Stationary non-equilibrium measure for a dynamics with two temperatures and two widely different time scales

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Abstract

Multibath generalizations of Langevin dynamics for multiple degrees of freedom with different temperatures and time-scales have been proposed some time ago as possible regularizations for the dynamics of spin-glasses. More recently it has been noted that the stationary non-equilibrium measure of this stochastic process is intimately related to Guerra’s hierarchical probabilistic construction in the framework of the rigorous derivation of Parisi’s solution for the Sherrington-Kirkpatrick model. In this contribution we discuss the time-dependent solution of the two-temperatures Fokker-Planck equation and provide a rigorous analysis of its convergence towards the non-equilibrium stationary measure for widely different time scales. Our proof rests on the validity of suitable log-Sobolev inequalities for conditional and marginal distributions of the limiting measure, and under these hypothesis is valid in any finite dimensions. We discuss a few examples of systems where the log-Sobolev inequalities are satisfied through usual simple, though not optimal, criteria. In particular, our estimates for the rates of convergence have the right order of magnitude for the exactly solvable case of quadratic potentials, and the analysis is also applicable to a spin-glass model with slowly varying external magnetic fields used to dynamically generate Guerra’s construction.

Introduction

The multibath model is a generalization of Langevin dynamics with many time-scales and satisfying a fluctuation-dissipation relation with a different temperature for each time-scale. In the simplest incarnation of the model we have a pair of Langevin equations in the overdamped limit:

\[
\begin{align*}
\lambda_1 \, dx_1 &= -\nabla_1 V(x_1, x_2) \, dt + \sqrt{2D_1} \, dW_1 \\
\lambda_2 \, dx_2 &= -\nabla_2 V(x_1, x_2) \, dt + \sqrt{2D_2} \, dW_2
\end{align*}
\]

(1)
where $\lambda_1, \lambda_2 > 0$ are friction coefficients which set two time-scales, $-\nabla_1 V(x_1, x_2)$, $-\nabla_2 V(x_1, x_2)$ are conservative forces arising from a (time-independent) potential $V(x_1, x_2)$ which couples two parts of a system with configurations denoted respectively by $x_1 \in \mathbb{R}^{d_1}$, $x_2 \in \mathbb{R}^{d_2}$, and $\sqrt{2D_1} dW_1$, $\sqrt{2D_2} dW_2$ are white noises with diffusion coefficients $D_1, D_2 > 0$ ($W_1, W_2$ are independent standard Brownian motions on $\mathbb{R}^{d_1}, \mathbb{R}^{d_2}$ respectively). We adjust the friction and diffusion coefficients so that the fluctuation-dissipation relation with temperatures $\beta_1^{-1}$ and $\beta_2^{-1}$ holds for each degree of freedom separately. In other words we set

$$D_1 \equiv \beta_1^{-1} \lambda_1, \quad D_2 \equiv \beta_2^{-1} \lambda_2.$$  

By a standard application of Ito calculus, the probability density function $\rho = \rho_t(x_1, x_2)$ describing the system configuration at time $t$ must verify the following Fokker-Planck (FP) equation:

$$\partial_t \rho = \frac{1}{\lambda_1} \nabla_1 \cdot \left( \frac{1}{\beta_1} \nabla_1 \rho + \rho \nabla_1 V \right) + \frac{1}{\lambda_2} \nabla_2 \cdot \left( \frac{1}{\beta_2} \nabla_2 \rho + \rho \nabla_2 V \right).$$  

The goal is to study the joint probability distribution $\rho$ of the stochastic process $(x_1(t), x_2(t))$, with particular interest in its long time behavior ($t \to \infty$) and its approach to a non-equilibrium stationary distribution when the two time scales are widely different ($\lambda_1/\lambda_2 \to 0$). In this work we develop this theory rigorously under suitable hypothesis on the potential $V$.

In the context of spin glasses, multibath models such as (1) have been first considered by Horner [1], [2], Coolen, Penney, Sherrington [3, 4], Dotsenko, Franz, Mézard [5], and Allahverdyan, Nieuwenhuizen, Saakian [6, 7] as a mean to regularize the long-time behaviour of mean field spin glasses dynamics, where the couplings are given a very slow dynamics instead of being quenched. Roughly speaking, the dynamics is of the form (1) with $x_1$ representing the annealed spins and $x_2$ the “quasi-quenched” couplings between spins. In model (1) is studied per se and a stationary measure is heuristically derived in the limit of widely different time scales (as well as corrections to this limit for restricted potentials). In [8] it is shown that the resulting (non-Gibbsian) stationary measure can be used to characterize out of equilibrium systems in the limit of small entropy production displaying a natural interpretation in terms of effective temperatures.

The concept of effective temperatures depending on the time-scales in non-equilibrium systems was first discussed by Hohenberg and Shraiman [10] for turbulent flows, by Cugliandolo, Kurchan and Peliti [9, 11] for spin glasses (see also [12]), and more recently for jamming [13]. In these contexts the notion of effective temperature for each degree of freedom separately. In other words we set

$$D_1 \equiv \beta_1^{-1} \lambda_1, \quad D_2 \equiv \beta_2^{-1} \lambda_2.$$  

Let us summarize the heuristic argument [8, 17] for the derivation of the non-equilibrium stationary distribution for widely separated time-scales $\lambda_2 \gg \lambda_1$; in other words when the $x_1$ variables thermalize much more quickly than the $x_2$ variables. The joint distribution of the system at time $t$ can be written as a product of conditional times marginal distributions $\rho_t(x_1, x_2) = \rho_t^{(1)}(x_1|x_2) \rho_t^{(2)}(x_2)$. The $x_1$ variables quickly equilibrate on a time scale $t = O(\lambda_1)$ such that the $x_2$ variables appear as

\[ \rho_t^{(1)}(x_1|x_2) = \frac{1}{\beta_1} \exp \left( -\beta_1 V(x_1, x_2) \right) \]  

for $x_1$ and

\[ \rho_t^{(2)}(x_2) = \frac{1}{\beta_2} \exp \left( -\beta_2 V(x_1, x_2) \right) \]  

for $x_2$.

We refer the interested reader to [14, 15, 16] for comprehensive reviews of such theories which form one more motivation for the analysis of model (1).
quenched, thus on this time scale the conditional distribution $\rho^{(1)}_t(x_1|x_2)$ tends to $\rho^{(1)}_{\star}(x_1|x_2) = e^{-\beta_1 V(x_1,x_2)} / Z_1(x_2)$ where $Z_1(x_2) = \int dx_1 e^{-\beta_1 V(x_1,x_2)}$. For $t \gg \lambda_1$, the $x_2$ variables are subject to the average force under the measure $\rho^{(2)}_t(x_1|x_2)$, i.e.,

$$\langle \nabla_2 V(x_1, x_2) \rangle_{\rho^{(2)}_t} = -\beta_1^{-1} \nabla_2 \log Z_1(x_2).$$

The associated effective potential is thus $F(x_2) = -\beta_1^{-1} \log Z_1(x_2)$. Therefore on a time scale $t = O(\lambda_2)$ the marginal distribution $\rho^{(2)}_t(x_2)$ tends to $\rho^{(2)}_{\star}(x_2) = e^{-\beta_2 F(x_2)} / Z_2$ where $Z_2 = \int dx_2 e^{-\beta_2 F(x_2)}$. Putting pieces together this argument suggests that for $t \gg \lambda_2 \gg \lambda_1$:

$$\rho_1(x_1, x_2) \to \rho^{(1)}_{\star}(x_1|x_2) \rho^{(2)}_{\star}(x_2) = \frac{e^{-\beta_1 V(x_1,x_2)} \left( \int dx'_1 e^{-\beta_1 V(x'_1,x_2)} \right)^{\frac{2_2}{\beta_1}}}{\int dx'_1 \left( \int dx'_2 e^{-\beta_1 V(x'_1,x'_2)} \right)^{\frac{2_2}{\beta_1}}}. \quad (5)$$

The measure $\rho_1(x_1, x_2) = \rho^{(1)}_t(x_1|x_2) \rho^{(2)}_t(x_2)$ defined by equation (5) differs from the Gibbs measure unless $\beta_1 = \beta_2$, in which case it reduces to the Gibbs measure with potential $V(x_1, x_2)$.

The heuristic argument above can be generalized to systems where more than two families of degrees of freedom have widely separated time-scales and different temperatures [18] and the analogous of expression (3) is studied, in the hypothesis of a quenched, thus on this time scale the conditional distribution $\rho^{(1)}_t(x_1|x_2)$ tends to $\rho^{(1)}_{\star}(x_1|x_2) = e^{-\beta_1 V(x_1,x_2)} / Z_1(x_2)$ where $Z_1(x_2) = \int dx_1 e^{-\beta_1 V(x_1,x_2)}$. For $t \gg \lambda_1$, the $x_2$ variables are subject to the average force under the measure $\rho^{(2)}_t(x_1|x_2)$, i.e.,

$$\langle \nabla_2 V(x_1, x_2) \rangle_{\rho^{(2)}_t} = -\beta_1^{-1} \nabla_2 \log Z_1(x_2).$$

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$$\rho_1(x_1, x_2) \to \rho^{(1)}_{\star}(x_1|x_2) \rho^{(2)}_{\star}(x_2) = \frac{e^{-\beta_1 V(x_1,x_2)} \left( \int dx'_1 e^{-\beta_1 V(x'_1,x_2)} \right)^{\frac{2_2}{\beta_1}}}{\int dx'_1 \left( \int dx'_2 e^{-\beta_1 V(x'_1,x'_2)} \right)^{\frac{2_2}{\beta_1}}}. \quad (5)$$

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The heuristic argument above can be generalized to systems where more than two families of degrees of freedom have widely separated time-scales and different temperatures [18] and the analogous of expression (3) is obtained through a hierarchy of conditional averages. As it turns out, the resulting measure is at the core of Guerra’s interpolation scheme [19] which leads to the Talagrand’s proof [20] of the celebrated Parisi solution [21, 22] for the Sherrington-Kirkpatrick model. We also mention that the hierarchical structure of the measure, as well the Parisi solution itself, can be represented in terms of Ruelle Probability Cascades [23, 24], a central object in the mathematical theory of replica symmetry breaking in mean-field spin glasses [25, 26]. Some aspects of the connection between multibath models and replica symmetry breaking has been recently investigated in [18].

The main motivation of this work is to provide a general rigorous framework for the dynamical derivation of the stationary measure in the limit of widely separated time scales. In particular we give a rigorous proof of (3) with precise convergence rates for a suitable class of potentials. Full hypothesis and statements are found in Section 4 and here we informally summarize the main results. Since the change of variables $(t, \lambda_1, \lambda_2) \mapsto (\frac{t}{\lambda}, 1, \frac{\lambda}{\lambda})$ does not affect the solution of equation (3), throughout the paper we will assume without loss of generality

$$\lambda_1 \equiv 1, \quad \lambda_2 \equiv \lambda. \quad (6)$$

The probability solution $\rho_{1,\lambda}(x_1, x_2)$ of the two-temperatures Fokker-Planck equation (3) is studied, in the hypothesis of a confining potential $V$ such that the “conditional potential” $x_1 \mapsto V(x_1, x_2)$ and the effective potential $F(x_2)$ both verify a logarithmic Sobolev inequality. For example, one can keep in mind the case of $V = V_c + V_0$, where $V_c$ is strongly convex and $V_0$ is any bounded perturbation. We will also assume that $V$ is smooth and has polynomial growth together with its derivatives and that the initial probability density $\rho_1$ at time $t = 0$ is bounded and has all finite moments. Our main result states that the Kullback-Leibler (KL) divergence $D_{KL}(\rho_{1,\lambda} \parallel \rho_\lambda)$ tends to zero as $t/\lambda \to +\infty, \lambda \to +\infty$. Thanks to Pinsker’s inequality the same holds true for
the total variation distance $\| \rho_{t, \lambda} - \rho_x \|_{\text{TV}}$. Furthermore, the KL divergence can be decomposed as $D_{KL}(\rho_{t, \lambda} \| \rho_x) = D_1(t, \lambda) + D_2(t, \lambda)$ where $D_1(t, \lambda)$ is the KL divergence between conditional distributions $\rho^{(1)}_{t, \lambda}(\cdot | x_2), \rho^{(2)}_{t, \lambda}(\cdot | x_2)$ averaged over $\rho^{(2)}_{t, \lambda}$, and $D_2(t, \lambda)$ is the KL divergence between marginal distributions $\rho^{(1)}_{t, \lambda}, \rho^{(2)}_x$. Theorem 1.7 provides precise estimates for $D_1(t, \lambda)$ and $D_2(t, \lambda)$ separately, proving exponential decay on two widely different time scales - up to time-independent residuals of order $\lambda^{-1}$.

Except for these residuals, $D_1(t, \lambda)$ decays exponentially fast in time on a time scale of order one, while $D_2(t, \lambda)$ decays exponentially fast on a much larger time scale of order $\lambda$. The convergence rates of $D_1(t, \lambda)$ and $D_2(t, \lambda)$ are given in terms of the constants characterizing the aforementioned logarithmic Sobolev inequalities.

Section 2 is devoted to the analysis of a simple criterion for logarithmic Sobolev inequalities (LSI’s). LSI’s are functional inequalities related to the approach to equilibrium of the usual one-temperature Fokker-Planck equation (this concept is well reviewed in [27]). The classical Bakry-Émery-Holley-Stroock criterion guarantees the validity of a LSI for the Gibbs measure $e^{-\beta V}/Z$ if the potential $V$ is the sum of a strongly convex contribution and a bounded perturbation. Here we show, by means of the Brascamp-Lieb concentration inequality [28, 29], that the same condition on $V$ turns out to guarantee the validity of LSI’s for both the conditional measure $\rho^{(1)}_t(x_1 | x_2) = e^{-\beta_1 V(x_1, x_2)}/Z_1(x_2)$ and the marginal measure $\rho^{(2)}_t(x_2) = e^{-\beta_2 F(x_2)}/Z_2$.

It is worth remarking that the Bakry-Émery-Holley-Stroock criterion provides (in general) convergence rates that are exponentially slow in the size of the system (i.e., dimensions $d_1, d_2$), unless the potential $V$ is strictly convex. In the latter fundamental case the related logarithmic Sobolev constant can be independent of the dimension.

Investigating for more physical criteria for the two LSI’s of interest (i.e., on the conditional and marginal distributions) is an interesting problem that goes beyond the scope of the present work.

In Section 3 we give concrete examples for which Theorem 1.7 applies. First, in Subsection 3.1 we analyze the exactly solvable case of the two-temperatures FP equation with a quadratic potential: this boils down to a multidimensional Ornstein-Uhlenbeck process. The non-equilibrium stationary measure as well as interesting properties such as associated rotational current densities have been discussed in detail in [30] (for all $\lambda$ and dimensions $d_1 = d_2 = 1$). Here we provide the explicit formulas for the whole time-dependence of $\rho_{t, \lambda}$ and its approach to $\rho_x$ when $\lambda > 1$ and $d_1 = d_2 = 1$, showing that our estimates give the correct time scales for approaching the stationary distribution as well as the correct order of magnitude $O(\lambda^{-1})$ for the residual KL divergences. The second example in Subsection 3.2 is a spin glass model where the spin degrees of freedom as well as the external magnetic fields are given a dynamics in the spirit of [17, 18]. The third example (Subsection 3.3) comes from a paradigm in modern high-dimensional inference and estimation theory, namely the problem of estimating a rank-one matrix which is observed through a noisy channel (see [31] for a general review on the subject and a discussion of applications).

Let us summarize the methods of proof detailed in Section 4. To analyze the approach to stationarity of the solution of two-temperatures FP equation (with detailed estimates on the rates of approach) we employ the method of free energy dissipation inequalities (Theorems 4.16, 4.20) which is well adapted to the physics of the problem. In the one-temperature case [27] one looks at the time-derivative of the KL divergence between the dynamical solution of the FP equation with potential $V$.
and the equilibrium (Gibbs) measure. The KL divergence coincides (up to a multiplicative constant given by the temperature) with the difference between the free energy functional of the dynamical measure and the free energy at equilibrium. Thus minus time-derivative of the KL divergence (up to temperature) is represented by the free energy dissipation functional. For a suitable potential $V$ (or more precisely for the associated Gibbs measure $e^{-\beta V} / Z(\beta)$) the LSI states that the KL divergence is smaller than a constant times the free energy dissipation. Then the LSI provides a closed differential inequality for the KL divergence. This is called a “free energy dissipation inequality” and its solution yields the trend to equilibrium. In the present case of a two-temperatures FP equation, we use two LSI’s for the conditional and marginal measures $\rho_1^*(t), \rho_2^*$ to deduce dissipation inequalities for the conditional and marginal KL divergences $D_1(t, \lambda), D_2(t, \lambda)$ respectively. This requires a careful analysis of the time-derivatives of the KL divergences providing a control of the remainders of order $O(\lambda^{-1})$, indeed - unlike the one-temperature case - the two-temperatures stationary measure depends on the friction coefficient $\lambda$ and $\rho_*$ turns out to be only the first order of its expansion. Our analysis relies: (i) on the fact that the marginal measure $\rho_{1\lambda}(x_2)$ satisfies a suitable Fokker-Planck equation itself (Subsection 4.2); (ii) on the observation that every moment of both measures $\rho_{1\lambda}(x_1, x_2)$ and $\rho_1^*(x_1|x_2) \rho_{1\lambda}(x_2)$ is bounded uniformly in $t$ and $\lambda$ (Subsection 4.1); (iii) and on a careful application of integration by parts and Pinsker inequality (Subsections 4.4, 4.5). The most technical aspects of the proof are postponed to the Appendix (Section 5). The uniform bound on moments is obtained by Lyapunov methods described in Subsections 5.2, 5.3.

There exists a large literature on the computation of solutions of generalized FP equations with two (or more) widely different time scales at finite times. We mention the book [33] and references therein for a rigorous approach using techniques of averaging and homogeneization. We are not aware that these results cover the long-time behaviour and stationary measure studied here.

There are not many instances in out-of-equilibrium Statistical Mechanics when one can determine stationary measures. Much work has been devoted to transport problems in extended systems coupled to many reservoirs at different temperatures. We refer to the review [34] and references therein for this realm. The multibath model treated here is a different (and easier) problem that does not involve spatial transport (or heat flux) in an extended system. Nevertheless it is of interest that under suitable assumptions on the potential the model is amenable to a rigorous analysis and the stationary non-equilibrium distribution (5) can be completely determined exactly at least in the limit of widely separated scales. Although the existence of the stationary distribution is guaranteed for any $\lambda$ (thanks to general theorems from the theory of parabolic partial differential equations), its explicit expression is unknown for finite $\lambda$ (except for the exactly solvable case of a quadratic potential). Presumably the computation of large $\lambda$ corrections to the measure $\rho_*$ in (5) is amenable to rigorous analysis but this is at the moment an open problem. Another compelling outlook is the extension to non-convex potentials in infinite dimension which is reasonably possible at least in specific phases (e.g., high temperature).

1This quantity is also called entropy dissipation functional or relative Fisher information.
1 Definitions and main result

Let $V : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$ be a potential that couples two parts of a system whose configurations are denoted by $x \equiv (x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \equiv \mathbb{R}^d$. We assume that $V$ is smooth and has polynomial growth together with all its derivatives, precisely:

(A1) $V \in C^\infty(\mathbb{R}^d)$;

(A2) for every multi-index $\nu \in \mathbb{N}^d$ there exist constants (depending on $\nu$) $r_1, r_2, \gamma_0, \gamma_1, \gamma_2 \in [0, \infty)$ such that for all $x = (x_1, x_2) \in \mathbb{R}^d$

$$|D^\nu V(x)| \leq \gamma_0 + \gamma_1 |x_1|^{r_1} + \gamma_2 |x_2|^{r_2} .$$

$D^\nu$ is the usual multi-index notation for partial derivatives with respect to $x$. We also assume that $V$ is confining in the following sense:

(A3) there exist $a_1, a_2 \in (0, \infty), a_0 \in [0, \infty)$ such that for all $x \in \mathbb{R}^d$

$$V(x) \geq a_1 |x_1|^2 + a_2 |x_2|^2 - a_0 ;$$

(A4) there exist $a_1, a_2 \in (0, \infty), a_0 \in [0, \infty)$ such that for all $x \in \mathbb{R}^d$

$$x \cdot \nabla V(x) \geq a_1 |x_1|^2 + a_2 |x_2|^2 - a_0 ;$$

(A5) there exist $a_1 \in (0, \infty), a_0, a_2, p \in [0, \infty)$ such that for all $x \in \mathbb{R}^d$

$$x_1 \cdot \nabla_1 V(x) \geq a_1 |x_1|^2 - a_2 |x_2|^p - a_0 .$$

In this paper $\nabla_1$ denotes the nabla operator with respect to the $x_1$ variables only, an analogous convention is adopted for $\nabla_2$, while $\nabla$ denotes the nabla operator with respect to $x = (x_1, x_2)$.

As explained in the Introduction, we are interested in the long time behaviour of the system when the two subsystems have widely different time scales (to fix ideas, the variables $x_2$ evolve much more slowly than the variables $x_1$). Heuristic arguments provide an explicit expression for the stationary measure in this limit. Given $\beta_1, \beta_2 > 0$ we introduce the probability density $\rho_* : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to (0, \infty)$

$$\rho_*(x_1, x_2) \equiv e^{-\beta_1 V(x_1, x_2)} Z_1(x_2)^{\beta_2 - 1} Z_2^{-1} ,$$

where

$$Z_1(x_2) \equiv \int_{\mathbb{R}^{d_1}} e^{-\beta_1 V(x_1, x_2)} \, dx_1 \quad Z_2 \equiv \int_{\mathbb{R}^{d_2}} Z_1(x_2)^{\beta_2} \, dx_2 .$$

Remark 1.1. Assumption (A3) guarantees that for every $r \in [0, \infty)$

$$\int_{\mathbb{R}^{d_1}} |x_1|^r e^{-\beta_1 V(x_1, x_2)} \, dx_1 < \infty ,$$

$$\int_{\mathbb{R}^{d_2}} |x_2|^r \left( \int_{\mathbb{R}^{d_1}} e^{-\beta_1 V(x_1, x_2)} \, dx_1 \right)^{\beta_2} \, dx_2 < \infty ,$$

hence $\rho_*$ is a suitably normalized probability measure and all moments are finite.
We also introduce an effective potential $F : \mathbb{R}^{d_2} \to \mathbb{R}$

$$F(x_2) \equiv -\frac{1}{\beta_1} \log Z_1(x_2),$$

which for a given configuration of the slow variables $x_2$ coincides with the free energy at equilibrium of the fast variables $x_1$. One may better understand the measure $\rho_*$ decomposing it as the product of its conditional and marginal measure:

$$\rho_*(x_1, x_2) = \rho_*^{(1)}(x_1|x_2) \rho_*^{(2)}(x_2).$$

It is easy to check that the conditional measure of $x_1$ given $x_2$ is the Gibbs measure with potential $x_1 \mapsto V(x_1, x_2)$ and inverse temperature $\beta_1$:

$$\rho_*^{(1)}(x_1|x_2) \equiv \frac{1}{Z_1(x_2)} e^{-\beta_1 V(x_1, x_2)};$$

while the marginal measure of $x_2$ is the Gibbs measure with effective potential $F$ and inverse temperature $\beta_2$:

$$\rho_*^{(2)}(x_2) \equiv \frac{1}{Z_2} e^{-\beta_2 F(x_2)}.$$

We will use a bracket notation for the conditional (Gibbs) expectation of any suitable observable $O$:

$$\langle O \rangle_* (x_2) \equiv \int_{\mathbb{R}^{d_1}} O(x_1, x_2) \rho_*^{(1)}(x_1|x_2) \, dx_1.$$

Our main result is based on two logarithmic Sobolev inequalities for the conditional and marginal measures $\rho_*^{(1)}, \rho_*^{(2)}$. Precisely we assume that:

(LS1) there exists $c_1 = c_1(\beta_1, V) \in (0, \infty)$ such that for every $x_2 \in \mathbb{R}^{d_2}$

$$\int_{\mathbb{R}^{d_1}} \pi(x_1) \log \frac{\pi(x_1)}{\rho_*^{(1)}(x_1|x_2)} \, dx_1 \leq \frac{1}{2\beta_1 c_1} \int_{\mathbb{R}^{d_1}} \pi(x_1) \left| \nabla_1 \log \frac{\pi(x_1)}{\rho_*^{(1)}(x_1|x_2)} \right|^2 \, dx_1$$

for every probability density $\pi \in C^1(\mathbb{R}^{d_1})$ such that $\pi > 0$, $\int_{\mathbb{R}^{d_1}} \pi(x) \, dx_1 = 1$, and $\pi \log \frac{\pi}{\rho_*^{(1)}(x_2)} \in L^1(\mathbb{R}^{d_1})$;

(LS2) there exists $c_2 = c_2(\beta_2, V) \in (0, \infty)$ such that

$$\int_{\mathbb{R}^{d_2}} \pi(x_2) \log \frac{\pi(x_2)}{\rho_*^{(2)}(x_2)} \, dx_2 \leq \frac{1}{2\beta_2 c_2} \int_{\mathbb{R}^{d_2}} \pi(x_2) \left| \nabla_2 \log \frac{\pi(x_2)}{\rho_*^{(2)}(x_2)} \right|^2 \, dx_2$$

for every probability density $\pi \in C^1(\mathbb{R}^{d_2})$ such that $\pi > 0$, $\int_{\mathbb{R}^{d_2}} \pi(x_2) \, dx_2 = 1$, and $\pi \log \frac{\pi}{\rho_*^{(2)}(x_2)} \in L^1(\mathbb{R}^{d_2})$.

In Section 2 we prove that (LS1), (LS2) hold true for potentials given by a sum of strongly convex part and bounded perturbation. In Section 3 we discuss various examples and the dependence of logarithmic Sobolev constants on $d_1, d_2$. The study of improved criteria for (LS1), (LS2) is beyond the scope of this work and is postponed to future work.
We are going to study the evolution of the system according to SDE (1) with diffusion coefficients given by (2),(6) through a rigorous analysis of its probability distribution described by Fokker-Planck equation (3). We assume that at initial time \( t = 0 \) the system configuration is distributed according to a probability measure \( \mu_I \) having density \( \rho_I \) with respect to the Lebesgue measure. We set

\[
\rho_I^{(i)}(x_2) \equiv \int_{\mathbb{R}^d_1} \rho_I(x_1, x_2) \, dx_1, \quad \rho_I^{(i)}(x_1| x_2) \equiv \frac{\rho_I(x_1, x_2)}{\rho_I^{(2)}(x_2)}.
\]

We make the following conditions are satisfied:

(B1) \( \rho_I \) is bounded on \( \mathbb{R}^d \);

(B2) \( \rho_I^{(2)} \) is bounded on \( \mathbb{R}^{d_2} \);

(B3) for every \( r \in [0, \infty) \)

\[
\int_{\mathbb{R}^d} \rho_I(x_1, x_2) (|x_1|^r + |x_2|^r) \, dx_1 \, dx_2 < \infty.
\]

The initial probability density \( \rho_I \) can be zero in some regions. On the other hand, any initial measure \( \mu_I \) which is not absolutely continuous with respect to the Lebesgue measure has been excluded, since its Kullback-Leibler divergence from \( \rho_\ast \) would not be finite.

Given \( \beta_1, \beta_2, \lambda > 0 \), we introduce the following differential operator

\[
L \varphi \equiv \left( \frac{1}{\beta_1} \nabla_1^2 \varphi - \nabla_1 \varphi \cdot \nabla_1 V \right) + \frac{1}{\lambda} \left( \frac{1}{\beta_2} \nabla_2^2 \varphi - \nabla_2 \varphi \cdot \nabla_2 V \right)
\]

and its formal adjoint

\[
L^\ast \rho \equiv \nabla_1 \cdot \left( \frac{1}{\beta_1} \nabla_1 \rho + \rho \nabla_1 V \right) + \frac{1}{\lambda} \nabla_2 \cdot \left( \frac{1}{\beta_2} \nabla_2 \rho + \rho \nabla_2 V \right).
\]

To compact the notation we also introduce the diagonal matrices

\[
\Lambda \equiv \begin{pmatrix} I_{d_1} & 0 \\ 0 & \lambda I_{d_2} \end{pmatrix}, \quad \beta \equiv \begin{pmatrix} \beta_1 I_{d_1} & 0 \\ 0 & \beta_2 I_{d_2} \end{pmatrix}
\]

and rewrite

\[
L \varphi = \nabla \cdot \left( (\Lambda \beta)^{-1} \nabla \varphi \right) - \Lambda^{-1} \nabla V \cdot \nabla \varphi,
\]

\[
L^\ast \rho = \nabla \cdot \left( (\Lambda \beta)^{-1} \nabla \rho + \Lambda^{-1} \nabla V \rho \right).
\]

Sometimes we will write \( L_\lambda, L^\ast_\lambda \) instead of \( L, L^\ast \) to stress the dependence of these differential operators on the friction coefficient \( \lambda \).

From the general theory of Fokker-Planck PDEs we have the following

**Theorem 1.2 (Existence and uniqueness).** Suppose that \( V \) verifies assumptions (A1)-(A4) and \( \mu_I \) verifies assumption (B3). Then there exists a unique collection \( \mu \equiv (\mu_t)_{t \geq 0} \) of Borel probability measures on \( \mathbb{R}^d \) such that

\[
\int_{\mathbb{R}^d} \varphi(x) \, \mu_t(dx) - \int_{\mathbb{R}^d} \varphi(x) \, \mu_0(dx) = \int_0^t \int_{\mathbb{R}^d} L \varphi(x) \, \mu_s(dx) \, ds
\]
for every $t > 0$ and every $\varphi \in C^2_c(\mathbb{R}^d)$. Moreover there exists a continuous density function $\rho : \mathbb{R}^d \times (0, \infty) \to (0, \infty)$ such that for every $t > 0$

$$
\mu_t(dx) = \rho_t(x) \, dx.
$$

(30)

The notations $\rho, \rho_t(x)$ will be maintained all along the paper. The notation $\rho_{t,\lambda}(x)$ is sometimes preferred to $\rho_t(x)$ in order to stress the dependence on the friction coefficient $\lambda$. The dependence on $\beta_1, \beta_2, \rho, V$ will be always implicit.

**Remark 1.3.** With the introduced notation the Fokker-Planck equation (3) with friction coefficients given by (6) writes as

$$
\partial_t \rho = L^* \rho.
$$

(31)

Equation (29) is a weak formulation of the latter equation with initial measure $\mu_I = \rho_I \, dx$.

**Remark 1.4.** For $\varphi \in C^2_c(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} L \varphi(x) \rho_t(x) \, dx$ is a continuous function of $s > 0$ by continuity of $\rho$ and dominated convergence $|L \varphi| \rho_s \leq |L \varphi| \max_{s \in [s-\delta, s+\delta]} \rho_s \in L^1(\mathbb{R}^d)$. The Fundamental Theorem of Calculus applies and identity (29) entails that

$$
\int_{\mathbb{R}^d} \varphi(x) \rho_t(x) \, dx \quad \text{is a } C^1\text{-function of } t > 0,
$$

and

$$
\int_{\mathbb{R}^d} \varphi(x) \rho_t(x) \, dx \xrightarrow{t \to 0} \int_{\mathbb{R}^d} \varphi(x) \rho_I(x) \, dx.
$$

(32)

(33)

For Theorem 1.2 we refer to [32], where the problem is studied in greater generality: see in particular Sections 6.1, 6.3, 6.4, 6.6, 9.1, 9.3, and 9.4 therein. It may be worth remarking that in our case, thanks to the existence of a continuous density $\rho$, equation (29) holds true for every $t > 0$, not only for almost every $t$. We also notice that if $\mu_t$ are sub-probability measures satisfying equation (29) then $\mu_t$ are actually probability measures, as can be seen by choosing a sequence of test functions $\varphi_N \in C^2_c(\mathbb{R}^d)$ such that $\varphi_N(x) = 1$ for $|x| \leq N$ and $\varphi_N, L \varphi_N$ uniformly bounded. In [32] several conditions guaranteeing the uniqueness of a (sub-)probability solution are provided; among them the condition $\Lambda^{-1} \nabla V \in L^1(\mathbb{R}^d \times (0,T), \mu_t(dx)dt)$ for all $T > 0$, which is discussed in Subsection 4.1.

**Theorem 1.5 (Regularity).** In the hypothesis of Theorem 1.2 we have:

i. $\rho \in C^\infty_c(\mathbb{R}^d \times (0, \infty))$, hence it is a strong solution of Fokker-Planck equation (31);

ii. $\| \rho_t - \rho_I \|_{L^1(\mathbb{R}^d)} \to 0$ as $t \to 0$.

Theorem 1.5 can be obtained adapting the bootstrap argument sketched in [35]. The proof is postponed to Subsection 5.4 in the Appendix.

We will focus on the Kullback-Leibler (KL) divergence of the dynamical measure $\rho_{t,\lambda}$ from the reference measure $\rho_*$. It turns out to be convenient to split the analysis
into a first contribution given by the conditional measures and a second one given by
the marginal measures. We set

$$\rho^{(2)}_t(x_2) \equiv \int_{\mathbb{R}^d_1} \rho_t(x_1, x_2) \, dx_1 ,$$

and $\rho^{(1)} : \mathbb{R}^d_1 \times \mathbb{R}^d_2 \times (0, \infty) \to (0, \infty)$

$$\rho^{(1)}_t(x_1|x_2) \equiv \frac{\rho_t(x_1, x_2)}{\rho^{(2)}_t(x_2)} .$$

A bracket notation for conditional (dynamical) expectations will be used for any
suitable observable $O$:

$$\langle O \rangle_t(x_2) \equiv \int_{\mathbb{R}^d_1} O(x_1, x_2) \, \rho^{(1)}_t(x_1|x_2) \, dx_1 .$$

Along the paper the notations $\rho^{(1)}_t(x_1|x_2) , \rho^{(2)}_t(x_2) , \langle \cdot \rangle_{t,\lambda}$ will often replace $\rho^{(1)}_t(x_1|x_2) , \rho^{(2)}_t(x_2) , \langle \cdot \rangle_t$ in order to stress the dependence on the friction coefficient $\lambda$. We set:

$$D_1(t, \lambda) \equiv \int_{\mathbb{R}^d} \rho_{t,\lambda}(x_1, x_2) \, \log \frac{\rho^{(1)}_{t,\lambda}(x_1|x_2)}{\rho^{(2)}_{t,\lambda}(x_2)} \, dx_1 dx_2 \geq 0 ,$$

$$D_2(t, \lambda) \equiv \int_{\mathbb{R}^d_2} \rho^{(2)}_{t,\lambda}(x_2) \, \log \frac{\rho^{(2)}_{t,\lambda}(x_2)}{\rho^{(2)}_{t,\lambda}(x_2)} \, dx_2 \geq 0 .$$

A standard computation shows that the KL divergence between the joint densities $\rho_{t,\lambda} , \rho_*$ is

$$D(t, \lambda) \equiv \int_{\mathbb{R}^d} \rho_{t,\lambda}(x) \, \log \frac{\rho_{t,\lambda}(x)}{\rho_*(x)} \, dx = D_1(t, \lambda) + D_2(t, \lambda) .$$

We also set:

$$D_{1I} \equiv \int_{\mathbb{R}^d} \rho_{t}(x_1, x_2) \, \log \frac{\rho^{(1)}_{t}(x_1|x_2)}{\rho^{(2)}_{t}(x_2)} \, dx_1 dx_2 ,$$

$$D_{2I} \equiv \int_{\mathbb{R}^d_2} \rho^{(2)}_{t}(x_2) \, \log \frac{\rho^{(2)}_{t}(x_2)}{\rho^{(2)}_{t}(x_2)} \, dx_2 .$$

Remark 1.6. Assumptions (B1)-(B3) imply that $\rho_t , \rho_t^{(2)}$ have finite differential entropies:

$$\int_{\mathbb{R}^d} \rho_t(x) \, | \log \rho_t(x)|\, dx < \infty , \quad \int_{\mathbb{R}^d_2} \rho_t^{(2)}(x_2) \, | \log \rho_t^{(2)}(x_2)|\, dx_2 < \infty$$

(one can mimic the last part of the Proof of Proposition 4.4 in Subsection 5.2). Therefore the integrals defining (40), (41) are absolutely convergent. From the results
stated in Subsections 4.1, 5.2 it will be clear that the integrals defining (37), (38) are
also absolutely convergent.
We can finally state the main result of the present paper.

**Theorem 1.7** (Main result). Let $\beta_1, \beta_2 > 0$. Suppose that $V$ verifies assumptions (A1)-(A5), logarithmic Sobolev inequalities (LS1), (LS2) hold true, and $p_t$ verifies assumptions (B1)-(B3). Then there exist finite non-negative constants $c_0 = c_0(\beta_1, \beta_2, p_t, V)$, $\tilde{c}_0 = \tilde{c}_0(\beta_1, \beta_2, p_t, V)$ such that the following facts hold true.

i) For all $t > 0$, $\lambda \geq 1$ we have

$$D_1(t, \lambda) \leq D_{11} e^{-2c_1 t} + R_1(t) \frac{1}{\lambda}$$

with bounded remainder

$$R_1(t) \equiv (1 - e^{-2c_1 t}) \frac{c_0}{2c_1}.$$  

The rate $c_1 = c_1(\beta_1, V)$ is the uniform constant that verifies (LS1). A suitable choice for $c_0$ can be obtained following Remark 4.17.

ii) For all $t > 0$, $\lambda \geq 1 \vee c_2 c_1^{-1}$, $\epsilon > 0$, $\eta \in (0, 2)$ we have

$$D_2(t, \lambda) \leq D_{21} e^{-\eta c_2 \epsilon t/\lambda} + \frac{R_2(t, \lambda, \eta)}{2 - \eta} + \frac{R_3(t, \lambda, \eta)}{2 - \eta} \epsilon$$

with bounded remainders

$$R_2(t, \lambda, \eta) \equiv (e^{-\eta c_2 \epsilon t/\lambda} - e^{-2c_1 t}) \frac{D_{11} + \frac{c_0}{2c_1}}{2c_1 - \frac{2\eta}{\lambda}} + (1 - e^{-\eta c_2 \epsilon t/\lambda}) \frac{c_0}{2\eta c_1 c_2}$$

$$R_3(t, \lambda, \eta) \equiv (1 - e^{-\eta c_2 \epsilon t/\lambda}) \frac{\tilde{c}_0}{\eta c_2}.$$  

The rate $c_2 = c_2(\beta_2, V)$ is the constant that verifies (LS2). A suitable choice for $\tilde{c}_0$ is given by Remark 4.27.

By the previous inequalities, two different time scales emerge in the convergence:

$$D_1(t, \lambda) \to 0 \quad \text{as} \quad t \to \infty, \quad \lambda \to \infty,$$

$$D_2(t, \lambda) \to 0 \quad \text{as} \quad t \lambda^{-1} \to \infty, \quad \lambda \to \infty.$$  

In particular by (49) it follows that $D(t, \lambda) \to 0$ as $t \lambda^{-1} \to \infty$ and $\lambda \to \infty$.

**Remark 1.8.** The convergence rate of total variation distance can be deduced by Csiszár-Kullback-Pinsker inequality. Precisely:

$$\|p_{t, \lambda} - p_*\|_{TV} \leq \sqrt{2} D(t, \lambda) = \sqrt{2} \sqrt{D_1(t, \lambda) + D_2(t, \lambda)}.$$  

**Remark 1.9.** Let us consider the two-temperatures free energy functional for a potential $V$ introduced by [5]:

$$F_{\beta_1, \beta_2}(\pi) \equiv \int_{\mathbb{R}^d} \pi(x) \left( V(x) + \beta_1^{-1} \log \pi^{(1)}(x_1|x_2) + \beta_2^{-1} \log \pi^{(2)}(x_2) \right) dx$$  

for every probability density $\pi$ on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ with conditional and marginal distributions $\pi^{(1)}$, $\pi^{(2)}$ respectively. It is easy to check that

$$F_{\beta_1, \beta_2}(p_{t, \lambda}) - F_{\beta_1, \beta_2}(p_*) = \beta_1^{-1} D_1(t, \lambda) + \beta_2^{-1} D_2(t, \lambda).$$

Theorem 1.7 immediately applies to this difference of free energy functionals.
2 A simple criterion for logarithmic Sobolev inequalities (LS1), (LS2) for conditional and marginal measures $\rho^{(1)}_\pi$, $\rho^{(2)}_\pi$

In this section we first recall a fundamental criterion due to Bakry-Émery and Holley-Stroock providing logarithmic Sobolev inequalities for a suitable class of Gibbs measures. Then, using a Brascamp-Lieb concentration inequality, we derive a simple criterion that provides logarithmic Sobolev inequalities for the conditional and marginal of our reference measure $\rho$, ensuring that assumptions (LS1), (LS2) hold true.

In general a probability density $\rho : \mathbb{R}^n \to (0, \infty)$ is said to verify a logarithmic Sobolev inequality (LS1) with constant $C > 0$ if:

$$\int_{\mathbb{R}^n} \pi(x) \log \frac{\pi(x)}{\rho_G(x)} \, dx \leq \frac{1}{2C} \int_{\mathbb{R}^n} \pi(x) \left| \nabla \log \frac{\pi(x)}{\rho_G(x)} \right|^2 \, dx$$

for every probability density $\pi$ suitably regular, e.g., $\pi \in C^1(\mathbb{R}^n)$ such that $\pi > 0$, $\int_{\mathbb{R}^n} \pi(x) \, dx = 1$, $\pi \log \frac{\pi}{\rho} \in L^1(\mathbb{R}^n)$.

Remark 2.1. The l.h.s. of (53) is the Kullback-Leibler divergence of the measure $\pi$ from $\rho_G$, while the integral on the r.h.s. is the so-called relative Fisher information. Let us think to $\rho_G$ as the Gibbs measure with potential $U$ and inverse temperature $\beta > 0$, that is

$$\rho_G(x) = \frac{1}{Z} e^{-\beta U(x)}, \quad Z \equiv \int_{\mathbb{R}^n} e^{-\beta U(x)} \, dx.$$

Introducing the free energy functional

$$\mathcal{F}(\pi) \equiv \int_{\mathbb{R}^n} \pi(x) \left( U(x) + \beta^{-1} \log \pi(x) \right) \, dx,$$

the KL divergence up to a factor $\beta$ coincide with the difference between the free energy associated to the measure $\pi$ and the equilibrium free energy:

$$\int_{\mathbb{R}^n} \pi(x) \log \frac{\pi(x)}{\rho_G(x)} \, dx = \beta \left( \mathcal{F}(\pi) - \mathcal{F}(\rho_G) \right).$$

The Fisher information on the other hand rewrites as

$$\int_{\mathbb{R}^n} \pi(x) \left| \nabla \log \frac{\pi(x)}{\rho_G(x)} \right|^2 \, dx = \beta \int_{\mathbb{R}^n} \pi(x) \left| \nabla U(x) + \beta^{-1} \nabla \log \pi(x) \right|^2 \, dx$$

and is related to the dissipation of free energy $\mathcal{F}$ along a standard one-temperature Langevin dynamics in the potential $V$. [27, 30]

Remark 2.2. Setting $f(r) \equiv r \log r - r + 1 \geq 0$ for $r \geq 0$, the KL divergence also rewrites as

$$\int_{\mathbb{R}^n} \pi(x) \log \frac{\pi(x)}{\rho_G(x)} \, dx = \int_{\mathbb{R}^n} \rho_G(x) f \left( \frac{\pi(x)}{\rho_G(x)} \right) \, dx,$$

while the Fisher information rewrites as

$$\int_{\mathbb{R}^n} \pi(x) \left| \nabla \log \frac{\pi(z)}{\rho_G(z)} \right|^2 \, dx = 4 \int_{\mathbb{R}^n} \rho_G(x) \left| \nabla \sqrt{\frac{\pi(x)}{\rho_G(x)}} \right|^2 \, dx.$$
The expressions on the r.h.s. of (58), (59) are well defined integrals for all $\pi \geq 0$ such that $\int_{\mathbb{R}^n} \pi(x) \, dx = 1$ and $\sqrt{\pi/\rho \pi G} \in W_{1,2}^{1,2}(\mathbb{R}^n)$. In fact this allows to extend the LSI to a larger class of densities $\pi$ (see [38]).

Logarithmic Sobolev inequalities were initially proved for Gaussian measures [37, 38]. A fundamental criterion was then provided by Bakry-Èmery:

**Theorem 2.3** (Bakry-Èmery [39]). Suppose $U \in C^2(\mathbb{R}^n)$ is strongly convex, namely there exists $\alpha > 0$ such that

$$\phi^T \text{Hess} U(x) \phi \geq \alpha |\phi|^2$$

for all $\phi \in \mathbb{R}^n$, $x \in \mathbb{R}^n$. Then the Gibbs measure $\rho_G = e^{-\beta U}/Z$ satisfies the LSI (53) with constant $C = \beta \alpha$.

Holley-Stroock perturbation lemma gives a simple way to extend the class of measures satisfying a LSI:

**Theorem 2.4** (Holley-Stroock [40]). Suppose

$$U = U_0 + U_b,$$

where $e^{-\beta U_0}/\int e^{-\beta U_0(x)} \, dx$ satisfies the LSI with constant $C_0$ and $U_b$ is bounded. Then the Gibbs measure $\rho_G = e^{-\beta U}/Z$ satisfies the LSI (53) with constant $C = C_0 e^{-\beta \text{osc}(U_b)}$, where $\text{osc}(U_b) = \sup(U_b) - \inf(U_b)$.

A review about logarithmic Sobolev inequalities and their relation with the approach to equilibrium of single-temperature Fokker-Planck equations can be found in [27]. In [36] the approach to equilibrium of a single-temperature dynamics in a strongly convex potential is studied. By analysing the free energy dissipation and the free energy dissipation rate with a careful application of integration by parts, a generalised class of convex Sobolev inequalities is obtained, recovering in particular the Bakry-Èmery criterion. We refer to [36] for a proof of Theorems 2.3, 2.4 adopting this dynamical point of view. It is worth noticing that the point of view of the present paper is reversed with respect to [36]: having in our hands logarithmic Sobolev inequalities for the conditional and marginal measures $\rho^{(1)}$, $\rho^{(2)}$ allows to study the long time behaviour of the two-temperatures dynamics and prove Theorem 1.7.

Now, suppose that our potential on $\mathbb{R}^d$ writes as:

$$V = V_c + V_b$$

where:

(C1) $V_c \in C^2(\mathbb{R}^d)$ and there exists $\alpha > 0$ such that for all $x \in \mathbb{R}^d$

$$\text{Hess } V_c(x) \geq \alpha I_d;$$

(C2) $V_b$ is bounded on $\mathbb{R}^d$;

(C3) the first and second order derivatives of $V_c$ are bounded by polynomial functions of $|x|$. 

13
Proposition 2.5. Let $\rho$ stant $\rho$ immediately that the Gibbs measure $(LS1), (LS2)$ assumptions are verified. Precisely the following facts hold true.

On the other hand we have derivatives with respect to the variables $x_1$ (resp. $x_2$) only. Denote by Hess the $d_1 \times d_2$ matrix of second derivatives with respect to one variable in $x_1$ and one in $x_2$.

**Proposition 2.5.** Let $V \in C^2(\mathbb{R}^d)$, $V = V_c + V_b$ satisfying (C1)-(C3). Then assumptions (LS1), (LS2) are verified. Precisely the following facts hold true.

i) There exists $\alpha_1 \geq \alpha$ such that for all $(x_1, x_2) \in \mathbb{R}^d$

$$\text{Hess}_{11} V_c(x_1, x_2) \geq \alpha_1 I_{d_1}$$

and for every $x_2 \in \mathbb{R}^{d_2}$ the conditional measure $\rho^{(1)}_s(\cdot|x_2)$ satisfies a LSI with uniform constant $\beta_1 \alpha_1 = \beta_1 \alpha_1 e^{-\alpha \text{osc}(V_c)}$:

$$\int_{\mathbb{R}^{d_1}} \pi(x_1) \log \frac{\pi(x_1)}{\rho^{(1)}_s(x_1|x_2)} \, dx_1 \leq \frac{1}{2 \beta_1 \alpha_1} \int_{\mathbb{R}^{d_1}} \pi(x_1) \left| \nabla_1 \log \frac{\pi(x_1)}{\rho^{(1)}_s(x_1|x_2)} \right|^2 \, dx_1$$

for every $\pi \in C^1(\mathbb{R}^{d_1})$ such that $\pi > 0$, $\int_{\mathbb{R}^{d_1}} \pi(x_1) \, dx_1 = 1$, $\pi \log \frac{\pi}{\rho^{(1)}_s(x_2)} \in L^1(\mathbb{R}^{d_1})$.

ii) There exists $\alpha_2 \geq \alpha$ such that for all $(x_1, x_2) \in \mathbb{R}^d$

$$\left( \text{Hess}_{22} V_c - (\text{Hess}_{12} V_c) \nabla (\text{Hess}_{11} V_c)^{-1} \text{Hess}_{12} V_c \right)(x_1, x_2) \geq \alpha_2 I_{d_2}$$

and the marginal measure $\rho^{(2)}_s(\cdot|x_2)$ satisfies a LSI with constant $\beta_2 \alpha_2 = \beta_2 \alpha_2 e^{-\alpha \text{osc}(V_c)}$:

$$\int_{\mathbb{R}^{d_2}} \pi(x_2) \log \frac{\pi(x_2)}{\rho^{(2)}_s(x_2)} \, dx_2 \leq \frac{1}{2 \beta_2 \alpha_2} \int_{\mathbb{R}^{d_2}} \pi(x_2) \left| \nabla_2 \log \frac{\pi(x_2)}{\rho^{(2)}_s(x_2)} \right|^2 \, dx_2$$

for every $\pi \in C^1(\mathbb{R}^{d_2})$ such that $\pi > 0$, $\int_{\mathbb{R}^{d_2}} \pi(x_2) \, dx_2 = 1$, $\pi \log \frac{\pi}{\rho^{(2)}_s} \in L^1(\mathbb{R}^{d_2})$.

The proof of part i) is quite straightforward.

**Proof of Proposition 2.5, part i).** Given $x_2 \in \mathbb{R}^{d_2}$, $\rho^{(1)}_s(\cdot|x_2)$ is the Gibbs measure on $\mathbb{R}^{d_1}$ associated with potential $x_1 \mapsto V(x_1, x_2)$ and inverse temperature $\beta_1$. For all $x_1, \phi \in \mathbb{R}^{d_1}$ we have

$$\phi^T \text{Hess}_{11} V_c(x_1, x_2) \phi = \tilde{\phi}^T \text{Hess}_{11} V_c(x_1, x_2) \tilde{\phi} \geq \alpha \lvert \tilde{\phi} \rvert^2 = \alpha \lvert \phi \rvert^2,$$

setting $\tilde{\phi} \equiv \begin{pmatrix} \phi \\ 0 \end{pmatrix} \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and using hypothesis (C1). Thus inequality (61) holds true choosing $\alpha_1 \equiv \alpha$ (however in specific cases one can have a better choice for $\alpha_1$).

On the other hand we have

$$\sup_{x_1 \in \mathbb{R}^{d_1}} V_b(x_1, x_2) - \inf_{x_1 \in \mathbb{R}^{d_1}} V_b(x_1, x_2) \leq \sup_{x_1 \in \mathbb{R}^{d_1}} V_b - \inf_{x_1 \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} V_b = \text{osc}(V_b).$$

Therefore the claim follows by Theorems 2.3, 2.4.

\qed
The proof of part ii) relies on Brascamp-Lieb concentration inequality for log-concave measures and requires the following algebraic lemma.

**Lemma 2.6.** Let $A$ be a $d \times d$ invertible symmetric block matrix:

$$A \equiv \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix}. \tag{70}$$

The Schur complement of $A_{11}$ in $A$ is the $d_2 \times d_2$ invertible symmetric matrix defined by

$$S \equiv A_{22} - A_{12}^T A_{11}^{-1} A_{12}. \tag{71}$$

Suppose that there exists $\alpha > 0$ such that

$$A \geq \alpha I_d, \tag{72}$$

then

$$S \geq \alpha I_{d_2}. \tag{73}$$

**Proof.** Matrix $A$ is similar to a block-diagonal matrix, indeed it is easy to check that

$$A = P D P^T \tag{74}$$

with

$$P \equiv \begin{pmatrix} I_{d_1} & 0 \\ A_{12}^T A_{11}^{-1} & I_{d_2} \end{pmatrix}, \quad D \equiv \begin{pmatrix} A_{11} & 0 \\ 0 & S \end{pmatrix}. \tag{75}$$

Now let $\psi \in \mathbb{R}^{d_2}$. Set $\tilde{\psi} \equiv \begin{pmatrix} -A_{11}^{-1} A_{12} \psi \\ \psi \end{pmatrix} \in \mathbb{R}^d$ observing that $P^T \tilde{\psi} = \begin{pmatrix} 0 \\ \psi \end{pmatrix} \in \mathbb{R}^d$. Therefore:

$$\tilde{\psi}^T A \tilde{\psi} = \tilde{\psi}^T P D P^T \tilde{\psi} = \psi^T S \psi. \tag{76}$$

On the other hand using hypothesis (72)

$$\tilde{\psi}^T A \tilde{\psi} \geq \alpha |\tilde{\psi}|^2 \geq \alpha |\psi|^2. \tag{77}$$

By arbitrariness of $\psi \in \mathbb{R}^{d_2}$ this proves thesis (73). \qed

**Proof of Proposition 2.5 part ii).** $\rho^{(2)}_* \equiv \rho^{(2)}_\beta_2$ is the Gibbs measure on $\mathbb{R}^{d_2}$ with potential $F$ defined by (15) and inverse temperature $\beta_2$. $F$ rewrites as

$$F = F_c + F_b \tag{78}$$

where:

$$F_c(x_2) \equiv -\frac{1}{\beta_1} \log \int_{\mathbb{R}^{d_1}} e^{-\beta_1 V_c(x_1, x_2)} dx_1, \tag{79}$$

$$F_b(x_2) \equiv -\frac{1}{\beta_1} \log \langle e^{-\beta_1 V_b} \rangle_c, \tag{80}$$

and $\langle \cdot \rangle_c$ denotes the following conditional expectation given $x_2$:

$$\langle O \rangle_c \equiv \frac{\int_{\mathbb{R}^{d_1}} O(x_1, x_2) e^{-\beta_1 V_c(x_1, x_2)} dx_1}{\int_{\mathbb{R}^{d_1}} e^{-\beta_1 V_c(x_1, x_2)} dx_1} \tag{81}$$
for every observable $O$ such that $O(x_1, x_2) \in L^1(e^{-\beta V_c(x_1, x_2)})$. Clearly from (80) we have:

$$\sup_{\mathbb{R}^d} F_b - \inf_{\mathbb{R}^d} F_b \leq \sup_{\mathbb{R}^d} V_b - \inf_{\mathbb{R}^d} V_b = \text{osc}(V_b).$$

(82)

On the other hand, from (79) we have:

$$\text{Hess}_{22} F_c = \langle \text{Hess}_{22} V_c \rangle_c - \beta_1 \langle \nabla_2 V_c \nabla_2^T V_c \rangle_c + \beta_1 \langle \nabla_2 V_c \rangle_c \langle \nabla_2^T V_c \rangle_c,$$

(83)

indeed hypothesis (C3) guarantees that the derivatives w.r.t. $x_2$ can be taken inside the integral w.r.t. $x_1$. Then for every $\psi \in \mathbb{R}^{d_2}$,

$$\psi^T \text{Hess}_{22} F_c \psi = \langle \psi^T \text{Hess}_{22} V_c \psi \rangle_c - \beta_1 \left( \langle \psi^T \nabla_2 V_c \rangle_c^2 \right) + \beta_1 \langle \psi^T \nabla_2 V_c \rangle_c^2.$$

(84)

Brascamp-Lieb concentration inequality applies to the measure $\langle \cdot \rangle_c$ since the potential $x_1 \mapsto V_c(x_1, x_2)$ is convex by hypothesis (C1), hence:

$$\langle \psi^T \nabla_2 V_c \rangle_c^2 - \langle \psi^T \nabla_2 V_c \rangle_c^2 \leq \langle \nabla_1^T (\psi^T \nabla_2 V_c) \beta_1 \text{Hess}_{11} V_c \rangle_c \langle \nabla_1 (\psi^T \nabla_2 V_c) \rangle_c \langle \beta_1 \text{Hess}_{12} V_c \rangle_c \langle \beta_1 \text{Hess}_{12} V_c \rangle_c.$$

(85)

Plugging inequality (85) into (84) and using Lemma 2.6 together with hypothesis (C1), we obtain:

$$\psi^T \text{Hess}_{22} F_c \psi \geq \langle \psi^T \text{Hess}_{22} V_c - (\text{Hess}_{12} V_c)^T (\text{Hess}_{11} V_c)^{-1} \text{Hess}_{12} V_c \rangle_c \geq \alpha |\psi|^2,$$

(86)

for every $\psi \in \mathbb{R}^{d_2}$. Therefore inequality (86) is verified taking $\alpha_2 \equiv \alpha$ (in specific cases one can have a better choice for $\alpha_2$). The claim then follows by Theorems 2.3, 2.4.

**Remark 2.7.** Suppose $V = V_c + V_b$ under conditions (C1), (C2). If moreover $\nabla V_b$ is bounded then assumptions (A3), (A4) are automatically satisfied. Indeed the strongly convex potential $V_c$ has a (unique) minimum point $x_c \in \mathbb{R}^d$ and a second order expansion of $V_c$ at $x_c$ shows that for all $x \in \mathbb{R}^d$

$$V_c(x) \geq \frac{\alpha}{2} |x - x_c|^2,$$

(87)

while a first order expansion of $\nabla V_c$ at 0 shows that

$$x \cdot (\nabla V_c(x) - \nabla V_c(0)) \geq \alpha |x|^2.$$

(88)

## 3 Examples

### 3.1 Quadratic potential

Positive definite quadratic potentials (in any dimensions $d_1, d_2$) clearly fit the hypothesis of Theorem 1.7. On the other hand explicit computation can be performed.
for quadratic potentials. Here we consider the case of a positive definite quadratic potential in dimensions $d_1 = d_2 = 1$. The stationary non-equilibrium measure and associated current densities have been analyzed for all $\lambda$ in [30]. Here we provide the solution of the two-temperatures Fokker-Planck equation for all times $t$ and compute explicit expansions for large $\lambda$. A comparison between explicitly computed convergence rates and estimates given by Theorem 1.7 shows that the latter are sharp for quadratic potentials.

For $(x, y) \in \mathbb{R} \times \mathbb{R}$ we consider the two-dimensional quadratic potential

$$V(x, y) \equiv \frac{1}{2} (x, y) A \begin{pmatrix} x \\ y \end{pmatrix},$$

where $A$ is a positive-definite symmetric matrix

$$A \equiv \begin{pmatrix} a & c \\ c & b \end{pmatrix} \quad (a > 0, \ ab - c^2 > 0).$$

Thanks to the quadratic nature of the potential, one can check that the measure $\rho_*$ defined by (11) becomes a centered Gaussian distribution with covariance matrix

$$\Sigma \equiv \begin{pmatrix} \frac{1}{\beta_1} + \frac{1}{\beta_2} \frac{c^2}{\det A} a & -\frac{1}{\beta_2} \frac{c}{\det A} a \\ -\frac{1}{\beta_2} \frac{c}{\det A} & \frac{1}{\beta_2} \frac{a}{\det A} \end{pmatrix}.$$  

(91)

The SDE (1) with diffusion coefficients defined by (2), (6) is an Ornstein-Uhlenbeck process, and rewrites as

$$\begin{pmatrix} dx(t) \\ dy(t) \end{pmatrix} = -\Gamma \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} dt + \Delta \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix}$$

(92)

with

$$\Gamma \equiv a^{-1} A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \quad \Delta \equiv \sqrt{2a^{-1} \beta^{-1}} = \begin{pmatrix} \sqrt{\frac{2}{\beta_1}} & 0 \\ 0 & \sqrt{\frac{2}{\beta_2}} \end{pmatrix}$$

(93)

The fundamental solution $\rho_\ast(x, y)$ of the Fokker-Planck equation associated to (92), is a Gaussian distribution of mean $\mu(t)$ and covariance matrix $\Omega(t)$ defined by

$$\mu(t) \equiv e^{-t\Gamma} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad \Omega(t) \equiv \int_0^t e^{-(t-s)\Gamma} \Delta \Delta^T e^{-(t-s)\Gamma^T} ds.$$  

(94)

where $(x_0, y_0)$ represents the initial condition. We can compute (94) exactly and deduce the behavior for large $\lambda$. This is the object of the next proposition and proof.

Let us denote by $\gamma_{1,2}$ the eigenvalues of the matrix $\Gamma$:

$$\gamma_{1,2} \equiv \frac{(a + b\lambda^{-1}) \pm \sqrt{(a + b\lambda^{-1})^2 - 4\lambda^{-1} \det A}}{2}.$$  

(95)

We note that $\gamma_1, \gamma_2 > 0$ for all $\lambda > 0$ and compute their expansions as $\lambda \to \infty$:

$$\gamma_1 = a + O(\lambda^{-1}), \quad \gamma_2 = \frac{\det A}{a} \lambda^{-1} + O(\lambda^{-2}).$$

(96)
Proposition 3.1. Let $\Omega(t) = (\Omega_{ij}(t))_{i,j=1,2}$ and $\mu(t) = (\mu_i(t))_{i=1,2}$ be defined by (94). Let $\Sigma = (\Sigma_{ij})_{i,j=1,2}$ defined by (91). As $\lambda \to \infty$ we have the following expansions uniformly in $t > 0$:

$$
\begin{align*}
\Omega_{11}(t) &= \Sigma_{11} - \frac{1}{\beta_2} \frac{c^2}{\sigma^2} \det A e^{-2\gamma t} - \frac{1}{\beta_1 a} e^{-2\gamma t} + O(\lambda^{-1}) \\
\Omega_{22}(t) &= \Sigma_{22} (1 - e^{-2\gamma t}) + O(\lambda^{-1}) \\
\Omega_{12}(t) &= \Omega_{21}(t) = \Sigma_{12} (1 - e^{-2\gamma t}) + O(\lambda^{-1})
\end{align*}
$$

and:

$$
\begin{align*}
\mu_1(t) &= -y_0 e^{-\gamma t} + \left(y_0 + \frac{a}{c} \gamma_0\right) e^{-\gamma t} + O(\lambda^{-1}) \\
\mu_2(t) &= \frac{a}{c} y_0 e^{-\gamma t} + O(\lambda^{-1}) .
\end{align*}
$$

Proof. A direct computation shows that

$$
e^\tau \Delta = \frac{\sqrt{2}}{v_2 - v_1} \left( \frac{1}{\sqrt{\beta_1}} (e^{-\gamma_1 \tau} v_2 - e^{-\gamma_1 \tau} v_1) - \frac{1}{\sqrt{\beta_2}} (e^{-\gamma_2 \tau} v_2 - e^{-\gamma_2 \tau} v_1) \right)
$$

for every $\tau \in \mathbb{R}$, setting $v_{1,2} \equiv \frac{\gamma_{1,2} - a}{c}$. Then using definition (94) one obtains:

$$
\begin{align*}
\Omega_{11}(t) &= \frac{r_1}{2\gamma_2} \left(1 - e^{-2\gamma_2 t}\right) + \frac{r_2}{2\gamma_2} \left(1 - e^{-2\gamma_2 t}\right) - \frac{2r_{12}}{\gamma_1 + \gamma_2} (1 - e^{-(\gamma_1 + \gamma_2) t}) \\
\Omega_{12}(t) &= \frac{v_1 r_1}{2\gamma_2} \left(1 - e^{-2\gamma_1 t}\right) + \frac{v_2 r_2}{2\gamma_2} \left(1 - e^{-2\gamma_1 t}\right) - \frac{(v_1 + v_2) r_{12}}{\gamma_1 + \gamma_2} (1 - e^{-(\gamma_1 + \gamma_2) t}) \\
\Omega_{22}(t) &= \frac{v_2^2 r_1}{2\gamma_2} \left(1 - e^{-2\gamma_1 t}\right) + \frac{v_1^2 r_2}{2\gamma_2} \left(1 - e^{-2\gamma_1 t}\right) - \frac{2v_1 v_2 r_{12}}{\gamma_1 + \gamma_2} (1 - e^{-(\gamma_1 + \gamma_2) t})
\end{align*}
$$

where

$$
\begin{align*}
\gamma_1 &\equiv \frac{2}{(v_2 - v_1)^2} \left( \frac{v_2^2}{\beta_1} + \frac{1}{\lambda \beta_2} \right) \\
\gamma_2 &\equiv \frac{2}{(v_2 - v_1)^2} \left( \frac{v_1^2}{\beta_1} + \frac{1}{\lambda \beta_2} \right) \\
\gamma_{12} &\equiv \frac{2}{(v_2 - v_1)^2} \left( \frac{v_1 v_2}{\beta_1} + \frac{1}{\lambda \beta_2} \right)
\end{align*}
$$

All in all we have obtained for $\Omega(t)$ an expression of the form

$$
\Omega(t) = M_0 + M_1 e^{-2\gamma_1 t} + M_2 e^{-2\gamma_2 t} + M_3 e^{-(\gamma_1 + \gamma_2) t}
$$

where $(M_k)_{0 \leq k \leq 3}$ are suitable time-independent $2 \times 2$ matrices. Expansions of the matrices $M_k$ for large $\lambda$ can be obtained using expansions (95) of $\gamma_1, \gamma_2$. Identities (97) then follows using also the obvious bounds $0 < e^{-\gamma_1 \tau} < 1$ for any $t > 0$. Identities (98) for $\mu(t)$ are obtained in a similar way.
The previous proposition can be used to obtain the behaviour of the KL divergence between the solution \( \rho_{t,\lambda} \) and the measure \( \rho_* \) for large \( \lambda \). More precisely we are interested in the quantities \( D_1(t,\lambda), D_2(t,\lambda) \) defined by (37), (35) respectively.

**Proposition 3.2.**

\[
\begin{align*}
D_1(t,\lambda) &= \frac{1}{2} \left\{ -\log \left( 1 - e^{-2\gamma_1 t} \right) - e^{-2\gamma_1 t} + e^{-2\gamma_1 t} \beta_1 a \left( y_0 + \frac{a}{c} x_0 \right)^2 \right\} + O(\lambda^{-1}) \\
D_2(t,\lambda) &= \frac{1}{2} \left\{ -\log \left( 1 - e^{-2\gamma_2 t} \right) - e^{-2\gamma_2 t} + e^{-2\gamma_2 t} \beta_2 \frac{a^2 y_0^2}{c^2} \frac{\det A}{a} \right\} + O(\lambda^{-1})
\end{align*}
\]

(103) (104)

**Proof.** Let \( p_1, p_2 \) be two 1-dimensional Gaussian distributions with means \( u_1, u_2 \) and variances \( \sigma_1, \sigma_2 \) respectively. Their KL divergence reads

\[
D_{KL}(p_1||p_2) = \frac{1}{2} \left\{ \log \frac{\sigma_2}{\sigma_1} - 1 + \frac{\sigma_1}{\sigma_2} + (u_2 - u_1)\sigma_2^{-1}(u_2 - u_1) \right\}.
\]

(105)

Since \( \rho_* \), \( \rho_t \) are both 2-dimensional Gaussian distributions, their conditional and marginal measures are Gaussian too and are easy to compute from (91), (100) respectively:

- \( \rho_*^{(1)}(\cdot|x_2) \) has mean \( \Sigma_{11} \Sigma_{22}^{-1} x_2 \) and variance \( \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} = \frac{1}{\gamma_1 a} \)
- \( \rho_*^{(2)}(\cdot|x_2) \) has mean \( \mu_1(t) + \Omega_{11}(t)\Omega_{22}(t)^{-1}(x_2 - \mu_2(t)) \) and variance \( \Omega_{11}(t) - \Omega_{12}(t)\Omega_{22}(t)^{-1}\Omega_{21}(t) \)
- \( \rho_*^{(3)} \) has mean 0 and variance \( \Sigma_{22} = \frac{1}{\gamma_2 a} \frac{\det A}{\det A} \)
- \( \rho_*^{(4)} \) has mean \( \mu_2(t) \) and variance \( \Omega_{22}(t) \).

Then combining (91) and Proposition 3.1 with expression (105) one obtains the claim.

\[ \Box \]

Note that by Proposition 3.2 it follows that for every \( \tau_0 > 0 \) there exists finite positive constants \( C_1(\tau_0), C_2(\tau_0) \) such that

\[
\begin{align*}
D_1(t,\lambda) &\leq C_1(\tau_0) e^{-2\gamma_1 t} + O(\lambda^{-1}), \quad \gamma_1 t \geq \tau_0 \\
D_2(t,\lambda) &\leq C_2(\tau_0) e^{-2\gamma_2 t} + O(\lambda^{-1}), \quad \gamma_2 t \geq \tau_0
\end{align*}
\]

(106) (107)

The constants \( C_1, C_2(\tau_0) \) blow up as \( \tau_0 \to 0 \) indeed we have chosen the Dirac delta as initial distribution and its KL divergence from \( \rho_* \) is infinite. Now one can see that for a quadratic potential the estimates of Theorem 1.7 give the right decay rates. It suffices to recall that any Gaussian density with variance \( \sigma \) verifies a logarithmic Sobolev inequality with best constant \( C = \frac{1}{\sigma} \), hence \( \rho_*^{(1)}, \rho_*^{(2)} \) verify assumptions (LS1), (LS2) with best constants \( \beta_1 c_1 = \beta_1 a, \beta_2 c_2 = \beta_2 \frac{\det A}{a} \) respectively (compare with the expansion (36) of \( \gamma_1, \gamma_2 \)).
3.2 Spin glass (dynamical approach to Guerra’s scheme)

We discuss a multibath spin-glass model with fast soft spins and slow external magnetic fields. This kind of model has been discussed recently in [17, 18] where it is argued on heuristic grounds that Guerra’s hierarchical measure for Replica Symmetry Breaking (RSB) appears as the stationary measure of the multibath Langevin dynamics. We show that our analysis is applicable for a finite system. The assumptions required by Theorem 1.7 - in particular the Bakry-Émery-Holley-Stroock criterion analysed in Section 2 - are checked to hold with probability (with respect to quenched couplings) exponentially close to one in a suitable region of parameters. Unfortunately, with this criterion we do not expect better than exponentially small convergence rate in the system size or dimension. For specific phases (e.g., high-temperatures) better phase-dependent criteria for the validity of log-Sobolev inequalities would yield more physical results (e.g., at high temperatures one could presumably use decay-of-correlations criteria). This however goes beyond the scope of the present work. There exists a vast physics literature on the large size limit for mean-field spin glass dynamics, we refer the interested reader to [16] for a survey and we mention the recent works [41, 42, 43] for rigorous results on the subject.

Let
\[ s = (s_i)_{i=1...N} \in \mathbb{R}^N \]

be fast soft-spin variables, \( y = (y_i)_{i=1...N} \in \mathbb{R}^N \) slow external fields, and \( J = (J_{ij})_{i,j=1...N} \) quenched couplings chosen as independent standard Gaussian random variables. The system is described by the following (random) Hamiltonian:

\[
V(s, y) = -\sqrt{\Delta} \sum_{i,j=1}^{N} J_{ij} s_i s_j - \sqrt{\Delta_0} \sum_{i=1}^{N} y_i s_i + \frac{A}{2} \sum_{i=1}^{N} (s_i^2 - 1)^2 + \frac{B}{2} \sum_{i=1}^{N} y_i^2
\]  

(108)

where \( \Delta_0, \Delta > 0 \) tune the one-body, two-bodies interactions respectively, \( A > 0 \) pushes the soft-spins towards hard-spins \( \pm 1 \), and \( B > 0 \). Clearly the potential \( V \) satisfies assumptions (A1), (A2) being a polynomial in \( s, y \). In order to verify logarithmic Sobolev inequalities (LS1), (LS2) - and also assumptions (A3), (A4) - we can write \( V = V_c + V_b \) and check hypothesis (C1), (C2) (see Section 2). For this purpose let us consider a convex function \( \eta \in C^2(\mathbb{R}) \) such that

\[
\eta(x) = \begin{cases} 
  x^2 & \text{for } |x| < x_0 - \delta \\
  (x^2 - 1)^2 & \text{for } |x| > x_0 + \delta 
\end{cases}
\]  

(109)

where \( x_0 \equiv \frac{1+\sqrt{5}}{2} \) is the largest point where the above quadratic and quartic curves intersect and \( \delta \equiv \frac{1}{10} \). It is possible to determine \( \eta \) on the interval \([x_0 - \delta, x_0 + \delta]\) by an interpolating polynomial of degree 5 and it turns out

\[
\eta'(x) \geq 2x, \quad \eta''(x) \geq 2 \quad \forall x \in \mathbb{R}.
\]  

(110)

We set

\[
V_b(s) \equiv \frac{A}{2} \sum_i (s_i^2 - 1)^2 - \frac{A}{2} \sum_i \eta(s_i),
\]  

(111)

\[
V_c(s, y) \equiv V(s, y) - V_b(s).
\]  

(112)
\(V_b \in C^2(\mathbb{R}^d)\) is bounded since it has compact support, and
\[
\text{osc}(V_b) = \frac{A}{2} N. \tag{113}
\]

Now let us compute the Hessian quadratic form of \(V_c\) for \((\phi, \psi) = (\phi_1, \ldots, \phi_N, \psi_1, \ldots, \psi_N) \in \mathbb{R}^{2N} :\)
\[
(\phi, \psi)^T \text{ Hess} \ V_c(s, y) (\phi, \psi) = \]
\[
\frac{A}{2} \sum_i \eta''(s_i) \phi_i^2 - \frac{\sqrt{\Delta}}{\sqrt{N}} \sum_{i,j} J_{ij} \phi_i \phi_j - 2 \sqrt{\Delta_0} \sum_i \phi_i \psi_i + B \sum_i \psi_i^2. \tag{114}
\]

Observe that
\[
\sum_{i,j} J_{ij} \phi_i \phi_j \leq \sigma_{\text{max}}(J) \sum_i \phi_i^2 \tag{115}
\]
where \(\sigma_{\text{max}}(J)\) denotes the largest singular value of the random matrix \(J\), and by general concentration inequalities (see, e.g., Theorem 3.1.1 in [44]) we have:
\[
P\left(\frac{1}{\sqrt{N}} \sigma_{\text{max}}(J) \leq \sqrt{2 + \tau} \right) \geq 1 - e^{-\frac{1}{2} \tau^2 N} \equiv p_{\tau,N} \tag{116}
\]
for every \(\tau > 0\). Therefore, with probability greater than \(p_{\tau,N}\) the following lower bound is valid for all \((\phi, \psi) \in \mathbb{R}^{2N} :\)
\[
(\phi, \psi)^T \text{ Hess} \ V_c(s, y) (\phi, \psi) \geq \]
\[
\frac{A}{2} \sum_i \eta''(s_i) \phi_i^2 - K_\tau \sum_i \phi_i^2 - 2 \sqrt{\Delta_0} \sum_i \phi_i \psi_i + B \sum_i \psi_i^2 \geq \]
\[
\alpha |(\phi, \psi)|^2 \tag{117}
\]
where \(K_\tau \equiv (\sqrt{2 + \tau})\sqrt{\Delta}\) and \(\alpha > 0\) is the smallest eigenvalue\(^2\) of the \(2 \times 2\) positive-definite matrix \(\begin{pmatrix} A - K_\tau & -\sqrt{\Delta_0} \\ -\sqrt{\Delta_0} & B \end{pmatrix}\), provided
\[
A > K_\tau + \frac{\Delta_0}{B}. \tag{118}
\]

This proves that hypothesis (C1) holds true for the random potential \(V_c\) with probability larger than \(p_{\tau,N}\), provided condition [118] is verified. To conclude we claim that \(V\) satisfies also assumption (A5) with probability larger than \(p_{\tau,N}\) under the same condition:
\[
s \cdot \nabla V_c(s, y) = -\frac{\sqrt{\Delta}}{\sqrt{N}} \sum_{i,j} J_{ij} s_i s_j - \sqrt{\Delta_0} \sum_i y_i s_i + \frac{A}{2} \sum_i \eta'(s_i) s_i \geq \]
\[
-K_\tau \sum_i s_i^2 - \sqrt{\Delta_0} \sum_i y_i s_i + \frac{A}{2} \sum_i \eta'(s_i) s_i \tag{119}
\]
\[
\geq \left( A - K_\tau - \frac{\Delta_0}{B} \right) |s|^2 - \frac{B}{4} |y|^2.
\]
Since \(s \cdot \nabla V_c(s)\) is bounded, inequality [119] proves the claim.
\(^2\alpha = A - K_\tau + B - \sqrt{(A - K_\tau + B)^2 - 4((A - K_\tau)B - \Delta_0)}\)
3.3 High-dimensional inference (rank-one matrix estimation)

An important paradigm in modern high-dimensional inference is the problem of low rank-matrix estimation [31]. In its simplest rank-one incarnation, the statistician is given a noisy version of a large rank-one $N_1 \times N_2$ matrix with elements $u_i^* v_j^*$ determined by latent (hidden) vectors $u^* \in \mathbb{R}^{N_1}$, $v^* \in \mathbb{R}^{N_2}$ with independent sub-Gaussian random components. Given the data matrix

$$J_{ij} = \sqrt{\Delta} \frac{u_i^* v_j^*}{N} + Z_{ij},$$

(120)

where $Z_{ij}$ are independent standard Gaussian variables, $N = N_1 + N_2$, and $\Delta > 0$ is the signal-to-noise ratio, the task of the statistician is to estimate the latent matrix $u^* v^*$. When prior distributions of latent vectors are known, estimates given by averages under the Bayes posterior distribution are optimal estimators (in the sense that they minimize the mean-square-error). As it turns out, the Bayes posterior in nothing else than the Gibbs distribution of a spin-glass with a replica symmetric solution (we refer to [45, 46, 47] for the rigorous analysis of such models). However, when priors are unknown to the statistician, other estimators may be better and it is of interest to depart from the standard Bayes posterior or Gibbs distributions. The Langevin multibath dynamics may serve as a useful estimation algorithm based on a non-Bayesian (or non-Gibbsian) distribution. Whether the two-temperatures Langevin dynamics is a useful algorithmic approach to such problems is still an interesting open question in itself. Without entering into more details here, this motivates the study of the multibath dynamics (11) for the following spin-glass potential

$$V(u, v) = \frac{1}{2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \left( J_{ij} - \sqrt{\frac{\Delta}{N}} u_i v_j \right)^2 +$$

$$+ \sum_{i=1}^{N_1} \left( \frac{a}{2} u_i^4 + \frac{A}{12} u_i^4 \right) + \sum_{j=1}^{N_2} \left( \frac{b}{2} v_j^4 + \frac{B}{12} v_j^4 \right),$$

(121)

where the random data matrix $J_{ij}$ is generated according to (120) and $a, b, A, B > 0$. In this setting the data matrix is entirely quenched and degrees of freedom $u \in \mathbb{R}^{N_1}$, $v \in \mathbb{R}^{N_2}$ undergo Langevin dynamics with different temperatures and timescales. We show that the assumptions required by Theorem 1.7 - in particular the Bakry-Emery criterion analysed in Section 2 - hold in a low signal-to-noise ratio regime with probability exponentially close to one (with respect to the quenched data matrix). In this regime the convergence rate is constant with respect to the size of the system. It is an open problem to use more detailed criteria in order to check the log-Sobolev inequalities for high signal-to-noise ratios.

The potential $V$ satisfies assumptions (A1), (A2) since it is a polynomial in the
variables $u,v$. For $(\phi,\psi) = (\phi_1,\ldots,\phi_{N_1},\psi_1,\ldots,\psi_{N_2}) \in \mathbb{R}^N$ let us compute:

$$(\phi,\psi)^T \text{Hess } V(u,v) (\phi,\psi) = \frac{\Delta}{N} \left( |\phi|^{2} |v|^{2} + |u|^{2} |\psi|^{2} + 4 (u \cdot \phi) (v \cdot \psi) \right) +$$

$$-2 \sqrt{\frac{\Delta}{N}} \sum_{i,j} J_{ij} \phi_i \psi_j +$$

$$+ \sum_{i} (a + A u_i^2) \phi_i^2 + \sum_{j} (b + B v_j^2) \psi_j^2. \quad \text{(122)}$$

It is convenient to rewrite the random interaction term using expression (120):

$$\sqrt{\frac{\Delta}{N}} \sum_{i,j} J_{ij} \phi_i \psi_j = \frac{\Delta}{N} \sum_{i,j} u_i^* \phi_i v_j^* \psi_j + \sqrt{\frac{\Delta}{N}} \sum_{i,j} Z_{ij} \phi_i \psi_j \quad \text{(123)}$$

$$\leq \frac{\Delta}{N} |u^*| |v^*| |\phi| |\psi| + \sqrt{\frac{\Delta}{N}} \sigma_{\max}(Z) |\phi| |\psi|,$$

where $\sigma_{\max}(Z)$ denotes the largest singular value of the $N_1 \times N_2$ random matrix $Z$. By general concentration inequalities (see, e.g., Corollary 3.35 in [44] and Theorem 3.1.1 in [18]), for any $\tau_0, \tau_1, \tau_2 > 0$ we have:

$$\mathbb{P} \left( \frac{1}{\sqrt{N_1}} |u^*| \leq \sqrt{\mathbb{E}(u_1^*)^2 + \tau_1} \right) \geq 1 - e^{-k_1 \tau_1^2 N}, \quad \text{(124)}$$

$$\mathbb{P} \left( \frac{1}{\sqrt{N_2}} |v^*| \leq \sqrt{\mathbb{E}(v_1^*)^2 + \tau_2} \right) \geq 1 - e^{-k_2 \tau_2^2 N}, \quad \text{(125)}$$

$$\mathbb{P} \left( \frac{1}{\sqrt{N}} \sigma_{\max}(Z) \leq \sqrt{1 - \gamma} \right) \geq 1 - e^{-\frac{1}{2} \gamma} N, \quad \text{(126)}$$

where we set $\gamma \equiv \frac{\tau_1}{\sqrt{N}}$ and the constants $k_1, k_2 > 0$ depend only on the (sub-Gaussian) distributions of $u_1^*, v_1^*$. Therefore with probability larger than $p_{r,N} \equiv 1 - e^{-\frac{1}{2} \gamma} N - e^{-k_1 \tau_1^2 N} - e^{-k_2 \tau_2^2 N}$ the following lower bound holds true for every $(\phi,\psi) \in \mathbb{R}^N$:

$$(\phi,\psi)^T \text{Hess } V(u,v) (\phi,\psi) \geq \frac{\Delta}{N} \left( |\phi|^{2} |v|^{2} + |u|^{2} |\psi|^{2} + 4 (u \cdot \phi) (v \cdot \psi) \right) +$$

$$- 2 K_{r} |\phi| |\psi| +$$

$$+ \sum_{i} (a + A u_i^2) \phi_i^2 + \sum_{j} (b + B v_j^2) \psi_j^2, \quad \text{(127)}$$

with

$$K_{r} \equiv \Delta \gamma (1-\gamma) \left( \sqrt{\mathbb{E}(u_1^*)^2 + \tau_1} \right) \left( \sqrt{\mathbb{E}(v_1^*)^2 + \tau_2} \right) + \sqrt{\Delta} \left( \sqrt{1 - \gamma} \right). \quad \text{(128)}$$

The r.h.s. of (127) may split in three terms. The first one is:

$$\frac{\Delta}{N} \left( |\phi|^{2} |v|^{2} + |u|^{2} |\psi|^{2} + 2 (u \cdot \phi) (v \cdot \psi) \right) \geq \frac{\Delta}{N} \left( |\phi| |v| - |u| |\psi| \right)^2 \geq 0. \quad \text{(129)}$$
The second one is:

$$\frac{\Delta}{N} 2(u \cdot \phi)(v \cdot \psi) + \sum_i A u_i^2 \phi_i^2 + \sum_j B v_j^2 \psi_j^2 =$$

$$\frac{1}{N} \sum_{i,j} \left( 2\Delta u_i \phi_i v_j \psi_j + \frac{A}{1-\gamma} u_i^2 \phi_i^2 + \frac{B}{\gamma} v_j^2 \psi_j^2 \right) \geq 0$$

(130)

provided

$$AB \geq \gamma (1-\gamma) \Delta^2.$$  

(131)

The last term is:

$$-2K_\tau |\phi| |\psi| + a |\phi|^2 + b |\psi|^2 \geq \alpha (|\phi|^2 + |\psi|^2)$$

(132)

where \(\alpha > 0\) is the smallest eigenvalue of the positive-definite matrix \(\begin{pmatrix} a & -K_\tau \\ -K_\tau & b \end{pmatrix}\).

Plugging inequalities (129), (130), (132) into (127) we conclude that with probability larger than \(p_{\tau,N}\)

$$\text{Hess} V(u,v) \geq \alpha I$$

(134)

provided conditions (131), (133) are satisfied. Assumptions (LS1), (LS2) and (A3), (A4) follow. It remains to prove that assumption (A5) is also verified. With probability larger than \(p_{\tau,N}\) we have for all \((u,v) \in \mathbb{R}^N\)

$$u \cdot \nabla u V(u,v) = \frac{\Delta}{N} |u|^2 |v|^2 - \sqrt{\frac{\Delta}{N}} \sum_{i,j} J_{ij} u_i v_j + a |u|^2 + \frac{A}{3} \sum_i u_i^4$$

$$\geq -K_\tau |u| |v| + a |u|^2$$

$$\geq -\frac{b}{4} |v|^2,$$

(135)

provided condition (133) holds true.

4 Proof of Theorem 1.7 and Further Results

4.1 Integrability of \(D'V\), \(D'F\), \(\log \rho\), \(\nabla \log \rho\)

The results stated in this Subsection will be extensively used in the following and are interesting per se. Any product of derivatives of the potential \(V\) has uniformly bounded expectations for all \(t > 0\), \(\lambda \geq 1\). These expectations may be either taken with respect to the dynamical measure \(\rho_{t,\lambda}\) or to the measure \(\rho^{(1)}_{t,\lambda}\) where the fast variables are at equilibrium. As a consequence, expectations of derivatives of the effective potential \(F = -\beta_1^{-1} \log \int e^{-\beta_1 V} dx_1\) with respect to the measure \(\rho_{t,\lambda}^{(2)}\) are also uniformly bounded for all \(t > 0\), \(\lambda \geq 1\). Finally, by general results \(\log \rho_{t,\lambda}\) and \(\nabla \log \rho_{t,\lambda}\) turn out to be square-integrable with respect to \(\rho_{t,\lambda}\).
Proof. Suppose that $V$ verifies assumptions (A1)-(A5) and $\rho_t$ verifies assumption (B3). Then for every finite collection of multi-indices $\nu_1,\ldots,\nu_n \in \mathbb{N}^d$ and exponents $s_1,\ldots,s_n \in [0,\infty)$

$$\sup_{t>0,\lambda\geq 1} \int_{\mathbb{R}^d} \prod_{i=1}^n |D^{\nu_i}V(x)|^{s_i} \rho_{t,\lambda}(x) \, dx < \infty . \quad (136)$$

**Theorem 4.2.** Suppose that $V$ verifies assumptions (A1)-(A4) and $\rho_t$ verifies assumption (B3). Then for every finite collection of multi-indices $\nu_1,\ldots,\nu_n \in \mathbb{N}^d$ and exponents $s_1,\ldots,s_n \in [0,\infty)$

$$\sup_{t>0,\lambda\geq 1} \int_{\mathbb{R}^d} \prod_{i=1}^n |D^{\nu_i}V(x_1,x_2)|^{s_i} \rho^{(1)}_{\lambda}(x_1|x_2) \rho^{(2)}_{t,\lambda}(x_2) \, dx_1 \, dx_2 < \infty . \quad (137)$$

As a consequence, for every $\nu \in \mathbb{N}^{d_2}$, $s \in [0,\infty)$

$$\sup_{t>0,\lambda\geq 1} \int_{\mathbb{R}^{d_2}} |D^\nu F(x_2)|^{s} \rho^{(2)}_{t,\lambda}(x_2) \, dx_2 < \infty . \quad (138)$$

Proofs of Theorems 4.1, 4.2 are postponed to Subsections 4.2, 5.3 in the Appendix. It will be shown that expectations of any polynomial are uniformly bounded in $t > 0$, $\lambda \geq 1$ and more explicit bounds will be provided.

**Corollary 4.3.** Suppose that $V$ verifies assumptions (A1)-(A5) and $\rho_t$ verifies assumption (B3). Then:

$$\sup_{t>0,\lambda\geq 1} \int_{\mathbb{R}^d} |\log \rho^{(1)}_{\nu}(x_1|x_2)|^2 \rho_{t,\lambda}(x) \, dx < \infty$$

$$\sup_{t>0,\lambda\geq 1} \int_{\mathbb{R}^d_2} |\log \rho^{(2)}_{\nu}(x_2)|^2 \rho^{(2)}_{t,\lambda}(x_2) \, dx_2 < \infty$$

$$\sup_{t>0,\lambda\geq 1} \int_{\mathbb{R}^d} |\nabla \log \rho^{(1)}_{\nu}(x_1|x_2)|^2 \rho_{t,\lambda}(x) \, dx < \infty$$

$$\sup_{t>0,\lambda\geq 1} \int_{\mathbb{R}^d} |\nabla F(x_2)| \rho^{(2)}_{t,\lambda}(x_2) \, dx_2 < \infty . \quad (139)$$

Proof. It follows by Theorems 4.1, 4.2 since:

$$\log \rho^{(1)}_{\nu}(x_1|x_2) = -\beta_1 \left( V(x_1,x_2) - F(x_2) \right) ,$$

$$\log \rho^{(2)}_{\nu}(x_2) = -\beta_2 F(x_2) - \log Z_2 .$$

□

**Proposition 4.4 (Log integrability).** Suppose that $V$ verifies assumptions (A1)-(A4) and $\rho_t$ verifies assumptions (B1), (B3). Then for every $T > 0$ we have $\|\rho\|_{L^\infty(\mathbb{R}^d \times [0,T])} < \infty$ and

$$\sup_{t \in (0,T]} \int_{\mathbb{R}^d} |\log \rho_t(x)|^2 \rho_t(x) \, dx < \infty . \quad (140)$$
Theorem 4.5 (Logarithmic gradient integrability). Suppose that $V$ verifies assumptions (A1)-(A4) and $\rho_I$ verifies assumption (B1), (B3). Then $\rho_t \in W^{1,1}(\mathbb{R}^d)$ for a.e. $t > 0$ and
\[
\int_0^T \int_{\mathbb{R}^d} |\nabla \log \rho_t(x)|^2 \rho_t(x) \, dx \, dt < \infty \tag{141}
\]
for all $T \in (0, \infty)$. As a consequence there exists a subset $N = N(\lambda, \beta_1, \beta_2, \rho_I, V)$ of $(0, \infty)$ of zero Lebesgue measure such that for all $t \in (0, \infty) \setminus N$
\[
\int_{\mathbb{R}^d} |\nabla \log \rho_t(x)|^2 \rho_t(x) \, dx < \infty . \tag{142}
\]

Proofs of Proposition 4.4 and Theorem 4.5 are postponed to Subsection 5.2 of the Appendix. They will mainly refer to general results contained in Chapter 7 of [32].

4.2 Fokker-Planck equation for the marginal $\rho^{(2)}_{t,\lambda}$

In this Subsection the marginal probability density $\rho^{(2)}_{t}(x_2) = \int_{\mathbb{R}^d} \rho_t(x_1, x_2) \, dx_1$ is proved to satisfy in turn a Fokker-Planck equation with time-dependent drift:
\[
\partial_t \rho^{(2)} = \frac{1}{\lambda} \nabla_2 \cdot \left( \frac{1}{\beta_2} \nabla_2 \rho^{(2)} + \rho^{(2)} \langle \nabla_2 V \rangle_t \right) , \tag{143}
\]
where the bracket $\langle \cdot \rangle_t$ has been defined by (36) as the conditional expectation with respect to $\rho^{(1)}_{t}(x_1|x_2)$. Regularity and integrability results for $\rho^{(2)}$ and $\rho^{(1)}$ are then stated. Let us rewrite the Fokker-Planck operator (24) as
\[
L = L_1 + \frac{1}{\lambda} L_2 \tag{144}
\]
where we set:
\[
L_1 \varphi \equiv \frac{1}{\beta_1} \nabla_1^2 \varphi - \nabla_1 V \cdot \nabla_1 \varphi , \tag{145}
\]
\[
L_2 \varphi \equiv \frac{1}{\beta_2} \nabla_2^2 \varphi - \nabla_2 V \cdot \nabla_2 \varphi . \tag{146}
\]

We introduce a time-dependent Fokker-Planck operator on $\mathbb{R}^{d_2}$:
\[
\langle L_2 \rangle \psi \equiv \frac{1}{\beta_2} \nabla_2^2 \psi - \langle \nabla_2 V \rangle_t \cdot \nabla_2 \psi . \tag{147}
\]

Theorem 4.6. In the hypothesis of Theorem 1.2, the marginal measure $\mu^{(2)}_t \equiv \rho^{(2)}_{t}(dx_2)$ defined by (31) verifies
\[
\int_{\mathbb{R}^{d_2}} \psi(x_2) \mu^{(2)}_t(dx_2) - \int_{\mathbb{R}^{d_2}} \psi(x_2) \mu^{(2)}_0(dx_2) = \frac{1}{\lambda} \int_0^t \int_{\mathbb{R}^{d_2}} \langle L_2 \rangle_s \psi(x_2) \mu^{(2)}_s(dx_2) \, ds \tag{148}
\]
for every $\psi \in C^2_c(\mathbb{R}^{d_2})$, every $t > 0$, setting $\mu^{(2)}_t \equiv \rho^{(2)}_{t}(dx_2)$.

Remark 4.7. We can view equation (143) as a linear Fokker-Planck equation for $\rho^{(2)}$, with $\rho^{(1)}$ given. Equation (148) is a weak formulation of (143).
Proof of Theorem 4.6. Let ψ ∈ C^2(ℝ^d). Consider a sequence ζ_N ∈ C^2(ℝ^d) such that ζ_N(x_1) = 1 for |x_1| ≤ N and ζ_N, ∇_1ζ_N, Hess_1ζ_N are uniformly bounded for all N ∈ ℕ. Set

\[ \varphi_N(x_1, x_2) = ζ_N(x_1) ψ(x_2). \]  

(149)

Now ϕ_N ∈ C^2(ℝ^d), hence by equation (29)

\[ \int_{ℝ^d} ϕ_N(x) \rho_t(x) dx - \int_{ℝ^d} ϕ_N(x) \rho_0(x) dx = \int_0^t \int_{ℝ^d} L ϕ_N(x) \rho_s(x) dx ds. \]  

(150)

For every x = (x_1, x_2) ∈ ℝ^d we have ϕ_N(x) → ψ(x_2) as N → ∞ and

\[ L ϕ_N(x) = (L_1ζ_N(x)) ψ(x_2) + \frac{1}{\lambda} ζ_N(x_1) (L_2 ψ)(x) \to \frac{1}{\lambda} (L_2 ψ)(x). \]  

(151)

Since ∇_1V ∈ L^1(ℝ^d × (0, t), ρ_s(x) dx ds) by Theorem 4.2 (actually Corollary 5.9 suffices), there is dominated convergence. Therefore letting N → ∞ identity (150) gives

\[ \int_{ℝ^d} ψ(x_2) \rho_t(x) dx - \int_{ℝ^d} ψ(x_2) \rho_0(x) dx = \frac{1}{\lambda} \int_0^t \int_{ℝ^d} (L_2 ψ)(x) \rho_s(x) dx ds. \]  

(152)

Now integrate with respect to x_1 first, as ψ depends on x_2 only. Since ∇_2V ∈ L^1(ℝ^d × (0, t), ρ_s(x) dx ds) by Theorem 4.2 (Proposition 5.7, Fubini theorem applies. Therefore equation (152) rewrites as (148).

Theorem 4.8 (Marginal regularity). In the hypothesis of Theorem 4.2 we have:

i. \( \rho_t^{(2)} \in C^∞(ℝ^2 × (0, ∞)) \), hence it is a strong solution of Fokker-Planck equation (143);

ii. \( \| \rho_t^{(2)} - ρ_1^{(2)} \|_{L^1(ℝ^2)} \to 0 \) as \( t \to 0 \).

Theorem 4.8 has been obtained modifying the bootstrap argument in 35 and relies on the integrability results of Subsection 3.14. Its proof is postponed to Subsection 5.5 in the Appendix.

Proposition 4.9 (Marginal log integrability). Suppose that V verifies assumptions (A1)-(A4) and \( ρ_t \) verifies assumptions (B2), (B3). Then for every \( T > 0 \) we have \( \| ρ_t^{(2)} \|_{L^∞(ℝ^2 × (0, T))} < ∞ \) and

\[ \sup_{t ∈ (0, T)} \int_{ℝ^2} \| log ρ_t^{(2)}(x_2) \|^2 ρ_t^{(2)}(x_2) dx_2 < ∞. \]  

(153)

Theorem 4.10 (Marginal logarithmic gradient integrability). Suppose that V verifies assumptions (A1)-(A4) and \( ρ_t \) verifies assumptions (B2), (B3). Then \( ρ_t^{(3)} ∈ W^{1,1}(ℝ^2) \) for a.e. \( t > 0 \) and

\[ \int_0^T \int_{ℝ^2} \| ∇_2 log ρ_t^{(2)}(x_2) \|^2 ρ_t^{(2)}(x_2) dx_2 dt < ∞ \]  

(154)

for all \( T ∈ (0, ∞) \). As a consequence there exists a subset \( N = N(λ, β_1, β_2, ρ_1, V) \) of \( (0, ∞) \) of zero Lebesgue measure such that for all \( t ∈ (0, ∞) \setminus N\n
\[ \int_{ℝ^2} \| ∇_2 log ρ_t^{(2)}(x_2) \|^2 ρ_t^{(2)}(x_2) dx_2 < ∞. \]  

(155)
Proofs of Proposition 4.9 and Theorem 4.10 are postponed to Subsection 5.2 in the Appendix. Since \( \rho^{(1)} \) is itself solution of a suitable Fokker-Planck equation, general results of \([32]\) applies.

**Corollary 4.11** (Conditional regularity). In the hypothesis of Theorem 4.7, the conditional density \( \rho^{(1)} \) belongs to \( C^\infty(\mathbb{R}^d_1 \times \mathbb{R}^d_2 \times (0, \infty)) \).

**Proof.** It follows immediately combining Propositions 4.4, 4.9.

**Corollary 4.12** (Conditional log integrability). Suppose that \( V \) verifies assumptions (A1)-(A4) and \( \rho_t \) verifies assumption (B1)-(B3). Then for every \( T > 0 \)

\[
\sup_{t \in (0,T)} \int_{\mathbb{R}^d} | \log \rho_t^{(1)}(x_1|x_2) |^2 \rho_t(x_1,x_2) \, dx_1 \, dx_2 < \infty .
\]  

(156)

**Proof.** It follows immediately combining Propositions 4.14, 4.10.

**Corollary 4.13** (Conditional logarithmic gradient integrability). Suppose that \( V \) verifies assumptions (A1)-(A4) and \( \rho_t \) verifies assumption (B1)-(B3). Then \( \rho_t^{(1)} \) belongs to \( W^{1,1}(\mathbb{R}^d, \rho_t^{(2)} \, dx) \) for a.e. \( t > 0 \) and

\[
\int_0^T \int_{\mathbb{R}^d} | \nabla \log \rho_t^{(1)}(x_1|x_2) |^2 \rho_t(x_1,x_2) \, dx_1 \, dx_2 \, dt < \infty .
\]  

(157)

As a consequence \( T \in (0, \infty) \) and there exists a subset \( N = N(\lambda, \beta_1, \beta_2, \rho_1, V) \) of \( (0, \infty) \) of zero Lebesgue measure such that for all \( t \in (0, \infty) \) \( N \)

\[
\int_{\mathbb{R}^d} | \nabla \log \rho_t^{(1)}(x_1|x_2) |^2 \rho_t(x_1,x_2) \, dx_1 \, dx_2 < \infty .
\]  

(158)

**Proof.** It follows immediately combining Theorems 4.14, 4.10.

### 4.3 Joint and marginal entropy dissipation

**Proposition 4.14.** Suppose that \( V \) verifies assumptions (A1)-(A4) and \( \rho_t \) verifies assumptions (B1), (B3). Then for every \( t, \lambda > 0 \)

\[
D(t, \lambda) = D_1 = - \int_{\mathbb{R}^d} \nabla^{-1}(\beta^{-1} \nabla \log \rho_{s,\lambda} + \nabla V) \cdot \nabla \log \frac{\rho_{s,\lambda}(x)}{\rho_s}(x) \, dx \, ds .
\]  

(159)

As a consequence, the function \( t \mapsto D(t, \lambda) \) extended with \( D_1 \) at \( t = 0 \) is absolutely continuous on compact subsets of \( [0, \infty) \) and there exists a subset of zero Lebesgue measure \( N_\lambda = N_\lambda(\beta_1, \beta_2, \rho_1, V) \) of \( (0, \infty) \) such that for all \( t \in (0, \infty) \) \( N_\lambda \)

\[
\frac{d}{dt} D(t, \lambda) = \int_{\mathbb{R}^d} \nabla^{-1}(\beta^{-1} \nabla \log \rho_{t,\lambda} + \nabla V) \cdot \nabla \log \frac{\rho_{t,\lambda}(x)}{\rho_s}(x) \, dx .
\]  

(160)

Identity (159) can be formally obtained by differentiating the KL divergence \( (39) \) with respect to time \( t \), taking the derivative inside the integral, using the Fokker-Planck equation \( \partial_t \rho = L^* \rho \), and integrating by parts. A rigorous proof of Proposition 4.14 relies on the integrability results of Subsection 4.3 and on the regularity Theorem 4.5 and is postponed to the Appendix.
Proposition 4.15. Suppose that $V$ verifies assumptions (A1)-(A4) and $p_1$ verifies assumptions (B1)-(B3). Then for every $t, \lambda > 0$

$$D_2(t, \lambda) - D_{21} =$$

$$= -\frac{1}{\lambda} \int_0^t \int_{\mathbb{R}^d} \left( \frac{1}{\beta_2} \nabla_2 \log \frac{\rho_{t,\lambda}^{(2)}}{\rho_2^{(2)}}(x_2) \cdot \nabla_2 \log \frac{\rho_{t,\lambda}^{(2)}}{\rho_2^{(2)}}(x_2) \right) \, dx_2 \, ds .$$

(161)

As a consequence, the function $t \mapsto D_2(t, \lambda)$ extended with $D_{21}$ at $t = 0$ is absolutely continuous on compact subsets of $[0, \infty)$ and there exists a subset of zero Lebesgue measure $N_\lambda = N_\lambda(\beta_1, \beta_2, \rho_1, V)$ of $(0, \infty)$ such that for all $t \in (0, \infty) \setminus N_\lambda$

$$\frac{d}{dt} D_2(t, \lambda) = -\frac{1}{\lambda} \int_{\mathbb{R}^d} \left( \frac{1}{\beta_2} \nabla_2 \log \frac{\rho_{t,\lambda}^{(2)}}{\rho_2^{(2)}}(x_2) \cdot \nabla_2 \log \frac{\rho_{t,\lambda}^{(2)}}{\rho_2^{(2)}}(x_2) \right) \, dx_2 .$$

(162)

Proposition 4.15 is analogous to the previous one, relies on the fact that $\rho^{(2)}$ satisfies in turn a suitable Fokker-Planck equation (Theorem 4.6). Its proof is postponed to the Appendix and uses the integrability results of Subsection 4.1 and the regularity theorem 4.8.

4.4 Convergence of the conditional measure $\rho_{t,\lambda}^{(2)}$ to $\rho_*^{(2)}$: proof of Theorem 1.7 part i)

We are ready to prove our main Theorem 1.7. This Subsection is devoted to study the conditional measure, while the next one will devoted to the marginal measure. In both Subsections $N_\lambda$ will denote a zero Lebesgue measure subset of $(0, \infty)$ which may depend on $\lambda, \beta_1, \beta_2, \rho_1, V$: one may take the intersection of the two sets introduced in Propositions 4.14, 4.15.

Theorem 4.16. Suppose that $V$ verifies assumptions (A1)-(A5), the logarithmic Sobolev inequality (LS1) holds true with constant $c_1 = c_1(\beta_1, V)$, and $p_1$ verifies assumptions (B1)-(B3). Then there exists a finite non-negative constant $c_0 = c_0(\beta_1, \beta_2, \rho_1, V)$ such that for all $\lambda \geq 1$, $t \in (0, \infty) \setminus N_\lambda$

$$\frac{d}{dt} D_1(t, \lambda) \leq -2c_1 D_1(t, \lambda) + \frac{c_0}{\lambda} .$$

(163)

A suitable choice for $c_0$ is to take the supremum over $t > 0$, $\lambda \geq 1$ of the following integral:

$$\int_{\mathbb{R}^d} \left( \frac{\beta_2}{4} - \beta_1 \right) |\nabla_2 V(x)|^2 + \frac{\beta_2^2}{4\beta_2} |\nabla_2 F(x_2)|^2 - \left( \frac{\beta_1}{2} - 1 \right) \nabla_2 V(x) \cdot \nabla_2 F(x_2) +$$

$$+ \frac{\beta_1}{\beta_2} \nabla_2^2 V(x) - \frac{\beta_1}{\beta_2} \nabla_2^2 F(x_2) \right) \rho_{t,\lambda}(x) \, dx .$$

(164)

Remark 4.17. According to Theorem 4.16, $c_0$ can be evaluated providing an upper bound for expression (164) independent of $t > 0$, $\lambda \geq 1$. In practice this can be done
as follows. Bound $|\nabla_2 V|^2$, $\nabla_2^2 V$ by expressions of type $\gamma_0 + \gamma_1 |x_1|^{r_1} + \gamma_2 |x_2|^{r_2}$, which is possible by assumption (A2). Observe that

$$\nabla_2 F = \langle \nabla_2 V \rangle,$$

(165)

$$\nabla_2^2 F = \langle \nabla_2^2 V \rangle - \beta_1 \langle |\nabla_2 V|^2 \rangle + \beta_1 \langle \nabla_2 V \rangle^2,$$

(166)

and use Proposition 5.11 to bound $\langle |x_1|^{r_1} \rangle$ by $C_0 + C_1 |x_2|^s$. Then use Corollary 5.8 and Proposition 5.10 to bound all the terms of type $\int |x_2|^r \rho_{t,\lambda}(x) \, dx$, $\int |x_2|^s \rho_{t,\lambda}(x) \, dx$, $\int |x_1|^{r_1} \rho_{t,\lambda}(x) \, dx$ uniformly with respect to $t, \lambda$. Finally plug the obtained bounds in (a suitable rearrangement of) expression (164).

**Proof of Theorem 4.16.** Let $\lambda \geq 1$, $t \in (0, \infty) \setminus N_\lambda$. It is a standard fact that the joint KL divergence (39) splits into the sum of a conditional and a marginal contribution, hence

$$\frac{d}{dt} D_2(t, \lambda) = \frac{d}{dt} D(t, \lambda) - \frac{d}{dt} D_1(t, \lambda).$$

(167)

On the other hand we can express $\frac{d}{dt} D_1$ and $\frac{d}{dt} D_2$ according to Propositions 4.14

(168)

It takes a moment to convince oneself that the integral on the r.h.s. of identity (160) splits into five contributions:

$$\frac{d}{dt} D(t, \lambda) = -I_1(t, \lambda) + \frac{1}{\lambda} \left( I_2^{(11)} + I_2^{(12)} + I_2^{(21)} + I_2^{(22)} \right)(t, \lambda)$$

(169)

where

$$I_1(t, \lambda) \equiv \int_{\mathbb{R}^d} \left( \frac{1}{\beta_1} \nabla_1 \log \rho_{t,\lambda}^{(1)}(x_1 | x_2) + \nabla_1 V(x) \right) \cdot \nabla_1 \log \rho_{t,\lambda}(x_1 | x_2) \rho_{t,\lambda}(x) \, dx,$$

(169)

$$I_2^{(11)}(t, \lambda) \equiv \int_{\mathbb{R}^d} \left( \frac{1}{\beta_2} \nabla_2 \log \rho_{t,\lambda}^{(1)}(x_1 | x_2) + \nabla_2 V(x) \right) \cdot \nabla_2 \log \rho_{t,\lambda}^{(1)}(x_1 | x_2) \rho_{t,\lambda}(x) \, dx,$$

(170)

$$I_2^{(22)}(t, \lambda) \equiv \int_{\mathbb{R}^d} \left( \frac{1}{\beta_2} \nabla_2 \log \rho_{t,\lambda}^{(2)}(x_2) + \nabla_2 V(x) \right) \cdot \nabla_2 \log \rho_{t,\lambda}^{(2)}(x_2) \rho_{t,\lambda}(x) \, dx,$$

(171)

$$I_2^{(12)}(t, \lambda) \equiv \int_{\mathbb{R}^d} \frac{1}{\beta_2} \nabla_2 \log \rho_{t,\lambda}^{(1)}(x_1 | x_2) \cdot \nabla_2 \log \rho_{t,\lambda}^{(2)}(x_2) \rho_{t,\lambda}(x) \, dx,$$

(172)

$$I_2^{(21)}(t, \lambda) \equiv \int_{\mathbb{R}^d} \frac{1}{\beta_2} \nabla_2 \log \rho_{t,\lambda}^{(2)}(x_2) \cdot \nabla_2 \log \rho_{t,\lambda}^{(1)}(x_1 | x_2) \rho_{t,\lambda}(x) \, dx.$$
Therefore identities (167), (168), (174) entail that
\[
\frac{d}{dt} D_1(t, \lambda) = -I_1(t, \lambda) - \frac{1}{\lambda} \left( I_2^{(11)} + I_2^{(12)} + I_2^{(21)} \right)(t, \lambda) .
\] (175)

The first integral can be easily estimated thanks to the logarithmic Sobolev inequality for $\rho_{t,\lambda}^{(i)}$. Indeed $I_1$ rewrites as
\[
I_1(t, \lambda) = \frac{1}{\beta_1} \int_{\mathbb{R}^d} \left| \nabla_1 \log \frac{\rho_{t,\lambda}^{(i)}}{\rho_{t,\lambda}^{(1)}}(x_1|x_2) \right|^2 \rho_{t,\lambda}(x) \, dx .
\] (176)

Integrating with respect to $x_1$ first and using assumption (LS1) with $\pi \equiv \rho_{t,\lambda}^{(i)}(\cdot|x_2)$ one obtains:
\[
I_1(t, \lambda) \geq 2c_1 D_1(t, \lambda) .
\] (177)

Now we have to control the remainder $I_2^{(11)} + I_2^{(12)} + I_2^{(21)}$ uniformly for $t \in (0, \infty) \setminus N_\lambda$ and $\lambda \geq 1$. We can rewrite:
\[
I_2^{(11)}(t, \lambda) = \int_{\mathbb{R}^d} \left( \frac{1}{\beta_2} \nabla_2 \log \rho_{t,\lambda}^{(i)}(x_1|x_2) + \nabla_2 V(x) \right) \cdot \left( \nabla_2 \log \rho_{t,\lambda}^{(i)}(x_1|x_2) + \beta_1 \left( \nabla_2 V(x) - \nabla_2 F(x_2) \right) \right) \rho_{t,\lambda}(x) \, dx
\] (178)

and we claim:
\[
I_2^{(12)}(t, \lambda) = 0 ,
\] (179)
\[
I_2^{(21)}(t, \lambda) = -\frac{\beta_1}{\beta_2} \int_{\mathbb{R}^d} \left( \nabla_2^2 V(x) + \nabla_2 V(x) \cdot \nabla_2 \log \rho_{t,\lambda}^{(i)}(x_1|x_2) - \nabla_2^2 F(x_2) \right) \rho_{t,\lambda}(x) \, dx .
\] (180)

The sum of equations (178), (179), (180) gives
\[
\left( I_2^{(11)} + I_2^{(12)} + I_2^{(21)} \right)(t, \lambda) = \int_{\mathbb{R}^d} R_{t,\lambda}(x) \rho_{t,\lambda}(x) \, dx
\] (181)

with
\[
R_{t,\lambda} \equiv \frac{1}{\beta_2} \left| \nabla_2 \log \rho_{t,\lambda}^{(i)} \right|^2 + \nabla_2 \log \rho_{t,\lambda}^{(i)} \cdot \nabla_2 V - \frac{\beta_1}{\beta_2} \nabla_2 \log \rho_{t,\lambda}^{(i)} \cdot \nabla_2 F + \beta_1 \nabla_2 V^2 - \nabla_2^2 F - \frac{\beta_1}{\beta_2} \nabla_2^2 V + \frac{\beta_1}{\beta_2} \nabla_2^2 F .
\] (182)

We can use the expansion of the square $\left| a \nabla_2 \log \rho_{t,\lambda}^{(i)} + b \nabla_2 V - c \nabla_2 F \right|^2 \geq 0$ with $a \equiv \beta_2^{-\frac{1}{2}}$, $b \equiv \frac{1}{2} \beta_2^{-\frac{1}{2}}$, $c \equiv \frac{1}{2} \beta_1 \beta_2^{-\frac{1}{2}}$ in order to bound from below the terms containing $\nabla_2 \log \rho_{t,\lambda}^{(i)}$. In this way we obtain:
\[
R_{t,\lambda} \geq -\left( \frac{\beta_2}{4} - \beta_1 \right) \left| \nabla_2 V \right|^2 - \frac{\beta_2^2}{4 \beta_2} \left| \nabla_2 F \right|^2 + \left( \frac{\beta_1}{2} - 1 \right) \nabla_2 V \cdot \nabla_2 F + \frac{\beta_1}{\beta_2} \nabla_2^2 V + \frac{\beta_1}{\beta_2} \nabla_2^2 F ,
\] (183)
where the latter lower bound does not depend on \( t, \lambda \) anymore. Plugging (183) into (181) and using Theorems 4.1, 4.2 proves that there exists a finite non-negative constant \( c_0 = c_0(\beta_1, \beta_2, \rho_1, V) \) such that

\[
\inf_{\lambda \geq 1, t > 0, t \notin N_\lambda} (J_{\lambda}^{(11)} + J_{\lambda}^{(12)} + J_{\lambda}^{(21)})(t, \lambda) \geq -c_0. \tag{184}
\]

Finally, plugging inequalities (177), (184) into expression (175) gives

\[
d_t D_1(t, \lambda) \leq -2c_1 D_1(t, \lambda) + \frac{c_0}{\lambda}. \tag{185}
\]

To conclude the proof it remains to prove claims (179), (180). First we have:

\[
I_{\lambda}^{(12)}(t, \lambda) = \frac{1}{\beta_2} \int_{\mathbb{R}^2} \left[ \int_{\mathbb{R}^2} \nabla_2 \rho_1^{(2)}(x_1|x_2) \, dx_1 \right] \cdot \nabla_2 \log \frac{\rho_1^{(2)}(x_2)}{\rho_2^{(2)}(x_2)} \rho_1^{(2)}(x_2) \, dx_2 = 0, \tag{186}
\]

where the first identity is due to Fubini theorem and the last one holds true because the term inside square brackets vanishes for almost every \( x_2 \) (see Lemma 4.18 in the following). Secondly, we have:

\[
I_{\lambda}^{(21)}(t, \lambda) = \frac{\beta_1}{\beta_2} \int_{\mathbb{R}^2} \nabla_2 \rho_1^{(2)}(x_2) \cdot \left[ \int_{\mathbb{R}^2} \nabla_2 \log \frac{\rho_1^{(2)}(x_1|x_2)}{\rho_2^{(2)}(x_2)} \rho_1^{(2)}(x_1|x_2) \, dx_1 \right] \, dx_2
\]

\[
= -\frac{\beta_1}{\beta_2} \int_{\mathbb{R}^2} \rho_1^{(2)}(x_2) \left[ \int_{\mathbb{R}^2} (\nabla_2 V(x) - \nabla_2 F(x_2)) \rho_1^{(2)}(x_1|x_2) \, dx_1 \right] \, dx_2
\]

\[
= -\frac{\beta_1}{\beta_2} \int_{\mathbb{R}^2} \nabla_2 V(x) \rho_1^{(2)}(x_1|x_2) + \nabla_2 F(x_2) \rho_1^{(2)}(x_1|x_2) \, dx_1 - \nabla_2^2 F(x_2) \right] \, dx_2, \tag{187}
\]

where the first identity is due to Fubini theorem, the second one is due to Lemma 4.18 and the last one is essentially integration by parts (justified by Lemma 4.19 in the following). Using again Fubini theorem identity (180) is finally obtained and both claims are proven.

**Lemma 4.18.** Under the hypothesis of Theorem 4.10, for a.e. \( x_2 \in \mathbb{R}^2, t > 0 \)

\[
\int_{\mathbb{R}^2} \nabla_2 \rho_1^{(1)}(x_1|x_2) \, dx_1 = 0. \tag{188}
\]

**Proof.** This is equivalent to prove that

\[
\int_{\mathbb{R}^2} \psi(x_2) \left[ \int_{\mathbb{R}^2} \nabla_2 \rho_1^{(1)}(x_1|x_2) \, dx_1 \right] \, dx_2 = 0 \tag{189}
\]
for all $\psi \in C_c^\infty(\mathbb{R}^d)$, a.e. $t > 0$. By Corollary 4.13, $\rho_t^{(1)} \in W^{1,1}(\mathbb{R}^d, \rho_t^{(2)} \, dx_1 dx_2)$ which is included in $W^{1,1}(\mathbb{R}^d, \psi \, dx_1 dx_2)$. Therefore by Fubini theorem and integration by parts we have

$$\int_{\mathbb{R}^d} \psi(x_2) \int_{\mathbb{R}^d} \nabla_2 \rho_t^{(1)}(x_1 | x_2) \, dx_1 \, dx_2 = \int_{\mathbb{R}^d} \psi(x_2) \nabla_2 \rho_t^{(1)}(x_2 | x_2) \, dx =$$

$$= - \int_{\mathbb{R}^d} \nabla_2 \psi(x_2) \rho_t^{(1)}(x_2 | x_2) \, dx_2 .$$

(190)

since $\int_{\mathbb{R}^d} \rho_t^{(1)}(x_1 | x_2) \, dx_1 = 1$ for every $x_2 \in \mathbb{R}^d$. As $\psi$ has compact support, the r.h.s. of (190) vanishes, concluding the proof.

\[ \Box \]

**Lemma 4.19.** Under the hypothesis of Theorem 4.10, let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ such that $\Phi \in L^2(\rho_1), \nabla_2 \cdot \Phi \in L^1(\rho_1)$ for all $t > 0$. Then for a.e. $t > 0$

$$\int_{\mathbb{R}^d} \nabla_2 \rho_t^{(1)}(x_2) \cdot \int_{\mathbb{R}^d} \Phi(x) \, \rho_t^{(1)}(x_1 | x_2) \, dx_1 \, dx_2 =$$

$$= - \int_{\mathbb{R}^d} \rho_t^{(2)}(x_2) \int_{\mathbb{R}^d} (\nabla_2 \cdot \Phi(x) \rho_t^{(1)}(x_1 | x_2) + \Phi(x) \cdot \nabla_2 \rho_t^{(1)}(x_1 | x_2)) \, dx_1 \, dx_2 .$$

(191)

**Proof.** To shorten the notation set $f(x) \equiv \Phi(x) \rho_t^{(1)}(x_1 | x_2), g(x) \equiv \rho_t^{(2)}(x_2)$. Observe that $f g, f \cdot \nabla_2 g, g \nabla_2 \cdot f$ belong to $L^1(\mathbb{R}^d)$ for a.e. $t > 0$, indeed:

$$\int_{\mathbb{R}^d} |f \cdot \nabla_2 g|(x) \, dx = \int_{\mathbb{R}^d} |\Phi(x) \cdot \nabla_2 \log \rho_t^{(2)}(x_2)| \, \rho_t(x) \, dx \leq$$

$$\leq \left( \int_{\mathbb{R}^d} |\Phi(x)|^2 \rho_t(x) \, dx \right)^\frac{1}{2} \left( \int_{\mathbb{R}^d} |\nabla_2 \log \rho_t^{(2)}(x_2)|^2 \rho_t(x) \, dx \right)^\frac{1}{2} < \infty ;$$

(192)

$$\int_{\mathbb{R}^d} |g \nabla_2 \cdot f|(x) \, dx = \int_{\mathbb{R}^d} |\nabla_2 \cdot \Phi(x) \rho_t^{(1)}(x_1 | x_2) + \Phi(x) \cdot \nabla_2 \rho_t^{(1)}(x_1 | x_2)| \, \rho_t^{(2)}(x_2) \, dx \leq$$

$$\leq \int_{\mathbb{R}^d} |\nabla_2 \cdot \Phi(x)| \, \rho_t(x) \, dx + \left( \int_{\mathbb{R}^d} |\Phi(x)|^2 \rho_t(x) \, dx \right)^\frac{1}{2} \left( \int_{\mathbb{R}^d} |\nabla_2 \log \rho_t^{(1)}(x_1 | x_2)|^2 \rho_t(x) \, dx \right)^\frac{1}{2}$$

$$< \infty .$$

(193)

Now consider a sequence $\varphi_N \in C_c^\infty(\mathbb{R}^d)$ such that $\varphi_N(x) = 1$ for $|x| \leq N$ and $\varphi_N, \nabla_2 \varphi_N$ are uniformly bounded for $N \in \mathbb{N}$. Integration by parts gives

$$\int_{\mathbb{R}^d} \nabla_2 \cdot (f g)(x) \, \varphi_N(x) \, dx = - \int_{\mathbb{R}^d} (f g)(x) \cdot \nabla_2 \varphi_N(x) \, dx .$$

(194)

Letting $N \to \infty$ by dominate convergence we obtain

$$\int_{\mathbb{R}^d} \nabla_2 \cdot (f g)(x) \, dx = 0 ,$$

(195)

which rewrites as

$$\int_{\mathbb{R}^d} (f \cdot \nabla_2 g)(x) \, dx = - \int_{\mathbb{R}^d} (g \nabla_2 \cdot f)(x) \, dx .$$

(196)

Identity (191) finally follows by Fubini theorem.

\[ \Box \]
Proof of Theorem 1.7 part i). It follows by Theorem 4.10 and Gronwall Lemma 5.4 since the map $t \mapsto D_1(t, \lambda)$ is absolutely continuous on compact subsets of $[0, \infty)$ (Proposition 4.14).

4.5 Convergence of the marginal measure $\rho^{(2)}_{\lambda} \text{ to } \rho^{(2)}$: proof of Theorem 1.7 part ii)

Theorem 4.20. Suppose that $V$ verifies assumptions (A1)-(A5), logarithmic Sobolev inequality (LS2) holds true with constant $c_2 = c_2(\beta_2, V)$ and $\rho$ verifies assumptions (B1)-(B3). Then there exists a finite non-negative constant $\tilde{c}_0 = \tilde{c}_0(\beta_1, \beta_2, \rho_1, V)$ such that for all $\lambda \geq 1$, $t \in (0, \infty) \setminus N_\lambda$, $\eta \in (0, 2)$, $\epsilon > 0$

$$\lambda \frac{d}{dt} D_2(t, \lambda) \leq -\eta c_2 D_2(t, \lambda) + \frac{1}{\epsilon(2-\eta)} D_1(t, \lambda) + \frac{\tilde{c}_0}{2} \epsilon .$$  \hspace{1cm} (197)

A suitable choice for $\tilde{c}_0$ is to take the supremum over $t > 0$, $\lambda \geq 1$ of the following quantity:

$$\frac{9}{2} \beta_2^2 \left( \int_{\mathbb{R}^d} |\nabla V|^4 \rho_{t, \lambda}(x) \, dx + \int_{\mathbb{R}^d} |\nabla V|^4 \rho^{(1)}_s(x_1|x_2) \rho^{(2)}_{t, \lambda}(x_2) \, dx \right).$$  \hspace{1cm} (198)

Remark 4.21. According to Theorem 4.20, $\tilde{c}_0$ can be evaluated providing an upper bound for expression (198) independent of $t > 0$, $\lambda \geq 1$. In practice we can compute a bound of type

$$|\nabla V|^4 \leq \gamma_0 + \gamma_1 |x_1|^\tau + \gamma_2 |x_2|^\tau^2,$$  \hspace{1cm} (199)

which is possible by assumption (A2). We use Proposition 5.11 to bound $|\langle x_1 |^{\tau}| \leq C_0 + C_1 |x_2|^{\tau}$. Then we use Corollary 5.8 in order to bound $\int |x_2|^\tau \rho_{t, \lambda}(x) \, dx \leq M_\sigma$ for $\sigma = r_2, s$, and we use Proposition 5.10 to bound $\int |x_1|^\tau \rho_{t, \lambda}(x) \, dx \leq M'_\sigma$. Finally we can plug the obtained bounds into expression (198).

Proof. Let $\lambda \geq 1$, $t \in (0, \infty) \setminus N_\lambda$. Proposition 4.15 provides expression (162) for $\frac{d}{dt} D_2$. Adding and subtracting a term $-\frac{\beta_2}{\rho_2} \nabla V \log \rho^{(2)}(x_2) = \nabla V \log \rho^{(2)}(x)$ shows that

$$\lambda \frac{d}{dt} D_2(t, \lambda) = -(J_1 + J_2)(t, \lambda)$$  \hspace{1cm} (200)

where

$$J_1(t, \lambda) \equiv \frac{1}{\beta_2} \int_{\mathbb{R}^d} \left( \nabla V \log \frac{\rho^{(2)}_{t, \lambda}(x_2)}{\rho^{(2)}_s(x_2)} \right)^2 \rho^{(2)}_{t, \lambda}(x_2) \, dx_2 ,$$  \hspace{1cm} (201)

$$J_2(t, \lambda) \equiv \int_{\mathbb{R}^d} \left( \langle \nabla V \rangle_{t, \lambda} - \nabla V \log \frac{\rho^{(2)}_{t, \lambda}(x_2)}{\rho^{(2)}_s(x_2)} \right) \cdot \nabla V \log \frac{\rho^{(2)}_{t, \lambda}(x_2)}{\rho^{(2)}_s(x_2)} \rho^{(2)}_{t, \lambda}(x_2) \, dx_2 .$$  \hspace{1cm} (202)

Using Cauchy-Schwarz inequality a further term of type $J_1$ can be extracted from $J_2$. Precisely we have:

$$|J_2(t, \lambda)| \leq (\beta_2 J_1(t, \lambda))^\frac{1}{2} J_3(t, \lambda)^\frac{1}{2} \leq \frac{\eta}{2} J_1(t, \lambda) + \frac{\beta_2}{2\eta} J_3(t, \lambda)$$  \hspace{1cm} (203)
for every $\eta > 0$, where

$$J_3(t, \lambda) \equiv \int_{\mathbb{R}^d} \left| \langle \nabla_2 V \rangle_{t,\lambda} - \nabla_2 F \right|^2(x_2) \rho_{t,\lambda}^{(2)}(x_2) \, dx_2 .$$

Choosing $\eta \in (0, 2)$ and plugging (203) into (200) gives

$$\lambda \frac{d}{dt} D_2(t, \lambda) \leq - \left(1 - \frac{\eta}{2}\right) J_1(t, \lambda) + \frac{\beta_2}{2\eta} J_3(t, \lambda) .$$

Now, $J_1$ can be estimated using the logarithmic Sobolev inequality for $\rho_{t,\lambda}^{(2)}$. Indeed by assumption (LS2) taking $\pi \equiv \rho_{t,\lambda}^{(2)}$ we have

$$J_1(t, \lambda) \geq 2 c_2 D_2(t, \lambda) .$$

In order to estimate $J_3$, let us start by observing that $\nabla_2 F = \langle \nabla_2 V \rangle_*$, hence

$$\left| \langle \nabla_2 V \rangle_{t,\lambda} - \nabla_2 F \right|(x_2) \leq \int_{\mathbb{R}^{d_2}} \left| \langle \nabla_2 V \rangle(x) \right| \left| \rho_{t,\lambda}^{(1)} - \rho_{*}^{(1)} \right|(x_1|x_2) \, dx_1 .$$

The Csiszár-Kullback-Pinsker inequality provides an upper bound for the total variation distance in terms of the KL divergence:

$$\int_{\mathbb{R}^{d_1}} \left| \rho_{t,\lambda}^{(1)} - \rho_{*}^{(1)} \right|(x_1|x_2) \, dx_1 \leq 2 \int_{\mathbb{R}^{d_1}} \log \frac{\rho_{t,\lambda}^{(1)}}{\rho_{*}^{(1)}}(x_1|x_2) \rho_{t,\lambda}^{(1)}(x_1|x_2) \, dx_1 \right)^{\frac{1}{2}} ,$$

hence squaring and integrating against $\rho_{t,\lambda}^{(2)}(x_2)$ on both sides of the inequality we find

$$\int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} \left| \rho_{t,\lambda}^{(1)} - \rho_{*}^{(1)} \right|(x_1|x_2) \, dx_1 \right)^2 \rho_{t,\lambda}^{(2)}(x_2) \, dx_2 \leq 2 D_1(t, \lambda) .$$

In order to be able to apply the latter inequality, we split $J_3$ in three terms and use the uniform integrability of $|\nabla_2 V|^2$. By (207) for every $\epsilon > 0$ we have

$$\left| \langle \nabla_2 V \rangle_{t,\lambda} - \nabla_2 F \right|(x_2) \leq (T_1 + T_2 + T_3)(t, \lambda, x_2)$$

where:

$$T_1(t, \lambda, x_2) \equiv \int_{\mathbb{R}^d} |\nabla_2 V(x)| \ I \left( |\nabla_2 V(x)| \leq \epsilon^{-\frac{1}{2}} \right) \left| \rho_{t,\lambda}^{(1)} - \rho_{*}^{(1)} \right|(x_1|x_2) \, dx_1 ,$$

$$T_2(t, \lambda, x_2) \equiv \int_{\mathbb{R}^d} |\nabla_2 V(x)| \ I \left( |\nabla_2 V(x)| > \epsilon^{-\frac{1}{2}} \right) \rho_{t,\lambda}^{(1)}(x_1|x_2) \, dx_1 ,$$

$$T_3(x_2) \equiv \int_{\mathbb{R}^d} |\nabla_2 V(x)| \ I \left( |\nabla_2 V(x)| > \epsilon^{-\frac{1}{2}} \right) \rho_{*}^{(1)}(x_1|x_2) \, dx_1 .$$

Clearly:

$$T_1(t, \lambda, x_2) \leq \epsilon^{-\frac{1}{2}} \int_{\mathbb{R}^{d_1}} \left| \rho_{t,\lambda}^{(1)} - \rho_{*}^{(1)} \right|(x_1|x_2) \, dx_1 ,$$

$$T_2(t, \lambda, x_2) \leq \epsilon^{\frac{1}{2}} \int_{\mathbb{R}^{d_1}} |\nabla_2 V(x)|^2 \rho_{t,\lambda}^{(1)}(x_1|x_2) \, dx_1 .$$
Finally, plugging estimates (206), (219) into (205) shows that
\[
J_3(t, \lambda) \leq 6 \epsilon^{-1} D_1(t, \lambda) + 3 \epsilon \int_{\mathbb{R}^d} |\nabla V(x)|^4 \rho_{t, \lambda}(x) \, dx + \\
+ 3 \epsilon \int_{\mathbb{R}^d} |\nabla V(x)|^4 \rho_{t, \lambda}^{(1)}(x_1|x_2) \rho_{t, \lambda}^{(2)}(x_2) \, dx.
\]

By Theorems 4.1, 4.2 we have (163), (197) rewrite as:
\[
\text{continuous on compact subsets of } [0, \infty). 
\]

Plugging inequalities (214)-(216) into (210), (204), then using Csiszár-Kullback-Pinsker inequality (209) and Jensen inequality we obtain:
\[
J_3(t, \lambda) \leq 6 \epsilon^{-1} D_1(t, \lambda) + 3 \epsilon \int_{\mathbb{R}^d} |\nabla V(x)|^4 \rho_{t, \lambda}(x) \, dx + \\
+ 3 \epsilon \int_{\mathbb{R}^d} |\nabla V(x)|^4 \rho_{t, \lambda}^{(1)}(x_1|x_2) \rho_{t, \lambda}^{(2)}(x_2) \, dx.
\]

By Theorems 4.1, 4.2 we have
\[
\sup_{t>0, \lambda \geq 1} \int_{\mathbb{R}^d} |\nabla V(x)|^4 \left( \rho_{t, \lambda} + \rho_{t, \lambda}^{(1)} \rho_{t, \lambda}^{(2)} \right)(x) \, dx \equiv \bar{c}_0 < \infty,
\]

hence
\[
J_3(t, \lambda) \leq 6 \epsilon^{-1} D_1(t, \lambda) + 3 \epsilon \bar{c}_0 \epsilon.
\]

Finally, plugging estimates (209), (210) into (206) shows that
\[
\lambda \frac{d}{dt} D_2(t, \lambda) \leq -(2 - \eta) c_2 D_2(t, \lambda) + \frac{3\beta_2}{\eta} D_1(t, \lambda) + \frac{3\beta_2 \bar{c}_0}{2\eta} \epsilon.
\]

Renaming $2 - \eta \equiv \eta'$, $\frac{\epsilon}{\eta' c_2} \equiv \epsilon'$, and $\frac{9\beta_2^2}{\eta^2} \bar{c}_0 \equiv \bar{c}'_0$ we find the desired inequality (197). \qed

**Proof of Theorem 1.7 part ii)**. It follows by Theorems 4.16, 4.20 using the extended Gronwall Lemma 5.6, since the maps $t \mapsto D_1(t, \lambda), t \mapsto D_2(t, \lambda)$ are absolutely continuous on compact subsets of $[0, \infty)$ (Propositions 4.13, 4.15). Indeed inequalities (163), (197) rewrite as:
\[
\frac{d}{dt} \left( \begin{array}{c} D_1(t, \lambda) \\ D_2(t, \lambda) \end{array} \right) \leq -C_\lambda \left( \begin{array}{c} D_1(t, \lambda) \\ D_2(t, \lambda) \end{array} \right) + B_\lambda
\]
for a.e. $t > 0$, where $\leq$ denotes componentwise inequality and
\[
C_\lambda \equiv \left( \begin{array}{cc} -2c_1 & 0 \\ 1 & -\frac{\eta c_2}{\lambda} \end{array} \right), \quad B_\lambda \equiv \left( \begin{array}{c} \frac{\eta c_2}{(2-\eta)\lambda} \\ \frac{\bar{c}'_0 \epsilon}{(2-\eta)\lambda} \end{array} \right).
\]

Therefore by Lemma 5.6 in the Appendix we have
\[
\left( \begin{array}{c} D_1(t, \lambda) \\ D_2(t, \lambda) \end{array} \right) \leq e^{-t C_\lambda} \left( \begin{array}{c} D_{11} \\ D_{21} \end{array} \right) + (I_2 - e^{-t C_\lambda}) C_\lambda^{-1} B_\lambda
\]
for all $t > 0$. For $\lambda \geq (\eta c_2)/(2c_1)$ standard computations show that
\[
e^{-t C_\lambda} = \left( \begin{array}{cc} e^{-2c_1 t} & 0 \\ e^{-\eta c_2 t} & e^{-\eta c_2 t/\lambda} \end{array} \right),
\]
\[
C_\lambda^{-1} B_\lambda = \left( \begin{array}{cc} \frac{\bar{c}'_0}{\eta c_2} e^{-\eta c_2 t/\lambda} + \frac{\bar{c}'_0 \epsilon}{\eta c_2 (2-\eta)} \\ 2\eta c_2 (2-\eta) e^{-\eta c_2 t/\lambda} \end{array} \right),
\]
and can check that the second component of inequality (223) gives precisely (19). \qed
Remark 4.22. Since any positive power of $|\nabla^2 V|$ is integrable (Theorems 4.1, 4.2), inequality \[ (197) \] holds true also if we replace the term $\tilde{c}_0 \epsilon$ therein by $\tilde{c}_0(r) \epsilon^r$ for any $r > 0$. Indeed the proof of Theorem 4.20 can be modified using the following upper bound for the integral $T_2$:

\[
T_2(t, \lambda, x_2) \leq \epsilon^2 \int_{\mathbb{R}^d} |\nabla^2 V(x)|^{r+1} \rho_{L,\lambda}^{(1)}(x_1 | x_2) \, dx_1 ,
\]

and analogously for $T_3$. Then following the same proof one obtains the desired result with

\[
\tilde{c}_0(r) \equiv \sup_{t>0, \lambda \geq 1} \int_{\mathbb{R}^d} |\nabla^2 V(x)|^{2r+2} \left( \rho_{L,\lambda}^{(1)} + \rho_{L,\lambda}^{(2)} \right)(x) \, dx < \infty .
\]

As a consequence also inequality \[ (15) \] in Theorem 4.7 modifies replacing the term $R_3(t, \lambda, \eta \epsilon)$ therein by $R_3(t, \lambda, \eta, r \epsilon)$, where

\[
R_3(t, \lambda, \eta, r) \equiv \left( 1 - e^{-\eta \epsilon t/\lambda} \right) \frac{\tilde{c}_0(r)}{\eta \epsilon^2} .
\]

5 Appendix

5.1 Gronwall-type inequalities

This Subsection of the Appendix is devoted to Gronwall inequality and its applications to Fokker-Planck equation. In particular we show how the expectation of an observable $v$ at any time $t$ can be controlled by means of a simple Lyapunov condition on $L v$, in the spirit of Section 7.1 in [32]. A generalization of Gronwall inequality to systems of differential inequalities is also discussed. These results are extensively used along the paper.

We start by the following:

**Proposition 5.1.** Let $v \in C^2(\mathbb{R}^d) \cap L^1(\rho_1)$, $C_0, C_1 \in \mathbb{R}$ such that for all $x \in \mathbb{R}^d$

\[ L v(x) \leq C_0 + C_1 v(x) . \]

Suppose that for all $T > 0$

\[
\sup_{t \in (0, T)} \int_{\mathbb{R}^d} |v(x)| \rho_t(x) \, dx < \infty , \quad \sup_{t \in (0, T)} \int_{\mathbb{R}^d} |L v(x)| \rho_t(x) \, dx < \infty , \quad \sup_{t \in (0, T)} \int_{\mathbb{R}^d} |\nabla v(x)| \rho_t(x) \, dx < \infty .
\]

Then for all $t > 0$ we have:

\[
\int_{\mathbb{R}^d} v(x) \rho_t(x) \, dx \leq e^{C_1 t} \int_{\mathbb{R}^d} v(x) \rho_t(x) \, dx + \left( e^{C_1 t} - 1 \right) \frac{C_0}{C_1} .
\]

**Proof.** Consider a sequence $\psi_N \in C^2(\mathbb{R}^d)$ such that

\[
\psi_N(x) = \begin{cases} 1 & \text{for } |x| \leq N \\ 0 & \text{for } |x| \geq N + 1 \end{cases}
\]
and $\psi_N$, $\nabla \psi_N$, Hess $\psi_N$ are uniformly bounded. We have $\varphi_N \equiv v \psi_N \in C_c^2(\mathbb{R}^d)$, hence by identity (230)

$$\int_{\mathbb{R}^d} \varphi_N(x) \rho_t(x) \, dx - \int_{\mathbb{R}^d} \varphi_N(x) \rho_t(x) \, dx = \int_0^t \int_{\mathbb{R}^d} L \varphi_N(x) \rho_{s, \lambda}(x) \, dx \, ds .$$

(233)

Now,

$$L \varphi_N = L v \psi_N + v L \psi_N + 2 \nabla v \cdot (\Lambda \beta)^{-1} \nabla \psi_N$$

(234)

hence hypothesis (230) guarantees that there is dominated convergence in (233). Precisely letting $N \to \infty$ we find

$$\int_{\mathbb{R}^d} v(x) \rho_t(x) \, dx - \int_{\mathbb{R}^d} v(x) \rho_I(x) \, dx = \int_0^t \int_{\mathbb{R}^d} L v(x) \rho_{s, \lambda}(x) \, dx \, ds .$$

(235)

As a consequence $t \mapsto \int_{\mathbb{R}^d} v(x) \rho_t(x) \, dx$ is absolutely continuous on compact subsets of $[0, \infty)$ and there exists

$$\frac{d}{dt} \int_{\mathbb{R}^d} v(x) \rho_t(x) \, dx = \int_{\mathbb{R}^d} L v(x) \rho_t(x) \, dx$$

(236)

for a.e. $t > 0$. Then by hypothesis (229) we have

$$\frac{d}{dt} \int_{\mathbb{R}^d} v(x) \rho_t(x) \, dx \leq C_0 + C_1 \int_{\mathbb{R}^d} v(x) \rho_t(x) \, dx .$$

(237)

Thesis (231) finally follows applying Gronwall inequality (see Lemma 5.4).

The previous result can be extended to observables $v$ that are not known to be integrable a priori:

**Proposition 5.2.** Let $v \in C^2(\mathbb{R}^d) \cap L^1(\rho_I)$, $C_0, C_1 \in \mathbb{R}$ such that inequality (229) holds true for all $x \in \mathbb{R}^d$. Suppose $v \geq 0$ and

$$v(x) \to \infty \quad \text{as} \quad |x| \to \infty .$$

(238)

Then inequality (231) holds true for all $t > 0$.

Proposition 5.2 is essentially a reinterpretation of Theorem 7.1.1 in [32] for the Fokker-Planck operator $L$ defined by (24). Note in particular that thanks to the regularity of our setting, we do not need any assumption on the signs of $C_0, C_1$ unlike in [32].

**Proof.** Let $\zeta_N \in C^2([0, \infty))$ such that

$$\zeta_N(r) = \begin{cases} r & \text{for } r \leq N - 1, \\ N & \text{for } r \geq N + 1 \end{cases}$$

(239)

$0 \leq \zeta_N \leq 1$ and $\zeta_N'' \leq 0$. Observe that $\varphi_N \equiv \zeta_N \circ v - N \in C_c^2(\mathbb{R}^d)$ since $v(x) \to \infty$ as $|x| \to \infty$, hence applying identities (234)-(222) to $\varphi_N$ we find:

$$\frac{d}{dt} \int_{\mathbb{R}^d} \zeta_N(v(x)) \rho_t(x) \, dx = \int_{\mathbb{R}^d} L(\zeta_N \circ v)(x) \rho_t(x) \, dx$$

(240)
for all $t > 0$ and
\[
\int_{\mathbb{R}^d} \zeta_N(v(x)) \rho_t(x) \, dx \mathop{\longrightarrow}_{t \to 0} \int_{\mathbb{R}^d} \zeta_N(v(x)) \rho_t(x) \, dx. \tag{241}
\]
Now observe that, since $\nabla (\zeta_N \circ v) = (\zeta'_N \circ v) \nabla v$ and $\text{Hess}(\zeta_N \circ v) = (\zeta''_N \circ v) \nabla v \nabla^T v + (\zeta'_N \circ v) \text{Hess } v$, by concavity of $\zeta_N$ we have
\[
L(\zeta_N \circ v) \leq (\zeta'_N \circ v) L v. \tag{242}
\]
Using the properties of $\zeta_N$ one can check that $\zeta'_N(r) r \leq \zeta_N(r)$ for all $r \geq 0$. Therefore using hypothesis (229) it follows that
\[
L(\zeta_N \circ v) \leq C_0 + C_1 (\zeta_N \circ v). \tag{243}
\]
Substituting estimate (243) into (240) we find
\[
\frac{d}{dt} \int_{\mathbb{R}^d} \zeta_N(v(x)) \rho_t(x) \, dx \leq C_0 + C_1 \int_{\mathbb{R}^d} \zeta_N(v(x)) \rho_t(x) \, dx. \tag{244}
\]
Therefore applying Gronwall inequality (see Lemma 5.4), from (244), (241) we obtain
\[
\int_{\mathbb{R}^d} \zeta_N(v(x)) \rho_t(x) \, dx \leq e^{C_1 t} \int_{\mathbb{R}^d} \zeta_N(v(x)) \rho_t(x) \, dx + \left( e^{C_1 t} - 1 \right) \frac{C_0}{C_1}. \tag{245}
\]
Finally, the thesis follows by monotone convergence letting $N \to \infty$. \hfill $\Box$

Remark 5.3. The hypothesis of Propositions 5.1, 5.2 can be weakened. Instead of bound (229), it suffices to assume:
\[
L v(x) \leq u_0(x) + C_1 v(x) \tag{246}
\]
where the function $u_0 \in \cap_{t > 0} L^1(\rho_t)$ is such that
\[
\sup_{t > 0} \int_{\mathbb{R}^d} u(x) \rho_t(x) \, dx \leq C_0. \tag{247}
\]
For completeness, let us briefly recall the classical Gronwall inequality used in the proofs of the previous Propositions.

Lemma 5.4 (Gronwall inequality). Let $f : [0, \infty) \to \mathbb{R}$ be absolutely continuous on compact sets. Let $a, b : [0, \infty) \to \mathbb{R}$ continuous such that
\[
f'(t) \leq a(t) f(t) + b(t) \tag{248}
\]
for almost every $t > 0$. Then setting
\[
e(t) \equiv \exp \int_0^t a(s) \, ds, \tag{249}
\]
we have for all $t \geq 0$
\[
f(t) \leq e(t) f(0) + e(t) \int_0^t b(s) \frac{e(s)}{e(s)} \, ds. \tag{250}
\]
In particular if $a, b$ are constant, inequality (250) becomes:
\[
f(t) \leq e^{at} f(0) + (e^{at} - 1) \frac{b}{a}. \tag{251}
\]
Proof. At $t = 0$ inequality (250) is trivial. Let $t > 0$ and set $\phi(t) \equiv f(t)/e(t)$. $\phi$ is absolutely continuous on compact sets and using (248) we find

$$
\phi'(t) = \frac{f'(t) - f(t) a(t)}{e(t)} \leq \frac{b(t)}{e(t)}.
$$

(252)

for a.e. $t > 0$. Then integrating on the interval $[0, t]$ we obtain

$$
\phi(t) - \phi(0) = \int_0^t \phi'(s) \, ds \leq \int_0^t \frac{b(s)}{e(s)} \, ds,
$$

(253)

which proves (250) by definition of $\phi$.

Remark 5.5. The right-hand side of (250) is the solution of the O.D.E. $y' = ay + b$ with initial condition $y(0) = f(0)$. In Lemma 5.4 there are no assumptions on the signs of $f$, $a$, $b$.

In the next Lemma we deal with a system of differential inequalities, extending the classical Gronwall inequality. This result was essentially due to [49], nevertheless for the sake of completeness we give a proof here adapting Lemma E.4 in [50] to the two-dimensional case with two absolutely continuous functions. The symbol $\preceq$ will denote a componentwise inequality, namely we write $(u_1, u_2) \preceq (v_1, v_2)$ meaning “$u_1 \leq v_1$ and $u_2 \leq v_2$”.

Lemma 5.6 (Comparison lemma for systems of differential inequalities). Let $f_1, f_2 : [0, \infty) \to \mathbb{R}$ be absolutely continuous functions on compact sets; let $\mathbf{f} \equiv (f_1, f_2)$. Let $A \equiv (A_1, A_2) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ be a Lipschitz continuous function such that

$$
f'(t) \preceq A(t, \mathbf{f}(t))
$$

(254)

for almost every $t > 0$. Suppose that:

i. the map $\xi_2 \mapsto A_1(t, \xi_1, \xi_2)$ is non-decreasing on $\mathbb{R}$, for every $(t, \xi_1) \in [0, \infty) \times \mathbb{R}$;

ii. the map $\xi_1 \mapsto A_2(t, \xi_1, \xi_2)$ is non-decreasing on $\mathbb{R}$, for every $(t, \xi_2) \in [0, \infty) \times \mathbb{R}$.

Then we have

$$
f(t) \preceq \mathbf{g}(t)
$$

(255)

for all $t \geq 0$, where $\mathbf{g}$ is the unique solution of the Cauchy problem

$$
\begin{cases}
\mathbf{g}'(t) = A(t, \mathbf{g}(t)) \\
\mathbf{g}(0) = \mathbf{f}(0)
\end{cases}
$$

(256)

In particular if $A$ is linear and time-independent, namely $A(t, \xi) = A \xi + B$ for a suitable $2 \times 2$ real matrix $A \equiv \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ with $A_{12}, A_{21} \geq 0$ and a suitable vector $B \equiv (B_1, B_2) \in \mathbb{R}^2$, then inequality (250) becomes:

$$
f(t) \preceq e^{tA} \mathbf{f}(0) + (e^{tA} - I_2) A^{-1}B.
$$

(257)
Proof. The Cauchy problem (256) admits a unique solution, which depends continuously on perturbations of \( A \). Precisely, for every \( T > 0, \epsilon > 0 \) there exists \( \delta = \delta(T, \epsilon) > 0 \) such that

\[
\sup_{t \in [0, T]} |g(t) - g(\delta)(t)| < \epsilon ,
\]

where \( g_\delta \) is the unique solution of the following perturbed Cauchy problem:

\[
\begin{cases}
(g_\delta)'(t) = A(t, g_\delta(t)) + \delta \\
g_\delta(0) = f(0)
\end{cases}
\]

We claim that \( f(t) \preceq g_\delta(t) \ \forall t \in [0, T] \); therefore using (258) we will obtain

\[
f(t) \preceq g(t) + \epsilon \ \forall t \in [0, T]
\]

and the thesis (255) will follow by arbitrariness of \( T > 0, \epsilon > 0 \). In order to prove claim (260), suppose by contradiction that there exists \( t \in [0, T] \) such that \( f(t) \not\preceq g_\delta(t) \).

Then set

\[
t_* = \inf \left\{ t \in [0, T] : f(t) \not\preceq g_\delta(t) \right\}
\]

and using continuity of the functions \( f, g(\delta) \), observe that:

i. \( t_* \in (0, T) \), since \( f(0) = g_\delta(0) \);

ii. \( f(t) \preceq g_\delta(t) \) for all \( t \in [0, t_*] \);

iii. \( \exists t_* \in (t_*, T] \) such that \( (f_1(t) > g_\delta(t) \ \forall t \in (t_*, t_*)) \) or \( (f_2(t) > g_\delta(t) \ \forall t \in (t_*, t_*)) \).

To fix ideas, assume that in last condition the first eventuality is verified. Then by continuity again, we have:

iv. \( f_1(t_*) = g_\delta(t_*) \).

In particular, for every \( t \in (t_*, t_*) \)

\[
\frac{f_1(t) - f_1(t_*)}{t - t_*} > \frac{g_\delta(t) - g_\delta(t_*)}{t - t_*} .
\]

Now, by absolute continuity of \( f_1 \) and hypothesis (254) we have:

\[
\frac{f_1(t) - f_1(t_*)}{t - t_*} = \frac{1}{t - t_*} \int_{t_*}^{t} f_1(s) \, ds \leq \frac{1}{t - t_*} \int_{t_*}^{t} A_1(s, f(s)) \, ds \rightarrow A_1(t_*, f(t_*))
\]

where the limit holds true by continuity of \( A_1 \) and \( f \). On the other hand:

\[
\frac{g_\delta(t) - g_\delta(t_*)}{t - t_*} \rightarrow (g_\delta)'(t_*) = A_1(t_*, g_\delta(t_*)) + \delta \geq A_1(t_*, f(t_*)) + \delta
\]

where the latter inequality follows by the monotonicity hypothesis on \( A_1 \). Equations (263), (264), (265) entail the contradiction \( A_1(t_*, f(t_*)) \geq A_1(t_*, f(t_*)) + \delta \). This proves the claim. \( \square \)
5.2 Expectations of polynomials with respect to $\rho_t, \lambda$: proof of Theorem 4.1

Assumption (A2) provides polynomial bounds on the derivatives of $V$. Theorem 4.1 will follow if we show that any positive power of the variables $|x_1|, |x_2|$ has uniformly bounded expectations in the measure $\rho_{t, \lambda}$ for all $t > 0, \lambda \geq 1$. Because of the different time scales in the evolution of the variables $x_1, x_2$, we have to start by the following auxiliary

**Proposition 5.7.** For $x = (x_1, x_2) \in \mathbb{R}^d$ let

$$v_\lambda(x) \equiv \frac{1}{\lambda} |x_1|^2 + |x_2|^2.$$  \hspace{1cm} (266)

In the hypothesis of Theorem 1.2, for every $r \in [1, \infty)$ there exists a finite non-negative constant $M_{2r} = M_{2r}(\beta_1, \beta_2, \rho_I, V)$ such that for all $t > 0, \lambda \geq 1$

$$\int_{\mathbb{R}^d} v_\lambda(x)^r \rho_{t, \lambda}(x) \, dx \leq M_{2r}.$$ \hspace{1cm} (267)

In particular by assumption (A4) there exist $a \in (0, \infty), a_0 \in [0, \infty)$ such that for every $x \in \mathbb{R}^d$

$$x \cdot \nabla V(x) \geq a |x|^2 - a_0,$$ \hspace{1cm} (268)

then a suitable choice for $M_{2r}$ is given by

$$M_{2r} \equiv \int_{\mathbb{R}^d} |x|^{2r} \rho_1(x) \, dx + \frac{2^r (r-1)^{r-1}}{a^r r^r} \left( a_0 + \frac{d_1}{\beta_1} + \frac{d_2}{\beta_2} + \frac{2(r-1)}{\beta_1 \wedge \beta_2} \right)^r.$$ \hspace{1cm} (269)

**Proof of Proposition 5.7.** We are going to apply Proposition 5.2 to $v_\lambda^r$. Computing first and second order derivatives one finds

$$L_\lambda (v_\lambda(x)^r) = -\frac{2r}{\lambda} v_\lambda(x)^{r-1} x \cdot \nabla V(x) + \frac{2r}{\lambda} v_\lambda(x)^{r-1} \left( \frac{d_1}{\beta_1} + \frac{d_2}{\beta_2} \right) +$$

$$+ \frac{2r}{\lambda} v_\lambda(x)^{r-2} 2(r-1) \left( \frac{|x_1|^2}{\beta_1} + \frac{|x_2|^2}{\beta_2} \right)$$ \hspace{1cm} (270)

$$\leq -\frac{2r}{\lambda} v_\lambda(x)^{r-1} (x \cdot \nabla V(x) - k_r),$$

where we set $k_r \equiv \frac{d_1}{\beta_1} + \frac{d_2}{\beta_2} + \frac{2(r-1)}{\beta_1 \wedge \beta_2}$. Using assumption (A4) and taking $\lambda \geq 1$ we have for all $x \in \mathbb{R}^d$

$$x \cdot \nabla V(x) \geq a |x|^2 - a_0 \geq a v_\lambda(x) - a_0.$$ \hspace{1cm} (271)

Hence:

$$L_\lambda (v_\lambda(x)^r) \leq -\frac{2r}{\lambda} v_\lambda(x)^{r-1} \left( a v_\lambda(x) - a_0 - k_r \right)$$

$$\leq -\frac{2r}{\lambda} \left( \frac{a}{2} v_\lambda(x)^r - m_r \right)$$ \hspace{1cm} (272)
where we set $-m_r \equiv \min_{\xi \geq 0} \left( \frac{\xi}{a} v^r - m_r v^{r-1} \right) = -\left( \frac{2(r-1)}{a} \right)^{r-1} \left( \frac{m_r}{a} \right)^r$. Finally, from inequality (272) and Proposition 5.7 it follows that for all $t > r$ where for $\xi$ Proposition 5.10.

In the hypothesis of Theorem 1.2 plus Corollary 5.9.

Let $\xi$ be defined as in Corollary 5.8.

Proof. It follows by Proposition 5.7.

It follows by Proposition 5.7 since $1 - \frac{1}{2} \beta_1 + \frac{1}{2} \beta_2 M_2$.

Proof. It follows by Proposition 5.7 since $|x_2|^r \leq v_\chi(x_1, x_2)\xi$ for all $(x_1, x_2) \in \mathbb{R}^d$. In addition if $r \in [0, 2)$, one has $\xi \leq 1 - \frac{1}{2} \xi + \frac{1}{2} \xi^2$ for all $\xi \geq 0$.

Corollary 5.9. Let $r \in [0, \infty)$. In the hypothesis of Theorem 1.2 we have for all $\lambda \geq 1$

$$\sup_{t > 0, \lambda \geq 1} \int_{\mathbb{R}^d} |x_1|^r \rho_{t, \lambda}(x) \, dx \leq \lambda M_r,$$

where $M_r$ is defined as in Corollary 5.8.

Proof. It follows by Proposition 5.7 since $|x_1|^r \leq \lambda v_\chi(x_1, x_2)\xi$.

Proposition 5.10. In the hypothesis of Theorem 1.2 plus (A5), for every $r \in [0, \infty)$ there exists a finite non-negative constant $M'_r = M'_r(\beta_1, \beta_2, \rho_1, V)$ such that

$$\sup_{t > 0, \lambda \geq 1} \int_{\mathbb{R}^d} |x_1|^r \rho_{t, \lambda}(x) \, dx \leq M'_r.$$

In particular by assumption (A5) there exist $a_1 \in (0, \infty)$, $a_0, a_2, p, \in [0, \infty)$ such that for every $(x_1, x_2) \in \mathbb{R}^d$

$$x_1 \cdot \nabla_1 V(x_1, x_2) \geq a_1 |x_1|^2 - a_2 |x_2|^p - a_0,$$

then a suitable choice for $M'_r$ is given by:

$$M'_r \equiv \begin{cases} \int_{\mathbb{R}^d} |x_1|^r \rho(x) \, dx + \frac{1}{a_1} (a_2 M_{\frac{4}{a_1}} + r m_r) & \text{if } r > 2 \\ \int_{\mathbb{R}^d} |x_1|^2 \rho(x) \, dx + \frac{1}{a_1} (a_2 M_p + a_0 + \frac{d}{p}) & \text{if } r = 2 \\ 1 - \frac{\xi}{2} + \frac{\xi}{2} M'_r & \text{if } r < 2 \end{cases}$$

where $M_{\frac{4}{a_1}}$ is defined by Corollary 5.8 and $m_r \equiv - \min_{\xi > 0} \left( \frac{\xi}{a} v^r - a_2 \frac{r-2}{r} \xi^{r-1} - (a_0 + \frac{d}{p} \xi^{-2}) \xi^{-2} \right)$.
Proof. Let $r > 2$. We want to apply Proposition [5.1] to

$$v(x) \equiv |x|^r. \quad (280)$$

First of all observe that $v$ satisfies the integrability hypothesis (239) thanks to Corollary [5.9] and assumption (A2) for $\nabla V$. Now, computing first and second derivatives we have

$$L v(x) = -r |x|^{r-2} (x_1 \cdot \nabla_1 V(x) - k_r), \quad (281)$$

setting $k_r \equiv \frac{d_1 + r - 2}{a_1}$. By assumption (A5)

$$x_1 \cdot \nabla_1 V(x) \geq a_1 |x|^2 - a_2 |x|^p - a_0, \quad (282)$$

hence

$$L v(x) \leq -r \left( a_1 |x|^r - a_2 |x|^{r-2} |x|^p - (a_0 + k_r) |x|^{r-2} \right). \quad (283)$$

Young’s inequality guarantees that

$$|x|^{r-2} |x|^p \leq \frac{1}{\sigma} |x|^{(r-2)\sigma} + \frac{1}{\tau} |x|^p \tau$$

for every $\sigma, \tau > 1$, $\sigma^{-1} + \tau^{-1} = 1$. For example, $\sigma \equiv \frac{1}{r-2}$ ensures $(r-2) \sigma < r$ and enforces $\tau \equiv \frac{1}{2}$. For this choice of $\sigma, \tau$ we obtain:

$$L v(x) \leq -r \left( a_1 |x|^r - a_2 \frac{r-2}{r-1} |x|^{r-1} - a_2 \frac{2}{r} |x|^p - (a_0 + k_r) |x|^{r-2} \right) \leq -r \left( \frac{a_1}{2} |x|^r - \frac{2a_2}{r} |x|^\frac{r}{2} - m_r \right) \quad (285)$$

where we set $-m_r \equiv \min_{t \geq 0} \left( \frac{2a_2}{r} \xi^r - a_2 \frac{r-2}{r-1} \xi^{r-1} - (a_0 + k_r) \xi^{r-2} \right)$. By Corollary [5.8] we know that for all $t > 0$, $\lambda \geq 1$

$$\int_{\mathbb{R}^d} |x|^\frac{r}{2} \rho_t(x,x_2) \; dx_1 \; dx_2 \leq M_{pr/2}. \quad (286)$$

Therefore by Proposition [5.1] and Remark [5.3] we have for all $t > 0$, $\lambda \geq 1$

$$\int_{\mathbb{R}^d} |x|^r \rho_t(x,x_2) \; dx \leq e^{-\frac{r}{ra_1}} \int_{\mathbb{R}^d} |x|^r \rho_t(x) \; dx + \left( 1 - e^{-\frac{r}{ra_1}} \right) \frac{4a_2 M_{pr/2} + r m_r}{ra_1} \leq \int_{\mathbb{R}^d} |x|^r \rho_t(x) \; dx + \frac{4a_2 M_{pr/2} + r m_r}{ra_1}, \quad (287)$$

which concludes the proof in the case $r > 2$. If $r = 2$, Young inequality is not needed and inequality (283) suffices to apply Proposition [5.1] and Remark [5.3] obtaining for all $t > 0$, $\lambda \geq 1$

$$\int_{\mathbb{R}^d} |x|^2 \rho_t(x,x_2) \; dx \leq e^{-2a_1 t} \int_{\mathbb{R}^d} |x|^2 \rho_t(x) \; dx + \left( 1 - e^{-2a_1 t} \right) \frac{a_2 M_p + k_2}{a_1} \leq \int_{\mathbb{R}^d} |x|^2 \rho_t(x) \; dx + \frac{a_2 M_p + k_2}{a_1}. \quad (288)$$

Finally, if $r < 2$ we can just use the bound $|x|^r \leq 1 - \frac{r}{2} + \frac{r}{2} |x|^2$ and come back to the previous case. \[\square\]
Proof of Theorem 4.1. It follows by combining assumption (A2) with Corollary 5.8 and Proposition 5.10.

Proof of Proposition 4.4. We rely on Corollary 7.3.8 in [32] that ensures
\[ \|\rho\|_{L^\infty(R^d \times (0,T))} < \infty, \] (289)
provided \( \rho_1 \) is bounded on \( R^d \) (assumption (B1)) and
\[ \int_0^T \int_{R^d} |\nabla V(x)|^{d+3} \rho_t(x) \, dx \, dt < \infty, \] (290)
which in turn follows from Corollaries 5.8, 5.9 together with assumption (A2). We also make use of the inequality \(|\log \xi|^2 \xi \leq 4 \sqrt{\xi}\) for \( \xi \in [0,1] \). Therefore:
\[ \int \{\rho_t \geq 1\} |\log \rho_t(x)|^2 \rho_t(x) \, dx \leq (\log \|\rho\|_{L^\infty(R^d \times (0,T))})^2; \] (291)
\[ \int \{\rho_t \leq e^{-|x|}\} |\log \rho_t(x)|^2 \rho_t(x) \, dx \leq \int_{R^d} e^{-|\frac{|x|}{\sigma}} \, dx; \] (292)
\[ \int \{e^{-|x|} < \rho_t < 1\} |\log \rho_t(x)|^2 \rho_t(x) \, dx \leq \int_{R^d} |x|^2 \rho_t(x) \, dx. \] (293)
The r.h.s. of (291) is finite by (289). The r.h.s. of (293) has finite supremum over \( t \in (0,T) \) by Corollaries 5.8, 5.9. This concludes the proof.

Proof of Theorem 4.5. We refer to Theorem 7.4.1 in [32]. It ensures that (141) is a consequence of two integrability conditions:
\[ \int_0^T \int_{R^d} |\nabla V(x)|^2 \rho_t(x) \, dx \, dt < \infty, \] (294)
\[ \int_0^T \int_{R^d} \log^2 (1,|x|) \rho_t(x) \, dx \, dt < \infty \] (295)
which in turn follow from Corollaries 5.8, 5.9.

Proof of Proposition 4.9. Since \( \rho^{(2)} \) is solution of a suitable Fokker-Planck equation with drift \( \frac{1}{\sqrt{2}} (\nabla_2 V)_{t} \) (Theorem 4.6), Corollary 7.3.8 in [32] ensures that
\[ \|\rho^{(2)}\|_{L^\infty(R^{d_2} \times (0,T))} < \infty, \] (296)
provided \( \rho^{(2)}_1 \) is bounded on \( R^{d_2} \) (assumption (B2)) and
\[ \int_0^T \int_{R^{d_2}} |(\nabla_2 V)_t(x_2)|^{d+3} \rho^{(2)}_t(x_2) \, dx_2 \, dt < \infty, \] (297)
which is true by Corollaries 5.8, 5.9 and Jensen inequality. The proof is then concluded by miming the Proof of Proposition 4.4 above.
Proof of Theorem 4.10. Being \( \rho^{(2)} \) solution of a suitable Fokker-Planck equation, we may refer to Theorem 7.4.1 in [32]. The latter ensures that (154) follows from two integrability conditions:

\[
\int_0^T \int_{\mathbb{R}^d} |(\nabla_2 V)_t(x_2)|^2 \rho_t^{(2)}(x_2) \, dx_2 \, dt < \infty , \tag{298}
\]

\[
\int_0^T \int_{\mathbb{R}^d} \log^2 \max(1,|x_2|) \rho_t^{(2)}(x_2) \, dx_2 \, dt < \infty \tag{299}
\]

which in turn follow from Corollaries 5.8, 5.9.

5.3 Expectations of polynomials with respect to \( \rho_s^{(2)} \rho_{t,\lambda}^{(2)} \): proof of Theorem 4.2

Assumption (A2) ensures polynomial bounds for the derivatives of \( V \). We show that the conditional expectation of a polynomial in \( |x_1| \) with respect to the measure \( \rho_{s,t}^{(1)}(x_1|x_2) \) is bounded by a polynomial in \( |x_2| \). Then from Subsection 5.2 we already know that this quantity has uniformly bounded expectations with respect to the measure \( \rho_{s,t}^{(2)} \) for all \( t > 0, \lambda \geq 1 \), hence we can prove Theorem 4.2. In particular the assertion about derivatives of the effective potential \( F \) follows as they can be expressed in terms of conditional expectation of products of derivatives of \( V \).

Proposition 5.11. Suppose that \( V \) verifies assumptions (A2) for \( \nu = 0 \) and (A3). Then for every \( r \in [0,\infty) \) there exist \( s_r = s_r(V) \in [0,\infty) \) and two finite non-negative constants \( C_{0, r} = C_{0, r}(\beta_1, V), C_{1, r} = C_{1, r}(V) \) such that for every \( x_2 \in \mathbb{R}^d_2 \)

\[
\int_{\mathbb{R}^d_1} |x_1|^r \rho_x^{(1)}(x_1|x_2) \, dx_1 \leq C_{0, r} + C_{1, r} |x_2|^{s_r} . \tag{300}
\]

In particular by assumptions (A2) for \( \nu = 0 \) and (A3) there exist \( a_1, a_2 \in (0, \infty), a_0 \in [0, \infty), m_1, m_2 \in [2, \infty), \gamma_0, \gamma_1, \gamma_2 \in [0, \infty) \) such that for every \( (x_1, x_2) \in \mathbb{R}^d \)

\[
a_1 |x_1|^2 + a_2 |x_2|^2 - a_0 \leq V(x_1, x_2) \leq \gamma_1 |x_1|^{m_1} + \gamma_2 |x_2|^{m_2} + \gamma_0 , \tag{301}
\]

then a suitable choice for \( s_r, C_{0, r}, C_{1, r} \) is given by:

\[
s_r \equiv r \frac{m_2}{2} , \quad C_{1, r} \equiv \left( \frac{2\gamma_2}{a_1} \right)^{\frac{r}{2}}, \quad C_{0, r} \equiv \frac{(\beta_1 \gamma_1)^{\frac{r}{2}} \Gamma(\frac{r+1}{2})}{2 \left( \frac{\beta_1 a_1}{2} \right)^{\frac{r}{2}} \Gamma(1 + \frac{r}{m_1})} e^{\beta_1 (a_0 + \gamma_0)} . \tag{302}
\]

Proof. Let \( \Delta > 0 \) and consider the set

\[
A_{x_2} \equiv \left\{ x_1 \in \mathbb{R}^d_1 \mid |x_1|^2 \geq \Delta |x_2|^{m_2} \right\} . \tag{303}
\]

for every \( x_2 \in \mathbb{R}^d_2 \). Integrating over its complementary set we have

\[
\int_{\mathbb{R}^d_1 \setminus A_{x_2}} |x_1|^r \rho_x^{(1)}(x_1|x_2) \, dx_1 \leq \Delta^{\frac{r}{2}} |x_2|^\frac{m_2}{2} . \tag{304}
\]
Now we evaluate the contribution of the integral over $A_{x_2}$. By assumption (A3)

\[
\int_{A_{x_2}} |x_1|^r e^{-\beta_1 V(x_1, x_2)} \, dx_1 \leq \int_{A_{x_2}} |x_1|^r e^{-\beta_1 (a_1 |x_1|^2 + a_2 |x_2|^2 - a_0)} \, dx_1 \leq I_r e^{-\beta_1 \left( \frac{a_1}{r} \Delta |x_2|^m + a_2 |x_2|^2 \right)} ,
\]

where we set

\[
I_r \equiv \int_{\mathbb{R}^d} |x_1|^r e^{-\beta_1 \left( \frac{a_1}{r} |x_1|^2 - a_0 \right)} \, dx_1 = |S_{d-1}| \frac{\Gamma \left( \frac{r+1}{2} \right)}{\left( \frac{a_1}{2r} \right)^{\frac{r}{2}}} e^{\beta_1 a_0} .
\]

On the other hand, by assumption (A2) with $\nu = 0$,

\[
\int_{\mathbb{R}^d} e^{-\beta_1 V(x_1, x_2)} \, dx_1 \geq \int_{\mathbb{R}^d} e^{-\beta_1 \left( \gamma_1 |x_1|^{n_1} + \gamma_2 |x_2|^{m_2} + \gamma_0 \right)} \, dx_1 = J_r e^{-\beta_1 \gamma_2 |x_2|^m} ,
\]

where we set

\[
J_r \equiv \int_{\mathbb{R}^d} e^{-\beta_1 \left( \gamma_1 |x_1|^{n_1} + \gamma_0 \right)} \, dx_1 = |S_{d-1}| \frac{\Gamma \left( 1 + \frac{1}{m_1} \right)}{\left( \beta_1 \gamma_1 \right)^{\frac{1}{m_1}}} e^{-\beta_1 \gamma_0} .
\]

By inequalities (305), (307) it follows that:

\[
\int_{A_{x_2}} |x_1|^r \rho_r^{(1)}(x_1|x_2) \, dx_1 = \frac{\int_{A_{x_2}} |x_1|^r e^{-\beta_1 V(x_1, x_2)} \, dx_1}{\int_{\mathbb{R}^d} e^{-\beta_1 V(x_1, x_2)} \, dx_1} \leq \frac{I_r}{J_r} e^{-\beta_1 \left( \frac{a_1}{r} \Delta |x_2|^m + a_2 |x_2|^2 - \gamma_2 |x_2|^m \right)} \leq \frac{I_r}{J_r}
\]

where the last inequality holds true for any $\Delta > \frac{a_2}{a_1}$. Finally, summing inequalities (304), (309) concludes the proof. \qed

Remark 5.12. If the potential $V(x_1, x_2)$ grows faster than quadratically in $x_1$, the exponent $s_r$ in Proposition 5.11 can be improved. Precisely if there are $n_1 \in [2, \infty)$, $b_2, b_1, b_0 \in [0, \infty)$ such that

\[
V(x_1, x_2) \geq b_1 |x_1|^{n_1} - b_2 |x_2|^{m_2} - b_0
\]

for all $(x_1, x_2) \in \mathbb{R}^d$, then inequality (300) holds true with

\[
s_r \equiv \frac{m_2}{n_1} r
\]
changing also the constants $C_{1,r}$, $C_{0,r}$ accordingly. Indeed the proof of Proposition 5.11 can be suitably modified, starting by considering the set

$$\tilde{A}_{x_2} \equiv \left\{ x_1 \in \mathbb{R}^{d_1} \mid |x_1|^{n_1} \geq \Delta |x_2|^{m_2} \right\}.$$  

(312)

**Proof of Theorem 5.2** Equation (137) follows combining assumption (A2) with Proposition 5.11 and Corollary 5.8.

Equation (138) for $|v| = 0$ follows from assumptions (A2), (A3) which guarantee:

$$a_1 |x_1|^2 + a_2 |x_2|^2 - a_0 \leq V(x_1, x_2) \leq \gamma_1 |x_1|^r + \gamma_2 |x_2|^r + \gamma_0.$$  

(313)

Therefore:

$$F(x_2) = -\frac{1}{\beta_1} \log \int_{\mathbb{R}^{d_2}} e^{-\beta_1 V(x_1, x_2)} \, dx_1 \left\{ \begin{array}{ll} \leq C_0 + \gamma_2 |x_2|^r \\ \geq C_1 + a_2 |x_2|^2 \end{array} \right.$$  

(314)

where $C_0 \equiv \gamma_0 - \frac{1}{\beta_1} \int_{\mathbb{R}^{d_1}} e^{-\beta_1 \gamma_1 |x_1|^r} \, dx_1$, $C_1 \equiv -a_0 - \frac{1}{\beta_1} \int_{\mathbb{R}^{d_1}} e^{-\beta_1 |x_1|^2} \, dx_1$ are finite constants. Inequalities (314) combined with Corollary 5.8 ensure that

$$\sup_{\lambda > 0, A \geq 1} \int_{\mathbb{R}^{d_2}} |F(x_2)|^\alpha \rho_{t, A}(x_2) \, dx_2 < \infty.$$  

(315)

Finally, equation (138) for $|v| \geq 1$ is an application of the multivariate Faà Di Bruno formula (see (51) and references therein):

$$D^\nu_{x_2} F(x_2) = \sum_{\pi \in P([\nu])} \frac{(-1)^{|\pi|} (|\pi| - 1)!}{\beta_1} \prod_{A \in \pi} \sum_{\sigma \in P(A)} (-\beta_1)^{|\sigma|} \int_{\mathbb{R}^{d_2}} \prod_{B \in \sigma} D^{|B|}_{x_2} V(x_1, x_2) \rho_{t, A}(x_1 | x_2) \, dx_1$$  

(316)

where: for a multi-index $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{N}^d \setminus \{0\}$, $[\nu]$ denotes the set where each number from 1 to $d_2$ is repeated in a distinguishable way according to its multiplicity encoded in $\nu$, i.e., $\{1^{(\nu_1)}, \ldots, d_2^{(\nu_{d_2})}\}$; conversely for a set $B \subseteq [\nu]$, $[B]$ denotes the multi-index $\{ |i : 1^{(\nu_i)} \in B|, \ldots, |i : d_2^{(\nu_{d_2})} \in B| \} \in \mathbb{N}^d$; $P(A)$ denotes the set of partitions of a set $A$. Assuming without loss of generality $s \geq 1$, equation (138) for $|v| \geq 1$ then follows from expressions (316) using Jensen inequality and equation (317).

5.4 **Regularity of $\rho$: proof of Theorem 1.5**

In this and all the following Subsections we will denote

$$A \equiv (A \beta)^{-1} = \begin{pmatrix} \frac{1}{\alpha^2} I_{d_1} & 0 \\ 0 & \frac{1}{\beta^2} I_{d_2} \end{pmatrix}, \quad b(x) \equiv \Lambda^{-1} \nabla V(x) = \begin{pmatrix} \nabla_1 V(x) \\ \frac{\beta}{\alpha} \nabla_2 V(x) \end{pmatrix}.$$  

(317)

The Fokker-Planck operator (27) rewrites as

$$L \varphi = \nabla \cdot (A \nabla \varphi) - b \cdot \nabla \varphi.$$  

(318)
This compact notation is convenient due to the generality of results.

In the present Subsection we prove the regularity of $\rho$ following the argument sketched in [35]. Instead of the standard heat kernel used there, we introduce

$$G_t(x) \equiv \frac{1}{(4\pi t)^{\frac{d}{2}} (\det A)^{\frac{1}{2}}} \exp \left( -\frac{x^T A^{-1} x}{4t} \right)$$  \hspace{1cm} (319)

for $x \in \mathbb{R}^d$, $t > 0$.

**Remark 5.13.** $G_t(x)$ is the fundamental solution of the diffusion equation

$$\partial_t G_t(x) = \nabla \cdot (A \nabla G_t(x)) \, .$$  \hspace{1cm} (320)

Precisely we have:

$$\nabla G_t(x) = -\frac{A^{-1} x}{2t} G_t(x) \, ,$$  \hspace{1cm} (321)

$$\nabla \cdot (A \nabla G_t(x)) = \left( -\frac{d}{2t} + \frac{A^{-1} x}{4t^2} \right) G_t(x) = \partial_t G_t(x) \, .$$  \hspace{1cm} (322)

Moreover, for all $f \in L^1(\mathbb{R}^d)$ we have:

$$\|f \ast G_t - f\|_{L^1(\mathbb{R}^d)} \xrightarrow{t \to 0} 0 \, ,$$  \hspace{1cm} (323)

where $\ast$ denotes the convolution with respect to space variables only:

$$(f \ast G_t)(x) \equiv \int_{\mathbb{R}^d} G_t(x-y) f(y) \, dy \, .$$  \hspace{1cm} (324)

**Lemma 5.14.** Let $p \geq 1$. For all $t > 0$ we have

$$\|G_t\|_{L^p(\mathbb{R}^d)} = t^{-\frac{d}{2}} (p-1) \|G_1\|_{L^p(\mathbb{R}^d)} \, ,$$  \hspace{1cm} (325)

$$\|\nabla G_t\|_{L^p(\mathbb{R}^d)} = t^{-\frac{d}{2}} (1-\frac{1}{p}) - \frac{1}{2} \|\nabla G_1\|_{L^p(\mathbb{R}^d)} \, .$$  \hspace{1cm} (326)

**Proof.** Direct computation using (319), (321) and performing the change of variable $x' = x/\sqrt{t}$. \hfill \square

**Lemma 5.15.** Let us denote

$$(f \ast G_t)(x) \equiv \int_{-\infty}^{t} (f_s \ast G_{t-s})(x) \, ds \, .$$  \hspace{1cm} (327)

Let $p > 1$. There exist constants $c_1 = c_1(p,d)$, $c_2 = c_2(p,d) \in (0,\infty)$ such that:

$$\|\partial_t(f \ast G)\|_{L^p(\mathbb{R}^d \times \mathbb{R})} + \sum_{i,j=1}^{d} \|\partial_{x_i} \partial_{x_j} (f \ast G)\|_{L^p(\mathbb{R}^d \times \mathbb{R})} \leq c_1 \|f\|_{L^p(\mathbb{R}^d \times \mathbb{R})}$$  \hspace{1cm} (328)

for every $f \in L^p(\mathbb{R}^d \times \mathbb{R})$, and:

$$\|\partial_t(\phi \ast G)\|_{L^p(\mathbb{R}^d \times (0,\infty))} + \sum_{i,j=1}^{d} \|\partial_{x_i} \partial_{x_j} (\phi \ast G)\|_{L^p(\mathbb{R}^d \times (0,\infty))} \leq c_2 \sum_{i=1}^{d} \|\partial_{x_i} \phi\|_{L^p(\mathbb{R}^d)}$$  \hspace{1cm} (329)

for every $\phi \in W^{1,p}(\mathbb{R}^d)$.
Proof. We refer to [52] Chapter IV Section 3, where the estimates (3.1), (3.2) coincide respectively with (328), (329) provided the matrix $A$ is replaced by the identity in the definition of $G$. The general case of a symmetric positive definite matrix $A$ can be deduced by performing the change of variable $y' = A^{-\frac{1}{2}}y$ in the convolution. Notice that in the case of interest for the present paper $A$ is diagonal and the constants $c_1, c_2$ do not depend on its entries. \hfill $\square$

Remark 5.16. The weak formulation (29) of FP equation extends to the following:

$$
\int_{\mathbb{R}^d} \varphi_t(y) \rho_t(y) \, dy - \int_{\mathbb{R}^d} \varphi_{t_0}(y) \rho_{t_0}(y) \, dy = \int_{t_0}^t \int_{\mathbb{R}^d} \left( \partial_s \varphi_s + L \varphi_s \right)(y) \rho_s(y) \, dy \, ds ,
$$

for every $t > t_0 > 0$, for every $\varphi \in C^{2,1}(\mathbb{R}^d \times (0, \infty))$ such that $\text{supp} \varphi_s \subset K$ compact subset of $\mathbb{R}^d$ for all $s > 0$. See, e.g., Proposition 6.1.2 and following remarks in [32].

Proof of Theorem 1.5 part i. Let $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $\delta \in (0, 1)$. Let $x \in \mathbb{R}^d$, $0 < t - \frac{1}{2} < t_0 < t_1 < t < T' < T$ and set

$$
\varphi_s(y) \equiv G_{\delta + s - t}(x - y) \eta(y)
$$

for all $y \in \mathbb{R}^d$, $s \in (0, t]$. Since $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times (0, t])$, $\text{supp} \varphi_s \subseteq \text{supp} \eta$, identity (330) holds true. Computing $L \varphi$ and using (320), one obtains:

$$
\left( (\eta \rho_t) \ast G_\delta \right)(x) = \left( (\eta \rho_{t_0}) \ast G_{\delta + t - t_0} \right)(x) +
$$

$$
+ \int_{t_0}^t \left( (L \eta \rho_s) \ast G_{\delta + s - t} \right)(x) \, ds + \int_{t_0}^t \left( (M \eta \rho_s) \ast \nabla G_{\delta + s - t} \right)(x) \, ds ,
$$

(332)

setting

$$
M \eta(y) \equiv -2A \nabla \eta(y) + b(y) \eta(y).
$$

(333)

As $\delta \to 0$ we have $(\eta \rho_t) \ast G_\delta \to \eta \rho_t$ in $L^1(\mathbb{R}^d)$ and also in $L^p(\mathbb{R}^d \times (t_0, T))$ by dominate convergence (this will be clear adapting the following argument with $p = 1$). Thus:

$$
(\eta \rho_t)(x) = \left( (\eta \rho_{t_0}) \ast G_{t - t_0} \right)(x) +
$$

$$
+ \int_{t_0}^t \left( (L \eta \rho_s) \ast G_{t - s} \right)(x) \, ds + \int_{t_0}^t \left( (M \eta \rho_s) \ast \nabla G_{t - s} \right)(x) \, ds ,
$$

(334)

for almost every $(x, t) \in \mathbb{R}^d \times (t_0, T)$.

\footnote{Actually (3.2) in [52] is expressed in terms of the Slobodeckij seminorm and is stronger than (329) here.}
Now let $p \in (1, \frac{d}{d-1})$. Using Young’s convolution inequality in (334) we find:

$$
\| \eta \rho_t \|_{L^p(\mathbb{R}^d)} \leq \| \eta \rho_{t_0} \|_{L^1(\mathbb{R}^d)} \| G_{t-t_0} \|_{L^p(\mathbb{R}^d)} + \\
+ \int_{t_0}^t \| \nabla (\eta \rho_s) \|_{L^1(\mathbb{R}^d)} \| G_{t-s} \|_{L^p(\mathbb{R}^d)} \, ds + \\
+ \int_{t_0}^t \| M \eta \rho_s \|_{L^1(\mathbb{R}^d)} \| \nabla G_{t-s} \|_{L^p(\mathbb{R}^d)} \, ds.
$$

(335)

Then, using Lemma 5.14 we obtain:

$$
\| \eta \rho_t \|_{L^p(\mathbb{R}^d)} \leq \| \eta \|_{\infty} \| G_1 \|_{L^p(\mathbb{R}^d)} (t-t_0)^{-\frac{d}{p} (p-1)} + \\
+ \| \nabla \|_{\infty} \| G_1 \|_{L^p(\mathbb{R}^d)} \frac{(t-t_0)^{1-\frac{d}{p} (p-1)}}{1 - \frac{d}{2} (p-1)} + \\
+ \| M \eta \|_{\infty} \| \nabla G_1 \|_{L^p(\mathbb{R}^d)} \frac{(t-t_0)^{\frac{d}{2} (1-\frac{1}{p})}}{\frac{d}{2} - \frac{d}{2} (1-\frac{1}{p})},
$$

(336)

since our choice of $p$ guarantees $1 - \frac{d}{2} (p-1) > 0$ and $\frac{1}{2} - \frac{d}{2} (1-\frac{1}{p}) > 0$. Observing that $\| f \|_{L^p(\mathbb{R}^d \times (t_1, T))} = \int_{t_1}^T \| f_t \|_{L^p(\mathbb{R}^d)} \, dt$, inequality (336) shows that $\eta \rho \in L^p(\mathbb{R}^d \times (t_1, T))$. By arbitrariness of $\eta, t_1, T$, one concludes

$$
\rho \in L^p_{\text{loc}}(\mathbb{R}^d \times (0, \infty))
$$

(337)

(without knowing a priori the continuity of $\rho$). In (335) a bootstrap argument that uses equation (334) iteratively together with Lemmas 5.14 5.15 is invoked. Let us detail the first steps. With the notation introduced in Lemma 5.14 equation (334) rewrites as

$$
(\eta \rho_t)(x) = \left( (\eta \rho_{t_0}) * G_{t-t_0} \right)(x) + \left( (\nabla \eta \rho \, \chi) \otimes G \right)_t(x) + \\
+ \sum_{k=0}^d \left( (M \eta \rho \, \chi) \otimes \partial_x G \right)_t(x) + R_t(x)
$$

(338)

where: $\chi \in C^\infty(\mathbb{R})$ is such that $\chi_s = 1$ for $t_0 \leq s < T$, $\chi_s = 0$ for $s \leq t_0$ or $s \geq T$, and we set

$$
R_t(x) \equiv \int_{t_0}^t \left( (\nabla \eta \rho \, \chi_s) * G_{t-s} + (M \eta \rho \, \chi_s) * \nabla G_{t-s} \right)(x) \, ds,
$$

(339)

$M \eta = (M \eta)_{k=1}^d, \quad x = (x_k)_{k=1}^d \in \mathbb{R}^d$. We claim that $\partial_x (\eta \rho)$ for $i = 1, \ldots, d$ exist in $L^p_{\text{loc}}(\mathbb{R}^d \times (t_1, T))$. The first and the last term on the r.h.s. of (338) cannot cause any problem because the kernel $G$ remains far from its singularity. Let us focus on the other terms. By Young’s convolution inequality and Lemma 5.14 we have:

$$
\| (\nabla \eta \rho \, \chi) \otimes \partial_x G \|_{L^p(\mathbb{R}^d)} \leq \int_{t_1}^t \| \nabla \eta \rho \|_{L^p(\mathbb{R}^d)} \| \partial_x G \|_{L^p(\mathbb{R}^d)} \, ds
$$

$$
\leq \| \nabla \eta \|_{\infty} \| \partial_x G \|_{L^p(\mathbb{R}^d)} \frac{(t-t_0)^{\frac{d}{2} (1-\frac{1}{p})}}{1 - \frac{d}{2} (p-1)},
$$

(340)

51
hence \((L \eta \rho \chi) \odot \partial_x G \in L^p(\mathbb{R}^d \times (t_1, T'))\). By Lemma 5.14 we have:

\[
\| (M_k \eta \rho \chi) \odot \partial_x \partial_x G \|_{L^p(\mathbb{R}^d \times (t_1, T'))} \leq c_1 \| M_k \eta \rho \chi \|_{L^p(\mathbb{R}^d \times \mathbb{R})} \\
\leq c_1 \| \chi \|_{L^\infty} \| M_k \eta \rho \|_{L^p(\mathbb{R}^d \times (t_1, T'))},
\]

(341)

which is finite since we know (337). Therefore (340), (341) used in (338) entail that it exists \(\partial_{x_i}(\eta \rho) \in L^p(\mathbb{R}^d \times (t_1, T'))\). By arbitrariness of \(\eta, t_1, T'\) it follows

\[
\partial_{x_i} \rho \in L^p_{\text{loc}}(\mathbb{R}^d \times (0, \infty)) .
\]

(342)

Now we claim that \(\partial_t(\eta \rho)\) exists in \(L^p_{\text{loc}}(\mathbb{R}^d \times (t_1, T'))\). By Lemma 5.15 we have:

\[
\| \partial_t ((L \eta \rho \chi) \odot G) \|_{L^p(\mathbb{R}^d \times (t_1, T'))} \leq c_1 \| \chi \|_{L^\infty} \| L \eta \rho \|_{L^p(\mathbb{R}^d \times (t_1, T'))},
\]

(343)

\[
\| \partial_t ((M_k \eta \rho \chi) \odot \partial_x G) \|_{L^p(\mathbb{R}^d \times (t_1, T'))} \leq c_1 \| \chi \|_{L^\infty} \| \partial_x (M_k \eta \rho) \|_{L^p(\mathbb{R}^d \times (t_1, T'))},
\]

(344)

which are finite by (337), (341). Using the previous inequalities in (338) proves the claim. By arbitrariness of \(\eta, t_1, T'\) it follows

\[
\partial_t \rho \in L^p_{\text{loc}}(\mathbb{R}^d \times (0, \infty)) .
\]

(345)

In the same way one proves that \(\partial_{x_i} \partial_x \rho \in L^p_{\text{loc}}(\mathbb{R}^d \times (0, \infty))\). Iterating this argument up to space-time derivatives of any order (since the coefficients of \(L, M\) are smooth) one obtains

\[
\rho \in W^{r, p}_{\text{loc}}(\mathbb{R}^d \times (0, \infty))
\]

(346)

for all \(r \in \mathbb{N}\). Then general Sobolev inequalities entail \(\rho \in C^\infty(\mathbb{R}^d \times (0, \infty))\), concluding the proof.

\[\square\]

**Proof of Theorem 1.5 part ii.** Let \(t_0 \to 0\) in equation (334). By property (33) and uniform continuity of \(G\) on compact subsets of \(\mathbb{R}^d \times (0, \infty)\) we have:

\[
(\eta \rho_t)(x) = ((\eta \rho_t) \ast G_t)(x) + \int_0^t ((L \eta \rho_s) \ast G_{t-s})(x) \, ds + \\
+ \int_0^t ((M \eta \rho_s) \ast \nabla G_{t-s})(x) \, ds.
\]

(347)

Then Young’s convolution inequality and Lemma 5.14 entail

\[
\| \eta \rho_t - (\eta \rho_t) \ast G_t \|_{L^1(\mathbb{R}^d)} \leq \| L \eta \|_{L^\infty} \| G_1 \|_{L^1(\mathbb{R}^d)} t + \| M \eta \|_{L^\infty} \| \nabla G_1 \|_{L^1(\mathbb{R}^d)} 2\sqrt{t} \quad \overset{t \to 0}{\longrightarrow} 0 .
\]

(348)

On the other hand \((\eta \rho_t) \ast G_t \to \eta \rho_t\) in \(L^1(\mathbb{R}^d)\) as \(t \to 0\), hence

\[
\| \eta \rho_t - \eta \rho_t \|_{L^1(\mathbb{R}^d)} \overset{t \to 0}{\longrightarrow} 0 .
\]

(349)
Now, consider a sequence $\eta_N \in C^\infty_c(\mathbb{R}^d)$ such that $\eta_N(x) = 1$ for $|x| \leq N$, $0 \leq \eta_N \leq 1$. We have:

$$
\|\rho_t - \rho_1\|_{L^1(\mathbb{R}^d)} \leq \frac{S}{N} + \|\eta_N \rho_t - \eta_N \rho_1\|_{L^1(\mathbb{R}^d)} + \frac{S_0}{N} \quad \text{as} \quad t \to 0 \quad \frac{S + S_0}{N} \quad \tag{350}
$$

where:

$$
S \equiv \sup_{t \in (0,T)} \int_{\mathbb{R}^d} |x| \rho_t(x) \, dx, \quad S_0 \equiv \int_{\mathbb{R}^d} |x| \rho_1(x) \, dx \quad \tag{351}
$$

are finite by Proposition 5.7 and assumption (B3) respectively. Inequality (350) concludes the proof by arbitrariness of $N \in \mathbb{N}$.

5.5 Regularity of $\rho^{(2)}$: proof of Theorem 4.8

In this Subsection the marginal $\rho^{(2)}_t(x_2) = \int_{\mathbb{R}^d} \rho_t(x_1, x_2) \, dx_1$ is shown to be in $C^\infty(\mathbb{R}^d \times (0, \infty))$. By Theorem 4.4 we already know that $\rho$ is smooth, hence suitable pointwise estimates for its derivatives would be sufficient to conclude. Another way would be to appeal to the fact that $\rho^{(2)}$ is itself weak solution of a Fokker-Planck equation (Theorem 4.10), but the drift coefficient of this equation involves $(\nabla_2 V)_t(x_2) = \int_{\mathbb{R}^d} \nabla V(x_1, x_2) \rho_t(x_1, x_2) \, dx_1$ which a priori is also not known to be smooth. We will modify the bootstrap argument of the previous Subsection (sketched in (355)), obtaining that both $\rho^{(2)}$, $(\nabla_2 V)_t$ are smooth. Precisely we consider the class of functions $\eta \in \mathcal{C}$ defined by the following two conditions:

i. $\eta \in C^\infty(\mathbb{R}^d)$;

ii. for every multi-index $\nu \in \mathbb{N}^d$ there exist $r_1, r_2, \gamma_0, \gamma_1, \gamma_2 \in [0, \infty)$ such that

$$
|D^\nu \eta(x_1, x_2)| \leq \gamma_0 + \gamma_1 |x_1|^{r_1} + \gamma_2 |x_2|^{r_2} \quad \tag{352}
$$

for all $(x_1, x_2) \in \mathbb{R}^d$.

Setting

$$
(\mathcal{I}_t \eta)(x_2) \equiv \int_{\mathbb{R}^d} \eta(x_1, x_2) \, \rho_t(x_1, x_2) \, dx_1, \quad \tag{353}
$$

we will prove that

$$
\mathcal{I} \eta \in C^\infty(\mathbb{R}^d \times (0, \infty)) \quad \forall \eta \in \mathcal{C}. \quad \tag{354}
$$

In particular it will follow that $\rho^{(2)}_t(x_2)$ and $(\nabla_2 V)_t(x_2)$ are in $C^\infty(\mathbb{R}^d \times (0, \infty))$ (by taking respectively $\eta \equiv 1$, $\eta \equiv \nabla_2 V$).

Remark 5.17. In order to build a bootstrap argument, it is essential to notice that the class of functions $\mathcal{C}$ is closed under sum, product and differentiation. Moreover by assumptions (A1), (A2) the coefficient $b = \Lambda^{-1} \nabla V$ - which defines the differential operators $L, M$ (cf. 513, 260) - belongs to $\mathcal{C}$. We conclude that if $\eta \in \mathcal{C}$, then also $L \eta, M_k \eta \in \mathcal{C}$ for all $k = 1, \ldots, d$.

Remark 5.18. If $\eta \in \mathcal{C}$, by Proposition 5.7 we know that

$$
\sup_{t \in (0,T)} \int_{\mathbb{R}^d} |\eta(x)| \, \rho_t(x) \, dx < \infty. \quad \tag{355}
$$

In particular $\mathcal{I}_t \eta$ is well defined and belongs to $L^1(\mathbb{R}^d)$.
Remark 5.19. Since the diffusion matrix $A = (\Lambda \beta)^{-1}$ is diagonal, the kernel $G$ introduced in (341) writes as the product

$$G_t(x_1, x_2) = G_t^{(1)}(x_1) G_t^{(2)}(x_2) ,$$

where

$$G_t^{(1)}(x_1) \equiv \left( \frac{\beta_1}{4 \pi t} \right)^{d_1} \exp \left( - \frac{\beta_1}{4 t} |x_1|^2 \right) ,$$

$$G_t^{(2)}(x_1) \equiv \left( \frac{\beta_2}{4 \pi t} \right)^{d_2} \exp \left( - \frac{\beta_2}{4 t} |x_2|^2 \right).$$

Clearly $\int_{\mathbb{R}^{d_1}} G_t^{(1)}(x_1) \, dx_1 = \int_{\mathbb{R}^{d_2}} G_t^{(2)}(x_2) \, dx_2 = 1$ for all $t > 0$. For $f, g \in L^1(\mathbb{R}^{d_2})$ let us denote

$$(f \ast_2 g)(x_2) \equiv \int_{\mathbb{R}^{d_2}} g(x_2 - y_2) f(y_2) \, dy_2 .$$

If $\eta \in C$, by Fubini theorem we get:

$$\int_{\mathbb{R}^{d_1}} ((\eta \rho_t) \ast G_\tau)(x_1, x_2) \, dx_1 = \left( (\mathcal{I}_t \eta) \ast_2 G_t^{(2)} \right)(x_2) ,$$

$$\int_{\mathbb{R}^{d_1}} ((\eta \rho_t) \ast \partial_1 G_\tau)(x_1, x_2) \, dx_1 = 0 ,$$

$$\int_{\mathbb{R}^{d_1}} ((\eta \rho_t) \ast \partial_2 G_\tau)(x_1, x_2) \, dx_1 = \left( (\mathcal{I}_t \eta) \ast_2 \partial_2 G_t^{(2)} \right)(x_2) ,$$

where $\partial_1, \partial_2$ denote any partial derivative w.r.t. one variable in $x_1, x_2$ respectively.

Proof of Theorem 4.8 part i. Let $\eta \in C$. We claim that $\eta$ can be plugged into equation (344). To prove this, consider a sequence $\psi_N \in C_0^\infty(\mathbb{R}^d)$ such that $\psi_N(x) = 1$ for $|x| \leq N$ and $\nabla \psi_N$, Hess $\psi_N$ are uniformly bounded for all $N \in \mathbb{N}$. Since $\eta \psi_N \in C_c^\infty(\mathbb{R}^d)$, it can be plugged into equation (334) in place of $\eta$. Letting $N \to \infty$ our claim can be proved by dominate convergence using Young’s convolution inequality, Remark 5.19 and Lemma 5.13. Now, integrating equation (334) with respect to $x_1 \in \mathbb{R}^{d_1}$ and using Remark 5.19 we obtain:

$$\mathcal{I}_t \eta(x_2) = \left( (\mathcal{I}_t \eta) \ast_2 G_t^{(2)} \right)(x_2) +$$

$$+ \int_{t_0}^t \left( (\mathcal{I}_s \mathcal{L} \eta) \ast_2 G_t^{(2)} \right)(x_2) \, ds +$$

$$+ \int_{t_0}^t \left( (\mathcal{I}_s M_2 \eta) \ast_2 \nabla_2 G_t^{(2)} \right)(x_2) \, ds$$

for almost every $(x_2, t) \in \mathbb{R}^{d_2} \times (t_0, T)$, setting:

$$M_2 \eta \equiv - \frac{2}{\beta_2 \lambda} \nabla_2 \eta + \frac{1}{\lambda} \eta \nabla_2 V .$$

54
Let $p \in (1, \frac{d_2}{d_2 - 1})$. Using Young’s convolution inequality and Lemma 5.14 (with $d_2, G^{(2)}$ in place of $d, G$) into equation (365), we find:

$$
\| I_{t_0} \eta \|_{L^p(R^{d_2})} \leq \| I_{t_0} \eta \|_{L^1(R^{d_2})} \| G^{(2)}_1 \|_{L^p(R^{d_2})} (t - t_0)^{-\frac{d_2}{p} (p-1)} + \\
+ \int_{t_0}^t \| I_s \eta \|_{L^1(R^{d_2})} \| G^{(2)}_1 \|_{L^p(R^{d_2})} (t - s)^{-\frac{d_2}{p} (p-1)} ds + \\
+ \int_{t_0}^t \| I_s M_2 \eta \|_{L^1(R^{d_2})} \| \nabla_2 G^{(2)}_1 \|_{L^p(R^{d_2})} (t - s)^{-\frac{d_2}{p} (1 - \frac{d_2}{p})} ds,
$$

(365)

hence, by Remark 5.18 and our choice of $p$, we have

$$
\| I_{t_0} \eta \|_{L^p(R^{d_2})} \leq \| I_{t_0} \eta \|_{L^1(R^{d_2})} \| G^{(2)}_1 \|_{L^p(R^{d_2})} (t - t_0)^{-\frac{d_2}{p} (p-1)} + \\
+ \sup_{s \in (0, T)} \| L \eta \rho_s \|_{L^1(R^{d_2})} \| G^{(2)}_1 \|_{L^p(R^{d_2})} \frac{(t - t_0)^{-\frac{d_2}{p} (1 - \frac{d_2}{p})}}{1 - \frac{d_2}{p} (1 - \frac{d_2}{p})} + \\
+ \sup_{s \in (0, T)} \| M_2 \eta \rho_s \|_{L^1(R^{d_2})} \| \nabla_2 G^{(2)}_1 \|_{L^p(R^{d_2})} \frac{(t - t_0)^{-\frac{d_2}{p} (1 - \frac{d_2}{p})}}{1 - \frac{d_2}{p} (1 - \frac{d_2}{p})}
$$

(366)

and the r.h.s. is finite. Since $\| I_{t_0} \eta \|_{L^p(R^{d_2})} = \int_{t_0}^T \| I_{t_0} \eta \|_{L^p(R^{d_2})} dt$, it follows that

$$
I \eta \in L^p(R^{d_2} \times (t_1, T))
$$

(367)

for all $r \in \mathbb{N}$. We skip the details, which are similar to Proof of Theorem 4.8 part ii. Let us just notice that the argument can be iterated up to space-time derivatives of any order thanks to Remarks 5.14, 5.18. Finally general Sobolev inequalities entail that

$$
I \eta \in \mathcal{W}^{r,p}(R^{d_2} \times (t_1, T))
$$

(368)

Proof of Theorem 4.8 part ii. It follows immediately by Theorem 4.8 since $\| \rho \|_{L^p(R^{d_2})} \leq \| \rho - \rho \|_{L^1(R^{d_2})}$.

5.6 Proof of Propositions 4.14, 4.15

The expression of entropy dissipation given by Proposition 4.14 is essentially an application of weak FP equation (29) for $\rho$ to the particular test function $\varphi \equiv \log(\rho / \rho_e)$. This is possible thanks to the regularity and integrability results given by Theorems 4.7, 4.14, 4.15 and Proposition 4.13.

Similarly, Proposition 4.15 gives an expression for the marginal entropy dissipation, relying on the fact that the marginal $\rho^{(2)}$ satisfies FP equation (138). This time one would like to choose $\varphi \equiv \log(\rho^{(2)} / \rho_e^{(2)})$ as a test function. The proof is based on regularity and integrability results given by Theorems 4.8, 4.14, 4.15 and Proposition 4.13.
Proof of Proposition 4.14.

For every \( \psi \in C^2(\mathbb{R}^d \times (0, \infty)) \) such that \( \operatorname{supp} \psi \subset K \) compact subset of \( \mathbb{R}^d \) for all \( t > 0 \). The contribution in \( \psi \) coming from the diffusion term of \( L \) can be integrated by parts:

\[
\int_{\mathbb{R}^d} \nabla \cdot (A \nabla \psi) \rho_s(x) \, dx = - \int_{\mathbb{R}^d} A \nabla \psi \cdot \nabla \rho_s(x) \, dx
\]

for almost every \( s \in (t_0, t) \), since \( \rho_s \in W^{1,1}(\mathbb{R}^d) \) by Theorem 1.5 and \( \psi \in C_c^2(\mathbb{R}^d) \). Using \( (318), (369) \) into \( (330) \) we obtain:

\[
\int_{\mathbb{R}^d} \varphi_t(x) \rho_t(x) \, dx - \int_{\mathbb{R}^d} \varphi_{t_0}(x) \rho_{t_0}(x) \, dx = \\
= \int_{t_0}^t \int_{\mathbb{R}^d} (\partial_s \psi - (A \nabla \log \rho_s + b) \cdot \nabla \psi) \rho_s(x) \, dx \, ds
\]

Remark 5.20. We will rewrite the weak formulation \( (330) \) of FP equation. Let \( \psi \in C^2(\mathbb{R}^d \times (0, \infty)) \) such that \( \operatorname{supp} \psi \subset \text{compact subset of } \mathbb{R}^d \) for all \( t > 0 \). The contribution in \( \psi \) coming from the diffusion term of \( L \) can be integrated by parts:

\[
\int_{\mathbb{R}^d} \nabla \cdot (A \nabla \psi_s(x)) \rho_s(x) \, dx = - \int_{\mathbb{R}^d} A \nabla \psi_s(x) \cdot \nabla \rho_s(x) \, dx
\]

On the r.h.s. of \( (372) \), by Fubini theorem and fundamental theorem of Calculus we have:

\[
\int_{t_0}^t \int_{\mathbb{R}^d} \partial_s \psi_{N,s} \rho_s(x) \, dx \, ds = \int_{t_0}^t \int_{\mathbb{R}^d} \psi_N(x) \partial_s \rho_s(x) \, dx \, ds = \\
= \int_{\mathbb{R}^d} \psi_N(x) (\rho_t(x) - \rho_s(x)) \, dx \xrightarrow{n \to \infty} \int_{\mathbb{R}^d} \psi_N(x) (\rho_t(x) - \rho_t(x)) \, dx
\]

Now let \( N \to \infty \). By Proposition 4.4 and Remark 1.6 we have dominate convergence in the following:

\[
\int_{\mathbb{R}^d} \varphi_{N,t}(x) \rho_t(x) \, dx \xrightarrow{N \to \infty} \int_{\mathbb{R}^d} \log \frac{\rho_t(x)}{\rho_s(x)} \rho_t(x) \, dx
\]
\[
\int_{\mathbb{R}^d} \psi_N(x) \log \frac{\rho_t(x)}{\rho_s(x)} \rho_t(x) \, dx \xrightarrow{N \to \infty} \int_{\mathbb{R}^d} \log \frac{\rho_t(x)}{\rho_s(x)} \rho_t(x) \, dx.
\]

(376)

Also:

\[
\int_{\mathbb{R}^d} \psi_N(x) (\rho_t(x) - \rho_1(x)) \, dx \xrightarrow{N \to \infty} \int_{\mathbb{R}^d} (\rho_t(x) - \rho_1(x)) \, dx = 0.
\]

(377)

Finally:

\[
\int_0^t \int_{\mathbb{R}^d} (A \nabla \log \rho_s + b)(x) \cdot \nabla \varphi_{N,s}(x) \rho_s(x) \, dx \, ds = \\
= \int_0^t \int_{\mathbb{R}^d} (A \nabla \log \rho_s + b)(x) \cdot \nabla \psi_N(x) \log \frac{\rho_s(x)}{\rho_s(x)} \rho_s(x) \, dx \, ds + \\
+ \int_0^t \int_{\mathbb{R}^d} (A \nabla \log \rho_s + b)(x) \cdot \nabla \log \frac{\rho_s(x)}{\rho_s(x)} \psi_N(x) \rho_s(x) \, dx \, ds \xrightarrow{N \to \infty} \\
\xrightarrow{N \to \infty} 0 + \int_0^t \int_{\mathbb{R}^d} (A \nabla \log \rho_s + b)(x) \cdot \nabla \log \frac{\rho_s(x)}{\rho_s(x)} \rho_s(x) \, dx \, ds,
\]

(378)

where dominate convergence is ensured by Theorems 4.1, 4.5, and Proposition 4.4.

The previous computations show that letting \(n \to \infty\) first and then \(N \to \infty\) in (372) we obtain:

\[
\int_{\mathbb{R}^d} \log \frac{\rho_t(x)}{\rho_s(x)} \rho_t(x) \, dx = \int_{\mathbb{R}^d} \log \frac{\rho_t(x)}{\rho_s(x)} \rho_t(x) \, dx = \\
= - \int_0^t \int_{\mathbb{R}^d} (A \nabla \log \rho_s + b)(x) \cdot \nabla \log \frac{\rho_s(x)}{\rho_s(x)} \rho_s(x) \, dx \, ds
\]

(379)

which concludes the proof.

**Proof of Proposition 4.15.** By Theorem 4.6 \(\rho_2\) satisfies a Fokker-Planck equation. In particular, for \(\varphi \in C^{2,1}_c(\mathbb{R}^d \times (0, \infty))\) such that \(\text{supp} \varphi \subset K\) compact subset of \(\mathbb{R}^d\), and for \(t > t_0 > 0\), identity (148) extends to the following one:

\[
\int_{\mathbb{R}^d} \varphi_t(x_2) \rho_t^{(2)}(x_2) \, dx_2 - \int_{\mathbb{R}^d} \varphi_{t_0}(x) \rho_{t_0}^{(2)}(x_2) \, dx_2 = \\
= \int_{t_0}^t \int_{\mathbb{R}^d} \left( \partial_s \varphi_s + \frac{1}{\lambda} \langle L_2 \rangle_s \varphi_s \right)(x_2) \rho_s^{(2)}(x_2) \, dx_2 \, ds
\]

(380)

(see Proposition 6.1.2 and following remarks in [32]). Integrating by parts the diffusion term of \(\langle L_2 \rangle_s\) as in Remark 5.16 we obtain:

\[
\int_{\mathbb{R}^d} \varphi_t(x_2) \rho_t^{(2)}(x_2) \, dx_2 - \int_{\mathbb{R}^d} \varphi_{t_0}(x_2) \rho_{t_0}^{(2)}(x_2) \, dx_2 = \\
= \int_{t_0}^t \int_{\mathbb{R}^d} \left( \partial_s \varphi_s - \frac{1}{\lambda} \left( \frac{1}{\beta_2} \nabla_2 \log \rho_s^{(2)} + \langle \nabla_2 V \rangle_s \cdot \nabla_2 \varphi_s \right)(x_2) \rho_s^{(2)}(x_2) \, dx_2 \, ds.
\]

(381)
Now let us consider a sequence $\psi_N \in C_c^\infty(\mathbb{R}^d)$ such that $\psi_N(x_2) = 1$ for $|x_2| \leq N$ and $\psi_N, \nabla \psi_N$ are uniformly bounded for all $N \in \mathbb{N}$. Set

$$\varphi_{N,t}(x_2) \equiv \psi_N(x_2) \log \frac{\rho^{(2)}_t(x_2)}{\rho^{(2)}_s(x_2)}.$$  \hspace{1cm} (382)

By Theorem 4.8 $\rho^{(2)} \in C^\infty(\mathbb{R}^d \times (0, \infty))$, hence identity (381) applies to $\varphi \equiv \varphi_N$. We let $t_0 \to 0$ along a suitable sequence such that $\rho_{t_0} \to \rho_1$ a.e. (Theorem 4.8), then we let $N \to \infty$. Proceeding as in the previous Proof of Proposition 4.14, the integrability results of Theorems 4.1, 4.10, and Proposition 4.9 guarantee that:

$$\int_{\mathbb{R}^d} \log \frac{\rho^{(2)}_t(x_2)}{\rho^{(2)}_s(x_2)} \rho^{(2)}_t(x_2) \, dx_2 - \int_{\mathbb{R}^d} \log \frac{\rho^{(2)}_1(x_2)}{\rho^{(2)}_s(x_2)} \rho^{(2)}_1(x_2) \, dx_2 =$$

$$= -\frac{1}{\lambda_0} \int_{t_0}^t \int_{\mathbb{R}^d} \left( \frac{1}{\beta_2} \nabla_2 \log \rho^{(2)}_s + (\nabla_2 V)_s \right)(x_2) \cdot \nabla_2 \log \frac{\rho^{(2)}_t(x_2)}{\rho^{(2)}_s(x_2)} \rho^{(2)}_s(x_2) \, dx_2 \, ds$$

(383)

concluding the proof.

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