FLAT MODEL STRUCTURES FOR NONUNITAL ALGEBRAS AND HIGHER K-THEORY

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Abstract. We prove the existence of a Quillen Flat Model Structure in the category of unbounded complexes of h-unitary modules over a nonunital ring (or a $k$-algebra, with $k$ a field). This model structure provides a natural framework where a Morita-invariant homological algebra for these nonunital rings can be developed. And it is compatible with the usual tensor product of complexes. The Waldhausen category associated to its cofibrations allows to develop a Morita invariant excisive higher $K$-theory for nonunital algebras.

1. Introduction.

Let $A$ be a nonunital algebra (or ring). A classical question in Homotopy Theory is to find a 'good definition' of $K$-theory and cyclic type homology for this type of rings and algebras. Namely, it is always possible to embed a nonunital ring $A$ as a two-sided ideal of a unital ring $R$ (for instance, by choosing $\tilde{A} = \mathbb{Z} \times A$ to be the ring obtained by adjoining an identity to $A$). Thus, it is possible to define the notions of $K$-theory and cyclic type homologies for $A$ in terms of this ring $R$. But this embedding of $A$ into a unital ring $R$ is not unique. And therefore, the different choices of $R$ give rise to different definitions of homology theories and $K$-theory for $A$. This problem is known in the literature as the 'excision problem' in the different theories. In [35], Wodzicki proved that if $R$ is a unitary $k$-pure extension of $A$, then it satisfies the excision property for Cyclic, Bar or Hochschild homology if and only if $A$ is an $H$-unital $k$-algebra, in the sense that its Bar homology $HB_*(A, V) = 0$ for any $k$-module $V$ (see e.g. [35] Theorem 3.1)]. This result extended to algebras over commutative rings his remarkable result showing that if a (nonunital) ring $A$ satisfies the excision property in rational algebraic $K$-theory, then the $\mathbb{Q}$-algebra $A \otimes \mathbb{Q}$ verifies the excision property in Cyclic homology [35] (see also [22, 32]). But this $H$-unitality condition is rarely satisfied in practice and thus, many authors have extended these ideas and techniques to more general settings. For instance, Cuntz and Quillen have proven in [8, 9, 10] that arbitrary extensions of $k$-algebras ($k \supset \mathbb{Q}$ a field) with $k$-linear section satisfy excision in periodic cyclic cohomology. And Weibel in [34] constructed a homotopy invariant algebraic $K$-theory satisfying excision and cohomologic descent. The obstruction for the classical $K$-theory excision has also been studied in [5] (see also [6]).

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In the present paper, we adopt the definition of $h$-unitary modules given by Suslin and Wodzicki (cf. [32, 7.3(ii)]). And we define the category $\mathfrak{M}(A)$ of $h$-unitary left $A$-modules as the full subcategory of all left $A$-modules consisting on those modules $M$ satisfying that $\text{Tor}_n^A(\mathbb{Z}, M) = 0$ (or $\text{Tor}_n^A(k, M) = 0$, if $A$ is an algebra over a ground field $k$). At first sight, this definition of $\mathfrak{M}(A)$ depends on the choosing embedding of $A$ as a two-sided ideal of the unital ring $R$. And therefore we will refer to it as the $h$-unitary module category associated to the pair $(R, A)$. But we will show in the last section of the paper that this $h$-unitary module category is independent of $R$.

We prove in Theorem 3.1 that any unitary left $A$-module $M$ (in particular, the regular module $A$) has a (unique up to isomorphism) cover by an $h$-unitary module (equivalently, a minimal right approximation in the sense of Auslander [3]) by an $h$-unitary module. And then, we show that there is a very satisfactory version of relative homological algebra in the category of unbounded complexes of $h$-unitary modules. Namely, we prove in Section 3 that the class of all flat $h$-unitary $A$-modules imposes a cofibrantly generated Model Structure (see [27]) in the category $\mathbb{C}(\mathfrak{M}(A))$ of all complexes in $\mathfrak{M}(A)$. Let us note that, if we consider in $\mathbb{C}(\tilde{A})$ the cofibrantly generated Flat Model Structure constructed in [17], then the bounded flat complexes which are generating cofibrations form a small Waldhausen category $S$ in the sense of [33]. And the subset $T \subset S$ of generating cofibrations which are flat $h$-unitary is a Waldhausen subcategory of it, since any cofibration in our model structure on $\mathbb{C}(\mathfrak{M}(A))$ is a cofibration in $\mathbb{C}(\tilde{A})$. Therefore we can use the general construction of $K$-theory on a Waldhausen category (see [33, Chapter IV]) to define a $K$-theory on $A$. This $K$-theory is going to be excisive in the sense of [35] by our results in the last section. In particular, we deduce that the group $K_0T$ is independent of the choice of $S$ (see [33, Chapter II, Theorem 9.2.2]). The explicit construction of these $K$-groups in terms of our model structure will be developed in a forthcoming paper. We would like to stress that our construction does not restrict just to $H$-unital algebras, but it applies to any nonunital algebra (or ring) without the $h$-unitality condition.

Unfortunately, the category of $h$-unitary modules is far from being a monoidal category and therefore, we cannot expect to construct a monoidal model structure on $\mathbb{C}(\mathfrak{M}(A))$. We solve this problem in Section 4, where we consider the wider subcategory $\mathfrak{F}(A)$ of $\tilde{A}$-$\text{Mod}$ consisting of all firm modules in the sense of [28]. The usual tensor product of $A$-modules is an endofunctor in $\mathfrak{F}(A)$. And we show in Theorem 4.1 that this tensor product in $\mathfrak{F}(A)$ induces a unitless monoidal structure in $\mathbb{C}(\mathfrak{F}(A))$ which is compatible with the model structure we have induced in $\mathbb{C}(\mathfrak{M}(A))$.

At this point, the question of when the additive category of $h$-unitary modules is abelian naturally arises. We show in Section 2 that this category is always an accessible category in the sense of [2, 24]. Actually, the constructions developed in this section will be critical for proving our main results.

Let us finally remark that, although the category of $h$-unitary modules over nonunital rings extends the category of (left) $R$-modules over a unital ring $R$ (actually, if $R$ is unital the three categories do coincide), there are important problems for developing a notion of homological algebra in these categories. The main reason is that neither of these categories contains in general enough projectives. In order to solve this problem, we prove that both categories have enough flat modules (note
that the notion of firm and h-unitary modules do coincide in the case of flat modules). That is, that any h-unitary module (resp. firm module) is the homomorphic image of an h-unitary flat module (resp. a firm flat module). Thus we can define flat resolutions of h-unitary or firm modules. However, it is not possible to define a good homotopy relation among these flat resolutions in the category of unbounded chain complexes. And this means that these resolutions do not allow to uniquely define torsion functors in the corresponding homotopy categories. We solve this problem by introducing a good homotopy theory in the chain complexes category of h-unitary modules. Namely, we prove that it is possible to impose an h-unitary flat Quillen Model Structure (cf. [21, 27]) in the category of h-unitary modules. Our proof of the existence of this Quillen Model Structure uses Hovey’s criteria (cf. [20]) relating Cotorsion Pairs and Model Category Structures. The flatness condition in this model structure ensures its compatibility respect to the monoidal structure in the larger category \( C(F(A)) \) induced by the endofunctor given by the usual tensor product of complexes. Moreover we deduce that h-unitary flat modules are preserved under equivalences given by a Morita context. As an application to this construction, if we take the nonunital algebra to be a \( C^* \)-algebra, our model category structure provides a solution to a question posed by Hovey in [21, Problem 8.4].

Along this paper, all rings will be associative and non necessarily unitary. Although we state our results for nonunital rings \( A \), we can assume (as it often occurs in practice) that our nonunital rings are algebras over some field \( k \) (so \( \tilde{A} \) is the \( k \)-algebra \( k \ltimes A \) obtained by adjoining an identity to \( A \)). We refer to [18, 21, 22, 33] for any undefined notion on cotorsion pairs, Model Structures or K-theory used in the text.

2. Main Lemmas.

Let \( \mathcal{A} \) be a Grothendieck category and \( \mathcal{L} \), a class of objects of \( \mathcal{A} \) closed under isomorphisms. A well-ordered direct system of objects of \( \mathcal{A} \) \( (X_\alpha | \alpha \leq \lambda) \) is said to be continuous if \( X_0 = 0 \), \( X_\alpha \subseteq X_{\alpha+1} \) and, for each limit ordinal \( \beta \leq \lambda \), we have that \( X_\beta = \cup_{\alpha < \beta} X_\alpha \). An object \( X \) of \( \mathcal{A} \) is called \( \mathcal{L} \)-filtered provided that \( X = X_\lambda \) for some well-ordered direct system \( (X_\alpha | \alpha \leq \lambda) \) such that \( X_{\alpha+1}/X_\alpha \) is in \( \mathcal{L} \) whenever \( \alpha + 1 \leq \lambda \).

**Notation 1.** Let \( D \) be a class of objects of a Grothendieck category \( \mathcal{A} \). We will denote by \( D^\perp \) the class of all objects \( Y \) of \( \mathcal{A} \) such that \( \text{Ext}^1(D,Y) = 0 \) for every \( D \in D \). Similarly, \( ^\perp D \) will denote the class of those objects \( Z \in \mathcal{A} \) such that \( \text{Ext}^1(Z,D) = 0 \) for every \( D \in D \).

We recall that a pair \((\mathcal{F}, \mathcal{C})\) of classes of objects of \( \mathcal{A} \) is called a cotorsion pair if \( \mathcal{F}^\perp = \mathcal{C} \) and \( ^\perp \mathcal{C} = \mathcal{F} \) (see e.g. [15]). The cotorsion pair is said to have enough injectives (resp. enough projectives) if, for every \( Y \) in \( \mathcal{A} \), there exists an exact sequence \( 0 \rightarrow Y \rightarrow C \rightarrow F \rightarrow 0 \) (resp. for every \( Z \in \mathcal{A} \) there exists an exact sequence \( 0 \rightarrow C' \rightarrow F' \rightarrow Z \rightarrow 0 \)) where \( F,F' \in \mathcal{F} \) and \( C,C' \in \mathcal{C} \). The cotorsion pair \((\mathcal{F}, \mathcal{C})\) is complete if it has enough injectives and projectives. We will say that \((\mathcal{F}, \mathcal{C})\) is functorially complete when these sequences can be chosen in a functorial manner (depending on \( Y \) and \( Z \)) (see [20, Definition 2.3]). Finally, a cotorsion pair \((\mathcal{F}, \mathcal{F}^\perp)\) is said to be cogenerated by a set \( S \subseteq \mathcal{F} \) if \( S^\perp = \mathcal{F}^\perp \). From results in [14] we get the following Theorem.
Theorem 2.1. Let $\mathcal{F}$ be a class of objects of a Grothendieck category $\mathcal{A}$ which is closed under direct sums, extensions and well ordered direct limits. Suppose that the pair $(\mathcal{F}, \mathcal{F}^\perp)$ is cogenerated by a set. If $\mathcal{F}$ contains a generator of $\mathcal{A}$, then the pair $(\mathcal{F}, \mathcal{F}^\perp)$ is a complete cotorsion pair.

Proof. It follows from [14, Theorem 2.5] that the pair $(\mathcal{F}, \mathcal{F}^\perp)$ has enough injectives. To show that the pair also has enough projectives, we will adapt the arguments of [31, Lemmas 2.2 and 2.3]. Let $M$ be any object of $\mathcal{A}$. There exists a short exact sequence $0 \to K \to G \to M \to 0$ with $G \in \mathcal{F}$. Since $(\mathcal{F}, \mathcal{F}^\perp)$ has enough injectives there also exists a short exact sequence $0 \to K \to C \to F \to 0$, with $F \in \mathcal{F}$ and $C \in \mathcal{F}^\perp$. Let us construct the pushout diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \\
\downarrow \\
K \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
C \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
F \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
0
\end{array}
\]

and let us note that $Q \in \mathcal{F}$, since $\mathcal{F}$ is closed under extensions. Thus, the short exact sequence $0 \to C \to Q \to M \to 0$ shows that $(\mathcal{F}, \mathcal{F}^\perp)$ has enough projectives, i.e. it is a complete pair. Finally if $M \in \mathcal{F}^\perp$ then the previous short exact sequence splits. In particular, $M \in \mathcal{F}$ and hence $(\mathcal{F}, \mathcal{F}^\perp)$ is a cotorsion pair. \qed

Let us fix a nonunital ring $A$. And let us denote by $\tilde{A} = \mathbb{Z} \ltimes A$ the unital ring obtained from $A$ by adjoining an identity. A left $A$-module $M$ is said to be h-unitary if the evaluation map $\mu : A \otimes_{\tilde{A}} M \to M$ given by $\mu(a \otimes m) = am$ is an isomorphism and $\text{Tor}_i^A(A, M) = 0, \forall i \geq 1$ or, equivalently, if $\text{Tor}_n^A(\mathbb{Z}, M) = 0, \forall n \geq 0$. Let us note that this definition generalizes the corresponding one given in [32, 7.2 (ii)] for modules over nonunital $k$-algebras. Results in Section 5 show that the category of h-unitary modules remains invariant if we replace $\tilde{A}$ by any other unitary extension of the ring $A$ (i.e., by any other unitary ring $R$ containing $A$ as a two-sided ideal). Therefore, from now on, we will denote by $R$ any unitary extension ring of $A$ and we will implicitly assume, if necessary, that $R = \tilde{A}$. Other concept which is independent of the chosen unitary extension ring of $A$ is that of firm module. A left $A$-module $M$ is said to be firm if $\mu$ is an isomorphism or, equivalently, if $\text{Tor}_i^A(\mathbb{Z}, M) = 0$ for $i = 0, 1$. We denote by $\mathcal{M}(A)$ and $\mathcal{M}(A^{op})$ (resp. $\mathcal{F}(A)$ and $\mathcal{F}(A^{op})$) the categories of left and right h-unitary $A$-modules (resp. left and right firm modules). We summarize next the main properties of these subcategories of $R - \text{Mod}$. We recall that a morphism in a category is called a monocokernel
monocokernels, epikernels and extensions in
instance, by Bican, El Bashir and Enochs for proving the exis
tence of flat covers of

Remark 2.1.
i,j sequence with respect to T or
subset

Lemma 2.2.

Lemma 2.1.

\[ \operatorname{cokernel} \text{ of a monomorphism (resp., epimorphism).} \]

\[ g \mid_{AP} \text{is surjective (since } AM = M \text{) and hence, there exists an endomorphi}

f : P \to P \text{ such that } g \circ f = g \text{ and } f(P) \subseteq AP. \text{ Let } F = \lim_{\to n \in \omega} P_n, \text{ where } P_n = P

\text{ and } P_n \to P_{n+1} = f, \forall n \geq 0. \text{ If } F \text{ is a flat } R\text{-module by construction and } AF = F

\text{ since } f(P_n) \subseteq AP_{n+1}. \text{ In particular, } F \text{ is an h-unitary module. As } g : P_n \to M \text{ is surjective}
\forall n \geq 0, \text{ the induced map } F \to M \text{ is also surjective.}

The category } \mathcal{A}(A) \text{ is clearly cocomplete because it is closed under coproducts}
\text{(since coproducts commute with tensor products) and it is easy to check that for}
\text{every morphism } f : M \to N, \text{ coker}(f) \text{ computed in } R\text{-Mod is actually in } \mathcal{A}(A).

Finally, let } 0 \to M_1 \to M_2 \to M_3 \to 0 \text{ be any short exact sequence in } R\text{-Mod}
\text{ such that } M_i, M_j \in \mathcal{M}(A), \text{ for } i, j \in \{1, 2, 3\}, i \neq j. \text{ Then by taking the long exact}
\text{sequence with respect to } \text{Tor}^R_n(\mathbb{Z}, ?) \text{ we get that } M_k \in \mathcal{M}(A), \text{ with } k \in \{1, 2, 3\}, k \neq
i, j. \quad \square

\text{Remarks 2.1.}

(1) \text{ Let us note that the category } \mathcal{A}(A) \text{ is also closed under}
\text{extensions and epikernels in } R\text{-Mod.}

(2) \text{ We do not know whether the category } \mathcal{M}(A) \text{ is cocomplete in general.}

Our next lemma extends to sets of arbitrary cardinality a well-known result that
\text{belongs to the folklore of the Theory of Purity of modules. It has been used, for
instance, by Bican, El Bashir and Enochs for proving the existence of flat covers of}
\text{modules (see [4]). We denote, as usual, the cardinality of a set } N \text{ by } |N|.

\text{Lemma 2.2. Let } R \text{ be a unitary ring and } M, \text{ a unitary left } R\text{-module. Every}
\text{subset } N \text{ of } M \text{ is contained in a pure submodule of } M \text{ of cardinality at most } \kappa = \max \{|R|, |N|, \aleph_0\}.

\text{Proof. Let } N_0 \text{ be the submodule of } M \text{ generated by } N. \text{ Note that } N = \sum_{y \in N} Ry
\text{ and therefore, } |N_0| \leq \sum_{y \in N} |Ry| \leq |N| \cdot |R| = \kappa.

\text{Consider now the short exact sequence of modules}
\[ 0 \to N_0 \xrightarrow{u} M \xrightarrow{p} M/N_0 \to 0 \]
\text{and let } \Omega_0 \text{ be the subset of } \cup_{n,m \in N}[\text{Hom}_R(R^n, N_0) \times \text{Hom}_R(R^m, R^n)] \text{ consisting on those pairs of morphisms (h, v) such that there exists a g_{(h,v)} : R^m \to M \text{ with}}
g_{(h,v)} \circ v = u \circ h. \text{ Note that}
\[ |\Omega_0| \leq |\cup_{n,m \in N}[\text{Hom}_R(R^n, N_0) \times \text{Hom}_R(R^m, R^n)]| =
\sum_{n,m \in N} \text{Hom}_R(R^n, N_0) \times \text{Hom}_R(R^m, R^n) =
\sum_{n,m \in N} |N_0^n \times R^{n \times m}| \leq \max \{|R|, |N|, \aleph_0\}. \]
\text{And let } N_1 \text{ be the submodule of } M \text{ generated by}
\[ N_0 \cup \{ \text{Im } g_{(h,v)} \mid (h,v) \in \Omega_0 \}. \]
Then \(|\operatorname{Im}(h)| \leq \max \{ |R|, |N|, N_0 \}\) for any \((h, v) \in \Omega_\ell\) and therefore, \(N_1\) has at most \(8 \times 8 = 64\) generators. In particular, this means that again \(|N_1| \leq 8\), since \(8 \geq |R|\).

Let us now replace \(N_0\) by \(N_1\) and let us construct \(N_2\) in a similar way. Following this method, we will get, by induction on \(N\), an infinite ascending chain \(\{N_k\}_{k \in \mathbb{N}}\) of submodules of \(M\) of cardinality bounded by \(N\). Let \(L = \bigcup_{k \in \mathbb{N}} N_k\). Clearly \(|L|\) is also bounded by \(N\).

We claim that \(L\) is pure in \(M\). To check it, let \(\phi : F \rightarrow M/L\) be a homomorphism from a finitely presented module to \(M/L\) and let us show that there exists a morphism \(t : F \rightarrow M\) such that \(\phi = p \circ t\), where \(p : M \rightarrow M/L\) is the canonical projection. \(F\) is always the cokernel of a homomorphism \(v : R^n \rightarrow R^m\). By projectivity, there exist homomorphisms \(g : R^m \rightarrow M\) and \(h : R^n \rightarrow L\) such that the following diagram commutes

\[
\begin{array}{ccc}
R^n & \xrightarrow{v} & R^m \\
\downarrow h & & \downarrow g \\
0 & \xrightarrow{u} & L \xrightarrow{p} M \xrightarrow{\phi} M/L \rightarrow 0
\end{array}
\]

As \(\operatorname{Im}(h)\) is a finitely generated submodule of \(L = \bigcup_{k \in \mathbb{N}} N_k\), there exists a \(k \in \mathbb{N}\) such that \(\operatorname{Im}(h) \subseteq N_k\). Therefore, the element \((h, v) \in \Omega_{k+1}\) and, by construction, this means that there exists a homomorphism \(g_{(h,v)} : R^m \rightarrow N_{k+1} \subseteq L\) such that \(g_{(h,v)} \circ v = h\). But then \((g - g_{(h,v)}) \circ v = 0\) and thus, there exists a \(t : F \rightarrow M\) such that \(t \circ q = g - g_{(h,v)}\). In particular, \(p \circ t \circ q = p \circ (g - g_{(h,v)}) = \phi \circ q = p \circ u \circ g_{(h,v)} = \phi \circ q\). Therefore, \(\phi = p \circ t\), since \(q\) is an epimorphism. \(\square\)

Let us now consider the category \(\mathcal{M}(A)\) of left h-unitary modules. Let us recall that, by Lemma 2.1, \(\mathcal{M}(A)\) has an h-unitary flat generator \(G\).

**Lemma 2.3.** Let \(G\) be an h-unitary flat generator of \(\mathcal{M}(A)\) and let us fix a cardinal number \(N \geq \max \{ |G|, N_0 \}\). Let \(M\) be an h-unitary \(A\)-module and \(N\), a subset of \(M\) of cardinality bounded by \(N\). Then \(N\) embeds in a pure h-unitary submodule \(L\) of \(M\) such that \(|L| \leq N\).

**Proof.** Let us denote by \(\rho : A \otimes_R M \rightarrow M\) the structural homomorphism. Note that \(\rho\) is an isomorphism since \(M\) is h-unitary. Let us construct \(L\) by induction on \(n \in \mathbb{N}\). By the above Lemma, there exists a pure \(R\)-submodule \(N_0 \subset M\) containing \(N\) such that \(|N_0| \leq N\). Therefore, we have the following diagram

\[
\begin{array}{ccc}
A \otimes_R N_0 & \xrightarrow{\rho} & N_0 \\
\downarrow & & \downarrow \\
A \otimes_R M & \xrightarrow{\rho} & M
\end{array}
\]

in which the vertical arrows are monomorphisms, since \(N_0\) is pure in \(M\). Let us choose, for any \(x \in N_0\), elements \(m_i^1, \ldots, m_i^{r_i} \in M\) and elements \(a_i^1, \ldots, a_i^{k_i} \in A\) such that \(\rho \left( \sum_{i=1}^{k_i} a_i^i \otimes m_i^i \right) = x\). And let us call \(K_1 = N_0 \cup \{ m_1^1, \ldots, m_{k_1}^{r_1} \}\).

Note that \(|K_1|\) is also bounded by \(N\).

On the other hand, as \(G\) is a generator of \(\mathcal{M}(A)\), there exists a homomorphism \(f_y : G \rightarrow M\) and an element \(z_y \in G\) such that \(y = f_y(z_y)\), for any \(y \in K_1\). Let \(K_1' = \sum_{y \in K_1} f_y(G)\). Then \(K_1'\) is a submodule of \(M\) containing \(K_1\) and thus, also containing \(N_0\). Moreover, it is straightforward to check that \(|K_1'|\) is also bounded by \(N\).
Let us now replace \( N \) by \( K'_{\lambda} \) and continue the construction. Set finally \( L = \bigcup_{n \in \mathbb{N}} N_n = \bigcup_{n \in \mathbb{N}} K'_{\lambda} \). Clearly \( L \) is a pure submodule of \( M \), since it is the union of the chain \( \{ N_n \}_{n \in \mathbb{N}} \) of pure submodules of \( M \). From this it is easy to deduce that \( N \subseteq M \) is also \( h \)-unitary. And \( |L| \leq \aleph_0 \times \aleph = \aleph \).

We recall that a category \( C \) is called \( \aleph \)-accessible (where \( \aleph \) is an infinite regular cardinal number) if it has \( \aleph \)-direct limits and there exists a set \( C_0 \) of \( \aleph \)-presentable objects of \( C \) such that any other object of \( C \) is (isomorphic to) an \( \aleph \)-direct limit of morphisms among objects in \( C_0 \). Where an \( \aleph \)-direct limit of morphisms is the direct limit of a directed set of morphisms \( \{ f_{ij} \}_I \) satisfying that for any subset \( I_0 \subseteq I \) of cardinality strictly smaller than \( \aleph \), there exists an \( i_0 \in I \) such that \( i \leq i_0 \) for any \( i \in I_0 \). And an object \( C \in C \) is called \( \aleph \)-presentable if the functor \( \text{Hom}(C, -) : C \to \text{Set} \) commutes with \( \aleph \)-direct limits (see e.g. [2, 24]). A category is simply called accessible if it is \( \aleph \)-accessible for some infinite regular cardinal \( \aleph \).

**Corollary 2.1.** The category of \( h \)-unitary left modules over a nonunital ring is an accessible additive category.

**Proof.** The category is clearly additive and it has \( \lambda \)-direct limits, for any infinite regular cardinal \( \lambda \), since the tensor product is a right exact additive functor commuting with direct limits. On the other hand, the above result shows that, for any cardinal \( \aleph \geq \max \{|G|, \aleph_0\} \), any object in the category is the \( (\aleph^+) \)-direct union of its \( h \)-unitary submodules of cardinality bounded by \( \aleph \). Moreover, it is easy to check that, if we choose \( \aleph \geq |A| \), the above submodules are \( \aleph^+ \)-presentable. Finally, let us note that \( \aleph^+ \) is regular, since it is a successor cardinal. \( \square \)

**Remark 2.2.**

Let us fix a generator \( G \) of \( \mathfrak{M}(A) \) and let \( \aleph \geq \max \{|G|, \aleph_0\} \). Let us call \( G_\aleph \) the direct sum of all isomorphism classes of \( h \)-unitary modules of cardinality bounded by \( \aleph \). The proof of the above corollary shows that \( G_\aleph \) is also a generator of \( \mathfrak{M}(A) \). Moreover, \( G_\aleph \) has the following interesting property: any \( h \)-unitary module is an \( (\aleph^+) \)-directed union of \( h \)-unitary pure submodules isomorphic to direct summands of \( G_\aleph \).

3. A Quillen Model Structure on \( \mathcal{C}(\mathfrak{M}(A)) \).

The main purpose of this section will be to impose a Quillen Model Structure (see [21, 27] for its definition) in the category \( \mathcal{C}(\mathfrak{M}(A)) \) of unbounded chain complexes of \( h \)-unitary left \( A \)-modules in terms of \( h \)-unitary flat modules. Our proof is based on Hovey’s Theorem [20] which relates Complete Cotorsion pairs and Model Structures in the corresponding category of unbounded complexes.

The additive category \( \mathfrak{M}(A) \) may not be abelian in general, but it can be obviously embedded as a full subcategory of the abelian category \( R-\text{Mod} \). As \( \mathfrak{M}(A) \) is closed under extensions in \( R-\text{Mod} \) (Lemma [2.1]), we may consider the proper class \( \mathcal{P} \) in \( \mathfrak{M}(A) \) consisting of all short exact sequences in \( R-\text{Mod} \), \( 0 \to M \to N \to P \to 0 \) with \( M, N, P \in \mathfrak{M}(A) \). And then define as in ([23, Chapter XII, Theorem 4.1]), the relative extension groups \( \text{Ext}^n_{\mathfrak{M}(A)}(M, N) \), for all \( M, N \in \mathfrak{M}(A) \) and \( n \geq 0 \), as well as natural transformations of bifunctors \( \text{Ext}^n_{\mathfrak{M}(A)}(?, ?) \to \text{Ext}^n_{\mathfrak{M}(A)}(?, ?) \), for all \( n \geq 0 \). Note that these natural transformations are monomorphisms for \( n = 1 \), and isomorphisms for \( n = 0 \).
We recall from the above section that an h-unitary module $A$ is flat if it is flat in $\tilde{A} - \text{Mod}$. Let us denote by $F$ the class of all h-unitary flat $A$-modules. We will show in our last section (Theorem 5.1) that this class does not depend on the chosen embedding of $A$ into a unital ring $R$. Let us also note that a flat $R$-module $F$ belongs to $F$ if and only if $A \cdot F = F$.

Let us denote by $\mathcal{C}$ the class of all h-unitary modules which are Ext-orthogonal to $F$, that is, $\mathcal{C} = \{ C \in \mathcal{M}(A) : \text{Ext}_1^{\mathcal{M}(A)}(F, C) = 0, \forall F \in F \}$. Modules in $\mathcal{C}$ will be called h-unitary cotorsion modules.

The problem of the previous approach is that these relative Ext groups are computed in terms of extensions in $R - \text{Mod}$. But we would like to be able to compute them intrinsically by resolutions and coresolutions in $\mathcal{M}(A)$. On the other hand, it is implicit in [35] that h-unitary flat modules are the right choice for computing all kinds of Morita invariant homology functors. So our approach will be to use the class $P$ as the proper class of exact sequences in $\mathcal{M}(A)$, but considering h-unitary flat and h-unitary cotorsion modules in $\mathcal{M}(A)$ as the basic bricks for constructing resolutions and coresolutions in $\mathcal{M}(A)$ respectively.

From now on, when we refer to exact sequences in $\mathcal{M}(A)$ we will mean exact sequences in $P$. We begin by proving that every $R$-module can be approximated in a minimal fashion by a unique up to isomorphism h-unitary module.

**Theorem 3.1.** Let $M$ be any $R$-module. Then there exists an h-unitary module $M_H$ and a morphism $\varphi : M_H \to M$ satisfying that:

1. For every morphism $\psi : N_H \to M$ with $N_H \in \mathcal{M}(A)$ there exists a morphism $f : N_H \to M_H$ such that $\varphi \circ f = \psi$.
2. Every morphism $g : N_H \to N_H$ such that $\varphi \circ g = \varphi$ is an automorphism.

Such $\varphi : M_H \to M$ is necessarily unique up to isomorphisms and it is called the the $\mathcal{M}(A)$-cover of $M$.

**Proof.** The proof is similar to [13, Theorem 4.1]. We just need to use Quillen’s small object argument (see [27, Lemma II.3.3]) and use the fact that $\mathcal{M}(A)$ is closed under extensions, well ordered direct limits and direct sums. Then we may apply Lemma 2.1 and the same arguments of [13, Theorem 4.1] show that every $R$-module $M$ has an $\mathcal{M}(A)$-cover.

**Theorem 3.2.** The pair $(F, C)$ is a complete cotorsion pair in $\mathcal{M}(A)$.

**Proof.** Let $F$ be an h-unitary flat module. By Lemma 2.1 for any $y \in F$ there exists a pure and h-unitary submodule $S \subseteq F$, with $y \in S$ and $|S| \leq \aleph$ (where $\aleph$ is the infinite cardinal obtained in Lemma 2.1). Hence, $S$ and $F/S$ are also h-unitary flat modules. If $S_\aleph$ denotes the set of all isomorphism classes of h-unitary flat modules of cardinality bounded by $\aleph$, we get that each h-unitary flat module $F$ is $S_\aleph$-filtered. It follows from [11, Lemma 1] that the set $S_\aleph$ cogenerates the pair $(F, F^\perp)$. Therefore by Theorem 2.1 the pair $(F, F^\perp)$ is complete. Furthermore, by Lemma 2.1 $\mathcal{M}(A)$ is closed under extensions, monocokernels and epikernels so it follows that in fact the pair $(F, C)$ in $\mathcal{M}(A)$ is complete.

Let us finally check that $(F, C)$ is a cotorsion pair. Let $M$ be an h-unitary module such that $M \in C$. Since $(F, C)$ has enough projectives, there exists a short exact
sequence of h-unitary modules

\[ 0 \to K \to F \to M \to 0 \]
such that \( F \in \mathcal{F} \) and \( K \in \mathcal{C} \). Thus, this short exact sequence splits and \( M \) is a direct summand of \( F \). In particular, \( M \in \mathcal{F} \).

Conversely assume that \( N \in \mathcal{F}^{\perp} \) and is h-unitary. Then \( \text{Ext}^1_R(F, N) = 0 \), for all \( F \in \mathcal{F} \). As \( \text{Ext}^1_{\mathcal{M}(A)}(?, ?) \) is a subfunctor of \( \text{Ext}^1_R(?, ?) \) we get that \( \text{Ext}^1_{\mathcal{M}(A)}(F, N) = 0 \). Thus \( N \in \mathcal{C} \).

We will denote the category of unbounded complexes of h-unitary modules by \( \mathbb{C}(\mathcal{M}(A)) \). That is, those complexes

\[ M = \cdots \to M^{i-1} \xrightarrow{\delta^{i-1}} M^i \xrightarrow{\delta^i} M^{i+1} \to \cdots \]
such that \( 0 \to \ker \delta^i \to M^i \to \text{Im} \delta^i \to 0 \) in \( \mathcal{P} \), \( \forall i \in \mathbb{Z} \). We shall denote by \( Z_iM \) the \( i \)th cycle module \( \ker(\delta^i) \) and by \( B_iM \), the \( i \)th boundary module \( \text{Im}(\delta^{i-1}) \).

Recall that the tensor and \( \text{Hom}_R \) functors on \( \mathcal{M}(A) \) can be canonically extended to \( \mathbb{C}(\mathcal{M}(A)) \) as follows: If \( M \) and \( N \) are complexes of h-unitary modules, we call \( \text{Hom}(M, N) \) the complex of abelian groups satisfying that

\[ \text{Hom}(M, N)^n = \prod_{t \in \mathbb{Z}} \text{Hom}_R(M^t, N^{n+t}) \]

and such that if \( f \in \text{Hom}(M, N)^n \), then

\[ (\delta^n f)^m = \delta^{n+m}_N \circ f^m - (-1)^n f^{m+1} \circ \delta^m_M. \]

The tensor product of a complex of right h-unitary modules \( M \) and a complex of left h-unitary modules \( N \) is the complex of abelian groups \( M \otimes N \) with \( (M \otimes N)^n = \oplus_{t \in \mathbb{Z}} M^t \otimes_R N^{n-t} \) and

\[ \delta(x \otimes y) = \delta_M(x) \otimes y + (-1)^t x \otimes \delta_N^{n-t}(y), \]

for \( x \in M^t \) and \( y \in N^{n-t} \).

\( \mathbb{C}(\mathcal{M}(A)) \) is an additive category, since so is \( \mathcal{M}(A) \). A complex \( M \) such that \( Z_iM = B_iM, \forall i \in \mathbb{Z} \) will be called exact. We shall denote by \( \mathcal{E} \) the class of all exact complexes. If \( G \) is an h-unitary flat generator of \( \mathcal{M}(A) \) then the family of complexes \( \{ S^n(G) : n \in \mathbb{Z} \} \) generates the category \( \mathbb{C}(\mathcal{M}(A)) \). Where \( S^n(G) \) denotes the complex with \( G \) in the \( n \)th position and 0 in all other places. If \( M \) is an h-unitary module we will denote by \( D^a(M) \) the complex \( \cdots \to 0 \to M \xrightarrow{id} M \to 0 \to \cdots \) with \( M \) in positions \( n - 1 \) and \( n \).

We recall from [17] the following definitions. A complex of h-unitary modules \( C \) is said to be \( \mathcal{F} \)-cotorsion if \( C \) is exact and \( Z_nC \in \mathcal{C} \), for all \( n \in \mathbb{Z} \). A complex \( F \) is called a dg-h-unitary flat complex if \( F^n \) is an h-unitary flat module for any \( n \in \mathbb{Z} \) and, for every \( \mathcal{F} \)-cotorsion complex \( C \), \( \text{Hom}(F, C) \) is an exact complex in \( \mathbb{Z} \text{-Mod} \). Dually, we define h-unitary flat complexes as those complexes \( X \) such that \( X \) is exact and \( Z_nX \in \mathcal{F}, \forall n \in \mathbb{Z} \). And dg-cotorsion complexes, as those \( Y \) such that \( Y_n \in \mathcal{C} \) and \( \text{Hom}(X, Y) \) is exact, for any h-unitary flat complex \( X \). We shall denote by \( \mathcal{F} \) and \( \mathcal{C} \) the classes of h-unitary flat complexes and \( \mathcal{F} \)-cotorsion complexes, respectively. And by \( \text{dgF} \) and \( \text{dgC} \), the classes of dg-h-unitary flat and dg-cotorsion complexes of h-unitary modules. It is clear that if \( F \) is an h-unitary flat module, then \( S^n(F) \in \text{dgF} \) for all \( n \in \mathbb{Z} \). In particular, \( \text{dgF} \) contains the previous family of generators of \( \mathbb{C}(\mathcal{M}(A)) \).
In order to obtain a Model Structure in \( C(\mathcal{M}(A)) \), we are going to apply Hovey’s Theorem relating Cotorsion Pairs and Model Category Structures.

**Theorem 3.3.** [20, Theorem 2.2] Let \( C(\mathcal{M}(A)) \) be the category of chain complexes of \( h \)-unitary modules. Let \( \mathcal{E} \) be the class of exact complexes in \( \mathcal{M}(A) \). If \( (dg\bar{\mathcal{F}}, dg\bar{\mathcal{C}} \cap \mathcal{E}) \) and \( (dg\bar{\mathcal{C}}, \bar{\mathcal{C}}) \) are cotorsion pairs, then there exists a Model Structure on \( C(\mathcal{M}(A)) \). In this Model Structure, the weak equivalences are the homology isomorphisms, the cofibrations are the monomorphisms whose cokernels are in \( dg\bar{\mathcal{F}} \), and the fibrations are the epimorphisms whose kernels are in \( dg\bar{\mathcal{C}} \).

**Remark 3.1.** The above theorem is proved in [20, Theorem 2.2] under the assumption that the considered category is Abelian. We do not know if the category \( \mathcal{M}(A) \) is Abelian, but one can check that it satisfies all the conditions used in the proof of [20, Theorem 2.2]. Namely, it is an additive category having a generator and it is closed under extensions, monocokernels, epicokernels and direct limits.

In order to apply the above criterium, we will need to prove the following:

1. The pairs \( (\bar{\mathcal{F}}, dg\bar{\mathcal{C}}) \) and \( (dg\bar{\mathcal{F}}, \bar{\mathcal{C}}) \) are cotorsion pairs.
2. The pairs \( (\bar{\mathcal{F}}, dg\bar{\mathcal{C}}) \) and \( (dg\bar{\mathcal{F}}, \bar{\mathcal{C}}) \) are complete.
3. \( dg\bar{\mathcal{F}} \cap \mathcal{E} = \bar{\mathcal{F}} \) and \( dg\bar{\mathcal{C}} \cap \mathcal{E} = \bar{\mathcal{C}} \)

Before proving all these conditions, we need to fix the statements of [17, Lemma 3.8(7),(8)].

**Lemma 3.1.** Let \( M, N \) be two complexes of \( C(\mathcal{M}(A)) \) and \( C \), an object of \( \mathcal{M}(A) \). Then there exist canonical monomorphisms of abelian groups

\[
0 \to \text{Ext}^1_{\mathcal{M}(A)}(C, Z_n N) \to \text{Ext}^1_{C(\mathcal{M}(A))}(S^n(C), N)
\]

and

\[
0 \to \text{Ext}^1_{\mathcal{M}(A)}(M_n/B_n M, C) \to \text{Ext}^1_{C(\mathcal{M}(A))}(M, S^n(C))
\]

**Proof.** Let

\[
0 \to Z_n N \to T \to C \to 0
\]

be any extension in \( \text{Ext}^1_{\mathcal{M}(A)}(C, Z_n N) \), and let us construct the pushout of the inclusions \( Z_n N \to N_n \) and \( Z_n N \to T \).
We have a commutative diagram

\[
\begin{array}{ccccccccc}
0 & 0 \\
\downarrow & & \downarrow \\
0 & \to & Z_n \to & T & \to & C & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & N_n & \xrightarrow{g} & Q & \xrightarrow{h} & C & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
B_{n+1}N & \xrightarrow{B_{n+1}N} & B_{n+1}N \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

We have a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & N_{n-1} & \xrightarrow{\delta^{n-1}} & N_{n-1} & \to & 0 & \to & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \to & N_n & \xrightarrow{g} & Q & \xrightarrow{\delta^n} & C & \to & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \to & N_{n+1} & \xrightarrow{h} & C & \to & 0 & \to & 0 \\
\end{array}
\]

and hence, we get an extension \( \xi \equiv 0 \to N \to H \to S^n(C) \to 0 \) in \( \Ext^1_{C(2R(A))}(S^n(C), N) \).

This defines a map

\[
\Ext^1_{2R(A)}(C, Z_n N) \to \Ext^1_{C(2R(A))}(S^n(C), N).
\]

Clearly this map is a morphism of abelian groups. Let us check that it is injective. Assume that \( \xi \) splits and \( r : C \to Q \) is the corresponding excision in the \( n \)th component of \( \xi \). We get that \( h \circ r = 0 \) by the commutativity of the diagram. Thus, \( \text{Im}(r) \subseteq T \). Hence \( 0 \to Z_n N \to T \to C \to 0 \) splits. The proof of the second monomorphism is dual. \( \Box \)

**Remark 3.2.** The monomorphisms that appear in Lemma 3.1 are not isomorphisms in general, as claimed in \([17]\). For instance, if we consider the category \( C(R) \) of unbounded complexes of \( R \)-modules over a ring with identity \( R \), then \( \Ext^1_R(P, -) = 0 \) for any projective \( R \)-module \( P \). But the functor \( \Ext^1_{C(R)}(S^n(P), -) \neq 0 \), since \( S^n(P) \) is a dg-projective complex that is not projective. The same holds for our second monomorphism if we choose an injective \( R \)-module.
Let us now prove Condition (1) in Hovey’s criteria.

**Lemma 3.2.** The pairs \((\tilde{F}, dg\tilde{C})\) and \((dg\tilde{F}, \tilde{C})\) are cotorsion pairs in \(\mathcal{C}(\mathfrak{M}(A))\).

**Proof.** The cotorsion pair \((F, C)\) in \(\mathfrak{M}(A)\) is complete (by Theorem 3.2). So the result follows from [17 Proposition 3.6] applied to the pair \((\tilde{F}, \tilde{C})\) (by realizing that the part of the proof of [17 Proposition 3.6] involving [17 Lemma 3.8(7),(8)] can be replaced by Lemma 3.1). 

We check now Condition (2).

**Theorem 3.4.** The pair \((dg\tilde{F}, \tilde{C})\) is complete in \(\mathcal{C}(\mathfrak{M}(A))\).

**Proof.** We already know, by the proof of Theorem 3.2, that the pair \((\tilde{F}, \tilde{C})\) is cogenerated by the set \(S_n\). Let \(G\) be the h-unitary flat generator of \(\mathfrak{M}(A)\). We claim that the cotorsion pair \((dg\tilde{F}, \tilde{C})\) is cogenerated by the set \(\mathcal{L} = \{S^m(F) : m \in \mathbb{Z}, F \in S_n \cup \{G\}\}\).

It is easy to check that \(\mathcal{L} \subseteq dg\tilde{F}\), since \(S^l(F)_{l} \in \tilde{F}\), for any \(l \in \mathbb{Z}\) and any \(F \in S_n \cup \{G\}\). Moreover, for every exact complex \(M \in \tilde{C}\), \(\text{Hom}(S^l(F), M)\) is the complex

\[
\cdots \rightarrow \text{Hom}(F, M^l) \rightarrow \text{Hom}(F, M^{l+1}) \rightarrow \cdots
\]

which is obviously exact because \(Z_n M, B_n M \in C\). Therefore, \(\mathcal{L}^\perp = (dg\tilde{F})^\perp = \tilde{C}\).

Let us now show the converse. Let \(N \in \mathcal{L}^\perp\). We must check that \(N\) is exact and \(Z_n N \in C\). We first show that \(N\) is exact. This is equivalent to prove that any morphism \(S^n(G) \rightarrow N\) can be extended to a morphism \(D^n(G) \rightarrow N\), for every \(n \in \mathbb{Z}\). But this fact follows from the short exact sequence

\[
0 \rightarrow S^n(G) \rightarrow D^n(G) \rightarrow S^{n-1}(G) \rightarrow 0
\]

since \(\text{Ext}^1(S^{n-1}(G), N) = 0\).

Let us now check that \(Z_n N \in C\). We only need to show that \(\text{Ext}^1(\mathfrak{M}(A), F, Z_n N) = 0\), for any \(n \in \mathbb{Z}\) and any \(F \in S_n\), since \(S_n\) cogenerates the cotorsion pair \((F, C)\). By Lemma 3.1 we have a monomorphism of abelian groups

\[
0 \rightarrow \text{Ext}_C^1(\mathfrak{M}(A), Z_n N) \rightarrow \text{Ext}_C^1(\mathfrak{M}(A), S^n(F), N)
\]

and, as we are assuming that \(\text{Ext}_C^1(\mathfrak{M}(A), S^n(F), N) = 0\), we get that \(Z_n N \in C\). Hence the cotorsion pair \((dg\tilde{F}, \tilde{C})\) is complete by Theorem 2.1. 

Now, we will prove that \((\tilde{F}, dg\tilde{C})\) is complete. We will first need the following lemma. If \(L = (L_i)_{i \in \mathbb{Z}}\) is a complex, we will denote by \(|L|\) the cardinality of \(\bigoplus_{i \in \mathbb{Z}} L_i\).

**Lemma 3.3.** Let \(G\) be an h-unitary flat generator of \(\mathfrak{M}(A)\) and let us fix \(K \geq \max\{|G|, n_0\}\). For any complex \(F \in \tilde{F}\) and any element \(x \in F^k\) \((k \in \mathbb{Z}\) arbitrary), there exists a subcomplex \(L\) of \(F\) such that \(x \in L^k, |L| \leq K\) and \(L, F/L\) are in \(\tilde{F}\).

**Proof.** We may assume without loss of generality that \(k = 0\) and \(x \in F^0\). By Lemma 2.1 there exists a morphism \(h_x : G \rightarrow F^0\) and an element \(z_x \in G\) such that \(h(z_x) = x\). Consider then the exact complex

\[
(S1) \quad \cdots \rightarrow A_1^{-2} \rightarrow A_1^{-1} \delta^{-1} \rightarrow h_x(G) \delta^0 \rightarrow \delta^0(h_x(G)) \delta^1 \rightarrow 0
\]

where \(A_1^{-1}\) is the submodule of \(F^{-1}\) constructed as follows. We know that \(|h_x(G)| \leq K\), since \(|G| \leq K\). So we may find an \(A_1^{-1} \subseteq F^{-1}\) such that \(|A_1^{-1}| \leq K\) and
\[ \delta^{-1}(A_1^{-1}) = \ker(\delta^0|_{h_x(G)}) \]. Let us now choose an \( A_i^{-2} \leq F^{-2} \) with \( |A_i^{-2}| \leq 8 \) and \( \delta^{-2}(A_i^{-1}) = \ker(\delta^{-1}|_{A_i^{-1}}) \). We recursively repeat this argument for constructing all the \( A_n^{-n} \).

On the other hand, \( \ker(\delta^0|_{h_x(G)}) \leq \ker \delta^0 \) and we know that \( |\ker(\delta^0|_{h_x(G)})| \leq 8 \). So by Lemma 2.3 \( \ker(\delta^0|_{h_x(G)}) \) can be embedded in a pure and h-unitary submodule \( S_0^0 \) of \( \ker \delta^0 \) in such a way that \( |S_0^0| \leq 8 \). Consider the exact complex

\[ (S2) \quad \cdots \to A_2^{-2} \xrightarrow{\delta^{-2}} A_2^{-1} \xrightarrow{\delta^{-1}} h_x(G) + S_2^0 \xrightarrow{\delta^0} h_x(G) \xrightarrow{\delta^1} 0 \]

where the \( A_n^{-n} \)'s are constructed as above. Clearly \( \ker(\delta^0|_{h_x(G)} + S_2^0) = S_2^0 \), which is a pure and h-unitary submodule of \( \ker \delta^0 \). Moreover, \( |h_x(G) + S_2^0| \leq 8 + 8 = 8 \).

As \( \delta^0(h_x(G)) \subseteq \ker \delta^1 \), we can embed \( \delta^0(h_x(G)) \) in a pure and h-unitary submodule \( S_3^0 \) of \( \ker \delta^3 \) with \( |S_3^0| \leq 8 \), since \( |\delta^0(h_x(G))| \leq 8 \). Let us consider the exact complex

\[ (S3) \quad \cdots \to A_3^{-2} \xrightarrow{\delta^{-2}} A_3^{-1} \xrightarrow{\delta^{-1}} A_3^0 \xrightarrow{\delta^0} S_3^1 \xrightarrow{\delta^1} 0. \]

We have again that \( \ker(\delta^1|_{S_3^1}) = S_3^1 \), which is a pure and h-unitary submodule of \( \ker \delta^1 \).

We turn over and find a pure and h-unitary submodule \( S_4^0 \leq \ker \delta^0 \) with \( |S_4^0| \leq 8 \) and \( S_3^0 \supseteq \ker(\delta^0|_{A_4^0}) \). And then construct \( A_4^{-1} \leq F^{-1} \) such that

\[ (S4) \quad \cdots \to A_4^{-2} \xrightarrow{\delta^{-2}} A_4^{-1} \xrightarrow{\delta^{-1}} A_4^0 + S_4^0 \xrightarrow{\delta^0} S_3^1 \xrightarrow{\delta^1} 0 \]

is exact. Once more, \( \ker(\delta^0|_{A_4^0 + S_4^0}) = S_4^0 \leq \ker \delta^0 \) is a pure and h-unitary submodule.

Then we can find a pure and h-unitary submodule \( S_5^{-1} \leq \ker \delta^{-1} \) with \( |S_5^{-1}| \leq 8 \) and \( \ker(\delta^{-1}|_{A_5^{-1}}) \subseteq S_5^{-1} \). Let us now consider the exact complex

\[ (S5) \quad \cdots \to A_5^{-2} \xrightarrow{\delta^{-2}} A_5^{-1} + S_5^{-1} \xrightarrow{\delta^{-1}} A_5^0 + S_4^0 \xrightarrow{\delta^0} S_3^1 \xrightarrow{\delta^1} 0, \]

in which \( \ker(\delta^{-1}|_{A_5^{-1} + S_5^{-1}}) = S_5^{-1} \leq \ker \delta^{-1} \) is pure and h-unitary.

Our next step will be to find a pure and h-unitary submodule \( S_6^{-2} \leq \ker \delta^{-2} \) such that \( |S_6^{-2}| \leq 8 \) and \( \ker(\delta^{-2}|_{A_6^{-2}}) \subseteq S_6^{-2} \). And then consider the exact complex

\[ (S6) \quad \cdots \to A_6^{-3} \xrightarrow{\delta^{-3}} A_5^{-2} + S_6^{-2} \xrightarrow{\delta^{-2}} A_4^{-1} + S_5^{-1} \xrightarrow{\delta^{-1}} A_5^0 + S_4^0 \xrightarrow{\delta^0} S_3^1 \xrightarrow{\delta^1} 0 \]

in which \( \ker(\delta^{-2}|_{A_6^{-2} + S_6^{-2}}) = S_6^{-2} \leq \ker \delta^{-2} \) is a pure and h-unitary submodule.

Finally, we can prove by induction that, for any \( n \geq 4 \), we may construct an exact complex

\[ (Sn) \quad \cdots \xrightarrow{\delta^{-n+2}} A_{n-2}^{-n+3} \xrightarrow{\delta^{-n+4}} T_{n-3}^{-n+4} \xrightarrow{\delta^{-n+5}} T_{n-2}^{-n+5} \cdots \xrightarrow{\delta^{-1}} T_0^0 \xrightarrow{\delta^0} T_1^1 \xrightarrow{\delta^1} 0 \]

such that \( \ker(\delta^{-n+j}|_{T_{n-j}^{-j}}) \) is a pure and h-unitary submodule of \( \ker \delta^{-n+j} \forall j \geq 4 \) and all the terms have cardinality bounded by 8.

Let \( L = \lim_{n \to \infty} (Sn) \). It is straightforward to check that \( L \) is an exact complex

(since it is a direct limit of exact complexes). And that \( \ker(\delta^i|_{L_i}) \) is a pure and h-unitary submodule of \( \ker \delta^i \forall i \leq 1 \). Furthermore \( |L^i| \leq 8 \cdot 8 = 8 \) for any \( i \leq 1 \) and thus, \( |L| \leq 8 \).

This complex

\[ L = \cdots \xrightarrow{\delta^{-1}} L^1 \xrightarrow{\delta^1} L^{i+1} \xrightarrow{\delta^{i+1}} \cdots \xrightarrow{\delta^{-1}} L^0 \xrightarrow{\delta^0} L^1 \xrightarrow{\delta^1} L^2 \xrightarrow{\delta^2} 0 \cdots, \]
is a subcomplex of $F$ which satisfies that $x \in L^0$ and that $\ker(\delta^i|_{L^i})$ is a pure and h-unitary submodule of $\ker \delta^i$ for any $i \in \mathbb{Z}$. In particular, $\ker(\delta^i|_{L^i})$ is an h-unitary flat module for any $i \in \mathbb{Z}$ (since so is $\ker \delta^i$). Therefore, the complex $L$ is an h-unitary flat subcomplex of $F$ satisfying that $|L| \leq 8$.

To finish the proof we only need to check that $F/L = (F^i/L^i, \overline{\delta}^i)$ is an h-unitary flat complex. An easy computation shows that $\ker \overline{\delta}^i = \ker(\delta^i)/\ker(\delta^i|_{L^i})$. But, by construction, $\ker(\delta^i|_{L^i})$ is a pure and h-unitary submodule of $\ker \delta^i$, $\forall i \in \mathbb{Z}$. So $\ker(\delta^i)/\ker(\delta^i|_{L^i})$ is h-unitary and flat for any $i \in \mathbb{Z}$. Finally, $F/L$ is exact since both $F$ and $L$ are exact. Thus, $F/L$ is an h-unitary and flat complex. 

**Theorem 3.5.** The pair $(\overline{F}, dg\overline{C})$ is complete.

**Proof.** Let $F$ be an h-unitary and flat complex and choose an $x \in F^i$. We can find, by the above result, a subcomplex $L \subseteq \overline{F}$ of $F$ such that $x \in L^i$, $|L| \leq 8$ and the quotient complex $F/L$ is h-unitary and flat. Then we can mimic the proof of Theorem 3.2 for showing that the cotorsion pair $(\overline{F}, dg\overline{C})$ is complete. □

Let us denote by $J : \mathfrak{F}(A) \hookrightarrow R-\text{Mod}$ the embedding functor. As $\mathfrak{F}(A)$ has a generator (see Lemma 2.1) and $J$ preserves small colimits, the Special Adjoint Functor Theorem (see e.g. [12]) ensures the existence of a right adjoint functor $D : R-\text{Mod} \rightarrow \mathfrak{F}(A)$ of $J$. Thus, $\mathfrak{F}(A)$ is a coreflective subcategory of $R-\text{Mod}$ and limits in $\mathfrak{F}(A)$ are computed by applying the functor $D$ to the usual limits in $R-\text{Mod}$ (see e.g. [2]).

**Lemma 3.4.** $dg\overline{F} \cap \mathcal{E} = \overline{F}$ and $dg\overline{C} \cap \mathcal{E} = \overline{C}$ in $C(\mathfrak{M}(A))$.

**Proof.** Let us choose $X \in \overline{F}$, and $Y \in \overline{C}$. Then any map from $X$ to $Y$ is homotopic to zero and so $Hom(X,Y)$ is exact. Therefore $X \in dg\overline{F} \cap \mathcal{E}$. Conversely, suppose that $X \in dg\overline{F}$ is exact. We have to show that the h-unitary cycle module $Z_nX$ is flat, $\forall n \in \mathbb{Z}$. Let $C$ be any cotorsion complex in $C(R)$. As the embedding functor $J : C(\mathfrak{F}(A)) \rightarrow C(R)$ is exact (and thus, $D(C) \in \overline{C}$), we deduce that its right adjoint functor $D$ preserves cotorsion complexes and therefore, $Hom_{C(R)}(J(X),C) \cong Hom_{C(\mathfrak{F}(A))}(X,D(C))$ is exact. Hence, $J(X)$ is a dg-flat and exact complex in $R-\text{Mod}$. By the arguments used in the proof of [16, Lemma 4.4.8], we deduce that $J(X)$ is a flat complex and so, $Z_nJ(X) = Z_nX$ is flat. Therefore $X \in \overline{F}$.

Let us now check that $dg\overline{C} \cap \mathcal{E} = \overline{C}$. As before, we have that $\overline{C} \subseteq dg\overline{C} \cap \mathcal{E}$. Conversely, assume that $Y \in dg\overline{C}$ is exact. We must check that $Y \in \overline{C}$. By Theorem 3.2 there exists a short exact sequence

$$0 \rightarrow Y \rightarrow C \rightarrow H \rightarrow 0$$

with $C \in \overline{C}$ and $H \in dg\overline{F}$. Let us note that $H$ is exact, since so are $Y$ and $C$. And $H \in \overline{F}$ because $H \in dg\overline{F}$. But then the above short exact sequence splits, by Lemma 3.2 So $Y$ is a direct summand of $C$. And this means that $Y \in \overline{C}$, since $\overline{C}$ is closed under direct summands. □

4. A Unitless Monoidal Structure in $C(\mathfrak{F}(A))$

We devote this section to show that the Flat Model Structure defined in last section on $C(\mathfrak{M}(A))$ is compatible with the tensor product in $C(\mathfrak{F}(A))$ inherited
from the tensor product in $\mathcal{F}(A)$. Note that we may assume that any left $R$-module is also a right $R$-module by [22].

Let us note that $\mathcal{F}(A)$ with the induced tensor product of $R$–Mod has all the structure of a symmetric monoidal category apart from the unit object. This is known in the literature as a unitless monoidal category (see [19]). Moreover we can get a closed structure in $\mathcal{F}(A)$ by applying the functor $D$ after the usual internal Hom functor of $R$–Mod. So $\mathcal{F}(A)$ is a unitless closed symmetric monoidal category. This imposes a canonical unitless closed symmetric monoidal structure in $C(\mathcal{F}(A))$.

In order to prove that the model structure defined in $C(M(A))$ in last section is compatible with this tensor product of $C(\mathcal{F}(A))$, we need to prove the following technical Lemma. Recall that the class $\mathcal{L}$ appearing on it was introduced in the proof of Theorem 3.4.

**Lemma 4.1.** Let $M \in dg\tilde{\mathcal{F}}$. Then $M$ is a direct summand of an $\mathcal{L}$-filtered $dg$-h-unitary flat complex.

**Proof.** We know that $(dg\tilde{\mathcal{F}}, \tilde{\mathcal{C}})$ is a cotorsion pair in $C(M(A))$ cogenerated by $\mathcal{L}$, by Theorem 3.4. Thus, $dg\tilde{\mathcal{F}} = \perp(\mathcal{L}^\perp)$ and $\tilde{\mathcal{C}} = \mathcal{L}^\perp$. By [11], for every complex of h-unitary modules $K$, there exists a short exact sequence

$$0 \to K \to Y \to Z \to 0$$

with $Y \in \mathcal{L}^\perp = \tilde{\mathcal{C}}$ and $Z$, an $\mathcal{L}$-filtered complex (in particular, $Z \in dg\tilde{\mathcal{F}}$).

We are going to finish the proof by adapting [31, Lemmas 2.2 and 2.3]. Given any $Mdg\tilde{\mathcal{F}}$, there exists a short exact sequence

$$0 \to K \to \oplus_{n \in \mathbb{Z}} S^n(G) \to M \to 0$$

where $G$ is an h-unitary flat generator of $M(A)$. Now let

$$0 \to K \to Y \to Z \to 0$$

be any short exact sequence with $Y \in \tilde{\mathcal{C}}$ and $Z$, $\mathcal{L}$-filtered. Let us construct the pushout

$$
\begin{array}{cccccccc}
0 & & 0 \\
& & \\
& & \\
0 & \rightarrow & K & \rightarrow & \oplus_{n \in \mathbb{Z}} S^n(G) & \rightarrow & M & \rightarrow & 0 \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
0 & \rightarrow & Y & \rightarrow & Q & \rightarrow & M & \rightarrow & 0 \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
Z & \rightarrow & Z \\
& & & & & & & & \\
& & & & & & & & \\
0 & & 0
\end{array}
$$
As $\oplus_{n \in \mathbb{Z}} S^n(G)$ is trivially $\mathcal{L}$-filtered and $Z$ is $\mathcal{L}$-filtered, we see that $Q$ is also $\mathcal{L}$-filtered. And clearly $Y \in \mathcal{C} = \mathcal{L}^\perp$. Hence, as $M \in dg\mathcal{F} = \perp(\mathcal{L}^\perp)$, we get that the short exact sequence

$$0 \to Y \to Q \to M \to 0$$

splits. And thus, $M$ is a direct summand of the $\mathcal{L}$-filtered complex $Q$. \hfill \Box

We can now state the main result of this section.

**Theorem 4.1.** The model structure induced on $\mathcal{C}(M(A))$ by the flat cotorsion pair $(\mathcal{F}, \mathcal{C})$ is compatible with the symmetric closed unitless monoidal structure of $\mathcal{C}(\mathfrak{R}(A))$ induced by the usual tensor product of chain complexes.

**Proof.** We are going to use [20, Theorem 7.2], so we will need to check that

1. Every cofibration is pure.
2. If $X, Y \in dg\mathcal{F}$, then $X \otimes Y \in dg\mathcal{F}$.
3. If $X, Y \in dg\mathcal{F}$ and one of the them belongs to $\mathcal{F}$, then $X \otimes Y \in \mathcal{F}$.

As any cofibration is a monomorphism with cokernel in $dg\mathcal{F}$, it follows that it is a short exact sequence with h-unitary flat cokernel. Thus, it is a pure exact sequence. This shows that (1) holds.

Let us choose $X, Y \in dg\mathcal{F}$. By Lemma 4.1.1 we only need to check condition (2) for complexes of the form $S^n(F)$, with $F$ a h-unitary flat module. But in this case, $S^n(F) \otimes S^m(F') \cong S^{n+m}(F \otimes F')$, for any other h-unitary flat module $F'$. And, as the tensor product of two h-unitary flat modules is clearly h-unitary and flat, we conclude that $X \otimes Y$ is a direct summand of the direct limit of the directed system of complexes $S^i(G \otimes G')$ in $dg\mathcal{F}$ and hence, it is also complex in $dg\mathcal{F}$.

Let us now check condition (3). Suppose that $X \in \mathcal{F}$ and $Y \in dg\mathcal{F}$. $X \otimes Y$ is exact, because $X$ is exact. And $X \otimes Y \in dg\mathcal{F}$, by the previous remarks. Therefore $X \otimes Y \in \mathcal{F}$ by Lemma 3.4. \hfill \Box

5. Morita Invariance

We finish this paper by showing the invariance of the previous constructions under the equivalences induced by the Morita contexts considered in [28]. Namely, the definitions of $\mathfrak{M}(A)$ and $\mathfrak{R}(A)$ apparently depend on the fact that we fixed the embedding of $A$ in the extended ring $\hat{A}$. But we are going to prove in this section that we get isomorphic definitions of $\mathfrak{M}(A)$ and $\mathfrak{R}(A)$ if we consider any other embedding of $A$ as a two-sided ideal of a unital ring $R$.

**Theorem 5.1.** Let $A$ be a two-sided ideal of a unital ring $R$. Let us denote by $\mathfrak{M}(A)_R$, $\mathcal{F}_R$ and $\mathcal{C}_R$ the categories of h-unitary $R$-modules, h-unitary flat $R$-modules and h-unitary cotorsion $R$-modules respectively. Then there exist canonical equivalences $\mathfrak{M}(A)_R \cong \mathfrak{M}(A)$, $\mathcal{F}_R \cong \mathcal{F}$ and $\mathcal{C}_R \cong \mathcal{C}$.

**Proof.** Let $M \in \mathfrak{M}(A)_R$. By the comments of the previous sections, there exists an $\mathcal{F}_R$-resolution of $M$

$$\cdots \to F_1 \to F_0 \to M \to 0.$$ 

On the other hand, it is clear that $A \otimes \hat{A} M \cong A \otimes_R M$ for all $R$-module $M$ such that $AM = M$ (in particular for all $M \in \mathfrak{M}(A)_R$). Hence $\text{Tor}_j^\hat{A}(A, M) = 0, \forall j \geq 1$
and so $M \in \mathcal{M}(A)$. We get then that, for a given $R$-module $M$, $M \in \mathcal{M}(A)_R$ if and only if $M \in \mathcal{M}(A)$.

Conversely, given $M$ in $\mathcal{M}(A)$, we have that $A \otimes \bar{A} M \cong M$. So $A \otimes \bar{A} M$ has an $R$-module structure given by $r(a \otimes m) = ra \otimes m$ and therefore, $M$ has a unique $R$-module structure extending its $\bar{A}$-module structure. By the preceding paragraph it follows that $M \in \mathcal{M}(A)_R$. Hence we have a one-to-one correspondence between $h$-unitary module structures on any abelian group for the pairs $(R, A)$ and $(\bar{A}, A)$.

Now let us choose $F \in \mathcal{F}$. We have that $AF = F$. So there is a canonical isomorphism $M \otimes \bar{A} F \cong M \otimes_R F$ for all right $R$-module $M$. It follows that $F \in \mathcal{F}_R$.

Conversely, assume that $F' \in \mathcal{F}_R$ and let $N \to M$ be an injection of right $\bar{A}$-modules. We get an exact sequence of right $R$-modules

$$\text{Tor}^1_{\bar{A}}(M/N, R) \to N \otimes_{\bar{A}} R \to M \otimes_{\bar{A}} R.$$

But it is easy to check that $\text{Tor}^1_{\bar{A}}(M/N, R)A = 0$. So applying the exact functor $- \otimes_R F'$ to the previous exact sequence we get that $N \otimes_{\bar{A}} F' \to M \otimes_{\bar{A}} F'$ is injective and hence, $F' \in \mathcal{F}$.

The analogous statement for the subcategories $\mathcal{C}_R$ and $\mathcal{C}$ is now easy to prove by using the preceding paragraph and noting that an exact sequence $0 \to N \to M \to M/N \to 0$ in $\mathcal{M}(A)$ is splitting if and only if it splits in $\mathcal{M}(A)_R$. \(\square\)

**Corollary 5.1.** The previous $h$-unitary flat model structure on $\mathbb{C}(\mathcal{M}(A))$ is independent of the embedding of $A$ into a unital ring.

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