DECIDABILITY AND INDEPENDENCE OF CONJUGACY PROBLEMS
IN FINITELY PRESENTED MONOIDS

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Abstract. There have been several attempts to extend the notion of conjugacy from groups to monoids. The aim of this paper is to study the decidability and independence of conjugacy problems for three of these notions (which we will denote by $\sim_p$, $\sim_o$, and $\sim_c$) in certain classes of finitely presented monoids. We will show that in the class of polycyclic monoids, $p$-conjugacy is "almost" transitive, $c$ is strictly included in $p$, and the $p$- and $c$-conjugacy problems are decidable with linear complexity. For other classes of monoids, the situation is more complicated. We show that there exists a monoid $M$ defined by a finite complete presentation such that the $c$-conjugacy problem for $M$ is undecidable, and that for finitely presented monoids, the $c$-conjugacy problem and the word problem are independent, as are the $c$-conjucacy and $p$-conjugacy problems.

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1. Introduction

The well-known notion of conjugacy from group theory can be extended to monoids in many different ways. The authors dealt with four notions of conjugacy in monoids in [1, 2]. The present paper can be considered an extension of this work. Any generalization of the conjugacy relation to general monoids must avoid inverses. One of the possible formulations, spread by Lallement [20] for a free monoid $M$, was the following relation:

$$a \sim_p b \iff \exists u, v \in M : a = uv \text{ and } b = vu.$$ (Lallement credited the idea of the relation $\sim_p$ to Lyndon and Schützenberger [22].) If $M$ is a free monoid, then $\sim_p$ is an equivalence relation on $M$ [20, Corollary 5.2], and so it can be regarded as a conjugacy in $M$. In a general monoid $M$, the relation $\sim_p$ is reflexive and symmetric, but not transitive. The transitive closure $\sim^*_p$ of $\sim_p$ has been defined as a conjugacy relation in a general semigroup [13,18,19]. (If $a \sim_p b$ in a general monoid, we say that $a$ and $b$ are primarily conjugate [19], hence our subscript in $\sim_p$).

Another relation that can serve as a conjugacy in any monoid is defined as follows:

$$a \sim_o b \iff \exists g, h \in M : ag = gb \text{ and } bh = ha.$$ (This relation was defined by Otto for monoids presented by finite Thue systems [30], but it is an equivalence relation in any monoid. Its drawback – as a candidate for a conjugacy for general monoids – is that it reduces to the universal relation $M \times M$ for any monoid $M$ that has a zero.

To remedy the latter problem, three authors of the present paper introduced a new notion of conjugacy [2], which retains Otto’s concept for monoids without zero, but does not reduce to $M \times M$ if $M$ has a zero. The main idea was to restrict the set from which conjugators can be chosen. For a monoid $M$ with zero and $a \in M \setminus \{0\}$, let $\mathcal{P}(a)$ be the set $\{g \in M : (\forall m \in M) mg = 0 \Rightarrow mna = 0\}$, and define $\mathcal{P}(0)$ to be $\{0\}$. If $M$ has no zero, we agree that $\mathcal{P}(a) = M$, for every $a \in M$. Following [2], we define a relation $\sim_c$ on any monoid $M$ by

$$a \sim_c b \iff \exists g \in \mathcal{P}(a) \exists h \in \mathcal{P}(b) : ag = gb \text{ and } bh = ha.$$ The relation $\sim_c$ is an equivalence relation on an arbitrary monoid $M$. Moreover, if $M$ is a monoid without zero, then $\sim_c = \sim_o$; and if $M$ is a free monoid, then $\sim_c = \sim_o = \sim_p$. In the case when $M$ has
a zero, the conjugacy class of 0 with respect to \( \sim_c \) is \{0\}. Throughout the paper we shall refer to \( \sim_i \), where \( i \in \{p, o, c\} \), as \( i \)-conjugacy.

The aim of this paper is to study the decidability and independence of the \( i \)-conjugacy problems in some classes of finitely presented monoids.

It is well-known that the conjugacy problem for finitely presented groups is undecidable; that is, there exists a finitely presented group for which the conjugacy problem is undecidable [27]. The relations \( \sim_p \), \( \sim_o \), and \( \sim_c \) reduce to group conjugacy when a monoid is a group. It follows that the \( i \)-conjugacy problem, for \( i \in \{p, o, c\} \), is also undecidable. However, it is of interest to study decidability of the \( i \)-conjugacy problems in particular classes of finitely presented monoids.

First, we consider the class of polycyclic monoids, which are finitely presented monoids with zero. The polycyclic monoids \( P_n \), with \( n \geq 2 \), were first introduced by Nivat and Perrot [26], and later rediscovered by Cuntz in the context of the theory of \( C^* \)-algebras [11] Section 1]. (Within the theory of \( C^* \)-algebras, the polycyclic monoids are often referred to as Cuntz inverse semigroups.) The polycyclic monoids appear to be related to the idea of self-similarity [14]. For example, the polycyclic monoid \( P_2 \) can be represented by partial injective maps on the Cantor set: its two generators, \( p_1 \) and \( p_2 \), map, respectively, the left and right hand sides of the Cantor set, to the whole Cantor set.

These monoids can also be characterized as the syntactic monoid of the restricted Dyck language on a set of cardinality \( n \), that is, the language that consists of all correct bracket sequences of \( n \) types of brackets. The study of representations of the polycyclic monoids naturally connects with the study of its conjugacy relations [17,21]. In [21], the classification of the ‘proper closed inverse submonoids’ of \( P_n \) depends on the study of its conjugacy classes.

In Section 5, we characterize \( p \)-conjugacy and \( c \)-conjugacy in the polycyclic monoids, and conclude that \( \sim_c \subset \sim_p \). (For sets \( A \) and \( B \), we write \( A \subset B \) if \( A \) is a proper subset of \( B \).) We then show that the \( p \)-conjugacy and \( c \)-conjugacy problems are decidable for polycyclic monoids, and that, given words \( a \) and \( b \), testing whether or not \( a \sim_i b \), for \( i \in \{p, c\} \), can be done linearly on the lengths of \( a \) and \( b \). Note that in a polycyclic monoid \( P_n \), the relation \( \sim_0 \) is universal since \( P_n \) has a zero.

These positive results obtained for polycyclic monoids concerning the decidability and complexity of the conjugacy problems cannot be extended to the general finitely presented monoids.

In Section 4 we study decidability results. In particular, we show that there exists a monoid \( M \) defined by a finite complete presentation such that the \( c \)-conjugacy problem for \( M \) is undecidable (Proposition 1.2).

In Section 5 we study independence results. The word problem for groups is undecidable [23,28,31]. However, for groups, the word problem is reducible to the conjugacy problem [30] page 225], hence if the conjugacy problem for a group \( G \) is decidable, then the word problem for \( G \) is also decidable. Therefore, the word problem and the conjugacy problem for groups are not independent.

The situation for monoids is different. Osipova [29] has proved that for finitely presented monoids, the word problem, the \( p \)-conjugacy problem, and the \( c \)-conjugacy problem are pairwise independent. We show that for finitely presented monoids, the word problem and the \( c \)-conjugacy problem are independent (Theorem 5.2), and that the \( p \)-conjugacy problem and the \( c \)-conjugacy problem are also independent (Theorem 5.3). We do not know if the \( c \)-conjugacy problem and the \( c \)-conjugacy problem are independent.

We conclude the paper with Section 6 that lists open problems regarding the conjugacies under discussion.

2. Background

In this section we will formulate the main concepts needed in the following sections. For further background on the free monoid, see [15]; for presentations, see [12,32]; and, for rewriting systems, see [7].

**Alphabets and words.** Let \( \Sigma \) be a non-empty set, called an alphabet. We denote by \( \Sigma^* \) the set of finite strings (called words) of elements of \( \Sigma \), including the empty word \( 1 \). For \( w \in \Sigma^* \) and \( a \in \Sigma \), we denote by \( |w| \) the length of the word \( w \) and by \( |w|_a \) the number of occurrences of \( a \) in \( w \). For example, if \( \Sigma = \{a, b, c\} \) and \( w = aabba \in \Sigma^* \), then \( |w| = 5 \), \( |w|_a = 3 \), and \( |w|_c = 0 \).
A non-empty word \( z \) is said to be a factor of \( w \in \Sigma^* \) if \( w = uzv \), for some words \( u, v \in \Sigma^* \). If \( w, u, \) and \( v \) are words with \( w = uv \), then \( u \) is called a prefix of \( w \) and \( v \) a suffix of \( w \); the word \( u \) is said to be a proper prefix of \( w \) if \( v \) is non-empty (the notion of proper suffix is dual). Two words \( u \) and \( v \) are said to be prefix-comparable if either \( u \) is a prefix of \( v \) or \( v \) is a prefix of \( u \).

**Rewriting systems.** Any subset \( R \) of \( \Sigma^* \times \Sigma^* \) is called a rewriting system (or a Thue system) on \( \Sigma \). An element \( (x, y) \) of \( R \), also commonly denoted \( x \sim y \), is called a rewriting rule. If \( (x, y) \in R \) and \( u, v \in \Sigma^* \), we say that \( uxv \) reduces to \( uye \) and we write \( uxv \rightarrow uye \). A word \( w \) is said to be irreducible if there is no \( w' \in \Sigma^* \), such that \( w \rightarrow w' \). We denote by \( \rightarrow \) the reflexive and transitive closure of \( \rightarrow \).

A rewriting system \( R \) on \( \Sigma \) is special if every element of \( R \) is of the form \( (x, 1) \) with \( x \neq 1 \); it is monadic if every element of \( R \) is of the form \( (x, y) \) with \( y \in \Sigma \cup \{1\} \) and \( |x| > |y| \); it is length reducing if \( |x| > |y| \) for all \( (x, y) \in R \); it is noetherian if there is no infinite sequence \( w_1, w_2, w_3, \ldots \) of words in \( \Sigma^* \) such that \( w_1 \rightarrow w_2 \rightarrow w_3 \rightarrow \cdots \); it is confluent if for all \( u, v, w \in \Sigma^* \), if \( u \rightarrow v \) and \( u \rightarrow w \), then there exists \( z \in \Sigma^* \) such that \( v \rightarrow z \) and \( w \rightarrow z \); and \( R \) is complete if it is both noetherian and confluent. Note that if \( R \) is special or monadic, then it is length reducing, and if \( R \) is length reducing, then it is noetherian.

**Monoid presentations.** Every rewriting system \( R \) on \( \Sigma \) defines a monoid. The set \( \Sigma^* \) with concatenation of words as multiplication is a monoid, called the free monoid on \( \Sigma \). Denote by \( p_R \) the smallest congruence on \( \Sigma^* \) containing \( R \) (called the Thue congruence). We denote by \( M(\Sigma; R) \) the quotient monoid \( \Sigma^*/p_R \). The elements of \( M(\Sigma; R) \) are the congruence classes \([u]_M = \{ w : w p_R u \} \), where \( u \in \Sigma^* \). Whenever possible and when it is clear from the context, we shall omit the brackets to denote congruence classes, and thus identify words with the elements of the monoid that they represent.

Suppose \( M \) is any monoid such that \( M \cong M(\Sigma; R) \) (that is, \( M \) is isomorphic to \( M(\Sigma; R) \)). Then the pair \( (\Sigma; R) \) is a presentation of \( M \) with generators \( \Sigma \) and defining relations \( R \), and we say that \( M \) is defined by \( (\Sigma; R) \) or simply by \( R \). A presentation \( (\Sigma; R) \) is said to be finite if both \( \Sigma \) and \( R \) are finite. A monoid \( M \) defined by a finite presentation is called finitely presented.

**Definition 2.1.** Let \( M = M(\Sigma; R) \) be a finitely presented monoid, and let \( \sim_i \) be one of the conjugacy relations under consideration \( (i \in \{p, a, c\}) \). We say that the \( i \)-conjugacy problem for \( M \) is decidable if there is an algorithm that given any pair \( (u, v) \) of words in \( \Sigma^* \), returns YES if \([u]_M \sim_i [v]_M \) and NO otherwise. If such an algorithm does not exist, we say that the \( i \)-conjugacy problem for \( M \) is undecidable. We have an analogous definition of the decidability of the word problem for \( M \), in which case we check if \([u]_M = [v]_M \).

**Monoids with zero: rewriting systems and presentations.** Consider a rewriting system \( R \) defined on a set \( \Sigma_0 = \Sigma \cup \{0\} \), where \( 0 \) is a symbol not in \( \Sigma \), and a set \( R_0 \) of rewriting rules of the form \((x0, 0)\), \((0x, 0)\) and \((00, 0)\), for any \( x \in \Sigma \). The monoid \( T = M(\Sigma_0; R \cup R_0) \) is a monoid with zero \([0]_T \). For simplicity, we refer to the pair \( (\Sigma_0; R) \) as a monoid-with-zero presentation of \( T \). Notice that the monoid presentation \( (\Sigma_0; R \cup R_0) \) is finite or monadic when \( (\Sigma_0; R) \) is finite or monadic, respectively.

If a monoid \( M \) is defined by a presentation \( (\Sigma; R) \) then the monoid \( M_0 \), obtained from \( M \) by adding a zero element, is defined by the monoid-with-zero presentation \( (\Sigma; R) \). Observe that \([0]_M \) is the zero in \( M_0 \) and that \( M_0 \neq [0]_{M_0} \). Regarding these presentations, we can deduce by [3 Proposition 3.1] that if the rewriting system \( R \) on \( \Sigma \) is complete, then so is the new rewriting system \( R \cup R_0 \) on \( \Sigma_0 \).

Throughout the text we refer to a presentation as noetherian, confluent, complete, monadic, etc., whenever the associated rewriting system has the respective property.

3. **Conjugacy in the polycyclic monoids**

In this section, we study \( p \)-conjugacy and \( c \)-conjugacy in the class of polycyclic monoids, an important class of inverse monoids. A monoid \( M \) is called an inverse monoid if for every \( a \in M \), there exists a unique \( a^{-1} \in M \) (an inverse of \( a \)) such that \( aa^{-1}a = a \) and \( a^{-1}aa^{-1} = a^{-1} \) [15, p. 145].

In general, \( p \)-conjugacy is not transitive in inverse semigroups. For instance, by [9, Proposition 4.2], \( p \)-conjugacy is not transitive in free inverse monoids. We will show that in the polycyclic monoids, \( p \)-conjugacy is transitive for the elements not \( \sim_p \)-related to zero, and that \( \sim_c \subset \sim_p \).

We note that in the polycyclic monoids, \( \sim_c \) is the universal relation since every polycyclic monoid has a zero.
3.1. General properties of the polycyclic monoids. Let \( n \geq 2 \). Consider a set \( A_n = \{ p_1, \ldots, p_n \} \) and denote by \( A_n^{-1} \) a disjoint copy \( \{ p_1^{-1}, \ldots, p_n^{-1} \} \). Let \( \Sigma = A_n \cup A_n^{-1} \). Given any \( x \in \Sigma \), we define \( x^{-1} \) to be \( p_i^{-1} \) if \( x = p_i \in A_n \), and to be \( p_i \) if \( x = p_i^{-1} \in A_n^{-1} \). This notation can be extended to \( \Sigma^* \) by setting \((xw)^{-1} = w^{-1}x^{-1}\), for every \( x \in \Sigma \) and \( w \in \Sigma^* \), and \( 1^{-1} = 1 \).

Denote by \( R \) the set of rewriting rules on \( \Sigma_0 = \Sigma \cup \{ 0 \} \) of the form \( p_i^{-1}p_i = 1 \), for \( i \in \{ 1, \ldots, n \} \), and of the form \( p_i^{-1}p_j = 0 \), for \( i, j \in \{ 1, \ldots, n \} \) and \( i \neq j \). Consider the monoid \( P_n \) defined by the monoid-with-zero presentation \( (\Sigma_0; R) \). The monoid \( P_n \) is called the **polycyclic monoid** on \( n \) generators. Notice that the given presentation of \( P_n \) is monadic, and thus length reducing.

An irreducible element (with respect to \( R \)) cannot have a factor of the form \( p_i^{-1}p_j \), for any \( i, j \in \{ 1, \ldots, n \} \). Thus, irreducible elements have the form \( yx^{-1} \), where \( y, x \in A_n^* \), or the form \( 0 \). It is well known (e.g., [21, subsection 9.3]) that every nonzero element \( w \) of \( P_n \) has a **unique irreducible representation** \( \overline{w} \) of the form \( yx^{-1} \) with \( y, x \in A_n^* \). Therefore, irreducible elements are in one-to-one correspondence with elements of the polycyclic monoid. We deduce the following:

**Lemma 3.1.** The monoid-with-zero presentation \( (\Sigma_0; R) \) of the polycyclic monoid \( P_n \) is finite and complete.

Whenever we write \( a = yx^{-1} \), it will be understood that \( x, y \in A_n^* \). Hereafter, we shall identify irreducible elements with the elements of the polycyclic monoid that they represent.

We will frequently use the following lemma, which follows from the unique representation of the nonzero elements of \( P_n \).

**Lemma 3.2.** Consider nonzero elements \( yx^{-1} \) and \( vu^{-1} \) of \( P_n \). Then:

1. \( yx^{-1} \cdot vu^{-1} \neq 0 \) if \( x \) and \( v \) are prefix-comparable;
2. if \( yx^{-1} \cdot vu^{-1} \neq 0 \), then
   \[
   yx^{-1} \cdot vu^{-1} = \begin{cases} 
   yzu^{-1} & \text{if } v = xz, \\
   y(uz)^{-1} & \text{if } x = vz.
   \end{cases}
   \]
3. \( y = v \) in \( P_n \) iff \( y = v \) in \( A_n^* \), and \( x^{-1} = u^{-1} \) in \( P_n \) iff \( x = u \) in \( A_n^* \).

An irreducible word is said to be **cyclically reduced** if it is empty or zero, or if its first letter \( c \) and its last letter \( d \) satisfy \( c \neq d^{-1} \). Every nonzero irreducible word can be written in the form \( r y x^{-1} r^{-1} \), with \( r \in A_n^* \) and \( yx^{-1} \) a cyclically reduced word. From any irreducible word \( a \) we compute a cyclically reduced word \( \bar{a} \) in the following way: if \( a \) is cyclically reduced, we let \( \bar{a} \) be equal to \( a \); otherwise, \( a = r y x^{-1} r^{-1} \) as above, so we let \( \bar{a} \) be the (possibly empty) cyclically reduced word \( yx^{-1} \). We obtain the following fact for any nonzero irreducible word \( a \in P_n \):

\[
(3.4) \quad a = r\bar{a}r^{-1} \quad \text{for some word } r \in A_n^*.
\]

For each nonzero element \( a = yx^{-1} \in P_n \), denote by \( \rho(a) \) the irreducible word obtained from \( x^{-1}y \).

We also set \( \rho(0) = 0 \). Let \( a = yx^{-1} \in P_n \). We record the following facts about \( \bar{a} \) and \( \rho(a) \):

(a) \( \rho(a) \) is \( x^{-1}y \) reduced in \( P_n \);
(b) \( \bar{a} \) is \( x^{-1}y \) reduced in \( (\Sigma, R_1) \), where \( R_1 = \{ (p_i^{-1}p_i, 1) : i \in \{ 1, \ldots, n \} \} \);
(c) \( \rho(a) \) is cyclically reduced;
(d) \( \rho(a) = \bar{a} \in A_n^* \) if \( x \) is a prefix of \( y \); \( \rho(a) = \bar{a} \in (A_n^{-1})^* \) if \( y \) is a prefix of \( x \); and \( \rho(a) = \rho(\bar{a}) = 0 \) otherwise.

For example, if \( a = p_1p_2p_3^{-1}p_1^{-1} \), then \( \bar{a} = p_2p_3^{-1} \) and \( \rho(a) = 0 \).

The following lemma can be easily deduced.

**Lemma 3.3.** For all \( a = pq^{-1} \cdot rs^{-1} \in P_n \), the cyclically reduced word \( \bar{a} \) is given by

\[
\begin{align*}
lt^{-1} & \quad \text{if } r = qt \text{ and } p = sl, \\
lt^{-1} & \quad \text{if } r = qt \text{ and } s = pl, \\
lt^{-1} & \quad \text{if } q = rt \text{ and } p = sl, \\
lt^{-1} & \quad \text{if } q = rt \text{ and } s = pl.
\end{align*}
\]
3.2. **p-conjugacy in** $P_n$. We first observe that for every $a \in P_n$, $a \sim_p \tilde{a}$ and $a \sim_p \rho(a)$ (by the definitions of $\tilde{a}$ and $\rho(a)$).

**Lemma 3.4.** Let $a \in P_n$. Then $a \sim_p 0$ if and only if $\rho(a) = 0$.

**Proof.** Suppose that $a \sim_p 0$. If $a = 0$ then $\rho(a) = 0$. Suppose $a \neq 0$. Then $0 \neq a = pq^{-1} \cdot rs^{-1}$ and $0 = rs^{-1} \cdot pq^{-1}$, for some $p, q, r, s \in A_n^\ast$. The latter equality implies that $p$ and $s$ are not prefix-comparable (by Lemma 3.2). And the former implies that $r = qt$ or $q = rt$, for some $t \in A_n^\ast$. Hence, $a = pts^{-1}$ (if $r = qt$) or $a = p(st)^{-1}$ (if $q = rt$). Suppose that $a = pts^{-1}$. If $pt$ is a prefix of $s$, then also $p$ is a prefix of $s$, and if $s$ is a prefix of $pt$, then either $s$ is a prefix of $p$ or $p$ is a prefix of $s$. It follows that neither $pt$ is a prefix of $s$ nor $s$ is a prefix of $pt$, which implies $\rho(a) = 0$. By a similar argument, we obtain $\rho(a) = 0$ if $a = p(st)^{-1}$. The converse follows from the fact that $a \sim_p \rho(a)$. □

**Lemma 3.5.** Let $a$ and $b$ be nonzero elements of $P_n$ with $\rho(a) = \rho(b) = 0$. Then $a \sim_p b$ if and only if $	ilde{a} = b$.

**Proof.** Suppose that $a \sim_p b$. Then there exist elements $x, y, u, v \in A_n^\ast$ such that $a = xy^{-1} \cdot vu^{-1}$ and $b = vu^{-1} \cdot yx^{-1}$. Since $a \neq 0$, we know that $x$ and $v$ are prefix-comparable. Similarly, since $b \neq 0$, $x$ and $y$ are prefix-comparable.

Suppose first that $x$ is a prefix of $v$ and $u$ is a prefix of $y$, that is, $v = xp$ and $y = uq$ for some $p, q \in A_n^\ast$. Hence $a = uq^{-1}xpu^{-1} = uqpu^{-1}$, and so $\rho(a) = qp \neq 0$, which is a contradiction. Similarly, we obtain a contradiction if we assume that $v$ is a prefix of $x$ and $y$ is a prefix of $u$.

Suppose that $x$ is a prefix of $v$ and that $y$ is a prefix of $u$, that is, $v = xp$ and $y = uq$ for some $p, q \in A_n^\ast$. Then $a = xy^{-1}xpu^{-1} = y(pq^{-1})y^{-1}$ and $b = x(pq^{-1})x^{-1}$. Thus $\tilde{a} = pq^{-1} = b$ as required. In a similar way we obtain the intended result if $v$ is a prefix of $x$ and $u$ is a prefix of $y$.

The converse follows easily by first noticing that $a = r\tilde{a}r^{-1}$ and $b = s\tilde{b}s^{-1}$, for some $r, s \in A_n^\ast$ by 3.4. If $\tilde{a} = \tilde{b}$ we get the required result since $a = r\tilde{a}s^{-1} \cdot s\tilde{b}r^{-1}$ and $b = s\tilde{b}s^{-1}$. □

The following theorem characterizes $p$-conjugacy in $P_n$.

**Theorem 3.6.** Let $a, b \in P_n$. Then $a \sim_p b$ if and only if one of the following conditions is satisfied:

(a) $a = \rho(b) = 0$ or $\rho(a) = b = 0$;
(b) $\rho(a) = \rho(b) = 0$ and $\tilde{a} = \tilde{b}$;
(c) $\tilde{a}, \tilde{b} \in A_n^\ast$ and $\tilde{a} \sim_p \tilde{b}$ in the free monoid $A_n^\ast$; or
(d) $\tilde{a}, \tilde{b} \in (A_n^\ast)^\ast$ and $\tilde{a} \sim_p \tilde{b}$ in the free monoid $(A_n^\ast)^\ast$.

**Proof.** Suppose that $a \sim_p b$. If $a = 0$ or $b = 0$, then (a) holds by Lemma 3.4. Now assume that $a$ and $b$ are nonzero elements. Then, there are $p, q, r, s \in A_n^\ast$ such that $a = pq^{-1} \cdot rs^{-1}$ and $b = rs^{-1} \cdot pq^{-1}$.

By Lemma 3.3 we have: if $r = qt$ and $p = sl$, then $\tilde{a} = lt$ and $\tilde{b} = tl$, so (c) holds; if $q = rt$ and $s = pl$, then $\tilde{a} = (lt)^{-1}$ and $\tilde{b} = (tl)^{-1}$, so (d) holds; if $r = qt$ and $s = pl$, then $\tilde{a} = \tilde{b} = tl^{-1}$; and if $q = rt$ and $p = sl$, then $\tilde{a} = \tilde{b} = lt^{-1}$. In the last two cases, we have $\tilde{a} = \tilde{b}$, and so $\rho(a) = \rho(b)$. Thus, in those cases, either (c) or (d) holds (if $\rho(a) = \rho(b) \neq 0$), or (b) holds by Lemma 3.4 if $\rho(a) = \rho(b) = 0$.

Conversely, if (a) holds then $a \sim_p b$ by Lemma 3.4 and if (b) holds then $a \sim_p b$ by Lemma 3.5.

Suppose that (c) holds and let $\tilde{a} = uv$ and $\tilde{b} = vu$, where $u, v \in A_n^\ast$. Then $a = puqp^{-1}$ and $b = vquq^{-1}$ for some $p, q \in A_n^\ast$ by 3.4. Hence $a = puqp^{-1} \cdot qvq^{-1}$ and $b = qvq^{-1} \cdot puq^{-1}$, and so $a \sim_p b$ as required.

The proof in the case when (d) holds is similar. □

It is worth noting that the nonzero idempotents of $P_n$ form a single $p$-conjugacy class. Indeed, the nonzero idempotents have the form $xx^{-1}$, and $\tilde{a}$ is 1 if and only if $a$ is an idempotent. So, they form a single $p$-conjugacy class by Theorem 3.6.

We recall that $\sim_p$ is transitive in any free monoid. For the polycyclic monoid, we have the following result.

**Theorem 3.7.** In the polycyclic monoid $P_n$, we have:

1. for all $a, b, c \in P_n$ with $b \neq 0$, if $a \sim_p b$ and $b \sim_p c$, then $a \sim_p c$;
2. $\sim_p \circ \sim_p = \sim_p^\ast$. 5.
Proof. To prove (a), let \( a, b, c \in P_n \) with \( b \neq 0 \). Suppose that \( a \sim_p b \sim_p c \). If \( \rho(b) \neq 0 \) then, by Theorem 3.6 either \( \tilde{a}, \tilde{b}, \tilde{c} \in A_n^* \) or \( \tilde{a}, \tilde{b}, \tilde{c} \in (A_n^{-1})^* \), and \( a \sim_p c \) follows since \( \sim_p \) is transitive in the free monoid. Suppose that \( \rho(b) = 0 \). Then, by Theorem 3.6 \( \rho(a) = \rho(c) = 0 \). Thus, if \( a = 0 \) or \( c = 0 \), then \( a \sim_p c \). Suppose that \( a \neq 0 \) and \( c \neq 0 \). Then, by Theorem 3.6 \( \tilde{a} = \tilde{b} = \tilde{c} \), and so, again by Theorem 3.6 \( a \sim_p c \).

Statement (2) follows from (1).

The relations \( \sim_p \) and \( \sim_p^* \) are not equal in \( P_n \). For example, consider the polycyclic monoid \( P_2 \) with \( A_2 = \{x, y\} \). Then, for \( a = xx^{-1} \) and \( c = yy^{-1} \) in \( P_2 \), \( a \sim_p 0 \sim_p c \), so \( a \sim_p^* \), but \( (a, c) \notin \sim_p \) by Theorem 3.6.

### 3.3. \( c \)-conjugacy in \( P_n \)

**Lemma 3.8.** For all \( x, y \in A_n^* \), \( P(yx^{-1}) = \{rs^{-1} : r \text{ is a prefix of } x\} \).

**Proof.** Let \( rs^{-1} \in P(yx^{-1}) \). Then \( yx^{-1} \cdot rs^{-1} \neq 0 \), so \( r \) and \( x \) are prefix-comparable. Suppose that \( x \) is a proper prefix of \( r \), that is, \( r = xp_t \) for some \( p_t \in A_n = \{p_1, \ldots, p_n\} \) and \( t \in A_n^* \). Let \( j \in \{1, \ldots, n\} \) with \( j \neq i \). Then \( (y_{p_j})^{-1} \cdot yx^{-1} = (x_{p_j})^{-1} \neq 0 \), while \( (y_{p_j})^{-1} \cdot yx^{-1} = (xp_j)^{-1} \cdot rs^{-1} = 0 \) since neither \( xp_j \) is a prefix of \( r \), nor \( r \) is a prefix of \( xp_j \). This contradicts the hypothesis that \( rs^{-1} \in P(yx^{-1}) \). Therefore, \( r \) is a prefix of \( x \).

Now, let \( rs^{-1} \) be an element of \( P_n \), and assume that \( r \) is a prefix of \( x \). Then \( x = rz \) for some \( z \in A_n^* \), which gives \( yx^{-1} \cdot rs^{-1} = y(sz)^{-1} \). Thus, for every \( vu^{-1} \in P_n, vu^{-1} \cdot yx^{-1} \cdot rs^{-1} = vu^{-1} \cdot y(sz)^{-1} = 0 \) iff \( vu^{-1} \cdot yx^{-1} = 0 \) (see Lemma 3.2). Thus \( rs^{-1} \in P(yx^{-1}) \).

The following theorem characterizes \( c \)-conjugacy in \( P_n \).

**Theorem 3.9.** Let \( a, b \in P_n \). Then \( a \sim_c b \) if and only if one of the following conditions is satisfied:

- (a) \( a = b = 0 \);
- (b) \( \tilde{a} = \tilde{b}; \) or
- (c) \( \tilde{a}, \tilde{b} \in (A_n^{-1})^* \) and \( \tilde{a} \sim_c \tilde{b} \) in the free monoid \((A_n^{-1})^*\).

**Proof.** Suppose that \( a \sim_c b \). If \( a = 0 \) or \( b = 0 \), then (a) holds since \([0]_c = \{0\} \). Suppose \( a, b \neq 0 \). Then, there exist \( x, y, u, v \in A_n^* \) such that \( a = yx^{-1} \) and \( b = vu^{-1} \); and there exist \( r, s \in A_n^* \) such that \( rs^{-1} \in P(yx^{-1}) \) and \( yx^{-1} \cdot rs^{-1} = rs^{-1} \cdot vu^{-1} \). By Lemma 3.8 \( x = rz \) for some \( z \in A_n^* \). By Lemma 3.3 \( s \) and \( v \) are prefix-comparable.

Suppose \( s = uw \) for some \( w \in A_n^* \). Then \( yx^{-1} \cdot rs^{-1} = yz^{-1}r^{-1}r^{-1}rs^{-1} = y(sz)^{-1} \) and \( rs^{-1} \cdot vu^{-1} = rs^{-1}swu^{-1} = ruw^{-1}. \) Thus, since \( yx^{-1} \cdot rs^{-1} = rs^{-1}vu^{-1} \), we have \( y(sz)^{-1} = ruw^{-1} \), and so \( y = rw \) and \( u = sz \). Hence, \( a = yx^{-1} = ruw(z)^{-1} = r(wz)^{-1}r^{-1} \) and \( b = vu^{-1} = sw(z)^{-1} = s(wz)^{-1}s^{-1} \), and so (b) holds.

Suppose \( s = vw \) for some \( w \in A_n^* \). Then \( yx^{-1} \cdot rs^{-1} = y(sz)^{-1} \) (as in the previous case) and \( rs^{-1} \cdot vu^{-1} = rw^{-1}v^{-1}vu^{-1} = r(wu)^{-1} \). Thus, \( y(sz)^{-1} = r(wu)^{-1} \), and so \( y = r \) and \( sz = uw \). Since \( s = vw \), we have \( vuzz = uw \), which implies that \( u = vt \) for some \( t \in A_n^* \). Thus, \( u = vt \), which implies \( vuzz = vtw \), and so \( uz = tw \). By Corollary 5.2, \( tw = uz \) implies \( t \sim_p z \) in \( A_n^* \).

Further, \( \tilde{a} = yx^{-1} = r(zt)^{-1} = rz^{-1}r^{-1} = z^{-1} \) and \( \tilde{b} = vz^{-1} = v(tz)^{-1} = vt^{-1}v^{-1} = t^{-1} \). Hence \( \tilde{a}, \tilde{b} \in (A_n^{-1})^* \). Since \( t \sim_p z \) in the free monoid \( A_n^* \), we have \( z^{-1} \sim_p t \) in \((A_n^{-1})^* \), and so \( \tilde{a} \sim_c \tilde{b} \) in \((A_n^{-1})^* \). Hence (c) holds.

Conversely, if (a) holds, then clearly \( a \sim_c b \). Suppose that (b) holds, that is, \( \tilde{a} = \tilde{b} \). Let \( r, s \in A_n^* \) be such that \( a = r\tilde{a}r^{-1} \) and \( b = s\tilde{b}s^{-1} \). Then, by Lemma 3.8 \( rs^{-1} \in P(a), sr^{-1} \in P(b), \) and

\[
\begin{align*}
a \cdot rs^{-1} &= \tilde{a} \sim_p r^{-1}rs^{-1} = \tilde{a} \sim_p rs^{-1}s^{-1} = rs^{-1}, \\
b \cdot sr^{-1} &= \tilde{b} \sim_p srs^{-1} = \tilde{b} \sim_p sr^{-1}r^{-1}rs^{-1} = sr^{-1}.
\end{align*}
\]

Hence \( a \sim_c b \). Now, suppose that (c) holds. Since \( \tilde{a}, \tilde{b} \in (A_n^{-1})^* \), then letting \( t^{-1} = \tilde{a} \) and \( z^{-1} = \tilde{b} \) we have \( a = y(t^{-1})^{-1} = y(yt)^{-1} \) and \( b = v(z^{-1})^{-1} = v(vz)^{-1} \) for some \( y, v \in A_n^* \). Moreover, we have
Lemma 3.11. An irreducible element is true for the p-p
Decidability and complexity of conjugacy in P.

The inclusion \( \sim \subseteq \sim_p \) follows by Theorems 3.6, 3.9. To show that \( \sim \) is properly contained in \( \sim_p \), consider two distinct generators \( x \) and \( y \) in \( A_n \). Let \( a = xxy^{-1} \) and \( b = yyxy^{-1} \) in \( P_n \). Then \( \tilde{a} = xy \) and \( \tilde{b} = yx \). Hence \( \tilde{a} \sim \tilde{b} \) in the free monoid \( A_n \), and so \( \tilde{a} \sim \tilde{b} \) in \( P_n \) by Theorem 3.6. On the other hand, none of (a), (b), or (c) of Theorem 3.9 holds for \( a \) and \( b \), and so \( a \neq \sim \tilde{b} \) in \( P_n \). \( \square \)

3.4. Decidability and complexity of conjugacy in \( P_n \). It is known that for free monoids, the p-conjugacy problem is decidable in linear time [1, Theorem 2.5]. We will show that the same result is true for the p-conjugacy and e-conjugacy problems for the polycyclic monoids.

The following lemma is a special case of [3, Theorem 4.1].

Lemma 3.11. Let \( (\Sigma_0; R) \) be the monoid-with-zero presentation of \( P_n \), and let \( w \in \Sigma_0^* \). Then the irreducible element \( \tilde{w} \in \Sigma_0^* \) such that \( \tilde{w} \mapsto \tilde{w} \) in \( P_n \) can be computed in time \( O(|w|) \).

(For more details on the big-O notation used in Lemma 3.11 and more generally for basic notions on complexity theory, see [33, Section 7].)

Lemma 3.12. Let \( a \) be an irreducible word of \( P_n \). Then the words \( \tilde{a} \) and \( \rho(a) \), can be computed in time \( O(|a|) \).

Proof. The result is obvious if \( a = 0 \). Let \( a = yx^{-1} \). To compute \( \tilde{a} \) proceed as follows:

1. Compute the word \( x^{-1}y \).
2. Reduce \( x^{-1}y \) to an irreducible word \( u^{-1}v \) in \( (\Sigma, R_1) \) (see (b) above);
3. Output the word \( \tilde{a} = vu^{-1} \).

To compute \( \rho(a) \) proceed in the same way to obtain the word \( vu^{-1} \), and next proceed as follows:

4. If \( v \) and \( u \) are non-empty, then output \( \rho(a) = 0 \), otherwise output \( \rho(a) = \tilde{a} \).

We show that each stage of this algorithm uses \( O(|a|) \) steps, and so the result holds. For the first stage, it is sufficient to scan through the word \( yx^{-1} \) (from left to right), detect the first symbol in \( (A_n^{\sim p})^* \), and output the symbols of \( x^{-1} \) followed by the symbols of \( y \). This requires \( O(|a|) \) steps. The third stage is similar. For the second stage, since \( R_1 \) is length reducing, we conclude by [3, Theorem 4.1] that \( \tilde{a} \) can be computed in \( O(|a|) \) steps. Checking if a word is empty can be done in constant time, and so \( \rho(a) \) can be computed in linear time as well. \( \square \)

Theorem 3.13. Let \( (\Sigma_0; R) \) be the monoid-with-zero presentation of \( P_n \), and let \( i \in \{p, c\} \). Then, given two words \( x, y \in \Sigma_0^* \), it can be tested in time \( O(m) \), where \( m = \max\{|x|, |y|\} \), whether or not \( x \sim_i y \) holds in \( P_n \).

Proof. Let \( x, y \in \Sigma_0^* \). By Lemma 3.11 the irreducible words \( \tilde{x} = a \) and \( \tilde{y} = b \) can be computed in time \( O(m) \), where \( m = \max\{|x|, |y|\} \). Note that \( |a| \leq |x| \) and \( |b| \leq |y| \). By Lemma 3.12 each of the words \( \tilde{a}, \tilde{b}, \rho(a), \) and \( \rho(b) \) can be computed in time \( O(m) \).

According to Theorems 3.6 and 3.9, in order to check whether or not \( x \sim_i y \) holds it suffices to compute \( a, b, \tilde{a}, \tilde{b}, \rho(a), \) and \( \rho(b) \), and check whether or not they are equal (as words) or p-conjugate (in the free monoid). Since the p-conjugacy problem in the free monoids is decidable in linear time, we deduce the desired result. \( \square \)
In this section, we discuss the decidability of \( i \)-conjugacy problems in some classes of finitely presented monoids.

**Separation of conjugacies.** Let \( M \) be a monoid without zero. Consider the monoid \( M^0 \) obtained from \( M \) by adjoining a zero element. It is immediate that \( \sim_0 \) is the universal relation in \( M^0 \), while \( \sim_c \) is not universal in \( M^0 \). Now, \( M^0 \) has no zero divisors, and hence any two given elements \( a \) and \( b \) of \( M \) are \( c \)-conjugate in \( M^0 \) if and only if they are \( o \)-conjugate in \( M \). Therefore, \( \sim_c^{M^0} = \sim_0^M \cup \{(0,0)\} \) in \( M^0 \). Similarly, for \( p \)-conjugacy we have \( \sim_p^{M^0} = \sim_p^M \cup \{(0,0)\} \). Thus, if we identify a monoid \( M \) for which \( \sim_0 \neq \sim_p \) in \( M \), we then immediately obtain an example of a monoid \( M^0 \) where \( \sim_c \neq \sim_p \), \( \sim_0 \neq \sim_p \), and \( \sim_0 \neq \sim_c \). To find such a monoid (within a certain class of rewriting systems), consider the following example from \([25, \text{Example 2.2}]\).

**Example 4.1.** Let \( M \) be the monoid defined by the monadic and confluent presentation \((\Sigma; R)\) with \( \Sigma = \{a, b, c\} \) and \( R = \{(ab, b), (cb, b)\} \). As explained in \([33, \text{Example 2.2}]\), we have \( bac \sim_p ba \), but clearly \( bac \neq ba \). Therefore, \( \sim_0 \neq \sim_p \) in \( M \), and hence the relations \( \sim_0, \sim_p \) and \( \sim_c \) are pairwise distinct in \( M^0 \).

We deduce that for monoids defined by monadic presentations, the relations \( \sim_c, \sim_p \) and \( \sim_0 \) may be different, even when such systems are also finite and confluent.

**Finite complete presentations.** Narendran and Otto \([25, \text{Lemma 3.6}]\) constructed a finite complete presentation \((\Sigma; R)\) such that the \( o \)-conjugacy problem is undecidable for the monoid \( M = M(\Sigma; R) \). Using the above observation, we obtain the following result.

**Proposition 4.2.** There is a monoid defined by a finite complete presentation for which the \( o \)-conjugacy problem is undecidable.

**Proof.** Consider the monoid \( M^0 \) obtained from the monoid \( M \) defined by Narendran and Otto in \([25, \text{page 35}]\) which has undecidable \( o \)-conjugacy problem. Since \( M \) is defined by a finite complete presentation, the monoid \( M^0 \) is also defined by a finite complete presentation by \([3, \text{Proposition 3.1}]\). It can be seen that \( M \) does not have a zero. Thus \( \sim_c^{M^0} = \sim_0^M \cup \{(0,0)\} \), and hence \( M^0 \) has undecidable \( c \)-conjugacy problem. \( \square \)

**Special presentations.** It is easy to see that a monoid defined by a special presentation has a zero if and only if it is trivial. Hence, within this class we have \( \sim_c = \sim_0 \). Zhang \([34, \text{Theorem 3.2}]\) proved that in every monoid \( M \) defined by a special presentation, the relations \( \sim_p \) and \( \sim_0 \) also coincide. Otto \([30, \text{Theorem 3.8}]\) proved that if \( M \) is a monoid defined by a finite, special, and confluent presentation, then the \( o \)-conjugacy problem for \( M \) is decidable (and so the \( p \)-conjugacy and \( c \)-conjugacy problems are also decidable for \( M \)).

**One-relator monoids.** A monoid \( M \) is called a one-relator monoid if it admits a finite presentation with one defining relation, which we will write as \((\Sigma; u = v)\) instead of \((\Sigma, \{(u, v)\})\). Many decision problems have been studied in the class of one-relator monoids. For example, it is decidable whether a one-relator monoid has a zero \([8, \text{Proposition 14}]\). Moreover, a one-relator monoid \( M \) containing a zero admits a presentation \( \langle \{a\}; a^{k+1} = a^k \rangle \), where \( k \) is a positive integer \([8, \text{the proof of Proposition 14}]\). It is easy to check that in this monoid \( \sim_p = \sim_c = \{(x, x) : x \in M\} \) and \( \sim_0 = M \times M \).

By the foregoing argument, if \( M \) is a one-relator monoid with a zero, then the \( c \)-conjugacy and \( o \)-conjugacy problems for \( M \) are decidable. If \( M \) has no zero, then \( \sim_c = \sim_0 \). Therefore, the \( c \)-conjugacy problem for such an \( M \) is decidable if and only if the \( o \)-conjugacy problem for \( M \) is decidable.

Some specific results concerning the decidability of the \( o \)-conjugacy problem for this class can be found in \([34, 35]\).

5. Independence in finitely presented monoids

In this section, we prove that for finitely presented monoids, the word problem and the \( c \)-conjugacy problem are independent, and that the \( p \)-conjugacy problem and the \( c \)-conjugacy problem are independent.
Definition 5.1. Decision problems $P_1$ and $P_2$ are independent if there exist finitely presented monoids $M_1$ and $M_2$ such that for $M_1$, $P_1$ is decidable and $P_2$ is undecidable; and for $M_2$, $P_2$ is decidable and $P_1$ is undecidable.

Theorem 5.2. For finitely presented monoids, the word problem and the c-conjugacy problem are independent.

Proof. First, there are finitely presented groups with decidable word problem but undecidable conjugacy problem [6,10]. Let $G$ be a finitely presented group. A finite group presentation of $G$ can be effectively converted to a finite (special) monoid presentation $(\Sigma; R)$ such that $G \cong M(\Sigma; R)$. It follows that there is a monoid $M$ defined by a finite presentation for which the word problem is decidable and the c-conjugacy problem is undecidable.

We will construct a finitely presented monoid for which the converse is true. Let $G = M(\Sigma; R)$ be a finitely presented group with undecidable word problem (see [28]), where $(\Sigma; R)$ is a monoid presentation. Let $a$ and $b$ be symbols not in $\Sigma$, and let $M = M(A; T)$ be the monoid defined by the presentation $(A; T)$, where

$$A = \Sigma \cup \{a, b\},$$

$$T = R \cup \{(xa, ax) : x \in \Sigma\} \cup \{(bx, b) : x \in \Sigma \cup \{a\}\} \cup \{(xb, b) : x \in \Sigma\} \cup \{(aa, a)\}.$$

Notice that $G$ is a subgroup of $M$. The word problem for $M$ is undecidable (since otherwise it would be decidable for $G$). It is easy to see that $M$ has no zero and that each congruence class $[u] = [u]_M$ has a representative of the form $a^p$, $aw$, $ab^p$, or $w$, where $p$ is a positive integer and $w \in \Sigma^*$. 

Observe that whenever a rewriting rule from $T$ is applied to a word in $A^*$, the number of occurrences of $b$ does not change. Thus, for all $u_1, u_2 \in A^*$, if $[u_1] = [u_2]$, then $[u_1]b = [u_2]b$. Let $[u], [v] \in M$. Suppose $[u] \sim_c [v]$. Then $[u][t] = [t][v]$ for some $t \in A^*$. Thus $[ut] = [tv]$, and so $[u]b = [v]b$ by the foregoing observation.

Conversely, suppose $[u]b = [v]b$. If $|u|_b = |v|_b = 0$, then $[u] \sim_c [v]$ since $[u][ab] = [ab] = [ab][u]$ and $[v][ab] = [ab] = [ab][u]$. Suppose $|u|_b = |v|_b > 0$. If $[u] = [v]$, then $[u] \sim_c [v]$. Suppose $[u] \neq [v]$. Then $[u] = [u][b] = [ab][u]$ and $[v] = [v][b] = [ab][v]$, or vice versa. We may assume that $[u] = [ab]$ and $[v] = [ab][u]$. Then $[u] \sim_c [v]$ since $[u][b] = [b][v]$, and $[v][a] = [ab][u]$. We have proved that for all $u, v \in A^*$, $[u] \sim_c [v]$ if and only if $[u]b = [v]b$. Hence the c-conjugacy problem for $M$ is decidable.

Theorem 5.3. For finitely presented monoids, the p-conjugacy problem and the c-conjugacy problem are independent.

Proof. Let $M = M(A; T)$ be the monoid from the proof of Theorem 5.2. For $w \in \Sigma^*$, we will write $[w] = [w]_M$ for the element of the monoid $M$, and $[w]_G$ for the element of the group $G$.

Let $u, v \in \Sigma^*$. Suppose $[u] \sim_p [v]$, that is, $[u] = [s][t]$ and $[v] = [t][s]$ for some $s, t \in A^*$. The words $s$ and $t$ cannot contain $b$ since in the presentation $(A; T)$ a word with $b$ cannot be reduced to a word without $b$. But then $s$ and $t$ cannot contain $a$ either since a word with $a$ cannot be reduced to a word without $a$ unless $b$ is also present. It follows that $[u]_G = [s]_G[t]_G$ and $[v]_G = [t]_G[s]_G$, and so $[u]_G \sim_p [v]_G$.

We have proved that for all $u, v \in \Sigma^*$, if $[u] \sim_p [v]$ in $M$, then $[u]_G \sim_p [v]_G$ in $G$. The converse is clearly true. Since $\sim_p$ in $G$ is the group conjugacy and $G$ has undecidable word problem (and so undecidable conjugacy problem), it follows that the p-conjugacy problem for $M$ is undecidable. We have already established in the proof of Theorem 5.2 that the c-conjugacy problem for $M$ is decidable.

We will now present a monoid that has decidable p-conjugacy problem and undecidable c-conjugacy problem. Osipova [29] showed that there exists a finitely presented monoid $M$ that has decidable p-conjugacy problem and undecidable l-conjugacy problem, where the l-conjugacy stands for the following relation $\sim_l$: given $a, b \in M$, $a \sim_l b$ if and only if there exists $g \in M$ such that $ag = gb$. Osipova’s proof follows the following steps (we use the original notation): (i) she considers a finitely presented monoid $\Pi_1 = M(\mathcal{U}_1; B_0)$ with undecidable p-conjugacy problem; (ii) she extends the alphabet $\mathcal{U}_1$ to $\mathcal{U}_2 = \mathcal{U}_1 \cup \{c, d, e_1, \ldots, e_m\}$, where $m = |\mathcal{U}_1| + 2|B_0|$, and builds a new finitely presented monoid $\Pi_2 = M(\mathcal{U}_2; B_0)$; (iii) she shows [29] Lemma 4 that for all words $Q, R \in \mathcal{U}_2^*$, $Q \sim_p R$ in $\Pi_1$ if and only if there exists $X \in \mathcal{U}_2^*$ such that $XcQd = cRdX$ in $\Pi_2$; (iv) she concludes [29] Theorem 2.
that the \(L\)-conjugacy problem for \(\Pi_3\) is undecidable; (v) she shows \cite{25} Theorem 3] that the \(p\)-conjugacy problem for \(\Pi_3\) is decidable.

Now, notice that \(\sim_p\) is symmetric, and hence, by \cite{25} Lemma 4], for all words \(Q, R \in \mathcal{U}_1\), we have \(Q \sim_p R\) in \(\Pi_1\) if and only if there exist \(X, Y \in \mathcal{U}_3^*\) such that \(XcQd = cRdX\) and \(YcRd = cQdY\) in \(\Pi_3\). Equivalently, \(Q \sim_p R\) in \(\Pi_1\) if and only if \(cQd \sim_c cRd\) in \(\Pi_3\). Therefore, \(\Pi_3\) has undecidable \(o\)-conjugacy problem.

The set \(\mathcal{B}_3\) of \(\Pi_3\) has rewriting rules of the form \((c, \mathcal{G}_1, ce_i), (e_i b_j, b_j e_i),\) and \((e_i d_i, g_i^0 d_i g_i^1),\) where \(i = 1, \ldots, m\) and \(j = 1, \ldots, n\), the \(b_j\) are the letters of the alphabet \(\mathcal{U}_1\), and the \(G_i\) and \(G_i^0\) are fixed words in \(\mathcal{U}_3^*\) \cite{29} pages 70 and 71]. From the form of these rules, we can easily deduce that any two words in \(\mathcal{U}_3^*\) that are equal in \(\Pi_3\) have the same number of occurrences of the letter \(c\). Therefore, \(\Pi_3\) does not have a zero since the zero element, say \([z]\), would satisfy the identity \([z][c] = [z]\), contradicting the above observation. Hence \(\sim_o = \sim_c\), and hence \(\Pi_3\) has undecidable \(c\)-conjugacy problem.

We do not know if the \(c\)-conjugacy problem and the \(o\)-conjugacy problem are independent for finitely presented monoids. Consider a finitely presented monoid \(M\) without zero that has undecidable \(c\)-conjugacy problem. Let \(M^0\) be the monoid \(M\) with a zero 0 adjoined. Then \(M^0\) is finitely presented and the \(c\)-conjugacy problem for \(M^0\) is undecidable (since for all \(a, b \in M\), \(a \sim_c b\) in \(M^0\) if and only if \(a \sim o b\) in \(M\)). On the other hand, the \(o\)-conjugacy problem for \(M^0\) is decidable since \(\sim_o = M^0 \times M^0\).

Now, suppose \(M\) is a finitely presented monoid that has decidable \(c\)-conjugacy problem. Then, if we could prove that \(M\) has a zero, then the algorithm that always says YES would decide if \([v]\), for all \([v]\), \(v \in M\), \(M^0\). Further, if we could prove that \(M\) has no zero, then the algorithm that works for \(\sim_c\) would also work for \(\sim_o\). However, suppose that the statement “\(M\) has a zero” can neither be proved nor disproved. Then it is conceivable that no algorithm for \(o\)-conjugacy problem in \(M\) exists, that is, that \(o\)-conjugacy problem is undecidable for \(M\).

6. Open problems

We conclude this paper with some natural questions related to conjugacy and presentations. As we have noticed in Section \(\Pi_3\), the independence of the \(c\)-conjugacy and \(o\)-conjugacy problems is related to the decidability of a monoid having a zero. Hence whether the \(o\)-conjugacy and \(c\)-conjugacy problems are independent hinges on the answer to the following question.

**Problem 6.1.** Does there exist a finitely presented monoid \(M\) for which it is undecidable if it has a zero, the \(o\)-conjugacy problem for \(M\) is undecidable, and the \(c\)-conjugacy problem for \(M\) is decidable?

The word problem is decidable for certain restricted classes of finitely presented monoids, in particular those admitting a finite complete presentation. It is then natural to consider this property as a useful tool in proving decidability results. In the class of monoids defined by finite, length-reducing, and confluent rewriting systems, the \(o\)-conjugacy problem is decidable \cite{24} Corollary 2.7. It is also decidable if such monoids have a zero. However, the \(p\)-conjugacy problem is undecidable in this class \cite{25} Corollary 2.4].

**Problem 6.2.** Is the \(c\)-conjugacy problem decidable for the class of monoids defined by finite, length-reducing, and confluent rewriting systems?

This problem could be approached by first considering the class of finite monadic confluent rewriting systems, as it is the case of polycyclic monoids.

**Problem 6.3.** Is the \(c\)-conjugacy \(p\)-conjugacy problem decidable for the class of monoids defined by finite, monadic, and confluent rewriting systems?

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