The scale dependent intermittency exponents in developed hydrodynamic turbulence are calculated assuming a natural hierarchy of correlations in the turbulence. The major correlations are taken into account explicitly, while the remaining small correlations are considered as perturbations. The results agree very well with the currently available experimental data.

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• **INTRODUCTION:** The energy dissipation in developed hydrodynamic turbulence is strongly non-uniform in space (and time). This property is usually referred to as an intermittency. A reason for intermittency was already present in the original Kolmogorov-Obukhov model of turbulence, as it was uncovered by the well-known comment of Landau [1]. The major process in developed turbulence is the energy transfer from turbulent motions of large spatial scales (where the energy is basically contained) to the smallest scales (where the dissipation occurs). The transferred energy flux becomes more and more spatially non-uniform as it goes to smaller and smaller scales, which ultimately leads to pronounced intermittency. A reason for this is that the velocity field at smaller scales is capable of changing in time much faster than at larger scales. It implies that the velocity field at smaller scales is capable of changing in time much faster than at larger scales. It implies that the velocity field is a quasi-stationary (in statistical sense) picture of turbulence. The velocity field has enough to be established at smaller scales before a noticeable change in the local large-scale flow occurs. Therefore, the statistical characteristics of smaller-scale motions depend just on the local (in space and time) larger-scale motion. Then, there is no smoothing mechanism for spatial fluctuations in the density of energy flux and those are amplified as the flux goes from larger to smaller scales. The quantitative description for such an intermittent amplification has not been developed yet because of the complexity of the problem.

An idea is suggested below on how to overcome the difficulties, assuming that not all of the correlations causing the intermittency are equally important. It appears to be possible to derive and solve explicitly an equation that takes into account the major correlations. After a renormalization, a local flux-conservation constrain leads to an equation that has (in addition the Kolmogorov solution) an intermittent solution. The latter agree very well with experiment, which supports the employed assumptions about hierarchy of correlations.

• **ENERGY FLUX AND DISSIPATION:** The problem of intermittency can be considered quantitatively in the terms of energy and flux densities in $(\hat{k}, \hat{r})$-space. Provided the energy is pumped at the smallest wave numbers $k \sim k_0$, damped at the largest $k \sim k_d \gg k_0$ and there is neither pumping nor damping at all intermediate $k$, the densities of energy and flux satisfy in the latter – inertial – range the continuity equation. It can be averaged over directions of $\hat{k}$ and over some domain of fixed shape in $\hat{r}$-space. The result contains a surface integral (over the boundary of domain), that describes the spatial transfer of energy from the domain to its neighborhood, and an integral over the domain, that represents the averaged density of energy flux from smaller to larger $k$. The latter quantity is denoted below as $\epsilon(\hat{R}, x, k)$, where $\hat{R}$ and $x$ define the spatial position and linear size of the domain; the time variable and variables describing the shape of domain are omitted for simplicity. One could expect that the average flux density $\epsilon(\hat{R}, x, k)$ tends at $k \gg 1/x$ (but $k$ is inside the inertial range) to a $k$-independent limit. Such a limit, if it exists, should be identified with the average density of energy dissipation in given domain at given moment of time, which quantity is denoted below as $\epsilon(\hat{R})$.

Statistical properties of random fields $\epsilon(\hat{R}, x, k)$ and $\epsilon(\hat{R}, x)$ may be sensitive to the shape of domain used for the averaging of energy flux and dissipation densities. Therefore, for comparison with experiments, the same shapes as there must be employed in the theory as well. Because of the technical reasons, the experiments usually deal with the so called “one-dimensional cuts”, obtained by means of small probes that record the longitudinal velocity of fluid in fixed points passed by the stream. In some recent experiments the transverse component of velocity is measured, but the currently available database contains primarily records of the longitudinal velocity. When such data are processed, the density of energy dissipation rate is substituted usually by the square of the velocity derivative along the record.

• **BASIC EQUATIONS:** In order to describe the amplification of fluctuations in density of energy flux from larger to smaller scales, consider the following conditional probabilities. Let $G_{n}(M) dM$ be the probability to find the flux $\epsilon(\hat{R}, x/2^n, 2^n / x)$ in the range $(M, M + dM)\epsilon$, provided $\epsilon(\hat{R}, x, 1/x) = \epsilon$. The ensemble averaging means here and further the $\hat{R}$-space averaging. The probability density
$G_n(M)$ does not depend on $x$ (inside the inertial range). It is a scale-invariance hypothesis. The $c$-dependence of $G_n(M)$ (which would imply a correlation between the local amplification coefficient $M$ and energy flux density $c$) is assumed to be negligible in a zero-order approximation. When correlation between the consequent amplification coefficients is also negligible, the moments $G_{n,q} \equiv \int dM M^q G_n(M)$ depend on $n$ exponentially:

$$G_{n,q} = G_{1,q}^n \equiv G_q^n.$$  \hspace{1cm} (1)

Let $g_n(M)dM$ be the probability to find the flux $\epsilon(\vec{R}, x, 2^n/x)$ in the range $(M, M + dM)$, provided $\epsilon(\vec{R}, x, 1/x) = \epsilon$. The consequent amplification coefficients of this kind are strongly correlated, so that $n$-dependence of $g_n(M)$ is not trivial. There is, however, a relation between the probability densities $g_n(M)$ and $G_1(M) \equiv G(M)$. To find it, consider the “one dimensional cut” of length $x$ as the sum of its halves (which length is $x/2$). The conditional probability to find the flux $\epsilon(\vec{R}, x/2, 2/x)$ in the range $(M, M + dM)\epsilon$, provided $\epsilon(\vec{R}, x, 1/x) = \epsilon$, is $G(M)dM$. It relates to each half of the segment $x$, when the dependence on the position of the half inside the segment is negligible. The amplification coefficients for the halves virtually do not affect each other, because of the strongly fluctuating short-scale spatial transfer of the energy in directions transverse to the segment. Then, the following recurrent relation holds up to small corrections:

$$g_n(M) = \int dM_1 dM_2 dM_3 dM_4 G(M_1)G(M_2) \times$$
$$g_{n-1}(M_3)g_{n-1}(M_4)\delta (M - (M_1M_3 + M_2M_4)/2).$$  \hspace{1cm} (2)

It can be reduced to the recurrent relation for moments

$$g_{n,q} \equiv \int dM M^q g_n(M),$$

that looks as

$$g_{n,q} = \sum_{q_1 = 0}^{q-1} q! G_{q_1} G_{q-q_1} g_{n-1,q_1} g_{n-1,q-q_1}. $$  \hspace{1cm} (3)

Eq. (3) with “initial conditions” $g_{0,q} = g_{n,0} = 1$ allows one to express $g_{n,q}$ in the terms of $G_q$ for all positive integer $n$ and $q$.

The $n$-dependent moments of energy flux densities averaged over segment $x$ and over its part $x/2^n$ at the same wave number $k_n = 2^n/x$ would be equal to each other for a spatially uniform flux. For an intermittent energy flux, the ratio

$$G_{n,q}/g_{n,q} \equiv 2^{n\mu_{n,q}}$$  \hspace{1cm} (4)

increases in $n$ (which indicates the amplification of fluctuations). According to (1) and (3), the “intermittency exponents” $\mu_{n,q}$ are given by

$$\mu_{n,q} = \log_2 G_q - n^{-1} \log_2 g_{n,q}. $$  \hspace{1cm} (5)

**Explicit Solution of Eq. (3):** Equation (3) can be solved explicitly, and thus the “intermittency exponents” $\mu_{n,q}$ can be expressed in the terms of $n$-independent functions $G_q$. Provided the latter satisfy restriction $G_q < 2^{q-1}$, the solution $g_{n,q}$ tends to finite limit $g_q$ as $n$ tends to infinity. Then, the “intermittency exponents” $\mu_{n,q}$ tend to $\mu_q = \log_2 G_q$ as $n$ tends to infinity, and the above restriction on $G_q$ is equivalent to $\mu_q < q - 1$. Explicit formulas for $g_{n,q}$ that solve (3) can be derived consequently for $q = 1, 2, ...$

For $q = 1$, taking into account that $G_0 = G_1 = 1$, one gets from (3) $g_{n,1} = 1$, and hence $\mu_{n,1} = 0$.

For $q = 2$ the solution of (3) is

$$g_{n,2} = g_2 + c_2 (G_2/2)^n;$$

$$g_2 = 1/(2 - G_2) , c_2 = 1 - g_2.$$  \hspace{1cm} (6)

For $q = 3$ the solution is

$$g_{n,3} = g_3 + c_3 (G_3/4)^n + c_{32} (G_2/2)^n;$$

$$g_3 = 3G_{32}/8 - G_4, c_32 = 3G_{32}/8 - G_4 , c_3 = 1 - g_3 - c_{32}.$$  \hspace{1cm} (7)

For $q = 4$ it is

$$g_{n,4} = g_4 + c_4 (G_4/8)^n + c_{43} (G_3/4)^n + c_{42} (G_2/2)^n;$$

$$g_4 = 4G_{32}/8 + G_4, c_{43} = 4G_{32}/8 + G_4 , c_{42} = 3G_2/8 - G_4;$$

$$c_4 = 1 - g_4 - c_{43} - c_{42} - c_{42}. $$  \hspace{1cm} (8)

For $q = 5$:

$$g_{n,5} = g_5 + c_5 (G_5/16)^n + c_{54} (G_4/8)^n + c_{53} (G_3/4)^n + c_{52} (G_2/2)^n + c_{532} (G_2G_3/8)^n + c_{522} (G_2/2)^n;$$

$$g_5 = 5G_{42}/8 - G_5 , c_{54} = 5G_{52}/8 + 2G_4 - G_5 ,$$

$$c_{53} = 5G_{42}/8 + 2G_4 - G_5 , c_{532} = 5G_{42}/8 + 2G_4 - G_5;$$

$$c_5 = 1 - g_5 - c_{54} - c_{53} - c_{52} - c_{532} - c_{522}.$$  \hspace{1cm} (9)

The similar (but longer) formulas exist for $q = 6, 7, 8, ...$ In particular, the $n$-independent function $g_q$ is given in general by

$$g_q = \sum_{q_1 = 1}^{q-1} q! G_{q_1} G_{q-q_1} g_{q_1} g_{q-q_1}.$$  \hspace{1cm} (10)

Consider now the alternative option $G_q > 2^{q-1}$. Under such a condition, $g_{n,q}$ does not tend to a finite limit as $n$ tends to infinity, but increases exponentially like $g_{n,q} \propto (G_q/2^{q-1})^n$. It follows then from (3) that the intermittency exponent $\mu_{n,q}$ tends to $q - 1$ as $n$ tends to infinity (and one can see that $\mu_{n,q}$ approaches to $q - 1$ from below). There is a simple physical interpretation
of such a behavior. It indicates that the strongest fluctuations of $\epsilon(\vec{R}, x)$ increase faster than $1/x$ as $x$ tends to zero (so that the density of energy dissipation rate has singularities non-integrable over the one-dimensional cuts). According to the estimate $\epsilon(\vec{R}, x) \sim \tilde{v}(\vec{R}, x)^3/x$ (where $\tilde{v}(\vec{R}, x)$ is a smoothed velocity variation in a fluctuation of size $x$ around point $\vec{R}$), it implies that the sweep $\tilde{v}(x)$ of $\tilde{v}(\vec{R}, x)$ variation in $\vec{R}$-space increases when the fluctuation size $x$ tends to zero. Thus, the alternative $G_q > 2^{q-1}$ corresponds to blow-up of velocity itself, rather than just its gradients, in Euler fluid.

In spite of many efforts applied to the problem, there is no clear theoretical answer yet to the question, whether velocity blows-up or not in Euler fluid. Practical answer can be extracted from the experimental data on intermittency, provided the collected statistics is sufficient to detect the strongest blow-up. The currently available database supports assumption $\mu_q \ll q - 1$ for $q = 2, 3, 4, \ldots$.

**DEGENERATE SOLUTIONS:** The above formulas for $g_{n,q}$ contain all possible items of the kind $(G_q/2^{q-1})^n$, $q = 2, 3, \ldots$ and their products. There is however a special class of solutions that contain only the items with $q = 2$. It corresponds to the physical situation when only one major kind of correlations exists, that dominates over all other correlations. For such degenerate solutions, all quantities $G_q$ (or $\mu_q = \log_2 G_q$) can be expressed in the terms of $\mu_2$. In the limit of infinitely small $\mu_2$, the expression looks as

$$\mu_q \approx \mu_2 q(q - 1)/2.$$  \(11\)

The dimensionality $D_q = 3 + \mu_q - q \mu_2'$ corresponding to \( \mu_2 \) is $D_q \approx 3 - \mu_2 q^2/2$. It would turn into zero at $q = q_M \approx (6/\mu_2)^{1/2}$, if $\mu_2'$ would be applicable there, – which is not so, as the applicability condition $\mu_q \ll 1$ is violated at $q \sim q_M$. In fact, the applicability condition is a little bit more soft quantitatively, it is $G_q - 1 \ll 1$. The slightly improved formula \( \mu_2 \) is

$$G_q - 1 \approx (G_2^{q-1} - 1) q/2.$$  \(12\)

It appears to be of a reasonable accuracy for real turbulence up to $q = 5$ (see more accurate formulas below for the comparison).

**INTERMITTENCY EXPONENTS:** For a moderately small $\mu_2$ that occurs in real turbulence, the above formulas require a modification. An appropriate formula for $\mu_q$ is derived from very general heuristic reasons in \( \mu_2 \) (where the major mechanism of intermittency is supposed to be blow-up of velocity gradients in Euler fluid, leading to point singularities). This formula looks as

$$\mu_q = \gamma q [\text{arsh} (q/a) - \text{arsh} (1/a)], \quad q \leq q_M,$$  \(13\)

$$\mu_q = \alpha_0 q - 3, \quad (\alpha_0 = \mu'_0), \quad q \geq q_M;$$  \(14\)

$$q_M = 3/\gamma \left[1/2 + (1/4 + (\gamma/3)^2)^{1/2}\right]^{1/2},$$  \(15\)

where \text{arsh} $= \text{ln}(\xi + \sqrt{1+\xi^2})$ is the hyperbolic arcsin, prime signifies $q$-derivative, $\gamma$ and $a$ are some numbers. For $a \gg q$, \( \mu_3 \) reduces to \( \mu_2 = 2\gamma/a \).

There is a connection between parameters $\gamma$ and $a$, so that $\mu_q$ can be expressed in the terms of $\mu_2$ for realistic values of the latter parameter as well. The extra condition has the form

$$F(\alpha_M, \alpha_0) \equiv 2^{3-\alpha_0} - 2^{-\alpha_M} - 7 = o(1)$$  \(16\)

(which is easy to understand in the framework of a cubic shell model). The quantity $\alpha_q = \mu'_0$ can be referred to as the “singularity exponent”. When $q$ increases from zero to $q_M$, $\alpha_q$ increases from its smallest value $\alpha_0 < 0$ to its largest value $\alpha_M = \alpha_0$. The exponent $\alpha_M$ corresponds to the strongest possible amplification of $\epsilon$ in a “hot spot”, i.e., to the strongest blow-up in Euler fluid. The exponent $\alpha_0 < 0$ corresponds to the background that goes down to compensate for amplification of $\epsilon$ in the hot spot and to secure the energy flux conservation. The flux conservation condition around the strongest hot spot gives

$$F(\alpha_M, \alpha_0) \approx \mu_2^2.$$  \(17\)

For comparison between the theory and experiment, the above definition of intermittency exponents $\mu_{n,q}$ (selected to get explicit analytical formulas) must be slightly modified and also small effects neglected in the derivation of eq. \( \mu_2 \) must be taken into account. The different definitions of intermittency exponents can be identified with each other, as long as only scale-independent intermittency exponents $\mu_q$ (corresponding to infinite scale ratio) are considered. The really measured intermittency exponents should be considered taking into account the finite scale-ratio effects which are sensitive to the definition. In particular, the above definition must be distinguished from that used in \( \mu_2 \). The latter deals with the amplification coefficients $M(\vec{R}, x, y) \equiv \epsilon(\vec{R}, y)/\epsilon(\vec{R}, x)$, and defines intermittency exponents in a way equivalent to

$$\overline{M(\vec{R}, x, x/2^n)^n} = 2^n \mu_{n,q}.$$  \(18\)

Here bar signifies the $\vec{R}$-space averaging. The result does not depend on $x$ due to the scale invariance.

The modification can be neglected for infinitely small $\mu_2$. For a moderately small $\mu_2$, the quantities $\mu_{n,q}$ defined by \( \mu_2 \) are expressed in the terms of $\mu_2$ by

$$\mu_{n,q} \approx \mu_q - \sqrt{n} (\log_2 g_{n,q})^C, \quad C \approx 1 + \ln(1 - \mu_2).$$  \(19\)
(where $\mu_q$ is given by $[13]$ and $g_{n,q}$ is the above exact solution of eq.$(3)$ with $G_q = 2^{\mu_q}$). Formula $[19]$ is valid for $n = 1, 2, \ldots$ and $q = 2, 3, \ldots$.

Noteworthy, that small correlations between the local values of energy flux density $\epsilon$ and amplification coefficient $M$ lead to a non-zero intermittency exponent $\mu_{n,1}$ (which otherwise is zero and which usually is assumed to be zero). It follows from the definition of $M$ rewritten in the form $\epsilon(\vec{R}, y) \equiv \epsilon(\vec{r}, x)M(\vec{R}, x, y)$. Space averaging of this relation gives

$$\bar{\tau}(1 - \bar{M}) = \left(\bar{\epsilon} - \bar{\tau}\right)(\bar{M} - \bar{M})$$

where $\bar{\tau}$ does not depend on $x$ (or $y$) due to the total flux conservation. If correlations between $\epsilon$ and $M$ were absent, the right hand side of $[20]$ would be zero. It would imply $\bar{M} = 1$ and $\mu_{n,1} = 0$, in agreement with eq.$[3]$ that neglects considered correlations. Positive correlations between $\epsilon$ and $M$, i.e., a positive right-hand side of $[20]$, implies that $\bar{M} < 1$, which means in turn $\mu_{n,1} < 0$. According to $[3]$, the $q$-dependence of the intermittency exponents is given by

$$\mu_{n,q} = \gamma_n q \left[\text{arsinh}(q/a_n) - \text{arsinh}(1/a_n)\right] + \mu_{n,1} q$$

$$q \leq q_n,M = 3/\gamma_n \left[1/2 + \left(1/4 + (a_n\gamma_n/3)^2\right)^{1/2}\right]^{1/2}$$

(which corresponds to the dimensionality $D_{n,q} \equiv 3 + \mu_{n,q} - g_{n,q}^\mu = 3 - \gamma_n q^2/\sqrt{q^2 + a_n^2}$, so that $D_{n,q,M} = 0$).

It can be verified now that these formulas (have not been contained in comparison with experiment), but do have both the Kolmogorov and the intermittent solutions.

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