Conformally Coupled Scalar in Rotating Black Hole Spacetimes

Finnian Gray,1,2∗ Ian Holst,1,2† David Kubizňák,1,2‡ Gloria Odak,1,2§ Dalila M. Pirvu,1,2¶ and Tales Rick Perche1,2,3,**

1Perimeter Institute, 31 Caroline Street North, Waterloo, ON, N2L 2Y5, Canada
2Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1
3Instituto de Física Teórica, Universidade Estadual Paulista, São Paulo, São Paulo, 01140-070, Brazil

(Dated: February 12, 2020)

We demonstrate separability of conformally coupled scalar field equation in general (off-shell) Kerr–NUT–AdS spacetimes in all dimensions. The separability is intrinsically characterized by the existence of a complete set of mutually commuting conformal wave operators that can be constructed from a hidden symmetry of the principal Killing–Yano tensor. By token of conformal symmetry, the separability also works for any Weyl rescaled (off-shell) metrics. This is especially interesting in four dimensions where it guarantees separability of a conformally coupled scalar field in the general Plebański–Demiański spacetime.

I. INTRODUCTION

The four-dimensional Kerr black hole geometry possesses many remarkable properties. Among these perhaps the most intriguing is the separability of various test field equations in the Kerr black hole background. The history of “separatists” began in 1968 when Carter demonstrated that the massive Hamilton–Jacobi and Klein–Gordon equations can both be solved by a method of separation of variables [1, 2]. Soon after that the massless wave equations for vector and tensor perturbations by Teukolsky [3, 4], the massive neutrino equations by Chandrasekhar [5] and Unruh [6], the massive Dirac equation by Teukolsky [3, 4], the massless vector and tensor perturbations demonstrated in [16], the Dirac equation was separated in [17], and the harmonic p-form equations in 4, 5, 6, 7, and prefactor η is chosen so that the equation enjoys conformal symmetry, see e.g. Appendix D in [24]. As we shall demonstrate such a separability is intrinsically characterized by the existence of a complete set of commuting operators that are constructed from the tower of Killing tensors and Killing vectors generated from the principal Killing–Yano tensor of the Kerr–NUT–AdS geometry. Exploiting the conformal symmetry, the demonstrated separability remains valid for any Weyl rescaled metrics and in particular implies the separability of the conformal scalar equation in the general class of four-dimensional Plebański–Demiański spacetimes.

Our paper is organized as follows: the general off-shell Kerr–NUT–AdS spacetime and its basic properties are introduced in Sec. IV. Separability of the conformal wave equation and its intrinsic characterization in this spacetime are studied in Sec. V. The associated separability in Weyl scaled metrics is briefly discussed in Sec. VI. Sec. VII is devoted to a concrete application of these results to four-dimensional spacetimes of the Plebański–Demiański class. We summarize in Sec. VIII.
II. KERR–NUT–ADS SPACETIMES

The canonical metric describing the off-shell Kerr–NUT–(AdS) geometry in \( d = 2n + \varepsilon \) number of dimensions (with \( \varepsilon = 0 \) in even and \( \varepsilon = 1 \) in odd dimensions) reads

\[
g = g_{\alpha\beta} dx^\alpha dx^\beta = \sum_{\mu=1}^{n} \left[ \frac{U_\mu}{X_\mu} dx^2_{\mu} + \frac{X_\mu}{U_\mu} \left( \sum_{j=0}^{n-1} A^{(j)}_\mu d\psi_j \right)^2 \right] + \frac{\varepsilon c}{A^{(n)}} \left( \sum_{k=0}^{n} A^{(k)} d\psi_k \right)^2. \tag{2}
\]

The coordinates \( y^a = \{ x_\mu, \psi_k \} \) naturally split into two sets: Killing\-coordinates \( \psi_k \) \((k = 0, \ldots, n-1, \varepsilon)\) associated with the explicit symmetries, and radial and longitudinal coordinates \( x_\mu \) \((\mu = 1, \ldots, n)\) labelling the orbits of Killing symmetries. The functions \( A^{(k)}, A^{(j)}, \) and \( U_\mu \) are ‘symmetric polynomials’ of the coordinates \( x_\mu \), and are defined by:

\[
A^{(k)} = \sum_{\nu_1, \ldots, \nu_k=1}^{n} x_{\nu_1} \cdots x_{\nu_k}, \quad A^{(j)} = \sum_{\nu_1, \ldots, \nu_j=1}^{n} x_{\nu_1} \cdots x_{\nu_j}, \quad U_\mu = \prod_{\nu=1}^{\mu} (x_\nu^2 - x_\mu^2), \quad U = \prod_{\mu, \nu=1}^{n} (x_\mu^2 - x_\nu^2) = \det(A^{(j)}),
\]

\[
\text{where we have fixed } A^{(0)} = 1 = A^{(0)}. \tag{3}
\]

Each metric function \( X_\mu \) is an unspecified function of a single coordinate \( x_\mu \):

\[
X_\mu = X_\mu(x_\mu). \tag{4}
\]

The coordinates \( x_\mu \) are determined by the principal Killing–Yano tensor: the co–

\[
Y(x_\mu, \psi_k). \tag{5}
\]

The inverse metric takes the form:

\[
g^{-1} = \sum_{\mu=1}^{n} \left[ \frac{X_\mu}{U_\mu} \delta_{x_\mu} + \frac{X_\mu}{U_\mu} \left( \sum_{k=0}^{n-1} \frac{(-x^2)^{n-1-k}}{U_\mu} \partial_\psi_k \right)^2 \right] + \frac{\varepsilon c}{A^{(n)} U_\mu} \delta_{\psi_n}, \tag{5}
\]

while the square root of the determinant of the metric reads

\[
\sqrt{|g|} = (cA^{(n)})^\frac{\varepsilon}{2} U. \tag{6}
\]

Despite the complexity of the metric the Ricci scalar (calculated in \[22\]) is fairly simple given by a sum

\[
R = \sum_{\mu=1}^{n} \frac{r_\mu}{U_\mu}, \tag{7}
\]

of the functions

\[
r_\mu = -X''_{\mu} - \frac{2 \varepsilon c X'}{X_{\mu}} - \frac{2 \varepsilon c}{X_{\mu}}. \tag{8}
\]

Importantly, each \( r_\mu \) only depends on a single variable \( x_{\mu} \).

The above off-shell Kerr–NUT–AdS spacetime is the most general metric that admits the principal Killing–Yano tensor \[22, 23\]. This is a non-degenerate closed conformal Killing–Yano 2-form \( h \) obeying the equation

\[
\nabla_a h_{bc} = g_{ab} \xi_c - g_{ac} \xi_b, \quad \xi^a = \frac{1}{D-1} \nabla_b h^{ba}. \tag{9}
\]

Explicitly, the principal Killing–Yano tensor is given by

\[
h = db, \quad b = \frac{1}{2} \sum_{k=0}^{n-1} A^{(k+1)} d\psi_k. \tag{10}
\]

It generates towers of explicit and hidden symmetries, see \[13\]. Namely, we obtain the following tower of closed conformal Killing–Yano tensors:

\[
h^{(j)} = \frac{1}{j!} h \wedge \cdots \wedge h. \tag{11}
\]

Their Hodge duals are Killing–Yano tensors \( f^{(j)} = *h^{(j)} \), and their square gives rise to a tower of rank-2 Killing tensors:

\[
k^{ab} = \frac{1}{(d-2j-1)!} f^{(j)a}_{\nu_1 \cdots \nu_{d-2j-1}} f^{(j)b\nu_1 \cdots \nu_{d-2j-1}}. \tag{12}
\]

The latter take the following explicit form \((j = 0, \ldots, n-1)\):

\[
k^{ab} = \sum_{\mu=1}^{n} A^{(j)}_{\mu} \left[ \frac{X_\mu}{U_\mu} \partial_\psi_{\mu} + \frac{U_\mu}{X_\mu} \left( \sum_{k=0}^{n-1} \frac{(-x^2)^{n-1-k}}{U_\mu} \partial_\psi_k \right)^2 \right] + \frac{\varepsilon c A^{(n)}_{\mu}}{U_\mu} \partial_\psi_n, \tag{13}
\]

and generate the tower of Killing vectors:

\[
l^{(j)} = k^{(j)} \cdot \xi = \partial_\psi, \tag{14}
\]

with an additional Killing vector in odd dimensions, \( l^{(n)} = \partial_\psi \). Note that the \( j = 0 \) Killing tensor is just the inverse metric \( A^{(n)} \), and the zeroth Killing vector is the primary Killing vector, \( l^{(0)} = \xi = \partial_\psi \).

Since all of the symmetries are generated by this single object \( h \), they all mutually Schouten–Nijenhuis commute (See \[12\] for details):

\[
[l^{(i)}, k^{(j)}] = 0, \quad [l^{(i)} l^{(j)}] = 0, \quad [k^{(i)}, k^{(j)}] = k^{(i)}_c \nabla^c k^{(j)} - k^{(i)}_c \nabla^c k^{(j)} = 0. \tag{15}
\]

The canonical coordinates \( \{ x_{\mu}, \psi_k \} \) are thus completely determined by the principal Killing–Yano tensor: the coordinates \( \psi_k \) are the Killing coordinates \[13\] and the coordinates \( x_{\mu} \) are the ‘eigenvalues’ of \( h \), see \[15\] for more details.
Let us finally mention that when the vacuum Einstein equations are imposed, $G_{ab} + A_{ab} = 0$, the metric functions $X_\mu$ take the following explicit ‘polynomial’ form:

$$X_\mu = \sum_{k=\varepsilon}^n c_k x_\mu^{2k} - 2b_\mu x_\mu^{\varepsilon} - \frac{\varepsilon c}{x_\mu^2},$$

where $c_n$ is related to the cosmological constant by $\Lambda = \frac{1}{2} (-1)^d (d-1)(d-2)/c_n$, while other parameters $c_k, b_\mu, \varepsilon$ are related to rotations, mass, and NUT charges. With these we recover the on-shell Kerr–NUT–AdS metrics constructed by Chen, Lü, and Pope [13]. However, in what follows we will not restrict to this specific case and we are going to work with the general off-shell Kerr–NUT–AdS metrics.

### III. SEPARABILITY OF CONFORMAL WAVE EQUATION

#### Conformal operators

In order to separate the conformally coupled scalar field equation (11), let us first consider the following operators:

$$\tilde{K}_{(j)} = \nabla_a k^{ab} \nabla_b,$$

which explicit action on a scalar $\Phi$ reads

$$\tilde{K}_{(j)} \Phi = \nabla_a k^{ab} \nabla_b \Phi = \frac{1}{\sqrt{|g|}} \partial_\mu \left( \sqrt{|g|} k^{ab} \partial_b \Phi \right).$$

To find the coordinate form of these operators, we use [13] to obtain

$$\tilde{K}_{(j)} \Phi = \sum_{\mu=1}^n \frac{1}{\sqrt{|g|}} \partial_\mu \left( \sqrt{|g|} \frac{A^{(j)}_\mu}{U_\mu} X_\mu \partial_\mu \Phi \right)$$

$$+ \sum_{\mu=1}^n A^{(j)}_\mu \left( \sum_{k=0}^{n-1+\varepsilon} (-x_\mu^2)^{n-1-k} \partial_k \right)^2 \Phi$$

$$+ \varepsilon \frac{A^{(j)}_\mu}{A^{(n)}_\mu} \partial_\mu^2 \Phi,$$

where we have abbreviated $\partial_\mu = \partial_{x_\mu}, \partial_k = \partial_{\psi_k},$ and $\partial_n = \partial_{\psi_n}$. Employing the expression for the metric determinant [11] and the fact that neither $A^{(j)}_\mu$ nor $U_\mu$ depend on coordinate $x_\mu$, we have

$$\frac{1}{\sqrt{|g|}} \partial_\mu \left( \sqrt{|g|} \frac{A^{(j)}_\mu}{U_\mu} X_\mu \partial_\mu \Phi \right)$$

$$= \frac{A^{(j)}_\mu \partial_\mu (c A^{(n)})^{\frac{1}{n}} X_\mu \partial_\mu \Phi}{(c A^{(n)})^{\frac{1}{n}}}$$

$$= \frac{A^{(j)}_\mu}{U_\mu} \left[ \partial_\mu (X_\mu \partial_\mu \Phi) + \varepsilon \frac{X_\mu}{x_\mu} \partial_\mu \Phi \right].$$

Finally using the following identity [16]:

$$\frac{A^{(j)}_\mu}{A^{(n)}_\mu} \left[ \partial_\mu (X_\mu \partial_\mu \Phi) + \varepsilon \frac{X_\mu}{x_\mu} \partial_\mu \Phi \right]$$

we arrive at the following explicit form of these operators:

$$\tilde{K}_{(j)} \Phi = \sum_{\mu=1}^n \frac{A^{(j)}_\mu}{U_\mu} \tilde{K}_{(\mu)} \Phi,$$

where each $\tilde{K}_{(\mu)}$ involves only one coordinate $x_\mu$ and reads

$$\tilde{K}_{(\mu)} \Phi = \partial_\mu (X_\mu \partial_\mu \Phi) + \frac{1}{X_\mu} \left( \sum_{k=0}^{n-1+\varepsilon} (-x_\mu^2)^n x_\mu \partial_k \right)^2$$

$$+ \varepsilon \frac{X_\mu}{x_\mu} \partial_\mu^2 \Phi,$$

which is the form derived in [20].

Let us next consider the following scalar functions

$$R_{(j)} = \sum_{\mu=1}^n A^{(j)}_\mu r_\mu,$$

where $r_\mu$ is given by Eq. (8), and upgrade the operators $\tilde{K}_{(j)}$ above to the following “conformal operators”:

$$K_{(j)} = \tilde{K}_{(j)} - \eta R_{(j)}.$$

We immediately find

$$K_{(j)} = \sum_{\mu=1}^n A^{(j)}_\mu K_{(\mu)} ,$$

where

$$K_{(\mu)} = \partial_\mu (X_\mu \partial_\mu \Phi) + \frac{1}{X_\mu} \left( \sum_{k=0}^{n-1+\varepsilon} (-x_\mu^2)^n x_\mu \partial_k \right)^2$$

$$+ \eta r_\mu + \varepsilon \frac{X_\mu}{x_\mu} \partial_\mu^2 \Phi.$$

#### Separability

Since $\tilde{K}_{(0)} = \square$, the conformally coupled scalar field equation (11) can be written as

$$K_{(0)} \Phi = 0.$$

Slightly more generally, we can include a mass term and consider an equation

$$(K_{(0)} - m^2) \Phi = 0.$$

To separate this equation we seek the solution in the multiplicative separated form,

$$\Phi = \prod_{\mu=1}^n (Z_\mu(x_\mu) \prod_{k=0}^{n-1+\varepsilon} e^{i \Phi_k \psi_k}),$$

where $Z_\mu(x_\mu)$ are given by

$$Z_\mu(x_\mu) = \frac{1}{x_\mu^{\varepsilon}} \left( \frac{X_\mu}{x_\mu} \partial_\mu \Phi \right).$$
where \( \Psi_k \) are (Killing vector) separation constants and each of the \( Z_\mu \) is a function of the single corresponding variable \( x_\mu \) only. With this ansatz we have

\[
\partial_k \Phi = i \Psi_k \Phi, \quad \partial_\mu \Phi = \frac{Z'_\mu}{Z_\mu} \Phi, \quad \partial^2_\mu \Phi = \frac{Z''_\mu}{Z_\mu} \Phi, \tag{31}
\]

which allows us to rewrite (25) in the following form:

\[
\Phi \sum_{\mu=1}^{n} \frac{G_\mu}{U_\mu} = 0, \tag{32}
\]

where \( G_\mu = G_\mu(x_\mu) \) are function of one variable only

\[
G_\mu = X_\mu \frac{Z''_\mu}{Z_\mu} + X'_\mu \frac{Z'_\mu}{Z_\mu} - \frac{1}{X_\mu} \left( \sum_{k=0}^{n-1+\varepsilon} (x_\mu^{n-1-k} \Psi_k) \right)^2 - \eta \rho - \frac{\varepsilon}{c a x_\mu} \Psi_n^2 + \frac{\varepsilon}{x_\mu} X_\mu \frac{Z'_\mu}{Z_\mu} - m^2 (x_\mu^{n-1} - 1). \tag{33}
\]

Here we have used another identity (16)

\[
1 = \sum_{\mu=1}^{n} \frac{(-x_\mu^{n-1})}{U_\mu}. \tag{34}
\]

Now let us use the following ‘separability lemma’ (16, 27)

**Lemma.** The most general solution of

\[
\sum_{\mu=1}^{n} \frac{f_\mu(x_\mu)}{U_\mu} = 0, \tag{35}
\]

where \( U_\mu \) is defined in (3), is given by

\[
f_\mu = \sum_{k=1}^{n} C_k (-x_\mu^{n-1-k}), \tag{36}
\]

where \( C_j \) are arbitrary (separation) constants.

Thus, we see that the most general solution of (22) is

\[
G_\mu = \sum_{k=1}^{n-1} C_k (-x_\mu^{n-1-k}). \tag{37}
\]

That is, the equation (29) is satisfied for our ansatz (30) provided the functions \( Z_\mu = Z_\mu(x_\mu) \) satisfy the following ordinary differential equations:

\[
Z''_\mu + X'_\mu \frac{Z'_\mu}{x_\mu} - X_\mu \frac{Z''_\mu}{x_\mu} = \sum_{k=0}^{n-1+\varepsilon} (x_\mu^{n-1-k} \Psi_k)^2 - \eta \rho - \frac{\varepsilon}{c a x_\mu} \Psi_n^2 + \sum_{k=0}^{n-1} C_k (-x_\mu^{n-1-k}) = 0, \tag{38}
\]

where we have set \( C_0 = m^2 \). When the coefficient \( \eta \) is set to zero, we recover the result from (16) on separability of the massive Klein–Gordon equation in the off-shell Kerr–NUT–AdS spacetime in canonical coordinates. On the other hand, setting \( \eta = 0 \) we have successfully separated the conformal equation (1) in these spacetimes.

### Commuting operators

Following (26) let us now show that the above demonstrated separability can be ‘justified’ by the existence of a complete set of mutually commuting operators. This set consists of the above constructed conformal operators \( K_j \) and the Killing vector operators \( L(j) \):

\[
K_j = \nabla_a k^{ab} \nabla_b - \eta \rho, \tag{39}
\]

\[
L(j) = -i \partial_j \nabla_a \tag{40}
\]

To show that these operators mutually commute, we consider their explicit form

\[
L(j) = -i \frac{\partial}{\partial \psi_j}, \tag{41}
\]

\[
K_j = \sum_{\mu=1}^{n} A^{(j)}_\mu \frac{U_\mu}{U_\mu} \tag{42}
\]

where \( K_\mu \) were derived above and are given by equation (27). Obviously, we have

\[
[K_k], L(l)] = 0, \quad [L(k), L(l)] = 0. \tag{43}
\]

To show that also

\[
[K_k], K_l] = 0 \tag{44}
\]

we can reuse the argument presented in (26). First, note that for \( \mu \neq \nu \) we have \( [K_\mu], K_\nu] = 0 \) because these operators depend on different \( x_\mu \neq x_\nu \) and so any derivatives terms will commute. Next we can employ the first of the following identities (16):

\[
\sum_{k=0}^{n-1} A^{(k)}_\nu (x_\mu^{n-1-k}) = \delta_\mu^\nu, \quad \sum_{k=1}^{n-1} (x_\mu^{n-1-k}) A^{(k)}_\nu = \delta_\mu^\nu \tag{45}
\]

to “invert” the expression in (42) to write

\[
K_\mu = \sum_{k=0}^{n-1} (-x_\mu^{n-1-k}) K_{(k)} \tag{46}
\]

Thus using \( [K_\mu], (-x_\mu^{n-1-k}) = 0 \) for \( \mu \neq \nu \) we can express the commutation of the \( K_\mu \)’s as

\[
0 = [K_\mu], K_\nu] = \sum_{k=1}^{n-1} (-x_\mu^{n-1-k} (-x_\nu^{n-1-k}) K_{(k)} \tag{47}
\]

In particular as the \( (-x_\mu^{n-1-k}) \) are non-vanishing in general this shows that \( [K_\mu], K_{(k)}] = 0 \), as required.

Of course, the separated solution above is nothing else than the “common eigenfunction” of these operators and the separation constants \( \{ \Psi_k, C_j \} \) are the corresponding eigenvalues, that is, for our solution (30) obeying (38) we have

\[
K(j) \Phi = C_j \Phi, \tag{48}
\]

\[
L(j) \Phi = \Psi_j \Phi \tag{49}
\]
To see the former, we write

\[
\frac{1}{\Phi} K_{(j)} \Phi = \frac{1}{\Phi} \sum_{\mu=1}^{n} \frac{A^{(j)}_{\mu}}{U_{\mu}} K_{(\mu)} \Phi
\]

\[
= \sum_{\mu=1}^{n} \frac{A^{(j)}_{\mu}}{U_{\mu}} \left( G_{\mu} + m^2 (-x_{\mu}^2)^{n-1} \right)
\]

\[
= \sum_{\mu=1}^{n} \frac{A^{(j)}_{\mu}}{U_{\mu}} \sum_{k=0}^{n-1} C_k (-x_{\mu}^2)^{n-1-k}
\]

\[
= \sum_{k=0}^{n-1} C_k \sum_{\mu=1}^{n} \frac{A^{(j)}_{\mu}}{U_{\mu}} (-x_{\mu}^2)^{n-1-k} = C_j ,
\]

where we have subsequently used \( 12 \), \( 37 \), and the second identity \( 15 \).

Now with the separability of the conformal wave equation guaranteed we can turn to applications involving metrics conformally related to general metric \( \eta \).

**IV. SEPARABILITY IN WEYL RESCALED METRICS**

The equation \( 1 \) enjoys a conformal symmetry. This means that under a Weyl scaling of the metric,

\[
g \rightarrow \tilde{g} = \Omega^2 g ,
\]

we have \( 24 \)

\[
(\tilde{\Box} - \eta \tilde{R}) \Omega^{1-d/2} \Phi = \Omega^{-1-d/2} (\Box - \eta R) \Phi .
\]

The corresponding form in \( \eta \) is that under a Weyl scaling of the metric \( g \) of the spacetime with metric \( g \),

\[
\tilde{\Phi} = \Omega^{1-d/2} \Phi
\]

is a solution of \( 1 \) in the spacetime with metric \( \tilde{g} \).

In particular, this implies that in any spacetime \( \tilde{g} \) related to the off-shell Kerr–NUT–AdS metric by the Weyl transformation, we can find a solution of the corresponding conformal equation \( 1 \) in the form \( 53 \), where \( \Phi \) is the separated solution of \( 39 \) and functions \( Z_{\mu} \) obey \( 35 \). Strictly speaking, due to the pre-factor \( \Omega^{1-d/2} \) the corresponding solution \( 53 \) is no longer formally written in a multiplicative separation form and the corresponding separability is called \( \text{R-separability.} \)

Let us also note that this result is non-trivial as the principal tensor no longer exists in the Weyl scaled metrics and consequently only towers of conformal hidden symmetries (as opposed to full hidden symmetries) exist in the Weyl rescaled spacetimes. Specifically, if \( \omega \) is a conformal Killing–Yano \( p \)-form in spacetime with \( g \), then \( \tilde{\omega} = \Omega^{p+1} \omega \) is a conformal Killing–Yano \( p \)-form in spacetime with \( \tilde{g} \). In particular

\[
\tilde{h} = \Omega^3 h
\]

is a new principal conformal Killing–Yano tensor, which however need no longer be closed and is a much weaker structure. This implies that each Killing tensor, generated from \( j \) copies of \( h \) with \( j + 1 \) contractions with the inverse metric, c.f. \( 12 \), becomes a conformal Killing tensor:

\[
\tilde{K}_{(j)}^{ab} = K_{(j)}^{ab} ,
\]

and the former explicit symmetries become conformal Killing vectors, \( \tilde{l}_{(j)}^a = l_{(j)}^a \). It would be interesting to study how these symmetries can directly be applied to guarantee separability of conformal wave equations in these spacetimes.

**V. FOUR-DIMENSIONAL EXAMPLES**

**Carter’s spacetime**

To apply the above machinery, let us now specify to \( d = 4 \) dimensions. Upon the Wick rotation of one of the \( x_{\mu} \) coordinates,

\[
\psi_0 = \tau , \quad \psi_1 = \psi , \quad x_1 = y , \quad x_2 = ir ,
\]

and setting

\[
X_1 = -\Delta_y , \quad X_2 = -\Delta_r ,
\]

\[
U_2 = \Sigma = r^2 + y^2 = -U_1 ,
\]

the off-shell Kerr–NUT–AdS spacetime yields the off-shell Lorentzian Carter’s metric \( 2 \),

\[
g = -\frac{\Delta_r}{\Sigma} d\tau + y^2 d\psi)^2 + \frac{\Delta_y}{\Sigma} (d\tau - r^2 d\psi)^2
\]

\[+ \frac{\Sigma}{\Delta_r} d\tau^2 + \frac{\Sigma}{\Delta_y} dy^2 ,
\]

with arbitrary

\[
\Delta_r = \Delta_r (r) , \quad \Delta_y = \Delta_y (y) ,
\]

the principal Killing–Yano tensor given by

\[
h = y dy \wedge (d\tau - r^2 d\psi) - r dr \wedge (d\tau + y^2 d\psi) ,
\]

and the following Ricci scalar:

\[
R = -\frac{\Delta_r'' + \Delta_y''}{\Sigma} .
\]

The conformal scalar field equation \( 11 \) reduces to

\[
(\Box - \frac{R}{6}) \Phi = 0 .
\]

Its solution can be found in a separable form,

\[
\Phi = Z(r) Y(y) e^{i\omega \tau} e^{i\phi \psi} ,
\]
where functions \( Z \) and \( Y \) satisfy the following ordinary differential equations:

\[
(\Delta_r Z')' + Z\left(\frac{1}{\Delta_r}(\Psi + r^2\omega)^2 + \frac{\Delta''}{6} - C\right) = 0, \tag{64}
\]

\[
(\Delta_y Y')' + Y\left(-\frac{1}{\Delta_y}(\Psi - y^2\omega)^2 + \frac{\Delta''}{6} + C\right) = 0. \tag{65}
\]

Of course, this result remains valid for the on-shell Carter spacetime \([2]\), a solution to the Einstein–Maxwell–

\[R \text{-separated form} \]

The following form:

\[
\text{in the original on-shell Plebański–Demiański metric} \ [28], \text{for which the metric functions} \ \Delta_r \text{and} \ \Delta_y \\text{take the following specific form:}
\]

\[
\Delta_r = (r^2 + a^2)(1 - \Lambda r^2/3) - 2mr + r^2 + g^2, \tag{66}
\]

\[
\Delta_y = (a^2 - y^2)(1 + \Lambda g^2/3) + 2ny. \tag{67}
\]

Here, \( e \) and \( g \) are electric and magnetic charges, and \( m, a, n \) are related to mass, rotation, and NUT charge parameters, while the metric is accompanied by the \( U(1) \) gauge potential

\[
A = -\frac{er}{\Sigma}(d\tau + y^2\,d\psi) - \frac{gy}{\Sigma}(d\tau - r^2\,d\psi). \tag{68}
\]

Plebański–Demiański class

Another, more general, class of 4-dimensional black hole spacetimes is encoded in the Plebański–Demiański

\[\text{spacetime} \ [28]. \text{The off-shell metric is given by}
\]

\[
\hat{g} = \Omega^2 g, \tag{69}
\]

where \( g \) is given in \([28]\) and the conformal prefactor takes the following form:

\[
\Omega = \frac{1}{1 - yr}. \tag{70}
\]

By the above theory, this spacetime admits a solution of the conformal equation \([22]\), which can be found in the R-separated form

\[
\Phi = \frac{1}{\Omega} Z(r)Y(y)e^{i\omega \tau}e^{i\Psi \psi}, \tag{71}
\]

where functions \( Z \) and \( Y \) obey the ordinary differential equations \([24]\).

One particular example of a spacetime in this class is the original on-shell Plebański–Demiański metric \([28]\), for which the metric functions \( \Delta_r \) and \( \Delta_y \) take the following specific form:

\[
\Delta_r = k + e^2 + g^2 - 2mr + e^2 - 2mr^2 - (k + \Lambda/3)r^4, \tag{72}
\]

\[
\Delta_y = k + 2ny - ey^2 + 2my^3 - (k + e^2 + g^2 + \Lambda/3)y^4. \tag{73}
\]

where \( e, g, n, k, m, \) and \( \epsilon \) are free parameters that are related to the electric and magnetic charges, NUT parameter, rotation, mass, and acceleration. Due to the conformal invariance of Maxwell equations in \(4d\), the gauge potential remains given by \([63]\). In this special case, the separability of the conformal scalar equation follows from the results presented in \([29]\), see also \([30]\) for its intrinsic characterization.

Another example of a spacetime which belongs to the off-shell Plebański–Demiański class is the hairy black hole solution constructed in \([31, 32]\), see also \([33]\) for a more general spacetime that can be written in the form \([65]\) with a more general conformal pre-factor.

VI. SUMMARY

In this paper we have separated the conformal wave equation in general off-shell Kerr–NUT–AdS spacetimes in all dimensions, generalizing the work \([10]\) on separability of the massive Klein–Gordon equation in these spacetimes. Let us emphasize that although the two results formally coincide in vacuum with cosmological constant – for the on-shell Kerr–NUT–AdS spacetime \([12]\) – they are very different for a more general matter content.

We have further shown that the demonstrated separability can be intrinsically characterized by a complete set of mutually commuting operators. To the leading order in derivatives, these operators are constructed from Killing tensors and Killing vectors generated from the hidden symmetry of the off-shell Kerr–NUT–AdS spacetime encoded in the principal Killing–Yano tensor. The second order operators also pick up an “anomalous” absolute term, see \([24]\) and \([25]\), which in the case of the original conformal wave operator is simply given by the Ricci scalar of the spacetime and guarantees the conformal invariance of the corresponding equation. It is plausible to conjecture that also for other operators \( K_{(j)} \) \( (j = 1, \ldots, n - 1) \) these anomalous terms ensure some kind of conformal symmetry. Unfortunately, at the moment we only have a coordinate expression for these correction terms and cannot study conformal properties of these operators until a covariant expression for the anomalous terms is found. This issue certainly deserves attention in the future.

We have also discussed the Weyl rescaled metrics and shown how our results imply separability of the conformal wave equations in those spacetimes. As a concrete application we have considered the most general type D spacetime described by the Plebański–Demiański family and constructed the associated R-separated test field solution of the conformal wave equation. We expect that this construction applies to a wide class of solutions with various matter content, similar to what happens in four dimensions \([28, 31, 53]\).

ACKNOWLEDGEMENTS

We would like to thank Gang Xu and Dan Wohns for organizing the PSI Winter School where this project was mostly completed, with special thanks to Robert who
kept smiles on our faces, and the PSI program for facilitating this research. D.P. is the 2019/20 recipient of the Emmy Noether Scholarship, a PSI Honorary Award supported by the Emmy Noether Circle to help increase the number of women in physics and mathematical physics. D.P. was supported in part by the Natural Sciences and Engineering Research Council of Canada. Research at Perimeter Institute is supported in part by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Economic Development, Job Creation and Trade.

[1] B. Carter, Global structure of the Kerr family of gravitational fields, Phys. Rev. 174 (1968) 1559.
[2] B. Carter, Hamilton–Jacobi and Schrödinger separable solutions of Einstein’s equations, Commun. Math. Phys. 10 (1968) 280.
[3] S. A. Teukolsky, Rotating black holes - separable wave equations for gravitational and electromagnetic perturbations, Phys. Rev. Lett. 29 (1972) 1114.
[4] S. A. Teukolsky, Perturbations of a rotating black hole. I. Fundamental equations for gravitational and electromagnetic and neutrino field perturbations, Astrophys. J. 185 (1973) 635.
[5] W. G. Unruh, Separability of the neutrino equations in a Kerr background, Phys. Rev. Lett. 31 (1973) 1265.
[6] S. Chandrasekhar, The solution of Dirac’s equation in Kerr geometry, Proc. R. Soc. Lond., Ser A 349 (1976) 571.
[7] D. N. Page, Dirac equation around a charged, rotating black hole, Phys. Rev. D 14 (1976) 1509.
[8] N. Kamran, Separation of variables for the rarita-schwinger equation on all type d vacuum backgrounds, Journal of mathematical physics 26 (1985) 1740.
[9] R. Penrose, Naked singularities, Ann. Phys. (N.Y.) 224 (1973) 125.
[10] R. Floyd, The dynamics of Kerr fields, Ph.D. thesis, London University, London, 1973.
[11] R. C. Myers and M. J. Perry, Black holes in higher dimensional space-times, Ann. Phys. (N.Y.) 172 (1986) 304.
[12] G. W. Gibbons, H. Lu, D. N. Page and C. N. Pope, The General Kerr-de Sitter metrics in all dimensions, J. Geom. Phys. 53 (2005) 49 hep-th/0404008.
[13] W. Chen, H. Lu and C. N. Pope, General Kerr-NUT-AdS metrics in all dimensions, Class. Quant. Grav. 23 (2006) 5323 hep-th/0604125.
[14] D. Kubizňák and V. P. Frolov, Hidden symmetry of higher dimensional Kerr-NUT-AdS spacetimes, Class. Quantum Grav. 24 (2007) F1 hep-th/0701014.
[15] V. Frolov, P. Krtouš and D. Kubizňák, Black holes, hidden symmetries, and complete integrability, Living Rev. Rel. 20 (2017) 6 1705.05482.
[16] V. P. Frolov, P. Krtouš and D. Kubizňák, Separability of Hamilton-Jacobi and Klein-Gordon equations in general Kerr-NUT-AdS spacetimes, JHEP 0702 (2007) 005 hep-th/0611245.
[17] T. Oota and Y. Yasui, Separability of Dirac equation in higher dimensional Kerr-NUT-de Sitter spacetime, Phys. Lett. B659 (2008) 688 0711.0079.
[18] O. Lunin, Maxwell’s equations in the Myers-Perry geometry, JHEP 12 (2017) 138 1706.05756.
[19] V. P. Frolov, P. Krtouš, D. Kubizňák and J. E. Santos, Massive Vector Fields in Rotating Black-Hole Spacetimes: Separability and Quasinormal Modes, Phys. Rev. Lett. 120 (2018) 231103 1804.09030.
[20] O. Lunin, Excitations of the Myers-Perry Black Holes, JHEP 10 (2019) 030 1907.03920.
[21] T. Oota and Y. Yasui, Separability of Gravitational Perturbation in Generalized Kerr-NUT-de Sitter Spacetime, Int. J. Mod. Phys. A25 (2010) 3055 0812.1633.
[22] T. Houri, T. Oota and Y. Yasui, Closed conformal Killing-Yano tensor and Kerr-NUT-de Sitter spacetime uniqueness, Phys. Lett. B 656 (2007) 214 0708.1368.
[23] P. Krtouš, V. P. Frolov and D. Kubizňák, Hidden Symmetries of Higher Dimensional Black Holes and Uniqueness of the Kerr-NUT-(A)dS spacetime, Phys. Rev. D78 (2008) 064022 0804.4705.
[24] R. M. Wald, General Relativity, Chicago Univ. Pr., Chicago, USA, 1984.
[25] N. Hamamoto, T. Houri, T. Oota and Y. Yasui, Kerr-NUT-de Sitter curvature in all dimensions, J. Phys. A40 (2007) F177 hep-th/0611285.
[26] A. Sergyeyev and P. Krtouš, Complete set of commuting symmetry operators for Klein-Gordon equation in generalized higher-dimensional Kerr-NUT-(A)dS spaces, Phys. Rev. D 77 (2008) 044033 0711.4623.
[27] P. Krtouš, Electromagnetic field in higher-dimensional black-hole spacetimes, Phys. Rev. D76 (2007) 084035 0707.0002.
[28] J. F. Plebański and M. Demiański, Rotating charged and uniformly accelerated mass in general relativity, Ann. Phys. (N.Y.) 98 (1976) 98.
[29] A. L. Dudley and J. Finley III, Separation of wave equations for perturbations of general type-d space-times, Physical Review Letters 38 (1977) 1505.
[30] N. Kamran and R. McLenaghan, Separation of variables and symmetry operators for the conformally invariant klein-gordon equation on curved spacetime, letters in mathematical physics 9 (1985) 65.
[31] C. Charmousis, T. Kolyvaris and E. Papantonopoulos, Charged C-metric with conformally coupled scalar field, Class. Quant. Grav. 26 (2009) 175012 0906.3568.
[32] A. Anabalon and H. Maeda, New Charged Black Holes with Conformal Scalar Hair, Phys. Rev. D81 (2010) 041501 0907.0219.
[33] A. Anabalon, Exact Black Holes and Universality in the Backreaction of non-linear Sigma Models with a potential in (A)dS, JHEP 06 (2012) 127 1204.2720.