Mean Motion Resonances at High Eccentricities: The 2:1 and the 3:2 Interior Resonances

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Received 2017 February 2; revised 2017 May 24; accepted 2017 May 29; published 2017 June 23

Abstract

Mean motion resonances (MMRs) play an important role in the formation and evolution of planetary systems and have significantly influenced the orbital properties and distribution of planets and minor planets in the solar system and in exoplanetary systems. Most previous theoretical analyses have focused on the low- to moderate-eccentricity regime, but with new discoveries of high-eccentricity resonant minor planets and even exoplanets, there is increasing motivation to examine MMRs in the high-eccentricity regime. Here we report on a study of the high-eccentricity regime of MMRs in the circular planar restricted three-body problem. Numerical analyses of the 2:1 and the 3:2 interior resonances are carried out for a wide range of planet-to-star mass ratio $\mu$, and for a wide range of eccentricity of the test particle. The surface-of-section technique is used to study the phase space structure near resonances. We find that new stable libration zones appear at higher eccentricity at libration centers that are shifted from those at low eccentricities. We provide physically intuitive explanations for these transitions in phase space, and we present novel results on the mass and eccentricity dependence of the resonance widths. Our results show that MMRs have sizable libration zones at high eccentricities, comparable to those at lower eccentricities.

Key words: celestial mechanics – chaos – Kuiper belt: general – minor planets, asteroids; general – planets and satellites: dynamical evolution and stability

1. Introduction

There are many examples of stable mean motion resonances (MMRs) in our solar system (Peale 1976, 1999; Malhotra & Williams 1997). Unstable MMRs are also of great significance; for example, the Kirkwood Gaps in the asteroid belt are linked to the unstable and chaotic MMRs with Jupiter (Moons 1996), and the long-term stability of our planetary system is related to the role of MMRs (Murray & Holman 1999). Moreover, there is growing recognition of the role of MMRs in the dynamics of exoplanetary systems (Marcy et al. 2001; Lee & Peale 2002; Fabrycky et al. 2014; Petigura et al. 2015; Mills et al. 2016; Nelson et al. 2016). The phase space structure near MMRs is quite complex, and there is a considerable literature on their mathematical analysis. The planetary three-body problem, with $(m_1 + m_2 + m_3)$, is the usual starting point of such studies. Although it does not describe the full complexities of real planetary systems, it remains a very useful approximation and provides important insights for the phase space structure of MMRs.

The special case of the circular planar restricted three-body problem is particularly important in studies of the dynamics of small bodies in planetary systems. In this special case, two massive bodies, $m_1 \gg m_2, m_3$, move in circular orbits about their center of mass, and a massless third body ($m_3 = 0$, the test particle) moves in the massive bodies’ orbital plane. This problem admits an integral of the particle’s motion, the Jacobi integral $C_J = 2(\Omega L - E)$, where $\Omega$ is the angular velocity of the massive bodies in their circular orbits and $E$ and $L$ are the specific energy and specific angular momentum of the particle. Theoretical analysis of MMRs, based on perturbation theory, in this special case can be found in Murray & Dermott (1999). In particular, we mention the analytical results for the widths of the interior MMRs of Jupiter for low to moderate eccentricities of the test particle (Murray & Dermott 1999); some discrepancies between these analytical results and numerical results using surfaces of section were discussed by Winter & Murray (1997). Semi-analytical perturbative methods have been used to map the widths of stable resonance libration zones for a wider range of eccentricity, for the major exterior MMRs of Neptune (Morbidelli et al. 1995); however, in this case, non-perturbative numerical analysis shows that the semi-analytical results overestimate the stable resonance zones and give poor accuracy at eccentricities exceeding ~0.2 (Malhotra 1996). Meanwhile, recent observations find that high-eccentricity MMRs may not be uncommon in our solar system. Michel et al. (2000) noted the importance of MMRs in the dynamics of planet-crossing asteroids in the inner solar system, and the presence of resonant libration behavior for high eccentricities has been noted in several studies (Nesvorný et al. 2002; Mardling 2016, and references therein). Chiang et al. (2003) noted the surprising existence of high-eccentricity, $e \gtrsim 0.4$, Kuiper Belt objects in the exterior 2:5 MMR of Neptune. Recently it has been suggested that some of the most distant Kuiper Belt objects may be strongly affected by MMRs with unseen distant planet (Batygin & Brown 2016; Malhotra et al. 2016); these distant Kuiper Belt objects have eccentricities exceeding 0.70.

Our goal is to understand how the widths of stable libration zones of MMRs behave at very high eccentricities. The problem is sufficiently challenging that we limit the present work to only two important cases, the interior 2:1 and 3:2 MMRs in the circular planar restricted three-body model. We use non-perturbative numerical analysis to calculate the resonance widths for test particle eccentricities as high as 0.99. The resonance libration regions are visualized in surfaces of section. In contrast to the common expectation that resonance overlap leads to chaos and loss of stable resonant librations at high eccentricities, we report that large new stable libration zones reappear at higher eccentricity, albeit at shifted libration centers. We report several novel results, including the transitions of the resonance libration

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centers with increasing eccentricity, the variation of resonance width with eccentricity, and its dependence on the mass ratio $\mu = m_2/(m_1 + m_2)$. Our results would be of interest in a wide range of applications of the restricted three-body problem in planetary systems.

2. Surfaces of Section

In the planar circular restricted three-body problem, we refer to the most massive body, of mass $m_1$, as the “Sun,” and the secondary mass, $m_2$, as the “planet,” and the third (massless) body as the “test particle.” All the bodies are restricted to move in a common plane, the $(x, y)$ plane. We adopt the natural units for this problem: the unit of length is the orbital separation of $m_1$ and $m_2$, the unit of time is their orbital period divided by $2\pi$, and the unit of mass is $m_1 + m_2 \simeq m_1$; with these units the constant of gravitation is unity, and the orbital angular velocity of $m_1$ and $m_2$ about their common center of mass is also unity. Then, in a rotating reference frame, of constant unit angular velocity and origin at the center of mass, both $m_1$ and $m_2$ remain at fixed positions, ($-\mu, 0$) and $(1 - \mu, 0)$, respectively, where $\mu = m_2/(m_1 + m_2)$. The equations of motion of the test particle can be written as

\[
\dot{x} = 2\dot{y} + x - \frac{(1 - \mu)(x + \mu)}{r_1^3} - \frac{\mu(x - 1 + \mu)}{r_2^3},
\]

\[
\dot{y} = -2\dot{x} + y - \frac{(1 - \mu)y}{r_1^3} - \frac{\mu y}{r_2^3},
\]

where $r_1$ and $r_2$ are the test particle’s distances to the Sun and the planet, respectively,

\[
r_1 = [(x + \mu)^2 + y^2]^{1/2},
\]

\[
r_2 = [(x - 1 + \mu)^2 + y^2]^{1/2}.
\]

The Jacobi constant can be derived from the dynamic equations as

\[
C_1 = x^2 + y^2 - \dot{x}^2 - \dot{y}^2 + \frac{2(1 - \mu)}{r_1} + \frac{2\mu}{r_2},
\]

which can also be expressed in terms of orbital elements as

\[
C_1 = \frac{1}{a} + 2\sqrt{a(1 - e^2)(1 - \mu)} + O(\mu),
\]

where $a$ and $e$ are semimajor axis and eccentricity of the particle’s osculating orbit about the Sun.

The motion of the test particle takes place in the four-dimensional phase space $(x, y, \dot{x}, \dot{y})$. With the constraint of the Jacobi constant, the test particle motion is constrained to a three-dimensional surface in this phase space. To visualize the phase space structure in a two-dimensional space, we compute “surfaces of section” near the MMRs of interest. A surface of section is analogous to the stroboscopic plot of the phase space variables of a periodically forced nonlinear oscillator (Lichtenberg & Lieberman 1983). Our implementation for the two degrees of freedom of our test particle dynamics is as follows. On a continuous phase space trajectory of the test particle with specified initial conditions (and with a specified value of the Jacobi constant), we identify its intersections with the surface of section with the following condition,

\[
\dot{r}_1 = 0, \quad \dot{r}_1 > 0.
\]

That is, the phase space trajectory of the particle is represented by the collection of its phase space variables $(x, \dot{x}, y, \dot{y})$ recorded at its pericenter passages. The phase space structure near the MMR is obtained by computing this representation for many test particles with different initial conditions in the neighborhood of the MMR but the same value of the Jacobi constant. However, rather than making the usual two-dimensional plots of $(x, \dot{x})$ or $(y, \dot{y})$, we transform the phase space variables to osculating orbital elements and make plots of the orbital parameters $(a, \psi)$, where $a$ is the test particle semimajor axis and $\psi$ is the angular separation of the planet and the particle at periapsis, as illustrated in Figure 1.

Successive points on the surface of section that are confined to smooth curves in the $(a, \psi)$ plane indicate a quasi-periodic stable orbit of the test particle, whereas successive points filling an area in this plane indicate a chaotic orbit. A closed curve on which $\psi$ changes sign indicates an orbit in resonant libration. The stable resonant zones (and their widths in semimajor axis) are then readily visualized in such surfaces of section.

The detailed procedure for producing the surfaces of section is as follows. First, for a specified initial value of $a$ and $e$, we calculate the Jacobi constant $C_1$ using Equation (6). (This first value of $a$ is always chosen as the resonant value, $a_{res}$, see Equation (14) below.) Next, with a specified initial value of $\psi$, we calculate the initial position and velocity vectors $(x_p, y_p)$, $(\dot{x}_p, \dot{y}_p)$ of the test particle at pericenter in the rotating frame:

\[
(x_p, y_p) = r_p (\cos(\psi) - \mu, -\sin(\psi)),
\]

\[
(\dot{x}_p, \dot{y}_p) = v_p (\sin(\psi), \cos(\psi)),
\]

where $r_p = a(1 - e)$ and $v_p = (\dot{x}_p^2 + \dot{y}_p^2)^{1/2}$ is calculated with the help of Equation (5),

\[
v_p = \sqrt{\dot{x}_p^2 + \dot{y}_p^2 + \frac{2(1 - \mu)}{r_1} + \frac{2\mu}{r_2} - C_1}.
\]

With these initial values of the position and velocity vectors, we numerically integrate Equations (1) and (2) to obtain the phase space trajectory of the test particle. Typical integration times run to 60,000 orbital periods of the test particle,
equivalent to approximately 188,500 units of time for 2:1 MMR and 251,300 units of time for 3:2 MMR. During the integration, we identify each pericenter passage occurrence (by testing for the condition Equation (7)), and compute the values of $a$ and $\psi$ at each pericenter passage,

$$\psi = \arctan(y_p, x_p + \mu),$$

$$a = -\frac{1}{2E},$$

where $E$ is the orbital energy of the test particle,

$$E = \frac{1}{2} (\dot{X}^2 + \dot{Y}^2) - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2},$$

where $[\dot{X}, \dot{Y}]$ is the velocity vector in the barycentric inertial frame. For additional trajectories in the same surface of section, we keep the same value of $C_i$ but use new initial values of $a$ and $\psi$ (close to $a_{res}$), and compute $e$ according to Equation (6); with these, we compute new initial values of $(x_p, y_p)$ and $(\dot{x}_p, \dot{y}_p)$. We repeat this procedure several times to fill in the surface of section in the MMR neighborhood.

The neighborhood of an MMR is usually identified as a small range of the semimajor axis near the exact resonant value in which a critical resonant argument can librate. A j: k MMR occurs when the test particle completes $j$ orbits around the Sun while the planet completes $k$ orbits, where $j$ and $k$ are mutually prime numbers. The resonant value of the semimajor axis of the test particle is

$$a_{res} = \left( \frac{k}{j} \right)^{2/3}. \quad (14)$$

When the eccentricity $e$ is also given, the Jacobi constant for an eccentric resonant orbit can be calculated using Equation (6) with $a_{res}$ and $e$. Thus, the surfaces of section in the neighborhood of an MMR can be characterized by either their fixed value of the Jacobi constant or by the combination $(a_{res}, e)$. In the following section, we label each different surface of section with simply the specific value $e$, since $a_{res}$ is also fixed for each resonance.

The critical resonant argument (in the circular planar restricted three-body problem) is defined as

$$\phi = j\lambda' - k\lambda - (j - k)\varpi,$$

where $\lambda$ and $\lambda'$ are mean longitudes of the test particle and the planet, and $\varpi$ is the longitude of pericenter of the test particle. In a stable resonance zone, the resonant argument will librate about a central value, either $0^\circ$ or $180^\circ$. (It is worth noting that asymmetric librations, in which the center of libration of the resonant argument is different from $0^\circ$ or $180^\circ$, can occur in certain exterior MMRs (Beauge 1994; Malhotra 1996).) When the test particle is at its pericenter, $\lambda = \varpi$, we have

$$\phi = j(\lambda' - \varpi). \quad (16)$$

We can see that the resonant argument is $j$ times the angular element $\psi$ that we defined for the surface of section.

In the present study, we focus on the interior 2:1 and 3:2 MMRs at high eccentricities, $0.05 \leq e \leq 0.99$, and for values of mass ratio $\mu$ ranging from $1 \times 10^{-5}$ to $3 \times 10^{-3}$. For the phase space structure at very low eccentricities, $e \lesssim 0.1$, we refer the reader to Murray & Dermott (1999).

### 3. MMR Zones

#### 3.1. Interior 2:1 MMR

A test particle near the 2:1 MMR with the planet has a semimajor axis value close to the resonant value,

$$a_{res} = \left( \frac{1}{2} \right)^{2/3} = 0.630. \quad (17)$$

For an elliptic orbit, the maximum aphelion distance is

$$\lim_{e \to 1} a_{res} (1 + e) = 1.260. \quad (18)$$

This means that for high eccentricities, $e > a_{res}^{-1} - 1 \approx 0.59$, the test particle orbit will cross the orbit of the planet.

We generated surfaces of section for initial values of the planet semimajor axis $a$ around the resonant $a_{res}$, various initial values of its angular separation from the planet, $\psi$, and for a range of initial eccentricities. Results are shown in Figure 2 for one value of the mass ratio, $\mu = 3 \times 10^{-5}$. We observe that when the eccentricity is low (Figure 2(a)), there are two stable islands in the surface of section, which are centered at $\psi = 0^\circ$ and $\psi = 180^\circ$; the boundaries of these stable islands are smooth and no significant chaotic regions exist. The fixed points at the centers of the stable islands have significant meaning: they correspond to the stable periodic orbits of the exact 2:1 resonance with the planet (Hadjidemetriou & Voyatzis 2000). Moreover, the periodic orbits appear as pairs, one stable (at the center of the stable islands) and one unstable (at the crossing points of the boundaries of the stable islands). These periodic orbits all have the same shape but are distinguished from each other by the different (but fixed) angular separation $\psi$ of the planet and the test particle at pericenter. Inside the stable islands we see the smooth curves around the stable fixed points; these correspond to stable quasi-periodic orbits in which $\psi$ exhibits librations about the central value. The smooth curves exterior to the stable islands are orbits in which $\psi$ smoothly circulates over the entire range $0^\circ$–$360^\circ$. Since the eccentricity is low, these orbits are far from the planet and close encounters will not occur. At higher eccentricity, the widths (measured as the range $\Delta \varpi$ where $\psi$ librates) of the stable islands grow; we observe chaotic regions near their boundaries, and these chaotic regions expand with increasing eccentricity (Figures 2(b)–(d)). In these chaotic regions outside the stable libration islands, the points in the surface of section have a superficially quasi-regular appearance but they do not stay confined to a smooth curve; rather they intermittently drift and jump around and the particle has frequent close encounters with the planet. When the eccentricity is higher than a certain critical value, $e_c \approx 0.59$, we observe that new stable islands appear in the surface of section, and the number of the stable islands doubles from two to four (Figure 2(b)). The appearance of new stable islands indicates a new set of periodic orbits of the 2:1 MMR. These new periodic orbits have the same shape as the old ones, but have a different angular separation of the planet and the test particle at pericenter. The new islands are centered at $\psi = 90^\circ$ and $\psi = 270^\circ$, in between the old islands. At higher values of eccentricity, the new stable islands continue to grow, while the old ones shrink. We find that the new stable islands have a size comparable to the old ones when the eccentricity approaches 0.99 (Figure 2(d)).

For other values of the mass ratio, $\mu$, the evolution of the surface of section with eccentricity is qualitatively the same, although the sizes of the stable islands vary with $\mu$. The critical
values of the transition eccentricity, $e_c$, also vary, but only very modestly, with $\mu$. Table 1 lists the critical values $e_c$ for the values of $\mu$ that we investigated.

From a dynamical point of view, the transition in the phase space structure (the doubling of the stable islands) is associated with the bifurcation of a periodic orbit from a collision orbit (Hadjidemetriou & Voyatzis 2000; Voyatzis & Kotoulas 2005). Here we provide a different and more intuitive explanation by considering the geometry of the resonant orbits in the rotating frame, as illustrated in Figure 3. In the rotating frame with origin at the center of mass, the position of the Sun is fixed at a distance $\mu$ from the origin and the planet is at unit distance apart from the Sun and at distance $1 - \mu$ from the origin. The closed blue curve in Figure 3 shows the shape of the exact resonant (periodic) orbit in the rotating frame. For each of the fixed points in the surfaces-of-section, there is a particular relative orientation of the resonant orbit and the (fixed) location of the Sun and planet in the rotating frame. On the red circle of radius $1 - \mu$, we denote with a black dot the planet location corresponding to the geometry of each of the stable centers of libration visible in the surface of section; and we denote with a black cross the planet location corresponding to the geometry of each of the unstable fixed points visible in the surface of section. Note that for a given eccentricity, all the periodic orbits have the same shape, but are distinguished from each other by their different orientations relative to the fixed locations of the Sun and planet in the rotating frame.

When the eccentricity is low, the particle orbit is completely inside the orbit of the planet and its aphelion is far from the orbit of the planet. In this case, the perturbation of the low-mass planet is weaker, and there is no chaotic region in the surface of section. From Figure 3 we can also see that the test particle path in the rotating frame has a two-fold symmetry, so the two centers of the stable islands have a step of $180^\circ$ in $\psi$. The stable resonance geometry is one in which the angular position of the planet is near one of the two locations denoted by the black dots, i.e., with $\psi = 0^\circ$ or $\psi = 180^\circ$; in this geometry, the planet perturbation on the particle is minimized. At higher eccentricity, the aphelion of the test particle is closer to the orbit of the planet, so the perturbation of the planet on the test particle also becomes stronger. The chaos regions appear in

| Mass Ratio, $\mu$ | $e_m$ | $e_c$ |
|------------------|------|------|
| $1 \times 10^{-5}$ | 0.59 | 0.590 |
| $3 \times 10^{-5}$ | 0.58 | 0.593 |
| $1 \times 10^{-4}$ | 0.57 | 0.602 |
| $3 \times 10^{-4}$ | 0.48 | 0.612 |
| $1 \times 10^{-3}$ | 0.42 | 0.648 |
| $3 \times 10^{-3}$ | 0.31 | 0.709 |

Note. $e_m$ represents the eccentricity where the first resonance zone has its maximum width, $\Delta e$; $e_c$ is the value of the eccentricity above which the second resonance zone is present in the phase space.
At eccentricity higher than $e_c = a_{\text{res}} - 1 = 0.59$, the test particle orbit is planet-crossing (its aphelion is larger than the planet orbital radius), and new stable islands appear in the surface of section. Once the aphelion of the test particle orbit exceeds the orbit radius of the planet, we observe in Figures 3(b)–(d) that the two aphelion lobes of the particle path are cut by the planet orbit. This is where the new stable islands appear in the surface of section, i.e., centered at $\psi = 90^\circ$ and $\psi = 270^\circ$. The length of the arc of the planet orbit that is inside the lobe of the test particle orbit in the rotating frame reflects the range of $\psi$ for the new stable islands. (The intersection points, denoted by the cross in the figure, represent the location of the planet at the new unstable fixed points visible in the surface of section.) For higher eccentricity, this length of the arc also increases. In this way, the new stable islands grow and expand at the expense of the old stable islands. For eccentricities near $e = 0.99$, the new stable islands are of comparable size with the old islands.

In each surface of section that we generated, we measured the resonance width, $\Delta a$, defined as the maximum range of $a$ about the resonant value, $a_{\text{res}}$, for which $\psi$ librates about a stable center. The key to measuring $\Delta a$ is to find the boundary of the stable islands. We obtained the value of $\Delta a$ by using an iterative method. We begin with a moderate but not so small step in the initial value of $a - a_{\text{res}}$ to find the upper boundary of the stable islands in the surface of section. Then we reduce the step to narrow down the upper boundary to a fifth decimal place accuracy. In the same way, we locate the lower boundary of the stable islands. Then, we compute $\Delta a$ as the difference between the upper and lower boundaries. We note that in most cases, the boundaries of the stable resonance zones are easy to identify, but in a few cases the surface of section presents a chain of numerous very small islands of secondary resonances close to the resonance boundary; these make the main resonance boundary rather fuzzy and difficult to identify. In these cases, we regard the secondary resonance chains as part of the chaotic zone because there are still some chaotic regions in between these chains of islands, and we estimate the boundary of the stable resonance zone just interior to these chains of islands.

As noted above, the resonance width depends upon both $\mu$ and the test particle eccentricity. Figure 4 plots the width of the 2:1 MMR for a test particle eccentricity in the range 0.05–0.99 for three values of $\mu$. Since there are two dynamically distinct sets of stable islands in the surface of section, we measured them separately. The blue lines plot the widths of the stable islands centered at $\psi = 0^\circ$, $180^\circ$ in the surface of section; we call these the “first resonance zone.” The red lines plot the widths of the stable islands centered at $\psi = 90^\circ$, $270^\circ$; these exist for eccentricities exceeding $e_c \approx 0.59$, and we call them the “second resonance zone.” In these plots, we observe that the first resonance zone increases in width with increasing eccentricity up to a maximum width, and then decreases with increasing eccentricity. The maximum width of the first resonance zone is an increasing function of the mass ratio, $\mu$.

Table 1 lists the values of the eccentricity, $e_m$, where the first resonance zone has its maximum width, for each mass ratio $\mu$ that we investigated. We find that $e_m$ decreases with increasing $\mu$, i.e., the maximum width occurs at lower eccentricity for larger $\mu$. In Figure 4 we can see that the first resonance zone reaches its maximum width just before the second resonance zone appears. The width of the second resonance zone, once it appears, initially increases rapidly, but then more gradually with increasing eccentricity; it reaches maximum width as the eccentricity approaches $e = 0.99$. We can also note that the condition for the existence of the second resonance zone for $e > e_c \approx 0.59$ depends mainly on the orbital geometry, and only weakly on the mass ratio $\mu$.

To simplify the quantification of the mass dependence of the resonance widths, $\Delta a$, of the first and second resonance zones, we postulate that in each case, $\Delta a$ has a power-law relation with the mass of the planet, and we empirically fit the values of $\Delta a$ (measured from the surfaces of section) to such a function,

$$\Delta a = \alpha \mu^\beta,$$

allowing that the parameters $\alpha$ and $\beta$ depend upon the test particle eccentricity. This power-law relation is motivated by the analytical perturbation theoretic results, which predict that for low eccentricities, the power-law index is 1/2 for both the 2:1 and the 3:2 MMRs (Murray & Dermott 1999).

In Figure 5 we plot the stable libration width, $\Delta a$, of the first resonance zone as a function the mass ratio $\mu$ for several fixed values of the eccentricity. For the lower eccentricities, $e = 0.2$ and $e = 0.3$, the aphelion of the test particle does not exceed

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2 We remark that the “second resonance zone” identified here appears at high eccentricities in the circular restricted three-body problem; it should not be confused with “secondary resonance,” a phenomenon that has been identified in the elliptic restricted three-body model (Wisdom 1985; Henrard et al. 1995) and in the non-restricted three-body problem with oblate primary (Tittmore & Wisdom 1989; Malhotra 1990).
the orbit radius of the planet and only the first resonance zone exists. In this low-eccentricity regime, for low-mass ratio, the width of stable resonance zones increases with increasing eccentricity; the boundaries of the stable islands in the surfaces of section are very smooth and no chaotic region exists; however, for a high-mass ratio, \( e \gtrsim (2-3) \times 10^{-4} \), chaotic regions appear in the surfaces of section but the power-law relation still holds. The best-fit power-law relations are found to be \( \Delta a = 1.18\mu^{0.499} \) and \( \Delta a = 1.62\mu^{0.504} \) for \( e = 0.2 \) and \( e = 0.3 \), respectively; the power-law index is near 1/2. For the higher eccentricities, \( e = 0.6 \) and \( e = 0.8 \), the aphelion of the particle exceeds the orbit radius of the planet, and the stable islands in the surfaces of section are surrounded by chaotic regions even for a low-mass ratio. In this high-eccentricity regime, the first resonance zone width shrinks with increasing eccentricity. The best-fit power-law relations are \( \Delta a = 0.977\mu^{0.397} \) and \( \Delta a = 0.709\mu^{0.393} \) for \( e = 0.6 \) and \( e = 0.8 \), respectively; the power-law index is near 2/5.

In Figure 6 we plot the stable libration width, \( \Delta a \), of the second resonance zone as a function the mass ratio \( \mu \) for \( e = 0.7 \) and \( e = 0.9 \). In this eccentricity regime, the width of the second resonance zone does not vary strongly with eccentricity (as can be seen in Figure 4). The best-fit power-law relations are found to be \( \Delta a = 0.225\mu^{0.322} \) and \( \Delta a = 0.373\mu^{0.367} \) for \( e = 0.7 \) and \( e = 0.9 \), respectively. The power-law index is higher for the higher eccentricity. We note that for an eccentricity near 0.6–0.7, there is a rollover in the slope at larger \( \mu \), \( \mu \gtrsim 3 \times 10^{-4} \) (Figures 5(b) and (6)). This rollover is associated with the existence of significant chaotic regions surrounding the resonance zones. We emphasize that the power-law relation is a simple empirical model that fits our numerical results quite well for \( \mu \lesssim 3 \times 10^{-4} \), but diverges at larger \( \mu \). Although this rollover causes the power law to be invalid at larger \( \mu \), the power-law relation between \( \Delta a \) and \( \mu \) is still a useful tool to give us an intuitive understanding about how resonance widths change with \( \mu \) from small to large (but not so large), over a range pertinent to planetary dynamics. A more sophisticated model-fitting is needed in a future study to find a more accurate and global relation between \( \Delta a \) and \( \mu \) and \( e \).

Table 2 summarizes the results for the best-fit power-law relations for the widths of the first and second resonance zones for different eccentricities. The dependence on the eccentricity is not described well as a simple power law or even a piece-wise power law, and we did not attempt to fit a functional form for it.

### 3.2. Interior 3:2 MMR

A test particle near the 3:2 MMR with the planet has a semimajor axis

\[
    a_{\text{res}} = \left( \frac{2}{3} \right)^{2/3} = 0.763.
\]
For an elliptic orbit, the maximum of the aphelion is
\[ \lim_{e \to 1} a_{\text{res}}(1 + e) = 1.526, \] (21)

therefore, the test particle orbit is planet-crossing when the eccentricity exceeds \( a_{\text{res}}^{-1} - 1 \approx 0.31. \)

We computed the surfaces of section for the 3:2 MMR for \( 0.05 \leq e \leq 0.99 \) and for \( 1 \times 10^{-5} \leq \mu \leq 3 \times 10^{-3}. \) Results are shown in Figure 7 for \( \mu = 3 \times 10^{-5}. \) We can see that the 3:2 MMR phase space is more complex than the 2:1 case. There are three transitions in the evolution of the surfaces of section with increasing eccentricity. When the eccentricity is low (Figure 7(a)), there are three stable islands in the surface of section that are centered at \( \psi = 0^0, 120^0, \) and \( 240^0; \) their boundaries are smooth, and no chaotic regions exist. The fixed points at the centers of the stable islands correspond to the stable periodic orbits at the exact 3:2 resonance with the planet. Inside the stable islands we see the smooth curves that correspond to quasi-periodic orbits librating about the exact resonant orbit. As the eccentricity approaches the first critical value, \( e_{c1} = 0.33, \) the widths of these three stable islands grow and chaotic regions appear and expand in the surface of section (Figure 7(b)). When the eccentricity exceeds \( e_{c1}, \) the number of stable islands doubles from three to six (Figure 7(c)); the three new stable islands appear in the surface of section centered at \( \psi = 60^0, 180^0, \) and \( 300^0. \) The appearance of new stable islands indicates a new set of periodic orbits of the 3:2 MMR. These new periodic orbits have different angular separations, \( \psi, \) of the planet and the test particle at pericenter, and they have different orientations relative to the fixed position of the Sun and planet in the rotating frame. At higher values of eccentricity, the new stable islands keep growing, while the old ones shrink. From Figure 7(d) we can see that when the eccentricity is 0.60, the new stable islands have a size comparable with the old ones. As the eccentricity increases further, the old stable islands continue to shrink and the new ones continue to grow. When the eccentricity exceeds a second critical value, \( e_{c2}, \) the old stable islands disappear and only the new islands exist in the surface of section. The number of the stable islands becomes three again, but the centers of these islands are shifted from the original ones at low eccentricity (Figure 7(f)). At even higher eccentricity, exceeding a third critical value, \( e_{c3}, \) another transition occurs as the original stable islands (centered at \( \psi = 0^0, 120^0, \) and \( 240^0) \) reappear and the number of the islands becomes six again (Figure 7(g)). The values of the critical eccentricity at the transitions in the phase space structure are resolved with a finer grid in the eccentricity values (equivalently, Jacobi constant values) of the surfaces of section. For a mass ratio \( \mu = 3 \times 10^{-5}, \) the transitions occur at these critical values of the eccentricity: \( e_{c1} = 0.33, e_{c2} = 0.87, \) and \( e_{c3} = 0.91. \) For other mass ratios \( \mu, \) the critical values of eccentricity are slightly different, but the evolution of the phase space structure is the same. The critical eccentricities for different mass ratios \( \mu \) are listed in Table 3. We note that \( e_{c1} \) and \( e_{c3} \) increase, whereas \( e_{c2} \) decreases with increasing \( \mu; \) for \( \mu \gtrsim (2-3) \times 10^{-3}, \) there is no third transition.

The transitions with increasing eccentricity that we find in the surfaces of section can be understood by studying the geometry of the resonant orbits in the rotating frame. This is illustrated in Figure 8. The meaning of the curves and marks in Figure 8 is the same as in previous section. When the eccentricity is low, the particle’s resonant orbit is completely inside the orbit of the planet and its aphelion is far from the planet. In these cases, the perturbation of the planet is weak, and there is no chaotic region in the surface of section. From
Figure 7. Surfaces of section in ($a$, $\psi$) near the interior 3:2 MMR for the mass ratio $\mu = 3 \times 10^{-3}$. Each panel has a fixed value of the Jacobi constant, parametrized by the combination of the resonance value of the semimajor axis, $a_{\text{res}} = 0.763$, and the indicated value of the eccentricity.
Figure 8 we can also see that the test particle’s resonant orbit has a three-fold symmetry in the rotating frame, so the three centers of the stable islands in the corresponding surfaces of section have spacings of 120° in $\psi$. The stable resonance geometry is one in which the angular position of the planet is near one of the two locations denoted by the black dots, i.e., with $\psi = 0^\circ$, $\psi = 120^\circ$, or $\psi = 240^\circ$; in this geometry, the planet perturbation on the particle is minimized. For higher eccentricity, the aphelion of the test particle approaches the orbit of the planet, so the perturbation of the planet on the test particle also increases; chaotic regions appear in the surface of section (Figure 8(a)). When the eccentricity exceeds $e_1$, the test particle orbit is planet-crossing and new stable islands appear in the surface of section. (The particle becomes planet-crossing when the eccentricity exceeds 0.31, however, for $\mu = 1 \times 10^{-5}$, the new stable islands do not appear in the surfaces of section until the eccentricity exceeds $e_1 = 0.32$.) Once the aphelion of the test particle orbit exceeds the orbit of the planet, the three lobes of the configuration of the particle orbit will intersect the planet orbit; the planet orbit is cut into six arcs. (The intersection points, denoted by the crosses in the figure, represent the location of the planet at the new unstable fixed points visible in the surface of section.) These are where new stable islands appear in the surface of section (Figures 8(b) and 7(c)). The length of the arc of the planet orbit that is enclosed by each of the three lobes of the test particle orbit (in the rotating frame) limits the range of $\psi$ for the new stable islands in the surface of section, whereas the lengths of arc outside the lobes correspond to the range of $\psi$ in the old stable islands; the chaotic regions in the surface of section reflect the chaotic orbits near the intersection points where the particle would have close encounters with the planet. As the eccentricity becomes higher, the length of the arc enclosed by each lobe also becomes larger, whereas the length of arc outside the lobes shrinks. So the new stable islands grow and expand at the expense of the range of the old stable islands (Figure 8(c)). As the eccentricity approaches 0.90, the three lobes intersect each other, and their intersection points approach the orbit of the planet (Figures 8(d) and (e)). At this eccentricity, the trace of the resonant orbit in the rotating frame again defines only three arcs, but these three are the arcs enclosed by the lobes. This means that the original three islands have disappeared and only the new stable islands exist. At even higher eccentricity, when the self-intersections of the lobes occur outside the orbit of the planet, as in Figure 8(f), we see that the planet orbit is again cut into six arcs, so the “old” stable islands centered at $\psi = 0^\circ$, $120^\circ$, $240^\circ$ reappear, albeit with smaller sizes.

Using the same method as for 2:1 MMR, we measured the resonance libration widths of the 3:2 MMR for different eccentricities, 0.05 $\leq e \leq$ 0.99, and mass ratios $\mu$. For fixed values of $\mu$, we plot in Figure 9 the stable libration regions for an eccentricity ranging from 0.05 to 0.99. Regions inside the blue lines are the stable islands centered at $\psi = 0^\circ$, $120^\circ$, and $240^\circ$ in the surface of section, while regions inside the red lines are stable islands centered at $60^\circ$, $180^\circ$, and $300^\circ$. We refer to the former as the “first resonance zone” and the latter as the “second resonance zone.” For low eccentricities, the first resonance zone increases in width with increasing eccentricity, reaches a maximum width, then decreases in width until it vanishes near $e = e_2$, and then begins to increase again. Once it appears near $e = e_1$, the second resonance zone width increases and reaches maximum width when the first resonance zone vanishes near $e_2$, and then decreases. From Figure 9 we can see that the first resonance zone reaches its maximum width just before the second resonance zone appears. Table 3 lists the values of the eccentricity, $e_m$, where the first resonance zone has its maximum width, for each mass ratio $\mu$ that we investigated. We find that $e_m$ decreases with increasing $\mu$, i.e., the maximum width occurs at lower eccentricity for higher $\mu$.

The maximum of the resonance width is monotonically larger for a higher mass ratio $\mu$. The second resonance zone reaches its maximum width when the eccentricity is around $e = 0.90$, which is also when the width of the first resonance zone is vanishingly small.

As before, we postulate that for a fixed eccentricity, the resonance width $\Delta \alpha$ has a power-law relation with the mass of the planet (Equation (20)). (Similar to the case of the 2:1 MMR, the dependence on the eccentricity is not described well as a power law, and we did not attempt to fit a functional form for it.) The results for the first resonance zone are shown in Figure 10. For $e = 0.1$ and $e = 0.2$, we find that the best-fit power-law relations are $\Delta \alpha = 1.61 \mu^{0.511}$ and $\Delta \alpha = 2.41 \mu^{0.500}$, respectively, which shows that the power-law index is near 1/2 (Figure 10(a)). For higher eccentricity, $e = 0.4$ and $e = 0.8$, the best-fit power-law relations are $\Delta \alpha = 0.375 \mu^{0.329}$ and $\Delta \alpha = 0.182 \mu^{0.327}$, respectively, which shows that the power-law index is near 1/3 (Figure 10(b)), different from that at lower eccentricity. It is worth noting that for these higher eccentricities, the resonance width $\Delta \alpha$ becomes smaller as the eccentricity increases.

For the second resonance zone, Figure 11 shows the results for the power-law relation for $e = 0.5$ and $e = 0.9$; the best-fit power-law relations are found to be $\Delta \alpha = 0.181 \mu^{0.303}$ and $\Delta \alpha = 0.882 \mu^{0.422}$, respectively. The power-law index is different for different eccentricities, higher for higher eccentricity.

We note that for both resonance zones, there is a rollover in the slope of the log $\Delta \alpha$–log $\mu$ curve at larger $\mu$; we can see this drop-off in the tail of the data in Figures 10 and 11. The reason for the rollover appears to be that when the mass ratio is high enough, the chaotic regions around the stable islands limit the growth of the stable libration region. In computing the reported best-fit power-law fits, we excluded the data for the high-mass ratios, $\mu \geq (2–3) \times 10^{-4}$.

Table 4 summarizes the results for the best-fit power-law relations for the first and second resonance zones for different eccentricities.

| Mass Ratio, $\mu$ | $e_m$ | $e_1$ | $e_2$ | $e_3$ |
|-------------------|-------|-------|-------|-------|
| $1 \times 10^{-5}$ | 0.28  | 0.320 | 0.885 | 0.906 |
| $3 \times 10^{-5}$ | 0.25  | 0.332 | 0.877 | 0.912 |
| $1 \times 10^{-4}$ | 0.22  | 0.356 | 0.862 | 0.925 |
| $3 \times 10^{-4}$ | 0.18  | 0.395 | 0.835 | 0.945 |
| $1 \times 10^{-3}$ | 0.15  | 0.465 | 0.777 | 0.958 |
| $3 \times 10^{-3}$ | 0.14  | 0.551 | 0.741 | ...  |

Note. $e_m$ represents the eccentricity where the first resonance zone has its maximum width, $\Delta \alpha$, and $e_1$, $e_2$, $e_3$ are the transition eccentricities when the stable islands in phase space change structure.
4. Summary and Discussion

In most previous studies, the widths of MMRs have been investigated in the low-eccentricity regime with perturbative analytical methods and it has been commonly assumed that the stable resonance widths vanish as MMRs become unstable and chaotic at high eccentricities owing to resonance overlap (Murray & Dermott 1999). In the present study, we carried out non-perturbative numerical analyses of MMRs in the circular planar restricted three-body problem for a wide range of eccentricities $e$ and mass ratios $\mu$. A surface of section was introduced to study the phase space structure near resonances. We located the resonance libration centers and computed the widths of stable zones of the interior 2:1 and 3:2 MMRs. Our investigation shows that stable MMR zones change with increasing eccentricity, with one or more transitions of the dominant stable islands, but sizable stable libration zones still exist at high eccentricities.
For the case of the interior 2:1 MMR, when the eccentricity is lower than the planet-crossing value, the surface of section shows a chain of two stable islands centered at $\psi = 0^\circ$ and $180^\circ$, where $\psi$ is the angular separation of the planet from the test particle at pericenter. With increasing eccentricity, the width, $\Delta a$, of the stable libration zones expands even as chaotic regions appear near the resonance separatrix. The stable resonance width reaches a maximum and then decreases with increasing eccentricity; the maximum width occurs at eccentricity values slightly below the planet-crossing eccentricity. A new chain of two stable islands centered at $\psi = 90^\circ$, $270^\circ$ appears when the eccentricity reaches a critical value, which is close to but somewhat higher than the planet-crossing eccentricity. The widths of the new libration islands grow, while the old ones shrink with increasing eccentricity. The critical eccentricities for different values of $\mu$ are listed in Table 1.

For the case of the interior 3:2 MMR, the centers of the stable libration zones are located at $\psi = 0^\circ$, $120^\circ$, and $240^\circ$ when the eccentricity is low. With increasing eccentricity, the stable islands expand and the separatrix becomes a chaotic layer. When the eccentricity exceeds a critical value, a new chain of two stable islands centered at $\psi = 90^\circ$, $270^\circ$ appears when the eccentricity reaches a critical value, which is close to but somewhat higher than the planet-crossing eccentricity. The widths of the new libration islands grow, while the old ones shrink with increasing eccentricity. The critical eccentricities for different values of $\mu$ are listed in Table 1.

For the case of the interior 3:2 MMR, the centers of the stable libration zones are located at $\psi = 0^\circ$, $120^\circ$, and $240^\circ$ when the eccentricity is low. With increasing eccentricity, the stable islands expand and the separatrix becomes a chaotic layer. When the eccentricity exceeds a critical value, a new chain of three stable islands centered at $\psi = 60^\circ$, $180^\circ$, and $300^\circ$. The new stable islands grow and expand, while the old ones shrink with increasing eccentricity. The three old stable zones vanish at a second critical value of the eccentricity, but they reappear and grow again when the eccentricity becomes
higher. So, with increasing eccentricity, there are three transitions in the phase space structure of the 3:2 MMR. The critical eccentricities for different values of $\mu$ are listed in Table 3.

We have shown that these transitions are related mainly to the geometry of the trace of the particle resonant orbit in the rotating frame; they depend only weakly on the mass ratio $\mu$. For an eccentricity below the first critical transition, the chain of stable islands in the surfaces of section corresponds to stable resonant librations about zero of the usual resonant angle, $\phi = 2\lambda - \lambda - \omega$ and $\phi = 3\lambda - 2\lambda - \omega$ for the 2:1 and 3:2 MMRs, respectively. Physically, these correspond to conjunctions of the planet and test particle when the particle is near pericenter; they can be called “pericentric librations.” The first transition occurs when the particle orbit becomes a planet-crossing orbit and a new chain of stable islands appears in the phase space; these correspond to librations of the resonant angle $\phi$ about 180°. Physically, these are stable librations of conjunctions of the planet and test particle when the particle is near apocenter; they can be called “apo-centric librations.” To our knowledge, this stable “apo-centric libration” in the interior MMRs at high eccentricity has not been reported in previous studies. Additionally, for the 3:2 MMR case, we have shown that a second and a third transition occurs when the self-intersections of the trace of the test particle resonant orbit in the rotating frame occur at heliocentric distance exceeding the orbit radius of the planet. Each of these transitions makes the number of the stable islands halve or double, as can be seen in Figures 2 and 7. We showed how these transitions can be understood physically by relating to the geometry of the eccentric resonant orbits in the rotating frame (Figures 3 and 8).

The non-perturbative numerical analyses show that high-eccentricity MMRs have stable libration zones that are nearly as large as those of low-eccentricity MMRs. We measured the resonant widths of the libration zones in the surfaces of section. Examples are plotted in Figures 4 and 9. We fit an empirical power-law relation between the resonant widths and the mass of the planet, $\Delta a = \alpha \mu^{\beta}$, for mass ratio $\mu \lesssim 3 \times 10^{-4}$; for a higher mass ratio, the resonance widths deviate from this simple power-law model. For the first resonance zone (pericentric librations) of the 2:1 MMR, the power-law index $\beta$ is close to 0.50 and 0.40, respectively, for eccentricities lower and higher than $e_m$, where $e_m$ is the value of eccentricity at maximum resonance width. For the first resonance zone (pericentric librations) of the 3:2 MMR, we find that $\beta$ is also close to 0.50 for eccentricities below $e_m$, but for higher eccentricity, it is close to 0.33. (The values of $e_m$ depend upon the mass ratio $\mu$ and are listed in Tables 1 and 3.) The second resonance zone (apo-centric librations) exists only for eccentricities exceeding the planet-crossing value; we find that $\alpha$ and $\beta$ both increase monotonically with increasing eccentricity. The results are summarized in Tables 2 and 4.

Murray & Dermott (1999) provide a plot of the resonance libration widths, $\Delta a$, of a few interior MMRs of Jupiter, using a perturbative analysis of the circular planar restricted three-body model (see their Figure 8.7). Their calculation is restricted to $\mu = 9.55 \times 10^{-4}$ and to the eccentricity range of 0–0.3 (pertinent to the Sun–Jupiter–asteroid problem). Our results for the interior 2:1 and 3:2 MMRs for $\mu = 10^{-3}$ agree fairly well with these analytical results over the eccentricity range 0.05–0.3, although the analytical theory somewhat overestimates the resonance width of the 3:2 MMR (possibly because it does not account for the chaotic boundary). More generally, we note that the results of analytical perturbation theory predict that the resonance widths (of first order resonances such as the 2:1 and 3:2) increase with particle eccentricity and planet mass as $\Delta a \sim (e \mu)^{1/2}$ (Murray & Dermott 1999). Our non-perturbative numerical results confirm this behavior for small $\mu$ and low eccentricities, $\mu \lesssim 10^{-3}$ and $0.05 \lesssim e \lesssim 0.1$. For larger $\mu$ and eccentricities up to $e_m$, the mass dependence is as expected, $\Delta a \sim \mu^{1/2}$, but the eccentricity dependence is more complex. For higher eccentricities, $e > e_m$, the resonance widths of the pericentric librations decrease with increasing eccentricity and have a weaker mass dependence than the theoretical prediction. The low-order perturbative analysis entirely misses the transitions in the resonance phase space and the appearance of the apocentric libration zone at higher eccentricities.

In the present work, we have investigated only the 2:1 and 3:2 interior MMRs. In a future study, it would be of interest to extend this investigation to other MMRs, and also extend it to higher mass ratios for applications to binary stars, binary planets, and binary asteroids.

We thank an anonymous referee for comments that improved this paper. X.W. acknowledges funding from the National Basic Research Program of China (973 Program) (2012CB720000) and the China Scholarship Council. R.M. acknowledges funding from NASA (grant NNX14AG93G) and NSF (grant AST-1312498).

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