Robust output tracking of a class of non-affine systems

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ABSTRACT
This paper considers the robust output tracking problem for a class of uncertain non-affine systems. The state-space equations of these systems have a non-affine quadratic polynomial structure. In order to design the output tracking controller, first the error dynamical equations are constructed. Then, a novel sliding mode controller is designed for robust stabilization of the error dynamical equations. For this purpose, a proper sliding manifold which is a function of error vector is suggested. According to upper and lower bounds of uncertainties, two quadratic polynomials are built and with respect to the location of the roots of the given polynomials, the new sliding mode control law is obtained. The proposed controller can conquer the uncertainties and guarantees the asymptotic convergence of the system output toward the wanted time-varying reference signal. Finally, in order to verify the theoretical results, the proposed method is applied to the magnetic ball levitation system. Computer simulations demonstrate the efficiency of the proposed method.

1. Introduction
Non-affine systems are an important class of nonlinear systems. These systems are nonlinear with respect to control input \( \dot{x} = f(x) + g(x,u) \) while affine systems are systems which are affine in control input and control input appears linearly in these systems (like \( \dot{x} = f(x) + g(x)u \)). There exist many practical systems which are non-affine in the control input. For example: the model of aircraft dynamics (Boškovic, Chen, & Mehra, 2004), underwater vehicles (Geranmehr & Nekoo, 2015), active magnetic bearings (Tombul, Banks, & Akturk, 2009); electromagnetic levitation (Gutierrez & Ro, 2005) and etc (Rios, Punta, & Fridman, 2017; Shi, Dong, Xue, Chen, & Zhi, 2017; Song & Song, 2014). Because of the complex structure of non-affine systems, the Lyapunov-based controllers have not been fully studied for these systems and most of the designed controllers for the non-affine systems are based on the non-model based methods like fuzzy or neural network controllers (Chien, Wang, Leu, & Lee, 2011, 2015; Gao, 2017; Labiod & Guerra, 2010). On the other hand, the analytical control methods like backstepping, control Lyapunov function (CLF) and sliding mode, have been basically introduced for affine systems. In some of papers, the authors proposed some transformations to convert the non-affine systems into affine structure. In order to find these transformations, the mathematical model of the system should be exactly known, thus these methods suffer from uncertainties in state-space equations and may lead to weak performance and even system instability (Tombul et al., 2009).

Although the design of nonlinear analytical controllers for non-affine systems is very valuable, only few papers exist that deal with nonlinear Lyapunov-based control methods (Binazadeh, Shafiei, & Rahgoshay, 2015; Binazadeh & Rahgoshay, 2016; Moulay & Perruquetti, 2005; Shiriaev & Fradkov, 2000). In the meantime, design of nonlinear stabilizing controllers which are robust to system uncertainties is more complex for non-affine systems. To the best of author knowledge, there is no paper that deals with the problem of robust output tracking of time-varying reference signals for uncertain non-affine systems by the analytical approach.

In this paper, the problem of robust output tracking is studied for a class of uncertain non-affine systems. The considered class of non-affine system is called non-affine quadratic polynomial systems. It’s because that the quadratic term of control input appears in system model. Several systems can be described by non-affine quadratic polynomial models (Binazadeh & Rahgoshay, 2016; Gutierrez & Ro, 2005; Tombul et al., 2009). In other to solve the tracking problem, the error dynamical equations are constructed. Then, a novel sliding mode controller is designed which guarantees the asymptotically output tracking in the presence of uncertainties and/or...
external disturbances. For this purpose, a proper sliding manifold which is a function of error vector is constructed and according to the upper and lower bounds of uncertainties, two quadratic polynomials are achieved. Moreover, with respect to the location of the roots of the proposed polynomials, the new sliding mode control law is obtained which has essential difference in comparison with the classical sliding mode controllers. The proposed controller guarantees the output tracking of time-varying reference signal and also conquers the effect of the uncertainties. Finally, in order to show the applicability of the proposed method, a practical system (magnetic ball suspension system) is considered for computer simulations. Magnetic suspension systems have been used in many fields such as super-fast magnetic trains, magnetic wind turbines, rotating magnetic systems and high precision positioning systems. These systems can be controlled in two ways: voltage mode control and current mode control. However, the designed controllers based on voltage mode suffer from some inefficiencies which can decrease system’s precision and performance (Gutierrez & Ro, 2005). The current mode control relaxes these problems, however it leads to non-affine quadratic polynomial structure. This paper considers the current mode control for output tracking of magnetic ball suspension system. Simulation results demonstrate the efficiency of the proposed controller.

2. Problem formulation

Consider the following uncertain non-affine quadratic polynomial system:

\[
\begin{align*}
\dot{x}_i(t) &= x_{i+1}(t) \quad 1 \leq i \leq n - 1 \\
\dot{x}_n(t) &= f(x, t) + g(x, t)u + h(x, t)u^2(t) + \Delta f(x, t) + d(t) \\
y(t) &= [c_1 \ c_2 \ldots \ c_n]x(t)
\end{align*}
\]

(1)

where \(x(t) = [x_1 \ x_2 \ldots \ x_n]^T \in D\) is the state vector \((D \subset R^n)\) and \(\{0\} \in D\) where \(R^n\) is the set of all \(n\)-dimensional real vectors and \(D\) is an open subspace of \(R^n\) which contains the origin). \(C = [c_1 \ c_2 \ldots \ c_n]\) is a row vector \((c_i \in R)\) where \(R\) is the set of real numbers. Furthermore, \(f(x, t), g(x, t)\) and \(h(x, t)\) are nonlinear Lipschitz functions in \(x\) on \(D\) and piecewise continuous in \(t\). Moreover, \(\delta(x, t) = \Delta f(x, t) + d(t)\) is an unknown nonlinear function \((\delta(x, t) \in \delta)\) which presents the unknown model uncertainty (i.e. \(\Delta f(x, t)\)) and external disturbance of the system (i.e. \(d(t)\)).

If \(m\) be the maximum value of \(i\) such that \(c_i \neq 0\) (for \(i = 1, 2, \ldots, n\)), therefore, \(y = \sum_{i=1}^{m} c_i x_i\) where \(m \leq n\). Let us define the parameter \(\rho\) as \(\rho = n - m + 1\). Therefore \(\rho\) is an integer number \((\rho \geq 1)\) which shows the relative degree of the system (1).

Note: Please note that if \(c_i \neq 0\) then, \(m = n\) and \(\rho = 1\).

The goal is design of a new sliding mode control law for the system (1) such that the output \(y(t)\) tracks a time-varying reference signal \((r(t))\), and \(y(t) - r(t)\) converges to zero.

Assumption 1: The time-varying reference signal \((r(t))\) is sufficiently smooth and its derivatives \((\dot{r}(t), \ddot{r}(t) \ldots, r^{(\rho)}(t))\) all are bounded for all \(t \geq 0\). Furthermore, \(r^{(\rho)}(t)\) is a piecewise continuous function in \(t\).

Assumption 2: The uncertainty term \(\Delta f(x, t)\) is assumed to be bounded by

\[|\Delta f(x, t)| \leq \eta_1\]

where \(\eta_1\) is a known positive constant.

Assumption 3: The external disturbance \(d(t)\) is assumed to be bounded by

\[|d(t)| \leq \eta_2\]

where \(\eta_2\) is a known positive constant.

3. The proposed technique

In this section, a novel approach is suggested for design of sliding mode controller for non-affine system (1). For this purpose, first an adequate sliding manifold is designed to achieve the control objective (robust tracking). Then, a discontinuous control law is proposed which forces the system trajectories to reach the sliding manifold in finite-time and stay on it for all future times.

The error vector is defined as follows:

\[
\tilde{e}(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_\rho(t) \end{bmatrix} = \begin{bmatrix} r(t) - y(t) \\ \dot{r}(t) - \dot{y}(t) \\ \vdots \\ r^{(\rho-1)}(t) - y^{(\rho-1)}(t) \end{bmatrix}
\]

(2)

Therefore, the dynamical equations of the error vector are in the following form:

\[
\begin{align*}
\dot{e}_1 &= e_2 \\
\vdots \\
\dot{e}_{\rho-1} &= e_\rho \\
\dot{e}_\rho &= r^{(\rho)}(t) - y^{(\rho)}(t)
\end{align*}
\]

where

\[y^{(\rho)}(t) = \sum_{i=1}^{m} c_i x_i^{(\rho)}(t)\]

Related to values of \(m, \rho\) and equations (1), one has

\[
\sum_{i=1}^{m} c_i x_i^{(\rho)}(t) = \sum_{i=1}^{m-1} c_i x_{i+\rho} + c_m (f + gu + hu^2 + \Delta f + d)
\]
Therefore:

\[
\begin{align*}
\dot{e}_1 &= e_2 \\
& \vdots \\
\dot{e}_{p-1} &= e_p 
\end{align*}
\]  

(3a)

\[\dot{e}_p = r^{(p)}(t) - \sum_{i=1}^{m-1} c_i x_{i+p} - c_m (f + gu + hu^2 + \Delta f + d)\]  

(3b)

Asymptotic stabilization of the above time-varying non-affine system results in output tracking for system (1). First, start with the subsystem (3a). Design \(e_p\) as:

\[e_p = -(k_1 e_1 + \ldots + k_{p-1} e_{p-1})\]

(4)

Then,

\[
\begin{bmatrix}
\dot{e}_1 \\
\dot{e}_2 \\
\vdots \\
\dot{e}_{p-1}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots \\
-k_1 & -k_2 & -k_3 & \ldots & -k_{p-1}
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2 \\
\vdots \\
e_{p-1}
\end{bmatrix}
\]

\[\Lambda \begin{bmatrix}
e_1 \\
e_2 \\
\vdots \\
e_{p-1}
\end{bmatrix} =
\begin{bmatrix}
\end{bmatrix}
\]

(5)

where \(\Lambda = \lambda^{p-1} + k_{p-1} \lambda^{p-2} + \ldots + k_1\) is the characteristic polynomial of the above system. If \(k_1\) to \(k_{p-1}\) are chosen such that this polynomial is Hurwitz, then the motion that is governed by (5) is asymptotically stable and \((e_1, \ldots, e_{p-1}) \to 0\). Therefore, the following manifold is a proper sliding manifold.

\[S = (k_1 e_1 + \ldots + k_{p-1} e_{p-1}) + e_p\]

(6)

Since, \(S = 0\) is equivalent to \(e_p = -(k_1 e_1 + \ldots + k_{p-1} e_{p-1})\); therefore, if the trajectories of the system stay on the proposed manifold (6), the motion that is governed by (5) is asymptotically stable. Moreover, \((e_1, \ldots, e_{p-1}) \to 0\); results in \(e_p \to 0\) which guarantees \(\vec{e}(t)\) tends to zero as \(t\) tends to infinity. By time differentiating of (6), one has;

\[\dot{S} = (k_1 \dot{e}_1 + \ldots + k_{p-1} \dot{e}_{p-1}) + \dot{e}_p\]

(7)

According to the error dynamical system (3a) and (3b),

\[\dot{S} = (k_1 e_2 + \ldots + k_{p-1} e_p) + r^{(p)}(t) - \sum_{i=1}^{m-1} c_i x_{i+p} - c_m (f + gu + hu^2 + \Delta f + d)\]

\[= \sum_{i=1}^{\rho-1} k_i e_{i+1} - \sum_{i=1}^{m-1} c_i x_{i+p} - c_m (f + gu + hu^2 + \Delta f + d) + r^{(p)}(t)\]

(8)

In the first order sliding mode method, by considering the Lyapunov function for the dynamical equation \(S\) as \(V(S) = 0.5 S^2\) the control law should be designed such that the condition \(V = -\gamma |S| = \gamma V^0.5\) (where \(\gamma\) is a positive constant) satisfied. This condition, which is called the reaching law, ensures that all trajectories starting off the manifold \(S = 0\), reach it in a finite time and those on the manifold cannot leave it (Yousefi & Binazadeh, 2016). Design of a control law which leads to \(SS \leq -\gamma |S|\) is easy and straightforward for affine systems; however it is not a simple task for non-affine ones. In system (1), if \(h = 0\), then system is affine and \(\dot{S} = \sum_{i=1}^{\rho-1} k_i e_{i+1} - \sum_{i=1}^{m-1} c_i x_{i+p} - c_m (f + gu + \Delta f + d) + r^{(p)}(t)\) where according to the theory of classical sliding mode method the following control law guarantees that \(SS \leq -\gamma |S|\) where \(h = 0\) (Khalil, 2014):

\[u = u_{eq-classical} + u_{classical}\]

\[u_{eq-classical} = \frac{1}{c mg} \left( \sum_{i=1}^{\rho-1} k_i e_{i+1} - \sum_{i=1}^{m-1} c_i x_{i+p} - c_m (f + gu + \Delta f + d) + r^{(p)}(t) \right)\]

\[u_{classical} = -\beta(x) \text{sgn}(s);\]

(9)

where

\[
\frac{\Delta f + d}{g} \leq \rho(x)
\]

\(\rho(x)\) is known positive function and

\[\beta(x) \geq \rho(x) + \beta_0; \beta_0 > 0\]

For \(h \neq 0\), the problem is more complicated and can’t be solved classically. Thus, an altered idea is suggested in this paper. For this purpose, consider the upper and lower bounds of \(\Delta f\) and \(d\) (refer to Assumptions 2 and 3), the following expressions are satisfied for \(\dot{S}\):

\[\dot{S} \leq \sum_{i=1}^{\rho-1} k_i e_{i+1} + r^{(p)}(t) - \sum_{i=1}^{m-1} c_i x_{i+p} - c_m (f + gu + hu^2 + \Delta f + d) + |c_m| (\eta_1 + \eta_2)\]

(10)
\[ \dot{S} \geq \sum_{i=1}^{\rho-1} k_i e_{i+1} + \rho(\dot{\rho})(t) - \sum_{i=1}^{m-1} c_i x_{i+\rho} - c_m (f + gu + hu^2) - |c_m| (\eta_1 + \eta_2) \]  \quad (11)

In the above inequalities, the expressions in the right-hand sides are quadratic polynomials with respect to \( u \). In order to satisfy the reaching law two cases may be considered.

**Case 1:** The system trajectories starting off the sliding manifold and \( S(t_0) > 0 \).

In order to satisfy the reaching condition, the control input \( u \) should be designed such that \( \dot{S} \leq -\gamma \). For this purpose, it is sufficient that the quadratic polynomial in the right-hand side of (10) has a negative value. This expression has the following two roots \( \alpha_1(x) \) and \( \alpha_2(x) \) (assume \( \alpha_1(x) < \alpha_2(x) \)).

\[ \alpha_{1,2}(x) = \frac{-b \pm \sqrt{\Delta}}{2a}, \quad \Delta = b^2 - 4ac \]  \quad (12)

where
\[
\begin{align*}
  a &= -c_m h \\
  b &= -c_m g \\
  c &= \sum_{i=1}^{\rho-1} k_i e_{i+1} + \rho(\dot{\rho}) - \sum_{i=1}^{m-1} c_i x_{i+\rho} - c_m f + |c_m| (\eta_1 + \eta_2)
\end{align*}
\]

If there exists \( D \in \mathbb{R}^n \), \( \{0\} \in D \) where \( \Delta \) is positive on \( D \), then according to the location of \( \alpha_{1,2}(x) \) the control law \( u = \psi(x) \) may be designed such that the sign of the quadratic polynomial in the right-hand side of (10) be negative and therefore the system trajectories reach the sliding manifold \( S = 0 \) in finite time and stay on it which guarantees \( \dot{e}(t) \) tends to zero.

**Case 2:** The system trajectories starting off the sliding manifold and \( S(t_0) < 0 \).

In this case, the control input should be designed such that \( \dot{S} \geq +\gamma \). For this purpose, it is sufficient that the quadratic polynomial in the right-hand side of (11) has a positive value. Similar to the discussion in the previous case, this polynomial may also have two real roots and the control law, \( u = \bar{\psi}(x) \) may be designed with respect to the location of real roots such that the sign of the mentioned quadratic polynomial be positive.

Consequently, the proposed approach results in the following structure for the controller.

\[
\begin{align*}
  u(x) &= \begin{cases} 
    \psi(x) & S > 0 \\
    \bar{\psi}(x) & S < 0 
  \end{cases}
\end{align*}
\]

**Remark 1:** Like the classical controller (9); the proposed control law may also be divided into continuous and discontinuous components to reduce chattering. In what follows, this idea is used in controller design for the magnetic ball levitation system.

### 4. Robust tracking of the non-affine magnetic ball levitation system

In this section, the proposed idea is applied to design of a robust control law for output tracking of a magnetic ball levitation system in the current mode.

#### 4.1. Mathematical model of magnetic ball levitation system in the current mode

Consider the state-space equations of the magnetic ball levitation system in the current mode:

\[
\begin{align*}
  x_1 &= x_2 \\
  x_2 &= g_0 - \frac{k_0}{(l_0 + x_1 + x_{ref})^2} u^2 + \delta(t, x) \\
  y &= x_1 
\end{align*}
\]  \quad (13)

where \( x_1 \) is the position of the magnetic ball with respect to the reference position (i.e. \( x_{ref} \), refer to Figure 1), \( x_2 \) is the velocity of the magnetic ball and \( u \) is the coil current. Moreover, \( \delta(t, x) = \Delta f + d \) is an unknown function consists of all uncertainties and external disturbance and \( k_0 \) and \( l_0 \) are the nominal values of the physical parameters of the magnetic ball levitation system and \( g_0 \) is the gravitational constant.

It is easily observed that the relative degree of the system (13) is equal to 2 (i.e. \( \rho = 2 \)). Thus, the error vector will be defined as follows:

\[
\begin{align*}
  \bar{e} &= \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} x_1 - r(t) \\ x_1 - \dot{r}(t) \end{bmatrix} = \begin{bmatrix} x_1 - r(t) \\ x_2 - \dot{r}(t) \end{bmatrix} 
\end{align*}
\]  \quad (14)

**Figure 1.** Magnetic ball levitation system.
If \( \bar{e} \) can be asymptotically stabilized, the position of the suspended ball will track the time-varying reference signal \( r(t) \). Based on the error vector, the state-space equations of the error dynamical system are as follows:

\[
\begin{align*}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= g_0 - \frac{k_0}{(l_0 + x_1 + x_{\text{ref}})^2} u^2 + \delta(t, x) - \bar{r}(t)
\end{align*}
\]  

(15)

where \( \bar{r}(t) \) is bounded and \( \mu \) is its upper bound (|\( \bar{r}(t) \)| \( \leq \mu \)). Now, the sliding manifold is selected as \( S = e_2 + M e_1 \) (where \( M \) is a positive constant). Differentiation of \( S \) yields to:

\[
\dot{S} = \dot{e}_2 + M \dot{e}_1 = g_0 - \frac{k_0}{(l_0 + x_1 + x_{\text{ref}})^2} u^2 + \delta(t, x) - \bar{r}(t) + M e_2 \tag{16}
\]

The sliding mode control law may be divided to continuous and discrete parts were the continuous part (\( u_{eq}(x) \)) is designed such that the known terms in \( \dot{S} \)-equation can be deleted. Then the discrete part (\( \nu(x) \)) may be designed such that the reaching law (\( \dot{S} \leq -\gamma |S| \)) satisfied. Division of controller to continuous and discrete parts results in discrete controller with lower amplitude which also reduces the chattering. Therefore, the following control law is considered where \( \nu(x) \) should be designed.

\[
u(x) = (l_0 + x_1 + x_{\text{ref}}) \sqrt{\frac{M e_2 + g_0 - \bar{r}(t)}{u_{eq}(x)}} + \nu(x) \tag{17}\]

(17)

It should be considered that controller (17) may work properly if the following expression is satisfied:

\[
M e_2 + g_0 - \bar{r}(t) \geq 0 \Rightarrow e_2 \geq \frac{\bar{r}(t) - g_0}{M} \tag{18}\]

By substituting Eq. (17) in Eq. (16), one has:

\[
\dot{S} = -\frac{k_0}{(l_0 + x_1 + x_{\text{ref}})^2} u^2 + \left( -\frac{2}{(l_0 + x_1 + x_{\text{ref}})} \sqrt{k_0(M e_2 + g_0 - \bar{r}(t))} \right) u + \delta(t, x) \tag{19}\]

assuming that |\( \delta(t, x) \)| \( \leq \eta \) and also \( \eta < g_0 \), the following inequalities are satisfied for \( \dot{S} \):

\[
\dot{S} \leq -\frac{k_0}{(l_0 + x_1 + x_{\text{ref}})^2} u^2 + \left( -\frac{2}{(l_0 + x_1 + x_{\text{ref}})} \sqrt{k_0(M e_2 + g_0 - \bar{r}(t))} \right) u + \eta \tag{20}\]

As mentioned in the previous section two cases may be considered:

**Case (1) \( S(t_0) > 0 \):** The right-hand side of the inequality (20) is a quadratic polynomial function with respect to \( \nu(x) \) and with the following coefficients:

\[
\begin{align*}
a(x) &= -\frac{k_0}{(l_0 + x_1 + x_{\text{ref}})^2} \\
b(x) &= \left( -\frac{2}{(l_0 + x_1 + x_{\text{ref}})} \sqrt{k_0(M e_2 + g_0 - \bar{r}(t))} \right) \\
c(x) &= \eta
\end{align*}
\]

(22)

Since, the term \( a(x) \) is always negative, the parabola is convex and \( \Delta \) is obtained as:

\[
\Delta = b^2 - 4ac = \frac{4k_0(M e_2 + g_0 - \bar{r}(t) + \eta)}{(l_0 + x_1 + x_{\text{ref}})^2}
\]

In order to have two real roots, \( \Delta \) should be positive. Therefore, the following condition should be satisfied:

\[
e_2 > \frac{-g_0 - \bar{r}(t) + \eta}{M} \tag{23}\]

For reference signal with admissible upper bound (i.e. |\( \bar{r}(t) \)| \( < g_0 + \eta \)) the above inequality is satisfied and therefore the right-hand side polynomial in (20) has two real roots (\( \alpha_1(x) \) and \( \alpha_2(x) \), where \( \alpha_2(x) > \alpha_1(x) \)). By choosing \( \nu(x) > \alpha_2(x) \) or \( \nu(x) < \alpha_1(x) \), the right-hand side term of (20) will be negative. Therefore, it guarantees that \( \dot{S} \) is negative and therefore the trajectories starting off the sliding manifold reach it in finite-time and the reaching condition satisfies for Case (1). For this purpose \( \nu(x) \) can be considered as follows:

\[
\nu = \alpha_2(x) + \varphi(x) \tag{24}\]

where \( \varphi(x) \) is a positive definite function which is added to the \( \alpha_2(x) \) to ensure that \( \nu(x) > \alpha_2(x) \). Moreover, \( \alpha_2(x) \) or the greater root of the mentioned polynomial is:

\[
\alpha_2(x) = \frac{1}{k_0} (-l_0 + x_1 + x_{\text{ref}}) \sqrt{\frac{k_0(M e_2 + g_0 - \bar{r}(t) + l_0 + x_1 + x_{\text{ref}})}{\sqrt{k_0(M e_2 + g_0 - \bar{r}(t) + \eta)}}} \tag{25}\]

**Case (2) \( S(t_0) < 0 \):** Regarding to the inequality (21), the term \( \nu(x) \) should be designed in a way that the polynomial in the right-hand side of the this inequality...
becomes positive which warrants that $\dot{S}$ is positive and therefore the trajectories reach the sliding manifold ($S = 0$) in finite-time. For this case $\Delta$ is defined as follows:

$$\Delta = \frac{4k_0(Me_2 + g_0 - \dot{r}(t) - \eta)}{(l_0 + x_1 + x_{\text{ref}})^2} \tag{26}$$

In order to have two real roots, $\Delta$ should be positive. Therefore, the following condition should be satisfied for case (2);

$$e_2 > -\frac{g_0 + \eta + \dot{r}(t)}{M} \tag{27}$$

By satisfaction of the above condition, the right-hand side polynomial in (21) has two real roots ($\bar{\alpha}_1(x)$ and $\bar{\alpha}_2(x)$ where $\bar{\alpha}_2(x) > \bar{\alpha}_1(x)$). To keep this polynomial positive, the term $\nu(x)$ should be selected between $\bar{\alpha}_1(x)$ and $\bar{\alpha}_2(x)$ ($\bar{\alpha}_1(x) < \nu < \bar{\alpha}_2(x)$). Therefore, $\nu(x)$ can be defined as:

$$\nu(x) = \bar{\alpha}_2(x) - \psi(x) \tag{28}$$

By choosing small positive function $\varphi(x)$, it can be ensured that $\bar{\alpha}_1(x) < \nu(x) < \bar{\alpha}_2(x)$. In this equation, $\bar{\alpha}_2(x)$ is the greater root of the right-hand side polynomial in (21) and is equal to:

$$\bar{\alpha}_2(x) = \frac{1}{k_0}(-l_0 + x_1 + x_{\text{ref}}) \sqrt{k_0(Me_2 + g_0 - \dot{r}(t))}$$

$$+ |l_0 + x_1 + x_{\text{ref}}| \sqrt{k_0(Me_2 + g_0 - \dot{r}(t) - \eta)} \tag{29}$$

Therefore, $\nu(x)$ is derived as follows:

$$\nu(x) = \begin{cases} 
\psi(x) & S > 0 \\
\dot{\psi}(x) & S < 0 
\end{cases} \tag{30}$$

where

$$\psi(x) = \alpha_1(x) + \varphi(x)$$

$$\dot{\psi}(x) = \alpha_1(x) - \varphi(x) \tag{31}$$

Consequently the following controller guarantees that the position of the magnetic ball tracks the time-varying signal $r(t)$:

$$u(x) = u_{\text{eq}}(x) + \nu(x)$$

**Remark 2:** According to Lyapunov stability theorem, the region of stability analysis of system (15) (i.e. the open set $D \in \mathbb{R}^2$) should include the origin. Regarding the inequalities (23) and (27) and since $-g_0 + \eta + \dot{r}(t) > -g_0 - \eta + \dot{r}(t)$ thus for reference signal with admissible upper bound $\mu$ (i.e. $|\dot{r}(t)| \leq \mu < g_0 - \eta$) the right-sides of (23) and (27) are both negative. Since $g_0 - \eta$ should be positive, thus uncertainties with the upper bound $\eta < g_0$ are admissible and therefore there exists the open set $D \in \mathbb{R}^2$ ($D = \{(e_1, e_2) \in \mathbb{R}^2 \text{ where } e_1 \in \mathbb{R} \text{ and } |e_2| < g_0 - \eta - \mu \}$) where $(0,0) \in D$. In this region the condition (18) is also satisfied.

**Remark 3:** The expression (30) may right as follows:

$$u(x) = \frac{\psi(x) + \dot{\psi}(x)}{2} + \frac{\psi(x) - \dot{\psi}(x)}{2} \text{sgn}(S)$$

In order to avoid the chattering behavior $\text{sgn}(S)$ may be substituted with its smooth approximation like the saturation function with high slope which is common in literature (Binazadeh & Yousefi, 2017; Chenarani & Binazadeh, 2017).

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5. Computer simulations

In this section computer simulations are done to show the performance of the proposed controller. The considered time-varying reference signal is a quasi-pulsed signal. The parameter are considered as $k_0 = 1$, $l_0 = 0.01$, $x_{\text{ref}} = 0.01 \text{ m}$, $g = 9.81 \text{ m/s}^2$, $M = 5$ and $\delta(t, x) = 0.5 \sin(10t)$.

Figures 2–4 show the time history of the output, the sliding manifold, and control signal, respectively. Simulations show the ability of the proposed controller in the robust output tracking.

6. Conclusion

In this paper, the robust output tracking problem was considered for non-affine quadratic polynomial systems. Dynamical equations of the error vector were constructed and a novel sliding mode controller was designed which guarantees the robust output tracking. Moreover, the proposed method was applied on a practical quadratic polynomial system (magnetic ball levitation system in the current mode). Computer simulations illustrated the efficiency of the proposed controller.

Disclosure statement

No potential conflict of interest was reported by the authors.

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