PARABOLIC BUNDLES AND PARABOLIC HIGGS BUNDLES

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Abstract. This is a survey article about parabolic bundles and parabolic Higgs bundles.

1. Introduction

The main aim of this article is to give a sample survey about Parabolic bundles and Parabolic Higgs bundles. This is an expanded version of my survey talk at the conference NS@50 held at Chennai Mathematical Institute during the month of October 2015. There will be many glaring omissions which is partially due to time constraint and mainly due to my limited knowledge of the subject.

2. Parabolic vector bundles on a Riemann Surface

Definition 2.1. Let $X$ be smooth irreducible curve over an algebraically closed field $K$. Let $p_1, \ldots, p_n$ be fixed finite set of closed points of $X$. A parabolic vector bundle on $X$ with parabolic structure at $p_1, \ldots, p_n$ is a vector bundle $W$ on $X$ together with the following data at each $p = p_i$,

a) a flag

$$W_p = F_1W_p \supset F_2W_p \supset \ldots \supset F_rW_p,$$

of subspaces of the vector space $W_p$, the fiber of $W$ at $p$

b) and real weights

$$\alpha_1, \ldots, \alpha_r$$

attached to $F_1W_p, \ldots, F_rW_p$ such that $0 \leq \alpha_1 < \ldots < \alpha_r < 1$.

Remark 2.2. 0) By abuse of notation we say that $W$ is a parabolic vector bundle with parabolic structure at $p_1, \ldots, p_n$.

1) The numbers $k_1 = \dim F_1W_p - \dim F_2W_p, \ldots, k_r = \dim F_rW_p$ are called the multiplicities of $\alpha_1, \ldots, \alpha_r$.

2) A quasi-parabolic structure on $W$ at $p_1, \ldots, p_n$ is just the condition a) in Definition 2.1 above at each $p = p_i$.

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Definition 2.3. Let $W_1$ and $W_2$ be two parabolic vector bundles on $X$ with parabolic structure at $p_1, \ldots, p_n$. A morphism $G : W_1 \to W_2$ is a vector bundle map from $W_1$ to $W_2$ such that for any $p \in \{p_1, \ldots, p_n\}$, if we denote by $g_p$ the linear map induced by $G$ on the fibers at $p$, we have

$$g_p(F_i(W_1)_p) \subset F_{j+1}(W_2)_p$$

whenever $\alpha_i > \beta_j$, where $\alpha_i$ (resp. $\beta_j$) weights of $W_1$ (resp. $W_2$).

Definition 2.4. Let $W$ be a parabolic vector bundle on $X$ with parabolic weights $\alpha_{1,i}, \ldots, \alpha_{n,i}$ with multiplicities $k_{1,i}, \ldots, k_{r_i,i}$ at $p_i$ for $i = 1, \ldots, n$. Then the parabolic degree of $W$ is defined by

$$\text{Para deg}(W) = \deg(W) + \sum_i(\Sigma_j k_{j,i} \alpha_{j,i})$$

Definition 2.5. Given a parabolic vector bundle $W$ and given any sub vector bundle $V$ and quotient bundle $V'$ then one defines in a natural way a parabolic structure on $V$ and $V'$ as follows: for $p \in \{p_1, \ldots, p_n\}$ the flag on $V_p$ (resp. $V'_p$) is induced by taking intersection (resp. quotient) with the flag of $W_p$ and weight attached for the subspace $F_k(V_p)$ (resp. $F_k(V'_p)$) is $\beta_k = \alpha_i$ where $i$ is the largest (resp. largest) integer such that $F_k(V_p) \subset F_{i}(W_p)$ (resp. $F_i(W_p) \to F_k(V_p)$ is onto).

Definition 2.6. A parabolic vector bundle $V$ is said to be Parabolic stable (resp. Parabolic semi-stable) if for every parabolic sub-bundle $W$ of $V$ we have:

$$\frac{\text{Para deg}W}{\text{rank}(W)} < \frac{\text{Para deg}V}{\text{rank}(V)}$$

(resp. $\leq$).

For details we refer to [8]

3. Narasimhan-Seshadri type result for Parabolic bundles

Theorem 3.1. Mehta-Seshadri Theorem: (1) Let $X$ be a smooth projective curve over an algebraically closed field with $g(X) \geq 2$. Let $S$ be the set of all parabolic semi-stable bundles of rank $k$, with fixed parabolic structure at a given point $p \in X$, fixed weights $0 < \alpha_1 < \ldots < \alpha_r < 1$ of fixed multiplicities, fixed degree $d$ and parabolic degree $0$. Assume $\alpha_i$ are all rational. Two bundles $W, W' \in S$ are termed ”equivalent” if $grW = grW'$. Then the set of equivalence classes of $S$ carries in a natural way the structure of a normal projective variety of dimension $k^2(g + 1) + 1 + \text{dim}F$ where $F$ is the flag variety determined by the multiplicities of the parabolic structure at $p$.

(2) Assume that the field is field of complex numbers and $\Gamma$ be a discrete subgroup of $\text{PSL}(2, \mathbb{R})$ with a single equivalence class of cusp in $\mathbb{R} \cup \infty$ for its action on the upper half plane $H$. Let

$$H^+ = H \cup \{Q \in \mathbb{R} | \text{Q is a cusp for } \Gamma\}$$

Then $X = H^+/\Gamma$ is a compact Riemann Surface Fix a cusp $Q \in H^+$ and $P \in X$ be the corresponding point. Let $\Gamma_Q$ be the stabilizer. Then the above
variety is isomorphic to the set of equivalence classes of unitary representations of $\Gamma$ with the image of the generator of $\Gamma_Q$ being conjugate to the diagonal matrix $\exp(2\pi i\alpha_1), \ldots, \exp(2\pi i\alpha_r)$ where each $\alpha_i$ is repeated $k_i$ times. In particular, irreducible representations correspond to parabolic stable bundles.

**Applications of Mehta-Seshadri Theorem**

The notion of parabolic structure has many applications in algebraic geometry. For example, they appear in Hecke correspondences of Narasimhan and Ramanan [12]. It also appears in the work of Nagaraj and Seshadri on moduli of torsion sheaves on curve which is a union of two smooth curves meeting at point [10]. Parabolic bundles have had interesting applications even outside algebraic geometry, in topology and physics. One can find more details about the application of parabolic bundles by searching in the "web".

**Generalizations**

1) *Parabolic vector bundles on Higher dimension varieties.*

Maruyama and Yokogawa generalize the notion of parabolic bundles and various associated notions from curves to non-singular projective varieties of arbitrary dimension [9]. The parabolic data now reside over an effective divisors. They proceed to construct a coarse moduli scheme for stable parabolic sheaves over a non-singular projective variety.

2) *Generalized parabolic vector bundles.*

Usha Bhosle, Narasimhan and Ramadas (see [14], [11]) generalized the notion of parabolic bundles to what are called Generalized parabolic bundles and constructed moduli space of such bundles. Generalized parabolic bundles are useful in the study of torsion free sheaves on nodal curves. This notion is used by Usha Bhosle, Narasimhan and Ramadas, Sun Xiaotao, Nagaraj and Seshadri, Ivan Kausz and many others in their work (see [5], [10], [11], [14], [15], [16]).

3) *Parabolic Principal bundles.*

Several attempts were made to generalize the concept of Parabolic bundles to $G$ bundles, where $G$ is a reductive algebraic group. Laszlo and Sorger defined the notion of Parabolic $G$ bundles and studied the Picard group of moduli stack of such bundles on a projective smooth curve (see [7]). Balaji, Biswas and Nagaraj defined Parabolic $G$ bundle as a functor from category of $G$ modules to category of Parabolic Vector bundles satisfying some conditions (following Nori’s approach to principal $G$ bundles) and latter as a ramified geometric objects (see, [1], [2]).

Let $X$ be an irreducible smooth projective algebraic curve of genus $g \geq 2$ over the ground field $\mathbb{C}$, and let $G$ be a semisimple simply connected algebraic group. Balaji and Seshadri introduce the notion of semistable
and stable parahoric torsors under a certain Bruhat-Tits parahoric group scheme $G$ over $X$. They construct the moduli space of semistable parahoric $G$-torsors; and identified the underlying topological space of this moduli space with certain spaces of homomorphisms of Fuchsian groups into a maximal compact subgroup of $G$. The results give a generalization of the earlier results of Mehta and Seshadri on parabolic vector bundles (See [3]).

4. Parabolic Higgs bundles

Motivation: Ordinary Higgs bundles on a Riemann Surface were introduced by Hitchin. Recall, if $X$ is a compact Riemann, then Higgs bundle on $X$ is a pair $(V, \theta)$ where $V$ is a vector bundle on $X$ and $\theta : V \to V \otimes \Omega^1_X$ a homomorphism of bundles, where $\Omega^1_X$ is the bundle of holomorphic 1-forms on $X$. There is a notion (semi-) stability associated these pairs which is natural generalization of same notions for a vector bundle. Moduli space of Higgs bundles exists as quasi projective variety. There is an analogous Narasimhan-Seshadri theorem which sets up correspondence between certain Higgs bundles on $X$ and linear representations of the fundamental group of $X$. In order to generalize Mehta-Seshadri correspondence one need to extend the definition of Higgs bundle to the Parabolic case.

Parabolic Higgs bundles

Parabolic Higgs bundle on compact connected Riemann $X$ is a pair $(V, \theta)$ where $V$ is a parabolic vector bundle on $X$ and $\theta$ is as indicated in the $\theta : V \to V \otimes \Omega^1_X$ a homomorphism of bundles which at each parabolic point preserves the flag.

One can define (semi-)stability of Parabolic Higgs bundles and moduli of such objects were constructed by Konno Hiroshi, Maruyama-Yokogawa and many others. Analog of Mehta-Seshadri theorem holds in this context (see [6], [9] [17]).

In [13] Nitsure generalized the notion of Higgs bundles to what are now called Hitchin Pairs and constructed the moduli spaces stable Hitchin Pairs. This notion of Hitchin pairs has been generalized to parabolic Hitchin pairs. The moduli spaces of Parabolic Higgs bundles is still an active area of research.

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