ON THE AXISYMMETRIC FORCE-FREE PULSAR MAGNETOSPHERE

Dmitri A. Uzdensky
Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106; uzdensky@kitp.ucsb.edu

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ABSTRACT

We investigate the axisymmetric magnetosphere of an aligned rotating magnetic dipole surrounded by an ideal force-free plasma. We concentrate on the magnetic field structure around the point of intersection of the separatrix between the open- and closed-field line regions and the equatorial plane. We first study the case in which this intersection point is located at the light cylinder. We find that in this case the separatrix equilibrium condition implies that all the poloidal current must return to the pulsar in the open-field region, i.e., there should be no finite current carried by the separatrix/equator current sheet. We then perform an asymptotic analysis of the pulsar equation near the intersection point and find a unique, self-similar solution; however, a light surface inevitably emerges right outside the light cylinder. We then perform a similar analysis for the situation in which the intersection point lies somewhere inside the light cylinder, in which case a finite current flowing along the separatrix and the equator is allowed. We find a very simple behavior in this case, characterized by a 90° angle between the separatrix and the equator and by a finite vertical field in the closed-field region. Finally, we discuss the implications of our results for global numerical studies of pulsar magnetospheres.

Subject headings: magnetic fields — MHD — pulsars: general — stars: neutron

1. INTRODUCTION

The structure of the pulsar magnetosphere has been an active area of research for more that 30 years, starting with the pioneering work by Goldreich & Julian (1969). Despite the fact that real astrophysical radio pulsars are believed to be oblique rotating magnetic dipoles, much of the theoretical effort has been devoted to a significantly simpler case of an aligned rotating magnetic dipole (i.e., a “non-pulsing pulsar”), in which case the problem becomes stationary and axisymmetric. The situation becomes even more tractable if one also assumes the space around the pulsar to be filled with a plasma that, on the one hand, is dense enough to short-circuit the longitudinal electric fields, thus providing the basis for the ideal-magnetohydrodynamic (MHD) approximation, and, on the other hand, is at the same time tenuous enough for all the nonelectromagnetic forces to be negligible, thus enabling one to regard the plasma as force-free.

All these simplifications have led early on to the derivation, simultaneously by several researchers, of the main equation governing the structure of the magnetosphere, the so-called pulsar equation (Scharlemann & Wagoner 1973; Michel 1973b; Okamoto 1974), which is essentially a special-relativistic generalization of the well-known force-free Grad-Shafranov equation. This equation is a quasilinear, elliptic, second-order partial differential equation (PDE) with a regular singular surface, the so-called light cylinder (LC). Despite the fact that this, in general nonlinear, equation was first derived 30 years ago, most of the attempts to solve it, both analytical and numerical, have been limited, until very recently, to its linear special cases and to the region inside the LC (e.g., Scharlemann & Wagoner 1973; Michel 1973a; Beskin, Gurevich, & Istomin 1983; Beskin & Malyshkin 1998). These self-imposed restrictions can of course be attributed to the considerations of mathematical convenience and tractability; however, they are not very well physically motivated and thus may be too restrictive to be relevant to real astrophysical systems.

It was only in the late 1990s that the most general nonlinear problem, involving the regions on both sides of the LC, came under serious theoretical investigation. In their pioneering work, Contopoulos, Kazanas, & Fendt (1999, hereafter CKF99) obtained the first (and apparently unique) numerical solution of the general problem, albeit with a rather poor spatial resolution. Their approach was later used by Ogura & Kojima (2003), who obtained essentially the same results but with a higher numerical resolution. Both of these groups have focused on the global structure of the solution; consequently, they have not paid enough attention, in our opinion, to some key but subtle issues regarding the separatrix between the open- and closed-field line regions and, especially, the point of intersection of this separatrix with the equatorial plane. As a result, the separatrix has not, we believe, been treated correctly, and, in particular, the separatrix equilibrium condition (Okamoto 1974; Lyubarskii 1990) has not been satisfied close to the intersection point. We thus suspect that their solution is probably not quite correct.

In this paper we try to clarify how the magnetic field near the separatrix and especially around the separatrix-equator intersection point should be treated, as this question appears to be extremely important for setting up the correct boundary conditions for the global problem. In § 2, we review the basic equations describing the ideal MHD force-free magnetosphere of an aligned rotating dipole; we pay special attention to the so-called light cylinder regularity condition and discuss how this condition could be used to find the correct form of the poloidal-current function and, simultaneously, to obtain the solution that passes smoothly across the LC. Then, in § 3, we investigate some general properties of the magnetic field near the putative separatrix between the open- and closed-field regions; we use the separatrix equilibrium condition to show...
that, if the separatrix intersects the equator at the LC (as has been assumed in the numerical simulations by CKF99 and Ogura & Kojima 2003), then the poloidal current and hence the toroidal magnetic field have to vanish on the last open-field line above the separatrix. We then use these findings in § 4 to construct a unique, self-similar, asymptotic solution of the pulsar equation in the vicinity of such an intersection point; in particular, we find all the power-law exponents describing the field near this point and also the angle between the separatrix and the equator. At the end of this section, we show that a light surface (LS; a surface where the electric field becomes equal to the magnetic field and therefore the particle drift velocity reaches the speed of light) has to appear right outside the LC (it starts at the intersection point and extends outward at a finite angle with respect to the equator). We thus conclude that the only way to get a force-free solution that would be valid at least some finite distance beyond the LC is to consider the case of a separatrix-equator intersection point lying some finite distance inside the LC. We investigate the magnetic field structure around such a point in § 5. We find that in this case the poloidal-current function may stay finite, but its derivative with respect to the poloidal flux has to go to zero on the last open-field line; we then find the asymptotic behavior of the magnetic field around this point corresponding to this case. Finally, we present our conclusions in § 6, where we also discuss the implications of our results for the past and future numerical studies of the axisymmetric pulsar magnetospheres.

2. BASIC EQUATIONS

We consider the magnetosphere of an aligned rotating magnetic dipole under the assumptions of stationarity, axial symmetry with respect to the rotation axis Z and reflection symmetry with respect to the equator Z = 0. We include the effects of special relativity but ignore general relativistic effects (i.e., work in Euclidean space). We assume that the dipole is surrounded by a very tenuous but highly conducting plasma. The first of these assumptions enables us to neglect all nonelectromagnetic forces (i.e., gravity, inertial, and pressure forces) and thus to conclude that the structure of the magnetosphere is governed by the relativistic force-free equation

\[ \rho E + \frac{j \times B}{c} = 0. \tag{1} \]

Here the electric charge density \( \rho \) and the electric current density \( j \) are related to the electric and magnetic fields through the steady state Maxwell equations:

\[ \rho = \frac{1}{4\pi} \mathbf{v} \cdot \mathbf{E}, \tag{2} \]

\[ j = \frac{c}{4\pi} \mathbf{v} \times \mathbf{B}. \tag{3} \]

The second of our assumptions concerning the magnetospheric plasma, i.e., the assumption of infinite conductivity, enables us to use ideal MHD:

\[ E + \frac{v \times B}{c} = 0. \tag{4} \]

These equations (with the appropriate boundary conditions) should be sufficient for determining the structure of the magnetosphere.

As is well known, an axisymmetric magnetic field \( \mathbf{B} \) can be described in terms of two functions, the poloidal magnetic flux (per 1 radian in the azimuthal direction) \( \Psi \) and the poloidal current \( \mathbf{I} \), as

\[ \mathbf{B} = \nabla \Psi \times \mathbf{v} + \mathbf{I} \nabla \phi, \tag{5} \]

where \( \phi \) is the azimuthal (or toroidal) angle. In cylindrical coordinates \((R, \phi, Z)\), the magnetic field components then are

\[ B_R = -\frac{1}{R} \frac{\partial \Psi}{\partial Z}, \tag{6} \]

\[ B_Z = \frac{1}{R} \frac{\partial \Psi}{\partial R}, \tag{7} \]

\[ B_\phi = \frac{I}{R}. \tag{8} \]

Next, by applying Faraday’s law \( \partial \mathbf{B}/\partial t = -\mathbf{v} \times \mathbf{E} \) in a steady state, we see that \( \mathbf{v} \times \mathbf{E} = 0 \), and hence \( \mathbf{E} = -\nabla \Psi \); together with the assumption of axial symmetry, this gives

\[ E_\phi = 0; \tag{9} \]

that is, the electric field is purely poloidal. Since, as follows from equations (1) or (4), the electric field must be perpendicular to the magnetic field, we can write it as

\[ E = \frac{R \Omega}{c} B_{pol} \times \hat{\phi} = -\frac{\Omega}{c} \nabla \Psi, \tag{10} \]

where \( B_{pol} = \nabla \Psi \times \mathbf{v} \phi \) is the poloidal magnetic field and \( \phi \) is the unit vector in the \( \phi \)-direction. By substituting this relationship into \( \mathbf{v} \times \mathbf{E} = 0 \), we find that \( \Omega \), the angular velocity of magnetic field lines, is constant along the field lines, \( \Omega = \Omega(\Psi) \). This is the well-known Ferraro isorotation law.

In the problem we are interested in, all the field lines are tied, at least at one end, to the surface of the pulsar, which is assumed to be rotating as a solid body with some uniform angular velocity \( \Omega_s \). We also assume that our ideal-MHD assumption is valid all the way up to the pulsar surface, i.e., that there is no appreciable gap between this surface and the ideal-MHD region. Therefore, from now on, we assume that all the field lines rotate with the same angular velocity, \( \Omega(\Psi) = \Omega_s = \text{const} \). \( \tag{11} \)

From equation (10) we find

\[ |E| = |B_{pol}| x, \tag{12} \]

where

\[ x \equiv \frac{R \Omega}{c} = \frac{R}{R_{LC}} \tag{13} \]

is the cylindrical radius normalized to the radius of the LC, \( R_{LC} \equiv c/\Omega \). Thus, we see that \( |E| < |B_{pol}| \) inside the LC and \( |E| > |B_{pol}| \) outside the LC.

Now let us turn to the force-free equation (1). The toroidal component of this equation, together with equation

1 The case where \( \Omega(\Psi) \neq \Omega_s \) was considered, e.g., by Beskin et al. (1983) and by Beskin & Malyshkin (1998).
the surface of the star, the symmetry axis, the separatrix function set up the problem properly. We believe that the correct (Beskin 1997), one then needs two boundary conditions to one singular surface, the LC. According to general theory (Scharlemann & Wagoner 1973; Michel 1973b; Okamoto 1974), also known in literature as the relativistic force-free Grad-Shafranov equation (see, e.g., Beskin 1997). It is an elliptic, second-order PDE for the flux function \( \Psi(x, z) \); the left-hand side (LHS) of this equation is linear, whereas the RHS is, in general, nonlinear.

One very important feature of equation (15) is that it has a regular singular surface at the LC when \( x = 1 \). Both indicial (or characteristic) exponents for this equation are equal to zero, and so a general solution of equation (15) diverges logarithmically at this surface (see, e.g., Bender & Orszag 1978). On physical grounds, one imposes an additional condition that both the function \( \Psi(x, z) \) and its first and second derivatives remain finite near the LC. For any given function \( I(\Psi) \), it is indeed possible to find such nondivergent solutions separately on each side of the LC. Then, as can be seen from equation (15) itself, the contribution of the first term on the LHS [the term proportional to \((1-x^2)^3\)] vanishes as \( x \rightarrow 1 \), and thus such nondivergent solutions on both sides of the LC have to satisfy the well-known LC regularity condition (see, e.g., Scharlemann & Wagoner 1973; Okamoto 1974; Beskin 1997):

\[
\frac{\partial \Psi}{\partial x} (x = 1, z) = \frac{1}{2} H'(\Psi) . \tag{20}
\]

This condition is very important and merits a few extra words of discussion. Our force-free equation with a constant field line angular velocity \( \Omega \) is a second-order equation with one, a priori unknown integral of motion, \( I(\Psi) \), and one singular surface, the LC. According to general theory (Beskin 1997), one then needs two boundary conditions to set up the problem properly. We believe that the correct approach here is to set the boundary conditions for the function \( \Psi \) at both the inner boundary (inside the LC: on the surface of the star, the symmetry axis, the separatrix between the open- and closed-field line regions, and maybe the equator) and whatever outer boundary (outside the LC: the equator and infinity or the LS). At the same time the poloidal current function \( I(\Psi) \) should not be prescribed explicitly at any boundary of the domain; instead, it is to be determined from the matching condition at the singular surface, i.e., the LC (see below).

Thus, our position differs from that of Beskin (see Beskin et al. 1983 and Beskin 1997), who proposed prescribing \( I(\Psi) \) explicitly at the surface of the pulsar. Indeed, if one considers the region inside the LC only, then upon prescribing both \( \Psi \) and \( I(\Psi) \) on the inner boundary (the pulsar surface, etc.), one should be able to obtain a solution that is regular at the LC by using the regularity condition (20) at the other boundary of this inner region, i.e., at the LC. Thus, the solution in this inner region is then completely determined; as such, it is totally independent of what happens outside the LC. This does not seem physical; indeed, one should then be able to take this solution and continue it smoothly across the LC, thus prescribing both \( \Psi(1, z) \) and \( \Psi(1, z) \) at the outer side of the LC. But then the problem of finding a solution in the region outside the LC becomes overdetermined, as we now have two conditions at \( x = 1 \), in addition to any conditions at the outer boundary. For example, in the very important particular case of the domain under consideration extending all the way to radial infinity, one cannot actually prescribe any specific boundary conditions at infinity because equation (15) has a regular singularity there. Instead, however, one imposes a regularity condition at infinity. In spherical polar coordinates\(^2\) \((r, \theta, \phi)\), this regularity condition can be written as

\[
\Psi_{\theta \theta} + \Psi_\theta \cot \theta = \frac{H'(\Psi)}{\sin^2 \theta}, \quad r \to \infty \tag{21}
\]

This regularity condition has a very simple physical meaning: it is the condition of the force balance in the \( \theta \)-direction between the toroidal magnetic field \( B_\theta \) and the poloidal electric field \( E_\theta \) (which both become much larger than the poloidal magnetic field \( B_{pol} \approx B_\phi \) as one approaches infinity). Thus, in this case the region outside the LC does not have any boundary conditions at all (apart from the condition \( \Psi = \text{const} \) at the equator) but has two regularity conditions, one at the LC and the other at infinity. These two regularity conditions are already sufficient to uniquely determine the solution \( \Psi(x > 1, z) \) for a given \( I(\Psi) \). Therefore, if, in addition to these two regularity conditions, one also tries to impose the function \( \Psi(1, z) \) along the LC, the system becomes overdetermined. Thus, we come to the conclusion that prescribing \( I(\Psi) \) at the inner boundary is not appropriate. One could, however, try to use the outer-boundary condition (or the regularity condition at infinity) as the condition that fixes \( I(\Psi) \). To do this, one first sets the inner-boundary conditions, picks an initial guess for \( I(\Psi) \), and uses equation (20) to obtain a regular solution inside the LC; one then continues this solution smoothly across the LC and solves equation (15) in the outer region as an initial-value problem (with both \( \Psi \) and \( \partial_\theta \Psi \) specified at \( x = 1 \)). As a result, one gets a mismatch at the outer boundary between the obtained solution and the desired outer boundary condition (or, in the case of an infinite domain,\(^2\) Note that in the rest of the paper we use a totally different set of polar coordinates defined in the vicinity of the separatrix intersection point, for which we use the same notation \((r, \theta)\). We hope that this does not cause any confusion.)
one would presumably fail to get a convergent solution satisfying the regularity condition [21]). Then one iterates with respect to \( I(\Psi) \) until this outer-boundary mismatch is zero (or until a solution regular at infinity is achieved). In reality, however, such an approach may not be practical, as it involves solving an initial-value problem for an elliptic equation, which is not a well-posed problem.

Instead, we advocate the approach adopted by CKF99. In this approach one considers the regions inside the LC \((x < 1)\) and outside the LC \((x > 1)\) separately. First, one makes an initial guess for \( I(\Psi) \) and prescribes the corresponding boundary conditions for \( \Psi \) at the inner boundary of the region inside the LC and at the outer boundary of the region outside the LC (or, if this outer boundary is at the radial infinity, imposes the regularity condition [21] there). Then, one uses, again separately in each region, the regularity condition (20) in lieu of a boundary condition at the LC (in practice, one can think of eq. [20] as a mixed-type Dirichlet–von Neumann boundary condition relating the value of \( \Psi \) at the surface \( x = 1 \) to the value of its derivative normal to this surface). Thus, one obtains two solutions, one inside and the other outside the LC, that correspond to the same function \( I(\Psi) \) and are both regular at \( x = 1 \). However, these solutions also depend on their respective boundary or regularity conditions set at the inner/outer boundaries of the domain. Hence, in general, although they are both regular at the LC, these solutions are not going to coincide at \( x = 1 \). The mismatch \( \Delta \Psi_{LC}(z) \) depends on both the chosen function \( I(\Psi) \) and the inner and outer boundary conditions. One then iterates with respect to \( I(\Psi) \) in order to minimize this mismatch. We believe that for a given set of boundary conditions there should be only one choice of \( I(\Psi) \) for which the mismatch \( \Delta \Psi_{LC} \) vanishes and hence the function \( \Psi \) becomes continuous along the entire LC, i.e., \( \Psi_{in}(x \rightarrow 1, z) = \Psi_{out}(x \rightarrow 1, z) \). Once this special function \( I(\Psi) \) is found, it then follows from equation (20), which is satisfied separately on both sides of the LC, that \( \partial_z \Psi \) is also continuous, and so the entire solution passes smoothly across the LC. Thus, one can say that the function \( I(\Psi) \) is determined by the regularity condition (20) applied separately on both sides of the LC, plus the condition of matching of the two solutions. Another, equivalent way to put it is to say that \( I(\Psi) \) is determined by the (nontrivial!) requirement that the derivative \( \partial_z \Psi(x = 1, z) \) actually exist at the LC [whereby \( I(\Psi) \) is expressed in terms of this derivative via eq. (20)].

This approach has been implemented successfully in the pioneering numerical work by CKF99 and then subsequently by Ogura & Kojima (2003). Both these groups have used a relaxation procedure to arrive at \( I(\Psi) \) that corresponded to a unique solution that was both continuous and smooth at the LC.

3. BEHAVIOR NEAR THE SEPARATRIX Y-POINT: GENERAL CONSIDERATIONS

The main focus of this paper is the behavior of the magnetic field in the vicinity of the point of intersection of the separatrix \( \Psi = \Psi_0 \) between the region of closed-field lines (region I) and the region of open-field lines (region II) with the equator \( z = 0 \) (see Fig. 1). We generally call this point the separatrix intersection point. Our interest in this nontrivial problem is fueled by the belief that understanding the key features of this behavior is absolutely crucial to devising the proper boundary conditions and providing verification benchmarks for any future global numerical investigations of a force-free pulsar magnetosphere. At the same time, the treatment of the magnetic field around this very special point is expected to require a certain degree of subtlety and delicacy, as noted, for example, by Beskin et al. (1983) and Ogura & Kojima (2003).

Among the questions that need to be answered are the following: What is the radial position \( x_0 \) of the intersection point (namely, does this point lie at the LC, \( x_0 = 1 \), or inside the LC, \( x_0 < 1 \)), and what is the angle \( \theta_0 \) at which the separatrix approaches the equator at this point, the three a priori possibilities being \( \theta_0 = 0 \) (in which case we call this point the cusp point), \( 0 < \theta_0 < \pi/2 \) (the Y-point), and \( \theta_0 = \pi/2 \) (the T-point)?

In our analysis, we assume that equations (1) and (4), which describe our system, are valid almost everywhere in the vicinity of the separatrix intersection point, i.e., everywhere with perhaps the exception of measure-zero regions. Thus, we allow for the presence in our system of current sheets of infinitesimal thickness, across which the magnetic field can experience a finite jump. We limit our consideration to the situation in which such current sheets can be present only along the separatrix between regions I and II for \( x < x_0 \) and along the equatorial \((z = 0)\) separatrix between the upper and lower open-field regions for \( x > x_0 \). Thus, we have one current sheet that lies on the equator at \( x > x_0 \) and at \( x = x_0 \) splits into two symmetrical current sheets (one in each hemisphere) lying along the separatrix \( \Psi = \Psi_s \) (see, e.g., Okamato 1974). At the same time, we assume that our equations apply everywhere else at least in some region around the intersection point, including the portion of this region that lies outside the LC (in the case \( x_0 = 1 \)).

Let us first assume that the separatrix between closed and open-field line regions reaches the LC (i.e., \( x_0 = 1 \)), and let us consider the separatrix equilibrium condition (e.g., Okamato 1974; Lyubarskii 1990). Indeed, whether or not the separatrix contains a current sheet (which we assume to be infinitesimally thin), it must satisfy the condition of force balance, which is obtained by integrating equation (1)
across the separatrix:

$$(B^2 - E^2)^I = (B^2 - E^2)^{II}.$$  \hfill (22)

(Note that strictly speaking, eq. [1] may not be valid inside a current sheet, as other forces may also be important in the pressure balance there. However, gravity and the plasma inertia are small because the separatrix current layer is assumed to be very thin; and the plasma pressure gradient, while not necessarily small inside the current layer, gives, when integrated across the layer, just the difference between the values of the pressure on both sides outside the separatrix, both of which are small by assumption.) By using equation (12) and also the fact that $B_0 = 0$ in region I, we can rewrite condition (22) as

$$\left\{ \left[ B_{pol}^I(l) \right]^2 - \left[ B_{pol}^{II}(l) \right]^2 \right\} (1 - x^2) = \frac{1}{R_{LC}^2} \frac{I^2(\Psi_s)}{x^2},$$ \hfill (23)

where $l$ marks the distance from the intersection point ($z = 0$) along the separatrix.

Now, as the separatrix approaches the LC ($x \rightarrow 1$, $l \rightarrow 0$), we get

$$2(1-x)\left\{ \left[ B_{pol}^I(l) \right]^2 - \left[ B_{pol}^{II}(l) \right]^2 \right\} = \frac{I^2}{R_{LC}^2},$$ \hfill (24)

where we have defined

$$I_s \equiv \lim_{\Psi \rightarrow \Psi_s} I(\Psi, \Psi_s)$$ \hfill (25)

as the value of the poloidal current of the last open-field line $\Psi_s$ (above the separatrix current sheet). Since $I_s$ is constant along this last open-field line and hence is independent of $l$, we immediately see that in order to satisfy equation (24) in the limit $x \rightarrow 1$, we must have either of the following:

1. $I_s = 0$; i.e., all poloidal current that flows out of the pulsar must return back to the pulsar along open magnetic field lines, with no finite poloidal current flowing along the separatrix.3

Simultaneously, from equation (24) it then follows that $B_{pol}^I(l) = B_{pol}^{II}(l)$, and hence no finite toroidal current flows along the separatrix either (we make a very natural assumption that $B_{pol}^I$ is in the same direction as $B_{pol}^{II}$). Thus, the separatrix in this case is actually not a current sheet.

2. A finite $I_s \neq 0$ but a divergent poloidal magnetic field in the closed-field region, $B_{pol}^I(l) \sim [1 - x(l)]^{-1/2} \rightarrow \infty$, as $l \rightarrow 0$, $x(l) \rightarrow 1$. Note that this situation cannot be dismissed automatically; indeed, close to the LC the magnitude of the electric field becomes very close to that of the poloidal magnetic field, with the infinitely strong electric forces balancing out the infinitely strong magnetic ones to a higher degree. Then the separatrix current sheet would have to carry back to the star a finite poloidal current $I_s$ (thus closing the poloidal current circuit) and also a toroidal corotation current with the surface current density $I_s(l)$

divergent near the LC:

$$I_s(l) \sim |B_{pol}^I - B_{pol}^{II}| \sim 1/[1 - x(l)]^{1/2} \rightarrow \infty;$$

there would also have to be a divergent surface charge density on the separatrix:

$$\sigma(l) \sim |E_{pol}^I - E_{pol}^{II}| \sim 1/[1 - x(l)]^{1/2} \rightarrow \infty.$$

Note that these conclusions, derived from the separatrix equilibrium condition near the LC, are valid regardless of the angle $\theta_0$ between the separatrix and the equator at the intersection point.

In our analysis in the next section, we assume that there are no infinitely strong fields anywhere in the system and thus dismiss situation (2) ($x_0 = 1$, $\Omega = \text{const}$, finite $I_s \neq 0$) as unphysical. We thus concentrate our attention on situation (1) ($I_s = 0$). We try to analyze the structure of the pulsar equation (15) and to obtain the asymptotic solution of this equation in the vicinity of the intersection point ($x_0 = 1$, $z_0 = 0$).

4. BEHAVIOR NEAR THE SEPARATRIX Y-POINT: ASYMPTOTIC ANALYSIS

In this section we perform an asymptotic analysis of the pulsar equation (15) in the vicinity of the separatrix intersection point, under the assumption that it is located at the LC, i.e., $x_0 = 1$, $z_0 = 0$. Our approach is actually very similar to the analysis of the magnetic field near the endpoint of a non-relativistic reconnecting current layer, performed previously by Uzdensky & Kulsrud (1997).

In addition to assuming $x_0 = 1$, we make use, in this section, of the following two assumptions:

1. The intersection point is a finite-angle $Y$-point, i.e., the separatrix approaches the equator at an angle $\theta_0$, $0 < \theta_0 < \pi/2$, where we measure angles from the radial vector lying on the equator and directed toward the star, as shown in Figure 1.

2. The poloidal current is fully closed in the open-field region, i.e., $I_s = I(\Psi = \Psi_s) = 0$; this assumption is motivated by the arguments presented in the previous section.

Furthermore, we also assume symmetry with respect to the equator and therefore consider only the upper half-space, $z \geq 0$.

When considering the $x$-dependent coefficients in equation (15) in the vicinity of the Y-point, we can, to the lowest order in $|x - 1|$, replace $x$ by 1 everywhere except where it appears in a combination like $1 - x$; thus, we can rewrite this equation as

$$2\xi \left( \frac{\partial^2 \Psi}{\partial \xi^2} + \frac{\partial^2 \Psi}{\partial z^2} \right) + 2 \frac{\partial \Psi}{\partial \xi} = -II'(\Psi),$$ \hfill (26)

where we introduce a new coordinate, $\xi \equiv 1 - x$. Instead of using coordinates $(\xi, z)$, it is actually more convenient to use polar coordinates $(r, \theta)$, defined by

$$\xi = 1 - x = r \cos \theta, \quad z = r \sin \theta.$$ \hfill (27)

We can rewrite the pulsar equation (26) in these coordinates

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3 This scenario is in agreement with the findings by Okamoto (1974) and also with the particular case $\beta_0 = 0$ of the compatibility condition by Beskin et al. (1983) and Beskin & Malyskina (1998).
as

\[
\left( \frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2} \right) r \cos \theta + 2 \frac{\partial \Psi}{\partial r} \cos \theta - \frac{\partial \Psi}{\partial \theta} \sin \theta
\]

\[
= -\frac{1}{2} II'(\Psi) . \quad (28)
\]

Now we need to solve this equation separately in region I (region of closed-field lines, \(0 \leq \theta \leq \theta_0\)) and in region II (region of open-field lines, \(\theta_0 \leq \theta \leq \pi\)) and match the two solutions together at the separatrix \(\theta = \theta_0\). Since in this paper we are dealing only with the local structure of the magnetic flux generality, adopt the convention of counting the poloidal magnetic field near the \(Y\)-point, we can, without any loss of generality, adopt the convention of counting the poloidal magnetic flux \(\Psi\) from the separatrix \(\theta = \theta_0\) (instead of the usual convention of counting \(\Psi\) from the rotation axis). Thus, we set \(\Psi = 0\) and, correspondingly, \(\Psi > 0\) in region I and \(\Psi < 0\) in region II.

Very close to the \(Y\)-point, i.e., at distances much smaller than the LC radius \((r \ll 1)\), the system lacks any natural length scale. Hence, we can expect the radial dependence of \(\Psi\) to be a power law, which enables us to make the following self-similar Ansatz:

\[
\Psi_I(r, \theta) \equiv \Psi(r, \theta \leq \theta_0) = r^{\alpha_1} f(\theta) , \quad (29)
\]

\[
\Psi_{II}(r, \theta) \equiv \Psi(r, \theta > \theta_0) = -r^{\alpha_2} g(\theta) , \quad (30)
\]

[we put a minus sign in eq. (30) in order to have \(g(\theta) \geq 0\).

Then, the poloidal magnetic field components are given by

\[
B^I_r = \frac{1}{R_{\text{LC}}} \frac{\partial \Psi_I}{\partial \theta} \frac{1}{r^{\alpha_1-1}} f'(\theta) , \quad (31)
\]

\[
B^I_\theta = -\frac{1}{R_{\text{LC}}} \frac{\partial \Psi_I}{\partial r} = -\frac{1}{R_{\text{LC}}} \alpha_1 r^{\alpha_1-1} f'(\theta) , \quad (32)
\]

in region I, and similarly

\[
B^{II}_r = -\frac{1}{R_{\text{LC}}} r^{\alpha_1-1} g'(\theta) , \quad (33)
\]

\[
B^{II}_\theta = \frac{1}{R_{\text{LC}}} \alpha_2 r^{\alpha_2-1} g(\theta) , \quad (34)
\]

in region II. Note that the condition that the magnetic field does not diverge near \(r = 0\) imposes the restriction

\[
\alpha_1, \quad \alpha_2 \geq 1 , \quad (35)
\]

whereas for the case in which the poloidal magnetic flux function in region I diverges as \(1/\sqrt{r}\) (i.e., case [2] in § 3), we would have \(\alpha_1 = 1/2\).

As for the poloidal current function \(I(\Psi)\) that appears on the RHS of equation (28), it takes very different forms in regions I and II. In region I there should be no toroidal field, so

\[
I(\Psi > 0) \equiv 0 . \quad (36)
\]

In the open-field region II, \(I(\Psi)\) is not zero but does approach zero in the limit \(\Psi \to 0\). Since there is no natural a priori magnetic field scale in the vicinity of the \(Y\)-point, we again employ a self-similar Ansatz, i.e., assume that \(I\) scales as a power of \(|\Psi|\) near \(\Psi = 0:\)

\[
I(\Psi < 0) = g(-\Psi)^\beta , \quad \beta > 0 , \quad (37)
\]

so that the RHS of equation (28) becomes

\[
-\frac{1}{2} II'(\Psi < 0) = \kappa(-\Psi)^{2\beta-1} , \quad (38)
\]

where \(\kappa \equiv \beta q^2/2 > 0\).

Upon substituting relationships (29) and (36) (for the closed-field region) and (30) and (38) (for the open-field region) into the main equation (28), we obtain two ordinary differential equations (ODEs) for the functions \(f(\theta)\) and \(g(\theta)\), along with a relationship between the power-law indices \(\alpha_2\) and \(\beta\). In particular, in region I we get a homogeneous, linear, second-order ODE for \(f(\theta)\),

\[
f''(\theta) - f'(\theta) \tan \theta + \alpha_1 (\alpha_1 + 1) f(\theta) = 0 , \quad (39)
\]

which is to be supplemented by two boundary conditions:

\[
f'(\theta = 0) = 0 , \quad f(\theta_0) = 0 . \quad (40)
\]

Similarly, in region II we get a nonlinear second-order ODE for \(g(\theta)\):

\[
g''(\theta) \cos \theta - g'(\theta) \sin \theta + \alpha_2 (\alpha_2 + 1) g(\theta) \cos \theta = -\kappa g^{1/(\alpha_2)}(\theta) , \quad (41)
\]

together with the relationship

\[
\beta = 1 - \frac{1}{2\alpha_2} , \quad (42)
\]

which follows from the requirement that the LHS of equation (28) scale as the same power of \(r\) as the RHS, i.e., \(\alpha_2 - 1 = (2/3 - 1)\alpha_2\). (At this point, however, we have to remark that one cannot a priori exclude the possibility \(\beta > 1 - 1/2\alpha_2\); in such a case the poloidal current would go to zero near the separatrix so rapidly that its contribution to the pulsar equation would become completely negligible. However, as we discuss later in this section, it is possible to show that no continuous solutions exist in this case.)

The boundary conditions for equation (41) are

\[
g(\theta_0) = 0 = g(\theta = \pi) . \quad (43)
\]

Note that the constant \(\kappa\) that appears on the RHS of equation (41) is in fact unimportant; it just sets the scale of variation of the function \(g(\theta)\) and, hence, the overall scale of the magnetic field strength. Thus, we can rescale \(\kappa\) away by incorporating it into the solution; we do this by defining a new variable,

\[
G(\theta) \equiv \kappa^{-\alpha_2} g(\theta) . \quad (44)
\]

Then equation (41) becomes

\[
G''(\theta) \cos \theta - G'(\theta) \sin \theta + \alpha_2 (\alpha_2 + 1) G(\theta) \cos \theta = -G^{1/(\alpha_2)}(\theta) , \quad (45)
\]

with the same boundary conditions,

\[
G(\theta_0) = 0 = G(\theta = \pi) . \quad (46)
\]

Thus, we see that in our problem the magnetic field structure near the \(Y\)-point is completely characterized by four finite, dimensionless parameters: the separatrix angle \(\theta_0\) and the three power-law indices \(\alpha_1, \alpha_2, \beta\). Our goal is to obtain a unique solution of our problem, that is, to determine the values of these parameters and simultaneously determine the functions \(f(\theta)\) and \(g(\theta)\). In fact, equation (42) already
gives us one relationship between the power exponents, but we still need three more relationships to fix all four parameters. Therefore, we now discuss the conditions that help us obtain these three additional relationships.

The second condition (the first being eq. [42]) that links our dimensionless parameters is rather obvious; it is the condition of force balance across the separatrix \( \theta = \theta_0 \), and it gives us a relationship between \( \alpha_1 \) and \( \alpha_2 \). We have already discussed this condition in \( \S \) 3 (see eqs. [22]–[24]); with \( |E| = x|B_\text{pol}|, I(\Psi > 0) = 0 \), and \( L_s = 0 \), this condition can be written simply as

\[
[B_\text{pol}^2(r, \theta_0)]^I = [B_\text{pol}^2(r, \theta_0)]^II. \tag{47}
\]

Since in our \((r, \theta)\)-coordinates the separatrix is a radial line \( \theta = \theta_0 \), and since we expect the poloidal magnetic field on the two sides of the separatrix to be in the same direction, condition (47) simply means that \( B_\text{pol}(r, \theta_0) = B_\text{pol}^0(r, \theta_0) \).

Then, using equations (31)–(34), we immediately obtain a second relationship between the dimensionless parameters

\[
\alpha_2 \equiv \alpha \tag{48}
\]

and also a relationship between the normalizations of the functions \( f(\theta) \) and \( g(\theta) \), cast in terms of their derivatives at \( \theta = \theta_0 \),

\[
f'(\theta_0) = -g'(\theta_0). \tag{49}
\]

The remaining two relationships between the dimensionless parameters come from the properties of equations (39) and (45) themselves and cannot, unfortunately, be written out as explicit algebraic equations containing these parameters. This fact, however, can prevent us neither from describing and discussing the physical and mathematical conditions on which these two additional relationships are based nor from using these conditions to obtain the actual unique values of the parameters.

First, we show that there is a unique one-to-one relationship between the separatrix angle \( \theta_0 \) and the power exponent \( \alpha \). This relationship comes from analyzing equation (39) for the closed-field region I. For any given \( \theta_0 \), this homogeneous linear equation with boundary conditions in equation (40) is an eigenvalue problem for the coefficient \( \alpha = \alpha(\alpha + 1) \geq 2 \).

It is in fact very easy to see why this must be the case. Indeed, the boundary conditions in equation (40) do not give us any scale for \( f \); if some function \( f(\theta) \) is a solution of problems (39) and (40), then \( C f(\theta) \) with an arbitrary multiplier \( C \) will also be a valid solution. Thus, the normalization of \( f(\theta) \) is arbitrary: if, for given \( \theta_0 \) and \( \alpha \), a nontrivial solution of problems (39) and (40) exists, then there will be infinitely many such solutions; in particular, there will be a solution with \( f(0) = 1 \). Thus, we can impose an additional boundary condition \( f(0) = 1 \). Then, however, we have three boundary conditions for a second-order differential equation, which means that for arbitrary \( \theta_0 \) and \( \alpha \) the system is overdetermined. Hence, such a solution will not exist for all values of \( \theta_0 \) and \( \alpha \); the condition that it actually does exist gives us a certain relationship between \( \theta_0 \) and \( \alpha \): for a given \( \theta_0 \) we would get an infinite discrete spectrum of the allowed values of \( \alpha \). Of these, however, we are interested only in the lowest one, because we want a solution without direction reversals of the magnetic field, i.e., with \( f \geq 0 \) everywhere.

From the practical point of view, the easiest way to determine the relationship between \( \alpha \) and \( \theta_0 \) is the following. We scan over the values of \( \alpha \); for each given \( \alpha \) we use our free-dom of normalization of \( f(\theta) \) to set \( f(0) = 1 \). Thus, we now have an initial-value problem with two conditions at \( \theta = 0 \): \( f(0) = 1 \) and \( f'(0) = 0 \). With these two conditions and with the value of \( \alpha = \alpha(\alpha + 1) \) given, we can integrate equation (39) forward in \( \theta \) (we do it numerically, using a second-order Runge-Kutta scheme) until \( f(\theta) \) becomes zero. This point is then declared the correct value of the separatrix angle \( \theta_0 \) for the given \( \alpha \). The function \( \theta_0(\alpha) \) we have obtained as a result of this procedure is shown in Figure 2, whereas Figure 3 shows the behavior of the function \( f(\theta) \) for several selected values of \( \alpha \). We see that \( \theta_0(\alpha) \) is a monotonically decreasing function with \( \theta_0(\alpha = 1) = 0.986 \ldots \) rad [instead of \( \theta_0(1) = \pi/2 \), which one would get if the Y-point were at a finite distance inside the LC!]. Also note that the value \( \theta_0(\alpha = 2) = 0.614 \ldots \) rad \( \approx 35^\circ \) agrees with the Michel (1973a) separatrix angle of \( \approx 35^\circ \), corresponding to the case of zero poloidal current (see also Beskin et al. 1983); this is
because in his analysis he had in fact assumed that \( \alpha = 2 \) and then derived the corresponding value of \( \theta_0 \). The asymptotic behavior of \( \theta_0(\alpha) \) in the limit \( \alpha \to \infty \) is \( \theta_0 \to \pi/2\alpha \), which is in fact very easy to obtain analytically. Indeed, as \( \alpha \to \infty \), we expect \( \theta_0 \ll 1 \), and then we can neglect the \( f'\tan \theta \) term in equation (39). As a result, we get a simple harmonic equation \( f''(\theta) + af(\theta) = 0 \), with \( f'(0) = 0 \), \( f(0) = 0 \); the solution of this equation, positive everywhere in the domain \( 0 \leq \theta < \theta_0 \), is \( f(\theta) = \cos(\pi \theta/2\theta_0) \); it corresponds to \( \alpha \simeq \sqrt{a} = \pi/2\theta_0 \). Actually, the exact analytical fit for our numerical curve \( \theta_0(\alpha) \) appears to be

\[
\theta_0(\alpha) = \frac{\pi}{2} \frac{1}{\sqrt{\alpha(\alpha + 1) + b}},
\]

with \( b \simeq 0.54 \ldots \).

With the dependence \( \theta_0(\alpha) \) thus determined and with \( \beta \) and \( \alpha_2 \) related to \( \alpha \) via equations (42) and (48), respectively, we now have everything expressed in terms of \( \alpha \).

The fourth (and final!) condition that helps us determine \( \alpha \) and hence \( \theta_0 \) and the rest of the power exponents is the LC regularity condition. This condition states that at the LC \( \theta = \pi/2 \) (which is a regular singular point for eq. [45]), the function \( G(\theta) \) should be regular; that is, it should be a continuously differentiable function, with finite first and second derivatives. This condition can be written as

\[
G'(\frac{\pi}{2}) = G^{-1/(\alpha)}(\frac{\pi}{2}),
\]

and it is just a particular manifestation of the general LC regularity condition (20). We use this condition to fix the unique value of \( \alpha \) (see the general discussion of this issue at the end of §2). Here is how we do it in practice.

First, we divide region II into two subregions: region II′ (inside the LC: \( \theta_0 \leq \theta \leq \pi/2 \)) and region II′′ (outside the LC: \( \pi/2 \leq \theta \leq \pi \)). Then we scan over \( \alpha \); for each value of \( \alpha \), we first determine the corresponding value of \( \theta_0 \) using the procedure outlined above. Once \( \theta_0 \) for a given \( \alpha \) is found, we solve equation (45) separately in regions II′ and II′′ (again using a numerical shooting method in conjunction with the second-order Runge-Kutta integration scheme). In each of these regions we use regularity condition (51) at \( \theta = \pi/2 \) and one of the boundary conditions in equation (46) at the other end of the region, i.e., \( G(\theta_0) = 0 \) in region II′ and \( G(\pi) = 0 \) in region II′′. Note that the use of the regularity condition (51) guarantees that the solutions in each of the two regions are regular. However, in general (i.e., for an arbitrarily chosen \( \alpha \)), the two solutions obtained in this manner do not match each other at the LC, the mismatch \( \Delta G_{LC} = G''(\pi/2) - G''(\pi/2) \) [and hence the mismatch in \( G'(\pi/2) \) related to \( \Delta G_{LC} \) via eq. (51)] being \( \alpha \)-dependent. We find that there is only one, special value of \( \alpha \) for which \( \Delta G_{LC} \) vanishes, and the solution continues smoothly across the LC as one passes from region II′ into region II′′. This special value, which we call \( \alpha_0 \), is declared the correct value of \( \alpha \). Numerically, we find

\[
\alpha = \alpha_0 \approx 1.5045 \ldots,
\]

and, correspondingly,

\[
\beta \approx 0.668 \ldots, \quad \theta_0 \approx 0.76 \text{rad}.
\]

This is the way in which the unique correct solution of the problem is obtained. The functions \( f(\theta) \) and

**Fig. 4.**—Function \( f(\theta) \) representing the magnetic flux function in the region of closed-field lines (region I) for \( \alpha = \alpha_0 = 1.50446 \ldots \)

**Fig. 5.**—Function \( G(\theta) = \kappa^{-\alpha}g(\theta) \) corresponding to this value of \( \alpha \) are plotted in Figures 4 and 5, respectively.

Now let us make a little digression and step back to discuss the possibility of finding a solution in the case \( \beta > 1 - 1/2\alpha_2 \). Recall that in this case the dominant balance of the lowest-order (in \( r \)) terms in the pulsar equation does not include the poloidal-current contribution \( -II' '(\Psi)/2 \). At the same time, note that relationships (48) and (49), derived from the separatrix force-balance condition, still hold. Hence, the solution in region II would have to satisfy a linear equation that is identical to equation (39) for \( f(\theta) \) on the other side of the separatrix. This fact, combined with condition (49), implies that the solution just has to pass smoothly across \( \theta = \theta_0 \), and thus this point (which is just an ordinary point for eq. [39]) does not have any special
significance; it is just the point where the solution changes sign. Thus, the general solution in this case takes the form
\[ \Psi = r^j f(\theta), \]
where \( f(\theta) \) satisfies the homogeneous linear equation (39) in the entire domain \([0, \pi] \) with the homogeneous boundary conditions \( f(0) = 0 \) and \( f(\pi) = 0 \). In addition, the solution has to pass smoothly across the regular singular point \( \theta = \pi/2 \) (the LC); the regularity condition at this point is simply \( f'(\pi/2) = 0 \). Such a solution would start positive with a zero derivative at \( \theta = 0 \), then decrease and change sign at some angle \( \theta_0 < \pi/2 \), reach a minimum at the LC [since \( f'(\pi/2) = 0 \), and finally increase and go back to zero at the equator \( \theta = \pi \). However, the extra regularity condition at \( \theta = \pi/2 \) makes the system overdetermined. Indeed, counting two boundary conditions at \( \theta = 0, \pi \), an arbitrary normalization of the solution [e.g., \( f(0) = 1 \)], and the LC regularity condition, we now have four conditions for our second-order equation with one free parameter \( \alpha \).

Thus, unless we are somehow extremely lucky, the solution of this problem does not exist. One can show that this is indeed the case; it turns out that a solution of equation (39) in the region inside the LC \([0, \pi/2] \) satisfying both the boundary condition \( f'(0) = 0 \) and the LC regularity condition \( f'(\pi/2) = 0 \) exists only for even values of \( \alpha = 2, 4, 6, \ldots \), whereas a solution in region \([\pi/2, \pi] \) satisfying both the boundary condition \( f(\pi) = 0 \) and the LC regularity condition exists only for odd values of \( \alpha = 1, 3, 5, \ldots \). This demonstrates that, in the case \( \beta > 1 - 1/2\alpha_2 \), it is impossible to obtain a solution that is continuous at the LC. This leaves us with the case \( \beta = 1 - 1/2\alpha_2 \) (for which we have just found a unique suitable solution) as the only possibility.

Thus, we have managed to obtain a unique asymptotic solution of the ideal-MHD, force-free system in equations (1)–(4) in the vicinity of the separatrix intersection point \((x_0 = 1, z_0 = 0) \). This solution, characterized by dimensionless parameters (52) and (53), satisfies the separatrix equilibrium condition and is continuous and smooth at the LC. Let us now discuss the physical relevance of our solution to the pulsar problem.

Note that the obtained unique solution of our problem has a finite value of the derivative \( G'(\theta) \) at the equator \( \theta = \pi \), namely, \( G' (\pi) = -2.02 \), and thus has a nonzero radial magnetic field just above (and just below) the equatorial current sheet. This actually spells bad news for the applicability of our solution. Indeed, one more condition that needs to be satisfied for the solution to be physically relevant and that we have not discussed so far is the condition
\[ |B|^2 > |E|^2. \]

With \( |E| \) related to \( |B_{pol}| \) via equation (12), this condition can be expressed as
\[ B_{pol}^2 > |B_{pol}|^2(x^2 - 1). \]

We see that this condition is satisfied trivially everywhere inside the LC but is not automatically satisfied outside the LC. In particular, in our solution, and in fact in any solution characterized by \( I_\ell = 0 \), the toroidal field on the last open-field line \( \Psi = \Psi_\ell \) is zero, whereas the poloidal magnetic (and hence electric) field is not zero, as our explicit solution shows. This means that inequality (55) is violated immediately outside the LC as one moves along the equator. More generally, this inequality is violated beyond the so-called light surface (defined as the surface where \( |B| = |E| \)) that emanates from the Y-point. It is in fact not difficult to find the location of this LS in our solution.

Indeed, according to equations (38) and (40), we have
\[ B_{pol}^2 = \frac{1}{R_{LC}^2} r^{2\alpha-2} \{ \alpha^2 g^2(\theta) + |g'|^2 \}, \]
and, according to equations (19), (37), and (42),
\[ B_{pol}^2 = \frac{1}{R_{LC}^2} \frac{q^2}{s^2} (\Psi) B_{pol}^2 \approx \frac{1}{R_{LC}^2} r^{2\alpha-2} g^{2-1/\alpha}(\theta). \]

Using result (56), we can rewrite this as
\[ B_{pol}^2 = B_{pol}^2 \frac{2r}{\beta} \frac{G^{2-1/\alpha}(\theta)}{\alpha^2 G^2(\theta) + |G'|^2}. \]

Upon comparing this result with
\[ E^2 - B_{pol}^2 = (x^2 - 1) B_{pol}^2 = -2r \cos \theta B_{pol}^2, \]
we conclude that the LS \( B_{pol}^2 = E^2 - B_{pol}^2 \) emanates from the Y-point at a finite angle \( \theta_{LS} \), which is determined from the algebraic equation [provided that the solution \( G(\theta) \) is known]
\[ 2\alpha \frac{G^{2-1/\alpha}(\theta_{LS})}{2\alpha - 1} = -\{ \alpha^2 G^2(\theta_{LS}) + |G'(\theta_{LS})|^2 \} \cos \theta_{LS}. \]

For our solution described by parameters (52) and (53), we find
\[ \theta_{LS} \approx 2.07 \text{ rad}, \]
which corresponds to the angle \( \pi - \theta_{LS} \approx 62^\circ \).

Beyond the LS \( \theta > \theta_{LS} \), the force-free equation (1) is not applicable. The solution we have obtained in this section is valid in the region \( \theta < \theta_{LS} \), but, because the process of obtaining this solution also involved the region \( \theta > \theta_{LS} \) [e.g., via the boundary condition \( g(\pi) = 0 \)], our solution may not be the correct solution of the overall problem.

In fact, we can make an even more general statement: any relevant (to our problem) finite-field magnetic configuration with the separatrix intersecting the equator at the LC cannot be almost everywhere (that is, again, everywhere except at a finite number of infinitesimally thin current sheets) ideal and force-free beyond the LC! Indeed, as we saw previously, if one insists on having the separatrix intersection point at the LC, then one has to contend either with having \( |B| \) divergent near \( x = 1 \) or with having \( I_\ell = I(\Psi_\ell) \). The latter case, however, is characterized by zero toroidal magnetic field everywhere along the last open-field line \( \Psi = \Psi_\ell \). Provided that the poloidal (radial) magnetic field does not vanish identically near the equatorial current sheet, it then follows that \( E^2 - B_{pol}^2 = (x^2 - 1) B_{pol}^2 > 0 \), and so the force-free equation (1) becomes inapplicable. And, as our particular solution shows, the condition \( B^2 - E^2 > 0 \) is violated not only on some singular surfaces of measure zero, such as the equatorial current sheet (that would be acceptable to us at this stage, since we do expect our simple force-free, ideal-MHD assumptions to break down there anyway), but also in a volume of nonzero extent in both poloidal directions, e.g., in the region \( \theta_{LS} < \theta \leq \pi, r > 0 \).

Finally, we would like to add that our finding of an LS near the LC points toward the separatrix intersection point
as a possible site of efficient particle acceleration; thus, it may be of great importance to the so-called σ-problem and to the models of pulsar emission generation (V. S. Beskin 2003, private communication).

5. PROPERTIES OF SOLUTIONS WITH THE SEPARATRIX INTERSECTING THE EQUATOR INSIDE THE LIGHT CYLINDER

From the results of the previous section, we have to conclude that if one is to keep the hope of finding a force-free solution that would be valid all the way up to infinity (in the limit of vanishing plasma density), or at least to some finite distance beyond the LC, then one has to make concessions regarding the anticipated location of the separatrix/equator intersection point. In other words, the requirement that \( B^2 > E^2 \) outside the LC means that \( I_x \) must be nonzero, and this, in turn, forces one to conclude that the intersection point must be located at some finite distance inside the LC, i.e., \( x_0 < 1 \). This scenario has actually been the subject of several studies in the past 20 years (e.g., Beskin et al. 1983; Lyubarskii 1990; Beskin & Malyshkin 1998). According to equation (23), the corresponding finite-\( I \) configuration has to be characterized by \( B^2_\text{ext} \), staying finite in the limit \( x \to x_0 \), and, therefore, the separatrix should approach the equator at a right angle, \( \theta_0 = \pi/2 \), corresponding to a T-point (see Lyubarskii 1990).

Let us now ask what other characteristic features should such a solution possess. Here we present a few simple facts that can be gleaned immediately by inspecting the force-free equation (15) and its regularity condition (20). First, since beyond \( x = x_0 \) the last open-field line, \( \Psi = \Psi_x \), is assumed to lie along the equator \( z = 0 \), the derivative \( \partial \Psi / \partial x \) should be zero everywhere along this line (for \( x > x_0 \)). In particular, we should have \( \partial \Psi / \partial x (x = 1, z = 0) = 0 \), but then it follows from equation (20) that either \( I(\Psi_x) \) or \( I'(\Psi_x) \) has to be zero. Since here we are considering the case \( I_x \neq 0 \), we conclude that \( I(\Psi) \) has to approach \( \Psi_x \), with a zero slope, \( I'(\Psi_x) = 0 \). Thus, we see that \( I(\Psi) \neq 0 \) cannot be a simple current function, unless it is constant everywhere. Next, by substituting the result \( I'(\Psi_x) = 0 \), together with \( \Psi(x > x_0, z = 0) = \Psi_x = \text{const} \), into equation (15), we find that \((1 - x^2)\partial^2 \Psi / \partial z^2 (x, z = 0) \equiv 0 \) for all \( x > x_0 \). This means that the radial magnetic field also approaches the equator \( z = 0 \) with a zero slope, \( \partial B_z / \partial z = 0 \).

Now let us see what one can deduce by investigating the regularity condition (20) just above the equatorial current sheet. According to what we have just established, above the equator the function \( I(\Psi) \) can be expanded as

\[
I(\Psi < 0) = I_x + q(-\Psi)^\beta \quad (\beta > 1) \quad \text{and} \quad I'(\Psi < 0) = -\beta q L_x (-\Psi)^{\beta - 1} \quad \text{as} \quad x > x_0.
\]

Now the radial magnetic field just above the equator is, in general, not zero, so we can expand \( \Psi(x, z \to 0) \) to lowest order in \( z \)

\[
\Psi(x, z) = z \Psi_1(x) + \ldots, \quad (64)
\]

where \( \Psi_1(x) \) is in fact just the rescaled electric field above the equator, as can be seen from equations (10) and (17):

\[
\Psi_1(x) = \frac{\partial \Psi}{\partial z} (x, z = 0) = -x R_{\text{LS}} B_R (x, z = 0) = -R_{\text{LS}}^2 E_z (x, z = 0). \quad (65)
\]

Upon substituting equations (63) and (64) into the regularity condition (20), we get

\[
z \Psi'_1 (x = 1) = -\frac{1}{2} \beta q L_x (-\Psi)^{\beta - 1} = \frac{1}{2} \beta q L_x z^{\beta - 1} [-\Psi_1(1)]^{\beta - 1}, \quad (66)
\]

and, therefore,

\[
\beta = 2, \quad (67)
\]

unless \( \Psi'_1(1) \equiv \partial_x \partial_z \Psi(1, 0) \neq 0 \) [and \( \beta > 2 \) if \( \Psi'_1(1) = 0 \)].

We can now use this result to analyze the magnetic field in the vicinity of the point \((x_0, 0)\). Equation (15) in the vicinity of this point can be written as

\[
(1 - x_0^2)^2 \left( \frac{\partial^2 \Psi}{\partial z^2} \right) + \frac{1 + x_0^2}{x_0} \frac{\partial \Psi}{\partial z} = -II'(\Psi), \quad (68)
\]

where \( \zeta = x_0 - x \). Now, just as we did in § 4, we can make a transition to the polar coordinates: \( \zeta = r \cos \theta, z = r \sin \theta \). In these polar coordinates equation (68) becomes

\[
(1 - x_0^2)^2 \left( \frac{\partial^2 \Psi}{\partial r^2} + \frac{\partial^2 \Psi}{\partial \theta^2} \right) + \frac{1 + x_0^2}{x_0} \frac{\partial \Psi}{\partial r} \cos \theta - \frac{\partial \Psi}{\partial \theta} \sin \theta = -II'(\Psi), \quad (69)
\]

Again, similarly to what we did in § 4 (see eq. [30]), let us assume that the magnetic flux function in region II (the region of open-field lines, \( \theta > \theta_0 = \pi/2 \)) is a power law of \( r \) near the point \((x_0, 0)\):

\[
\Psi(r, \theta) = -g(\theta) r^{\alpha_2}, \quad \alpha_2 > 1. \quad (70)
\]

Then, using equations (63) and (67), we can express the RHS of equation (69) as

\[
\text{RHS} = -II'(\Psi) = -2q I_x \Psi = 2q I_x g(\theta) r^{\alpha_2}, \quad (71)
\]

whereas the LHS of this equation can be expressed as

\[
\text{LHS} = -(1 - x_0^2)^2 r^{\alpha_2 - 2} \left[ \alpha_2 g(\theta) + \alpha_2 (\alpha_2 - 1) g'(\theta) + g''(\theta) \right] - \frac{1 + x_0^2}{x_0} \rho^{\alpha_2 - 1} \left[ \alpha_2 g(\theta) \cos \theta - g'(\theta) \sin \theta \right]. \quad (72)
\]

In this expression, the terms of lowest order in \( r \) are those proportional to \( r^{\alpha_2 - 2} \); at the same time we see from equation (71) that the RHS of equation (69) scales with \( r^{\alpha_2} \). Therefore, we see that in this case of the separatrix intersecting the equator inside the LC, \( x_0 < 1 \), the contribution of the RHS is completely negligible (as \( r \to 0 \)).

Thus, the dominant balance in equation (69) dictates that the terms of order \( r^{\alpha_2 - 2} \) should balance each other, and hence we get a very simple homogeneous, linear ODE for
the function \( g(\theta) \):

\[
g''(\theta) + \alpha_2^2 g(\theta) = 0. \tag{73}
\]

The boundary conditions for this equation are

\[
g(\theta_0 = \pi/2) = 0 = g(\pi), \tag{74}
\]

and so the solution with the lowest \( \alpha_2 \) (corresponding to the simplest magnetic field topology in this region) is obviously

\[
g(\theta) = -g_{\text{max}} \sin 2\theta, \quad \pi/2 \leq \theta \leq \pi, \tag{75}
\]

and, correspondingly,

\[
\alpha_2 = 2. \tag{76}
\]

Thus, \( \Psi_I(r, \theta) = g_{\text{max}} r^2 \sin 2\theta \). The fact that we have been able to find such a simple nondivergent solution near the T-point, whereas Lyubarskii (1990) has found only a logarithmically divergent one, can be explained by noting that, as we have shown in the beginning of this section, \( H''(\psi) \) must be equal to zero according to the LC regularity condition, whereas Lyubarskii assumed that \( I(\psi) \sim \psi \) and hence had a finite, nonvanishing \( H''(\psi) \).

As for region I (the region of closed-field lines), the magnetic field there, to lowest order in \( r \), is purely vertical and finite at \( x \rightarrow x_0 \), so we can write

\[
\Psi_I(r, \theta) = f_{\text{max}} \xi = f_{\text{max}} r \cos \theta, \tag{77}
\]

where \( f_{\text{max}} \) is related to \( I_z \) via the separatrix equilibrium condition (23):

\[
f_{\text{max}} = \frac{I_z}{x_0 \sqrt{1 - x_0^2}}. \tag{78}
\]

We express hope that these simple findings will be helpful in setting up or checking the correctness of future numerical attempts to solve this nontrivial problem.

6. DISCUSSION AND CONCLUSIONS

In this paper we have considered the axisymmetric force-free magnetosphere of an aligned rotating magnetic dipole, under the additional assumptions of ideal MHD and the uniformity of the field line angular velocity \( \Omega(\psi) = \text{const.} \). This fundamental model problem is of great importance to any attempts to understand the workings of radio pulsars.

More specifically, we have focused most of our attention on the structure of the magnetic field in the vicinity of the point of intersection of the separatrix (between the closed- and open-field regions) and the equator. We call this point the separatrix intersection point. The unique singular nature of this point makes it play an extremely important role in the overall global problem; in particular, without a thorough understanding of the subtleties of the magnetic field behavior near this point, it is impossible to prescribe the correct global boundary conditions in any sensible way.

We start, however, by discussing (in § 2) the basic equations governing the global force-free pulsar magnetosphere. We give special attention to the role played by the LC regularity condition in determining the poloidal electric current. After this general discussion, the rest of the paper is devoted entirely to the analysis of the separatrix intersection point.

We first consider the separatrix equilibrium condition in the vicinity of this point (see § 3) and find that if it lies at the LC, then all the poloidal current \( I(\psi) \) has to return back to the pulsar in the open-field region above the equator; i.e., there should be no singular current running in a current sheet along the equator and the separatrix. We then perform (in § 4) an asymptotic analysis of the relativistic Grad-Shafranov, or pulsar, equation in the vicinity of such an intersection point located at the LC. We find a unique self-similar solution that can be described by the power law \( \psi \sim r^\alpha \), where \( r \) is the distance from the intersection point and \( \alpha \approx 1.5045 \), and by the equator-separatrix angle \( \theta_0 \approx 0.76 \) rad. However, a further analysis of this solution in the region outside the LC shows that an LS (a surface where \( \mathbf{B} = \mathbf{E} \)) appears just outside the LC; in particular, we find that this LS originates right at the intersection point and extends outward at a finite angle with respect to the equator. The appearance of an LS in this case is, of course, not surprising, taking into account the fact that (in this case of the separatrix intersection point lying at the LC) the poloidal electric current \( I \) and hence the toroidal magnetic field have to become zero on the last open-field line. We therefore conclude that the only possibility for an ideal-MHD, force-free magnetosphere above the putative equatorial current sheet to extend at least some finite distance beyond the LC is for the separatrix intersection point to be located inside, as supposed to right at, the LC. Of course, we understand that it is impossible to find the exact location of this point from our local analysis. Instead, it must be determined self-consistently as a part of the global solution of the pulsar equation. Thus, we suggest that any future numerical investigations of the axisymmetric pulsar magnetosphere should incorporate the determination of the correct value of \( x_0 \) as one of their goals, without making any a priori assumptions in this regard.

These findings give us the motivation to consider the case in which the intersection point lies at some finite distance inside the LC. In § 5 we examine the behavior of the function \( I(\psi) \) near the last open-field line, \( \psi = \psi_\text{t} \) (above the equatorial current sheet), for this case; we find that the derivative \( I'(\psi) \) has to go to zero as \( \psi \rightarrow \psi_\text{t} \), whereas the current itself can approach a finite value \( I_\text{t} \neq 0 \). We then perform an asymptotic analysis of the magnetic field around the intersection point; it is essentially a somewhat simplified and trivialized analog of our analysis in § 4. We find that the separatrix approaches the equator at a right angle, \( \theta_0 = \pi/2 \), and that the field in the region of open-field lines behaves simply as \( \psi(r, \theta > \pi/2) \sim -r^2 \sin 2\theta \), while in the closed-field region the magnetic field is simply vertical and finite near the intersection point (we call this configuration the T-point).

Finally, we would like to make several remarks regarding the numerical simulations by CKF99 and Ogura & Kojima (2003):

1. The midplane boundary conditions \( \partial \psi = 0 \), \( R < R_\text{LS} \); \( \psi = \text{const.} \), \( R > R_\text{LS} \) adopted by both groups have automatically assumed that the separatrix intersection point lies at the LC and thus have precluded them from even considering the possibility \( x_0 < 1 \). It is interesting to note, however, that from the magnetic contour plots presented by
Ogura & Kojima (2003) it does seem that the actual intersection point lies a little bit inside the LC.

2. Whereas CKF99 have acknowledged that, because of the separatrix equilibrium condition, a discontinuity of $B_{\psi}$ across the separatrix would lead to a discontinuity in $B_{\text{pol}}$, Ogura & Kojima (2003) have not even mentioned this condition. Thus, it is not clear whether the separatrix equilibrium condition has been satisfied in their simulations. In the light of our present findings and of the fact that both of these groups have found $I_s \neq 0$, we suspect that the equilibrium condition has not, in fact, been satisfied in their studies, at least close to the separatrix intersection point. CKF99 have managed to deal with this problem in some artificial numerical way (by effectively smoothing out the discontinuity). They have thus obtained a smaller value for $\Psi_s$ than Ogura & Kojima (2003), who have seemingly just ignored the issue.

3. Ogura & Kojima (2003) have claimed that they had found $|E| > |B|$ at some finite distance outside the LC, whereas CKF99 have reported that in their solution $|E| < |B|$ everywhere. The origin of this discrepancy is not quite clear. It may be attributed to the difference in numerical resolution, although neither of the two groups have conducted very extensive convergence studies. In addition, as suggested by the referee of this paper, the difference may be due to the influence of the outer boundary conditions in the Ogura & Kojima (2003) solution. Indeed, Ogura & Kojima have had to deal with a computational domain of a finite size and have required the magnetic field to be horizontal at the outer boundary; this rather artificial condition may have affected the global structure of the solution. CKF99, on the other hand, have used a computational domain extending to infinity (by variable transformation), and their solutions have asymptotically approached Michel’s monopole solution.

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