INTEGRABILITY AND SYMMETRY ALGEBRA
ASSOCIATED WITH $N = 2$ KP FLOWS

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Abstract

We show the complete integrability of $N = 2$ nonstandard KP flows establishing the biHamiltonian structures. One of Hamiltonian structures is shown to be isomorphic to the nonlinear $N = 2 \hat{W}_{\infty}$ algebra with the bosonic sector having $\hat{W}_{1+\infty} \oplus \hat{W}_{\infty}$ structure. A consistent free field representation of the super conformal algebra is obtained. The bosonic generators are found to be an admixture of free fermions and free complex bosons, unlike the linear one. The fermionic generators become exponential in free fields, in general.

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1. Introduction

The close relationship between the conformal algebras and the rich symmetry associated with integrable systems is well understood. The Hamiltonian structures of the integrable hierarchies have been found to be isomorphic to the various higher spin conformal algebras at the classical level. This was realized when it was shown that the \( W_n \) algebra incorporates in its classical limit the Hamiltonian structure of the nonlinear integrable systems i.e. the generalized \( n^{th} \) KdV hierarchy \([1, 2, 3]\). This technique of obtaining classical conformal algebras through the integrable hierarchies is indeed a powerful one and its importance was recognized when the existence of a new higher spin conformal algebra was realized at the classical level \([4]\). Prior to this all the known conformal algebras were obtained first in terms of free fields via the bootstrap approach and subsequently their relation to the symmetries of the integrable hierarchies was obtained.

Generalized KdV hierarchy, in general, is significant in its own right because of its rich symmetry structures. But it becomes physically more relevant since the equations of motion and symmetries of 2D quantum gravity \([5]\) can be formulated in terms of KdV-like equations.

The KP hierarchy incorporates all KdV hierarchies \([6]\) and thus there exists the distinct possibility that it is the bedrock of 2D quantum gravity \([7]\). It was believed that the large \( n \) limit of \( W_n \) algebra, namely \( W_{1+\infty} \) and \( W_{\infty} \) algebras would provide the necessary framework in this direction. The naive approach to the large \( n \) limit, however, gives rise to linear algebras which are truly infinite dimensional symmetry algebras containing all the conformal spins \([8]\), but the fact that they are linear prevents them from being the right candidates for universal algebras, there being no straightforward mechanism which effectively truncates the spin content of these algebras and produces the nonlinear features of \( W_n \). A nonlinear realization of the \( W \) algebra namely \( \hat{W}_{\infty} \) algebra was obtained by the bootstrap approach and identified with the
2nd Hamiltonian structure of the KP hierarchy \cite{9,10}. It is a universal $W$ algebra containing all $W_n$ algebras. This was obtained by associating the symmetry algebra of the $SL(2,R)/U(1)$ coset model with the $\hat{W}_\infty$ algebra characterized by the label $k$ and then by showing that the symmetry algebra truncates to $W_n$ algebra for $k = -n$ \cite{11}.

Subsequently Manin and Radul \cite{12} provided the supersymmetric extension of the KP hierarchy, and this was based on the odd parity superLax formulation. But construction of the Hamiltonian structure for odd parity Lax operator, following the Drinfeld Sokolov formalism, is not well understood yet \cite{13}. Later on, an even parity Lax operator associated with supersymmetric KP hierarchy was obtained \cite{14,15} and a supersymmetric extension of the linear $W$ algebra was realized \cite{13}. The connection of the $N = 2$ super KdV hierarchies with affine Lie algebras was demonstrated by Inami and Kanno \cite{17,18} which is a step forward towards an $N = 2$ super analogue of the Drinfeld Sokolov formulation. This indicates that there ought to be consistent $N = 2$ superLax formulation of the super KP hierarchy which should be Hamiltonian with respect to the super Gelfand Dikii bracket of the second kind and also should reduce to the Lax operators considered by Inami and Kanno under suitable reduction. Consequently, the existence of a nonlinear realization of the super $\hat{W}_\infty$ algebra \cite{4} may be shown through the super KP formulation. Unlike all other known conformal algebras, there is no bootstrap approach to find the higher spin extension of $N = 2$ superconformal algebra, namely the super $\hat{W}_\infty$ algebra through the free field representations.

While the bosonic KP hierarchy and their connection to matrix model have been studied extensively, not much is known about the higher spin extension of $N = 2$ superconformal symmetry. Moreover, conventional formulation of the supersymmetric matrix model failed to describe nonperturbative superstring theory; it gives nothing but the ordinary matrix model \cite{19}. The super KP formalism may throw some light in this direction even if no super symmetric extension of the matrix model can be formulated.
Renewed interest in the study of $N = 2$ and $N = 4$ supersymmetry in the context of quantum gravity and their connections with integrable systems, opened up a series of studies relating to the symmetry structures of $N = 2$ and $N = 4$ supersymmetric integrable models [20]. A major breakthrough in recent times is the nonperturbative solution of $N = 2$ super Yang-Mills equation and their connection with integrable systems [21]. Investigations have also been made in recent times to obtain $\tau$ functions for supersymmetric integrable hierarchies [22]. Motivated by these works, we intend to further explore the super $\hat{W}_\infty$ algebra and particularly its free field representation, the significance of which cannot be overestimated since the underlying representation of a conformal field theory is essentially a free field representation. Moreover it plays a major role in the classification of various conformal algebras. Further, the quantization of classical symmetry algebras becomes straightforward in terms of free fields.

In this paper, we show that nonstandard supersymmetric KP flows following the Gelfand Dikii method, are biHamiltonian. We further show that one of the Hamiltonian structures is a candidate for a higher spin extension of $N = 2$ superconformal algebra and has the required number of spin fields and the bosonic sector of the algebra has the right structure with two commuting sets of bosonic generators. We will also obtain the free field representation of the generators which turns out to be nonlinear. The generators in the fermionic sector are exponential in the free fields. It will be apparent that in the bosonic sector the representation of generators are not trivial extensions of linear representation. However, unlike the linear symmetry algebras, all the generators cannot be expressed in terms of the free fields in a closed form. A few of the lower order generators are explicitly written down and an algorithm for constructing higher order generators will be indicated. This brings us closer to establishing that super $\hat{W}_\infty$ is a universal symmetry containing all finite dimensional bosonic and $N = 2$ supersymmetric $W$ algebras.

The organization of the paper is as follows. In section 2 we introduce
the $N = 2$ super KP model and obtain its biHamiltonian structures through
the Gelfand Dikii method. This establishes the complete integrability of
the system. In section 3 we show that the second Hamiltonian structure of
$N = 2$ super KP hierarchy exhibits the appropriate structure of a super $\hat{W}_\infty$
algebra. In particular, the bosonic sector of this nonlinear algebra is shown to
possess the required $W_{1+\infty} \oplus W_\infty$ structure. We obtain a nonlinear free field
representation of the bosonic and fermionic generators in a suitable basis in
section 4. Unlike the linear representation of the bosonic generators which
was in terms of bilinears of free fields – either fermionic or bosonic; here the
consistent free field representation of these algebras comprises an admixture
of free fermions and free complex scalar fields. Section 5 is the concluding
one.

2. $N = 2$ Super KP Hierarchy and biHamiltonian structures

In an earlier work [4], it was shown that with an even parity superLax op-
erator the Hamiltonian structure leading to a nonlinear super $\hat{W}_\infty$ algebra
becomes local, thus making it a right candidate for a universal symmetry
algebra containing all finite dimensional bosonic as well as $N = 2$ super-

symmetric algebras. For completeness and future reference we mention the
explicit forms of the Hamiltonian structure and the dynamical equations of
the $N = 2$ super KP hierarchy. The super Lax operator of the associated
$N = 2$ super KP hierarchy is given by

$$ L = D^2 + \sum_{i=0}^{\infty} u_{i-1}(X) D^{-i} \quad (2.1) $$

where, $D$ is the superderivative with $D^2 = d/dx$ and $u_{i-1}(X)$ are superfields
in $X = (x, \theta)$ space, $\theta$ being Grassmann odd coordinate. The grading of
$u_{i-1}(X)$ is $|u_{i-1}| = i$ so that $u_{2i-1}$ are bosonic superfields, whereas $u_{2i}$ are
fermionic ones.

The local Poisson bracket algebra among the coefficient fields \( u_{i-1}(X) \) can be obtained following the method of Gelfand and Dikii [4]. This has the explicit form

\[
\{ u_{j-1}(X), u_{k-1}(Y) \}_2 = \left[ -\sum_{m=0}^{j+1} \binom{j+1}{m} (-1)^{j+k+m+1} u_{j+k-m} D^m \right.
\]

\[
+ \sum_{m=0}^{j+1} \binom{k+1}{m} (-1)^{j+k+m}(j+1)(m+1) D^m u_{j+k-m}
\]

\[
+ \sum_{m=0}^{j+1} \binom{k+1}{m} \sum_{l=0}^{k} \left( \binom{j}{m+1} \binom{k}{l+1} - \binom{j-1}{m} \binom{k-1}{l} \right) (-1)^{(j+k+m+1)} u_{j+k-m} D^m
\]

\[
\times u_{j+k-m-l-2} D^{m+l+1} u_{k-l-2}
\]

\[
+ \sum_{n=0}^{k-1} \binom{k-n-1}{l} (-1)^{(j+k+l+1)(n+k+1)} u_{n-1} D^l u_{j+k-n-l-2}
\]

\[
- \sum_{m=0}^{j-1} \sum_{n=0}^{k-1} \sum_{l=0}^{k-1} \binom{j-1}{m} \binom{n+l-1}{l} (-1)^{(j+k+m+l+1) + (n+l+1)} u_{n-1} D^m
\]

\[
\times u_{j+k-m-l-2} D^{m+l} u_{n-1} \] \[ \Delta(X - Y) \quad (2.2) \]

Notice that the symmetry algebra (2.2) possesses the following features of interest.

1. The algebra is antisymmetric and satisfies the Jacobi identity.
2. The lowest subalgebra contains two super fields, namely \( u_{-1}(X) \) and \( u_0(X) \) and becomes isomorphic to the classical analogue of the \( N = 2 \) super conformal algebra.
3. The Hamiltonian structure along with conserved quantites [4] provides a set of dynamical equations of the \( N = 2 \) KP hierarchy consistent with the nonstandard flow equation, namely

\[
\frac{dL}{dt_n} = [L_{>0}, L] \quad (2.3)
\]

where the super Lax operator \( L \) is given in (2.4). In (2.3) ‘\( > 0 \)’ implies the +ve part of \( L^n \) without \( D^0 \) term. The significance of nonstandard flow
equation becomes apparent from the explicit forms of the following set of dynamical equations. In fact, the nonstandard flows provide the nontrivial dynamics to the the lowest superfield $u_{-1}$ which is instrumental in making the Poisson bracket structure (2.2) local. The evolution equations corresponding to the lowest three time-flows of the hierarchy which follow from (2.1,2.2) are given below for completeness and for future reference.

\[
\frac{du_{i-1}}{dt_1} = u_{i-1}^{[2]}
\]

\[
\frac{du_{i-1}}{dt_2} = 2u_{i+1}^{[2]} + u_{i-1}^{[4]} + 2u_0u_{i-1}^{[1]} + 2u_{-1}u_{i-1}^{[2]} - 2 \left[ \frac{i+1}{i} \right] u_{i-1}^{[1]}
\]

\[
-2(1 + (-1)^i)u_0u_i + 2 \sum_{m=0}^{i-1} \left[ \frac{i}{m+1} \right] (-1)^{i+1+[-m/2]}u_{i-m-1}u_0^{[m+1]}
\]

\[
+2 \sum_{m=0}^{i-1} \left[ \frac{i+1}{m+2} \right] (-1)^{[m/2]}u_{i-m-1}u_0^{[m+2]}
\]

\[
\frac{du_{i-1}}{dt_3} = 3u_{i+3}^{[2]} + 3u_{i+1}^{[4]} + u_{i-1}^{[6]} + 6u_{-1}u_{i+1}^{[2]} + 3u_{-1}u_{i-1}^{[4]}
\]

\[
-3 \left[ \frac{i+3}{1} \right] u_{i+2}u_{i-1}^{[1]} + 3 \left[ \frac{i+3}{2} \right] u_{i+1}u_{i-1}^{[2]} + 3 \left[ \frac{i+3}{3} \right] u_{i}u_{i-1}^{[3]}
\]

\[
-3(1 + (-1)^i)u_0u_{i+2} + 3u_0u_{i+1}^{[1]} - 3(-1)^i u_0u_{i}^{[2]} + 3u_0u_{i-1}^{[3]}
\]

\[
+3 \left[ \frac{i+2}{1} \right] (-1)^{i}u_{i+1}u_0^{[1]} - 3 \left[ \frac{i+2}{2} \right] (-1)^{i}u_{i}u_0^{[2]}
\]

\[
+3(u_1 + 2u_{i+1}^{[2]} + u_{i-1}^{[2]} + u_{i-1}^{[2]})u_{i-1}^{[1]} - 3(1 + (-1)^i)(u_2 + 2u_{-1}u_0 + u_{0}^{[2]})u_{i-1}
\]

\[
-3 \sum_{m=0}^{i-1} \left[ \frac{i+3}{m+4} \right] (-1)^{[m/2]}u_{i-m-1}u_0^{[m+4]}
\]

\[
-3 \sum_{m=0}^{i-1} \left[ \frac{i+2}{m+3} \right] (-1)^{i+[-m/2]}u_{i-m-1}u_0^{[m+3]}
\]

\[
+3 \sum_{m=0}^{i-1} \left[ \frac{i+1}{m+2} \right] (-1)^{[m/2]}u_{i-m-1}(u_1 + u_{i-1}^{[2]} + u_{i-1}^{[2]})^{[m+2]}
\]
\[+3 \sum_{m=0}^{i-1} \left( \frac{i}{m+1} \right) (-1)^{i+[m/2]} u_{i-m-1}(u_2 + 2u_{-1}u_0 + u_0^{[2][m+1]} \right) (2.4)\]

In order to show the dynamical equations associated with the nonstandard flows are completely integrable, we show the existence of another Hamiltonian structure making the \(N = 2\) supersymmetric KP hierarchy biHamiltonian. The super Gelfand Dikii bracket of the first kind is defined by

\[\{F_P(L), F_Q(L)\}_1 = -Tr ([P, Q] L) = -Tr ([L, P] Q) \tag{2.5}\]

where \(P\) and \(Q\) are the auxiliary fields defined as

\[P = \sum_{j=-2}^{\infty} D^j p_j \quad ; \quad Q = \sum_{j=-2}^{\infty} D^j q_j \tag{2.6}\]

with the grading \(|p_j| = |q_j| = j\) so that the linear functional \(F_P(L)\) (and similarly \(F_Q(L)\)) becomes

\[F_P(L) = Tr(LP) = \sum_{i=0}^{\infty} \int dX (-1)^{i+1} u_{i-1}(X)p_{i-1}(X) \tag{2.7}\]

Consequently the L.H.S. of (2.3) becomes

\[\{F_P(L), F_Q(L)\} = \sum_{i,j=0}^{\infty} \int dX \int dY (-1)^{i+j} p_{i-1}(X) \{u_{i-1}(X), u_{j-1}(Y)\} q_{j-1}(Y) \tag{2.8}\]

Notice that (2.5) does not involve terms like \(p_{-2}\) and \(q_{-2}\) since the superfields begin from \(u_{-1}(X)\) in the Lax operator (2.1). This consistency is ensured by setting the coefficient of the \(D\) term in the commutator \([L, P]\) to zero. This leads to the constraint

\[p_{-1} = \sum_{j=-1}^{\infty} p_{2j+2}^{[2j+2]} + \sum_{i,m=0}^{\infty} (-1)^i u_{i-1}^{[2m]} p_{i+2m+1}^{[2m]} + \sum_{j=0}^{j-1} \sum_{m=0}^{j-1} \left( \frac{j}{m} \right) (-1)^{j(m+1)} (p_j u_{j-m-2})^{[m]} \tag{2.9}\]
Using the constraint (2.9) we obtain from (2.8) the following Poisson bracket among the superfields

\[ \{ u_{i-1}(X), u_{j-1}(Y) \} = \left[ -\delta_{i,0} D^{j+1} + (-1)^{i/2} \delta_{j,0} D^{i+1} \right. \]

\[ + \sum_{m=0}^{i+j-1} (-1)^{j(m+1)+[m/2]} \left[ \begin{array}{c} i-1 \cr j \end{array} \right] u_{i+j-m-2} D^m \]

\[ - \sum_{m=0}^{i+j-1} (-1)^{(i+1)m+j(m+1)} \left[ \begin{array}{c} j-1 \cr m \end{array} \right] D^m u_{i+j-m-2} \] \Delta(X - Y) \]  

(2.10)

It is seen however, that the first Hamiltonian structure above does not correctly reproduce the equations of motion (2.4). This inconsistency arises because the superLax operator considered is not a pure differential operator and was observed also in other cases involving pseudo-differential operators [9, 15]. This indicates, like previous cases, a modification of Hamilton’s equation and consequently the Hamiltonian structure is required. If we modify Hamilton’s equation of motion to the form given below,

\[ \frac{du_{i-1}(X)}{dt_n} = \{ u_{i-1}(X), H_{n+1} \} \]

\[ -(-1)^i \int dY \{ u_{i-1}(X), u_{-1}(Y) \} sResL^n(Y) \]

\[ -(-1)^i \int dY \{ u_{i-1}(X), u_0(Y) \} sRes \left( L^n D^{-1}(Y) \right) \]  

(2.11)

where

\[ sResL^n(X) = \sum_{j=0}^{\infty} \delta_{j,0} \frac{\delta H_{n+1}}{\delta u_{j-1}(X)} \]

\[ = \sum_{j=0}^{\infty} \sum_{l=0}^{j-1} \left[ \begin{array}{c} j-1 \cr l+1 \end{array} \right] D^{j-1-l} u_{j-l-2}(X) \frac{\delta H_{n+1}}{\delta u_{j-1}(X)} \]  

(2.12)

and

\[ sResL^n D^{-1}(X) = \sum_{j=0}^{\infty} (-1)^j D^j \frac{\delta H_{n+1}}{\delta u_j(X)} \]
\[
\sum_{j=0}^{\infty} \sum_{l=0}^{j-1} \left\{ (-1)^j D^{-2} u_{j-l-2}(X) D_X^l + \left[ \frac{j-1}{l} \right] \right. \\
\times \left. (-1)^{(j+1)} D_X^{l-2} u_{j-l-2}(X) \right\} \frac{\delta H_{n+1}}{\delta u_{j-1}(X)}, \quad (2.13)
\]

it reproduces the equations of motion correctly. Substituting (2.12) and (2.13) in (2.11), the equation of motion can be rewritten in the form

\[
\frac{du_{i-1}(X)}{dt_n} = \{u_{i-1}(X), H_{n+1}\}_1
\]

(2.14)

which eventually leads us to the correct form of the first Hamiltonian structure as

\[
\{u_{i-1}(X), u_{j-1}(Y)\}_1 = \\
\left[ \sum_{m=0}^{i+j-1} \left\{ \left[ \begin{array}{c} i-1 \\ m \end{array} \right] \left( -1 \right)^{(i+m+1)+[m/2]} u_{i+j-m-2} D^m + \left[ \begin{array}{c} j-1 \\ m \end{array} \right] \left( -1 \right)^{(m+1)+[m/2]} D^m u_{i+j-m-2} \right\} \\
+ \sum_{m=0}^{i-1} \sum_{l=0}^{j-1} \left\{ \left[ \begin{array}{c} j-1 \\ l+1 \end{array} \right] \left( -1 \right)^{(m+1)+[m/2]} D^m u_{i-m-2} D^{l-1} u_{j-l-2} \\
- \left[ \begin{array}{c} i-1 \\ m+1 \end{array} \right] \left( -1 \right)^{-[m/2]} u_{i-m-2} D^{m-1} u_{j-l-2} D^l \\
+ \left[ \begin{array}{c} i-1 \\ m \end{array} \right] \left[ \begin{array}{c} j-1 \\ l+1 \end{array} \right] \left( -1 \right)^{(i+1)+[m/2]} u_{i-m-2} D^{m+l-1} u_{j-l-2} \\
+ \left[ \begin{array}{c} i-1 \\ m+1 \end{array} \right] \left[ \begin{array}{c} j-1 \\ l \end{array} \right] \left( -1 \right)^{(l+1)(j+1)+[-m/2]} u_{i-m-2} D^{m+l-1} u_{j-l-2} \right\} \Delta(X - Y) \quad (2.15)
\]

which provides the dynamical equations of the hierarchy associated with the super Lax operator (2.1) and the nonstandard flows (2.3). It is evident that the above Hamiltonian structure is manifestly antisymmetric and satisfies the Jacobi identity. But this Hamiltonian structure, unlike the earlier one (2.2),
is nonlocal and may not be associated with the conformal symmetry. This feature is noticed in other supersymmetric integrable hierarchies also that one of the two Hamiltonian structures becomes nonlocal [15]. The existence of two Hamiltonian structures, nonetheless, confirms the complete integrability of the even parity super KP hierarchy.

3. Nonlinear Super $\hat{W}_\infty$ Algebra

In this section we show that the local superalgebra (2.2) obtained in the previous section is a higher spin extension of $N = 2$ conformal algebra containing all conformal spins. The nonlinear nature of superalgebra endows it with rich algebraic structures.

If the super fields are expressed in component form as

$$
\begin{align*}
    u_{2i-1}(X) &= u_{2i-1}^b(x) + \theta u_{2i-1}^f(x), \\
    u_{2i}(X) &= u_{2i}^f(x) + \theta u_{2i}^b(x)
\end{align*}
$$

(3.1)

the odd bosonic fields $u_{2i-1}^b(x)$ have conformal weights $i + 1$, whereas the even bosonic fields $u_{2i}^b(x)$ have conformal weights $i + 2$ ($i = 0, 1, 2, \ldots$), with respect to the stress tensor

$$
T(x) = u_0^b(x) - \frac{1}{2} \partial_x u_{-1}^b(x)
$$

(3.2)

On the other hand, both odd fermionic fields $u_{2i-1}^f(x)$ and even fermionic fields $u_{2i}^f(x)$ have conformal weights $i + \frac{3}{2}$ ($i = 0, 1, 2, \ldots$) with respect to the same stress tensor (3.2). The stress tensor $T(x)$ belongs to the $N = 2$ conformal algebra being a subalgebra of (3.2). The conformal weights of the component fields are evident from the following relations.

$$
\{T(x), u_{2i-1}^b(y)\} = [(i + 1)u_{2i-1}^b(y)\partial_y + (u_{2i-1}^b(y))']
$$

$$
- \sum_{m=0}^{i-2} (-1)^m \left( \begin{array}{c}
    i \\
    m + 2
\end{array} \right) u_{2i-2m-3}^b(y)\partial_y^{m+2}
$$

10
\[-\frac{1}{2} \sum_{m=0}^{i-1} (-1)^m \left( \frac{i}{m+1} \right) u_{2i-2m-3}^b(y) \partial_y^{m+2} \delta(x-y) \]

\[\{T(x), u_{2i}^b(y)\} = \left( (i+2) u_{2i}^b(y) \partial_y + (u_{2i}^b(y))' \right) \]

\[- \sum_{m=0}^{i-2} (-1)^m \left( \frac{i}{m+2} \right) u_{2i-2m-2}^b(y) \partial_y^{m+2} \delta(x-y) \]

\[\{T(x), u_{2i-1}^f(y)\} = \left( (i+3) u_{2i-1}^f(y) \partial_y + (u_{2i-1}^f(y))' \right) \]

\[- \sum_{m=0}^{i-2} (-1)^m \left( \frac{i}{m+2} \right) u_{2i-2m-2}^f(y) \partial_y^{m+2} \delta(x-y) \]

\[\{T(x), u_{2i-1}^f(y)\} = \left( (i+3) u_{2i-1}^f(y) \partial_y + (u_{2i-1}^f(y))' \right) \]

This ensures the presence of a nonlinear supersymmetric conformal algebra in the Hamiltonian structure (2.2) of the $\mathcal{N} = 2$ super KP hierarchy. The Poisson brackets among all the component fields are given in appendix A in a basis which will be defined later.

The $W_{1+\infty} \oplus W_{\infty}$ structure of the bosonic sector, however, is not apparent in our case from the Poisson bracket between two types of bosons, $u_{2i-1}^b$ and $u_{2i}^b$. This is in contrast to the other supersymmetric algebras. We shall establish that the bosonic sector of this algebra, indeed, has the required $W_{1+\infty} \oplus W_{\infty}$ structure \[\text{[16]}\]. This step is crucial in obtaining the free field representation of the generators. To carry out this program, a suitable basis is required. Notice that the odd bosons themselves $u_{2i-1}$ form a closed algebra (A5) and consequently for the odd bosons the new set of generators may be constructed from a linear combination of the fields as considered in \[\text{[14]},\]
Since the odd bosons form a closed algebra among themselves, the algebra of the new set of generators are also closed and constitute the $\hat{W}_{1+\infty}$ algebra. Using the Poisson bracket amongst the component fields given in Appendix A, we obtain following Poisson brackets for the lower order odd boson generators.

\[
\{\tilde{W}_1, \tilde{W}_1\} = 0 \\
\{\tilde{W}_2, \tilde{W}_1\} = \tilde{W}_1 \delta(x - y) \\
\{\tilde{W}_2, \tilde{W}_2\} = \left[2\tilde{W}_2 \partial + \tilde{W}_2'\right] \delta(x - y) \\
\{\tilde{W}_3, \tilde{W}_1\} = 2\tilde{W}_2 \delta(x - y) \\
\{\tilde{W}_3, \tilde{W}_2\} = \left[3\tilde{W}_3 \partial + \tilde{W}_3' + \frac{1}{6} \tilde{W}_1 \partial^3\right] \delta(x - y) \\
\{\tilde{W}_3, \tilde{W}_3\} = \left[4\tilde{W}_4 \partial + 2\tilde{W}_4' + \frac{3}{5} \tilde{W}_2'' \partial + 2\tilde{W}_3 \tilde{W}_1 \partial - \frac{1}{3} \tilde{W}_1'' \tilde{W}_1 \partial \\
-2\tilde{W}_2 \partial + \frac{1}{2} \tilde{W}_1 \partial + \tilde{W}_2' \partial^2 + 2\tilde{W}_3 \partial^3 + \frac{2}{15} \tilde{W}_2''\right] \delta(x - y) \\
+\tilde{W}_3' \tilde{W}_1 - \frac{1}{6} \tilde{W}_1'' \tilde{W}_1 - 2\tilde{W}_2 \tilde{W}_2 + \tilde{W}_3 \tilde{W}_1' + \frac{1}{3} \tilde{W}_1'' \tilde{W}_1'\right] \delta(x - y) \\
\{\tilde{W}_4, \tilde{W}_1\} = \left[3\tilde{W}_3 \partial + \frac{1}{10} \tilde{W}_1'' \partial - \frac{3}{10} \tilde{W}_1' \partial^2 + \frac{1}{10} \tilde{W}_1 \partial^3\right] \delta(x - y) \\
\{\tilde{W}_4, \tilde{W}_2\} = \left[4\tilde{W}_4 \partial + \tilde{W}_4' + \frac{7}{10} \tilde{W}_2 \partial^3\right] \delta(x - y) \tag{3.5}
\]

In the even boson, $u_{2i}^b$ sector, however, the Poisson bracket relation is complex and it appears that the generators neither form a closed algebra nor do they commute with the odd bosons. Apparently, therefore, the direct sum structure is not maintained as required for the the supersymmetric $W$ algebra. This problem can be circumvented and for the even bosons also a suitable basis with these desirable properties can be obtained. The first step in making them commute is to redefine the even bosons as a linear
combination of odd and even bosons of equal spin as in \[14\].

\[ v^b_{2j} = u^b_{2j+1} + u^b_{2j} \]  

(3.6)

and similarly we choose a linear combination of generators for the odd fermions

\[ v^f_{2j-1} = u^f_{2j-1} - u^f_{2j} \]  

(3.7)

In the linear super $W_\infty$ algebra, this is sufficient to ensure commutation between odd and even bosons, and thereby establish the $W_{1+\infty} \oplus W_\infty$ structure, but in this case it is observed that the odd bosons commute with only the lowest spin even boson generator i.e.

\[ \{ u^b_{2j-1}, v^b_0 \} = 0 \]  

(3.8)

and the Poisson brackets with higher even bosons $v^b_{2j}$ ($j \neq 0$) are nonzero. Interestingly, nonlinear combinations of bosons and fermions exist which commute with all odd bosons. This may be achieved, as the second step, by taking the most general combinations of the fields of the appropriate conformal spin and the coefficient of the terms may be determined by allowing them to commute with the odd bosons. This procedure can be carried out for all even boson generators thereby yielding the mutually commuting set of generators. The explicit expressions of a few even boson generators are given below.

\[
\begin{align*}
W_2 &= v^b_0 \\
W_3 &= v^b_2 + \frac{1}{2}v^b_0 + u^b_0 v^b_0 + u^f_0 v^f_1 \\
W_4 &= v^b_4 + v^b_2 + \frac{1}{5}v^b_0 + 2v^b_0 u^b_1 + u^f_2 v^f_1 \\
&\quad + u^f_0 v^f_1 + u^f_0 v^b_1 + u^f_0 v^f_1 + v^b_2 u^b_1 + u^f_0 u^b_1 + u^f_0 v^f_1
\end{align*}
\]

(3.9)

and so on. The distinguishing character of this set is that spin 3 and higher generators are nonlinear. For the spin 3 generator it is a bilinear combination
of bosonic as well as spin $\frac{3}{2}$ generators. For the spin 4 generator, this combination is more involved, having terms trilinear in the fields. This indicates that for higher spin generators, the basis becomes more and more nonlinear. But most importantly, these generators are such that the desired property is exhibited, namely

$$\{u^b_{2j-1}, W_2\} = 0$$
$$\{u^b_{2j-1}, W_3\} = 0$$
$$\{u^b_{2j-1}, W_4\} = 0$$

(3.10)

i.e. the new set of even boson generators commute with all the odd bosons. Moreover the $W$ boson generators produce the requisite form of $\hat{W}_\infty$ algebra as can be observed from the following Poisson bracket relations.

$$\{W_2, W_2\} = [-2W_2\partial - W'_2] \delta(x - y)$$
$$\{W_3, W_2\} = [-3W_3\partial - W'_3] \delta(x - y)$$
$$\{W_3, W_3\} = \left[-4W_4\partial - 2W_2^2\partial - \frac{9}{20}W_2''\partial - 2W'_4 - (W_2^2)'ight.$$
$$\left. - \frac{1}{10}W_4'' - \frac{3}{4}W_2'\partial^2 - \frac{1}{2}W_2\partial^3\right] \delta(x - y)$$
$$\{W_4, W_2\} = \left[-4W_4\partial - W'_4 - \frac{2}{5}W_2\partial^3\right] \delta(x - y)$$

(3.11)

This algebra is isomorphic to classical analogue of the $\hat{W}_\infty$ algebra [11]. The procedure outlined above, although straightforward, becomes extremely difficult to use in obtaining the still higher spin generators and to show the closure of the algebra explicitly. To obtain the generators of higher spin we employ a different strategy. The $W_4$ generator, for example, may be obtained straightforwardly from the $\{W_3, W_3\}$ Poisson bracket algebra (3.11) by ensuring the closure of the algebra following the classical analogue of the $\hat{W}_\infty$ algebra [11]. Importantly the $W_4$ generator thus obtained matches with that of (3.9), which commutes with the odd bosons (3.10) and the form is
unique. Similarly, the explicit form of the $W_5$ generator may be obtained from the Poisson bracket relation, $\{W_4, W_3\}$ by ensuring the closure of the algebra following the classical analogue of $\hat{W}_\infty$ algebra. It is found that the leading term of the $\{W_4, W_3\}$ algebra becomes $v_b^6$. This, indeed, ensures the presence of the $W_5$ generator in the algebra. The $W_5$ generator exhibits the explicit form

\[
W_5 = v_b^6 + \frac{3}{2}v_4^b + \frac{9}{14}v_2^b + \frac{1}{14}v_0^{b''} + 3u_{-1}^b v_4^b + 3u_{-1}^b v_2^b + 3u_{-1}^b v_2^b \\
+ \frac{3}{2}u_{-1}^b v_0^b + u_{-1}^b v_0^b - u_{-1}^b v_0^b + u_{-1}^b v_0^b + \frac{1}{7}(u_{-1}^b v_0^b)'' \\
+ u_{-1}^f v_1^f + u_{-1}^f v_1^f + u_{0}^f v_3 + \frac{3}{2}u_{0}^f v_1^f - \frac{1}{2}u_{0}^f v_1^f + \frac{1}{2}u_{0}^f v_1^f \\
+ \frac{3}{2}u_{-1}^f v_{-1}^f - u_{-1}^f v_{-1}^f + u_{0}^f v_1^f + \frac{1}{7}(u_{-1}^f v_{-1}^f)'' + \frac{1}{2}u_{-1}^f u_{0}^f v_{-1}^f - \frac{1}{2}u_{-1}^f u_{0}^f v_{-1}^f \\
+ \frac{3}{2}u_{-1}^f u_{0}^f v_{-1}^f + u_{-1}^f u_{0}^f v_{-1}^f + 2u_{-1}^f u_{0}^f v_{-1}^f + 2u_{-1}^f u_{0}^f v_{-1}^f + u_{1}^f u_{0}^f v_{-1}^f \tag{3.12}
\]

It is found by explicit calculation that the $W_5$ generator commutes with the odd bosons, $u_{2j-1}^b$. The spin six generator $W_6$, in a similar way, may be obtained from the Poisson bracket relation, $\{W_4, W_4\}$, whose leading term turns out to be $v_b^8$. Thus the first three generators may be obtained by the bootstrap approach and from spin four onwards all the generators may be found following the above procedure. This strategy of obtaining bosonic higher spin generators evidently guarantees the closure of the algebra being isomorphic to the classical $\hat{W}_\infty$ algebra. We have checked up to spin six generators explicitly. But to obtain the explicit forms of all higher generators becomes very difficult, although the strategy is quite clear. Moreover, this strategy also ensures that in the bosonic limit the super $\hat{W}_\infty$ reduces to the $\hat{W}_\infty$ algebra. In this way, we may establish the $\hat{W}_{1+\infty} \oplus \hat{W}_\infty$ structure in the bosonic sector of the super $\hat{W}_\infty$ algebra.

In the fermionic sector both the fermions, $u_{2i}^f$ and $v_{2i-1}^f$ form closed algebras ($A_2, A_4$) separately. This reveals that a linear representation such as the one in $[\mathbb{F}]$ always exists for the fermions. We shall, however, show
the existence a nonlinear basis for both the fermions in conjunction with the odd bosons. This follows from the observation that the even fermions, \( u^{f}_{2i} \) as well as the odd fermions, \( v^{f}_{2i-1} \) and the odd bosons also satisfy separately a subalgebra (see appendix A). Nonetheless, it turns out that nonlinear basis has an interesting consequence. Both the fermions satisfy identical algebras in the nonlinear basis with an added advantage of generating the minimal algebra.

The new basis for the even fermion generators may be constructed as a suitable nonlinear combination of even fermions \( u^{f}_{2i} \) and odd bosons \( u^{b}_{2i-1} \) in the form

\[
\begin{align*}
\tilde{J}_{3/2} &= -u^{f}_{0} \\
\tilde{J}_{5/2} &= -u^{f}_{2} - u^{f}_{0} + u^{b}_{-1}u^{f}_{0} \\
\tilde{J}_{7/2} &= -u^{f}_{4} - u^{f}_{2} - \frac{5}{4}u^{f''}_{0} + u^{b}_{-1}u^{f}_{2} + u^{b}_{1}u^{f}_{0} + u^{b'}_{-1}u^{f}_{0} - u^{b^2}u^{f}_{0}
\end{align*}
\]  

(3.13)

Thus the new set of generators become more and more nonlinear as in the even boson case. But it is seen that the nonlinear basis for the even fermions (3.13) can be recast in terms of the bilinears of the generators only as

\[
\begin{align*}
\tilde{J}_{3/2} &= -u^{f}_{0} \\
\tilde{J}_{5/2} &= -u^{f}_{2} + \tilde{J}^{3/2}_{3/2} - \tilde{W}_{1}\tilde{J}_{3/2} \\
\tilde{J}_{7/2} &= -u^{f}_{4} + \tilde{J}^{5/2}_{5/2} + \frac{1}{4}\tilde{J}^{3/2}_{3/2} - \tilde{W}_{1}\tilde{J}_{5/2} + 2\tilde{W}_{1}\tilde{J}^{3/2}_{3/2}
\end{align*}
\]  

(3.14)

This demonstrates that all the even fermions, in general, can be written as bilinears in \( \tilde{J} \) and \( \tilde{W} \) having the form

\[
\begin{align*}
\tilde{J}_{n+3/2} &= -u^{f}_{2n} + \frac{2^{-n}(n-1)!}{(2n-1)!!} \sum_{l=0}^{n-1} \binom{n}{l} \binom{n + l + 1}{l + 1} j^{(n-l)}_{l+3/2} \\
&- \sum_{m=0}^{n-1} \sum_{k=0}^{n-m-1} \sum_{l=0}^{m} B_{kl} C^{m}_{ml} (-1)^{k+l} j^{(n-m-l-1)}_{k+3/2} \tilde{W}_{l+1}^{m-l}
\end{align*}
\]  

(3.15)
\( n = 0, 1, 2, \ldots \), where \( B_{kl}^n \) and \( C_{ml}^n \) are the c-number coefficients. While the values of \( C_{ml}^n \) can be easily extracted from (3.3) by determining \( u_{2i-1}^b \) in terms of \( \tilde{W}_{i+1} \), it appears that there is no straightforward procedure to determine the explicit expressions of the \( B_{kl}^n \) for arbitrary spins. The closed algebras among the even fermions and odd bosons strongly corroborates the existence of a nonlinear basis for all higher spin generators and thereby ensuring the coefficients \( B_{kl}^n \) can always be determined for all higher spin generators. The even fermions of higher spins may be obtained from (A1) and (A9) through the Poisson brackets of the even fermions \( u_{2k}^f \) with the \( W_2 \) and \( \tilde{W}_2 \) generators,

\[
\{ W_2, u_{2k}^f \} = -u_{2k+2}^f - \sum_{m=0}^{k} \binom{k + 1}{m + 1} \partial^{m+1} u_{2k-2m}^f \tag{3.16}
\]

and

\[
\{ \tilde{W}_2, u_{2k}^f \} = -u_{2k+2}^f - \frac{1}{2} u_{2k}^f + \frac{1}{2} u_{2k}^f \partial \\
+ \sum_{l=0}^{k} \binom{k}{l} u_{0}^f \partial_l u_{2k-2l-1}^b - \sum_{l=0}^{k} \binom{k}{l} u_{-1}^b \partial_l u_{2k-2l}^f \tag{3.17}
\]

The presence of the \( u_{2k+2}^f \) term in (3.16,3.17) clearly confirms that the next higher spin generator can be generated from the preceding one. Finally, (3.16) and (3.17) together with the closure of the fermion generators determine the higher spin generators explicitly. As a consequence, the closure of the algebra among the even fermions is ensured in the nonlinear basis. To show the closure property, the algebra among a few even fermion generators are given below.

\[
\{ \tilde{J}_{3/2}, \tilde{J}_{3/2} \} = 0 \\
\{ \tilde{J}_{5/2}, \tilde{J}_{3/2} \} = 0 \\
\{ \tilde{J}_{5/2}, \tilde{J}_{5/2} \} = 2\tilde{J}_{3/2}^r \tilde{J}_{3/2} \delta(x - y) \\
\{ \tilde{J}_{7/2}, \tilde{J}_{3/2} \} = -\tilde{J}_{3/2}^r \tilde{J}_{3/2} \delta(x - y) \\
\{ \tilde{J}_{7/2}, \tilde{J}_{5/2} \} = [-3\tilde{J}_{5/2}^r \tilde{J}_{3/2} + 3 \tilde{J}_{5/2} \tilde{J}_{3/2} \delta + 2\tilde{J}_{3/2}^r \tilde{J}_{3/2} \partial] \delta(x - y)
\]
\[
\{ \bar{J}_{7/2}, J_{7/2} \} = \left[ 6 \bar{J}'_{5/2} \bar{J}_{5/2} - 3 \bar{J}'_{5/2} \bar{J}'_{5/2} + 3 \bar{J}_{5/2} \bar{J}'_{5/2} + \frac{3}{4} \bar{J}'_{3/2} \bar{J}'_{3/2} - \frac{5}{4} \bar{J}'_{3/2} \bar{J}_{3/2} - \frac{5}{2} \bar{J}'_{3/2} \bar{J}_{3/2} \partial - \frac{5}{2} \bar{J}'_{3/2} \bar{J}_{3/2} \partial^2 \right] \delta(x - y)
\] (3.18)

It is interesting to note that unlike the linear algebra, the generators do not commute among themselves. This is a significant departure from the linear representation and compels one to make the free field representations nonlinear as will be observed later. This difference becomes evident from the spin 5/2 generator onwards. For example, the Poisson bracket of the \( \bar{J}_{5/2} \) generator with itself becomes nonzero. In fact, the self brackets of the generators, in general, cannot be made to zero by any change of basis. Thus the nonlinear algebra cannot be reduced trivially to the linear one. The algebra of \( \bar{W} \) and \( W \) bosons with even fermions are given in appendix B.

The arguments regarding the existence of the nonlinear basis for the even fermion generators stated above are also valid for the odd fermions since the odd fermions \( v^f_{2i-1} \) and the odd bosons \( u^b_{2i-1} \) form a closed algebra among themselves (A3,A4,A5). We can therefore generate a nonlinear basis for the odd fermions \( v^f_{2i-1} \) as well in terms of odd fermions and the odd bosons \( u^b_{2i-1} \). For example a consistent nonlinear basis for a few odd fermions may be identified as

\[
\begin{align*}
J_{3/2} &= v^f_{-1} \\
J_{5/2} &= -v^f_{1} + u^b_{-1} v^f_{-1} \\
J_{7/2} &= v^f_{3} + v^f_{1} + \frac{5}{4} v^f_{-1} - u^b_{-1} v^f_{-1} - u^b_{1} v^f_{-1} - u^b_{1} v^f_{-1} + u^b_{1} v^f_{-1}
\end{align*}
\] (3.19)

It is straightforward to observe that the generators in (3.19) can be rewritten as bilinear combinations of \( \bar{W} \) and \( J \) like the even fermions. This can also be argued from the fact that odd fermions and odd bosons form a subalgebra and thus can be generalised for other higher spin generators. The higher spin generators, however, may be obtained from the lower ones following the
Poisson bracket relations resulting from (A3,A8)

\[ \{ W_2, v_{2k-1}^f \} = v_{2k+1}^f - v_{2k-1}^f \partial + u_{-1}^b v_{2k-1}^f - u_{2k-1}^b v_{-1}^f \]  \hspace{1cm} (3.20)

\[ \{ \tilde{W}_2, v_{2k-1}^f \} = v_{2k+1}^f - \sum_{l=0}^{k-1} \left( \frac{k}{l} \right) u_{-1}^b \partial^l u_{2k-2l-3}^f \\
+ \left[ \sum_{m=0}^{k-1} \left( \frac{k}{m+1} \right) + \frac{1}{2} \sum_{m=0}^{k} \left( \frac{k}{m} \right) \right] \partial^{m+1} v_{2k-2m-1}^f \]  \hspace{1cm} (3.21)

and by demanding the closure of the algebra among the odd fermions. To demonstrate the identical algebra among the odd fermions and the even fermions in the nonlinear basis, we give below the algebra among a few odd fermion generators explicitly.

\[ \{ J_{3/2}, J_{3/2} \} = 0 \]
\[ \{ J_{5/2}, J_{3/2} \} = 0 \]
\[ \{ J_{5/2}, J_{5/2} \} = 2 J_{3/2}^f J_{3/2} \delta(x - y) \]
\[ \{ J_{7/2}, J_{3/2} \} = - J_{3/2}^f J_{3/2} \delta(x - y) \]
\[ \{ J_{7/2}, J_{5/2} \} = \left[ -3 J_{5/2}^f J_{5/2} - 3 J_{5/2} J_{3/2} \partial + 2 J_{3/2}^f J_{3/2} \partial \right] \delta(x - y) \]
\[ \{ J_{7/2}, J_{7/2} \} = \left[ 6 J_{5/2}^f J_{5/2} - 3 J_{5/2} J_{3/2} + 3 J_{5/2} J_{3/2}^f + \frac{3}{4} J_{3/2}^f J_{3/2}^f \right. \\
\left. - \frac{5}{4} J_{3/2}^f J_{3/2} - \frac{5}{2} J_{3/2}^f J_{3/2} \partial - \frac{5}{2} J_{3/2} J_{3/2} \partial^2 \right] \delta(x - y) \]  \hspace{1cm} (3.22)

The algebra of odd fermions with even fermions as well as with the bosons are also given in appendix B.

The super $\hat{W}_\infty$ algebra, in its own right, deserves to be a candidate for a universal algebra, unifying all finite dimensional bosonic $W$ algebras as well as supersymmetric $W$ algebras. The presence of classical analogue of $\hat{W}_\infty$ algebra and a direct sum basis in the bosonic sector guarantees that all finite dimensional bosonic $W$ algebras can be obtained under suitable truncation. Since the super $\hat{W}_\infty$ algebra is a higher spin extension of $N = 2$ conformal
algebra, it is expected to contain all finite dimensional $N = 2$ supersymmetric $W$ algebras, like the bosonic universal algebra. But a systematic analysis of the truncation of super $\hat{W}_\infty$ algebra through some non-compact coset model is yet to be studied.

4. Free-Field Representation

In this section we construct a consistent free field representation of the generators discussed in the earlier section. We will show that all the generators can be represented by the free complex bosons, $\phi(x, t)$ and $\bar{\phi}(x, t)$ and free fermions $\psi(x, t)$ and $\psi^*(x, t)$, which satisfy the following Poisson bracket algebras

\[
\{\psi^*(x), \psi(y)\} = \delta(x - y) \quad (4.1)
\]

and

\[
\{\partial\phi(x), \partial\bar{\phi}(y)\} = \partial_x \delta(x - y) \quad (4.2)
\]

The nontrivial Poisson bracket algebras among the even fermions $\tilde{J}_{n+3/2} (3.13, 3.14, 3.15)$ as well as odd fermions $J_{n+3/2} (3.19)$ ensure a significant departure of the free field representations from the linear ones [16]. We will see that the representations of the fermion generators become not only nonlinear, and in general exponential, but also a suitable combinations of both types of bosons and fermions. This makes the free field representation distinct and important. In the bosonic sector such a nontrivial change in the free field representation is not apparent from the Poisson bracket algebras of the generators. But it will be observed that the nontrivial Poisson bracket algebras of the fermions become responsible for a nontrivial change in the free field representation of the bosons over the linear ones.

In order to reproduce the the Poisson brackets for the even fermion generators, the free field representation of all the generators, in general, turns out to be exponential of the boson fields. The explicit forms of the represent-
tation a few even fermions may be given in order to observe the change in
the fermion sector.

\[ \tilde{J}_{3/2} = \psi^* \partial \phi e^{i\epsilon \phi_2} \]  
\[ \tilde{J}_{5/2} = \left[ \psi' \partial \phi + \psi^*(a \partial \phi + b \bar{\partial} \bar{\phi}) \partial \phi - b \psi \psi^* \psi' \right] e^{i\epsilon \phi_2} \]  
\[ \tilde{J}_{7/2} = \frac{5}{4} (\psi^* \partial \phi e^{i\epsilon \phi_2})'' - \psi' (\partial \phi e^{i\epsilon \phi_2})' + b \psi \psi^* \psi' e^{i\epsilon \phi_2} \]  
\[ + \psi' (a \partial \phi + b \bar{\partial} \bar{\phi})^2 \partial \phi e^{i\epsilon \phi_2} + b \psi^* \psi' e^{i\epsilon \phi_2} \]  
\[ + 2b^2 \psi \psi^* \psi' \partial \phi e^{i\epsilon \phi_2} + 3ab \psi \psi^* \psi' \partial \phi e^{i\epsilon \phi_2} \]  

where, \( a, b \) and \( \epsilon \) are real parameters and \( \phi_2 = \frac{1}{2i}(\phi - \bar{\phi}) \). Notice that
the significant change of the free field representation becomes obvious from
\( \tilde{J}_{5/2} \) onwards. This is due to the presence of both \( \psi \) and \( \psi^* \) fields in the
representations, the presence of both the fields being essential to reproduce
the algebra (3.18). In fact, the presence of both kinds of fermions, \( \psi \) and \( \psi^* \) is
inevitable to reproduce the nonzero algebras consistently in the even fermion
sector. An algorithmic procedure can be developed following (3.16,3.17) to
reproduce the free field representation of all even fermion generators, but it
involves explicit representations of the \( W_2 \) and \( \tilde{W}_2 \) generators, which will be
obtained later on. It is important to note that in order to reproduce the
even fermion algebras (3.18), the consistency condition demands that all the
parameters are not independent, but are related by

\[ \frac{b \epsilon}{\sqrt{2}} - \frac{a \epsilon}{\sqrt{2}} - ab = 1 \]  

Evidently, two more parameters still remain arbitrary, which cannot be fixed
at the classical level. The quantization of the algebra may fix the arbitrary
parameters through its central charge. The relation (4.6) dictates that all the
parameters \( a, b \) and \( \epsilon \) cannot be set to be zero simultaneously. This implies
the non-linear free field representation cannot be reduced to the linear one
trivially, which is a crucial observation and will also be seen in the odd fermion sector.

The similar Poisson bracket structures of the odd fermions $J_{n+\frac{3}{2}}$ and the even fermions $\tilde{J}_{n+\frac{3}{2}}$, however, indicate the similar free field representations of the generators for the odd fermions. We give below the representations of the odd fermions up to spin $\frac{7}{2}$. The other higher spin generators can be constructed in a similar fashion as in the even fermion case. The representations of the odd fermions may be given as

\begin{align}
J_{3/2} &= -\psi \partial \bar{\phi} e^{-i\phi_2} \\
J_{5/2} &= \left[ -\psi' \partial \bar{\phi} + \psi (a\partial \bar{\phi} + b\partial \bar{\phi}) \partial \bar{\phi} + a\psi^* \psi' \right] e^{-i\phi_2} \\
J_{7/2} &= -\frac{5}{4}(\psi \partial \bar{\phi} e^{-i\phi_2})'' + \psi' (\partial \bar{\phi} e^{-i\phi_2})' - a\psi^* \psi' (e^{-i\phi_2})' \\
&\quad -\psi (a\partial \bar{\phi} + b\partial \bar{\phi}) \partial \bar{\phi} e^{-i\phi_2} - \psi (a\partial \bar{\phi} + b\partial \bar{\phi}) (\partial \bar{\phi} e^{-i\phi_2})' \\
&\quad -\psi (a\partial \bar{\phi} + b\partial \bar{\phi})^2 \partial \bar{\phi} e^{-i\phi_2} - a\psi^* \psi' e^{-i\phi_2} \\
&\quad + 2a^2 \psi^* \psi' \partial \bar{\phi} e^{-i\phi_2} - 3ab \psi^* \psi' \partial \bar{\phi} e^{-i\phi_2} \\
&\quad - \psi (a\partial \bar{\phi} + b\partial \bar{\phi}) \partial \bar{\phi} e^{-i\phi_2} - \psi (a\partial \bar{\phi} + b\partial \bar{\phi}) (\partial \bar{\phi} e^{-i\phi_2})' \\
&\quad - \psi (a\partial \bar{\phi} + b\partial \bar{\phi})^2 \partial \bar{\phi} e^{-i\phi_2} - a\psi^* \psi' e^{-i\phi_2} \\
&\quad + 2a^2 \psi^* \psi' \partial \bar{\phi} e^{-i\phi_2} - 3ab \psi^* \psi' \partial \bar{\phi} e^{-i\phi_2} \\
&\quad + 2a^2 \psi^* \psi' \partial \bar{\phi} e^{-i\phi_2} - 3ab \psi^* \psi' \partial \bar{\phi} e^{-i\phi_2} \\
&\quad + 2a^2 \psi^* \psi' \partial \bar{\phi} e^{-i\phi_2} - 3ab \psi^* \psi' \partial \bar{\phi} e^{-i\phi_2} \quad (4.7)
\end{align}

The complex nature of the representations of the odd and even fermions ensure that the odd and fermions generators cannot be represented by a pair of free fields and their complex conjugates, like the linear one. The nonzero values of $a$ and $b$ make the representations different from each other. The representations of other higher spin generators may follow from (3.20,3.21) like the even fermion case.

In the bosonic sector, the free field representations become more involved. To be precise, in the linear representation [16], the $W_{1+\infty}$ algebra was realized in terms of the bilinears of a free fermion field and its conjugate whereas the $W_{\infty}$ algebra was realized from the bilinears of a free complex scalar field and its conjugate. This had the advantage of automatically ensuring the direct sum basis of the generators of these algebras. But such a simple representation cannot be considered in the present case. This, in turn, leads to an inconsistency in the fermion sectors and consequently demands for a
significant change in the free field representations in the bosonic sector.

For the odd bosons we have the following consistent representation. The linear part of the representation may be written in terms of the fermion bilinears. Thus the lowest one is identical to that of the linear representation, namely

\[ \tilde{W}_1 = -\psi^* \psi \]  

(4.10)

On the other hand in spin 2, we have trilinear terms,

\[ \tilde{W}_2 = -\frac{1}{2}(\psi^* \psi - \psi^* \psi') - \psi^* \psi(a\partial \phi + b\partial \bar{\phi}) \]

(4.11)

and this is the most general form of the spin 2 generator, but involving complex boson fields. From spin three onwards the representation becomes complicated having more and more nonlinear combinations of the free fields. We will therefore follow a different strategy from spin three onwards. Following a similar procedure as in the odd boson case, we can obtain the spin 2 generator of the other sector. The most general form of the \( W_2 \) generator is

\[ W_2 = -\partial \phi \partial \bar{\phi} - \psi^* \psi(a\partial \phi + b\partial \bar{\phi}) \]

(4.12)

which commutes with odd bosons. For both the spin 2 generators the last term turns out to be trilinear and more so this is the only possible term that exists at the spin 2 level being a mixture of bosonic and fermionic fields.

For higher spin generators, however, the representations of both types of the bosonic generators may be obtained from the leading order terms of the Poisson brackets, \( \{\tilde{J}_{n+3/2}, J_{3/2}\} \), the \( \tilde{J}_{n+3/2} \) being given in (3.15). This will immediately follow from the Poisson bracket relation (A7),

\[ \{u_{2n}, v_{-1}^{f}\} = \left((u_{2n+1}^b - u_{2n}^b) - \sum_{m=0}^{n} \binom{n+1}{m+1} (-1)^m u_{2n-2m-1}^b \partial^{m+1}\right) \delta(x - y) \]

(4.13)

The leading order terms in (4.13) evidently will be a combination of \( (\tilde{W}_{j+2} - W_{j+2}) \) term and the suitable combinations of lower order spin terms. The representation of \( W_{j+2} \) generator and consequently the \( \tilde{W}_{j+2} \) generator may be
obtained explicitly by exploiting the commuting property of $W$ and $\tilde{W}$ generators and from the Poisson bracket relation $\{W_{j+2}, W_2\}$. The consistency of these representations may be checked by comparing the algebra among the other bosonic generators. For example, the free field representations $\tilde{W}_3$ and $W_3$ generators may be obtained as follows. The Poisson bracket $\{\tilde{J}_{5/2}, J_{3/2}\}$ using the free field representations of the fermion generators $\tilde{J}_{5/2}$ and $J_{3/2}$ (4.4,4.7) is found to be

$$\{\tilde{J}_{5/2}, J_{3/2}\} = \left[ -\partial\phi\partial^2\phi + \frac{\epsilon}{\sqrt{2}}\partial\phi\partial\bar{\phi}(\partial\phi - \partial\bar{\phi}) - \partial\phi\partial\bar{\phi}(a\partial\phi + b\partial\bar{\phi}) \\
-\psi^*\psi' - 2a\psi^*\psi'\partial\phi - b\psi^*\psi'\partial\bar{\phi} + \psi^*\psi\partial\bar{\phi} \\
+ a\frac{\epsilon}{\sqrt{2}}\psi^*\psi(\partial\phi)^2 - b\frac{\epsilon}{\sqrt{2}}\psi^*\psi(\partial\bar{\phi})^2 - b\psi^*\psi\partial^2\phi \\
- \left(\partial\phi\partial\bar{\phi} + \psi^*\psi + 2\psi^*\psi(a\partial\phi + b\partial\bar{\phi})\right)\partial\delta(x - y) \right]$$

(4.14)

comparing the same with the Poisson bracket of generators $\tilde{J}_{5/2}$ and $J_{3/2}$ (B14) allows us to determine the free field representation of $W_3 - \tilde{W}_3$ from the leading order term since the free field representation of the non leading order terms are already known. Explicitly,

$$W_3 - \tilde{W}_3 = \frac{1}{2}[\partial^2\phi\partial\bar{\phi} - \partial\phi\partial^2\phi] + (-a + \frac{\epsilon}{\sqrt{2}})(\partial\phi)^2\partial\bar{\phi} - (b + \frac{\epsilon}{\sqrt{2}})\partial\phi(\partial\bar{\phi})^2 \\
+ \frac{1}{2}\psi^*\psi(a\partial^2\phi - b\partial^2\bar{\phi}) - \psi^*\psi(a\partial\phi + b\partial\bar{\phi})^2 + 2\partial\phi\partial\bar{\phi}\psi^*\psi \\
+ \frac{\epsilon}{\sqrt{2}}\psi^*\psi[a(\partial\phi)^2 - b(\partial\bar{\phi})^2] - \frac{1}{2}(\psi^*\psi + \psi^*\psi')(a\partial\phi - b\partial\bar{\phi}) \\
+ \frac{1}{6}(\psi^*\psi - 4\psi^*\psi' + \psi^*\psi'') + \psi^*\psi(a\partial\phi + b\partial\bar{\phi})^2 \\
+ (\psi^*\psi - \psi^*\psi')(a\partial\phi + b\partial\bar{\phi})$$

(4.15)

Next, we determine the Poisson bracket between $W_3 - \tilde{W}_3$ and $W_2$. Since the odd and even boson generators commute (B10), this operation allows one to find $W_3$ from (B11) and hence $\tilde{W}_3$. This procedure eventually leads to the
following explicit forms of spin 3 generators

\[
\tilde{W}_3 = -\frac{1}{6}(\psi''\psi - 4\psi'\psi' + \psi\psi'') - \psi\psi'(a\partial\phi + b\partial\bar{\phi})^2 \\
- (\psi'\psi - \psi'\psi')(a\partial\phi + b\partial\bar{\phi}) \\
(4.16)
\]

\[
W_3 = \frac{1}{2}[\partial^2\phi\partial\bar{\phi} - \partial\phi\partial^2\bar{\phi}] + (-a + \frac{\epsilon}{\sqrt{2}})(\partial\phi)^2\partial\bar{\phi} - (b + \frac{\epsilon}{\sqrt{2}})\partial\phi(\partial\bar{\phi})^2 \\
+ \frac{1}{2}\psi^\ast\psi(a\partial^2\phi - b\partial^2\bar{\phi}) - \psi^*\psi(a\partial\phi + b\partial\bar{\phi})^2 + 2\partial\phi\partial\bar{\phi}\psi^\ast\psi \\
+ \frac{\epsilon}{\sqrt{2}}\psi^\ast\psi[a(\partial\phi)^2 - b(\partial\bar{\phi})^2] - \frac{1}{2}(\psi'\psi + \psi'\psi')(a\partial\phi - b\partial\bar{\phi}) \\
(4.17)
\]

where \(a\) and \(b\) are the same parameters, already introduced in the fermionic sectors. Notice that the spin 3 generators acquire a complex structure and possess terms quadrilinear in free fields. In a similar manner both the spin four generators may be obtained from the Poisson bracket \(\{\tilde{J}_{7/2}, J_{3/2}\}\). This constitutes an algorithmic procedure by means of which free field representations of the higher spin generators may be constructed. The relation (4.6) dictates the both the parameters \(a\) and \(b\) cannot be set to be zero simultaneously, making the representation essentially nonlinear. The presence of an admixture of the bosonic and fermionic terms, on the other hand, makes the representation of bosonic sector nontrivial, unlike the linear case. Importantly, even with the presence of the fermionic and bosonic fields together, we observe that the odd bosons commute with the even bosons. All the higher spin generators may be constructed from the algebra amongst the even and odd fermionic generators (4.13) and consistency of these representations may be checked by comparing the algebra among the bosonic generators. But the explicit forms of the higher spin generators in terms of the free fields become more and more complicated as observed from the spin 3 generators (4.16, 4.17). However, the strategy is quite clear.
5. Conclusion

In this paper we have shown that $N = 2$ KP hierarchy associated with non-standard flows are biHamiltonian, one of the Hamiltonian structures being nonlocal. To show the existence of biHamiltonian structures is not straightforward, as it is an intricate process to obtain the correct Poisson bracket which makes the $N = 2$ KP flows Hamiltonian. Since one of the Hamiltonian structures is local, it becomes a candidate for a nonlinear super $\hat{\mathcal{W}}_\infty$ algebra which is a higher spin extension of $N = 2$ superconformal algebra. The bosonic sector correctly reproduces the $\hat{\mathcal{W}}_{1+\infty} \oplus \hat{\mathcal{W}}_\infty$ structure with an appropriate choice of basis. To be explicit, in the even boson sector the basis becomes highly nontrivial and nonlinear. But we have evoked a novel strategy to obtain all the generators. Consequently, the $\hat{\mathcal{W}}_\infty$ algebra becomes isomorphic to the classical analogue of the nonlinear symmetry considered in [1]. This ensures that the nonlinear super $\hat{\mathcal{W}}_\infty$ algebra under suitable reduction truncates and gives rise to all finite dimensional bosonic algebras.

In the fermionic sector, the odd and even fermions also form closed algebras among themselves in a suitable basis. It turns out that the algebra among both kinds of fermions becomes distinctly different from the linear algebra and more so they form identical algebra among themselves. The super $\hat{\mathcal{W}}_\infty$ algebra thus deserves to be a universal algebra unifying all finite dimensional bosonic as well as fermionic $W$ algebras.

The free field representations of the $N = 2$ nonlinear super $\hat{\mathcal{W}}_\infty$ algebra are obtained in terms of free complex bosons and fermions. These representations cannot be reduced to the linear one trivially. This is due the constraint condition (4.6). Moreover, the representation of the bosonic generators in terms of the free fields possesses a more complex structure having an admixture of complex bosons as well as fermions. But at the same time the odd and the even bosonic generators mutually commute with each other maintaining $\hat{\mathcal{W}}_{1+\infty} \oplus \hat{\mathcal{W}}_\infty$ structure. This is a nontrivial generalisation in
contrast to the linear representation of the $N = 2$ super $\hat{W}_\infty$ algebra. In the fermionic sector the most general representations become exponential in terms of the free fields. The free field representations of the $N = 2$ nonlinear super $\hat{W}_\infty$ algebra, in fact, is a major breakthrough in classifying $N = 2$ super conformal algebras.
Appendix A

Poisson bracket algebra amongst the component fields $u_{2i}^b$, $u_{2i-1}^b$, $u_{2i}^f$ and $u_{2i-1}^f$.

\[
\{ u_{2j-1}^b(x), u_{2k}^f(y) \} = \left[ - \sum_{m=0}^{j-1} \binom{j}{m} (-1)^m u_{2j+2k-2m} \partial^m 
\right. \\
+ \sum_{m=0}^{j-1} \sum_{l=0}^{k} \binom{j-1}{m} \binom{k}{l} (-1)^m u_{2j-2m-2l-1}^b \partial^{m+l} u_{2k}^b \\
\left. - \sum_{m=0}^{j-1} \sum_{l=0}^{k-1} \binom{j}{m+1} \binom{k}{l+1} (-1)^m u_{2j-2m-3l-2}^b \partial^{m+l+1} u_{2k}^b \\
+ \sum_{n=0}^{k-1} \sum_{l=0}^{k-n-1} \binom{k-n-1}{l} u_{2n}^f \partial^l u_{2j+2k-2n-2l-2}^b \\
- \sum_{m=0}^{j+k-n-l-1} \sum_{n=0}^{k-n} \sum_{l=0}^{k-n-1} \binom{j-1}{m} \binom{n+l-1}{l} (-1)^m \\
\times u_{2j+2k-2m-2n-2l-2}^f \partial^{m+l} u_{2n-1}^b \big] \delta(x-y) \tag{A1}
\]

\[
\{ u_{2j}^f(x), u_{2k}^f(y) \} = \\
\left[ - \sum_{m=0}^{j-1} \sum_{l=0}^{k-1} \binom{j}{m+1} \binom{k}{l+1} (-1)^m u_{2j-2m-2l-2}^f \partial^{m+l+1} u_{2k}^f \\
- \sum_{n=0}^{k-1} \sum_{l=0}^{k-n-1} \binom{k-n-1}{l} u_{2n}^f \partial^l u_{2j+2k-2n-2l-2}^f \\
+ \sum_{m=0}^{j+k-n-l-1} \sum_{n=0}^{k-n} \sum_{l=0}^{k-n-1} \binom{j}{m} \binom{n+l}{l} \\
\times (-1)^m u_{2j+2k-2m-2n-2l-2}^f \partial^{m+l} u_{2n}^f \big] \delta(x-y) \tag{A2}
\]
\[
\begin{align*}
\{u_{2j-1}^b, v_{2k-1}^f\} &= \left[ \sum_{m=0}^{k} \binom{k}{m} \partial^m v_{2j+2k-2m-1}^f \right] \\
&+ \sum_{m=0}^{j-1} \sum_{l=0}^{k-1} \left\{ \binom{j-1}{m} \binom{k-1}{l} - \binom{j}{m+1} \binom{k}{l+1} \right\} \\
& \times (-1)^m u_{2j-2m-3}^b \partial^{m+l+1} v_{2k-2l-3}^f \\
&+ \sum_{n=0}^{k-1} \sum_{l=0}^{k-n-1} \binom{k-n-1}{l} u_{2n-1}^b \partial^l v_{2j+2k-2n-2l-3}^f \\
&- \sum_{m=0}^{j+k-n-1} \sum_{n=0}^{k-1} \sum_{l=0}^{k-n-1} \binom{j-1}{m} \binom{n+l-1}{l} \\
& \times (-1)^m u_{2j+2k-2n-2l-3}^b \partial^l v_{2n-1}^f \delta(x - y)
\end{align*}
\]

(A3)

\[
\begin{align*}
\{v_{2j-1}^f, v_{2k-1}^f\} &= \left[ \sum_{m=0}^{j-1} \sum_{l=0}^{k-1} \left\{ \binom{j-1}{m} \binom{k-1}{l} - \binom{j}{m+1} \binom{k}{l+1} \right\} \\
& \times (-1)^m v_{2j-2m-3}^f \partial^{m+l+1} v_{2k-2l-3}^f \\
&+ \sum_{n=0}^{k-1} \sum_{l=0}^{k-n-1} \binom{k-n-1}{l} v_{2n-1}^f \partial^l v_{2j+2k-2n-2l-3}^f \\
&- \sum_{m=0}^{j+k-n-1} \sum_{n=0}^{k-1} \sum_{l=0}^{k-n-1} \binom{j-1}{m} \binom{n+l-1}{l} \\
& \times (-1)^m v_{2j+2k-2n-2l-3}^f \partial^l v_{2n-1}^f \delta(x - y)
\end{align*}
\]

(A4)
\[
\left\{u^b_{2j-1}(x), u^b_{2k-1}(y)\right\} = \begin{bmatrix} -\sum_{m=0}^{j} \binom{j}{m} (-1)^m u^b_{2j+2k-2m-1} \partial^m \\
+ \sum_{m=0}^{k} \binom{k}{m} \partial^m u^b_{2j+2k-2m-1} \\
+ \sum_{m=0}^{j-1} \sum_{l=0}^{k-1} \left\{ \binom{j-1}{m} \binom{k-1}{l} - \binom{j}{m+1} \binom{k}{l+1} \right\} \\
\times (-1)^m u^b_{2j-2m-3} \partial^{m+l+1} u^b_{2k-2l-3} \\
+ \sum_{n=0}^{j+k-n-l-1} \sum_{l=0}^{k-n-1} \binom{j-1}{m} \binom{n+l-1}{l} \\
\times (-1)^m u^b_{2j+2k-2n-2l-3} \partial^{m+l} u^b_{2n-1} \right] \delta(x-y) \tag{A5}\]

\[
\left\{v^b_{2j}(x), v^b_{2k}(y)\right\} = \\
\begin{bmatrix} \sum_{m=0}^{j+1} \binom{j+1}{m} (-1)^m v^b_{2j+2k-2m+2} \partial^m - \sum_{m=0}^{k+1} \binom{k+1}{m} \partial^m v^b_{2j+2k-2m+2} \\
- \sum_{m=0}^{j-1} \sum_{l=0}^{k-1} \binom{j}{m+1} \binom{k}{l+1} (-1)^m v^b_{2j-2m-2} \partial^{m+l+1} v^b_{2k-2l-2} \\
- \sum_{l=0}^{k} \binom{k}{l} u^b_{l-1} \partial^l v^b_{2j+2k-2l} - \sum_{l=0}^{k} \binom{k}{l} u^f_{2j} \partial^l v^f_{2k-2l-1} \\
+ \sum_{m=0}^{j} \binom{j}{m} (-1)^m v^b_{2j+2k-2m} \partial^m u^b_{l-1} - \sum_{m=0}^{j} \binom{j}{m} (-1)^m v^f_{2j-2m-1} \partial^m u^f_{2k} \\
- \sum_{n=0}^{k-1} \sum_{l=0}^{k-n-1} \binom{k-n-1}{l+1} v^b_{2n} \partial^l v^b_{2j+2k-2n-2l-2} \\
+ \sum_{n=0}^{j+k-n-l-1} \sum_{l=0}^{k-n-1} \binom{j}{m} \binom{n+l}{l} \\
\times (-1)^m v^b_{2j+2k-2n-2l-2} \partial^{m+l} v^b_{2n} \right] \delta(x-y) \tag{A6}\]
\]
\[ \begin{align*}
\left\{ u_{2j}^f(x), v_{2k-1}^f(y) \right\} &= \\
&= \left[ \sum_{m=0}^{j+1} \binom{j+1}{m} (-1)^m u_{2j+2k-2m+1}^b \partial^m - \sum_{m=0}^{k} \binom{k}{m} \partial^m v_{2j+2k-2m}^b \right] \\
&\quad - \sum_{m=0}^{j-1} \sum_{l=0}^{k-1} \binom{j}{m+1} \binom{k}{l+1} (-1)^m u_{2j+2k-2m-2}^f \partial^{m+l+1} v_{2k-2l-3}^f \\
&\quad - \sum_{m=0}^{j} \binom{j}{m} (-1)^m u_{2j+2k-2m-1}^b \partial^m u_{2k-1}^b \\
&\quad - \sum_{m=0}^{j} \binom{j}{m} (-1)^m v_{2j+2k-2m-1}^f \partial^m u_{2k-1}^b \\
&\quad - \sum_{n=0}^{k-1} \sum_{l=0}^{k-n-1} \binom{k-n-1}{l} v_{2n-1}^f \partial^l v_{2j+2k-2n-2l-2}^b \\
&\quad + \sum_{m=0}^{j+k-n-l-1} \sum_{n=0}^{k-n-l-1} \sum_{l=0}^{j} \binom{j}{m} \binom{n+1}{l} \\
&\quad \times (-1)^m v_{2j+2k-2m-2n-2l-3}^f \partial^{m+l+1} v_{2n}^b \] \\
&\times (x-y) \quad \text{(A7)}
\end{align*} \]

\[ \begin{align*}
\left\{ v_{2j}^b(x), v_{2k-1}^f(y) \right\} &= \left[ \sum_{m=0}^{j+1} \binom{j+1}{m} (-1)^m v_{2j+2k-2m+1}^f \partial^m \\
&\quad - \sum_{m=0}^{j-1} \sum_{l=0}^{k-1} \binom{j}{m+1} \binom{k}{l+1} (-1)^m v_{2j+2k-2m-2}^f \partial^{m+l+1} v_{2k-2l-3}^f \\
&\quad + \sum_{m=0}^{j} \binom{j}{m} (-1)^m v_{2j+2k-2m-1}^f \partial^m u_{2k-1}^b \\
&\quad - \sum_{m=0}^{j} \binom{j}{m} (-1)^m v_{2j+2k-2m-1}^b \partial^m u_{2k-1}^b \\
&\quad - \sum_{n=0}^{k-1} \sum_{l=0}^{k-n-1} \binom{k-n-1}{l} v_{2n-1}^f \partial^l v_{2j+2k-2n-2l-2}^b \\
&\quad + \sum_{m=0}^{j+k-n-l-1} \sum_{n=0}^{k-n-l-1} \sum_{l=0}^{j} \binom{j}{m} \binom{n+1}{l} \\
&\quad \times (-1)^m v_{2j+2k-2m-2n-2l-3}^f \partial^{m+l+1} v_{2n}^b \] \\
&\times (x-y) \quad \text{(A8)}
\end{align*} \]
\[\begin{align*} \{ v^b_{2j}(x), u^f_{2k}(y) \} &= \left[ -\sum_{m=0}^{k+1} \binom{k+1}{m} \partial^m u^f_{2j+2k-2m+2} \right. \\
&\quad - \sum_{m=0}^{j-1} \sum_{l=0}^{k-1} \binom{j}{m+1} \binom{k}{l+1} (-1)^m v^b_{2j-2m-2} \partial^{m+l+1} u^f_{2k-2l-2} \\
&\quad + \sum_{l=0}^{k} \binom{k}{l} u^f_{2j} \partial^l u^b_{2k-2l-1} - \sum_{l=0}^{k} \binom{k}{l} u^b_{2j} \partial^l u^f_{2k-2l} \\
&\quad - \sum_{n=0}^{k-1} \sum_{l=0}^{k-n-1} \binom{k-n-1}{l} v^b_{2n} \partial^l u^f_{2j+2k-2n-2l-2} \\
&\quad + \sum_{m=0}^{j-k-n-1} \sum_{n=0}^{k-n-1} \sum_{l=0}^{k-n-1} \binom{j-1}{m} \binom{n+l}{l} \\
&\left. \times (-1)^m v^b_{2j+2k-2m-2n-2l-2} \partial^{m+l} u^f_{2n} \right] \delta(x - y) \tag{A9} \end{align*}\]

\[\begin{align*} \{ u^b_{2j-1}(x), v^b_{2k}(y) \} &= \left[ -\sum_{m=0}^{j-1} \sum_{l=0}^{k} \binom{j-1}{m} \binom{k}{l} (-1)^m u^f_{2j-2m-2} \partial^{m+l} v^b_{2k-2l-1} \right. \\
&\quad - \sum_{m=0}^{j-1} \sum_{l=0}^{k-1} \binom{j}{m+1} \binom{k}{l+1} (-1)^m u^b_{2j-2m-3} \partial^{m+l+1} v^b_{2k-2l-2} \\
&\quad - \sum_{n=0}^{k-1} \sum_{l=0}^{k-n-1} \binom{k-n-1}{l} u^f_{2n} \partial^l v^b_{2j+2k-2n-2l-3} \\
&\quad + \sum_{m=0}^{j-k-n-1} \sum_{n=0}^{k-n-1} \sum_{l=0}^{k-n-1} \binom{j-1}{m} \binom{n+l-1}{l} \\
&\left. \times (-1)^m u^f_{2j+2k-2m-2n-2l-2} \partial^{m+l} v^b_{2n-1} \right] \delta(x - y) \tag{A10} \end{align*}\]
Appendix B

The Poisson brackets between odd bosons and even fermions are:

\[
\{\bar{W}_1, \tilde{J}_{3/2}\} = -\tilde{J}_{3/2} \delta(x-y) \tag{B1}
\]

\[
\{\bar{W}_2, \tilde{J}_{3/2}\} = \left[-\tilde{J}_{5/2} + \frac{1}{2} \tilde{J}_{3/2}' + \frac{1}{2} \tilde{J}_{3/2} \partial - \bar{W}_1 \tilde{J}_{3/2}\right] \delta(x-y) \tag{B2}
\]

\[
\{\bar{W}_1, \tilde{J}_{5/2}\} = \left[-\tilde{J}_{5/2} + \tilde{J}_{5/2}' + \tilde{J}_{3/2} \partial\right] \delta(x-y) \tag{B3}
\]

\[
\{\bar{W}_2, \tilde{J}_{5/2}\} = \left[-\tilde{J}_{7/2} + \frac{1}{2} \tilde{J}_{5/2}' + \frac{3}{4} \tilde{J}_{3/2}' - \bar{W}_1 \tilde{J}_{5/2}\right.
\]
\[
+ \frac{3}{2} \tilde{J}_{5/2} \partial - \tilde{J}_{3/2} \partial - \frac{1}{2} \tilde{J}_{3/2} \partial^2 \right] \delta(x-y) \tag{B4}
\]

The Poisson brackets between odd bosons and odd fermions are:

\[
\{\bar{W}_1, J_{3/2}\} = J_{3/2} \delta(x-y) \tag{B5}
\]

\[
\{\bar{W}_2, J_{3/2}\} = \left[-J_{5/2} + \frac{1}{2} J_{3/2}' + \frac{1}{2} J_{3/2} \partial + \bar{W}_1 J_{3/2}\right] \delta(x-y) \tag{B6}
\]

\[
\{\bar{W}_1, J_{5/2}\} = \left[J_{3/2} - J_{3/2}' - J_{3/2} \partial\right] \delta(x-y) \tag{B7}
\]

\[
\{\bar{W}_2, J_{5/2}\} = \left[-J_{7/2} + \frac{1}{2} J_{5/2}' + \frac{3}{4} J_{3/2}' + \bar{W}_1 J_{5/2}\right.
\]
\[
+ \frac{3}{2} J_{5/2} \partial - J_{3/2} \partial - \frac{1}{2} J_{3/2} \partial^2 \right] \delta(x-y) \tag{B8}
\]

The Poisson brackets between even bosons and even fermions are:

\[
\{W_1, \tilde{J}_{3/2}\} = \left[-\tilde{J}_{5/2} - \tilde{J}_{3/2} \partial - \bar{W}_1 \tilde{J}_{3/2}\right] \delta(x-y) \tag{B9}
\]

\[
\{W_2, \tilde{J}_{5/2}\} = \left[-\tilde{J}_{7/2} - \tilde{J}_{5/2}' + \frac{5}{4} \tilde{J}_{3/2}' - \bar{W}_1 \tilde{J}_{5/2} - \tilde{J}_{5/2} \partial\right] \delta(x-y) \tag{B10}
\]

The Poisson brackets between even bosons and odd fermions are:

\[
\{W_1, J_{3/2}\} = \left[-J_{5/2} - J_{3/2} \partial + \bar{W}_1 J_{3/2}\right] \delta(x-y) \tag{B11}
\]

\[
\{W_2, J_{5/2}\} = \left[-J_{7/2} - J_{5/2}' + \frac{5}{4} J_{3/2}' + \bar{W}_1 J_{5/2} - J_{5/2} \partial\right] \delta(x-y) \tag{B12}
\]
The Poisson bracket between even fermions and odd fermions are:

\[
\{ \tilde{J}_{3/2}, J_{3/2} \} = \left[ -\tilde{W}_2 + W_2 + \frac{1}{2} \tilde{W}'_1 + \tilde{W}_1 \partial \right] \delta(x - y) \quad (B13)
\]

\[
\{ \tilde{J}_{3/2}, J_{5/2} \} = \left[ \tilde{W}_3 - W_3 - \tilde{W}'_2 - \frac{1}{2} W'_2 - \tilde{W}_2 \tilde{W}_1 + 2 W_2 \tilde{W}_1 \\
+ \frac{3}{2} \tilde{W}'_1 \tilde{W}_1 + \frac{1}{3} \tilde{W}'''_1 - \left( \tilde{W}'_2 + W_2 - \frac{1}{2} \tilde{W}'_1 - \tilde{W}^2_1 \right) \partial \right] \delta(x - y) \quad (B14)
\]

\[
\{ \tilde{J}_{5/2}, J_{3/2} \} = \left[ -\tilde{W}_3 + W_3 + \frac{1}{2} W'_2 + \tilde{W}_2 \tilde{W}_1 - 2 W_2 \tilde{W}_1 \\
- \frac{1}{2} \tilde{W}'_1 \tilde{W}_1 + \frac{1}{6} \tilde{W}'''_1 + \left( \tilde{W}_2 + W_2 + \frac{1}{2} \tilde{W}'_1 - \tilde{W}^2_1 \right) \partial \right] \delta(x - y) \quad (B15)
\]

\[
\{ \tilde{J}_{5/2}, J_{5/2} \} = \left[ \tilde{W}_4 - W_4 - \frac{1}{2} \tilde{W}'_3 - W'_3 - \tilde{W}_3 \tilde{W}_1 + 3 W_2 \tilde{W}_1 \\
+ W_2 \tilde{W}_2 + \frac{3}{2} W'_2 \tilde{W}_1 + \frac{1}{6} \tilde{W}'_2 \tilde{W}_1 + \frac{3}{2} W'_2 \tilde{W}'_1 + \tilde{W}_2 \tilde{W}'_1 - 3 W_2 \tilde{W}'_1 \\
+ \tilde{W}_2 \tilde{W}^2_1 - \frac{3}{10} W'''_2 - \frac{1}{10} \tilde{W}'''_2 - \tilde{W}'_2 + \frac{1}{12} \tilde{W}''''_1 \tilde{W}_1 + \frac{3}{4} \tilde{W}^2_1 \\
+ \frac{1}{12} \tilde{W}'''_2 - \frac{3}{2} \tilde{W}'_2 \tilde{W}'_1 - J'_{3/2} \tilde{J}_{3/2} - \left( \tilde{W}_3 + 2 W_3 - 3 W_2 \tilde{W}_1 - 2 \tilde{W}_2 \tilde{W}_1 \\
- \frac{1}{6} \tilde{W}'_1 + W'_2 - \tilde{W}'^3_1 + J_{3/2} \tilde{J}_{3/2} \right) \partial - W_2 \partial^2 \right] \delta(x - y) \quad (B16)
\]

and so on.
References

[1] A.B.Zamolodchikov, Theor. Math. Phys. 65 (1985) 1205;
    A.B.Zamolodchikov and V.A.Fateev, Nucl. Phys. B 280 [FS18] (1987)
    644;
    V.A.Fateev and S.L.Lykyanov Int. J. Mod. Phys. A 3(1988) 507.

[2] I.M.Gelfand and L.A.Dikii, Russ. Math. Surv. 30 (1975) 77; Funct. Anal.
    Appl. 10 (1976) 259.

[3] V.G.Drinfel'd and V.V.Sokolov, Lie algebra and Equations of Kortweg-
    De Vries type (Plenum, 1985) p. 1975-2036.

[4] S.Ghosh and S.K.Paul Phys. Lett. B 341 (1995) 293.

[5] P.Di Francesco, P.Ginsparg and J.Zinn-Justin, 2D Gravity and Random
    Matrices, Phys. Rep. 254 (1995) 1;
    F.Toppan, hep-th/9506113 and the references therein;
    A.Morozov, hep-th/9502091;
    A.Mironov, Int. J. Mod. Phys. A9 (1994) 4355.

[6] M.Sato, RIMS Kokyuroku 439 (1981) 30;
    E.Date, M.Jimbo, M.Kashiwara and T.Miwa, in Proc. RIMS Symp.
    on Nonlinear Integrable Systems, eds. M.Jimbo and T.Miwa (World
    Scientific, 1983);
    G.Segal and G.Wilson, Publ. IHES 61 (1985) 1. a

[7] M.Fukuma, H.Kawai and R.Nakayama, Int. J. Mod. Phys. A 6 (1991)
    1385;
    M.A.Awada and S.J.Sin, Int. J. Mod. Phys. A 7 (1992) 4791.

[8] C.Pope, L.Romans, and X.Shen, Nucl. Phys. B 339 (1990) 191; Phys.
    Lett. B 236 (1990) 173;
I. Bakas, Phys. Lett. B 228 (1989) 57; Comm. Math. Phys. 134 (1990) 487;
F.Yu and Y.S.Wu, Phys. Lett. B 263 (1991) 109.

[9] I.Bakas and E.Kiritsis, Nucl. Phys. B 343 (1990) 185.

[10] A.Das, W.-J.Huang and S.Panda, Phys. Lett. B 271 (1991) 109;
F.Yu and Y.-S.Wu, Nucl. Phys. B 373 (1992) 713.

[11] I.Bakas and E.Kiritsis, Int. J. Mod. Phys. A 7, Suppl. 1A (1992) 55.

[12] Yu.I.Manin and A.O.Radul, Comm. Math. Phys. 98 (1985) 65.

[13] S.Panda and S.Roy, Phys. Lett. B 291 (1992) 77.

[14] F.Yu, Nucl. Phys. B 375 (1992) 173.

[15] J.Barcelos-Neto, S.Ghosh and S.Roy, J. Math. Phys. 36 (1995) 258.

[16] E.Bergshoeff, C.Pope, L.Romans, E.Sezgin, and X.Shen, Phys.Lett. B 245 (1990) 447.

[17] T.Inami and H.Kanno, Nucl. Phys. B 359 (1990) 185.

[18] T.Inami and H.Kanno, Int. J. Mod. Phys. A 7, Suppl. 1A (1992) 419
and references therein.

[19] L.Alvarez-Gaume and J.L.Manes CERN preprint CERN-TH.6067/91.

[20] H.Aratyn, A.Das and C.Rasinariu, hep-th/9704119;
E.Ivanov, S.Krivonos and F.Toppan, hep-th/9703224;
F.Delduc, F.Gieres, S.Gourmelon and S.Thiesen, Int. J. Mod. Phys. A14
(1999) 4043;
O.Lechtenfeld and A.Sorin, solv-int/9907021;
K.Takasaki, hep-th/9905224;
G.Falqui, C.Reina and A.Zampa, nlin.SI/0001052.
[21] N.Seiberg and E.Witten, Nucl. Phys. B 426 (1994) 19; Nucl. Phys. B 431 (1994) 484;
    R.Y.Donagi, alg-geom/9705010.

[22] H.Aratyn, E.Nissimov, S.Pascheva, solv-int/9808004.
    I.N.McArthur and C.M.Yung, Mod. Phys. Lett. A 18 (1993) 1739.