\alpha)}$. Moreover, for $x \in X$,
\[ x \in \overline{D(A^\alpha)} \Leftrightarrow \lim_{t \to \infty} t^\alpha (t + A)^{-\alpha} x = x \Leftrightarrow \lim_{t \to \infty} t^\alpha (t + A)^{-\alpha} x \text{ exists} \]
\[ \Leftrightarrow \lim_{t \to \infty} A^\alpha (t + A)^{-\alpha} x = 0 \Leftrightarrow \lim_{t \to \infty} A^\alpha (t + A)^{-\alpha} x \text{ exists} \]
\[ \Leftrightarrow A^\alpha (\lambda + A)^{-\alpha} x \in \overline{D(A^\alpha)} \text{ for all (or, some) } \lambda > 0. \]

(ii) $R(A) = \overline{R(A^\alpha)}$. Moreover, for $x \in X$,
\[ x \in \overline{R(A^\alpha)} \Leftrightarrow \lim_{t \to 0} t^\alpha (t + A)^{-\alpha} x = x \Leftrightarrow \lim_{t \to 0} t^\alpha (t + A)^{-\alpha} x = 0 \]
\[ \Leftrightarrow \lambda^\alpha (\lambda + A)^{-\alpha} x \in \overline{R(A^\alpha)} \text{ for all (or, some) } \lambda > 0. \]

(iii) $\text{Ker}(A) = \overline{\text{Ker}(A^\alpha)} = \text{Ker}(A^\alpha (t + A)^{-\alpha})$ for $t > 0$. Moreover, for $x \in X$,
\[ x \in \text{Ker}(A^\alpha) \Leftrightarrow A^\alpha (t + A)^{-\alpha} x = 0 \Leftrightarrow \lim_{t \to 0} A^\alpha (t + A)^{-\alpha} x = 0 \]
\[ \Leftrightarrow t^\alpha (t + A)^{-\alpha} x \equiv x \Leftrightarrow \lim_{t \to 0} t^\alpha (t + A)^{-\alpha} x = x. \]

(iv) For $x \in X$, we have
\[ x \in \overline{R(A^\alpha)} \oplus \text{Ker}(A^\alpha) \Leftrightarrow \lim_{t \to 0} A^\alpha (t + A)^{-\alpha} x \text{ exists} \]
\[ \Leftrightarrow \lim_{t \to 0} t^\alpha (t + A)^{-\alpha} x \text{ exists}. \]

Moreover, $\lim_{t \to 0} t^\alpha (t + A)^{-\alpha} x = x_0$ and $\lim_{t \to 0} A^\alpha (t + A)^{-\alpha} x = x_1$ whenever $x = x_0 + x_1$ with $x_0 \in \text{Ker}(A^\alpha)$ and $x_1 \in \overline{R(A^\alpha)}$.

**Proof.** (i) Thanks to [65, Theorem 6.1.1 (ii)], it remains to verify that
\[ \lim_{t \to \infty} A^\alpha (t + A)^{-\alpha} x \text{ exists } \Rightarrow \lim_{t \to \infty} A^\alpha (t + A)^{-\alpha} x = 0, \]
\[ \lim_{t \to \infty} t^\alpha (t + A)^{-\alpha} x \text{ exists } \Rightarrow \lim_{t \to \infty} t^\alpha (t + A)^{-\alpha} x = x \]
and

\[(2.26) \quad x \in \overline{D(A^\alpha)} \iff A^\alpha(\lambda + A)^{-\alpha}x \in \overline{D(A^\alpha)} \text{ for all (or, some) } \lambda > 0.\]

Indeed, \[(2.24)\] is a simple consequence of the uniform boundedness of \(\{t^\alpha(t + A)^{-\alpha}\}_{t > 0}\) (see Lemma \[2.2\] (i) above) and the closedness of \(A^\alpha\).

Now we verify \[(2.26)\]. Let \(x \in X\) such that the limit \(\lim_{t \to \infty} t^\alpha(t + A)^{-\alpha}x\) exists. From \[(2.11)\] and \[(2.23)\] it follows that

\[x - t^\alpha(t + A)^{-\alpha}x = Ax_t, \quad t > 0,\]

where

\[x_t = \frac{\Gamma(n)}{\Gamma(n - \alpha)} \sum_{k=0}^{n-1} \int_0^\infty \frac{\lambda^\alpha}{(1 + \lambda)^n} \left( \frac{t(1 + \lambda)}{\lambda} \right)^k \left[ \frac{t(1 + \lambda)}{\lambda} + A \right]^{-(k+1)} \frac{d\lambda}{\lambda}\]

with \(\Re \alpha < n \in \mathbb{N}\). Thanks to the dominated convergence theorem, from the uniform boundedness of \(\{t(t + A)^{-1}\}_{t > 0}\) (see, Example \[2.1\] (b) above) it follows that \(x_t \to 0\) as \(t \to \infty\), so that \(Ax_t \to 0\) as \(t \to \infty\) due to the closedness of \(A\), i.e., \(t^\alpha(t + A)^{-\alpha}x \to x\) as \(t \to \infty\). Thus, we have verified \[(2.26)\].

As for \[(2.26)\], it suffices to verify that

\[(2.27) \quad x \in \overline{D(A^\alpha)} \iff A^\alpha(\lambda + A)^{-\alpha}x \in \overline{D(A^\alpha)} \text{ for all (or, some) } \lambda > 0\]

due to the fact that \(\overline{D(A^\alpha)} = \overline{D(A)}\). To this end, let \(x \in X\) and fix an integer \(n > \Re \alpha\). From \[(2.11)\] and \[(2.22)\] it follows that

\[A^\alpha(\lambda + A)^{-\alpha}x - x = c \sum_{k=1}^{n} \int_0^\infty \frac{\mu^\alpha}{(1 + \mu)^n} (-1)^k \left( \frac{\lambda \mu}{1 + \mu} \right)^k \left( \frac{\lambda \mu}{1 + \mu} + A \right)^{-k} \frac{d\mu}{\mu},\]

where \(c = \frac{\Gamma(n)}{\Gamma(n - \alpha)}\). By \[3\] Proposition 1.1.7 we conclude that the integrals given in the right-hand side of the last equality are all in \(\overline{D(A)}\), so that \(A^\alpha(\lambda + A)^{-\alpha}x - x \in \overline{D(A)}\). Thus, we have verified \[(2.27)\].

(ii) Analogous to \[(2.26)\], we can verify that

\[x \in \overline{R(A^\alpha)} \iff \lambda^\alpha(\lambda + A)^{-\alpha}x \in \overline{R(A^\alpha)} \text{ for all (or, some) } \lambda > 0.\]

The other statements have been verified in \[65\] Theorem 6.1.1 (iii)],

(iii) Recall that

\[(2.28) \quad \ker(A) = \ker(A^n), \quad n \in \mathbb{N},\]

as shown in \[65\] Corollary 1.1.6. A direct proof of \[(2.28)\] can be given as follows. Indeed, it is trivial that \(\ker(A) \subset \ker(A^n)\) for each \(n \in \mathbb{N}\). Conversely, it suffices to verify that \(\ker(A^n) \subset \ker(A)\). Let \(x \in \ker(A^n)\). By using \(x = (t + A)(t + A)^{-1}x\) with \(t > 0\) we have

\[Ax = tA(t + A)^{-1}x + (t + A)^{-1}A^2x = tA(t + A)^{-1}x.\]

Applying \(t \to 0\) in the last equality yields \(Ax = 0\) since the last term converges to zero as \(t \to 0\) due to the uniform boundedness of \(\{A(t + A)^{-1}\}_{t > 0}\). Thus, we have verified \[(2.28)\].

Now we verify the first equality of (iii), i.e.,

\[(2.29) \quad \ker(A) = \ker(A^\alpha).\]

To this end, fix an integer \(n > \Re \alpha\). It is clear that \(\ker(A^n) \subset \ker(A^\alpha) = \ker(A)\) due to \[(2.7)\] and \[(2.28)\]. Conversely, let \(x \in \ker(A)\). It is clear that \(x \in \ker(A^\alpha)\).
due to (2.28), and hence, $A^\alpha x = 0$ due to the representation (2.11). This implies that $x \in Ker(A^\alpha)$, and hence, $Ker(A) \subset Ker(A^\alpha)$. Thus, we have verified (2.29).

Next we verify the second equality of (iii), i.e.,

$$
(2.30) \quad Ker(A^\alpha) = Ker\left(A^\alpha(t + A)^{-\alpha}\right), \quad t > 0.
$$

Indeed, by using (2.22) and the commutativity between $A$ and $(t + A)^{-1}$, we have $Ker(A(t + A)^{-1}) = Ker(A)$ for each $t > 0$. Applying (2.29) to the operator $A(t + A)^{-1}$ yields (2.30), immediately.

And finally, we verify the desired equivalences of (iii). It is trivial that

$$
\text{if } x \in Ker(A^\alpha) \Rightarrow A^\alpha(t + A)^{-\alpha}x = 0 \Rightarrow \lim_{t \to 0} A^\alpha(t + A)^{-\alpha}x = 0.
$$

Now fix $x \in X$ such that $\lim_{t \to 0} A^\alpha(t + A)^{-\alpha}x = 0$. Note that

$$
x \leftarrow x - A^\alpha(t + A)^{-\alpha}x
$$

where the equality follows from (2.11) and (2.29), while $A\eta \to 0$ as $t \to 0^+$ by the dominated convergence theorem. By use of the closedness of $A$ we conclude that $x \in D(A)$ and $Ax = 0$, i.e., $x \in Ker(A)$. Applying (2.29) yields $x \in Ker(A) = Ker(A^\alpha)$. Thus, we have verified that

$$
x \in Ker(A^\alpha) \Leftrightarrow A^\alpha(t + A)^{-\alpha}x = 0 \Rightarrow \lim_{t \to 0} A^\alpha(t + A)^{-\alpha}x = 0.
$$

By using analogous arguments, we can also verify the equivalence

$$
x \in Ker(A^\alpha) \Leftrightarrow t^\alpha(t + A)^{-\alpha}x \equiv x \Leftrightarrow \lim_{t \to 0} t^\alpha(t + A)^{-\alpha}x = x.
$$

Thus, we have verified the statement (iii).

(iv) We first verify the equivalence

$$
(2.31) \quad x \in R(A^\alpha) \ominus Ker(A^\alpha) \Leftrightarrow \lim_{t \to 0^+} A^\alpha(t + A)^{-\alpha}x \text{ exists}.
$$

Fix $x \in R(A^\alpha) \ominus Ker(A^\alpha)$, and write $x = x_1 + x_2$ with $x_1 \in R(A^\alpha)$ and $x_2 \in Ker(A^\alpha)$. Thanks to the statement (ii), there exists the limit $\lim_{t \to 0} A^\alpha(t + A)^{-\alpha}x_2$. More precisely,

$$
\lim_{t \to 0} A^\alpha(t + A)^{-\alpha}x = \lim_{t \to 0} A^\alpha(t + A)^{-\alpha}x_1 = x_1.
$$

Conversely, fix $x \in X$ such that the limit $\lim_{t \to 0} A^\alpha(t + A)^{-\alpha}x$ exists. Write

$$
y := \lim_{t \to 0} A^\alpha(t + A)^{-\alpha}x.
$$

It suffices to verify $x - y \in Ker(A^\alpha)$ due to the fact that $y \in R(A^\alpha)$. To this end, write $x_t := A^\alpha(t + A)^{-\alpha}x$ with $t > 0$. From (2.11) and (2.22) it follows that

$$
x - x_t = C_\alpha \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} \int_0^\infty \frac{\lambda^\alpha}{(1 + \lambda)^n} \lambda t \left(\frac{\lambda t}{1 + \lambda} + A\right)^{-k} \frac{d\lambda}{\lambda},
$$

where $C_\alpha = \frac{\Gamma(n) \Gamma(n - \alpha)}{\Gamma(n - \alpha) \Gamma(n - \alpha)}$. By using (2.1) and (2.3) we conclude that $x - x_t \in D(A)$ and $A(x - x_t) \to 0$ as $t \to 0$. Applying the closedness of $A$ yields $x - y \in Ker(A)$, and hence, $x - y \in Ker(A^\alpha)$ due to (2.29). Thus, we have verified (2.31).
Proof. By induction, it is routine to verify that the last equality yields (2.32), immediately. The equalities (2.33) and (2.35) are direct consequences of the fundamental theorem of calculus. Applying (2.35) to (2.32) can be verified analogously. Moreover, thanks to the statements (ii) and (iii), it can be seen that \( \lim_{t \to 0^+} t^\alpha(t + A)^{-\alpha} x = x_0 \) and \( \lim_{t \to 0^+} A^\alpha(t + A)^{-\alpha} x = x_1 \) whenever \( x = x_0 + x_1 \) with \( x_0 \in \text{Ker}(A^{\alpha}) \) and \( x_1 \in \overline{R}(A^{\alpha}) \). The proof is complete. \( \square \)

In the following lemma, we present some reproducing formulas for the fractional powers of non-negative operators, which will be used to construct Besov spaces associated with operators in Section 3 below.

**Lemma 2.5.** Let \( A \) be non-negative on \( X \), and let \( \alpha, \beta \in \mathbb{C}_+^* \) and \( \lambda, \mu > 0 \). The following statements hold.

(i) If \( \text{Re} \alpha > 0 \) and \( \beta = m \in \mathbb{N} \) then, for \( x \in \overline{D}(A) \),

\[
\lambda^{\alpha+\beta} A^\beta(\lambda + A)^{-\alpha-\beta} x = (\alpha + \beta) \int_0^\lambda t^{\alpha+\beta} A^{\beta+1}(t + A)^{-\alpha-\beta-1} x \frac{dt}{t} + \mu^{\alpha+\beta} A^\beta(\mu + A)^{-\alpha-\beta} x.
\]

(ii) For \( x \in X \),

\[
\lambda^{\alpha+\beta} A^\beta(\lambda + A)^{-\alpha-\beta} x = (\alpha + \beta) \int_0^\lambda t^{\alpha+\beta} A^{\beta+1}(t + A)^{-\alpha-\beta-1} x \frac{dt}{t}
\]

for \( x \in X \); and if \( \beta = 0 \), then (2.34) holds for \( x \in \text{R}(A) \).

(iv) For \( x \in X \),

\[
A^\beta(\lambda + A)^{-\alpha-\beta} x = (\alpha + \beta) \int_0^\lambda A^\beta(t + A)^{-\alpha-\beta-1} x dt + A^\beta(\mu + A)^{-\alpha-\beta} x.
\]

(v) If \( \text{Re} \alpha > 0 \), then

\[
A^\beta(\lambda + A)^{-\alpha-\beta} x = (\alpha + \beta) \int_0^\lambda A^\beta(t + A)^{-\alpha-\beta-1} x dt
\]

for \( x \in X \); and if \( \alpha = 0 < \text{Re} \beta \), then (2.36) holds for \( x \in \overline{D}(A) \).

**Proof.** By induction, it is routine to verify that

\[
\frac{d}{dt} \left\{ \sum_{k=0}^{m-1} \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)\Gamma(k+1)} [A(t + A)^{-1}]^k t^\alpha(t + A)^{-\alpha} x \right\}
\]

\[
= \frac{\Gamma(\alpha + m)}{\Gamma(\alpha)\Gamma(m)} t^{\alpha-1} A^m(t + A)^{-\alpha-m} x.
\]

Since \( t^\alpha A^m(t + A)^{-\alpha-m} x \to 0 \) as \( t \to \infty \) due to Lemma 2.4 (i), integrating the last equality yields (2.32), immediately. The equalities (2.33) and (2.35) are direct consequences of the fundamental theorem of calculus. Applying \( \mu \to 0 \) to (2.33)
exists the limit \( \lim_{N \to \infty} x \) regularity of that (2.11) admits a more general version. More precisely, on the one hand, the \( D \) operators goes back to A. V. Balakrishnan [10], while H. Komatsu [51, Theorem i.e., a vector-valued version of the Euler integral (1.5), for the fractional powers of \( D \) description for the fractional domain \( A \) due to [49, Lemma 2.1], where (2.32) is given for special values \( x \) satisfies \( \Re \alpha > s > -1 \) and \( \alpha < n \in \mathbb{N} \). Also see (2.42) below for an alternative version of the reproducing formula for fractional powers of operators.

2.3. Representations of fractional powers. Given an operator \( A \) on a Banach space \( X \), a fundamental matter in the theory of fractional powers of operators is to provide explicit representations for the fractional power \( A^\alpha \) and give exact description for the fractional domain \( D(A^\alpha) \). As mentioned above, the use of (2.11), i.e., a vector-valued version of the Euler integral (1.5), for the fractional powers of operators goes back to A. V. Balakrishnan [10], while H. Komatsu [51, Theorem 2.10] characterized the fractional domain \( D(A^\alpha) \) for the case in which \( D(A) = X \) and \( \alpha \in \mathbb{C}_+ \) with \( \Re \alpha \notin \mathbb{N} \).

Let \( A \) be non-negative on \( X \) and let \( \alpha \in \mathbb{C}_+ \) with \( \Re \alpha < n \in \mathbb{N} \). Recall that (2.11) admits a more general version. More precisely, on the one hand, the regularity of \( x \) in (2.11) can be weaken from \( x \in D(A^m) \) to \( x \in X \) such that there exists the limit \( \lim_{N \to \infty} \int_0^N \lambda^\alpha A^n(\lambda + A)^{-n} \, \frac{d\lambda}{\lambda} \) (see [51, Theorem 2.10]). On the other hand, (2.11) can be extended from integers \( n > \Re \alpha \) to complex numbers \( \beta \) satisfying \( \Re \beta > \Re \alpha \), as shown in [65, Corollary 5.1.13, (5.17)]. For the sake of the reader’s convenience, we reformulate [65, Corollary 5.1.13] as follows.

**Lemma 2.7.** Let \( A \) be non-negative on \( X \) and let \( \alpha, \beta \in \mathbb{C}_+ \) with \( \Re \alpha < \Re \beta \). Fix \( x \in X \) such that there exists the limit
\[
\lim_{N \to \infty} \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta - \alpha)} \int_0^N \lambda^\alpha A^\beta(\lambda + A)^{-\beta} \, d\lambda = y.
\]
Then \( x \in D(A^\alpha) \) and \( A^\alpha x = y \).

In particular, the limit (2.37) exists for \( x \in X \) satisfying
\[
\int_0^\infty \|\lambda^\alpha A^\beta(\lambda + A)^{-\beta} x\| \frac{d\lambda}{\lambda} < \infty.
\]
Thus, we have the following consequence of Lemma 2.7.

**Corollary 2.8.** Let \( A \) be non-negative on \( X \) and let \( \alpha, \beta \in \mathbb{C}_+ \) with \( \Re \alpha < \Re \beta \). Fix \( x \in X \) such that (2.38) holds. Then \( x \in D(A^\alpha) \) and
\[
A^\alpha x = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta - \alpha)} \int_0^\infty \lambda^\alpha A^\beta(\lambda + A)^{-\beta} x \frac{d\lambda}{\lambda}.
\]

By use of (2.31) and (2.38), it is easy to verify that (2.38) is satisfied whenever \( x \in D(A^{\alpha+\epsilon}) \). Thus, we have the following simple consequence of Corollary 2.8. Also see [23, Lemma 3.1].

**Corollary 2.9.** Let \( A \) be non-negative on \( X \), and let \( \alpha, \beta \in \mathbb{C}_+ \) with \( \Re \alpha < \Re \beta \). Then (2.39) holds for \( x \in D(A^{\alpha+\epsilon}) \) with \( \epsilon > 0 \).
Compared with Lemma 2.7 above, the following lemma reads that the limit (2.37) exists whenever \( x \in D(A^\alpha) \) with \( \alpha \in \mathbb{C}_+ \), possibly except for the case in which \( \text{Re}\alpha \in \mathbb{N} \) and \( \text{Im}\alpha \neq 0 \), as shown in [25, Theorem 3.5, (i)\( \Rightarrow\) (iii)].

**Lemma 2.10.** Let \( A \) be non-negative on \( X \) and let \( \alpha, \beta \in \mathbb{C}_+ \) with \( \text{Re}\alpha < \text{Re}\beta \). If \( x \in D(A^\alpha) \), then there exists the limit (2.37) and \( A^\alpha x = y \), possibly except for the case in which \( \text{Re}\alpha \in \mathbb{N} \) and \( \text{Im}\alpha \neq 0 \).

**Remark 2.11.** Here we say a few more words on Lemma 2.10. Recall that, as for the three statements (i), (ii) and (iii) given in [25, Theorem 3.5], the equivalence (i)\( \Leftrightarrow\) (ii) and the implication (iii)\( \Rightarrow\) (i) both hold, while the implication (i)\( \Rightarrow\) (iii) was merely verified for the case in which \( \text{Re}\alpha \notin \mathbb{N} \) or \( \alpha = n \in \mathbb{N} \) (since, in Step II of the proof given in [25, Theorem 3.5], the use of integration by parts is possible merely in the case \( \text{Re}\alpha = 1 \) with \( \text{Im}\alpha = 0 \), i.e., \( \alpha = 1 \)). Thus, whether Lemma 2.10 above holds in the case \( \alpha = 1 + it \) with \( n \in \mathbb{N} \) and \( n \neq 0 \) \( \forall t \in \mathbb{R} \) is still an open problem. We also refer the reader to [51, Theorem 2.10] and [65, Theorem 6.1.3 and Remark 6.1.1] for more information on this topic.

Analogous to (2.12), if \( A \) is injective, replacing \( A \) by \( A^{-1} \) in (2.39) yields

\[
A^{-\alpha}x = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta - \alpha)} \int_0^\infty \lambda^{-\alpha} \lambda^{\beta} (\lambda + A)^{-\beta} x \frac{d\lambda}{\lambda}, \quad x \in R(A^\alpha^*).
\]

In particular, (2.40) holds for each \( x \in X \) if \( A \) is positive.

The next result is new, which states that both (2.39) of positive powers \( A^\alpha \) and (2.40) of negative powers \( A^{-\alpha} \) can be reformulated as a unified one.

**Proposition 2.12.** Let \( A \) be non-negative on \( X \) and let \( z \in C \) and \( \alpha, \beta \in \mathbb{C}_+ \) such that \( -\text{Re}\alpha < \text{Re}z < \text{Re}\beta \). The following statements hold.

(i) If \( \text{Re}z > 0 \), then (1.2) holds for \( x \in D(A^{z+\epsilon}) \) with \( \epsilon > 0 \).

(ii) If \( \text{Re}z < 0 \), then (1.2) holds for \( x \in R(A^{-z+\epsilon}) \) with \( \epsilon > 0 \) whenever \( A \) is injective.

(iii) If \( \text{Re}z = 0 \) then (1.2) holds for \( x \in R(A^{-z+\epsilon}) \cap D(A^{z+\epsilon}) \) with \( \epsilon > 0 \) whenever \( A \) is injective.

**Proof.** (i) Let \( \text{Re}z > 0 \) and fix \( x \in D(A^{z+\epsilon}) \) with \( 0 < \epsilon < \text{Re}\beta - \text{Re}z \). First, the integral given in the right-hand side of (1.2) absolutely converges. This can be verified by using (2.11), (2.17) and the fact that \( \text{Re}\beta > \text{Re}z + \epsilon \). More precisely,

\[
\int_0^\infty \|\lambda^{z+\alpha} A^\beta (\lambda + A)^{-\alpha-\beta} x\| \frac{d\lambda}{\lambda} \lesssim \|x\| + \|A^{z+\epsilon} x\|.
\]

Next we verify (1.2). To this end, write \( A_t := A + t \) with \( t > 0 \), and fix \( \gamma \in \mathbb{C}_+ \) such that \( \text{Re}\gamma > \text{Re} \alpha + \text{Re}\beta \). Applying (2.40) to the operator \( \lambda + A_t \) yields

\[
\int_0^\infty \lambda^{z+\alpha} A_t^\beta (\lambda + A_t)^{-\alpha-\beta} x \frac{d\lambda}{\lambda} = C \int_0^\infty \lambda^{\gamma-\alpha-\beta} (\mu + A_t)^{-\gamma} x \frac{d\mu}{\mu} \frac{d\lambda}{\lambda},
\]

where \( C = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)\Gamma(\gamma-\alpha-\beta)} \). Applying the change of variable \( \sigma = \lambda + \mu \) and exchanging the order of integration yields

\[
\int_0^\infty \lambda^{z+\alpha} A_t^\beta (\lambda + A_t)^{-\alpha-\beta} x \frac{d\lambda}{\lambda} = C \int_0^\infty \sigma^{z-\beta} \sigma^{\gamma} A_t^\beta (\sigma + A_t)^{-\gamma} x \frac{d\sigma}{\sigma},
\]
where $C = \frac{\Gamma(\gamma)\Gamma(\epsilon + \alpha)}{\Gamma(\alpha + \beta)\Gamma(\beta - \alpha)}$. Thanks to the closedness of $A_t^\beta$, by [5, Proposition 1.1.7] we conclude that

$$\int_0^\infty \lambda^{\alpha - \beta} A_t^\beta (\lambda + A_t)^{-\alpha - \beta} x \frac{d\lambda}{\lambda} = CA_t^\beta \int_0^\infty \sigma^{\alpha - \beta} \sigma^{\gamma} (\sigma + A_t)^{-\gamma} x \frac{d\sigma}{\sigma},$$

where $C = \frac{\Gamma(\gamma)\Gamma(\epsilon + \alpha)}{\Gamma(\alpha + \beta)\Gamma(\beta - \alpha)}$. Since $A_t$ is positive, from (2.40) it follows that

$$\int_0^\infty \lambda^{\alpha} A_t^\beta (\lambda + A_t)^{-\alpha - \beta} x \frac{d\lambda}{\lambda} = CA_t^\beta A_t^\gamma = CA_t^\gamma,$$

where $C = \frac{\Gamma(\epsilon + \beta)\Gamma(\beta - z)}{\Gamma(\alpha + \beta)\Gamma(\beta - z)}$. Thus, we have verified that

$$(2.41) \quad A_t^\gamma x = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha + z)\Gamma(\beta - z)} \int_0^\infty \lambda^{\alpha} A_t^\beta (\lambda + A_t)^{-\alpha - \beta} x \frac{d\lambda}{\lambda}.$$  

Note that $A_t^\gamma x = \lim_{t \to 0} A_t^\gamma x$ due to the fact that $x \in D(A_t^{\epsilon}) \subset D(A_t^\gamma)$. Thanks to the dominated convergence theorem, applying $t \to 0$ to (2.41) yields (1.2), immediately.

(ii) The statement is a simple consequence of (i). More precisely, let $\text{Re} z < 0$ and fix $x \in R(A_t^{\gamma + \epsilon})$ with $0 < \epsilon < \text{Re} z + \text{Re} \alpha$. Note that $-\text{Re} \beta < \text{Re} (\epsilon - z) < \text{Re} \alpha$ and $R(A_t^{\gamma + \epsilon}) = D((A_t^{-\gamma})^{-\epsilon})$. Applying (i) to the $(\epsilon z)$-power of $A_t^{-\gamma}$ yields

$$A_t^\gamma x = \frac{\Gamma(\alpha + \beta)}{\Gamma(\beta - z)\Gamma(\alpha + z)} \int_0^\infty \mu^{\alpha - \beta} (A_t^{-\gamma})^\alpha (\mu + A_t^{-\gamma})^{-\beta - \alpha} x \frac{d\mu}{\mu}$$

$$= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha + z)\Gamma(\beta - z)} \int_0^\infty \lambda^{\alpha} A_t^\beta (\lambda + A_t)^{-\alpha - \beta} x \frac{d\lambda}{\lambda}.$$  

Thus, we have verified (1.2) for $\text{Re} z < 0$.

(iii) Let $z = it$ with $t \in \mathbb{R}$ and fix $x \in D(A_t^{\epsilon + it}) \cap R(A_t^{\epsilon + it})$ with $0 < \epsilon < \min\{1, \text{Re} \alpha, \text{Re} \beta\}$. Thanks to (2.1) and (2.2), it is easy to verify the absolute convergence of the integral given in the right-hand side of (1.2) due to the fact that $0 < \epsilon < \min\{\text{Re} \alpha, \text{Re} \beta\}$. By using additivity, we obtain from (2.1) that

$$A_t^{\epsilon + it} x = (1 + A_t^2) A_t^{-\epsilon} A_t^{\epsilon + it} (1 + A_t)^{-2} x.$$  

Applying (i) to the positive power $A_t^{\epsilon + it}$ of $A$ with $-\text{Re} (\alpha - \epsilon) < \text{Re} (\epsilon + it) < \text{Re} (\epsilon + \beta)$ yields

$$A_t^{\epsilon + it} x = C(1 + A_t^2) A_t^{-\epsilon} \int_0^\infty \lambda^{\epsilon + it} \lambda^{\alpha - \epsilon} A_t^\beta (\lambda + A_t)^{-\alpha - \beta} (1 + A_t)^{-2} x \frac{d\lambda}{\lambda},$$

where $C = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha + it)\Gamma(\beta - it)}$. Thanks to the fact that $0 < \epsilon < \min\{1, \text{Re} \alpha, \text{Re} \beta\}$ and the closedness of $(1 + A_t^2) A_t^{-\epsilon}$, applying [5, Proposition 1.1.7] to the operator $(1 + A_t^2) A_t^{-\epsilon}$ yields (1.2), immediately. The proof is complete.

**Remark 2.13.** By Proposition 2.12(iii), specifying $z = 0$ to (1.2) yields

$$(2.42) \quad x = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \lambda^{\alpha} A_t^\beta (\lambda + A_t)^{-\alpha - \beta} x \frac{d\lambda}{\lambda}, \quad x \in D(A_t^\epsilon) \cap R(A_t^\epsilon),$$

for $\epsilon > 0$. This (homogeneous) reproducing formula will be used to establish interpolations for abstract Besov spaces in Section 5 below.

Furthermore, we present some representations for resolvents of fractional powers in the following lemma. In particular, the representation (2.42) below will be used to establish smoothness reiteration for abstract Besov spaces in Section 5.
Lemma 2.14. Let $A$ be non-negative on $X$ and let $0 < \alpha < 1$. Then $A^\alpha$ is non-negative with the non-negativity constant $M_{A^\alpha} \leq M_A$. More precisely, for $\lambda > 0$,

\begin{equation}
(\lambda + A^\alpha)^{-1} = C \int_0^\infty \frac{\mu^{\alpha+1}}{\lambda^2 + 2\mu \cos \pi \alpha + \mu^{2\alpha} (\mu + A)^{-1}} \frac{d\mu}{\mu}
\end{equation}

and

\begin{equation}
A^\alpha (\lambda + A^\alpha)^{-1} = C \int_0^\infty \frac{\lambda \mu^\alpha}{\lambda^2 + 2\lambda \mu \cos \pi \alpha + \mu^{2\alpha} (\mu + A)^{-1}} \frac{d\mu}{\mu},
\end{equation}

where $C = \frac{1}{\Gamma(\alpha) \Gamma(1 - \alpha)}$ and $M_A$ is the non-negativity constant of $A$ given by (2.1).

Proof. It is routine to verify (2.43) by using the vector-valued version of the Cauchy integral formula (see, for example, [50, Proposition 10.2], [65, Proposition 5.3.2(5.24)] or [38, Remark 3.1.16]). Applying (2.1) and (1.6) to (2.43) gives $M_{A^\alpha} \leq M_A$, immediately. Moreover, from (2.2) and (2.43) it follows that

\begin{equation}
A^\alpha (\lambda + A^\alpha)^{-1} = I - C \int_0^\infty \frac{\lambda \mu^\alpha}{\lambda^2 + 2\lambda \mu \cos \pi \alpha + \mu^{2\alpha} (\mu + A)^{-1}} \frac{d\mu}{\mu}
\end{equation}

\begin{equation}
= C \int_0^\infty \frac{\lambda \mu^\alpha}{\lambda^2 + 2\lambda \mu \cos \pi \alpha + \mu^{2\alpha} (\mu + A)^{-1}} \frac{d\mu}{\mu}, \quad \lambda > 0,
\end{equation}

the desired (2.44). The proof is complete. \(\square\)

Finally, in contrast to the fractional powers of non-negative operators on Banach spaces, a complete account of the theory of fractional powers of operators is beyond the scope of our work, and we refer the reader to, for instance, [32, 83, 39] (for fractional powers associated with the fractional calculus), [30, 85, 74, 31, 56, 87] (for fractional powers of operators with polynomially bounded resolvent), [66] (for fractional powers of almost non-negative operators), [65, Section 5.7] (for fractional powers of non-negative (multivalued) operators in Fréchet spaces), and [8, 20, 43, 78, 61] for some applications associated with the fractional Laplacians, fractional Schrödinger operators and some other fractional operators.

3. Besov spaces associated with operators

This section is devoted to the construction of Besov spaces associated with non-negative operators in the framework of Banach spaces. In this section, $(X, \|\cdot\|)$ is a Banach space and $A$ is a non-negative operator (not necessarily defined densely or with dense range) on $X$.

3.1. Subspaces of Besov type. One of the advantages of the classical Besov spaces is that they provide continuous scales for the smoothness of function spaces. From the operator-theoretic point of view, the classical Besov spaces $B_{p,q}^s(\mathbb{R}^n)$ admit a close relation with the well known Laplacian on $L^p(\mathbb{R}^n)$ (see Examples 6.11 and 6.12 below or [84, Section 2.12.2]). Compared with (2.37), (2.38) provides a quantitative estimate, i.e., $\lambda^\alpha A^\beta (\lambda + A)^{-\beta} x \in L^1((0,\infty); d\lambda/\lambda)$, for the fractional domain $D(A^\alpha)$. The quantitative estimate (2.38) can also be refined via an additional size index $1 \leq q \leq \infty$, i.e., $\lambda^\alpha A^\beta (\lambda + A)^{-\beta} x \in L^q((0,\infty); d\lambda/\lambda)$ (see (1.1) for $\beta = n \in \mathbb{N}$). This is the main idea of H. Komatsu [51] on abstract Besov spaces from the interpolation point of view.
In order to give unified quantitative estimates of fractional domains for a full range \( s \in \mathbb{R} \) and \( 0 < q \leq \infty \), we start from the unified representation \((1.2)\) and turn to the following dyadic series instead of the Lebesgue integrals.

**Definition 3.1.** Let \( 0 < q \leq \infty \) and \( s \in \mathbb{R} \). Fix \( k \in \mathbb{Z} \) and \( \alpha, \beta \in \mathbb{C}_+ \) such that \( -\text{Re} \, \alpha < s < \text{Re} \, \beta \). The space \( R_{q,X}^{s,A}(k, \alpha, \beta) \) is defined by

\[
R_{q,X}^{s,A}(k, \alpha, \beta) := \left\{ x \in D(A) : \sum_{i=k}^{\infty} \left\| 2^{i(s+\alpha)} A^\beta (2^i + A)^{-\alpha-\beta} x \right\|^q < \infty \right\},
\]

endowed with the semi-quasinorm

\[
|x|_{R_{q,X}^{s,A}(k, \alpha, \beta)} := \left\{ \sum_{i=k}^{\infty} \left\| 2^{i(s+\alpha)} A^\beta (2^i + A)^{-\alpha-\beta} x \right\|^q \right\}^{1/q}
\]

(with the usual modification when \( q = \infty \)).

**Remark 3.2.** It is necessary to point out that the subspace \( R_{q,X}^{s,A}(k, \alpha, \beta) \) is not trivial and that the semi-quasinorm \( |\cdot|_{R_{q,X}^{s,A}(k, \alpha, \beta)} \) is indeed a quasi-norm in some special cases. More precisely,

(i) It is clear that \( D(A^\beta) \subseteq R_{q,X}^{s,A}(k, \alpha, \beta) \) if \( s \geq 0 \) and that \( R_{q,X}^{s,A}(k, \alpha, \beta) = D(A) \) if \( s < 0 \) due to the uniform boundedness of \( \{2^{j\alpha} (2^j + A)^{-\alpha}\}_{j \geq k} \) and \( \{A^\beta (2^j + A)^{-\beta}\}_{j \geq k} \) as shown in Lemma 2.2 (ii), so that the space \( R_{q,X}^{s,A}(k, \alpha, \beta) \) is non-zero for a non-trivial operator \( A \).

(ii) The semi-quasinorm \( |\cdot|_{R_{q,X}^{s,A}(k, \alpha, \beta)} \) is indeed a quasi-norm if \( \beta = 0 \) or \( A \) is injective. Indeed, the case when \( \beta = 0 \) is trivial while the case when \( A \) is injective can be seen by observing that

\[
|x|_{R_{q,X}^{s,A}(k, \alpha, \beta)} = 0 \iff A^\beta (2^j + A)^{-\beta} x = 0 \iff x \in \text{Ker}(A),
\]

where the last equivalence follows from Lemma 2.4 (iii) above.

It is easy to verify that the space \( R_{q,X}^{s,A}(k, \alpha, \beta) \) is independent of the choice of \( k \) in the sense of equivalent semi-quasinorms.

**Lemma 3.3.** Let \( 0 < q \leq \infty \) and \( s \in \mathbb{R} \). Fix \( k \in \mathbb{Z} \) and \( \alpha, \beta \in \mathbb{C}_+ \) such that \( -\text{Re} \, \alpha < s < \text{Re} \, \beta \). Then \( R_{q,X}^{s,A}(k, \alpha, \beta) = R_{q,X}^{s,A}(k', \alpha, \beta) \) from the set-theoretic point of view and the equivalence

\[
|x|_{R_{q,X}^{s,A}(k, \alpha, \beta)} \simeq |x|_{R_{q,X}^{s,A}(k', \alpha, \beta)}
\]

holds for each \( k' \in \mathbb{Z} \).

**Proof.** It suffices to verify that

\[
|x|_{R_{q,X}^{s,A}(k, \alpha, \beta)} \simeq |x|_{R_{q,X}^{s,A}(k+1, \alpha, \beta)}.
\]

Indeed, it is trivial that

\[
|x|_{R_{q,X}^{s,A}(k+1, \alpha, \beta)} \leq |x|_{R_{q,X}^{s,A}(k, \alpha, \beta)},
\]

and hence, it remains to verify that

\[
|x|_{R_{q,X}^{s,A}(k, \alpha, \beta)} \leq |x|_{R_{q,X}^{s,A}(k+1, \alpha, \beta)}.
\]
If $0 < q < \infty$, applying (1.3) yields
\[|x| R_{q,X}^{s,A}(k,\alpha,\beta) \leq C_{1/q} \left[ |x| R_{q,X}^{s,A}(k+1,\alpha,\beta) + \|2^{k(s+\alpha)} A^\beta (2^k + A)^{-\alpha - \beta} x\| \right], \]
while applying (2.18) with $c = 2$ to $(2^k + A)^{\alpha + \beta}(2^k + A)^{-\alpha - \beta}$ yields
\[\|2^{k(s+\alpha)} A^\beta (2^k + A)^{-\alpha - \beta} x\| = 2^{-\alpha(s+\text{Re}\alpha)} \| (2^k + A)^{\alpha + \beta}(2^k + A)^{-\alpha - \beta} \| \]
(3.3)
where $C = 2^{-\alpha(s+\text{Re}\alpha)} C_{\alpha+\beta,N}(L_A + 2M_A)^N$ with $\text{Re}(\alpha + \beta) < N \in \mathbb{N}$ and $C_{\alpha+\beta,N}$ given by (2.14). This implies that
\[|x| R_{q,X}^{s,A}(k,\alpha,\beta) \leq C_{1/q} \left[ |x| R_{q,X}^{s,A}(k+1,\alpha,\beta) + C \|2^{(k+1)(s+\alpha)} A^\beta (2^{k+1} + A)^{-\alpha - \beta} \| \right] \]
\[\leq C_{1/q} \max\{1, C\} \cdot |x| R_{q,X}^{s,A}(k+1,\alpha,\beta), \]
where $C_{1/q}$ and $C$ are given in (1.3) and (3.3), respectively. Thus, we have verified (3.2). The proof is complete.

The case when $q = \infty$ can be verified analogously. More precisely, by using (3.3) again we have
\[|x| R_{q,X}^{s,A}(k,\alpha,\beta) = \sup_{j \geq k} \|2^{j(s+\alpha)} A^\beta (2^j + A)^{-\alpha - \beta} \| \]
\[\leq \max\left\{ \|2^{k(s+\alpha)} A^\beta (2^k + A)^{-\alpha - \beta} \|, |x| R_{q,X}^{s,A}(k+1,\alpha,\beta) \right\} \]
\[\leq C |x| R_{q,X}^{s,A}(k+1,\alpha,\beta), \]
where $C$ is the constant given in (3.3). Thus, we have verified (3.2). The proof is complete.

Thanks to Lemmas 2.4 and 2.5, we can verify that the space $R_{q,X}^{s,A}(k,\alpha,\beta)$ is also independent of the choice of $\alpha$ in the sense of equivalent semi-quasinorms.

**Lemma 3.4.** Let $0 < q \leq \infty$ and $s \in \mathbb{R}$. Fix $k \in \mathbb{Z}$ and $\alpha, \beta \in \mathbb{C}_+^*$ such that $-\text{Re} \alpha < s < \text{Re} \beta$. Then $R_{q,X}^{s,A}(k,\alpha,\beta) = R_{q,X}^{s,A}(k,\alpha',\beta)$ from the set-theoretic point of view and
\[|x| R_{q,X}^{s,A}(k,\alpha,\beta) \simeq |x| R_{q,X}^{s,A}(k,\alpha',\beta) \]
for each $\alpha' \in \mathbb{C}_+^*$ satisfying $-\text{Re} \alpha' < s$.

**Proof.** It suffices to verify (3.4) in the case $\text{Re} \alpha \leq \text{Re} \alpha'$, otherwise the conclusion follows from the symmetry of $\alpha$ and $\alpha'$. Indeed, it suffices to verify the following two inequalities. One is
\[|x| R_{q,X}^{s,A}(k,\alpha''',\beta) \lesssim |x| R_{q,X}^{s,A}(k,\alpha,\beta) \]
for each $\alpha'' \in \mathbb{C}_+^*$ satisfying $\text{Re} \alpha'' > \text{Re} \alpha$ and the other is
\[|x| R_{q,X}^{s,A}(k,\alpha,\beta) \lesssim |x| R_{q,X}^{s,A}(k,\alpha+1,\beta). \]
If this is the case, from (3.6) and (3.5) it follows that
\[|x| R_{q,X}^{s,A}(k,\alpha,\beta) \lesssim |x| R_{q,X}^{s,A}(k,\alpha+1,\beta) \]
\[\lesssim |x| R_{q,X}^{s,A}(k,\alpha',\beta) \]
\[ \leq |x|_{R_{q,X}^s(k,\alpha' + 1, \beta)} \leq |x|_{R_{q,X}^s(k,\alpha, \beta)} \]

whenever \( \text{Re} \, \alpha = \text{Re} \, \alpha' \) and that
\[ |x|_{R_{q,X}^s(k,\alpha, \beta)} \leq |x|_{R_{q,X}^s(k,\alpha + n, \beta)} \leq |x|_{R_{q,X}^s(k,\alpha', \beta)} \leq |x|_{R_{q,X}^s(k,\alpha, \beta)} \]

whenever \( \text{Re} \, \alpha < \text{Re} \, \alpha' \), where \( n \in \mathbb{N} \) with \( \text{Re} \, \alpha' < \text{Re} \, \alpha + n \). Thus, we have verified (3.4) by using (3.5) and (3.6).

It remains to verify (3.5) and (3.6). Indeed, (3.5) is a direct consequence of the uniform boundedness of the family \( \{2j^{(\alpha' - \alpha)}(2^j + A)^{-\alpha' - \alpha}\}_{j \geq k} \) due to (2.16) and it remains to verify (3.6). We merely verify (3.6) in the case \( 0 < q < \infty \) and the case when \( q = \infty \) can be verified analogously. To this end, let \( 0 < q < \infty \). From (2.30) it follows that
\[ |x|_{R_{q,X}^s(k,\alpha, \beta)} \leq |\alpha + \beta| \left( \sum_{i=k}^{\infty} \left[ \int_2^\infty \|2^i(s+\alpha)\alpha \beta (t + A)^{-\alpha - \beta - 1}x\| \, dt \right]^{1/q} \right)^{1/q}. \]

Rewriting \( J_2^\infty = \sum_{r=1}^\infty J_2^{r+1} \) and applying the estimate (2.19) with \( c = 1 \) yields
\[
(3.7) \quad |x|_{R_{q,X}^s(k,\alpha, \beta)} \leq CJ,
\]
where \( C = |\alpha + \beta| C_{\alpha + \beta + 1, N} (M_A + L_A)^N \) with \( \text{Re} \, \alpha + \text{Re} \, \beta + 1 \leq N \in \mathbb{N} \) and \( C_{\alpha + \beta + 1, N} \) given by (2.14) and
\[ J = \left\{ \sum_{i=k}^{\infty} \left[ \sum_{r=i}^{\infty} \left[ 2^i(s+\alpha)\alpha \beta (2^r + A)^{-\alpha - \beta - 1}x \right]^q \right]^{1/q} \right\}^{1/q}. \]

It can be verified that
\[
(3.8) \quad J \leq |x|_{R_{q,X}^s(k,\alpha + 1, \beta)}.
\]

Indeed, if \( 0 < q \leq 1 \), applying the classical inequality (1.4) and exchanging the order of summation yields
\[ J \leq \left( \sum_{r=k}^{\infty} \sum_{i=k}^{\infty} \left\| 2^i(s+\alpha)\alpha \beta (2^r + A)^{-\alpha - \beta - 1}x \right\|^q \right)^{1/q} \]
\[ \leq \left[ 1 - 2^{-(s+\text{Re} \, \alpha)q} \right]^{-1/q} |x|_{R_{q,X}^s(k,\alpha + 1, \beta)} \]

And if \( 1 < q < \infty \), rewriring \( I \) as
\[ J = \left\{ \sum_{i=k}^{\infty} 2^{r(s+\text{Re} \, \alpha)q} \left[ \sum_{r=i}^{\infty} 2^r(\delta - \text{Re} \, \alpha) \left\| 2^r(s+\alpha - \delta + 1)A^\delta (2^r + A)^{-\alpha - \beta - 1}x \right\|^q \right]^{1/q} \right\}^{1/q} \]
with an auxiliary constant \( \delta \in (0, s + \text{Re} \, \alpha) \) and applying the H"older inequality to the inner bracket yields
\[ J \leq \frac{1}{1 - 2^{(\delta - s+\text{Re} \, \alpha)q}} \left( \sum_{i=k}^{\infty} \sum_{r=i}^{\infty} \left\| 2^r(\delta - s+\alpha - \delta + 1)A^\delta (2^r + A)^{-\alpha - \beta - 1}x \right\|^q \right)^{1/q}. \]

Exchanging the order of summation yields
\[ J \leq \frac{1}{1 - 2^{(\delta - s+\text{Re} \, \alpha)q}} \left( \sum_{i=k}^{\infty} \sum_{r=i}^{\infty} \left\| 2^r(\delta - s+\alpha - \delta + 1)A^\delta (2^r + A)^{-\alpha - \beta - 1}x \right\|^q \right)^{1/q}. \]
Thus, we have verified (3.8). Finally, from (3.7) and (3.8) the desired inequality follows immediately. The proof is complete. 

For a relatively satisfactory independence of the space $R_{q,X}^{s,A}(k,\alpha,\beta)$ with respect to $\beta$, however, an auxiliary term $\| (2^k + A)^{-\alpha} x \|$ seems to be indispensable. The next result is crucial to the theory of inhomogeneous Besov space $s$ associated with non-negative operators.

**Lemma 3.5.** Let $0 < q \leq \infty$ and $s \in \mathbb{R}$, and let $k \in \mathbb{Z}$ and $\alpha, \beta \in \mathbb{C}^*_+$ such that $-\Re \alpha < s < \Re \beta$. Then $R_{q,X}^{s,A}(k,\alpha,\beta) = R_{q,X}^{s,A}(k',\alpha',\beta')$ from the set-theoretic point of view and the equivalence

$$(3.9) \quad \| (2^k + A)^{-\alpha} x \| + |x| R_{q,X}^{s,A}(k,\alpha,\beta) \simeq \| (2^{k'} + A)^{-\alpha'} x \| + |x| R_{q,X}^{s,A}(k',\alpha',\beta'),$$

holds for all $\alpha, \beta \in \mathbb{C}^*_+$ satisfying $-\Re \alpha' < s < \Re \beta'$ and $k' \in \mathbb{Z}$.

**Proof.** Fix $k' \in \mathbb{Z}$ and let $\alpha', \beta' \in \mathbb{C}^*_+$ such that $-\Re \alpha' < s < \Re \beta'$. It suffices to verify that

$$(3.10) \quad \| (2^k + A)^{-\alpha} x \| + |x| R_{q,X}^{s,A}(k,\alpha,\beta) \simeq \| (2^{k'} + A)^{-\alpha'} x \| + |x| R_{q,X}^{s,A}(k',\alpha',\beta'),$$

and

$$(3.11) \quad \| (2^k + A)^{-\alpha} x \| + |x| R_{q,X}^{s,A}(k,\alpha,\beta) \simeq \| (2^{k'} + A)^{-\alpha'} x \| + |x| R_{q,X}^{s,A}(k',\alpha',\beta'),$$

First we verify (3.10). Thanks to (3.11), it suffices to verify that

$$\| (2^k + A)^{-\alpha} x \| \simeq \| (2^{k'} + A)^{-\alpha'} x \|. $$

Indeed, note that $(2^k + A)(2^{k'} + A)^{-1}$ is bounded on $X$ due to (2.41) and (2.43), so is $(2^k + A)^{\alpha}(2^{k'} + A)^{-\alpha}$ as the $\alpha$-power of $(2^k + A)(2^{k'} + A)^{-1}$ given by (2.6) above (also, see (2.18) above). This implies that

$$\| (2^k + A)^{-\alpha} x \| \leq \| (2^{k'} + A)^{\alpha}(2^{k'} + A)^{-\alpha} \| \cdot \| (2^{k'} + A)^{-\alpha'} x \| \leq \| (2^{k'} + A)^{-\alpha} x \|.$$ 

The inverse inequality can be verified analogously. Thus, we have verified (3.10).

Next we verify (3.11). Analogous to (3.6), it suffices to verify that

$$|x| R_{q,X}^{s,A}(k,\alpha,\beta) \lesssim |x| R_{q,X}^{s,A}(k,\alpha,\beta+1) \lesssim \| (2^k + A)^{-\alpha} x \| + |x| R_{q,X}^{s,A}(k,\alpha,\beta+1)$$

for $\Re \beta' > \Re \beta$. Indeed, the former is a direct consequence of (2.17). The latter can be verified by using Lemmas 2.2 and 2.5. We merely give a proof in the case $0 < q < \infty$ and the case when $q = \infty$ can be proved analogously. To this end, fix $x \in D(A)$ such that $|x| R_{q,X}^{s,A}(k,\alpha,\beta+1) < \infty$. Applying (2.33) with $\lambda = 2^k$ and $\mu = 2^k$ yields

$$|x| R_{q,X}^{s,A}(k,\alpha,\beta) \leq |\alpha + \beta| \left\{ \sum_{i=k}^{\infty} 2^{i(s-\Re \beta)} \int_{2^i}^{2^{i+1}} \| t^{\alpha+\beta+1} A^{-\alpha-\beta-1} x \| dt \right\}^{1/q}$$

$$+ 2^{i(s-\Re \beta) + k(\Re \alpha + \Re \beta)} \| A^{-\alpha} (2^k + A)^{-\alpha} x \| q.$$
Rewriting $J_{2^k}^x = \sum_{r=k}^{i-1} J_{2^r}^x$ and applying (1.33) and (2.19) with $c = 1$ yields

$$|x|_{R_{q,X}^{\alpha,A}(k,\alpha,\beta)} \lesssim J_1 + J_2,$$

where

$$J_1 = \left\{ \sum_{k=1}^{\infty} \left( \sum_{r=k}^{i-1} \left\| 2^{i(s-\beta)} 2^{r(\alpha+\beta)} A^{\beta+1}(2^r + A)^{-\alpha-\beta-1} x \right\| \right) \right\}^{1/q}$$

and

$$J_2 = \left\{ \sum_{k=1}^{\infty} 2^{i(s-\alpha+\alpha+\beta+\beta)} q \left\| A^{\beta}(2^k + A)^{-\alpha-\beta-1} x \right\| \right\}^{1/q}.$$

Analogous to (3.8), by using the classical inequality (1.4) for $0 < q \leq 1$ and the Hölder inequality for $1 < q < \infty$ we have

$$J_1 \lesssim |x|_{R_{q,X}^{\alpha,A}(k,\alpha,\beta+1)}.$$

By (2.17) we also have

$$J_2 \lesssim \left\| (2^k + A)^{-\alpha} x \right\|$$

due to the fact that $s - \Re \beta < 0$. Then it follows from (3.13) that

$$|x|_{R_{q,X}^{\alpha,A}(k,\alpha,\beta)} \lesssim \left\| (2^k + A)^{-\alpha} x \right\| + |x|_{R_{q,X}^{\alpha,A}(k,\alpha,\beta+1)},$$

which is the desired inequality. Thus, we have verified (3.11).

Finally, we verify (3.12). To this end, we may assume that $\Re \alpha \leq \Re \alpha'$, otherwise the desired conclusion follows from the symmetry of $\alpha$ and $\alpha'$. Moreover, it suffices to verify the following two inequalities for some special indices. One is

$$\left\| (2^k + A)^{-\alpha} x \right\| + |x|_{R_{q,X}^{\alpha,A}(k,\alpha,\beta)} \lesssim \left\| (2^k + A)^{-(\alpha+1)} x \right\| + |x|_{R_{q,X}^{\alpha,A}(k,\alpha+1,\beta)}$$

for all $\alpha_1, \alpha_2 \in \mathbb{C}_+$ with $\Re \alpha_1 > \Re \alpha_2$, and the other is

$$\left\| (2^k + A)^{-\alpha} x \right\| + |x|_{R_{q,X}^{\alpha,A}(k,\alpha,\beta)} \lesssim \left\| (2^k + A)^{-(\alpha+1)} x \right\| + |x|_{R_{q,X}^{\alpha,A}(k,\alpha+1,\beta)}.$$

If this is the case, from (3.15) and (3.14) it follows that

$$\left\| (2^k + A)^{-\alpha} x \right\| + |x|_{R_{q,X}^{\alpha,A}(k,\alpha,\beta)} \lesssim \left\| (2^k + A)^{-(\alpha+1)} x \right\| + |x|_{R_{q,X}^{\alpha,A}(k,\alpha+1,\beta)}$$

whenever $\Re \alpha = \Re \alpha'$ and that

$$\left\| (2^k + A)^{-\alpha} x \right\| + |x|_{R_{q,X}^{\alpha,A}(k,\alpha,\beta)} \lesssim \left\| (2^k + A)^{-(\alpha+n)} x \right\| + |x|_{R_{q,X}^{\alpha,A}(k,\alpha+n,\beta)}$$

whenever $\Re \alpha < \Re \alpha'$, where $n \in \mathbb{N}$ satisfying $\Re \alpha + n > \Re \alpha'$. Thus, we have verified (3.12) by using (3.15) and (3.14).
It remains to verify (3.14) and (3.15). Indeed, (3.14) is a direct consequence of (2.10) and it is sufficient to verify (3.15). Thanks to (3.11), it suffices to verify (3.15) for the case in which \( \beta \) with \( \text{Im} \beta = -\text{Im} \alpha \) and \( \text{Re} \beta = n - \text{Re} \alpha \), where \( n \) is an integer large enough such that \( n > s + \text{Re} \alpha \). To this end, let \( x \in D(A) \) such that \( |x|_{R_{\alpha}^{\beta}(k, \alpha + 1, \beta)} < \infty \). Note that \( \alpha + \beta = n \in \mathbb{N} \) due to the hypothesis, and hence, by using (2.22) with \( \alpha \) and \( m \) replaced by \( \alpha + 1 \) and \( \alpha + \beta \), respectively, we have

\[
\| (2^k + A)^{-\alpha} x \| \lesssim \| (2^k + A)^{-(\alpha + 1)} x \| + \int_{2^k}^{\infty} \| A^{\beta} (\mu + A)^{-\alpha - \beta - 1} x \| d\mu.
\]

Decomposing the integral \( \int_{2^k}^{\infty} \sum_{i=k}^{\infty} \int_{2^i}^{2^{i+1}} \) and applying (2.19) yields

\[
\| (2^k + A)^{-\alpha} x \| \leq \| (2^k + A)^{-(\alpha + 1)} x \| + \sum_{i=k}^{\infty} 2^{-i(s + \text{Re} \alpha)} \| 2^i \beta \| A^{\beta} (2^i + A)^{-(\alpha + 1)} x \| \lesssim \| (2^k + A)^{-(\alpha + 1)} x \| + |x|_{R_{\alpha}^{\beta}(k, \alpha + 1, \beta)}.
\]

where the last inequality follows from the monotonicity of spaces \( \ell_q \) if \( 0 < q \leq 1 \), from the Hölder inequality if \( 1 < q < \infty \) and from the convergence of the series \( \sum_{i=k}^{\infty} 2^{-i(s + \text{Re} \alpha)} \) if \( q = \infty \), respectively. Thus, we have verified that

(3.16) \[ \| (2^k + A)^{-\alpha} x \| \lesssim \| (2^k + A)^{-(\alpha + 1)} x \| + |x|_{R_{\alpha}^{\beta}(k, \alpha + 1, \beta)}. \]

Finally, by using (3.16) and (2.23) (or, more precisely, (3.21)) we obtain the desired inequality (3.15) immediately. The proof is complete. \( \square \)

3.2. Inhomogeneous Besov spaces. Recall that a vector space \( \mathcal{X} \) on \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \) is said to be a quasi-normed space on \( \mathbb{K} \) if there is a functional \( \| \cdot \|_{\mathcal{X}} \) on \( \mathcal{X} \) which satisfies the following three conditions (a), (b) and (c):

(a) \( \| x \|_{\mathcal{X}} \geq 0 \) for \( x \in \mathcal{X} \), and \( \| x \|_{\mathcal{X}} = 0 \) if and only if \( x = 0 \),

(b) \( \| \alpha x \|_{\mathcal{X}} = |\alpha| \| x \|_{\mathcal{X}} \) for \( \alpha \in \mathbb{K} \) and \( x \in \mathcal{X} \),

(c) there is a constant \( K \geq 1 \) such that

(3.17) \[ \| x + y \|_{\mathcal{X}} \leq K(\| x \|_{\mathcal{X}} + \| y \|_{\mathcal{X}}), \quad x, y \in \mathcal{X}. \]

By the Aoki-Rolewicz theorem [14, 75], for each quasi-normed space \( (\mathcal{X}, \| \cdot \|_{\mathcal{X}}) \), there is an equivalent quasi-norm \( \| \cdot \|_{\mathcal{X}} \) on \( \mathcal{X} \) which satisfies the \( p \)-subadditivity, i.e.,

\[ \| x + y \|_{\mathcal{X}}^p \leq \| x \|_{\mathcal{X}}^p + \| y \|_{\mathcal{X}}^p, \quad x, y \in \mathcal{X}, \]

for some \( 0 < p \leq 1 \). More precisely, one can take \( p = \ln 2/(\ln K + \ln 2) \) with \( K \), the modulus of concavity given in (3.17), and define an equivalent quasi-norm \( \| \cdot \|_{\mathcal{X}} \) on \( \mathcal{X} \) satisfying the \( p \)-subadditivity in the following way:

(3.18) \[ \| x \|_{\mathcal{X}} := \inf \left\{ \left( \sum_{i=1}^{n} \| x_i \|_{\mathcal{X}}^p \right)^{1/p} : x = \sum_{i=1}^{n} x_i, x_i \in \mathcal{X}, i = 1, 2, \ldots, n \in \mathbb{N} \right\} \]

for \( x \in \mathcal{X} \) (see [16, Theorems 1.2 and 1.3]). The metric topology on \((\mathcal{X}, \| \cdot \|_{\mathcal{X}})\), induced by \( \| \cdot \|_{\mathcal{X}} \), can be defined by \( d(x, y) := \| x - y \|_{\mathcal{X}} \) for \( x, y \in \mathcal{X} \), where \( \| \cdot \|_{\mathcal{X}} \) is the equivalent quasi-norm on \( \mathcal{X} \) given by (3.18). Therefore, every quasi-normed space admits a completion in the sense of equivalent quasi-norms. Moreover, a quasi-normed space \( \mathcal{X} \) is said to be a quasi-Banach space if it is complete with
Proof. By Lemma 3.5 above, it remains to verify that $R^s_A$ is a quasi-norm on $R^s_A(k,\alpha,\beta)$ due to the fact that $\|O^k + A\|^{-\alpha} \|x\|_X$ is a norm on the Banach space $X$. Therefore, there is a metric topology on $R^s_A(k,\alpha,\beta)$ induced by the quasi-norm $\|\|\cdot\|_{B^s_A(k,\alpha,\beta)}$ as we discussed in the last paragraph. More precisely, a metric on $R^s_A(k,\alpha,\beta)$ can be defined by
\begin{equation}
\|x - y\|_{B^s_A(k,\alpha,\beta)} = \|x - y\|_{B^s_A(k,\alpha,\beta)}^{(p)}, \quad x, y \in R^s_A(k,\alpha,\beta),
\end{equation}
where $\|\cdot\|_{B^s_A(k,\alpha,\beta)}$ is the equivalent quasi-norm given by $[3.13]$ with $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ replaced by $(R^s_A(k,\alpha,\beta), \|\cdot\|_{B^s_A(k,\alpha,\beta)})$.

Unfortunately, the space $R^s_A(k,\alpha,\beta)$ is possibly not complete with respect to the metric $d$ given by $[3.20]$, even in some classical examples (like $A = -\Delta_p$, the negative of the Laplacian on $L^p(\mathbb{R}^n)$). We turn to the completion of $R^s_A(k,\alpha,\beta)$ with respect to the quasi-norm $\|\|B^s_A(k,\alpha,\beta)\|$ and define (inhomogeneous) Besov spaces associated with $A$ as follows.

**Definition 3.6.** Let $0 < q < \infty$ and $s \in \mathbb{R}$, and fix $k \in \mathbb{Z}$ and $\alpha, \beta \in \mathbb{C}_+$ satisfying $-\Re \alpha < s < \Re \beta$. The inhomogeneous Besov space $B^s_A(k,\alpha,\beta)$ associated with $A$ is defined as the completion of $R^s_A(k,\alpha,\beta)$ with respect to the metric $d$ given in $[3.20]$ (the quasi-norm on $B^s_A(k,\alpha,\beta)$ is still denoted by $\|\|B^s_A(k,\alpha,\beta)\|$ for convenience).

**Remark 3.7.** As shown in Lemma 3.9, the quasi-norm $\|\|B^s_A(k,\alpha,\beta)\|$ given by $[3.19]$ is independent of the choice of $k, \alpha$ and $\beta$ in the sense of equivalent quasi-norms. We shall not distinguish between equivalent quasi-norms of a given quasi-Banach space, and therefore we write $B^s_A$ instead of $B^s_A(k,\alpha,\beta)$ as well as $\|\|B^s_A\|$ instead of $\|\|B^s_A(k,\alpha,\beta)\|$ unless otherwise the indices $k, \alpha$ and $\beta$ need to be explicitly specified.

The next result reveals that the inhomogeneous Besov space $B^s_A$ is indeed a subspace of the underlying space $X$ whenever $s > 0$.

**Proposition 3.8.** Let $s > 0$ and $0 < q \leq \infty$. Then $B^s_A(k,\alpha,\beta) = R^s_A(k,\alpha,\beta)$ for each $k \in \mathbb{Z}$, $\alpha \in \mathbb{C}_+$ and $\beta \in \mathbb{C}_+$ satisfying $-\Re \alpha < s < \Re \beta$.

**Proof.** Let $k \in \mathbb{Z}$ and fix $\alpha \in \mathbb{C}_+$ and $\beta \in \mathbb{C}_+$ such that $-\Re \alpha < s < \Re \beta$. It suffices to verify that $R^s_A(k,\alpha,\beta)$ is complete with respect to $\|\|B^s_A(k,\alpha,\beta)\|$. By Lemma 3.5 above, it remains to verify that $R^s_A(0,0,\beta)$ is complete with respect to $\|\|B^s_A(0,0,\beta)\|$. To this end, suppose that $\{x_n\} \subset \overline{D(A)}$ is a Cauchy sequence with respect to $\|\|B^s_A(0,0,\beta)\|$. Let $\epsilon > 0$ and fix $N \in \mathbb{N}$ such that
\begin{equation}
\|x_m - x_n\|_{B^s_A(0,0,\beta)} < \epsilon, \quad m, n \geq N.
\end{equation}
By \[3.19\], it is clear that \(\{x_n\}\) is also Cauchy in \(X\), and hence, there is an \(x \in \overline{D(A)}\) such that \(x_n \to x\) as \(n \to \infty\). Applying \(n \to \infty\) to \(3.21\) yields

\[
\|x_m - x\| + |x_m - x|_{R^\alpha,A(X)}(0,0,\beta) \leq \epsilon, \quad m \geq N,
\]

and hence,

\[
|x|_{R^\alpha,A(X)}(0,0,\beta) \leq |x - x_N|_{R^\alpha,A(X)}(0,0,\beta) + |x_N|_{R^\alpha,A(X)}(0,0,\beta) < \infty.
\]

This implies that \(x \in R^\alpha,A(X)\) and \(x_n \to x\) with respect to the quasi-norm \(\|\cdot\|_{R^\alpha,A(X)}\). The proof is complete. \(\square\)

Now we give a connection between the fractional domains \(D(A^\alpha)\) and Besov spaces \(B^\alpha,A(X)\), which will be used to establish interpolation spaces for abstract Besov spaces in Section 5 below.

**Proposition 3.9.** Let \(0 < q \leq \infty\), and let \(s, s' > 0\) and \(\alpha \in \mathbb{C}_+\) with \(0 < s < \Re \alpha < s' < \infty\). Then

\[
(3.22) \quad B^\alpha,A(0) \subset D(A^\alpha) \subset B^\alpha,A(X).
\]

**Proof.** The inclusion

\[
D(A^\alpha) \subset B^\alpha,A(X)
\]

follows from the non-negativity of \(A\). Indeed, let \(x \in D(A^\alpha)\). Fix \(\beta \in \mathbb{C}_+\) with \(\Re \alpha < \Re \beta\). From \(2.16^*\) and \(2.17^*\) it follows that

\[
\|x\|_{R^\alpha,A(X)}(0,0,\beta) = \sup_{j \geq 0} \|2^ja^\beta(2^j + A)^{-\beta}x\| \leq C\|A^\alpha x\|
\]

for \(q = \infty\) and

\[
\|x\|_{R^\alpha,A(X)}(0,0,\beta) = \sum_{j=0}^\infty \|2^ja^\beta(2^j + A)^{-\beta}x\|^q \leq \frac{C}{1 - 2^{(s-\Re \alpha)q}}\|A^\alpha x\|
\]

for \(0 < q < \infty\), where \(C = C_{\alpha,n}C_{\beta-\alpha,m}M_A^nL_{\alpha}^m\) with \(C_{\alpha,n}, C_{\beta-\alpha,m}\) given by \(2.14\) and \(M_A\) and \(L_A\) given by \(2.3\) and \(2.3\), respectively. This implies that \(x \in B^\alpha,A(X)\).

It remains to verify that

\[
(3.23) \quad B^\alpha,A(0) \subset D(A^\alpha).
\]

Let \(x \in B^\alpha,A(X)\). We now verify that \(x \in D(A^\alpha)\). Thanks to Corollary \(2.8\) it suffices to verify that

\[
(3.24) \quad \int_0^\infty \|\lambda^\alpha A^\beta(\lambda + A)^{-\beta}x\| \frac{d\lambda}{\lambda} < \infty,
\]

where \(\beta \in \mathbb{C}_+\) with \(\Re \alpha < \Re \beta\). To this end, fix \(\beta \in \mathbb{C}_+\) such that \(\Re \alpha < \Re \beta\). For the part \(j_0^\infty\), from \(2.17^*\) it follows that

\[
\int_{j_0}^\infty \|\lambda^\alpha A^\beta(\lambda + A)^{-\beta}x\| \frac{d\lambda}{\lambda} \leq \frac{C_{\beta,m}L_{\alpha}}{\Re \alpha} \|x\|,
\]

and
where $C_{\beta,m}$ is given by (2.13) and $L_A$ is the non-negativity constant $A$ given by (2.3). As for the part $\int_1^\infty$, rewriting $\int_1^\infty = \sum_{j=0}^\infty \int_{2^j}^{2^{j+1}}$ and applying (2.19) with $c = 1$ yields

$$\int_1^\infty \| \lambda^n A^\beta (\lambda + A)^{-\beta} x \| \, d\lambda \leq \sum_{j=0}^\infty \left\| 2^{jn} A^\beta (2^j + A)^{-\beta} x \right\|$$

$$= \sum_{j=0}^\infty \left\| 2^{j(n-s')} 2^{js'} A^\beta (2^j + A)^{-\beta} x \right\| < |x|_{R^{s',\beta}_q(0,0,\beta)}$$

where the last inequality follows from (1.9) in the case $0 < q \leq 1$ and from the Hölder inequality in the case $1 < q < \infty$ (the case when $q = \infty$ is a direct consequence of the estimate (2.19) with $c = 1$). Thus, we have verified (3.24). The proof is complete.

A density property of the inhomogeneous Besov spaces associated with non-negative operators is given as follows.

**Proposition 3.10.** Let $s \in \mathbb{R}$ and $0 < q < \infty$. Then $D(A^\beta)$ is dense in $B^{s,\beta}_q$ for each $\beta \in \mathbb{C}_+$ satisfying $\text{Re } \beta > |s|$.

**Proof.** Since the Besov space $B^{s,\beta}_q$ is the completion of $R^{s,\beta}_q$ with respect to $\| \cdot \|_{B^{s,\beta}_q}$, it suffices to show that $D(A^\beta)$ is dense in $R^{s,\beta}_q$ with respect to $\| \cdot \|_{B^{s,\beta}_q}$. To this end, let $x \in R^{s,\beta}_q$, fix $\beta \in \mathbb{C}_+$ such that $\text{Re } \beta > |s|$ and write

$$x_n := n^\beta (n + A)^{-\beta} x \in D(A^\beta), \quad n \in \mathbb{N}.$$ 

It suffices to verify that $\{x_n\}$ converges to $x$ in $B^{s,\beta}_q$. Indeed, fix $\epsilon > 0$. Thanks to the fact that $\|x\|_{B^{s,\beta}_q} < \infty$, there exists a $K \in \mathbb{N}$ such that

$$\left\{ \sum_{i=K}^{\infty} \left\| 2^{i(s+\beta)} A^\beta (2^i + A)^{-2\beta} x \right\|^q \right\}^{1/q} < \frac{\epsilon}{4C_1 1/q D_K},$$

where $\text{Re } \beta < m \in \mathbb{N}$ and $C_1, C_\beta,m$ and $M_A$ are the three constants given in (1.8), (2.14) and (2.1), respectively. Furthermore, since $x_n \to x$ in $X$ as $n \to \infty$ due to Lemma 2.4 (i), there exists an $N \in \mathbb{N}$ such that

$$\|x_n - x\| < \frac{\epsilon}{4C_1 1/q D_K}, \quad n \geq N,$$

where $D_K = \max \{d_K, \| (1 + A)^{-\beta} \| \}$ with

$$d_K = \left\{ \sum_{i=0}^{K-1} \left\| 2^{i(s+\beta)} A^\beta (2^i + A)^{-2\beta} \right\|^q \right\}^{1/q},$$

and hence,

$$\| (1 + A)^{-\beta} (x_n - x) \| < \frac{\epsilon}{4}, \quad n \geq N.$$ 

From (3.25), (3.26) and (3.27) it follows that

$$\| x_n - x \|_{B^{s,\beta}_q} \leq \| (1 + A)^{-\beta} (x_n - x) \|$$
and hence, \( \{x_n\} \subset D(A^\beta) \) converges to \( x \) as \( n \to \infty \) in \( B_{q,X}^{s,A} \). Thus, we have verified that \( D(A^\beta) \) is dense in \( R_{q,X}^{s,A} \) with respect to \( \| \cdot \|_{B_{q,X}^{s,A}} \). The proof is complete. \( \Box \)

In addition to the (inhomogeneous) Besov spaces associated with non-negative operators given in Definition 3.6, an alternative version of abstract (inhomogeneous) Besov spaces is given as follows.

**Definition 3.11.** Let \( s \in \mathbb{R} \) and \( 0 < q \leq \infty \), and let \( A \) be injective. The inhomogeneous Besov space \( \tilde{B}_{q,X}^{s,A} := \tilde{B}_{q,X}^{s,A}(k,\alpha,\beta) \) associated with \( A \) is defined as the completion of the subspace

\[
\tilde{R}_{q,X}^{s,A}(k,\alpha,\beta) := \left\{ x \in R(A) : \sum_{i=-\infty}^{k} \| 2^{i(s+\alpha)} A^\beta (2^i + A)^{-\alpha-\beta} x \|^q < \infty \right\}
\]

with respect to the quasi-norm

\[
\| x \|_{\tilde{B}_{q,X}^{s,A}} := \| A^\beta (Q^k + A)^{-\beta} x \| + \left\{ \sum_{i=-\infty}^{k} \| 2^{i(s+\alpha)} A^\beta (2^i + A)^{-\alpha-\beta} x \|^q \right\}^{1/q}
\]

(with the usual modification if \( q = \infty \)), where \( k \in \mathbb{Z} \) and \( \alpha \in \mathbb{C}_+^* \) and \( \beta \in \mathbb{C}_+ \) satisfying \( \text{Re} \alpha < s < \text{Re} \beta \).

**Remark 3.12.** Analogous to \( B_{q,X}^{s,A} \), it can be verified that \( \tilde{B}_{q,X}^{s,A} \) is also independent of the choice of indices \( k, \alpha \), and \( \beta \), and hence, \( \tilde{B}_{q,X}^{s,A} \) is well defined as a quasi-Banach space (Banach space) for each \( 0 < q \leq \infty \) \((1 \leq q \leq \infty \)). In particular, the inhomogeneous Besov space \( \tilde{B}_{q,X}^{s,A} \) with \( s < 0 \) and \( 1 \leq q \leq \infty \) coincides with the Komatsu space \( R_{q,X}^{s,A}(\alpha) \) [22, Definitions 2.1 and 2.4].

The next result reveals a connection between Besov spaces \( \tilde{B}_{q,X}^{s,A} \) and \( B_{q,X}^{s,A} \).

**Proposition 3.13.** Let \( A \) be injective, and let \( s \in \mathbb{R} \) and \( 0 < q \leq \infty \). Then \( \tilde{B}_{q,X}^{-s,A} = B_{q,X}^{s,A-1} \) in the sense of equivalent quasi-norms. In particular, \( \tilde{B}_{q,X}^{0,A} = B_{q,X}^{s,A-1} \) in the sense of equivalent quasi-norms.

**Proof.** Thanks to the density argument, it suffices to verify that \( R_{q,X}^{s,A} = \tilde{R}_{q,X}^{s,A} \) from the set-theoretic point of view and that

\[
\| x \|_{\tilde{B}_{q,X}^{-s,A}} \simeq \| x \|_{B_{q,X}^{s,A-1}}, \quad x \in B_{q,X}^{s,A}.
\]
Indeed, let \( x \in R_{q,X}^{s,A} \) and fix \( k \in \mathbb{Z} \) and \( \alpha, \beta \in \mathbb{C}_+ \) such that \( -\Re \alpha < s < \Re \beta \). By Definitions 3.6 and 3.11, we have
\[
\| x \|_{B_{q,X}^{s,A}} \simeq \| A^\alpha (2^k + A)^{-\alpha} x \|^{1/q} \\
+ \left\{ \sum_{i=-\infty}^{k} \| 2^{i(\alpha+s)} A^\alpha (2^i + A)^{-\alpha-\beta} x \| \right\}^{1/q} \\
= 2^{-k \Re \alpha} \| (2^{-k} + A^{-1})^{-\alpha} x \|^{1/q} \\
+ \left\{ \sum_{i=-k}^{\infty} \| 2^{i(s+\alpha)} A^{-\beta} (2^i + A^{-1})^{-\alpha-\beta} x \| \right\}^{1/q} \simeq \| x \|_{\dot{B}_{q,X}^{s,A-1}} ,
\]
the desired conclusion. The proof is complete. \(
\square
\)

In the next subsection we will introduce the homogeneous Besov space associated with non-negative operators. Further properties of inhomogeneous Besov spaces will be given in Sections 4 and 5 subsequently.

3.3. Homogeneous Besov spaces. Let \( A \) be an injective non-negative operator on \( X \). Analogous to the inhomogeneous Besov spaces associated with \( A \), in order to define homogeneous Besov spaces associated with \( A \), we start with the following subspace \( \dot{R}_{q,X}^{s,A}(\alpha, \beta) \).

**Definition 3.14.** Let \( 0 < q \leq \infty \) and \( s \in \mathbb{R} \), and let \( \alpha \in \mathbb{C}_+ \) and \( \beta \in \mathbb{C}_+ \) such that \( -\Re \alpha < s < \Re \beta \). The space \( \dot{R}_{q,X}^{s,A}(\alpha, \beta) \) is defined by
\[
\dot{R}_{q,X}^{s,A}(\alpha, \beta) := \left\{ x \in D(A) \cap R(A) : |x|_{\dot{R}_{q,X}^{s,A}(\alpha, \beta)} < \infty \right\},
\]
endowed with the quasi-norm
\[
|x|_{\dot{R}_{q,X}^{s,A}(\alpha, \beta)} := \left\{ \sum_{i=-\infty}^{\infty} \| 2^{i(s+\alpha)} A^\beta (2^i + A)^{-\alpha-\beta} x \| \right\}^{1/q}
\]
(with the usual modification if \( q = \infty \)).

Analogous to Lemma 3.5 above, we can verify that the space \( \dot{R}_{q,X}^{s,A}(\alpha, \beta) \) is independent of the choice of indices \( \alpha \) and \( \beta \) in the sense of equivalent quasi-norms.

**Lemma 3.15.** Let \( s \in \mathbb{R} \) and \( 0 < q \leq \infty \), and let \( \alpha \in \mathbb{C}_+ \) and \( \beta \in \mathbb{C}_+ \) such that \( -\Re \alpha < s < \Re \beta \). Then \( \dot{R}_{q,X}^{s,A}(\alpha, \beta) = \dot{R}_{q,X}^{s,A}(\alpha', \beta') \) in the sense of equivalent quasi-norms for all \( \alpha' \in \mathbb{C}_+ \) and \( \beta' \in \mathbb{C}_+ \) satisfying \( -\Re \alpha' < s < \Re \beta' \).

**Proof.** By analogous arguments given in the first paragraph of the proof of Lemma 3.4 it suffices to verify that
\[
|x|_{\dot{R}_{q,X}^{s,A}(\alpha, \beta)} \lesssim |x|_{\dot{R}_{q,X}^{s,A}(\alpha+1, \beta)}
\]
and that
\[
|x|_{\dot{R}_{q,X}^{s,A}(\alpha, \beta)} \lesssim |x|_{\dot{R}_{q,X}^{s,A}(\alpha, \beta+1)} .
\]
Moreover, we merely verify the conclusion in the case \( 0 < q < \infty \) and the case when \( q = \infty \) can be verified analogously.
First we verify (3.28). To this end, let $x \in R_{q,A}^{\alpha+1,\beta}$. Note that $x \in D(A)$ by Definition 3.14. If $\alpha = 0$ and that $(2^t + A)^{-\alpha} x \in D(A^\alpha) \subset D(A)$ if $\text{Re}\, \alpha > 0$, and hence, by using (2.36) with $\lambda = 2^i$ we have

$$|x|_{R_{q,A}^{\alpha+1,\beta}} \leq |\alpha + \beta| \left\{ \sum_{i=-\infty}^{\infty} \left[ 2^i (s + \text{Re}\, \alpha) \int_{2^i}^{\infty} \| A^{\beta} (t + A)^{-\alpha-\beta-1} x \| \, dt \right] \right\}^{1/q}.$$  

Rewriting $\int_{2^i}^{\infty} = \sum_{r=i}^{\infty} \frac{2^r}{2^i}$ and applying (2.19) yields

$$|x|_{R_{q,A}^{\alpha+1,\beta}} \leq C \left\{ \sum_{i=-\infty}^{\infty} \left[ \sum_{r=i}^{\infty} 2^i (s + \text{Re}\, \alpha) 2^r \left\| A^{\beta} (2^r + A)^{-\alpha-\beta-1} x \right\| \right] \right\}^{1/q} = CK,$$

where $C = |\alpha + \beta| C_{\alpha+\beta+1,N} (L_A + 2M_A)^N$ with $\text{Re}\, (\alpha + \beta + 1) < N \in \mathbb{N}$ and $C_{\alpha+\beta+1,N}$ given by (2.14) and

$$K = \left\{ \sum_{i=-\infty}^{\infty} \left[ \sum_{r=i}^{\infty} 2^i (s + \text{Re}\, \alpha) 2^r \left\| A^{\beta} (2^r + A)^{-\alpha-\beta-1} x \right\| \right] \right\}^{1/q}.$$

It can be verified that

$$K \lesssim |x|_{R_{q,A}^{\alpha+1,\beta}}.$$  

Indeed, if $0 < q \leq 1$, applying (1.4) to the inner summation of $K$ and exchanging the order of summation yields

$$K \leq \frac{1}{[1 - 2^{-\text{Re}\, (s + \alpha \epsilon) q}]^{1/q}} |x|_{R_{q,A}^{\alpha+1,\beta}}.$$

And if $1 < q < \infty$, rewriting $K$ as

$$K = \left\{ \sum_{i=-\infty}^{\infty} 2^i (s + \text{Re}\, \alpha) q \left[ \sum_{r=i}^{\infty} 2^{-r(s + \alpha - \epsilon)} \left\| A^{\beta} (2^r + A)^{-\alpha-\beta-1} x \right\| \right] \right\}^{1/q}$$

with $0 < \epsilon < s + \text{Re}\, \alpha$ and applying the Hölder inequality to the inner summation yields

$$K \leq \frac{1}{[1 - 2^{-\text{Re}\, (s + \alpha \epsilon) q}]^{1/q}} \frac{1}{(1 - 2^{-\epsilon q})^{1/q}} |x|_{R_{q,A}^{\alpha+1,\beta}}.$$

Thus, we have verified (3.30).

Next we verify (3.29). To this end, let $x \in R_{q,A}^{\alpha+1,\beta}$. By letting $\lambda = 2^i$ in (2.34) we have

$$|x|_{R_{q,A}^{\alpha+1,\beta}} \leq |\alpha + \beta| \left\{ \sum_{i=-\infty}^{\infty} \left[ \int_{0}^{2^i} \left\| 2^i (s - \beta) t^{\alpha + \beta} A^{\beta+1} (t + A)^{-\alpha-\beta-1} x \right\| \frac{dt}{t} \right] \right\}^{1/q}.$$  

Rewriting $\int_{0}^{2^i} = \sum_{r=-\infty}^{i-1} \frac{2^r}{2^i}$ and applying (2.19) to the integrands yields

$$|x|_{R_{q,A}^{\alpha+1,\beta}} \lesssim J,$$
where
\[
J = \left\{ \sum_{i=-\infty}^{\infty} \left[ \sum_{r=-\infty}^{i-1} \left\| 2^i (s-\beta) 2^r (\alpha+\beta) A^{\beta+1} (2^r + A)^{-\alpha-\beta-1} x \right\| \right]^{q/2} \right\}^{1/q}.
\]

It remains to verify that
\[
(3.31) \quad J \lesssim |x| R^{s,A}_{q,X} (\alpha, \beta+1).
\]

Indeed, if \(0 < q \leq 1\), applying (1.3) to the inner summation of \(J\) and exchanging the order of summation yields
\[
J \leq \left\{ \sum_{i=-\infty}^{\infty} \left[ \sum_{r=-\infty}^{i-1} \left\| 2^i (s-\beta) 2^r (\alpha+\beta) A^{\beta+1} (2^r + A)^{-\alpha-\beta-1} x \right\| \right] \right\}^{1/q} = \frac{2^{s-\text{Re} \beta}}{[1 - 2(s - \text{Re} \beta q)]^{1/q}} |x| R^{s,A}_{q,X} (\alpha, \beta+1);
\]

If \(1 < q < \infty\), rewriting the part \(J\) as
\[
J = \left\{ \sum_{i=-\infty}^{\infty} 2^i (s-\beta) q \left[ \sum_{r=-\infty}^{i-1} \left\| 2^r 2^r (\alpha+\beta-\epsilon) A^{\beta+1} (2^r + A)^{-\alpha-\beta-1} x \right\| \right] \right\}^{1/q},
\]

with \(0 < \epsilon < \text{Re} \beta - s\), applying the Hölder inequality to the inner summation of \(J\) and exchanging the order of summation yields
\[
J \leq \frac{1}{(2q' - 1)^{1/q}} \frac{2^{s-\text{Re} \beta + \epsilon}}{[1 - 2(s - \text{Re} \beta + \epsilon q)]^{1/q}} |x| R^{s,A}_{q,X} (\alpha, \beta+1).
\]

Thus, we have verified (3.31). The proof is complete. \(\square\)

Thanks to Lemma 3.15, we can now define the homogeneous Besov spaces associated with non-negative operators as follows.

**Definition 3.16.** Let \(0 < q \leq \infty\) and \(s \in \mathbb{R}\). The homogeneous Besov space \(\dot{B}^{s,A}_{q,X} := \dot{B}^{s,A}_{q,X} (\alpha, \beta)\) associated with \(A\) is defined as the completion of \(\dot{R}^{s,A}_{q,X} (\alpha, \beta)\) with respect to the quasi-norm
\[
\| \cdot \|_{\dot{B}^{s,A}_{q,X}} := \| \cdot \|_{\dot{R}^{s,A}_{q,X} (\alpha, \beta)},
\]
\[
(3.32)
\]

where \(\alpha \in \mathbb{C}^*_+\) and \(\beta \in \mathbb{C}^*_+\) satisfying \(-\text{Re} \alpha < s < \text{Re} \beta\).

**Remark 3.17.** By Lemma 3.15, the Besov space \(\dot{B}^{s,A}_{q,X}\) is well defined as a quasi-Banach space (Banach space) for each \(0 < q \leq \infty\) (\(1 \leq q \leq \infty\)). In particular, for \(q \geq 1\) and \(s \in \mathbb{R}\), the Besov space \(\dot{B}^{s,A}_{q,X}\) coincides with the space \(\dot{B}^{s}_{q}\), with \(\phi(x) = x^s\), defined by T. Matsumoto and T. Ogawa [67, Definition 2.8]. Also, see Remark 4.3 below for more information.

Compared with inhomogeneous Besov spaces as in Proposition 3.13, the next result for homogeneous Besov spaces is of independent interest.

**Proposition 3.18.** Let \(0 < q \leq \infty\) and \(s \in \mathbb{R}\). Then \(\dot{B}^{-s,A}_{q,X} = \dot{B}^{s,A^{-1}}_{q,X}\) in the sense of equivalent quasi-norms whenever \(A\) is injective.
Proof. Let $A$ be injective. It suffices to show that $\hat{B}^{s,A}_{q,X}(\alpha, \beta) = \hat{B}^{s,A-1}_{q,X}(\alpha, \beta)$ in the sense of equivalent quasi-norms for fixed $\alpha \in C^*_+$ and $\beta \in \mathbb{C}_+$ satisfying $-\text{Re} \alpha < s < \text{Re} \beta$. To this end, let $\alpha \in C^*_+$ and $\beta \in \mathbb{C}_+$ satisfying $-\text{Re} \alpha < s < \text{Re} \beta$. It is clear that $-\text{Re} \beta < -s < \text{Re} \alpha$, and hence,

$$|x|_{\hat{B}^{s,A}_{q,X}(\alpha, \beta)} = \left\{ \sum_{i=-\infty}^{\infty} \left\| 2^{i+s+\beta} A^\alpha (2^i + A)^{-\alpha-\beta} x \right\|^q \right\}^{1/q} = \left\{ \sum_{i=-\infty}^{\infty} \left\| 2^{i+s-\beta} A^\alpha (2^{-i} + A)^{-\alpha-\beta} x \right\|^q \right\}^{1/q} = |x|_{\hat{B}^{s,A-1}_{q,X}(\alpha, \beta)},$$

the conclusion desired. The proof is complete. \hfill $\square$

Finally, we present a relationship between the inhomogeneous Besov spaces and homogeneous Besov spaces to end this section.

**Proposition 3.19.** Let $0 < q \leq \infty$ and $s > 0$. Then $B^{s,A}_{q,X} = \hat{B}^{s,A}_{q,X} \cap X$ whenever $D(A) = R(A) = X$.

**Proof.** Let $k = \alpha = 0$ and $\beta \in \mathbb{C}_+$ with $s < \text{Re} \beta$. It is clear that $B^{s,A}_{q,X} \subset X$ by Proposition 3.8. Moreover, it is also clear that $|x|_{\hat{B}^{s,A}_{q,X}(0,0,\beta)} < \infty \iff |x|_{\hat{B}^{s,A}_{q,X}(0,\beta)} < \infty$ due to the fact that $s > 0$. Thus, $B^{s,A}_{q,X} = \hat{B}^{s,A}_{q,X} \cap X$. \hfill $\square$

### 4. Regular properties of Besov spaces

This section is devoted to basic properties of Besov spaces associated with non-negative operators, including quasi-norm equivalence, continuous embedding and translation invariance. As in the last section, $(X, \| \cdot \|)$ is a Banach space and $A$ is a non-negative operator on $X$ in this section.

#### 4.1. Quasi-norm equivalence

The semi-quasinorm $|x|_{\hat{B}^{s,A}_{q,X}(k,\alpha,\beta)}$ given in Definition 3.11 can also be characterized by use of the Lebesgue integrals. For convenience, we write

$$|x|_{\hat{B}^{s,A}_{q,X}(k,\alpha,\beta)}^c := \left\{ \int_{2^k}^{\infty} \left\| t^{s+\alpha} A^\beta (t + A)^{-\alpha-\beta} x \right\|^q \frac{dt}{t} \right\}^{1/q},$$

with the usual modification when $q = \infty$.

**Lemma 4.1.** Let $0 < q \leq \infty$ and $s \in \mathbb{R}$. Fix $k \in \mathbb{Z}$ and $\alpha, \beta \in C^*_+$ such that $-\text{Re} \alpha < s < \text{Re} \beta$, and let $x \in D(A)$. Then $x \in R^{s,A}_{q,X}(k,\alpha,\beta)$ if and only if $|x|_{\hat{B}^{s,A}_{q,X}(k,\alpha,\beta)}^c < \infty$. If this is the case,

$$|x|_{\hat{B}^{s,A}_{q,X}(k,\alpha,\beta)} \simeq |x|_{\hat{B}^{s,A}_{q,X}(k,\alpha,\beta)}^c.$$ 

**Proof.** First we verify the statement in the case $0 < q < \infty$. Let $|x|_{\hat{B}^{s,A}_{q,X}(k,\alpha,\beta)} < \infty$. From (4.1) it follows that

$$|x|_{\hat{B}^{s,A}_{q,X}(k,\alpha,\beta)}^c = \left\{ \sum_{i=k}^{\infty} \int_{2^i}^{2^{i+1}} t^{(s+\text{Re} \alpha)q-1} \left\| A^\beta (t + A)^{-\alpha-\beta} x \right\|^q dt \right\}^{1/q}$$

with the usual modification when $q = \infty$. 

From (4.1) it follows that

$$|x|_{\hat{B}^{s,A}_{q,X}(k,\alpha,\beta)}\leq \left\{ \sum_{i=k}^{\infty} \int_{2^i}^{2^{i+1}} t^{(s+\text{Re} \alpha)q-1} \left\| A^\beta (t + A)^{-\alpha-\beta} x \right\|^q dt \right\}^{1/q}$$

for $0 < q < \infty$.
\[ \left\{ \sum_{i=k}^{\infty} \int_{2^i}^{2^{i+1}} t^{(s + \text{Re} \alpha)q - 1} \| A^\beta (2^i + A)^{-\alpha - \beta} x \|^q dt \right\}^{1/q} \]

\[ = \left[ \frac{2^{(s + \text{Re} \alpha)q - 1}}{(s + \text{Re} \alpha)q} \right]^{1/q} \cdot |x|_{\dot{B}^{s,A}_{q,\infty}(k,\alpha,\beta)}^c, \]

where the last inequality follows from (2.19).

Conversely, let \( |x|_{\dot{B}^{s,A}_{q,\infty}(k,\alpha,\beta)}^c < \infty \). Applying the equality

\[ \frac{s + \text{Re} \alpha}{2s + \text{Re} \alpha - 1} \int_{2^i}^{2^{i+1}} t^{s + \text{Re} \alpha - 1} 2^{-i(s + \text{Re} \alpha)} dt = 1 \]

yields

\[ |x|_{\dot{B}^{s,A}_{q,\infty}(k,\alpha,\beta)}^c \lesssim \left\{ \sum_{i=k+1}^{\infty} 2^{i(s + \text{Re} \alpha)q} \| A^\beta (2^i + A)^{-\alpha - \beta} x \|^q \right\}^{1/q} \]

\[ = \left[ \frac{(s + \text{Re} \alpha)q}{2^{(s + \text{Re} \alpha)q} - 1} \right]^{1/q} \left\{ \sum_{i=k+1}^{\infty} \int_{2^i}^{2^{i+1}} \| t^{s + \alpha} A^\beta (2^i + A)^{-\alpha - \beta} x \|^q \frac{dt}{t} \right\}^{1/q}. \]

Applying (2.19) with \( c = 2 \) yields

\[ |x|_{\dot{B}^{s,A}_{q,\infty}(k,\alpha,\beta)}^c \lesssim \left\{ \sum_{i=k+1}^{\infty} \int_{2^i}^{2^{i+1}} \| t^{s + \alpha} A^\beta (t + A)^{-\alpha - \beta} x \|^q \frac{dt}{t} \right\}^{1/q} \]

\[ = \left\{ \int_{2^{k+1}}^{\infty} \| t^{s + \alpha} A^\beta (t + A)^{-\alpha - \beta} x \|^q \frac{dt}{t} \right\}^{1/q} \lesssim |x|_{\dot{B}^{s,A}_{q,\infty}(k,\alpha,\beta)}^c. \]

Thus, we have verified the statement for \( 0 < q < \infty \).

Next we verify the statement for \( q = \infty \). Indeed, it is clear that

\[ \sup_{i \geq k} 2^i \cdot \| 2^i A^\beta (2^i + A)^{-\alpha - \beta} x \| \leq \sup_{\lambda \geq 2^k} \lambda^s \cdot \| \lambda^\alpha A^\beta (\lambda + A)^{-\alpha - \beta} x \| \]

and that

\[ \sup_{\lambda \geq 2^k} \lambda^s \cdot \| \lambda^\alpha A^\beta (\lambda + A)^{-\alpha - \beta} x \| \]

\[ = \sup_{i \geq k} \sup_{\lambda \in [2^i, 2^{i+1}]} \| \lambda^{s+\alpha} A^\beta (\lambda + A)^{-\alpha - \beta} x \| \]

\[ \lesssim \sup_{i \geq k} \| 2^{(s+\alpha)} A^\beta (2^i + A)^{-\alpha - \beta} x \|, \]

from which we obtain

\[ |x|_{\dot{B}^{s,A}_{q,\infty}(k,\alpha,\beta)}^c \lesssim |x|_{\dot{B}^{s,A}_{\infty,\infty}(k,\alpha,\beta)}^c, \]

immediately. The proof is complete. \qed

Thanks to Lemma 3.1 we can now give an equivalent characterization of the inhomogeneous Besov spaces as follows.
Theorem 4.2. Let $s \in \mathbb{R}$ and $0 < q \leq \infty$. Fix $k \in \mathbb{Z}$ and fix $\alpha, \beta \in \mathbb{C}_+$ such that $-\Re \alpha < s < \Re \beta$. Then $B_{q,X}^{s,A}$ is the completion of $D^{s,A}$ with respect to the quasi-norm $\| \cdot \|_{B_{q,X}^{s,A}}$ given by

$$\|x\|_{B_{q,X}^{s,A}} := \| (2^k + A)^{-\alpha} x \| + \| x \|_{R_{q,X}^{s,A}(k, \alpha, \beta)},$$

where $\| \cdot \|_{R_{q,X}^{s,A}(k, \alpha, \beta)}$ is given in (4.7) above.

Remark 4.3. Let $0 < q \leq \infty$ and $s \in \mathbb{R}$. We have the following explicit characterizations of the Besov space $B_{q,X}^{s,A}$.

(i) Let $s > 0$. Fix $k = \alpha = 0 < s < \Re \beta$. It is clear that $B_{q,X}^{s,A} \subset X$ and that

$$\|x\|_{B_{q,X}^{s,A}} \simeq \|x\| + \left\{ \sum_{i=0}^{\infty} \| 2^{i\alpha} A^\beta (2^i + A)^{-\beta} x \| \right\}^{1/q},$$

$$\simeq \|x\| + \left\{ \sum_{i \in \mathbb{Z}} \| 2^{i\alpha} A^\beta (2^i + A)^{-\beta} x \| \right\}^{1/q}, \quad x \in B_{q,X}^{s,A},$$

where the last equivalence follows from the estimate (2.17) and the fact that $s > 0$. Moreover, by Lemma 4.1 we have

$$\|x\|_{B_{q,X}^{s,A}} \simeq \|x\| + \left\{ \int_{1}^{\infty} \| t^\alpha A^\beta (t + A)^{-\beta} x \| q dt \right\}^{1/q}$$

$$\simeq \|x\| + \left\{ \int_{0}^{\infty} \| t^\alpha A^\beta (t + A)^{-\beta} x \| q dt \right\}^{1/q}, \quad x \in B_{q,X}^{s,A}.$$
4.2. Continuous embedding. By $X_1 \hookrightarrow X_2$ we denote that the quasi-normed space $X_1$ is continuously embedded into the quasi-normed space $X_2$, i.e., $X_1 \subset X_2$ and $\|x\|_{X_2} \leq \|x\|_{X_1}$ for each $x \in X_1$.

Motivated by the continuous embedding of the classical Besov spaces on $\mathbb{R}^n$ (see [54, P.47, (5) and (7) and P.244, Section 5.2.5 Miscellaneous Properties]), we have the following continuous embedding for abstract Besov spaces.

**Proposition 4.4.** Let $0 < q \leq \infty$ and $s \in \mathbb{R}$. The following statements hold.

(i) $B^{s,A}_{q,X} \hookrightarrow B^{s,A}_{q,X}$ for all $0 < q \leq q_1 \leq \infty$.

(ii) $B^{-s,A}_{p,X} \hookrightarrow B^{s,A}_{q,X}$ for all $-\infty < s < s_1 < \infty$ and $0 < p \leq \infty$.

(iii) $B^{s,A}_{q,X} \hookrightarrow B^{s,A}_{q,X}$ for all $0 < q \leq q_1 \leq \infty$.

**Proof.** (i) Let $q \leq q_1 \leq \infty$. Fix $\alpha, \beta \in \mathbb{C}_+^*$ such that $-\Re \alpha < s < \Re \beta$. Since $R^{s,A}_{q,X}(0, \alpha, \beta)$ is dense in $B^{s,A}_{q,X}(0, \alpha, \beta)$, there is a sequence $\{x_n\} \subset R^{s,A}_{q,X}(0, \alpha, \beta)$ such that $x_n \to \tilde{x}$ with respect to $\|\cdot\|_{B^{s,A}_{q,X}(0, \alpha, \beta)}$. Furthermore, it can be seen that the mapping $\tau : \tilde{x} \mapsto \tilde{x}^*$ is injective and bounded from $B^{s,A}_{q,X}(0, \alpha, \beta)$ to $B^{s,A}_{q,X}(0, \alpha, \beta)$ by observing that $\|x\|_{B^{s,A}_{q,X}(0, \alpha, \beta)} = \lim_{n \to \infty} \|x_n\|_{B^{s,A}_{q,X}(0, \alpha, \beta)}$.

Therefore, $B^{s,A}_{q,X}(0, \alpha, \beta)$ is continuously embedded into $B^{s,A}_{q,X}(0, \alpha, \beta)$.

(ii) Let $s < s_1 < \infty$. First we verify the conclusion in the case $p = q$. To this end, fix $\alpha, \beta \in \mathbb{C}_+^*$ such that $-\Re \alpha < s < s_1 < \Re \beta$ and let $\tilde{x} \in B^{s,A}_{q,X}(0, \alpha, \beta)$. Then there is a sequence $\{x_n\} \subset R^{s,A}_{q,X}(0, \alpha, \beta)$ such that $x_n \to \tilde{x}$ in $B^{s,A}_{q,X}(0, \alpha, \beta)$. For each $x \in R^{s,A}_{q,X}(0, \alpha, \beta)$, it can be seen that

$$|x|_{R^{s,A}_{q,X}(0, \alpha, \beta)} = \left( \sum_{i=0}^{\infty} \|2^{i(s-s_1)}2^{i(s_1+\alpha)}A^\beta(2^i + A)^{-\alpha-\beta}x\|^q \right)^{1/q} \leq \left( \sum_{i=0}^{\infty} \|2^{i(s_1+\alpha)}A^\beta(2^i + A)^{-\alpha-\beta}x\|^q \right)^{1/q} = |x|_{R^{s,A}_{q,X}(0, \alpha, \beta)}.$$
This implies that \( \{x_n\} \) is also a Cauchy sequence with respect to \( \|\cdot\|_{B_{q,X}^{s,A}(0,\alpha,\beta)} \) and hence, there is a unique \( \tilde{x}^* \in B_{q,X}^{s,A}(0,\alpha,\beta) \) such that \( x_n \to \tilde{x}^* \) in \( B_{q,X}^{s,A}(0,\alpha,\beta) \). It can be seen that the mapping \( \tilde{x} \mapsto \tilde{x}^* \) is injective and bounded from \( B_{q,X}^{s,A}(0,\alpha,\beta) \) to \( B_{q,X}^{s,A}(0,\alpha,\beta) \) by observing that

\[
\|x\|_{B_{q,X}^{s,A}(0,\alpha,\beta)} = \lim_{n \to \infty} \|x_n\|_{B_{q,X}^{s,A}(0,\alpha,\beta)} \leq \lim_{n \to \infty} \|x\|_{B_{q,X}^{s,A}(0,\alpha,\beta)} = \|x\|_{B_{q,X}^{s,A}(0,\alpha,\beta)}. 
\]

Therefore, \( B_{q,X}^{s,A}(0,\alpha,\beta) \) is continuously embedded into \( B_{q,X}^{s,A}(0,\alpha,\beta) \).

Next we verify that \( B_{q,X}^{s,A} \hookrightarrow B_{p,X}^{s,A} \) for \( q \leq p \leq \infty \). Indeed, by the discussion above and (i), it is clear that

\[
B_{q,X}^{s,A} \hookrightarrow B_{q,X}^{s,A} \hookrightarrow B_{p,X}^{s,A}.
\]

It remains to verify the desired embedding for \( 0 < p < q \). Fix \( \alpha, \beta \in \mathbb{C}_+ \) such that \( -\Re \alpha < s < s_1 < \Re \beta \). Analogous to the discussion above, it suffices to verify that

\[
(4.2) \quad |x|_{B_{p,X}^{s,A}(0,\alpha,\beta)} \lesssim |x|_{B_{q,X}^{s,A}(0,\alpha,\beta)}, \quad x \in R_{q,X}^{s,A}(0,\alpha,\beta).
\]

Indeed, it is clear that

\[
|x|_{R_{p,X}^{s,A}(0,\alpha,\beta)} = \left( \sum_{i=0}^{\infty} 2^{2i(s-s_1)p} \|2^{2i} 2^{i\alpha} A^\beta (2^i + 1) - \alpha - \beta \| x \|^p \right)^{1/p}.
\]

Applying the Hölder inequality with respect to the index \( q/p > 1 \) yields

\[
|x|_{R_{p,X}^{s,A}(0,\alpha,\beta)} \leq \left( \sum_{i=0}^{\infty} 2^{i(s-s_1)p(q/p)} \right)^{1/(q/p)} \left( \sum_{i=0}^{\infty} \|2^{2i} 2^{i\alpha} A^\beta (2^i + 1) - \alpha - \beta \| x \|^p(q/p) \right)^{1/(q/p)} = C \|x\|_{R_{q,X}^{s,A}(0,\alpha,\beta)},
\]

where \( C = \left[ \frac{1}{1 - 2^{(s-s_1)p(q/p)}} \right]^{1/p} \). Thus, we have verified (ii), and therefore \( B_{q,X}^{s,A}(0,\alpha,\beta) \) is continuously embedded into \( B_{q,X}^{s,A}(0,\alpha,\beta) \).

(iii) The statement is a direct consequence of the monotonicity of the \( \ell_q \) spaces. Indeed, let \( q < q_1 \leq \infty \), and fix \( x \in B_{q_1,X}^{s,A} \). Then there is a sequence \( \{x_n\} \subset R_{q,X}^{s,A} \) such that \( \{x_n\} \) is Cauchy with respect to \( \|\cdot\|_{B_q^{s,A}} \), an equivalent quasi-norm on \( B_{q,X}^{s,A} \) given by (3.18). Note that

\[
\|x\|_{B_{q_1,X}^{s,A}} = \lim_{n \to \infty} \|x_n\|_{B_{q_1,X}^{s,A}} \leq \lim_{n \to \infty} \|x_n\|_{B_{q,X}^{s,A}} = \|x\|_{B_{q,X}^{s,A}}
\]

due to the monotonicity of the sequence spaces \( \ell_q \), from which the desired conclusion follows immediately. The proof is complete. \( \square \)
4.3. **Translation invariance.** Another property of interest for inhomogeneous Besov spaces associated with non-negative operators is that they are translation invariant with respect to the underlying operators. More precisely, we have the following proposition.

**Proposition 4.5.** Let $0 < q \leq \infty$ and $s \in \mathbb{R}$. Then $B^{s,A+\epsilon}_{q,X} = B^{s,A}_{q,X}$ in the sense of equivalent quasi-norms for each $\epsilon > 0$.

**Proof.** Let $\epsilon > 0$, take $\alpha, \beta \in \mathbb{C}_+^*$ such that $-\Re \alpha < s < \Re \beta$, and fix $k \in \mathbb{Z}$ such that $\epsilon < 2^k$. It suffices to verify that

\[
\|x\|_{B^{s,A+\epsilon}_{q,X}(k,\alpha,\beta)} \lesssim \|x\|_{B^{s,A}_{q,X}(k,\alpha,\beta)},
\]

and that

\[
\|x\|_{B^{s,A+\epsilon}_{q,X}(k,\alpha,\beta)} \lesssim \|x\|_{B^{s,A}_{q,X}(k,\alpha,\beta)}.
\]

First we verify the equivalence

\[
\|(2^k + A)^{-\alpha}x\| \simeq \|(2^k + A + \epsilon)^{-\alpha}x\|.
\]

Indeed, applying \((2.18)\) with $c = 1$ yields

\[
\|(2^k + A + \epsilon)^{-\alpha}x\| \leq \|(2^k + A)^\alpha (2^k + \epsilon + A)^{-\alpha}x\| \cdot \|(2^k + A)^{-\alpha}x\| \leq C_{\alpha,n}(L_A + M_A)^\alpha \|(2^k + A)^{-\alpha}x\|.
\]

Thanks to the fact that $\epsilon < 2^k$, applying \((2.18)\) with $c = 2$ yields

\[
\|(2^k + A)^{-\alpha}x\| \leq \|(2^k + \epsilon + A)^\alpha (2^k + A)^{-\alpha}x\| \cdot \|(2^k + A + \epsilon)^{-\alpha}x\| \leq C_{\alpha,n}(L_A + 2M_A)^\alpha \|(2^k + A + \epsilon)^{-\alpha}x\|,
\]

where $C_{\alpha,n}$ is given by \((2.14)\) with $\Re \alpha < n \in \mathbb{N}$, and $M_A$ and $L_A$ are the constants given by \((2.1)\) and \((2.3)\), respectively. Thus, we have verified \((4.5)\).

Next we verify \((4.3)\). Thanks to \((4.5)\), it suffices to verify that

\[
|x|_{R^{s,A}_{q,X}(k,\alpha,\beta)} \lesssim |x|_{R^{s,A+\epsilon}_{q,X}(k,\alpha,\beta)}.
\]

We merely verify \((4.6)\) in the case $0 < q < \infty$ while the case when $q = \infty$ can be verified analogously. To this end, applying \((2.17)\) to $A^\beta(\epsilon + A)^{-\beta}$ and applying \((2.18)\) with $c = 2$ to $(2^j + \epsilon + A)^{\alpha+\beta}(2^j + A)^{-(\alpha+\beta)}$ yields

\[
|x|_{R^{s,A}_{q,X}(k,\alpha,\beta)} = \left\{ \sum_{j=k}^{\infty} \| 2^j \|_2 \| 2^j A^\beta (2^j + A)^{-(\alpha+\beta)} \|_q \right\}^{1/q} \leq \left\{ \sum_{j=k}^{\infty} \| A^\beta (\epsilon + A)^{-\beta} \| \cdot \|(2^j + \epsilon + A)^{\alpha+\beta}(2^j + A)^{-(\alpha+\beta)} \| \cdot \| 2^j (s+\alpha) (A + \epsilon)^\beta (2^j + A + \epsilon)^{-(\alpha+\beta)} \|_q \right\}^{1/q} \leq C_{\beta,m}C_{\alpha+\beta,N}L_A^m (L_A + 2M_A)^N |x|_{R^{s,A+\epsilon}_{q,X}(k,\alpha,\beta)},
\]

where $C_{\beta,m}$ and $C_{\alpha+\beta,N}$ are given by \((2.14)\) with $\Re \alpha < m \in \mathbb{N}$ and $\Re (\alpha + \beta) < N \in \mathbb{N}$, respectively, and $M_A$ and $L_A$ are the non-negativity constants of $A$ given by \((2.1)\) and \((2.3)\), respectively. Thus, we have verified \((4.6)\).
It remains to verify (4.4). By Lemma 3.5 (the independence of Besov quasi-norms with respect to the index $\beta$), it suffices to verify (4.4) for the case in which $\beta = m \in \mathbb{Z}$. Moreover, we may assume that $0 < q < \infty$ and the case when $q = \infty$ can be verified analogously. Note that

$$\|x\|_{0,\beta} = \|x\|_{\beta_{\text{norms}}}$$

where

$$\|x\|_{\beta_{\text{norms}}} = \left\{ \sum_{j=k}^{\infty} \|2^{j}2^{j}\alpha(A + \epsilon)^{m}(2^{j} + A + \epsilon)^{-\alpha - m}x\|^{q} \right\}^{1/q}$$

Applying (2.18) yields

$$\|x\|_{\beta_{\text{norms}}} = \left\{ \sum_{j=k}^{\infty} \sum_{i=0}^{m} \left( \frac{m}{i} \right)^{i} \|2^{j(i + \alpha)(2^{j} + A + \epsilon)^{-\alpha - m}x\|^{q} \right\}^{1/q}$$

Decomposing $\alpha + m = (m - i) + (i + \alpha)$ yields

$$\|x\|_{\beta_{\text{norms}}} = \left\{ \sum_{j=k}^{\infty} \sum_{i=0}^{m} \left( \frac{m}{i} \right)^{i} \|2^{j(i + \alpha)(2^{j} + A + \epsilon)^{-\alpha - m}x\|^{q} \right\}^{1/q}$$

Applying of (2.1) yields

$$\|2^{j(i + \alpha)(2^{j} + A + \epsilon)^{-\alpha - m}x\|^{q} \right\}^{1/q}$$

Applying (2.18) with $c = 1$ yields

$$\|2^{j(i + \alpha)(2^{j} + A + \epsilon)^{-\alpha - m}x\|^{q} \right\}^{1/q}$$

where $C_{i + \alpha,N}$ is given by (2.14) with $\text{Re}(i + \alpha) < N \in \mathbb{N}$. Applying (4.7), (4.8) and (4.9) yields

$$\|x\|_{\beta_{\text{norms}}} = \left\{ \sum_{j=k}^{\infty} \sum_{i=0}^{m} \left( \frac{m}{i} \right)^{i} \|2^{j(i + \alpha)(2^{j} + A + \epsilon)^{-\alpha - m}x\|^{q} \right\}^{1/q}$$

Applying (2.18) with $c = 1$ yields

$$\|x\|_{\beta_{\text{norms}}} = \left\{ \sum_{j=k}^{\infty} \sum_{i=0}^{m} \left( \frac{m}{i} \right)^{i} \|2^{j(i + \alpha)(2^{j} + A + \epsilon)^{-\alpha - m}x\|^{q} \right\}^{1/q}$$

Applying (2.18) with $c = 1$ yields

$$\|x\|_{\beta_{\text{norms}}} = \left\{ \sum_{j=k}^{\infty} \sum_{i=0}^{m} \left( \frac{m}{i} \right)^{i} \|2^{j(i + \alpha)(2^{j} + A + \epsilon)^{-\alpha - m}x\|^{q} \right\}^{1/q}$$

where $C = 1$ due to the Minkowski inequality for $1 < q < \infty$ and $C = 2^{m(1/q - 1)}$ due to (1.3) for $0 < q \leq 1$. Applying (4.5) and (4.10) yields

$$\|x\|_{\beta_{\text{norms}}} \lesssim \|2^{j} + A\|^{-\alpha}x$$
Applying Lemma 3.5 (the independence of Besov quasi-norms with respect to the index $\beta$) yields

$$\|x\|_{B_{q,X}^{s,2\gamma}((k,\alpha,m)} \lesssim \sum_{i=0}^{m} \binom{m}{i} (\epsilon M)^{m-i} C_{i+\alpha,N} (L_A + M_A)^N \|x\|_{B_{q,X}^{s,i-m\cdot A}(k,\alpha,i)}.$$  

Since $s+i-m \leq s$, by Proposition 4.4 (ii) we obtain that

$$\|x\|_{B_{q,X}^{s,2\gamma}((k,\alpha,m)} \lesssim \sum_{i=0}^{m} \binom{m}{i} (\epsilon M)^{m-i} C_{i+\alpha,N} (L_A + M_A)^N \cdot \|x\|_{B_{q,X}^{s,\gamma}((k,\alpha,m)}.$$  

Thus, we have verified (4.4). The proof is complete. $\square$

## 5. Fractional powers on Besov spaces

This section is devoted to the connections between Besov spaces associated with non-negative operators and fractional powers of the underlying operators, including the lifting property, smoothness reiteration and interpolation. As in the last two sections, $(X,\|\cdot\|)$ is a Banach space and $A$ is a non-negative operator on $X$.

### 5.1. Lifting property

Recall that $x \in B_{q,X}^{s,A}$ if and only if $x \in D(A^{\gamma})$ and $A^{\gamma}x \in B_{q,X}^{s-\Re \gamma,A}$ whenever $0 < \Re \gamma < s$ and $1 \leq q \leq \infty$ (see [51, Theorem 2.6]). We can show further that the fractional power $A^{\gamma}$ is continuous from $B_{q,X}^{s,A}$ to $B_{q,X}^{s-\Re \gamma,A}$, even if $0 < q < 1$. Furthermore, it can be verified that $\|A^{\gamma}\|_{B_{q,X}^{s-\Re \gamma,A}}$ is indeed an equivalent quasi-norm on $B_{q,X}^{s,A}$ whenever $A$ is positive.

**Lemma 5.1.** Let $0 < \Re \gamma < s$ and $0 < q \leq \infty$. The following statements hold.

(i) $x \in B_{q,X}^{s,A}$ if and only if $x \in D(A^{\gamma})$ and $A^{\gamma}x \in B_{q,X}^{s-\Re \gamma,A}$.

(ii) $A^{\gamma}$ is continuous from $B_{q,X}^{s,A}$ to $B_{q,X}^{s-\Re \gamma,A}$.

**Proof.** (i) Sufficiency. Suppose that $x \in D(A^{\gamma})$ and $A^{\gamma}x \in B_{q,X}^{s-\Re \gamma,A}$. Fix $\alpha, \beta \geq 0$ such that $-\alpha < s < \beta$. It is clear that $x \in B_{q,X}^{s,A}$ due to the fact that

$$\sum_{i=0}^{\infty} \|2^{i(s+\alpha)} A^\beta (2^i + A)^{-(\alpha+\beta)} x\|_{q}^{1/q} \leq \|A^{\gamma}x|_{R_{q,X}^{s,A}(0,\alpha+\gamma,\beta-\gamma)} < \infty.$$  

Necessity. Let $x \in B_{q,X}^{s,A}$. First we verify that $x \in D(A^{\gamma})$. To this end, fix $\alpha, \beta, \gamma \geq 0$ such that $-\alpha < \Re \gamma < \gamma' < s < \beta$. In order to verify that $x \in D(A^{\gamma})$, it suffices to show that

$$\int_0^\infty \|t^{\gamma'} t^\alpha A^\beta (t+A)^{-(\alpha-\beta)} x\| \frac{dt}{t} < \infty.$$  

\[ \tag{5.1} \]
More precisely, if this is the case, then \( x \in D(A^\gamma) \) by Corollary 2.8 while \( D(A^\gamma) \subset D(A^\gamma) \) due to the fact that \( \gamma' > \Re \gamma, \) and hence, \( x \in D(A^\gamma). \) We now verify (5.1). On the one hand, from (2.16)* and (2.17)* it follows that

\[
\int_0^1 \left \| t^\gamma t^\alpha A^\beta (t + A)^{-\alpha - \beta} x \right \| \frac{dt}{t} \leq C_{\alpha, n} C_{\alpha, m} M_A^n L_A^n \| x \| / \gamma' < \infty
\]

(5.2)

where \( \alpha < n \in \mathbb{N}, \beta < m \in \mathbb{N} \) and \( C_{\alpha, m} \) and \( C_{\beta, n} \) are given by (2.14). Furthermore, we can show that it remains to verify (5.3). Indeed, rewriting

\[
\int_1^\infty \left \| t^\gamma t^\alpha A^\beta (t + A)^{-\alpha - \beta} x \right \| \frac{dt}{t} \leq \sum_{i=0}^{\infty} \left \| 2^{i(\alpha + \gamma)} A^\beta (2^i + A)^{-\alpha - \beta} x \right \|
\]

(5.4)

where \( C = C_{\alpha + \beta, N} (L_A + 2 M_A)^N (2^{\gamma' + \alpha} - 1) / (\gamma' + \alpha) \) with \( \alpha + \beta \leq N \in \mathbb{N} \) and \( C_{\alpha, \beta, N} \) given by (2.14). Furthermore, we can show that

\[
J := \sum_{i=0}^{\infty} \left \| 2^{i(\alpha + \gamma)} A^\beta (2^i + A)^{-\alpha - \beta} x \right \| \leq |x| R_{*_{\gamma, \alpha}}^{\gamma, A}(0, \alpha, \beta),
\]

(5.5)

and hence, the desired inequality (5.3) follows from (5.4) and (5.5), immediately. Thus, it is sufficient to verify (5.5). Indeed, if \( 0 < q < 1 \), applying the fact that \( \gamma' < s \) yields

\[
J \leq \sum_{i=0}^{\infty} \left \| 2^{(s + \gamma)} A^\gamma (2^i + A)^{-\alpha - \beta} x \right \|^{1/q}
\]

(5.6)

\[
= \left \{ \sum_{i=0}^{\infty} \left \| 2^{i(\alpha + \gamma)} A^\beta (2^i + A)^{-\alpha - \beta} x \right \|^q \right \}^{1/q} = |x| R_{*_{\gamma, \alpha}}^{\gamma, A}(0, \alpha, \beta),
\]

where the last inequality follows from (1.4). If \( 1 < q < \infty \), applying the Hölder inequality yields

\[
J \leq \left \{ \sum_{i=0}^{\infty} 2^{(\gamma - s)q} \right \}^{1/q} \left \{ \sum_{i=0}^{\infty} \left \| 2^{i(\alpha + \gamma)} A^\beta (2^i + A)^{-\alpha - \beta} x \right \|^q \right \}^{1/q}
\]

(5.7)

\[
= \left [ \frac{1}{1 - 2^{(\gamma - s)q}} \right ]^{1/q} \left \| 2^{i(\gamma - s)} A^\gamma (2^i + A)^{-\alpha - \beta} x \right \| = \left [ \frac{1}{1 - 2^{\gamma - s}} \right ]^{1/q} \left \| 2^{i(\gamma - s)} A^\gamma (2^i + A)^{-\alpha - \beta} x \right \|
\]

And if \( q = \infty \), it is clear that

\[
J \leq \left \{ \sup_{i \geq 0} 2^{i(\gamma - s)} \right \} \left \| 2^{i\gamma} A^\beta (2^i + A)^{-\alpha - \beta} x \right \| = \left [ \frac{1}{1 - 2^{\gamma - s}} \right ] \left \| 2^{i(\gamma - s)} A^\gamma (2^i + A)^{-\alpha - \beta} x \right \|.
\]

By (5.6), (5.7) and (5.8), we obtain (5.5), immediately. Thus, we have verified that \( x \in D(A^\gamma). \)
Next we verify that $A^\gamma x \in B^s_{q,X}$. To this end, fix $\alpha, \beta \geq 0$ such that $\alpha > 2\Re \gamma$ and $-(\alpha - 2\Re \gamma) < s < \beta$. On the one hand, from $(2.17)^+$ it follows that
\[
\| (2^k + A)^{- (\alpha - \gamma)} A^\gamma x \| \leq C_{\gamma,n} L^n A \| (2^k + A)^{-(\alpha - 2\gamma)} x \|
\]
where $0 < \Re \gamma < n \in \mathbb{N}$ and $C_{\gamma,n}$ and $L_A$ are given by $(2.14)$ and $(2.3)$, respectively. On the other hand, it is clear that
\[
\left\{ \sum_{i=0}^{\infty} \| 2^i (s - \gamma) , 2^i (\alpha - \gamma) A^{\beta - \gamma} (2^i + A)^{-(\alpha + \beta - 2\gamma)} A^\gamma x \| q \right\}^{1/q} = |x| R^\gamma_{s,A} (0, \alpha - 2\gamma, \beta).
\]
Thus, we have $A^\gamma x \in B^s_{q,X}$ by the definition of inhomogeneous Besov spaces (see Definition 3.6 and Remark 3.7 above).

(ii) Fix $\alpha, \beta > 0$ such that $\alpha > 2\Re \gamma$ and $-(\alpha - 2\Re \gamma) < s < \beta$. From $(5.9)$ it follows that
\[
\| A^\gamma x \|_{B^s_{q,X}} = \| (2^k + A)^{-(\alpha - \gamma)} A^\gamma x \| + |x| R^\gamma_{s,A} (0, \alpha - \beta - \gamma)
\]
\[
\leq C_{\gamma,n} L^n A \| (2^k + A)^{- (\alpha - 2\gamma)} x \| + |x| R^\gamma_{s,A} (0, \alpha - 2\gamma, \beta) \lesssim \| x \|_{B^s_{q,X}}.
\]
Thus, $A^\gamma$ is continuous from $B^s_{q,X}$ to $B^{s-Re \gamma}_{q,X}$. The proof is complete.

Thanks to Lemma 5.1 above, we can now give an alternative equivalent quasi-norm on $B^s_{q,X}$ by the use of $\| A^\gamma \|_{B^{s-Re \gamma}_{q,X}}$ whenever $A$ is positive on $X$.

**Theorem 5.2.** Let $A$ be positive on $X$, and let $s > 0$ and $0 < q \leq \infty$. The following statements hold.

(i) $A^{-\gamma}$ is continuous from $B^s_{q,X}$ to $B^{s-Re \gamma}_{q,X}$ for each $\gamma \in \mathbb{C}_+$.

(ii) $\| A^{-\gamma} \cdot \|_{B^{s-Re \gamma}_{q,X}}$ is an equivalent quasi-norm on $B^s_{q,X}$ for each $\gamma \in \mathbb{C}_+$.

(iii) $\| A^{-\gamma} \cdot \|_{B^{s-Re \gamma}_{q,X}}$ is an equivalent quasi-norm on $B^s_{q,X}$ for each $\gamma \in \mathbb{C}_+$ satisfying $0 < \Re \gamma < s$.

**Proof.** (i) Let $x \in B^s_{q,X}$ and $\gamma \in \mathbb{C}_+$. Fix $k \in \mathbb{Z}$, $\alpha, \beta \geq 0$ such that $\beta > s$ and $\alpha > \Re \gamma$ (so that $-\alpha < s < \beta$ and $-(\alpha - \Re \gamma) < s + \Re \gamma < \beta < \Re \gamma$). By the definition of the quasi-norm on Besov space $B^s_{q,X}$, we have
\[
\| A^{-\gamma} x \|_{B^{s-Re \gamma}_{q,X} (k, \alpha - \gamma, \beta + \gamma)} = \| (1 + 2^k A^{-\gamma}) (2^k + A)^{-\alpha} x \|
\]
\[
\leq C \| x \|_{B^s_{q,X} (k, \alpha, \beta)}.
\]
where $C = (1 + 2^k \| A^{-\gamma} \|)$. Thus, $A^{-\gamma}$ is continuous from $B^s_{q,X}$ to $B^{s-Re \gamma}_{q,X}$.

(ii) The statement is a simple consequence of $(5.10)$ and $(5.11)$. Indeed,
\[
\| A^{-\gamma} x \|_{B^{s-Re \gamma}_{q,X}} \lesssim \| x \|_{B^s_{q,X}} = \| A^\gamma A^{-\gamma} x \|_{B^{s-Re \gamma-Re \gamma}_{q,X}} \lesssim \| A^{-\gamma} x \|_{B^{s-Re \gamma}_{q,X}}.
\]

(iii) The statement is a simple consequence of $(5.11)$ and $(5.10)$. Indeed,
\[
\| A^\gamma x \|_{B^{s-Re \gamma}_{q,X}} \lesssim \| x \|_{B^s_{q,X}} = \| A^{-\gamma} A^\gamma x \|_{B^{s-Re \gamma-Re \gamma}_{q,X}} \lesssim \| A^\gamma x \|_{B^{s-Re \gamma}_{q,X}}.
\]
The proof is complete.
5.2. **Smoothness reiteration.** The smoothness of abstract Besov spaces has a closely relation to the fractional powers of the underlying non-negative operators. By using the so-called smoothness reiteration, we can give more explicit estimates of the quasi-norms on abstract Besov spaces.

We will give the main result of this subsection in Theorem 5.5 below based on the following two lemmas.

**Lemma 5.3.** Let $0 < \alpha < 1$ and let

$$f(t) := 1 + 2t \cos \pi \alpha + t^2, \quad t > 0.$$  

Then, for given $t > 0$, we have

$$f(u) \leq K_\alpha f(t), \quad t/2 \leq u \leq t,$$  

where

$$K_\alpha = \begin{cases} 1, & 0 < \alpha \leq 1/2, \\ \frac{1}{4}(1 + \frac{3}{\sin^2 \pi \alpha}), & 1/2 < \alpha < 1. \end{cases}$$  

**Proof:** The statement is obvious when $0 < \alpha \leq 1/2$ since $f$ is increasing on $(0, \infty)$. Thus, it remains to verify (5.12) in the case $1/2 < \alpha < 1$. Fix $t > 0$ and write

$$g_t(u) := \frac{f(u)}{f(t)}, \quad u > 0.$$  

It suffices to verify that

$$\sup_{t/2 \leq u \leq t} g_t(u) \leq \frac{1}{4}(1 + \frac{3}{\sin^2 \pi \alpha}).$$  

To this end, applying $g'_t(u) = 0$ yields $u = -\cos \pi \alpha$, so that $g_t$ is decreasing on $(0, -\cos \pi \alpha)$ while increasing on $(-\cos \pi \alpha, \infty)$. If $t/2 \geq -\cos \pi \alpha$, we have

$$\sup_{t/2 \leq u \leq t} g_t(u) \leq g_t(t) = 1 < \frac{1}{4}(1 + \frac{3}{\sin^2 \pi \alpha})$$  

since $g_t$ is increasing on $(t/2, t)$. If $t/2 < -\cos \pi \alpha \leq t$, we have

$$\sup_{t/2 \leq u \leq t} g_t(u) \leq \max\{g_t(t), g_t(t/2)\} = \max\{1, g_t(t/2)\}$$  

since $g_t$ is decreasing on $(t/2, -\cos \pi \alpha)$ and increasing on $(-\cos \pi \alpha, t)$. Observe that

$$g_t(t/2) = \frac{1}{4} \left(1 + \frac{3 + 2t \cos \pi \alpha}{1 + 2t \cos \pi \alpha + t^2} \right) \leq \frac{1}{4} \left(1 + \frac{3}{1 + 2t \cos \pi \alpha + t^2} \right) \leq \frac{1}{4} \left(1 + \frac{3}{\sin^2 \pi \alpha} \right).$$  

deondefactthatcos\pi\alpha<0. Byusing(5.14)and(5.15)weobtain(5.13), immediately. And if $t < -\cos \pi \alpha$, it can be seen from (5.15) that

$$\sup_{t/2 \leq u \leq t} g_t(u) \leq g_t(t/2) \leq \frac{1}{4} \left(1 + \frac{3}{\sin^2 \pi \alpha} \right)$$  

since $g_t$ is decreasing on $(t/2, t)$. Thus, we have verified (5.13). The proof is complete. $\square$
Lemma 5.4. Let $0 < q \leq \infty$ and let $0 < s, \alpha < 1$. Define $T : \ell_q \to \ell_q$ by

$$ Ta := b, \ a = \{a_j\} \in \ell_q, $$

where $b = \{b_j\}$ with

$$ b_j = \sum_{i=-\infty}^{\infty} \frac{(2^{-j}\lambda^s_a)^{1-s}}{1 + 2(2^{-j}\lambda^s_a) \cos \pi \alpha + (2^{-j}\lambda^s_a)^2} \cdot a_i, \ j \in \mathbb{Z}. $$

Then $T$ is bounded on $\ell_q$.

Proof. First we verify the statement in the case $0 < q \leq 1$. Fix $a = \{a_j\} \in \ell_q$. It follows from (1.4) that

$$ \|Ta\|_{\ell_q} \leq \left\{ \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \left| \frac{(2^{-j}\lambda^s_a)^{1-s}}{1 + 2(2^{-j}\lambda^s_a) \cos \pi \alpha + (2^{-j}\lambda^s_a)^2} a_i \right|^q \right\}^{1/q} = \left\{ \sum_{i=-\infty}^{\infty} |a_i|^q \sum_{j=-\infty}^{\infty} \left[ \frac{(2^{-j}\lambda^s_a)^{1-s}}{1 + 2(2^{-j}\lambda^s_a) \cos \pi \alpha + (2^{-j}\lambda^s_a)^2} \right]^q \right\}^{1/q}. $$

In order to verify the $\ell_q$-boundedness of $T$, it is sufficient to verify that

$$ \sup_{i \in \mathbb{Z}} \sum_{j=-\infty}^{\infty} \left[ \frac{(2^{-j}\lambda^s_a)^{1-s}}{1 + 2(2^{-j}\lambda^s_a) \cos \pi \alpha + (2^{-j}\lambda^s_a)^2} \right]^q < \infty. $$

To this end, by using the identity $\int_{2^j}^{2^{j+1}} 2^{-j} d\lambda = 1$ for $j \in \mathbb{Z}$ we have

$$ \sup_{i \in \mathbb{Z}} \sum_{j=-\infty}^{\infty} \left[ \frac{(2^{-j}\lambda^s_a)^{1-s}}{1 + 2(2^{-j}\lambda^s_a) \cos \pi \alpha + (2^{-j}\lambda^s_a)^2} \right]^q \leq 2^{(1-s)q+1} \sup_{i \in \mathbb{Z}} \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \left[ \frac{(2^{-j}\lambda^s_a)^{1-s}}{1 + 2(2^{-j}\lambda^s_a) \cos \pi \alpha + (2^{-j}\lambda^s_a)^2} \right]^q \frac{d\lambda}{\lambda}, $$

where the last inequality follows from the fact that $\lambda \leq 2^{j+1}$. Since $2^{-j}\lambda^s_a/2 \leq \lambda^{-1}2^{s \alpha} < 2^{-j}\lambda^s_a$, applying (5.12) with $t = 2^{-j}\lambda^s_a$ to the integrand of the last integel yields

$$ \sup_{i \in \mathbb{Z}} \sum_{j=-\infty}^{\infty} \left[ \frac{(2^{-j}\lambda^s_a)^{1-s}}{1 + 2(2^{-j}\lambda^s_a) \cos \pi \alpha + (2^{-j}\lambda^s_a)^2} \right]^q \leq K_\alpha 2^{(1-s)q+1} \sup_{i \in \mathbb{Z}} \int_0^{\infty} \left[ \frac{(\lambda^{-1}2^{s \alpha})^{1-s}}{1 + 2(\lambda^{-1}2^{s \alpha}) \cos \pi \alpha + (\lambda^{-1}2^{s \alpha})^2} \right]^q \frac{d\lambda}{\lambda}, $$

from which the desired (5.16) follows immediately since the last integral converges due to the fact that $0 < s < 1$. More precisely,

$$ \|Ta\|_{\ell_q} \leq C\|a\|_{\ell_q}, $$
where \( C = (JK_\alpha)^{1/q} 2^{1-s+1/q} \) with
\[
J = \int_0^\infty \left( \frac{\mu^{1-s}}{1 + 2\mu \cos \pi \alpha + \mu^2} \right)^q \frac{d\mu}{\mu}.
\]
Thus, we have verified the \( \ell_q \)-boundedness of \( T \) for \( 0 < q \leq 1 \).

Next we verify the \( \ell_q \)-boundedness of \( T \) for \( q = \infty \). Indeed, analogous to (5.16), by using
\[
\int_{2^{-i}}^{2^i} \frac{1}{2^i} d\lambda = 1
\]
and the estimate (5.12) we can conclude that
\[
\sup_{j \in \mathbb{Z}} \sum_{i=-\infty}^{\infty} (2^{-j} 2^i)^{1-s} (2^{-j} 2^i)^{1-s} \leq \alpha^{-1} K_\alpha 2^{\alpha(1-s)+1} \int_0^\infty \frac{\mu^{1-s}}{1 + 2\mu \cos \pi \alpha + \mu^2} \frac{d\mu}{\mu} := D_T < \infty,
\]
where \( K_\alpha \) is the constant given in (5.12). This implies that
\[
\| Ta \|_{l_\infty} = \sup_{j \in \mathbb{Z}} |b_j| \leq D_T \cdot \sup_{i \in \mathbb{Z}} |a_i| = D_T \| a \|_{l_\infty},
\]
the desired \( \ell_\infty \)-boundedness of \( T \).

Finally, the \( \ell_q \)-boundedness of \( T \) for \( 1 < q < \infty \) is a direct consequence of the well-known Marcinkiewicz interpolation theorem due to the \( \ell_1 \)-boundedness and \( \ell_\infty \)-boundedness of the (sub)linear operator \( T \). The proof is complete. \( \square \)

We can now establish the smoothness reiteration for Besov spaces \( B^{s,A}_{q,X} \), which was first discussed by H. Komatsu [51, Theorem 3.2] via integral transforms in the case \( 1 \leq q \leq \infty \) and subsequently described by M. Haase [37, Corollary 7.3] by using functional calculi in the case \( 1 \leq q \leq \infty \) as well. Here we give a unified approach to the smoothness reiteration for Besov spaces with a full range of \( 0 < q \leq \infty \). Moreover, recall that a closed linear operator on a Banach spaces is non-negative if and only if it is sectorial.

**Theorem 5.5.** Let \( s > 0 \), \( 0 < q \leq \infty \) and \( 0 \leq \omega < \pi \), and let \( A \) be sectorial of angle \( \omega \). Then
\[
B^{s,\alpha}_{q,X} = B^{s\alpha,A}_{q,X}, \quad 0 < \alpha < \pi/\omega,
\]
in the sense of equivalent quasi-norms.

**Proof.** It suffices to verify the statement in the case \( 0 < \alpha < 1 \), because otherwise we have \( A = (A^\alpha)^{1/\alpha} \) with \( 0 < 1/\alpha < 1 \). Moreover, thanks to Lemma 5.1 (i) we may assume that \( s \) is sufficiently small, say \( 0 < s < 1 \). Clearly, it needs merely to verify that
\[
\| x \|_{B^{s,\alpha}_{q,X}} \lesssim \| x \|_{B^{s\alpha,A}_{q,X}}
\]
and that
\[
\| x \|_{B^{s\alpha,A}_{q,X}} \lesssim \| x \|_{B^{s,\alpha}_{q,X}}.
\]
First we verify (5.18). To this end, let \( x \in B^{s\alpha,A}_{q,X} \). From (2.44) it follows that
\[
|x|_{B^{s\alpha,A}_{q,X}(0,\alpha)} = \left\{ \sum_{j=0}^{\infty} 2^{js} A^\alpha (2^j A^\alpha)^{-1} x \right\}^{1/q}
\]
Since it can be verified that \( C \) where

\[
\text{(5.20)}
\]

in Lemma 5.4 above. More precisely, applying \( a \) which is the desired inequality (5.21). Moreover, from (5.20) and (5.21) it follows that

\[
\text{Thus, we have verified (5.18).}
\]

Next we verify (5.19). It suffices to verify (5.19) in the case \( \alpha = 1/m \) with an odd integer \( m \), i.e.,

\[
\text{(5.22)}
\]

since in a general case \( 0 < \alpha < 1 \) we can fix an odd integer \( m \) large enough such that \( 1/m < \alpha \) and, by using (5.22) and (5.18), obtain that

\[
\text{which is the desired inequality (5.19).}
\]
It remains to verify (5.22). Let \( m \) be an odd integer and let \( x \in B_{q,X}^{s,A/\lambda} \). If \( 0 < q < \infty \), by Lemma 4.1 we have
\[
|x|_{R_{q,X}^{s/m,\lambda}(k,0,1)}^c \simeq |x|_{R_{q,X}^{s/m,\lambda}(k,0,1)} = \left( \int_2^\infty \| t^{s/m} A(t + A)^{-1} x \| q \frac{dt}{t} \right)^{1/q} \leq \left( \int_0^\infty \| t^{s/m} A(t + A)^{-1} x \| q \frac{dt}{t} \right)^{1/q} = \left( \int_0^\infty \| t^{s/m} \prod_{i=1}^m A^{1/m}(z_i t^{1/m} + A^{1/m})^{-1} x \| q \frac{dt}{t} \right)^{1/q},
\]
where \( z_i' \)s are all roots of \((-z)^m = -1\) with \( z_1 = 1 \). Note that the family \( \{A^{1/m}(z_i t + A^{1/m})^{-1}\}_{t > 0} \) is uniformly bounded for each \( i = 2, 3, \ldots, m \). By Lemma 4.1 again we have
\[
|x|_{R_{q,X}^{s/m,\lambda}} \lesssim \left( \int_0^\infty \| t^{s/m} A^{1/m}(t^{1/m} + A^{1/m})^{-1} x \| q \frac{dt}{t} \right)^{1/q} = m^{1/q} |x|_{R_{q,X}^{s,A/\lambda}} \simeq |x|_{R_{q,X}^{s,A/\lambda}},
\]
from which the desired inequality (5.22) follows immediately. And if \( q = \infty \), by Lemma 4.1 we also have
\[
\|x\|_{B_{\infty,X}^{s,A/\lambda}} = \|x\| + \sup_{j \geq 0} \| 2^{js/m} A(2^j + A)^{-1} x \| \lesssim \|x\| + \sup_{\lambda > 0} \| \lambda^{s/m} A(\lambda + A)^{-1} x \| \lesssim \|x\| + \sup_{\lambda > 0} \| \lambda^{s/m} A^{1/m}(\lambda^{1/m} + A^{1/m})^{-1} x \| = \|x\| + \sup_{\mu > 0} \| \mu^s A^{1/m}(\mu + A^{1/m})^{-1} x \| = \|x\|_{B_{\infty,X}^{s,A/\lambda}},
\]
which is the desired inequality (5.22). The proof is complete. \( \square \)

The following continuous embedding of Besov spaces associated with fractional powers operators is a direct consequence of Theorem 5.5.

**Corollary 5.6.** Let \( A \) be non-negative on \( X \) with sectorial angle \( \theta \), and let \( s > 0 \) and \( 0 < q \leq \infty \). Then, for \( 0 < \alpha \leq \beta < \pi/\theta \),
\[
B_{q,X}^{s,\alpha} \hookrightarrow B_{q,X}^{s,\beta}.
\]

5.3. **Interpolation spaces.** Let \((X_0, \| \cdot \|_{X_0})\) and \((X_1, \| \cdot \|_{X_1})\) be a couple of quasi-normed spaces continuously embedded into a topological vector space. Recall that, for each \( x \in X_0 + X_1 \), the \( K \)-functional \( K(t, x) \) is defined by
\[
K(t, x) := \inf_{x_0 + x_1 = x} (\|x_0\|_{X_0} + t \|x_1\|_{X_1}), \quad x_i \in X_i, \ i = 0, 1,
\]
with \( 0 < t < \infty \). It is easy to see that \( K(t, \cdot) \) is a quasi-norm on \( X_0 + X_1 \) for given \( t > 0 \) and that \( K(\cdot, x) \) is a non-negative, increasing and concave function for each \( x \) fixed.
Let $0 < \theta < 1$ and $0 < q \leq \infty$. Recall that the interpolation space $(X_0, X_1)_{\theta,q}$ between $X_0$ and $X_1$ is given by

$$(X_0, X_1)_{\theta,q} := \left\{ x \in X_0 + X_1 : \int_0^\infty \left( t^{-\theta} K(t, x) \right)^{q \frac{dt}{t}} < \infty \right\}$$

and that $(X_0, X_1)_{\theta,q}$ is a quasi-Banach space endowed with the quasi-norm

$$(5.23) \quad \|x\|_{(X_0, X_1)_{\theta,q}} := \left\{ \int_0^\infty \left( t^{-\theta} K(t, x) \right)^{q \frac{dt}{t}} \right\}^{1/q}.$$  

Also, recall the following equivalent quasi-norm on interpolation spaces (see \[41, Theorem 5.1\]):

$$(5.24) \quad \|x\|_{(X_0, X_1)_{\theta,q}} \simeq \|x\|_{(X_0, X_1)_{\theta,q}}^* := \inf_{x_0(t) + x_1(t) \equiv x} (B_0 + B_1),$$

where

$$B_0 = \left\{ \int_0^\infty \left( t^{-\theta} \|x_0(t)\|_{X_0} \right)^{q \frac{dt}{t}} \right\}^{1/q},$$

$$B_1 = \left\{ \int_0^\infty \left( t^{1-\theta} \|x_1(t)\|_{X_1} \right)^{q \frac{dt}{t}} \right\}^{1/q}.$$  

Now we give the main result of this subsection, which states that the abstract Besov spaces can be characterized by the interpolation spaces even in the case $0 < q < 1$.

**Theorem 5.7.** Let $0 < q \leq \infty$, $0 < \theta < 1$ and $\alpha > 0$. Then $(X, D(A^\alpha))_{\theta,q} = B_{q,X}^{\theta\alpha,A}$ in the sense of equivalent quasi-norms.

**Proof.** Thanks to Proposition \[4.5\] we may suppose that $0 \in \rho(A)$ without loss of generality. It suffices to verify that

$$(5.25) \quad \|x\|_{(X, D(A^\alpha))_{\theta,q}} \lesssim \|x\|_{B_{q,X}^{\theta\alpha,A}}$$

and that

$$(5.26) \quad \|x\|_{B_{q,X}^{\theta\alpha,A}} \lesssim \|x\|_{(X, D(A^\alpha))_{\theta,q}}.$$  

First we verify (5.26). We merely give a proof of (5.26) in the case $0 < q < \infty$ and the case when $q = \infty$ can be verified analogously. To this end, write $X_0 := X$ and $X_1 = D(A^\alpha)$ endowed with the graph norm, i.e., $\|x\|_{X_1} = \|A^\alpha x\|$ for $x \in D(A^\alpha)$. Let $x \in B_{q,X}^{\theta\alpha,A}$ and write

$$u(\lambda) := \frac{\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(\alpha)} \lambda^\alpha A^\alpha (\lambda + A)^{-2\alpha} x.$$  

Moreover, let $x_0$ and $x_1$ be two simple functions defined by

$$x_0(t) := \int_2^\infty u(\lambda) \frac{d\lambda}{\lambda}, \quad x_1(t) := \int_0^2 u(\lambda) \frac{d\lambda}{\lambda}, \quad 2^{-j+1}\alpha < t \leq 2^{-j}\alpha, \quad j \in \mathbb{Z}.$$  

On the one hand, by Proposition \[3.9\] it can be seen that $x \in D(A^\epsilon)$ for $0 < \epsilon < \theta\alpha$. Since $0 \in \rho(A)$ due to the hypothesis, from \[22\] it follows that

$$x = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)\Gamma(\alpha)} \int_0^\infty \lambda^\alpha A^\alpha (\lambda + A)^{-2\alpha} x \frac{d\lambda}{\lambda} = x_0(t) + x_1(t), \quad t > 0.$$
On the other hand, by using the change of variable \( t = s^{-\alpha} \) and decomposition \( \int_0^\infty = \sum_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} \), we can rewrite \( \| x \|_{(X_0, X_1)_{q, \alpha}} \) as
\[
\| x \|_{(X_0, X_1)_{q, \alpha}} = \alpha^{1/\alpha} \left\{ \sum_{j = -\infty}^\infty \int_{2^j}^{2^{j+1}} \left( s^{-\alpha} K(s^{-\alpha}, x) \right) \frac{q^\alpha}{s} \right\}^{1/q}.
\]
Since \( K(\cdot, x) \) is increasing, we have
\[
\| x \|_{(X_0, X_1)_{q, \alpha}} \leq \alpha^{1/q} \left( \int_{2^{-1}}^{2^1} \left( s^{-\alpha} K(s^{-\alpha}, x) \right) \frac{q^\alpha}{s} \right)^{1/q}.
\]
By the definition of \( K(t, x) \) we have
\[
\| x \|_{(X_0, X_1)_{q, \alpha}} \leq C \left\{ \sum_{j = -\infty}^\infty \left[ 2^j \left( \left\| x \right\|_{0, (2^{-1})^\alpha} + 2^{-j} \left\| A\alpha x \right\|_{1, (2^{-1})^\alpha} \right) \right]^{1/q} \right\}^{1/q},
\]
where \( C = \frac{\alpha^{1/q} (2^{\alpha - 1})}{q^\alpha} \). By the classical inequality (1.3) and the definitions of \( x_0(t) \) and \( x_1(t) \) we have
\[
\| x \|_{(X_0, X_1)_{q, \alpha}} \leq \left\{ \sum_{j = -\infty}^\infty \left[ 2^j \sum_{i = j}^\infty \left( \left\| x \right\|_{0, (2^i)^\alpha} + 2^{-j} \left\| A\alpha x \right\|_{1, (2^i)^\alpha} \right) \right]^{1/q} \right\}^{1/q}.
\]
And by using the estimate (2.19) with \( c = 1 \) we have further that
\[
\| x \|_{(X_0, X_1)_{q, \alpha}} \lesssim J_1 + J_2,
\]
where
\[
J_1 = \left\{ \sum_{j = -\infty}^\infty \left[ 2^j \sum_{i = j}^\infty \left\| 2^i A\alpha (2^i + A) - 2^{-2\alpha} x \right\| \right]^{1/q} \right\}^{1/q},
\]
\[
J_2 = \left\{ \sum_{j = -\infty}^\infty \left[ 2^j \sum_{i = j}^\infty \left\| 2^i A\alpha (2^i + A) - 2^{-2\alpha} x \right\| \right]^{1/q} \right\}^{1/q}.
\]
It can be verified that both \( J_1 \) and \( J_2 \) are dominated by \( \| x \|_{B_q^0, A} \). Indeed, if \( 0 < q \leq 1 \), by using (1.3) we have
\[
J_1 \leq \left\{ \sum_{j = -\infty}^\infty \sum_{i = j}^\infty \left\| 2^j A\alpha (2^i + A) - 2^{-2\alpha} x \right\|^q \right\}^{1/q}.
\]
\[
= \left\{ \sum_{i = -\infty}^\infty \sum_{j = -\infty}^1 \left\| 2^j A\alpha (2^i + A) - 2^{-2\alpha} x \right\|^q \right\}^{1/q}.
\]
\[
= \left( \frac{1}{1 - 2^{-q^\alpha}} \right)^{1/q} \left\{ \sum_{i = -\infty}^\infty \left\| 2^i A\alpha (2^i + A) - 2^{-2\alpha} x \right\|^q \right\}^{1/q}.
\]
\[
\lesssim \| x \|_{B_q^0, A} \leq \| x \|_{B_q^0, A},
\]
where
Applying the classical inequality \((1.3)\) yields

Indeed, applying the change of variable \((5.27)\) we also have

\[
J_2 \leq \left\{ \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{j-1} \| 2^{j(\theta-1)\alpha} 2^{i\alpha} A(2^j + A)^{-2\alpha} x \|^q \right\}^{1/q}
\]

\[
= \left\{ \sum_{i=-\infty}^{\infty} \sum_{j=i+1}^{\infty} \| 2^{j(\theta-1)\alpha} 2^{i\alpha} A(2^j + A)^{-2\alpha} x \|^q \right\}^{1/q}
\]

\[
= \left[ \frac{1}{2(1-\theta)\alpha q - 1} \right]^{1/q} \left\{ \sum_{i=-\infty}^{\infty} 2^{i(\theta-1)\alpha} \| 2^{i\alpha} A(2^j + A)^{-2\alpha} x \|^q \right\}^{1/q}
\]

\[
\simeq \| x \|_{B^{\theta \alpha,\alpha}_{q,X,k,0,2\alpha}} \leq \| x \|_{B^{\theta \alpha,\alpha}_{q,X}}.
\]

If \(1 < q < \infty\), choosing \(\epsilon \in (0,\theta\alpha)\) and applying the Hölder inequality yields

\[
J_1 = \left\{ \sum_{j=-\infty}^{\infty} 2^{j\theta\alpha} \left( \sum_{i=j}^{\infty} 2^{-i\epsilon} \| 2^{i\alpha} A(2^j + A)^{-2\alpha} x \|^q \right)^{q/q'} \right\}^{1/q}
\]

\[
\leq \left\{ \sum_{j=-\infty}^{\infty} 2^{j\theta\alpha} \left( \sum_{i=j}^{\infty} 2^{-i\epsilon} \| 2^{i\alpha} A(2^j + A)^{-2\alpha} x \|^q \right)^{q'/q} \right\}^{1/q}
\]

\[
= \left( \frac{1}{1 - 2^{-(\theta-1)\alpha} q} \right)^{1/q'} \left[ \frac{1}{1 - 2^{-(\theta-1)\alpha} q} \right]^{1/q} \left\{ \sum_{i=-\infty}^{\infty} 2^{i\theta\alpha} \| 2^{i\alpha} A(2^j + A)^{-2\alpha} x \|^q \right\}^{1/q}
\]

\[
\simeq \| x \|_{B^{\theta \alpha,\alpha}_{q,X,k,0,2\alpha}} \leq \| x \|_{B^{\theta \alpha,\alpha}_{q,X}}.
\]

Thus, we have verified \((5.25)\).

Next we verify \((5.26)\). We merely give a proof of \((5.26)\) in the case \(0 < q < \infty\) and the case when \(q = \infty\) can be verified analogously. To this end, let \(x \in (X, D(A^\alpha))_{\theta,q}\). The proof of \((5.26)\) is divided into the following four steps.

**Step I.** It can be verified that

\[
(5.27) \quad \left\{ \int_0^\infty \| s^{\theta\alpha} A^\alpha (s + A)^{-\alpha} x \| \frac{ds}{s} \right\}^{1/q} \lesssim \| x \|_{(X, D(A^\alpha))_{\theta,q}}.
\]

Indeed, applying the change of variable \(s = t^{-1/\alpha}\) yields

\[
\left\{ \int_0^\infty \| s^{\theta\alpha} A^\alpha (s + A)^{-\alpha} x \| \frac{ds}{s} \right\}^{1/q} \simeq \left( \int_0^\infty \| t^{-\theta} A^\alpha (t^{-1/\alpha} + A)^{-\alpha} x \| \frac{dt}{t} \right)^{1/q}.
\]

Let \(x_0 : (0, \infty) \mapsto X\) and \(x_1 : (0, \infty) \mapsto D(A^\alpha)\) such that \(x_0(t) + x_1(t) \equiv x\). Applying the classical inequality \((1.3)\) yields

\[
\left\{ \int_0^\infty \| s^{\theta\alpha} A^\alpha (s + A)^{-\alpha} x \| \frac{ds}{s} \right\}^{1/q}
\]
Indeed, by Lemma 4.1 (also, see Remark 4.3 (i)), we have

\[ \int_0^\infty \| t^{-\theta} A^\alpha (t^{-1/\alpha} + A)^{-\alpha} \| \frac{dt}{t} \] **1/q**

Thus, we have verified (5.27).

Applying (5.27) yields (5.28), immediately. More precisely,

\[ \| x \|_{(X,D(A^\alpha))_{\theta,\alpha}} \leq \| x \|_{(X,D(A^\alpha))_{\theta,\alpha}}. \]

Step II. It can be verified that

\[ (x)_{p,a}^{\theta_{p,a}(0,0,\alpha)} \leq \| x \|_{(X,D(A^\alpha))_{\theta,\alpha}}. \]

Indeed, by Lemma 4.1 (also, see Remark 4.3 (i)), we have

\[ \| x \|_{(X,D(A^\alpha))_{\theta,\alpha}} \leq \left( \int_0^\infty \| t^{-\theta} A^\alpha (t^{-1/\alpha} + A)^{-\alpha} \| \frac{dt}{t} \right)^{1/q}. \]

Applying (5.28) yields (5.29), immediately.

Step III. It can also be verified that

\[ \| x \| \leq \| x \|_{(X,D(A^\alpha))_{\theta,\alpha}}. \]

To this end, observe that \( x \in B_{q,\lambda}^{\theta_{p,a}(0,0,\alpha)} \) due to (5.28), and therefore \( x \in D(A^\epsilon) \) for \( 0 < \epsilon < \theta \alpha \) by Proposition 3.9 above. Thanks to \( 0 \in \rho(A) \), by using (2.13) we have

\[ (x)_{p,a}^{\theta_{p,a}(0,0,\alpha)} \leq \left( \int_0^\infty \| t^{-\theta} A^\alpha (t^{-1/\alpha} + A)^{-\alpha} \| \frac{dt}{t} \right)^{1/q}. \]

If \( 0 < q \leq 1 \), by using the decomposition \( \int_0^\infty = \sum_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} \) and dyadic estimate (2.19) we obtain from (5.30) that

\[ \| x \| \leq \sum_{j \in \mathbb{Z}} \| 2^{2j} A^\alpha (2^j + A)^{-2\alpha} \| \leq \left( \sum_{j \in \mathbb{Z}} \| 2^{2j} A^\alpha (2^j + A)^{-2\alpha} \| \right)^{1/q}, \]

where the last inequality follows from (1.3). From (1.3), it follows that

\[ \| x \| \leq \left( \sum_{j=0}^\infty \| 2^{2j} A^\alpha (2^j + A)^{-2\alpha} \|^{1/q} \right) + \left( \sum_{j=-\infty}^{-1} \| 2^{2j} A^\alpha (2^j + A)^{-2\alpha} \|^{1/q} \right). \]
Applying (2.16) to \( \|2^{j\alpha}(2^j + A)^{-\alpha}\| \) yields
\[
\left\{ \sum_{j=0}^{\infty} \|2^{j\alpha} A^\alpha(2^j + A)^{-2\alpha}\|^q \right\}^{1/q} \lesssim \left\{ \sum_{j=0}^{\infty} \|A^\alpha(2^j + A)^{-\alpha}\|^q \right\}^{1/q} \leq \left\{ \sum_{j=0}^{\infty} \|2^{j\beta\alpha} A^\alpha(2^j + A)^{-\alpha}\|^q \right\}^{1/q},
\]
while applying (2.16) and (2.17) to \( \|2^{j\alpha(1-\theta)} A^\theta(2^j + A)^{-\alpha}\| \) yields
\[
\left\{ \sum_{j=-\infty}^{-1} \|2^{j\alpha} A^\alpha(2^j + A)^{-2\alpha}\|^q \right\}^{1/q} \lesssim \|A^{-\theta\alpha}\| \left\{ \sum_{j=0}^{\infty} \|2^{j\theta\alpha} A^\alpha(2^j + A)^{-\alpha}\|^q \right\}^{1/q},
\]
due to the fact that \( 0 \in \rho(A) \). This implies that
\[
\|x\| \lesssim \left\{ \sum_{j=0}^{\infty} \|2^{j\beta\alpha} A^\alpha(2^j + A)^{-\alpha}\|^q \right\}^{1/q},
\]
Applying the identity \( \int_2^{2^{j+1}} 2^{-j} d\lambda = 1 \) and the dyadic estimate (2.19) yields
\[
\|x\| \lesssim \left\{ \int_1^{\infty} \|\lambda^{\alpha\theta} A^\alpha(\lambda + A)^{-\alpha}\|^q \frac{d\lambda}{\lambda} \right\}^{1/q} \leq \left\{ \int_0^{\infty} \|\lambda^{\beta\alpha} A^\alpha(\lambda + A)^{-\alpha}\|^q \frac{d\lambda}{\lambda} \right\}^{1/q}.
\]
Finally, applying (5.27) yields the desired (5.29), immediately.

And if \( 1 < q < \infty \), from (5.31) it follows that
\[
\|x\| \lesssim \int_0^{\infty} \|\lambda^{\alpha(1-\theta)}(\lambda + A)^{-\alpha}\| \cdot \|\lambda^{\beta\alpha} A^\alpha(\lambda + A)^{-\alpha} x\| \frac{d\lambda}{\lambda}.
\]
Thanks to the H"older inequality, it can be seen that
\[
\|x\| \lesssim \left\{ \int_0^{\infty} \|\lambda^{\beta\alpha} A^\alpha(\lambda + A)^{-\alpha} x\|^q \frac{d\lambda}{\lambda} \right\}^{1/q}
\]
by observing that
\[
\int_0^{\infty} \|\lambda^{\alpha(1-\theta)}(\lambda + A)^{-\alpha}\|^q' \frac{d\lambda}{\lambda} < \infty
\]
due to the estimates (2.16) and (2.17) and the fact that \( 0 \in \rho(A) \). Again, applying (5.27) yields the desired (5.29), immediately.

Step IV. By using (5.28) and (5.29), we obtain the desired (5.26), immediately. The proof is complete. \( \square \)

Remark 5.8. Theorems 5.2, 5.5 and 5.7 are the main results of this paper, which are novel in the sense equivalent (quasi-)norms even in the case \( 1 \leq q \leq \infty \). More precisely, let \( s > 0 \) and \( 1 \leq q \leq \infty \).

(i) By Remark 4.3 (i) and Theorem 5.2, we obtain [51, Theorem 2.6], immediately. Moreover, Theorem 5.2 also improves [51, Theorem 2.6] in the sense of equivalent norms for \( 1 \leq q \leq \infty \).

(ii) By Remark 4.3 (i) and Theorem 5.7, we obtain [51, Theorem 3.1], immediately. Moreover, Theorem 5.7 also improves [51, Theorem 3.1] in the sense of equivalent norms for \( 1 \leq q \leq \infty \).
(iii) By Remark 1.3 (i) and Theorems 5.7 and 5.8, we obtain Corollary 7.3 (a), immediately. Moreover, Theorems 5.7 and 5.8 also improve Corollary 7.3 (b) in the sense of equivalent norms for $1 \leq q \leq \infty$.

6. COMPARISON OF CLASSICAL AND NEW BESOV SPACES

It is natural to compare our new Besov spaces with the classical Besov spaces and other known Besov spaces associated with concrete operators, for instance, Besov spaces constructed in the frame of extrapolation spaces [67] and Besov spaces associated with $0$-sectorial operators [57] or heat kernels [17].

6.1. Extrapolation method. Let $A$ be a non-negative operator on a Banach space $(X, \|\cdot\|_X)$. Write $X_0 := D(A)$ endowed with the norm $\|\cdot\|_{X_0} := \|\cdot\|_X$, and let $A_0 := A|_{X_0}$, the part of $A$ in $X_0$. It can be verified that $A_0$ is non-negative on $X_0$ with dense domain. Indeed, for each $x \in X_0$ fixed, write $x_n := n(n + A)^{-1}x$ with $n \in \mathbb{N}$. It is clear that $x_n \in D(A)$ and

$$Ax_n = nA(n + A)^{-1}x = n(A + n - n)(n + A)^{-1}x = nx - n^2(n + A)^{-1}x \in X_0$$

due to the fact that $nx \in D(A) = X_0$ and $n^2(n + A)^{-1}x \in D(A) \subset X_0$, and hence, $x_n \in D(A_0)$ for each $n \in \mathbb{N}$. Since $x_n \to x$ in $X$ as $n \to \infty$ due to the fact that $x \in X_0 = D(A)$, it follows that $D(A_0) = X_0$. Moreover, the non-negativity of $A_0$ follows from that of $A$, immediately. More precisely,

$$\sup_{\lambda > 0} \|\lambda(A + A_0)^{-1}\| \leq M_A,$$

where $M_A$ is the non-negativity constant of $A$ given in (2.1).

Now let $X_{-1}$ be the completion of $X_0$ with respect to the norm $\|\cdot\|_{-1} := \|((1 + A_0)^{-1})\|_{X_0}$. It is clear that, for each $\lambda > 0$, $(\lambda + A_0)^{-1}$ admits a bounded extension $J_{\lambda, -1}$ on $X_{-1}$ and

$$\sup_{\lambda > 0} \|\lambda J_{\lambda, -1}\| \leq M_A.$$}

Furthermore, by the definition of the bounded extension, it is easy to verify that the operator $J_{\lambda, -1}$ is injective for each $\lambda > 0$, and hence, $A_{-1} := J_{\lambda, -1}^{-1} - \lambda$ is non-negative $X_{-1}$ with dense domain. By recursion, one can define an extrapolation space $X_{-l}$ and a non-negative operator $A_{-l}$ with dense domain in $X_{-l}$ for each $l \geq 2$. Thus, we have obtained a series of Banach spaces $\{X_{-l}\}_{l \in \mathbb{N}_0}$, where $X_{-l}$ is densely embedded in $X_{-l-1}$ for each $l \in \mathbb{N}_0$.

Note that $(1 + A_0)^{-1}$ is bounded from $X_0$ to $R((1 + A_0)^{-1}) = (1 + A_0)^{-1}X_0 \subset X_0$ and that $X_0$ is the completion of $R((1 + A_0)^{-1})$ with respect to $\|\cdot\|_{-1}$. Thus, the extension $J_{-1, -1}$ of $(1 + A_0)^{-1}$ is bounded from $X_{-1}$ to $X_0$, and hence,

$$D(A_{-1}) = D(1 + A_{-1}) = D(J_{-1, -1}^{-1}) = R(J_{-1, -1}) = X_0.$$}

In general, by the recursive construction, it can be seen that $D(A_{-l}) = X_{-l+1}$ for each $l \in \mathbb{N}$ and that $A_{-l} = A_{-l-1}D(A_{-l})$ for each $l \in \mathbb{N}_0$.

Observe that the operators $A_{-n}$ on $X_{-n}$ are consistent with each other, and therefore there is a well defined operator $\mathcal{A}$ on

$$X_{-\infty} := \bigcup_{n \in \mathbb{N}_0} X_{-n}.$$
Clearly, $A|X_\infty = A_{-n}$ for each $n \in \mathbb{N}_0$. Moreover, for given $t > 0$, it can be seen that $A^m(t + A)^{-m-l}\tilde{x} \in X_0$ for $\tilde{x} \in X_{-l}$ by observing that $(t + A)^{-l}$ maps $X_{-l}$ into $X_0$. More precisely,

(6.1) \quad A^m(t + A)^{-m-l}\tilde{x} = A^m(t + A)^{-m}(t + A)^{-l}\tilde{x}, \quad \tilde{x} \in X_{-l}.

We can now give an extrapolation version of abstract Besov spaces in the following way.

**Definition 6.1.** Let $0 < q \leq \infty$ and $s \in \mathbb{R}$. Let $k \in \mathbb{Z}$ and let $l$ and $m$ be two non-negative integers such that $-l < s < m$. The inhomogeneous Besov space $B^{s,A}_{q,X}$ is defined by

$$B^{s,A}_{q,X} := \{\tilde{x} \in X_{-l} : |\tilde{x}|_{\mathcal{R}^{s,A}_{q,X}(k,l,m)} < \infty\}$$

endowed with the quasi-norm

$$\|\tilde{x}\|_{B^{s,A}_{q,X}} := \|\|2^k + A\|^{-l}\tilde{x}\|_{X} + |\tilde{x}|_{\mathcal{R}^{s,A}_{q,X}(k,l,m)}\|^{1/q}$$

where

$$|\tilde{x}|_{\mathcal{R}^{s,A}_{q,X}(k,l,m)} := \left\{ \sum_{j=k}^{\infty} \left\|2^j 2^l A^m(2^j + A)^{-m-l}\tilde{x}\right\|_{X}^{q}\right\}^{1/q}$$

(with the usual modification if $q = \infty$).

A homogeneous version of abstract Besov spaces via the extrapolation is given as follows.

**Definition 6.2.** Let $0 < q \leq \infty$ and $s \in \mathbb{R}$, and let $l$ and $m$ be two non-negative integers such that $-l < s < m$. The homogeneous Besov space $\dot{B}^{s,A}_{q,X}$ is defined by

$$\dot{B}^{s,A}_{q,X} := \left\{\tilde{x} \in X_{-l} : \sum_{j=-\infty}^{\infty} \left\|2^j 2^l A^m(2^j + A)^{-m-l}\tilde{x}\right\|_{X}^{q}\right\}^{1/q},$$

endowed with the quasi-norm

$$\|\tilde{x}\|_{\dot{B}^{s,A}_{q,X}} := \left\{ \sum_{j=-\infty}^{\infty} \left\|2^j 2^l A^m(2^j + A)^{-m-l}\tilde{x}\right\|_{X}^{q}\right\}^{1/q}$$

(with the usual modification if $q = \infty$).

**Remark 6.3.** Analogous to Theorem 4.2 (more precisely, Remark 4.3 above), it can be verified that $\dot{B}^{s,A}_{q,X}$ admits an equivalent quasi-norm of continuous type, and therefore $\dot{B}^{s,A}_{q,X}$ coincides with $\dot{B}^{\phi}_{X,s}$ with $\phi(\lambda) = \lambda^s$ ($\lambda > 0$) for $1 \leq q \leq \infty$ and $s \in \mathbb{R}$, where $\dot{B}^{\phi}_{X,s}$ is the homogeneous Besov space due to T. Matsumoto and T. Ogawa [67, Definitions 2.8 and 2.10].

**Proposition 6.4.** Let $0 < q \leq \infty$ and $s \in \mathbb{R}$. Then $B^{s,A}_{q,X} = \dot{B}^{s,A}_{q,X}$ in the sense of equivalent quasi-norms.

**Proof.** We first verify that

(6.2) \quad B^{s,A}_{q,X} \subset \dot{B}^{s,A}_{q,X},

and

(6.3) \quad \|\tilde{x}\|_{B^{s,A}_{q,X}} \simeq \|\tilde{x}\|_{\dot{B}^{s,A}_{q,X}}, \quad x \in B^{s,A}_{q,X}.$
To this end, let \( \tilde{x} \in B_{q,A}^{r,s}(k,l,m) \). Fix \( k \in \mathbb{Z} \) and let \( m \) and \( l \) be non-negative integers such that \(-l < s < m\). By Definitions 3.6 and 3.1, \( \tilde{x} = \{x_n\} \) for some \( \{x_n\} \subset R_{q,A}^{s,A}(k,l,m) \) which is Cauchy with respect to the Besov quasi-norm \( \| \cdot \|_{B_{q,A}^{r,s}(k,l,m)} \) given by (3.19). This implies that, for each \( \epsilon > 0 \) fixed, there is an \( N \in \mathbb{N} \) such that

\[
\|x_n - x_{n'}\|_{B_{q,A}^{r,s}(k,l,m)} = \|(2^k + A)^{-l}(x_n - x_{n'})\|_X < \epsilon, \quad n, n' \geq N.
\]

(6.4)

From (6.4) it is clear that \( \{x_n\} \) is Cauchy with respect to the norm \( \| (2^k + A)^{-l} \|_X \), and hence, \( x \in X_{-l} \) by observing that \( X_{-l} \) coincides with the completion of \( X_0 = D(A) \) with respect to the norm \( \| (2^k + A)^{-l} \cdot x_0 = (2^k + A)^{-l} \cdot X \).

We now verify that

(6.5) \[ |\tilde{x}|_{R_{q,A}^{r,s}(k,l,m)} < \infty. \]

Fix \( \epsilon > 0 \). From (6.1) and (6.4) it follows that

\[
\left\{ \sum_{j=k}^{K} \|2^{j(s+l)}A^m(2^j + A)^{-m-l}(x_n - x_{n'})\|_X^q \right\}^{1/q} \leq \epsilon, \quad n, n' \geq N,
\]

for each integer \( K \geq k \). Applying \( n' \to \infty \) yields

\[
\left\{ \sum_{j=k}^{K} \|2^{j(s+l)}A^m(2^j + A)^{-m-l}(x_n - \tilde{x})\|_X^q \right\}^{1/q} \leq \epsilon, \quad n \geq N,
\]

for each integer \( K \geq k \) since \( (2^j + A)^{-l}(x_n - x_{n'}) \) converges to \( (2^j + A)^{-l}(x_n - \tilde{x}) \) as \( n' \to \infty \) due to (6.4). Applying \( K \to \infty \) yields

(6.6) \[ |x_n - \tilde{x}|_{R_{q,A}^{r,s}(k,l,m)} = \left\{ \sum_{j=k}^{\infty} \|2^{j(s+l)}A^m(2^j + A)^{-m-l}(x_n - \tilde{x})\|_X^q \right\}^{1/q} \leq \epsilon \]

for \( n \geq N \). From (1.3) and (6.6) it follows that

\[
|\tilde{x}|_{R_{q,A}^{r,s}(k,l,m)} < |\tilde{x} - x_n|_{R_{q,A}^{r,s}(k,l,m)} + |x_n|_{R_{q,A}^{r,s}(k,l,m)} < \infty,
\]

which is the desired (6.3). Thus, we have verified (6.2).

Moreover, observing that \( x_n \to \tilde{x} \) as \( n \to \infty \) in \( X_{-l} \) as mentioned above and that \( x_n \to \tilde{x} \) as \( n \to \infty \) in \( R_{q,A}^{s,A}(k,l,m) \) due to (6.6), we conclude that

\[
\| \tilde{x} \|_{B_{q,A}^{r,s}} = \lim_{n \to \infty} \| x_n \|_{B_{q,A}^{r,s}} = \lim_{n \to \infty} \| x_n \|_{B_{q,A}^{r,s}} \approx \| \tilde{x} \|_{B_{q,A}^{r,s}},
\]

which is the desired (6.3).
Therefore, from (1.3) it follows that

\[
\mathcal{B}^{s,A}_{q,X} \subset B^{s,A}_{q,X}.
\]

Clearly, it suffices to verify that \( R^{s,A}_{q,X}(k,l,m) \) is dense in \( B^{s,A}_{q,X} \). Indeed, let \( \tilde{x} \in B^{s,A}_{q,X} \), i.e., \( \tilde{x} \in X_{-l} \) and

\[
|\tilde{x}|_{R^{s,A}_{q,X}(k,l,m)} = \left\{ \sum_{j=k}^{\infty} \left\| 2^{j(s+l)} A^m (2^j + A)^{-m-l} \tilde{x} \right\|_X^q \right\}^{1/q} < \infty.
\]

Write \( P_n := n(n+A)^{-1} \) with \( n \in \mathbb{N} \). It can be seen that \( P^l_n \tilde{x} \in D(A) \) for each \( n \in \mathbb{N} \) by observing that \((t+A)^{-1}\) maps \( X_{-l} \) into \( X_0 = D(A) \) for each \( t > 0 \), and therefore \( P^l_n \tilde{x} \in R^{s,A}_{q,X}(k,l,m) \) for each \( n \in \mathbb{N} \) due to (6.8) and the uniform boundedness of \( \{n(n+A)^{-1}\}_{n \in \mathbb{N}} \). Thanks to Lemma 2.4 (i), it can also be seen that

\[
\|(2^k + A)^{-1}(P^l_n \tilde{x} - \tilde{x})\|_X = \|(P^l_n - I)(2^k + A)^{-1} \tilde{x}\|_X \to 0
\]
as \( n \to \infty \) by observing the fact that \((2^k + A)^{-1} \tilde{x} \in D(A) \). Also, by Lemma 2.4 (i) we conclude that

\[
\left\{ \sum_{j=k}^{\infty} \left\| 2^{j(s+l)} A^m (2^j + A)^{-m-l} (P^l_n \tilde{x} - \tilde{x}) \right\|_X^q \right\}^{1/q} \to 0, \quad n \to \infty.
\]

More precisely, fix \( \epsilon > 0 \). On the one hand, thanks to (6.8), there is an integer \( K \) large enough such that

\[
\left\{ \sum_{j=k}^{K-1} \left\| 2^{j(s+l)} A^m (2^j + A)^{-m-l} \tilde{x} \right\|_X^q \right\}^{1/q} < \frac{\epsilon}{2CC_1/q},
\]

where \( C = \sup_{n \in \mathbb{N}} \|P^l_n - I\| \) and \( C_{1/q} \) is the constant given in (1.3). On the other hand, thanks to Lemma 2.4 (i), there is an integer \( N \) such that

\[
\left\{ \sum_{j=0}^{K-1} \left\| (P^l_n - I)2^{j(s+l)} A^m (2^j + A)^{-m-l} \tilde{x} \right\|_X^q \right\}^{1/q} < \frac{\epsilon}{2C_{1/q}}, \quad n \geq N,
\]
due to the fact that \( 2^{j(s+l)} A^m (2^j + A)^{-m-l} \tilde{x} \in D(A) \) for each \( j = 0, 1, \ldots, K - 1 \). Therefore, from (1.3) it follows that

\[
\left\{ \sum_{j=k}^{\infty} \left\| (P^l_n - I)2^{j(s+l)} A^m (2^j + A)^{-m-l} \tilde{x} \right\|_X^q \right\}^{1/q}
\]

\[
= \left\{ \sum_{j=0}^{\infty} \left\| (P^l_n - I)2^{j(s+l)} A^m (2^j + A)^{-m-l} \tilde{x} \right\|_X^q \right\}^{1/q}
\]

\[
\leq C_{1/q} \left\{ \sum_{j=0}^{K-1} \left\| (P^l_n - I)2^{j(s+l)} A^m (2^j + A)^{-m-l} \tilde{x} \right\|_X^q \right\}^{1/q}
\]

\[
+ C_{1/q} \left\{ \sum_{j=K}^{\infty} \left\| (P^l_n - I)2^{j(s+l)} A^m (2^j + A)^{-m-l} \tilde{x} \right\|_X^q \right\}^{1/q}.
\]
Applying the uniform boundedness of \( \{ P_n^l - I \}_{n \in \mathbb{N}} \) yields
\[
\left\{ \sum_{j=k}^{\infty} \| 2^j s^l \| A^m (2^j + A)^{-m-l} (P_n^l - I) \| X \right\}^{1/q} \\
\leq C_{1/q} \left\{ \sum_{j=0}^{K-1} \| (P_n^l - I) 2^j s^l \| A^m (2^j + A)^{-m-l} \| X \right\}^{1/q} \\
+ C_{1/q} \sup_{n \in \mathbb{N}} \| P_n^l - I \| \left\{ \sum_{j=K}^{\infty} \| 2^j s^l \| A^m (2^j + A)^{-m-l} \| X \right\}^{1/q} \\
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad n \geq N,
\]
from which (6.10) follows immediately. From (6.9) and (6.10) it can be seen that
\( P_n^l \xrightarrow{n \to \infty} \tilde{x} \), and hence, \( R^{s^l A} \) is dense in \( B^{s^l A} \). Thus, we have verified (6.7). The proof is complete. \( \square \)

6.2. Functional calculus approach. Let \( X \) be a Banach space and let \( 0 \leq \omega < \pi \).
Recall that a closed linear operator \( A : D(A) \subset X \to X \) is sectorial of angle \( \omega \) if \( \sigma(A) \subset \Sigma_\omega \) and
\[
\sup \{ \| z R(z, A) \| : z \in \mathbb{C} \setminus \Sigma_{\omega'} \} < \infty
\]
for each \( \omega' \in (\omega, \pi) \). If this is the case, the value
\[
\omega_A := \min \{ \omega \in [0, \pi] : A \in S(\omega) \}
\]
is called the spectral angle of \( A \).

Let \( A \) be a closed linear operator on a Banach space \( X \). Recall that \( A \) is non-negative if and only if it is sectorial (with spectral angle \( \omega_A \leq \pi - \arcsin 1/M_A \)), where \( M_A \) is the non-negativity constant of operator \( A \) (see, [65, Proposition 1.2.1]). Thus, the theory of Besov spaces associated with non-negative operators on Banach spaces can also be developed by using the approach of functional calculus.

Let’s recall some preliminaries of the so-called primary functional calculus of sectorial operators. Let \( \omega \in (0, \pi) \) and \( A \) be sectorial with spectral angle \( \omega_A < \omega \). Let \( H^\infty(\Sigma_\omega) \) be the space of bounded holomorphic function on \( \Sigma_\omega \) and write
\[
H^\infty_0(\Sigma_\omega) := \{ f \in H^\infty(\Sigma_\omega) : \exists C, \epsilon > 0 \text{ s.t. } |f(z)| \leq C \min \{|z|^\epsilon, |z|^{-\epsilon}\} \}.
\]
For each \( \psi \in H^\infty_0(\Sigma_\omega) \), define
\[
\psi(A) := \frac{1}{2\pi i} \int_{\partial \Sigma_{\omega'}} \psi(z)(z - A)^{-1} dz,
\]
where \( \partial \Sigma_{\omega'} \) is the boundary of a sector \( \Sigma_{\omega'} \) with \( \omega' \in (\omega, \pi) \), oriented counterclockwise. By the Cauchy integral theorem, the integral in the right-hand side of the last equality is independent of the choice of \( \omega' \), and hence, \( \psi(A) \) is well defined as a bounded linear operator on the Banach space \( X \). More precisely, the calculus
\[
\Phi : H^\infty_0(\Sigma_\omega) \to B(X)
\]
is a homomorphism of algebras. Furthermore, one can enlarge the algebra \( H^\infty_0(\Sigma_\omega) \) as
\[
\mathcal{E}(\Sigma_\omega) := H^\infty_0(\Sigma_\omega) \oplus \mathbb{C} \frac{1}{1 + z} \oplus \mathbb{C}1,
\]
and define
\[ f(A) := \psi(A) + c(1 + A)^{-1} + d \]
for each function \( f(z) := \psi(z) + \frac{c}{1+z} + d \in \mathcal{E}(\Sigma_{\omega}). \) Such an extended calculus
\[ \Phi : \mathcal{E}(\Sigma_{\omega}) \to B(X) \]
is also a homomorphism of algebras [38 Page 34, (2.7)], and we refer to [38, Chapters 1 and 2] for more information on the primary functional calculus for sectorial operators.

**Proposition 6.5.** [37, Theorem 1] Let \( A \) be a sectorial operator of angle \( \omega \), and let \( \omega' \in (\omega, \pi) \) and \( \Re \alpha > 0 \). Take a function \( 0 \neq \psi \in \mathcal{E}(\Sigma_{\omega'}) \) such that \( z^{-\alpha} \psi(z) \in \mathcal{E}(\Sigma_{\omega'}) \). Then
\[ \left( X, D(A) \right)_{\theta,q} = \left\{ x \in X : \int_0^\infty \left\| t^{\theta\alpha} \psi(tA)x \right\| q \frac{dt}{t} < \infty \right\} \]
with the equivalence of norms
\[ \| x \|_{\left( X, D(A) \right)_{\theta,q}} \sim \| x \| + \left\{ \int_0^\infty \left\| t^{\theta\alpha} \psi(tA)x \right\| q \frac{dt}{t} \right\}^{1/q} \]
for all \( \theta \in (0, 1) \) and \( 1 \leq q \leq \infty \).

In particular, for given \( \theta \in (0, 1) \) and \( 1 \leq q \leq \infty \), applying \( \psi(z) = z^\alpha (1 + z)^{-\beta} \) with \( 0 < \Re \alpha \leq \Re \beta < \infty \) to Proposition 6.5 above yields
\[ \left( X, D(A) \right)_{\theta,q} = \left\{ x \in X : \int_0^\infty \left\| t^{\theta\alpha} \cdot t^{\beta-\alpha} A^\alpha (t + A)^{-\beta} x \right\| q \frac{dt}{t} < \infty \right\} \]
with the equivalence of norms
\[ \| x \|_{\left( X, D(A) \right)_{\theta,q}} \sim \| x \| + \left\{ \int_0^\infty \left\| t^{\theta\alpha} \cdot t^{\beta-\alpha} A^\alpha (t + A)^{-\beta} x \right\| q \frac{dt}{t} \right\}^{1/q} \]
(see, [37 subsection 7.3]). Obviously, the norm given in the right-hand side of (6.11) is equivalent to our Besov norm \( \| \cdot \|_{B_{\theta,q}^{\alpha,\omega}(k,\beta-\alpha,\alpha)} \) as that given in (3.19) due to Theorem 4.2 above.

An alternative approach to Besov spaces associated with operators is the so-called Mihlin functional calculus [57, Definition 3.3], which is constructed in the frame of sectorial operators with particular angle \( \omega = 0 \) (i.e., 0-sectorial operator for short) and bounded holomorphic functions \( f \) satisfying the following Mihlin condition,
\[ \sup_{t>0} \left| t^k f^{(k)}(t) \right| < \infty, \quad k = 0, 1, \ldots, N. \]
More precisely, let \( \alpha > 0 \) and write
\[ \mathcal{M}^\alpha := \{ f : \mathbb{R}_+ \to \mathbb{C} : f(e^s) \in B_{\infty,1}^\alpha \}, \]
endowed with the norm \( \| f \|_{\mathcal{M}^\alpha} := \| f_x \|_{B_{\infty,1}^\alpha} \). Let \( A \) be a sectorial operator of angle \( \omega = 0 \). Recall that \( A \) has a bounded \( \mathcal{M}^\alpha \) calculus if there is a constant \( C > 0 \) such that
\[ \| f(A) \| \leq C \| f \|_{\mathcal{M}^\alpha}, \quad f \in \bigcap_{0<\omega<\pi} H^\alpha(\Sigma_{\omega}) \cap \mathcal{M}^\alpha, \]
Proposition 6.6. [57] Let \( A \) be a sectorial operator of angle \( \omega = 0 \) with an \( M_1^\alpha \) calculus, and let \( 1 \leq q \leq \infty \). Furthermore, let \( f : (0, \infty) \to \mathbb{C} \) be a function satisfying \( \sum_{k\in\mathbb{Z}} \| f(2^{-k}) \|_{M_1^\alpha} 2^{-ks} < \infty \) and \( f^{-1} \hat{\varphi}_0 \in M_1^\alpha \). Then, with standard modification for \( q = \infty \), these spaces are independent of the choice of partition of unity and index \( N \), so that they are well defined as Banach spaces. We refer to the reader to [57, Section 5] for more information.

**Proposition 6.6.** [57] Theorems 5.2 and 5.3 and Remark 5.4] Let \( A \) be a sectorial operator of angle \( \omega = 0 \) with an \( M_1^\alpha \) calculus, and let \( 1 \leq q \leq \infty \). Furthermore, let \( f : (0, \infty) \to \mathbb{C} \) be a function satisfying \( \sum_{k\in\mathbb{Z}} \| f(2^{-k}) \|_{M_1^\alpha} 2^{-ks} < \infty \) and \( f^{-1} \hat{\varphi}_0 \in M_1^\alpha \). Then, with standard modification for \( q = \infty \), the following two statements hold:

1. For the homogeneous Besov space \( B_q^s(A) \) with \( s \in \mathbb{R} \), we have the norm equivalence

\[
\| x \|_{B_q^s(A)} \simeq \left\{ \int_0^\infty t^{-sq} \| f(tA)x \|^q \frac{dt}{t} \right\}^{1/q}.
\]

2. For the inhomogeneous Besov space \( B_q^s(A) \) with \( s > 0 \), we have the norm equivalence

\[
\| x \|_{B_q^s(A)} \simeq \| x \| + \left\{ \int_0^\infty t^{-sq} \| f(tA)x \|^q \frac{dt}{t} \right\}^{1/q}.
\]
Let $s \in \mathbb{R}$ and fix $0 \leq \alpha \leq \beta < \infty$ such that $0 < \alpha - s < \beta < \infty$. Applying $f(t) = t^\alpha (1 + t)^{-\beta}$ in Proposition 6.6 yields

$$\|x\|_{\dot{B}_q^s(A)} \simeq \left\{ \int_0^\infty t^{-sq} \left\| t^\alpha A^\alpha (1 + tA)^{-\beta} x \right\|^q \frac{dt}{t} \right\}^{1/q}$$

which coincides with our Besov norm $\|x\|_{\dot{B}_{q,X}^s(\beta-\alpha,\alpha)}$ as that given in (3.32). On the other hand, if $s > 0$, again taking $0 \leq \alpha \leq \beta < \infty$ such that $0 < \alpha - s < \beta < \infty$ and applying $f(t) = t^\alpha (1 + t)^{-\beta}$ in Proposition 6.6 yields

$$\|x\|_{B_q^s(\dot{A})} \simeq \left\{ \int_0^\infty t^{-sq} \left\| t^\alpha A^\alpha (1 + tA)^{-\beta} x \right\|^q \frac{dt}{t} \right\}^{1/q}$$

which coincides with our Besov norm $\|x\|_{B_{q,X}^s(0,\beta-\alpha,\alpha)}$ given by (3.19).

### 6.3. Semigroups revisited

Let $0 < \theta \leq \pi/2$, and let $A$ be the negative generator of a bounded analytic semigroup $T$ of angle $\theta$ on $X$. It is well known that $A$ is non-negative (more precisely, sectorial of angle $\pi/2 - \theta$) on $X$ with dense domain. By a routine calculation of complex integrals, one can verify that

$$\sup_{t > 0} \| (tA)^\alpha T(t) \| < \infty$$

for $\alpha \in \mathbb{C}_+$ (see [50] Theorem 12.1). Let $\alpha \in \mathbb{C}_+$ and $s > 0$, and write $C_T := \sup_{t \geq 0} \| T(t) \|$. From (6.12) it can be seen that

$$\| (tA)^\alpha T(t) \| \leq 2^\alpha C_T \| (sA)^\alpha T(s) \|, \quad s \leq t \leq 2s,$$

by observing the well known semigroup property, i.e., $T(t + s) = T(t)T(s)$ for $t, s \geq 0$. Moreover, by a standard calculation of real integrals, one can also verify a semigroup version of Lemma 2.7 (see [51] Theorem 5.4). In particular, one has

$$A^\alpha x = \frac{1}{\Gamma(\beta - \alpha)} \int_0^\infty t^{-\alpha}(tA)^\beta T(t)x \frac{dt}{t}$$

for $x \in D(A^{\alpha+\epsilon})$ with $\epsilon > 0$.

Let $s > 0$ and $0 < q \leq \infty$. For $k \in \mathbb{Z}$ and $\beta \in \mathbb{C}_+$ with $s < \text{Re} \beta$, we define $\dot{B}_{q,X}^{s,T}(k,\beta)$ to be the completion of $X$ with respect to the quasi-norm

$$\|x\|_{\dot{B}_{q,X}^{s,T}(k,\beta)} := \|x\| + \|x\|_{\dot{B}_{q,X}^{s,T}(k,\beta)},$$

where

$$\|x\|_{\dot{B}_{q,X}^{s,T}(k,\beta)} := \left\{ \sum_{j=k}^{\infty} \left\| 2^{j(s-\beta)} A^\beta T(2^{-j})x \right\|^q \right\}^{1/q}.$$

The following two propositions state that the resolvent of the underlying operator $A$ can be replaced by the semigroup generated by $A$ in the description of abstract Besov spaces.
Proposition 6.7. Let \( s > 0 \) and \( 0 < q \leq \infty \) and let \( A \) be the negative generator of a bounded analytic semigroup \( T \) on \( X \). Fix \( k \in \mathbb{Z} \) and \( \beta \in \mathbb{C}_+ \) such that \( s < \text{Re} \beta \). Then, for \( x \in X \),
\[
\|x\|_{\tilde{B}^s_{q,X}} := \|x\|_{\tilde{B}^s_{q,X}(k,\beta)} \lesssim \|x\|_{\tilde{B}^s_{q,X}},
\]
where \( \|\cdot\|_{\tilde{B}^s_{q,X}(k,\beta)} \) is the quasi-norm on \( X \) given by (6.17). Therefore, the quasi-Banach space \( \tilde{B}^s_{q,X}(k,\beta) \) is independent of the choice of \( k \) and \( \beta \) and
\[
\tilde{B}^s_{q,X} := \tilde{B}^s_{q,X}(k,\beta) = B^{s,a}
\]
in the sense of equivalent quasi-norms.

Proof. First we verify that
\[
(6.17)
\]
Let \( x \in X \) such that \( \|x\|_{\tilde{B}^s_{q,X}} < \infty \) and fix an integer \( m > \text{Re} \beta \). Applying (2.39) to \( A^\beta \) yields
\[
|x|_{\tilde{B}^s_{q,X}(k,\beta)} = C \left\{ \sum_{j=k}^{\infty} \left( \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} 2^{j(s-\beta)} \lambda^\beta A^m (\lambda + A)^{-m} T(2^{-j}) x \frac{d\lambda}{\lambda} \right)^q \right\}^{1/q},
\]
where \( C = \frac{|\Gamma(m)|}{\Gamma(2(\beta - m))} \). Rewriting the integral \( \int_0^\infty = \sum_{i=-\infty}^\infty \int_{2^i}^{2^{i+1}} \) yields
\[
|x|_{\tilde{B}^s_{q,X}(k,\beta)} \lesssim \left\{ \sum_{j=k}^{\infty} \left[ \sum_{i=-\infty}^{\infty} \int_{2^i}^{2^{i+1}} \|2^{j(s-\beta)} \lambda^\beta A^m (\lambda + A)^{-m} T(2^{-j}) x\| \frac{d\lambda}{\lambda} \right]^q \right\}^{1/q}.
\]
Applying the estimate (2.19) with \( c = 1 \) yields
\[
|x|_{\tilde{B}^s_{q,X}(k,\beta)} \lesssim J,
\]
where
\[
J = \left\{ \sum_{j=k}^{\infty} \left[ \sum_{i=-\infty}^{\infty} \|2^{j(s-\beta)} 2^i \lambda^\beta A^m (2^i + A)^{-m} T(2^{-j}) x\|^q \right] \right\}^{1/q}.
\]
It can be verified that
\[
(6.17)
\]
We merely verify (6.17) in the case \( 0 < q \leq 1 \) and the case when \( 1 < q \leq \infty \) can be verified analogously. To this end, let \( 0 < q \leq 1 \). It follows from (1.4) that
\[
J \lesssim \|x\|_{\tilde{B}^s_{q,X}},
\]
where
\[
J_1 + J_2 + J_3,
\]
\[
J_1 = \left\{ \sum_{j=k}^{\infty} \sum_{i=-\infty}^{k-1} \|2^{j(s-\beta)} 2^i \lambda^\beta A^m (2^i + A)^{-m} T(2^{-j}) x\|^q \right\}^{1/q},
\]
\[
J_2 = \left\{ \sum_{j=k}^{\infty} \sum_{i=-\infty}^{k} \|2^{j(s-\beta)} 2^i \lambda^\beta A^m (2^i + A)^{-m} T(2^{-j}) x\|^q \right\}^{1/q},
\]
\[
J_3 = \left\{ \sum_{j=k}^{\infty} \sum_{i=-\infty}^{k-1} \|2^{j(s-\beta)} 2^i \lambda^\beta A^m (2^i + A)^{-m} T(2^{-j}) x\|^q \right\}^{1/q}.
\]
\[ J_2 = \left\{ \sum_{j=k}^{\infty} \sum_{i=k}^{j-1} \left\| 2^j (s-\beta) 2^i \beta A^m (2^i + A)^{-m} T(2^{-j}) x \right\|^q \right\}^{1/q}, \]
\[ J_3 = \left\{ \sum_{j=k}^{\infty} \sum_{i=j}^{\infty} \left\| 2^j (s-\beta) 2^i \beta A^m (2^i + A)^{-m} T(2^{-j}) x \right\|^q \right\}^{1/q}. \]

As for the part \( J_1 \), applying (2.17) to \( A^m (2^i + A)^{-m} \) yields
\[ J_1 \lesssim \left\{ \sum_{j=k}^{\infty} \sum_{i=-\infty}^{k-1} \left\| 2^j (s-\beta) 2^i \beta T(2^{-j}) x \right\|^q \right\}^{1/q} \lesssim \| x \|, \]
where the last inequality follows from the uniform boundedness of \( \{ T(t) \}_{t \geq 0} \) and the fact that \( 0 < s < \text{Re} \beta \). As for the part \( J_2 \), exchanging the order of summation yields
\[ J_2 = \left\{ \sum_{i=k}^{\infty} \sum_{j=i+1}^{\infty} \left\| 2^j (s-\beta) 2^i \beta A^m (2^i + A)^{-m} T(2^{-j}) x \right\|^q \right\}^{1/q} \]
\[ \lesssim \left\{ \sum_{i=k}^{\infty} \left[ \sum_{j=i+1}^{\infty} 2^j (s-\beta) \left\| 2^i \beta A^m (2^i + A)^{-m} x \right\|^q \right] \right\}^{1/q} \]
\[ \lesssim \left\{ \sum_{i=k}^{\infty} \left\| 2^i \beta A^m (2^i + A)^{-m} x \right\|^q \right\}^{1/q} = \| x \|_{R_{q,x}^s (k,0,m)}, \]

And as for the part \( J_3 \), exchanging the order of summation yields
\[ J_3 = \left\{ \sum_{i=k}^{\infty} \sum_{j=k}^{i} \left\| 2^j (s-\beta) 2^i \beta A^m (2^i + A)^{-m} T(2^{-j}) x \right\|^q \right\}^{1/q} \]
\[ = \left\{ \sum_{i=k}^{\infty} \sum_{j=k}^{i} \left\| 2^j \beta A^m (2^i + A)^{-m} x \right\|^q \right\}^{1/q}. \]

Thanks to (6.12), applying the uniform boundedness of \( (2^{-j} A)^{\beta} T(2^{-j}) \) yields
\[ J_3 \lesssim \left\{ \sum_{i=k}^{\infty} \left( \sum_{j=k}^{i} 2^{j q} \right) \left\| 2^i \beta A^m (2^i + A)^{-m} x \right\|^q \right\}^{1/q} \]
\[ \lesssim \left\{ \sum_{i=k}^{\infty} \left\| 2^i (s+\beta) A^{m-\beta} (2^i + A)^{-m} x \right\|^q \right\}^{1/q} = \| x \|_{R_{q,x}^s (k,\beta,m-\beta)}, \]

This implies that
\[ J \leq J_1 + J_2 + J_3 \lesssim \| x \|_{R_{q,x}^s}, \]
which is the desired (6.17). Thus, we have verified that (6.16).

Next we verify that
\[ (6.18) \quad \| x \|_{R_{q,x}^s (k,0,m)} \lesssim \| x \|_{R_{q,x}^s (k,m)} + \| x \|_{R_{q,x}^s (k,2m)}. \]
for each integer $m > \text{Re} \beta$. To this end, fix an integer $m > \text{Re} \beta$ and let $x \in X$ such that $|x|_{R^+_{q,X}(k,m)}$ and $|x|_{\tilde{R}^+_{q,X}(k,2m)}$ are both finite. From (6.14) it follows that

$$
|x|_{R^+_{q,X}(k,0,m)} \leq \frac{1}{\Gamma(m)} \left\{ \sum_{j=k}^\infty \left[ \int_0^\infty \|2^{js}t^m A^{2m} T(t)(2^j + A)^{-m} x\| \frac{dt}{t} \right]^q \right\}^{1/q}
$$

$$
= \frac{1}{\Gamma(m)} \left\{ \sum_{j=k}^\infty \left[ \int_0^\infty \|2^{js} t^{-m} A^{2m} T(t^{-1})(2^j + A)^{-m} x\| \frac{dt}{t} \right]^q \right\}^{1/q}.
$$

Rewriting $\int_0^\infty = \sum_{i=-\infty}^{\infty} \int_{2^{i-1}}^{2^i}$ and applying (6.13) to $t^{-m} A^m T(2^{-j})$ yields

$$
|x|_{R^+_{q,X}} \lesssim K,
$$

where

$$
K = \left\{ \sum_{j=k}^\infty \left[ \sum_{i=-\infty}^{\infty} \|2^{js} t^{-m} A^{2m} T(2^{-i})(2^j + A)^{-m} x\| \right] q \right\}^{1/q}.
$$

In order to verify (6.18), it needs merely to verify that

$$
(6.19) \quad K \lesssim \|x\| + |x|_{R^+_{q,X}(k,m)} + |x|_{\tilde{R}^+_{q,X}(k,2m)}.
$$

We merely verify (6.19) in the case $0 < q \leq 1$ and the case when $1 < q \leq \infty$ can be verified analogously. To this end, let $0 < q \leq 1$. It follows from (1.4) that

$$
K \leq \left\{ \sum_{j=k}^\infty \sum_{i=-\infty}^{\infty} \|2^{js} t^{-m} A^{2m} T(2^{-i})(2^j + A)^{-m} x\|^q \right\}^{1/q}
$$

$$
= K_1 + K_2 + K_3,
$$

where

$$
K_1 = \left\{ \sum_{j=k}^\infty \sum_{i=-\infty}^{j-1} \|2^{js} t^{-m} A^{2m} T(2^{-i})(2^j + A)^{-m} x\|^q \right\}^{1/q},
$$

$$
K_2 = \left\{ \sum_{j=k}^\infty \sum_{i=-\infty}^{j-1} \|2^{js} t^{-m} A^{2m} T(2^{-i})(2^j + A)^{-m} x\|^q \right\}^{1/q},
$$

$$
K_3 = \left\{ \sum_{j=k}^\infty \sum_{i=-\infty}^{\infty} \|2^{js} t^{-m} A^{2m} T(2^{-i})(2^j + A)^{-m} x\|^q \right\}^{1/q}.
$$

As for the part $K_1$, applying (6.12) to $(2^{-i} A)^{2m} T(2^{-i})$ yields

$$
K_1 = \left\{ \sum_{j=k}^\infty \sum_{i=-\infty}^{j-1} \|2^{js} t^m (2^{-i} A)^{2m} T(2^{-i})(2^j + A)^{-m} x\|^q \right\}^{1/q}
$$

$$
\lesssim \left\{ \sum_{j=k}^\infty \sum_{i=-\infty}^{j-1} \|2^{j(s-m)} t^m (2^{-i} A)^2 m (2^j + A)^{-m} x\|^q \right\}^{1/q} \lesssim \|x\|.
$$
where the last inequality follows from the uniform boundedness of $2^j m (2^j + A)^{-m}$ due to (2.16). As for the pre-$K_2$, exchanging the order of summation yields and applying (2.16) to $2^j m (2^j + A)^{-m}$ yields

$$K_2 = \left\{ \sum_{i=k}^{\infty} \sum_{j=i+1}^{\infty} \left| 2^{j(s-m)} 2^{-im} A^{2m} T(2^{-i}) 2^{jm} (2^j + A)^{-m} x \right|^q \right\}^{1/q}$$

$$\lesssim \left\{ \sum_{i=k}^{\infty} \left[ \sum_{j=i+1}^{\infty} 2^{j(s-m)q} \left\| 2^{-im} A^{2m} T(2^{-i}) x \right\|^q \right] \right\}^{1/q}$$

$$\approx \left\{ \sum_{i=k}^{\infty} \left( \sum_{j=i+1}^{\infty} 2^{j(s-2m)} A^{2m} T(2^{-i}) x \right\|^q \right\}^{1/q} \lesssim |x| R_{q, X}^{\ast, T}(k, 2m).$$

And as for the part $K_3$, exchanging the order of summation yields and applying (2.17) to $A^{m} (2^j + A)^{-m}$ yields

$$K_3 = \left\{ \sum_{i=k}^{\infty} \sum_{j=k}^{i} \left| 2^{i s} 2^{-im} A^{2m} T(2^{-i}) (2^j + A)^{-m} x \right|^q \right\}^{1/q}$$

$$\lesssim \left\{ \sum_{i=k}^{\infty} \left( \sum_{j=k}^{i} 2^{j s q} \left\| 2^{-im} A^{m} T(2^{-i}) x \right\|^q \right) \right\}^{1/q} \lesssim |x| R_{q, X}^{\ast, T}(k, m).$$

This implies that

$$K \lesssim K_1 + K_2 + K_3 \lesssim \|x\| + |x| R_{q, X}^{\ast, T}(k, 0, m) + |x| R_{q, X}^{\ast, T}(k, 2m),$$

which is the desired (6.19). Thus, we have verified (6.18).

Finally, from (6.18) and (6.19) it follows that

$$\|x\|_{B_{q, X}^{\ast, A}} \approx \|x\| + |x| R_{q, X}^{\ast, A}(k, 0, m)$$

$$\lesssim \|x\| + |x| R_{q, X}^{\ast, T}(k, m) + |x| R_{q, X}^{\ast, T}(k, 2m)$$

$$\lesssim \|x\| B_{q, X}^{\ast, T}(k, m) + \|x\| B_{q, X}^{\ast, T}(k, m) \lesssim \|x\|_{B_{q, X}^{\ast, A}},$$

from which the desired conclusion follows immediately. The proof is complete. \(\square\)

A homogeneous version of Proposition 6.7 is given as follows.

Proposition 6.8. Let $s > 0$ and $0 < q \leq \infty$ and let $A$ be the negative generator of a bounded analytic semigroup $T$ on $X$. Fix $\beta \in \mathbb{C}_+$ such that $s < \Re \beta$ and let $x \in \overline{R(A)}$. If $A$ is injective, then

$$|x| R_{q, X}^{\ast, A}(0, \beta) \approx \left\{ \sum_{j=-\infty}^{\infty} \left\| 2^{j s} (2^{-j} A)^{\beta} T(2^{-j}) x \right\|^q_X \right\}^{1/q}$$

$$= \left\{ \sum_{j=-\infty}^{\infty} \left\| 2^{-j s} (2^j A)^{\beta} T(2^j) x \right\|^q_X \right\}^{1/q} := |x| R_{q, X}^{\ast, T}(\beta).$$

If this is the case, $\hat{B}_{q, X}^{\ast, A}$ is the completion of $R_{q, X}^{\ast, A}(0, \beta)$ with respect to the quasi-norm $\| \cdot \|_{R_{q, X}^{\ast, T}(\beta)}$. 
Proof. Assume that $A$ is injective. Let $x \in \mathcal{R}(A)$ and fix an integer $m > \text{Re} \beta$. Let $|x|_{\mathcal{R}_q^s(0,\beta)} < \infty$. Analogous to (6.17), by using (2.39) to $A^\beta$ and (2.19) with $c = 1$, we conclude that

$$|x|_{\mathcal{R}_q^s(0,\beta)} < J,$$

where

$$J = \left\{ \sum_{j=-\infty}^{\infty} \left[ \sum_{i=-\infty}^{\infty} \|2^{i-s-\beta}i^\beta A^m (2^{i} + A)^{-m} \mathcal{T}(2^{-j}) x \| \right] \right\}^{1/q}.$$

By using the decomposition $\sum_{i=-\infty}^{\infty} = \sum_{i=-\infty}^{j-1} + \sum_{i=j}^{\infty}$ and applying the classical inequality (1.3) for $0 < q \leq 1$ and the Hölder inequality for $1 < q < \infty$ (the case when $q = \infty$ can be verified directly by use of the corresponding estimates) and we can conclude that

$$J \lesssim |x|_{\mathcal{R}_q^s(0,\beta)}.$$ 

This implies that

$$|x|_{\mathcal{R}_q^s(0,\beta)} \lesssim |x|_{\mathcal{R}_q^s(0,\beta)}.$$ 

Conversely, let $|x|_{\mathcal{R}_q^s(0,\beta)} < \infty$. By using (6.14) and (6.13) we have

$$|x|_{\mathcal{R}_q^s(0,\beta)} \lesssim K,$$

where

$$K = \left\{ \sum_{j=-\infty}^{\infty} \left[ \sum_{i=-\infty}^{\infty} \|2^{i-s}2^{-\beta} A^m \mathcal{T}(2^{i}) (2^j + A)^{-m} x \| \right] \right\}^{1/q}.$$ 

Analogously, by using the decomposition $\sum_{i=-\infty}^{\infty} = \sum_{i=-\infty}^{j-1} + \sum_{i=j}^{\infty}$ and applying the classical inequality (1.3) for $0 < q \leq 1$ and the Hölder inequality for $1 < q < \infty$ (the case when $q = \infty$ can be verified directly by use of the corresponding estimates) and we can conclude that

$$K \lesssim |x|_{\mathcal{R}_q^s(0,\beta)}.$$ 

This implies that

$$|x|_{\mathcal{R}_q^s(0,\beta)} \lesssim |x|_{\mathcal{R}_q^s(0,\beta)}.$$ 

The proof is complete. \hfill $\square$

Let $A$ be the negative generator of a bounded $\mathcal{C}_0$-semigroup $T$ on $X$. It is well known that $-A^\alpha$ generates a bounded analytic semigroup $\mathcal{T}_\alpha$ of angle $\frac{\pi}{2}(1 - \alpha)$ for each $0 < \alpha < 1$ (sometimes we call $\mathcal{T}_\alpha$ the subordinated semigroup of $T$). More precisely,

$$\mathcal{T}_\alpha(t) = \int_0^t k_\alpha(t, s) T(s) \, ds, \quad t > 0,$$

where $k_\alpha(\cdot, \cdot)$ is the so-called subordinated kernel given by

$$k_\alpha(s, t) = \frac{1}{\pi} \int_0^\infty \sin(rs \sin \pi \alpha) \exp(-rt - r^\alpha s \cos \pi \alpha) \, dr, \quad s, t > 0.$$
See [62, Corollary 3.3 (a)]. In particular, if \( T \) is bounded analytic of angle \( \theta \) for some \( 0 < \theta \leq \pi/2 \) then \(-A^\alpha \) also generates a bounded analytic semigroup \( T_\alpha \) of angle \( \frac{\pi}{2} - \alpha \left( \frac{\pi}{2} - \theta \right) \) for each \( 1 < \alpha < \frac{\pi}{\pi - 2\theta} \) (see [62, Proposition 3.5 (a)]). The following corollary is a direct consequence of Theorem 5.5 and Proposition 6.7.

**Corollary 6.9.** Let \( s > 0 \) and \( 0 < q \leq \infty \) and let \( A \) be the negative generator of a bounded \( C_0 \)-semigroup \( T \) on \( X \). The following statements hold.

(i) For \( 0 < \alpha < 1 \), \( B_{q,X}^{s,A} \) is the completion of \( X \) with respect to \( \|\cdot\|_{B_{q,X}^{s,A}} \), where \( T_\alpha \) is the bounded analytic semigroup generated by \(-A^\alpha\).

(ii) If \( T \) is bounded analytic of angle \( \theta \) for some \( 0 < \theta \leq \pi/2 \), then \( B_{q,X}^{s,A} \) is the completion of \( X \) with respect to \( \|\cdot\|_{B_{q,X}^{s,A}} \) for each \( 0 < \alpha < \frac{\pi}{\pi - 2\theta} \), where \( T_\alpha \) is the bounded analytic semigroup generated by \(-A^\alpha\).

**Remark 6.10.** By Proposition 6.7 and Remark 4.3 (i), we obtain directly [31, Theorem 5.3] for \( s > 0 \) and \( 1 \leq q \leq \infty \). Also, see [7, Chapter 1, Section 4] for a homogeneous version for abstract Besov norms with \( 0 < s < 1 \) and \( 1 \leq q \leq \infty \). Moreover, thanks to Corollary 6.9, one can obtain a variety of abstract Besov spaces by specifying bounded analytic semigroups or their subordinated semigroups on concrete function spaces. See Examples 6.11 and 6.12 below for the classical Besov spaces associated with the fractional Laplacians. We also refer the reader to [26] for some applications of the classical Besov spaces associated with the fractional Laplacians, [29, 19, 23] for Besov spaces associated with Schrödinger operators, [51, Definition 4.1 and Theorem 4.3] for Besov spaces associated with bounded \( C_0 \)-semigroups on Banach spaces, and [3] and references therein for Besov spaces associated with heat semigroup on Dirichlet spaces.

Finally, we present some explicit examples to illustrate that, to some extent, the fractional powers of (unbounded) operators are indeed a generalization of some singular integrals operators (for example, the Riesz potential and Bessel potential) on function spaces, and therefore we achieve the classical Besov spaces \( B_{p,q}^s(\mathbb{R}^n) \) by applying the Laplacian to abstract Besov spaces.

Let \( n \in \mathbb{N} \) and \( 1 < p < \infty \), and let \( \Delta_p \) be the Laplacian on \( L^p(\mathbb{R}^n) \) with maximal domain, i.e.,

\[
\Delta_p f := \Delta f,
\]

\[
D(\Delta_p) := \{ f \in L^p(\mathbb{R}^n) : \Delta f \in L^p(\mathbb{R}^n) \},
\]

where \( \Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} \) is understood in the distributional sense. It is well known that \( \Delta_p \) in injective and non-negative on \( L^p(\mathbb{R}^n) \), and therefore \( \Delta_p \) has dense domain and dense range due to the fact that \( L^p(\mathbb{R}^n) \) is reflexive (see [65, Corollary 1.1.4 (v)] or [38, Proposition 2.1.1 (h)]). Moreover, the inverse \( \Delta_p^{-1} \) of the Laplacian \( \Delta_p \) is an unbounded operator on \( L^p(\mathbb{R}^n) \), so that \( 0 \in \sigma(\Delta_p) \) (see the Riesz potential below).

Thanks to the non-negativity and injectivity of \(-\Delta_p\), the fractional power \((-\Delta_p)^\alpha\) of \(-\Delta_p\), i.e., the so-called fractional Laplacian, can be defined for each \( \alpha \in \mathbb{C} \) via the Balakrishnan-Komatsu operator as shown in Section 2.2 above. Alternatively, the fractional Laplacian also can be characterized via the Fourier transform (see, for example, [79, Chapter V]) or suitable harmonic extension [22]. For more information on representations and estimates of the fractional Laplacian, we refer the reader to, for example, [76, 44, 27, 34, 70, 2, 16, 60, 1, 24, 63] and references therein.
Now we turn to explicit representations of the fractional Laplacian via singular integral operators. The following representations (i), (ii) and (iii) are quite standard and we refer the reader to [79, Chapter V] and [65, Section 12.2] for more information on the Riesz potentials and Bessel potentials.

(i) Let $0 < s < 2$. The fractional Laplacian $(-\Delta_p)^{s/2}$ can be written as a singular integral operator in the following way:

$$((-\Delta_p)^{s/2} f)(x) = c_{n,s} \text{p.v.} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+s}} \, dy, \quad x \in \mathbb{R}^n,$$

where $f \in \mathcal{S}(\mathbb{R}^n)$ and $c_{n,s} = \frac{2^n \Gamma(n/2)}{\pi^{n/2} \Gamma(-s/2)}$ is a normalization constant. Moreover, $(-\Delta_p)^{s/2}$ admits a bounded extension to $L^p(\mathbb{R}^n)$ for $1 < p < n/s$ and $1/q = 1/p - s/n$.

(ii) Let $0 < s < n$. It can be verified that $(-\Delta_p)^{-s/2} = I_s$, i.e., the so-called Riesz potential which is given by

$$(I_s f)(x) = d_{n,s} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-s}} \, dy, \quad x \in \mathbb{R}^n,$$

where $f \in \mathcal{S}(\mathbb{R}^n)$ and $d_{n,s} = \frac{\Gamma(n/2)}{2^n \pi^{n/2} \Gamma(s/2)}$ is a normalization constant. Moreover, $(-\Delta_p)^{-s/2}$ admits a bounded $L^p-L^q$ extension for $1 < p < n/s$ and $1/q = 1/p - s/n$.

(iii) It is clear that the translation $I - \Delta_p$ of $-\Delta_p$ is positive, so that $(I - \Delta_p)^{-s/2}$ is bounded on $L^p(\mathbb{R}^n)$ for each $s \in \mathbb{C}_+$. More precisely, $(I - \Delta_p)^{-s/2} = B_s$ for $s \in \mathbb{C}_+$, where $B_s$ is the so-called Bessel potential defined by $B_s f = G_s * f$ with the Bessel kernel $G_s$ given by

$$G_s(x) = \frac{1}{(4\pi)^{s/2} \Gamma(s/2)} \int_0^{\infty} e^{-|x|^2/4y} y^{\frac{n-s}{2} - 1} \, dy, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Moreover, applying $A = -\Delta_p$ to (1.2) yields a unified resolvent representation of the fractional Laplacian, as shown in (iv) below.

(iv) Let $\alpha, \beta \geq 0$ such that $-n < -\alpha < s < \beta$. From (1.2) it follows that

$$(-\Delta_p)^{s/2} f = C \int_0^{\infty} \lambda^{s/2} \lambda^\alpha (-\Delta_p)^\beta (\lambda - \Delta_p)^{-\alpha - \beta} f \, d\lambda,$$

for $f \in \mathcal{S}(\mathbb{R}^n)$, where $C = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha + s) \Gamma(\beta - s)}$. In particular, we can obtain (i) and (ii) above by specifying indices $\alpha$ and $\beta$.

It is well known that $D(\Delta_p) = W^{2,p}(\mathbb{R}^n)$, i.e., the Sobolev space on $\mathbb{R}^n$. In general, it is also well known that $D((-\Delta_p)^{\alpha/2}) = W^{\alpha,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for $\alpha \in \mathbb{C}_+$, where $W^{\alpha,p}(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ are the so-called fractional Sobolev space and Bessel potential space, respectively (see [65, Section 12.3]). Moreover, applying the negative Laplacian and fractional Laplacian to our abstract Besov spaces gives the classical Besov spaces, as shown in Examples [6.11 and 6.12 below.

**Example 6.11** (Gaussian semigroup). Let $s > 0$, $0 < q \leq \infty$ and $1 < p < \infty$. It is well known that $\Delta_p$ is the generator of the Gaussian semigroup, a bounded analytic semigroup of angle $\pi/2$, on $L^p(\mathbb{R}^n)$. Applying $-\Delta_p$ to Theorem [5.5] and Proposition [6.7] yields the classical Besov spaces on $\mathbb{R}^n$, i.e.,

$$B^{ \alpha}(-\Delta_p)^\alpha_{q, L^p(\mathbb{R}^n)} = B^{2\alpha s}_{p,q}(\mathbb{R}^n), \quad \alpha > 0.$$
(in the sense of equivalent quasi-norms) due to the fact that $B_{q,L^p}(\mathbb{R}^n) = B_{p,q}^s(\mathbb{R}^n)$ (see [51, Section 2.12.2, Theorem (i)]).

In particular, applying $\alpha = 1/2$ in Example 6.11 yields the well-known description of $B_{p,q}^s(\mathbb{R}^n)$ via the Poisson semigroup (also, see [51, Section 2.12.2, Theorem (ii)]). Moreover, observe that

Example 6.12 (Poisson semigroup). Let $s > 0$, $0 < q \leq \infty$ and $1 < p < \infty$. Write $A_p := -H_\partialx$, where $H$ is the Hilbert transform on $L^p(\mathbb{R}^n)$. It is known that $A_p$ is the square root of the Laplacian, i.e., $A_p = -(\Delta p)^{1/2}$ with $D(A_p) = W^{1,p}(\mathbb{R}^n)$ and that $A_p$ is the generator of the Poisson semigroup on $L^p(\mathbb{R}^n)$, a bounded analytic semigroup of angle $\pi/2$. Applying $-A_p$ to Theorem 5.5 and Example 6.11 also yields the classical Besov spaces on $\mathbb{R}^n$, i.e.,

$$B_{q,L^p}(\mathbb{R}^n) = B_{p,q}^s(\mathbb{R}^n) = B_{p,q}^{s/2,-\Delta_p} = B_{p,q}^s(\mathbb{R}^n).$$

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