Fractional Top Trading Cycle

on the Full Preference Domain*

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Abstract

Efficiency and fairness are two desiderata in market design. Fairness requires randomization in many environments. Observing the inadequacy of Top Trading Cycle (TTC) to incorporate randomization, Yu and Zhang (2020) propose the class of Fractional TTC mechanisms to solve random allocation problems efficiently and fairly. The assumption of strict preferences in the paper restricts the application scope. This paper extends Fractional TTC to the full preference domain in which agents can be indifferent between objects. Efficiency and fairness of Fractional TTC are preserved. As a corollary, we obtain an extension of the probabilistic serial mechanism in the house allocation model to the full preference domain. Our extension does not require any knowledge beyond elementary computation.

Keywords: fractional top trading cycle; fractional endowment; weak preferences; house allocation; probabilistic serial

JEL Classification: C71, C78, D71

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1 Introduction

Efficiency and fairness are two desiderata in market design. In many allocation problems, fairness requires the use of randomization because of indivisibility of resources. Through allocating probabilities, we restore symmetry between agents and get a measure of fairness. Among the few successful matching mechanisms, Gale’s Top Trading Cycle (TTC; Shapley and Scarf, 1974) is best known for being efficient to solve deterministic allocation problems. By trading all efficiency-enhancing cycles, TTC excludes any further Pareto improvement over its assignment. However, TTC is no longer efficient when randomization is incorporated into its procedure through randomizing endowments or randomly breaking priority ties.\(^1\) To solve this problem, Yu and Zhang (2020) propose the class of Fractional Top Trading Cycle mechanisms (FTTC). FTTC extends TTC to solve random allocation problems efficiently and fairly. However, though the assumption of strict preferences in that paper covers many applications, there are still many environments where the natural preference domain is the full preference domain in which preferences can be weak.\(^2\) To maintain efficiency, preference ties cannot be broken arbitrarily before running FTTC. This paper extends FTTC to the full preference domain, and maintains its efficiency and fairness.

On the strict preference domain, we present FTTC in the fractional endowment exchange (FEE) model, which is a direct extension of Shapley and Scarf’s housing market model. In the model an agent may own fractions of multiple objects, and an object may be owned by multiple agents. The idea of trading cycles in TTC cannot be directly extended to the model. In FTTC agents report favorite objects step by step as in TTC. Our innovation is to use linear equations to define how agents trade endowments at each step. The equations satisfy a balanced trade condition, which requires that at each step the amount of favorite object obtained by each agent be equal to the amount of endowments lost by the agent. By connecting the equations to the closed Leontief input-output model (Leontief, 1941), we prove that the solution to the equations exists. This ensures that FTTC is well-defined.

We add parameters to the equations to control fairness. The parameters determine how the owners of each object divide the right of using the object to trade with the others.

A simple example can illustrate why preference ties cannot be arbitrarily broken. Let us

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\(^1\)For example, in the house allocation model, TTC with uniformly random endowments is equivalent to the Random Priority mechanism (Abdulkadiroğlu and Sönmez, 1998), which is not ex-ante efficient (Bogomolnaia and Moulin, 2001).  
\(^2\)See Bogomolnaia et al. (2005); Erdil and Ergin (2017) for arguments for why the full preference domain naturally appears in many environments.
consider two agents $i, j$ who own equal divisions of two objects $a, b$. Agent $i$ is indifferent between $a$ and $b$, but $j$ strictly prefers $a$ to $b$. If after breaking ties we let $i$ strictly prefer $a$ to $b$, then in the only individually rational assignment $i, j$ will keep their endowments. But for true preferences, the only individually rational and efficient assignment is the one in which $i$ obtains $b$ and $j$ obtains $a$.

Our idea to solve weak preferences is to utilize endowment exchange in FTTC. At any step, if an object has been used up by all of its owners in the trading process, but some agent who obtains an amount of the object finds that a remaining object is as good as the former object, we let the agent label his consumption of the former object as an endowment that is available for trading. In the following trading process, the balanced trade condition ensures that when the agent loses an amount of the object from his consumption, he will be compensated by obtaining an equal amount of indifferent objects. The agent will withdraw the label when there no longer exists an indifferent available object. In the above example, suppose at the first step $i, j$ obtain their own endowments $1/2a$ through self-trading. After that, $a$ is exhausted, but $i$ labels his consumption $1/2a$ as available for trading because there remains an indifferent object $b$. At the second step, $i$ demands $b$ and $j$ demands $a$. So $i$ will not only obtain his endowment $1/2b$ through self-trading, but also obtain $1/2b$ through exchanging endowments with $j$. We obtain the unique desirable assignment in the example.

Of course, the example illustrates a simple case. In general cases, when an agent labels his consumption of an object as available for trading, this may induce another agent to label his consumption of another object as available for trading, and so on until a chain appears. To maintain efficiency, it is crucial to find all such chains in the procedure of FTTC. So in the definition of FTTC in this paper, we add a labeling stage at the beginning of each step to find all such chains. Other than that, the definition remains almost same as on the strict preference domain. We prove that FTTC remains to be individually rational and sd-efficient, and the conditions imposed on the parameters in FTTC to ensure various fairness axioms on the strict preference domain remain to work on the full preference domain.

The literature since Hylland and Zeckhauser (1979) has studied random allocation intensively in the house allocation model. The model implicitly assumes that agents collectively own all objects, and no agent is favored over any other. So we regard the model as a special case of FEE in which agents own equal divisions of all objects. On the strict preference domain, we have shown that every FTTC coincides with a simultaneous eating algorithm of Bogomolnaia and Moulin (2001), and a subclass of FTTC that treats all agents equally coincides the Probabilistic Serial mechanism (PS). This means that our definition of FTTC
in this paper subsumes an extension of PS to the full preference domain. The extension can be described as an eating algorithm and maintains efficiency and fairness of PS. Its only deviation from PS is that when an agent labels his consumption of an object as available for the others to consume and some others are indeed consuming his consumption, we instantly increase his eating rate following the rule we call “you request my house - I get your rate”.\textsuperscript{3} This rule is the degeneration of the balanced trade condition to eating algorithms. An extreme case of weak preferences is the dichotomous preference domain.\textsuperscript{4} On such domain we show that our extension of PS finds the egalitarian solution proposed by Bogomolnaia and Moulin (2004). Katta and Sethuraman (2006) propose an algorithm to find the egalitarian solution through solving a network flow problem. By iteratively applying the algorithm, they obtain an extension of PS to the full preference domain. Comparing with their algorithm, ours does not require any knowledge beyond elementary computation.

On the strict preference domain, we extend FTTC to school choice with coarse priorities. At each step, among remaining students, only those of highest priority at each school can use the seats of the school to trade with the others, and for fairness we let them use equal fractions of the seats of the school to trade with the others. We can extend this mechanism to the full preference domain as we did in the FEE model. We omit the details. We are different from Erdil and Ergin (2017) who focus on stable matchings in two-sided matching with weak preferences on both sides. Because FTTC reduces to TTC in the housing market model, our definition of FTTC also subsumes an extension of TTC to the full preference domain. The extension resembles Jaramillo and Manjunath (2012) and Alcalde-Unzu and Molis (2011), and thus can preserve strategy-proofness of TTC if parameters in its definition are properly chosen. We also omit the details.

The rest of the paper is organized as follows. Section 2 presents the FEE model. Section 3 briefly revisits the definition of FTTC on the strict preference domain, and then presents our definition of FTTC on the full preference domain. Section 4 shows that efficiency and fairness of FTTC are preserved on the full preference domain. Section 5 applies FTTC to the house allocation model. FTTC finds the egalitarian solution on the dichotomous preference domain, and extends PS to the full preference domain.

\textsuperscript{3}Yu and Zhang (2020) use this rate-adjusting rule to obtain an extension of PS to the house allocation with existing tenants model.

\textsuperscript{4}Each agent regards each object as either acceptable or unacceptable, and regards the objects in the same class as indifferent.
2 Fractional Endowment Exchange Model

A fractional endowment exchange (FEE) problem is a four-tuple \((I, O, \succeq_I, \omega)\) in which

- \(I\) is a finite set of agents;
- \(O\) is a finite set of objects;
- \(\succeq_I = \{\succeq_i\}_{i \in I}\) is the preference profile of agents;
- \(\omega = (\omega_{i,o})_{i \in I, o \in O}\) is the endowment matrix.

For each agent \(i\), \(\omega_i = (\omega_{i,o})_{o \in O}\) denotes \(i\)'s endowments, with \(\omega_{i,o} \in [0, 1]\) being the amount (probability share) of \(o \in O\) owned by \(i\). Let \(q_o = \sum_{i \in I} \omega_{i,o}\) denote the total amount of \(o \in O\) in the market, which is an integer. Each agent demands one object and his total amount of endowments is no more than one; that is, \(\sum_{o \in O} \omega_{i,o} \leq 1\). Each agent has a preference relation \(\succeq_i\) over objects. \(\succeq_i\) is complete and transitive, but needs not to be strict. Let \(\succ_i\) and \(\sim_i\) respectively denote the asymmetric and the symmetric components of \(\succeq_i\). The housing market problem is a special case of the FEE model if \(|I| = |O|\) and \(\omega\) is a permutation matrix. The house allocation problem can be regarded as a special case of the FEE model if \(|O| = |I|\) and \(\omega_{i,o} = 1/|I|\) for all \(i \in I\) and all \(o \in O\).

A lottery is a vector \(l \in \mathbb{R}_{+}^{|O|}\) such that \(\sum_{o \in O} l_o \leq 1\). A lottery \(l\) weakly (first-order) stochastically dominates another lottery \(l'\) for agent \(i\), denoted by \(l \succeq_i l'\), if \(\sum_{o \in O} l'_{o} \geq \sum_{o \in O} l_{o}\) for all \(o \in O\). If the inequality is strict for some \(o\), \(l\) strictly stochastically dominates \(l'\), denoted by \(l \succ_i l'\). We denote by \(l \sim_i l'\) if \(l \succeq_i l'\) and \(l' \succeq_i l\).

An assignment is a matrix \(p = (p_{i,o})_{i \in I, o \in O} \in \mathbb{R}_{+}^{|I| \times |O|}\) such that \(\sum_{o \in O} p_{i,o} \leq 1\) for all \(i \in I\). Each \(p_{i,o}\) is the amount of \(o\) assigned to \(i\). The row vector \(p_i = (p_{i,o})_{o \in O}\) is the lottery assigned to \(i\). If all elements of \(p\) are integers, \(p\) is a deterministic assignment. The Birkhoff-von Neumann theorem and its generalization (Birkhoff, 1946; Von Neumann, 1953; Kojima and Manea, 2010) guarantee that every assignment is a convex combination of deterministic assignments. An assignment \(p\) is ex-post efficient if it can be written as a convex combination of Pareto efficient deterministic assignments. An assignment \(p\) strictly stochastically dominates another assignment \(p'\), denoted by \(p \succ_i p'\), if \(p_i \succeq_i p_i'\) for all \(i\) and \(p_j \succ_j p_j'\) for some \(j\). An assignment \(p\) is sd-efficient if it is never strictly stochastically dominated. It is individually rational (IR) if \(p_i \succeq_i \omega_i\) for all \(i \in I\). IR implies that \(\sum_{o \in O} p_{i,o} = \sum_{o \in O} \omega_{i,o}\) for all \(i \in I\).

\(^5\)If an agent has more endowments than his demand, when his demand is satisfied in our mechanisms, his residual endowments can be inherited by the remaining agents. For simplicity we do not discuss inheritance.
We define four fairness axioms. From the weakest to the strongest, they are equal treatment of equals (ETE), equal-endowment no envy (EENE), bounded envy (BE), and envy-freeness (EF). ETE and EENE require fairness among “equal” agents. BE is proposed by Yu and Zhang (2020) to require fairness among any agents. It requires that if an agent is envied by another agent, then the envy is bounded by the former agent’s advantage in endowments. EF is the strongest axiom that eliminates envy between any two agents. It is compatible with IR in special cases of the FEE model, but they are incompatible in general cases.

Formally, an assignment $p$ satisfies

- ETE if for all $i, j \in I$ such that $\omega_i = \omega_j$ and $\succ_i \equiv \succ_j$, $p_i = p_j$;
- EENE if for all $i, j \in I$ such that $\omega_i = \omega_j$, $p_i \succ i^sd p_j$ and $p_j \succ j^sd p_i$;
- BE if for all $i, j \in I$, $\max_{o \in O} \left[ \sum_{o' \succ o} p_{j,o'} - \sum_{o' \succ o} p_{i,o'} \right] \leq \sum_{o \in O: \omega_j,o > \omega_i,o} (\omega_j,o - \omega_i,o)$;
- EF if for all $i, j \in I$, $p_i \succ i^sd p_j$ and $p_j \succ j^sd p_i$.

We denote an FEE problem by its preference profile when the other elements are fixed. Let $R$ denote the set of all complete and transitive preference relations. A mechanism $\varphi$ finds an assignment $\varphi(\succ I)$ for each $\succ I \in R|I|$. The lottery assigned to each $i \in I$ in $\varphi(\succ I)$ is denoted by $\varphi_i(\succ I)$. A mechanism satisfies an efficiency or fairness axiom if its found assignments satisfy the axiom.

We say an agent $i$ weakly manipulates a mechanism $\varphi$ at $\succ I$ by reporting $\succ' \in R \setminus \{\succ I\}$ if $\varphi_i(\succ I) \succ i^sd \varphi_i(\succ' I, \succ I)$. We say $i$ strongly manipulates $\varphi$ at $\succ I$ by reporting $\succ'$ if $\varphi_i(\succ' I, \succ I) \succ i^sd \varphi_i(\succ I)$. $\varphi$ is (weakly) strategy-proof if it is never (strongly) manipulated.

3 FTTC on the full preference domain

3.1 Strict preference domain

To understand our definition of FTTC on the full preference domain, we revisit the definition on the strict preference domain. When preferences are strict, at each step of FTTC, each remaining agent reports his unique favorite remaining object. A parameterized linear equation system describes how agents trade endowments at each step. By solving the equations, we obtain the amount of favorite object each agent obtains and the amount of endowments he loses. Parameters in the equations control fairness.
We define some notations to describe the equations. These notations will also be used in the next subsection. At the end of step \(d\), let \(I(d)\) and \(O(d)\) denote the set of remaining agents and the set of remaining objects respectively (\(I(0) = I\) and \(O(0) = O\)); let \(\omega(d) = (\omega_{i,o}(d))_{i \in I, o \in O}\) denote the matrix of remaining endowments. At step \(d\), each \(i \in I(d-1)\) reports his favorite object among \(O(d-1)\), denoted by \(o_i(d)\). At step \(d\), let \(x_i(d)\) denote the amount of \(o_i(d)\) assigned to \(i \in I(d-1)\), and let \(x_o(d)\) denote the amount of \(o \in O(d-1)\) assigned to all agents. So \(x_o(d)\) is also the total amount of \(o\) lost by its owners from their endowments at step \(d\). We use a parameter \(\lambda_{i,o}(d)\) denote the proportion of \(x_o(d)\) that is lost by \(i\) from his endowments. That is, \(i\) loses \(\lambda_{i,o}(d)x_o(d)\) of \(o\) at step \(d\). We write all such parameters into a matrix \(\lambda(d) = (\lambda_{i,o}(d))_{i \in I(d-1), o \in O(d-1)}\), and call it ratio matrix. The matrix controls how the owners of each object divide the right of using the object to trade with the others at step \(d\). For all \(i \in I(d-1)\) and all \(o \in O(d-1)\), \(\sum_{i \in I(d-1)} \lambda_{i,o}(d) = 1\) and \(\lambda_{i,o}(d) > 0\) only if \(\omega_{i,o}(d-1) > 0\). We use another parameter \(\beta_{i,o}(d)\) to control the maximum amount of \(o \in O(d-1)\) that each \(i \in I(d-1)\) can lose at step \(d\). So \(0 \leq \beta_{i,o}(d) \leq \omega_{i,o}(d-1)\).

We write them into another matrix \(\beta(d) = (\beta_{i,o}(d))_{i \in I(d-1), o \in O(d-1)}\), and call it quota matrix.

At each step \(d\), we solve the equations

\[
\begin{cases}
  x_o(d) = \sum_{i \in I(d-1); o_i(d) = o} x_i(d) & \text{for all } o \in O(d-1), \\
  x_i(d) = \sum_{o \in O(d-1)} \lambda_{i,o}(d)x_o(d) & \text{for all } i \in I(d-1),
\end{cases}
\]

subject to the constraints

\[
\lambda_{i,o}(d)x_o(d) \leq \beta_{i,o}(d) \quad \text{for all } i \in I(d-1) \text{ and all } o \in O(d-1).
\]

The first equation of (1) is obtained by the definition of \(x\), while the second equation of (1) describes balanced trade among agents. Denote the maximum solution to (1) subject to (2) by \(x^*(d)\). In Yu and Zhang (2020) we prove that, given \(\lambda(d)\) and \(\beta(d)\), because the coefficient matrix of (1) is stochastic (i.e., its every column sums to one), the solution \(x^*(d)\) at each step \(d\) exists. So FTTC is well-defined. The fact that agents obtain favorite objects step by step and they trade endowments in a balanced way straightforwardly implies that FTTC is IR and sd-efficient. When \(\lambda(d)\) is properly chosen (as presented in Section 4), FTTC can satisfy any of ETE, EENE, and BF.

### 3.2 Full preference domain

With weak preferences, we can run FTTC after breaking preference ties. But as explained in Introduction, any preference-independent tie-breaking rule can cause efficiency loss. Our
method is to let agents label some of their consumptions as endowments available for trading when they find other available indifferent objects. The balanced trade condition ensures that when they lose an amount of consumptions, they will be compensated by obtaining an equal amount of indifferent objects. We have briefly explain this method through a simple example in Introduction. Below we use another example to explain it more clearly.

**Example 1.** Consider three agents \( \{1, 2, 3\} \) and three objects \( \{a, b, c\} \). Agents have equal endowments \( (1/3a, 1/3b, 1/3c) \) and the following preferences:

\[
\begin{array}{ccc}
\preceq_1 & \preceq_2 & \preceq_3 \\
\{a, b\} & a & a \\
 & b & c \\
 & c & b \\
\end{array}
\]

- **Step one:** Agent 1 points to \( a, b \). Agents 2 and 3 point to \( a \). Suppose agents obtain equal amounts of favorite objects and lose equal amounts of endowments; in particular, 1 obtains equal amounts of \( a, b \). So after this step, 1 obtains \( (1/5a, 1/5b) \), 2 and 3 each obtain \( 2/5a \), and each agent loses \( (1/3a, 1/15b) \). Object \( a \) is used up.

- **Step two:** Because 1 is indifferent between \( a \) and \( b \), and \( b \) has not been used up, 1 labels his consumption of \( a \) as an endowment available for trading. So 1 points to \( b \) and 2, 3 point to \( a \). Suppose 2, 3 obtain equal amounts of \( a \). Then each of 2, 3 obtains \( 1/10a \), and 1 obtains \( 3/10b \). Note that 1 loses his consumption \( 1/5a \) to obtain an additional \( 1/5b \). His net consumption amount is \( 3/10 - 1/5 = 1/10 \), which is equal to that of 2, 3.

- **Step three:** Because 1’s consumption of \( a \) has been exhausted at step two, 1, 2 point to \( b \) and 3 points to \( c \). Suppose agents obtain equal amounts of objects. Then 1, 2 each obtain \( 1/4b \), and 3 obtains \( 1/4c \).

- **Step four:** All agents point to \( c \), and each obtains \( 1/4c \).

The mechanism finds the following assignment, which is the unique IR, sd-efficient and envy-free assignment in this example:

\[
\begin{array}{ccc}
1 & 2 & 3 \\
3/4b & 1/2a & 1/2a \\
1/4c & 1/4b & 1/2c \\
& & 1/4c \\
\end{array}
\]
As mentioned in Introduction, after an agent labels his consumption of some object as an endowment available for trading, it may induce a chain of the other agents also to label their consumptions as endowments available for trading. Every chain will look like

\[ i_{m+1} \rightarrow o_m \rightarrow i_m \rightarrow \cdots \rightarrow o_1 \rightarrow i_1 \rightarrow o_0 , \]

where \( o_0 \) is an object that has been exhausted in the trading process, while the other objects in the chain have been exhausted. For each \( k = 1, \ldots, m, \) \( i_k \) is indifferent between \( o_{k-1} \) and \( o_k \), and labels his consumption of \( o_k \) as a new endowment. The last agent \( i_{m+1} \) strictly prefers \( o_m \) to all remaining endowments and most prefers \( o_m \) among all new endowments. Finding such chains are crucial for maintaining sd-efficiency of FTTC.

**FTTC on the full preference domain**

**Notations:** \( O(d), \omega(d), x_i(d), \) and \( x_o(d) \) are defined as in Section 3.1. Let \( p(d) = (p_{i,o}(d))_{i \in I, o \in O} \) denote the assignment found by the end of step \( d \).

**Step** \( d \geq 1 \): Every step consists of three stages.

1. **Labeling**
   - **Round 1:** If any \( i \in I \) is indifferent between any \( o \in O(d-1) \) and any \( o' \in O \setminus O(d-1) \) with \( p_{i,o'}(d-1) > 0 \), label \( o' \) as an endowment and let \( o' \) point to \( i \). Denote the set of such \( i \) by \( L_1(d-1) \). For each \( i \in L_1(d-1) \), let \( \tilde{O}_1(d-1) \) denote the set of objects labeled by \( i \). Let \( \tilde{O}_1(d-1) = \bigcup_{i \in L_1(d-1)} \tilde{O}_i(d-1) \).
   - **Round 2:** If any \( i \in I \setminus L_1(d-1) \) is indifferent between any \( o \in \tilde{O}_1(d-1) \) and any \( o' \in O \setminus [O(d-1) \cup \tilde{O}_1(d-1)] \) with \( p_{i,o'}(d-1) > 0 \), label \( o' \) as an endowment and let \( o' \) point to \( i \). Denote the set of such \( i \) by \( L_2(d-1) \). For each \( i \in L_2(d-1) \), let \( \tilde{O}_2(d-1) \) denote the set of objects labeled by \( i \). Let \( \tilde{O}_2(d-1) = \bigcup_{i \in L_2(d-1)} \tilde{O}_i(d-1) \).
   - **Round n:** If any \( i \in I \setminus \bigcup_{k=1}^{n-1} L_k(d-1) \) is indifferent between any \( o \in \tilde{O}_{n-1}(d-1) \) and any \( o' \in O \setminus [O(d-1) \cup \tilde{O}_1(d-1) \cup \cdots \cup \tilde{O}_{n-1}(d-1)] \) with \( p_{i,o'}(d-1) > 0 \), label \( o' \) as an endowment and let \( o' \) point to \( i \). Denote the set of such \( i \) by \( L_n(d-1) \). For each \( i \in L_n(d-1) \), let \( \tilde{O}_n(d-1) \) denote the set of objects labeled by \( i \). Let \( \tilde{O}_n(d-1) = \bigcup_{i \in L_n(d-1)} \tilde{O}_i(d-1) \).
Since there are finite agents and finite objects, the above procedure must stop in finite rounds. Suppose it stops in \( n \) rounds. Let \( L(d - 1) = \cup_{k=1}^{n} L_k(d - 1) \), \( \hat{O}(d - 1) = \bigcup_{i \in L(d-1)} \hat{O}_i(d - 1) \), and \( \hat{O}(d - 1) = O(d - 1) \cup \hat{O}(d - 1) \). So \( \hat{O}(d - 1) \) is the set of objects that are available in the trading process at step \( d \).

2. Pointing

Define \( I(d - 1) = L(d - 1) \cup \{ i \in I : \sum_{o \in O} \omega_{i,o}(d - 1) > 0 \} \) to be the set of active agents. These agents can join the trading process at step \( d \).

- Round 1: For every \( i \in I(d - 1) \), if \( i \)'s favorite objects among \( \hat{O}(d - 1) \) include objects from \( O(d - 1) \), let \( i \) point to all of his favorite objects from \( O(d - 1) \). Denote the set of such agents by \( P_1(d) \).
- Round 2: For every \( i \in I(d - 1) \setminus P_1(d) \), if \( i \)'s favorite objects among \( \hat{O}(d - 1) \) include objects from \( \hat{O}_1(d - 1) \), let \( i \) point to all of his favorite objects from \( \hat{O}_1(d - 1) \). Denote by the set of such agents by \( P_2(d) \).

:\
- Round \( m \): For every \( i \in I(d - 1) \setminus [\cup_{k=1}^{m-1} P_k(d)] \), if \( i \)'s favorite objects among \( \hat{O}(d - 1) \) include objects from \( \hat{O}_{m-1}(d - 1) \), let \( i \) point to all of his favorite objects from \( \hat{O}_{m-1}(d - 1) \). Denote the set of such agents by \( P_m(d) \).

Since there are finite agents and finite objects, the above procedure must stop in finite rounds. Suppose it stops in \( m \) rounds. Then it must be that \( m \leq n + 1 \). For every \( i \in I(d - 1) \), let \( A_i(d) \) denote the set of objects pointed by \( i \).

3. Trading

We choose a ratio matrix \( \lambda(d) = (\lambda_{i,o}(d))_{i \in I(d-1), o \in \hat{O}(d-1)} \) and a quota matrix \( \beta(d) = (\beta_{i,o}(d))_{i \in I(d-1), o \in O(d-1)} \) as we did on the strict preference domain. The only difference is that now the ratio matrix includes parameters for \( \hat{O}(d - 1) \). For all \( o \in \hat{O}(d - 1) \), we require that \( \sum_{i \in I(d-1)} \lambda_{i,o}(d) = 1 \) and \( \lambda_{i,o}(d) > 0 \) only if \( o \in \hat{O}_i(d - 1) \).

Because an agent may point to several objects, we introduce another nonnegative matrix \( \gamma(d) = (\gamma_{i,o}(d))_{i \in I(d-1), o \in \hat{O}(d-1)} \) to control how each agent \( i \) divides his demand among the objects in \( A_i(d) \). We call \( \gamma(d) \) division matrix and require that, for all \( i \in I(d - 1) \), \( \sum_{o \in \hat{O}(d-1)} \gamma_{i,o}(d) = 1 \), and \( \gamma_{i,o}(d) > 0 \) only if \( o \in A_i(d) \).

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\(^6\)Every agent will appear in at most one round of the Labeling stage.
Let $\mathbf{x}^*(d) = (x^*_a(d))_{a \in I(d-1) \cup O(d-1)}$ be the maximum solution to the equation system:

$$
\begin{align*}
  &x_o(d) = \sum_{i \in I(d-1): o \in A_i(d)} \gamma_{i,o}(d)x_i(d) \quad \text{for all } o \in \overline{O}(d-1), \\
  &x_i(d) = \sum_{o \in \overline{O}(d-1)} \lambda_{i,o}(d)x_o(d) \quad \text{for all } i \in I(d-1),
\end{align*}
$$

subject to the constraints

$$
\begin{align*}
  &\lambda_{i,o}(d)x_o(d) \leq \beta_{i,o}(d) \quad \text{for all } i \in I(d-1) \text{ and all } o \in O(d-1), \\
  &\lambda_{i,o}(d)x_o(d) \leq p_{i,o}(d-1) \quad \text{for all } i \in I(d-1) \text{ and all } o \in \tilde{O}(d-1).
\end{align*}
$$

For all $i \in I(d-1)$ and all $o \in O$, let

$$
\omega_{i,o}(d) = \begin{cases}
  \omega_{i,o}(d-1) - \lambda_{i,o}(d)x_o(d) & \text{if } o \in O(d-1), \\
  0 & \text{otherwise},
\end{cases}
$$

and

$$
p_{i,o}(d) = \begin{cases}
  p_{i,o}(d-1) - \lambda_{i,o}(d)x_o(d) & \text{if } o \in \tilde{O}(d), \\
  p_{i,o}(d-1) + \gamma_{i,o}(d)x_i(d) & \text{if } o \in A_i(d), \\
  p_{i,o}(d-1) & \text{otherwise}.
\end{cases}
$$

For all $i \in I \setminus I(d-1)$, $\omega_{i}(d) = \omega_{i}(d-1)$ and $p_{i}(d) = p_{i}(d-1)$.

Let $O(d) = \{ o \in O(d-1) : \sum_{i \in I} \omega_{i,o}(d) > 0 \}$. If $O(d)$ is empty, stop the algorithm. Otherwise, go to step $d+1$.

Because the coefficient matrix of the equation system (3) is still stochastic, the maximum solution $\mathbf{x}^*(d)$ at each step exists. By choosing different parameter values, we obtain different FTTC mechanisms on the full preference domain.

To facilitate our discussion in remaining sections, define

$$
x_i^c(d) = \sum_{o \in \overline{O}(d-1)} \lambda_{i,o}(d)x_o(d),
$$

$$
x_i^a(d) = \sum_{o \in O(d-1)} \lambda_{i,o}(d)x_o(d).
$$

In words, $x_i^c(d)$ denotes the amount of consumptions that $i$ loses at step $d$, and $x_i^a(d)$ denotes the amount of $i$’s net consumption at step $d$. So $x_i(d) = x_i^c(d) + x_i^a(d)$. By losing $x_i^c(d)$ of consumptions, $i$ obtains an equal amount $x_i^c(d)$ of indifferent objects. What matters for $i$’s welfare is the net consumption $x_i^a(d)$. 


4 Efficiency and Fairness

We show that IR, efficiency and fairness of FTTC on the strict preference domain remain to hold on the full preference domain.

Recall that at each step $d$, $\overline{O}(d-1) = O(d-1) \cup \tilde{O}(d-1)$ is the set of objects available for trading. We prove a lemma stating that $\overline{O}(d-1)$ weakly shrinks in the procedure of FTTC. It means that once an object becomes unavailable for trading at some step, it remains unavailable at following steps. This feature is crucial for maintaining desirable properties of FTTC on the full preference domain.

Lemma 1. For any step $d \geq 1$, $\overline{O}(d) \subseteq \overline{O}(d-1)$.

Proof. By definition, $O(d) \subseteq O(d-1) \subseteq \overline{O}(d-1)$. So we only need to prove that $\tilde{O}(d) \subseteq \overline{O}(d-1)$. We enumerate the objects in $\tilde{O}(d)$ to prove this result. Define $E(d) = O(d-1) \setminus O(d)$ to be the set of objects that are exhausted in the trading process at step $d$. In the labeling stage of step $d+1$, we know that there exists $n \in \mathbb{N}$ such that $\tilde{O}(d) = \cup_{k=1}^{n} \tilde{O}_{k}(d)$.

Base step. For every $o' \in \tilde{O}_{1}(d)$, if $o' \in E(d)$, then $o' \in O(d-1) \subseteq \overline{O}(d-1)$. If $o' \notin E(d)$, then $o' \notin O(d-1)$. So $o'$ is exhausted before step $d$. The fact that $o' \in \tilde{O}_{1}(d)$ means that some agent $i$ is indifferent between $o'$ and a distinct object $o \in O(d)$. Since $o \in O(d) \subseteq O(d-1)$, $i$’s consumption of $o'$ must be labeled as available for trading at step $d$. So $o' \in \overline{O}(d-1)$. Thus, $\tilde{O}_{1}(d) \subseteq \overline{O}(d-1)$.

Inductive step. Suppose for all $\ell = 1, \ldots, k-1$, $\tilde{O}_{\ell}(d) \subseteq \overline{O}(d-1)$. For every $o' \in \tilde{O}_{k}(d)$, if $o' \in E(d)$, then $o' \in O(d-1) \subseteq \overline{O}(d-1)$. If $o' \notin E(d)$, then $o' \notin O(d-1)$. So $o'$ is exhausted before step $d$. The fact that $o' \in \tilde{O}_{k}(d)$ means that some agent $i$ is indifferent between $o'$ and a distinct object $o \in \tilde{O}_{k-1}(d)$. Since $o \in \tilde{O}_{k-1}(d) \subseteq \overline{O}(d-1)$, $i$’s consumption of $o'$ must be labeled as available for trading at step $d$. So $o' \in \overline{O}(d-1)$. Thus, $\tilde{O}_{k}(d) \subseteq \overline{O}(d-1)$.

By induction, for all $1 \leq \ell \leq n$, $\tilde{O}_{\ell}(d) \subseteq \overline{O}(d-1)$. So $\tilde{O}(d) \subseteq \overline{O}(d-1)$. □

We prove that FTTC remains to be IR and sd-efficient.

Proposition 1. FTTC on the full preference domain is individually rational and sd-efficient.

Proof. (IR) At every step $d$, $i$’s net consumption stochastically dominates the endowments he loses. So IR is obvious.

(Sd-efficiency) Suppose for some preference profile, the assignment found by some FTTC is not sd-efficient. Then there must exist $k \geq 2$ agents who need not be distinct, denoted by $i_1, i_2, \ldots, i_k$, and $k$ objects in the lotteries they obtain, denoted by $o_1, o_2, \ldots, o_k$, such that
if the $k$ agents trade an amount of the $k$ objects in their consumptions as indicated by the following cycle, none of them becomes worse off and some becomes strictly better off:

$$i_1 \rightarrow o_2 \rightarrow i_2 \rightarrow o_3 \rightarrow i_3 \rightarrow \cdots \rightarrow o_k \rightarrow i_k \rightarrow o_1 \rightarrow i_1.$$  

By trading the cycle, $i_1$ obtains an amount of $o_2$, $i_2$ obtains an amount of $o_3$, and so on. Without loss of generality, assume that $i_1$ is strictly better off. This means that $i_1$ strictly prefers $o_2$ to $o_1$. Suppose in the FTTC procedure, $i_1$ starts consuming $o_1$ at step $d$. Then it must be that $o_1 \in \overline{O}(d-1)$ and $o_2 \notin \overline{O}(d-1)$. Consider agent $i_2$. Assume that $i_2$ starts consuming $o_2$ at step $d'$. There are two cases:

- If $i_2$ strictly prefers $o_3$ to $o_2$, then it must be that $o_2 \in \overline{O}(d'-1)$ and $o_3 \notin \overline{O}(d'-1)$. Since $o_2 \notin \overline{O}(d-1)$, by Lemma 1, it must be that $d' < d$, and so $o_3 \notin \overline{O}(d-1)$.

- If $i_2$ is indifferent between $o_3$ and $o_2$, since $o_2 \notin \overline{O}(d-1)$, it must be that $o_3 \notin \overline{O}(d-1)$; otherwise, given $o_3 \in \overline{O}(d-1)$, $i$ should label his consumption of $o_2$ as available.

So in any case we get $o_3 \notin \overline{O}(d-1)$. By applying the above arguments inductively to the remaining agents and objects in the cycle, we get $o_1 \notin \overline{O}(d-1)$, which is a contradiction.

Yu and Zhang (2020) present conditions on $\lambda(d)$ to satisfy various fairness axioms. We show that they still ensure fairness on the full preference domain.

**Definition 1.** An FTTC satisfies

1. stepwise equal treatment of equals (stepwise ETE) if at every step $d$,

   $$\omega_i(d-1) = \omega_j(d-1), \bar{O}_i(d-1) = \bar{O}_j(d-1) \text{ and } A_i(d) = A_j(d) \implies \lambda_{i}(d) = \lambda_{j}(d).$$

2. stepwise equal-endowment equal treatment (stepwise EEET) if at every step $d$,

   $$\omega_i(d-1) = \omega_j(d-1) \implies \lambda_{i,o}(d) = \lambda_{j,o}(d) \text{ for all } o \in O(d-1).$$

3. bounded advantage if at every step $d$,

   $$\omega_{i,o}(d-1) \geq \omega_{j,o}(d-1) \implies \lambda_{i,o}(d) \geq \lambda_{j,o}(d) \text{ and } \omega_{i,o}(d) \geq \omega_{j,o}(d).$$

**Proposition 2.** (1) An FTTC satisfying stepwise ETE satisfies ETE;

(2) An FTTC satisfying stepwise EEET satisfies EEEN;

(3) An FTTC satisfying bounded advantage satisfies BE.
Proof. (1) For any two agents \( i, j \) with \( \omega_i = \omega_j \) and \( \succeq_i = \succeq_j \), stepwise ETE implies that at every step, \( i \) and \( j \) label the same set of consumptions as available, point to the same set of favorite objects, and obtain equal consumptions. So ETE is satisfied.

(2) For any two agents \( i, j \), at any step \( d \), if \( \omega_i(d - 1) = \omega_j(d - 1) \), by stepwise EEET, \( \lambda_{i,o}(d) = \lambda_{j,o}(d) \) for all \( o \in O(d - 1) \). It implies that \( \omega_i(d) = \omega_j(d) \) and \( x_i^n(d) = x_j^n(d) \). So \( i, j \) have equal amounts of net consumptions at step \( d \) and their remaining endowments are still equal after step \( d \). Now if \( i, j \) have equal endowments (i.e., \( \omega_i = \omega_j \)), then \( i, j \) must have equal amounts of net consumptions throughout the procedure of FTTC. Because at every step \( i, j \) point to their respective favorite objects, in the found assignment there must be no envy between them.

(3) Let \( p \) denote the assignment found by any FTTC satisfying bounded advantage. For any distinct \( i, j \in I \), let \( o^* \) be the solution to \( \max_{o \in O} \left[ \sum_{o' \succeq_i o} p_{j,o'} - \sum_{o' \succeq_i o} p_{i,o'} \right] \). Let \( d \) be the earliest step after which all objects in \( \{ o \in O : o \succeq_i o^* \} \) become unavailabe. That is, \( \{ o \in O : o \succeq_i o^* \} \cap \overline{O}(d) = \emptyset \) and \( \{ o \in O : o \succeq_i o^* \} \cap \overline{O}(d - 1) \neq \emptyset \). By Lemma 1, \( \{ o \in O : o \succeq_i o^* \} \cap \overline{O}(d') = \emptyset \) for all \( d' \geq d \). So,

\[
\sum_{o \succeq_i o^*} p_{i,o} = \sum_{d' = 1}^{d} x_i^n(d') = \sum_{o \in O} \left( \omega_{i,o} - \omega_{i,o}(d) \right),
\]

\[
\sum_{o \succeq_i o^*} p_{j,o} \leq \sum_{d' = 1}^{d} x_j^n(d') = \sum_{o \in O} \left( \omega_{j,o} - \omega_{j,o}(d) \right).
\]

For all \( o \in O \) such that \( \omega_{i,o} \geq \omega_{j,o} \), bounded advantage implies that for all \( 1 \leq d' \leq d \), \( \omega_{i,o}(d') \geq \omega_{j,o}(d') \) and \( \lambda_{i,o}(d') \geq \lambda_{j,o}(d') \). So,

\[
\omega_{j,o} - \omega_{j,o}(d) = \sum_{d' = 1}^{d} \lambda_{j,o}(d') x_o(d') \leq \sum_{d' = 1}^{d} \lambda_{i,o}(d') x_o(d') = \omega_{i,o} - \omega_{i,o}(d),
\]

or equivalently,

\[
(\omega_{j,o} - \omega_{j,o}(d)) - (\omega_{i,o} - \omega_{i,o}(d)) \leq 0.
\]

For all \( o \in O \) such that \( \omega_{i,o} < \omega_{j,o} \), bounded advantage implies that for all \( 1 \leq d' \leq d \), \( \omega_{i,o}(d') \leq \omega_{j,o}(d') \) and \( \lambda_{i,o}(d') \leq \lambda_{j,o}(d') \). In particular, \( \omega_{i,o}(d) \leq \omega_{j,o}(d) \). So,

\[
(\omega_{j,o} - \omega_{j,o}(d)) - (\omega_{i,o} - \omega_{i,o}(d)) \leq \omega_{j,o} - \omega_{i,o}.
\]

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Therefore,
\[
\sum_{o \succcurlyeq o^*} p_{j,o} - \sum_{o \preccurlyeq o^*} p_{i,o} \leq \sum_{o \in O} \left( (\omega_{j,o} - \omega_{j,o}(d)) - (\omega_{i,o} - \omega_{i,o}(d)) \right)
\]
\[
= \sum_{o \in O; \omega_{i,o} \geq \omega_{j,o}} \left( (\omega_{j,o} - \omega_{j,o}(d)) - (\omega_{i,o} - \omega_{i,o}(d)) \right)
\]
\[
+ \sum_{o \in O; \omega_{i,o} < \omega_{j,o}} \left( (\omega_{j,o} - \omega_{j,o}(d)) - (\omega_{i,o} - \omega_{i,o}(d)) \right)
\]
\[
\leq \sum_{o \in O; \omega_{i,o} < \omega_{j,o}} (\omega_{j,o} - \omega_{i,o}).
\]

So \( p \) satisfies BE.

As examples, Yu and Zhang (2020) present three fair FTTC and connect their fairness motivations to classical solution rules in the bankruptcy problem. All of the three FTTC satisfy bounded advantage, and thus BE. One of them is called equal-FTTC and denoted by \( \mathcal{T}^e \). Its idea is to let the remaining owners of each object at each step use equal amounts of the object to trade with the others. Formally, it uses the following parameters:

\[
\lambda^e_{i,o}(d) = \begin{cases} 
1 & \text{if } \omega_{i,o}(d-1) > 0, \\
0 & \text{if } \omega_{i,o}(d-1) = 0,
\end{cases}
\]

\[
\beta^e_{i,o}(d) = \omega_{i,o}(d-1).
\]

Another is called proportional-FTTC and denoted by \( \mathcal{T}^p \). Its idea is to let the remaining owners of each object at each step use amounts proportional to their endowments of the object to trade with the others. Formally, it uses the following parameters:

\[
\lambda^p_{i,o}(d) = \frac{\omega_{i,o}(d-1)}{\sum_{j \in I(d-1)} \omega_{j,o}(d-1)},
\]

\[
\beta^p_{i,o}(d) = \omega_{i,o}(d-1).
\]

The third FTTC, omitted here, can be found in our other paper.\(^7\) To extend them to the full preference domain, we can choose the parameters \( \lambda(d) = (\lambda_{i,o}(d))_{i \in I(d-1), o \in \bar{O}(d-1)} \) and \( \gamma(d) \) arbitrarily. The extensions will still satisfy bounded advantage. Because of their intuitive fairness, they are appealing candidates in applications when market designers want to choose an FTTC.

\(^7\)At each step of the third FTTC, among the remaining owners of each object only those who own the most amount of object can use the object to trade with the others, and the amount they can use is no more than the difference between the most amount and the second most amount.
5 House allocation

In the house allocation model, a number of objects $O$ are to be assigned to an equal number of agents $I$. In Yu and Zhang (2020) we regard the model as a special case of FEE in which agents own equal divisions of all objects. We have explained that, on the strict preference domain, every FTTC coincides with a simultaneous eating algorithm (SEA) defined by Bogomolnaia and Moulin (2001), and every FTTC satisfying stepwise EEET coincides with PS. This means that in the house allocation model, every FTTC satisfying stepwise EEET defined in this paper (e.g., $\mathcal{T}^e$ and $\mathcal{T}^p$) is an extension of PS to the full preference domain. Because agents have equal endowments, such FTTC satisfies envy-freeness.

**Proposition 3.** In the house allocation model, every FTTC satisfying stepwise EEET is an extension of PS to the full preference domain. The extension is sd-efficient and envy-free.

Actually, every FTTC satisfying stepwise EEET can be described as an SEA that deviates from PS in two respects. First, agents label their consumptions as available for others to consume when they find indifferent available objects in the market. Second, when an agent’s labeled consumption is being consumed by the others, his eating rate is instantly increased, following the rule we call “you request my house - I get your rate”. Formally, at any time $t \in [0, 1]$, any agent $i$’s eating rate is defined to be

$$s_i(t) = 1 + \sum_{o \in \hat{O}_i(t)} \left[ \lambda_{i,o}(t) \sum_{j \in I : o \in A_j(t)} \gamma_{j,o}(t)s_j(t) \right].$$

In words, $i$’s eating rate is increased by an amount that is equal to the total rate at which his labeled consumptions are being consumed by the others.

The dichotomous preference domain is an extreme case of weak preferences. On such domain, all Pareto efficient deterministic assignments assign the same number of objects to agents, and sd-efficiency coincides with ex-post efficiency for random assignments. Each agent’s welfare in an assignment is simply measured by the amount of acceptable objects he obtains. Bogomolnaia and Moulin (2004) propose an efficient and fair welfare distribution called *egalitarian solution*. Its idea is to maximize agents’ total welfare and at the same time equalize their welfare as much as possible. Katta and Sethuraman (2006) show that the egalitarian solution can be found by an algorithm that computes a lexicographically optimal flow in a network. By iteratively applying the algorithm, Katta and Sethuraman propose an extension of PS to the full preference domain. In the next subsection we prove that any FTTC satisfying stepwise EEET finds the egalitarian solution on the dichotomous
preference domain. This clarifies the relation between our extension of PS with Katta and Sethuraman’s. The merit of our extension is that it remains to have an SEA description, and it does not require any knowledge beyond elementary computation.

5.1 Dichotomous preferences

For every $i \in I$, let $C_i$ denote the set of $i$’s acceptable objects. For every nonempty $Y \subseteq I$ and nonempty $O' \subset O$, define

$$\Gamma(Y,O') = \left( \bigcup_{i \in Y} C_i \right) \cap O'$$

to be the set of objects from $O'$ that are acceptable to at least one agent in $Y$. Given any economy, the Gallai-Edmonds Decomposition Lemma (Bogomolnaia and Moulin, 2004) states that it can be decomposed into three subproblems. In the first subproblem there is a perfect match between objects and agents so that each agent obtains an acceptable object. In the second subproblem there are oversupply of acceptable objects for any subset of agents, so every agent can also obtain an acceptable object. But in the third subproblem there is shortage of acceptable objects. BM’s egalitarian solution is proposed to solve the third subproblem. So we restrict attention to economies belonging to the third type. We assume that every object is acceptable to at least one agent (that is, $\Gamma(I,O) = O$), and for every nonempty $O' \subset O$, $|\{i \in I : C_i \cap O' \neq \emptyset\}| > |O'|$.

The egalitarian solution is defined through finding a sequence of bottleneck sets of agents. The first bottleneck set is defined to be

$$X_1^* = \arg \min_{Y \subseteq I} \frac{\Gamma(Y,O)}{|Y|}.$$

(5)

When there are multiple solutions to the above problem, let $X_1^*$ be the solution of largest cardinality. $\Gamma(X_1^*,O)$ are assigned to $X_1^*$ fairly such that each $i \in X_1^*$ obtains $\frac{|\Gamma(X_1^*,O)|}{|X_1^*|}$ of acceptable objects.

The second bottleneck set is found in the same way as (5) among the remaining agents $Z_1 = I \setminus X_1^*$ and remaining objects $P_1 = O \setminus \Gamma(X_1^*,O)$. In general, the $k$-th bottleneck set is defined to be

$$X_k^* = \arg \min_{Y \subseteq Z_{k-1}} \frac{|\Gamma(Y,P_{k-1})|}{|Y|}.$$

8When both sides are agents as in the model of Bogomolnaia and Moulin (2004), the second and the third subproblems are symmetric.

9The union of two solutions is still a solution. So $X_1^*$ is unique.
When there are multiple solutions, let $X^*_k$ be the solution of largest cardinality. In the egalitarian solution, every $i \in X^*_k$ obtains $\frac{|\Gamma(X^*_k, P^*_k-1)|}{|X^*_1|}$ of acceptable objects. Let $X^*_1, X^*_2, \ldots, X^*_m$ be the sequence of bottleneck sets.

We prove that the above sequence of bottleneck sets is implicitly found in the procedure of any FTTC satisfying EEET. Recall that $O(d) = O(d-1) \cup \tilde{O}(d)\) and $O(0) = O$. By Lemma 1, $\tilde{O}(d) \subset O(d-1)$ for every step $d$. Let $d_1, d_2, \ldots, d_n$ be the sequence of steps in the procedure of FTTC such that, for all $k = 1, \ldots, n$,

$$\tilde{O}(d_k - 1) \setminus \tilde{O}(d_k) \neq \emptyset.$$ 

That is, $d_1$ is the first step after which some objects (i.e., $O \setminus \tilde{O}(d_1)$) become unavailable for trading at following steps. The other steps are interpreted similarly. $d_n$ is the last step after which all objects are unavailable. We prove that each $\tilde{O}(d_k - 1) \setminus \tilde{O}(d_k)$ is a bottleneck set defined by Bogomolnaia and Moulin (2004).

**Lemma 2.** $m = n$, and for all $k = 1, \ldots, n$

$$\tilde{O}(d_k - 1) \setminus \tilde{O}(d_k) = \Gamma(X^*_k, P^*_k),$$

where $P^*_0 = O$ and $P^*_k = P^*_{k-1} \setminus \Gamma(X^*_k, P^*_k) = \tilde{O}(d_k)$.

**Proof.** We prove the lemma by induction.

**Base case.** We first prove that $\tilde{O}(d_1 - 1) \setminus \tilde{O}(d_1) = \Gamma(X^*_1, O)$. Let $p$ denote the assignment found by any FTTC satisfying EEET, and $p(d)$ denote the assignment found by the end of step $d$. Define $X_1 = \{i \in I : p_{i,o}(d) > 0 \text{ for some } o \in \tilde{O}(d_1 - 1) \setminus \tilde{O}(d_1)\}$. Because all objects in $\tilde{O}(d_1 - 1) \setminus \tilde{O}(d_1)$ are assigned to agents at the end of step $d$ and they are no longer available after step $d$, it must be that $\tilde{O}(d_1 - 1) \setminus \tilde{O}(d_1) \subset \Gamma(X_1, O)$. Suppose there exist $i \in X_1$ and $o \in C_i$ such that $o \not\in \tilde{O}(d_1 - 1) \setminus \tilde{O}(d_1)$. Then, $o \in \tilde{O}(d_1)$. But it implies that $i$ should label his consumption of the objects in $\tilde{O}(d_1 - 1) \setminus \tilde{O}(d_1)$ as available at the beginning of step $d_1 + 1$, which is a contradiction. So,

$$\tilde{O}(d_1 - 1) \setminus \tilde{O}(d_1) = \Gamma(X_1, O).$$

Define

$$t_1 = \frac{|\Gamma(X_1, O)|}{|X_1|}.$$ 

Because no objects become unavailable before step $d_1$, no agent changes the objects he points to before step $d_1$. Then stepwise EEET implies that agents obtain equal amounts of
net consumptions at every step. So at the end of step $d_1$, every agent must obtain a total amount $t_1$ of acceptable objects.

For any nonempty $Y \subset X_1$, $\Gamma(Y, O) \subseteq \Gamma(X_1, O) = \overline{O}(d_1 - 1) \setminus O(d_1)$. So,

$$\frac{|\Gamma(Y, O)|}{|Y|} = t_1.$$ 

For any nonempty $Y \subset I$ such that $Y \setminus X_1 \neq \emptyset$, it must be that for every $j \in Y \setminus X_1$, $C_j \cap O(d_1) \neq \emptyset$, since otherwise $j \in X_1$. Because every $i \in Y$ obtains an amount $t$ of acceptable objects at the end of step $d_1$ and every $j \in Y \setminus X_1$ has acceptable objects that are still available after step $d_1$, it must be that

$$\frac{|\Gamma(Y, O)|}{|Y|} > t_1.$$ 

So

$$X_1 = \arg \min_{Y \subset I} \frac{|\Gamma(Y, O)|}{|Y|},$$

and $X_1$ is the solution of largest cardinality. It means that $X_1 = X_1^*.$

**Induction step.** Suppose for all $k = 2, \ldots, \ell$ ($\ell \leq n - 1$), $\overline{O}(d_k - 1) \setminus O(d_k) = \Gamma(X_k^*, P_k^*)$.

We prove that $\overline{O}(d_{\ell + 1} - 1) \setminus O(d_{\ell + 1}) = \Gamma(X_{\ell + 1}^*, P_{\ell + 1}^*)$ where $P_{\ell + 1}^* = \overline{O}(d_{\ell + 1}) = \overline{O}(d_{\ell + 1} - 1)$. Define $Z_{\ell} = I \setminus \bigcup_{k=1}^{\ell} X_k^*$, and $X_{\ell + 1} = \{i \in I : p_{i,o}(d_{\ell + 1}) > 0 \text{ for some } o \in \overline{O}(d_{\ell + 1} - 1) \setminus \overline{O}(d_{\ell + 1})\}$. Because all objects in $O \setminus \overline{O}(d_{\ell + 1} - 1)$ are assigned to the agents in $\bigcup_{k=1}^{\ell} X_k^*$, it must be that $X_{\ell + 1} \subset Z_{\ell}$ and $\overline{O}(d_{\ell + 1} - 1) \setminus \overline{O}(d_{\ell + 1}) \subset \Gamma(X_{\ell + 1}, \overline{O}(d_{\ell + 1} - 1))$.

Suppose there exist $i \in X_{\ell + 1}$ and $o \in C_i$ such that $o \in \overline{O}(d_{\ell + 1})$. Then it implies that $i$ should label his consumption of the objects in $\overline{O}(d_{\ell + 1} - 1) \setminus \overline{O}(d_{\ell + 1})$ as available at the beginning of step $d_{\ell + 1} + 1$, which is a contradiction. So

$$\overline{O}(d_{\ell + 1} - 1) \setminus \overline{O}(d_{\ell + 1}) = \Gamma(X_{\ell + 1}, \overline{O}(d_{\ell + 1} - 1)).$$

Then we can use similar arguments as in the base case to prove that

$$X_{\ell + 1} = \arg \min_{Y \subset Z_{\ell}} \frac{|\Gamma(Y, \overline{O}(d_{\ell + 1} - 1))|}{|Y|},$$

and $X_{\ell + 1}$ is the solution of largest cardinality. It means that $X_{\ell + 1} = X_{\ell + 1}^*$.

With Lemma 2, the following proposition is immediate.

**Proposition 4.** On the dichotomous preference domain, any FTTC satisfying stepwise EEET finds the egalitarian solution of Bogomolnaia and Moulin (2004).
Bogomolnaia and Moulin (2004) have proved that any mechanism finding the egalitarian solution is strategy-proof on the dichotomous preference domain.\textsuperscript{10}

Bogomolnaia and Moulin (2001) shows the sd-inefficiency of the Random Priority mechanism (RP). On the dichotomous preference domain, RP will proceed as follows. It first generates an ordering of agents uniformly at random, and then lets agents in the ordering sequentially choose favorite assignments. Every agent chooses a set of favorite assignments from those being chosen by previous agents. Because sd-efficiency coincides with ex-post efficiency, RP becomes desirable: It is sd-efficient and strategy-proof, and seems to fair because the ordering of agents is uniformly random. However, noting that this randomization is preference-independent, when it interacts with preferences through letting agents picking objects (assignments), the fairness of the final assignment becomes not transparent. This is also true for TTC with randomized endowments or randomized priorities. Example 2 shows that RP does not find the egalitarian solution on the dichotomous preference domain.

**Example 2.** Consider five agents \(\{1, 2, 3, 4, 5\}\) and three objects \(\{o_1, o_2, o_3\}\). Agents’ dichotomous preferences are shown by letting every agent point to all of his acceptable objects in the following graph.

\[1\rightarrow o_1, 2\rightarrow o_1, 3\rightarrow o_2, 4\rightarrow o_2, 5\rightarrow o_3\]

Any FTTC satisfying stepwise EEET finds an egalitarian assignment:

\[
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
1/2o_1 & 1/2o_1 & 2/3o_2 & 1/3o_2 & 2/3o_3 \\
& & & 1/3o_3 \\
\end{array}
\]

But RP finds the following assignment:

\[
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
9/20o_1 & 9/20o_1 & 1/10o_1 & 4/10o_2 & 7/10o_3 \\
& & & 6/10o_2 & 3/10o_3 \\
\end{array}
\]

Comparing with the RP assignment, the egalitarian assignment equalizes agents’ welfare as much as possible.

\textsuperscript{10}Bogomolnaia and Moulin (2004) prove that any such mechanism is group strategy-proof, meaning that no group of agents can jointly manipulate the mechanism.
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