Cardy states as idempotents of fusion ring in string field theory

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Abstract

With some assumptions, the algebra between Ishibashi states in string field theory can be reduced to a commutative ring. From this viewpoint, Cardy states can be identified with its idempotents. The algebra can be identified with a fusion ring for the rational conformal field theory and a group ring for the orbifold. This observation supports our previous observation that boundary states satisfy a universal idempotency relation under closed string star product.

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Commutative rings In this short note, we consider projection operators of two types of commutative rings which are related to the conformal field theory. The first one is the group ring \( C[\Gamma] \) based on a discrete finite group \( \Gamma \). It is defined as a vector space with \(|\Gamma|\) dimensions. If we write its basis as \( e_g (g \in \Gamma) \), generic elements can be expanded as \( \lambda = \sum_{g \in \Gamma} \lambda_g e_g \), \( \lambda_g \in \mathbb{C} \). It is equipped with a natural product structure,

\[
\lambda \star \mu = \sum_{g, g' \in \Gamma} \lambda_g \mu_{g'} e_{gg'} = \sum_{g_1, g_2 = g} \left( \sum_{\gamma = g_1 \gamma_2} \lambda_{\gamma_1} \mu_{\gamma_2} \right) e_g
\]

where \( gg' \) is the product in the group \( \Gamma \).

When the group \( \Gamma \) is non-abelian, we combine the basis which belong to the same conjugacy class. Namely instead of treating the basis \( e_g \) separately, we take a combination \( e_i = \sum_{g \in C_i} e_g \) as the basis where \( C_i \) is a conjugacy class. While the group ring becomes noncommutative, the combined basis satisfies a commutative algebra,

\[
e_i \star e_j = \sum_k N_{ij}^k e_k.
\]

The structure constant \( N_{ij}^k \) is non-negative integer and can be written in terms of the characters of the irreducible representations of \( \Gamma \) as \( \mathbb{1} \),

\[
N_{ij}^k = \frac{1}{|\Gamma|} \sum_{\alpha \in \text{irreps.}} r_i r_j c^{(\alpha)}_{\text{irr}}(\zeta_j^{(\alpha)} / \zeta_1^{(\alpha)}),
\]

where \( c^{(\alpha)}_{\text{irr}} \) is the character of an irreducible representation \( \alpha \) for the elements in the conjugacy class \( C_i \). \( r_j \) is the number of elements in \( C_i \). We take \( C_1 \) as the conjugacy class that consists only of the identity element. This formula is analogous to Verlinde formula \( \mathbb{5} \) in the following.

Another algebra in consideration is the fusion ring \( \mathbb{2} \) of rational conformal field theories (RCFT). Suppose there exist \( n \) primary fields \( \phi_i (i = 1, \ldots, n) \). The fusion ring for RCFT consists of the \( n \) dimensional vector space with base \( e_i (i = 1, \ldots, n) \) together with a product structure for the basis,

\[
\left( \sum_i \lambda_i e_i \right) \star \left( \sum_j \mu_j e_j \right) = \sum_k \left( \sum_{ij} \lambda_i \mu_j N_{ij}^k \right) e_k,
\]

where \( N_{ij}^k \) is Verlinde’s fusion coefficient \( \mathbb{2} \) defined by the modular transformation matrices,

\[
N_{ij}^k = \sum_l \frac{S_{ij}^l S_{ji} S_{il}^{*}}{S_{ll}}.
\]

\( S \) is symmetric, unitary and satisfies \((S^2)_{ij} = \delta_{ij}\) where \( i^* \) is a charge conjugation of \( i \).

In both cases, the product \( \star \) satisfies the associativity. For the group ring, this is obvious. For the fusion ring, it reduces to a property of the fusion multiplicity,

\[
\sum_l N_{ij}^l N_{lk}^m = \sum_l N_{jk}^l N_{li}^m,
\]
which can be proved directly from Eq.(5) and the unitarity of $S$.

**Results from closed string field theory**  Our motivation to consider the commutative rings comes from string field theory. In our previous papers [4, 5], we have derived properties of the boundary states with respect to the star product of closed string field theories [6, 7].

1. Suppose we have two boundary states which satisfies,

$$ (L_n - \tilde{L}_{-n}) |B_i\rangle = 0, \quad (i = 1, 2). $$

Then a state which is created by HIKKO’s star product [6] which specifies interactions of closed strings satisfies the same equation, $$(L_n - \tilde{L}_{-n})(|B_1\rangle \ast |B_2\rangle) = 0.$$ This claim is proved in the background independent fashion and should be applied to any conformal field theories.

2. The second observation is the idempotency of boundary states. It was proved in flat background that boundary states $|x^\perp, F\rangle$ of $D_p$-branes with transverse coordinates $x^\perp$ and flux $F$ satisfy

$$ |x^\perp, F\rangle_{\alpha_1} \ast |y^\perp, F\rangle_{\alpha_2} = [\mathcal{C}(\alpha_1, \alpha_2)]^{1/d(d-p-1)} d^{d-1}/2 (x^\perp - y^\perp)|y^\perp, F\rangle_{\alpha_1 + \alpha_2}, $$

where $\alpha_i$ are parameters which specify the length of overlapping closed strings at the vertex. The coefficient $\mathcal{C}$ is given in appendix B of Ref. [5], $\mathcal{C}(\alpha_1, \alpha_2) = T^{-1/8}|\alpha_1\alpha_2\alpha_3|^{1/24} (\alpha_3 = -\alpha_1 - \alpha_2)$. Here we write only the contribution from the matter part.\footnote{In the definition of $\ast$ for each boson, we absorbed a factor $\mu^{1/12}$ where $\mu = e^{-\tau_0} \sum_{r=1}^3 \alpha_r^{-1}$, $\tau_0 = \sum_{r=1}^3 \alpha_r \log |\alpha_r|$. Including ghost sector, we should replace $[\mathcal{C}(\alpha_1, \alpha_2)]^{d}$ with $[\mathcal{C}(\alpha_1, \alpha_2)]^{2d-2} e_0^+$ in Eq.(5).}

$Ishibashi and Cardy state$  A convenient basis of the states which satisfy Eq.(7) is Ishibashi state [8]. In a generic CFT, there exists a state $|i\rangle$ for each primary field $\phi_i$ which satisfies Eq.(7) and,

$$ \langle \langle i | \tilde{q}^{1/2}(L_0 + \tilde{L}_0 - \frac{d-2}{4}) |j\rangle \rangle = \delta_{ij} \chi_j(\tilde{q}), \quad \tilde{q} = \exp(-2\pi i / \tau), $$

where $\chi_j$ is the character for the highest weight representation associated with the primary field $\phi_j$. Since Ishibashi states give the basis of the Hilbert space that satisfies Eq.(7), we can expand the star product between them,

$$ |i\rangle_{\alpha_1} \ast |j\rangle_{\alpha_2} = \sum_k C_{ij}^k(\alpha_1, \alpha_2) |k\rangle_{\alpha_1 + \alpha_2}, $$

where the coefficient $C_{ij}^k(\alpha_1, \alpha_2)$ is a $c$-number. Here the $\ast$ product is HIKKO’s one. It is commutative and non-associative but satisfies Jacobi identity instead.
The conformal invariance of the boundary (7) is, actually, not enough to define a consistent boundary state. We have to impose Cardy condition, in order to have well-defined open string sector \( \mathcal{O} \) (see also Refs. [10, 11] for reviews). We take a linear combination of Ishibashi states, \( |\alpha\rangle = \sum_i \psi^i_\alpha |i\rangle \), and calculate the inner product between them. After we perform a modular transformation,

\[
\langle \alpha | \tilde{q}^{L_0 + \bar{L}_0 - \frac{c}{24}} | \beta \rangle = \sum_i (\psi^i_\alpha)^* \psi^i_\beta \chi_i(\tilde{q}) = \sum_j n_{\alpha\beta}^j \chi_j(q), \quad n_{\alpha\beta}^j := \sum_i (\psi^i_\alpha)^* \psi^i_\beta S_{ji}^*, \tag{11}
\]

where \( \chi_i(q) = S_{ij} \chi_j(\tilde{q}), \quad (q = e^{2\pi i\tau}). \) In order to have a well-defined open string Hilbert space, the coefficient \( n_{\alpha\beta}^j \) must be non-negative integer. This gives a set of quadratic constraints for the coefficients \( \psi^i_\alpha \). A famous family of solutions found by Cardy is

\[
\psi^i_\alpha = \frac{S_{i1}}{\sqrt{S_{11}}}. \tag{12}
\]

With this choice, the multiplicities \( n_{\alpha\beta}^j \) coincides with the fusion rule coefficient \( N_{\alpha\beta}^j \) in Eq.(5).

For a systematic study of other choices of \( \psi \), see for example, Ref. [10].

The purpose of this note is to study if the idempotency relation is consistent with Cardy condition (11). While both equations are quadratic, they look rather different. To compare the two conditions, we first solve the idempotency relation using Eq.(10) and compare them with the solutions of Cardy condition. As we see below, they are essentially the same.

**Reduced SFT algebra for Ishibashi state** In the following, we conjecture that the structure constant \( C_{ijk}(\alpha_1, \alpha_2) \) in Eq.(10) will take a factorized form \( [C(\alpha_1, \alpha_2)]^c R_{ij}^k \) where \( R_{ij}^k \) is a constant which is independent of \( \alpha_i \) and \( c \) is the central charge of the conformal field theory.

This assumption was explicitly proved for the flat background [4, 5] where Ishibashi state is the momentum eigenstate in the transverse direction. By an explicit computation in oscillator formulation, we found \( |p^+_{1}\rangle_{\alpha_1} * |p^+_{2}\rangle_{\alpha_2} = [C(\alpha_1, \alpha_2)]^d |p^+_{1} + p^+_{2}\rangle_{\alpha_1 + \alpha_2} \). As a more nontrivial background, we have studied numerically the \( \mathbb{Z}_2 \) orbifold. We have confirmed the factorization while we need to change the definition of the factor \( \mu \) which appeared in footnote [1] in the string vertex which involves the twisted sectors. The detail will be published elsewhere.

From the constant part of the structure constant, one can define an algebra \( e_i * e_j = \sum_k R_{ij}^k e_k \) which is independent of \( \alpha_i \) where \( e_i \) corresponds to \( |i\rangle \) in the reduced algebra. Since HIKKO’s product \( * \) is commutative, the reduced product \( \star \) is commutative. Moreover we assume that the reduced product \( \star \) is associative for Ishibashi states which can be guessed from the associativity in the flat background. In the following, we call the simplified algebra as **reduced SFT algebra**.

In the following, we fix \( R_{ij}^k \) for two CFT models. The first one is the orbifold \( M/\Gamma \) with the discrete group \( \Gamma \) (see, for example, a review paper [11] and references therein). In this case, the
The Ishibashi state $|g\rangle$ has a label $g \in \Gamma$ which specifies the twisted boundary condition,

$$(X(\sigma + 2\pi) - g \cdot X(\sigma))|g\rangle = 0. \quad (13)$$

In this note, we consider a situation where there is only one boundary state which satisfies Eq. (7) for each twisted sector for simplicity. The star product $|g\rangle \star |g'\rangle$ belongs to the twisted sector of $gg'$. The reduced algebra therefore should be written in the form $e_g \star e_{g'} = R(g,g')e_{gg'}$. The associativity of the $\star$ product imposes a condition on the coefficient $R(g,g')$:

$$R(gh,k)R(g,h) = R(g,hk)R(h,k). \quad (14)$$

A change of the normalization of the basis $e_g \rightarrow \beta(g)e_g$ will change the above coefficient: $R(g,h) \rightarrow \frac{\beta(gh)}{\beta(g)\beta(h)}R(g,h)$. The nontrivial solutions of Eq. (14) with this identification are parametrized by the cohomology group $H^2(\Gamma, U(1))$. This is the discrete torsion factor in the orbifold models [12]. In this note, we consider the situation where there is no such cohomology where one can consistently put $R(g,h) = 1$ up to the normalization factor of the boundary state. The reduced SFT algebra is identical to group ring $C[\Gamma]$ in Eq. (1). When $\Gamma$ is non-commutative, it is known that we have to combine the Ishibashi states of the same conjugacy class. The reduced SFT algebra is thus the same as Eq. (2).

Our second example is the rational conformal field theory where we have only finite number of primary states. We need to impose on the coefficient $R_{ij}^k$ a few constraints (i) if the coefficient $N_{ij}^k$ vanishes for a set $(ijk)$, the corresponding $R_{ij}^k$ should also vanish (ii) it needs to satisfy the associativity condition similar to Eq. (6). At this moment, we do not know the analogue of the cohomology group which would fix $R_{ij}^k$ up to the normalization constant as in the group ring case. However, since $N_{ij}^k$ itself explicitly solves them, we will focus on the fusion ring [4] without further verification from the string field theory. The outcome is, as we will see, natural and it supports this assumption.

To summarize, up to some ambiguities, we regard the commutative rings (1), (4) as the reduced SFT algebras of the Ishibashi states. We note that if a projector of the reduced algebra is found, for example in the form $\sum_i \lambda_ie_i$, the same combination $\sum_i \lambda_i|i\rangle$ becomes an idempotent of the full string field algebra.

**A simple example of projectors in the group ring** We consider the projection operators for the star product $\star$ for these commutative rings. Before writing down somewhat abstract formula from the beginning, we start from a simple example which illuminates the basic structure. We consider the case $\Gamma = Z_3$. We write its elements as $\{1, g, g^2\}$ where $g^3 = 1$. We can solve the

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2This condition is also imposed by Jacobi identity for HIKKO’s $\star$ product without assuming the associativity.
algebraic equation \( \lambda \star \lambda = \lambda \) with \( \lambda = \alpha_0 e_1 + \alpha_1 e_g + \alpha_2 e_{g^2} \). It has three quadratic equations for three unknown variables. It therefore has \( 2^3 = 8 \) solutions,

\[
(\alpha_0, \alpha_1, \alpha_2) = (0, 0, 0), \quad \frac{1}{3}(1, 1, 1), \quad \frac{1}{3}(1, \omega, \omega^2), \quad \frac{1}{3}(1, \omega^2, \omega), \\
\frac{1}{3}(2, -1, -1), \quad \frac{1}{3}(2, -\omega, -\omega^2), \quad \frac{1}{3}(2, -\omega^2, -\omega), \quad (1, 0, 0),
\]

with \( \omega = e^{2\pi i/3} \). The first line is a trivial solution. We will identify the three solutions in the second line with the fractional D-branes in \( Z_3 \) orbifold. They are written in terms of Ishibashi states as,

\[
|\alpha \rangle = \alpha_0 |1\rangle + \alpha_1 |g\rangle + \alpha_2 |g^2\rangle
\]

(15)

where \( |1\rangle \) (resp. \( |g\rangle \)) is the boundary state for the untwisted (g-twisted) sector. After this interpretation, the solutions in the third line correspond to the combinations of two different fractional D-branes and the fourth line corresponds to the non-fractional D-brane that corresponds to the regular representation of \( Z_3 \).

General projectors of group (fusion) ring and Cardy states

We write down generic idempotents of the commutative rings and compare them with Cardy states. As we have seen in \( Z_3 \) example, there exist \( 2^n \) solutions to the idempotency relation where \( n \) is the number of the conjugacy classes (for group ring) or that of the primary fields (for RCFT). In both cases, the explicit form of the idempotents is known. A novelty here is the comparison with the Cardy states.

For the group ring, there exists an idempotent for each irreducible representation of \( \Gamma \).

\[
P^{(\alpha)} = \frac{d_\alpha}{|\Gamma|} \sum_{i \in \text{Class}} \zeta_i^{(\alpha)} e_i ,
\]

(16)

where \( d_\alpha = \zeta_1^{(\alpha)} \) is the dimension of the irreducible representation \([\Pi]\). For the fusion ring \([3]\),

\[
P^{(\alpha)} = S_{1\alpha}^* \sum_{i \in \text{Primary}} S_{i\alpha} e_i .
\]

(17)

In both cases, they satisfy the idempotency relation, \( P^{(\alpha)} \star P^{(\beta)} = \delta_{\alpha\beta} P^{(\beta)} \). Note that we use the charge conjugacy symmetry \( N_{i\alpha j}^{*\beta} k^* = N_{ij}^{\alpha \beta} k \) in the proof. Because of the orthogonality, \( 2^n \) general solutions are immediately obtained, \( P = \sum_\alpha \epsilon_\alpha P^{(\alpha)} \) where \( \epsilon_\alpha = 0 \) or \( 1 \). Proof of the idempotency relation reduces to the orthogonality of the group character for the first case and the unitarity of the modular transformation matrices for the second. In both cases, the most essential property is that the structure constant is written in Verlinde’s form \([3,5]\).
We compare these expressions with the Cardy states. For the orbifold \([\mathbb{Z}_2]\),

\[
|\alpha\rangle = \frac{1}{\sqrt{|\Gamma|}} \sum_{i} \sqrt{\sigma_i} \xi_{i}^{(\alpha)} |i\rangle, \quad |i\rangle := \sum_{g \in \mathcal{C}_i} |g\rangle.
\]  

(18)

where \(\sigma_i\) is a factor which appears in the modular transformation of the character for the orbifold.

For the generic modular character \(\chi^h_g(q) = \text{Tr}_{H_g} h q^{L_0 - c/24}\), we write their modular transformation as \(\chi^h_g(q) = \sigma(g, h) \chi^h_q(q)\). \(\sigma_i\) is defined as \(\sigma(e, g)\) for \(g \in \mathcal{C}_i\). For the RCFT, Cardy’s solution is written as,

\[
|\alpha\rangle = \sum_{i \in \text{Primary}} \frac{S_{\alpha i}}{S_{11}} |i\rangle.
\]  

In either case, if we identify the base \(e_i\) of group (resp. fusion) ring with the normalized Ishibashi state \(\sqrt{\sigma_i} |i\rangle\) (resp. \((S_{11})^{-1/2} |i\rangle\)), Cardy states are the idempotents.

### Discussion

In this short note, we show that the Cardy states can be interpreted as the idempotents of the group (fusion) ring. It supports our conjecture \([5]\) that every consistent boundary states satisfy a universal nonlinear relation, the idempotency relation with respect to the star product of the closed string field theory. We note that our discussion for RCFT can be formally extended to the generic CFT with an infinite number of primary fields.

We have to admit that our argument is so far weak from the viewpoint of closed string field theory. Namely we need to assume that the factorization of the coefficients \(\mathcal{C}^{(\alpha_1, \alpha_2)}_{ij k}\) in Eq.(10) without a detailed analysis of the string vertex. For the generic CFT, the explicit proof is rather difficult since there exists no oscillator definition of three string vertex for the generic background. As we mentioned, however, an explicit formula is known \([13]\) for the \(\mathbb{Z}_2\) orbifold. In our future publication, we will present our explicit evaluation of the coupling constant and present a proof of our conjecture in this note.

We compare our discussion with the more popular arguments of the description of D-branes \([14]\). In a spirit that D-branes should be described by projectors, our discussion is basically the same as theirs. However, while these arguments are based on the open string, ours comes from the closed string. It induces some differences. First, the algebra is non-commutative in open string approaches, it is usually commutative in our discussion. While the projectors in the open string theory pick up localized solitons in the non-commutative space-time, ours specify the representations that characterize the D-branes. The scope of these two approaches seems to be mutually complementary but not contradictory.

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