Small Contractions of Symplectic 4-folds

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Abstract: We classify small contractions of (holomorphically) symplectic 4-folds.

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1 Introduction

Definition 1. A symplectic manifold is defined to be a complex algebraic or analytic manifold $X$ of dimension $2n$, which carries a (holomorphic) symplectic 2-form $\sigma \in H^0(X, \Omega^2_X)$, that is $\sigma$ is closed and anywhere nondegenerate, which means $d\sigma = 0$ and $\sigma^{\wedge n}$ vanishes nowhere.

We note that the form $\sigma^{\wedge n}$ trivializes the canonical bundle $K_X$ so if $X$ is compact and simply-connected then it is a Calabi-Yau variety. The symplectic form $\sigma$ defines an isomorphism $\hat{\sigma} : T_X \to \Omega^1_X$ given by the formula $\hat{\sigma}(v)(w) = \sigma(v, w)$; we will call it a $\sigma$-duality. A reduced but possibly reducible subvariety $M \subset X$ is called Lagrangian if any component $M$ has dimension $n$ and at any smooth point $x$ of $M$ the form $\sigma$ is trivial on the tangent space $T_xM$.

We want to understand the local structure of birational morphisms of projective symplectic manifolds. For this purpose we introduce the following definition.
Definition 2. Let \( \varphi : X \to Y \) be a birational projective morphism (of complex algebraic varieties or complex analytic spaces) where \( X \) is a symplectic manifold and where \( Y \) is normal. We say that \( \varphi : X \to Y \) is a symplectic contraction.

We note that \( K_X = \varphi^* K_Y \) so \( \varphi \) is a crepant (or log terminal) contraction in the sense of the Minimal Model Program. In terms of the Program crepant contractions form a natural extension of Fano-Mori contractions for which \( -K_X \) is \( \varphi \)-ample (for a discussion on these see e.g. [AW2]). As it will turn out, symplectic contractions have some features which make them somewhat similar to Fano-Mori contractions.

In what follows we will frequently shrink both the domain as well as the target of the contraction in order to understand its local structure. Sometimes we will stress it by referring to them as to local contractions. Here is an example of a symplectic contraction.

Example. Let \( X = T^* \mathbb{P}^2 = \text{Spec}_{\mathbb{C}}(\bigoplus_{m \geq 0} S^m T_{\mathbb{P}^2}) \) be the cotangent bundle of \( \mathbb{P}^2 \). Then \( X \) is a smooth variety of dimension 4 which carries a natural symplectic form. The bundle \( T_{\mathbb{P}^2} \) is very ample, and if we set \( Y = \text{Spec} \bigoplus_{m \geq 0} H^0(\mathbb{P}^2, S^m T_{\mathbb{P}^2}) \) then \( Y \) is a variety of dimension 4 and we have a birational morphism \( \varphi : X \to Y \), defined by the evaluation of global sections

\[
\bigoplus_{m \geq 0} H^0(\mathbb{P}^2, S^m T_{\mathbb{P}^2}) \to \bigoplus_{m \geq 0} S^m T_{\mathbb{P}^2}
\]

which contracts precisely the zero section of \( T^* \mathbb{P}^2 \) to a single point. The morphism \( \varphi : X \to Y \) is a symplectic contraction and we refer to it as the collapsing of the zero section in the cotangent bundle of \( \mathbb{P}^2 \).

Let us note that conversely, if a symplectic 4-fold \( X \) contains a subvariety \( E \cong \mathbb{P}^2 \) then a formal neighborhood of \( E \) is isomorphic to the formal neighborhood of the zero section of \( T^* \mathbb{P}^2 \). Indeed, it is easy to verify that \( E \) is a Lagrangian subvariety with normal bundle isomorphic to \( T_{\mathbb{P}^2} = \Omega_{\mathbb{P}^2} \). Since for \( i \geq 0 \) we have the vanishing \( H^1(\mathbb{P}^2, T_{\mathbb{P}^2} \otimes S^i(T_{\mathbb{P}^2})) = H^1(\mathbb{P}^2, \Omega_{\mathbb{P}^2} \otimes S^i(T_{\mathbb{P}^2})) = 0 \) then by Grauert and Hironaka-Ross (see [Mo2, 3.33], [AW2, 3.7]) an analytic neighborhood is uniquely defined.

The variety \( Y \) in our example is equal to the affine cone over the projective variety \( \mathbb{P}(T_{\mathbb{P}^2}) \subset \mathbb{P}^7 \) which is a hyperplane section of the Segre embedding \( \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8 \). The morphism \( \varphi \) is a resolution of the vertex singularity which is the graph of a rational morphism obtained from the projection \( \mathbb{P}(T_{\mathbb{P}^2}) \subset \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2 \). The two projections lead to two non-equivalent resolutions \( \varphi : X \to Y \) and \( \varphi' : X' \to Y \) which are dominated by a simple blow-up of the vertex.

More generally, let \( X \) be a symplectic manifold of dimension 4 and suppose \( X \) contains a \( \mathbb{P}^2 \). Let \( \beta : Z \to X \) be the blow up of \( X \) along \( \mathbb{P}^2 \). As we noted above, in an analytic neighborhood of the exceptional divisor of \( \beta \) the manifold \( Z \) is isomorphic to the cotangent bundle of \( \mathbb{P}^2 \) blown-up along zero section. Thus there exists another blow-down map \( \beta' : Z \to X' \), where \( X' \) is again a symplectic manifold with a symplectic form \( \sigma' \) which coincides with \( \sigma \) outside of the exceptional locus of the transformation. We note that although the above arguments are performed in analytic category their algebraic counterpart holds whenever one assumes that the \( \mathbb{P}^2 \) in question is an isolated positive-dimensional fiber of a symplectic contraction \( \varphi : X \to Y \). Indeed, in such a case the blow-down \( \beta' : Z \to X' \) exists by the relative cone and contraction theorems, see e.g. [KM, 3.25] or [KMM, 4-2-1].

Definition 3. The birational map \( \phi : X \dashrightarrow X' \) constructed above is called the Mukai flop.
The main result of the paper is the following.

**Theorem 1.1.** Let $\varphi : X \to Y$ be a symplectic contraction with $Y$ quasiprojective. Suppose that $\dim X = 4$ and that $\varphi$ is small (i.e. it does not contract a divisor). Then $\varphi$ is local analytically isomorphic to the collapsing of the zero section in the cotangent bundle of $\mathbb{P}^2$ and therefore it admits a Mukai flop.

We note that in view of the preceding discussion the actual contents of the theorem is as follows: if $\varphi$ is a small contraction of a symplectic 4-fold then the exceptional locus of $\varphi$ may be just an analytic space, we put it here in order to use deformation results: [Wie2, Thm. 2.1] and [Kl, Thm. 1.3], which we need for the crucial observation 2.7.

Burns, Hu and Luo observed in [BHL] that the above theorem allows to understand birational transformations of symplectic 4-folds. A complete proof of this result will be given in a forthcoming paper by the first named author. Using [1] and the Minimal Model Program, in particular [Co, Prop. 2.7], and standard arguments providing termination of log-flips, one gets the following.

**Theorem 1.2.** Let $\phi : X \dasharrow X'$ be a birational map of two smooth projective 4-folds which is isomorphism in codimension 1. Suppose that $X$ is symplectic. Then $X'$ is symplectic as well and $\phi$ can be factorised in a finite sequence of Mukai flops.

The paper is organised as follows. Firstly we recall results which we use in the course of the proof of 1.1 and restate them in the suitable context. This concerns results about rational curves, vanishings, normal surfaces with many rational curves and local existence of a vector field on a symplectic manifold. The actual proof of 1.1 is divided into four steps: first we reduce the arguments to the case when $\varphi$ is an elementary contraction. Then we use deformation of rational curves, as developed by Mori and Kollár, in order to understand 1-dimensional singular locus of $E$: as the result we prove that $E$ is irreducible and homeomorphic to $\mathbb{P}^2$ in codimension 1. Using this we are able to produce locally near $E$ a line bundle $\mathcal{O}(1)$ for which we apply Kawamata base-point-free arguments. Finally we analyse a non-normal del Pezzo surface, appearing in a classification list by Reid, which is left as a possible exception by the previous argument.

Recently (September 2001) we were informed that Cho, Miyaoka and Shepherd-Barron have obtained a proof of a version of 1.1 valid in arbitrary dimension, see [CMSB]. Their method however is substantially different from ours.

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## 2 Preliminaries

Let $\varphi : X \to Y$ be a morphism of analytic spaces, which is a symplectic contraction. The morphism $\varphi$ is then a crepant or log-extremal contraction in the sense of the Minimal Model Program and as
such it admits several special properties which we want to discuss in this preliminary section. For the fundamental results of the Program we refer to [KMM], [KM] and [Ka] in the analytic set-up. When dealing with rational curves we will use the language and notation consistent with [Ko].

Let \( E = \bigcup E_i \) denote the exceptional locus of the symplectic contraction with \( E_i \) being its irreducible components with the reduced structure. In the present paper we shall discuss the case when \( E \) is contracted to an isolated singular point \( y \in Y. \) We will call such a \( \phi \) a very small symplectic contraction. The following result can be found in [Wie1, Ch. 4] or [Kl, Sec. 4], we sketch its proof after 2.21.

**Lemma 2.1.** Let \( \phi : X \to Y \) be a very small symplectic contraction with exceptional locus \( E = \bigcup E_i. \) Then the components \( E_i \) are projective varieties of dimension \( \dim X/2 \) and they are Lagrangian subvarieties of \( X. \)

By the Ionescu estimate, see [Ko, IV.2.6], in the above case \( E_i \) have the least possible dimension allowed for the components of the exceptional locus of a symplectic contraction and therefore we call it very small contraction. If \( \dim X = 4 \) then a very small contraction is the same as a small contraction in terms of the Minimal Model Program. By \( \mu_i : E_i \to E_i \subset X \) we will denote its normalization.

Although we are primarily interested in small symplectic contractions of 4-folds some of our preliminary arguments will be valid in arbitrary dimension. We shall examine the structure of \( E \) using two essential tools: deformation of rational curves and vanishing theorems.

### 2.1 Rational curves

By Mori [Mo] and Kawamata [Ka], the exceptional locus of a crepant contraction is covered by rational curves. By \( \text{Hom}(\mathbb{P}^1, X) \) we denote the scheme parametrizing morphisms from \( \mathbb{P}^1 \) into \( X, \) its points are named after morphisms they are associated to. For \( f \in \text{Hom}(\mathbb{P}^1, X) \) we can estimate the dimension of all components of the parametrizing scheme from below by \( \dim X + \deg(f^*T_X), \) see [Ko, II.1.2]. However, as it was observed by Bogomolov, for symplectic manifolds one gets actually a better estimate, see [Ra] and [Wie1].

**Proposition 2.2.** Let \( X \) be a symplectic manifold of dimension \( n. \) Suppose that either \( X \) is projective or it admits a symplectic contraction onto an affine variety. Then the dimension of every component of the scheme \( \text{Hom}(\mathbb{P}^1, X) \) is equal to \( n + 1 \) at least.

Let us sketch an argument, for details we refer to [Wie1, Ch. 3] or to [Wie2, Sec. 2]. We consider a 1-parameter deformation of \( X \) which does not extend the class of \( f(\mathbb{P}^1). \) That is, in the compact case by a result of Bogomolov, Tian and Todorov, while in the non-compact case by [Ko, Thm. 1.3], see also [KV, Thm. 3.6], there exists a 1-parameter family of deformations of \( X \) with the total space \( \mathcal{X} = \bigcup \mathcal{X}_t \) such that given \( f : \mathbb{P}^1 \to X = \mathcal{X}_0 \) no deformation of it extends to \( \mathcal{X}_t \) for \( t \neq 0. \) Hence we have the equality \( \text{Hom}(\mathbb{P}^1, X) = \text{Hom}(\mathbb{P}^1, \mathcal{X}) \) and thus the general estimate applied to \( \text{Hom}(\mathbb{P}^1, \mathcal{X}) \) gives the above estimate for \( \text{Hom}(\mathbb{P}^1, X). \)

Let \( F : \text{Hom}(\mathbb{P}^1, X) \times \mathbb{P}^1 \to X \) denote the evaluation morphism \( F(f, p) = f(p). \) Consider an \( \text{Aut}(\mathbb{P}^1) \)-invariant subset \( V \subset \text{Hom}(\mathbb{P}^1, X). \) By \( F_V \) we denote the restriction of the evaluation to \( V \times \mathbb{P}^1 \) and by \( \text{Locus}(V) \) the closure of the image of \( F_V. \) If \( x \in F_V(V) \) then by \( V_x \) we denote \( V \cap \text{Hom}(\mathbb{P}^1, X; 0 \to x) \) and by \( \text{Locus}(V_x) \) the closure of \( F(V_x). \) We shall use the notions of an unsplit and generically unsplit subset of \( \text{Hom}(\mathbb{P}^1, X), \) as defined in [Ko, IV.2] (in case of a
generically unsplit irreducible component of $\text{Hom}(\mathbb{P}^1, X)$ we will consider its open and non-empty
subset parametrizing birationally rational curves). Moreover the term unsplit family will also
be used for a subset of $\text{Hom}(\mathbb{P}^1, X; 0 \rightarrow x)$ with the obvious change that the subset has to be
$\text{Aut}(\mathbb{P}^1; 0)$-invariant.

**Definition 4.** Let $X$ be a projective variety with an ample divisor $H$ and let $W \subset \text{Hom}(\mathbb{P}^1, X)$ be
an irreducible subset.

(i) By $\deg_H W$ we denote the degree of the $f^*H$ over $\mathbb{P}^1$ for a $f \in W$.

(ii) We say that $W$ dominates a subset $E \subset X$ if $\deg_H W > 0$ and $\text{Locus}(W) = E$.

(iii) We call $W$ a minimal dominating for $E$ if it satisfies (ii) and $\deg_H W$ is minimal among all
components of $\text{Hom}(\mathbb{P}^1, X)$ which dominate $E$.

We note that a minimal dominating $V$ is generically unsplit in the sense of [Ko, IV.2]. However,
these two notions are not equivalent. In fact, if $W$ is minimal in the above sense then for a general
$x \in \text{Locus}(V)$ the set $W_x = W \cap \text{Hom}(\mathbb{P}^1, X; 0 \rightarrow x)$ is unsplit; such $W$ may be called locally
unsplit. If $X$ is obtained by blowing up $\mathbb{P}^2$ at a point then the lift-up of a (parametrization of a)
general line produces a generically unsplit family which is not locally unsplit in the above sense.

**Lemma 2.3.** Let $\varphi : X \rightarrow Y$ be a very small symplectic contraction a with the exceptional
locus $E = \bigcup E_i$. Suppose that an irreducible component $W \subset \text{Hom}(\mathbb{P}^1, X)$ parametrizes curves
contracted by $\varphi$ and assume moreover that either $W$ is generically unsplit or $\dim X = 4$ and $W$
contains a morphism which is birational onto its image. Then there exists a (unique) irreducible
component $E_W$ of $E$ which is dominated by $W$.

**Proof.** The proof follows from the above dimension estimate: we are supposed to get
$\dim \text{Locus}(V) \geq \dim X/2$. If $\dim X = 4$ then we just observe that $\dim V \geq \dim \text{Aut}(\mathbb{P}^1)$ so
$\dim \text{Locus}(V) > 1$. In arbitrary dimension we use [Ko, IV.2].

We recall that morphisms can be pushed-forward naturally. That is given a morphism $\mu : M \rightarrow N$
we have $\mu_* : \text{Hom}(\mathbb{P}^1, M) \rightarrow \text{Hom}(\mathbb{P}^1, N)$ defined as $\mu_*(f) = \mu \circ f$. If $\mu$ is the normalization then
for dominating subsets of $\text{Hom}(\mathbb{P}^1, N)$ the push-forward operation can be (generically) inverted.

**Lemma 2.4.** Let $\mu : \hat{M} \rightarrow M$ be the normalization of a projective irreducible variety and let
$W \subset \text{Hom}(\mathbb{P}^1, M)$ be an irreducible subset which dominates $M$. Then there exists a naturally
defined subset $\hat{W}$ of $\text{Hom}(\mathbb{P}^1, \hat{M})$ which dominates $\hat{M}$ and $\mu_* : \hat{W} \rightarrow W$ is proper and surjective.
Moreover $\mu_*$ and the operation $W \mapsto \hat{W}$ define bijection between dominating components and $W$
is minimal dominating component for $M$ if and only if $\hat{W}$ is minimal dominating component for
$\hat{M}$.

**Proof.** Let $\mu_W : \hat{W} \rightarrow W$ be the normalization of $W$. Since $W$ dominates $M$ the evaluation
$F_W : W \times \mathbb{P}^1 \rightarrow M$ can be lifted up to $\hat{F}_W : \hat{W} \times \mathbb{P}^1 \rightarrow \hat{M}$, so that
$\mu \circ F_W = F_W \circ (\mu_W \times \text{id}_{\mathbb{P}^1})$. Thus by the universality of $\hat{W}$ we have a morphism $\psi : \hat{W} \rightarrow \text{Hom}(\mathbb{P}^1, \hat{M})$
such that $\psi \times \text{id}_{\mathbb{P}^1}$ factors $\hat{F}_W$. We define $\hat{W}$ as the image of $\psi$ and because we have
$\mu_W = \mu_* \circ \psi$ then the properties of $\hat{W}$ follow.

The above procedure will be called a lift-up to the normalization and will be applied to
normalizations $\mu_i : E_i \rightarrow E_i$ of components of $E$. Similarly, by considering curves passing through
a fixed point we obtain families dominating normalizations and passing through a fixed smooth
point. Thus we get the following.
Corollary 2.5. Let $\varphi : X \to Y$ be a very small symplectic contraction with the exceptional locus $E = \bigcup_i E_i$. Then the normalization of each irreducible component $\hat{E}_i \to E_i$ is rationally connected and it is a rational surface if $\dim X = 4$.

Proof. Use [Ko, IV.2.6.1] to get $\dim \text{Locus}(V_x) = \dim X/2$ for $V$ generically unsplit and a general $x \in E_i = \text{Locus}(V)$. Thus we can lift up $V_x$ to a family of curves dominating $\hat{E}_i$.

In the course of the present paper we shall need the following definition, see [FOV, Sect.3].

Definition 5. Let $Z$ be a variety or an analytic space with a closed point $z$. The space $Z$ is called $d$-connected at $z$ if any component of $Z$ containing $z$ is of dimension bigger than $d$ and for any subspace $T \subset Z$ containing $z$ of dimension smaller than $d$ there exists a small neighborhood $U$ of $z$ in $Z$ such that $U \setminus T$ is connected.

Now the following estimate on $d$-connectedness of Hom is obtained exactly as the dimension estimate.

Proposition 2.6. Let $X$ be a projective variety which contains a rational curve $f : \mathbb{P}^1 \to C \subset X$. Then the scheme (or the analytic space) $\text{Hom}(\mathbb{P}^1, X)$ is $\dim X + \deg(f^*TX) - 1$ connected at $f$.

Proof. By [Ko, I.2.8 and proof of I.2.16] the Hom scheme is defined locally by $\dim H^1(\mathbb{P}^1, f^*TX)$ equations in a smooth space of dimension $\dim H^0(\mathbb{P}^1, f^*TX)$ thus the proposition follows by [FOV, 3.1.13] and Riemann-Roch.

In the case of a symplectic manifold the argument of extending $X$ by taking its 1-parameter deformation (which we presented above arguing for 2.2) applies and the actual estimate is better.

Corollary 2.7. Let $X$ be a symplectic 2n-fold. Suppose that either $X$ is projective or it admits a symplectic contraction onto an affine variety. Then $\text{Hom}(\mathbb{P}^1, X)$ is $n$-connected at any point.

2.2 Vanishings

We need the following vanishing due to Grauert-Riemenschneider, Kodaira, Kawamata and Viehweg, see [KMM, Sect. 1-2].

Theorem 2.8. Let $X$ be a smooth variety and $\varphi : X \to Y$ a birational proper morphism. If $L$ is a line bundle such that $K_X + L$ is $\varphi$-big and nef then $R^i_\varphi L = 0$ for $i > 0$. In particular if $\varphi$ is a Fano-Mori or crepant contraction then $R^i_\varphi O_X = 0$ for $i > 0$.

The vanishing is needed, among other things, for the following.

Lemma 2.9. Let $\varphi : X \to Y$ be a Fano-Mori or crepant birational contraction of a smooth variety with the exceptional locus $E = \bigcup_i E_i$ which is contracted to a point $y \in Y$. Then after possible shrinking both $X$ and $Y$ to an analytic neighborhood of $E$ and $y$, respectively, we have $\text{Pic}(X) \cong H^2(E, \mathbb{Z})$.

Proof. Firstly, because of the above vanishing, we may shrink $X$ so that $H^i(X, O_X) = 0$, for $i > 0$, and therefore the Chern class map $\text{Pic}(X) \to H^2(X, \mathbb{Z})$ becomes an isomorphism. Secondly, again possibly shrinking $X$, we get $E$ a deformation retract of $X$, by [Ko], and therefore $H^i(E, \mathbb{Z}) \cong H^i(X, \mathbb{Z})$. Combining these two we are done.
We shall need the following.

**Proposition 2.10.** Let \( \varphi : X \to Y \) be a very small contraction of a symplectic manifold of dimension \( 2n \). Let \( E = \bigcup E_i \) be the exceptional locus. By \( \mu_i : \tilde{E}_i \to E_i \) we denote the normalization of irreducible components. Then

\[
H^n(E_i, \mathcal{O}_{E_i}) = H^n(\tilde{E}_i, \mathcal{O}_{\tilde{E}_i}) = 0
\]

**Proof.** The vanishing for \( E_i \) is in [AW2, 1.7]. Then the vanishing for the normalization follows by cohomology of the sequence \( 0 \to \mathcal{O}_{E_i} \to (\mu_i)_* \mathcal{O}_{\tilde{E}_i} \to \mathcal{Q} \to 0 \), where the support of \( \mathcal{Q} \) is of dimension smaller than \( n \).

**Corollary 2.11.** Let \( \varphi : X \to Y \) be a small contraction of a symplectic 4-fold with the exceptional locus \( E = \bigcup E_i \). Then the normalization of each irreducible component is a rational surface with quotient singularities.

**Proof.** Let \( \pi_i : \hat{E}_i \to \tilde{E}_i \) be a desingularisation. Then, by rationality 2.3 we have \( H^j(\hat{E}_i, \mathcal{O}_{\hat{E}_i}) = 0 \) for \( j = 1, 2 \) and by 2.10 we get additionally \( H^2(\hat{E}_i, \mathcal{O}_{\hat{E}_i}) = 0 \) so \( R_2^j \pi_i \mathcal{O}_{\tilde{E}_i} = 0 \) by Leray spectral sequence for \( \pi_i \). Thus \( E_i \) has rational singularities which are quotient singularities as well.

### 2.3 Normal surfaces with many rational curves

The main result of this section is a 2-dimensional version of a result by Cho, Miyaoka and Shepherd-Barron announced in earlier versions of [CMSB]. We include the proof to make the present paper self-contained. We begin by stating a characterization of \( \mathbb{P}^2 \) obtained in the course of the proof of the main theorem of [Ka2], see step 2.2 of the proof.

**Proposition 2.12.** Let \( S \) be a normal projective surface with an ample divisor \( H \). Let \( f : \mathbb{P}^1 \to S \) be a morphism whose degree \( \deg(f^*H) \) with respect to \( H \) is minimal but positive. If for some \( p \in \mathbb{P}^1 \), with \( f(p) \in S \setminus \text{Sing}(S) \), we have \( \dim_f \text{Hom}(\mathbb{P}^1, S; p \mapsto f(p)) \geq 3 \) then \( S \cong \mathbb{P}^2 \).

We need to extend this characterization to quotients of \( \mathbb{P}^2 \).

**Theorem 2.13.** Let \( S \) be a normal projective surface with quotient singularities. Suppose that for a smooth point \( s \in S \) there exists a component \( W_s \) of \( \text{Hom}(\mathbb{P}^1, S; 0 \mapsto s) \) which is unsplit and \( \dim W_s \geq 3 \). Then the fundamental group \( G = \pi_1(S \setminus \text{Sing}(S)) \) acts algebraically on \( \mathbb{P}^2 \) and \( S \) is the quotient \( \mathbb{P}^2/G \) with the quotient morphism \( \nu : \mathbb{P}^2 \to S \) which is smooth covering outside the inverse image of singularities of \( S \).

In the course of the proof we will construct the morphism \( \nu \). We note that once the covering \( \nu : \mathbb{P}^2 \to S \), which is smooth outside \( \nu^{-1}(\text{Sing}(S)) \), is constructed then the rest of the theorem follows. Indeed, its restriction \( \mathbb{P}^2 \setminus \nu^{-1}(\text{Sing}(S)) \to S \setminus \text{Sing}(S) \) is then the universal cover of the smooth locus of \( S \). Thus, if \( G \) is the Galois group of this covering then it acts on \( \mathbb{P}^2 \) and \( S \cong \mathbb{P}^2/G \).

We follow Kollár’s book [Ka, pp. 108–112] and construct a normal family of rational curves \( V \) (we drop Kollár’s subscript “”) which is obtained from the normalization of \( W_s \) as a quotient by the action of \( \text{Aut}(\mathbb{P}^1, 0) \) with the universal \( \mathbb{P}^1 \) bundle \( \pi_U : U \to V \) which admits the evaluation morphism \( F_V : U \to S \). Points of \( V \) will be denoted by classes of morphisms, that is by \( [f] \), where \( f \in W_s \). The bundle \( U \to V \) has a section \( V_0 \subset U \) which is contracted by \( F_V \) to \( s \). Since the family
$W_s$ is unsplit the quotient $V$ is proper and the morphism $F$ is finite-to-one outside $V_0$ hence it is surjective.

After having made this preliminary construction, to which we will refer in the course of the proof, we state a somewhat more general observation.

**Lemma 2.14.** Let $B$ be a germ of a smooth curve (a small disc in the analytic set-up) with the closed (central) point $b$. Consider a morphism $F : B \times \mathbb{P}^1 \to X$ into a smooth variety such that $\dim(\text{im} F) = 2$. By $f$ let us denote $F_{b\times \mathbb{P}^1}$. Let $p_1, \ldots, p_k \in \mathbb{P}^1$ be such that $F(B \times p_i) = x_i$. Then the quotient $(f^*T_X)/T_{\mathbb{P}^1}$ has a (generically) non-zero section vanishing at points $p_i$.

**Proof.** Let $t$ be a local coordinate in $B$ with $b = \{ t = 0 \}$ and $v$ a nonvanishing vector field. Via the tangent morphism $T_{B \times \mathbb{P}^1} = T_B \times T_{\mathbb{P}^1} \to F^*(T_X)$ the field $v$ gives a non-zero section $s$ in $F^*(T_X)/T_{\mathbb{P}^1}$. The section vanishes along $B \times p_i$. Let $m$ denote the multiplicity with which it vanishes along $b \times \mathbb{P}^1$ ($m$ may be zero as well). Then $s/v^m$ does not vanish identically along $b \times \mathbb{P}^1$ and it is zero along $B \times p_i$ so its restriction to $b \times \mathbb{P}^1$ is what we are looking for.

**Corollary 2.15.** In the notation introduced above let $f \in W_s$ be such that $f^*(\mathbb{P}^1)$ is contained in the smooth locus of $S$. Then $f^*(T_S) = O(2) \oplus O(1)$ and consequently $f$ is immersion (that is $f^*(\mathbb{P}^1)$ has no cusp but possibly nodes), $W_s$ is smooth at $f$, and the evaluation $F_{W_s}$ is of maximal rank (smooth) along $f \times (\mathbb{P}^1 - \{0\})$. In particular $V$ is smooth at $[f]$, and $F_V : U \to S$ is étale over $\pi_{U'}^1([f]) \setminus V_0$.

**Proof.** Since $W_s$ is unsplit $\deg(f^*(T_S)) \leq 3$. Next, by the previous lemma, $f^*(T_S) = O(2) \oplus O(1)$. The rest follows by [Ko, II.3.4] and [Ko, II.2.16].

Now we want to have a version of the above corollary also to $f$ such that $f(\mathbb{P}^1)$ meets singularities of $S$. For this purpose we need a general, somewhat technical, observation which holds in the analytic category.

**Lemma 2.16.** Let $X$ be a normal complex variety with isolated quotient singularities. Let $C = \mathbb{P}^1$ and let $\Delta$ denote a small disc around $0 \in \mathbb{C}$. Let $F : \Delta \times C \to X$ be a holomorphic morphism such that $F(\Delta \times C \setminus \bigcup_i \{(0, p_i)\})$ is contained in the smooth locus of $X$. By $f$ we denote $F|_{(0 \times C)}$ and by $x_i$ we denote $F(0, p_i)$. Suppose that $f$ is birational onto its image. Then, after possibly shrinking $\Delta$ to a smaller disc $\Delta' \ni 0$ we can choose an open (analytic) subset $U \subset X$ such that $F(\Delta' \times C) \subset U$ and there exists a complex manifold $V$ and a holomorphic morphism $\pi : V \to U$ which satisfies the following conditions:

(i) $\pi$ is finite and smooth outside $\bigcup_i \pi^{-1}(0, x_i)$,
(ii) there exists a morphism $F' : \Delta' \times C \to V$ such that $\pi \circ F' = F|_{\Delta' \times C}$.

**Proof.** First we claim that we can find an étale cover $V' \to U$ of a neighborhood of $f(C)$ so that after lifting $F$ to $F' : \Delta' \times C \to V'$ the resulting $f' : C \to U'$ is bijective onto its image. In other words we want to separate branches of nodes of $f(C)$. (We can make our argument assuming that $U$ is smooth since at this stage we can embed all data into a smooth variety.) So take $U$ to be a small tubular neighborhood of $f(C)$ such that if $x = f(p_i)$, with $i = 1, \ldots, k$, is a multiple point then there exists a neighborhood $U_x$ (say a ball) such that $U \cap U_x = U_{x_1}^1 \cup \ldots \cup U_{x_k}^1$ and each $U_{x_i}^1$ is a tubular neighborhood of the respective branch of $C$. That is $C^i = f^{-1}(U_{x_i}^1)$ are disjoint neighborhoods of $p_i$, for $i = 1, \ldots, k$, and each $U_{x_i}^1$ is a tubular neighborhood of $f(C^i)$. We can moreover assume that
Theorem 2.18. Let 

\[ S \to \mathbb{P}^2 \]

be a normal projective surface with quotient singularities and \( H \) an ample divisor over \( S \). Assume that \( W \subset \text{Hom}(\mathbb{P}^1, S) \) is a minimal dominating component for \( S \). If 

\[ \dim W \geq 5 \]

then \( S \cong \mathbb{P}^2/\pi_1(S \setminus \text{Sing}(S)) \) is as described in the conclusion of 2.13 and moreover the quotient map \( \nu : \mathbb{P}^2 \to S \) induces a surjective morphism of components of Hom-schemes \( \nu_* : W_1 \to W \), where \( W_1 \subset \text{Hom}^{(\mathbb{P}^1, \mathbb{P}^2)} \) parametrizes lines. In particular such \( W \) is unique. If moreover any \( f \in W \) is birational onto the image then \( S \cong \mathbb{P}^2 \).

Proof. By what we have noticed above \( W_s = W \cap \text{Hom}(\mathbb{P}^1, S; 0 \to s) \) is unsplit for a general choice of \( s \in S \) and of dimension \( \geq 3 \). Thus 2.13 applies and actually its proof produces lifting of curves.
parametrized by $W_s$ to lines on $\mathbb{P}^2$. Thus it remains to prove the last statement of the theorem. To this end we note that the action of $G = \pi_1(S \setminus \text{Sing}(S))$ on $\mathbb{P}^2$ induces a dual action on $(\mathbb{P}^2)^*$ which parametrizes lines on $\mathbb{P}^2$. (The quotient $(\mathbb{P}^2)^*/G$ can be identified in Chow($S$) as a component parametrizing images of lines and the morphism $\nu_*: (\mathbb{P}^2)^* \to (\mathbb{P}^2)^*/G$ is the push-forward of cycles.) In particular, if $G_i \subset G$ is the stabilizer of a line $l \in (\mathbb{P}^2)^*$ then the morphism $\nu_l$ is of degree equal to $|G_i|$. Thus, if $\nu_l$ is birational then $(\mathbb{P}^2)^* \to (\mathbb{P}^2)^*/G$ is unramified covering hence it is isomorphism and we are done.

Having completed the proof of \ref{2.13} we derive the following conclusion for the proof of our main theorem.

**Corollary 2.19.** Let $\phi: X \to Y$ be a small contraction of a symplectic 4-fold with the exceptional locus $E = \bigcup_i E_i$. Then the normalization of each irreducible component $\mu_i: \hat{E}_i \to E_i \subset X$ admits a finite morphism $\nu_i: \mathbb{P}^2 \to \hat{E}_i$ which has the properties described in \ref{2.13} and \ref{2.18}.

**Proof.** In view of \ref{2.2} \ref{2.3} \ref{2.4} and \ref{2.13} the description of normalization follows. \hfill $\square$

### 2.4 A vector field

We begin this section by discussing a general fact related to properties of Fano-Mori and crepant contractions. This property was also observed by Campagna and Flenner [CF].

**Proposition 2.20.** Let $\phi: X \to Y \ni y$ be a local analytic Fano-Mori or crepant contraction and let $\sigma \in \Omega^1_X$ be a closed holomorphic form. After possibly shrinking $X$ to a smaller neighborhood of $\phi^{-1}(y)$ there exists a 1-form $\alpha \in H^0(X, \Omega^1_X)$, such that $d\alpha = \sigma$.

**Proof.** We consider the analytic de Rham complex over $X$:

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \xrightarrow{d_0} \Omega^1_X \xrightarrow{d_1} \Omega^2_X \xrightarrow{d_2} \Omega^3_X \xrightarrow{d_3} \cdots$$

Let us set $\mathcal{K} = \ker(d_0) = \text{im}(d_0)$. In order to prove the proposition we have to show that $H^i(X, \mathcal{K}) \to H^i(X, \Omega^1_X)$ is injective. After possibly shrinking $X$ so that its image is Stein we have vanishing $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. This implies that the boundary map $\delta: H^1(X, \mathcal{K}) \to H^2(X, \mathbb{C})$ is an isomorphism.

Now we look at the exponential sequence $0 \to \mathbb{Z}_X \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0$ to find out, again by $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$, that the boundary $\delta': H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z}_X)$ is an isomorphism too. In fact $\delta$ and $\delta'$ are related by a commutative square appearing in the following diagram

$$\begin{array}{ccc}
H^1(X, \mathcal{O}_X^\times) & \to & H^1(X, \mathcal{K}) \to H^1(X, \Omega^1_X) \\
\downarrow s' & & \downarrow \delta' \\
H^2(X, \mathbb{Z}) & \xrightarrow{\otimes \mathbb{C}} & H^2(X, \mathbb{C}).
\end{array}$$

where the composition of the upper row arrows is the Chern class $c: H^1(X, \mathcal{O}_X^\times) = \text{Pic} X \to H^1(X, \Omega^1_X)$, with $c(g) = dg/g = d(\log(g))$, see [Ha, II Ex. 7.4]. Thus we will be done if $c$ is injective.
Suppose $L \in \text{Pic } X$ is a line bundle such that $c(L) = 0$. The definition of $c$ is functorial hence descends to any curve contracted by $\varphi$ so $L$ is numerically trivial. This, by Kawamata-Shokurov base point freeness theorem, see [KMM, 3-1-1, 3-1-2] or [KM, 3.24], implies that $L$ is actually trivial.

At this point let us sketch a proof of the dimension estimate on the components of the exceptional locus of a symplectic contraction $2.1$, see [Wie, Ch. 4].

**Proof of 2.1.** The proof comes by combining the two tools which we have explained above. From the dimension estimate on Hom, 2.2, we get Ionescu’s inequality, see [Ko, IV.2.6]: $\dim E_i \geq n$. On the other hand by 2.20, after shrinking $X$ to a small neighborhood of $E$ we see that the cohomology class of $\sigma$ in de Rham cohomology of $X$ is trivial. Let $\hat{E_i} \rightarrow E_i$ be a projective resolution of singularities of a component $E_i \subseteq E$. The pullback of the form $\sigma$ is zero in the cohomology of $\hat{E_i}$ hence, by Hodge theory on $\hat{E_i}$, the pull-back is zero 2-form itself. Thus for any smooth point $x \in E_i$ the linear space $T_x E_i \subseteq T_x E$ is isotropic for $\sigma$, hence 2.1.

The 2.20 will work together with the following results which we formulate in a somewhat more general context.

Let $X$ be a symplectic 2n-fold with the symplectic form $\sigma$. Suppose that there is a 1-form $\alpha$ such that $d\alpha = \sigma$. Let $\xi$ be the vector field on $X$ which is via $\sigma$ dual to $\alpha$. Let $M \subseteq X$ be a (reduced, but possibly reducible) Lagrangian subvariety such that for some resolution $M' \rightarrow M$ we have $H^0(M', \Theta_{M'}) = 0$. Then the following two results hold.

**Proposition 2.21.** The vector field $\xi$ preserves $M$, that is, it induces a derivation $\zeta \in H^0(M, \Theta_M)$.

**Proof.** Let $\mathcal{I} \subseteq \mathcal{O}_X$ be the ideal sheaf defining $M$. Since $M$ is Lagrangian, there is a commutative diagram

$$
\begin{array}{cccc}
\mathcal{I}/\mathcal{I}^2 & \rightarrow & \Omega_X|_M & \xrightarrow{a} & \Omega_M & \rightarrow 0 \\
\downarrow & & \|\hat{\sigma} & \downarrow & \downarrow \\
0 & \rightarrow & \Theta_M & \xrightarrow{b} & (\mathcal{I}/\mathcal{I}^2)^\vee \\
\end{array}
$$

where $\zeta = \hat{\sigma}(a)$. Suppose $a(\alpha)$ is nonzero at some generic point of $M$. Then $a(\alpha)$ induces a nonzero section of $\Omega_{M'}$. But by assumption $H^0(M', \Theta_{M'}) = 0$. Therefore $a(\alpha)$ is zero at all generic points of $M$. Since $(\mathcal{I}/\mathcal{I}^2)^\vee$ is torsion free it follows that $b(\xi)$ is zero and therefore $\xi$ lifts to $\zeta \in H^0(M, \Theta_M)$.

The following result is due to Dmitri Kaledin.

**Proposition 2.22.** In the above situation if the derivation $\zeta$ is zero on $M$ then $M$ is smooth.

**Proof.** $M$ is Lagrangian, hence of dimension $n$. If $\zeta$ is zero on $M$, then the image of $\alpha$ in $\Omega_X|_M$ is zero and therefore $\alpha$ lifts to a global section of $\Theta_X \otimes \mathcal{I}$, where $\mathcal{I} \subseteq \mathcal{O}_X$ is the ideal sheaf defining $M$. Let $p \in M$ be a point. Then $\alpha_p \in \Omega_{X,p}$ is of the form $\alpha = \sum a_idx_i$ with $a_i \in \mathcal{I}_p$ and $b_i \in \mathcal{O}_{X,p}$. Since $\sigma$ is symplectic,

$$
\sigma^{\wedge n} = (d\alpha)^{\wedge n} = \sum da_{i_1} \wedge db_{i_1} \wedge \cdots \wedge da_{i_n} \wedge db_{i_n}
$$

is a non vanishing section of $\omega_X$. Let $m_p \subseteq \mathcal{O}_{X,p}$ be the maximal ideal. If $M$ is non-smooth at $p$, then the image of $\mathcal{I}_p \rightarrow m_p/m^2_p = \Omega_X \otimes k(p)$ is at most $(n-1)$-dimensional. Therefore $da_{i_1} \wedge \cdots \wedge da_{i_n} \mod m_p = 0$ for all $i_1, \ldots, i_n$. Therefore $\sigma^{\wedge n}$ vanishes at $p$. Contradiction.
As an application of the above we prove the following key ingredient to the proof of the main theorem.

**Proposition 2.23.** Let \( \varphi : X \to Y \) be a small contraction of a symplectic 4-fold with the exceptional locus \( E = \bigcup_i E_i \). Let \( \text{Sing}(E) \) denote the singular locus of \( E \). Then the normalization of each irreducible component \( \mu_i : \tilde{E}_i \to E_i \) admits a uniquely defined finite morphism \( \nu_i : \mathbb{P}^2 \to \tilde{E}_i \) which is unramified in codimension 1. Moreover the inverse image of \( \text{Sing}(E) \) under the composition \( \psi_i : \mathbb{P}^2 \to \tilde{E}_i \to E_i \subset E \) consists of points and rational curves.

**Proof.** In view of 2.19 we are only to prove the statement about the inverse image of the singular set. By 2.20 we can construct on \( X \) a 1-form \( \alpha \) such that \( d\alpha = \sigma \).

Suppose first that \( \alpha \) does not vanish identically on the components \( E_i \). Then by 2.21 it produces non-trivial differentiation \( \zeta \) on \( E_i \). The differentiation \( \zeta \) lifts up to \( \tilde{E}_i \) and because the morphism \( \mathbb{P}^2 \to \tilde{E}_i \) is étale outside the inverse image of singularities of \( E_i \) it lifts up to \( \mathbb{P}^2 \) too. Another explanation is as follows: by the uniqueness of the construction of the morphism \( \mathbb{P}^2 \to \tilde{E}_i \to E_i \subset E \subset X \) the action of a 1-parameter group on \( X \) associated to the vector field \( \xi \) lifts up to \( \mathbb{P}^2 \). Now by its very construction the differentiation (or the 1-parameter group action) must preserve the inverse image of the singular set of \( E \). Thus we are only to note that among curves on \( \mathbb{P}^2 \) only rational have this feature.

Now suppose that \( \alpha \) vanishes on a component \( E_i \) then by 2.22 \( E_i = \mathbb{P}^2 \) and it cannot be identically zero on any components of \( E \) meeting \( E_i \). Thus by the previous part the common locus consist of isolated points or rational curves.

\[ \square \]

### 3 Proof of Theorem 1.1

From now in this section on we deal exclusively with small contractions of symplectic 4-folds. That is, by \( \varphi : X \to Y \) we will denote a small contraction of a symplectic 4-fold with a connected exceptional locus \( E = \bigcup_i E_i \) contracted to an isolated singular point \( y \in Y \).

At this point, in view of 2.23 we have some information on the normalization of components \( E_i \) as well as on the locus of their common points and singularities. We want to prove that actually there is only one component and it is \( \mathbb{P}^2 \). Our intermediate task will be the irreducibility of \( E \) and 3.3 which says that the normalization of \( E \) is a homeomorphism to \( \mathbb{P}^2 \) in codimension 1. Because of 2.23 the line bundle \( \mathcal{O}(1) \) from \( \mathbb{P}^2 \) extends over \( X \) and Kawamata’s base point free technique can be applied. This idea was suggested to us by Daniel Huybrechts and Manfred Lehn.

#### 3.1 Reduction to \( \rho(X/Y) = 1 \)

Let \( E = F_1 \cup \cdots \cup F_r \) be the decomposition of \( E \) into 1-connected components, that is each \( F_i \) is 1-connected and \( F_i \) meets \( F_j \) in a finite number of points , see [FVO, 3.1.5].

**Lemma 3.1.** After possibly shrinking \( X \), the rank of \( \text{Pic}(X) \) is \( r \) and for any \( i = 1, \ldots, r \) there is a contraction \( \varphi_i : X \to Y_i \) which factorizes \( \varphi : X \to Y \) and such that the exceptional set of \( \varphi_i \) is precisely \( F_i \).

**Proof.** Firstly we note that because of 2.19 all curves in \( F_i \) are numerically equivalent, that is \( \dim N_1(F_i) = 1 \). Since \( F_i \) meet in dimension 0 the natural surjection \( \coprod_i F_i \to E \) implies an isomorphism \( \bigoplus_i N_1(F_i) \cong N_1(E) \) under which the cone \( NE(E) \) of effective 1-cycles in \( E \) is generated.
by classes of effective cycles in $F_i$. Thus $N_1(E)$ is of dimension $r$ and $NE(E)$ is simplicial with rays generated by classes of curves in $F_i$. On the other hand by 2.3, after possibly shrinking $X$, the inclusion $E \subset X$ implies the isomorphism $N^1(X/Y) \cong N^1(E)$. Thus $\rho(X/Y) = r$ and the rest of the lemma follows by the contraction theorem, see e.g. [KM, 3.7].

\[ \square \]

**Proposition 3.2.** Reduction to $\rho(X/Y) = 1$: Suppose that Theorem 1.1 holds with the additional assumption $\rho(X/Y) = 1$ (or with equivalent assumption that $E$ is 1-connected). Then it is true without this assumption as well.

**Proof.** Let $f : X \to Y$ be a contraction as in Theorem 1.1. Suppose that $\rho(X/Y) = r > 1$. Let $F_1, \ldots, F_r$ be as above. Then by 3.1 we obtain for each $i = 1, \ldots, r$ a contraction $f_i : X \to Y_i$, which contracts just $F_i$. Note that $\rho(X/Y_i) = 1$. Therefore we can apply Theorem 1.1 with the additional assumption $\rho = 1$ to the contraction morphism $f_i : X \to Y_i$ to conclude that $F_i \cong \mathbb{P}^2$.

Therefore all exceptional components of $E$ are $\mathbb{P}^2$’s and they meet only in finitely many points. Let $E_1$ and $E_2$ be two irreducible components with nonempty intersection. Let $X'$ be obtained from $X$ by the Mukai flop of $E_1$. Then the strict transform $E_2'$ of $E_2$ is a nontrivial blow up of a $\mathbb{P}^2$. On the other hand there is a contraction $X' \to Y$ satisfying the conditions of Theorem 1.1 and $E_2'$ is an exceptional component. Therefore, by 2.13 the surface $E_2'$ cannot be a nontrivial blow up of $\mathbb{P}^2$. Contradiction.

\[ \square \]

### 3.2 Connectivity arguments

In view of Proposition 3.2 we assume from now on that $\rho(X/Y) = 1$ which, by 3.1, is the same as saying that $E$ is connected in codimension 1. By $L$ let us denote the positive generator of $Pic(X/Y)$. Let $E = E_1 \cup \cdots \cup E_n$ be the decomposition into irreducible components. For each $i = 1, \ldots, n$, let $\mu_i : \bar{E}_i \to E_i$ be the normalization and let $\nu_i : \mathbb{P}^2 \to \mathbb{P}^2/G_i = \bar{E}_i$ be the quotient map given by 2.13 and let $\psi_i : \mathbb{P}^2 \to E_i$ be the composition of these two maps. By $(\nu_i)_*$ and $(\mu_i)_*$ we denote the natural morphism of Hom-schemes: $(\nu_i)_*(f) = \nu \circ f$, $(\mu_i)_*(f) = \mu \circ f$.

The task of the present section, which is proving that $E$ is homeomorphic to $\mathbb{P}^2$ in codimension 1, is achieved in three steps. The main tool used in each step is the connectivity feature 2.7.

**Proposition 3.3.** The normalization $\bar{E}_i$ of any component is isomorphic to $\mathbb{P}^2$.

**Proof.** Let $W_i \subset \text{Hom}(\mathbb{P}^1, X)$ be a minimal component with respect to $L$ dominating $E_i$. Using the normalization we lift up $W_i$ to $\bar{W}_i \subset \text{Hom}(\mathbb{P}^1, \bar{E}_i)$ which is minimal and dominating for $\bar{E}_i$ and thus by 2.18 it is unique and the image $(\nu_i)_*(W^1)$ of $W^1 \subset \text{Hom}(\mathbb{P}^1, \mathbb{P}^2)$ parametrizes lines. Suppose that $\nu_i$ is not isomorphism. We will prove that this implies that $\text{Hom}(\mathbb{P}^1, X)$ is not 4-connected at some point hence we will arrive to contradiction with 2.7.

According to 2.13 there exists $f \in W^1$ such that the degree of the morphism $\pi_i \circ f = \mu_i \circ \nu_i \circ f$ is bigger than 1, say it is of degree $d$. Let $g \in \text{Hom}(\mathbb{P}^2, X)$ denote the normalization of the image of $\psi_i \circ f$. We have a degree $d$ covering $\pi_d : \mathbb{P}^1 \to \mathbb{P}^1$ such that $g \circ \pi_d = \psi_i \circ f$. Let $W_g \subset \text{Hom}(\mathbb{P}^1, X)$ be a component containing $g$, then by 2.3 there exists a component $E_j \neq E_i$ which is dominated by $W_g$. Now taking composition $W_g \ni h \to h \circ \pi_d \in \text{Hom}(\mathbb{P}^1, X)$ we obtain a subset of $\text{Hom}(\mathbb{P}^1, X)$ which dominates $E_j$ and which contains also $\psi_i \circ f \in W_i$. Thus, apart of $W_i$ we have at least one irreducible component of $\text{Hom}(\mathbb{P}^1, X)$ which contains $\psi_i \circ f$.

Now in order to prove that $\text{Hom}(\mathbb{P}^1, X)$ is not 4-connected at $\psi_i \circ f$ it is enough to show that if $h$ is a small deformation of $\psi_i \circ f$ which is in $W_i \setminus \psi_i \circ f \circ \text{Aut}(\mathbb{P}^1)$ then $h$ is not contained in any
other irreducible component of \( \text{Hom}(\mathbb{P}^1, X) \). If however it was the case then we would get another component of \( \text{Hom}(\mathbb{P}^1, X) \) which dominates \( E_i \) and which is of the same degree with respect to \( H \) as the component \( W_i \). This contradicts the uniqueness statement of \( \text{Proposition } 3.18 \). \( \square \)

**Proposition 3.4.** \( E \) is irreducible.

**Proof.** Suppose that the proposition is not true. Since \( E \) is 1-connected we can choose \( C \subset E \), a curve which is common to components \( E_1, \ldots, E_r \) where \( r > 1 \). Let \( f_1 : \mathbb{P}^1 \to C \subset X \) be the normalization. It is uniquely defined up to the action of \( \text{Aut}(\mathbb{P}^1) \). Let \( W_1 \) be an irreducible component of \( \text{Hom}(\mathbb{P}^1, X) \) which contains \( f_1 \). Then, because of \( \text{Proposition } 3.4 \) the evaluation map \( F_{W_1} \) dominates one of the components of \( E \), say \( E_1 \). Let \( E_2 \) be another component of \( E \) which contains \( C \) as well. Let \( μ_2 : \mathbb{P}^2 \to E_2 \subset X \) be the normalization, \( \text{Proposition } 3.3 \). Take a curve \( C_2 \subset \mathbb{P}^2 \) which is mapped to \( C \). By \( \text{Proposition } 3.3 \) the curve \( C_2 \) is rational so let \( f_2 : \mathbb{P}^1 \to C_2 \) be its normalization; by \( \text{Proposition } 3.3 \) denote the (unique) component containing \( f_2 \). Let \( W_2 = (μ_2)_*(W) \) be its image which dominates \( E_2 \). As in the proof of \( \text{Proposition } 3.3 \) we note that \( (μ_2)_*(f_2) = μ_2 \circ f_2 \) apart of being contained in \( W_2 \) is contained also in another component of \( \text{Hom}(\mathbb{P}^1, X) \) which dominates \( E_1 \) and which contains morphisms from \( W_1 \) composed with a finite morphism \( π_d : \mathbb{P}^1 \to \mathbb{P}^1 \) such that \( f_1 \circ π_d = μ_2 \circ f_2 \)

We claim that \( \text{Hom}(\mathbb{P}^1, X) \) is not 4-connected at \( μ_2 \circ f_2 \), more precisely that it is not connected after removing \( μ_2 \circ f_2 \circ \text{Aut}(\mathbb{P}^1) \). Indeed, otherwise there would exist small deformations of \( μ_2 \circ f_2 \) in \( W_2 \) which were contained in a component of \( \text{Hom}(\mathbb{P}^1, X) \) different from \( W_2 \). Thus we would get a component \( W_3 \subset \text{Hom}(\mathbb{P}^1, X) \) dominating \( E_2 \). Lifting \( W_3 \) to \( \tilde{W}_3 \subset \text{Hom}(\hat{\mathbb{P}}^1, \hat{\mathbb{P}}^2) \) we would get a component meeting \( \tilde{W}_2 \). This impossible because \( \text{Hom}(\mathbb{P}^1, \mathbb{P}^2) \) is smooth. \( \square \)

**Proposition 3.5.** The normalization map \( μ : \mathbb{P}^2 \to E \) is a homeomorphism in codimension 1.

**Proof.** Let \( Σ = \{ x \in E : |μ^{-1}(x)| > 1 \} \) be the set over which \( μ \) is not bijective. We argue by contradiction. Let us assume that \( Σ \) contains a 1-dimensional component \( C_0 \). Suppose that \( C_1, \ldots, C_k \), with \( k \geq 1 \), are curves which are mapped via \( μ \) to \( C_0 \); by \( \text{Proposition } 3.3 \) they are all rational curves. Let \( f_0 : \mathbb{P}^1 \to C_0 \subset X \) denote the normalization. Choose an irreducible component \( W_0 \subset \text{Hom}(\mathbb{P}^1, X) \) which contains \( f_0 \). Then \( W_0 \) dominates \( E \) and thus we can lift it to component \( W_1 \subset \text{Hom}(\mathbb{P}^1, \mathbb{P}^2) \) (recall that this means that \( μ_*(W_1) = W_0 \). If \( f_1 \in W_1 \) is such \( μ_*(f_1) = f_0 \) then \( f_1 \) maps \( \mathbb{P}^1 \) birationally onto a rational curve in \( \mathbb{P}^2 \), say \( C_1 \), and \( μ : C_1 \to C_0 \) is birational. Hence \( k \geq 2 \) and moreover \( \text{deg}C_i = \text{deg}(μ_{μ(C_i)}) \cdot \text{deg}C_1 \).

After re-numerating curves we may assume that \( \text{deg}C_2 \) is the smallest among \( \text{deg}C_2, \ldots, \text{deg}C_k \). Let \( W_2 \subset \text{Hom}(\mathbb{P}^1, \mathbb{P}^2) \) be the set parametrizing curves of degree \( \text{deg}C_2 \) (recall that any component of \( \text{Hom}(\mathbb{P}^1, \mathbb{P}^2) \) parametrizing morphisms of a given degree is smooth and connected), so that the normalization \( f_2 : \mathbb{P}^1 \to C_2 \subset \mathbb{P}^2 \) is in \( W_2 \). Let \( d = \text{deg}C_2/\text{deg}C_1 \), then exists the unique degree \( d \) cover \( π_d : \mathbb{P}^1 \to \mathbb{P}^1 \) such that \( μ \circ f_1 \circ π_d = f_0 \circ π_d = μ \circ f_2 \) and clearly \( f_1 \circ π_d \neq f_2 \).

Let \( W_2 = μ_*(W_2) \), then \( W_2 \) is a connected component of \( \text{Hom}(\mathbb{P}^1, X) \) (note that because of \( \text{Proposition } 3.3 \) any irreducible component of \( \text{Hom}(\mathbb{P}^1, X) \) dominates \( E \) so can be lift up to a component of \( \text{Hom}(\mathbb{P}^1, \mathbb{P}^2) \)). We claim that \( W_2 \) is not 4-connected (in the analytic topology !!) at \( μ \circ f_2 \). Let us take a small deformation of \( f_2 \), call it \( h \), which is not contained in \( f_2 \circ \text{Aut}(\mathbb{P}^1) \). Then the generic point of \( h(\mathbb{P}^1) \) is in the set where \( μ \) an isomorphism is and therefore \( μ \circ h \) lifts up to \( h \), that is \( μ^{-1}_*(μ(h)) = h \). Thus, if we take a small analytic neighbourhood \( \mathcal{U} \ni μ \circ f_2 \), such that \( μ^{-1}_*(\mathcal{U}) \)
decomposes into disjoint small neighborhoods of the inverse images \( \mu^{-1}(\mu \circ f_2) \ni f_2, f_1 \circ \pi_d \) then \( U \setminus \mu \circ f_2 \circ \text{Aut}(\mathbb{P}^1) \) will be disconnected.

### 3.3 Non-vanishing

In this section we will use base-point-free techniques, due to Kawamata, Kollár. In [AW1] the method was developed for studying local contractions, in the present situation we use [Me] for reference as it is particularly applicable in our case.

**Proposition 3.6.** There is a line bundle \( O_X(1) \) on \( X \), such that \( \nu^*O_X(1) \cong O_{\mathbb{P}^2}(1) \).

**Proof.** The exponential sequences on \( X \), \( E \) and \( \mathbb{P}^2 \) together with the vanishing of the higher cohomology groups of \( O_X \) and \( O_{\mathbb{P}^2} \) give the following diagram coming from the normalization \( \mu \):

\[
\begin{array}{ccc}
\text{Pic} X & \rightarrow & \text{Pic} E \\
\| & & \| \\
H^2(X, \mathbb{Z}) & \xrightarrow{i^*} & H^2(E, \mathbb{Z}) & \xrightarrow{\mu^*} & H^2(\mathbb{P}^2, \mathbb{Z})
\end{array}
\]

By 2.9, \( i^* \) is an isomorphism. By 3.5, \( \mu^* \) is an isomorphism. Therefore \( \text{Pic} X \rightarrow \text{Pic} \mathbb{P}^2 \) is an isomorphism.

**Proposition 3.7.** After possibly shrinking \( X \) the bundle \( O_X(1) \) has a section that doesn’t vanish on \( E \). If we denote its zero set by \( X' \) then \( X' \) is smooth.

**Proof.** As in [AW1, Claim 3.1], we find a \( \mathbb{Q} \)-divisor \( D \) on \( X \), which is a pull back from \( Y' \) and such that \( (X, D) \) is klt outside \( E \) and lc at \( E \) (see [KM] 2.34 e.g. for definition). Let \( W \subset X \) be a minimal center of lc singularities of \( (X, D) \) (see e.g. [Me] for a definition). By construction \( W \subset E \). By Kawamata [Ka4, 1.6], see also [Me, Thm. 1.12], \( W \) is normal. Since \( E \) may be assumed to be non-normal, it follows that \( \dim W \leq 1 \). We can now apply [Me, Lemma 2.2.ii] (using Mella’s notation we have in our case \( r = 0 \) and \( \gamma = 0 \)) to conclude that there exists a section of \( O_X(1) \) which doesn’t vanish identically on \( W \). This finishes the first part, to prove the second part it is enough to check smoothness along \( X' \cap E \). Let \( p \in X' \cap E \) be a point. Since \( E \) normalizes to a \( \mathbb{P}^2 \) we find a line in \( \mathbb{P}^2 \) with image \( C \) in \( E \) such that \( p \in C \) and \( C \not\subset X' \). By construction, the intersection number \( X' \cdot C \) in \( X \) is 1. This implies that both \( X' \) and \( C \) are smooth at \( p \).

Let \( \varphi' : X' \rightarrow Y' \) be the normalized restriction of \( \varphi \) to \( X' \). Let \( O_X(1) \) be the restriction of \( O_X(1) \) to \( X' \). Note that by adjunction, \( K_{X'} = O_{X'}(1) \). Let \( F = E \cap X' \subset X \). In \( X' \) the set \( F \) is \( \varphi' \)-exceptional. Note that \( F \) is irreducible and it is the image of a line in \( E = \mathbb{P}^2 \).

**Proposition 3.8.** In the above situation one of the two cases holds:

(i) \( F \) is smooth.

(ii) \( O_X(1) \) has a section that does not vanish on \( F \).

**Proof.** Again, by [AW1, Claim 3.1] we find a \( \mathbb{Q} \)-divisor \( D \) on \( X' \), which is a pull back from \( Y' \) and such that \( (X', D) \) is lc outside \( F \) and lc inside \( F \). Let \( W \) be a minimal center of lc singularities.
Again, by Kawamata [Ka4], see [Mc, Thm 1.12], \( W \) is normal. If \( \dim W = 1 \), then it follows that \( F \) is normal hence it is smooth and we are in case (i).

Let’s suppose that \( \dim W = 0 \). We now wish to apply [Mc, Lemma 2.3.iii]. Using Mella’s notation, we have \( \gamma = 0 \), \( r = -1 \), \( w = 1 \). Therefore there exists a section of \( \mathcal{O}_X^1(1) \) that doesn’t vanish on \( F \). This gives Case (ii).

Now we will discuss the cases.

**Proposition 3.9.** Suppose Proposition 3.8.i holds. Then \( E \) is regular in codimension 1.

*Proof.* Let \( l \subset \mathbb{P}^2 \) be the line such that \( \mu(l) = F \). We claim that \( \mu : \mathbb{P}^2 \to E \subset X \) is an immersion along \( l \). So we check the derivative map \( (T_{\mathbb{P}^2})_l = T_l \oplus N_{l/\mathbb{P}^2} \to (T_X)_{|F} = T_F \oplus N_{F/X} \). Since \( l \to F \) is an isomorphism it is enough to verify \( N_{l/\mathbb{P}^2} \to N_{F/X} \). Since however \( F \subset X \) thus we can consider the composition \( N_{l/\mathbb{P}^2} \to N_{F/X} \to (N_{X'/X})_{|F} \) which is an isomorphism. \( \square \)

The proof of the following proposition is postponed until the next section.

**Proposition 3.10.** Suppose Proposition 3.8.ii holds. Then \( E \) is regular in codimension 1.

Assuming 3.10 in order to finish the proof of Theorem 1.1 it is enough to prove the following result.

**Proposition 3.11.** Suppose \( E \) is regular in codimension 1. Let \( \mathcal{I} \subset \mathcal{O}_X \) be the ideal sheaf defining \( E \). Then

(i) \( H^2(E, \mathcal{I}/\mathcal{I}^{r+1}) = 0 \) for each \( r \geq 1 \),

(ii) \( H^1(E, \mathcal{O}_E) = 0 \),

(iii) \( E \) is normal.

*Proof.* Let \( \mu : \mathbb{P}^2 \to E \) be the normalization map.

(i) There is a natural surjective map \( S^r(\mathcal{I}/\mathcal{I}^2) \to \mathcal{I}^r/\mathcal{I}^{r+1} \). Therefore it is enough to show that \( H^2(E, S^r(\mathcal{I}/\mathcal{I}^2)) = 0 \). But since \( E \) is Lagrangian, the sheaves \( \mu_* S^r E \) and \( S^r(\mathcal{I}/\mathcal{I}^2) \) are isomorphic over the smooth locus of \( E \), in particular they are isomorphic in codimension 1. We note that any two sheaves that are isomorphic in codimension 1 on have the same top cohomology class. Therefore it is enough to show that \( H^2(E, \mu_* S^r T_{\mathbb{P}^2}) \) is zero. But this follows from the well know fact that \( H^2(\mathbb{P}^2, S^r T_{\mathbb{P}^2}) = 0 \) for all \( r > 0 \).

(ii) Let \( E_r \) be the closed subscheme of \( X \), defined by the ideal sheaf \( \mathcal{I}^r \). The obstruction to lifting an element of \( H^1(E_r, \mathcal{O}_{E_r}) \) to \( H^1(E_{r+1}, \mathcal{O}_{E_{r+1}}) \) lies in \( H^2(E, \mathcal{I}^r/\mathcal{I}^{r+1}) \), which is zero. This implies that the restriction map \( \lim H^1(E_r, \mathcal{O}_{E_r}) \to H^1(E, \mathcal{O}_E) \) is surjective. By the Theorem on formal functions [Ha, II.11.1], the left hand side is isomorphic to \( (R^1 f_* \mathcal{O}_X)_0 \), which is zero by 2.8. Therefore \( H^1(E, \mathcal{O}_E) = 0 \).

(iii) We define the sheaf \( Q \) on \( E \) to be the quotient

\[
0 \to \mathcal{O}_E \to \mu_* \mathcal{O}_{\mathbb{P}^2} \to Q \to 0
\]

Then \( Q \) has support precisely on the locus of points where \( \mu \) is not an isomorphism, that is \( Q \) has only zero-dimensional support. Now take cohomology in above sequence to obtain

\[
0 \to H^0(E, \mathcal{O}_E) \to H^0(E, \mu_* \mathcal{O}_{\mathbb{P}^2}) \to H^0(E, Q) \to H^1(E, \mathcal{O}_E).
\]

We have \( h^0(E, \mathcal{O}_E) = 1 \) and \( h^0(E, \mu_* \mathcal{O}_{\mathbb{P}^2}) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 1 \). By (ii), we have \( h^1(E, \mathcal{O}_E) = 0 \). This implies \( h^0(E, Q) = 0 \) and consequently \( Q = 0 \). This implies that \( \mathbb{P}^2 \cong E \). \( \square \)
3.4 A non-normal surface

We assume from now on in this section that Proposition 3.8.ii holds and moreover, in order to derive the contradiction with 3.10 that $E$ is not regular in codimension 1 so that in particular $O_E(1)$ is not spanned.

Lemma 3.12. In the situation of 3.8.ii the line bundle $O_X(2)$ is spanned.

Proof. Since $H^1(O_X(1)) = 0$, the restriction map $H^0(O_X(2)) \to H^0(O_X'(2))$ is surjective. Therefore it is enough to show that $O_X'(2)$ is spanned. By assumption, $O_X'(1)$ has a section that does not vanish on $F$. Let $X'' \subset X'$ be its zero locus. Then $X''$ is affine hence $O_X''(2)$ is spanned. Therefore it is enough to show that the restriction map $H^0(O_X'(2)) \to H^0(O_X''(2))$ is surjective. For that it is enough to show that $H^1(O_X'(1)) = 0$, which in turn follows from the exact sequence $H^1(O_X'(1)) \to H^1(O_X'(1)) \to H^2(O_X)$ and the vanishing of $H^3(O_X(1))$ and $H^2(O_X)$. □

Now we use the sections of $O_X(1)$ and $O_X(2)$ to analyse the geometry of $E$.

We pull-back sections of $O_E(a)$ to sections of $O_{P^2}(a)$. By 3.6 and 3.8.i, $O_E(1)$ has two sections, their pull-backs are denoted by $x, y \in H^0(P^2, O_{P^2}(1))$. Therefore $O_E(2)$ admits three sections $x^2, xy, y^2 \in H^0(P^2, O_{P^2}(2))$. Moreover, by 3.12 $O_E(2)$ admits a section $w \in H^0(P^2, O_{P^2}(2))$, such that $(x^2, xy, y^2, w)$ do not vanish simultaneously. Let $z \in H^0(P^2, O_{P^2}(1))$, such that $[x, y, z]$ form a homogeneous coordinate system of $P^2$. Then $w$ is a quadratic polynomial in $x, y, z$. Let $\pi : P^2 \to P^3$ be the morphism defined by $[x, y, z] \mapsto [x^2, xy, y^2, w]$; the image of $\pi$ is the singular quadric cone $Q \subset P^3$. By the construction the morphism $\pi$ factors through the normalization $\mu$ so that we have $\eta : E \to Q$ which is a $2 : 1$ covering. Thus $E$ is “sandwiched” $P^2 \to E \to Q$ between two known surfaces.

Let $e_0 \in E$ be the unique base point of $O_E(1)$. Then $e_0$ is a smooth point $E$ (the sections of $O_E(1)$ pull-back to local coordinates around the inverse image of $e_0$) and it is mapped to the vertex $q_0$ of $Q$. Let us blow-up $Q$, respectively $E$ and $P^2$, along $q_0, e_0$ and $\mu^{-1}(e_0)$ to $Q', E'$ and $P'$ and denote the exceptional curves by $A_Q, A_E$ and $A_P$, respectively. The induced morphism of the blow-ups we denote by $\mu', \pi'$ and $\eta'$, respectively. Then $Q'$ is rational ruled with the exceptional curve $A_Q$ and the fiber of the ruling denoted by $F_Q$. If $O_{P'}(1)$ denote the pullback of $O_{P^2}(1)$ then $O_{E'}(1) = (\pi')^*(A_Q + 2F_Q)$, moreover $(\pi')^*(A_Q) = 2A_P$. The branching divisor of $\pi' : P' \to Q'$ consist of $A_Q$ and a curve $B_Q$, disjoint from $A_Q$, which is in $|O_{Q'}(A_Q + 2F_Q)|$, the latter can be checked by adjunction. Therefore $\pi'_*O_{P'} = O_{Q'} \oplus O_{Q'}(-A_Q - F_Q)$. By $B_P \subset P'$ let us denote the component of the ramification of $\pi'$ which is over $B_Q$; it is a lift-up of a line $B \subset P^2$.

The sequence $P' \to E' \to Q'$ can be described in terms of inclusions of $O_{Q'}$ algebras: $O_{Q'} \to \eta'_*O_{E'} \to \pi'_*O_{P'}$. The trace splits off the trivial factor $O_{Q'}$ in $\pi'_*O_{P'}$ as well as in $\eta'_*O_{E'}$, so that we can write $\eta'_*O_{E'} = O_{Q'} \oplus \mathcal{J}$ for some rank 1 sheaf of $O_{Q'}$ modules which admits morphism $\mathcal{J} \otimes \mathcal{J} \to O_{Q'}$ coming from the multiplication in $O_{Q'}$ algebra structure. Let $S$ be the reflexivisation of $\eta'_*O_{E'}$. Since $\eta'_*O_{E'}$ is torsion free we have an inclusion $\eta'_*O_{E'} \to S$ which extends to $O_{Q'} \to \eta'_*O_{E'} \to S \to \pi'_*O_{P'}$.

Moreover, we have the splitting $S = O_{Q'} \oplus S_1$. Being reflexive of rank 1 the sheaf $S_1$ is a line bundle and since it is the reflexivisation of $\mathcal{J}$ it admits the unique morphism $S_1 \otimes S_1 \to O_{Q'}$ which gives a natural $O_{Q'}$ algebra structure on $S$. Thus we can set $S' = Spec_{Q'}S$ and we have a sequence of surjective morphism associated to the above inclusions of sheaves $P' \to S' \to E' \to Q'$.
Since $\mathbb{P}' \to E'$ is an isomorphism around $A_2 \to A_E$ the image of $A_2$ in $S'$ is a $(-1)$-curve $A_S$ which can be blow-down to a smooth point on a surface $\beta : S' \to S$. Thus we get a sequence

$$\mathbb{P}^2 \to S \to E \to Q$$

and $S$ is a “partial normalization”. By the construction the resulting morphism $\alpha : S \to E$ is an isomorphism in codimension 1 and $S$ satisfies the Serre’s condition $S_2$ hence it is the $S_2$-ization of $E$ c.f. [Re]. By $O_S(1)$ let us denote the pull-back of $O_E(1)$ and by $B_S$ the (reduced) image of $B$, the ramification divisor of $\pi$.

**Lemma 3.13.** The surface $S$ is a del Pezzo non-normal surface: it has locally complete intersection singularities (hence it is Cohen-Macaulay), its non-normal locus is $B_S$ and its dualising sheaf is $O_S(-1)$.

**Proof.** The surface $S'$ is produced as divisor in the total space of the line bundle $S_1$ given by the section coming from the multiplication $S_1 \otimes S_1 \to O_Q$, hence $S$ has only locally complete intersection singularities. Since outside $B$ and $A_S$ the surface $S'$ was obtained by factorizing a local analytic isomorphism coming from $\mathbb{P}' \to Q'$ it is smooth; the smoothness of $S$ at the image of $A_S$ comes from the construction. Thus it remains to compute the dualising sheaf of $S$. For this we compute the line bundle $S_1$: by the inclusion $S_1 \to O_Q(-A_Q-F_Q)$ we know that $S_1 = O(-A_Q-F_Q-D)$ where $D$ is an effective and non-zero divisor (we assume that $E$ is not regular in codimension 1 !) such that $D \cap A_Q = \emptyset$ and thus $D \in |d(A_Q+2F_Q)|$ where $d > 0$. On the other hand by the vanishing 2.10 we know that $H^2(S, O_S) = 0$ hence also $H^2(Q', S_1) = 0$. Comparing this with the previous observation and using duality we get $H^0(Q', S'_1 \otimes K_{Q'}) = H^0(Q', O_{Q'}((d-1)A_Q + (2d-3)F_Q) = 0$

Therefore $d = 1$. Now, denoting by $\gamma$ the morphism $S' \to Q'$ we can compute $K_{S'}$ by adjunction:

$$K_{S'} = \gamma^*(K_{Q'} + 2A_Q + 3F_Q) = \gamma^*(-F_Q) = 1/2 : \gamma^*A_Q - 1/2 : \gamma^*(A_Q + 2F_Q) = A_S - \beta^*(O_S(1))$$

Hence $K_S = O_S(-1)$.

The surface $S$ can be found in the list of [Re], in particular it can be written explicitly as the hypersurface $(z^2 = y^3)$ in the weighted projective space $\mathbb{P}(1, 1, 2, 3)$ with homogeneous coordinates $[x_1, x_2, y, z]$. This allows to compute its sheaf of differentials.

**Lemma 3.14.** Let $S$ be the above surface with the non-normal locus at $B_S \cong \mathbb{P}^1$. Then $\Omega_S|_{B_S} = O_{\mathbb{P}^1(-2)} \oplus O_{\mathbb{P}^1(-2)} \oplus O_{\mathbb{P}^1(-3)}$

**Proof.** In the above situation, let $\mathbb{P}$ denote the weighted projective space $\mathbb{P}(1, 1, 2, 3)$ The line $B_S \subset \mathbb{P}$ is then given by the equations $y = z = 0$ and it is contained in the smooth locus of $\mathbb{P}$. Thus we can verify that $\Omega_{\mathbb{P}}|_{B_S} = O_{\mathbb{P}^1(-2)} \oplus O_{\mathbb{P}^1(-2)} \oplus O_{\mathbb{P}^1(-3)}$. On the other hand restricting the exact sequence $I_{B_S}/I_{B_S}^2 \to \Omega_{\mathbb{P}}|_{B_S} \to \Omega_S|_{B_S} \to 0$ to the line $B_S$ we get the isomorphism $\Omega_{\mathbb{P}}|_{B_S} \to \Omega_S|_{B_S}$. 

**Proof of 3.10** We argue by contradiction. If $E$ is not normal in codimension 1 then the partial normalization $\alpha : S \to E$ is an isomorphism in codimension 1, and the induced morphism of differentials $\alpha^*\Omega_{E|B_S} \to \Omega_{S|B_S}$ is an isomorphism at the generic point of the line $B_S$. Moreover we have a surjective map $\Omega_X \to \Omega_E$ coming from the inclusion $E \to X$. Therefore we obtain a map $\alpha^*\Omega_X \to \Omega_{E|L}$ which is generically surjective. By the previous lemma it follows that $\alpha^*\Omega_X \cong \bigoplus_{i=1}^4 O(a_i)$ with $a_1 \leq a_2 \leq a_3 \leq a_4$ and such that $a_3 < 0$. But since $X$ is symplectic, we have $a_1 = -a_4$ and $a_2 = -a_3$. Contradiction.

This finishes the proof of Theorem [1.1].

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