A Short Survey of Cyclic Cohomology

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Dedicated with admiration and affection to Alain Connes

Abstract. This is a short survey of some aspects of Alain Connes’ contributions to cyclic cohomology theory in the course of his work on noncommutative geometry over the past 30 years.

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1. Introduction

Cyclic cohomology was discovered by Alain Connes no later than 1981 and in fact it was announced in that year in a conference in Oberwolfach [5]. I have reproduced the text of his abstract below. As it appears in his report, one of Connes’ main motivations to introduce cyclic cohomology theory came from index theory on foliated spaces. Let $(V,F)$ be a compact foliated manifold and let $V/F$ denote the space of leaves of $(V,F)$. This space, with its natural quotient topology, is, in general, a highly singular space and in noncommutative geometry one usually replaces the quotient space $V/F$ with a noncommutative algebra $A = C^*(V,F)$ called the foliation algebra of $(V,F)$. It is the convolution algebra of the holonomy groupoid of the foliation and is a $C^*$-algebra. It has a dense subalgebra $A = C^\infty(V,F)$ which plays the role of the algebra of smooth functions on $V/F$. Let $D$ be a transversally elliptic operator on $(V,F)$. The analytic index of $D$, $\text{index}(D) \in K_0(A)$, is an element of the $K$-theory of $A$. This should be compared with the family

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index theorem \[1\] where the analytic index of a family of fiberwise elliptic operators is an element of the $K$-theory of the base. Connes realized that to identify this class by a cohomological expression it would be necessary to have a noncommutative analogue of the Chern character, i.e., a map from $K_0(A)$ to a, then unknown, cohomology theory for the noncommutative algebra $A$. This theory, now known as cyclic cohomology, would then play the role of the noncommutative analogue of de Rham homology of currents for smooth manifolds. Its dual version, cyclic homology, corresponds, in the commutative case, to de Rham cohomology.

Connes arrived at his definition of cyclic cohomology by a careful analysis of the algebraic structures deeply hidden in the (super)traces of products of commutators of operators. These expressions are directly defined in terms of an elliptic operator and its parametrix and give the index of the operator when paired with a $K$-theory class. In his own words \[5\]:

“The transverse elliptic theory for foliations requires as a preliminary step a purely algebraic work, of computing for a noncommutative algebra $A$ the cohomology of the following complex: $n$-cochains are multilinear functions $\varphi(f^0, \ldots, f^n)$ of $f^0, \ldots, f^n \in A$ where

\[
\varphi(f^1, \ldots, f^0) = (-1)^n \varphi(f^0, \ldots, f^n)
\]

and the boundary is

\[
b\varphi(f^0, \ldots, f^{n+1}) = \varphi(f^0 f^1, \ldots, f^{n+1}) - \varphi(f^0, f^1 f^2, \ldots, f^{n+1}) + \cdots + (-1)^{n+1} \varphi(f^{n+1} f^0, \ldots, f^n).
\]

The basic class associated to a transversally elliptic operator, for $A = \text{the algebra of the foliation}$, is given by:

\[
\varphi(f^0, \ldots, f^n) = \text{Trace} \left( \varepsilon F[f, f^0] [F, f^1] \cdots [F, f^n] \right), \quad f^i \in A
\]

where

\[
F = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

and $Q$ is a parametrix of $P$. An operation

\[
S : H^n(A) \rightarrow H^{n+2}(A)
\]

is constructed as well as a pairing

\[
K(A) \times H(A) \rightarrow \mathbb{C}
\]

where $K(A)$ is the algebraic $K$-theory of $A$. It gives the index of the operator from its associated class $\varphi$. Moreover $\langle e, \varphi \rangle = \langle e, S\varphi \rangle$, so that the important group to determine is the inductive limit $H_p = \text{Lim} H^n(A)$ for the map $S$. Using the tools of homological algebra the groups $H^n(A, A^*)$ of Hochschild cohomology with coefficients in the bimodule $A^*$ are easier to determine and the solution of the problem is obtained in two steps:

1) the construction of a map

\[
B : H^n(A, A^*) \rightarrow H^{n-1}(A)
\]

and the proof of a long exact sequence

\[
\cdots \rightarrow H^n(A, A^*) \xrightarrow{B} H^{n-1}(A) \xrightarrow{S} H^{n+1}(A) \xrightarrow{L} H^{n+1}(A, A^*) \rightarrow \cdots
\]

where $L$ is the obvious map from the cohomology of the above complex to the Hochschild cohomology;
2) the construction of a spectral sequence with $E_2$ term given by the cohomology of the degree $-1$ differential $I \circ B$ on the Hochschild groups $H^n(A, A^*)$ and which converges strongly to a graded group associated to the inductive limit.

This purely algebraic theory is then used. For $A = C^\infty(V)$ one gets the de Rham homology of currents, and for the pseudo-torus, i.e. the algebra of the Kronecker foliation, one finds that the Hochschild cohomology depends on the Diophantine nature of the rotation number while the above theory gives $H_p^0$ of dimension 2 and $H_p^1$ of dimension 2, as expected, but from some remarkable cancellations."

A full exposition of these results later appeared in two IHES preprints [6], and were eventually published as [9]. With the appearance of [9] one could say that the first stage of the development of noncommutative geometry and specially cyclic cohomology reached a stage of maturity. In the next few sections I shall try to give a quick and concise survey of some aspects of cyclic cohomology theory as they were developed in [9]. The last two sections are devoted to developments in the subject after [9] arising from the work of Connes.

It is a distinct honor and a great pleasure to dedicate this short survey of cyclic cohomology theory as a small token of our friendship to Alain Connes, the originator of the subject, on the occasion of his 60th birthday. It inevitably only covers part of what has been done by Alain in this very important corner of noncommutative geometry. It is impossible to cover everything, and in particular I have left out many important topics developed by him including, among others, the Godbillon-Vey invariant and type III factors [8], the transverse fundamental class for foliations [8], the Novikov conjecture for hyperbolic groups [18], entire cyclic cohomology [10], and multiplicative characteristic classes [12]. Finally I would like to thank Farzad Fathi zadeh for carefully reading the text and for several useful comments, and Arthur Greenspoon who kindly edited the whole text.

2. Cyclic cohomology

Cyclic cohomology can be defined in several ways, each shedding light on a different aspect of it. Its original definition [5, 9] was through a remarkable subcomplex of the Hochschild complex that we recall first. By algebra in this paper we mean an associative algebra over $\mathbb{C}$. For an algebra $A$ let

$$C^n(A) = \text{Hom}(A^{\otimes(n+1)}, \mathbb{C}), \quad n = 0, 1, \ldots,$$

denote the space of $(n+1)$-linear functionals on $A$. These are our $n$-cochains. The Hochschild differential $b : C^n(A) \to C^{n+1}(A)$ is defined as

$$(b\varphi)(a_0, \ldots, a_{n+1}) = \sum_{i=0}^{n} (-1)^i \varphi(a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1})$$

$$+ (-1)^{n+1} \varphi(a_{n+1} a_0, \ldots, a_n).$$

The cohomology of the complex $(C^*(A), b)$ is the Hochschild cohomology of $A$ with coefficients in the bimodule $A^*$.

The following definition is fundamental and marks the departure from Hochschild cohomology in [5, 9]:

**Definition 2.1.** An $n$-cochain $\varphi \in C^n(A)$ is called cyclic if

$$\varphi(a_n, a_0, \ldots, a_{n-1}) = (-1)^n \varphi(a_0, a_1, \ldots, a_n)$$
for all \(a_0, \ldots, a_n\) in \(A\). The space of cyclic \(n\)-cochains will be denoted by \(C^n_\lambda(A)\).

Just why, of all possible symmetry conditions on cochains, the cyclic property is a reasonable choice is at first glance not at all clear.

**Lemma 2.1.** The space of cyclic cochains is invariant under the action of \(b\), i.e., \(bC^n_\lambda(A) \subset C^{n+1}_\lambda(A)\) for all \(n \geq 0\).

To see this one introduces the operators \(\lambda: C^n(A) \rightarrow C^n(A)\) and \(b': C^n(A) \rightarrow C^{n+1}(A)\) by

\[
(\lambda \varphi)(a_0, \ldots, a_n) = (-1)^n \varphi(a_n, a_0, \ldots, a_{n-1}),
\]

\[
(b' \varphi)(a_0, \ldots, a_{n+1}) = \sum_{i=0}^{n} (-1)^i \varphi(a_0, \ldots, a_i a_i+1, \ldots, a_{n+1}),
\]

and checks that \((1 - \lambda)b = b'(1 - \lambda)\). Since \(C^n_\lambda(A) = \text{Ker}(1 - \lambda)\), the lemma is proved.

We therefore have a subcomplex of the Hochschild complex, called the cyclic complex of \(A\):

\[
\begin{array}{cccc}
C^0_\lambda(A) & \xrightarrow{b} & C^1_\lambda(A) & \xrightarrow{b} & C^2_\lambda(A) & \xrightarrow{b} & \cdots
\end{array}
\]

**Definition 2.2.** The cohomology of the cyclic complex \((1)\) is the cyclic cohomology of \(A\) and will be denoted by \(HC^n(A)\), \(n = 0, 1, 2, \ldots\).

And that is Connes’ first definition of cyclic cohomology. A cocycle for the cyclic cohomology group \(HC^n(A)\) is called a cyclic \(n\)-cocycle on \(A\). It is an \((n+1)\)-linear functional \(\varphi\) on \(A\) which satisfies the two conditions:

\[
(1 - \lambda)\varphi = 0, \quad \text{and} \quad b\varphi = 0.
\]

The inclusion of complexes

\[
(C^*_\lambda(A), b) \hookrightarrow (C^*(A), b)
\]

induces a map \(I\) from cyclic cohomology to Hochschild cohomology:

\[
I: HC^n(A) \rightarrow HH^n(A), \quad n = 0, 1, 2, \ldots
\]

A closer inspection of the long exact sequence associated to \((2)\), yields Connes’ long exact sequence relating Hochschild cohomology to cyclic cohomology. This is however easier said than done. The reason is that to identify the cohomology of the quotient one must use another long exact sequence, and combine the two long exact sequences to obtain the result. To simplify the notation, let us denote the Hochschild and cyclic complexes by \(C\) and \(C_\lambda\), respectively. Then \((2)\) gives us an exact sequence of complexes

\[
\begin{array}{cccc}
0 & \rightarrow & C_\lambda & \xrightarrow{\pi} & C/C_\lambda & \rightarrow & 0
\end{array}
\]

Its associated long exact sequence is

\[
\begin{array}{cccc}
\cdots & \rightarrow & HC^n(A) & \rightarrow & HH^n(A) & \rightarrow & H^n(C/C_\lambda) & \rightarrow & HC^{n+1}(A) & \rightarrow & \cdots
\end{array}
\]

We need to identify the cohomology groups \(H^n(C/C_\lambda)\). To this end, consider the short exact sequence of complexes

\[
\begin{array}{cccc}
0 & \rightarrow & C/C_\lambda & \xrightarrow{1-\lambda} & (C, b') & \xrightarrow{N} & C_\lambda & \rightarrow & 0
\end{array}
\]
where the operator $N$ is defined by

$$N = 1 + \lambda + \lambda^2 + \cdots + \lambda^n : C^n \to C^n.$$ 

The relations $(1 - \lambda)b = b'(1 - \lambda)$, $N(1 - \lambda) = (1 - \lambda)N = 0$, and $bN = Nb'$ show that $1 - \lambda$ and $N$ are morphisms of complexes in (5). As for the exactness of (5), the only nontrivial part is to show that $\ker(N) \subseteq \text{im}(1 - \lambda)$, which can be verified. Assuming $A$ is unital, the middle complex $(C, b')$ in (5) can be shown to be exact with a contracting homotopy $s : C^n \to C^{n-1}$ defined by $(s\varphi)(a_0, \ldots, a_{n-1}) = (-1)^{n-1}\varphi(a_0, \ldots, a_{n-1}, 1)$, which satisfies $b's + sb' = \text{id}$. The long exact sequence associated to (5) looks like

$$\cdots \to H^n(C/C_\lambda) \to H^0_b(C) \to HC^n(A) \to H^{n+1}(C/C_\lambda) \to H^0_{b'}(C) \to \cdots.$$

Since $H^0_b(C) = 0$ for all $n$, it follows that the connecting homomorphism

$$(7) \quad \delta : HC^{n-1}(A) \to H^n(C/C_\lambda)$$

is an isomorphism for all $n \geq 0$. Using this in (4), we obtain Connes’ long exact sequence relating Hochschild and cyclic cohomology:

$$\cdots \to HC^n(A) \xrightarrow{\iota} HH^n(A) \xrightarrow{B} HC^{n-1}(A) \xrightarrow{S} HC^{n+1}(A) \to \cdots.$$

The operators $B$ and $S$ play a prominent role in noncommutative geometry. As we shall see, the operator $B$ is the analogue of de Rham’s differential in the noncommutative world, while the periodicity operator $S$ is closely related to Bott periodicity in topological $K$-theory. Remarkably, there is a formula for $B$ on the level of cochains given by $B = NB_0$, where $B_0 : C^n \to C^{n-1}$ is defined by

$$B_0\varphi(a_0, \ldots, a_{n-1}) = \varphi(1, a_0, \ldots, a_{n-1}) - (-1)^n\varphi(a_0, \ldots, a_{n-1}, 1).$$

Using the relations $(1 - \lambda)b = b'(1 - \lambda)$, $(1 - \lambda)N = N(1 - \lambda) = 0$, $bN = Nb'$, and $sb' + bs = 1$, one shows that

$$(9) \quad bB + Bb = 0, \quad \text{and} \quad B^2 = 0.$$

Using the periodicity operator $S$, the periodic cyclic cohomology of $A$ is then defined as the direct limit of cyclic cohomology groups under the operator $S$:

$$HP^i(A) := \text{Lim} HC^{2n+i}(A), \quad i = 0, 1.$$

Notice that since $S$ has degree 2, there are only two periodic groups. These periodic groups have better stability properties compared to cyclic cohomology groups. For example, they are homotopy invariant, and they pair with $K$-theory.

A much deeper relationship between Hochschild and cyclic cohomology groups is encoded in Connes’ $(b, B)$-bicomplex and the associated Connes spectral sequence that we shall briefly recall now. Consider the relations (9). The $(b, B)$-bicomplex
of a unital algebra \( A \), denoted by \( \mathcal{B}(A) \), is the bicomplex

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots \\
C^2(A) & \xrightarrow{B} & C^1(A) & \xrightarrow{B} & C^0(A) \\
\uparrow b & & \uparrow b & & \\
C^1(A) & \xrightarrow{B} & C^0(A) \\
\uparrow b & & \\
C^0(A)
\end{array}
\]

As usual, there are two spectral sequences attached to this bicomplex. The following fundamental result of Connes [9] shows that the spectral sequence obtained from filtration by rows converges to cyclic cohomology. Notice that the \( E_1 \) term of this spectral sequence is the Hochschild cohomology of \( A \).

**Theorem 2.1.** The map \( \varphi \mapsto (0, \ldots, 0, \varphi) \) is a quasi-isomorphism of complexes

\[
(C^*_\lambda(A), b) \rightarrow (\text{Tot} \mathcal{B}(A), b + B).
\]

This is a consequence of the vanishing of the \( E_2 \) term of the second spectral sequence (filtration by columns) of \( \mathcal{B}(A) \). To prove this, Connes considers the short exact sequence of \( b \)-complexes

\[
0 \rightarrow \text{Im } B \rightarrow \text{Ker } B \rightarrow \text{Ker } B/\text{Im } B \rightarrow 0,
\]

and proves that ([9], Lemma 41), the induced map

\[
H^b(\text{Im } B) \rightarrow H^b(\text{Ker } B)
\]

is an isomorphism. This is a very technical result. It follows that \( H^b(\text{Ker } B/\text{Im } B) \) vanishes. To take care of the first column one appeals to the fact that \( \text{Im } B \simeq \text{Ker } (1 - \lambda) \) is the space of cyclic cochains.

We give an alternative proof of Theorem (2.1) above. To this end, consider the cyclic bicomplex \( \mathcal{C}(A) \) defined by

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots \\
C^2(A) & \xrightarrow{1 - \lambda} & C^2(A) & \xrightarrow{N} & C^2(A) & \xrightarrow{1 - \lambda} & \cdots \\
\uparrow b & & \uparrow -b' & & \uparrow b & & \\
C^1(A) & \xrightarrow{1 - \lambda} & C^1(A) & \xrightarrow{N} & C^1(A) & \xrightarrow{1 - \lambda} & \cdots \\
\uparrow b & & \uparrow -b' & & \uparrow b & & \\
C^0(A) & \xrightarrow{1 - \lambda} & C^0(A) & \xrightarrow{N} & C^0(A) & \xrightarrow{1 - \lambda} & \cdots
\end{array}
\]

The total cohomology of \( \mathcal{C}(A) \) is isomorphic to cyclic cohomology:

\[
H^n(\text{Tot } \mathcal{C}(A)) \simeq HC^n(A), \quad n \geq 0.
\]

This is a consequence of the simple fact that the rows of \( \mathcal{C}(A) \) are exact except in degree zero, where their cohomology coincides with the cyclic complex \((C^*_\lambda(A), b)\).
So it suffices to show that $\text{Tot} B(A)$ and $\text{Tot} C(A)$ are quasi-isomorphic. This can be done by explicit formulas. Consider the chain maps

$$I : \text{Tot} B(A) \to \text{Tot} C(A), \quad I = \text{id} + Ns,$$

$$J : \text{Tot} C(A) \to \text{Tot} B(A), \quad J = \text{id} + sN.$$ 

It can be directly verified that the following operators define chain homotopy equivalences:

$$g : \text{Tot} B(A) \to \text{Tot} B(A), \quad g = Ns^2 B_0,$$

$$h : \text{Tot} C(A) \to \text{Tot} C(A), \quad h = s.$$

To give an example of an application of the spectral sequence of Theorem 2.1, let me recall Connes’ computation of the continuous cyclic cohomology of the topological algebra $A = C^\infty(M)$, i.e., the algebra of smooth complex valued functions on a closed smooth $n$-dimensional manifold $M$. This example is important since, apart from its applications, it clearly shows that cyclic cohomology is a noncommutative analogue of de Rham homology.

The continuous analogues of Hochschild and cyclic cohomology for topological algebras are defined as follows. Let $A$ be a topological algebra. A continuous cochain on $A$ is a jointly continuous multilinear map $\varphi : A \times A \times \cdots \times A \to C$. By working with just continuous cochains, as opposed to all cochains, one obtains the continuous analogues of Hochschild and cyclic cohomology groups. In working with algebras of smooth functions (both in the commutative and noncommutative case), it is essential to use this continuous analogue.

The topology of $C^\infty(M)$ is defined by the sequence of seminorms

$$\|f\|_n = \sup |\partial^\alpha f|, \quad |\alpha| \leq n,$$

where the supremum is over a fixed, finite, coordinate cover for $M$. Under this topology, $C^\infty(M)$ is a locally convex, in fact nuclear, topological algebra. Similarly one topologizes the space of $p$-forms on $M$ for all $p \geq 0$. Let

$$\Omega_p M := \text{Hom}_{\text{cont}}(\Omega^p M, C)$$

denote the continuous dual of the space of $p$-forms on $M$. Elements of $\Omega_p M$ are called de Rham $p$-currents. By dualizing the de Rham differential $d$, we obtain a differential $d^\ast : \Omega_p M \to \Omega_{p-1} M$, and a complex, called the de Rham complex of currents on $M$:

$$\Omega_0 M \xleftarrow{d^\ast} \Omega_1 M \xleftarrow{d^\ast} \Omega_2 M \xleftarrow{d^\ast} \cdots.$$ 

The homology of this complex is the de Rham homology of $M$ and we denote it by $H^*_{\text{dR}}(M)$.

It is easy to check that for any $m$-current $C$, closed or not, the cochain

$$\varphi_C(f_0, f_1, \ldots, f_m) := \langle C, f_0 df_1 \cdots df_m \rangle,$$

is a continuous Hochschild cocycle on $C^\infty(M)$. Now if $C$ is closed, then one checks that $\varphi_C$ is a cyclic $m$-cocycle on $C^\infty(M)$. Thus we obtain natural maps

$$\Omega_m M \to \text{HH}^m_{\text{cont}}(C^\infty(M))$$

and

$$Z_m M \to \text{HC}^m_{\text{cont}}(C^\infty(M)),$$
where $Z_m(M) \subset \Omega_m M$ is the space of closed $m$-currents on $M$. For example, if $M$ is oriented and $C$ represents its orientation class, then

$$\varphi_C(f_0, f_1, \ldots, f_n) = \int_M f_0 df_1 \cdots df_n,$$

which is easily checked to be a cyclic $n$-cocycle on $A$.

In [9], using an explicit resolution, Connes shows that (11) is a quasi-isomorphism. Thus we have a natural isomorphism between space of de Rham currents on $M$ and the continuous Hochschild cohomology of $C^\infty(M)$:

$$\text{HH}^i_{\text{cont}}(C^\infty(M)) \simeq \Omega_i M \quad i = 0, 1, \ldots$$

To compute the continuous cyclic homology of $A$, one first observes that under the isomorphism (14) the operator $B$ corresponds to the de Rham differential $d^*$. More precisely, for each integer $n \geq 0$ there is a commutative diagram:

$$\begin{array}{ccc}
\Omega_{n+1} M & \xrightarrow{\mu} & C^{n+1}(A) \\
\downarrow d^* & & \downarrow B \\
\Omega_n M & \xrightarrow{\mu} & C^n(A)
\end{array}$$

where $\mu(C) = \varphi_C$ and $\varphi_C$ is defined by (10). Then, using the spectral sequence of Theorem (2.1) and the isomorphism (14), Connes obtains [9]:

$$HC^{n}_{\text{cont}}(C^\infty(M)) \simeq Z_n(M) \oplus H_{n-2}^{\text{dR}}(M) \oplus \cdots \oplus H_k^{\text{dR}}(M),$$

where $k = 0$ if $n$ is even and $k = 1$ is $n$ is odd. For the continuous periodic cyclic cohomology he obtains

$$HP^k_{\text{cont}}(C^\infty(M)) \simeq \bigoplus_i H_{2i+k}^{\text{dR}}(M), \quad k = 0, 1.$$

We shall also briefly recall Connes’ computation of the Hochschild and cyclic cohomology of smooth noncommutative tori [9]. This result already appeared in Connes’ Oberwolfach report [5]. When $\theta$ is rational, the smooth noncommutative torus $A_\theta$ can be shown to be Morita equivalent to $C^\infty(T^2)$, the algebra of smooth functions on the 2-torus. One can then use Morita invariance of Hochschild and cyclic cohomology to reduce the computation of these groups to those for the algebra $C^\infty(T^2)$. This takes care of rational $\theta$. So we can assume $\theta$ is irrational and we denote the generators of $A_\theta$ by $U$ and $V$ with the relation $VU = \lambda UV$, where $\lambda = e^{2\pi i \theta}$.

Recall that an irrational number $\theta$ is said to satisfy a Diophantine condition if $|1 - \lambda^n|^{-1} = O(n^k)$ for some positive integer $k$.

**Proposition 2.1.** ([9]) Let $\theta \notin \mathbb{Q}$. Then

a) One has $HH^0(A_\theta) = \mathbb{C}$,

b) If $\theta$ satisfies a Diophantine condition then $HH^1(A_\theta)$ is 2-dimensional for $i = 1$ and is 1-dimensional for $i = 2$,

c) If $\theta$ does not satisfy a Diophantine condition, then $HH^i(A_\theta)$ are infinite dimensional non-Hausdorff spaces for $i = 1, 2$.

Remarkably, for all values of $\theta$, the periodic cyclic cohomology is finite dimensional and is given by

$$HP^0(A_\theta) = \mathbb{C}^2, \quad HP^1(A_\theta) = \mathbb{C}^2.$$
An explicit basis for these groups are given by cyclic 1-cocycles
\[ \varphi_1(a_0, a_1) = \tau(a_0 \delta_1(a_1)), \quad \text{and} \quad \varphi_1(a_0, a_1) = \tau(a_0 \delta_2(a_1)) \]
and cyclic 2-cocycles
\[ \varphi(a_0, a_1, a_2) = \tau(a_0 \delta_1(a_1) \delta_2(a_2) - \delta_2(a_1) \delta_1(a_2)), \quad \text{and} \quad S\tau, \]
where \( \delta_1, \delta_2 : \mathcal{A}_\theta \to \mathcal{A}_\theta \) are the canonical derivations defined by
\[ \delta_1(\sum a_{mn} U^m V^n) = \sum ma_{mn} U^m V^n, \quad \delta_2(U^m V^n) = \sum n a_{mn} U^m V^n, \]
and \( \tau : \mathcal{A}_\theta \to \mathbb{C} \) is the canonical trace.

A noncommutative generalization of formulas like (13) was introduced in [9] and played an important role in the development of cyclic cohomology theory in general. It gives a geometric meaning to the notion of a cyclic cocycle over an algebra and goes as follows. Let \( (\Omega, d) \) be a differential graded algebra. A closed graded trace of dimension \( n \) on \( (\Omega, d) \) is a linear map
\[ \int : \Omega^n \to \mathbb{C} \]
such that
\[ \int d\omega = 0, \quad \text{and} \quad \int [\omega_1, \omega_2] = 0, \]
for all \( \omega \) in \( \Omega^{n-1} \), \( \omega_1 \) in \( \Omega^i \), \( \omega_2 \) in \( \Omega^j \) and \( i + j = n \). An \( n \) dimensional cycle over an algebra \( \mathcal{A} \) is a triple \( (\Omega, f, \rho) \), where \( f \) is an \( n \)-dimensional closed graded trace on \( (\Omega, d) \) and \( \rho : \mathcal{A} \to \Omega_0 \) is an algebra homomorphism. Given a cycle \( (\Omega, f, \rho) \) over \( \mathcal{A} \), its character is the cyclic \( n \)-cocycle on \( \mathcal{A} \) defined by
\[ \varphi(a_0, a_1, \ldots, a_n) = \int \rho(a_0) d\rho(a_1) \cdots d\rho(a_n). \]
Conversely one shows that all cyclic cocycles are obtained in this way.

Once one has the definition of cyclic cohomology, it is not difficult to formulate a dual notion of cyclic homology and a pairing between the two. Let \( C_n(\mathcal{A}) = \mathcal{A}^\otimes (n+1). \) The analogues of the operators \( b, b' \) and \( \lambda \) are easily defined on \( C_*(\mathcal{A}) \) and are usually denoted by the same letters, as we do here. For example \( b : C_n(\mathcal{A}) \to C_{n-1}(\mathcal{A}) \) is defined by
\[ b(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \]
\[ + (-1)^n (a_n a_0 \otimes \cdots \otimes a_{n-1}). \]
Let
\[ C_n^\lambda(\mathcal{A}) := C_n(\mathcal{A}) / \text{Im}(1 - \lambda). \]
The relation \( (1 - \lambda)b' = b(1 - \lambda) \) shows that the operator \( b \) is well defined on \( C_n^\lambda(\mathcal{A}) \). The complex \( (C_n^\lambda(\mathcal{A}), b) \) is called the homological cyclic complex of \( \mathcal{A} \) and its homology, denoted by \( HC_n^\lambda(\mathcal{A}), n = 0, 1, \ldots, \) is the cyclic homology of \( \mathcal{A} \). The evaluation map \( \langle \varphi, (a_0 \otimes \cdots \otimes a_n) \rangle \to \varphi(a_0, \ldots, a_n) \) clearly defines a degree zero pairing \( HC^*[\mathcal{A}] \otimes HC_*\mathcal{A} \to \mathbb{C} \). Many results of cyclic cohomology theory, such as Connes' long exact sequence and spectral sequence, and Morita invariance, continue to hold for cyclic homology theory with basically the same proofs.

Another important idea of Connes in the 1980's was the introduction of entire cyclic cohomology of Banach algebras [10]. This allows one to deal with algebras of
functions on infinite dimensional (noncommutative) spaces such as those appearing
in constructive quantum field theory. These algebras typically don’t carry finitely
summable Fredholm modules, but in some cases have so-called $\theta$-summable Fred-
holm modules. In [10] Connes extends the definition of Chern character to such
Fredholm modules with values in entire cyclic cohomology.

After the appearance of [9], cyclic (co)homology theory took on many lives
and was further developed along distinct lines, including a purely algebraic one,
with a big impact on algebraic $K$-theory. The cyclic cohomology of many algebras
was later computed including the very important case of group algebras [2] and
groupoid algebras. For many of these more algebraic aspects of the theory we refer
to [33, 32] and references therein.

3. From $K$-homology to cyclic cohomology

As I said in the introduction, Connes’ original motivation for the development
of cyclic cohomology was to give a receptacle for a noncommutative Chern character
map on the $K$-homology of noncommutative algebras. The cycles of $K$-homology
can be represented by, even or odd, Fredholm modules. Here we just focus on the
odd case, and we refer to [9, 13] for the even case. Given a Hilbert space $H$, let
$L(H)$ denote the algebra of bounded linear operators on $H$, and $K(H)$ denote the
algebra of compact operators. Also, for $1 \leq p < \infty$, let $L^p(H)$ denote the Schatten
ideal of $p$-summable operators. By definition, $T \in L^p(H)$ if $|T|^p$ is a trace class
operator.

**Definition 3.1.** An odd Fredholm module over a unital algebra $A$ is a pair $(H, F)$
where
1. $H$ is a Hilbert space endowed with a representation
   $\pi : A \rightarrow L(H),$
2. $F \in L(H)$ is a bounded selfadjoint operator with $F^2 = I$,
3. For all $a \in A$ we have
   \begin{equation}
   [F, \pi(a)] = F\pi(a) - \pi(a)F \in K(H).
   \end{equation}

A Fredholm module $(H, F)$ is called $p$-summable if, instead of (20), we have the stronger condition:
\begin{equation}
[F, \pi(a)] \in L^p(H)
\end{equation}
for all $a \in A$.

To give a simple example, let $A = C(S^1)$ be the algebra of continuous functions
on the circle and let $A$ act on $H = L^2(S^1)$ as multiplication operators. Let $F(e_n) = e_n$ if $n \geq 0$ and $F(e_n) = -e_n$ for $n < 0$, where $e_n(x) = e^{2\pi inx}, n \in \mathbb{Z}$, denotes the standard orthonormal basis of $H$. Clearly $F$ is selfadjoint and $F^2 = I$. To show that $[F, \pi(f)]$ is a compact operator for all $f \in C(S^1)$, notice that if $f = \sum_{|n| \leq N} a_n e_n$ is a finite trigonometric sum then $[F, \pi(f)]$ is a finite rank operator and hence is compact. In general we can uniformly approximate a continuous function by a trigonometric sum and show that the commutator is compact for any continuous $f$. This shows that $(H, F)$ is an odd Fredholm module over $C(S^1)$. This Fredholm module is not $p$-summable for any $1 \leq p < \infty$. If we restrict it to the subalgebra $C^\infty(S^1)$ of smooth functions, then it can be checked that $(H, F)$ is in fact $p$-summable for all $p > 1$, but is not 1-summable even in this case.
Now let me describe Connes’ noncommutative Chern character from $K$-homology to cyclic cohomology. Let $(\mathcal{H}, F)$ be an odd $p$-summable Fredholm module over an algebra $\mathcal{A}$. For any odd integer $2n - 1$ such that $2n \geq p$, Connes defines a cyclic $(2n - 1)$-cocycle $\varphi_{2n-1}$ on $\mathcal{A}$ by \cite{9}

\begin{equation}
\varphi_{2n-1}(a_0, a_1, \ldots, a_{2n-1}) = \text{Tr} (F[F, a_0][F, a_1] \cdots [F, a_{2n-1}]),
\end{equation}

where $\text{Tr}$ denotes the operator trace and instead of $\pi(a)$ we simply write $a$. Notice that by our $p$-summability assumption, each commutator is in $L^p(\mathcal{H})$ and hence, by Hölder inequality for Schatten class operators, their product is in fact a trace class operator as soon as $2n \geq p$. One checks by a direct computation that $\varphi_{2n-1}$ is a cyclic cocycle.

The next proposition shows that these cyclic cocycles are related to each other via the periodicity $S$-operator of cyclic cohomology. This is probably how Connes came across the periodicity operator $S$ in the first place.

**Proposition 3.1.** For all $n$ with $2n \geq p$ we have

$$S \varphi_{2n-1} = -(n + \frac{1}{2}) \varphi_{2n+1}.$$  

By rescaling $\varphi_{2n-1}$'s, one obtains a well defined element in the periodic cyclic cohomology. The (unstable) odd Connes-Chern character $\text{Ch}^{2n-1} = \text{Ch}^{2n-1}(\mathcal{H}, F)$ of an odd finitely summable Fredholm module $(\mathcal{H}, F)$ over $\mathcal{A}$ is defined by rescaling the cocycles $\varphi_{2n-1}$ appropriately. Let

$$\text{Ch}^{2n-1}(a_0, \ldots, a_{2n-1}) := (-1)^n 2(n - \frac{1}{2}) \cdots \frac{1}{2} \text{Tr} (F[F, a_0][F, a_1] \cdots [F, a_{2n-1}]).$$

**Definition 3.2.** The Connes-Chern character of an odd $p$-summable Fredholm module $(\mathcal{H}, F)$ over an algebra $\mathcal{A}$ is the class of the cyclic cocycle $\text{Ch}^{2n-1}$ in the odd periodic cyclic cohomology group $H^{p\text{odd}}(\mathcal{A})$.

By the above Proposition, the class of $\text{Ch}^{2n-1}$ in $H^{p\text{odd}}(\mathcal{A})$ is independent of the choice of $n$.

Let us compute the character of the Fredholm module of the above Example with $\mathcal{A} = C^\infty(S^1)$. By the above definition, $\text{Ch}^{1}(\mathcal{H}, F) = [\varphi_1]$ is the class of the following cyclic 1-cocycle in $H^{p\text{odd}}(\mathcal{A})$:

$$\varphi_1(f_0, f_1) = \text{Tr} (F[F, f_0][F, f_1]).$$

One can identify this cyclic cocycle with a local formula. We claim that

$$\varphi_1(f_0, f_1) = \frac{4}{2\pi i} \int f_0 df_1, \quad \text{for all } f_0, f_1 \in \mathcal{A}.$$  

By linearity, it suffices to check this relation for basis elements $f_0 = e_m, f_1 = e_n$ for all $m, n \in \mathbb{Z}$, which is easy to do.

The duality, that is, the bilinear pairing, between $K$-theory and $K$-homology is defined through the Fredholm index. More precisely there is an index pairing between odd (resp. even) Fredholm modules over $\mathcal{A}$ and the algebraic $K$-theory group $K_1^{\text{alg}}(\mathcal{A})$ (resp. $K_0(\mathcal{A})$). We shall describe it only in the odd case at hand. Let $(\mathcal{H}, F)$ be an odd Fredholm module over $\mathcal{A}$ and let $U \in \mathcal{A}^\times$ be an invertible element in $\mathcal{A}$. Let $P = \frac{F + U}{2} : \mathcal{H} \to \mathcal{H}$ be the projection operator defined by $F$. One checks that the operator

$$PUP : P\mathcal{H} \to P\mathcal{H}$$
is a Fredholm operator. In fact, using the compactness of commutators \([F,a]\), one checks that \(PU^{-1}P\) is an inverse for \(PUP\) modulo compact operators, which of course implies that \(PUP\) is a Fredholm operator. The index pairing is then defined as
\[
\langle (\mathcal{H}, F), [U] \rangle := \text{index} (PUP),
\]
where the index on the right hand side is the Fredholm index. If the invertible \(U\) happens to be in \(M_n(A)\) we can apply this definition to the algebra \(M_n(A)\) by noticing that \((\mathcal{H} \otimes \mathbb{C}^n, F \otimes 1)\) is a Fredholm module over \(M_n(A)\) in a natural way. The resulting map can be shown to induce a well defined additive map
\[
\langle (\mathcal{H}, F), - \rangle : K_1^\text{alg}(A) \to \mathbb{C}.
\]
Notice that this map is purely topological in the sense that to define it we did not have to impose any finite summability, i.e., smoothness, condition on the Fredholm module.

Going back to our example and choosing \(f : S^1 \to GL_1(\mathbb{C})\) a continuous function on \(S^1\) representing an element of \(K_1^\text{alg}(C(S^1))\), the index pairing \(\langle ([\mathcal{H}, F]), [f] \rangle = \text{index} (PfP)\) can be explicitly calculated. In fact in this case a simple homotopy argument gives the index of the Toeplitz operator \(PfP : PH \to PH\) in terms of the winding number of \(f\) around the origin:
\[
\langle ([\mathcal{H}, F]), [f] \rangle = -W(f,0).
\]
Of course, when \(f\) is smooth the winding number can be computed by integrating the 1-form \(\frac{1}{2\pi i} \frac{dz}{z}\) over the curve defined by \(f\):
\[
W(f,0) = \frac{1}{2\pi i} \int f^{-1} df = \frac{1}{2\pi i} \varphi(f^{-1}, f)
\]
where \(\varphi\) is the cyclic 1-cocycle on \(C^\infty(S^1)\) defined by \(\varphi(f,g) = \int f dg\). This is a special case of a very general index formula proved by Connes \cite{9} in a fully noncommutative situation:

**Proposition 3.2.** Let \((\mathcal{H}, F)\) be an odd \(p\)-summable Fredholm module over an algebra \(A\) and let \(2n - 1\) be an odd integer such that \(2n \geq p\). If \(u\) is an invertible element in \(A\) then
\[
\text{index} (PuP) = \frac{(-1)^n}{2^{2n}} \varphi_{2n-1}(u^{-1}, u, \ldots, u^{-1}, u),
\]
where the cyclic cocycle \(\varphi_{2n-1}\) is defined by
\[
\varphi_{2n-1}(a_0, a_1, \ldots, a_{2n-1}) = \text{Tr} (F[F,a_0][F,a_1] \cdots [F,a_{2n-1}]).
\]

The above index formula can be expressed in a more conceptual manner once Connes' Chern character in \(K\)-theory is introduced. In \cite{4, 9}, Connes shows that the Chern-Weil definition of Chern character on topological \(K\)-theory admits a vast generalization to a noncommutative setting. For a noncommutative algebra \(A\) and each integer \(n \geq 0\), he defined pairings between cyclic cohomology and \(K\)-theory:
\[
HC^{2n}(A) \otimes K_0(A) \to \mathbb{C}, \quad HC^{2n+1}(A) \otimes K_1^\text{alg}(A) \to \mathbb{C}.
\]
These pairings are compatible with the periodicity operator \(S\) in cyclic cohomology in the sense that
\[
\langle [\varphi], [e] \rangle = \langle S[\varphi], [e] \rangle,
\]
for all cyclic cocycles $\varphi$ and $K$-theory classes $[e]$, and thus induce a pairing

$$HP^i(A) \otimes K_i^{alg}(A) \rightarrow \mathbb{C}, \quad i = 0, 1$$

between periodic cyclic cohomology and $K$-theory.

We briefly recall its definition. Let $\varphi$ be a cyclic $2n$-cocycle on $A$ and let $e \in M_k(A)$ be an idempotent representing a class in $K_0(A)$. The pairing $HC^{2n}(A) \otimes K_0(A) \rightarrow \mathbb{C}$ is defined by

$$(24) \quad \langle [\varphi], [e] \rangle = (n!)^{-1} \tilde{\varphi}(e, \ldots, e),$$

where $\tilde{\varphi}$ is the ‘extension’ of $\varphi$ to $M_k(A)$ defined by the formula

$$(25) \quad \tilde{\varphi}(m_0 \otimes a_0, \ldots, m_{2n} \otimes a_{2n}) = \text{tr}(m_0 \cdots m_{2n}) \varphi(a_0, \ldots, a_{2n}).$$

It can be shown that $\tilde{\varphi}$ is a cyclic $n$-cocycle as well.

The formulas in the odd case are as follows. Given an invertible matrix $u \in M_k(A)$, representing a class in $K_1^{alg}(A)$, and an odd cyclic $(2n-1)$-cocycle $\varphi$ on $A$, the pairing is given by

$$(26) \quad \langle [\varphi], [u] \rangle := 2^{-2n+1} \left( \frac{1}{n-\frac{1}{2}} \right) \tilde{\varphi}(u^{-1} - 1, u - 1, \ldots, u^{-1} - 1, u - 1).$$

Any cyclic cocycle can be represented by a normalized cocycle for which $\varphi(a_0, \ldots, a_n) = 0$ if $a_i = 1$ for some $i$. When $\varphi$ is normalized, formula (26) reduces to a particularly simple form:

$$(27) \quad \langle [\varphi], [u] \rangle = 2^{-2n+1} \left( \frac{1}{n-\frac{1}{2}} \right) \tilde{\varphi}(u^{-1}, u, \ldots, u^{-1}, u).$$

Using the pairing $HC^{2n-1}(A) \otimes K_1^{alg}(A) \rightarrow \mathbb{C}$ and the definition of $\text{Ch}^{2n-1}(H, F)$, the above index formula in Proposition (3.2) can be written as

$$(28) \quad \text{index}(PuP) = \langle \text{Ch}^{2n-1}(H, F), [u] \rangle,$$

or in its stable form

$$\text{index}(PuP) = \langle \text{Ch}^{\text{odd}}(H, F), [u] \rangle.$$
the standard abelian homological algebra is not applicable here. Let $k$ be a unital commutative ring. In [7], an abelian category $\Lambda_k$ of cyclic $k$-modules is defined that can be thought of as the ‘abelianization’ of the category of $k$-algebras. Cyclic cohomology is then shown to be the derived functor of the functor of traces, as we shall explain in this section. More generally Connes defined the notion of a cyclic object in an abelian category and its cyclic cohomology [7].

Later developments proved that this extension of cyclic cohomology was of great significance. Apart from earlier applications, we should mention the recent work [16] where the abelian category of cyclic modules plays a role similar to that of the category of motives for noncommutative geometry. Another recent example is the cyclic cohomology of Hopf algebras [20, 21, 30, 31], which cannot be defined as the cyclic cohomology of an algebra or a coalgebra but only as the cyclic cohomology of a cyclic module naturally attached to the given Hopf algebra and a coefficient system (see the last section for more on Hopf cyclic cohomology). Let us briefly sketch the definition of the cyclic category $\Lambda$.

Recall that the simplicial category $\Delta$ is a small category whose objects are the totally ordered sets

$$[n] = \{0 < 1 < \cdots < n\}, \quad n = 0, 1, 2, \ldots,$$

and whose morphisms $f : [n] \to [m]$ are order preserving, i.e. monotone non-decreasing, maps $f : \{0, 1, \ldots, n\} \to \{0, 1, \ldots, m\}$. Of particular interest among the morphisms of $\Delta$ are faces $\delta_i$ and degeneracies $\sigma_j$,

$$\delta_i : [n - 1] \to [n], \quad i = 0, 1, \ldots, n,$$

$$\sigma_j : [n + 1] \to [n], \quad j = 0, 1, \ldots, n.$$

By definition $\delta_i$ is the unique injective morphism missing $i$ and $\sigma_j$ is the unique surjective morphism identifying $j$ with $j + 1$.

The cyclic category $\Lambda$ has the same set of objects as $\Delta$ and in fact contains $\Delta$ as a subcategory. Morphisms of $\Lambda$ are generated by simplicial morphisms and new morphisms $\tau_n : [n] \to [n], n \geq 0$, defined by $\tau_n(i) = i + 1$ for $0 \leq i < n$ and $\tau_n(n) = 0$. We have the following extra relations:

$$\tau_n \delta_i = \delta_{i-1} \tau_{n-1}, \quad \tau_n \delta_0 = \delta_n,$$

$$\tau_n \sigma_i = \sigma_{i+1} \tau_{n+1}, \quad \tau_n \sigma_0 = \sigma_n \tau_{n+1}^2,$$

$$\tau_{n+1} = id.$$

It can be shown that the classifying space $B\Lambda$ of the small category $\Lambda$ is homotopy equivalent to the classifying space of the circle $S^1$ [7].

A cyclic object in a category $C$ is a functor $\Lambda^{op} \to C$. A cocyclic object in $C$ is a functor $\Lambda \to C$. For any commutative unital ring $k$, we denote the category of cyclic $k$-modules by $\Lambda_k$. A morphism of cyclic $k$-modules is a natural transformation between the corresponding functors. It is clear that $\Lambda_k$ is an abelian category. More generally, if $\mathcal{A}$ is an abelian category then the category $\Lambda \mathcal{A}$ of cyclic objects in $\mathcal{A}$ is itself an abelian category.

Let $\text{Alg}_k$ denote the category of unital $k$-algebras and unital algebra homomorphisms. There is a functor

$$\sharp : \text{Alg}_k \to \Lambda_k, \quad A \mapsto A^\sharp,$$

defined by

$$A_n^\sharp = A^\otimes (n+1), \quad n \geq 0,$$
with face, degeneracy and cyclic operators given by
\[
\delta_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_ia_{i+1} \otimes \cdots \otimes a_n,
\]
\[
\delta_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_na_0 \otimes a_1 \otimes \cdots \otimes a_{n-1},
\]
\[
\sigma_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes \cdots \otimes a_n,
\]
\[
\tau_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}.
\]
A unital algebra map \( f : A \to B \) induces a morphism of cyclic modules \( f^\# : A^\# \to B^\# \) by
\[
f^\#(a_0 \otimes \cdots \otimes a_n) = (f(a_0) \otimes \cdots \otimes f(a_n)).
\]

This functor \( \# \) embeds the non-additive category of \( k \)-algebras into the abelian category of cyclic \( k \)-modules. A first main observation of [7] is that
\[
\text{Hom}_{A_k}(A^\#, k^\#) \simeq T(A),
\]
where \( T(A) \) is the space of traces from \( A \to k \). To a trace \( \tau \) one associate the cyclic map \( (f_n)_{n \geq 0} \), where
\[
f_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \tau(a_0a_1\cdots a_n), \quad n \geq 0.
\]
It can be easily shown that this defines a one to one correspondence.

Now we can state the following fundamental theorem of Connes [7] which greatly extends the above observation and shows that cyclic cohomology is a derived functor, in fact an Ext functor, provided that we work in the category of cyclic modules:

**Theorem 4.1.** Let \( k \) be a field of characteristic zero. For any unital \( k \)-algebra \( A \), there is a canonical isomorphism
\[
HC^n(A) \simeq \text{Ext}^n_{A_k}(A^\#, k^\#), \quad \text{for all } n \geq 0.
\]

Apart from their applications in the study of cyclic cohomology of algebras and Hopf algebras (about the latter see the next section), cyclic modules have also come to play an important role in applications of noncommutative geometry to number theory. They play a role similar to that of motives in algebraic geometry. Let me briefly explain this point.

The program outlined by Connes, Consani and Marcolli in [16] aims at creating an environment where something like Weil’s proof of the Riemann hypothesis for function fields can be repeated in the characteristic zero case. Among other things, they produce an analogue of the Frobenius automorphism in characteristic zero in this paper. Since Connes’ trace formula is over the noncommutative ad`eles class space [14], the geometric setting is that of noncommutative geometry and they must go far beyond what is done so far in noncommutative geometry and import many ideas from modern algebraic geometry to noncommutative geometry. To achieve this, as a first step, good analogues of étale cohomology, the category of motives, and correspondences in noncommutative geometry must be introduced. Happily it turns out that Connes’ category of cyclic modules and the closely related bivariant cyclic homology, as well as \( KK \)-theory, are quite useful in this regard.

The construction of the Frobenius in characteristic zero follows a very general process that combines cyclic homology with quantum statistical mechanics in a novel way. Starting from a pair \((A, \varphi)\) of an algebra and a state \( \varphi \) (a noncommutative space endowed with a ‘probability measure’), they proceed by invoking the canonical one-parameter group of automorphisms \( \sigma = \sigma_\varphi \) and consider the extremal
equilibrium states $\Sigma_\beta$ at inverse temperatures $\beta > 1$. Under suitable conditions, there is an algebra map

$$\rho : A \ltimes_{\sigma} \mathbb{R} \to S(\Sigma_\beta \times \mathbb{R}_+^* \otimes \mathcal{L}),$$

where $\mathcal{L}$ denotes the algebra of trace class operators. The cyclic module $D(A, \varphi)$ is defined as the cokernel of the induced map by $\text{Tr} \circ \rho$ on the cyclic modules of these two algebras. The dual multiplicative group $\mathbb{R}_+^*$ acts on $D(A, \varphi)$ and, in examples coming from number theory, replaces Frobenius in characteristic zero. The three steps involved in the construction of $D(A, \varphi)$ are called cooling, distillation, and dual action in the paper.

A remarkable property of the cyclic category $\Lambda$, not shared by the simplicial category, is its self-duality in the sense that there is a natural isomorphism of categories $\Lambda \cong \Lambda^{\text{op}}$. Roughly speaking, the duality functor $\Lambda^{\text{op}} \to \Lambda$ acts as the identity on objects of $\Lambda$ and exchanges face and degeneracy operators while sending the cyclic operator to its inverse. Thus to a cyclic (resp. cocyclic) module one can associate a cocyclic (resp. cyclic) module by applying the duality isomorphism. This duality plays an important role in Hopf cyclic cohomology.

5. The local index formula and beyond

In practice, computing Connes-Chern characters defined by formulas like (22) is rather difficult since they involve the ordinary operator trace and are non-local. Thus one needs to compute the class of this cyclic cocycle by a local formula. This is rather similar to passing from the McKean-Singer formula for the index of an elliptic operator to a local cohomological formula involving integrating a locally defined differential form, i.e., the Atiyah-Singer index formula. The solution of this problem was arrived at in two stages. First, in [13], Connes gave a partial answer by giving a local formula for the Hochschild class of the Chern character, and then Connes and Moscovici gave a formula that captures the full cyclic cohomology class of the character by a local formula [19]. Broadly speaking, the ideas involved amount to going from noncommutative differential topology to noncommutative spectral geometry, and need the introduction of two new concepts.

In the first place, a noncommutative analogue of integration was found by Connes by replacing the operator trace by the Dixmier trace [11], and, secondly, one refines the topological notion of Fredholm module by the metric notion of a spectral triple, or $K$-cycles as they were originally named in [13]. Developing the necessary tools to handle this local index formula, shaped, more or less, the second stage of the development of noncommutative geometry after the appearance of the landmark papers [9]. One can say that while in its first stage noncommutative geometry was influenced by differential and algebraic topology, especially index theory, the Novikov conjecture and the Baum-Connes conjecture, in this second stage it was chiefly informed by spectral geometry.

We start with a quick review of the Dixmier trace and the noncommutative integral, following [13] closely. For a compact operator $T$, let $\mu_n(T), n = 1, 2, \ldots$, denote the sequence of eigenvalues of $|T| = (T^*T)^{1/2}$ written in decreasing order. Thus, by the minimax principle, $\mu_1(T) = ||T||$, and in general

$$\mu_n(T) = \inf \ ||T|_V||, \quad n \geq 1,$$

where the infimum is over the set of subspaces of codimension $n - 1$, and $T|_V$ denotes the restriction of $T$ to the subspace $V$. The natural domain of the Dixmier
trace is the set of operators
\[ \mathcal{L}^{1,\infty}(\mathcal{H}) := \{ T \in \mathcal{K}(\mathcal{H}); \sum_{1}^{N} \mu_n(T) = O(\log N) \}. \]

Notice that trace class operators are automatically in \( \mathcal{L}^{1,\infty}(\mathcal{H}) \). The Dixmier trace of an operator \( T \in \mathcal{L}^{1,\infty}(\mathcal{H}) \) measures the logarithmic divergence of its ordinary trace. More precisely, we are interested in taking some kind of limit of the bounded sequence
\[
\sigma_N(T) = \frac{\sum_{1}^{N} \mu_n(T)}{\log N}
\]
as \( N \to \infty \). The problem of course is that, while by our assumption the sequence is bounded, the usual limit may not exists and must be replaced by a carefully chosen 'generalized limit'.

To this end, let \( \text{Trace}_\Lambda(T), \Lambda \in [1, \infty), \) be the piecewise affine interpolation of the partial trace function \( \text{Trace}_\Lambda(T) = \sum_{1}^{N} \mu_n(T) \). Recall that a state on a \( C^* \)-algebra is a non-zero positive linear functional on the algebra. Let \( \omega : C_b(\mathbb{R}, \infty) \to \mathbb{C} \) be a normalized state on the algebra of bounded continuous functions on \( [\varepsilon, \infty) \) such that \( \omega(f) = 0 \) for all \( f \) vanishing at \( \infty \). Now, using \( \omega \), the Dixmier trace of a positive operator \( T \in \mathcal{L}^{1,\infty}(\mathcal{H}) \) is defined as
\[
\text{Tr}_\omega(T) := \omega(\tau_\Lambda(T)),
\]
where
\[
\tau_\Lambda(T) = \frac{1}{\log \Lambda} \int_{\varepsilon}^{\Lambda} \frac{\text{Trace}_r(T)}{\log r} \, dr
\]
is the Cesàro mean of the function \( \frac{\text{Trace}_r(T)}{\log r} \) over the multiplicative group \( \mathbb{R}_+^* \). One then extends \( \text{Tr}_\omega \) to all of \( \mathcal{L}^{1,\infty}(\mathcal{H}) \) by linearity.

The resulting linear functional \( \text{Tr}_\omega \) is a positive trace on \( \mathcal{L}^{1,\infty}(\mathcal{H}) \). It is easy to see from its definition that if \( T \) actually happens to be a trace class operator then \( \text{Tr}_\omega(T) = 0 \) for all \( \omega \), i.e., the Dixmier trace is invariant under perturbations by trace class operators. This is a very useful property and makes \( \text{Tr}_\omega \) a flexible tool in computations. The Dixmier trace, \( \text{Tr}_\omega \), in general depends on the limiting procedure \( \omega \); however, for the class of operators \( T \) for which \( \lim_{\Lambda \to \infty} \tau_\Lambda(T) \) exits, it is independent of the choice of \( \omega \) and is equal to \( \lim_{\Lambda \to \infty} \tau_\Lambda(T) \). One of the main results proved in [11] is that if \( M \) is a closed \( n \)-dimensional manifold, \( E \) is a smooth vector bundle on \( M \), \( P \) is a pseudodifferential operator of order \( -n \) acting between \( L^2 \)-sections of \( E \), and \( H = L^2(M, E) \), then \( P \in \mathcal{L}^{1,\infty}(\mathcal{H}) \) and, for any choice of \( \omega \), \( \text{Tr}_\omega(P) = n^{-1} \text{Res}(P) \). Here \( \text{Res} \) denotes Wodzicki's noncommutative residue. For example, if \( D \) is an elliptic first order differential operator, \( |D|^{-n} \) is a pseudodifferential operator of order \( -n \) and, for any bounded operator \( a \), the Dixmier trace \( \text{Tr}_\omega(a|D|^{-n}) \) is independent of the choice of \( \omega \).

The second ingredient of the local index formula is the notion of spectral triple [13]. Spectral triples provide a refinement of Fredholm modules. Going from Fredholm modules to spectral triples is similar to going from the conformal class of a Riemannian metric to the metric itself. Spectral triples simultaneously provide a notion of Dirac operator in noncommutative geometry, as well as a Riemannian type distance function for noncommutative spaces.

To motivate the definition of a spectral triple, we recall that the Dirac operator \( \slashed{D} \) on a compact Riemannian Spin\(^c \) manifold acts as an unbounded selfadjoint
operator on the Hilbert space $L^2(M, S)$ of $L^2$-spinors on the manifold $M$. If we let $C^\infty(M)$ act on $L^2(M, S)$ by multiplication operators, then one can check that for any smooth function $f$, the commutator $[D, f] = Df - fD$ extends to a bounded operator on $L^2(M, S)$. Now the geodesic distance $d$ on $M$ can be recovered from the following distance formula of Connes [13]:

$$d(p, q) = \text{Sup}\{|f(p) - f(q)|; \|D, f\| \leq 1\}, \quad \forall p, q \in M.$$ 

The triple $(C^\infty(M), L^2(M, S), \mathcal{B})$ is a commutative example of a spectral triple. Its general definition, in the odd case, is as follows. This definition should be compared with Definition (3.1).

**Definition 5.1.** Let $A$ be a unital algebra. An odd spectral triple on $A$ is a triple $(A, \mathcal{H}, D)$ consisting of a Hilbert space $\mathcal{H}$, a selfadjoint unbounded operator $D : \text{Dom}(D) \subset \mathcal{H} \to \mathcal{H}$ with compact resolvent, i.e., $(D + \lambda)^{-1} \in \mathcal{K}(\mathcal{H})$, for all $\lambda \notin \mathbb{R}$, and a representation $\pi : A \to \mathcal{L}(\mathcal{H})$ of $A$ such that for all $a \in A$, the commutator $[D, \pi(a)]$ is defined on $\text{Dom}(D)$ and extends to a bounded operator on $\mathcal{H}$.

The finite summability assumption (21) for Fredholm modules has a finer analogue for spectral triples. For simplicity we shall assume that $D$ is invertible (in general, since $\text{Ker} D$ is finite dimensional, by restricting to its orthogonal complement we can always reduce to this case). A spectral triple is called finitely summable if for some $n \geq 1$

$$(29) \quad |D|^{-n} \in L^{1, \infty}(\mathcal{H}).$$

A simple example of an odd spectral triple is $(C^\infty(S^1), L^2(S^1), D)$, where $D$ is the unique selfadjoint extension of the operator $-i \frac{d}{dx}$. Eigenvalues of $|D|$ are $|n|, n \in \mathbb{Z}$, which shows that, if we restrict $D$ to the orthogonal complement of constant functions, then $|D|^{-1} \in L^{1, \infty}(L^2(S^1))$.

Given a spectral triple $(A, \mathcal{H}, D)$, one obtains a Fredholm module $(A, \mathcal{H}, F)$ by choosing $F = \text{Sign}(D) = D|D|^{-1}$. Connes’ Hochschild character formula gives a local expression for the Hochschild class of the Connes-Chern character of $(A, \mathcal{H}, F)$ in terms of $D$ itself. For this one has to assume that the spectral triple $(A, \mathcal{H}, D)$ is regular in the sense that for all $a \in A$,

$$a \text{ and } [D, a] \in \cap \text{Dom}(\delta^k)$$

where the derivation $\delta$ is given by $\delta(x) = |D||x|.$

Now, assuming (29) holds, Connes defines an $(n + 1)$—linear functional $\varphi$ on $A$ by

$$\varphi(a^0, a^1, \ldots, a^n) = \text{Tr}_\omega(a^0[D, a^1] \cdots [D, a^n]|D|^{-n}).$$

It can be shown that $\varphi$ is a Hochschild $n$-cocycle on $A$. We recall that a Hochschild $n$-cycle $c \in Z_n(A, A)$ is an element $c = \sum a^0 \otimes a^1 \otimes \cdots \otimes a^n \in A^{\otimes(n+1)}$ such that its Hochschild boundary $b(c) = 0$, where $b$ is defined by (13). The following result, known as Connes’ Hochschild character formula, computes the Hochschild class of the Chern character by a local formula, i.e., in terms of $\varphi$:

**Theorem 5.1.** Let $(A, \mathcal{H}, D)$ be a regular spectral triple. Let $F = \text{Sign}(D)$ denote the sign of $D$ and $\tau_n \in HC^n(A)$ denote the Connes-Chern character of $(\mathcal{H}, F)$. For every $n$-dimensional Hochschild cycle $c = \sum a^0 \otimes a^1 \otimes \cdots \otimes a^n \in Z_n(A, A)$, one has

$$\langle \tau_n, c \rangle = \sum \varphi(a^0, a^1, \ldots, a^n).$$
Identifying the full cyclic cohomology class of the Connes-Chern character of 
\((A, H, D)\) by a local formula is the content of Connes-Moscovici’s local index for-
mula. For this we have to assume the spectral triple satisfies another technical 
condition. Let \(B\) denote the subalgebra of \(L(H)\) generated by operators \(\delta^k(a)\) and 
\(\delta^k([D, a]), k \geq 1\). A spectral triple is said to have a discrete \textit{dimension spectrum} \(\Sigma\) 
if \(\Sigma \subset \mathbb{C}\) is discrete and for any \(b \in B\) the function 
\[
g_b(z) = \text{Trace}(b|D|^{-z}), \quad \Re z > n,
\]
extends to a holomorphic function on \(\mathbb{C} \setminus \Sigma\). It is further assumed that \(\Sigma\) is simple 
in the sense that \(g_b(z)\) has only simple poles in \(\Sigma\).

The local index formula of Connes and Moscovici \cite{ConnesMoscovici} is given by the following 
Theorem (we have used the formulation in \cite{ConnesMoscovici15}):

**Theorem 5.2.** 1. The equality

\[
\int P = \text{Res}_{z=0} \text{Trace}(P|D|^{-z})
\]
defines a trace on the algebra generated by \(A, [D, A],\text{ and } |D|^z, z \in \mathbb{C}\).

2. There are only a finite number of non-zero terms in the following formula which 
defines the odd components \((\varphi_n)_{n=1,3,...}\) of an odd cyclic cocycle in the 
(b, B) bi-complex of \(A\): For each odd integer \(n\) let

\[
\varphi_n(a^0, \ldots, a^n) := \sum_k c_{n,k} \int a^0[D, a^1]^{(k_1)} \cdots [D, a^n]^{(k_n)}|D|^{-n-2|k|}
\]
where \(T^{(k)} := \nabla^k\) and \(\nabla(T) = D^2T - TD^2\), \(k\) is a multi-index, \(|k| = k_1 + \cdots + k_n\) and

\[
c_{n,k} := (-1)^{|k|} \sqrt{2\pi}! (k_1! \cdots k_n!)^{-1} ((k_1 + 1) \cdots (k_1 + k_2 + \cdots k_n))^{-1} \Gamma(|k| + \frac{n}{2}).
\]

3. The pairing of the cyclic cohomology class \((\varphi_n) \in HC^*(A)\) with \(K_1(A)\) gives the 
Fredholm index of \(D\) with coefficients in \(K_1(A)\).

As is indicated in part 1) of the above Theorem, a regular spectral triple neces-
sarily defines a trace on its underlying algebra by the formula \(a \in A \mapsto 
\text{Res}_{z=0} \text{Trace}(a|D|^{-z})\). Thus, to deal with ‘type III algebras’ which carry no non-
trivial traces, the notion of spectral triple must be modified. In \cite{ConnesMoscovici25} Connes 
and Moscovici define a notion of \textit{twisted spectral triple}, where the twist is afforded by 
an algebra automorphism (related to the modular automorphism group). More 
precisely, one postulates that there exists an automorphism \(\sigma\) of \(A\) such that the 
twisted commutators

\[
[D, a]_\sigma := Da - \sigma(a)D
\]
are bounded operators for all \(a \in A\). They show that, in the twisted case, the 
Dixmier trace induces a twisted trace on the algebra \(A\), but surprisingly, under 
some regularity conditions, the Connes-Chern character of the phase space lands 
in ordinary cyclic cohomology. Thus its pairing with ordinary K-theory makes 
sense, and it can be recovered as the index of Fredholm operators. This suggests 
the significance of developing a local index formula for twisted spectral triples, \textit{i.e.} 
finding a formula for a cocycle, cohomologous to the Connes-Chern character in 
the \((b, B)\)-bicomplex, which is given in terms of twisted commutators and residue 
functionals. I believe that this new theme of twisted spectral triples, and type III 
noncommutative geometry in general, will dominate the subject in near future.
For example, very recently a local index formula has been proved for a class of twisted spectral triples by Henri Moscovici [34] that can be found in the present volume. This class is obtained by twisting an ordinary spectral triple \((A, H, D)\) by a subgroup \(G\) of conformal similarities of the triple, i.e. the set of all unitary operators \(U \in \mathcal{U}(H)\) such that \(UAA^* = A\), and \(UDU^* = \mu(U)D\), with \(\mu(U) > 0\). It is shown that the crossed product algebra \(A \rtimes G\) admits an automorphism \(\sigma\), given by the formula \(\sigma(aU) = \mu(U)^{-1}aU\), for all \(a \in A, U \in G\), and \((A \rtimes G, H, D)\) is a twisted spectral triple. The analogue of the noncommutative residue on the circle, for algebras of formal twisted pseudodifferential symbols, is constructed in [27].

A very recent development related to (twisted) spectral triples is the noncommutative Gauss-Bonnet theorem of Connes and Tretkoff for the noncommutative two-torus \(A_\theta\) [26]. In classical geometry a spectral zeta function is associated to the Laplacian \(\Delta_g = d^*d\) of a Riemann surface with metric \(g\):

\[
\zeta(s) = \sum_j \lambda_j^{-s}, \quad \text{Re}(s) > 1,
\]

where the \(\lambda_j\)’s are the nonzero eigenvalues of \(\Delta_g\). This zeta function has a meromorphic continuation with no pole at 0, and the Gauss-Bonnet theorem for surfaces can be expressed as

\[
\zeta(0) + \text{Card}\{j|\lambda_j = 0\} = \frac{1}{12\pi} \int_\Sigma R = \frac{1}{6} \chi(\Sigma),
\]

where \(R\) is the curvature and \(\chi(\Sigma)\) is the Euler-Poincaré characteristic.

It is this formulation of the Gauss-Bonnet theorem in spectral terms that admits a generalization to noncommutative geometry. Let \(A_\theta\) denote the \(C^*\)-algebra of the noncommutative torus with parameter \(\theta \in \mathbb{R} \setminus \mathbb{Q}\) and let \(\tau : A_\theta \to \mathbb{C}\) denote its faithful normalized trace. One can define an inner product

\[
\langle a, b \rangle = \tau(b^*a), \quad a, b \in A_\theta,
\]

and complete \(A_\theta\) with respect to this inner product to obtain a Hilbert space \(\mathcal{H}_\theta\). More generally, for any smooth selfadjoint element \(h = h^* \in A_\theta\) one defines an inner product \(\langle a, b \rangle_\varphi = \tau(b^*ae^{-h})\), where the positive linear functional \(\varphi = \varphi_h\) is defined by

\[
\varphi(a) = \tau(ae^{-h}), \quad a \in A_\theta.
\]

Let \(\mathcal{H}_\varphi\) denote the completion of \(A_\theta\) with respect to this conformally equivalent metric.

Using the canonical derivations \(\delta_1\) and \(\delta_2\) of \(A_\theta\), one can introduce a complex structure on \(A_\theta\) by defining

\[
\vartheta = \delta_1 + i\delta_2, \quad \vartheta^* = \delta_1 - i\delta_2.
\]

These operators can be considered as unbounded operators on \(\mathcal{H}_0\) and \(\vartheta^*\) is the adjoint of \(\vartheta\). Then the unperturbed Laplacian on \(A_\theta\) is given by

\[
\Delta = \vartheta^*\vartheta = \delta_1^2 + \delta_2^2.
\]

In general we can consider the unperturbed operator \(\vartheta = \delta_1 + i\delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}\), where \(\mathcal{H}^{(1,0)}\) is the completion of the linear span of elements of the form \(a \delta_1 b\) with \(a, b \in A_\theta^\infty\). Let \(\vartheta^*_\varphi\) denote its adjoint. Then the Laplacian for the conformally equivalent metric \(\langle a, b \rangle_\varphi\) is given by \(\Delta' = \vartheta^*_\varphi\vartheta\).
In \[26\], Connes and Tretkoff show that the value at 0 of the zeta function associated to this Laplacian \(\Delta'\) is an invariant of the conformal class of the metric on \(A_\theta\), i.e. of \(h\). A natural problem here is to extend this result by considering the most general complex structure on \(A_\theta\) of the form \(\partial = \delta_1 + \tau \delta_2\), where \(\tau\) is a complex number with \(\text{Im}(\tau) > 0\). This problem is now solved in full generality in \[28\].

6. Hopf cyclic cohomology

A major development in cyclic cohomology theory in the last ten years was the introduction of Hopf cyclic cohomology for Hopf algebras by Connes and Moscovici \[20\]. As we saw in Section 5, the local index formula gives the Connes-Chern character of a regular spectral triple \((A, \mathcal{H}, D)\) as a cyclic cocycle in the \((b, B)\)-bicomplex of the algebra \(A\). For spectral triples of interest in transverse geometry \[20\], this cocycle is differentiable in the sense that it is in the image of the Connes-Moscovici characteristic map \(\chi_\tau\) defined below \(31\), with \(H = H_1\) a Hopf algebra and \(A = A_\Gamma\), a noncommutative algebra, whose definitions we shall recall in this section. To identify this cyclic cocycle in terms of characteristic classes of foliations, they realized that it would be extremely helpful to show that it is the image of a polynomial in some universal cocycles for a cohomology theory for a universal Hopf algebra, and this gave birth to Hopf cyclic cohomology and to the universal Hopf algebra \(H = H_1\). This is similar to the situation for classical characteristic classes of manifolds, which are pullbacks of group cohomology classes.

The Connes-Moscovici characteristic map can be formulated in general terms as follows. Let \(H\) be a Hopf algebra acting as quantum symmetries of an algebra \(A\), i.e., \(A\) is a left \(H\)-module, and the algebra structure of \(A\) is compatible with the coalgebra structure of \(H\) in the sense that the multiplication \(A \times A \to A\) and the unit map \(\mathbb{C} \to A\) of \(A\) are morphisms of \(H\)-modules. A common terminology to describe this situation is to say that \(A\) is a left \(H\)-module algebra. Using Sweedler’s notation for the coproduct of \(H\), \(\Delta(h) = h(1) \otimes h(2)\) (summation is understood), this latter compatibility condition can be expressed as

\[ h(ab) = h^{(1)}(a)h^{(2)}(b), \quad \text{and} \quad h(1) = \varepsilon(h)1, \]

for all \(h \in H\) and \(a, b \in A\). In general one should think of such actions of Hopf algebras as the noncommutative geometry analogue of the action of differential operators on a manifold.

It is also important to extend the notion of trace to allow twisted traces, such as KMS states in quantum statistical mechanics, as well as the idea of invariance of a (twisted) trace. The general setting introduced in \[20\] is the following. Let \(\delta : H \to \mathbb{C}\) be a character of \(H\), i.e. a unital algebra map, and \(\sigma \in H\) be a grouplike element, i.e. it satisfies \(\Delta \sigma = \sigma \otimes \sigma\). A linear map \(\tau : A \to \mathbb{C}\) is called \(\delta\)-invariant if for all \(h \in H\) and \(a \in A\),

\[ \tau(h(a)) = \delta(h)\tau(a), \]

and is called a \(\sigma\)-trace if for all \(a, b\) in \(A\),

\[ \tau(ab) = \tau(b\sigma(a)). \]

Now for \(a, b \in A\), let

\[ \langle a, b \rangle := \tau(ab). \]
Let $\tau$ be a $\sigma$-trace on $\mathcal{A}$. Then $\tau$ is $\delta$-invariant if and only if the integration by parts formula holds. That is, for all $h \in H$ and $a, b \in \mathcal{A}$,

\begin{equation}
\langle h(a), b \rangle = \langle a, \tilde{S}_\delta(h)(b) \rangle.
\end{equation}

(30)

Here $S$ denotes the antipode of $H$ and the $\delta$-twisted antipode $\tilde{S}_\delta : H \to H$ is defined by $\tilde{S}_\delta = \delta * S$, i.e.

$$
\tilde{S}_\delta(h) = \delta(h^{(1)})S(h^{(2)}),
$$

for all $h \in H$. Loosely speaking, [20] says that the formal adjoint of the differential operator $h$ is $\tilde{S}_\delta(h)$. Following [20, 21], we say that $(\delta, \sigma)$ is a modular pair if $\delta(\sigma) = 1$, and a modular pair in involution if in addition we have

$$
\tilde{S}_\delta^2(h) = \sigma h \sigma^{-1},
$$

for all $h$ in $H$. The importance of this notion will become clear in the next paragraph.

Now, for each $n \geq 0$, the Connes-Moscovici characteristic map

\begin{equation}
\chi_\tau : H^\otimes n \longrightarrow C^n(\mathcal{A}),
\end{equation}

(31)

is defined by

$$
\chi_\tau(h_1 \otimes \cdots \otimes h_n)(a_0 \otimes \cdots \otimes a_n) = \tau(a_0 h_1(a_1) \cdots h_n(a_n)).
$$

Notice that the right hand side of (31) is the cocyclic module that (its cohomology) defines the cyclic cohomology of the algebra $\mathcal{A}$. The main question about (31) is whether one can promote the collection of linear spaces $\{H^\otimes n\}_{n \geq 0}$ to a cocyclic module such that the characteristic map $\chi_\tau$ turns into a morphism of cocyclic modules. We recall that the face, degeneracy, and cyclic operators for $\{C^n(\mathcal{A})\}_{n \geq 0}$ are defined by:

$$
\begin{align*}
\delta_i &\varphi(a_0, \ldots, a_{n+1}) = \varphi(a_0, \ldots, a_i a_{i+1}, \ldots, a_{n+1}), & i = 0, \ldots, n, \\
\delta_{n+1} &\varphi(a_0, \ldots, a_{n+1}) = \varphi(a_{n+1} a_0, a_1, \ldots, a_n), \\
\sigma_i &\varphi(a_0, \ldots, a_n) = \varphi(a_0, \ldots, a_i, 1, \ldots, a_n), & i = 0, \ldots, n, \\
\tau_n &\varphi(a_0, \ldots, a_n) = \varphi(a_n, a_0, \ldots, a_{n-1}).
\end{align*}
$$

The relation $h(ab) = h^{(1)}(a)h^{(2)}(b)$ shows that, in order for $\chi_\tau$ to be compatible with face operators, the face operators $\delta_i$ on $H^\otimes n$, at least for $0 \leq i < n$, must involve the coproduct of $H$. In fact if we define, for $0 \leq i \leq n$, $\delta_i^n : H^\otimes n \to H^\otimes (n+1)$, by

$$
\begin{align*}
\delta_0(h_1 \otimes \cdots \otimes h_n) &= 1 \otimes h_1 \otimes \cdots \otimes h_n, \\
\delta_i(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes h_i^{(1)} \otimes h_i^{(2)} \otimes \cdots \otimes h_n, \\
\delta_{n+1}(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes h_n \otimes \sigma,
\end{align*}
$$

then we have, for all $i = 0, 1, \ldots, n$

$$
\chi_\tau \delta_i = \delta_i \chi_\tau.
$$

Notice that the last relation is a consequence of the $\sigma$-trace property of $\tau$. Similarly, the relation $h(1_A) = \varepsilon(h)1_A$ shows that the degeneracy operators on $H^\otimes n$ should involve the counit of $H$. We thus define

$$
\sigma_i(h_1 \otimes \cdots \otimes h_n) = h_1 \otimes \cdots \otimes \varepsilon(h_i) \otimes \cdots \otimes h_n.
$$
It is very hard, on the other hand, to come up with a correct formula for the cyclic operator $\tau_n : H^\otimes n \to H^\otimes n$. Compatibility with $\chi$ demands that

$$\tau(a_0 \tau_n(h_1 \otimes \cdots \otimes h_n)(a_1 \otimes \cdots \otimes a_n)) = \tau(a_n h_1(a_0) h_2(a_1) \cdots h_n(a_n-1)),$$

for all $a_i$’s and $h_i$’s. For $n = 1$, the integration by parts formula (20), combined with the $\sigma$-trace property of $\tau$, shows that

$$\tau(a_1 h(a_0)) = \tau(h(a_0) \sigma(a_1)) = \tau(a_0 \tilde{S}\delta(h) \sigma(a_1)).$$

This suggests that we should define $\tau_1 : H \to H$ by

$$\tau_1(h) = \tilde{S}\delta(h) \sigma.$$

Note that the required cyclicity condition for $\tau_1$, $\tau_1^2 = 1$, is equivalent to the involution condition $\tilde{S}\delta(h) = \sigma h \sigma^{-1}$ for the pair $(\delta, \sigma)$. This line of reasoning can be extended to all $n \geq 0$ and gives us:

$$\tau(a_n h_1(a_0) \cdots h_n(a_n-1)) = \tau(h_1(a_0) \cdots h_n(a_n-1) \sigma(a_n))$$

$$= \tau(a_0 \tilde{S}\delta(h_1)(h_2(a_1) \cdots h_n(a_n-1) \sigma(a_n)))$$

$$= \tau(a_0 \tilde{S}\delta(h_1) \cdot (h_2 \otimes \cdots \otimes h_n \otimes \sigma)(a_1 \otimes \cdots \otimes a_n)).$$

This suggests that the Hopf-cyclic operator $\tau_n : H^\otimes n \to H^\otimes n$ should be defined as

$$\tau_n(h_1 \otimes \cdots \otimes h_n) = \tilde{S}\delta(h_1) \cdot (h_2 \otimes \cdots \otimes h_n \otimes \sigma),$$

where $\cdot$ denotes the diagonal action defined by

$$h \cdot (h_1 \otimes \cdots \otimes h_n) := h^{(1)} h_1 \otimes h^{(2)} h_2 \otimes \cdots \otimes h^{(n)} h_n.$$

The remarkable fact, proved by Connes and Moscovici [20, 21], is that endowed with the above face, degeneracy, and cyclic operators, $\{H^\otimes n\}_{n \geq 0}$ is a cocyclic module. The proof is a very clever and complicated tour de force of Hopf algebra identities.

The resulting cyclic cohomology groups, which depend on the choice of a modular pair in involution $(\delta, \sigma)$, are denoted by $HC^n_{(\delta, \sigma)}(H)$, $n = 0, 1, \ldots$. The characteristic map $\chi$ clearly induces a map between corresponding cyclic cohomology groups

$$\chi : HC^n_{(\delta, \sigma)}(H) \to HC^n(A).$$

Under this map Hopf cyclic cocycles are mapped to cyclic cocycles on $A$. Very many of the interesting cyclic cocycles in noncommutative geometry are obtained in this fashion. Using the above discussed cocyclic module structure of $\{H^\otimes n\}_{n \geq 0}$, we see that a Hopf cyclic $n$-cocycle is an element $x \in H^\otimes n$ which satisfies the relations

$$bx = 0, \quad (1-\lambda)x = 0,$$

where $b : H^\otimes n \to H^\otimes (n+1)$ and $\lambda : H^\otimes n \to H^\otimes n$ are defined by

$$b(h^1 \otimes \cdots \otimes h^n) = 1 \otimes h_1 \otimes \cdots \otimes h_n$$

$$+ \sum_{i=1}^n (-1)^i h_1 \otimes \cdots \otimes h_1^{(1)} \otimes h_i^{(2)} \otimes \cdots \otimes h_n$$

$$+ (-1)^{n+1} h_1 \otimes \cdots \otimes h_n \otimes \sigma,$$

$$\lambda(h_1 \otimes \cdots \otimes h_n) = (-1)^n \tilde{S}\delta(h_1) \cdot (h_2 \otimes \cdots \otimes h_n \otimes \sigma).$$

The characteristic map $\chi$ has its origins in Connes’ earlier work on noncommutative differential geometry [4], and on his work on the transverse fundamental
class of foliations [8]. In fact in these papers some interesting cyclic cocycles were defined in the context of actions of Lie algebras and (Lie) groups. Both examples can be shown to be special cases of the characteristic map. For example let \( A = A_\theta \) denote the smooth algebra of coordinates for the noncommutative torus with parameter \( \theta \in \mathbb{R} \). The abelian Lie algebra \( \mathbb{R}^2 \) acts on \( A_\theta \) via canonical derivations \( \delta_1 \) and \( \delta_2 \). The standard trace \( \tau \) on \( A_\theta \) is invariant under the action of \( \mathbb{R}^2 \), i.e., we have \( \tau(\delta_1(a)) = \tau(\delta_2(a)) = 0 \) for all \( a \in A_\theta \). Then one can directly check that under the characteristic map [31] the two dimensional generator of the Lie algebra homology of \( \mathbb{R}^2 \) is mapped to the following cyclic 2-cocycle on \( A_\theta \) first defined in [4]:
\[
\varphi(a_0, a_1, a_2) = \frac{1}{2\pi i} \tau(a_0(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))).
\]

For a second example let \( G \) be a discrete group and \( \epsilon \) be a normalized group \( n \)-cocycle on \( G \) with trivial coefficients. Here by normalized we mean \( c(g_1, \ldots, g_n) = 0 \) if \( g_i = e \) for some \( i \). Then one checks that the following is a cyclic \( n \)-cocycle on the group algebra \( \mathbb{C}G \) [8]:
\[
\varphi(g_0, g_1 \ldots, g_n) = \begin{cases} c(g_1, g_2 \ldots, g_n) & \text{if } g_0 g_1 \ldots g_n = 1 \\ 0 & \text{otherwise} \end{cases}
\]

After an appropriate dual version of Hopf cyclic cohomology is defined, one can show that this cyclic cocycle can also be defined via [31].

The most sophisticated example of the characteristic map [31], so far, involves the Connes-Moscovici Hopf algebra \( H_1 \) and its action on algebras of interest in transverse geometry. In fact, as we shall see, \( H_1 \) acts as quantum symmetries of various objects of interest in noncommutative geometry, like the frame bundle of the ‘space’ of leaves of codimension one foliations or the ‘space’ of modular forms modulo the action of Hecke correspondences.

To describe \( H_1 \), let \( g_{aff} \) denote the Lie algebra of the group of affine transformations of the line with linear basis \( X \) and \( Y \) and the relation \( [Y, X] = X \). Let \( g \) be an abelian Lie algebra with basis \( \{ \delta_n; \ n = 1, 2, \ldots \} \). Its universal enveloping algebra \( U(g) \) should be regarded as the continuous dual of the pro-unipotent group of orientation preserving diffeomorphisms \( \varphi \) of \( \mathbb{R} \) with \( \varphi(0) = 0 \) and \( \varphi'(0) = 1 \). It is easily seen that \( g_{aff} \) acts on \( g \) via
\[
[Y, \delta_n] = n \delta_n, \quad [X, \delta_n] = \delta_{n+1},
\]
for all \( n \). Let \( g_{CM} := g_{aff} \rtimes g \) be the corresponding semidirect product Lie algebra. As an algebra, \( H_1 \) coincides with the universal enveloping algebra of the Lie algebra \( g_{CM} \). Thus \( H_1 \) is the universal algebra generated by \( \{ X, Y, \delta_n; n = 1, 2, \ldots \} \) subject to the relations
\[
[Y, X] = X, \quad [Y, \delta_n] = n \delta_n, \quad [X, \delta_n] = \delta_{n+1}, \quad [\delta_k, \delta_l] = 0,
\]
for \( n, k, l = 1, 2, \ldots \). We let the counit of \( H_1 \) coincide with the counit of \( U(g_{CM}) \). Its coproduct and antipode, however, will be certain deformations of the coproduct and antipode of \( U(g_{CM}) \) as follows. Using the universal property of \( U(g_{CM}) \), one checks that the relations
\[
\Delta Y = Y \otimes 1 + 1 \otimes Y, \quad \Delta \delta_1 = \delta_1 \otimes 1 + 1 \otimes \delta_1,
\]
\[
\Delta X = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y,
\]
determine a unique algebra map \( \Delta : H_1 \to H_1 \otimes H_1 \). Note that \( \Delta \) is not cocommutative and it differs from the coproduct of the enveloping algebra \( U(g_{CM}) \). Similarly,
one checks that there is a unique antialgebra map, the antipode, \( S : \mathcal{H}_1 \to \mathcal{H}_1 \) determined by the relations
\[
S(Y) = -Y, \quad S(X) = -X + \delta_1 Y, \quad S(\delta_1) = -\delta_1.
\]

The first realization of \( \mathcal{H}_1 \) was through its action as quantum symmetries of the ‘frame bundle’ of the noncommutative space of leaves of codimension one foliations. More precisely, given a codimension one foliation \((V, \mathcal{F})\), let \( M \) be a smooth transversal for \((V, \mathcal{F})\). Let \( A = C_0^\infty(F^+ M) \) denote the algebra of smooth functions with compact support on the bundle of positively oriented frames on \( M \) and let \( \Gamma \subset Diff^+(M) \) denote the holonomy group of \((V, \mathcal{F})\). One has a natural prolongation of the action of \( \Gamma \) to \( F^+(M) \) by
\[
\varphi(y, y_1) = (\varphi(y), \varphi'(y)(y_1)).
\]

Let \( A_\Gamma = C_0^\infty(F^+ M) \rtimes \Gamma \) denote the corresponding crossed product algebra. Thus the elements of \( A_\Gamma \) consist of finite linear combinations (over \( \mathbb{C} \)) of terms \( fU^*_{\varphi} \) with \( f \in C_0^\infty(F^+ M) \) and \( \varphi \in \Gamma \). Its product is defined by
\[
fU^*_{\varphi} \cdot gU^*_{\psi} = (f \cdot \varphi(g))U^*_{\psi \varphi}.
\]

There is an action of \( \mathcal{H}_1 \) on \( A_\Gamma \), given by [20, 23]:
\[
Y(fU^*_{\varphi}) = y_1 \frac{\partial f}{\partial y_1} U^*_{\varphi}, \quad X(fU^*_{\varphi}) = y_1 \frac{\partial f}{\partial y_1} U^*_{\varphi},
\]
\[
\delta_n(fU^*_{\varphi}) = y_1^n \frac{d^n}{dy^n} (\log \frac{d\varphi}{dy}) fU^*_{\varphi}.
\]

Once these formulas are given, it can be checked, by a long computation, that \( A_\Gamma \) is indeed an \( \mathcal{H}_1 \)-module algebra. To define the corresponding characteristic map, Connes and Moscovici defined a modular pair in involution \((\delta, 1)\) on \( \mathcal{H}_1 \) and a \( \delta \)-invariant trace on \( A_\Gamma \) as we shall describe next.

Let \( \delta \) denote the unique extension of the modular character
\[
\delta : \mathfrak{g}_{aff} \to \mathbb{R}, \quad \delta(X) = 1, \quad \delta(Y) = 0,
\]
to a character \( \delta : U(\mathfrak{g}_{aff}) \to \mathbb{C} \). There is a unique extension of \( \delta \) to a character, denoted by the same symbol \( \delta : \mathcal{H}_1 \to \mathbb{C} \). Indeed, the relations \([Y, \delta_n] = n\delta_n\) show that we must have \( \delta(n) = 0 \), for \( n = 1, 2, \ldots \). One can then check that these relations are compatible with the algebra structure of \( \mathcal{H}_1 \). The algebra \( A_\Gamma = C_0^\infty(F^+ (M) \rtimes \Gamma \) admits a \( \delta \)-invariant trace \( \tau : A_\Gamma \to \mathbb{C} \) given by [20]:
\[
\tau(fU^*_{\varphi}) = \int_{F^+(M)} f(y, y_1) \frac{dydy_1}{y_1}, \quad \text{if } \varphi = 1,
\]
and \( \tau(fU^*_{\varphi}) = 0 \), otherwise. Now, using the \( \delta \)-invariant trace \( \tau \) and the above defined action \( \mathcal{H}_1 \otimes A_\Gamma \to A_\Gamma \), the characteristic map [31] takes the form
\[
\chi_{\tau} : HC^*(\delta, 1)(\mathcal{H}_1) \to HC^*(A_\Gamma).
\]

This map plays a fundamental role in transverse index theory in [20].

The Hopf algebra \( \mathcal{H}_1 \) shows its beautiful head in number theory as well. To give an indication of this, I shall briefly discuss the modular Hecke algebras and actions of \( \mathcal{H}_1 \) on them as they were introduced by Connes and Moscovici in [23, 24]. For each \( N \geq 1 \), let
\[
\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) ; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}
\]
denote the level $N$ congruence subgroup of $\Gamma(1) = SL(2, \mathbb{Z})$. Let $\mathcal{M}_k(\Gamma(N))$ denote the space of modular forms of level $N$ and weight $k$ and

$$\mathcal{M}(\Gamma(N)) := \bigoplus_k \mathcal{M}_k(\Gamma(N))$$

be the graded algebra of modular forms of level $N$. Finally, let

$$\mathcal{M} := \lim_{\rightarrow N} \mathcal{M}(\Gamma(N))$$

denote the algebra of modular forms of all levels, where the inductive system is defined by divisibility. The group

$$G^+(\mathbb{Q}) := GL^+(2, \mathbb{Q}),$$

acts on $\mathcal{M}$ through its usual action on functions on the upper half-plane (with corresponding weight):

$$(f, \alpha) \mapsto f|_k \alpha(z) = \det(\alpha)^{k/2} (cz + d)^{-k} f(\alpha \cdot z),$$

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \alpha \cdot z = \frac{az + b}{cz + d}.$$  

The simplest modular Hecke algebra is the crossed-product algebra

$$\mathcal{A} = \mathcal{A}_{G^+(\mathbb{Q})} := \mathcal{M} \rtimes G^+(\mathbb{Q}).$$

Elements of this (noncommutative) algebra will be denoted by finite sums $\sum f_U^* \gamma$, $f \in \mathcal{M}$, $\gamma \in G^+(\mathbb{Q})$. $\mathcal{A}$ can be thought of as the algebra of noncommutative co-ordinates on the noncommutative quotient space of modular forms modulo Hecke correspondences.

Now consider the operator $X$ of degree two on the space of modular forms defined by

$$X := \frac{1}{2\pi i} \frac{d}{dz} - \frac{1}{12\pi i} \frac{d}{dz} (log \Delta) \cdot Y,$$

where

$$\Delta(z) = (2\pi)^{12} \eta^{24}(z) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q = e^{2\pi i z},$$

$\eta$ is the Dedekind eta function, and $Y$ is the grading operator

$$Y(f) = \frac{k}{2} f, \quad \text{for all } f \in \mathcal{M}_k.$$  

It is shown in [23] that there is a unique action of $\mathcal{H}_1$ on $\mathcal{A}_{G^+(\mathbb{Q})}$ determined by

$$X(fU^*_\gamma) = X(f)U^*_\gamma, \quad Y(fU^*_\gamma) = Y(f)U^*_\gamma,$$

$$\delta_1(fU^*_\gamma) = \mu_\gamma \cdot f(U^*_\gamma),$$

where

$$\mu_\gamma(z) = \frac{1}{2\pi i} \frac{d}{dz} \log \frac{\Delta/\gamma}{\Delta}.$$  

This action is compatible with the algebra structure, i.e., $\mathcal{A}_{G^+(\mathbb{Q})}$ is an $\mathcal{H}_1$-module algebra. Thus one can think of $\mathcal{H}_1$ as quantum symmetries of the noncommutative space represented by $\mathcal{A}_{G^+(\mathbb{Q})}$.

More generally, for any congruence subgroup $\Gamma$, an algebra $A(\Gamma)$ is constructed in [23] that contains as subalgebras both the algebra of $\Gamma$-modular forms and the Hecke ring at level $\Gamma$. There is also a corresponding action of $\mathcal{H}_1$ on $A(\Gamma)$. 
The Hopf cyclic cohomology groups $HC_n^{(\delta,\sigma)}(H)$ are computed in several cases in [20]. Of particular interest for applications to transverse index theory and number theory is the (periodic) cyclic cohomology of $H_1$. It is shown in [20] that the periodic groups $HP_n^{(\delta,1)}(H_1)$ are canonically isomorphic to the Gelfand-Fuchs cohomology, with trivial coefficients, of the Lie algebra $a_1$ of formal vector fields on the line:

$$HP^{(\delta,1)}_n(H_1) = \bigoplus_{i \geq 0} H^{*+2i}_{GF}(a_1, \mathbb{C}).$$

This result is very significant in that it relates the Gelfand-Fuchs construction of characteristic classes of smooth manifolds to a noncommutative geometric construction of the same via $H_1$. Connes and Moscovici also identified certain interesting elements in the Hopf cyclic cohomology of $H_1$. For example, it can be directly checked that the elements $\delta_2 := \delta_2 - \frac{1}{2} \delta_1$ and $\delta_1$ are Hopf cyclic 1-cocycles for $H_1$, and

$$F := X \otimes Y - Y \otimes X - \delta_1 Y \otimes Y$$

is a Hopf cyclic 2-cocycle. Under the characteristic map [31] and for $A = A_F$ these Hopf cyclic cocycles are mapped to the Schwarzian derivative, the Godbillon-Vey cocycle, and the transverse fundamental class of Connes [8], respectively. See [24] for detailed calculations as well as relations with modular forms and modular Hecke algebras. Very recently the unstable cyclic cohomology groups of $H_1$, and a series of other Hopf algebras attached to pseudogroups of geometric structures, were fully computed in [35, 36]. In particular it is shown that the groups $HC^{n}_{(\delta,\sigma)}(H_1)$ are finite dimensional for all $n$.

The notion of modular pair in involution $(\delta, \sigma)$ for a Hopf algebra might seem rather ad hoc at a first glance. This is in fact not the case and the concept is very natural and fundamental. For example, it is shown in [21] that coribbon Hopf algebras and compact quantum groups are endowed with canonical modular pairs of the form $(\delta, 1)$ and, dually, ribbon Hopf algebras have canonical modular pairs of the type $(1, \sigma)$. The fundamental importance of modular pairs in involution was further elucidated when Hopf cyclic cohomology with coefficients was introduced in [30, 31]. It turns out that some very stringent conditions have to be imposed on an $H$-module $M$ in order for $M$ to serve as a coefficient (local system) for Hopf cyclic cohomology theory. Such modules are called stable anti-Yetter-Drinfeld modules. More precisely, a (left-left) anti-Yetter-Drinfeld $H$-module is a left $H$-module $M$ which is simultaneously a left $H$-comodule such that

$$\rho(hm) = h^{(1)}m^{(-1)}S(h^{(3)}) \otimes h^{(2)}m^{(0)},$$

for all $h \in H$ and $m \in M$. Here $\rho : M \to H \otimes M$, $\rho(m) = m^{(-1)} \otimes m^{(0)}$ is the comodule structure map of $M$. $M$ is called stable if in addition we have

$$m^{(-1)}m^{(0)} = m,$$

for all $m \in M$. Given a stable anti-Yetter-Drinfeld (SAYD) module $M$ over $H$, one can then define the Hopf cyclic cohomology of $H$ with coefficients in $M$. One-dimensional SAYD modules correspond to Connes-Moscovici’s modular pairs in involution. More precisely, there is a one-to-one correspondence between modular pairs in involution $(\delta, \sigma)$ on $H$ and SAYD module structures on $M = \mathbb{C}$, the ground field, defined by

$$h.r = \delta(h)r, \quad r \mapsto \sigma \otimes r,$$
for all $h \in H$ and $r \in \mathbb{C}$. Thus a modular pair in involution can be regarded as an ‘equivariant line bundle’ over the noncommutative space represented by the Hopf algebra $H$.

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