STATIONARY, MARKOV, STOCHASTIC PROCESSES WITH POLYNOMIAL CONDITIONAL MOMENTS AND CONTINUOUS PATHS

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ABSTRACT. We are studying stationary random processes with conditional polynomial moments that allow a continuous path modification. Processes with continuous path modification, are important because they are relatively easy to simulate. One does not have to care about the distribution of their jumps which is always difficult to find. Among these processes with the continuous path are the Ornstein-Uhlenbeck process, the Gamma process, the process with Arcsin or Wigner margins and the Theta functions as the transition densities and others. We give a simple criterion for the stationary process to have a continuous path modification expressed in terms of skewness and excess kurtosis of the marginal distribution.

1. INTRODUCTION

Inspired by the excellent paper of Ray [19] we decided to return to the problem of path continuity of the stationary Markov processes. In the meantime, there appeared new papers and notions. Among them is the class stochastic processes with polynomial conditional moments. We confine ourselves to the subclass of stationary Markov processes. We do so since within this subclass we have identifiability of the marginal distribution, consequently the existence of the set of polynomials that are orthonormal with respect to this distribution and what is more we have the property that all its conditional moments are polynomials in the conditioning random variable.

In fact, the theory of polynomial Markov processes (that is the Markov processes with polynomial conditional moments) has two independent sources. Starting with the paper [11], which was inspired by the applications of the polynomial processes in mathematical finance, the theory of \( m \)-polynomial processes started being developed. In this theory, the finite \( m \) leading to the assumption that the property of having polynomial conditional moments is restricted only to polynomial of a degree less or equal than \( m \). On the other hand, inspired by the series of papers [4], [3], [7], [6], [8], [7], [9], of W. Bryc, J. Wesołowski and sometimes W. Matysiak on quadratic harnesses, the

Date: May 2022.

2020 Mathematics Subject Classification. [2020] Primary 60G10, 60G17 Secondary 60J35, 60G44.

Key words and phrases. stationary Markov processes, conditional moments, Lancaster type expansion, Gamma process, Ornstein-Uhlenbeck process, Arcsine distribution, Semicircle distribution, Hermite, Laguerre, Chebyshev polynomials, Theta functions, moments’ sequences.

The author is very grateful to the unknown referee for pointing out misprints and small mistakes as well as suggesting some simplifications in proofs. The author is also grateful to the managing editor for indicating unknown to the author positions of the literature.
author was impressed by the existence within this theory the very useful families of polynomial martingales. And thus wanted to examine the family of processes where such a family of martingales appears. The restriction imposed by the assumptions of being a harness or a quadratic harness would be imposed later, within the theory of polynomial processes so far developed.

As shown in the series of papers [25], [27] and [28] the assumption of polynomial conditional moments and the assumption of Markovian stationarity lead to a very specific form of the transition probability. Namely, these assumptions allow its Lancaster-type expansion of the transition probability. As remarked recently in [30] all bivariate distributions that satisfy some mild technical conditions and have the property that all their conditional moments are polynomials in the conditioning random variable can be expanded in a Lancaster-type series. The fact that we have such expansion at our disposal, leads to a deeper insight into such processes. More precisely, we are able to define families of polynomial and orthogonal martingales. Moreover, having a guaranteed existence of all moments, we can refer to the simple Kolmogorov theorem as the primary tool in examining the path continuity of the process under consideration. Recall that the Kolmogorov continuity theorem assures the path continuity provided certain conditions expressed in terms of some moments of the bivariate distribution are satisfied. Of course, we lose generality and the simplest statement concerning path continuity practically guarantees that the paths of the process are of all of $r^{\text{th}}$ Hölder class of continuity, most commonly for $r < 1/4$. Hence, we lose cases with the continuous path of at most $r^{\text{th}}$ Hölder class for $r < 1/4$. But the simplicity of consideration and conditions is the price. In my opinion worth to be paid. How much are we getting? As mentioned above, very simple conditions that assure continuous path modification, and due to an assumed form of the bivariate (i.e., consequently transitional) probability a deep insight into the structure and behavior of the process.

We analyze several examples of marginal distribution. In some cases we prove that they lead to continuous path modifications with $r^{\text{th}}$ Hölder class of continuity with $r < 1/2$ and present related transitional densities. They are the Normal case, which is known and presented for the completeness of the paper and the Gamma case which is believed to be new. There are however two examples concerning special cases of beta distribution, i.e., arcsine and semicircle distributions which lead to transitional densities of the form of some combinations of the Jacobi Theta function and families of orthogonal martingales of the form (2.6) with coefficients $\alpha_n \sim n^2$.

The paper is organized as follows. First, we fix notation, then we recall some facts from the theory of stochastic processes and the theory of probability. In particular, we recall the notion of the stationary stochastic process with polynomials conditional moments, the main object of the research presented in this paper. Then we formulate some necessary conditions, expressed in terms of the first four moments of the marginal distribution that allow continuous path modification of the process with this given marginal. Finally, we present examples of such stationary processes that allow continuous path modification. These include Gaussian, Gamma, Laplace, Semicircle, Arcsine and $q-$Normal processes. The name of the process refers naturally, to the name of the marginal distribution.
2. Notation and the basic facts

Let us start with the following remarks concerning notation. We will be considering only probability measures, that is, nonnegative measures that integrate over their supports to 1. Moreover, the integrals with respect to such measure $\mu$, will be exchangeably denoted either traditionally as $\int f(x)d\mu(x)$ or as $Ef(X)$. Here we denote by $\mu$ the so-called distribution of the random variable $X$. More precisely $\mu$ is defined by the following formula:

$$P(X \leq x) = \mu((-\infty, x]) =: \int_{-\infty}^{x} d\mu(x).$$

In the above-mentioned formulae, $f : \text{supp } X \to \mathbb{R}$, denoted a $\mu-$measurable function and $\text{supp }$ denotes support of the random variable $X$, which in this case means support of its distribution. Since we will be considering Markov stochastic processes the majority of measures considered will be at most 2–dimensional. Moreover, we will be often using the so-called tower property in integration with respect to such 2–dimensional measure. Namely, if $X = (Y_1, Y_2)$ then

$$Ef(Y_1, Y_2) = \int_{\text{supp}(X)} f(y_1, y_2)d\mu(y_1, y_2)$$

$$= \int_{\text{supp}(Y_2)} \int_{\text{supp}(Y_1)} f(y_1, \cdot)d\mu_{Y_1|Y_2}(y_1|y_2)d\lambda_{Y_2}(y_2) = E(Ef(Y_1, Y_2)|Y_2)).$$

Here $\lambda_{Y_2}(y_2)$ denotes a marginal measure of the random variable $Y_2$ and $d\mu_{Y_1|Y_2}(y_1|y_2)$ denotes the so-called conditional measure of $Y_1$ given $Y_2 = y_2$. The existence of such a measure is guaranteed by the theory of measure at $\lambda_{Y_2}$- almost every point of $\text{supp } Y_2$. Moreover, $Ef(Y_1, Y_2)|Y_2)$ denotes the so-called conditional expectation of a random variable $f(Y_1, Y_2)$ given $Y_2$.

There were four incentives to write this paper.

The first one is the so-called continuity Kolmogrov Theorem that reads the following (see e.g. [20], p.51):

**Theorem 1.** Let $(S,d)$ be a metric space and let $X : [0, \infty) \times \Omega \to S$ be a stochastic process. Suppose, that for all $T > 0$ there exist 3 positive constants $\alpha, \beta, K$ such that $\forall 0 \leq s, t \leq T$

$$Ed^\alpha (X_s, X_t) \leq K|s - t|^{1+\beta}.$$ 

Then, there exists a modification $\tilde{X}$ of $X$ that has continuous paths, $\forall t \geq 0 : P(X_t = \tilde{X}_t) = 1$. Moreover, every path of $\tilde{X}$ is $\delta$–Hölder, for $\delta \in (0, \beta/\gamma)$.

Note, that by a continuous mapping of time $s : [0, \infty) \to \mathbb{R}$, like, for example, $s(x) = \log x, x \geq 0$, that doesn’t affect the continuity of the paths of the stochastic process, we can extend the formulation of the above-mentioned theorem to the stochastic process defined for all real $t$.

In the sequel we will use the following notation concerning Gaussian variables. Namely, if the vector $(X, Y)^T$ has bivariate normal distribution with $EX = m_1$, $EY = m_2$, var$(x) = \sigma_1^2$, var$(Y) = \sigma_2^2$ and cov$(X, Y) = r$ then we will write $(X, Y)^T \sim N(m_1, m_2; \sigma_1^2, \sigma_2^2, r)$. In case of the one-dimensional Gaussian distribution we write $X \sim N(m; \sigma^2)$ when $EX = m$ and var$(X) = \sigma^2$. From now on the symbol $\sim$ will also mean ”has distribution” i.e. $X \sim \mu$ means that $X$ has distribution defined by function $\mu$. 

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Hence, another incentive for writing this paper is the following auxiliary, well-known result that we prove for the sake of completeness of the paper:

**Lemma 1.** Let $(X, Y)$ have bivariate Gaussian distribution $N(0, 0; 1, 1, \rho)$.

Then

$$E(X - Y)^{2k} = \frac{(2k)!}{k!}(1 - \rho)^k.$$ 

*Proof suggested by the referee.* We know that if $(X, Y)$ is bivariate Normal (Gaussian) then all its linear transformations. In particular $X - Y$ has Normal distribution.

Since $E(X - Y) = 0$ and $E(X - Y)^2 = 2(1 - \rho)$ by our assumptions, we deduce that $X - Y \sim N(0; 2(1 - \rho))$. Now recalling that if $X \sim N(0; \sigma^2)$ then $E X^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \text{ is odd} \\ (n - 1)!! & \text{if } n \text{ is even} \end{cases}$, we get:

$$E(X - Y)^{2k} = 2^k(1 - \rho)^k(2k - 1)!! = \frac{(2k)!}{k!} \gamma^k,$$

where $\gamma = 1 - \exp(-\alpha t)$. Thus, the Ornstein-Uhlenbeck process allows modification with the continuous path. This is an obvious fact since Ornstein-Uhlenbeck process is a continuous transformation of the Wiener process. It is however not so obvious that the paths are of the $\gamma$-Hölder class with $\gamma < \frac{1}{2}$.

The third incentive for writing this paper were the following two results. The first one is the so-called formula for the expansion of the 2-dimensional distribution $dG(x, y)$ of, say $(X_\tau, X_{\tau+t})$, where $X_\tau$ and $X_{\tau+t}$ belong to some normalized Ornstein-Uhlenbeck process. Namely, we have the following Lancaster-type expansion

$$(2.1) \quad dG(x, y) = \frac{1}{2\pi} \exp(-\frac{x^2 + y^2}{2}) \sum_{j=0}^{\infty} H_j(x) H_j(y) \exp(-j\alpha t)/j! dx dy,$$

where $H_j(x)/\sqrt{j!}$ are orthonormal with respect to the measure with the density $\frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$. (2.1) is a simple modification of the so-called Mehler expansion given e.g., in [22], formula (1.2). Now, we can easily deduce that this example has a nice feature that for every $k$, conditions for having a continuous path modification are expressed in terms of moments of the marginal distribution. One knows that $H_j$ are the so-called monic (that is with 1 as the coefficient by $x^j$) Hermite polynomials of the probabilistic type.

The fourth result that spurred to write this paper is slightly more complicated and requires a small introduction, but is basically simple and concerns, so to say, a generalization of expansion (2.1).

As stated above, in this paper we will be examining the path continuity of the processes defined on the whole real line. But out of all defined so stochastic processes, we will confine our considerations to stationary stochastic processes additionally having the property of possessing polynomial conditional moments. The class of Markov stochastic processes having polynomial conditional moments has been described and analyzed in the series of papers [25], [27] and [28]. Recently yet
another property of such class of stochastic processes has been added. Namely, in [30] it has been shown that under some regularity conditions, the two-dimensional distributions of a Markov stochastic process with the property that all its conditional moments are polynomials of the conditional random variable, must be of the Lancaster-type, that will be explained and defined below. The example of a Lancaster-type distribution is given by (2.1), above. The term Lancaster-type refers to the series of papers of H.O. Lancaster [15], [17], [16], [18], where those types of expansions were introduced and studied.

Since there are several points in fixing notation and exposing necessary assumptions that will enable necessary regularity, let’s present them first.

Let \( X = (X_t)_{t \in \mathbb{R}} \) be a real stochastic process defined on some probability space \( (\Omega, \mathcal{F}, P) \). By the stationary Markov processes we mean those Markov processes \( X = (X_t)_{t \in \mathbb{R}} \) that have marginal distributions that do not depend on the time parameter and the property that the conditional distributions of say \( X_t \) given \( X_s \) does depend only on \( t - s \).

We will assume that \( \forall n \in N, t \in \mathbb{R} : E|X_t|^n < \infty \). More precisely, we assume that the distributions of \( X_t \) will be identifiable by their moments. This assumption is a slightly stronger assumption than the existence of all moments. For example, it is known that if \( \exists \beta > 0 : \int \exp(\beta|x|)d\mu(x) < \infty \), then the measure \( \mu \) is identifiable by its moments. Here \( \mu \) denotes the distribution of \( X_0 \). In fact, there exist other conditions assuring this. For details see e.g. [21].

In [27] one considers the general case of the cardinality of \( \text{supp} \mu \). But in order to avoid unnecessary complications, we will confine ourselves to the infinite number of points of the set \( \text{supp} \mu \), i.e., infinite cardinality of \( \text{supp} \mu \).

To fix further notation, let us denote \( \mathcal{F}_{\leq s} = \sigma(X_r : r \in (-\infty, s] \cap \mathbb{R}) \), \( \mathcal{F}_{\geq s} = \sigma(X_r : r \in [s, \infty) \cap \mathbb{R}) \) and \( \mathcal{F}_{s,u} = \sigma(X_r : r \notin (s, u), r \in \mathbb{R}) \).

Let us also denote by \( \mu(\cdot) \) and by \( \eta(\cdot | y, \tau) \) respectively marginal stationary distribution and transition distribution of our Markov process. That is \( P(X_t \in A) = \int_A \mu(dx) \) and \( P(X_{t+\tau} \in A | X_t = y) = \int_A \eta(dx | y, \tau) \). Stationarity of \( X \) means thus that \( \forall T \ni \tau \neq 0, B \in \mathcal{B} (\mathcal{B} \text{ denotes here Borel } \sigma-\text{field}) \)

\[
\mu(B) = \int \eta(B | y, \tau)\mu(dy).
\]

By \( L_2(\mu) \) let us denote the space spanned by the real functions that are square-integrable (more precisely by the set of equivalence classes) with respect to \( \mu \) i.e.

\[
L_2(\mu) = \{ f : \mathbb{R} \rightarrow \mathbb{R}, \int |f|^2d\mu < \infty \}.
\]

Our assumption of the existence of all moments and their ability to define the unique underlying measure (uniqueness of the related moment problem) of \( X_0 \) in terms of \( L_2(\mu) \), implies that there exists a set of orthogonal polynomials that constitute the orthogonal base of this space. This statement is based on the properties of real Hilbert spaces found in any textbook on mathematical analysis or in particular in [21] or [1]. Let us denote these polynomials by \( \{h_n\}_{n \geq -1} \). Additionally, let us assume that polynomials \( h_n \) are orthonormal and \( h_{-1}(x) = 0, h_0(x) = 1 \). Thus we will assume that for all \( i, j \geq 0 \):

\[
(2.2) \quad \int h_i(x)h_j(x)d\mu(x) = \delta_{ij},
\]

where, as usually, \( \delta_{ij} \) denotes Kronecker’s delta.
Having introduced $\mathcal{F}_{\leq s}$, $\mathcal{F}_{\geq s}$ and $\mathcal{F}_{s,u}$ we can specify more precisely the classes of Markov processes that we are considering in this paper. Processes with polynomial conditional moments are those for which the following condition holds for all $n \geq 0$ and $s \in \mathbb{R}$

$$E(X^n_t | \mathcal{F}_{\leq s}) = P_n(X_s | t, s) \text{ a.s.},$$

where $P_n$ denotes here polynomial of degree not exceeding $n$ in $X_s$ with coefficients depending on $t$ and $s$.

Note that if the above-mentioned condition holds only for $n = 1$ and $P_1(x | t, s) = x$ then we deal with a martingale.

Occasionally, there will appear processes that are also harnesses. So let us define a $n-$harness by the following condition:

$$E(X^n_t | \mathcal{F}_{s,u}) = R_n(X_s, X_u | s, t, u) \text{ a.s.},$$

where $R_n$ is a polynomial of degree not exceeding in $X_s$ and $X_u$ with coefficients depending on $s, t,$ and $u$.

Hammersley in 1967 introduced the notion of harness in the paper [12] by considering 1–harnesses. W. Bryc, J. Wesolowski and W. Matysiak considered quadratic harnesses that are both 1- and 2–harnesses according to the above-mentioned definition.

Thus, the class of Markov processes that we consider, is a class of stochastic processes that satisfies some mild technical assumptions that were described and interpreted in [27] and moreover satisfying the following conditions: $\forall t \in \mathbb{T}, n \in \mathbb{N} : E(X^n_t) = m_n$ and $\forall n \geq 1, s < t$ :

\[
(2.3) \quad E(X^n_t | \mathcal{F}_{\leq s}) = Q_n(X_s, t - s) \text{ a.s.,}
\]

where $Q_n(x, t - s)$ is a polynomial of order not exceeding $n$ in $x$.

It has been shown in [27] that under the above-mentioned regularity assumptions and also under the following assumption that $\eta \ll \mu$ and

\[
(2.4) \quad \int \left( \frac{d\eta}{d\mu} \right)^2 d\mu < \infty,
\]

where, as above, $\mu(dx)$ and $\eta(dx | y, t)$ denote respectively marginal and transitional measures of $X$, the following expansion holds

\[
(2.5) \quad \frac{dn}{d\mu}(x | y, t) = \sum_{n \geq 0} \exp(-\alpha_n t) h_n(x) h_n(y).
\]

In this formula, we will set $\alpha_0 = 0$ and there appear certain positive constants $\{\alpha_i\}_{i \geq 1}$ whose existence is guaranteed by the mentioned above technical assumptions. The constants $\{\alpha_i\}_{i \geq 1}$ allow to define orthogonal martingales defined by the formula:

\[
(2.6) \quad M_n(X_t, t) = \exp(\alpha_n t) h_n(X_t), \quad n \geq 1.
\]

The class of such stationary stochastic process will be briefly called SMPR. More precisely, since from (2.5) it follows that such processes are completely characterized by the distribution $\mu$ and a sequence $\{\alpha_n\}$ of positive numbers. We will write to denote such a process $X = \{X_t\}_{t \in \mathbb{R}} = \text{SMPR}(\{\alpha_n\}, \mu)$. 
Thus, under the above-mentioned assumption, the two-dimensional distribution of say \((X_\tau, X_{\tau+t})\) is given by the formula

\[
\rho(dx, dy) = d\mu(x)d\mu(y) \sum_{n \geq 0} \exp(-\alpha_n t) h_n(x)h_n(y).
\]

**Remark 1.** Notice, that from the fact that \(M_n(X_t, t)\) is a martingale, it follows that

\[
E(M_n(X_{t+t}, t+\tau)|\mathcal{F}_{\leq \tau}) = M_n(X_\tau, \tau),
\]

hence, following (2.6), we see that

\[
E(h_n(X_{t+t})|\mathcal{F}_{\leq \tau}) = \exp(-\alpha_n t) h_n(X_\tau),
\]
a.s. mod \(d\mu\).

Now, let us recall that in [24] the following numbers \(\{c_{j,k}\}_{j \geq 0, 0 \leq n \leq j}\) were introduced and analyzed. Their interpretation is the following:

\[
x^j = \sum_{n=0}^{j} c_{j,n}h_n(x).
\]

for all \(j \geq 0\). Let us set \(c_{j,n} = 0\) for \(n > j\). Let us also denote by \(L_k\) the following lower-triangular matrix \(\begin{bmatrix} c_{j,n} \end{bmatrix}_{j=0}^{\infty} k, n=0,\ldots,k\). It has been remarked in [24] (Propositions 1 and 2) that

\[
L_kL_k^T = M_k,
\]

where \(M_k\) is the moment matrix, i.e., \(M_k = [m_{i+j}]_{i=0,\ldots,k, j=0,\ldots,k}\), where \(m_j = \int x^j d\mu(x)\) that is equal to \(j\)-th moment of the distribution \(\mu\). Hence, the coefficients \(c_{j,n}\) can be computed directly from the moments’ matrix. Besides, we know by ([24], Proposition 1(iii)):

\[
\sum_{n=0}^{\min(j,k)} c_{j,n}c_{k,n} = m_{j+k}.
\]

for all \(j, k \geq 0\). We have the following observation:

**Remark 2.**

\[
E(X_{t+t}^j|\mathcal{F}_{\leq \tau}) = E(\sum_{n=0}^{j} c_{j,n}h_n(X_{t+t})|\mathcal{F}_{\leq \tau}) = \sum_{n=0}^{j} c_{j,n} \exp(-\alpha_n t) h_n(X_\tau)
\]

\[
= X_\tau^j - \sum_{n=1}^{j} c_{j,n}(1 - \exp(-\alpha_n t)) h_n(X_\tau),
\]

mod \(d\mu\), since we have \(\alpha_0 = 0\).

In the sequel, we will use the following almost trivial lemma. It has been presented in assertion 6 of Proposition 1 in [32]. Since it is important in the present context, we will present its generalized version once more with its simple proof.

**Lemma 2.** Suppose that a sequence \(\{b_n\}_{n \geq 0}\) is a positive moment sequence.

i) Suppose that \(b_0 = 1\), and \(b_{4m+2} = b_{2m+1}^2\) for some \(m \geq 0\). Then \(\forall n \geq 0 : b_n = b_{2m+1}^{n/(2m+1)}\).
ii) Suppose that \(b_0 = 1\) and \(b_{2m} = b_{2m}^2\) for some \(m \geq 1\). Let us set \(p = (b_1 + b_{2m}^{1/(2m)})/(2b_{2m}^{1/(2m)})\). Then for all \(n \geq 0\): \(b_{2n} = b_{2n}^{m/n}\) and \(b_{2n+1} = b_{2(n+1)/2m}^{1/(2m)} - b_{2m}^{1/(2m)}(1 - p)\).

**Proof.** Let \(X\) denote a random variable whose moments are \(b_n\), that is \(EX^n = b_n\). In the case i) we have \(E(X^{4m+2}) = (E(X^{2m+1}))^2\). That is \(E(X^{2m+1} - EX^{2m+1})^2 = \text{var}(X^{2m+1}) = 0\). But this equality means that the distribution of \(X\) is a one-point distribution, i.e. \(P(X = b_{2m+1}^{1/(2m+1)}) = 1\). In the case ii) we deduce that \(E(X^{2m} - EX^{2m})^2 = 0\). That is that the distribution of \(X^2\) is a one point distribution. One can find that \(X^2 = b_{2m}^{1/(2m)}\) and that \(P(X = b_{2m}^{1/(2m)}) = p = 1 - P(X = -b_{2m}^{1/(2m)})\), for some \(p \in [0, 1]\). But it means that \(b_1 = EX = b_{2m}^{1/(2m)} - b_{2m}^{1/(2m)}(1 - p)\). Parameter \(p\) can be be found to be equal to \((b_1 + b_{2m}^{1/(2m)})/(2b_{2m}^{1/(2m)})\). Then and \(b_{2n+1}\) is a \(2n + 1\)-th moment of such variable that is \(b_{2m+1}^{1/(2m)} - b_{2m}^{1/(2m)}(1 - p)\).

**Remark 3.** 1) It has been remarked in \([30]\), Corollary 1) that the sequence \(\{\exp(\alpha_n t)\}_{n \geq 0}\) must be such that 1) \(\sum_{n \geq 0} \exp(-2\alpha_n t) < \infty\) for all \(t \in \mathbb{R}\). 2) If \(\text{supp} \mu\) is unbounded, then \(\{\exp(-\alpha_n t)\}_{n \geq 0}\) must be a moment sequence. Moreover, since all \(\{\alpha_n\}_{n \geq 1}\) are positive then we deduce that the support of the measure with respect to which \(\exp(-\alpha_n t)\) is a moment sequence must have support contained in \([0, 1]\). Consequently, not only the matrix \(\{\exp(-\alpha_{i+j} + t)\}_{0 \leq i, j \leq n}\) but also the matrix \(\{\exp(-\alpha_{i+j+1} + t)\}_{0 \leq i, j \leq n}\) must be nonnegative definite. In particular, we deduce that

\[
2\alpha_{2n+1} \geq \alpha_{2n} + \alpha_{2n+2} \text{ and } 2\alpha_{2n} \geq \alpha_{2n-1} + \alpha_{2n+1}.
\]

But this means that for all \(m \geq 0\) we have

\[
2\alpha_m \geq \alpha_{m-1} + \alpha_{m+1},
\]

which means that the sequence \(\{\alpha_n\}_{n \geq 0}\) must be a concave sequence. We must also have for all \(i, n \geq 0, i + n \geq 1\)

\[
\alpha_{2n} + \alpha_{2i} \leq 2\alpha_{n+i} \text{ and } \alpha_{2i+1} + \alpha_{2i+2n+1} \leq 2\alpha_{2i+2n}.
\]

The last inequality follows the fact that for \(\{\exp(-\alpha_n t)\}_{n \geq 0}\) being a moment sequence is equivalent to the fact that for all \(m\) the matrices \(\{\exp(-\alpha_{i+j} + t)\}_{0 \leq i, j \leq m}\) must be nonnegative definite. Thus, in particular, all central \(2 \times 2\) minors must be nonnegative. Now some such minors are the following ones:

\[
\begin{bmatrix}
\exp(-\alpha_{2i}) & \exp(-\alpha_{2i+m} + t) \\
\exp(-\alpha_{2i+m}) & \exp(-\alpha_{2m+2i} + t)
\end{bmatrix} \text{ and } 
\begin{bmatrix}
\exp(-\alpha_{2i+1}) & \exp(-\alpha_{2i+2n} + t) \\
\exp(-\alpha_{2i+2n}) & \exp(-\alpha_{2i+2n+1} + t)
\end{bmatrix}.
\]

In the first of these matrices we set \(n = i + m\).

**Remark 4.** Notice also that if the support of the measure \(\mu\) is unbounded and we are able to show that \(\alpha_2 = 2\alpha_1\) then, as it follows from assertion 6 of Proposition 1. we must have \(\alpha_n = n\alpha_1\) for all \(n \geq 1\). Hence, consequently, \(\text{SMPR}(\{\alpha_n\}, \mu)\) is additionally a harness (for details see \([27]\)).

3. Necessary conditions

Having said this, we can state that the paper is dedicated to defining conditions under which a given \(\text{SMPR}(\{\alpha_n\}, \mu)\) allows continuous path modification.
Thus we can calculate the quantities $E|X_{\tau} - X_{\tau+t}|^n$ that are needed in order to apply Kolmogorov’s continuity theorem and examine the dependence of these quantities on $t$.

Of course, in order to simplify calculations, we consider only $\alpha = 2k$ for some natural $k$.

It turns out that for further analysis, we will need the following moments defined by the formula

\[(3.1) \quad \int x^j h_n(x) d\mu(x) = EX^j h_n(X).\]

Notice that we have

\[EX^j h_n(X) = c_{j,n},\]

where $c_{j,n}$ are defined above, by (2.8). Let us observe that from the definition of the orthogonal polynomials it follows that $\forall 0 \leq j < n$ we have $EX^j h_n(X) = 0$. Thus, in the sequel we will be using parameters $c_{j,n}$ set as 0 for $j < n$.

**Remark 5.** Notice that regardless of the boundedness of the support of marginal measure $\mu$ we observe that the following sequence

\[(3.2) \quad \left\{ m_{2j} - \sum_{n=1}^{j} (1 - \exp(-\alpha_n t)) c_{j,n}^2 \right\}_{j \geq 0} \]

must be a moment sequence for every $t > 0$. This simple observation follows the fact that

\[E(X_{\tau} X_{\tau+t})^j = \int \int (xy)^j \rho(dx, dy),\]

where $\rho(dx, dy)$ is given by (2.7). Then we use the definition of coefficients $c_{j,n}$ and the fact that $\sum_{n=0}^{j} c_{j,n}^2 = m_{2j}$.

Now we can formulate sufficient conditions for the process $X = SMP R(\{\alpha_n\}, \mu)$ to allow continuous path modification. Namely, we have the following lemma.

**Lemma 3.** Let $X = \{X_t\}_{t \in \mathbb{R}} = SMP R(\{\alpha_n\}, \mu)$ with polynomial $\{h_n(x)\}$ orthonormal with respect to $\mu$. Let us define numbers $\{c_{j,n}\}_{j \geq 0, 0 \leq n \leq j}$ by (3.1). If for some $k \geq 1$ and $r > 1$ the following conditions hold:

\[(3.3) \quad \sum_{n=1}^{2k} \alpha_n^s \sum_{j=n}^{2k} (-1)^j \binom{2k}{j} c_{j,n} c_{2k-j,n} = 0,\]

for $s = 1, \ldots, r$, then the process $X$ allows continuous path modification and the so-modified process has paths of the $m$–th Holder class where $m < \frac{r}{2}$. 

Proof. Let us start with the following calculation:

(3.4) \[ E(X_{t+\tau} - X_{\tau})^m = \sum_{j=0}^{m} (-1)^j \binom{m}{j} EX_t^{m-j} E(X_{t+\tau}^j | F_{\leq t}) = \]

\[ \sum_{j=0}^{m} (-1)^j \binom{m}{j} EX_t^{m-j} (X_t^m - \sum_{n=1}^{j} (1 - \exp(-\alpha_n t)) c_{j,n} h_n(X_t) = \]

\[ = - \sum_{n=1}^{m} (1 - \exp(-\alpha_n t)) \sum_{j=0}^{m} (-1)^j \binom{m}{j} c_{j,n} c_{m-j,n} \]

\[ = \sum_{k=1}^{r} (-1)^{k-1} k! \sum_{n=1}^{m} \alpha_n \sum_{j=0}^{m} (-1)^j \binom{m}{j} c_{j,n} c_{m-j,n} + O(t^{r+1}). \]

Now we take \( m = 2k \) and see that \( E |X_{t+\tau} - X_{\tau}|^{2k} \approx O(|t|^{r+1}) \). Now we can apply Kolmogorov’s Theorem [1]. \( \square \)

Notice that, the crucial from the point of view of this paper, numbers \( c_{j,n} \) can be expressed in terms of moments of the marginal distribution \( \mu \). Since the first two particular cases of \( k \) are the most important let us find numbers \( c_{j,n} \) for \( n = 0, 1, 2 \) and \( j = 0, 1, 2, 3, 4 \). The conditions presented in (3.3) are somewhat difficult to satisfy in the general case and have rather a theoretical character. The most important case is the case \( k = 2 \). Notice that then we have only two constants \( \alpha_1 \) and \( \alpha_2 \) involved. They are thus the most important from the point of view of the continuity of the paths of the analyzed process. Hence, let us analyze this case in more detail.

Proposition 1. Let \( X \sim \mu \). Let us denote \( EX = \nu \), \( m_j = E(X - \nu)^j \), and \( h_j(x) \) the polynomial of degree \( j \) that is orthonormal with respect to the measure \( \mu \). Let the numbers \( c_{j,n} \) be defined by (3.4). Then:

i) \( c_{j,0} = EX^j = \sum_{k=0}^{j} \binom{j}{k} \nu^k m_{j-k} \), with an obvious fact that \( m_1 = 0 \),

ii) \( c_{2,1} = m_2, c_{1,1} = (m_3 + 2m_2) / \sqrt{m_2}, c_{3,1} = (m_4 + 3m_3 + 3\nu m_2) / \sqrt{m_2} \), consequently we have \( E(X_t - X_{t+\tau})^2 = 2m_2(1 - \exp(-\alpha_1 t)) \).

iii) \( c_{2,2} = \sqrt{m_4 m_2 - m_3^2 - m_2^2} / \sqrt{m_2} \), consequently we have

\[ E(X_{t+\tau})^4 = 2(m_4 + 3m_2^2) - 2\exp(-\alpha_2 t)(4m_2 m_1 - 3m_3^2) / m_2 + 6\exp(-\alpha_2 t) (m_2 m_4 - m_3^2 - m_2^2) / m_2. \]

Proof. i) Since \( h_0(x) = 1 \), we must have \( c_{j,0} = EX^j h_0(X) = E(X - \nu + \nu)^j = \sum_{k=0}^{j} \binom{j}{k} \nu^k m_{j-k} \) with \( c_{1,0} = \nu \). Moreover, we have following (3.4)

\[ E(X_{t+\tau} - X_{t+\tau})^2 = 2c_{2,0} - 2(c_{1,0}^2 + \exp(-\alpha_1 t) m_2) = 2m_2 - 2m_2 \exp(-\alpha_1 t). \]
ii) We start with the well-known fact that polynomials that are orthogonal with respect to the measure \( \mu \) are given by the formula

\[
(3.5) \quad \eta_n(x) = a_n \det \begin{bmatrix} 1 & EX & \cdots & EX^n \\ EX & EX^2 & \cdots & EX^{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x & \cdots & x^n \end{bmatrix},
\]

for some constants \( a_n \). Hence, orthonormal polynomials \( h_n(x) \) are given as \( b_n \eta_n(x) \) for suitably chosen constants \( b_n \). Thus we have \( h_1(x) = b_1 \det \begin{bmatrix} 1 \\ 1 \\ \nu \\ \nu \end{bmatrix} = b_1(x - \nu) \).

Further, we have to have \( Eh_1^2(X) = 1 \), so \( b_1 = 1/\sqrt{m_2} \). Hence, we have \( c_{1,1} = (EX^1(X - \nu))/\sqrt{V} = (EX^2 - \nu EX^1)/\sqrt{m_2} \). In particular, we have: \( c_{1,1} = \sqrt{m_2} \), \( c_{2,1} = (m_3 + 2 \nu m_2)/\sqrt{m_2} \) and \( c_{3,1} = (m_4 + 6 \nu m_2 + 3 \nu^2 m_2)/\sqrt{m_2} \).

iii) Using (3.5), we get

\[
h_2(x) = b_2 \det \begin{bmatrix} 1 & \nu & m_2 + c^2 \\ \nu & m_2 + \nu^2 & m_3 + 3 \nu m_2 + \nu^3 \\ 1 & x & x^2 \end{bmatrix}
= b_2(m_2(x^2 - m_2 - \nu^2) - (m_3 + 2 \nu m_2)(x - \nu)).
\]

Since we are interested in orthonormal polynomials \( h_2 \) we have to calculate \( Eh_2(X)^2 \). Simple, but the lengthy calculation gives

\[
Eh_2(X)^2 = b_2^2 m_2(m_2 m_4 - m_3^3 - m_2^3).
\]

Hence, we have \( b_2 = 1/(\sqrt{m_2(m_2 m_4 - m_3^3 - m_2^3)}) \). Now, we have \( c_{2,2} = b_2 EX^2 h_2(X) = b_2(m_2 m_4 - m_3^3 - m_2^3) = \sqrt{(m_2 m_4 - m_3^3 - m_2^3)/\sqrt{m_2}} \).

**Corollary 1.** Under the assumptions about Lemma 3 and upon applying Kolmogorov Theorem 4, we get

\[
E(X_\tau - X_{\tau + t})^4 = t(\alpha_1(8m_4 m_2 - 6m_3^3)/m_2 - 6\alpha_2(m_2 m_4 - m_3^3 - m_2^3)) + O(t^2).
\]

Consequently, the process allows continuous path modification if only

\[
\frac{4m_4 m_2 - 3m_3^2}{3(m_2 m_4 - m_3^3 - m_2^3)} > 0
\]

and

\[
(3.6) \quad \alpha_2 = \alpha_1 \frac{4m_4 m_2 - 3m_3^2}{3(m_2 m_4 - m_3^3 - m_2^3)}.
\]

Now let’s introduce, popular in mathematical statistics, parameters \( \kappa \)-kurtosis (excess kurtosis more precisely) defined as \( \kappa = m_4/m_2^2 - 3 \) and Fisher’s skewness parameter \( s \) defined as \( s = m_3/m_2^{3/2} \). (3.7) takes now, the following form:

\[
\alpha_2 = \alpha_1 \frac{12 + 4\kappa - 3s^2}{6 + 3\kappa - 3s^2}.
\]

Since the ratio of \( \alpha_2/\alpha_1 \) is important in sorting off stationary processes which do not allow continuous path modification (at least as far as our simple theory is concerned), let us introduce the following parameter
**Definition 1.** The following coefficient

\[ C_c = \frac{12 + 4\kappa - 3s^2}{6 + 3\kappa - 3s^2}, \]

will be called **continuity coefficient**. Above, as before, \( s \) denotes the skewness while \( \kappa \) the excess kurtosis of the marginal measure \( \mu \) of the SMPR(\( \{\alpha_n\}, \mu \)) that we are analyzing.

Since, as mentioned earlier, if only the support of the marginal measure \( \mu \) is unbounded the sequence \( \{\exp(-\alpha_n t)\} \) must be a moment sequence.

**Corollary 2.** If the support of the measure \( \mu \) is unbounded and \( 3s^2 = 2\kappa \) then the only SMPR(\( \{\alpha_n\}, \mu \)) process having continuous path modifications is the one with \( \alpha_n = n\alpha_1 \). From [27] it follows that such process is also a harness.

**Proof.** Notice that, following (3.7), when \( 3s^2 = 2\kappa \) we must have \( \alpha_2 = 2\alpha_1 \). Now, by Lemma 2 we have \( \alpha_n = n\alpha_1 \). \( \square \)

Now let us show some examples and consider particular cases.

### 4. Examples

**Example 1 (Gaussian case.)** Although it was analyses at the beginning of the paper, we are returning to this example, once more just to illustrate the theory developed above on this very simple example of relatively simple calculations. Let us consider \( d\mu(x) = \exp(-x^2/2)dx/\sqrt{2\pi} \). Consequently, polynomials \( h_n(x) \) are the so-called probabilistic Hermite polynomials satisfying the following three-term recurrence

\[ h_{n+1}(x) = x h_n(x) - nh_{n-1}(x), \]

with \( h_{-1}(x) = 0 \) and \( h_0(x) = 1 \). It is a common knowledge, that for \( \forall n \geq 0 \):

\[ x^j = j! \sum_{m=0}^{\lfloor j/2 \rfloor} \frac{1}{2^m m! (j-2m)!} h_{j-2m}(x). \]

Consequently, the numbers \( c_{j,n} \) are given by

\[ c_{j,n} = \begin{cases} 0 & \text{if } n > j \text{ or } j-n \text{ odd} \\ \frac{n!}{2^{(j-n)/2}((j-n)/2)!}\sqrt{n} & \text{if } j-n \text{ is even} \end{cases} \]

There is \( \sqrt{n!} \) in the denominator since polynomials \( h_n \) are not orthonormal. They become orthonormal after dividing \( n \)-th polynomial by \( \sqrt{n!} \). We have \( \kappa = 0 \) and \( s = 0 \) so \( \alpha_2 = 2\alpha_1 \). From the Remark [4] it follows that we must have \( \alpha_n = n\alpha_1 \) and consequently we must be dealing with the Ornstein-Uhlenbeck (OU) process. We can thus conclude that the Ornstein-Uhlenbeck process is the only SMPR process with Gaussian marginals that allow continuous path modification.

**Example 2 (Gamma distribution.)** Let us consider the Gamma distribution with rate parameter zero and shape parameter \( \beta > 0 \), i.e., the distribution with the following density:

\[ f_g(x, \beta) = x^{\beta-1} \exp(-x)/\Gamma(\beta), \]

for \( x > 0 \) and 0 otherwise. In order to simplify notation let us introduce the so-called raising factorial, which is the following function:

\[ (x)^{(n)} = x(x+1)\ldots(x+n-1), \]
defined for all complex \( x \). Notice, that we have for all \( x \neq 0 \):

\[
(x)^{(n)} = \frac{\Gamma(x+n)}{\Gamma(x)}.
\]

It is well known that if \( X \sim f_g(x, \beta) \), then \( EX^j = (\beta)^{(j)} \). Thus we have \( m_2 = \beta \), \( m_3 = 2\beta \), \( m_4 = 3\beta(2 + \beta) \). Hence, the skewness parameter \( s \) is equal to \( 2/\sqrt{\beta} \) and the kurtosis \( \kappa = 3(2 + \beta)/\beta - 3 = 6/\beta \). Consequently, the parameter \( \alpha_2/\alpha_1 \) is equal to

\[
\frac{12 + 4\kappa - 3s^2}{6 + 3\kappa - 3s^2} = \frac{12 + 24/\beta - 12/\beta}{6 + 18/\beta - 12/\beta} = 2.
\]

Since the support of Gamma distribution is unbounded, we deduce that \( \forall n \geq 1 : \alpha_n = n\alpha_1 \) and following \[27\] (Thm. 2) such process is also a harness.

Moreover, we know also that the polynomials \( h_n(x|\beta) \) are proportional to the so-called generalized Laguerre polynomials \( L_n^{(\beta)} \), defined by the following recurrence relation

\[
L_{n+1}^{(\beta)} = \frac{2n + \beta - x}{n+1} L_n^{(\beta)} - \frac{n+1 - 1}{n+1} L_{n-1}^{(\beta)},
\]

with \( L_{-1}^{(\beta)} = 0 \) and \( L_0^{(\beta)} = 1 \). We also know that

\[
\frac{1}{\Gamma(\beta)} \int_0^\infty L_n^{(\beta)}(x)L_m^{(\beta)}(x)x^{\beta-1} \exp(-x)dx = \frac{\Gamma(n+\beta)}{n!} \delta_{nm}.
\]

Thus

\[
h_n(x|\beta) = \sqrt{\frac{n!}{\Gamma(n+\beta)}} L_n^{(\beta)}(x).
\]

Moreover, we also know the exact form of the two-dimensional density of such \( SMPR(\{n\alpha_1\}, f_g) \) process. It has complicated form and is commonly known under the name Hardy-Hille formula. Namely, we have

\[
f_g(x|\beta)f_g(y|\beta) \sum_{j \geq 0} h_j(x|\beta)h_j(y|\beta) \exp(-n\alpha_1 t) \times \frac{1}{(1 - \exp(-\alpha_1 t)(xy\exp(-\alpha_1 t))^{(\beta-1)/2}} \exp(-(x + y) \\
\times \frac{\exp(-\alpha_1 t)}{1 - \exp(-\alpha_1 t)} I_{\beta-1} \left( \frac{2\exp(-\alpha_1 t/2)\sqrt{xy}}{1 - \exp(-\alpha_1 t)} \right),
\]

where \( I\) denotes modified Bessel function of the first kind. Recently, it has been shown in \[31\] that, as in the Gaussian case, we have

\[
E(X_{t} - X_{t+\tau})^{2k} = \frac{(2k)!}{k!} (\beta)^{(k)} (1 - \exp(-\alpha_1 t))^{k}.
\]

Hence, the gamma process with transition distribution given above is as smooth as the Ornstein-Uhlenbeck process.

**Example 3 (Laplace distribution.)**. Laplace distribution is defined by its density given by the formula \( f_L(x) = \exp(-|x|)/2 \), for \( x \in \mathbb{R} \). It is also known that it is a symmetric distribution, hence all its odd moments are equal to 0 while all even moments, say of degree 2n are equal to (2n)! . Consequently, we have \( m_2 = 2 \), \( m_3 = 0 \) and \( m_4 = 24 \), hence \( \kappa = 3 \), and \( s = 0 \). Thus, we have

\[
\frac{\alpha_2/\alpha_1}{6 + 3 \times 3} = 8/5 < 2.
\]
Let us denote by \( \{h_n\}_{n\geq 0} \) the family of polynomials orthonormal with respect to the measure with the density \( f_L \). Thus, for every moment sequence \( \{1, \exp(-\alpha_1 t), \exp(-\alpha_2 t), \ldots\} \), such that the bivariate function:

\[
1 + \sum_{j\geq 1} \exp(-\alpha_j t) h_j(x) h_j(y),
\]

with \( \alpha_2 = 8\alpha_1/5 \) is nonnegative for all \( t \), the SMPR(\( \{\alpha_n\}, f_L \)) allows continuous path modification with \( E(X_{t+\tau} - X_\tau)^4 = O(t^2) \).

Interestingly, if we calculate (with the help of Mathematica) the coefficients \( \{c_{i,n}\}_{i=0,...,6,n=0,...,i} \), which can be easily done using moments of the Laplace distribution, and then solving system of equations (3.3) we get

\[
\alpha_2/\alpha_1 = (35-\sqrt{105})/28, \quad \alpha_3/\alpha_1 = (15-\sqrt{105})/12.
\]

Consequently, every SMPR(\( \{\alpha_n\}, f_L \)) with these parameters \( \alpha_2, \alpha_3 \) (for which naturally function (4.2) is nonnegative), would allow continuous path modification and \( E(X_{t+\tau} - X_\tau)^6 = O(t^6) \).

We can continue this procedure. There are however some numerical problems since there is not known a nice general form of polynomials that are orthonormal with respect to the Laplace distribution.

**Example 4 (Beta distribution.).** So first let us start with the general case of the Beta distribution. Under this name function, two distributions related to one another. Namely, primarily the distribution with the density

\[
f_\beta(x|\alpha, \beta) = x^{\gamma-1}(1-x)^{\beta-1}/B(\gamma, \beta),
\]

for \( x \in (0, 1) \) and \( \gamma, \beta > 0 \), where \( B(\gamma, \beta) = \Gamma(\gamma)\Gamma(\beta)/\Gamma(\gamma+\beta) \) is the so-called beta function. Often under the name Beta distribution also works the distribution with the following density

\[
f_B(x|\gamma, \beta) = 2^{1-\gamma-\beta}(1+x)^{\gamma-1}(1-x)^{\beta-1}/B(\gamma, \beta),
\]

for \( x \in (-1, 1) \) and \( \gamma, \beta > 0 \). As one can easily notice if random variable \( X \sim f_\beta \) then \( Y = 2X - 1 \) has density \( f_B \). It is also known that the so-called Jacobi polynomials are orthogonal with respect to the measure with the density \( f_B \). In particular, we know that

\[
f_B(x|1/2, 1/2) = \frac{1}{\pi \sqrt{1-x^2}},
\]

that is, we are dealing with the so-called arcsine distribution while \( f_B(x|3/2, 3/2) = \frac{2}{\pi} \sqrt{1-x^2} \) is known under the name Wigner or semicircle distribution. It is known (see e.g. [13]) that the skewness of the beta distribution is equal to:

\[
s = \frac{2(\beta - \gamma) \sqrt{\gamma + \beta + 1}}{(\gamma + \beta + 2)\sqrt{\gamma\beta}},
\]

while the excess kurtosis is given by the formula:

\[
\kappa = \frac{6((\gamma - \beta)^2(\gamma + \beta + 1) - (\gamma + \beta + 2)\gamma\beta)}{\gamma\beta(\gamma + \beta + 2)(\gamma + \beta + 3)}.
\]

Now

\[
\frac{12 + 4\kappa - 3s^2}{6 + 3\kappa - 3s^2} = \frac{2(\gamma + \beta + 1)}{\gamma + \beta} = 2 + \frac{2}{\gamma + \beta}.
\]
Example 5 (Arcsine distribution.). Arcsine distribution is an example of the beta distribution with parameters $\gamma = 1/2$, $\beta = 1/2$. Let us recall that in this case, the sequence of polynomials orthogonal with respect to the measure with the density $f_B(1/2, 1/2)$ are the so-called Chebyshev polynomials of the first kind $\{T_n(x)\}_{n \geq 1}$. Following [11], let us recall the most important properties of these polynomials. They are defined by the following three-term recurrence

$$2xT_n(x) = T_{n+1}(x) + T_{n-1}(x),$$

with $T_0(x) = 1$ and $T_1(x) = x$. What is however important for our purposes, is the following property of polynomials $\{T_n\}_{n \geq 0}$. Namely, we have:

$$T_n(\cos \varphi) = \cos(n\varphi).$$

for $n \geq 0$ and $\varphi \in \mathbb{R}$. Moreover, it is known that

$$\frac{1}{\pi} \int_{-1}^{1} T_n(x)T_m(x) \frac{1}{\sqrt{1 - x^2}} dx = \left\{ \begin{array}{ll} 1 & \text{if } n = m = 0 \\ \delta_{mn}/2 & \text{if } n + m > 0. \end{array} \right.$$
Thus, returning to variables $x$ and $y$, we get:

$$1 + 2 \sum_{n \geq 1} \rho^{n^2} T_n(x) T_n(y) = \frac{1}{2} \left( \theta(\mu; (\arccos x) - (\arccos y)) + \theta(\mu; (\arccos x) + (\arccos y)) \right).$$

Setting $\rho = \exp(-\alpha_1 t)$ in (4.4) and multiplying by, say $\frac{1}{\sqrt{1-y^2}}$, we get a transitional density of $X_{t+\tau}$ given $X_\tau = y$. Moreover, we have for some $\alpha_1 > 0$, all $n \geq 0$ and almost all (mod $\mu$) $y \in [-1, 1]$ the following relationships

$$E(T'_n(X_{t+\tau})|X_\tau = y) = \exp(-\alpha_1 t n^2) T_n(y).$$

**Remark 6.** Let us note that we have $E(X_{t+\tau} - X_\tau)^4 = O(t^2)$ as it follows directly from [4.3]. However, we know the parameters $c_{i,n}$ defined above since we have:

$$x^j = 2^{1-j} \sum_{\substack{j=0, \text{mod } 2 \leq n \leq j}} \binom{j}{(j-n)/2} h_n(x) \equiv \sum_{n=0}^{j} c_{i,n} h_n(x),$$

where the $'$ above the sum means that when $n = 0$ then the appropriate coefficient is divided by 2. Thus consequently, we can, using Mathematica, check that for $k = 3, 4, 5$ we have also $E(X_{t+\tau} - X_\tau)^{2k} = O(t^k)$. Thus, one can express the following conjecture:

**Conjecture 1.** Let us consider $X = \text{SMPR}\left(\{n^2 \alpha_1\}, f_B(x|1/2, 1/2)\right)$. Then for all $k \geq 1$ we have $E(X_{t+\tau} - X_\tau)^{2k} = O(t^k)$.

Thus, this process is as 'smooth' as the Ornstein-Uhlenbeck process. It is not, however, a harness.

**Remark 7.** Notice also, that the sequence $\{\exp(-n^2 \alpha_1)\}_{n \geq 0}$ is not a moments’ sequence. This is so, for example, the matrix $\begin{bmatrix} 1 & \exp(-\alpha_1 t) \\ \exp(-\alpha_1 t) & \exp(-4\alpha_1 t) \end{bmatrix}$ is not nonnegative definite. Recall, that this feature of $\text{SMPR}(\{\alpha_n\}, \mu)$, as pointed out in [30], has been allowed because the support of the measure $\mu$ is bounded in this case.

**Example 6 (Semicircle distribution.).** As mentioned above, the semicircle or Wigner distribution is another example of a beta distribution, this time we with $\alpha = 3/2$ and $\beta = 3/2$ which leads to the following distribution with the density

$$f_B(x|3/2, 3/2) = f_W(x) = \frac{2}{\pi} \sqrt{1-x^2}.$$ 

Further, we know that the family of polynomials that are orthogonal with respect to the Wigner measure is the family of Chebyshev polynomials of the second kind, that is the family $\{U_n(x)\}_{n \geq 0}$, satisfying the following three-term recurrence

$$2xU_n(x) = U_{n-1}(x) + U_{n+1}(x),$$

with $U_{-1}(x) = 0$, $U_0(x) = 1$. Besides, we also have:

$$\int_{-1}^{1} U_n(x) U_m(x) f_W(x) dx = \delta_{nm}.$$ 

Hence, the family $\{U_n\}$ constitutes an orthonormal family.
Now notice that setting \( \gamma = \beta = 3/2 \) in (4.4) we get \( c_0 = 8/3 \). Further finding that
\[
x = U_1(x)/2, \quad x^2 = \frac{1}{22}(U_2(x) + 1), \quad x^3 = \frac{1}{25}(U_3(x) + 2U_1(x)),
\]
\[
x^4 = \frac{1}{24}(U_4(x) + 3U_2(x) + 2), \quad x^5 = \frac{1}{25}(U_5(x) + 4U_3(x) + 5U_1(x)),
\]
\[
x^6 = \frac{1}{26}(U_6(x) + 5U_4(x) + 9U_2(x) + 5),
\]
\[
x^7 = \frac{1}{27}(U_7(x) + 6U_5(x) + 14U_3(x) + 14U_1(x)),
\]
\[
x^8 = \frac{1}{28}(U_8(x) + 7U_6(x) + 20U_4(x)/2^6 + 28U_2(x) + 14),
\]
we can, using formula (3.3), find that \( \alpha_3/\alpha_1 = 5, \alpha_4/\alpha_1 = 8 \). So generalizing we will consider the following function
\[
f(x, y|\alpha) = \frac{4}{\pi^2} \sqrt{(1 - x^2)(1 - y^2)} \sum_{n \geq 0} \exp(-\alpha n(n + 2)/3)U_n(x)U_n(y),
\]
as the candidate for the density of the bivariate measure \( \rho \) according to the formula (2.4). First of all, notice that \( \int_{-1}^{1} \int_{-1}^{1} f(x, y|\alpha)dx\,dy = 1 \) for all \( \alpha > 0 \). Thus it remains to show that \( f(x, y|\alpha) \geq 0 \) for all \( x, y \in [-1, 1] \). In order to simplify notation let us consider an auxiliary function:
\[
g(x, y|\alpha) = \sum_{n \geq 0} \rho^{n(n+1)} U_n(x)U_n(y).
\]
We immediately notice that, since we have \( n(n + 2) = (n + 1)^2 - 1, U_n(\cos x) = \frac{\sin(n+1)x}{\sin x} \) and \( 2\sin(\gamma) \sin(\phi) = \cos(\gamma - \phi) - \cos(\gamma + \phi) \), that
\[
g(\cos \gamma, \cos \phi|\alpha) = \frac{1}{4\rho \sin(\gamma) \sin(\phi)} \left( 2 \sum_{n=1}^\infty \rho^{n^2} \cos(n(\gamma - \phi)) - 2 \sum_{n=1}^\infty \rho^{n^2} \cos(n(\gamma + \phi)) \right)
\]
\[= \frac{1}{4\rho \sin(\gamma) \sin(\phi)} (\theta(\rho; (\gamma - \phi)/2) - \theta(\rho; (\gamma + \phi)/2)).
\]
Now notice that the following (4.6) function \( \theta(\rho; \gamma/2) \) behaves like a cosine function for positive \( \rho \) and \( \gamma \in (0, \pi) \) in the sense that it increases and decreases on exactly the same intervals. In particular, it means that both functions have derivatives of the same signs. This leads to the conclusion that
\[
\frac{\theta(\rho; \gamma/2) - \theta(\rho; \phi/2)}{\cos \gamma - \cos \phi} \geq 0,
\]
for \( \rho \in (0, 1) \) and \( \gamma, \phi \in (0, \pi) \). But we have
\[
g(\cos \gamma, \cos \phi|\alpha) = \frac{\theta(\rho; (\gamma - \phi)/2) - \theta(\rho; (\gamma + \phi)/2)}{2\rho(\cos(\gamma - \phi) - \cos(\gamma + \phi))}.
\]
Returning to variables \( x \) and \( y \) we get:
\[
g(x, y|\alpha) = \left( \frac{\theta(\rho; (\arccos(x) - \arccos(y))/2) - \theta(\rho; (\arccos(x) + \arccos(y))/2)}{4\rho \sqrt{(1 - x^2)(1 - y^2)}} \right) \cdot
\]

consequently, transition probabilities density is equal to:

\[ \eta(x|y, t) = \frac{\theta(e^{-\alpha t}; \arccos(x) - \arccos(y))/2 - \theta(e^{-\alpha t}; \arccos(x) + \arccos(y))/2}{4 \exp(-\alpha t) \sqrt{1 - y^2}} \]

and we have the following relationships

\[ E(U_n(X_{t+\tau})|X_t = y) = \exp(-\alpha tn(n + 2)) U_n(y), \]

for some \( \alpha > 0 \), \( n \geq 0 \) and almost all \((\text{mod } \mu)\) \( y \in \text{supp } \mu \).

**Remark 8.** Let us note that we have \( E(X_{t+\tau} - X_\tau)^4 = O(t^2) \) as it follows directly from \([4,4]\). However, we can find \( c_{i,n} \) defined above for several \( i \) and \( n \) numerically performing Cholesky decomposition of the moment matrix. The \( 2n \)-th moment of the semicircle distribution is known and is equal to, depending on the radius \( r \), is equal \( r^{2n} C_n/4^n \). Here \( C_n \) denotes \( n \)-th Catalan number and by the Wigner distribution with radius \( r \) we mean the one with the following density defined for \( |x| \leq r \):

\[ f_W(x|r) = \frac{2}{\pi r^2} \sqrt{r^2 - x^2}. \]

Thus consequently, we can, using Mathematica, check that for \( k = 3, 4, 5 \) that also \( E(X_{t+\tau} - X_\tau)^{2k} = O(t^k) \). Thus one can utter the following conjecture:

**Conjecture 2.** Let us consider \( X = SM_{PR}\{1(n(n + 2)\alpha)\}, f_B(x|3/2, 3/2) \). Then for all \( k \geq 1 \) we have \( E(X_{t+\tau} - X_\tau)^{2k} = O(t^k) \).

**Example 7** (q-Normal distribution.). \( q \)-Normal is in fact a family of distributions indexed by a parameter \( q \in [-1, 1] \). Its properties have been described in many papers in particular the following review paper \([29]\) where this distribution and the others related to it are presented and various their applications in probability theory and combinatorics are presented. It is defined as follows.

For \( q = -1 \), it is a discrete \( 2 \)- point distribution, which assigns values \( 1/2 \) to \(-1\) and \( 1 \).

For \( q \in (-1, 1) \), it has density given by

\[ f_N(x|q) = \frac{\sqrt{1 - q}}{2\pi\sqrt{4 - (1 - q)x^2}} \prod_{k=0}^{\infty} ((1 + q)^k - (1 - q)x^2 q^k) \prod_{k=0}^{\infty} (1 - q^{k+1}), \]

for \( |x| \leq \frac{2}{\sqrt{1-q}} \). In particular \( f_H(x|0) = \frac{1}{\pi x} \sqrt{4-x^2} \), for \( |x| \leq 2 \). Hence, it is a Wigner distribution with a radius 2. For \( q = 1 \), the \( q \)-Gaussian distribution is the Normal distribution with parameters 0 and 1.

It is also known that the family of polynomials orthogonal with respect to \( q \)-Normal distribution are the so-called \( q \)-Hermite polynomials satisfying the following three-term recurrence

\[ H_{n+1}(x|q) = x H_n(x|q) - [n]_q H_{n-1}(x|q), \]

with \( H_{-1}(x|q) = 0 \) and \( H_0(x|q) = 1 \), where \([n]_q = 1 + \ldots + q^{n-1}\). It is also known that

\[ \int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}} H_n(x|q) H_m(x|q) f_N(x|q) dx = \delta_m [n]_q!, \]

where \([n]_q = \prod_{i=0}^{n} [i]_q \). Recall that \(-1 < q \leq 1\) and that by the \( q \)-Normal distribution, we mean the one defined by the following density. Following \([29]\) (Proposition...
2) we know that $m_4(q) = 2 + q$, $m_3(q) = 0$ and $m_2(q) = 1$. Hence, $\alpha = 0$ and $\kappa = q - 1$. Consequently, we have

$$C_c = \frac{\alpha_2}{\alpha_1} = \frac{12 + 4(q - 1)}{6 + 3(q - 1)} = \frac{8 + 4q}{3 + 3q}.$$ 

This parameter is greater than 2 for all $q < 1$. Recall that $q = 1$ refers to the Gaussian case that was already analyzed. Then, as it can be seen we have $C_c = 2$. Since for all cases other than 1 the possible values of parameter $q$, the support of the measure $f_N$ is bounded, equal to $S_q = [-2/\sqrt{1-q}, 2/\sqrt{1-q}]$ the sequence $\{\exp(-\alpha_n t)\}$ may not be a moment sequence. Hence, one can expect that for all choices of the sequence $\{\alpha_n\}$ such that $\alpha_2 = \frac{8 + 4q}{3 + 3q} \alpha_1$ we can expect path continuity of the given SMPR process. The only thing is to select the sequence of positive numbers $\{\alpha_n\}_{n \geq 1}$ with $\alpha_2$ given above in such a way that the function given by (2.5) is nonnegative for all $|x|, |y| \leq \frac{1}{\sqrt{1-q}}$.

Note that the case $q = 0$ refers to the Wigner case also analyzed above (although with a different radius). What is more, as shown in [2] to each classical $q$-Normal (Gaussian) process one can define a related process defined within the non-commutative probability setting. Among those related processes defined within the non-commutative probability setting, the case $q = 0$ refers to the so-called free probability. Thus, one can expect that our analysis concerning Wigner marginal distribution presented above can help to construct free probability process having a continuous path modification.

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