Low-Energy Effective Field Theory of Lepton-Proton Bremsstrahlung

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Abstract

We calculate the cross section for the lepton-proton bremsstrahlung process \( l + p \to l' + p + \gamma \) in effective field theory. This process corresponds to an undetected background signal for the proposed MUSE experiment at PSI. MUSE is designed to measure elastic scattering of low-energy electrons and muons off a proton target in order to extract a precise value for the proton’s r.m.s. radius. We show that the commonly used peaking approximation, which is used to evaluate the radiative tail for the elastic cross section, is not applicable for muon proton scattering at the low-energy MUSE kinematics. We also correct a misprint in a commonly cited review article.

1 Introduction

Recent high precision experimental determinations of the proton’s r.m.s. radius have produced results which are not consistent with earlier results [1][2]. The proton radius puzzle refers to the contrasting results obtained between the proton’s electric charge radii extracted from the Lamb shift of muonic hydrogen atoms and those extracted from electron-proton scattering measurements, (see, e.g., Refs. [3][4].) In order to resolve this issue, a number of newly commissioned experiments are underway, along with proposals of redoing the lepton proton scattering measurements at low momentum transfers. The latter includes the MUon proton Scattering Experiment (MUSE) [6]. The MUSE collaboration proposes to measure the elastic differential cross sections for \( e^\pm p \) and \( \mu^\pm p \) scattering at very low momentum transfers with an anticipated accuracy of a few tenths of a percent. This should allow for a very precise determination of the slope of the proton’s electric form factor \( G_E \) and thereby extract a value for the proton’s r.m.s. radius squared, \( \langle r_p^2 \rangle = \frac{6}{Q^2} \frac{\partial G_E(Q^2)}{\partial Q^2} |_{Q^2=0} \), where \( Q^2 \) is four momentum transfer. The MUSE collaboration aims for an r.m.s. radius error of
about 0.01 fm. However, in MUSE only the lepton scattering angle $\theta$ is detected. The final scattered lepton energy $E'_l$ is not measured, nor are the bremsstrahlung photons. The bremsstrahlung photons constitute an integral part of the lepton-proton elastic scattering, and is one of the principal sources of uncertainties in the accurate measurement of the momentum transfers. In order to determine the proton charge radius, the data analysis necessarily needs to correct for this radiation process.

The lepton beam momenta considered by MUSE are of the order of the muon mass [6, 7]. A particular concern is the standard radiative correction procedure, which makes use of the so-called peaking approximation, (see e.g., Refs, [8–11] for reviews)\(^1\) This approximation assumes that the bremsstrahlung photons are emitted either along the incident beam direction, or in the direction of the scattered final lepton momentum. The validity of this approximation normally relies on elastic scattering of highly relativistic particles, like electrons. This is, however, questionable when either the particle energy is comparable to its mass, e.g., in the case of low-energy muon scattering in MUSE, or for inelastic scatterings with large energy loss (30-40%) from the incident projectiles due to bremsstrahlung [11]. The purpose of this paper is to evaluate the bremsstrahlung processes from electron and muon scattering off a proton at low energies in a model independent formalism. We show in a pedagogical manner the radical differences between the electron and muon angular bremsstrahlung spectra. We also correct a misprint in Eq.(B.5) of Ref. [11], which is important in the low-energy lepton scattering processes.

At low energies, which includes the kinematic region for MUSE, hadrons are the relevant degrees of freedom, and the dynamics is governed by chiral symmetry requirements. Heavy Baryon Chiral Perturbation Theory ($\chi$PT) is a low-energy hadronic effective field theory (EFT), which incorporates the underlying symmetries and symmetry breaking patterns of QCD. In $\chi$PT the evaluation of observables follow well-defined chiral power counting rules, which determines the dominant leading order (LO) contributions, as well as the next-to-leading order (NLO) and higher order corrections to observables in a perturbative scheme. For example, in $\chi$PT the proton’s r.m.s. radius enters at a higher chiral order where pion loops at the proton-photon vertex enter the calculation, see e.g., Ref. [12]. Furthermore, $\chi$PT naturally includes the photon-hadron coupling in a gauge invariant way.

As mentioned, the goal of the present paper is to provide a pedagogical and model independent presentation of the bremsstrahlung process in a low-energy EFT framework. We will show how the lepton mass will crucially influence the photon’s angular distribution, and we will discuss how the radiative tail cross section $d\sigma/dp'd\Omega'$ depends on the outgoing lepton’s momentum $p'$ and mass $m_l$. We demonstrate that the peaking approximation is not applicable for muon scattering at MUSE momenta. In fact, our results corroborate the analysis in Ref. [13], where it was shown that the peaking approximation, which is predominantly valid in the zero-mass limit or for very high-momentum transfers, becomes questionable at lower energies and could lead to significant errors in estimating the low-energy radiative cross sections. In this paper we will for clarity only

\(^1\)Ref. [8] nicely explain the distinctions between radiative corrections and the radiative bremsstrahlung tail of the elastic process involving lepton scattering.
evaluate the LO $\chi$PT contribution of the bremsstrahlung process. The paper is organized as follows: In section 2 we present a brief description of the lepton-proton bremsstrahlung process and the associated kinematics in the context of $\chi$PT, which allows a systematic analysis of the radiative and proton recoil correction order by order in a perturbative framework. In section 3 we present and discuss our LO results before drawing some conclusions.

2 Low-Energy Lepton-Proton Bremsstrahlung

In our evaluation the relativistic lepton-current \[ J_\mu^l(Q) = e\bar{u}_l(p')\gamma^\mu u_l(p), \] where $e = \sqrt{4\pi\alpha}$ and the four-momentum transfer to the proton is $Q = p - p'$, and where the lepton mass $m_l$ is included in all our expressions. The hadronic current is derived from the $\chi$PT Lagrangian. In $\chi$PT it is assumed that the LO terms give the dominant contributions to the amplitude, while the higher chiral orders contribute smaller corrections to the LO amplitude. The effective $\chi$PT Lagrangian $L_\chi$ is expanded in increasing chiral order as

\[ L_\chi = L_{\chi}^{(0)} + L_{\chi}^{(1)} + \cdots, \]

where the superscript in $L^{(\nu)}$ denotes interaction terms of chiral order $\nu$. The proton mass, $m_p$, is large, of the order of the chiral scale $\Lambda_\chi \sim 4\pi f_\pi \sim 1$ GeV, where $f_\pi = 93$ MeV is the pion decay constant. An integral part of $\chi$PT is the expansion of the Lagrangian in powers of $m^{-1}_p$, where at LO, i.e., $L_{\chi}^{(0)}$, the proton is assumed static ($m_p \to \infty$). Please note that in our evaluation the $m^{-1}_p$ corrections to the bremsstrahlung process have two different origin, namely, the *kinematic* phase-space corrections, as seen in Eqs.(5) and (6) below, and the *dynamical* $m^{-1}_p$ corrections that arise from the photon-proton interaction in the $L_{\chi}^{(1)}$ terms of the Lagrangian. In particular, it should be mentioned that the anomalous magnetic moments of the nucleons enter the $\chi$PT formalism at NLO through $L_{\chi}^{(1)}$.

In this paper we limit ourselves to LO, i.e., $\nu = 0$, and the explicit expression for $L_{\chi}^{(0)}$ relevant for the processes under study is $[12,15]$

\[ L_{\pi p}^{(0)} = \psi_p^\dagger(i v \cdot D + g_A S \cdot u)\psi_p. \]

Here $\psi_p$ denotes the heavy nucleon field and $g_A = 1.26$ is the nucleon axial vector coupling constant. It is convenient to choose the proton four-velocity to be $v^\mu = (1, \vec{0})$ which determines the covariant proton spin operator to be $S^\mu = (0, \frac{1}{2}\vec{\sigma})$. The covariant derivative $D_\mu$ in Eq.(3) is defined by

\[ D_\mu = \partial_\mu + \Gamma_\mu - iv_\mu^{(s)}; \quad \text{where} \quad \Gamma_\mu = \frac{1}{2}[u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - il_\mu)u^\dagger], \quad \text{and} \quad u_\mu = iu^\dagger\nabla_\mu Uu^\dagger; \quad \text{where} \quad \nabla_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu. \]

\[ \text{By LO, we mean the correction terms that are leading order in chiral power counting, which however, includes the kinematic recoil corrections up to } O(m^{-1}_p) \text{ (see, next section for a clarification).} \]
The photon field $A_\mu(x)$ is the only external source field in this work. The iso-scalar source (the iso-scalar part of the photon field) is therefore $v^{(s)}_\mu(x) = -\frac{e}{2} A_\mu(x)$, and, the chiral connection, $\Gamma_\mu$ and vielbein, $u_\mu$, which include the external iso-vector sources $r_\mu(x)$ and $l_\mu(x)$, become $r_\mu(x) = l_\mu(x) = -e\frac{2}{3} A_\mu(x)$. Generally, the $U$-field depends non-linearly on the pion field $\pi$, and in the so-called “sigma”-gauge has the following form: $U(x) \equiv u^2(x) = \sqrt{1 - \pi^2/f^2_\pi + i\tau \cdot \pi / f_\pi}$. However, in our study $U(x) = 1$ in the absence of explicit pions at the given chiral order we are working.

We further adopt the Coulomb gauge, $\epsilon \cdot v = 0$, where $\epsilon^\mu$ is the outgoing photon polarization four-vector. This implies that in $\chi$PT the bremsstrahlung photon from the proton diagrams, (C) and (D) in Fig. 1, do not contribute to the bremsstrahlung at LO. The first non-trivial contributions of photon radiation from protons arise from the NLO interactions specified in $\mathcal{L}_{\pi N}^{(1)}$. In other words, our LO analysis only includes contributions from the lepton bremsstrahlung.

We evaluate the Feynman diagrams (A) and (B) in Fig. 1, and denote the incident and scattered lepton four-momenta as $p = (E_l, \vec{p})$ and $p' = (E'_l, \vec{p'})$, respectively, where, e.g., $E_i = \sqrt{m_l^2 + \vec{p}^2}$. The corresponding proton four-momenta are $P_p = (E_p, \vec{P}_p)$ and $P'_p = (E'_p, \vec{P}'_p)$, and the outgoing photon has the four-momentum $k = (E_\gamma, \vec{k})$. Furthermore, the lepton scattering angle is $\theta$, i.e., $\vec{p} \cdot \vec{p}' = |\vec{p}| |\vec{p}'| \cos \theta$, and $q = (Q - k)$ is the four-momentum transferred to the proton when the lepton is radiating.

In $\chi$PT the non-relativistic heavy proton momentum satisfies re-parameterization invariance [16], and takes the form $P'^\mu_p = m_p v^\mu + p'^\mu_p$, where the momentum, $p^2_p \ll m_p^2$. This means that the incident proton kinetic energy to lowest order in $m_p^{-1}$ becomes $v \cdot p_p = \frac{p^2_p}{2m_p} + \cdots$, and similarly for the final
recoiling proton. The bremsstrahlung cross section in the lab. frame (with \( \vec{p}_p = 0 \)) can be written as

\[
\frac{1}{4} \sum |M_{br}|^2 = \frac{1}{4} \sum |M_A + M_B|^2 = 128\pi^3 \alpha^3 (m_p + E_p)(m_p + E'_p)
\]

where in the phase-space expression (including the \( \delta \)-function) we expand the recoil proton energy as

\[
E'_p = \sqrt{m_p^2 + (\vec{p}'_p)^2} = m_p + \frac{(\vec{p} - \vec{p}' - \vec{k})^2}{2m_p} + \ldots .
\]

A straightforward evaluation of the two Feynman diagrams (A) and (B), shown in Fig. 1, leads to the following LO expression of the bremsstrahlung amplitude squared:

\[
\frac{1}{4} \sum |M_{br}|^2 = \frac{1}{4} \sum |M_A + M_B|^2 = 128\pi^3 \alpha^3 (m_p + E_p)(m_p + E'_p)
\]

\[
\times \left\{ \frac{4}{q^4[(p' + k)^2 - m_f^2]^2} \left[ m_f^2 \vec{p}' \cdot \vec{p}' + m_f^2 E'_f E_\gamma + m_f^2 E'_f E_l + m_f^2 E_l E_\gamma - m_f^2 \vec{k} \cdot \vec{p}' - E'_f E_l \vec{k} \cdot \vec{p}' \\
+ (\vec{k} \cdot \vec{p}')(\vec{k} \cdot \vec{p}) + m_4 + m_3 \vec{k} \cdot \vec{p}' - E'_f E_f E_l + (\vec{k} \cdot \vec{p}')(\vec{k} \cdot \vec{p}) \right] \\
- \frac{4}{q^4[(p' - k)^2 - m_f^2]^2} \left[ m_f^2 \vec{p}' \cdot \vec{p}' - m_f^2 \vec{k} \cdot \vec{p}' + m_f^2 \vec{k} \cdot \vec{p}' - E'_f E_f E_l + (\vec{k} \cdot \vec{p}')(\vec{k} \cdot \vec{p}) \right] \\
- 8 \left[ m_f^2 E'_f E_l + E'_f E'_f E'_l - m_f^2 \vec{p}' \cdot \vec{p}' + E'_f \vec{k} \cdot \vec{p}' - E'_f \vec{k} \cdot \vec{p}' - (\vec{p} \cdot \vec{p}')^2 + (\vec{p} \cdot \vec{p}')(\vec{k} \cdot \vec{p} - \vec{k} \cdot \vec{p}') \right] \right\}.
\]

To evaluate the cross section, it is convenient to define our reference frame such that the momentum transfer, \( \vec{Q} = \vec{p} - \vec{p}' \) is directed along the z-axis \([11]\), while the lepton momenta, \( \vec{p} \) and \( \vec{p}' \), lie in xz-plane, as shown in Fig. 2. The pertinent angles are defined as follows:

\[
\vec{k} \cdot \vec{p}' = E_\gamma |\vec{p}'| (\cos \gamma \cos \alpha + \sin \alpha \sin \gamma \cos \phi_\gamma),
\]

\[
\vec{k} \cdot \vec{p} = E_\gamma |\vec{p}| (\cos \zeta \cos \alpha + \sin \alpha \sin \zeta \cos \phi_\gamma).
\]

The lepton scattering angle \( \theta = \gamma - \zeta \) in our coordinate system. When the lepton radiates a photon the square of the four-momentum transferred to the proton is

\[
q^2 = 2 \left[ m_f^2 - E_l E_f' + |\vec{p}'||\vec{p}'| \cos \theta - E_\gamma (E_l - E_f') \right] + E_\gamma \cos \alpha \sqrt{|\vec{p}'|^2 + |\vec{p}'|^2 - 2|\vec{p}'||\vec{p}'| \cos \theta},
\]

which is independent of \( \phi_\gamma \) in our reference frame as suggested in Ref. [11]. This choice of reference frame readily allows the analytical \( \phi_\gamma \) integration.

In our final expression for the differential cross section \( d^3\sigma^{(LO)}/(d|\vec{p}'| \ d\Omega') d\cos \theta \), Eq. (12), we for convenience define two angle dependent parameters, \( a = (1 - \beta' \cos \phi_\gamma)/(\beta' \sin \phi_\gamma) \) and \( b = (1 - \beta \cos \phi_\gamma)/(\beta \sin \phi_\gamma) \). Here \( \beta = |\vec{p}|/E_l \) and \( \beta' = |\vec{p}'|/E_f' \) are the incoming and outgoing
lepton velocities, respectively. In this equation, we also define the magnitudes of the incoming and outgoing lepton three-momenta as $p = |\vec{p}|$ and $p' = |\vec{p}'|$, respectively. To obtain our final expression, we integrate the cross section in Eq. (5) over the photon energies, $E_\gamma$, which means that the infrared singularity will appear when the momentum $p'$ is close to its maximal allowed value. To account for the proton recoil corrections in the kinematics we include the $m_p^{-1}$ terms in the photon energy $E_\gamma$ given by the delta-function in Eq. (5). We define $E_\gamma^0 = E_l - E'_l$, which gives the following expression for $E_\gamma$,

$$E_\gamma \approx E_\gamma^0 - \frac{K}{m_p},$$  \hspace{1cm} (10)

where $K = \frac{1}{2} [ |\vec{p}|^2 + |\vec{p}'|^2 - 2 |\vec{p}| |\vec{p}'| \cos \theta + (E_\gamma^0)^2 - 2 E_\gamma^0 \cos \alpha \sqrt{|\vec{p}|^2 + |\vec{p}'|^2 - 2 |\vec{p}| |\vec{p}'| \cos \theta}].$ When integrating over $E_\gamma$, the expression in the delta-function also introduces a factor $\left(1 - \frac{Z}{m_p}\right)$ in the phase-space to lowest order in $m_p^{-1}$, where $Z = E_\gamma^0 - \cos \alpha \sqrt{|\vec{p}|^2 + |\vec{p}'|^2 - 2 |\vec{p}| |\vec{p}'| \cos \theta}. \hspace{1cm} (11)$

Furthermore, the $m_p^{-1}$ correction in the photon energy affects the expression for the four-momentum transfer $q^2$, which can be written as $q^2 \approx (q^0)^2 + \frac{\kappa}{m_p}$, where $(q^0)^2 = 2 \left[ m_p^2 - E_l E'_l + |\vec{p}'| |\vec{p}'| \cos \theta - E_\gamma^0 (E_l - E'_l) + E_\gamma^0 \cos \alpha \sqrt{|\vec{p}|^2 + |\vec{p}'|^2 - 2 |\vec{p}| |\vec{p}'| \cos \theta} \right]$, and $\kappa = 2K \left[ E_l - E'_l - \cos \alpha \sqrt{|\vec{p}|^2 + |\vec{p}'|^2 - 2 |\vec{p}| |\vec{p}'| \cos \theta} \right]$. In our final expression we include the $m_p^{-1}$ kinematic terms, i.e., we include the $m_p^{-1}$ corrections for $E_\gamma$, $q^2$, as well as the above given phase-space factor in the LO expression for the cross section, Eq. (12) below.

As a pedagogical, qualitative survey of the bremsstrahlung process at energies not much larger than the muon mass, we initially consider the static proton limit ($m_p \to \infty$). Later we compare these preliminary results with the ones obtained by including the kinematic $m_p^{-1}$ corrections in the phase space factor including the delta-function expression for $E_\gamma$ in Eq. (13). In other words, we first
evaluate the LO cross section in the static proton limit \((m_p \to \infty)\), which is expressed as
\[
\frac{d^3 \sigma^{(LO)}}{dp^2 dp'^2 d\cos \alpha} = \frac{(\pi \beta')^2}{2 \pi^2 q^4} \left(1 - \frac{Z}{m_p}\right) E_\gamma \left\{ - \int_0^{2\pi} d\phi_\gamma \frac{1}{(a - \cos \phi_\gamma)^2} \left( E'_i E'_j E'_k (\beta' \sin \alpha \sin \gamma)^2 \right) \right. \\
\times \left[ m_i^4 - E_i E'_j (\cos \alpha \cos \gamma + \sin \alpha \sin \gamma \cos \phi_\gamma) + m_i^2 E_i E'_j \left( \cos \alpha \cos \gamma + \sin \alpha \sin \gamma \cos \phi_\gamma \right) \right] \left. \right|_{\text{dir}(\gamma)} \\
- \int_0^{2\pi} d\phi_\gamma \frac{1}{(b - \cos \phi_\gamma)^2} \left( E'_i E'_j E'_k (\beta \sin \alpha \sin \gamma)^2 \right) \right. \\
\times \left[ m_i^4 - E_i E'_j E'_k - m_i^2 E'_j E'_k + m_i^2 E_i E'_j - m_i^2 E_i E'_k \right] \\
+ m_i^2 p'_p \cos \theta - m_i^2 p'_p E_\gamma \left( \cos \alpha \cos \gamma + \sin \alpha \sin \gamma \cos \phi_\gamma \right) \\
- p'_E E'_i \left( \cos \alpha \cos \gamma + \sin \alpha \sin \gamma \cos \phi_\gamma \right) + m_i^2 p E_\gamma \left( \cos \alpha \cos \gamma + \sin \alpha \sin \gamma \cos \phi_\gamma \right) \\
+ m_i^2 p E_\gamma \left( \cos \alpha \cos \gamma + \sin \alpha \sin \gamma \cos \phi_\gamma \right) + p'^2 E_\gamma \left( \cos \alpha \cos \gamma + \sin \alpha \sin \gamma \cos \phi_\gamma \right) \\
+ \int_0^{2\pi} d\phi_\gamma \frac{1}{(a - \cos \phi_\gamma)(b - \cos \phi_\gamma)} \left( \frac{2}{E'_i E'_j E'_k (\beta \sin \alpha \sin \gamma)^2} \right) \\
\times \left[ m_i^2 E_i E'_j + E'_i E'_j - m_i^2 E'_j - m_i^2 p' p \cos \theta - p'^2 \cos \theta \right] \\
+ p' E'_i \left( \cos \alpha \cos \gamma + \sin \alpha \sin \gamma \cos \phi_\gamma \right) + p'^2 E_\gamma \cos \theta \left( \cos \alpha \cos \gamma + \sin \alpha \sin \gamma \cos \phi_\gamma \right) \\
- p'^2 E_\gamma \cos \theta \left( \cos \alpha \cos \gamma + \sin \alpha \sin \gamma \cos \phi_\gamma \right) \right\} . \tag{12}
\]

The terms within the first and second square brackets, i.e., \(\ldots\left|_{\text{dir}(\gamma)}\right.\) and \(\ldots\left|_{\text{dir}(\zeta)}\right.\), represent the contributions from the “direct” terms [matrix element squared of diagram (B) and (A), respectively]
of real photon emissions from the outgoing and incoming leptons, respectively. The third square bracket \([\cdots]_{\text{int}}\) represents the “interference” contribution of diagrams (A) and (B).

3 Results and Discussion

The MUSE collaboration [6] proposes the scattering of lepton off proton at the following three beam momenta, \(p = 115, 153\) and \(210\) MeV/c. As discussed, MUSE is designed to count the number of scattered leptons at a given scattering angle \(\theta\) for any value of the scattered lepton four-momentum \(p'\) larger than a certain minimum value. We shall discuss the dependence of the cross section on the photon angle \(\alpha\) and the lepton momentum \(|\vec{p}'|\). First, however, we analyse the \(q^2\) dependence of the bremsstrahlung process.

In order to extract a precise value for the proton r.m.s. radius, one needs to know accurately the \(Q^2\) dependence of the proton form factor. To LO in \(\chi\)PT, however, the momentum transfer to the proton is \(\vec{q} = \vec{Q} - \vec{k}\). For a given scattering angle \(\theta\), Eq.(9) shows that \(q^2\) is a function of \(|\vec{p}'|\), the lepton scattering angle \(\theta\), and the photon angle \(\alpha\). Although the bremsstrahlung process for muon scattering at a given angle \(\theta\), constitutes a small correction to the cross section, the process introduces a non-negligible \(q^2\) value uncertainty. Thus, we find it important to examine the \(q^2\) dependence on \(|\vec{p}'|\) and \(\alpha\) in order to guessimate the uncertainty given by the bremsstrahlung process. We find that for given angles, \(\theta\) and \(\alpha\), and \(|\vec{p}'| < m_l \lesssim |\vec{p}'|\), the square momentum transfer, \(-q^2\), becomes linear in \(|\vec{p}'|\) with a negative slope for forward scattering angles \(\theta < \pi/2\). Furthermore, for given angles, \(\theta\) and \(\alpha\), \(-q^2\) as a function of \(|\vec{p}'|\) shows an overall quadratic behavior with a local minimum for a small \(|\vec{p}'|\) value of about 50 MeV for \(|\vec{p}'| = 210\) MeV/c. These small \(q^2\) values for \(|\vec{p}'|\) below 100 MeV/c will produce significant effects on the differential cross section \(d\sigma/dp'd\Omega\), as discussed later. Note that the MUSE collaboration is expected to detect a range of \(|\vec{p}'|\) momenta down to \(|\vec{p}'|\) of the order 50 - 20 MeV/c.

In Fig. 3, we show the differential cross section, Eq.(11) versus the cosine of the outgoing photon angle \(\alpha\), for the three MUSE incoming momenta, \(p = |\vec{p}'| = 210, 153, 115\) MeV/c. For the bremsstrahlung process the outgoing lepton momentum can be chosen arbitrarily in the range \(0 \leq p' < p\). We only display the results for \(p' = 30, 100\) and \(200\) MeV/c for three forward angles: \(\theta = 15^\circ, 30^\circ\) and \(60^\circ\). The differential cross section shows that the commonly used peaking approximation [10, 11] is very well satisfied for the electron even at these low electron momenta \(p\) and \(p'\). The double-peak structure is a distinctive feature of the radiative angular spectrum for ultra-relativistic particles. The peaks occur for photon angle \(\alpha\) close to the angle \(\zeta\) (the \(\zeta\) peak) for the incoming electron momentum, and the angle \(\gamma\) (the \(\gamma\) peak) for outgoing electron momentum, as defined in Fig.2. It may be noted that for \(\theta = 15^\circ\) for both the lower plots (as well as for \(\theta = 30^\circ\) and \(60^\circ\) in the lower right plot) three peaks are generated with the \(\zeta\) peak being the dominant one. In each case the rightmost peak-like structure very close to \(\cos \alpha = 1\), as shown in the insert in the lower right graph in Fig.2, can be attributed to the behavior of the very small \(q^2\) values at the
Figure 3: The bremsstrahlung differential cross section versus $\cos \alpha$ for electron scattering for three incident momenta $p = |\vec{p}|$ are displayed. For each $p$ just one value for the outgoing electron momentum $p' = |\vec{p}'|$ is plotted. In the two l.h.s. and the bottom right plots, the solid (red) curves correspond to $\theta = 15^\circ$, the dotted (blue) curves to $\theta = 30^\circ$, and the dashed (orange) curves to $\theta = 60^\circ$. The insert in the lower right graph shows the dominant $\zeta$ peak and the additional third peak very close to $\cos \alpha = 1$. These cross sections are evaluated in the static proton limit ($m_p \to \infty$). The top right graph shows our leading order (LO) evaluations, without (i.e., static, Eq. (11)), and with the proton recoil terms (including $O(m_p^{-1})$) in the phase space, (i.e., recoil Eq. (12)). The third (dashed) curve shows the corresponding result using the corrected expression for Eq. (B.5) in Ref. [11] (see text).
\(\alpha\) angle close to zero (confer Fig. 11 in Ref. [11]). As expected from a classical bremsstrahlung angular spectrum, see e.g., Ref. [11], for the relativistic electrons the emitted photons get collimated close to the direction of the electron momentum. Some observations are in order:

- The separation between the peaks increase with increasing scattering angle \(\theta = \gamma - \zeta\),

- For a given scattering angle \(\theta\) and three momentum transfer \(|\vec{Q}| = |\vec{p} - \vec{p}'|\) the differential cross section decreases with increasing incident momentum \(|\vec{p}|\).

- The differential cross section decreases with increasing scattering angle \(\theta\), for fixed \(p\) and \(p'\).

In contrast, the \(\cos \alpha\) dependence of the bremsstrahlung cross section for incoming muons is very different, as is shown in Fig. 4. The initial muon momenta are not much larger than the muon mass, and the bremsstrahlung differential cross section versus \(\cos \alpha\) has a broad angular spectrum. It is obvious that the peaking approximation does not apply for muon proton scattering at MUSE energies. We note that, as expected, the muon bremsstrahlung differential cross section for static protons (\(m_p \rightarrow \infty\)) is reduced by roughly two orders of magnitude compared to the corresponding electron cross sections for the same kinematic specification.

Thus, a comparison of the plots in Figs. 3 and 4 demonstrates that the so-called peaking approximations [11], a widely used practical recipe for data analysis incorporating radiative corrections, while viable for the electron scattering at the low-momentum MUSE kinematics, can not be applicable for muon scattering at MUSE energies. Furthermore, our calculation shows that the "interference" term contribution to the total differential bremsstrahlung cross section, Eq. (12), is large for both the electron and muon scattering provided the value for \(p'\) is not too small for the electron bremsstrahlung process. In particular, the broad angular interference contribution to the cross section, labelled \([\ldots]_{\text{int}}\) in Eq. (12), is dominant for muon scattering.

Zooming in on each peak in the electron angular \(\cos \alpha\) dependence reveals the existence of a (3D) cone-like sub-structure, as displayed in Fig. 5. It may be recalled that for a charged relativistic particle with an acceleration parallel to its velocity \(\vec{\beta}\), the angular intensity distribution of the classical radiation corresponds to a cone with maximal opening angle \(\sim O(\sqrt{1 - \beta^2})\) with respect to the direction of motion \(\vec{\beta}\). The dashed vertical lines in Fig. 5 correspond in our reference system (Fig. 2) to the expected directions of the incoming and outgoing leptons. The effect of bremsstrahlung radiation results in the lepton recoiling away in a slightly different direction, leading to the characteristic cone-like feature for each peak with the vertex along the expected axis of the radiation cone. Unlike the electron bremsstrahlung cross section versus \(\cos \alpha\) dependence, in the muon case we observe a significant interference effects between the two broad angular peaks, labelled \([\ldots]_{\text{dir}(\gamma)}\) and \([\ldots]_{\text{dir}(\zeta)}\), and the interference contribution, labelled \([\ldots]_{\text{int}}\) in Eq. (12), which cause deviations of each minimum from the expected directions (vertically (red) dashed lines). Our graphic demonstrations support part of the analysis presented in Ref. [13], e.g., Fig.4 in Ref. [13], where radiative corrections to \((e,e'p)\) coincident experiments were discussed. It is, however, notable that if we reduce the value of the muon mass from its physical value, there is a steady reduction of this observed mismatch between the vertical red dashed lines and the cone minimums.
Figure 4: The bremsstrahlung differential cross sections versus \( \cos \alpha \) for muon scattering for three incident momenta \( p = |\vec{p}| \) are displayed. For each \( p \) one just value for the outgoing muon momentum \( p' = |\vec{p}'| \) is plotted. In the two l.h.s. and the bottom right plots, the solid (red) curves correspond to \( \theta = 15^\circ \), the dotted (blue) curves to \( \theta = 30^\circ \), and the dashed (orange) curves to \( \theta = 60^\circ \). These cross sections are evaluated in the static proton limit \( (m_p \rightarrow \infty) \). The top right graph shows our leading order (LO) evaluations, without (i.e, static, Eq. (11)), and with the proton recoil terms (including \( \mathcal{O}(m_p^{-1}) \)) in the phase space, (i.e., recoil Eq. (12)). The third (dashed) curve shows the corresponding result using the corrected expression for Eq. (B.5) in Ref. [11] (see text).
Figure 5: Plots indicating photon emissions from the leptons distributed within a shallow cone about their incident or scattered directions in the static proton approximation. The left plot is for electrons whereas the right plot is for muons. Both plots correspond to $p = 210$ MeV/c and $p' = 200$ MeV/c, and for the lepton scattering angle $\theta = 120^\circ$. The red dashed lines indicate the directions of the incident and scattered leptons, i.e., $\cos \zeta$ and $\cos \gamma$, respectively.

The “radiative tails” cross sections for the electron and muon scattering for the MUSE specified values of the incoming lepton momenta, $p = |\vec{p}| = 210, 153$ and $115$ MeV/c are displayed in Figs. 6 and 7, respectively. We only display the plots for forward scattering angles, $\theta = 15^\circ$, $30^\circ$, and $60^\circ$. The bremsstrahlung cross section versus $p'$ is plotted from $0.1$ MeV/c up to $p'_{\text{max}} = p'_{\text{elastic}} - \Delta p'$, where we have chosen $\Delta p' = 1$ MeV/c in order to avoid the IR singularity. As $p'$ tends to zero the differential cross section goes to zero for the both muon and electron cases, although the logarithmic scale does not reflect this aspect. In particular, for the electron tail spectrum, the cross section reaches a local maximum before going to zero, as evident from Fig. 6. For a qualitative understanding of the cause of the local maximum in the cross section we consider the results presented in Fig. 7 and then artificially lower the value of the muon mass from it’s physical value, $m_\mu = 105.7$ MeV, which is normally used in our numerical evaluations of the cross section. We find that for a muon mass of about $30$ MeV, the cross section develops a “shoulder”, which for an even smaller muon mass value develops into a local maximum and starts to resemble the cross section for electron scattering in Fig. 6. Thus, at sufficiently small values of the muon mass, the muon cross section has a distinct local maximum in the cross section for small $p'$ values, as discussed earlier.

In comparing our results with the expression Eq. (B.5) in Ref. [11], we find that Eq. (B.5) has to be corrected as follows. The very first energy factor, $(E_p/E_s)$, multiplying the integral in the expression of Eq. (B.5) should have been $(\vec{p}^2/(E_pE_s))$. This factor reduces to the the energy factor in Eq. (B.5) provided we neglect the lepton mass. Incorporating this correction and by ignoring the proton form factors (anomalous magnetic moment also excluded) in Eq. (B.5) of Ref. [11], we find only a nominal difference with our cross sections for the static proton approximation, given by

\[
\frac{d^3\sigma}{d^3p' d\Omega'} = \frac{\alpha}{2\pi} \frac{210 \text{ MeV}}{c^2} \frac{200 \text{ MeV}}{c^2} \frac{120^\circ}{\pi} \left(10^{-5} \text{mb/GeV})\right).
\]
\[ \theta = 15^\circ \]
\[ \theta = 30^\circ \]
\[ \theta = 60^\circ \]

\[ p = 210 \text{ MeV}/c \]
\[ p = 153 \text{ MeV}/c \]
\[ p = 115 \text{ MeV}/c \]

**Figure 6:** The “radiative tail spectrum” cross section is plotted as a function of the scattered electron momentum \(|\vec{p}'|\) for incoming electron momenta specified by the MUSE. In the top left graph where \(|\vec{p}'| = 210 \text{ MeV}/c\), we plot the cross section in the static proton limit (\(m_p \to \infty\)) for \(\theta = 15^\circ\) solid (red) curve, 30° dotted (blue) curve, and 60° dashed (orange) curve. The top right graph shows the cross section at an electron scattering angle \(\theta = 30^\circ\) for the three incoming MUSE specified momenta, \(p = 210\) (solid), 153 (dotted) and 115 (dashed) MeV/c in the limit \(m_p \to \infty\). The bottom graph compares our static and recoil LO evaluations, Eqs. (11) and (12), respectively, with the results obtained using the corrected expression for Eq. (B.5) in Ref. [11] (see text).
Figure 7: The “radiative tail spectrum”: muon cross section is plotted as a function of the scattered muon momentum $|\vec{p}'|$. See text and the caption in Fig. 6 for details.
Eq. (11). However, once we include the $m_p^{-1}$ corrections in the phase space and the δ-function, our Eq. (12), the differences with Ref. [11] indeed becomes very small, as displayed in Fig. 6. Note that Ref. [11] treated both the leptons and the nucleon in a relativistic framework, whereas in our LO χPT the nucleon is treated either as being static or non-relativistic in the phase space expression.

In Figs. 6 and 7 the bremsstrahlung cross sections diverges as the maximal value $p'_{\text{max}}$ approaches the elastic lepton-proton scattering value $p^\prime_{\text{elastic}}$. As mentioned this is due to the infrared divergence (IR) in the bremsstrahlung process where the bremsstrahlung photon energy goes to zero. As is demonstrated in, e.g., Refs. [9,11], the cross section is free from the IR divergences, provided the radiative corrections are included in the calculation. In χPT one can systematically evaluate the virtual photon loop corrections as well as the bremsstrahlung processes from the leptons and protons. The IR divergences, which appear in bremsstrahlung and in the photon loop evaluations, will cancel order by order in the chiral expansion. For the elastic lepton-proton scattering the virtual photon loops along with the so-called two-photon exchange (TPE) contributions will introduce ultraviolet (UV) divergences. These divergences can be treated systematically in χPT using dimensional regularization. Such an evaluation has not been done in the context of low-energy lepton-proton scattering to date, and would require the introduction of low-energy constants (LECs) in the χPT Lagrangian. Fortunately, at the order of our interest in this work, these LECs are known and can be taken from earlier χPT works, see e.g., Ref. [17]. Such a systematic evaluation of the radiative corrections, which is beyond the scope of the present discussion, shall be pursued in a future work.

The purpose of this work is to present a short qualitative but yet pedagogical evaluation and discussion of a scenario where a large change in the angular spectrum of the lepton-proton bremsstrahlung process can be expected at typical momenta not much larger than the muon mass. The importance of such a study is very relevant to MUSE experimental program where high precision muon-proton scatterings at very low-momentum transfers will be pursued to ultimately probe the unexpected large discrepancy of the proton charge radius previously observed from muonic hydrogen measurements. Our analysis demonstrates that a non-standard treatment of the bremsstrahlung corrections for muon scattering must be carefully thought through by the MUSE collaboration. As shown in Fig. 6 the radiative tail cross section for $p' < 50$ MeV has a local maximum mainly due to the non-zero electron mass. Furthermore, we correct for a misprint in the overall energy factor in the review article [11]. Since MUSE only detects the lepton scattering angle $\theta$ and does not determine the value of the outgoing lepton momentum $|\vec{p}'|$, care must be taken in the analysis of the MUSE data when one corrects for the radiative corrections. In our LO χPT evaluation the bremsstrahlung from the proton does not contribute. The bremsstrahlung process from the proton (diagrams (C) and (D) in Fig. 1) will only appear at NLO in χPT, where the sub-leading interactions present in the chiral Lagrangian at chiral orders $\nu = 1,2$ will contribute to the matrix element. The proton r.m.s. radius also enters the evaluation via a LEC, meaning the low-momentum aspects of the proton form factors naturally enters the χPT calculation at

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4The nucleon anomalous magnetic moment, which enters at NLO in χPT, are excluded in our LO evaluations.
sub-leading orders. We intend to evaluate the NLO bremsstrahlung cross section as well as the radiative corrections in a future work.

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References

[1] R. Pohl, et al., The size of the proton, Nature 466 (2010) 213.
[2] R. Pohl, R. Gilman, G. A. Miller, G. A. Nille, K. Pachucki, Annu. Rev. Nucl. Part. Science, 63, 175 (2013).
[3] J. Bernauer et al. (A1 Collaboration), Phys. Rev. Lett. 105, 242001 (2010). arXiv:1007.5076
[4] X. Zhan et al., Phys. Lett. B 705, 59 (2011). arXiv:1102.0318
[5] M. Mihovilović et al., Phys. Lett. B 771, 194 (2017).
[6] R. Gilman, et al., Studying the Proton “Radius” Puzzle with μp Elastic Scattering, “White Paper” on the Paul Scherer Institute MUSE experiment: R-12-01.1, arXiv:1303.2160 (2013).
[7] Lin Li, Study of Systematic Uncertainties in the Muonic Scattering Experiment, Ph.D. Thesis Proposal, University of South Carolina.
[8] L. Maximon, Comments on Radiative corrections, Rev. Mod. Phys. 41 193 (1969).
[9] L. Maximon and J. Tjon, Radiative corrections to electron-proton scattering, Phys. Rev. C 62, 054320 (2000).
[10] Y.-S. Tsai, Radiative corrections to Electron-Proton Scattering, Phys. Rev. 122, 1898 (1961).
[11] L. W. Mo and Y.-S. Tsai, Radiative corrections to Elastic and Inelastic ep and μp Scattering, Rev. Mod. Phys. 41 205 (1969).
[12] V. Bernard, N. Kaiser and U.-G. Meißner, Chiral Dynamics In Nucleons and Nuclei, Int. J. Mod. Phys. Rev. E 4, (1995) 193.
[13] R. Ent, et al., Radiative corrections for (e, e′p) reactions at GeV energies, Phys. Rev. C 64, 054610 (2001).
[14] J. D. Bjorken and S.D. Drell, “Relativistic Quantum Fields”, McGraw-Hill Book Comp., N.Y., 1965.
[15] N. Fettes, Ph.D Dissertation, Univ. Bonn (2000), unpublished.
[16] A. V. Manohar and M. Luke, Phys. Lett. B 286, 348 (1992).

[17] S. Ando et al., Phys. Lett. B 595, 250 (2004).