Entropy of the Conditional Expectation under Gaussian Noise

Arda Atalik
Department of Electrical-Electronics Engineering
Bilkent University
Ankara, TR 06800
arda.atalik@bilkent.edu.tr

Alper Köse
Department of Electrical-Electronics Engineering
Bogazici University
Istanbul, TR 34342
alper.kose@boun.edu.tr

Michael Gastpar
School of Computer and Communication Sciences
École Polytechnique Fédérale de Lausanne
Lausanne, CH 1015
michael.gastpar@epfl.ch

Abstract
This paper considers an additive Gaussian noise channel with arbitrarily distributed finite variance input signals. It studies the differential entropy of the minimum mean-square error (MMSE) estimator and provides a new lower bound which connects the entropy of the input, output, and conditional mean. That is, the sum of entropies of the conditional mean and output is always greater than or equal to twice the input entropy. Various other properties such as upper bounds, asymptotics, Taylor series expansion, and connection to Fisher Information are obtained. An application of the lower bound in the remote-source coding problem is discussed, and extensions of the lower and upper bounds to the vector Gaussian channel are given.

Keywords Differential entropy · conditional mean estimator · Gaussian noise · remote source coding problem

1 Introduction and Motivation

In estimation theory, mean-squared error (MSE) is one of the most commonly used error metrics and plays a significant role in many real-life applications ranging from signal detection in communications to regression problems in machine learning. As one aims at minimizing a cost function in an estimation problem, it is of great importance to find the estimator which leads to the minimum mean-square error (MMSE). A basic yet essential result in estimation theory is the optimality of the conditional mean in MSE sense, i.e., the MMSE estimate of $X$ observing $Y = y$ is found by the conditional mean of $X$ given $Y = y$.

In this paper, we consider the additive Gaussian noise model, i.e., $Y = X + W$ where $X$ is the input having an arbitrary continuous distribution with finite variance $\sigma^2_X$, $W$ is the zero-mean additive Gaussian noise with variance $\sigma^2_W$, and $Y$ is the observation.\footnote{Note that the model in Fig. 1 and the model in \cite{1} differ in the definition of conditional mean. The results we propose change slightly with the latter model.} The model is illustrated in Fig. 1.
Entropy of the Conditional Expectation under Gaussian Noise

\[ X \xrightarrow{+} Y = X + W \]

\[ W \sim \mathcal{N}(0, \sigma_W^2) \]

Figure 1: The Observation Model

Estimation in Gaussian noise has been a central topic at the intersection of estimation and information theory for the last sixty years, and the behavior of the conditional mean is well-studied. Earlier work in Gaussian noise includes but not limited to Tweedie’s Formula [2], which connects the conditional mean to the score function and Hatsell-Nolte Identity [3], which relates the conditional mean and variance, and Brown’s Identity [4], which shows a connection between the MMSE and Fisher Information.

In [1], a fundamental derivative identity connecting the mutual information and MMSE is discovered and in [5], it has been further explored. Properties of MMSE such as monotonicity, convexity, and infinite differentiability as a function of snr have been shown in [6], while its functional properties as a function of input-output distribution have been analyzed in [7]. Recently, in [8], the authors have focused on the derivatives of the conditional mean with respect to the observation, and many previously known identities in the literature have been recovered.

1.1 Contribution and Outline

- In Section 2, we provide a new lower bound which relates the entropy of the conditional mean to the input and output entropy.

\[ h(\mathbb{E}[X | Y]) \geq 2h(X) - h(Y) \]  

(1)

- In Section 3, we study further properties of the entropy of conditional mean such as upper bounds, Taylor series expansion, low-and-high input variance asymptotics, and connection to Fisher Information. For different input distributions, the bounds on \( h(\mathbb{E}[X | Y]) \) are illustrated.

- In Section 4, an application of the lower bound in the remote source coding [9] is investigated.

- In Section 5, the lower and upper bounds are extended to the vector additive Gaussian noise model.

- Section 6 concludes the paper and suggests some future directions.

1.2 Notation

We use uppercase letters \( X, Y \) to denote random variables, lowercase letters \( x, y \) to denote their realizations, and boldface uppercase letters \( \mathbf{X}, \mathbf{Y} \) for matrices and random vectors. Given a square-integrable, absolutely continuous random variable \( X \) with density \( p_X(x) \), its variance \( \text{Var}(X) \) is denoted as \( \sigma_X^2 \), and its differential entropy is

\[ h(X) = - \int p_X(x) \log p_X(x) \, dx. \]  

(2)

The entropy power of \( X \) is \( N(X) = \frac{e^{2h(X)}}{2 \pi e^2} \), mutual information is \( I(X; Y) = h(Y) - h(Y | X) = h(X) - h(X | Y) \) and its Fisher information is \( J(X) = \int p_X(x) \left( \frac{d}{dx} \log p_X(x) \right)^2 \, dx \). Denote the conditional expectation and variance of \( X \) given \( Y \) as

\[ \mathbb{E}[X | Y] \quad \text{and} \quad \text{Var}(X | Y) \]  

(3)

and its corresponding mean-square error as

\[ \text{mmse}(X | Y) = \mathbb{E}[\text{Var}(X | Y)] = \mathbb{E}[(X - \mathbb{E}[X | Y])^2]. \]  

(4)

To show the dependence on the observation, \( V, V(Y); S, S(Y) \) are used interchangeably. Whenever the input distribution is fixed, every function of interest is written as a function of \( \sigma_X^2 \) and \( \sigma_W^2 \).
1.3 Entropy of the Conditional Mean

The probability density function of the conditional mean is calculated explicitly in [8], but in general, it may not be easy to calculate for arbitrary input distributions. The following identity is useful as it omits the calculation of the density of $E[X|Y]$.

Lemma 1 (Calculation of $h(E[X|Y])$):

$$h(E[X|Y]) = h(Y) + \mathbb{E}\left[\log\left(\frac{1}{\sigma_W^2}\Var(X|Y)\right)\right]$$

Proof. See Appendix A.

2 Main Results

The entropy of the conditional expectation shows up as a lower/upper bound in certain multi-terminal information theory problems as shown in [9]. Thus, it is useful to derive tight bounds on $h(V)$ to obtain further insights. In general, deriving upper bounds to the entropy is not difficult using the maximum entropy argument, while lower bounds are less trivial and they might require extra assumptions on the input distribution such as log-concavity. Using the main result of [10], one obtains $h(V) \geq \frac{1}{2} \log \left(\frac{4}{\sigma_V^2}\Var(X|Y)\right)$ as long as the density of $V$ is log-concave, which might be tedious to check.

For the Gaussian noise, we present a new lower bound which combines the maximum entropy argument in the conditional setting with the identity (5), and is applicable regardless of the input distribution.

Theorem 2 (A Lower Bound of Entropy): Let $X$ be an arbitrary continuous random variable with finite variance and $Y = X + W$, where $W$ is Gaussian and independent of $X$. Then,

$$h(E[X|Y]) \geq 2h(X) - h(Y).$$

Furthermore, equality is achieved if and only if $X$ is Gaussian.

Proof. Since $h(X|Y = y) \leq \frac{1}{2} \log \left(\frac{2\pi e\Var(X|Y = y)}{\sigma_Y^2}\right)$ for every $y$, taking the expectation of both sides, one obtains

$$\mathbb{E}\left[\log(\Var(X|Y))\right] \geq 2 \left(h(X|Y) - \frac{1}{2} \log(2\pi e)\right) - h(W) + \log \sigma_W$$

$$= 2 \left(h(X|Y) - h(W) + \log \sigma_W\right)$$

$$= 2 \left(h(X) - h(Y) + \log \sigma_W\right)$$

where (i) follows by the definition of mutual information

$$I(X;Y) = h(Y) - h(W) = h(X) - h(X|Y).$$

Combining (5) and (i), the $\log \sigma_W$ term disappears and the desired result follows, in which the equality argument is due to the maximum entropy property of Gaussian distribution.

To the best of our knowledge, (6) is not known in the literature except the equality in Gaussian input was observed in [9]. As an immediate application, (6) can be used to compare the tightness of rate-distortion lower bounds in remote source coding under Gaussian noise, which is explained in Section 4.

3 Further Properties of the Entropy and Entropy Power of Conditional Mean

3.1 Upper Bounds

By the concavity of the logarithm, one can use Jensen’s Inequality to derive an upper bound.
Lemma 3 (Upper Bounds of Entropy):

\[ h(V) = h(Y) + \mathbb{E} \left[ \log \text{Var} \left( X \mid Y \right) \right] - \log \sigma_W^2 \]

(ii) \leq h(Y) + \log \mathbb{E} \left[ \text{Var} \left( X \mid Y \right) \right] - \log \sigma_W^2

(iii) = h(Y) + \log \text{mmse} \left( X \mid Y \right) - \log \sigma_W^2

(iv) \leq h(Y) + \log \left( \frac{\sigma_X^4}{\sigma_X^2 + \sigma_W^2} \right)

(v) \leq \frac{1}{2} \log \left( 2\pi e \frac{\sigma_X^4}{\sigma_X^2 + \sigma_W^2} \right)

where (ii) follows from the Jensen’s Inequality, (iii) is by definition of mmse, and (iv) and (v) follow from the maximization of mmse and entropy, respectively. Each inequality is satisfied with equality if and only if the input \( X \) is Gaussian. That is, for Gaussian inputs, \( \text{Var} \left( X \mid Y \right) \) is constant almost surely, and mmse and entropy are maximized \([6]\). From (v), it is evident that the maximum entropy of the conditional mean is achieved when the input \( X \) is Gaussian even though it minimizes the variance of \( \mathbb{E} \left[ X \mid Y \right] \).

Remark 4 (Maximization of \( h(V) \)):

\[ \max_{p_X} h(V) = \frac{1}{2} \log \left( 2\pi e \frac{\sigma_X^4}{\sigma_X^2 + \sigma_W^2} \right) \]

is achieved when \( X \sim N(0, \sigma_X^2) \).

Using the definition of the entropy power, upper and lower bounds immediately follow from (iii), (v), and (vi).

Remark 5 (Bounds on the Entropy Power):

\[ \frac{N^2(X)}{N(Y)} \leq \frac{\text{mmse} \left( X \mid Y \right)^2}{\sigma_W^2} \leq \frac{\sigma_X^4}{\sigma_X^2 + \sigma_W^2} \]

When the input \( X \) is Gaussian, the upper and lower bounds are satisfied with equality. In Fig. 2 the bounds are illustrated for the case when the input \( X \) follows Exponential and Uniform distribution.\(^2\)

Observe that the lower bound (10) is equivalent to

\[ h(Y) - h(X) \geq -\frac{1}{2} \mathbb{E} \left[ \log \text{Var} \left( X \mid Y \right) \right] + \log \sigma_W \]

which is guaranteed to be as tight as the trivial lower bound \( \frac{1}{2} \log \frac{\sigma_X^2 + \sigma_W^2}{\sigma_X^2} \), i.e., the worst noise is Gaussian for the Gaussian input. A comparison between this representation and the classical EPI is given in Appendix C.

3.2 Taylor Series Expansion

In this subsection, we provide Taylor’s expansion to \( \mathbb{E} \left[ \log \left( \text{Var} \left( X \mid Y \right) \right) \right] \). Assume that all moments of \( \text{Var} \left( X \mid Y \right) \) are finite. Denote the \( k^{th} \) central moment of \( S = \text{Var} \left( X \mid Y \right) \) by \( c_k \), i.e., \( \mathbb{E} \left[ (S - \mathbb{E}[S])^k \right] = c_k \) and mmse \( (X \mid Y) \) by mmse. Since \( \log(\cdot) \) is sufficiently differentiable, we have

\[ \mathbb{E} \left[ \log S \right] = \mathbb{E} \left[ \log \left( \mathbb{E}[S] + S - \mathbb{E}[S] \right) \right] \]

\[ = \sum_{k=0}^{\infty} \frac{c_k}{k!} \log^{(k)} \left( \mathbb{E}[S] \right) \]

\[ = \log \text{mmse} + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k \text{mmse}^k} c_k. \]

\(^2\)Note that \( h(Y), h(V), \) and mmse are calculated numerically as there are no closed-form expressions. We also calculated the bounds for other input distributions, e.g., Laplace and Triangular, and observed that there are no visible gaps. Without loss of generality, the noise variance is set to unity in both simulations.
Figure 2: $h(V)$ and the bounds (6), (iii), (iv): the bounds are tighter for input distributions close to Gaussian in terms of Kullback-Leibler divergence.

Using (5), Taylor expansion of $h(V)$ follows.

$$h(V) = h(Y) + \log \left( \frac{\text{mmse}}{\sigma_W^2} \right) + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k \text{mmse}^k} c_k$$

(22)

Observe that neglecting the summation in (22), we recover the upper bound in (iv). It is viable to obtain approximations to $h(V)$ by truncating the summation at $k = \hat{k}$, e.g., $\hat{k} = 2$ yields

$$h(V) \approx h(Y) + \log \left( \frac{\text{mmse}}{\sigma_W^2} \right) - \frac{\text{Var} (\text{Var} (X | Y))}{2 \text{mmse}^2}.$$  

(23)

### 3.3 Input Variance Asymptotics

We fix the noise variance $\sigma_W^2$, and study low and high $\sigma_X^2$ asymptotics of $h(V)$ and $N(V)$.

**Lemma 6 (Low $\sigma_X^2$ Asymptotics):** As $\sigma_X^2 \to 0^+$

$$h(V) \sim \log \sigma_X^2$$

(24)

$$N(V) \sim \sigma_X^4$$

(25)
i.e., \( \lim_{\sigma_X^2 \to 0^+} \frac{h(V)}{\log \sigma_X^2} = 1 \) and \( \lim_{\sigma_X^2 \to 0^+} \frac{N(V)}{\sigma_X^2} = 1 \).

**Proof.** Observe that as \( \sigma_X^2 \) approaches \( 0^+ \), since \( Y \) converges to \( \mathbb{E}[X] + W \) almost surely, the first term in the right hand side of (5) approaches to the differential entropy of the noise, i.e., \( h(Y) = -\int p_Y(y) \log p_Y(y) dy \to -\int p_W(y) \log p_W(y) dy = h(W) \). For the second term, it is easy to check that \( \text{Var}(X|Y) \) converges to \( \text{Var}(X) = \sigma_X^2 \) almost surely, which implies that \( \mathbb{E}[\log \text{Var}(X|Y)] \to \log \sigma_X^2 \). Combining these, we obtain (24), which is equivalent to (25). \( \square \)

**Remark 7 (High \( \sigma_X^2 \) Asymptotics):** As \( \sigma_X^2 \to \infty \)

\[
\begin{align*}
    h(V) &= O(\log \sigma_X^2) \\
    N(V) &= O(\sigma_X^2)
\end{align*}
\]

**Proof.** Proof directly follows from [v], i.e., \( h(V) \leq \frac{1}{2} \log (2\pi e) + \frac{1}{2} \log \left( \frac{\sigma_X^4}{\sigma_X^4 + \sigma_W^4} \right) + \frac{1}{2} \log \sigma_X^2 \) and \( N(V) \leq \frac{\sigma_X^4}{\sigma_X^4 + \sigma_W^4} \) \( \square \)

### 3.4 Connection to Fisher Information

A lower bound for the Fisher Information of \( V \) directly follows from Stam’s Inequality [11].

\[
J(V) \geq \frac{1}{N(V)}
\]

(28)

Combining (17) and (28), lower bounds for \( J(V) \), which do not require the entropy of \( V \), can be proposed:

\[
J(V) \geq \frac{1}{N(Y)} \frac{\sigma_W^4}{\text{mmse}(X|Y)^2} \geq \frac{\sigma_X^2 + \sigma_W^2}{\sigma_X^2}
\]

(29)

in which the equality is achieved when the input is Gaussian.

### 4 Application: Lower Bounds of Rate Distortion Function in the Remote Source Coding Problem

An important application of inequality (6) can be found in the remote source coding problem addressed in [9] in which the observation noise is assumed to be Gaussian and the underlying source \( X \), not necessarily Gaussian, has finite differential entropy. The authors provide two different lower bounds for the remote rate-distortion function, i.e.,

\[
R_X^R(D) \geq \frac{1}{2} \log^+ \frac{N(V)}{D} + \frac{1}{2} \log^+ \frac{N(Y)}{N(Y) - \frac{N(X)}{D} N(W)}
\]

(30)

and

\[
R_X^R(D) \geq \frac{1}{2} \log^+ \frac{N(X)}{D} + \frac{1}{2} \log^+ \frac{N(Y)}{N(Y) - \frac{N(X)}{D} \sigma_W^2}
\]

(31)

where \( D > \mathbb{E}[\{X - V\}^2] \) and \( \log^+ x = \max\{0, \log x\} \). At the time of their writing, a comparison between the right-hand sides (RHS) of (30) and (31) was not available. We prove that the RHS of (30) is always greater than or equal to the RHS of (31).

**Proposition 8 (A comparison of lower bounds for the remote rate-distortion function):**

\[
\frac{1}{2} \log^+ \frac{N(V)}{D} + \frac{1}{2} \log^+ \frac{N(Y)}{N(Y) - \frac{N(X)}{D} N(W)} \geq \frac{1}{2} \log^+ \frac{N(X)}{D} + \frac{1}{2} \log^+ \frac{N(Y)}{N(Y) - \frac{N(X)}{D} \sigma_W^2}
\]

(32)

Furthermore, equality is achieved if and only if \( X \) is Gaussian.
Proof. Note that $N(W) = \sigma_W^2$, hence the second term in the RHS of (30) is always greater than or equal to the second term in (31) since
\[ N(Y) \geq N(X) \] 
and
\[ N(Y) > N(Y) - \frac{N(X)}{D} N(W). \] 
(34)

To compare the first terms in the inequalities, observe that if $D \geq N(X) \geq N(W)$, both $\log + \frac{N(Y)}{D}$ and $\log + \frac{N(X)}{D}$ are equal to 0, so the lower bound in (30) is greater. If $N(X) \geq N(V) > D$, since $N(V) N(Y) \geq (N(X))^2$ from Proposition (6), it is again guaranteed that the lower bound in (30) is greater. Finally, the last possible and a little more complicated case is $N(X) > D \geq N(V)$. In this case the lower bounds in (30) and (31) to be compared become
\[
\frac{DN(Y)}{DN(Y) - N(X)\sigma_W^2} \leq \frac{(N(X))^2}{DN(Y) - N(X)\sigma_W^2}
\]
(35)
where it is enough to compare $DN(Y)$ and $(N(X))^2$. Taking the logarithm of both sides, we get
\[
2h(Y) + \log 2\pi e D \leq 4h(X).
\]
Note that we have assumed $D \geq N(V)$ for this case at the beginning. This is equivalent to the following inequality.
\[
\log 2\pi e D \geq 2h(V)
\]
(37)
Combining (6), (36) and (37), one obtains,
\[
2h(Y) + \log 2\pi e D \geq 2h(Y) + 2h(V) \geq 4h(X).
\]
(38)
Therefore, we conclude that the lower bound in (30) is always greater than or equal to the lower bound in (31).

5 Extension to Vector Case

In this section, we consider the extension of the main result (6) and the upper bound (ii) under the vector Gaussian noise model, i.e., the input-output relationship is governed by
\[ Y = X + W \]
(39)
where $W \in \mathbb{R}^n$ is a zero mean Gaussian random vector with positive-definite covariance matrix $K_W$. It is assumed that $X$ and $W$ are independent, and the only assumption on $X$ is that its covariance matrix $K_X$ is full-rank, i.e., $X$ is non-degenerate. Denote the conditional variance matrix by $\text{Var} (X \mid Y) \triangleq \mathbb{E} [XX^T \mid Y] - \mathbb{E} [X \mid Y] \mathbb{E} [X^T \mid Y]$, the MMSE matrix by $\text{MMSE}(X \mid Y) \triangleq \mathbb{E} [\text{Var} (X \mid Y)]$, and the Jacobian matrix of a transformation $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ by $J \psi(y)$ with $i^{th}$ row, $j^{th}$ column element being $\frac{\partial \psi_j}{\partial y_i}$. The result of this section is that the same lower bound (6) is valid under (39).

Proposition 9 (A Lower Bound of Entropy in Vector Case):
\[ h(V) \geq 2h(X) - h(Y) \]
(40)

Proof. Since $I(X; Y) = h(Y) - h(W) = h(X) - h(X \mid Y)$, (40) is equivalent to
\[ h(X \mid Y) \leq h(W) - \frac{1}{2} h(Y) + \frac{1}{2} h(V) \]
(41)
\[ = \frac{1}{2} \log (2\pi e)^n \det K_W - \frac{1}{2} h(Y) + \frac{1}{2} h(V). \]
(42)
By the maximum entropy argument,
\[ h(X \mid Y) \leq \frac{1}{2} \mathbb{E} \left[ \log (2\pi e)^n \det \text{Var} (X \mid Y) \right]. \]
(43)
Hence, it is sufficient to show that
\[
\frac{1}{2} \mathbb{E} \left[ \log ((2\pi e)^n \det \text{Var}(X \mid Y)) \right] \leq \frac{1}{2} \log ((2\pi e)^n \det K_W) - \frac{1}{2} h(Y) + \frac{1}{2} h(V)
\]
which is equivalent to
\[
\mathbb{E} \left[ \log \det (K_W^{-1} \text{Var}(X \mid Y)) \right] \leq h(V) - h(Y).
\]
By the change of variables formula, \( h(V) = h(Y) + \mathbb{E} [\log \det J_Y(V)] \) provided that the transformation \( y \mapsto \mathbb{E} [X \mid Y = y] \) is a bijection.

Thus, it remains to justify \( \mathbb{E} [\log \det J_Y(V)] \geq \mathbb{E} [\log \det (K_W^{-1} \text{Var}(X \mid Y))] \) and \( y \mapsto \mathbb{E} [X \mid Y = y] \) is a bijection. Under Gaussian noise, the variance identity of Hatsell and Nolte \cite{8} guarantees both conditions provided that \( K_X \) is full-rank.

By the concavity of the log determinant, Jensen’s Inequality gives an upper bound to \( h(V) \).

**Lemma 10 (Upper Bounds of Entropy in Vector Case):**

\[
\begin{align*}
\bi{vii} h(V) &\overset{(vi)}{=} h(Y) + \mathbb{E} [\log \det (K_W^{-1} \text{Var}(X \mid Y))] \\
&\overset{(vii)}{\leq} h(Y) + \log \det \mathbb{E} [\text{Var}(X \mid Y)] - \log \det K_W \\
&\overset{(viii)}{=} h(Y) + \log \det \text{MMSE}(X \mid Y) - \log \det K_W \\
&\overset{(ix)}{\leq} \frac{1}{2} \log ((2\pi e)^n \det (K_X + K_W)) + \log \det \text{MMSE}(X \mid Y) - \log \det K_W
\end{align*}
\]

where \( (vi) \) follows from Hatsell and Nolte Identity \cite{8} combined with the change of variables, \( (vii) \) follows from Jensen’s Inequality, \( (viii) \) is by the definition of MMSE matrix and \( (ix) \) is by the maximum entropy argument.

\section{Conclusion and Outlook}

In this work, we looked at the differential entropy of the conditional mean under Gaussian noise. Specifically, we proposed a new lower bound \( h(\mathbb{E} [X \mid Y]) + \hat{h}(Y) \geq 2h(X) \) and demonstrated its application in the remote source coding problem. Furthermore, we derived upper bounds, obtained asymptotics, and provided Taylor series expansion to \( h(\mathbb{E} [X \mid Y]) \). In the last section, we extended the lower and upper bounds to the vector case. An interesting future direction is to apply these findings to the CEO problem \cite{9} and multi-terminal detection problems such as hypothesis testing against independence \cite{12}.
Appendix A  Calculation of the Entropy of the Conditional Mean

Lemma 11 (Entropy of Transformations of a Random Variable): Let $x \mapsto \varphi(x)$ be a transformation on $\mathbb{R}$ and $X$ be an absolutely continuous random variable with finite entropy. Then, the entropy of $\varphi(X)$ satisfies the following upper bound.

$$h(\varphi(X)) \leq h(X) + \mathbb{E} \left[ \log \left| \frac{d\varphi}{dX} \right| \right]$$

(49)

where the equality is achieved if and only if the transformation $x \mapsto \varphi(x)$ is a bijection.

Observe that by Hatsell-Nolte Identity, $\sigma_W^2 \frac{d\mathbb{E}[X|Y]}{\sigma_Y^2} = \text{Var}(X|Y) > 0$ almost surely for all $\sigma_X^2 > 0$, which implies that $y \mapsto \mathbb{E}[X|Y = y]$ is strictly increasing for all non-degenerate input $X$. Thus, the equality in (49) is achieved provided that $\sigma_X^2 > 0$, which is a reasonable assumption in all applications. Using (49), we immediately obtain

$$h(\mathbb{E}[X|Y]) = h(Y) + \mathbb{E} \left[ \log \left| \frac{d\mathbb{E}[X|Y]}{dY} \right| \right]$$

(50)

$$= h(Y) + \mathbb{E} \left[ \log \left( \frac{1}{\sigma_W^2} \text{Var}(X|Y) \right) \right].$$

(51)

Appendix B  Table of Quantities for Gaussian Input

|             | $X$                      | $Y$                      | $\mathbb{E}[X|Y]$ | $\text{Var}(X|Y)$ | $\text{mmse}(X|Y)$ |
|-------------|--------------------------|--------------------------|-------------------|-------------------|-------------------|
| $h(\cdot)$  | $\frac{1}{2} \log (2\pi e \sigma_X^2)$ | $\frac{1}{2} \log (2\pi e (\sigma_X^2 + \sigma_W^2))$ | $\frac{1}{2} \log \left( \frac{2\pi e \sigma_X^2}{\sigma_X^2 + \sigma_W^2} \right)$ |                   |                   |
| $N(\cdot)$  | $\sigma_X^2$             | $\sigma_X^2 + \sigma_W^2$ | $\frac{\sigma_X^2}{\sigma_X^2 + \sigma_W^2}$ |                   |                   |
| $J(\cdot)$  | $\frac{1}{\sigma_X^2}$  | $\frac{1}{\sigma_X^2 + \sigma_W^2}$ | $\frac{\sigma_X^2 + \sigma_W^2}{\sigma_X^2}$ |                   |                   |
| $\mathbb{E}[X|Y]$ | $\frac{\sigma_X^2}{\sigma_X^2 + \sigma_W^2} Y$ | $\frac{\sigma_X^2}{\sigma_X^2 + \sigma_W^2} Y$ | $\frac{\sigma_X^2}{\sigma_X^2 + \sigma_W^2}$ |                   |                   |
| $\text{Var}(X|Y)$ | $\frac{\sigma_X^2}{\sigma_X^2 + \sigma_W^2} Y$ | $\frac{\sigma_X^2}{\sigma_X^2 + \sigma_W^2} Y$ | $\frac{\sigma_X^2}{\sigma_X^2 + \sigma_W^2}$ |                   |                   |
| $\text{mmse}(X|Y)$ | $\frac{\sigma_X^2}{\sigma_X^2 + \sigma_W^2} Y$ | $\frac{\sigma_X^2}{\sigma_X^2 + \sigma_W^2} Y$ | $\frac{\sigma_X^2}{\sigma_X^2 + \sigma_W^2}$ |                   |                   |

Table 1: Values of different quantities when $X$ is a Gaussian random variable with $\text{Var}(X) = \sigma_X^2$ and $\text{Var}(W) = \sigma_W^2$

Appendix C  Comparison with the Classical EPI

It is tempting to compare (18) with the classical EPI as they both provide lower bounds on the entropy of the output $Y$, i.e., EPI states $h(Y) \geq \frac{1}{2} \log (\exp (2h(X)) + 2\pi e)$ and (18) guarantees $h(Y) \geq h(X) - \frac{1}{2} \mathbb{E} \left[ \log \text{Var}(X|Y) \right]$ for $\sigma_W = 1$. Note that these two bounds are equal when the input is Gaussian.

In Fig. 3, we compare these bounds when the input $X$ is an Exponential and Uniform random variable. Two observations are immediate. As the input variance increases, (18) becomes tighter than EPI, and the gap between two bounds at high input variance increases as the input distribution becomes distant from Gaussian in terms of Kullback-Leibler divergence.
References

[1] D. Guo, S. Shamai, and S. Verdu, “Mutual information and minimum mean-square error in gaussian channels,” *IEEE Trans. Inf. Theory*, vol. 51, no. 4, pp. 1261–1282, Apr. 2005.

[2] H. Robbins, “An empirical Bayes approach to statistics,” in *Proc. Third Berkeley Symp. Math Statist. Probab.*, vol. 1, pp. 157–163, 1956.

[3] C. Hatsell and L. Nolte, “Some geometric properties of the likelihood ratio (corresp.),” *IEEE Trans. Inf. Theory*, vol. 17, no. 5, pp. 616–618, 1971.

[4] L. D. Brown, “Admissible estimators, recurrent diffusions, and insoluble boundary value problems,” *Ann. Math. Statist.*, vol. 42, no. 3, pp. 855–903, Jun. 1971.

[5] D. Palomar and S. Verdu, “Gradient of mutual information in linear vector gaussian channels,” *IEEE Trans. Inf. Theory*, vol. 52, no. 1, pp. 141–154, Jan. 2006.

[6] D. Guo, Y. Wu, S. Shamai, and S. Verdú, “Estimation in gaussian noise: Properties of the minimum mean-square error,” *IEEE Trans. Inf. Theory*, vol. 57, no. 4, pp. 2371–2385, Apr. 2011.

[7] Y. Wu and S. Verdu, “Functional properties of minimum mean-square error and mutual information,” *IEEE Trans. Inf. Theory*, vol. 58, no. 3, pp. 1289–1301, Mar. 2012.

[8] A. Dytso, H. V. Poor, and S. S. Shitz, “A general derivative identity for the conditional mean estimator in gaussian noise and some applications,” in *Proc. IEEE Int. Symp. Inf. Theory*, (Los Angeles, CA, USA), pp. 1183–1188, Jun. 2020.

[9] K. Eswaran and M. Gastpar, “Remote source coding under gaussian noise: Dueling roles of power and entropy power,” *IEEE Trans. Inf. Theory*, vol. 65, no. 7, pp. 4486–4498, Jul. 2019.

[10] A. Marsiglietti and V. Kostina, “A lower bound on the differential entropy of log-concave random vectors with applications,” *Entropy*, vol. 20, no. 3, p. 185, Mar. 2018.

[11] A. J. Stam, “Some inequalities satisfied by the quantities of information of fisher and shannon,” *Inf. Control*, vol. 2, no. 2, pp. 101–112, 1959.
[12] A. Zaidi, “Hypothesis testing against independence under Gaussian noise,” in Proc. IEEE Int. Symp. Inf. Theory, (Los Angeles, CA, USA), pp. 1–5, Jun. 2020.

[13] A. Dytso and H. V. Poor, “Estimation in poisson noise: Properties of the conditional mean estimator,” IEEE Transactions on Information Theory, vol. 66, no. 7, p. 4304–4323, 2020.