On the classification of binary self-dual $[44, 22, 8]$ codes with an automorphism of order 3 or 7

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Abstract

All binary self-dual $[44, 22, 8]$ codes with an automorphism of order 3 or 7 are classified. In this way we complete the classification of extremal self-dual codes of length 44 having an automorphism of odd prime order.

1 Introduction

Let $\mathbb{F}_q^n$ be the $n$-dimensional vector space over the field $\mathbb{F}_q$ of $q$ elements. A linear $[n, k]$ code $C$ is a $k$-dimensional subspace of $\mathbb{F}_q^n$. The elements of $C$ are called codewords. The weight of a vector $v \in \mathbb{F}_q^n$ (denoted by wt$(v)$) is the number of its non-zero coordinates. The minimum weight $d$ of $C$ is the smallest weight among all nonzero weights of codewords of $C$; a code $C$ with minimum weight $d$ is called an $[n, k, d]$ code. A matrix whose rows form a basis of $C$ is called a generator matrix of this code. The weight enumerator $W(y)$ of a code $C$ is given by

$$W(y) = \sum_{i=0}^{n} A_i y^i$$

where $A_i$ is the number of codewords of weight $i$ in $C$. Two binary codes are called equivalent if one can be obtained from the other by a permutation of coordinates. The permutation $\sigma \in S_n$ is an automorphism of $C$, if $C = \sigma(C)$ and the set of all automorphisms of $C$ forms a group called the automorphism group of $C$, which is denoted by $Aut(C)$ in this paper.

Let $(u, v) \in \mathbb{F}_q^n$ for $u, v \in \mathbb{F}_q^n$ be an inner product in $\mathbb{F}_q^n$. The dual code of an $[n, k]$ code $C$ is $C^\perp = \{ u \in \mathbb{F}_q^n \mid (u, v) = 0 \text{ for all } v \in C \}$ and $C^\perp$ is a linear $[n, n-k]$ code. If $C \subseteq C^\perp$, $C$ is termed self-orthogonal, and if $C = C^\perp$, $C$ is self-dual. We call a binary code self-complementary if it contains the all-ones vector. Every binary self-dual code is self-complementary. If $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n) \in \mathbb{F}_2^n$ then $(u, v) = \sum_{i=1}^{n} u_i v_i \in \mathbb{F}_2$.

It was shown in [15] that the minimum weight $d$ of a binary self-dual code of length $n$ is bounded by $d \leq 4[n/24] + 4$, unless $n \equiv 22 \pmod{24}$ when $d \leq 4[n/24] + 6$. We call a self-dual code meeting this upper bound extremal.
In this paper, we consider extremal binary self-dual [44, 22, 8] codes. All the odd primes \( p \) dividing the order of the automorphism group of such a code are 11, 7, 5, and 3 [20]. The codes with automorphisms of order 11 and 5 are classified in [20], [21], [3], and [4]. Unfortunately we noticed that there are some omissions in the classification of the codes with automorphisms of order 7 given in [16]. That’s why we focus on the automorphisms of orders 3 and 7, and we complete the classification of [44, 22, 8] self-dual codes having an automorphism of odd prime order.

As in the case of binary self-dual [42, 21, 8] codes with an automorphism of order 3, there are five different possibilities for the number of independent cycles in the decomposition of the automorphism, namely 6, 8, 10, 12, and 14 [5]. Codes with automorphisms of order 3 with 6 and 14 independent 3-cycles are considered but not classified in [4] and [17], respectively. In this paper, we give the classification of all self-dual [44, 22, 8] codes having an automorphism of order 3 or 7. To do that we apply the method for constructing binary self-dual codes via an automorphism of odd prime order developed in [8] and [18]. We give a short description of this method in Section 2. In Section 3 and Section 4 we classify the extremal self-dual codes of length 44 with an automorphism of order 3 and 7, respectively. In Section 5 we present the full classification of the self-dual [44, 22, 8] codes having automorphisms of odd prime order, and offer some open problems.

The weight enumerators of the extremal self-dual codes of length 44 are known (see [7]):

\[
W_{44,1}(y) = 1 + (44 + 4\beta)y^8 + (976 - 8\beta)y^{10} + (12289 - 20\beta)y^{12} + \ldots
\]

for \( 10 \leq \beta \leq 122 \) and

\[
W_{44,2}(y) = 1 + (44 + 4\beta)y^8 + (1232 - 8\beta)y^{10} + (10241 - 20\beta)y^{12} + \ldots
\]

for \( 0 \leq \beta \leq 154 \).

Codes exist for \( W_{44,1} \) when \( \beta = 10, \ldots, 68, 70, 72, 74, 82, 86, 90, 122 \) and for \( W_{44,2} \) when \( \beta = 0, \ldots, 56, 58, \ldots, 62, 64, 66, 68, 70, 72, 74, 76, 82, 86, 90, 104, 154 \) (see [9]).

\[2\] Construction Method

Let \( C \) be a binary self-dual code of length \( n = 44 \) with an automorphism \( \sigma \) of prime order \( p \geq 3 \) with exactly \( c \) independent \( p \)-cycles and \( f = 44 - cp \) fixed points in its decomposition. We may assume that

\[
\sigma = (1, 2, \cdots, p)(p+1, p+2, \cdots, 2p) \cdots (p(c-1) + 1, p(c-1) + 2, \cdots, pc),
\]

and say that \( \sigma \) is of type \( p\cdot(c, f) \).

Denote the cycles of \( \sigma \) by \( \Omega_1, \ldots, \Omega_c \), and the fixed points by \( \Omega_{c+1}, \ldots, \Omega_{c+f} \). Let

\[
E_\sigma(C) = \{ v \in C \mid v\sigma = v \}
\]

and

\[
E_\sigma(C) = \{ v \in C \mid \text{wt}(v|\Omega_i) \equiv 0 \pmod{2}, i = 1, \cdots, c+f \},
\]

where \( v|\Omega_i \) is the restriction of \( v \) on \( \Omega_i \).
Theorem 1 [8] The self-dual code \( C \) is a direct sum of the subcodes \( F_\sigma(C) \) and \( E_\sigma(C) \). These subcodes have dimensions \( \frac{c+f}{2} \) and \( \frac{(p-1)}{2} \), respectively.

Thus each choice of the codes \( F_\sigma(C) \) and \( E_\sigma(C) \) determines a self-dual code \( C \). So for a given length all self-dual codes with an automorphism \( \sigma \) can be obtained.

We have that \( v \in F_\sigma(C) \) if and only if \( v \in C \) and \( v \) is constant on each cycle. Let \( \pi : F_\sigma(C) \to \mathbb{F}_2^{c+f} \) be the projection map where if \( v \in F_\sigma(C) \), \( \pi(v) = (v_{i}) \) for some \( j \in \Omega_{i}, \)

Let \( \sigma \) be a primitive root modulo \( p \). Then the following theorem holds:

**Theorem 2** [19] A binary \([n, n/2]\) code \( C \) with an automorphism \( \sigma \) is self-dual if and only if the following two conditions hold:

(i) \( C_\pi = \pi(F_\sigma(C)) \) is a binary self-dual code of length \( c+f \),

(ii) for every two vectors \( u, v \) from \( C_\varphi = \varphi(E_\sigma(C)^*) \) we have

\[
u_1(x)v_1(x^{-1}) + u_2(x)v_2(x^{-1}) + \cdots + u_c(x)v_c(x^{-1}) = 0.
\]

Let \( x^p - 1 = (x-1)h_1(x) \cdots h_s(x) \), where \( h_1, \ldots, h_s \) are irreducible binary polynomials.

If \( g_j(x) = (x^p - 1)/h_j(x) \), and \( I_j = \langle g_j(x) \rangle \) is the ideal in \( \mathcal{R}_p \), generated by \( g_j(x) \), then \( I_j \) is a fields with \( 2^{deg(h_j(x))} \) elements, \( j = 1, 2, \ldots, s \), and \( \mathcal{P} = I_1 \oplus I_2 \oplus \cdots \oplus I_s \). [13]

**Lemma 3** [19] Let \( M_j = \{ u \in \varphi(E_\sigma(C)^*) | u_i \in I_j, \ i = 1, 2, \ldots, c \} \), \( j = 1, 2, \ldots, s \). Then

1) \( M_j \) is a linear space over \( I_j, \ j = 1, 2, \ldots, s \);

2) \( C_\varphi = \varphi(E_\sigma(C)^*) = M_1 \oplus M_2 \oplus \cdots \oplus M_s \) (direct sum of \( \mathcal{P} \)-submodules);

3) If \( C \) is a self-dual code, then \( \sum_{j=1}^{s} \dim M_j = cs/2 \).

In the case, when \( 2 \) is a primitive root modulo \( p \), \( \mathcal{P} \) is a field with \( 2^{p-1} \) elements and the following theorem holds:

**Theorem 4** [8] Let \( 2 \) be a primitive root modulo \( p \). Then the binary code \( C \) with an automorphism \( \sigma \) is self-dual iff the following two conditions hold:

(i) \( C_\pi \) is a self-dual binary code of length \( c+f \);

(ii) \( C_\varphi \) is a self-dual code of length \( c \) over the field \( \mathcal{P} \) under the inner product \( (u, v) = \sum_{i=1}^{c} u_i v_i^{(p-1)/2} \).
Let $B$, respectively $D$, be the largest subcode of $C_\pi$ whose support is contained entirely in the left $c$, respectively, right $f$, coordinates. Suppose $B$ and $D$ have dimensions $k_1$ and $k_2$, respectively. Let $k_3 = k - k_1 - k_2$. Then there exists a generator matrix for $C_\pi$ in the form

$$G_\pi = \begin{pmatrix} B & O \\ O & D \\ E & F \end{pmatrix}$$

(1)

where $B$ is a $k_1 \times c$ matrix with $\text{gen}(B) = [B \ O]$, $D$ is a $k_2 \times f$ matrix with $\text{gen}(D) = [O \ D]$, $O$ is the appropriate size zero matrix, and $[E \ F]$ is a $k_3 \times n$ matrix. Let $B^*$ be the code of length $c$ generated by $B$, $B_E$ the code of length $c$ generated by the rows of $B$ and $E$, $D^*$ the code of length $f$ generated by $D$, and $D_F$ the code of length $f$ generated by the rows of $D$ and $F$. The following theorem is a modification of Theorem 2 from [12].

**Theorem 5** With the notations of the previous paragraph

(i) $k_3 = \text{rank}(E) = \text{rank}(F)$,

(ii) $k_2 = k + k_1 - c = k_1 + \frac{f - c}{2}$, and

(iii) $B_E^\perp = B^*$ and $D_F^\perp = D^*$.

### 3 Extremal Self-Dual Codes of Length 44 with an Automorphism of Order 3

Using Theorem 4 as 2 is a primitive root modulo 3, $\mathcal{P}$ is a field with 4 elements. We have that $\mathcal{P} = \{0, e = x + x^2, w = 1 + x^2, w^2 = 1 + x\} \cong \mathbb{F}_4$ where $e$ is the identity of $\mathcal{P}$. In this case $C_\varphi$ is a (Hermitian) self-dual code of length $c$ over the quaternary field $\mathcal{P}$ under the inner product $(u, v) = \sum_{i=1}^{c} u_i v_i^2$. Since the minimum distance of $E_\sigma(C)$ is at least 8, this Hermitian code should have minimum distance at least 4.

To classify the codes, we need additional conditions for equivalence. That's why we use the following theorem:

**Theorem 6** [18] The following transformations preserve the decomposition and send the code $C$ to an equivalent one:

(i) a permutation of the fixed coordinates;

(ii) a permutation of the 3-cycles coordinates;

(iii) a substitution $x \rightarrow x^2$ in $C_\varphi$ and

(iv) a cyclic shift to each 3-cycle independently.
3.1 Codes with an automorphism of type 3-(6, 26)

The extremal self-dual [44, 22, 8] codes having an automorphism of type 3-(6, 26) are considered in [4] but the author didn’t succeed to classify all codes. We do this classification now. Generator matrices of the codes $C_\varphi$ and $E_\sigma(C)^*$ are presented in [4]. In the same paper, it is also proved that $C_\pi$ is a binary self-dual [32, 16, $\geq 4$] code with a generator matrix

$$G_\pi = \begin{pmatrix} 0 & D \\ I_6 & F \end{pmatrix}$$

where $D$ generates a [26, 10, 8] self-orthogonal code $D^*$, and $D_F$ is its dual code. The code $D^*$ cannot be self-complementary (see [4]). According to [2], there are 1768 inequivalent [26, 10, 8] self-orthogonal codes. Using as $D$ generator matrices of those codes which are not self-complementary, we obtain the self-dual [44, 22, 8] codes invariant under the given permutation. To test them for equivalence, we use the program Q-EXTENSION [1]. The weight enumerators of the constructed codes are listed in Table 1.

**Theorem 7** There are exactly 15621 self-dual [44, 22, 8] codes having an automorphism of type 3-(6, 26).

Table 1: Extremal self-dual [44, 22, 8] codes having an automorphism of type 3-(6, 26)

| $\beta$ | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |
|--------|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $W_1$  | -  | -  | -  | -  | -  | 4  | 16 | 33 | 31 | 59 | 62 | 82 | 79 | 72 |
| $W_2$  | 11 | 26 | 58 | 201| 342| 433| 505| 462| 677| 685| 717| 599| 611|
| $\beta$| 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 |
| $W_1$  | 47 | 72 | 48 | 51 | 51 | 68 | 54 | 64 | 39 | 54 | 38 | 38 | 29 |
| $W_2$  | 463| 490| 452| 485| 654| 724| 674| 851| 558| 530| 430| 438| 327|
| $\beta$| 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 |
| $W_1$  | 32 | 32 | 28 | 35 | 66 | 49 | 51 | 41 | 40 | 33 | 39 | 29 | 33 |
| $W_2$  | 328| 238| 194| 120| 140| 72 | 89 | 43 | 85 | 13 | 46 | 5  | 27 |
| $\beta$| 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 |
| $W_1$  | 17 | 24 | 8  | 18 | 4  | 15 | 4  | 7  | 1  | 5  | 1  | 2  | 3  |
| $W_2$  | 5  | 21 | 6  | 11 | -  | 15 | 1  | 6  | 1  | 7  | -  | 2  | -  |
| $\beta$| 66 | 67 | 68 | 70 | 72 | 74 | 76 | 82 | 86 | 90 | 104| 122| 154|
| $W_1$  | 5  | 2  | 1  | 2  | 1  | 2  | -  | 1  | 1  | 1  | -  | 1  | -  |
| $W_2$  | 1  | -  | 1  | 1  | 3  | 4  | 2  | 2  | 1  | 1  | 1  | 1  | -  |

3.2 Codes with an automorphism of type 3-(8, 20)

Up to equivalence, a unique Hermitian quaternary [8, 4, 4] code exists (see [11]). So up to equivalence we have a unique subcode $E_\sigma(C)^*$. The code $C_\pi$ is a binary self-dual [28, 14, $\geq 4$] code with a generator matrix $G_\pi$ given in (1) where $B$ and $D$ generate self-orthogonal
[8, k₁ ≥ 4] and [20, k₁ + 6, ≥ 8] codes, respectively. Since 0 ≤ k₁ ≤ 4, D* is a binary self-orthogonal [20, 6 ≤ k₂ ≤ 10, ≥ 8] code. All optimal binary self-orthogonal codes of length 20 are classified in [4]. There are exactly 23 inequivalent [20, 6, 8] self-orthogonal codes, four inequivalent [20, 7, 8] self-orthogonal codes, and a unique [20, 8, 8] self-orthogonal code. Hence k₁ ≤ 2.

In the case k₁ = 2 we obtain only two inequivalent extremal codes of length 44, both with weight enumerator W₂, respectively for β = 68 and β = 76. For k₁ = 1, there exist 52 self-dual [44, 22, 8] codes, and for k₁ = 0, the inequivalent codes number 5399. Their weight enumerators are listed in Table 2.

**Theorem 8** There are exactly 5453 self-dual [44, 22, 8] codes having an automorphism of type 3-(8,20).

**Remark:** The extremal self-dual [44, 22, 8] codes invariant under a permutation of type 3-(8,20) are considered independently in [10]. The author of that paper has classified all extremal self-dual codes which have an automorphism of order 3 with 8 independent 3-cycles.

| β  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| W₁ | -  | -  | -  | -  | -  | -  | 2  | -  | -  | 5  | 5  | 3  | 9  | 16 | 8  |
| W₂ | 2  | -  | 3  | 10 | 8  | 27 | 47 | 81 | 157| 174| 330| 395| 442| 481| 560|

| β  | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| W₁ | 16 | 28 | 69 | 36 | 39 | 27 | 60 | 29 | 55 | 26 | 34 | 15 | 25 | 15 |    |
| W₂ | 442| 432| 307| 298| 140| 172| 79 | 69 | 41 | 56 | 29 | 55 | 26 | 34 | 37 |

| β  | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 49 | 50 | 52 | 53 | 68 | 76 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| W₁ | 15 | 3  | 8  | 4  | 6  | 1  | 4  | -  | 3  | 1  | 1  | -  | 1  | -  | -  |
| W₂ | 18 | -  | 9  | 4  | 6  | 3  | 4  | -  | 3  | -  | 3  | 1  | -  | 1  | 1  |

**3.3 Codes with an automorphism of type 3-(10, 14)**

In this case C₂ is a Hermitian self-dual [10, 5, 4] code and by [11] is equivalent to either E₁₀ or B₁₀. As in [5], we can fix the generator matrix of the subcode E₉(C)* in the following two forms, respectively:

\[
\begin{pmatrix}
01101110110100000000000000000000 \\
10110101101000000000000000000000 \\
00000011110111101000000000000000 \\
00000011110111101000000000000000 \\
00000000000011110111101000000000 \\
00000000000011110111101000000000 \\
000000000000000000000111101110111 \\
000000000000000000000111101110111 \\
0110000111000110001100011000110011 \\
1010000110000101000010001001100111 \\
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
01101110110100000000000000000000 \\
10110110110010000000000000000000 \\
00011101111010100000000000000000 \\
00011101111010100000000000000000 \\
00000000000001111011101010000000 \\
00000000000001111011101010000000 \\
000000000000000000000111101110111 \\
000000000000000000000111101110111 \\
011011101101000000000111101110111 \\
0101001111000101000010001001100111 \\
\end{pmatrix}
\]
The code $C_\pi$ has parameters $[24, 12, \geq 4]$. There are exactly thirty inequivalent such codes, namely $E_8^3$, $E_{16} \oplus E_8$, $F_{16} \oplus E_8$, $E_{12}^3$, and the indecomposable codes denoted by $A_{24}, B_{24}, \ldots, Z_{24}$ in [6]. All codes have minimum weight 4 except the extended Golay code $G_{24}$ with minimum weight 8 and the code $Z_{24}$ with minimum weight 6. We use the generator matrices of the codes given in [14]. For any weight 4 vector in $C_\pi$ at most two nonzero coordinates may be fixed points. An examination of the vectors of weight 4 in the listed codes eliminates 23 of them. By investigation of all alternatives for a choice of the 3-cycle coordinates in the remaining codes $G_{24}$, $R_{24}$, $U_{24}$, $W_{24}$, $X_{24}$, $Y_{24}$ and $Z_{24}$ we obtain, up to equivalence, all possibilities for the generator matrix of the code $C_\pi$.

Let $C_\pi$ be $R_{24}$. There is a unique possibility for the choice of the 3-cycle coordinates up to equivalence. The generator matrix of $C_\pi$ in this case can be fixed in the form

$$G_\pi(R_{24}) = \begin{pmatrix}
1100000000 & 11000000000000 \\
0110000000 & 01100000000000 \\
0001100000 & 00011000000000 \\
0000011000 & 00000110000000 \\
0000001100 & 00000011000000 \\
0000000110 & 00000001100000 \\
1001000000 & 10010000000111 \\
1000010000 & 10000100001100 \\
0000000111 & 00000000101100 \\
1110000000 & 00000000001111 \\
0011100000 & 00000000001110 \\
0000000011 & 00000000110000 \\
0000001100 & 00000011000000 \\
0000110000 & 00001100000000 \\
0001100000 & 00011000000000 \\
0110000000 & 01100000000000 \\
1001000000 & 10010000001111 \\
0000000011 & 00000000110000 \\
0000000001 & 00000000001111
\end{pmatrix}. $$

Let $\tau$ be a permutation of the ten cycle coordinates in $G_\pi(R_{24})$. Denote by $C_\tau$ the self-dual [44, 22] code determined by $C_\pi$ and the matrix $\tau(G_\pi(R_{24}))$.

We consider the products of transformations (ii), (iii) and (iv) from Theorem 6 which preserve the quaternary code $C_\varphi$. Their permutation parts form a subgroup of the symmetric group $S_{10}$ which we denote by $L$. Let $S = Stab(R_{24})$ be the stabilizer of the automorphism group of the code generated by $G_\pi(R_{24})$ on the set of the fixed points. It is easy to prove that if $\tau_1$ and $\tau_2$ are permutations from the group $S_{10}$, the codes $C_{\tau_1}$ and $C_{\tau_2}$ are equivalent iff the double cosets $S\tau_1L$ and $S\tau_2L$ coincide. In our case $Stab(R_{24}) = \{(7, 8)(9, 10), (7, 9, 10), (7, 9)(8, 10), (7, 10), (5, 6), (4, 6, 5), (2, 3), (1, 3, 2), (1, 4)(2, 5)(3, 6)\}$.

When $C_\varphi = B_{10}$ we found in [5] a subgroup of the group $L$ generated by the permutations $(3, 4)(8, 9), (1, 2)(3, 4), (1, 3)(2, 4), (6, 7)(8, 9), (6, 8)(7, 9)$ and $(1, 6)(2, 7)(3, 8)(4, 9)(5, 10)$. So we obtain four [44, 22, 8] self-dual codes: $C_{B_{10}}^{ind}$, $C_{B_{10}}^{(567)}$, $C_{B_{10}}^{(36754)}$ and $C_{B_{10}}^{(368574)}$. These codes have weight enumerator $W_{44,1}$ with $\beta = 60, 33, 30$ and 21 and automorphism groups of orders $2^7 \cdot 3^4$, $2^4 \cdot 3^3$, 72 and 48, respectively.

When $C_\varphi = E_{10}$ the group $L = \{(1, 3, 5, 7, 9)(2, 4, 6, 8, 10), (1, 2)(3, 4), (1, 3)(2, 4), (9, 10)\}$. We obtain seven [44, 22, 8] self-dual codes $C_{E_{10}}^\tau$ for $\tau \in \{\text{id}, (4, 5, 6, 7), (4, 5, 7)(6, 9, 8), (2, 3, 5, 4), (2, 3, 5, 4)(6, 7), (2, 3, 5, 7, 4)(6, 9, 8), (6, 7)\}$. These codes have also weight enumerator $W_{44,1}$ with $\beta = 42, 30, 36, 24, 42, 30$ and 21 and automorphism groups of orders $2^{10} \cdot 3, 24, 192, 36, 2^7 \cdot 3^2$, again 24 and 720, respectively.
In this way from all the cases for $C_\pi$ we constructed 1865 inequivalent $[44,22,8]$ self-dual codes with weight enumerator $W_{44,1}$ for $\beta = 10, \ldots, 52, 54, 55, 60, 62, 65$ and 6873 codes with weight enumerator $W_{44,2}$ for $\beta = 3, \ldots, 36, 38, 42, 45, 46, 50$ and 52. The calculations for these results were done with the GAP Version 4r4 software system and the program Q-Extension [1]. The results are summarized in Tables 3 and 4.

Table 3: Extremal self-dual $[44,22,8]$ codes having an automorphism of type 3-(10,14)

| $W_{44,1}$ | $W_{44,2}$ | $W_{44,1}$ | $W_{44,2}$ | $W_{44,1}$ | $W_{44,2}$ |
|------------|------------|------------|------------|------------|------------|
| $C_{24},B_{10}$ | 3 | 12 | $U_{24},E_{10}$ | 74 | 49 | $Y_{24},B_{10}$ | 136 | 746 |
| $C_{24},E_{10}$ | 6 | 25 | $W_{24},B_{10}$ | 71 | 11 | $Y_{24},E_{10}$ | 456 | 2764 |
| $R_{24},B_{10}$ | 4 | - | $W_{24},E_{10}$ | 188 | 33 | $Z_{24},B_{10}$ | 71 | 541 |
| $R_{24},E_{10}$ | 7 | - | $X_{24},B_{10}$ | 161 | 224 | $Z_{24},E_{10}$ | 207 | 1824 |
| $U_{24},B_{10}$ | 29 | 19 | $X_{24},E_{10}$ | 459 | 635 |  |

**Theorem 9** There are exactly 8738 inequivalent self-dual $[44,22,8]$ codes having an automorphism of type 3-(10,14).

Table 4: Extremal self-dual $[44,22,8]$ codes having an automorphism of type 3-(10,14)

| $\beta$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---------|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| $W_1$   | - | - | - | - | - | - | - | 1  | 2  | 11 | 49 | 63 | 25 | 114 |
| $W_2$   | 1 | 3 | 31 | 31 | 93 | 143 | 183 | 377 | 428 | 560 | 622 | 552 | 755 | 510 |
| $\beta$ | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| $W_1$   | 97 | 51 | 159 | 134 | 71 | 157 | 99 | 63 | 129 | 81 | 49 | 90 | 61 | 41 |
| $W_2$   | 411 | 585 | 270 | 223 | 321 | 145 | 96 | 176 | 35 | 71 | 64 | 32 | 13 | 57 |
| $\beta$ | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 |
| $W_1$   | 55 | 31 | 28 | 41 | 21 | 16 | 22 | 21 | 11 | 14 | 11 | 10 | 12 | 4  |
| $W_2$   | 7  | 23 | 16 | 11 | 3  | 8  | -  | 9  | -  | -  | 4  | -  | -  | -  |
| $\beta$ | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 54 | 55 | 60 | 62 | 65 |  |
| $W_1$   | 4  | 4  | 1  | 1  | 1  | 2  | 1  | 2  | 1  | 1  | 1  | 1  | 1  |
| $W_2$   | 1  | 1  | -  | -  | -  | 1  | -  | 1  | -  | -  | -  | -  | -  |-

3.4 Codes with an automorphism of type 3-(12,8)

In this case $C_{\varphi}$ is a quaternary Hermitian self-dual code of length 12 with minimum weight at least 4. There exist exactly five inequivalent quaternary self-dual $[12,6,4]$ codes, denoted by $d_{12}, 2d_{6}, 3d_{4}, e_6 \oplus e_6,$ and $e_7 + e_5$ in [11].

The code $C_\pi$ is a binary self-dual $[20,10,\geq 4]$ code. There are exactly seven such codes, namely $d_{12} + d_8, d_{12} + e_8, d_{20}, d_{5}^3, f_2, d_{8}^2 + d_4,$ and $e_7^2 + d_6$ [6]. Each choice for the fixed
points can lead to a different subcode $F_\sigma(C)$. We have considered all possibilities for each of these seven codes, and found exactly 7 inequivalent codes for $d_{12} + d_8$, one code for $d_{12} + e_8$, one code for $d_{20}$, 10 codes for $d_1^3$, 26 codes for $d_8^2 + f_2$, 18 codes for $d_8^2 + d_4$, and 3 codes for $e_7^2 + d_6$. Denote these codes by $H_{i,j}$, for $i = 1, 2, \ldots, 7$.

By the method used in Section 3.3, considering the permutation parts of the products of transformations (ii), (iii) and (iv) from Theorem 6 and the stabilizer of the automorphism group of the codes $H_{i,j}$ on the fixed points, we classified all codes up to equivalence. There are exactly 122787 inequivalent codes. Their weight enumerators are of type $W_{44,1}$ for $\beta = 10, \ldots, 68, 70, 72, 74, 82, 86, 90, 122$ and of type $W_{44,2}$ for $\beta = 0, \ldots, 56, 58, \ldots, 62, 64, 66, 68, 70, 72, 74, 76, 82, 86, 90, 104, 154$. The values obtained for $\beta$ are listed in Table 5.

**Theorem 10** There are exactly 122787 inequivalent self-dual [44, 22, 8] codes having an automorphism of type 3-(12, 8).

| $\beta$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 |
|---------|----|----|----|----|----|----|----|----|----|----|----|----|
| $W_1$   |    |    |    |    |    |    |    |    |    |    |    |    |
| $W_2$   | 7  | 151| 594| 1434| 2178| 3468| 5793| 7034| 6881| 9434| 10031| 6906 |

| $\beta$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 |
|---------|----|----|----|----|----|----|----|----|----|----|----|----|
| $W_1$   | 313| 1915| 1072| 655| 2141| 1105| 912| 1770| 1029| 736| 1338| 666 |
| $W_2$   | 8502| 7975| 5072| 4805| 5111| 2549| 2552| 2438| 1692| 1176| 1609| 778 |

| $\beta$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 |
|---------|----|----|----|----|----|----|----|----|----|----|----|----|
| $W_1$   | 642| 731| 511| 382| 568| 286| 286| 263| 236| 161| 179| 99 |
| $W_2$   | 773| 745| 532| 311| 484| 204| 242| 169| 217| 65 | 176| 32 |

| $\beta$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 |
|---------|----|----|----|----|----|----|----|----|----|----|----|----|
| $W_1$   | 126| 87 | 88 | 55 | 69 | 38 | 52 | 28 | 48 | 17 | 32 | 10 |
| $W_2$   | 73 | 42 | 68 | 30 | 44 | 29 | 30 | 21 | 21 | 9  | 26 | 10 |

| $\beta$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 |
|---------|----|----|----|----|----|----|----|----|----|----|----|----|
| $W_1$   | 18 | 7  | 19 | 9  | 15 | 5  | 7  | 3  | 11 | 4  | 9  | 5  |
| $W_2$   | 14 | 7  | 17 | 3  | 15 | 4  | 9  | 6  | 13 | -  | 11 | 1  |

| $\beta$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 |
|---------|----|----|----|----|----|----|----|----|----|----|----|----|
| $W_1$   | 3  | 1  | 2  | 1  | 2  | 3  | 4  | 2  | 1  | 2  | 1  | 2  |
| $W_2$   | 6  | 1  | 5  | -  | 2  | -  | 1  | -  | 1  | 1  | 3  | 4  |

| $\beta$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 |
|---------|----|----|----|----|----|----|----|----|----|----|----|----|
| $W_1$   | 76 | 82 | 86 | 90 | 104| 122| 154|
| $W_2$   | -  | 1  | 1  | 1  | -  | -  | -  |
3.5 Codes with an automorphism of type 3-(14, 2)

The code $C_{\pi}$ in this case is a self-dual [16, 8, 4] code. There are exactly three such codes, namely $d_8^2$, $d_{16}$, and $e_8^2$ [6]. We consider their generator matrices in the form

$$G_1 = gen(d_8^2) = \begin{pmatrix} 1000000011000000 \\ 0100000011000000 \\ 0010000000001110 \\ 0001000000001111 \\ 0000100011001011 \\ 0000010011000111 \\ 0000001011011100 \end{pmatrix}, \quad G_2 = gen(d_{16}) = \begin{pmatrix} 1111000000000000 \\ 0011110000000000 \\ 0001111000000000 \\ 0000111100000000 \\ 0000011100000000 \\ 0000001111000000 \\ 0000000111100000 \end{pmatrix},$$

and $G_3 = gen(e_8^2) = \begin{pmatrix} HO \\ OH \end{pmatrix}$, where $H = (I_4|J + I_4)$, $J$ is the all-ones 4 × 4 matrix and $O$ is the 8 × 8 zero matrix. We have to consider permutations on these generator matrices that can lead to different subcodes $F_{\pi}(C)$. From all possibilities for each of these codes we have found exactly 7 different cases for $C_{\pi}$ which can produce inequivalent codes $C$, namely $G_1$, $G_1^{3,16}$, $G_2$, $G_2^{3,16}$, $G_3$, and $G_3^{3,16}$.

The code $C_{\pi}$ is a quaternary Hermitian self-dual [14, 7, 4] code. There are exactly 10 such codes, namely $d_{14}$, $2e_7$, $d_8 + e_5 + f_1$, $2e_5 + d_4$, $d_8 + d_6$, $2d_6 + f_2$, $d_6 + 2d_4$, $3d_4 + f_2$, $2d_4 + 18$, and $q_{14}$ [11].

Again, considering the permutation parts of the products of transformations (ii), (iii) and (iv) from Theorem [6] and the stabilizer of the automorphism group of the codes $C_{\pi}$ on the fixed points, we classified all codes up to equivalence.

When $C_{\pi} = d_{16}$ all codes have weight enumerators $W_{44,1}$ for $\beta = 11, 14, 17, 20, 23, 26, 29, 32, 35, 38, 41, 44, 53, 62, and 65$. When $C_{\pi} = e_8^2$ the weight enumerators are $W_{44,1}$ for $\beta = 10, 13, 16, 19, 22, 25, 28, 31, 34, 37, 40, 43, 46, 49, 52, and 58$. Lastly, when $C_{\pi} = d_8^2$ we constructed codes with weight enumerator $W_{44,2}$ for $\beta = 1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 25, 26, 28, 29, 31, 32, 34, 35, 37, 38, 40, 41, 43, 44, 46, 52, and 55. The total number of all self-dual [44, 22, 8] codes, having an automorphism of type 3-(14, 2) is 243927. The results are presented in Tables 6 and 7.

\textbf{Theorem 11} There are exactly 243927 inequivalent self-dual [44, 22, 8] codes having an automorphism of type 3-(14, 2).

3.6 All self-dual [44, 22, 8] codes with an automorphism of order 3

Here we summarize all obtained results for the extremal self-dual codes of length 44 having an automorphism of order 3. To test the codes for equivalence, we used the program Q-EXTENSION. The classification result is given in the following theorem.
Table 6: Extremal self-dual $[44,22,8]$ codes having an automorphism of type $3-(14,2)$

| $\beta$ | $d_{14}$ | $2e_7$ | $d_8 + e_5 + f_1$ | $2e_5 + d_4$ | $d_8 + d_6$ |
|-------|---------|--------|-------------------|--------------|-------------|
| $d_{16}$ | 7       | 33     | 66                | 26           | 144         |
| $e_8$  | 9       | 20     | 77                | 26           | 197         |
| $d_{8}^+$ | 114     | 876    | 2907              | 490          | 6148        |
|        | $2d_6 + f_2$ | $d_6 + 2d_4$ | $3d_4 + f_2$ | $2d_4 + 1_8$ | $q_{14}$ |
| $d_{16}$ | 573     | 384    | 2040              | 1663         | 1191        |
| $e_8$  | 735     | 496    | 2830              | 2225         | 1561        |
| $d_{8}^+$ | 25841   | 14639  | 84081             | 60246        | 34520       |

Table 7: Extremal self-dual $[44,22,8]$ codes having an automorphism of type $3-(14,2)$

| $\beta$ | 1  | 2  | 4  | 5  | 7  | 8  | 10 | 11 | 13 | 14 |
|-------|----|----|----|----|----|----|----|----|----|----|
| $W_1$ | -  | -  | -  | -  | -  | -  | 704| 984| 1912| 1537|
| $W_2$ | 4565 | 4374 | 21709 | 15796 | 35653 | 26242 | 33236 | 22914 | 21064 | 14322 |
| $\beta$ | 16 | 17 | 19 | 20 | 22 | 23 | 25 | 26 | 28 | 29 |
| $W_1$ | 2006 | 1281 | 1447 | 1008 | 978 | 493 | 480 | 384 | 295 | 147 |
| $W_2$ | 10879 | 6663 | 4407 | 3053 | 1866 | 992 | 621 | 521 | 344 | 152 |
| $\beta$ | 31 | 32 | 34 | 35 | 37 | 38 | 40 | 41 | 43 | 44 |
| $W_1$ | 123 | 124 | 98 | 29 | 17 | 54 | 21 | 8  | 9  | 18 |
| $W_2$ | 109 | 88  | 85 | 19 | 16 | 24 | 14 | 9  | 4  | 2  |
| $\beta$ | 46 | 49 | 52 | 53 | 55 | 58 | 62 | 65 |    |    |
| $W_1$ | 7  | 2  | 3  | 1  | -  | 1  | 1  | 2  |    |    |
| $W_2$ | 4  | 4  | -  | 2  | -  | -  | -  | -  |    |    |

**Theorem 12** There are exactly 394916 inequivalent self-dual $[44,22,8]$ codes having an automorphism of order 3.

We list the number of the codes with different weight enumerators in Table 8. For $\beta \geq 67$, all codes have simultaneously automorphisms of type $3-(12,8)$ and also automorphisms of type $3-(6,26)$. This proves that the orders of the automorphism groups of these codes are multiples of 9. We give these orders in Table 13. All codes with $\beta \geq 63$ have automorphisms of type $3-(6,26)$. All seven codes with $\beta = 0$ have automorphisms of type $3-(12,8)$. The full automorphism group for four of them is the cyclic group of order 3, and the other three codes have automorphism groups of order 12.
Table 8: All extremal self-dual [44, 22, 8] codes having an automorphism of order 3

| β  | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|----|----|----|----|----|----|----|----|----|----|----|----|
| W₁ | -  | -  | -  | -  | -  | -  | -  | -  | -  | -  | 1487 |
| W₂ | 7  | 4713 | 4968 | 1435 | 23881 | 19271 | 5824 | 42768 | 33242 | 9617 | 43614 |
| β  | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
| W₁ | 1539 | 324 | 3860 | 2659 | 680 | 4248 | 2471 | 972 | 3385 | 2182 | 851 |
| W₂ | 30231 | 9070 | 29668 | 19954 | 5666 | 16669 | 9965 | 3804 | 7898 | 5880 | 2440 |
| β  | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 |
| W₁ | 2523 | 1327 | 807 | 1428 | 1080 | 512 | 1051 | 558 | 431 | 515 | 504 |
| W₂ | 4798 | 2963 | 2095 | 2250 | 1985 | 978 | 1465 | 870 | 849 | 965 | 1048 |
| β  | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 |
| W₁ | 266 | 396 | 201 | 213 | 176 | 205 | 95 | 131 | 86 | 92 | 79 |
| W₂ | 761 | 1082 | 606 | 609 | 477 | 526 | 348 | 366 | 258 | 221 | 134 |
| β  | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 |
| W₁ | 115 | 66 | 86 | 47 | 55 | 39 | 55 | 34 | 43 | 18 | 28 |
| W₂ | 151 | 73 | 102 | 44 | 87 | 15 | 51 | 5 | 30 | 5 | 23 |
| β  | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 |
| W₁ | 8 | 25 | 5 | 20 | 5 | 7 | 1 | 6 | 1 | 2 | 3 |
| W₂ | 6 | 15 | - | 17 | 1 | 6 | 1 | 7 | - | 2 | - |
| β  | 66 | 67 | 68 | 70 | 72 | 74 | 76 | 82 | 86 | 90 | 104 |
| W₁ | 5 | 2 | 1 | 2 | 1 | 2 | - | 1 | 1 | - | - |
| W₂ | 1 | - | 1 | 1 | 3 | 4 | 2 | 2 | 1 | 1 | 1 |
| β  | 122 | 154 |
| W₁ | 1 | - |
| W₂ | - | 1 |

4 Extremal Self-Dual Codes of Length 44 with an automorphism of order 7

If σ is an automorphism of a binary self-dual [44, 22, 8] code of order 7, then σ is of type 7-(3, 23) or 7-(6, 2) [9].

Let \( h_1(x) = (x^3 + x + 1) \) and \( h_2(x) = (x^3 + x^2 + 1) \). As \( x^7 - 1 = (x - 1)h_1(x)h_2(x) \), we have \( P = I_1 \oplus I_2 \), where \( I_j \) is an irreducible cyclic code of length 7 with parity-check polynomial \( h_j(x) \), \( j = 1, 2 \). According Lemma 3, \( C_\varphi = M_1 \oplus M_2 \), where \( M_j = \{ u \in C_\varphi \mid u_i \in I_j, i = 1, \ldots, c \} \) is a linear code over the field \( I_j \) and \( \dim I_j M_j = c \). The polynomials \( e_1 = x^4 + x^2 + x + 1 \) and \( e_2 = x^6 + x^5 + x^3 + 1 \) generate the ideals \( I_1 \) and \( I_2 \) defined above. Any nonzero element of \( I_j = \{ 0, e_j, xe_j, \ldots, x^6 e_j \} \), \( j = 1, 2 \) generates a binary cyclic [7, 4, 3] code. Since the minimum weight of the code \( C \) is 8, every vector of \( C_\varphi \) must contain at least two nonzero coordinates. Hence the minimum weight of \( M_j \) is at least 2, \( j = 1, 2 \).

The transformation \( x \to x^{-1} \) interchanges \( e_1 \) and \( e_2 \). The orthogonal condition (ii) from Theorem 2 implies that once chosen, \( M_1 \) determines \( M_2 \) and the whole \( C_\varphi \). So we can
assumed, without loss of generality, that \( \dim_I M_1 \leq \dim_I M_2 \), and we can examine only \( M_1 \).

4.1 Codes with an automorphism of type 7-(3, 23)

Let \( C \) be a binary self-dual \([44, 22, 8]\) code having an automorphism of type 7-(3, 23). Then we have \( \dim_I M_1 + \dim_I M_2 = 3 \). Since the minimum weight of \( M_2 \) is at least 2, we have \( 1 \leq \dim_I M_1 \leq \dim_I M_2 \leq 2 \). Hence \( \dim_I M_1 = 1 \) and \( \dim_I M_2 = 2 \). Then \( M_2 \) is an MDS \([3, 2, 2]\) code over the field \( \mathbb{F}_2 \) and according to condition (ii) from Theorem 2, \( M_1 = \langle (e_1, e_1, e_1) \rangle \) and \( M_2 = \langle (e_2, e_2, 0), (0, e_2, e_2) \rangle \).

In this case \( C_\pi \) is a binary self-dual code of length 26. If \( v = (1100 \ldots 0) \in C_\pi \) then \( \pi^{-1}(v) + (\phi^{-1}(e_2, e_2, 0), 00 \ldots 0) \) will be a codeword from \( C \) of weight 6 which contradicts the minimum weight of \( C \). Hence in the notations of Theorem \( \ref{thm:mds} \), \( k_1 = 0, k_2 = 10, k_3 = 3 \), and \( \text{gen } C_\pi = \begin{pmatrix} 0 & D \\ E & F \end{pmatrix} \), where the matrix \( D \) generates a \([23, 10, \geq 8]\) binary self-orthogonal code. There are three such codes and their generator matrices are given in \([2]\). We take \( E = I_3 \), and we determine the matrix \( F \) using the condition (iii) of Theorem \( \ref{thm:mds} \). For each of the three cases there is a unique possibility for the matrix \( F \), up to equivalence. We obtain the codes \( C_{7,1} \) with weight enumerator \( W_{44,1} \) for \( \beta = 122 \), \( C_{7,2} \) with weight enumerator \( W_{44,2} \) for \( \beta = 104 \), and \( C_{7,3} \) with weight enumerator \( W_{44,2} \) for \( \beta = 154 \). The orders of their automorphism groups are 3251404800 = \( 2^{22} \cdot 3^4 \cdot 5^2 \cdot 7^2 \), 116121600 = \( 2^{13} \cdot 3^4 \cdot 5^2 \cdot 7 \), and 786839961600 = \( 2^{16} \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11^2 \), respectively. All of these codes have automorphisms of order 5 and are known from \([3]\).

**Theorem 13** There are exactly three inequivalent binary \([44, 22, 8]\) codes having an automorphism of type 7-(3, 23).

4.2 Codes with an automorphism of type 7-(6, 2).

Let \( C \) be a binary self-dual \([44, 22, 8]\) code having an automorphism of type 7-(6, 2). Now \( C_\pi \) is a binary \([8, 4]\) self-dual code equivalent either to \( C_{6,2}^4 \) or the extended Hamming code \( E_8 \), generated by the matrices \( G_1 = (I_4 | I_4) \) and \( G_2 = (I_4 | J + I_4) \), respectively where \( I_4 \) is the \( 4 \times 4 \) identity matrix and \( J \) is the all-ones \( 4 \times 4 \) matrix.

In this case \( \dim_I M_1 + \dim_I M_2 = 6 \) and \( 1 \leq \dim_I M_1 \leq \dim_I M_2 \leq 5 \). Hence \( \dim_I M_1 = 1, 2, \) or 3.

**Case I:** \( \dim_I M_1 = 1, \dim_I M_2 = 5 \). It follows that \( M_2 \) is an MDS \([6, 5, 2]\) code, and \( M_1 = \langle (e_1, e_1, e_1, e_1, e_1) \rangle \). If \( C_\pi = C_{6,2}^4 \), then \( C_\pi \) contains a codeword \( v = (v_1, 00) \) such that \( \text{wt}(v_1) = 2 \). Since \( M_2 \) is an MDS code, it contains a codeword \( w \) of weight 2 with the same support as \( v_1 \). But then the codeword \( \pi^{-1}(v) + (\phi^{-1}(w), 00) \in C \) has weight 6 - a contradiction. Therefore \( C_\pi = E_8 \). Fixing the codes \( M_1 \) and \( M_2 \) and considering all binary codes equivalent to \( E_8 \), we found only one \([44, 22, 8]\) code with weight enumerator \( W_{44,1} \) for \( \beta = 38 \) and \(|\text{Aut}(C)| = 8064\).
Case II: $\dim_{I_1} M_1 = 2, \dim_{I_2} M_2 = 4$. We can take
\[
\text{gen}(M_1) = \begin{pmatrix} e_1 & 0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ 0 & e_1 & \alpha_5 & \beta_1 & \beta_2 & \beta_3 \end{pmatrix},
\]
where $\alpha_i \in \{0, e_1\}, i = 1, \cdots, 5$, and $\beta_i \in I_1, i = 1, 2, 3$. Considering all such matrices we obtain nine possibilities such that the minimum weight of $M_1$ is $\geq 2$, up to equivalence. Here $\text{gen}(M_1)$ is written in the form $(I_2 | A)$, where $A$ is one of the following matrices:

\[
A_1 = \begin{pmatrix} e_1 & 0 & 0 & 0 \\ e_1 & e_1 & e_1 & e_1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} e_1 & e_1 & 0 & 0 \\ e_1 & e_1 & e_1 & e_1 \end{pmatrix}, \quad A_7 = \begin{pmatrix} e_1 & e_1 & e_1 & 0 \\ e_1 & e_1 & x e_1 & e_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} e_1 & e_1 & 0 & 0 \\ 0 & 0 & e_1 & e_1 \end{pmatrix}, \quad A_5 = \begin{pmatrix} e_1 & e_1 & e_1 & 0 \\ 0 & e_1 & e_1 & e_1 \end{pmatrix}, \quad A_8 = \begin{pmatrix} e_1 & e_1 & e_1 & 0 \\ e_1 & e_1 & x e_1 & e_1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} e_1 & e_1 & 0 & 0 \\ 0 & e_1 & e_1 & e_1 \end{pmatrix}, \quad A_6 = \begin{pmatrix} e_1 & e_1 & e_1 & 0 \\ 0 & e_1 & e_1 & e_1 \end{pmatrix}, \quad A_9 = \begin{pmatrix} e_1 & e_1 & e_1 & 0 \\ e_1 & e_1 & x e_1 & e_1 \end{pmatrix}.
\]

In the case $C_\pi = C_{1/2}$, denote by $A_{id}$ the $[44,22,8]$ code determined by $(I_2 | A_4)$ and $C_\pi = \tau(G_1)$. There are 21 inequivalent codes, namely $A_{id}^1, A_{id}^2, A_{id}^3, A_{id}^4, A_{id}^5, A_{id}^6, A_{id}^7, A_{id}^8, A_{id}^9, A_{id}^{10}, A_{id}^{11}, A_{id}^{12}, A_{id}^{13}, A_{id}^{14}, A_{id}^{15}, A_{id}^{16}, A_{id}^{17}, A_{id}^{18}, A_{id}^{19}, A_{id}^{20}, A_{id}^{21}$. The code $A_{id}^2$ has an automorphism group of order 786839961600 and is equivalent to the code $C_{7,3}$ constructed above.

In the case $C_\pi = E_3$, denote by $B_{id}$ the $[44,22,8]$ code determined by $(I_2 | A_4)$ and $C_\pi = \tau(G_2)$. There are 19 inequivalent codes, namely $B_{id}^1, B_{id}^2, B_{id}^3, B_{id}^4, B_{id}^5, B_{id}^6, B_{id}^7, B_{id}^8, B_{id}^9, B_{id}^{10}, B_{id}^{11}, B_{id}^{12}, B_{id}^{13}, B_{id}^{14}, B_{id}^{15}, B_{id}^{16}, B_{id}^{17}, B_{id}^{18}, B_{id}^{19}$. The code $B_{id}^2$ is equivalent to $C_{7,2}$, constructed in the previous section.

Case III: $\dim_{I_1} M_1 = \dim_{I_2} M_2 = 3$. Then
\[
\text{gen}(M_1) = \begin{pmatrix} e_1 & 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & e_1 & 0 & \alpha_4 & \beta_1 & \beta_2 \\ 0 & 0 & e_1 & \alpha_5 & \beta_3 & \beta_4 \end{pmatrix},
\]
where $\alpha_i \in \{0, e_1\}, i = 1, \cdots, 5$, and $\beta_i \in I_1, i = 1, 2, 3, 4$. There are 18 inequivalent codes $M_1$ with minimum weight $\geq 2$. We can fix the generator matrices for $M_1$ and $M_2$ and consider all possibilities for $C_\pi$.

When $C_\pi = E_8$ we obtain 64 inequivalent codes with $W_{44,1}$ for $\beta = 10, 17, 24, 31, 38, 52, 122$. In the case $C_\pi = C_{1/2}$ we obtain 87 inequivalent codes with $W_{44,2}$ for $\beta = 0, 7, 14, 21, 28, 35, 42, 56, 154$. The codes with $\beta = 122$ and $\beta = 154$ are equivalent to $C_{7,1}$ and $C_{7,3}$, respectively.

Theorem 14 There are exactly 191 inequivalent $[44,22,8]$ codes having an automorphism of order 7.
Table 9: Automorphism groups of self-dual \([44, 22, 8]\) codes for \(C_\pi = E_8\)

| \(|\text{Aut}(C)|\) | 7  | 14 | 21 | 28 | 42 | 56 | 84 | 112 | 126 | 168 |
|---|---|---|---|---|---|---|---|---|---|---|
| \# codes | 13 | 35 | 1 | 5 | 9 | 2 | 2 | 2 | 1 | 1 |
| \(|\text{Aut}(C)|\) | 252 | 336 | 672 | 1344 | 2688 | 5376 | 8064 | 64512 | 3251404800 |
| \# codes | 1 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |

Table 10: Automorphism groups of self-dual \([44, 22, 8]\) codes for \(C_\pi = C_2^4\)

| \(|\text{Aut}(C)|\) | 7  | 14 | 28 | 42 | 56 | 112 | 168 | 336 |
|---|---|---|---|---|---|---|---|---|
| Number of codes | 49 | 33 | 4 | 1 | 3 | 2 | 1 | 2 |
| \(|\text{Aut}(C)|\) | 672 | 1344 | 2688 | 10752 | 21504 | 43008 | 786839961600 |
| Number of codes | 1 | 2 | 3 | 1 | 1 | 3 | 1 |

The orders of the automorphism groups of these codes are presented in Tables 9 and 10. The weight enumerators of the constructed codes are listed in Table 11.

5 Summary

The self-dual \([44, 22, 8]\) codes having automorphisms of order 11 are classified in [21] and [20]. The codes invariant under an automorphism of order 5 are presented in [3] and [4]. Summarizing these classifications and the results from the previous sections, we obtain the following theorem.

**Theorem 15** There are exactly 395555 inequivalent self-dual \([44, 22, 8]\) codes having an automorphism of odd prime order.

All constructed codes with \(\beta \geq 43\) have automorphisms of order 3. In Table 12 we list the number of codes having an automorphism of odd prime order according to their weight enumerators but only for these values of \(\beta\) for which there are also codes having automorphisms of order 5, 7 or 11, but not 3. For the other values of \(\beta\) the number of all extremal self-dual codes having an automorphism of odd prime order is the same as in Table

Table 11: Weight enumerators of self-dual \([44, 22, 8]\) codes having an automorphism of order 7

| \(\beta\) in \(W_{44,1}\) | 10 | 17 | 24 | 31 | 38 | 52 | 59 | 122 |
|---|---|---|---|---|---|---|---|---|
| Number of codes | 23 | 19 | 14 | 12 | 9 | 4 | 1 | 1 |
| \(\beta\) in \(W_{44,2}\) | 0 | 7 | 14 | 21 | 28 | 35 | 42 | 56 | 104 | 154 |
| Number of codes | 27 | 29 | 32 | 5 | 7 | 1 | 4 | 1 | 1 | 1 |
We can send the generator matrices of the obtained codes by e-mail to everybody who is interested.

Table 12: Self-dual \([44,22,8]\) codes having an automorphism of odd prime order

| \(\beta\) | 0  | 4  | 5  | 7  | 9  | 10 | 11 | 12 |
|------------|----|----|----|----|----|----|----|----|
| \(W_1\)   | -  | -  | -  | -  | -  | 1506 | 1539 | 397 |
| \(W_2\)   | 54 | 23926 | 19293 | 42796 | 9658 | 43639 | 30237 | 9070 |
| \(\beta\) | 14 | 15 | 17 | 19 | 20 | 21 | 22 | 24 |
| \(W_1\)   | 2659 | 680 | 2549 | 3385 | 2182 | 851 | 2561 | 820 |
| \(W_2\)   | 20026 | 5672 | 9965 | 7909 | 5888 | 2445 | 4802 | 2117 |
| \(\beta\) | 25 | 27 | 28 | 29 | 30 | 31 | 32 |
| \(W_1\)   | 1428 | 528 | 1051 | 558 | 431 | 525 | 523 |
| \(W_2\)   | 2251 | 978 | 1470 | 872 | 852 | 965 | 1048 |
| \(\beta\) | 34 | 35 | 37 | 38 | 42 | 44 |
| \(W_1\)   | 396 | 201 | 179 | 207 | 96 | 115 |
| \(W_2\)   | 1090 | 607 | 477 | 526 | 221 | 153 |

Looking at the tables, one can notice that there is only one code for \(\beta = 154\). This code has a large automorphism group - its order is \(2^{16} \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11^2 = 786839961600\). The same is the situation with the codes for \(\beta = 122\) and \(\beta = 104\). These two codes have automorphism groups of orders \(2^{15} \cdot 3^4 \cdot 5^2 \cdot 7^2 = 3251404800\) and \(2^{13} \cdot 3^4 \cdot 5^2 \cdot 7 = 116121600\), respectively.

In Table 13 we present the orders of the automorphism groups of the codes with \(\beta \geq 67\). All these orders are multiples of \(288 = 9 \cdot 2^5\). Actually, all 12 codes with automorphism groups of orders bigger than 400000 have weight enumerators of both types with \(\beta \geq 67\) and are given in Table 13. We list the number of codes \(C\) with full automorphism groups of orders \(6000 < \vert Aut(C)\vert < 400000\) and \(\vert Aut(C)\vert \neq 2^s\), in Table 14. The code with the largest automorphism group (order 368640) which is not listed in Table 13 has weight enumerator \(W_{44,1}\) with \(\beta = 42\). Actually, the full automorphism group for most of the codes (exactly 309666) is the cyclic group of order 3. These codes have weight enumerators of both types with \(\beta \leq 42\).

Looking at the weight enumerators of the extremal codes of length 44 constructed up to now, the following open problems arise:

1. Prove that there are not extremal self-dual \([44,22,8]\) codes with weight enumerator \(W_{44,1}\) for \(\beta = 69, 71, 73, 75, \ldots, 81, 83, 84, 85, 87, 88, 89, 91, \ldots, 121\), or \(W_{44,2}\) for \(\beta = 57, 63, 65, 67, 69, 71, 73, 75, 77, \ldots, 81, 83, 84, 85, 87, 88, 89, 91, \ldots, 103, 105, \ldots, 153\).

2. Are the constructed codes with weight enumerators \(W_{44,1}\) for \(\beta = 61, 63, 68, 72, 82, 86, 90, 122\), and \(W_{44,2}\) for \(\beta = 59, 61, 66, 68, 70, 86, 90, 104, 154\), the unique examples for their weight enumerators?

3. Which of these codes have connections with combinatorial designs?
Table 13: The orders of the automorphism groups of the self-dual $[44, 22, 8]$ codes with $\beta \geq 67$

| $\beta$ | 67 | 68 | 70 | 72 | 74 | 76 |
|---------|----|----|----|----|----|----|
| $\sharp$ codes with $W_{44,1}$ | 2  | 1  | 2  | 1  | 2  | -  |
| $|Aut(C)|$ | 2592 | 5184 | 13824 | 6912 | 6912 | -   |
| $|Aut(C)|$ | 2304 | 18432 | 73728 |  |  |  |

| $\sharp$ codes with $W_{44,2}$ | -  | 1  | 1  | 3  | 4  | 2  |
| $|Aut(C)|$ | -  | 207360 | 69120 | 92160 | 69120 | 207360 |
| $|Aut(C)|$ | -  | 184320 | 14745600 | 23040 | 331776-2 | 165888 |

| $\beta$ | 82 | 86 | 90 | 104 | 122 | 154 |
| $\sharp$ codes with $W_{44,1}$ | 1  | 1  | 1  | -  | 1  | -  |
| $|Aut(C)|$ | 7372800 | 1105920 | 2211840 | -  | 3251404800 |  |

| $|Aut(C)|$ | 663552 | 1105920 | 14745600 | 116121600 | -  | 786839961600 |

Acknowledgements

The authors would like to acknowledge the many helpful suggestions of the anonymous reviewers. We also thank the Editor of this Journal.

The first author thanks the Department of Algebra and Geometry at Magdeburg University for its hospitality while this work was completed, and the Alexander von Humboldt Foundation for its support.

Table 14: Number of the self-dual $[44, 22, 8]$ codes with $6000 < |Aut(C)| < 400000$

| $|Aut(C)|$ | 368640 | 331776 | 207360 | 184320 | 165888 | 98304 | 92160 |
| Number of codes | 1  | 2  | 2  | 1  | 1  | 1  | 1  |

| $|Aut(C)|$ | 73728 | 69120 | 64512 | 61440 | 55296 | 46080 | 43008 |
| Number of codes | 2  | 2  | 1  | 2  | 1  | 1  | 3  |

| $|Aut(C)|$ | 36864 | 34560 | 21504 | 18432 | 15552 | 13824 | 12288 |
| Number of codes | 4  | 2  | 1  | 8  | 1  | 2  | 11 |

| $|Aut(C)|$ | 11520 | 10752 | 10368 | 9216 | 8064 | 6912 | 6144 |
| Number of codes | 1  | 1  | 1  | 6  | 1  | 6  | 35 |
References

[1] Bouyukliev, I. (2007) 'About the code equivalence’, in Advances in Coding Theory and Cryptology, T. Shaska, W. C. Huffman, D. Joyner, V. Ustimenko: Series on Coding Theory and Cryptology, World Scientific Publishing, Hackensack, NJ.

[2] Bouyukliev, I., Bouyuklieva, S., Gulliver, T.A. and Östergård, P. (2006) 'Classification of optimal binary self-orthogonal codes', J. Combin. Math. and Combin. Comput., Vol. 59, pp.33–87.

[3] Bouyuklieva, S. (1997) 'New extremal self-dual codes of lengths 42 and 44', IEEE Trans. Inform. Theory, Vol. 43, pp.1607–1612.

[4] Bouyuklieva, S. (2004) 'Some optimal self-orthogonal and self-dual codes', Discrete Mathematics, Vol. 287, pp.1–10.

[5] Bouyuklieva, S., Yankov, N. and Russeva, R. (2007) 'Classification of the binary self-dual [42,21,8] codes having an automorphism of order 3', Finite Fields and Their Applications, Vol.13, pp.605–615.

[6] Conway, J.H., Pless, V. and Sloane, N.J.A. (1992) 'The binary self-dual codes of length up to 32: a revised enumeration', Journal of Comb. Theory, Ser.A, Vol. 60, pp.183–195.

[7] Conway, J.H. and Sloane, N.J.A. (1990) 'A new upper bound on the minimal distance of self-dual codes', IEEE Trans. Info. Theory, Vol. 36, pp.1319–1333.

[8] Huffman, W.C. (1982) 'Automorphisms of codes with application to extremal doubly-even codes of length 48', IEEE Trans. Info. Theory, Vol. 28, pp.511–521.

[9] Huffman, W.C. (2005) 'On the classification and enumeration of self-dual codes', Finite Fields Appl., Vol. 11, pp.451–490.

[10] Hyun Jin Kim (2010) 'Self-dual codes with automorphism of order 3 having 8 cycles’, Designs, Codes and Cryptography, Vol. 57, pp.329–346.

[11] MacWilliams, F.J., Odlyzko, A.M., Sloane, N.J.A. and Ward, H.N. (1997) 'Self-Dual Codes over GF(4), Journal of Comb. Theory, Ser.A, Vol. 25, pp.288–318.

[12] Pless, V. (1998) 'Coding constructions', in Handbook of Coding Theory, V.S. Pless and W.C. Huffman, eds., Elsevier, Amsterdam, pp.141–176.

[13] Pless, V. and Huffman, W.C. (1998) Handbook of Coding Theory, Elsevier, Amsterdam.

[14] Pless, V. and Sloane, N.J.A. (1975) 'On the classification and enumeration of self-dual codes’, Journ. Combin. Theory, ser. A, Vol. 18, pp.313–335.
[15] Rains, E.M. (1998) 'Shadow bounds for self-dual-codes', *IEEE Trans. Inform. Theory*, Vol. 44, pp.134–139.

[16] Yankov, N. and Russeva, R. (2008) 'Classification of the Binary Self-Dual [44, 22, 8] Codes with Automorphisms of Order 7', *Proceedings of the 37th Conference of the UBM*, Bulgaria, pp.239–244.

[17] Yankov, N. (2007) 'Extremal self-dual [44,22,8] codes with automorphism of order 3 with 14 cycles', *Proceedings of the International Workshop on Optimal Codes and Related Topics (OCRT)*, Bulgaria, pp.249–254.

[18] Yorgov, V. (1987) 'A method for constructing inequivalent self-dual codes with applications to length 56', *IEEE Trans. Info. Theory*, Vol. 33, pp.77–82.

[19] Yorgov, V. (1983) 'Binary self-dual codes with an automorphism of odd order', *Problems Inform.Transm.*, Vol. 4, pp.13–24 (in Russian).

[20] Yorgov, V. (1993) 'New extremal singly-even self-dual codes of length 44’, *Proceedings of the Sixth Joint Swedish -Russian Intern. Workshop on Inform. Theory (Sweden)*, pp.372–375.

[21] Yorgov, V. and Russeva, R. (1993) 'Two extremal codes of length 42 and 44’, *Problems Inform.Transm.*, Vol. 29, pp.385–388.