Multiplicative deconvolution estimator based on a ridge approach

Sergio Brenner Miguel\textsuperscript{a,*}

\textsuperscript{a}Institut für Angewandte Mathematik, MATHEMATIKON, Im Neuenheimer Feld 205, D-69120 Heidelberg, Germany

Abstract

We study the non-parametric estimation of an unknown density \( f \) with support on \( \mathbb{R}^+ \) based on an i.i.d. sample with multiplicative measurement errors. The proposed fully-data driven procedure consists of the estimation of the Mellin transform of the density \( f \) and a regularisation of the inverse of the Mellin transform by a ridge approach. The upcoming bias-variance trade-off is dealt with by a data-driven choice of the ridge parameter. In order to discuss the bias term, we consider the Mellin-Sobolev spaces which characterise the regularity of the unknown density \( f \) through the decay of its Mellin transform. Additionally, we show minimax-optimality over Mellin-Sobolev spaces of the ridge density estimator.

Keywords: Density estimation, multiplicative measurement errors, Mellin transform, ridge estimator, minimax theory, inverse problem, adaptation

2020 MSC: Primary 62G05, secondary 62G07, 62C20

1. Introduction

In this paper we are interested in estimating the unknown density \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) of a positive random variable \( X \) given independent and identically distributed (i.i.d.) copies of \( Y = XU \), where \( X \) and \( U \) are independent of each other and \( U \) has a known density \( g : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \). In this setting the density \( f_Y : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) of \( Y \) is given by

\[
f_Y(y) = (f * g)(y) := \int_{\mathbb{R}^+} f(x)g(y/x)x^{-1}dx \quad \forall y \in \mathbb{R}^+.
\]

Here * denotes multiplicative convolution. The estimation of \( f \) using an i.i.d. sample \( Y_1, \ldots, Y_n \) from \( f_Y \) is thus an inverse problem called multiplicative deconvolution.

This particular model was studied by [6]. Inspired by the work [4], the authors of [6] introduced an estimator based on the estimation of the Mellin transform of the unknown density \( f \) and a spectral cut-off regularisation of the inverse of the Mellin transform. In [4] a pointwise kernel density estimator was proposed and investigated, while the authors of [6] studied the global risk of the density estimation. For the model of multiplicative measurement, the multivariate case of global density estimation, respectively the univariate case of global survival function estimation, was considered by [5], respectively [7], based on a spectral cut-off approach.

In this work, we will borrow the ridge approach from the additive deconvolution literature, for instance used by [12] and [14], to build a new density estimator and compare it with the spectral cut-off estimator proposed by [6]. The contribution of this work to the existing literature is the inclusion of the ridge approach and the comparison to the spectral cut-off approach. We discuss in which situations the corresponding estimators are comparable, respectively when the ridge approach is favourable. Furthermore, the ridge approach can be used for future works considering oscillatory error densities or unknown error densities, compare [12] and [14].

\textsuperscript{*}Corresponding author. Email address: brennermiguel@math.uni-heidelberg.de

Preprint submitted to arXiv August 4, 2021
1.1. Related works

The model of multiplicative measurement errors was motivated in the work of \cite{4} as a generalisation of several models, for instance the multiplicative censoring model or the stochastic volatility model.\cite{17} and \cite{18} introduce and analyse intensively multiplicative censoring, which corresponds to the particular multiplicative deconvolution problem with multiplicative error $U$ uniformly distributed on $[0, 1]$. This model is often applied in survival analysis as explained and motivated in \cite{10}. The estimation of the cumulative distribution function of $X$ is discussed in \cite{19} and \cite{2}. Series expansion methods are studied in \cite{11} treating the model as an inverse problem. The density estimation in a multiplicative censoring model is considered in \cite{8} using a kernel estimator and a convolution power kernel estimator. Assuming an uniform error distribution on an interval $[1 - a, 1 + a]$ for $a \in (0, 1)$, \cite{9} analyse a projection density estimator with respect to the Laguerre basis. \cite{3} investigate a beta-distributed error model. The density estimation in a multiplicative censoring model is considered in \cite{8} using a kernel estimator and a convolution power kernel estimator. Assuming an uniform error distribution on an interval $[1 - a, 1 + a]$ for $a \in (0, 1)$, \cite{9} analyse a projection density estimator with respect to the Laguerre basis. \cite{3} investigate a beta-distributed error model. In the work of \cite{4}, the authors used the Mellin transform to construct a kernel estimator for the pointwise density estimation. Moreover, they point out that the following widely used naive approach is a special case of their estimation method. The density estimation in a multiplicative censoring model is considered in \cite{8} using a kernel estimator and a convolution power kernel estimator. Assuming an uniform error distribution on an interval $[1 - a, 1 + a]$ for $a \in (0, 1)$, \cite{9} analyse a projection density estimator with respect to the Laguerre basis. \cite{3} investigate a beta-distributed error model.

1.2. Organisation

The paper is organised as follows. In Section 1 we recapitulate the definition of the Mellin transform and collect further properties of the Mellin transform, which will be used in the upcoming theory, are collected in Appendix A. In Section 2 we will show that the ridge density estimator is minimax-optimal over the Mellin-Sobolev spaces, by stating an upper bound and using the lower bound result given in \cite{5}. A data-driven procedure based on a Goldenshlider-Lepski method is described and analysed in Section 3. Finally, results of a simulation study are reported in section 4 which visualize the reasonable finite sample performance of our estimators.

1.3. The spectral cut-off and ridge estimator

We define for any weight function $\omega : \mathbb{R} \to \mathbb{R}_+$ the corresponding weighted norm by $\|h\|_{\omega} := \int_{0}^{\infty} |h(x)|^2 \omega(x) dx$ for a measurable, complex-valued function $h$. Denote by $L^2(\mathbb{R}_+, \omega)$ the set of all measurable, complex-valued functions with finite $\|\cdot\|_{\omega}$-norm and by $\langle h_1, h_2 \rangle_{\omega} := \int_{0}^{\infty} h_1(x) \overline{h_2(x)} \omega(x) dx$ for $h_1, h_2 \in L^2(\mathbb{R}_+, \omega)$ the corresponding weighted scalar product. Similarly, define $L^1(\mathbb{R}) := \{ h : \mathbb{R} \to \mathbb{C} \text{ measurable} : \|h\|_{\omega} := \int_{0}^{\infty} h(t) dt < \infty \}$ and $L^1(\Omega, \omega) := \{ h : \Omega \to \mathbb{C} : \|h\|_{\omega} := \int_{\Omega} |h(x)| \omega(x) dx < \infty \}$. In the introduction we already mentioned that the density $f_Y$ of $Y_1$ can be written as the multiplicative convolution of the densities $f$ and $g$. We will now define this convolution in a more general setting. Let $c \in \mathbb{R}$. For two functions $h_1, h_2 \in L^1(\mathbb{R}_+, x^{-c})$, where we use the notation $x^{-c}$ for the weight function $x \mapsto x^{-c}$, we define the multiplicative convolution $h_1 * h_2$ of $h_1$ and $h_2$ by

$$\langle h_1 * h_2 \rangle(y) := \int_{0}^{\infty} h_1(y/x) h_2(x) x^{-c} dx, \quad y \in \mathbb{R}_+. \tag{1}$$

In fact, one can show that the function $h_1 * h_2$ is well-defined, $h_1 * h_2 = h_2 * h_1$ and $h_1 * h_2 \in L^1(\mathbb{R}_+, x^{-c})$, compare \cite{5}. It is worth pointing out, that the definition of the multiplicative convolution in equation (1) is independent of the model parameter $c \in \mathbb{R}$. We also know for densities $h_1, h_2$ that $h_1, h_2 \in L^1(\mathbb{R}_+, x^{\alpha})$. If additionally $h_1 \in L^2(\mathbb{R}_+, x^{2\alpha})$ then $h_1 * h_2 \in L^2(\mathbb{R}_+, x^{2\alpha})$.

Mellin transform. We will now define the Mellin transform for $L^1(\mathbb{R}_+, x^{-c})$ functions and present the convolution theorem. Further properties of the Mellin transform, which will be used in the upcoming theory, are collected in Appendix A.
Appendix A: Proof sketches of these properties can be found in [6], respectively [5]. Let \( h_1 \in L^1(\mathbb{R}, x^{-1}) \). Then, we define the Mellin transform of \( h_1 \) at the development point \( c \in \mathbb{R} \) as the function \( \mathcal{M}_c[h_1] : \mathbb{R} \to \mathbb{C} \) with

\[
\mathcal{M}_c[h_1](t) := \int_0^{\infty} x^{-1+ct}h_1(x)dx, \quad t \in \mathbb{R}.
\]

The key property of the Mellin transform, which makes it so appealing for the use of multiplicative deconvolution, is the so-called convolution theorem, that is, for \( h_1, h_2 \in L^1(\mathbb{R}_+, x^{-1}) \),

\[
\mathcal{M}_c[h_1 * h_2](t) = \mathcal{M}_c[h_1](t)\mathcal{M}_c[h_2](t), \quad t \in \mathbb{R}.
\]

Let us now revisit the definition of the spectral cut-off estimator.

\textbf{Spectral-cut off estimator.} The family of spectral cut-off estimator \( \{\tilde{f}_k\}_{k \in \mathbb{N}} \) proposed by [6], respectively [5], is based on the estimation of the Mellin transform of \( f_\epsilon \) and a spectral cut-off regularisation of the inverse Mellin transform.

Given the sample \( (Y_j)_{j \in \mathbb{N}} \) with \( \|a\| := [1, a] \cap \mathbb{N} \) for any \( a \in \mathbb{N} \), an unbiased estimator of \( \mathcal{M}_c[f_\epsilon](t) \), \( t \in \mathbb{R} \), is given by the empirical Mellin transform

\[
\tilde{\mathcal{M}}(t) := n^{-1} \sum_{j \in \mathbb{N}} Y_j^{-1+it}, \quad t \in \mathbb{R}
\]

if \( \mathbb{E}[Y_j^{-1}] < \infty \) for \( c \in \mathbb{R} \). Exploiting the convolution theorem, eq. (3), under the assumption that \( \mathcal{M}_c[g](t) \neq 0 \) we can define the unbiased estimator \( \tilde{\mathcal{M}}(t)/\mathcal{M}_c[g](t) \) of \( \mathcal{M}_c[f_\epsilon](t) \) for \( t \in \mathbb{R} \). To construct an estimator of the unknown density \( f_\epsilon \), the authors of [5] used a spectral-cut off approach. That is, for \( k \in \mathbb{N} \), we assume that \( \mathbb{E}[g_j^2]\mathcal{M}_c[g_j]^2 \in L^2(\mathbb{R}) \), then we can ensure that \( \mathbb{E}[g_j^2]\mathcal{M}_c[g_j] \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) since \( \mathcal{M}_c(t) \leq \mathcal{M}_c(0) < \infty \) almost surely. Now, the spectral cut-off density estimator \( \tilde{f}_k \) can be defined by

\[
\tilde{f}_k(x) := \mathcal{M}_c^{-1}[\mathbb{I}_{[-k, k]}\tilde{\mathcal{M}}/\mathcal{M}_c[g]](x) = \frac{1}{2\pi} \int_{-k}^{k} x^{-c-it} \frac{\tilde{\mathcal{M}}(t)}{\mathcal{M}_c[g](t)} dt, \quad x \in \mathbb{R}_+.
\]

Here we used two minor assumptions on the error density \( g \), that is,

\[
\forall t \in \mathbb{R} : \mathcal{M}_c[g(t)] \neq 0 \quad \text{and} \quad \forall k \in \mathbb{R}_+ : \mathbb{I}_{[-k, k]}\mathcal{M}_c[g]^2 \in L^2(\mathbb{R}). \quad \text{([G0])}
\]

This assumption implies that the Mellin transform of \( g \) does not approach zero too fast. Although this assumption is fulfilled for a large class of error densities, we will now show that one can define an estimator for an even weaker assumption on the error density. An intense study of this estimator, including the minimax optimality and data-driven choice of the parameter \( k \in \mathbb{N}_+ \), can be found in [6].

\textbf{Ridge estimator.} Inspired by the work of [14] and [12], let \( r, \xi \geq 0 \) such that \( t \mapsto \mathcal{M}_c[g(t)](1 + |t|)^{-\xi(t+2)} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \). Then for any \( k \in \mathbb{R}_+ \) we define the function \( R_{k, \xi} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) by

\[
R_{k, \xi}(t) := \frac{\mathcal{M}_c[g(-t)]\mathcal{M}_c[g(t)]^\xi}{\max(\mathcal{M}_c[g](t), k^{-1}(1 + |t|)^\xi)^{r+2}}, \quad t \in \mathbb{R},
\]

and the set \( G_k := \{ t \in \mathbb{R} : k^{-1}(1 + |t|)^\xi > \mathcal{M}_c[g](t) \} \). Now for all \( t \in G_k \) holds \( R_{k, \xi}(t) = \mathcal{M}_c[g(t)]^{-1} \). We define next the ridge density estimator \( \tilde{f}_{k, \xi} \) by \( \tilde{f}_{k, \xi} := \mathcal{M}_c^{-1}[\mathcal{M}_c R_{k, \xi}] \). In fact, it can be written explicitly for \( x \in \mathbb{R}_+ \) as

\[
\tilde{f}_{k, \xi}(x) = \frac{1}{2\pi} \int_{-k}^{k} x^{-c-it} \frac{\tilde{\mathcal{M}}(t)}{\mathcal{M}_c[g](t)} dt = \frac{1}{2\pi} \int_{G_k} x^{-c-it} \frac{\tilde{\mathcal{M}}(t)}{\mathcal{M}_c[g](t)} dt + \frac{1}{2\pi} \int_{G_k} x^{-c-it} \frac{\tilde{\mathcal{M}}(t)}{\mathcal{M}_c[g](t)} dt.
\]

By the construction of \( G_k \) the quotient \( \tilde{\mathcal{M}}(t)/\mathcal{M}_c[g](t) \) in the integrand in eq. (5) is well-defined even without assumption [G0].
2. Minimax theory

In this section, we will see that an even milder assumption on the error density \( g \) than \([G1]\) is sufficient to ensure that the presented ridge estimator is consistent. We finish this Section 2 by showing that the estimator is minimax optimal over the Mellin-Sobolev ellipsoids. We denote by \( \mathbb{E}_f^n \) the expectation corresponding to the distribution of \((Y)_{i=1}^{n}\). Respectively we define \( \mathbb{E}_f := \mathbb{E}_f^n \) and \( \mathbb{E}_y, \mathbb{E}_f \).

2.1. General consistency

Although the sequence \((G_k)_{k \in \mathbb{N}}\) is obviously nested, that is \( G_{k+1} \subseteq G_k \) for all \( k \in \mathbb{N} \), we want to stress out that the squared bias, \( \|f - \mathbb{E}_f^n(\hat{f}_{k,x})\|_2^2 \), of \( \hat{f}_{k,x} \) defined in eq. \(5\), might not tend to zero for \( k \) going to infinity. For instance, one may consider the case where \( \mathcal{M}_c[g] \) vanishes on an open, nonempty set \( A \subseteq \mathbb{R} \) and \( \mathcal{M}_c[f] \) does not vanish on \( A \). A more sophisticated discussion about identifiability and consistency in the context of additive deconvolution problems can be found in the work of \([14]\). The discussion there can be transferred to the case of multiplicative deconvolution problems. Based on the discussion presented in \([14]\), we will give a minimal assumption to ensure that we can define a consistent estimator using the ridge approach. We will from now on assume, that the Mellin transform of \( g \) is almost nonzero everywhere, that is,

\[
\lambda(t \in \mathbb{R} : \mathcal{M}_c[g](t) = 0) = 0.
\] ([G-1])

Under the assumption \([G-1]\) we can use the dominated convergence theorem to show that the bias \( \|f - \mathbb{E}_f^n(\hat{f}_{k,x})\|_2^2 \) vanishes for \( k \) going to infinity. Further, it is worth stressing out that for \( k \in \mathbb{N} \) and \( t \in \mathbb{R} \) we have \( R_{k+1,x}(t) \geq R_{k,x}(t) \). We then get the following results whose proofs is postponed to Appendix B.

** Proposition 1.** Let \( c \in \mathbb{R} \) such that \( f \in L^2(\mathbb{R}^+, x^{2c-1}) \) and \( \sigma_c := \mathbb{E}_f(Y_1^{2(c-1)}) < \infty \). Then for any \( r, \xi \geq 0 \) with \( \mathcal{M}_c[g]^{-1}(1 + |t|)^{-\xi-2} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) we have

\[
\mathbb{E}_f^n(\|f - \hat{f}_{k,x}\|_2^2) \leq \frac{1}{2n} \| \mathbb{E}G_c \mathcal{M}_c[f] \|_2^2 + \frac{\sigma_c}{2n} \| R_{k,x} \|_2^2
\]

where \( G_k := \{ t \in \mathbb{R} : k^{-1}(1 + |t|)^{\xi} > \mathcal{M}_c[g](t) \} \) and \( \hat{f}_{k,x} \) is defined in equation \(5\). If additionally \([G-1]\) holds and \((k_n)_{n \in \mathbb{N}}\) satisfies \( k_n \to \infty \) and \( n^{-1} \| R_{k_n,x} \|_2^2 \to 0 \) for \( n \to \infty \) then

\[
\mathbb{E}_f^n(\|\hat{f}_{k,x} - f\|_2^2) \to 0
\]

for \( n \to \infty \).

Although the assumptions on \( \xi, k_n, r \geq 0 \) in Proposition 1 seem to be rather technical, we will see that they are fulfilled when considering more precise classes of error densities, so-called smooth error densities. Before we define this family of error densities let us shortly comment on the consistency of the presented estimator.

**Remark 1** (Strong consistency). In Proposition 1 we have seen that we can determine a set of assumptions which ensures by application of the Markov inequality, that \( \|\hat{f}_{k,x} - f\|_2^2 \to 0 \) in probability. Here, we needed the additional assumption that \( f \in L^2(\mathbb{R}, x^{2c-1}) \) and \( \sigma_c := \mathbb{E}_f(Y_1^{2(c-1)}) < \infty \) to construct the estimator and show its properties. A less restrictive metric which can be considered would be the \( L^1(\mathbb{R}^+, x^d) \)-metric, since for any density, \( f \in L^1(\mathbb{R}^+, x^d) \) holds. Further, the Mellin transform developed in \( c = 1 \) is well-defined for any density \( f \). In the book of \([14]\) they proposed an estimator \( \hat{f}_V \) of the density \( f_V : \mathbb{R} \to \mathbb{R} \) of a real random variable \( V \) given i.i.d. copies of \( Z = V + \varepsilon \) where \( V \) and \( \varepsilon \) are stochastically independent. They were able to show that their estimator \( \hat{f} \) is strongly consistent in the \( L^1(\mathbb{R}) \)-sense, that is, \( \|f - \hat{f}\|_{L^1(\mathbb{R})} \to 0 \) almost surely. Given the log transformed data, \( \log(Y) = \log(X) + \log(\varepsilon) \), we can use the estimator \( \hat{f}_V \) for \( V = \log(X) \) and deduce the estimator \( \hat{f}_V(x) := \hat{f}_V(\log(x))x^{-1} \) for any \( x \in \mathbb{R}^+ \). Then \( \|f - \hat{f}\|_{L^1(\mathbb{R}^+, x^d)} = \|\hat{f}_V - f\|_{L^1(\mathbb{R})} \), implying that the estimator \( \hat{f}_V \) is strongly consistent in the \( L^1(\mathbb{R}^+, x^d) \). Although it might be tempting generalise this result for the \( L^1(\mathbb{R}^+, x^d) \)-distance for any \( c \in \mathbb{R} \), it would need an additional moment assumption on \( f \) which contradicts the idea of considering the most general case.
2.2. Noise assumption

Up to now, we have only assumed that the Mellin transform of the error density \( g \) does not vanish almost everywhere, i.e. [G-1]. To develop the minimax theory for the estimator \( \hat{f}_{k,x} \) we will specify the class of considered error density \( g \) through an assumption on the decay of its corresponding Mellin transform \( M_g \). This assumption will allow us to determine the growth of the variance term more precisely. In the context of additive deconvolution problems, compare [11], densities whose Fourier transform decay polynomially are called smooth error densities. To stay in this way of speaking we say that an error density \( g \) is a smooth error density if there exists \( \varepsilon, C, \gamma \in \mathbb{R}_+ \) such that

\[
  c_g(1 + r^2)^{-\gamma/2} \leq |M_g(t)| \leq C_g(1 + r^2)^{-\gamma/2}, \quad t \in \mathbb{R}.
\]

(G1)

This assumption on the error density was also considered in the works of [4], [6] and [5]. We focus on to the case where \( \xi = 0 \), and use the abbreviation \( f_k := f_{k,0,r} \), respectively \( R_k := R_{k,0,r} \). Then under the assumption of Proposition [1] and assumption [G1] we can show that for each \( r > 0 \lor (\gamma^{-1} - 1) \) there exists a constant \( C_{g,r} > 0 \) such that

\[
  n^{-1}\|R_d\|^2_\mathbb{R} \leq C_{g,r}k^{2\gamma+1}n^{-1},
\]

which leads to the following corollary whose proof can be found in Appendix B. Here \( a \lor b := \max(a, b) \) for \( a, b \in \mathbb{R} \).

Corollary 1. Let the assumptions of Proposition [1] and [G1] be fulfilled. Then for \( r > 0 \lor (\gamma^{-1} - 1) \),

\[
  \mathbb{E}_g^n(\|\hat{f} - f_k\|^2_{L^2_{\mathbb{R}_c}}) \leq \frac{1}{2\pi} \|L_{C_g}M_{f}\|^2_\mathbb{R} + \gamma^c k^{2\gamma+1}n^{-1}.
\]

If one chooses \( k_n \in \mathbb{N} \) such that \( k_n^{2\gamma+1}n^{-1} \to 0 \) and \( k_n \to \infty \) then \( \mathbb{E}_g^n(\|\hat{f} - f_k\|^2_{L^2_{\mathbb{R}_c}}) \to 0 \) for \( n \to \infty \).

Considering the bound of the variance term, a choice of \( k_n \in \mathbb{N} \) increasing slowly in \( n \), would imply a faster decay of the variance term. On the other hand, the opposite effect on the bias term can be observed. In fact, to balance both terms, an assumption on the decay of the Mellin transform of the unknown density \( f \) is needed. In the non-parametric Statistics and in the inverse problem community this is usually done by considering regularity spaces.

2.3. The Mellin-Sobolev space

We will now introduce the so-called Mellin-Sobolev spaces, which are, for instance, considered by [6] for the case \( c = 1 \) and [5] for the multivariate case. In the work of [6] their connection to regularity properties, in terms of analytical properties, and their connection to the Fourier-Sobolev spaces are intensely studied. For \( s, L \in \mathbb{R}_+ \) and \( c \in \mathbb{R} \) we define the Mellin-Sobolev spaces by

\[
  \mathcal{W}_c^s(\mathbb{R}_+^s) := \left\{ f \in L^2(\mathbb{R}_+, s^{2c-1}) : \|f\|^2_{L^s_{\mathbb{R}_+}} := \|(1 + r^2)^{c/2}M_f\|^2_\mathbb{R} \leq \infty \right\}
\]

and their corresponding ellipsoids by \( \mathcal{W}_c^s(L) := \{ f \in \mathcal{W}_c^s(\mathbb{R}_+) : \|f\|^2_{L^s_{\mathbb{R}_+}} \leq L \} \). Then for \( f \in \mathcal{W}_c^s(L) \) and under assumption [G1] we can show that \( \|L_{C_g}M_f\|^2_\mathbb{R} \leq C(g, L, s)k^{-2s}\gamma \). Since \( f \) is a density and to control the variance term, it is natural to consider the following subset of \( \mathcal{W}_c^s(L) \),

\[
  \mathcal{D}_c^s(L) := \{ f \in \mathcal{W}_c^s(L) : f \text{ is a density}, \mathbb{E}_f(\chi_{\mathbb{R}_+^{2c-1}}) \leq L \}.
\]

Then we can state the following theorem whose proof is postponed to Appendix B.

Theorem 1 (Upper bound over \( \mathbb{D}_c^s(L) \). Let \( c \in \mathbb{R}, s, L \in \mathbb{R}_+ \) and \( \mathbb{E}_g(U_{-1}^{2c-1}) < \infty \). Let further [G1] be fulfilled and \( r > 0 \lor (\gamma^{-1} - 1) \). Then the choice \( k_o := n^{(2s+2\gamma+1)} \) leads to

\[
  \sup_{f \in \mathbb{D}_c^s(L)} \mathbb{E}_g^n(\|\hat{f} - f_k\|^2_{L^2_{\mathbb{R}_c}}) \leq C_{g,L}n^{-2s(2s+2\gamma+1)}.
\]

A similar result was presented by the authors [6] and [5] showing that for the spectral cut-off estimator \( \hat{f}_k \) the choice \( k_o := n^{1/(2s+2\gamma+1)} \) leads to the same rate of \( n^{-2s/(2s+2\gamma+1)} \) uniformly over the classes \( \mathbb{D}_c^s(L) \). For the case \( c = 1 \) the authors of [6] presented a lower bound result, showing that in many cases the rate given in Theorem 1 is the minimax rate for the density estimation \( f \) given the i.i.d. sample \( (Y_j)_{j \in \mathbb{N}} \). For the multivariate case, the author in [5] has generalised the proof for all \( c > 0 \). The following Theorem follows as a special case of the lower bound presented in [5] for the dimension \( d = 1 \) and its proof is thus omitted.
Theorem 2 (Lower bound over $\mathbb{D}^s(L)$). Let $s, \gamma \in \mathbb{N}$, $c > 0$ and assume that [GI] holds. Additionally, assume that $g(x) = 0$ for $x > 1$ and that there exists constants $c_g, C_g$ such that

$$c_g(1 + r^2)^{-\gamma/2} \leq |M_{\xi}(g)(t)| \leq C_g(1 + r^2)^{-\gamma/2}, \quad t \in \mathbb{R},$$

where $\overline{c} = 1/2$ for $c > 1/2$ and $\overline{c} = 0$ for $c \in (0, 1/2)$.

Then there exist constants $C_{g, s}, L_{g, s} > 0$ such that for all $L \geq L_{s, g, s}, n \in \mathbb{N}$ and for any estimator $\hat{f}$ of $f$ based on an i.i.d. sample $(Y_i)_{i \in [n]}$

$$\sup_{f \in \mathbb{D}^s(L)} \mathbb{E}_{\mathcal{F}_f}(\|\hat{f} - f\|_2^2) \geq C_{g, s} n^{-2s/(2s+2)}.$$

We want to emphasize that the additional assumption on the error densities is for technical reasons. To ensure that $M_{\xi}(g)$ is well-defined, we need to additionally assume that $\mathbb{E}_x(U_1^{-1/2}) < \infty$ for the case of $c > 1/2$. If $c \in (0, 1/2]$, then $\mathbb{E}_x(U_1^{-1}) < \infty$ follows from $\mathbb{E}_x(U_2^{-2})$, compare Proposition [2].

In the work of [3] the authors showed that for the case of $c = 1$ the spectral cut-off estimator $\hat{f}_c$, defined in eq. (4) is minimax optimal for some examples of error densities. In fact, they stressed out that for Beta-distributed $U_1$, considered for instance by [3], all assumption on $g$ are fulfilled.

3. Data-driven method

In Section 2 we determined a choice of the parameter $\xi, k, r \geq 0$ such that the resulting ridge estimator $\hat{f}_{\xi, k, r}$ is consistent, see Corollary [1]. Setting $\xi = 0$ we additionally found a choice of the parameter $k \in \mathbb{R}_+$ which makes the estimator minimax optimal over the Mellin-Sobolev ellipsoids $\mathbb{D}^s(L)$, compare Theorem [1]. We want to emphasize that the latter choice of $k \in \mathbb{R}_+$ might not be explicitly dependent on the exact unknown density $f$ but is still dependent on its regularity parameter $s \in \mathbb{R}_+$ which again is unknown.

We will now present a data-driven version of the estimator $\hat{f}_{\xi, k}$ only dependent on the sample $(Y_i)_{i \in [n]}$. For the data-driven choice of $k \in \mathbb{R}_+$ we will use a version of the Goldenshtuler-Lepski method. That is, we will define the random functions $\hat{A}, \hat{V} : \mathbb{R}_+ \to \mathbb{R}_+$ for $k \in \mathbb{R}_+$ by

$$\hat{A}(k) := \sup_{\xi \in \mathbb{R}_+} (\|\hat{f}_{\xi / \hat{A}(k)}\|_2^2 - \chi_1 \hat{V}(k))$$

and $\hat{V}(k) := 2\gamma/\gamma(\|R\|_2^2 n^{-1})$ for $\tilde{\chi} \in \mathbb{R}_+$ and $\mathcal{K}_c := \{k \in \mathbb{N} : \|R\|_2^2 n \leq 1\}$. Here $a \wedge b := \min(a, b)$ and $a_+ := \max(a, 0)$ for any real numbers $a, b \in \mathbb{R}$.

Generally, the random function $\hat{V}$ is an empirical version of $V(k) := \sigma_\gamma(\|R\|_2^2 n^{-1})$ which mimics the behaviour of the variance term, compare Proposition [1]. Analogously, $\hat{A}$ is an empirical version of $A(k) := \sup_{\xi \in \mathbb{R}_+} (\|\hat{f}_{\xi} - \hat{f}_{\xi / \hat{A}(k)}\|_2^2 - \chi_1 V(k))$, which behaves like the bias term. For $\hat{\chi}_2 \geq \hat{\chi}_1$ we then set

$$\hat{k} := \arg \min_{k \in \mathcal{K}_c} \hat{A}(k) + \hat{V}(\hat{k}).$$

Then we can show the following result where we denote by $\|h\|_\infty$ the essential supremum of a measurable function $h : \mathbb{R} \to \mathbb{C}$ and $\|h\|_{\infty, \chi, \gamma}$ the essential supremum of $x \mapsto x^{2\gamma - 1} h(x)$.

Theorem 3. Let $c \in \mathbb{R}$ and $f \in \mathbb{L}^2(\mathbb{R}_+, x^{2\gamma - 1})$. Assume that $\mathbb{E}_f(Y_1^{5(\epsilon - 1)}) < \infty$, $\|g\|_{\infty, \chi, \gamma} < \infty$ and [GI] is fulfilled. Then for $\hat{\chi}_2 \geq \hat{\chi}_1 \geq 72$,

$$\mathbb{E}_{\hat{f}_k}(\|\hat{f}_k - f\|_2^2) \leq C_1 \inf_{k \in \mathcal{K}_c} (\|\hat{f}_{\xi} M_{\xi}(f)\|_2^2 + V(k)) + \frac{C_2}{n},$$

where $C_1$ is a positive constant depending on $\hat{\chi}_2, \hat{\chi}_1$ and $C_2$ is a positive constant depending on $\mathbb{E}_{\hat{f}_k}(Y_1^{5(\epsilon - 1)}), \|g\|_{\infty, \chi, \gamma}$, $g$ and $r$.

Assuming now that the density lies in a Mellin-Sobolev ellipsoid, we can deduce directly the following corollary whose proof is thus omitted.
Corollary 2. Let \( c \in \mathbb{R}, s, L \in \mathbb{R}_+ \) and \( f \in D_c^c(L) \). Assume further that \( \mathbb{E}_{\hat{f}^c}(Y_1^{c(1)}) < \infty, \|g\|_{\infty,c-1} < \infty \) and \([G1]\) is fulfilled. Then for \( \chi_2 \geq \chi_1 \geq 72, \)

\[
\mathbb{E}_{\hat{f}^c}(\|f_k - f\|^2_{\infty}) \leq C(L, s, r, g, \mathbb{E}_{\hat{f}^c}(X_1^{c(1)})) n^{-2s/(2s+2y+1)}
\]

where \( C(L, s, r, g, \mathbb{E}_{\hat{f}^c}(X_1^{c(1)})) \) is a positive constant depending on \( L, s, r, g \) and \( \mathbb{E}_{\hat{f}^c}(X_1^{c(1)}) \).

Conclusion

Let us summarise the presented results of the ridge estimator \( \hat{f}_{\chi,\hat{\rho}} \) in comparison to the properties of the spectral cut-off estimator \( \hat{f}_{\chi} \), considered by [6] and [5]. For the definition of the estimator, the spectral cut-off estimator needs the assumption \([G0]\). This assumption already implies the existence of a consistent version of the spectral cut-off estimator. For the definition of the ridge estimator the assumption \([G0]\) is not necessary. Nevertheless, in order to show that there exists a consistent version of the ridge estimator, we needed assumption \([G-1]\), which is weaker than \([G0]\). In this scenario, the estimator \( \hat{f}_{\chi,\hat{\rho}} \) seems to be favourable if one aims to consider minimal assumptions on the error density, for instance to construct a strong consistent estimator, compare Remark 1. As soon as we are interested in developing the minimax theory of the estimators, the assumption \([G1]\) is natural to be considered. It is worth pointing out, that \([G1]\) implies \([G0]\) and therefore \([G-1]\). Here the assumptions of Proposition 1, which are needed for the minimax optimality of both estimators, are identical to the assumptions of [5]. Thus none of the estimators seem to be more favourable in terms of minimax-optimality. Again, for the data-driven estimators \( \hat{f}_k \) and \( \hat{f}_{\chi,\hat{\rho}} \), proposed by [6], the assumptions on the error densities are identical. Here it should be mentioned that the authors [6] have proven the case \( c = 1 \). The general case for \( c \in \mathbb{R} \) can be easily shown using the same strategies as in the proof of Theorem 3. In total, we can say that for the construction of an estimator with minimal assumption on the error density \( g \), the ridge estimator seems to be favourable, in the sense, that it requires weaker assumptions on \( g \). As soon as we consider smooth error densities, that is under assumption \([G1]\), neither the ridge estimator nor the spectral cut-off estimator seems to be more favourable in terms of minimax-optimality and data-driven estimation.

4. Numerical study

In this section, we illustrate the behaviour of the data-driven ridge estimator \( \hat{f}_{\chi,\hat{\rho}} \) presented in eq. 5 and 6 and compare it with the spectral cut-off estimator \( \hat{f}_{\chi,\hat{\rho}} \) presented in [6], where

\[
\tilde{k} = \arg \min_{k \in \mathbb{R}_+} \|\hat{f}(\tilde{k}, \hat{\rho})\|^2_{\infty} + \tilde{\rho}n(k)
\]

with \( \tilde{\rho}n(k) := 2\chi\tilde{\rho}\|I_{[k,\tilde{k}]}M_{[g]^{-1}}\|^2_{\infty}/(2\pi n) \) and \( \tilde{\rho} := \{ k \in \mathbb{N} : \|I_{[\tilde{k},\tilde{k}]}M_{[g]^{-1}}\|^2_{\infty} \leq 2\pi n \}. \) To do so, we use the following examples for the unknown density \( f \),

(i) **Beta Distribution:** \( f(x) = B(2, 5)^{-1}x(1-x)^{4}I_{[0,1]}(x), x \in \mathbb{R}_+ \),

(ii) **Log-Gamma Distribution:** \( f(x) = 5^4\Gamma(5)^{-1}x^{-6}\log(x)^4I_{[1,\infty)}(x), x \in \mathbb{R}_+ \),

(iii) **Gamma Distribution:** \( f(x) = \Gamma(5)^{-1}x^{4}\exp(-x)I_{[0,\infty)}(x), x \in \mathbb{R}_+ \), and

(iv) **Log-Normal Distribution:** \( f(x) = (0.08\pi x^2)^{-1/2}\exp(-\log(x)^2/0.08)I_{[0,\infty)}(x), x \in \mathbb{R}_+ \).

A detailed discussion of these examples in terms of the decay of their Mellin transform can be found in [5]. To visualize the behaviour of the estimator, we use the following examples of error densities \( g \),

a) **Symmetric noise:** \( g(x) = I_{[0.5,1.5]}(x), x \in \mathbb{R}_+ \), and

b) **Beta Distribution:** \( g(x) = 2xI_{[0,1]}(x), x \in \mathbb{R}_+ \).

Here it is worth pointing out that the example a) and b) fulfill \([G1]\) with \( \gamma = 1 \) and \( \gamma = 2 \). By minimising an integrated weighted squared error over a family of histogram densities with randomly drawn partitions and weights we select for a) \( \chi_1 = \chi_2 = 72 \) for \( \hat{f}_{\chi,\hat{\rho}} \) and \( \chi = 5 \) for \( \hat{f}_{\chi,\hat{\rho}} \). For the case b) we choose \( \chi_1 = \chi_2 = 6 \) and \( \chi = 3 \). In both cases, we have set \( r = 2 \).
Fig. 1: The estimator $\hat{f}$ (top) and $\tilde{f}$ (bottom) is depicted for 50 Monte-Carlo simulations with sample size $n = 2000$ in the case $(i)$ under the error density $a$ (left) and $b$ (right) for $c = 1$. The true density $f$ is given by the black curve while the red curve is the point-wise empirical median of the 50 estimates.

Figure 1 shows that both estimators behave similarly. As suggested by the theory, the reconstruction of the density $f$ from the observation $(Y_j)_{j\in\mathbb{N}}$ seems to be less difficult if the error variable is uniformly distributed, case $a)$, than if the error variable is Beta distributed, case $b)$.
Fig. 2: The estimator \( \hat{f} \quad \hat{k} \) (top) and \( \tilde{f} \tilde{k} \) (bottom) is depicted for 50 Monte-Carlo simulations with sample size \( n = 2000 \) in the case (i) under the error density \( a_0 \) for \( c = 0 \) (left), \( c = 1/2 \) (middle) and \( c = 1 \) (right). The true density \( f \) is given by the black curve while the red curve is the point-wise empirical median of the 50 estimates.

Again we see that both estimators react analogously to varying values of the model parameter \( c \in \mathbb{R} \). Looking at the medians in Figure 2, for \( c = 0 \) the median seems to be closer to the true density for smaller values of \( x \in \mathbb{R}^+ \). For \( c = 1 \) the opposite effects seems to occur. For \( c = 1/2 \), the case of the unweighted \( L^2 \)-distance, such effects cannot be observed. Regarding the risk, this seems natural as the weight function for \( c = 0 \) is monotonically decreasing, while for \( c = 1 \) it is monotonically increasing.

| Case | Sample size | (i) 500 | (i) 2000 | (ii) 500 | (ii) 2000 | (iii) 500 | (iii) 2000 | (iv) 500 | (iv) 2000 |
|------|-------------|--------|----------|----------|----------|----------|----------|---------|---------|
| a)   | Ridge       | 0.94   | 0.31     | 2.17     | 1.54     | 0.63     | 0.17     | 7.13    | 2.38    |
|      | Spectral    | 1.10   | 0.38     | 2.03     | 1.26     | 0.52     | 0.16     | 15.07   | 2.34    |
| b)   | Ridge       | 2.32   | 1.43     | 5.90     | 3.81     | 1.19     | 0.47     | 25.84   | 11.03   |
|      | Spectral    | 3.95   | 1.56     | 10.63    | 7.12     | 1.52     | 0.84     | 33.95   | 13.45   |

Table 1: The entries showcase the MISE (scaled by a factor of 100) obtained by Monte-Carlo simulations each with 500 iterations. We take a look at different densities \( f \) and \( g \), two distinct sample sizes and for both estimators \( \hat{f} \hat{k} \) and \( \tilde{f} \tilde{k} \) we set \( c = 1 \).

### Appendix A. Preliminary

We will start by defining the Mellin transform for square-integrable functions \( h \in L^2(\mathbb{R}^+, x^{2c-1}) \) and collect some of its major properties. Proof sketches for all the mentioned results can be found in [6], respectively [5].

**The Mellin transform.** To define the Mellin transform of a square-integrable function, that is for \( h \in L^2(\mathbb{R}^+, x^{2c-1}) \), we make use of the definition of the Fourier-Plancherel transform. To do so, let \( \varphi : \mathbb{R} \to \mathbb{R}^+, x \mapsto \exp(-2\pi x) \) and
\(\varphi^{-1} : \mathbb{R} \to \mathbb{R}\) be its inverse. Then, as diffeomorphisms, \(\varphi, \varphi^{-1}\) map Lebesgue null sets on Lebesgue null sets. Thus the isomorphism \(\Phi_c : \mathbb{L}^2(\mathbb{R}_+, x^{2\nu-1}) \to \mathbb{L}^2(\mathbb{R}), h \mapsto \varphi^{-1}(\varphi \circ h)\) is well-defined. Moreover, let \(\Phi^{-1}_c : \mathbb{L}^2(\mathbb{R}) \to \mathbb{L}^2(\mathbb{R}_+, x^{2\nu-1})\) be its inverse. Then for \(h \in \mathbb{L}^2(\mathbb{R}_+, x^{2\nu-1})\) we define the Mellin transform of \(h\) developed in \(c \in \mathbb{R}\) by

\[
\mathcal{M}_c[h](t) := (2\pi i)^{\nu} \Phi_c[h](t), \quad t \in \mathbb{R},
\]

where \(\mathcal{F} : \mathbb{L}^2(\mathbb{R}) \to \mathbb{L}^2(\mathbb{R}), H \mapsto (t \mapsto \mathcal{F}[H](t) := \lim_{\delta \to \infty} \int_{-\delta}^{\delta} \exp(-2\pi itx)H(x)dx)\) is the Plancherel-Fourier transform. Due to this definition several properties of the Mellin transform can be deduced from the well-known theory of Fourier transforms. In the case that \(h \in \mathbb{L}^1(\mathbb{R}_+, x^{-1}) \cap \mathbb{L}^2(\mathbb{R}_+, x^{2\nu-1})\) we have

\[
\mathcal{M}_c[h](t) = \int_0^\infty x^{-1+\nu} h(x)dx, \quad t \in \mathbb{R}
\]

which coincides with the usual notion of Mellin transforms as considered in [15].

Now, due to the construction of the operator \(\mathcal{M}_c : \mathbb{L}^2(\mathbb{R}_+, x^{2\nu-1}) \to \mathbb{L}^2(\mathbb{R})\) it can easily be seen that it is an isomorphism. We denote by \(\mathcal{M}_c^{-1} : \mathbb{L}^2(\mathbb{R}) \to \mathbb{L}^2(\mathbb{R}_+, x^{2\nu-1})\) its inverse. If additionally to \(H \in \mathbb{L}^2(\mathbb{R}), H \in \mathbb{L}^1(\mathbb{R})\), we can express the inverse Mellin transform explicitly through

\[
\mathcal{M}_c^{-1}[H](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x^{-1-\nu} H(t)dt, \quad x \in \mathbb{R}_+.
\]

Furthermore, we can directly show that a Plancherel-type equation holds for the Mellin transform, that is for all \(h_1, h_2 \in \mathbb{L}(\mathbb{R}_+, x^{2\nu-1})\),

\[
(h_1, h_2)_{x^{\nu-1}} = (2\pi)^{-1} \langle \mathcal{M}_c[h_1], \mathcal{M}_c[h_2] \rangle \quad \text{whence} \quad ||h||_{2, \nu-1}^2 = (2\pi)^{-1} ||\mathcal{M}_c[h]||_2^2.
\]

**Useful Inequality.** The following inequality is due to [16], the formulation can be found for example in [13].

**Lemma 1.** (Talagrand’s inequality) Let \(X_1, \ldots, X_n\) be independent \(\mathbb{Z}\)-valued random variables and let

\[
\bar{v}_h = n^{-1} \sum_{i=1}^n [v_h(X_i) - \mathbb{E}(v_h(X_i))]
\]

for \(v_h\) belonging to a countable class \(\{v_h, h \in \mathcal{H}\}\) of measurable functions. Then,

\[
\mathbb{E}(\sup_{h \in \mathcal{H}} \bar{v}_h^2) - 6\Psi^2/\nu \leq C \left\langle \frac{\tau}{n} \psi^2 \exp \left(\frac{-n\Psi^2}{6\tau}\right) + \frac{\psi^2}{n^2} \exp \left(\frac{-Kn\Psi}{\psi}\right) \right\rangle (A.4)
\]

with numerical constants \(K = (\sqrt{2} - 1)/(2\sqrt{2})\) and \(C > 0\) and where

\[
\sup_{h \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^n \mathbb{V}(v_h(X_i)) \leq \tau.
\]

**Remark 2.** Keeping the bound Eq. (A.4) in mind, let us specify particular choices \(K\), in fact \(K \geq \frac{1}{100}\). The next bound is now an immediate consequence,

\[
\mathbb{E}(\sup_{h \in \mathcal{H}} \bar{v}_h^2) - 6\Psi^2/\nu \leq C \left\langle \frac{\tau}{n} \psi^2 \exp \left(\frac{-n\Psi^2}{6\tau}\right) + \frac{\psi^2}{n^2} \exp \left(\frac{-n\Psi}{100\psi}\right) \right\rangle (A.5)
\]

In the sequel we will make use of the slightly simplified bounds Eq. (A.5) rather than Eq. (A.4).
Appendix B. Proofs of Section 2

Proof of Proposition 7 First we see that
\[ E_{\rho_f}^n \| f - \hat{f}_{\rho^f} \|_{L^2_{\rho^f}}^2 = \| f - \hat{f}_{\rho^f} \|_{L^2_{\rho^f}}^2 + \frac{1}{2\pi} \int_{-\infty}^{\infty} \| \hat{f}_{\rho^f} \|_{L^2_{\rho^f}}^2 dt \]
using the Plancherel equality, compare eq [A.3] and the Fubini-Tonelli theorem. Considering the bias term, we have for \( t \in G_k \), \( R_{k,\rho^f}(t) = M_t(g)(t)^{-1} \). On the other hand, for \( t \in \mathbb{R} \) we have
\[ |R_{k,\rho^f}(t)| \leq \max(|M_t(g)(t)|, |t|/\gamma)^{-1} \leq |M_t(g)(t)|^{-1}. \]
Now the Plancherel equality implies
\[ \| f - E_{\rho_f}^n \|_{L^2_{\rho^f}}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |M_t(g)(t)R_{k,\rho^f}(t) - 1|^2 |M_t(f)(t)|^2 dt \]
which proves the proposition.

Proof of Corollary 7 To show the corollary, it is sufficient to show that \( \| R_k \|_{L^2}^2 \leq C_{g,k} k^{2\gamma-1} \). In detail, we have
\[ \| R_k \|_{L^2}^2 = \| R_k \|_{G_1}^2 + \| R_k \|_{G_2}^2 = \| \mathbb{I}_{G_1} M_t(g)^{1+\gamma} \|_{L^2}^2 + \| \mathbb{I}_{G_2} M_t(g) \|_{L^2}^2 \]
using the assumption [G1] and for \( r > 0 \) \((\gamma - 1) \). The latter restriction ensures that \( M_t(g)^{1+\gamma} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \). Since \( g \) fulfill [G1] we can find positive constants \( C_{g,1}, C_{g,2} > 0 \) only depending on \( g \) such that the sets \( G_{k,i} := \mathbb{R} \setminus [-C_{g,k}^{-\gamma}G_{k,1}, C_{g,k}^{-\gamma}G_{k,2}] \) for \( i = 1, 2 \) satisfy the inclusion relationship
\[ G_{k,1} \subseteq G_k \subseteq G_{k,2}. \]
For the first summand we get that
\[ k^{2(r+2)} \| \mathbb{I}_{G_{k,1}} M_t(g)^{1+\gamma} \|_{L^2}^2 \leq C(g, L, r) k^{2(r+2)} \int_{C_{k,1}^{k^{1+\gamma}}} t^{-2\gamma(r+1)} dt = C_{g,k} k^{2\gamma-1} \]
since \( \gamma(r + 1) > 1 \). For the second summand we get
\[ \| \mathbb{I}_{G_{k,2}} M_t(g)^{1+\gamma} \|_{L^2}^2 \leq \int_{C_{k,2}^{k^{1+\gamma}}} |M_t(g)(t)|^{-2} dt \leq C_{g} k^{2\gamma-1}. \]

Proof of Theorem 7 First we see that for \( f \in \mathbb{D}_n^0(L) \)
\[ \| \mathbb{I}_{G_1} M_t(f) \|_{L^2}^2 \leq \| \mathbb{I}_{G_2} M_t(f) \|_{L^2}^2 \leq \frac{1}{\pi} \int_{C_{k,2}^{k^{1+\gamma}}} |M_t(f)(t)|^2 dt \leq C(g, L) k^{-2\gamma} \]
staying in the notation of the proof of Corollary 7 Further, we have that \( \sigma_c = E_f(A^{2(1-c)}) E_q(U^{2(1-c)}_1) \leq C(L, g) \). In total we get
\[ E_{\rho_f}^n (\| f - \hat{f}_{\rho^f} \|_{L^2_{\rho^f}}^2) \leq C(g, L) r(k^{-2\gamma} + k^{2\gamma-1} n^{-1}), \]
where both summands are balanced by the choice \( k_0 = n^{1/(2(2\gamma+1))} \).
Appendix C. Proofs of Section 3

Proof of Theorem 3. The proof can be split in two main steps. The first one using a sequence of elementary steps to find a controllable upper bound for the risk of the data-driven estimator. In the second step, we use mainly the Talagrand inequality to show the claim of the theorem. These two steps are expressed through the following lemmata which we state first and prove afterwards.

Lemma 2. Under the assumptions of Theorem 3, we have for any $k \in \mathcal{K}_0$,

$$
\mathbb{E}_f^n (\|f_k - f\|_2^2) \leq C(\chi_1, \chi_2) \left( (\|f_k - f\|_2^2 + V(k)) + 108\mathbb{E}_f^n (\sup_{k \in \mathcal{K}_c} (\|f_k - f\|_2^2 - \frac{\chi_1}{6} V(k')) \right)
$$

for positive constants $C(\chi_1, \chi_2)$ and $C(\chi_1)$ only depending on $\chi_1$ and $\chi_2$ and $f_k := \mathbb{E}_f^n (f_k)$.

To be able to apply the Talagrand inequality on the term $\mathbb{E}_f^n (\sup_{k \in \mathcal{K}_c} (\|f_k - f\|_2^2 - \frac{\chi_1}{6} V(k'))$ we need to split the process first. To do so, let us define the set $U := \{ h \in \mathbb{L}^2(\mathbb{R}^+, x^{2\alpha-1}) : \|h\|_2 \leq 1 \}$. Then for $k \in \mathcal{K}_0$ we have $\|f_k - f\|_2 = \sup_{h \in U} \langle f_k - f, h \rangle_{2\alpha-1}$ where

$$
\langle f_k - f, h \rangle_{2\alpha-1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathcal{M}_k(t) - \mathbb{E}_f^n (\mathcal{M}_k(t))) R_{k, \alpha}(t) |h|(-t) dt
$$

by application of the Plancherel equation, eq. A.3. Now for a positive sequence $(c_i)_{i \in \mathbb{N}}$ we decompose the empirical Mellin transform $\mathcal{M}_k(t), t \in \mathbb{R}$, into

$$
\mathcal{M}_k(t) := n^{-1} \sum_{j=1}^{n} \frac{c_j}{Y_j} I_{\{\alpha, \omega\}}(Y_j) + n^{-1} \sum_{j=1}^{n} \frac{c_j}{Y_j} I_{\{\alpha, \omega\}}(Y_j)
$$

$$
= \mathcal{M}_{k,1}(t) + \mathcal{M}_{k,2}(t).
$$

Setting

$$
v_{k,i}(h) := \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathcal{M}_{k,i}(t) - \mathbb{E}_f^n (\mathcal{M}_{k,i}(t))) R_{k, \alpha}(t) |h|(-t) dt
$$

for $h \in U, i \in \{1, 2\}$, we can deduce that

$$
\mathbb{E}_f^n (\sup_{k \in \mathcal{K}_c} (\|f_k - f\|_2^2 - \frac{\chi_1}{6} V(k))) \leq 2\mathbb{E}_f^n (\sup_{k \in \mathcal{K}_c, h \in U} v_{k,1}(h)^2 - \frac{\chi_1}{12} V(k)) + 2\mathbb{E}_f^n (\sup_{h \in U} v_{k,2}(h)^2). \tag{C.1}
$$

This decomposition and the following Lemma then proves the claim.

Lemma 3. Under the assumptions of Theorem 3,

(i) $\mathbb{E}_f^n (\sup_{k \in \mathcal{K}_c, h \in U} v_{k,1}(h)^2 - \frac{\chi_1}{12} V(k)) \leq \frac{C(g, r, \mathbb{E}_f (X_1^{2(\varepsilon-1)})}{n}$,

(ii) $\mathbb{E}_f^n (\sup_{h \in U} v_{k,2}(h)^2) \leq \frac{C(\sigma^*, \mathbb{E}_f (Y_1^{4(\varepsilon-1)})}{n}$ and

(iii) $\mathbb{E}_f^n (\sup_{k \in \mathcal{K}_c} (\mathcal{M}_k(t) - V(k))) \leq \frac{C(\sigma^*, \mathbb{E}_f (Y_1^{4(\varepsilon-1)})}{n}$.

$\square$

12
Proof of Lemma If \( \chi_2 \geq \chi_1 \) and by the definition of \( \tilde{k} \) follows for any \( k \in \mathcal{K}_n \),
\[
\|f - \hat{f}_k\|_{\infty}^2 \leq 3\|f - \hat{f}_k\|_{2,\infty}^2 + 3\|\hat{f}_k - \hat{f}_k\|_{2,\infty}^2 + 3\|\hat{f}_k - \hat{f}_k\|_{2,\infty}^2.
\]
To simplify the notation, let us set \( \chi := (\chi_1 + \chi_2)/2 \). Let us now have a closer look at \( \tilde{A}(k) \). From
\[
\|f - \hat{f}_k\|_{2,\infty}^2 \leq 3\|f - \hat{f}_k\|_{2,\infty}^2 + 3\|\hat{f}_k - \hat{f}_k\|_{2,\infty}^2 + 3\|\hat{f}_k - \hat{f}_k\|_{2,\infty}^2
\]
we conclude by a straightforward calculation that
\[
\tilde{A}(k) \leq 6 \max_{k \in \mathcal{K}_n} \left( \|f - \hat{f}_k\|_{2,\infty}^2 - \frac{\chi_1}{6} V(k') \right) + 3\|f - \hat{f}_k\|_{2,\infty}^2 + \chi_1 \max_{k \in \mathcal{K}_n} (V(k') - \tilde{V}(k')).
\]
This implies
\[
\mathbb{E}^n_{f_t} \left( \mathbb{E}^n_{f_t} \left( \sup_{k \in \mathcal{K}_n} \left( |\hat{M}_n, \hat{M}_n| \right) \right) \right) \leq \sum_{k=1}^{K_n} \mathbb{E}^n_{f_t} \left( \mathbb{E}^n_{f_t} \left( \sup_{k \in \mathcal{K}_n} \left( |\hat{M}_n, \hat{M}_n| \right) \right) \right)
\]
where \( K_n := \max(\mathcal{K}_n) \). To apply now the Talagrand inequality, compare Lemma 1 to each summand we need to determine the constants \( \Psi^2, \psi^2 \) and \( r \) first. Staying in the notation of the Talagrand inequality, we set for \( h \in \mathbb{R} \),
\[
v(y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-1+it} \mathbf{1}_{\{0,\infty\}}(y) R_k(t) M_L(h)(-t) dt, \quad y \in \mathbb{R}.
\]
Now applying Cauchy-Schwarz inequality
\[
v_{1,1}^2 \leq \frac{\|h\|_{2,\infty}^2}{2\pi} \int_{-\infty}^{\infty} |\hat{M}_L(1) - \mathbb{E}^n_{f_t}(\hat{M}_L(1))|^2 |R_k(t)|^2 dt \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{M}_L(1) - \mathbb{E}^n_{f_t}(\hat{M}_L(1))|^2 |R_k(t)|^2 dt \leq \sigma_t \|R_k\|_{L^2}^2.
\]
Since \( h \in \mathbb{R} \). We deduce that
\[
\mathbb{E}^n_{f_t} (\sup_{k \in \mathcal{K}} v_{1,1}(h)^2) \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbb{E}^n_{f_t} (|\hat{M}_L(1) - \mathbb{E}^n_{f_t}(\hat{M}_L(1))|^2) |R_k(t)|^2 dt \leq \sigma_t \|R_k\|_{L^2}^2.
\]
compare proof of Proposition 1. For \( y > 0 \) we have \( |v_{1,1}(y)|^2 \leq c_2^2 \|R_k\|_{L^2}^2 |\mathbb{M}_L(h)|^2/(2\pi) \leq c_2^2 \|R_k\|_{L^2}^2 =: \psi^2 \) since \( h \in \mathbb{R} \). Additionally, we have for any \( h \in \mathbb{R} \) that \( \mathbb{V}_{1,1}(v_{1,1}(y_1)) \leq \mathbb{E}^n_{f_t}(v_{1,1}(y_1))^2 \leq \|f_t\|_{L^{2,\infty}} |v_{1,1}|_{L^{2,\infty}}^2. \) More precisely, we see that
\[
y^{2-1} \int_0^\infty f(x)g(y/x)x^{-1} dx \leq \|g\|_{L^{2,\infty}} \mathbb{E}^n(\mathbb{X}^{2k-1}), \quad y \in \mathbb{R}.
\]
Next, we have
\[ \|v_k\|_{L^2}^2 \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{M}_k[h](t)|^2 |R_k(t)|^2 dt \leq \|R_k\|_{L^\infty}^2 \frac{1}{2\pi} \|\mathcal{M}_k[h]\|_{L^2}^2 \leq \|R_k\|_{L^\infty}^2 \]
which implies the choice \( \tau := \|g\|_{L^{\infty}} (c_2) = \frac{1}{2} \|\mathcal{M}_k[h]\|_{L^2}^2 \). Applying now the Talagrand inequality we get
\[
\mathbb{E}_f (\sup_{h \in U} v_{k,1}(h)^2) \leq \sum_{k=1}^{n} \mathbb{E}_f (\sup_{h \in U} v_{k,1}(h)^2) \leq \frac{C_{f_k}}{n} \left( \|R_k\|_{L^\infty}^2 \|\mathcal{M}_k[h]\|_{L^2}^2 + n^{-1} \right)
\]
for the choice \( c_n := \sqrt{\frac{\log(n)}{100}} \). Following the same step as in the proof of \( \text{(i)} \) we can state that \( K_n \leq C_{g,n} n^{\theta/(2\gamma+1)} \leq C_{g,n} n^1 \). For \( \chi_1 \geq 72 \) we can conclude that
\[
\mathbb{E}_f (\sup_{h \in U} v_{k,1}(h)^2) \leq \frac{C_{f_k}}{n} \left( \|R_k\|_{L^\infty}^2 \|\mathcal{M}_k[h]\|_{L^2}^2 + n^{-1} \right)
\]

Now it can easily be seen that there exists constants \( c_{g,r}, c_{g,r} > 0 \) such that \( c_{g,r} k^{2\gamma-1} \leq \|R_k\|_{L^2}^2 \leq c_{g,r} k^{2\gamma-1} \) using \( [G1] \). By simple calculus, one can show that \( \|R_k\|_{L^2}^2 \leq C_{g,r} n^2 \). Since \( (k^2 \exp(-C_{f_k} k^{2\gamma}))_{k \in \mathbb{N}} \) is summable we can deduce that \( \mathbb{E}_f (\sup_{h \in U} v_{k,1}(h)^2 - \frac{k^2}{12} V(k)) \leq C_{f_k} n^{-1} \).

Let us now show part (ii): For any \( h \in U \) and \( k \in \mathcal{K}_n \) we get
\[
v_{k,2}(h)^2 \leq \frac{\|h\|_{L^2}^2}{2\pi} \int_{-\infty}^{\infty} |\hat{\mathcal{M}}_k[h](t)|^2 |R_k(t)|^2 dt \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{\mathcal{M}}_k[h](t)|^2 |\hat{\mathcal{M}}_k[h](t)|^2 |R_k(t)|^2 dt
\]
and thus, since \( R_k(t) \geq R(t) \) for all \( t \in \mathbb{R} \) and \( k \geq \ell \),
\[
\mathbb{E}_f (\sup_{k \in \mathcal{K}_n} v_{k,2}(h)^2) \leq \frac{1}{n} \|R_k\|_{L^2}^2 \mathbb{E}_f (V_k)^{2(c-1)} 1_{\{c_n > 0\}}(Y_k^{2(c-1)}).
\]

Now by definition of \( \mathcal{K}_n \) we know that \( \|R_k\|_{L^2}^2 c_n^{-1} \leq 1 \). We deduce that for any \( p > 0 \)
\[
\mathbb{E}_f (\sup_{k \in \mathcal{K}_n} v_{k,2}(h)^2) \leq \frac{C(p) \mathbb{E}_f (V_k)^{2(c-1)} 1_{\{c_n > 0\}}(Y_k^{2(c-1)})}{n}
\]
choosing \( p = 3 \) and by the definition of \( (c_n)_{n \in \mathbb{N}} \).

Part (iii): First we see that for any \( k \in \mathcal{K}_n \) \( (V_k - \hat{V}(k))_+ = \|R_k\|_{L^2}^2 n^{-1} (\sigma_c - 2\bar{\sigma}_c)_+ \leq (\sigma_c - 2\bar{\sigma}_c)_+ \). On \( \Omega = [\sigma_c - \sigma_c] \leq \sigma_c/2 \) we have \( \bar{\sigma}_c - \sigma_c = \frac{3}{4} \sigma_c \). This implies
\[
\mathbb{E}_f (\sup_{k \in \mathcal{K}_n} (V_k - \hat{V}(k))_+) \leq \frac{c_n}{n} \mathbb{E}_f (V_k)^{2(c-1)} 1_{\{c_n > 0\}}(Y_k^{2(c-1)}) \leq \frac{4\bar{\sigma}_c}{\sigma_c} \]
applying the Cauchy Schwartz inequality and the Markov inequality. Now the last inequality implies the claim. \( \Box \)

References

[1] K. E. Andersen, M. B. Hansen, Multiplicative censoring: density estimation by a series expansion approach, J. Statist. Plann. Inference 98 (2001) 137–155.
[2] M. Asgharian, D. B. Wolfson, Asymptotic behavior of the unconditional NPMLE of the length-biased survivor function from right censored prevalent cohort data, Ann. Statist. 33 (2005) 2109–2131.

[3] D. Belomestny, F. Comte, V. Genon-Catalot, Nonparametric Laguerre estimation in the multiplicative censoring model, Electron. J. Stat. 10 (2016) 3114–3152.

[4] D. Belomestny, A. Goldenshluger, Nonparametric density estimation from observations with multiplicative measurement errors, Ann. Inst. Henri Poincaré Probab. Stat. 56 (2020) 36–67.

[5] S. Brenner Miguel, Anisotropic spectral cut-off estimation under multiplicative measurement errors, arXiv preprint arXiv:2107.02120 (2021).

[6] S. Brenner Miguel, F. Comte, J. Johannes, Spectral cut-off regularisation for density estimation under multiplicative measurement errors, Electronic Journal of Statistics 15 (2021) 3551 – 3573.

[7] S. Brenner Miguel, Phandoidaen, Multiplicative deconvolution in survival analysis under dependency, arXiv preprint arXiv:2107.05267 (2021).

[8] E. Brunel, F. Comte, V. Genon-Catalot, Nonparametric density and survival function estimation in the multiplicative censoring model, TEST 25 (2016) 570–590.

[9] F. Comte, C. Dion, Nonparametric estimation in a multiplicative censoring model with symmetric noise, J. Nonparametr. Stat. 28 (2016) 768–801.

[10] B. van Es, C. A. J. Klaassen, K. Oudshoorn, Survival analysis under cross-sectional sampling: length bias and multiplicative censoring, volume 91, Prague Workshop on Perspectives in Modern Statistical Inference: Parametrics, Semi-parametrics, Non-parametrics (1998).

[11] J. Fan, On the optimal rates of convergence for nonparametric deconvolution problems, Ann. Statist. 19 (1991) 1257–1272.

[12] P. Hall, A. Meister, A ridge-parameter approach to deconvolution, Ann. Statist. 35 (2007) 1535–1558.

[13] T. Klein, E. Rio, Concentration around the mean for maxima of empirical processes, Ann. Probab. 33 (2005) 1060–1077.

[14] A. Meister, Deconvolution problems in nonparametric statistics, volume 193 of Lecture Notes in Statistics, Springer-Verlag, Berlin, 2009.

[15] R. B. Paris, D. Kaminski, Asymptotics and Mellin-Barnes integrals, volume 85 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2001.

[16] M. Talagrand, New concentration inequalities in product spaces, Invent. Math. 126 (1996) 505–563.

[17] Y. Vardi, Multiplicative censoring, renewal processes, deconvolution and decreasing density: nonparametric estimation, Biometrika 76 (1989) 751–761.

[18] Y. Vardi, C.-H. Zhang, Large sample study of empirical distributions in a random-multiplicative censoring model, Ann. Statist. 20 (1992) 1022–1039.