NOWHERE DENSE GRAPH CLASSES AND DIMENSION

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Abstract. Nowhere dense graph classes provide one of the least restrictive notions of sparsity for graphs. Several equivalent characterizations of nowhere dense classes have been obtained over the years, using a wide range of combinatorial objects. In this paper we establish a new characterization of nowhere dense classes, in terms of poset dimension: A monotone graph class is nowhere dense if and only if for every $h \geq 1$ and every $\varepsilon > 0$, posets of height at most $h$ with $n$ elements and whose cover graphs are in the class have dimension $O(n^\varepsilon)$. 

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1. Introduction

A class of graphs is nowhere dense if for every $r \geq 1$, there exists $t \geq 1$ such that no graph in the class contains a subdivision of the complete graph $K_t$ where each edge is subdivided at most $r$ times as a subgraph. Examples of nowhere dense classes include most sparse graph classes studied in the literature, such as planar graphs, graphs with bounded treewidth, graphs excluding a fixed (topological) minor, graphs with bounded maximum degree, graphs that can be drawn in the plane with a bounded number of crossings per edges, and more generally graph classes with bounded expansion.

At first sight, being nowhere dense might seem a weak requirement for a graph class to satisfy. Yet, this notion captures just enough structure to allow solving a wide range of algorithmic problems efficiently: In their landmark paper, Grohe, Kreutzer, and Siebertz [6] proved for instance that every first-order property can be decided in almost linear time on graphs belonging to a fixed nowhere dense class.

One reason nowhere dense classes attracted much attention in recent years is the realization that they can be characterized in several, seemingly different ways. Algorithmic applications in turn typically build on the ‘right’ characterization for the problem at hand and sometimes rely on multiple ones, such as in the proof of Grohe et al. [6]. Nowhere dense classes were characterized in terms of shallow minor densities [21] and consequently in terms of generalized coloring numbers (by results from [28]), low tree-depth colorings [21] (by results from [18]), and subgraph densities in shallow minors [20]; they were also characterized in terms of quasi-uniform wideness [19, 15, 24], the so-called splitter game [6], sparse neighborhood covers [6], neighborhood complexity [3], the model theoretical notion of stability [1], as well as existence of particular analytic limit objects [17]. The reader is referred to the survey on nowhere dense classes by Grohe, Kreutzer, and Siebertz [5] for an overview of the different characterizations, and to the textbook by Nešetřil and Ossona de Mendez [22] for a more general overview of the various notions of sparsity for graphs (see also [23]).

The main contribution of this paper is a new characterization of nowhere dense classes that brings together graph structure theory and the combinatorics of posets. Informally, we show that the property of being nowhere dense can be captured by looking at the dimension of posets whose order diagrams are in the class (when seen as graphs).

Recall that the dimension $\dim(P)$ of a poset $P$ is the least integer $d$ such that elements of $P$ can be embedded into $\mathbb{R}^d$ in such a way that $x < y$ in $P$ if and only if the point of $x$ is below the point of $y$ with respect to the product order of $\mathbb{R}^d$. Dimension is a key measure of a poset’s complexity.

The standard way of representing a poset is to draw its diagram: First, we draw each element as a point in the plane, in such a way that if $a < b$ in the poset then $a$ is drawn below $b$. Then, for each relation $a < b$ in the poset not implied by transitivity (these are called cover relations), we draw a $y$-monotone curve going from $a$ up to $b$. The diagram implicitly defines a corresponding undirected graph, where edges correspond to pairs.
of elements in a cover relation. This is the cover graph of the poset. Let us also recall that the height of a poset is the maximum size of a chain in the poset (a set of pairwise comparable elements).

Recall that a monotone class means a class closed under taking subgraphs. Our main result is the following theorem.

**Theorem 1.** Let $C$ be a monotone class of graphs. Then $C$ is nowhere dense if and only if for every integer $h \geq 1$ and real number $\varepsilon > 0$, $n$-element posets of height at most $h$ whose cover graphs are in $C$ have dimension $O(n^\varepsilon)$.

This result is the latest step in a series of recent works connecting poset dimension with graph structure theory. This line of research began with the following result of Streib and Trotter [25]: For every fixed $h \geq 1$, posets of height $h$ with a planar cover graph have bounded dimension. That is, the dimension of posets with planar cover graphs is bounded from above by a function of their height. This is a remarkable theorem, because in general bounding the height of a poset does not bound its dimension, as shown for instance by the height-2 posets called standard examples, depicted in Figure 1 (left). Requiring the cover graph to be planar does not guarantee any bound on the dimension either, as shown by Kelly’s construction [12] of posets with planar cover graphs containing large standard examples as induced subposets (Figure 1, right). Thus, it is the combination of the two ingredients, bounded height and planarity, that allow the dimension be bounded.

Soon afterwards, it was shown in a sequence of papers that requiring the cover graph to be planar in the Streib-Trotter result could be relaxed: Posets have dimension upper bounded by a function of their height if their cover graphs

- have bounded treewidth, bounded genus, or more generally exclude an apex-graph as minor [8];
- exclude a fixed graph as a (topological) minor [27, 16];
- belong to a fixed class with bounded expansion [11].
A class of graphs has \textit{bounded expansion} if for every $r \geq 1$, there exists $c \geq 0$ such that no graph in the class contains a subdivision of a graph with average degree at least $c$ where each edge is subdivided at most $r$ times as a subgraph. This is a particular case of nowhere dense classes. In [11], it is conjectured that bounded expansion is in fact the right answer to the question of when is dimension bounded by a function of the height:

\textbf{Conjecture 2 ([11])}. A monotone class of graphs $\mathcal{C}$ has bounded expansion if and only if for every fixed $h \geq 1$, posets of height at most $h$ whose cover graphs are in $\mathcal{C}$ have bounded dimension.

While the result of [11] shows the forward direction of the conjecture, the backward direction remains surprisingly (and frustratingly) open. By contrast, showing the backward direction of Theorem 1 is a straightforward matter (see Section 4). The non-trivial part of our theorem is that $n$-element posets of bounded height with cover graphs in a nowhere dense class have dimension $O(n^\varepsilon)$ for all $\varepsilon > 0$.

Characterizations of nowhere dense classes often go hand in hand with characterizations of classes with bounded expansion. Zhu established a close connection between weak coloring numbers and densities of bounded depth minors, which he used to characterize bounded expansion class [28]. The characterization of nowhere dense classes in terms of shallow minor densities [21] consequently led to a similar characterization of nowhere dense classes. It follows that a class

- has bounded expansion if and only if for every $r \geq 0$, there exists $c \geq 1$ such that every graph in the class has weak $r$-coloring number at most $c$;
- is nowhere dense if and only if for every $r \geq 0$ and every $\varepsilon > 0$, every $n$-vertex graph in the class has weak $r$-coloring number $O(n^\varepsilon)$.

Weak coloring numbers were originally introduced by Kierstead and Yang [13] as a generalization of the degeneracy of a graph (also known as the coloring number). As they play an important role in this paper, let us first recall their definition before pursuing further. Let $G$ be a graph. Consider some linear order $\pi$ on its vertices; it will be convenient to see $\pi$ as ordering the vertices of $G$ from left to right. Write $x <_\pi y$ if $x$ is to the left of $y$ in $\pi$. Given a path $Q$ in $G$, we denote by $\ell(Q)$ the leftmost vertex of $Q$ w.r.t. $\pi$. Given a vertex $v$ in $G$ and an integer $r \geq 0$, we say that $u \in V(G)$ is \textit{weakly $r$-reachable from} $v$ w.r.t. $\pi$ if there exists a path $Q$ of length at most $r$ from $v$ to $u$ in $G$ such that $\ell(Q) = u$. We let $\text{WReach}_r^\pi[v]$ denote the set of weakly $r$-reachable vertices from $v$ w.r.t. $\pi$ (note that this set contains $v$ for all $r \geq 0$). The \textit{weak $r$-coloring number} $\text{wcol}_r(G)$ of $G$ is defined as

$$\text{wcol}_r(G) := \min_{\pi} \max_{v \in V(G)} |\text{WReach}_r^\pi[v]|.$$  

As a consequence of the characterizations in terms of shallow minor densities, it is a common feature of several characterizations in the literature that bounded expansion and nowhere dense classes can be characterized using the same graph invariants, but requiring $O(1)$ and $O(n^\varepsilon)$ $\forall \varepsilon > 0$ bounds on the invariants respectively. Thus, it is natural conjecture the statement of Theorem 1, and indeed it appears as a conjecture
in [11]. (We note that it was originally Dan Kráľ who suggested to the first author to try and show Theorem 1 right after the result in [11] was obtained; however, the proof techniques from [11] were not well tailored to this job.)

The novelty of our approach in this paper is that we bound the dimension of a poset using weak coloring numbers of its cover graph. Indeed, the general message that we aim to convey is that dimension works surprisingly well with weak coloring numbers. We give a first illustration of this principle with the following theorem:

**Theorem 3.** Let $P$ be a poset of height at most $h$, let $G$ denote its cover graph, and let $c := w_{col_{3h-3}}(G)$. Then

$$\dim(P) \leq 4^c.$$ 

By Zhu's theorem, if we restrict ourselves to posets with cover graphs $G$ in a fixed class $C$ with bounded expansion, then $w_{col_{3h-3}}(G)$ is bounded by a function of $h$. Thus Theorem 3 implies the theorem from [11] for classes with bounded expansion. However, the proof of Theorem 3 is much simpler and implies better bounds on the dimension that those following from previous works (see the discussion in Section 3). We see this as a first sign that weak coloring numbers are the right tool to use in this context.

Going back to nowhere dense classes, we remark that the $4^c$ bound in Theorem 3 unfortunately falls short of implying the forward direction of Theorem 1. Indeed, if the cover graph $G$ has $n$ vertices and belongs to a nowhere dense class, we only know that $w_{col_{3h-3}}(G) \in O(n^\varepsilon)$ for every $\varepsilon > 0$. Thus from the theorem we only deduce that $\dim(P) \leq 4^{O(n^\varepsilon)}$ for every $\varepsilon > 0$, which is a vacuous statement since $\dim(P) \leq n$ always holds.

In order to address this shortcoming, we developed a second upper bound on the dimension of a height-$h$ poset in terms of the weak $w(h)$-coloring number of its cover graph $G$ (for some function $w$) and another invariant of $G$. This extra invariant is the smallest integer $t$ such that $G$ does not contain an $\leq s(h)$-subdivision of $K_t$ as a subgraph (for some function $s$). The key aspect of our bound is that, for fixed $h$ and $t$, it depends polynomially on the weak $w(h)$-coloring number that is being considered. Its precise statement is as follows. (Let us remark that the particular values $w(h) := 4h - 4$ and $s(h) := 2h - 3$ used in the theorem are not important for our purposes, any functions $w$ and $s$ would have been enough, as will be apparent in the discussion below.)

**Theorem 4.** There exists a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for every $h \geq 1$ and $t \geq 1$, every poset $P$ of height at most $h$ whose cover graph $G$ contains no $\leq (2h - 3)$-subdivision of $K_t$ as a subgraph satisfies

$$\dim(P) \leq (4c)^{f(h,t)},$$

where $c := w_{col_{4h-4}}(G)$.

Recall that for every nowhere dense graph class $C$ and every $r \geq 1$, there exists $t \geq 1$ such that no graph in $G$ contains an $\leq r$-subdivision of $K_t$ as a subgraph. Hence, Theorem 4 implies the following corollary.
Corollary 5. For every nowhere dense class of graphs \( C \), there exists a function \( g : \mathbb{N} \to \mathbb{N} \) such that every poset \( P \) of height at most \( h \) whose cover graph \( G \) is in \( C \) satisfies
\[
\dim(P) \leq (4c)^{g(h)},
\]
where \( c := \text{wcol}_{4h-4}(G) \).

For every integer \( h \geq 1 \) and real number \( \varepsilon > 0 \), this in turn gives a bound of \( \mathcal{O}(n^\varepsilon) \) on the dimension of \( n \)-element posets of height at most \( h \) whose cover graphs \( G \) are in \( C \). Indeed, if we take \( \varepsilon' := \varepsilon/g(h) \), Zhu’s theorem tells us that \( \text{wcol}_{4h-4}(G) \in \mathcal{O}(n^{\varepsilon'}) \), and hence \( \dim(P) \in \mathcal{O}(n^{g(h)\varepsilon'}) = \mathcal{O}(n^\varepsilon) \) by the corollary. Therefore, this establishes the forward direction of Theorem 1.

Let us also point out that Corollary 5 provides yet another proof of the theorem from [11] for classes with bounded expansion, since \( \text{wcol}_{4h-4}(G) \) is bounded by a function of \( h \) only when \( C \) has bounded expansion. However, the proof is more involved than that of Theorem 3 and the resulting bound on the dimension is typically larger. Indeed, the bound in Theorem 4 is mostly interesting when the weak coloring number under consideration grows with the number of vertices.

Our proof of Theorem 4 takes its roots in the alternative proof due to Micek and Wiechert [16] of Walczak’s theorem [27], that bounded-height posets whose cover graphs exclude \( K_t \) as a topological minor have bounded dimension. This proof is essentially an iterative algorithm which, if the dimension is large enough (as a function of the height), explicitly builds a subdivision of \( K_t \), one branch vertex at a time. This is very similar in appearance to what we would like to show, namely that if the dimension is too big, then the cover graph contains a subdivision of \( K_t \) where each edge is subdivided a bounded number of times (by a function of the height).

However, the main difficulty in adapting the proof in [16] is that each iteration involves an ‘unfolding’ step, where a ‘layer’ of the current poset that has large dimension is identified. We skip the formal definition of the unfolding operation as it is not needed in this paper, and limit ourselves to an illustration (Figure 2) and the following graph coloring analogy: If a graph has large chromatic number then one can do a breadth first search from any vertex \( v \) and find a layer inducing a subgraph that still has large chromatic number (up to a factor 2). Unfolding is a similar operation w.r.t. dimension for posets. Typically, one has no control on how far the layer with large chromatic number / large dimension is from the starting vertex \( v \). This in turn means that one cannot control the lengths of the paths between branch vertices in the subdivision of \( K_t \) that is being built in [16].

We circumvented this difficulty as follows. The heart of our proof is a new technique, Lemma 15, based on weak coloring numbers. Informally, the lemma shows that in a height-\( h \) poset of large dimension there exists a special element \( q \) such that, if we unfold from \( q \), then the very first layer still has relatively large dimension. (We remark that this is not exactly what the lemma says—in particular, it is not stated in terms of the unfolding operation—but this is the underlying idea.) Here, ‘relatively large’ means roughly within a factor \( c \), where \( c \) is the weak \((4h - 4)\)-coloring number of the cover.
graph. We can then focus on that particular layer and iterate. This way, if the dimension is large enough initially, we can apply our lemma iteratively a bounded number of times before losing steam. The constant $f(h,t)$ in the theorem statement stands essentially for the number of such iterations. Choosing $f(h,t)$ large enough as a function of $h$ and $t$, this eventually allows us to build a subdivision of $K_t$ where each edge is subdivided a bounded number of times (at most $2h - 3$ times), as desired, by adapting the algorithmic proof of [16] to work with the lemma.

The paper is organized as follows. We begin with some background material on posets and their dimension in Section 2. Then we prove Theorem 3 in Section 3 and discuss the improved bounds it implies for special cases that were studied in the literature, such as for posets with planar cover graphs and posets with cover graphs of bounded treewidth. We note that the material in Section 3 is not needed for our characterization of nowhere dense classes and thus can be freely skipped. (On the other hand, it is a good warmup for the rest of the paper.) We proceed in Section 4 with the statement and proof of our key lemma, Lemma 15, and then prove Theorem 4 using it.

2. BACKGROUND ON POSETS AND THEIR DIMENSION

In this section, we define the few terms and notations regarding posets and their dimension that will be needed in the proofs, and that were not already defined in the introduction. The reader interested in learning more about poset dimension is referred to Trotter’s textbook [26] on the subject.

Let $P$ be a poset. A poset $Q$ is an induced subposet of $P$ if $Q$ is obtained by selecting a subset $X$ of the elements of $P$ together with all their relations, that is, given $a, b \in X$, we have $a \leq b$ in $Q$ if and only if $a \leq b$ in $P$. A chain $X$ of $P$ is said to be a covering chain if the elements of $X$ can be enumerated as $x_1, x_2, \ldots, x_k$ in such a way that $x_i < x_{i+1}$ is a cover relation in $P$ for each $i \in [k-1]$. (We use the notation $[n] := \{1, \ldots, n\}$.)
The \textit{upset} $U(x)$ of an element $x \in P$ is the set of all elements $y \in P$ such that $x \leq y$ in $P$. If we reverse the relation $x \leq y$ into $x \geq y$, we get the definition of the \textit{downset} of $x$, which we denote by $D(x)$.

An \textit{incomparable pair} of $P$ is an ordered pair $(x, y)$ of elements of $P$ that are incomparable in $P$. We denote by $\text{Inc}(P)$ the set of incomparable pairs of $P$. Let $I \subseteq \text{Inc}(P)$ be a non-empty set of incomparable pairs of $P$. We say that $I$ is \textit{reversible} if there is a linear extension $L$ of $P$ reversing each pair of $I$, that is, we have $x > y$ in $L$ for every $(x, y) \in I$. We denote by $\text{dim}(I)$ the least integer $d$ such that $I$ can be partitioned into $d$ reversible sets. We will use the convention that $\text{dim}(I) = 1$ when $I$ is an empty set. As is well known, the dimension $\text{dim}(P)$ of $P$ can equivalently be defined as $\text{dim}(\text{Inc}(P))$, that is, the least integer $d$ such that the set of all incomparable pairs of $P$ can be partitioned into $d$ reversible sets. This is the definition that we will use in the proofs.

A sequence $(x_1, y_1), \ldots, (x_k, y_k)$ of incomparable pairs of $P$ with $k \geq 2$ is said to be an \textit{alternating cycle of size $k$} if $x_i \leq y_{i+1}$ in $P$ for all $i \in \{1, \ldots, k\}$ (cyclically, so $x_k \leq y_1$ in $P$ is required). (We remark that possibly $x_i = y_{i+1}$ for some $i$'s.) Observe that if $(x_1, y_1), \ldots, (x_k, y_k)$ is a alternating cycle in $P$, then this set of incomparable pairs cannot be reversed by a linear extension $L$ of $P$. Indeed, otherwise we would have $y_i < x_i < y_{i+1}$ in $L$ for each $i \in \{1, 2, \ldots, k\}$ cyclically, which cannot hold. Hence, alternating cycles are not reversible. The converse is also true, as is well known: A set $I$ of incomparable pairs of a poset $P$ is reversible if and only if $I$ contains no alternating cycles.

The following non-standard definition will be convenient in the proofs: Given a set $I$ of incomparable pairs of $P$, we define the \textit{convex hull} $\text{conv}(I)$ of $I$ as the subposet of $P$ induced by the set $\{z \in P : \exists (x, y), (x', y') \in I \text{ s.t. } x \leq z \leq y'\}$.

We end this section by recalling the definition of the standard examples alluded to in the introduction: The \textit{standard example $S_m$} ($m \geq 2$) is the height-2 poset consisting of $m$ minimal elements $a_1, \ldots, a_m$ and $m$ maximal elements $b_1, \ldots, b_m$, and the relations $a_i < b_j$ for all $i, j \in [m]$ with $i \neq j$. It is well known (and an easy exercise) that the dimension of $S_m$ is exactly $m$.

\section{A Simple Bound in Terms of Weak Coloring Numbers}

Let us begin with a standard observation about weak reachability.

\textbf{Observation 6.} Let $G$ be a graph and let $\pi$ be a linear order on its vertices. If $x, y, z$ are three (non-necessarily distinct) vertices of $G$ such that $x$ is weakly $k$-reachable from $z$ w.r.t. $\pi$, and $y$ is weakly $\ell$-reachable from $z$, then either $x$ is weakly $(k + \ell)$-reachable from $y$, or $y$ is weakly $(k + \ell)$-reachable from $x$.

\textit{Proof.} Consider a path $Q$ from $z$ to $x$ witnessing that $x$ is weakly $k$-reachable from $z$, and a path $Q'$ from $z$ to $y$ witnessing that $y$ is weakly $\ell$-reachable from $z$. The union of $Q$ and $Q'$ contains a path $Q''$ connecting $x$ to $y$ of length at most $k + \ell$. Since $x$ is the leftmost vertex of $Q$ in $\pi$ and $y$ is the leftmost vertex of $Q'$ in $\pi$, we have that one
of $x$ and $y$ is the leftmost vertex of $Q''$ in $\pi$. This proves that one of them is weakly $(k + \ell)$-reachable from the other. 

In this section we prove the following single-exponential bound on the dimension of a height-$h$ poset in terms of the weak $(3h - 3)$-coloring number of its cover graph:

**Theorem 3.** Let $P$ be a poset of height at most $h$, let $G$ denote its cover graph, and let $c := \text{wcol}_{3h-3}(G)$. Then

$$\dim(P) \leq 4^c.$$  

**Proof.** Let $\pi$ be a linear order on the elements of $P$ such that

$$\text{WReach}_{3h-3}[x] \leq c$$

for each $x \in P$. Here and in the rest of the proof, weak reachability is to be interpreted w.r.t. the cover graph $G$ and the ordering $\pi$.

First, we greedily color the elements of $P$ using the ordering $\pi$ from left to right. When element $x$ is about to be colored, we give $x$ the smallest color $\phi(x)$ in $[c]$ that is not used for elements of $\text{WReach}_{3h-3}[x] \setminus \{x\}$. (Note that by (1) at least one color is available.)

The resulting greedy coloring $\phi$ has the following property.

**Claim 7.** Let $x \in P$. If $y, z \in \text{WReach}_{h-1}[x]$ are distinct, then $\phi(y) \neq \phi(z)$.  

**Proof.** For every two vertices $x, y \in \text{WReach}_{h-1}[x]$, we know by Observation 6 that $x$ is weakly $(2h - 2)$-reachable from $y$, or the other way round. In both cases we have that $\phi(x) \neq \phi(y)$. \qed

In the proof, we will focus on elements $y$ in $\text{WReach}_{h-1}[x]$ that are weakly reachable from $x$ via covering chains that either start at $x$ and end in $y$, or the other way round. This leads us to introduce the *weakly reachable upset* $\text{WU}[x]$ and the *weakly reachable downset* $\text{WD}[x]$ of $x$:

$$\text{WU}[x] := \{y \in U(x) \mid \exists \text{ covering chain } Q \text{ from } x \text{ to } y \text{ such that } \ell(Q) = y\},$$

$$\text{WD}[x] := \{y \in D(x) \mid \exists \text{ covering chain } Q \text{ from } y \text{ to } x \text{ such that } \ell(Q) = y\}.$$  

Note that these two sets are disjoint and that $\text{WU}[x] \cup \text{WD}[x] \subseteq \text{WReach}_{h-1}[x]$. By Claim 7, this implies that the elements in $\text{WU}[x]$ and in $\text{WD}[x]$ have all different colors.  

If $X$ is a set of elements of $P$, we write $\phi(X)$ for the set of colors $\{\phi(x) \mid x \in X\}$. Given an element $x$ of $P$ and a color $i \in \phi(\text{WU}[x])$, there is a unique element in $\text{WU}[x]$ with color $i$; let us denote it by $\text{wu}_i(x)$. Similarly, given $i \in \phi(\text{WD}[x])$, we let $\text{wd}_i(x)$ denote the unique element in $\text{WD}[x]$ with color $i$.

The plan of the proof is simple: We assign a signature to each incomparable pair of $P$, in such a way that the number of distinct signatures is at most $4^c$, and that for each signature $\sigma$, the set of incomparable pairs with signature $\sigma$ is reversible. This shows that $P$ has dimension at most $4^c$, as desired.
Signatures are defined as follows. First, given \((x, y) \in \text{Inc}(P)\), we define a vector 
\[ \tau(x, y) = (\tau_i(x, y))_{i \in [\ell]}, \]
where
\[ \tau_i(x, y) := \begin{cases} 1 & \text{if } i \in \phi(WU[x]) \cap \phi(WD[y]) \text{ and } wu_i(x) <_\pi w_d_i(y), \\ 0 & \text{otherwise.} \end{cases} \]
The signature \(\sigma(x, y)\) of the pair \((x, y)\) is a 3-tuple defined as follows:
\[ \sigma(x, y) := (\phi(WU[x]), \phi(WU[x]) \cap \phi(WD[y]), \tau(x, y)). \]

Let also \(\Sigma := \{\sigma(x, y) \mid (x, y) \in \text{Inc}(P)\}\).

**Claim 8.** Let \(\sigma \in \Sigma\). Then the set of incomparable pairs \((x, y)\) of \(P\) with \(\sigma(x, y) = \sigma\) is reversible.

**Proof.** Arguing by contradiction, suppose that this set is not reversible. Then it contains an alternating cycle \((x_1, y_1), \ldots, (x_k, y_k)\). Suppose that \(\sigma = (A, B, \tau)\). Thus \(\phi(WU[x_j]) = A, \phi(WU[x_j]) \cap \phi(WD[y_j]) = B,\) and \(\tau(x_j, y_j) = \tau\) for each \(j \in [k]\).

For each \(j \in [k]\), consider all covering chains witnessing the comparability \(x_j \leq y_{j+1}\) in \(P\) (indices are taken cyclically) and choose one such covering chain \(Q_j\) such that \(\ell(Q_j)\) is as to the left as possible w.r.t. \(\pi\). Let \(q_j := \ell(Q_j)\). Clearly,
\[ x_j \leq q_j \leq y_{j+1} \text{ in } P \quad \text{and} \quad q_j \in WU[x_j] \cap WD[y_{j+1}] \] (2)
for each \(j \in [k]\). It follows that \(\phi(q_j) \in \phi(WU[x_j]) = A = \phi(WU[x_{j+1}])\), and also \(\phi(q_j) \in \phi(WD[y_{j+1}])\), for each \(j \in [k]\). Thus, \(\phi(q_j) \in B = \phi(WU[x_{j+1}]) \cap \phi(WD[y_{j+1}])\) for each \(j \in [k]\). Hence, letting \(C := \{\phi(q_j) \mid j \in [k]\}\), we have \(C \subseteq B\).

Without loss of generality we may assume that \(q_1\) is leftmost w.r.t. \(\pi\) among the \(q_j\)'s. Let \(t := \phi(q_1)\). By (2),
\[ wu_t(x_1) = q_1 = w_d_t(y_2). \]
Let us now show that \(\tau_t(x_2, y_2) = 0\). Consider the element \(q_2\). Note that \(q_1 \neq q_2\), since otherwise \(x_2 \leq q_2 = q_1 \leq y_2\) in \(P\) by (2). Hence \(q_1 <_\pi q_2\) by our assumption on \(q_1\). Let \(t' := \phi(q_2)\). If \(t' = t\) then \(w_d_t(y_2) = q_1 <_\pi q_2 = wu_t(x_2)\), and we directly obtain that \(\tau_t(x_2, y_2) = 0\).

Suppose thus that \(t' \neq t\). Arguing by contradiction, let us assume that \(\tau_t(x_2, y_2) = 1\), that is, that
\[ wu_t(x_2) <_\pi w_d_t(y_2) = q_1 <_\pi q_2. \]
(See Figure 3.) It follows that \(w_d_t(y_3) \neq wu_t(x_2)\), since otherwise there would be a covering chain witnessing \(x_2 \leq y_3\) in \(P\) containing \(wu_t(x_2)\). However, this would contradict the fact that \(Q_2\) was chosen so that its element \(\ell(Q_2) = q_2\) is as to the left as possible in \(\pi\). (Here and in the rest of the proof, let us point out that \(y_3 = y_1\) in case \(k = 2\), which makes no difference in the arguments.)

Next, we aim to show that one of \(w_d_t(y_3), wu_t(x_2)\) is weakly \((3h - 3)\)-reachable from the other w.r.t. \(\pi\). Since \(wu_t(x_2) \in WU[x_2]\), there is a covering chain \(Q'\) from \(x_2\) to \(wu_t(x_2)\)
in $G$ such that $\ell(Q') = wu_t(x_2)$. Similarly, there is a covering chain $Q''$ from $wd_t(y_3)$ to $y_3$ in $G$ such that $\ell(Q'') = wd_t(y_3)$.

Now, let us consider $Q_2$, $Q'$, and $Q''$ as paths in the cover graph $G$. Observe that their union contains a path $Q$ of length at most $3h - 3$ connecting $wu_t(x_2)$ to $wd_t(y_3)$. (Recall that each covering chain contains at most $h$ elements.) Moreover, since $wu_t(x_2) <_\pi q_2 = \ell(Q_2)$, we see that the leftmost element $\ell(Q)$ of $Q$ is either $wu_t(x_2)$ or $wd_t(y_3)$.

If $\ell(Q) = wu_t(x_2)$, then $Q$ witnesses the fact that $wu_t(x_2) \in \text{WReach}^{\pi}_{3h-3}[wd_t(y_3)]$. If, on the other hand, $\ell(Q) = wd_t(y_3)$, then we obtain that $wd_t(y_3) \in \text{WReach}^{\pi}_{3h-3}[wu_t(x_2)]$. Since $\phi(wu_t(x_2)) = \phi(wd_t(y_3))$ and $wd_t(y_3) \neq wu_t(x_2)$, in both cases this contradicts Claim 7. It follows that $\tau_t(x_2, y_2) = 0$, as claimed.

Applying a symmetric argument with $q_k$ instead of $q_2$, and where the roles of $x$'s and $y$'s are exchanged, we similarly deduce that $\tau_t(x_1, y_1) = 1$. This contradicts the fact that all pairs of the alternating cycle have the same signature $\sigma$. This completes the proof of the claim. \hfill $\Box$

It remains to bound the number of signatures in $\Sigma$. A naive counting already shows that $|\Sigma| \leq 2^c \cdot 2^c \cdot 2^c = 8^c$. However, we can do better as follows. Let $\sigma = (A, B, \tau) \in \Sigma$. Note that, by definition, $B \subseteq A \subseteq 2^c$. Moreover, for each color $i \in [c]$ we can only have $\tau_i = 1$ if $i \in B$. Hence, for each color $i \in [c]$, exactly one of the following four options holds:

1. $i \notin A$,
2. $i \in A$ and $i \notin B$,
3. $i \in B$ and $\tau_i = 1$,
4. $i \in B$ and $\tau_i = 0$.

It follows that we can encode each signature in $\Sigma$ by a vector of length $c$ with entries from the set $\{1, 2, 3, 4\}$. Therefore, $|\Sigma| \leq 4^c$, which concludes the proof of the theorem. \hfill $\Box$

Let us discuss some applications of Theorem 3, starting with posets whose cover graphs have bounded genus. It was shown by van den Heuvel, Ossona de Mendez, Quiroz,
Rabinovich, and Siebertz [7] that
\[ \text{wcol}_r(G) \leq \left( 2g + \binom{r + 2}{2} \right) \cdot (2r + 1) \]
for every graph \( G \) with genus \( g \). Combining this inequality with Theorem 3, we obtain the following upper bound.

**Corollary 9.** For every poset \( P \) of height at most \( h \) whose cover graph has genus \( g \),
\[ \dim(P) \leq 4^{(2g + \binom{3h - 1}{2})(6h - 5)}. \]

For fixed genus, this is a \( 2^{O(h^3)} \) bound on the dimension. In particular, this improves on the previous best bound for posets with planar cover graphs [8], which was doubly exponential in the height.

It is in fact conjectured that posets with planar cover graphs have dimension at most linear in their height. This was recently proved [10] for posets whose diagrams can be drawn in a planar way; these posets form a strict subclass of posets with planar cover graphs. Regarding the latter, Kozik, Micek, and Trotter recently announced that they could prove a polynomial bound on the dimension. Let us remark that it is rather remarkable that linear or polynomial bounds can be obtained when assuming that the poset has a planar diagram or a planar cover graph, respectively. Indeed, for the slightly larger class of posets with \( K_5 \)-minor-free cover graphs, constructions show that the dimension can already be exponential in the height, as shown in [10]. (This also follows from Theorem 11 below, applied with \( t = 3 \).)

We continue our discussion with graphs of bounded treewidth. Let us first quickly recall the definitions of tree decompositions and treewidth (see e.g. Diestel [2] for an introduction to this topic). A **tree decomposition** of a graph \( G \) is a pair consisting of a tree \( T \) and a collection \( \{ B_x \subseteq V(G) \mid x \in V(T) \} \) of sets of vertices of \( G \) called **bags**, one for each node of \( T \), satisfying:

- each vertex \( v \in V(G) \) is contained in at least one bag;
- for each edge \( uv \in E(G) \), there is a bag containing both \( u \) and \( v \), and
- for each vertex \( v \in V(G) \), the set of nodes \( x \in V(T) \) such that \( v \in B_x \) induces a subtree of \( T \).

The **width** of the tree decomposition is \( \max_{x \in V(T)} |B_x| - 1 \). The **treewidth** of \( G \) is the minimum width of a tree decomposition of \( G \).

Grohe, Kreutzer, Rabinovich, Siebertz, and Konstantinos Stavropoulos [4] showed that
\[ \text{wcol}_r(G) \leq \binom{t + r}{r} \]  \hspace{1cm} (3)
for every graph \( G \) of treewidth \( t \). Combining Theorem 3 with (3), we obtain a single exponential bound.
Corollary 10. For every poset $P$ of height at most $h$ with a cover graph of treewidth $t$, 
$$\dim(P) \leq 4^{\left(\frac{h^3 - 3}{t}\right)}.$$ 

For fixed $t$, this is a $2^{\Omega(h^3)}$ bound on the dimension, which improves on the doubly exponential bound in [8]. Surprisingly, this upper bound turns out to be essentially best possible:

Theorem 11. Let $t \geq 3$ be fixed. For each $h \geq 4$, there exists a poset $P$ of height at most $h$ whose cover graph has treewidth at most $t$, and such that 
$$\dim(P) \geq 2^{\Omega(h^{(t-1)/2})}.$$ 

This theorem will be implied by the following slightly more technical theorem, which is an extension of the construction for treewidth 3 in [10]. In this theorem, we use the following terminology: If a poset $P$ is such that the sets of its minimal and maximal elements induce a standard example $S_k$ then we call a vertical pair each of the $k$ pairs $(a, b)$ with $a$ a minimal element, $b$ a maximal element, and $(a, b) \in \text{Inc}(P)$.

Theorem 12. For every $h \geq 1$ and $t \geq 1$, there exists a poset $P_{h,t}$ and a tree decomposition of its cover graph such that

(i) $P_{h,t}$ has height $2h$;
(ii) the minimal and maximal elements of $P_{h,t}$ induce the standard example $S_k$ with $k = 2^{(h+t-1)}$;
(iii) the tree decomposition has width at most $2t + 1$, and
(iv) for each vertical pair $(a, b)$ in $P_{h,t}$ there is a bag of the tree decomposition containing both $a$ and $b$.

Proof. We prove the theorem by induction on $h$ and $t$. Let us first deal with the case $h = 1$, which serves as the base cases for the induction. If $h = 1$, then it is easy to see that letting $P_{1,t}$ be the standard example $S_2$ fulfills the desired conditions. For the tree decomposition, it suffices to take a tree consisting of a single node whose bag contains all four vertices. (We note that we could in fact take $P_{1,t} := S_{2t+1}$ and increase slightly the bound in (ii) but the gain is negligible.)

Next, for the inductive case, suppose that $h \geq 2$. We treat separately the cases $t = 1$ and $t \geq 2$.

First, suppose that $t = 1$. The poset $P_{h,1}$ is defined using the inductive construction illustrated in Figure 4 (left): We start with the poset $P_{h-1,1}$, and for each vertical pair $(a, b)$ in $P_{h-1,1}$ we introduce four elements $x_1, x_2, y_1, y_2$ forming a standard example $S_2$ (with $x$’s and $y$’s being minimal and maximal elements, respectively). Then we add the relations $x_1 < a$ and $x_2 < a$, and $b < y_1$ and $b < y_2$, and take the transitive closure. This defines the poset $P_{h,1}$. It is easy to see that the height of $P_{h,1}$ is exactly the height of $P_{h-1,1}$ plus 2, which implies (i). It is also easily checked that the number of minimal (maximal) elements in $P_{h-1,1}$ is twice the number in $P_{h-1,1}$, which was $2^{h-1}$, and that the union of minimal and maximal elements induce the standard example $S_{2h}$, showing (ii).
Now, consider a tree decomposition of the cover graph of $P_{h-1,1}$ satisfying (iii) and (iv). For each vertical pair $(a, b)$ of $P_{h-1,1}$, consider a node $z$ of the tree whose bag $B_z$ contains both $a$ and $b$, and let $x_1, x_2, y_1, y_2$ be the four elements introduced when considering $(a, b)$ in the definition of $P_{h,1}$. Extend the tree decomposition by adding three new nodes $z', z'_1, z'_2$ with bags $B_{z'} := \{a, b, y_1, y_2\}$, $B_{z'_1} := \{a, x_1, y_1, y_2\}$, $B_{z'_2} := \{a, x_2, y_1, y_2\}$, and adding the three edges $zz', zz'_1, zz'_2$ to the tree, as illustrated in Figure 5. Clearly, once this extension is done for each vertical pair of $P_{h-1,1}$, the resulting tree decomposition of the cover graph of $P_{h,1}$ satisfies (iii) and (iv).

Next, suppose that $t \geq 2$. We start with a copy of $P_{h-1,t}$ and let $(a_1, b_1), \ldots, (a_t, b_t)$ denote its vertical pairs. For each vertical pair $(a_i, b_i)$, we introduce a copy $P_i$ of $P_{h,t-1}$, and add the relation $x < a_i$ for each minimal element $x$ of $P_i$, and the relation $b_i < y$ for each maximal element $y$ of $P_i$; see Figure 4 (right). Then $P_{h,t}$ is obtained by taking the transitive closure of this construction.

Observe that $P_{h,t}$ has height $2h$, thus (i) holds. Moreover, the minimal and maximal elements of $P_{h,t}$ induce a standard example $S_k$ with

$$k = 2^{h+1-t} \cdot 2^{(h+1-t) - 1} = 2^{h+1-t}$$

by the induction hypothesis, showing (ii).

Next, consider the tree decomposition of the cover graph of $P_{h-1,t}$ given by the induction hypothesis. We extend this tree decomposition by doing the following for each vertical pair $(a_i, b_i)$ of $P_{h-1,t}$: Consider a node $z^i$ of the tree whose bag contains both $a_i$ and $b_i$. Take the tree decomposition of the cover graph of $P_i$ given by the induction hypothesis and denote its tree by $T^i$ (on a new set of nodes). Then add an edge between the node $z^i$ and an arbitrary node of $T^i$, and add $a_i$ and $b_i$ to every bag of nodes coming from $T^i$. It is easily checked that this defines a tree decomposition of the cover graph of $P_{h,t}$, of width at most $2t + 1$, such that for each vertical pair $(a, b)$ of $P_{h,t}$ there is bag containing both $a$ and $b$. Therefore, properties (iii) and (iv) are satisfied.

\[ \square \]

**Proof of Theorem 11.** Let $t \geq 3$ and $h \geq 4$. Then we set $h' := \lfloor h/2 \rfloor$ and $t' = \lceil (t-1)/2 \rceil$. With these values, the poset $P_{h',t'}$ from Theorem 12 has height $2h' \leq h$ and its cover graph has treewidth at most $2t' + 1 \leq t$. Moreover,

$$\dim(P_{h',t'}) \geq 2^{2h'/h'} = 2^{\Omega(h')} = 2^{\Omega(h/(t-1)/2))}.$$
We pursue with the case of posets whose cover graphs exclude $K_t$ as a minor. It was shown by van den Heuvel et al. [7] that

$$\text{wcol}_r(G) \leq \binom{r + t - 2}{t - 2} \cdot (t - 3)(2r + 1) \in O(r^{t-1})$$

for every graph $G$ excluding $K_t$ as a minor. Together with Theorem 3, this yields the following improvement on the previous best bound [16], which was doubly exponential in the height (for fixed $t$).

**Corollary 13.** For every poset $P$ of height at most $h$ whose cover graph excludes $K_t$ as a minor,

$$\dim(P) \leq 4^{(3h+1-t-5) \cdot (t-3)(6h-5)}.$$  

For a fixed integer $t \geq 5$, this $2^{O(h^{t-1})}$ bound is again essentially best possible by Theorem 11 (using an upper bound of $t - 2 \geq 3$ on the treewidth), because graphs of treewidth at most $t - 2$ cannot contain $K_t$ as a minor. On the other hand, it is no coincidence that we cannot use Theorem 11 in this way when $t \leq 4$: Indeed, posets whose cover graphs exclude $K_4$ as a minor have dimension bounded by a universal constant (at most 1276), irrespectively of their height [9].

Regarding graphs $G$ that exclude $K_t$ as a topological minor, it is implicitly proven in the work of Kreutzer, Pilipczuk, Rabinovich, and Siebertz [14] that these graphs satisfy

$$\text{wcol}_r(G) \leq 2^{O(r \log r)}$$

when $t$ is fixed. Combining this inequality with Theorem 3 we get a slight improvement upon the bound derived in [16], however the resulting bound remains doubly exponential:

**Corollary 14.** Let $t \geq 1$ be a fixed integer. Then, every poset $P$ of height at most $h$ whose cover graph excludes $K_t$ as a topological minor satisfies

$$\dim(P) \leq 2^{2^{O(h \log h)}}.$$  

4. **Nowhere Dense Classes**

In this section we prove Theorem 1, which we restate for the reader’s convenience.
Theorem 1. Let $C$ be a monotone class of graphs. Then $C$ is nowhere dense if and only if for every integer $h \geq 1$ and real number $\varepsilon > 0$, $n$-element posets of height at most $h$ whose cover graphs are in $C$ have dimension $O(n^\varepsilon)$.

Let us start with the backward direction, which is fairly easy. We prove the contrapositive. Thus let $C$ be a monotone graph class which is not nowhere dense (such a class is said to be somewhere dense). Our aim is to prove that there exist $h \geq 1$ and $\varepsilon > 0$ such that there are $n$-element posets of height at most $h$ with dimension $\Omega(n^\varepsilon)$ whose cover graphs are in $C$.

Since $C$ is somewhere dense, there exists an integer $r \geq 0$ (depending on $C$) such that for every $t \geq 1$ there is a graph $G \in C$ containing an $\leq r$-subdivision of $K_t$ as a subgraph. Since $C$ is closed under taking subgraphs, this means that for every $m \geq 2$, the class $C$ contains a graph $G_m$ that is an $\leq r$-subdivision of the cover graph of the standard example $S_m$. Notice that $G_m$ has at most $rm^2 + 2m$ vertices. Now it is easy to see that $G_m$ is also the cover graph of a poset $P_m$ of height at most $r + 2$ containing $S_m$ as an induced subposet (simply perform the edge subdivisions on the diagram of $S_m$ in the obvious way). Let $n$ be the number of elements of $P_m$. The poset $P_m$ has dimension at least $m$, and thus its dimension is $\Omega(\sqrt{n})$ since $n \leq rm^2 + 2m$. Hence, we obtain the desired conclusion with $h := r + 2$ and $\varepsilon := 1/2$. This completes the proof of the backward direction.

The forward direction is harder to prove. We begin with our key lemma, which will be iteratively applied in the algorithmic part of the proof.

Lemma 15. Let $P$ be a poset of height $h$ with cover graph $G$, let $I \subseteq \text{Inc}(P)$, and let $c := \text{wcol}_{4h-4}(G)$. Then there exists an element $q \in P$ such that the set $I' := \{(x, y) \in I : q \leq y \text{ in } P\}$ satisfies

$$\dim(I') \geq \dim(I)/c - 3.$$ 

Proof. Fix a linear order $\pi$ of the vertices of $G$ witnessing $\text{wcol}_{4h-4}(G) \leq c$. Here and in the rest of the proof, weak reachability is to be interpreted w.r.t. the cover graph $G$ and the ordering $\pi$.

Let $\phi$ be a greedy vertex coloring of $G$ obtained by considering the vertices one by one according to $\pi$, and assigning to each vertex $z$ a color $\phi(z) \in [c]$ different from all the colors used on vertices in $\text{WReach}_{4h-4}^\pi[z] = \{z\}$.

Claim 16. For every $z \in P$, all the vertices in $\text{WReach}_{2h-2}^\pi[z]$ have different $\phi$-colors.

Proof. For every two vertices $x, y \in \text{WReach}_{2h-2}^\pi[z]$, we know by Observation 6 that $x$ is weakly $(4h - 4)$-reachable from $y$, or the other way round. In both cases we have that $\phi(x) \neq \phi(y)$. □

For each $z \in P$, we denote by $\tau(z) \in [c]$ the color of the element in the downset $D(z)$ of $z$ that is leftmost w.r.t. $\pi$. Note that this element is weakly $(h - 1)$-reachable from $z$. 
Given a color $i \in [c]$, we let $w_i(z)$ denote the unique element of $\text{WReach}^2_{h-2}[z]$ colored $i$ if there is one, and leave $w_i(z)$ undefined otherwise. Observe that $w_{\tau(z)}(z)$ is the leftmost element in the downset of $z$. In particular, $w_{\tau(z)}(z) \leq z$ in $P$.

**Claim 17.** If $x \leq y$ in $P$, then $w_{\tau(y)}(x) = w_{\tau(y)}(y)$.

**Proof.** Let $Q_1$ and $Q_2$ be covering chains witnessing that $x \leq y$ and $w_{\tau(y)}(y) \leq y$ in $P$, respectively. Then the union of $Q_1$ and $Q_2$ contains a path $Q$ from $x$ to $w_{\tau(y)}(y)$ in $G$ as both paths meet in $y$. Since all elements of $Q$ lie in the downset of $y$, it follows that $w_{\tau(y)}(y)$ is the leftmost element of $Q$ w.r.t. $\pi$. Since $Q_1$ and $Q_2$ each have length at most $h-1$, this implies that $w_{\tau(y)}(y)$ is weakly $(2h-2)$-reachable from $x$, implying in turn that $w_{\tau(y)}(y)$ is the unique element with color $\tau(y)$ in $\text{WReach}^2_{2h-2}[x]$. This shows $w_{\tau(y)}(x) = w_{\tau(y)}(y)$.

Let us now define some properties of incomparable pairs. Given $(x, y) \in I$, we let

$$
\alpha(x, y) = \begin{cases} 
1 & \text{if } w_{\tau(y)}(x) = w_{\tau(y)}(y) \\
2 & \text{if } w_{\tau(y)}(x) <_\pi w_{\tau(y)}(y) \\
3 & \text{if } w_{\tau(y)}(x) >_\pi w_{\tau(y)}(y) \\
4 & \text{if } w_{\tau(y)}(x) \text{ is not defined.}
\end{cases}
$$

Then, we define the *signature* $\sigma(x, y)$ of the pair $(x, y)$ to be the pair

$$
\sigma(x, y) := (\tau(y), \alpha(x, y)).
$$

For each color $\tau \in [c]$ and value $\alpha \in [4]$, let $J_{\tau,\alpha}$ be the set of incomparable pairs $(x, y) \in I$ such that $\sigma(x, y) = (\tau, \alpha)$. Note that the sets $J_{\tau,\alpha}$ form a partition of $I$.

**Claim 18.** For each color $\tau \in [c]$, the sets $J_{\tau,2}$ and $J_{\tau,3}$ are reversible.

**Proof.** We show that $J_{\tau,2}$ is reversible, the proof for $J_{\tau,3}$ is symmetric. Suppose for a contradiction that $J_{\tau,2}$ is not reversible. Then it contains an alternating cycle $(x_1, y_1), \ldots, (x_k, y_k)$. In particular, for every $i \in [k]$ we have $x_i \leq y_{i+1}$ in $P$ (cyclically). By Claim 17 we obtain that $w_{\tau}(x_i) = w_{\tau}(y_{i+1})$. However, by our signature function this implies $w_{\tau}(y_{i+1}) = w_{\tau}(x_i) <_\pi w_{\tau}(y_i)$, which clearly cannot hold cyclically for all $i \in [k]$. \hfill $\Box$

**Claim 19.** For each color $\tau \in [c]$, the set $J_{\tau,4}$ is reversible.

**Proof.** Arguing by contradiction, suppose that $J_{\tau,4}$ is not reversible, and let $(x_1, y_1), \ldots, (x_k, y_k)$ denote an alternating cycle. Since $x_1 \leq y_2$ in $P$, we have that $w_{\tau(y_2)}(x_1) = w_{\tau(y_2)}(y_2)$ by Claim 17. However, $\tau(y_2) = \tau = \tau(y_1)$, which contradicts the fact that $w_{\tau(y_1)}(x_1)$ is not defined. \hfill $\Box$

Since

$$
I = \bigcup_{\tau \in [c], \alpha \in [4]} J_{\tau,\alpha}
$$
the previous claims imply that
\[
\dim(I) \leq \sum_{\tau \in [c]} \dim(J_{\tau,1}) + \sum_{\tau \in [c]} \dim(J_{\tau,2}) + \sum_{\tau \in [c]} \dim(J_{\tau,3}) + \sum_{\tau \in [c]} \dim(J_{\tau,4})
\]
\[
\leq \sum_{\tau \in [c]} \dim(J_{\tau,1}) + 3c.
\]
It follows that there exists a color \( \tau \in [c] \) such that \( \dim(J_{\tau,1}) \geq \dim(I)/c - 3 \). In the rest of the proof we focus on the set \( J_{\tau,1} \). Thus, denoting this set by \( I_{\tau} \), we have
\[
\dim(I_{\tau}) \geq \dim(I)/c - 3.
\]
(4)

Given an element \( p \in P \), we denote by \( I_{\tau,p} \) the set of incomparable pairs \( (x,y) \in I_{\tau} \) such that \( p = w_{\tau}(x) = w_{\tau}(y) \). Note that the sets \( I_{\tau,p} \ (p \in P) \) partition \( I_{\tau} \).

**Claim 20.** \( \dim(I_{\tau}) = \max_{p \in P} \dim(I_{\tau,p}) \).

**Proof.** Let \( d := \max_{p \in P} \dim(I_{\tau,p}) \). Note that \( \dim(I_{\tau}) \geq d \) since \( I_{\tau,p} \subseteq I_{\tau} \) for each \( p \in P \). Thus it remains to show that \( \dim(I_{\tau}) \leq d \).

For each \( p \in P \), there exists a partition of \( I_{\tau,p} \) into at most \( d \) reversible sets. Let \( I_{\tau,p}^1, \ldots, I_{\tau,p}^d \) be disjoint reversible sets such that
\[
I_{\tau,p} = \bigcup_{j \in [d]} I_{\tau,p}^j,
\]
some sets being possibly empty. We claim that the set \( \bigcup_{p \in P} I_{\tau,p}^j \) is reversible for each \( j \in [d] \). Arguing by contradiction, suppose that for some \( j \in [d] \) this set is not reversible. Then it contains an alternating cycle \( (x_1, y_1), \ldots, (x_k, y_k) \). Applying Claim 17 for \( x_i \leq y_{i+1} \) in \( P \) (with \( i \in [k] \)), we obtain that \( w_{\tau}(x_i) = w_{\tau}(y_{i+1}) \), which by the signatures of these pairs implies that \( w_{\tau}(y_i) = w_{\tau}(y_{i+1}) \). As this holds cyclically, there is \( p \in P \) such that \( p = w_{\tau}(y_i) \) for every \( i \in [k] \). However, this implies that \( (x_1, y_1), \ldots, (x_k, y_k) \) is an alternating cycle in \( I_{\tau,q}^j \), which is a contradiction since this set is reversible by assumption.

Thus, \( \bigcup_{p \in P} I_{\tau,p}^j \) is reversible for each \( j \in [d] \). Since \( I_{\tau} = \bigcup_{j \in [d]} \bigcup_{p \in P} I_{\tau,p}^j \), it follows that \( \dim(I_{\tau}) \leq d \), as desired. \( \square \)

Now we can complete the proof of the lemma. Let \( q \in P \) be an element witnessing the maximum value in the right-hand side of the equation in Claim 20. Clearly, \( I_{\tau,q} \subseteq \{(x,y) \in I \mid q \leq y \text{ in } P\} \). Since
\[
\dim(I_{\tau,q}) = \dim(I_{\tau}) \geq \dim(I)/3 - c,
\]
this completes the proof of the lemma. \( \square \)

We are now ready to prove the forward direction of Theorem 1. As discussed in the introduction, it follows from the following bound on the dimension of a poset combined with Zhu’s characterization of nowhere dense classes.

Theorem 4. There exists a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for every $h \geq 1$ and $t \geq 1$, every poset $P$ of height at most $h$ whose cover graph $G$ contains no $\leq (2h - 3)$-subdivision of $K_t$ as a subgraph satisfies

$$\dim(P) \leq (4c)^{f(h,t)},$$

where $c := \text{wcol}_{4h-4}(G)$.

Proof. Let $h \geq 1$ and $t \geq 1$. We prove the theorem with the following value for $f(h,t)$:

$$f(h,t) := \left( \binom{m + h}{h} \right), \quad \text{where } m := \left( \frac{t}{2} \right)^{h}. $$

Let thus $P$ be a poset of height at most $h$, let $G$ denote its cover graph, and let $c := \text{wcol}_{4h-4}(G)$. We prove the contrapositive. That is, we assume that

$$\dim(P) > (4c)^{f(h,t)},$$

and our goal is to show that $G$ contains a $\leq (2h - 3)$-subdivision of $K_t$ as a subgraph. For technical reasons, we will need to suppose also that $4c > t$. This can be assumed without loss of generality, because if not then $4\text{wcol}_{3h-3}(G) \leq 4\text{wcol}_{4h-4}(G) \leq t$, and hence $\dim(P) \leq 4^{t/4} \leq (4c)^{f(h,t)}$ by Theorem 3.\(^1\)

The proof is divided in two phases.

**Phase 1.** We define the value of an antichain $S$ of size at most $m$ in $P$ to be the vector of heights of elements in $S$ ordered in non-increasing order and padded at the end with 0-entries so that the vector is of size exactly $m$. Note that $\left( \binom{m + h}{h} \right)$ is the number of size-$m$ vectors with entries in $\{0, 1, \ldots, h\}$ ordered in non-increasing order. We enumerate these vectors in lexicographic order with numbers from 0 to $\left( \binom{m + h}{h} \right) - 1 = f(h,t) - 1$. Let $\text{value}(S)$ denote the number of the value of $S$ in this enumeration. Notice that $\text{value}(\emptyset) = 0$.

We start with the empty antichain $S := \emptyset$ and apply an iterative algorithm, Algorithm 1, that increases the value of $S$ at each iteration. At all times $S$ will be an antichain of size at most $m$, and when the algorithm stops $S$ will have size exactly $m$.

Let us show that the following three properties are invariants of the while loop. That is, these properties hold at the beginning of each iteration of the loop, and thus in particular they hold when the algorithm stops since $S$ and $I$ are not updated during the last iteration.

1. $\dim(I) > (4c)^{f(h,t) - \text{value}(S)}$,
2. $S$ is an antichain of size at most $m$, and

\(^1\)The reader might object that this makes the proof dependent on Theorem 3, while we claimed in the introduction that it was not. In order to address this perfectly valid point, let us mention that one could choose instead to add the extra assumption that $4c > t$ in the statement of Theorem 4; this does not change the fact that it implies the forward direction of Theorem 1 (in combination with Zhu’s theorem). However, it seemed rather artificial to do so, since the theorem remains true without this technical assumption.
We will also prove that \( \text{value}(S) \) increases each time \( S \) is updated, and that element \( q \) can always be found in line 4.

The proof is by induction on the number of iterations of the \textbf{while} loop. Observe that the above three properties are true at the beginning, thus the base case holds.

Next, consider the inductive case. Consider the sets \( S \) and \( I \) at the beginning of an iteration of the loop. Element \( q \) can always be found in line 4 of the algorithm, as we now explain. Consider the poset \( Q := P - \bigcup_{s \in S} \text{D}(s) \), and let \( I_Q := I \cap \text{Inc}(Q) \). First, we claim that \( I - I_Q \) is reversible in \( P \). Arguing by contradiction, suppose that this set contains an alternating cycle \( (x_1, y_1), \ldots, (x_k, y_k) \). Since \( Q \) is an induced subposet of \( P \), for each \( i \in [k] \), at least one \( x_i \) and \( y_i \) must be in \( \bigcup_{s \in S} \text{D}(s) \) (otherwise, \( (x_i, y_i) \) would be an incomparable pair of \( Q \)). We cannot have \( x_i \in \bigcup_{s \in S} \text{D}(s) \), because otherwise \( x_i \leq s \) in \( P \) for some \( s \in S \), and since \( s \leq y_i \) in \( P \) by invariant (1.3) this would contradict the fact that \( x_i \) and \( y_i \) are incomparable. Thus, \( y_i \in \bigcup_{s \in S} \text{D}(s) \). However, since \( x_{i-1} \leq y_i \) in \( P \) (taking indices mod \( k \), it follows that \( x_{i-1} \in \bigcup_{s \in S} \text{D}(s) \), a contradiction. Hence, \( I - I_Q \) is reversible, as claimed. It follows

\[
\dim(I_Q) \geq \dim(I) - 1. \tag{5}
\]

Applying Lemma 15 on poset \( Q \) and set \( I_Q \), we obtain an element \( q \in Q \) such that \( \dim_Q \{ (x, y) \in I_Q : q \leq y \text{ in } Q \} \geq \dim_Q(I_Q)/c - 3 \). (The subscript \( Q \) indicates that dimension is computed w.r.t. \( Q \).) Here we use that \( Q \) has height at most that of \( P \), and thus at most \( h \), and that the cover graph \( G_Q \) of \( Q \) is an (induced) subgraph of \( G \), and thus \( \text{wcol}_h(G_Q) \leq \text{wcol}_h(G) \leq c \). It only remains to point out that \( \dim_Q(I_Q) = \dim(I_Q) \) because \( Q \) is an induced subposet of \( P \) (that is, a subset of \( I_Q \) is an alternating cycle in \( Q \) if and only if it is one in \( P \)), and similarly \( \dim_Q \{ (x, y) \in I_Q : q \leq y \text{ in } Q \} = \)}
\[
\dim \left( \{ (x, y) \in I_Q : q \leq y \text{ in } P \} \right) \geq \dim(I_Q) / c - 3 \\
\geq (\dim(I) - 1) / c - 3 \\
\geq \dim(I) / c - 4,
\]
as desired. This completes the proof that element \( q \) in line 4 of the algorithm can be found. (We remark that a subtlety of the algorithm is that it gives priority to elements \( q \) such that \( D(q) \cap S \neq \emptyset \), which need not always exist.)

Let \( I' := \{ (x, y) \in I : q \leq y \text{ in } P \} \), and \( S' := (S - D(q)) \cup \{ q \} \). First, observe that \( \text{value}(S') > \text{value}(S) \), since either \( D(q) \cap S \neq \emptyset \) and the height of \( q \) is strictly larger than the heights of all the elements in \( D(q) \cap S \). Or \( D(q) \cap S = \emptyset \) but then \( S' = S \cup \{ q \} \).

By the induction hypothesis, we had \( \dim(I) > (4c)^{f(h,t)-\text{value}(S)} \). Now, by the above discussion,
\[
\dim(I') \geq \dim\left( \{ (x, y) \in I_Q : q \leq y \text{ in } P \} \right) \\
\geq \dim(I) / c - 4 \\
\geq (4c)^{f(h,t)-\text{value}(S)} / c - 4 \\
\geq 4 \cdot (4c)^{f(h,t)-\text{value}(S)-1} - 4 \\
\geq (4c)^{f(h,t)-\text{value}(S')}.
\]
For the last inequality we use that \( \text{value}(S) < \text{value}(S') < f(h,t) \). Thus, (1.1) holds at the end of the iteration.

The fact that \( S' \) is an antichain is immediate, since \( q \notin \bigcup_{s \in S} D(s) \) and all the elements in \( S \) that are below \( q \) in \( P \) are removed from \( S' \).

Finally, note that the \texttt{return} instruction ensures that \( |S| \leq m \) all the time. This completes the proof of (1.2).

Item (1.3) holds for \( I' \) as it held for \( I \) by the induction hypothesis, and we keep in \( I' \) only these pairs \( (x, y) \in I \) such that \( q \leq y \) in \( P \). This concludes the proof that the three properties are invariants of the \texttt{while} loop. Note also that since \( \text{value}(S) \) increases each time \( S \) is updated, and since \( \text{value}(S) \) can take only finitely many values, the algorithm will stop eventually.

Let \( S \) and \( I \) now denote the sets returned by Algorithm 1 when it stops. Let
\[
d := (4c)^{f(h,t)-\text{value}(S)}.
\]
Note that \( |S| = m \). Furthermore, there was no way to choose \( q \) in line 4 so that \( D(q) \cap S \neq \emptyset \). This implies the following property, which will be useful in Phase 2.

\textbf{Claim 21.} The following two properties hold:

(i) \( \dim(I) > d, \) and
(ii) for every element \( q \) such that there exists \( s \in S \) with \( s < q \) in \( P \),
\[
\dim\left(\{(x, y) \in I \mid q \leq y \text{ in } P\}\right) \leq d/4c.
\]

**Phase 2.** During the second phase we first run \( t \) rounds in which we specify \( t \) vertices of \( G \). These will be the branch vertices of the subdivision of \( K_t \) that we aim to construct.

While running the loop of Phase 2, we maintain a pair \((V, R)\) with \( V \subseteq P \) and \( R \subseteq S \), which satisfies the following three invariants after the \( j \)-th iteration (\( j \in \{0, 1, \ldots, t\}\)):

1. \( |V| = j \) and \( |R| \geq m^{(1/h)^j} \).
2. \( V \cap R = \emptyset \), and
3. for every \( v \in V \) and \( r \in R \), there is \( p \in P \) such that \( p \) is covered by \( v \) in \( P \) and \( D(p) \cap R = \{r\} \).

(In item (2.3), we say that \( p \) is covered by \( v \) if \( p < v \) is a cover relation in \( P \).) At the beginning we start with the pair \((V, R) := (\emptyset, S)\). Clearly, invariants (2.1)–(2.3) are satisfied for \( j = 0 \).

We now describe the \( j \)-th iteration (\( 1 \leq j \leq t \)). Let \((V, R)\) be the pair satisfying the invariants after the \((j - 1)\)-th iteration. Our main task is to find a new vertex that we could add to \( V \). Our starting point will be an arbitrarily chosen vertex in the set
\[
\{y \mid (x, y) \in I\} - \bigcup_{v \in V} U(v).
\]

Then, we will refine our choice iteratively according to a simple criterion. In order to start the process, we first need to show that the above set is not empty. This can be seen as follows. It follows from Invariant (2.3) that, for every \( v \in V \), there is \( r \in R \subseteq S \) such that \( r < v \) in \( P \). Hence, by Claim 21
\[
\dim\left(\left\{(x, y) \in I \mid y \notin \bigcup_{v \in V} U(v)\right\}\right) \geq \dim(I) - \sum_{v \in V} \dim(\{(x, y) \in I \mid v \leq y \text{ in } P\})
\]
\[
> d - |V| \cdot d/(4c)
\]
\[
> 0.
\]

(This is the place in the proof where we use our assumption that \( 4c > t \), which implies the last inequality.) It follows that the left-hand side is at least 2. Therefore, \( \{y \mid (x, y) \in I\} - \bigcup_{v \in V} U(v) \) is not empty. (Indeed, the dimension of an empty set of incomparable pairs is 1.) Choose some element \( y \) in this set.

Now starting from the element \( y \) we go down along cover relations in the poset \( P \). Initially we set \( v := y \), and as long as there is a element \( x \in P \) such that \( x \) is covered by \( v \) in \( P \) and
\[
|D(x) \cap R| > |D(v) \cap R|/m^{(1/h)^j},
\]
we update \( v := x \).

\[\text{(6)}\]

To clarify, ‘after the 0-th iteration’ means right at the beginning; the first iteration is iteration 1.
We deduce that this proves (2.1). When \( x \) were done and hence this finishes the \( j \)-th iteration of the loop.

Consider now the set \( Z \) consisting of all elements that are covered by \( v \) in \( P \). Since \( v \notin R \), we have \( D(v) \cap R \subseteq \bigcup_{z \in Z} D(z) \). Let \( Z' \) be an inclusion-wise minimal subset of \( Z \) such that \( D(v) \cap R \subseteq \bigcup_{z \in Z'} D(z) \). The minimality of \( Z' \) allows us to fix for every \( z \in Z' \) an element \( r_z \in D(v) \cap R \) such that \( r_z \in D(z) \) and \( r_z \notin D(z') \) for every \( z' \in Z' - \{ z \} \).

Finally, we update our maintained pair to \((V', R')\), where

\[
V' := V \cup \{ v \} \quad \text{and} \quad R' := \{ r_z \mid z \in Z' \}.
\]

This finishes the \( j \)-th iteration of the loop.

**Claim 22.** The pair \((V', R')\) fulfills the invariants (2.1)–(2.3).

**Proof.** First of all, we show that \( v \notin V \), and thus \( |V'| = j \). Recall that we have chosen \( y \) such that \( w \notin y \) in \( P \) for every \( w \in V \). On the other hand, by our procedure we have \( v \leq y \) in \( P \) and hence \( v \notin V \).

Next, we want to show that the size of \( R' \) is large enough, that is, that it satisfies the lower bound in (2.1). Since elements \( r_z, r_{z'} \) are distinct for distinct \( z, z' \in Z' \), we have \( |R'| = |Z'| \). Moreover,

\[
|D(v) \cap R| = \left| \bigcup_{z \in Z'} D(z) \cap R \right| \leq \sum_{z \in Z'} |D(z) \cap R| \leq |Z'| \cdot \frac{|D(v) \cap R|}{m^{(1/h)^j}}.
\]

We deduce that

\[
|R'| = |Z'| \geq m^{(1/h)^j}.
\]

This proves (2.1).

Invariant (2.2) holds since \( V \) is disjoint from \( R \) and since \( v \) is not contained in \( R \), and thus not in \( R' \) either.

It remains to verify (2.3) for \((V', R')\). Since \( R' \subseteq R \), we only need to check this property for the new vertex \( v \). Consider an element \( r \in R' \). By the definition of \( R' \) there is \( z \in Z' \) such that \( r = r_z \). Recalling the way we defined \( r_z \), we obtain \( D(z) \cap R' = \{ r_z \} \). This completes the verification of the invariants for \((V', R')\). \( \square \)
For the rest of the proof let \((V, R)\) denote the pair satisfying the invariants (2.1)–(2.3) after the \(t\)-th iteration of Phase 2.

We are ready to connect the vertices of \(V\), which serve as the branch vertices of the subdivision of \(K_t\) we are building, by internally vertex-disjoint paths. By (2.1) we have

\[ |V| = t \quad \text{and} \quad |R| \geq \left(\frac{t}{2}\right). \]

Thus, for each unordered pair \(\{v_1, v_2\} \subseteq V\) we can choose a corresponding element \(r_{\{v_1, v_2\}} \in R\) in such a way that all chosen elements are distinct. Furthermore, by Invariant (2.3), there are cover relations \(p_1 < v_1\) and \(p_2 < v_2\) in \(P\) such that \(D(p_1) \cap R = D(p_2) \cap R = \{r_{\{v_1, v_2\}}\}\). Let

\[ r_{\{v_1, v_2\}} = u_1 < u_2 < \cdots < u_k < p_1 < v_1 \quad \text{and} \quad r_{\{v_1, v_2\}} = w_1 < w_2 < \cdots < w_\ell < p_2 < v_2 \]

be covering chains in \(P\). Clearly, the union of the two covering chains contains a path connecting the vertices \(v_1\) and \(v_2\) in \(G\); fix such a path \(Q_{\{v_1, v_2\}}\) for the unordered pair \(\{v_1, v_2\}\). Observe that the path \(Q_{\{v_1, v_2\}}\) has length at most \(2h - 2\).

Connecting all other pairs of vertices in \(V\) in a similar way, we claim that the union of these paths forms a subdivision of \(K_t\). All we need to prove is that whenever there is \(z \in Q_{\{v_1, v_2\}} \cap Q_{\{v_1', v_2'\}}\) for distinct sets \(\{v_1, v_2\}, \{v_1', v_2'\} \subseteq V\), then \(z\) is an endpoint of both paths. Suppose to the contrary that \(z\) is an internal vertex of one path, say of \(Q_{\{v_1, v_2\}}\). By our construction, there are cover relations \(p_1 < v_1\) and \(p_2 < v_2\) in \(P\) with \(D(p_1) \cap R = D(p_2) \cap R = \{r_{\{v_1, v_2\}}\}\). Furthermore, we have \(z \leq p_1\) or \(z \leq p_2\) in \(P\). Say \(z \leq p_2\) without loss of generality. From \(z \in Q_{\{v_1', v_2'\}}\) we deduce that \(r_{\{v_1', v_2'\}} \leq z\) in \(P\), which implies that \(r_{\{v_1', v_2'\}} \leq p_1\) in \(P\). However, it follows that \(r_{\{v_1', v_2'\}} \in D(p_1) \cap R = \{r_{\{v_1, v_2\}}\}\) and hence \(r_{\{v_1, v_2\}} = r_{\{v_1', v_2'\}}\), a contradiction to our construction. We conclude that both paths \(Q_{\{v_1, v_2\}}\) and \(Q_{\{v_1', v_2'\}}\) are indeed internally disjoint.

Finally, since all the paths \(Q_{\{v_1, v_2\}}\) \((\{v_1, v_2\} \subseteq V)\) have length at most \(2h - 2\), this shows the existence of a \(\leq (2h - 3)\)-subdivision of \(K_t\) in \(G\), as desired. This completes the proof of Theorem 4.

We remark that no effort has been made to optimize the bound in Theorem 4.

\[ \square \]

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NOWHERE DENSE GRAPH CLASSES AND DIMENSION

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