TOPOLOGY OF RANDOM
RIGHT ANGLED ARTIN GROUPS

ARMINDO COSTA AND MICHAEL FARBER

Abstract. In this paper we study topological invariants of a class of random groups. Namely, we study right angled Artin groups associated to random graphs and investigate their Betti numbers, cohomological dimension and topological complexity. The latter is a numerical homotopy invariant reflecting complexity of motion planning algorithms in robotics. We show that the topological complexity of a random right angled Artin group assumes, with probability tending to one, at most three values, when $n \to \infty$. We use a result of Cohen and Pruidze which expresses the topological complexity of right angled Artin groups in combinatorial terms. Our proof deals with the existence of bi-cliques in random graphs.

1. Introduction

Given a finite graph (i.e. a one-dimensional simplicial complex) $\Gamma$ with vertex set $V$ and with the set of edges $E$ one associates to it a right angled Artin group (RAAG) (also known as a graph group)

$G_\Gamma = \{v \in V; vw = wv \text{ iff } (v, w) \in E\},$

see [4], [16]. In the case when $\Gamma$ is a complete graph $G_\Gamma$ is a free abelian group of rank $n = |V|$; in the other extreme, when $\Gamma$ has no edges the group $G_\Gamma$ is the free group of rank $n$. In general $G_\Gamma$ interpolates between the free and free abelian groups.

In this paper we are interested in right angled Artin groups associated to random graphs $\Gamma$. We adopt one of the basic Erdős - Rényi models of random graphs in which each edge of the complete graph on $n$ vertices is included with probability $0 < p < 1$ independently of all other edges. In other words, we consider the probability space $\Omega_n$ of all $2^{\binom{n}{2}}$ subgraphs of the complete graph on $n$ vertices $\{1, 2, \ldots, n\}$ and the probability
that a specific graph $\Gamma \in \Omega_n$ appears as a result of a random process equals
\begin{equation}
P(\Gamma) = p^{E_\Gamma}(1-p)^{\binom{n}{2}-E_\Gamma},
\end{equation}
where $E_\Gamma$ denotes the number of edges of $\Gamma$, see [12].

In the sequel we examine statistics of various topological invariants of the group $G_\Gamma$ associated to a random graph. Each of such invariants is a random function and it is quite natural to ask about its mathematical expectation and distribution function. We assume that $n \to \infty$ and seek results of asymptotic nature.

Various probabilistic approaches to group theory can be found in [11] and [18].

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2. Betti numbers of random graph groups

First we remind the well-known construction of an aspherical complex $K_\Gamma$ with fundamental group $G_\Gamma$. We refer to [4] and [16] for proofs and more detail.

Let $V = V_\Gamma$ denote the set of vertices of the graph $\Gamma$. The torus $T^n$ where $n = |V|$ can be identified with the set of all functions $\phi : V \to S^1$. The support \text{supp}$(\phi) \subset V$ of a function $\phi : V \to S^1$ is defined as the set of vertices $v \in V$ such that $\phi(v) \neq 1$. One defines $K_\Gamma \subset T^n$ to be the set of all functions $\phi$ such that their support \text{supp}$(\phi)$ generates a complete subgraph of $\Gamma$, i.e. any two vertices of the support are connected by an edge in $\Gamma$. It is known [4], [16] that $K_\Gamma$ (viewed with the induced topology) is aspherical and its fundamental group is $\Gamma$.

Fix the cell decomposition of $S^1$ consisting of a 0-cell $1 \in S^1$ and a 1-cell $S^1 - \{1\}$. Then $T^n$ inherits a cell decomposition with cells in one-to-one correspondence with subsets of $V$. In this decomposition $K_\Gamma \subset T^n$ is a cell subcomplex; the cells of $K_\Gamma$ are in 1-1 correspondence with complete subgraphs of $\Gamma$. Namely, given a subset $S \subset V$ one considers the set $e_S$ of all functions $\phi : V \to S^1$ with support equal $S$; clearly $e_S$ is a cell of dimension $|S|$.

The cohomology algebra of $K_\Gamma$ with integral coefficients is the quotient
\begin{equation}
H^*(K_\Gamma; \mathbb{Z}) \simeq E(v_1, \ldots, v_n)/J_\Gamma
\end{equation}
where $E(v_1, \ldots, v_n)$ is the exterior algebra generated by degree one classes corresponding to the vertices $V = \{v_1, \ldots, v_n\}$ of $\Gamma$ and the ideal $J_\Gamma$ is generated by the degree two monomials $vw$ such that the corresponding vertices $v, w$ are not connected by an edge.
In particular any product \( v_{i_1} v_{i_2} \ldots v_{i_r} \) vanishes iff the corresponding vertices \( \{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\} \) do not form a complete subgraph of \( \Gamma \).

One obtains: for an integer \( r \geq 2 \) the \( r \)-th Betti number \( b_r(G_{\Gamma}) = b_r(K_{\Gamma}) \) equals the number of complete subgraphs of size \( r \) in \( \Gamma \). Note that \( b_0(G_{\Gamma}) = 1 \) and \( b_1(G_{\Gamma}) = n \) for any graph \( \Gamma \).

**Lemma 1.** The expectation of the \( r \)-th Betti number of the group \( G_{\Gamma} \) of a random graph \( \Gamma \), where \( r \geq 2 \), equals

\[
E(b_r(G_{\Gamma})) = \binom{n}{r} p^r.
\]

**Proof.** As explained above we must find the number of maximal complete subgraphs of size \( r \) in \( \Gamma \in \Omega_n \). For a subset \( S \subset \{1, \ldots, n\} \) with \( |S| = r \) consider the random variable \( I_S : \Omega_n \rightarrow \{0, 1\} \) which equals 1 on a graph \( \Gamma \in \Omega_n \) iff \( S \) forms a complete subgraph in \( \Gamma \). Then \( E(I_S) = p^r \) and \( \sum_S I_S \) is the number of all complete subgraphs on \( r \) vertices. This shows that \( E(\sum_S I_S) \) is as stated. \( \square \)

Now we assume that \( r \) (the dimension) is fixed and \( p \) may depend on \( n \). Asymptotically, the expectation of \( b_r(G_{\Gamma}) \) can be written as

\[
E(b_r(G_{\Gamma})) \sim \frac{1}{r!} \left[ n p^{r-1} \right]^r.
\]

The expectation has a positive limit for \( n \to \infty \) if and only if

\[
np^{\frac{r-1}{2}} \rightarrow c > 0.
\]

Under this condition the expectation \( E(b_r(G_{\Gamma})) \) converges to \( \frac{c^r}{r!} \).

Note that the convergence (4) to a positive limit may happen for one dimension \( r \) only.

Moreover, assuming (4), the distribution of \( b_r : \Omega \rightarrow \mathbb{Z} \) converges to the Poisson distribution with expectation

\[
\lambda = \frac{c^r}{r!}.
\]

see below. Theorem 2 is an interpretation of a theorem of Schürger [17] about complete subgraphs in random graphs. More general results concerning the containment of a specific graph in a random graph were established by by Ballobás (1981) and Karoński and Ruciński (1983). See Theorem 3.19 from [12].

**Theorem 2.** Fix an integer \( r > 1 \) and consider the function of \( r \)-th Betti number of the associated graph group,

\[
b_r : \Omega_n \rightarrow \mathbb{Z}, \quad b_r(\Gamma) = b_r(G_{\Gamma}),
\]
as a random function of a random graph. If the limit (4) exists and is positive then for any integer \( k = 0, 1, \ldots \) the probability
\[
P(b_r(G_\Gamma) = k)
\]
converges (as \( n \to \infty \)) to
\[
e^{-\lambda} \frac{\lambda^k}{k!}
\]
where \( \lambda \) is given by formula (5).

In other words, Theorem 2 claims that the limiting distribution is Poisson with mean \( \lambda \).

**Example 3.** Consider the following examples illustrating the previous Theorem.

Suppose that \( r = 2 \) and \( p = \frac{4}{n^2} \). Then \( c = 2 \), \( \lambda = 2 \), and for any integer \( k = 0, 1, \ldots \) the probability that \( b_2(G_\Gamma) = k \) converges to \( \frac{2^k}{e^{2k}} \) as \( n \to \infty \).

As another example, assume that \( r = 3 \) and \( p = \frac{6}{n^3} \). Then \( \lambda = 36 \) and the probability that \( b_3(G_\Gamma) = k \) converges to \( \frac{36^k}{e^{36k}} \) as \( n \to \infty \).

3. **Cohomological dimension of random graph groups**

The cohomological dimension of \( G_\Gamma \) equals the size of the maximal clique in \( \Gamma \); this follows from the discussion of section [2]. Recall that a clique in a graph is defined as a maximal complete subgraph. The clique number \( \text{cl}(\Gamma) \) of a graph \( \Gamma \) is the maximal order of a clique in \( \Gamma \).

There are many results in the literature about the clique number of random graphs; we may interpret these results as statements about the cohomological dimension of graph groups build out of random graphs. Matula [14], [15] discovered that for fixed values of \( p \) the distribution of the clique number of a random graph is highly concentrated in the sense that almost all random graphs have about the same clique number. These results were developed further by Bollobás and Erdős [2]; see the monographs of B. Bollobás [3] and of N. Alon and J. Spencer [1].

Below we restate a result of Matula [15] as a statement about cohomological dimension of random graph groups.

Denote
\[
z(n, p) = 2 \log_q n - 2 \log_q \log_q n + 2 \log_q (e/2) + 1,
\]
where \( q = p^{-1} \).

**Theorem 4.** Fix an arbitrary \( \epsilon > 0 \). Then
\[
[z(n, p) - \epsilon] \leq \text{cd}(G_\Gamma) \leq [z(n, p) + \epsilon],
\]
asymptotically almost surely (a.a.s). In other words, the probability that a graph $\Gamma \in \Omega_n$ does not satisfy inequality (7) tends to zero when $n$ tends to infinity.

Here $\lfloor x \rfloor$ denotes the largest integer not exceeding $x$. We may assume that $\epsilon < 1/2$; then the integers $\lfloor z(n, p) - \epsilon \rfloor$ and $\lfloor z(n, p) + \epsilon \rfloor$ either coincide or differ by 1.

Thus, according Theorem \ref{thm:cohom} the cohomological dimension $\text{cd}(G_\Gamma)$ for a random graph $\Gamma$ takes on one of at most two values depending on $n$ and $p$, with probability approaching 1 as $n \to \infty$.

The next Lemma is a technical result which will be used later in this paper.

Lemma 5. Fix $\epsilon > 0$ and let $\ell = \lfloor z(n, p) - \epsilon \rfloor$. Then
\[
\ell^{-1} \cdot \binom{n}{\ell} p^{(\ell)} \to \infty
\]
as $n \to \infty$.

Proof. One has $\ell = \lfloor z(n, p) - \epsilon \rfloor \leq z(n, p) - \epsilon$ and therefore
\[
p^{(\ell)} \geq \left( p^{(z(n, p) - \epsilon - 1)/2} \right)^r
\]
\[
= \left( p^{\log q n - \log p \log q n + \log q/(e/2)} \right)^r
\]
\[
= \left( \frac{2C \log q n}{en} \right)^r, \quad \text{where} \quad C = q^{\epsilon/2} > 1.
\]

On the other hand, using Stirling’s formula, we have
\[
\binom{n}{r} = c_n \cdot \left( \frac{n}{r} \right)^r e^{-r} r^{-1/2}
\]
where $c_n$ and $c_n^{-1}$ are bounded. Therefore,
\[
\ell^{-1} \cdot \binom{n}{r} p^{(\ell)} \geq \ell^{-1} c_n \left( \frac{n}{\ell} \right)^r r^{-1/2} \left( \frac{2C \log q n}{en} \right)^r
\]
\[
\begin{aligned}
&= \left( C \cdot \frac{2 \log q n}{r} \right)^r \cdot r^{-3/2} \cdot c_n \\
&\geq C^r \cdot r^{-3/2} \cdot c_n.
\end{aligned}
\]
Clearly, $C^r \cdot r^{-3/2} \cdot c_n$ tends to infinity since $C > 1$. This completes the proof. \nobreak\hfill $\Box$

4. Motion planning algorithms and the concept of topological complexity

Given a mechanical system, a motion planning algorithm is a function which assigns to any pair of states of the system, an initial state and a desired state, a continuous motion of the system starting at the
initial state and ending at the desired state. The design of effective motion planning algorithms is one of the challenges of modern robotics, see [13]. Motion planning algorithms are applicable in various situations when the system is autonomous and operates in a fully or partially known environment.

The complexity of motion planning algorithms is measured by a numerical invariant $\text{TC}(X)$ which depends on the homotopy type of the configuration space $X$ of the system [7]. This invariant is defined as the Schwarz genus (also known as the “sectional category”) of the path-space fibration

$$p : PX \to X \times X.$$  \hspace{1cm} (8)

Here $PX$ is the space of all continuous paths $\gamma : [0, 1] \to X$ equipped with the compact-open topology and $p(\gamma) = (\gamma(0), \gamma(1))$ is the map associating to a path $\gamma : [0, 1] \to X$ its pair of endpoints. Explicitly, $\text{TC}(X)$ is the smallest integer $k$ such that $X \times X$ admits an open cover $U_1 \cup U_2 \cup \cdots \cup U_k = X \times X$ with the property that there exists a continuous section $U_i \to PX$ of (8) for each $i = 1, \ldots, k$. If $X$ is an Euclidean neighborhood retract then $\text{TC}(X)$ can be equivalently characterized as the minimal integer $k$ such that there exists a section $s : X \times X \to PX$ of the fibration $p$ with the property that $X \times X$ can be represented as the union of $k$ mutually disjoint locally compact sets

$$X \times X = G_1 \cup \cdots \cup G_k$$

such that the restriction $s|G_i$ is continuous for $i = 1, \ldots, k$, see [10], Proposition 4.2. A section $s$ as above represents a motion planning algorithm: given a pair $(A, B) \in X \times X$ the image $s(A, B) \in PX$ is a continuous motion of the system starting at the state $A$ and ending at the state $B$.

Intuitively, the topological complexity $\text{TC}(X)$ can be understood as a measure of the navigational complexity of the topological space $X$; it is the minimal number of continuous rules which are needed to describe a motion planning algorithm in $X$.

The invariant $\text{TC}(X)$ admits an upper bound in terms of the dimension of the configuration space $X$,

$$\text{TC}(X) \leq 2 \dim(X) + 1$$ \hspace{1cm} (9)

see [7], Theorem 4. There are many examples when inequality (9) is sharp: take for instance $X = T^n \# T^n$, the connected sum of two copies of a torus, having the topological complexity $\text{TC}(X) = 2n + 1$. However for any simply connected space $X$ one has a more powerful upper bound

$$\text{TC}(X) \leq \dim(X) + 1,$$ \hspace{1cm} (10)
see [8]. The latter inequality is sharp for any simply connected closed symplectic manifold $X$, see [9].

There are many other examples when the inequality (9) is not sharp. It was established in [5] that for any finite cell complex $X$ with $\pi_1(X) = \mathbb{Z}_2$ one has

$$\text{TC}(X) \leq 2\dim(X).$$

(11)

For example $\text{TC}(\mathbb{R}P^n) \leq 2n$ for all $n$; moreover, $\text{TC}(\mathbb{R}P^n) = 2n$ if and only if $n$ is a power of 2, see Corollary 14 of [9].

The main result of this paper states that the inequality (9) is asymptotically very close to be an equality in the case of Eilenberg-MacLane spaces of random graph groups.

This statement is consistent with the general philosophy asserting that topological invariants of “random spaces” are “simpler” than their deterministic analogues.

5. The main result

Consider the probability space $\Omega_n$ of random graphs on $n$ vertices with probability given by formula (1). For any $\Gamma \in \Omega_n$ consider the corresponding Eilenberg-MacLane complex $K_{\Gamma}$ (see §2) and its topological complexity $\text{TC}(K_{\Gamma})$, as defined in the previous section.

**Theorem 6.** Fix an arbitrary $0 < \epsilon < 1/2$. Then for any random graph $\Gamma \in \Omega_n$ one has

$$2 \cdot \lfloor z(n, p) - \epsilon \rfloor + 1 \leq \text{TC}(K_{\Gamma}) \leq 2 \cdot \lfloor z(n, p) + \epsilon \rfloor + 1,$$

(12) 

asymptotically almost surely, where $z(n, p)$ is given by formula (6). In other words, probability that a graph $\Gamma \in \Omega_n$ does not satisfy inequality (12) tends to zero when $n$ tends to infinity.

It is clear that the integers on the left and on the right of inequality (12) differ at most by 2. Hence Theorem 6 determines the value of the topological complexity $\text{TC}(G_{\Gamma})$ for a random graph with ambiguity of at most 2. Comparing with the result of Theorem 4 we obtain

**Corollary 7.** For a random graph $\Gamma \in \Omega_n$ one has

$$2 \cdot \text{cd}(\Gamma) - 1 \leq \text{TC}(K_{\Gamma}) \leq 2 \cdot \text{cd}(\Gamma) + 1,$$

(13) 

asymptotically almost surely.

Note that we have

$$\text{cat}(K_{\Gamma}) = \text{cd}(\Gamma) + 1$$

for the Lusternik-Schnirelmann category, as it is easy to see.

The rest is this section is devoted to the proof of Theorem 6.
By an \((r, r)\) bi-clique in a graph \(\Gamma\) we understand an ordered pair consisting of two disjoint complete subgraphs of \(\Gamma\) on \(r\) vertices. To specify an \((r, r)\) bi-clique one has to determine an \(r\)-element subset \(S\) of the set of vertices of \(\Gamma\) and an \(r\)-element subset \(T\) in the complement \(V - S\) such that the induced graphs on \(S\) and \(T\) are complete.

We know from sections §2, 3 that \(\text{cd}(G_\Gamma) \geq r\) if and only if \(\Gamma\) contains an \(r\)-clique, i.e. a maximal complete subgraph on \(r\) vertices. By Theorem of Cohen and Pruidze [6] one has \(\text{TC}(K_\Gamma) \geq 2r + 1\) if \(\Gamma\) contains an \((r, r)\) bi-clique.

In the rest of this section we set
\[
r = \lfloor z(n, p) - \epsilon \rfloor.
\]
Theorem [6] follows once we have shown that a random graph \(\Gamma \in \Omega_n\) contains an \((r, r)\) bi-clique a.a.s. The right hand side of the inequality (12) follows from the general upper bound (10) and from the right hand side of (7).

Let \(r > 0\) be an integer and let \(X : \Omega_n \to \mathbb{Z}\) denote the number of \((r, r)\) bi-cliques in random graph. Our goal is to show that \(X > 0\) asymptotically almost surely, i.e.
\[
\mathbb{P}(X > 0) \to 1, \quad \text{for} \quad n \to \infty.
\]
The proof of (14) will use the second moment method and will be based on the inequality
\[
\mathbb{P}(X > 0) \geq \frac{(EX)^2}{\mathbb{E}(X^2)}
\]
see [12], page 54. Thus, our statement follows once we show that
\[
\frac{\mathbb{E}(X^2)}{(EX)^2} \to 1 \quad \text{as} \quad n \to \infty.
\]

Let \(S\) and \(T\) be disjoint \(r\)-element subsets of the set of vertices of \(K_n\) and let
\[
I_{(S,T)} : \Omega_n \to \{0, 1\}
\]
denote the function which equals 1 on a graph \(\Gamma \in \Omega_n\) if and only if \(S\) and \(T\) form a bi-clique in \(\Gamma\). Then
\[
X = \sum_{(S,T)} I_{(S,T)}
\]
where the sum is taken over all ordered pairs of disjoint \(r\)-element subsets of \(\{1, 2, \ldots, n\}\). Note that one obviously has
\[
\mathbb{E}(I_{(S,T)}) = p^2(\epsilon)
\]
and thus

\[ E(X) = \binom{n}{r,r} p^{2\binom{r}{2}}, \]

where

\[ \binom{n}{r,r} = \frac{n!}{r! \cdot r! \cdot (n-2r)!} \]

denotes the multinomial coefficient. Similarly,

\[ X^2 = \sum I_{(S,T)} \cdot I_{(S',T')}. \]  

(17) Here \((S,T)\) and \((S',T')\) run over all ordered pairs of disjoint \(r\)-element subsets of the set of vertices \(\{1, \ldots, n\}\). Denoting

\[
\begin{align*}
  a &= |S \cap S'|, \\
  b &= |T \cap S'|, \\
  c &= |S \cap T'|, \\
  d &= |T \cap T'|,
\end{align*}
\]

(see Figure 1) we find

\[ E(I_{(S,T)} \cdot I_{(S',T')}) = p^{4\binom{r}{2} - \binom{a}{2} - \binom{b}{2} - \binom{c}{2} - \binom{d}{2}}. \]  

(18) Therefore taking into account (17) one obtains the following expression

\[ \frac{E(X^2)}{E(X)^2} = \sum_{\alpha \in D} F_{\alpha} \cdot q^{L(\alpha)} = \sum_{\alpha \in D} T_{\alpha}. \]  

(19) Here

\[ \alpha = (a, b, c, d) \in \mathbb{Z}^4 \]

denotes a vector and \(D\) is the set of all vectors \(\alpha = (a, b, c, d)\) with nonnegative integer components satisfying the inequalities

\[ a + b \leq r, \quad a + c \leq r, \quad c + d \leq r, \quad b + d \leq r, \]  

(20)
see Figure 1. In formula (19) the coefficient $F_\alpha$ is given by

$$F_\alpha = \frac{(r_a, c)(r_b, d)(n-2r_{r-a-b, r-c-d})}{n_{r,r}}$$

and

$$L(\alpha) = \left(\begin{array}{c} a \\ 2 \end{array}\right) + \left(\begin{array}{c} b \\ 2 \end{array}\right) + \left(\begin{array}{c} c \\ 2 \end{array}\right) + \left(\begin{array}{c} d \\ 2 \end{array}\right), \quad q = p^{-1},$$

while

$$T_\alpha = F_\alpha \cdot q^{L(\alpha)}.$$

Let $m(x, y)$ denotes $\max\{x, y\}$. Then inequalities (20) can be rewritten in a simple form as

$$m(a, d) + m(b, c) \leq r.$$ 

Next we mention the symmetry of the problem. There are two commuting involutions

$$\beta, \gamma : D \rightarrow D, \quad \beta^2 = 1 = \gamma^2,$$

where

$$\beta(a) = b, \quad \beta(c) = d, \quad \gamma(a) = c, \quad \gamma(b) = d.$$ 

These two involutions generate an action of the group $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ of $D$ which preserves both functions $T_\alpha$ and $L(\alpha)$. This action is transitive on the four coordinates.

Recall that our goal is to show that the sum (19) tends to 1 as $n \rightarrow \infty$. Note that

$$\sum_{\alpha \in D} F_\alpha = 1$$

for obvious reasons. Observe also that the term $F_0$ corresponding to $\alpha = (0, 0, 0, 0) \in D$ equals

$$F_0 = \frac{(n-2r)}{n_{r,r}} = \prod_{k=0}^{2r-1} \left(1 - \frac{2r}{n-k}\right) \geq \left(1 - \frac{2r}{n-2r+1}\right)^{2r} \geq 1 - \frac{4r^2}{n-2r+1}.$$ 

Hence we see that $F_0 \rightarrow 1$ as $n \rightarrow \infty$. Therefore, the sum of all coefficients $F_\alpha$ with $\alpha \neq 0$ tends to zero. However the value of the second factor $q^{L(\alpha)}$ becomes increasingly high when the coordinates of $\alpha$ grow.
As an example, consider the term of (19) corresponding to \( \alpha = (r, 0, 0, r) \). Then \( F_\alpha = \binom{n}{r, r}^{-1} \), \( L(\alpha) = 2 \binom{r}{2} \) and

\[
T_\alpha = F_\alpha \cdot q^{L(\alpha)} = \frac{1}{\binom{n-r}{r, r}} q^{2 \binom{r}{2}} \sim \left[ \binom{n}{r} p^{\binom{r}{2}} \right]^{-2}.
\]

Thus we obtain

\[
(26) \quad r^2 T_{(r,0,0,r)} = o(1),
\]

by Lemma 5.

As another example consider the term with \( \alpha = (r, 0, 0, 0) \). Then

\[
T_\alpha = F_\alpha q^{L(\alpha)} = \frac{(r-2)}{r} \frac{n-r}{(r, r)} q^{(r-1)} \sim \left[ \binom{n-r}{r} p^{\binom{r}{2}} \right]^{-1}.
\]

In this case we have

\[
(27) \quad r T_{(r,0,0,0)} = o(1),
\]

by Lemma 5.

The term \( T_\alpha \) with \( \alpha = (1, 0, 0, 0) \) satisfies

\[
(28) \quad T_{(1,0,0,0)} \leq \frac{r^2}{n}
\]

as one easily checks.

Next we consider \( T_\alpha \) with \( \alpha = (r - 1, 0, 0, 0) \). One has

\[
T_\alpha = r \cdot \frac{(n-2r)}{r} \binom{r-1}{r, r} q^{(r-1)} \sim r(n - 2r) \frac{(n-2r)}{r} \binom{r-1}{r, r} q^{(r-1)}
\]

\[
\sim \frac{r(n - 2r)}{r} \binom{r-1}{r, r} q^{(r-1)} \sim np^{r-1} \cdot \left[ \frac{r}{\binom{n}{r} p^{\binom{r}{2}}} \right]
\]

\[
\leq C' np^{r-1} \sim C \log_2 \frac{n}{n}
\]

for some constants \( C, C' \); here we have used Lemma 5. Thus, we have the inequality

\[
(29) \quad T_{(r-1,0,0,0)} \leq C \cdot \log_2 \frac{n}{n}.
\]

Using similar arguments one obtains

\[
(30) \quad T_{(r-1,0,0,r-1)} \leq C'' \cdot \log_2 \frac{n}{r^2},
\]

\[\text{In this paper the symbol } a_n \sim b_n \text{ means that the sequences } a_n b_n^{-1} \text{ and } a_n^{-1} b_n \text{ are bounded.}\]
where $C''$ is a constant independent of $n$.

As a summary of the above discussion of examples we can make the following claim which will be referred to later:

If $\alpha$ is either $(1, 0, 0, 0)$, or $(r - 1, 0, 0, 0)$, or $(r - 1, 0, 0, r - 1)$ then

$$(31) \quad r^4 T_\alpha = o(1).$$

Recall that

$$r = \lfloor 2 \log_q n - 2 \log_q \log_q n + 2 \log_q (e/2) + 1 - \epsilon \rfloor,$$

and in particular $r \leq 2 \log_q n$. Fix $\lambda$ satisfying the inequality

$$(32) \quad 0 < \lambda < \frac{1}{1 + eq}$$

and split the set of all integers in $[0, r]$ into three subsets

$$S_\lambda = \{ x \in \mathbb{N}; 0 \leq x \leq (1 - \lambda) \log_q n \},$$
$$I_\lambda = \{ x \in \mathbb{N}; (1 - \lambda) \log_q n < x < (1 + \lambda) \log_q n \},$$
$$L_\lambda = \{ x \in \mathbb{N}; (1 - \lambda) \log_q n \leq x \leq r \}.$$

Integers lying in $S_\lambda$, $I_\lambda$, and $L_\lambda$ will be called “small”, “intermediate” and “large”, correspondingly.

Suppose that $\alpha' \in D$ is obtained from $\alpha = (a, b, c, d) \in D$ by increasing of one of the coordinates by 1, say, $\alpha' = (a + 1, b, c, d)$. Then the ratio of the corresponding terms of sum (19) equals

$$\frac{T_{\alpha'}}{T_\alpha} = \frac{(r - a - b)(r - a - c)}{(a + 1)(n - 4r + \ell + 1)} \cdot q^a,$$

where $\ell = \ell(\alpha) = a + b + c + d$. Clearly, one has

$$n/2 \leq n - 4r + \ell + 1 \leq n,$$

assuming that $n$ is large enough. Hence we obtain

$$(33) \quad A \cdot q^a \leq \frac{T_{\alpha'}}{T_\alpha} \leq 2 \cdot A \cdot q^a$$

where

$$(34) \quad A = \frac{(r - a - b)(r - a - c)}{(a + 1)n}.$$

If $a \in S_\lambda$ is small then $q^a \leq n^{1-\lambda}$, $A \leq \frac{r^2}{n^3}$ and

$$Aq^a \leq \frac{r^2}{n^{\lambda}}$$

tends to zero as $n \to \infty$. Hence the ratio which appears in (33) is less than 1 for $n$ large enough.
If \( a \in L_\lambda \) is large then \( q^a \geq n^{1+\lambda} \), \( A \geq \frac{1}{2 n \log_q n} \) and hence
\[
A q^a \geq \frac{n^\lambda}{2 \log_q n}
\]
tends to infinity for \( n \to \infty \). This gives the following statement:

**Lemma 8.** There exists a constant \( N > 0 \) such that for all \( n \geq N \) the following is true: (1) If \( \alpha' \in D \) is obtained from \( \alpha \in D \) by adding 1 to one of its coordinates which is small (see above) then
\[
(35) \quad T_\alpha > T_{\alpha'}.
\]
(2) If \( \alpha' \in D \) is obtained from \( \alpha \in D \) by adding 1 to one of its coordinates which is large then
\[
(36) \quad T_\alpha < T_{\alpha'}.
\]

![Figure 2](image)

**Figure 2.** Schematic representation of behavior of \( T_\alpha \) with respect to small \( a \in S_\lambda \) and large \( a \in L_\lambda \) coordinates.

Figure 2 illustrates Lemma 8. Next we analyze the case when one increases an intermediate index.

**Lemma 9.** There exists a constant \( N > 0 \) such that for all \( n \geq N \) the following is true: Suppose that \( \alpha' = (a+1, b, c, d) \in D \) is obtained from \( \alpha = (a, b, c, d) \in D \) by adding 1 to one of its coordinates. If \( a \leq r/2 \) and \( m(b, c) \notin S_\lambda \), then
\[
(37) \quad T_\alpha > T_{\alpha'}.
\]

**Proof.** Without loss of generality we may assume that \( a \in I_\lambda \) since the case \( a \in S_\lambda \) is covered by Lemma 8. Then our assumptions imply that \( m(b, c) \in I_\lambda \), and therefore by symmetry we may assume that \( b \in I_\lambda \). Our goal is to estimate the value of \( A \) given by (34). We have
\[
a + b > 2(1 - \lambda) \log_q n
\]
and since $r < 2 \log_q n$ we obtain
\begin{equation}
(38) \quad r - a - b < 2 \lambda \log_q n
\end{equation}
and thus the numerator in (34) satisfies
\[
(r - a - b)(r - a - c) < 4 \lambda \log^2_q n.
\]
To estimate the denominator we observe that $a < r/2$ implies
\[
q^a \leq eq \cdot n \log q n.
\]
Since $a + 1 \geq (1 - \lambda) \log_q n$ we obtain
\begin{equation}
(39) \quad 2Aq^a \leq \frac{4 \lambda \log^2_q n}{(1 - \lambda) \log_q n} \cdot \frac{eq}{2} \cdot \frac{n}{\log q n} \cdot \frac{1}{n} = \frac{4 \lambda}{1 - \lambda} \cdot \frac{eq}{2} < 1;
\end{equation}
the last inequality uses our assumption (32). This completes the proof of statement (1). \quad \Box

Lemma 10. For $n$ sufficiently large and $\alpha = (a, 0, 0, d) \in D$ with $1 \leq a \leq r - 1$, one has
\begin{equation}
(40) \quad T_\alpha \leq \max\{T_{(1, 0, 0, d)}, T_{(r - 1, 0, 0, d)}\}.
\end{equation}
Proof. The assertion of the Lemma follows from Lemma 8 in the case when either $a \in S_\lambda$ or $a \in L_\lambda$. Hence we may assume below that $\alpha = (a, 0, 0, d)$ where $a \in I_\lambda$.

Denote $\alpha' = (a + 1, 0, 0, d)$ and $\alpha'' = (a + 2, 0, 0, d)$. Then
\[
\frac{T_{\alpha''}}{T_{\alpha'}} = \left(\frac{r - a - 1}{r - a}\right)^2 \cdot \frac{(a + 1)}{(a + 2)} \cdot \frac{n - 4r + a + d + 1}{n - 4r + a + d + 2} \cdot q.
\]
In the RHS of this formula two bracketed factors tend to 1 as $n \to \infty$; besides $q > 1$. Hence for $n > N$ large enough one has
\begin{equation}
(41) \quad \frac{T_\alpha T_{\alpha''}}{T_{\alpha'}^2} > 1.
\end{equation}
This proves that $\log_q(T_\alpha)$ is convex as function of $a \in I_\lambda$. By Lemma 8, this function increases for $a \in S_\lambda$ and decreases for $a \in L_\lambda$. This implies (40).
\quad \Box

Now we are able to complete the proof of Theorem 6. Recall that we have to show that the sum $\sum_{\alpha \in D'} T_\alpha$ tends to 0 as $n \to \infty$ where $D' = D - \{(0, 0, 0, 0)\}$. Consider the subset $\tilde{D} \subset D$ consisting of vectors with at least one coordinate equal $r$. Each $\alpha \in \tilde{D}$ has the form $\alpha = (r, 0, 0, d)$ (up to symmetry) where $d = 0, \ldots, r$. Applying Lemma 10 we obtain that
\[
T_\alpha \leq \max\{T_{(r, 0, 0, 0)}, T_{(r, 0, 0, r)}\}.
\]
Since the cardinality of $\tilde{D}$ does not exceed $5r$, we obtain, using (26) and (27), that
\begin{equation}
(42) \quad \sum_{\alpha \in \tilde{D}} T_\alpha = o(1).
\end{equation}

Each vector $\alpha \in D'$ may have at most two large coordinates. Decompose
\[ D' - \tilde{D} = D'_0 \cup D'_1 \cup D'_2, \]
where $D'_i$ denotes the set all vectors in $\tilde{D}$ having exactly $i$ large coordinates, $i = 0, 1, 2$.

Suppose that $\alpha \in D'_2$. Without loss of generality we may assume that $a$ and $d$ are large and $b$ and $c$ are small, i.e. $a, d \in L_\lambda$, $b, c \in S_\lambda$. Applying Lemma 8 we obtain $T_\alpha \leq T(a,0,0,d)$. Since $a \neq r \neq d$ we may engage Lemma 10 to obtain
\begin{equation}
(43) \quad T_\alpha \leq \max\{T(1,0,0,r-1), T(r-1,0,0,r-1)\}.
\end{equation}

Now, taking into account (26), (27) and (31), we obtain
\begin{equation}
(44) \quad \sum_{\alpha \in D'_2} T_\alpha = o(1).
\end{equation}

Consider now the sum $\sum_{\alpha \in D'_1} T_\alpha$. In this case the vector $\alpha = (a, b, c, d)$ contains one large index. Assume that $a$ is large. Then $b, c$ must be small and applying Lemma 8 and Lemma 10 we obtain
\[ T_\alpha \leq T(a,0,0,d) \leq T(r-1,0,0,d) \leq \max\{T(r-1,0,0,0), T(r-1,0,0,r-1)\}. \]

Now (31) implies that
\begin{equation}
(45) \quad \sum_{\alpha \in D'_1} T_\alpha = o(1).
\end{equation}

Next we show that for any $\alpha \in D'_0$ one has
\begin{equation}
(46) \quad T_\alpha \leq \max\{T(1,0,0,0), T(r-1,0,0,0), T(r-1,0,0,r-1)\}
\end{equation}
which in view of (31) would imply that
\begin{equation}
(47) \quad \sum_{\alpha \in D'_0} T_\alpha = o(1).
\end{equation}

The combination of (42), (44), (45) and (47) give Theorem 6.

To prove (46) consider $\alpha = (a, b, c, d) \in D'_0$. Note that coordinates $a, b, c, d$ can be either small or intermediate. Assume first that all coordinates $a, b, c, d$ are small. Then $T_\alpha \leq T(1,0,0,0)$ (by Lemma 8) implying (46).
Suppose now that exactly one of the coordinates of \( \alpha \) is intermediate. If \( a \) is intermediate and \( b, c, d \) are small then

\[
T_\alpha \leq T_{(a,0,0,0)} \leq \max\{T_{(1,0,0,0)}, T_{(r-1,0,0,0)}\}
\]

(by Lemma 8 and Lemma 10) proving (46).

Suppose now that two coordinates of \( \alpha \) are intermediate. Taking into account symmetry (the action of \( G \) on \( D \), see above), this case can be subdivided into two subcases: (i) \( a \) and \( b \) are intermediate and (ii) \( a \) and \( d \) are intermediate. In the subcase (i), since \( a + b \leq r \), either \( a \leq r/2 \), or \( b \leq r/2 \) and we may apply Lemma 9. Assuming that \( a \leq r/2 \) we obtain

\[
T_\alpha \leq T_{(0,b,0,0)} \leq \max\{T_{(1,0,0,0)}, T_{(r-1,0,0,0)}\},
\]

implying (46). In the subcase (ii), we know that \( b, c \) are small hence \( T_\alpha \leq T_{(a,0,0,d)} \) and application of Lemma 10 gives (46).

In the remaining case when \( \alpha \in D'_0 \) has three or four intermediate indices we know that at least two of these indices are \( \leq r/2 \) and by Lemma 9 one has

\[
T_\alpha \leq T_{\alpha'}
\]

where \( \alpha' \) is obtained from \( \alpha \) by replacing by zeros two coordinates which were \( \leq r/2 \). To estimate \( T_{\alpha'} \) one applies Lemma 10 leading again to (46). This completes the proof of Theorem 6.

\[\square\]

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Department of Mathematical Sciences, Durham University, South Road, Durham, DH1 3LE, UK
E-mail address: a.e.costa@durham.ac.uk
E-mail address: michael.farber@durham.ac.uk