HEAT–STRUCTURE INTERACTION WITH VISCOELASTIC DAMPING: ANALYTICITY WITH SHARP ANALYTIC SECTOR, EXPONENTIAL DECAY, FRACTIONAL POWERS

IRENA LASIECKA
Department of Mathematical Sciences
University of Memphis, Memphis, TN 38152, USA
IBS, Polish Academy of Sciences, Warsaw, Poland

ROBERTO TRIGGIANI∗
Department of Mathematical Sciences
University of Memphis, Memphis, TN 38152, USA

(Communicated by Igor Kukavica)

Abstract. We consider a heat–structure interaction model where the structure is subject to viscoelastic (strong) damping. This is a preliminary step toward the study of a fluid–structure interaction model where the heat equation is replaced by the linear version of the Navier–Stokes equation as it arises in applications. We prove four main results: analyticity of the corresponding contraction semigroup (which cannot follow by perturbation); sharp location of the spectrum of its generator, which does not have compact resolvent, and has the point λ = −1 in its continuous spectrum; exponential decay of the semigroup with sharp decay rate; finally, a characterization of the domains of fractional power related to the generator.

1. Introduction and statement of main result.

1.1. Introduction. We proceed to describe the canonical heat–structure PDE model of the present paper: its distinctive feature with respect to past literature on truly fluid-structure models such as [2, 3], [4] – [10], [30, 35] (involving actually the dynamic Stokes problems in place of the heat equation) is that it possesses a viscoelastic (‘strong’) damping term of the structure. It is a first study which is preliminarily focused on the boundary (interface) homogeneous problem because of space constraints. It is followed [40] by the ultimate case of interest, where the heat–structure PDE system is subject to a boundary control acting at the interface between the two media: the heat component and the structure component. The control may be either of Neumann-type or of Dirichlet-type, depending on the specific interface condition on which it acts, see (4) below. For this boundary non-homogeneous case, we study in [40] the corresponding (optimal) boundary → interior regularity, from which solution to corresponding optimal control problems,
as well as min-max game theory problems will follow as a specialization of the abstract PDE-based theory in [31]. In turn, this project is the first step toward the more realistic fluid–structure PDE model which has the more challenging dynamic Stokes equation in place of the n-dimensional heat equation [32, p. 121], [19]. It will be treated in a subsequent publication. Such model arises in biological applications [12, 13]. In the present work, we shall consider the structure immersed in the fluid, while leaving to other efforts the study of the fluid within a structure (such as blood running inside arteries).

The above references study the fluid-structure model (with dynamic Stokes equation involving pressure) with ‘static’ interface (justified in [19] to be appropriate under the assumption of small, rapid oscillations of the structure). For well-posedness result in the challenging case of moving interface, we quote [22, 25].

Thus, throughout, \( \Omega_f \subseteq \mathbb{R}^n \), \( n = 2 \) or 3, will denote the bounded domain on which the heat component of the coupled PDE system evolves. Its boundary will be denoted here as \( \partial \Omega_f = \Gamma_s \cup \Gamma_f \), \( \Gamma_s \cap \Gamma_f = \emptyset \), with each boundary piece being sufficiently smooth. Moreover, the geometry \( \Omega_s \), immersed within \( \Omega_f \), will be the domain on which the structural component evolves with time. As configured then, the coupling between the two distinct fluid and elastic dynamics occurs across boundary interface \( \Gamma_s = \partial \Omega_s \); see Figure 1. In addition, the unit normal vector \( \nu(x) \) will be directed away from \( \Omega_f \); thus on \( \Gamma_s \), toward \( \Omega_s \). (This specification of the direction of \( \nu \) will influence the computations to be done below.)

\[
\begin{align*}
\text{FIG. 1: THE FLUID–STRUCTURE INTERACTION}
\end{align*}
\]

On this geometry in Figure 1, we thus consider the following fluid-structure PDE model in solution variables \( u = [u_1(t, x), u_2(t, x), \ldots, u_n(t, x)] \) (the heat component here replacing the usual velocity field), and \( w = [w_1(t, x), w_2(t, x), \ldots, w_n(t, x)] \) (the structural displacement field):

\[
\begin{align*}
\text{(PDE)} & \quad \begin{cases}
    u_t - \Delta u = 0 & \text{in } (0, T) \times \Omega_f; \\
    w_{tt} - \Delta w - \Delta w_t + bw = 0 & \text{in } (0, T) \times \Omega_s;
\end{cases} \\
\text{(BC)} & \quad \begin{cases}
    u|_{\Gamma_f} = 0 & \text{on } (0, T) \times \Gamma_f; \\
    u = w_t & \text{on } (0, T) \times \Gamma_s; \\
    \frac{\partial(w + w_t)}{\partial \nu} = \frac{\partial u}{\partial \nu} & \text{on } (0, T) \times \Gamma_s;
\end{cases} \\
\text{(IC)} & \quad [w(0, \cdot), w_t(0, \cdot), u(0, \cdot)] = [w_0, w_1, u_0] \in H_b.
\end{align*}
\]

The constant \( b \) in (1b) will take up either the value \( b = 0 \), or else the value \( b = 1 \), as explained below. Accordingly, the space of well-posedness is taken to be the
finite energy space

\[ H_b = \begin{cases} \mathcal{H}^1(\Omega_s) \times \mathbb{R} \times \mathcal{L}^2(\Omega_s) \times \mathcal{L}^2(\Omega_f), & b = 0; \\ \mathcal{H}^1(\Omega_s) \times \mathcal{L}^2(\Omega_s) \times \mathcal{L}^2(\Omega_f), & b = 1, \end{cases} \]  

(2a)

\[ H_b \] is a Hilbert space with the following norm inducing inner product, where \( (f, g)_\Omega = \int_\Omega f \hat{g} \, d\Omega \):

\[
\begin{bmatrix}
    v_1 \\
    v_2 \\
    f
\end{bmatrix} \in H_b \quad \Rightarrow \quad
\begin{bmatrix}
    \langle \nabla v_1, \nabla \tilde{v}_1 \rangle_{\Omega_s} + \langle v_2, \tilde{v}_2 \rangle_{\Omega_s} + \langle f, \tilde{f} \rangle_{\Omega_f} \\
    \langle \nabla v_1, \nabla \tilde{v}_1 \rangle_{\Omega_s} + \langle v_1, \tilde{v}_1 \rangle_{\Omega_s} + \langle v_2, \tilde{v}_2 \rangle_{\Omega_s} + \langle f, \tilde{f} \rangle_{\Omega_f}
\end{bmatrix} = 0,
\]

(3a)

the first line for \( b = 0 \), the second for \( b = 1 \). In (2a), the space \( \mathcal{H}^1(\Omega_s) / \mathbb{R} = \mathcal{H}^1(\Omega_s) / \text{const} \) is endowed with the gradient norm.

1.2. The non-homogeneous case. As already noted, a companion publication [40] deals with the ‘boundary’ (interface) non-homogeneous problem, whereby the homogeneous boundary condition (B.C.) in (1e), respectively (1d), will be replaced by the following boundary non-homogeneous conditions, respectively:

\[
\text{either } \frac{\partial(w + w_t)}{\partial \nu} = \frac{\partial u}{\partial \nu} + g; \text{ or else } u = w_t + g \text{ on } (0, T) \times \Gamma_s. \tag{4}
\]

Here, \( g \) is a control function of Neumann-type, respectively, of Dirichlet-type. Writing each problem in the abstract form \( \dot{x} = Ax + B_N g \) (Neumann case) or \( \dot{x} = Ax + B_D g \) (Dirichlet case), with explicit operators \( B_N \) and \( B_D \), reference [40] shows two main preliminary results: (1) that \( A^{-1/2}B_N \) is a bounded operator from functions at the interface measured in the \( \mathcal{L}^2 \)-norm to the energy space \( H_0 \); (2) that \( A^{-1/2}B_D \) is a bounded operator from functions at the interface measured this time in the \( \mathcal{H}^2 \)-norm to the energy space \( H_0 \). In the much more challenging Neumann case, the proof that \( A^{-1/2}B_N \in \mathcal{L}(\mathcal{L}^2(\Gamma_s), H_0) \) requires explicit knowledge of the \( \mathcal{D}(\langle -A \rangle^{\theta}) \), or at least of a suitable subspace thereof [41]. In Theorem 1.5 below, we give a characterization in fact of \( \mathcal{D}(\langle -A \rangle^{\theta}) \), \( 0 < \theta < 1 \). Henceforth, we shall restrict the present paper to the homogeneous case \( g = 0 \), while referring to [40] for the case \( g \neq 0 \).

1.3. Abstract model for the free dynamics (1a–f) (i.e. \( g = 0 \) in (4).) The operator \( A \). We rewrite problem (1a–e) as a first-order equation

\[
\frac{d}{dt} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ \Delta - bI & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = A \begin{bmatrix} w \\ w_t \\ u \end{bmatrix}, \tag{5}
\]

where we introduce the operator \( A : H_b \supset \mathcal{D}(A) \rightarrow H_b \)

\[
A \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ \Delta - bI & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) - bv_1 \\ \Delta h \end{bmatrix}, \tag{6}
\]

for \( \{v_1, v_2, h\} \in \mathcal{D}(A) \). A description of \( \mathcal{D}(A) \) is as follows (we write it explicitly only in the case \( b = 0 \), which is the one to which we shall restrict below):
\(v_1 \in \mathcal{H}^1(\Omega_x)/\mathbb{R}; v_2 \in \mathcal{H}^1(\Omega_x)/\mathbb{R}, \text{so that } v_2|_{\Gamma_x} = h|_{\Gamma_x} \in \mathcal{H}^{1/2}(\Gamma_x);\)
\[
\Delta(v_1 + v_2) \in \mathcal{L}^2(\Omega_x);
\]  
\((7a)\)

\(h \in \mathcal{H}^1(\Omega_f), \Delta h \in \mathcal{L}^2(\Omega_f), h|_{\Gamma_f} \equiv 0, h|_{\Gamma_x} = v_2|_{\Gamma_x} \in \mathcal{H}^{1/2}(\Gamma_x);\)
\[
\frac{\partial h}{\partial \nu}|_{\Gamma_x} = \frac{\partial(v_1 + v_2)}{\partial \nu} \bigg|_{\Gamma_x} \in \mathcal{H}^{-1/2}(\Gamma_x).
\]  
\((7b)\)

We now justify \((7a-b)\). Minimal preliminary conditions for \(\{v_1, v_2, h\} \in \mathcal{D}(\mathcal{A})\) on \(\mathcal{H}_{b=0}\) are:
\[
\begin{cases}
(v_1, v_2, h) \in \mathcal{H}_{b=0}, \text{ actually } v_2 \in \mathcal{H}^1(\Omega_x)/\mathbb{R}, \text{hence } v_2|_{\Gamma_x} \in \mathcal{H}^{1/2}(\Gamma_x); \\
\Delta(v_1 + v_2) \in \mathcal{L}^2(\Omega_x), \Delta h \in \mathcal{L}^2(\Omega_f), h|_{\Gamma_f} \equiv 0.
\end{cases}
\]  
\((8a)\) \(8b)\)

But then, \(\Delta h \in \mathcal{L}^2(\Omega_f)\) along with \(h|_{\Gamma_f} \equiv 0\) and \(h|_{\Gamma_x} = v_2|_{\Gamma_x} \in \mathcal{H}^{1/2}(\Gamma_x)\) yields \(h \in \mathcal{H}^1(\Omega_f)\), as desired, by elliptic theory; hence \(|\nabla h| \in \mathcal{L}^2(\Omega_f)\). Moreover, the claimed Neumann boundary regularity \(\frac{\partial h}{\partial \nu}|_{\Gamma_x} \in \mathcal{H}^{-1/2}(\Gamma_x)\) follows from the identity
\[
\int_{\Omega_f} \Delta h \varphi d\Omega_f = \int_{\Gamma_x} \frac{\partial h}{\partial \nu} \varphi d\Gamma_x - \int_{\Omega_f} \nabla h \cdot \nabla \varphi d\Omega_f = \int_{\Omega_f} f \varphi d\Omega_f,
\]  
\((9)\)

with \(\Delta h = f \in \mathcal{L}^2(\Omega_f)\), test function \(\varphi \in \mathcal{H}^1(\Omega_f), \varphi|_{\Gamma_f} \equiv 0, \text{so that the last two}\)
\(\text{interior integral terms in } (9) \text{ are well defined, and then so is the boundary term on}\)
\(\Gamma_x \text{ with } \varphi \in \mathcal{H}^{1/2}(\Gamma_x). \text{ By surjectivity of the trace } [34], \varphi \text{ runs over all of } \mathcal{H}^{1/2}(\Gamma_x).\)

Then \(\frac{\partial h}{\partial \nu} \in \mathcal{H}^{-1/2}(\Gamma_x)\), as desired.

Remark 1.1. The above description of \(\mathcal{D}(\mathcal{A})\) in \((7a-b)\) shows that the point
\(v_1, v_2, h \in \mathcal{D}(\mathcal{A})\) enjoys a smoothing of regularity by one Sobolev unit—from
\(\mathcal{L}^2(\cdot)\) to \(\mathcal{H}^1(\cdot)\) — but only of the coordinates \(v_2\) and \(h\), with respect to the original
finite energy state space \(\mathcal{H}_0\) in \((2a)\). In contrast, the first coordinate \(v_1\) experiences
no smoothing: it is in \(\mathcal{H}^1(\Omega_x)\), the first coordinate component of the space \(\mathcal{H}_0\). This
amounts to the fact that \(\mathcal{A}\) has non-compact resolvent \(\mathcal{R}(\lambda, \mathcal{A})\) on \(\mathcal{H}_0\). Consis-
tently, we shall see below in Proposition 2.4 that the point \(\lambda = -1\) belongs to the
continuous spectrum of \(\mathcal{A} : -1 \in \sigma_c(\mathcal{A}).\) Henceforth, it will be convenient to focus
on the case \(b = 0\), as the case \(b = 1\) is a cosmetic variation of it. The space \(\mathcal{H}_{b=0}\)
will be henceforth denoted simply by \(\mathcal{H}_0\).

1.4. Main results of the free dynamics \((1a-f)\), case \(b = 0\). Our first main
result in the homogeneous case \((g = 0)\) is that the operator \(\mathcal{A}\) in \((6)-(7a-b)\) generates
a s.c. contraction semigroup on the energy space \(\mathcal{H}_0\) which moreover is analytic
(holomorphic) with maximal sector of analyticity, modulo a finite translation. More
precisely, the location of the spectrum \(\sigma(\mathcal{A})\) of \(\mathcal{A}\) in peculiar: conservatively, it is
contained in an infinite key-shaped set \(\mathcal{K}\):
\[
\sigma(\mathcal{A}) \subset (-\infty, -2) \cup \{S_{r=1}(x_0)\} \cap \mathcal{K} \equiv \mathcal{K}
\]  
\((10a)\)

\(x_0 = \text{point } \{-1, 0\} \text{ in } \mathbb{C}, \text{ or } \mathbb{C} \setminus \mathcal{K} \subset \rho(\mathcal{A}) = \text{resolvent set of } \mathcal{A}
\]  
\((10b)\)

where \(S_{r=1}(x_0)\) is the open disk centered at the point \(x_0 = \{-1, 0\}\) and of radius
1, and \(S_{r_0}(0)\) is the open disk centered at the origin of radius \(r_0 > 0\). In addition,
we shall see that the spectrum \(\sigma(\mathcal{A})\) of the operator \(\mathcal{A}\), which does not have
compact resolvent, contains the point $-1$ in the continuous spectrum. We refer to Theorem 1.4 and Proposition 2.3. Similar results hold for the adjoint $A^*$ of $A$ to be characterized next.

**Theorem 1.1 (The adjoint $A^*$ of $A$).** The $H_0$-adjoint of the operator $A$ defined in (6) (for $b = 0$) and (7) is given by

$$
A^* \begin{bmatrix}
\tilde{v}_1 \\
\tilde{v}_2 \\
\tilde{h}
\end{bmatrix} = \begin{bmatrix}
0 & -I & 0 \\
-\Delta & \Delta & 0 \\
0 & 0 & \Delta
\end{bmatrix} \begin{bmatrix}
\tilde{v}_1 \\
\tilde{v}_2 \\
\tilde{h}
\end{bmatrix} = \begin{bmatrix}
-\tilde{v}_2 \\
\Delta(\tilde{v}_2 - \tilde{v}_1) \\
\Delta \tilde{h}
\end{bmatrix},
$$

(11)

with $D(A^*)$ described as follows (compare with $D(A)$ in (7a–b)): \( \{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\} \in D(A^*) \) means:

(i) \( \tilde{v}_1 \in H^1(\Omega_s)/\mathbb{R}; \tilde{v}_2 \in H^1(\Omega_s)/\mathbb{R}, \) so that \( \tilde{v}_2|_{\Gamma_s} = \tilde{h}|_{\Gamma_s} \in H^{1/2}(\Gamma_s); \)

\( \Delta(\tilde{v}_2 - \tilde{v}_1) \in L^2(\Omega_s); \)

(12a)

(ii) \( \tilde{h} \in H^1(\Omega_f), \Delta \tilde{h} \in L^2(\Omega_f), \tilde{h}|_{\Gamma_f} \equiv 0, \tilde{h}|_{\Gamma_s} = \tilde{v}_2|_{\Gamma_s} \in H^{1/2}(\Gamma_s); \)

\( \frac{\partial \tilde{h}}{\partial \nu}|_{\Gamma_s} = \frac{\partial (\tilde{v}_2 - \tilde{v}_1)}{\partial \nu}|_{\Gamma_s} \in H^{-1/2}(\Gamma_s). \)

(12b)

The computational proof is given in Appendix A.

On the space $H_0$ introduce the following bounded, symmetric operator

$$
T \equiv \begin{bmatrix}
I & 0 & 0 \\
0 & -I & 0 \\
0 & 0 & -I
\end{bmatrix} \equiv T^* \text{ on } H_0.
$$

(13)

Then, one may verify the following properties:

(i) \( T^2 = \text{identity on } H_0. \)

(14)

(ii) \( T : D(A) \overset{\text{onto}}{\longrightarrow} D(A^*); \quad T = T^{-1} : D(A^*) \overset{\text{onto}}{\longrightarrow} D(A) \)

(15a)

\( T D(A) \equiv D(A^*); \quad T D(A^*) \equiv D(A) \)

(15b)

(iii) \( T A = A^* T = A^* T^* = (T A)^* \text{ on } D(A) \)

\( A = T^{-1} A^* T \text{ on } D(A) \) (similarity)

(16a)

(16b)

and $T A$ is self-adjoint with domain $D(A)$;

\[ S_{r=1}(x_0) \]

\[ S_{r_0}(0) \]

**FIG 2:** The set $K$
Theorem 1.2

(iii) The operator $A$ is boundedly invertible on $H_0$: $A^{-1} \in \mathcal{L}(H_0)$. (A more precise result is given in Lemma 2.1 below.) Thus, there exists a disk $S_{r_0}$ of the complex plane $\mathbb{C}$, centered at the origin and of suitably small radius $r_0 > 0$ such that $S_{r_0} \subset \rho(A)$, the resolvent set of $A$. Similarly for the operator $A^*$.
(iv) Hence, \(A\) is maximal dissipative on \(H_0\). By the Lumer–Phillips Theorem, \(A\) generates a s.c. \((C_0-\) contraction semigroup \(e^{At}\) on \(H_0\):

\[
\begin{bmatrix}
w_0 \\
w_1 \\
u_0
\end{bmatrix} \in H_0 \rightarrow \begin{bmatrix} w(t) \\
w(t) \\
u(t)
\end{bmatrix} \equiv e^{At} \begin{bmatrix} w_0 \\
w_1 \\
u_0
\end{bmatrix} \in C([0,T];H_0) .
\quad (23)
\]

(v) The same generation results hold also for \(A^*\).

The proof is given in Section 2.

**Theorem 1.3** (Dissipative energy identity; further regularity of \((1a-f)\)). With reference to problem \((1a-f)\), whose semigroup well-posedness is guaranteed by Theorem 1.2, define the energies for \(\{w,w_t,u\} \in H_0\):

\[
E_u(t) = \int_{\Omega_f} u^2(t) d\Omega_f; \quad E_w(t) = \int_{\Omega_s} \left| w^2(t) + |\nabla w(t)|^2 \right| d\Omega_s , \quad t > 0 .
\quad (24)
\]

(a) Then, the following identities hold true for \(t > 0\) and \(\{w_0,w_1,u_0\} \in D(A)\), so that \([w(t),w_t(t),u(t)] \in C([0,T];D(A))\):

\[
E_u(t) + 2 \int_0^t \int_{\Omega_f} |\nabla u|^2 d\Omega_f d\tau = E_u(0) + 2 \int_0^t \int_{\Gamma_s} \frac{\partial u}{\partial \nu} u d\Gamma_s d\tau .
\quad (25)
\]

(b) (Regularity properties for \(\{w_0,w_1,u_0\} \in H_0\) complementing \((23)\).) The following regularity properties of problem \((1a-f)\) hold true, all continuously with respect to \(\{w_0,w_1,u_0\} \in H_0\):

\[
w \in C([0,T];H^1(\Omega_s));
\quad (28)
\]

\[
w_t \in C([0,T];L^2(\Omega_s)) \cap L^2(0,T; H^1(\Omega_s)/\mathbb{R})
\quad (29)
\]

\[
u \in C([0,T];L^2(\Omega_f)) \cap L^2(0,T; H^1(\Omega_f));
\quad (30)
\]

\[
u|_{\Gamma_s} \in L^2(0,T; H^{\frac{1}{2}}(\Gamma_s)); \quad \frac{\partial u}{\partial \nu}|_{\Gamma_s} \in L^2(0,T; H^{-\frac{1}{2}}(\Gamma_s)).
\quad (31)
\]

**Proof.** (a1) We multiply the \(u\)-problem in \((1a)\) by \(u\) and integrate over \(\int_0^t \int_{\Omega_f}\), using \(\frac{1}{2} \frac{d}{dt} (u^2) = u w_t\), Green’s First Theorem with \(|u|_{\Gamma_f} \equiv 0\), and obtain \((25)\).

(a2) We multiply the \(w\)-problem in \((1b)\) by \(w_t\) and integrate over \(\int_0^t \int_{\Omega_f}\), using \(\frac{1}{2} \frac{d}{dt} (w_t^2) = w_t w_{tt}\), Green’s First Theorem with unit normal \(\nu\) inward, as well as
\[ \frac{1}{2} \frac{\partial}{\partial t} |\nabla w|^2 = \nabla w \cdot \nabla w_t, \] finally the B.C. \( u = w_t \) on \( \Gamma_s \), and \( \frac{\partial(w+w_t)}{\partial \nu} = \frac{\partial u}{\partial \nu} \) on \( \Gamma_s \). This way we obtain (26).

(a3) Summing up (a1) and (a2) yields (a3) first for I.C. in \( D(A) \), next in \( H_0 \) by density and continuous extension.

(b) The interior regularity in \( C([0, T]; \cdot) \) in (28)–(30) is an explicit restatement of (23). The interior regularity in \( L^2(0, T; \cdot) \) in (29), (30) is a consequence of identity (27), whereby then the Dirichlet boundary regularity of \( u|_{\Gamma_s} \) in the LHS of (21) follows by trace theory. In turn, such result on \( u|_{\Gamma_s} \), used with the property that \( \int_0^t \int_{\Gamma_s} \frac{\partial u}{\partial \nu} u \, d\Gamma_s \, ds \in L^2(0, T) \) yields then the Neumann boundary regularity in (31).

1.5. Orientation on analyticity. A first main result of the present paper is Theorem 1.4 below that claims that the s.c. semigroup \( e^{At} \) asserted by Theorem 1.2 is, in fact, analytic on the space \( H_0 \), and in fact, with the triangular sector containing the spectrum \( \sigma(A) \) of its generator \( A \) that reduces itself to the infinite axis \( (\infty, -2) \) for \( \Re \lambda < -2 \). A more precise statement on the location of the spectrum \( \sigma(A) \) of \( A \) is given in (10).

Analyticity per se is not surprising in view of the following motivating considerations.

A motivating result. (a) Analyticity. The following is a very special case of a much more general result (noted below) for which we refer to [15], [16], (see also [31, Appendix 3B of Chapter 3, pp 285-296], [17], [18]). These references solve and improve upon the conjectures posted in [14]. Let \( A \) be a positive, self-adjoint operator on the Hilbert space \( Y \). On it, consider the following abstract equation

\[ \ddot{x} + Ax + A\dot{x} = 0; \quad \text{or} \quad \frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \mathbb{A} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}; \] (32)

\[ \mathbb{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A & -A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -A(x_1 + x_2) \end{bmatrix}; \] (33a)

\[ D(\mathbb{A}) = \left\{ [x_1, x_2] \in E \equiv D(A^{\frac{1}{2}}) \times Y : x_2 \in D(A^{\frac{1}{2}}), x_1 + x_2 \in D(A) \right\}. \] (33b)

The operator \( \mathbb{A} \) is dissipative and with domain (33b) is closed and generates a s.c. contraction semigroup \( e^{kt} \) on the finite energy space \( E \equiv D(A^{\frac{1}{2}}) \times Y \), which moreover is analytic on \( E \). Thus, the second-order dynamic (32) with strong ‘structural’ damping is parabolic-like. Indeed, [15], [16], (see also [31, Appendix 3B of Chapter 3, pp 285-296]) show the more general result, and more useful in application to mixed PDEs-problems, that analyticity holds true if in equation (32) the damping term \( A\dot{x} \) is replaced by \( B\dot{x} \), where \( B \) is another positive self-adjoint operator (which needs not commute with \( A \)) which is comparable with \( A^\alpha \), \( \frac{1}{2} \leq \alpha \leq 1 \), in the sense of inner product: \( \rho_1 A^\alpha \leq B \leq \rho_2 A^\alpha, 0 < \rho_1 < \rho_2 < +\infty \).

(b) The spectrum of \( \mathbb{A} \). Reference [16, Appendix A, Lemma A.1, p45] shows that the spectrum \( \sigma(\mathbb{A}) \) of the operator \( \mathbb{A} \) defined in (32)–(33b) (case \( \alpha = 1 \)) has the following features assuming that the positive self-adjoint operator \( A \) has compact resolvent on \( Y \):

The spectrum of \( \mathbb{A} \) consists of two branches of eigenvalues \( \lambda_n^{\pm} \):

\[ \lambda_n^{+, -} = -\mu_n \pm \frac{\mu_n}{2} \sqrt{\frac{\mu_n - \overline{\lambda}}{\mu_n}}, \] (34)
solutions of the algebraic equation \( \lambda^2 + \mu_n \lambda + \mu_n = 0 \), where \( \{\mu_n\}_{n=1}^\infty \) are the eigenvalues of the positive self-adjoint operator \( A : 0 < \mu_1 < \cdots < \mu_n \to +\infty \). The
branch $\lambda_n^- \searrow -\infty$ monotonically. The branch $\lambda_n^+ \nearrow -1$ monotonically. Moreover, the point $\lambda = -1$ belongs to the continuous spectrum $\sigma_c(A)$ of the operator $A$. The operator $A$ does not have compact resolvent on the finite energy space $E$, even though $A$ has compact resolvent on $Y$.

By looking at the operator $A$ in (6), the above abstract result for equation (33a) suggests, or makes one surmise, that the homogeneous problem (1a–e) is the coupling of ‘two parabolic problems’ and hence generates an analytic semigroup $e^{At}$ ($A$ in (6), (7)) on the finite energy space $H_0$ in (2). Of course, the above considerations are purely indicative and qualitatively suggestive, as the Laplacian $\Delta$ in (1b) has coupled, high-level, non-homogeneous interface boundary conditions which constitute the crux of the matter to be resolved before making the assertion of analyticity of problem (1). At any rate analyticity cannot follow by a perturbation argument.

We have already noted in Remark 1.1 that the operator $A$ in (6), (7) does not have compact resolvent on the finite energy space $H_0$. We shall see in Proposition 2.4 below that $\lambda = -1$ is a point in the continuous spectrum of the operator $A$: $-1 \in \sigma_c(A)$. This result coupled with the location of the spectrum of $A$ anticipated in (10a) (that is $\sigma(A) \subset (-\infty, -2) \cup \{S_{r=1}([-1,0]) \setminus S_{r=0}(0)\} \equiv K$) make one expect that, qualitatively, the spectrum of $A$ is like the spectrum of the operator $A$ in (33a), with one branch of eigenvalues being negative and going to $-\infty$, and the other branch going to the point $-1$ of the continuous spectrum. 

Our first main result is

**Theorem 1.4.** (i) The generator $A$ in (6), (7) (with $b = 0$) of the s.c. contraction semigroup $e^{At}$ asserted by Theorem 1.2 satisfies the following resolvent condition

$$
\|(i\omega I - A)^{-1}\|_{L(H_0)} = \|R(i\omega, A)\|_{L(H_0)} \leq \frac{c}{|\omega|},
$$

where $|\omega| \geq \delta_0 > 0$ arbitrarily small, (35)

with $\omega \in \mathbb{R}$. Hence, the s.c. semigroup $e^{At}$ is analytic on the finite energy space $H_0$, $t > 0$, [31, Thm 3E.3, p 334].

(ii) More precisely, the resolvent operator $R(\lambda, A) = (\lambda I - A)^{-1}$ of the generator $A$ in (6), (7) (with $b = 0$), satisfies the following estimate

$$
\|R(\lambda, A)\|_{L(H_0)} \leq \frac{C}{|\lambda|}, \text{ for all } \lambda \in \mathbb{C} \setminus K
$$

where $K$ is the (infinite) key-shaped set defined in (10a), Fig 2

$$(36a)$$

with $S_{r=1}(x_0)$ the open disk centered at the point $x_0 = \{-1, 0\}$ and of radius 1; and $S_{r=0}$ defined in Theorem 1.2(iii).

(iii) The spectrum $\sigma(A)$ of $A$ is confined within the set $K$; in particular

$$
\text{Re}\sigma(A) \subset (-\infty, -\delta], \text{ for some } \delta > 0.
$$

(iv) Complementing (35) we have that the resolvent $R(\cdot, A)$ is uniformly bounded on the imaginary axis

$$
\|R(i\omega, A)\|_{L(H_0)} \leq c, \quad \forall \omega \in \mathbb{R}.
$$

Hence, the s.c. analytic semigroup $e^{At}$ is uniformly exponentially stable on $H_0$: there exist constants $M \geq 1$, $\delta > 0$, such that [38]

$$
\|e^{At}\|_{L(H_0)} \leq Me^{-\delta t}, \quad t \geq 0.
$$

(39)
For s.c. analytic semigroups such as $e^{At}$, domains of fractional powers of $(-A)$ are very important. For instance, a special subcase of the following result allows one to obtain optimal regularity results of the non-homogeneous mixed problem (1a-d) with forcing term $g \in L^2(0,T : L^2(\Gamma_s))$ acting at the interface $\Gamma_s$ in the Neumann B.C. as in (4) (LHS).

**Theorem 1.5.** With reference to the operator $A$ in (6), (7a-b) (say with $b = 0$) and its adjoint $A^*$ in (10), (11a-b), we have

$$
D((-A)^{1/2}) = D((-A^*)^{1/2}) \equiv V = \left\{ v_1, v_2, h : v_1 \in H^1(\Omega_s)/\mathbb{R}; 
\vphantom{\rho\theta}\right.

\begin{align*}
v_2 &\in H^1(\Omega_s)/\mathbb{R}; & h &\in H^1(\Omega_f); & h|_{\Gamma_f} &\equiv 0; & h|_{\Gamma_s} = v_2|_{\Gamma_s}\right\}
\end{align*}

(40)

A proof is given in Section 4. We note that the well-known sufficient condition of [33, Theorem 6.1, p.238] implying $D((-A)^{1/2}) = D((-A^*)^{1/2})$ for an operator $-A$ defined by a bilinear form $a(\cdot, \cdot) : a(u, v) = -(Au, v)$ continuous on a space $V$ is not applicable in the present case, as assumption (i) of [33, Theorem 6.1, p.238] is not fulfilled.

**Remark 1.2.** Theorem 1.5 provides explicit identification as well as coincidence of $D((-A)^{1/2})$ and $D((-A^*)^{1/2})$ -the domains of the square roots of the maximal accretive operators $(-A)$ and $(-A^*)$-with $A$ the operator in (6), (7), with strongly coupled boundary conditions, as it arises in the original physical model. Such property of coincidence is usually referred to as Kato’s problem. It is well-known that in 1961, T. Kato [23] showed that for a maximal accretive operator $Q$, one always has $D(Q^{\theta}) \equiv D(Q^{\theta})$ for $\theta \in [0, 1/2)$. He also conjectured that the coincidence should also obtain for $\theta = 1/2$ for regularly accretive operators. One year later, J.L. Lions [32], page 240, provided a counterexample for $\theta = 1/2$, by considering a first order differential operator $Q_1 = \frac{d}{dx}$ on $L^2(0, \infty)$, with boundary condition $u(0) = 0$. Then such accretive operator $Q_1$ has $D(Q_1^{1/2})$ which is strictly contained in $D(Q_1^{1+1/2})$. Since then, there has been intensive work on the Kato’s problem. In 1972, A. McIntosh [36] provided another counterexample where $D(Q_2^{1/2}) \neq D(Q_2^{1+1/2})$ for a maximal accretive operator arising from a bilinear form. A positive solution of the Kato’s square root problem for second order elliptic operators in $\mathbb{R}^n$ was given in [1] in 2002. In contrast, our maximal accretive operator $(-A)$ which satisfies (43b) [32, footnote p. 233] has strongly coupled boundary conditions which are not amenable to classical interpolation theorems. The proof of our result relies on some devices in order to incorporate the coupled boundary conditions in a way suitable for interpolation.

**Remark 1.3.** Either by our direct proof in Section 4, or else by interpolation between $H$ and $D((-A)^{1/2})$ and between $D((-A)^{1/2})$ and $D(A)$, one may readily obtain $D((-A)^{\theta}), \theta \in (0, 1)$.

**Remark 1.4.** For $\Phi = \{v_1, v_2, h\}, \tilde{\Phi} = \{\tilde{v}_1, \tilde{v}_2, \tilde{h}\} \in V$ (in (40)), define the sesquilinear form

$$
a(\Phi, \tilde{\Phi}) \equiv -\langle \nabla v_2, \nabla \tilde{v}_1 \rangle + \langle \nabla v_1, \nabla \tilde{v}_2 \rangle + \langle \nabla v_2, \nabla \tilde{v}_2 \rangle + \langle \nabla h, \nabla \tilde{h} \rangle
$$

(41)

which is continuous on the space $V$ defined in (40).

The RHS of (41) is the opposite of the RHS of identity (45) below, which will be derived for $\Phi = \{v_1, v_2, h\} \in D(A)$ and $\tilde{\Phi} \in V$. By density and continuity, we can
extend the validity of identity (45) to all of $\Phi \in V$ as well, and thus write
\[
a(\Phi, \tilde{\Phi}) \equiv (-A)\Phi, \tilde{\Phi})_H = - (\nabla v_2, \nabla \tilde{v}_1) + (\nabla v_1, \nabla \tilde{v}_2) + (\nabla v_2, \nabla \tilde{v}_2) + (\nabla h, \nabla \tilde{h})
\]
for $\Phi, \tilde{\Phi} \in V$.

In particular, recalling (21), we can write for $\Phi = \{v_1, v_2, h\} \in D(A)$
\[
\text{Re} \left( \left[ (-A) + I \right] \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right)_H = \text{Re} \ a(\Phi, \tilde{\Phi}) + \|\Phi\|_H^2 \tag{43a}
\]
\[
= \|\nabla v_1\|^2 + \|\nabla v_2\|^2 + \|\nabla h\|^2 + \|v_2\|^2 + \|h\|^2 
\]
\[
\geq \|\nabla v_1\|^2 + \|\nabla v_2\|^2 + \|\nabla h\|^2 \equiv \|\{v_1, v_2, h\}\|^2_V \tag{43b}
\]
so that the sesquilinear form $a(\cdot, \cdot)$ is regular accretive [33, p233, footnote]. However, as noted above, the setting of [32] on bilinear forms and trace spaces cannot be invoked to obtain the characterizations (40).

2. Proof of Theorem 1.2 ($b = 0$).

**Lemma 2.1. (Bilinear form, dissipativity of $A$, strict accretivity of $[(-A) + I]$)**

Let $\{v_1, v_2, h\} \in D(A)$ given by (7a-b) and $\{\tilde{v}_1, \tilde{v}_2, \tilde{h}\} \in V \supset D(A)$, where
\[
V = \left\{ \{\tilde{v}_1, \tilde{v}_2, \tilde{h}\} \in H^1(\Omega_s)/\mathbb{R} \times H^1(\Omega_s)/\mathbb{R} \times H^1(\Omega_f); \; \tilde{h}|_{\Gamma_f} \equiv 0; \; \tilde{h}|_{\Gamma_s} = \tilde{v}_2|_{\Gamma_s} \right\} \tag{44}
\]
Then: (i)
\[
\left( A \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{h} \end{bmatrix} \right)_H = (\nabla v_2, \nabla \tilde{v}_1)_{\Omega_s} - (\nabla v_1, \nabla \tilde{v}_2)_{\Omega_s} - (\nabla v_2, \nabla \tilde{v}_2)_{\Omega_s} - (\nabla h, \nabla \tilde{h})_{\Omega_f}; \tag{45}
\]
in the respective $L^2$–inner products on $\Omega_s$ or $\Omega_f$.

(ii) in particular, for $[v_1, v_2, h] \in D(A)$:
\[
\left( A \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right)_H = (\nabla v_2, \nabla v_1)_{\Omega_s} - (\nabla v_1, \nabla v_2)_{\Omega_s} - \|\nabla v_2\|^2_{\Omega_s} - \|\nabla h\|^2_{\Omega_f}; \tag{46a}
\]
\[
= 2 \text{Im}(\nabla v_2, \nabla v_1)_{\Omega_s} - \|\nabla v_2\|^2_{\Omega_s} - \|\nabla h\|^2_{\Omega_f}. \tag{46b}
\]

Refer also to Remark 1.2.

(iii) If we specialize further and let $e = [v_1, v_2, h] \in D(A)$ be a normalized (in $H_0$) eigenvector of $A$ corresponding to the eigenvalue $\lambda : A e = \lambda e$, then $\lambda = \text{RHS}$ of (46a); and $\lambda$ need not be real.

(iv) for $[v_1, v_2, h] \in D(A)$:
\[
\text{Re} \left( A \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right)_H = - \|\nabla v_2\|^2_{\Omega_s} - \|\nabla h\|^2_{\Omega_f}, \tag{46b}
\]
(v) **Strict accretivity property of the operator** \([-\mathcal{A} + I]\). Still for \([v_1, v_2, h] \in D(\mathcal{A})\):

\[
\text{Re} \left( [-\mathcal{A} + I] \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right)_{\mathcal{H}_0} = \| \nabla v_1 \|_{\Omega}^2 + \| \nabla v_2 \|_{\Omega}^2 + \| \nabla h \|_{\Omega_j}^2 + \| v_2 \|_{\Omega}^2 + \| h \|_{\Omega_j}^2
\]

\[
\geq \| \nabla v_1 \|_{\Omega}^2 + \| \nabla v_2 \|_{\Omega}^2 + \| \nabla h \|_{\Omega_j}^2
\]

\[
= \| \{ v_1, v_2, h \} \|_{\mathcal{H}}^2
\]

(47)

**Proof.** (i) Let \(\{v_1, v_2, h\} \in D(\mathcal{A})\) and \(\{\tilde{v}_1, \tilde{v}_2, \tilde{h}\} \in V\) (in (44)).

To show (45), we compute recalling the inner product (3a) for \(b = 0\) in \(\mathcal{H}_0\), Green’s First Theorem, \(\tilde{h} |_{\Omega_j} = 0\) by (44), and the unit normal \(\nu\) being inward to \(\Omega_s\):

\[
\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{h} \end{bmatrix} \right)_{\mathcal{H}_0} = \left( \begin{bmatrix} 0 & I & 0 \\ \Delta & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \right)_{\mathcal{H}_0}
\]

\[
= (\nabla v_2, \nabla \tilde{v}_1)_{\Omega} + (\Delta (v_1 + v_2) + (\Delta h, \tilde{h}))_{\Omega_j}
\]

\[
= (\nabla v_2, \nabla \tilde{v}_1)_{\Omega} - \int_{\Gamma_s} \frac{\partial (v_1 + v_2)}{\partial \nu} \tilde{v}_2 d\Gamma_s
\]

\[
- (\nabla (v_1 + v_2), \nabla \tilde{v}_2)_{\Omega} + \int_{\Gamma_s} \frac{\partial h}{\partial \nu} \tilde{h} d\Gamma_s
\]

\[
- (\nabla h, \nabla \tilde{h})_{\Omega_j}
\]

(48)

since, moreover, \(\tilde{h} = \tilde{v}_2\) on \(\Gamma_s\) by (44) and \(\frac{\partial (v_1 + v_2)}{\partial \nu} = \frac{\partial h}{\partial \nu}\) on \(\Gamma_s\) by (7b) and the two boundary integral terms cancel out. Then (48) yields (45).

(ii), (iii), (iv). Results (46a), (46b), (47) readily follow from (45). In particular, (20), (21) are established for \(\mathcal{A}\). \qed}

**Lemma 2.2.** (i) Consider the operator \(\mathcal{A}\) in (6), (7), with \(b = 0\) on the space \(\mathcal{H}_{b=0}\) in (2a), as well as its adjoint \(\mathcal{A}^*\) given by (11), (12). Then the point \(\lambda = 0\) is neither an eigenvalue of \(\mathcal{A}\) nor an eigenvalue of \(\mathcal{A}^*\). Thus, if \(\sigma_p, \sigma_r\) denote point and residual spectrum, we have

\[
0 \notin \sigma_p(\mathcal{A}), \quad 0 \notin \sigma_r(\mathcal{A}^*), \quad 0 \notin \sigma_r(\mathcal{A}).
\]

(49)

(ii) In fact, \(\lambda = 0\) is in the resolvent set \(\rho(\mathcal{A})\) of \(\mathcal{A}\), and in the resolvent set \(\rho(\mathcal{A}^*)\) of \(\mathcal{A}^*\):

\[
0 \in \rho(\mathcal{A}), \quad 0 \in \rho(\mathcal{A}^*), \quad \mathcal{A}^{-1} \in \mathcal{L}(\mathcal{H}_0), \quad \mathcal{A}^{*-1} \in \mathcal{L}(\mathcal{H}_0).
\]

(50)

More precisely, given \(\{v_1^*, v_2^*, h^*\} \in \mathcal{H}_b, b = 0, b = 1\) in (2), the unique solution \(\{v_1, v_2, h\} \in D(\mathcal{A})\) of

\[
\mathcal{A} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} \Delta (v_1 + v_2) - bv_1 \\ v_2 \\ \Delta h \end{bmatrix} = \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}.
\]

(51)
is given explicitly by
\[ v_1 = -v_1^* + (-A_{N,s} - bI)^{-1}(-bv_1^* + v_2^*) + N_s \left\{ \frac{\partial}{\partial \nu} \left[ -A_{D,f} h^* + D_f(v_1^*|_{\Gamma_s}) \right] \right\} \in \left\{ \begin{array}{ll} H^1(\Omega_f)/\mathbb{R} & b = 0 \\ H_1(\Omega_s) & b = 1; \end{array} \right\} \]  
\[ v_2 = v_1^* \in \left\{ \begin{array}{ll} H^1(\Omega_s)/\mathbb{R} & b = 0 \\ H_1(\Omega_s) & b = 1; \end{array} \right\} \]  
\[ h = -A_{D,f}^{-1} h^* + D_f(v_1^*|_{\Gamma_s}) \in H^1(\Omega_f); \]  
(53)

In operator form, we have for the case of interest (ii):
\[ \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = A^{-1} \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} = \begin{bmatrix} -v_1^* - A_{N,s}^{-1} v_2^* + N_s \left\{ \frac{\partial}{\partial \nu} \left[ -A_{D,f}^{-1} h^* + D_f(v_1^*|_{\Gamma_s}) \right] \right\} \\ v_1^* \\ -A_{D,f}^{-1} h^* + D_f(v_1^*|_{\Gamma_s}) \end{bmatrix} (54a) \]
\[ = \begin{bmatrix} -I + N_s \frac{\partial}{\partial \nu} D_f(\cdot|_{\Gamma_s}) & -A_{N,s}^{-1} & -N_s \frac{\partial}{\partial \nu} A_{D,f}^{-1} \\ -A_{D,f}^{-1} & 0 & 0 \\ D_f(\cdot|_{\Gamma_s}) & 0 & -A_{D,f}^{-1} \end{bmatrix} \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} (54b) \]
\[ \|v_1, v_2, h\|_{H^1} \leq c\|v_1^*, v_2^*, h^*\|_{H^1}, (54c) \]

where the operators \( A_{N,s}, A_{D,f}, D_f, N_s \), are defined in the proof below.

(iii) If, on the other hand, in the case \( b = 0 \), the operator \( A \) in (6) is viewed in the the space \( H^1(\Omega_s) \times L^2(\Omega_s) \times L^2(\Omega_f) \) with full \( H^1 \)-norm on the first component, then \( \lambda = 0 \) is an eigenvalue of such \( A \), with corresponding eigenvector [1,0,0].

Proof. (i): We consider the operator \( A \) fixed. The analysis of \( A^* \) is similar, or we use the properties (14)-(19). Let \( Ax = 0 \) for \( x = [v_1, v_2, h] \in D(A) \). Then, (6) yields \( v_2 \equiv 0 \) a.e. in \( \Omega_s \), hence \( v_2|_{\Gamma_s} \equiv 0 \) a.e. in \( \Gamma_s \), as well as (*) \( \Delta v_1 - bv_1 = 0 \) in \( L^2(\Omega_s) \) and \( \Delta h = 0 \) in \( L^2(\Omega_f) \). The latter result, along with the required B.C. \( h|_{\Gamma_f} = 0 \) and \( h|_{\Gamma_s} = v_2|_{\Gamma_s} = 0 \) implies \( h = 0 \) in \( L^2(\Omega_f) \). Hence, the above \( v_1 \)-equation (*) is accompanied by the B.C. \( \frac{\partial}{\partial \nu} |_{\Gamma_s} = \frac{\partial}{\partial \nu} |_{\Gamma_s} = 0 \), and hence it implies: \( v_1 \equiv 0 \) for \( b = 1 \); and \( v_1 \equiv 0 \) for \( b = 0 \). Thus, \( \lambda = 0 \) is not an eigenvalue in case \( b = 1 \): \( v_1 = v_2 = h = 0 \). In case \( b = 0 \), the function \( f \equiv 1 \) is not in \( H^1(\Omega_s)/\mathbb{R} \), and again \( \lambda = 0 \) is not an eigenvalue of \( A \) for \( b = 0 \) on the space \( H_0 \) in (2a). Instead, on the space \( H^1(\Omega_s) \times L^2(\Omega_s) \times L^2(\Omega_f) \) the operator \( A \) in (6) has \( \lambda = 0 \) as an eigenvalue with a one-dimensional eigenspace spanned by the vector [1,0,0]. Finally, if \( \lambda = 0 \) were in \( \sigma_p(A^*) \), then we would have \( \lambda \in \sigma_p(A^*) \) by [39, p 292], a contradiction.

(ii) Identity (51) yields
\[ v_2 = v_1^* \] (as in (53)): \[ \begin{cases} \Delta h = h^* \in L(\Omega_f); \\ h|_{\Gamma_f} = 0, h|_{\Gamma_s} = v_2|_{\Gamma_s} = v_1^*|_{\Gamma_s} \in H^1(\Gamma_s), \end{cases} \]  
(55a)
\[ \text{and the } h\text{-problem in (55) yields the solution } h \text{ in (53), where} \]
\[ -A_{D,f} \varphi = \Delta \varphi \text{ in } \Omega_f; \quad \varphi \in D(A_{D,f}) = H^2(\Omega_f) \cap H^1(\Omega_f); \]  
(56)
\[ D_f : H^s(\partial \Omega_f) \to H^{s+\frac{1}{2}}(\Omega_f), s \in \mathbb{R} : \quad D_f \mu = \psi \iff \begin{cases} \Delta \psi = 0 \text{ in } \Omega_f; \\ \psi|_{\Gamma_f} = 0, \quad \psi|_{\Gamma_s} = \mu. \end{cases} \]  

(57a)  

Moreover, (51), \( v_2 = v_1^* \) in (55) and (7b) yield by (53)  
\[
\begin{align*}
&\Delta(v_1 + v_2) - bv_1 = v_2, \quad \text{or} \quad \Delta(v_1 + v_1^*) - b(v_1 + v_1^*) = -bv_1^* + v_2^*; \\
&\left. \frac{\partial(v_1 + v_2)}{\partial \nu} \right|_{\Gamma_s} = \left. \frac{\partial h}{\partial \nu} \right|_{\Gamma_s}, \quad \text{or} \quad \left. \frac{\partial(v_1 + v_1^*)}{\partial \nu} \right|_{\Gamma_s} = \left. \frac{\partial h}{\partial \nu} \right|_{\Gamma_s} \in H^{-\frac{1}{2}}(\Gamma_s). 
\end{align*}
\]

(58a)  

Then, the solution of problem (58) is given by (52) via (53), where  
\[ -A_{N,s} \varphi = \Delta \varphi \text{ in } \Omega_s; \quad D(A_{N,s}) = \left\{ \varphi \in H^2(\Omega_s) \setminus \mathbb{R}, \quad \frac{\partial \varphi}{\partial \nu}|_{\Gamma_s} = 0 \right\}; \]  

(59)  

\[ N_s : H^s(\Gamma_s) \to \begin{cases} H^{s+\frac{1}{2}}(\Omega_s) / \mathbb{R}, & b = 0 \\
H^{s+\frac{1}{2}}(\Omega_s), & b = 1 \end{cases}; \quad N_s \mu = \psi \iff \begin{cases} \Delta \psi = 0 \text{ in } \Omega_s \\
\left. \frac{\partial \psi}{\partial \nu} \right|_{\Gamma_s} = \mu. \end{cases} \]  

(60a)  

Lemma 2.2 is proved.  

We next present a result which will be much extended in Theorem 1.4(ii), Eqn.(36), to be proved in Section 3. The proof here is simpler.  

**Proposition 2.3.** In the complex plane \( \mathbb{C} \), consider the open disk \( \mathbb{S}_{r=1}(x_0) \), centered at the point \( x_0 = (-1,0) \) and of radius \( r = 1 \). Let \( \mathbb{S}_{r=1}(x_0) \) be its closed complement in \( \mathbb{C} \) (it suffices in \( \mathbb{C}^* \) = \{ \lambda : Re \lambda \leq 0 \} by Theorem 1.2). Then, the operator \( A \) in (6), (7), \((b = 0)\) has no eigenvalue \( \lambda = \alpha + i\omega \) with \( \omega \neq 0 \) in \( \mathbb{S}_{r=1}(x_0) \equiv \{ (\alpha, \omega) : \alpha^2 + 2\alpha + \omega^2 \geq 0 \} \). Same conclusion for the operator \( A^* \). Hence [39, p. 282], the points \( \lambda = \alpha + i\omega, \omega \neq 0 \) in \( \mathbb{S}_{r=1}(x_0) \) do not belong to the residual spectrum \( \sigma_r(A) \) of \( A \).  

**Proof.** Step 1. Consider the eigenvalue/eigenvector equation for \( [v_1, v_2, h] \in D(A) \) for \( b = 0 \):  
\[ A \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ \Delta & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = (\alpha + i\omega) \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \]  

(61)  

with \( \alpha < 0, \omega \in \mathbb{R} \); or explicitly,  
\[
\begin{align*}
v_2 &= (\alpha + i\omega)v_1, \\
\Delta(v_1 + v_2) &= (\alpha + i\omega)v_2, \\
\Delta h &= (\alpha + i\omega)h.
\end{align*}
\]

(62a)  

(62b)  

(62c)  

**Step 2.** Multiply Eqn. (62c) by \( \bar{h} \), integrate over \( \Omega_f \), use Green’s First Theorem on \( \int_{\Omega_f} \Delta h \bar{h} \, d(\Omega_f) \) along with the B.C. \( h|_{\Gamma_f} = 0 \) in \( D(A) \) to obtain (all norms are \( L^2 \)-norms on their respective domains)  
\[ \int_{\Gamma_s} \left. \frac{\partial h}{\partial \nu} \right|_{\Gamma_s} \, d\Gamma_s - ||\nabla h||^2 = (\alpha + i\omega)||h||^2. \]  

(63)
Next, multiply Eqn. (62b) by \( \overline{v}_2 \), integrate over \( \Omega_s \), use the Green’s First Theorem on
\[
\int_{\Omega_s} \Delta (v_1 + v_2) \overline{v}_2 \, d\Omega_s \quad \text{with unit normal vector} \ \nu \quad \text{inward to} \quad \Omega_s \quad \text{to obtain by use the BCs (7a-b)}.
\]
\[
- \int_{\Gamma_s} \frac{\partial h}{\partial \nu} \, n \, d\Gamma_s - \| \nabla v_2 \|^2 - (\alpha - i \omega) \| \nabla v_1 \|^2 = (\alpha + i \omega) \| v_2 \|^2. \quad (64)
\]

To obtain (64), we have also invoked \( \nabla v_2 = (\alpha + i \omega) \nabla v_1 \) from (62a) on the term \( (\nabla v_1, \nabla v_2) = (\alpha - i \omega) \| \nabla v_1 \|^2 \). Sum up (63) and (64) to obtain, after a cancellation of the boundary terms
\[
0 = \| \nabla v_2 \|^2 + \| \nabla h \|^2 + \alpha \| \nabla v_1 \|^2 + \| v_2 \|^2 + \| h \|^2 \] 
\[+ i \omega [\| v_2 \|^2 + \| h \|^2 - \| \nabla v_1 \|^2]. \quad (65)
\]

**Step 3.** Taking the Real part and the Imaginary part of identity (65) yields (recall \( \alpha < 0 \))
\[
\| \nabla v_2 \|^2 + \| \nabla h \|^2 + \alpha \| \nabla v_1 \|^2 + \| v_2 \|^2 + \| h \|^2 = 0; \quad (66)
\]
\[
\omega \| v_2 \|^2 + \| h \|^2 - \| \nabla v_1 \|^2 = 0. \quad (67)
\]

For \( \omega \neq 0 \), we obtain from (67) and (62a),
\[
\| \nabla v_1 \|^2 = \| v_2 \|^2 + \| h \|^2; \quad \| \nabla v_2 \|^2 = (\alpha^2 + \omega^2) \| \nabla v_1 \|^2, \quad \omega \neq 0 \]
(68)

which substituted in (66) yields
\[
[\omega^2 + \alpha^2 + 2\alpha] \| \nabla v_1 \|^2 + \| \nabla h \|^2 = 0, \quad \omega \neq 0 \quad (69)
\]

**First case.** Assume first that
\[
\omega^2 + \alpha^2 + 2\alpha > 0, \quad \omega \neq 0; \quad \text{i.e.,} \quad \lambda = \alpha + i \omega, \quad \omega \neq 0 \quad \text{lies outside the closed disk} \ \overline{S_{r=1}(x_0)},
\]
(70)

with center \( x_0 = \{-1, 0\} \) and radius \( r = 1 \). Then identity (69) implies
\[
\begin{align*}
\| \nabla v_1 \| &= 0, \quad \| \nabla h \| = 0, \quad \text{hence} \ h = \text{const} = 0 \quad \text{in} \quad L^2(\Omega_f) \quad (71a) \\
v_1 &= \text{const} = 0 \quad \text{in} \quad H^1(\Omega_s)/\mathbb{R}, \\
v_2 &= (\alpha + i \omega)v_1 = 0 \quad \text{in} \quad L^2(\Omega_s), \quad \alpha + i \omega \neq 0, \quad \alpha < 0
\end{align*}
\]
(71b)

as \( h|_{\Gamma_s} = 0 \) (and \( h|_{\Gamma_s} = v_2|_{\Gamma_s} = (\alpha + i \omega)v_1|_{\Gamma_s} = 0 \)). In conclusion, we obtain \( v_1 = v_2 = h = 0 \) in \( H^0 \) for \( \lambda = \alpha + i \omega, \quad \alpha < 0, \quad \omega \neq 0, \quad \overline{S_{r=1}(x_0)} \)

**Second case.** Assume next that \( \lambda = \alpha + i \omega, \) still \( \omega \neq 0, \) lies on the circumference of the disk \( S_{r=1}(x_0) \); that is, it satisfies
\[
\alpha^2 + 2\alpha + \omega^2 = 0 \quad (72)
\]

Then equation (69), which now reads \( \| \nabla h \| = 0 \), implies again \( h = \text{const} = 0 \) in \( L^2(\Omega_f) \), hence via (62a):
\[
\frac{\partial h}{\partial \nu}|_{\Gamma_s} \equiv 0, \quad 0 \equiv h|_{\Gamma_s} = v_2|_{\Gamma_s}, \quad v_1|_{\Gamma_s} \equiv 0. \quad (73)
\]
Thus, we can solve uniquely for $v$ for any $\{\alpha \in \mathbb{C}\}$, $S\rho$ in (6), (7) for $b = 0$.

Returning to (62b) augmented by the relevant B.C., we obtain via (62a) and (73)

$$\Delta v_1 = \frac{(\alpha + i\omega)^2}{(1 + \alpha) + i\omega} v_1$$

or

$$\Delta v_1 = 2\alpha v_1 \quad \text{in} \quad \Omega_s$$

as $1 + \alpha + i\omega \neq 0$. Indeed, we readily verifies by use of (72) that $\frac{(\alpha + i\omega)^2}{(1 + \alpha) + i\omega} = 2\alpha$. By uniqueness ([24, p 75]) of the overdetermined elliptic problem in the RHS of (74a-b), one obtains $v_1 \equiv 0$ in $L^2(\Omega_s)$, hence $v_2 \equiv 0$ in $L^2(\Omega_s)$. In conclusion: points on the circumference of the disk $S_{r_0}(x_0)$, thus satisfying (72), are not eigenvalues of the operator $A$ in (6), (7) for $b = 0$.

Proposition 2.3 is established.

Proposition 2.4. For the operator $A$ in (6), (7) with $b = 0$ on $H_0$, the point $\{\alpha = -1, \omega = 0\}$ is not an eigenvalue: $\lambda = -1 \notin \sigma_p(A)$. In fact, $-1 \in \sigma_c(A)$, the continuous spectrum of $A$.

Proof. Eqn. (62a) yields $v_2 = -v_1$ or $v_1 + v_2 = 0$. Then (62b) gives $\Delta(v_1 + v_2) = 0 = -v_2$, thus $v_1 = v_2 = 0$ in $L^2(\Omega_s)$. Finally, (62c) and the required B.C. imply

$$\begin{align*}
\Delta h &= -h \quad \text{in} \quad L^2(\Omega_f); \\
h|_{\Gamma_f} &= 0, \quad h|_{\Gamma_s} &= v_2|_{\Gamma_s} = 0,
\end{align*}$$

and the uniqueness ([24, p 75]) of the above over-determined problem yields $h = 0$ in $L^2(\Omega_f)$, as desired. In conclusion, $v_1 = v_2 = h = 0$, so that $(I + A)$ is injective in $H_b$ and $\lambda = -1 \notin \sigma_p(A)$.

We now show that the point $-1 \notin \sigma_c(A)$. Writing $(I + A)\Phi = \Phi^*$ for $\Phi = [v_1, v_2, h^*] \in \text{Range of } (I + A)$ and solving for $\Phi = [v_1, v_2, h] \in D(A)$, we have explicitly:

$$\begin{align*}
v_1 + v_2 &= v_1^*; \\
v_2 &= v_2^* - \Delta v_1^*; \\
\Delta h + h &= h^* \quad \text{in} \quad L^2(\Omega_f).
\end{align*}$$

We now pick $v_1^* \in H^1(\Omega_s), v_2^* \in H^1(\Omega_s), h^* \in L^2(\Omega_f)$. We then solve (76) uniquely and obtain $v_2 \in H^1(\Omega_s)$, hence $v_2|_{\Gamma_s} \in H^1(\Gamma_s), v_1 \in H^1(\Omega_s)$, and $h \in H^1(\Omega_f)$. Thus, we can solve uniquely for $[v_1, v_2, h] \in D(A)$, as the BC are also satisfied, for any $\{v_1^*, v_2^*, h^*\}$ in a dense subspace $H^1(\Omega_s) \times H^1(\Omega_s) \times L^2(\Omega_f)$ of $H_b$ in the Range of $(I + A) = D((I + A)^{-1})$. Thus, $-1 \notin \sigma_c(A)$ and finally $-1 \in \sigma_c(A)$, as the map $(I + A)^{-1}$ (which is well-defined on Range of $(I + A)$) is not continuous on $H_b$.

3. Proof of Theorem 1.4: Analyticity, location of the spectrum $\sigma(A)$ within the set $K$, exponential decay.

3.0. Orientation. (a) We have already established that the operator $A$ in (6), (7), possesses the following two features: (i) it is the generator of a s.c. $(C_0)$ semigroup $e^{At}$ of contractions on the finite energy space $H_0$, henceforth denoted simply by $H_b$ (Theorem 1.2); (ii) $0 \in \rho(A)$, the resolvent set of $A$, and hence there is a small open disk $S_{r_0}$ in the complex plane centered at the origin and of small radius $r_0 > 0$, that is all contained in $\rho(A) : S_{r_0} \subset \rho(A)$ (Lemma 2.2 (ii)). Accordingly, to conclude
that $e^{At}$ is, moreover, analytic on $H$, all we need to show [31, Thm. 3E.3, p. 334] is that ($\mathcal{A}$ has no spectrum on the imaginary axis, and):

$$\|R(i\omega, \mathcal{A})\|_{L(H)} \leq \frac{C}{|\omega|}, \quad \forall |\omega| \geq \text{some } \omega_0 > 0. \tag{77}$$

Then, the proof in [31, p. 335] establishes that, in fact, for a suitable constant $M > 0$, we have

$$\|R(\lambda, \mathcal{A})\|_{L(H)} \leq M|\lambda|, \quad \forall \lambda \in \Sigma_{\theta_1}, \tag{78a}$$

$$\Sigma_{\theta_1} = \left\{ \lambda \in \mathbb{C} : 0 \leq |\arg \lambda| \leq \frac{\pi}{2} + \theta_1 \right\}, \tag{78b}$$

where one may take the angle $\theta_1$, $0 < \theta_1 < \frac{\pi}{2}$, such that $\tan \left(\frac{\pi}{2} - \theta_1\right) = \frac{\xi}{\rho}$, with $C$ the constant in (77), for an arbitrary fixed constant $0 < \rho < 1$. We seek the ‘largest’ possible angle $\theta_1 < \frac{\pi}{2}$, at least after moving the vertex of the triangular sector in a nearby point. In our case, this nearby point will be $x_0 = \{-1, 0\}$; in which case, with vertex on $x_0 = \{-1, 0\}$ the angle $\theta_1$ will be arbitrarily close to $\frac{\pi}{2}$. In this section, we shall establish a resolvent estimate such as (78a) for all $\lambda \in \mathbb{C}\setminus \mathcal{K}$, namely

$$\left\{ \begin{array}{l}
\|\lambda R(\lambda, \mathcal{A})\|_{L(H)} \leq \text{const}, \quad \forall \lambda \in \mathbb{C}\setminus \mathcal{K}; \\
equivalently (since \mathcal{A}R(\lambda, \mathcal{A}) = -I + \lambda R(\lambda, \mathcal{A})) \\
\|AR(\lambda, \mathcal{A})\|_{L(H)} \leq \text{const}, \quad \forall \lambda \in \mathbb{C}\setminus \mathcal{K}; 
\end{array} \right. \tag{79a}$$

$$\mathcal{K}$$ being the infinite key-shaped set defined in (36b); whereby, moving the vertex of the triangular sector of analyticity to coincide with the point $x_0 = \{-1, 0\}$, the corresponding angle $\theta_1$ is arbitrarily close to $\frac{\pi}{2}$ (Fig 3.1).

(b) Then (79) and $S_{\rho_0} \subset \rho(\mathcal{A})$ will imply (Section 3.2) that the real part of the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ is confined inside the negative axis $(-\infty, -r_0]$. The direct passage from (79) to (78) is exhibited in Remark 3.1 below. Moreover, our proof below, once specialized with $\Re \lambda = 0$, $\lambda = i\omega$, $\omega \in \mathbb{R}$, will yield (through simplified computations in Remark 3.2 below) the establishment of inequality (77) for any $\omega_0 > 0$. This result, combined with $S_{\rho_0} \subset \rho(\mathcal{A})$ will allow us to conclude that

$$\|R(i\omega, \mathcal{A})\|_{L(H)} \leq \text{const}, \quad \omega \in \mathbb{R}. \tag{80}$$

Then, (80) will imply [38] uniform stabilization of the analytic semigroup $e^{At}$: there exists constants $M \geq 1$, $\delta > 0$ such that

$$\|e^{At}\|_{L(H)} \leq Me^{-\delta t}, \quad t \geq 0, \tag{81a}$$
and hence that
\[ \sigma(A) \in (-\infty; -\delta]. \tag{81b} \]

In conclusion, the present section establishes three results: (a) analyticity of the semigroup \( e^{At} \); (b) location of the spectrum \( \sigma(A) \) of \( A \), in Subsection 3.1; and (c) exponential stability (81a) in Subsection 3.2.

3.1. Analyticity of \( e^{At} \), with \( \sigma(A) \subset \mathcal{K} \). Step 1. Given \( \{v_1^*, v_2^*, h^*\} \in H \), constants \( \alpha < 0 \) and \( \omega \in \mathbb{R} \setminus \{0\} \), we seek to solve the equation

\[
((\alpha + i\omega)I - A) \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} (\alpha + i\omega)I - \begin{bmatrix} 0 & I & 0 \\ \Delta & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}
\]

in terms of \( \{v_1, v_2, h\} \in D(A) \) uniquely, and establish, in fact, the analyticity estimate (79a). For \( \lambda = \alpha + i\omega \in \rho(A) \), we have

\[
\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = R(\lambda, A) \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}; \quad AR(\lambda, A) \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ \Delta & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}
\]

\[ = \begin{bmatrix} v_2 \\ \Delta(v_1 + v_2) \\ \Delta h \end{bmatrix}. \tag{83} \]

We see that the analyticity condition (79) is equivalent to showing the following estimates (all norms are \( L_2 \)-norms on the respective domains): there exists a constant \( C > 0 \) such that

\[
\left\{ \begin{array}{l}
\|\nabla v_2\|_{L_2}^2 + \|\Delta(v_1 + v_2)\|_{L_2}^2 + \|\Delta h\|_{L_2}^2 \\
\|\Delta v_1\|_{L_2}^2 + \|v_2\|_{L_2}^2 + \|h^*\|_{L_2}^2
\end{array} \right\} \leq C \left\{ \begin{array}{l}
\|\nabla v_1^*\|_{L_2^\alpha}^2 + \|v_2^*\|_{L_2^\alpha}^2 + \|h^*\|_{L_2^\alpha}^2
\end{array} \right\}
\]

for all \( \lambda = \alpha + i\omega \in \rho(A) \setminus \mathcal{K} \); that is, outside the set \( \mathcal{K} \) defined in (36b) (Fig 2)

\[ \text{(84a)} \]

This is what we shall show below. Explicitly (82) is rewritten as

\[
\left\{ \begin{array}{l}
(\alpha + i\omega)v_1 - v_2 = v_1^* \in H^1(\Omega_s)/R; \\
(\alpha + i\omega)v_2 - \Delta(v_1 + v_2) = v_2^* \in L^2(\Omega_s); \\
(\alpha + i\omega)h - \Delta h = h^* \in L^2(\Omega_f).
\end{array} \right. \tag{85a,b,c} \]

Step 2. Henceforth, to streamline the notation, \( \| \cdot \| \), respectively \( (\cdot, \cdot) \) will denote the \( L^2(\cdot) \)-norm, respectively the complex inner product on either the set \( \Omega_s \) or the set \( \Omega_f \). No ambiguity is likely to occur. We take the \( L^2(\Omega_f) \)-inner product of Eqn. (85c) against \( \Delta h \), use Green’s First Theorem to evaluate \( \int_{\Omega_f} h \Delta \overline{h} d\Omega_f \), recall the B.C. \( h|_{\Gamma_f} = 0 \) in \( D(A) \) and obtain

\[
(\alpha + i\omega) \int_{\Gamma_s} h \frac{\partial \overline{h}}{\partial v} d\Gamma_s - (\alpha + i\omega)\|\nabla h\|^2 - \|\Delta h\|^2 = (h^*, \Delta h). \tag{86} \]

Similarly, we take the \( L^2(\Omega_s) \)-inner product of (85b) against \( \Delta(v_1 + v_2) \), use Green’s First Theorem to evaluate \( \int_{\Omega_s} v_2 \Delta(v_1 + v_2) d\Omega_s \), recalling that the normal
vector $\nu$ is inward w.r.t. $\Omega_s$, and obtain

$$-(\alpha + i\omega) \int_{\Gamma_s} v_2 \frac{\partial (v_1 + v_2)}{\partial \nu} \, d\Gamma_s - (\alpha + i\omega)(\nabla v_2, \nabla (v_1 + v_2)) - \|\Delta (v_1 + v_2)\|^2 = (v_2^*, \Delta (v_1 + v_2)).$$

(87)

Invoking now the B.C. $h|_{\Gamma_s} = v_2|_{\Gamma_s}$ and $\frac{\partial (v_1 + v_2)}{\partial \nu}|_{\Gamma_s} = \frac{\partial h}{\partial \nu}|_{\Gamma_s}$ in $D(A)$ (see (7b)), we rewrite (87) as

$$-(\alpha + i\omega) \int_{\Gamma_s} h \frac{\partial h}{\partial \nu} \, d\Gamma_s - (\alpha + i\omega)||\nabla v_2||^2 - (\alpha + i\omega)(\nabla v_2, \nabla v_1) - ||\Delta (v_1 + v_2)||^2 = (v_2^*, \Delta (v_1 + v_2)).$$

(88)

Summing up (86) and (88) yields after a cancellation of the boundary terms

$$-(\alpha + i\omega)[||\nabla v_2||^2 + \|\nabla h||^2] = ||\Delta (v_1 + v_2)||^2 + ||\Delta h||^2 + (\alpha + i\omega)(\nabla v_2, \nabla v_1) + (v_2^*, \Delta (v_1 + v_2)) + (h^*, \Delta h).$$

(89)

We now return to (85a), multiply by $(\alpha - i\omega) \neq 0$, and rewrite the result as $v_1 = [(\alpha - i\omega)/(\alpha^2 + \omega^2)](v_2 + v_2^*)$, which introduced in the third term on the RHS of (89) yields

$$(\alpha + i\omega)(\nabla v_2, \nabla v_1) = \frac{(\alpha + i\omega)^2}{\alpha^2 + \omega^2} [||\nabla v_2||^2 + (\nabla v_2, \nabla v_2^*)]$$

(90a)

$$= \frac{(\alpha^2 - \omega^2) + 2\alpha \omega}{\alpha^2 + \omega^2} ||\nabla v_2||^2 + \frac{(\alpha + i\omega)^2}{\alpha^2 + \omega^2} (\nabla v_2, \nabla v_2^*)$$

(90b)

Substituting (90b) into (89), we obtain the final identity

$$||\Delta (v_1 + v_2)||^2 + ||\Delta h||^2 + i\left\{ \frac{2\alpha}{\alpha^2 + \omega^2} \right\} \left\{ ||\nabla v_2||^2 + ||\nabla h||^2 \right\} = -\alpha + \frac{\omega^2 - \alpha^2}{\alpha^2 + \omega^2} ||\nabla v_2||^2 - \alpha ||\nabla h||^2 - (v_2^*, \Delta (v_1 + v_2)) - (h^*, \Delta h)$$

(91)

Step 3. We take the real part of identity (91), thus obtaining the new identity:

$$||\Delta (v_1 + v_2)||^2 + ||\Delta h||^2 = -\alpha + \frac{\omega^2 - \alpha^2}{\alpha^2 + \omega^2} ||\nabla v_2||^2 - \alpha ||\nabla h||^2 - \text{Re}\left\{ \frac{(\alpha + i\omega)^2}{\alpha^2 + \omega^2} (\nabla v_2, \nabla v_2^*) \right\} - \text{Re}(v_2^*, \Delta (v_1 + v_2)) - \text{Re}(h^*, \Delta h)$$

(92)

We estimate the RHS of (92), noticing that $|\frac{\omega^2 - \alpha^2}{\alpha^2 + \omega^2}| \leq 1$, $|\frac{(\alpha + i\omega)^2}{\alpha^2 + \omega^2}| \equiv 1$, thus obtaining the inequality

$$||\Delta (v_1 + v_2)||^2 + ||\Delta h||^2 \leq (|\alpha| + 1 + \epsilon_1)||\nabla v_2||^2 + |\alpha||\nabla h||^2 + \epsilon \left\{ ||\Delta (v_1 + v_2)||^2 + ||\Delta h||^2 \right\} + C_{\epsilon_1} \left\{ ||\nabla v_2^*||^2 + ||v_2^*||^2 + ||h^*||^2 \right\}$$

(93)
or taking $|\alpha| \geq r_0 > 0$, with $r_0$ fixed but arbitrarily small, $(1 + \epsilon_1) < \frac{1}{r_0}|\alpha|$, and setting $k_0 = 1 + \frac{1}{r_0}$, we obtain

$$
(1 - \epsilon) \left[ \left| \Delta(v_1 + v_2) \right|^2 + \left| \Delta h \right|^2 \right] \leq \left( |\alpha| + 1 + \epsilon_1 \right) \left| \nabla v_2 \right|^2 + |\alpha| \left| \nabla h \right|^2 \\
+ C_{\epsilon_1} \left\{ \left| \nabla v_1^* \right|^2 + \left| v_2^* \right|^2 + \left| h^* \right|^2 \right\} \\
\leq k_0 |\alpha| \left| \nabla v_2 \right|^2 + |\alpha| \left| \nabla h \right|^2 \\
+ C_{\epsilon_1} \left\{ \left| \nabla v_1^* \right|^2 + \left| v_2^* \right|^2 + \left| h^* \right|^2 \right\} 
$$

(94)

**Step 4.** We now take the imaginary part of identity (91), thus obtaining the new identity

$$
\omega \left\{ 1 + \frac{2\alpha}{\alpha^2 + \omega^2} \right\} \left| \nabla v_2 \right|^2 + |\nabla h|^2 \\
= - \Im \left\{ \frac{(\alpha + i\omega)^2}{\alpha^2 + \omega^2} (\nabla v_2, \nabla v_1^*) \right\} - \Im \left\{ \langle v_2^*, \Delta(v_1 + v_2) \rangle + \langle h^*, \Delta h \rangle \right\} 
$$

(95)

With $\epsilon > 0$ arbitrary assume that

$$
0 < \epsilon < \left[ 1 + \frac{2\alpha}{\alpha^2 + \omega^2} \right] 
$$

(96)

that is, that the point $\{\alpha, \omega\}$ lies outside the disk $(\alpha + 1)^2 + \omega^2 = 1$

Then taking the absolute value of both sides of (95), using (96) as well as

$$
\left| \frac{(\alpha + i\omega)^2}{\alpha^2 + \omega^2} \right| = 1, \text{ we obtain}
$$

$$
\epsilon |\omega| \left| \nabla v_2 \right|^2 + |\omega| \left| \nabla h \right|^2 \\
\leq \frac{\epsilon^2}{2} \left| \nabla v_2 \right|^2 + C_{\epsilon_1} \left| \nabla v_1^* \right|^2 + \epsilon^3 \left[ \left| \Delta(v_1 + v_2) \right|^2 + \left| \Delta h \right|^2 \right] + C_{\epsilon_2} \left[ \left| v_2^* \right|^2 + \left| h^* \right|^2 \right] 
$$

(97)

We then obtain from (97)

$$
0 < \frac{\epsilon |\omega|}{2} \left| \nabla v_2 \right|^2 + |\omega| \left| \nabla h \right|^2 \leq \left[ \epsilon |\omega| - \frac{\epsilon^2}{2} \right] \left| \nabla v_2 \right|^2 + |\omega| \left| \nabla h \right|^2 \\
\leq \epsilon^3 \left[ \left| \Delta(v_1 + v_2) \right|^2 + \left| \Delta h \right|^2 \right] + C_{\epsilon} \left[ \left| \nabla v_1^* \right|^2 + \left| v_2^* \right|^2 + \left| h^* \right|^2 \right] 
$$

(98)

where the LHS of (98) is valid for all $|\omega|$ s.t.

$$
\frac{\epsilon |\omega|}{2} + \frac{\epsilon^2}{2} \leq \epsilon |\omega|; \quad \text{or} \quad 0 < \epsilon < |\omega|. 
$$

(99)

That is, (98) implies

$$
\left\{ \begin{array}{l}
|\omega| \left| \nabla h \right|^2 \leq \epsilon^3 \left[ \left| \Delta(v_1 + v_2) \right|^2 + \left| \Delta h \right|^2 \right] + C_{\epsilon} \left[ \left| \nabla v_1^* \right|^2 + \left| v_2^* \right|^2 + \left| h^* \right|^2 \right] \\
|\omega| \left| \nabla v_2 \right|^2 \leq 2\epsilon^2 \left[ \left| \Delta(v_1 + v_2) \right|^2 + \left| \Delta h \right|^2 \right] + C_{\epsilon} \left[ \left| \nabla v_1^* \right|^2 + \left| v_2^* \right|^2 + \left| h^* \right|^2 \right]
\end{array} \right. 
$$

(100)
Step 5. We now impose $|\alpha| / |\omega| \leq \frac{1}{\epsilon}$, or $\epsilon |\alpha| \leq |\omega|$ and rewrite (94), also by means of (100), as

\[
(1 - \epsilon) \left[ \|\Delta(v_1 + v_2)\|^2 + \|\Delta h\|^2 \right] 
\leq \frac{k_0}{\epsilon} |\omega| \|\nabla v_2\|^2 + \frac{|\omega|}{\epsilon} \|\nabla h\|^2 
\leq (2k_0\epsilon + \epsilon^2) \left[ \|\Delta(v_1 + v_2)\|^2 + \|\Delta h\|^2 \right] + \tilde{C}_\epsilon \left\{ \|\nabla v_1^\ast\|^2 + \|v_2^\ast\|^2 + \|h^\ast\|^2 \right\},
\]

finally

\[
(1 - \epsilon - 2k_0\epsilon - \epsilon^2) \left[ \|\Delta(v_1 + v_2)\|^2 + \|\Delta h\|^2 \right] \leq \tilde{C}_\epsilon \left\{ \|\nabla v_1^\ast\|^2 + \|v_2^\ast\|^2 + \|h^\ast\|^2 \right\}.
\]

Taking $\epsilon > 0$ sufficiently small, we obtain from (101)

\[
\left\{ \begin{array}{l}
\|\Delta(v_1 + v_2)\|^2 + \|\Delta h\|^2 \leq C_\epsilon \left\{ \|\nabla v_1^\ast\|^2 + \|v_2^\ast\|^2 + \|h^\ast\|^2 \right\};
\end{array} \right.
\]

valid for all points $\{\alpha, \omega\}$ satisfying (96) and $0 < r_0 \leq |\alpha|, \epsilon r_0 \leq \epsilon |\alpha| \leq |\omega|$. so that (99) is a-fortiori satisfied. Substituting (102) in (98), where in its LHS we use $r_0 \epsilon^2 / 2 < r_0 \epsilon < |\omega|$, produces the desired estimate

\[
\|\nabla v_2\|^2 + \|\nabla h\|^2 \leq C_\epsilon \left\{ \|\nabla v_1^\ast\|^2 + \|v_2^\ast\|^2 + \|h^\ast\|^2 \right\}.
\]

Hence, summing up (102) and (103a) yields the desired final estimate

\[
\left\| \|\Delta(v_1 + v_2)\|^2 + \|\Delta h\|^2 + \|\nabla v_2\|^2 + \|\nabla h\|^2 \right\| \leq \text{const} \{ \|\nabla v_1^\ast\|^2 + \|v_2^\ast\|^2 + \|h^\ast\|^2 \}
\]

for all points $\{\alpha, \omega\}$ satisfying (96) and (102). Since $\epsilon > 0$ and $r_0$ in these two relations are arbitrary, we conclude that (estimate (102) and hence the conclusive) estimate (103b) hold(s) true for all points $\{\alpha, \omega\}, \alpha < 0, \text{outside the disk: } (\alpha + 1)^2 + \omega^2 = 1, \text{with } \omega \neq 0$. This is precisely the conclusion (79b); that is, conclusion (36). We have thus proved parts (ii) (hence part (i)) (analyticity of $e^{At}$ on $H$) and part (iii) (location of the spectrum $\sigma(A)$) of Theorem 1.4.

\[\square\]

Remark 3.1. (direct passage from (79) to (78)) Returning to Eqns. (85a–c), we obtain with $\lambda = \alpha + i\omega$:

\[
\begin{align*}
|\lambda| \|\nabla v_1\| - \|\nabla v_2^\ast\| &\leq \|\lambda \nabla v_1 - \nabla v_1^\ast\| = \|\nabla v_2\|; \\
|\lambda| \|v_2\| - \|v_2^\ast\| &\leq \|\lambda v_2 - v_2^\ast\| = \|\Delta(v_1 + v_2)\|; \\
|\lambda| \|h\| - \|h^\ast\| &\leq \|\lambda h - h^\ast\| = \|\Delta h\|.
\end{align*}
\]

Hence, summing up,

\[
|\lambda| \left\{ \|\nabla v_1\| + \|v_2\| + \|h\| \right\} \leq \|\nabla v_2\| + \|\Delta(v_1 + v_2)\| + \|\Delta h\| + \|\nabla v_2^\ast\| + \|v_2^\ast\| + \|h^\ast\|
\]

(by (103)) \leq C \left\{ \|\nabla v_1^\ast\| + \|v_2^\ast\| + \|h^\ast\| \right\},
\]

for $\lambda$ satisfying (96) and (102). In short, in view of (82) and (2a) for $H_{\alpha=0} = H$, estimate (107) says that

\[
\begin{vmatrix}
v_1 \\
v_2 \\
h
\end{vmatrix}_H = R(\lambda, A) \begin{vmatrix}
v_1^\ast \\
v_2^\ast \\
h^\ast
\end{vmatrix}_H \leq C_{k_0, \omega, \alpha} \begin{vmatrix}
v_1^\ast \\
v_2^\ast \\
h^\ast
\end{vmatrix}_H,
\]

for all such $\lambda = \alpha + i\omega$. Theorem 1.4(ii), Eqn. (36), is proved.
Remark 3.2 (Specialization to the case $\alpha = 0$). We specialize the above computations to the case $\alpha = 0$, $\lambda = i\omega$, to obtain:

(a) The counterpart of identity (92) (real part) is

\[
\|\Delta(v_1 + v_2)\|^2 + \|\Delta h\|^2 = \|\nabla v_2\|^2 + \text{Re}(\nabla v_2, \nabla v_1^*).
\]

which then yields

\[
(1 - \epsilon) [\|\nabla (v_1 + v_2)\|^2 + \|\Delta h\|^2] \leq \epsilon [\|\Delta (v_1 + v_2)\|^2 + \|\Delta h\|^2]
\]

Thus, (111) implies the estimate

\[
\|\Delta (v_1 + v_2)\|^2 + \|\Delta h\|^2 \leq \left( \frac{\epsilon}{|\omega| - \epsilon} \right) \left( \|\Delta (v_1 + v_2)\|^2 + \|\Delta h\|^2 \right)
\]

(b) The counterpart of identity (95) (imaginary part) is

\[
\omega \left[ \|\nabla v_2\|^2 + \|\nabla h\|^2 \right] = \text{Im}(\nabla v_2, \nabla v_1^*) - \text{Im}(v_2^*, \Delta (v_1 + v_2)) - \text{Im}(h^*, \Delta h).
\]

Thus, (111) implies the estimate

\[
\|\nabla v_2\|^2 + \|\nabla h\|^2 \leq \left( \frac{\epsilon}{|\omega| - \epsilon} \right) \left[ \|\Delta (v_1 + v_2)\|^2 + \|\Delta h\|^2 \right]
\]

(c) Use of inequality (112) into the RHS of inequality (110) for the $\nabla v_2$-term yields

\[
\left[ 1 - \epsilon - \frac{\epsilon(1 + \epsilon)}{|\omega| - \epsilon} \right] [\|\Delta (v_1 + v_2)\|^2 + \|\Delta h\|^2]
\]

or taking $|\omega| - \epsilon \geq \omega_0 > 0$, hence $\frac{1}{2} < \left[ 1 - \epsilon - \frac{\epsilon(1 + \epsilon)}{\omega_0} \right] < \left[ 1 - \epsilon - \frac{\epsilon(1 + \epsilon)}{|\omega| - \epsilon} \right] :$

\[
\|\Delta (v_1 + v_2)\|^2 + \|\Delta h\|^2 \leq \text{const}_{\omega_0} \left\{ \|\nabla v_1^*\|^2 + \|\nabla v_2^*\|^2 \right\} + \|\Delta h\|^2, \quad \omega_0 > 0
\]

which is the counterpart of (102).

(d) Finally, returning to (112) and using here (109), we obtain

\[
\|\nabla v_2\|^2 + \|\nabla h\|^2 \leq \text{const}_{\omega_0} \left\{ \|\nabla v_1^*\|^2 + \|\nabla v_2^*\|^2 \right\} + \|\Delta h\|^2
\]

which is the counterpart of estimate (103a).

(e) Summing up (114) and (115), we obtain the counterpart of (103b) = (79b) for $\alpha = 0$, i.e.

\[
\left\{ \|\mathcal{A} (i\omega, \mathcal{A})\|_{L(H_0)} \leq C, \quad \forall \omega \geq \omega_0 > 0 \text{ arbitrary,} \right. \quad \text{equivalently}
\]

\[
\left\{ \|\mathcal{A} (i\omega, \mathcal{A})\|_{L(H_0)} \leq \frac{C}{|\omega|}, \quad \forall \omega \geq \omega_0 > 0 \text{ arbitrary}, \right. \quad \text{equivalently}
\]

where (117) can also be obtained through the argument in Remark 3.1 (with $\alpha = 0$). Theorem 1.4(i) is established.

Remark 3.3. Suppose, in the present computations, that $[v_1^*, v_2^*, h^*] = 0$, so that the present problem (82) (explicitly, (85)) becomes the eigenvalue/vector problem (61) (explicitly, (62)) of Section 2. Then, the real part identity (92) specializes to (recall $\alpha < 0$):

\[
\|\Delta (v_1 + v_2)\|^2 + \|\Delta h\|^2 = \left( -\alpha \right) \left[ \|\nabla v_2\|^2 + \|\nabla h\|^2 \right] + \frac{\omega^2 - \alpha^2}{\alpha^2 + \omega^2} \|\nabla v_2\|^2,
\]
while the imaginary part identity (95) specializes to
\[ \omega \left\{ \left[ 1 + \frac{2\alpha}{\alpha^2 + \omega^2} \right] \| \nabla v_2 \|^2 + \| \nabla h \|^2 \right\} = 0. \] (118)
Thus, for
\[ \omega \neq 0 \quad \text{and} \quad \omega^2 + \alpha^2 + 2\alpha > 0, \] (119)
Eqn. (118) implies \( \nabla v_2 \equiv 0, v_2 \equiv \text{const}; \nabla h \equiv 0, h \equiv \text{const} = 0 \) by use of \( h \big|_{\Gamma_f} = 0 \). Then \( v_2 \big|_{\Gamma_s} \equiv h \big|_{\Gamma_s} \equiv 0 \). Accordingly then, \( v_2 \equiv \text{const} \equiv 0 \) in \( \Omega_s \) and (62a) then implies \( v_1 \equiv 0 \) in \( \Omega_s \). Next, if \( \omega \neq 0 \) and \( \omega^2 + \alpha^2 + 2\alpha = 0 \), then again \( h = 0 \) and the argument at the end of proof of the Proposition 2.2 shows again \( v \neq \omega \) that the resolvent is uniformly bounded on the imaginary axis \( i\mathbb{L} \).

**3.2. Exponential stability.** The resolvent bound (116) combined with \( A^{-1} \in \mathcal{L}(\mathbb{H}) \) by (54), hence \( S_{\Gamma_0} \subset \rho(A) \) by Theorem 1.2(ii), Fig 3.1 allows one to conclude that the resolvent is uniformly bounded on the imaginary axis \( i\mathbb{R} \):
\[ \| R(i\omega, A) \|_{\mathcal{L}(\mathbb{H})} \leq \text{const}, \] (120)
as claimed in (38). Hence, [38] the s.c. analytic semigroup \( e^{At} \) is, moreover, (uniformly) exponentially bounded: There exist constants \( M \geq 1, \delta > 0 \), such that
\[ \| e^{At} \|_{\mathcal{L}(\mathbb{H})} \leq Me^{-\delta t}, \quad t \geq 0, \quad \text{any} \quad \delta > r_0 \] (121)
by (37). This proves Theorem 1.4(iii), Eqn. (39).

Theorem 1.4 is fully proved.

**4. Proof of Theorem 1.5. Step 1.** We recall from Section 2 the following operators:
(i) the Dirichlet map \( D_f \) on \( \Omega_f \), defined by (57a-b);
(ii) the Neumann map \( N_s \) on \( \Omega_s \), defined by (60a-b);
(iii) the positive self-adjoint operator \( A_{D,f} : \mathbb{L}^2(\Omega_f) \supset \mathcal{D}(A_{D,f}) \to \mathbb{L}^2(\Omega_f) \) defined by (56), where we now additionally note that [20], [21], [29]
\[ \mathcal{D}(A_{D,f}^{1/2}) = \mathbb{H}^1_0(\Omega_f) = \{ \phi \in \mathbb{H}^1(\Omega_f) : \phi|_{\partial\Omega_f} = 0 \}; \] (122)
(iv) the positive self-adjoint operator \( A_{N,s} : \mathbb{L}^2(\Omega_s) \supset \mathcal{D}(A_{N,s}) \to \mathbb{L}^2(\Omega_s) \) defined by (59), where we now additionally note that [20], [21], [29]
\[ \mathcal{D}(A_{N,s}^{1/2}) = \mathbb{H}^1(\Omega_s) \setminus \mathbb{R}. \] (123)
Below we shall deal with the triple: \( v_1 \in \mathbb{H}^1(\Omega_s) \setminus \mathbb{R}, v_2 \in \mathbb{H}^1(\Omega_s) \setminus \mathbb{R}, h \in \mathbb{H}^1(\Omega_f) \).
We further note that
(v)
\[ h - D_f(v_2|_{\Gamma_s}) \in \mathcal{D}(A_{D,f}) \quad \text{means via} \quad (56), (57a-b) \] (124)
\[ \begin{align*}
\Delta(h - D_f(v_2|_{\Gamma_s})) &= \Delta h \in \mathbb{L}^2(\Omega_f) \\
\left[ h - D_f(v_2|_{\Gamma_s}) \right]_{\Gamma_f} &= h|_{\Gamma_f} = 0 \\
\left[ h - D_f(v_2|_{\Gamma_s}) \right]_{\Gamma_s} &= h|_{\Gamma_s} - v_2|_{\Gamma_s} = 0, \quad \text{or} \quad h|_{\Gamma_s} = v_2|_{\Gamma_s} \in \mathbb{H}^{1/2}(\Gamma_s)
\end{align*} \] (125a-b-c)
so that by (7b)
\[
\frac{\partial h}{\partial \nu} \bigg|_{\Gamma_s} \in H^{-1/2}(\Gamma_s);
\] (126)

(vi)
\[
\left[ (v_1 + v_2) - N_s \left( \frac{\partial h}{\partial \nu} \bigg|_{\Gamma_s} \right) \right] \in \mathcal{D}(A_{N,s}) \quad \text{means via (59), (60a – b)}
\] (127)

\[
\begin{cases}
\Delta \left[ (v_1 + v_2) - N_s \left( \frac{\partial h}{\partial \nu} \bigg|_{\Gamma_s} \right) \right] = \Delta (v_1 + v_2) \in L^2(\Omega_s) \\
\frac{\partial}{\partial \nu} \left[ (v_1 + v_2) - N_s \left( \frac{\partial h}{\partial \nu} \bigg|_{\Gamma_s} \right) \right] = \frac{\partial (v_1 + v_2)}{\partial \nu} \bigg|_{\Gamma_s} - \frac{\partial h}{\partial \nu} \bigg|_{\Gamma_s} = 0
\end{cases}
\] (128a-b)

(128c)

**Step 2.** With \( \{v_1, v_2, h\} \in H^1(\Omega_s) \setminus \mathbb{R} \times H^1(\Omega_s) \setminus \mathbb{R} \times H^1(\Omega_f) \), we introduce the following positive self-adjoint operator \( \mathbb{H} \supset \mathcal{D}(A) \rightarrow \mathbb{H} \), acting on the variables
\[
\mathbb{H} = \left\{ v_1, v_2, h, \left[ (v_1 + v_2) - N_s \left( \frac{\partial h}{\partial \nu} \bigg|_{\Gamma_s} \right) \right], \left[ h - D_f(v_2 |_{\Gamma_s}) \right] \right\}:
\]

\[
\begin{bmatrix}
I_s & I_s & A_{N,s} \\
I_s & I_f & A_{D,f}
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
h \\
(v_1 + v_2) - N_s \left( \frac{\partial h}{\partial \nu} \bigg|_{\Gamma_s} \right) \\
(h - D_f(v_2 |_{\Gamma_s})
\end{bmatrix}
\]

(129)

where \( I_s \) is the identity on \( H^1(\Omega_s) \setminus \mathbb{R} \) and \( I_f \) is the identity on \( H^1(\Omega_f) \).

\( \mathbb{H} \equiv H^1(\Omega_s) \setminus \mathbb{R} \times L^2(\Omega_s) \times L^2(\Omega_f) \times L^2(\Omega_s) \times L^2(\Omega_f) \equiv H \times L^2(\Omega_s) \times L^2(\Omega_f) \) (130)
\[ D(\mathcal{A}) = \begin{cases} v_1 \in H^1(\Omega_s) \setminus \mathbb{R}, & v_2 \in H^1(\Omega_s) \setminus \mathbb{R}, & h \in H^1(\Omega_f) : \\ (v_1 + v_2) - N_s \left( \frac{\partial h}{\partial \nu} \right) |_{\Gamma_s} \in D(A_{N,s}), & h - D_f(v_2 |_{\Gamma_s}) \in D(A_{D,f}) \end{cases} \] (131)

\[ = \begin{cases} v_1 \in H^1(\Omega_s) \setminus \mathbb{R}, & v_2 \in H^1(\Omega_s) \setminus \mathbb{R}, & h \in H^1(\Omega_f) : \\ (i) \Delta(v_1 + v_2) \in L^2(\Omega_s), & \frac{\partial(v_1 + v_2)}{\partial \nu} |_{\Gamma_s} = \frac{\partial h}{\partial \nu} |_{\Gamma_s} \in H^{-1/2}(\Gamma_s); \\ (ii) \Delta h \in L^2(\Omega_f), & h |_{\Gamma_f} = 0, & h |_{\Gamma_s} = v_2 |_{\Gamma_s} \in H^{1/2}(\Gamma_s) \end{cases} \] (132)

where in going from (131) to (132) we have invoked (124),(125),(127),(128). Thus, recalling the operator \( \mathcal{A} \) in (6), (7a-b), we conclude from (132) that
\[ D(\mathcal{A}) = D(\mathcal{A}). \] (133)

**Step 3.** We consider just the case \( \theta = \frac{1}{2} \). For the diagonal positive self-adjoint operator \( \mathcal{A} \) in (4), its (positive) square root is

\[ A^{1/2} = \begin{bmatrix} v_1 & v_2 & h \\ (v_1 + v_2) - N_s \left( \frac{\partial h}{\partial \nu} \right) |_{\Gamma_s} & h - D_f(v_2 |_{\Gamma_s}) \end{bmatrix} \]

\[ = \begin{bmatrix} I_s & I_f \\ A_{N,s}^{1/2} & A_{D,f}^{1/2} \end{bmatrix} \begin{bmatrix} v_1 & v_2 & h \\ (v_1 + v_2) - N_s \left( \frac{\partial h}{\partial \nu} \right) |_{\Gamma_s} & h - D_f(v_2 |_{\Gamma_s}) \end{bmatrix} \] (134)

\[ D(A^{1/2}) = \begin{cases} v_1 \in H^1(\Omega_s) \setminus \mathbb{R}, & v_2 \in H^1(\Omega_s) \setminus \mathbb{R}, & h \in H^1(\Omega_f) : \\ (i) (v_1 + v_2) - N_s \left( \frac{\partial h}{\partial \nu} \right) |_{\Gamma_s} \in D(A_{N,s}^{1/2}) = H^1(\Omega_s) \setminus \mathbb{R} \\ (ii) h - D_f(v_2 |_{\Gamma_s}) \in D(A_{D,f}^{1/2}) = H^1_0(\Omega_f) \end{cases}, \] (135)

invoking (123), (122). We recall again (126), (60) with \( s = -\frac{1}{2} \) whereby \( N_s \left( \frac{\partial h}{\partial \nu} \right) |_{\Gamma_s} \in H^1(\Omega_s) \setminus \mathbb{R} \) and conclude that (i) imposes no new conditions on \( v_1, v_2 \). Likewise, we recall (57) with \( s = \frac{1}{2} \) and \( v_2 |_{\Gamma_s} \in H^{1/2}(\Gamma_s) \) whereby \( D_f(v_2 |_{\Gamma_s}) \in H^1(\Gamma_f) \), and conclude that (ii) only imposes by definition of \( D_f \) that \( h |_{\Gamma_f} \equiv 0 \) and \( h |_{\Gamma_s} = v_2 |_{\Gamma_s} \).
Thus, (135) is equivalently re-written as
\[ D(\mathcal{A}^{1/2}) = \{ v_1 \in H^1(\Omega_s) \setminus \mathbb{R}, \quad v_2 \in H^1(\Omega_s) \setminus \mathbb{R}, \quad h \in H^1(\Omega_f) : \]
\[ h|_{\Gamma_f} = 0, \quad h|_{\Gamma_s} = v_2|_{\Gamma_s} \}. \]  

(136)

**Step 4.** Since \( \mathcal{A} \) is maximal dissipative on \( H \) and \( \mathcal{A}^{-1} \in \mathcal{L}(H) \) (Lemma 2.2(ii)), we have [11, Prop. 6.1, p 171], [31, p 5]
\[ D((-\mathcal{A})^{1/2}) \equiv [D(\mathcal{A}) : H]_{1/2}. \]  

(137)

**Step 5.** Finally, from (133) and (137) we have with \( \mathcal{A} \) positive self-adjoint on \( H \) in terms of the triple \( \{ v_1, v_2, h \} \):
\[ D(\mathcal{A}^{1/2}) = [D(\mathcal{A}) : H]_{1/2} = [D(\mathcal{A}) : H]_{1/2} = D((-\mathcal{A})^{1/2}) \]
and so \( D((-\mathcal{A})^{1/2}) \) is given by (136). This proves (40) of Theorem 1.5 for \( D((-\mathcal{A})^{1/2}) \).

The proof for \( D((-\mathcal{A}^*)^{1/2}) \) is similar using now \( D(\mathcal{A}^*) \) in (12a-b).

**Appendix A: Proof of Theorem 1.1 \((b = 0)\) on \( H_0 \).** Let \( \{ w_1, w_2, u \} \in D(\mathcal{A}) \), hence subject to the conditions in (7a–b), and let \( \{ v_1, v_2, h \} \in \text{H}_0 = \text{H}_{b=0} \), subject to the conditions in (11a–b). In particular, the following B.C. are satisfied:
\[ \frac{\partial(w_1 + w_2)}{\partial \nu} \bigg|_{\Gamma_s} = \frac{\partial u}{\partial \nu} \bigg|_{\Gamma_s} ; \quad u|_{\Gamma_f} = 0; \quad u|_{\Gamma_s} = w_2|_{\Gamma_s}; \]  

(A.1)
\[ \frac{\partial(v_2 - v_1)}{\partial \nu} \bigg|_{\Gamma_s} = \frac{\partial h}{\partial \nu} \bigg|_{\Gamma_s} ; \quad h|_{\Gamma_f} = 0; \quad h|_{\Gamma_s} = v_2|_{\Gamma_s}. \]  

(A.2)

We then compute, recalling (3a) (throughout, (\cdot, \cdot) denotes an \( L^2(\cdot, \cdot) \)-inner product
\[ (A \begin{bmatrix} w_1 \\ v_1 \\ w_2 \\ v_2 \\ u \\ h \end{bmatrix}, \begin{bmatrix} w_1 \\ v_1 \\ w_2 \\ v_2 \\ u \\ h \end{bmatrix})_{H_0} = \begin{bmatrix} 0 & I & 0 \\ \Delta & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix})_{H_0} \]
\[ = (\nabla w_2, \nabla v_1) + (\Delta(w_1 + w_2), v_2) + (\Delta u, h). \]  

(A.3)

By Green’s Second Theorem on \( \Omega_f \), recalling \( u|_{\Gamma_f} = 0, \ h|_{\Gamma_f} = 0 \), we obtain
\[ (\Delta u, h) = (u, \Delta h) + \int_{\Gamma_f} \frac{\partial u}{\partial \nu} \overrightarrow{h} \, d\Gamma_s - \int_{\Gamma_s} u \frac{\partial \overrightarrow{h}}{\partial \nu} \, d\Gamma_s. \]  

(A.5)

Similarly, by Green’s Second Theorem on \( \Omega_s \), recalling that the unit normal vector \( \nu \) is inward w.r.t. \( \Omega_s \), as well as \( \frac{\partial(w_1 + w_2)}{\partial \nu} = \frac{\partial u}{\partial \nu} \) on \( \Gamma_s \), and \( v_2|_{\Gamma_s} = h|_{\Gamma_s}, \ w_2|_{\Gamma_s} = u|_{\Gamma_s} \), we obtain
\[ (\Delta(w_1 + w_2), v_2) = (w_1 + w_2, \Delta v_2) - \int_{\Gamma_s} \frac{\partial u}{\partial \nu} \overrightarrow{h} \, d\Gamma_s + \int_{\Gamma_s} (w_1 + u) \frac{\partial \overrightarrow{h}}{\partial \nu} \, d\Gamma_s. \]  

(A.6)
Substituting (A.5), (A.6) into the RHS of identity (A.4) yields

\[
\text{RHS of (A.4)} = (\nabla w_2, \nabla v_1) + (w_1 + w_2, \Delta v_2) + (u, \Delta h)
\]

\[
= \int_{\Gamma_s} \frac{\partial u}{\partial \nu} \, \tilde{h} \, d\Gamma_s + \int_{\Gamma_s} w_1 \frac{\partial \tilde{v}_2}{\partial \nu} \, d\Gamma_s + \int_{\Gamma_s} u \frac{\partial \tilde{v}_2}{\partial \nu} \, d\Gamma_s
\]

\[
+ \int_{\Gamma_s} \frac{\partial u}{\partial \nu} \, \tilde{h} \, d\Gamma_s - \int_{\Gamma_s} u \frac{\partial \tilde{h}}{\partial \nu} \, d\Gamma_s,
\]

(A.7)

where two boundary integrals cancel out. By Green’s First Theorem with \(w_2|_{\Gamma_s} = u|_{\Gamma_s}\) from (A.2)

\[
(w_1, \Delta v_2) = \int_{\Omega_s} \Delta \tilde{v}_2 \, w_1 \, d\Omega_s = -\int_{\Gamma_s} \frac{\partial \tilde{v}_2}{\partial \nu} \, w_1 \, d\Gamma_s - (\nabla w_1, \nabla v_2);
\]

(A.8)

\[
(w_2, \Delta v_1) = \int_{\Omega_s} \Delta \tilde{v}_1 \, w_2 \, d\Omega_s = -\int_{\Gamma_s} \frac{\partial \tilde{v}_1}{\partial \nu} \, w_2 \, d\Gamma_s - (\nabla w_2, \nabla v_1).
\]

(A.9)

Substitute \((w_1, \Delta v_2)\) from (A.8) and \((\nabla w_2, \nabla v_1)\) from (A.9) into the RHS of (A.7) to obtain

\[
\text{RHS of (A.4)} = - (w_2, \Delta v_1) - (\nabla w_1, \nabla v_2) + (w_2, \Delta v_2) + (u, \Delta h)
\]

\[
= \int_{\Gamma_s} \frac{\partial \tilde{v}_1}{\partial \nu} \, u \, d\Gamma_s - \int_{\Gamma_s} \frac{\partial \tilde{v}_2}{\partial \nu} \, w_1 \, d\Gamma_s
\]

\[
+ \int_{\Gamma_s} \frac{\partial \tilde{v}_2}{\partial \nu} \, d\Gamma_s + \int_{\Gamma_s} u \frac{\partial \tilde{v}_2}{\partial \nu} \, d\Gamma_s - \int_{\Gamma_s} u \frac{\partial \tilde{h}}{\partial \nu} \, d\Gamma_s.
\]

(A.10)

\[
= (w_2, \Delta (v_2 - v_1)) - (\nabla w_1, \nabla v_2) + (u, \Delta h),
\]

(A.11)

after imposing also the B.C. \(\frac{\partial (v_2 - v_1)}{\partial \nu}|_{\Gamma_s} = \frac{\partial h}{\partial \nu}|_{\Gamma_s}\) from (A.2). In conclusion, we obtain

\[
\left( A \begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right)_{H_0} = - (\nabla w_1, \nabla v_2) + (w_2, \Delta (v_2 - v_1)) + (u, \Delta h)
\]

(A.12)

\[
= \left( \begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix}, \begin{bmatrix} 0 & -I & 0 \\ -\Delta & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right)_{H_0},
\]

(A.13)

and Theorem 1.1 is proved.

**Acknowledgments.** The authors wish to thank the referees. The research of LL. and R.T. was partially supported by the National Science Foundation under grant DMS-1434941, and by the Air Force Office of Scientific Research under grant FA9550-09-1-0459. R.T. thanks the Instytut Badan Systemowych PAN, Warsaw, Poland, for collaboration and hospitality under project UMO-2014/15/B/ST1/00067.

**REFERENCES**

[1] P. Ausher, S. Hofmann, M. Lacey, A. McIntosh and P. Tehamitchian, The solution of the Kato square root problem for second order elliptic operators in \(R^n\), *Annals of Mathematics*, **156** (2002), 633–654.

[2] G. Avalos and M. Dvorak, A new maximality argument for a coupled fluid-structure interaction, with implications for a divergence-free finite element method, *Applications Mathematics*, **35** (2008), 259–280.
[3] G. Avalos, I. Lasiecka and R. Triggiani, Higher regularity of a coupled parabolic-hyperbolic fluid-structure interactive system, invited paper, special issue of *Georgian Mathematical Journal*, 15 (2008), 403–437; dedicated to the memory of J. L. Lions; J. Mawhin, editor.

[4] G. Avalos and R. Triggiani, The coupled PDE-system arising in fluid-structure interaction. Part I: Explicit semigroup generator and its spectral properties, *AMS Contemporary Mathematics, Fluids and Waves*, 440 (2007), 15–55.

[5] G. Avalos and R. Triggiani, Uniform stabilization of a coupled PDE system arising in fluidstructure interaction with boundary dissipation at the interface, *Discr. & Cont. Dynam. Systems*, 22 (2008), 817–833 (invited paper).

[6] G. Avalos and R. Triggiani, Semigroup well-posedness in the energy space of a parabolic-hyperbolic coupled Stokes-Lamé PDE system, *Discr. & Cont. Dynam. Systems DCDS-S*, 2 (2009), 417–448.

[7] G. Avalos and R. Triggiani, A coupled parabolic-hyperbolic Stokes-Lamé PDE system: Limit behavior of the resolvent operator on the imaginary axis, *Applicable Analysis*, 88 (2009), 1357–1396.

[8] G. Avalos and R. Triggiani, Boundary feedback stabilization of a coupled parabolic-hyperbolic Stokes-Lamé PDE system, *J. Evol. Eqns.*, 9 (2009), 341–370.

[9] G. Avalos and R. Triggiani, Rational decay rates for a PDE heat-structure interaction: A frequency domain approach, *Evolution Equations and Control Theory*, 2 (2013), 233–253.

[10] G. Avalos and R. Triggiani, Fluid–structure interaction with and without internal dissipation of the structure: A contrast in stability, *Evolution Equations and Control Theory*, 2 (2013), 563–598, special issue by invitation on the occasion of W. Littman’s retirement.

[11] A. Bensoussan, G. Da Prato, M. Delfour and S. Mitter, *Representation and Control of Infinite Dimensional Systems*, 2nd edition, Birkhauser, 2007, 575 pages.

[12] S. Canic, A. Mikelic and J. Tambača, A two-dimensional effective model describing fluidstructure interaction in blood flow: analysis, simulation and experimental validation, *Compte Rendus Mechanique Acad. Sci. Paris*, 333 (2005), 867–883.

[13] S. Canic, D. Lamponi, A. Mikelic and J. Tambača, Self-consistent effective equations modeling blood flow in medium-t-large compliant arteries, *Multiscale Model. Simul.*, 3 (2005), 559–596.

[14] G. Chen and D.L. Russel, A mathematical model for linear elastic systems with structural damping, *Quart. Appl. Math.*, (1982), 433–454.

[15] S. Chen and R. Triggiani, Proof of two conjectures of G. Chen and D. L. Russell on structural damping for elastic systems: The case $\alpha = 1/2$, *Springer-Verlag Lecture Notes in Mathematics*, 1354 (1988), 234–256. *Proceedings of Seminar on Approximation and Optimization*, University of Havana, Cuba (January 1987).

[16] S. Chen and R. Triggiani, Proof of extensions of two conjectures on structural damping for elastic systems: The case $1/2 \leq \alpha \leq 1$, *Pacific J. Math.*, 136 (1989), 15–55.

[17] S. Chen and R. Triggiani, Characterization of domains of fractional powers of certain operators arising in elastic systems, and applications, *J. Diff. Eqns.*, 88 (1990), 279–293.

[18] S. Chen and R. Triggiani, Gevrey class semigroups arising from elastic systems with gentle perturbation, *Proceedings Amer. Math. Soc.*, 110 (1990), 401–415.

[19] Q. Du, M. D. Gunzburger, L. S. Hou and J. Lee, Analysis of a linear fluid-structure interaction problem, *Discr. Dynam. Sys.,* 9 (2003), 633–650.

[20] D. Fujiwara, Concrete characterization of the domains of fractional powers of some elliptic differential operators of the second order, *Proc. Japan Acad.*, 43 (1967), 82–86.

[21] P. Grisvard, Characterization de quelques espaces d’ interpolation, *Arch. Pat. Mech. Anal.*, 25 (1967), 40–63.

[22] M. Ignatova, I. Kukavica, I. Lasiecka and A. Tuffaha, On well-posedness and small data global existence for an interface damped free boundary fluid-structure model, *Nonlinearity* 27 (2014), 467–499.

[23] T. Kato, Fractional powers of dissipative operators, *J.Math.Soc. Japan*, 13 (1961), 246–274.

[24] V. Komornik, *Exact controllability and stabilization. The multiplier method*, Masson, Paris; John Wiley & Sons Ltd, Chichester (1994), 156 pp.

[25] I. Kukavica and A. Tuffaha, Regularity of solutions to a free boundary problem of fluidstructure interaction, *Indiana Univ. Math. J.*, 61 (2012), 1817–1859.

[26] I. Kukavica, A. Tuffaha and M. Ziane, Strong solutions to a nonlinear fluid structure interaction system, *J. Differential Equations*, 247 (2009), 1452–1478.

[27] I. Kukavica, A. Tuffaha and M. Ziane, Strong solutions for a fluid structure interaction system, *Adv. Differential Equations*, 15 (2010), 231–254.
[28] I. Kukavica, A. Tuffaha and M. Ziane, Strong solutions to a Navier-Stokes-Lamé system on a domain with a non-flat boundary, Nonlinearity, 24 (2011), 159–176.
[29] I. Lasiecka, Unified theory for abstract parabolic boundary problems—a semigroup approach, Appl. Math. & Optimiz., 6 (1980), 31–62.
[30] I. Lasiecka and Y. Lu, Stabilization of a fluid structure interaction with nonlinear damping, Control Cybernet., 42 (2013), 155–181.
[31] I. Lasiecka and R. Triggiani, Control Theory for Partial Differential Equations: Continuous and Approximation Theories I, Abstract Parabolic Systems, Encyclopedia of Mathematics and Its Applications Series, Cambridge University Press, January 2000.
[32] J.L. Lions, Quelques Methodes de Resolution des Problemes aux Limites Nonlinearies, Dunod. Paris, 1969.
[33] J.L. Lions, Espaces d’interpolation et domaines de puissances fractionnaires d’operateurs. J. Math Soc., 14 (1962), 233–241.
[34] J.L. Lions and E. Magenes, Nonhomogeneous Boundary Value Problems and Applications, Vol. I., Springer-Verlag, (1972), 357 pp.
[35] Y. Lu, Uniform stabilization to equilibrium of a nonlinear fluid-structure interaction model, Nonlinear Anal. Real World Appl., 25 (2015), 51–63.
[36] A. McIntosh, On the comparability of $A^{1/2}$ and $A^*^{1/2}$, Proceedings AMS, 32 (1972), 430–434.
[37] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer Verlag, 1983.
[38] J. Pruss, On the spectrum of $C_0$ semigroups, Transactions of the American Mathematical Society, 284 (1984), 847–857.
[39] A. Taylor, and D. Lay, Introduction to Functional Analysis, 2nd edition, 1980, Wiley.
[40] R. Triggiani, A heat-viscoelastic structure interaction model with Neumann or Dirichlet boundary control at the interface: optimal regularity, control theoretic implications, Applied Mathematics and Optimization, special issue in memory of A.V. Balakrishnan, to appear.
[41] R. Triggiani, A matrix-valued generator $A$ with strong boundary coupling: a critical subspace of $D((-A)^{1/2})$ and $D((-A^*)^{1/2})$ and implications, Evolution Equations and Control Theory, vol 5, No.1, March 2016.

Received November 2015; revised February 2016.

E-mail address: lasiecka@memphis.edu
E-mail address: rtriggani@memphis.edu