COMPUTING INVARIANTS OF THE WEIL REPRESENTATION

STEPHAN EHLEN AND NILS-PETER SKORUPPA

Abstract. We propose an algorithm for computing bases and dimensions of spaces of invariants of Weil representations of $\text{SL}_2(\mathbb{Z})$ associated to finite quadratic modules. We prove that these spaces are defined over $\mathbb{Z}$, and that their dimension remains stable if we replace the base field by suitable finite prime fields.

1. Introduction

Weil representations associated to finite abelian groups $A$ equipped with a non-degenerate quadratic form $Q$ provide a fundamental tool in the theory of automorphic forms. They are at the basis of the theory of automorphic products, the theory of Jacobi forms or Siegel modular forms of singular and critical weight, and they find also applications in other disciplines like coding theory or quantum field theory. Of particular interest among the mentioned applications is the space $\mathbb{C}[A]^G$ of invariants of the Weil representations of $G = \text{SL}_2(\mathbb{Z})$ associated to a given finite quadratic module $(A, Q)$. Despite the importance of $\mathbb{C}[A]^G$ for the indicated applications neither any explicit closed formula is known for the dimension of $\mathbb{C}[A]^G$ nor any useful description\(^1\) of its elements.

The purpose of the present note is to discuss questions related to the computation of the dimension and a basis of $\mathbb{C}[A]^G$ for a given finite quadratic module $(A, Q)$. In particular, we develop an algorithm (Algorithm 4.2) for computing a basis of $\mathbb{C}[A]^G$ which we also implemented and ran successfully in various examples. We mention two results of this article which might be of independent interest. First, we prove that $\mathbb{C}[A]^G$ always possesses a basis whose elements are in $\mathbb{Z}[A]$ (Theorem 3.3). Second, if a finite prime field $\mathbb{F}_\ell$ contains the $N$th root of unity, where $N$ is the level of $(A, Q)$, then the Weil representation can also be defined on $\mathbb{F}_\ell[A]$. We prove that then $\dim \mathbb{C}[A]^G = \dim \mathbb{F}_\ell[A]^G$ (except for possibly $(N, \ell) = (2, 3)$). Our algorithm

\(^1\)However, if $(A, Q)$ possesses a self-dual isotropic subgroup $U$ then the characteristic function of $U$ is quickly checked to be an invariant, and one can show that, in the case that $(A, Q)$ possesses self-dual subgroups, the characteristic functions of the self-dual isotropic subgroups span in fact the space $\mathbb{C}[A]^G$ (A proof of this will be given in [Sko16]). But an arbitrary finite quadratic module does not necessarily possess self-dual isotropic subgroups and still admits nonzero invariants if its order is big enough.
has already been used successfully to compute the dimension of spaces of vector valued cusp forms of weight 2 and 3/2 in [BEF16], where a classification of all lattices of signature (2, n) without obstructions to the existence of weakly holomorphic modular forms of weight $1 - \frac{n}{2}$ for the associated Weil representation was given.

The plan of this note is as follows. In Section 2 we recall the basic definitions and facts from the theory of finite quadratic modules and its associated Weil representations. In Section 3 we prove some basic facts about the space of invariants $\mathbb{C}[A]^G$. Most of the material of this section is probably known to specialists. However, since it is often difficult to find suitable references we decided to include this section. To our knowledge Theorem 3.3 is new, which shows that the space of invariants $\mathbb{C}[A]^G$ is in fact defined over $\mathbb{Z}$. In Section 4 we explain our algorithm for computing a basis for $\mathbb{C}[A]^G$, and we discuss some improvements. In Section 5 we consider the reduction of Weil representations modulo suitable primes $\ell$ and prove that the dimension of the space of invariants remains stable under reduction. This interesting fact can be used to improve the run-time of our algorithm. Finally, in Section 6 we provide tables of dimensions for quadratic modules of small order.

2. Finite quadratic modules and Weil representations

A finite quadratic module (also called a finite quadratic form or discriminant form in the literature) is a pair $\mathfrak{A} = (A, Q)$ consisting of a finite abelian group $A$ together with a $\mathbb{Q}/\mathbb{Z}$-valued non-degenerate quadratic form $Q$ on $A$. The bilinear form corresponding to $Q$ is defined as

$$Q(x, y) := Q(x + y) - Q(x) - Q(y).$$

The quadratic form $Q$ is called non-degenerate if $Q(\cdot, \cdot)$ is non-degenerate, i.e. if there exists no $x \in A \setminus \{0\}$, such that $Q(x, y) = 0$ for all $y \in A$. Two finite quadratic modules $\mathfrak{A} = (A, Q)$ and $\mathfrak{B} = (B, R)$ are called isomorphic if there exists an isomorphism of groups $f : A \to B$ such that $Q = R \circ f$.

The theory of finite quadratic modules has a long history; see e.g. [Wal63], [Wal72], [Nik79] and the upcoming [Sko16].

If $\mathfrak{L} = (L, \beta)$ is an even lattice, the quadratic form $\beta$ on $L$ induces a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form $Q$ on the discriminant group $L'/L$ of $\mathfrak{L}$. The pair $D_{\mathfrak{L}} := (L'/L, Q)$ defines a finite quadratic module, which we call the discriminant module of $\mathfrak{L}$. According to [Wal63, Thm. (6)], any finite quadratic module can be obtained as the discriminant module of an even lattice $\mathfrak{L}$. If $\mathfrak{A} = (A, Q)$ is a finite quadratic module and $\mathfrak{L}$ a lattice whose discriminant module is isomorphic to $\mathfrak{A}$, then the difference $b^+ - b^-$ of the real signature $(b^+, b^-)$ of $\mathfrak{L}$ is already determined modulo 8 by $\mathfrak{A}$. Namely, by Milgram’s formula [MH73, p. 127] one has

$$\frac{1}{\sqrt{\text{card}(A)}} \sum_{x \in A} e(Q(x)) = e((b^+ - b^-)/8),$$
where we use \( e(z) = e^{2\pi i z} \) for \( z \in \mathbb{C} \). We call
\[
\text{sig}(\mathfrak{A}) := b^+ - b^- \mod 8 \in \mathbb{Z}/8\mathbb{Z}
\]
the \textit{signature of} \( \mathfrak{A} \). The number
\[
N = \min \{ n \in \mathbb{Z}_{>0} \mid nQ(x) \in \mathbb{Z} \text{ for all } x \in A \}
\]
is called the \textit{level of} \( \mathfrak{A} \).

The metaplectic extension \( \text{Mp}_2(\mathbb{Z}) \) of \( \text{SL}_2(\mathbb{Z}) \) (i.e. the nontrivial twofold central extension of \( \text{SL}_2(\mathbb{Z}) \)) can be realized as the group of pairs \((M, \phi(\tau))\), where \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}) \) and \( \phi \) is a holomorphic function on the complex upper half plane \( \mathbb{H} \) with \( \phi(\tau)^2 = c\tau + d \) (see e.g. [Shi73]). The group \( \text{SL}_2(\mathbb{Z}) \) is generated by
\[
T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},
\]
and the group \( \text{Mp}_2(\mathbb{Z}) \) is generated by \( T^* := (T, 1) \) and \( S^* = (S, \sqrt{7}) \) with relations \( S^{*2} = (S^{*}T^*)^3 = \zeta \), where \( \zeta = \left( \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, i \right) \) is the standard generator of the center of \( \text{Mp}_2(\mathbb{Z}) \).

The Weil representation \( \rho_\mathfrak{A} \) associated to \( \mathfrak{A} \) is a representation of \( \text{Mp}_2(\mathbb{Z}) \) on the group algebra \( \mathbb{C}[A] \). Here, and throughout, we denote the standard basis of \( \mathbb{C}[A] \) by \((\epsilon_x)_{x \in A}\). The action of \( \rho_\mathfrak{A} \) can then be given in terms of the generators \( S, T \in \text{Mp}_2(\mathbb{Z}) \) as follows:
\[
\rho_\mathfrak{A}(T^*)\epsilon_x = e(Q(x))\epsilon_x,
\]
\[
\rho_\mathfrak{A}(S^*)\epsilon_x = \frac{e(-\text{sig}(\mathfrak{A})/8)}{\sqrt{\text{card}(A)}} \sum_{y \in A} e(-Q(x,y)) \epsilon_y.
\]

We shall sometimes simply write \( \alpha.v \) for \( \rho_\mathfrak{A}(\alpha)v \), i.e. we consider \( \mathbb{C}[A] \) as \( \text{Mp}_2(\mathbb{Z}) \)-module via the action \( (\alpha, v) \mapsto \rho_\mathfrak{A}(\alpha)v \). For details of the theory of Weil representations attached to finite quadratic modules we refer the reader to [BS17], [Nob76], [NW76], [Sko16], [Str13a].

The kernel of the projection of \( \text{Mp}_2(\mathbb{Z}) \) onto its first coordinate is the subgroup generated by \((1, -1)\). It is easily checked that \( \rho_\mathfrak{A}((1, -1)) = \rho_\mathfrak{A}(S^*)^4 \) acts as multiplication by \( e(\text{sig}(\mathfrak{A})/2) \). This simple observation has two immediate consequences. First of all, the space of invariants \( \mathbb{C}[A]^{\text{Mp}_2(\mathbb{Z})} \), i.e. the subspace of elements \( v \) in \( \mathbb{C}[A] \) fixed by \( \text{Mp}_2(\mathbb{Z}) \), reduces to \( \{0\} \) unless \( \text{sig}(\mathfrak{A}) \) is even. Secondly, \( \rho_\mathfrak{A} \) descends to a representation of \( \text{SL}_2(\mathbb{Z}) \) if and only \( \text{sig}(\mathfrak{A}) \) is even. Note, that \( \text{sig}(\mathfrak{A}) \) is always even if the level of \( \mathfrak{A} \) is odd as follows from Milgram’s formula.

3. \textbf{Invariants}

Let \( \mathfrak{A} = (A, Q) \) be a finite quadratic module of level \( N \). We shall assume in this section that \( \text{sig}(\mathfrak{A}) \) is even. As we saw at the end of the last section the space of invariants is otherwise zero. The representation \( \rho_\mathfrak{A} \) then descends to a representation of \( \text{SL}_2(\mathbb{Z}) \) and, even more, factors through a representation of the finite group \( \Gamma(N) \backslash \text{SL}_2(\mathbb{Z}) \), i.e. of the group
\[
G_N := \text{SL}_2(\mathbb{Z}/N\mathbb{Z}).
\]
We will denote this representation also by $\rho_\mathfrak{A}$.

An easy closed and explicit formula for the dimension of $\mathbb{C}[A]^{G_N}$ is not known for general $\mathfrak{A}$. Of course, orthogonality of group characters yields

$$\dim \mathbb{C}[A]^{G_N} = 1 \cdot \text{card}(G_N) \sum_{g \in G_N} \text{tr}(\rho_\mathfrak{A}(g)).$$

While it is therefore in principle possible to compute the dimension of $\mathbb{C}[A]^{G_N}$, there are two obstructions in practice. First of all, the size of the sum on the right can become very large. More precisely, the number of conjugacy classes of $G_N$ is asymptotically equal to $N$ for increasing $N$ (see [Nob76, Tabelle 2]). Secondly, the straight-forward formulas for $\text{tr}(\rho_\mathfrak{A}(g))$ which follow from explicit formulas for the matrix coefficients of $\rho_\mathfrak{A}(g)$ involve trigonometric sums with about $\text{card}(A)^2$ many terms (see e.g. [Str13a, Theorem 6.4]).

The following proposition implies that we can compute the invariants or the dimension of the space of invariants “locally”, i.e. for every $p$-component of $A$ separately. For a given prime $p$, denote the $p$-subgroup of $A$ by $A_p$. It is quickly verified that $\mathfrak{A}_p := (A_p, Q|_{A_p})$ is again a finite quadratic module. Moreover, the decomposition $A = \bigoplus_{p | \text{card}(A)} A_p$ of $A$ as sum over its $p$-subgroups $A_p$ induces an orthogonal direct sum decomposition of $\mathfrak{A}$. We also decompose $G_N$ as a product

$$G \cong \prod_{p^{\nu} \| N} G_{p^{\nu}}$$

with $G_{p^{\nu}} := \text{SL}_2(\mathbb{Z}/p^{\nu}\mathbb{Z})$ via the Chinese remainder theorem. In this way $\bigotimes_{p | N} \mathbb{C}[A_p]$ becomes a $G_N$-module in the obvious way. For this, we note that the set of primes dividing $N$ is equal to the set of primes dividing $\text{card}(A)$.

**Proposition 3.1.** Let $A = \bigoplus_{p^{\nu} | N} A_p$ be the decomposition of $A$ as sum over its $p$-subgroups $A_p$. Then $\mathfrak{c}_{p^{\nu} \| N} \mapsto \otimes_p \mathfrak{c}_{a_p}$ defines via linear extension an isomorphism of $G$-modules

$$\mathbb{C}[A] \cong \bigotimes_{p^{\nu} | N} \mathbb{C}[A_p].$$

Under this isomorphism we have

$$\mathbb{C}[A]^{G_N} \cong \bigotimes_{p^{\nu} | N} \mathbb{C}[A_p]^{G_{p^{\nu}}}.$$  

**Remark 3.2.** The proposition implies in particular

$$\dim \mathbb{C}[A]^{G_N} = \prod_{p^{\nu} | N} \dim \mathbb{C}[A_p]^{G_{p^{\nu}}}.$$  

However, in [BS17] a much simpler formula is given, which expresses the traces of the Weil representations in terms of the natural invariants for the conjugacy classes of $\text{SL}_2(\mathbb{Z})$. 
Proof of Proposition 3.1. The given map clearly defines an isomorphism of complex vector spaces. That this map commutes with the action of $G_N$, where $G_N$ acts component-wise on the right-hand side, as described above, is easily checked using the formulas for the $S$ and $T$-action. It follows that

$$\text{tr}(g, \mathbb{C}[A]) = \prod_p \text{tr}(g_p, \mathbb{C}[A_p])$$

for all $g = \otimes_p g_p$ in $G_N$, which implies, in particular, the second statement via orthogonality of group characters. □

A natural problem is to determine the field or ring of definition of the space $\mathbb{C}[A]^{G_N}$. From the formulas defining $\rho_A$, it is clear that $\mathbb{C}[A]^{G_N}$ is defined over the cyclotomic field $K_N$. However, it turns out that the invariants are in fact defined over the field of rational numbers, as we shall see in a moment. This will allow us in Section 5 to compute a basis for $\mathbb{C}[A]^{G_N}$ by doing the computations in $\mathbb{F}_\ell[A]$ for suitable sufficiently large primes $\ell$.

**Theorem 3.3.** The space $\mathbb{C}[A]^{G_N}$ is defined over $\mathbb{Z}$.

For the proof we need some preparations.

**Lemma 3.4.** For any $g = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ in $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$ and $x$ in $A$, one has

$$\rho_A(g)e_x = \chi_A(d) e(bdQ(x)) e_{dx},$$

where $\chi_A(d) = \sigma_d(w)/w$ with $w = \sum_{x \in A} e(Q(x))$.

A careful analysis of $\chi_A$ yields

$$\chi_A(d) = \begin{cases} \frac{d}{\text{card}(A)} & \text{if card}(A) \text{ is odd}, \\ \frac{d}{\text{card}(A)} \left(\frac{-d}{d}\right)^s & \text{if card}(A) \text{ is even}, \end{cases}$$

where $s = \frac{\text{sig}(A)}{2}$ (see e.g. [Str13b, Lemma 3.9]). However, we shall not need this formula.

**Proof of Lemma 3.4.** Since $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & bd \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & bd \\ 0 & 1 \end{bmatrix} e_x = e(bdQ(x)) e_x$

it suffices to consider the action of $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$. For this we write

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = S^{-1} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} S \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} S \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and apply the formulas for the action of $S$ and $T$ to obtain

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} e_x = \gamma e_{dx},$$

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3We say that a subspace $V$ of $\mathbb{C}[A]$ is defined over the ring $R$ if it possesses a basis whose elements are in $R[A]$.

4For this one needs that $e(-\text{sig}(A)/8)/\sqrt{\text{card}(A)}$ is in $K_N$, which can be read off from Milgram’s formula.
where
\[ \gamma = \sigma_d(w)/w, \quad w = \sum_{x \in A} e(Q(x)). \]
(Here we used Milgram’s formula). This proves the lemma. \( \square \)

For any \( s \) in \((\mathbb{Z}/N\mathbb{Z})^\times\), let \( \sigma_s \) denote the automorphism of \( K_N \) which sends each \( N \)th root of unity \( z \) to \( z^N \). For any endomorphism \( f \) of \( \mathbb{C}[A] \) which leaves invariant \( K_N[A] \), say \( f \mathbf{e}_x = \sum_{y \in A} f(x, y) \mathbf{e}_y \) with \( f(x, y) \) in \( K_N \), we use \( \sigma_s(f) \) for the endomorphism of \( \mathbb{C}[A] \) such that
\[ \sigma_s(f) \mathbf{e}_x = \sum_{y \in A} \sigma_s(f(x, y)) \mathbf{e}_y. \]

Note that \( f \mapsto \sigma_s(f) \) defines an automorphism of the ring of endomorphisms of \( \mathbb{C}[A] \) which leave invariant \( K_N[A] \).

**Lemma 3.5.** For any \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) in \( G_N \), one has
\[ \sigma_s(\rho A (\begin{bmatrix} a & b \\ c & d \end{bmatrix})) = \rho A (\begin{bmatrix} a s^{-1} & s b \\ c & d \end{bmatrix}) \].

**Proof.** Both sides of the claimed identity are multiplicative in \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) (for this note that the map \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & s b \\ c & d \end{bmatrix} \) defines a automorphism of \( G_N \)). It suffices therefore to prove the claimed formula for the generators \( T \) and \( S \) of \( G_N \). For \( T \) the formula can be read off immediately from the formula for the action of \( T \). For \( S \) one has on the one hand side for any \( x \) in \( A \)
\[ \sigma_s(\rho A (S)) \mathbf{e}_x = \sigma_s(w) \sum_{y \in A} e(-sQ(x, y)) \mathbf{e}_y, \]
where \( w = e(-\text{sig}(A)/8)/\sqrt{\text{card}(A)} = \sum_{x \in A} e(-Q(x))/\text{card}(A) \). On the other hand, \( \begin{bmatrix} 0 & -s \\ s^{-1} & 0 \end{bmatrix} = S \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix} \), and hence, using Lemma 3.4,
\[ \rho A (\begin{bmatrix} 0 & s \\ s^{-1} & 0 \end{bmatrix}) \mathbf{e}_x = \chi A(s)w \sum_{y \in A} e(-Q(sx, y)) \mathbf{e}_y. \]
But \( \sigma_s(w)/w = \chi A(s) \), which implies the claimed formula. \( \square \)

**Proof of Theorem 3.3.** The \( G_N \)-invariant projection \( \mathcal{P} : \mathbb{C}[A] \to \mathbb{C}[A]^{G_N} \) is given by the formula
\[ \mathcal{P} = \frac{1}{\text{card}(G_N)} \sum_{g \in G_N} \rho A(g). \]
It suffices to show that, for any \( x \) in \( A \), we have \( \mathcal{P} \mathbf{e}_x = \sum_y \mathcal{P}(x, y) \mathbf{e}_y \) with rational numbers \( \mathcal{P}(x, y) \), in other words, that we have, for any \( s \) in \((\mathbb{Z}/N\mathbb{Z})^\times\) the identity \( \sigma_s(\mathcal{P}) = \mathcal{P} \). But this follows from Lemma 3.5 and the fact that \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a s^{-1} & s b \\ c & d \end{bmatrix} \) permutes the elements of \( G_N \). This proves the theorem. \( \square \)
4. The algorithm

In this section we explain our algorithm for computing a basis for the space of invariants. We then discuss various easy and natural improvements. We fix a finite quadratic module \( \mathcal{A} = (A, Q) \) of level \( N \), and assume that \( \operatorname{sig}(\mathcal{A}) \) is even (since otherwise the space of invariants of the associated Weil representation is trivial). The Weil representation \( \rho_\mathcal{A} \) is then a representation of \( G = \text{SL}_2(\mathbb{Z}) \), which factors even through a representation of \( G_N = \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \). Define

\[
\text{Iso}(\mathcal{A}) := \{ x \in A : Q(x) = 0 \},
\]

and, for \( v \in \mathbb{C}[A] \),

\[
\text{supp}(v) := \{ x \in A : v(x) \neq 0 \}.
\]

Note that, for any \( G \)-submodule \( M \) of \( \mathbb{C}[A] \), we have

\[
M^T := \{ v \in M : \rho_\mathcal{A}(T)v = v \} = \{ v \in M : \text{supp}(v) \subseteq \text{Iso}(\mathcal{A}) \}
\]

as follows immediately from the formula for the action of \( T \) in Section 2. Our algorithm is based on the following observation.

**Proposition 4.1.** Let \( M \) be a \( G \)-submodule of \( \mathbb{C}[A] \). Then

\[
M^G = (1 + \rho_\mathcal{A}(S) + \rho_\mathcal{A}(S)^2 + \rho_\mathcal{A}(S)^3) (M^T) \cap M^T.
\]

**Proof.** An element \( v \) of \( M \) is invariant under all of \( G \) if it is invariant under the generators \( T \) and \( S \) of \( G \), i.e. if it is contained in \( M^T \) and the set \( M^S \) of vectors invariant under \( \rho_\mathcal{A}(S) \). Since \( S^4 = 1 \) we have \( M^S = \text{Tr}_S(M) \), where

\[
\text{Tr}_S = 1 + \rho_\mathcal{A}(S) + \rho_\mathcal{A}(S)^2 + \rho_\mathcal{A}(S)^3.
\]

But \( M^G \subseteq M^T \), hence \( M^G = \text{Tr}_S(M^G) \subseteq \text{Tr}_S(M^T) \), and therefore

\[
M^G = M^G \cap M^T \subseteq \text{Tr}_S(M^T) \cap M^T.
\]

The proposition is now obvious. \( \square \)

The Proposition is quickly converted into a first version of our algorithm:

**Algorithm 4.1.** (Computing a basis for the space \( \mathbb{C}[A]^G \) of invariants)

1. Find the isotropic elements \( a_1, \ldots, a_m \) and the non-isotropic elements \( b_1, \ldots, b_n \) in \( A \).
2. Compute the \((m + n) \times m\) matrix \( H \) such that

\[
(L\epsilon_{a_1}, \ldots, L\epsilon_{a_m}) = (\epsilon_{a_1}, \ldots, \epsilon_{a_m}, \epsilon_{b_1}, \ldots, \epsilon_{b_n})H,
\]

where \( L = 1 + \rho_\mathcal{A}(S) + \rho_\mathcal{A}(S)^2 + \rho_\mathcal{A}(S)^3 \).
3. Let \( U \) and \( V \) be the matrices obtained by extraction the first \( m \) and the last \( n \) rows of \( H \), respectively.
4. Compute a basis \( \mathcal{Y} \) for the space of vectors \( x \) such that \( Vx = 0 \).
5. Return a basis \( \mathcal{X} \) for the space of all \( Ux \), where \( x \) runs through the basis \( \mathcal{Y} \).
For implementing this algorithm we need, first of all, to decide over which field $K$ we would like to do the computations. One possibility is to use floating point numbers to do a literal implementation using the field of complex numbers. However, the matrix coefficients of $\rho_N(S)$ with respect to the natural basis of $\mathbb{C}[A]$ are elements of the $N$th cyclotomic field $K_N$. Hence it is reasonable to the calculations over $K_N = \mathbb{Q}[x]/(\phi_N)$, where $\phi_N$ is the $N$th cyclotomic polynomial. Another choice for $K$ will be discussed in Section 5.

There are two easy improvements which can help to reduce the computing time. The first one is due to the following observation.

**Proposition 4.2.** The subspaces $\mathbb{C}[A]^+ \text{ and } \mathbb{C}[A]^-$ of even and odd functions are $G$-submodules of $\mathbb{C}[A]$. Let $\epsilon = (-1)^{\text{sig}(3)}/2$. Then $\mathbb{C}[A]^G = (\mathbb{C}[A]^\epsilon)^G$ and $(\mathbb{C}[A]^{\epsilon})^G = \{0\}$.

**Proof.** The first statement follows immediately from the observation that the map $\epsilon_a \rightarrow \epsilon_{-a}$ intertwines with the action of $S$ and $T$, and hence with the action of $G$, as is obvious from the formulas for the action of $S$ and $T$.

For the proof of the second statement we note that $S^2 \epsilon_a = \epsilon_{-a}$ which is again an immediate consequence of the formula for the action of $S$. In other words, any invariant $v$ satisfies $v(a) = (S^2v)(a) = \epsilon v(-a)$ for all $a$ in $A$. \qed

Let $\rho_N^\pm : G \rightarrow \text{GL}(\mathbb{C}[A]^\pm)$ afforded by the $G$-modules $\mathbb{C}[A]^\pm$. As we saw in the proof of the preceding proposition $S^2$ acts on $\mathbb{C}[A]^\epsilon$ ($\epsilon = (-1)^{\text{sig}(3)}/2$) as identity, i.e. $\rho_N^\epsilon(S^2) = 1$. Using this Propositions (4.1), (4.2) imply

$$\mathbb{C}[A]^G = \mathbb{C}[A]^\epsilon = \{v \in (1 + \rho_N^\epsilon(S)) (\mathbb{C}[A]^\epsilon) : \text{supp}(v) \subseteq \text{Iso}(\Omega)\}.$$ 

A basis for $\mathbb{C}[A]^\epsilon$ is obtained by replacing in the standard basis $\epsilon_a$ by $\epsilon_a' = \frac{1}{2} (\epsilon_a + \epsilon_{-a})$ and omitting all zeroes and all duplicated vectors. This leads to the following modified algorithm.

**Algorithm 4.2.** *(Modified algorithm for computing a basis for the space of invariants)*

1. As in Algorithm 4.1.

2. a Construct the basis $\epsilon_{a_i}'$, $\epsilon_{b_j}'$ ($1 \leq i \leq m'$, $1 \leq j \leq n'$) of $\mathbb{C}[A]^\epsilon$ obtained from the standard basis $\epsilon_{a_i}$, $\epsilon_{b_j}$ by (anti-)symmetrizing, suppressing zeroes and duplicates, and after possibly renumbering the $a_i$ and $b_j$.

2. b Compute the $(m' + n') \times n'$ matrix $H'$ such that

$$(L \epsilon_{a_1}', \ldots, L \epsilon_{a_{m'}}') = (\epsilon_{a_1}', \ldots, \epsilon_{a_{m'}}', \epsilon_{b_1}', \ldots, \epsilon_{b_{n'}}')H',$$

where $L = 1 + \rho_N^\epsilon(S)$.

3. As in Algorithm 4.1 with $H$, $m$, $n$ replaced by $H'$, $m'$, $n'$.

The dimension of $\mathbb{C}[A]^+$ equals $\frac{1}{2} (\text{card}(A) + \text{card}(A[2]))$, where $A[2]$ denotes the subgroup of elements annihilated by “multiplication by 2”. Note that $A[2] = \{0\}$ if card $(A)$ is odd. Therefore the size of $H'$ is about half of
the size of $H$ in Algorithm 4.1. Also note that $H'$ has entries in the totally real subfield $K_N^+$ of $K_N$. This implies that $\mathbb{C}[A]^G$ is in fact defined over $K_N^+$ and we can perform our computations over $K_N^+$ instead of $K_N$.

To implement the algorithm, we still need an explicit formula for the entries of the matrix $H' = (h_{ij})$, where $1 \leq i, j \leq m' + n'$. We just write $x_i = a_i$ for $1 \leq i \leq m'$ and $x_i = b_{i - m'}$ for $m' < i \leq m' + n'$ for the elements of $A$. By a straightforward calculation, we obtain

$$h_{ij} = f_i^{-1} \left< \rho_A(S) e_{x_j}^\epsilon + e_{x_i}^\epsilon, e_{x_i}^\epsilon \right>$$

$$= \frac{e(-\operatorname{sig}(A)/8)}{2f_i\sqrt{\operatorname{card}(A)}} (e(-Q(x_j, x_i)) + e(Q(x_j, x_i))) + \delta_{i,j},$$

where $f_i = \langle e_{x_i}^\epsilon, e_{x_i}^\epsilon \rangle$ and $\langle \cdot, \cdot \rangle$ denotes the standard hermitean inner product on $\mathbb{C}[A]$ (conjugate-linear in the second component), such that $\langle e_{x}^\epsilon, e_y^\epsilon \rangle = \delta_{x,y}$. Note that $f_i = \frac{1}{2}$ if $x_i \neq -x_i$ and $f_i = 1$, otherwise.

Given a finite quadratic module the exact value of quantity $\operatorname{sig}(\mathfrak{A})$ is not immediately clear. For finding the $\epsilon$ of the preceding proposition the following is helpful.

**Proposition 4.3.** For odd $\operatorname{card}(A)$ one has

$$(-1)^{\operatorname{sig}(\mathfrak{A})/2} = \left( \frac{-1}{\operatorname{card}(A)} \right).$$

**Proof.** Indeed, directly from the formula for the $S$-action we obtain $S^2 e_x = (-1)^{\operatorname{sig}(\mathfrak{A})/2} e_{-x}$. On the other hand $S^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and therefore we obtain by Lemma 3.4 that $S^2 e_x = \chi\mathfrak{A}(-1)e_{-x}$. For odd card $(A)$ it is then easy to deduce from the formula of the lemma for $\chi\mathfrak{A}$ that $\chi\mathfrak{A}(-1) = \left( \frac{-1}{\operatorname{card}(A)} \right)$ (see also the remark after Lemma 3.4).

The second possible improvement is the factorization into local components as explained in Proposition 3.1. We compute first the local components $\mathfrak{A}_p := (A_p, Q|_{A_p})$, and apply then Algorithm 4.2 to the finite quadratic modules $\mathfrak{A}_p$. If the number of different primes in card $(A)$ is large this reduces the run-time of our algorithm enormously. Indeed, the two bottle necks of our algorithm are the search for the isotropic elements in $A$ and the computation of the kernel of a matrix of size $m \times \operatorname{card}(A)$, where $m$ is the number of isotropic elements of $A$. If card $(A)$ contains more than two different primes, say card $(A) = p_1^{k_1} \cdots p_r^{k_r}$ with $r \geq 2$, then it takes $p_1^{k_1} \cdots p_r^{k_r}$ many search steps to find all isotropic elements in $A$, whereas an application of Proposition 3.1 allows us to dispense with $p_1^{k_1} + \cdots + p_r^{k_r}$ many search steps to find eventually all invariants of $A$. A similar comparison applies to the size of the matrices in our algorithm when run either directly on $A$ or else separately on the $p$-parts $A_{p_j}$. 

5. Reduction mod $\ell$

In this section we fix again a finite quadratic module $\mathfrak{A} = (A, Q)$ of level $N$. Let $\ell$ denote a prime such that $\ell \equiv 1 \mod N$. Then $Q_\ell$ contains the $N$th roots of unity, hence the $N$th cyclotomic field. Accordingly, we can consider $\rho_N$ as a representation of $G_N = \text{SL}_2(\mathbb{Z}/N\mathbb{Z})$ taking values in $\text{GL}(Q_\ell[A])$, and $Q_\ell[A]$ as $G_N$-module. From the formulas for the action of $S$ and $T$ on $Q_\ell[A]$ it is clear that $Z_\ell[A]$ is invariant under $G_N$, and that the $Z_\ell$-rank of $Z_\ell[A]^{G_N}$ equals the dimension of $\mathbb{C}[A]^{G_N}$.

For computing the rank of $Z_\ell[A]^{G_N}$ it is natural to consider the reduction modulo $\ell$ of $Z_\ell[A]$. More precisely, note that $\ell Z_\ell[A]$ is a $G_N$-submodule of $Z_\ell[A]$, so that we have the exact sequence of $G_N$-modules

$$0 \rightarrow \ell Z_\ell[A] \rightarrow Z_\ell[A] \rightarrow F_\ell[A] \rightarrow 0,$$

where $r$ denotes the reduction map $r(f) : a \mapsto f(a) + \ell Z_\ell$. Here the action of $G_N$ on $F_\ell[A] \cong Z_\ell[A]/\ell Z_\ell[A]$ is the one induced by the action on $Z_\ell[A]$.

**Theorem 5.1.** Suppose that $(N, \ell) \neq (2, 3)$. Then

$$\dim_{Q_\ell} Q_\ell[A]^{G_N} = \dim_{F_\ell} F_\ell[A]^{G_N}.$$

**Remark 5.2.** Numerical computed examples suggest that the theorem is also true for $N = 2$ and $\ell = 3$. However, we did not try to pursue this further.

**Proof of Theorem 5.1.** From the short exact sequence preceding the theorem we obtain the long exact sequence in cohomology

$$0 \rightarrow \ell Z_\ell[A]^{G_N} \rightarrow Z_\ell[A]^{G_N} \xrightarrow{r} F_\ell[A]^{G_N} \rightarrow H^1(G_N, \ell Z_\ell[A]) \rightarrow \ldots.$$

We shall show in a moment that the order of $G_N$ is a unit of $Z_\ell$. Hence, the cohomology group $H^1(G_N, \ell Z_\ell[A])$ is trivial [Bro82, Corollary 10.2]. It follows then that $F_\ell[A]^{G_N} \cong Z_\ell[A]^{G_N}/\ell Z_\ell[A]^{G_N}$. Since $Z_\ell[A]^{G_N}$ is free we conclude that $\dim_{F_\ell} F_\ell[A]^{G_N}$ equals the $Z_\ell$-rank of $Z_\ell[A]^{G_N}$, which implies the proposition.

For proving that $\text{card } (G)_N$ is not divisible by $\ell$, first note that $\ell \equiv 1 \mod N$ implies that $\ell > N$. Then, recall that the order of $G_N = \text{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is given by

$$\text{card } (G) = N^3 \prod_{p|N} \frac{p^2 - 1}{p^2}.$$

Hence, if $\ell | \text{card } (G)$, we have that there is a prime $p | N$, such that $\ell | p + 1$ or $\ell | p - 1$. However, $p - 1 < N < \ell$ and thus the only possibility is $\ell = p + 1$ and $N = p$. Since $\ell$ and $p$ are primes we conclude $N = 2$ and $\ell = 3$, which we excluded in the statement of the proposition.

The results on reduction modulo $\ell$ are not only interesting from a theoretical point of view. Our implementation profits tremendously from reduction modulo a suitable prime $\ell$ as it speeds up the calculation in practice. The
reason is that there are highly optimized libraries for computation with matrices over finite fields (and/or over the integers) available. For instance, in sage (which uses the linbox library default), computing the nullity of a random $200 \times 200$ matrix with entries in a cyclotomic field $\mathbb{Q}(\zeta_{11})$ takes about 4 seconds on our test machine, whereas computing the nullity of a $2000 \times 2000$ matrix over $\mathbb{F}_{23}$ takes about 600 milliseconds. This immediately speeds up the computation of the dimension of $\mathbb{C}[A]^G$ although it does not give a basis for $\mathbb{C}[A]^G$.

6. Tables

Tables 1 to 6 list the values $s = \text{sig}(A)$ and dimension $d = \dim \mathbb{C}[A]^{\text{SL}_2(\mathbb{Z})}$ for various $p$-modules $A = (A, Q)$, where $p = 2, 3, 5$. We use genus symbols for denoting isomorphism classes of finite quadratic modules (see [Sko16, BEF16]). In short, for a power $q$ of an odd prime $p$ and a nonzero integer $d$ the symbol $q^d$ stands for the quadratic module
\[
\left( \left( \mathbb{Z}/q\mathbb{Z} \right)^k, \frac{x_1^2 + \cdots + x_{k-1}^2 + ax_k^2}{q} \right),
\]
where $k = |d|$ and $a$ is an integer such that $\left( \frac{2a}{p} \right) = \text{sign}(d)$. For a 2-power $q = 2^e$, we have the following symbols: We write $q^d_a$ for the module
\[
\left( \left( \mathbb{Z}/q\mathbb{Z} \right)^k, \frac{x_1^2 + \cdots + x_{k-1}^2 + ax_k^2}{2q} \right),
\]
with $k = |d|$ and $\left( \frac{a}{2} \right) = \text{sign}(d)$. We normalize $a$ to be contained in the set $\{1, 3, 5, 7\}$ and if $q = 2$, we take $a \in \{1, 7\}$. Finally, we write $q^{+2k}$ for
\[
\left( \left( \mathbb{Z}/q\mathbb{Z} \right)^{2k}, \frac{x_1x_2 + \cdots + x_{k-1}x_k}{q} \right),
\]
and $q^{-2k}$ for
\[
\left( \left( \mathbb{Z}/q\mathbb{Z} \right)^{2k}, \frac{x_1x_2 + \cdots + x_{k-3}x_{k-2} + x_{k-1}^2 + 2x_{k-1}x_k + x_k^2}{q} \right).
\]
The concatenation of such symbols stands for the direct sum of the corresponding modules. For instance, $3^{-1}9^{+1}27^{-2}$ denotes the finite quadratic module
\[
\left( \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z} \times (\mathbb{Z}/27\mathbb{Z})^2, \frac{x^2}{3} - \frac{y^2}{9} + \frac{z^2 - w^2}{27} \right).
\]
It can be shown that every finite quadratic $p$-module is isomorphic to a module which can be described by such symbols, and that this description is essentially unique (up to some ambiguities for $p = 2$). For details of this we refer to [Sko16].
For the computations we used \([S+13]\), the additional package \([S+16]\) and our implementation of Algorithm 4.2, which is available as part of the package \([Ehl16]\).

\begin{table}
\centering
\caption{Dimension \(d = \dim_{\mathbb{C}} \mathbb{C}[A]^G\) for some 2-modules of even signature \(s\)}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\(A\) & \(s = 0\) & \(A\) & \(s = 4\) & \(A\) & \(s = 0\) \\
\hline
2+2 & 2 & 2-2 & 0 & 4+2 & 3 & 4-8 & 1191 & 0 \\
2+4 & 5 & 2-4 & 1 & 4+4 & 16 & 2_0 & 1 & 0 \\
2+6 & 15 & 2-6 & 7 & 4+6 & 141 & 2_2 & 0 & 2 \\
2+8 & 51 & 2-8 & 35 & 4+8 & 1711 & 2_4 & 2 & 0 \\
2+10 & 187 & 2-10 & 155 & 4-2 & 1 & 2_4 & 0 & 4 \\
2+12 & 715 & 2-12 & 651 & 4-4 & 6 & 2_6 & 0 & 6 \\
2+14 & 2795 & 2-14 & 2667 & 4-6 & 73 & 2_6 & 5 & 0 \\
\hline
\end{tabular}
\end{table}

\begin{table}
\centering
\caption{Dimension \(d = \dim_{\mathbb{C}} \mathbb{C}[A]^G\) for some 2-modules of even signature \(s\)}
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\(A\) & \(s = 0\) & \(A\) & \(s = 4\) & \(A\) & \(s = 2\) \\
\hline
2+2_{4+2} & 8 & 2+2_{8-2} & 1 & 2+4_{2+2} & 0 \\
2+2_{4+2} & 4 & 2+2_{8-2} & 0 & 2+4_{2+2} & 0 \\
2+4_{4+2} & 25 & 2+4_{8-2} & 7 & 4_2 & 1 \\
2+2_{8+2} & 11 & 2+4_{8-2} & 1 & 2+2_{4+4} & 4 \\
2+4_{4+2} & 11 & 4+2_{8-2} & 2 & 2+2_{4+4} & 4 \\
2+1_{4+28+1} & 4 & 4_{2}^{2-8} & 2 & 4_2^{2+8} & 3 \\
2+1_{4+28+1} & 4 & 2+2_{4-8} & 2 & 4_1^{3+16} & 1 \\
2+2_{8+2} & 6 & 2+2_{4-8} & 2 & 4_7^{1+16} & 1 \\
\hline
\end{tabular}
\end{table}
### Table 3. Dimension $d = \dim_{C}[A]^G$ for some 3-modules of signature $s$

| $A$ | $d$ | $A$ | $d$ | $A$ | $d$ | $A$ | $d$ |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $s = 6$ | | $s = 2$ | | $s = 0$ | | |
| $3^{+1}$ | 0 | $3^{-1}$ | 0 | $9^{+1}$ | 1 | $27^{+1}$ | 0 | 6 |
| $3^{-2}$ | 2 | $3^{+2}$ | 0 | $9^{+2}$ | 1 | $27^{+2}$ | 0 | 4 |
| $3^{+3}$ | 1 | $3^{-3}$ | 1 | $9^{+3}$ | 5 | $27^{+3}$ | 5 | 2 |
| $3^{-4}$ | 1 | $3^{+4}$ | 7 | $9^{+4}$ | 33 | $27^{+1}$ | 0 | 2 |
| $3^{+5}$ | 10 | $3^{-5}$ | 10 | $9^{+5}$ | 121 | $27^{-1}$ | 0 | 2 |
| $3^{-6}$ | 40 | $3^{+6}$ | 22 | $9^{-1}$ | 1 | $27^{-3}$ | 5 | 6 |
| $3^{+7}$ | 91 | $3^{-7}$ | 91 | $9^{-2}$ | 3 | $81^{+1}$ | 1 | 0 |
| $3^{-8}$ | 247 | $3^{+8}$ | 301 | $9^{-3}$ | 5 | $81^{+2}$ | 1 | 0 |
| $3^{+9}$ | 820 | $3^{-9}$ | 820 | $9^{-4}$ | 11 | $81^{-1}$ | 1 | 0 |
| $3^{-10}$ | 2542 | $3^{+10}$ | 2380 | $9^{-5}$ | 121 | $81^{-2}$ | 5 | 0 |

### Table 4. Dimension $d = \dim_{C}[A]^G$ for some 3-modules of signature $s$

| $A$ | $d$ | $A$ | $d$ | $A$ | $d$ | $A$ | $d$ |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $s = 6$ | | $s = 2$ | | $s = 6$ | | $s = 2$ | |
| $3^{+1}27^{-1}$ | 2 | $3^{+1}27^{-1}$ | 0 | $3^{+1}243^{-1}$ | 2 | $3^{+1}243^{-1}$ | 0 |
| $3^{-2}27^{-1}$ | 1 | $3^{-2}27^{-1}$ | 1 | $3^{-2}243^{-1}$ | 1 | $3^{-2}243^{-1}$ | 1 |
| $3^{+3}27^{-1}$ | 1 | $3^{+3}27^{-1}$ | 7 | $3^{+3}243^{-1}$ | 1 | $3^{+3}243^{-1}$ | 7 |
| $3^{-4}27^{-1}$ | 10 | $3^{-4}27^{-1}$ | 10 | $3^{-4}243^{-1}$ | 10 | $3^{-4}243^{-1}$ | 10 |
| $3^{+5}27^{-1}$ | 40 | $3^{+5}27^{-1}$ | 22 | $3^{+5}243^{-1}$ | 40 | $3^{+5}243^{-1}$ | 40 |
| $3^{-1}27^{-1}$ | 2 | $3^{-1}27^{-1}$ | 0 | $3^{-1}243^{-1}$ | 2 | $3^{-1}243^{-1}$ | 0 |
| $3^{+2}27^{-1}$ | 1 | $3^{+2}27^{-1}$ | 1 | $3^{+2}243^{-1}$ | 1 | $3^{+2}243^{-1}$ | 1 |
| $3^{-3}27^{-1}$ | 1 | $3^{-3}27^{-1}$ | 7 | $3^{-3}243^{-1}$ | 1 | $3^{-3}243^{-1}$ | 7 |
| $3^{+3}27^{-1}$ | 10 | $3^{+3}27^{-1}$ | 10 | $3^{+3}243^{-1}$ | 10 | $3^{+3}243^{-1}$ | 10 |
| $3^{-4}27^{-1}$ | 40 | $3^{-4}27^{-1}$ | 22 | $3^{-4}243^{-1}$ | 40 | $3^{-4}243^{-1}$ | 40 |
Table 5. Dimension $d = \dim_C \mathbb{C}[A]^G$ for some 5-modules of signature $s$

| \(A\) | \(d\) | \(A\) | \(d\) | \(A\) | \(d\) | \(A\) | \(d\) |
|---|---|---|---|---|---|---|---|
| 5^{+1} | 0 | 5^{-1} | 0 | 25^{+1} | 1 | 125^{+1} | 0 |
| 5^{-2} | 0 | 5^{+2} | 2 | 25^{+2} | 3 | 125^{+2} | 4 |
| 5^{+3} | 1 | 5^{-3} | 1 | 25^{+3} | 7 | 125^{-1} | 0 |
| 5^{-4} | 1 | 5^{+4} | 11 | 25^{-1} | 1 | 125^{-2} | 0 |
| 5^{+5} | 26 | 5^{-5} | 26 | 25^{-2} | 1 |   |   |
| 5^{+6} | 106 | 5^{-6} | 156 | 25^{-3} | 7 |   |   |
| 5^{+7} | 651 | 5^{-7} | 651 |   |   |   |   |

Table 6. Dimension $d = \dim_C \mathbb{C}[A]^G$ for some 5-modules of signature $s$

| \(A\) | \(d\) | \(A\) | \(d\) | \(A\) | \(d\) | \(A\) | \(d\) |
|---|---|---|---|---|---|---|---|
| 5^{+1}125^{-1} | 0 | 5^{-1}125^{+1} | 0 | 5^{+1}125^{+1} | 2 | 5^{-1}125^{-1} | 2 |
| 5^{-2}125^{-1} | 1 | 5^{+2}125^{+1} | 1 | 5^{-2}125^{+1} | 1 | 5^{+2}125^{-1} | 1 |
| 5^{+3}125^{-1} | 1 | 5^{-3}125^{+1} | 1 | 5^{+3}125^{+1} | 11 | 5^{-3}125^{-1} | 11 |
| 5^{-4}125^{-1} | 26 | 5^{+4}125^{+1} | 26 | 5^{-4}125^{+1} | 26 | 5^{+4}125^{-1} | 26 |
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E-mail address: stephan.ehlen@math.uni-koeln.de

University of Cologne, Mathematisches Institut, Weyertal 86-90, D-50931 Cologne, Germany

E-mail address: nils.skoruppa@gmail.de

Universität Siegen, Fachbereich Mathematik, Walter-Flex-Str. 3, D-57072 Siegen, Germany