THE 3-DIMENSIONAL COMPLEX PROJECTIVE SPACE ADMITS NO SPECIAL GENERIC MAPS

NAOKI KITAZAWA

Abstract. The main theorem of the present paper is that the 3-dimensional complex projective space does not admit special generic maps. Special generic maps are generalized versions of Morse functions on spheres with exactly two singular points. The canonical projections of unit spheres are of the class. The paper mainly focuses on special generic maps on 6-dimensional closed and simply-connected manifolds.

The differentiable structures of spheres admitting special generic maps are known to be restricted strongly in general. Special generic maps on closed and simply-connected manifolds and projective spaces have been studied by various people including the author. The existence or non-existence and construction are main problems. Studies on such maps on closed and simply-connected manifolds whose dimensions are greater than 5 have been difficult.

1. Introduction.

What are special generic maps? In short, Morse functions on spheres with exactly two singular points are regarded as simplest special generic maps. Canonical projections of unit spheres are also special generic.

We define a special generic map. First we introduce terminologies and notation on (smooth) manifolds and maps. For a positive integer $k$, $\mathbb{R}^k$ denotes the $k$-dimensional Euclidean space. We regard this as a natural smooth manifold with a standard Euclidean metric as a Riemannian metric. For $x \in \mathbb{R}^k$, $||x||$ denotes the distance between $x$ and the origin $0 \in \mathbb{R}^k$. For a positive integer $k$, $D^k := \{x \mid ||x|| \leq 1\} \subset \mathbb{R}^k$ denotes the $k$-dimensional unit disk, which is a $k$-dimensional compact, connected and smooth closed submanifold. $S^{k-1} := \{x \mid ||x|| = 1\} \subset \mathbb{R}^k$ denotes the $(k-1)$-dimensional unit sphere, which is a $(k-1)$-dimensional closed smooth submanifold with no boundary. The 0-dimensional unit sphere is a two-point set endowed with the discrete topology where $k = 1$ and $k - 1 = 0$. The $(k-1)$-dimensional unit sphere is connected for $k - 1 \geq 1$.

It is well-known that every smooth manifold has the structure of a canonical PL manifold. We regard smooth manifolds as the canonical PL manifolds.

For a manifold or a polyhedron $X$, let $\dim X$ denote the dimension of $X$.

Let $c : X \to Y$ be a smooth map from a smooth manifold $X$ into another smooth manifold $Y$. A singular point $p \in X$ of a smooth map $c : X \to Y$ is a point where the rank of the differential $dc_p$ is smaller than $\min\{\dim X, \dim Y\}$. The singular set $S(c)$ of $c$ is defined as the set of all singular points of $c$.

Key words and phrases. Special generic maps. (Co)homology. Projective spaces. Closed and simply-connected manifolds.

2020 Mathematics Subject Classification: Primary 57R45. Secondary 57R19.
A diffeomorphism is defined as a smooth map between two manifolds which is a homeomorphism with no singular points. A diffeomorphism from a manifold X onto the same manifold is said to be a diffeomorphism on X. The diffeomorphism group of a smooth manifold X is the group of all diffeomorphisms on X, topologized with the so-called Whitney $C^\infty$ topology. Whitney $C^\infty$ topologies are well-known to be natural topologies on sets of smooth maps between two manifolds. Consult [8] for example.

Two smooth manifolds are diffeomorphic if there exists a diffeomorphism from a manifold to the other manifold. A homotopy sphere means a smooth manifold which is homeomorphic to a sphere. A standard sphere means a homotopy sphere which is diffeomorphic to a unit sphere. An exotic sphere means a homotopy sphere which is not a standard sphere. A standard disk is a smooth manifold diffeomorphic to a unit disk.

[20] is a pioneering paper on an exotic sphere. This shows 7-dimensional exotic spheres. [10] is on homotopy spheres whose dimensions are greater than 4. In short, exotic spheres are classified via abstract algebraic topological theory in dimensions greater than 4. It is well-known that exotic spheres do not exist in dimensions 1, 2, 3, 5 and 6. The existence or the non-existence of 4-dimensional exotic spheres is an open problem. A smooth manifold which is homeomorphic to a unit disk and not diffeomorphic to any unit disk is still undiscovered. If it exists, then it must be 4-dimensional.

**Definition 1.** A smooth map $c : X \to Y$ from a closed smooth manifold X into another smooth manifold Y with no boundary satisfying dim $X \geq$ dim $Y$ is a special generic map if at each singular point $p$, it is locally represented by the form

$$(x_1, \cdots, x_{\dim X}) \mapsto (x_1, \cdots, x_{\dim Y - 1}, \sum_{k=\dim Y}^{\dim X} x_k^2)$$

for suitable coordinates.

The differentiable structures of spheres admitting special generic maps are known to be restricted strongly in general. The existence or the non-existence and construction of special generic maps on closed and simply-connected manifolds and projective spaces have been studied by various people including the author.

**Main Theorem.** The 3-dimensional complex projective space does not admit special generic maps into Euclidean spaces.

This manifold is also a 6-dimensional closed and simply-connected manifold. Note that classifications of closed and simply-connected manifolds whose dimensions are greater than 4 have a long history. [35, 40, 41] are on classifications of 6-dimensional closed and simply-connected manifolds.

The goal of the present paper is to show this. In the second section, we review existing studies on special generic maps such as fundamental properties and the existence or the non-existence of special generic maps on projective spaces and closed and simply-connected manifolds. It may be important to review special generic maps on homotopy spheres. However we omit this considering the main content of the present paper. We only introduce [3, 25, 26, 37, 38] and preprints [16, 17, 18, 19] by the author, which review related theory. In the third section, we show a new result as Theorem 3 and have Main Theorem as Corollary 2. The
present paper includes similar descriptions as some of descriptions in \[18\]. \[18\] also studies special generic maps on 6-dimensional closed and simply-connected manifolds.

We show a new closely related theorem or Theorem 4. This is regarded as an extension of a main theorem or Main Theorem 3 of \[18\].

2. SOME EXISTING STUDIES ON SPECIAL GENERIC MAPS.

**Proposition 1.** For a special generic map in Definition 1, we have the following properties.

1. The singular set is an \((n-1)\)-dimensional smooth closed submanifold with no boundary.
2. The restriction of the map to the singular set is a smooth immersion.
3. For suitable coordinates, around each singular point, the map is represented as the product map of a Morse function and the identity map on a small open neighborhood of the singular point where the neighborhood is considered in the singular set.

Hereafter, we use terminologies and notions on bundles such as fibers, structure groups, projections, sections and trivial bundles without rigorous expositions for example. A bundle is said to be a smooth bundle if its fiber is a smooth manifold and its structure group is a subgroup of the diffeomorphism group. A smooth bundle is a linear bundle whose fiber is an Euclidean space, a unit sphere, or a unit disk and whose structure group consists of linear transformations. We also explicitly or implicitly apply arguments on bundles and characteristic classes of linear bundles. For linear bundles and more general bundles, see \[21, 32\] for example.

**Proposition 2** (E. g. \[25\]). Let \(m \geq n \geq 1\) be integers. For a special generic map \(f : M \to N\) on an \(m\)-dimensional closed manifold \(M\) into an \(n\)-dimensional manifold \(N\) with no boundary, the following properties hold.

1. There exists an \(n\)-dimensional compact manifold \(W_f\) and a smooth immersion \(\tilde{f} : W_f \to N\).
2. There exists a smooth surjection \(q_f : M \to W_f\) and \(f = \tilde{f} \circ q_f\).
3. \(q_f\) maps the singular set \(S(f)\) of \(f\) onto the boundary \(\partial W_f \subset W_f\) as a diffeomorphism.
4. We have the following two bundles.
   a. For some small collar neighborhood \(N(\partial W_f) \subset W_f\), the composition of the map \(q_f|_{q_f^{-1}(\partial W_f)}\) onto \(N(\partial W_f)\) with the canonical projection to \(\partial W_f\) is the projection of a linear bundle whose fiber is the \((m-n+1)\)-dimensional unit disk.
   b. The restriction of \(q_f\) to the preimage of \(W_f - \text{Int } N(\partial W_f)\) is the projection of a smooth bundle over \(W_f - \text{Int } N(\partial W_f)\) whose fiber is an \((m-n)\)-dimensional standard sphere.

This is explicitly presented in \[25\] as a proposition in the case \(m > n \geq 1\). The theory of special generic maps from \(m\)-dimensional closed manifolds into \(n\)-dimensional manifolds with no boundaries with \(m = n \geq 1\) is essentially the theory of Eliashberg (\[5\]).

Let \(m \geq n \geq 1\) be integers again. Conversely, if we have a smooth immersion \(\tilde{f} : W_f \to N\) of an \(n\)-dimensional compact smooth manifold \(W_f\) into
an \( n \)-dimensional manifold \( N \) with no boundary, we have a special generic map \( f_0 : M_0 \to N \) on a suitable \( m \)-dimensional closed manifold \( M_0 \) into \( N \) satisfying the following properties.

1. There exists an \( n \)-dimensional compact manifold \( W_{f_0} \) and a smooth immersion \( f_0 : W_{f_0} \to N \).
2. There exists a diffeomorphism \( \phi : W_f \to W_{f_0} \) satisfying \( \tilde{f} = f_0 \circ \phi \).
3. There exists a smooth surjection \( q_{f_0} : M \to W_{f_0} \) and \( f_0 = f_0 \circ q_{f_0} \).
4. \( q_{f_0} \) maps the singular set \( S(f_0) \) of \( f_0 \) onto the boundary \( \partial W_{f_0} \subset W_{f_0} \) as a diffeomorphism.
5. We have the following two bundles.
   a) For some small collar neighborhood \( N(\partial W_{f_0}) \subset W_{f_0} \), the composition of the map \( q_{f_0}|_{q_0^{-1}(\partial W_{f_0})} \) onto \( N(\partial W_{f_0}) \) with the canonical projection to \( \partial W_{f_0} \) is the projection of a trivial linear bundle whose fiber is the \((m - n + 1)\)-dimensional unit disk.
   b) The restriction of \( q_{f_0} \) to the preimage of \( \partial W_{f_0} = \text{Int} N(\partial W_{f_0}) \) is the projection of a trivial smooth bundle over \( W_{f_0} = \text{Int} N(\partial W_{f_0}) \) whose fiber is an \((m - n)\)-dimensional standard sphere.

This property is introduced as propositions in most of articles by the author. Hereafter, for a finite set \( X \), \(|X|\) denotes the size of \( X \).

Example 1. Let \( m \geq n \geq 2 \) be integers. Let \( \{S^{k_j} \times S^{m-k_j}\}_{j \in J} \) be a family of finitely many products of two unit spheres where \( k_j \) is an integer satisfying \( 1 \leq k_j \leq n - 1 \). We consider a connected sum of these \( |J| \) manifolds in the smooth category. Let \( M_0 \) be an \( m \)-dimensional closed and connected manifold diffeomorphic to the obtained manifold. We have a special generic map \( f_0 \) so that \( W_{f_0} \) is diffeomorphic to a manifold represented as a boundary connected sum of \( |J| \) manifolds in \( \{S^{k_j} \times D^{m-k_j}\}_{j \in J} \) just before. Of course the boundary connected sum is considered in the smooth category. Furthermore, we can take \( N = \mathbb{R}^n \) and \( f_0|_{S(f_0)} \) as an embedding.

Proposition 3 (E. g. [25]). Let \( m \geq n \geq 1 \) be integers. For a special generic map \( f : M \to N \) on an \( m \)-dimensional closed and connected manifold \( M \) into an \( n \)-dimensional manifold \( N \) with no boundary, there exists an \((m + 1)\)-dimensional compact and connected PL manifold \( W \) whose boundary is \( M \) and which collapses to \( W_f \) where \( W_f \) is an \( n \)-dimensional compact and smooth manifold "\( W_f \) in Proposition 2" and identified with a suitable subpolyhedron of \( W \). Furthermore, for the canonical inclusion \( i_M : M \to W \) and a PL map \( r : W \to W_f \) giving a collapsing to \( W_f \), \( q_f = r \circ i_M \) holds. If \( m - n = 1, 2, 3 \) in addition, then \( W \) can be chosen as a smooth manifold and \( r \) as a smooth map.

Let \( m \geq n \geq 1 \) be integers again. Conversely, if we have a smooth immersion \( \tilde{f} : W_f \to N \) of an \( n \)-dimensional compact smooth manifold \( W_f \) into an \( n \)-dimensional manifold \( N \) with no boundary, we have a suitable special generic map \( f_0 : M_0 \to N \) on a suitable \( m \)-dimensional closed manifold \( M_0 \) into \( N \) satisfying the properties just after Proposition 2 and we can have an \((m + 1)\)-dimensional compact and connected smooth manifold \( W \) and a smooth map \( r : W \to W_f \) as in Proposition 3. For more general propositions of this type, see [30] and see papers [12, 13, 14] by the author.
The following two theorems give some characterizations of manifolds admitting special generic maps where the classes of the manifolds and the dimensions of the Euclidean spaces of the targets are fixed.

**Theorem 1.** Let $m \geq n \geq 1$ be integers. We have the following characterizations where connected sums are considered in the smooth category.

1. ([25]) Let $m \geq 2$. An $m$-dimensional closed and connected manifold admits a special generic map into $\mathbb{R}^2$ if and only if either of the following two holds.
   (a) $M$ is a homotopy sphere which is not a 4-dimensional exotic sphere.
   (b) $M$ is diffeomorphic to a manifold represented as a connected sum of the total spaces of smooth bundles over $S^1$ whose fibers are diffeomorphic to homotopy spheres which are not 4-dimensional exotic spheres.

2. ([25]) Let $m = 4, 5$. An $m$-dimensional closed and simply-connected manifold admits a special generic map into $\mathbb{R}^3$ if and only if either of the following two holds.
   (a) $M$ is a standard sphere.
   (b) $M$ is diffeomorphic to a manifold represented as a connected sum of the total spaces of linear bundles over $S^2$ whose fibers are diffeomorphic to $S^3$.

3. ([22]) Let $m = 5$. An $m$-dimensional closed and simply-connected manifold admits a special generic map into $\mathbb{R}^4$ if and only if either of the following two holds.
   (a) $M$ is a standard sphere.
   (b) $M$ is diffeomorphic to a manifold represented as a connected sum of the total spaces of linear bundles over $S^2$ whose fibers are diffeomorphic to $S^3$.

We introduce notation and terminologies on homology groups, cohomology groups and rings and homotopy groups. For elementary or advanced theory, consult [9] for example.

Let $(X, X')$ be a pair of topological spaces satisfying $X' \subset X$ where $X'$ can be the empty set. Let $A$ be a commutative ring. The homology group (cohomology group and ring) of the pair $(X, X')$ of topological spaces satisfying $X' \subset X$ is defined and denoted by $H_*(X, X'; A)$ (resp. $H^*(X, X'; A)$) where the coefficient ring is $A$. If $A$ is isomorphic to the integer ring, denoted by $\mathbb{Z}$ (resp. the rational ring, denoted by $\mathbb{Q}$), then the homology group and the cohomology group and ring are called the integral (resp. rational) homology group and the integral (resp. rational) cohomology group and ring, respectively. The $k$-th homology group (cohomology group) is denoted by $H_k(X, X'; A)$ (resp. $H^k(X, X'; A)$). If $A$ is isomorphic to $\mathbb{Z}$ (resp. $\mathbb{Q}$), then we add "integral" (resp. "rational") after "$k$-th" as before. If $X'$ is empty, then we may omit ",$X'$" in the notation and the homology group (cohomology group and ring) of the pair $(X, X')$ is also called the homology group (resp. cohomology group and ring) of $X$. We can define similarly for $k$-th homology groups and cohomology groups and rings and integral or rational ones. (Co)homology classes of $(X, X')$ (or $X$) are elements of the (resp. co)homology groups. If the degree is $k$ for each element, then we add "$k$-th" as for the homology group, cohomology group and ring. We can define similarly for integral or rational homology groups and cohomology groups and rings.

The $k$-th homotopy group of a topological space $X$ is denoted by $\pi_k(X)$. 
Let \((X_1, X_1')\) and \((X_2, X_2')\) be pairs of topological spaces satisfying \(X_1' \subset X_1\) and \(X_2' \subset X_2\) where the second topological spaces of the pairs can be empty. For a continuous map \(c : X_1 \to X_2\) satisfying \(c(X_1') \subset X_2'\), \(c_* : H_*(X_1, X_1'; A) \to H_*(X_2, X_2'; A)\), and \(c^* : H^*(X_2, X_2'; A) \to H^*(X_1, X_1'; A)\) denote canonically induced homomorphisms. For a continuous map \(c : X_1 \to X_2\), \(c_* : \pi_k(X_1) \to \pi_k(X_2)\) also denotes the induced homomorphism between the homotopy groups of degree \(k\).

The cup products for a pair \(c_1, c_2 \in H^*(X; A)\) and a family \(\{c_j\}_{j=1}^l \subset H^*(X; A)\) of \(l > 0\) cohomology classes are important. \(c_1 \cup c_2\) and \(\cup_{j=1}^l c_j\) denote them.

**Theorem 2 ([15]).** Let \(m > n \geq 1\) be integers. Let \(l > 0\) be another integer. Let \(M\) be an \(m\)-dimensional closed and connected manifold. Let \(A\) be a commutative ring.

Let there exist a sequence \(\{a_j\}_{j=1}^l \subset H^*(M; A)\) satisfying the following three.

- The cup product \(\cup_{j=1}^l a_j\) is not zero.
- The degree of each class in \(\{a_j\}_{j=1}^l\) is smaller than or equal to \(m - n\).
- The sum of the degrees for the \(l\) classes in \(\{a_j\}_{j=1}^l\) is greater than or equal to \(n\).

Then \(M\) does not admit special generic maps into any \(n\)-dimensional connected manifold which is not compact and which has no boundary.

We review a proof without expositions on a *handle* and its *index* for polyhedra including PL manifolds.

**Proof.** Suppose that \(M\) admits a special generic map into an \(n\)-dimensional connected manifold \(N\) which is not compact and which has no boundary. We can take an \((m + 1)\)-dimensional compact and connected PL manifold \(W\) as in Proposition 3. \(W_f\) is a compact smooth manifold smoothly immersed into the connected and non-compact manifold \(N\) with no boundary. As a result it is simple homotopy equivalent to an \((n - 1)\)-dimensional compact and connected polyhedron. \(W\) is simple homotopy equivalent to \(W_f\). \(W\) is shown to be a PL manifold obtained by attaching handle to \(M \times \{0\} \subset M \times [-1, 0]\) whose indices are greater than \((m + 1) - \dim W_f = m - n + 1\). We can take a unique cohomology class \(b_j \in H^*(W; A)\) satisfying \(a_j = i_M^* b_j\) where \(i_M\) is as in Proposition 3. \(W\) has the simple homotopy type of an \((n - 1)\)-dimensional polyhedron. This means that the cup product \(\cup_{j=1}^l a_j\) is zero, which contradicts the assumption. This completes the proof. \(\square\)

**Corollary 1 ([17]).** Let \(m \geq n \geq 1\) be integers. Let \(N\) be an \(n\)-dimensional connected manifold which is not compact and which has no boundary. We have the following two.

1. Suppose that \(m > n\) in addition. Then the \(m\)-dimensional real projective space does not admit special generic maps into \(N\).

2. Suppose also that \(n < m - 1\) and that \(m\) is an even integer. The \(\mathbb{RP}^m\)-dimensional complex projective space, which is also an \(m\)-dimensional closed, simply-connected and smooth manifold, does not admit special generic maps into \(N\).

In Corollary 1 in the case \(m = n\), [5, 11] produce useful tools for example. Our exposition on Corollary 2 (Main Theorem) and Remarks 2 and 3 refer to some of the theory.
Remark 1. A slide [38] of a related talk in a conference shows a proof of Corollary 1 for the 7-dimensional real projective space. This is based on the theory on 7-dimensional homotopy spheres and special generic maps on them, referred in the first section, before the appearance of [17]. After [17] appeared, [39] announced another proof. It investigates restrictions on the torsion group of the integral homology group for a closed and connected smooth manifold whose rational homology group is isomorphic to that of a sphere. Note also that the dimension 7 is the smallest dimension where we have discovered exotic spheres. [20] is a pioneering work, followed by [4] and [10] for example.

For other related studies on special generic maps, see [2, 7, 28, 29, 31] for example.

3. Proofs of our Main Theorems.

Before we prove Main Theorems, we shortly refer to several notions.

The fundamental class of a compact, connected and oriented smooth or PL manifold $Y$ is the canonically defined $(\dim Y)$-th homology class. This is defined as the generator of the group $H_{\dim Y}(Y, \partial Y; A)$, isomorphic to $A$, and compatible with the orientation where $A$ is a commutative ring having the identity element different from the zero element. Let $i_{Y,X}: Y \to X$ be a smooth or PL embedding satisfying $i_{Y,X}(\partial Y) \subset \partial X$ and $i_{Y,X}(\operatorname{Int} Y) \subset \operatorname{Int} X$. In other words, $Y$ is properly embedded into $X$. Let $h$ be a homology class in $H^*(X, Y; A)$. If the value of the homomorphism $i_{Y,X}^*$ induced by the smooth or PL embedding $i_{Y,X}: Y \to X$ at the fundamental class of $Y$ is $h$, then $h$ is said to be represented by the oriented submanifold $Y$. For notions here, consult [9] again for example.

The following theorem is also one of our main theorems.

**Theorem 3.** Let $m > 5$ be an integer. An $m$-dimensional closed and simply-connected smooth manifold whose 2nd integral cohomology group is isomorphic to $\mathbb{Z}$ and which has a generator $u \in H^2(M; \mathbb{Z})$ satisfying the following conditions does not admit special generic maps into $\mathbb{R}^5$.

- The cup product $u \cup u$ is a 4-th integral cohomology class of infinite order.
- The cup product $u \cup u$ is not divisible by any integer greater than 1.

Spin bundles are real vector bundles or linear bundles which are orientable and whose 2nd Stiefel-Whitney classes are zero. 2nd Stiefel-Whitney classes are 2nd cohomology classes with coefficient ring $\mathbb{Z}/2\mathbb{Z}$, which is a field of order 2. We omit more precise expositions on them. Spin manifolds are orientable smooth manifolds whose tangent bundles are spin. Consult [21] again for example.

We also need fundamental or advanced arguments on singularity theory of differentiable maps and theory of PL or smooth manifolds. For example we encounter generic immersions, embeddings and maps. We omit precise expositions in the present paper. See [8] for related theory for example.

A proof of Theorem 3. For the proof of the present theorem, only the assumption $m = 6$ is essentially new. This reviews several main theorems and their proofs of [18]. Our exposition here may be a bit different from ones there.

Suppose that a special generic map $f: M \to \mathbb{R}^5$ exists. We investigate the topology of $W_f$ in Proposition 2. This is a 5-dimensional compact and connected manifold smoothly immersed into $\mathbb{R}^5$ via $f: W_f \to \mathbb{R}^5$. Proposition 3 and the proof of Theorem 2 yield the triviality of $\pi_1(W_f)$. See also related arguments in
Suppose that the rank of $H_2(W_f; \mathbb{Z})$ is greater than 1. We have two 2nd integral homology classes $e_1, e_2 \in H_2(W_f; \mathbb{Z})$ satisfying the following conditions.

- $e_i$ is not divisible by any integer greater than 1 for $i = 1, 2$.
- $e_1$ and $e_2$ are mutually independent and of infinite order.

$W_f$ is simply-connected. These classes are represented by oriented smooth submanifolds given by the smooth embeddings $e_i$ of the 2-dimensional standard sphere into $W_f$. We consider a smooth homotopy $E_i : S^2 \times [0, 1] \to W_f$ from the original embedding $e_i$ of the 2-dimensional standard sphere into $W_f \subset W_f$ to another smooth embedding satisfying the following properties. We have this due to the structure of the special generic map $f$ and the smooth surjection $q_f : M \to W_f$.

- There exists a finite subset $J_i \subset (0, 1)$.
- The restriction of $E_i$ to each subset of the form $S^2 \times \{p\} \subset S^2 \times [0, 1]$ is a smooth embedding the intersection of the boundary $\partial W_f$ and whose image is a finite set.
- The restriction of $E_i$ to each connected component of $S^2 \times ([0, 1] - J_i)$ is regarded as a smooth isotopy.
- There exists a smooth embedding $s_i : S^2 \to M$ satisfying $q_f \circ s_i = e_i'$ where $e_i' : S^2 \to W_f$ is defined by $e_i'(x) := E_i(x, 1)$.

$s_i$ is regarded as a variant of the section for the original bundle. We can see that two mutually independent 2nd integral homology classes are represented by the oriented smooth submanifolds given by the smooth embeddings $s_1$ and $s_2$. They are of infinite order. This contradicts the assumption on the rank of $H^2(M; \mathbb{Z})$.

Suppose that the rank of $H_2(W_f; \mathbb{Z})$ is 1. We have a similar variant $s : S^2 \to M$ of a section of the bundle over a smoothly embedded 2-dimensional standard sphere in $Int W_f$. A 2nd integral homology class $e \in H_2(W_f; \mathbb{Z})$ satisfying the following conditions is represented by the sphere in $Int W_f \subset W_f$.

- $e$ is not divisible by any integer greater than 1.
- $e$ is of infinite order.

We can define the dual $e^* \in H^2(W_f; \mathbb{Z})$ of $e$. By Poincaré duality theorem, we can take a 3-dimensional compact and oriented submanifold $Y$ in $W_f$ and the Poincaré dual to $e^*$ is represented by this. We can take $Y$ so that $q_f^{-1}(Y)$ is regarded as a 4-dimensional closed manifold of the domain of a special generic map whose image is $Y$ and the manifold of whose target is a 3-dimensional smooth manifold $Y'' \supset Y$ with no boundary. $W_f$ is simply-connected and this with the structure of the special generic map yields the fact that $q_f^{-1}(Y)$ is orientable. We can also see this from the fact that $M, W_f$ and $Y$ are orientable. By arguments on intersections and Poincaré duality, an integral homology class $e_s \in H_2(M; \mathbb{Z})$ of infinite order is represented by an oriented submanifold given by $s : S^2 \to M$ and the class is not divisible by any integer greater than 1. Its Poincaré dual is represented by the oriented submanifold $q_f^{-1}(Y)$. We consider so-called generic self-intersections of $Y$.
and \(q_f^{-1}(Y)\). \(Y\) is a spin manifold since it is 3-dimensional, compact and orientable. 
\(W_f\) is smoothly immersed into \(\mathbb{R}^5\) and thus spin. By arguments on spin bundles and manifolds, a generic self-intersection of \(Y\) can be regarded as the union of the empty sets or 1-dimensional smoothly and properly embedded submanifolds chosen for all homology classes of \(H_1(Y; \mathbb{Z}/2\mathbb{Z})\). This satisfies the following conditions.

- For each homology class of \(H_1(Y; \mathbb{Z}/2\mathbb{Z})\), it is the empty set or represented by each connected component of the chosen submanifold.
- For each homology class of \(H_1(Y; \mathbb{Z}/2\mathbb{Z})\), the chosen submanifold is a disjoint union consisting of only segments or only circles.
- For each homology class of \(H_1(Y; \mathbb{Z}/2\mathbb{Z})\), the chosen submanifold is a disjoint union of segments or circles of an even number.

Note that if the coefficient ring for the homology groups is \(\mathbb{Z}/2\mathbb{Z}\), then we do not need orientations for compact submanifolds. The structure of special generic maps yields the following arguments where the coefficient ring for the homology groups is \(\mathbb{Z}/2\mathbb{Z}\). We can argue as follows.

- A generic self-intersection of \(q_f^{-1}(Y)\) is the union of manifolds diffeomorphic to \(S^1 \times S^{m-5}\) of an even number and \((m-4)\)-dimensional homotopy spheres of an even number. \(W_f\) is simply-connected and the preimage of each circle in \(Y\) and \(\text{Int } W_f \subset W_f\) is the total space of a trivial smooth bundle. The \((m-4)\)-dimensional homotopy spheres are the preimages of the connected components diffeomorphic to closed intervals in \(Y\) smoothly and properly embedded: the interiors are embedded in \(\text{Int } Y\) and \(\text{Int } W_f\) and the boundaries are in \(\partial Y\) and \(\partial W_f\).
- For each of the \((m-4)\)-dimensional closed and connected manifolds before, consider the value of the homomorphism induced by the inclusion into \(M\) at the fundamental class of the compact and connected manifold of the domain. Take the sum for all of them. Then we have zero.

For arguments here, see [18, 19] for example. We can take the dual \(e_s^*\) of \(e_s\). This yields that the cup product \(e_s^* \cup e_s^*\) is divisible by 2. This is a contradiction.

The rank of \(H_2(W_f; \mathbb{Z})\) is thus 0. The given cohomology class \(u\) is the dual of a 2nd integral homology class \(v\). \(v\) is represented by a smooth embedding \(e_0: S^2 \to M\). Assume that \(q_f \circ e_0(S^2) \subset \text{Int } W_f\). \(\partial W_f\) is a 4-dimensional, closed and orientable smooth manifold. If \(m \geq 7\), then we can take a suitable embedding \(e_0\) satisfying this by the condition on the dimensions of the manifolds. \(q_f \circ e_0\) can be regarded as a null-homotopic map as a map into \(\text{Int } W_f\). The fact that \(W_f\) is simply-connected plays an important role here. This yields the fact that \(e_0 : S^2 \to M\) is null-homotopic. This is a contradiction.

This completes the proof for the case \(m \geq 7\) since we can always regard that \(q_f \circ e_0(S^2) \subset \text{Int } W_f\) holds by the assumption on the dimensions.

Hereafter assume that \(m = 6\).

We can do so that \(q_f \circ e_0(S^2) \cap \partial W_f\) is a finite set. Assume also that this finite set cannot be empty. We investigate the self-intersection of \(\partial W_f\) as before. By Proposition 2 with the facts that \(W_f\) is simply-connected and that \(H^3(W_f; \mathbb{Z})\) is zero, we have that the linear bundle whose fiber is the 2-dimensional unit disk in the properties is trivial. This means that the self-intersection can be empty. Thus we can also regard that \(q_f \circ e_0(S^2) \cap \partial W_f\) is the empty set. This is a contradiction.

This completes the proof.

\(\Box\)
We can easily see the following by a fundamental topological property of the complex projective space.

**Corollary 2 (Main Theorem).** The 3-dimensional complex projective space does not admit special generic maps into any Euclidian space.

**Proof.** Theorem 2 or Corollary 1 (2) yields the fact that the 3-dimensional complex projective space does not admit special generic maps into $\mathbb{R}^n$ for $n = 1, 2, 3, 4$. This does not admit ones into $\mathbb{R}^6$ by [5]. This says that a closed and orientable manifold admits a special generic map into the Euclidean space of the same dimension if and only if the so-called *Whitney sum* of the tangent bundle and a trivial linear bundle whose fiber is $\mathbb{R}$ is trivial. Theorem 3 completes the proof. □

**Remark 2.** The 1-dimensional complex projective space is a 2-dimensional standard sphere. The 2-dimensional complex projective space does not admit special generic maps into $\mathbb{R}^n$ for $n = 1, 2, 3$ by Theorem 1 (1) and (2), Theorem 2 and Corollary 1 (2). It does not admit ones into $\mathbb{R}^4$, which follows from the theory [5] or [11].

According to [11], if a closed manifold whose Euler number is odd admits a special generic map into an Euclidean space, then the dimension of the Euclidean space must be 1, 3 or 7. This theorem is, more generally, one for *fold* maps: the class of fold maps is a natural extension of the class of Morse functions and that of special generic maps.

A *fold* map is, in short, defined as a smooth map locally represented as the product maps of the Morse functions and the identity maps on open disks. [8] presents some singularity theory on singular points of fold maps and more general generic smooth maps. [34, 36] are pioneering studies on fold maps or more general smooth maps into the plane on closed manifolds whose dimensions are greater than or equal to 2. [12, 13, 14, 15] are on *round* fold maps, defined as fold maps whose singular value sets are embedded concentric spheres and defined first by the author in [12, 13, 14].

Last, we go back to the complex projective spaces. For general complex projective spaces, the existence or the non-existence of special generic maps there into one-dimensional lower Euclidean spaces is an open problem closely related to our study.

**Remark 3.** The existence and non-existence of fold maps on closed manifolds have been natural, important, difficult and attractive problems. Morse functions always exist. Fold maps exist on a closed manifold whose dimension is greater than or equal to 2 if the Euler number is even. Eliashberg’s theory [5, 6] has solved problems in considerable cases. The triviality of the Whitney sum of the tangent bundle and a trivial linear bundle whose fiber is $\mathbb{R}$ is a strong sufficient condition for the existence. Later [1] has given generalized answers.

For projective spaces, such problems are still difficult to solve in considerable cases. For example, we do not know the existence or the non-existence of fold maps into $\mathbb{R}^5$ on the 3-dimensional complex projective space. We know the existence for $n = 1, 2, 3, 4$. For this, see also construction of explicit fold maps on the total spaces of smooth bundles over standard spheres in [12, 13, 15] for example. For the 2-dimensional projective space, we know the non-existence of fold maps into $\mathbb{R}^n$ for $n = 2, 3, 4$. For the case $n = 3$, see also [27].

For related studies, see [23, 24], for example.
The following theorem is a main theorem of [18] or Main Theorem 1 there if we replace \( m = 6 \) by \( m \geq 7 \).

**Theorem 4.** For a closed and simply-connected manifold \( M \) of dimension \( m = 6 \) admits a special generic map \( f : M \to \mathbb{R}^5 \) whose singular set \( S(f) \) is connected, the cup product \( c_1 \cup c_2 \) vanishes for any pair \( c_1, c_2 \in H^2(M; \mathbb{Z}) \) of integral cohomology classes.

**Proof.** We apply some arguments on Proposition 3.10 of [25]. According to an exact sequence there, the sum of the ranks of \( H_2(W_f; \mathbb{Z}) \) and \( H_2(W_f; \mathbb{Z}) \) is smaller than that of \( H_2(M; \mathbb{Z}) \). \( H_1(W_f; \mathbb{Z}) \) is zero since \( W_f \) is simply-connected as presented in the proof of Theorem 3 for example and \( H^1(W_f, \partial W_f; \mathbb{Z}) \) is isomorphic to \( H_4(W_f; \mathbb{Z}) \) by virtue of Poincaré duality theorem in our case. \( H_3(W_f, \partial W_f; \mathbb{Z}) \) is zero from the assumption that \( \partial W_f = q_f(S(f)) \) is connected and the homology exact sequence for the pair \( (W_f, \partial W_f) \). Thus \( q_f^*: H_2(M; \mathbb{Z}) \to H_2(W_f; \mathbb{Z}) \) is an isomorphism. Let \( l \) denote the rank of \( H_2(W_f; \mathbb{Z}) \). We have a family \( \{Y_j\}_{j=1}^l \) of \( l \) 3-dimensional compact, connected and orientable manifolds smoothly embedded in \( W_f \) in a way similar to that of \( Y \) in the proof of Main Theorem. The integral homology classes represented by these 3-dimensional manifolds are regarded as the Poincaré duals to the duals of 2nd integral homology classes of \( H_2(W_f; \mathbb{Z}) \). These 2nd integral homology classes are taken as ones forming a basis of a free subgroup of rank \( l \) in \( H_2(W_f; \mathbb{Z}) \). We have a family \( \{q_f^{-1}(Y_j)\} \) of 4-dimensional closed, connected and orientable manifolds. The integral homology classes represented by these manifolds are also regarded as the Poincaré duals to the duals of 2nd integral homology classes of \( H_2(M; \mathbb{Z}) \). These 2nd integral homology classes are taken as ones forming a basis of a free subgroup of rank \( l \) in \( H_2(M; \mathbb{Z}) \). As done in the proof of our Theorem 3 and that of Main Theorem 1 of [18], we investigate (generic) intersections of two submanifolds in \( \{q_f^{-1}(Y_j)\} \) or (generic) self-intersections of single submanifolds there. The dimensions of these resulting submanifolds are \( (m - 2) + (m - 2) - m = 6 - 2 + 6 - 2 - 6 = 2 \).

We discuss the homology class represented by some connected component of such a submanifold. Showing that the zero class \( 0 \in H_2(M; \mathbb{Z}) \) is represented by such a connected submanifold completes our proof by virtue of the facts on the ranks of \( H_2(M; \mathbb{Z}) \) and \( H_2(W_f; \mathbb{Z}) \) and Poincaré duality. Most of these arguments are based on the proof of Main Theorem 1 of [18].

This 2-dimensional submanifold is the disjoint union of finitely many copies of the torus \( S^1 \times S^1 \) of an even number and 2-dimensional spheres of an even number. The element \( 0 \) is represented by each of the former tori due to the fact that \( W_f \) is simply-connected. These tori are the preimages of circles smoothly embedded in \( \text{Int} \ Y \) and \( \text{Int} \ W_f \).

Each of the 2-dimensional spheres here is the preimage of a closed interval smoothly and properly embedded in \( Y \): the interior is embedded into \( \text{Int} \ Y \) and \( \text{Int} W_f \) and the boundary is into \( \partial Y \) and \( \partial W_f \).

\( W_f \) is simply-connected and \( \partial W_f \) is connected. This yields a smooth homotopy \( H_t : S^2 \times [0, 1] \to W_f \) from the composition of the original embedding of each 2-dimensional sphere here into \( M \) with \( q_f \) to a constant map to a point in \( \partial W_f \). For \( 0 \leq t \leq 1 \), define \( S_{t, f}(t) \) as the set of all points in \( S^2 \) such that the pairs of the points and \( t \) are mapped to \( \partial W_f \) by the homotopy. We can take the homotopy \( H_t \) satisfying \( S_{t, f}(t_1) \subset S_{t, f}(t_2) \) for \( 0 < t_1 < t_2 < 1 \). Thus the original embedding of the 2-dimensional sphere into \( M \) is smoothly null-homotopic.
This shows the vanishing of the cup product of two 2nd integral cohomology classes for any pair. This completes the proof.

4. ACKNOWLEDGEMENT.

The author is a member of the project supported by JSPS KAKENHI Grant Number JP17H06128 "Innovative research of geometric topology and singularities of differentiable mappings" (Principal investigator: Osamu Saeki). The present study is also supported by the project. We declare that data essentially supporting the present study are all in the present paper.

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Institute of Mathematics for Industry, Kyushu University, 744 Motooka, Nishi-ku Fukuoka 819-0395, Japan, TEL (Office): +81-92-802-4402, FAX (Office): +81-92-802-4405,

Email address: n-kitazawa@imi.kyushu-u.ac.jp

Webpage: https://naokikitazawa.github.io/NaokiKitazawa.html