Januarials were defined by Graham Higman in his last series of lectures. In this paper we answer some questions posed by Higman in these lectures.

AMS Classification[2010]: Primary 05C25; Secondary 20G40

Keywords: Coset diagrams, Januarials, Triangle groups, Hecke groups and genus.

1 Introduction

Coset diagrams were introduced by Graham Higman as a diagram representing an action of a particular group on a set or a space. Several papers, namely [2, 3, 6, 10, 11] appeared in which these coset diagrams were used to prove interesting topological and group theoretical results. Q. Mushtaq [8] laid the foundations of coset diagrams for the action of the modular group on finite Galois fields and real quadratic irrational number fields.

Graham Higman introduced the concept of januarials in his lectures at Oxford in 2000, which turned out to be his last – see [1]. In these he posed several questions. This paper answers two of them – see Proposition [8] and Theorem [11] below.

2 Januarials

Let $\Delta = \Delta(2, k, \ell)$ be the abstract group with presentation

$$\Delta(2, k, \ell) = \langle x, y : x^2 = y^k = (xy)^\ell = 1 \rangle$$

where $k, \ell \in \mathbb{Z}^{\geq 2} \cup \infty$. Such groups are called triangle groups as they act discretely on the 2-sphere, Euclidean plane or hyperbolic plane with fundamental region a triangle. The $\Delta(2, k, \infty)$ are called Hecke groups; in particular when $k = 3$ we obtain the classical modular group.
Suppose that we have an action of $\Delta$ on some set $S$. A coset graph $\Gamma$ is a directed graph in the plane depicting the action as follows: the vertices of the graph are the elements of the set $S$; if two vertices are transposed by the action of $x$ then there is an undirected $x$-edge connecting them; vertices fixed by $x$ have no $x$-edge incident with them; there is a directed $y$-edge from $u$ to $v$ when $u$ is sent to $v$ under the action of $y$.

Now associate to each vertex of the coset graph a permutation of the vertices incident to it as in Figure 1.

Then by [12, Theorem 6.47] we have a uniquely defined 2-cell embedding of $\Gamma$ into an orientable surface where the 2-cells have boundary labels $y^n$, with $n$ dividing $k$, or $(xy)^m$, with $m$ dividing $\ell$. Call such 2-cells $y$-faces or $xy$-faces respectively.

The result is called a coset diagram. The picture above shows a $\Gamma$ depicting the action of $\Delta(2,3,3)$ on a set of size six and the resulting diagram embedded in the 2-sphere.

From now we can omit the orientations on the $y$-edges of a coset diagram with the understanding that the boundary edges of each $y$-face are oriented anticlockwise around the face.

Suppose we have an action of some $\Delta$ on a set $S$ such that there are two orbits of $\langle xy \rangle$, each consisting of $|S|/2$ elements. A januarial $J$ is the resulting coset diagram. In particular, a januarial is a diagram with two $xy$-faces. The examples in Figure 1 and 2 are thus januarials. For a nice introduction to januarials, see [1], from which the following concepts are taken.

Let $S_1$ and $S_2$ be the discs that result from taking the closures of the two $xy$-faces.

It is convenient to collapse superfluous structure in a graph or diagram. Let $J$ be a januarial arising from the coset graph $\Gamma$. Collapsing all the $y$-faces to a point gives the companion graph $\Gamma'$ and the companion diagram $J'$. Let $S_1', S_2'$ be the result of applying this to the discs $S_1, S_2$. 

Figure 1: 2-cell embedding using the method described by [12, Theorem 6.47]
Let $g$ be the genus of the surface in which the januarial $J$ is embedded. Let $R_i$ be a small closed neighbourhood of $S'_i$ in $J'$. Then $R_i$ is a closed surface with boundary. Let $g_i$ be its genus and $h_i$ is the number of connected components in the boundary.

The common graph $\Upsilon$ is the intersection $S'_1 \cap S'_2$. The following is in [1, Lemma 3.5]:

**Lemma 1** Let $P_1$ (resp. $P_2$) be the set of all paths that traverse successive edges in $\Upsilon$ in the directions they are traversed by $S'_1$ (resp. $S'_2$) in such a way that whenever such a path reaches a vertex, it continues along the right-most of the other edges in $\Upsilon$ incident with that vertex. (The next edge is necessarily traversed by $S_1$ (resp. $S_2$) in that direction.) All such paths close up into circuits, and $P_1$ (resp. $P_2$) partitions $\Upsilon$, in the sense that the union of the circuits is $\Upsilon$ and no two share an edge. The cardinality of $P_1$ (resp. $P_2$) is $h_2$ (resp. $h_1$).

A januarial is of simple type if $\Upsilon$ is composed of $h$ disjoint simple circuits. In this case $h = h_1 = h_2$ and we denote it by $(h, g_1, g_2)$. Otherwise, the januarial is of general type $((h_1, g_1), (h_2, g_2))$.

The following are Lemmas 3.1, 3.4 and 3.6 of [1]:

**Lemma 2** Twice the genus $g$ of a januarial equals the number $E$ of $x$-edges which are not loops minus the number $V$ of $y$-faces.

**Lemma 3** The genus of a januarial $J$ of simple type is $g = g_1 + g_2 + h - 1$.

**Lemma 4** The $g_i$ satisfy $2 - 2g_i = V_i - E_i + h_i + 1$ where $V_i$ and $E_i$ denote the number of vertices and edges in the subgraph of $\Gamma'$ visited by $S'_i$.

In this paper we first generalize the formula in lemma 3 for obtaining the genus of a januarial of simple type to that of general type. Then we address the questions posed by Graham Higman.
Problem 5 For a given $k$, what are the possible values for and interrelationships between $h, g_1$ and $g_2$ for januarials of simple type? And a similar question for januarial of general type.

Are there arbitrarily large values of $k$ for which there exist examples with $h = 1$?

In [4] we describe a method that constructs a januarial from an action of $\Delta_k$ on the projective line $PL(F_p)$ over the field of order $p$, with the property that $xy$ acts as permutation of $PL(F_p)$ of finite order $\ell$. Necessarily $2\ell = p + 1$ for a januarial. It turns out that there is a polynomial

$$f_\ell = \begin{cases} 
\left(\frac{\ell - 1}{2}\right)\theta^{(\ell-1)/2} - \left(\frac{\ell - 2}{1}\right)\theta^{(\ell-3)/2} + \left(\frac{\ell - 3}{2}\right)\theta^{(\ell-5)/2} - \ldots & \text{for odd } \ell \\
\left(\frac{\ell - 1}{2}\right)\theta^{\ell/2-1} - \left(\frac{\ell - 2}{1}\right)\theta^{\ell/2-3} + \left(\frac{\ell - 3}{2}\right)\theta^{\ell/2-5} - \ldots & \text{for even } \ell
\end{cases}$$

certain roots of which give linear fractional transformations of $PL(F_p)$. For, let $\theta$ be a root of $f_\ell$ that is not a root of $f_{\ell/s}$ where $s \mid \ell$. Let $X, Y$ be the linear fractional transformations of $PL(F_p)$ given by

$$X : z \mapsto \frac{az + cd}{cz - a} \quad \text{and} \quad Y : z \mapsto \frac{ez + fd}{fz + b - e}$$

where $a, b, c, d, e, f \in \mathbb{Z}$, with $\nabla = -(a^2 + dc^2) \neq 0$, $r = a(2e - b) + 2dcf$ and $1 + df^2 + e^2 - eb = 0$ and $\theta\nabla = r^2$.

Then the action of $\Delta_k$ on the $p + 1$ points of $PL(F_p)$ gives a januarial. Write $\bar{x}$ and $\bar{y}$ for the permutations induced by $X$ and $Y$. We say that this januarial has been constructed from the Hecke group $\Delta_k$.

In [1] a technique is proposed for finding januarials. Associates are constructed of known coset diagrams and investigated. Q. Mushtaq and S. Mehwish [4] used the method of parametrization described in [8, 7] to construct januarials from Hecke groups. In this paper we construct januarials for the natural action of some permutation groups that are homomorphic images of triangle groups.

3 Relationship between $h_1, h_2, g_1$ and $g_2$

We now obtain the generalized formula for the genus of a januarial of general type:

Lemma 6 Let $\alpha = v - e$ where $v$ is the number of vertices in $\Upsilon$ and $e$ is the number of edges. Then the genus of a general type januarial $((h_1, g_1), (h_2, g_2))$ is

$$g_1 + g_2 + (h_1 + h_2 + \alpha)/2 - 1.$$
Proof: By Lemma 2 the genus $g$ of a januarial is $g = \frac{1}{2}(E - V)$ and by Lemma 4 $g_i = \frac{1}{2}(E_i - V_i - h_i + 1)$. Since $E = E_1 + E_2 - e$ and $V = V_1 + V_2 - v$, we get $g = \frac{1}{2}(E_1 + E_2 - e - (V_1 + V_2 - v))$. Simple calculations using the above lead to the required result $g = g_1 + g_2 + \frac{1}{2}(h_1 + h_2 + \alpha) - 1$. □

In [4] it is proved that the genus $g$ of januarials constructed from Hecke Groups is given by

$$g = -\frac{p + 1 - \eta_y}{2k} + \frac{1}{4}(p + 1 - 2\eta_y - \eta_x),$$

where $\eta_x$ and $\eta_y$ are the number of fixed points of $x$ and $y$ respectively. This equation implies that $k$ and $p$ are proportional. Moreover, the number of fixed points of $x$ and $y$ are the same for fixed values of $k$ and the prime $p$. Thus:

Lemma 7 The genus of januarials constructed from Hecke groups for fixed values of $k$ and the prime $p$, is a fixed value $g_{pk}$.

From lemmas [3, 4, and 7] we have:

Proposition 8 When a januarial is constructed from a Hecke Group with $k$ and the prime $p$ fixed, we have

1. $g_1 + g_2 + h = g_{pk} + 1$ for januarials of simple type and,

2. $g_1 + g_2 + (h_1 + h_2 + \alpha)/2 = g_{pk} + 1$ for januarials of general type.

That is, for any $p$ and $k$ the sums $g_1 + g_2 + h$ and $g_1 + g_2 + (h_1 + h_2 + \alpha)/2$ are constant in these two cases.

4 Januarials with a simple disjoint circuit

Analyzing januarials constructed from Hecke groups as described at the end of Section 2 it is observed that for higher values of $k$ the chances of getting a januarial with simple disjoint circuits decreases. Moreover the value of $p$ affects the number of circuits in the januarial. Therefore finding a simple januarial with $h = 1$ from a Hecke group becomes almost impossible as we move to higher values of $k$. These remarks are made precise in this section.

Proposition 9 The valency of each vertex of the subgraph $\Upsilon$ of a $k$-januarial is even.

Proof: Each vertex in the subgraph $\Upsilon$ is a $y$-face of the $k$-januarial and the edges are $x$-monogons. Let $E$ be an $x$-monogon between the vertices $U$ and $V$ of $\Upsilon$.

Since $\Upsilon = S'_1 \cap S'_2$, the $x$-monogon $E$ lies both in the restriction of $S'_1$ to $\Upsilon$ and in the restriction of $S'_2$ to $\Upsilon$. If in the restriction of $S'_1$ to $\Upsilon$ the monogon...
Figure 3: A diagram for \( \Delta(17, 17, 8) \). Note that there is an \( x \)-edge connecting the vertices 4 and 13.

\( E \) takes \( U \) to \( V \) then in the restriction of \( S'_2 \) to \( \Upsilon \) we have that \( E \) takes \( V \) to \( U \). In other words \( E \) is unidirectional in both the restrictions.

Now for the vertex \( U \) in the restriction of \( S'_1 \) to \( \Upsilon \) the monogon \( E \) is an outgoing edge for some circuit in \( P_1 \). There must then be a unidirectional incoming edge \( E' \) at \( U \) in that circuit.

This implies that at any vertex of \( \Upsilon \), incident edges exist pairwise. So the valency of each vertex of the subgraph \( \Upsilon \) of a \( k \)-januarial is even.

\[ \square \]

**Theorem 10** Every 3-januarial is of simple type.

**Proof:** By Proposition 9 at any vertex of the subgraph \( \Upsilon \) of a 3-januarial only two edges can be incident while moving along the orbit: one pointing towards the vertex and the other pointing away from the vertex. So only disjoint simple circuits will be formed in the subgraph \( \Upsilon \), giving it a simple 3-januarial.

\[ \square \]

When \( k \geq 4 \) the likelihood of getting simple \( k \)-januarials decreases expeditiously. For an example we construct an 8-januarial from the Hecke group \( \Delta_8 \) acting on \( PL(F_{17}) \). We have \( \ell = 9 \) and hence

\[ f_9 = 10^4 - 80^3 + 210^2 - 200^1 + 5. \]

The roots of \( f_9 \) yielding januarials are 9, 15 and 16. For an 8-januarial, if we consider the root \( \theta = 9 \) we get the linear fractional transformations

\[ X : z \mapsto \frac{z + 10}{10z - 1} \quad \text{and} \quad Y : z \mapsto \frac{4}{4z + 8}. \]
The action of $X$ and $Y$ on $PL(F_{17})$ gives

$$
\bar{x} = (0, 7)(1, 5)(2, 6)(3, 11)(4, 13)(8, 14)(9, 10, 16)(12, \infty)(15)
$$

$$
\bar{y} = (0, 9, 14, 16, 1, 6, 15, \infty)(2, 13, 8, 12, 11, 4, 3, 7)(10)
$$

$$
\bar{x}\bar{y} = (0, 4, 7, 19, 13, 1, 17, 5, 22, 11, 14, 18)(\infty, 20, 6, 16, 8, 3, 9, 15, 10, 2, 12, 21).
$$

Hence the coset diagram $D(17, 17, 8)$ is given by Figure 3. The two partitions of $\Upsilon$ into circuits is drawn in Figure 4.

From the above example it follows that the action of Hecke groups $\Delta_k$ for larger $k$’s on smaller prime fields do not yield simple januarials. This leads to the question of whether there exist examples of simple januarials for larger values of $k$. If so, then this answers Higman’s question of whether there are arbitrarily large values of $k$ for which there exist examples with $h = 1$:

**Theorem 11** There exist simple januarials with $h = 1$ for all values of $k$.

By taking into consideration the definition of januarials and our requirement that we have one simple disjoint circuit in the common graph, in the proof below we construct the $k$-januarials for even and odd $k$ separately.

**Proof:** Consider the diagram in Figure 5. It is easy to see it as a permutation action of the subgroup generated by

$$
\bar{x} = (1, 3k/2 + 1)(k/2 + 1, k + 1) \text{ and } \bar{y} = (1, 2, \ldots, k)(k + 1, k + 2, \ldots, 2k)
$$
of $S_{2k}$.

This is a $k$-januarial for even values of $k$. The companion graph and the two partitions of the subgraph $\Upsilon$ are shown in Figure 6.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig6.png}
\caption{Figure 6:}
\end{figure}

This shows that the januarial is of simple type with $h = 1$. For odd values of $k$ consider the diagram in Figure 7 which shows a permutation action of $\Delta(2, k, 2k)$ on $4k$ points.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig7.png}
\caption{Figure 7:}
\end{figure}

It is a $k$-januarial and the two faces of $xy$ are labeled by $1 \ 2 \ 3 \ldots \ 2k$ and $2k + 1 \ 2k + 2 \ 2k + 3 \ldots \ 4k$. The companion graph and the two partitions of the subgraph $\Upsilon$ are shown in Figure 8.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig8.png}
\caption{Figure 8:}
\end{figure}
This shows that the januarial is of simple type with $h = 1$. □

These januarials are the result of actions of $\Delta(2, k, \ell)$ on the set of $2k$ and $4k$ points for even and odd values of $k$ respectively, which can easily be extended to larger finite sets in many ways. For example, we can insert $k$-polygons in the diagrams above while maintaining the symmetry. This technique is also used by Q. Mushtaq and F. Shaheen in [6]. Thus, there exist many more januarials with the required condition.

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