EULERIAN SUMMATION OPERATORS AND A REMARKABLE FAMILY OF POLYNOMIALS

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Abstract. We give several families of polynomials which are related by Eulerian summation operators. They satisfy interesting combinatorial properties like being integer-valued at integral points. This involves nearby-symmetries and a recursion for the values at half-integral points. We also obtain identities for super Catalan numbers.

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1. Introduction

Define the function $A: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ by $A(k, l) = a(k, l)^2$, where

$$a(k, l) = \sum_{\nu=0}^{k} \binom{l}{\nu} \binom{l-1+k-\nu}{l-1}.$$ 

In this paper we study numbers $P(m, n)$ which satisfy the summation equations

$$P(m, n + 1) + 2 \cdot P(m, n) + P(m, n - 1) = A(n, m),$$

for all $m \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and all $n \in \{2, 3, \ldots\}$. It is worthwhile noting that once having found one solution $P(m, n)$ of these equations, any other solution $G(m, n)$ is given by $G(m, n) = P(m, n) + (-1)^n(c_1(m)n + c_0(m))$, where $c_0(m)$ and $c_1(m)$ are complex numbers depending only on $m$. Because, the difference $B(m, n) = G(m, n) - P(m, n)$ clearly satisfies the trivial summation equations

$$B(m, n + 1) + 2 \cdot B(m, n) + B(m, n - 1) = 0,$$

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\[1The switching of $m$ and $n$ on the right hand side is due to other equalities explained later on.\]
which exactly have the solutions $B(m, n) = (-1)^n(c_1 n + c_0)$ with $c_1, c_0 \in \mathbb{C}$ for each single $m$. Notice further that $A(n, m)$ for fixed $m$ is polynomial in $n$ of degree $2(m - 1)$. The summation operator

$$S f(x) = f(x + \frac{1}{2}) + f(x - \frac{1}{2})$$

is bijective on the polynomial ring $\mathbb{C}[x]$. Hence there exists exactly one family of polynomials, which by abuse of notation we call $P(m, x)$, such that the polynomials

$$S^2 P(m, x) = P(m, x) + 2 \cdot P(m, x) + P(m, x - 1)$$

at each $x = n \in \mathbb{N}$ have values $S^2 P(m, n) = A(m, n)$. From now on we will denote by $P(m, n)$ the special solution of (1) given by the values of these polynomials $P(m, x)$ at places $x = n \in \mathbb{N}$. This solution has a number of interesting properties of which we collect the two most important ones here. First, it describes an integer-valued function $P : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$, and for $n \geq m$ the values $P(m, n)$ are indeed natural numbers. This is shown in Corollary 2.2 and Remark 2.3 (iii). Second, consider the summation equation in the first variable

$$\hat{P}(m + 1, n) + 2 \cdot \hat{P}(m, n) + \hat{P}(m - 1, n) = A(m, n).$$

By virtue of formula (1), the values $\hat{P}(m, n) = P(n, m) + (-1)^m(c_1(n)m + c_0(n))$ give a solution of (2) for all $c_0(n), c_1(n) \in \mathbb{C}$. In Theorem 3.1 we show that indeed $\hat{P}(m, n)$ also is a solution of (2). It follows that $P(m, n)$ is nearly symmetric for all $m, n \in \mathbb{N}$,

$$P(m, n) = P(n, m) + (-1)^m(c_1(n)m + c_0(n)),$$

and we show $c_1(n) = (-1)^{n-1}$ and $c_0(n) = (-1)^n \cdot n$. Hence, $P(m, n)$ is almost a polynomial in the first variable, too. For the first values $P(m, n)$ see Table 1.

The strategy of this paper is reverse to the above exposition. In Proposition 2.1 we iteratively define a family of polynomials $P(m, x)$ imposing a number of properties on them. Then we determine an explicit formula for these $P(m, x)$ and give the results on integer values and nearby symmetry in Section 2. In Theorem 3.1 we define two polynomials $A_1(m, x)$ and $A_2(x, n)$ which both interpolate the values $A(m, n)$. Here $A_2(\cdot, m)$ is the polynomial interpretation of $A(\cdot, m)$ used above. By counting arguments we show the first variable summation equation

$$P(m + 1, x) + 2P(m, x) + P(m - 1, x) = A_1(m, x).$$

This implies (2) for all $x = n \in \mathbb{N}$ and the second variable summation equation $P(m, x + 1) + 2P(m, x) + P(m, x - 1) = A_2(x, m)$ for all $x = n \in \mathbb{N}$. Both sides being polynomials, the equation must hold for all $x$. As a consequence, we obtain a polynomial identity $m^2 A_1(m, x) = x^2 A_2(x, m)$ (Proposition 3.3), respectively the symmetry

$$m^2 A(m, n) = n^2 A(n, m).$$
The numbers $P(m,n)$ arise as dimensions of certain $\text{GL}(n|n)$-modules, where $\text{GL}(n|n)$ is a general linear super group. See [3] for details. To our surprise their fascinating combinatorial properties have not been studied in the literature so far.

The values $P(m,n)$ satisfy nice summation equations in the first and in the second variable. One may ask whether this also holds for the mixed summation equation. Define the family $Q(m,x)$ of polynomials by the images of $P(m,x)$ under the summation operator

$$Q(m,x) = P(m,x) + P(m,x-1) + P(m-1,x) + P(m-1,x-1).$$

We study the combinatorial properties of the values $Q(m,n)$ in section [5]. Like for the construction of $P(m,x)$, in Proposition [5.1] we impose properties on a family $Q(m,x)$ of polynomials defining them iteratively, and show that the right hand sides of (3) satisfy these properties. For the summation operator $S$ it holds

$$Q(m,x+rac{1}{2}) = S(P(m,x) + P(m-1,x)).$$

This suggests that the values at half-integral numbers of all the polynomials involved should allow a description. We give one by Proposition [5.4] and a recursion process. This also justifies the mixing of the summation operators $S$ and $\tilde{S}$

$$\tilde{S}f(x) = f(x+1) + f(x).$$

We also obtain the identity

$$Q(m,x) = \tilde{E}(A_2(x,m) + A_2(x,m-1)),$$

where the Euler operator $\tilde{E} = \tilde{S}^{-1}$ is the inverse on polynomials of the operator $\tilde{S}$.

We get a new polynomial identity from this in Proposition [5.5]. This involves the preimage $\tilde{E}(A_2(x,m))$ for which we have to compute the polynomials

$$F(x,\nu,n) = \tilde{E}\left(\left[\frac{x}{n}\right] \cdot \left[\frac{x-\nu}{n}\right]\right),$$

where $\left[\frac{x}{n}\right] = \frac{1}{m}x(x-1)\cdots(x-(n-1))$. This is the purpose of Section [4]. We give a method for finding the preimage $\tilde{E}(f)$ for a polynomial $f$ in case that a series of subsequent values $f(0), f(1), \ldots, f(\text{deg}f)$ is given. This is a result parallel to Euler’s summation formula for the solution $\tilde{f}$ of the difference operator $\tilde{f}(x+1) = f(x)$ (see [2, 11.10]). But because the inverse of the difference operator is a discrete integration operator, whereas $\tilde{E}$ is not, our formula in Proposition [4.1] is more bulky. The constant coefficients $c(\nu,n-1) = F(0,\nu,n-1)$ of the above polynomials satisfy two recursion formulas themselves (Proposition [4.4])

$$c(\nu,n) = c(\nu-2,n) + c(\nu-1,n-1),$$
and
\[ c(\nu, n) = -c(\nu - 2, n) + \frac{\nu}{n}c(\nu - 1, n - 1). \]

They are given by Gessel’s \([1]\) super Catalan numbers
\[ C(m, k) = \frac{(2m)!/(2k)!}{2^m k!(m+k)!}, \]
\[ c(\nu, n) = \frac{(-1)^{\mu}2^{2n}}{2m} \cdot C(n - \mu, \mu), \]
if \(\nu\) is of the form \(\nu = n - 2\mu\). If \(\nu\) is not of this form, then \(c(\nu, n)\) is zero. In Corollary \([4,5]\) we give some identities for super Catalan numbers which we obtain from the above construction.

2. A FAMILY OF POLYNOMIALS

**Proposition 2.1.** For \(m = 0, 1, 2, \ldots\) there is a unique family of polynomials \(P(m, x)\) in \(\mathbb{Q}[x]\) with the following properties.

(i) \(P(0, x) = 0\).

(ii) \(\deg_x P(m, x) \leq 2(m - 1)\) for all \(m > 0\).

(iii) \(P(m, x) = P(m, -x)\) holds for all \(m \in \mathbb{N}_0\).

(iv) The function \(f(m, n) = P(m, n) + (-1)^{m+n} \cdot m\) is a symmetric function on \(\mathbb{N} \times \mathbb{N}\), i.e. \(f(m, n) = f(n, m)\).

**Proof of Proposition 2.1.** We show that the properties (i)–(iv) uniquely define the polynomials \(P(m, x)\) by recursion. For \(m = 0\) the polynomial \(P(0, x) = 0\) is fixed by property (i). For \(m = 1\), by (ii) we know \(P(1, x) = c\) is a constant polynomial the constant \(c\) being given by \(P(1, 0) = c\). By (iv) we see
\[ P(1, 0) + (-1)^{1+0} \cdot 1 = P(0, 1) + (-1)^{0+1} \cdot 0, \]
so \(c = P(0, 1) + 1 = 1\). Assuming \(P(k, x)\) to be constructed for \(0 \leq k \leq m\) we obtain by property (iv) the following values of \(P(m + 1, x)\)
\[ P(m + 1, k) = P(k, m + 1) + (-1)^{m+k}(m + 1 - k). \]
Using (iii) we find \(P(m + 1, -k) = P(m + 1, k)\) and we thus have fixed the values \(P(m + 1, x)\) at the \(2m + 1\) places \(x \in \{-m, \ldots, 0, \ldots, m\}\). But by (ii) the degree of \(P(m + 1, x)\) is at most \(2m\), hence \(P(m + 1, x)\) is the unique interpolation polynomial of degree \(2m\) for the above values. \(\square\)

For example, condition (iv) together with (i) implies
\[ P(m, 0) = (-1)^{m-1} \cdot m, \]
as well as
\[ P(m, 1) = 1 + (-1)^{m}(m - 1). \]
In particular
\[ P(0, x) = 0, \]
\[ P(1, x) = 1, \]
\[ P(2, x) = 4x^2 - 2, \]
\[ P(3, x) = 4x^4 - 8x^2 + 3, \]
\[ P(4, x) = \frac{16}{9}x^6 - \frac{56}{9}x^4 + \frac{112}{9}x^2 - 4, \]
\[ P(5, x) = \frac{4}{9}x^8 - \frac{16}{9}x^6 + \frac{92}{9}x^4 - \frac{152}{9}x^2 + 5, \]
\[ P(6, x) = \frac{16}{225}x^{10} - \frac{8}{45}x^8 + \frac{848}{225}x^6 - \frac{592}{45}x^4 + \frac{1612}{75}x^2 - 6, \]
\[ P(7, x) = \frac{16}{2025}x^{12} + \frac{32}{2025}x^{10} + \frac{596}{675}x^8 - \frac{794}{2025}x^6 + \frac{34696}{2025}x^4 - \frac{5872}{225}x^2 + 7, \]
\[ P(8, x) = \frac{64}{99225}x^{14} + \frac{32}{4725}x^{12} + \frac{64}{405}x^{10} - \frac{46384}{99225}x^8 + \frac{27968}{4725}x^6 - \frac{41312}{2025}x^4 + \frac{339392}{11025}x^2 - 8. \]

The proof of Proposition 2.1 shows that the values \( P(m, k) \) for all \( k = -m, \ldots, m \) are integers for all \( m \geq 0 \). Hence by (iv), for an integer \( j > 0 \) the value
\[ P(m, m + j) = P(m + j, m) + (-1)^j \cdot j \]
also is integral. This proves

**Corollary 2.2.** The function \( P : \mathbb{N} \times \mathbb{N} \to \mathbb{Z} \) defined by the values \( P(m, n) \) on natural numbers of the family of polynomials \( P(m, x) \) in Proposition 2.1 is integer-valued.

Let \( m \) be a natural number. For integers \( 0 \leq \mu \leq m - 1 \) put
\[ \mu^* = m - 1 - \mu. \]

For integers \( 0 \leq \nu, \mu \leq m - 1 \) we define the polynomials
\[ t(\nu, \mu, m; x) = \prod_{k=1}^{\mu} (x + \nu - \mu + k) \cdot \prod_{l=1}^{\nu} (x - 1 - \mu + l) = (x + \nu) \cdots (x + \nu - \mu + 1) \cdot (x + \nu - \mu - 1) \cdots (x - \mu). \]

**Proposition 2.3.** The polynomials
\[ P(m, x) = \sum_{\nu, \mu=0}^{m-1} \frac{t(\nu, \mu, m; x) \cdot t(\mu^*, \nu^*, m; x)}{\nu!\nu^!*\mu!\mu^!} \]
for \( m > 0 \), and \( P(0, x) = 0 \) satisfy the properties of Proposition 2.1.

**Proof of Proposition 2.3.** By definition, condition (i) of Proposition 2.1 is satisfied. For the summands of \( P(m, x) \) we have for all \( \nu, \mu \)
\[ \deg_x t(\nu, \mu, m; x) \cdot t(\mu^*, \nu^*, m; x) = \nu + \mu + \nu^* + \mu^* = 2(m - 1), \]
Table 1. Initial values of the function $P: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Z}$.

| $P(m, n)$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|
| 1         | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |
| 2         | 2   | 14  | 34  | 98  | 142 | 194 | 254 |
| 3         | −1  | 35  | 151 | 588 | 24196| 75324| 194820| 441340 |
| 4         | 4   | 60  | 900 | 5884| 24197| 75322| 1947542| 74307834 |
| 5         | −3  | 101 | 2301| 76429| 24197| 151805| 676198| 74307834 |
| 6         | 6   | 138 | 4902| 75322| 676198| 4160778| 2376699| 573785096 |
| 7         | −5  | 199 | 9211| 194823| 2376699| 19475142| 118493179| 3465441272 |
| 8         | 8   | 248 | 15880| 74307832| 74307832| 573785096| 3465441272 |

so the same holds true for $P(m, x)$. Hence property (ii) holds. Obviously,

$$t(\nu, \mu, m; -x) = (-1)^{\nu+\mu} \cdot t(\mu, \nu, m; x),$$

so condition (iii) follows

$$P(m, -x) = \sum_{\mu, \nu=0}^{m-1} (-1)^{2(m-1)} \frac{t(\mu, \nu, m; x) \cdot t(\nu^*, \mu^*, m; x)}{\nu! \nu^*! \mu! \mu^*!} = P(m, x).$$

In order to prove (iv), which is trivial for $m = n$, we assume $n > m$ without loss of generality. Substituting $\mu \mapsto m - 1 - \mu$ we may write

$$P(m, n) = \sum_{\nu^*, \mu^*=0}^{m-1} \frac{t(\nu, \mu^*, m; n) \cdot t(\mu, \nu^*, m; n)}{\nu! \nu^*! \mu! \mu^*!}.$$

Notice that for $n \leq \mu^*$ the value

$$t(\nu, \mu^*, m; n) = (n+\nu) \cdots \{(n+\nu - \mu^* + 1) \cdot (n+\nu - \mu^* - 1) \cdots (n-\mu^*)$$

is zero unless $n + \nu - \mu^* = 0$, where the value is $(-1)^n \nu! \nu^*!$. Similarly, $t(\nu, \mu^*, m; n)$ is zero for $n \leq \nu^*$ unless $n + \nu - \mu^* = 0$, in which case it is $(-1)^\nu \mu^!\nu^*!$. So we obtain

$$P(m, n) = \sum_{\nu^*, \mu^*=0}^{m-1} \frac{t(\nu, \mu^*, m; n) \cdot t(\mu, \nu^*, m; n)}{\nu! \nu^*! \mu! \mu^*!} + \sum_{\nu+\mu=m-1-n}^{m-1} (-1)^{\nu+\mu}.$$

In this expression, the second sum is $(-1)^{m+n-1}(m-n)$. Substituting $i = m - 1 - \nu$ and $j = m - 1 - \mu$ the first sum becomes

$$\sum_{i, j=0}^{n-1} \frac{t(i, n-1-j, n; m) \cdot t(j, n-1-i, n; m)}{i! (n-1-i)! j! (n-1-j)!} = P(n, m).$$

So for $n > m$

$$P(m, n) = P(n, m) + (-1)^{m+n-1}(m-n).$$

Hence condition (iv) of Proposition 2.1 holds for all integers $m, n > 0$. □
**Definition 2.4.** For integers $\alpha$ and $\beta$ define the natural number

$$
D_n(\alpha + 1,\beta) = \begin{cases}
\frac{n^\alpha}{\alpha+\beta+1}\binom{n+\alpha}{\alpha}^{(n-1)} & \text{if } \alpha \geq 0 \text{ and } 0 \leq \beta \leq n - 1 \\
0 & \text{else}
\end{cases}
$$

**Remark 2.5.**

1. We have seen $\deg_x P(m,x) = 2(m - 1)$. So property (ii) of Proposition 2.1 can be sharpened as

\[(ii') \deg_x P(m,x) = 2(m - 1) \text{ for all } m > 0.\]

2. Fixing the first variable, the function $P(m,x)$ is polynomial in $x$ by definition. By property (iv) of Proposition 2.1 the values $P(m,n)$ are nearly symmetric

$$
P(n,m) = P(m,n) + (-1)^{m+n}(m-n),
$$

Hence for fixed $n \in \mathbb{N}$ the function $P(m,n)$ is almost a polynomial of degree $2(n-1)$ in the first variable $m$.

3. For integers $n > 0$ there is an appealing presentation of $t(\nu,\mu;m;n)$

$$
t(\nu,\mu;m;n) = \begin{cases}
\frac{n^\nu}{(n+\nu-\mu)^2} & \text{if } n + \nu - \mu \neq 0 \\
(-1)^\nu & \text{if } n + \nu - \mu = 0
\end{cases}
$$

In particular, for integers $n \geq m > 0$ we obtain

$$
P(m,n) = \sum_{\nu,\mu=0}^{m-1} D_n(\nu + 1,n - 1 - \mu) \cdot D_n((m - 1 - \mu) + 1,n - 1 - (m - 1 - \nu)).
$$

Equivalently, using Definition 2.4 for integers $n \geq m > 0$

$$
P(m,n) = \sum_{\nu,\mu=0}^{m-1} D_n(\nu + 1,n - 1 - \mu) \cdot D_n((m - 1 - \mu) + 1,n - 1 - (m - 1 - \nu)).
$$

As of two natural numbers, any single summand of $P(m,n)$ is a natural number for the integers $n \geq m > 0$. Hence the values $P(m,n)$ are natural numbers for all integers $n \geq m$. In general, for integers $m,n > 0$ define the numbers

$$
\tilde{P}(m,n) = \sum_{\nu,\mu=0}^{m-1} D_n(\nu + 1,n - 1 - \mu) \cdot D_n((m - 1 - \mu) + 1,n - 1 - (m - 1 - \nu)).
$$

Hence $P(m,n) = \tilde{P}(m,n)$ holds for $n \geq m > 0$, whereas for $n < m$ we obtain

$$
P(m,n) = \tilde{P}(m,n) + (-1)^{m+n-1}(m-n).
$$

On the other hand, we know $P(m,n) = P(n,m) + (-1)^{m+n-1}(m-n)$ by property (iv). It follows for $n < m$

$$
\tilde{P}(m,n) = P(n,m) = \tilde{P}(n,m).
$$

Hence $\tilde{P}(m,n)$ is symmetric.
3. Summation operators

We define the summation operator $S$ acting on a function $f$

$$Sf(x) = f(x + \frac{1}{2}) + f(x - \frac{1}{2}).$$

On polynomials $S$ acts by $S(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} 2^{-k}(1 + (-1)^k)$. The preimages $S^{-1}(x^n)$ can be obtained uniquely by recursion starting with $S^{-1}(0) = 0$ and $S^{-1}(1) = \frac{1}{2}$. Hence $S$ is bijective on polynomial rings over fields of characteristic $\neq 2$. Noticing

$$S^2 f(x) = f(x + 1) + 2 \cdot f(x) + f(x - 1),$$

there is a natural understanding of the definition of the summation operator $S^2$ on functions $f$ on the integers, i.e. on sequences.

Let denote

$$\left[\begin{array}{c} x \\ k \end{array}\right] = \frac{x(x-1)\cdots(x-k+1)}{k!}$$

the polynomial of degree $k$ interpolating the binomial coefficient $\binom{n}{k} = \left[\begin{array}{c} n \\ k \end{array}\right]$ for integers $n \geq 0$. In particular, $\left[\begin{array}{c} n \\ 0 \end{array}\right] = 1$ is the constant polynomial.

The following theorem shows that the images of the summation operator $S^2$ applied to the nearly symmetric function $P(m, n)$ of Corollary 2.2, once in the first and once in the second variable, behave similarly.

**Theorem 3.1.** Let $P : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ be the function defined in Corollary 2.2. Let $S_1$ and $S_2$ be the summation operators in the first and second variable, respectively, and consider the function $A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by

$$S_1^2 P(m, n) = P(m + 1, n) + 2 \cdot P(m, n) + P(m - 1, n) = A(m, n).$$

Then it holds

$$S_2^2 P(m, n) = P(m, n + 1) + 2 \cdot P(m, n) + P(m, n - 1) = A(n, m).$$

For all integers $m = 1, 2, \ldots$ define polynomials of degree $2m$

$$A_1(m, x) = \left(\sum_{\nu=0}^{m} \left[\begin{array}{c} x - 1 + \nu \\ \nu \end{array}\right] \left[\begin{array}{c} x \\ m - \nu \end{array}\right]\right)^2.$$

For all integers $n = 1, 2, \ldots$ define polynomials of degree $2(n-1)$

$$A_2(x, n) = \left(\sum_{\nu=0}^{n} \binom{n}{\nu} \left[\begin{array}{c} x - \nu + n - 1 \\ n - 1 \end{array}\right]\right)^2.$$

Then for all integers $n, m > 0$ the following holds

$$A_1(m, n) = A(m, n) = A_2(m, n).$$

Let $P(m, x)$ be the family of polynomials of Proposition 2.1 defining the numbers $P(m, n)$. Then for all integers $m > 0$

$$S_1^2 P(m, x) = P(m + 1, x) + 2 \cdot P(m, x) + P(m - 1, x) = A_1(m, x).$$
Table 2. Initial values of the function $A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

| $A(m, n)$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|-----------|------|------|------|------|------|------|------|------|
| 1         | 4    | 16   | 36   | 64   | 100  | 144  | 196  | 256  |
| 2         | 4    | 64   | 324  | 1024  | 2500 | 5184 | 9604 | 16384 |
| 3         | 4    | 144  | 7744 | 36864 | 28900 | 85264 | 213444 | 473344 |
| 4         | 4    | 256  | 4356 | 1444  | 7744 | 28900 | 85264 | 213444 |
| 5         | 4    | 400  | 10404| 129600| 1004004| 5588496| 24423364| 729432064 |
| 6         | 4    | 576  | 21316| 369664| 3880900| 28472896| 159820164| 729432064 |
| 7         | 4    | 784  | 39204| 906304| 12460900| 117418896| 830246596| 4687319296 |
| 8         | 4    | 1024 | 66564| 1982464| 34692100| 410305536| 3588728836| 24706809856 |

Furthermore, for all integers $m > 0$ we have

$$S_2^2 P(n, x) = P(n, x + 1) + 2 \cdot P(n, x) + P(n, x - 1) = A_2(x, n).$$

Before we prove Theorem 3.1 let us include the following remarks. Consider the symmetric numbers $\tilde{P}(m, n)$ defined in Remark 2.5 (iii). For $n > m$ they satisfy $\tilde{P}(m + i, n) = P(m + i, n)$ for $i = -1, 0, 1$. Similarly for $n < m$, they satisfy $\tilde{P}(m + i, n) = P(n, m + i) = P(m + i, n) + (-1)^{m+i+n}(m+i-n)$ for $i = -1, 0, 1$ by Proposition 2.1 (iv). For $m = n$ we obtain $\tilde{P}(n + i, n) = P(n + i, n)$ for $i = -1, 0, 1$, but $\tilde{P}(n + 1, n) = P(n + 1, n) - 1$. By Theorem 3.1 it follows for all $m \neq n$

$$\tilde{P}(m + 1, n) + 2 \cdot \tilde{P}(m, n) + \tilde{P}(m - 1, n) = A(m, n),$$

whereas

$$\tilde{P}(n + 1, n) + 2 \cdot \tilde{P}(n, n) + \tilde{P}(n - 1, n) + 1 = A(n, n).$$

This proves the following corollary.

**Corollary 3.2.** For all integers $m, n > 0$ the symmetric numbers

$$\tilde{P}(m, n) = \sum_{\nu, \mu=0}^{\min\{n-1,m-1\}} \frac{n^2}{(n + \nu - \mu)^2} \left( \begin{array}{c} n + \nu \\ \nu \end{array} \right) \left( \begin{array}{c} n - 1 \\ \mu \end{array} \right) \left( \begin{array}{c} n + m - 1 - \mu \\ m - 1 - \mu \end{array} \right) \left( \begin{array}{c} n - 1 \\ m - 1 - \nu \end{array} \right)$$

satisfy the summation equation

$$S_1^2 \tilde{P}(m, n) + \delta_{m,n} = A(m, n).$$
We include a list of polynomials $A_1(m, x)$ and $A_2(x, n)$,

\begin{align*}
A_1(1, x) &= 4x^2, & A_2(1, x) &= 4, \\
A_1(2, x) &= 4x^4, & A_2(2, x) &= 16x^2, \\
A_1(3, x) &= \frac{16}{9} x^2(x^2 + \frac{1}{2})^2, & A_2(3, x) &= 16(x^2 + \frac{1}{2})^2, \\
A_1(4, x) &= \frac{4}{9}x^2(x^2 + 2)^2, & A_2(4, x) &= \frac{64}{9}x^2(x^2 + 2)^2, \\
A_1(5, x) &= \frac{16}{225}x^2(x^4 + 5x^2 + \frac{3}{2})^2, & A_2(5, x) &= \frac{16}{9}(x^4 + 5x^2 + \frac{3}{2})^2, \\
A_1(6, x) &= \frac{16}{2025}x^4(x^4 + 10x^2 + \frac{23}{2})^2, & A_2(6, x) &= \frac{64}{225}x^2(x^4 + 10x^2 + \frac{23}{2})^2, \\
A_1(7, x) &= \frac{64}{99225}x^4(x^n + 35 x^4 + 49x^2 + \frac{45}{4})^2, & A_2(7, x) &= \frac{64}{2025}(x^6 + \frac{35}{2}x^4 + \frac{49}{4}x^2 + \frac{45}{4})^2, \\
A_1(8, x) &= \frac{4}{99225}x^4(x^2 + 6)^2(x^4 + 22x^2 + 22)^2, & A_2(8, x) &= \frac{256}{99225}x^2(x^2 + 6)^2(x^4 + 22x^2 + 22)^2.
\end{align*}

The polynomials $A_1(m, x)$ and $A_2(x, n)$ satisfy the following properties.

Proposition 3.3. \hspace{1em} (a) For all $m > 0$ there is an identity of polynomials

\[ m^2 \cdot A_1(m, x) = x^2 \cdot A_2(x, m). \]

In particular, $m^2 A(m, n) = n^2 A(n, m)$ is a symmetric function on $\mathbb{N}^2$.

(b) The polynomial

\[ A_2(x, n) = \frac{2^{2n}}{(n-1)!^2} \cdot x^{2(n-1)} + \cdots + (1 + (-1)^{n-1})^2 \]

of degree $2(n-1)$ is even $A_2(x, n) = A_2(-x, n)$. Its value at $x = 1$ is $A_2(1, n) = 4n^2$.

Proof of Proposition 3.3. The $\nu$-th summand of the sum in $A_2(x, m)$ is

\[
\binom{m}{\nu} \left[ \frac{x - \nu + m - 1}{m - 1} \right] = m \cdot \frac{(m - \nu + x - 1) \cdots (1 + x - 1) (x - 1) \cdots (x - (\nu - 1))}{(m - \nu)!},
\]

so

\[
\frac{x}{m} \cdot \binom{m}{\nu} \left[ \frac{x - \nu + m - 1}{m - 1} \right] = \binom{m - \nu + x - 1}{m - \nu - 1} \left[ \frac{x}{\nu} \right]
\]

is the $(m - \nu)$-th summand of the sum in $A_1(m, x)$. Part (a) follows.

Property (iii) of the family $P(m, x)$ (cf. Proposition 2.1) of being even polynomials $P(m, x) = P(m, -x)$ is inherited by their images $S_1^2 P(m, x) = A_1(m, x)$ under the summation operator. By part (a) of this proposition, it carries over to $A_2(x, m) = A_2(-x, m)$. The leading term of the polynomial $\binom{x}{n-1}$ is $\frac{1}{(n-1)!} x^{n-1}$. So the leading term of $A_2(x, n)$ is

\[
\left( \sum_{\nu=0}^{n} \binom{n}{\nu} \right)^2 x^{2(n-1)} = \frac{2^{2n}}{(n-1)!^2} \cdot x^{2(n-1)}.
\]
Evaluating $A_2(x, n)$ at $x = 0$ reduces the sum to the terms for $\nu = 0$ and $n$, so $A_2(0, n) = (1 + (-1)^{n-1})^2$. Similarly, at $x = 1$ only the terms for $\nu = 0$ and $1$ are non-zero, and we obtain $A_2(1, n) = 4n^2$. \hfill \qed

We now prove Theorem 3.1.

**Proof of Theorem 3.1.** We show (6) for at integer places $x = n > m$. Then, as both the image $S_2^2 P(m, x)$ and $A_1(m, x)$ are polynomials, they must be equal. Let $n > m$ be an integer. By changing the summation index we obtain

$$A_1(m, n) = \left( \sum_{\nu=0}^{m} \binom{n-1+\nu}{n-1} \binom{n}{m-\nu} \right) \cdot \left( \sum_{\mu=0}^{m} \binom{n-1+m-\mu}{n-1} \binom{n}{\mu} \right).$$

Let

$$a(\nu, \mu, m, n) = \binom{n-1+\nu}{\nu} \binom{n}{m-\nu} \binom{n-1+m-\mu}{m-\mu} \binom{n}{\mu},$$

so that $A_1(m, n) = \sum_{\nu, \mu=0}^{m} a(\nu, \mu, m, n)$. On the other hand, using the notation of Remark 2.5 (iii),

$$S_2^2 P(m, n) = \sum_{\nu, \mu=0}^{m} D_n(\nu+1, n-1-\mu) \cdot D_n(m-\mu+1, n-1-m+\nu)$$

$$+ 2 \cdot \sum_{\nu, \mu=0}^{m-1} D_n(\nu+1, n-1-\mu) \cdot D_n(m-\mu, n-1-m+\nu)$$

$$+ \sum_{\nu, \mu=0}^{m-2} D_n(\nu+1, n-1-\mu) \cdot D_n(m-2-\mu+1, n-1-m-2+\nu).$$

An index shift $\nu \mapsto \nu + 1, \mu \mapsto \mu + 1$ in the last sum and in one of the two second sums yields

$$S_2^2 P(m, n) = \sum_{\nu, \mu=0}^{m} D_n(\nu+1, n-1-\mu) \cdot D_n(m-\mu+1, n-1-m+\nu)$$

$$+ \sum_{\nu, \mu=0}^{m-1} D_n(\nu+1, n-1-\mu) \cdot D_n(m-\mu, n-m+\nu)$$

$$+ \sum_{\nu, \mu=1}^{m} D_n(\nu, n-\mu) \cdot D_n(m-\mu+1, n-1-m+\nu)$$

$$+ \sum_{\nu, \mu=1}^{m-1} D_n(\nu, n-\mu) \cdot D_n(m-1-\mu+1, n-m+\nu).$$

Hence equation (6) follows from Lemma 3.4. The identity $A_1(m, n) = A_2(m, n)$ for integers $m, n > 0$ is obvious by substituting $\nu \mapsto m - \nu$ in $A_1(m, n)$ and then extending the sums in both $A_1(m, n)$ and $A_2(m, n)$ to $\max\{m, n\}$. Notice that $A_2(m, n) = A(m, n)$ by definition of $A$ in the introduction, so formula (6) is
proved. Hence equation \((7)\) holds for all integers \(x = m > 0\) and for all \(n > 0\) by the nearby symmetry of \(P(m, n)\). Exchanging \(n\) and \(m\) we obtain \((5)\). Therefore, both being polynomials of degree \(2(n - 1)\) in \(x\), the functions \(A_2(x, n)\) and \(S_1^2 P(n, x)\) coincide.

**Lemma 3.4.** For \(n > m\) the numbers \(a(\nu, \mu, m, n)\) defined in formula \((8)\) satisfy

\[ a(\nu, \mu, m, n) = b(\nu, \mu, m, n), \]

where \(b(\nu, \mu, m, n)\) denotes the following sum

\[
D_n(\nu + 1, n - 1 - \mu) \cdot D_n(m - \mu + 1, n - 1 - m + \nu) \\
+ (1 - \delta_{\nu,m}) \cdot (1 - \delta_{\mu,m}) \cdot D_n(\nu + 1, n - 1 - \mu) \cdot D_n(m - \mu, n - m + \nu) \\
+ (1 - \delta_{\nu,0}) \cdot (1 - \delta_{\mu,0}) \cdot D_n(\nu, n - \mu) \cdot D_n(m - \mu + 1, n - 1 - m + \nu) \\
+ (1 - \delta_{\nu,0}) \cdot (1 - \delta_{\mu,0})(1 - \delta_{\nu,m}) \cdot (1 - \delta_{\mu,m}) \cdot D_n(\nu, n - \mu) \cdot D_n(m - \mu, n - m + \nu).
\]

**Proof of Lemma 3.4.** Using the definition of \(D_n(\alpha + 1, \beta)\) (see \((2.4)\)), the lemma follows by straightforward calculations distinguishing the cases \(\nu, \mu\) equal to 0, \(m\), or generic. We exemplify this in the generic case \(0 < \nu, \mu < m\). The sum \(b(\nu, \mu, m, n)\) then is

\[
\frac{n^2}{(n + \nu - \mu)^2} \left(\frac{n + \nu}{\nu}\right) \left(\frac{n - 1}{n - 1 - \mu}\right) \left(\frac{n + m - \mu}{m - \mu}\right) \left(\frac{n - 1}{n - 1 - m + \nu}\right) \\
+ \frac{n^2}{(n + \nu - \mu)^2} \left(\frac{n + \nu}{\nu}\right) \left(\frac{n - 1}{n - 1 - \mu}\right) \left(\frac{n + m - \mu}{m - \mu}\right) \left(\frac{n - 1}{n - m + \nu}\right) \\
+ \frac{n^2}{(n + \nu - \mu)^2} \left(\frac{n + \nu - 1}{\nu - 1}\right) \left(\frac{n - 1}{n - \mu}\right) \left(\frac{n + m - \mu}{m - \mu}\right) \left(\frac{n - 1}{n - 1 - m + \nu}\right) \\
+ \frac{n^2}{(n + \nu - \mu)^2} \left(\frac{n + \nu - 1}{\nu - 1}\right) \left(\frac{n - 1}{n - \mu}\right) \left(\frac{n + m - \mu}{m - \mu}\right) \left(\frac{n - 1}{n - m + \nu}\right).
\]

Summing the first and the second line as well as the third and forth we obtain

\[
\frac{n^3}{(n + \nu - \mu)} \left[ \left(\frac{n + \nu}{\nu}\right) \left(\frac{n - 1}{n - 1 - \mu}\right) + \left(\frac{n + \nu - 1}{\nu - 1}\right) \left(\frac{n - 1}{n - \mu}\right) \right],
\]

which is

\[
\frac{n^2 \cdot (n + m - \mu - 1)! (n + \nu - 1)!}{\nu! (m - \nu)! (n - m + \nu)! \mu! (m - \mu)! (n - \mu)!}.
\]

Expanding by \((n - 1)!^2\) we see that this fraction is equal to \(a(\nu, \mu, m, n)\).

\[ \square \]

4. **Euler operator**

In this section we study the summation operator \(\tilde{S}\) defined by

\[ \tilde{S}f(x) = f(x + 1) + f(x). \]

On polynomial rings the operators \(S\) and \(\tilde{S}\) are related by \(\tilde{S}f(x) = Sf(x + \frac{1}{2})\). The Euler operator \(\tilde{E}\), that is the operator inverse to \(\tilde{S}\), is given on polynomials...
by recursion, \( \tilde{E}(x^0) = \frac{1}{2} \), and for \( n > 0 \)
\[
\tilde{E}(x^n) = \frac{1}{2} x^n - \frac{1}{2} \sum_{j=0}^{n-1} \binom{n}{j} \tilde{E}(x^j).
\]
Indeed
\[
\tilde{S}(\frac{1}{2}x^n) = \frac{1}{2}(x+1)^n + \frac{1}{2} x^n = x^n + \frac{1}{2} \sum_{j=0}^{n-1} \binom{n}{j} x^j.
\]
We may also determine the polynomials \( \tilde{E}(x^n) \) by their generating series. Define polynomials \( e_n(x) \) by
\[
\sum_{n=0}^{\infty} e_n(x) \frac{t^n}{n!} = e^{xt}.
\]
It holds \( \tilde{S}(e^{xt}) = e^{xt} \), hence the polynomials satisfy
\[
\tilde{S}e_n(x) = x^n.
\]
The polynomials \( e_n(x) = \tilde{E}(x^n) \) are the well-known Euler polynomials. In particular, \( e_1(x) = \frac{1}{2}x - \frac{1}{2} \), \( e_2(x) = \frac{1}{3}x^2 - \frac{1}{2}x \), \( e_3(x) = \frac{1}{2}x^3 - \frac{3}{4}x^2 + \frac{1}{8} \), and \( e_4(x) = \frac{1}{2}x^4 - x^3 + \frac{1}{2}x \). The values \( E_{2m} = 2^{2m+1}c_{2m}(\frac{1}{2}) \) are the alternating Euler numbers, \( \tilde{E}_0 = 1 \), \( \tilde{E}_2 = -1 \), \( E_4 = 5 \), \( E_6 = -61 \), etc. Whereas for all \( m \geq 0 \) it holds \( e_{2m+1}(\frac{1}{2}) = 0 \).
Define the second order Euler polynomials \( e_n^{[2]}(x) \) by the generating series
\[
\sum_{n=0}^{\infty} e_n^{[2]}(x) \frac{t^n}{n!} = \frac{e^{xt}}{(e^t + 1)^2}.
\]
It holds \( (\tilde{S})^2e_n^{[2]}(x) = \tilde{S}e_n(x) = x^n \). The polynomials \( e_n^{[2]}(x) \) are determined from the first order ones \( e_n^{[1]} = e_n(x) \) by Cauchy product expansion
\[
e_n^{[2]}(x) = \sum_{k=0}^{n} \binom{n}{k} e_{n-k}(x)e_k(0).
\]
On the other hand, we may determine \( \tilde{E}(f) \) for a polynomial \( f \) as follows. Define a sequence \( F_k \), \( k \in \mathbb{N}_0 \), by \( F_0 = 0 \), and for all \( k \geq 0 \)
\[
F_{k+1} = f(k) - F_k.
\]
It holds
\[
F_{k+1} + F_k = f(k),
\]
so the sequence \( F_k \) is a solution of the sequence of discrete equations \( \tilde{S}F_k = f(k) \). Any other solution differs from \( F_k \) only by a sequence \( (-1)^k \cdot c \) for some constant \( c \). Let \( G(x) \) be the interpolation polynomial of degree at most \( n = \deg f \) of the values \( G(k) = F_k \) for \( k = 0, \ldots, n \). It is given by Lagrange interpolation
\[
G(x) = \sum_{j=0}^{n} F_j \cdot \prod_{k=0, k \neq j}^{n} \frac{x - k}{j - k} = \sum_{j=0}^{n} \frac{(-1)^{n-j}F_j}{j!(n-j)!} \prod_{k=0, k \neq j}^{n} (x - k)
\]
and satisfies the summation equation
\[ G(k+1) + G(k) = f(k) \]
for \( k = 0, 1, \ldots n-1 \). In order to obtain the polynomial solution \( F(x) = \tilde{E}(f(x)) \) we must add a multiple of the polynomial \( B_n(x) \) of degree \( n \) interpolating the values \( B_n(k) = (-1)^k \) for \( k = 0, 1, \ldots, n \),
\[ F(x) = G(x) + c \cdot B_n(x), \tag{9} \]
where the constant \( c \) is determined by the summation equation at \( x = n \)
\[ F(n+1) + F(n) = G(n+1) + G(n) + c \cdot (B_n(n+1) + B_n(n)) = f(n). \]
Because the polynomial \( B_n(x) \) is given explicitly in Lagrange form
\[ B_n(x) = (-1)^n \sum_{j=0}^{n} \frac{1}{j!(n-j)!} \prod_{k=0,k\neq j}^{n} (x-k), \]
it follows
\[ B_n(n+1) + B_n(n) = (-1)^n((2^{n+1} - 1) + 1) = (-1)^n2^{n+1}. \]
We obtain
\[ (-1)^n2^{n+1} \cdot c = f(n) - G(n) - G(n+1) \]
\[ = \sum_{j=0}^{n} (-1)^{n-j} \left( f(j) - \binom{n+1}{j} F_j \right) \]
\[ = (-1)^n \sum_{j=0}^{n} \left( (-1)^j f(j) + \binom{n+1}{j} \sum_{k=0}^{j-1} (-1)^k f(k) \right), \]
or equivalently
\[ c = \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^k f(k) \sum_{j=k+1}^{n+1} \binom{n+1}{j}. \tag{10} \]
We summarize.

**Proposition 4.1.** The preimage \( F = \tilde{E}(f) \) of the polynomial \( f \) of degree \( n \) under the summation operator \( \tilde{S} \) is the polynomial
\[ F(x) = \sum_{j=0}^{n} \frac{c + (-1)^{n-j} E_j}{j!(n-j)!} \prod_{k=0,k\neq j}^{n} (x-k), \]
where the constant \( c = (-1)^n F(0) \) is given by (10), and the coefficients \( F_j \) are determined by the recursion \( F_0 = 0 \), and \( F_{k+1} = f(k) - F_k \) for \( k = 0, \ldots, n \).

Writing
\[ f(x) = a_n x^n + \cdots + a_1 x + a_0, \]
for the solution polynomial it follows
\[ F(x) = \frac{a_n}{2} x^n + \cdots. \]
Comparing this with the highest coefficient of (9) we obtain
\[ \frac{a_n \cdot n!}{2} = (-1)^n 2^n \cdot c + (-1)^n \sum_{j=0}^{n} \binom{n}{j} (-1)^j F_j , \]
which by expanding \( F_j = \sum_{k=0}^{j-1} (-1)^{j-1-k} f(k) \) is equivalent to a second formula for the constant \( c \)
\[ (11) \quad c = \frac{(-1)^n n!}{2^{n+1}} \cdot a_n + \frac{1}{2^n} \sum_{k=0}^{n} (-1)^k f(k) \sum_{j=k+1}^{n} \binom{n}{j} . \]
Simplifying the identity (10) = (11) yields the well-known expression for the leading coefficient \( a_n \) of the polynomial \( f \)
\[ (12) \quad (-1)^n n! \cdot a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k) . \]
By this and \( (-1)^n c = F(0) \) being the constant coefficient of \( F \), a number of non-obvious combinatorial identities arise.

**Example 4.2.**

(a) Let \( f(x) = x^n \). We obtain \( F_j = \sum_{l=0}^{j-1} (-1)^{j-1-l} l^n \) as well as
\[ c = \frac{1}{2^{n+1}} \sum_{l=0}^{n} (-1)^l l^n \sum_{i=l+1}^{n+1} \binom{n+1}{i} . \]
For the coefficients \( F_j + (-1)^j c \) of the solution polynomial
\[ F(x) = \sum_{j=0}^{n} (F_j + (-1)^j c) \prod_{k \neq j} \frac{x - k}{j - k} \]
we obtain
\[ F_j + (-1)^j c = (-1)^j \left( \sum_{l=0}^{n} (-1)^l l^n \sum_{i=l+1}^{n+1} \binom{n+1}{i} - \sum_{l=0}^{j-1} (-1)^l l^n \right) \]
\[ = \frac{(-1)^j}{2^{n+1}} \left( \sum_{l=j}^{n} (-1)^l l^n \sum_{i=l+1}^{n+1} \binom{n+1}{i} - \sum_{l=0}^{j-1} (-1)^l l^n \sum_{i=0}^{l} \binom{n+1}{i} \right) . \]
Computing the leading coefficient of \( F(x) = \frac{1}{2^n} x^n + \ldots \) from the above formula we obtain the following identity. For all integers \( n > 0 \) it holds true
\[ (-1)^n 2^n n! = \sum_{j=0}^{n} \binom{n}{j} \left( \sum_{l=j}^{n} (-1)^l l^n \sum_{i=l+1}^{n+1} \binom{n+1}{i} - \sum_{l=0}^{j-1} (-1)^l l^n \sum_{i=0}^{l} \binom{n+1}{i} \right) . \]
(b) Let
\[ f(x) = \left[ \frac{x}{n-1} \right] = \frac{1}{(n-1)!} \cdot x(x-1) \cdots (x-(n-1)+1) . \]
Then the polynomial solution $F$ of the summation equation $\tilde{S} F(x) = f(x)$ is

$$F(x) = \frac{1}{2^n (n-1)!} \sum_{j=0}^{n-1} \binom{n-1}{j} \prod_{k=0, k \neq j}^{n-1} (x - k).$$

The values $F_k, k = 0, \ldots, n-1$ in the consideration above are zero. Hence the associated interpolation polynomial $G$ of degree at most $n-1$ of these values is zero, too. It follows $F(x) = c \cdot B_{n-1}(x)$, where the constant $c$ given by (10) is

$$c = \frac{1}{2^n} (-1)^{n-1} f(n-1) \binom{n}{n} = \frac{(-1)^{n-1}}{2^n}. $$

We apply the above method once more. We will use the notion of super Catalan numbers given by Gessel [1].

**Definition 4.3.** For integers $m, k \geq 0$ define the super Catalan number

$$C(m, k) = \frac{(2m)! \cdot (2k)!}{2 \cdot m! \cdot k! \cdot (m+k)!}. $$

Super Catalan numbers are integers and satisfy the summation equation [1, p. 191]

(13) $$C(m+1, k) + C(m, k+1) = 4 \cdot C(m, k).$$

**Proposition 4.4.** (a) For $\nu = 0, 1, \ldots, n$ let

$$f(x, \nu, n) = \left[ \begin{array}{c} x \\ n \end{array} \right] \left[ \begin{array}{c} x - \nu \\ n \end{array} \right].$$

Then the polynomial solution of the equation $\tilde{S} F(x, \nu, n) = f(x, \nu, n)$ is

$$F(x, \nu, n) = \sum_{j=0}^{2n} (F_j(\nu, n) + (-1)^j c(\nu, n)) \prod_{k=0, k \neq j}^{2n} \frac{x - k}{j - k},$$

where

$$F_j(\nu, n) = (-1)^{j-1} \sum_{k=0}^{j-1} (-1)^k \binom{k}{n} \binom{k - \nu}{n}.$$ 

Here the constants $c(\nu, n)$ satisfy the following two recursion formulas for $2 \leq \nu \leq n$

(14) $$c(\nu - 2, n) = c(\nu, n) - c(\nu - 1, n - 1),$$

and

(15) $$c(\nu - 2, n) = -c(\nu, n) + \frac{\nu}{n} c(\nu - 1, n - 1).$$

For $0 \leq \nu \leq n$ they are explicitly given by

(16) $$c(\nu, n) = \begin{cases} 0, & \text{if } \nu = n - 1 - 2\mu \\
\frac{(-1)^{\nu}}{2^{\nu}} \cdot C(n - \mu, \mu), & \text{if } \nu = n - 2\mu \end{cases}.$$
(b) For
\[ f(x, n + 1, n) = \binom{x}{n} \binom{x - n - 1}{n} \]
the polynomial solution of the equation \( \hat{S}F(x, n + 1, n) = f(x, n + 1, n) \) is
\[ F(x, \nu, n) = \frac{1}{2} \sum_{j=0}^{n} (-1)^j \prod_{k=0, k \neq j}^{2n} \frac{x - k}{j - k} + \frac{1}{2} \sum_{j=n+1}^{2n} (-1)^{j+1} \prod_{k=0, k \neq j}^{2n} \frac{x - k}{j - k}. \]

For example we obtain
\[ F(x, n, n) = C(n, 0) \cdot B_{2n}(x), \]
while
\[ F(x, n - 1, n) = \binom{2n - 1}{n} \cdot \binom{x}{2n}. \]

We emphasize that the values of \( F(x, \nu, n) \) at \( x = k \) for \( 0 \leq k \leq 2n \) are explicitly given by
\[ F(k, \nu, n) = (-1)^k c(\nu, n) + \sum_{j=n+\nu}^{k-1} (-1)^{k-1+j} \binom{j}{n} \binom{j - \nu}{n}. \]

In particular, for \( \nu = n - 1 - 2\mu \) the polynomial \( F(x, \nu, n) \) has zeros in \( x = 0, 1, \ldots, n - \nu \).

**Corollary 4.5.** The super Catalan numbers \( C(n, \mu) \) satisfy the following identities.

(a) For all \( 0 \leq \mu \leq n \)
\[ 2 \cdot C(n, 0) = \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k \binom{k}{n} \binom{k - \mu}{n}. \]

(b) For all \( 0 \leq \mu \leq \lfloor \frac{n-1}{2} \rfloor \)
\[ C(n, 0) = \sum_{k=0}^{2n} (-1)^{k+1} \binom{k}{n} \binom{k + 1 + 2\mu - n}{n} \sum_{j=k+1}^{2n} \binom{2n}{j}, \]
whereas for all \( 0 \leq \mu \leq \lfloor \frac{n}{2} \rfloor \)
\[ (-1)^\mu \cdot C(n - \mu, \mu) - C(n, 0) = \sum_{k=0}^{2n} (-1)^k \binom{k}{n} \binom{k + 2\mu - n}{n} \sum_{j=k+1}^{2n} \binom{2n}{j}. \]

Notice that in each sum, the summands actually are zero for \( 0 \leq k \leq n + \mu \).

**Proof.** Part (a) is given by identity \( [12] \) for the leading coefficients of the polynomials \( f(x, \nu, n) \). Part (b) is given by formula \( [11] \) for the constants \( c(\nu, n) \) of the polynomials \( f(x, \nu, n) \) which are also determined by Proposition \( [\ref{prop:4.4}] \). Part (b) follows in particular from equation \( [19] \) below by inserting the special values of \( c(\nu - 2, n) \) given in Proposition \( [\ref{prop:4.4}] \). \( \square \)
Proof of Proposition 4.4. Part (a): Notice that 
\[ f(x, \nu, n) = \frac{1}{n!^2} \prod_{j=0}^{\nu-1} (x - j)(x - n - j) \cdot \prod_{k=1}^{n-\nu} (x - (\nu - 1) - k)^2. \]

Hence, in Proposition 4.1, the series \( F_j = F_j(\nu, n) \) is given by 
\[ F_j(\nu, n) = (-1)^{j-1} \sum_{k=0}^{\nu-1} (-1)^k \binom{k}{\nu} \binom{k - \nu}{n}, \]
where the summands vanish for \( k < n + \nu \). In particular, \( F_j(\nu, n) = 0 \) for \( 0 \leq j \leq n + \nu \). Let 
\[ G(x, \nu, n) = \sum_{j=0}^{2n} F_j(\nu, n) \prod_{k \neq j} \frac{x - k}{j - k} = \sum_{j=0}^{2n} F_j(\nu, n) \prod_{k \neq j} \frac{x - k}{j - k} \]
be the associated interpolation polynomial of degree 2\( n \). We obtain the polynomial solution 
\[ F(x, \nu, n) = \sum_{j=0}^{2n} (F_j(\nu, n) + (-1)^j c(\nu, n)) \prod_{k \neq j} \frac{x - k}{j - k}, \]
where the constant \( c(\nu, n) \) is determined by (11) 
\[ (2n)! \frac{1}{2^{2n+1}n!^2} + \sum_{k=n+\nu}^{2n} (-1)^k \binom{k}{\nu} \binom{k - \nu}{n} \sum_{j=k+1}^{2n} \binom{2n}{j}. \]
Evaluating this formula for \( \nu = n \) we obtain 
\[ c(n, n) = \frac{(2n)!}{2^{2n+1}n!^2} = \frac{1}{2^{2n}} \cdot C(n, 0). \]
Evaluation at \( \nu = n - 1 \) yields 
\[ c(n - 1, n) = \frac{(2n)!}{2^{2n+1}n!^2} - \frac{1}{2^{2n}} \binom{2n - 1}{n} = 0. \]
We use these special values to prove (16) by increasing induction on \( n \) and decreasing induction on \( \nu \) using recursion formula (15). But first observe that recursion formula (14) follows from recursion formula (13) for super Catalan numbers once the explicit values (16) hold true. For the above induction we have to prove (15). Observe that 
\[ \binom{k+1}{n} \binom{k+1-n}{n-1} - \binom{k}{n} \binom{k+1-n}{n-1} = \frac{n+\nu}{n} \binom{k}{n-1} \binom{k+1-n}{n-1}. \]
Accordingly, 
\[ \binom{k}{n} \binom{k-\nu+2}{n} = \binom{k+1}{n} \binom{k+1-n}{n-1} + \frac{\nu}{n} \binom{k}{n} \binom{k-\nu}{n-1}. \]
Using the definition of the super Catalan number, by (17) we know

\begin{equation}
2^{2n} c(\nu - 2, n) = C(n, 0) + \sum_{k=n+\nu-2}^{2n} (-1)^k \binom{k}{n} \binom{k - \nu + 2}{n} \sum_{j=k+1}^{2n} \binom{2n}{j}.
\end{equation}

We split this expression according to the identity for binomial coefficients (18). For the first part we obtain

\begin{align*}
C(n, 0) + \sum_{k=n+\nu-2}^{2n} (-1)^k \binom{k + 1}{n} \binom{k + 1 - \nu}{n} & \sum_{j=k+1}^{2n} \binom{2n}{j} \\
= C(n, 0) & - \sum_{k=n+\nu}^{2n+1} (-1)^k \binom{k}{n} \binom{k - \nu}{n} \sum_{j=k}^{2n} \binom{2n}{j} \\
= -C(n, 0) & - \sum_{k=n+\nu}^{2n} (-1)^k \binom{k}{n} \binom{k - \nu}{n} \sum_{j=k+1}^{2n} \binom{2n}{j} \\
= (-1) \cdot 2^{2n} \cdot c(\nu, n),
\end{align*}

where we used formula (12) for the highest coefficient \( (n!)^{-2} \) of the polynomial \( f(x, \nu, n) \). For the second part we obtain

\begin{align*}
\sum_{k=n+\nu-2}^{2n} (-1)^k \binom{k}{n-1} \binom{k - (\nu - 1)}{n-1} & \sum_{j=k+1}^{2n} \binom{2n}{j} \\
= 2 \cdot \sum_{k=n-1+(\nu-1)}^{2(n-1)} (-1)^k \binom{k}{n-1} \binom{k - (\nu - 1)}{n-1} \sum_{j=k+1}^{2n-1} \binom{2n-1}{j} \\
+ \sum_{k=n-1+(\nu-1)}^{2n} \binom{2n-1}{k} (-1)^k \binom{k}{n-1} \binom{k - (\nu - 1)}{n-1} \\
= 4 \cdot 2^{2(n-1)} \cdot c(\nu - 1, n - 1) + 2C(n - 1, 0) \\
+ \sum_{k=n-1+(\nu-1)}^{2n-1} \binom{2(n-1)}{k-1} (-1)^k \binom{k}{n-1} \binom{k - (\nu - 1)}{n-1} \\
= 4 \cdot 2^{2(n-1)} \cdot c(\nu - 1, n - 1) + 2 \cdot C(n - 1, 0) - 2 \cdot C(n - 1, 0) \\
= 2^{2n} \cdot c(\nu - 1, n - 1),
\end{align*}
where we used that the sum
\[
\sum_{k=n-1+\nu-1}^{2n-1} \binom{2(n-1)}{k}(-1)^k \binom{k}{n-1}(k-(\nu-1))
\]
equals up to the constant \((2(n-1))!\) the leading coefficient of the polynomial
\[g(x) = f(x+1, \nu-1, n-1).\]
So putting the two parts together keeping in mind \([18]\), we obtain
\[
2^{2n}c(\nu-2, n) = (-1) \cdot 2^{2n}c(\nu, n) + \frac{9^{2n} \cdot \nu}{n} \cdot c(\nu-1, n-1).
\]
We have proved recursion formula \([15]\). In order to finish the induction argument, by hypothesis we assume that \([16]\) holds true for \(c(\nu, n)\) as well as for \(c(\nu-1, n-1)\).
If \(\nu\) is of the form \(\nu = n-1 - 2\mu\) these two constants are zero, so \([15]\) implies \(c(\nu-2, n) = 0\). If \(\nu = n-2\mu\) we obtain for the right hand side of \([15]\)
\[
2^{-2n}(-1)^{\mu+1}\left(C(n-\mu, \mu) - \frac{n-2\mu}{n-1} \cdot 4 \cdot C(n-1, \mu, \mu)\right)
\]
equals \[
\frac{(-1)^{\mu+1}(2(n-1-\mu))!(2\mu)!}{2^{2n}(n-1-\mu)!\mu!n!}(2(2n+1-2\mu)-4(n-2\mu))
\]
\[
\frac{(-1)^{\mu+1}(2(n-1-\mu))!(2\mu)!2(2\mu+1)(\mu+1)}{2^{-2n}(n-1-\mu)!((\mu+1))!n!}
\]
\[
\frac{(-1)^{\mu+1}}{2^{2n}}C(n-(\mu+1), \mu+1),
\]
which must equal the left hand side \(c(n-2(\mu+1), n)\) of \([15]\).
Part (b): We proceed again by Proposition \([1.1]\) to obtain the values
\[
F_j(n+1, n) = \sum_{k=0}^{j-1} (-1)^{j-1-k} f(k, n+1, n) = \begin{cases} 0 & \text{if } j = 0, \ldots, n \\ (-1)^{j-1} & \text{if } j = n+1, \ldots, 2n \end{cases},
\]
as well as the constant
\[
c = 2^{-2n}\left(\frac{(2n)!}{2n!^2} + \sum_{k=0}^{2n} (-1)^k f(k, n+1, n) \sum_{j=k+1}^{2n} \binom{2n}{j}\right)
\]
\[
= 2^{-2n}\left(\frac{1}{2} \left(\frac{2n}{n}\right)^2 + \sum_{j=n+1}^{2n} \binom{2n}{j}\right) = \frac{1}{2}.
\]
This leads to the solution polynomial

\[ F(x, n+1, n) = \sum_{j=0}^{2n} (F_j + (-1)^j c) \cdot \prod_{k=0, k \neq j}^{2n} \frac{x - k}{j - k} \]

\[ = \frac{1}{2} \sum_{j=0}^{n} (-1)^j \prod_{k=0, k \neq j}^{n} \frac{x - k}{j - k} + \frac{1}{2} \sum_{j=n+1}^{2n} (-1)^{j+1} \prod_{k=0, k \neq j}^{n} \frac{x - k}{j - k}. \]

\[ \square \]

5. Mixed summation operator

We study the action of the mixed summation operator \( \tilde{S}_1 \tilde{S}_2 \) in two variables on the functions \( P(n, m) \).

**Proposition 5.1.** For \( m = 1, 2, 3, \ldots \) there is a unique family of polynomials \( Q(m, x) \) in \( \mathbb{Q}[x] \) satisfying the following properties.

(i) For all \( m \) the degree of the polynomial is \( \deg_x Q(m, x) = 2(m - 1) \).

(ii) For all \( m \) the leading coefficient of \( Q(m, x) \) is \( \frac{2^{2m-1}}{(m-1)!^2} \).

(iii) \( Q(m, -x) = Q(m, x + 1) \) holds for all \( m \in \mathbb{N} \).

(iv) The function \( Q(m, n) \) is symmetric on \( \mathbb{N} \times \mathbb{N} \), i.e. \( Q(m, n) = Q(n, m) \).

**Proof.** Similarly to Proposition 2.1 properties (i)–(iv) uniquely determine the family of polynomials by recursion. By properties (i) and (ii), \( Q(1, x) = 2 \) is constant. By property (iii), \( Q(m, x) \) is an even polynomial in \( x - \frac{1}{2} \), so by (ii)

\[ Q(m, x) = \frac{2^{2m-1}}{(m-1)!^2} (x - \frac{1}{2})^{2(m-1)} + \sum_{j=0}^{m-2} a_{j,m} (x - \frac{1}{2})^{2j}. \]

Assuming by recursion that the polynomials \( Q(k, x) \) are defined for all \( k = 1, \ldots, m - 1 \), the values \( Q(m, k) \) are determined by (iv) for \( k = 1, \ldots, m - 1 \). This fixes \( m - 1 \) values of the polynomial \( \sum_{j=0}^{m-2} a_{j,m} y^j \) in \( y = (x - \frac{1}{2})^2 \) of degree at most \( m - 2 \). Hence the coefficients \( a_{j,m}, j = 0, \ldots, m - 2 \) are uniquely determined, and so is \( Q(m, x) \). For example, \( Q(2, x) = 8(x - \frac{1}{2})^2 \). \( \square \)
Table 3. Initial values of the function $Q : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$.

| $Q(m,n)$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|----------|-----|-----|-----|-----|-----|-----|-----|-----|
| 1        | 2   | 2   | 2   | 2   | 2   | 2   | 2   | 2   |
| 2        | 2   | 18  | 50  | 98  | 162 | 242 | 338 | 450 |
| 3        | 2   | 50  | 338 | 1250| 3362| 7442| 14450|25538|
| 4        | 2   | 98  | 1250| 7938| 33282|106722|284258|661250|
| 5        | 2   | 162 | 3362| 206082|3323042|927522|3323042|10044162|
| 6        | 2   | 242 | 7442| 106722|284258|661250|25538|103190978|
| 7        | 2   | 338 | 14450|284258|3323042|7442|14450|25538|
| 8        | 2   | 450 | 25538|661250|25538|103190978|786061250|4731504642|

In particular we obtain

$$Q(1,x) = 2,$$

$$Q(2,x) = 8(x - \frac{1}{2})^2,$$

$$Q(3,x) = 8((x - \frac{1}{2})^2 + \frac{5}{4}),$$

$$Q(4,x) = \frac{32}{9}((x - \frac{1}{2})^2((x - \frac{1}{2})^2 + \frac{5}{4}),$$

$$Q(5,x) = \frac{8}{9}((x - \frac{1}{2})^4 + \frac{7}{2}(x - \frac{1}{2})^2 + \frac{9}{16}),$$

$$Q(6,x) = \frac{32}{225}((x - \frac{1}{2})^2((x - \frac{1}{2})^4 + \frac{15}{2}(x - \frac{1}{2})^2 + \frac{299}{16}),$$

$$Q(7,x) = \frac{32}{2025}((x - \frac{1}{2})^2 + \frac{9}{4})^2((x - \frac{1}{2})^4 + \frac{23}{2}(x - \frac{1}{2})^2 + \frac{25}{16}),$$

$$Q(8,x) = \frac{128}{99225}((x - \frac{1}{2})^2((x - \frac{1}{2})^6 + \frac{91}{4}(x - \frac{1}{2})^4 + \frac{1519}{16}(x - \frac{1}{2})^2 + \frac{3429}{64}).$$

Proposition 5.2. The family of polynomials $Q(m,x)$ of Proposition 5.1 is given by

$$Q(m,x) = P(m,x) + P(m,x - 1) + P(m - 1,x) + P(m - 1,x - 1),$$

where $P(m,x)$ are the polynomials defined in Proposition 2.3. The polynomials have the following properties.

(a) The constant coefficient $Q(m,0) = 2$ equals two for all $m \geq 1$.

(b) The values $Q(m,n)$ define a symmetric function $Q : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$.

(c) For all $m \geq 1$ the polynomial $Q(m,x)$ is even in $x - \frac{1}{2}$

$$Q(m,x) = \frac{2^{2m-1}}{(m-1)!}(x - \frac{1}{2})^2 + \cdots + a_{1,m}(x - \frac{1}{2})^2 + a_{0,m}.$$

(d) For the Euler operator $\tilde{E}$ and the polynomials $A_2(x,m)$ defined in Theorem 3.1 we obtain

$$Q(m,x) = \tilde{E}(A_2(x,m) + A_2(x,m - 1)).$$
In Proposition 5.4 (c) we determine the constant coefficient $a_{0,m}$ from Proposition 5.2 (c). By numerical evidence for $1 \leq m \leq 100$, we suppose that the polynomial $2 \cdot Q(m, x)$ is a square for all $m$.

**Proof of Proposition 5.2.** The polynomials $P(m, x)$ satisfy the properties of Proposition 2.1, and by Remark 2.5(i) it holds $\deg_x P(m, x) = 2(m - 1)$ for all $m \geq 1$. Hence the sums $Q(m, x) = P(m, x) + P(m, x-1) + P(m-1, x) + P(m-1, x-1)$ satisfy properties (i),(iii), and (iv) of Proposition 5.1. From the definition of $P(m, x)$ for $m > 0$ we determine the leading coefficient of $P(m, x)$

$$
\sum_{\nu,\mu=0}^{m-1} \frac{1}{\nu!(m-1-\nu)!\mu!(m-1-\mu)!} = \left( \frac{2^{m-1}}{(m-1)!} \right)^2.
$$

It follows that the leading coefficient of $Q(m, x)$ is twice this number, hence equals $\frac{2^{2m-1}}{(m-1)!}$. Property (a) follows from Proposition 5.1. $Q(m, 0) = Q(m, 1) = Q(1, m) = 2$. Property (b) follows because all the values $P(m, n)$ are integers, see Corollary 2.2. The shape of the polynomials $Q(m, x)$ given in (c) is provided again by Proposition 5.1. Recall from Theorem 3.1

$$
S^2 P(n, x) = P(n, x+1) + 2P(n, x) + P(n, x-1) = A_2(x, n).
$$

For the shifted summation operator $\tilde{S}P(n, x) = P(n, x+1) + P(n, x)$ we therefore obtain

$$
\tilde{S} \left( P(n, x) + P(n, x-1) \right) = A_2(x, n).
$$

Hence the polynomials $Q(m, x)$ in question are also given by the Euler operator $E = \tilde{S}^{-1}$

$$
Q(m, x) = E \left( A_2(x, m) + A_2(x, m-1) \right).
$$

Using of the summation operator $S f(x) = f(x+\frac{1}{2})+f(x-\frac{1}{2})$, the explicit formula for the polynomials $Q(m, x)$ given in Proposition 5.2 can be reformulated

$$
Q(m, x + \frac{1}{2}) = S \left( P(m, x) + P(m - 1, x) \right).
$$

This suggests that all the families of polynomials we have defined in this paper should have interesting properties at half-integral places. We illustrate this by determining their values in $x = \frac{1}{2}$. We need the following lemma.

**Lemma 5.3.** For $m = 1, 2, 3, \ldots$ define the polynomials

$$
a_1(m, x) = \sum_{\nu=0}^{m} \left[ \begin{array}{c} x-1+
u \\ \nu \end{array} \right] \left[ \begin{array}{c} x \\ m-\nu \end{array} \right].
$$

Then the values in $x = \frac{1}{2}$ are given by

$$
a_1(2n, \frac{1}{2}) = a_1(2n+1, \frac{1}{2}) = \left[ \begin{array}{c} n-\frac{1}{2} \\ n \end{array} \right]
$$

for all $n \in \mathbb{N}_0$. 

Proof of Lemma 5.3. We first notice that
\[
\left[\frac{n - \frac{1}{2}}{n}\right] = \frac{1}{2^{2n}} \binom{2n}{n} = (-1)^n \left\lfloor -\frac{1}{2} n \right\rfloor.
\]
So the generating series of \(\left[\frac{n - \frac{1}{2}}{n}\right]\) is given by the Taylor series
\[
\sum_{n=0}^\infty x^n \left[\frac{n - \frac{1}{2}}{n}\right] = \sum_{n=0}^\infty (-x)^n \left[\frac{-\frac{1}{2}}{n}\right] = (1 - x)^{-\frac{1}{2}},
\]
while
\[
(1 + x)^{\frac{1}{2}} = \sum_{n=0}^\infty x^n \left[\frac{n}{2}\right].
\]
By Cauchy product expansion we obtain the generating series for \(a_1(m, \frac{1}{2})\)
\[
\sqrt{\frac{1 + x}{1 - x}} = \sum_{m=0}^\infty x^m \sum_{\nu=0}^m \left[\frac{1}{2}\nu\right] \left[\frac{m - \nu - \frac{1}{2}}{m - \nu}\right] = \sum_{m=0}^\infty x^m a_1(m, \frac{1}{2}).
\]
On the other hand, by the Taylor series above
\[
\frac{1}{\sqrt{1 - x^2}} = \sum_{n=0}^\infty x^{2n} \left[\frac{n - \frac{1}{2}}{n}\right]
\]
we obtain the expansion
\[
\sqrt{\frac{1 + x}{1 - x}} = \frac{1 + x}{\sqrt{1 - x^2}} = \sum_{n=0}^\infty \left(x^{2n} + x^{2n+1}\right) \left[\frac{n - \frac{1}{2}}{n}\right].
\]
Comparing coefficients, the lemma is proved. \(\square\)

**Proposition 5.4.**  
(a) For the polynomials \(A_1(m, x)\) and \(A_2(x, m)\) of Theorem 3.1 the values in \(x = \frac{1}{2}\) are given by
\[
A_1(2n, \frac{1}{2}) = A_1(2n + 1, \frac{1}{2}) = \left[\frac{n - \frac{1}{2}}{n}\right]^2,
\]
respectively
\[
A_2(\frac{1}{2}, m) = (2m)^2 A_1(m, \frac{1}{2}).
\]
(b) For all \(n \in \mathbb{N}_0\) define the rational number
\[
r(n) = \sum_{k=0}^n \left[\frac{k - \frac{1}{2}}{k}\right]^2 = \sum_{k=0}^n \left(\frac{1}{2^{2k}} \binom{2k}{k}\right)^2.
\]
Then the values in \(x = \frac{1}{2}\) of the polynomials \(P(m, x)\) of Proposition 2.1 are given by the recursion formula
\[
P(2n + 1, \frac{1}{2}) = r(n) = (-1) \cdot P(2n + 2, \frac{1}{2}).
\]
(c) The values in \( x = \frac{1}{2} \) of the polynomials \( Q(m, x) \) are given by \( Q(2n, \frac{1}{2}) = 0 \), respectively

\[
Q(2n + 1, \frac{1}{2}) = 2 \left[ n - \frac{1}{2} \right]^2.
\]

Proof of Proposition 5.4. Because \( A_1(m, x) = a_1(m, x)^2 \) the first identity of part (a) follows from Lemma 5.3. For the second identity recall from Proposition 3.3 (a) that \( m^2 A_1(m, x) = x^2 A_2(x, m) \). For part (b) we proceed by induction. By the list of \( P(m, x) \) following Proposition 2.1 it holds \( P(1, \frac{1}{2}) = 1 = r(0) = -P(2, \frac{1}{2}) \) and \( P(3, \frac{1}{2}) = \frac{5}{4} = r(1) = P(4, \frac{1}{2}) \). Assume formula (21) holds true for all \( n < N \). Then by identity (1) of Theorem 3.1

\[
P(2N + 1, \frac{1}{2}) + 2P(2N, \frac{1}{2}) + P(2N - 1, \frac{1}{2}) = A_1(2N, \frac{1}{2}),
\]

it follows

\[
P(2N + 1, \frac{1}{2}) = \left[ N - \frac{1}{2} \right] + 2r(N - 1) - r(N - 1) = r(N).
\]

Similarly, from

\[
P(2N + 2, \frac{1}{2}) + 2P(2N + 1, \frac{1}{2}) + P(2N, \frac{1}{2}) = A_1(2N + 1, \frac{1}{2})
\]

we deduce

\[
P(2N + 2, \frac{1}{2}) = \left[ N - \frac{1}{2} \right] - 2r(N) + r(N - 1) = -r(N).
\]

Concerning part (c), by (20) it holds

\[
Q(m, \frac{1}{2}) = P(m, \frac{1}{2}) + P(m, -\frac{1}{2}) + P(m - 1, \frac{1}{2}) + P(m - 1, -\frac{1}{2}).
\]

Recalling the polynomials \( P(m, x) \) are even functions, we obtain

\[
Q(m, \frac{1}{2}) = 2(P(m, \frac{1}{2}) + P(m - 1, \frac{1}{2})).
\]

By part (b), this is zero in case \( m = 2n \) is even, whereas in case \( m = 2n + 1 \) we obtain \( Q(2n + 1, \frac{1}{2}) = 2(r(n) - r(n - 1)) = 2 \left[ n - \frac{1}{2} \right]^2 \). \( \square \)

By the set of initial values \( P(m, \frac{1}{2}) = P(m, -\frac{1}{2}), A_1(m, \frac{1}{2}), \) and \( A_2(\frac{1}{2}, m) \) for all \( m \in \mathbb{N} \), we obtain a recursion for the values \( P(m, \frac{2k+1}{2}), A_1(m, \frac{2k+1}{2}), \) and \( A_2(\frac{2k+1}{2}, m) \) for all \( k \in \mathbb{N} \) as follows. Identity (7) implies for all \( m \in \mathbb{N} \)

\[
P(m, \frac{2k+1}{2}) = A_2(\frac{2k+1}{2}, m) - 2P(m, \frac{2k+1}{2}) - P(m, \frac{2k+3}{2}).
\]

Inserting this into identity (3) yields for all \( m \in \mathbb{N} \)

\[
A_1(m, \frac{2k+1}{2}) = P(m + 1, \frac{2k+1}{2}) + 2P(m, \frac{2k+1}{2}) + P(m - 1, \frac{2k+1}{2}).
\]
Then by Proposition 3.3 (a)
\[ A_2 \left( \frac{2k+1}{2}, m \right) = \left( \frac{2m}{2k+1} \right)^2 A_1 \left( m, \frac{2k+1}{2} \right), \]
which closes the recursion cycle. By (20), we obtain the values \( Q(m, \frac{2k+1}{2}) \) as well.

We remark another polynomial identity which arises from the above considerations for the polynomials \( Q(m, x) \). We rearrange
\[
A_2(x, m) = \left( \sum_{\mu=0}^{m} \left[ \frac{x + m - 1 - \mu}{m - 1} \right] \binom{m}{\mu} \right)^2 \\
= \sum_{\mu=0}^{m} \binom{m}{\mu}^2 f(x + m - 1 - \mu, 0, m - 1) \\
+ 2 \cdot \sum_{0 \leq \nu < \mu \leq m} \binom{m}{\nu} \binom{m}{\mu} f(x + m - 1 - \nu, \mu - \nu, m - 1).
\]
Here the polynomials \( f(x, \nu, m - 1) \) were defined in Proposition 4.4 where we determined their preimages under \( \tilde{S} \). Hence we may read off the following proposition.

**Proposition 5.5.** For the polynomials \( A_2(x, m) \) defined in Theorem 3.1 it holds true
\[
\tilde{E}(A_2(x, m)) = \sum_{\mu=0}^{m} \binom{m}{\mu}^2 F(x + m - 1 - \mu, 0, m - 1) \\
+ 2 \cdot \sum_{0 \leq \nu < \mu \leq m} \binom{m}{\nu} \binom{m}{\mu} F(x + m - 1 - \nu, \mu - \nu, m - 1),
\]
where the polynomials \( F(x, \nu, m - 1) \) were defined in Proposition 4.4.

**Corollary 5.6.** There is a non-trivial polynomial identity given by \( P(m, x) + P(m, x - 1) = \tilde{E}(A_2(x, m)) \),
\[
\sum_{\nu, \mu=0}^{m-1} \frac{t(\nu, \mu, m; x)t(\mu^*, \nu^*, m; x) + t(\nu, \mu, m; x - 1)t(\mu^*, \nu^*, m; x - 1)}{\nu!\nu^*!\mu!\mu^*!} \\
= \sum_{\mu=0}^{m} \binom{m}{\mu}^2 F(x + m - 1 - \mu, 0, m - 1) \\
+ 2 \cdot \sum_{0 \leq \nu < \mu \leq m} \binom{m}{\nu} \binom{m}{\mu} F(x + m - 1 - \nu, \mu - \nu, m - 1). \]


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