A limiting weak type estimate for capacitary maximal function

Une estimation de type faible limite pour la fonction maximale capacitaire

Jie Xiao, Ning Zhang

Department of Mathematics and Statistics, Memorial University, St. John’s, NL A1C 5S7, Canada

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A B S T R A C T

A capacitary analogue of the limiting weak type estimate of P. Janakiraman for the Hardy–Littlewood maximal function of an $L^1(\mathbb{R}^n)$-function (cf. [5,6]) is discovered.

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R É S U M É

Pour l’analogue en termes de capacités de la fonction maximale de Hardy–Littlewood, on démontre une estimation de type faible limite correspondant à celle de P. Janakiraman.

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1. Statement of theorem

For an $L^1_{loc}$-integrable function $f$ on $\mathbb{R}^n$, $n \geq 1$, let $Mf(x) = \sup_{x \in B} (L^1(\mathbb{R}^n))^{-1} \int_B |f(y)| \, dy$ denote the Hardy–Littlewood maximal function of $f$ at $x \in \mathbb{R}^n$, where the supremum is taken over all Euclidean balls $B$ containing $x$ and $L^1(\mathbb{R}^n)$ stands for the $n$-dimensional Lebesgue measure of $B$. Among several results of [5,6], P. Janakiraman obtained the following fundamental limit:

$$\lim_{\lambda \to 0} \lambda L^1(\{x \in \mathbb{R}^n: Mf(x) > \lambda\}) = \|f\|_{L^1} = \int_{\mathbb{R}^n} |f(y)| \, dy \quad \forall f \in L^1(\mathbb{R}^n).$$

This note studies the limiting weak-type estimate for a capacity. To be more precise, recall that a set function $C(\cdot)$ on $\mathbb{R}^n$ is said to be a capacity (cf. [2,3]) provided that:

$$\begin{cases}
C(\emptyset) = 0; \\
0 \leq C(A) \leq \infty \quad \forall A \subseteq \mathbb{R}^n; \\
C(A) \leq C(B) \quad \forall A \subseteq B \subseteq \mathbb{R}^n; \\
C\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} C(A_i) \quad \forall A_i \subseteq \mathbb{R}^n.
\end{cases}$$

© Project supported by NSERC of Canada as well as by URP of Memorial University, Canada.
E-mail addresses: jxiao@mun.ca (J. Xiao), nz7701@mun.ca, nzhang2@ualberta.ca (N. Zhang).
1 Current address: Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada.
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For a given capacity \( C(\cdot) \) let:

\[
M_C f(x) = \sup_{x \in B} \frac{1}{C(B)} \int_B |f(y)| \, dy
\]

be the capacitary maximal function of an \( L^1_{\text{loc}} \)-integrable function \( f \) at \( x \) for which the supremum ranges over all Euclidean balls \( B \) containing \( x \); see also [7].

In order to establish a capacitary analogue of the last limit formula for \( f \in L^1(\mathbb{R}^n) \), we are required to make the following natural assumptions:

- **Assumption 1** – the capacity \( C(B(x, r)) \) of the ball \( B(x, r) \) centered at \( x \) with radius \( r \) is a function depending on \( r \) only, but also the capacity \( C([x]) \) of the set \([x]\) of a single point \( x \in \mathbb{R}^n \) equals 0.

- **Assumption 2** – there are two nonnegative functions \( \phi \) and \( \psi \) on \((0, \infty)\) such that:

\[
\begin{align*}
\phi(t)C(E) &\leq \psi(t)C(E) \quad \forall t > 0 \quad \text{and} \quad tE = \{tx \in \mathbb{R}^n : x \in E \subseteq \mathbb{R}^n\}; \\
\lim_{t \to 0} \phi(t) = 0 = \lim_{t \to 0} \psi(t) \quad \text{and} \quad \lim_{t \to 0} \psi(t)/\phi(t) = \tau \in (0, \infty).
\end{align*}
\]

**Theorem 1.1.** Under the above-mentioned two assumptions, one has:

\[
\lim_{\lambda \to 0} \lambda C \left( \{x \in \mathbb{R}^n : M_C f(x) > \lambda \} \right) \approx \|f\|_1 \quad \forall f \in L^1(\mathbb{R}^n).
\]

*Here and henceforth, \( X \approx Y \) means that there is a constant \( c > 0 \) independent of \( X \) and \( Y \) such that \( c^{-1}Y \leq X \leq cY \).*

Note that the \((0, n] \ni (n - \lambda)\)-dimensional Hausdorff content \( A^{\infty}_{(n-\lambda)} \) and the \( 1 < p \)-variational capacity obey Assumptions 1–2 (cf. [1,9]). So, an application of Theorem 1.1 to \( C = A^{\infty}_{(n-\lambda)} \) actually reveals that the real interpolation between \( L^1(\mathbb{R}^n) \) and the Morrey space \( \mathcal{L}^{1,1}(\mathbb{R}^n) \) (of all functions \( f \) with \( M_{A^{\infty}_{(n-\lambda)}} f \in L^\infty(\mathbb{R}^n) \)):

\[
\|f\|_{(L^1, L^\infty)^{1-p, p}} \approx \|M_{A^{\infty}_{(n-\lambda)}} f\|_{L^p(A^{\infty}_{(n-\lambda)})} \approx \left( \int_0^\infty A^{\infty}_{(n-\lambda)} \left( \{x \in \mathbb{R}^n : M_{A^{\infty}_{(n-\lambda)}} f(x) > t \} \right) dt \right)^{1/p}
\]

established in [8, Theorem 3] is a natural extension of the classical real interpolation \( \|f\|_{(L^1, L^\infty)^{1-p, p}} \approx \|M f\|_{L^p} \).

2. Four lemmas

To prove Theorem 1.1, we will always suppose that \( C(\cdot) \) is a capacity obeying Assumptions 1–2 above, but also need four lemmas based on the following capacitary maximal function \( M_C \nu \) of a finite nonnegative Borel measure \( \nu \) on \( \mathbb{R}^n \):

\[
M_C \nu(x) = \sup_{B \ni x} \frac{\nu(B)}{C(B)} \quad \forall x \in \mathbb{R}^n,
\]

where the supremum is taken over all balls \( B \subseteq \mathbb{R}^n \) containing \( x \).

**Lemma 2.1.** If \( \delta_0 \) is the delta measure at the origin, then \( \lambda C([x \in \mathbb{R}^n : M_C \delta_0(x) > \lambda]) = 1 \).

**Proof.** According to the definition of the delta measure and Assumptions 1–2, we have:

\[
M_C \delta_0(x) = \frac{1}{C(B(x, |x|))} \quad \forall |x| \neq 0.
\]

Now, if \( x \) obeys \( M_C \delta_0(x) > \lambda \), then \( \lambda C(B(x, |x|)) < 1 \). Note that if \( C(B(0, r)) = \frac{1}{r^p} \) equals \( \frac{1}{\lambda^p} \), then one has the following property:

\[
\begin{align*}
C(B(x, |x|)) &< \frac{1}{\lambda} \quad \forall |x| < r; \\
C(B(x, |x|)) &= \frac{1}{\lambda} \quad \forall |x| = r; \\
C(B(x, |x|)) &> \frac{1}{\lambda} \quad \forall |x| > r.
\end{align*}
\]

Thus, \( |x| \in \mathbb{R}^n : M_C \delta_0(x) > \lambda \) = \( B(0, r) \), and consequently, \( \lambda C([x \in \mathbb{R}^n : M_C \delta_0(x) > \lambda]) = C(B(0, r)) = \lambda^{-1} \). \( \square \)
Lemma 2.2. If \( \nu \) is a finite nonnegative Borel measure on \( \mathbb{R}^n \) with \( \nu(\mathbb{R}^n) = 1 \), then \( \lambda \lim_{t \to 0} C(\{x \in \mathbb{R}^n: M_C \nu(t) > \lambda\}) = 1 \), where \( t > 0 \); \( \nu(t) = \nu(\{x \in \mathbb{R}^n: 1/t \leq |x| \leq 1/t \}) \).\( \lambda \leq \frac{\eta}{C(B(x, |x| - \epsilon))} \) is true.

Proof. For two positive numbers \( \epsilon \) and \( \eta \), choose \( \epsilon_1 \) small relative to both \( \epsilon \) and \( \eta \), but also let \( t \) be small and the induced \( \epsilon_t \) be such that: \( \nu_t(B(0, \epsilon_t)) > 1 - \epsilon; \ \epsilon_t = 3^{-1} \epsilon_1 \); \( \lim_{t \to 0} \epsilon_t = 0; \ \epsilon < \eta C(B(0, \epsilon_1)) \). Now, if:

\[
E_{1, \lambda}^t = \left\{ x \in \mathbb{R}^n \setminus B(0, \epsilon_1): \lambda < M_C \nu_t(x) \leq \frac{1}{C(B(x, |x| - \epsilon_t))} \right\};
\]

\[
E_{2, \lambda}^t = \left\{ x \in \mathbb{R}^n \setminus B(0, \epsilon_1): \max \left\{ \lambda, \frac{1}{C(B(x, |x| - \epsilon_t))} \right\} < M_C \nu_t(x) \right\},
\]

then \( E_{1, \lambda}^t \cup E_{2, \lambda}^t \cup B(0, \epsilon_1) = \{ x \in \mathbb{R}^n : M_C \nu_t(x) > \lambda \} \).

On the one hand, for such \( x \in E_{1, \lambda}^t \) and \( \forall \tilde{r} > 0 \), one has:

\[
\frac{\nu_t(B(x, \tilde{r}))}{C(B(x, |x| - \epsilon_t))} \leq \frac{1}{C(B(x, |x| - \epsilon_t))} < M_C \nu_t(x).
\]

Additionally, since for any \( r_1, r_2 \) satisfying \( 0 \leq r_1 \leq r_2 \), one has \( C(B(x, r_1)) \leq C(B(x, r_2)) \), one gets \( C(B(x, r)) \) is an increasing function with respect to \( r \). There exists \( r < |x| - \epsilon_t \) such that:

\[
\frac{\nu_t(B(x, r))}{C(B(x, |x| - \epsilon_t))} \leq \frac{\nu_t(B(x, r))}{C(B(x, r))} \leq M_C \nu_t(x),
\]

and hence by Assumption 1, for any \( x \in E_{2, \lambda}^t \), there exists \( r_1 > 0 \) such that \( r_1 < |x| - \epsilon_t \) and \( \lambda \leq \nu_t(B(x, r_1))/C(B(x, r)) \). By the Wiener covering lemma, there exists a disjoint collection of such balls \( B_i = B(x_i, r_i) \) and a constant \( \alpha > 0 \) such that \( \bigcup_i B_i \leq E_{2, \lambda}^t \subset \bigcup_i \alpha B_i \). Therefore, we get a constant \( \gamma > 0 \), which only depends on \( \alpha \), such that:

\[
C(E_{2, \lambda}^t) \leq \gamma \sum_i C(B_i) \leq \gamma \sum_i \frac{\nu_t(B_i)}{\lambda} \leq \frac{\gamma \epsilon}{\lambda},
\]

thanks to \( B_i \cap B(0, \epsilon_1) = \emptyset \) and \( 1 - \nu_t(B(0, \epsilon_1)) < \epsilon \).

On the other hand, if \( x \in E_{1, \lambda}^t \), then:

\[
\frac{1 - \epsilon}{C(B(x, |x| + \epsilon_t))} \leq \frac{\nu_t(B(x, |x| + \epsilon_t))}{C(B(x, |x| + \epsilon_t))} \leq M_C \nu_t(x) \leq \frac{1}{C(B(x, |x| - \epsilon_t))}.
\]

Since

\[
\lim_{t \to 0} \left( \frac{1}{C(B(x, |x| + \epsilon_t))} - \frac{1}{C(B(x, |x| - \epsilon_t))} \right) = 0 \quad \text{and} \quad \lim_{t \to 0} \left( \frac{1}{C(B(x, |x| + \epsilon_t))} - \frac{1}{C(B(x, |x|))} \right) = 0,
\]

for \( \eta > 0 \) there exists \( T > 0 \) such that:

\[
|M_C \nu_t(t) - M_C \delta_0| < \eta + \frac{\epsilon}{C(B(0, \epsilon_1))} < \eta + \frac{\epsilon}{C(B(0, \epsilon_1))} < 2 \eta \quad \forall t \in (0, T).
\]

Note that:

\[
M_C \delta_0(x) - 2 \eta \leq M_C \nu_t \leq M_C \delta_0(x) + 2 \eta \quad \forall x \in E_{1, \lambda}^t.
\]

Thus:

\[
\left\{ x \in \mathbb{R}^n : M_C \delta_0(x) > \lambda + 2 \eta \right\} \subseteq E_{1, \lambda}^t \subseteq \left\{ x \in \mathbb{R}^n : M_C \delta_0(x) > \lambda + 2 \eta \right\}.
\]

This in turn implies:

\[
C(\{x \in \mathbb{R}^n : M_C \nu_t(x) > \lambda\}) \leq C(\{x \in \mathbb{R}^n : M_C \delta_0(x) > \lambda + 2 \eta\}) \leq C(\{x \in \mathbb{R}^n : M_C \nu_t(x) > \lambda\} \cap (\mathbb{R}^n \setminus B(0, \epsilon_1))).
\]

Now, an application of Lemma 2.1 yields:

\[
\frac{1}{\lambda + 2 \eta} \leq C(\{x \in \mathbb{R}^n : M_C \nu_t(x) > \lambda\} \cap (\mathbb{R}^n \setminus B(0, \epsilon_1))) \leq \frac{1}{\lambda - 2 \eta} + \frac{\gamma \epsilon}{\lambda}.
\]

Letting \( t \to 0 \) and using Assumption 1, we get \( \lim_{t \to 0} C(\{x \in \mathbb{R}^n : M_C \nu_t(x) > \lambda\}) = \lambda^{-1} \). □

Lemma 2.3. If \( \nu \) is a nonnegative Borel measure on \( \mathbb{R}^n \), then \( M_C \nu(x) \) is upper semi-continuous.
Proof. According to the definition of $M_v(x)$, there exists a radius $r$ corresponding to $M_v(x) > \lambda > 0$ such that $v(B(x, r))/C(B(x, r)) > \lambda$. For a slightly larger number $s$ with $\lambda + \delta > s > r$, we have $v(B(x, r))/C(B(x, s)) > \lambda$. Then applying Assumption 1, one gets that for any $z$ satisfying $|z - x| < \delta$, $M_v(z) > v(B(z, s))/C(B(z, s)) > v(B(x, r))/C(B(x, s)) > \lambda$. Whence finding that $\{x \in \mathbb{R}^n : M_v(x) > \lambda\}$ is open, as desired. \(\Box\)

Lemma 2.4. If $v$ is a finite nonnegative Borel measure on $\mathbb{R}^n$, then there exists a constant $\gamma > 0$ such that $\gamma \lambda \in (\gamma / \lambda) v(\mathbb{R}^n)$.

Proof. Following the argument for [4, p. 39, Theorem 5.6], we set $E_\lambda = \{x \in \mathbb{R}^n : M_v(x) > \lambda\}$, and then select a $\nu$-measurable set $E \subseteq E_\lambda$ with $\nu(E) < \infty$. Lemma 2.3 proves that $E_\lambda$ is open. Therefore, for each $x \in E$, there exists an $x$-related ball $B_x$ such that $v(B_x)/C(B_x) > \lambda$. A slight modification of the proof of [4, p. 39, Lemma 5.7] applied to the collection of balls $\{B_x\}_{x \in E}$, and Assumption 2 show that we can find a sub-collection of disjoint balls $\{B_i\}$ and a constant $\gamma > 0$ such that:

$$C(E) \leq \gamma \sum_i C(B_i) \leq \sum_i \frac{\nu(B_i)}{\lambda} \leq \frac{\nu(\mathbb{R}^n)}{\lambda}.$$

Note that $E$ is an arbitrary subset of $E_\lambda$. Thereby, we can take the supremum over all such $E$ and then get $C(E_\lambda) \leq (\gamma / \lambda) v(\mathbb{R}^n)$. \(\Box\)

3. Proof of theorem

First of all, suppose that $v$ is a finite nonnegative Borel measure on $\mathbb{R}^n$ with $v(\mathbb{R}^n) = 1$. According to the definition of the capacitary maximal function, we have:

$$M_v(x) = \sup_{r > 0} \frac{v(B(x, r))}{C(B(x, r))} = \sup_{r > 0} \frac{v(B(x, t))}{C(B(x, t))}.$$

From Assumption 2 it follows that:

$$M_v(x) \leq \frac{M_v(x)}{\psi(t)} \leq \frac{M_v(x)}{\phi(t)} \frac{\phi(t)}{\psi(t)} \frac{\lambda \psi(t)}{\phi(t)} C\left(\left\{x \in \mathbb{R}^n : M_v(x) > \lambda \psi(t)\right\}\right) \leq \lambda \psi(t) C\left(\left\{x \in \mathbb{R}^n : M_v(x) > \lambda \psi(t)\right\}\right).$$

The last inclusions give that:

$$\frac{\psi(t)}{\phi(t)} \lambda \psi(t) C\left(\left\{x \in \mathbb{R}^n : M_v(x) > \lambda \psi(t)\right\}\right) \leq \lambda \psi(t) C\left(\left\{x \in \mathbb{R}^n : M_v(x) > \lambda \psi(t)\right\}\right)$$

$$\leq \lambda C\left(\left\{x \in \mathbb{R}^n : M_v(x) > \lambda \psi(t)\right\}\right) = \lambda C\left(\left\{x \in \mathbb{R}^n : M_v(x/t) > \lambda \phi(t)\right\}\right) \leq \lambda C\left(\left\{x \in \mathbb{R}^n : M_v(x/t) > \lambda \phi(t)\right\}\right)$$

$$\leq \lambda C\left(\left\{x \in \mathbb{R}^n : M_v(x/t) > \lambda \phi(t)\right\}\right) \leq \lambda C\left(\left\{x \in \mathbb{R}^n : M_v(x/t) > \lambda \phi(t)\right\}\right) \leq \lambda C\left(\left\{x \in \mathbb{R}^n : M_v(x/t) > \lambda \phi(t)\right\}\right) \leq \psi(t) \lambda \phi(t) C\left(\left\{x \in \mathbb{R}^n : M_v(x/t) > \lambda \phi(t)\right\}\right).$$

These estimates and Lemma 2.2, plus applying Assumption 2 and letting $t \rightarrow 0$, in turns derive:

$$\tau^{-1} \leq \lim_{\lambda \rightarrow 0} \lambda C\left(\left\{x \in \mathbb{R}^n : M_v(x) > \lambda\right\}\right) \leq \lim_{\lambda \rightarrow 0} \lambda C\left(\left\{x \in \mathbb{R}^n : M_v(x) > \lambda\right\}\right) \leq \tau.$$

Next, let $h(\lambda) = \lambda C\left(\left\{x \in \mathbb{R}^n : M_v(x) > \lambda\right\}\right)$. By Lemma 2.4 and the last estimate for both the limit inferior and the limit superior, there exist two constants $A > 0$ and $\lambda_0 > 0$ such that $A \leq h(\lambda) \leq \gamma \forall \lambda \in (0, \lambda_0)$. Moreover, for any given $\varepsilon > 0$, choose a sequence $\{y_i = \left(\frac{1}{N} - \varepsilon\right)^N\}_{i=1}^\infty$, where $N$ is a natural number satisfying $\left(\frac{1}{N} - \varepsilon\right)^N < 1$. Then, there exists an integer $N_0 \geq 1$, such that $y_{N_0} < \lambda_0$. Hence, for any $n > m > N_0$, we have:
\[ |h(y_m) - h(y_n)| \leq |y_m C(\{ x \in \mathbb{R}^n : M_C v(x) > y_m \}) - y_n C(\{ x \in \mathbb{R}^n : M_C v(x) > y_n \})| \]
\[ \leq |y_m - y_n| C(\{ x \in \mathbb{R}^n : M_C v(x) > y_m \}) \]
\[ + |y_n| C(\{ x \in \mathbb{R}^n : M_C v(x) > y_m \}) - C(\{ x \in \mathbb{R}^n : M_C v(x) > y_n \})| \]
\[ \leq |y_m - y_n| \frac{y}{y_m} + |y_n - \frac{A}{y_m}| \]
\[ \leq \gamma \left( 1 - \frac{y_n}{y_m} \right) + \left( \gamma - \frac{A}{y_m} \right) |h(y_i) - D| \]
\[ \leq \gamma \left( 1 - \frac{y_i + 1}{y_i} \right) + \left( \gamma - \frac{A y_i}{y_i} \right) |h(y_i) - D| \]
\[ \leq (\gamma N + 1) \epsilon. \]

Consequently, \((h(y_i))\) is a Cauchy sequence, \(D = \lim_{i \to \infty} h(y_i)\) exists. Note that for any small \(\lambda\), there exists a large \(i\) such that \(y_i < \lambda < y_i\). Thereby, from the triangle inequality, it follows that if \(i\) is large enough, then:

\[ |h(\lambda) - D| \leq |h(\lambda) - h(y_i)| + |h(y_i) - D| \]
\[ \leq |y_i - \lambda| \frac{y}{y_i} + A \frac{y}{y_i} |h(y_i) - D| \]
\[ \leq \gamma \left( 1 - \frac{y_i + 1}{y_i} \right) + \left( \gamma - \frac{A y_i}{y_i} \right) |h(y_i) - D| \]
\[ \leq (\gamma N + 1) \epsilon. \]

This in turn implies that \(\lim_{\lambda \to 0} \lambda C(\{ x \in \mathbb{R}^n : M_C v(x) > \lambda \})\) exists, and consequently, \(\tau^{-1} \leq \lim_{\lambda \to 0} \lambda C(\{ x \in \mathbb{R}^n : M_C v(x) > \lambda \})\) holds.

Finally, upon employing the given \(L^1(\mathbb{R}^n)\) function \(f\) with \(\|f\|_1 > 0\) to produce a finite nonnegative measure \(\nu\) with \(\nu(\mathbb{R}^n) = 1\) via

\[ \nu(E) = \frac{1}{\|f\|_1} \int_E |f(y)| \, dy \quad \forall E \subseteq \mathbb{R}^n, \]

we obtain:

\[ \lim_{\lambda \to 0} \lambda C(\{ x \in \mathbb{R}^n : M_C f(x) > \lambda \|f\|_1 \}) = 1, \]

thereby getting:

\[ \lim_{\lambda \to 0} \lambda \|f\|_1 C(\{ x \in \mathbb{R}^n : M_C f(x) > \lambda \|f\|_1 \}) \approx \|f\|_1. \]

By setting \(\lambda = \|f\|_1\) in the last estimate, we reach the desired result.

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