On time-dependent quasi-exactly solvable problems

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Abstract

In this paper we demonstrate that there exists a close relationship between quasi-exactly solvable quantum models and two special classes of classical dynamical systems. One of these systems can be considered a natural generalization of the multi-particle Calogero-Moser model and the second one is a classical matrix model.

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1 Introduction

A quantum mechanical model is called *quasi-exactly solvable* (QES) if a finite number of energy levels and the corresponding wavefunctions of the model can be constructed explicitly. One possible way of studying the QES models and their solutions in the one-dimensional case is based on a reformulation of the spectral equations in terms of the wavefunction zeros. It can be shown that, in all the cases when the number of wavefunction zeros is finite (and this is just the case of QES models), the problem of reconstructing solutions of the stationary Schrödinger equation becomes purely algebraic and is reduced to the determination of their positions in the complex plane. These positions (which hereafter we shall denote by $\xi_i$) obey the system of algebraic equations

$$\sum_{k=1, k \neq i}^{M} \frac{\hbar}{\xi_i - \xi_k} + F(\xi_i) = 0, \quad i = 1, \ldots, M \tag{1}$$

where $F(\xi)$ is a rational function in which all the information of a QES model is contained. In turns out that equations of the type (1) are typical not only for QES models but appear in many branches of mathematical physics and this fact enables one to establish a deep relationship between QES models and many other seemingly unrelated models. For example, QES models turn out to be equivalent to completely integrable Gaudin spin chains \[1, 10\] (for which system (1) plays the role of the Bethe ansatz equations for the coordinates of elementary spin excitations), to random matrix models \[4, 5\] (for which system (1) determines the distribution of eigenvalues of large random matrices), and also to purely classical models of 2-dimensional electostatics \[1, 2, 6\] respectively hydrostastics of point vortices \[3\]. For both the latter models system (1) determines the equilibrium positions of pointwise classical objects (the charged Coulomb particles or resp. vortices) in an external (electrostatic or resp. hydrostatic) field.

Up to now only the stationary solutions of QES Schrödinger equations have been considered from this point of view. The aim of the present paper is to consider time dependent solutions for QES models and to show how they can equivalently be described as dynamical equations for the motion of wavefunction zeros \[4\]. This enables one to reveal three classical (complex) dynamical systems closely related to quantum QES models. One of these classical systems does not have an immediate physical interpretation, but the two remaining ones are very interesting from both the physical and mathematical point of view. The point is that one of these two sys-

\[4\] We mean here just the solutions of evolution equations for standard QES models with time-independent potentials. Note that QES models with time-dependent potentials were discussed (from different point of view) in paper [1].
tems is a natural generalization of the famous classical Calogero-Moser multi-particle system and the second one is a classical matrix model from which the first system can be obtained by means of Olshanetsky and Perelomov’s projection method [13].

Strictly speaking, the idea of studying classical dynamical systems describing the motion of zeros of solutions of linear differential equations belongs to Calogero and is far from being new. Papers devoted to the investigation of such systems appear in the literature rather frequently. The goal of our paper is to apply Calogero and Olshanetsky-Perelomov methods to a concrete class of quantum QES models and to derive the associated classical models. The most interesting and quite unexpected result which we intend to present here is that the potentials of the resulting classical matrix models turn out to almost coincide with the potentials of the initial quantum QES ones (up to terms of order $\hbar$). This fact may hint to the existence of a certain non-standard “quantization procedure” relating the classical multi-particle systems of Calogero-Moser type to QES models of one-dimensional quantum mechanics.

The paper is organized as follows: in Section 2 we remind the reader of the basic facts concerning the simplest QES model in the stationary case. In Section 3 we derive the explicit form of solutions for this model in the time-dependent case and the corresponding system of evolution equations for the wavefunction zeros. In Section 4 we show that solutions of this system can be considered as “soliton like solutions” of a multi-particle classical system of the Calogero-Moser type. The matrix version of this system is discussed in Section 5. In Section 6 we discuss the limiting case when the number of algebraically calculable states in the QES model tends to infinity. The last section is devoted to a discussion of our results.

2 The simplest QES model

The simplest QES model is the sextic anharmonic oscillator. Its potential has the following form

$$V(x) = \frac{x^2}{2} (ax^2 + b)^2 - \hbar a \left( M + \frac{3}{2} \right) x^2,$$

where $a$ and $b$ are real parameters with $a > 0$, and $M$ is an arbitrary non-negative integer. It can be shown [8] that for any given $M \geq 0$, the stationary Schrödinger equation

$$\left( -\frac{\hbar^2}{2} \frac{d^2}{dx^2} + V(x) \right) \Psi(x) = E \Psi(x)$$

for model (2) admits solutions of the form

$$\Psi(x) = \prod_{i=1}^{M} (x - \xi_i) \exp \left[ -\frac{1}{\hbar} \left( \frac{bx^2}{2} + ax^4 \right) \right],$$
\[ E = \hbar b \left( M + \frac{1}{2} \right) + \hbar a \sum_{i=1}^{M} \xi_i^2, \]  

(5)

where the numbers \( \xi_i, i = 1, \ldots, M \) (playing the role of the wavefunction zeros) satisfy the system of numerical equations

\[ \sum_{k=1, k \neq i}^{M} \frac{\hbar}{\xi_i - \xi_k} = b \xi_i + a \xi_i^3, \quad i = 1, \ldots, M \]

(6)

with the following additional condition

\[ \sum_{i=1}^{M} \xi_i = 0. \]

(7)

Note that the form of system (6) coincides exactly with that of the famous Bethe Ansatz equations appearing in the theory of completely integrable Gaudin models \[9, 10\]. Therefore we hereafter will call (6) the Bethe Ansatz equations.

It is not difficult to show that for any fixed \( M \) the Bethe Ansatz equations (6)-(7) have

\[ N_{\text{sol}} = \left[ \frac{M}{2} \right] + 1 \]

(8)
solutions. This means that for the sextic anharmonic oscillator we can find \( N_{\text{sol}} \) wavefunctions \( \Psi_i(x) \) and \( N_{\text{sol}} \) corresponding energy levels \( E_i \) by means of purely algebraic methods.

Note also that model (2) can be considered a deformation of the simple harmonic oscillator with the potential

\[ V_0(x) = \frac{b^2 x^2}{2}. \]

(9)

The role of the deformation parameter is played by \( a \). It is not difficult to see that after taking \( a = 0 \) in formulas (4)-(6) they reduce to the well known formulas describing the solution of the simple harmonic oscillator in terms of wavefunction zeros \[1, 12\]. In this case the number \( M \) can be considered a free parameter because it is not any longer correlated with the form of the potential.

### 3 Evolution equations

Let us next consider the time-dependent Schroedinger equation

\[ i\hbar \frac{\partial \Psi(x,t)}{\partial t} = \left\{ \frac{\hbar^2}{2} \frac{\partial^2}{\partial x^2} + V(x) \right\} \Psi(x,t). \]

(10)
for potential (2). It is obvious that any linear combination of the stationary solutions

\[ \Psi(x, t) = \sum_{n=1}^{N_{sol}} c_n \Psi_n(x) e^{-iE_n t/h} = \]

\[ = \left( \sum_{n=1}^{N_{sol}} c_n \prod_{i=1}^M (x - \xi_i) e^{-iE_n t/h} \right) \exp \left[ -\frac{1}{\hbar} \left( \frac{bx^2}{2} + \frac{ax^4}{4} \right) \right] \] (11)
gives a certain dynamical solution of equation (10). Remember that the pre-exponential factor in (4) is always a polynomial of degree \( M \). Therefore also the pre-exponential factor in (11) is a certain polynomial and hence its zeros are functions of time. This enables one to write

\[ \Psi(x, t) = C(t) \prod_{i=1}^M (x - \xi_i(t)) \exp \left[ -\frac{1}{\hbar} \left( \frac{bx^2}{2} + \frac{ax^4}{4} \right) \right]. \] (12)

Formula (12) enables one to derive the evolution equations for the functions \( \xi_i(t) \) and \( C(t) \). For this it is convenient to rewrite equation (10) in the form

\[ V(x) = i\hbar \frac{\partial}{\partial t} \ln \Psi(x, t) + \frac{\hbar^2}{2} \left\{ \left( \frac{\partial}{\partial x} \ln \Psi(x, t) \right)^2 + \frac{\partial^2}{\partial x^2} \ln \Psi(x, t) \right\}. \] (13)

Substituting expressions (2) and (12) into equation (13) we obtain after some algebra the relation

\[ i\hbar \frac{C(t)}{C(t)} - \frac{h}{M} \sum_{i=1}^M \xi_i^2(t) - hax \sum_{i=1}^M \xi_i(t) - \hbar b \left( M + \frac{1}{2} \right) + \]

\[ - \sum_{i=1}^M \frac{\hbar}{x - \xi_i(t)} \left( i\dot{\xi}_i(t) - \sum_{k=1, k \neq i}^M \frac{\hbar}{\xi_i(t) - \xi_k(t)} \right) + b\xi_i(t) + a\xi_i^3(t) = 0 \] (14)

from which it immediately follows that the functions \( \xi_i(t) \) must satisfy the following system of first order differential equations

\[ i\dot{\xi}_i(t) = \sum_{k=1, k \neq i}^M \frac{\hbar}{\xi_i(t) - \xi_k(t)} - b\xi_i(t) - a\xi_i^3(t), \quad i = 1, \ldots, M \] (15)

supplemented by the condition

\[ \sum_{i=1}^M \xi_i(t) = 0. \] (16)

For the function \( C(t) \) we obtain on the other hand

\[ C(t) = \exp \left[ -i \left( a \sum_{i=1}^M \int \xi_i^2(t) dt + b \left( M + \frac{1}{2} \right) t \right) \right]. \] (17)
The system (15)-(16) is the dynamical extension of the stationary system (6)-(7). Since the functions \( \xi_i(t) \) are assumed to be complex, we face a typical example of a complex dynamical system of dimension \( M \). It is remarkable that this system can be rewritten in the following “potential” form

\[
i \dot{\xi}_i(t) = -\frac{\partial}{\partial \xi_i(t)} U(\xi(t)),
\]

where

\[
U(\xi) = -\hbar \sum_{k=1, k \neq i}^{M} \ln(\xi_i(t) - \xi_k(t)) + \frac{b}{2} \sum_{i=1}^{M} \xi_i^2(t) + \frac{a}{4} \sum_{i=1}^{M} \xi_i^4(t)
\]

is playing the role of a complex potential.

### 4 Complex multi-particle systems

Despite the fact that system (18) cannot be directly derived from the Lagrange principle, it is possible to relate it to a certain Lagrangian system in the following way. For this, consider the complex “Lagrangian”

\[
L(\xi, \dot{\xi}) = \frac{1}{2} \sum_{i=1}^{M} \dot{\xi}_i^2(t) - \frac{1}{2} W(\xi)
\]

with

\[
W(\xi) = \sum_{i=1}^{M} \left( \frac{\partial U(\xi)}{\partial \xi_i} \right)^2.
\]

It is not difficult to see that this Lagrangian (20) can be represented in the form

\[
L(\xi, \dot{\xi}) = -\frac{1}{2} \sum_{i=1}^{M} \Lambda_i^2(\xi, \dot{\xi}) + i \frac{d}{dt} U(\xi),
\]

where

\[
\Lambda_i(\xi, \dot{\xi}) = i \dot{\xi}_i(t) - \sum_{k=1, k \neq i}^{M} \frac{\hbar}{\xi_i(t) - \xi_k(t)} + b \xi_i(t) + a \xi_i^3(t).
\]

The total time derivative in (22) can be omitted because it does not affect the form of the equations of motion. The form of these equations derived from the action principle reads then

\[
\sum_{i=1}^{M} \left[ \frac{d}{dt} \left( \Lambda_i(\xi, \dot{\xi}) \frac{\partial \Lambda_i(\xi, \dot{\xi})}{\partial \xi_n} \right) - \Lambda_i(\xi, \dot{\xi}) \frac{\partial \Lambda_i(\xi, \dot{\xi})}{\partial \xi_n} \right] = 0, \quad n = 1, \ldots, M.
\]

From (24) it immediately follows that the solutions

\[
\Lambda_i(\xi, \dot{\xi}) = 0, \quad i = 1, \ldots, M
\]
are automatically solutions of the dynamical equations for the Lagrangian (24). Note that system (25) exactly coincides with equations (15). To derive the form of the Lagrangian (20) it is sufficient to substitute expression (19) into (20). This gives

\[
L(\xi, \dot{\xi}) = \frac{1}{2} \sum_{i=1}^{M} \dot{\xi}_i^2(t) - \sum_{i=1}^{M} \sum_{k=1, k \neq i}^{M} \frac{\hbar^2}{2} (\xi_i(t) - \xi_k(t))^2 + \\
- \sum_{i=1}^{M} \frac{\xi_i^2(t)}{2} \left( a\xi_i^2(t) + b \right)^2 + h a \left( M - \frac{3}{2} \right) \sum_{i=1}^{M} \xi_i^2(t).
\]

(26)

We have obtained a complex Lagrangian system supplemented with the additional constraint (16). This system can obviously be regarded a deformation of a complexified version of the famous Calogero-Moser system with Lagrangian

\[
L_0(\xi, \dot{\xi}) = \frac{1}{2} \sum_{i=1}^{M} \dot{\xi}_i^2(t) - \sum_{i=1}^{M} \sum_{k=1, k \neq i}^{M} \frac{\hbar^2}{2} (\xi_i(t) - \xi_k(t))^2 - \frac{b^2}{2} \sum_{i=1}^{M} \xi_i^2(t).
\]

(27)

Note that the above construction is very similar to the one usually used in constructing the soliton solutions of some classical dynamical equations. Therefore it is natural to interpret the solutions of equation (25) describing the motion of wave-function zeros in quantum QES model (2) as solitons of the classical multi-particle system with Lagrangian (20). The relation between the undeformed theories of the simple harmonic oscillator and the Calogero-Moser system is well known in the literature [11, 12].

5 Classical matrix model of the sextic anharmonic oscillator

Let \( X \) be a \( M \times M \) traceless complex matrix whose entries are considered as dynamical variables of a certain complex dynamical system. The Lagrangian of this system can be chosen in the form

\[
\mathcal{L}(X, \dot{X}) = \frac{1}{2} \text{Tr} X^2 - \text{Tr} V(X)
\]

(28)

with \( V(X) \) given by formula (4). The corresponding dynamical equations then read

\[
\ddot{X} = -\frac{\partial}{\partial X} V(X).
\]

(29)

It is known that any complex matrix with non-coinciding eigenvalues can be reduced to diagonal form by means of an appropriate similarity transformation

\[
X = S \Xi S^{-1}
\]

(30)
where we used the notation \( \Xi = \text{diag}\{\xi_1(t), \ldots, \xi_M(t)\} \) with \( \xi_1(t) + \cdots + \xi_M(t) = 0 \).

Taking the time derivative of (30) we obtain

\[
\dot{X} = SLS^{-1},
\]

(31)

where

\[
L = \dot{\Xi} + [M, \Xi]
\]

(32)

and

\[
M = S^{-1}\dot{S}.
\]

(33)

Differentiating (31) once more and using (29) and (30) we get the equation

\[
\dot{L} + [M, L] = -\frac{\partial}{\partial \Xi} V(\Xi)
\]

(34)

which together with (32) is equivalent to system (29). Note that (34) has the form of a deformed Lax equation. A relation between equations (29) and (34) can also be established in the opposite direction: Assume that we have two matrices \( M \) and \( L \) satisfying (34) and (32). Let \( S \) be a solution of the linear evolution equation

\[
\dot{S} = SM.
\]

(35)

Then the function \( X \) in (30) is a solution of equation (29). It is easy to check that the matrices \( L \) and \( M \) with components

\[
L_{ii} = \dot{\xi}_i(t), \quad L_{ik} = \frac{i\hbar}{\xi_i(t) - \xi_k(t)},
\]

\[
M_{ii} = \sum_{k=1, k \neq i}^{M} \frac{-i\hbar}{(\xi_i(t) - \xi_k(t))^2}, \quad M_{ik} = \frac{-i\hbar}{(\xi_i(t) - \xi_k(t))^2}
\]

(36)

satisfy the system (32), (34) provided the functions \( \xi_i(t) \) are solutions of the deformed Calogero-Moser system (29). As mentioned above for this it is sufficient that \( \xi_i(t) \) be solutions of system (15)-(16) describing the motion of the wavefunction zeros in the quantum QES model with potential (2).

From the above reasonings it follows that we essentially established a relationship between the quantum QES model with Hamiltonian

\[
H = \frac{P^2}{2} + \frac{x^2}{2} \left(ax^2 + b\right)^2 - \hbar a \left(M + \frac{3}{2}\right) x^2
\]

(37)

and the classical matrix model with Hamiltonian

\[
\mathcal{H} = \text{Tr} \left( \frac{P^2}{2} + \frac{X^2}{2} \left(aX^2 + b\right)^2 - \hbar a \left(M - \frac{3}{2}\right) X^2 \right).
\]

(38)
6 The large $M$ limit

It is remarkable that the hamiltonians of both the quantum and classical models (37) and (38) essentially coincide. The difference is of order $\hbar a$. Let us demonstrate now that this difference also disappears if we take a proper large $M$ limit of these models.

The necessity for taking the limit $M \to \infty$ in formulas (37) and (38) arises from the following reasonings. For finite $M$ it is not correct to speak of an equivalence of models (37) and (38). The point is that for finite $M$ model (37) describes the evolution of only a certain finite superposition of quantum states (because the model (37) is quasi-exactly solvable). However, if $M$ tends to infinity, then the number of stationary states in this superposition also tends to infinity and fills all the spectrum of the model. Only in this case we can say that models (37) and (38) are equivalent. But how to proceed to the large $M$ limit? It is clear that if we simply take $M = \infty$ in formulas (37) and (38) then we obtain a meaningless (minus) infinity. In order to get a finite expression we must take into account that the parameters $a$ and $b$ may also depend on $M$. Choosing this dependence according to the conditions

$$ab = g, \quad \frac{b^2}{2} - \hbar a M = \frac{\omega^2}{2}$$

in which $g$ and $\omega$ are fixed (positive) numbers, we find that in the large $M$ limit the parameter $a$ must behave as $a \sim M^{-1/3}$. This means that in the limit $M \to \infty$ the terms containing $a^2$ and $a$ (i.e. the sextic term and also the harmonic term responsible for the difference between the models) disappear and we obtain the two models

$$H = \frac{p^2}{2} + \frac{\omega^2 x^2}{2} + gx^4$$

and

$$\mathcal{H} = \text{Tr} \left( \frac{p^2}{2} + \frac{\omega^2 X^2}{2} + gX^4 \right)$$

with exactly coinciding classical and quantum hamiltonians. Note however that model (40) is a one-particle quantum model while model (41) is a classical model of traceless infinite matrices. The models are equivalent in the sense that the soliton-like solutions in the classical model (41) describe the evolution of the wavefunctions in the quantum model (40) and vice versa.

7 Discussion

When speaking of a relationship between quantum and classical mechanics one usually thinks of a pair of quantum and classical models related to each other by a certain quantization – dequantization procedure. The hamiltonians of these models,
considered as functions of coordinates and momenta, formally coincide up to terms of order $\hbar$. However, the mathematical meaning of the coordinates and momenta is essentially different in the quantum and classical case. In the quantum case they are operators in Hilbert space (infinite matrices) while in the classical case – they are simply numbers. Correspondingly, the number of quantities (degrees of freedom) necessary to fix uniquely the state of a dynamical system is infinitely larger in the quantum case compared to the classical one. In this sense the transition to the classical limit is equivalent to freezing infinitely many degrees of freedom of the quantum system.

There are however several examples where the correspondence between the quantum and classical model is not approximate but exact and is not related to any limiting procedure and any lose of degrees of freedom. The idea underlying such examples is based on a proper parametrization of the wavefunction by an infinite number of parameters depending on time and considered as canonically conjugated dynamical variables. If such a parametrization is found then we can associate with a given quantum model a certain classical one which in this case should necessarily be infinite-dimensional. It is quite clear that a priori there are no reasons for any relation between the forms of the corresponding quantum and classical hamiltonians. In general, they may be of absolutely different nature. Consider a simple but most instructive example. Let us take the evolution equation for a certain quantum model with hamiltonian $H$

$$i\hbar \partial_t \Psi = H \Psi \quad (42)$$

Since the wavefunction is complex, $\Psi = Q + iP$, we can rewrite equation (42) in real form

$$\hbar \partial_t Q = HP, \quad \hbar \partial_t P = -HQ, \quad (43)$$

which, after introducing the functional

$$\mathcal{H}(Q, P) = \frac{1}{\hbar} (Q, HQ) + \frac{1}{\hbar} (P, HP), \quad (44)$$

can in turn be rewritten in the form of the classical Hamilton-equations:

$$\partial_t Q = \frac{\delta \mathcal{H}(Q, P)}{\delta P}, \quad \partial_t P = -\frac{\delta \mathcal{H}(Q, P)}{\delta Q}. \quad (45)$$

We see that irrespective of the form of the initial quantum model, the resulting classical one describes an infinite-dimensional coupled harmonic oscillator.

In this paper we have found examples for an exact relationship between quantum and classical models. The main distinguished feature of these examples is that the potentials of the initial quantum and the resulting classical models exactly coincide. In this case the construction of the classical counterpart of a given quantum model (in
our case the quartic or harmonic oscillator) is extremely simple. One should simply replace the operators of coordinate and momentum entering into the quantum model by infinite traceless complex matrices and, after this, take the trace of this matrix hamiltonian. What we obtain will be just the hamiltonian of a classical model whose solutions contain the complete information of the dynamics of the wavefunctions in the initial quantum model. The origin of this coincidence is not clear to us at the moment but it is quite obvious that it cannot be accidental. It would be tempting to conjecture that any one-dimensional quantum model with hamiltonian

\[ H = \frac{p^2}{2} + V(x) \] (46)

is somehow equivalent to the classical infinite and traceless complex matrix model with the same hamiltonian

\[ \mathcal{H} = \text{Tr} \left( \frac{P^2}{2} + V(X) \right) \] (47)

We have an idea how to check this conjecture at least for models with polynomial anharmonicity and hope to publish the results in the near future.

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