SEMILINEAR ELLIPTIC EQUATIONS ADMITTING SIMILARITY TRANSFORMATIONS

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Abstract. In this paper we study the equation

\[-\Delta u + \rho^{-2-\alpha} h(\rho^\alpha u) = 0\]

in a smooth bounded domain \(\Omega\) where \(\rho(x) = \text{dist}(x, \partial\Omega)\), \(\alpha > 0\) and \(h\) is a nondecreasing function which satisfies Keller-Osserman condition. We introduce a condition on \(h\) which implies that the equation is subcritical, i.e. the corresponding boundary value problem is well posed with respect to data given by finite measures. Under additional assumptions on \(h\) we show that this condition is necessary as well as sufficient. We also discuss b.v. problems with data given by positive unbounded measures. Our results extend results of [10] treating equations of the form

\[-\Delta u + \rho^\beta u^q = 0\]

with \(q > 1\), \(\beta > -2\).

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1. Introduction

In this article we consider equations of the type

\[-\Delta u + H(\rho, u) = 0\]

in \(\Omega\),

where \(\Omega\) is a \(C^2\) domain in \(\mathbb{R}^N\), \(N > 1\),

\(H(\rho, u) = \rho^{-2-\alpha} h(\rho^\alpha u), \quad \rho(x) = \text{dist}(x, \partial\Omega)\)

and \(\alpha > 0\). With respect to the nonlinearity \(h\) we assume:

(i) \(h \in C^1(\mathbb{R}), \ h(0) = 0, \ h(t) > 0\) for \(t > 0\),

(ii) \(h\) is a convex and odd function,

(iii) \(\int_{\Omega \cap B_R(0)} h(c\rho^\alpha)\rho^{-(1+\alpha)}dx < \infty\) \(\forall c > 0, \ R > 0\).

Note that (i) and (ii) imply that \(h\) is nondecreasing.

More general equations such as

\[-\Delta u + g \circ u = 0\]

in \(\Omega\),
where \( g \circ u(x) = g(x, u(x)) \) and various special cases have been studied intensively over the last twenty years (see [9], [10] and the references therein). In the case of equations with absorption, a basic set of assumptions on \( g \) is

\[
\begin{align*}
(i) & \quad g(x, \cdot) \in C^1(\mathbb{R}), \quad g(x, 0) = 0, \quad g(x, t) > 0 \text{ for } t > 0, \\
(ii) & \quad g(x, \cdot) \text{ is nondecreasing and odd}, \\
(iii) & \quad g(\cdot, t) \in L^1(\Omega \cap B_R(0), \rho) \quad \forall \ t \in \mathbb{R} \text{ and } \forall \ R > 0,
\end{align*}
\]

The assumption that \( g(x, \cdot) \) is odd is often omitted. However it simplifies the presentation without affecting the treatment in an essential way. We note that conditions (1.3) imply that \( g = H \) satisfies (1.5).

The family of functions satisfying (1.5) will be denoted by \( G_0(\Omega) \).

Equations of the form (1.1) (with \( \rho(x) \) replaced by \(|x|\)) have been introduced by Bandle and Marcus [3]. These equations admit a similarity transformation. A special case of (1.1), namely,

\[
H(\rho, t) = \rho^{\gamma}|t|^q \text{sign } t, \quad \gamma > -2
\]

has been extensively studied. The case \( \gamma = 0 \) received much attention; the problem with arbitrary \( \gamma > -2 \) was studied in [10]. For the case \( \gamma > 0 \) see [6] and references therein. Equation (1.2) includes the special case \( H(\rho, t) = t^p + \rho^q t^q \) where \( p, q > 1, \gamma > -2 \).

**Notation.** We denote by \( \mathcal{M}(\partial \Omega) \) the family of set functions \( \nu \) on \( \mathcal{B}(\partial \Omega) \), such that, for every compact set \( E \subset \partial \Omega \), \( \nu 1_E \) is a finite measure. Thus, if \( \partial \Omega \) is compact then \( \mathcal{M}(\partial \Omega) \) is the set of finite Borel measures on \( \partial \Omega \).

Put

\[
\tilde{L}^1(\Omega) = \{ u \in L^1_{\text{loc}}(\Omega) : u \in L^1(\Omega \cap B_R(0)) \forall R > 0 \}
\]

and

\[
C^2_0(\bar{\Omega}) := \{ \phi \in C^2(\bar{\Omega}) : \phi = 0 \text{ on } \partial \Omega \}.
\]

Assume that \( \Omega \) is a \( C^2 \) domain (not necessarily bounded) and let \( \nu \in \mathcal{M}(\partial \Omega) \).

A function \( u \) is a solution of equation (1.4) if \( u \) and \( g \circ u \) are locally integrable in \( \Omega \) and the equation is satisfied in the distribution sense.

A function \( u \) is a solution of the boundary value problem

\[
-\Delta u + g \circ u = 0 \quad \text{in } \Omega; \quad u = \nu \quad \text{on } \partial \Omega,
\]

where \( \nu \) is a Radon measure on \( \partial \Omega \), if \( u \) and \( (g \circ u) \rho \) belong to \( \tilde{L}^1(\Omega) \) and

\[
\int_{\Omega} (-u \Delta \phi + (g \circ u) \phi) \, dx = -\int_{\partial \Omega} \frac{\partial \phi}{\partial n} \, d\nu \quad \text{for every } \phi \in C^2_0(\bar{\Omega})\text{ with bounded support.}
\]

The set of all measures \( \nu \in \mathcal{M}(\partial \Omega) \) such that (1.6) possesses a solution is denoted by \( \mathcal{M}^\theta(\partial \Omega) \). It is well-known that when a solution exists it is unique, [9].
Definition 1.1. Let $f: \mathbb{R} \to \mathbb{R}$ be an odd function and satisfy the following assumptions:

$$f \in C^1(\mathbb{R}), \quad f(0) = 0, \quad f(t) > 0 \text{ for } t > 0.$$  

We say that $f$ satisfies Keller-Osserman condition (KO) if

$$\int_a^{\infty} F(s)^{-\frac{1}{2}} ds < \infty \quad \forall a > 0,$$

where $F(s) = \int_0^s f(t) dt$.

If $g(x,t) \in \mathcal{G}_0$ we say that it satisfies the global KO condition if there exists $f$ as above that satisfies the KO condition and

$$f(|t|) \leq |g(x,t)| \quad x \in \Omega, \quad t \in \mathbb{R}.$$  

We say that $g$ satisfies the local KO condition if, for every domain $\Omega' \subset \Omega$ there exists $f = f^{\Omega'} \in C(\mathbb{R})$ that satisfies the KO condition and

$$f(|t|) \leq |g(x,t)| \quad x \in \Omega', \quad t \in \mathbb{R}.$$  

Remark 1.1. If $h$ satisfies the KO condition then the function $H$ given by

$$H(t) = \int_0^t f(s)ds$$

satisfies the local KO condition. Therefore, in this case, the family of solutions of (1.1) is uniformly bounded in compact subsets of $\Omega$.

Definition 1.2. Let $u \in W^{1,p}_{\text{loc}}(\Omega)$ for some $p > 1$. We say that a measure $\nu \in M(\partial \Omega)$ is the boundary trace of $u$ on $\partial \Omega$ if, for every uniform $C^2$ exhaustion $\{\Omega_n\}$,

$$(1.9) \quad \int_{\partial \Omega_n} u^{1_{\partial \Omega_n}} \phi dS \to \int_{\partial \Omega} \phi d\nu$$

for every compactly supported $\phi \in C(\bar{\Omega})$. (Here $u^{1_{\partial \Omega_n}}$ denotes the Sobolev trace.)

Similarly we define the boundary trace on a relatively open set $A \subset \partial \Omega$. A measure $\nu \in M(A)$ is the boundary trace of $u$ on $A$ if (1.9) holds for every $\phi \in C(\bar{\Omega})$ such that $\text{supp} \phi$ is a compact subset of $\Omega \cup A$. In the case of positive solutions we slightly extend this definition to include positive Radon measures on $A$.

Lemma 1.3. Let $g \in \mathcal{G}_0$ and let $u$ be a positive solution of (1.4). Let $O$ be an open set in $\mathbb{R}^N$ such that $O \cap \partial \Omega \neq \emptyset$. If

$$\int_{\Omega \cap O} g(x,u(x)) \rho(x) dx < \infty.$$  

Then $u$ has a boundary trace on $O \cap \partial \Omega$.

Notation. We denote by $\mathcal{B}_{\text{reg}}(\partial \Omega)$ the space of positive regular Borel measure on $\partial \Omega$.

Note that a measure $\nu \in \mathcal{B}_{\text{reg}}(\partial \Omega)$ need not be a Radon measure; it may blow up on compact sets. However the outer regularity implies that, for each measure $\nu$ in this space there exists a closed set $S_\nu$, called the blow up...
set of $\nu$, such that $\nu(A) = \infty$ for every non-empty Borel set $A \subset S_{\nu}$ and the restriction of $\nu$ to $R_{\nu} := \partial \Omega \setminus S_{\nu}$ is a Radon measure.

Next we extend the notion of boundary trace to positive solutions of (1.4) that may not have a finite boundary trace.

**Definition 1.4.** Let $u$ be a positive solution of (1.4). We say that $u$ has a (generalized) boundary trace $\nu \in B_{\text{reg}}(\partial \Omega)$ if:

(a) $\nu|_{R_{\nu}}$ is the boundary trace of $u$ on the relatively open set $R_{\nu}$ (in the sense of Definition 1.2) and

(b) for every open set $O$ in $\mathbb{R}^N$ such that $O \cap S_{\nu} \neq \emptyset$ and every uniform $C^2$ exhaustion $\{\Omega_n\}$ (see [9, Definition 1.3.1])

\[ \int_{\partial \Omega_n \cap O} u \, dS \to \infty. \] (1.11)

**Definition 1.5.** Let $u \in C^2(\Omega)$ be a positive solution of (1.4). A point $z \in \partial \Omega$ is called a regular boundary point of $u$ if there exists an open neighborhood $U$ of $z$ such that

\[ \int_{U \cap \Omega} (g \circ u) \rho \, dx < \infty. \] (1.12)

The set of regular boundary points of $u$ will be denoted by $R(u)$. Its complement $S(u) := \partial \Omega \setminus R(u)$ is the singular boundary set of $u$. A point $y \in S(u)$ is called a strongly singular point of $u$.

Clearly $R(u)$ is relatively open; therefore $S(u)$ is closed.

Here we recall [9][10, Theorem 1.1].

**Theorem 1.6.** Let $\Omega$ be a domain whose boundary is a manifold of class $C^2$, not necessarily compact. Suppose that $g \in G_0$, satisfies the local KO condition. Also assume that (1.4) possesses a barrier at every point of $\partial \Omega$. Then every positive solution $u$ of (1.4) possesses a (generalized) boundary trace given by a positive measure $\nu \in B_{\text{reg}}(\partial \Omega)$.

$S(u)$ coincides with the blow up set of $\nu$ so that $\nu$ is a Radon measure on $R(u)$.

For the definition of barrier and the conditions of its existence see Definition 2.1 and Proposition 2.5.

**Definition 1.7.** A nonlinearity $g \in G_0$ is called subcritical if problem (1.6) possesses a solution for every $\nu \in M(\partial \Omega)$.

In this article we focus on the boundary value problem

\[ \begin{align*}
-\Delta u + H(\rho, u) &= 0 \quad \text{in } \Omega; \\
u &= \nu \quad \text{on } \partial \Omega,
\end{align*} \] (1.13)

where $\nu$ is a regular Borel measure, $H$ is as in (1.2) and $h$ is assumed to satisfy (1.3).

Following is a description of our main results concerning this problem.
Theorem 1.8. Let $\Omega$ be a bounded domain of class $C^2$ and $H$ is defined as in (1.2). Assume that $h$ satisfies (1.3) (i),(ii) and

$$(1.14) \quad H(\rho, cP(\cdot, y)) \in L^1(\Omega; \rho) \quad \forall y \in \partial \Omega, \quad \forall c \in \mathbb{R}.$$ 

where $P(x, y)$ is the Poisson kernel of $-\Delta$ in $\Omega$.

Then $H$ is subcritical and the following assertion holds:

Assume that there exists $\epsilon > 0$ such that

$$(1.15) \quad \limsup_{t \to 0} \frac{h(t)}{t^{1+\epsilon}} \leq \infty.$$ 

Let $\{\nu_k\}$ be a sequence in $\mathcal{M}(\partial \Omega)$ converging weakly to a measure $\mu$. Let $v$ (resp. $v_k$) denote the solution of (1.13) with $\nu = \mu$ (resp. $\nu = \nu_k$). Then $v_k \to v$ in $L^1(\Omega)$ and $H(\rho, v_k) \to H(\rho, v)$ in $L^1(\Omega, \rho)$.

Remark 1.8 Let $y \in \partial \Omega$ and put

$$C_R = \{x \in \Omega \cap B_R(y) : |x - y| \leq 2\rho(x)\}.$$ 

Assume that $R$ is sufficiently small so that $\overline{C_R} \subset \Omega \cup \{y\}$. Then (1.14) implies

$$\int_{C_R} \rho^{-(1+\alpha)} h(c \rho^\alpha P(x, y)) dx < \infty,$$

which in turn implies

$$(1.16) \quad \int_0^1 t^N - \alpha - 2h\left(ct^{\alpha-N+1}\right) dt < \infty.$$ 

Actually, in a bounded $C^2$ domain (1.16) is equivalent to (1.14) (see Section 4). (1.16) implies that, $\alpha \neq N - 1$ and

$$\begin{align*}
(a) & \quad \int_a^\infty h(t^{-1}) dt < \infty \quad \text{if } \alpha > N - 1 \\
(b) & \quad \int_0^a h(t^{-1}) dt < \infty \quad \text{if } 0 < \alpha < N - 1,
\end{align*}$$

for every $a > 0$. Consequently, if $h$ satisfies (1.3):

$$(1.18) \quad \text{Condition (1.14) implies } \alpha > N - 1.$$ 

Indeed, as $h$ is convex on $[0, \infty)$, $h(0) = 0$ and $h$ is nondecreasing, it follows that $s \to h(s)/s$ is nondecreasing. Thus $h(s) \geq h(1)s$ for $s > 1$ and, by assumption, $h(1) > 0$. Therefore $\int_0^1 h(t^{-1}) dt = \infty$. This rules out (1.17)(b) so that $\alpha > N - 1$.

In Section 4 we show that, under some additional assumptions on $h$, the condition $\alpha > N - 1$ is necessary and sufficient for $H$ to be subcritical; in particular it is equivalent to the subcriticality condition (1.14).

If $h(t) = t^q$, $q > 1$ then $H = \rho^\beta t^q$ where $\beta = \alpha(q - 1) - 2$. In this case, by (10), $H$ is subcritical if and only if $q < \frac{N+1+\beta}{\alpha}$, i.e., $\alpha > N - 1$. 

Definition 1.9. We say that \( h \) satisfies \( \Delta_2 \) condition if there exists a constant \( c > 0 \) such that
\[
h(a + b) \leq c(h(a) + h(b)) \quad \forall a > 0, b > 0.
\]

Theorem 1.10. Let \( \Omega \) be a bounded domain, \( y \in \partial \Omega \) and \( H \) be defined as in (1.2). Suppose that \( h \) satisfies KO condition, \( \Delta_2 \) condition, (1.3) and \( H \) satisfies the global barrier condition. If \( \alpha \leq N - 1 \) and \( u \) is a nonnegative solution of (1.1) vanishing on \( \partial \Omega \setminus \{y\} \), then \( u = 0 \) in \( \Omega \).

Notation. Assume that \( H \) is subcritical. If \( y \in \partial \Omega \) denote by \( u_{k,y} \) the unique solution of (1.13) with \( \nu = k\delta_y \).

Definition 1.12. Let \( y \in \partial \Omega \). If \( u \) is a positive solution of (1.1) such that \( u = 0 \) on \( \partial \Omega \setminus \{y\} \) and \( y \in S(u) \) we say that \( u \) has a strong isolated singularity at \( y \).

Theorem 1.11 implies that \( u_{\infty,y} \) is the smallest such solution.

For the next statement we need some additional notation. Let \( \Omega \) be a \( C^2 \) domain and let \( y \in \partial \Omega \). We denote by \( n_y \) the unit normal to \( \partial \Omega \) at \( y \) pointing outward. We denote by \( R_y \) the rotation that maps the vector \( e_1 = (1, 0, \ldots, 0) \) to \( -n_y \).

Theorem 1.13. Let \( \Omega \) be a bounded \( C^2 \) domain, \( y \in \partial \Omega \). Suppose that \( H \) and \( h \) satisfy the assumptions of Theorem 1.11. In addition assume that \( h \) is strictly convex.

Then
(i) equation (1.1) possesses a unique solution with strong isolated singularity at \( y \). This solution, denoted by \( U_y \), satisfies
\[
\lim_{r \to 0} r^\alpha U_y(y + r\sigma) = w(R_y(\sigma)),
\]
where \( w \) is the solution to the problem
\[
\Delta_{S^{N-1}} w + \lambda w - (\sigma \cdot e_1)^{(2+\alpha)} h((\sigma \cdot e_1)^\alpha w) = 0 \quad \text{in} \quad S_{+}^{N-1},
\]
\[
w = 0 \quad \text{on} \quad \partial S_{+}^{N-1},
\]
\[
\lambda = \alpha - (N - 1), \quad e_1 = (1, 0, \ldots, 0) \quad \text{and} \quad S_{+}^{N-1} = \{ x \in S^{N-1} : x_1 > 0 \}. \text{The convergence is uniform in compact subsets of} \quad S_{+}^{N-1}.\]
(ii) There exists a positive constant $C$, depending on $N, \alpha, h$, $C^2$ characteristic of $\Omega$ but independent of $y$, such that
\begin{equation}
C^{-1}|x - y|^{-\alpha - 1}\rho(x) \leq U_y(x) \leq C^1|x - y|^{-\alpha - 1}\rho(x) \quad \forall x \in \Omega.
\end{equation}

**Theorem 1.14.** Let $\Omega$ be a bounded $C^2$ domain. Under the assumptions of Theorem 1.13 problem $(1.13)$ possesses a unique positive solution for every $\nu \in B_{\text{reg}}(\partial \Omega)$. Furthermore, if $F$ is a closed subset of $\partial \Omega$ then the unique solution, $U_F$, with the boundary trace $\infty X_F$ satisfies,
\begin{equation}
c^{-1}\text{dist}(x,F)^{-\alpha - 1}\rho(x) \leq U_F(x) \leq c\text{dist}(x,F)^{-\alpha - 1}\rho(x) \quad \forall x \in \Omega.
\end{equation}

Some examples of nonlinearities for which the above theorem holds:

\begin{enumerate}
\item[(i)] $H(x,t) = \rho^\beta |t|^q \text{sign } t, \quad q > 1, \beta > -2,$
\item[(ii)] $H(x,t) = \rho^\beta |t|^q \ln(1 + \rho^\alpha |t|) \text{sign } t, \quad q > 1, \beta = -2 + \alpha(q - 1),$
\item[(iii)] $H(x,t) = \rho^{-2}|t| \exp(\rho^\alpha|t|) \text{sign } t.$
\end{enumerate}

The first example was treated in [10] and similar results have been obtained. However the estimates corresponding to $(1.23)$ and $(1.24)$ have been established only for $\beta \geq 0$.

The paper is organized as follows: In section 2 we present some definitions and preliminary results. In section 3 we derive global estimates for solutions of $(1.1)$. Section 4 is devoted to the proof of Theorem 1.8 and Theorem 1.10. We establish a lower estimate of the singular solution in Section 5. Section 6 contains the proof of existence and uniqueness of solutions with strong isolated singularity, first in half space and then in bounded domain. In section 7 we prove Theorem 1.14.

### 2. Definitions and preliminary results

Let $\Omega$ be a $C^2$ domain in $\mathbb{R}^N$. We use our notation:
\begin{equation}
\Sigma_\beta = \{ x \in \Omega : \rho(x) = \beta \}, \quad D_\beta = \{ x \in \Omega : \rho(x) > \beta \}, \quad \Omega_\beta = \Omega \setminus D_\beta.
\end{equation}

$n_x$ denotes the unit outward normal at $x \in \partial \Omega$. It is known that there exists $\beta_0 > 0$ such that:

(a) For every point $x \in \bar{\Omega}_{\beta_0}$, there exists a unique point $\sigma(x) \in \partial \Omega$ such that $|x - \sigma(x)| = \rho(x)$, i.e.
\begin{equation}
x = \sigma(x) - \rho(x)n_{\sigma(x)}, \quad \lim_{x \to \sigma(x)} \nabla \rho(x) = -n_{\sigma(x)}.
\end{equation}

(b) The mapping $x \mapsto \rho(x)$ and $x \mapsto \sigma(x)$ belong to $C^2(\bar{\Omega}_{\beta_0})$ and $C^1(\Omega_{\beta_0})$ respectively.

(c) The mapping $x \mapsto (\rho(x), \sigma(x))$ is a $C^2$ diffeomorphism from $\bar{\Omega}_{\beta_0}$ to $[0, \beta_0] \times \partial \Omega$.

Thus $(\rho, \sigma)$ may serve as a set of coordinate in one sided neighborhood of the boundary and are called flow coordinates of $\Omega$. 
Definition 2.1. Let \( z \in \partial \Omega \). We say that there exists a g-barrier at \( z \) if there exists \( r(z) > 0 \) such that for every \( 0 < r \leq r(z) \), (1.4) possesses a positive solution \( w = w_{r,z} \) in \( B_r(z) \cap \Omega \) such that

\[
\begin{align*}
(2.2) \\
(i) & \quad w \in C(B_r(z) \cap \Omega) \quad w = 0 \text{ on } \partial \Omega \cap B_r(z), \\
(ii) & \quad \lim_{x \to y} w(x) = \infty \quad \forall y \in \partial B_r(z) \cap \Omega.
\end{align*}
\]

We say that \( g \) satisfies the global barrier condition if:

(i) A g-barrier exists at every point of \( \partial \Omega \).

(ii) There exists a number \( \bar{r} > 0 \) such that \( r(z) \geq \bar{r} \) \( \forall z \in \partial \Omega \).

(iii) If \( w = w_{r,z} \), then \( ||w||_{C^2(\bar{\Omega} \cap B_{\bar{r}}(z))} \leq C \) where \( C \) is a constant independent of \( z \).

Remark: If \( g \in G_0 \) and satisfies global KO condition then a g-barrier exists at every point \( z \in \partial \Omega \).

In fact one can get a stronger implication:

Lemma 2.2. If \( \Omega \) is a domain uniformly of class \( C^2 \) and \( g \in G_0 \) satisfies global KO condition in \( \Omega \) then \( g \) satisfies the global barrier condition.

For proof see [9, Lemma 3.1.10].

Definition 2.3. Let \( \Omega \) be a bounded domain and \( g \in G_0 \). A measure \( \nu \in \mathcal{M}(\partial \Omega) \) is called g-admissible if

\[
\begin{align*}
g \circ \mathbb{P}_\Omega(\nu) & \in L^1(\Omega, \rho) \quad \text{and} \quad g \circ ( - \mathbb{P}_\Omega(\nu) ) \in L^1(\Omega, \rho),
\end{align*}
\]

where

\[
\mathbb{P}_\Omega(\nu)(x) = \int_{\partial \Omega} P(x, y) d\nu(y),
\]

and \( P \) is the Poisson kernel of \(-\Delta \) in \( \Omega \).

It is known from [10] Lemma 4.4 that if \( \Omega \) is a bounded domain, \( g \in G_0 \) and \( \nu \) is g-admissible, then problem (1.6) possesses a unique solution.

It is known from [6] that, equation (1.4) possesses a global barrier when \( g(x, t) = \rho^\alpha |t|^{q-1} t \) where \( \alpha > 0 \) and \( q > 1 \). The notion of global barrier condition (see Definition 2.1) did not appear in their work but the construction of their proof establishes the fact that the nonlinearity \( g \) given by \( \rho^\alpha |t|^{q-1} t \), with \( \alpha > 0 \) and \( q > 1 \), satisfies global barrier condition. Also note that when \( -2 < \alpha \leq 0 \), existence of global barrier follows from the fact that \( g \) satisfies global KO condition. In the next proposition we establish a sufficient condition for the nonlinearity \( g \in G_0 \) to satisfy a global barrier condition.

Definition 2.4. A possibly unbounded domain \( \Omega \) is uniformly of class \( C^2 \) if it satisfies the following conditions:

(i) There exists \( r_\Omega > 0 \) such that, for every \( X \in \partial \Omega \) there exists a set of coordinates \( \xi_\xi = \xi^X \) and a function \( F_X \in C^2(\mathbb{R}^{N-1}) \) such that \( F_X(0) = 0 \), \( \nabla F_X(0) = 0 \) and

\[
\Omega \cap B_{r_\Omega}(X) = \{ \xi : |\xi| < r_\Omega, \xi_\xi > F_X(\xi_2, \cdots, \xi_N) \}.
\]
(ii) The set \( \{ F_X : X \in \partial \Omega \} \) can be chosen so that
\[
||\partial \Omega||_{C^2} := \sup\{||F_X||_{C^2(B_{\Omega}(0))} : X \in \partial \Omega \} < \infty
\]
and there exists \( \kappa \in C(0,1) \) such that \( D^2F_X \) has modulus of continuity \( \kappa \) for every \( X \in \partial \Omega \). The pair \( (r_\Omega, ||\partial \Omega||_{C^2}) \) is called a \( C^2 \) characteristic of \( \Omega \).

**Proposition 2.5.** Let \( \Omega \) be a domain (not necessarily bounded), uniformly of class \( C^2 \). We assume that \( g \in G_0 \) satisfies local KO condition and there exists \( C, T > 0 \) such that
\[
(2.3) \quad g(x,t) > C \rho^{\alpha_0} t^\Gamma \quad \text{when} \quad t > T,
\]
for some \( \alpha_0 \geq 0 \) and \( \Gamma > 1 \). Let \( z \in \partial \Omega \), \( 0 < r < \frac{\beta_0}{2} \). Then there exists a positive solution \( W = W_{z,r} \) of (1.4) in \( B_r(z) \cap \Omega \) such that
\[
(2.4) \quad \begin{align*}
& (i) \quad W \in C(B_r(z) \cap \Omega), \quad W = 0 \text{ on } \partial \Omega \cap B_r(z), \\
& (ii) \quad \lim_{x \to y} W(x) = \infty \quad \forall y \in \partial B_r(z) \cap \Omega.
\end{align*}
\]
Furthermore, if \( r = \frac{3\beta_0}{4} \), the norm of the solution \( W_{z,r} \) in \( C^2(\Omega \cap B_{\frac{3\beta_0}{4}}(z)) \) is bounded by a constant depending only on \( N, \alpha_0 \) and the \( C^2 \) characteristics of \( \Omega \) but not on \( z \).

A proof of the proposition is provided in the Appendix.

**Definition 2.6.** Let \( F \) be a compact subset of \( \partial \Omega \) and \( g \in G_0 \). Denote by \( \mathcal{V}_F \) the family of all non-negative solutions of (1.4) such that \( u \in C(\Omega \setminus F) \) and \( u = 0 \) on \( \partial \Omega \setminus F \).

We say that \( U \) is the maximal solution relative to \( F \) if \( U \in \mathcal{V}_F \) and \( U \) dominates every solution in \( \mathcal{V}_F \).

**Lemma 2.7.** Suppose that \( g \in G_0 \) satisfies the local KO condition. Let \( F \) be a compact subset of \( \partial \Omega \) such that \( \mathcal{V}_F \) is not empty. Then \( U_F := \sup \mathcal{V}_F \) belongs to \( \mathcal{V}_F \). Thus \( U_F \) is the maximal solution relative to \( F \).

**Proof.** See [9, Lemma 3.2.3].

**Corollary 2.8.** Let the function \( h \) in (1.2) satisfy (1.3) and KO condition. In addition, assume that (1.1) possesses a global barrier and \( F \) is a compact subset of \( \partial \Omega \) such that \( \mathcal{V}_F \) is not empty. Then there exists a positive solution \( U_F \) of (1.1) that vanishes on \( \partial \Omega \setminus F \) and dominates every solution with this property.

**Proof.** Let \( u_n \) be the solution of (1.1) with \( u_n = n \) on \( F \) and \( u_n = 0 \) on \( \partial \Omega \setminus F \). Since \( h \) satisfies KO condition, applying Remark 1.1 we obtain \( u := \lim_{n \to \infty} u_n \) exists. Let \( z \in \partial \Omega \setminus F \) and \( 0 < r < \frac{\beta_0}{2} \). Then as (1.1) possesses a global barrier using Proposition 2.5, we obtain
\[
(1.1) \quad u_n \leq W_{z,r} \quad \text{in } B_{\frac{\beta_0}{2}}(z) \cap \Omega \quad \forall n.
\]
Thus $u$ vanishes on $\partial \Omega \setminus F$. Now define $V_F$ and $U_F$ as in (2.6). Therefore applying Lemma 2.7, the result follows.

3. The similarity transformation and global estimates of the solutions

A basic tool in our presentation is a similarity transformation associated with (1.1), denoted by $T^\alpha_a$, $a > 0$, given by

$$T^\alpha_a u(x) = a^\alpha u(ax).$$

If $u$ is a weak solution of (1.1) in $\Omega$ then $T^\alpha_a u$ is a weak solution of this equation in $\frac{1}{a} \Omega$. If $\Omega = \frac{1}{a} \Omega$ and $u = T^\alpha_a u$ for every $a > 0$, we say that $u$ is a self-similar solution.

The next lemma establishes a global estimate of solutions of (1.1) assuming the Keller-Osserman condition.

**Lemma 3.1.** Let the function $h$ in (1.2) satisfy the conditions (i) and (ii) in (1.3) and KO condition. Then there exists a constant $C = C(N, \alpha, h)$ such that for every $\alpha > 0$ and every solution $u$ of (1.1),

$$|u(x)| \leq C \rho(x)^{-\alpha}.$$  

**Proof.** If $u$ is a solution of (1.1) then, by Kato’s inequality [9, Proposition 1.5.4], $|u|$ is a subsolution; therefore it is sufficient to prove the lemma for $u > 0$.

We fix a point $x \in \Omega$ and denote

$$R = \rho(x)/2, \quad \Omega' = B_R(x).$$

Then

$$R \leq \rho(y) \leq 3R \quad \forall y \in \Omega'.$$

Using this fact and the monotonicity of $h$ we obtain

$$(-2-\alpha) h(R^\alpha u) \leq \rho^{-2-\alpha} h(\rho^\alpha u) \leq R^{-2-\alpha} h(3^\alpha R^\alpha u) \text{ in } \Omega'.$$

Since $u$ satisfies (1.1) and $\Omega' \subset \Omega$,

$$-\Delta u + (3R)^{-2-\alpha} h(R^\alpha u) \leq 0 \quad \text{in } \Omega'.$$

Let $v$ be the weak solution of the boundary value problem

$$-\Delta v + (3R)^{-2-\alpha} h(R^\alpha v) = 0 \quad \text{in } \Omega', \quad v = u \quad \text{on } \partial \Omega'.$$

By the comparison principle, $u \leq v$ in $\Omega'$.

Put $c_1 := (3R)^{-2-\alpha}$ and $c_2 := R^\alpha$. Since $h$ satisfies the KO condition, $\tilde{h}(\cdot) := c_1 h(c_2 \cdot)$ also satisfies this condition. Denote,

$$H_1(t) = \int_0^t h(s)ds, \quad \psi(a) = \int_a^\infty \frac{ds}{(2H_1(s))^{\frac{1}{2}}} \quad \forall a > 0, \quad \phi = \psi^{-1}.$$  

Define $\tilde{H}_1, \tilde{\psi}$ and $\tilde{\phi}$ in the same way with $h$ replaced by $\tilde{h}$. 

As \( v \) satisfies (3.6),
\[
v(y) \leq \tilde{\phi}(\text{dist}(y, \partial \Omega')) \quad \text{in} \quad \Omega',
\]
(see [7], [4]). A simple calculation yields
\[
\tilde{\phi}(s) = \frac{1}{c_2} \phi(\sqrt{c_1 c_2 s}), \quad s > 0.
\]
Therefore
\[
v(x) \leq \tilde{\phi}(R) \leq \tilde{\phi}(\frac{\rho(x)}{3}) = \frac{1}{c_2} \phi(\sqrt{c_1 c_2 \frac{\rho(x)}{3}}).
\]
Here we use (3.3) and the fact that \( \tilde{\phi} \) is decreasing. Substituting the values of \( c_1 \) and \( c_2 \), using again (3.3) and the fact that \( R = \rho(x)/2 \) we obtain
\[
v(x) \leq C \rho(x)^{-\alpha}
\]
where \( C = 2^\alpha \phi(\frac{3}{3} - \frac{\alpha}{2}) \). Since \( u \leq v \) in \( \Omega' \), (3.2) follows. \( \square \)

Assuming existence of barrier at every point of \( \partial \Omega \) we can improve the inequality in Lemma 3.1.

**Proposition 3.2.** Let \( \Omega \) be a domain (not necessarily bounded) uniformly of class \( C^2 \) and \( F \) be a compact subset of \( \partial \Omega \). In addition, assume that the function \( h \) in (1.2) satisfies (1.3) and (1.1) possesses a global barrier. Then there exists a constant \( C \) depending only on \( N, \alpha, h \) and the \( C^2 \) characteristic of \( \Omega \) such that for every solution \( u \) of (1.1) vanishing on \( \partial \Omega \setminus F \),
\[
|u(x)| \leq C \rho(x) \text{dist}(x, F)^{-\alpha-1}
\]
for every \( x \in \Omega \) with \( \text{dist}(x, F) \leq (1 + \beta_0)^{-1} \). If \( \Omega \) is bounded then (3.8) holds for every \( x \in \Omega \).

**Proof.** If \( u \) is a solution then by Kato’s inequality \( |u| \) is a subsolution and the existence of barrier guarantees that the smallest solution above \( |u| \) vanishes on \( \partial \Omega \setminus F \). Therefore it is enough to prove the proposition for \( u > 0 \). Let \( z \in \partial \Omega \setminus F \) and \( \gamma(z) := \frac{1}{\sqrt{c_1 c_2}} \text{dist}(z, F) \). Now if \( u \) is a solution to (1.1) then \( u_z(x) := \gamma(z)^{\alpha} u(\gamma(z) x) \) is a solution to
\[
-\Delta u_z + \rho_\gamma(x)^{-2-\alpha} h(\rho_\gamma(x)^{\alpha} u_z(x)) = 0 \quad \text{in} \quad \frac{1}{\gamma(z)} \Omega.
\]
where \( \rho_\gamma(x) = \text{dist}\left(x, \partial \frac{\Omega}{\gamma(z)}\right) \)

Let \( r = \frac{3\beta_0}{4} \min(1, \frac{1}{\gamma(z)}) \). We assume \( \gamma(z) < 1 \) so that \( r = \frac{3\beta_0}{4} \). Then the solution \( W_{z,r} \) mentioned in Proposition 2.5 satisfies
\[
u_z < W_{z,r} \quad \text{in} \quad B_{\frac{\beta_0}{2}}(z) \cap \frac{1}{\gamma(z)} \Omega.
\]
Thus \( u_z \) is bounded in \( B_{\frac{\beta_0}{2}}(z) \cap \frac{1}{\gamma(z)} \Omega \) by a constant \( C \) depending only on \( N, \alpha \) and the \( C^2 \) characteristic of \( \frac{1}{\gamma(z)} \Omega \). Since \( \gamma(z) < 1 \), \( C^2 \) characteristic of \( \Omega \) is also \( C^2 \) characteristic of \( \frac{1}{\gamma(z)} \Omega \). Therefore the constant \( C \) can be chosen
independent of \( z \). Now applying the mean value theorem on \( h \) and using Lemma 3.1 we note that, there exists \( c_x \in (0, \rho_\gamma(x)^a u_z) \subset (0, C) \) with

\[
0 = -\Delta u_z + [\rho_\gamma(x)^{-2} h'(c_x)]u_z \quad \text{in} \quad \frac{1}{\gamma(z)} \Omega.
\]

Note that as \( h \) is non-decreasing and \( C^1, h'(c_x) \) is non-negative and uniformly bounded in \( \frac{1}{\gamma(z)} \Omega \). Let \( v_z \) denote the solution of

\[
-\Delta v + V(x)v = 0 \quad \text{in} \quad B_{\frac{1}{\gamma(z)}} \cap \frac{1}{\gamma(z)} \Omega
\]
\[
v = u_z \quad \text{on} \quad \partial(\frac{1}{\gamma(z)} \Omega \cap B_{\frac{1}{\gamma(z)}})
\]

where \( V(x) = h'(c_x) \) which is non-negative and uniformly bounded. Therefore as \( u_z \) is uniformly bounded on \( \partial(\frac{1}{\gamma(z)} \Omega \cap B_{\frac{1}{\gamma(z)}}) \), applying weak maximum principle we obtain \( v_z \) is uniformly bounded in \( \frac{1}{\gamma(z)} \Omega \cap B_{\frac{1}{\gamma(z)}} \). Now as \( v_z = 0 \) in \( \partial \Omega \cap B_r(z) \), applying Hopf’s lemma we obtain

\[
v_z(x) \leq C_1 \rho_\gamma(x) \quad \forall \ x \in B_{\frac{1}{\gamma(z)}} \cap \frac{1}{\gamma(z)} \Omega.
\]

Also note that, from (3.10) and (3.11) we obtain that \( u_z \) is a subsolution of (3.11). Therefore \( u_z \leq v_z \) in \( B_{\frac{1}{\gamma(z)}} \cap \frac{1}{\gamma(z)} \Omega \). Hence using (3.12) and the definition of \( u_z \) we obtain

\[
u(x) \leq C_1 \dist(x, \partial \Omega) \gamma(z)^{-\alpha - 1} \quad \forall \ x \in B_{\gamma(z)\frac{1}{\beta_0}} \cap \Omega.
\]

Let \( x \in \Omega_{\beta_0} \) and assume that

\[
\rho(x) = \dist(x, \partial \Omega) \leq \beta_0 \dist(x, F), \quad \dist(x, F) < \frac{1}{1 + \beta_0}.
\]

Let \( z \) be the unique point on \( \partial \Omega \) such that \( \dist(x, z) = \rho(x) \). Then

\[
2\gamma(z) \geq \dist(x, F) - \rho(x) \geq (1 - \beta_0) \dist(x, F).
\]

Therefore

\[
u(x) \leq C_1 \rho(x) \left( \frac{1}{2} (1 - \beta_0) \dist(x, F) \right)^{-\alpha - 1}.
\]

Now if \( \rho(x) > \beta_0 \dist(x, F) \), by Lemma 3.1

\[
u(x) \leq C \rho(x)^{-\alpha} \leq C \rho(x) \left( \beta_0 \dist(x, F) \right)^{-\alpha - 1}.
\]

Thus the proposition holds for every \( x \in \Omega_{\beta_0} \) such that \( \dist(x, F) < (1 + \beta_0)^{-1} \). Now if \( \Omega \) is bounded, by maximum principle \( u \) is bounded in \( \{ x \in \Omega : \dist(x, F) \geq (1 + \beta_0)^{-1} \} \). Therefore by maximum principle (3.8) is true in this set and therefore for every \( x \in \Omega \). \( \square \)
4. Subcriticality

In this section we consider the boundary value problem \((1.13)\), \(\nu \in \mathcal{M}(\partial \Omega)\). \(H\) is defined as in \((1.2)\) and \(h\) is assumed to satisfy \((1.3)\) and the KO condition. In addition, we assume that \((1.1)\) possesses a global barrier and \(\Omega\) is a bounded domain.

For \(p > 1\), \(L^p_w(\Omega, \tau)\) denotes the weak \(L^p\) space with the norm

\[
\|f\|_{L^p_w(\Omega, \tau)} = \sup_{\omega \subset \Omega, \omega \text{ measurable}, 0 < \tau(\omega) < \infty} \left\{ \int_{\omega} |f| \left( \frac{d\tau}{\tau(\omega)^{\frac{1}{p}} \rho^\beta} \right) : \omega \subset \Omega, \omega \text{ measurable} \right\}
\]

where \(\frac{1}{p} + \frac{1}{p'} = 1\).

**Notation.** We denote by \(P(\nu)\) the solution of

\[
\Delta u = 0 \quad \text{in } \Omega, \quad u = \nu \quad \text{on } \partial \Omega.
\]

**Lemma 4.1.** The mapping

\[
P : \mathcal{M}(\partial \Omega) \mapsto L^{\frac{N+\beta}{N-\beta}}(\Omega, \rho^\beta) \quad \forall \beta \geq -1
\]

is continuous relative to the norm topologies.

**Proof.** See [9, Lemma 2.3.3(ii)]. \(\square\)

**Theorem 4.2.** Let \(f \in C(\mathbb{R})\) and be a monotone increasing function such that

\[
f(t) \to 0 \quad \text{as} \quad t \to 0
\]

\(\tau\) be a positive measure in \(\mathcal{M}(\Omega)\). Assume that, for some \(p \in (1, \infty)\),

\[
\int_0^1 f(r^{-\frac{1}{p}}) dr < \infty.
\]

If \(\{w_n\}\) is a bounded sequence in \(L^p_w(\Omega, \tau)\) then \(\{f \circ w_n\}\) is uniformly integrable in \(L^1(\Omega, \tau)\). Furthermore the modulus of uniform integrability depends only on the bound of the sequence, the \(C^2\) characteristic of \(\Omega\) and \(\text{diam } \Omega\).

**Proof.** See [9, Thm. 2.3.4]. \(\square\)

**Notation.** \(\mathcal{M}(\Omega, \rho)\) denotes the space of signed Radon measures \(\mu\) in \(\Omega\) such that

\[
\rho \mu \in \mathcal{M}(\Omega) \quad \text{where} \quad \rho(x) := \text{dist} (x, \partial \Omega).
\]

The norm of a measure \(\mu \in \mathcal{M}(\Omega, \rho)\) is given by

\[
||\mu||_{\Omega, \rho} = \int_{\Omega} \rho \, d|\mu|.
\]
Proof of Theorem 1.8 Let \( \nu \in \mathcal{M}(\partial \Omega) \). As \( h \) is convex and odd we have,

\[
\| \rho^{-2-\alpha} h(\pm \rho \mathbb{P}_\Omega(|\nu|)) \|_{L^1(\Omega, \rho)} = \int_\Omega \rho^{-1(\alpha)} h(\rho \int_{\partial \Omega} |\nu|(\partial \Omega) \rho^0 P(x, y) d|\nu|(y)) dx
\]

\[
= \int_\Omega \rho^{-1(\alpha)} h(\int_{\partial \Omega} |\nu|(\partial \Omega) \rho^0 P(x, y) d|\nu|(y)) dx
\]

\[
\leq \int_\Omega \rho^{-1(\alpha)} \left[ \int_{\partial \Omega} h(|\nu|(\partial \Omega) \rho^0 P(x, y)) d|\nu|(y) \right] dx
\]

\[
= \int_{\partial \Omega} \int_\Omega \rho^{-1(\alpha)} h(|\nu|(\partial \Omega) \rho^0 P(x, y)) dx d|\nu|(y).
\]

Since \( \Omega \) is a bounded domain of class \( C^2 \),

\[
c_1^{-1} |x - y|^{1-N} \leq P(x, y) \leq c_1 |x - y|^{1-N} \quad \forall (x, y) \in (\Omega \times \partial \Omega),
\]

where the constant \( c_1 \) depends only on the \( C^2 \) characteristic of \( \Omega \) and its diameter.

Therefore (1.14) is equivalent to

\[
\int_\Omega \rho^{-2-\alpha} h(c \rho^0 |x - y|^{1-N}) \rho dx < \infty,
\]

for every \( c > 0 \) and every \( y \in \partial \Omega \). Passing to spherical coordinates we see that (4.3) implies

\[
\int_0^1 s^{-\alpha+1} h(c s) s^N ds < \infty \quad \forall c > 0.
\]

In fact this inequality is equivalent to (4.4) and therefore to (1.14). By a substitution of variables this inequality reduces to (1.17)(a). Hence, (4.4) implies that, for every \( M > 0 \) there exists a constant \( c(M) \), depending only on \( M \), the \( C^2 \) characteristic of \( \Omega \) and its diameter, such that

\[
\| \rho^{-2-\alpha} h(c \rho^0 P(\cdot, y)) \|_{L^1(\Omega, \rho)} \leq c(M) \quad \forall y \in \partial \Omega, \forall c \in [-M, M].
\]

Finally (4.2) and (4.5) imply that problem (1.13) has a solution for every \( \nu \in \mathcal{M}(\partial \Omega) \).

We turn to the proof of the second assertion of the theorem.

Assertion 1. Let \( \{\nu_k\} \) be a bounded sequence in \( \mathcal{M}(\partial \Omega) \) and put \( u_k := \mathbb{P}(\nu_k) \). Then \( \{H(\rho, u_k)\} \) is uniformly integrable in \( L^1(\Omega, \rho) \).

We may and shall assume that \( \nu_k \) is a positive measure for each \( k \).
Since \( \{\nu_k\} \) is bounded in \( \mathcal{M}(\partial \Omega) \) and \((1.14)\) implies that \( \alpha > N - 1 \) (see \((1.18)\)),

\[
\rho^\alpha u_k(x) = \int_{\Omega} P(x, y) \rho^\alpha(x) dv_k(y) \\
\leq C \int_{\Omega} |x - y|^{-N+1} \rho(x)^\alpha dv_k(y) \\
\leq C \int_{\Omega} |x - y|^{-N+1+\alpha} dv_k(y) < C_1.
\]

Thus \( \{\rho^\alpha u_k\} \) is uniformly bounded in \( \Omega \), say by \( C_1 \). By assumption \((1.15)\) there exists \( M' \) such that

\[
h(t) \leq M't^{1+\epsilon} \quad \forall \ t \in [0, C].
\]

Consequently

\[
\rho^{-1-\alpha} h(\rho^\alpha u_k) \leq M'\rho^{-1+\epsilon}\alpha u_k^{1+\epsilon}.
\]

Hence to prove the assertion it’s enough to show that \( u_k^{1+\epsilon} \) is uniformly integrable in \( L^1(\Omega, \rho^{\alpha-1}) \). As \( \alpha > N - 1 \) we have

\[
\frac{(N - 1)(1 + \epsilon)}{N + \epsilon\alpha - 1} < 1.
\]

Let \( f \) be the function given by \( f(s) = s^{1+\epsilon}, \ s > 0 \). Then,

\[(4.6) \quad \int_0^1 f(r^{-\frac{N-1}{N+\epsilon\alpha-1}})dr < \infty.\]

By Lemma \((4.1)\) \( \{u_k\} \) is bounded in \( L^{\frac{N+\beta}{N-1}}(\Omega, \rho^\beta) \) for every \( \beta > -1 \). Choose \( \beta = \epsilon\alpha - 1 \). Then, by \( \{(4.6)\} \) and Theorem \((4.2)\) with \( p = \frac{N+\epsilon\alpha-1}{N-1} \) it follows that \( \{f(u_k)\} \) is uniformly integrable in \( L^1(\Omega, \rho^{\alpha-1}) \). This proves the assertion.

Now, let \( \{\nu_k\} \), \( \mu \) and \( v_k \) be as in the second part of the theorem. As \( v_k \leq u_k \) and \( \{H(\rho, u_k)\} \) is uniformly integrable (and therefore bounded) in \( L^1(\Omega, \rho) \) it follows that \( \{H(\rho, v_k)\} \) is bounded in \( L^1(\Omega, \rho) \). Hence \( \{\Delta v_k\} \) is bounded in this space and consequently \( \{v_k\} \) is bounded in \( W^{1,p}_{loc}(\Omega) \) for every \( p \in [1, \frac{N}{N-1}) \). Therefore there exists \( v \in L^1(\Omega) \) such that, up to a subsequence, \( v_k \rightharpoonup v \) a.e in \( \Omega \). This fact and the uniform integrability of \( \{H(\rho, v_k)\} \) in \( L^1(\Omega, \rho) \) imply that \( H(\rho, v_k) \to H(\rho, v) \) in this space. As \( \nu_k \rightharpoonup \mu \) it follows that \( u_k \to \mathbb{P}[\mu] \) in \( L^1(\Omega) \). Therefore, since \( v_k \leq u_k \) and \( v_k \rightharpoonup v \) a.e in \( \Omega \), it follows that \( v_k \to v \) in \( L^1(\Omega) \). These facts together with the weak convergence \( \nu_k \rightharpoonup \mu \) imply that \( v \) is the weak solution of problem \((1.13)\) with \( \nu = \mu \).

\[\square\]

**Remark.** The proof of Theorem \((1.8)\) actually yields a stronger version of the second part:

**Theorem 4.3.** Assume that \((1.17)\) and \((1.15)\) hold. Let \( \{\Omega_k\} \) be a sequence of \( C^2 \) domains with a uniform \( C^2 \) characteristic such that \( \Omega_k \subset \)
Let $v_k$ denote the solution of
\[-\Delta v + H(\rho_k, v) = 0 \quad \text{in } \Omega_k,\]
\[v = \nu_k \quad \text{in } \partial \Omega_k,\]
where $\rho_k(x) = \text{dist}(x, \partial \Omega_k)$ for $x \in \Omega_k$. Finally let $v$ be the solution of
\[(1.13)\]
Considering $v_k$ (resp. $v$) as functions in $B_R(0)$, defined by zero outside $\Omega_k$ (resp. $\Omega$) we have: (a) $v_k \rightarrow v$ in $L^1(B_R(0))$ and uniformly in compact subsets of $\Omega$ and (b) $\{H(\rho, v_k)\}$ is uniformly integrable in the sense that, for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that, for every Borel set $E \subset B_R(0)$,
\[m(E) < \epsilon \implies \int_{E \cap \Omega_k} H(\rho_k, v_k) \, dx < \delta(\epsilon).\]

Proof of Theorem 1.10: Since $|u|$ is a subsolution, it’s enough to prove the theorem in the case $u \geq 0$. Without loss of generality let us assume that $y = 0$.

We claim that
\[(4.7)\]
\[H(\rho, u) \in L^1(\Omega, \rho) \quad \text{and} \quad u \in L^1(\Omega).\]
To prove the claim, let $\eta$ be a function in $C^2(\mathbb{R})$ such that
\[0 \leq \eta \leq 1, \quad \eta(t) = 0 \quad \text{for} \quad t < 1, \quad \eta(t) = 1 \quad \text{for} \quad t > 2.\]
Further let $\phi$ be the solution of
\[-\Delta \phi = 1 \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \partial \Omega.\]
Given $\epsilon > 0$, set $\zeta_\epsilon = \eta(|x|/\epsilon)\phi$. Thus $\zeta_\epsilon \in C^2(\Omega)$ and vanishes on $\partial \Omega$ and in a neighborhood of origin. Therefore we have
\[(4.8)\]
\[\int_{\Omega} -u\Delta \zeta_\epsilon + H(\rho, u)\zeta_\epsilon = 0.\]
Let $E_\epsilon = \{x \in \Omega : \epsilon < |x| < 2\epsilon\}$. Then a straight forward calculation yields that
\[(4.9)\]
\[\int_{\Omega} u\Delta \zeta_\epsilon \leq C_1 \int_{E_\epsilon} u \epsilon^{-1} - \int_{\Omega} u(x)\eta(|x|/\epsilon)\]
where $C_1$ is a constant independent of $\epsilon$. By Proposition 3.2, $u(x) \leq C \rho(x)|x|^{-\alpha - 1}$. Therefore in $E_\epsilon$, $u(x) \leq C \epsilon^{-\alpha}$. Consequently,
\[(4.10)\]
\[\int_{E_\epsilon} u \epsilon^{-1} \leq C_2 \epsilon^{N-\alpha-1} \leq C_3\]
where $C_3$ is a constant independent of $\epsilon$. Therefore combining (4.8), (4.9) and (4.10) we obtain,

$$\int_{\Omega} (u(x) + H(\rho, u)\phi) \eta(\frac{|x|}{\epsilon}) < C_4$$

where $C_4$ is a constant independent of $\epsilon$. Now as $\phi = O(\rho)$, letting $\epsilon \to 0$ to the previous inequality and applying Fatou’s lemma we obtain (4.7). Hence the claim follows.

Now suppose that $u > 0$. By Corollary 2.8, let $U$ be the maximal solution of (1.1) vanishing on $\partial \Omega \setminus \{0\}$. Since $U$ satisfies (4.7), $U$ must have boundary trace $c_0\delta$, for some $c_0 > 0$. Clearly $2U$ is a supersolution with the boundary trace $2c\delta_0$. Therefore the largest solution dominated by $2U$, say $U'$, has the same trace $2c\delta_0$ which is impossible as $U$ is the maximal solution vanishing on $\partial \Omega \setminus \{0\}$. □

**Corollary 4.4.** Let $\Omega$ be a bounded $C^2$ domain, $H$ be defined as in (1.2) and $H$ satisfies global barrier condition. Assume that $h$ satisfies (1.3), (1.15), KO condition and $\Delta_2$ condition. Then $H$ is subcritical if and only if $\alpha > N - 1$.

**Proof.** By Theorem 1.10 $\alpha \leq N - 1$ implies $H$ is not subcritical. On the other hand Theorem 1.8 implies that if condition (1.14) is satisfied then $H$ is subcritical. Therefore it is enough to prove that when $\alpha > N - 1$, condition (1.14) always holds.

We will prove this by negation. Suppose there exists a $y \in \partial \Omega$ such that $\int_{\Omega} \rho^{-1+\alpha} h(\rho^\alpha P(x, y)) dx = \infty$. Since $\rho(x) \leq |x-y|$, by (1.3) we have

$$\infty = \int_{\Omega} \rho^{-1+\alpha} h(\rho^\alpha P(x, y)) dx \leq \int_{\Omega} \rho^{-1+\alpha} h(c_1 \rho^\alpha |x-y|^{1-N}) dx$$

$$\leq \int_{\Omega} \rho^{-1+\alpha} h(c_1 \rho^{\alpha+1-N}) dx$$

$$\leq c_2 \int_{0}^{a} r^{-1-\alpha} h(c_1 r^{\alpha+1-N}) r^{N-1} dr$$

$$\leq c_3 \int_{0}^{b} \frac{h(r)}{r^2} dr \quad (as \ \alpha > N - 1)$$

where $b = c_1 a^{\alpha+1-N}$ and $c_3$ is a positive constant. Therefore $\int_{0}^{b} \frac{h(r)}{r^2} dr = \infty$ but this is impossible since we assumed (1.15). Hence condition (1.14) holds which implies that $H$ is subcritical if and only $\alpha > N - 1$. □

5. **Lower estimate of the singular solution with point singularity**

In this section we study (1.1) when $H$ satisfies the subcriticality condition (1.14) and $h$ satisfies the KO condition.

**Proof of Theorem 1.11:** Here we assume that the set of coordinates in $\mathbb{R}^N$ is positioned so that $y$ is the origin and the hyperplane $x_1 = 0$ is tangent to $\partial \Omega$ at $y$ with the positive $x_1$-axis pointing into the domain.
Suppose that $u$ is a positive solution of (1.1) and define
\[ \hat{H}(\rho,t) := \frac{H(\rho,t)}{t} \quad \forall t > 0. \]

**Assertion 1.** Under the assumptions of the theorem, there exists a sequence \( \{\xi_n\} \subset \Omega \) converging to the origin such that
\[ u(\xi_n)\rho(\xi_n)^{N-1} \to \infty. \tag{5.1} \]

By negation, if the assertion is not valid there exists \( R \in (0,1) \) and a constant \( C \) such that
\[ u(x) < C\rho(x)^{1-N} \quad \forall x \in B_R(0) \cap \Omega. \tag{5.2} \]
As \( h \) is convex, non-decreasing and \( h(0) = 0 \), it follows that \( t \mapsto \frac{h(t)}{t} \) is non-decreasing on \((0,\infty)\). Therefore
\[ \hat{H}(\rho,u) = \frac{\rho^{-2}h(\rho^\alpha u)}{\rho^\alpha u} \leq \frac{\rho^{-2}h(C\rho^{\alpha+1-N})}{C\rho^{\alpha+1-N}} = \frac{1}{C}\rho^{N-\alpha-3}h(C\rho^{-\alpha+1}) \]
in \( Q := \Omega \cap B_R(0) \). Thus
\[ f_u(t) := \sup_{\Sigma t \cap B_R(0)} \hat{H} \leq \frac{1}{C}t^{N-\alpha-3}h(Ct^{\alpha-1}) \]
and by (1.16),
\[ \int_0^1 tf_u(t)dt < \infty. \]
Consequently, by [9, Lemma 3.1.16] \( u \in L^1(Q) \) and \( H(\rho,u) \in L^1(Q,\rho) \). Therefore \( 0 \in \mathcal{R}(u) \), which contradicts the assumption that \( 0 \in S(u) \).

Next we observe that
\[ \hat{H} = \frac{\rho^{-2}h(\rho^\alpha u)}{\rho^\alpha u} \leq C_1\rho^{-2} \quad \text{in } \Omega \tag{5.3} \]
Indeed by Lemma \([5.1] \rho^\alpha u \leq C \) where \( C \) depends only on \( N,\alpha,h \). Therefore (5.3) follows from the fact that
\[ \sup_{0<t<C} \frac{h(t)}{t} < \infty. \]

Let \( B_n = B_{\rho(\xi_n)/2}(\xi_n) \). As \( u \) satisfies the equation \(-\Delta u + \hat{H}u = 0\), (5.3) and the classical Harnack inequality imply that there exists a constant \( \tilde{c} \) such that
\[ \sup_{B_n} u \leq \tilde{c} \inf_{B_n} u. \tag{5.4} \]

We assume that \( |\xi_n| < \beta_0/4 \). Put
\[ \gamma_n = \rho(\xi_n), \quad \eta_n = \sigma(\xi_n), \quad D_n = \{x \in \Omega: \rho(x) > \gamma_n\}, \quad V_n = B_n \cap \Sigma_{\gamma_n}. \]
Then by (5.4) and (5.1)

\[ b_n := \int_{V_n} u \, dS \geq c_1(\Omega)(\sup_{B_n} u)^{\gamma_n N - 1} \to \infty. \]

Let \( f_{n,k} \) be a function on \( \Sigma_\gamma_n \) given by,

\[ f_{n,k} = (k/b_n)u \text{ in } V_n, \quad f_{n,k} = 0 \text{ in } \Sigma_\gamma_n \setminus V_n. \]

Then, by (5.5),

\[ f_{n,k} \leq kc_3(\Omega)\gamma_n^{1-N} \quad \text{and} \quad f_{n,k} \, dS_{\Sigma_\gamma_n} \rightharpoonup k\delta_0 \text{ weakly relative to } C(\bar{\Omega}). \]

where \( dS_{\Sigma_\gamma_n} \) denotes the surface element on \( \Sigma_\gamma_n \).

Let \( w_{n,k} \) denote the solution of the boundary value problem

\[ -\Delta w + H(\rho, w) = 0 \quad \text{in } D_n \]

\[ w = f_{n,k} \quad \text{on } \partial D_n. \]

Given \( k > 0 \) pick \( n(k) \) such that

\[ b_n \geq k \quad \text{in } V_n \quad \forall \, n \geq n(k). \]

Then \( f_{n,k} \leq u \) on \( \Sigma_\gamma_n \) and consequently

\[ w_{n,k} \leq u \quad \text{in } D_n \quad \forall \, n \geq n(k). \]

Further, by (5.7),

\[ P(x, \eta_n) \geq c_2(\Omega)\gamma_n^{1-N} \geq c_3(\Omega)f_{n,k}(x)/k \quad \forall \, x \in V_n. \]

Therefore, by the maximum principle,

\[ (k/c_3)P(x, \eta_n) \geq w_{n,k}(x) \quad \forall \, x \in D_n. \]

By the argument employed in the proof of Theorem 1.8 the sequences \( \{P(\cdot, \eta_n)\} \) and \( \{H(\rho, (k/c_3)P(\cdot, \eta_n))\} \) are uniformly integrable in \( L^1(\Omega) \) and \( L^1(\Omega; \rho) \) respectively. In view of (5.10) this implies that the sequences \( \{w_{n,k}\}_{n=1}^\infty \) and \( \{H(\rho, w_{n,k})\}_{n=1}^\infty \) are uniformly integrable in \( L^1(\Omega) \) and \( L^1(\Omega; \rho) \) respectively. (Here we refer to the extension of \( w_{n,k} \) by zero outside \( D_n \).) By a standard argument, a subsequence of \( \{w_{n,k}\}_{n=1}^\infty \) converges locally uniformly in \( \Omega \) to a function \( w \). Since \( w_{n,k} \) satisfies (5.8) and \( \{f_{n,k}\} \) converges weakly as stated in (5.7), it follows that \( w \) is the (unique) solution of the problem

\[ -\Delta w + H(\rho, w) = 0 \quad \text{in } \Omega \]

\[ w = k\delta_0 \quad \text{on } \partial \Omega, \]

i.e., \( w = u_{0,k} \). Finally, (5.9) implies that \( u \geq u_{0,k} \). As \( k \) was arbitrary we obtain (1.20). \( \square \)
6. The very singular solution

In this section we study (1.1) when \( H \) satisfies the subcriticality condition (1.14), \( h \) satisfies (1.3) and the KO condition and (1.1) possesses a global barrier. These conditions will be assumed without further mention.

Let \( U_y \) denote the space of positive solutions of (1.1) such that
\[
(6.1) \quad u \in C(\Omega \setminus \{y\}), \quad u = 0 \text{ on } \partial \Omega \setminus \{y\}.
\]
Define
\[
(6.2) \quad U_{\infty,y} = \sup \{ u \in U_y \}.
\]
Then by Corollary 2.8, \( U_{\infty,y} \) is a solution of (1.1) and it satisfies (6.1).

6.1. In half-space.

**Theorem 6.1.** Let \( \Omega = \mathbb{R}^N_+ = \{ x_1 > 0 \} \). In addition to the basic conditions, assume that \( h \) satisfies (1.15), (1.14) and is strictly convex near zero. Then, for \( y \in \partial \mathbb{R}^N_+ \), there exists a unique very singular solution \( U_y \) at \( y \),
\[
(6.3) \quad U_y(x) = r^{-\alpha} w(\sigma) \quad \text{where} \quad |x - y| = r, \quad \sigma = \frac{x - y}{r}, \quad \forall \ x \in \mathbb{R}^N_+,
\]
and \( w \) is the (unique) solution to (1.22).

**Proof.** Without loss of generality we assume that \( y = 0 \). Let \( U_{\infty,0} \) be defined as in (6.2) and let \( T_a^\alpha \), \( a > 0 \), be the similarity transformation defined in (3.1). Our basic assumptions imply that \( U_{\infty,0} \) is a positive solution vanishing on \( \partial \Omega \setminus \{0\} \). Therefore \( T_a^\alpha U_{\infty,0} \) is again a solution of (1.1) in \( \mathbb{R}^N_+ \) which vanishes on \( \partial \Omega \setminus \{0\} \). Since \( u \mapsto T_a^\alpha u \) is an order preserving \( 1-1 \) mapping from \( U_0 \) onto itself it follows that \( T_a^\alpha U_{\infty,0} = U_{\infty,0} \), i.e., \( U_{\infty,0} \) is self-similar. Therefore
\[
(6.4) \quad U_{\infty,0}(x) = a^{-\alpha} U_{\infty,0}(\frac{x}{a}) \quad \forall \ x \in \mathbb{R}^N_+, \quad \forall a > 0.
\]
This can be rewritten in the form
\[
(6.5) \quad \begin{cases} U_{\infty,0}(x) = r^{-\alpha} w(\sigma) \quad \text{where} \quad |x| = r, \quad \sigma = \frac{x}{r}, \quad \forall \ x \in \mathbb{R}^N_+ \\ w(\sigma) = U_{\infty,0}(\sigma) \quad \forall \ \sigma \in S_{N-1}^+ \end{cases}
\]
A straightforward computation yields that \( w \) is a solution of (1.22). Our next aim is to show that (1.22) has a unique solution. If \( \tilde{w} \) is another solution then we can easily check that \( v = r^{-\alpha} \tilde{w} \) will be a solution of (1.1) in \( \mathbb{R}^N_+ \). Therefore \( v \leq U_{\infty,0} \) and \( \tilde{w} \leq w \). By a simple computation, (1.22) implies
\[
0 = \int_{S_{N-1}^+} (\sigma \cdot e_1)^{-(2+\alpha)} \left[ \tilde{w} h((\sigma \cdot e_1)^\alpha w) - wh((\sigma \cdot e_1)^\alpha \tilde{w}) \right] \\
= \int_{S_{N-1}^+} (\sigma \cdot e_1)^{-2} \left[ h((\sigma \cdot e_1)^\alpha w) - h((\sigma \cdot e_1)^\alpha \tilde{w}) \right] \tilde{w}.
\]
As $h$ satisfies (1.3), \( \frac{h(t)}{t} \) is nondecreasing. Therefore (6.6) implies
\[
(6.7) \quad \frac{h((\sigma \cdot e_1)^\alpha w)}{(\sigma \cdot e_1)^\alpha w} = \frac{h((\sigma \cdot e_1)^\alpha \tilde{w})}{(\sigma \cdot e_1)^\alpha \tilde{w}} \quad \forall \sigma \in S_{N-1}^N.
\]
As $h$ is strictly convex and $h(0) = 0$, \( \frac{h(t)}{t} \) is strictly increasing. Therefore (6.7) implies that $w = \tilde{w}$.

Recall that condition (1.14) implies that $\alpha > N - 1$ (see (1.18)). Therefore, by (6.5),
\[
\|\rho H(\rho, U_{\infty,0})\|_{L^1(B_1(0) \cap \{x_1 > 0\})} = \int_{S_{N-1}^N} \int_0^1 \rho^{-(1+\alpha)} h(\frac{\rho}{r} w(\sigma)) r^{N-1} dr d\sigma 
\geq \int_{S_{N-1}^N \cap [\rho > r/2]} \int_0^1 r^{-(1+\alpha)} h((2)^{-\alpha} w(\sigma)) r^{N-1} dr d\sigma = \infty.
\]
Thus $U_{\infty,0}$ is a very singular solution at 0.

By Theorem 1.11, $u_{\infty,0}$ (see (1.19)) is the smallest very singular solution at 0. Clearly $u_{\infty,0}$ is self-similar; therefore (6.5) holds for $u_{\infty,0}$ as well. The uniqueness of the solution of (1.22) implies that $u_{\infty,0} = U_{\infty,0}$ so that the very singular solution is unique. \(\square\)

6.2. In general domain.

**Definition 6.2.** Let $\Omega$ be a $C^2$ domain, $y \in \partial \Omega$ and $u$ a function in $\Omega$. We say that $u(x) \to \ell$ as $x \to y$ non-tangentially if, for every fixed $c > 0$,
\[
\lim_{x \to y \atop -n y \cdot (x-y) > c} u(x) = \ell.
\]

**Proposition 6.3.** Let $\Omega$ be a $C^2$ domain, not necessarily bounded and assume that $h$ satisfies all the assumptions of Theorem 6.1. For $y \in \partial \Omega$ let $u_{y,\infty}$ be defined as in (1.19).

Then both $U_{y,\infty}$ and $u_{y,\infty}$ satisfy (1.21), uniformly in compact subsets of $S_{N-1}^N$. Consequently
\[
(6.8) \quad \frac{u_{y,\infty}}{U_{y,\infty}} \to 1 \quad \text{as} \quad x \to y \quad \text{non-tangentially in} \ \Omega.
\]

**Proof.** Without loss of generality we assume that $y = 0$ and that the set of coordinates is positioned so that the hyperplane $x_1 = 0$ is tangent to $\partial \Omega$ at 0 with the positive $x_1$-axis pointing into the domain.

Let $u$ stand for either $u_{0,\infty}$ or $U_{0,\infty}$. For every $\alpha > 0$, let $u_{k}^\alpha$ denote the solution of (1.1) in $\Omega^\alpha := \frac{1}{\alpha} \Omega$ with boundary trace $k\delta_0$, extended to $\mathbb{R}^N \setminus \{0\}$, by setting it zero outside $\Omega^\alpha \cup \{0\}$. Further denote
\[
u_{\infty}^\alpha = \lim_{k \to \infty} u_k^\alpha
\]
and observe that
\[
u_{\infty}^\alpha = T_\alpha u_{0,\infty},
\]
$T^a_n$ being defined as in \((3.1)\). Let \(\{a_n\}\) be a sequence of positive numbers converging to zero. In view of the local Keller - Osserman condition and the global barrier condition, we can extract a subsequence (still denoted by \(\{a_n\}\)) such that
\[
(6.10) \quad u_k^{a_n} \to v_k, \quad u_\infty^{a_n} \to V
\]
uniformly in compact subsets of \(\mathbb{R}^N \setminus \{0\}\) and \(v_k\) and \(V\) are solutions of \((1.1)\) in \(\mathbb{R}^N_+\) that vanish on \(\partial \mathbb{R}^N_+\setminus \{0\}\).

**Assertion 1.** \(v_k = u_k^0\), namely, the solution of \((1.1)\) in \(\mathbb{R}^N_+\) with boundary trace \(k\delta_0\).

Let \(s > 0\) and put \(\Omega^{a,s} := \Omega^a \cap B_s(0)\). Denote by \(v_k^{a,s}\) the solution of the problem,
\[
(6.11) \quad -\Delta v + H(\rho_a, v) = 0 \quad \text{in} \quad \Omega^{a,s},
\]
\[
v = k\delta_0 \quad \text{on} \quad \partial \Omega^{a,s},
\]
where \(\rho_a(x) = \text{dist}(x, \partial \Omega^a)\). Clearly
\[
(6.12) \quad v_k^a \geq v_k^{a,s} \quad \text{in} \quad \Omega^{a,s}.
\]
Keeping \(s\) fixed, \(\Omega^{a,s} \to \mathbb{R}^N_+ \cap B_s(0)\) as \(a \to 0\). By Theorem 4.3,
\[
\lim_{a \to 0} v_k^{a,s} = v_k^{0,s}
\]
where \(v_k^{0,s}\) is the solution of \((1.1)\) in \(\mathbb{R}^N_+ \cap B_s(0)\) with boundary data \(k\delta_0\).

Since \(v_k^{a,s} \leq u_k^{a_n}\) it follows that \(v_k^{0,s} \leq v_k^{a_n}\) for every \(s > 0\). Clearly \(v_k^{0,s} \to u_k^0\) as \(s \to \infty\). Therefore
\[
(6.13) \quad u_0^k \leq v_k.
\]
On the other hand, it is easily verified that \(v_k \leq u_k^0\). This proves Assertion 1.

Now \(V\) is a solution of \((1.1)\) in \(\mathbb{R}^N_+\) which vanishes on \(\partial \mathbb{R}^N_+ \setminus \{0\}\). Furthermore, \(V \geq v_k = u_k^0\) for every \(k > 0\). Therefore \(V\) is a very singular solution of \((1.1)\) in \(\mathbb{R}^N_+\). By Theorem 6.1 (uniqueness) \(V = U_0\). Since the limit is independent of the sequence we conclude that
\[
(6.14) \quad \lim_{a \to 0} u_\infty^a(x) = U_0(x) = |x|^{-\frac{2}{q+1}} \omega(x/|x|)
\]
uniformly in compact subsets of \(\mathbb{R}^N_+\).

Let \(S_1\) be a compact subset of the open half sphere \(S^{N-1}_+\) and let \(\tilde{r}\) be a positive number such that
\[
\{x : |x| \leq \tilde{r}, \ x/|x| \in S_1\} \subset \Omega.
\]
Then \(S_1 \subset \frac{1}{a} \Omega\) for \(0 < a \leq \tilde{r}\). As the convergence is uniform on \(S_1\), \((6.14)\) implies,
\[
\lim_{a \to 0} a^{\frac{2}{q+1}} u(ax) = \omega(x) \quad \text{uniformly} \ \forall x \in S_1
\]
which is equivalent to \((1.21)\). \(\square\)
The next proposition is a version of the boundary Harnack principle due to Ancona [1]. Given \( s, s' > 0 \) denote by \( T(s, \gamma) \) the cylinder,

\[
T(s, \gamma) = \{ \xi = (\xi', \xi_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : |\xi'| < s, -\gamma s < \xi_N < \gamma s \}.
\]

**Proposition 6.4.** Let \( s, \gamma \) be positive numbers and let \( f \) be a Lipschitz function on \( \mathbb{R}^{N-1} \) with Lipschitz constant \( c_f \) such that

\[
f(0) = 0, \quad c_f \leq \gamma / 10.
\]

Denote

\[
D = \{ (\xi', \xi_N) \in T(s, \gamma) : f(\xi') < \xi_N \}
\]

\[
\Gamma = \{ (\xi', \xi_N) \in T(s, \gamma) : f(\xi') = \xi_N \}
\]

Let \( V \in L^\infty_{\text{loc}}(D) \) be a non-negative function and \( c_V \) a positive constant such that

\[
V(x) \leq c_V \rho_\Gamma(x)^{-2}, \quad \rho_\Gamma(x) = \text{dist}(x, \Gamma) \quad \forall x \in D
\]

and denote \( L^V := -\Delta + V \). Let \( u \) be a positive \( L^V \)-harmonic function (i.e. \( L^V u = 0 \)) in \( D \) such that \( u \) is continuous in \( D \cup \Gamma \) and \( u = 0 \) on \( \Gamma \). Denote by \( G^V \) the Green kernel for \( L^V \) in \( D \). Then there exists a constant \( \gamma \) depending only on \( N, c_V \) and \( \gamma \) such that

\[
c^{-1}s^{N-2}G^V(x, A') \leq \frac{u(x)}{u(A)} \leq cs^{N-2}G^V(x, A') \quad \forall x \in D \cap T(s/2, \gamma),
\]

where \( A = (0, \ldots, 0, \gamma s/2) \) and \( A' = \frac{3}{4}A \).

The following is a consequence of [2, Theorem 9.1]. We use the notation of the previous proposition.

**Proposition 6.5.** Let \( D \) be as in Proposition 6.4 and let \( V \) be a non-negative function in \( D \) such that \( V \in L^\infty(E) \) for every set \( E \subset D \) that is bounded away from \( \Gamma \). Further assume that there exist \( \epsilon > 0 \) and \( c(\epsilon) > 0 \) such that

\[
V(x) \leq c \rho_\Gamma(x)^{-2+\epsilon},
\]

Let \( G^V \) (resp. \( G^0 \)) denote the Green kernel for \( L^V \) (resp. \( -\Delta \)) in \( D \). Then there exists a constant \( c_G \) depending only on \( N, \gamma, \epsilon \) and \( c(\epsilon) \) such that

\[
c_G^{-1}G^0 \leq G^V \leq c_GG^0 \quad \text{in } D \cap T(s/2, \gamma).
\]

As \( G^0(x, A') \sim \rho(x)|x - A'|^{-(N-1)} \sim s^{-(N-1)}\rho(x) \) in \( D \cap T(s/2, \gamma) \), combining the previous two propositions we obtain,

**Corollary 6.6.** Let \( V_1, V_2 \) be two non-negative functions in \( D \) satisfying the assumptions of Proposition 6.5. Let \( u_i \) be a positive \( L^{V_i} \) harmonic function
in $D$ such that $u_i$ is continuous in $D \cup \Gamma$ and $u = 0$ on $\Gamma$, $i = 1, 2$. Then there exists a constant $c, C$ depending only on $N, \gamma, \epsilon$ and $c(\epsilon)$ such that

\begin{equation}
(i) \quad C^{-1} \frac{\rho(x)}{s} \leq \frac{u_i(x)}{u_i(A)} \leq C \frac{\rho(x)}{s} \quad \forall x \in D \cap T(s/2, \gamma/2), i = 1, 2,
\end{equation}

\begin{equation}
(ii) \quad c^{-1} \frac{u_1(A)}{u_2(A)} \leq \frac{u_1(x)}{u_2(x)} \leq c \frac{u_1(A)}{u_2(A)} \quad \forall x \in D \cap T(s/2, \gamma).
\end{equation}

where $A = (0, \cdots, 0, \gamma s/2)$.

**Proof of Theorem 1.13.** We assume that the set of coordinates is positioned so that $y = 0$ and the hyperplane $x_N = 0$ is tangent to $\partial \Omega$ at $0$ with the positive $x_N$–axis pointing into the domain.

Let $U_{\infty,0}$ and $u_{0,\infty}$ be as in Proposition 6.3. By Theorem 1.11, $u_{\infty,0}$ is the minimal very singular solution at $0$. Therefore in order to establish uniqueness of the very singular solution it is enough to show that $U_{\infty,0} = u_{\infty,0}$.

By Proposition 6.3, for every $\beta > 1$ there exists a constant $c_\beta > 0$ such that,

\begin{equation}
(6.22) \quad c_\beta^{-1} |x|^{-1-\alpha} \rho(x) \leq u_{\infty,0}(x) \leq U_{\infty,0}(x) \leq c_\beta |x|^{-1-\alpha} \rho(x)
\end{equation}

in the truncated cone

$$E_\beta := \{ x \in \Omega : |x| \leq \beta \rho(x), |x| < r_\Omega \},$$

where $(r_\Omega, M_\Omega)$ is the $C^2$ characteristic of $\Omega$. Hence

\begin{equation}
(6.23) \quad u_{\infty,0} \leq U_{\infty,0} \leq c_1(\beta) u_{\infty,0} \quad \text{in } E_\beta.
\end{equation}

Put

$$\tilde{E}_\beta := \{ x \in \Omega : \beta \rho(x) \leq |x| < \tilde{s}/4 \}.$$

Next we show that, for an appropriate choice of $\beta$ and $\tilde{s}$, inequality (6.23) (with a different constant) holds in $\tilde{E}_\beta$ as well. In this part of the proof we make use of Corollary 6.6.

Let $f \in C^2(\mathbb{R}^{N-1})$ be a function such that $f(0) = 0$, $\nabla f(0) = 0$ and, for some $s_0 \in (0, r_\Omega)$ and $\gamma_0 \in (0, 1)$,

$$\Omega \cap T(s_0, \gamma_0) = \{ x = (x', x_N) : |x'| < s_0, f(x') < x_N < \gamma_0 s_0 \} := D_0.$$

For $s \in (0, s_0]$ denote,

$$\gamma(s) = 10 \sup_{|x'| < s} |\nabla f(x')|.$$

Note that $\gamma(s) \downarrow 0$ as $s \downarrow 0$ and choose $\tilde{s} \in (0, s_0/2)$ such that

$$\tilde{\gamma} := \gamma(\tilde{s}) \leq \gamma_0/4.$$
Next we show that, for $\beta$ sufficiently large the following condition holds: for every $\zeta \in \Gamma_0 := \partial \Omega \cap B_{\tilde{s}/2}(0)$,

\begin{align}
(6.24) \\
(i) & \quad \eta_\zeta := \zeta - \frac{1}{4}\tilde{\gamma}|\zeta|\mathbf{n}_\zeta \in E_\beta, \\
(ii) & \quad \tilde{E}_\beta \subset \cup_{\zeta \in \Gamma_0} [\zeta, \eta_\zeta],
\end{align}

where $[x, y]$ denotes the linear segment connecting the points $x, y$.

As $\rho(\eta_\zeta) = \frac{1}{4}\tilde{\gamma}|\zeta|$, (6.24) (i) is equivalent to

\begin{align}
(6.25) \\
|\eta_\zeta| \leq \beta \tilde{\gamma}|\zeta|/4 \quad \forall \zeta \in \Gamma_0.
\end{align}

Since $|\eta_\zeta| \leq (1 + \tilde{\gamma}/4)|\zeta|$, (6.25) holds for $\beta \geq 4(1 + \tilde{\gamma}/4)/\tilde{\gamma}$.

Therefore it is sufficient to show that, for $\beta$ sufficiently large,

\begin{align}
(6.26) \\
\beta \rho(x) \leq |x| < \tilde{s}/4 \implies \rho(x) \leq \frac{1}{4}\tilde{\gamma}|\sigma(x)|.
\end{align}

If $\{P_k\}$ is a sequence of points in $\tilde{E}_\beta$ such that $|P_k|/|\sigma(P_k)| \to \infty$ then $P_k \to 0$. But, if $\beta > 1$,

$$
\lim_{k \to \infty} \frac{\rho(P_k)^2 + |\sigma(P_k)|^2}{|P_k|^2} = 1
$$

and consequently, $\rho(P_k)/|P_k| \to 1$, which is impossible when $\beta > 1$.

In continuation we assume that $\beta$ has been chosen so that (6.24) holds. For every $P \in \Gamma_1 := \partial \Omega \cap B_{\tilde{s}/4}(0) \setminus \{0\}$, denote by $\xi^P$ the Euclidean coordinates centered at $P$, such that the hyperplane $\xi^P_N = 0$ is tangent to $\partial \Omega$ at $P$ and the positive $\xi^P_N$ axis is in the direction of $-\mathbf{n}_P$. Let $T^\Omega_P(s, \gamma)$ be defined as in (6.15) with $\xi = \xi^P$. Note that, for $a > 0$, $P \in \partial \Omega$,

$$
T^\Omega_{P/a}(s/a, \gamma) = \frac{1}{a}T^\Omega_P(s, \gamma), \quad \mathbf{n}_{P/a} = \mathbf{n}_P,
$$

where $\mathbf{n}_{P/a}$ denotes the unit normal vector to $\frac{1}{a}\partial \Omega$ at the point $P/a$. Put

$$
D_P = T^\Omega_P([P/2, \tilde{\gamma}]) \cap \Omega.
$$

Let $u$ be a positive solution of (1.1) in $\Omega$ and let $a > 0$. Denote $v^a = T^\alpha_a u$ where $T^\alpha_a$ is the similarity transformation defined in (3.1). Then $v^a$ is a solution of (1.1) in $\frac{1}{a}\Omega$ vanishing continuously on $\frac{1}{a}\partial \Omega \setminus \{0\}$. In particular, if $a = |P|/2$ then $w_P := v^a$ satisfies (1.1) (with $\rho$ replaced by $\rho_P(x) = \text{dist}(x, \frac{2}{P}\partial \Omega)$) in

$$
D'_P = \frac{2}{|P|}D_P = T^\prime_P(1, \tilde{\gamma}) \cap \Omega
$$
where
\[ T_P(1, \tilde{\gamma}) := \frac{2}{|P|} T_P^\Omega(|P|/2, \tilde{\gamma}) = T_{1/P}^\Omega(1, \tilde{\gamma}) \]
and \( w_P = 0 \) on \( \Gamma_P = D'_P \cap \frac{2}{|P|}\partial\Omega \).

Put \( V_P = H(\rho_P, w_P)/w_P \). We claim that \( V_P \) satisfies
\[ V_P(x) \leq c\rho_P(x)^{-2+\epsilon} \quad \text{in } D'_P \]
with \( \epsilon \) as in \((1.15)\). Indeed,
\[ V_P = \rho_P^{-\gamma} \frac{h(\rho_P^2 w_P)}{\rho_P^2 w_P} \leq c\rho_P^{-2}(\rho_P w_P)^\epsilon. \]

In view of the uniform barrier condition, \( w_P \) is bounded in \( D'_P \) by a constant depending only on \( N, \alpha, h \) and the \( C^2 \) characteristic of \( \Omega \). Therefore the last inequality implies \((6.28)\).

Denote
\[ w_{1,P} = T_\alpha^\alpha u_{0,\infty}, \quad a = \frac{|P|}{2}. \]

Note that
\[ w_{1,P}(2x/|P|) = \left(\frac{|P|}{2}\right)^\alpha u_{0,\infty}(x) \quad \forall x \in \Omega. \]

By \((6.22)\) and \((6.24)\)(i), \( \eta_P = P - \frac{1}{4}\tilde{\gamma}|P|n_P \in E_\beta \) and consequently
\[ w_{1,P}(2\eta_P/|P|) = \left(\frac{|P|}{2}\right)^\alpha u_{0,\infty}(\eta_P) \geq c_1(\beta)(\frac{|P|}{2})^\alpha |\eta_P|^{-\alpha-1} \rho(\eta_P). \]

The point \( \eta'_P = 2\eta_P/|P| \) is in the same position relative to \( T_P(1, \tilde{\gamma}) \) as the point \( A \) (defined in Proposition 6.4) relative to \( T(s, \gamma) \). Therefore applying \((6.21)\)(i) we have,
\[ w_{1,P}(z) \geq c_P c_1(\beta) \left(\frac{|P|}{2}\right)^\alpha |\eta_P|^{-\alpha-1} \rho_P(\eta_P) \rho_P(z) \quad \forall z \in D'_P \cap T_{\tilde{\gamma}/2}^\Omega \left(1/2, \tilde{\gamma}/2\right). \]

As \( \rho(\eta_P) = \tilde{\gamma}|P| \) and \( |\eta_P| = \frac{|P|}{2} |\eta_P'| \sim \frac{|P|}{2} |z| \) in \( D'_P \cap T_{\tilde{\gamma}/2}^\Omega \left(1/2, \tilde{\gamma}/2\right) \), we obtain from \((6.29)\)
\[ w_{1,P}(z) \geq c_P c_2(\beta) |P|^\alpha \left(\frac{|P|}{2} |z|\right)^{-\alpha-1} |P| \tilde{\gamma} \rho_P(z) \quad \forall z \in D'_P \cap T_{\tilde{\gamma}/2}^\Omega \left(1/2, \tilde{\gamma}/2\right), \]
i.e.
\[ w_{1,P}(z) \geq c_3(\beta) |z|^{-\alpha-1} \rho(z) \quad \forall z \in D'_P \cap T_{\tilde{\gamma}/2}^\Omega \left(1/2, \tilde{\gamma}/2\right). \]

Since \( \Omega \) is uniformly of class \( C^2 \), we may choose \( c_P \) to be independent of \( P \in \Gamma_1 \). Hence there exists a constant \( c_4(\beta) \) such that,
\[ u_{0,\infty}(x) \geq c_4(\beta) |x|^{-\alpha-1} \rho(x) \quad \forall x \in D_P \cap T_P(|P|/4, \tilde{\gamma}/2). \]
This inequality and (6.24)(ii) imply that

\[ u_{0,\infty}(x) \geq c_5(\beta) \rho(x) |x|^{-\alpha - 1} \quad \forall x \in \frac{1}{2}\bar{E}_\beta. \]

Combining this inequality with (6.22) we conclude that there exists \( r, c \) positive such that

\[ u_{0,\infty}(x) \geq c \rho(x) |x|^{-\alpha - 1} \quad \forall x \in B_r(0) \cap \Omega. \]

Finally combining this inequality with Proposition 3.2 we obtain

\[ c \rho(x) |x|^{-\alpha - 1} \leq u_{0,\infty}(x) \leq C \rho(x) |x|^{-\alpha - 1} \]

for every \( x \in B_r(0) \cap \Omega \).

**Claim:** (6.30) holds for every \( x \in \Omega \).

Indeed if \( x \in \bar{\Omega} \setminus B_r(0) \), then \( |x| > r \). Now choose \( z \in \partial \Omega \setminus B_r(0) \) and consider the cylinder \( T(\frac{x}{2}, \tilde{\gamma}) \) centered at \( z \). Then the point \( \eta_z = z - \frac{r \tilde{\gamma}}{2} n_z \) is in the same position relative to \( T(\frac{x}{2}, \tilde{\gamma}) \) as the point \( A \) (defined in Proposition 6.4) relative to \( T(\frac{x}{2}, \tilde{\gamma}) \). Therefore by Corollary 6.6 (in particular (6.21)(i) applied in \( D \cap T(\frac{x}{2}, \tilde{\gamma}) \)) we obtain,

\[ u_{0,\infty}(x) \geq c z \rho(x) \frac{r}{r} u_{0,\infty}(\eta_z) \quad \forall x \in D \cap T(\frac{r}{2}, \tilde{\gamma} \frac{r}{2}). \]

As \( \Omega \) is a bounded \( C^2 \) domain, we may choose \( c_z \) independent of \( z \in \partial \Omega \setminus B_r(0) \). Also note that for every \( z \in \partial \Omega \setminus B_r(0) \), corresponding \( \eta_z \) belongs to \( \Gamma_{\frac{r}{2}} := \{ x \in \Omega : \rho(x) = \frac{r}{2} \} \). Therefore by maximum principle,

\[ \inf_{z \in \partial \Omega \setminus B_r(0)} u_{0,\infty}(\eta_z) \geq c_1(r) > 0. \]

As \( \Omega \) is bounded, we can cover \( \{ x \in \Omega \setminus B_r(0) : \rho(x) \leq \frac{r}{2} \} \) by a finite number of cylinders. Hence from (6.31) we conclude that there exists a constant \( c = c(r) > 0 \) such that

\[ u_{0,\infty}(x) \geq c(r) \rho(x) \quad \forall x \in \Omega \setminus B_r(0), \quad \rho(x) \leq \frac{r}{2}. \]

By above inequality and Proposition 3.2

\[ c(r) \rho(x) \leq u_{0,\infty}(x) \leq U_{0,\infty}(x) \leq C(r) \rho(x) \quad \forall x \in \Omega \setminus B_r(0). \]

Therefore (6.30) with a constant \( C \) depending on the parameters mentioned in assertion (ii) of Theorem 1.13 holds for every \( x \in \Omega \). Hence there exists a positive constant \( c \) such that

\[ u_{0,\infty} \leq U_{0,\infty} \leq cu_{0,\infty} \quad \text{in } \Omega. \]

Therefore, as \( h \) is convex, a standard argument introduced in [8], implies that

\[ U_{0,\infty}(x) = u_{0,\infty}(x). \]
7. The generalized boundary value problem

In this section we study the generalized boundary value problem:

\[-\Delta u + H(\rho, u) = 0, \quad \text{in } \Omega,\]
\[u = \nu \quad \text{on } \partial\Omega,\]
\[u > 0 \quad \text{in } \Omega,\]

where $\nu \in \mathcal{B}_{\text{reg}}(\partial\Omega)$, $H(\rho, t)$ is given by (1.2) and satisfies (1.14) and $\Omega$ is a bounded $C^2$ domain. Our goal is to prove the existence and uniqueness of the solution for this problem as stated in Theorem 1.14.

**Proof of Theorem 1.14**  Existence follows from [9, Theorem 3.3.1] (also see [11, Theorem 4.16]) because, in the subcritical case, conditions (i) and (ii) are satisfied by any measure in $\mathcal{B}_{\text{reg}}(\partial\Omega)$.

Define $F := S_{\nu}$. By [9, Theorem 3.3.1] it is enough to prove the uniqueness result in the case $\nu = \infty X_F$.

Next we show that, for every compact set $F \subset \partial\Omega$, the maximal solution $U_F$ is the unique solution of problem (7.1) with trace $\nu = \infty X_F$. We begin by constructing the minimal solution with this trace.

Let $\{x_n\} \subset F$ be a sequence dense in $F$ and put

$$\nu_k = k \sum_{i=1}^k \delta_{x_i}.$$ 

Let $u_k$ be the unique solution of (1.1) with boundary trace $\nu_k$. Thus the sequence $\{u_k\}$ is increasing and since $h$ satisfies KO condition, using Remark 1.1 we obtain, $\{u_k\}$ is uniformly bounded in compact subset of $\Omega$. Therefore

$V_F := \lim_{k \to \infty} u_k$

is a solution of (1.1) which is $\infty$ on the set $\{x_n\}_{i=1}^n$ and vanishes on $\partial\Omega \setminus F$. Now if $U_{x_n}$ is the unique very singular solution at $x_n$, then

$$V_F \geq U_{x_{n}}, \quad n = 1, 2, \ldots.$$ 

Therefore $x_n \subset S(V_F) \subset F$, now as $S(V_F)$ is closed it follows that $S(V_F) = F$. Therefore $V_F$ is a solution to the problem (7.1) with boundary trace $\nu = \infty X_F$.

Now if $u$ is any positive solution of (1.1) such that $S(u) = F$, then by Theorem 1.11 we obtain

$$u \geq u_k \quad \forall \ x \in F.$$ 

Hence $u \geq u_k$ and that implies $u \geq V_F$. Therefore $V_F$ is the minimal solution with boundary trace $\infty X_F$. It remains to prove that if $U_F$ is the maximal solution vanishing on $\partial\Omega \setminus F$ then

$$U_F = V_F.$$ 

Let $x \in \Omega$ and $\text{dist}(x, F) \leq \frac{\beta_0}{2}$, where $\beta_0$ is defined as in the beginning of Section 2. Now choose $y \in F$ such that $|x-y| = \text{dist}(x, F)$. Now as $U_y$ is the
unique very singular solution at \( y \), we have \( U_y \leq V_F \). Therefore by (1.23) we obtain,
\[
C^{-1}|x-y|^{-\alpha-1} \rho(x) \leq U_y \leq V_F.
\]
By Proposition 3.2 we also obtain,
\[
V_F(x) \leq U_F(x) \leq C_1 \text{dist}(x,F)^{-\alpha-1} \rho(x).
\]
Hence there exists a positive constant \( C_2 \) (depending only on \( N, \alpha, h, C^2 \) characteristic of \( \Omega \)) such that
\[
(7.2) \quad C^{-2} \text{dist}(x,F)^{-\alpha-1} \rho(x) \leq V_F(x) \leq U_F(x) \leq C_2 \text{dist}(x,F)^{-\alpha-1} \rho(x)
\]
for every \( x \in \Omega \) such that \( \text{dist}(x,F) < \frac{\beta_0}{2} \). By the same argument that was used to prove (6.32), we deduce that (7.2), with possibly a larger constant, is valid in the entire domain. Thus there exists a constant \( c > 1 \) such that
\[
V_F \leq U_F \leq cV_F \quad \text{in } \Omega.
\]
As before this implies \( U_F = V_F \). \( \square \)

8. Appendix

Proof of Proposition 2.5: Define
\[
\tilde{g}(x,t) := \max\{g(x,t), C\rho^\alpha t^\Gamma\}.
\]
Thus
\[
(8.1) \quad \tilde{g}(x,t) \geq C\rho^\alpha t^\Gamma \quad \forall \ t.
\]
From the result of Du and Guo [6], it is known that (1.4) possesses a global barrier \( W \), which satisfies (i) and (ii) in Proposition 2.5 when \( g(x,t) = \rho^\alpha t^\beta \), \( \alpha > 0 \) and \( q > 1 \). Therefore, thanks to (8.1), it is easy to check that
\[
-\Delta u + \tilde{g} \circ u = 0 \quad \text{in } \Omega,
\]
and \( \tilde{V}_M \) be the solution of
\[
(8.2) \quad -\Delta u + \tilde{g} \circ u = 0 \quad \text{in } \Omega \cap B_r(z),
\]
with
\[
(8.3) \quad \tilde{V}_M = \begin{cases} 0 & \text{on } \partial\Omega \cap B_r(z) \\ M & \text{on } \Omega \cap \partial B_r(z). \end{cases}
\]
In addition, assume that \( V_M \) be the solution of
\[
(8.4) \quad -\Delta u + g \circ u = 0 \quad \text{in } \Omega \cap B_r(z),
\]
where \( V_M \) satisfies the boundary data (8.3). Now note that
\[
g(x,t) \leq \tilde{g}(x,t) \leq g(x,t) + g_1(x,t),
\]
where \( g_1(x,t) = C\rho^\alpha \min\{t,T\}^\Gamma \). Hence \( W_M \leq \hat{V}_M \leq V_M \), where \( W_M \) is the solution of
\[
(8.5) \quad -\Delta u + (g + g_1) \circ u = 0 \quad \text{in } \cap B_r(z)
\]
and $W_M$ satisfies (8.3). Let $W^M_0$ be the solution of
\begin{equation}
-\Delta W^M_0 = g_1 \circ W_M \quad \text{in } \Omega \cap B_r(z); \quad W^M_0 = 0 \quad \text{on } \partial(\Omega \cap B_r(z)).
\end{equation}

Note that, $g_1$ is bounded in $\Omega \cap B_r(z)$, which implies $g_1 \circ W_M$ is uniformly
bounded. Hence there exists $C_1 > 0$, independent of $M$, such that $W^M_0 < C_1$. Combining (8.5) and (8.6) we obtain
\begin{equation}
-\Delta(W_M + W^M_0) + g \circ (W_M + W^M_0) \geq 0 \quad \text{in } \Omega \cap B_r(z),
\end{equation}
and
\[ \tilde{W}_M + W^M_0 = \begin{cases} 0 & \text{on } \partial \Omega \cap B_r(z) \\ M & \text{on } \Omega \cap \partial B_r(z). \end{cases} \]

Hence $V_M \leq W_M + W^M_0 \leq W_M + C_1$. As $g$ satisfies local KO condition,
$\tilde{g}$ satisfies the same. Therefore passing to the limit we obtain, $\lim_{M \to \infty} \tilde{V}_M$
exists and the limit is a barrier for the equation $-\Delta u + \tilde{g} \circ u = 0$ \text{ at } z$. Now as $W_M \leq \tilde{V}_M$, we obtain, $\lim_{M \to \infty} W_M = V$. Finally, as $V_M \leq W_M + C_1$, we can conclude that $\lim_{M \to \infty} V_M = V$ is a barrier of (1.4) at $z$.
\[ \square \]

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