FRÉCHET GENERALIZED TRAJECTORIES AND MINIMIZERS FOR VARIATIONAL PROBLEMS OF LOW COERCIVITY

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Abstract. We address consecutively two problems. First we introduce a class of so called Fréchet generalized controls for a multi-input control-affine system with non-commuting controlled vector fields. For each control of the class one is able to define a unique generalized trajectory, and the input-to-trajectory map turns out to be continuous with respect to the Fréchet metric. On the other side, the class of generalized controls is broad enough to settle the second problem, which is proving existence of generalized minimizers of Lagrange variational problem with functionals of low (in particular linear) growth. Besides we study possibility of Lavrentiev-type gap between the infima of the functionals in the spaces of ordinary and generalized controls.

1. Introduction

1.1. Lagrange optimal control problem: classical setting. Consider a Lagrange optimal control problem with control-affine dynamics:

\[ J(x, u) = \int_0^1 L(x(t), u(t)) \, dt \to \min, \]

\[ \dot{x}(t) = f(x(t)) + \sum_{i=1}^k g_i(x(t))u_i(t) = f(x(t)) + G(x(t))u(t) \quad \text{a.e. } t \in [0, 1], \]

\[ x(0) = x^0, \quad x(1) = x^1. \]

We assume the vector fields \( f, g_i, \ i = 1, 2, \ldots, k \) to be locally Lipschitz in \( \mathbb{R}^n \), and the function \( (x, u) \mapsto L(x, u) \) to be continuous in \( \mathbb{R}^n+k \), and convex with respect to \( u \).

Regarding existence of minimizers for this problem, the classical approach, pioneered by L. Tonelli and D. Hilbert more than a century ago, (see monograph for historical remarks and bibliography) introduced the following assumptions for the Lagrangian \( L \):

A) convexity of the Lagrangian with respect to \( u \) for each fixed \( x \);

B) boundedness of the Lagrangian from below and superlinear growth of the Lagrangian as \( |u| \to \infty \).

Besides, one must require existence of an admissible trajectory of the controlled dynamics satisfying the boundary conditions.

These assumptions guarantee existence of a minimizing control \( \tilde{u}(t) \in L^1_1[0, 1] \), which we call ordinary minimizing control.

It is well known, at least since the work of L.C. Young in the 1930’s, that without convexity of \( L \) in \( u \), ordinary minimizing controls of the Lagrange problem may
cease to exist and the minimum can be achieved by so-called relaxed controls. By now, a rich theory of relaxed controls is developed (see [30]). We will not deal with relaxed controls, assuming below the convexity assumption (A) to hold. Instead, we will weaken the condition of superlinear growth of the Lagrangian, as $|u| \to \infty$.

1.2. Weakening growth assumption and generalized minimizers. If instead of superlinear growth assumption (B) we assume merely linear growth

\[ L(x,u) \geq a + b|u|, \quad a \in \mathbb{R}, \quad b > 0, \]

then existence of ordinary minimizer for the problem (1)–(3) may cease, as the following simple example shows.

**Example 1** (transfer with minimal fuel consumption). Consider the optimal control problem

\[
\begin{align*}
J(u) &= \int_0^1 |u(t)| dt \to \min, \\
\dot{x}(t) &= x(t) + u(t), \quad u \in \mathbb{R}, \quad x(0) = 0, \quad x(1) = e,
\end{align*}
\]

which describes transfer of a point on a line, with the minimized cost, seen as fuel consumption for such transfer.

From the differential system and the boundary conditions we get

\[ e = x(1) = e \int_0^1 e^{-\tau} u(\tau) d\tau \]

and then for any control $u(\cdot) \in L_1[0,1]$, compatible with the boundary condition (5),

\[
\int_0^1 |u(\tau)| d\tau > \left| \int_0^1 e^{-\tau} u(\tau) d\tau \right| = 1.
\]

On the other side for a sequence of needle-like controls

\[ u_i(\tau) = \bar{u}_i \chi_{[0,1/i]}(\tau), \quad \bar{u}_i = 1/(1 - e^{-1/i}), \]

which are compatible with the boundary condition, there holds $J(u_i) \to 1$ as $i \to \infty$.

It is easy to see that the sequence $\{u_i\}$ converges in $W_{-1,1}$-norm to the Dirac measure or optimal impulsive generalized control $\tilde{u} = \delta(\tau)$; the corresponding generalized trajectory is a discontinuous function $\tilde{x}(\tau) = e$, $\forall \tau > 0$, with $\tilde{x}(0) = 0$. □

If the Lagrangian $L$ has linear growth with respect to control, then each sequence of minimizing controls $\{u_i\}_{i \in \mathbb{N}}$ will be bounded in $L_1$-norm. If the fields $f, g_1, \ldots, g_k$ have linear or sublinear growth with respect to the state variables, then the corresponding sequence of trajectories $\{x_{u_i}\}_{i \in \mathbb{N}}$ is bounded in total variation. Helly’s selection theorem [8] guarantees that there is a function $x : [0, 1] \to \mathbb{R}^n$ of bounded variation (not necessarily continuous), such that $x_{u_i}$ converges pointwise to $x$ at every point of continuity of $x$. It is reasonable to conjecture that there is a space of generalized trajectories including discontinuous curves, and a space of generalized controls including impulses, for which the problem (1)–(3) admits a solution.
1.3. Generalized and impulsive controls in involutive and non-involutive cases. Study of optimal impulsive controls for linear systems has been initiated in the 1950’s, particularly for applications in spacecraft dynamics. Later, a more general nonlinear theory has been developed; it englobes the problem \([1]–[3]\) for the cases, where the controlled vector fields \(\{g^1, \ldots, g^k\}\) in \([2]\) form an involutive system.

It turns out that in such cases one can provide the space of 'ordinary', controls \(u(\cdot)\) (say \(L^1_1[0,T]\)) and of the trajectories \(x(\cdot)\) with weak topologies, for which one can still guarantee uniform continuity of the input-to-trajectory map \(u(\cdot) \mapsto x(\cdot)\). Then one can extend this map by continuity onto the topological completion of the space of controls, which contains distributions.

Results obtained for nonlinear control systems by this approach since the 1970’s, can be found in \([4, 15, 22, 26]\). In particular the method allows to extend the input-to-trajectory map onto the space \(W^{-1,\infty}\) of generalized derivatives of measurable essentially bounded functions, with generalized trajectories belonging to \(L^\infty\). Some representation formulae for the generalized trajectories via the generalized primitives of the inputs can be found in \([26]\).

In the linear-quadratic case, this approach allows for the extension of the input-to-trajectory map and the cost functional. Indeed, linear-quadratic Lagrange problems admit a generalized minimizer in some Sobolev space of sufficiently large negative index, provided the boundary conditions can be satisfied and the quadratic functional is bounded from below \([12, 33]\).

Problems with the continuous extension of the input-to-trajectory map which arise in the non-involutive case have been identified in the 1950’s (see \([16]\)). It has been proved in \([15]\) that involutivity of the system of controlled vector fields is necessary for continuity of the map in the weak topology – a property coined in \([15]\) as vibrocorrectness.

To see, why vibrocorrectness fails in the non-involutive case, look at the following simple example.

**Example 2.** Consider the system
\[
\begin{align*}
\dot{x}_1 &= u_1, \\
\dot{x}_2 &= u_2, \\
\dot{x}_3 &= x_2u_1,
\end{align*}
\]
and three bi-dimensional controls, which are concatenations of needles:
\[
\begin{align*}
u^{1,\varepsilon}(t) &= \left(\frac{1}{\varepsilon} \chi_{[0,\varepsilon]}(t), \frac{1}{\varepsilon} \chi_{[\varepsilon,2\varepsilon]}(t)\right), \\
u^{2,\varepsilon}(t) &= \left(\frac{1}{\varepsilon} \chi_{[0,\varepsilon]}(t), \frac{1}{\varepsilon} \chi_{[0,\varepsilon]}(t)\right), \\
u^{3,\varepsilon}(t) &= \left(\frac{1}{\varepsilon} \chi_{[\varepsilon,2\varepsilon]}(t), \frac{1}{\varepsilon} \chi_{[0,\varepsilon]}(t)\right).
\end{align*}
\]

For \(\varepsilon \to 0^+\), all the concatenations tend in \(W^2_{1,1}\) to the bi-dimensional impulsive control \(u(t) = (\delta(t), \delta(t))\), while the corresponding trajectories converge pointwise to different discontinuous curves with \(x(0^+) = (1, 1, 0)\), \(x(0^+) = (1, 1, \frac{1}{2})\), and \(x(0^+) = (1, 1, 1)\) respectively. □

Thus, in the noninvolutive case an extension of input-to-trajectory map onto classical spaces of distributions and/or Sobolev spaces of negative order seems to be impossible.
One approach to the study of noninvolutive systems with impulsive controls proceeds by construction of an appropriate Lie extension of the original system \cite{6,14}. The extension is a new system such that: (i) the extended system of controlled fields is involutive, (ii) all the trajectories of the original system are trajectories of the new system, and (iii) the trajectories of the extended system can be approximated by trajectories of the original system. This reduces the noninvolutive case to the involutive and, after some further transformation, to the commutative case. However, any relation between controls of the extended system and controls of the original system is indirect.

An alternative approach providing a unique extension of the input-to-trajectory map is one of the main issues treated in this contribution.

1.4. Time-reparametrization and ”graph completion” techniques in the noncommutative case. For the noncommutative case, a different approach has been adopted. It is based on a technique of time reparametrization introduced by R.W. Rischel \cite{25} and J. Warga \cite{29}, and further developed by other authors \cite{1,2,9,20,21,23,28,30,31}. For a detailed monography and further references, see \cite{20}. The approach proceeds by introducing a new independent variable with respect to which the trajectories become absolutely continuous. This creates an auxiliary control system which includes time as an additional state variable.

Several authors \cite{1,2,9,20,21,23,28} use the auxiliary system to obtain representations of generalized solutions of \cite{2} by solutions of systems having Radon measures as generalized controls and (right-continuous) functions of bounded variation as generalized trajectories. The definitions introduced have a ’sequential form’: couples \((x(\cdot), U(\cdot))\) of functions of bounded variation, which are correspondingly the generalized trajectory and the primitive of generalized control are weak* limits in \(BV\) of couples \((x_n(\cdot), U_n(\cdot))\) of classical trajectories \(x_n\) and primitives \(U_n\) of classical controls \(u_n\) which generate \(x_n(\cdot)\), with \(\sup_n \|u_n(\cdot)\|_{L^1} < \infty\). It is known that in the scope of this approach for the same \(U\), different sequences \(x_n(\cdot)\), driven by different \(U_n\) may converge to different limits, i.e., each generalized input defines a ’funnel’ of generalized trajectories, rather than a well defined unique trajectory.

A different line of argument has been followed in \cite{5}. Any function \(x : [0,1] \to \mathbb{R}^n\) can be identified with its graph, that is the set \(\Gamma_x = \{(t,x(t)) : t \in [0,1]\} \subset \mathbb{R}^{1+n}\). If the function \(x\) is not continuous, then its graph is not connected. However, if the total variation of \(x\) is finite, then there is a graph completion of \(\Gamma_x\) which is connected. In \cite{5}, each control \(u \in L^1_{k}[0,1]\) is identified with the graph of its primitive \(U(t) = \int_0^t u(\tau) d\tau\). The spaces of generalized controls and generalized trajectories are spaces of graph completions of functions of bounded variation.

The input-to-trajectory map is shown to be continuous over sets of generalized controls equibounded in variation provided with an appropriate metric, into the space of generalized trajectories provided with the Hausdorff metric over the graph completions of generalized primitives.

1.5. Fréchet curves approach to the noncommutative case. In what regards the construction of generalized inputs and trajectories, our approach is rather close to the one of \cite{5}. It is easy to observe that the graph completions introduced in \cite{5} are Fréchet curves \cite{10,17}, and the metric introduced in the space of generalized controls is the classical Fréchet metric.
We prove a stronger version of the main result in [5]: the input-to-trajectory map is continuous with respect to a strengthened Fréchet metric in both the domain and the image. Notice that the Fréchet metric is topologically stronger than the Hausdorff metric. Since ordinary controls are densely embedded in the space of Fréchet curves, this proves existence and uniqueness of a continuous extension of the input-to-trajectory map into the space of generalized controls. This map admits a simple representation in the form of an input-to-trajectory map of an equivalent auxiliary system.

1.6. Fréchet generalized minimizers for Lagrange problems with functionals of linear growth. This continuity result together with the representations of generalized trajectories contributes to proper extension of the cost functional (1) onto the space of Fréchet generalized controls. Thus, we extend the Lagrange variational problem (1)–(3) onto the class of Fréchet generalized controls and trajectories so that

- the cost functional (1) is lower semicontinuous in the space of Fréchet generalized controls and, under linear growth assumption for the integrand, the problem possesses a Fréchet generalized minimizer;
- there may exist a (Lavrentiev-type) gap between the infimum of the cost functional in $L^1_{	ext{loc}}[0,1]$ and the infimum in the space of generalized controls;
- one can formulate regularity conditions which preclude occurrence of the gap.

We do not claim finding the weakest topology in the space of controls, which provides continuity of input-to-trajectory map. In fact, the study in [15] indicates that, under lack of involutivity, the weakest topology should depend on the structure of the Lie algebra generated by the the vector fields $f,g_1,\ldots,g_k$. The topology, we introduce does not depend on it. However, it allows for a proper extension of Lagrange variational problems onto a set of generalized controls, which is broad enough to guarantee existence of generalized minimizers for integral functionals of low (in particular of linear) growth.

1.7. Structure of the paper. This paper is organized as follows. In Section 2 we discuss the spaces of Fréchet curves and their topologies. We prove the continuous canonical selection theorem (Theorem 9), and introduce the spaces of generalized controls and generalized trajectories. Section 3 deals with the definition of the generalized input-to-trajectory map. Section 4 discusses the auxiliary problem. In Section 5 we discuss the extension of the cost functional and its properties. Existence of minimizers for the extended problems is settled in Section 6. Possible occurrence of a Lavrentiev gap is discussed in Section 7. In Section 8 we present an example of a problem with an integrand of linear growth, whose minimizers are all generalized. The proofs of some technical results are collected in the appendix (Section 9).

2. Fréchet generalized controls and generalized paths

The goal of this Section is to introduce the spaces of generalized controls and generalized paths. Subsection 2.1 contains definitions and some basic facts about Fréchet curves. Subsection 2.2 contains the key theorem of continuous canonical selection (Theorem 9). Subsection 2.3 specialises on Fréchet curves defined in
space-time. In Subsections 2.4 and 2.5, we give definitions of what we call the spaces of Fréchet generalized controls and generalized paths.

2.1. Fréchet curves. Various slightly different definitions of Fréchet curves can be found in the literature [10, 17]. In this paper we consider curves that are rectifiable and oriented. Such curves admit absolutely continuous parameterizations, which is a natural requirement when dealing with ordinary differential equations. We allow Fréchet curves to be parameterized by non-compact intervals, which is a convenient way to account for solutions of (2) with a blow up time in the interval \([0, 1]\).

Below we state the exact definitions and basic properties.

We say that a set \(\gamma \subset \mathbb{R}^n\) is a parameterized curve if there is an absolutely continuous function \(g : [0, +\infty] \to \mathbb{R}^n\) such that \(\gamma = g([0, +\infty])\). A parameterization provides the curve with a terminal point \(g(+\infty)\) only if a finite limit \(\lim_{t \to +\infty} g(t)\) exists. In that case we don’t distinguish between \(\gamma\) and \(\gamma \cup \{g(+\infty)\}\).

Definition 3. Two absolutely continuous curves \(g_1, g_2 : [0, +\infty] \to \mathbb{R}^n\) are equivalent if

\[
\inf_{\alpha \in \mathcal{T}} \|g_1 - g_2 \circ \alpha\|_{L_\infty[0, +\infty]} = 0,
\]

where \(\mathcal{T}\) denotes the set of monotonically increasing absolutely continuous bijections \(\alpha : [0, +\infty] \to [0, +\infty]\) admitting absolutely continuous inverse. \(\square\)

The following Lemma relates the previous definition with alternative formulations; its proof can be found in Appendix (Subsection 9.1).

Lemma 4. Two absolutely continuous parameterizations \(g_1, g_2 : [0, +\infty] \to \mathbb{R}^n\) are equivalent if and only if there are absolutely continuous nondecreasing functions \(\alpha_1, \alpha_2 : [0, +\infty] \to [0, +\infty]\) satisfying the following conditions:

a) \(g_1 \circ \alpha_1(t) = g_2 \circ \alpha_2(t)\) \(\forall t \in [0, +\infty]\);

b) \(\alpha_1(0) = \alpha_2(0) = 0\) and \(\alpha_i([0, +\infty]) = [0, +\infty]\) for at least one \(i \in \{1, 2\}\);

c) If \(\alpha_i(\infty) = T < +\infty\), then \(g_i(t) = g_i(T^-)\) for every \(t \geq T\). \(\square\)

Definition 3 introduces an equivalence relation. The equivalence class of a function \(g : [0, +\infty] \to \mathbb{R}^n\) is

\[
[g] = \left\{ h \in AC\left([0, +\infty], \mathbb{R}^n\right) : \inf_{\alpha \in \mathcal{T}} \|g - h \circ \alpha\|_{L_\infty[0, +\infty]} = 0 \right\},
\]

and is called absolutely continuous Fréchet curve in \(\mathbb{R}^n\), or for the sake of brevity, Fréchet curve; each \(\tilde{g} \in [g]\) will be called either a representative or parameterization of \(g\), depending on context.

The space of Fréchet curves in \(\mathbb{R}^n\) is provided with the Fréchet metric

\[
d\left([g_1], [g_2]\right) = \inf_{\alpha \in \mathcal{T}} \|g_1 - g_2 \circ \alpha\|_{L_\infty[0, +\infty]}.
\]

Parameterizations by bounded intervals, i.e., absolutely continuous functions \(g : [0, T] \to \mathbb{R}^n\) or \(g : [0, T] \to \mathbb{R}^n\), with \(T < +\infty\) are included in the present definition of Fréchet curves: \([g]\) stays for \([g \circ \alpha]\) where \(\alpha : [0, +\infty] \to [0, T]\) is any monotonically increasing absolutely continuous bijection with absolutely continuous inverse.

For every subset \(A \subset \mathbb{R}^n\) and any Fréchet curve \([g]\), we set

\[
A \cap [g] = \{ g(t) : t \in [0, +\infty], g(t) \in A \}.
\]
We say that \( A \cap [g] \) is a segment if the set \( \{ t \geq 0 : g(t) \in A \} \) is an interval; according to aforesaid, a nonempty segment is also a Fréchet curve.

For every nondecreasing \( \alpha : [0, +\infty] \to [0, +\infty] \), we introduce the function \( \alpha^\# : [0, +\infty] \to [0, +\infty] \), defined as
\[
\alpha^\#(t) = \sup \{ s \geq 0 : \alpha(s) \leq t \} \quad t \in [0, +\infty].
\]
If \( \alpha \) is continuous, then \( \alpha^\# \) is the right-inverse of \( \alpha \), that is, \( \alpha \circ \alpha^\#(t) = t \) for every \( t < \alpha(+\infty) \).

**Lemma 5.** Let \( \alpha : [0, +\infty] \to [0, +\infty] \) be nondecreasing absolutely continuous, and \( g : [0, +\infty] \to \mathbb{R}^n \) be absolutely continuous. Then:

a) \( \alpha \circ \alpha^\#(t) > 0 \) a.e. on \( [\alpha(0), \alpha(\infty)] \).
b) \( g \circ \alpha^\# \) is absolutely continuous in \( [\alpha(0), \alpha(\infty)] \) if and only if the set \( \{ t \geq 0 : \dot{\alpha}(t) = 0, \ g(t) \neq 0 \} \) has zero Lebesgue measure.
c) If \( g \circ \alpha^\# \) is absolutely continuous in \( [\alpha(0), \alpha(\infty)] \), then
\[
\left( \frac{d}{dt} (g \circ \alpha^\#) (t) \right) = \frac{\dot{g}}{\dot{\alpha}} \circ \alpha^\#(t) \quad \text{for a.e. } t \in [\alpha(0), \alpha(\infty)].
\]
d) If \( g \circ \alpha^\# \) is absolutely continuous in \( [\alpha(0), \alpha(\infty)] \) and \( g \) is constant on each interval \( [\alpha(0), \alpha(\infty)] \), then \( g \circ \alpha^\# \in [g] \). \( \square \)

**Proof.** See Subsection 3.2

An absolutely continuous parameterization \( g : [0, +\infty] \to \mathbb{R}^n \) generates an arc-length function \( \ell_g : [0, +\infty] \to [0, +\infty] \), defined as
\[
\ell_g(t) = \int_0^t |\dot{g}(s)| \, ds \quad \forall t \geq 0.
\]

By Lemma 5, the function \( g \circ \ell^\#_g \in [g] \) and \( |\frac{d}{dt} (g \circ \ell^\#_g) (t)\| = 1 \). Further, \( \tilde{g} \circ \ell^\#_g = g \circ \ell^\# \) for every \( \tilde{g} \in [g] \). That is, the transformation \( [g] \mapsto g \circ \ell^\# \) does not depend on the particular \( g \) representative of \([g] \). This transformation selects one particular element of the class \([g] \). Since it plays an important role in our approach we introduce the following definition:

**Definition 6.** We call \( g \circ \ell^\#_g \) the canonical representative or canonical parameterization of \([g] \), and the mapping \([g] \mapsto g \circ \ell^\#_g \) is the canonical selector. \( \square \)

The total length of a Fréchet curve \([g] \), \( \ell_g(\infty) \), does not depend on the particular parameterization \( g \). We call a Fréchet curve \([g] \) infinite if \( \ell_g(\infty) = +\infty \). Otherwise, we call \([g] \) finite. The space of finite Fréchet curves can be provided with the strengthened Fréchet metric
\[
d^+( [g_1], [g_2] ) = d([g_1], [g_2]) + |\ell_{g_1}(\infty) - \ell_{g_2}(\infty)|.
\]
This metric provides a stronger topology than the Fréchet metric, as can be seen from the following example.

**Example 7.** For the sequence
\[
g_i(t) = \frac{1}{i} \left( \cos(2^it), \sin(2^it) \right) \quad t \in [0, 1], \ i \in \mathbb{N},
\]
\([g_i] \) converges to \([0] \) with respect to the Fréchet metric \( d \). However \( \ell_{g_i}(1) = i \) and therefore \([g_i] \) does not converge in the metric \( d^+ \). \( \square \)
Every finite Fréchet curve \([g]\) has a well defined terminal point \(g(\infty)\). Thus, we adopt the following

**Convention 1.** Any parameterization of a finite Fréchet curve by a compact interval \(g : [0, T] \rightarrow \mathbb{R}^n\) is extended to the interval \([0, +\infty[\) by
\[
g(t) = g(T) \quad \forall t \geq T.
\]
In particular, a canonical parameterization \(t \mapsto g \circ \ell^g_\#(t)\) is extended to the interval \([0, +\infty[\) by
\[
g \circ \ell^g_\#(t) = g(\infty) \quad \forall t \geq \ell_g(\infty). \Box
\]

2.2. **Continuity of canonical selector.** The space of Fréchet curves consists of sets (classes of curves), in which we introduced the strengthened Fréchet metric. A crucial fact is that choosing the canonical representative of each class provides us with a \(C_0\)-continuous selector.

**Proposition 8.** The canonical selector \([g] \mapsto g \circ \ell^g_\#\) is a continuous one-to-one mapping of the space of finite Fréchet curves in \(\mathbb{R}^n\) provided with the strengthened Fréchet metric \(d^+\) into the space \(AC^n[0, +\infty[\) provided with the topology of \(C_0\)-convergence. \(\square\)

**Proof.** Since for each \(g\) the canonical representative \(g \circ \ell^g_\#\) belongs to \([g]\) and is uniquely defined, then the correspondence \([g] \mapsto g \circ \ell^g_\#\) is one-to-one. It remains to prove that the canonical selector is continuous.

Fix a finite Fréchet curve \([\gamma]\), with canonical representative \(\gamma\), and pick a small \(\varepsilon > 0\). There exists a partition \(0 = t_0 < t_1 < t_2 < \ldots < t_N < +\infty\) such that
\[
\sum_{i=1}^N |\gamma(t_i) - \gamma(t_{i-1})| > \ell_\gamma(\infty) - \varepsilon, \quad t_N > \ell_\gamma(\infty) + \varepsilon.
\]
This implies that the length of any segment \(\gamma|_{[t_j, t_j+k]}\) admits the bounds
\[
(9) \quad \sum_{i=1}^k |\gamma(t_{j+i}) - \gamma(t_{j+i-1})| \leq \ell_\gamma|_{[t_j, t_j+k]} \leq \sum_{i=1}^k |\gamma(t_{j+i}) - \gamma(t_{j+i-1})| + \varepsilon.
\]
Pick an arbitrary Fréchet curve \([g]\) such that
\[
(10) \quad d^+([g], [\gamma]) < \frac{\varepsilon}{N},
\]
and let \(g\) be the canonical representative of \([g]\). The bound \(10\) implies
\[
(11) \quad |\ell_g(\infty) - \ell_\gamma(\infty)| < \frac{\varepsilon}{N}, \quad \|g \circ \alpha - \gamma\|_{L_\infty} < \frac{\varepsilon}{N},
\]
for some (absolutely continuous monotonically increasing) function \(\alpha \in \mathcal{T}\). Without loss of generality, we may assume that \(\alpha(t) = t\) for every \(t \geq t_N > \max\{\ell_g(\infty), \ell_\gamma(\infty)\}\).

Let \(\theta_i = \alpha(t_i)\) for \(i = 0, 1, 2, \ldots, N\). By \(11\), we have
\[
(12) \quad |g(\theta_i) - \gamma(t_i)| < \frac{\varepsilon}{N} \quad \text{for } i = 0, 1, 2, \ldots, N.
\]
Let
\[
M = \|g - \gamma\|_{L_\infty[0, +\infty[} = \max\{|g(t) - \gamma(t) : t \in [0, t_N]\} = |g(\hat{t}) - \gamma(\hat{t})|.
\]
We may add the point \(\hat{t}\) to the partition and (with a small abuse of notation) think that \(\hat{t} = t_k\) for some \(k \in \{0, 1, 2, \ldots, N\}\).
Thus, \( \| \) implies 

\[ \ell_g(\infty) = \theta_k + \ell_g(\infty) - \ell_g(\theta_k) = \theta_k + \ell_g|_{|\nu_k, o\infty[} \geq \theta_k + \sum_{i = k+1}^{N} |g(\theta_i) - g(\theta_{i-1})| \geq \]

\[ \geq \theta_k + \sum_{i = k+1}^{N} (|\gamma(t_i) - \gamma(t_{i-1})| - |g(\theta_i) - g(\theta_{i-1})| - |g(\theta_{i-1} - \gamma(t_{i-1})|) \geq \]

and by \((12)\):

\[ \ell_g(\infty) \geq \theta_k + \sum_{i = k+1}^{N} |\gamma(t_i) - \gamma(t_{i-1})| - 2\varepsilon. \]

By virtue of \((9)\), we get

\[ (13) \quad \ell_g(\infty) \geq \theta_k + \ell_\gamma|_{|\nu_k, o\infty[} - 3\varepsilon = \ell_\gamma(\infty) + \theta_k - t_k - 3\varepsilon. \]

Similar computation, based on \((12)\) and \((9)\), yields

\[ \ell_g(\infty) = \ell_g(\theta_k) + \ell_g(\infty) - \theta_k \geq \sum_{i = 1}^{k} |g(\theta_i) - g(\theta_{i-1})| + \ell_g(\infty) - \theta_k \geq \]

\[ \geq \sum_{i = 1}^{k} |\gamma(t_i) - \gamma(t_{i-1})| - 2\varepsilon + \ell_g(\infty) - \theta_k \geq \ell_\gamma(\infty) + \ell_\gamma(t_k) - \theta_k - 4\varepsilon = \]

\[ = \ell_\gamma(\infty) + t_k - \theta_k - 4\varepsilon. \]

Joining \((13)\) and \((14)\), one concludes

\[ |\theta_k - t_k| < \ell_g(\infty) - \ell_\gamma(\infty) + 4\varepsilon. \]

Finally, we have the estimate

\[ |\theta_k - t_k| = |\ell_g(\theta_k) - \ell_g(t_k)| \geq |g(\theta_k) - g(t_k)| = |g \circ \alpha(t_k) - \gamma(t_k)| \geq \]

\[ \geq |\gamma(t_k) - g(t_k)| - |g \circ \alpha(t_k) - \gamma(t_k)| \geq M - \frac{\varepsilon}{N}. \]

Thus, \((10)\) implies \( \| g - \gamma \|_{L^\infty} = M < 5\varepsilon. \)

It is a bit surprising that the continuity property of the canonical selector can be strengthened to \( W^{n}_{1,p}[0, +\infty[ \). 

**Theorem 9.** The canonical selector \( [g] \mapsto g \circ \ell^g_\# \) is a continuous map from the space of finite Fréchet curves in \( \mathbb{R}^{n} \) provided with the strengthened Fréchet metric \( d^+ \) into the Sobolev space \( W^{n}_{1,p}[0, +\infty[ \) for each \( p \in [1, +\infty[. \)

**Remark 10.** Since we are dealing with finite Fréchet curves, each canonical element’s derivative is supported in some compact interval. Therefore, for each pair of finite Fréchet curves \([g], [h]\), there is some \( T < +\infty \) such that

\[ \left\| g \circ \ell^g_\# - h \circ \ell^h_\# \right\|_{W^{n}_{1,p}[0, +\infty[} = \left\| g \circ \ell^g_\# - h \circ \ell^h_\# \right\|_{W^{n}_{1,p}[0, T[}. \]

However, we cannot fix a priori one such \( T \) for every finite \([g], [h]\). □

**Proof.** Fix \([\gamma]\), a finite Fréchet curve with canonical representative \( \gamma \). Let a sequence of finite Fréchet curves \( \{[\gamma_i]\}_{i \in \mathbb{N}} \), with canonical representatives \( \gamma_i, i \in \mathbb{N} \), converge to \([\gamma]\): \( \lim_{i \to \infty} d^+ ([\gamma_i], [\gamma]) = 0. \)
As far as the lengths of $\gamma_i, i \in \mathbb{N}$ are bounded by some $T$, then the interval $[0, T]$ contains the supports of $\dot{\gamma}_i, \dot{\gamma}_i, i \in \mathbb{N}$. According to Proposition $8$, $\lim_{i \to \infty} \|\gamma_i - \gamma\|_{L_\infty[0, T]} = 0$. One wishes to prove that $\lim_{i \to \infty} \|\dot{\gamma}_i - \dot{\gamma}\|_{L_p[0, T]} = 0$, and hence

$$\lim_{i \to \infty} \|\gamma_i - \gamma\|_{W_{1,p}^p[0, T]} = 0.$$

First, we show that $\dot{\gamma}_i$ converges to $\dot{\gamma}$ in the weak* topology of $L_\infty^n[0, T]$. Indeed, seeing $\dot{\gamma}_i$ as a functional on $L_1^n[0, T]$, we note that

$$\forall t \in [0, T], \forall v \in \mathbb{R}^n: \quad \langle \gamma_i(t), v \rangle = \int_0^t \langle \dot{\gamma}_i(s), v \rangle ds = \left\langle \dot{\gamma}_i, v \chi_{[0,t]} \right\rangle$$

is the result of the action of the functional $\dot{\gamma}_i$ on the vector-function $s \mapsto v \chi_{[0,t]}(s)$. As far as the space of linear combinations of the functions $v \chi_{[0,t]}(\cdot)$ is dense in $L_1^n[0, T]$, and $L_\infty$-norms of $\dot{\gamma}_i$ are bounded by $1$, we conclude that

$$\lim_{i \to \infty} \int_0^T \langle \dot{\gamma}_i, \varphi \rangle dt = \int_0^T \langle \dot{\gamma}, \varphi \rangle dt, \quad \forall \varphi \in L_1^n[0, T].$$

Since $L_\infty^n[0, T] \subset L_1^n[0, T], \forall q \in [1, +\infty]$, this shows that $\dot{\gamma}_i \rightharpoonup \dot{\gamma}$ weakly in $L_p^n[0, T]$ for every $p \in [1, +\infty]$.

Note that from $d^+$-convergence of $\gamma_i$ to $\gamma$, it follows that

$$\lim \|\dot{\gamma}_i\|_{L_p^n[0, T]} = \lim \|\dot{\gamma}_i\|_{L_\infty^n[0, T]} = \|\dot{\gamma}\|_{L_p^n[0, T]} \quad \forall p \in [1, +\infty].$$

Therefore, the Radon-Riesz theorem guarantees that

$$\lim \|\dot{\gamma}_i - \dot{\gamma}\|_{L_p^n[0, T]} = 0 \quad \forall p \in [1, +\infty],$$

which implies $\lim \|\dot{\gamma}_i - \dot{\gamma}\|_{L_1^n[0, T]} = 0$.

\medskip

2.3. \textbf{Fréchet curves in space-time}. Let $\mathcal{Y}_n$ denote the set of absolutely continuous functions $(\theta, y): [0, +\infty] \to \mathbb{R}^{1+n}$ such that

\begin{align*}
\theta(0) &= 0, \quad \dot{\theta}(t) \geq 0 \quad \text{a.e. } t \geq 0, \\
\ell(\theta, y)(\infty) &= +\infty.
\end{align*}

The first coordinate $\theta$ represents time. Thus, each $(\theta, y) \in \mathcal{Y}_n$ is a parameterization of a curve in space-time, defined in the time interval $[0, \theta(\infty)]$.

The condition $15$ reflects the fact that time should be a monotonically increasing variable. We don’t require it to be strictly increasing because we are interested in jumps and impulses, i.e., processes that evolve instantaneously.

The condition $16$ means that the time interval $[0, \theta(\infty)]$ is maximal, that is, the curve parameterized by $(\theta, y)$ cannot be prolonged beyond the time $\theta(\infty) \in [0, +\infty]$.

For each $T > 0$, let $\mathcal{Y}_{n,T}$ be the set of all $(\theta, y) \in \mathcal{Y}_n$ such that $\theta(\infty) > T$, i.e., the set of all absolutely continuous parameterizations of curves in space-time, which are well defined on the compact time interval $[0, T]$.

For each $(\theta, y) \in \mathcal{Y}_n$, define $(\theta_T, y_T)$ as

\begin{align*}
(\theta_T, y_T)(t) &= (\theta, y)(t) \quad \text{for } t \in [0, \theta^*(T)], \\
(\theta_T, y_T)(t) &= (T, y(\theta^*(T))) \quad \text{for } t \geq \theta^*(T);
\end{align*}

in particular $(\theta_T, y_T)$ coincides with $(\theta, y)$ if $\theta^*(T) = +\infty$. 

Now we introduce a family of semimetrics in \( \mathcal{Y}_n \), \( \{\rho_T\}_{T \in [0, +\infty)} \), defined as

\[
\rho_T \left( (\theta, y), (\tilde{\theta}, \tilde{y}) \right) = \left\| (\theta_T, y_T) - (\tilde{\theta}_T, \tilde{y}_T) \right\|_{L_\infty [0, +\infty)}.
\]

Each \( \rho_T \) becomes a metric if we don’t distinguish between \( (\theta, y), (\tilde{\theta}, \tilde{y}) \in \mathcal{Y}_n \) such that \( \theta^#(T) = \tilde{\theta}^#(T) \) and \( (\theta, y) \) coincides with \( (\tilde{\theta}, \tilde{y}) \) in \([0, \theta^#(T)]\).

**Convention 2.** When dealing with the metric \( \rho_T \), we identify \( (\theta, y) \in \mathcal{Y}_n \) with the corresponding \( (\theta_T, y_T) \), defined in \((17)\). □

Let \( \mathcal{F}_n \) and \( \mathcal{F}_{n,T} \) be the spaces of Fréchet curves in \( \mathbb{R}^{1+n} \) corresponding to \( \mathcal{Y}_n \) and \( \mathcal{Y}_{n,T} \), that is

\[
\mathcal{F}_n = \{[(\theta, y)] : (\theta, y) \in \mathcal{Y}_n\}, \quad \mathcal{F}_{n,T} = \{[(\theta, y)] : (\theta, y) \in \mathcal{Y}_{n,T}\}.
\]

Due to condition \((16)\), \( \mathcal{F}_n \) is a set of infinite Fréchet curves.

Each semimetric \( \rho_T \) induces a semimetric \( d_T \) in the space \( \mathcal{F}_n \):

\[
d_T \left( [(\theta_1, y_1)], [(\theta_2, y_2)] \right) =
\]

\[
= \inf \left\{ \rho_T \left( (\hat{\theta}_1, \hat{y}_1), (\hat{\theta}_2, \hat{y}_2) \right) : (\hat{\theta}_i, \hat{y}_i) \in [(\theta_i, y_i)], \ i = 1, 2 \right\}
\]

By **Convention 2**, \( d_T \) becomes a metric; \( d_T \) \( \{[(\theta_1, y_1)], [(\theta_2, y_2)]\} \) coincides with the Fréchet distance between the segments \([(\theta_1, y_1)] \cap ([0, T] \times \mathbb{R}^n) \) and \([(\theta_2, y_2)] \cap ([0, T] \times \mathbb{R}^n) \), for any \([(\theta_1, y_1)], [(\theta_2, y_2)] \in \mathcal{F}_n \). When dealing with the metric \( d_T \) we identify each \([(\theta, y)] \in \mathcal{F}_n \) with the segment \([(\theta, y)] \cap ([0, T] \times \mathbb{R}^n) \).

Consider an absolutely continuous function \( x : [0, T] \rightarrow \mathbb{R}^n \), where \( T \in [0, +\infty) \) is maximal in the sense that \( x \) does not admit an absolutely continuous extension onto any interval \([0, \tilde{T}] \) with \( \tilde{T} > T \). Then, the function \( t \mapsto (t, x(t)) \in \mathbb{R}^{1+n} \) is an element of \( \mathcal{Y}_n \) and hence \([(t, x)]\) is an element of \( \mathcal{F}_n \). The correspondences \( x \mapsto (t, x) \) and \( x \mapsto [(t, x)] \) are one-to-one.

Conversely, the Lemma 4 implies that for every \([(\theta, y)] \in \mathcal{F}_n \), the function \( y \circ \theta^# : [0, \theta^#(\infty)] \rightarrow \mathbb{R}^n \) does not depend on the particular representative of \([(\theta, y)]\). In particular, \( y \circ \theta^# = x \) for every \((\theta, y) \in [(t, x)]\). Following this argument, we identify each absolutely continuous function \( x \) defined on a maximal interval with the corresponding Fréchet curve \([(t, x)]\).

For every \([(\theta, y)] \in \mathcal{F}_n \), the function \( y \circ \theta^# \) has bounded variation on compact subintervals of \([0, \theta^#(\infty)]\). Due to Lemma 5, \( y \circ \theta^# \) is absolutely continuous if and only if the set \( \{ t \geq 0 : \dot{\theta}(t) = 0, \ \dot{y}(t) \neq 0 \} \) has zero Lebesgue measure.

The following proposition shows that every function of locally bounded variation can be 'lifted' to a Fréchet curve by virtue of the transformation \([(\theta, y)] \mapsto y \circ \theta^# \).

**Proposition 11.** If a function \( x : [0, T] \mapsto \mathbb{R}^n \) has finite variation on every compact subinterval of \([0, T]\), then there exists \((\theta, y) \in \mathcal{Y}_n \) such that \( x(t) = y \circ \theta^#(t) \) for every \( t \in [0, T] \), a continuity point of \( x \). □

**Proof.** See Appendix (Subsection 9.3). □

Note that the mapping \([(\theta, y)] \mapsto y \circ \theta^# \) is not one-to-one. Indeed for each function of locally bounded variation \( x : [0, T] \mapsto \mathbb{R}^n \), there are infinitely many \([(\theta, y)] \in \mathcal{F}_n \), such that \( y \circ \theta^# = x \). To see this, consider the following example.
Example 12. For a discontinuous function $x : [0, +\infty) \to \mathbb{R}^2$ defined as
\[
x_1(t) = x_2(t) = 0, \quad \text{for } t < 1,
\]
\[
x_1(t) = 0, \quad x_2(t) = 1, \quad \text{for } t \geq 1.
\]
Every Fréchet curve $[(\theta, y)]$ in space-time, consisting of the concatenation of the following arcs

i. the segment of straight line from the point $(0, 0, 0)$ to the point $(1, 0, 0)$;

ii. an absolutely continuous Fréchet curve from the point $(1, 0, 0)$ to the point $(1, 0, 1)$, contained in the plane $\{ (1, x_1, x_2) : (x_1, x_2) \in \mathbb{R}^2 \}$;

iii. the ray $\{(1 + t, 0, 1), t \geq 0 \}$
satisfies $y \circ \theta^\#(t) = x(t) \forall t \geq 0$ (see Figure 1). □

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The discontinuous function $x$ (solid line) and some Fréchet curves satisfying $y \circ \theta^\# = x$ (different dashed lines).}
\end{figure}

Remark 13. The graph of a function $x : [0, T] \to \mathbb{R}^n$ is the set $\Gamma_x = \{ (t, x(t)) : t \in [0, T] \}$. If $x(t)$ is absolutely continuous, then the function $t \mapsto (t, x(t))$ is absolutely continuous parameterization of $\Gamma_x$. Thus, Fréchet curves in space-time can be seen as generalizations of absolutely continuous graphs to the class of functions of locally bounded variation. □

Fréchet curves in space-time coincide with graph completions in the terminology of [5].

2.4. Fréchet generalized controls. By ordinary control, we understand any locally integrable function $u : [0, +\infty) \to \mathbb{R}^k$. The control system (2) can be represented as
\[
(18) \quad \dot{x}(t) = f(x(t)) + G(x(t))U(t),
\]
where $U(t) = \int_0^t u(s)ds$.

The function $t \mapsto (t, U(t))$ is a representative of the Fréchet curve $[(t, U)]$. By construction $U(0) = 0$; besides local integrability of $u$ guarantees that $(t, U) \in Y_{k,T}$, and thus $[(t, U)] \in F_{k,T}$, for every $T \in [0, +\infty[$.

For each $T \in [0, +\infty[,$ we introduce the set

$$Y_{k,T}^0 = \{ (V, W) \in Y_{k,T} : W(0) = 0 \},$$

and define the space of Fréchet generalized controls on the interval $[0, T]$ to be the set of Fréchet curves, whose representatives belong to $Y_{k,T}^0$, that is

$$F_{k,T}^0 = \{ [(V, W)] \in F_{k,T} : W(0) = 0 \}.$$  

According to Convention 2, we identify each generalized control $[(V, W)] \in F_{k,T}^0$ with it’s segment $[(V, W)] \cap ([0, T] \times \mathbb{R}^k)$, and provide this space with the strengthened Fréchet metric:

$$d_T^+( [(V, W), (V', W')] ) = d_T ( [(V, W), (V', W') ] ) + 
\frac{1}{2} \left| \ell_{(V_1, W_1)} (V_1^#(T)) - \ell_{(V_2, W_2)} (V_2^#(T)) \right|.$$  

A Fréchet curve $[(V, W)] \in F_{k,T}^0$ coincides with an ordinary control in the interval $[0, T]$ if and only if the set $\{ t \in [0, T] : W(t) \neq 0 \}$ has zero Lebesgue measure. In that case, $U = W \circ V^\#$ is absolutely continuous, and the corresponding ordinary control is $u(t) = \frac{d}{dt} (W \circ V^#)(t)$.

The following proposition demonstrates that every generalized control can be approximated by sequences of ordinary controls.

**Proposition 14.** The space of ordinary controls \( \{(t, \int_0^t u(s)ds) : u \in L_1[0, T] \} \) is dense in \( \left( F_{k,T}^0, d_T^+ \right) \).

**Proof.** We introduce in $Y_{k,T}$ a semimetric $\rho_T^+$:

$$\rho_T^+ ((V_1, W_1), (V_2, W_2)) = \rho_T ((V_1, W_1), (V_2, W_2)) + 
\frac{1}{2} \left| \ell_{(V_1, W_1)} (V_1^#(T)) - \ell_{(V_2, W_2)} (V_2^#(T)) \right|. \quad (19)$$

Fix arbitrary $(V, W) \in Y_{k,T}^0$. For each $\varepsilon > 0$, let

$$V_\varepsilon(t) = \begin{cases} 
\frac{\varepsilon}{\max(V, \varepsilon)} \int_0^t \max(\dot{V}, \frac{\varepsilon}{\max(V, \varepsilon)})d\tau, & \text{for } t \leq V^#(T), \\
T + t - V^#(T) & \text{for } t > V^#(T).
\end{cases}$$

$V_\varepsilon$ admits absolutely continuous inverse and therefore $U_\varepsilon(t) = W \circ V_\varepsilon^{-1}(t)$ is absolutely continuous with $U_\varepsilon(0) = 0$.

One can check that $V_\varepsilon^{-1}(T) = V^#(T)$, $V_\varepsilon$ converges to $V$ uniformly in $[0, V^#(T)]$, as $\varepsilon \to 0^+$, and

$$\lim_{\varepsilon \to 0^+} \ell_{(V_\varepsilon, W)} (V^#(T)) = \ell_{(V, W)} (V^#(T)).$$

Therefore, $\lim_{\varepsilon \to 0^+} \rho_T^+ ((V_\varepsilon, W), (V, W)) = 0$, which implies $\lim_{\varepsilon \to 0^+} d_T^+ ([(t, U_\varepsilon), [(V, W)]) = 0$.

The space of generalized controls has the following compactness property:
**Proposition 15.** Every sequence of ordinary controls bounded in $L^1_{k}[0,T]$ admits a subsequence $\{u_i\}_{i \in \mathbb{N}}$ such that $\{(t,U_i)\}_{i \in \mathbb{N}}$ converges in $(F^0_{k,T},d^+_T)$. □

**Proof.** Let a sequence $\{u_i \in L^1_{k}[0,T]\}_{i \in \mathbb{N}}$ be such that

\begin{equation}
\|u_i\|_{L^1_{k}[0,T]} \leq M, \quad \forall i \in \mathbb{N}.
\end{equation}

For each $i \in \mathbb{N}$, let $(V_i, W_i) = \left(\ell_{(t,U_i)}^{-1}(U_i \circ \ell_{(t,U_i)}^{-1})\right)$ be the canonical parameterization of $[(t,U_i)]$. By (20), $\ell_{(t,U_i)}(T) \leq M + T, \forall i \in \mathbb{N}$. The sequence $\{(V_i, W_i)\}_{i \in \mathbb{N}}$ is uniformly bounded and equicontinuous in $[0, T + M]$. Therefore by the Ascoli-Arzelà theorem, it admits a uniformly converging subsequence. It follows that the corresponding subsequence $\{[(V_i, W_i)]\}_{j \in \mathbb{N}}$ converges in $(F^0_{k,T},d^+_T)$. □

Let us revise Example 2.

**Example 16.** Consider the controls $u^i_{\epsilon}, i = 1, 2, 3$ from Example 2. Let $U^i_{\epsilon}$ be the respective primitives. It can be shown that the limits

$\lim_{\epsilon \to 0^+} [(t, U^i_{\epsilon})], \quad i = 1, 2, 3,$

exist in $(F^0_{n,T},d^+_T)$, and differ on the segment from the point $(0, 0, 0)$ to the point $(0, 1, 1)$, as shown on Figure 2. One can directly compute

\begin{align*}
d^+_T([(V_1, W_1)]), [(V_2, W_2)]) &= 2 - \frac{1}{\sqrt{2}}, \\
d^+_T([(V_1, W_1)], [(V_3, W_3)]) &= 1. \quad \Box
\end{align*}

**Figure 2.** The generalized controls from Example 16: $[(V_1, W_1)]$ (solid line), $[(V_2, W_2)]$ (dashed line), and $[(V_3, W_3)]$ (doted line).
The ability to characterize impulses by a path in space-time provides a "resolution" of an impulse, which takes place in zero time, as Example 16 illustrates. This feature is crucial for dealing with the case, where the controlled vector fields in the system (2) do not commute.

2.5. Fréchet generalized paths. We call ordinary path any absolutely continuous function \( x : [0, T] \rightarrow \mathbb{R}^n \), with \( T \in [0, +\infty) \) being maximal in the sense that \( x \) does not admit an absolutely continuous extension onto any interval \([0, \hat{T}] \supset [0, T] \).

Using the notation of Section 2.3, \( x \mapsto (t, x) \) and \( x \mapsto [(t, x)] \) are one-to-one mappings from the space of ordinary trajectories into \( \mathcal{Y}_n \) and \( \mathcal{F}_n \), respectively.

We identify each ordinary path \( x \) with the corresponding Fréchet curve \( [(t, x)] \in \mathcal{F}_n \), and define the space of generalized paths to be the space \( \mathcal{F}_n \), provided with the metric \( d_T \) (with \( T \in [0, +\infty] \) fixed).

Since the metric \( d_T \) is fixed, we follow Convention 2 and identify each generalized path \( [(\theta, y)] \in \mathcal{F}_n \) with its segment \( [(\theta, y)] \cap [0, T] \times \mathbb{R}^n \). In particular, any ordinary path \( x : [0, \hat{T}] \rightarrow \mathbb{R}^n \), with \( \hat{T} > T \), is identified with its restriction to the interval \([0, T] \).

According to Section 2.3, \( \mathcal{F}_{n,T} \) is the space of generalized paths defined on the time interval \([0, T] \) and extendable beyond \([0, T] \). Taking into account the conventions above, for any ordinary path \( x, (t, x) \) is a representative of some \( [(\theta, y)] \in \mathcal{F}_{n,T} \) if and only if \( x \in AC^n[0, T] \). That is, we identify the space of ordinary paths that can be extended beyond \([0, T] \) with the set
\[
\{[(t, x)] : x \in AC^n[0, T]\} \subset \mathcal{F}_{n,T}.
\]

For generalized paths, we have an analogue of Proposition 14

**Proposition 17.** The space of ordinary paths defined in the interval \([0, T] \), \( \{[(t, x)] : x \in AC^n[0, T]\} \) is dense in \( \mathcal{F}_{n,T}, d_T^\ast \) (and therefore, it is also dense in \( \mathcal{F}_{n,T}, d_T \)).

**Proof.** A trivial adaptation of the proof of Proposition 14.

\[\square\]

3. The input-to-trajectory map

This section is centered on Theorem 19, which establishes existence and uniqueness of a continuous extension of the input-to-trajectory map onto the space of Fréchet generalized controls. The extended map takes values in the space of generalized paths.

We use capital letters for the elements of \( \mathcal{Y}_{0,k,T} \) and small letters for their first-order derivatives (i.e., \( (v, w) = (\dot{V}, W) \), with \((V, W) \in \mathcal{Y}_{0,k,T} \)).

For each locally integrable \( u : [0, +\infty] \rightarrow \mathbb{R}^k \), let \( x_u \) denote the corresponding trajectory of the system (2), starting at a point \( x_u(0) = x^0 \), defined on a maximal interval. We will show that the input-to-trajectory mapping \( u \mapsto x_u \) defines a unique mapping \( [(t, U)] \mapsto [(t, x_u)] \), which in its turn admits a unique continuous extension onto \( \mathcal{F}_{0,k,T} \).

There is the following intuition behind this. Fix an ordinary control \( u \), and suppose that \( x_u \) is well defined in the interval \([0, T] \). Pick a representative \((V, W) \in [(t, U)] \), and let \( y = x_u \circ V \). It follows that
\[
y(0) = x_0, \quad \dot{y}(t) = f(y(t))v(t) + G(y(t))w(t) \quad \text{a.e.} \; t \in [0, V^\#(T)].
\]
The mapping Proposition 18. thus providing a "natural" extension of the input-to-trajectory map.

Proof. Fix a generalized control \( \hat{V} \), then one may expect that \((V, y_{(v,w)}) \in [(t, x_u)]\) for every \((V, W) \in [(t, U)]\).

The following Proposition shows that this is also true for generalized controls, thus providing a "natural" extension of the input-to-trajectory map.

**Proposition 18.** The mapping

\[
((V, W)) \in \mathcal{F}_{k,T}^0 \mapsto [(V, y_{(v,w)})] \in \mathcal{F}_n
\]

is properly defined, i.e., it does not depend on the choice of \((V, W) \in [(V, W)]\).

Proof. Fix a generalized control \([(\hat{V}, \hat{W})] \in \mathcal{F}_{k,T}^0\) and the canonical representative \((\hat{V}, \hat{W})\). For any \((V, W) \in [(\hat{V}, \hat{W})]\),

\[
(\hat{V}, \hat{W}) = (V, W) \circ \ell_{(V,W)}^#, \quad (\hat{v}, \hat{w}) = \frac{(v, w)}{\sqrt{v^2 + |w|^2}} \circ \ell_{(V,W)}^#,
\]

and it follows that

\[
(V, y_{(v,w)}) \circ \ell_{(V,W)}^# = \left(V \circ \ell_{(V,W)}^#, y_{(v,w)} \circ \ell_{(V,W)}^#\right) = \left(\hat{V}, y_{(v,w)} \circ \ell_{(V,W)}^#\right).
\]

Since \(\{t : y_{(v,w)}(t) \neq 0\}\) has zero Lebesgue measure, the Lemma guarantees that the function \(t \mapsto y_{(v,w)} \circ \ell_{(V,W)}^#(t)\) is absolutely continuous and

\[
\frac{d}{dt} \left(y_{(v,w)} \circ \ell_{(V,W)}^#\right) = \left(\hat{y}_{(v,w)} \circ \ell_{(V,W)}^#\right) \cdot \frac{1}{\sqrt{v^2 + |w|^2}} \circ \ell_{(V,W)}^# = \frac{v}{\sqrt{v^2 + |w|^2}} \circ \ell_{(V,W)}^# + G \left(y_{(v,w)} \circ \ell_{(V,W)}^#\right) \cdot \frac{w}{\sqrt{v^2 + |w|^2}} \circ \ell_{(V,W)}^# =
\]

\[
= f \left(y_{(v,w)} \circ \ell_{(V,W)}^#\right) \hat{v} + G \left(y_{(v,w)} \circ \ell_{(V,W)}^#\right) \hat{w},
\]

then \(y_{(\hat{v}, \hat{w})} = y_{(v,w)} \circ \ell_{(V,W)}^#\), and therefore, \([(V, y_{(v,w)})] = [(\hat{V}, y_{(v,w)})]\).

To formulate the result on continuity of the input-to-trajectory map we introduce the set

\[
\mathcal{W}_T = \{[(V, W)] \in \mathcal{F}_{k,T}^0 : [(V, y_{(v,w)})] \in \mathcal{F}_{n,T}\},
\]

of generalized controls, such that the generalized trajectory assigned to them by \(f^0\) is well defined on the time interval \([0, T]\).

The following is a stronger version of Theorem 2, Corollary 1 in [5].

**Theorem 19.** The set \(\mathcal{W}_T\) is an open subset of \(\left(\mathcal{F}_{k,T}^0, d_T^*\right)\).

The mapping \([(V, W)] \in \mathcal{W}_T \mapsto [(V, y_{(v,w)})] \in \mathcal{F}_{n,T}\) is the unique extension of the input-to-trajectory map that is continuous with respect to the metrics \(d_T^*\) in the domain and in the image. □

**Proof.** The transformation \([(V, W)] \mapsto [(V, y_{(v,w)})]\) can be decomposed into a chain of mappings

\[
[(V, W)] \in \mathcal{F}_{k,T}^0 \mapsto (V, W) \in W_{1,1}^{1+k}[0, +\infty[ \mapsto (V, y_{(v,w)}) \in Y_{n,T} \mapsto [(V, y_{(v,w)})] \in \mathcal{F}_{n,T},
\]

where the first transformation is the canonical selector, which is continuous by Theorem 5.
We provide the space \( Y_{n,T} \) with the metric \( \rho_1^+ \) defined in (19). By definition, 
\[
d^+_T ([(\theta_1, y_1)], [(\theta_2, y_2)]) \leq \rho_1^+ ((\theta_1, y_1), (\theta_2, y_2)) \quad \text{for every} \quad (\theta_1, y_1), (\theta_2, y_2) \in Y_{n,T}.
\]
Hence, the last transformation in (23) is continuous.

Fix a generalized control \([(V, W)] \in \mathcal{W}_T\). Under Conventions 1 and 2, the support of \((\tilde{v}, \tilde{w})\) is contained in the compact interval \([0, V^#(T)]\), and for every \([(V, W)] \in \mathcal{F}_{k,T}^0\) such that \(d_T^+ \left( [(V, W)], [(\tilde{V}, \tilde{W})] \right) < \varepsilon\), the support of \((v, w)\) is contained in \([0, \tilde{V}^#(T) + \varepsilon]\).

By standard continuity result, there is some \(\varepsilon > 0\) such that the trajectory of the system (21) is well defined on the interval \([0, V^#(T) + \varepsilon]\) for every \((V, W) \in W_{1,1}^{1+k}[0, +\infty]\) such that \(\|(v, w) - (\tilde{v}, \tilde{w})\|_{L_{1+k}^{1+k}[0, +\infty]} < \varepsilon\).

Since the input-to-trajectory map of system (21) \((v, w) \mapsto y_{(v,w)}\) is continuous with respect to the norms \(\|\cdot\|_{L_{1+k}^{1+k}[0, V^#(T)+\varepsilon]}\) in the domain and \(\|\cdot\|_{W_{1,1}^{1+k}[0, V^#(T)+\varepsilon]}\) in the image, the Theorem follows.

We compare Theorem 19 with the corresponding result (Theorem 2, Corollary 1) of [5]. There, the generalized controls have equibounded variations and the metric in the space of impulses is (in our terminology) the Fréchet metric. The topology in the space of generalized trajectories is defined by the Hausdorff metric.

By introducing the strengthened Fréchet metric \(d^+\), we automatically require convergence of the full variations, and therefore guarantee equiboundedness of converging sequences of inputs. The main difference lies in the fact that we prove continuity of the input-to-trajectory map when the space of generalized trajectories is provided with the \(d^+\) metric instead of the weaker Hausdorff metric. Continuity of the canonical selector (Theorem 3) is essential for this result.

To see that the topology generated by \(d^+\) is strictly stronger than the topology generated by the Hausdorff metric, consider the following simple example:

**Example 20.** Consider two curves \([(\theta, y)], [(\tilde{\theta}, \tilde{y})] \in \mathcal{F}_{2,T}\), with canonical elements \((\theta, y), (\tilde{\theta}, \tilde{y})\). Suppose that \((\theta(t), y(t)) = (\tilde{\theta}(t), \tilde{y}(t))\) for \(t \geq 2\pi\) and

\[
(\theta(t), y(t)) = (0, \cos t, \sin t), \quad (\tilde{\theta}(t), \tilde{y}(t)) = (0, \cos(2\pi - t), \sin(2\pi - t)) \quad \text{for} \quad t \in [0, 2\pi].
\]

It is simple to check that the Hausdorff distance between \([(\theta, y)]\) and \([(\tilde{\theta}, \tilde{y})]\) is zero but \(d \left( [(\theta, y)], [(\tilde{\theta}, \tilde{y})] \right) = d^+ \left( [(\theta, y)], [(\tilde{\theta}, \tilde{y})] \right) = 2\). □

To illustrate the extended input-to-trajectory mapping, we return to Example 2.

**Example 21.** Consider Example 2. By Theorem 19, the input-to-trajectory map is well defined. To compute the generalized trajectories corresponding to the generalized controls in the example, notice that here the system (21) reduces to

\[
\dot{y}_1 = w_1, \quad \dot{y}_2 = w_2, \quad \dot{y}_3 = y_2 w_2.
\]
Let \((V_i, W_i), \ i = 1, 2, 3\) be the parameterizations by length of the generalized controls \([(V_i, W_i)], \ i = 1, 2, 3\) in Example 16. Then,

\[
\begin{align*}
(v_1, w_1)(t) &= (0, 1, 0) \chi_{[0,1]}(t) + (0, 0, 1) \chi_{[1,2]}(t) + (1, 0, 0) \chi_{[2,\infty)}(t), \\
(v_2, w_2)(t) &= \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \chi_{[0,\sqrt{2}]}(t) + (1, 0, 0) \chi_{[\sqrt{2},\infty)}(t), \\
(v_3, w_3)(t) &= (0, 0, 1) \chi_{[0,1]}(t) + (0, 1, 0) \chi_{[1,2]}(t) + (1, 0, 0) \chi_{[2,\infty)}(t).
\end{align*}
\]

Therefore,

\[
\begin{align*}
(V_1, y(v_1, w_1))(t) &= (0, t, 0) \chi_{[0,1]}(t) + (0, 1, t - 1, 0) \chi_{[1,2]}(t) + (t - 2, 1, 1, 0) \chi_{[2,\infty)}(t), \\
(V_2, y(v_2, w_2))(t) &= \left(0, \frac{t}{\sqrt{2}}, \frac{t^2}{4} \right) \chi_{[0,\sqrt{2}]}(t) + \left(t - \sqrt{2}, 1, \frac{1}{2} \right) \chi_{[\sqrt{2},\infty)}(t), \\
(V_3, y(v_3, w_3))(t) &= (0, 0, t, 0) \chi_{[0,1]}(t) + (0, t - 1, 1, t - 1) \chi_{[1,2]}(t) + (t - 2, 1, 1, 1) \chi_{[2,\infty)}(t)
\end{align*}
\]

are parameterizations of the generalized trajectories corresponding to \([(V_i, W_i)], \ i = 1, 2, 3\), respectively (see Figure 3). Notice that

\[
y(v_1, w_1) \circ V_1^#(t) = (1, 1, 0), \quad y(v_2, w_2) \circ V_2^#(t) = \left(1, 1, \frac{1}{2}\right),
\]

\[
y(v_3, w_3) \circ V_3^#(t) = (1, 1, 1),
\]

for every \(t > 0\). □
To compare our approach with the one developed by Miller and Rubinovich \[20\], we formulate the following result, which shows that every generalized trajectory of system \[2\], as defined in \[20\] coincides with some \(y_{(v,w)} \circ V^\#\), with \((V,W) \in Y_{k,T}^0\).

**Proposition 22.** Consider a sequence of ordinary controls \(\{u_i \in L^1_t[0, T]\}_{i \in \mathbb{N}}\), equibounded in \(L^1_t[0, T]\)-norm, such that the corresponding sequence of trajectories \(\{x_{u_i}\}_{i \in \mathbb{N}}\) is equibounded in \(L^\infty[0, T]\)-norm.

There is a subsequence \(\{u_{i_j}\}\) such that \([(t, U_{i_j})]\) converges towards some \([\{(V,W)\}] \in F_{k,T}^0\), and \(x_{u_{i_j}}(t)\) converges pointwise to \(y_{(v,w)} \circ V^\#(t)\) in \([0, T]\), with the possible exception of a countable set of points. \(\square\)

**Proof.** The Proposition \[15\] guarantees existence of a convergent subsequence \(\{[(t, U_i)]\}_{i \in \mathbb{N}}\). Thus, we can assume without loss of generality that \(\{((t, U_i))\}_{i \in \mathbb{N}}\) converges to some \([\{(V,W)\}] \in F_{k,T}^0\). Let \((V,W)\) be the canonical parameterization of \([\{(V,W)\}]\), and \((V_i,W_i)\) be the canonical parameterization of \([(t, U_i)]\), for \(i \in \mathbb{N}\). Notice that \(x_{u_i} = y_{(v,w)} \circ V_i^\#\) for every \(i \in \mathbb{N}\).

Since the sequence \(\{x_{u_i}\}_{i \in \mathbb{N}}\) is equibounded, we can assume that the vector fields \(f, g_1, \ldots, g_k\) have compact support. In that case, the sequence \(\{(V_i,y_{(v,w)})\}\) is uniformly Lipschitz and converges uniformly towards \((V,y_{(v,w)})\) in the interval \([0,1+V^\#(T)]\). Then, for any \(t \in [0, T]\):

\[
\left| y_i \circ V_i^\#(t) - y \circ V^\#(t) \right| \leq \left| y_i \circ V_i^\#(t) - y_i \circ V^\#(t) \right| + \left| (y_i - y) \circ V_i^\#(t) \right| \\
\leq C \left| V_i^\#(t) - V^\#(t) \right| + \| y_i - y \|_{L^\infty[0,1+V^\#(T)]}.
\]

Since

\[
\liminf_{i \to \infty} V_i^\#(t) \geq V^\#(t^-), \quad \limsup_{i \to \infty} V_i^\#(t) \leq V^\#(t^+), \quad \forall t \in [0, T],
\]

we see that \(\lim_{i \to \infty} x_{u_i}(t) = y \circ V^\#(t)\), with exceptions only at the points of discontinuity of \(V^\#\). This set is at most countable, and the result follows. \(\square\)

### 4. Auxiliary Problem

Coming back to the optimal control problem \[1\]–\[3\], we show that parameterization by the arc length of the curves \([(t,U)]\) results in its equivalent reformulation. Canonical parameterizations where introduced in \[5\], for Cauchy problems. Here we extend the analysis to Lagrange problems \[1\]–\[3\]. A similar reparameterization was introduced in \[20\].

For each ordinary control \(u \in L^1_t[0, 1]\), the length function \(\ell_{(t,U)}\) coincides with the function \(\tau_u : [0, +\infty[ \to [0, +\infty]\), given by

\[
(24) \quad \tau_u(t) = \int_0^t \sqrt{1 + |u|^2} \, ds,
\]

and the canonical parameterization of the curve \([(t,U)]\) is the function \((V,W) = (\tau_u^{-1}, U \circ \tau_u^{-1})\). The function \(\tau_u\) admits an absolutely continuous inverse \(\tau_u^{-1}\) with

\[
\frac{d}{dt} \tau_u^{-1} = \frac{1}{\sqrt{1 + |u|^2}} \circ \tau_u^{-1}.
\]
The reparameterized trajectory \( x_u \circ \tau_u^{-1} \) coincides with \( y(v, w) \), the unique solution of (21) with

\[
(v, w) = \left( \frac{1}{\sqrt{1+|u|^2}} \circ \tau_u^{-1}, \frac{u}{\sqrt{1+|u|^2}} \circ \tau_u^{-1} \right).
\]

By the change of variable \( t = \tau_u^{-1} \), the cost functional (1) becomes

\[
\int_0^1 L(x_u, u) dt = \int_0^{\tau_u(1)} L(y(v, w), u \circ \tau_u^{-1}) \frac{1}{\sqrt{1+|u|^2}} \circ \tau_u^{-1} dt = \int_0^{\tau_u(1)} L(y(v, w), \frac{w}{v}) v dt.
\]

Conversely, for any \( T \in [0, +\infty] \) and any measurable function \( t \mapsto (v(t), w(t)) \in \mathbb{R}^{1+k} \) satisfying

\[
v(t) > 0, \quad v(t)^2 + |w(t)|^2 = 1 \quad \text{a.e. } t \in [0, T], \quad \int_0^T v(t) dt = 1,
\]

there is a unique \( u \in L^1_T[0, 1] \) satisfying (25).

Therefore, the problem (1)–(3) is equivalent to

\[
I(v, w, T) = \int_0^T L \left( y(t), \frac{w(t)}{v(t)} \right) v(t) dt \rightarrow \min,
\]

\[
\dot{\theta}(t) = v(t), \quad \dot{y}(t) = f(y(t))v(t) + G(y(t))w(t),
\]

\[
v(t) > 0, \quad v(t)^2 + |w(t)|^2 = 1 \quad \text{a.e. } t \in [0, T],
\]

\[
\theta(0) = 0, \quad \theta(T) = 1, \quad y(0) = x^0, \quad y(T) = x^1,
\]

with free \( T \in [0, +\infty[. \)

It is crucial that the integrand in (26) is parametric in the terminology of L.C. Young \[32\], i.e. invariant with respect to the dilation \((v, w) \rightarrow (\kappa v, \kappa w), \kappa \in \mathbb{R}_+ \). This will later allow us to extend the functional (26) onto the class of Fréchet generalized controls.

The proof of the following Lemma is given in Appendix (Subsection 9.4).

**Lemma 23.** 1. For every \( y \in \mathbb{R}^n \), the function \((v, w) \mapsto L \left( y, \frac{w}{v} \right) v \) is convex in \([0, +\infty[ \times \mathbb{R}^k \).

2. For every \((\hat{y}, \hat{v}, \hat{w}) \in \mathbb{R}^n \times [0, +\infty[ \times \mathbb{R}^k \),

\[
\liminf_{(y, v, w) \rightarrow (\hat{y}, \hat{v}, \hat{w}) \atop v > 0} L \left( y, \frac{w}{v} \right) v = \liminf_{(v, w) \rightarrow (\hat{v}, \hat{w}) \atop v > 0} L \left( \hat{y}, \frac{\hat{w}}{\hat{v}} \right) v = \lim_{v \rightarrow \hat{v}, \atop v > 0} L \left( \hat{y}, \frac{\hat{w}}{\hat{v}} \right) v. \quad \square
\]

We define the new Lagrangian on \( \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^k \):

\[
\lambda(y, v, w) = \begin{cases} 
L \left( y, \frac{w}{v} \right) v, & \text{for } v > 0, \\
\lim_{\eta \rightarrow 0^+} L \left( y, \frac{w}{v} \right) \eta, & \text{for } v = 0
\end{cases}
\]

**Lemma 23** implies

**Corollary 24.** The function \( \lambda(y, v, w) \), defined by (30) is the lower semicontinuous envelope of the function \((y, v, w) \mapsto L \left( y, \frac{w}{v} \right) v \). \quad \square

**Remark 25.** Since the lower semicontinuous envelope of a convex function is convex, we conclude that \( \lambda \) is convex with respect to \((v, w) \in [0, +\infty[ \times \mathbb{R}^k \). \quad \square
Replacing the condition \( v(t) > 0 \) by \( v(t) \geq 0 \) in (28), one obtains a problem with controls taking values in the compact set \( \{(v, w) \in \mathbb{R}^{1+k} : v \geq 0, v^2 + |w|^2 = 1 \} \). This is the so-called compactification technique, started probably in [11]. For a recent contribution, see [13]. By relaxing the control values to the convex hull \( B^+_k \) of this set, we introduce the relaxed problem

\[
\tilde{I}(v, w, T) = \int_0^T \lambda(y(t), v(t), w(t)) \, dt \to \min,
\]

\[
(31)
\]

\[
\dot{\theta}(t) = v(t), \quad \dot{y}(t) = f(y(t))v(t) + G(y(t))w(t),
\]

\[
(32)
\]

\[
(v(t), w(t)) \in B^+_k = \{(v, w) | v \geq 0, v^2 + |w|^2 \leq 1 \} \quad \text{a.e. } t \in [0, T],
\]

\[
(33)
\]

\[
\theta(0) = 0, \quad \theta(T) = 1, \quad y(0) = x^0, \quad y(T) = x^1,
\]

with free \( T \in [0, +\infty[. \)

5. The cost functional for Fréchet generalized trajectories and controls

The argument used above shows that for an ordinary control \( u(t) \), and the graph of its primitive \( (t, U(t)) \), one gets for each \( (V, W) \) :

\[
J(x_u, u) = \int_0^V \lambda^{(1)} \left( y_{(v,w)}, v, w \right) \, d\tau.
\]

This suggests that the cost functional (1) can be extended onto the space \( F_{0,k,T} \) and this extension should coincide with the functional

\[
I([(V, W)]) = \int_0^{V^{(1)}} \lambda \left( y_{(v,w)}, v, w \right) \, d\tau \quad [(V, W)] \in W_1.
\]

First, note that the functional (35) is properly defined, that is:

**Proposition 26.** The mapping

\[
[(V, W)] \in W_1 \mapsto \int_0^{V^{(1)}} \lambda \left( y_{(v,w)}, v, w \right) \, d\tau
\]

does not depend on a particular representative \( (V, W) \) \( \in [(V, W)] \). □

**Proof.** Follow the argument of the proof of Proposition 18. □

The following property of the functional (35) holds:

**Proposition 27.** The functional \( I([(V, W)]) = \int_0^{V^{(1)}} \lambda(y_{(v,w)}, v, w) \, dt \) is lower semicontinuous in the space of generalized controls \( \left( F_{0,k,T}^0, d_T^+ \right) \). □

**Proof.** For any \( 0 < \delta < \varepsilon < 1 \) and \( y \in \mathbb{R}^n \), \( (v, w) \in B^+_k \) (satisfying the constraints (33)), we get

\[
L \left( y, \frac{w}{v+\varepsilon} \right) (v+\varepsilon) = L \left( y, \frac{v+\delta}{v+\varepsilon} \frac{w}{v+\delta} + \frac{\varepsilon-\delta}{v+\varepsilon} 0 \right) (v+\varepsilon) \leq L \left( y, \frac{w}{v+\delta} \right) (v+\delta) + L(y,0) (\varepsilon-\delta).
\]
Passing to the limit at the right-hand side, as $\delta \to 0^+$, we invoke Lemma 23 to conclude
\[
\lambda(y, v, w) \geq L \left( y, \frac{w}{v+\varepsilon} \right) (v + \varepsilon) - L(y, 0)\varepsilon,
\]
and thus $\lambda$ is bounded from below on compact sets.

Fix a sequence $\{(V_i, W_i)\} \in F_{k,T}^0$ converging to $(V, W)$ with respect to the metric $d^+_j$, and let $\{(V_i, W_i)\}_{i \in \mathbb{N}}$, $(V, W)$ be the respective canonical representatives.

By Theorem 19, $y(v_i, w_i)$ converges uniformly to $y(v, w)$. By the Theorem 9 $(v_i, w_i)$ converges to $(v, w)$ with respect to the $L_1$-norm. Therefore, there is a subsequence $(v_{i_j}, w_{i_j})$ converging pointwise almost everywhere to $(v, w)$.

Using Fatou’s Lemma:
\[
\liminf_{j \to \infty} \int I([(V_{i_j}, W_{i_j})]) = \liminf_{j \to \infty} \int_0^{V_{i_j}^\#(T)} \lambda(y_{v_{i_j}, w_{i_j}}, v_{i_j}, w_{i_j}) dt \geq \int_0^{V^\#(T)} \liminf_{j \to \infty} \lambda(y(v_{i_j}, v), v_{i_j}, w_{i_j}) dt \geq \int_0^{V^\#(T)} \lambda(y(v, v), v, w) dt = I([(V, W)]).
\]

In order to extend the optimal control problem (1)–(3) onto the class of Fréchet generalized controls $F_{k,1}^0$, let us recall some of previously obtained results:

- The functional $[(V, W)] \mapsto I([(V, W)])$ is lower semicontinuous in $F_{k,1}^0$ and
  \[ I([(t, u)]) = J(x_u, u), \quad \forall u \in L_1[k, 1]. \]
- The input-to-trajectory map $[(V, W)] \mapsto [(V, y(v, w))]$ is the unique continuous extension of the input-to-trajectory of system (2). A generalized trajectory has a well defined endpoint given by $y(v, w) \circ V^\#(1)$. This point does not depend on a particular $(V, y(v, w)) \in [(V, y(v, w))]$; it is the $x$-component of the point, at which the Fréchet curve $[(V, y(v, w))]$ crosses (leaves) the hyperplane \{$(t, x) \in \mathbb{R}^{1+n} : t = 1$\}.

Basing on these considerations, we introduce the extended problem:
\[
\begin{align*}
(36) 
I([(V, W)]) & \to \text{min}, \\
(37) 
[(V, W)] & \in F_{k,1}^0, \quad y(v, w) \circ V^\#(T) = x^1.
\end{align*}
\]

This problem is equivalent to the relaxed problem (31)–(34), in the following sense:

i) If $(v, w)$ is an optimal control for the problem (31)–(34), then $[(V, W)] = \left[ \int_0^1 (v, w) dt \right]$ is optimal for the problem (36)–(37) and the corresponding generalized trajectory is $[(V,y(v, w))]$.

ii) If $[(V, W)]$ is optimal for the problem (36)–(37) and $(V, W)$ is its canonical representative, then $(v, w) = (V, W)$ is an optimal control for the problem (31)–(34).

6. Existence of Fréchet generalized minimizers for integrands with linear growth

Following the aforesaid, the problem (1)–(3) admits a generalized solution if and only if the problem (31)–(34) admits a solution. In this Section we prove existence of minimizer for (31)–(34).
To this end we start with a classical Ascoli-Arzelà-Filippov argument to obtain a general necessary and sufficient condition (Proposition 31) for existence of minimizer for the relaxed problem. Then we prove that this condition is satisfied in the case of the integrand of linear growth in control variable(s).

Consider the set $B^+_k$ defined by (33) and let for each $y \in \mathbb{R}^n$:

$$Q(y) = \left\{ (\phi,v,f(y)v + G(y)w) : (v,w) \in B^+, \phi \geq \lambda(y,v,w) \right\}.$$  

We pass to the differential inclusion form of the problem (31)–(34), following the scheme developed in [7]:

$$C(T) \to \min,$$

$$\left( \dot{C}(t), \dot{\theta}(t), \dot{y}(t) \right) \in Q(y(t)) \quad \text{a.e. } t \in [0,T],$$

$$C(0) = 0, \quad \theta(0) = 0, \quad \theta(T) = 1,$$

$$y(0) = x^0, \quad y(T) = x^1,$$

with free $T \in [0, +\infty]$.

For each $y \in \mathbb{R}^n$, $\varepsilon > 0$, let $Q(B_\varepsilon(y)) = \bigcup_{|z-y|<\varepsilon} Q(z)$. The following Lemma shows that the differential inclusion (39) is continuous.

**Lemma 28.** For every $y \in \mathbb{R}^n$, $Q(y) = \bigcap_{\varepsilon>0} Q(B_\varepsilon(y))$. □

**Proof.** Fix $(\phi, V) \in \bigcap_{\varepsilon>0} Q(B_\varepsilon(y))$. By definition, there is a sequence $$\left\{ (y_i,v_i,w_i) \in B_\varepsilon(y) \times B^+ \right\}_{i \in \mathbb{N}}$$ such that

$$V = (v_i,f(y_i)v_i + G(y_i)w_i), \quad \phi \geq \lambda(y_i,v_i,w_i)$$

for every $i \in \mathbb{N}$.

Since $\left\{ (v_i,w_i) \right\}$ is bounded, we can assume, passing to a subsequence, that it converges to some $(v,w) \in B^+$. By continuity of $f, G$, we have $V = (v,f(y)v + G(y)w)$. By Lemma 23 $\lambda(y,v,w) \leq \liminf \lambda(y_i,v_i,w_i) \leq \phi$. Therefore, $(\phi,V) \in Q(y)$. □

The classical Filippov selection theorem requires the Lagrangian to be continuous, a condition that is not guaranteed for the auxiliary Lagrangian $\lambda$. However we manage to prove that Filippov’s theorem holds for $\lambda$.

**Proposition 29.** Fix $(C, \theta, y)$, a trajectory of the differential inclusion (39), defined in the interval $[0,T]$.

There is a measurable control $(v,w) : [0,T] \to B^+$ such that

$$\dot{\theta}(t) = v(t), \quad \dot{y}(t) = f(y(t))v(t) + G(y(t))w(t)$$

$$\dot{C}(t) \geq \lambda(y(t),v(t),w(t)),$$

for a.e. $t \in [0,T]$. □

**Proof.** See Appendix (Section 9.5). □

**Corollary 30.** If $\left\{ (v_i,w_i) \right\}_{i \in \mathbb{N}}$ is a minimizing sequence for the problem (31)–(34), then

$$\left\{ \left( -\frac{d}{dt} \int_0^t \lambda(y(v_i,w_i),v_i,w_i)dt, V_i, y(v_i,w_i) \right) \right\}_{i \in \mathbb{N}}$$
is a minimizing sequence for the problem \((38)-(41)\).

The problem \((31)-(34)\) admits a solution if and only if the problem \((38)-(41)\) does. □

Proof. Due to the Proposition 29, if \((42)\) fails to be a minimizing sequence for the problem \((38)-(41)\), then there would exist an admissible control \((\hat{v}, \hat{w})\), for which

\[
\int_0^{\hat{v}} \lambda(y(\hat{v}, \hat{w}, v, w)) dt < \liminf \int_0^{v_i} \lambda(y(v_i, w_i, v_i, w_i)) dt,
\]

which is a contradiction.

It follows that, whenever \((v, w)\) is a solution for the problem \((31)-(34)\), then \((\int (\cdot) \lambda(y(v, w, v, w)) dt, V, y(v, w))\) must be solution for the problem \((38)-(41)\).

Now, suppose that the problem \((38)-(41)\) admits a solution \((C, \theta, y)\). By Proposition 29, there is an admissible control \((v, w)\) such that \(I(v, w) \leq C(T)\). As far as the infimum of problem \((31)-(34)\) cannot be strictly less than the minimum of \((38)-(41)\), it follows that \((v, w)\) must be a solution for \((31)-(34)\). □

The following Proposition provides necessary and sufficient condition for existence of solution of the problem \((38)-(41)\).

Proposition 31. The problem \((38)-(41)\) admits a solution if and only if it admits a minimizing sequence \(\{(C_i, \theta_i, y_i, T_i)\}_{i \in \mathbb{N}}\) for which the sequences \(\{T_i\}_{i \in \mathbb{N}}, \left\{\|(C_i, \theta_i, y_i)\|_{L^\infty[0, T_i]}\right\}_{i \in \mathbb{N}}\) are bounded. □

Proof. The condition is clearly necessary.

To verify sufficiency, note that such a sequence is bounded and equicontinuous. Therefore, the Ascoli-Arzelà Theorem guarantees existence of a subsequence, converging to a limit. Lemma 28 guarantees that the limit solves the differential inclusion \((39)\) and hence the limit is optimal. □

The Corollary 30 and the Proposition 31 immediately imply the following:

Corollary 32. The relaxed problem \((31)-(34)\) admits a solution if and only if it has finite infimum and admits a minimizing sequence \(\{(v_i, w_i)\}\) for which the sequences \(\{T_i\}_{i \in \mathbb{N}}, \left\{\|y(v_i, w_i)\|_{L^\infty[0, T_i]}\right\}_{i \in \mathbb{N}}\) are bounded. □

Using this Corollary, we will prove existence of generalized solutions when the Lagrangian \((1)\) has linear growth with respect to controls:

Proposition 33. Suppose the following conditions hold:

i) There are constants \(a \in \mathbb{R}, b > 0\) such that

\[
L(x, u) \geq a + b|u| \quad \forall (x, u) \in \mathbb{R}^{n+k}.
\]
ii) There are constants $\hat{a}, \hat{b} < +\infty$ such that
\[
|G(x)| \leq (\hat{a} + \hat{b}|x|)u + \hat{b}L(x, u),
\]
\[
|f(x)| \leq \hat{a} + \hat{b}(|x| + L(x, u)) \quad \forall (x, u) \in \mathbb{R}^{n+k}.
\]
Then, the relaxed problem (31)–(34) admits a minimizer, i.e., the original problem (1)–(3) admits a Fréchet generalized minimizer.

Proof. Adding a suitable constant to the Lagrangian $L$, we may replace the conditions (i), (ii) by

i') There is a constant $b > 0$ such that
\[
L(x, u) \geq b(1 + |u|) \quad \forall (x, u) \in \mathbb{R}^{n+k}.
\]

ii') There is a constant $\hat{b} < +\infty$ such that
\[
|G(x)| \leq \hat{b}(|x - x^0||u| + L(x, u)),
\]
\[
|f(x)| \leq \hat{b}(|x - x^0| + L(x, u)) \quad \forall (x, u) \in \mathbb{R}^{n+k}.
\]

Fix $\{(v_i, w_i)\}$, a minimizing sequence for the problem (31)–(34). Due to Propositions 18 and 26, $(v_i, w_i) \circ \ell^\#(V, W)$ is also a minimizing sequence. Thus, we can assume that
\[
v_i(t)^2 + |w_i(t)|^2 = 1 \quad \text{a.e. } t \geq 0, \forall i \in \mathbb{N}.
\]
In that case, the condition i') guarantees that $\lambda(y, v_i, w_i) \geq \frac{b}{\sqrt{2}} \sqrt{v_i^2 + |w_i|^2} = \frac{b}{\sqrt{2}}$. Hence $\hat{I}(v_i, w_i, T_i) \geq \frac{b}{\sqrt{2}} T_i$ and therefore the infimum of the problem is finite and the sequence $\{T_i\}$ is bounded.

From the condition (ii'), we get
\[
|y_{(v_i, w_i)}(t) - x^0| \leq \int_0^t |f(y_{(v_i, w_i)}))v_i + |G(y_{(v_i, w_i)})w_i|\,d\tau \leq \hat{b}|y_{(v_i, w_i)} - x^0| + \hat{b}\lambda(y_{(v_i, w_i)}, v_i, w_i)\,d\tau \leq 2\hat{b}\hat{I}(v_i, w_i, T_i) + 2\hat{b}\int_0^t |y_{(v_i, w_i)} - x^0|\,d\tau,
\]
and by Gronwall's Lemma, the sequence $\{\|y_{(v_i, w_i)}\|_{L_\infty[0, T_i]}\}$ is bounded. \hfill \Box

7. Lavrentiev Gap for Ordinary and Generalized Controls

We briefly discuss, what we call Lavrentiev gap for the classes of ordinary and Fréchet generalized controls.

We say that the functional $I$ exhibits an $L^k_1[0, 1] \cdot F_{k,1}^0$ Lavrentiev(-type) gap, if
\[
\inf_{u \in L^k_1[0, 1]} I ([\{t, U\}]) > \inf_{[(V, W)] \in F_{k,1}^0} I ([\{V, W\}]).
\]
This definition is complete only after we specify how to deal with the boundary conditions (3). One possibility is to consider approximations of generalized controls by ordinary controls that satisfy exactly the boundary conditions. That is, to take the infima over the $u \in L^k_1[0, 1]$ satisfying (3) and over the $[(V, W)] \in F_{k,1}^0$, satisfying (3). In alternative, we may consider approximations of generalized controls by ordinary controls that satisfy approximately the boundary conditions.

We adopt this last point of view, which leads to the
Definition 34. The functional $I$ exhibits an $L^1_k[0,1]-F^0_k$ Lavrentiev gap, if
\[
\lim_{\varepsilon \to 0^+} \inf_{u \in L^1_k[0,1]} \frac{I((t,U))}{|x_2(t) - x_1(t)|} = \inf_{(V,W) \in F^0_k} I([(V,W)]) \geq 0.
\]

The original Lavrentiev phenomenon has been studied in the classical problem of the calculus of variations, where simple examples with a $W_{1,\infty}-W_{1,1}$ gap are known \cite{3, 19}. Some generalizations can be found in \cite{27}. Therefore, the occurrence of a $L^1_k[0,1]-F^0_k$ gap is not surprising; the following example shows that such gap is a real possibility.

Example 35. Consider the optimal control problem
\[
J(u) = \int_0^1 |x_1(t)| + h(x_1(t), u(t)) dt \to \min,
\]
\[
\dot{x}_1 = x_1 + x_2, \quad \dot{x}_2 = u, \quad x(0) = (0, -1), \quad x(1) = (0, 0),
\]
with
\[
h(x_1, u) = \begin{cases} 
\max \left( |u| - \frac{1}{\sqrt{|x_1|}}, 0 \right) & \text{for } x_1 \neq 0, \\
0 & \text{for } x_1 = 0.
\end{cases}
\]

Note that the integrand $|x_1| + h(x_1, u)$ is a continuous function. The problem is equivalent to
\begin{align*}
&I(v, w) = \int_0^1 |y_1(t)| + h(y_1(t), w(t)) v dt \to \min, \quad T - \text{free}, \\
&\dot{y}_1 = (y_1 + y_2) v, \quad \dot{y}_2 = w, \quad \dot{V} = v, \quad v \geq 0, \quad v^2 + w^2 = 1, \\
&y_1(0) = 0, \quad y_2(0) = -1, \quad V(0) = 0, \quad V(T) = 1, \quad y_1(T) = y_2(T) = 0.
\end{align*}

The control $(\dot{v}, \dot{w}) = (0,1)\chi_{[0,1]} + (1,0)\chi_{[1,\infty]}$, $T = 2$, satisfies the boundary condition and $I(\dot{v}, \dot{w}) = 0$. Thus, it is optimal. It corresponds to a generalized control containing an impulse which is optimal for the initial problem.

We will show that for the problem \cite{14, 16} there is a constant $C > 0$ such that $I(v, w) > C$ whenever $v(t) > 0$ almost everywhere, $V(T) = 1$, and $|y_2(T)|$ is sufficiently small. i.e. whenever a control in the original problem is ordinary and generates a trajectory with endpoint in a neighbourhood of the boundary condition $x(1) = (0, 0)$.

Fix an arbitrary triple $(v, w, T)$ with $v(t) > 0$ and $v(t)^2 + w(t)^2 = 1$ for a.e. $t \geq 0$, such that $V(T) = 1$ and $y_2(T) > -\frac{1}{2}$.

Let $T_1 = \min \{ t \in [0, T] : y_2(t) = -\frac{1}{2} \}$, hence $-1 \leq y_2(t) \leq -1/2$ on $[0, T_1]$.

Given that $|\dot{y}_2(t)| = |w(t)| < 1$ we conclude $T_1 > 1/2$.

Then for $t \in [0, T_1]$:
\begin{align*}
&y_1(t) = \int_0^t e^{\int_s^t \tau d\tau} v(s)y_2(s) ds < 0, \\
\text{and} \quad &\dot{y}_1(t) = v(t)(y_1(t) + y_2(t)) < 0. \quad \text{Hence} \quad \dot{y}_1 = v(t)y_1(t) + y_2(t) < v(t)y_2(t), \quad \text{and}
\end{align*}
\begin{align*}
&|y_1(t)| = -y_1(t) \geq \int_0^t -y_2(t)v(t) dt \geq \frac{1}{2} \int_0^t v(t) dt = \frac{1}{2} V(t) \quad \forall t \in [0, T_1].
\end{align*}

Then
\begin{align*}
\int_0^T |y_1(t)|v(t) dt \geq \int_0^{T_1} \frac{1}{2} V(t)v(t) dt = \frac{1}{4} (V(T))^2,
\end{align*}
and from (44)
\[
I(v, w) \geq \frac{1}{4}(V(T_1))^2 + \int_0^{T_2} \left( |w(t)| - \sqrt{\frac{2}{V(t)}}v(t) \right) dt \geq \frac{1}{4}(V(T_1))^2 + \frac{1}{2} - \sqrt{\frac{V(T_1)}{2}};
\]

one notes that \( \int_0^{T_1} |w(t)| dt \geq |y_2(T_1) - y_2(0)| = \frac{1}{2} \).

Given that \( V(T_1) \in [0, 1] \) we conclude that
\[
I(v, w) \geq \min_{z \in [0, 1]} \left( \frac{1}{4}z^2 + \frac{1}{2} - \sqrt{\frac{z}{2}} \right) = \frac{1}{2} - \frac{3}{28/3} \geq 0.0275. \quad \Box
\]

Now, we present some conditions that exclude a \( L_1^1[0, 1] - F_{k,1}^0 \) Lavrentiev gap.

**Proposition 36.** If the auxiliary Lagrangian \( \lambda \) is continuous in \( \mathbb{R}^n \times B_k^+ \), (see (33)), then the problem (1)-3 does not exhibit Lavrentiev gap. \( \Box \)

**Proof.** Pick a generalized control \( [(V, W)] \) with canonical element \( (V, W) \), and \( T \in [0, +\infty[ \), satisfying the boundary condition \( y_{(v, w)}(T) = x^1 \), \( V(T) = 1 \).

For each \( \varepsilon > 0 \), let
\[
(V_\varepsilon(t), W_\varepsilon(t)) = \left( V(t) \frac{t}{1 + \varepsilon}, W(t) \frac{t}{1 + \varepsilon} \right), \quad t \geq 0,
\]
and let \( T_\varepsilon \) be the unique \( t \) solving \( V_\varepsilon(t) = 1 \). Then, \( [(V_\varepsilon, W_\varepsilon)] \) is an ordinary control and the Lebesgue’s dominated convergence theorem guarantees that
\[
\lim_{\varepsilon \to 0^+} \int_0^{T_\varepsilon} \lambda \left( y_{(v_\varepsilon, w_\varepsilon)}, v_\varepsilon, w_\varepsilon \right) dt = \int_0^T \lambda \left( y_{(v, w)}, v, w \right) dt.
\]

\( \Box \)

The following example shows that there are problems in which the Fréchet generalized minimizer contain jumps along discontinuities of the auxiliary Lagrangian and yet have no Lavrentiev gap. Thus, continuity of the auxiliary Lagrangian is not a necessary condition to exclude existence of gap.

**Example 37.** Consider the optimal control problem
\[
J(u) = \int_0^1 |x_1(t)|^\alpha u(t)^2 dt \to \min,
\]
\[
\dot{x}_1 = x_1 + x_2, \quad \dot{x}_2 = u, \quad x(0) = (0, -1), \quad x(1) = (0, 0),
\]
with \( \alpha > 0 \) constant.

The auxiliary Lagrangian is
\[
\lambda(y_1, y_2, v, w) = \lambda(y_1, v, w) = \begin{cases} |y_1|^\alpha \frac{w^2}{v}, & \text{if } v \neq 0, \\ 0, & \text{if } w = 0 \text{ or } y_1 = 0, \\ +\infty, & \text{if } v = 0, y_1 \neq 0, w \neq 0. \end{cases}
\]

Clearly, it is discontinuous at the points \( (0, 0, w), w \in \mathbb{R} \), for every positive \( \alpha \). The auxiliary problem is
\[
I(v, w) = \int_0^T \lambda(y_1, v, w) dt \to \min,
\]
\[
\dot{y}_1 = (y_1 + y_2)v, \quad \dot{y}_2 = w, \quad v \geq 0, \quad v^2 + w^2 = 1,
\]
\[
y(0) = (0, -1), \quad V(T) = 1, \quad y(T) = (0, 0).\]
The control \((\tilde{v}, \tilde{w}) = (0, 1)\chi_{[0,1]} + (1, 0)\chi_{[1,\infty]}\) satisfies the boundary condition with \(T = 2\) and \(I(\tilde{v}, \tilde{w}) = 0\). Thus, it is optimal. It corresponds to a Fréchet generalized control with an impulse at \(t = 0\).

Now, consider the approximation of the generalized minimizer by ordinary controls corresponding to \((v, \tilde{w}) = (\eta, 1)\chi_{[0,1]} + (1, 0)\chi_{[1,\infty]}\). A simple computation shows that \(V_\eta(2 - \eta) = 1\), \(y_{(v,\tilde{w})}(2 - \eta) = O(\eta)\) and \(I(v, \tilde{w}) = O(\eta^{\alpha - 1})\). Thus, the problem has no Lavrentiev gap when \(\alpha > 1\).

The argument breaks down for \(\alpha \leq 1\). Indeed, it can be shown that the problem has a Lavrentiev gap when \(\alpha \in [0,1]\).

The Proposition 36 has the following immediate corollary.

**Corollary 38.** Suppose that the Lagrangian can be written as

\[
L(x, u) = L_1(x) + L_2(x, u), \quad \forall (x, u) \in \mathbb{R}^{n+k},
\]

with \(u \mapsto L_2(x, u)\) positively homogeneous of degree 1 for every \(x \in \mathbb{R}^n\). Then, the problem (1)–(3) has no Lavrentiev gap in the sense of Definition 34. □

**Proof.** If the assumption holds, then \(\lambda(y, v, w) = L_1(y)v + L_2(y, w)\). □

**Remark 39.** Linear growth of the Lagrangian with respect to control does not guarantee lack of Lavrentiev gap.

To see this, consider the same dynamics and boundary conditions as in Example 35 and introduce the modified functional

\[
\tilde{J}(u) = \int_0^1 |x_1(t)| + h(x_1(t), u(t)) + \varepsilon |u(t)| dt,
\]

with \(\varepsilon > 0\), a small constant. Existence of generalized minimizer is guaranteed by Proposition 35.

The inequality \(\tilde{J}(u) \geq C\) holds for every ordinary control satisfying the boundary condition. However, for the generalized minimizer given in Example 35 we have \(\tilde{I}(\tilde{v}, \tilde{w}) = \varepsilon\), and therefore

\[
\inf_{[\tilde{(v, w)}] \in J_{\tilde{u}, 1}} \tilde{I}([\tilde{(v, w)}]) < \inf_{u \in L_1[0,1]} \tilde{J}(u)
\]

for sufficiently small \(\varepsilon > 0\). □

We conclude this section with two further cases where Lavrentiev gap cannot occur.

**Proposition 40.** If \(L(x, u) = L_1(x) + L_2(u)\) with \(L_1\) continuous and \(L_2\) convex, then the optimal control problem does not have a Lavrentiev gap. □

**Proof.** Since \(y_{(v+\eta,w)} \rightarrow y_{(v,w)}\) uniformly when \(\eta \rightarrow 0^+\), and

\[
\left( L_1(y_{(v+\eta,w)}) - L_2\left(\frac{w}{v+\eta}\right) \right)(v + \eta) \leq \lambda(y_{(v,w)}, v, w) + \left( L_1(y_{(v+\eta,w)}) - L_1(y_{(v,w)}) \right)(v + \eta) + L_2(0)\eta,
\]

The result follows from Lebesgue’s dominated convergence theorem. □

**Proposition 41.** If \(f \equiv 0\) (i.e., the system (2) has no drift), then the problem (1)–(3) has no Lavrentiev gap in the sense of Definition 34. □
Proof. If the system (2) has no drift, then $y(v,w) = y(\tilde{v},w)$ for every $v, \tilde{v}, w$. Since $L(y, \frac{w}{v}) (v + \varepsilon) \leq \lambda (y, v, w) + \varepsilon L(y, 0)$, it follows that
\[
\lim_{\varepsilon \to 0^+} I(V + \varepsilon t, W) = I(V, W)
\]
for every generalized control $[(V, W)]$. \qed

8. Example

We provide an example of a Lagrange variational problem with a functional of linear growth, for which the minimum is attained at a generalized minimizer.

The set of all horizontal curves in the Heisenberg group can be identified with the set of trajectories of the control system
\[
\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = 2x_2 u_1 - 2x_1 u_2.
\]
(49)

By adding a smooth drift $f$, one obtains a control-affine system
\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{pmatrix} = \begin{pmatrix}
f_1(x) \\
f_2(x) \\
f_3(x)
\end{pmatrix} v + \begin{pmatrix}
1 \\
0 \\
2x_2
\end{pmatrix} u_1 + \begin{pmatrix}
0 \\
1 \\
-2x_1
\end{pmatrix} u_2
\]
(50)

We wish to minimize the functional
\[
J(u_1, u_2) = \int_0^1 \sqrt{1 + u_1^2 + u_2^2} \, dt,
\]
under the boundary conditions $x(0) = \bar{x}, \ x(1) = \bar{x}$.

This problem satisfies the assumptions of Proposition 33, provided $f$ does not have supralinear growth with respect to $x$. Therefore, it has a generalized solution in the class $F_{0,1}^2$.

The auxiliary Lagrangian is
\[
\lambda(y, v, w) = v \sqrt{1 + \left(\frac{w_1}{v}\right)^2 + \left(\frac{w_2}{v}\right)^2} = \sqrt{v^2 + w_1^2 + w_2^2}.
\]
Therefore, the Proposition 36 guarantees that the problem (50)–(51) does not have a $L^2[0,1]-F_{0,1}^2$ Lavrentiev gap.

The extension of the problem (50)–(51) is equivalent to the problem
\[
T \to \min,
\]
\[
\begin{pmatrix}
\dot{y}_1 \\
\dot{y}_2 \\
\dot{y}_3
\end{pmatrix} = \begin{pmatrix}
f_1(y) \\
f_2(y) \\
f_3(y)
\end{pmatrix} v + \begin{pmatrix}
1 \\
0 \\
2y_2
\end{pmatrix} w_1 + \begin{pmatrix}
0 \\
1 \\
-2y_1
\end{pmatrix} w_2, \quad \dot{V} = v,
\]
(53)

$v \geq 0, \ v^2 + w_1^2 + w_2^2 = 1,$
(54)

$y(0) = \bar{x}, \ V(0) = 0, \ V(T) = 1, \ y(T) = \bar{x}.$
(55)

Optimal controls for this problem satisfy the Pontryagin maximum principle with Hamiltonian
\[
H = (\lambda_1 f_1(y) + \lambda_2 f_2(y) + \lambda_3 f_3(y) + \lambda_4) v + (\lambda_1 + \lambda_3 y_2) w_1 + (\lambda_2 - \lambda_3 y_1) w_2.
\]
An optimal trajectory of the problem (50)–(51) exhibits a jump if there is an interval where the corresponding extremal of the problem (52)–(55) satisfies
\[
\lambda_1 f_1(y) + \lambda_2 f_2(y) + \lambda_3 f_3(y) + \lambda_4 \leq 0,
\]
and hence, by the Pontryagin maximum principle, \( v(t) = 0 \). Jump paths are sub-Riemannian geodesics of the Heisenberg group.

The presence or absence of jumps in optimal solutions depends on the drift vector field \( f \).

8.1. **Constant drift.** For example, if the drift is a constant vector field of the form \( f \equiv (0, 0, C) \), one can easily conclude, that all optimal trajectories are continuous.

Indeed in this case the Hamiltonian amounts to

\[
H = (\lambda_3 C + \lambda_4) v + (\lambda_1 + \lambda_3 y_2) w_1 + (\lambda_2 - \lambda_3 y_1) w_2,
\]

and the extremals satisfy \( \dot{\lambda}_3 = \dot{\lambda}_4 \equiv 0 \), \( v(t) = \max\{0, \lambda_3 C + \lambda_4\} \). It follows that \( \lambda_3 C + \lambda_4 \) is constant and, if the constant is positive, then \( v(\cdot) \) does not vanish and the extremal trajectory is continuous, or, if it is non-positive, then \( v(\cdot) \) vanishes identically. The latter possibility is incompatible with the condition \( \int_0^T v(s)ds = 1 \).

8.2. **Case of linear drift.** Contrasting with the case above, for the linear drift vector field \( f(x) = (0, 0, -x_3) \) optimal trajectories may have a jump. This happens, for example, for the boundary conditions \( \overline{x} = 0 \), \( \overline{y} = (0, 0, C) \), whenever \( C > 0 \) is large enough. We prove that in this case the optimal trajectory consists of an analytic arc in the interval \( [0, 1] \) and a final jump at \( t = 1 \).

Let us write the equations of Pontryagin Maximum Principle with the Hamiltonian

\[
H = (\lambda_4 - \lambda_3 y_3) v + (\lambda_1 + 2\lambda_3 y_2) w_1 + (\lambda_2 - 2\lambda_3 y_1) w_2.
\]

The adjoint vector satisfies the system

\[
\dot{\lambda}_1 = 2\lambda_3 w_2, \quad \dot{\lambda}_2 = -2\lambda_3 w_1, \quad \dot{\lambda}_3 = \lambda_3 v, \quad \dot{\lambda}_4 = 0.
\]

As far as Hamiltonian \( H \) is homogeneous, we may consider the abnormal case \( H \equiv 0 \) and the normal one: \( H \equiv 1 \).

**Lemma 42.** Abnormal extremals for this problem are trivial: \( y(t) \equiv 0 \). □

**Proof.** The identity \( H \equiv 0 \) implies

\[
\lambda_1 + 2\lambda_3 y_2 \equiv 0, \quad \lambda_2 - 2\lambda_3 y_1 \equiv 0, \quad \lambda_4 - \lambda_3 y_3 \leq 0.
\]

Differentiating the first two equalities, one obtains

\[
\lambda_3(y_2 v + 2w_2) = \lambda_3(y_1 v + 2w_1) = 0.
\]

Besides

\[
\lambda_3(t) = e^{V(t)}\lambda_3(0),
\]

and if \( \lambda_3(t) \) vanishes at a point, then \( \lambda_3 \equiv 0 \) and by (57) \( \lambda_1 \equiv \lambda_2 \equiv 0 \), \( \lambda_4 < 0 \) and \( v \equiv 0 \), meaning that the end-point condition \( V(T) = 1 \) can not be achieved.

If \( y_2 v + 2w_2 = y_1 v + 2w_1 \equiv 0 \), then \( 1 = v^2 + w_1^2 + w_2^2 = v^2(1 + y_1^2/4 + y_2^2/4) \) and \( v = \frac{2}{\sqrt{4+y_1^2+y_2^2}} \) is absolutely continuous with

\[
\frac{dv}{dt} = 2 \frac{y_1^2+y_2^2}{(4+y_1^2+y_2^2)^2} = \frac{1}{2} v^2(1 - v^2).
\]

Besides \( v(0) = \sqrt{\frac{2}{4+(y_1(0))^2 + (y_2(0))^2}} = 1 \), and hence \( v(t) \equiv 1, w_1(t) = w_2(t) \equiv 0 \), which results in a trivial trajectory \( y \equiv 0 \) □
Now, consider an extremal \((y, \lambda) = (y_1, y_2, y_3, \lambda_1, \lambda_2, \lambda_3, \lambda_4)\) such that \(H \equiv 1\), \(y(0) = 0\) and \(y(T) = (0, 0, C)\), \(C > 1\). The extremal controls are

\[
(59) \quad v = \max(0, \lambda_4 - \lambda_3 y_3), \quad w_1 = \lambda_1 + 2\lambda_3 y_2, \quad w_2 = \lambda_2 - 2\lambda_3 y_1.
\]

We will use the following three lemmata.

**Lemma 43.** For the imposed boundary conditions, the extremal control \(v(\cdot)\) is monotonously decreasing and \(0 < \lambda_4 \leq 1\). \(\square\)

**Proof.** Let \(\sigma_v(t) = \lambda_4 - \lambda_3(t)y_3(t)\) be the switching function, which determines \(v(t)\) along the extremal. From (59) and the dynamics (53), (56), we have

\[
\frac{d}{dt}(\lambda_1 y_2 - \lambda_2 y_1) = 2\lambda_3 w_2 y_1 + \lambda_1 w_2 + 2\lambda_3 w_1 y_1 - \lambda_2 w_1 = w_1 w_2 - w_2 w_1 = 0,
\]

and therefore

\[
(60) \quad \lambda_1 y_2 - \lambda_2 y_1 \equiv 0
\]

along any extremal trajectory with \(y(0) = 0\). Further,

\[
\frac{d}{dt}(\lambda_3 y_3) = \lambda_3 v y_3 + \lambda_3 (-y_3 v + 2 y_2 w_1 - 2 y_1 w_2) = 2\lambda_3 (\lambda_1 y_2 - \lambda_2 y_1) + 4\lambda_3^2 (y_1^2 + y_2^2) = 4\lambda_3^2 (y_1^2 + y_2^2) \geq 0.
\]

Therefore, \(\frac{d}{dt}(\lambda_4 - \lambda_3 y_3) = -\frac{d}{dt}(\lambda_3 y_3) \leq 0\) and hence the extremal control \(v\) is a monotonically decreasing function. Since the boundary condition (55) requires \(v\) to be positive in some interval, we see that \(\lambda_4 > 0\). The equality \(H \equiv 1\) implies \(\lambda_4 = \lambda_4 - \lambda_3(0)y_3(0) \leq 1\). \(\square\)

**Lemma 44.** The trajectories of the system

\[
\dot{y}_1 = w_1, \quad \dot{y}_2 = w_2, \quad \dot{y}_3 = -y_3 v + 2y_2 w_1 - 2y_1 w_2, \quad y(0) = 0,
\]

are invariant with respect to the dilation

\[
(v, w_1, w_2, y_1, y_2, y_3) \rightarrow (v, \eta w_1, \eta w_2, \eta y_1, \eta y_2, \eta^2 y_3).\]

**Proof.** A direct verification. \(\square\)

**Lemma 45.** For each \(T \geq 0\) the attainable set of the system (53) is bounded; in particular the time \(T_C\), needed to attain the point \((0, 0, C)\), grows to \(+\infty\) as \(C \rightarrow +\infty\). \(\square\)

**Proof.** Notice that the right-hand side of (53) is bounded by a linear function of \(|y|\). \(\square\)

Now, let \((\hat{y}, \hat{\lambda})\) be an extremal satisfying the boundary conditions (55), and let \((\hat{v}, \hat{w}) = (\hat{v}, \hat{w}_1, \hat{w}_2)\) be the corresponding extremal control. Assume the extremal value of the functional to be \(\hat{T}\) and \(\hat{v}(t) > 0\) on \([0, \hat{T}]\). We will prove that if \(C > 0\) is large enough, the extremal cannot be optimal.

We proceed by showing that there is a control \((v, w)\) with \(v \geq 0\) such that the corresponding trajectory of (53) satisfies \(V(\hat{T}) = 1, y(\hat{T}) = (0, 0, C)\), and

\[
\int_0^{\hat{T}} \sqrt{v^2 + w_1^2 + w_2^2} dt < \hat{T}.
\]
This inequality requires that \( v^2 + w_1^2 + w_2^2 \neq 1 \), but the Propositions 18 and 26 show that \((v, w)\) can be transformed by a time reparameterization into a control satisfying (54) and (55) for \( T = \int_0^T \sqrt{v^2 + w_1^2 + w_2^2} \, dt \) and therefore \( \hat{T} \) is not minimal.

For each \( \varepsilon \in ]0, \hat{T}[ \), let \( a = \int_{\hat{T} - \varepsilon}^\hat{T} \hat{v} \, dt \). We avoid the notation \( a(\varepsilon) \), but keep the dependence on \( \varepsilon \) in mind. In particular, \( 0 < a \leq \varepsilon \).

Fix \( \varepsilon \) and consider the modified control \( \hat{v} \), defined as

\[
\hat{v}(t) = \begin{cases} 
\hat{v}(t) + 1, & t \in [0, a], \\
\hat{v}(t), & t \in [a, \hat{T} - \varepsilon[, \\
0, & t \in [\hat{T} - \varepsilon, \hat{T}],
\end{cases}
\]

and let \( \hat{y} \) be the trajectory of the system (53) for the control \((\hat{v}, \hat{w})\).

Then, \( \hat{V}(\hat{T}) = \hat{V}(\hat{T}) = 1 \), \( \hat{y}_1 = y_1 \), and \( \hat{y}_2 = y_2 \). Further, for any \( t \geq 0 \):

\[
\hat{y}_3(t) - \hat{y}(t) = \int_0^t \hat{y}_3v - \hat{y}_3\hat{v} \, d\tau = \int_0^t - (\hat{y}_3 - \hat{y}_3)\hat{v} + \hat{y}_3(\hat{v} - \hat{v}) \, d\tau.
\]

It follows that

\[
\hat{y}_3(\hat{T}) - \hat{y}(\hat{T}) = e^{-1} \int_0^{\hat{T}} e^{\hat{V}(\hat{T})} \hat{y}_3(\hat{v} - \hat{v}) \, d\tau = - \int_0^a \int_0^{\hat{T} - \varepsilon} e^{\hat{V}(\hat{T})} \hat{y}_3 \, d\tau + \int_0^{\hat{T} - \varepsilon} \hat{y}_3 \, d\tau.
\]

Since \( |\hat{y}_3(t)| \leq 2t^2 \) and \( \lim_{t \to \hat{T}} \hat{y}_3(t) = C \), there is a constant \( k \in ]0, +\infty[ \) such that

\[
(61) \quad \hat{y}_3(\hat{T}) > C(1 + a - k \varepsilon a),
\]

for every \( C \in ]0, +\infty[ \) and every sufficiently small \( \varepsilon > 0 \).

Let \( \eta = \sqrt{\frac{C}{\hat{y}_3(\hat{T})}} \). Due to Lemma 44, the control \((\hat{v}, \hat{w}) = (\hat{v}, \eta \hat{w}_1, \eta \hat{w}_2)\) satisfies the boundary conditions (55), and we estimate the functional

\[
\int_0^{\hat{T}} \sqrt{v^2 + w_1^2 + w_2^2} \, dt = \int_0^a \sqrt{1 + \hat{v}^2} \, dt + \int_a^{\hat{T} - \varepsilon} \sqrt{\hat{v}^2 + \eta^2(1 - \hat{v}^2)} \, dt + \int_{\hat{T} - \varepsilon}^{\hat{T}} \sqrt{\eta^2(1 - \hat{v}^2)} \, dt 
\]

\[\leq \int_0^a \sqrt{1 + \hat{v}^2} \, dt + \int_0^a \sqrt{1 + 2\hat{v} + \hat{v}^2} \, dt + \int_a^{\hat{T} - \varepsilon} \sqrt{\hat{v}^2 + \eta^2(1 - \hat{v}^2)} \, dt.
\]

Since \( 1 + \hat{v} \leq 3 \), the second integral is bounded by \( \sqrt{3}a \) and therefore

\[
\int_0^a \sqrt{1 + \hat{v}^2} \, dt \leq \int_0^a \sqrt{1 - (1 - \eta^2)(1 - \hat{v}^2)} \, dt + \sqrt{3}a 
\]

\[\leq \int_0^a \sqrt{1 - \frac{1 - \eta^2}{2} (1 - \hat{v}^2)} \, dt + \sqrt{3}a \leq \frac{1 - \eta^2}{2} \int_0^a 1 - \hat{v} \, dt + 3a = \hat{T} - \frac{1 - \eta^2}{2} (\hat{T} - 1) + \sqrt{3}a
\]

Since (61) implies \( 1 - \eta^2 > \frac{1 - k \varepsilon}{1 + (1 - k \varepsilon)a} \), the estimate above yields

\[
\int_0^{\hat{T}} \sqrt{v^2 + w_1^2 + w_2^2} \, dt < \hat{T} - \left( \frac{\hat{T} - 1}{2} + \frac{1 - k \varepsilon}{1 + (1 - k \varepsilon)a} - \sqrt{3} \right) a < \hat{T}.
\]
any functions (uniformly in compact intervals towards some absolutely continuous nondecreasing
belong to $T$ provided $\epsilon > 0$ is sufficiently small and $\hat{T} > 1 + 2\sqrt{3}$. Due to Lemma 45, this last
condition holds for every sufficiently large $C > 0$. For such $C$ no extremal satisfying $\hat{v} > 0$ in $[0, \hat{T}]$ can be optimal.

9. Appendix: proofs of technical results

9.1. Proof of Lemma 4

Proof. Suppose that (6) holds and pick a sequence $\{\beta_i \in T\}_{i \in \mathbb{N}}$ such that

$$\lim_{i \to \infty} \|g_1 - g_2 \circ \beta_i\|_{L^\infty[0, +\infty]} = 0.$$ 

For each $i \in \mathbb{N}$, let $\alpha_{1,i}$ denote the inverse function of $t \mapsto t + \beta_i(t)$, and let

$$\alpha_{2,i} = \beta_i \circ \alpha_{1,i}.$$ 

Since $\alpha_{1,i} + \alpha_{2,i} = Id$, it follows that $\alpha_1 + \alpha_2 = Id$ and therefore (b) holds.

Suppose that $\alpha_1(\infty) = T < +\infty$. Due to continuity of $g_1$, $g_1(T^-) = g_1(T)$. For any $t > T$, and any $i \in \mathbb{N}$:

$$\left|g_1(T) - g_1(t)\right| = \left|g_1(T) - g_1 \circ \alpha_{1,i} \left(\alpha_{1,i}^{-1}(t)\right)\right| \leq \left|g_1(T) - g_2 \circ \alpha_{2,i} \left(\alpha_{1,i}^{-1}(t)\right)\right| + \left|g_2 \circ \alpha_{2,i} \left(\alpha_{1,i}^{-1}(t)\right) - g_1 \circ \alpha_{1,i} \left(\alpha_{1,i}^{-1}(t)\right)\right| \leq \left|g_1(T) - g_2 \circ \alpha_{2,i} \left(\alpha_{1,i}^{-1}(t)\right)\right| + \left\|g_2 \circ \alpha_{2,i} - g_1 \circ \alpha_{1,i}\right\|_{L^\infty[0, +\infty]}.$$

By assumption, $\lim_{i \to \infty} \alpha_{1,i}^{-1}(t) = +\infty$ and therefore $\lim_{i \to \infty} \alpha_{2,i} \left(\alpha_{1,i}^{-1}(t)\right) = +\infty$. Since

the condition (a) implies that $\lim_{s \to +\infty} g_2(s) = g_1(T)$, (c) holds.

Now, suppose there are $\alpha_1, \alpha_2$ satisfying (a), (b), and (c).

First, consider the case where $\alpha_1([0, +\infty[) = \alpha_2([0, +\infty[) = [0, +\infty[$. Then, there is a sequence $\{T_j\}_{j \in \mathbb{N}}$ such that

$$\lim_{j \to \infty} T_j = +\infty, \quad \text{and} \quad \alpha_i(T_j) < \alpha_i(T_{j+1}) \quad \forall j \in \mathbb{N}, \quad i = 1, 2.$$

For any sequence $\{\varepsilon_j \in ]0, 1]\}_{j \in \mathbb{N}}$, the functions

$$\alpha_i^\varepsilon(t) = \sum_{j=1}^\infty \left(\alpha_i(T_{j-1}) + (1 - \varepsilon_j)(\alpha_i(t) - \alpha_i(T_{j-1})) + \varepsilon_j \frac{\alpha_i(T_j) - \alpha_i(T_{j-1})}{T_j - T_{j-1}}(t - T_{j-1})\right) \chi_{[T_{j-1}, T_j]}(t)$$

belong to $\mathcal{T}$ and therefore $\alpha_i^\varepsilon \circ (\alpha_i^\varepsilon)^{-1} \in \mathcal{T}$. Also,

$$\left|\alpha_i^\varepsilon(t) - \alpha_i(t)\right| \leq \sum_{j=1}^\infty \varepsilon_j \left|\alpha_i(T_j) - \alpha_i(T_{j-1})\right| \chi_{[T_{j-1}, T_j]}(t) \quad \forall t \geq 0.$$
Since \( g_1, g_2 \) are uniformly continuous in compact intervals, for every \( \delta > 0 \) there is some sequence \( \{ \varepsilon_j \in ]0, 1[ \}_{j \in \mathbb{N}} \) such that \( \| g_1 \circ \alpha_1^j - g_2 \circ \alpha_2^j \|_{L^\infty[0, +\infty[} < \delta \). Since
\[
\| g_1 \circ \alpha_1^j - g_2 \circ \alpha_2^j \|_{L^\infty[0, +\infty[} = \| g_1 - g_2 \circ \alpha_2 \circ (\alpha_1^j)^{-1} \|_{L^\infty[0, +\infty[},
\]
we see that \( \text{[6]} \) holds.

In the case where \( \alpha_1([0, +\infty[) = [0, +\infty[ \) and \( \alpha_2(\infty) = T < +\infty \), there is a sequence \( \{ T_j \}_{j \in \mathbb{N}} \) such that
\[
\lim T_j = +\infty, \quad \text{and} \quad \alpha_1(T_j) < \alpha_1(T_{j+1}) \ \forall j \in \mathbb{N}.
\]
Then we can apply a similar argument to the functions
\[
\alpha_1^j(t) = \sum_{j=1}^{\infty} \left( \alpha_1(T_{j-1}) + (1 - \varepsilon_j)(\alpha_1(t) - \alpha_1(T_{j-1})) + \varepsilon_j \frac{\alpha_1(T_j) - \alpha_1(T_{j-1})}{T_j - T_{j-1}}(t - T_{j-1}) \right) \chi_{[T_{j-1}, T_j]}(t),
\]
\[
\alpha_2^j(t) = \alpha_2(t) + \varepsilon_j t,
\]
and this completes the proof. \( \square \)

9.2. Proof of Lemma \[5\]

**Proof.** To prove (a): Pick \( t \in [\alpha(0), \alpha(\infty)[ \), and let \( \hat{\theta} = \alpha^\#(t) \). By continuity of \( \alpha \), there is some \( s \in ]0, +\infty[ \) such that \( t = \alpha(s) \), and \( \hat{\theta} = \max \{ \theta : \alpha(\theta) = \alpha(s) \} \), that is, \( \alpha(\hat{\theta}) = \alpha(s) = t \). The equality \( \dot{\alpha} \circ \alpha^\#(t) = 0 \) reduces to \( \dot{\alpha}(\hat{\theta}) = 0 \). Therefore, \( \dot{\alpha} \circ \alpha^\#(t) = 0 \) implies \( t \in \alpha(\{ \theta : \dot{\alpha} = 0 \}) \). Since this set has zero Lebesgue measure, we proved (a).

To prove (b) and (c): Let \( A = \{ t : \dot{\alpha}(t) = 0, \hat{g}(t) \neq 0 \} \), and let \( \mu \) denote the Lebesgue measure.

For each \( \varepsilon > 0 \) there is a sequence of intervals \( \{ [a_i, b_i] \}_{i \in \mathbb{N}} \) such that
\[
A \subset \bigcup_{i=1}^{\infty} [a_i, b_i], \quad \sum_{i=1}^{\infty} (b_i - a_i) < \mu(A) + \varepsilon, \quad \sum_{i=1}^{\infty} (\alpha(b_i) - \alpha(a_i)) < \varepsilon.
\]
Fix \( \varepsilon \) and a sequence as above and let
\[
t_i = \alpha(b_i), \quad s_i = \alpha(a_i) - \frac{\varepsilon}{2i}, \quad i \in \mathbb{N}.
\]
Notice that \( \sum_{i=1}^{\infty} (t_i - s_i) < 2\varepsilon \) and \( \alpha^\#(t_i) \geq b_i, \alpha^\#(s_i) < a_i \) for every \( i \in \mathbb{N} \).

Therefore,
\[
\sum_{i=1}^{\infty} \int_{\alpha^\#(s_i)}^{\alpha^\#(t_i)} |\hat{g}(s)|ds \geq \sum_{i=1}^{\infty} \int_{a_i}^{b_i} |\hat{g}(s)|ds \geq \int_A |\hat{g}(s)|ds.
\]
Thus, \( g \circ \alpha^\# \) cannot be absolutely continuous when \( \mu(A) > 0 \).

Now, suppose that \( \mu(A) = 0 \). In order to prove that \( g \circ \alpha^\# \) is absolutely continuous and satisfies \( \text{[5]} \), we only need to consider the case where \( g \) is scalar. Taking the decomposition \( g = g^+ - g^- \), where \( g^+(t) = g(0) + \int_0^t \max(0, \hat{g}(s))ds \) and \( g^-(t) = \int_0^t \max(0, -\hat{g}(s))ds \), we only need to consider the case where \( g : [0, +\infty[ \to \mathbb{R} \) and \( \hat{g}(s) \geq 0 \) for a.e. \( s \geq 0 \).
Fix $t_1 < t_2$ with $t_1 \geq 0$, $t_2 < \alpha(\infty)$, and fix $T \in [\alpha^#(t_2), +\infty[$. For each $i \in \mathbb{N}$, let 
\[ \alpha_i(s) = \sup \left\{ \alpha(\bar{s}) + \frac{s - \bar{s}}{i} : \bar{s} \in [0, s] \right\} \quad \forall s \in [0, T], \]
and let $B_i = \{ s \in [0, S] \mid \alpha(s) < \alpha_i(s) \}$. Since $\dot{g}(s) = 0$ for a.e. $s \in [\alpha^#(t^-), \alpha^#(t)]$, and $\lim_{i \to \infty} \alpha_i^{-1}(t) = \alpha^#(t^-)$, we have 
\[ g \circ \alpha^#(t_2) - g \circ \alpha^#(t_1) = \int_{\alpha_i^{-1}(t_2)}^{\alpha_i^{-1}(t_1)} \dot{g} ds = 0. \]

Since $\int_{\alpha_i^{-1}(t_1), \alpha_i^{-1}(t_2)} \dot{g} ds = 0$, it follows that 
\[ g \circ \alpha^#(t_2) - g \circ \alpha^#(t_1) = \lim_{i \to \infty} \int_{\alpha_i^{-1}(t_1), \alpha_i^{-1}(t_2)} \dot{g} ds. \]

Notice that $\dot{\alpha}_i$ is absolutely continuous and $\dot{\alpha}_i = \dot{\alpha}\chi_{B_i^c} + \frac{1}{2} \chi_{B_i} \geq \frac{1}{2}$. Therefore, 
\[ g \circ \alpha^#(t_2) - g \circ \alpha^#(t_1) = \lim_{i \to \infty} \int_{[t_1, t_2] \setminus \alpha_i(B_i)} \frac{\dot{g}}{\dot{\alpha}_i} \circ \alpha_i^{-1} ds = \]
\[ = \lim_{i \to \infty} \int_{[t_1, t_2] \setminus \alpha_i(B_i)} \frac{\dot{g}}{\dot{\alpha}} \circ \alpha^# ds. \]

Since $B_{i+1} \subset B_i$ and the set $\alpha_i(B_i)$ has Lebesgue measure no greater than $\frac{T}{T}$, the Lebesgue monotone convergence theorem guarantees that 
\[ g \circ \alpha^#(t_2) - g \circ \alpha^#(t_1) = \int_{t_1}^{t_2} \frac{\dot{g}}{\dot{\alpha}} \circ \alpha^# ds. \]
Thus, $g \circ \alpha^#$ is absolutely continuous and satisfies (d).

To prove (d): Notice that $\alpha(s) = \alpha(t)$ for every $s \in [t, \alpha^# \circ \alpha(t)]$. Since the set \{ $t : \dot{\alpha}(t) = 0$, $\dot{g}(t) \neq 0$ \} has zero Lebesgue measure, we see that $g(s) = g(t)$ for every $s \in [t, \alpha^# \circ \alpha(t)]$. In particular, $g \circ \alpha^# \circ \alpha(t) = g(t)$. Thus, the result follows from Lemma 3.

9.3. Proof of Proposition 11

Proof: Without loss of generality, we can assume that $x$ has finite variation in the interval $[0, T]$ and it is constant in $[T, +\infty[$.

Consider a sequence of partitions of the interval $[0, T]$
\[ P_k = \{ 0 = t_{k,0} < t_{k,1} < \cdots < t_{k,k} = T \} \quad k \in \mathbb{N}, \]
such that $P_k \subset P_{k+1}$ for every $k \in \mathbb{N}$, and $\bigcup_{k \in \mathbb{N}} P_k$ is dense in $[0, T]$. Let $x_k : [0, +\infty[ \to \mathbb{R}^n$ be the piecewise linear function interpolating the points $x(t_{k,i})$, $i = 0, 1, \ldots, k$ and $x_k(t) = x(T)$ for every $t > T$. Then, $\{(t_k, y_k) = \left( \ell_{t,x_k}^{-1}, x_k \circ \ell_{t,x_k}^{-1} \right) \}_{k \in \mathbb{N}}$ is a sequence in $\mathcal{Y}_n$. 


The length of the graph of $x_k$ on the interval $[0, T]$ is

$$\ell_{(t, x_k)}(T) = \sum_{i=1}^{k} \sqrt{(t_i - t_{i-1})^2 + |x(t_i) - x(t_{i-1})|^2} \leq T + V_{[0, T]}(x).$$

Thus, the sequence $\{(\theta_k, y_k)\}$ is uniformly bounded and equicontinuous on the interval $[0, T + V_{[0, T]}(x)]$, and the Ascoli-Arzelà theorem guarantees that it has a subsequence converging uniformly uniformly towards some $(\theta, y) \in \mathcal{Y}_n$. Without loss of generality, we assume that this subsequence is $\{(\theta_k, y_k)\}$.

Notice that $\{\theta_k^{-1}(t)\}$ may fail to converge towards $\theta^\#(t)$ if $t$ is a discontinuity point of $\theta^\#$. Instead, we take the sequence $\tilde{\theta}_k = \left(\theta_k - \|\theta_k - \theta\|_{L\infty[0, T + V_{[0, T]}(x)]}\right)^+$. Notice that $\{(\tilde{\theta}_k, y_k)\}$ converges uniformly towards $(\theta, y)$ and $\lim_{k \to \infty} \tilde{\theta}_k^\#(t) = \theta^\#(t)$ for every $t \in [0, T]$. Therefore,

$$\lim_{k \to \infty} y_k \circ \tilde{\theta}_k^\#(t) = y \circ \theta^\#(t) \quad \forall t \in [0, T].$$

Now, suppose that $x$ is continuous at the point $t \in [0, T]$. By continuity, for every $\varepsilon > 0$ there is some $\delta > 0$ such that $|x(\tau) - x(t)| < \varepsilon$ for every $\tau \in [t - \delta, t + \delta]$. This implies $|x_k(\tau) - x(t)| < \varepsilon$ for every sufficiently large $k$ and every $\tau \in [t - \frac{\delta}{2}, t + \frac{\delta}{2}]$, because then $x_k(\tau)$ is a convex combination of points in $B_k(x(t))$. Thus,

$$y \circ \theta^\#(t) = \lim_{k \to \infty} y_k \circ \tilde{\theta}_k^\#(t) = \lim_{k \to \infty} x_k \left(t + \|\theta_k - \theta\|_{L\infty[0, T + V_{[0, T]}(x)]}\right) = x(t).$$

\[\square\]

9.4. Proof of Lemma [23].

Proof. Notice that

$$L \left( y, \frac{\lambda w + (1 - \lambda)\hat{w}}{\lambda v + (1 - \lambda)\hat{v}} \right) \left(\lambda v + (1 - \lambda)\hat{v}\right) =
\begin{align*}
&= L \left( y, \frac{\lambda v}{\lambda v + (1 - \lambda)\hat{v}} \frac{w}{v} + \frac{(1 - \lambda)\hat{v}}{\lambda v + (1 - \lambda)\hat{v}} \frac{\hat{w}}{v} \right) \left(\lambda v + (1 - \lambda)\hat{v}\right) \\
&\leq \frac{\lambda v}{\lambda v + (1 - \lambda)\hat{v}} L \left( y, \frac{w}{v} \right) + \frac{(1 - \lambda)\hat{v}}{\lambda v + (1 - \lambda)\hat{v}} L \left( y, \frac{\hat{w}}{v} \right) \\
&= \lambda L \left( y, \frac{w}{v} \right) v + (1 - \lambda) L \left( y, \frac{\hat{w}}{v} \right) \hat{v}.
\end{align*}$$

Therefore, $(v, w) \mapsto L \left( y, \frac{w}{v} \right) v$ is convex.

The inequality

$$\lim_{v \to 0} \inf_{(y, v, w) \to (\hat{y}, \hat{v}, \hat{w})} L \left( y, \frac{w}{v} \right) v \leq \lim_{v \to 0} \inf_{(y, v, w) \to (\hat{y}, \hat{v})} L \left( y, \frac{w}{v} \right) v \leq \lim_{v \to 0} \inf_{(y, v, w) \to (\hat{y}, \hat{v}), \hat{v} \to \hat{\varepsilon}} L \left( \hat{y}, \frac{\hat{w}}{v} \right) v$$

holds trivially. Therefore, we only need to prove that

$$\limsup_{v \to 0} L \left( \hat{y}, \frac{\hat{w}}{v} \right) v \leq \lim_{v \to 0} \inf_{(y, v, w) \to (\hat{y}, \hat{v}, \hat{w})} L \left( y, \frac{w}{v} \right) v.$$
Due to continuity of $L$, this inequality holds for every $\dot{v} > 0$. Suppose $\dot{v} = 0$ and fix $b < \limsup_{v \to 0^+} L\left(\check{y}, \frac{w}{v}\right) v$, and $\varepsilon > 0$. Then, we can pick $a \in [0, \varepsilon]$ such that $L\left(\check{y}, \frac{w}{a}\right) > b \frac{1}{a}$. By continuity of $L$, there is some $\delta > 0$ such that

$$L\left(y, \frac{w}{a}\right) > \frac{b}{a}, \quad \text{and} \quad |L(y, w) - L(\check{y}, \check{w})| < \varepsilon$$

for every $(y, w)$ such that $|y - \check{y}| < \delta$ and $|w - \check{w}| < \delta$.

Due to convexity of $w \mapsto L(y, w)$, we have

$$L\left(y, \frac{w}{v}\right) \geq L(y, w) + \frac{L\left(y, \frac{w}{v}\right) - L(y, w)}{\frac{1}{v} - 1} \frac{1}{v} - 1 = L(y, w) + \frac{a}{1 - a} \left(L\left(y, \frac{w}{a}\right) - L(y, w)\right) \frac{1 - v}{v} \quad \forall v \in [0, a].$$

Using the estimates (63), this yields

$$L\left(y, \frac{w}{v}\right) v \geq (L(\check{y}, \check{w}) - \varepsilon) v + \frac{a}{1 - a} \left(\frac{b}{a} - L(\check{y}, \check{w}) - \varepsilon\right) (1 - v) = 1 - v \frac{a}{1 - a} b + L(\check{y}, \check{w}) \left(v - a \frac{1 - v}{1 - a}\right) - \varepsilon \left(v + a \frac{1 - v}{1 - a}\right),$$

that is,

$$\liminf_{v \to 0^+} L\left(y, \frac{w}{v}\right) v \geq \frac{1}{1 - a} b - (L(\check{y}, \check{w}) + \varepsilon) \frac{a}{1 - a}.$$

Making $\varepsilon$ tend to zero and $b$ tend to $\limsup_{v \to 0^+} L\left(\check{y}, \frac{w}{v}\right) v$, this implies (62). \hfill \Box

9.5. Proof of Proposition 29

Proof. Fix $(C, \theta, y)$, a trajectory of the differential inclusion (39), and let $V_i = \left(\hat{\theta}(t), \hat{y}(t)\right)$ for almost every $t \in [0, T]$.

For each compact set $K \subset \mathbb{R}^{1+k}$, consider the function $F_K : [0, T] \to \mathbb{R}$, defined almost everywhere by

$$F_K(t) = \inf \left\{ \lambda(y(t), v, w) : (v, w) \in B^+ \cap K, (v, f(y(t))v + G(y(t))w) = V_i \right\},$$

being understood that $\inf \emptyset = +\infty$.

First, we show that the functions $F_K$ are measurable.

For any set $A \subset \mathbb{R}^k$ and any $\varepsilon > 0$, let $B_\varepsilon(A) = \bigcup_{x \in A} B_\varepsilon(x)$. Then, lower semicontinuity of $\lambda$ implies that for any $\alpha \in \mathbb{R},$

$$F_K^{-1}([-\infty, \alpha[) = \{ t : \exists (v, w) \in B^+ \cap K, (v, f(y(t))v + G(y(t))w) = V_i, \lambda(y(t), v, w) < \alpha \} = \bigcap_{i \in \mathbb{N}} \{ t : \exists (v, w) \in B_\varepsilon^-(B^+ \cap K), (v, f(y(t))v + G(y(t))w) - V_i < \frac{1}{i}, \lambda(y(t), v, w) < \alpha \}.$$
Due to Lemma 23, this is
\[ F_K^{-1}([-\infty, \alpha]) = \bigcap_{t \in \mathbb{N}} \bigcup_{v \in \mathbb{Q} \cap [0,1]} \left\{ t : \exists w \in \mathbb{R}^n, (v, w) \in B_{\frac{1}{t}} (B^+ \cap K), \right. \\
\left. |(v, f(y(t))v + G(y(t))w) - V_i| < \frac{1}{t}, (y(t), \frac{w}{v}) v < \alpha \right\}. \]

Due to continuity of \( L \), this further reduces to
\[ F_K^{-1}([-\infty, \alpha]) = \bigcap_{i \in \mathbb{N}} \bigcup_{w \in \mathbb{Q}^n \cap [0,1]} \bigcup_{(v, w) \in B_{\frac{1}{t}} (B^+ \cap K)} \left\{ t : |(v, f(y(t))v + G(y(t))w) - V_i| < \frac{1}{t}, \right. \\
\left. L(y(t), \frac{w}{v}) v < \alpha \right\}. \]

Since \( y, V \) are measurable and \( f, G, L \) are continuous, it follows that \( F_K \) is measurable.

Now, we construct a sequence \( \{A_i\}_{i \in \mathbb{N}} \) with the following properties:

(a) Each \( A_i = \{A_{i,1}, A_{i,2}, \ldots, A_{i,h_i}\} \) is a finite ordered collection of measurable subsets of \( B^+ \);

(b) All the members of each collection \( A_i \) are pairwise disjoint, \( B^+ = \bigcup_{A \in A_i} A \), and each element of \( A_i \) is contained in a ball of radius \( \frac{1}{t} \);

(c) For any \( i < j \), every element of \( A_j \) is a subset of some element of \( A_i \). All elements of \( A_j \) that are contained in \( A_{i,h} \) precede (in the order of \( A_i \)) any element of \( A_j \) contained in \( A_{i,h+1} \).

To see that such sequences exist, let \( A_0 = \{B^+\} \). For each \( i \in \mathbb{N} \), let \( A_i = \{B_{i,1}, B_{i,2}, \ldots, B_{i,j_i}\} \), a finite cover of \( B^+ \) by balls of radius \( \frac{1}{t} \), and let
\[ C_{i,h} = B_{i,h} \setminus \bigcup_{j<h} B_{i,j}, \quad h = 1, 2, \ldots, j_i. \]

For each \( i \in \mathbb{N} \), let \( A_i \) be the collection of intersections
\[ A \cap C_{i,h}, \quad A \in A_{i-1}, \quad h = 1, 2, \ldots, j_i, \]

ordered in any way such that any \( C_{i,h} \cap A_{i-1,l} \) precedes every \( C_{i,s} \cap A_{i-1,l+1} \) (discard empty intersections). So, \( \{A_i\}_{i \in \mathbb{N}} \) satisfies (a)–(c).

Fix a sequence \( \{A_i\}_{i \in \mathbb{N}} \) as above and for each \( i \in \mathbb{N}, \quad j \in \{1, 2, \ldots, j_i\} \), fix \((v, w)_{i,j} = (v_{i,j}, w_{i,j}) \in A_{i,j}\). For each \( i \in \mathbb{N} \), consider a function \( j(i, \cdot) : [0, T] \to \mathbb{N} \) defined almost everywhere by
\[ j(i, t) = \min \left\{ h \in \{1, 2, \ldots, j_i\} : F^{-1}_{A_{i,h}}(t) = F_{B^+}(t) \right\}, \]
and consider the sequence \( \{(v_i, w_i) : [0, T] \to B^+\}_{i \in \mathbb{N}} \) defined as
\[ (v_i, w_i)(t) = (v, w)_{i,j(i,t)}, \quad i \in \mathbb{N}, \quad t \in [0, T]. \]

Notice that \((v, w)([0, T]) \subset \{(v, w)_{i,j}, j \in \{1, 2, \ldots, j_i\}\} \) is a finite set and
\[ \{t : (v_i, w_i)(t) = (v, w)_{i,j}\} = \left\{ t : F^{-1}_{A_{i,j}}(t) = F_{B^+}(t) \right\} \setminus \bigcup_{h<j} \left\{ t : F^{-1}_{A_{i,h}}(t) = F_{B^+}(t) \right\}. \]

Therefore, measurability of \( F_K \) guarantees measurability of \((v_i, w_i)\).

For almost every \( t \in [0, T] \), we have:
\[ (v_i(t), f(y(t))v_i(t) + G(y(t))w_i(t)) = V_i \quad \forall i \in \mathbb{N}, \quad \text{and} \]
\[ \{(v_i, w_i)(t)\}_{i \in \mathbb{N}} \text{ is a Cauchy sequence}. \]
Thus, $(v,w)(t) = \lim_{i \to \infty} (v_i, w_i)(t)$ is a measurable function satisfying
\[
(v(t), f(y(t))v(t) + G(y(t))w(t)) = V_t \quad \text{a.e. } t \in [0,T].
\]
Lower semicontinuity of $\lambda$ and (64) imply that
\[
\lambda(y(t), v(t), w(t)) = \inf \{ \lambda(y(t), \tilde{v}, \tilde{w}) : (\tilde{v}, f(y(t))\tilde{v} + G(y(t))\tilde{w}) = V_t \} \leq \dot{C}(t)
\]
for almost every $t \in [0,T]$. \hfill $\Box$

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