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CHAOS, REGULARITY, AND NOISE IN
SELF-GRAVITATING SYSTEMS

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Abstract

This paper summarises a number of new, potentially significant, results, obtained
recently by the author and his collaborators, which impact on various issues related
to the gravitational $N$-body problem, both Newtonianly and in the context of general
relativity.
1 Introduction and motivation

The overall objective of the research reported herein is the application of ideas and techniques from modern nonlinear dynamics and nonequilibrium statistical mechanics to self-gravitating systems, both Newtonianly and in the context of general relativity, with particular emphasis on the gravitational $N$-body problem. The basic motivation for this research is a desire to identify some of the physical processes which can play a role in determining the structure and evolution of self-gravitating systems. The results described here will, for specificity, typically be formulated in the language of galactic dynamics. However, it should be evident that they also have potential implications in other settings as well, including, e.g., statistical quantum field theory in the early Universe.

It should be stressed that the principal focus here is not on the detailed modeling of any specific class of astronomical object where, in particular, other nongravitational effects, such as dissipative hydrodynamics, can be important. However, the results reported here should find relatively direct applications to the study of systems like elliptical and lenticular galaxies, which are known to be gas poor, albeit not as gas poor as they were ten years ago.

In approaching the study of self-gravitating systems, there are several different approaches which one might adopt. At the most fundamental level, one can attack the full $N$-body problem, either by performing and analysing numerical simulations or by proving (hopefully useful) mathematical theorems which provide insights into the qualitative character of the evolution. In either case, the focus here is not on solar system type problems, involving a relatively small number of masses, one or two of which are much larger than the others. Rather, the principal focus is on collections of large numbers $N$ of objects, comparable in mass, in particular the problem of the $N \to \infty$ limit.

Conventional wisdom says that, in this large $N$ limit, such a collection of comparable masses can be described by a collisionless Boltzmann equation, i.e., what the mathematician would term the Vlasov-Poisson or Vlasov-Einstein system. Such a description involves the assumption that, Newtonianly, particles follow trajectories in the self-consistent gravitational potential associated with the average mass distribution. Relativistically, one supposes that the particles follow geodesics in a spacetime, the form of which is determined by the stress energy tensor associated with the average mass distribution. In this general context, two things need to be done, namely (1) to determine precisely the conditions under which such a mean field description is justified and then (2) to understand the qualitative and quantitative implications of this description.
In this connection, it is important to stress that, although the Vlasov-Poisson system was first formulated by Jeans in the context of galactic dynamics\cite{1} more than twenty years before it was formulated in plasma physics\cite{2}, the gravitational equation is substantially less well motivated than is the plasma analogue. In particular, the absence of shielding prevents a systematic derivation except for the special case of cosmological systems, which are nearly homogeneous and of finite age.\cite{3} The problem becomes especially acute for the case of relativistic systems, the point being that the derivation of the corresponding Vlasov-Maxwell system relies heavily on the linearity of Maxwell’s equations\cite{4}, whereas Einstein’s equation is nonlinear.

A yet simpler tact involves the consideration of orbits in a fixed potential. The idea here is to specify one’s favourite potential, chosen to represent the bulk potential of some physical object, to study in detail the form of orbits in this potential, and only at the end of the day to incorporate the fact that the potential must be determined self-consistently by the mass distribution associated with the orbits themselves.

If one chooses to focus simply on an average potential, in the context of a Vlasov description or otherwise, there remains the important question of determining precisely, both qualitatively and quantitatively, what sorts of effects have been ignored. In other words, there remains the problem of structural stability. Only to the extent that these additional effects have been appropriately identified and understood can one say that one has a satisfactory understanding of so-called “collisionless stellar dynamics.”

The aim of this talk is to illustrate each of the preceding aspects by explaining several new results that have been derived during the past three or four years. Section 2 focuses on the full Newtonian $N$-body problem. Section 3 then turns to the collisionless Boltzmann equation. Section 4 addresses several issues related to the problem of orbits in a fixed potential, and Section 5 concludes with a discussion of some aspects of the problem of structural stability.

## 2 The gravitational $N$-body problem

Detailed numerical simulations over the past thirty years have established that, given generic initial conditions corresponding to a bound configuration, a self-gravitating system of point masses will typically exhibit a rapid approach, on a characteristic crossing time $t_{cr}$, towards a statistical quasi-equilibrium where, in a time-averaged sense, the system only shows subsequent systematic variability on substantially longer timescales. Moreover, there has evolved a substantial and detailed conventional wisdom which serves, at least roughly, to determine what kinds of initial data give rise to
what kinds of final quasi-equilibria. However, despite these impressive successes, one does not completely understand this process. Indeed, at a truly fundamental level there is no real understanding of why this happens.

To obtain some insights into this basic question, it is natural to identify various microscopic and mesoscopic phenomena which could perhaps conspire to produce the qualitative macroscopic evolution that is observed in numerical experiments. The aim here is to identify two such phenomena.

1. Viewed microscopically in the many particle phase space, Newtonian \( N \)-body simulations exhibit an exponentially sensitive dependence on the specific choice of initial conditions. Specify unperturbed initial data, \( \{ r^u_A(0) \ p^u_A(0) \} \), \( (A = 1, ..., N) \), and perform a simulation. Then specify perturbed initial data, \( \{ r^p_A(0) \ p^p_A(0) \} \), and repeat. What one discovers thereby is that, generically, quantities like the total \( N \)-particle configuration space perturbation

\[
\sum_{A=1}^{N} |\delta r_A(t)|^2 \equiv \sum_{A=1}^{N} |r^p_A(t) - r^u_A(t)|^2
\]

will typically grow exponentially in time \( t \) on a relatively short time scale, even though the unperturbed and perturbed evolution are essentially identical at the macroscopic level.

This fact was first observed by S. Ulam in the early 1960's, and the classic paper on the subject is by Miller. However, only in the last several years, with the advent of improved computers, has this instability been studied systematically in complete and gory detail.

The net result of these investigations is that this instability is an exceedingly robust phenomenon, which proceeds generically on a characteristic crossing time \( t_{cr} \), independent of many/most details. In particular, the timescale associated with this instability is independent of the detailed choice of initial data and the detailed choice of the initial perturbations, as well as the specific diagnostics used to quantify the growth of the perturbation – e.g., configuration or momentum space perturbation, the total \( N \)-particle perturbation or the perturbation of “typical” or “representative” particles. The timescale is also insensitive to a possible distribution of masses, provided that everything is not dominated by a few particularly massive particles.

More significantly, the simulations also suggest strongly that the rate is insensitive to the total particle number \( N \), provided at least that \( N \gg 2 \). Thus, e.g., for \( 200 \leq N \leq 4000 \) one observes no appreciable changes if everything is scaled in terms of an \( N \)-dependent characteristic time \( t_{cr} \). In particular, there is no sense in which the instability appears to “turn off” in the limit of large \( N \). Indeed, Goodman \textit{et al} have predicted that the instability should accelerate for large \( N \), with the character-
istic timescale \( t_* \) scaling as \( t_* \sim t_{cr}/\ln N \). Interestingly, \( t_{cr} \) is the same timescale on which generic initial data evolve towards a macroscopic quasi-equilibrium.

2. **Gravitational \( N \)-body simulations evidence a considerable “memory.”** Viewed microscopically or mesoscopically, the quasi-equilibrium does not correspond to a particularly well-shuffled state. Many aspects of the initial data for individual particles are forgotten, but other aspects are in fact remembered. The strongest correlation between initial and final conditions, from which all others may perhaps derive, is between the initial and final values of the binding energy. Both for *isolated systems* approaching a quasi-equilibrium and for *collisions* between pairs of objects, e.g., spherical polytropes and other axisymmetric or triaxial near-equilibria, there is a strong correlation between the initial and final values of the binding energy.

Specifically, one observes that particles initially with high binding energy tend to end up with high binding energy, low with low, and intermediate with intermediate, even for an evolution that involves rapid, violent changes in the bulk potential, so that there is no obvious sense in which the binding energy should behave as an adiabatic invariant.\(^{[11, 12, 13, 14]}\) This phenomenon can be quantified at a coarse-grained mesoscopic level, through various binnings of the orbital data.\(^{[13, 14]}\) Alternatively, it can be quantified at the completely microscopic level through the computation of a rank correlation between the initial and final *orderings* of particles in terms of their binding energies.\(^{[14]}\) Such a computation shows that there exist strong, albeit not complete, correlations between initial and final conditions. The absence of a complete correlation is at least partly “collisional” in origin, but appears to persist even in the limit of large \( N \), where, according to conventional wisdom, the system should be essentially collisionless in character.

To summarise: In the gravitational \( N \)-body problem one is confronted with a system that exhibits a rapid macroscopic evolution towards a statistical quasi-equilibrium. Viewed microscopically, this evolution is characterised by an exponentially sensitive dependence on the specific choice of initial conditions. However, despite this sensitive dependence, the quasi-equilibrium does not correspond, either microscopically or mesoscopically, to a particularly well-shuffled state.

### 3 The collisionless Boltzmann equation

The principal message of this section is that the *collisionless Boltzmann equation*, i.e., a mean field Vlasov description, is *a constrained Hamiltonian system*. This fact was first established by Morrison\(^{[13]}\) for the electrostatic Vlasov-Poisson system, and is equally true for the analogous gravitational Vlasov-Poisson system. In this setting, the
The fundamental dynamical variable is the one-particle distribution function, \( f(x, p, m) \), evaluated at a fixed time \( t \), which is interpreted as a number density of particles of mass \( m \) at the spatial point \( x \) with conjugate momentum \( p \). The gravitational potential \( \Phi(t) \) is viewed as a functional of \( f(t) \), constructed in terms of an appropriate Green function. The phase space is then the infinite-dimensional space of distribution functions. The Hamiltonian character of the evolution is manifest through the identification of a Hamiltonian function \( H[f] \) and a cosymplectic structure \( \{ \dots \} \), in terms of which the Vlasov equation takes the form \( \partial f / \partial t = \{ f, H \} \).

A generalisation to the spherically symmetric Vlasov-Einstein system is completely straightforward. [16] The crucial point is that, for spherical systems, the gravitational degrees of freedom are not triggered: given appropriate boundary conditions, one can view the spacetime metric \( g_{ab} \) at any given time \( t \) as a functional of the distribution function \( f \) at that same \( t \).

The full Vlasov-Einstein system is substantially more complicated, since, in the general case, one must incorporate the field degrees of freedom. However, the analysis still turns out to be straightforward, at least in principle. [17] The basic formulation is analogous to the Hamiltonian formulation of the Vlasov-Maxwell system, [18, 19] the nonlinearity of the Einstein equation not playing a significant role.

Working in the context of the ADM formulation of general relativity, there are now three different dynamical variables, each defined on \( t = \text{const} \) hypersurfaces, namely

1. the distribution function, \( f(x, p, m) \),
2. the spatial three-metric, \( h_{ab}(x) \), and
3. the conjugate field momentum, \( \Pi^{ab}(x) \).

The natural phase space is the infinite-dimensional space coordinatised by these three variables.

In this case the cosymplectic structure given as the sum of two pieces, namely

1. the functional Poisson bracket of vacuum gravity and
2. the matter bracket appropriate for the spherically symmetric Vlasov-Einstein system (which coincides also with the bracket for the Vlasov-Poisson system).

Explicitly, for two functions \( F[f, h_{ab}, \Pi^{ab}] \) and \( G[f, h_{ab}, \Pi^{ab}] \),

\[
\langle F, G \rangle = 16\pi \int d^3x \left( \frac{\delta F}{\delta h_{ab}} \frac{\delta G}{\delta \Pi^{ab}} - \frac{\delta G}{\delta h_{ab}} \frac{\delta F}{\delta \Pi^{ab}} \right) + \int d^3x d^3p d\mu \frac{\delta F}{\delta f} \frac{\delta F}{\delta f},
\]

where \( \delta / \delta X \) denotes a functional derivative with respect to the variable \( X \) and

\[
[f, g] = \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x^i} \frac{\partial f}{\partial p_i}
\]

denotes an ordinary three-dimensional Poisson bracket.

To give meaning to variations \( \delta X \), one requires a rule identifying particle coordinates \( \{ x^a, p_a \} \) and \( \{ x'^a, p'_a \} \) in two nearby cotangent bundles. In the context of this
3+1 formulation, it is natural to identify spatial coordinates and conjugate momenta, as well as time $t$ and mass $m$, i.e.,

$$x' = x \quad p' = p \quad t' = t \quad m' = m.$$ (4)

However, other choices are also possible.\[20, 21\]

The Hamiltonian $H = H_G + H_M = \int d^3x H_G + \int d^3x H_M$ is also given as the sum of two pieces, namely (1) the ADM Hamiltonian $H_G$ of vacuum gravity, i.e.,

$$H_G = \frac{1}{16\pi} h^{1/2} \left\{ N \left[ -(3) R + h^{-1}(\Pi^{ab}\Pi_{ab} - \frac{1}{2}\Pi^2) \right] - 2N_b[D_a(h^{-1/2}\Pi^{ab})] + 2D_a(h^{-1/2}N_b\Pi^{ab}) \right\}$$ (5)

and (2) a matter Hamiltonian $H_M$ constructed as the integral of the local energy density, i.e.,

$$H_M = \int d\Gamma fE = \int N h^{1/2} d^3x T^t_t.$$ (6)

Here $d\Gamma = d^3xd^3p dm$ denotes the covariant seven-dimensional volume element on a $t = \text{const}$ hypersurface, $D_a$ a covariant derivative on the hypersurface, and $E(x, p, m) = |p_t|$ the particle energy. $N$ and $N_a$ correspond respectively to the lapse function and shift vector.

This formulation reproduces the Vlasov-Einstein system in the sense that the equation $\partial F/\partial t = \langle F, H \rangle$ for arbitrary functions $F[f, h_{ab}, \Pi^{ab}]$ is equivalent to the Vlasov-Einstein system in its usual 3+1 form: The distribution function $f$ satisfies $\partial f/\partial t = [E, f]$, and $\partial h_{ab}/\partial t$ and $\partial \Pi^{ab}/\partial t$ both satisfy the appropriate 3+1 Einstein equations sourced by $T^a_b[f]$. For the spherically symmetric case, with the metric viewed as a functional of $f$, the first term in the bracket $\langle F, G \rangle$ disappears and the Hamiltonian $H$ reduces to the ADM mass, $H_{ADM}$, realised as a volume integral in the natural fashion facilitated by Schwarzschild coordinates.

Such a Hamiltonian formulation of the collisionless Boltzmann equation is in fact an example of a much more general result. Specifically, consider any Hamiltonian theory for a system with multiple degrees of freedom, and then construct the associated mean field description appropriate in the limit that correlations amongst the degrees of freedom are negligible (i.e., a statistical description in which the full $N$-“particle” distribution function is approximated as factorising into a product of reduced one-“particle” distribution functions). There is then a precise mathematical sense in which the mean field description of this Hamiltonian system is itself Hamiltonian.\[22\]

The fact that the collisionless Boltzmann equation, or any other mean field theory, is Hamiltonian is significant in that a Hamiltonian evolution is much more restricted
than a non-Hamiltonian evolution. However, of more pragmatic importance perhaps is the fact that this Hamiltonian formulation permits, for the first time, a clear geometric approach to the problem of stability for general equilibrium solutions to the Vlasov-Einstein system. Here an “equilibrium solution” \( \{ f_0, h_{ab}^0, \Pi_{ab}^0 \} \) entails a stationary matter distribution, corresponding to a spacetime that admits a timelike Killing field.

The crucial fact facilitating this geometric approach is that every such equilibrium is an energy extremal, so that the first variation \( \delta^{(1)} H \) vanishes identically for perturbations \( \{ \delta f, \delta h_{ab}, \delta \Pi_{ab} \} \) which satisfy the constraints. The field constraints are enforced by restricting attention to perturbed initial data for which \( \delta H/\delta N = \delta H/\delta N^a = 0 \). The matter constraints, a reflection of Liouville’s Theorem, imply that the perturbed distribution function must be generated from the unperturbed \( f_0 \) via a canonical transformation in terms of some generating function \( h \), i.e., \( f_0 + \delta f = \exp(\{ h, \cdot \}) f_0 = f_0 + \{ h, f_0 \} + \frac{1}{2} \{ h, \{ h, f_0 \} \} + ..... \)

The fact that the equilibrium is an energy extremal implies that stability hinges on the sign of the second variation \( \delta^{(2)} H \). Indeed, modulo infinite-dimensional technicalities the situation is analogous to the problem of stability for mechanical Hamiltonian systems. If \( \delta^{(2)} H > 0 \) for all infinitesimal perturbations, linear stability is guaranteed. Alternatively, if \( \delta^{(2)} H \) is indeterminate, one cannot necessarily infer a linear instability, but one does expect at least nonlinear instability and/or instability in the presence of dissipation.\(^{[23]}\)

Indeed, one can actually prove that generic rotating axisymmetric equilibria are always unstable towards dissipation, as provided, e.g., by the emission of gravitational radiation.\(^{[24]}\) This is the collisionless analogue of the theorem\(^{[25]}\) that all rotating perfect fluid stars are unstable. Neither for collisionless nor collisional systems is there any guarantee that the timescale associated with this instability is sufficiently short to be of interest astronomically. However, it is significant that general relativity triggers a generic instability which, apparently, is insensitive to the form of the self-gravitating matter. The astronomical implications of this instability are currently under investigation.

This general approach to stability can also be adapted to the consideration of steady-state equilibria, such as an homogeneous and isotropic Friedman cosmology, where it provides an interesting derivation of the Jeans instability.\(^{[26]}\) Viewed Newtonianly, such an expanding Universe corresponds in the “inertial” frame to a system characterised by a time-independent Hamiltonian which finds itself in a time-dependent steady state. This explicit time-dependence can be removed by effecting a time-dependent canonical transformation into the average “comoving” frame. This
transformation leads to a new time-dependent Hamiltonian \( H(t) \). However, the first variation \( \delta^{(1)} H \) vanishes identically, and the second variation \( \delta^{(2)} H \) can be shown to satisfy \( d\delta^{(2)} H/dt \leq 0 \). A simple Liapounov argument therefore guarantees that the existence of negative energy perturbations \( \delta^{(2)} H < 0 \) for sufficiently long wavelengths must imply an instability: If the system be perturbed in such a fashion that \( \delta^{(2)} H < 0 \), the energy can only become more negative, so that the “distance” from equilibrium, as probed by the magnitude of \( \delta^{(2)} H \), can only increase.

4 Transient Ensemble Dynamics

In studying the properties of orbits in a fixed potential, it is natural to apply the technology of nonlinear dynamics, as has been developed over the past several decades. However, in many settings involving gravitational systems, the utilisation of this technology may require a new twist. The standard formulation of nonlinear dynamics typically involves a theory of asymptotic orbital dynamics, which focuses primarily, if not exclusively, on the long time behaviour of individual orbits. However, for many astronomical systems this may not be appropriate.

For example, in terms of their natural timescale, galaxies are relatively young objects, only \( \sim 100 - 200 \) crossing times \( t_{cr} \) in age, so that it is not at all obvious that an asymptotic \( t \to \infty \) limit is well motivated physically. Thus, e.g., standard estimates of Liapounov exponents typically require integrations for times \( t \geq 10^4 t_{cr} \), a period that is orders of magnitude longer than the age of the Universe, \( t_U \). Moreover, it is arguably true that, in many astronomical systems, individual orbits are not the fundamental objects of interest. It is, for example, obvious that one cannot track individual orbits of stars within a galaxy. All that one can detect are properties like the overall brightness distribution which reflect the contributions of many stars. Similarly, it is evident that one must focus on collections of orbits if he or she wishes ultimately to address the problem of self-consistency.

For these reasons, it would seem more natural to consider instead a theory of transient ensemble dynamics, which focuses on the statistical properties of ensembles of orbits, restricting attention exclusively to short timescales, \( t < t_H \), and recognising that much of the observed behaviour may be intrinsically transient in character.

This distinction may not be all that important for integrable, or near-integrable potentials, which contain only regular orbits. However, there is no reason to assume that the bulk potential associated with a self-gravitating system is integrable, or even near-integrable, and there are indications from numerical experiments that objects like rotating elliptical galaxies and barred spiral galaxies may admit large numbers
of stochastic, or chaotic, orbits.\[27, 28\]

Roughly, regular orbits correspond to orbits that have regular shapes and are characterised by simple topologies, e.g., box orbits in two dimensions with trajectories that resemble a Lissajous figure or tube orbits in three dimensions that are restricted to a region with the topology of a torus. By contrast, stochastic orbits are manifestly irregular in shape and appear to “run all over” phase space. Unlike regular orbits, stochastic orbits exhibit an exponentially sensitive dependence on initial conditions, as manifest by the fact that such orbits are characterised by a positive Liapounov exponent.\[29\]

The crucial point to be illustrated below is that, at least when considering stochastic orbits, the transient ensemble perspective can prove extremely important.

Consider, for example, the scattering of photons incident on a multi-black hole system; or similarly, consider a star moving in a nonspherical potential, supposing that the star is unbound but that, owing to the shape of the potential, it can only escape in certain directions. For each of these problems, one discovers that, in certain phase space regions, the direction and time of escape to infinity exhibits a very sensitive dependence on initial conditions, in fact a fractal dependence, this being an example of what the nonlinear dynamicist would call chaotic scattering.\[30\]

Naively, one might anticipate that this complex microscopic behaviour would lead to an equally complex description when considering the evolution of ensembles of orbits. However, this is not always the case. At least in certain cases, one observes instead striking regularities, which lead to a simple scaling behaviour,\[31\] that may actually be universal.\[32\] The very fact that the microscopic evolution is complex appears to be responsible for the fact that the macroscopic evolution is very simple.

As a concrete example, consider orbits in the two-dimensional potential $V(x, y) = \frac{1}{2}(x^2 + y^2) - \epsilon x^2 y^2$, holding fixed the value of the energy $E$ and studying as a function of $\epsilon$ the evolution of orbits initially localised in a small phase space region. For $\epsilon$ below a critical value $\epsilon_1 = 1/(4E)$, escape is impossible energetically. For values of $\epsilon$ slightly above $\epsilon_1$, escape is possible energetically, but only very few particles escape on short timescales and the time of escape exhibits no striking regularities, except that the escape probability eventually appears to decay towards zero. However, for $\epsilon$ above another critical value $\epsilon_2$ (only determined numerically), one sees the onset of striking scaling behaviour:

1) After the decay of initial transients, the escape probability per unit time approaches a constant value $P_\infty(\epsilon)$, which is independent of initial conditions, i.e., the location of the phase space region from which the initial ensemble was chosen. Moreover, this rate scales as $P_\infty(\epsilon) \sim (\epsilon - \epsilon_2)^\alpha$ for a critical exponent $\alpha$. 

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2) For fixed size of the initial phase space region probed by the ensemble, the time $T_\infty$ required to converge to $P_\infty$ also depends on $\epsilon$ and satisfies $T_\infty(\epsilon) \sim (\epsilon - \epsilon_2)^{-\beta}$.

3) For fixed $\epsilon$, the convergence time $T_\infty$ depends on the linear size $R$ of the phase space region that was probed initially, satisfying $T_\infty(R) \sim R^{-\delta}$.

Moreover, to within statistical errors $\alpha - \beta - \delta \approx 0$. In a certain sense, the qualitative change in behaviour at $\epsilon = \epsilon_2$ is like a phase transition, complete with a critical “slowing down” as the “order parameter” $\epsilon - \epsilon_2 \to 0$.

As another example in which the transient ensemble perspective is important, consider the behaviour of ensembles of orbits of fixed energy $E$, evolving in a two-dimensional time-independent potential $V(x, y)$, which admits both regular and stochastic orbits. If $V(x, y)$ is bounded from below and diverges at infinity, the constant energy hypersurfaces will be compact, so that the notion of “equilibrium” is well defined. One might therefore anticipate that generic ensembles of initial conditions will evolve towards an invariant distribution corresponding to a statistical equilibrium.

To test this hypothesis, one can select localised ensembles of initial conditions of fixed $E$, corresponding to stochastic orbits initially located far from any regular phase space regions, and then evolve these initial data into the future. At least for certain potentials, one then observes that the orbits do indeed disperse in such a fashion as to exhibit a coarse-grained evolution towards a quasi-equilibrium, which is at least approximately time-independent. This approach is, moreover, exponential in time and characterised by a rate $\Lambda(E)$ which is independent of the specific choice of initial data. The characteristic timescale $t_* = \Lambda^{-1}$ is typically $\ll 100t_{cr}$, so that, in “physical units” for a galaxy, $t_* \ll t_H$.

One also observes that the rate $\Lambda(E)$ is comparable in magnitude to the value of the Liapounov exponent $\chi(E)$, which probes the average rate of instability exhibited by orbits of energy $E$. There is, moreover, a direct correlation between $\Lambda$ and $\chi$ in the sense that, e.g., both have similar curvatures when viewed as functions of $E$. This is particularly tantalising in view of the fact that, for the $N$-body problem, the timescale associated with the approach towards a statistical quasi-equilibrium is comparable in magnitude to the timescale associated with the instability towards small changes in initial conditions.

Despite these regularities, there is no guarantee that this apparent equilibrium coincides with the true invariant distribution, and, in general, it will not! Astronomers are well acquainted with the fact that collisionless equilibria do not constitute true $N$-body equilibria, which must incorporate discreteness effects that become important on sufficiently long timescales. However, the key point here is different, and more fundamental: Even motion in a smooth two-dimensional potential may not yield a
uniform approach towards a true statistical equilibrium. The explanation of the discrepancy between the true and approximate equilibria is simple. Viewed over sufficiently long timescales, there are only two different classes of orbits, namely regular orbits, with vanishing Liapounov exponent $\chi$, and stochastic orbits, for which $\chi > 0$. The distinction between these two classes is, moreover, absolute, since members of the two different classes are separated by invariant KAM tori. If, e.g., one were to compute a surface of section, plotting coordinates $x$ and $p_x$ for successive intersections of the $y = 0$ hyperplane, he or she would generically find islands of regularity embedded in a surrounding stochastic sea.

However, this is not the whole story. Lurking in the shallows of the stochastic sea, slightly away from the shore, are cantori, these corresponding to fractured KAM tori, associated with the breakdown of integrability, which contain a cantor set of holes. The point then is that these cantori serve as partial barriers that divide the stochastic orbits into two subclasses, namely confined, or sticky, stochastic orbits which are trapped near the regular islands, and unconfined, or filling, stochastic orbits which travel unimpeded throughout the rest of the stochastic sea.

Because of the holes in the cantori, these barriers are not absolute, so that orbits can in fact change from one class to another via so-called intrinsic diffusion. However, this process is a slow one, requiring orbits to wend their way through a maze (cf. the “turnstile model” of MacKay, Meiss, and Percival, so that the characteristic timescale is typically $\gg 100 t_{cr}$, i.e., much longer than the age of the Universe.

What this implies is that, on short times, ensembles of orbits initially outside the cantori will evolve towards a near-invariant distribution which uniformly populates the filling regions, but avoids the confined regions. The situation is analogous to the classical effusion problem. Consider two evacuated cavities connected one with another by an extremely narrow conduit, and suppose that gas is inserted into one of the cavities. If the conduit be sufficiently narrow, the timescale on which gas effuses from one cavity to the other will be much longer than the timescale on which the gas spreads to fill the original cavity. This implies, however, that, even though the true equilibrium corresponds to a uniform density concentration throughout both cavities, one can speak meaningful of a shorter time quasi-equilibrium, in which the original cavity is populated uniformly and the other is essentially empty.

Significantly, these two different populations of stochastic orbits are fundamentally dissimilar in terms of their stability properties as well as where they are located in phase space. Although both sticky and filling stochastic orbits are exponentially unstable, there is a precise sense in which the sticky orbits are less unstable overall than are the filling orbits. Specifically, if one computes local Liapounov exponents.
\( \chi(\Delta t) \), for different ensembles of stochastic orbits, integrating for some relatively short interval \( \Delta t \), he or she will find\cite{34, 36} that the typical \( \chi(\Delta t) \) for a sticky orbit is substantially smaller than the typical \( \chi(\Delta t) \) for a filling orbit. Indeed, the composite distribution of local Liapounov exponents (i.e., distribution of instability timescales) generated from a sampling of the true invariant measure appears to be given, at least approximately, as a sum of two different near-Gaussian distributions with unequal means.

It should perhaps be noted explicitly that the general conclusions recounted in this Section have been observed for several different potentials, with rather different symmetries, including (1) the dihedral \( D_4 \) potential of Armbruster, Guckenheimer, and Kim,\cite{40} (2) the sixth order truncation of the three-particle Toda\cite{41} lattice potential, and (3) a generalised anisotropic Kepler potential of the form

\[
V(x, y) = -\frac{1}{(1 + x^2 + y^2)^{1/2}} - \frac{m}{(1 + x^2 + ay^2)^{1/2}},
\]

with constant \( m \) and \( a \), for \( E < 0 \). The fact that these diverse potentials, which are fundamentally different in appearance, yield similar conclusions, both qualitatively and semi-quantitatively, would suggest strongly that these conclusions are robust, depending only on such topological features as the existence of KAM tori and cantori.

The existence of confined stochastic orbits is of potential importance astronomically because such orbits can help (the theorist) support various sorts of structures, e.g., bars in a spiral galaxy. It is natural to assume that, in systems like galaxies, regular orbits serve to provide the skeleton to support various structures. However, because of resonance overlap one may find that, near corotation and other resonances, the desired regular orbits do not exist, even though sticky stochastic orbits are present.

Finally, it should be stressed that one can observe similar short time “zones of avoidance” in higher dimensional systems as well. The key point physically is that just because a region of phase space is connected, so that orbits can pass throughout the entire region, does not mean that all of the region will be accessed on comparable timescales.

5 Structural stability of the smooth potential approximation

The collisionless Boltzmann equation is a Hamiltonian system which neglects various realistic non-Hamiltonian irregularities that must be present in any self-gravitating system. One obvious point is that such a Vlasov description neglects entirely all
discreteness effects, i.e., “collisions,” by idealising the system as a continuum, rather than a collection of nearly point mass objects. Viewed in the $N$-particle phase space, the statistical description of an isolated $N$-body evolution is of course Hamiltonian. However, when projected into the reduced one-particle phase space, any allowance for particle-particle correlations that transcend a mean field description necessarily breaks the Hamiltonian constraints.[22] Another point, perhaps less obvious but equally important, is that a Vlasov description also neglects any couplings to an external environment. In the past, astronomers have been wont oftentimes to pretend that galaxies exist in splendid isolation but, over the past several decades, it has become increasing evident that such an approximation may not be justified.[42]

Detailed modeling of these sorts of perturbing influences may prove extremely complex. In particular, an external environment can give rise to a variety of different effects characterised by a broad range of timescales. Those influences proceeding on timescales $\sim t_{\text{cr}}$ will be particularly complicated, in that the details of their effects may depend very sensitively on the details of the environment. However, there is a well-established paradigm in statistical physics,[43, 44] dating back to the beginning of the century,[45] which would suggest that irregularities proceeding on shorter timescales, $\ll t_{\text{cr}}$, can oftentimes be modelled as friction and noise, related via a fluctuation-dissipation theorem. This idea underlies, for example, Chandrasekhar’s original formulation[46] of so-called “collisional stellar dynamics.”

It is therefore natural to investigate the structural stability of Hamiltonian trajectories towards the effects of friction and noise. This was done[47, 48] by effecting large numbers of *Langevin simulations*, in which the deterministic equations of motion were perturbed by allowing for (1) a dynamical friction $-\eta p$, which serves systematically to remove energy from the orbits and (2) random kicks, modeled as white noise with temperature, or mean squared velocity, $\Theta$, which serve systematically to pump energy back into the orbits. As a first simple test, $\eta$ was assumed to be constant, in which case the fluctuation-dissipation theorem implies that the noise must be additive, rather than multiplicative.[50, 51]

Thus, in units with particle mass $m = 1$, one is led explicitly to equations of motion of the form

$$\frac{dr}{dt} = p \quad \text{and} \quad \frac{dp}{dt} = -\nabla V(r) - \eta p + F,$$

(8)

where $F$ is characterised completely by its statistical properties. Here, e.g., component by component,

$$\langle F_i(t) \rangle = 0 \quad \text{and} \quad \langle F_i(t_1)F_j(t_2) \rangle = 2\Theta\eta\delta_{ij}\delta_D(t_1 - t_2),$$

(9)
where the angular brackets denote a statistical average. The idea is to effect large numbers of different realisations of the same initial conditions, and to analyse these realisations to extract statistical properties.

It is well known that even very weak friction and noise will eventually become important on sufficiently long timescales. In particular, one knows that, on the natural timescale $t_R \sim \eta^{-1}$, these effects will try to force the system to evolve towards a thermal state. The question of relevance here is quite different: *Can the friction and noise have substantial effects already on much shorter timescales $\lesssim t_R$?* The conventional wisdom in astronomy is that the answer to this is: *no!* For example, the standard assumption that “collisionality” is irrelevant in a galaxy relies completely on the observation that the natural timescale associated with binary encounters is much longer than the age of the Universe.[46]

The Langevin simulations were effected for total times $t \leq 200t_{cr}$, and involved friction and noise corresponding to a broad range of characteristic timescales, $10^3 \leq t_R/t_{cr} \leq 10^{12}$. The most significant conclusions derived from these simulations are the following:

When viewed in terms of the *collisionless invariants*, i.e., the quantities that are conserved in the absence of the friction and noise, these perturbing influences only serve to induce a classical *diffusion process*, with the unperturbed and perturbed orbits diverging significantly only on a timescale $t_R \sim \eta^{-1}$. Thus, in particular,

$$\delta E_{rms}^2 \equiv \langle |E_{unp} - E_{per}|^2 \rangle = A(E)E\Theta\eta t,$$

(10)

where $A(E)$ is a slowly varying function of $E$ with magnitude of order unity. In this sense, the conventional wisdom is confirmed.

However, when viewed in *configuration space* or *momentum space*, the effects are more complicated, and actually depend on orbit class. Unperturbed and perturbed regular orbits only diverge as a power law in time, i.e., $\delta r_{rms}, \delta p_{rms} \sim t^p$, so that, once again, one only gets macroscopic deviations after a time $t_R \sim \eta^{-1}$. However, unperturbed and perturbed stochastic orbits diverge exponentially at a rate set by the Liapounov exponent $\chi$, so that, even for very weak friction and noise, one gets macroscopic deviations within a few crossing times. In particular, when considering ensembles of stochastic initial conditions, one observes a simple scaling

$$\delta r_{rms}, \delta p_{rms} \sim (\Theta \eta)^{1/2}\exp[X(E)t],$$

(11)

where $X$ is comparable to, but slightly larger than, the Liapounov exponent $\chi$.

This exponential divergence is easy to understand[52] and the functional dependence on $\Theta, \eta,$ and $\chi$ can actually be derived theoretically.[53] The obvious point is
that the unperturbed deterministic trajectory is an unstable stochastic orbit, so that even the tiniest perturbing influences will tend to grow exponentially. The average rate of instability is given by $\chi$ and, as such, moments like $\delta r_{\text{rms}}$ should grow at a rate $X \sim \chi$. That $X$ is slightly larger than $\chi$ is a reflection of the fact that, for different noisy realisations, one sees somewhat different local Liapounov exponents, and that the total $\delta r_{\text{rms}}$ will be dominated by those noisy realisations for which the rate of instability is above average.

This argument might suggest that, although the unperturbed and perturbed trajectories exhibit a rapid pointwise divergence, their statistical properties should be virtually identical. Specifically, one might anticipate that, on short times, the only effect of the friction and noise is to continually displace the trajectory from one stochastic orbit to another with essentially the same statistical properties. This, however, is false. Under certain circumstances, even very weak friction and noise can also alter the statistical properties of ensembles of stochastic orbits on relatively short times $\ll 100 t_{\text{cr}}$. Specifically, one observes that such perturbing influences can dramatically accelerate the rate of penetration through cantori by providing an additional source of extrinsic diffusion.

Provided that the friction and noise are sufficiently weak, on short timescales the energy $E$ is almost conserved, so that one can speak meaningfully of an evolution restricted to an “almost constant energy hypersurface.” Suppose now that, for some energy $E$, this hypersurface contains large measures of both sticky and filling stochastic orbits. If, for this energy, the near-invariant distribution described in Section 4 is evolved into the future, allowing for even very weak friction and noise, one then observes a rapid ($t \ll 100 t_{\text{cr}}$) systematic evolution towards a new noisy near-invariant distribution which is (1) quite different from the deterministic near-invariant distribution and (2) much closer to the true deterministic invariant distribution. In this sense, it appears that, on timescales short compared the timescale $t_R$ on which the system would evolve towards a thermal state, the principal effect of the friction and noise is to accelerate the approach towards a deterministic invariant distribution which, in the absence of these perturbing influences, would only have been realised on much longer timescales.

The key point in all of this is that friction and noise can induce changes in orbit class, from filling to confined stochastic, and vice versa. Moreover, when the deterministic invariant distribution contains large measures of both sticky and filling orbits, such transitions can happen within a time $t < 100 t_{\text{cr}}$, even for very weak friction and noise. Visual inspection of $\sim 2.5 \times 10^4$ orbits in several different potentials leads to the following conclusions.
Typically, for a relaxation time $t_R$ as long as $10^{12}t_{cr}$, not many such changes are observed within a time $t \sim 100t_{cr}$. However, if $t_R$ be reduced to a value $\sim 10^9t_{cr}$, transitions begin to become more frequent, and, even for $t_R$ as large as $\sim 10^6t_{cr}$, transitions are quite common, occurring for $> 50\%$ of orbits within a time $t \sim 100t_{cr}$. If the amplitude of the friction and noise are further increased, one finds that, for $t_R \sim 10^3t_{cr}$, transitions are so common that the distinction between filling and confined becomes essentially meaningless. The distinction between regular and stochastic is more robust. Only for $t_R$ as small as $\sim 10^3t_{cr}$ are significant numbers of transitions between regular and stochastic orbits observed within a time as short as $t \sim 100t_{cr}$.

The fact that even very weak friction and noise, with $t_R \sim 10^6t_{cr} - 10^9t_{cr}$, can significantly alter the statistical properties of ensembles of orbits on timescales $t < t_H \sim 100t_{cr}$ has direct astronomical implications since, e.g., for galaxies, the timescale $t_R$ for discreteness effects, i.e., “collisionality” is $\sim 10^6t_{cr}$! The natural timescale associated with external perturbations is less easily estimated, but may well be even shorter.

To summarise, it is evident that even very weak friction and noise can alter both the pointwise and the statistical properties of stochastic orbits in a nonintegrable potential on relative short timescales $\ll t_R$. In particular, such effects may be manifest already on time scales much shorter than the time on which numerical errors in a simulation can accumulate. This fact has direct and immediate implications for the problem of “shadowing” for numerical orbits.[54, 55] Physicists, mathematicians, and astronomers are often worried[56, 57] about whether numerical simulations performed on a computer, which incorporate roundoff and/or truncation error, can correctly shadow the evolution of some model system described by a simple set of deterministic differential equations. However, it would also seem relevant[58] to worry about whether the “real world,” replete with other sorts of irregularities, can shadow either the model system or its numerical realisations. In this regard, one final remark is in order: Rather than being an impediment to realistic modeling, in certain cases numerical noise may actually be a good thing, in that it may capture, at least qualitatively, some of the effects of small perturbing influences to which real systems are always subjected.

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