QCD ON A TREE

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Abstract

A model is proposed which can be regarded as a mean field approximation for pure lattice QCD and chiral field. It always possesses a phase transition between a strong-coupling phase (where it reduces to a one-plaquette integral) and a non-trivial weak-coupling one. For the U(N) gauge group, it is equivalent to some hermitian multi-matrix model. This analogy allows for determining possible large $N$ critical regimes thus generalizing the Gross-Witten phase transition in the one-plaquette model.
1 Introduction

The recent few years have seen the considerable development of the planar-diagram technique connected mostly with the matrix models of 2D gravity. Unfortunately, it seems that all those achievements has brought no new insights into large $N$ gauge theory, which was the original motivation for the method [1]. Nevertheless, as was demonstrated by the exact solution of QCD$_2$ on a sphere [2], a reduction to a hermitian matrix model can be very profitable technically.

In the present paper I establish a connection between a special class of hermitian multi-matrix models and a mean field approximation for pure lattice QCD. It enables for using the saddle-point technique for the former in the rather new framework. The most interesting phenomenon here is, probably, a large $N$ phase transition of the Gross-Witten type [3], which should have some stringy interpretation. For the one-plaquette model, the connection with 2D gravity was discussed in a number of papers [4].

The standard QCD mean field (MF), as was proposed by K.Wilson [5] (for review see [6]), suffers from the obvious drawback of being gauge dependent. I suggest a purely geometrical approach to the problem, which avoids the step of gauge fixing. This model, however, shares all limitations of any MF approximation at the price of being, in principle, soluble. One can say it learns nothing about QCD itself. Unfortunately, the same could be said about any other model solved so far.

The starting point is to substitute a regular D-dimensional lattice by an infinite (Cayley) tree constructed of two-dimensional plaquettes. The plaquettes are glued along their edges so that the tree is a simply connected covering of the lattice. Therefore, gauge theory defined on such a tree can be regarded as a MF approximation for lattice QCD in D dimensions.

A similar idea was put forward in Ref. [7] where gauge theory on a Cayley tree made of cubes was considered. As far as phase structures of lattice models are concerned, the cube-made Cayley tree might provide better accuracy than the plaquette-made one (although it is not clear a priori). However, the latter enjoys the property of being soluble in the large $N$ limit by a saddle-point technique, while the former can hardly be handled for continuous gauge groups. Another nice feature of the model under consideration is that it includes both chiral field and gauge theory on equal footing, actually interpolating between them.
The transfer-matrix formalism is very simple in our case. Let us consider a tree with one root, *i.e.*, to leave one of gauge variables, *u*, not integrated. Then, the corresponding partition function $I_V(u)$ for the rooted tree of a large volume $V$ obeys the equation

$$I_{pmV+1}(u) = \int \prod_{k=1}^{p} dx_k \; K(u \prod_{k=1}^{p} x_k) \prod_{k=1}^{p} I_{V}^{m}(x_k)$$

(1)

where the tree is assumed to be made of $(p+1)$-sided polygons, $m + 1$ on each link. If $p = 3$, it is a covering of the $D = \frac{m+3}{2}$ dimensional hypercubic lattice. The value $p = 1$ corresponds to the case when a plaquette-made tree degenerates into an ordinary one constructed of one-dimensional links. Hence, we have a model interpolating between the spin and gauge MF approximations. For the $SU(N)$ group, it interpolates between $D_{CF} = \frac{m+1}{2}$ dimensional chiral field and $D_{GT} = D_{CF} + 1$ lattice gauge theory.

The Boltzmann weight, $K(x)$, is a real positive class function. Two standard choices of it are: the Wilson one

$$K(x) = \exp \frac{N}{2g^2} \text{tr} (x + x^+)$$

(2)

and the heat-kernel

$$K(x) = \sum_r d_r e^{-\frac{g^2}{N} C_r} \chi_r(x)$$

(3)

where $g^2$ is a bare gauge coupling; $\chi_r(x)$ is a character of an irrep $r$; $C_r$ is a second Casimir; $d_r = \chi_r(I)$ is the dimension of $r$.

In the thermodynamical limit, the conveniently normalized quantity

$$J(u) = \lim_{V \to \infty} e^{-\frac{(V+1)}{mp} f} I_V(u)$$

(4)

(where $f$ is a free energy per volume) obeys the equation

$$J(u) = \int \prod_{k=1}^{p} dx_k \; K(u \prod_{k=1}^{p} x_k) \prod_{k=1}^{p} J^{m}(x_k)$$

(5)

and the free energy is given by the relation

$$f(g^2) = -\frac{pm - 1}{m + 1} \log \int du \; J^{m+1}(u)$$

(6)
Eq. (5) always has the trivial solution
\[ J_{\text{sc}} \equiv j_0 = \left[ \int dx \ K(x) \right]^{\frac{1}{mp-1}} \] (7)
In this case, the free energy coincides with the one-plaquette-model one
\[ f_{\text{sc}}(g^2) = \log \int dx \ K(x) \] (8)
This is the strong-coupling phase of the model.
In the weak-coupling phase, \( J(x) \) is a non-trivial class function:
\[ J(x) = \sum_r j_r \chi_r(x) \] (9)
where the sum runs over all irreps of a gauge group. Eq. (8) can be rewritten in terms of the Fourier coefficients \( j_r \) as
\[ j_r = \lambda_r \left[ \frac{1}{d_r} \sum_{s_1, \ldots, s_m} j_{s_k} \int dx \chi_r(x) \prod_{k=1}^{m} \chi_{s_k}(x) \right]^p \] (10)
where \( \lambda_r \) are Fourier coefficients of the Boltzmann weight.
The free energy can be rewritten as the single sum over representations
\[ f(g^2) = -\frac{pm - 1}{m + 1} \log \sum_r d_r j_r \left( \frac{j_r}{\lambda_r} \right)^{\frac{1}{p}} \] (11)
A glueball spectrum can be determined from the eigenvalue problem:
\[ e^{-m_G v_r} = \sum_t M_{rt} v_t \] (12)
where the analog of the transfer matrix in our case is simply
\[ M_{rt} = \lambda_r \left[ \frac{1}{d_r} \sum_{s_1, \ldots, s_m} j_{s_k} \int dx \chi_r(x) \prod_{k=1}^{m} \chi_{s_k}(x) \right]^{p-1} \]
\[ \frac{1}{d_r} \sum_{s_1, \ldots, s_{m-1}} j_{s_k} \int dx \chi_r(x) \chi_{t}(x) \prod_{k=1}^{m-1} \chi_{s_k}(x) \] (13)
Eq. (12) always has the trivial eigenvalue \( m_G = 0 \), which corresponds to the identity operator (i.e., the partition function). The number of possible
excitations in the system is equal to the number of irreducible representations of a gauge group. In the weak-coupling phase they are all excited. However, non of them can become massless and, hence, there is no continuum limit associated with the model.

2 Phase transition

Let us assume that, at some critical value $g_2^*$, the Fourier coefficients of all non-trivial representations in Eq. (3) vanish (i.e., the weak-coupling solution transforms smoothly into the strong-coupling one). Then, in the vicinity, $\Delta g^2 = g_2^* - g^2 \ll 1$, the most essential contribution comes from the fundamental representation, $r = 1$,

$$j_0 = O(1); \quad j_1 = O(\Delta g^2); \quad j_r = o(\Delta g^2), \text{ for } r \neq 0, 1 \quad (14)$$

After expanding Eq. (10) in $j_1$, one finds

$$j_0 = \lambda_0 j_0^{mp} + O(j_1^2)$$

$$j_1 = \lambda_1 [\frac{m}{d_1} j_0^{m-1} j_1]^p + \ldots \quad (15)$$

From which it follows that, for $p \neq 1$, the smooth transition between the strong and weak-coupling solutions is impossible. For $p = 1$, the critical value $g_2^*$ is determined by the equation

$$\frac{1}{m} = \frac{\lambda_1(g_2^*)}{d_1\lambda_0(g_2^*)} \quad (16)$$

For $SU(N)$ with the Wilson weight, one finds

$$\frac{1}{m} = \frac{1}{N} \langle \text{tr} U \rangle_{N \to \infty} = \frac{1}{g_2^2} \quad (17)$$

Hence, the phase transition takes place when the one-plaquette model is in its strong-coupling regime (i.e., above the Gross-Witten critical point). Also notice that, in the $m \to \infty$ limit, the strong-coupling phase disappears. For the heat-kernel, one finds simply $g_2^2 = \log m$ independent of $N$. 

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The lowest two representations play a special role in the $p = 1$ case. Therefore, the simplest $Z_2$ model may be rather instructive as far as the nature of the phase transition is concerned. In this case one can represent

$$J(x) = j_0 + j_1 x = \rho e^{x\phi}$$  \hspace{1cm} (18)

where $x = \pm 1$ is a $Z_2$ variable and

$$j_0 = \rho \cosh \phi \quad j_1 = \rho \sinh \phi$$  \hspace{1cm} (19)

Let me choose $\lambda_0 = 1$, then Eq. (10) takes the form

$$\rho \cosh \phi = (\rho^m \cosh m\phi)^p$$
$$\rho \sinh \phi = \lambda_1 (\rho^m \sinh m\phi)^p$$

or, equivalently,

$$\tanh \phi = \lambda_1 \tanh^p m\phi \quad \rho = \left( \frac{\cosh \phi}{\cosh^p m\phi} \right)^{\frac{1}{mp-1}}$$  \hspace{1cm} (21)

Hence, a solution can always be determined from some algebraic equation with respect to $x = \tanh \phi$.

For example, for $m = 3$ and $p = 1$ (2D Ising), one finds the equation

$$x = \lambda_1^{3} + x^3$$  \hspace{1cm} (22)

from which

$$x = \sqrt[3]{\frac{3\lambda_1 - 1}{3 - \lambda_1}} \quad f = \frac{1}{2} \log \frac{8\lambda_1^3}{6\lambda_1 - \lambda_1^2 - 1}$$  \hspace{1cm} (23)

And the transition is of the second order: $f'(\frac{1}{3}) = 0$. This is the case, if $p = 1$, for arbitrary $m$ and all compact groups. In general, it can be easily proven by using the representation (11) for the free energy.

In the $Z_2$ model, there is only one excitation for which, in the previous example ($m = 3$, $p = 1$), one finds the mass $m_G = \log \frac{4\lambda_1}{(1-\lambda_1)^2}$. One can easily show that, at the critical point for all $m$, $m_G(\lambda_1^*) = \log m$. Therefore, the transition does not produce any continuum limit.

Of course, when $p = 1$, one just repeats the standard spin MF approximation [8]. If $p \neq 1$, the phase transition between the strong and weak-coupling
phases is of the first order. Eq. (21) has obviously no real solutions for $\lambda_1$ small enough. Hence, the weak coupling branch disappears somewhere being already meta-stable. The first-order transition point can be determined from the equation

$$\frac{p+1}{j_0^p} + \lambda_1 \frac{p+1}{j_1^p} = 1$$

which is quite a standard numerical problem.

3 Connection with multi-matrix models

The Fourier representation (11) is not convenient for the investigation of the weak-coupling phase. In this case, the original matrix variables are more suitable. Let us choose the Boltzmann weight in the form of the $U(N)$ heat-kernel:

$$K(x) = \sum_r d_r e^{-\frac{g^2}{2N} C_r} \chi_r(x)$$

where, in terms of the highest weight components ($m_1 \geq m_2 \geq \ldots \geq m_N$),

$$C_r = \sum_{k=1}^{N} \left( m_k - k + N \right)^2$$

is a conveniently redefined second Casimir. For diagonal matrices, $x = e^{i\alpha}$, one has the Weyl formula for characters

$$\chi_r(e^{i\alpha}) = \frac{\Delta_r(e^{i\alpha})}{\Delta(e^{i\alpha})}$$

where

$$\Delta_r(e^{i\alpha}) = \det_{(j,k)} e^{i(m_k-k+N)\alpha_j}$$

$$\Delta(e^{i\alpha}) = \det_{(j,k)} e^{i(N-k)\alpha_j} = \prod_{j<k} (e^{i\alpha_j} - e^{i\alpha_k})$$

$$d_r = \chi_r(I) = \prod_{j<k} \left( 1 + \frac{m_j - m_k}{k - j} \right)$$
is the dimension of an irrep \( r \).

Let us substitute (25) in (5) and integrate over angular parts of all variables:

\[
\int \prod_{i=1}^{p} dS_i \ K(e^{i\alpha} \prod_{k=1}^{p} S_k e^{i\beta_k} S_k^+) = \sum_r d_r e^{-\frac{g^2}{2N} C_r} \chi_r(e^{i\alpha}) \prod_{k=1}^{p} \chi_r(e^{i\beta_k}) =
\]

\[
\left( \prod_{n=1}^{N-1} n! \right)^{p-1} \frac{1}{\Delta(e^{i\alpha})} \sum_{\ell_1 > \ell_2 > \ldots > \ell_N} [\Delta(\ell)]^{1-p} \exp \left( - \frac{g^2}{2N} \sum_{k=1}^{N} (\ell_k^2) \det e^{i\ell_k \alpha_j} \right)
\]

\[
\prod_{i=1}^{p} \frac{\det e^{i\ell_i \beta_k}}{\Delta(e^{i\beta_k})} = \frac{1}{N!} \left( \prod_{n=1}^{N-1} n! \right)^{p-1} \frac{1}{\Delta(e^{i\alpha})} \prod_{k=1}^{N} \left( \sum_{n_k = -\infty}^{+\infty} \int_{-\infty}^{+\infty} d\lambda_k \right) [\Delta(\lambda)]^{1-p}
\]

\[
\exp \left[ - \frac{g^2}{2N} \sum_{k=1}^{N} \lambda_k^2 + i \sum_{k=1}^{N} \lambda_k (\alpha_k + 2\pi n_k) \right] \prod_{i=1}^{p} \frac{\det e^{i\lambda_i \beta_k}}{\Delta(e^{i\beta_k})}
\]

The last equality holds owing to Poisson’s formula. Now, Eq. (5) takes the form (after renormalizing \( J(x) \))

\[
J(e^{i\alpha}) = \frac{1}{\Delta(e^{i\alpha})} \prod_{k=1}^{N} \left( \int_{-\infty}^{+\infty} d\lambda_k \right) \Delta(\lambda) \exp \left[ - \frac{g^2 N}{2} \sum_{k=1}^{N} \lambda_k^2 \right. + iN \sum_{k=1}^{N} \lambda_k (\alpha_k + 2\pi n_k) \left. \left[ \int_{0}^{2\pi} \prod_{k=1}^{N} d\beta_k \ \Delta(e^{i\beta}) \frac{e^{iN \sum_{k=1}^{N} \lambda_k \beta_k}}{\Delta(\lambda)} J^m(e^{i\beta}) \right] \right]^{p}
\]

Let us introduce a new function \( F(\alpha) \) such that

\[
\Delta(e^{i\alpha}) J(e^{i\alpha}) = \sum_{\{n \in \mathbb{Z}^N\}} \Delta(\alpha + 2\pi n) F(\alpha + 2\pi n)
\]

which obeys the equation

\[
F(\alpha) = \frac{1}{\Delta(\alpha)} \int_{-\infty}^{+\infty} \prod_{k=1}^{N} d\lambda_k \ \Delta(\lambda) \exp \left[ - \frac{g^2 N}{2} \sum_{k=1}^{N} \lambda_k^2 + iN \sum_{k=1}^{N} \lambda_k \alpha_k \right. \left. \left[ \int_{-\infty}^{+\infty} \prod_{k=1}^{N} d\beta_k \ \Delta(\beta) \frac{e^{iN \sum_{k=1}^{N} \lambda_k \beta_k}}{\Delta(\lambda)} J^{m-1}(e^{i\beta}) F(\beta) \right] \right]^{p}
\]
If one introduces hermitian matrices $A, \Lambda, B$ having eigenvalues $\alpha, \lambda, \beta$ correspondingly, then one can rewrite Eq. (32) as the matrix integral equation

$$F(A) = \int d^N \Lambda e^{-\frac{1}{2}g^2 N \text{tr} \Lambda^2 + i N \text{tr} \Lambda A} \left[ \int d^N B e^{i N \text{tr} \Lambda B + V[B]} F(B) \right]^p$$

$$= \left( \frac{2\pi}{g} \right)^{N^2} \int \prod_{i=1}^p d^N B_i e^{-\frac{N}{2g^2} \text{tr} (A + \sum B_i)^2 + \sum V[B_i]} \prod_{i=1}^p F(B_i)$$

where the effective potential is determined by the equation

$$V[B] = (m - 1) \log \sum_{\{n \in \mathbb{Z}^N\}} \frac{\Delta(\beta + 2\pi n)}{\Delta(e^{i\beta})} F(\beta + 2\pi n)$$

$V[B]$ is a symmetric non-singular function of eigenvalues, hence, can be expanded in Schur polynomials. It is easily checked that, if $m = 1$, $F(\beta)$ is Gaussian and $J(e^{i\beta})$ is given by the heat-kernel in the Dowker form [9].

## 4 Large $N$ solution

If $p = 1$, nothing prevents one from introducing an arbitrary potential, $N \text{tr} U(B)$, for the chiral field. In this case, what one ends up with is a unitary analog of the Bethe-tree matrix model considered in Ref. [10]. One can use a similar technique in both cases. The first step is to write down the self-consistency equation for the resolvent matrix [11]:

$$F(A) = \left( \frac{2\pi}{g} \right)^{N^2} \int d^N B e^{-\frac{N}{2g^2} \text{tr} (A + B)^2 + V[B] - N \text{tr} U(B)} F(B) = \left( \frac{2\pi}{g} \right)^{N^2}$$

$$\int d^N B \frac{1}{N} \text{tr} \left[ \left( z + A + \frac{g^2}{N} \frac{\partial}{\partial A} \right) \frac{1}{z - B} \right] e^{-\frac{N}{2g^2} \text{tr} (A + B)^2 + V[B] - N \text{tr} U(B)} F(B)$$

$$= \frac{1}{N} \sum_{k=1}^N \left\{ (z + \frac{g^2}{N} \frac{\partial}{\partial \alpha_k} + \alpha_k) G_k(z) + \frac{g^2}{N} \sum_{j \neq k} \frac{G_k(z) - G_j(z)}{\alpha_k - \alpha_j} \right\}$$

(35)
where

\[ G_k(z) = \left( \frac{2\pi}{g} \right) \frac{N^2}{g} \int dB \left[ \frac{1}{z - B} \right]_k e^{-\frac{N}{2\pi} \text{tr}(A+B)^2 + V[B] - N\text{tr}U(B)} F(B) \] (36)

are diagonal matrix elements of the resolvent matrix. This equation is a recursive relation for moments of \( B \).

Let us introduce two functions

\[ f(x) = \left( \frac{1}{N} \text{tr} \frac{1}{x - A} \right) \sim \int dy \frac{\rho(y)}{x - y} \]

\[ F(x, z) = \left( \frac{1}{N} \text{tr} \frac{1}{x - A} \frac{1}{z - B} \right) \sim \int dy \frac{\rho(y)W(y, z)}{x - y} \] (37)

\( \rho(y) \) is a density of eigenvalues; \( W(y, z) \) is a real function on a support of \( \rho(y) \). As we are looking for a homogeneous ground state, \( f(x) \) is the same at all sites of the tree and \( F(x, z) \) is symmetric: \( F(x, z) = F(z, x) \).

From Eq. (35) it follows that, at \( N = \infty \), \( W(x, z) \) obeys the equation

\[ (z + g^2w(x) + x)W(x, z) + g^2 \int dy \rho(y) \frac{W(x, z) - W(y, z)}{x - y} = 1 \] (38)

where

\[ w(x) = \lim_{N \to \infty} \frac{1}{N} \frac{\partial}{\partial \alpha_k} \log F[A] \bigg|_{\alpha_k = x} = \frac{\partial}{\partial x} \frac{\delta}{\delta \rho(x)} \lim_{N \to \infty} \frac{1}{N^2} \log F \] (39)

As was first noticed in Ref. [12], equations of this type can be solved by the Riemann-Hilbert method. The outcome of which is the following integral representation

\[ F(x, z) = 1 - \exp \int \frac{dy}{2\pi i} \frac{1}{x - y} \log \frac{z - u_+(y)}{z - u_-(y)} \] (40)

where, in our case,

\[ u_\pm(x) = -x - g^2(w(x) + \text{Re} f(x) \pm i \text{Im} f(x)) \] (41)
and the integral goes along a support of $\text{Im} \ f(y) = \pi \rho(y)$. Off the support of $\rho(x)$, $\text{Re} \ f(x)$ and $\text{Im} \ f(x)$ continue analytically as two independent holomorphic functions. Eq. (40) makes sense as a set of recursive integral relations obtained by expanding both sides in inverse powers of $x$ and $z$ (see Ref. [10] for details).

As $F(x, z)$ is symmetric, the following equation holds

$$u_+(u_-(x)) = x$$

(42)

The function $w(x)$ can be determined from the saddle-point equation

$$2 \text{Re} \ f(x) + 2w(x) + \frac{\partial}{\partial x} \frac{\delta}{\delta \rho(x)} \lim_{N \to \infty} \frac{1}{N^2} V[x] - U'(x) = 0$$

(43)

which is, in general, a complicated non-linear integral equation. However, for $g^2$ small enough, we can neglect in the large $N$ limit all non-trivial winding numbers in Eq. (34) (i.e., put $n_k = 0 \ \forall k$), then Eq. (43) takes the simple form

$$2 \text{Re} \ f(x) + (m+1)w(x) - (m-1) \int dy \rho(y) \left( \frac{1}{2} \cot \frac{x-y}{2} - \frac{1}{x-y} \right) - U'(x) = 0$$

(44)

or

$$w(x) = \frac{1}{m+1} \left\{ U'(x) - 2 \text{Re} \ f(x) + (m-1) \sum_{n=-\infty}^{+\infty} \left[ f(x - 2\pi n) + \frac{1}{2\pi n} \right] \right\}$$

(45)

The prime means that the $n = 0$ term in the sum is omitted. This formula makes sense only when a support of $\rho(x)$ lies inside the interval $(-\pi, +\pi)$, i.e., when there is a gap in the eigenvalue distribution of original unitary matrices.

If $p > 1$, any local potential for unitary variables spoils the gauge invariance of the model. However, one can introduce a non-gaussian potential for the auxiliary field $\Lambda$ in Eq. (33). It corresponds to taking an arbitrary Boltzmann weight. In this case, one cannot easily integrate out the auxiliary field. Nevertheless, our method can be generalized to the inhomogeneous system.
Now, one has two different external field problems and has to introduce three functions

\[ f(x) = \left( \frac{1}{N} \text{tr} \frac{1}{x - B} \right)_{N \to \infty} = \int dy \frac{\rho(y)}{x - y} \]

\[ \varphi(z) = \left( \frac{1}{N} \text{tr} \frac{1}{z - \Lambda} \right)_{N \to \infty} = \int d\lambda \frac{\eta(\lambda)}{z - \lambda} \]

\[ F(x, z) = \left( \frac{1}{N} \text{tr} \frac{1}{x - B \ z - \Lambda} \right)_{N \to \infty} = \int dy \frac{\rho(y)W(y, z)}{x - y} = \int d\lambda \frac{\eta(\lambda)\Omega(\lambda, x)}{z - \lambda} \]

Both \( W(x, z) \) and \( \Omega(\lambda, x) \) obey equations analogous to (38) and \( F(x, z) \) has two different integral representations of the type (40). For example,

\[ I(\lambda) = \int dB \ e^{iN \text{tr} \Lambda B + V[B]} F(B) = \]

\[ \int dB \frac{1}{N} \text{tr} \left[ \left( x + i \frac{\partial}{N \partial \lambda} \right) \frac{1}{x - B} \right] e^{iN \text{tr} \Lambda B + V[B]} F(B) = \]

\[ \frac{1}{N} \sum_{k=1}^{N} \left\{ \left( x + i \frac{\partial}{N \partial \lambda_k} \right) G_k(x) + \frac{i}{N} \sum_{j \neq k} \frac{G_k(x) - G_j(x)}{\lambda_k - \lambda_j} \right\} \]

where \( G_k(x) \) are diagonal elements of the resolvent matrix as in Eq. (36). Then one finds

\[ (x + i\omega(\lambda))\Omega(\lambda, x) + i \int d\mu \eta(\mu) \frac{\Omega(\lambda, x) - \Omega(\mu, x)}{\lambda - \mu} = 1 \]

where

\[ \omega(\lambda) = \lim_{N \to \infty} \frac{1}{N} \frac{\partial}{\partial \lambda_k} \log I(\Lambda) \bigg|_{\lambda_k = \lambda} = \frac{\partial}{\partial \lambda} \frac{\delta}{\delta \eta(\lambda)} \lim_{N \to \infty} \frac{1}{N^2} I(\Lambda) \]

The saddle-point equation with respect to \( \Lambda \) gives

\[ 2\text{Re} \varphi(\lambda) - U'(\lambda) + (p + 1)\omega(\lambda) = 0 \]

where \( U(\lambda) \) is an arbitrary, in principle, potential.
Thus, one finds for $F(x, z)$ the representation

$$F(x, z) = 1 - \exp \int \frac{d\lambda}{2\pi i} \frac{1}{z - \lambda} \log \frac{x - v_+(\lambda)}{x - v_-(\lambda)}$$

(51)

where

$$v_\pm(\lambda) = \frac{1}{i} \left\{ \frac{1}{p + 1} U''(\lambda) + \frac{p - 1}{p + 1} \text{Re} \varphi(\lambda) \pm i \text{Im} \varphi(\lambda) \right\}$$

(52)

In a close analogy, one finds

$$(z + iw(x))W(x, z) + i \int dy \rho(y) \frac{W(x, z) - W(y, z)}{x - y} = 1$$

(53)

and the saddle-point equation for a distribution with a gap:

$$2\text{Re} f(x) + (m + 1)w(x) - (m - 1) \sum_{n=-\infty}^{+\infty} \left[ f(x - 2\pi n) + \frac{1}{2\pi n} \right] = 0$$

(54)

Hence,

$$F(x, z) = 1 - \exp \int \frac{dy}{2\pi i} \frac{1}{x - y} \log \frac{z - u_+(y)}{z - u_-(y)}$$

(55)

where

$$u_\pm(x) = \frac{1}{i} \left\{ \frac{m - 1}{m + 1} \text{Re} f(x) + \frac{m - 1}{m + 1} \sum_{n=-\infty}^{+\infty} \left[ f(x - 2\pi n) + \frac{1}{2\pi n} \right] \pm i \text{Im} f(x) \right\}$$

(56)

And, instead of Eq. (42), there are two equations

$$u_+(v_-(x)) = x \quad u_-(v_+(x)) = x$$

(57)

5 Critical regimes

Following Ref. [10], one can determine possible large $N$ critical regimes. Eqs. (42) and (57) allows, in principle, for constructing the functions $f(x)$.
and \( \varphi(x) \) thus solving the model at large \( N \). Practically, it is a very complicated problem. However, universal behavior is determined by a scaling of the imaginary parts of \( f(x) \) and \( \varphi(x) \) near their edges, which are, in general, branching points:

\[
f(x) = f_{\text{reg}}(x) + c(x - x_0)^{1+\gamma} \ldots \quad \varphi(\lambda) = \varphi_{\text{reg}}(\lambda) + c'(\lambda - \lambda_0)^{1+\gamma} \ldots
\]

\( f_{\text{reg}}(x) \) and \( \varphi_{\text{reg}}(x) \) are regular parts of the functions at the branching points.

Let us start with the \( p = 1 \) model. We are interested in the situation when the edges of the distribution of eigenvalues of original unitary matrices collide. It corresponds to the case when the edges of \( \text{Im} f(x) \) in Eq. (41) touch branching points of \( f(x \pm 2\pi) \). Let us expand all quantities in Eq. (41) near one of the collision points (by redefining the variables one can always place it at the origin):

\[
u_\pm(x) = ax + b\frac{m-1}{m+1}(\cos \pi \gamma + 1)e^{-i\pi \gamma}x^{1+\gamma} + ib\sin \pi \gamma e^{-i\pi \gamma}x^{1+\gamma} + \ldots
\]

By substituting it in Eq. (42), one obtains two equations

\[
a^2 = 1
\]

and

\[
\frac{m-1}{m+1}(\cos \pi \gamma + 1) - i\sin \pi \gamma + a^\gamma \left[ \frac{m-1}{m+1}(\cos \pi \gamma + 1) + i\sin \pi \gamma \right] = 0
\]

The simplest possibility is \( a = 1 \), when one finds

\[
\cos \pi \gamma = -1
\]

which is the standard Gross-Witten singularity always possible in the model. By tuning the potential \( U(x) \), one can reach a multi-critical point when \( a = -1 \), then there are two branches: (i) \( a^\gamma = e^{i\pi \gamma} \) yielding the equation for \( \gamma \)

\[
\cos \pi \gamma = \frac{1}{m}
\]
and (ii) $a^\gamma = e^{-i\pi \gamma}$ yielding

$$\cos \pi \gamma = m$$

(64)

Let me remind a reader that $m$ is connected with the effective space dimension as $m = 2D - 1$. The obvious duality $m \to \frac{1}{m}$ is just a manifestation of the duality $D \to \frac{D}{2D-1}$ in the Bethe-tree matrix model found in Ref. [10]. The solution fits two soluble cases: $m = -1$, which corresponds to the one-plaquette model, and $m = 1$, which is just a one-dimensional matrix chain. The case $m = 0$ corresponds to a two-matrix model (as we have introduced an arbitrary potential, it is not simply reducible to a one-matrix integral any more).

If $p > 1$, there are two functions obeying Eq. (57). Expanding them as

$$u_\pm(x) = ax - b \frac{m - 1}{m + 1} (\cos \pi \gamma + 1) e^{-i\pi \gamma} x^{1+\gamma} \pm ib \sin \pi \gamma e^{-i\pi \gamma} x^{1+\gamma} + \ldots$$

$$v_\pm(x) = a'x - b' \frac{p - 1}{p + 1} \cos \pi \gamma e^{-i\pi \gamma} x^{1+\gamma} \pm ib' \sin \pi \gamma e^{-i\pi \gamma} x^{1+\gamma} + \ldots$$

and substituting the expansions in Eqs. (57), one finds that $aa' = 1$ and

$$a' b \frac{m - 1}{m + 1} (\cos \pi \gamma + 1) \pm i \sin \pi \gamma - b' a^{1+\gamma} \left[ - \frac{p - 1}{p + 1} \cos \pi \gamma \pm i \sin \pi \gamma \right] = 0$$

(65)

This is a degenerate system of linear equations for the real and imaginary parts. Equating to zero the corresponding determinant, one finds

$$\cos \pi \gamma = -\frac{1}{1 + \frac{p-1}{p+1} \frac{m+1}{m-1}}$$

(67)

At $p = 1$, this formula reproduces Eq. (62) and, at $m = 1$, gives $\gamma = \frac{1}{2}$, which is the simplest allowed singularity in a general matrix model.

The critical regime (67) corresponds to the situation when the potential for the $\Lambda$ variable becomes critical (in the standard matrix-model sense) at the same moment as the edges of the density for the unitary matrices collide. It is a multi-critical point with respect to the Gross-Witten singularity, which is always allowed and corresponds to the case where $u_+(x)$ has a quadratic extremum matching with a square-root branching point of $v_-(x)$. 


6 Discussion

Our method allows for determining possible large $N$ critical regimes without really solving a model. It means that some critical points can correspond to non-stable or non-unitary models. It was indeed the case for the Bethe-tree matrix model at $D > 1$, where critical potentials appeared to be, in general, complex \[10\]. As our MF approximation for chiral field is quite similar to the one for hermitian field, one can expect that the $m > 1$ branch (\[63\]) for the $p = 1$ model exists only owing to the formal duality, in the sense of analytical continuation into an unphysical region of parameters. On the other hand, the scaling (\[67\]) in the gauge MF model should be quite sensible for $p > 1$ and $m > 1$ as corresponds to the simplest reachable singularity. However, to determine a critical form of the Boltzmann weight is the crux of our approach.

Usually, the Gross-Witten transition is considered as a pure lattice artifact having no physical meaning. However, it is a very general phenomenon taking place in any model reducible to a saddle-point problem for a unitary-matrix-valued master field.

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