ON A WEIGHTED SUM OF MULTIPLE T-VALUES OF FIXED WEIGHT AND DEPTH

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(Received 13 January 2021; accepted 1 February 2021; first published online 19 March 2021)

Abstract

The multiple $T$-value, which is a variant of the multiple zeta value of level two, was introduced by Kaneko and Tsumura [‘Zeta functions connecting multiple zeta values and poly-Bernoulli numbers’, in: Various Aspects of Multiple Zeta Functions, Advanced Studies in Pure Mathematics, 84 (Mathematical Society of Japan, Tokyo, 2020), 181–204]. We show that the generating function of a weighted sum of multiple $T$-values of fixed weight and depth is given in terms of the multiple $T$-values of depth one by solving a differential equation of Heun type.

2020 Mathematics subject classification: primary 11M32; secondary 33C90.

Keywords and phrases: Heun’s equation, hypergeometric function, multiple $T$-value.

1. Introduction

We call a tuple of positive integers an index. The weight, depth and height of an index $k = (k_1, \ldots, k_n)$ are defined by $k = \sum_{j=1}^n k_j$, $n$ and $s = |\{j \mid k_j \geq 2\}|$, respectively. An index $k = (k_1, \ldots, k_n)$ is called admissible if $k_n \geq 2$. Denote by $I(k, n, s)$ the set of indices of weight $k$, depth $n$ and height $s$, and by $I^0(k, n, s)$ the subset of $I(k, n, s)$ consisting of admissible indices.

In this paper we consider a generating function of the multiple $T$-value introduced by Kaneko and Tsumura [4] as a variant of the multiple zeta value of level two. For an admissible index $k = (k_1, \ldots, k_n)$, the multiple $T$-value is defined by

$$T(k) = 2^n \sum_{0 < m_1 < \cdots < m_n \atop m_i \equiv i \mod 2 \text{ for all } i} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}.$$

Our result is the following formula for the generating function of the weighted sum of the multiple $T$-values of fixed weight and depth.

**Theorem 1.1.** We have

$$1 - \sum_{k \geq n \geq 1} \left( \sum_{s=1}^n \frac{1}{2^s} \sum_{k \in I^0(k,n,s)} T(k) \right) x^{k-n} y^n = \exp \left( \sum_{n \geq 2} \frac{T(n)}{2n} (x^n + y^n - (x+y)^n) \right). \quad (1.1)$$

This work was partially supported by JSPS KAKENHI grant number 18K03233.

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398
It is known that the $\mathbb{Q}$-vector space spanned by multiple $T$-values is closed with respect to multiplication. Hence we obtain linear relations among the multiple $T$-values from Theorem 1.1 in principle by expanding the products of the $T(n)$’s in the Taylor expansion of the right side of (1.1). For example, by calculating the coefficients of $x^{k-2}y^2$ for $k \geq 3$, we find that

$$-\left(\frac{1}{2} T(1, k-1) + \frac{1}{4} \sum_{j=2}^{k-2} T(k-j, j)\right) = -\frac{k-1}{4} T(k) + \frac{1}{8} \sum_{j=2}^{k-2} T(j)T(k-j).$$

Combining this identity with the shuffle product relation (see [3, Equation (4.8)])

$$\frac{1}{2} \sum_{j=2}^{k-2} T(j)T(k-j) = (2^{k-2} - 2)T(1, k-1) + \sum_{j=2}^{k-2} (2^{j-1} - 1)T(k-j, j),$$

we reproduce the weighted sum formula

$$\sum_{j=2}^{k-1} 2^{j-1}T(k-j, j) = (k - 1)T(k)$$

due to Kaneko and Tsumura [3, Theorem 3.2]. See also [1] and the references therein for recent results on weighted sums of variants of multiple zeta values.

The proof of (1.1) is quite similar to that by Ohno and Zagier [5] for a formula for the generating function of the sum of the multiple zeta values

$$\zeta(k) = \sum_{0 < m_1, \ldots, m_n} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}$$

of fixed weight, depth and height. They consider the function

$$\Phi_0(x, y, z; t) = \sum_{k \geq 1, s \geq 1} \left( \sum_{k \in \mathbb{N}(k,n,s)} L_k(t) \right) x^{k-n-s} y^{n-s} z^{s-1},$$

where $L_k(t)$ is the multiple polylogarithm of one variable

$$L_k(t) = \sum_{0 < m_1, \ldots, m_n} \frac{t^{m_n}}{m_1^{k_1} \cdots m_n^{k_n}},$$

and show that it is the unique solution of the differential equation

$$t(1 - t) \frac{d^2 \Phi_0}{dt^2} + ((1-x)(1-y) - yt) \frac{d \Phi_0}{dt} + (xy - z) \Phi_0 = 1$$

which is holomorphic in a neighbourhood of $t = 0$ and satisfies $\Phi_0(0) = 0$. The solution is written in terms of the hypergeometric function $F(\alpha, \beta, \gamma; t)$. Using the Gauss summation formula

$$F(\alpha, \beta, \gamma; t) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)},$$
we see that the generating function $\Phi_0(x, y, z; 1)$ of the multiple zeta values is written in terms of Riemann zeta values $\zeta(n)$ with $n \geq 2$.

We will carry out the same calculation for the multiple $T$-values. In this case we have to solve Heun’s equation, which is a Fuchsian equation of second order with four singularities. Although no closed explicit formula for solutions of Heun’s general equation is known, there are several methods to find solutions under special conditions of its parameters. For our generating function in Theorem 1.1, the technique found by Ishkhanyan and Suominen [2] works, and we obtain the formula (1.1) for the multiple $T$-values in the same way as for the multiple zeta values.

2. Proof of Theorem 1.1

In [6] Sasaki defines the polylogarithm of level two. In the proof of Theorem 1.1 we use its multiple version introduced by Kaneko and Tsumura [4]. For an index $k = (k_1, \ldots, k_n)$, the multiple version of the polylogarithm of level two is defined by

$$\text{Ath}(k; t) = \sum_{0 < m_1 < \cdots < m_n \atop m_i \equiv i \text{ mod } 2 \text{ for all } i} \frac{t^{m_n}}{m_1^{k_1} \cdots m_n^{k_n}}.$$  

For the empty index $k = \emptyset$, we set $\text{Ath}(\emptyset; t) = 1$. The function $\text{Ath}(k; t)$ is holomorphic in the open disc $|t| < 1$ and, if $k$ is admissible, continuous on the closed disc $|t| \leq 1$. We have

$$d \frac{d}{dt} \text{Ath}(k; t) = \begin{cases} \frac{1}{t} \text{Ath}(k_1, \ldots, k_{n-1}, k_n - 1; t) & \text{for } k_n \geq 2, \\
\frac{1}{1 - t^2} \text{Ath}(k_1, \ldots, k_{n-1}; t) & \text{for } k_n = 1, \end{cases} \quad (2.1)$$

for any nonempty index $k = (k_1, \ldots, k_n)$.

For $n \geq 1$ and $k, s \geq 0$, we set

$$G(k, n, s; t) = 2^n \sum_{k \in I(k, n, s)} \text{Ath}(k; t), \quad G_0(k, n, s; t) = 2^n \sum_{k \in I_0(k, n, s)} \text{Ath}(k; t).$$

If the range of the sum is empty, the right side is zero. Note that

$$G_0(k, n, s; 1) = \sum_{k \in I_0(k, n, s)} T(k). \quad (2.2)$$

From (2.1) we see that

$$\frac{d}{dt} G_0(k, n, s; t)$$

$$= \frac{1}{t} (G_0(k - 1, n, s; t) + G(k - 1, n, s - 1; t) - G_0(k - 1, n, s - 1; t)),$$  

$$\frac{d}{dt} (G(k, n, s; t) - G_0(k, n, s; t)) = \frac{2}{1 - t^2} G(k - 1, n - 1, s; t), \quad (2.4)$$
The right side of (2.6) converges uniformly on the closed disc (ii).

For any index \( n \)

\[
G(k, 0, s; t) = \begin{cases} 
1 & \text{for } (k, s) = (0, 0), \\
0 & \text{otherwise}.
\end{cases}
\]

Now we consider the generating functions

\[
\Phi(t) = \Phi(x, y, z; t) = 1 + \sum_{k,n \geq 1 \atop s \geq 0} \phi(k, n, s; t)x^{k-n-s}y^{n-s}z^s,
\]

(2.5)

\[
\Phi_0(t) = \Phi_0(x, y, z; t) = \sum_{n,k,s \geq 1} G_0(k, n, s; t)x^{k-n-s}y^{n-s}z^{s-1}.
\]

(2.6)

Although we set \( z = xy/2 \) to derive (1.1), we keep the parameter \( z \) for a while.

**Lemma 2.1.** Suppose that \(|x| < 1/2, |y| < 1/4\) and \(|z| < |xy|\).

(i) The right side of (2.5) converges uniformly on the closed disc \(|t| \leq 1/2\).

(ii) The right side of (2.6) converges uniformly on the closed disc \(|t| \leq 1\).

**Proof.** Assume that \(|t| \leq 1/2\). Since \( m \leq 2^n \) for \( m \geq 1 \),

\[
|\text{Ath}(k; t)| \leq T(k_1, \ldots, k_{n-1}, k_n + 1) \leq T(1, \ldots, 1, 2)
\]

for any index \( k = (k_1, \ldots, k_n) \) of depth \( n \). Using the relation

\[
T(1, \ldots, 1, 2) = T(n + 1),
\]

which is a special case of the duality of the multiple \( T \)-values proved in [3], and the inequality \( T(n + 1) \leq 2 \), we obtain

\[
|G(k, n, s; t)| \leq 2^{n+1}|I(k, n, s)| \leq 2^{n+1} \sum_{s=0}^{n} |I(k, n, s)| = 2^{n+1} \binom{k-1}{n-1}
\]

for \( n \geq 1 \). Under the assumption on \( x, y \) and \( z \),

\[
\sum_{k,n \geq 1 \atop s \geq 0} 2^{n+1} \binom{k-1}{n-1}x^{k-n-s}y^{n-s}z^s = \frac{4|y|}{(1-|z/xy|)(1-|x|-2|y|)} < +\infty.
\]

Therefore, \( \Phi(x, y, z; t) \) converges uniformly on \(|t| \leq 1/2\), which completes the proof of (i). If \( \text{Ath}(k; t) \) is admissible, \(|\text{Ath}(k; t)| \leq 2\) on the closed disc \(|t| \leq 1\), which implies (ii). \( \square \)

Because of Lemma 2.1, term-by-term differentiation is allowed and we see that

\[
\frac{d}{dt} \Phi_0(t) = \frac{x}{t} \Phi_0(t) + \frac{1}{yt}(\Phi(t) - 1 - z \Phi_0(t)),
\]

\[
\frac{d}{dt}(\Phi(t) - z \Phi_0(t)) = \frac{2y}{1-t^2} \Phi(t),
\]
by using (2.3) and (2.4). Eliminating \( \Phi(t) \) we obtain

\[
(1 - t^2)\Phi'_0 + [(1 - x)(1 - t^2) - 2ty] \Phi'_0 + 2(xy - z) \Phi_0 = 0.
\]

Set

\[
u(t) = u(x, y, z; t) = 1 - (xy - z) \Phi_0(x, y, z; t).
\tag{2.7}

Then \( u(t) \) is the unique solution of the homogeneous equation

\[
(1 - t^2)u'' + [(1 - x)(1 - t^2) - 2ty] u' + 2(xy - z) u = 0
\tag{2.8}

which is regular in the neighbourhood of \( t = 0 \) and satisfies \( u(0) = 1 \). From (2.2) and Lemma 2.1(ii),

\[
\lim_{t \to 1-0} u(x, y, xy/2; t) = 1 - \sum_{k > n \geq 1} \left( \sum_{s=1}^{n} \frac{1}{2^s} \sum_{k \in I_k(k, n, x)} T(k) \right) x^{k-n} y^n,
\tag{2.9}

which is the left side of (1.1).

The Equation (2.8) is Heun’s equation with singularities at \( t = 0, 1, \infty \) and \(-1\). In such a case, Heun’s equation can be solved if the parameters satisfy a special condition, as discussed in [2]. We change the dependent variable \( u \)

\[
v(t) = t^{1+x} (1-t)^y (t^{-2} u(t))',
\]

where \( t^x \) and \( (1-t)^y \) are the principal values. Then \( v(t) \) is the unique solution of the equation

\[
(1 - t^2)v'' - \{x(1 - t^2) - 2(y - 1)t^2\} v' - \{(2z - xy) + y(y + x - 1)t\} v = 0
\tag{2.10}

which is holomorphic in a neighbourhood of \( t = 0 \) and satisfies \( v(0) = -x \). Substituting the power series expansion \( v(t) = -x \sum_{n=0}^{\infty} a_n t^n \) with \( a_0 = 1 \), we obtain the three-term recursion relation

\[
(n + 2)(n + 1 - x)a_{n+2} + (xy - 2z)a_{n+1} - (n - y)(n + 1 - x - y)a_n = 0
\]

for \( n \geq 0 \). Now we consider the case where

\[
z = xy/2.
\]

Then \( a_n = 0 \) if \( n \) is odd, and

\[
a_{2n} = \frac{(-y/2)_n((1-x-y)/2)_n}{n!(1-x)/2)_n}
\]

for \( n \geq 0 \), where \((c)_n\) is the shifted factorial \((c)_n = \prod_{j=0}^{n-1}(c+j)\). Thus we obtain the solution

\[
v(t) = -xF\left(-\frac{y}{2}, 1 - x - y; \frac{1}{2}, \frac{1}{2}; t^2\right),
\]

where \( F(\alpha, \beta, \gamma; t) \) is the hypergeometric function

\[
F(\alpha, \beta, \gamma; t) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n} t^n.
\]
For the time being, we assume that
\[ \text{Re} x < 0 \quad \text{and} \quad 0 < \text{Re} y < 1. \] (2.11)

Then
\[
\begin{align*}
  u(x, y, xy/2; t) &= -x t^x \int_0^t ds \, s^{-1-x}(1 - s)^{-y} F\left(-\frac{y}{2}, -\frac{1-x-y}{2}; \frac{1-x}{2}; s\right) \\
  &= -x t^x \sum_{m \geq 0} \frac{(-y/2)_m((1-x-y)/2)_m}{m! ((1-x)/2)_m} \int_0^t ds \, s^{2m-1-x}(1 - s)^{-y} \\
  &= -x \sum_{m \geq 0} t^{2m} \frac{(-y/2)_m((1-x-y)/2)_m}{m! ((1-x)/2)_m} \int_0^1 ds \, s^{2m-1-x}(1 - ts)^{-y} \quad (2.12)
\end{align*}
\]

in the open disc \(|t| < 1\). Thus we obtain the formula for the holomorphic solution \(u(t)\) of the differential equation (2.8) of Heun type in the case of \(z = xy/2\).

To prove Theorem 1.1, it suffices to calculate the limit of \(u(x, y, xy/2; t)\) as \(t \to 1 - 0\) because of (2.9).

**Lemma 2.2.** Under the assumption (2.11), the right side of (2.12) converges uniformly on the interval \(0 \leq t \leq 1\).

**Proof.** For \(0 \leq t \leq 1\) and \(m \geq 0\),
\[
\left| t^{2m} \frac{(-y/2)_m((1-x-y)/2)_m}{m! ((1-x)/2)_m} \int_0^1 ds \, s^{2m-1-x}(1 - ts)^{-y} \right| \leq \left| \frac{(-y/2)_m((1-x-y)/2)_m}{m! ((1-x)/2)_m} \right| \int_0^1 ds \, s^{-1-\text{Re} x}(1 - s)^{-\text{Re} y}
\]
under the assumption (2.11). Since the series \(F(\alpha, \beta, \gamma; 1)\) converges absolutely if \(\text{Re}(\alpha + \beta - \gamma) < 0\), the infinite sum
\[
\sum_{m \geq 0} \left| \frac{(-y/2)_m((1-x-y)/2)_m}{m! ((1-x)/2)_m} \right|
\]
converges if \(y\) satisfies (2.11). Hence the right side of (2.12) passes the Weierstrass \(M\)-test on \(0 \leq t \leq 1\). \(\square\)

Thus we obtain
\[
\lim_{t \to 1 - 0} u(x, y, xy/2; t) = -x \sum_{m \geq 0} \frac{(-y/2)_m((1-x-y)/2)_m}{m! ((1-x)/2)_m} \frac{\Gamma(2m - x)\Gamma(1-y)}{\Gamma(2m + 1 - x - y)}
\]
\[
= \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} \sum_{m \geq 0} \frac{(-y/2)_m(-x/2)_m}{m! (1 - (x+y)/2)_m}
\]
\[
= \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} F\left(-\frac{y}{2}, -\frac{x}{2}, -\frac{1}{2}; \frac{1-x+y}{2}; 1\right)
\]
\[
= \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} \frac{\Gamma(1-(x+y)/2)}{\Gamma(1/2)\Gamma(1-y/2)}.
\] (2.13)
Using
\[ \Gamma(1-t) = \exp\left(\gamma t + \sum_{n \geq 2} \frac{\zeta(n)}{n} t^n\right), \]
where \( \gamma \) is Euler’s constant, and
\[ T(n) = 2\left(1 - \frac{1}{2n}\right)\zeta(n), \]
we see that the right side of (2.13) is equal to that of (1.1), which is holomorphic in a neighbourhood of \((x, y) = (0, 0)\). Therefore, we can omit the assumption (2.11). This completes the proof of Theorem 1.1.

**Remark 2.3.** Equation (2.10) can be solved explicitly also in the case of \( z = 0 \), where the solution is \( v(t) = -x(1 + t)^y \). From this we reproduce the formula for the generating function of height-one \( T \)-values
\[
1 - \sum_{k \geq n \geq 1} T(1, \ldots, 1, k - n + 1)x^{k-n}y^n = \frac{2\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)}F(1-x, 1-y, 1-x-y; -1)
\]
proved by Kaneko and Tsumura [3]. We omit the details of the calculation.

**Acknowledgements**

The author is very grateful to Professor Yasuo Ohno for a discussion on the problem dealt with in this article and for encouragement, and to Professor Masanobu Kaneko for valuable comments. He also thanks the referees for suggestions that improved this paper.

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