Quantum Cohomology of Partial Flag Manifolds and a Residue Formula for Their Intersection Parings

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Abstract

As a generalization of our previous paper [GK], we formulate a residue formula and some simple behaviors of equivariant quantum cohomology applying to compute the quantum cohomology of partial flag manifolds $F_{k_1,\ldots,k_l}$ with a try to give a rigorous definition of equivariant quantum cohomology.

1 Introduction

Recently, the great effort of establishing Gromov-Witten invariants, which arise in Mirror Symmetry of Calabi-Yau manifolds and give quantum cohomology of compact semi-positive symplectic manifolds, is succeeded by Ruan, Ruan-Tian, McDuff-Salamon and Kontsevich [R][RT][MS][K].

But so far, a complete description of quantum ring structure by a linear basis are known only for complex projective space $\mathbb{P}^n$. For Grassmannian, there is an answer as a family of algebra parameterized by generators, Chern classes of the tautological bundle of ordinary cohomology, in papers [W][P] and in [ST] with rigorous explanation of all details. In [GK], like the case of Grassmannian, quantum cohomology of complete flag manifolds is predicted by relations of Borel’s generators introducing equivariant quantum cohomology due to Givental or equivariant Gromov-Witten invariants. The assumption therein to prove the prediction is that the equivariant quantum cohomology of compact Kahler manifolds is well-defined, associative and a weighted-homogeneous ordinary equivariant $q$-deformation of ordinary cohomology.

The advantage of equivariant version of quantum cohomology is its behaviors simply under product, restriction and induction operation which we explain in section 4 (c.f. [GK]).

Using those properties under the assumption, equivariant quantum cohomology is well-defined, and stable maps due to Kontsevich [K], we give a computation of the equivariant quantum cohomology of partial flag manifolds as a family of algebra parameterized by some generators. Our hope is to prove or check all details of a rigorous definition of equivariant quantum cohomology elsewhere. By some observations in residue properties and intersection parings of complete flag manifolds, we derive a general theorem for those of some manifolds which present their quantum cohomology as regular functions on complete intersections, including partial flag manifolds.

1 Actually they did it in different fields of manifolds.
2 Also, some cases of lower dimensional manifolds are known.
Let $F_{k_1,k_2,\ldots,k_l}$ or simply $F$ denote the manifold of partial flags

$$C^{k_1} \subset C^{k_1+k_2} \subset \ldots \subset C^{k_1+\ldots+k_l}$$

in $\mathbb{C}^n$ where $n = k_1 + \cdots + k_l$ and integers $k_i > 0$. To describe the (equivariant) cohomology of it, let us consider, for total Chern class $c = 1 + c_1 + \ldots + c_k$ ($c_i =$i-th Chern class), $k \times k$ matrix $A(c)$ which is

$$
\begin{pmatrix}
c_1 & c_2 & \cdots & c_k \\
-1 & 0 & \cdots & 0 \\
0 & -1 & \cdots & 0 \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & -1
\end{pmatrix}.
$$

Let $A(c^1, c^2, \ldots, c^l) = \text{diag}(A(c^1), \ldots, A(c^l))$ which is

$$
\begin{pmatrix}
A(c^1) & 0 & \cdots & 0 \\
0 & \ddots & \cdots & 0 \\
0 & \cdots & \cdots & A(c^l)
\end{pmatrix}
$$

where $c^i = 1 + c_1^i + \ldots + c_k^i$, is the total Chern class of the vector bundle with fiber $(\mathbb{C}^{k_i}/\mathbb{C}^{k_i-1})^*$ over $F_{k_1,k_2,\ldots,k_l}$ and $c_i$ is the $i$-th Chern class of the universal $U(n)$-bundle over the classifying space $BU(n)$.

The equivariant cohomology algebra $H^*_U(F_{k_1,k_2,\ldots,k_l})$ of flag manifolds $F_{k_1,\ldots,k_l}$ as $U(n)$-spaces is canonically isomorphic to the quotient of the polynomial algebra

$$\mathbb{Z}[c_1, \ldots, c^1_{k_1}, c_1^2, \ldots, c^2_{k_2}, \ldots, c^l_{k_l}, c_1, \ldots, c_n]$$

by the ideal generated by the $n$ coefficients of $f(\lambda) - g(\lambda)$ where $f$ and $g$ are the characteristic polynomials of matrices $A(c^1, \ldots, c^l)$ and $A(c)$ respectively.

For quantum cohomology, consider nonnegative determinant line bundles of fibers $(\mathbb{C}^{k_1+\ldots+k_l})^* = \text{Hom}(\mathbb{C}^{k_1+\ldots+k_l}, \mathbb{C})$ and the duals $q_i$ in $H_{1,1}(F) \cap H_2(F, \mathbb{Q})$ of their first Chern classes.

**Theorem I** The equivariant quantum cohomology of partial flag manifold $F_{k_1,\ldots,k_l}$, where $k_1 + \cdots + k_l = n$, is canonically isomorphic to the quotient of the polynomial algebra

$$\mathbb{Q}[c_1, \ldots, c^1_{k_1}, \ldots, c^l_{k_l}, q_1, \ldots, q_{l-1}, c_1, \ldots, c_l]$$

by the ideal generated by the coefficients of

$$(*) \cdots \det(A(c^1, \ldots, c^l) + C_{k_1,\ldots,k_l}(q) + \lambda) - \det(A(c) + \lambda)
$$

where $C_{k_1,\ldots,k_l}(q)$ is a matrix with 0 entities except $(-1)^{k_i} q_i$ at positions $(k_1 + \cdots + k_{i-1} + 1, k_1 + \cdots + k_{i+1})$ and $-1$ at positions $(k_1 + \cdots + k_i + 1, k_1 + \cdots + k_i + 1)$ for $i = 1, \ldots, l - 1$.

Taking $c_i = 0$ in the above relations, we get the quantum cohomology of partial flag manifolds.
For the volume generating function\footnote{A correlation function in topological \( \sigma \) model.} of partial flag manifolds which is defined by \( \Psi(\alpha) = \sum_N \sum_d \frac{1}{N!} d^{\Psi_d}(\alpha, \ldots, \alpha) \) for \( \alpha \in H^*(X) \) and the notation \( \Psi_d \)\footnote{\( \Psi_d = \tilde{\Phi}_d \) in \([R]\)} in [MS], we formulate a general theorem for some \( G \)-space \( X \).

Ordinary cohomology classes will be called geometric classes.

**Theorem II** Suppose \( X \) be a positive symplectic manifold and \( QH^*_G(X) \) is generated by geometric even degree generators \( p_1, \ldots, p_n \) with only \( n \) algebraically independent relations—or regular sequence in each \( q \)\footnote{\( \Psi_d = \tilde{\Phi}_d \) in \([R]\)}—
\[
\Sigma_1(p, q) - c_1, \ldots, \Sigma_n(p, q) - c_n
\]
where \( q \) are parameters from quantum cohomology and \( c_i, i = 1, \ldots, n \) are \( G \)-characteristic classes in \( H_G^*(pt) = \mathbb{Q}[c_1, \ldots, c_n] \). Then the volume generating function is the global residue integral
\[
\frac{a}{(2\pi\sqrt{-1})^n} \int \frac{\exp(z, p) dp_1 \wedge \cdots \wedge dp_n}{(\Sigma_1(p, q) - c_1) \cdots (\Sigma_n(p, q) - c_n)}
\]
for some suitable number \( a \). In particular, \( a=1 \) for \( U(n) \)-space partial flag manifolds with respect to the relations in theorem I. Hence a partial answer of Gromov-Witten invariants of genus zero is obtained for those generators of cohomology of partial flag manifolds.

When this paper was in preparation, we learned that A. Astashkevich and V. Sadov obtained the same result in computation of quantum cohomology of partial flag manifolds [AS].

**Conventions.** For the sake of simplicity, assume \( H^*(X, \mathbb{Q}) \) denoted by \( H^*(X) \), has even degree cohomology classes. Hence it is convenient to count all dimensions and degrees in complex units—one half of those in ordinary (real) units throughout this paper except when we say even degrees.

**Thanks.** I would like to express my sincere gratitude to my thesis advisor Alexander Givental for teaching me mirror symmetry phenomena and for encouraging me to produce this paper. I would also like to thank Hung-wen Chang for numerous helpful discussions.

## 2 A Construction of Multiplications

Let \( R \ (R') \) be a commutative \( \mathbb{Q} \)-algebra such that unity \( 1 \in \mathbb{Q} \subset \mathbb{Q} \cdot 1 \subset R(R') \) and let \( A \) be a finite dimensional \( R \)-free module with a nondegenerate symmetric \( R \)-valued \( R \)-bilinear pairing \( < \cdot | \cdot > \) and unity \( 1 \in \mathbb{Q} \subset R \subset A \) and \( a \in A \). Suppose also \( A \) has two basis \( \{ w_i \} \) and \( \{ \tilde{w}_i \} \) such that \( < w_i | \tilde{w}_j > = \delta_{i,j} \)

In this set-up, we have the following useful observations.

1) Suppose \( A \) is an \( R \)-algebra with \( ra = ar \) for any \( r \in R \) and \( a \in A \). Then it has an \( R \)-multilinear triple-pairing \( < | \cdot | \cdot > \) defined by
\[
1) \cdots < a | b | c > = < ab | c > .
\]
The invariant property of $R$-valued product so that $A$ becomes a Frobenius algebra\(^5\) over $R$ is equivalent to a condition

$$\sum_i <a|b|w_i><\tilde{w}_i|c> = \sum_i <a|w_i><b|\tilde{w}_i>.$$  

The associativity of algebra $A$ is equivalent to condition

$$\sum_i <a|b|w_i><\tilde{w}_i|c|d> = \sum_i <b|c|w_i><a|\tilde{w}_i|d>.$$  

ii) Suppose $A$ has a $R$-valued triple pairing satisfying 2) and 3), then it becomes an unique Frobenius algebra over $R$ satisfying 1)

iii) Let $f : R \to R'$ be $\mathbb{Q}$-algebra homomorphism such that it induces a Frobenius algebra over $R'$ from $A$ in ii). Then $a \cdot_R b = a \cdot_{R'} b$.

**Example.** Suppose a differentiable map from a compact oriented manifold $X$ to another $Y$ induces a $H^*(Y)$-free algebra $H^*(X)$ with two $H^*(Y)$-basis $\{w_i\}$ and $\{\tilde{w}_i\}$ such that $<w_i|\tilde{w}_j>=\delta_{i,j}$. Then after a forgetting of $H^*(Y)$-module structure, $H^*(Y)$-algebra $H^*(X)$ is $\mathbb{Q}$-algebra $H^*(X)$.

iv) If $f : A \to B$ be a $\mathbb{Q}$-algebra homomorphism from $A$ to $B$, sending $R$ to $R'$, such that the induced map $A \otimes_R R' \to B$ is an $R'$-linear isomorphism, then they are isomorphic as $R'$-algebras.

**Example.** From maps between compact oriented manifolds

$$
\begin{array}{ccc}
X & \leftarrow & X' \\
\downarrow & & \downarrow \\
Y & \leftarrow & Y'
\end{array}
$$

we have maps between rings

$$
\begin{array}{ccc}
A = H^*(X) & \longrightarrow & B = H^*(X') \\
\downarrow & & \downarrow \\
R = H^*(Y) & \longrightarrow & R' = H^*(Y')
\end{array}
$$

Then $H^*(X')$ as $H^*(Y')$-algebra is isomorphic to $H^*(X) \otimes_{H^*(Y)} H^*(Y')$ if the condition in iv) is satisfied.

v) Consider Frobenius algebras $A$ and $B$ with a $\mathbb{Q}$ linear map $f : A \to B$. If $R'f(A) = B$, $f(R) \subset R'$ and

$$
\begin{array}{ccc}
A^\otimes N & \xrightarrow{f^\otimes N} & B^\otimes N \\
\downarrow & & \downarrow \\
R & \xrightarrow{f} & R'
\end{array}
$$

\(^{5}\text{See [D] for definition.}\)
is commutative for $N = 2, 3$ where vertical maps are pairings, then $f$ is a $\mathbb{Q}$-algebra homomorphism.

Proof.

$$< f(a)f(b)|w > = < f(a)f(b)|\sum_i r_i'f(c_i) > = \sum_i r_i'< f(a)|f(b)f(c_i) >$$

$$= \sum_i r_i'< a|b|c_i > = \sum_i r_i'< f(ab)|f(c) >= < f(ab)|w >.$$

\[\square\]

3 Quantum Cohomology

3.1 Naive definition. Let $X$ be a compact connected Kahler manifold with the positive first Chern class $c_1(TX)$ of the tangent bundle. As an $\mathbb{Q}$-free module, the quantum cohomology of $X$, by definition, is the tensor product of ordinary cohomology and $\mathbb{Q}[q]$ where $q = (q_1, \ldots, q_s)$ is a basis of the space $Q(X)$ generated by the closed Kahler cone of $X$, which is in $H^{1,1}(X) \cap H^2(X, \mathbb{Z})$. But their ring structure is quite different from ordinary one $H^*(X)$ which can be defined, as in the context of the preceeding section, by the intersections of three cocycles $x, y$ and $w$ with the Poincare pairing $< \cdot, \cdot > = \int_X \cdot \wedge \cdot$:

$$< x \cdot y, w > = < x, y, w > \text{ by notation}$$

$$= a \cap b \cap c$$

$$= \int_X x \wedge y \wedge w$$

where cycles $a, b$ and $c$ are denoted the Poincare dual of $x, y$ and $w$.

In more detail, for the diagonal class $\sum_i \alpha_i \otimes \beta_i \in H^*(X \times X) = H^*(X) \otimes H^*(X)$, which is Poincare-dual to the homology class of the diagonal $X \subset X \times X$, the product is defined by

$$x \cdot y = \sum_i < x, y, \alpha_i > \beta_i.$$

To give instanton corrections, let us think of the space $\mathcal{M}_f$ of holomorphic maps from $\mathbb{C}P^1$ to $X$ with given type $d = (d_1, \ldots, d_n) \in H_2(X, \mathbb{Z}) \cap H_{1,1}(X)$ with respect to a coordinate $q_i$. Then the ring structure of quantum cohomology $QH^*(X, \mathbb{Q})$ as a $\mathbb{Q}[q]$-algebra is, by definition, given by a triple product

$$< x \ast y, w > = < x|y|w > \text{ by notation}$$

$$= \sum_d \varphi \in \mathcal{M}_f \text{ such that } \varphi(t) \in \varphi(\infty) \text{ and } \varphi(\infty) \in ] \pm q^d$$

$$= \sum_d q^d \int_{\mathcal{M}_f} \varphi_0^*(x) \wedge \varphi_1^*(y) \wedge \varphi_{\infty}^*(w)$$

where $\varphi_0, \varphi_1$ and $\varphi_{\infty}$ are evaluation maps from $\mathcal{M}_f$ to $X$ at $0, 1$ and $\infty$. 5
3.2 Remarks. The difficulties of the rigorous definition of quantum cohomology come from when the $\mathbb{C}$-dimension of the moduli space $\mathcal{M}_f$ is $c(d) + \dim X$ as predicted by the Riemann-Roch theorem where $c(d)$ is the pull back of the first Chern class of $X$ by a holomorphic map in $\mathcal{M}_f$, giving a nice compactification of the moduli space $\mathcal{M}_f$ and showing associativity which was highly nontrivial and overcome by Ruan and Tian for semi-positive symplectic manifolds using an inhomogeneous term in Cauchy-Riemann equation [RT]. Recently also McDuff and Salamon [MS] give a complete definition by Gromov compactification. There is a beautiful algebro-geometric approach achieved by Kontsevich and Manin [KM][M] for projective manifolds and analytic symplectic manifolds using stable maps, stable curves and vertical fundamental classes. Here we will follow the moduli space defined by Kontsevich and Manin [KM] since, according to Kontsevich [K], the moduli space of stable maps to convex manifolds—by definition, they are manifolds such that for any stable map $f$, $H^1(C, f^*(TX)) = 0$—are compact and smooth as stacks* and their motivic axiom seems easily applied to equivariant version. So, throughout the paper, we consider a convex manifold $X$ and the moduli space $\overline{\mathcal{M}}_{0,3}(X, d)$ of stable maps in $[M]$ which is the collection of all algebraic map from a connected compact reduced curve $C$ with genus zero, marked three distinct points $x_1, x_2, x_3$ or symbolically $0, 1, \infty$ and at most ordinary double singular points to $X$ such that every irreducible component of $C$ which maps to a point must have at least 3 special (i.e. marked or ordinary double singular) points on its normalization.

Conventions. For shorthand, $\overline{\mathcal{M}}_d$ and $\bar{a}_i$ will be denoted $\overline{\mathcal{M}}_{0,3}(X, d)$ and $\varphi_{x_i}(a_i), a_i \in H^*(X)$ respectively. Note that $\overline{\mathcal{M}}_0 = X$ so that we get a “$q$-deformation” ring structure.

3.3 Volume generating functions. For even degree basis or generators $p_1, \cdots, p_M$ of ordinary cohomology ring, we can define the volume generating functions $\Psi(z_1, \cdots, z_M)$ introduced by Givental [GK];

$$\Psi(z_1, \cdots, z_M) = \int_X \exp(z_1 p_1 + \cdots + z_M p_M)$$

where exponentials are obtained from quantum multiplications in $X$. Then, we have another equivalent definition of quantum cohomology of $X$ by $\mathbb{Q}[p, q]$ modulo an ideal generated by polynomials $R(p, q)$ such that $R(\frac{\partial}{\partial z}, q)\Psi(z, q) = 0$. We shall see why they are equivalent definitions in subsection 5.5.

3.4 Grading. If we give a degree of $q$ the first Chern class of the pull back bundle over $\mathbb{C}P^1$ of the tangent bundle of $X$ by a holomorphic map represented by $q$, then $QH^*(X)$ will be a $\mathbb{Z}$-graded algebra.

4 Equivariant quantum cohomology

4.1 Definition. The classical equivariant cohomology $H^*_G(X)$ of a manifold $X$ on which $G$ acts is the ordinary cohomology of the homotopy quotient $X_G$—or $X^G$ later for partial flag manifolds—of $X$ and $EG$, i.e. $X \times_G EG$ as a $H^*(BG)$-module. In greater detail, consider the associated fiber bundle $X_G \to BG$ with fiber $X$ from a fixed universal principal $G$-bundle $EG \to BG$. Then the ordinary cohomology of $X_G$ as a $H^*(BG) = H^*_G(pt)$-module by the pull back of the

*For example, homogeneous projective varieties are convex.
projection $\pi$ is called the equivariant cohomology of $G$-space $X$. Suppose $H^*_G(X)$ is a $H^*(BG)$-free module canonically isomorphic to $H^*(X) \otimes H^*_G(pt)$, then we may consider the ordinary equivariant cohomology as a Frobenius algebra over coefficients $H^*_G(pt)$ provided with equivariant $N$-intersection $H^*_G(pt)$-indices, namely for $p_1, \ldots, p_n \in H^*(X_G)$—$H^*(BG)$-module—and a cycle $C$ in $BG$, $< p_1, \ldots, p_N > [C] = (p_1 \cdots p_N)[\pi^{-1}(C)]$ which is $H^*_G(pt)$-multi-linear.\footnote{I have learned this method only from Givental.}

Recall some facts on equivariant cohomology to fully understand our definition [AB][G][GK].

i) If $G$-action on $X$ is Hamiltonian for a symplectic form or a Kahler form, then $H^*_G(X)$ is a $H^*(BG)$-free module isomorphic to $H^*_G(pt) \otimes H^*(X)$ so that $H^*_G(X)$ is canonically isomorphic to $H^*_G(pt) \otimes H^*_G(pt)$ $H^*_G(X)$ for a Lie subgroup $G' \subset G$.

ii) An $G$-action from an algebraic $G_C$-action on a complex projective manifold is always Hamiltonian.

iii) $H^*_G(X) \overset{\cong}{\longrightarrow} H^*_G(pt) \otimes H^*(X)$

$\pi \downarrow$

$H^*_G(pt) \overset{\cong}{\longrightarrow} H^*_G(pt) \otimes H^*(pt)$

integration over $X$

is commutative where $\pi \downarrow$ is denoted the push forward of $\pi : X_G \to BG$.

iv) The pairing

$H^*_G(X) \otimes H^*_G(X) \overset{\text{product}}{\longrightarrow} H^*_G(X) \overset{\text{push forward}}{\longrightarrow} H^*_G(pt)$

is nondegenerate and its associated determinant is in scalar $\mathbb{Q}$.

Suppose $X$ is a compact convex Kahler manifold provided with a holomorphic action of compact Lie group $G \subset G_C$. The additive structure of equivariant quantum cohomology $QH^*_G(X)$ of $G$-space $X$ is $H^*_G(X) \otimes \mathbb{Q}[q]$ where $q$ is a basis of $Q(X)$.

Then the quantum ring structure of $QH^*_G(X)$ as $H^*_G(pt) \otimes \mathbb{Q}[q]$-module is, by definition [GK], given by triple products of three cycles $a, b$ and $c$ and their Poincare-duals $x, y$ and $w$;

$< x *_{X_G} y, c > [C] = < x | y | w > [C]$ by notation

$= \sum_{\phi \in \mathcal{M}_d} \sum_{\phi(0) = 0, \phi(\kappa) \in [1]} \pm q^d$

$= \sum_d q^d \int_{\mathcal{M}_d} \varphi_0^*(x) \wedge \varphi_1^*(y) \wedge \varphi_\infty^*(w)$

where $a, b$ and $c$ are finite co-dimension cycles in $X_G$ and $\mathcal{M}_d[C]$ is the space of all stable maps from from stable curves of genus zero marked three points $0, 1, \infty$ to $X_G$ of type $d$ such that their image under the projection $\pi$ are points at the finite dimensional cycle $C$ in $BG$—let us call them vertical stable maps. Notice that $H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ is a sublattice $\subset H^2(X_G)$ so that we can canonically count types of vertical stable maps by $d$.\footnote{I have learned this method only from Givental.}
The diagrams

\[
\overline{\mathcal{M}}_d(X)_G = \text{Claim} \quad \overline{\mathcal{M}}_d(X_G) \xrightarrow{\pi^d} BG \\
H^*(\overline{\mathcal{M}}_d(X)_G)^{\otimes 3} \xrightarrow{\pi^d} H^*_G(pt)
\]

should give an almost complete picture for our definition, where \( \overline{\mathcal{M}}_d(X)_G \) is the homotopic quotient of the moduli space \( \overline{\mathcal{M}}_d \) of genus zero stable maps of genus zero and with three marked points 0, 1, \( \infty \) to \( X \). Here we assume that the above picture gives an associative equivariant (quantum cohomology) ring structure.

In [GK] simple behaviors of volume generating functions formulated. But here we would like to point out the simple behaviors of equivariant quantum cohomology:

4.2. It is a \( H^*_G(pt) \otimes \mathbb{Q}[q] \subset QH^*_G(X) \)-module.

4.3 Product. Let \( X' \) and \( X'' \) be compact Kahler \( G' \)-and \( G'' \)-spaces respectively, then the quantum cohomology of \( G' \times G'' \)-space \( X' \times X'' \) is the tensor product of those of \( X' \) and \( X'' \) since \( H^*_G(pt) \otimes H^*_{G'}(pt) \) is canonically \( H^*_{G' \times G''}(pt) \).

4.4 Restriction. Let \( X \) be a compact Kahler \( G \)-space and \( G' \subset G \) be a Lie subgroup. Considering \( X \) as a \( G' \)-space, we obtain an \( X \)-bundle \( X' \to BG' \) induced, as a bundle, from \( X \to BG \) by means of the natural map \( \pi : BG' \to BG \) of classifying spaces and corresponding map of total spaces \( \zeta : X' \to X \) with the fiber \( G/G' \). In this setup we have the following lemma:

**Lemma** \( QH^*_G(X) \) is canonically isomorphic to \( H^*_G(pt) \otimes H^*_G(X) \) as \( H^*_G(pt) \)-algebras.

**Proof.** It is enough to show that the induced \( \mathbb{Q} \)-linear map \( \zeta^* \) is a quantum ring homomorphism by iv) in section 2 and i) in 4.1. From v) in section 2, we only need to prove that

\[
\begin{array}{ccc}
QH^*_G(X)^{\otimes 3} & \xleftarrow{\zeta^* \otimes 3} & QH^*_G(X)^{\otimes 3} \\
3\text{-intersection index} & & 3\text{-intersection index} \\
H^*_G(pt) \otimes \mathbb{Q}[q] & \xleftarrow{\zeta^*} & H^*_G(pt) \otimes \mathbb{Q}[q]
\end{array}
\]

is commutative. But it follows because for \( a_1, a_2, a_3 \in H^*_G(X) \) and corresponding forms \( \tilde{a}_1, \tilde{a}_2, \tilde{a}_3 \in H^*_G(\overline{\mathcal{M}}_d) = H^*(\mathcal{M}_G) \),

\[
\pi_1(\zeta^* a_1 \zeta^* a_2 \zeta^* a_3) = \sum_d q^d \pi_1(\zeta^* a_1 \zeta^* a_2 \zeta^* a_3) \\
= \sum_d q^d \pi_1(\zeta^* (\tilde{a}_1 \tilde{a}_2 \tilde{a}_3)) \\
= \sum_d q^d \zeta^* \overline{d}(\tilde{a}_1 \tilde{a}_2 \tilde{a}_3) \\
= \zeta^* \overline{p}(a_1 a_2 a_3)
\]

*One might convince oneself by the functorial nature of equivariant version.
where \( \pi_d \) and \( p_d \) are push forwards as in a commutative diagram

\[
\begin{align*}
H^*(\overline{M}_d(X)_{G'}) & \leftarrow H^*(\overline{M}_d(X)_{G}) \\
\pi_d \uparrow & \downarrow \\
H^*_G(pt) & \leftarrow H^*_G(pt).
\end{align*}
\]

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5 Computation

In this section $G = U(n)$, $X = F_{k_1, \ldots, k_l}$, $n = k_1 + \ldots + k_l$ and $k = k_1 + \ldots + k_l$.

5.1. First, let us specify a basis $\{q_i\}$ of $H_{1,1}$ of the flag manifold by the duals of first Chern classes of the duals of the determinant line bundles with fibers $\mathbb{C}^{k_1 + \ldots + k_i}$. Since partial flag manifolds $F_{k_1, \ldots, k_l}$ are $U(n)$-Kahler manifolds, $F_{k_1, \ldots, k_l}^U(n) = F_{k_1, \ldots, k_i}^U(n-k) \times F_{k_i+1, \ldots, k_l}^U(n-k)$, integral homology of $H_{1,1}(F_{k_1, \ldots, k_l}) = \mathbb{Z}q_1 \oplus \ldots \oplus \mathbb{Z}q_{i-1}$, integral homology of $H_{1,1}(F_{k_i+1, \ldots, k_l}) = \mathbb{Z}q_{i+1} \oplus \ldots \oplus \mathbb{Z}q_l$ and they have the simple behavior of the equivariant cohomology, i.e. they are $H_{U(n)}^*(pt, \mathbb{Z})$-free module canonically isomorphic to $H_{U(n)}^*(pt) \times H^*(X)$, we can state

** Proposition 1.** Restriction to identity group.

$$QH^*(F_{k_1, \ldots, k_l}) \cong QH_{id}^*(F_{k_1, \ldots, k_l}) \cong QH_{U(n)}^*(F_{k_1, \ldots, k_l}) / < H_G^+(pt) >$$

where $id$ is the trivial group and $< H_G^+(pt) >$ is the ideal generated by positive degree elements.

2. Product.

$$QH_{U(n)}^*(F_{k_1, \ldots, k_i} \times F_{k_i+1, \ldots, k_l}) \cong QH_{U(n)}^*(F_{k_1, \ldots, k_i}) \otimes QH_{U(n)}^*(F_{k_i+1, \ldots, k_l}).$$

3. Induction. As $H_{U(n)}^*(pt)$-algebras,

$$QH_{U(n)}^*(F_{k_1, \ldots, k_l}) \otimes \mathbb{Q}[q_1, \ldots, q_l] \cong QH_{U(n)}^*(F_{k_1, \ldots, k_i} \times F_{k_i+1, \ldots, k_l})$$

where $q_i$ is dropped.

5.2 Theorem. ([W][P][ST]) For Grassmannian $G(n, k) = F_{k,n-k}$ the theorem in the introduction holds.

** Proof.** Since ordinary equivariant cohomology has relations $\det A_{k,n-k}(c_1^1, c_2^2) = \det A_n(c)$ and we have quantum correction only on top degree, namely $c_k^1 c_{n-k}^2 = (-1)^k q$ by Witten [W], the proof follows. □

5.3 Lemma. (c.f. [GK]) For $l > 2$, suppose that a quasi-homogeneous relation of the form

$$(\lambda^{k_1} + c_1^1 \lambda^{k_1-1} + \ldots + c_{k_1}^1) \cdots (\lambda^{k_l} + c_1^l \lambda^{k_l-1} + \ldots + c_{k_l}^l) - (\lambda^n + c_1^l \lambda^{n-1} + \ldots + c_n)$$

is satisfied in the quantum equivariant cohomology algebra of partial flag manifolds $F_{k_1, \ldots, k_l}$ modulo $q_i$ for each $i = 1 \cdots l - 1$. Then this relation holds identically (i.e. for all $q$).

** Proof.** Since each line bundles $\bigwedge^* \mathbb{C}^{k_1 + \ldots + k_l}$ over the flag manifold is the induced bundle of the dual of the determinant of universal bundle over Grassmannian $G(n, k)$, we have deg $q_i = k_i + k_{i+1}$.  

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Hence LHS-RHS of the above equation is divisible by \(q_1 \cdots q_{l-1}\) of degree \(> n\) which completes the proof. \(\square\)

### 5.4 Proof of relations in theorem I.

We shall use induction on \(l\) in \(F_{k_1, \ldots, k_l}\). When \(l=1\), it is the case of Grassmannian which has shown in theorem 5.2. When \(l > 1\) it suffices to show by lemma 5.3 that the relation \((*)\) in theorem I from the introduction holds modulo each \(q_i\) in order to get the relation. But this follows from statements 2 and 3 in the proposition and induction hypothesis. The statement 1 in proposition yields relations of the non-equivariant version of theorem I.

### 5.5 No more relations.

Using the induction on degree, one can prove that \(\{c_{11}, \ldots, c_{k_11}, \ldots, c_{1l}, \ldots, c_{kl}\}\) generate the equivariant quantum cohomology and there are no more relations counting ranks over \(\mathbb{Q}\) (see [BT][ST] for detail). One may use a formula for Poincare series to count ranks over \(\mathbb{Q}\) explicitly which depend only on degrees of generators and polynomial relations (see [BT] for detail). Now we are done.

### 6 Residue formulas for volume generating functions

#### 6.1 Proof of theorem II for non-equivariant case.

Let \(X\) be a compact positive Kahler manifold with \(H^*(X)\) generated by even degree elements \(x_1, \ldots, x_n\). Suppose \(\mathbb{C}[x, q]\) modulo weighted homogeneous polynomials \(f_1(x, q), \ldots, f_n(x, q)\) presents the quantum cohomology of some manifold where \(\deg c_i = d_i\) which is the half of the usual one and \(f = f(x, q), \ldots, f_n(x, q)\) is a regular sequence (or equivalently defines a complete intersection) for each parameter \(q\)—for each complex value of \(q\)—and \(x = (x_1, \ldots, x_n) q = (q_1, \ldots, q_s)\). Then we have the following properties of residues and regular sequences in graded rings [BT][GH][T].

- Global Duality says the pairing \((\varphi, \phi) = \text{Res}_f(\varphi\phi) = \int \frac{\varphi \phi dx_1 \cdots dx_n}{f_1 \cdots f_n}\) is nondegenerate on the ring \(\mathbb{C}[x]/\langle f \rangle\), where \(\langle f \rangle\) is the ideal generated by \(f_1, \ldots, f_n\).

- Global Residue Theorem on weighted projective space \(\mathbb{P}^n_{d_1, d_2, \ldots, d_n}\), which is a variety so that the singular set is of co-dimension at least 2 and on which Stokes' Theorem can be applied, implies \(\text{Res}_f(g) = 0\) if \(\deg g \leq \deg f - \sum_i d_i - 1\), \(\deg f = \sum \deg f_i\).

- \(\text{Res}_f(g) = 0\) if \(\deg g \geq \dim X\) since \(f\) is computed from the cohomology of a manifold.

- \(\dim \mathbb{C}X = \deg f - \sum_i d_i\) since \(f\) is a regular sequence.

- Residues algebraically depend on parameters \(q\).

- Since \(\text{Res}(\frac{df}{f})\) are integers, \(\text{Res}(\frac{df}{f}) = \text{Res}(\frac{df(x,0)}{f(x,0)}) = \text{local Res}(\frac{df(x,0)}{f(x,0)})\) at zero which is \(\dim \mathbb{C}O/f(x,0) = \dim \mathbb{C}[x]/f(x,0)\) = Euler number of \(X\) where \(O\) is the local ring of holomorphic functions at \(0 \in \mathbb{C}^n\).

- Computing the rank of a graded ring by a method in [BT],

\[
\prod_i \frac{\deg f_i}{\deg x_i} = \dim \mathbb{C}[x]/\langle f(x,0) \rangle
\]

which is the Euler number of \(X\).
Thus, we conclude that the evaluation map $\langle \cdot \rangle$, which is the integration over $X$, on the quantum cohomology of $X$ is nothing but the residue map with respect to $f$ up to a constant multiplication—note that $\langle \frac{\partial f(x,q)}{\partial x} \rangle$ does not depend on $q$ by degree counting where $\left| \frac{\partial f}{\partial x} \right|$ is the determinant of the Jacobian of $f = (f_1, \ldots, f_n)$ with respect to $x = (x_1, \ldots, x_n)$.

6.2 Computing the constant term. First, notice that $\Sigma_1, \ldots, \Sigma_n$ are algebraically independent—which is a result of a well-known fact that the elementary symmetric polynomials are algebraically independent—so that they form a regular sequence for each $q$. To compute the constant for partial flag manifolds and $\Sigma_1, \ldots, \Sigma_n$, i.e. to prove Theorem II for non-equivariant cases of partial flag manifolds, let us consider the following obvious fiber bundles and the corresponding integrations along fibers

$$
\begin{array}{ccc}
F(n) & F_{k_1} \times \cdots \times F_{k_l} & H^*(F(n)) \\
\downarrow & \downarrow \pi_* & \downarrow <>
\\
\text{pt} & \text{pt} & \mathbb{C}
\end{array}
$$

where $F(n) = F_{1, \ldots, 1}$. Let use $u = (u_1, \ldots, u_n)$ instead of $c_1, \ldots, c_n$’s for complete flag manifolds. We know $n! = < |\frac{\partial f}{\partial u}| >'$ by the residue formula of volume generating functions at [GK]. $n! = < |\frac{\partial f}{\partial c_1}| > \pi_* |\frac{\partial c_1}{\partial u}| = k_1! \cdots k_l! < |\frac{\partial f}{\partial c_l}| >$; consequently, $< |\frac{\partial f}{\partial c_l}| > = \text{Euler number of } F_{k_1, \ldots, k_l}$ and Theorem II for non-equivariant cases.

6.3 Proof of theorem II. Let $V$ be the global residue

$$\frac{a}{(2\pi \sqrt{-1})^n} \int \frac{\exp(z,p)dp_1 \wedge \cdots \wedge dp_n}{(\Sigma_1(p,q) - c_1) \cdots (\Sigma_n(p,q) - c_n)}$$

and let $\Psi$ be the volume generating function of $X$ with respect to even degree generators $p_1, \ldots, p_n$. Then $\Psi = V$ if $c_1 = \ldots = c_n = 0$ by subsection 6.1. But since $(\Sigma_i(\frac{\partial}{\partial z}, q) - c_i)V(Psi) = 0$ for $i = 1, \ldots, n$ and $V$ is holomorphic on variable $c_i$ by a well-known property—trace formula—of residues, $V$ is determined by leading terms in expansion with respect to $c_1, \ldots, c_n$ and hence equal to $\Psi$.

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