Local, Deterministic Hidden Variable Theories Based on a Loophole in Bell’s Theorem

Vladimir Z. Nuri
<vznuri@netcom.com>
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Abstract: This paper furthers the long historical examination of and debate on the foundations of quantum mechanics (QM) by presenting two local hidden variable (LHV) rules in the context of the EPRB experiment which violate Bell’s inequality, but which are nevertheless local and deterministic under reasonable definitions of the terms, and coincide approximately with the conventional QM prediction. The theories are based on the general idea of probabilistic detection of particles depending on an interaction of hidden variables within the measuring device and particle, and relate mathematically to Fourier analysis. The crucial discrepancy of variations in the hidden variable distribution based on relative polarizer orientations is isolated which invalidates assumptions in Bell-type theorems. The first theory can be analyzed completely symbolically whereas the second was analyzed using numerical methods. The properties of the second in particular are shown to be approximately consistent with the reported results and uncertainties in all three Aspect experiments. Variation in the total photon pairs detected over orientations is shown to be a basic characteristic of these theories. Some comments on the relevance of active vs. passive locality are made. Two sections consider these ideas relative to energy conservation and the measurement problem (collapse of the wavefunction). One section proposes new experiments.

1 Introduction

The famous debate about the completeness of quantum mechanics (QM) as a physical theory dates at least to the publication of Einstein’s ‘EPR’ paper in 1935 [1], with earlier seeds in the Solvay physics conferences where Einstein debated Bohr via carefully contrived gedanken experiments. Almost seven decades of intense debate, including major theoretical contributions by Bohm [7] and Bell [3], and experimental ones by Aspect [1, 2, 3], have clarified but not resolved the issue in the minds of physicists. Wick [17] is a good informal exposition focusing on historical details of the Bell inequality and Aspect experiments, and Baggott [4] does so with more emphasis on the mathematical formalism.
Greenstein and Zajonc [12] focus on the paradoxes of QM with useful derivations. I found these sources very useful for the multilayer analysis (ranging from the level of mathematical formalism to historical background) that follows.

A dissatisfaction and skepticism for the adequacy of the Copenhagen interpretation promoted by Bohr has emerged in recent times, observed by e.g. Bernstein [6] pp. 38-46. Entire books have been dedicated to the topic of re-examination of the foundations of QM, particularly relative to the EPR experiment, such those of Wick and Greenstein. A consensus has emerged that the foundations of QM are exceptionally solid relative to the history of scientific theories but that there might nevertheless be some deeper order yet waiting to be uncovered. This theme has been explored aggressively by Bohm [8]. The prescription of the Copenhagen interpretation is, to paraphrase informally, “there is no deeper reality.” The intensity with which hidden variable theories have been pursued, particularly in recent decades, indicate the lack of widespread confidence and adherence to this ideology. The interpretation, taken to an extreme, can become the equivalent of an unscientific self-fulfilling prophecy: nothing further will be found if nothing further is sought.

Wick in particular describes how a continual series of modifications and exceptions have been made to the conclusions of Bell-type analysis of the EPRB experiment, based on the feedback between theoreticians and experimenters, in a sort of continuous ongoing game of “find the loophole.” The search has been motivated by so-called “confused realists” looking for ways of preserving their philosophy despite baffling experimental results. After long study myself of the aforementioned excellent expositions and some careful tinkering I have solidified what I consider an excellent candidate for a *bona fide* theoretical breakthrough, which is at least worthy of further exploration.

Within the fruitful context and framework of the EPRB correlated particle problem, I’ve found two variations on a hidden variable theory that is local and deterministic while still predicting the same results of QM (those experimentally observed by Aspect). These are extensions of some empirical results simultaneously considered and observed by David Elm (see the acknowledgments in the final section). This paper will describe this theory, first dissecting and revealing its consistency and plausibility, particularly relative to the careful and established results of Aspect, with a later section devoted to some further speculation on its properties.

### 2 Example LHV theory

Greenstein and Zajonc have a nice mathematical analysis and derivation of a sample local hidden variable (LHV) theory in chapter 5 [12], pp. 119-122. In what follows I will borrow heavily from their presentation. A very similar version can be found in Baggott [4] (pp. 125-130). Each reference also derives Bell’s theorem in this context. In what follows I will presume the reader is familiar with the basic EPRB setup and Bell’s derivation. Merely to set up a framework that can be utilized similarly for the new theory, I will show the process of
derivation for this familiar ‘naive’ or ‘toy’ hidden variable theory sometimes considered in the literature (the first case I am aware of is in [4]).

An elementary example of a hidden variable theory to describe the EPRB experiment based on an objective reality might be as follows. Two Stern-Gerlach analyzers at opposite ends of the source are aligned relative to directions specified by the vectors $\hat{a}$ and $\hat{b}$. The ‘particle’ is ejected from the source with a “spin direction” corresponding to the $V_a$ vector on the $\hat{a}$ detector side and $V_b$ for the $\hat{b}$ detector. In the anticorrelated spin source case, $V_b = -V_a$ (see Figure 1). These vectors $V_a$ and $V_b$ are the “hidden variables” that precisely determine the eventual spin measurements. The detectors will respond either “spin up” or “spin down” as defined by two quantities $A$ and $B$ equal to $+1$ or $-1$ respectively. The product $AB$ of the two random variables (in the sense of being statistical quantities) will have an expected value over a large number of experiments as predicted by QM, $E_{QM}$. According to the theory,

$$E_{QM}(\hat{a}, \hat{b}) = -\cos(\phi)$$

(1)

where $\phi$ is the angle of separation between the two detectors, i.e. the angle between vectors $\hat{a}$ and $\hat{b}$.

As Greenstein and Zajonc observe, a notable, basic, and natural parallel of the hidden variable theory exists to an earlier era of scientific inquiry in which it was proven fundamental and essential. The existence of atoms was not completely definitive during the development of statistical mechanics laws for the behavior of gases. Scientists of the era conjectured the existence of hidden variables such as atomic velocities to explain macroscopic observable quantities such as pressure, and the tremendously successful accuracy of the ensuing theories was taken as strong evidence in favor of the atomic hypothesis. Hence, pressure properties (measurable) can be derived from the overall sum or average distribution of atom velocities (hidden). The statistical nature of the model is then seen and realized as not intrinsic to the underlying phenomena but related to lack of knowledge of hidden variables, the crucial explanation that is instead denied by the Copenhagen interpretation in the case of QM. In fact, even Planck’s breakthrough of quantifying energy transfer could be considered a successful early hidden-variable theory.
Table 1: The function $AB(\theta)$

| $\theta_n$       | $\theta_{n+1}$ | $A$ | $B$ | $AB$ |
|------------------|----------------|-----|-----|------|
| $-\pi/2$         | to $-\pi/2 + \phi$ | $+$ | $+$ | $+1$ |
| $-\pi/2 + \phi$  | to $\pi/2$     | $+$ | $-$ | $-1$ |
| $\pi/2 + \phi$   | to $\pi/2 + \phi$ | $-$ | $+$ | $+1$ |
| $\pi/2 + \phi$   | to $3\pi/2$    | $-$ | $-$ | $-1$ |

Analogously to the technique above, to compute the expected value of $AB$ relative to the sample theory, it is necessary to calculate the product $AB$ for each possible value of the hidden variable and then average over all possible values to obtain the result, here labelled $E_{HV}(\hat{a}, \hat{b})$. Assume the hidden variable is randomly distributed over all orientations during measurements, and that the detector actually registers the sign of the projection of the spin vector along its axis. From the figure, $A = \text{sign}[\cos(\theta)]$ and $B = \text{sign}[\cos(180 + \theta - \phi)] = \text{sign}[- \cos(\theta - \phi)]$; hence $AB = \text{sign}[- \cos(\theta) \cos(\theta - \phi)]$. The average (expected) value of $AB$ over all possible values of the hidden variable $\theta$ is:

$$E_{HV}(\hat{a}, \hat{b}) = \frac{1}{2\pi} \int_{-\pi/2}^{3\pi/2} AB d\theta$$

The integral is simple to compute because the product $AB$ takes on exclusively the constant values $+1$ or $-1$ over the four-part interval as indicated in Table 1. The end formula, which is not consistent with QM, is not relevant to any further considerations here but since following derivations have parallels to the calculation I will include it here:

$$E_{HV}(\hat{a}, \hat{b}) = \frac{1}{2\pi} \left\{ \begin{array}{l}
(1) \left[ \left( -\frac{\pi}{2} + \phi \right) - \left( -\frac{3\pi}{2} \right) \right] \\
(1) \left[ \left( \frac{\pi}{2} \right) - \left( -\frac{\pi}{2} + \phi \right) \right] \\
(1) \left[ \left( \frac{3\pi}{2} + \phi \right) - \left( \frac{\pi}{2} + \phi \right) \right] \\
(1) \left[ \left( \frac{\pi}{2} + \phi \right) - \left( \frac{3\pi}{2} + \phi \right) \right] \\
\end{array} \right\} = \frac{2}{\pi} \phi - 1$$

3 Novel LHV theories overview

This section introduces the underlying framework of new theories in this paper. In general, I will be presenting two LHV theories that are based on a probabilistic detection of a particle at each detector as a function of a hidden variable. So, consider a function $f(\lambda')$, where $\lambda'$ is the hidden variable. This function will model the key aspects of the hidden variable theory as follows.

Typically the LHV theories in the literature consider a hidden variable $\lambda$, given in ‘absolute’ coordinates, as the parameter to the spin detection functions $f_a(\hat{a}, \lambda), f_b(\hat{b}, \lambda)$ where $\hat{a}, \hat{b}$ are the absolute orientations of the analyzers. Yet in contrast one can assume a local coordinate (angle $\lambda'$) of the hidden variable relative to each detector orientation without loss of generality. In the example
in the previous section, \( \theta \) and \( \theta - \phi \) are the angles of the hidden variable relative to detectors \( A \) and \( B \) respectively. This will allow the major simplification of looking for a single detection function \( f(\lambda') \) common to each detector such that the expectation value of the correlation associated with the detection function gives the predictions made by QM. In this formulation \( f(\lambda') \) is different in that not only is it given in local coordinates, but also in that it gives the probability of detection for a given value of \( \lambda' \) during \( dt \), i.e. analogous to a probability density function, such that integration is required to give the cumulative probability of detection.

Again without loss of generality, assume the range of the hidden variable is \( 0 \leq \lambda' \leq 2\pi \). For symmetry purposes let the function be periodic over this interval. Now as alluded, consider a case where \( f(\lambda') \) does not directly give the detected spin orientation at a given detector depending on whether (say, for detector \( A \) with orientation \( \hat{a} \)) a function \( f_a(\hat{a}, \lambda) \) is \( \pm 1 \), the basic and common formulation considered in the literature to date by e.g. [12] p. 123 and originally by Bell [5]. Instead, the following scheme will be used. \( |f(\lambda')| \), i.e. the absolute value of \( f(\lambda') \), will give the probability the particle is detected; hence \( |f(\lambda')| \leq 1 \). If \( f(\lambda') > 0 \), the particle will be “spin up” if detected; similarly the “spin down” case is implied by \( f(\lambda') < 0 \).

The prediction of QM as derived by Clauser et. al. in their seminal 1969 paper for a photon-based version of Bell’s experiment is that the number of particles detected simultaneously at each detector should be constant over different orientations of the analyzers. Or, equivalently, the rate of photons detected per time from a steady source should be constant. This rate is given by the variable \( R_0 \) in their paper which will be considered very closely later herein. For the \( f(\lambda') \) formulation considered here, however, the condition implies the following.

If \( \theta \) is the hidden variable, then \( |f(\theta)f(\theta - \phi)| \) gives the probability a particle is detected simultaneously by both detectors, i.e. a pair is measured at some instant \( dt \). Integrating the probability density function over all hidden variable values yields the cumulative probability function, which should be constant, say \( t \), for all relative orientations of the detectors \( \phi \):

\[
t(\phi) = \int_0^{2\pi} |f(\theta)f(\theta - \phi)| \theta = t \tag{4}
\]

However, I will be considering versions of \( f(\lambda') \) for which it is only true that \( t(\phi) \approx t \).

The expected value of the correlation function can be computed as follows. One can view the function \( f(\lambda') \) as the product of the spin value \( \pm 1 \) with the absolute probability of detection. Then

\[
c(\phi) = \int_0^{2\pi} f(\theta)f(\theta - \phi) d\theta \tag{5}
\]

must give the expected correlation function for all \( \phi \), the product of detected spins over all \( \theta \), not normalized to the number of pairs detected. The expected value of the correlation function is normalized to the total pairs detected,
$t(\phi)$ in Eq. 4, i.e. $E_{HV'} = c(\phi)/t(\phi)$. The prediction of QM for spin-anticorrelated pairs [12] p. 118 is that $E_{HV'} = -\cos(\phi)$. One can summarize all of the above by writing

$$E_{HV'} = \frac{c(\phi)}{t(\phi)} = \frac{\int_{0}^{2\pi} f(\theta)f(\theta - \phi)d\theta}{\int_{0}^{2\pi} |f(\theta)f(\theta - \phi)|d\theta} = -\cos(\phi).$$

(6)

with the condition that the denominator $t(\phi)$ is approximately equal to some constant $t$ over all orientations $\phi$.

4 Fourier transform relevance

The integrals in Eq. 6 have interesting mathematical properties; they can be computed using the theory of Fourier transforms. Let $F(u) = \mathcal{F}\{f(x)\}$ denote the Fourier transform operator on a function $f(x)$. The convolution of two functions $f(x)$ and $g(x)$, denoted by $f(x) * g(x)$, is defined by the integral

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\alpha)g(x - \alpha)d\alpha$$

(7)

Define the autoconvolution of a function $f(x)$ as $f(x) * f(x)$. Then the numerator and denominator of Eq. 3 are autoconvolutions, and Fourier theory states that if $f(x)$ has the Fourier transform $F(u) = \mathcal{F}\{f(x)\}$, then $f(x) * f(x)$ has the Fourier transform $F(u)^2$. This can be written

$$f(x) * f(x) \iff F(u)^2$$

(8)

In other words, to compute the numerator of Eq. 3, one can take the Fourier transform of the function $f(\lambda')$, square it, and then take the inverse Fourier transform:

$$c(\phi) = \mathcal{F}^{-1}\{\mathcal{F}\{f(\lambda')\}^2}\}$$

(9)

The denominator can be computed analogously using an $f'(\lambda') = |f(\lambda')|$. I will not give a mathematical proof of these properties, which follow straightforwardly from basic Fourier theory, but I have verified these properties numerically for the two hidden variable theories considered herein. The technique relies on the periodicity of the function assumed above and it being ‘even’, i.e. $f(\lambda') = f(-\lambda')$.

5 Preliminary critique

The two theories in this paper do not exactly reproduce the predictions of QM. They differ slightly in (a) the correlation expectation curve $E_{HV'} \approx E_{QM}$ (Eqs. 3, 4), and do not predict (b) a strictly unvarying rate of particle pairs detected over all orientations, the function $t(\phi) \approx t$ (Eq. 4). I do not take ‘theory i’ below seriously because of the very large variation in (b), the total particles detected over different orientations, even though it is conceivably consistent with the first
Aspect experiment \[1\]. It is interesting for its tractable and analytic mathematical properties (results can be expressed in closed form) that demonstrate potential implications of these types of theories, and furthermore is very similar to the second theory \[\text{II}\].

However for ‘theory \[\text{II}\]’, as discussed below there is very near agreement for both (a) and (b) to QM predictions, possibly even with the experimental uncertainty bounds of all three Aspect experiments, including the latter two \[\text{2, 3}\] that measured total pairs detected. Disregarding experimental error, however, admittedly the prospects for perfect theoretical agreement on both (a) and (b) via further tinkering with these types of theories (namely those based on probabilistic detection of the particles based on the hidden variable) are doubtful.

Specifically regarding (b), I suspect there is a simple proof that \(t(\phi)\) cannot be constant. Consider the following informal argument. As noted above, the denominator \(t(\phi)\) of Eq. 6 represents an ‘autoconvolution’. Hence the problem reduces to finding a function for which its autoconvolution is a constant, \(t(\phi) = t\). Or in other words, any solution is among the functions \(g(\lambda')\) such that \(t(\phi) = \mathcal{F}^{-1}\{\mathcal{F}\{g(\lambda')\}^2\} = t\) (from the definition of autoconvolution). Using the algebraic properties of Fourier transforms, inverting this gives the unique function \(g(\lambda') = \mathcal{F}\{\mathcal{F}^{-1}\{t\}^{1/2}\}\). But any sequence of Fourier transforms or inverses of a constant must be a constant. This would correspond to a situation in which the probability of detection of a particle doesn’t vary for \(\lambda'\), precisely the case under study.

Regarding (a), it is not clear to me that there is no variation of this type of hidden variable theory that will not give \(E_{\text{HV}} = E_{\text{QM}}\) exactly even though \(t(\phi) \approx t\). I have empirically tested many functions \(f(\lambda')\) numerically and haven’t found anything that can be ‘tweaked’ to arbitrary close degrees of agreement, but neither have I found a proof that no such functions exist, although such a proof might be tricky. Theory \[\text{II}\] below is the best compromise in simultaneously satisfying (a) and (b) I have found to date.

Clearly for these theories there is some relationship between the nearness of approximations of \(t(\phi)\) and \(E_{\text{HV}}(\phi)\) to QM predictions. I do not know whether a ‘better’ theory than \[\text{II}\] is possible, in the sense that a different detection function could yield even better approximations. However using the Fourier theory techniques mentioned earlier I speculate it would be possible to derive an expression for the tradeoff between these two parameters that would give a definitive answer. I want to emphasize that even if it can be shown that the Aspect data is unequivocally incompatible with theory \[\text{II}\], a new variation based on similar principles as \[\text{I, II}\] (i.e. probabilistic detection) probably cannot necessarily be ruled out without such a “tradeoff formula.”

6 The loophole

Finding any function \(f(\lambda')\) such that Eq. 6 holds would give a LHV for the aspects of QM embodied in the particle-pair spin-correlation experiments, and hence it is of paramount importance. It is not immediately obvious that Bell’s
theorem applies to this situation, in which there is a variable detection probability at each detector dependent on the hidden variable, although the theory appears local so far.

I will argue that this theory can be both deterministic, local, and not covered by Bell’s theorem (that is, it nevertheless violates his inequality in close accord with QM) based on a remarkable loophole not previously considered by other researchers. I am not suggesting that Bell’s analysis is mistaken for the cases he considers, only that there is a very tricky subtlety in his mathematical definition of ‘locality’ that is unexpected but can be cleverly exploited. Bell’s assumption in his theorem is that there exists a hidden variable $\lambda$ that precisely determines the spin detected at each detector. Even if there is only probabilistic particle detection as I consider, it would seem one can consider the subset of cases in which both particles are detected. Then it appears to me relevance of the theorem hinges on what determines whether the particle is detected as follows.

Consider the case where there is some hidden property within the particle, $\lambda_d$, that determines whether it is registered by either detector. In other words it has a domain of values denoted by the set $\lambda_d \in D$, for which there are subsets $D_a \subset D$ and $D_b \subset D$ that overlap, $D_{ab} = D_a \cap D_b \neq \emptyset$, which comprises the pairs for which both particles are detected. Then there is some subset of its hidden values, namely $D_{ab}$, for which one can apply Bell’s theorem utilizing new functions $f_a(\hat{a}, m_a(\hat{a} - \lambda))$, $f_b(\hat{b}, m_b(\hat{b} - \lambda))$ that are exclusively $\pm 1$ over the domain $\lambda_d \in D_{ab}$, where $m_a(\lambda')$, $m_b(\lambda')$ are local mapping functions. In this case $m_a(\lambda') = \hat{a} - \lambda'$, $m_b(\lambda') = \hat{b} - \lambda'$, and one can successfully convert or reduce to the standard Bell forms $f_a(\hat{a}, \lambda)$, $f_b(\hat{b}, \lambda)$.

Next, consider the case where there are hidden variables $\lambda_a, \lambda_b$ within each detector that determine whether the particles are detected. If the distribution of $\lambda$ stays the same over all orientations even in this case, then Bell’s theorem applies as well.

Now consider the case where hidden variables in each detector $\lambda_a, \lambda_b$ in conjunction with $\lambda$ determine whether particles are detected. In other words, deterministic detection functions with values $\pm 1$ are $f_a(\hat{a}, \lambda_a, \lambda)$ and $f_b(\hat{b}, \lambda_b, \lambda)$, which are consistent with Eq. 6. Even though this dependence can result from exclusively local interactions, in this situation Bell’s theorem is apparently not applicable. Evidently there exist no local mapping functions $m_a(\lambda_a, \lambda) = m_b(\lambda_b, \lambda)$ to obtain Bell equivalence via a single hidden variable with an identical distribution over orientations except in trivial cases in which the distributions of $\lambda_a, \lambda_b$ are independent of $\lambda$.

Another way of stating this problem is that the Bell theorem assumes a distribution for the hidden variable $\lambda$ that does not vary over polarizer orientations. Greenstein [12, p. 123 says “Bell’s theorem can be extended to the case of a nonuniform distribution of $\lambda$s” (without citation or proof; Bell [5], p. 106 asserts the same). My clarification and qualification is that it can be nonuniform, but must be an identical nonuniform distribution over all orientations. If each detector contains hidden variables $\lambda_a, \lambda_b$ that, in combination with the particle hidden variable property $\lambda$, determine whether the particle is detected,
the distribution of $\lambda$ over the particle pairs detected can vary for different polaronizer orientations, even though there are no nonlocal physical events. In Bell’s original paper [5] this limitation is in the form of the distribution function $\rho(\lambda)$, assumed dependent on a particle-centric hidden property $\lambda$ only, p. 15.

I have arrived at the previous ideas but do not have rigorous formal proof for the exact reason Bell’s theorem is violated and therefore inapplicable in case of probabilistic particle detection based on the hidden variable, although the above sketch might be transformed into one by others. Nevertheless the following LHV theory descriptions will provide numerical proof of the violation of one simple version of Bell’s theorem commonly cited (see Greenstein [12] p. 122, Bell [5] pp. 18,38):

$$|E_{HV}(\hat{a}, \hat{b}) - E_{HV}(\hat{a}, \hat{c})| \leq 1 + E_{HV}(\hat{b}, \hat{c})$$  \hspace{1cm} (10)

One might argue the Bell theorem is not applicable in the following theories not due to a varying hidden variable distribution but instead simply because of the approximations (a) $E_{HV}(\phi) \approx E_{QM}$ and/or (b) $t(\phi) \approx t$. The logical interplay and intertwined dependency of the implications involved are subtle and in my opinion worth further close scrutiny by others. For example, do (a) and/or (b) imply a varying hidden variable distribution? It might seem the whole question is mute if it can be shown that the class of theories based on detection as a function of the hidden variable cannot be consistent with either theoretical or experimental physical results, but I would argue their study would still shed light on the precise scope of Bell’s theorems. A section immediately following the mathematical details critiques their plausibility, particularly relative to the Aspect experiments.

7 Hidden variable theory 1

This section will consider a more sophisticated hidden variable theory that actually produces effects that could be consistent with the first Aspect experiment, [1]. Consider a simple detection function based on the projection of the hidden variable $\lambda$ with the detector axis, and the corresponding correlation and total count functions (Eqs. 5, 4).

\begin{align*}
    f_1(\lambda') &= \cos(\lambda') \hspace{1cm} (11) \\
    c_1(\phi) &= \int_0^{2\pi} \cos(\theta) \cos(\theta - \phi) d\theta \hspace{1cm} (12) \\
    t_1(\phi) &= \int_0^{2\pi} |\cos(\theta) \cos(\theta - \phi)| d\theta \hspace{1cm} (13) \\
    c'_1(\theta, \phi) &= \cos(\theta, \phi) \cos(\theta - \phi) \hspace{1cm} (14)
\end{align*}

The sign of $c'_1(\theta, \phi)$ (Eq. 14) gives the expected correlation product of spins detected, $-1$ for opposite spins and $+1$ for the same. The absolute value $|c'_1(\theta, \phi)|$ gives the probability of detecting the pair. These two functions are graphed in Fig. 2. From the figure, it is clear the distribution of $\lambda$ (i.e. $\theta$) changes over polarizer orientations $\phi$. The function $|c'_1(\theta, \phi)|$ is periodic on the interval $[-\pi/2, \pi/2]$, with minima of zero at those points and $\theta = \pi/2 + \phi$, where no pairs
Figure 2: LHV theory \( c'_1(\theta, \phi) \) and \(|c'_1(\theta, \phi)|\) for \( \phi = \frac{\pi}{3} \) (Eq. 11)

Figure 3: LHV theory \( c_\pm(\phi) \) and \( t_1(\phi)/2 \) (Eqs. 15)

are detected; the maxima of one is at \( \frac{\phi}{2} \) where the pair is always detected as correlated.

Eq. 13 can be integrated symbolically even with the absolute value complication using the trick of separation into the intervals \([-\frac{\pi}{2}, -\frac{\pi}{2} + \phi]\) where \( c'_1(\theta, \phi) \) is negative and \([-\frac{\pi}{2} + \phi, \frac{\pi}{2}]\) where it is positive, and doubling the value of the integral for the full interval \([-\frac{\pi}{2}, \frac{\pi}{2}]\) based on the periodicity. The separation in fact is equivalent to dealing with the anticorrelated and correlated cases separately, respectively. In other words, let the functions \( c_-(\phi), c_+(\phi) \) represent the probability \( 0 \leq c_\pm(\phi) \leq 1 \) of detecting an anticorrelated or correlated pair relative to \( \phi \), respectively, over the half-period. Then

\[
\begin{align*}
c_-(\phi) &= \int_{-\pi/2}^{-\pi/2+\phi} c'_1(\theta, \phi)d\theta \\
c_+(\phi) &= \int_{-\pi/2+\phi}^{\pi/2} c'_1(\theta, \phi)d\theta \\
t_1(\phi)/2 &= \int_{-\pi/2}^{-\pi/2+\phi} |c'_1(\theta, \phi)|d\theta + \int_{-\pi/2+\phi}^{\pi/2} |c'_1(\theta, \phi)|d\theta \\
&= c_-(\phi) + c_+(\phi)
\end{align*}
\]

Symbolic integration and simplification via Mathematica software yields

\[
\begin{align*}
2c_-(\phi) &= \sin(\phi) - \phi \cos(\phi) \\
2c_+(\phi) &= \sin(\phi) + (\pi - \phi) \cos(\phi) \\
t_1(\phi) &= 2 \sin(\phi) + (\pi - 2\phi) \cos(\phi) \quad (16)
\end{align*}
\]

A graph of the two functions \( c_\pm(\phi) \) and their sum \( t_1(\phi)/2 \) appears in Fig. 3. The total pair count curve has both upward and downward concavity. The
maximum deviation from the mean of $\frac{\pi}{2} + \frac{1}{2} = 1.2853$ is $\pm (\frac{\pi}{2} - \frac{1}{2}) = \pm .2853$ or $\pm 22.3\%$, with a standard deviation of 15.7\% for an equidistant 50 point sample over the interval—clearly completely physically incorrect. However, it does show exactly how a purely local theory can exhibit fluctuations in the total photon pairs detected over polarizer orientations, even with a constant flux into each detector during the interval $dt$.

Conveniently, $c_1(\phi)/2 = c_+ (\phi) - c_-(\phi)$, so $c_1(\phi) = -\pi \cos(\phi)$. Hence for this theory (Eq. 6),

$$E'_{HV} = \frac{c_1(\phi)}{t_1(\phi)} = -\frac{\pi \cos(\phi)}{2 \sin(\phi) + (\pi - 2\phi) \cos(\phi)} \quad (17)$$

A comparison graph of $E'_{HV}(\phi)$ vs. $E_{QM}(\phi)$ is given in Fig. 4. The maximum absolute difference is $\pm 19.8\%$ with a standard deviation of 12.7\% for an equidistant 50 point sample over the interval. The graph is striking in that it reproduces the concavity of the QM prediction, in contrast to the earlier naive hidden variable theory (Eq. 3) which predicts a straight line.

The Bell formula inequalities can now be computed and the violation demonstrated based on Eq. (10), say for $a = 0, b = \frac{\pi}{3}, c = \frac{2\pi}{3}$ with $E'_{HV}(a,b) = E'_{HV}(b-a)$:

$$|E'_{HV}(0,\frac{\pi}{3}) - E'_{HV}(0,\frac{2\pi}{3})| = 1.39277$$

$$1 + E'_{HV}(\frac{\pi}{3},\frac{2\pi}{3}) = 0.30362 \quad (18)$$

8 Hidden variable theory II

A slight variation on theory i above gives significantly improved results. This section will consider a model for the case where correlated spin particles emanate from a source, with a reduction of the theory to the anticorrelated case requiring only some simple geometric transformations (namely translation and scaling of the hidden variable). Consider the function

$$f_2(\lambda') = \cos(\lambda')^{1/|e|} \quad (19)$$

where $e$ is the base of natural logarithms, $e \approx 2.7183$, and $1/e \approx .3679$, and the definition of exponentiation in this function is adjusted for negative values in
the following way:

\[
|a|^b = \begin{cases} 
  a^b & \text{if } a \geq 0 \\
  -(a)^b & \text{if } a < 0 
\end{cases}
\] (20)

A graph of \(f_2(\lambda')\) vs. \(f_1(\lambda')\) (Eq. 11) can be found in Fig. 5. The unnormalized correlation (Eq. 5) and total flux functions (Eq. 4) are defined (analogously to Eqs. 12, 13) as

\[
c_2(\phi) = \int_{0}^{2\pi} \cos(\theta)^{1/|e|} \cos(\theta - \phi)^{1/|e|} d\theta \\
t_2(\phi) = \int_{0}^{2\pi} |\cos(\theta)^{1/|e|} \cos(\theta - \phi)^{1/|e|}| d\theta
\] (21)

(again with the specialized exponentiation operator).

These integrals almost certainly have no closed-form solution yielding a precise symbolic integration. However numerical integration is not difficult. I wrote a simple program in Mathematica that used rectangular integration over 50 equally-spaced points for \(0 \leq \theta \leq \pi\) and the half-interval \(0 \leq \phi \leq \pi\) (utilizing symmetry). To test it I applied it to formula \(f_1(\lambda') = \cos(\lambda')\) (Eq. 11) and found an identical result to Eqs. 16 with negligible discrepancy due to numerical inaccuracy.

The results of the integration and normalization \(E''_{\text{HV}}(\phi) = c_2(\phi)/t_2(\phi)\) for Eqs. 21, 22 are shown in Fig. 6 with \(t_2(\phi)/2, E'_{\text{QM}}\) in the same graph. For correlated particles, \(E'_{\text{QM}}(\phi) = \cos(\phi)\) where here \(\phi\) is twice the angular difference between polarizer orientations (see e.g. Greenstein [12], p. 136). More impressively for this theory II, the total particle pairs detected is closer to a constant (the mean \(t_2/2 = 2.07\)). Most remarkable is the very near equivalence to the
QM prediction $E_{QM}'$. The differences in these two key measures are shown in Fig. 7. $E''_{HV}$ differs from $E'_{QM}$ by a maximum of $\pm 1.2\%$ with a standard deviation of $0.8\%$ (sampled over the 50 points). $t_2$ differs from the mean $\overline{t_2}$ by a maximum of $\pm 5.7\%$ with a standard deviation of $3.7\%$. In contrast to the first theory (Fig. 3) the total pair detection count curve appears to be concave upward only.

Because of the close correspondence between $E''_{HV}, E'_{QM}$ the theory II function $f_2$ violates the Bell inequality almost exactly in accordance with QM.

9 Experimental plausibility

The meticulous experiments of Aspect [1, 2, 3] are extremely useful and crucial constraints for investigating the plausibility of any alternative QM theory, particularly any supposedly LHV varieties. The experiments are widely regarded as definitive proof in favor of conventional QM, or at least the impossibility of any local hidden variable theories. Nevertheless the results above suggest a review of the exact experimental findings of the papers would be reasonable.

Consider the first Aspect experiment, [1]. In this experiment, because of a polarizer setup that either blocked or passed photons of a particular polarization, assumptions had to be made about the equivalence of the absence of detected photons to the existence of photons of opposite polarization from those detected. That is, the experiment could not detect photons of both ‘horizontally’ and ‘vertically’ oriented spins at each detector. The effect of this is that the experiment cannot measure total photons detected over all orientations. Aspect et. al. then must measure the maximum photon rate detected for aligned polarizers, i.e. $R_0$, and assume it stays constant over all orientations.

But viewing Eq. 17 of theory i, it is apparent that the theory will predict correct results if the expectation value is not normalized to the varying total number of photon pairs over all orientations. Eq. 16 will give a $t_{max} = \pi$ measured in the aligned case, and $c_1(\phi)/t_{max} = -\cos(\phi)$. I regard this only as an interesting theoretical curiosity, however, given that the later Aspect experiments indeed measured constant total photon pairs detected.

Nevertheless the second Aspect experiment [2] had experimental uncertainties that are conceivably consistent with theory II. They reported a $\pm 1\%$ vari-
ation in the detection efficiency of the polarizers and a variation from $E'_{QM}$ “better than 1%.” These two uncertainties combined might be enough to be within the bounds of theory $\Pi$.

Most importantly Aspect et. al. conspicuously did not report the crucial uncertainty in the ‘constant’ fourfold-sum coincident detection rate (or simply, the photon pair rate) $R_{\pm\pm}(\hat{a}, \hat{b})$, although they do state the rate itself was ‘typically’ 80 s$^{-1}$ (\cite{aspect}, p. 93). I would consider it striking if it varied in a curve similar to that of Fig. 5.

A final factor in these experiments that might support LHV theory $\Pi$ is a theoretical distinction that has been asserted by Santos in \cite{santos}. He insists that the expression for the “depolarization factor” derived by Clauser et. al. in \cite{clauser} is in fact dependent on the polarizer angle difference as well as the detector light cone angle, $F(\theta, \varphi)$, whereas their expression is only dependent on the latter. (Note that Santos renamed some variables rather confusingly relative to their paper and my own. In his case $\theta$ is the polarizer angle difference and $\varphi$ is the detector light cone half-angle. Clauser et. al. use $\theta$ to refer to the light cone half-angle and $\varphi$ to the polarizer angle difference.) If there is fundamental disagreement on this point about ‘depolarization’ related to polarizer angles then it is conceivable that even theory $\Pi$ could be more than a theoretical curiosity.

\section{10 Active vs. passive locality}

Diverse researchers have formulated increasingly rigorous and precise qualifications on Bell’s theorem and isolated its key assumptions since its origin. Theories based on probabilistic detection dependent on hidden variables are not specifically addressed in general but surely fall under known general categories, although their exact classification is a delicate issue. Even though they are approximate they may raise issues and reveal potentials.

An excellent technical analysis of Bell’s theorem’s precise nature is given by Faris in an appendix to Wick \cite{faris}, pp. 227-279. He frames the distinction observed in the previous section in that “QM cannot be reduced to probability on a single probability model. . . . The first Bell theorem rules out the possibility that QM is described by a unique probability measure.” However, a ‘family’ of probability measures depending on the measurement “indeed is possible” (p. 270). He indicates that Bell was aware of the exception and formulated a “second theorem” intended to make more manifest these hidden assumptions, apparently referring to \cite{bell} pp. 105-110.

Faris makes the distinction between active and passive locality, the two crucial prerequisites for the new theorem. A violation of active locality would support a nonlocal physics capable of superluminal signalling. On the other hand, Faris writes on the other condition, “If deterministic passive locality were violated, it would be possible that there would be randomness at the measurement stage with no origin in any preparation stage, yet which would still maintain the perfect correspondence between events at distant locations” (p. 272).

(Determinism is implied by passive locality, p. 244.)
The theories of this paper appear to me to violate ‘passive locality.’ If so I object to misleadingly labelling the latter property, “deterministic passive locality,” either a ‘local’ or ‘deterministic’ constraint, because the theories violate it (via the probability of detection in the detectors) yet are fully local and deterministic. The ‘randomness’ referred to by Faris may be that the probability of detection of particles is dependent on an interaction of hidden variables within both the measuring device and the particles. This would be not so much ‘non-local’ as it is ‘nonreductionistic’ such that any model that takes only the device or the particle properties into account separately must invoke counterintuitive and unrealistic ‘nonlocal’ effects. Wick [17] writes p. 220,

> The existence of the apparatus, it seems, transforms the hidden variables describing the particle pair. Either these variables are altered deterministically depending on the settings of both analyzers (violating active locality on a hidden level) or Nature throws dice in a way depending on both (violating passive locality).

But theory II of this paper exhibits a varying distribution of hidden variables over orientations even via a local and deterministic physics, yet yields close agreement with QM predictions. Faris writes on p. 278, “a violation of passive locality would only mean that dependence between simultaneous events at distant locations need have no explanation in terms of prior events. This is not clearly ruled out, but it is not evident how to construct such a theory.” One might qualify and rephrase this as “…no explanation in terms of prior events based on a reductionistic perspective.” Eq. 6 is a mathematical constraint sufficient for such a theory, and the LHV theories I,II satisfy it to varying degrees of approximation.

Others have isolated many of the subtle prerequisites of the Bell theorems in theory and in practice. A contrived exception to exhibiting the Bell inequality in practice that depends on detector efficiency upon which a LHV theory can be constructed is noted in a comprehensive 1978 review article by Clauser and Shimony (p. 1913). They write, “Although the selection is done locally, it does have the appearance of being highly artificial and, indeed, almost conspirational against the experimenter.” My findings are that generally the LHV theories I,II are novel relative to prior literature overall, and my opinion in contrast is that they are not at all physically implausible.

11 Beam density fluctuation

Clauser and Shimony consider various qualifications such as the ‘CHSH’ and weaker ‘CH’ assumptions about the probabilities of detecting particles relative to polarizer equipment, [14] p. 1904. Santos [13] constructed a mini-theory for the aligned polarizer case based on a violation of the latter CH “no enhancement” hypothesis \( P(\lambda, a) \leq 1 \) but not related to detector efficiency. He defines a photon density function in terms of two hidden variables \( \rho(\lambda_1, \lambda_2) \), analogous to my own use of a probability density function, where \( \lambda_1, \lambda_2 \) are associated with
the separate detectors and detection, for say $a$, is proportional to the product $\rho(\lambda_1, \lambda_2)P(\lambda_1, \hat{a})$, which he apparently presumes can be given in terms of a function of the detector variable $\lambda_1$ and the particle variable $\lambda$. However, he doesn’t explain this in particular, i.e. how such a density might come about based on exclusively local effects.

There is some further discussion of $\rho(\lambda)$ on pp. 105-110 of Bell [3]. He insists that “the difficulty would not arise... if $\rho_1$ were allowed to depend on $b$, or $\rho_2$ on $a$. Such a dependence would not only be of mysteriously long range, but also, for the case presented, would have to propagate faster than light. The correlations of QM are not explicable in terms of local causes.” I disagree that a $\rho_1(\lambda, \lambda_a, \lambda_b)$ would necessarily be nonlocal, with theory II again as the counterexample. (Although this unexpected dependence is certainly counterintuitively nonreductionistic.) The paper seems to me to get caught up in the crucial distinction between correlation and causality:

Now surely it would be very remarkable if the choice of program in Lille proved to be a causal factor in Lyon, or if the choice of program in Lyon proved to be a causal factor in Lille. It would be very remarkable, that is to say, if $\rho_1$... had to depend on $b$, or $\rho_2$ on $a$. But, according to QM, situations presenting just such a dilemma can be contrived.

But a function $\rho_1(\lambda, \lambda_a, \lambda_b)$ may embody a correlation or bias that is implicit in the selection sample, without implying causality, even when it appears the chosen sample ought not to have biased properties. This is apparently the “passive locality” assumption which as indicated might be more accurately termed “passive reductionism.”

In general it appears to me prior research has not explicitly isolated the crucial distinctions between detection functions that depend on detector-based hidden variables interacting in a dependent way with the particle property $\lambda$, although Santos seems to have unconsciously invoked it. The difficulty is aggravated in that very subtle formal differences between equivalent and inequivalent mathematical formulations do not make the differences obvious.

12 Energy conservation

David Elm (see the final section) has collected some objections to a theory based on a variation in detection depending on the hidden variable. One main objection by his correspondents is that such a theory would violate conservation of energy. Looking closer at this claim, in general conservation of energy in physical theories can be stated in several different ways:

\[
E_{\text{in}} = E_{\text{out}} \tag{23}
\]
\[
E_{\text{in}} \geq E_{\text{out}} \tag{24}
\]
\[
E_{\text{in}} = E_{\text{out}} + s \tag{25}
\]
(23) implies all energy is precisely accounted for. (24) and (25) are equivalent, where the former suggests that energy is dissipated in the system that isn’t measured, and the latter implies that it is contained in some internal property that cannot be directly measured, say s, such as entropy. A supposed “violation of energy conservation” with novel LHV theories is only true in the sense that (23) is not applicable if not all energy can be accounted for because some particles are not always detected. But this is not a physical impossibility, because (24) or (25) may still apply, and indeed are routinely invoked to describe virtually all physical experiments.

Generally I would say that a “violation of energy conservation” (in the sense of the laws of thermodynamics) only really occurs if it can be shown that $E_{\text{out}} > E_{\text{in}}$, neither of which (24) or (25) imply. That is, it is not at all necessary to account for all energy fluctuations exactly, except unless the theory claims to be complete, in which case the inequality (24) would indeed seem to be inadequate. But this would be a problem for QM, not any LHV theory that implicitly insists that conventional QM is incomplete. Actually, for this reason, any experiment that could show energy that was consistently not accounted for by QM would tend to support the existence of any other theory (25) that somehow accounted for unmeasurable dissipation ‘s’ at least with theoretical satisfaction.

Theories 1,II indeed postulate unmeasured energy in the form of particles (photons) released from the source but not detected. An ‘independent’ calculation of energy in the experiment could possibly reveal the discrepancy. However typically in experimental arrangements no such information can be determined; the energy released by the source can only be inferred by the flux into detectors.

13 Collapse of the wavefunction

Finally, a short idea speculating on the measurement problem relative to theories 1,II is apropos here. The crucial nonreductionistic but not nonlocal interaction between experimental apparatus and the measured object is revealed by these theories. In the particle-pair correlation experiments, hidden variables $\lambda_1, \lambda_2$ within each detector might interact with the hidden variable $\lambda$ within the particle to determine detection probabilities, even locally, and in conjunction lead to, but not cause, a nonlocal correlation. (Equivalently, the distribution of values for the particle-based hidden variable $\lambda$ varies depending on orientation for the detected subset of pairs.)

Perhaps after this interaction, the hidden variable $\lambda$ within the particle is fixed to the measured state. This might be sufficient to explain the notoriously mysterious “collapse of the wavefunction” that has plagued QM thought since its inception. In other words, in the language of decoherence (see e.g. Greenstein [12] ch. 8), prior to the measurement the state of the system is a nonlocal superposition of $\lambda_1, \lambda_2, \lambda$, but the measurement leads to a mixture in which $\lambda$ ‘collapses’ to some measured state if detected.

Somewhat related to this, some researchers are pursuing LHV theories based on a return to a locally realist theory of the electromagnetic wave with nonclas-
sical fluctuations of the Zero Point Field (ZPF) that would be regarded as ‘noise’ in QM theories \cite{13, 14, 15}. I am partial toward these efforts for which theory II may be compatible.

14 Proposals

As stated, Aspect observed $\pm 1\%$ detection efficiency variations over orientations in the second experiment \cite{2}, which might actually be related to photon density variations as predicted by theory II, Fig. 6. They also observed $\pm 1\%$ variation in $E_{\text{expt}} \approx E_{\text{QM}}$. Their combined reported uncertainties are approximately large enough such that the results could be considered consistent with LHV theory II.

Based on the ideas in this paper and the Aspect reports I have a few suggestions for physical experiments:

- The existing data for the Aspect experiments could be reexamined and reinterpreted in light of these findings since they did not report the crucial degree of variation in the ‘constant’ fourfold-sum coincident detection rate (or simply, photon pair rate) $R_{\pm\pm}(\vec{a}, \vec{b})$. In particular this key measurement of total photon pair counts could be plotted over orientation. If any trend is observed, it could be favorable toward theory II.

- Or, for photons, new experiments could try to gain greater accuracy than the existing Aspect experiments which are widely considered definitive. Some would not consider this important, but precision in other physical constants has been pursued at great effort to distinguish between competing theories, such as the measurement of the magnetic moment of the electron to verify the predictions of quantum electrodynamics (QED). Arguably the locality of QM is at least as fundamental and paramount.

- New Bell-type experiments that involve particles with masses instead of massless photons might be performed. In some ways the similarity of predictions of QM for these two seemingly widely divergent situations is remarkable. Unfortunately experimental intractability is high.

15 Acknowledgements

Here I will give a brief footnote about the historical priority of some of these ideas for future reference. In 1997 I speculated that a rule based on a hidden variable that determined the probability of a particle being detected might simultaneously explain the results and exhibit both determinism and locality. Toward this end I found the excellent intuition and guidance of Wick’s book particularly absorbing, and the straightforward mathematical derivations of Greenstein very helpful. I intended to experiment empirically with different formulas, starting with the simplest possible I could imagine, in which the probability of detection was proportional to the projection of the hidden variable direction vector onto the detector direction vector.
In 1998, after an extensive search of the web prior to any tinkering, I discovered David Elm’s results (web site: www.tiac.net/users/davidelm/epr.htm; email: <davidelm@tiac.net>) which give empirical evidence that the simple projection rule $f_1$ is sufficient in some sense. However, at the time of writing, Elm describes his results in somewhat simplistic and obscure terms of a game he calls “Circles and Shadows.”

Elm apparently tested a projection rule similar to $f_1$ using computer simulations, although I couldn’t verify this except in retrospect. The information in his description is not presented in a mathematical format and was somewhat unclear and not entirely sufficient for me to understand his formulation, and he did not include his code on the web site or respond to an email query on my part to obtain it. Hence I couldn’t verify he was referring to the simple projection rule until I obtained similar results by somewhat independent analysis, and seeing in retrospect our expectation curves (which he plotted) were approximately the same in each case.

Elm frames his results as a demonstration of erroneous reasoning by Bell. I prefer the point of view advanced above that Bell was correct in his derivations but a subtle loophole in his definitions of ‘local’ and ‘deterministic’ can lead to a new theory, although the viewpoints are essentially identical.

On the $R(\phi)/R_0$ issue, Elm writes: “…the ‘simple logic’ used by Bell seems to me to contain a vital flaw. I believe the mistake is manifested in the way this reasoning causes one to scale of the data [sic] to plot it against the graph of the upper limit as put forth by Bell.” He writes later:

It seems more logical that the detection of each photon arriving at each polarizer is simply a probability based on the relative angles between the polarizer and the spin vector of the photon.

Elm states this would imply “a substantial amount of events would be missing from the data” whereas “Bell’s inequality is only applicable if it includes all the data.” However, Elm doesn’t note that the second experiment of Aspect [2] that detected constant total photon pairs for all polarizer orientations would tend to reject the simple theory in which total pairs detected varies tremendously. In fact Elm’s account is not aware at all of the key constraint on any LHV theory verified by that experiment that total photon pairs detected should be roughly constant. Hence I think his arguments are mostly relevant (and limited) to the first Aspect experiment in which the lack of measured events is equated with the existence of anticorrelated particle pairs, in spite of his insistence that “all EPR experiments have used this faulty scaling as the basis of their determination of the validity, or lack of, the local reality views…”

In consideration of all this I would say that we have independently discovered at least the implication of probabilistic yet local detection violating Bell’s theorem, and possibly also the same basic principle for the simple projection rule $f_1$; all justifiable credit goes to Elm for his prior informal and empirical work in this area. I am grateful for his communication of the knowledge and perspectives his experiments led to, which were explored, according to his web site, possibly as early as 1993. Various objections he posted to his web site by
respondents were also helpful in gauging the legitimacy of the theory and likely reactions, on the energy conservation discrepancy in particular. The mathematical analysis of the simple projection rule $f_1$, the advanced projection rule $f_2$, and the following observations and speculations are my own.

Finally, I would like to thank V. S. for various resources and encouragement, and Wolfram Research for the Mathematica software which was used extensively throughout the paper.

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