COUPLING OF TWO CONFORMAL FIELD THEORIES AND
NAKAJIMA-YOSHIOKA BLOW-UP EQUATIONS

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Abstract. We study the vertex operator algebras which naturally arises in relation to
the Nakajima-Yoshioka blow-up equations.

1. Introduction

1.1. Denote by $M(r,N)$ the moduli space of framed torsion free sheaves on $\mathbb{CP}^2$ of rank
$r$, $c_1 = 0$, $c_2 = N$. This space is a smooth partial compactification of moduli space of
$U(r)$ instantons.

There is a natural action of the $r + 2$ dimensional torus $T$ on the $M(r,N)$: $(\mathbb{C}^*)^2$ acts
on the base $\mathbb{CP}^2$ and $(\mathbb{C}^*)^r$ acts on the framing at the infinity. The Nekrasov partition
function for pure Yang-Mills theory is defined as the equivariant volume:

$$Z(\epsilon_1, \epsilon_2, \vec{a}; q) = \sum_{N=0}^{\infty} q^N \int_{M(r,N)} 1,$$

where $\vec{a} = (a_1, \ldots, a_r)$ and $\epsilon_1, \epsilon_2, a_1, \ldots, a_r$ are the coordinates on the $t = \text{Lie} T$. The last
integrals can be computed by localisation method and equal to the sum of contributions
of torus fixed points (which are labeled by $r$-tuple of Young diagrams $\lambda_1, \ldots, \lambda_r$).

Nakajima and Yoshioka in paper [21] proved so called blow-up equations for the function
$Z(\epsilon_1, \epsilon_2, \vec{a}; q)$. For the $r = 1, 2$ this equations have form

$$Z(\epsilon_1, \epsilon_2, \vec{a}; q) = Z(\epsilon_1, \epsilon_2 - \epsilon_1, a; q) \cdot Z(\epsilon_1 - \epsilon_2, \epsilon_2, a; q)$$

($1.1$)

$$Z(\epsilon_1, \epsilon_2, a_1, a_2; q) = \sum_{k \in \mathbb{Z}} \frac{q^{k^2}}{l_k} Z(\epsilon_1, \epsilon_2 - \epsilon_1, a_1 + k\epsilon_1, a_2 - k\epsilon_1; q) \cdot Z(\epsilon_1 - \epsilon_2, \epsilon_2, a_1 + k\epsilon_2, a_2 - k\epsilon_2; q)$$

($1.2$)

The geometrical meaning of these equations is the relation between $M$ and $\widehat{M}$ — the
moduli space of framed torsion free sheaves on the blow up of $\mathbb{CP}^2$. The shifted parameters
$\epsilon_1, \epsilon_2 - \epsilon_1$) and $\epsilon_1 - \epsilon_2, \epsilon_2$ are weights of the torus action on the tangent space of two torus
fixed points on the blow up of $\mathbb{CP}^2 \subset \mathbb{CP}^2$.

For the $r = 1$ the function $Z(\epsilon_1, \epsilon_2, \vec{a}; q) = \exp(q/\epsilon_1\epsilon_2)$ (see [21] Sec 5]) so the equation
($1.1$) is trivial. But for the $r = 2$ the function $Z(\epsilon_1, \epsilon_2, a_1, a_2; q)$ coincide with the certain
limit of the four point conformal block (due to the AGT conjecture [1]) for the conformal
field theory (CFT) with the central charge $c = 1 + \frac{6(\epsilon_1 + \epsilon_2)^2}{\epsilon_1\epsilon_2}$. Therefore the equation

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suggests the relation in conformal field theory. This relation is a main purpose of the paper.

1.2. We will mainly consider the $r = 2$ case which corresponds to the CFT with the Virasoro algebra symmetry. Denote by $V_{\Delta,c}$ the Verma module over the Virasoro algebra and by $L_{\Delta,c}$ its irreducible quotient. The vacuum module $L_{0,c}$ has the structure of the vacuum module of the Virasoro vertex operator algebra. The corresponding operators are stress-energy tensor $T(z)$, its derivatives and products. We parametrize central charge $c = c(b) = 1 + 6(b + b^{-1})^2$ and denote this vertex operator algebra as $M_b$ similar to the notation for the minimal models. The representations $L_{\Delta,c}$ become modules over the algebra $M_b$.

We know from the AGT conjecture that the left hand side of (1.2) is a conformal block\(^1\) in $M_b$, where $b = \sqrt{\epsilon_1/\epsilon_2}$. On the right hand side we have linear combination of the conformal blocks in $M_{b_1} \otimes M_{b_2}$, for the appropriate $b_1 = b/\sqrt{b^2 - 1}$, $b_2 = \sqrt{1 - b^2}$. It appears that right hand side is a conformal block for the vertex algebra $A_b$. This algebra $A_b$ appears to be an extension of the $M_{b_1} \otimes M_{b_2}$ by the field $\Phi_{1,3} \cdot \Phi_{3,1}$ (in notation of [3]).

Geometrically two Virasoro algebras $M_{b_1}$ and $M_{b_2}$ correspond to two torus fixed points on the blow up of $\mathbb{C}^2$. Algebraically one can extend the product $M_{b_1} \otimes M_{b_2}$ due to the relation between central charges. In our case this relation is $b_1^2 + b_2^{-2} = -1$, more general relations of this kind will be discussed in the Conclusion.

The identity (1.2) means the relation between vertex algebras $M_b$ and $A_b$. We prove for generic $b$ that $A_b \cong M_b \otimes U$ for certain conformal field theory $U$, see Theorem 2.1.

This theory $U$ is one of the main objects of the paper. It appears that $U$ can be constructed in terms of one free bosonic field $\varphi(z)$. More precisely as a vertex operator algebra $U$ is isomorphic to lattice algebra $V_\sqrt{2} \mathbb{Z}$ or affine Lie algebra $\widehat{\mathfrak{sl}}(2)$ on the level 1. But $U$ have nonstandard (deformed) stress-energy tensor, its central charge equals to -5. see eq. (2.3).

The algebra $U$ contains two commuting Virasoro subalgebras with central charges $\frac{-22}{5}$ and $\frac{-3}{5}$. This central charges corresponds to minimal models $(2, 5)$ and $(5, 3)$. The sum of the corresponding stress-energy tensors $T_{2/5} + T_{3/5}$ equals to the full stress-energy tensor $T_U$. As consequence we can decompose the $U$ as the tensor product of minimal models see Theorem 2.4 and character identities (2.9).

1.3. The paper is organized as follows. In the Section 2 we state the main results of the paper (which were shortly described above). The next section is devoted to proofs. The main tool of the proof is a Drinfeld-Sokolov reduction of the representation of $\widehat{\mathfrak{sl}}(2)_1 \oplus \widehat{\mathfrak{sl}}(2)_k$ with respect to diagonal $\widehat{\mathfrak{sl}}(2)_{k+1}$. Some arguments were based on the explicit computations which were made by use Akira Fujitsu ope.math package [15].

In the Section 4 we discuss the combinatorial meaning of the character identities (2.9) between the characters of the $\widehat{\mathfrak{sl}}(2)$ of the level 1 and minimal models $(2, 5)$ and $(5, 3)$. In the Section 5 we show how to use our results to the blow-up equations.

In the Conclusion we discuss possible generalizations of the product $M_{b_1} \otimes M_{b_2}$ and theory $U$.

\(^1\)actually, the Whittaker-Gaiotto limit of the conformal block
2. Results

2.1. Vertex operator algebras. We will use the language of vertex operator algebras, (VOA for short) (see e.g. [13]). Recall that vector space $V$ is called vacuum representation of VOA if any vector $v \in V$ corresponds to the power series of operators $Y(v; z) = \sum Y_n z^{-n}$, $Y \in \text{End}(V)$. This correspondence $v \leftrightarrow Y(v; z)$ is called the operator-state correspondence. In the definition of vertex operator algebra the correspondence $v \leftrightarrow Y(v; z)$ should satisfy certain conditions: vacuum axiom, translation axiom and locality axiom.

Recall that the vertex algebra $V$ is called conformal if there exist non-zero conformal vector $\omega \in V$ such that corresponding power series of operators $T(z)$ satisfy:

$$T(z)T(w) = \frac{c}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial T(w) + \text{reg}.$$  \hfill (2.1)

The corresponding $T(z)$ is called stress-energy tensor, parameter $c$ is called the central charge. If we expand $T(z)$ into power series $T(z) = \sum_n L_n z^{-n-2}$ then equation (2.1) is equivalent to the Virasoro algebra (Vir for short) relations:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}c\delta_{n,-m}.$$  

2.2. Algebra $\mathcal{U}$. First we recall the construction of the lattice vertex algebra for one-dimensional lattice $\sqrt{2} \cdot \mathbb{Z}$ (see [13] Sec 5.2). Let $a_n$ be generators of the Heisenberg algebra:

$$[a_n, a_m] = n\delta_{m+n,0}.$$  

It is convenient to consider operators $a_n$ as modes of the bosonic field $\varphi(z)$:

$$\varphi(z) = \sum_{n \in \mathbb{Z}} \frac{a_n}{n}z^{-n} + a_0 \log z + \hat{Q},$$  \hfill (2.2)

where the operator $\hat{Q}$ is conjugate to the operator $\hat{P} = a_0$, i.e. satisfy the relation: $[\hat{P}, \hat{Q}] = 1$. The relations of the Heisenberg algebra can be rewritten in terms of operator product expansion:

$$\varphi(z)\varphi(w) = \log(z-w) + \text{reg}.$$  

We will contract such notation to $\varphi(z)\varphi(w) \sim \log(z-w)$ below.

Denote by $F_\lambda$ the Fock representation of the Heisenberg algebra with the highest weight vector $v_\lambda$:

$$a_n v_\lambda = 0 \text{ for } n > 0, \quad a_0 v_\lambda = \lambda v_\lambda.$$  

Denote by $S_\lambda$ the shift operator $S_\lambda: F_\mu \to F_{\mu+\lambda}$ defined by

$$S_\lambda v_\mu = v_{\mu+\lambda}, \quad [S_\lambda, a_n] = 0, \text{ for } n \neq 0.$$  

Actually $S_\lambda$ is just an exponent $\exp(\lambda\hat{Q})$. 

The direct sum $V_{\sqrt{Z}} := \bigoplus_{k \in \mathbb{Z}} F_{k \sqrt{2}}$ has the structure of the vertex operator algebra. This algebra is called the lattice algebra for the lattice $\sqrt{2} \cdot \mathbb{Z}$. Under the operator-state correspondence the highest weight vectors $v_\lambda$, $\lambda = k \sqrt{2}$ correspond to

$$Y(v_\lambda; z) := e^{\lambda \varphi} := S_\lambda z^{\lambda_0} \exp \left( \lambda \sum_{n \in \mathbb{Z}_{>0}} \frac{a_{-n} z^n}{n} \right) \exp \left( \lambda \sum_{n \in \mathbb{Z}_{>0}} \frac{a_n z^{-n}}{-n} \right),$$

Here and below : $\ldots$ : denotes the creation-annihilation normal ordering. For the more general vectors of the form $v = a_{-m} \cdots a_{-1} v_\lambda$ the corresponding operator have the form:

$$Y(v; z) =: (\partial^m \varphi)^n \cdots (\partial \varphi) e^{\lambda \varphi} :$$

For the more details about this construction see [13, Sec. 5.2].

The algebra $V_{\sqrt{Z}}$ is isomorphic to the vertex operator algebra of affine Lie algebra $\widehat{\mathfrak{sl}}(2)$ on the level 1. We denote the standard generators of $\widehat{\mathfrak{sl}}(2) = \mathfrak{sl}(2) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} K$ by $e_n = e \otimes t^n$, $f_n = f \otimes t^n$, $h_n = h \otimes t^n$, and the central element by $K$. We denote by $L_{h,k}$ the irreducible module of the $\widehat{\mathfrak{sl}}(2)$ algebra generated by the highest vector $v$ such that:

$$e_n v = 0, \text{ for } n \geq 0; \quad f_n v = h_n v = 0, \text{ for } n > 0; \quad h_0 v = hv, \quad K v = kv.$$  

The value $k$ of the central element is called the level of the representation.

The module $L_{0,k}$ has the structure of the vacuum module of the VOA. This algebra for generic $k$ will be used in the next section. If $k = 1$ then the VOA $L_{0,1}$ is isomorphic to the lattice algebra $V_{\sqrt{2}}$. The action of the generators of $\widehat{\mathfrak{sl}}(2)$ is defined by the formulas:

$$\sum_{n \in \mathbb{Z}} e_n z^{-n-1} := e^{\sqrt{2} \varphi} ; \quad \sum_{n \in \mathbb{Z}} f_n z^{-n-1} := e^{-\sqrt{2} \varphi} ; \quad \sum_{n \in \mathbb{Z}} h_n z^{-n-1} = \sqrt{2} \partial \varphi(z).$$

The standard conformal vector for the algebra $V_{\sqrt{2}} = L_{0,1}$ is $\omega_0 = \frac{1}{2} a_{-1}^2 v_0$, the corresponding stress-energy tensor equals $T(z) = \frac{1}{2} (\partial \varphi)^2$, and has the central charge 1. This VOA have two representations: the vacuum representation and the second one $\bigoplus_{k \in \mathbb{Z} + \frac{1}{2}} F_{k \sqrt{2}} = L_{1,1}$. Their characters i.e. traces of $q^{L_0}$ are:

$$\chi(L_{0,1}) = \chi \left( \bigoplus_{k \in \mathbb{Z}} F_{k \sqrt{2}} \right) = \sum_{k \in \mathbb{Z}} q^{k^2} (q)_\infty = 1 + 3q + 4q^2 + 7q^4 + \ldots,$$

$$\chi(L_{1,1}) = \chi \left( \bigoplus_{k \in \mathbb{Z} + \frac{1}{2}} F_{k \sqrt{2}} \right) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} q^{k^2} (q)_\infty = 2q^{1/4} + 2q^{5/4} + 6q^{9/4} + \ldots,$$

where we used that $L_0 v_\lambda = (\lambda^2/2) v_\lambda$.

There are other possible conformal vectors in $V_{\sqrt{2}}$. Namely the local operators $\frac{1}{2} (\partial \varphi)^2 + u (\partial^2 \varphi)$ satisfy stress-energy relation (2.1) with the central charge $c = 1 - 12u^2$. The addition of $u (\partial^2 \varphi)$ changes the $L_0$ operator to $L_0 - ua_0$. In particular the eigenvalues of the new $L_0$ are integers if and only if $u \in \frac{1}{\sqrt{2}} \mathbb{Z}$.

Now we can define $U$. 
**Definition 2.1.** The conformal algebra $\mathcal{U}$ coincide with the $V_{\sqrt{2}z}$ as the operator algebra, but the stress-energy tensor is modified:

$$T_{\mathcal{U}} = \frac{1}{2} (\partial \varphi)^2 + \frac{1}{\sqrt{2}} (\partial^2 \varphi) + \epsilon \left( 2 (\partial \varphi)^2 e^{\sqrt{2} \varphi} + \sqrt{2} (\partial^2 \varphi) e^{\sqrt{2} \varphi} \right) = \frac{1}{2} \partial_z \varphi(z)^2 + \frac{1}{\sqrt{2}} \partial^2_z \varphi(z) + \epsilon \partial^2_z e(z), \quad \epsilon \neq 0 \quad (2.3)$$

It is clear that the conformal algebras $\mathcal{U}$ are isomorphic for different values $\epsilon \neq 0$. For $\epsilon = 0$ $T_{\mathcal{U}}(z)$ has the from discussed above form for $u = \frac{1}{\sqrt{2}}$ and central charge $-5$.

The additional term in (2.3) corresponds to vector $2e_{-3}v_0 = (2a_{-1} + \sqrt{2}a_{-2})v_{\sqrt{2}}$. The eigenvalue of the operator $L_0 - \frac{1}{\sqrt{2}}a_0$ on this vector equals to 2. The OPE of corresponding operator $\partial^2 e(z)$ has no singular terms (since $e(z)e(w) \sim (z - w)^2$). Therefore $T_{\mathcal{U}}$ defined in (2.3) satisfy stress-energy tensor OPE (2.1) with the central charge $a_\mathcal{U} = -5$.

We will call this algebra by Urod conformal operator algebra. This algebra is one of the main objects of the paper.

The representations of $\mathcal{U}$ are the same as the representations of $V_{\sqrt{2}z}$: $U_0 = \bigoplus_{k \in \mathbb{Z}} F_{k \sqrt{2}}$ and $U_1 = \bigoplus_{k \in \mathbb{Z}+1/2} F_{k \sqrt{2}}$, but with the different characters due to shift of the $L_0$ operator. Schematically this shift of the grading is represented on the following picture:

![Diagram of grading shift](image)

**Figure 1.** The basic vectors with the lowest $L_0$ grading. The left part correspond to the vacuum representation of $V_{\sqrt{2}z}$, the right part correspond to the vacuum representation of $\mathcal{U}$. Dotted curved arrows shows the shift of the $L_0$ grading to $L_0 - \frac{1}{\sqrt{2}}a_0$.

Their characters have the form:

$$\chi(U_0) = \text{Tr} q^{L_0} |_{U_0} = \sum_{k \in \mathbb{Z}} q^{k^2-k} \frac{q^{k^2}}{(q)_\infty} = 2 + 2q + 6q^2 + 8q^3 + \ldots, \quad (2.4)$$

$$\chi(U_1) = \text{Tr} q^{L_0} |_{U_1} = \sum_{k \in \mathbb{Z}+\frac{1}{2}} q^{k^2-k} \frac{q^{k^2}}{(q)_\infty} = q^{-1/4} + 3q^{3/4} + 4q^{7/4} + \ldots.$$
Remark 2.1. It is interesting to note that \( \chi(U_0) = q^{-1/4}\chi(L_{1,1}) \) and \( \chi(U_1) = q^{-1/4}\chi(L_{0,1}) \).

2.3. Main theorem. Denote by \( V_{\Delta,c} \) the Verma module of the Virasoro algebra generated by the highest weight vector \( v \):

\[
L_n v = 0, \text{ for } n > 0 \quad L_0 v = \Delta v, \quad Cv = cv.
\]

By \( \mathbb{L}_{\Delta,c} \) denote its irreducible quotient. It is convenient to parametrize \( \Delta \) and \( c \) as

\[
\Delta = \Delta(P, b) = \frac{(b^{-1} + b)^2}{4} - P^2, \quad c = c(b) = 1 + 6(b^{-1} + b)^2 \tag{2.5}
\]

We denote the corresponding irreducible representation as \( \mathbb{L}_{\Delta,c} \). Then for \( P \notin \{ P_{m,n} \} \) or \( P = P_{m,n}, mn \leq 0 \) the representation \( \mathbb{L}_{\Delta,c} \) is isomorphic to the Verma module and have the character \( \text{Tr}q^{L_0} |_{\mathbb{L}_{\Delta,c}} = q^{\Delta}/(q)_\infty \), where \( (q)_\infty = \prod_{k=1}^{\infty} (1-q^k) \) (for the reference see e.g. [11] or [16]). For \( P = P_{m,n}, mn > 0 \) the Verma module contains the singular vector on the level \( mn \). The irreducible representation is a quotient of Verma module by the submodule generated by this singular vector. The character of this module (which we denote by \( \mathbb{L}_{(m,n)} \)) reads:

\[
\chi_{m,n}(q) = \text{Tr}q^{L_0} |_{\mathbb{L}_{(m,n)}} = q^{\Delta - q^{mn}}/(q)_\infty
\]

If \( P = P_{1,1} \) then \( \Delta = 0 \) and the corresponding irreducible representation \( \mathbb{L}_{(1,1)} \) has the structure of conformal operator algebra. The conformal vector is \( L_{-2}v \) and the stress-energy tensor has the form \( T(z) = \sum_n L_n z^{-n-2} \).

The vector spaces \( \mathbb{L}_{p,b} \) are the representations of this vertex algebra. We will denote this algebra by \( \mathcal{M}_b \) and by \( T_b \) its stress-energy energy tensor

**Theorem 2.1.** a) Let \( b_1 = b/\sqrt{1-b^2}, \ b_2 = \sqrt{b^2-1}, \ b \) is generic. Then the vector space

\[
\mathcal{A}_b = \bigoplus_{n \in 2\mathbb{N}-1} \mathbb{L}_{(1, n)}^{b_1} \otimes \mathbb{L}_{(n, 1)}^{b_2}
\]

has a structure of the vacuum module of the conformal algebra with the stress-energy tensor \( T_{b_1} + T_{b_2} \).

b) This algebra is isomorphic to \( \mathcal{U} \otimes \mathcal{M}_b \), with the stress-energy tensor \( T_U + T_b \).

**Remark 2.2.** The parameters \( b_1 \) and \( b_2 \) from Theorem 2.1 satisfy relation:

\[
b_1^2 + b_2^2 = -1
\]

This relation was already mentioned in the Introduction.

**Remark 2.3.** The direct sum \( \bigoplus_{n \in \mathbb{N}} \mathbb{L}_{(1, n)}^{b_1} \otimes \mathbb{L}_{(n, 1)}^{b_2} \) form a representation of the \( \mathcal{A}_b \). This remark can be viewed as an addition to the part a) of the Theorem 2.1. One can say that the whole sum \( \bigoplus_{n \in \mathbb{N}} \mathbb{L}_{(1, n)}^{b_1} \otimes \mathbb{L}_{(n, 1)}^{b_2} \) is an operator algebra but nonlocal in sense that the fractional powers like \((z-w)^{1/2}\) appears in the OPE.
Thus we have an equality of the highest weights:

\[ c_1 + c_2 = (1 + 6\left(\frac{1}{b_1} + b_1\right)^2) + \left(1 + 6\left(\frac{1}{b_2} + b_2\right)^2\right) = -5 + \left(1 + 6\left(\frac{1}{b} + b\right)^2\right) = c_{\mathcal{U}} + c \]

Second, the character of the vacuum representation of the algebra \( \mathcal{A}_b \) equals to:

\[
\chi(\mathcal{A}_b) = \sum_{n \in \mathbb{Z}} \chi_{1,n}^b \cdot \chi_{n,1}^b = \sum_{n \in \mathbb{Z}} q^{(n-2)^2-1} \left(\frac{1-q}{q}\right)^2 = \chi(U_0) \cdot \frac{1-q}{(q)_{\infty}}, \quad (2.7)
\]

where the last equality is an easy combinatorial statement. The equality (2.7) can be rewritten as: \( \chi(\mathcal{A}_b) = \chi(U_0) \cdot \chi_{1,1}^b \) i.e. equality of characters of vacuum modules from the Theorem 2.1. Similarly

\[
\chi \left( \bigoplus_{n \in \mathbb{Z}} \mathbb{L}_{(1,n)}^b \otimes \mathbb{L}_{(n,1)}^b \right) = \chi(U_1) \cdot \chi_{1,1}^b
\]

Now we consider the representations of the algebra \( \mathcal{A}_b \). Taking to account the isomorphism \( \mathcal{A}_b = \mathcal{U} \otimes \mathcal{M}_b \) we have two obvious representations \( U_0 \otimes \mathbb{L}_{P,b} \) and \( U_1 \otimes \mathbb{L}_{P,b} \). The next theorem describes the action of \( T_1(z) \) and \( T_2(z) \) on these modules.

**Theorem 2.2.** Let \( P \not\in \{P_{m,n}\} \), \( P_1 = \sqrt{b-1/(b-1-b)}P \), \( P_2 = \sqrt{b/(b-b-1)}P \). Then the modules \( U_0 \otimes \mathbb{L}_{P,b} \) and \( U_1 \otimes \mathbb{L}_{P,b} \) have the following decomposition with respect to the subalgebra \( \mathcal{M}_{b_1} \otimes \mathcal{M}_{b_2} \)

\[
U_1 \otimes \mathbb{L}_{P,b} = \bigoplus_{k \in \mathbb{Z}} \mathbb{L}_{(p_1+kb_1)b_1}^b \otimes \mathbb{L}_{(p_2+kb_2^{-1})b_2},
\]

\[
U_0 \otimes \mathbb{L}_{P,b} = \bigoplus_{k \in \mathbb{Z} + \frac{1}{2}} \mathbb{L}_{(p_1+kb_1)b_1}^b \otimes \mathbb{L}_{(p_2+kb_2^{-1})b_2}. \quad (2.8)
\]

One can easily check the coincidence of the characters of the modules in (2.8).

**2.4. Minimal models.** In the previous subsection we have considered the general values of the Virasoro central charge \( c \). Now let \( b_{p/p'}^2 = -p/p' \), \( p, p' \in \mathbb{N} \), \( (p, p') = 1 \). The central charge equals \( c_{p/p'} = 1 - 6\left(\frac{p-p'}{pp'}\right)^2 \).

The singular values of \( P \) (defined in (2.6)) possess the symmetry \( P_{m,n} = -P_{p-m,p'-n} \). Thus we have an equality of the highest weights:

\[
\Delta(P_{m,n}, b_{p/p'}) = \Delta(P_{p-m,p'-n}, b_{p/p'}).
\]

Therefore for \( 1 \leq m \leq p, 1 \leq n \leq p' \) the corresponding Verma module \( V_{\Delta,c} \) contains two singular vectors on the level \( mn \) and \( (p - m)(p' - n) \). For example the irreducible module \( \mathbb{L}_{1,(1,1)}^b \) is the quotient of the Verma module \( V_{0,c} \) by the submodule generated by singular vectors of the level 1 and \( (p - 1)(p' - 1) \).
The minimal model $\mathbb{M}_{(1,1)}$ has the structure of the vacuum module of the VOA which is called the minimal model. We will denote it by $\mathcal{M}_{p/p'}$. This VOA is rational, the only representations are $\mathbb{L}_{(m,n)}^b$ for $1 \leq m \leq p, 1 \leq n \leq p'$ with the identification $\mathbb{L}_{(m,n)}^{b/p/p'} = \mathbb{L}_{(p-m, p-n)}^{b/p/p'}$ mentioned before. We will denote such representations as $\mathbb{L}_{(m,n)}^{p/p'}$.

We want to state analogues of the Theorems 2.1 and 2.2 for the minimal models. Note that if $b_1^2 = -p/p'$ then $b_2^2 = -(p + p')/p'$ i.e. they correspond to the minimal models $\mathcal{M}_{p/(p+p')}$ and $\mathcal{M}_{(p+p')/p}$.

**Theorem 2.3.** a) The vector space

$$\bigoplus_{n \equiv 1 \mod 2} \mathbb{L}_{(1,n)}^{p/(p+p')} \otimes \mathbb{L}_{(n,1)}^{(p+p')/p'}$$

has a structure of the vacuum module of the conformal algebra with the stress-energy tensor $T_{p/(p+p')} + T_{(p+p')/p'}$. This algebra is isomorphic to $\mathcal{U} \otimes \mathcal{M}_{p/p'}$, with the stress-energy tensor $T_{\mathcal{U}} + T_{p/p'}$.

b) The algebra $\mathcal{U} \otimes \mathcal{M}_{p/p'}$ has natural representation $U_0 \otimes \mathbb{L}_{(m,m')}^{p/p'}$ and $U_1 \otimes \mathbb{L}_{(m,m')}^{p/p'}$. These modules have the following decomposition:

$$U_0 \otimes \mathbb{L}_{(m,m')}^{p/p'} \cong \bigoplus_{1 \leq n \leq p+p'-1 \atop n \equiv 1 \mod 2} \mathbb{L}_{(m,n)}^{p/(p+p')} \otimes \mathbb{L}_{(n,m')}^{(p+p')/p'}, \quad U_1 \otimes \mathbb{L}_{(m,m')}^{p/p'} \cong \bigoplus_{1 \leq n \leq p+p'-1 \atop n \equiv m+m' \mod 2} \mathbb{L}_{(m,n)}^{p/(p+p')} \otimes \mathbb{L}_{(n,m')}^{(p+p')/p'}.$$

Denote by $\chi_{(m,n)}^{p/p'}$ the character of irreducible module $\mathbb{L}_{(m,n)}^{p/p'}$. The following combinatorial identities follows from the 2.3:

$$\chi(U_0) \cdot \chi_{(m,n)}^{p/p'} = \sum_{1 \leq n \leq p+p'-1 \atop n \equiv 1 \mod 2} \chi_{(m,n)}^{p/(p+p')} \cdot \chi_{(n,m')}^{(p+p')/p'}, \quad \chi(U_1) \cdot \chi_{(m,n)}^{p/p'} = \sum_{1 \leq n \leq p+p'-1 \atop n \equiv m+m' \mod 2} \chi_{(m,n)}^{p/(p+p')} \otimes \chi_{(n,m')}^{(p+p')/p'}.$$

Theorem 2.3 have remarkable particular case. Let $(p, p') = (2, 3)$ then central charge $c_{2/3} = 0$. The minimal model $\mathcal{M}_{2/3}$ only has one representation $\mathbb{L}_{(1,1)}^{2/3}$ which is trivial representation. Therefore factor $\mathcal{M}_{2/3}$ can be omitted.

**Theorem 2.4.** The Urod algebra $\mathcal{U}$ has the subalgebra $\mathcal{M}_{2/5} \otimes \mathcal{M}_{5/3}$. The representations $U_0$ and $U_1$ have the decomposition:

$$U_0 = \left( \mathbb{L}_{(1,1)}^{5/3} \otimes \mathbb{L}_{(1,1)}^{2/5} \right) \bigoplus \left( \mathbb{L}_{(1,3)}^{2/5} \otimes \mathbb{L}_{(3,1)}^{5/3} \right), \quad U_1 = \left( \mathbb{L}_{(1,2)}^{2/5} \otimes \mathbb{L}_{(2,1)}^{5/3} \right) \bigoplus \left( \mathbb{L}_{(1,4)}^{2/5} \otimes \mathbb{L}_{(4,1)}^{5/3} \right).$$

Two commuting Virasoro algebras can be constructed explicitly in terms of Heisenberg algebra:

$$T_{2/5} = -\frac{1}{10\epsilon} e^{-\sqrt{2}\varphi} + \frac{1}{5} (\partial \varphi)^2 + \frac{3}{5\sqrt{2}} (\partial^2 \varphi) + \frac{12\epsilon}{5} (\partial \varphi)^2 e^{-\sqrt{2}\varphi} + \frac{3\sqrt{2}\epsilon}{5} (\partial^2 \varphi) e^{-\sqrt{2}\varphi} - \frac{12\epsilon^2}{5} e^{2\sqrt{2}\varphi},$$

$$T_{5/3} = \frac{1}{10\epsilon} e^{-\sqrt{2}\varphi} + \frac{3}{10} (\partial \varphi)^2 + \frac{2}{5\sqrt{2}} (\partial^2 \varphi) - \frac{2\epsilon}{5} (\partial \varphi)^2 e^{\sqrt{2}\varphi} + \frac{2\sqrt{2}\epsilon}{5} (\partial^2 \varphi) e^{\sqrt{2}\varphi} + \frac{12\epsilon^2}{5} e^{2\sqrt{2}\varphi}. $$
Direct calculation shows that $T_{2/5}$ and $T_{5/3}$ commute and satisfy (2.1) with the central charges $c_{2/5} = \frac{-22}{5}$ and $c_{5/3} = \frac{-3}{5}$ correspondingly. It is clear that

$$T_u = T_{2/5} + T_{5/3}.$$  

**Remark 2.4.** One can try to find the general stress-energy tensor $T(z)$ in ansatz:

$$T(z) = \alpha e^{-\sqrt{2}\varphi} + \beta_1 (\partial \varphi)^2 + \beta_2 (\partial^2 \varphi) + \gamma_1 (\partial \varphi)^2 e^{\sqrt{2}\varphi} + \gamma_2 (\partial^2 \varphi)e^{\sqrt{2}\varphi} + \delta e^{2\sqrt{2}\varphi}.$$  

The solutions of the equation (2.1) are $T_u$, $T_{2/5}$, $T_{5/3}$, standard solution $T(z) = \frac{1}{2}(\partial \varphi)^2 + u(\partial^2 \varphi)$ and another two deformations $T^{(1)}(z) = \alpha e^{-\sqrt{2}\varphi} + \frac{1}{2}(\partial \varphi)^2$ and $T^{(2)}(z) = \frac{1}{2}(\partial \varphi)^2 + \frac{3}{2\sqrt{2}} (\partial^2 \varphi) + \delta e^{2\sqrt{2}\varphi}$. The corresponding central charges equals $c_u = -5$, $c_{2/5} = -\frac{22}{5}$, $c_{5/3} = -\frac{3}{5}$, $c = 1 - 12u^2$, $c^{(1)} = 1$, $c^{(2)} = -\frac{35}{2}$. We can conclude that formulas for $T_{2/5}$, $T_{5/3}$ above are quite distinguished.

At the end of the subsection we remark two identities:

$$\chi(L_{0,1}) = q^{-1/4} \left( \chi^{2/5}_{(1,2)} \cdot \chi^{5/3}_{(2,1)} + \chi^{2/5}_{(1,4)} \cdot \chi^{5/3}_{(4,1)} \right),$$  

$$\chi(L_{1,1}) = q^{-1/4} \left( \chi^{2/5}_{(1,1)} \cdot \chi^{5/3}_{(1,1)} + \chi^{2/5}_{(1,3)} \cdot \chi^{5/3}_{(3,1)} \right).$$  

(2.9)
These identities link characters of the Level 1 representations \( \hat{\mathfrak{sl}}(2) \) and characters of minimal models and follow from the Theorem 2.4 and Remark 2.1. These identities will be discussed in Section 3 from the combinatorial point of view.

2.5. \( c = -5 \) description. We can also study \( \mathcal{U} \) as the Virasoro algebra representation with the central charge \( c_{\mathcal{U}} = -5 \). The corresponding parameter \( b_{\mathcal{U}} = \frac{1 + \sqrt{5}}{2} \) (see parameterization (2.5)) is generic by means \( b_{\mathcal{U}}^2 \notin \mathbb{Q} \). The unique singular vector in Verma module \( V_{(m,n)}^{b_{\mathcal{U}}} \) has \( L_0 \) grading \( \Delta(P_{m,n}, b_{\mathcal{U}}) + mn = \Delta(P_{m,-n}, b_{\mathcal{U}}) \). These fact can be also written in a short exact sequence:

\[
0 \rightarrow \mathbb{L}_{(m,-n)}^{b_{\mathcal{U}}} \rightarrow V_{(m,n)}^{b_{\mathcal{U}}} \rightarrow \mathbb{L}_{(m,n)}^{b_{\mathcal{U}}} \rightarrow 0
\]

We also need the projective modules in the BGG category \( \mathcal{O} \) (3) Virasoro representation with central charge \(-5\). By \( P_{m,n} \) we denote the unique projective module such that \( \text{Hom}(P_{m,n}, \mathbb{L}_{m,n}) \neq 0 \). These projective modules have the description similar to the projective modules for the \( \mathfrak{sl}(2) \). For \( mn \geq 0 \) we have an isomorphism \( P_{(m,n)}^{b_{\mathcal{U}}} \cong V_{(m,n)}^{b_{\mathcal{U}}} \cdot \). If \( mn < 0 \) then \( P_{(m,-n)}^{b_{\mathcal{U}}} \) is defined by a short exact sequence:

\[
0 \rightarrow V_{(m,n)}^{b_{\mathcal{U}}} \rightarrow P_{(m,-n)}^{b_{\mathcal{U}}} \rightarrow \mathbb{L}_{(m,-n)}^{b_{\mathcal{U}}} \rightarrow 0.
\]

“Theorem”. The modules \( \mathbb{L}_{(1,1)}^{2/5} \otimes \mathbb{L}_{(1,1)}^{5/3} \), \( 1 \leq n \leq 4 \) have the following decomposition with respect to diagonal Virasoro \( L_n = L_n^{2/5} + L_n^{5/3} \):

\[
\begin{align*}
\mathbb{L}_{(1,1)}^{2/5} \otimes \mathbb{L}_{(1,1)}^{5/3} & \cong \bigoplus_{n \in 2\mathbb{N} - 1} \mathbb{L}_{(n,n)}^{b_{\mathcal{U}}} , \\
\mathbb{L}_{(1,2)}^{2/5} \otimes \mathbb{L}_{(2,1)}^{5/3} & \cong P_{(0,0)}^{b_{\mathcal{U}}} \bigoplus \bigoplus_{n \in 2\mathbb{N}} P_{(n,-n)}^{b_{\mathcal{U}}} , \\
\mathbb{L}_{(1,3)}^{2/5} \otimes \mathbb{L}_{(3,1)}^{5/3} & \cong \bigoplus_{n \in 2\mathbb{N} - 1} P_{(n,-n)}^{b_{\mathcal{U}}} , \\
\mathbb{L}_{(1,4)}^{2/5} \otimes \mathbb{L}_{(4,1)}^{5/3} & \cong \bigoplus_{n \in 2\mathbb{N}} \mathbb{L}_{(n,n)}^{b_{\mathcal{U}}} ,
\end{align*}
\]

Due to Theorem 2.4 the first row gives the decomposition of \( U_0 \) and the second row gives the decomposition of \( U_1 \).

We will not prove this fact in this paper (and therefore we called it “Theorem”).

3. CONSTRUCTIONS AND PROOFS

3.1. Quantum Hamiltonian reduction of \( \hat{\mathfrak{sl}}(2) \). For the reader convenience we first recall the quantum Hamiltonian (or the Drinfeld-Sokolov) reduction of \( \hat{\mathfrak{sl}}(2) \), see e.g. [13] for the reference. We denote by \( \mathcal{V}_{h,k} \) the Verma module of the \( \hat{\mathfrak{sl}}(2) \), and by \( \mathcal{L}_{h,k} \) its irreducible quotient.

Let \( V \) be a representation of VOA \( \mathcal{L}_{0,k} \). It is convenient to consider \( \hat{\mathfrak{sl}}(2) \) generators as modes of the fields \( e(z), f(z), h(z) \):

\[
e(z) = \sum_{n \in \mathbb{Z}} e_n z^{-n-1}, \quad h(z) = \sum_{n \in \mathbb{Z}} h_n z^{-n-1}, \quad f(z) = \sum_{n \in \mathbb{Z}} f_n z^{-n-1}
\]
The relations of the $\mathfrak{sl}(2)_k$ can be written in terms of OPE
\[
\begin{align*}
    h(z)e(w) &\sim \frac{2e(w)}{z-w}, \quad h(z)f(w) \sim -\frac{2f(w)}{z-w}, \quad h(z)h(w) \sim \frac{2k}{(z-w)^2} \\
    e(z)f(w) &\sim \frac{h(w)}{(z-w)^2} + \frac{h(w)}{z-w}, \quad e(z)e(w) \sim 0, \quad f(z)f(w) \sim 0
\end{align*}
\]
The stress-energy tensor is given by the Sugavara formula:
\[
T_{\text{Sug}}(z) = \frac{1}{2(k+2)} :\left(\frac{1}{2}h^2(z) + e(z)f(z) + f(z)e(z)\right): 
\]
The central charge of $T_{\text{Sug}}$ equals $c_{\text{Sug}} = \frac{3k}{k+3}$.

Introduce the anticommuting operators $\psi_n, \psi_n^*$ with the relations:
\[
\psi(z) = \sum_n \psi_n z^{-n}, \quad \psi^*(z) = \sum_n \psi_n^* z^{-n}, \quad (\psi(z)\psi^*(w) \sim 1/(z-w), \quad (\psi(z)\psi^*(w) \sim 0)
\]
Operators $\psi_n, \psi_n^*$ generates Clifford algebra. By $\Lambda$ we denote the Fock representation generated by vector $v$
\[
\psi_n v = 0 \text{ for } n \geq 0, \quad \psi_n^* v = 0 \text{ for } n > 0.
\]
We introduce the grading on $\Lambda$ by $\deg(v) = 0, \deg(\psi_n) = 1, \deg(\psi_n^*) = -1$. The operator
\[
Q(A) = \oint_{|z-w|=1} (e(z) + 1)\psi(z)A(w)
\]
acts on the space $V \otimes \Lambda$. It is easy to see that $Q^2 = 0$. We denote by $H^i_{\text{DS}}(V)$ the cohomology of the complex $(V \otimes \Lambda, Q)$, where $i$ stands for the grading on $\Lambda$. These cohomology are called quantum Hamiltonian (or the Drinfeld-Sokolov) reduction of the $V$.

The following proposition is standard (see e.g. [13 Sec. 15.1.8])

**Proposition 3.1.** If $V$ has a structure of VOA then $H_{\text{DS}}(V \otimes \Lambda, Q)$ has a structure of VOA.

**Theorem 3.1** (e.g. [13 Ch. 15]). Let $V \cong \mathcal{L}_{0,k}$, $k \neq -2$. Then $H^i_{\text{DS}}(\mathcal{L}_{0,k}) = 0$ for $i \neq 0$ and $H^0_{\text{DS}}(\mathcal{L}_{0,k})$ is isomorphic to the vacuum representation of the algebra Virasoro with the central charge $c$, where $c = c(b_{k+2}/1), b_{k+2}/1 = \sqrt{-(k+2)}$.

The stress-energy tensor which generates the Virasoro symmetry reads:
\[
T_{\text{DS}}(z) = T_{\text{Sug}}(z) + \frac{1}{2} \partial_z h(z) - \psi(z)\partial\psi^*(z).
\]
This current $T_{\text{DS}}$ commutes with $Q$ (and corresponds to the vector in the cohomology $H^0_{\text{DS}}(\mathcal{L}_{0,k}))$. Therefore, this Virasoro algebra acts on $H_{\text{DS}}(V)$ for any level $k$ representation of $\mathfrak{sl}(2)$.

We say that the pair $(h, k)$ is generic there is no $m, n \in \mathbb{Z}_{\geq 0}$ such that $h + m(k+2) = n$ or $k - h + m(k+2) = n$. These conditions are equivalent to the fact Shapovalov form on Verma module $V_{h,k}$ is non degenerate [17]. Therefore the pair $(h, k)$ is generic if and only if the Verma module $V_{h,k}$ is irreducible.

Another important example is generic $k$ and integer $h = n \in \mathbb{Z}_{>0}$. In this case the irreducible module $\mathcal{L}_{h,k}$ is integrable with respect to $\mathfrak{sl}(2)$ generated by $e, h, f$. 
Theorem 3.2 ([9]). Let $b = b_{k+2/1} = \sqrt{-(k + 2)}$.

a) Let $n \in \mathbb{Z}_{\geq 0}$, $V \cong \mathcal{L}_{n,k}$. Then $H^i_{DS}(\mathcal{L}_{h,k}) = 0$, for $i \neq 0$ and $H^0_{DS}(\mathcal{L}_{n,k}) = \mathbb{I}^b_{(n+1,1)}$.

b) Let $(h, k)$ be generic, $V \cong \mathcal{L}_{h,k}$. Then $H^1_{DS}(\mathcal{L}_{h,k}) = 0$, for $i \neq 0$ and $H^0_{DS}(\mathcal{L}_{h,k}) = \mathbb{I}^P_{r,b}$, where $P = \frac{b^+(h+1)+b}{2}$.

3.2. Coset. Assume that $k \not\in \mathbb{Q}$.

Consider the tensor product of two $\hat{\mathfrak{sl}}(2)$ modules $\mathcal{L}_{i,1} \otimes \mathcal{L}_{h,k}$, where $i = 1, 2$. There is an action of the algebra $\hat{\mathfrak{sl}}(2) \otimes \hat{\mathfrak{sl}}(2)$ on this space, we denote by $e_{n}^{(1)}, h_{n}^{(1)}, f_{n}^{(1)}$ the generators of the first factor and by $e_{n}^{(2)}, h_{n}^{(2)}, f_{n}^{(2)}$ the generators of the second factor.

The space $\mathcal{L}_{i,1} \otimes \mathcal{L}_{h,k}$ becomes a level $k + 1$ representation under the diagonal action of $\hat{\mathfrak{sl}}(2)$: $e_{n}^{\Delta} = e_{n}^{(1)} + e_{n}^{(2)}, h_{n}^{\Delta} = h_{n}^{(1)} + h_{n}^{(2)}, f_{n}^{\Delta} = f_{n}^{(1)} + f_{n}^{(2)}$. It was noticed in [19] that there is a Virasoro algebra which commute with an action of $\hat{\mathfrak{sl}}(2)^{\Delta}$:

$$T_{\text{Coset}} = T^{(1)}_{\text{Sug}} + T^{(2)}_{\text{Sug}} - T^{\Delta}_{\text{Sug}}$$

This Virasoro algebra is called coset Virasoro algebra, its central charge equals $c = c(b_{k+2/k+3})$, where $b_{k+2/k+3} = \sqrt{-(k+2)}/(k+3)$.

Proposition 3.2. Let $b = b_{k+2/k+3}$.

a) The tensor product of the vacuum modules have the decomposition as a $\text{Vir} \oplus \hat{\mathfrak{sl}}(2)$ module:

$$\mathcal{L}_{0,1} \otimes \mathcal{L}_{0,k} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{I}^{b_{k+2/k+3}}_{(1,2n+1)} \otimes \mathcal{L}_{2n,k+1},$$

b) Let $(h, k)$ be generic, $P = \frac{(1+b)}{2 \sqrt{-(k+2)(k+3)}}$. Then we have decomposition of the $\mathcal{L}_{i,1} \otimes \mathcal{L}_{h,k}$ as a $\text{Vir} \oplus \hat{\mathfrak{sl}}(2)$ module:

$$\mathcal{L}_{0,1} \otimes \mathcal{L}_{h,k} = \bigoplus_{n \in \mathbb{Z}} \mathbb{I}^{p+nb,b}_{p,n+b,1} \otimes \mathcal{L}_{h+2n,k+1}, \quad \mathcal{L}_{1,1} \otimes \mathcal{L}_{h,k} = \bigoplus_{n \in \mathbb{Z}_{\geq 2}} \mathbb{I}^{p+nb}_{p+n+2} \otimes \mathcal{L}_{h+2n,k+1},$$

Now we can prove Theorem 2.1 a). We construct VOA algebra $\mathcal{A}_b$ by the Drinfeld-Sokolov reductions to the space $V = \mathcal{L}_{0,1} \otimes \mathcal{L}_{0,k}$, where $b = \sqrt{-(k+2)}$. Using the Proposition 3.2 and Theorem 3.2 we get:

$$H^\Delta_{DS}(\mathcal{L}_{0,1} \otimes \mathcal{L}_{0,k}) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{I}^{b_{k+2/k+3}}_{(1,2n+1)} \otimes H_{DS}(\mathcal{L}_{2n,k+1}) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{I}^{b_{k+2/k+3}}_{(1,2n+1)} \otimes \mathbb{I}^{b_{k+3/1}}_{(2n+1),1} \quad (3.1)$$

This space has a VOA structure due to the Proposition 3.1. If we put $b = \sqrt{-(k+2)}$, then $b_1 = b_{k+2/k+3}, b_2 = b_{k+3/1}$ and we obtain the statement of the Theorem 2.1 a).

The total stress-energy tensor is given by the formula:

$$T_{\text{tot}}(z) = T_{\text{Coset}}(z) + T^\Delta_{DS}(z) = T^{(1)}_{\text{Sug}}(z) + T^{(2)}_{\text{Sug}}(z) + \frac{1}{2} \partial_z (h^1(z) + h^2(z)) - \psi(z) \partial \psi^*(z)$$

3.3. Main proof. Recall that level 1 representations $\mathcal{L}_{i,1}$ can be constructed in terms of one field $\varphi(z)$ (see subsection 2.2).
Introduce the differential $Q_\epsilon$:

$$Q_\epsilon(A) = \oint_{|z-w|=1} (\epsilon e^{(1)}(z) + e^{(2)}(z) + 1)\psi(z)A(w)$$

If $\epsilon = 1$ then $Q_1$ coincides with the Drinfeld-Sokolov differential for $\widehat{\mathfrak{sl}}(2)^\Delta$. Actually for any $\epsilon \neq 0$ we can rescale $e^{(1)}$ and $f^{(1)}$ by $\epsilon$ and again get the Drinfeld-Sokolov differential for $\widehat{\mathfrak{sl}}(2)^\Delta$. But for $\epsilon = 0$ the situation is changed and $Q_0$ becomes the Drinfeld-Sokolov differential for $\widehat{\mathfrak{sl}}(2)^{(2)}$.

Therefore the cohomology space of $Q_0$ has the form: $H^{(2)}_{DS}(\mathcal{L}_{0,k}) = \mathcal{L}_{0,1} \otimes \mathbb{L}_{(1,1)}^b$, where $b = \sqrt{-(k+2)}$. The stress-energy tensor for $\mathcal{L}_{0,1}$ natural for this construction should have the form:

$$T_{\mathcal{L}_{0,1}} = T_{\text{tot}} - T^{(2)}_{DS} = \frac{1}{2}(\partial \varphi)^2 + \frac{1}{\sqrt{2}}\partial^2 \varphi.$$

We can compute the cohomology of $Q_\epsilon = Q_0 + \epsilon \oint e^{(1)}(z)\psi(z)$ by use of spectral sequence. The $E^1$ term of this sequence equals to the cohomology of $Q_0$. Since the Drinfeld-Sokolov cohomology $H^{(1)}_{DS}$ vanishes for $i \neq 0$ then the spectral sequence degenerates in $E^1$ term.

Thus $\mathcal{L}_{0,1} \otimes \mathbb{L}_{(1,1)}^b$ is isomorphic to $\bigoplus_{n \in \mathbb{Z} \geq 0} \mathbb{L}_{(1,2n+1)}^{b_{k+2/k+3}} \otimes \mathbb{L}_{(2n+1),1}^{b_{k+3/1}}$ as a $L_0$ graded vector space. But the representative of the cohomology classes are deformed like $A \rightarrow A + \epsilon A_1 + e^2 A_2 + \ldots$. The theorem 2.1 b) states that the $E^\infty = \bigoplus_{n \in \mathbb{Z} \geq 0} \mathbb{L}_{(1,2n+1)}^{b_{k+2/k+3}} \otimes \mathbb{L}_{(2n+1),1}^{b_{k+3/1}}$ is isomorphic to $E^1 = \mathcal{L}_{0,1} \otimes \mathbb{L}_{(1,1)}^b$ as the vertex operator algebra but has a different chiral structure.

Probably it will be better to deduce this isomorphism of VOA from the vanishing of the corresponding infinitesimal deformation. We will prove the isomorphism by direct calculation.

In order to study the deformation of the representative of cohomology classes we denote $\partial \chi = \psi \psi^* - \frac{1}{2} h^{(2)}$. The field $\partial \chi$ has the properties:

$$Q_0(\partial \chi) = -\psi, \quad \partial \chi(z)\partial \chi(w) = \frac{K + 2}{2(z - w)^2}.$$

Now we can give formulas for the representatives in the cohomology of $Q_\epsilon$. Denote $\partial \tilde{\varphi} = \partial \varphi - \epsilon \sqrt{2} \partial \chi e^{v^2}$. Then:

$$Q_\epsilon(\partial \tilde{\varphi}) = 0, \quad \partial \tilde{\varphi}(z)\partial \tilde{\varphi}(w) = \frac{1}{(z - w)^2}.$$

In other words the $\partial \tilde{\varphi}$ is free bosonic field in the cohomology of $Q_\epsilon$. The stress-energy tensor of the $\mathbb{L}_{(1,1)}^b$ which we denote by $T^{(2)}_{DS}$ is deformed to:

$$\tilde{T} = T^{(2)}_{DS} + \epsilon \left(\sqrt{2} \cdot \partial \chi \partial \varphi - \frac{K+1}{2} \cdot \left(2(\partial \varphi)^2 + \sqrt{2}(\partial^2 \varphi)\right)\right) \cdot e^{v^2} - \epsilon^2 \frac{K+2}{2} \cdot e^{2v^2}.$$

This current $\tilde{T}(z)$ belongs to the cohomology $Q_\epsilon(\tilde{T}) = 0$, commutes with $\tilde{\varphi}$ and satisfies Virasoro OPE with the central charge $c = c(b)$. The stress-energy tensor for the field $\tilde{\varphi}$
can be found as $T_{\text{tot}} - \tilde{T}$:

$$T_u = \frac{1}{2} (\partial \varphi)^2 + \frac{1}{\sqrt{2}} (\partial^2 \varphi) - \epsilon \left( \sqrt{2} \cdot \partial \chi \partial \varphi - \frac{K+1}{2} \cdot (2(\partial \varphi)^2 + \sqrt{2}(\partial^2 \varphi)) \right) e^{\sqrt{2} \varphi} + \epsilon^2 \frac{K+2}{2} e^{2\sqrt{2} \varphi}$$

$$= \frac{1}{2} (\partial \varphi)^2 + \frac{1}{\sqrt{2}} (\partial^2 \varphi) + \epsilon \frac{K+1}{2} \cdot (2(\partial \varphi)^2 + \sqrt{2}(\partial^2 \varphi)) e^{\sqrt{2} \varphi}.$$  

This last formula for $T_u$ coincides with eq. (2.3) after rescaling of $\epsilon$.

**Remark 3.1.** The representations of the algebra $A_b$ can be constructed by use of Drinfeld-Sokolov reduction too. This construction proves the Theorem 2.2. We take the representations of the VOA $\mathcal{M}_b$ too. This construction proves the Theorem 2.2.

Now we slightly change the notation:

$$H^A_{\text{DS}} (\mathcal{L}_{0,1} \otimes \mathcal{L}_{h,k}) = \bigoplus_{n \in \mathbb{Z}} \mathbb{L}_{\varphi_1 + nb_1, b_1} \otimes H^A_{\text{DS}} (\mathcal{L}_{h+2n,k+1}) = \bigoplus_{n \in \mathbb{Z}} \mathbb{L}_{\varphi_1 + nb_1, b_1} \otimes \mathbb{L}_{\varphi_2 + nb_2^{-1}, b_2},$$

$$H^A_{\text{DS}} (\mathcal{L}_{1,1} \otimes \mathcal{L}_{h,k}) = \bigoplus_{n \in \mathbb{Z} + \frac{1}{2}} \mathbb{L}_{\varphi_1 + nb_1, b_1} \otimes H^A_{\text{DS}} (\mathcal{L}_{h+2n,k+1}) = \bigoplus_{n \in \mathbb{Z} + \frac{1}{2}} \mathbb{L}_{\varphi_1 + nb_1, b_1} \otimes \mathbb{L}_{\varphi_2 + nb_2^{-1}, b_2}.$$  

Now we slightly change the notation:

$$P_1 = \tilde{P}_1 + b_1/2 = \frac{h - k - 1}{2\sqrt{-(k+3)(k+2)}}, \quad P_2 = \tilde{P}_2 + b_2^{-1}/2 = \frac{h - k - 1}{2\sqrt{-(k+3)}}.$$  

Then we have the following decompositions for the representations:

$$\bigoplus_{n \in \mathbb{Z}} \mathbb{L}_{(v_1 + b_1), b_1} \otimes \mathbb{L}_{(v_2 + b_2^{-1})}, b_2,$$

$$\bigoplus_{n \in \mathbb{Z} + \frac{1}{2}} \mathbb{L}_{(v_1 + b_1), b_1} \otimes \mathbb{L}_{(v_2 + b_2^{-1})}, b_2.$$  

The isomorphism from Theorem 2.2 follows from the equality of characters.

**Remark 3.2.** The isomorphism of VOA in Theorem 2.1 can be made more explicit. This theorem states in particular that one can find in $\mathcal{U} \otimes \mathcal{M}_b$ two commuting Virasoro algebras with central charges $c(b_1)$ and $c(b_2)$. In other words one can express $T_{b_1}$ and $T_{b_2}$ in terms of free field $\varphi$, exponents $e^{n\sqrt{2} \varphi}$ and stress-energy tensor $T_b$. The formulas looks:

$$T_{b_1} = \frac{b + b^{-1}}{2(b - b^{-1})} e^{-\sqrt{2} \varphi} + \frac{b}{2(b - b^{-1})} (\partial \varphi)^2 - \frac{b^{-1}}{b - b^{-1}} \partial^2 \varphi - \frac{1 + 2b^{-2}}{b^2 - b^{-2}} e^{\sqrt{2} \varphi}$$

$$- \frac{\sqrt{2}b^{-1} \epsilon e^{\sqrt{2} \varphi}}{b - b^{-1}} - \frac{2 \epsilon^2}{b^2 - b^{-2}} e^{2\sqrt{2} \varphi} - \frac{b^{-1}}{b - b^{-1}} T_b - \frac{2 \epsilon}{b^2 - b^{-2}} T_b e^{\sqrt{2} \varphi}, \quad (3.2)$$

$$T_{b_2} = - \frac{b + b^{-1}}{2(b - b^{-1})} e^{-\sqrt{2} \varphi} - \frac{b^{-1}}{2(b - b^{-1})} (\partial \varphi)^2 + \frac{b}{\sqrt{2}(b - b^{-1})} \partial^2 \varphi + \frac{2b^2 + 1}{b^2 - b^{-2}} e^{\sqrt{2} \varphi}$$

$$+ \frac{\sqrt{2} \epsilon}{b - b^{-1}} (\partial^2 \varphi) e^{\sqrt{2} \varphi} + \frac{2 \epsilon}{b^2 - b^{-2}} e^{2\sqrt{2} \varphi} + \frac{b}{b - b^{-1}} T_b + \frac{2 \epsilon}{b^2 - b^{-2}} T_b e^{\sqrt{2} \varphi}, \quad (3.3)$$

It is easy to see that $T_{b_1} + T_{b_2} = T_b + T_u$.

If $b = b_{2/3}$ then $T_b = 0$ and these formulas reduces to formulas for $T_{2/3}$ and $T_{5/3}$ from Subsection 2.3.
Remark 3.3. For the proof of the Theorem 2.3 we can consider the tensor product $\mathcal{L}_{i,1} \otimes \mathcal{L}_{h,k}$, where $k = -2 + p/p'$, and $\mathcal{L}_{h,k}$ is an admissible representation. Then we combine the results on the Drinfeld-Sokolov reduction for $\hat{\mathfrak{sl}}(2)$, the results for the coset and explicit formulas from this subsection. For the coset we need the analogue of the Proposition 3.2 for the admissible representations, see [18, eq. [7]].

3.4. Proof of Theorem 2.4 The Theorem 2.4 follows from the more general Theorem 2.3. In this subsection we give another, more direct proof of this fact.

Let $V$ be the highest weight representation of the Virasoro algebra. Recall that the triple $(A, B, C)$ is called asymptotic dimension of $V$ if:

$$\text{Tr} e^{-2\pi t} \sim A \cdot t^B \cdot \exp \left( \frac{\pi C}{12t} \right) \text{ where } t \to 0.$$ 

The $C$ is also called the effective central charge.

It is known [18] that the level 1 representations of $\hat{\mathfrak{sl}}(2)$ have effective central charge $C = 1$. Therefore the modules $U_0$ and $U_1$ have the effective central charge $C = 1$.

These spaces are representations of the algebra Vir $\oplus$ Vir due to formulas for $T_{2/5}$ and $T_{5/3}$ in Subsection 2.4. The only Virasoro representations with the effective central charge less then 1 are the minimal model representations [18]. Therefore $U_0$ and $U_1$ decompose into direct sum of the tensor products of the $(2/5)$ and $(5/3)$ minimal model representations (since there is no Ext’s between minimal models representations).

The conformal dimensions of these minimal model representations are:

$$\Delta(P_{1,1}, b_{2/5}) = \Delta(P_{1,4}, b_{2/5}) = 0, \quad \Delta(P_{1,2}, b_{2/5}) = \Delta(P_{1,3}, b_{2/5}) = -\frac{1}{5},$$

$$\Delta(P_{1,1}, b_{5/3}) = 0, \quad \Delta(P_{2,1}, b_{5/3}) = -\frac{1}{20}, \quad \Delta(P_{3,1}, b_{5/3}) = \frac{1}{5}, \quad \Delta(P_{4,1}, b_{5/3}) = \frac{3}{4}.$$ 

The eigenvalues of $L_0$ on $U_0$ belongs to $\mathbb{Z}$ and the eigenvalues on $U_1$ belongs to $\mathbb{Z} - \frac{1}{4}$. Therefore $U_0$ is decomposed into the sum of the representations $\mathbb{L}_{2/5}^{(1,1)} \otimes \mathbb{L}_{5/3}^{(1,1)}, \mathbb{L}_{2/5}^{(1,3)} \otimes \mathbb{L}_{5/3}^{(3,1)}$ and $U_0$ is decomposed into the sum of the representations $\mathbb{L}_{2/5}^{(1,2)} \otimes \mathbb{L}_{5/3}^{(2,1)}, \mathbb{L}_{2/5}^{(2,1)} \otimes \mathbb{L}_{5/3}^{(1,4)}$.

Comparing the first terms in the $q$-expansion of the characters we get the Theorem 2.4.

Remark 3.4. Denote by $\Phi_{1,n} \Phi_{n,1}$ the operator which corresponds to highest weight vector of the representation $\mathbb{L}_{2/5}^{(1,n)} \otimes \mathbb{L}_{5/3}^{(n,1)}$. In terms of free field $\varphi$ these operators have form:

$$\Phi_{1,1} \Phi_{1,1} = \text{Id}, \quad \Phi_{1,2} \Phi_{2,1} = e^{\sqrt{1/2}\varphi}, \quad \Phi_{1,3} \Phi_{3,1} = \text{Id} + 2\epsilon e^{\sqrt{2}\varphi},$$

$$\Phi_{1,4} \Phi_{4,1} = e^{-\sqrt{1/2}\varphi} + 2\sqrt{2}\epsilon \partial \varphi e^{\sqrt{1/2}\varphi} + 2\epsilon^2 e^{3\sqrt{1/2}\varphi}.$$ 

4. COMBINATORICS

Recall, that $\mathcal{L}_{h,k}$ denotes the irreducible highest weight representation of $\hat{\mathfrak{sl}}(2)$. In this section we will consider the representations $\mathcal{L}_{i,k}$ where $k \in \mathbb{N}, l \in \mathbb{Z}$ and $0 \leq l \leq k$. These representations are integrable.

We call the function $f: \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ a $(l, k)$ configuration if

(1) $f(m) + f(m + 1) \leq k$
(2) \( f(2m + 1) = k - l, \ f(2m) = l, \) for \( m \ll 0 \)
(3) \( f(m) = 0, \) for \( m \gg 0 \)

The set of such configurations we denote by \( \Sigma_{l,k} \). By \( f_n \) we denote so called extremal configurations:

\[
f_n(m) = \begin{cases} 
0 & \text{if } m > n; \\
l & \text{if } m \leq n, \ m \text{ is even}; \\
k - l & \text{if } m \leq n, \ m \text{ is odd}.
\end{cases}
\]

In the table below we represent configuration \( f_{2n} \):

| \( f_{2n}(m) \) | \( \cdots \) | \( k - l \) | \( l \) | \( k - l \) | \( l \) | \( 0 \) | \( 0 \) | \( 0 \) | \( \cdots \) |
|------------------|----------------|----------|----------|----------|----------|------|------|------|----------|
| \( m \)          | \( \cdots \)    | \( 2n - 3 \) | \( 2n - 2 \) | \( 2n - 1 \) | \( 2n \) | \( 2n + 1 \) | \( 2n + 2 \) | \( \cdots \) |

By \( v \) we denote the highest weight vector of \( \mathcal{L}_{l,k} \). Extremal vectors \( v_n \in \mathcal{L}_{l,k}, \ n \in \mathbb{Z} \) defined by the relations:

\[
v_0 = v, \quad v_{2n} = (e_{-2n})^lv_{2n-1}, \quad v_{2n-1} = (e_{-2n+1})^{k-l}v_{2n-2},
\]

(4.1)

The Weyl group of \( \widehat{sl}(2) \) acts on \( \mathcal{L}_{l,k} \) and the set of vectors \( \{v_n\} \) is an orbit of the highest weight vector \( v \) under the action of this group. We formally define \( v_{-\infty} \) such that:

\[
v_{2n} = (e_{-2n})^1(e_{-2n+1})^{k-l}(e_{-2n+2})^1(e_{-2n+3})^{k-l} \cdots (e_{-2m})^1(e_{-2m+1})^{k-l} \cdots v_{-\infty}
\]

\[
v_{2n-1} = (e_{-2n+1})^{k-l}(e_{-2n+2})^1(e_{-2n+3})^{k-l}(e_{-2n+4})^1 \cdots (e_{-2m})^1(e_{-2m+1})^{k-l} \cdots v_{-\infty}
\]

Clearly these formulas are agree with the equations (4.1). Due to condition (1) for any \( f \in \Sigma_{l,k} \) there exist \( n \in \mathbb{Z} \) such that for any \( m > n, \ f(-2m) = l, \ f(-2m - 1) = k - l. \) Then \( f \) differs from \( f_n \) only in finite number of \( m \) and we can define the semiinfinite product \( \prod (e_{-m})^{f(m)}v_{\infty} \) by use of action of finite product of \( e_m \) on \( v_n \).

**Theorem 4.1 ([12] Prop 2.6.1').** The vectors of the form \( \prod e_{-m}^{f(m)}v_{\infty} \) form a basis in the space \( \mathcal{L}_{l,k} \)

As a consequence we can find the character of \( \mathcal{L}_{l,k} \). For any \( f \in \Sigma_{l,k} \) we define the \( q \)-weight:

\[
w_q(f) = -\sum_{m < 0} (2m + 1)(k - l - f(2m + 1)) - \sum_{m < 0} 2m(l - f(2m)) + \sum_{m \geq 0} mf(m)
\]

Due to conditions (2) and (3) this sum is finite. Clearly it is just the difference between \( L_0 \) gradings of the \( \prod (e_{-m})^{f(m)}v_{\infty} \) and \( v_0 \). Since \( L_0 v = \frac{t(t+1)}{4(k+2)}v \) we have:

\[
\chi(\mathcal{L}_{l,k}) = q^{\frac{t(t+1)}{3(k+2)}} \sum_{f \in \Sigma_{l,k}} q^{w_q(f)}.
\]

Now we come to the main point of the subsection. We decompose the set \( \Sigma_{l,k} \) as \( \Sigma_{l,k} = \sqcup \Sigma'^{r}_{l,k} \), where \( \Sigma'^{r}_{l,k} \) consists of \((l,k)\) configurations such that \( f(0) = r \). It is clear that \( \Sigma'^{r}_{l,k} = \Sigma^+, k-r + \Sigma^{r,-}, k-r \), where \( \Sigma^+, k-r \) consists of functions \( f : \mathbb{N} \to \mathbb{Z}_{\geq 0} \) such that
$f(1) \leq k-r$ and conditions (1) and (3) hold and $\Sigma_{l,k}^{r-k}$ consists of functions $f: -\mathbb{N} \rightarrow \mathbb{Z}_{\geq 0}$ such that $f(-1) \leq k - r$ and conditions (1) and (2) hold. Therefore:

$$q^{-\frac{4(k+2)}{3(k+2)}} \cdot \chi(L_{l,k}) = \sum_{f \in \Sigma_{l,k}} q^{w_q(f)} = \sum_{0 \leq r \leq k} \left( \sum_{f \in \Sigma_{k}^{r-k}} q^{w_q(f)} \right) \cdot \left( \sum_{f \in \Sigma_{l,k}^{-r}} q^{w_q(f)} \right) \quad (4.2)$$

It was proven in [10, Prop. 5] that the characters of the $(2,2k + 3)$ minimal model representations have the form:

$$\chi_{1,r}^{2/2k+3} = q^{\Delta(P_{1,r},b_{2/2k+3})} \sum_{f \in \Sigma_{1,r}^{-1}} q^{w_q(f)} = q^{\Delta(P_{1,r},b_{2/2k+1})} \sum_{n_1,n_2,...,n_k} q^{\sum_{i,j=1}^{k} n_i n_j \min(i,j) + \sum_{j=1}^{k} (j-r+1) n_j} \frac{(q)_{n_1} \cdots (q)_{n_k}},$$

where $b_{p/p'} = -p/p'$ and $(q)_n = \prod_{j=1}^{n}(1-q^j)$. This is the algebraic meaning of the first multipliers in (4.2).

For the second multipliers let us consider $k = 1$. In this case sums over $\Sigma_{l,k}^{r}$ can be simply rewritten in “fermionic” form. Due to e.g. [8, eq. (8) and Theorem 2.3] these fermionic formulas equal to the characters of the representations of $(3,5)$ minimal model:

$$\chi_{1,1}^{3/5} = q^{\Delta(P_{1,1},b_{3/5})} \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_{2n}} = q^{\Delta(P_{1,1},b_{3/5})} \sum_{f \in \Sigma_{1,1}^{-1}} q^{w_q(f)},$$

$$\chi_{1,2}^{3/5} = q^{\Delta(P_{1,2},b_{3/5})} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_{2n}} = q^{\Delta(P_{1,2},b_{3/5})} \sum_{f \in \Sigma_{1,1}^{-1}} q^{w_q(f)},$$

$$\chi_{1,3}^{3/5} = q^{\Delta(P_{1,3},b_{3/5})} \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_{2n+1}} = q^{\Delta(P_{1,3},b_{3/5})} \sum_{f \in \Sigma_{1,1}^{-1}} q^{w_q(f)},$$

$$\chi_{1,4}^{3/5} = q^{\Delta(P_{1,4},b_{3/5})} \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q)_{2n+1}} = q^{\Delta(P_{1,4},b_{3/5})} \sum_{f \in \Sigma_{1,1}^{-1}} q^{w_q(f)}.$$

Therefore, for $k = 1$ first multipliers in (4.2) are equal to the characters of $(2,5)$ minimal model (due to (4.3)) and the second equal to characters of $(3,5)$ minimal model. So for $k = 1$ the combinatorial identity (4.2) is equivalent to (2.9).

For $k > 1$ the sums over $\Sigma_{l,k}^{r}$ should coincide with certain coset characters. There supposed to exist a generalization of the Theorem [2,4] which relates Urod algebra for $\hat{\mathfrak{sl}}(2)$ on the level $k$ and product of minimal model $\mathcal{M}_{2/2k+3}$ and these coset algebras.
5. Functional equations

In this section we use the results of the Section 2 for the functional equations on the conformal blocks. Then we explain the relation between these equations and and Nakajima-Yoshioka blow-up equations mentioned in the beginning of the paper (1.2).

5.1. Whittaker vector. First we need to recall the definition of the Whittaker (or Gaiotto) limit of conformal block.

For the Verma module $V_{P,b}$ the Whittaker vector $W_{P,b} = \sum_{N=0} w_{P,b,N} q^{N/2}$, where $w_{P,b,N} \in V_{P,b}$, $L_0 w_{P,b,N} = (\Delta + N) w_{P,b,N}$ defined by the equations:

$$L_1 w_{P,b,N} = w_{P,b,N-1}, \quad L_2 w_{P,b,N} = 0.$$ 

These equations can be simply rewritten as $L_1 W_{P,b} = q^{1/2} W_{P,b}$, $L_2 W_{P,b} = 0$. \[2\] It is easy to see that for generic $P,b$ the Whittaker vector $W_{P,b}$ exists and unique up to normalization. We will always use normalization of $W_{P,b}$ such that $\langle w_{P,b,0}, w_{P,b,0} \rangle = 1$, where $\langle \cdot, \cdot \rangle$ is a Shapovalov form in Verma module $V_{P,b}$.

The Whittaker limit of the four point conformal block is defined by:

$$\mathbb{F}(P,b; q) = \langle W_{P,b}, W_{P,b} \rangle = \sum_{N=0}^{\infty} \langle w_{P,b,N}, w_{P,b,N} \rangle q^{N} \quad (5.1)$$

Now we consider the representation $U_1 \otimes \mathbb{L}_{P,b}$ of the algebra $\mathcal{A}_b = \mathcal{U} \otimes \mathcal{M}_b$. Consider the vector $v \sqrt{1/2} \otimes W_{P,b}(q) \in U_1 \otimes \mathbb{L}_{P,b}$ (recall that after the shift of the grading $v \sqrt{1/2}$ become the highest vector of $U_1$). It follows from the formulas (3.2), (3.3) that:

$$L_1^{b_1} \left( v \sqrt{1/2} \otimes W_{P,b}(q) \right) = q^{1/2} \frac{b^{-1}}{b-1} \left( v \sqrt{1/2} \otimes W_{P,b}(q) \right), \quad L_2^{b_1} \left( v \sqrt{1/2} \otimes W_{P,b}(q) \right) = 0$$

$$L_1^{b_2} \left( v \sqrt{1/2} \otimes W_{P,b}(q) \right) = q^{1/2} \frac{b}{b-1} \left( v \sqrt{1/2} \otimes W_{P,b}(q) \right), \quad L_2^{b_2} \left( v \sqrt{1/2} \otimes W_{P,b}(q) \right) = 0.$$ 

Therefore, using the decomposition of $U_1 \otimes \mathbb{L}_{P,b}$ from Theorem 2.2 we have the decomposition:

$$v \sqrt{1/2} \otimes W_{P,b}(q) = \sum_{k \in \mathbb{Z}} \frac{q^{k^2/2}}{\sqrt{l_k(P,b)}} \left( W_{P_1+kb_1,b_1} (\beta_1 q) \otimes W_{P_2+kb_2^{-1},b_2} (\beta_2 q) \right), \quad (5.2)$$

where $\beta_1 = \frac{b^{-2}}{(b^{-1} - b)^2}$, $\beta_2 = \frac{b^2}{(b - b^{-1})^2}$, the degrees $k^2/2$ defined by the difference in $L_0$ grading (see (2.5))

$$k^2 = \Delta(P_1+kb_1,b_1) + \Delta(P_2+kb_2^{-1},b_2) - \Delta(P_1,b_1) + \Delta(P_2,b_2)$$

and $l_k(P,b)$ are unknown coefficient. Taking the norm of the right and left hand sides of (5.2) we get:

$$\mathbb{F}(P,b; q) = \sum_{k \in \mathbb{Z}} \frac{q^{k^2}}{l_k(P,b)} \cdot \mathbb{F}(P+kb_1,b_1; \beta_1 q) \cdot \mathbb{F}(P_2+kb_2^{-1},b_2; \beta_2 q) \quad (5.3)$$

\[2\] More general Whittaker vectors defined by the relations $L_1 W = \alpha W$, $L_2 W = \beta W$, for generic $\alpha$, $\beta$. The Whittaker vector for $\beta = 0$ used in our paper sometimes called the Gaiotto vector.
The factors \( l_k(P, b) \) in principle are determined by the equation (5.2). Clearly \( l_0(P, b) = 1 \).

5.2. Differential equations. Let us consider the operator \( H = bL_0^{b_1} + b^{-1}L_0^{b_2} \). It follows from the formulas (3.2), (3.3) that the corresponding local operator have the form:

\[
bT_{b_1} + b^{-1}T_{b_2} = \frac{b + b^{-1}}{2\epsilon} e^{-\sqrt{2}\varphi} + \frac{b + b^{-1}}{2} (\partial \varphi)^2 + (b + b^{-1})\epsilon (\partial \varphi)^2 e^{\sqrt{2}\varphi} - \frac{2\epsilon^2}{b + b^{-1}} e^{2\sqrt{2}\varphi} - \frac{2\epsilon}{b + b^{-1}} T_b e^{\sqrt{2}\varphi} \tag{5.4}
\]

Define the function \( \widehat{F} \) by:

\[
\widehat{F}(P, b; q, t) = \sum_{k=0}^{\infty} \frac{q^{k^2}}{l_k(P, b)} e^{i\Delta k} \mathbb{F} \left( P + kb_1, b_1; \beta_1 q e^{tb} \right) \cdot e^{i\beta_2 q e^{tb}} \mathbb{F} \left( P + kb_2^{-1}, b_2; \beta_2 q e^{tb} \right),
\]

where \( \Delta_k = \Delta(P_1 + kb_1, b_1) \) and \( \Delta_k = \Delta(P_2 + kb_2^{-1}, b_2) \). Clearly \( \widehat{F}_0(P, b; q) = \mathbb{F}(P, b; q) \).

In order to write analogues formulas for \( \widehat{F}_m(P, b; q) \) we will use generalized Hirota-differential [21]:

\[
(D_x^{(\epsilon_1, \epsilon_2)})^m (f \cdot g) = \left( \frac{d}{dy} \right)^m f(x + \epsilon_1 y)g(x + \epsilon_2 y) \bigg|_{y=0}
\]

Therefore:

\[
\widehat{F}_m(P, b; q) = \sum_{k \in \mathbb{Z}} \frac{q^{1/4 - \Delta(P, b)}}{l_k(P, b)} \left( D_{\log q}^{(b, b^{-1})} \right)^m \left( q^{\Delta k^2} \mathbb{F} \left( P + kb_1, b_1; \beta_1 q \right) \cdot q^{\Delta k^2} \mathbb{F} \left( P + kb_2^{-1}, b_2; \beta_2 q \right) \right),
\]

where we used that \( \Delta_k + \Delta_k^2 = \Delta(P, b) - 1/4 + k^2 \).

On the other hand we can use the formula (5.4) and apply \( H \) to \( v_{1/\sqrt{2}} \otimes W_{P, b} \). It is easy to see that:

\[
H \left( v_{\sqrt{1/2}} \otimes W_{P, b} \right) = \frac{b + b^{-1}}{4} \left( v_{\sqrt{1/2}} \otimes W_{P, b} \right) + \frac{-2\epsilon q^{1/2}}{b + b^{-1}} \left( v_{3/\sqrt{2}} \otimes W_{P, b} \right)
\]

But the vectors \( v_{3/\sqrt{2}} \) and \( v_{1/\sqrt{2}} \) are orthogonal, hence we have:

\[
\widehat{F}_1(P, b; q) = \left( v_{\sqrt{1/2}} \otimes W_{P, b}, H \left( v_{\sqrt{1/2}} \otimes W_{P, b} \right) \right) = \frac{b + b^{-1}}{4} \mathbb{F}(P, b; q) \tag{5.5}
\]
Similarly applying $H$ one can prove that:
\[
\hat{F}_2(P, b; q) = \left(\frac{b+b^{-1}}{4}\right)^2 F(P, b; q), \quad \hat{F}_3(P, b; q) = \left(\frac{b+b^{-1}}{4}\right)^3 F(P, b; q) \tag{5.6}
\]
\[
\hat{F}_4(P, b; q) = \left(\frac{b+b^{-1}}{4}\right)^4 - 2q F(P, b; q)
\]
\[
\hat{F}_5(P, b; q) = \left(\frac{b+b^{-1}}{4}\right)^5 - \frac{17}{2} (b+b^{-1})q F(P, b; q)
\]
\[
\hat{F}_6(P, b; q) = \left(\frac{b+b^{-1}}{4}\right)^6 - \frac{183(b+b^{-1})^2}{8} q F(P, b; q) + 8q^{3-\Delta(P, b)} \partial_q \left(q^{\Delta(P, b)} F(P, b; q)\right)
\]

One can easily see the general statement for the structure of the $\hat{F}_m$.

**Proposition 5.1.** $\hat{F}_m$ is a linear combination of the derivatives of $q^{-\Delta(P, b)} (\partial_q)^l \left(q^{\Delta(P, b)} F_0(P, b; q)\right)$ with coefficients which are polynomials in $q$ and $b+b^{-1}$.

This fact follows from the following two observations. First, in the $V_{r,b}$ module the scalar product $\langle W_{r,b}, L_{a_1} L_{a_2} \cdots L_{a_k} W_{r,b}\rangle$ where $a_1 \leq a_2 \leq \cdots \leq a_k$ vanishes if $a_1 < -1$ or $a_k > 1$. And the nonzero products equals:

\[
\langle W_{r,b}, L_{-1} L_0^l L_{-1}^l W_{r,b}\rangle = q^{(l+l')/2} q^{-\Delta(P, b)+l'} (\partial_q)^l \left(q^{\Delta(P, b)} F_0(P, b; q)\right).
\]

Second, we will have no $(b+b^{-1})$ in the denominator. The only possible origin of such denominators are the operators $-\frac{2e^2}{b+b^{-1}} e^{2\sqrt{\varphi}}$ and $-\frac{2b}{b+b^{-1}} T_b e^{\sqrt{\varphi}}$. But this operators increase $\varphi_0$ grading. Therefore in order to have non zero scalar product the actions of such operators should be accompanied by we the action of the operator $\frac{b+b^{-1}}{2e} e^{-\sqrt{\varphi}}$, which cancels the denominator.

**5.3. Geometric interpretation** Recall the notation from the beginning of the introduction. The $M(2, N)$ denotes the moduli space of framed torsion free sheaves of rank 2 on $\mathbb{C}P^2$, $Z(\epsilon_1, \epsilon_2, a; q)$ denotes the generating function of the equivariant volumes of $M(2, N)$, $\epsilon_1, \epsilon_2, a_1, a_2$ are equivariant parameters, $a = (a_1 - a_2) / 2$. The AGT relation for Whittaker limit reads:

\[
F\left(\frac{a}{\sqrt{\epsilon_1 \epsilon_2}}, \sqrt{\frac{\epsilon_1}{\epsilon_2}} \frac{q}{(\epsilon_1 \epsilon_2)^2}\right) = Z(\epsilon_1, \epsilon_2, a; q), \tag{5.7}
\]

This relation is proven now (e.g. follows from the more general results proved in [7], [2], [23], [20]). Using this identity we see that equation (5.3) is equivalent to Nakajima-Yoshioka blow up equation (1.2). The factors $l_k(P, b)$ are determined geometrically in [21] and have the form:

\[
l_k(P, b) = \prod_{i,j \geq 0, i+j < 2k} (-2P - ib - jb^{-1}) \prod_{i,j \geq 1, i+j < 2k} (2P + ib + jb^{-1}) \tag{5.8}
\]

The insertion of the operator $H$ geometrically equivalent to multiplication of the integrand by the $\mu(C) + (b+b^{-1})/4$, where $C \subset \mathbb{C}P^2$ is an exceptional divisor. The $\mu(C)$ is
a cohomology class on \(\hat{M}\) defined in [21] p. 22] and consists of bundles that restrict to \(C\) in a non-trivial way.

One can see that equations (5.5) and (5.6) are equivalent to the equation (6.14) in [21] for the \(r = 2\). The other equations should follow from the results of the paper [22].

6. Conclusion

In this paper we considered the very concrete vertex-operator algebras, with explicit formulas. Possible generalizations may provide some understanding of the subject.

6.1. One can ask whether the product \(\mathcal{M}_{b_1} \otimes \mathcal{M}_{b_2}\) can be extended by fields \(\Phi_{1,3} \otimes \Phi_{3,1}\) such that the result is well defined VOA. It can be argued that this can be done if the central charges are related by the equation \(b_1^2 + b_2^{-2} = n\), where \(n \in \mathbb{Z}\). By AGT duality the \(|n| > 1\) case corresponds to the instanton counting on the Hirzebruch surface [6].

More general VOA can be constructed as an extension of the product \(\mathcal{M}_{b_1} \otimes \mathcal{M}_{b_2} \otimes \cdots \otimes \mathcal{M}_{b_n}\), where central charges related by the relation \(b_i^2 + b_{i+1}^{-2} = n_i\), where \(n_i \in \mathbb{Z}\).

Let us mention two special cases. If \(n = -2\) then the extended product \(\mathcal{M}_{b_1} \otimes \mathcal{M}_{b_2}\) corresponds to the instanton counting on the resolution of \(\mathbb{C}^2/\mathbb{Z}_2\). These conformal theory have an isomorphic description similar Theorem [21] b). Namely the product \(\mathcal{M}_{b_1} \otimes \mathcal{M}_{b_2}\) extended by fields \(\Phi_{1,2} \otimes \Phi_{2,1}\) is isomorphic to the product \(\mathcal{F} \otimes \mathcal{S}\mathcal{M}_c\), where \(\mathcal{S}\mathcal{M}_c\) is a Super-Virasoro VOA and \(\mathcal{F}\) is the Majorana fermion algebra (see [4], and references therein).

If \(n = 0\) then the corresponding central charges are related by \(c_1 + c_2 = 26\). In this case one multiply by the ghost representation \(\Lambda^{bc}\) and compute BRST cohomology. These cohomology classes are physical states in the chiral part of the Liouville gravity.

6.2. The Urod algebra \(\mathcal{U}\) considered in this paper is a deformation of the \(\widehat{\mathfrak{sl}(2)}_1\) as a chiral algebra. It is natural to expect the existence of the such Urod deformations of the \(\widehat{\mathfrak{sl}}(r)_k\) for any \(k\) and \(r \in \mathbb{N}\). Clearly the Section 3 constructions (combination of coset construction and Drinfeld-Sokolov reduction) have such generalization.

The Urod algebra for \(\widehat{\mathfrak{sl}}(r)_1\) by AGT relation corresponds to the the blow up equations for the \(U(r)\) instantons. Similar to \(r = 2\) case this algebra have a subalgebra isomorphic to the product of two \(W_r\) algebra corresponding to minimal models \((r, 2r + 1)\) and \((2r + 1, r + 1)\). This implies the character identities, similar to (2.19). The corresponding identity for \(r = 3\) and vacuum representation reads:

\[
\chi(\mathcal{L}_{0,0,1}) = q^{-1} \left( \chi^{3/7}_{(0,0)} \cdot \chi^{7/4}_{(0,0)} + \chi^{3/7}_{(0,3\Lambda_1)} \cdot \chi^{7/4}_{(3\Lambda_1,0)} + \chi^{3/7}_{(0,3\Lambda_2)} \cdot \chi^{7/4}_{(3\Lambda_2,0)} + \chi^{3/7}_{(0,\Lambda_1+\Lambda_2)} \cdot \chi^{7/4}_{(\Lambda_1+\Lambda_2,0)} + \chi^{3/7}_{(0,2\Lambda_1+2\Lambda_2)} \cdot \chi^{7/4}_{(2\Lambda_1+2\Lambda_2,0)} \right),
\]

where \(\mathcal{L}_{0,0,1}\) is a vacuum level 1 representation of \(\widehat{\mathfrak{sl}(3)}\), \(\Lambda_1, \Lambda_2\) are fundamental \(\mathfrak{sl}(3)\) weights.

As another particular case we discuss the Urod algebra for \(\widehat{\mathfrak{sl}}(2)_k\), where \(k \in \mathbb{N}\). From the section 3 constructions and combinatorial arguments from the section 4 follow that the \(\widehat{\mathfrak{sl}}(2)_k\) Urod algebra contains a subalgebra which is isomorphic to the product of \((2, 2k + 1)\) Virasoro minimal model and coset minimal model. For example \(\widehat{\mathfrak{sl}}(2)_2\) Urod algebra has
a subalgebra which is the product of (2, 7) Virasoro algebra and (3, 7) Super Virasoro algebra. The geometrical meaning of these Urod algebras is unknown.

7. ACKNOWLEDGMENTS

We thank A. Belavin and H. Nakajima for interest to our work and discussions.

This work was supported by RFBR grants No.12-01-00836-a, 12-02-01092-a, 12-01-31236-mol-a, 12-02-33011-mol-a, 13-01-90614 and by the Russian Ministry of Education and Science under the grants 2012-1.5-12-000-1011-012 and 2012-1.1-12-000-1011-016, contract No.8410.

REFERENCES

[1] L. F. Alday, D. Gaiotto, Y. Tachikawa, Liouville Correlation Functions from Four-dimensional Gauge Theories, Lett. Math. Phys. 91 (2010) 167-197, [arXiv:0906.3219].
[2] V. A. Alba, V. A. Fateev, A. V. Litvinov, G. M. Tarnopolsky, On combinatorial expansion of the conformal blocks arising from AGT conjecture, Lett. Math. Phys. 98 (2011) 33-64, [arXiv:1012.1312].
[3] A. Belavin, A. Polyakov and A. Zamolodchikov, Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory. Nucl. Phys. B241 (1984), 333.
[4] A. Belavin, M. Bershtein, B. Feigin, A. Litvinov, G. Tarnopolsky Instanton moduli spaces and bases in coset conformal field theory. Comm. Math. Phys. 319 1, 269-301 (2013), [arXiv:1111.2803]
[5] J. Bernstein, I. Gel’fand, S. Gel’fand Category of \( g \)-modules Funkts. Anal. Prilozh., 10 2 (1976), 18.
[6] U. Bruzzo, R. Poghossian, and A. Tanzini, Poincaré Polynomial of Moduli Spaces of Framed Sheaves on (Stacky) Hirzebruch Surfaces, Commun. Math. Phys. 304 (2011) 395–409, [arXiv:0909.1458].
[7] V.A. Fateev, A.V. Litvinov On AGT conjecture JHEP 1002 (2010) 014 [arXiv:0912.0504].
[8] B. Feigin, O. Foda, T. Welsh, Andrews-Gordon type identities from combinations of Virasoro characters, Ramanujan J., 17 (1), (2008) 33-52 ; [arXiv:math-ph/0504014].
[9] B. Feigin, E. Frenkel Quantization of the Drinfeld-Sokolov reduction. Phys. Lett. B, 246(1-2) (1990) 75.
[10] B. Feigin, E. Frenkel Coinvariants of nilpotent subalgebras of the Virasoro algebra and partition identities. Adv. Sov. Math., 16, (1993), 139-148 [arXiv:hep-th/9301039].
[11] B. Feigin and D. Fuchs, Representations of the Virasoro algebra. Representations of Lie Groups and Related Topics, 465, Adv. Stud. Contemp. Math., 7, Gordon and Breach, New York, 1990.
[12] B.L. Feigin, A.V. Stoyanovsky, Quasi-particles models for the representation of Lie algebras and geometry of flag manifold, Funct. Anal. Appl. 28 (1994) 6890, [arXiv:hep-th/9308079].
[13] E. Frenkel and D. Ben-Zvi: Vertex Algebras and Algebraic Curves, Mathematical Surveys and Monographs 88, American Mathematical Society 2004
[14] I. B. Frenkel, V. G. Kac, Basic representations of affine Lie algebras and dual resonance models, Invent. Math. 62 (1980/81), no. 1.
[15] A. Fujitsu, ope.math: Operator product expansions in free field realizations of conformal field theory, Comput. Phys. Commun. 79 (1994) 78-99.
[16] K. Iohara, Y. Koga, Representation theory of the Virasoro algebra, Springer Monographs in Mathematics, London: Springer-Verlag London Ltd (2011)
[17] V. G. Kac, D. A. Kazhdan, Structure of representations with highest weight of infinite-dimensional Lie algebras, Adv. in Math. 34 1 (1979), 97-108.
[18] V. G. Kac, M. Wakimoto Modular invariant representations of infinite-dimensional Lie algebras and superalgebras Proc. Natl. Acad. Sci. USA, 85 (1988), 4956-4960.
[19] P. Goddard, A. Kent, D. Olive, Unitary representations of the Virasoro and super-Virasoro algebras Comm. Math. Phys. 103 1 (1986), 105.
[20] D. Maulik, A. Okounkov Quantum Groups and Quantum Cohomology [arXiv:1211.1287]
[21] H. Nakajima, K. Yoshioka Instanton counting on blowup. I. 4-dimensional pure gauge theory, Inventiones mathematicae 165 2 (2005), 313-355 [arXiv:math/0306198].
[22] H. Nakajima, K. Yoshioka *Perverse coherent sheaves on blow-up. III. Blow-up formula from wall-crossing*, Kyoto J. Math. 51 2 (2011), 263 [arXiv:0911.1773].

[23] O. Schiffmann, E. Vasserot, *Cherednik algebras, W algebras and the equivariant cohomology of the moduli space of instantons on \( \mathbb{A}^2 \)*, [arXiv:1202.2756].