Conformal-Frame (In)dependence of Cosmological Observations in Scalar-Tensor Theory

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We provide the correspondence between the variables in the Jordan frame and those in the Einstein frame in scalar-tensor gravity and consider the frame-(in)dependence of the cosmological observables. In particular, we show that the cosmological observables/relations (redshift, luminosity distance, temperature anisotropies) are frame-independent. We also study the frame-dependence of curvature perturbations and find that the curvature perturbations are conformal invariant if the perturbation is adiabatic and the entropy perturbation between matter and the Brans-Dicke scalar is vanishing. The relation among various definitions of curvature perturbations in the both frames is also discussed, and the condition for the equivalence is clarified.

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I. INTRODUCTION

The scalar-tensor gravity \cite{1} is usually formulated in the so-called Jordan frame where the metric tensor is minimally coupled to the matter sector. On the other hand, when we perform the conformal transformation of the metric, the action can be reduced to that of the Einstein gravity, but this time the metric is non-minimally coupled to the matter sector \cite{2}. So, the question is, "which frame describes the physics?", about which there has been a long debate.

This problem is attacked recently by Deruelle and Sasaki \cite{3} (see also \cite{4,5,6}) by comparing the observables in the Einstein gravity and those in the theory "conformally transformed" from the Einstein gravity. They found that the Hubble’s law holds in both frames and therefore the observational predictions are exactly the same in both frames although the physical interpretation (redshift and the evolution of the universe, for example) in each frame is different.

However, the conformal factor studied there is an arbitrary function of space-time and is not a dynamical field since there is no scalar gravitational degrees of freedom in general relativity. So, it is unclear to what extent their conclusion holds for the scalar-tensor gravity where the conformal factor corresponds to the scalar gravitational degrees of freedom of the theory and is truly a dynamical field. This is the problem we are going to solve. Our motivation is the same in spirit with \cite{2,3}: Since the conformal transformation is merely a change of variables or units, physical laws and any observables should not depend on the particular set of variables and should be conformal-frame independent.

In the following sections, we calculate in both frames the observables in classical gravitational theory which include the effective gravitational constant (Sec. II), redshift (Sec. III A.), the luminosity distance and the Hubble’s law (Sec. III B.), the Sachs-Wolfe effect (Sec. IV A.) and the adiabatic/isocurvature perturbation (Sec. IV B.) and confirm the frame independence or dependence. In short, we provide a Jordan-Einstein dictionary. In Sec. IV B. we also clarify the relation among the various definitions of the curvature perturbations in both frames and give the condition of the equivalence. Some of our results in Sec. III and IV overlap with the results in \cite{4,5,6}. We extend them to include the effective gravitational constant, the Hubble’s law and the relation among the various definitions of the curvature perturbations. Several applications of the conformal transformation are given in Sec. II C. and Sec. II D. We use the units of $c = \hbar = 1$.

II. JORDAN VS. EINSTEIN: THE CHANGE OF UNIT

The action of the scalar-tensor theories of gravity in the so-called Jordan frame is given by

\begin{equation}
S(\bar{g}_{\mu\nu}, \Phi, \psi) = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\bar{g}} \left( \Phi \bar{R} - \frac{\omega(\Phi)}{\Phi} (\bar{\nabla}\Phi)^2 - U(\Phi) \right) + S_m(\bar{g}_{\mu\nu}, \psi),
\end{equation}

where $\kappa^2 = 8\pi G_*$ is the bare gravitational constant, $\bar{g}_{\mu\nu}$, $\Phi$ are the Jordan-frame metric, the Brans-Dicke scalar field, respectively. $S_m$ is the matter action and $\psi$ denotes the matter field. The equations of motion derived from the action
Eq. (1) are given by

\[ \Phi \left( \tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} \right) = \kappa^2 \tilde{T}_{\mu\nu} + \frac{\omega(\Phi)}{\Phi} \left( \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} \tilde{g}_{\mu\nu} (\nabla \Phi)^2 \right) + \nabla_\mu \nabla_\nu \Phi - \tilde{g}_{\mu\nu} \Box \Phi - \frac{1}{2} \tilde{g}_{\mu\nu} U, \tag{2} \]

\[ \Box \Phi = \frac{1}{2 \omega(\Phi) + 3} \left( \kappa^2 \tilde{T} - \frac{d\omega(\Phi)}{d\Phi} (\nabla \Phi)^2 + \Phi \frac{dU}{d\Phi} - 2U \right), \tag{3} \]

where the energy-momentum tensor \( \tilde{T}_{\mu\nu} \) is defined by \( \sqrt{-\tilde{g}} \tilde{T}_{\mu\nu} = -2 \delta S_m / \delta \tilde{g}^{\mu\nu} \) and \( \tilde{T} = \tilde{T}_{\mu}^{\mu} \) is the trace, and the matter energy-momentum conservation is

\[ \nabla_\mu \tilde{T}_{\mu\nu} = 0. \tag{4} \]

There is another distinct frame so-called Einstein frame where the gravity action becomes that of the Einstein gravity with a minimally coupled scalar field. The Einstein frame metric \( g_{\mu\nu} \) is related to the Jordan frame metric by the conformal transformation

\[ \tilde{g}_{\mu\nu} = \frac{1}{\Phi} g_{\mu\nu} \equiv e^{2a} g_{\mu\nu}, \tag{5} \]

and the action can be rewritten as

\[ S(g_{\mu\nu}, \varphi, \psi) = \int d^4x \sqrt{-g} \left( \frac{R}{2\kappa^2} - \frac{1}{2} (\nabla \varphi)^2 - V(\varphi) \right) + S_m (e^{2a(\varphi)} g_{\mu\nu}, \psi), \tag{6} \]

where \( \varphi \) and \( V(\varphi) \) are defined by

\[ \frac{1}{4\Phi^2} \left( \frac{d\Phi}{d\varphi} \right)^2 = \left( \frac{da}{d\varphi} \right)^2 = \frac{\kappa^2}{2(2\omega(\Phi) + 3)}, \tag{7} \]

\[ V(\varphi) = \frac{U(\Phi)}{2\kappa^2 \Phi^2}. \tag{8} \]

The equations of motion are given by

\[ R_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} = \kappa^2 \left[ T_{\mu\nu} + \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} \tilde{g}_{\mu\nu} \left( (\nabla \varphi)^2 + 2V(\varphi) \right) \right], \tag{9} \]

\[ \Box \varphi - \frac{dV}{d\varphi} = -\frac{da}{d\varphi} T, \tag{10} \]

where the energy-momentum tensor \( T_{\mu\nu} \) is defined by \( \sqrt{-g} T_{\mu\nu} = -2 \delta S_m / \delta g^{\mu\nu} \) and \( T = T_{\mu}^{\mu} \) is the trace.

The energy-momentum tensor in the Einstein frame \( T_{\mu\nu} \) is related to that in the Jordan frame \( \tilde{T}_{\mu\nu} \) as

\[ \tilde{T}_{\mu\nu} = e^{-2a} T_{\mu\nu}, \tag{11} \]

and the matter energy-momentum conservation in the Einstein frame is given by

\[ \nabla_\mu T_{\mu\nu} = \frac{da}{d\varphi} (\partial_\nu \varphi) T. \tag{12} \]

We can directly observe that Eqs. (2), (3), and (4) are equivalent to Eqs. (9), (10), and (12) if we change variables \( \tilde{g}_{\mu\nu} = g_{\mu\nu} / \Phi = e^{2a} g_{\mu\nu} \) given in Eq. (5) and use the relations Eqs. (7), (8), and (11).

In the following we consider the motion of particle and photon, and compare the observed redshift in both frames.

### A. Particle motion

The action of test particle of rest mass \( \tilde{m} \) in the Jordan frame is given by

\[ S_p = \int \tilde{m} d\tilde{s}, \tag{13} \]
and the motion of the particle (with the 4-velocity $\tilde{u}^\mu$) in the Jordan frame is given by the familiar geodesic equation:

$$\tilde{u}^\nu \tilde{\nabla}_\nu \tilde{u}^\mu = 0.$$  \hfill (14)

Since

$$ds^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = e^{2a(\varphi)} ds^2,$$  \hfill (15)

the action in the Einstein frame is written as

$$S_p = \int \tilde{m} e^{a(\varphi)} ds \equiv \int m(\varphi) ds,$$  \hfill (16)

and hence the motion of the particle (with the 4-velocity $u^\mu = e^a(\varphi) \tilde{u}^\mu$) in the Einstein frame is given by

$$u^\nu \nabla_\nu u^\mu = -(u^\mu u^\nu + g^\mu\nu) \partial_\nu a(\varphi).$$  \hfill (17)

Eq. (15) determines the relation of the proper distance/time between both frames. The trajectory in the Einstein frame is not a "straight line". However, it is incorrect to state that "the (weak) equivalence principle is violated in the Einstein frame [8]", as is clear from studying the Newtonian limit given below. The weak equivalence principle is violated when the $\varphi$-dependence cannot be eliminated by the conformal transformation.

Note that the spatial velocity of matter (the velocity measured in the locally orthonormal frame) $\tilde{v}_i \equiv \tilde{u}_i / \tilde{u}_0$ is independent of the conformal-frame, $\tilde{v}_i = v_i$.

B. Photon

As is well-known, the classical electromagnetic action is conformally invariant:

$$S_{EM} = -\frac{1}{4} \int d^4 x \sqrt{-\tilde{g}} \tilde{g}^{\mu\rho} \tilde{g}^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} = -\frac{1}{4} \int d^4 x \sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma},$$  \hfill (18)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, so the photon (with the tangent vector $\tilde{k}^\mu(k^\mu)$ in the Jordan (Einstein) frame) follows the null geodesics in both frames:

$$\tilde{k}^\nu \tilde{\nabla}_\nu \tilde{k}^\mu = 0 = k^\nu \nabla_\nu k^\mu.$$  \hfill (19)

C. Newtonian limit and the Gravitational constant

Let us consider the Newtonian limit (weak gravity and slow motion and the weak material stresses) to determine the effective gravitational constant. We set $U(\Phi) = 0$ for simplicity. In the Jordan frame, the geodesic equation Eq. (14) yields the Newtonian equation of motion if

$$\tilde{g}_{00} \simeq -(1 + 2\tilde{\phi}), \quad \tilde{g}_{ij} \simeq \delta_{ij},$$  \hfill (20)

where $\tilde{\phi}$ is the Newtonian gravitational potential produced by the rest mass density $\tilde{\rho}$ as

$$\nabla^2 \tilde{\phi} = 4\pi \tilde{G} \tilde{\rho},$$  \hfill (21)

where $\tilde{G}$ is the effective gravitational constant determined later.

Adopting the post-Newtonian bookkeeping rule \ref{postnew}, we assign the Newtonian gravitational potential as O(2) quantity. Similarly, $v^i \sim \tilde{\rho} / \tilde{\rho} \sim O(2)$ and $|\partial / \partial t| / |\partial / \partial x| \sim O(1)$. The Newtonian limit (up to O(2)) of Eq. (3) becomes

$$\nabla^2 \Phi = -\frac{\kappa^2 \tilde{\rho}}{2\omega(\Phi) + 3},$$  \hfill (22)

and the solution up to O(2) can be written as

$$\Phi = \Phi_0 + \frac{\kappa^2}{4\pi(2\omega_0 + 3)} \int d^3 x' \tilde{\rho}(x') |x - x'|,$$  \hfill (23)
where $\Phi_0$ denotes the value of $\Phi$ at spatial infinity and we have already expanded $\omega(\Phi)$ and replaced $\omega(\Phi)$ with its asymptotic value $\omega_0 \equiv \omega(\Phi_0)$. Then, from Eq. (2) using Eq. (22) we obtain

$$\Phi_0 \nabla^2 \tilde{\phi} = \frac{\kappa^2}{2} \frac{2\omega_0 + 4}{2\omega_0 + 3} \rho,$$

(24)

Comparing this with Eq. (21), we can determine the effective gravitational constant \cite{10} \footnote{Our $\varphi$ is related to $\varphi_{\text{DEF}}$ used in \cite{10} via $\varphi = \sqrt{2 \varphi_{\text{DEF}} / \kappa}$.}

$$\tilde{G} = \frac{\kappa^2}{8\pi} \frac{2\omega_0 + 4}{2\omega_0 + 3} \Phi_0 = G_* \left[ e^{2a(\varphi)} \left( 1 + \frac{2}{\kappa^2} \left( \frac{da}{d\varphi} \right)^2 \right) \right]_{\varphi_0},$$

(25)

where $\varphi_0$ is the asymptotic value of $\varphi$ corresponding to $\Phi_0$ and we have used Eq. (4) and $\kappa^2 = 8\pi G_*$. This $\tilde{G}$ is the quantity which is measured by the Cavendish-type experiment. Since $\tilde{G}$ involves $\varphi$, the mass of a self-gravitating body depends on $\varphi$, and hence the strong equivalence principle is violated in the scalar-tensor theory.

Next, we consider the Newtonian limit in the Einstein frame. For the metric in the Newtonian limit, $g_{00} \simeq -(1 + 2\phi), g_{ij} \simeq \delta_{ij}$, the Newtonian limit of Eq. (9) and Eq. (10) become

$$\nabla^2 \phi = \frac{\kappa^2}{2} \rho,$$

(26)

$$\nabla^2 \varphi = \frac{da}{d\varphi} \rho.$$

(27)

Similar to $\Phi$, the solution of $\varphi$ up to $O(2)$ is given by

$$\varphi = \varphi_0 - \frac{1}{4\pi} \frac{da}{d\varphi} \left| \int d^3x' \frac{\rho(x')}{|x - x'|} \right|.$$

(28)

Then the particle equation of motion Eq. (17) in the Newtonian limit becomes

$$\frac{d^2x}{dt^2} = -\nabla \phi - \frac{da}{d\varphi} \nabla \varphi$$

$$= -\frac{\kappa^2}{8\pi} \left( 1 + \frac{2}{\kappa^2} \left( \frac{da}{d\varphi} \right)^2 \right) \int d^3x' \frac{(x - x') \rho(x')}{|x - x'|^3}. $$

(29)

The coefficient in front of the integral is to be regarded as the effective gravitational constant $G$ in the Einstein frame:

$$G = G_* \left[ 1 + \frac{2}{\kappa^2} \left( \frac{da}{d\varphi} \right)^2 \right]_{\varphi_0}. $$

(30)

$\tilde{G}$ and $G$ are related as $G = e^{-2a(\varphi_0)} \tilde{G}$ in accord with the fact that the gravitational constant has the dimension of $(\text{mass})^{-2}$ and the unit of mass changes by changing the frame according to Eq. (16). However, the transformation may be regarded as ”non-local” in the sense that the conformal transformation refers to the value of $\varphi$ at spatial infinity.

### III. COSMOLOGY IN SCALAR-TENSOR THEORY

#### A. Redshift

Now we consider the observed redshift in each frame. We calculate the redshift in the FRW spacetime. The line element in the Jordan frame is given by

$$ds^2 = -dt^2 + \tilde{R}(t)^2 (d\chi^2 + \sin^2 K \chi d\Omega^2) = \tilde{R}(\eta)^2 (-d\eta^2 + d\chi^2 + \sin^2 K \chi d\Omega^2),$$

(31)

$$ds^2 = -dt^2 + \tilde{R}(t)^2 (d\chi^2 + \sin^2 K \chi d\Omega^2) = \tilde{R}(\eta)^2 (-d\eta^2 + d\chi^2 + \sin^2 K \chi d\Omega^2).$$

(31)
where $\tilde{t}$ is the cosmic time and $\tilde{R}$ is the scale factor and $\eta$ is the conformal time. $\sin_K \chi = \sin \chi (\sinh \chi)$ for a closed ($K = 1$) (an open ($K = -1$) ) universe and $\sin_K \chi = \chi$ for a flat universe ($K = 0$). Since $d\tilde{s}^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu = e^{2a(\varphi)} ds^2$, the corresponding variables (without tilde) in the Einstein frame are given by

$$d\tilde{t} = e^{a(t)} dt, \quad \tilde{R}(\tilde{t}) = e^{a(t)} R(t). \quad (32)$$

First, we calculate the observed redshift in the Jordan frame. For the null geodesic with the tangent vector $\tilde{k}^\mu$, the frequency $\tilde{\nu}$ measured by the observer with 4-velocity $\tilde{u}^\mu$ is given by $\tilde{\nu} = -\tilde{k}^\mu \tilde{u}_\mu$. Then the observed redshift is given by

$$1 + \tilde{z} = \frac{\tilde{\nu}_S}{\tilde{\nu}_O} \quad (33)$$

where the subscript $S(O)$ denotes the quantity measured at the source (observer’s) position. The derivative of $\tilde{\nu}$ with respect to the affine parameter $\tilde{\lambda}$ is given by

$$\frac{d\tilde{\nu}}{d\tilde{\lambda}} = -\tilde{k}^\mu \tilde{H} \nabla_{\mu} \tilde{u}_\nu, \quad (34)$$

Since $\tilde{u}^\mu$ obeys the geodesics in the FRW universe, $\nabla_{\mu} \tilde{u}_\nu = \tilde{H}(\tilde{g}_{\mu\nu} + \tilde{u}_\mu \tilde{u}_\nu)$, where $\tilde{H} = \tilde{R}^{-1}(d\tilde{R}/d\tilde{t})$ is the Hubble parameter in the Jordan frame. So we have

$$\frac{d\tilde{\nu}}{d\tilde{\lambda}} = -\tilde{\nu}^2 \tilde{H}, \quad (35)$$

or

$$\frac{d\tilde{\nu}}{dt} = -\tilde{\nu} \tilde{H}. \quad (36)$$

Hence, we obtain the usual redshift law: $1 + \tilde{z} \propto \tilde{R}^{-1}$. Therefore, the observed redshift is given by

$$1 + \tilde{z} = \frac{\tilde{\nu}_S}{\tilde{\nu}_O} = \frac{\tilde{R}_O}{\tilde{R}_S}. \quad (37)$$

Next, we calculate the redshift in the Einstein frame. Now that the observer’s 4-velocity $u^\mu$ does not follow geodesics, and $\nabla_{\mu} u_\nu$ is given by $\nabla_{\mu} u_\nu = \tilde{H}(g_{\mu\nu} + u_\mu u_\nu) - u_\mu (u^\rho \nabla_{\rho} u_\nu)$, where $\tilde{H} = \tilde{R}^{-1}(d\tilde{R}/d\tilde{t})$. 2 So the derivative of $\nu$ with respect to the affine parameter $\lambda$ becomes

$$\frac{d\nu}{d\lambda} = -k^\mu k^\nu \nabla_{\mu} u_\nu, \quad (38)$$

Using Eq. (17), we find that the last term vanishes. Hence again $\nu \propto R^{-1}$ and

$$\frac{\nu_S}{\nu_O} = \frac{R_O}{R_S}. \quad (39)$$

So, considering Eq. (32), one may naively conclude that the observed is conformal-frame dependent. However, it is incorrect, since the observation is (and should be) made in the observer’s reference frame where the unit is in general different from that of the source frame. For example, from Eq. (32), for the same transition frequency $\tilde{\nu}$, the frequency measured in the unit of one frame is $\nu_1 = e^a|_1 \tilde{\nu}$ and the frequency measured in the unit of another frame is $\nu_2 = e^a|_2 \tilde{\nu}$, and the relation between them is

$$\frac{\nu_1}{\nu_2} = e^{a|_1}, \quad e^{a|_2}. \quad (40)$$

2 Note that $H$ is not the measurable Hubble parameter in the Einstein frame. See the discussion below.
The observed redshift is, more precisely, defined by the ratio of the source frequency in unit of the observer’s frame to the observed frequency. It is given by, taking into account the relation Eq. (40),

\[ 1 + z = \frac{e^a|_O}{e^a|_S} \frac{\nu_S}{\nu_O} = \frac{e^a|_O}{e^a|_S} \frac{R_O}{R_S} = \frac{R_O}{R_S} = 1 + \bar{z}, \]  

where we have used Eq. (39) in the second equality. Therefore, the observed redshift is conformal-frame independent.

B. Distances

1. Luminosity distance and Hubble’s Law

The luminosity distance in the Jordan frame \( \bar{d}_L \) is defined in terms of the source’s (absolute) luminosity \( \bar{L}_S \) and the observed flux \( \bar{f}_O \) as \( \bar{d}_L = \sqrt{\frac{\bar{L}_S}{4\pi R_O}} \). Since \( \bar{L}_S \) is redshifted to become the luminosity in the observer’s frame \( L_O = \bar{L}_S (R_S/R_O)^2 \) and \( \bar{f}_O = L_O / 4\pi R_O^2 \sin^2 K \chi \), \( \bar{d}_L \) is written in the usual form

\[ \bar{d}_L = R_O (1 + \bar{z}) \sin K \chi, \]  

where \( \chi \) is the comoving distance to the source and is given by

\[ \chi = \int_{r_S}^{\bar{r}_O} \frac{dt}{\bar{R}} = \int_0^{\bar{z}} \frac{d\bar{z}}{R_O \bar{H}(\bar{z})}. \]  

Taking the limit \( \bar{z} \to 0 \), we obtain the Hubble’s law:

\[ \bar{d}_L = \frac{\bar{z}}{\bar{H}_0}, \]  

where \( \bar{H}_0 = \bar{H}(0) \) is the present Hubble parameter.

Now we calculate the luminosity distance \( d_L \) in the Einstein frame. Remembering the discussion about the redshift, one should consistently use the unit of the observer’s frame to define the luminosity distance. Since the luminosity has the unit of mass divided by time, the source’s luminosity \( L_S \) measured by the observer’s unit is \( L_S (e^a|_O/e^a|_S)^2 \) and from Eq. (32), \( L_S = L_O (R_O/R_S)^2 \). Thus

\[ d_L = \sqrt{\frac{L_S}{4\pi f_O} \left( \frac{e^a|_O}{e^a|_S} \right)^2} = R_O \frac{R_O e^a|_O}{R_S e^a|_S} \sin K \chi = R_O (1 + z) \sin K \chi, \]  

where \( \chi \) is the same as Eq. (43) since the null geodesics is conformally invariant. Note that since \( R = e^{-a} \bar{R} \), \( d_L \) differs from \( \bar{d}_L \): \( d_L = e^{-a}|_O \bar{d}_L \). But this is legitimate because the unit is different between the frames and this is precisely the correspondence of length (dimensionful quantity) in the Einstein frame to that in the Jordan frame.

Again taking \( z \to 0 \), we find

\[ d_L = \frac{z}{e^a|_O \bar{H}_0} = \frac{z}{\bar{H}_{E0}}. \]  

From Eq. (32), the Einstein-frame Hubble parameter \( H_{E0} \) is given by

\[ H_{E0} = e^a|_O \bar{H}_0 = \frac{1}{R} \left. \frac{d\bar{R}}{dt} \right|_0 = \bar{H}_0 + \frac{da}{dt}|_0. \]  

\( H_{E0} \) differs from \( \bar{H}_0 \) by \( e^a|_O \) in accord with the fact that the Hubble parameter has the dimension of the inverse of time.

Note that since \( e^a|_O \) is merely a constant, it is absorbed into the definition of the absolute magnitude, and hence the magnitude-redshift relation is conformal-frame independent.

The correspondence between the variables in the Jordan frame and those in the Einstein frame is given in Table I.
quantity | in Jordan frame | in Einstein frame
---|---|---
metric | $\tilde{g}_{\mu\nu}$ | $g_{\mu\nu} = e^{-2a}\tilde{g}_{\mu\nu}$
time/length | $\tilde{d}\tilde{s}$ | $d\tilde{s} = e^{-a}d\tilde{s}$
scale factor | $\tilde{R}$ | $R = e^{-a}\tilde{R}$
(particle) mass | $\tilde{m}$ | $m = \tilde{m}e^{-a}$
effective gravitational constant | $\tilde{G}$ | $G = e^{-2a(\phi_0)}\tilde{G}$
energy density | $\tilde{\rho}$ | $\rho = e^{4a}\tilde{\rho}$
redshift | $\tilde{z}$ | $z = \tilde{z}$
Hubble parameter | $\tilde{H}$ | $H_E = H + da/dt = e^a\tilde{H}$
curvature perturbation | $\tilde{\zeta}$ | $\zeta = \tilde{\zeta}$

(if $\delta \rho / (d\rho / dt) = \delta a / (da / dt)$)

| TABLE I: Jordan-Einstein dictionary. The scaling relation of dimensionful quantity can be found by the change of unit of length, time, mass given in the second to the fourth three columns. $\tilde{G}$ is given by Eq. (25). |

2. **Angular Diameter Distance and the Reciprocity Relation**

For an object with the proper diameter $\tilde{D}$ and the angular diameter $\theta$, the angular diameter distance $\tilde{d}_A$ in the Jordan frame is defined by

$$\tilde{d}_A = \frac{\tilde{D}}{\theta} = \tilde{R}_S \sin K \chi = \frac{1}{1 + \tilde{z}} \tilde{R}_O \sin K \chi,$$

(48)

where we have used Eq. (37). The angular diameter distance is related to the luminosity distance through the so-called reciprocity relation [12], $\tilde{d}_L = (1 + \tilde{z})^2 \tilde{d}_A$, which holds as long as the photon energy-momentum conservation holds.

The discussion similar in the luminosity distance gives the proper diameter in the observer’s unit in the Einstein frame as

$$D = \frac{e^a}{e^a O} R_S \sin K \chi \theta.$$

(49)

The angular diameter distance in the Einstein frame is given by

$$d_A = \frac{e^a}{e^a O} R_S \sin K \chi = \frac{1}{1 + \tilde{z}} R_O \sin K \chi = e^{-a} |O| \tilde{d}_A.$$

(50)

Since the photon energy-momentum conservation also holds in the Einstein frame, the reciprocity relation also holds: $d_L = (1 + z)^2 d_A$.

C. **Horizon problem and inflation**

As an exercise, let us consider the horizon problem and the condition for inflation. The almost isotropy of the cosmic microwave background indicates that the Universe was already smooth at the decoupling time $\tilde{t}_{dec}$. The necessary condition for solving the horizon problem is that the distance light can travel between two times (say, from $\tilde{t}_i$ and $\tilde{t}_f(\lt \tilde{t}_{dec})$) is larger than the present Hubble distance. In the comoving coordinate, the condition is given by [13]

$$\int_{\tilde{t}_i}^{\tilde{t}_f} \frac{d\tilde{t}}{\tilde{R}} > \frac{1}{\tilde{R}_0 H_{0}}.$$

(51)

We rewrite the condition in terms of the Einstein-frame variables. Since from Eq. (52) and Eq. (47), $\tilde{R}_0 \tilde{H}_0 = R_0 H_{E0}$, the condition is conformal-frame independent

$$\int_{t_i}^{t_f} \frac{dt}{\tilde{R}} > \frac{1}{R_0 H_{E0}}.$$

(52)
In the comoving coordinate, the wave number of any mode remains constant with time. In the standard cosmology, since the comoving Hubble distance \((\bar{R}H)^{-1}\) increases with time, these modes must have all been outside the horizon before the decoupling. Then the necessary requirement to solve the horizon problem is that \((\bar{R}H)^{-1}\) must have decreased sometime in the past. The requirement is written as

\[
\frac{d}{dt} \left( \frac{1}{\bar{R}H} \right) = - \left( \frac{d\bar{R}}{dt} \right)^2 < 0
\] (53)

in the past, or \(d^2\bar{R}/dt^2 > 0\), which is nothing but the condition of inflation. Alternatively, we may consider the condition in the contracting universe \((\bar{H} < 0)\): \(d(-\bar{R}H)^{-1}/dt < 0\), or \(d^2\bar{R}/dt^2 < 0\).

In the Einstein frame, the comoving Hubble distance is \((\bar{R}H)^{-1} = (R \bar{H})^{-1}\), where \(H_E = H + da/dt\) (see Eq. (47)). Therefore, the requirement that the comoving Hubble distance decreases with time is then

\[
\frac{d}{dt} \left( \frac{1}{R \bar{H}} \right) < 0,
\] (54)

or \(d^2R/dt^2 + (dR/dt)(da/dt) + Rd^2a/dt^2 > 0\), and the opposite inequality for the contracting universe. Therefore, it is possible in principle to solve the horizon problem without accelerating expansion in the Einstein frame. An extreme example as such is the pre-big-bang scenario \cite{14} which corresponds to \(\omega = -1\) Brans-Dicke gravity. The solution in a flat universe is given by

\[
\bar{R} = (-\bar{t})^{-1/\sqrt{3}}, \quad \bar{H} = \frac{1}{\sqrt{3}(-\bar{t})}, \quad \Phi = (-\bar{t})^{1+\sqrt{3}},
\] (55)

for \(\bar{t} < 0\), whereas the universe contracts in the Einstein frame

\[
R = (-t)^{1/3}, \quad H = -\frac{1}{3(-t)}.
\] (56)

Although it is possible to solve the horizon problem, the pre-big-bang requires a fine-tuning of the initial conditions in order to account for the the flatness and homogeneity of our part of the universe \cite{15}.

D. Apparent Equation of State

As a second application, let us consider the equation of state of dark energy if \(\Phi\) (or \(\varphi\)) plays the role of dark energy. The basic equations in FRW universe models are given in Appendix.

An interaction between dark matter and dark energy can result in an effective dark energy equation of state of \(w < -1\) \cite{10}. In the Einstein frame, \(\varphi\) couples to (dark) matter, so the scalar-tensor gravity in the Einstein frame provides such an interaction.\(^3\) To see how \(w < -1\) happens, consider the situation where an observer (erroneously) interprets the observational data using the Einstein gravity with a minimal coupling to matter (that is, \(S_m(g_{\mu\nu}, \psi)\) in Eq. (40) instead of \(S_m(e^{2\varphi}g_{\mu\nu}, \psi)\)).\(^4\) Then, such an observer regards \(H\) as the "correct" Hubble parameter and assumes that the dark matter is non-interacting and its energy density redshifts as \(R^{-3}\). So, instead of Eq. (A5) the observer assumes the Friedmann equation (in a flat universe consisting of dark matter \(\rho_m\) and dark energy \(\rho_\varphi\)) as

\[
3H^2 = \kappa^2 \left( \rho_m \left( \frac{R_0}{R} \right)^3 + \rho_\varphi \right),
\] (57)

and by equating this with Eq. (A5), the he/she would infer an effective dark energy density with

\[
\rho_\varphi^{\text{eff}} = \rho_m - \rho_m \left( \frac{R_0}{R} \right)^3 + \rho_\varphi.
\] (58)

\(^3\) Note again that this does not immediately imply that the weak equivalence principle is violated. See the discussion of the Newtonian limit in Sec.II C.

\(^4\) In the Jordan frame language, “an observer interprets the observational data using the Einstein gravity with the metric \(\tilde{g}_{\mu\nu}\) instead of \(g_{\mu\nu}\), and the rest is the same.
Assuming that dark energy is also non-interacting, the observer determines the equation of state of dark energy \( w_{\text{eff}} \) by

\[
\frac{d\rho_{\phi}^{\text{eff}}}{dt} = -3H(1 + w_{\text{eff}})\rho_{\phi}^{\text{eff}}. \tag{59}
\]

From the time derivative of Eq. (58) and noting that

\[
\frac{d\rho_{m}}{dt} = -3H\rho_{m} + \frac{da}{dt}\rho_{m}, \tag{60}
\]

\[
\frac{d\rho_{\phi}}{dt} = -3H\left(\frac{d\phi}{dt}\right)^2 - \frac{da}{dt}\rho_{m} \equiv -3H(1 + w_{\phi})\rho_{\phi} - \frac{da}{dt}\rho_{m}, \tag{61}
\]

\( w_{\text{eff}} \) is given by

\[
w_{\text{eff}} = \frac{w_{\phi}\rho_{\phi}}{\rho_{m} - \rho_{m0}\left(\frac{R_{0}}{R}\right)^3 + \rho_{\phi}}. \tag{62}
\]

Therefore, \( w_{\text{eff}} < w_{\phi} \) is possible if \( \rho_{m} < \rho_{m0}\left(\frac{R_{0}}{R}\right)^3 \). Note that from Eq. (A4) (or Eq. (60)), \( \rho_{m} \) redshifts as

\[
\rho_{m} = e^{4a}\rho_{m0}e^{\frac{a}{R_{0}}\left(\frac{R_{0}}{R}\right)^3} = \rho_{m0}e^{\frac{a}{R_{0}}}\left(\frac{R_{0}}{R}\right)^3. \tag{63}
\]

Hence, if \( a(\phi) \) is an increasing function of time, then \( w_{\text{eff}} < w_{\phi} \) and \( w_{\text{eff}} \) can be \( w_{\text{eff}} < -1 \) if \( \phi \) slow-rolls so that \( w_{\phi} \sim -1 \). This can be easily realized with \( a(\phi) \) being an increasing function and \( V(\phi) \) being a decreasing function of \( \phi \).

IV. PERTURBED OBSERVABLES

A. Sachs-Wolfe effect

Next we consider perturbed observable quantities. First we consider the temperature anisotropies of the cosmic background radiation. First of all, we note that the distribution function \( f \) is conformally invariant because the number of photons is invariant: \( dN = fd^3x d^3p \) and that the collisionless Boltzmann equation is conformal-frame invariant \[5]:

\[
\frac{Df}{Dt} = \frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial x} + \frac{d\rho_{m}}{dt}\frac{\partial f}{\partial \rho_{m}} = \frac{1}{e^a} \frac{Df}{D\eta} = 0. \tag{64}
\]

Therefore, the temperature anisotropies should be invariant under the conformal transformation. In the following we shall verify the invariance in the analysis of linear perturbations.

In the Jordan frame, the fractional temperature fluctuation for a blackbody, \( \Theta = \delta T/T \), is defined in terms of the photon distribution function by

\[
f(x, \tilde{p}, \tilde{t}) = \frac{1}{\exp\left(\frac{\tilde{p}}{T(\Theta + \Theta_{\text{red}})}\right) - 1}, \tag{65}
\]

where \( \tilde{p} \) is the photon momentum. In the longitudinal gauge, the metric can be written as

\[
d\tilde{s}^2 = -(1 + 2\tilde{\Psi})d\tilde{t}^2 + \tilde{R}^2(1 + 2\tilde{\Phi})dx^2. \tag{66}
\]

The collisionless Boltzmann equation is then written as \[17\]

\[
\frac{d}{d\eta} (\Theta + \tilde{\Psi}) = \frac{\partial \tilde{\Psi}}{\partial \eta} - \frac{\partial \tilde{\Phi}}{\partial \eta}, \tag{67}
\]

where \( d\eta = d\tilde{t}/\tilde{R} \) is the conformal time.
We now consider the corresponding relation in the Einstein frame. Since the temperature has the dimension of mass, $\tilde{T}$ is related to the temperature in the Einstein frame $T$ as $\tilde{T} = e^{-a}T$. So

$$\tilde{\Theta} = \frac{\delta T}{\tilde{T}} - \delta a = \Theta - \delta a,$$  

(68)

where $\delta a = a(t, x) - a(t)$. On the other hand, considering $d\tilde{T} = e^{a(t)} dt$ and $\tilde{R}(\tilde{t}) = e^{a(t)} R(t)$, $\tilde{\Psi}$ and $\tilde{\Phi}$ are related to $\Psi$ and $\Phi$ (not to be confused with the Brans-Dicke scalar field) as [18]

$$\tilde{\Psi} = \Psi + \delta a,$$

(69)

$$\tilde{\Phi} = \Phi + \delta a.$$  

(70)

Therefore, as expected, the Boltzmann equation is conformal-frame independent.

$$\frac{d}{d\eta} (\Theta + \Psi) = \frac{\partial \Psi}{\partial \eta} - \frac{\partial \Phi}{\partial \eta},$$

(71)

and hence the temperature anisotropy is conformal-frame independent. One may also verify that the collisional term is also conformal-frame independent because the cross section scales as $e^{2a}$ and the momentum integral scales as $e^{-3a}$ and these in total give a factor $e^{-a}$ in accord with Eq. [19, 20].

B. Adiabatic and isocurvature perturbations

We then consider the adiabatic perturbation and the entropy perturbation, and ask whether these perturbations are conformal-frame independent.

The pressure perturbation can be split into adiabatic and entropic parts as [19] [20]

$$\delta \tilde{p} = c^2_s \delta \rho + \frac{d \tilde{p}}{d \tilde{t}} \tilde{\Gamma},$$

(72)

where $c^2_s \equiv (d\tilde{p}/d\tilde{t})/(d\tilde{p}/d\tilde{t})$ and the first term is the adiabatic part and the second term is entropy (the non-adiabatic) part.

1. Adiabaticity and Curvature Perturbations

First, we rewrite the gauge invariant total entropy perturbation $\tilde{\Gamma}$ defined by Eq. [20], which represents the displacement between hypersurfaces of uniform pressure and uniform energy density, in terms of the variables in the Einstein frame:

$$\tilde{\Gamma} = \frac{\delta \tilde{p}}{d\tilde{p}/d\tilde{t}} - \frac{\delta \tilde{\rho}}{d\tilde{\rho}/d\tilde{t}} = \frac{e^a(d\tilde{p}/d\tilde{t})}{((d\tilde{p}/d\tilde{t}) - 4(d\tilde{a}/d\tilde{t})\tilde{p})((d\tilde{p}/d\tilde{t}) - 4(d\tilde{a}/d\tilde{t})\tilde{p})} \left( \delta \tilde{p} - c^2_s \delta \rho - 4(d\tilde{a}/d\tilde{t})\tilde{p} \left( \delta \tilde{p} - w \delta \rho - 4\tilde{\rho} \delta \tilde{a}(w - c^2_s) \right) \right),$$

$$= \frac{e^a(d\tilde{p}/d\tilde{t})}{((d\tilde{p}/d\tilde{t}) - 4(d\tilde{a}/d\tilde{t})\tilde{p})((d\tilde{p}/d\tilde{t}) - 4(d\tilde{a}/d\tilde{t})\tilde{p})} \left( \left( d\tilde{p}/d\tilde{t} \right) \Gamma (1 - 4(d\tilde{a}/d\tilde{t})\tilde{p}) + 4(d\tilde{a}/d\tilde{t})\tilde{p} (w - c^2_s) \left( \frac{\delta \rho}{d\tilde{\rho}/d\tilde{t}} - \frac{\delta \tilde{a}}{d\tilde{a}/d\tilde{t}} \right) \right),$$

(73)

where $w \equiv \tilde{p}/\tilde{\rho} = p/\rho$ and $c^2_s \equiv (d\tilde{p}/d\tilde{t})/(d\tilde{p}/d\tilde{t})$. Therefore, when the entropy perturbation is vanishing in the Jordan frame $\tilde{\Gamma} = 0$, then $\Gamma = 0$ holds also in the Einstein frame if

$$w = c^2_s \quad \text{or} \quad \frac{\delta \rho}{d\tilde{\rho}/d\tilde{t}} = \frac{\delta \tilde{a}}{d\tilde{a}/d\tilde{t}} = \frac{\delta \varphi}{d\varphi/d\tilde{t}}.$$  

(74)

The former condition corresponds to $w = \text{const}$. For example, even if we assume the barotropic equation of state $\tilde{p} = \tilde{p}(\tilde{\rho})$ in the Jordan frame, the equation of state in the Einstein frame is then $p = e^{-4a} \tilde{p}(e^{4a} \rho) = p(\rho, \varphi)$ and the pressure becomes a function of the density and the scalar field $\varphi$.

On the other hand, the latter condition corresponds to the situation where the entropy perturbation between matter and $\varphi$ is vanishing. In fact, it can be shown that the curvature perturbation is conformal-frame invariant if
this condition holds and moreover it is constant if the adiabatic condition \((\Gamma = 0)\) holds \(^5\). The (gauge invariant) curvature perturbation on uniform-density hypersurfaces (in the Jordan frame) is defined by

\[
\tilde{\zeta} \equiv \Phi - \frac{\delta \rho}{\rho} \frac{d\rho}{dt} H, \tag{75}
\]

which is shown to be constant on large scales if the pressure perturbation is adiabatic \((\Gamma = 0)\) because the energy-momentum tensor in the Jordan frame is conserved \(^{20}\): \(\nabla_{\mu} T^\mu_{\nu} = 0\). Using the relation Eq. \((10)\) and Eq. \((A1)\), \(\tilde{\zeta}\) can be rewritten as (again not to be confused \(\Phi\) with the Brans-Dicke scalar field)

\[
\tilde{\zeta} = \Phi + \delta a + \frac{1}{3(1 + w)} \left( \frac{\delta \rho}{\rho} - 4 \delta a \right). \tag{76}
\]

Using Eq. \((A8)\), the curvature perturbation on comoving hypersurfaces in the Einstein frame can be written as

\[
\zeta = \Phi - \frac{\delta \rho}{\rho} \frac{d\rho}{dt} H = \Phi + \frac{\delta \rho}{3H(1 + w)\rho - (1 - 3w)(da/dt)\rho} H. \tag{77}
\]

Note that generally \(\zeta\) is not constant on large scales even if \(\Gamma = 0\) because the energy-momentum tensor in the Einstein frame is no longer conserved in general: \(\nabla_{\mu} T^\mu_{\nu} \neq 0\) (\(\tilde{\zeta}\) being constant on large scales if the pressure perturbation is adiabatic \((\Gamma = 0)\)). For perturbations obeying \(\delta \rho/(dp/dt) = \delta a/(da/dt)\), or, when \(w = 1/3\), the adiabatic perturbation in one frame remains adiabatic in another frame, and the curvature perturbations in the two frames are equal and constant. One may verify the constancy of \(\zeta\) by the direct calculation of the evolution of \(\zeta\) from Eq. \((12)\) as,

\[
\frac{d\zeta}{dt} = -H + \frac{da}{dt} \frac{dp}{dt} \frac{d\rho}{\rho + p} - \frac{1}{3R^2} \nabla^2 v
\]

\[
+ \frac{1}{3H} \frac{1 - 3w (da/dt)^2}{\delta a} \left[ H \left( \frac{d\delta a}{da} - \frac{d\delta \rho}{\rho} \right) \left( \frac{\delta a}{da} - \frac{\delta \rho}{\rho} \right) - \frac{\delta \rho}{\rho} \frac{d\delta a}{da} - \frac{\delta \rho}{\rho} \frac{d\delta \rho}{\rho} \right] \tag{79}
\]

where \(v\) is a velocity scalar perturbation. Thus, as long as the condition \((78)\) is satisfied and \(\Gamma = 0\), \(\zeta\) is conserved on superhorizon scales.

In sum, the notion of adiabaticity is not conformal-frame independent in general, though it is gauge-invariant \(^7\).

2. Isocurvature Perturbations

Let us also consider the isocurvature perturbations defined as

\[
\bar{S}_{ij} = 3 \left( \frac{\delta \bar{\rho}_i}{d\bar{\rho}_i/dt} - \frac{\delta \bar{\rho}_j}{d\bar{\rho}_j/dt} \right), \tag{80}
\]

which can be rewritten in terms of the quantities in the Einstein frame as

\[
\bar{S}_{ij} = \frac{3e^\alpha (dp_i/dt)(dp_j/dt)}{[(dp_i/dt) - 4(da/dt)\rho_i] [(dp_j/dt) - 4(da/dt)\rho_j]}
\]

\[
\left[ \frac{\delta \rho_i}{dp_i/dt} - \frac{\delta \rho_j}{dp_j/dt} - \frac{4(da/dt)\rho_j}{dp_j/dt} \left( \frac{\delta \rho_i}{dp_i/dt} - \frac{dp_i/dt}{dp_j/dt} \frac{\rho_i}{dp_j/dt} \frac{\delta \rho_j}{dp_j/dt} \right) - 4\delta a \left( \frac{\rho_i}{dp_i/dt} - \frac{\rho_j}{dp_j/dt} \right) \right] \tag{81}
\]

\(^5\) In the absence of matter (i.e. only the Brans-Dicke scalar is present), it is shown that the curvature perturbation on comoving hypersurfaces is invariant under the conformal transformation \(^{18,21,22}\).
with
\[
\frac{(dp_j/dt) \rho_s}{(dp_i/dt) \rho_j} \equiv \frac{\rho_i}{dp_i/dt} - \frac{\rho_j}{dp_j/dt} = -3 \frac{\rho_i \rho_j}{(dp_i/dt)(dp_j/dt)} (w_j - w_i) (H + da/dt).
\]  

(83)

Here, \( w_i \equiv p_i/\rho_i \) and we have assumed that the energy momentum tensor for each component \( i \) in the Jordan frame is conserved individually. It is manifest that \( S_{i,j} = 0 \) is equivalent to \( S_{ij} = 0 \) if and only if \( w_i = w_j \). Thus, the notion of the adiabaticity is not conformal-frame independent in general, though it is again gauge-invariant.

However we believe that this does not immediately imply that the entropy perturbation is not observable. Once we fix the frame to define the unit, it is still meaningful to talk about the entropy perturbation. The situation is similar to the fact that a statement like ‘Newton constant is changing with time’ is meaningless while the statement ‘Newton constant in SI units is changing with time’ does make sense [23].

3. Relation Among Various Curvature Perturbations

Here, let us also consider other definitions of curvature perturbations which are often discussed in the literatures and study the relation among them. The equation of motion in the Jordan frame [2] can be rewritten as

\[
\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} = \kappa^2 \tilde{S}_{\mu\nu},
\]

(84)

where the “effective” energy momentum tensor \( \tilde{S}_{\mu\nu} \) is defined by

\[
\tilde{S}_{\mu\nu} = \frac{\tilde{T}_{\mu\nu}}{\Phi} + \frac{1}{\kappa^2 \Phi} \left[ \frac{1}{2} \left( \frac{\omega(\Phi)}{\Phi} \right)^2 - \frac{3 \tilde{H}}{\Phi} - \frac{1}{2} \right]
+ \tilde{g}_{\mu\nu} \tilde{g}_{\nu} \Phi - \tilde{g}_{\mu\nu} \tilde{g}_{\nu} \Phi - \frac{1}{2} \tilde{g}_{\mu\nu} U \right]
+ 4 \left( \omega(a) + 1 \right) \partial_{\mu} a \partial_{\nu} a - 2 \tilde{\nabla}_{\mu} \tilde{\nabla}_{\nu} a + \tilde{g}_{\mu\nu} \left( -(2 \omega(a) + 4) \tilde{\nabla} a^2 + 2 \nabla a - \frac{1}{2} U e^{2a} \right) .
\]

(85)

(86)

From the Bianchi identity, this "effective" energy momentum tensor is also covariantly conserved, that is, \( \tilde{\nabla}_{\mu} \tilde{S}_{\mu} = 0 \). It should be noticed that the conservation of the original energy momentum tensor \( \tilde{\nabla}_{\mu} \tilde{T}_{\mu} = 0 \) in Eq. (41) together with the scalar equation of motion \( \Phi \) is equivalent to the conservation of this new energy momentum tensor.

Then, the following curvature perturbation is often used in the literatures [7, 24],

\[
\tilde{\zeta}_s \equiv \Phi - \frac{\delta \tilde{\rho}_s}{d \tilde{\rho}_s/dt} \tilde{H},
\]

(87)

where \( \tilde{\rho}_s \) and \( \delta \tilde{\rho}_s \) are the energy density and its fluctuation defined by the energy momentum tensor \( \tilde{S}_{\mu\nu} \) and are given by

\[
\tilde{\rho}_s = \frac{\tilde{\rho}}{\Phi} + \frac{1}{\kappa^2 \Phi} \left[ \frac{1}{2} \left( \frac{\omega(\Phi)}{\Phi} \right)^2 - 3 \tilde{H} \frac{d \Phi}{dt} + \frac{1}{2} U \right],
\]

(88)

\[
\delta \tilde{\rho}_s \equiv \frac{\delta \tilde{\rho}}{\Phi} + \frac{1}{\kappa^2 \Phi} \left[ \frac{1}{2} \left( \frac{\omega(\Phi)}{\Phi} \right)^2 + 2 \frac{d \Phi}{dt} \right] \tilde{H} \frac{d \Phi}{dt} + \frac{1}{2} U e^{2a} ,
\]

(89)

\[
\delta \tilde{\rho}_s \equiv \frac{\delta \tilde{\rho}}{\Phi} + \frac{1}{\kappa^2 \Phi} \left[ \frac{1}{2} \left( \frac{\omega(\Phi)}{\Phi} \right)^2 + 2 \frac{d \Phi}{dt} \right] \tilde{H} \frac{d \Phi}{dt} + \frac{1}{2} U e^{2a} ,
\]

(90)

\[
\delta \tilde{\rho}_s \equiv \frac{\delta \tilde{\rho}}{\Phi} + \frac{1}{\kappa^2 \Phi} \left[ \frac{1}{2} \left( \frac{\omega(\Phi)}{\Phi} \right)^2 + 2 \frac{d \Phi}{dt} \right] \tilde{H} \frac{d \Phi}{dt} + \frac{1}{2} U e^{2a} ,
\]

(91)
where $\tilde{\Phi}$ and $\tilde{\Psi}$ are the metric perturbations defined in Eq. (66).

Similarly, the equation of motion in the Einstein frame (9) can be rewritten as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 S_{\mu\nu},$$ (92)

where another energy momentum tensor $S_{\mu\nu}$ in the Einstein frame is defined by

$$S_{\mu\nu} = T_{\mu\nu} + \frac{4\omega(a) + 6}{\kappa^2} \partial_\mu a \partial_\nu a - \frac{1}{2}g_{\mu\nu} \left( \frac{4\omega(a) + 6}{\kappa^2} (\nabla a)^2 + 2V(a) \right).$$ (93)

(94)

From the Bianchi identity, this energy momentum tensor is covariantly conserved, that is, $\nabla_\mu S_{\mu\nu} = 0$, though the original energy momentum tensor is not conserved, $\nabla_\mu T_{\mu\nu} = \partial_\mu da \partial_\nu a$, as given in Eq. (12). However, $\nabla_\mu S_{\mu\nu} = 0$ can be derived by combining Eq. (12) and the scalar equation of motion (10).

Associated with this new energy momentum tensor, the following curvature perturbation is often used in the literatures (again not to be confused $\Phi$ with the Brans-Dicke scalar field),

$$\zeta_s \equiv \Phi - \frac{\delta \rho_s}{d\rho_s/dt} H,$$ (95)

where $\rho_s$ and $\delta \rho_s$ are the energy density and its fluctuation defined by the new energy momentum tensor $S_{\mu\nu}$ and are given by

$$\rho_s = \rho + \frac{2\omega + 6}{\kappa^2} \left( \frac{da}{dt} \right)^2 + V,$$ (96)

$$\delta \rho_s = \delta \rho + \frac{4\omega + 6}{\kappa^2} \left( \frac{da}{dt} \right)^2 + \frac{2 \omega}{\kappa^2} \left( \frac{da}{dt} \right)^2 \Psi + \frac{dV}{da} \delta a.$$ (97)

First of all, let us consider the relation between the two curvature perturbations $\zeta$ and $\zeta_s$ in the Einstein frame. The two curvature perturbations $\zeta$ and $\zeta_s$ are equal if and only if

$$\left[ \frac{4\omega + 6 d^2 a}{\kappa^2 dt^2} + \frac{2 d\omega}{d a} \left( \frac{da}{dt} \right)^2 + \frac{dV}{da} \right] \left( \frac{\delta \rho}{d\rho/dt} - \frac{\delta a}{da/dt} \right) - \frac{4\omega + 6}{\kappa^2} \left( \frac{da}{dt} \right)^2 \left[ \frac{da}{dt} \right] \frac{d^2 a}{dt^2} \delta a - \left( \frac{da}{dt} \right)^2 \Psi = 0.$$ (98)

This condition holds true if the following two conditions are satisfied

$$\frac{\delta \rho}{d\rho/dt} = \frac{\delta a}{da/dt},$$ (99)

$$\frac{da}{dt} \frac{d\delta a}{dt} - \frac{d^2 a}{dt^2} \delta a - \left( \frac{da}{dt} \right)^2 \Psi = 0.$$ (100)

The latter condition can be recast in terms of the canonical scalar field $\varphi$ into

$$\frac{d\varphi}{dt} \frac{d\delta \varphi}{dt} - \frac{d^2 \varphi}{dt^2} \delta \varphi - \left( \frac{d\varphi}{dt} \right)^2 \Psi = 0,$$ (101)

which implies that the intrinsic non-adiabatic pressure perturbation of $\varphi$ (or $a$) vanishes [25]. Hence, $\zeta = \zeta_s$ if $\delta \rho/(d\rho/dt) = \delta a/(da/dt)$ holds (that is, the entropy perturbation between matter and $\varphi$ is vanishing) and if the intrinsic entropy perturbation of $\varphi$ is vanishing.

The total non-adiabatic pressure (entropy) perturbations in the Einstein frame are also defined in two ways:

$$\delta P_{n,ad} = \frac{dP_{s,ad}}{dt} = \delta P_s - c_s^2 \delta \rho_s,$$ (102)

$$\delta P_{s,ad} = \frac{dP_{s,ad}}{dt} = \delta P_s - \zeta_{s,s}^2 \delta \rho_s,$$ (103)
where $c_*^2 = (dp/\dot{d}t)/(dp_s/\dot{d}t)$, $c_{s,s}^2 = (dp_s/\dot{d}t)/(dp_s/\dot{d}t)$, and

$$p_s = \rho + \frac{2\omega + 3}{\kappa^2} \left( \frac{\dot{d}a}{dt} \right)^2 - V,$$  \hspace{1cm} (104)

$$\delta p_s = \delta p + \frac{4\omega + 6}{\kappa^2} \left( \frac{d\delta a}{dt} \dot{d}t - \left( \frac{\dot{d}a}{dt} \right)^2 \Psi \right) + \frac{2\delta \omega}{\kappa^2 \delta a} \left( \frac{\dot{d}a}{dt} \right)^2 \delta a - \frac{dV}{\delta a} \delta a.$$  \hspace{1cm} (105)

Then, the total non-adiabatic pressure (entropy) perturbations are related as follows,

$$\frac{d\rho_s}{dt} \delta P_{s,nad} - \frac{d\rho}{dt} \delta P_{nad} = 2\frac{dV}{\delta a} \frac{d\delta a}{\delta a/dt} \dot{d}t \left( \frac{\delta p}{\delta a/dt} - \frac{\delta a}{\delta a/dt} \right) + \frac{4\omega + 6}{\kappa^2} \frac{d\delta a}{\delta a/dt} \dot{d}t \left( \frac{2\delta p}{\delta a - \delta \rho \delta a} \right) \left[ \frac{d\delta a}{\delta a/dt} \dot{d}t - \frac{d^2 a}{dt^2} \delta a - \left( \frac{\dot{d}a}{dt} \right)^2 \Psi \right]$$

$$\frac{d\delta a}{\delta a/dt} \dot{d}t - \frac{d^2 a}{dt^2} \delta a - \left( \frac{\dot{d}a}{dt} \right)^2 \Psi = 0.$$  \hspace{1cm} (106)

It is manifest that $\delta P_{s,nad} = 0$ and $\delta P_{nad} = 0$ are equivalent if the following two conditions are satisfied

$$\frac{\delta \rho}{\delta \rho/\dot{d}t} = \frac{\delta a}{\delta a/dt},$$  \hspace{1cm} (107)

$$\frac{d\delta a}{\delta a/dt} \dot{d}t - \frac{d^2 a}{dt^2} \delta a - \left( \frac{\dot{d}a}{dt} \right)^2 \Psi = 0.$$  \hspace{1cm} (108)

Next, let us consider the condition for $\bar{\zeta} = \bar{\zeta}_s$ in the Jordan frame. After some calculations, it is easily found that the two curvature perturbations $\zeta$ and $\bar{\zeta}_s$ in the Jordan frame are equal if and only if

$$6\bar{H} \left[ \frac{d^2 a}{dt^2} - 2(\omega + 1) \left( \frac{\dot{d}a}{dt} \right)^2 - \left( \bar{H} + \frac{d\bar{H}/\dot{d}t}{\bar{H}} \right) \right] \left( \frac{\delta a}{\delta a/dt} - \frac{\delta \rho}{\delta \rho/\dot{d}t} \right)$$

$$+ 2 \left( 2\omega + 3 \frac{\bar{H}}{\delta a/\dot{d}t} \right) \left[ \frac{d\delta a}{\delta a/dt} \dot{d}t - \frac{d^2 a}{dt^2} \delta a - \left( \frac{\dot{d}a}{dt} \right)^2 \Psi \right] - \frac{2}{R^2} \nabla^2 \delta a = 0.$$  \hspace{1cm} (109)

This condition is satisfied on superhorizon scales, where the spatial derivatives can be neglected, if the following two conditions are satisfied

$$\frac{\delta \rho}{\delta \rho/\dot{d}t} = \frac{\delta a}{\delta a/dt},$$  \hspace{1cm} (110)

$$\frac{d\delta a}{\delta a/dt} \dot{d}t - \frac{d^2 a}{dt^2} \delta a - \left( \frac{\dot{d}a}{dt} \right)^2 \Psi = 0.$$  \hspace{1cm} (111)

Hence, $\bar{\zeta} = \bar{\zeta}_s$ on superhorizon scales if $\delta \rho/(\dot{d}p/\dot{d}t) = \delta a/(\dot{d}a/\dot{d}t)$ holds (that is, the entropy perturbation between matter and $\varphi$ is vanishing) and if the intrinsic entropy perturbation of $a$ is vanishing. It should be noticed that the above two conditions are equivalent to

$$\frac{\delta \rho}{\delta \rho/\dot{d}t} = \frac{\delta a}{\delta a/dt},$$  \hspace{1cm} (112)

$$\frac{d\delta a}{\delta a/dt} \dot{d}t - \frac{d^2 a}{dt^2} \delta a - \left( \frac{\dot{d}a}{dt} \right)^2 \Psi = 0.$$  \hspace{1cm} (113)

Thus, as long as these conditions, which imply the vanishing of the entropy perturbation between matter and the Brans-Dicke scalar field and the vanishing of the intrinsic entropy perturbation of the Brans-Dicke scalar, are satisfied, all of the four curvature perturbations coincide,

$$\bar{\zeta}_s = \bar{\zeta} = \zeta = \zeta_s.$$  \hspace{1cm} (114)

on superhorizon scales.
Finally, we give explicit expressions for the total non-adiabatic pressure (entropy) perturbations in the Jordan frame. They are also defined in two ways:

$$\delta \tilde{P}_{n,ad} = \frac{d\tilde{p}}{dt} = \delta \tilde{p} - \tilde{c}_s^2 \tilde{\delta \tilde{p}}, \quad \delta \tilde{P}_{s,ad} = \frac{d\tilde{p}_s}{dt} = \delta \tilde{p}_s - \tilde{c}_s^2 \delta \tilde{p}_s,$$

where $\tilde{c}_s^2 = (d\tilde{p}/d\tilde{t})/(d\tilde{p}/d\tilde{t})$, $\tilde{c}_{s,ad}^2 = (d\tilde{p}_s/d\tilde{t})/(d\tilde{p}_s/d\tilde{t})$, and

$$\tilde{p}_s = \frac{\bar{p}}{\Phi} + \frac{1}{\kappa^2 \Phi} \left[ \frac{1}{2} \omega \left( \frac{d\Phi}{dt} \right)^2 + \frac{d^2 \Phi}{dt^2} + 2 \tilde{H} \frac{d\Phi}{dt} - \frac{1}{2} U \right]$$

$$\delta \tilde{p}_s = \frac{1}{\Phi} (\tilde{\delta \tilde{p}} - \tilde{p}_s \delta \Phi) + \frac{1}{\kappa^2 \Phi} \left[ \omega \left( \frac{d\tilde{\Phi}}{dt} \right)^2 + \frac{d^2 \tilde{\Phi}}{dt^2} - 2 \tilde{H} \frac{d\tilde{\Phi}}{dt} - 2 \Phi \right] + \frac{d^2 \tilde{\Phi}}{dt^2} + 2 \tilde{H} \frac{d\tilde{\Phi}}{dt}$$

$$-2 \left( \frac{d^2 \Phi}{dt^2} + 2 \tilde{H} \frac{d\Phi}{dt} \right) \tilde{\Psi} - \frac{d \tilde{\Phi}}{dt} - \frac{2 \tilde{H} \Phi}{d \Phi} - \frac{1}{2} \Phi \frac{d \Phi}{dt} - \frac{2 \tilde{H} \Phi}{d \Phi} - \frac{1}{2} \frac{d \Phi}{dt}$$

$$= e^{2\alpha} \left( \tilde{\delta \tilde{p}} + 2 \tilde{p} \delta \tilde{a} \right) + \frac{1}{\kappa^2} \left[ -2 \frac{d^2 \delta a}{dt^2} + 2 \left( \frac{\rho}{\tilde{H} \delta a/d\tilde{t}} \right)^2 \right] \tilde{\Psi} + 2 \frac{\rho}{\tilde{H} \delta a/d\tilde{t}} - \frac{\rho}{\tilde{H} \delta a/d\tilde{t}} + 4 \frac{\rho}{\tilde{H} \delta a/d\tilde{t}}$$

$$= e^{2\alpha} \left( \tilde{\delta \tilde{p}} + 2 \tilde{p} \delta \tilde{a} \right) + \frac{1}{\kappa^2} \left[ -2 \frac{d^2 \delta a}{dt^2} + 2 \left( \frac{\rho}{\tilde{H} \delta a/d\tilde{t}} \right)^2 \right] \tilde{\Psi} + 2 \frac{\rho}{\tilde{H} \delta a/d\tilde{t}} - \frac{\rho}{\til{H} \delta a/d\tilde{t}} + 4 \frac{\rho}{\til{H} \delta a/d\tilde{t}}$$

where $\tilde{\Phi}$ and $\tilde{\Psi}$ are the metric perturbations defined in Eq. (115). Then, the total non-adiabatic pressure (entropy) perturbations in the Jordan frame are related as follows,

$$\frac{d\tilde{p}_s}{dt} \delta \tilde{P}_{n,ad} - \frac{6}{\kappa^2} \frac{dH}{dt} e^{2\alpha} \delta \tilde{P}_{n,ad} = \frac{6}{\kappa^2} \frac{dH}{dt} e^{2\alpha} \left( \frac{d\tilde{p}}{dt} - \frac{2}{3} \left( \tilde{H} - \delta a/d\til{t} \right) \frac{d\rho}{dt} \right)$$

$$+ \frac{6}{\kappa^2} \frac{dH}{dt} \left[ \frac{4}{3} \til{H} \frac{d\rho}{d\til{a}/d\til{t}} \til{\nabla}^2 \delta a - \frac{2 \frac{d\rho}{d\til{a}/d\til{t}}}{\til{R}^2} + 2 \frac{d\rho}{d\til{a}/d\til{t}} \frac{d^2 \delta a}{d\til{t}^2} \frac{d\rho}{d\til{a}/d\til{t}} \right]$$

$$- \frac{d\tilde{p}_s}{dt} \left( \til{H} \delta a/d\til{t} \right) \left[ e^{2\alpha} \frac{d\til{p}}{dt} \left( \frac{d\til{\rho}}{d\til{a}/d\til{t}} \right) + \frac{2}{\kappa^2} \left( 2 \omega + 3 \til{H} \delta a/d\til{t} \right) \left( \frac{d\til{a}/d\til{a}/d\til{t}}{d\til{a}/d\til{t}} \right) - \frac{d^2 \delta a}{d\til{t}^2} \frac{d\rho}{d\til{a}/d\til{t}} \right]$$

Here we have used the following relations,

$$\frac{d\til{p}_s}{dt} = \frac{6}{\kappa^2} \frac{dH}{dt} e^{2\alpha} \left( \til{H} - \delta a/d\til{t} \right),$$

$$\frac{d\til{p}}{dt} = \frac{6}{\kappa^2} \frac{dH}{dt} \frac{d\til{H}}{d\til{a}/d\til{t}} \delta a + \frac{1}{\kappa^2} \frac{d^2 \til{H}}{d\til{a}/d\til{t}^2} \delta a + \frac{1}{\kappa^2} \frac{d^2 \til{H}}{d\til{a}/d\til{t}^2} \delta a$$

$$+ \frac{2}{\kappa^2} \left( 2 \omega + 3 \til{H} \delta a/d\til{t} \right) \left( \frac{d\til{a}/d\til{a}/d\til{t}}{d\til{a}/d\til{t}} \right) - \frac{d^2 \delta a}{d\til{t}^2} \frac{d\rho}{d\til{a}/d\til{t}} \right) - \frac{d^2 \delta a}{d\til{t}^2} \frac{d\rho}{d\til{a}/d\til{t}}.$$
It is manifest that $\delta \tilde{P}_{s,nad} = 0$ and $\delta \tilde{P}_{nad} = 0$ are equivalent on superhorizon scales if the following two (three) conditions are satisfied

\begin{align}
\frac{\delta \tilde{\rho}}{d\tilde{\rho}/dt} &= \frac{\delta a}{da/dt}, \quad (125) \\
\frac{da}{dt} \frac{d\delta a}{dt} - \frac{d^2 a}{dt^2} \delta a - \left(\frac{da}{dt}\right)^2 \tilde{\Psi} &= 0, \quad (126) \\
\frac{d}{dt} \left(\frac{da}{dt} \frac{d\delta a}{dt} - \frac{d^2 a}{dt^2} \delta a - \left(\frac{da}{dt}\right)^2 \tilde{\Psi}\right) &= 0. \quad (127)
\end{align}

V. SUMMARY

We have discussed the frame independence/dependence of the observable quantities between the Jordan and the Einstein frame in scalar-tensor gravity and have provided a Jordan-Einstein dictionary. We have found that dimensionless quantities and the relations of them (redshift, magnitude redshift relation, reciprocity relation, the Sachs-Wolfe effect) are independent of the conformal frame. The dimensional quantities such as the Hubble parameter and the luminosity/angular diameter distances, are related to each frame by a simple change of local units due to the conformal transformation, although for the effective gravitational constant the transformation involves the asymptotic value of the conformal factor.

As applications, we have considered the horizon problem and the dark energy equation of state. We have found that the condition for solving the horizon problem itself is conformal-frame independent. However, the condition that the comoving Hubble distance decreases differs in each frame and hence the detailed implementation of the mechanism to solve the horizon problem can differ depending on the frame. We have also considered the situation where an observer (erroneously) interprets the observational data using the Einstein gravity with a minimal coupling to matter although the true gravity theory is scalar-tensor gravity and have found that the apparent equation of state dark energy could be "phantom-like" $w < -1$.

Finally, we have studied the frame dependence of curvature perturbations. We have found that the absence of the non-adiabatic pressure and hence the absence of the isocurvature perturbations are conformal-frame dependent in general. There exist four types of curvature perturbations on uniform-matter density hypersurfaces depending on the definition of the (effective) energy momentum tensors $\tilde{T}_{\mu\nu}$ (or $T_{\mu\nu}$) and $\tilde{S}_{\mu\nu}$ (or $S_{\mu\nu}$). For the matter defined by the energy momentum tensors $\tilde{T}_{\mu\nu}$ and $T_{\mu\nu}$, the conformal invariance of the corresponding curvature perturbations on uniform-density hypersurfaces $\tilde{\zeta}$ and $\zeta$ holds if the entropy perturbation between matter and the Brans-Dicke scalar field is vanishing or the equation of state of matter is constant. For the matter defined by the energy momentum tensors $\tilde{S}_{\mu\nu}$ and $S_{\mu\nu}$, the corresponding curvature perturbations on uniform-density hypersurfaces $\tilde{\zeta}_s$ and $\zeta_s$ coincide with $\tilde{\zeta}$ and $\zeta$ if the entropy perturbation between matter and the Brans-Dicke scalar field is vanishing and if the intrinsic entropy perturbation of the Brans-Dicke scalar is vanishing.

Although our analysis in this paper was limited to classical theory, we would like to extend the analysis to quantum theory (see [26] for recent attempts) in order to investigate to what extent the conformal-frame independence persists in quantum theory [27].

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Appendix A: The equations of motion in FRW universe

In this appendix, we list the equations of motion for FRW universe models for completeness.
The equations of motion for FRW universe models in the Jordan frame are given by

\begin{align}
3\Phi \left( \frac{H^2}{2} + \frac{K}{R^2} \right) &= \kappa^2 \rho + \frac{\omega}{2\Phi} \left( \frac{d\Phi}{dt} \right)^2 - 3\frac{H}{2} \frac{d\Phi}{dt} + \frac{1}{2} U(\Phi), \\
\frac{d\Phi}{dt} \left( \frac{dH}{dt} - \frac{K}{2R^2} \right) &= -\frac{\kappa^2}{2} (\bar{\rho} + \bar{p}) - \frac{1}{2} \frac{d^2\Phi}{dt^2} + \frac{1}{2} \frac{H}{R^2} \frac{d\Phi}{dt} - \frac{\omega}{2\Phi} \left( \frac{d\Phi}{dt} \right)^2, \\
\frac{d^2\Phi}{dt^2} + 3\frac{H}{2} \frac{d\Phi}{dt} &= \frac{1}{2\omega + 3} \left( \kappa^2 (\bar{\rho} - 3\bar{p}) - \frac{d\omega}{d\Phi} \left( \frac{d\Phi}{dt} \right)^2 - \Phi \frac{dU}{d\Phi} + 2U \right),
\end{align}

and the energy-momentum conservation is given by

\begin{equation}
\frac{d\bar{\rho}}{dt} + 3\bar{H} (\bar{\rho} + \bar{p}) = 0.
\end{equation}

The equations of motion in the Einstein frame are given by

\begin{align}
3 \left( H^2 + \frac{K}{R^2} \right) &= \kappa^2 \left( \rho + \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 + V \right) \equiv \kappa^2 (\rho + \rho_\phi), \\
\frac{dH}{dt} - \frac{K}{2R^2} &= -\frac{\kappa^2}{2} \left( \rho + p + \left( \frac{d\phi}{dt} \right)^2 \right), \\
\frac{d^2\phi}{dt^2} + 3H \frac{d\phi}{dt} + \frac{dV}{d\phi} &= \frac{da}{d\phi} (-\rho + 3p),
\end{align}

and the energy-momentum conservation is given by

\begin{equation}
\frac{d\rho}{dt} + 3H (\rho + p) = \frac{d\phi}{dt} \frac{da}{d\phi} (-\rho + 3p).
\end{equation}

Considering the correspondence between the variables in the Jordan frame and those in the Einstein frame given by (see also Eq. (1), Eq. (3) and Eq. (32))

\begin{equation}
\bar{H} = e^{-a} H_E = e^{-a} \left( H + \frac{da}{dt} \right), \quad \bar{\rho} = e^{-4a} \rho, \quad \bar{p} = e^{-4a} p, \quad \Phi = e^{-2a},
\end{equation}

we verify that the equations of motion of the both frames are equivalent. Of course, this is a special case of the equivalence of the equations of motion in the both frames, as mentioned in the section II.
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