A NOTE ON THE CONVERGENCE OF ALMOST MINIMAL SETS

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ABSTRACT. In this paper, we will show that Hausdorff convergence and varifold convergence coincide on the class of almost minimal sets.

1. INTRODUCTION AND NOTATION

We see obvious in general Hausdorff distance convergence for a sequence of sets do not implies the varifold convergence of the associate sequence of varifolds. But in this paper, we will show that the implication is true in case that restrict on general quasiminimal sets with the total Hausdorff measure of the sequence tending to the total Hausdorff measure of the limit set. It is also true by replacing the total Hausdorff measure with the integral of an elliptic integrand. As a consequence, we may get that the Hausdorff convergence and varifold convergence coincide on almost minimal sets.

An $m$-varifold on an open subset $U \subseteq \mathbb{R}^n$ is a Radon measure on $U \times G(n, m)$. We denote by $V_m(U)$ the collection of $m$-varifolds on $U$. It can be equipped with a weak topology given by saying that $V_i \rightharpoonup V$ if

$$
\int \varphi \, dV_i \to \int \varphi \, dV,
$$

for all compactly supported, continuous real valued function $\varphi$ on $U \times G(n, m)$.

Given any varifold $V$, we can get a corresponding Radon measure $\|V\|$ on $U$ defined by

$$
\|V\|(A) = V(A \times G(n, m)), \quad \text{for } A \subseteq U.
$$

For any Borel regular measure $\mu$ on $U$ and $x \in U$, we let $\Theta^m_+(\mu, x)$ and $\Theta^m_-\mu, x)$ be the lower and upper $m$-density of $\mu$ at $x$, see [1], if they are equal, we will denote it by $\Theta^m(\mu, x)$, called the $m$-density. For any set $E \subseteq U$, $\Theta^m(E, x)$ is understood as the $m$-density of $H^m \cup E$ at $x$.

A subset $E \subseteq \mathbb{R}^n$ is called $m$-rectifiable, if there exists a sequence of Lipschitz mappings $f_i : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$
H^m (E \setminus f_i(\mathbb{R}^m)) = 0.
$$

$E$ is called purely $m$-unrectifiable (or $m$-irregular) if $H^m (E \cap F) = 0$ for any $m$-rectifiable set $F$, see for example [9, Definition 15.3] or [8, 3.2.14].

Let $E \subseteq \mathbb{R}^n$ be a $m$-rectifiable set, $x \in E$ be any point. An $m$-plane $\pi$ is called an approximate tangent plane if

$$
\limsup_{r \to 0} r^{-m} H^d (E \cap B(x, r)) > 0
$$

and for any $\varepsilon > 0$,

$$
\lim_{r \to 0} r^{-m} H^d (E \cap B(x, r) \setminus C(x, \pi, r, \varepsilon)) = 0,
$$

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where \( C(x, \pi, r, \varepsilon) = \{ y \in B(x, r) \mid \text{dist}(y - x, \pi) \leq \varepsilon |y - x| \} \). An \( m \)-plane \( \pi \) is called a (true) tangent plane if for any \( \varepsilon > 0 \), there exists \( r_\varepsilon > 0 \) such that
\[
E \cap B(x, r) \subseteq C(x, \pi, r, \varepsilon) \quad \text{for } 0 < r < r_\varepsilon.
\]

We will denote by \( \Tan(E, x) \) the tangent plane of \( E \) at \( x \), if it exists.

Let \( E \subseteq U \) be an \( m \)-rectifiable set. Then for \( \mathcal{H}^m \) almost every \( x \in E \), there is an unique approximate tangent plane of \( E \) at \( x \), see for example [8, Theorem 3.2.19] or [9, Theorem 15.11]. If additionally \( E \) is local Ahlfors regular, that is, there exists \( C \geq 1 \) and \( r_0 > 0 \) such that for any \( x \in E \), \( 0 < r < r_0 \) with \( B(x, 2r) \subseteq U \), we have that
\[
C^{-1}r^m \leq \mathcal{H}^m(E \cap B(x, r)) \leq Cr^m,
\]
then in this case, every approximate tangent plane is a true tangent plane.

Let \( E \subseteq U \) be any set such that \( \mathcal{H}^m(E \cap K) < \infty \) for any compact sets \( K \subseteq U \). We define the associates varifold \( v(E) \), by setting
\[
v(E)(\beta) = \int_{E_{rec}} \beta(x, \Tan(E_{rec}, x)) \, d\mathcal{H}^d(x) + \int_{E_{irr}} \beta(x, T) \, d\gamma_{n,m}(T) \, d\mathcal{H}^m(x)
\]
for any continues function \( \beta : U \times G(n, m) \to \mathbb{R} \) with compact support, where we decompose \( E \) as the union \( E_{rec} \cup E_{irr} \), \( E_{rec} \) is \( m \)-rectifiable, \( E_{irr} \) is purely \( m \)-unrectifiable, \( \gamma_{n,m} \) denotes the Haar measure on \( G(n, m) \).

On the power set of \( \mathbb{R}^n \), we define the normalized local Hausdorff distance \( d_{x, r} \) by the formula
\[
d_{x, r}(X, Y) = \frac{1}{r} \sup \{ \text{dist}(z, Y) : z \in X \cap B(x, r) \} + \frac{1}{r} \sup \{ \text{dist}(y, X) : y \in Y \cap B(x, r) \};
\]
a sequence \( E_k \subseteq U \) converges to a set \( E \subseteq U \) in local Hausdorff distance, by definition, we mean that for any \( x \in U \) and \( 0 < r < \text{dist}(x, U^c) \), \( d_{x, r}(E_k, E) \to 0 \) as \( k \to \infty \).

2. Convergence of Quasiminimal Sets

For any \( m \)-plane \( T \), we will denote by \( T_\perp \) the orthogonal projection of \( \mathbb{R}^n \) onto \( T \). For any \( x \in \mathbb{R}^n \) and \( r > 0 \), we denote by \( \mu_{x, r} : \mathbb{R}^n \to \mathbb{R}^n \) the mapping given be \( \mu_{x, r}(y) = r^{-1}(y - x) \). For any \( m \)-rectifiable set \( E \subseteq \mathbb{R}^n \) and mapping \( \varphi : E \to \mathbb{R}^n \), we will denote by \( \text{ap } J_m \varphi \) the approximate Jacobian of \( \varphi \), see [8, Theorem 3.2.22].

**Lemma 2.1.** Let \( \{ E_k \} \) be a sequence of \( m \)-rectifiable subsets in \( U \). Suppose that there is an \( m \)-rectifiable set \( E \subseteq U \) such that \( \mathcal{H}^m(E) = \lim_{k \to \infty} \mathcal{H}^m(E_k) < +\infty \), and for \( \mathcal{H}^d \)-a.e. \( x \in E \), by setting \( T = \Tan(E, x) \),
\[
\lim_{r \to 0} \lim_{k \to \infty} \mathcal{H}^m(T_\perp \circ \mu_{x, r}(E_k \cap B(x, r))) \geq \omega_m.
\]

Then we have that\[
v(E_k) \rightharpoonup v(E).
\]

**Proof.** We first prove that for any open set \( O \subseteq U \),
\[
\mathcal{H}^m(E \cap O) \leq \lim_{k \to \infty} \mathcal{H}^m(E_k \cap O).
\]

Since \( E \) is rectifiable, we have that for \( \mathcal{H}^m \)-a.e. \( x \in E \), denote by \( E^1 \) the collection of such point,
\[
\lim_{r \to 0} \frac{\mathcal{H}^m(E \cap B(x, r))}{\omega_m r^m} = 1.
\]
For any $\varepsilon > 0$ fixed, we can find $r_{x,1} > 0$ such that for any $0 < r < r_{x,1}$,

$$(1 - \varepsilon)\omega_m r_m^{m} \leq \mathcal{H}^m(E \cap B(x, r)) \leq (1 + \varepsilon)\omega_m r_m^{m}.$$ 

We get from (2.1) that there exist $r_{x,2} > 0$ and $k_x > 0$ such that

$$\mathcal{H}^m(E_k \cap B(x, r)) \geq r^m \mathcal{H}^m(P_{x,r} \circ \mu_{x,r}(E_k \cap B(x, r))) \geq (1 - \varepsilon)\omega_m r_m^{m}$$

for any $0 < r < r_{x,2}$ and $k \geq k_x$. We put $r_x = \min\{r_{x,1}, r_{x,2}\}$, then

$$\mathcal{H}^m(E_k \cap B(x, r)) \geq (1 - \varepsilon)(1 + \varepsilon)^{-1}\mathcal{H}^m(E \cap B(x, r)),$$

for any $0 < r < r_x$ and $k \geq k_x$.

We see that $\mathcal{B} = \{B(x, r) \subseteq \mathcal{O} : x \in E^1 \cap \mathcal{O}, 0 < r < r_x\}$ is a Vitali covering of $E^1 \cap \mathcal{O}$, thus there exists a countable many balls $\{B_i\}_{i \in I} \subseteq \mathcal{B}$, such that $B_i \cap B_j = \emptyset$ for $i, j \in I$, $i \neq j$, and

$$\mathcal{H}^m\left(E^1 \cap \mathcal{O} \setminus \bigcup_{i \in I} B_i\right) = 0.$$

We take $N > 0$ such that $\mathcal{H}^m(E^1 \cap \mathcal{O} \setminus \bigcup_{i=1}^{N} B_i) < \varepsilon$. Assume that $B_i = B(x_i, r_i)$, $i \in I$. Then we have that, for any $k \geq \max\{k_{x_i} : 1 \leq i \leq N\}$,

$$\mathcal{H}^m(E \cap \mathcal{O}) = \mathcal{H}^m(E^1 \cap \mathcal{O}) \leq \sum_{i=1}^{N} \mathcal{H}^m(E \cap B_i) + \varepsilon$$

$$\leq \frac{1 + \varepsilon}{1 - \varepsilon} \sum_{i=1}^{N} \mathcal{H}^m(E_k \cap B_i) + \varepsilon$$

$$\leq \frac{1 + \varepsilon}{1 - \varepsilon} \mathcal{H}^m(E_k \cap \mathcal{O}) + \varepsilon,$$

thus

$$\mathcal{H}^m(E \cap \mathcal{O}) \leq \frac{1 + \varepsilon}{1 - \varepsilon} \lim_{k \to \infty} \mathcal{H}^m(E_k \cap \mathcal{O}) + \varepsilon,$$

we let $\varepsilon$ tend to 0 to get that (2.2) holds.

Next, we show that, for any subsequence of $\{\nu(E_k)\}$, if it converge to some varifold $\nu$, then

$$\Theta^m(\|\nu\|, x) \geq 1 \text{ for } \mathcal{H}^m\text{-a.e. } x \in E,$$

and so that $\Theta(\|\nu\|, x) = 1$ for $\mathcal{H}^m\text{-a.e. } x \in E$ and $\|\nu\|(U \setminus E) = 0$. Indeed, we assume that $\nu(E_k) \rightharpoonup \nu$. Then for any $x \in E^1$, and any ball $B(x, r) \subseteq U$, we have that

$$\|\nu\|(B(x, r)) \geq \lim_{k \to \infty} \mathcal{H}^m(E_k \cap B(x, r)) \geq \mathcal{H}^m(E \cap B(x, r)),$$

thus

$$\Theta^m(\|\nu\|, x) \geq \Theta^m(E, x) \geq 1.$$

But $\mathcal{H}^m(E) = \lim_{k \to \infty} \mathcal{H}^m(E_k) = \|\nu\|(U)$, we have so that $\Theta(\|\nu\|, x) = 1$ for $\mathcal{H}^m\text{-a.e. } x \in E$ and $\|\nu\|(U \setminus E) = 0$.

Finally, we show that $\text{VarTan}(\nu) = \{\nu(\text{Tan}(E, x))\}$ for $\mathcal{H}^m\text{-a.e. } x \in E$. We will denote by $E^2$ the points $x$ in $E_1$ that $\Theta^m(\|\nu\|, x) = 1$ and $E$ has unique tangent $m$-plane at $x$. Then we see that $\mathcal{H}^m(E \setminus E^2) = 0$. For any $x \in E^2$, we have that

$$1 \leq \lim_{r \to 0+} \lim_{\varepsilon \to \infty} \frac{\mathcal{H}^m(T_{x,r} \circ \mu_{x,r}(E_k \cap B(x, r)))}{\omega_m} \leq \lim_{r \to 0+} \lim_{\varepsilon \to \infty} \frac{\mathcal{H}^m(E_k \cap B(x, r))}{\omega_m r_m^{m}} \leq \Theta^m(\|\nu\|, x),$$
but $\Theta^m(||V||, x) = 1$, we get so that

\[(2.5) \lim_{r \to 0 +} \lim_{\ell \to \infty} H^m(T, o \mu_{x,r}(E_{k\ell} \cap B(x, r))) = \lim_{r \to 0 +} \lim_{\ell \to \infty} H^m(\mu_{x,r}(E_{k\ell} \cap B(x, r))) = \omega_m.\]

We put $E_{k\ell, x, r} = \mu_{x,r}(E_{k\ell} \cap B(x, r))$, $T = \Tan(E, x)$ and $Q_\ell(y) = \Tan(E_{k\ell, x, r}, y)$ for $y \in E_{k\ell, x, r}$. Employing [1, 8.9 (3)], we have that

\[\|Q_\ell(y) - T\| = \|T^\perp \circ Q_\ell(y)\| = \sup_{v \in T, \|v\| = 1} |T^\perp(v)|,\]

so that we can find $v_1 \in T$, $v = 1$ such that $\|Q_\ell(y) - T\| = |T^\perp(v_1)|$. Let $v_1, v_2, \ldots, v_m$ be a unit orthogonal basis of $Q_\ell(y)$. Let $\Phi_\ell : E_{k\ell, x, r} \to T$ be defined by $\Phi_\ell(y) = T^\perp(y)$. Then

\[\text{ap} J_m \Phi_\ell(y) = ||T_1(v_1) \land T_2(v_2) \land \cdots \land T_m(v_m)|| \leq |T_1(v_1)|,\]

thus

\[\text{ap} J_m \Phi_\ell(y)^2 \leq 1 - \|Q_\ell(y) - T\|^2.\]

We get that

\[(2.6) \|Q_\ell(y) - T\|^2 \leq 1 - \text{ap} J_m \Phi_\ell(y)^2 \leq 2(1 - \text{ap} J_m \Phi_\ell(y)),\]

and by the Hölder's inequality and Theorem 3.2.22 in [8], we have that

\[\int_{E_{k\ell, x, r}} ||Q_\ell(y) - T|| dH^m(y) \leq 2H^m(E_{k\ell, x, r}) \int_{E_{k\ell, x, r}} (1 - \text{ap} J_m \Phi_\ell(y)) \leq 2H^m(E_{k\ell, x, r}) (H^m(E_{k\ell, x, r}) - H^m(P_T(E_{k\ell, x, r}))),\]

combine this with (2.5), we get that

\[(2.7) \lim_{r \to 0 +} \lim_{\ell \to \infty} \int_{E_{k\ell, x, r}} ||Q_\ell(y) - T|| dH^m(y) = 0.\]

For any $C \in \text{VarTan}(V, x)$, we assume that

\[C = \lim_{j \to \infty} (\mu_{x,r_j})_{\#} V,\]

where $\{r_j\}$ is a decreasing sequence which tend to 0. Then we get that

\[C = \lim_{j \to \infty} \lim_{k \to \infty} v(\mu_{x,r_j}(E_k)),\]

$C$ support on $T$, and

\[(2.8) C \cap B(0, 1) \times G(n, m) = \lim_{j \to \infty} \lim_{k \to \infty} v(F_{k, x, r_j}).\]

For any $\varphi \in C^\infty_c(\mathbb{R}^n \times G(n, m), \mathbb{R})$ supported on $B(0, 1) \times G(n, m)$, by (2.8), we have that

\[(2.9) C(\varphi) = \lim_{j \to \infty} \lim_{k \to \infty} \int_{E_{k\ell, x, r_j}} \varphi(y, Q_\ell(y)) dH^m(y),\]

we let $\psi \in C^\infty_c(\mathbb{R}^n, \mathbb{R})$ be defined by $\psi(x) = \varphi(x, T)$, then again by (2.8), we have that

\[(2.10) \lim_{j \to \infty} \lim_{k \to \infty} \int_{E_{k\ell, x, r_j}} \psi(x) dH^m(y) = \int_T \psi(y) dH^m(y) = v(T)(\varphi).\]
We get, from (2.7), so that
\[
|C(\varphi) - v(T)(\varphi)| \leq \lim_{j \to \infty} \lim_{t \to \infty} \int_{E_{k,t,x,r}} |\varphi(y, Q_{t}(y)) - \varphi(y, T)| \, d\mathcal{H}^m(y)
\]
\[
(2.11)
\]
\[
\leq \|D\varphi\|_{\infty} \lim_{j \to \infty} \lim_{t \to \infty} \int_{E_{k,t,x,r}} \|Q_{t}(y) - T\| \, d\mathcal{H}^m(y) = 0.
\]

Thus
\[
C \subseteq B(0, 1) \times G(n, m) = v(T) \subseteq B(0, 1) \times G(n, m),
\]
but both C and v(T) are cones, we have that C = v(T). □

Let E ⊆ U be given, and let B be an open ball such that \(\overline{B} \subseteq U\). A family of mappings \(\{\varphi_t\}_{0 \leq t \leq 1}\) from E to U is called a deformation of E in B if

- \(\varphi_0 = \text{id}_E\), \(\varphi_1(x) = x\) for \(x \in E \setminus B\), and
- \([0, 1] \times E \to U\) given by \(t \times x \to \varphi_t(x)\) is continuous.

By a deformation of E in U we mean a deformation of E in a ball which is contained in U.

**Definition 2.2.** For any nondecreasing function \(h : [0, +\infty) \to [0, +\infty]\), and number \(M \geq 1\), we denote by \(QM(U, M, h)\) the collection of relatively closed sets \(E \subseteq U\) which satisfy that

- \(\mathcal{H}^m(E)\) is locally finite, \(\mathcal{H}(E \cap B(x, r)) > 0\) for any \(x \in E\) and some \(r = r(x) > 0\),
- for any ball \(B = B(x, r)\) with \(\overline{B} \subseteq U\), and any deformation \(\{\varphi_t\}_{0 \leq t \leq 1}\) of \(E\) in \(B\), by setting \(W_t = \{y \in U : \varphi_t(y) \neq y\}\), we have that
  \[
  \mathcal{H}^m(E \cap W_1) \leq M \mathcal{H}^m(\mathcal{P}(E \cap W_1)) + h(r) r^m.
  \]

It is quite easy to see from the definition that, if \(M_1 \leq M_2\) and \(h_1 \leq h_2\), then
\[
QM(U, M_1, h_1) \subseteq QM(U, M_2, h_2).
\]

If the function \(h\) satisfies that \(h(t) = 0\) for \(t < \delta\), and \(h(t) = +\infty\) for \(t \geq \delta\), where \(\delta > 0\), then the sets in \(QM(U, M, h)\) are usual \((U, M, \delta)\)-quasiminimal sets, see for example Definition 2.4 in [3], and also Definition 1.9 in [6], but it is called \((U, M, \delta)\)-quasiminimizer. If \(h\) satisfies that \(h(t) = h \in [0, 1)\) is a constant for \(t < \delta\), and \(h(t) = +\infty\) for \(t \geq \delta\), where \(\delta > 0\), then \(QM(U, M, h)\) will be the general Almgren quasiminimal sets \(GAQ(M, \delta, U, h)\) defined in Definition 2.10 in [4]. A function \(h : [0, \infty) \to [0, \infty]\) is called a gauge function if \(h\) is a nondecreasing function with \(h(0+) = 0\). Note that if \(h\) is a gauge function, then \(QM(U, 1, h)\) will be the usual almost minimal sets, see for example Definition 4.3 in [4]. We see from Lemma 2.15 in [4] that every set in \(QM(U, M, h)\) is local Ahlfors regular in case \(h(0+)\) small enough, namely that (1.1) holds, and the constant \(C\) only depends on \(n\) and \(m\).

**Lemma 2.3.** Let \(\{E_k\} \subseteq QM(U, M_k, h_k)\) be a sequence. Suppose that \(E_k\) converge to some set E in U in local Hausdorff distance, \(M = \lim_{k \to \infty} M_k < +\infty\), and \(h = \lim_{k \to \infty} h_k\) satisfying that \(h(0+)\) is small enough. Then we have that

1. \(\mathcal{H}^m(E \cap O) \leq \lim_{k \to \infty} \mathcal{H}^m(E_k \cap O)\) for any open set \(O \subseteq U\);
2. \(E\) is \(m\)-rectifiable, \(E \in QM(U, M, h)\);
3. \(\lim_{k \to \infty} \mathcal{H}^m(E_k \cap K) \leq (1 + Ch(0+))M \mathcal{H}^m(E \cap K)\) for any compact set \(K \subseteq U\).

**Proof.** Indeed, (1) follows from Lemma 3.3 in [4]. The fact \(E \in QM(U, M, h)\) follows from Lemma 4.1 in [4]; and the rectifiability of \(E\) comes from the local uniform rectifiability of \(E\), which can be proved by adapting the proof of the local uniform rectifiability of quasiminimal sets (Theorem 2.11 in [6]) to generalized quasiminimal sets, see [4, p.81].
It follows from Lemma 3.12 in [4] that
\[ \lim_{k \to \infty} H^m(E_k \cap K) \leq (1 + C h(t)) \mu H^m(E \cap K), \]
for any compact set \( K \subseteq U \),
for \( t \) small enough which makes \( h(t) \) small enough, thus we let \( t \) tends to 0 to get the conclusion (3).
\[ \square \]

From above lemma, we see that \(QM(U, M, h)\) is compact under the locally Hausdorff distance. That is, for any sequence \( \{E_k\} \subseteq QM(U, M, h) \), there is a subsequence \( \{E_{k_n}\} \) which converges in local Hausdorff distance to some set in \( QM(U, M, h) \).

**Theorem 2.4.** Let \( \{E_k\} \) be a sequence of sets such that \( E_k \in QM(U, M_k, h_k) \). Suppose that \( E_k \to E \) in \( U \), \( M = \lim_{k \to \infty} M_k < +\infty \), \( h = \lim_{m \to \infty} h_m \) satisfy that \( h(0+) \) is small enough. If \( H^m(E) = \lim_{k \to \infty} H^m(E_k) < \infty \), then \( \nu(E_k) \to \nu(E) \).

**Proof.** By Lemma 2.3, we have that \( E \) is rectifiable. Thus for \( H^m\)-a.e. \( x \in E \), \( \Theta^m(E, x) = 1 \) and \( E \) has a tangent plane at \( x \), denote it by \( T_x \). Since \( E_k \to E \) in \( U \), and \( T_x = \text{Tan}(E, x) \), we get that for any \( \varepsilon > 0 \), there exist \( 0 < r_x < \text{dist}(x, U^c) \) and \( k_x > 0 \) such that for any \( 0 < r < r_x \) and \( k \geq k_x \), we have that
\[
\mu_{x,r}(E_k \cap B(x,r)) \subseteq T_x + B(0, \varepsilon)
\]
and
\[
(1 - \varepsilon)\omega m^m \leq H^m(E \cap B(x,r)) \leq (1 + \varepsilon)\omega m^m.
\]
Since \( H^m(E \cap \partial B(x,r)) = 0 \) for \( H^1\)-a.e. \( r > 0 \), we always put ourself in the case for such \( r \).

We put \( T = T_x \), \( E_{k,x,r} = \mu_{x,r}(E_k \cap B(x,r)) \) and define \( h_{k,r} \) by given \( h_{k,r}(t) = h_k(rt) \). Then we have that
\[
E_{k,x,r} \in QM(B(0,1), M_k, h_{k,r}).
\]

For any \( 0 < \varepsilon < 1 \), we let \( g : \mathbb{R} \to \mathbb{R} \) be a function of class \( C^\infty \) such that \( 0 \leq g \leq 1 \), \( g(t) = 1 \) for \( t \leq \varepsilon \), \( g(t) = 0 \) for \( t \geq 1 \), and \( \|Dg\| \leq 2/\varepsilon \). We define mapping \( T^g_k : \mathbb{R}^n \to \mathbb{R}^n \)
\[
T^g_k(x) = (1 - g(|x|))x + g(|x|)T_1(x).
\]

Then, by setting \( T^g = [T + B(0, \varepsilon)] \cap B(0, 1) \), we have that
\[
\text{Lip}(T^g_k |_{T^g}) \leq \|DT^g_k |_{T^g}\| \leq 2 + \frac{2\varepsilon}{\varepsilon} = 4.
\]

We claim that
\[
T^g_k(E_{k,x,r}) \supseteq T \cap B(0, 1 - 2\varepsilon).
\]

We proceed by contradiction for the claim. Assume \( y \in T \cap B(0, 1 - 2\varepsilon) \setminus T^g_k(E_{k,x,r}) \). Then there is a small ball \( B(y, \rho) \) such that \( T^g_k(E_{k,x,r}) \cap B(y, \rho) = 0 \). Let \( \Psi : \mathbb{R}^n \to \mathbb{R}^n \) be a mapping of class \( C^\infty \) such that \( \Psi(z) \in \partial B(0, 1 - 2\varepsilon) \) for \( z \in B(0, 1 - 2\varepsilon) \setminus B(y, \rho) \), and \( \Psi(z) = z \) for \( z \notin B(0, 1 - 2\varepsilon) \). Then, by setting \( A_{x,r,\varepsilon} = E_k \cap B(x, r) \setminus B(x, (1 - 2\varepsilon)r) \), we have that
\[
H^m(\Psi \circ T^g_k(E_{k,x,r})) = H^m(T^g_k(E_{k,x,r} \setminus B(0, 1 - 2\varepsilon))) \leq \text{Lip}(T^g_k |_{T^g})^m H^m(E_{k,x,r} \setminus B(0, 1 - 2\varepsilon)) \leq 4^m r^{-m} H^m(E_k \cap A_{x,r,\varepsilon}),
\]
\[
\leq 4^m r^{-m} H^m(E_k \cap A_{x,r,\varepsilon}),
\]
Proof. By Lemma 2.3, we have that for any open set \( E \) equipped with the topology deduced by the local Hausdorff distance and \( \varepsilon \) and \( m \),

\[
\lim_{k \to \infty} |M\mathcal{H}^m(E_k \cap A_{x,r,\varepsilon})| \leq 4^m \varepsilon_{2(m+1)} \varepsilon \cdot |M\mathcal{H}^m(E \cap A_{x,r,\varepsilon})|.
\]

Hence

\[
\mathcal{H}^m(E \cap B(x,r)) \leq \lim_{k \to \infty} \mathcal{H}^m(E_k \cap B(x,r)) = r^m \lim_{k \to \infty} \mathcal{H}^m(E_k,x,r)
\]

\[
\leq r^m \lim_{k \to \infty} (M^2 \mathcal{H}^m(E_k,x,r) + h_k(1))
\]

\[
\leq 4^m (1 + Ch(0+)) M^2 \mathcal{H}^m(E \cap A_{x,r,\varepsilon}) + h(0+)r^m.
\]

But \( \mathcal{H}^m(E \cap B(x,r)) \geq (1 - \varepsilon) \omega_m r^m \) and

\[
\mathcal{H}^m(E \cap A_{x,r,\varepsilon}) \leq (1 + \varepsilon) \omega_m r^m - (1 - \varepsilon) \omega_m (1 - 2\varepsilon)m^2 r^m \leq 2(m + 1)\varepsilon r^m,
\]

we get so that

\[
(1 - \varepsilon) \omega_m \leq 2(m + 1) \cdot 4^m (1 + Ch(0+)) M^2 \varepsilon + h(0+),
\]

but this is a contradiction when \( h(0+) \) is small enough and \( \varepsilon \) tends to 0, and we proved the claim.

By (2.16), we have that

\[
\mathcal{H}^m(T_0 \circ \mu_{x,r}(E_k \cap B(x,r))) \geq \mathcal{H}^m(T_0(E_k,x,r \cap B(0,1 - \varepsilon))) = \mathcal{H}^m(T_0(E_k,x,r \cap B(0,1 - \varepsilon)))
\]

\[
\geq \mathcal{H}^m(T_0(E_k,x,r) - \mathcal{H}^m(T_0(E_k,x,r \cap B(0,1 - \varepsilon)))
\]

\[
\geq \mathcal{H}^m(T_k \cap B(0,1 - 2\varepsilon)) - 4^m r^{-m} \mathcal{H}^m(E_k \cap A_{x,r,\varepsilon}),
\]

thus

\[
\lim_{k \to \infty} \mathcal{H}^m(T_0 \circ \mu_{x,r}(E_k \cap B(x,r))) \geq (1 - 2\varepsilon) \omega_m - 4^m r^{-m} (1 + Ch(0+)) M \mathcal{H}^m(E \cap A_{x,r,\varepsilon})
\]

\[
\geq (1 - 2\varepsilon) \omega_m - 4^m (1 + Ch(0+)) M \cdot 2(m + 1)\varepsilon,
\]

and

\[
\lim_{r \to 0} \lim_{k \to \infty} \mathcal{H}^m(T_0 \circ \mu_{x,r}(E_k \cap B(x,r))) \geq (1 - 2\varepsilon) \omega_m - 4^m (1 + Ch(0+)) M \cdot 2(m + 1)\varepsilon,
\]

we let \( \varepsilon \) tend to 0 to get that

\[
\lim_{r \to 0} \lim_{k \to \infty} \mathcal{H}^m(T_0 \circ \mu_{x,r}(E_k \cap B(x,r))) \geq \omega_m.
\]

Applying Lemma 2.1, we get the conclusion \( v(E_k) \to v(E) \).

\[ \square \]

**Corollary 2.5.** Let \( \{E_k\} \) be a sequence of sets such that \( E_k \in QM(U, M_k, h_k) \). Suppose that \( E_k \to E \), \( M = \lim_{k \to \infty} M_k = 1 \), \( h = \lim_{m \to \infty} h_m \) satisfy that \( h(0+) = 0 \). Then we have that \( v(E_k) \to v(E) \). In particular, for any gauge function \( h \), the mapping \( QM(U, 1, h) \to V_m(U) \) given by \( E \mapsto v(E) \) is a homeomorphism between its domain and image, where \( QM(U, 1, h) \) is equipped with the topology deduced by the local Hausdorff distance and \( V_m(U) \) is equipped with the weak topology.

**Proof.** By Lemma 2.3, we have that for any open set \( O \) and compact set \( K \),

\[
\mathcal{H}^m(E \cap O) \leq \lim_{k \to \infty} \mathcal{H}^m(E_k \cap O)
\]

and

\[
\mathcal{H}^m(E \cap K) \geq \lim_{k \to \infty} \mathcal{H}^m(E_k \cap K).
\]
If $O \subseteq U$ is an open set satisfying that $\overline{O} \subseteq U$ and $\mathcal{H}^m(E \cap \partial O) = 0$, then we have that

$$\mathcal{H}^m(E \cap \overline{O}) \geq \lim_{k \to \infty} \mathcal{H}^m(E_k \cap \overline{O}) \geq \lim_{k \to \infty} \mathcal{H}^m(E_k \cap O) \geq \mathcal{H}^m(E \cap O),$$

thus

$$\mathcal{H}^m(E \cap O) = \lim_{k \to \infty} \mathcal{H}^m(E_k \cap O).$$

For any $x \in U$, we see that $\mathcal{H}^m(E \cap \partial B(x, r)) = 0$ for $\mathcal{H}^1$-a.e. $r > 0$, we can find $r > 0$ so that $B(x, r) \subseteq U$, $\mathcal{H}^m(E \cap \partial B(x, r)) = 0$ and $\mathcal{H}^m(E \cap B(x, r)) < +\infty$, thus

$$\mathcal{H}^m(E \cap B(x, r)) = \lim_{k \to \infty} \mathcal{H}^m(E_k \cap B(x, r)).$$

By Theorem 2.4, we have that $v(E_k \cap B(x, r)) \to v(E \cap B(x, r))$. Hence

$$v(E_k) \to v(E).$$

\[\square\]

3. CONVERGENCE OF QUASIMINIMAL SETS INVOLVING ELLIPTIC INTEGRANDS

A function $F : \mathbb{R}^n \times G(n, m) \to (0, \infty)$ is called an integrand, if additionally $1 \leq \sup F / \inf F < +\infty$, then we say that $F$ is bounded. For any $x \in \mathbb{R}^n$, we define integrand $F^x$ be given $F^x(y, T) = F(x, T)$. We define the functional $\Phi_F : V(\mathbb{R}^n) \to \mathbb{R}$ by the formula

$$\Phi_F(V) = \int F(x, T) dV(x, T).$$

An integrand $F$ is called elliptic if there exists a continuous function $c : \mathbb{R}^n \to (0, \infty)$ such that for any $x \in \mathbb{R}^n$,

$$\Phi_{F^x}(S) - \Phi_{F^x}(D) \geq c(x)(\mathcal{H}^m(S) - \mathcal{H}^m(D))$$

whenever $D = T \cap B(0, 1)$ for some $T \in G(n, m)$ and $S$ is a compact $m$-rectifiable set which can not be mapped into $T \cap \partial B(0, 1)$ by any Lipschitz mapping which leaves $T \cap \partial B(0, 1)$ fixed, see [2]. $F$ is called semi-elliptic if it hold $\Phi_{F^x}(S) - \Phi_{F^x}(D) \geq 0$ instead of (3.1).

**Lemma 3.1.** Let $E_k$, $M_k$, $h_k$, $E$, $M$ and $h$ be the same as in Lemma 2.3, and let $F$ be a semi-elliptic integrand. Then, for any open set $O \subseteq U$, we have that

$$\Phi_F(E \cap O) \leq \lim_{k \to \infty} \Phi_F(E_k \cap O)$$

**Proof.** For a proof, see for example Theorem 25.7 in [5] or Theorem 2.5 in [7], so we omit it here. \[\square\]

**Theorem 3.2.** Let $\{E_k\}$ be a sequence of sets such that $E_k \in QM(U, M_k, h_k)$. Suppose that $E_k \to E$ in $U$, $M = \lim_{k \to \infty} M_k < +\infty$, and $h = \lim_{m \to \infty} h_m$ satisfy that $h(0+) \to \infty$ for some elliptic integrand $F$, then $v(E_k) \to v(E)$.

**Proof.** For any $x \in \mathbb{R}^n$ and $r > 0$, we define integrand $F_{x,r}$ by given

$$F_{x,r}(y, T) = F(\mu_{x,r}(y), T), \text{ for } (y, T) \in \mathbb{R}^n \times G(n, m).$$

Then $F_{x,r}$ is also elliptic, and $F^x = \lim_{r \to 0} F_{x,r}$. Since $\Phi_F(E) = \lim_{k \to \infty} \Phi_F(E_k) < \infty$, we see that

$$\Phi_{F_{x,r}}(\mu_{x,r}(E)) = \lim_{k \to \infty} \Phi_{F_{x,r}}(\mu_{x,r}(E_k)) < \infty.$$
We will put $U_{x,r} = \mu_{x,r}(U)$, $h_{k,r}(t) = h(rt)$, $B_{x,r} = B(x,r)$, $E_{k,x,r} = \mu_{x,r}(E_k \cap B(x,r))$, $E_{x,r} = \mu_{x,r}(E)$, and $B = B(0,1)$ for convenient. Then $\mu_{x,r}(E_k) \in QM(U_{x,r},M,h_{k,r})$ and $E_{k,x,r} \in QM(B,M,h_{k,r})$. By Lemma 3.1, we have that

$$\Phi_{F_{x,r}}(\mu_{x,r}(E) \cap (U_{x,r} \setminus \overline{B})) \leq \lim_{k \to \infty} \Phi_{F_{x,r}}(\mu_{x,r}(E_k) \cap (U_{x,r} \setminus \overline{B})),$$

thus

$$\Phi_{F_{x,r}}(\mu_{x,r}(E) \cap \overline{B}) \geq \lim_{k \to \infty} \Phi_{F_{x,r}}(\mu_{x,r}(E_k) \cap \overline{B}). \quad (3.2)$$

We see that for $H^m$-a.e $x \in E$, $\Tan(E,x) \exists$ and $\Theta^m(E,x) = 1$. For any $\varepsilon > 0$, we take $0 < r_\varepsilon < \dist(x,U^c)$ and $k_\varepsilon > 0$ such that, for any $0 < r < r_\varepsilon$ and $k \geq k_\varepsilon$,

$$\mu_{x,r}(E_k \cap B(x,r)) \subseteq \Tan(E,x) + B(0,\varepsilon) \quad (3.3)$$

and

$$\mu_{x,r}(E_k \cap B(x,r)) \subseteq \Tan(E,x) + B(0,\varepsilon) \quad (3.4)$$

Let $g_1 : \mathbb{R} \to \mathbb{R}$ be a function of class $C^\infty$ such that $0 \leq g_1 \leq 1$, $g_1(t) = 0$ for $t \in (-\infty, 1 - 3\varepsilon] \cup [1, +\infty)$, $g_1(t) = 1$ for $t \in [1 - 2\varepsilon, 1 - \varepsilon]$, and $\|Dg_1\| \leq 2/\varepsilon$. We let $\Pi^c : \mathbb{R}^n \to \mathbb{R}^n$ be the mapping defined by

$$\Pi^c(x) = (1 - g_1(|x|))x + g_1(|x|)T_2(x),$$

take $1 - 2\varepsilon < \rho < \sqrt{1 - 2\varepsilon}$ and $\overline{E_k} = \Pi^c(\overline{E_{k,x,r}} \cap \overline{B(0,\rho)})$. We claim that $\overline{E_k} \supseteq \partial B(0,\rho) \cap T$ and $\overline{E_k}$ cannot be mapped into $\partial B(0,\rho) \cap T$ by any Lipschitz mapping which leaves $\partial B(0,\rho) \cap T$ fixed, where $T = \Tan(E,x)$ and $k \geq k_\varepsilon$. Suppose for the sake of contradiction there is Lipschitz mapping $\varphi$ such that $\varphi \mid_{B(0,\rho)} = \text{id}$ and $\varphi(\overline{E_k}) \subseteq T \cap \partial B(0,\rho)$. Indeed, by putting $T^c = [T + B(0,\varepsilon)] \cap B(0,1)$ and $\overline{\varphi} = \varphi \circ \Pi^c \circ \mu_{x,r}$, we have that

$$\Lip(\Pi^c \mid_{T^c}) \leq 4,$$

and

$$H^m(\overline{\varphi}(E_k \cap B(x,r))) = H^m(\Pi^c \circ \mu_{x,r}(E_k \cap B(x,r)) \setminus B(0,\rho))$$

$$\leq 4^mr^{-m}H^m(\mu_{x,r}(E_k \cap B(x,r)) \setminus B(x,(1 - 3\varepsilon)r)), \quad \text{since} \quad \lim_{k \to \infty} H^m(\mu_{x,r}(E_k \cap A_\varepsilon)) \leq M(1 + Ch(0+))(3m + 3)\omega_m\varepsilon + h(r)r^m;$$

this contradict with the local Ahlfors regularity of $E$ in case $h(0+)$ small enough, and the claim is true.

We continue to do the estimation, in fact we would like to get the same estimation as in (2.5), then we use the same technique to get the varifold convergence. For convenient, we denote by $X \triangle Y$ the symmetric difference $(X \setminus Y) \cup (Y \setminus X)$ for any sets $X,Y \subseteq \mathbb{R}^n$. Then we have that

$$H^m(\Pi^c(\mu_{x,r}(E_k)) \Delta \mu_{x,r}(E_k)) \leq (\Lip(\Pi^c \mid_{T^c})^m + 1)H^m(\mu_{x,r}(E_k) \cap B \setminus B(0,1 - 3\varepsilon))$$

$$\leq (4^m + 1)r^{-m}(E_k \cap B(x,r) \setminus B(x,(1 - 3\varepsilon)r)),$$
Thus
$$\lim_{k \to \infty} \mathcal{H}^m \left( \Pi^c(\mu_{x,r}(E_k)) \triangle \mu_{x,r}(E_k) \right) \leq (4^m + 1) \cdot M(1 + Ch(0+))(3m + 2)\omega_m \varepsilon$$
and
$$\lim_{k \to \infty} \Phi_{F_{x,r}}(E_k) \leq \lim_{k \to \infty} \Phi_{F_{x,r}}(\Pi^c \circ \mu_{x,r}(E_k) \cap B)$$
(3.5)
$$\leq \lim_{k \to \infty} \Phi_{F_{x,r}}(E_{k,x,r}) + (\sup F)(4^m + 1)M(1 + Ch(0+))(3m + 2)\omega_m \varepsilon$$
$$\leq \Phi_{F_{x,r}}(\mu_{x,r}(E) \cap \overline{B}) + C_1 \varepsilon,$$
where $C_1 = (\sup F)(4^m + 1)M(1 + Ch(0+))(3m + 2)\omega_m$. On the other hand, we see from Theorem (1)(a) in Section 3.5 in [1] that for $\mathcal{H}^m$-a.e. $x \in E$,
$$\lim_{r \to 0} r^{-m} \int_{E \cap B(x,r)} F^x(\tan(E, y)) \, d\mathcal{H}^1(y) = F^x(\tan(E, x)) \omega_m,$$
we get that
$$\lim_{r \to 0} \Phi_{F^x}(\mu_{x,r}(E) \cap \overline{B}) = \lim_{r \to 0} \Phi_{F^x}(\mu_{x,r}(E) \cap B) = F^x(\tan(E, x)) \omega_m.$$ 
We put $\omega(x, r) = \sup \{|F(y, s) - F(x, s)| : |y - x| \leq r, s \in G(n, m)|$. Then $\omega(x, r) \to 0$ as $r \to 0$; and
$$|\Phi_{F^x}(\mu_{x,r}(E_k) \cap B) - \Phi_{F_{x,r}}(\mu_{x,r}(E_k) \cap B)| \leq \omega(x, r)r^{-m} \mathcal{H}^m(E_k \cap B(x, r)).$$
We get so that
$$\lim_{k \to \infty} \Phi_{F^x}(E_k) \leq \omega(x, r) \lim_{k \to \infty} \mathcal{H}^m(E_k) + \lim_{k \to \infty} \Phi_{F_{x,r}}(E_k)$$
$$\leq \Phi_{F_{x,r}}(E_{x,r} \cap \overline{B}) + C_1 \varepsilon + \omega(x, r) \lim_{k \to \infty} \mathcal{H}^m(E_k)$$
$$\leq \Phi_{F_{x}}(E_{x,r} \cap \overline{B}) + C_1 \varepsilon + \omega(x, r) \lim_{k \to \infty} \mathcal{H}^m(E_k).$$
Since $F$ is elliptic, we have that
$$\Phi_{F^x}(E_k) - \Phi_{F^x}(T \cap B(0, \rho)) \geq c(x)(\mathcal{H}^m(E_k) - \mathcal{H}^m(T \cap B(0, \rho))),$$
thus
$$\mathcal{H}^m(E_k) \leq \omega_m \rho^m + c(x)^{-1}(\Phi_{F^x}(E_k) - F(x, T) \omega_m \rho^m).$$
Hence
$$\lim_{k \to \infty} \mathcal{H}^m(E_k) \leq \omega_m \rho^m - c(x)^{-1}F(x, T)\omega_m \rho^m$$
$$+ c(x)^{-1}[\Phi_{F^x}(E_{x,r} \cap \overline{B}) + C_1 \varepsilon + \omega(x, r) \lim_{k \to \infty} \mathcal{H}^m(E_k)],$$
and
$$(1 - c(x)^{-1}\omega(x, r)) \lim_{k \to \infty} \mathcal{H}^m(E_k) \leq \omega_m \rho^m + c(x)^{-1}[\Phi_{F^x}(E_{x,r} \cap \overline{B}) - F(x, T) \omega_m \rho^m + C_1 \varepsilon]$$
$$\leq \omega_m \rho^m + c(x)^{-1}F(x, T)\omega_m (1 - \rho^m) + c(x)^{-1}C_1 \varepsilon$$
$$\leq \omega_m + c(x)^{-1}F(x, T)\omega_m \cdot 2m \varepsilon + c(x)^{-1}C_1 \varepsilon.$$ 
Thus
$$\mathcal{H}^m(\mu_{x,r}(E_k) \cap B) \leq \mathcal{H}^m(\mu_{x,r}(E_k) \cap B \setminus B(1 - 3\varepsilon)) + \mathcal{H}^m(\mu_{x,r}(E_k) \cap B(0, 1 - 3\varepsilon))$$
$$\leq \mathcal{H}^m(\mu_{x,r}(E_k) \cap B \setminus B(1 - 3\varepsilon)) + \mathcal{H}^m(\overline{E_k}),$$
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and
\[ \lim_{k \to \infty} \mathcal{H}^m(\mu_{x,r}(E_k) \cap B) \leq \lim_{k \to \infty} \mathcal{H}^m(E_k) + M(1 + Ch(0+))(3m + 2) \omega_m \varepsilon \]
\[ \leq [1 - c(x)^{-1}\omega(x, r)]^{-1}(\omega_m + c_1(x)\varepsilon) + C_2 \varepsilon, \]

where \( c_1(x) = c(x)^{-1}F(x, T)\omega_m \cdot 2m + c(x)^{-1}C_1 \), and \( C_2 = M(1 + Ch(0+))(3m + 2) \omega_m \). We get so that
\[ \lim_{r \to 0} \lim_{k \to \infty} \mathcal{H}^m(\mu_{x,r}(E_k) \cap B) \leq \omega_m + c_1(x)\varepsilon + C_2 \varepsilon, \]

let \( \varepsilon \) tend to 0, we get that
\[ \lim_{r \to 0} \lim_{k \to \infty} \mathcal{H}^m(\mu_{x,r}(E_k) \cap B) \leq \omega_m. \]

But we see from the proof of Theorem 2.4 that
\[ \lim_{r \to 0} \lim_{k \to \infty} \mathcal{H}^m(T_1 \circ \mu_{x,r}(E_k \cap B(x, r))) \geq \omega_m, \]

we get so that for \( \mathcal{H}^m \)-a.e. \( x \in E \).
\[ \lim_{r \to 0} \lim_{k \to \infty} \mathcal{H}^m(\mu_{x,r}(E_k \cap B(x, r))) = \lim_{r \to 0} \lim_{k \to \infty} \mathcal{H}^m(T_1 \circ \mu_{x,r}(E_k \cap B(x, r))) = \omega_m. \]

Then similar to the proof of Lemma 2.1, we conclude that
\[ v(E_k) \rightharpoonup v(E). \]

\[ \square \]

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