By applying a fractional $q$-calculus operator, we define the subclasses $S_n^{\alpha}(\lambda, \beta, b, q)$ and $G_n^{\alpha}(\lambda, \beta, b, q)$ of normalized analytic functions with complex order and negative coefficients. Among the results investigated for each of these function classes, we derive their associated coefficient estimates, radii of close-to-convexity, starlikeness and convexity, extreme points, and growth and distortion theorems.

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1. INTRODUCTION AND DEFINITIONS

Here, in this paper, we denote by $A(n)$ the class of functions of the following normalized form:

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in \mathbb{N}; \; \mathbb{N} := \{1, 2, 3, \ldots\}), \quad (1.1)$$

which are analytic in the open unit disk $U$ centered at the origin ($z = 0$) in the complex $z$-plane. We write $A(1) = A$. We also denote by $F(n)$ the subclass of $A(n)$ consisting of functions of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0; \; k \geq n + 1; \; n \in \mathbb{N}). \quad (1.2)$$

In our investigation, we make use of various operators of $q$-calculus and fractional $q$-calculus. For this purpose, we refer the reader to the various definitions, notations and conventions, which are considerably detailed in our earlier paper (see, for details, [22]; see also [8]).
For a fixed $\mu \in \mathbb{C}$, a set $D$ is called a $\mu$-geometric set if and only if both $z \in D$ and $\mu z \in D$. For a function $f$ defined on a $q$-geometric set, we make use of Jackson’s $q$-derivative and $q$-integral ($0 < q < 1$) of a function on a subset of $\mathbb{C}$, which are already introduced in several earlier investigations (see, for example, [2], [4], [6], [8], [9], [10], [14], [15], [16], [17], [21], [22] and [25]).

Now, for a complex-valued function $f(z)$, we introduce the fractional $q$-calculus operators as follows (see, for example, [12] and [13]; see also [1]).

**Definition 1** (Fractional $q$-integral operator). The fractional $q$-integral operator $I_{q,z}^\lambda$ of order $\lambda$ is defined, for a function $f(z)$, by

$$I_{q,z}^\lambda f(z) = D_{q,z}^{-\lambda} f(z) = \frac{1}{\Gamma_q(\lambda)} \int_0^z (z - tq)_{\lambda-1} f(t) dt \quad (\lambda > 0),$$

where the function $f(z)$ is analytic in a simply-connected region of the complex $z$-plane containing the origin. Here, and elsewhere in this paper, the $q$-binomial $(z - tq)_{\lambda-1}$ is given by

$$(z - tq)_{\lambda-1} = z^{\lambda-1} \prod_{k=0}^{\infty} \left[ \frac{1 - (tz^{-1})q^k}{1 - (tz^{-1})q^{\lambda+k-1}} \right] = z^{\lambda-1} \Phi_0(q^{1-\lambda}; \cdots; q, tz^{\lambda-1}).$$

**Remark 1.** The $q$-hypergeometric series $\Phi_0(\lambda; \cdots; q, z)$ is known to be single-valued when $|\arg(z)| < \pi$ (see, for example, [8]). Therefore, the $q$-binomial $(z - tq)_{\lambda-1}$ in (1.4) is single-valued when

$$|\arg(-tq^{\lambda^{-1}})| < \pi, \quad \left| \frac{tq^{\lambda}}{z} \right| < 1 \text{ and } |\arg(z)| < \pi.$$

**Definition 2** (Fractional $q$-derivative operator). The fractional $q$-derivative operator $D_{q,z}^\lambda$ of order $\lambda$ ($0 \leq \lambda < 1$) is defined, for a function $f(z)$, by

$$D_{q,z}^\lambda f(z) = D_{q,z} t^{-\lambda} f(z) = \frac{1}{\Gamma_q(1-\lambda)} D_q \int_0^z (z - tq)_{-\lambda} f(t) dt,$$

where $f(z)$ is suitably constrained and the multiplicity of $(z - tq)_{-\lambda}$ is removed as in Definition 1.

**Definition 3** (Extended fractional $q$-derivative operator). Under the hypotheses of Definition 2, for a function $f(z)$, the fractional $q$-derivative of order $\lambda$ is defined by

$$D_{q,z}^\lambda f(z) = D_{q,z}^m t^{-\lambda} f(z) = \prod_{k=0}^{m-1} (z - tq)_{-\lambda} f(t) dt,$$

Clearly, we have

$$D_{q,z}^\lambda z^n = \frac{\Gamma_q(n+1)}{\Gamma_q(n+1-\lambda)} z^{n-\lambda} \quad (\lambda \geq 0; n > -1).$$
Now, by using the operator $D^\lambda_{q,z}$, we define (for $-\infty < \lambda < 2$, $0 < q < 1$ and $z \in \mathbb{U}$) a $q$-differintegral operator $D^\lambda_{q,z} : \mathcal{T}(n) \to \mathcal{T}(n)$ as follows (see [12] and [13]):

$$\Omega^\lambda_{q,z} f(z) = \frac{\Gamma_q(2-\lambda)}{\Gamma_q(\lambda)} z^\lambda D^\lambda_{q,z} f(z) = z - \sum_{k=n+1}^{\infty} A_q(\lambda,k) a_k z^k \quad (1.7)$$

where

$$A_q(\lambda,k) = \frac{\Gamma_q(k+1)\Gamma_q(2-\lambda)}{\Gamma_q(2)\Gamma_q(k+1-\lambda)} \quad (1.8)$$

and $D^\lambda_{q,z} f(z)$ in (1.7) represents, respectively, the fractional $q$-integral of $f(z)$ of order $\lambda$ ($-\infty < \lambda < 0$) and the fractional $q$-derivative of $f(z)$ of order $\lambda$ ($0 \leq \lambda < 2$) (see, for details, [7, 18–20]). We note that some interesting special and limit cases of (1.7) were investigated in the earlier works by Owa and Srivastava [11] and by Srivastava and Owa (see [23] and [24]).

**Remark 2.** From (1.3), (1.7) and (1.8), we find that

$$\Omega^{-\lambda}_{q,z} f(z) = \frac{\Gamma_q(2+\lambda)}{\Gamma_q(2)} z^{-\lambda} D^{-\lambda}_{q,z} f(z) = \frac{\Gamma_q(2+\lambda)}{\Gamma_q(2)\Gamma_q(1-\lambda)} z^{-\lambda} D^{-\lambda}_{q,z} f(z)$$

$$= z - \sum_{k=n+1}^{\infty} A_q(-\lambda,k) a_k z^k. \quad (1.9)$$

where

$$A_q(-\lambda,k) = \frac{\Gamma_q(k+1)\Gamma_q(2+\lambda)}{\Gamma_q(2)\Gamma_q(k+1+\lambda)} \quad (\lambda > 0; \ 0 < q < 1). \quad (1.10)$$

**Definition 4.** A function $f(z) \in \mathcal{T}(n)$ is said to be in the function class:

$$\mathcal{S}^\alpha_n(\lambda, \beta, b, q) \quad (\lambda < 2; \ 0 \leq \alpha \leq 1; \ 0 < q < 1; \ \beta > 0; \ b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\})$$

if it satisfies the following condition:

$$\left| \frac{1}{b} \left( \frac{1}{b} (1-\alpha)z D_q(\Omega^\lambda_{q,z} f(z)) + \alpha z D_q(\Omega^\lambda_{q,z} f(z)) \right) \right| < \beta. \quad (1.11)$$

Some of the interesting particular cases of the function class $\mathcal{S}^\alpha_n(\lambda, \beta, b, q)$ are being recorded below:

(i) $\mathcal{S}^\alpha_n(\lambda, 1, b, q) = \mathcal{S}^\alpha_n(\lambda, b, q)$ (see [12]);

(ii) $\mathcal{S}^\alpha_n(0, \beta, b, q) = \mathcal{S}^\alpha_n(\beta, b, q)$, where

$$\mathcal{S}^\alpha_n(\beta, b, q) := \left\{ \text{f : f} \in \mathcal{T}(n) \quad \text{and} \right\}.$$
\[
\begin{aligned}
&\left| \frac{1}{b} \left( \frac{1 - \alpha}{(1 - \alpha) f(z) + a z^2 D_q f(z)} \right) \right| < \beta \bigg\}.
\end{aligned}
\]

(iii) \( \lim_{q \to 1^-} \mathcal{S}_n^a(\beta, b, q) = \mathcal{S}_n(b, \alpha, \beta) \) (see [3]);

(iv) \( \mathcal{S}_n^a(\lambda, \beta, b, q) = \mathcal{S}_n^a(\lambda, \beta, b, q) \), where

\[
\begin{aligned}
\mathcal{S}_n(b, \alpha, \beta) &:= \left\{ f : f \in \mathcal{T}(n) \text{ and } \left| \frac{1}{b} \left( \frac{z D_q (\Omega_{q,z} f(z)) - 1}{D_q (\Omega_{q,z} f(z))} \right) \right| < \beta \bigg\} ;
\end{aligned}
\]

(v) \( \lim_{q \to 1-} \mathcal{S}_n^a(\lambda, \beta, b, q) = \mathcal{K}_n(\lambda, b, \beta) \) (see [5] with \( p = 1 \));

(vi) \( \mathcal{S}_n^a(\lambda, \beta, b, q) = \mathcal{C}_n(\lambda, \beta, b, q) \), where

\[
\begin{aligned}
\mathcal{C}_n(\lambda, \beta, b, q) &:= \left\{ f : f \in \mathcal{T}(n) \text{ and } \left| \frac{1}{b} \left( \frac{z D_q^2 (\Omega_{q,z}^{\lambda} f(z)) - 1}{D_q (\Omega_{q,z}^{\lambda} f(z))} \right) \right| < \beta \bigg\} .
\end{aligned}
\]

**Definition 5.** A function \( f(z) \in \mathcal{T}(n) \) is in the function class

\( \mathcal{G}_n^a(\lambda, \beta, b, q) \) \( (\lambda < 2; 0 \leq \alpha \leq 1; 0 < q < 1; b \in \mathbb{C}^*; \beta > 0) \)

if it satisfies the following condition:

\[
\begin{aligned}
&\left| \frac{1}{b} \left( \frac{1 - \alpha}{(1 - \alpha) f(z) + a z^2 D_q f(z)} \right) \right| < \beta .
\end{aligned}
\]

We choose to note the following special case of the function class \( \mathcal{G}_n^a(\lambda, \beta, b, q) \):

(i) \( \mathcal{G}_n^a(0, \beta, b, q) = \mathcal{G}_n^a(\beta, b, q) \), where

\[
\begin{aligned}
\mathcal{G}_n^a(\beta, b, q) &:= \left\{ f : f \in \mathcal{T}(n) \text{ and } \left| \frac{1}{b} \left( D_q f(z) + a z D_q^2 f(z) - 1 \right) \right| < \beta \bigg\} ;
\end{aligned}
\]

(ii) \( \mathcal{G}_n^a(\lambda, 1, b, q) = \mathcal{R}_n^a(\lambda, b, q) \) (see [13]);

(iii) \( \mathcal{G}_n^a(0, \beta, b, q) = \mathcal{R}_n(\alpha, \beta, b, q) \) (see [13]);

(iv) \( \lim_{q \to 1-} \mathcal{G}_n^a(0, \beta, b, q) = \mathcal{R}_n(\alpha, \beta, b) \) (see [3]).

For each of the above-defined general function classes \( \mathcal{G}_n^a(\lambda, \beta, b, q) \) and \( \mathcal{G}_n^a(\lambda, \beta, b, q) \) of analytic functions with complex order and negative coefficients, we propose here to investigate the associated coefficient estimates, radii of close-to-convexity, starlikeness and convexity, extreme points, and growth and distortion theorems.
2. Properties of the function classes $S^n_\alpha(\lambda, \beta, b, q)$ and $G^n_\alpha(\lambda, \beta, b, q)$

Henceforth in this paper, unless otherwise mentioned, we assume that $\lambda < 2$, $0 < q < 1$, $b \in \mathbb{C}^*$, $\beta > 0$, $[\lambda]_q$ denotes the basic (or $q$-) number defined by

$$[\lambda]_q = \frac{1-q^\lambda}{1-q} \quad (|q| < 1).$$

(2.1)

which readily yields

$$[\lambda]_q = \frac{1-q^\lambda}{1-q} \to \lambda \quad (q \to 1-).$$

$A_q(\lambda, k)$ is given by (1.8), $f(z)$ is in the form (1.2) and $z \in \mathbb{U}$.

Theorem 1. The function $f(z) \in S^n_\alpha(\lambda, \beta, b, q)$ if and only if

$$\sum_{k=n+1}^{\infty} ([k]_q + \beta |b| - 1)[1 + \alpha([k]_q - 1)]A_q(\lambda, k)a_k \leq \beta |b|.$$

(2.2)

Proof. Let $f(z) \in S^n_\alpha(\lambda, \beta, b, q)$. Then, in view of (1.11) and (1.7), we readily find that

$$\Re\left( - \frac{\sum_{k=n+1}^{\infty} [1 + \alpha([k]_q - 1)]([k]_q - 1)A_q(\lambda, k)a_k z^{k-1}}{1 - \sum_{k=n+1}^{\infty} [1 + \alpha([k]_q - 1)]A_q(\lambda, k)a_k z^{k-1}} \right) > -\beta |b|.$$

(2.3)

Setting $z = r \ (0 \leq r < 1)$ in (2.3), we observe that the expression in the denominator of the left-hand side of (2.3) is positive for $r = 0$ and also for $0 < r < 1$. Thus, if we let $r \to 1^-$ through real values, (2.3) would lead us to (2.2).

Conversely, let (2.2) hold true and $|z| = 1$. We then find that

$$\frac{[1 - \alpha]z D_q(D_q \Omega_{q,z}^\lambda f(z)) + \alpha z D_q(\Omega_{q,z}^\lambda f(z))}{[1 - \alpha]D_q(\Omega_{q,z}^\lambda f(z)) + \alpha D_q(\Omega_{q,z}^\lambda f(z))} = 1 - \frac{\beta |b| \{1 - \sum_{k=n+1}^{\infty} [1 + \alpha([k]_q - 1)]A_q(\lambda, k)a_k\}}{1 - \sum_{k=n+1}^{\infty} [1 + \alpha([k]_q - 1)]A_q(\lambda, k)a_k} = \beta |b|.$$

Hence, by the Maximum Modulus Theorem, we conclude that $f(z) \in S^n_\alpha(\lambda, \beta, b, q)$, which completes the proof of Theorem 1. □

The following corollary follows easily from Theorem 1.
Corollary 1. Let \( f(z) \in \mathcal{S}_n^\alpha(\lambda, \beta, b, q) \). Then
\[
a_k \leq \frac{\beta |b|}{([k]_q + \beta |b| - 1)[1 + \alpha([k]_q - 1)]A_q(\lambda, k)} (k \geq n + 1).
\] (2.4)

The result is sharp for the function \( f(z) \) given (for \( k \geq n + 1 \)) by
\[
f(z) = z - \frac{\beta |b|}{([k]_q + \beta |b| - 1)[1 + \alpha([k]_q - 1)]A_q(\lambda, k)} z^k.
\] (2.5)

Putting \( \beta = 1 \) in Theorem 1, we have Corollary 2 below.

Corollary 2. Let \( f(z) \in \mathcal{S}_n^\alpha(\lambda, b, q) \). Then
\[
\sum_{k=n+1}^{\infty} ([k]_q + |b| - 1)[1 + \alpha([k]_q - 1)]A_q(\lambda, k)a_k \leq |b|.
\]

Corollary 3. Let \( f(z) \in \mathcal{S}_n^\alpha(\lambda, b, q) \). Then
\[
a_k \leq \frac{|b|}{([k]_q + |b| - 1)[1 + \alpha([k]_q - 1)]A_q(\lambda, k)} (k \geq n + 1).
\] (2.6)

The result is sharp for the function \( f(z) \) given by
\[
f(z) = z - \frac{|b|}{([k]_q + |b| - 1)[1 + \alpha([k]_q - 1)]A_q(\lambda, k)} z^k (k \geq n + 1).
\] (2.7)

It is not difficult to prove the following results. The details involved are being left as an exercise for the interested reader.

Theorem 2. The function \( f(z) \in \mathcal{S}_n^\alpha(\lambda, \beta, b, q) \) if and only if
\[
\sum_{k=n+1}^{\infty} [k]_q [1 + \alpha([k]_q - 1)]A_q(\lambda, k)a_k \leq \beta |b|.
\] (2.8)

Corollary 4. Let \( f(z) \in \mathcal{S}_n^\alpha(\lambda, \beta, b, q) \). Then
\[
a_k \leq \frac{\beta |b|}{[k]_q [1 + \alpha([k]_q - 1)]A_q(\lambda, k)} (k \geq n + 1).
\] (2.9)

The result is sharp for the function \( f(z) \) given by
\[
f(z) = z - \frac{\beta |b|}{[k]_q [1 + \alpha([k]_q - 1)]A_q(\lambda, k)} z^k (k \geq n + 1).
\] (2.10)

We now state (without proof) Theorem 3 below.

Theorem 3. If \( b_1, b_2 \in \mathbb{C}^* \) and \( |b_1| < |b_2| \), then
\[
\mathcal{S}_n^\alpha(\lambda, \beta, b_1, q) \subset \mathcal{S}_n^\alpha(\lambda, \beta, b_2, q).
\]

The following result can indeed be proven along the lines which we have already indicated above.
Theorem 4. If \( b_1, b_2 \in \mathbb{C}^* \) and \( |b_1| < |b_2| \), then
\[
\mathcal{F}_n^\alpha(\lambda, \beta, b_1, q) \subset \mathcal{F}_n^\alpha(\lambda, \beta, b_2, q). 
\] (2.9)

3. EXTREME POINTS FOR THE FUNCTION CLASSES \( \mathcal{F}_n^\alpha(\lambda, \beta, b, q) \) AND \( \mathcal{G}_n^\alpha(\lambda, \beta, b, q) \)

In this section, we first prove the following result.

Theorem 5. Let \( f_n(z) = z \) and
\[
f_k(z) = z - \frac{\beta |b|}{([k]_q + \beta |b|-1)[1 + \alpha([k]_q - 1)]A_q(\lambda, k)} z^k 
\] (3.1)

\( (k \geq n + 1) \).

Then the function \( f(z) \) is in the class \( \mathcal{F}_n^\alpha(\lambda, \beta, b, q) \) if and only if it can be expressed in the following form:

\[
f(z) = \sum_{k=n}^{\infty} \mu_k f_k(z), 
\] (3.2)

where

\[
\sum_{k=n}^{\infty} \mu_k = 1 \quad \text{and} \quad \mu_k \geq 0. 
\]

Proof. By assuming (3.2) to hold true, if we appropriately apply Theorem 1, it is not difficult to conclude that \( f(z) \in \mathcal{F}_n^\alpha(\lambda, \beta, b, q) \).

Conversely, upon letting \( f(z) \in \mathcal{F}_n^\alpha(\lambda, \beta, b, q) \), if we set

\[
\mu_k = \frac{([k]_q + \beta |b| -1)[1 + \alpha([k]_q - 1)]A_q(\lambda, k)}{\beta |b|} a_k 
\] (k \geq n + 1)

and

\[
\mu_n = 1 - \sum_{k=n+1}^{\infty} \mu_k. 
\]

we can easily see that \( f(z) \) can be expressed in the form (3.2). This completes the proof of Theorem 5.

Corollary 5. The extreme points of the function class \( \mathcal{F}_n^\alpha(\lambda, \beta, b, q) \) are the functions \( f_n(z) = z \) and \( f_k(z) \) \( (k \geq n + 1) \) given by (3.1).

Similarly, we can prove the following theorem.

Theorem 6. Let \( f_n(z) = z \) and
\[
f_k(z) = z - \frac{\beta |b|}{[k]_q[1 + \alpha([k]_q - 1)]A_q(\lambda, k)} z^k 
\] (3.3)

\( (k \geq n + 1) \).
Then the function \( f(z) \) is in the class \( S_\alpha^n(\lambda, \beta, b, q) \) if and only if it can be expressed in the form given by
\[
f(z) = \sum_{k=n}^{\infty} \mu_k f_k(z),
\]
where
\[
\sum_{k=n}^{\infty} \mu_k = 1 \quad \text{and} \quad \mu_k \geq 0.
\]

**Corollary 6.** The extreme points of the function class \( S_\alpha^n(\lambda, \beta, b, q) \) are the functions \( f_n(z) = z \) and \( f_k(z) \) \((k \geq n + 1)\) given by (3.3).

4. **Radii of Close-to-Convexity, Starlikeness and Convexity of the Function Class \( S_\alpha^n(\lambda, \beta, b, q) \)**

**Theorem 7.** Let \( f(z) \in S_\alpha^n(\lambda, \beta, b, q) \). Then \( f(z) \) is close-to-convex of order \( \rho \) \((0 \leq \rho < 1)\) in \(|z| < r_1\), where
\[
r_1 := \inf_{k\geq n+1} \left\{ \left(1 - \rho\right)([k]_q + \beta |b| - 1)[1 + \alpha([k]_q - 1)]A_q(\lambda, k) \right\}^{\frac{1}{k-1}}. \tag{4.1}
\]
The sharpness of this result is attained for the function \( f(z) \) given by (2.5).

**Proof.** By showing that
\[
|f'(z) - 1| \leq 1 - \rho \quad \text{for} \quad |z| < r_1,
\]
where \( r_1 \) is given by (4.1), we readily find that
\[
|f'(z) - 1| \leq 1 - \rho
\]
if
\[
\sum_{k=n+1}^{\infty} \frac{k}{1 - \rho} a_k |z|^{k-1} \leq 1. \tag{4.2}
\]
But, by Theorem 1, it is seen that (4.2) will hold true if \((k \geq n + 1)\)
\[
|z| \leq \left( \frac{(1 - \rho)([k]_q + \beta |b| - 1)[1 + \alpha([k]_q - 1)]A_q(\lambda, k)}{k \beta |b|} \right)^{\frac{1}{k-1}}.
\]
This completes the proof of Theorem 7. \(\square\)

By using arguments and analysis similar to those in the proof of Theorem 7, we can analogously derive Theorem 8 and Corollary 7 below.
Theorem 8. Let \( f(z) \in S_q^n(\lambda, \beta, b, q) \). Then the function \( f(z) \) is starlike of order \( \rho \) (\( 0 \leq \rho < 1 \)) in \( |z| < r_2 \), where
\[
r_2 := \inf_{k \geq n+1} \left\{ \left( 1 - \rho \right)(|k|_q + \beta |b| - 1)[1 + \alpha([k]_q - 1)]A_q(\lambda, k) \right\}^{\frac{1}{1 - \rho}}. \tag{4.3}
\]
The sharpness of this result is attained for the function \( f(z) \) given by (2.5).

Corollary 7. Let \( f(z) \in S_q^n(\lambda, \beta, b, q) \). Then the function \( f(z) \) is convex of order \( \rho \) (\( 0 \leq \rho < 1 \)) in \( |z| < r_3 \), where
\[
r_3 := \inf_{k \geq n+1} \left\{ \left( 1 - \rho \right)(|k|_q + \beta |b| - 1)[1 + \alpha([k]_q - 1)]A_q(\lambda, k) \right\}^{\frac{1}{1 - \rho}}. \tag{5.1}
\]
The sharpness of the result is attained for the function \( f(z) \) given by (2.5).

5. Growth and Distortion Theorems

For convenience in this section, for \( k \geq n+1 \), we shall henceforth use the following notations:
\[
\sigma_{k,\alpha}(\lambda, \beta, b, q) := ([k]_q + \beta |b| - 1)[1 + \alpha([k]_q - 1)]A_q(\lambda, k) \tag{5.2}
\]
and
\[
\phi_{k,\alpha}(\lambda, \beta, b, q) := [k]_q[1 + \alpha([k]_q - 1)]A_q(\lambda, k).
\]
We now prove the following which will be needed in our further investigation in this section.

Lemma 1. The sequence \( \{A_q(\lambda, k)\}_{k=n+1}^{\infty} \) is a decreasing sequence in \( k \) (\( k \geq n+1 \)) for \( \lambda < 2 \) and \( 0 < q < 1 \).

Proof. It follows from (1.8) and the recurrence relation:
\[
\Gamma_q(z+1) = [z]_q \Gamma_q(z)
\]
that
\[
\frac{A_q(\lambda, k+1)}{A_q(\lambda, k)} = \frac{\Gamma_q(k+2)\Gamma_q(k-\lambda+1)}{\Gamma_q(k+1)\Gamma_q(k-\lambda+2)} = \frac{[k+1]_q\Gamma_q(k+1)\Gamma_q(k-\lambda+1)}{\Gamma_q(k+1)[k-\lambda+1]_q\Gamma_q(k-\lambda+2)} = \frac{[k+1]_q}{[k-\lambda+1]_q}.
\]
It is sufficient to consider the value \( k = n+1 \). By using the definition (2.1) of the basic (or \( q \)-) number \( [\lambda]_q \) again, we thus find that
\[
\frac{A_q(\lambda, k+1)}{A_q(\lambda, k)} = \frac{[n+2]_q}{[n-\lambda+2]_q} = \frac{1-q^{n+2}}{1-q^{n-\lambda+2}} \quad (0 < q < 1; -\infty < \lambda < 2).
\]
The sequence \( \{A_q(\lambda, k)\}^\infty_{k=n+1} \) is a decreasing sequence in \( k \) if
\[
\frac{A_q(\lambda, k+1)}{A_q(\lambda, k)} < 1 \quad (k \geq n + 1),
\]
that is, if
\[
1 - q^n \frac{1 - q^{n+1}}{1 - q^{n-\lambda + 2}} < 1 \quad (0 < q < 1; \, -\infty < \lambda < 2),
\]
which implies that \( \lambda < 0 \). Thus \( \{A_q(\lambda, k)\}^\infty_{k=n+1} \) is a decreasing sequence in \( k \) \( (k \geq n + 1) \) for \( -\infty < \lambda < 2 \) and \( 0 < q < 1 \).

**Theorem 9.** Let \( f(z) \in S^n_\alpha(\lambda, \beta, b, q) \). Then
\[
|z| - \frac{\beta |b|}{\sigma_{n+1, a}(\lambda, \beta, b, q)} |z|^{n+1} \leq |f(z)| \leq |z| + \frac{\beta |b|}{\sigma_{n+1, a}(\lambda, \beta, b, q)} |z|^{n+1}. \tag{5.4}
\]
The result is sharp for the function \( f(z) \) given by
\[
f(z) = z - \frac{\beta |b|}{\sigma_{n+1, a}(\lambda, \beta, b, q)} z^{n+1}. \tag{5.5}
\]

**Proof.** Since \( f(z) \in S^n_\alpha(\lambda, \beta, b, q) \), in view of Theorem 1, we have
\[
\sigma_{n+1, a}(\lambda, \beta, b, q) \sum_{k=n+1}^\infty a_k \leq \sum_{k=n+1}^\infty \sigma_{k, a}(\lambda, \beta, b, q) a_k \leq \beta |b|,
\]
that is,
\[
\sum_{k=n+1}^\infty a_k \leq \frac{\beta |b|}{\sigma_{n+1, a}(\lambda, \beta, b, q)}. \tag{5.6}
\]
We thus obtain
\[
|f(z)| \geq |z| - \sum_{k=n+1}^\infty a_k |z|^k \geq |z| - |z|^{n+1} \sum_{k=n+1}^\infty a_k 
\]
\[
\geq |z| - \frac{\beta |b|}{\sigma_{n+1, a}(\lambda, \beta, b, q)} |z|^{n+1} \tag{5.7}
\]
and
\[
|f(z)| \leq |z| + \sum_{k=n+1}^\infty a_k |z|^k \leq |z| + |z|^{n+1} \sum_{k=n+1}^\infty a_k 
\]
\[
\leq |z| + \frac{\beta |b|}{\sigma_{n+1, a}(\lambda, \beta, b, q)} |z|^{n+1}. \tag{5.8}
\]
These last inequalities (5.7) and (5.8) complete the proof of Theorem 9. \( \square \)
Corollary 8. Under the hypothesis of Theorem 9, the function \( f(z) \) is included in a disk with center at the origin and radius \( r \) given by
\[
r = 1 + \frac{\beta |b|}{\sigma_{n+1,a}(\lambda, \beta, b, q)}.
\]

Similarly, we can prove the following distortion theorem for \( f(z) \in \mathcal{B}_n^\alpha(\lambda, \beta, b, q) \).

Theorem 10. Let \( f(z) \in \mathcal{B}_n^\alpha(\lambda, \beta, b, q) \) and let \( \phi_{k,a}(\lambda, \beta, b, q) \) be given by (5.2).

Then
\[
|z| - \frac{\beta |b|}{\phi_{n+1,a}(\lambda, \beta, b, q)} |z|^{n+1} \leq |f(z)| \leq |z| + \frac{\beta |b|}{\phi_{n+1,a}(\lambda, \beta, b, q)} |z|^{n+1}.
\] (5.9)

The result is sharp for the function \( f(z) \) given by
\[
f(z) = z - \frac{\beta |b|}{\phi_{n+1,a}(\lambda, \beta, b, q)} z^{n+1}.
\] (5.10)

Corollary 9. Under the hypothesis of Theorem 10, the function \( f(z) \) is included in a disk with its center at the origin and its radius \( r \) given by
\[
r = 1 + \frac{\beta |b|}{\phi_{n+1,a}(\lambda, \beta, b, q)}.
\]

A further distortion theorem involving the generalized fractional \( q \)-differintegral operator \( \Omega_{q,z}^\lambda \), defined by (1.7) is given by the following theorem.

Theorem 11. Let \( f(z) \in \mathcal{S}_n^\alpha(\lambda, \beta, b, q) \). Then
\[
|z| - \frac{\beta |b|}{([n+1]_q + \beta |b| - 1)(1 + \alpha([n+1]_q - 1))} |z|^{n+1}
\leq \left| \Omega_{q,z}^\lambda f(z) \right|
\leq |z| + \frac{\beta |b|}{([n+1]_q + \beta |b| - 1)(1 + \alpha([n+1]_q - 1))} |z|^{n+1}.
\] (5.11)

The result is sharp.

Proof. From the above Lemma 1, in conjunction with the equations (5.6) and (1.7), we have
\[
\left| \Omega_{q,z}^\lambda f(z) \right| \geq |z| - A_q(\lambda, n + 1) |z|^{n+1} \sum_{k=n+1}^{\infty} a_k
\geq |z| - \frac{\beta |b|}{([n+1]_q + \beta |b| - 1)(1 + \alpha([n+1]_q - 1))} |z|^{n+1}
\] (5.12)
and
\[
\left| \Omega_{q,z}^\lambda f(z) \right| \leq |z| + A_q(\lambda, n + 1) |z|^{n+1} \sum_{k=n+1}^{\infty} a_k
\]
\[ |z| + \frac{\beta |b|}{([n+1]_q + \beta |b|-1)[1 + \alpha([n+1]_q - 1)]} |z|^n. \quad (5.13) \]

The equalities in (5.11) are attained for the function \( f(z) \) given by
\[
D_{q,z}^\lambda f(z) = \frac{\Gamma_q(z)}{\Gamma_q(2-\lambda)} \left( 1 - \frac{\beta |b|}{([n+1]_q + \beta |b|-1)[1 + \alpha([n+1]_q - 1)]} |z|^n \right) \quad (5.14)
\]
or by the function \( f(z) \) given by (5.5). We have thus completed our demonstration of Theorem 11.

From Theorem 10 and (1.7), we have the following distortion inequality involving the fractional \( q \)-derivative operator \( D_{q,z}^\lambda \).

**Corollary 10.** Let \( f(z) \in \mathcal{S}_n^\alpha(\lambda, \beta, b, q) \). Then
\[
\frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)} |z|^{1-\lambda} \left( 1 - \frac{\beta |b|}{([n+1]_q + \beta |b|-1)[1 + \alpha([n+1]_q - 1)]} \right) |z|^n \leq |D_{q,z}^\lambda f(z)| \leq \frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)} |z|^{1-\lambda} \left( 1 + \frac{\beta |b|}{([n+1]_q + \beta |b|-1)[1 + \alpha([n+1]_q - 1)]} \right) |z|^n. \quad (5.15)
\]
The result is sharp for the function \( f(z) \) given by (5.5).

Upon setting \( \beta = 1 \) in Corollary 10, we get the following corollary which provided the corrected version of a result obtained by Purohit and Raina [12, Corollary 1].

**Corollary 11.** Let \( f(z) \in \mathcal{S}_n^\alpha(\lambda, \beta, b, q) \). Then
\[
\frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)} |z|^{1-\lambda} \left( 1 - \frac{|b|}{([n+1]_q + |b|-1)[1 + \alpha([n+1]_q - 1)]} \right) |z|^n \leq |D_{q,z}^\lambda f(z)| \leq \frac{\Gamma_q(2)}{\Gamma_q(2-\lambda)} |z|^{1-\lambda} \left( 1 + \frac{|b|}{([n+1]_q + |b|-1)[1 + \alpha([n+1]_q - 1)]} \right) |z|^n. \quad (5.16)
\]
The result is sharp for the function \( f(z) \) given by (5.5) with \( \beta = 1 \).

Also, in view of (1.9) or by virtue of (1.3), Theorem 10 gives the following distortion inequality involving the fractional \( q \)-integral operator \( I_{q,z}^\lambda \).

**Corollary 12.** Let \( f(z) \in \mathcal{S}_n^\alpha(\lambda, \beta, b, q) \). Then
\[
\frac{\Gamma_q(2)}{\Gamma_q(2+\lambda)} |z|^{1+\lambda} \left( 1 - \frac{\beta |b|}{([n+1]_q + \beta |b|-1)[1 + \alpha([n+1]_q - 1)]} \right) |z|^n
\]
\[ \begin{align*}
&\| I_{q,z} f(z) \| \leq \left| \frac{\Gamma_q(2)}{\Gamma_q(2 + \lambda)} |z|^{1+\lambda} \right| \cdot \left( 1 + \frac{\beta |b|}{(n+1)q + \beta |b|} |z|^n \right). \\
&\| I_{q,z} f(z) \| \leq \left| \frac{\Gamma_q(2)}{\Gamma_q(2 + \lambda)} |z|^{1+\lambda} \right| \cdot \left( 1 + \frac{\beta |b|}{(n+1)q + \beta |b|} |z|^n \right). 
\end{align*} \]

The result is sharp for the function \( f(z) \) given by (5.5).

Putting \( \beta = 1 \) in Corollary 12, we have the following result.

**Corollary 13.** Let \( f(z) \in S_n^q(\lambda, b, q) \). Then

\[ \frac{\Gamma_q(2)}{\Gamma_q(2 + \lambda)} |z|^{1+\lambda} \left( 1 - \frac{|b|}{(n+1)q + |b| - 1} |z|^n \right) \]

\[ \leq \left| I_{q,z} f(z) \right| \leq \left| \frac{\Gamma_q(2)}{\Gamma_q(2 + \lambda)} |z|^{1+\lambda} \right| \cdot \left( 1 + \frac{|b|}{(n+1)q + |b| - 1} |z|^n \right). \]

The result is sharp for the function \( f(z) \) given by (5.5) with \( \beta = 1 \) and \( \lambda \) replaced by \(-\lambda\).

**Theorem 12.** Let \( f(z) \in S_n^q(\lambda, \beta, b, q) \). Then

\[ \frac{\Gamma_q(2)}{\Gamma_q(2 + \lambda)} |z|^{1+\lambda} \left( 1 - \frac{\beta |b|}{(n+1)q + |b|} |z|^n \right) \]

\[ \leq \left| \Omega_{q,z} f(z) \right| \leq \left| \frac{\Gamma_q(2)}{\Gamma_q(2 + \lambda)} |z|^{1+\lambda} \right| \cdot \left( 1 + \frac{\beta |b|}{(n+1)q + |b|} |z|^n \right). \]

The result is sharp for the function \( f(z) \) given by

\[ D_{q,z} f(z) = \frac{\Gamma_q(2)}{\Gamma_q(2 - \lambda)} \left( 1 - \frac{\beta |b|}{(n+1)q + |b|} |z|^n \right) \]

or by the function \( f(z) \) given by (5.10).

Similarly, we can prove the following distortion inequalities for \( f(z) \in S_n^q(\lambda, \beta, b, q) \) involving the fractional \( q \)-derivative operator \( D_{q,z}^\lambda \) and the fractional \( q \)-integral operator \( I_{q,z}^\lambda \).

**Corollary 14.** Let \( f(z) \in G_n^q(\lambda, \beta, b, q) \). Then

\[ \frac{\Gamma_q(2)}{\Gamma_q(2 - \lambda)} |z|^{1-\lambda} \left( 1 - \frac{\beta |b|}{(n+1)q + |b|} |z|^n \right) \]

\[ \leq \left| D_{q,z} f(z) \right| \leq \left| \frac{\Gamma_q(2)}{\Gamma_q(2 - \lambda)} |z|^{1-\lambda} \right| \cdot \left( 1 + \frac{\beta |b|}{(n+1)q + |b|} |z|^n \right). \]
The result is sharp for the function $f(z)$ given by (5.10).

**Corollary 15.** Let $f(z) \in S^\alpha_n (\lambda, \beta, b, q)$. Then

\[
\frac{\Gamma_q(2)}{\Gamma_q(2 + \lambda)} |z|^{1 + \lambda} \left( 1 + \frac{\beta |b|}{[n + 1]_q [1 + \alpha([n + 1]_q - 1)]} |z|^n \right)
\]

is sharp for the function $f(z)$ given by (5.10).
6. Conclusion

In our present investigation, we applied various operators of $q$-calculus and fractional $q$-calculus in the study of two general subclasses $S_n^{\alpha}(\lambda, \beta, b, q)$ and $G_n^{\alpha}(\lambda, \beta, b, q)$ of normalized analytic functions with complex order and negative coefficients. For each of these function classes, we have derived their associated coefficient estimates, radii of close-to-convexity, starlikeness and convexity, extreme points, and growth and distortion theorems. Our main results and their new or known consequences are stated and proved as theorems and corollaries.

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