Module sectional category of products

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Abstract

Extending a result of Félix-Halperin-Lemaire on Lusternik-Schnirelmann category of products, we prove additivity of a rational approximation for Schwarz’s sectional category with respect to products of fibrations.

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Introduction

The sectional category \([11]\) (or Schwarz genus) of a fibration \(p: E \to X\) is the smallest integer \(n\) such that \(X\) admits a cover by open sets on each of which a local section for \(p\) exits. This homotopy invariant is a generalization of the well known Lusternik-Schnirelmann (LS) category \([9]\) of a path-connected space \(X\), \(\text{cat}(X)\), as it is the sectional category of the path fibration \(PX \to X, \alpha \mapsto \alpha(1)\), where \(PX\) is the space of paths starting at the base point.

One of the most important results of \([4]\) says that, if \(X\) and \(Y\) are simply connected rational spaces of finite type, then \(\text{cat}(X \times Y) = \text{cat}(X) + \text{cat}(Y)\). This was done through Hess’ theorem \([8]\) by proving the analogous result for

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a lower bound of LS category called module LS category.

Throughout this paper we will consider all spaces to be simply connected CW-complexes of finite type. We will also denote $f_0$ the rationalisation of a map $f$. As for LS category, there exists a lower bound of sectional category, called module sectional category [6], for which we have $\text{mcat}(X) = \text{msecat}(PX \to X)$. In this paper we prove

**Theorem 1.** Let $f$ and $g$ be two fibrations. If $f_0$ admits a homotopy retraction, then

$$\text{msecat}(f \times g) = \text{msecat}(f) + \text{msecat}(g).$$

Another important particular case of sectional category is Farber’s (higher) topological complexity [3, 10] of a space $X$, $\text{TC}_n(X) = \text{secat}(\pi_n)$, where the fibration $\pi_n: X^{[1,n]} \to X^n$ is such that $\pi_n(\alpha) = (\alpha(1), \alpha(2), \ldots, \alpha(n))$.

As a direct application of Theorem 1 the module invariant associated to (higher) topological complexity,

$$\text{mTC}_n(X) := \text{msecat}(\pi_n),$$

is additive:

**Corollary 2.** Let $X$ and $Y$ be two spaces. Then

$$\text{mTC}_n(X \times Y) = \text{mTC}_n(X) + \text{mTC}_n(Y).$$

The results given are an improvement of [2].

## 1 Preliminaries

This section contains a brief summary of the tools that will be used, see [5] for further details. Let $(A,d)$ be a commutative differential graded algebra over $\mathbb{Q}$ (cdga). An $(A,d)$-module is a chain complex $(M, d)$ together with a degree 0 action of $A$ verifying that $d(ax) = d(a)x + (-1)^{\deg(a)} ad(x)$. The module $M^\# = \text{hom}(M, \mathbb{Q})$ admits an $(A,d)$-module structure with action $(a\varphi)(x) = (-1)^{\deg(a)} \deg(\varphi) \varphi(ax)$ and differential $d(\varphi) = (-1)^{\deg(\varphi)} \varphi \circ d$. If $N$ is an $(A,d)$-modules, then the module $M \otimes_A N$ admits an $(A,d)$-module structure with action $a(m \otimes n) = (am) \otimes n$ and differential $d(m \otimes n) =$
A morphism of \((A,d)\)-modules \(\varphi: (M,d) \rightarrow (N,d)\) is said to have a homotopy retraction if there exists a commutative diagram of \((A,d)\)-modules,
\[
\begin{array}{ccc}
(M,d) & \xrightarrow{\text{Id}} & (M,d) \\
\varphi \downarrow & & \downarrow \\
(N,d) & \xrightarrow{\sim} & (P,d).
\end{array}
\]

We will use the following lemma which is an expression of one of the central ideas of \([4]\).

**Lemma 3.** Let \(\varphi: (A,d) \rightarrow (B,d)\) be a surjective cdga morphism with kernel \(K\) and \(A\) of finite type. The morphism \(\varphi\) admits a homotopy retraction of \((A,d)\)-modules if and only if for any \((A,d)\) semi-free resolution \(\eta: P \xrightarrow{\sim} A^\#\), the projection
\[
\varrho: P \rightarrow \frac{P}{K \cdot P},
\]
is injective in homology.

**Proof.** Suppose that \(\varphi\) admits a homotopy retraction of \((A,d)\)-module. This means that there exists a commutative diagram of \((A,d)\) module of the form
\[
\begin{array}{ccc}
A & \xrightarrow{\text{Id}_A} & A \\
\varphi \downarrow & & \downarrow r \\
B & \xrightarrow{\sim} & M
\end{array}
\]
where we can suppose that \(M\) is a \((A,d)\) semi-free resolution. Let now \(P \xrightarrow{\sim} A^\#\) be a \((A,d)\) semi-free resolution and apply \(- \otimes_A P\) to the diagram above. We get
\[
\begin{array}{ccc}
P & \xrightarrow{\text{Id}_P} & P \\
\downarrow & & \downarrow \\
B \otimes_A P & \xrightarrow{\sim} & M \otimes_A P.
\end{array}
\]
Since $B = \frac{A}{K}$ we have $B \otimes_A P = \frac{P}{K \cdot P}$ and the left hand morphism is the projection $\varrho: P \to \frac{P}{K \cdot P}$. The diagram shows that $\varrho$ admits a homotopy retraction of $(A, d)$-module and therefore that it is injective in homology.

Conversely, suppose that $\varrho$ is homology injective. Since $A$ is of finite type, $\eta^\# = \hom(\eta, \mathbb{Q}): A \to \hom(P, \mathbb{Q})$ is also a semi-free resolution. Since $\varrho$ is homology injective,

$\varrho^\# = \hom(\varrho, \mathbb{Q}): \hom\left(\frac{P}{K \cdot P}, \mathbb{Q}\right) \to \hom(P, \mathbb{Q})$

is homology surjective. There exists then a cycle $f \in \hom\left(\frac{P}{K \cdot P}, \mathbb{Q}\right)$ such that $[f \circ \varrho] = [z]$, where $z = \eta^\#(1)$. Now define the $(A, d)$-module morphism $\alpha: A \to \hom\left(\frac{P}{K \cdot P}, \mathbb{Q}\right)$ as $\alpha(1) = f$. Then $\varrho^\# \circ \alpha$ is homotopic to $\eta^\#$ and thus a quasi-isomorphism. To finish the proof, we observe that $K \cdot \hom(P, \mathbb{Q}) = \{0\}$ and thus we have a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{1d_A} & A \\
\varphi \downarrow & & \Downarrow{\varrho^\# \circ \alpha} \\
B & \xrightarrow{\alpha} & \hom\left(\frac{P}{K \cdot P}, \mathbb{Q}\right) \xrightarrow{\varrho^\#} \hom(P, \mathbb{Q}),
\end{array}
\]

which yields a homotopy retraction for $\varphi$ as $(A, d)$-modules. \hfill \Box

Let us denote by $p_n: J_X^n(E) \to X$ the join of $n + 1$ copies of a fibration $p: E \to X$. As it is well-known [11], $\secat(p) \leq n$ if and only if $p_n$ admits a homotopy section. By definition, $\msecat(p)$ is the smallest $n$ such that $A_{PL}(p_n)$ admits a homotopy retraction of $A_{PL}(X)$-modules, where $A_{PL}$ denotes Sullivan’s functor of piecewise linear forms [12].

Recall the following general characterization of $\msecat(f)$ from [3]. Let $(A, d) \to (A \otimes (\mathbb{Q} \oplus X), d)$ be a semi-free extension of $(A, d)$-module which is a model for $f$. For $x \in X$, write $dx = d_0 x + d_+ x \in A \oplus A \otimes X$. Then $\msecat(f)$ is the least $m$ such that the following $(A, d)$ semi-free extension admits a retraction of $(A, d)$-module:

$\hat{j}_m: (A, d) \to J_m = (A \otimes (\mathbb{Q} \oplus s^{-m} X^\otimes m+1), d)$. 

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Here \( d = d_0 + d_+ \) (in \( A \oplus A \otimes s^{-m}X^{\otimes m+1} \)) is given by

\[
d(s^{-m}x_0 \otimes \cdots \otimes x_m) = (-1)^k \sum_{k=1}^{m} (k|x_{m-k}|+k-1) d_0 x_0 \cdots d_0 x_m \\
+ \sum_{i=0}^{m} \sum_{j_i} (-1)^{|a_{ij_i}|+1}(|x_0|+\cdots+|x_{i-1}|+m) a_{ij_i} \otimes s^{-m}x_0 \otimes \cdots \otimes x_{ij_i} \otimes \cdots \otimes x_m,
\]

for \( x_0, \ldots, x_m \in X \) and \( d_+ x_i = \sum a_{ij_i} \otimes x_{ij_i} \) with \( a_{ij_i} \in A \) and \( x_{ij_i} \in X \).

Using the following notation (suggested by the standard rules of signs)

\[
s^{-m}x_0 \otimes \cdots \otimes d_+ x_i \otimes \cdots \otimes x_m := \sum_{j_i} \sigma_{ij_i} a_{ij_i} \otimes s^{-m}x_0 \otimes \cdots \otimes x_{ij_i} \otimes \cdots \otimes x_m
\]

we can write \( d_+ (s^{-m}x_0 \otimes \cdots \otimes x_m) \) as

\[
d_+ (s^{-m}x_0 \otimes \cdots \otimes x_m) = (-1)^m \sum_{i=0}^{m} \sum_{j_i} \tau_i s^{-m}x_0 \otimes \cdots \otimes d_+ x_i \otimes \cdots \otimes x_m,
\]

where \( \sigma_{ij_i} := (-1)^{|a_{ij_i}|+|x_0|+\cdots+|x_{i-1}|+m} \) and \( \tau_i := (-1)^{|x_0|+\cdots+|x_{i-1}|} \).

Now let \( f \) be a fibration such that \( f_0 \) admits a homotopy retraction. Then by [1], there exists a surjective model for \( f \), \( \varphi: A \rightarrow A_{K^m} \) (called s-model) such that \( \text{msecat}(f) \) is the smallest \( m \) for which the projection \( \rho_m: A \rightarrow A_{K^{m+1}} \) admits a homotopy retraction of \((A, d)\)-modules. We have

**Proposition 4.** Let \( f \) be a fibration such that \( f_0 \) admits a homotopy retraction, \( \varphi: A \rightarrow A_{K^m} \) and s-model for \( f \) and \((A, d) \rightarrow (A \otimes (Q \oplus X), d) \) a semifree model for \( f \), as in previous paragraphs. Let also \( \eta: P \overset{\cong}{\rightarrow} A^\# \) be an \((A, d)\) semi-free resolution. Then the following are equivalent

(i) \( \text{msecat}(f) \leq m \),

(ii) the morphism \( \text{Id}_P \otimes A j_m: P \rightarrow P \otimes (Q \oplus s^{-m}X^{\otimes m+1}) \) is injective in homology,

(iii) the projection \( P \rightarrow \frac{P}{K^{m+1}P} \) is injective in homology.
Proof. By [1], there is a diagram

\[
\begin{array}{ccc}
J_m & \xrightarrow{h_A} & A \\
\downarrow & & \downarrow \\
\cong & C & \xrightarrow{A_{K^{m+1}}} \\
\end{array}
\]

where the left hand triangle is commutative up to a homotopy of \((A, d)\)-modules and the right hand triangle is strictly commutative. Applying to previous diagram \(\text{Id}_P \otimes_A -\), we get the following diagram of \((A, d)\) module:

\[
\begin{array}{ccc}
P & \xrightarrow{P \otimes_A C} & P \\
\downarrow & & \downarrow \\
\cong & C & \xrightarrow{A_{K^{m+1}}} \\
\end{array}
\]

where the left hand triangle is commutative up to a homotopy of \((A, d)\)-module and the right hand triangle is strictly commutative. The result then follows from Lemma [3].

\[\square\]

2 The main result

Observe that Proposition [4] together with the strategy of [4] can be used to easily prove Theorem [1] provided that both fibrations admit a homotopy retraction. In line with our statement, we here present a proof of the additivity of module sectional category when only one of the fibrations admits homotopy retraction.

We first notice that one of the inequalities of Theorem [1] follows in general:

**Proposition 5.** Let \(p: E \to X\) and \(p': E' \to X'\) be two fibrations. We have

\[
\text{msecat}(p \times p') \leq \text{msecat}(p) + \text{msecat}(p').
\]

**Proof.** In [7] Pg. 26], a commutative diagram of the following form is constructed:

\[
\begin{array}{ccc}
J^n_X(E) \times J^m_{X'}(E') & \xrightarrow{\psi^{E, E'}_{n, m}} & J^{n+m}_{X \times X'}(E \times E') \\
\downarrow & & \downarrow \\
X \times X' & \xrightarrow{(p \times p')_{n+m}} & \text{msecat}(p \times p')
\end{array}
\]

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By applying $A_{PL}$ to this diagram, we can establish that, if $\text{msecat}(p) \leq n$ and $\text{msecat}(p') \leq m$ then $\text{msecat}(p \times p') \leq n + m$. \[
\]

Keeping the notation of Proposition 3 the differential in $(A \otimes (Q \oplus X), d)$ can be taken such that $d_0(x) \in K$. This implies that, in $P \otimes (Q \oplus s^{-m}X^{\otimes m+1})$, $d_0(s^{-m}X^{\otimes m+1}) \subset K^{m+1} \cdot P$. With this in mind we proceed to the

**Proof of Theorem 4.** Take an $s$-model for $f$, $\varphi$ and an $(A, d)$ semi-free extension of $\varphi$, $A \otimes (Q \oplus X)$, as in the previous section. Let also $(B, d) \rightarrow (B \otimes (Q \oplus Y), d)$ a semi-free model of $g$ (where $(B, d)$ is a cdga model of the base of $g$). Then $f \times g$ is modeled by the tensor product of the two semi-free extensions which gives a semi-free extension of $(A \otimes B, d)$-modules that we write as follows

$$A \otimes B \rightarrow A \otimes B \otimes (Q \oplus Z) \quad \text{where } Z = X \oplus Y \oplus X \otimes Y.$$ In order to prove the statement, we suppose $\text{msecat}(f) = m$ and $\text{msecat}(f \times g) \leq m + p$ and we establish that $\text{msecat}(g) \leq p$.

Let $P \rightarrow A^\#$ be an $(A, d)$ semi-free resolution. Since $\text{msecat}(f) = m$ we know from Proposition 4 that there exists $\Omega \in H(K^m \cdot P)$ which is not trivial in $H(P)$. Then there exist a cocyle $\omega \in K^m \cdot P$ representing $\Omega$ in $H(P)$ and $\theta \in P \otimes s^{-(m-1)}X^{\otimes m}$ such that $d\theta = \omega$. As a chain complex, we can write $P = \omega \cdot Q \oplus S$ where $d(S) \subset S$, and we define the following linear map of degree $-|\omega|$:

$$I_\omega : P \rightarrow Q, \quad I_\omega(\omega) = 1, \quad I_\omega(S) = 0.$$ This map commutes with differentials. Now write the element $\theta \in P \otimes s^{-(m-1)}X^{\otimes m}$ as

$$\theta = \sum_i m_i \otimes s^{-(m-1)}x_i$$ with $m_i \in P$ and $x_i \in X^{\otimes m}$. Since $d\theta = \omega$ we have $d_\omega \theta = 0$ and $d_0 \theta = \omega$.

Let $\psi : B \otimes (Q \oplus s^{-p}Y^{\otimes p+1}) \rightarrow P \otimes B \otimes (Q \oplus s^{-m-p}Z^{\otimes m+p+1})$ be the $B$-linear map of degree $|\omega|$ given by $\psi(1) = \omega \otimes 1$ and, for $y \in Y^{\otimes p+1}$,

$$\psi(s^{-p}y) = -(-1)^{p|\omega|} \sum_i (-1)^{(p+1)|m_i|} m_i \otimes 1 \otimes s^{-m-p}x_i \otimes y$$ and extended to $B \otimes (Q \oplus s^{-p}Y^{\otimes p+1})$ by the rule $\psi(b \cdot x) = (-1)^{|b|\omega} b \cdot \psi(x)$. Notice that the structure of $(B, d)$-module on $P \otimes B \otimes (Q \oplus s^{-m-p}Z^{\otimes m+p+1})$ is
given by $b \cdot (m \otimes b' \otimes z) = (-1)^{|m||b|} m \otimes bb' \otimes z$. In particular $\psi(b) = \omega \otimes b$. Let us now see that $\psi$ commutes with differentials, that is $\psi \circ d = (-1)^{|\omega|} d \circ \psi$. Since $\psi$ is $B$-linear and since $\omega$ is a cocycle we only have to see that

$$d\psi(s^{-p}y) = (-1)^{|\omega|} \psi(ds^{-p}y),$$

for each $y \in Y^{\otimes p+1}$. Writing the differential of $P \otimes B \otimes (Q \oplus s^{-m-p} Z^{\otimes m+p+1})$ as

$$d = d_0 + d_+ \in P \otimes B \oplus P \otimes B \otimes s^{-m-p} Z^{\otimes m+p+1}$$

we can check that

- $d_0 \psi(s^{-p}y) = (-1)^{|\omega|} \psi(d_0 s^{-p}y)$ using the fact that $d_0 \theta = \omega$, and
- $d_+ \psi(s^{-p}y) = (-1)^{|\omega|} \psi(d_+ s^{-p}y)$ using the fact that $d_+ \theta = 0$.

From $\text{msecat}(f \times g) \leq m + p$ we know that the morphism

$$j^{A \otimes B}_{m+p} : A \otimes B \rightarrow A \otimes B \otimes (Q \oplus s^{-m-p} Z^{\otimes m+p+1})$$

admits a retraction $r$ of $(A \otimes B, d)$-modules. Finally the composite

$$B \otimes (Q \oplus s^{-p} Y^{\otimes p+1}) \xrightarrow{\psi} P \otimes B \otimes (Q \oplus s^{-m-p} Z^{\otimes m+p+1}) \xrightarrow{P \otimes Ar} P \otimes B \xrightarrow{I \otimes \text{Id}} B$$

gives a morphism (of degree 0) of $(B, d)$-module which is a retraction for the inclusion $B \rightarrow B \otimes (Q \oplus s^{-p} Y^{\otimes p+1})$. This proves that $\text{msecat}(g) \leq p$. \hfill $\Box$

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