LOCALLY FINITE ADMISSIBLE SIMPLE LIE ALGEBRAS

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ABSTRACT. We introduce a class of Lie algebras called admissible Lie algebras. We show that a locally finite admissible simple Lie algebra contains a nonzero maximal toral subalgebra and the corresponding root system is an irreducible locally finite root system.

0. INTRODUCTION

In 1976, G.B. Seligman [Se] showed that finite dimensional simple Lie algebras containing a nonzero maximal toral subalgebra have a decomposition as

\[(\ast) \quad \mathcal{L} = (\mathcal{G} \otimes \mathcal{A}) \oplus (S \otimes \mathcal{B}) \oplus (V \otimes \mathcal{C}) \oplus \mathcal{D}\]

in which \(\mathcal{G}\) is a finite dimensional split simple Lie algebra and \(S\), \(V\) are specific irreducible \(\mathcal{G}\)-modules. The Lie algebra structure on \(\mathcal{L}\) induces an algebra structure on the vector space \(b := \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C}\) and \(\mathcal{D}\) is a Lie subalgebra of \(\mathcal{L}\) isomorphic to a specific subalgebra of inner derivations of \(b\). Following G.B. Seligman, S. Berman and R. Moody [BM] introduced the notion of a Lie algebra graded by an irreducible reduced finite root system \(R\) which is a Lie algebra containing a finite dimensional split simple Lie subalgebra \(\mathcal{G}\) of type \(R\) with a Cartan subalgebra \(\mathcal{H}\) such that \(\mathcal{L}\) has a weight space decomposition with respect to \(\mathcal{H}\) and as an algebra, \(\mathcal{L}\) is generated by weight spaces corresponding to "nonzero" weights. Next Allison, Benkart and Gao generalized [BM]'s definition to non reduced case by letting \(\mathcal{G}\) be a finite dimensional split simple Lie algebra of type \(B, C\) or \(D\) if \(R\) is of type \(BC\). In [BM], [BZ] and [ABC], the authors stated recognition theorems for root graded Lie algebras to classify such Lie algebras up to centrally isogeny, indeed any root graded Lie algebra \(\mathcal{L}\) has a decomposition as \((\ast)\) with a prescribed algebra structure on the so called coordinate algebra \(b := \mathcal{A} \oplus \mathcal{B} \oplus \mathcal{C}\). In 1999, N. Stumme studied locally finite Lie algebras containing a splitting Cartan subalgebra and called such algebras, locally finite split Lie algebras. She showed that, a reduced locally finite root system, the root system of a locally finite split semisimple Lie algebra, is the direct union of finite root subsystems of semisimple types. She also proved that a locally finite split simple Lie algebra is the direct union of finite dimensional simple subalgebras. Locally finite root systems appear in the theory of Kac-Moody Lie algebras as well, more precisely, countable irreducible reduced locally

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finite root systems are the root systems of infinite rank affine Lie algebras [K] §7.11. In 2003, O. Loos and E. Neher [LN] introduced locally finite root systems axiomatically and gave a complete description of these root systems. Next Neher [N] generalized the definition of a root graded Lie algebra to Lie algebras graded by a locally finite root system. In Neher’s sense, if \( R \) is a locally finite root system, \( S \) is a subsystem of \( R \) and \( \Lambda \) is an abelian group, an \((R, S, \Lambda)\)-graded Lie algebra is a compatible span_\(\mathbb{Z}(R)\)-graded and \( \Lambda \)-graded Lie algebra whose support with respect to \( \text{span}_\mathbb{Z}(R) \)-grading is contained in \( R \). For every \( 0 \neq \alpha \in S \), the homogeneous space \( L^0_\alpha \) contains a so called invertible element and \( L_0 = \sum_{0 \neq \alpha \in R} [L_\alpha, L_{-\alpha}] \). In this work, we first study the direct union \( \bigcup_{n \in \mathbb{N}} G_n \) of finite dimensional simple Lie algebras \( G_n, n \in \mathbb{N} \), containing a nonzero maximal toral subalgebra, these Lie algebras are \((R, R_{sdiv}, 0)\)-graded Lie algebras (see Definition [1.1]) in Neher’s sense for a locally finite root system \( R \). Here we study a class of Lie algebras, called admissible Lie algebras. An admissible Lie algebra \( \mathcal{L} \) has a nonzero toral subalgebra \( \mathcal{H} \) contained in the subalgebra of \( \mathcal{L} \) generated by the weight spaces such that for any nonzero weight vector \( x \), there is a weight vector \( y \) such that \( [x, y] \in \mathcal{H} \) and \( (x, [x, y], y) \) is an \( \mathfrak{sl}_2 \)-triple. We show that if the Lie algebra \( \mathcal{L} \) is locally finite simple Lie algebra whose weight spaces are finite dimensional, then \( \mathcal{H} \) is a maximal toral subalgebra, the root system of \( \mathcal{L} \) with respect to \( \mathcal{H} \) is a locally finite root system and \( \mathcal{L} \) is the direct union of finite dimensional simple subalgebras.

1. Admissible Lie Algebras

Throughout this work \( \mathbb{N} \) denotes the set of all nonnegative integers and \( \mathbb{F} \) is a field of characteristic zero. Unless otherwise mentioned, all vector spaces are considered over \( \mathbb{F} \). In the present paper, we denote the dual space of a vector space \( V \) by \( V^* \) and by \( GL(V) \), we mean the group of automorphisms of \( V \). For a matrix \( A, tr(a) \) denotes the trace of \( A \). Also for a Lie algebra \( \mathcal{L} \), we mean by \( Z(\mathcal{L}) \), the center of \( \mathcal{L} \) and if \( \mathcal{L} \) is finite dimensional, we denote the Killing form of \( \mathcal{L} \) by \( \kappa \). We also make a convention that for elements \( x_1, \ldots, x_m \) of a Lie algebra, by an expression of the form \([x_1, \ldots, x_m] \), we always mean \([x_1, [x_{m-1}, x_m]] \ldots \) to be zero.

Definition 1.1. Let \( \mathcal{V} \) be a nontrivial vector space and \( R \) be a subset of \( \mathcal{V} \), \( R \) is said to be a locally finite root system in \( \mathcal{V} \) if

(i) \( 0 \notin R \), \( R \) is locally finite and spans \( \mathcal{V} \),

(ii) for every \( \alpha \in R \), there exists \( \tilde{\alpha} \in \mathcal{V}^* \) such that \( \tilde{\alpha}(\alpha) = 2 \) and \( s_\alpha(\beta) \in R \) for \( \alpha, \beta \in R \) where \( s_\alpha : \mathcal{V} \to \mathcal{V} \) maps \( v \in \mathcal{V} \) to \( v - \tilde{\alpha}(v) \alpha \).

(iii) \( \tilde{\alpha}(\beta) \in \mathbb{Z} \), for \( \alpha, \beta \in R \).

Set \( R_{sdiv} := R \setminus \{ \alpha \in R \mid 2\alpha \in R \} \) and call it the semi-divisible subset of \( R \), the root system \( R \) is called reduced if \( R = R_{sdiv} \).

Suppose that \( R \) is a locally finite root system. A nonempty subset \( S \) of \( R \) is said to be a subsystem of \( R \) if \( s_\alpha(\beta) \in S \) for \( \alpha, \beta \in S \). Following [LN] §2.6, we say two roots \( \alpha, \beta \) are connected if there exist finitely many roots...
\( \alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n = \beta \) such that \( \tilde{\alpha}_{i+1}(\alpha_i) \neq 0 \), \( 1 \leq i \leq n-1 \). Connectedness is an equivalence relation on \( R \). The root system \( R \) is the disjoint union of its connected components. A nonempty subset \( X \) of \( R \) is called irreducible, if each two elements \( x, y \in X \) are connected and it is called closed if for \( x, y \in X \) with \( x + y \in R \), we would have \( x + y \in X \). It is easy to see that each connected component of the locally finite root system \( R \) is a closed subsystem of \( R \). Also using [LN, Cor. 3.15], \( R \) is a direct limit of its finite subsystems, and if \( R \) is irreducible, it is a direct limit of its irreducible finite subsystems.

**Definition 1.2.** Let \( \mathcal{H} \) be a Lie algebra. We say an \( \mathcal{H} \)-module \( M \) has a weight space decomposition with respect to \( \mathcal{H} \), if

\[ M = \oplus_{\alpha \in \mathcal{H}^*} M_{\alpha} \text{ where } M_{\alpha} := \{ x \in M \mid h \cdot x = \alpha(h)x; \ \forall h \in \mathcal{H} \}; \ \alpha \in \mathcal{H}^*. \]

The set \( R := \{ \alpha \in \mathcal{H}^* \setminus \{0\} \mid M_{\alpha} \neq \{0\} \} \) is called the set of weights of \( M \) with respect to \( \mathcal{H} \) and \( M_{\alpha}, \alpha \in R \), is called a weight space, also any element of \( M_{\alpha} \) is called a weight vector of weight \( \alpha \). If a Lie algebra \( L \) has a weight space decomposition with respect to a subalgebra of \( L \) via the adjoint representation, the set of weights is called the root system and weight spaces are called root spaces.

An easy verification proves the following lemma.

**Lemma 1.3.** Let \( \mathcal{H} \) be a Lie algebra and \( M \) be an \( \mathcal{H} \)-module admitting a weight space decomposition \( M = \oplus_{\alpha \in \mathcal{H}^*} M_{\alpha} \) with respect to \( \mathcal{H} \). Let \( T \) be a subalgebra of \( \mathcal{H} \) and take \( \pi : \mathcal{H}^* \rightarrow T^* \) to be defined by \( \pi(\alpha) = \alpha|_{T} \), the restriction of \( \alpha \) to \( T \). For \( \beta \in T^* \), define \( M'_{\beta} := \{ v \in M \mid t \cdot v = \beta(t)v; \ \forall t \in T \} \), then \( M = \oplus_{\beta \in T^*} M'_{\beta} \) and for \( \beta \in T^* \), \( M'_{\beta} = \bigoplus_{\{\alpha \in \mathcal{H}^*, \pi(\alpha)=\beta\}} M_{\alpha} \).

**Definition 1.4.** Let \( \mathcal{L} \) be a Lie algebra. A nontrivial subalgebra \( \mathcal{H} \) of \( \mathcal{L} \) is called a split toral subalgebra if \( \mathcal{L} \) as an \( \mathcal{H} \)-module, via the adjoint representation, has a weight space decomposition with respect to \( \mathcal{H} \). One can see that a split toral subalgebra of a Lie algebra is abelian. Throughout the present paper by a toral subalgebra, we always mean a split toral subalgebra.

Now suppose that \( \mathcal{L} \) is a Lie algebra containing a nonzero toral subalgebra \( \mathcal{H} \) with corresponding root system \( R \). The Lie algebra \( (\mathcal{L}, \mathcal{H}) \) (or \( \mathcal{L} \) for simplicity) is called admissible if \( \mathcal{L} \) satisfies the following property:

\[
(1.5) \quad \mathcal{H} \subseteq \sum_{\alpha \in R}[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \text{ and for } \alpha \in R, 0 \neq x \in \mathcal{L}_\alpha, \text{ there is } y \in \mathcal{L}_{-\alpha} \text{ such that } h := [x, y] \in \mathcal{H} \text{ and } (x, h, y) \text{ is an } \mathfrak{sl}_2 \text{-triple.}
\]

An element \( h \in \mathcal{H} \) is called a splitting element corresponding to \( \alpha \in R \) if there are \( x \in \mathcal{L}_\alpha \) and \( y \in \mathcal{L}_{-\alpha} \) such that \( (x, h := [x, y], y) \) is an \( \mathfrak{sl}_2 \)-triple. A subset \( \Delta \) of \( R \) is called connected with respect to a fix set \( \{ h_\alpha \mid \alpha \in R \} \) of splitting elements if for any \( \beta, \gamma \in \Delta \), there is a finite sequence \( \alpha_1, \ldots, \alpha_n \) of elements of \( \Delta \) such that \( \alpha_1 = \beta, \alpha_n = \gamma \) and \( \alpha_{i+1}(h_{\alpha_i}) \neq 0, 1 \leq i \leq n-1 \).
A root \( \alpha \in R \) is called \textit{integrable} if there are \( e_\alpha \in \mathcal{L}_\alpha, f_\alpha \in \mathcal{L}_{-\alpha} \) such that 
\[
 h_\alpha := [e_\alpha, f_\alpha] \in \mathcal{H}, \quad (e_\alpha, h_\alpha, f_\alpha) \text{ is an } \mathfrak{sl}_2\text{-triple and that } \text{ad}_{e_\alpha} \text{ and } \text{ad}_{f_\alpha} \text{ act locally nilpotently on } \mathcal{L}.
\] We denote by \( R_{\text{int}} \), the set of integrable roots of \( \mathcal{L} \) and note that if \( \mathcal{L} \) is a locally finite admissible Lie algebra, then \( R = R_{\text{int}} \).

A subset \( \Delta \) of \( R \) is called \textit{symmetric} if \( \Delta = -\Delta \) and it is called \textit{closed} if \( (\Delta + \Delta) \cap R \subseteq \Delta \).

\textbf{Example 1.6.} Let \( \mathcal{L} \) be a finite dimensional semisimple Lie algebra containing a maximal toral subalgebra \( \mathcal{T} \). Take \( \Phi \) to be the root system of \( \mathcal{L} \) with respect to \( \mathcal{T} \). Using [Se], Lem. I.3 and Lem. I.5, we get that \( (\mathcal{L}, \mathcal{T}) \) is an admissible Lie algebra. Moreover Corollary to Lemma I.4 of [Se] shows that for \( \beta \in \Phi \), there is a unique \( k_\beta \in \mathcal{T} \) with

\[
\beta(k_\beta) = 2 \quad \text{and} \quad k_\beta = [u, v] \quad \text{for some } u \in \mathcal{L}_\beta, v \in \mathcal{L}_{-\beta}.
\]

Using [Se], Lem. I.5 and Lem. I.6, we get that

\[
(1.7) \quad \mathcal{T} = \text{span}_F \{k_\beta \mid \beta \in \Phi \} \quad \text{and} \quad \mathcal{T}^* = \text{span}_F \{\beta \mid \beta \in \Phi \}.
\]

Following the proof of Lemma I.5 of [Se], we get that if \( \beta \in \Phi \), then \( \kappa(k_\beta, k_\beta) \neq 0 \) and for \( t \in \mathcal{T}, \beta(t) = 2\kappa(t, k_\beta)/\kappa(k_\beta, k_\beta) \), now \((1.7)\) implies that the linear transformation form \( \mathcal{T} \) to \( \mathcal{T}^* \) mapping \( t \mapsto \kappa(t, \cdot), t \in \mathcal{T}, \) is onto and so is one to one, this in particular implies that the Killing form restricted to \( \mathcal{T} \) is non-degenerate and that for \( \beta \in \Phi \), \( 2k_\beta/\kappa(k_\beta, k_\beta) \) is the unique element of \( \mathcal{T} \) representing \( \beta \) through the Killing form. Therefore one concludes that

\[
(1.8) \quad \text{if } \alpha, \beta_1, \ldots, \beta_n \in \Phi \text{ are such that } \alpha \in \text{span}_F \{\beta_1, \ldots, \beta_n\}, \text{ then } k_\alpha \in \text{span}_F \{k_{\beta_1}, \ldots, k_{\beta_n}\}.
\]

\textit{From now on we assume } \((\mathcal{L}, \mathcal{H})\) \textit{is an admissible Lie algebra with nonempty root system } \( R \).

Using \( \mathfrak{sl}_2\)-module theory, one can easily prove the following proposition.

\textbf{Proposition 1.9.} Let \( \alpha \in R \). Take \( e \in \mathcal{L}_\alpha \) and \( f \in \mathcal{L}_{-\alpha} \) to be such that 
\( (e, h := [e, f], f) \) is an \( \mathfrak{sl}_2\)-triple and set \( \mathfrak{g} := \text{span}_F \{e, h, f\} \). Suppose \( \text{ad}_e \) and \( \text{ad}_f \) act locally nilpotently on \( \mathcal{L} \), then the followings are satisfied:

(i) For \( \beta \in R, \beta(h) \in Z \) and \( \beta - \beta(h)\alpha \in R \).

(ii) If \( \beta \in R \) is such that \( \alpha + \beta \in R \), we have \( [e, \mathcal{L}_\beta] \neq \{0\} \), in particular, \( [\mathcal{L}_\alpha, \mathcal{L}_\beta] \neq \{0\} \).

(iii) For \( \beta \in R, \) \( \{k \in Z \mid \beta + k\alpha \in R \cup \{0\}\} \) is an interval.

(iv) For \( \beta \in R, \) if \( \beta(h) > 0 \), then \( \beta - \alpha \in R \cup \{0\} \) and if \( \beta(h) < 0 \), then \( \beta + \alpha \in R \cup \{0\} \).

(v) If \( k \in F \) and \( k\alpha \in R \), then \( k \in Z/2 \).

(vi) \( \{k\alpha \mid k \in F\} \cap R_{\text{int}} \subseteq \{\pm \alpha, \pm (1/2)\alpha, \pm 2\alpha\} \).

Now use Proposition \((1.7)\), (iv) together with the same argument as in [LN] \S10.2 to prove the following Corollary:
Corollary 1.10. If $R = R_{\text{int}}$, $R$ has the partial sum property, in the sense that if $n \in \mathbb{N} \setminus \{0\}$ and $\alpha, \alpha_1, \ldots, \alpha_n \in R \cup \{0\}$ are such that $\alpha + \cdots + \alpha_n = \beta$, there is a permutation $\pi$ of $\{1, \ldots, n\}$ such that $\alpha_{\pi(1)} + \cdots + \alpha_{\pi(i)} \in R \cup \{0\}$ for all $i \in \{1, \ldots, n\}$.

Proposition 1.11. Suppose that $R = R_{\text{int}}$.

(a) For an ideal $I$ of $\mathcal{L}$, set $R_I := \{\alpha \in R \mid I \cap \mathcal{L}_\alpha \neq \{0\}\}$, then $R_I$ is a symmetric closed subset of $R$ and $I = (I \cap \mathcal{L}_0) \oplus \sum_{\alpha \in R_I} (I \cap \mathcal{L}_\alpha) = (I \cap \mathcal{L}_0) \oplus \sum_{\alpha \in R_I} \mathcal{L}_\alpha$. Moreover if $I$ is simple as a Lie algebra, $I \cap \mathcal{L}_0 = \sum_{\alpha \in R_I} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]$.

(b) If $\mathcal{L}$ is semisimple (in the sense that it is a direct sum of simple ideals), then $\mathcal{L}_0 = \sum_{\alpha \in R}[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]$.

(c) Let $I$ be an ideal of $\mathcal{L}$. Define $\Delta := R \setminus R_I$ and set $J := \sum_{\alpha \in \Delta} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \oplus \sum_{\alpha \in \Delta} \mathcal{L}_\alpha$. If $\mathcal{L}_0 = \sum_{\alpha \in R}[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]$, then $J$ is an ideal of $\mathcal{L}$, $\mathcal{L} = I + J$ and $I \cap J = 0$.

Proof. (a) Since $I$ is an ideal of $\mathcal{L}$, we have $I = (I \cap \mathcal{L}_0) \oplus \sum_{\alpha \in R_I} (I \cap \mathcal{L}_\alpha)$ by [MP, Prop. 2.1.1]. Now if $\alpha \in R_I$ and $0 \neq x \in I \cap \mathcal{L}_\alpha$, then by (1.5), there exists $y \in \mathcal{L}_{-\alpha}$ such that $(x, h := [x, y], y)$ is an $\mathfrak{s}\mathfrak{l}_2$-triple. Now as $x \in I$, one gets that $h, y \in I$ and for all $z \in \mathcal{L}_\alpha$, $2z = \alpha(h)z = [h, z] \in I$. So we have

\begin{equation}
R_I = -R_I \quad \text{and} \quad \mathcal{L}_\alpha \subseteq I; \quad \alpha \in R_I.
\end{equation}

This in particular implies that $I = (I \cap \mathcal{L}_0) \oplus \sum_{\alpha \in R_I} \mathcal{L}_\alpha$. Now this together with Proposition (1.9) implies that $R_I$ is closed. For the last assertion, one can easily check that $I_c := \sum_{\alpha \in R_I} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \oplus \sum_{\alpha \in R_I} \mathcal{L}_\alpha$ is an ideal of $I$ and so we are done as $I$ is simple.

(b) Since $\mathcal{L}$ is semisimple, there are an index set $A$ and simple ideals $\mathcal{L}^i$, $i \in A$, of $\mathcal{L}$ such that $\mathcal{L} = \oplus_{i \in A} \mathcal{L}^i$. For $i \in A$, use the notation as in the previous part and set $R_i := R_{\text{cl}}$; since $\mathcal{L}^i$ is simple, part (a) implies that $\mathcal{L}^i = \sum_{\alpha \in R_i} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \oplus \sum_{\alpha \in R_i} \mathcal{L}_\alpha$. Now we are done as $\mathcal{L} = \oplus_{i \in A} \mathcal{L}^i$.

(c) Suppose $\alpha, \beta \in R$, then thanks to (1.12), we have

\begin{equation}
[\mathcal{L}_\alpha, \mathcal{L}_\beta] \subseteq \mathcal{L}_{\alpha + \beta} \cap I \quad \text{if} \quad \alpha \in R_I \quad \text{or} \quad \beta \in R_I
\end{equation}

(1.13)

\[[\mathcal{L}_{\alpha + \beta}, \mathcal{L}_{-\alpha}] \subseteq \mathcal{L}_\beta \cap I \quad \text{if} \quad \alpha \in R_I \quad \text{or} \quad \alpha + \beta \in R_I.\]

So using Proposition (1.9) (ii), we have

\begin{equation}
[\mathcal{L}_\alpha, \mathcal{L}_\beta] \subseteq \begin{cases}
I & \text{if} \ \alpha, \beta \in R_I, \\
J & \text{if} \ \alpha, \beta \in \Delta, \\
\{0\} & \text{if} \ \alpha \in R_I, \ \beta \in \Delta.
\end{cases}
\end{equation}

(1.14)

This in particular implies that

\begin{equation}
\left[ \sum_{\alpha \in R_I} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \right] + \sum_{\alpha \in R_I} \mathcal{L}_\alpha \cup \left[ \sum_{\beta \in \Delta} [\mathcal{L}_\beta, \mathcal{L}_{-\beta}] \right] = \{0\}.
\end{equation}

(1.15)
Lemma 1.21. For a subset $\Delta$ of $R$, set $\mathcal{L}_\Delta := \sum_{\alpha \in \Delta} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] + \sum_{\alpha \in \Delta} \mathcal{L}_\alpha$ and $\mathcal{H}_\Delta := \mathcal{H} \cap \sum_{\alpha \in \Delta} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]$. Suppose that $\Delta \subseteq R_{\text{int}}$ is symmetric and closed and take $\pi_\Delta : \mathcal{H}^* \to \mathcal{H}_\Delta^*$ to be defined by $\alpha \mapsto \alpha_{|\mathcal{H}_\Delta}$, $\alpha \in \mathcal{H}^*$. Then $\pi_\Delta$ restricted to $\Delta \cup \{0\}$ is injective. Also identifying $\Delta$ with $\pi_\Delta(\Delta)$, we have

Now (1.14), (1.15) imply that

$$[\mathcal{L}, J] = \sum_{\alpha \in R} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}], J = \sum_{\alpha \in \Delta} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}], J$$

(1.16)

which means that $J$ is an ideal of $\mathcal{L}$. Also as $\sum_{\alpha \in R} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \subseteq I$, one gets that

$$\mathcal{L} = I + J.$$  

Now suppose that $x \in I \cap J$, then $x \in I \cap \sum_{\alpha \in \Delta} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]$, so by (1.15),

$$[x, \sum_{\alpha \in R} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]] = \{0\}.  
(1.18)$$

On the other hand since $x \in I \cap \mathcal{L}_0$, we have

$$[x, \mathcal{L}_\alpha] = \{0\}; \quad \alpha \in \Delta.$$  

This in particular implies that $[x, \sum_{\alpha \in \Delta} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] + \sum_{\alpha \in \Delta} \mathcal{L}_\alpha] = \{0\}$. Now this together with (1.17) and (1.18) implies that $x \in Z(\mathcal{L})$.

Corollary 1.19. Suppose that $R = R_{\text{int}}$. Set $\mathcal{L}_c := \mathcal{L}_{0,0} \oplus \sum_{\alpha \in R} \mathcal{L}_\alpha$ where $\mathcal{L}_{0,0} := \sum_{\alpha \in R} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]$. Then $\mathcal{L}_c := \mathcal{L}_c/Z(\mathcal{L}_c)$ is a semisimple Lie algebra. In particular, if $\mathcal{L}$ is finite dimensional, $\mathcal{L}_c$ is semisimple.

Proof. It follows from (1.5) that the canonical projection map $\mathcal{L}_c \to \mathcal{L}_{cc}$ restricted to $\sum_{\alpha \in R} \mathcal{L}_\alpha$ is injective, so we identify $\sum_{\alpha \in R} \mathcal{L}_\alpha$ as a subspace of $\mathcal{L}_{cc}$, also it is easy to see that $(\mathcal{L}_{cc}, \frac{\mathcal{H} + Z(\mathcal{L}_c)}{Z(\mathcal{L}_c)})$ is an admissible Lie algebra whose root system can be identified with $R$. More precisely, we have the following weight space decomposition for $\mathcal{L}_{cc}$ with respect to $(\mathcal{H} + Z(\mathcal{L}_c))/Z(\mathcal{L}_c)$:

$$\mathcal{L}_{cc} = (\mathcal{L}_{cc})_0 \oplus \sum_{\alpha \in R} \mathcal{L}_{cc, \alpha}$$

(1.20)

$$\mathcal{L}_{cc,0} = \mathcal{L}_{0,0}/Z(\mathcal{L}_c), \mathcal{L}_{cc, \alpha} = \mathcal{L}_\alpha, \alpha \in R.$$  

Moreover we have $R = R_{\text{int}}$, $(\mathcal{L}_{cc})_0 = \sum_{\alpha \in R} ([\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] + Z(\mathcal{L}_{cc}) = \{0\}$. Using Proposition 1.11(c), we get that for any ideal $I$ of $\mathcal{L}_{cc}$, there is an ideal $J$ of $\mathcal{L}_{cc}$ such that $\mathcal{L} = I \oplus J$, this results in that $\mathcal{L}_{cc}$ is semisimple (see [H], §3.5). Now suppose $\mathcal{L}$ is finite dimensional and take $\mathfrak{r}$ to be the solvable radical of $\mathcal{L}_c$, then $Z(\mathcal{L}_c) \subseteq \mathfrak{r}$. Also as $\mathfrak{r}$ is solvable, we get that $\mathfrak{r}/Z(\mathcal{L}_c)$ is a solvable ideal of the semisimple Lie algebra $\mathcal{L}_{cc}$. So $\mathfrak{r}/Z(\mathcal{L}_c) = \{0\}$, i.e., $\mathfrak{r} = Z(\mathcal{L}_c)$. This means that $\mathcal{L}_c$ is a reductive Lie algebra, now as $\mathcal{L}_c$ is perfect, one concludes that $\mathcal{L}_c = [\mathcal{L}_c, \mathcal{L}_c]$ is semisimple. This completes the proof.

□
that \((\mathcal{L}_\Delta, \mathcal{H}_\Delta)\) is an admissible Lie algebra with root system \(\Delta\). If moreover \(\Delta\) is connected with respect to a fixed set of splitting elements, then \(\mathcal{L}_\Delta\) is a simple Lie subalgebra of \(\mathcal{L}\).

**Proof.** Set \(\pi := \pi_\Delta\). Suppose that \(\alpha, \beta \in \Delta \cup \{0\}\) and \(\pi(\alpha) = \pi(\beta)\). We first suppose that \(\gamma := \alpha - \beta \in R\), then since \(\Delta\) is symmetric and closed, we have \(\gamma \in \Delta\), so \((1.5)\) guarantees the existence of \(t \in \mathcal{H} \cap [\mathcal{L}_\gamma, \mathcal{L}_{-\gamma}] \subseteq \mathcal{H} \cap \mathcal{L}_\Delta = \mathcal{H}_\Delta\) with \(\gamma(t) = 2\), so \((\alpha - \beta)(t) = 2\) which contradicts the fact that \(\alpha \mid_{\mathcal{H}_\Delta} = \beta \mid_{\mathcal{H}_\Delta}\). Next suppose that \(\alpha - \beta \notin R \cup \{0\}\), then \(\alpha \neq 0\) and \(\beta \neq 0\). Since \(\alpha\) is integrable, there are \(e \in \mathcal{L}_\alpha, f \in \mathcal{L}_{-\alpha}\) such that 
\[(e, h := [e, f], f)\] is an \(\mathfrak{sl}_2\)-triple and \(\text{ad}\) act locally nilpotently on \(\mathcal{L}\). Now since \(\alpha - \beta \notin R \cup \{0\}\), Proposition \((1.9)(iv)\) implies that \(\beta(h) \leq 0\). This contradicts the fact that \(\pi(\alpha) = \pi(\beta)\) as \(h \in \mathcal{H} \cap [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \subseteq \mathcal{H}_\Delta, \alpha(h) = 2\) and \(\beta(h) \leq 0\). These all together imply that \(\pi \mid_{\Delta \cup \{0\}}\) is injective. Next we note that since \(\Delta\) is closed, \(\mathcal{L}_\Delta\) is a subalgebra of \(\mathcal{L}\). One also sees that \(\mathcal{L}_\Delta\) is an \(\mathcal{H}\)-submodule of \(\mathcal{H}\)-module \(\mathcal{L}\) admitting the weight space decomposition 
\[
\mathcal{L}_\Delta = \sum_{\alpha \in \Delta \cup \{0\}} (\mathcal{L}_\Delta)_\alpha \quad \text{with respect to } \mathcal{H} \quad \text{where } (\mathcal{L}_\Delta)_\alpha = \mathcal{L}_\alpha \quad \text{for } \alpha \in \Delta \\
\text{and } (\mathcal{L}_\Delta)_0 = \sum_{\alpha \in \Delta} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] .
\]
From Lemma \((1.3)\) we know that \(\mathcal{L}_\Delta\) admits a weight space decomposition 
\[
\mathcal{L}_\Delta = \bigoplus_{\pi(\alpha) \in \pi(\Delta \cup \{0\})} (\mathcal{L}_\Delta)_{\pi(\alpha)}
\]
with respect to \(\mathcal{H}_\Delta\) where for \(\alpha \in \Delta \cup \{0\}\), 
\[
(\mathcal{L}_\Delta)_{\pi(\alpha)} = \{x \in \mathcal{L}_\Delta \mid [h, x] = \alpha(h)x; \forall h \in \mathcal{H}_\Delta\} = \bigoplus_{\beta \in \Delta \cup \{0\}, \pi(\beta) = \pi(\alpha)} (\mathcal{L}_\Delta)_{\beta} .
\]
Now this together with the injectivity of \(\pi \mid_{\Delta \cup \{0\}}\) and the fact that \((\mathcal{L}, \mathcal{H})\) is an admissible Lie algebra implies that \((\mathcal{L}_\Delta, \mathcal{H}_\Delta)\) is an admissible Lie algebra with root system \(\Delta\). The last sentence follows from Proposition \((1.11)\) \(\square\)

**Proposition 1.22.** Suppose that \(\mathcal{L}\) is a semisimple Lie algebra and \(\mathcal{H}\) is a nonzero maximal toral subalgebra of \(\mathcal{L}\) such that \((\mathcal{L}, \mathcal{H})\) is an admissible Lie algebra with root system \(R = R_{\text{init}}\). Take \(I\) to be an index set such that 
\[
\mathcal{L} = \bigoplus_{i \in I} \mathcal{L}^i \quad \text{where for } i \in I, \mathcal{L}^i \text{ is an ideal of } \mathcal{L} \text{ which is simple as a Lie algebra.}
\]
Set \(R_i := \{\alpha \in R \mid \mathcal{L}_\alpha \subseteq \mathcal{L}^i\}\) and \(\mathcal{H}_i := \mathcal{L}^i \cap \mathcal{H}, i \in I,\) then

(a) \(\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i\),

(b) for \(i \in I, \mathcal{H}_i\) is a maximal toral subalgebra of \(\mathcal{L}^i\) and \((\mathcal{L}^i, \mathcal{H}_i)\) is an admissible Lie algebra with root system \(R_i\).

**Proof.** (a) We first mention that for \(i, j \in I\), we have 
\[(1.23) \quad [\mathcal{L}^i, \mathcal{L}^j] \subseteq \delta_{i,j} \mathcal{L}^i .
\]
Using Proposition \((1.11)(a)\), we get that for \(i \in I\), \(\mathcal{L}^i = \sum_{\alpha \in R_i} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \oplus \sum_{\alpha \in R_i} \mathcal{L}_\alpha\), and \(\mathcal{L}_0 = \sum_{i \in I} \sum_{\alpha \in R_i} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]\). For \(i \in I\), take \(\pi_i : \mathcal{L} \rightarrow \mathcal{L}^i\) to be the natural projection map and set \(T_i\) to be the image of \(\mathcal{H}\) under \(\pi_i\). Take \(T_i^\perp\) to be a subspace of \(T_i\) such that \(T_i = (\mathcal{H} \cap T_i) \oplus T_i^\perp\). Let \(h \in \mathcal{H}\), then there is a unique expression \(h = \sum_{t_i \in I} t_i\) with a finitely many nonzero terms \(t_i \in \mathcal{L}^i, i \in I\). Now fix \(i \in I\), using \((1.23)\), we have for \(x \in \mathcal{L}_\alpha\) with \(\alpha \in R_i\).
that

\[(1.24) \quad \pi_i(h), x = [t_i, x] = [\sum_{j \in I} t_j, x] = [h, x] = \alpha(h)x.\]

This in particular implies that

if \(h, h' \in \mathcal{H}\) and \(\pi_i(h) = \pi_i(h')\), then for \(\alpha \in R_i\), \(\alpha(h) = \alpha(h')\)

which allows us to define \(f_{t, \alpha}\) for \(t \in T'_i\) and \(\alpha \in R_i\) as follows:

\[(1.25) \quad f_{t, \alpha} := \alpha(h)\) where \(h \in \mathcal{H}\) is such that \(\pi_i(h) = t,\)

also define \(f_{t, \alpha} := 0\) if \(t \in T'_i\) and \(\alpha \in (R \cup \{0\}) \setminus R_i\). Next suppose that \(\alpha \in R \cup \{0\}\) and extend \(\alpha\) to the functional \(\alpha^i \in (\mathcal{H} \oplus T'_i)^*\) defined as follows:

\[(1.26) \quad \alpha^i(h + t) = \alpha(h) + f_{t, \alpha}; \quad h \in \mathcal{H}, t \in T'_i.\]

Now one can see that \(\mathcal{H} \oplus T'_i\) is a toral subalgebra of \(\mathcal{L}\) with corresponding root system \(\{\alpha^i \mid \alpha \in R\}\). But \(\mathcal{H} \oplus T'_i\) contains the maximal toral subalgebra \(\mathcal{H}\), therefore \(T'_i = \{0\}\) and so \(T_i = T_i \cap \mathcal{H}\) which implies that \(T_i \subseteq \mathcal{H}\).

This gives that \(T_i \subseteq \mathcal{H} \cap \mathcal{L}' = \mathcal{H}_i\), on the other hand, if \(h \in \mathcal{H}_i\), then \(h = \pi_i(h) \in T_i\), so we get that \(T_i = \mathcal{H}_i\). Now we have \(\mathcal{H} = \oplus_{i \in I} T_i\), and so \(\mathcal{H} = \oplus_{i \in I} \mathcal{H}_i\).

(b) Suppose that \(i \in I\). Using Preposition 1.11(a) and Lemma 1.21, we get that \((\mathcal{L}', \mathcal{H}_i)\) is an admissible Lie algebra with root system \(R_i\). Next we note that any toral subalgebra \(T_i\) of \(\mathcal{L}_i\) larger than \(\mathcal{H}_i\) would automatically be toral in \(\mathcal{L}\) and centralizes \(\mathcal{H}_j\) for \(j \in I \setminus \{i\}\). Now \(\mathcal{H} + T_i\) is a toral subalgebra of \(\mathcal{L}\) larger than \(\mathcal{H}\) which is a contradiction. This completes the proof. \(\square\)

**Lemma 1.27.** If \((\mathcal{L}, \mathcal{H})\) is a finite dimensional semi-simple admissible Lie algebra, then \(\mathcal{H}\) is a maximal toral subalgebra.

**Proof.** Let \(\mathcal{T}\) be a maximal toral subalgebra of \(\mathcal{L}\) containing \(\mathcal{H}\). We first suppose that \(\mathcal{L}\) is simple. Since \(\mathcal{T}\) is a toral subalgebra of \(\mathcal{L}, \mathcal{T} = \sum_{\beta \in \mathcal{T}^*} \mathcal{L}'_{\beta}\) where for \(\beta \in \mathcal{T}^*, \mathcal{L}'_{\beta} := \{x \in \mathcal{L} \mid [t, x] = \beta(t)x; \forall t \in \mathcal{T}\}\). Take \(\Phi\) to be the root system of \(\mathcal{L}\) with respect to \(\mathcal{T}\). Using Example 1.6 one knows that for \(\beta \in \Phi\), there is a unique element \(k_{\beta} \in \mathcal{T}\) satisfying

\[(1.28) \quad \beta(k_{\beta}) = 2 \text{ and } k_{\beta} = [u, v] \text{ for some } u \in \mathcal{L}'_{\beta}, v \in \mathcal{L}'_{-\beta}.\]

Using Lemma 1.3, we get that for \(\alpha \in R \cup \{0\}, \mathcal{L}_\alpha = \bigoplus_{\beta \in A_\alpha} \mathcal{L}'_{\beta}\) where \(A_\alpha := \{\beta \in \mathcal{T}^* \mid h = \alpha\}\). Now let \(\alpha \in R\) and \(\beta \in \Phi\) be such that \(\beta \in A_\alpha\). Suppose that \(0 \neq x \in \mathcal{L}'_{\beta} \subseteq \mathcal{L}_\alpha\), then 1.5 guarantees the existence of \(y \in \mathcal{L}_{-\alpha}\) such that \(h := [x, y] \in \mathcal{H}\) and \((x, h, y)\) is an \(\mathfrak{s}\mathfrak{l}_2\)-triple. Since \(y \in \mathcal{L}_{-\alpha} = \bigoplus_{\gamma \in A_\alpha} \mathcal{L}'_{-\gamma}\), we have that \(y = \sum_{\gamma \in A_\alpha} y_{-\gamma}\) with \(y_{-\gamma} \in \mathcal{L}'_{-\gamma}\). Now

\[h = [x, y] = [x, \sum_{\gamma \in A_\alpha} y_{-\gamma}] = \sum_{\gamma \in A_\alpha} [x, y_{-\gamma}]\]

and so \(h = [x, y_{-\beta}]\) as \(h \in \mathcal{H} \subseteq \mathcal{T} \subseteq \mathcal{L}_0\). Also as \(h \in \mathcal{H}\), \(\beta(h) = \alpha(h) = 2\). Now considering \(1.28\), we get from the uniqueness of \(k_{\beta}\) that \(k_{\beta} = h\), therefore we have proved

\[(1.29) \quad k_{\beta} \in \mathcal{H}; \quad \beta \in \Phi \text{ with } \beta \mid \mathcal{H} \neq 0.\]
Now set \( A := \{ \beta \in \Phi \mid \beta |_H \neq 0 \} \) and \( I := \bigoplus_{\beta \in A} \mathcal{L}'_{\beta} \oplus \bigoplus_{\beta \in A} [\mathcal{L}'_{\beta}, \mathcal{L}'_{-\beta}] \). We claim that \( I \) is an ideal of \( \mathcal{L} \). For this, it is enough to show that \( (\Phi + A) \cap \Phi \subseteq A \). Suppose that \( \gamma \in \Phi \) and \( \beta \in A \) are such that \( \gamma + \beta \in \Phi \). If \( \gamma |_H = 0 \), then \( \gamma + \beta |_H \neq 0 \) and so \( \gamma + \beta \in A \) and if \( \gamma |_H \neq 0 \), then \( (1.29) \) together with \( (1.29) \) implies that \( k_{\gamma + \beta} \in \mathcal{H} \), now as \( (\gamma + \beta)(k_{\gamma + \beta}) = 2 \neq 0 \), we get that \( \gamma + \beta |_H = 0 \), i.e., \( \gamma + \beta \in A \). This shows that \( I \) is an ideal of \( \mathcal{L} \), but \( \mathcal{L} \) is simple, so \( I = \mathcal{L} \) which in particular implies that \( A = \Phi \). Now \( (1.27) \) together with \( (1.29) \) implies that \( T \subseteq \mathcal{H} \) and so \( \mathcal{H} = T \) is a maximal toral subalgebra. Next suppose that \( \mathcal{L} = \bigoplus_{i=1}^n \mathcal{L}_i \) where for \( 1 \leq i \leq n \), \( \mathcal{L}_i \) is a simple ideal of \( \mathcal{L} \). So by Proposition \( (1.11) \) and Lemma \( (1.21) \), \( (\mathcal{L}_i, \mathcal{H} \cap \mathcal{L}_i) \) is a finite dimensional admissible simple Lie algebra and so using the first part of the proof, we get that \( \mathcal{H} \cap \mathcal{L}_i \) is a maximal toral subalgebra of \( \mathcal{L}_i \). Now if \( \mathcal{L} \) is a maximal toral subalgebra of \( \mathcal{L} \) containing \( \mathcal{H} \), Example \( (1.6) \) gets that \( (\mathcal{L}, T) \) is a semi-simple admissible Lie algebra. So Proposition \( (1.22) \) implies that \( T_i := \mathcal{L}_i \cap T \) is a maximal toral subalgebra of \( \mathcal{L}_i \) and that \( T = T_1 \oplus T_2 \oplus \cdots \oplus T_n \). But \( \mathcal{H} \cap \mathcal{L}_i \subseteq T_i \), therefore \( \mathcal{H} \cap \mathcal{L}_i = T_i \). Now we have \( \mathcal{T} = T_1 \oplus T_2 \oplus \cdots \oplus T_n \subseteq \mathcal{H} \) and so we are done.

**Proposition 1.30.** (a) Suppose that \( \Delta \) is a symmetric closed finite subset of \( R_{\text{int}} \) such that \( \mathcal{L}_\alpha \) is finite dimensional for any \( \alpha \in \Delta \), then for \( \alpha \in \Delta \), there is a unique \( h_\alpha \in \mathcal{H} \) such that

\[
\alpha(h_\alpha) = 2 \quad \text{and} \quad [x_\alpha, x_{-\alpha}] = h_\alpha \text{ for some } x_{\pm \alpha} \in \mathcal{L}_{\pm \alpha}.
\]

Moreover if \( \alpha, \beta_1, \ldots, \beta_m \in \Delta \) are such that \( \alpha \in \text{span}_F \{ \beta_1, \ldots, \beta_m \} \), then \( h_\alpha \in \text{span}_F \{ h_{\beta_i} \mid 1 \leq i \leq m \} \).

(b) If \( R = R_{\text{int}} \) and all the weight spaces are finite dimensional, then there is a unique set \( \{ h_\alpha \mid \alpha \in R \} \) of splitting elements which we refer to as the splitting subset of \( \mathcal{L} \).

**Proof.** (a) Set \( \mathcal{L}_\Delta := \bigoplus_{\beta \in \Delta} [\mathcal{L}_\beta, \mathcal{L}_{-\beta}] \oplus \bigoplus_{\beta \in \Delta} \mathcal{L}_\beta \) and \( \mathcal{H}_\Delta = \mathcal{H} \cap \mathcal{L}_\Delta \). Using Propositions \( (1.21) \) and Corollary \( (1.19) \), we get that \( (\mathcal{L}_\Delta, \mathcal{H}_\Delta) \) is a finite dimensional admissible semi-simple Lie subalgebra with root system \( \Delta \). Using Lemma \( (1.27) \) one gets that \( \mathcal{H}_\Delta \) is a maximal toral subalgebra of \( \mathcal{L}_\Delta \). Now the result follows from Example \( (1.6) \).

(b) Let \( \alpha \in R \) and take \( \Delta := R \cap F \alpha \). Using Proposition \( (1.9 \text{vi}) \) together with part (a), we are done.

**Proposition 1.31.** Suppose that \( (\mathcal{L}, \mathcal{H}) \) is a locally finite admissible Lie algebra with corresponding root system \( R \) such that all weight spaces are finite dimensional. Take \( \{ h_\alpha \mid \alpha \in R \} \) to be the splitting subset of \( \mathcal{L} \) and \( \mathcal{V} := \text{span}_F R \). For \( \alpha \in R \), define \( x_\alpha : \mathcal{V} \rightarrow \mathcal{V} \) by \( v \mapsto v(h_\alpha) \), \( v \in \mathcal{V} \) and \( s_\alpha : \mathcal{V} \rightarrow \mathcal{V} \) by \( v \mapsto v - \alpha(v) \alpha \), \( v \in \mathcal{V} \). For \( M \subseteq R \), set \( M_\pm := M \cup -M \) and take \( \mathcal{G}_M \) to be the subalgebra of \( \mathcal{L} \) generated by \( \cup_{\alpha \in M} \mathcal{L}_\alpha \), finally set \( \Delta_M := \{ \alpha \in R \mid \mathcal{G}_M \cap \mathcal{L}_\alpha \neq \{0\} \} \), then we have the followings:

(a) For \( M \subseteq R \), \( \Delta_M \) is a symmetric closed subset of \( R \), in particular \( \mathcal{L}_{\Delta_M} = \sum_{\alpha \in \Delta_M} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \oplus \sum_{\alpha \in \Delta_M} \mathcal{L}_\alpha \) is a Lie subalgebra of \( \mathcal{L} \).
(b) If \( M \) is a subset of \( R \) and \( \alpha, \beta \in \Delta_M \), then \( s_\alpha(\beta) \in \Delta_M \). Also if \( M \) is a finite subset of \( R \), \( \mathfrak{g}_M \) is a finite dimensional Lie subalgebra of \( \mathcal{L} \) and \( \Delta_M \) is a finite root system in its span where the reflection based on \( \alpha \in \Delta_M \) is \( s_\alpha \) restricted to \( \text{span}_R \Delta_M \).

(c) If \( \alpha, \beta \in R \), then \( \beta(h_\alpha) \in \mathbb{Z} \cap [-4,4] \).

(d) \( R \) is a locally finite root system in its span where the reflection based on \( \alpha \in R \) is \( s_\alpha \) and if \( \mathcal{L} \) is semi-simple, the necessary and sufficient condition for \( R \) to be irreducible is that \( \mathcal{L} \) is simple. Also for \( M \subseteq R \), \( \Delta_M \) is a closed subsystem of \( R \) which is irreducible if \( M \) is irreducible.

Proof. (a) We first recall that as \( \mathcal{L} \) is locally finite, for any weight vector \( x \), \( \text{ad}_x \) acts locally nilpotently on \( \mathcal{L} \). Now we show that \( \Delta_M \) is symmetric. One knows that each element of \( \mathfrak{g}_M \) is a sum of elements of the form \([x_1, \ldots, x_1] \) where \( n \in \mathbb{N} \setminus \{0\} \) and for \( 1 \leq i \leq n \), \( x_i \in L_{\alpha_i} \), for some \( \alpha_i \in M_\pm \). Now if \( \alpha \in \Delta_M \), \( \mathfrak{g}_M \cap L_\alpha \neq \{0\} \), so there are \( \alpha_1, \ldots, \alpha_n \in M_\pm \) such that \( \alpha = \sum_{i=1}^n \alpha_i \) and there are \( x_i \in L_{\alpha_i}, 1 \leq i \leq n \), such that \( 0 \neq x := [x_n, \ldots, x_1] \in \mathfrak{g}_M \cap L_\alpha \).

We use induction on \( n \) to show that \( \Delta_M \) is symmetric. If \( n = 1 \), then \( \alpha \in M_\pm \) and so \( -\alpha \in M_\pm \subseteq \Delta_M \). Now let \( \alpha \in \Delta_M \) be such that there are \( n \in \mathbb{N} \) such that \( \alpha_1, \ldots, \alpha_n \in M_\pm \) and \( x_i \in L_{\alpha_i}, 1 \leq i \leq n \), such that \( \alpha = \sum_{i=1}^n \alpha_i \) and \( 0 \neq x := [x_n, \ldots, x_1] \in \mathfrak{g}_M \cap L_\alpha \). Therefore \( 0 \neq [x_{n-1}, \ldots, x_1] \in \mathfrak{g}_M \cap L_\beta \) where \( \beta := \sum_{i=1}^{n-1} \alpha_i \). Now we have \( \beta, \alpha, \alpha = \beta + \alpha_n \in \Delta_M \subseteq R \). Using the induction hypothesis, we get that \( -\beta \in \Delta_M \) and so there is a nonzero \( y \in \mathcal{L}_- \cap \mathfrak{g}_M \). Now contemplating Proposition 1.9(ii), we have that \( \{0\} \neq \{y, \mathcal{L}_- \} \subseteq (\mathfrak{g}_M \cap \mathcal{L}_- \alpha_n - \beta) \) which shows that \( -\alpha \in \Delta_M \). Next we show that \( \Delta_M \) is closed. We first show that

\[
\text{(1.32)} \quad \text{if} \ \alpha \in \Delta_M \text{ and } \beta \in M_\pm \text{ such that } \alpha + \beta \in R \text{ and } x \text{ to be a nonzero element of } \mathfrak{g}_M \cap L_\alpha \text{. Using Lemma 1.9(ii), we get that } 0 \notin [x, \mathcal{L}_\alpha] \subseteq [\mathfrak{g}_M, \mathfrak{g}_M] \subseteq \mathfrak{g}_M \text{ and so we are done as } [x, \mathcal{L}_\beta] \subseteq \mathcal{L}_{\alpha + \beta} \text{. Now suppose } \alpha, \beta \in \Delta_M \text{ and } \alpha + \beta \in R \text{. Since } \alpha, \beta \in \Delta_M, \text{ there is } \{\alpha_i \ | \ 1 \leq i \leq n\} \cup \{\beta_j \ | \ 1 \leq j \leq m\} \subseteq M_\pm \text{ such that } \alpha = \sum_{i=1}^n \alpha_i \text{ and } \beta = \sum_{j=1}^m \beta_j \text{. For } 1 \leq i \leq m + n, \text{ set}
\gamma_i := \begin{cases} 
\alpha_i, & \text{if } 1 \leq i \leq n \\
\beta_{i-n}, & \text{if } n + 1 \leq i \leq n + m,
\end{cases}
\]

then \( \gamma_i \in M_\pm, 1 \leq i \leq m + n \) and since \( \alpha + \beta \in R \), \( \sum_{i=1}^{n+m} \gamma_i \in R \). Now by Corollary 1.10, there exists a permutation \( \pi \) on \( \{1, \ldots, m + n\} \) such that all partial sums \( \sum_{i=1}^t \gamma_{\pi(i)} \) \( R \cup \{0\}, 1 \leq t \leq m + n \) Now using \( \text{(1.32)} \) together with an induction process, we get that \( \alpha + \beta = \sum_{i=1}^{n+m} \gamma_i \in \Delta_M \). This means that \( \Delta_M \) is a closed. This completes the proof.

(b) Since \( \alpha, \beta \in \Delta_M \subseteq R \), by Proposition 1.9(i), \( \beta(h_\alpha) \in \mathbb{Z} \) and \( s_\alpha(\beta) \in R \). Now take \( e_\alpha \in \mathcal{L}_\alpha \) and \( f_\alpha \in \mathcal{L}_{-\alpha} \) to be such that \( h_\alpha = [e_\alpha, f_\alpha] \) and \( (e_\alpha, h_\alpha, f_\alpha) \) is an \( \mathfrak{sl}_2 \)-triple, setting \( \theta_\alpha := \text{exp(ad}_{e_\alpha})\text{exp(ad}_{-f_\alpha})\text{exp(ad}_{h_\alpha}) \), we get that \( \{0\} \neq \theta_\alpha(\mathcal{L}_\beta) \subseteq (\mathcal{L}_{\beta(\alpha)} \cap \mathcal{L}_{\Delta_M}) \) and so \( s_\alpha(\beta) \in \Delta_M \). Now if \( M \)
is a finite set, since $\mathcal{L}$ is locally finite, $\mathcal{G}_M$ is a finite dimensional subalgebra of $\mathcal{L}$ and so $\Delta_M$ is finite. This completes the proof.

(c) Set $M := \{\alpha, \beta\}$, by part (b), $\Delta_M$ is a finite root system in $\text{span}_F \Delta_M$ where the reflection based on $\gamma \in \Delta_M$ is defined by $v \mapsto v - \tilde{\gamma}(v)\gamma$, $v \in \text{span}_F \Delta_M$. Using the finite root system theory, we get that $\{\tilde{\gamma}(\eta) \mid \gamma, \eta \in \Delta_M\} \subseteq \mathbb{Z} \cap [-4, 4]$, in particular $\beta(h_\alpha) = \alpha(\beta) \in \mathbb{Z} \cap [-4, 4]$.

(d) Using Proposition 1.9(i), it is enough to prove that $R$ is locally finite. Suppose that $M$ is a finite subset of $R$, then by part (a), $\Delta := \Delta_M$ is a finite root system in $\text{span}_F \Delta$ where the reflection based on $\gamma \in \Delta_M$ is defined by $v \mapsto v - \tilde{\gamma}(v)\gamma$ for $v \in \text{span}_F \Delta$. Take $\Pi := \{\alpha_1, \ldots, \alpha_n\}$ to be a base of $\Delta$. One knows that the Cartan matrix $(\alpha_i(\alpha_j))$ is an invertible matrix. So for any choice of $\{k_1, \ldots, k_n\} \subseteq F$, the following system of equations 

$$\sum_{i=1}^{n} \tilde{\alpha}_j(\alpha_i)x_i = k_j; \quad 1 \leq j \leq n$$

has a unique solution. Next suppose that $(r_1, \ldots, r_n) \in \mathbb{F}^n$ is such that $\eta := \sum_{i=1}^{n} r_i \alpha_i \in R$, then for any $1 \leq j \leq n$,

$$\sum_{i=1}^{n} r_i \tilde{\alpha}_j(\alpha_i) = \sum_{i=1}^{n} r_i \alpha_i(h_{\alpha_j}) = \eta(h_{\alpha_j}).$$

This means that $(r_1, \ldots, r_n)$ is a solution for the following system of equations

$$\sum_{i=1}^{n} \tilde{\alpha}_j(\alpha_i)x_i = \eta(h_{\alpha_j}); \quad 1 \leq j \leq n.$$  

(1.33)

But we know that by part (c), $\eta(h_{\alpha_j}) \in [-4, 4] \cap \mathbb{Z}$, so there are a finitely many choice for $\eta(h_{\alpha_j})$, $1 \leq j \leq n$. This together with the fact that (1.33) has a unique solution implies that there are a finitely many choice for $(r_1, \ldots, r_n) \in \mathbb{F}^n$ such that $\sum_{i=1}^{n} r_i \alpha_i$ is a root. So $(\text{span}_F M) \cap R = (\text{span}_F \Delta) \cap R = (\text{span}_F \{\alpha_1, \ldots, \alpha_n\}) \cap R$ is a finite set. Now if $\mathcal{W}$ is a finite dimensional subspace of $\mathcal{V} = \text{span}_F R$, then there is a finite set $M$ of $R$ such that $\mathcal{W} \subseteq \text{span}_F M$, so $\mathcal{W} \cap R \subseteq (\text{span}_F M) \cap R$ and so $\mathcal{W} \cap R$ is a finite set. This shows that $R$ is a locally finite root system in $\mathcal{V}$. It follows easily from Proposition 1.11(a) that if $\mathcal{L}$ is semi-simple, $\mathcal{L}$ is simple if and only if $R$ is irreducible. Now let $M$ be a subset of $R$, parts (a) and (b) shows that $\Delta_M$ is a subsystem of $R$. Next suppose that $M$ is irreducible but $\Delta_M$ is not irreducible, then there is $\alpha \in \Delta_M$ such that $\alpha$ is connected to none of the elements of $M$, in particular,

$$\alpha(h_\beta) = 0; \quad \forall \beta \in M.$$  

(1.34)

But we know there are $\beta_1, \ldots, \beta_m \in M$ such that $\alpha = \sum_{i=1}^{m} \pm \beta_i$. Take $M' := \{\alpha, \beta_1, \ldots, \beta_m\}$, we know from parts (a) and (b) that $\Delta_{M'}$ is a finite closed subsystem of $R$, so Proposition 1.30 implies that $h_\alpha = \sum_{i=1}^{m} r_i h_{\beta_i}$ for
some \( r_i \in \mathbb{F} \), \( 1 \leq i \leq m \). This together with (1.34) implies that \( \alpha(h_\alpha) = 0 \) which is a contradiction. Therefore \( \Delta_M \) is irreducible. \( \square \)

**Proposition 1.35.** Suppose that \( \mathcal{L} \) is simple and locally finite such that all the weight spaces are finite dimensional, then \( \mathcal{H} \) is a maximal toral subalgebra of \( \mathcal{L} \).

**Proof.** Take \( \{ h_\alpha \mid \alpha \in R \} \) to be the splitting subset of \( \mathcal{L} \). By Proposition 1.31(d), \( R \) is an irreducible locally finite root system in its span where the reflection based on \( \alpha \in R \) is defined by \( v \mapsto v - v(h_\alpha)\alpha \), \( v \in \text{span}_\mathbb{F} R \). Now let \( T \) be a toral subalgebra of \( \mathcal{L} \) containing \( \mathcal{H} \). Take \( \Phi \) to be the root system of \( \mathcal{L} \) with respect to \( T \) and for \( \beta \in \Phi \cup \{ 0 \} \), denote by \( \mathcal{L}'_\beta \), the "weight space" of \( \mathcal{L} \) corresponding to \( \beta \). For \( \alpha \in R \cup \{ 0 \} \), set \( A_\alpha := \{ \beta \in \Phi \cup \{ 0 \} \mid \beta \mid_\mathfrak{n} = \alpha \} \), then by Lemma 1.3, we have \( \mathcal{L}_\alpha = \oplus_{\beta \in A_\alpha} \mathcal{L}'_\beta \). Now let \( t \in T \), since \( \mathcal{H} \subseteq T \) and \( T \) is abelian, \( t \in \mathcal{L}_0 \), but \( \mathcal{L} \) is simple, so Proposition 1.11(b) implies that \( t \in \sum_{\alpha \in R} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \). Therefore there is a finite subset \( M \) of \( R \) such that \( t \in \sum_{\alpha \in M} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \). Now take \( \Delta \) to be a finite irreducible closed subsystem of \( R \) containing \( M \) (see [LN, Cor. 3.16]). Using Lemma 1.21, we get that

\[
(1.36) \quad \mathcal{L}_\Delta = \sum_{\alpha \in \Delta} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \oplus \sum_{\alpha \in \Delta} \mathcal{L}_\alpha = \sum_{\alpha \in \Delta} \sum_{\beta, \gamma \in A_\alpha} [\mathcal{L}'_\beta, \mathcal{L}'_{-\gamma}] \oplus \sum_{\alpha \in \Delta} \sum_{\beta \in A_\alpha} \mathcal{L}'_\beta
\]

is a finite dimensional simple admissible Lie algebra. Now take \( T_\Delta := T \cap \sum_{\alpha \in \Delta} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \). It is read from (1.36) that \( \mathcal{L}_\Delta \) has a weight space decomposition with respect to \( T_\Delta \), in other words \( T_\Delta \) is a toral subalgebra of \( \mathcal{L}_\Delta \) containing \( \mathcal{H}_\Delta = \mathcal{H} \cap \mathcal{L}_\Delta \) which results in \( \mathcal{H}_\Delta = T_\Delta \) contemplating Lemmas 1.21 and 1.27. Now we have

\[
t \in T \cap \sum_{\alpha \in M} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \subseteq T \cap \sum_{\alpha \in \Delta} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] = T_\Delta = \mathcal{H}_\Delta \subseteq \mathcal{H}.
\]

This completes the proof. \( \square \)

Now one can use Propositions 1.35, 1.31(d) and Lemma 1.21 to get the following theorem:

**Theorem 1.37.** Suppose that \( (\mathcal{L}, \mathcal{H}) \) is a locally finite simple admissible Lie algebra whose weight spaces are finite dimensional. Take \( R \) to be the root system of \( \mathcal{L} \) with respect to \( \mathcal{H} \). Then \( R \) is an irreducible locally finite root system. Next take \( \{ R_i \mid i \in I \} \) to be the class of finite irreducible closed subsystems of \( R \). For \( i \in I \), set \( \mathcal{L}_i := \sum_{\alpha \in R_i} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] \oplus \sum_{\alpha \in R_i} \mathcal{L}_\alpha \) and \( \mathcal{H}_i := \mathcal{H} \cap \mathcal{L}_i \). Then \( \mathcal{H} \) is a maximal toral subalgebra of \( \mathcal{L} \) and for \( i \in I \), \( \mathcal{H}_i \) is a maximal toral subalgebra of \( \mathcal{L}_i \), also \( (\mathcal{L}_i, \mathcal{H}_i) \) is a finite dimensional simple admissible Lie algebra with root system \( R_i \). Moreover \( \{ R_i \mid i \in I \} \) and \( \{ \mathcal{L}_i \mid i \in I \} \) are directed systems with respect to inclusion, \( R \) is a direct limit of \( \{ R_i \mid i \in I \} \) and \( \mathcal{L} \) is a direct limit of \( \{ \mathcal{L}_i \mid i \in I \} \).
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