SOME IDENTITIES AND FORMULAS INVOLVING
GENERALIZED CATALAN NUMBERS

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A generalization of the Catalan numbers is considered. New results include binomial identities, recursive relations and a close formula for the multivariate generating function. A simple expression for the Catalan determinant is derived.

Key words: Catalan numbers, paths, generating functions

1. Introduction

By a path, we mean a finite sequence \( \{a_i\}_{0 \leq i \leq n} \) where \( a_i \in \mathbb{Z}^2 \) and either \( a_{i+1} = a_i + (1, 0) \) or \( a_{i+1} = a_i + (0, 1) \). That is to say, a walk from \( a_0 \) to \( a_n \) whose steps consist of horizontal or vertical positive unit movements.

The classical Catalan number is the number of paths from the origin to \((n,n)\) without crossing the diagonal. It is given by \( \frac{1}{n+1} \binom{2n}{n} \) and has many well-known interpretations (Cf [4]). In [7] we came across the following natural generalization:

Definition 1. Let \( 0 \leq m \leq n \). The generalized Catalan number is defined as the number of paths from \((0,-2m)\) to \((n-m,n-m)\) without crossing the line \( y = x \) and is denoted by \( C_{n,m} \).

For completeness, we give in [22] a proof of \( C_{n,m} = \frac{2m+1}{n+m+1} \frac{2n}{n+m} \). The number \( C_{n,m} \) has an ancient history. In [1] Bertrand studied \( a_{n,m} \), the number of paths from the origin to \((n,m)\) without crossing the diagonal, as the number of solutions to the ballot problem. One verifies that \( C_{n,m} = a_{n+m,n-m}, \quad n \geq m \).

(For another derivation, one can check that \( C_{n,m} = d_{1-2m}(1+n-m) \) for the \( d_{qk} \) defined in [5].)

We then move on to proving new identities and formulas.

In [3] we consider some variations of \( C_{n,m} \) and compute closed formulas for them. In [4] we derive some binomial identities by counting Catalan numbers and their...
variants. In §5 we obtain recursive relations for the generalized Catalan numbers, from which identities for the central binomial coefficients and an identity for certain sum of products of the generalized Catalan numbers are given. In the main section §6 we give a close formula for the multivariate generating function for the generalized Catalan numbers. In §7 more formulas are produced, including certain product of the generating functions. We end in the final section §8 with an elegant formula for the Catalan determinant.

2. Generalized Catalan numbers

For completeness, we give a direct derivation of a close formula for \( C_{n,m} \) although as remarked earlier it is already included in references such as [1] and [5].

**Theorem 2.** Given integers \( 0 \leq m \leq n \), we have

\[
C_{n,m} = \frac{2m+1}{n+m+1} \binom{2n}{n+m}.
\]

**Proof.** Fix \( 0 \leq m \leq n \). We let \( \Lambda \) be the set of paths from \((0, -2m)\) to \((n - m, n - m)\) and \( \Gamma \) be the subset of “bad” paths, i.e. those that cross the line \( y = x \). So \( C_{n,m} = |\Lambda| - |\Gamma| \).

Given a path \( P \in \Gamma \), let \( k \) be smallest such that \( P \) crosses \( y = x \) at \((k, k)\), i.e. both \((k, k)\) and \((k, k+1)\) belong to \( P \). Let \( Q \) be the path obtained from \( P \) by reflecting across the line \( y = x + 1 \) only the portion of \( P \) from \((0, -2m)\) to \((k, k+1)\). The result is a path from \((-2m - 1, 1)\) to \((n - m, n - m)\). See Figure 1 above. This is simply the André’s reflection method and one can easily see that it establishes a bijection between \( \Gamma \) and the set of paths from \((-2m - 1, 1)\) to \((n - m, n - m)\). Therefore

\[
|\Gamma| = \binom{2(n - m) - (-2m - 1 + 1)}{n - m - 1} = \binom{2n}{n - m - 1}.
\]

On the other hand,

\[
|\Lambda| = \binom{2(n - m) - (-2m)}{n - m} = \binom{2n}{n - m} = \binom{2n}{n + m}.
\]

Hence

\[
C_{n,m} = \binom{2n}{n + m} - \binom{2n}{n - m - 1}.
\]
and the conclusion follows by noticing that
\[
\binom{2n}{n-m-1} = \frac{n-m}{n+m+1} \binom{2n}{n-m} = \frac{n-m}{n+m+1} \binom{2n}{n+m}.
\]

**Corollary 3.** The probability of the event in the generalized ballot problem is
\[
\frac{2m+1}{n+m+1}.
\]

**Proof.** By shifting all paths 2m units up, it is clear that \(C_{n,m}\) is the number of paths from \((0,0)\) to \((n-m,n+m)\) without crossing \(y = x + 2m\). Then identify each vote for party A with \((0,1)\) and that for party B with \((1,0)\).

The conclusion follows by noticing that each possible ballot outcome in the problem corresponds to a path from \((0,0)\) to \((n-m,n+m)\), hence there are
\[
\binom{2n}{n+m}
\]
of them in total, and those ballot outcome with A leading B by at least \(2m\) votes corresponds to paths from \((0,0)\) to \((n-m,n+m)\) without crossing \(y = x + 2m\).

\[\square\]

3. SOME VARIATIONS

Superficially, the definition of \(C_{n,m}\) seems to be too restricted, since it considers only the even number \(2m\) and allowing touching \(y = x\). Here in this section we consider some variations and they can be computed in similar ways.

**Definition 4.** Let \(0 \leq m \leq n\).

\(\bar{C}_{n,m} := \text{the number of paths from } (0, -2m - 1) \text{ to } (n - m, n - m) \text{ without crossing } y = x.\)

\(\bar{D}_{n,m} := \text{the number of paths from } (0, -2m) \text{ to } (n - m, n - m) \text{ strictly below } y = x \text{ until reaching } (n - m, n - m).\)

\(\bar{C}_{n,m} := \text{the number of paths from } (0, -2m - 1) \text{ to } (n - m, n - m) \text{ strictly below } y = x \text{ until reaching } (n - m, n - m).\)

**Theorem 5.** Suppose \(0 \leq m \leq n\). Then

\[
\begin{align*}
(1) \quad \bar{C}_{n,m} &= \bar{D}_{n+1,m+1} = \frac{2m + 2}{n + m + 2} \binom{2n + 1}{n - m}; \\
(2) \quad \bar{D}_{n,m} &= \frac{2m}{n + m} \binom{2n - 1}{n - m}; \\
(3) \quad \bar{C}_{n,m} &= \bar{C}_{n,m} = \frac{2m + 1}{n + m + 1} \binom{2n}{n - m}.
\end{align*}
\]

**Proof.** We first prove (2). Notice that a path \(P\) from the set defining \(\bar{D}_{n,m}\) must pass \((n - m, n - m - 1)\) before reaching \((n - m, n - m)\). Similar to the proof of Theorem 2, \(P\) can be reflected across the line \(y = x\) to form a path \(Q\) from \((-2m, 0)\) to \((n - m, n - m - 1)\), as shown in Figure 2.
We let $\Lambda$ be the set of paths from $(0, -2m)$ to $(n - m, n - m - 1)$ and $\Gamma$ be the subset of “bad” paths, i.e. those that cross the line $y = x - 1$. Clearly, $|\Lambda| = \binom{2n - 1}{n - m}$. By the reflection trick used in the proof of Theorem 2 we can identify paths in $\Gamma$ with paths from $(-2m, 0)$ to $(n - m, n - m - 1)$.

Therefore $|\Gamma| = \binom{2n - 1}{n + m}$.

So $D_{n,m} = |\Lambda| - |\Gamma| = \binom{2n - 1}{n - m} - \binom{2n - 1}{n + m} = \frac{2m}{n + m} \binom{2n - 1}{n - m}$.

To prove (1), by shifting 1 unit down, we can identify paths defining $\bar{C}_{n,m}$ with paths from $(0, -2m - 2)$ to $(n - m, n - m)$ strictly below $y = x$ until reaching $(n - m, n - m)$. Therefore

$$\bar{C}_{n,m} = D_{n+1,m+1}$$

and the conclusion follows.

Now for (3), we notice that by shifting 1 unit up, paths defining $\bar{D}_{n,m}$ are identified with paths from $(0, -2m)$ to $(n - m, n - m)$ without crossing $y = x$. Therefore $\bar{D}_{n,m} = C_{n,m}$. □
4. IDENTITIES

In this section we derive some interesting identities which are corollaries of Theorem 2 or Theorem 5.

Corollary 6.

\[
\sum_{k=m}^{n-1} \frac{1}{2k+1} \binom{2k+1}{k-m} \binom{2(n-k)}{n-k} = \frac{2}{2m+1} \binom{2n}{n-m-1},
\]

where \(0 \leq m \leq n-1\).

Proof. A “bad” path \(P\) from \((0, -2m)\) to \((n-m, n-m)\), i.e. one that crosses \(y = x\), is a path from \((0, -2m)\) to some \((k, k)\), immediately before crossing \(y = x\), then proceed to \((k, k+1)\) and takes any path to \((n-m, n-m)\) as shown in Figure 3 below.

![Figure 3. Decomposition of a “bad” path P.](image-url)
The number of paths from \((0, -2m)\) to \((k, k) = ((m + k) - m, (m + k) - m)\) without crossing \(y = x\) is given by \(C_{m+k,m}\), while the number of paths from \((k, k+1)\) to \((n-m, n-m)\) is given by \(\left(\frac{2(n - m - k) - 1}{n - m - k}\right)\). Hence the number of the “bad” paths are

\[
\sum_{k=0}^{n-m-1} C_{m+k,m} \left(\frac{2(n - m - k) - 1}{n - m - k}\right) = \sum_{k=0}^{n-1} C_{k,m} \left(\frac{2(n - k) - 1}{n - k}\right) = \sum_{k=0}^{n-1} \frac{2m + 1}{k + m + 1} \frac{2k}{k + m} \left(\frac{2(n - k) - 1}{n - k}\right).
\]

Since this number is also \(\left(\frac{2n}{n-m}\right) - C_{n,m}\) the conclusion follows from the last part of the proof of Theorem 2.

\[\square\]

**Corollary 7.**

\[
(3) \quad \sum_{k=m}^{n} \frac{1}{k(n - k + 1)} \binom{2k}{k - m} \binom{2(n - k)}{n - k} = \frac{2m + 1}{m(n + m + 1)} \binom{2n}{n - m}
\]

where \(1 \leq m \leq n - 1\).

**Proof.** Note that a path from \((0, -2m)\) to \((n-m, n-m)\) without crossing the line \(y = x\) can be decomposed into a path from \((0, -2m)\) to the first touch at \(y = x\), some \((k, k)\), \(0 \leq k \leq n - m\), followed by a path from \((k, k)\), to \((n-m, n-m)\) without crossing the line \(y = x\), as shown in Figure 4 below.
The number of the former paths is $\mathcal{D}_{k+m,m}$ and the number for the latter is $\mathcal{C}_{n-m-k,0}$, therefore we have

\[(4) \quad \mathcal{C}_{n,m} = \sum_{k=0}^{n-m} \mathcal{D}_{k+m,m} \mathcal{C}_{n-m-k,0}.
\]

Hence, by Theorem 2 and Theorem 5

\[
\frac{2m+1}{n+m+1} \binom{2n}{n-m} = \sum_{k=0}^{n-m} \frac{2m}{k+2m} \binom{2(k+m)-1}{k} \frac{1}{n-m-k+1} \binom{2(n-m-k)}{n-m-k} \\
= \sum_{k=m}^{n} \frac{2m}{k+m} \binom{2k-1}{k-m} \frac{1}{n-k+1} \binom{2(n-k)}{n-k} \\
= \sum_{k=m}^{n} \frac{m}{k(n-k+1)} \binom{2k}{k-m} \binom{2(n-k)}{n-k},
\]

and equation (3) follows. \qed
Remark 8. It is easy to check that equation (2) can be transformed into the following form:

\[
\sum_{k=m}^{n} \frac{2n - 2k + 1}{(k + m + 1)(n - k + 1)} \binom{2k}{k-m} \binom{2(n-k)}{n-k} = \frac{(2n+1)(2n+2)}{(2m+1)(n+m+1)(n+m+2)} \binom{2n}{n-m}.
\]

Although equations (3) and (5) are similar, it is not clear how they are related.

Corollary 9.

\[
\sum_{h=0}^{n-m} \frac{k}{h+k} \binom{2h+k-1}{h} \binom{2n-2h-k}{n-m-h} = \binom{2n}{n-m}
\]

where \(0 \leq m \leq n-1\) and \(1 \leq k \leq 2m\).

Proof. The number of paths from \((0, -2m)\) to \((n-m, n-m)\) is \(\binom{2n}{n-m}\), the right side of the above equation (6).

For each \(1 \leq k \leq 2m\) we can decompose these paths into the path from \((0, -2m)\) first touching the line \(y = x - 2m + k\) at \((h, h - 2m + k)\), for some \(0 \leq h \leq n - m\), followed by a path from \((h, h - 2m + k)\) to \((n-m, n-m)\), as shown in Figure 5 below.

![Figure 5. The decomposition at \((h, h - 2m + k)\).](image-url)
The number of paths from \((h, h - 2m + k)\) to \((n - m, n - m)\) is \(\binom{2n - 2h - k}{n - m - h}\).

By shifting up \(2m - k\) units, we can see that a path from \((0, -2m)\) first touching the line \(y = x - 2m + k\) at \((h, h - 2m + k)\) corresponds to a path from \((0, -k)\) first touching the line \(y = x\) at \((h, h)\). Therefore, if \(k\) is even, the number is given by

\[
D_{\frac{h}{2}, \frac{k}{2}} = \frac{k}{h + k} \binom{2h + k - 1}{h};
\]

if \(k\) is odd, the number is given by

\[
\overline{D}_{\frac{h}{2}, \frac{k-1}{2}} = \frac{k}{h + k} \binom{2h + k - 1}{h};
\]

both producing the same number.

Summing the product \(\frac{k}{h + k} \binom{2h + k - 1}{h} \binom{2n - 2h - k}{n - m - h}\) over \(0 \leq h \leq n - m\), equation (6) follows.

\[\square\]

**Remark 10.** It appears that equation (6) actually holds for all \(k > 0\). The current proof doesn’t seem to work for \(k > 2m\).

## 5. Recursive relations and identities

In this section we will derive two recursive relations for the generalized Catalan numbers. As corollaries, we obtain some identities for the central binomial coefficients. The first recursive relation for the generalized Catalan numbers we consider is the following.

**Theorem 11.** For \(0 \leq m < n\), we have the following recursive relation

\[
C_{n,m} = \frac{n(2m + 1)}{(n - m)(m + 1)} \sum_{k=0}^{n-m-1} C_{k+m, m} C_{n-m-k-1, 0}.
\]

**Proof.** A path defining \(C_{n,m}\) is either a path touching the line \(y = x\) only once at the terminal point \((n - m, n - m)\) (the number of such is given by \(D_{n,m}\)) or a path touching some \((k, k)\), where \(0 \leq k \leq n - m - 1\) for the last time before hitting the terminal point \((n - m, n - m)\), as shown in Figure 6.
Figure 6. A path \( P \) touching \( y = x \) at \((k, k)\) for the last time before touching the terminal point \((n - m, n - m)\).

Write \((k, k)\) as \(((k + m) - m, (k + m) - m)\), we have

\[
C_{n,m} = D_{n,m} + \sum_{k=0}^{n-m-1} C_{k+m, m} C_{n-m-k-1, 0}.
\]

But

\[
D_{n,m} = \frac{2m}{n + m} \binom{2n - 1}{n - m} = \frac{m(n + m + 1)}{n(2m + 1)} C_{n,m},
\]

hence the result follows.

As a consequence, we have an identity for the \( n \)th central binomial coefficient.

**Corollary 12.** Let \( 0 \leq m < n \), then

\[
\binom{2n}{n} = \frac{n + m + 1}{m + 1} \left( \prod_{k=0}^{m} \frac{n+k}{n-k} \right) \sum_{k=0}^{n-m-1} C_{k+m, m} C_{n-m-k-1, 0}.
\]

**Proof.** By equation (7) in Theorem 11 we have:

\[
\frac{2m+1}{n+m+1} \frac{(2n)!}{(n+m)!(n-m)!} = \frac{n(2m+1)}{(n-m)(m+1)} \sum_{k=0}^{n-m-1} C_{k+m, m} C_{n-m-k-1, 0},
\]

i.e.

\[
\frac{(2n)!}{(n+m)!(n-m)!} = \frac{n+m+1}{m+1} \frac{n}{n-m} \sum_{k=0}^{n-m-1} C_{k+m, m} C_{n-m-k-1, 0}.
\]
hence the result follows if \( m = 0 \); for \( m > 0 \), we get

\[
\frac{(2n)!}{n!^2} = \frac{n + m + 1}{m + 1} \frac{n(n + 1) \cdots (n + m)}{n(n - 1) \cdots (n - m + 1)(n - m)} \sum_{k=0}^{n-m-1} \binom{k+m, m}{k, m} \binom{n-m-k-1, 0, 0}{n-m-k-1, 0},
\]

and equation (8) is proved. \( \square \)

The following is an identity to be used in §7.

**Corollary 13.** Let \( 1 \leq m < n \), then

\[
\sum_{k=0}^{n-m-1} \binom{k+m, m}{k, m} \binom{n-m-k-1, 0}{n-m-k-1, 0} = \frac{m + 1}{n!} \left( \prod_{k=1}^{m} (n - k) \right) \left( \prod_{k=m+2}^{n} (n + k) \right)
\]

\[
= \left( \prod_{k=1}^{m} \frac{n - k}{k} \right) \left( \prod_{k=m+2}^{n} \frac{n + k}{k} \right).
\]

**Proof.** This is just another form of equation (8) in Corollary 12 by noticing that

\[
\binom{2n}{n} = \prod_{k=1}^{n} \frac{n + k}{k}.
\]

\( \square \)

Now we consider another recursive relation for the generalized Catalan numbers.

**Theorem 14.** Let \( 0 \leq m \leq n \). Then

\[
\binom{n-m-1}{k, m} \binom{n, 0}{n, 0} = \frac{m + 1}{n!} \left( \prod_{k=1}^{m} (n - k) \right) \left( \prod_{k=m+2}^{n} (n + k) \right)
\]

\[
= \left( \prod_{k=1}^{m} \frac{n - k}{k} \right) \left( \prod_{k=m+2}^{n} \frac{n + k}{k} \right).
\]

**Proof.** From Theorem 5 in [7] we can obtain

\[
\mathcal{D}_{n,m} = \frac{n(n + m + 1)}{n(2m + 1)} \mathcal{C}_{n,m}.
\]

The result then follows from the identity (4) of Corollary 7 in [7]. \( \square \)

From the above we have another expression for the \( n \)th central binomial coefficient. The proof is similar to that of Corollary 12.

**Corollary 15.** Let \( 1 \leq m \leq n \), then

\[
\binom{2n}{n} = \frac{m(n - m)}{(2m + 1)^2} \left( \prod_{k=1}^{m+1} \frac{n + k}{n - k + 1} \right) \sum_{k=m}^{n} \frac{k + m + 1}{k} \binom{k, m}{k, m} \binom{n-m-k-1, 0}{n-m-k-1, 0}.
\]

\( \square \)
6. Generating functions

In this section we derive a formula for the generating function of generalized Catalan numbers. A comprehensive treatment of generating functions can be found in [3].

For fixed \( m \geq 0 \) we let \( \gamma_m(x) \) denote the generating function of \( C_{n+m,m} \), i.e. the power series

\[
\gamma_m(x) = \sum_{n=0}^{\infty} C_{n+m,m} x^n.
\]

It is well-known for the classical Catalan numbers \( C_{n,0} \) that

\[
\gamma_0(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.
\]

(See for example [4].)

We denote the corresponding multivariate generating function by:

\[
\Gamma(x, y) = \sum_{n,m=0}^{\infty} C_{n+m,m} x^n y^m.
\]

In [2] a formula for the generating function for \( a_{n,m} \) is given, i.e.

\[
a(x, y) := \sum_{n \geq m \geq 0} a_{n,m} x^m y^m = \frac{1 - y \gamma_0(xy)}{1 - x - y}.
\]

Using the relation \( C_{n,m} = a_{n+m,n-m} \), it can be verified that

\[
\Gamma(x, y) = \frac{1}{2} \left( a\left(\sqrt{y}, \frac{x}{\sqrt{y}}\right) + a\left(-\sqrt{y}, -\frac{x}{\sqrt{y}}\right)\right).
\]

However we have another derivation here:

**Theorem 16.**

\[
\Gamma(x, y) = \frac{\gamma_0(x)}{1 - y (\gamma_0(x))^2} \quad \text{i.e.} \quad \frac{x(1 - \sqrt{1 - 4x})}{2x(x + y) - y(1 - \sqrt{1 - 4x})}.
\]

□

We first need the following lemmas:

**Lemma 17.** Let \( n \geq 0 \) then

\[
C_{n+m+1,m+1} = \sum_{a+b+c=n} C_{a+m,m} C_{b,0} C_{c,0},
\]

where we sum over \( a, b, c \geq 0 \).
Proof. We count paths defining \( C_{n+m+1, m+1} \), i.e. paths from \((0, -2(m + 1))\) to \((n, n) = (n + m + 1 - (m + 1), n + m + 1 - (m + 1))\) without crossing \(y = x\), by using their first touch \((a, a - 1)\) and \((a + b, a + b)\) at the lines \(y = x - 1\) and \(y = x\) respectively as in Figure 7 below.

![Figure 7](image_url)

**Figure 7.** A path defining \( C_{n+m+1, m+1} \) whose first touch at \(y = x - 1\) is \((a, a - 1)\) and whose first touch at \(y = x\) is \((a + b, a + b)\) for some \(a, b \geq 0\) and \(a + b \leq n\).

The number of paths from \((0, -2m - 2)\) that first touch \(y = x - 1\) at \((a, a - 1)\) is the same as those reaching \((a, a - 2)\) without crossing \(y = x - 2\), i.e. the number is \(C_{a+m, m}\). Similarly, the number of paths from \((a, a - 1)\) first reaching \(y = x\) at \((a + b, a + b)\) is \(C_{b,0}\) and the number of paths from \((a + b, a + b)\) to \((n, n)\) without crossing \(y = x\) is \(C_{c,0}\), where \(c = n - a - b\), hence the result is proved. \(\Box\)

Now we have a recursive relation for the generating function \(\gamma_m(x)\):
Lemma 18. For \( m \geq 0 \), we have

\[
\gamma_{m+1}(x) = \gamma_m(x) \gamma_0(x)^2.
\]

Proof. We use Lemma 17:

\[
\gamma_{m+1}(x) = \sum_{n=0}^{\infty} C_{n+m+1, m+1} x^n
\]

\[
= \sum_{n=0}^{\infty} \sum_{a+b+c=n} C_{a+m, m} C_{b, 0} C_{c, 0} x^n
\]

\[
= \left( \sum_{n=0}^{\infty} C_{n+m, m} x^n \right) \left( \sum_{n=0}^{\infty} C_{n, 0} x^n \right)^2
\]

\[
= \gamma_m(x) \gamma_0(x)^2.
\]

\( \square \)

Now we prove Theorem 16 from Lemma 18. Note that:

\[
y \Gamma(x, y)(\gamma_0(x))^2 = \sum_{m=0}^{\infty} \gamma_m(x) \gamma_0(x)^2 y^{m+1}
\]

\[
= \sum_{m=0}^{\infty} \gamma_{m+1}(x) y^{m+1}
\]

\[
= \Gamma(x, y) - \gamma_0(x),
\]

from which the result follows.

7. More formulas

In this section we prove a few more formulas relating to the generating functions.

Theorem 19. Let \( m \geq 1 \). The generating function \( \gamma_m(x) \) is equal to

\[
\frac{2x}{1 - \sqrt{1 - 4x}} \left( 1 + \sum_{n=1}^{\infty} \left( \prod_{k=1}^{m} \frac{n + m - k + 1}{k} \right) \left( \prod_{k=m+2}^{\infty} \frac{n + m + k + 1}{k} \right) x^n \right).
\]
Proof. Replacing $n - m - 1$ by $n$ in Corollary 13 equation (9), we obtain for $n \geq 1$ that
\[
\sum_{k=0}^{n} \mathcal{C}_{k+m, m} \mathcal{C}_{n-k, 0} = \left( \prod_{k=1}^{m} \frac{n + m - k + 1}{k} \right) \left( \prod_{k=m+2}^{n+m+1} \frac{n + m + k + 1}{k} \right).
\]
Then the theorem follows from
\[
\gamma_m(x) \gamma_0(x) = \sum_{n=0}^{\infty} \sum_{k+h=n} \mathcal{C}_{k+m, m} \mathcal{C}_{h, 0} x^n
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \mathcal{C}_{k+m, m} \mathcal{C}_{n-k, 0} \right) x^n
= 1 + \sum_{n=1}^{\infty} \left( \prod_{k=1}^{m} \frac{n + m - k + 1}{k} \right) \left( \prod_{k=m+2}^{n+m+1} \frac{n + m + k + 1}{k} \right) x^n.
\]

We now the product $\gamma_m(x) \gamma_0(x)$ and the following power series:
\[
\theta(x) = \sum_{n=0}^{\infty} \frac{(n+1)(m+1)}{(n+m+1)(2m+1)} \mathcal{C}_{n+m+1, m} x^n.
\]

**Theorem 20.** Let $0 \leq m \leq n$. Then
\[
\theta(x) = \gamma_m(x) \gamma_0(x) = \gamma_m(x) \frac{1 - \sqrt{1 - 4x}}{2x}.
\]

Proof. From Theorem 13 equation (7) we have:
\[
\mathcal{C}_{n+m+1, m} = \frac{(n+m+1)(2m+1)}{(n+1)(m+1)} \sum_{k+h=n} \mathcal{C}_{k+m, m} \mathcal{C}_{h, 0},
\]
therefore
\[
\theta(x) = \sum_{n=0}^{\infty} \left( \sum_{k+h=n} \mathcal{C}_{k+m, m} \mathcal{C}_{h, 0} \right) x^n = \gamma(x) \frac{1 - \sqrt{1 - 4x}}{2x}.
\]

**Corollary 21.** For $1 \leq m \leq n$, we have
\[
\mathcal{C}_{n+m, m} = \frac{(n+m)(2m+1)}{n(m+1)} \left( \prod_{k=1}^{m} \frac{n + m - k}{k} \right) \left( \prod_{k=m+2}^{n+m+1} \frac{n + m + k}{k} \right).
\]

Proof. Compare the power series expansion of $\gamma_m(x) \gamma_0(x)$ given in the last line of the proof of Theorem 13 and the one given by $\theta(x)$. Then equation (17) is obtained by replacing $n + 1$ for $n$. 

\[\square\]
8. A BEAUTIFUL DETERMINANT

In this final section, we prove the following:

**Theorem 22.** For integer $N \geq 0$,

\[
\det \left[ \frac{(2i + 1)(2(i + j))!}{j!(2i + j + 1)!} \right]_{0 \leq i,j \leq N} = 2^{N(N+1)/2}.
\]

We suggest the name *Catalan determinant* for the above, for the entries of the matrix are the generalized Catalan numbers $C_{i+j,i}$.

First we need a slight modification of [8] Theorem 1. (See also [6].) An almost identical proof is given here for completeness.

**Lemma 23.** Let $f(x) = 1 + a_1 x + a_2 x^2 + \ldots$ be a formal power series and let $c_{i,j} = [x^j] f^{2i+1}(x)$, where $[x^i]$ denotes the coefficient of $x^j$ in the series. Then

\[
\det [c_{i,j}]_{0 \leq i,j \leq N} = (2a_1)^{\frac{N(N+1)}{2}}.
\]

**Proof.** Let

\[ C := [c_{i,j}]_{0 \leq i,j \leq N} \quad \text{and} \quad B := [b_{i,j}]_{0 \leq i,j \leq N} \quad \text{where} \quad b_{i,j} = (-1)^{i+j} \binom{i}{j}. \]

The $(i,k)$-entry of $BC$ is given by

\[
\sum_{j=0}^{N} b_{i,j} c_{j,k} = \left( -1 \right)^i \left[ x^k \right] \sum_{j=0}^{N} \left( -1 \right)^j \binom{i}{j} f^{2j+1}(x)
\]

\[ = \left( -1 \right)^i \left[ x^k \right] \sum_{j=0}^{i} \left( -1 \right)^j \binom{i}{j} f^{2j+1}(x) \quad \text{(since} \quad \binom{i}{j} = 0 \quad \text{for} \quad j > i)\]

\[ = \left( -1 \right)^i \left[ x^k \right] \left( 1 - f^2(x) \right)^i f(x)
\]

\[ = \left( -1 \right)^i \left[ x^k \right] \left( -2a_1 x + \ldots \right)^i (1 + a_1 x + \ldots), \]

which equals $(2a_1)^i$ if $k = i$ and equals 0 if $k < i$, hence $BC$ is an upper triangular matrix with diagonal entries $(2a_1)^i$. In particular

\[ \det BC = \prod_{i=0}^{N} (2a_1)^i = (2a_1)^{\frac{N(N+1)}{2}}. \]

But $B$ is lower triangular with diagonal entries 1, so $\det B = 1$ and the result follows. \qed
To prove Theorem 22 we let, as in §6, \( \gamma_i(x) \) be the generating function of \( C_{i+j,i} \), i.e. the power series
\[
\gamma_i(x) = \sum_{n=0}^{\infty} C_{i+j,i} x^n.
\]
Then
\[
\gamma_0(x) = \frac{1 - \sqrt{1 - 4x}}{2x} = 1 + x + 2x^2 + 5x^3 + 14x^4 + \ldots
\]
and Lemma 18 gives that
\[
\gamma_i(x) = \gamma_0^{2i+1}(x) \quad \text{for} \quad i \geq 0.
\]
Therefore
\[
C_{i+j,i} = [x^j]\gamma_i(x) = [x^j]\gamma_0^{2i+1}(x)
\]
and Theorem 22 is now proved by applying Lemma 23 to \( f(x) = \gamma_0(x) \).

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