Dirac operator as a random matrix and the quenched limit of QCD with chemical potential

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The behavior of quenched QCD at nonzero chemical potential $\mu$ has been a long-standing puzzle. An explicit solution is found using the random matrix approach to chiral symmetry breaking. At nonzero $\mu$ the quenched QCD is not a simple $n \to 0$ limit of a theory with $n$ quarks: a naive ‘replica trick’ fails. A limit that leads to the quenched QCD is that of a theory with $2n$ quarks: $n$ quarks with original action and $n$ quarks with conjugate action.

1. The puzzle

The main result of the present study is a resolution of a puzzle which existed since the middle of 80’s and has been one of the major obstacles towards our understanding of the behavior of QCD at finite baryon density. The contradiction is simple and acute. The quark chemical potential $\mu$ in QCD shifts the energy of a given quantum state by $3\mu$ times this state’s net baryon number. A transition at $T = 0$ should occur when $\mu = \mu_c \approx m_p/3$: the vacuum state without baryons is no longer the lowest energy state. It is believed that the chiral symmetry is restored in the new phase. Testing this prediction by Monte Carlo simulation is difficult since the determinant of the fermion matrix becomes complex at nonzero $\mu$. Nevertheless one can use quenched approximation in this case. Such Monte Carlo studies\textsuperscript{1} give consistently the critical value of $\mu_c \approx m_\pi/2$ — a clearly unphysical result, since the pion does not carry baryon number.

The present study is motivated by a desire to clarify the problem using random matrix approach. On an exactly solvable model of chiral symmetry breaking in QCD we demonstrate explicitly a general phenomenon related to the nature of the quenched limit. Namely, one can view the quenched theory as a somewhat abstract limit $n \to 0$ of QCD with variable number $n$ of quark flavors. Then if the limit is smooth the quenched theory is a reasonable approximation to the theory at finite $n$. In fact, this limit is not always smooth and can be different from the quenched theory as, e.g., in a theory of spin glasses. We show that the quenched limit is not smooth in QCD at finite $\mu$. Quenched theory is therefore not a good approximation to full QCD at finite $\mu$. This resolves the old contradiction. Moreover, we can describe a theory to which quenched QCD is a good approximation. It is a theory (suggested in\textsuperscript{2}) in which each quark has a conjugate partner, essentially, with opposite baryon number assignment.

2. $\langle \bar{\psi}\psi \rangle$ and $\rho$

We study analytical properties of the quark condensate $\langle \bar{\psi}\psi \rangle$ — the order parameter of the chiral symmetry breaking — as a function of the bare quark mass $m$ which we allow to go into the complex plane: $m \equiv z = x + iy$. Integrating out the quark degrees of freedom we can express $\langle \bar{\psi}\psi \rangle$ through the average of the inverse Dirac operator $D$: $\langle \bar{\psi}\psi \rangle = \langle \text{Tr} \ (z - D)^{-1} \rangle \equiv G$. In the infinite volume the value of the $\langle \bar{\psi}\psi \rangle$ in the chiral limit $z \to 0$ can be related to the density of small eigenvalues $\rho(\lambda)$ by the Banks-Casher formula: $\langle \bar{\psi}\psi \rangle = \pi \rho(0)$.

To generalize this well-known relation to our case we must consider the density per unit area of the complex plane: $\rho(x, y)$. The relation between $\rho$ and $G$ is straightforward: $G(x, y) = \int dx\, dy\rho(x', y')(z - z')^{-1}$. Viewing $\rho$ as a density

\footnote{This work was supported by NSF-PHY92-00148.}
of charges and \( \text{Re} G, - \text{Im} G \) as components of the electric field strength \( \vec{G} \) one easily inverts this relation:

\[
\rho = \frac{1}{2\pi} \Im \vec{G} \equiv \frac{1}{\pi} \text{Re} \frac{\partial G}{\partial z^*}.
\] (1)

The main point here is that \( \rho \) vanishes where \( G \) depends analytically on \( z \), the bare quark mass. For example, when \( \mu = 0 \) all eigenvalues of \( D \) lie on the imaginary axis. This means that \( G \) has a cut as on Fig. 1. The discontinuity across the cut is the signature of the chiral symmetry breaking and is given precisely by the Banks-Casher formula.

3. Random matrices

Now we would like to calculate \( G \) and \( \rho \). Since we are interested in the density of small eigenvalues of the Dirac operator one can expect that some simplification can be made. Indeed, it is by now well understood that these small eigenvalues are related to the zero modes of the Dirac operator in the instanton field background. The behavior of these small eigenvalues is rather universal and can be described by a simple random matrix model [3]. We choose chiral representation of the Dirac gamma-matrices. Then the matrix of the Dirac operator in a certain basis has the block-diagonal form:

\[
D = \begin{pmatrix} 0 & iX \\ iX^\dagger & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}.
\] (2)

Nonzero entries \( X \) are approximated by a complex random \( N \times N \) matrix with a Gaussian distribution: \( P(X) \sim \exp \{-N \text{Tr} XX^\dagger\} \). To adapt the model to our case we add a chemical potential: \( \mu \gamma_0 \).

To find \( G \) one calculates the quenched free energy:

\[
V_n = -\frac{1}{n} \ln(\det^n(z - D)),
\] (3)

and differentiates it with respect to the bare quark mass \( z \): \( G = -\partial V/\partial z \). Then \( \rho \) is given by:

\[
\rho = -\frac{1}{\pi} \text{Re} \frac{\partial^2 V}{\partial z \partial z^*}.
\] (4)

Figure 1. The evolution of cuts of the function \( G(z) \equiv \langle \bar{\psi} \psi \rangle \) with increasing \( \mu \) from (a) to (c).

A standard way to deal with the logarithm inside the averaging in (3) is the ‘replica trick’. One does the calculation for arbitrary number of fermion species \( n \):

\[
V_n = -\frac{1}{n} \ln(\det^n(z - D)),
\] (5)

and then takes \( n \to 0 \) hoping that the limit is smooth. Doing this calculation we find that, first of all, \( G \) is a holomorphic function of \( z \). It is given by a solution of a cubic equation:

\[
G((z + G)^2 - \mu^2) - (z + G) = 0.
\]

It is analytic except for branch points, where 2 solutions coincide, connected by cuts. The cuts follow lines where the value of \( \text{Re} [G^2 - \ln((z + G)^2 - \mu^2)] \) on two solutions coincide. The physical sheet is determined by an asymptotic condition at \( z \to \infty \):

\[
G \to 1/z, \text{ from the normalization of } \rho.
\]

The singularities of \( G \) are shown in Fig. 2. For \( \mu^2 \leq 1/8 \) the cut is on the imaginary axis. At \( \mu^2 > 1/8 \) the branch points bifurcate and the cuts go into the complex plane. Above a certain value of \( \mu^2_c = 0.278 \ldots \) the cut no longer goes through \( z = 0 \) — there is no discontinuity across the imaginary axis, i.e., the chiral symmetry is restored.

4. \( n \to 0 \) and quenched theory

One can now compare this replica trick calculation to the actual distribution of eigenvalues of the random matrix \( D \) in the quenched theory. It is obtained numerically, by plotting the eigenvalues for an ensemble of random matrices [3] on Fig. 2. First of all, the eigenvalues do not stay on the imaginary axis for any nonzero \( \mu \)!
spread in a blob of the $x$-width growing $\sim \mu^2$ at small $\mu$. Exactly the same behavior is observed in the quenched QCD calculations and leads to the puzzling conclusion that $\mu_c \approx m_\pi/2$ and vanishes in the chiral limit $m = 0$ because there is no discontinuity in $G$ across the imaginary axis when $\mu > 0$.

Now, however, we are able to find the explicit solution of the model and see the nature of this quenched blob. First of all, we must learn that the limit $n \to 0$ of QCD with $n$ quarks is not smooth and the quenched theory is not a good approximation to the full theory at $\mu > 0$. Second, we want to know what theory has quenched QCD as its $n \to 0$ limit. We find that it is a theory where each quark has a partner with the conjugate Dirac operator. The free energy reads:

$$V_{n,n} = -\frac{1}{n} \ln(\det^n(z - D)(z^* - D^\dagger)).$$ (6)

This statement can be made rigorous even beyond the random matrix approximation using equations (4) and:

$$\delta^2(z - \bar{\lambda}_i) = -\frac{1}{4\pi} \Delta \ln|z - \lambda_i|^2.$$ (7)

Naively, when $n \to 0$ the conjugate quarks decouple: $V_{n,n} \to V(z) + V^*(z^*)$, and $G = -\partial V/\partial z$ is holomorphic. In fact, the quarks $\psi$ and conjugate quarks $\chi$ mix for some values of parameters $z$ and $\mu$, i.e., precisely inside the blob. A condensate $\langle \bar{\chi}\psi \rangle$ develops nonzero vacuum expectation value. This condensate carries nonzero baryon number. It was indeed observed in a Monte Carlo simulation of the $SU(2)$ theory with quarks, which are self-conjugate in that case. We learn now that similar phenomenon occurs in the quenched theory. Calculating (6) we find the density of eigenvalues inside of the blob:

$$\rho = \frac{1}{4\pi} \left( \frac{x^2 + \mu^2}{(\mu^2 - x^2)^2} - 1 \right),$$ (7)

which one can check is in perfect agreement with the numerical data Fig. 2.

5. Conclusions

Using a simple random matrix model of chiral symmetry breaking we demonstrate a general phenomenon that at nonzero $\mu$ the limit $n \to 0$ of QCD with $n$ quarks is not smooth and does not coincide with the quenched theory. On the other hand, the quenched theory is a smooth limit of a theory where each quark has a conjugate partner. Another way of putting this is to notice that the difference between such a theory and QCD with $2n$ quarks is that the phase of the fermion determinant is omitted in the former. We learn that the phase of the determinant is extremely important for a simulation of full QCD at finite $\mu$. Without this phase the result $\mu_c = 0$ is natural, as happens in quenched QCD. This is due to the formation of the baryonic condensate, which breaks the (replica) chiral symmetry of quarks and conjugate quarks. The Goldstone boson associated with this breaking is the so-called baryonic pion — a bound state of a quark and a conjugate antiquark. The presence of this particle explains the result $\mu_c = m_\pi/2$.

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