A New Construction for LDPC Codes using Permutation Polynomials over Integer Rings

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Abstract

A new construction is proposed for low density parity check (LDPC) codes using quadratic permutation polynomials over finite integer rings. The associated graphs for the new codes have both algebraic and pseudo-random nature, and the new codes are quasi-cyclic. Graph isomorphisms and automorphisms are identified and used in an efficient search for good codes. Graphs with girth as large as 12 were found. Upper bounds on the minimum Hamming distance are found both analytically and algorithmically. The bounds indicate that the minimum distance grows with block length. Near-codewords are one of the causes for error floors in LDPC codes; the new construction provides a good framework for studying near-codewords in LDPC codes. Nine example codes are given, and computer simulation results show the excellent error performance of these codes. Finally, connections are made between this new LDPC construction and turbo codes using interleavers generated by quadratic permutation polynomials.

Index Terms

LDPC codes, interleaver, quadratic, permutation polynomial, algebraic, graph, isomorphism, automorphism, near-codeword.
I. Introduction

This paper addresses a new construction for the class of capacity-approaching low density parity check (LDPC) codes invented by Gallager [1]. However, we start by introducing a result in turbo coding [2], which motivated this work. Although the focus is on LDPC codes, we will establish a bridge between turbo code and LDPC code constructions using quadratic permutation polynomials (QPP) over finite integer rings. The algebraic structure and excellent error performance of the new construction are investigated.

Turbo codes using interleavers generated by permutation polynomials over finite integer rings [2] have proven to be both practical and yield excellent error performances [3]–[5]. Moreover, the construction allows a high degree of parallel processing of turbo decoding without memory access contention [3] during the flow of extrinsic information. To the best of our knowledge, this is the only known class of algebraic interleavers with a very simple description producing turbo codes with error rate performance meeting or exceeding all other algebraic and pseudo-random constructions at practical error rates [3] for a wide range of block lengths (256–4096-bit interleavers) such as in the 3GPP standard [6]. We believe that the excellent error rate performance is due to two main features of the interleaver: a “pseudo-randomness” property obtained by a non-linear nature [7] of permutation polynomials of degree two or larger; and, an algebraic structure allowing designs matched to the constituent convolutional codes [2]. The new construction for LDPC codes using permutation polynomials in this paper also enjoys this coexistence of pseudo-randomness and algebraic structure.

LDPC codes, like turbo codes, were initially designed with random constructions [8]–[11]. A “randomness” was perceived as an important feature for both types of codes. However, requirements such as ease of implementation, large girth for the associated graph, and better error performance quickly spawned other construction methods. Many good algebraic, combinatoric, and geometric constructions for LDPC codes have been proposed in the literature [12]–[16]. Additionally, good algorithmic constructions have also been proposed [17]. Most constructions focus on the maximization of the girth of the associated graph and employ computer simulations for validation. We follow the same path in this paper.

Recently, some attention has been given to the error floor of LDPC codes [18], [19]. As an example, the (2640, 1320) Margulis code [13] has a floor at a frame error rate (FER) of $10^{-6}$. The two main causes of the floors in LDPC codes are known to be a small minimum distance and low-weight near-codewords. Graph symmetries and automorphisms are key properties being used to investigate error floors in LDPC codes [18] below the reach of Monte Carlo simulations. For the (2640, 1320) Margulis code, rather than a poor minimum distance\(^1\), the main cause for its floor has been identified as low-weight near-codewords of the type (12, 4), i.e., near-codewords of weight 12 and syndrome weight 4. In our simulations, example codes for the new construction are given with no apparent error floors down to FER's close to $10^{-7}$. This does not mean the new codes do not have near-codewords; on the contrary, we have identified near-codewords in one of our example codes. However, their simple algebraic structure allows an easy identification of graph automorphisms and may be a valuable framework for the understanding of error floors in LDPC codes.

The existence of graph automorphisms of our new construction also implies that permutation polynomial-based LDPC codes are quasi-cyclic [21]. However, the new construction does not necessarily generate codes that are equivalent to the codes in [21]. It is known that the parity check matrices of quasi-cyclic codes can be written as adjoined circulant square matrices up to a code equivalence. A recent work [16] defines two subclasses of quasi-cyclic codes. Type I and II codes have equivalent parity check matrices whose circulant sub-matrices have row/column weights 0,1 and 0,1,2 respectively. The new construction generates codes of both types. Codes of type II were shown in [16] to have better minimum distance upper bounds than codes of type I, which is a superclass of the codes in [21]. We further observe a generalization of the result in [16] showing that codes generated by the new construction have upper bounds on the minimum distance potentially growing with the block length. This happens because the so-called “weight matrices” for the circulant sub-matrices of the parity check matrix have larger sizes (and consequently

\(^1\)However, the minimum distance of this code is also not impressive because there are codewords of Hamming weight 40 for this code [20].
smaller circulant sub-matrices) than the previously known constructions. Larger minimum distances are also confirmed by using the nearest nonzero codeword search (NNCS) method, which finds true codewords of low weight (with a good likelihood of yielding a codeword of lowest non-zero Hamming weight) in LDPC codes [20].

This paper is organized as follows. In section II, we define the new LDPC construction and review a result for quadratic permutation polynomials [2], [4] over the finite integer ring \(\mathbb{Z}_N\). The main results are derived in section III, and examples and computer simulation results are given in section IV. Finally, conclusions are discussed in section V.

II. LDPC Construction

Let \(G\) be a \((\lambda, \rho)\) regular bipartite graph. The graph \(G\) consists of \(n\) variable nodes \(\Lambda = \{v_0, v_1, \ldots, v_{n-1}\}\) with degree \(\lambda\) and \(r\) check nodes \(\Gamma = \{c_0, c_1, \ldots, c_{r-1}\}\) with degree \(\rho\). There are \(N = n\lambda = r\rho\) edges \(\Xi = \{e_0, e_1, \ldots, e_{N-1}\}\) in \(G\). Each edge \(e_i, 0 \leq i < N\) has a left-label \(i\) and right-label \(f(i)\), where \(f(\cdot)\) is a permutation on \(\{0, 1, 2, \ldots, N - 1\}\). Naturally, if the right-label of an edge \(e_i\) is \(0 \leq j < N\) then the left-label is \(i = f^{-1}(j) = g(j)\). i.e., \(g(\cdot)\) is the inverse permutation function of \(f(\cdot)\). Each variable node \(v_m, 0 \leq m < n\) is connected to \(\lambda\) edges whose left-labels are in the set \(\lambda_m = \{m\lambda, m\lambda + 1, \ldots, m\lambda + (\lambda - 1)\}\). Each check node \(c_m, 0 \leq m < r\) is connected to \(\rho\) edges whose right-labels are in the set \(\rho_m = \{m\rho, m\rho + 1, \ldots, m\rho + (\rho - 1)\}\). Thus every regular \((\lambda, \rho)\) graph \(G\) with \(N\) edges is completely and uniquely (up to a graph isomorphism) defined by a permutation \(f(\cdot)\) on \(\{0, 1, \ldots, N - 1\}\). In this paper, we investigate an LDPC construction when \(f(\cdot)\) is a quadratic permutation polynomial over integer rings [2], [4].

In this paper, let the set of primes be \(P = \{p_2 = 2, p_3 = 3, p_5 = 5, \ldots\}\). Then an integer \(N\) can be factored as \(N = \prod_{p_i \in P} p_i^{n_{N,i}}\), where \(p_i\)'s are distinct primes, \(n_{N,i} \geq 1\) for a finite number of \(i\) and \(n_{N,i} = 0\) otherwise. The necessary and sufficient condition for a quadratic polynomial \(f(x)\) to be a permutation polynomial is given in the following proposition.

**Proposition 1**: [4] [2] Let \(N = \prod_{p_i \in P} p_i^{n_{N,i}}\). The necessary and sufficient condition for a quadratic polynomial \(f(x) = f_1x + f_2x^2 \pmod{N}\) to be a permutation polynomial can be divided into two cases.

1) Either \(2 \nmid N\) or \(4 \nmid N\) (i.e., \(n_{N,2} \neq 1\))
   \[\gcd(f_1, N) = 1\] and \(f_2 = \prod_{p_i \in P} p_i^{n_{f_2,i}}, n_{f_2,i} \geq 1, \forall i\) such that \(n_{N,i} \geq 1\).

2) \(2 \mid N\) and \(4 \nmid N\) (i.e., \(n_{N,2} = 1\))
   \[f_1 + f_2\] is odd, \(\gcd(f_1, \frac{N}{2}) = 1\) and \(f_2 = \prod_{p_i \in P} p_i^{n_{f_2,i}}, n_{f_2,i} \geq 1, \forall i\) such that \(p_i \neq 2\) and \(n_{N,i} \geq 1\).

III. Design of Good LDPC Codes

We propose an efficient search for LDPC graphs with large girth by avoiding inspection of isomorphic graphs. Additionally, we only check the girth of a graph by computing the local girth starting from vertices that belong to different equivalence classes under a graph automorphism.

A. Isomorphic Graphs

The following two propositions identify quadratic permutation polynomials generating LDPC codes with isomorphic graphs.

**Proposition 2**: The graphs generated by \(f(x) = f_1x + f_2x^2\) and \(f'(x) = m\rho + f_1x + f_2x^2\), where \(m\) is any integer are isomorphic.

**Proof**: This is readily seen by the definition of the construction and it simply corresponds to a difference of a constant \(m\) modulo \(N/\rho\) in the indices \(c_i\) and \(c'_i, 0 \leq i < r\) of the check nodes of the two graphs.

**Proposition 3**: The graphs generated by \(f(x) = f_1x + f_2x^2\) and \(f'(x) = (f_1 + 2m\alpha f_2)x + f_2x^2\), where \(m\) is any integer and \(\alpha = \text{lcm}(\lambda, \rho)\) are isomorphic.
Proof: First we observe that a graph induced by \( f(x + m\alpha) \) is clearly isomorphic to the one induced by \( f(x) \) because it just corresponds to a different relabeling of the variable nodes. Developing \( f(x + m\alpha) \) we obtain

\[
    f(x + m\alpha) = f_1(x + m\alpha) + f_2(x + m\alpha)^2 = f_1x + f_2x^2 + 2m\alpha f_2x + \alpha m f_1 + \alpha^2 m^2 f_2
\]

and the proposition follows from Proposition 2. Proposition 3 implies that in the search for good QPP’s, the range of search for \( f_1 \) can be set from 1 to \( 2f_2\alpha \). This means a small \( f_2 \) may be advantageous because it reduces the search space for \( f_1 \). Proposition 3 can also be interpreted as a constrained design rule suggested in [2] for turbo codes. Conversely, the search range proposed therein can be interpreted as a special case of Proposition 3 in which \( \alpha = 1 \).

B. Automorphic Graphs

The nature of permutation polynomials makes the graph to have automorphisms. Hence the determination of the girth of the graph by exhaustive search is simplified by only examining trees starting from vertices in \( G \) that belong to different equivalence classes under graph automorphisms. We prove next a theorem showing a graph automorphism with the help of two lemmas.

Lemma 1: Let \( u = \gcd(2f_2, N) \). Then the set of edge left-labels

\[
    \theta_i = \left\{ i, i + \frac{N}{u}, i + \frac{2N}{u}, \ldots, i + \frac{(u-1)N}{u} \right\}
\]

and \( |\theta_i| = u \) for all \( i \) forms an equivalence class under the difference of mapped of right-labels.

Proof: This is seen by observing that

\[
    f(x + \gamma) - f(x) \equiv 2f_2x\gamma + f(\gamma) \pmod{N}
\]

and finding the solutions for

\[
    2f_2x \equiv 0 \pmod{N}.
\]

Lemma 2: Let \( t = \text{lcm}(N/u, \lambda)/\lambda \). Then the set variable nodes

\[
    \left\{ v_i, v_{i+t}, v_{i+2t}, \ldots, v_{i+(N/(t\lambda))-1} \right\}
\]

for all \( i \) forms an equivalence class under the difference of mapped right-label edges connected to them.

Proof: This is a direct consequence of Lemma 1.

Theorem 1: Let \( \beta = mt \) such that \( m \) is the smallest positive integer that makes \( \rho | f(mt) \). Then the set of variable nodes

\[
    \left\{ v_i, v_{i+\beta}, v_{i+2\beta}, \ldots, v_{i+(n/\beta)-1} \right\}
\]

for all \( i \) forms an equivalence class under graph automorphisms.

Proof: The theorem follows from Lemma 2.

Corollary 1: Let \( \gamma = \frac{\beta\lambda}{\rho} \). Then the set of check nodes

\[
    \left\{ c_i, c_{i+\gamma}, c_{i+2\gamma}, \ldots, c_{i+(r/\gamma)-1} \right\}
\]

for all \( i \) forms an equivalence class under graph automorphisms.

Proof: This follows from Theorem 1 and the natural periodicity of the variable and check node labels induced by the permutation polynomial.

Theorem 1 also leads to an intuitive design rule: by minimizing the number of equivalent classes, the code may look less uniform and more random. This is achieved by selecting \( f_2 \) as small as possible.
Following Proposition 1 we may select $f_2$ to be the product of every prime factor of $N$ repeated exactly once, say this number is $f_{2\text{min}}$. Moreover, increasing $f_2$ with more factors of $N$ also means $f_2$ approaches $N$, eventually becoming a multiple of $N$ and hence zero modulo $N$. The QPP then reduces to a linear permutation polynomial [22] losing the “randomness” that the non-linear quadratic polynomial provides. Our search for good coefficients confirmed this trend where the girth of the resulting graph increases with $f_2$ but only up to a certain point. Often, $f_{2\text{min}}$ is the value that maximizes the girth of the corresponding graph. A similar fact is described in [2] for turbo codes establishing a close tie between LDPC and turbo codes based on permutation polynomials. The main difference between the constructions of turbo codes and LDPC codes with the permutation polynomials approach is in the constraints: cycle-lengths of the constituent codes in turbo codes [2] and the degree distribution in LDPC codes.

C. Quasi-Cyclic Representation

Theorem 1 naturally implies the new codes are quasi-cyclic whose shifting constraint [23, p. 185] is $\beta$. Quasi-cyclic codes can have their generator and parity check matrices represented by circulant sub-matrices. Quasi-cyclic constructions are interesting because encoding can be performed by shift-registers [24, pp. 256–261]. Quasi-cyclic LDPC codes from circulant permutation matrices have some important limiting factors; the girth of the graph is at most 12 [21] and the minimum Hamming distance of the LDPC code is upper bounded by $(\lambda + 1)!$ [25]. These ideas have been recently generalized [16] when circulant sub-matrices are allowed to have row/column weights 0 and 1 (type I constraint) and row/column weights 0, 1 and 2 (type II constraint). Our construction generates both type I and II codes but are not equivalent to the construction in [21], which are of type I with circulant matrices of weight 1 only. The non-equivalence can be demonstrated by a counter example. The codes in [21] always generate rank deficient parity check matrices whereas we have observed that in general our construction yields full rank parity check matrices (examples are given in Section IV). We show next an example code that has constraints of type II.

The example code II in Table II is (3,6)-regular with size $(1008, 504)$. The plot of its parity check matrix is shown in Figure 1. A dark dot represents a 1 and 0 otherwise.

The matrix apparently lacks regularity. This is the “randomness” introduced by the quadratic permutation polynomial. However, the parity check matrix can be rearranged by grouping columns and rows according to their corresponding variable node and check node equivalence classes using Theorem I and Corollary II. Let the parity check matrix be $H = [x_0 x_1 \ldots x_{n-1}]$ where $x_i$’s are column vectors. Then define

$$H' = [x_0 x_0 + \beta \cdots x_0 + (n/\beta - 1)\beta, x_1 x_1 + \beta \cdots x_1 + (n/\beta - 1)\beta \cdots x_1 x_0 + (n/\beta - 1)\beta, \ldots]$$

$^2$To simplify the explanation, we are assuming only case 1) in Proposition I however, the procedure is easily generalized for case 2).
Let now $H' = [y_0, y_1, \cdots, y_{r-1}]^T$, where the $y_i$’s are column vectors and $T$ denotes matrix transposition. Then define 

$$H'' = [y_0 y_0 + \gamma \cdots y_0 + (n/\gamma - 1) \gamma, y_1 y_1 + \gamma \cdots y_1 + (n/\gamma - 1) \gamma \cdots y_{\gamma - 1} y_{\gamma - 1} + \gamma \cdots y_{\gamma - 1} + (n/\gamma - 1) \gamma]^T$$

These simple transformations of the parity check matrix result in a new matrix consisting of circulant sub-matrices of size $r/\gamma \times n/\beta$. In our example, the circulant sub-matrices are of size $84 \times 84$ and the resulting parity check matrix is shown in Fig. 2.

![Fig. 2. Parity check matrix for the (3,6)-regular (1008, 504) example code II in circulant form.](image)

This means that the new code is quite structured after all. More precisely, it can be said that the “pseudo-randomness” introduced by the non-linearity of the second degree permutation polynomial has been factored. We will see next where the “pseudo-randomness” gets condensed. The weight of the rows of the circulant sub-matrices in this example are 0, 1 and 2. This is depicted by a weight matrix $A$ in (1).

$$A = \begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

(1)

Observing matrix $A$, the “pseudo-randomness” is condensed in the arrangement of the circulant sub-matrices of $A$. Further, matrix $A$ has a more general form than the matrices obtained in [16], [25]. Both work demonstrate an upper bound on the minimum distance assuming that the number of rows in $A$ is $\lambda$. The recent work by Smarandache and Vontobel [16] shows that quasi-cyclic constructions of type II have larger minimum distance upper bounds than the ones for type I (a similar result for high-rate quasi-cyclic LDPC codes is reported in [26]). The upper bound [16, Theorem 2] and an example code are provided for $\lambda = 3$ achieving the upper bound of $d_{min} \leq 32$. Their theorem is stated as follows:

**Theorem 2 ([16]):** Let $C$ be a quasi-cyclic code with a $\lambda \times \rho$ weight matrix $A$.

$$d_{min} \leq \min_{S \subseteq \{1, \ldots, \rho\}} \sum_{S' \subseteq S} \sum_{\sigma \in \Pi} a_{\sigma(1),i_1} \cdots a_{\sigma(\lambda),i_\lambda}$$

where $\Pi$ is the set of all permutations of $\{1, \ldots, \lambda\}$ and $a_{x,y} \quad x = 1, 2, \ldots, \lambda \quad y = 1, 2, \ldots, \rho$ are the entries of $A$.

MacKay’s Theorem 2 in [25] can be interpreted as a special case of the previous theorem when all entries in $A$ are equal to 1. The important point in Theorem 2 is that by allowing different weights for the circulant matrices, a larger upper bound is obtained. We make another straightforward generalization, which improves the upper bound on the minimum distance even when the codes are of type I. By allowing
A to have a number of rows larger than λ (and columns larger than ρ) to accommodate our weight matrices, we obtain the following theorem.

**Theorem 3:** Let C be a quasi-cyclic code with a $s\lambda \times s\rho$ weight matrix A. Let $S \subseteq \{1, \ldots, s\lambda + 1\}$ and

$$\psi(S) = \sum_{S' \subseteq S} \sum_{s' \in S'} a_{s(1),i_1} \cdots a_{s(s\lambda),i_{s\lambda}}$$

where $\Pi$ is the set of all permutations of $\{1, \ldots, s\lambda\}$. Then

$$d_{\text{min}} \leq \min_{S \subseteq \{1, \ldots, s\rho\} |S| = s\lambda + 1, \psi(S) \neq 0} \psi(S)$$

**Proof:** This follows directly from Theorem 2 in [25] and Theorem 2 in [16]. However, it is important to exclude $\psi(S) = 0$, otherwise the upper bound easily incorrectly evaluates to zero for the new construction. This is because in the new construction, the weight matrix A is itself of low density as opposed to the ones in [16], [25] which are dense\(^3\). The event $\psi(S) = 0$ happens when the set $S = \{j_1, j_2, \ldots, j_{s\lambda+1}\}$ defines a sub-matrix $A(S) = [b_{j_1} b_{j_2} \ldots b_{j_{s\lambda+1}}]$ containing a row of zeros where $b_{j_i}$ is the $j_i$th column of A.

Applying Theorem 3 on matrix A, we obtain $d_{\text{min}} \leq 62$ for the new (1008, 504) code II, which is greater than the upper bound of 24 for all type I codes with $\lambda = 3$. It is also greater than the upper bound of 32 for type II codes with a dense weight matrix in [16].

Tighter bounds may be obtained by recursively applying Theorem 3 on the sub-matrices corresponding to the events $\psi(S) = 0$ but with the all-zeros row removed. As an example, in the previous matrix A, if $S = \{1, 2, 3, 4, 5, 6\}$ then $\psi(S) = 0$ because the corresponding matrix has the third row from the top all-zero. We can thus apply Theorem 3 on the matrix

\[
A' = \begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 1 & 2 & 1 & 0 \\
1 & 2 & 1 & 0 & 1 & 1
\end{bmatrix}
\]

However, the bound was not improved for this code.

Another upper bound on the minimum distance for code II was computed by using the nearest nonzero codeword search (NNCS) method in [20], which is likely to yield the nonzero codeword of lowest Hamming weight. Codewords of weight 44 have been found. Therefore Theorem 3 gives a loose upper bound for this code. Moreover, no codewords of low weight have been found by extensive computer simulations in Section IV via undetected errors. In summary, the combination of circulant matrices of weights 0, 1, and 2 together with a low density weight matrix gives good minimum distances for the new code construction. Another interesting example is code III. Its weight matrix is of type I, therefore the new construction generates both type I and II codes. Its weight matrix is of size $64 \times 128$, therefore the explicit computation of the minimum distance upper bound in Theorem 3 becomes complex, at least by a brute force. However, the NNCS method is still manageable giving an upper bound on the minimum distance of 78. We conjecture the minimum distance of the new construction grows linearly with the block length. This fact is supported by the upper bounds on the minimum distance computed for codes of several lengths as shown in Table I. We also observe that the tightness of the upper bound in Theorem 3 with the NNCS method gets looser with the block length. Finally, even the NNCS method has its limitations in terms of complexity. A full-run of the algorithm using a “two position bit reverse” is very costly for

\(^3\)Theorem 2 in [16] must also exclude the instances of $\psi(S) = 0$ to be strictly correct.
code V. The upper bound of 248 is only for a limited search with some optimization by exploiting the automorphism of the code.

D. A Search Procedure for Good Coefficients

We give an outline of an efficient search procedure for graphs with large girth using Propositions 3 and Theorem 1.

1) Choose the degree distribution \((\lambda, \rho)\)
2) Choose the code size \((n, k)\) and set the interleaver length \(N = n\lambda\)
3) Set \(f_2\) to be the product of every prime factor of \(N\) repeated exactly once
4) Search the \(f_1\) that maximizes the girth of the corresponding graph using Propositions 3 and Theorem 1
5) Try a larger \(f_2\) by including more factors of \(N\) and repeat the previous step. If no improvement in girth is noted, proceed to the next step otherwise repeat this step
6) Compute the rank of the parity check matrix to find the true dimension of the code

The search procedure quickly generates on a regular personal computer all codes in Table I.

IV. Example Codes and Simulation Results

Nine new QPP-based LDPC codes are listed in Table I. Most of the entries are self-explanatory. The columns \(d_{\min}\) and \(d_{T3}^{\min}\) represent upper bounds on the minimum distance computed using the NNCS method in [20] and Theorem 3 respectively. The only code that has rate not equal to exactly 1/2 is code VIII because its parity check matrix is rank deficient. We simulated codes II, IV and VI using BPSK modulation under an additive white Gaussian noise (AWGN) channel. We first simulated code II, which is (3,6)-regular and has size \((1008, 504)\). It was compared with a girth-8 progressive edge growth (PEG) code [17]. For a fair comparison with the curve in [17], we used the same number of 80 belief-propagation (BP) decoding iterations. At least 50 frame errors were counted per simulated point in all of our simulations unless otherwise noted. Simulation curves for bit error rate (BER) and frame error rate (FER) are shown in Figure 3. The new QPP code outperforms the PEG code at high signal-to-noise ratios (SNR’s).

| Code | \((\lambda, \rho)\) | \((n, k)\) | \(f(x)\) | \(N\) | Girth | \(\beta\) | \(d_{\min}\) | \(d_{T3}^{\min}\) |
|------|-------------------|------------|------------|------|-------|-------|------------|------------|
| I    | \((3, 6)\)        | \((504, 252)\) | \(5x + 210x^2\) | 1512 | 8     | 6     | 22         | 22         |
| II   | \((3, 6)\)        | \((1008, 504)\) | \(29x + 42x^2\) | 3024 | 8     | 12    | 44         | 62         |
| III  | \((3, 6)\)        | \((2048, 1024)\) | \(7x + 24x^2\) | 6144 | 8     | 128   | 78         | –          |
| IV   | \((3, 6)\)        | \((2432, 1216)\) | \(11x + 114x^2\) | 7296 | 10    | 32    | 90         | 344        |
| V    | \((3, 6)\)        | \((4096, 2048)\) | \(43x + 24x^2\) | 12288 | 10   | 256   | 248        | –          |
| VI   | \((3, 6)\)        | \((8192, 4096)\) | \(19x + 24x^2\) | 24576 | 10   | 512   | –          | –          |
| VII  | \((3, 6)\)        | \((16384, 8192)\) | \(7x + 24x^2\) | 49152 | 10   | 1024  | –          | –          |
| VII  | \((3, 6)\)        | \((32768, 16384)\) | \(7x + 48x^2\) | 98304 | 12   | 1024  | –          | –          |
| IX   | \((4, 8)\)        | \((1120, 562)\) | \(87x + 70x^2\) | 4480 | 8     | 8     | 40         | 96         |

* The true dimension has not been computed.

We next extensively simulated codes II, IV and VI using 200 BP decoding iterations. The BER and FER simulation curves are shown in Figure 4.

\(^4\)The inclusion of factors not present in \(N\) is in some cases effective. This was the case for code I in Table I.
A. Code II (3,6)-Regular (1008, 504)

For code II, we simulated over 168 million frames at 3.00dB and accounted for 77 detectable frame errors but no undetectable frame errors. We also found that among the detected frame errors there were 8 (10%) low-weight near-codewords of the type (12, 2), i.e., a near-codeword of weight 12 and syndrome of weight 2, respectively. Over 146 million frames were simulated at 3.10dB. Out of the 19 detectable frame errors, 8 (42%) were (12, 2) near-codewords. This is a significant increase in the percentage of low-weight near-codewords out of the detected frame errors compared with the results at the SNR of 3.00dB. This indicates an error floor to appear at an FER near $5 \times 10^{-8}$. Therefore, code II has a similar problem with near-codewords as the Margulis (2640, 1320) code [12] as reported in [19]. However, the floor is about one order of magnitude below that of the Margulis code.

B. Code IV (3,6)-Regular (2432, 1216)

Because code II has only half the length of the Margulis (2640, 1320) code studied in [19], code IV was simulated for a fair comparison. The results are very encouraging because there are no signs of an error floor down to an FER of $3 \times 10^{-7}$. Further, the following indicates that a floor must be much lower. Over 16 million frames were simulated at 2.25dB resulting in a total of 34 detected frame errors. Near-codeword and syndrome weights were mostly triple-digit except for 6 of them with the following weights: (16, 40), (47, 95), (80, 156), (83, 99), (84, 140), and (93, 125). Over 63 million frames were simulated at 2.35dB resulting in a total of 21 detected frame errors. Near-codeword and syndrome weights were mostly triple-digit except for 3 of them with the following weights: (45, 99), (84, 118), and (96, 128).
LDPC Code Simulations (200 maximum iterations)

Fig. 4. BER and FER curves for the QPP LDPC codes II, IV and VI using 200 BP iterations

C. Code VI (3,6)-Regular (8192,4096)

Code VI has a block size of approximately 8000 bits as all the codes compared at the Jet Propulsion Laboratory (JPL) in [27]. There are two (3,6)-regular codes compared in [27]: a random code and a PEG code. Both codes do not have structure and easy encoding and decoding methods. The PEG code outperforms the random code but the new code IV performs no worse than the PEG code. Further, encoding and decoding of the new code are simpler due to its algebraic structure. We observed no low-weight near-codewords at 1.70dB for over 3 million simulated frames and 28 detected frame errors ($FER = 8.9 \times 10^{-6}$). The number of frames simulated at 1.80dB was over 16 million with 3 detected frame errors ($FER = 1.7 \times 10^{-7}$). All detected frame errors at 1.70dB and 1.80dB had triple-digit near-codeword and syndrome weights.

V. CONCLUSIONS

We proposed a new construction for regular LDPC codes using quadratic permutation polynomials (QPP) over finite integer rings. It is one of the simplest known constructions and yet provides enough flexibility to generate a large family of good codes of practical interest. The new construction only requires the code size $(n,k)$, the degree distribution $(\lambda, \rho)$ of the associated bipartite graph with $N$ edges, and two integers representing the coefficients of a QPP $f(x) = f_1x + f_2x^2 \pmod{N}$ as the code defining parameters. The algebraic structure allows the identification of graph isomorphisms and automorphisms, which significantly simplify the search for good parameters for the coefficients of $f(x)$. The degree of $f(x)$ being two ensures a non-linearity in the graph structure, which we believe makes it as good as random
constructions with the additional advantages of an algebraic construction. LDPC codes with corresponding bipartite graphs with girth as large as 12 are easily obtained. Analytical and algorithmic upper bounds on the minimum distance were given and computed for the new codes. Although not formally proven, the new codes appear to have a minimum distance growing with the block length as opposed to a fixed small upper bound on the minimum distance of 24 for quasi-cyclic LDPC codes in [21]. Simulation results confirm that the new codes have excellent error rate performance. In particular, we have found neither undetected errors nor noticeable error floors down to frame error rates close to $10^{-7}$; therefore, the new QPP LDPC codes exceed the performance of other algebraic constructions such as the Margulis construction [12], [13]. We do expect an error floor around an FER of $10^{-8}$ for one of our codes caused by low-weight near-codewords. However, because the new codes have easily identifiable graph automorphisms, we believe a framework can be set for a further understanding of the important issue of error floors in LDPC codes due to low-weight near-codewords using the techniques in [18], [19].

We also narrowed the theoretical gap between LDPC codes and turbo codes under the unified method of permutation polynomial algebraic interleavers over integer rings. The designs of the permutation polynomials were very similar. The constraints for an LDPC code were the degree distribution, while for a turbo code the constraints were the cycle-length of the recursive constituent codes.

In this paper, only regular constructions have been demonstrated. However, an extension to irregular codes is possible by laying out vertex labels periodically according to their corresponding node degrees. This will be investigated in future work.

Finally, from the practical side, the new QPP LDPC codes have very attractive features. They can be encoded by shift-registers because they are quasi-cyclic, and a parity check matrix in the form of circulant sub-matrices is easily obtained. Moreover, the decoding allows a high degree of parallel processing without exhibiting memory access contention [3] caused by extrinsic information flow.

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REFERENCES

[1] R. G. Gallager, “Low density parity check codes,” IRE Trans. Inform. Theory, vol. IT-8, no. 1, pp. 21–28, Jan. 1962.
[2] J. Sun and O. Y. Takeshita, “Interleavers for turbo codes using permutation polynomials over integer rings,” IEEE Trans. on Inform. Theory, vol. 51, no. 1, pp. 101–119, Jan. 2005.
[3] O. Y. Takeshita, “On maximum contention-free interleavers and permutation polynomials over integer rings,” submitted to IEEE Trans. on Inform. Theory.
[4] J. Ryu and O. Y. Takeshita, “On quadratic inverses for quadratic permutation polynomials over integer rings,” submitted to IEEE Trans. on Inform. Theory.
[5] M. Cheng, M. Nakashima, J. Hamkins, B. Moision, and M. Barsoum, “A decoder architecture for high-speed free-space laser communications,” submitted to SPIE Remote Sensing Conference 2005.
[6] “The 3rd generation partnership project (3GPP),” http://www.3gpp.org.
[7] O. Y. Takeshita and D. J. Costello, Jr., “New deterministic interleaver designs for turbo codes,” IEEE Trans. on Inform. Theory, vol. 46, no. 6, pp. 1988–2006, Sept. 2000.
[8] D. J. C. MacKay, “Good error-correcting codes based on very sparse matrices,” IEEE Trans. on Inform. Theory, vol. 45, no. 2, pp. 399–431, Mar. 1999.
[9] T. J. Richardson, M. A. Shokrollahi, and R. L. Urbanke, “Design of capacity-approaching irregular low-density parity check codes,” IEEE Trans. Inform. Theory, vol. 47, no. 2, pp. 619–637, Feb. 2001.
[10] M. G. Luby, M. Mitzenmacher, M. A. Shokrollahi, and D. A. Spielman, “Improved low-density parity check codes using irregular graphs,” IEEE Trans. on Inform. Theory, vol. 47, no. 2, pp. 585–598, Feb. 2001.
[11] S. Chung, G. D. Forney, Jr. T. J. Richardson, and R. Urbanke, “On the design of low-density parity-check codes within 0.0045 dB of the Shannon limit,” IEEE Commun. Letters, vol. 5, no. 2, pp. 58–60, Feb. 2001.
[12] G. A. Margulis, “Explicit constructions of graphs without short cycles and low-density codes,” Combinatorica, vol. 2, no. 1, pp. 71–78, 1982.
[13] J. Rosenthal and P. Vontobel, “Constructions of LDPC codes using Ramanujan graphs and ideas from Margulis,” in Proc. 38th Annual Allerton Conference on Communication, Control, and Computing, Oct. 2000, pp. 248–257.
[14] Y. Kou, S. Lin, and M. Fossorier, “Low-density parity-check codes based on finite geometries: A rediscovery and new results,” *IEEE Trans. on Inform. Theory*, vol. 47, pp. 2711–2736, Nov. 2001.
[15] M. Fossorier, “Quasicyclic low density parity check codes,” in *Proc. 2003 IEEE International Symposium on Information Theory*, Yokohama, Japan, June 2003, p. 150.
[16] R. Smarandache and P. O. Vontobel, “On regular quasi-cyclic LDPC codes from binomials,” in *Proc. 2004 IEEE International Symposium on Information Theory*, June 2004, p. 274.
[17] X.-Y. Hu, E. Eleftheriou, and D. Arnold, “Progressive edge-growth Tanner graphs,” in *Proc. IEEE Globecom Conf.*, Nov. 2001, pp. 995–1001.
[18] T. J. Richardson, “Error floors of LDPC codes,” in *Proc. 41th Annual Allerton Conference on Communication, Control, and Computing*, Allerton, IL, Oct. 2003, pp. 1426–1435.
[19] D. J. C. MacKay and M. S. Postol, “Weaknesses of Margulis and Ramanujan-Margulis low-density parity-check codes,” *Electronic Notes in Theoretical Computer Science*, vol. 74, pp. 1–8, Oct. 2003.
[20] X.-Y. Hu, M. P. C. Fossorier, and E. Eleftheriou, “On the computation of the minimum distance of low-density parity-check codes,” in *Proc. IEEE International Conference on Communications*, June 2004, pp. 767–771.
[21] M. P. C. Fossorier, “Quasicyclic low-density parity-check codes from circulant permutation matrices,” *IEEE Trans. on Inform. Theory*, vol. 50, no. 8, pp. 1788–1793, Aug. 2004.
[22] O. Y. Takeshita and D. J. Costello, Jr., “On deterministic linear interleavers for turbo-codes,” in *Proc. 35th Annual Allerton Conference on Communication, Control, and Computing*, Allerton, IL, Sept. 1997, pp. 711–712.
[23] S. Lin and D. J. Costello, Jr., *Error Control Coding: Fundamentals and Applications*, Prentice-Hall, 204.
[24] W. W. Peterson and E. J. Weldon, Jr., *Error Correcting Codes*, MIT Press, 1972.
[25] D. J. C. MacKay and M. C. Davey, “Evaluation of Gallager codes for short block length and high rate applications,” in *Codes, Systems and Graphical Models*, B. Marcus and J. Rosenthal, Eds., vol. 123 of *IMA Volumes in Mathematics and its Applications*, pp. 113–130. Springer, New York, 2000.
[26] N. Kamiya, “Generators and minimum-weight codewords for a class of high-rate quasi-cyclic low-density parity-check codes,” Parma, Italy, Oct. 2004, pp. 540–545.
[27] K. Andrews, S. Dolinar, D. Divsalar, and J. Thorpe, “Design of low-density parity-check (LDPC) codes for deep-space applications,” IPN Progress Report 42-159, JPL, Nov. 2004.