Stationary states in two lane traffic: insights from kinetic theory

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Kinetics of dilute heterogeneous traffic on a two lane road is formulated in the framework of Ben-Naim Krapivsky model and stationary state properties are analytically derived in the asymptotic limit. The heterogeneity is introduced into the model as a quenched disorder in desired speeds of vehicles. The two-lane model assumes that each vehicle/platoon in a lane moves ballistically until it approaches a slow moving vehicle/platoon and then joins it. Vehicles in a platoon are assumed to escape the platoon at a constant rate by changing lanes after which they continue to move at their desired speeds. Each lane is assumed to have a different escape rate. As the stationary state is approached, the platoon density in the two lanes become equal, whereas the vehicle densities and fluxes are higher in the lane with lower escape rate. A majority of the vehicles enjoy a free-flow if the harmonic mean of the escape rates of the lanes is comparable to average initial flux on the road. The average platoon size is close to unity in the free-flow regime. If the harmonic mean is lower than the average initial flux, then vehicles with desired speeds lower than a characteristic speed $v^*$ still enjoy free-flow while those vehicles with desired speeds that are greater than $v^*$ experience congestion and form platoons behind the slower vehicles. The characteristic speed depends on the mean of escape times $(R = (R_1 + R_{-1})/2)$ of the two lanes (represented by 1 and -1) as $v^* \sim R^{-\mu_1}$, where $\mu$ is the exponent of the quenched disorder distribution for desired speed in the small speed limit. The average platoon size in a lane, when $v^* \ll 1$, is proportional to $R^{\mu_1}$ plus a lane dependent correction. Equations for the kinetics of platoon size distribution for two-lane traffic are also studied. It is shown that a stationary state with platoons as large as road length can occur only if the mean escape rate is independent of platoon size.

Keywords: Traffic flow, power laws, quenched disorders, Boltzmann equations, mult-lane traffic

I. INTRODUCTION

Research on the development of algorithms related to mobility of connected and automated vehicles (CAVs) has taken a front seat as it is expected that a traffic system with CAVs would help in efficient traffic management, reduce energy consumption and pollution, apart from other advantages like increasing safety etc.\textsuperscript{1,2} An important application of these algorithms for CAVs is cooperative driving automation (CDA)\textsuperscript{3,4} by which vehicles operate and move in a cooperative way by sharing information. CDA brings in a need to understand the Physics of heterogeneous traffic in a detailed way, especially the collective phenomena that occur in heterogeneous traffic, for building efficient algorithms to implement the CDA.

Models for traffic on a single lane predict the occurrence of interesting collective phenomena in the kinetics of relaxation to stationary state and in some physically observable quantities in the stationary state when passing is not allowed.\textsuperscript{5,6,13} The main reason for the emergence of the collective phenomena is heterogeneity in traffic (namely, differences in driving characteristics from vehicle to vehicle), typically modeled as quenched disorders in the parameters of the model.\textsuperscript{5,7,8,14,15,19,20,23} For instance, an interesting phenomenon occurs when the desired speeds of the drivers/vehicles in the traffic are heterogeneous and if passing is not allowed. Faster vehicles in the system form platoons behind the slower vehicles and the average platoon size grows with time as a power-law $\sim \mu \nu^{1/\mu + 2}$ until all the platoons coalesce and form a single giant platoon. $\mu$ is the exponent of the quenched disorder distribution in desired speeds of vehicles in the low-speed limit. The collective phenomena on a single lane are have been studied using various kinds of approaches of cellular-automata, car-following models\textsuperscript{9,16,17} and formalism was developed for a single lane. A more realistic case would be to consider two lanes with vehicles allowed to change lanes at any point on the road. The question of what happens in a two lane system, when lane changing is allowed, is not fully understood.

Ben Naim, Krapivsky and Redner developed a kinetic theory for which they could analytically derive the power-law for growth of platoon size with time.\textsuperscript{18} In a series of later work,\textsuperscript{20–23} Ben-Naim and Krapivsky (BK) included passing of vehicles in their formalism and came up with an equation similar to the Boltzmann equation. They studied the stationary states of the system and showed that the platoons of vehicles still form when the rate of escape from a platoon is smaller than the rate of formation of the platoon. The beauty of the formulation by BK is that, despite its simplifying assumptions, some analytical results could be obtained which give a qualitative understanding of the phenomenon. However, the formalism was developed for a single lane. A more realistic case would be to consider two lanes with vehicles allowed to change lanes at any point on the road. The nuances of a two-lane picture over a single-lane picture needs to be understood as that would provide insights into collective phenomena in road traffic in a more real-
istic way. Thus, in the present work we formulate the two-lane road problem within the framework laid out by the BK model and study its stationary states.

In Sec. [II] the kinetic equations for velocity distribution of platoons and vehicles for a two lane traffic are formulated. General forms for the velocity distributions in the stationary state are derived in Sec. [II] Conditions for a majority of vehicles to experience free flow and those for congested flow are derived in Sec. [IV] and various useful quantities are obtained in the asymptotic limits. In Sec. [VI] the equations for platoon size distribution in a two lane traffic system are formulated and the platoon phase transition is studied. The results are summarized

II. KINETIC THEORY FOR TWO LANE TRAFFIC FLOW

Consider an infinitely long two-lane freeway with lane changing allowed in any part of the road. Suppose the traffic is heterogeneous due to a quenched disorder in the (free-flow) desired speeds of vehicles. All vehicles, at \( t = 0 \), move with their desired speeds which are drawn independently from a bounded probability density \( P_0(V_0 = v) \). Assume that the length scale considered is sufficiently large that the probability distribution is accurately sampled by the vehicles in one unit of length. As the traffic flow evolves in time, formation and dissolution of platoons (clusters) of vehicles happens. A platoon’s size grows when a fast moving platoon joins a slow moving platoon and the two platoons merge; we will refer to this merging of platoons as a “collision” event (not to be confused with an actual collision between two vehicles). The platoons move ballistically between successive collisions. A platoon’s size drops when a vehicle in the platoon changes lanes, which we refer to as an “escape” event. After changing lanes the vehicle moves at its desired speed until it collides with another vehicle in that lane. The vehicles are assumed to instantaneously increase or decrease their speeds which is justified for a dilute system where the mean collision time is much larger than the mean relaxation time of speed.

We use the same dimensionless units as used by BK for consistency. The two lanes are indexed as \(-1\) for one of the lanes and \(1\) for the other; these indices appear as suffixes throughout the paper. In the dimensionless units, velocity is scaled by \( v_0 \) i.e., \( v/V_0 \rightarrow v \) where \( V_0 \) is the maximum free-flow speed available, i.e., \( v_0 \equiv \sup_v P_0(v) \). Let \( \rho_0 = (\rho_{01} + \rho_{02}) \) be the initial global density of vehicles on the road (total number of vehicles per unit length of the system), where \( \rho_{01} \) is the initial global density in lane \( i \). \( \rho_0 \) remains constant over time; it’s inverse, \( \bar{\rho}_0 = \rho_0^{-1} \), is the initial mean gap between vehicles on the road. We use \( \bar{x} \) to scale the spatial coordinates as \( x\bar{x}^{-1} = \bar{x} \rho_0 \rightarrow x \) to make it dimensionless. Similarly time is scaled as \( \rho_0 t \rightarrow t \) which renders \( t \) dimensionless. \( (\rho_0 t)^{-1} \) may be interpreted as initial mean platoon collision time. Let \( R_i \) be the mean escape time of vehicles in lane \( i \). In dimensionless units, \( \rho_0 R_i \) is the relative to the slowest vehicle in the system.

For book keeping purposes, we term those vehicles moving at their desired speed \( v \) as \( v \)-vehicles and those vehicles whose desired speed is \( v \) but are moving at a slower speed \( v' \) are termed as \( (v, v') \)-vehicles.

Let \( \Phi_i(v, t) (i \in \{-1,1\}) \) be the density of platoons (of any size) moving at speed \( v \) which are led by vehicles whose desired speed is \( v \) (i.e., \( v \)-vehicles). Thus, it also gives the density of \( v \)-vehicles moving at their desired speed \( v \). At time \( t = 0 \), we have that \( \Phi_i(v, 0) = P_0(v) \rho_0 \), implying that initially there are only single vehicles (platoons of size \( 1 \)) and all start moving at their desired speed. Let \( \Psi_i(v', t; v) \) be the density of \( (v, v') \)-vehicles at time \( t \). The speeds of these vehicles would have dropped as a result of collisions with slower platoons. The master equation describing the evolution of platoon density in lane \( i \) is

\[
\frac{\partial \Phi_i(v, t)_{\text{}}}{\partial t} = \frac{1}{R_{-i}} \int_0^\nu dv' \Psi_{-i}(v', t; v)
- \Phi_i(v, t) \int_0^\nu dv' (v' - v) \Phi_i(v', t).
\] (1)

The first term on the right-hand side (RHS) represents the rate at which \( \Phi_i(v, t) \) increases as a result of \( (v, v') \)-vehicles in the adjacent lane \((-i)\) changing lanes to increase their speed to their desired speed \( v \). \( R_{-i}^{-1} \) is the average rate at which vehicles leave lane \(-i\).

\[
\int_0^\nu dv' \Psi_{-i}(v', t; v) \text{ is the density of vehicles in lane } -i \text{ moving slower than their desired speed } v \text{ per unit time from lane } -i \text{ to lane } i.
\]

The second term represents the number of collisions per unit length per unit time in lane \( i \) between platoons moving at speed \( v \) and slower platoons ahead, thereby slowing down. The second term employs Boltzmann’s Stosszahlsatz assumption (a.k.a. molecular chaos). It essentially states that the rate of collision is proportional to the difference in speed, the higher the difference, the more likely the collision (or the higher the rate of collision). The master equation describing the evolution of slowed-down cars in lane \( i \) is

\[
\frac{\partial \Psi_i(v', t; v)}{\partial t} = -\frac{1}{R_i} \Psi_i(v', t; v)
- \Psi_i(v', t; v) \int_0^\nu dw(v' - w) \Phi_i(w, t)
+ \Phi_i(v', t) \int_0^\nu dw (w - v') \Psi_i(w, t; v)
+ \Phi_i(v, t) \Phi_i(v', t)(v - v').
\] (2)

The first term on the RHS represents the rate at which \( (v, v') \)-vehicles escape a platoon by changing from lane \( i \) to lane \(-i\). The second term represents collisions between \( (v, v') \)-vehicles and platoons moving at speeds lower than \( v' \). The third term represents collisions between platoons moving at a speed of \( v' \) and \((v, w)\)-vehicles for which
\( v' < w < v \). The fourth term is the collision rate between \( v \)-platoons and \( v' \)-platoons. Note that there is no term due to vehicles from lane \(-i\) as it is assumed that the vehicles attain their desired speed as soon as they change lane. Equations (1) and (2) are a set of four coupled equations that describe the evolution of a two lane traffic within the BKR model. The initial conditions for Equations (1) and (2) are \( \Phi_i(v, 0) = P_0(v)\rho_{0i} \) and \( \Psi_i(v', 0; v) = 0 \), respectively. The density of vehicles in lane \( i \) whose desired speed is \( v \) is

\[
\eta_i(v, t) = \Phi_i(v, t) + \int_0^v dv' \psi_i(v', t; v).
\]

The density of vehicles on any lane whose desired speed is \( v \) is \( \eta(v, t) = \frac{1}{2}(\eta_{-1}(v, t) + \eta_1(v, t)) \). From Equations (1) and (2) we have that

\[
\frac{\partial \eta_i(v, t)}{\partial t} = 0, \tag{4}
\]

which implies that

\[
\eta_{-1}(v, t) + \eta_1(v, t) = P_0(v) \tag{5}
\]

for all \( t \geq 0 \), confirming the conservation of the number of vehicles by the equations of the model. To show that (4) is true, we only need to note that the second and third terms in Eq. (2) cancel one another upon integrating \( v' \) over \((0, v)\), that is,

\[
-\int_0^v \int_0^{v'} dv' dw'(v' - w) \psi_i(v', t; v) \Phi_i(w, t) + \int_0^{v'} \int_0^v dv' dw(w - v') \psi_i(w, t; v) \Phi_i(v', t) = 0,
\]

since the domain of integration of the first integral, \( \{(v', w) : v' \in (0, v), w \in (0, v')\} \), is a triangular region in \( \mathbb{R}^2 \) that can be equivalently written as \( \{(w, v') : w \in (0, v), v' \in (w, v)\} \). For the first integral above, this equivalence implies that

\[
\int_0^v \int_0^{v'} dv' dw'(v' - w) \psi_i(v', t; v) \Phi_i(w, t) = \int_0^v dw dw'(v' - w) \psi_i(v', t; v) \Phi_i(w, t).
\]

A change of variable shows that the terms cancel one another as claimed. The remaining steps needed to demonstrate that Eq. (1) follow immediately from the definitions.

In the next section we study the stationary state behavior of the system of equations Eq. (1), Eq. (2) and the Eq. (5).

### III. THE STATIONARY STATE

As the stationary state is approached, “gain” becomes equal to “loss” and all the densities become time independent. Thus, in each of the Eq. (1) and Eq. (2), LHS tends to zero asymptotically in time. Therefore, we have in the stationary state

\[
\frac{1}{R_i} \int_0^v dv' \psi_i(v'; v) = \Phi_i(v) \int_0^v dv' (v' - v) \Phi_i(v') \tag{6}
\]

and

\[
\frac{1}{R_i} \psi_i(v'; v) = -\Psi_i(v'; v) \int_0^v dw (v' - w) \Phi_i(w) + \Phi_i(v') \int_0^v dw (w - v') \Psi_i(w; v) + \Phi_i(v) \Phi_i(v')(v - v'). \tag{7}
\]

Note that we have dropped the dependence on \( t \) in this stationary setting. Integrating Eq. (7) on both sides over \( v' \) between \( 0 \) and \( v \), we get

\[
\frac{1}{R_i} \int_0^v dv' \psi_i(v'; v) = \Phi_i(v) \int_0^v dv' (v' - v) \Phi_i(v'), \tag{8}
\]

where we have again used the fact that the first and second terms in Eq. (7) cancel each other upon integrating \( v' \) over \((0, v)\). For convenience, we define the auxiliary functions

\[
Q_i(v) = R_i^{-1} + \int_0^v dv' (v' - v) \Phi_i(v'), \tag{9}
\]

and

\[
\tilde{Q}_i(v'; v) = \int_0^{v'} dw (w - v') \Psi_i(w; v). \tag{10}
\]

Then using Eq. (8) and Eq. (3), we can write Eq. (5) as

\[
R_i \Phi_i(v) Q_1(v) + R_{-1} \Phi_{-1}(v) Q_{-1}(v) = P_0(v). \tag{11}
\]

Equating Eq. (8) and Eq. (6), we get

\[
\Phi_i(v) \xi_i(v) = \Phi_{-1}(v) \xi_{-1}(v) \tag{12}
\]

where \( \xi_i(v) = Q_i(v) - R_i^{-1} \) is the stationary collision rate. Eq. (8) and Eq. (6) mean that the escape rate and the collision rate on any lane must become equal in the stationary state. Noting that

\[
Q_i''(v) = \frac{\partial^2 Q_i(v)}{\partial v^2} = \Phi_i(v) \tag{13}
\]

and

\[
\tilde{Q}_i''(v'; v) = \frac{\partial^2 \tilde{Q}_i(v'; v)}{\partial v'^2} = \Psi_i(v'; v) \tag{14}
\]

with \( Q_i(0) = R_i^{-1} \), \( Q_i'(0) = 0 \), \( \tilde{Q}_i(v; v) = 0 \) and \( \frac{\partial \tilde{Q}_i(v; v)}{\partial v} = 0 \), Eq. (11) and Eq. (12) may be re-written as

\[
R_i Q_i'(v) Q_1(v) + R_{-1} Q_{-1}'(v) Q_{-1}(v) = P_0(v) \tag{15}
\]

and

\[
Q_i''(v) \xi_1(v) = Q_{-1}''(v) \xi_{-1}(v) \tag{16}
\]
Integrating the above equation using the boundary conditions, we get
\[ \Psi_i(v'; v) = \Phi_i(v) Q_i(v) \Phi_i(v') \int_{v'}^{v} \frac{du}{[Q_i(u)]^2}. \] (19)

Equations (15), (16) and (19) completely describe the stationary state of the system. Upon solving them, \( \Phi_i(v) \), \( \Psi_i(v'; v) \) can be obtained, and from these other quantities of interest may be obtained. A recipe for calculating a few quantities is as follows: the density of platoons in lane \( i \) is
\[ \rho_i = \int_0^1 dv \Phi_i(v) \] (20)
and the total platoon density is
\[ \rho = (\rho_1 + \rho_{-1})/2. \] (21)
The average platoon size \( \Lambda_i \) is
\[ \Lambda_i \sim \rho_i / \rho_i^c \] (22)
where
\[ \rho_i^c = \int_0^1 dv \ G_i(v). \] (23)
The average platoon speed is given by
\[ (v_i) = \rho_i^{-1} \int_0^1 dv \ v \ \Phi_i(v). \] (24)
The flux in lane \( i \) is given by
\[ J_i = \int_0^1 dv \ v \ G_i(v). \] (25)

IV. FREE FLOW AND CONGESTION

As seen in the single lane case without passing, even in dilute traffic, vehicles with higher desired speeds face congestion and those with lower desired speed always experience free-flow. When passing is allowed in the single lane, Ben Naim and Krapivsky showed that vehicles having their desired speed below a characteristic speed \( v^* \) experience free-flow while those having desired speed above \( v^* \) face congestion. Below we quantify the phenomenon for the two lane case with lane changing.

First we note that \( \Phi_i \) not only represents the density of platoons in lane \( i \) moving with speed \( v \) but it also represents the density of \( v \)-vehicles in lane \( i \). A majority of these vehicles experience free-flow in the stationary state if, in Eq. (11),
\[ 0 \sim R_1 \Phi_1(v) \xi_1(v) + R_{-1} \Phi_{-1}(v) \xi_{-1}(v) \ll \Phi_1(v) + \Phi_{-1}(v), \] (26)
which basically means that the total number of collision events experienced by \( v \)-vehicles on both the lanes over a time scale of their respective escape times is much less than the number of \( v \)-vehicles. Above condition occurs when \( R_1 \xi_1(v) \) and \( R_{-1} \xi_{-1}(v) \) are small. As \( \xi_i(v) \) is a monotonically increasing function of \( v \), the smallness of \( \xi_i(v) \) implies \( v \) has to be small. Thus the requirement that \( R_i \xi_i(v) \) has to be small implies that \( v \) has to be smaller than a characteristic speed \( v^* \) as shown below. When above condition is satisfied, Eq. (11) becomes
\[ \Phi_1(v) + \Phi_{-1}(v) \approx P_0(v). \] (27)

Using Eq. (12) in above equation, we get
\[ \Phi_i(v) \approx \frac{P_0(v) \xi_{-1}(v)}{\xi_i(v) + \xi_{-1}(v)} \] (28)
A solution which satisfies both Eq. (27) and Eq. (12) is \( \xi_i(v) = \xi_{-1}(v) \) which implies
\[ \Phi_1(v) = \Phi_{-1}(v) \approx P_0(v)/2. \] (29)

Considering the above forms of \( \Phi_1 \) and \( \Phi_{-1} \) as zeroth order terms of a perturbation series with \( (R_1 + R_{-1}) \xi_0(v) \) as a perturbation parameter, one may obtain higher order terms in the series for \( \Phi_1 \) and \( \Phi_{-1} \), where \( \xi_0(v) = \int_0^v dv' (v - v') P_0(v'). \) The choice of the perturbation parameter may be motivated by Eq. (26), which says \( (R_1 + R_{-1}) \xi_0(v) \sim 0. \) Using the zeroth order forms of \( \Phi_i \) in \( \xi_i \), in Eq. (11) and Eq. (12), the first order approximation turns out to be
\[ \Phi_i(v) \approx \frac{P_0(v)}{2} \left( 1 - \frac{R_1 + R_{-1}}{4} \xi_0(v) \right). \] (30)
When \( v \sim v^* \),
\[ R_1 \Phi_1(v) \xi_1(v) + R_{-1} \Phi_{-1}(v) \xi_{-1}(v) \sim \Phi_1(v) + \Phi_{-1}(v) \] (31)
Approximating \( \Phi_1(v) \sim \Phi_2(v) \sim P_0(v)/2 \), we get
\[ \int_0^{v^*} dv (v^* - v) P_0(v) \sim \left( \frac{R_1 + R_{-1}}{4} \right)^{-1}. \] (32)
If \( P_0(v) \sim A v^\mu \) for \( v \in [0, v^*] \) we see that
\[ v^* \sim \left( \frac{R_1 + R_{-1}}{4} \right)^{-1/(\mu+2)}, \] (33)
where $\mu > -1$ to ensure normalizability of $P_0(v)$. We may also arrive at the above form of $v^*$ by equating the perturbation parameter to unity. For those $v$-vehicles with $v > v^*$, the perturbation parameter is no longer small and the perturbation series becomes invalid. Both lanes have same $v^*$ as expected because of mixing of vehicles. If $v^* < 1$, then most of vehicles with their desired speed between $v^*$ and 1 (maximum speed) would have slowed down due to collisions and experience congestion as the stationary state is reached. The power-law decay of $v^*$ with $R_1 + R_{-1}$ would be faster for smaller values of $\mu$. $v^* > 1$ actually signifies that all the vehicles on the road would experience free-flow irrespective of their speed as the maximum available speed is 1. The conditions $v^* \sim 1$ separating the free-flow regime from the mixed regime (with both congested and free-flowing traffic) may be understood in a more illuminating way by writing $v^*$ in dimensional form, which is

$$v^* \sim (\rho_0 v_0)^{-1/(\mu+2)} \left(\frac{R_1 + R_{-1}}{4}\right)^{-1/(\mu+2)}$$

$$= \left(\frac{\rho_0 v_0}{2} \left(\frac{R_1 + R_{-1}}{2}\right)\right)^{-1/(\mu+2)}$$

Therefore,

$$v^* \sim 1 \implies 2 \frac{R_1 + R_{-1}}{\rho_0 v_0} \sim \frac{2}{R_1 + R_{-1}},$$

(35)

where $\rho_0 v_0/2$ is the lane averaged initial flux and $2/(R_1 + R_{-1})$ is harmonic mean of escape rates ($R_1^{-1}$ and $R_{-1}^{-1}$) of individual lanes. Hence, a majority of the vehicles in the system enjoy free-flow if the harmonic mean of the escape rates on the lanes is of the order of average initial flux. Therefore, if the initial density is very low and/or the maximum desired speed of the drivers is very small, then the system experiences a free-flow even if the escape rates on the lanes are low. On the other hand, if the initial density is high and the maximum desired speed is also high, the escape rates have to be high to maintain free-flow. Below we calculate some quantities of interest for both the cases $v^* > 1$ and $v^* < 1$.

**A. Case: $v^* > 1$**

The total platoon density $\rho_i$ in lane $i$ to first order in $(R_1 + R_{-1})$ is

$$\rho_i \approx \frac{1}{2} \frac{R_1 + R_{-1}}{4} \int_0^1 dv G_i(v)$$

Thus, mixing of vehicles between lanes due to lane changing equalizes the platoon density on both the lanes as the stationary state is reached.

Using $\Phi_i(v) \approx P_0(v)/2$ in $Q_i$ and noting that $(R_1 + R_{-1})\xi_0(v) \ll 1$, we find that $Q_i(v) \approx R_i^{-1}$ which on substitution into Eq. (19) gives

$$\Psi_i(v'; v) \approx \frac{R_i}{4} P_0(v) P_0(v')(v - v')$$

(37)

Using Eqs. (30) and (37) in Eq. (17) gives

$$G_i(v) \approx \frac{P_0(v)}{2} - \frac{R_i + R_{-1}}{8} P_0(v) \xi_0(v)$$

$$+ \frac{R_i}{4} P_0(v) \int_0^1 dv' (v' - v) P_0(v').$$

(38)

The total vehicle density in lane $i$ is

$$\rho_i^* = \int_0^1 dv G_i(v) \approx \frac{1}{2} + \frac{R_i - R_{-1}}{8} \int_0^1 dv P_0(v) \xi_0(v)$$

$$= \rho_{0i} + \frac{\rho_{0i} - \rho_{0i}}{2} + \frac{R_i - R_{-1}}{8} \int_0^1 dv P_0(v) \xi_0(v).$$

(39)

The average platoon size in lane $i$ is

$$\Lambda_i = \frac{\rho_i^*}{\rho_i} \approx 1 + \frac{3R_i + R_{-1}}{4} \int_0^1 dv P_0(v) \xi_0(v).$$

(40)

which is close to unity as $R_1$ and $R_2$ are small. The $G_i(v)$ obtained above can be used to calculate the flux on lane $i$ which turns out to be

$$J_i \approx \frac{J_0}{2} + \frac{R_i - R_{-1}}{8} A_0$$

$$= J_0 + \frac{\rho_{0i} - \rho_{0i}}{2} J_0 + \frac{R_i - R_{-1}}{8} A_0,$$

(41)

where $J_0$ is the first moment of $P_0(v)$ i.e.,

$$J_0 = \int_0^1 dv v P_0(v),$$

$$A_0 = \int_0^1 dv v P_0(v) \xi_0(v),$$

and

$$J_{0i} = \rho_{0i} \int_0^1 dv v P_0(v)$$

is the initial flux. Thus, the total vehicle density and the flux on each lane are different. For instance, if $\rho_{0i} > \rho_{0i}$ and $R_{-1} < R_1 < 1$, then Eq. (39) and Eq. (41) say that $\rho_i^* > \rho_{0i}^*$ and $J_i > J_{0i}$ which is physically expected as vehicles leave lane 1 at a slower rate than those entering lane 1.

To summarize, when $v^* > 1$, the system reaches to a stationary state in which both the lanes have same platoon density which is similar to that derived for a single lane with an effective initial velocity distribution $P_0(v)/2$ and effective escape time $(R_1 + R_{-1})/2$ i.e., the mean of the escape times of both the lanes. However, the vehicle density and flux on each road is different from that of the single lane case and thus the two lane case cannot be exactly mapped to the single lane case.
B. Case: \( v^* < 1 \)

Those \( v \)-vehicles with \( v < v^* \) experience free-flow and the above discussed perturbation series is valid for \( \Phi_i(v) \). For \( v \)-vehicles with \( v \gg v^* \),

\[
R_1 \Phi_1(v) \xi_1(v) + R_{-1} \Phi_{-1}(v) \xi_{-1}(v) \approx \Phi_1(v) + \Phi_{-1}(v). \tag{42}
\]

As \( v^* \ll 1 \) implies \( (R_1 + R_{-1}) \gg 1 \) from Eq.\((33)\). Using the above approximation and Eq.\((12)\) in Eq.\((11)\), \( \Phi_1(v) \) and \( \Phi_{-1}(v) \) get decoupled and satisfy

\[
(R_1 + R_{-1}) \Phi_2(v) \xi_2(v) \approx P_0(v) \tag{43}
\]

or

\[
\xi''_i(v) \approx P_0(v)(R_1 + R_{-1})^{-1} \approx P_0(v)v^{\sigma + 2}. \tag{44}
\]

In the above form, the equations for platoon density are same as those of single lane case with an effective escape rate \( (R_1 + R_{-1})^{-1} \). Looking at the equations, it may be inferred that \( \Phi_1 = \Phi_{-1} \) in this case as well. Ben Naim and Krapiisky derived an approximate form for \( \Phi_i \) using Eq.\((44)\) when

\[
P_0(v) = \frac{v^\mu}{\mu + 1}, \quad \text{for } v \in [0, 1], \tag{45}
\]

where \( \mu > -1 \) for \( P_0(v) \) to be normalizable. We derive it below for completeness and explain in detail the subtle approximations involved. Firstly, we re-write \( \xi_i(v) \) as

\[
\xi_i(v) = \int_0^v dv'(v - v') \Phi_i(v')
= \int_0^{v^*} dv'(v - v') \frac{P_0(v')}{2} + \int_{v^*}^v dv'(v - v') \Phi_i(v')
= \frac{v^{\mu + 1}}{2(\mu + 1)} \left\{ \frac{v}{\mu + 1} - \frac{v^*}{\mu + 2} \right\} + \tilde{\xi}_i(v), \tag{46}
\]

where

\[
\tilde{\xi}_i(v) = \int_{v^*}^v dv'(v - v') \Phi_i(v').
\]

and \( \xi''_i(v) = \Phi_i(v) \). Using Eq.\((46)\) in Eq.\((44)\) we get

\[
\xi''_i(v) \approx P_0(v)v^{\sigma + 2}. \tag{47}
\]

Substituting \( \tilde{\xi}_i \sim v^{\sigma^*}v^\delta \) in Eq.\((47)\), with some manipulation we get

\[
\left( \frac{v}{v^*} \right)^{2\delta - 2} - \left( \frac{v}{v^*} \right)^{\delta - 2} v^{2\sigma + 2\delta - 2}
+ \left( \frac{v}{v^*} \right)^{\delta - 1 - \frac{\sigma}{\mu + 1}} - \left( \frac{v}{v^*} \right)^{\delta - 2} v^{\sigma + \mu + \delta} \approx \left( \frac{v}{v^*} \right)^{\mu} v^{2\mu + 2}. \tag{48}
\]

In the above equation, equating the exponents of \( v^* \) on both sides we get the relation

\[\sigma + \delta = \mu + 2. \tag{49}\]

When \( v \gg v^* \) and \( \delta > 1 \), Eq.\((48)\) may be approximated as

\[
\left( \frac{v}{v^*} \right)^{2\delta - 2} v^{2\sigma + 2\delta - 2} \approx \left( \frac{v}{v^*} \right)^{\mu} v^{2\mu + 2}, \tag{50}
\]

from which we get \( \delta = (\mu + 2)/2 \) by equating the exponents of \( v/v^* \). Using it in Eq.\((49)\), we get \( \sigma = (\mu + 2)/2 \). The above relations are valid when \( \mu > 0 \) as \( \delta > 1 \). In a similar way, when \( v \gg v^* \) and \( \delta < 1 \), Eq.\((48)\) may be approximated as

\[
\left( \frac{v}{v^*} \right)^{\delta - 1} v^{\sigma + \mu + \delta} \approx \left( \frac{v}{v^*} \right)^{\mu} v^{2\mu + 2}. \tag{51}
\]

Equating the coefficients of \( v/v^* \) we get \( \delta = \mu + 1 \) which when substituted in Eq.\((49)\) gives \( \sigma = 1 \). These expressions are valid when \( \mu < 0 \) as \( \delta < 1 \). Summarizing the results,

\[
\Phi_i(v) \sim \begin{cases} 
R_1^{-1/2}v^{(\mu - 2)/2} & \mu \geq 0 \\
R_1^{-1/2}v^{(\mu + 2)/2} & \mu < 0 
\end{cases} \tag{52}
\]

where \( R = (R_1 + R_{-1})/4 \)

Let us look at the joint distribution on lane \( i \), \( \Phi_i(v'; v) \) in the limit \( v^* \ll 1 \). When \( v' < v^* \) and \( v < v^* \), by the same arguments given for platoon density we note that the approximations derived in free-flow limit are valid. Therefore,

\[
\Psi_i(v'; v) \approx \frac{R_0}{4}P_0(v)P_0(v')(v - v'). \tag{53}
\]

For the case \( v' < v^* \) and \( v > v^* \), substituting \( \Phi_i(v') \approx P_0(v)/2 \) and the expression in Eq.\((52)\) for \( \Phi_i(v) \) we get

\[
\Psi_i(v'; v) \approx (R^2 + R^\mu)v^{\mu - 1}v'v^\mu \tag{54}
\]

for any \( \mu > -1 \). For \( v^* > v^* \) and (obviously) \( v > v^* \), we get

\[
\Psi_i(v'; v) \sim \begin{cases} 
R_{1/2}v^{(\mu + 1)/2}v'v'^{\mu - 2/2} & (\mu \geq 0) \\
R_{1/2}v^{(\mu + 1)/2}v'^{\mu - 2/2} & (\mu < 0) 
\end{cases} \tag{55}
\]

The car density may be obtained by using above derived forms for \( \Phi_i(v) \) and \( \Psi_i(v'; v) \) in Eq.\((17)\). For \( v < v^* \) we get

\[
G_i(v) \sim \left( \frac{R}{R_1} \right)^{\mu + 1} \left( \frac{R}{R_1} \right)^{\mu + 1} \left( R^2 + R^\mu \right) v^{\mu} \tag{56}
\]

for any \( \mu > -1 \). Thus the car density increases enormously from its initial value for \( v < v^* \). Similarly for
\[ v \gg v^* \text{ and } \mu \geq 0, \text{ we get} \]
\[ G_i(v) \sim R^{-1/2} v^{\mu - 2/1} \]
\[ + R^{-1/2} \left[ \frac{v^{-(\mu/2+1)} - v^{\mu/2-1}}{(\mu + 1)^2} - \frac{v^{\mu/2-1}}{\mu + 1} \ln \left( \frac{1}{v} \right) \right], \quad (57) \]
\[ \text{and for } v \gg v^* \text{ and } \mu < 0 \]
\[ G_i(v) \sim R^{-1/(\mu+2)} v^{\mu - 1} \]
\[ + R^{\mu+2} \left[ \frac{v^{\mu-2}(1 - v^{\mu+1})}{\mu + 1} - \frac{v^{2\mu-2}(1 - v^\mu)}{\mu} \right]. \quad (58) \]

Therefore as \( v \to 1 \), the car density becomes the same as the platoon density and it is much smaller than the car density for \( v < v^* \).

The average platoon size in lane \( i \) may be estimated as
\[ \Lambda_i = \int_0^1 dv G_i(v) = \int_0^{v^*} dv G_i(v) + \int_{v^*}^1 dv G_i(v) \]
\[ \approx \int_0^{v^*} dv G_i(v) + \int_0^1 dv G_i(v) \sim \left( \frac{R_i}{R} \right)^{\frac{\mu+2}{\mu+1}} \rho_i^{\mu+1} \int_0^1 \Phi_i(v) \]. \quad (59) \]

On RHS of second line of above equation, the contribution to integrals above \( v > v^* \) has been neglected because it is negligible when compared to the contribution from \( v < v^* \). Thus, the average size of platoons increases significantly and depends on the escape time in the lane. If the escape times on both the lanes are of similar magnitude, then \( \Lambda_i \approx R_i^{\mu+2} \) for both the lanes. If \( R_j \gg R_i \gg 1 \), the leading dependence would be \( \Lambda_i \approx \Lambda_j \sim R_i^{\mu+2} \). Thus, the platoon size in each lane increases as a power-law of the largest escape time with the exponent \( \mu+1 \). It may be noted that the form of \( \Lambda_i \) is valid for any \( \mu > -1 \). Since the above argument does not use any information about \( G_i(v) \), \( \Phi_i(v) \) and \( P_0(v) \) for \( v > v^* \) except for the fact that \( G_i(v > v^*) \) and \( \Phi_i(v > v^*) \) are negligible in comparison to \( G_i(v < v^*) \) and \( P_i(v < v^*) \) respectively, it is valid for any \( P_0(v) \) which goes as \( v^\mu \) for \( v < v^* \). The power-law dependence of \( \Lambda_i \) may be related to the power-law observed in the single lane case without passing where the platoon size increases with time as \( R_i^{\mu+2} \) from which we may infer in the present case that the platoon size increases until a time scale of escape time is reached after which it saturates.

The flux in lane \( i \) is
\[ J_i = \int_0^1 dv v G_i(v) = \int_0^{v^*} dv v G_i(v) + \int_{v^*}^1 dv v G_i(v) \]
\[ \sim \left\{ \begin{array}{ll}
C_i v^{\mu+2} & \mu \geq 0 \\
C_i v^{\mu+2} & \mu < 0
\end{array} \right. \]
\[ \times \left\{ \begin{array}{ll}
\frac{R_i^{1/2}}{R} (1 - v^{\mu/2+1}) & \mu \geq 0 \\
\frac{R_i^{1/2}}{R} (1 - v^{\mu/2+1}) & \mu < 0
\end{array} \right. \]. \quad (60) \]

where
\[ C_i = \frac{R_i^{\mu+2}}{R_i^{\mu+2}} R_i^{\mu+2} + R_i^{2\mu+2} \] and we used the leading terms in Eq.(58) and Eq.(57) to derive the above expression.

### C. An example

Let’s plot the flux in each lane when \( P_0(v) \) is a beta distribution
\[ P_0(v) = \frac{1}{B(\mu - 1, \nu - 1)} v^\mu (1 - v)^\nu I_{[0,1]}(v), \quad (62) \]
where \( I_{[0,1]}(v) = 1 \) if \( v \in [0,1] \) and \( 0 \) otherwise and \( B(\mu - 1, \nu - 1) \) is the beta function with parameters \( \mu > -1 \) and \( \nu > -1 \). For ease of plotting we first write flux and related quantities in dimensional units. Dimensional quantities are denoted by a bar on the top (\( \bar{J} \) is dimensional form of \( J \)). For \( v^* \geq 1 \), i.e., \( \rho_0 \in \left( 0, \frac{4}{R_i R_{-i} v_0} \right) \),
\[ \bar{J}_i \sim \rho_0 v_0 J_0 \frac{\bar{R}_i - \bar{R}_{-i}}{8} A_0(\rho_0 v_0)^2 \quad (63) \]
and for \( v^* < 1 \) i.e., \( \rho_0 > \frac{4}{R_i R_{-i} v_0} \),
\[ \bar{J}_i \sim \left\{ \begin{array}{ll}
\rho_0 v_0 C_i v^{\mu+2} & \mu \geq 0 \\
\rho_0 v_0 C_i v^{\mu+2} & \mu < 0
\end{array} \right. \]
\[ \times \left\{ \begin{array}{ll}
\frac{R_i^{1/2}}{R} (1 - v^{\mu/2+1}) & \mu \geq 0 \\
\frac{R_i^{1/2}}{R} (1 - v^{\mu/2+1}) & \mu < 0
\end{array} \right. \]. \quad (64) \]

From Eq.(63) for \( \bar{J}_i \), it may be noted that, when \( \bar{R}_i = \bar{R}_{-i} \), only the first term remains and it is simply the product of arithmetic average of initial density and the expectation value of free-flow speed. While Eq.(64) shows a complicated dependence of \( \bar{J}_i \) on \( \rho_0 \), it can be shown that \( J \sim \rho_0^{\mu+2} \) leading order in \( \rho_0 \). Thus the \( J_i \) curve flattens as \( \rho_0 \) becomes greater than \( \rho^* \). Below we plot in Fig[1] \( \bar{J}_i \) for \( \mu = -0.9, \nu = 1 \), \( \bar{R}_i = 0.001hr \), \( \bar{R}_{-i} = 0.002hr \). It may be noted that in general kinetic theories are accurate only for low densities. So we depicted only the low density regime in the plot where the transition from free-flow phase in which all the vehicles experience free-flow to a platoon forming phase in which vehicles with high free-flow speeds form platoons behind slow moving vehicles.

### D. Relation to three-phase theory

Kerner et al.\[23,24\] analyzed a bulk of traffic flow data and noted that the flow has the following pattern across a bottleneck which has been put forward as three-phase theory: The traffic flow ahead of a bottleneck is in free-flow\( (F) \) phase and at the bottleneck the flow undergoes a phase transition into a synchronous phase\( (S) \) in which vehicles face congestion but move in a synchronous
manner thereby forming platoons. The $S$-phase spreads downstream as it forms until it becomes unstable resulting in a $F \rightarrow S$ transition followed by an $S \rightarrow J$ transition along the upstream direction across a bottleneck. While the whole process takes place in a transient state, under suitable conditions the system may reach a non-equilibrium stationary state maintaining the same flow pattern across the bottleneck.

To relate the present work to three-phase theory, we first have to note a few observations from our previous work in which we simulated heterogeneous traffic flow on a single lane using quenched-disordered Newell’s car-following model. The quenched-disorders are basically the parameters of the model drawn from static probability distributions. For Newell’s model, the parameters are the free-flow speed, the jam density and the backward-wave propagation speed. In addition to study of the emergence of power-laws in various quantities like platoon size etc., we also showed that the simulations could reproduce the traffic flow pattern as described in the three-phase theory. Basically, every slow-moving vehicle forms a moving bottleneck to fast-moving vehicles and thus, ahead a slow-moving vehicle, one may find a free-flowing traffic of which it is also a part. Behind the slow-moving vehicle, the fast-moving vehicles form a platoon and thereby experience a $S$-phase which spreads upstream within the platoon until the flow becomes unstable resulting in the $J - phase$. We described how the formation of $J - phase$ can be explained by the instability created by the reaction time of the drivers. If the drivers reacted instantaneously to any perturbations ahead, the $S$-flow would continue up to the end of the platoon. Further it has been observed that even in the $J$-phase, the platoons still exist if the instability is strong enough and the power-laws for the platoon size and quantities averaged over the platoon remain the same irrespective of whether the system is an $S$-phase or $J - phase$. This forms the basis for the validity of the emergent phenomena described by the kinetic theory used in the present work even though the model assumes zero reaction time of drivers to keep the analytical study tractable. In Fig.1 the flattening of the flux above $\rho^*$ is essentially due to platoon formation behind the slow-moving vehicles as described above in relation to three-phase theory.

V. PLATOON PHASE TRANSITION

The dependence of average platoon size on the escape time could be arrived at using just the kinetics of the speed distribution of platoons as shown in the previous section. However, it is desirable to obtain the size distribution of platoons which explains more about the structure of the system. Ben-Naim and Krapivsky derived the stationary size distribution for a single lane by further simplifying the kinetic equations for cluster size distribution assuming a constant collision rate which they called the Maxwell model. Later Isolatorov and Krapivsky studied a modified version in which only next to the leading car is allowed to escape the platoon and observed a phase-transition from a phase having smaller platoons of finite size and to a phase having infinitely long platoons (or platoons with size of the order of road length) at $R = 1$. They noted that violation of the sum-rule for vehicle conservation is a signature of the phase transition. Below we formulate a generalized set of equations for kinetics of platoon size distribution for the two lane case and study them to see if the mentioned violation of the sum rule occurs signalling the phase transition.

Let $P_i(m,t)$ be the density of platoons of size $m$ at time $t$ moving at any speed on lane $i$. The kinetic equations followed by $P_i(m,t)$ within the Maxwell model are

$$\frac{\partial P_i(m,t)}{\partial t} = \begin{cases} \frac{1}{R_i}[(\alpha(m-1) + 1)P_i(m+1,t) - (\alpha(m-2) + 1)P_i(m,t) - P_i(m,t)\rho_i] & m \geq 2 \\ \frac{1}{R_i}\sum_{n \geq 2}(\alpha(n-2) + 1)P_{i-n}(m,t) + P_i(2,t) & m = 1 \\ \frac{1}{R_i}\sum_{n \geq 2}(\alpha(n-2) + 1)P_{i-n}(m,t) - \rho_iP_i(1,t) & m = 1 \\ \end{cases}$$

In the above equations, $\alpha = 1$ implies that each of the follower vehicles in the platoon may independently chose to escape at a rate $R_i^{-1}$ (as used by Ben-Naim and Krapivsky) and $\alpha = 0$ implies only one vehicle, presumably the one next to the leader, may escape at a rate.
$R_i^{-1}$ (as used by Isplato and Krapivsky). Thus both the cases can be studied in one shot using the above set of equations. Further, $0 < \alpha < 1$ case may be interpreted as the fraction of the follower vehicles that may choose to escape the platoon at a rate $R_i^{-1}$. Eq. (65) may be arrived by starting from a more general equation for a joint speed-size distribution and integrating out the speed (see appendix). As stationary state is approached, the equation becomes,

$$P_i(m)\rho_i = \frac{1}{R_i}((\alpha(m - 1) + 1)P_i(m + 1) - (\alpha(m - 2) + 1)P_i(m)(1 - \delta_{m_1})) + \frac{1}{2}\sum_l P_i(l)P_i(m - l) + \sum_{m \geq 2}(\alpha(m - 2) + 1)P_i(m)\delta_{m_1}. \tag{66}$$

Further, in the stationary state, the number of vehicles escaping lane 1 per unit time must be equal to the number of vehicles escaping lane 2 per unit time. This can be seen by noting that total number of vehicles leaving lane $j$ per unit time is

$$\int_0^1 dv\int_0^1 dv'\Psi_i(v'; v) = \sum_{m \geq 2}(\alpha(m - 2) + 1)P_i(m)$$

and equating the LHS of Eq. (6) and the LHS of Eq. (8). Therefore,

$$\sum_{m \geq 2}(\alpha(m - 2) + 1)P_i(m)\frac{R_i}{R_i} = \sum_{m \geq 2}(\alpha(m - 2) + 1)P_{m-i}(m)\frac{R_{m-i}}{R_{m-i}}. \tag{67}$$

Thus, as the stationary state is approached, the equations in Eq. (66) get decoupled and we get

$$P_i(m)\rho_i = \frac{1}{R_i}((\alpha(m - 1) + 1)P_i(m + 1) - (\alpha(m - 2) + 1)P_i(m)(1 - \delta_{m_1})) + \frac{1}{2}\sum_l P_i(l)P_i(m - l) + \sum_{m \geq 2}(\alpha(m - 2) + 1)P_i(m)\delta_{m_1}, \tag{68}$$

which are exactly the equations obtained for a single lane with passing. The $\alpha = 0$ and $\alpha = 1$ cases have already been studied. Readers may refer to Ben-Naim and Krapivsky[22] and Isplato and Krapivsky[23] for details which we do not repeat here. We find some interesting points to note from the generalised set of equations which are explained below.

The above hierarchy of equations are usually solved using a generating function method which is a series of $P_i(m)$ whose coefficients have to be determined using Eq. (68). Since equations are the same for both the lanes,

we do not show the subscript indicating lane for convenience in the below derivation. Consider the generating function

$$G(z) = \sum_{m=1}^\infty (z - 1)^m F_m, \tag{69}$$

where $F_m = R P(m)$. Using the definition of $G(z)$ in Eq. (68) we get

$$\frac{G(z)^2}{2} + \alpha z(1-z)\frac{d}{dz}\left(\frac{G(z)^2}{2z}\right) + (1 - \alpha)\frac{(1-z)^2}{z} G(z) + (1 - 2\alpha)\frac{(1-z)^2}{z} F + \alpha(z - 1) R = 0, \tag{70}$$

where $F = \sum F_m$. The boundary condition is $G(1) = 0$. By definition, $G(0) = -F$ and $\frac{dG}{dz}|_{z=1} = R$ which are sum rules the solution of Eq. (70) is expected to satisfy. The first sum rule is satisfied by definition. The second sum rule is related to conservation of vehicles and is satisfied if there are no infinite size platoons in the system. When $\alpha = 0$, Eq. (70) becomes a quadratic equation whose physically acceptable solution is

$$G(z; \alpha = 0) = \frac{z - 1}{z}\left[1 - \sqrt{1 - 2zF}\right]. \tag{71}$$

The first sum rule (i.e., $G(0) = -F$) is trivially followed while the second sum rule (i.e., $\frac{dG}{dz}|_{z=1} = R$) is followed by $G(z; \alpha = 0)$ for $R \in [0, 1]$ (see Fig. 2). Violation of the second sum rule above $R = 1$ has been interpreted as formation of an infinitely long platoon (on an infinitely long lane[22]. Thus in this case there is a phase transition from a state with no platoons or small size platoons to a state with platoons whose size is of the order of the road length.

Interestingly, Eq. (68) has exact solution for $\alpha = 1/2$ as well. For $\alpha = 1/2$, Eq. (70) becomes

$$(1 - z)\frac{dG}{dz} + G^2 + (z - 1) R = 0. \tag{72}$$

Substituting $z = 1 - x$ and using the transformation $v(x) = G(x)/x$ and defining $u(x)$ such that $v(x) = -\frac{1}{u(x)}$, we have
we get the Bessel differential equation for \( u(z) \) which is

\[
    u''(x) + \frac{u'(x)}{x} - \frac{R}{x} u(x) = 0. \tag{73}
\]

The solution to above equation which satisfies the boundary condition for \( G \) is

\[
    u(x) = I_0(2\sqrt{Rx}), \tag{74}
\]

where \( I_0 \) is the zeroth order Bessel function of first kind. The \( G(z) \) obtained from above \( u(x) \) is

\[
    G(z) = -\sqrt{R(1-z)} \frac{I_1(2\sqrt{R(1-z)})}{I_0(2\sqrt{R(1-z)})}. \tag{75}
\]

The above solution satisfies both sum rules for all \( R \) indicating that there is no phase transition. See also, fig.3.

For any \( 0 < \alpha < 1 \), Eq.(68) (a form of Riccati equation) may be re-written as

\[
    \frac{d^2u}{dx^2} = \mathcal{R}(x) \frac{du}{dx} + S(x) u(x) = 0 \tag{76}
\]

where

\[
    \frac{1}{u} \frac{du}{dx} = \frac{G(x)}{2\alpha (1-x)},
\]

\[
    \mathcal{R}(x) = \frac{(1-2\alpha)}{\alpha} \frac{1}{1-x} - \frac{1}{x},
\]

and

\[
    S(x) = \frac{1-2\alpha}{2\alpha^2} \frac{F}{1-x} - \frac{R}{2\alpha x}
\]

and \( z = 1-x \). Since Eq.(76) is a second order linear equation, one may attempt a power-series solution for \( u(x) \). Let

\[
    u(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}. \tag{77}
\]

To check for the violation of the sum rule, we first note that

\[
    -\frac{dG}{dz} \bigg|_{z=1} = \frac{dG}{dx} \bigg|_{x=0} = -\frac{2a_1}{a_0}. \tag{78}
\]

Therefore, it is enough to determine \( a_0 \) and \( a_1 \) to obtain the above derivative. Substituting it in Eq.(76) and comparing coefficients of equal powers of \( x \), we get the \( a_1 \) values. The indicial equation gives \( k = 0 \). Equating the coefficient of \( x^k \) to zero, we get

\[
    a_1 = a_0 \frac{R}{2\alpha} \tag{79}
\]

which when used in Eq.(78), we get

\[
    \frac{dG}{dz} \bigg|_{z=1} = R. \tag{80}
\]

Hence, the sum rule regarding the conservation of vehicles is respected for any \( R \) when \( \alpha > 0 \) corroborating with the the result of \( \alpha = 1/2 \). Thus, we see that the phase-transition from a phase with small platoons to a phase with large platoons with size of the order of road length occurs only when the number of vehicles leaving the platoon per unit time does not increase with the size of the platoon. Another interesting point to note is that if \( R_1 > 1 > R_{-1} \) and \( \alpha = 0 \), we see that lane 1 would have very large size platoons while lane 2 has small platoons in the stationary state.

VI. SUMMARY AND DISCUSSION

We formulated a kinetic theory for a dilute two-lane traffic within the framework of Ben-Naim Krabivsky model and studied the stationary state properties of the system. We find that the platoon velocity distribution and the platoon density on both the lanes get equalized as the stationary state is reached whereas the vehicle velocity distribution and the vehicle density are different for each lane. Essentially, the lane with larger mean-escape time would have larger density of vehicles and a larger platoon size. Similar to the single lane case, the vehicles in the system may be characterised into two groups based on a characteristic speed \( v^* \) which is inversely proportional to mean of escape times on the lanes as \( v^* \sim \left((R_1 + R_{-1})/4\right)^{-1/(\mu+2)} \) where \( \mu \) is the exponent of the quenched disorder in speed \( P_0(v) \sim v^\mu \). If \( v^* \sim 1 \), then all (or a majority of) the vehicles in the system enjoy free-flow and the average platoon size is of the order of unity. If \( v^* < 1 \), then the vehicles with desired speed greater than \( v^* \) experience a congested flow and those with desired speed lesser than \( v^* \) experience free-flow on any lane. In this case, average platoon size is large (proportional to \( v^{-(\mu+1)} \)) and is lane dependent.

We also wrote down the equations for platoon size distribution for the two lane system and showed that the
equations can be exactly mapped to the single lane problem. We showed that the phase-transition, from a phase with small platoons to a phase with platoons as large as the road length, happens only if the escape rate of vehicles in a lane is independent of the size of the platoon.

Overall, the following differences emerge between a homogeneous traffic and a heterogeneous traffic: For a homogeneous traffic on a single lane, i.e., identical vehicles and identical drivers, the vehicle density is homogeneous in the stationary state. When the traffic is heterogeneous, at least due to different free-flow speeds adopted by drivers, the density in the stationary state is no more homogeneous. For a single lane road, the stationary state of a heterogeneous traffic has platoons on both sides and noting that

\[ \int_0^1 dv P_i(m, v, t) = P_i(m, t) \]  

(A4)

we get

\[ \frac{\partial P_i(m, v, t)}{\partial t} = \frac{P_i(m + 1, v, t) - P_i(m, v, t)}{R_i} - P_i(m, v, t) \int_0^1 dv' |(v - v')| P_i(v', t) \]

\[ + \sum_l P_i(l, v, t) \int_v^1 dv' (v' - v) P_i(m - l, v', t). \]  

(A1)

Using the second fundamental theorem of calculus, the collision integrals in the second and third terms can be re-written as

\[ \frac{\partial P_i(m, v, t)}{\partial t} = \frac{P_i(m + 1, v, t) - P_i(m, v, t)}{R_i} - \zeta_1(v) P_i(m, v, t) \int_0^1 dv' P_i(v', t) \]

\[ + \zeta_2(v) \sum_l P_i(l, v, t) \int_v^1 dv' P_i(m - l, v', t). \]  

(A2)

Maxwell’s model basically assumes that \( \zeta_1 \) and \( \zeta_2 \) are constant. Here we take them to be unity. Thus the equation becomes

\[ \frac{\partial P_i(m, v, t)}{\partial t} = \frac{P_i(m + 1, v, t) - P_i(m, v, t)}{R_i} - P_i(m, v, t) \int_0^1 dv' P_i(v', t) \]

\[ + \sum_l P_i(l, v, t) \int_v^1 dv' P_i(m - l, v', t). \]  

(A3)

Integrating over \( v \) on both sides and using Maxwell’s approximation and integrating out \( v \) as described above gives

\[ \frac{\partial P_i(1, v, t)}{\partial t} = \sum_{m \geq 2} \frac{P_i(m, v, t)}{R_j} + \frac{P_i(2, v, t)}{R_i} \]

\[ - P_i(1, v, t) \int_0^1 dv' |(v - v')| P_i(v', t) \]  

(A6)

which upon using Maxwell’s approximation and integrating out \( v \) as described above gives

\[ \frac{\partial P_i(1, v, t)}{\partial t} = \sum_{m \geq 2} \frac{P_i(m, v, t)}{R_j} + \frac{P_i(2, v, t)}{R_i} - P_i(1, v, t) \rho_i \]  

(A7)

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Appendix A:

Let \( P_i(m, v, t) \) be the density of platoons of size \( m \geq 2 \) moving with speed \( v \) at time \( t \) on lane \( i \). The kinetic equations followed by it are

Using the second fundamental theorem of calculus, the collision integrals in the second and third terms can be re-written as

\[ \frac{\partial P_i(m, v, t)}{\partial t} = \frac{P_i(m + 1, v, t) - P_i(m, v, t)}{R_i} - \zeta_1(v) P_i(m, v, t) \int_0^1 dv' P_i(v', t) \]

\[ + \zeta_2(v) \sum_l P_i(l, v, t) \int_v^1 dv' P_i(m - l, v', t). \]  

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