Invariance properties of random curves: an approach based on integral geometry

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Traveled lengths statistic is a key quantity for characterizing stochastic processes in bounded domains. For straight lines and diffusive random walks, the average length of the trajectories through the domain is independent of the random walk characteristics and depends only on the ratio of the volume domain over its surface, a behavior that has been recently observed experimentally for exponential jump processes. In this article, relying solely on geometrical considerations, we extend this remarkable property to all d-dimensional random curves of arbitrary lengths (finite or infinite), thus including all kind of random walks as well as fibers processes. Integral geometry will be central to establishing this universal property of random trajectories in bounded domains.

Keywords: random curves, Cauchy formula, integral geometry, kinematic formula

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I. INTRODUCTION

Numerous biological and physical phenomena perform a random walk process: bacteria, insects or particles follow a straight path for a certain distance at constant speed and then suddenly change direction before repeating the process\(^1\). Depending on the environment, the rapid change of direction may be isotropic or anisotropic (with an average-turning angle in this case\(^2\)) and stochastic\(^3\). Moreover, many other physical processes such as fibers processes (textile, carbon)\(^4\) or the behavior of a human moving in an area where WiFi is available\(^5\) are described by finite random curves. Observations of these processes, whether made under the microscope or in the natural environment, are realized in a finite portion of space which means that these observations are made in bounded domains. A central quantity to characterize the process is the average length of the traveled distance (ideally the whole distribution, but this quantity is most often out of reach\(^6\)). Under certain assumptions specified below, for purely diffusing system, it has been established that the average length of the trajectories through the bounded domain is independent of the random walk characteristics. This quantity depends only on the ratio of the volume domain over its surface and is given by\(^7,8\) (in this article we use the notation \(\langle X \rangle\) for the mean of \(X\))

\[
\langle L_{\text{diff}} \rangle = \frac{\eta_n V}{S} \quad \text{with} \quad \eta_n = \sqrt{n} \frac{\Gamma\left((n - 1)/2\right)}{\Gamma\left(n/2\right)} ,
\]

a constant depending only on the dimension \(n\), \(\eta_2 = \pi\) and \(\eta_3 = 4\). Equation (1) is a generalization to diffusive random walks of the celebrated Cauchy formula originally established for straight paths\(^9,10\). This remarkable relationship has many applications ranging from the description of biological species (insects motion\(^11,12\)) to optics, where experimental evidence of the mean path length invariance in scattering media has recently been reported\(^13,14\).

Let us be more specific about Cauchy’s formula and its generalizations. First established by Cauchy in two dimensions for straight lines intercepting a convex object, i.e. a chord, the relationship was rediscovered by Dirac during world war II\(^15\) and popularized under the name of ”chord method”, mainly among transport physicists\(^16,17\). Around the same time, mathematicians formalized its derivation and it is now a standard result of integral geometry\(^9,10\). This result is also well known in stereology\(^18\) and image analysis\(^19\) since probing structures with random lines allows characterizing a given material. This result concerns straight lines only. However, in 1981, based on Boltzmann’s linear equation and the detailed balance principle, Bardsley and Dubi showed that Cauchy’s formula applies more generally to stochastic paths of Pearson random walks with flight lengths exponentially distributed\(^7\) (see Fig. 1). For such Markovian processes, when the walkers started uniformly and isotropically on the surface of

![FIG. 1. Historically, Cauchy’s formula was first derived for chords [AB] and then extended to random walks [CD]. In this article, we will establish this formula for random curves (like the left path).](image-url)
generalized by resorting to the Feynman-Kac path-integral approach which allows explicit formulas to be derived in the presence of both absorption and branching. All these generalizations of Cauchy’s formula concern the memory-less exponential random walk, but many real-world stochastic transport phenomena, such as active particles in complex environments, are not governed by exponential processes. However, as has been proven recently, for isotropic random walks when the walker enters the domain with a length distribution compatible with equilibrium, Cauchy’s formula still holds. This result suggests that Cauchy’s formula should be valid for more general stochastic paths and the aim of this article is to establish the validity of this remarkable formula to all infinite random curves and also to extend the relationship to finite random curves.

Thus far, all generalizations of Cauchy’s formula are based either on the linear Boltzmann equation or probabilistic techniques such as backward equations or Feynman-Kac techniques. However, Cauchy’s formula is essentially geometric, and as such, any generalization should be able to be established only on geometric grounds. In this article, we will follow a purely geometric approach by resorting to integral geometry, a discipline that relates, among other things, random extended objects (lines, planes, geodesics, etc.) and their intersection with bounded domains.

The paper is organized as follows: after describing the general framework of our studies in section II, we derive Cauchy’s formula for random curves in the plane in section III. We distinguish two cases: the one where the curve is simple (without loops) and the one where it has loops. Then, in section IV relying on integral geometry in n-dimensional Euclidean space, we extend the generalization of Cauchy’s formula to all dimensions. Section V deals with the case of infinite curves (without or with loops). Finally, the section VI presents a discussion and some perspectives of this work. Along the path, we will also obtain probabilistic results such as the conditions for a convex domain to contain another one by introducing the notion of inclusion probability.

II. GENERAL FRAME

We consider a collection of independent random particles evolving freely in space. We further assume that the system is at equilibrium so that there is no privileged direction (the system is invariant under translations and rotations), and we wish to describe the behavior of these particles in a bounded portion of space. In practice, this corresponds to observing the particles in a small region of space compared to the extent of their trajectories, for example, bacteria under a microscope. Figure 2 shows an example of a planar fiber process and its observation window. The assumption that the random process is invariant under translations and rotations (which formally corresponds to the group of motions in n-dimensional Euclidean space) is the key hypothesis to obtain our findings. Indeed, this assumption allows us to use several advanced concepts of integral geometry, the mathematical field that accurately describe the measure of random geometric elements. For this purpose, we consider one copy of the random curve and the observation zone as two geometrical objects. Then, evaluating the average path length of the random curves in the observation area amounts to finding all the rigid transformations that bring the random curve into a hitting position with the observation zone. To be more precise, we consider two domains, a fixed one \(K_0\) (the observation window), a moving one \(K_1\), and the measure \(g\) of the set of motions such that \(gK_1 \cap K_0 \neq \emptyset\) (i.e. the set of motions that carries \(K_1\) to a position where it intersects \(K_0\)). As mentioned in the Introduction, we
start by deriving the generalization of Cauchy’s formula in the plane and then we will establish it in the general case, i.e. in the n-dimensional Euclidean space.

III. GENERALIZATION OF CAUCHY’S FORMULA IN THE PLANE

We begin by considering two bounded domains $K_0$ and $K_1$ in the plane. Each domain $K_i (i = 0, 1)$ is characterized by its area $F_i$, its perimeter $L_i$ and its Euler characteristic $\chi_i$. We assume that the observation window $K_0$ is fixed and that $K_1$ is moving. Besides, we denote by $dK_1$ the kinematic density of $K_1$. Then, the so-called fundamental kinematic formula states that

$$\int_{K_1 \cap K_0 \neq \emptyset} \chi(K_1 \cap K_0) dK_1 = 2\pi(F_0\chi_1 + F_1\chi_0) + L_0L_1,$$

where $\chi(K_1 \cap K_0)$ is the Euler characteristic of the intersection of $K_0$ and $K_1$. It is remarkable that the integral $\int_{K_1 \cap K_0 \neq \emptyset} \chi(K_1 \cap K_0) dK_1$ over all positions of $K_1$ can be expressed only in terms of $F_i$, $L_i$ and $\chi_i (i = 0, 1)$. The fundamental kinetic formula is due to Blaschke, see also Santaló, Chern or the recent book of Chirikjian for a somewhat less technical approach. Besides the fundamental kinetic formula, Santaló established that

$$\int_{K_1 \cap K_0 \neq \emptyset} L dK_1 = 2\pi F_0 L_1,$$

where $L$ denotes the length of the arc $\partial K_1$ that is interior to $K_0$. For the problem we are studying, $K_1$ is the geometric representation of a trajectory (which, let us recall, can be a segment, a fiber or a random walk, etc...) and as such, is a one-dimensional object, i.e. a curve. However, the previous relation concerns two-dimensional quantities. One way to transform the 2-dimensional bounded domain $K_1$ into a one-dimensional object is to consider $K_1$ as a two-dimensional object with a small thickness $\Delta$ and shrink $\Delta$ to 0. The procedure, named $\Delta$-thickening by Cowan, is shown in Fig. 3. An immediate consequence concerns the perimeter $L_1$ of a thickened curve of length $l$, which is $2l$ (up to a negligible error of order $\Delta$). The area is $F_1 = l\Delta$ and therefore tends to 0, as expected. For such a curve, the two previous equations become

$$\int_{K_1 \cap K_0 \neq \emptyset} \chi(K_1 \cap K_0) dK_1 = 2\pi F_0\chi_1 + 2lL_0,$$

and

$$\int_{K_1 \cap K_0 \neq \emptyset} L dK_1 = 4\pi F_0 l.$$

In Eq.(4), the Euler characteristic $\chi_1$ has a fixed value. However, the Euler characteristic in the left-hand side of Eq.(4) depends on the intersection of $K_0$ and $K_1$ and therefore requires some attention. First, we consider that $K_1$
is a fiber, i.e. a curve without loops. With this hypothesis, \( \chi_1 = 1 \) and the Euler characteristic of the intersection \( \chi(K_1 \cap K_0) \) is the number of pieces of \( K_1 \) inside \( K_0 \). Therefore, \( \int_{K_1 \cap K_0 \neq \emptyset} \chi(K_1 \cap K_0) \, dK_1 \) is the measure of the number of pieces of \( K_1 \) inside \( K_0 \). This counting procedure is coherent with the definition given for chords passing through non-convex bodies. Indeed, for straight lines, the multiple chord distribution function \( 34 \) which leads to the Cauchy formula for non-convex bodies \( 35 \) consists in considering each chord length segment inside the body for itself \( 36 \). By extending this definition to curve, we get that twice the expected value \( \langle L \rangle \) of \( K_1 \) inside \( K_0 \) is given the ratio of Eq.(5) over Eq.(4) (the factor 2 comes from the fact that the curve perimeter is twice its length),

\[
2\langle L \rangle = \frac{\int_{K_1 \cap K_0 \neq \emptyset} L \, dK_1}{\int_{K_1 \cap K_0 \neq \emptyset} \chi(K_1 \cap K_0) \, dK_1} = \frac{4\pi F_0 l}{2\pi F_0 + 2lL_0}
\]

that is

\[
\frac{1}{\langle L \rangle} = \frac{1}{l} + \frac{1}{\langle \sigma \rangle},
\]

where \( \langle \sigma \rangle \), the mean chord length through \( K_0 \), comes from the two-dimensional Cauchy formula \( \langle \sigma \rangle = \pi F_0 / L_0 \).

Note that no assumption is required on the observation window \( K_0 \) which can be convex, non-convex, with holes, or even made of several components. Equation \( 7 \) can be found in \( 37 \) where the relationship is established for convex bodies and line segments of fixed length. When the length of the random curve tends to infinity, then

\[
\langle L \rangle = \langle \sigma \rangle = \frac{\pi F_0}{L_0},
\]

a result that generalizes for infinite random curves the well-known Cauchy formula.

We can push our reasoning a step further by observing that the sole hypothesis leading to the above results is \( \chi_1 = 1 \). Now, a classical result of algebraic topology states that all contractile spaces have an Euler characteristic equal to 1, which, in addition to convex domains, also includes curves with ramifications in one dimension. Consequently, Eq.(7) remains valid for curves with ramifications (without loops) whose total length is equal to \( l \). Furthermore, it is worth noticing that we can draw a parallel between our approach and graph theory. Indeed, if we identify the trajectory of a ramified curve (still without loops) with a graph, then this graph is a tree (i.e. a connected graph containing no cycles) as shown in Fig. 4. The fact that the Euler characteristic of a tree (the number of vertices minus the number of edges) is equal to 1 shows that our approach is consistent with graph theory.

We now consider the case of trajectories with loops, starting with a simple loop. Such an object has an Euler characteristic \( \chi_1 = 0 \). On the one hand, having \( \chi_1 = 0 \) reduces the right-hand side of Eq.(4) to \( 2lL_0 \), but on the other hand, the interpretation of the left-hand side of Eq.(4) (the random integral) in terms of pieces of trajectory is no longer possible. Indeed, if a loop is entirely contained in \( K_0 \), then \( \chi(K_1 \cap K_0) = 0 \), and such a loop is not counted. We postpone the study of this somewhat puzzling case to the next paragraph. First, to avoid the difficulty of having loops entirely inside \( K_0 \), we assume that the minimal radius of curvature of \( K_1 \) is greater than

![Fig. 4. Schematic equivalence between a ramified random walk and a connected graph (8 vertices and 7 edges, so \( \chi = 1 \).](image)
the maximal radius of curvature of $K_0$. Under this hypothesis, the loop always intercepts $K_0$ once and we have $\chi(K_1 \cap K_0) = 1$, therefore, Eq. (10) reduces to

$$\int_{K_1 \cap K_0 \neq \varnothing} \chi(K_1 \cap K_0) d\beta = \int_{K_1 \cap K_0 \neq \varnothing} d\beta = 2L_0 ,$$

and since Eq. (9) is left unchanged, from Eq. (6) we can secure the following result: for loops whose minimal radius of curvature is greater than the maximal radius of curvature of the observation window then

$$\langle L \rangle = \frac{\pi F_0}{L_0} .$$

Thus, for this kind of trajectory, Cauchy’s formula is valid under weaker assumptions than in the case of curves where the assumption of infinite length is mandatory, as we have established earlier.

We now consider the case where the maximal radius of curvature of $K_1$ is smaller than the minimal radius of curvature of $K_0$. Such a loop can be entirely contained inside $K_0$ but cannot cross the border $\partial K_0$ more than once. To obtain the kinematic measure of such loops, we consider them as two-dimensional filled objects (of area $F_1$ and perimeter $L_1$) as shown in Fig. 5. We denote by $K_1$ the filled loop and by $\chi(K_1)$ its Euler characteristic.

By construction, the filled loop is a contractible object and $\chi(K_1) = 1$ and thanks to the hypothesis regarding the radii of curvature, we also have $\chi(K_1 \cap K_0) = 1$. Moreover, whether the loop is filled or not, its kinematic measure is the same. The same goes for the events $\{K_1 \cap K_0 \neq \varnothing\}$ and $\{K_1 \cap K_0 \neq \varnothing\}$, so

$$2\pi(F_0 + F_1) + L_0L_1 = \int_{K_1 \cap K_0 \neq \varnothing} \chi(K_1 \cap K_0) d\beta = \int_{K_1 \cap K_0 \neq \varnothing} 1 d\beta .$$

Since Eq. (9) is left unchanged, we have,

$$\langle L \rangle = \frac{\int_{K_1 \cap K_0 \neq \varnothing} L d\beta}{\int_{K_1 \cap K_0 \neq \varnothing} d\beta} = \frac{2\pi F_0 L_1}{2\pi(F_0 + F_1) + L_0L_1} .$$

Thus, for such loops, the average length of the trajectories inside the observation window is no longer given by the elegant Cauchy formula. Remark that for tiny loops, when $L_1 \rightarrow 0$ (and $F_1 \rightarrow 0$), Eq. (12) reduces to $\langle L \rangle \rightarrow L_1$ as expected. Indeed, for small loops (compared to the size of the observation window) most trajectories fall entirely inside the domain, and the contribution of the average length of the trajectories inside the observation window comes mainly from trajectories of length $L_1$. We can be more precise by calculating the conditional probability $p = P(K_0 \subseteq K_1 | K_1 \cap K_0 \neq \varnothing)$ that a trajectory falls entirely in the domain knowing that it has touched it. By analogy with the inclusion measure introduced in Ref. 38, we call this probability, the inclusion probability. To determine this quantity, we consider the two objects $K_1$ and $K_1$ and use the kinematic formula for both objects. For the filled loop $K_1$, the Eq. (11) reads

$$\int_{K_1 \cap K_0 \neq \varnothing} d\beta = 2\pi(F_0 + F_1) + L_0L_1 .$$

FIG. 5. From left to right: illustrations of a loop, a filled loop, an open loop and their intersections with $K_0$. The numbers indicate the Euler characteristic associated to each configuration: $\chi(K_1 \cap K_0)$ on the left, $\chi(K_1 \cap K_0)$ in the center, and $\chi(K_1 \cap K_0)$ on the right.
For the loop $K_1$, the kinematic formula Eq. (2) gives (recall that for a loop $F_1 = 0$, $\chi(K_1) = 0$ and the perimeter is twice the length of the loop)

$$\int_{K_1 \cap K_0 \neq \emptyset} \chi(K_1 \cap K_0) dK_1 = 2L_0L_1. \quad (14)$$

From the two previous equations, we obtain the mean value of the Euler characteristic of the intersection of $K_1$ and $K_0$

$$\langle \chi(K_1 \cap K_0) \rangle = \frac{\int_{K_1 \cap K_0 \neq \emptyset} \chi(K_1 \cap K_0) dK_1}{\int_{K_1 \cap K_0 \neq \emptyset} dK_1} = \frac{2L_0L_1}{2\pi(F_0 + F_1) + L_0L_1}. \quad (15)$$

This quantity can also be obtained by resorting to the probability $p$. Either the loop falls entirely inside $K_0$ with probability $p$ and $\chi(K_1 \cap K_0) = 0$, or it touches the border $\partial K_0$ with probability $1 - p$ and $\chi(K_1 \cap K_0) = 1$ since in this case due to the assumptions on the radii of curvature, the intersection of $K_1$ and $K_0$ is a one-piece curve. Therefore,

$$\langle \chi(K_1 \cap K_0) \rangle = p \times 0 + (1 - p) \times 1 = 1 - p. \quad (16)$$

By matching the two equations Eq. (15) and Eq. (16), we get

$$p = \frac{2\pi(F_0 + F_1) - L_0L_1}{2\pi(F_0 + F_1) + L_0L_1}. \quad (17)$$

The preceding relationship leads to the following interesting finding: The possibility of a transposition of $K_1$ in the interior of $K_0$ is given by $p > 0$, i.e. $2\pi(F_0 + F_1) - L_0L_1 > 0$, recovering Hadwiger celebrated result concerning the necessary conditions for a plane domain to be able to contain another.

For completeness, we can also obtain this result by considering the loop where we have deleted a point. We denote by $\tilde{K}_1$ this "open" loop. Such an object has zero area ($F_1 = 0$), a perimeter equal to $2L_1$, and since it is a contractible object, has an Euler characteristic $\chi(\tilde{K}_1)$ equal to 1. As before, the mean Euler characteristic of the intersection of $\tilde{K}_1$ and $K_0$ can be evaluated in two ways. The first method involves the kinematic formula, and the second the probability $p$ we wish to calculate. On the one hand, since a point has zero measure, the kinematic measures $d\tilde{K}_1$ and $dK_1$ are equal as well as the events $\{\tilde{K}_1 \cap K_0 \neq \emptyset\}$ and $\{K_1 \cap K_0 \neq \emptyset\}$. From the kinematic formula and Eq. (11) we have

$$\langle \chi(\tilde{K}_1 \cap K_0) \rangle = \frac{\int_{\tilde{K}_1 \cap K_0 \neq \emptyset} \chi(\tilde{K}_1 \cap K_0) d\tilde{K}_1}{\int_{\tilde{K}_1 \cap K_0 \neq \emptyset} d\tilde{K}_1} = \frac{2\pi F_0 + 2L_0L_1}{2\pi(F_0 + F_1) + L_0L_1}. \quad (18)$$

On the other hand, either the open loop falls entirely inside $K_0$ and in this case the contribution to $\langle \chi(\tilde{K}_1 \cap K_0) \rangle$ is 1, or it touches the boundary of $K_0$ with the probability $1 - p$. In the latter case, if the removed point is outside $K_0$ then the contribution to $\langle \chi(\tilde{K}_1 \cap K_0) \rangle$ is 1. Otherwise, because the removed point separates the curve inside $K_0$ into two distinct pieces (See Fig. 5), the contribution to $\langle \chi(\tilde{K}_1 \cap K_0) \rangle$ is 2. Moreover, since this point is uniformly chosen on the loop, the probability that it is inside $K_0$ is $\langle s \rangle/L_1$ where $\langle s \rangle$, the average arc, is the average length of $L$ when the open loop crosses $\partial K_0$. So, we have

$$\langle \chi(\tilde{K}_1 \cap K_0) \rangle = p \times 1 + (1 - p) \left[ \frac{\langle s \rangle}{L_1} \times 2 + \left( 1 - \frac{\langle s \rangle}{L_1} \right) \times 1 \right] = 1 + (1 - p) \frac{\langle s \rangle}{L_1}. \quad (19)$$

In addition, two mutually exclusive events contribute to the average length $\langle L \rangle$ of a loop: Either the loop is entirely contained in $K_0$ with probability $p$, or it intersects the boundary of $K_0$ with probability $1 - p$. In the first case the length is $L_1$ (the perimeter of the loop) and in the second it is on average $\langle s \rangle$, therefore

$$\langle L \rangle = p \times L_1 + (1 - p) \times \langle s \rangle. \quad (20)$$

We deduce that

$$p = 1 + \frac{\langle L \rangle}{L_1} - \langle \chi(K_0 \cap \tilde{K}_1) \rangle, \quad (21)$$

and using Eqs. (12) and (18) we recover Eq. (17).
Note that in the case of two circles of radii $R_0$ and $R_1$ with $R_0 > R_1$, the probability $p$ that the smaller circle lies entirely inside the larger one becomes

$$p = \left( \frac{R_0 - R_1}{R_0 + R_1} \right)^2. \quad (22)$$

Due to rotation invariance, as expected, $p$ is the ratio of the area of the circle radius $R_0 - R_1$ over the area of the circle of radius $R_0 + R_1$ as shown in Fig. 6. In appendix A, similar results are derived for 3-dimensional objects.

![Fig. 6](image)

FIG. 6. The inclusion probability $p = P[K_0 \subseteq K_1 | K_1 \cap K_0 \neq \emptyset]$ for two circles of radii $R_0$ and $R_1$ is the ratio of the shaded area of the circle of radius $R_0 - R_1$ (corresponding to the set of events $\{K_0 \subseteq K_1\}$) over the hatched area of the large circle of radius $R_0 + R_1$ (corresponding to the set of events $\{K_1 \cap K_0 \neq \emptyset\}$), i.e. $p = (R_0 - R_1)^2 / (R_0 + R_1)^2 \quad (\text{Eq.}(22))$.

So far, we have assumed that the length $l$ of the trajectories or of the loops is fixed. We now relax this hypothesis by considering that the length of the trajectories is distributed according to a given probability law $f(l)$ and we denote by $\langle l \rangle$ its expectation if it exists. Compared to the case where the lengths were fixed, we must now average over all possible realizations of the lengths according to $f(l)$. This means that a double average must be performed on both the kinematic density (as in the case of fixed lengths) and the process realizations.

We start by treating the case of fibers where $\chi_1 = 1$. Equation (4) which gives the measure of the number of pieces of $K_1$ inside $K_0$ becomes

$$\int_0^\infty dl f(l) \int_{K_1 \cap K_0 \neq \emptyset} \chi(K_1 \cap K_0) dK_1 = \int_0^\infty dl f(l) \left( 2\pi F_0 \chi_1 + 2lL_0 \right) = 2\pi F_0 + 2\langle l \rangle L_0. \quad (23)$$

Similarly, from Eq. (5) the average length $\langle L \rangle$ becomes

$$\int_0^\infty dl f(l) \int_{K_1 \cap K_0 \neq \emptyset} L dK_1 = \int_0^\infty dl f(l) 4\pi F_0 l = 4\pi F_0 \langle l \rangle. \quad (24)$$

As in the case of fixed length, the ratio of the two previous quantities gives twice the average length $\langle L \rangle$ of $K_1$ inside $K_0$, that is

$$\frac{1}{\langle L \rangle} = \frac{1}{\langle l \rangle} + \frac{1}{\langle \sigma \rangle}, \quad (25)$$

a result that generalizes Cauchy’s formula for trajectories distributed according to any probability law. An immediate consequence is that when the mean length of the trajectories diverges, Cauchy’s formula holds.

$$\langle L \rangle = \langle \sigma \rangle = \frac{\pi F_0}{L_0} \quad \text{when} \quad \langle l \rangle > \infty. \quad (26)$$

The case where trajectories are loops is treated the same way, taking care that the random distribution of loop lengths does not mix up large and small loops because in this case the generality of the results would be lost. We
first consider the case of large loops, assuming that the minimal radius of curvature of all \( K_1 \) is greater than the maximal radius of curvature of \( K_0 \). Under this assumption Eq. (9) becomes

\[
\int_0^\infty df(l) \int_{K_1 \cap K_0 \neq \emptyset} \chi(K_1 \cap K_0) dK_1 = \int_0^\infty df(l) 2dL_0 = 2\langle l \rangle L_0,
\]

and since twice the average length \( \langle L \rangle \) of \( K_1 \) inside \( K_0 \) is given by the ratio of Eq. (24) over Eq. (27), we obtain

\[
\langle L \rangle = \pi \frac{F_0}{L_0}.
\]

In the same vein, for small loops, assuming that the maximum radius of curvature of all the realizations of \( K_1 \) is smaller than the minimum radius of curvature of \( K_0 \), we secure the result for the inclusion probability, namely

\[
p = \frac{2\pi(F_0 + \langle F_1 \rangle)}{2\pi(F_0 + \langle F_1 \rangle) + L_0\langle L_1 \rangle}.
\]

where \( \langle L_1 \rangle \) and \( \langle F_1 \rangle \) denote respectively the mean perimeter and the mean area of the loop distribution.

For some applications in biology, physics or robotics, one may also need the average arc, i.e. the average length of the trajectories touching the boundary of the domain \( K_0 \) (which excludes the loops totally contained in \( K_0 \)) see\(^{39}\) and references therein. This quantity \( \langle s \rangle \) is easily obtained from the inclusion probability. Indeed, recall that

\[
\langle L \rangle = p \times L_1 + (1 - p) \times \langle s \rangle.
\]

Then, using Eqs. (12) and (17), this equation yields

\[
\langle s \rangle = \frac{L_1}{2} - \frac{\pi F_1}{L_0},
\]

in agreement with\(^{39}\) (Eq. 5). A careful reader might note that the assumptions for obtaining Eq. (30) are significantly different in the two papers. In\(^{39}\) the authors assume that the loop becomes small with respect to \( K_0 \). In the present work we do not assume that the loop becomes small with respect to \( K_0 \) but make the assumption that the maximum radius of curvature of the loop is smaller than the minimum radius of curvature of \( K_0 \). In both cases, the key ingredient is that the arc has only one component and thus two intersecting points with \( K_0 \). When this hypothesis is satisfied then Eq. (30) is an exact result.

Finally we can randomize the loops. Observing that the previous equations are all linear, we immediately deduce

\[
\langle s \rangle = \frac{\langle L_1 \rangle}{2} - \frac{\pi \langle F_1 \rangle}{L_0}.
\]

IV. GENERALIZATION OF CAUCHY’S FORMULA FOR RANDOM CURVES IN THE N-DIMENSIONAL EUCLIDEAN SPACE

We now focus on the n-dimensional Euclidean space, where the strategy for generalizing the Cauchy formula for curves will be identical to that in two dimensions. First, we establish the results for curves of fixed length by distinguishing the case without and with a loop, then we randomize the length. Thanks to the linearity of the equations, this second step can be carried out in a simple way. However, for this purpose, we need some important results of integral geometry. We recall them for the general case of two bounded domains then we apply these results to the particular case of a (possibly infinite) curve (i.e. a one-dimensional object living in \( \mathcal{R}^n \)) and a bounded domain (the observation window).

From Santaló\(^{39}\) let \( K_0 \) and \( K_1 \) be two domains in \( \mathcal{R}^n \) of volumes \( V_0 \) and \( V_1 \) bounded by the hypersurfaces \( \partial K_0 \) and \( \partial K_1 \), respectively. Assume that \( K_0 \) is fixed and that \( K_1 \) is moving with the kinematic density \( dK_1 \). \( M_0^n \) and \( M_1^n \) denote the \( i \)th integrals of the mean curvature of \( \partial K_0 \) and \( \partial K_1 \), respectively\(^{39}\). Then the kinematic formula in \( \mathcal{R}^n \) reads

\[
\int_{K_0 \cap K_1 \neq \emptyset} \chi(K_0 \cap K_1) dK_1 = O_1 \ldots O_{n-2} \left[ O_{n-1} \left( \chi(K_0) V_1 + \chi(K_1) V_0 \right) \right]
\]

\[
+ \frac{1}{n} \sum_{h=0}^{n-2} \left( \frac{n}{h + 1} \right) M_0^h M_1^{n-h-2}
\]

where \( \chi(K_0 \cap K_1) \) is the Euler characteristic of the intersection of \( K_0 \) and \( K_1 \). \( O_m \), the area of the m-dimensional unit sphere, is given by

\[
O_m = \frac{2\pi^{(m+1)/2}}{\Gamma((m + 1)/2)}.
\]
where $\Gamma$ denotes the gamma function. Remark that if $K_0$ is convex, $\chi(K_0) = 1$ and $\chi(K_0 \cap K_1) = 1$, in which case
\[
\int_{K_0 \cap K_1 \neq \emptyset} dK_1 = O_1 \ldots O_{n-2} \left[ O_{n-1} \left( V_1 + \chi(K_1)V_0 \right) + \frac{1}{n} \sum_{h=0}^{n-2} \left( \frac{n}{h+1} \right) M_0^h M_{n-h-2}^1 \right] (K_0 \text{ convex}).
\] (34)

We now need to specify the values of $M_i^1$ for a curve. To this end, we start by recalling some known properties for
a segment. Next, we will extend these results to curves. Let us consider a line segment of length $s$. For such an
object, the integrals of mean curvature $M_i^1$ can be found in Santaló’s book\footnote{5}
\[
\begin{aligned}
M_i &= 0 \quad (i = 1, 2, \ldots, n - 3), \\
M_{n-2} &= \frac{O_{n-2} s}{n-1}, \\
M_{n-1} &= \frac{n-1}{n}.
\end{aligned}
\] (35)

and thanks to the fact that the mean curvature integral is invariant under bending\footnote{5}, we can safely extend these
results to curves in $\mathbb{R}^n$. By plugging these values in the kinematic formula Eq.(32), we obtain
\[
\int_{K_0 \cap K_1 \neq \emptyset} \chi(K_0 \cap K_1) dK_1 = O_1 \ldots O_{n-2} \left[ O_{n-1} V_0 + \frac{1}{n} \left( \frac{n}{1} \right) M_0^0 M_{n-2}^1 \right].
\] (36)

Moreover, since the integral of the mean curvature $M_i^0$ of is $= F_0$ (the area of $\partial K_0$)\footnote{2}, we get
\[
\int_{K_0 \cap K_1 \neq \emptyset} \chi(K_0 \cap K_1) dK_1 = O_1 \ldots O_{n-2} \left[ O_{n-1} V_0 + F_0 \frac{O_{n-2}}{n-1} \right].
\] (37)

In addition to the kinematic formula, Santaló also established the expression of the integral of the intersection of
two manifolds in $\mathbb{R}^n$. More precisely, let $M^q$ be a fixed $q$-dimensional manifold and $M^r$ a moving one of
dimension $r$ having finite volume $\sigma_q(M^q)$ and $\sigma_r(M^r)$ respectively. Let $q + r \geq n$ and consider all
positions of $M^r$ such that $M^q \cap M^r \neq \emptyset$, then\footnote{2}
\[
\int_{M^q \cap M^r \neq \emptyset} \sigma_{q+r-n}(M^q \cap M^r) dK_1 = O_n \ldots O_1 O_{q+r-n} \frac{\sigma_q(M^q) \sigma_r(M^r)}{O_q O_r}.
\] (38)

For our purpose, $q = n$ (the observation window is an $n$-dimensional manifold) and $r = 1$ since the moving
manifold is a curve. With these parameters, the previous equation reduces
\[
\int_{M^n \cap M^1 \neq \emptyset} \sigma_1(M^n \cap M^1) dK_1 = O_{n-1} \ldots O_1 \sigma_n(M^n) \sigma_1(M^1).
\] (39)

The one-dimensional volume $\sigma_1$ of the intersection of $M^1 \cap M^n$ is nothing but the length of $M^1$ inside $M^n$, and
the previous relationship can be written with the more friendly notations (in the appendix B\footnote{2} we give another
derivation of this result by resorting to the integrals of mean curvatures)
\[
\int_{K_1 \cap K_0 \neq \emptyset} L dK_1 = O_{n-1} \ldots O_1 V_0 s.
\] (40)

Compared to the two-dimensional case discussed in paragraph \footnote{3} an advantage of this formal approach is to work
directly with the length of the curve and not with its perimeter. Indeed, by working directly with an object of
dimension 1, we do not need to introduce the factor 2 as in the two-dimensional case and the average length $\langle L \rangle$
of $K_1$ inside $K_0$ is directly given by the ratio of Eq.(40) over Eq.(37):
\[
\langle L \rangle = \frac{\int_{K_1 \cap K_0 \neq \emptyset} L dK_1}{\int_{K_1 \cap K_0 \neq \emptyset} \chi(K_0 \cap K_1) dK_1} = \frac{O_{n-1} \ldots O_1 V_0 s}{O_{n-2} \ldots O_1 \left[ O_{n-1} V_0 + \frac{O_{n-2} F_0}{n-1} \right]} = \frac{1}{s + \frac{O_{n-2} F_0}{O_{n-1} F_0}},
\] (41)

or
\[
\frac{1}{\langle L \rangle} = \frac{1}{s} + \frac{1}{\langle \sigma \rangle}.
\] (42)

In the previous equation we used the fact that $(n-1)O_{n-1} V_0/O_{n-2} F_0 = \eta_0 V_0/F_0$ is the mean chord length
through $K_0$ and is given by Cauchy’s formula in $\mathbb{R}^n$\footnote{2}. As in the two-dimensional case, we can relax the hypothesis regarding
the fixed length $s$ of the trajectories and make it random. Since Eq.(37) is linear in $s$, averaging over the trajectories
according to a density $f(s)$ can be done straightforwardly (the procedure is identical to that of the two-dimensional
case) and without further development, we state one of the central results of this article.
Theorem 1.
Let $K \in \mathbb{R}^n$ be a body of volume $V$, surface $S$ and mean chord length $\langle \sigma \rangle = \eta_n V / S$. If this body is exposed to a uniform isotropic field of random curves of mean length $\langle s \rangle$, then the mean length of the curves $\langle L \rangle$ in the body is given by the generalization of Cauchy’s formula:

$$\frac{1}{\langle L \rangle} = \frac{1}{\langle s \rangle} + \frac{1}{\langle \sigma \rangle}.$$  \hspace{1cm} (43)

Since no assumption is required regarding the shape of the body $K$, the theorem is valid for both convex and non-convex bodies. Besides, if the mean length of the trajectories diverges, we immediately obtain that

$$\langle L \rangle = \langle \sigma \rangle = \eta_n \frac{V}{S},$$  \hspace{1cm} (44)

so that the mean curve length in the observation zone satisfies the Cauchy formula\cite{35}.

We now turn our attention to the loop. For sake of simplicity, we only treat large loop trajectories whose minimum radius of curvature is greater than the maximum radius of curvature of $K_0$. This case is easy to handle because for a large loop, the average length of the loop in the observation zone does not depend on its length $s$, cf Eq.\cite{10}. Therefore, the randomization has no effect and we can state:

Theorem 2.
Let $K \in \mathbb{R}^n$ a body of volume $V$, surface $S$ and mean chord length $\langle \sigma \rangle = \eta_n V / S$. If this body is exposed to a uniform isotropic field of random loops whose minimum radius of curvature is greater than the maximum radius of curvature of $K$ then the average curve length $\langle L \rangle$ in the body is given by Cauchy’s formula:

$$\langle L \rangle = \eta_n \frac{V}{S}.$$  \hspace{1cm} (45)

V. GENERALIZATION OF CAUCHY’S FORMULA FOR INFINITE RANDOM TRAJECTORIES IN THE N-DIMENSIONAL EUCLIDEAN SPACE

We now address the central question concerning the average length traveled by stationary and microscopically reversible particles without interaction in a sub-domain. Particle trajectories are large compared to the size of the observation area and will be considered as infinite. Strictly speaking, by trajectory, we mean "a rectifiable curve of infinite length". Moreover, we restrict our study to particles that cannot give birth to other particles (no branching). In other words, we only consider simple curves. On the other hand, these trajectories can have loops, as it is highly probable in the two-dimensional case. Examples of such trajectories are shown in Fig.\cite{7} where some paths cross inside the observation window. To be consistent with the previous developments, we wish to count each re-entry in $K_0$ as an independent path (two paths for the red and blue curves and one path for the green curve shown in Fig.\cite{7}). However, the Euler characteristic is not the natural object for counting the number of parts when

\hspace{1cm} FIG. 7. Euler characteristic of different paths inside $K_0$: red path $\chi(\times \times) = 1$, green path $\chi(\bigcirc \bigcirc) = 0$ and blue path $\chi(\bigcirc \bigcirc \times) = -1$.
loops are involved. Indeed, inside $K_0$ the red trajectory has an "X" shape, which is a contractible object, and therefore has an Euler characteristic equal to 1. The situation is even worse for the green trajectory since a loop has an Euler characteristic equal to 0 and is not counted, not to mention the blue trajectory that has a negative Euler characteristic. To circumvent this spurious topological effect due to the representation of the trajectories as geometrical objects, we will count the number of points of intersection between the trajectories and the boundary of the sub-domain, taking advantage that for infinite trajectories the number of intersection points is twice the number of pieces of trajectory inside $K_0$. To this aim, we consider the boundary $\partial K_0$ of the observation window which is a manifold of dimension $n-1$ and then apply Eq. (38) with the parameters $q = n - 1$ and $r = 1$. With these parameters, $\sigma_0(M^1 \cap M^{n-1}) = N(K_1 \cap \partial K_0)$ denotes the number of intersection points between $K_1$ and $\partial K_0$. However, strictly speaking, Eq. (38) is valid for bounded domains. Therefore, we give a large but finite length $s$ to the trajectories (compared to the size of $K_0$) and then we will let $s$ go to infinity. Equation (38) reads

$$\int_{K_1 \cap K_0 \neq \emptyset} N(K_1 \cap \partial K_0) dK_1 = \frac{O_n \ldots O_1 O_0}{O_{n-1} O_1} F_0 s. \quad (46)$$

Since, $K_1$ is a long curve of length $s$ (that eventually goes to infinity) crossing $K_0$, $N(K_1 \cap \partial K_0)$ is twice the number of segments inside $K_0$ and the preceding equation becomes

$$\int_{K_1 \cap K_0 \neq \emptyset} 2 dK_1 = \frac{O_n \ldots O_1 O_0}{O_{n-1} O_1} F_0 s. \quad (47)$$

On the other hand, the length of the curves inside $K_0$ is still given by Eq. (40), that is

$$\int_{K_1 \cap K_0 \neq \emptyset} L dK_1 = O_{n-1} \ldots O_1 V_0 s. \quad (48)$$

From the two previous equations, we immediately get that

$$\langle L \rangle = \frac{\int_{K_1 \cap K_0 \neq \emptyset} L dK_1}{\int_{K_1 \cap K_0 \neq \emptyset} dK_1} = 2 \pi \frac{O_{n-1} V_0}{O_n} F_0 = \eta_n \frac{V_0}{F_0}. \quad (49)$$

Equation (49) is the generalization of Cauchy’s formula in $\mathbb{R}^n$ for infinite random curves, of which lines and random walks are special cases. We summarize the result of this paragraph by stating,

**Theorem 3.**

*Let $K \in \mathbb{R}^n$ a body of volume $V$, surface $S$ and mean chord length $\langle \sigma \rangle = \eta_n V/S$. If this body is traversed by a uniform isotropic field of random rectifiable trajectories of infinite length then the mean curve length $\langle L \rangle$ in the body given by Cauchy’s formula:

$$\langle L \rangle = \eta_n \frac{V}{S}. \quad (50)$$

VI. DISCUSSION AND PERSPECTIVES

In this article, based on integral geometry, we have established that the average traveled length of a collection of stationary and microscopically reversible non-interacting particles in a sub-domain is independent of the details of the trajectories and is given by the generalized Cauchy’s formula Eq. (13). This generalized Cauchy formula is therefore the signature that the process is at equilibrium and an observation of the average length traveled violating this result means that the process is not at equilibrium or that the particles interact in some way. Advanced concepts from integral geometry used in this article are essential for an in-depth understanding of the physical properties of independent particles in a sub-domain since they allow us to treat random curves, random walks, random lines or random segments in a unified way.

In addition, the integral geometric approach sheds new light on the modifications to be made to Cauchy’s formula during the passage of light between two diffusive media. Indeed, let us consider two media of index of refraction $n_1$ and $n_2$. A direct consequence of the Snell-Descartes law relates the density of rays entering medium 2 to that of medium 1 by $dM_2 = (n_1/n_2)dM_1$. This relationship shows that, depending on the ratio $n_1/n_2$, random walks entering into the second medium have a density measure more or less dense compared to that of the first medium. However, the ratio of two refractive indices is a constant and therefore does not change the probability measure. As
a result, Cauchy’s formula is modified only by the difference in light intensities between the two media. When the radiation field is statistically homogeneous and isotropic, light intensities between the first and the second media are related by \( I_2 = (n_2/n_1)^2 I_1 \). Therefore, Cauchy’s formula is modified by this factor in the second medium, leading to \( \langle L \rangle = (n_2/n_1)^2 \eta n V/S \), as recently noted in \cite{12} and confirmed in \cite{43}. We conclude by observing that if the random walks cross a succession of media with different refractive indices, having the same initial and final index of refraction, then Cauchy’s formula remains unchanged. This last remark extends to media of different refractive indexes an observation made in reference \cite{22} to sub-domains, revealing another facet of this fascinating universal formula.

**VII. ACKNOWLEDGEMENTS**

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Appendix A: Probability for a 3-dimensional convex domain to be able to contain another

In order to derive the inclusion probability, \( p = P(K_1 \subseteq K_0 | K_1 \cap K_0 \neq \emptyset) \) that a 3-dimensional convex object \( K_1 \) falls entirely in another convex domain \( K_0 \) knowing that it has touched it, we follow the same procedure as for the two-dimensional case. We apply the fundamental kinematic formula successively to the plain object \( K_1 \) (a volume), and to its border \( \partial K_1 \) (a surface). In the 3-dimensional space, the fundamental kinematic formula Eq.\((A2)\) becomes

\[
\int_{K_1 \cap K_0 \neq \emptyset} \chi(K_1 \cap K_0) dK_1 = O_1 \left[ O_2 \chi(K_0)V_1 + O_2 \chi(K_1)V_0 + \frac{1}{3} \sum_{h=0}^{1} \left( \frac{3}{h+1} M_0^h M_1^{1-h} \right) \right]
\]

(A1)

\[
= 8\pi^2 (\chi(K_0)V_1 + \chi(K_1)V_0) + 2\pi (F_0M_1 + F_1M_0)
\]

where \( V_i, F_i, \) and \( M_i \) denote the volume, the surface area and the integral of the mean curvature of the object \( K_i \). \( K_0 \) is a convex body, and \( \chi(K_0) = 1 \). First we consider \( K_1 \) also as a plain convex body: we thus have \( \chi(K_1) = 1 \) as well as \( \chi(K_1 \cap K_0) = 1 \). Therefore, Eq.\((A1)\) becomes

\[
\int_{K_1 \cap K_0 \neq \emptyset} dK_1 = 8\pi^2 (V_1 + V_0) + 2\pi (F_0M_1 + F_1M_0).
\]

(A2)

We now consider \( K_1 \) as a hollow body of thickness \( \Delta \to 0 \) (to avoid any confusion, we denote by \( K_1' \) this hollow object and by \( V_1', F_1', \) and \( M_1' \) its volume, its surface area, and the integral of its mean curvature). The kinematic densities \( dK_1 \) and \( dK_1' \) are equal, and the notation \( dK_1 \) is preferred. Similarly, since the events \( \{ K_1' \cap K_0 \neq \emptyset \} \) and \( \{ K_1 \cap K_0 \neq \emptyset \} \) are equal, we only use the notation \( K_1' \cap K_0 \). The volume of \( K_1' \) is equal to 0, moreover such an object is homeomorphic to a sphere whose Euler characteristic is equal to 2. In addition, up to a negligible error of order \( \Delta \), the contribution of the concave part of \( K_1' \) to the integral of the mean curvature is equal to minus that of the contribution of the convex part, so \( M_1' = (2\pi) \). Putting all together, Eq.\((A1)\) reads

\[
\int_{K_1 \cap K_0 \neq \emptyset} \chi(K_1' \cap K_0) dK_1 = 8\pi^2 (2V_0) + 2\pi (2F_1M_0).
\]

(A3)

and

\[
\langle \chi(K_1' \cap K_0) \rangle = \frac{\int_{K_1 \cap K_0 \neq \emptyset} \chi(K_1' \cap K_0) dK_1}{\int_{K_1 \cap K_0 \neq \emptyset} dK_1} = \frac{8\pi V_0 + 2\pi F_1M_0}{4\pi(V_0 + V_1) + F_1M_0 + F_0M_1}.
\]

(A4)

Without additional assumption, when \( K_1' \) crosses \( K_0 \) entirely, the intersection of \( K_1' \) and \( K_0 \) is topologically equivalent to that of a cylinder whose Euler characteristic is zero. Therefore, the stochastic integral of Eq.\((A3)\) does not count such a configuration. To circumvent this difficulty, we will assume that the maximum of the radii of curvature of \( K_1 \) is smaller than the minimum of those of \( K_0 \). With this additional hypothesis, either \( K_1' \) falls entirely inside \( K_0 \) and \( \chi(K_1' \cap K_0) = \chi(K_1) = 2 \) (this event has probability \( p \), since \( P(K_1' \subseteq K_0 | K_1 \cap K_0 \neq \emptyset) = P(K_1' \subseteq K_0 | K_1 \cap K_0 \neq \emptyset \) or crosses \( K_0 \) once with probability \( 1 - p \)). In this latter case, the intersection of \( K_1' \) and \( K_0 \) is homeomorphic to a portion of a sphere which is a contractile object and has an Euler characteristic equal to 1. Thus, in terms of probability, the mean value of the Euler characteristic satisfies the following relationship

\[
\langle \chi(K_1' \cap K_0) \rangle = p \times 2 + (1 - p) \times 1 = 1 + p.
\]

(A5)

By matching both expressions of \( \langle \chi(K_1' \cap K_0) \rangle \), Eq.\((A4)\) and Eq.\((A5)\), one obtains

\[
p = \frac{4\pi(V_0 - V_1) + (F_1M_0 - F_0M_1)}{4\pi(V_0 + V_1) + (F_1M_0 + F_0M_1)}.
\]

(A6)

The possibility of a transposition of \( K_1 \) inside \( K_0 \) is given by the condition \( p > 0 \), which is

\[
4\pi V_0 + F_1M_0 > 4\pi V_1 + F_0M_1.
\]

(A7)

Note that in the case of two spheres of radii \( R_0 \) and \( R_1 \) with \( R_0 > R_1 \), the probability \( p \) that the small sphere lies entirely inside the large one becomes

\[
p = \left( \frac{R_0 - R_1}{R_0 + R_1} \right)^3,
\]

(A8)

where we used the fact that for a sphere of radius \( R \), the integral of the mean curvature is \( M = 4\pi R^2 \). Due to the rotational invariance, as expected, \( p \) is the ratio of the volume of the sphere of radius \( R_0 - R_1 \) over the volume of the sphere of radius \( R_0 + R_1 \) as shown in Fig. 8.

The result obtained in this appendix is summarized as follows:
Theorem 4.
Let $K_0$ and $K_1$ be two smooth convex bodies in $\mathbb{R}^3$. We assume that $K_0$ is fixed and $gK_1$ is moving under the rigid motion $g$. Let $dg$ be the kinematic density in $\mathbb{R}^3$. We define the inclusion probability of the convex body $K_1$ contained in the convex body $K_0$ by

$$p[K_1 \subseteq K_0 | K_1 \cap K_0 \neq \emptyset] = \frac{\int_{\{g:gK_1 \subseteq K_0\}} dg}{\int_{\{g:gK_1 \cap K_0 \neq \emptyset\}} dg}.$$  \hspace{1cm} (A9)

When the minimal radius of curvature of $K_0$ is greater than the maximal radius of curvature of $K_1$, then the inclusion probability is given by

$$p[K_1 \subseteq K_0 | K_1 \cap K_0 \neq \emptyset] = \frac{4\pi(V_0 - V_1) + F_1 M_0 - F_0 M_1}{4\pi(V_0 + V_1) + F_1 M_0 + F_0 M_1}.$$  \hspace{1cm} (A10)

where $V_i$, $F_i$ and $M_i$ denote the volume, the surface area and the integral of the mean curvature of the convex object $K_i$.

Appendix B: Integrals of mean curvatures

In this appendix, we highlight the link between the results obtained in paragraph IV and the integrals of mean curvatures. To this aim, we will use additional results obtained by Santaló in the 70s. In addition to the kinematic formulas, Santaló has also established the expressions for the integrals of mean curvatures of hypersurfaces in $\mathbb{R}^n$, more precisely

$$\int_{K_1 \cap K_0 \neq \emptyset} M_{q-1}(\partial(K_1 \cap K_0))dK_1 = O_1 \ldots O_{n-2} \left[ O_{n-1} \left( V_1 M_{q-1}^0 + V_0 M_{q-1}^1 \right) \right]$$

$$+ \frac{(n-q)O_{q-1}}{O_{n-q-1}} \sum_{h=q}^{n-2} \left[ \frac{O_{q-h}^1}{(h+1)O_{n-h}O_{h+q-n}} M_{n-2-h}^1 M_{h+q-n}^0 \right].$$  \hspace{1cm} (B1)

which holds for $q = 1, 2, \ldots, n - 1$. Having in mind that $K_1$ is a curve in $\mathbb{R}^n$ (as well as its intersection with $K_0$), and that $M_{n-2}$ (Eq. (B5)) gives the information about its length, we apply the previous formula with $q = n - 1$ (recall that $V_1 = 0$)

$$\int_{K_1 \cap K_0 \neq \emptyset} M_{n-2}(\partial(K_1 \cap K_0))dK_1 = O_1 \ldots O_{n-2} \left[ O_{n-1} V_0 M_{n-2}^1 + O_{n-2} \sum_{h=1}^{n-2} \frac{(n-2)O_{n+1-h}O_h}{(h+1)O_{n-h}O_{h-1}} M_{n-2-h}^1 M_{h-1}^0 \right].$$  \hspace{1cm} (B2)
In the case of a curve of length $s$ in $\mathbb{R}^n$, since we have $M_i = 0$ for $i = 1, 2, \ldots, n - 3$, in the previous equation all the terms in the sum are equal to zero, and since $M_{n-2} = O_{n-2}/(n-1)$, we obtain

$$\int_{K_1 \cap K_0 \neq \emptyset} \frac{O_{n-2}}{n-1} L dK_1 = O_1 \cdots O_{n-2} O_{n-1} V_0 \frac{O_{n-2}}{n-1} s,$$

thus recovering Eq.(40) of the main text.

**Appendix C: One-chord distribution**

In this appendix, we derive a Cauchy-like formula in $\mathbb{R}^n$ for the one-chord distribution (see definition below)

$$\langle \sigma_{\text{OCD}} \rangle = \eta_n \frac{V_0}{F_0^*}, \quad \text{(C1)}$$

where $V_0$ is the volume of the object and $F_0^*$ its convex hull. As usual, $\eta_n$ is the dimensional constant $2\pi O_{n-1}/O_n$.

To this end, we consider random straight paths $K_1$ through a non-convex object $K_0$. In such a situation, as noted by Gill [23], the random line $K_1$ may cross $K_0$ more than one time. Consequently, there are several definitions for the chord length distribution: the multiple chord distribution (MCD) when considering each chord length segment inside $K_0$ for itself, and the one-chord distribution (OCD) when considering the chord as the sum of all length segments inside $K_0$. A straight line is a particular case of infinite curves, therefore we know from paragraph V that the average length of the MCD-chord is given by Cauchy’s formula Eq.(14) (see also [20] for a demonstration concerning straight lines only). However, in some situations, such as light scattering, it is the mean chord of the OCD distribution that is the relevant quantity and for which less is known, except in the two-dimensional case where the average value is given by the Cauchy-like formula

$$\langle \sigma_{\text{OCD}} \rangle = \pi \frac{S}{L^*}. \quad \text{(C2)}$$

Here $L^*$ is the perimeter of the convex hull of $K_0$ and $S$ its surface. A derivation of Eq.(C2) is given in Ref.[10]. Using integral geometry arguments similar to those we have employed, we will extend this result to $\mathbb{R}^n$. First, recall that Eq.(40) gives the length of the straight line inside $K_0$, that is

$$\int_{K_1 \cap K_0 \neq \emptyset} L dK_1 = O_{n-1} \cdots O_1 V_0 s. \quad \text{(C3)}$$

Formally this is an unbounded measure since $s \to \infty$. Nevertheless, this is not an issue since the averaging process involves the ratio of two similar unbounded measures. However, in order to be reasonably rigorous, we consider $s$ as very large compared to the size of $K_0$. Then we will let $s$ go to infinity. Instead of using directly the kinematic formula, we found out that it was more convenient to consider the boundary $\partial K_0$ of $K_0$, i.e. a manifold of dimension $n - 1$, and then apply Eq.(35) with the parameters $q = n - 1$ and $r = 1$. With these parameters, as we already noticed in paragraph V $\sigma_0(M_{n-1} \cap M^1) = N(K_1 \cap \partial K_0)$ denotes the number of points of the intersection of $\partial K_0$ and $K_1$, so that Eq.(35) can be written

$$\int_{K_1 \cap K_0 \neq \emptyset} N(K_1 \cap \partial K_0) dK_1 = \frac{O_n \cdots O_1 O_0}{O_{n-1} O_1} F_0^* s. \quad \text{(C4)}$$

Since $K_1$ is a straight line crossing $K_0$, $N(K_1 \cap \partial K_0)$ is twice the number of segments inside $K_0$. However, to obtain the average length of the sum of the lengths of the segments, we need to count the segment only once and this regardless of the number of times the line crosses $K_0$. One way to achieve this goal is to consider the convex hull $K_0^\ast$ of $K_0$. For the convex hull, except for a set of null measure, we have $N(K_1 \cap \partial K_0^\ast) = 2$ when $\{K_0^\ast \cap K_1 \neq \emptyset\}$ and obviously because $\{K_1 \cap K_0^\ast \neq \emptyset\} = \{K_1 \cap K_0 \neq \emptyset\}$ for straight lines, the preceding equation becomes

$$\int_{K_1 \cap K_0^\ast \neq \emptyset} 2 dK_1 = \int_{K_1 \cap K_0 \neq \emptyset} 2 dK_1 = \frac{O_n \cdots O_1 O_0}{O_{n-1} O_1} F_0^* s, \quad \text{(C5)}$$

where $F_0^*$ stands for the surface area of the convex hull of $K_0$. The average value of the OCD distribution is easily obtained from Eqs.(C3) and (C5). Indeed,

$$\langle \sigma_{\text{OCD}} \rangle = \frac{\int_{K_1 \cap K_0 \neq \emptyset} L dK_1}{\int_{K_1 \cap K_0 \neq \emptyset} dK_1} = \frac{V_0}{F_0^*}, \quad \text{(C6)}$$

which is the announced result.
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29. In this article, the term curve encompasses any one-dimensional rectifiable object of possibly infinite length, including fibers, loops, segments, random walks, etc. Only Brownian paths and Brownian-like paths (diffusion) are excluded due to their fractal nature.
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In appendix [C] we also consider another distribution, namely the one-chord distribution, where the sum of all chord segments on one straight line is the random variable.

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Recall that for the n-dimensional sphere $S^n$, we have

$$
\chi(S^n) = 1 + (-1)^n = \begin{cases} 
2 & \text{if } n \text{ is even}, \\
0 & \text{if } n \text{ is odd}.
\end{cases}
$$

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