Total correlations and mutual information

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In quantum information theory it is generally accepted that quantum mutual information is an information-theoretic measure of total correlations of a bipartite quantum state. We argue that there exist quantum states for which quantum mutual information cannot be considered as a measure of total correlations. Moreover, for these states we propose a different way of quantifying total correlations.

\[ Q(\rho^{AB}) \geq \frac{1}{2} I(\rho^{AB}), \]

If one assumes that statements (i), (ii) and (iii) hold for all \( \rho^{AB} \), then one concludes that entanglement of formation cannot be considered as a measure of quantum correlations. However, if one assumes that entanglement of formation is a measure of quantum correlations and statements (ii) and (iii) hold, then it can be shown that for these quantum states \( T(\rho^{AB}) \geq I(\rho^{AB}) \).

Moreover, it has been shown that for certain quantum states quantum correlations, as measured by entanglement of formation, exceed total correlations, as measured by quantum mutual information, \( Q(\rho^{AB}) \geq \frac{1}{2} I(\rho^{AB}) \). If one assumes that statements (i), (ii) and (iii) hold for all \( \rho^{AB} \), then one concludes that entanglement of formation cannot be considered as a measure of quantum correlations, if one assumes that entanglement of formation is a measure of quantum correlations and statements (ii) and (iii) hold, then it can be shown that for these quantum states \( T(\rho^{AB}) \geq I(\rho^{AB}) \).

Recently, it has been shown that for certain quantum states the quantum correlations, as measured by entanglement of formation, exceed half of the total correlations, as measured by quantum mutual information, \( Q(\rho^{AB}) \geq \frac{1}{2} I(\rho^{AB}) \). If one assumes that statements (i), (ii) and (iii) hold for all \( \rho^{AB} \), then one concludes that entanglement of formation cannot be considered as a measure of quantum correlations. However, if one assumes that entanglement of formation is a measure of quantum correlations and statements (ii) and (iii) hold, then one comes to conclusion that for these quantum states \( T(\rho^{AB}) \geq I(\rho^{AB}) \).

The above examples show that statement (i) may not be true for all bipartite quantum states. It is clear that if we assume that it is true, then we immediately conclude that quantum mutual information must be a measure of total correlations also in the case when the quantum state has only classical correlations, i.e.

\[ C(\rho^{AB}) = T(\rho^{AB}) = I(\rho^{AB}). \]
II. A TWO-QUBIT MIXED STATE

Assume that Alice and Bob share a pair of qubits in the following state

\[ \rho^{AB} = \alpha |00\rangle \langle 00| + (1 - \alpha) |11\rangle \langle 11|, \]

where \( \alpha \in (0, 1) \). This state cannot have quantum correlations because it is separable and qubit \( A \) (\( B \)) is in the state \( \rho^{A(B)} = \alpha |0\rangle \langle 0| + (1 - \alpha) |1\rangle \langle 1| \) which is a mixture of orthogonal states \( |0\rangle \) and \( |1\rangle \). Therefore, it is clear that the correlations between two orthogonal states of qubits \( A \) and \( B \) can be purely classical (see e.g. [2, 19, 20]).

Suppose now that Alice and Bob measure two observables \( M_A = a_0 |0\rangle \langle 0| + a_1 |1\rangle \langle 1| \) and \( M_B = b_0 |0\rangle \langle 0| + b_1 |1\rangle \langle 1| \), respectively. If the measurement outcome of \( M_A \) (\( M_B \)) is \( a_i \) (\( b_i \)), then qubit \( A \) (\( B \)) is certainly in the state \( |i\rangle \). Therefore, it is clear that the classical correlations between two orthogonal states of qubits \( A \) and \( B \) are simply correlations between two classical random variables \( A \) and \( B \) corresponding to the measurement outcomes of \( M_A \) and \( M_B \), respectively. Notice that \( A \) and \( B \) are random variables with alphabets \( A = \{a_0, a_1\} \), \( B = \{b_0, b_1\} \) and probability mass functions \( p^{A(B)} = [p_i^{A(B)}] \), where \( p_i^{A(B)} = \text{Tr}[|i\rangle \langle i| \rho^{A(B)}] \) denotes the probability that the measurement outcome of \( M_A \) (\( M_B \)) is \( a_i \) (\( b_i \)). In the case under consideration, it can be shown that

\[ p^A = (\alpha, 1 - \alpha), \]
\[ p^B = (\alpha, 1 - \alpha). \]

Assume now that first Alice performs a measurement of \( M_A \) and then Bob performs a measurement of \( M_B \). If the measurement outcome of \( M_A \) is \( a_i \), then the post-measurement state of the system is given by \( \rho_i^{AB} = (|i\rangle \langle i| \otimes I) \rho^{AB} (|i\rangle \langle i| \otimes I) / p_i^A \). Therefore, the conditional probability that Bob’s outcome is \( b_j \) provided that Alice’s was \( a_i \) is \( p_{ij}^{B|A} = \text{Tr}[(I \otimes |j\rangle \langle j|) \rho_i^{AB}] \) while the joint probability that the measurement outcome of \( M_A \) and \( M_B \) is \( a_i \) and \( b_j \), respectively is given by \( p_{ij}^{AB} = \text{Tr}[(|i\rangle \langle i| \otimes |j\rangle \langle j|) \rho^{AB}] \). In the case under consideration, it can be shown that

\[ p_{ij}^{B|A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]

and

\[ p_{ij}^{AB} = \begin{pmatrix} \alpha & 0 \\ 0 & 1 - \alpha \end{pmatrix}. \]

Thus, we see that random variables \( A \) and \( B \) are not independent and therefore there exist classical correlations between qubits \( A \) and \( B \) in the state \([2]\).

Assume now, according to the statement (i), that total correlations of a bipartite quantum state are quantified by quantum mutual information which is defined in formal analogy to classical mutual information as

\[ I(\rho^{AB}) = S(\rho^A) + S(\rho^B) - S(\rho^{AB}), \]

where \( S(\cdot) \) denotes the von Neumann entropy. Then, according to Eq. (6) the classical correlations between qubits \( A \) and \( B \) are measured by the quantum mutual information. In the case under consideration, it can be shown that \( I(\rho^{AB}) \) is just classical mutual information of random variables \( A \) and \( B \) given by

\[ I(A:B) = \sum_{i,j} p_{ij}^{AB} \log_2(p_{ij}^{AB} / p_i^A p_j^B) \]

\[ = -\alpha \log_2 \alpha - (1 - \alpha) \log_2 (1 - \alpha). \]

Therefore, we see that the classical correlations content of the quantum state \([2]\) can be arbitrarily small, as measured by classical mutual information, because \( \alpha \in (0, 1) \).

Now, we check if these correlations can be really arbitrarily small. From Eq. (7) it follows that if Alice’s measurement outcome is \( a_i \), then Bob’s one is \( b_i \), i.e. \( B \) is a one-to-one function of \( A \), \( B = f(A) \) (see Fig. 1). It means that the random variables \( A \) and \( B \) and what follows the states of qubits are perfectly correlated in the information-theoretic sense. Therefore, we see that in the general case quantum mutual information cannot be considered as a measure of total correlations in bipartite quantum states.

Now, we show why the classical mutual information \( I(A:B) \) does not capture all the correlations between random variables \( A \) and \( B \), except the case when \( \alpha = 1/2 \). We know that Shannon entropy of random variable \( B \), \( H(B) = -\sum_j p_j^B \log p_j^B \), is a measure of Alice’s \( a \) priori uncertainty about the measurement outcome of \( M_B \) and if the measurement outcome of \( M_A \) is \( a_i \), then Alice’s uncertainty about the measurement outcome of \( M_B \) is changed, preferably reduced, to \( H(B|A = a_i) = -\sum_j p_{ij}^{B|A} \log p_{ij}^{B|A} \). Therefore, the information she gained about the measurement outcome of \( M_B \) due to the measurement of \( M_A \) is given by \( H(B) - H(B|A = a_i) \). Thus, the average information gain about the measurement outcome of \( M_B \) due to the knowledge of the measurement outcome of \( M_A \) is \( \sum_i p_i^A (H(B) - H(B|A = a_i)) = H(B) - H(B|A) \), and it can be shown that it is equal to \( I(A:B) \). Therefore, we see that the average information gain about one random variable due to the knowledge of other one can be arbitrarily small although they are perfectly correlated. Thus it is clear that classical mutual information is not a measure of correlations between two random variables,
it is rather a measure of their mutual dependency.

This conclusion leads us to the following question: What is an information-theoretic measure of correlations between random variables $A$ and $B$? For a pair of random variables with identical probability mass functions, Cover and Thomas [30] define it in the following way

$$C(A, B) = \frac{I(A : B)}{H(A)}.$$ \hfill (11)

In the case under consideration, $p^A = p^B$ and it can be shown that $I(A : B) = H(A)$, therefore $C(A, B) = 1$, i.e. $A$ and $B$ are perfectly correlated for all $\alpha \in (0, 1)$. In the next section we show how to extend this definition to the case when the probability mass functions are not identical.

### III. A TWO-QUTRIT MIXED STATE

Assume now that Alice and Bob share a pair of qutrits in the following separable state

$$\rho^{AB} = \frac{1}{3}|11\rangle\langle 11| + \frac{1}{3}|20\rangle\langle 20| + \frac{1}{3}|22\rangle\langle 22|$$ \hfill (12)

in which orthogonal states of qutrits $A$ and $B$ are classically correlated. Notice that $p^A = \frac{1}{3}|1\rangle\langle 1| + \frac{2}{3}|2\rangle\langle 2|$ and $p^B = \frac{1}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1| + \frac{1}{3}|2\rangle\langle 2|$. Suppose now that Alice and Bob measure two observables $M_A = a_0|0\rangle\langle 0| + a_1|1\rangle\langle 1| + a_2|2\rangle\langle 2|$ and $M_B = b_0|0\rangle\langle 0| + b_1|1\rangle\langle 1| + b_2|2\rangle\langle 2|$. It can be easily shown that the probability mass functions $p^A$ and $p^B$ are not identical, and they are given by

$$p^A = (0, \frac{1}{3}, \frac{2}{3}), \quad p^B = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$ \hfill (13a) \hfill (13b)

Assume now that first Alice performs a measurement of $M_A$ and then Bob performs a measurement of $M_B$. It can be shown that

$$p^B_{01} = 0, \quad p^B_{11} = 1, \quad p^B_{21} = 0,$$ \hfill (14a)

$$p^B_{02} = \frac{1}{2}, \quad p^B_{12} = 0, \quad p^B_{22} = \frac{1}{2},$$ \hfill (14b)

and

$$p^{AB} = [p_{ij}^{AB}] = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & 0 & 0 \end{pmatrix}. \hfill (15)$$

Thus, we see that the random variables $A$ and $B$ are not independent, and from Eqs. (14) it follows that Bob’s measurement outcomes are correlated with Alice’s ones, but they are not perfectly correlated (see Fig. 2). Now, we show how to quantify these correlations. We know that for any two random variables $A$ and $B$ (i) $H(B|A) \leq H(B)$ with equality if and only if they are independent, and (ii) $H(B|A) \geq 0$ with equality if and only if $B = f(A)$, i.e. $A$ is perfectly correlated with $B$.

![FIG. 2: This diagram shows that the random variables $A$ and $B$ are not perfectly correlated in the information-theoretic sense.](image)

$$p^B B \quad p^{A\mid B} A \quad p^A$$

$$\begin{array}{c|c|c|c|c|c|}
\frac{1}{3} & 0 & 1 & 0 & 0 \\
\frac{1}{3} & 1 & 1 & 1 & \frac{1}{3} \\
\frac{1}{3} & 2 & 1 & 2 & \frac{2}{3} \\
\end{array}$$

![FIG. 3: This diagram shows the perfect correlations, in the information-theoretic sense, between random variables $A$ and $B$, $A = f(B)$, despite the fact that the mutual information $I(A : B) = H(A) \simeq 0.918$.](image)

in the information-theoretic sense. Therefore, it is clear that if $H(B) > 0$, then

$$1 \geq H(B|A)/H(B) \geq 0.$$ \hfill (16)

Notice that this inequality can be rewritten in the following form

$$0 \leq I(A : B)/H(B) \leq 1.$$ \hfill (17)

Therefore, the correlations between $A$ and $B$ can be measured by $I(A : B)/H(B)$. In the case under consideration, it can be shown that $I(A : B) = \log_2 3 - \frac{2}{3}$, $H(B) = \log_2 3$ and therefore

$$\frac{I(A : B)}{H(B)} = 1 - \frac{2}{3} \log_2 \frac{1}{3}. \hfill (18)$$

Assume now that first Bob performs a measurement of $M_B$ and then Alice performs a measurement of $M_A$. If the measurement outcome of $M_B$ is $b_i$, then the post-measurement state of the system is given by $\rho^{AB} = ([i] \otimes |i\rangle\langle i|)\rho^{AB}([i] \otimes |i\rangle\langle i|)/p_i^B$. Therefore, the conditional probability that Alice’s outcome is $a_j$ provided that Bob’s was $b_i$ is $p_{ij}^{AB} = \text{Tr}([i] \otimes |i\rangle\langle i|)\rho^{AB}([i] \otimes |i\rangle\langle i|)/p_i^B$. Therefore, the conditional probability of Alice’s outcome is $a_j$ provided that Bob’s was $b_i$ is $p_{ij}^{AB} = \text{Tr}([i] \otimes |i\rangle\langle i|)\rho^{AB}$. While the joint probability that the measurement outcome of $M_A$ and $M_B$ is $a_j$ and $b_i$, respectively, is given by $p_{ij}^{AB} = \text{Tr}([i] \otimes |i\rangle\langle i|)\rho^{AB}$. This case, the joint probabilities are given by $\text{Tr}$ while the conditional probabilities are as follows

$$p^{AB} = [p_{ij}^{AB}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \hfill (19)$$
Thus, we see that Alice’s measurement outcomes are perfectly correlated with Bob’s ones (see Fig. 3). Now, we explain why this is the case. We know that for any two random variables $A$ and $B$ (i) $H(A|B) \leq H(A)$ with equality if and only if they are independent, and (ii) $H(A|B) \geq 0$ with equality if and only if $A = f(B)$, i.e. $B$ is perfectly correlated with $A$. Therefore, it is clear that if $H(A) > 0$, then

$$1 \geq H(A|B)/H(A) \geq 0. \quad (20)$$

Notice that this inequality can be rewritten in the following form

$$0 \leq I(A : B)/H(A) \leq 1. \quad (21)$$

Therefore, the correlations between $A$ and $B$ can be measured by $I(A : B)/H(A)$. In the case under consideration, it can be shown that $I(A : B) = H(A)$ and therefore

$$I(A : B)/H(A) = 1. \quad (22)$$

Thus, we see that the correlations between two random variables $A$ and $B$ corresponding to the measurement outcomes of $M_A$ and $M_B$ depend on the temporal order of the measurements performed by Alice and Bob. Therefore, in order to capture all classical correlations that can be observed in the state (12), we propose to define a measure of correlations between two random variables $A$ and $B$ in the following way

$$C(A, B) = \text{Max} \left( \frac{I(A : B)}{H(A)}, \frac{I(A : B)}{H(B)} \right) = \frac{I(A : B)}{\text{Min}(H(A), H(B))}. \quad (23)$$

Notice that in the case when a state $\rho^{AB}$ has only classical correlations the Shannon entropies are just equal to the von Neumann entropies and from Eqs. (4) and (23) it follows that

$$T(\rho^{AB}) = \frac{I(\rho^{AB})}{\text{Min}(S(\rho^A), S(\rho^B))}. \quad (24)$$

Thus, we see that total correlations of a bipartite quantum state $\rho^{AB}$ should be quantified by (24) instead of quantum mutual information alone, at least for the classically correlated quantum states.

IV. CONCLUSION

In this paper, we have shown that for bipartite quantum systems there exist quantum states for which quantum mutual information cannot be considered as a proper measure of total correlations, understood as the correlations between the measurement outcomes of two local observables. Moreover, for these states we have proposed a different way of quantifying total correlations, which takes into account that the correlations can depend on the temporal order of the measurements.

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