Comment on “Kinetic theory for a mobile impurity in a degenerate Tonks-Girardeau gas”

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In a recent paper Gamayun, Lychkovskiy, and Cheianov studied the dynamics of a mobile impurity weakly coupled to a one-dimensional Tonks-Girardeau gas of strongly interacting bosons. Employing the Boltzmann equation approach, they arrived at the following conclusions: (i) a light impurity, being accelerated by a constant force, F, does not exhibit Bloch oscillations, which were predicted and studied in Refs. 2 3; (ii) a heavy impurity does undergo Bloch oscillations, accompanied by a drift with the velocity \( v_F = \mu F \). The mobility \( \mu \) may be expressed exactly 3 in terms of \( E_0(P, \rho) \). Result (ii), while not valid at exponentially small forces, indeed reflects an interesting intermediate-force behavior.

The origin of Bloch oscillations is most transparent for a weakly interacting Bose gas, described by the Gross-Pitaevskii (GP) equation. Its solution reveals that a repulsive impurity binds to a dark soliton – a region of depleted host gas. The resulting composite object (the “deplenton”) has a periodic dispersion curve and thus exhibits Bloch oscillations, if a sufficiently small force is applied to the impurity. In a strongly interacting Bose gas the GP approach is not applicable, but the bound-state formation still takes place. To illustrate this phenomenon, one may represent the Tonks-Girardeau gas of \( N \) hard-core bosons by free fermions created by \( c^\dagger_{p,F} \), weakly coupled to a quantum impurity with the coordinate \( x \), through the density-density interaction (1):

\[
\hat{H} = -\frac{1}{2m_i} \frac{\partial^2}{\partial x_i^2} + \sum_p \frac{p^2}{2m_h} c^\dagger_p c_p^\dagger + \frac{\rho^2}{m_h N} \sum_{p,q} c^\dagger_p c^\dagger_q e^{ixpr} ,
\]

where \( m_h \) is the mass of host particles, \( m_i \) is the impurity mass and \( 0 < \gamma \ll 1 \) is a dimensionless coupling constant.

Consider a state of the system with total momentum \( P > 0 \). If \( P < P_0 \equiv \min\{m_i v_F, k_F\} \), the low energy states are those where most of the momentum is carried by the impurity. Indeed, the impurity kinetic energy \( P^2/(2m_i) \) is less than that of soft particle-hole excitations above the Fermi sea \( \sim v_F P \). In the opposite limit \( P > P_0 \) the low energy states are those where hole excitations carry a significant fraction of the entire momentum \( P \). The many-body ground state adiabatically connects between these two limits, signaling strong impurity-hole hybridization at \( P > P_0 \). Indeed, consider a subspace of the full many-body space, which contains a single hole excitation with momentum \( 0 < k < 2k_F \) in addition to the impurity with momentum \( P - k \) (this restriction is justified in the limit \( \gamma \ll 1 \)). The basis vectors of this subspace are

\[
|k; P\rangle = e^{i(P-k)x} c^\dagger_{k_F} c_{k_F-k}|\text{Fermi Sea}\rangle. \tag{2}
\]

The corresponding Schrödinger equation

\[
\sum_{k'} \langle k'|H|k\rangle \psi_{P/k'}(k') = E\psi_P(k)
\]

takes the form of the two-particle problem with the attractive delta-interaction (formally the attraction arises from anti-commuting the fermionic operators in the last term in Eq. (1)).

\[
\frac{(P-k)^2}{2m_i} + E_h(k) \psi_P(k) - \frac{\gamma \rho}{m_h} \int_0^{2k_F} \frac{dk'}{2\pi} \psi_{P/k'}(k') = E\psi_P(k)
\]

where \( E_h(k) = v_F k - k^2/(2m_h) \) is the hole kinetic energy (we measure \( E \) relative to \( NE_F/3 + \gamma \rho^2/m_h \)). This problem admits a unique bound-state solution, whose energy \( E = E_b(P) \) is found from the integral equation

\[
\int_0^{2k_F} \frac{dk'}{(P-k')^2 + E_h(k') - E_b(P)} = \frac{2\pi m_h}{\gamma \rho}. \tag{4}
\]

Its solution represents a non-perturbative correction to the bare impurity dispersion and is completely missed in the Boltzmann equation treatment. We plot \( E_b(P) \), along with the continuum of the scattering states, in Fig. 1 for the case of light, \( \eta = m_i/m_h < 1 \), and heavy, \( \eta > 1 \) impurity. The gap \( \Delta \) between the bound-state and the continuum is found to be \( \Delta/E_F \sim \gamma^2 \eta/(1-\eta) \) for \( \eta < 1-\gamma/\pi^2 \) and \( P_0 \leq P \), while \( \Delta/E_F \sim \exp(-\pi^2(\eta-1)/\gamma) \) for \( \eta > 1 + \gamma/\pi^2 \). For an almost equal mass case \( [1-\eta] < \gamma/\pi^2 \), one finds \( \Delta/E_F \sim \gamma \). We also note that for \( \eta = 1 \), integrability of Eq. (1) allows access to the exact many-body ground state energy \( E_0(P \sim k_F) = E_F - \frac{2\pi^2 (P-k_F)^2}{3\gamma} \). Remarkably, as one may verify from Eq. (4), \( E_b(P \sim k_F) + \gamma \rho^2/m_h = E_0(P \sim k_F) \) for \( \gamma \ll 1 \), justifying our Hilbert space truncation.

The hard gap between the bound-state and the continuum is an artifact of restricting the particle in Eq. (2) to
be created right at the Fermi momentum $k_F$. Allowing for slight deviation $c^\dagger_{k_F} \to c^\dagger_{k_F+p}$, introduces interaction of the bound-state with low energy, $\sim v_F p$, excitations. It is known [4,6] that such interaction transforms the bound-state into the quasi bound-state with the power-law (instead of the pole) correlation function. These low energy excitations are responsible for radiation losses and thus for linear mobility $\mu$. They do not, however, destroy the quasi bound-state and associated Bloch oscillations at small applied force.

The Bloch oscillations are destroyed if a large enough force $F > F_{\text{max}}$ is applied to the impurity. The physics of this process is that of the Landau-Zener transition between the bound-state and the continuum at $P \approx P_0 = k_F \min\{\eta, 1\}$. One may thus estimate the crossover force as $F_{\text{max}} \sim \Delta^2/v$, where $v = v_F \min\{1, 1/\eta\}$. This leads to the following estimate for the maximal force, preserving (nearly) adiabatic bound-state dynamics

$$F_{\text{max}} \propto \frac{k_F^3}{m_i} \left\{ \begin{array}{ll} \left(\frac{\gamma^2}{\eta} - 1\right)^2, & \eta < 1 - \frac{\gamma}{2}; \\ \frac{\eta}{\eta - 1} \left(\frac{\eta - 1}{\eta - 2}\right)^2 e^{-2\eta^2(\eta - 1)/\eta}, & \eta > 1 + \frac{\gamma}{2}, \end{array} \right. \quad (5)$$

while for $|1 - \eta| < \gamma/\pi^2$, one finds $F_{\text{max}} \propto k_F^3 \gamma^2/m_h$. For $F < F_{\text{max}}$ both light and heavy impurities exhibit Bloch oscillations along with the drift [3], whose velocity scales linearly with the force $v_D = \mu F$.

In Refs. [4, 7] it was shown that for a heavy impurity away from the Tonks-Girardeau limit, there exists a phase transition at a critical value of the impurity mass: for $m_i < M_c$ the ground-state is a smooth function of momentum, while for $m_i > M_c$ the ground-state exhibits a cuspid singularity at momenta $P = (1 + 2n)k_F$ for integer $n$ (in the Tonks-Girardeau limit $M_c \to \infty$). In the latter case the impurity “overshoots” the intersection points at $P = (1 + 2n)k_F$ and has to emit phonons to reach the ground state. This leads to an enhanced dissipation [7] and thus to super-linear drift velocity

$$v_D \propto F^{1/(1+\alpha)},$$

where $\alpha \approx 2K - 1$ for $\gamma \ll 1$ and $K$ is the Luttinger parameter of the host.

Notice that in the Tonks-Girardeau limit the validity of the $v_D = \mu F$ response for $\eta > 1$ is limited to an exponentially small force [3]. This scale originates from the exponentially narrow region of momenta, where the bound-state exhibits the avoided crossing behavior, Fig. [7]. For $F > F_{\text{max}}$ the impurity overshoots the avoided crossing and follows the “wrong” parabola before emitting phonons and returning to the ground state. Thus, for $F > F_{\text{max}}$ one may apply Eq. (6) with $K = 1$ – appropriate for the Tonks gas. This leads to $v_D \propto \sqrt{F}$, in full agreement with Ref. [1]. An important extension of Ref. [1] is that the super-linear drift [6] is to be expected for moderately heavy impurities $m_h > m_i > M_c$ in an intermediate range of forces where the linear mobility $v_D = \mu F$ is inapplicable.

M. S. and A. K. were supported by DOE Contract No. DE-FG02-08ER46482. D. M. G. acknowledges support by the EPSRC.

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