Functional integral for non-Lagrangian systems

Denis Kochan
Department of Theoretical Physics, Comenius University, Bratislava, Slovakia

Theory Group, CERN, Genève, Switzerland

A novel functional integral formulation of quantum mechanics for non-Lagrangian systems is presented. The new approach, which we call “stringy quantization,” is based solely on classical equations of motion and is free of any ambiguity arising from Lagrangian and/or Hamiltonian formulation of the theory. The functionality of the proposed method is demonstrated on several examples. Special attention is paid to the stringy quantization of systems with a general $A$-power friction force $-\kappa \dot{q}^A$.

Results for $A = 1$ are compared with those obtained in the approaches by Caldirola-Kanai, Bateman and Kostin. Relations to the Caldeira-Leggett model and to the Feynman-Vernon approach are discussed as well.

Dedicated to my father on the occasion of his 60th birthdays.

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I. INTRODUCTION

Quantization is a phenomenon that changes our bright classical perspective into a bit uncertain and at first sight rather nonintuitive picture. This picture, however, is more rigorous than the classical one, possesses many fascinating features and has produced a lot of successful predictions.

The subtle problem of transition from classical to quantal attracts attention from the early days of quantum mechanics. Over the years various techniques and methods for solving this puzzle have been invented. Our aim is not to trace back the complete (hi)story of the milestone ideas in this field (for the review we refer to [1]). What we want to do is to give a concise exposition of the method we have developed. However, since we have generalized the original Feynman’s path integral approach, we will recapitulate this approach shortly in section II.

The main goal of our paper is to obtain a functional integral formula for the quantum propagator which would not refer to Lagrangian and/or Hamiltonian function. Quantum propagator is the probability amplitude $A(q_1,t_1 | q_0,t_0)$ for the transition of the system from the initial configuration $(q_0, t_0)$ to the final configuration $(q_1, t_1)$. We will derive a closed expression for this quantity starting from the given set of classical dynamical equations of motion.

The proposed method uses functional integration in the extended phase space, but instead of integration over path histories we introduce integration over stringy surfaces. This crucial element of our approach is explained in full detail in sections III and IV and in appendices A and B. We also make sure that whenever the system under consideration becomes Lagrangian, the stringy description reduces to the standard one with the Feynman path integral.

In sections V and VI some simple examples are scrutinized. It is well known that there are classical dynamical systems that cannot be described within the traditional Lagrangian or Hamiltonian framework. The stringy approach enables us to quantize them straightforwardly. In section V the quantization of a weakly non-Lagrangian system is performed. The transition amplitude is computed for a particle in a conservative field, whose motion is damped by a general $A$-power friction force $F = -\kappa \dot{q}^A$.

The stringy results for $A = 1$ are compared with the results obtained in the approaches by Caldirola-Kanai [2], Bateman [3] and Kostin [4].

One can argue against the stringy quantization of dissipative systems that it is unable to describe decoherence phenomena. This is true, but the same objection can be raised against the generally accepted heuristic approaches by the authors cited above, as well as those by Dekker [5], Razavy [6], Geicke [7] and others, simply because their kinematical and dynamical prerequisites are different from the prerequisites of the particle-plus-environment quantum models. However, a possible argument for the stringy quantization is that the particle-plus-environment models are not able to handle satisfactorily the case with the friction force $-\kappa \dot{q}^A$ for the general power $A$. Moreover, they describe a rather different phenomenon, namely the quantum Brownian motion for which the total force equals $-\kappa \dot{q} + \text{stochastic term}$.

Section VII contains the analysis of a curious two-dimensional Douglas system [8] which is strongly non-Lagrangian, i.e. not derivable from any sort of Lagrangian or Hamiltonian. This causes a fundamental problem for all conventional quantization methods, but can be dealt with in a rather transparent way in the

*Electronic address: kochan@fmph.uniba.sk*
stringy approach.

In section [VII] conclusion, discussion and outlook are collected. The section also includes some comments on the relation between the stringy quantization and the Caldeira-Leggett model of quantum Brownian motion [3], as well as the influence functional technique by Feynman and Vernon [10].

Some rather technical material is left to appendices. Appendix [A] is devoted to the stringy variational principle, which plays an important role in the motivation of our approach. In appendix [B] computational details concerning the surface functional integral for quantum friction force systems are presented.

Historically the first attempt at quantization based on the dynamical equations of motion belongs to Feynman, see [11]. A similar problem was considered by Wigner, the dynamical equations of motion belongs to Feynman, and their strategy is different. In our opinion, their approach fits much better the context of gauge field dy-

As Ludwig Faddeev noted during the Edward Witten’s talk at the Mathematical Physics Conference: From XX To XXI Century [20], quantization is not a science, quantization is an art. Let us believe that the quantization method proposed here will fit the Ludwig’s dictum and will be meaningful enough to be considered artistic.

II. FEYNMAN QUANTUM MECHANICS

According to Feynman [15], the probability amplitude of the transition of the system from the space-time configuration $\{q_0,t_0\}$ to another space-time configuration $\{q_1,t_1\}$ is

$$A(q_1,t_1 \mid q_0,t_0) = \frac{1}{N} \int [D\tilde{\gamma}] \exp \left\{ \frac{i}{\hbar} \int \sum dp - H dt \right\}. \quad (1)$$

Here the integral is taken over all paths (histories) $\tilde{\gamma}(t) = (q(t), \dot{q}(t), t)$ in the extended phase space [27], satisfying the conditions $q(t_0) = q_0$ and $q(t_1) = q_1$.

The preexponential factor $1/N$ in the expression [11] serves just the normalization. To fix it properly we impose two physical conditions on the transition amplitude. First we introduce an integral condition that ensures that the total probability is conserved,

$$\int dq_1 A(q_1,t_1 \mid q_0,t_0) A(q_1,t_1 \mid q_0,t_0) = \delta(q_0 - q_0). \quad (2a)$$

This specifies the absolute value of $N$. Then we add a constraint that expresses the obvious fact that no evolution takes place if the final time $t_1$ approaches the initial time $t_0$,

$$\lim_{t_1 \to t_0} A(q_1,t_1 \mid q_0,t_0) = \delta(q_1 - q_0). \quad (2b)$$

This determines the phase of $N$.

A miraculous consequence of the definition of the transition amplitude [1] (not an additional requirement!) is that it satisfies the evolutionary chain rule, or Chapman-Kolmogorov equation,

$$A(q_1,t_1 \mid q_0,t_0) = \int dq' A(q_1,t_1 \mid q',t) A(q',t \mid q_0,t_0). \quad (3)$$

The infinitesimal version of this formula is the celebrated Schrödinger equation.

Quantum states of the system are described by the square integrable functions with the standard Hilbert space structure. Physical observables are hermitian operators acting on such functions. Given the state $\Psi_0(q)$ at the initial moment $t_0$ one is able to predict the state $\Psi_1(q)$ at any later moment $t_1$ according to the formula

$$\Psi_1(q) \equiv \Psi_0(q). \quad (4)$$

In what follows it is assumed that the above concept of states, observables and quantum evolution is valid for the stringy quantization of non-Lagrangian systems as well. The only new element is a modified prescription for the evolutionary integral kernel $A(q_1,t_1 \mid q_0,t_0)$. The same kinematical prerequisites can be found also in other phenomenological approaches [2]–[7].

III. ONE STEP BEYOND FEYNMAN

A possible step beyond the theory summarized above consists in the elimination of the Hamiltonian function $H$ from formula [1]. The price to be paid is the replacement of the path integration by the surface functional integration.

Our aim is to construct the amplitude for the transition between $\{q_0,t_0\}$ and $\{q_1,t_1\}$ starting from the classical equations of motion (and not from the Hamiltonian function which provides them)

$$\frac{\partial \gamma^a}{\partial p_a} = m \gamma^a, \quad \frac{\partial H}{\partial q^a} \equiv F_a. \quad (5)$$

In the first set of equations, $m$ is the mass of the particle. We restricted ourselves to the simplest case of one particle, although it is trivial to generalize the theory to a system with an arbitrary number of particles. Note also that if the particle is unconstrained and we make use of Cartesian coordinates, the momenta $p^a$ defined in [5] reduce to $p_a$.

Suppose that there exists a unique classical trajectory in the extended phase space $\gamma(t) = (q(t), \dot{q}(t), t)$, connecting the points $(q_0,t_0)$ and $(q_1,t_1)$. Then we can assign to any other trajectory $\bar{\gamma}(t) = (\bar{q}(t), \bar{\dot{q}}(t), t)$, which enters the path integral in [11], two auxiliary curves

$$\lambda_0(s) = (q_0, \pi(s), t_0), \quad \lambda_1(s) = (q_1, \phi(s), t_1), \quad \text{where} \quad \pi(0) = p_{cl}(t_0), \quad \pi(1) = \bar{p}(t_0);$$

$$\lambda_0(s) = (q_0, \pi(s), t_0), \quad \lambda_1(s) = (q_1, \phi(s), t_1), \quad \text{where} \quad \phi(0) = p_{cl}(t_1), \quad \phi(1) = \bar{p}(t_1).$$
where $\partial \Sigma = \tilde{\gamma} - \lambda_1 - \gamma_{cl} + \lambda_0$ is a contour in the extended phase space consisting of four curves $\hat{\gamma}(t)$, $\gamma_{cl}(t)$, $\lambda_0(s)$, $\lambda_1(s)$. The first integral on the right hand side is the classical action $S_{cl}(q_1, t_1 | q_0, t_0)$ (its equivalent for the non-Lagrangian case will be specified later), while the second integral can be rewritten as

$$\int_{\partial \Sigma} p_a dq^a - H dt = \int_{\Sigma} dp_a \wedge (dq^a \frac{\partial H}{\partial p_a} dt) - \frac{\partial H}{\partial q^a} dq^a \wedge dt,$$

where $\Sigma$ is a surface spanning the contour $\partial \Sigma$, i.e., a map from the parametric space $(t, s) \in [t_0, t_1] \times [0, 1]$ to the extended phase space,

$$\Sigma : (t, s) \mapsto \Sigma(t, s) \equiv (q^a(t, s), p_a(t, s), t(t, s) = t),$$

satisfying

$$\Sigma(t, 0) = \gamma_{cl}(t) \quad \text{and} \quad \Sigma(t_0, s) = \lambda_0(s),$$

$$\Sigma(t, 1) = \hat{\gamma}(t) \quad \text{and} \quad \Sigma(t_1, s) = \lambda_1(s).$$

The surface $\Sigma$ can be viewed as a worldsheet of a string, therefore we will call the quantization method using such surfaces “stringy.” Partial derivatives of the Hamiltonian function entering the integral over $\Sigma$ can be eliminated with the help of the equations of motion $[\hat{\gamma}]$. By doing so, one transforms $[\hat{\gamma}]$ into the following form:

$$\int_{\tilde{\gamma}} p_a dq^a - H dt = S_{cl}(q_1, t_1 | q_0, t_0) + \int_{\Sigma} \Omega,$$

where the two-form $\Omega$ is defined as

$$\Omega = dp_a \wedge dq^a - \left(\frac{p^a}{m} dp_a - F_a dq^a\right) \wedge dt.$$

This two-form is an object in the extended phase space and its structure can be read out from the underlying equations of motion. Note that for non-potential forces the expression $(p^a/m) dp_a - F_a dq^a$ does not reduce to $dH$, so that the two-form $\Omega$ is not closed. This becomes essential in the next subsection.

It is obvious that for a given pair of histories $(\hat{\gamma}, \gamma_{cl})$ there exist infinitely many $\Sigma$-surfaces such that $\hat{\gamma} - \gamma_{cl} \subset \partial \Sigma$. All of them form a set which we will call $U_{(\hat{\gamma}, \gamma_{cl})}$. Since no $\Sigma$ is preferred and $\int_{\Sigma} \Omega$ is only boundary dependent, it is natural to average the exponent of $[\hat{\gamma}]$ over the whole stringy set $U_{(\hat{\gamma}, \gamma_{cl})}$. After doing so we obtain the identity

$$\exp\left\{\frac{i}{\hbar} \int_{\hat{\gamma}} p_a dq^a - H dt\right\} = \exp\left\{\frac{i}{\hbar} \int_{\Sigma} \left[\mathcal{D} \Sigma\right] \Omega\right\},$$

where $N_{\Sigma}$ is the cardinality of the stringy set $U_{(\hat{\gamma}, \gamma_{cl})}$, $N_{\Sigma} := \# U_{(\hat{\gamma}, \gamma_{cl})}$, and $\left[\mathcal{D} \Sigma\right]$ is a functional integration measure specified in Appendix B. If no topology-related problems arise in the extended phase space, the infinite constant $N_{\Sigma}$ is independent of the history $\hat{\gamma}$. Taking all this into account we can rewrite $[\hat{\gamma}]$ as

$$A(q_1, t_1 | q_0, t_0) = \frac{\exp[iS_{cl}]}{N_{\Sigma}} \int_{U_{(\hat{\gamma}, \gamma_{cl})}} \left[\mathcal{D} \Sigma\right] \exp\left\{\frac{i}{\hbar} \int_{\Sigma} \Omega\right\},$$

where the set $U_{(\gamma_{cl})}$ over which the functional integration is carried out contains all strings in the extended phase space which are anchored to the given classical trajectory $\gamma_{cl}$. Some surfaces from $U_{(\gamma_{cl})}$ are depicted in Figure 2. In $[\Sigma]$, the undetermined constant $1/N_{\Sigma}$ was absorbed into the overall preexponential factor $1/N$ and the path integral over $\hat{\gamma}$’s was converted into the surface functional integral, as promised earlier, using the identities

$$\int_{[\mathcal{D} \Sigma]} \left[\mathcal{D} \Sigma\right] \cdots = \int_{U_{(\gamma_{cl})} U_{\hat{\gamma}} U_{(\hat{\gamma}, \gamma_{cl})}} \left[\mathcal{D} \Sigma\right] \cdots \left[\mathcal{D} \Sigma\right].$$

IV. ONE MORE STEP BEYOND – NON-LAGRANGIAN SYSTEMS

In case the Hamiltonian function $H$ is given, the formula $[\hat{\gamma}]$ for the propagator is equivalent to the Feynman
formula \( \Omega \) we have started with. The surface functional integral \( \Omega \) is however completely free of \( U \) and requires just the knowledge of the classical equations of motion. This observation allows us to postulate \( \Omega \) as a quantization tool in situations in which one cannot use the standard Hamiltonian approach. We just have to relax the requirement of the closedness of the two-form \( \Omega \), following from the definition of \( \Omega \) for Hamiltonian systems. This relaxation is what is hidden behind the slightly provocative phrase one more step beyond in the title of this subsection.

The problematic part here is the definition of the classical action \( S_{cl}(q_1, t_1 \mid q_0, t_0) \) for non-Lagrangian systems. Later we will see that in some specific situations this quantity can be read out from the structure of the surface integral

\[
\int_{\Sigma} |d\Sigma| \exp \left\{ \frac{i}{\hbar} \int_{\Sigma} \Omega \right\}.
\]

Another possibility which comes to mind is to use the integrating factor of the generally non-closed two-form \( \Omega \). This means that one will look for a function \( f(q, p, t) \) on the extended phase space such that \( d(f\Omega) = 0 \). If the structure of the dynamical equations allows for such a function (i.e., the forces satisfy the Helmholtz condition) then we can define a local auxiliary Hamiltonian \( H_{aux} \) such that \( f\Omega = d(p_\alpha dq^\alpha - H_{aux} dt) \). Having \( H_{aux} \) we can define the auxiliary classical action \( S_{aux} \). The alternative approach which tries to identify \( S_{cl} \) with \( S_{aux} \) has, however, several substantial disadvantages. First of all, the procedure of finding \( f \) and \( S_{aux} \) is highly ambiguous. Furthermore, \( S_{aux} \) lacks symmetries that were originally present in the equations of motion. Because of these findings we do not follow this strategy hereinafter.

From the physical point of view dynamical equations seem to be more fundamental than their compact but ambiguous precursors, Hamiltonian and/or Lagrangian function, see [16]. Since \( S_{cl} \) requires just the knowledge of the equations of motion, it determines the transition amplitude in a completely new way.

Our proposal for the transition amplitude has appeared here out of thin air. Actually, we were rewriting \( S_{cl} \) in terms of stringy surfaces and then, when realizing that the integrated function can be written without any reference to the Hamiltonian, we postulated formula \( \Omega \) to be valid in general. However, this is not the whole story. The stringy functional quantization can be motivated also by the stringy variational principle. Having the dynamical equations and initial and final endpoints, one can form \( U_{\gamma cl} \) and \( \Omega \). Then using these objects one can introduce the stringy action functional

\[
S : U_{\gamma cl} \to \mathbb{R}, \quad \Sigma \mapsto S(\Sigma) := \int_{\Sigma} \Omega.
\]

This is a variational problem with varying boundaries, therefore the total variation has two terms. First one specifies the boundary and determines the initial equations of motion for \( \gamma_{cl} \). The second one specifies the bulk of the stationary world-sheet \( \Sigma \) (which turns out to be shrunk into \( \gamma_{cl} \) itself). Moreover, one immediately verifies that in the special case when \( \Omega = d(p_\alpha dq^\alpha - H dt) \), the stringy variational principle reduces (up to an additive constant) to the celebrated Hamilton least action principle. Of course, many subtleties were omitted here, but all of them can be found in [17], or in Appendix A. Thus, our variational principle enables us to perform the limit \( \hbar \to 0 \), in which we recover the original classical dynamics as required.

V. QUANTIZATION OF FRICTION FORCE SYSTEMS

To examine the functionality of the proposed quantization method let us first analyze the simplest friction force system. It consists of a particle with unit mass, moving in one dimension under the action of the conservative force \( F = -dU/dq \) and the friction force \(-\kappa q^4 \).

\[
\ddot{q} = -\kappa q^4 + F = \Omega = d(p_\alpha dq^\alpha - H dt) - \kappa q^4 dq \wedge dt.
\]
In this example the surface functional integral can be calculated explicitly (for more detail see [17], or Appendix B). In the course of calculation, the surface functional integral in the extended phase space reduces to path integral in the configuration space,

\[ \int [D\Sigma] \exp \left\{ \frac{i}{\hbar} \int_{t_0}^{t_1} \Omega \right\} \propto \exp \left\{ -\frac{i}{\hbar} \int_{t_0}^{t_1} \left( \frac{1}{2} \dot{q}_{cl}^2 - U(q_{cl}) - \kappa q_{cl} p_{cl}^A \right) dt \right\} \times \int [Dq] \exp \left\{ \frac{i}{\hbar} \int_{t_0}^{t_1} \left( \frac{1}{2} \dot{q}^2 - U(q) - \kappa q p^A \right) dt \right\}. \]  

(12)

This combined with the formula (3) suggests that it will be convenient to define the classical action as [31]

\[ S_{Acl}(q_1, t_1|q_0, t_0) = \int_{t_0}^{t_1} \left( \frac{1}{2} \dot{q}_{cl}^2 - U(q_{cl}) - \kappa q_{cl} p_{cl}^A \right) dt. \]  

(13)

When doing so, we obtain a controlled cancelation of \( \exp \left\{ \frac{i}{\hbar} S_{Acl} \right\} \) with the preexponential factor arising from (12). The final probability amplitude then assumes a compact and reasonable form:

\[ A_{Astr}(q_1, t_1|q_0, t_0) = \frac{1}{N_{Astr}} \int [Dq] \exp \left\{ \frac{i}{\hbar} \int_{t_0}^{t_1} \left( \frac{1}{2} \dot{q}^2 - U(q) - \kappa q p^A \right) dt \right\}. \]  

(14)

The preexponential factor \( 1/N_{Astr} \) can in principle be obtained by subjecting \( A_{Astr}(q_1, t_1|q_0, t_0) \) to the conditions [2].

Let us explain why one should consider the propagator formula (14) reasonable. In the path integral above there appears an unconventional “external source” term \( -\kappa q p^A \). Its appearance guarantees that the quantum dynamics governed by \( A_{Astr}(q_1, t_1|q_0, t_0) \) transforms into classical mechanics in the limit \( \hbar \rightarrow 0 \). This follows from the simple fact that the unique solution of the saddle point equation for the path integral (14),

\[ \ddot{q} = -\frac{dU}{dq} - \kappa \dot{q}_A^A, \]

which satisfies the given initial and final conditions \( q(t_0) = q_0 \) and \( q(t_1) = q_1 \), is the classical trajectory \( q_{cl}(t) \).

The external source term in (14) breaks the validity of the Chapman-Kolmogorov (memoryless) equation [32]. From the physical point of view it is a desired phenomenon. The microscopic origin of friction is some environmental interaction. This, however, was not accounted for here explicitly. What has been considered is some effective (macroscopic, phenomenological) interaction emerging on the classical level only. Microscopically the system is a part of a larger system and hence it should be affected by the memory effect.

It is not difficult to compute \( A_{Astr}(q_1, t_1|q_0, t_0) \) for the general power \( A \) and the potentials \( U(q) = 0 \) and \( U(q) = \frac{1}{2} \omega^2 q^2 \). In both cases we can carry out the path integration in (14) explicitly to obtain

\[ A_{Astr}(q_1, t_1|q_0, t_0) = \frac{1}{N_{Astr}} \exp \left\{ \frac{i}{\hbar} S_{Acl}(q_1, t_1|q_0, t_0) \right\}, \]  

(15)

where the non-Lagrangian action \( S_{Acl}(q_1, t_1|q_0, t_0) \) is given by the expression (13). An open problem is to determine the preexponential factor \( 1/N_{Astr}(t_1 - t_0, q_1, q_0) \) for all powers \( A \) except for \( A = 1 \) (at least the author is incapable to do that). If \( A \neq 1 \), the preexponential factor is apparently dependent not only on the time difference \( t_1 - t_0 \), but also on the endpoint positions \( q_0 \) and \( q_1 \). This hypothesis is supported by the results obtained in [18], where the path integral for a non-conservative force quadratic in velocity is calculated. The normalization prefactor which appears there is explicitly dependent on the endpoints positions as well as on \( t_1 - t_0 \).

In what follows we will restrict ourself to the case when the power \( A \) is equal to 1. This simplified setup enables us to compare the stringy approach with the Caldirola-Kanai, Bateman and Kostin approaches [33].

### A. Stringy versus Caldirola-Kanai approach

The dissipative system under consideration is weakly non-Lagrangian. This means that \( d \Omega \neq 0 \), but \( \Omega \) possesses a local integrator \( f \neq 0 \). Let us define the auxiliary Lagrangian function (one of plenty) as

\[ L_{CK}(q, t) = e^{\kappa t} \left[ \frac{1}{2} \dot{q}^2 - U(q) \right]. \]  

(16)

This Lagrangian is usually called CK-Lagrangian after its inventors P. Caldirola and E. Kanai [2]. Henceforward a damped free particle is considered only, i.e. the potential energy \( U(q) \) is supposed to be zero. It is immediately clear that the transition amplitude computed from (15) differs from the CK-amplitude

\[ A_{CK}(q_1, t_1|q_0, t_0) = \frac{1}{N_{CK}} \int [Dq] \exp \left\{ \frac{i}{\hbar} \int_{t_0}^{t_1} L_{CK}(q, t)dt \right\} \]  

\[ = \frac{1}{N_{CK}} \exp \left\{ \frac{i}{\hbar} S_{CK}(q_1, t_1|q_0, t_0) \right\}. \]

Indeed, the stringy and CK actions which enter the corresponding transition amplitudes are different,

\[ S_{1cl} = \frac{\kappa}{4} (q_1 - q_0) (q_0 + 3q_1) e^{-\kappa t_1} - (q_1 + 3q_0) e^{-\kappa t_0}, \]

\[ S_{CK} = \frac{\kappa}{2} (q_1 - q_0)^2. \]
Since both actions are quadratic functions of the endpoints $q_0$ and $q_1$, and since we require that the total probability is conserved, the preexponential factors for both transition amplitudes can be computed from the Van Vleck formula

$$\frac{1}{N} = \sqrt{\frac{i}{2\pi\hbar} \frac{\partial^2 S}{\partial q_1 \partial q_0}}.$$  

Using this formula we arrive at the normalized transition amplitudes

$$A_{str}(q_1, t_1|q_0, t_0) = \sqrt{\frac{\kappa}{4\pi i \hbar \tanh \frac{\kappa}{2}(t_1 - t_0)}} e^{i S_{str}}, \quad (17)$$

$$A_{CK}(q_1, t_1|q_0, t_0) = \sqrt{\frac{\kappa}{2\pi i \hbar (e^{-\kappa t_0} - e^{-\kappa t_1})}} e^{i S_{CK}}, \quad (18)$$

which trivially satisfy (2). Moreover, one immediately verifies that in the frictionless limit $\kappa \to 0$ both transition amplitudes coincide with the free particle propagator. A short inspection shows that the stringy propagator depends on $t_0$ and $t_1$ only through $t_1 - t_0$, but this is not the case for the CK propagator. The same observation holds for $S_{str}$ and $S_{CK}$. This is a typical feature for all auxiliary actions defined in terms of integrators of motion of Gaussian wave packets when compared to the CK one. The evolution of Gaussian wave packets

$$\Psi_0(q) \propto e^{-q^2 + i\frac{\alpha}{2}\frac{q^4}{\hbar}} \sim \Psi_1(q) = \int dq' \mathcal{A}(q, t_1 | q', t_0) \Psi_0(q'),$$

whose dynamics is governed by (17) and (18), is visualized in Figure 3. For more detail, see [22].

From the figure we can see that the stringy averaged momentum $\langle p \rangle$ becomes negative (i.e., meaningless) for $t_1 - t_0 > \ln 3 \kappa^{-1}$. Let us try to explain this peculiarity. It is well known that the relevancy of the classical solution $q_{cl}(t)$ of $\ddot{q} = -\kappa \dot{q}$ breaks down when the time $t_1 - t_0$ exceeds the relaxation time $\kappa^{-1}$. Since $q_{cl}$ is used in the derivation of the formula (17) for the stringy propagator, its applicability is automatically restricted as well. After this restriction is taken into account, the stringy evolution of $\langle p \rangle$ becomes more acceptable from the physical point of view than the evolution in the CK approach.

**B. Stringy versus Bateman approach**

The key element of the Bateman(-Morse-Feshbach) approach [23] are new subsidiary degrees of freedom introduced in addition to the initial ones, which are amplified rather than damped; thus, they evolve according to the time reversed dynamical equations. In our case we have:

$$\ddot{q} = -\kappa \dot{q} \quad \text{(damped)} \quad \text{AND} \quad \ddot{Q} = +\kappa \dot{Q} \quad \text{(amplified).}$$

These equations of motion can be derived from the least action principle with the quadratic and time independent Bateman(-Morse-Feshbach) Lagrangian:

$$L_B(q, \dot{q}, Q, \dot{Q}) = \dot{q} \dot{Q} + \frac{\kappa}{2} (\dot{q}^2 - Q^2). \quad (19)$$

The canonical as well as path integral quantization of the theory encounters various difficulties because non-normalizable states of the amplified system must be employed [23]. This problem will be discussed later, when the treatment of the auxiliary $Q$-degrees of freedom in this approach will be described. The path integral evaluation of the transition probability amplitude based on $L_B$ is straightforward [24]. The result is:

$$A_B(q_1, Q_1, t_1|q_0, Q_0, t_0) = \frac{\kappa}{4\pi \hbar \sinh \frac{\kappa T}{2}} \times$$

$$\times \exp \left\{ - \frac{i}{\hbar} \alpha \left[ Q_1 (q_1 - q_0 \beta_-) + Q_0 (q_0 - q_1 \beta_+) \right] \right\}, \quad (20)$$

where

$$\alpha = \frac{\kappa}{2} \frac{1}{\tanh \frac{\kappa T}{2}}, \quad \beta_\pm = \frac{e^{\pm \frac{\kappa T}{2}}}{\cosh \frac{\kappa T}{2}}, \quad T = t_1 - t_0.$$
must project out the nonphysical $Q$-degrees of freedom. When using the standard formula

$$A_B^{\text{tral}}(q_1,t_1|q_0,t_0) = \int dQ A_B(q_1,Q,t_1|q_0,Q,t_0),$$

we do not reproduce the free particle propagator for $\kappa = 0$ as desired. To overcome this trouble the following “repairing prescription” is introduced (for more detail see [25]): the amplitude for the damped (sub)system to pass from $|q_0\rangle$ to $|q_1\rangle$ within the time $T$ is equal to the amplitude for the whole system to pass from the nonphysical state $|q_0\rangle|\Psi_-\rangle$ to the nonphysical state $|q_1\rangle|\Psi_+\rangle$.

$$A_B^{\text{eff}}(q_1,t_1|q_0,t_0) := \text{Amp}\left(|q_0\rangle|\Psi_-\rangle \rightarrow |q_1\rangle|\Psi_+\rangle\right) =$$

$$= \int dQ dq' \Psi_+(Q') A_B(q_1,Q',t_1|q_0,Q,t_0) \Psi_-(Q), \quad (21)$$

where the non-normalizable $Q$-system states $|\Psi_\pm\rangle$ are chosen as

$$\Psi_\pm(Q) = \sqrt{i \cosh \frac{\kappa}{2} T} \sqrt{\frac{-2i\alpha}{\pi \hbar}} \exp\left\{ \frac{i}{\hbar} \alpha Q^2 \right\}.$$

Substituting (20) into (21) we obtain the following effective propagator for the damped (sub)system:

$$A_B^{\text{eff}}(q_1,t_1|q_0,t_0) := \frac{\kappa}{4\pi i \hbar \tanh \frac{\kappa}{2} T} \times$$

$$\times \exp\left\{ \frac{i}{\hbar} \frac{\alpha}{4} \left[ (q_1 - q_0 \beta_\pm)^2 + (q_0 - q_1 \beta_\pm)^2 \right] \right\}. \quad (22)$$

The form of the auxiliary states $|\Psi_+\rangle$ and $|\Psi_-\rangle$ guarantees that the effective Bateman propagator for the damped (sub)system satisfies the normalization condition (2).

The Bateman effective propagator (22) depends only on the time difference $T$, which favors it in comparison with the CK-propagator (18). On the other hand, the artificial “repairing procedure” involving nonphysical states discredits the Bateman approach with respect to the stringy one. The Gaussian wave packet characteristics obtained from (17) and (22) are drawn in Figure 4. As seen from the figure, the stringy quantization possesses again better qualitative features than the approach to which we have compared it.

### C. Stringy versus Kostin approach

A possible incorporation of the classical dynamics into the quantum one can be obtained by using Heisenberg equations for the position and momentum operators [34]:

$$\dot{\hat{q}} = \hat{p}, \quad \dot{\hat{p}} = F(\hat{q},\hat{p}),$$

where $\hat{q}$ and $\hat{p}$ obey $[\hat{q},\hat{p}] = i\hbar$ for all $t$. In the special case when $F(\hat{q},\hat{p}) = -dU(\hat{q})/d\hat{q} - \kappa\hat{p}$, Kostin found an equivalent description of the system by the Schrödinger equation [3]:

$$i\hbar \partial_t \Psi(q,t) = \left[ -\frac{\hbar^2}{2} \frac{d^2}{dq^2} + U(q) + \hbar K[\Psi] \right] \Psi(q,t), \quad (23)$$

where $K[\Psi]$ is the $\Psi$-dependent Kostin potential defined as

$$K[\Psi] = \frac{\kappa}{2} \left[ \ln \frac{\Psi(q,t)}{\Psi(q,t_0)} - \int dq' \frac{\Psi(q',t)}{\Psi(q',t_0)} \Psi(q',t) \right].$$

This is known as Kostin-Schrödinger(-Langevin) equation. In the Kostin’s paper a complete set of solutions of this equation in the special case of free particle ($U(q) = 0$) is presented. The set consists of the states $\Psi_{p_0}(q,t)$ labeled by the continuous quantum number $p_0$, the initial momentum of the particle. The state $\Psi_{p_0}(q,t)$ starts as a momentum eigenstate with the momentum $p_0$, and remains the momentum eigenstate also later, but with the decreasing momentum,

$$\dot{p} \Psi_{p_0}(q,t) = p_0 e^{-\kappa(t-t_0)} \Psi_{p_0}(q,t).$$

From the ensemble of non-localized Kostin’s states one would like to form normalized wave packet solutions, localized in configuration as well as momentum space. This is, however, impossible since the Kostin-Schrödinger equation is nonlinear; thus, the Kostin approach belongs to nonlinear quantum mechanics and lacks superposition principle. In Figure 5 we plot the expectation value of the momentum for the Kostin state $\Psi_{p_0}(q,t_1)$ and the stringy evolved state $\Psi(q,t_1) = \int dq' A_{\text{str}}(q,t_1|q',t_0) \Psi_{p_0}(q',t_0)$ for various values of $\kappa$ [35]. As $\kappa$ tends to zero, the stringy and Kostin results match each other.
VI. QUANTIZATION OF THE DOUGLAS SYSTEM

After we have dealt with the weak non-Lagrangeanity, let us consider the simplest strongly non-Lagrangean system. It was proposed by Jesse Douglas (one of the two winners of the first Fields Medals awarded in 1936) when studying the inverse problem of variational calculus [8]. The system is governed by the following dimensionless dynamical equations:

\[
\ddot{x} = -\dot{y}, \quad \ddot{y} = -y \quad \Leftrightarrow \quad p_x = \dot{x}, \quad p_y = -\dot{y}, \quad (24)
\]

A quick calculation shows that the associated two-form

\[
\Omega = d(p_x dx + p_y dy - \frac{p_x^2}{2} dt - \frac{p_y^2}{2} dt - y^2 dt) - p_y dx \wedge dt
\]

does not possess a non-trivial local integrator \( f \). Consequently, equations (24) cannot be obtained as Euler-Lagrange (Hamilton) equations.

From the standard point of view, this curious situation is stalemate: no Lagrangian ⇒ no Quantum Mechanics. However, the surface integral method offers a way to overcome this deadlock. Using the stringy functional integration in the extended phase space we get

\[
\int [\mathcal{D}\Sigma] \exp \left\{ \frac{i}{\hbar} \int \Sigma \right\} \propto \\
\times \exp \left\{ -\frac{i}{\hbar} \int_{t_0}^{t_1} \left( \frac{1}{2} \dot{x}_c^2 + \frac{1}{2} \dot{y}_c^2 - \frac{1}{2} y_c^2 - x_c p_y c\right) dt \right\} \\
\times \int [\mathcal{D}x \mathcal{D}y] \exp \left\{ \frac{i}{\hbar} \int_{t_0}^{t_1} \left( \frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{y}^2 - \frac{1}{2} y^2 - x p_y c\right) dt \right\}. (25)
\]

Here, obviously, \( (x_c(t), p_{x c}(t)) \) and \( (y_c(t), p_{y c}(t)) \) stand for the classical solutions of (21) matching the initial and final endpoints for which the quantum transition probability amplitude is sought for, \( x_c(t_0) = x_0, \ y_c(t_0) = y_0 \) and \( x_c(t_1) = x_1, \ y_c(t_1) = y_1 \). In the classical limit \( \hbar \to 0 \) we want the stringy propagator \( \Psi \) to maintain the properties of the original dynamical system. This guides us to define:

\[
S_{D c}(x_1, y_1, t_1|x_0, y_0, t_0) = \int_{t_0}^{t_1} \left\{ \frac{1}{2} \dot{x}_c^2 + \frac{1}{2} \dot{y}_c^2 - \frac{1}{2} y_c^2 - x_c p_y c\right\} dt.
\]

The exponential \( \exp \{ \frac{i}{\hbar} S_{D c} \} \) of the classical action defined in such a way cancels the prefactor in (25). Finally we arrive at the following transition amplitude for the Douglas system:

\[
A_D(x_1, y_1, t_1|x_0, y_0, t_0) = \frac{1}{N_D} \times \\
\times \int [\mathcal{D}x \mathcal{D}y] \exp \left\{ \frac{i}{\hbar} \int_{t_0}^{t_1} \left( \frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{y}^2 - \frac{1}{2} y^2 - x p_y c\right) dt \right\}. (26)
\]

The path integral in the configuration space we have constructed is quadratic and hence can be computed explicitly. The normalized transition amplitude is

\[
A_D(x_1, y_1, t_1|x_0, y_0, t_0) = \frac{e^{\frac{i}{\hbar} S_{D c}}}{2\pi i\hbar} \left| \frac{\partial^2 S_{D c}}{\partial x_1 \partial x_0} \right| \left| \frac{\partial^2 S_{D c}}{\partial y_1 \partial y_0} \right|^{\frac{1}{2}}
\]

\[
S_{D c} = \frac{(x_1 - x_0)^2}{2(t_1 - t_0)} + \tan^2 \left( \frac{t_1 - t_0}{2} \right) \left( \frac{y_1 + y_0}{2} \right)^2 \\
+ \tan \left( \frac{t_1 - t_0}{2} \right) (x_1 - x_0)(y_1 + y_0) - (x_1 y_1 - x_0 y_0) \\
+ \frac{3}{4 \sin (t_1 - t_0)} \left[ (y_1^2 + y_0^2) \cos (t_1 - t_0) - 2 y_1 y_0 \right] \\
- \frac{1}{4 \sin^2 (t_1 - t_0)} \left[ (y_1^2 + y_0^2) - 2 y_1 y_0 \cos (t_1 - t_0) \right].
\]

VII. CONCLUSION, DISCUSSION AND OUTLOOK

In the paper we have developed a new quantization method that generalizes the conventional path integral approach. Throughout the paper we considered only the nonrelativistic quantum mechanics of spinless systems. However, the generalization to the field theory is rather straightforward. We have just to pass from the space of particle positions to the space of field configurations.

The mathematical language of our exposition respects the Vladimir Arnold’s “principle of minimal generality.”
Formulas are mostly written down in one (local) chart. From this, however, one can ascend to a global, coordinate free description employing bundles and jet prolongations. In order that we did not cloud up the main idea of the paper, we did not follow such fluffy approach. However, in the future it could be useful to analyze obstructions to the implementation of our method which arise from the global properties of the underlying space-time geometry.

Special attention was paid to the stringy quantization of dissipative systems and of the Douglas system. The latter cannot be quantized by any known quantization technique while the former can. In fact, dissipative (friction force) systems were studied extensively in the past and several approaches to their quantization were proposed. The stringy quantization was compared with three of them, Caldirola-Kanai, Bateman and Kostin, and it was shown that it either gives better results (in the first two cases) or is simpler to apply (in the third case).

A. Stringy versus Caldeira-Leggett model

As mentioned in the introduction, the quantization proposed here shares a common property with the well established approaches of Caldirola-Kanai, Bateman, Kostin and others: they all fail to cover decoherence phenomena. On the other hand, as we have seen, the stringy quantization produces a quantum propagator that does not respect the memoryless Chapman-Kolmogorov equation. We believe that this essential feature of the theory is measuring/reflecting the non-Lagrangeanity of the system.

B. Surface functional integral versus Feynman-Vernon

The goal of our paper was to write down a functional integral formula for the quantum transition amplitude in terms of the underlying classical equation of motion. However, we did not eliminate the artificial notions of pure states and classical action for non-Lagrangian systems. In the surface functional integral \([\mathcal{D}\Sigma]\), these notions were implicitly present.

A possible alternative to our approach consists in expressing the transition probability (not the probability amplitude!) using the functional integral. The probability that the system evolves from the mixed state \(\rho_0(q_0, q'_0, t_0)\) at the time \(t_0\) to the mixed state \(\rho_1(q_1, q'_1, t_1)\) at some later time \(t_1\) is

\[
P_{\rho_0 \rightarrow \rho_1} \propto \int dq_0 dq'_0 dq_1 dq'_1 \rho_0(q_0, q'_0, t_0) \rho_1(q_1, q'_1, t_1) \times 
\times \int [\mathcal{D}\tilde{\gamma}] [\mathcal{D}\tilde{\gamma}'] \exp\left\{ -i \int_{t_0}^{t_1} (pdq - H dt) \right\}.
\]

Here \(\tilde{\gamma}\) and \(\tilde{\gamma}'\) are curves in the extended phase space whose \(q\)-projections connect \(q_0\) with \(q_1\) and \(q'_0\) with \(q'_1\) respectively. Using the Stokes theorem one is able to convert the difference of the line integrals of the one-form \(pdq - H dt\) into the surface integral of the two-form \(\Omega = d(pdq - H dt)\). Explicitly,

\[
\left( \int_{\tilde{\gamma}} - \int_{\tilde{\gamma}'} \right) pdq - H dt = - \int \Omega - \int_{\lambda_1}^{\lambda_0} \int_{\lambda_0}^{\lambda_1} pdq,
\]

where \(\Sigma\) represents again a map from the parametric space \((t, s) \in [t_0, t_1] \times [0, 1]\) to the extended phase space,

\[
\Sigma: (t, s) \mapsto (\Sigma(t, s) = (q_\Sigma(t, s), p_\Sigma(t, s), t_{\Sigma}(t, s) = t),
\]

such that \(d\Sigma = \tilde{\gamma}'(t) - \lambda_1(t - \tilde{\gamma}(t) + \lambda_0(s))\). The sideways \(\lambda\)-boundary curves \([\lambda]\) of \(\Sigma\)

\[
\lambda_0(s) = \Sigma(t_0, s) \quad \text{and} \quad \lambda_1(s) = \Sigma(t_1, s)
\]

live in the instant phase spaces of \(\mathbb{R}^{2n+1}[q^a, p_a, t]\), i.e. in the submanifolds \(\mathbb{R}^{2n}[q^a, p_a, t_0]\) and \(\mathbb{R}^{2n}[q^a, p_a, t_1]\). Moreover, as the intrinsic parameter \(s\) varies from 0 to 1, the \(q_\Sigma\)-components of \(\lambda_0(s)\) and \(\lambda_1(s)\) vary from \(q_0\) to \(q'_0\) and from \(q_1\) to \(q'_1\) respectively.

Let us denote by \(U\) the space of all \(\Sigma\)-maps for which

\[
q_\Sigma(t_0, 0) = q_0, \quad q_\Sigma(t_1, 0) = q_1, \quad \lambda_0(s) \subset \mathbb{R}^{2n}[q, p, t_0],
\]

\[
q_\Sigma(t_0, 1) = q'_0, \quad q_\Sigma(t_1, 1) = q'_1, \quad \lambda_1(s) \subset \mathbb{R}^{2n}[q, p, t_1].
\]

Since the right hand side of \((27)\) depends only on \(\Sigma\) and its two boundaries, the double path integral \(\int [\mathcal{D}\tilde{\gamma}][\mathcal{D}\tilde{\gamma}']\) … entering \(P_{\rho_0 \rightarrow \rho_1}\) can be rewritten as a surface functional integral \(\int [\mathcal{D}\Sigma]\) …

\[
P_{\rho_0 \rightarrow \rho_1} \propto \int dq_0 dq'_0 dq_1 dq'_1 \rho_0(q_0, q'_0, t_0) \rho_1(q_1, q'_1, t_1) \times 
\times \int [\mathcal{D}\Sigma] \exp\left\{ -i \int_{t_0}^{t_1} \left[ \Omega + \int_{\lambda_1}^{\lambda_0} (pdq) \right] \right\}. \quad (28)
\]
The formula we have just arrived at was derived under the assumption that $\Omega = d(pdq - Hdt)$. However, it is clear that the expression in the exponent is free of any reference to the Hamiltonian and requires just the classical equations of motion. Therefore it seems reasonable to postulate the probability formula (28) also for non-Lagrangian systems.

The approach we shortly presented here does not use the notion of pure states of a non-Lagrangian system. It resembles the Feynman-Vernon approach [10], in which one introduces the influence functional in the presence of dissipative forces. Our probability formula (28) uses the surface functional integral in the extended phase space, while in the Feynman-Vernon approach one just has to calculate a double path integral in the configuration space. There is a chance that after a certain discretization we will be able to convert the surface functional integral into the double path integral in the configuration space (see appendix B), and as a result, we will find the explicit form of the influence functional. Work on this topic is in progress.

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$A$. M. D. Ž.

Appendix A: Stringy variational principle

In section [11] we shortly outlined how the transition amplitude (9) can be related to the stringy variational principle, with the action $S$ defined in (11). In what follows we will perform a complete variation of $S$, with the domain extended from $U_{\gamma_0}$ to a wider class $U := \bigcup_{\gamma_0} U_{\gamma_1} U_{\gamma_0}$. The stringy set $U$ contains all surfaces in the extended phase space trapped between the submanifolds with fixed $(q_0, t_0)$ and $(q_1, t_1)$ (including separate histories, regarded as shrunk surfaces). In the course of variation the classical dynamics (9) we have had in the course of the work.

\[ A = \mu \partial_{\lambda} \gamma. \]

\[ \partial_{\gamma} W(d(q_0, t_0) + d(q_1, t_1)) = \int W \partial_{\Omega} d\Omega, \]

\[ \Omega = \int W \partial_{\Omega} d\Omega. \]

\[ \partial_{\Omega} W = W \partial_{\lambda} \gamma_0. \]

FIG. 6: Vector field $W$ acting infinitesimally in the extended phase space. As a result, the initial surface $\Sigma_{ext} \in U_{\gamma_0}$ moves to some nearby surface $\Sigma_{ext}' \in U_{\gamma_1'}$, and its boundary $\partial \Sigma_{ext} = \gamma_1 - \gamma_0 + \lambda_0$ is transported to a new boundary $\partial \Sigma_{ext}' = \gamma_1' - \gamma_0' + \lambda_0'$. Since the variational vector field is assumed to preserve $U$, $W$ stays tangential to the momentum submanifolds with fixed $(q_0, t_0)$ and $(q_1, t_1)$.
Thus, it must hold the curves $\gamma_{W}$ for all variational fields $W$ the integral is zero. As a result, to annihilate the boundary term for all variational fields $W$ we are forced to chose the curves $\gamma_{0} = (q_{0}(t), p_{0}(t), t)$ and $\gamma_{1} = (q_{1}(t), p_{1}(t), t)$ in the extended phase space in such a way that their instant tangent vectors annihilate $\Omega$ at any moment $t \in [t_{0}, t_{1}]$. Thus, it must hold

$$0 = \frac{d\gamma_{\lambda}}{dt} \cdot \Omega,$$

(A2a)

or, equivalently,

$$0 = (F_{a} - \dot{p}_{a})dq^{a} + \left( \dot{q}^{a} - \frac{p^{a}}{m} \right) dp_{a} + \left( \ddot{q}^{a} F_{a} - \dot{p}_{a} \frac{p_{a}}{m} \right) dt,$$

(A2b)

for both $\gamma_{0}$ and $\gamma_{1}$. Here one recognizes the dynamical equations [5] as desired.

From the boundary term analysis and the assumption about the uniqueness of the history $\gamma_{cl}$ which we adopted from the very beginning we can conclude that $\Sigma_{ext} \in \mathcal{U}(\gamma_{cl}, \gamma_{cl}) \subset \mathcal{U}_{\gamma_{cl}}$. Hence $\Sigma_{ext}$ has the topology of a closed string attached to the classical trajectory $\gamma_{cl}$. To specify its shape the second term in (A1) must be employed. The resulting variational equation, written in a coordinate-free notation, is

$$0 = d\Omega \left( \partial_{t_{1}} \Sigma_{ext}, \partial_{t_{1}} \Sigma_{ext}, \ldots \right).$$

(A3a)

This is equivalent to the following system of partial differential equations for the unknown functions $q^{a}(t, s)$ and $p_{a}(t, s)$ (indices $a$ and $k$ run from 1, $\ldots$, $n$):

$$0 = \frac{\partial q^{a}}{\partial s} \frac{\partial F_{k}}{\partial p_{a}},$$

$$0 = \frac{\partial q^{a}}{\partial s} \left( \frac{\partial F_{a}}{\partial q^{k}} - \frac{\partial F_{k}}{\partial q^{a}} \right) + \frac{\partial p_{k}}{\partial s} \frac{\partial F_{a}}{\partial p_{k}}.$$

(A3b)

One solution of these equations, satisfying all boundary conditions, is trivial. It is the shrunk surface $\Sigma_{ext}(t, s) = \gamma_{cl}(t)$. After recalling the assumption of uniqueness of $\gamma_{cl}$ once more, we can see that this is the only solution of (A3). If there existed a closed unshrunk extremal surface $\Sigma_{ext}$, the initial set of equations of motion [5] would have at least one one-parametrical family of classical solutions between the given pair of endpoints.

The variational principle we described above operates on a wider stringy class $\mathcal{U}$ than it is in fact necessary. The surface integral formula [9] requires just the restricted subset $\mathcal{U}_{\gamma_{cl}} \subset \mathcal{U}$. The transition to $\mathcal{U}_{\gamma_{cl}}$ is advisable for two reasons. First, in the Lagrangian case with $\Omega = d(pdq - Hdt)$ we obtain an equivalence (modulo additive constant) between the least action principle using $S$ and the standard Hamilton least action principle. Explicitly,

$$\Sigma \in \mathcal{U}(\gamma_{cl}) \subset \mathcal{U}_{\gamma_{cl}} \Rightarrow S(\Sigma) = \left( \int_{\gamma_{cl}} - \int_{\gamma_{cl}} \right) (pdq - Hdt).$$

Second, the wider class $\mathcal{U}$ contains plenty of degenerate (shrunk) surfaces, the histories $\tilde{\gamma}$. These are obviously stationary surfaces of $S$ when being varied within $\mathcal{U}$, since

$$S(\tilde{\gamma}_{W}) - S(\tilde{\gamma}) = \int_{\tilde{\gamma}} \Omega - \int_{\gamma} \Omega = 0 - 0 = 0$$

for any variational vector field $W$. However, only one of these histories, namely $\gamma_{cl}$, satisfies both equations [A2] and [A3] at the same time. This fictitious problem is avoided when one works from the beginning with the stringy subclass $\mathcal{U}_{\gamma_{cl}} \subset \mathcal{U}$.

Appendix B: Surface functional integral – computational details

Let us explain here in some detail how the surface functional integral is computed. To be as tangible as possible consider a one-dimensional system only, so that the extended phase space will be the three-dimensional space $\mathbb{R}^{3}[q, p, t]$ (more dimensions represent only a technical problem). The particle is supposed to move under the combined action of the potential $U(q)$ and the friction force with a general $A$-power law. The dynamical equations are

$$\dot{q} = p, \quad \dot{p} = -\frac{d}{dq} U(q) - \kappa \dot{p}^{A},$$

and in addition to them, we require that the particle satisfies the boundary conditions $q(t_{0}) = q_{0}$ and $q(t_{1}) = q_{1}$. According to the definition [8], the two-form $\Omega$ is

$$\Omega = d(pdq - \frac{1}{2} p^{2} dt - U(q)dt) - \kappa p^{A} dq \wedge dt.$$

(B1)

Our aim is to compute the surface functional integral [9] over the stringy set $\mathcal{U}_{\gamma_{cl}} = \bigcup_{\Sigma} \mathcal{U} \Sigma_{cl}$. The direct application of [10] and [B1] together with the Stokes theorem yields

$$\int_{\Sigma} [D\Sigma] \exp \left\{ \frac{i}{\hbar} \int_{\Sigma} \Omega \right\}$$

(B2)

$$= \int_{\Sigma} [D\tilde{\gamma}] \exp \left\{ \frac{i}{\hbar} \left( \int_{\gamma_{cl}} - \int_{\gamma_{cl}} \right) (pdq - \frac{1}{2} p^{2} dt - U(q) dt) \right\} \times$$

$$\times \int_{\Sigma} [D\Sigma] \exp \left\{ - \frac{i}{\hbar} \kappa \int_{\Sigma} p^{A} dq \wedge dt \right\}.$$

The nontrivial part of this is the functional integral over the stringy subset $\Sigma_{cl}(\gamma_{cl})$. As mentioned earlier, $\Sigma \in \mathcal{U}(\gamma_{cl})$ is a map

$$\Sigma : [t_{0}, t_{1}] \times [0, 1] \rightarrow \mathbb{R}^{3}[q, p, t], \quad (t, s) \mapsto \Sigma(t, s),$$

such that for all $t \in [t_{0}, t_{1}]$ and $s \in [0, 1]$ there holds

$$\Sigma(t, 0) = \gamma_{cl}(t) \quad \text{and} \quad \Sigma(t, s) \in \mathbb{R}^{3}[q_{0}, p_{0}, t], \quad \Sigma(t_{1}, s) \in \mathbb{R}^{3}[q_{1}, p_{1}]$$

(B3)
The boundary values are required to be consistent with the choice of an appropriate integration measure. For the reason that will become clear in a moment, we choose the measure as

\[
\int_\Sigma \Delta q_{\tau,\sigma} \epsilon = \frac{A K \Delta}{2 \pi i h} p_{\tau,\sigma}^A \left( q_{\tau,\sigma} - q_{\tau,0} \right) \text{d} p_{\tau,\sigma} \ .
\]

The first step, when dealing with the discretized functional integral

\[
\begin{align*}
\text{Int} & = \int_\Sigma \exp \left\{ -i \frac{\hbar}{\kappa} \int_0^\tau q^A \text{d} q \wedge \text{d} t \right\}, \\
& = \int_\Sigma \exp \left\{ -i \frac{\hbar}{\kappa} \sum_{\tau=1}^{K-1} \sum_{\sigma=1}^{L-1} \Delta \left[ q_{\tau,\sigma} - q_{\tau,0} \right] \right\}.
\end{align*}
\]

The next step is the integration over the internal stringy positions \( q_{\tau,\sigma} \). After performing this trivial integration we arrive at the expression

\[
\int = \exp \left\{ -i \frac{\hbar}{\kappa} \sum_{\tau=1}^{K-1} \Delta \left[ q_{\tau,\sigma} - q_{\tau,0} \right] \right\}.
\]

In the exponent there appears the discretized version of the integral

\[
\int_{t_0}^{t_1} \text{d} t \left\{ \tilde{q}(t) p^A_{\tau,\sigma}(t) - q_{\sigma}(t) p^A_{\tau,\sigma}(t) \right\},
\]

where \( \tilde{q}(t) \) stands for the \( q \)-projection of \( \tilde{\gamma}(t) \). Consequently, after returning back to the continuum limit we obtain the following important result:

\[
\int_\Sigma \Delta q_{\tau,\sigma} \epsilon = \frac{A K \Delta}{2 \pi i h} \left( p_{\tau,\sigma}^A - p_{\tau,0}^A \right) \ .
\]

After substituting (B6) into the initial formula (B2) we recover a path integral in the extended phase space. The integral is quadratic in momenta, with the discretized (standard) Liouville measure

\[
\int \left[ D \gamma \right] \equiv \int_{-\infty}^{\infty} \text{d} p_{\tau,\sigma} \prod_{\tau=1}^{K-1} \prod_{\sigma=1}^{L-1} \left[ \frac{d \tilde{p}_{\tau,\sigma}}{2 \pi i h} \right] \ .
\]
The integration over momenta can be carried out explicitly and we finally obtain a path integral in the configuration space only (tildes are removed from the position variables),

\[
\int [\mathcal{D}\Sigma] \exp \left\{ \frac{i}{\hbar} \int_{t_0}^{t_1} \Omega \right\} = \exp \left\{ -\frac{i}{\hbar} \int_{t_0}^{t_1} \left( \frac{1}{2} \dot{q}_c^2 - U(q_c) - \kappa q_c p_c^2 \right) dt \right\} \times \int [\mathcal{D}q] \exp \left\{ \frac{i}{\hbar} \int_{t_0}^{t_1} \left( \frac{1}{2} \dot{q}^2 - U(q) - \kappa q p^2 \right) dt \right\}. \tag{B7}
\]

This formula served us as a guide when introducing the classical action \(\{13\}\).

Let us make one final comment. In our functional measure \(\{B5\}\) no integration over the momenta \(p_{(0, \sigma)}\) and \(p_{(K, \sigma)}\) was prescribed. These momenta are however needed when the boundary of the discretized surface \(\Sigma \in \mathcal{U}_{\sigma, cl}\) is specified. They define the auxiliary curves \(\lambda_0(s)\) and \(\lambda_1(s)\) introduced in section \(\{11\}\). The curves were chosen completely arbitrarily, but as seen from \(\{B4\}\), the result is not affected by them at all. Hence, since everything substantial is independent of \(p_{(0, \sigma)}\) and \(p_{(K, \sigma)}\), it is justifiable to discard these quantities from the functional measure \([\mathcal{D}\Sigma]\). If we would not do that for some reason, they will integrate into an artificial infinite factor, which will be removed anyway after we apply the normalization conditions \(\{2\}\).

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Quantization of Non-Lagrangian Systems
Path integral for the damped harmonic oscillator,

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For the record, the extended phase space is locally described by the coordinates in the configuration space $q^a$, the canonically conjugated momenta $p_a$, and the time coordinate $t$ (the index $a$ runs from 1 to the number of degrees of freedom $n$). One assumes the standard Poisson brackets $\{p_a, q^b\} = -\delta^b_a$. In a more elevated language, the extended phase space is the space $T^* M \times \mathbb{R}$, where $M$ stands for the configuration space and $\mathbb{R}$ for time.

The integrals over $\lambda_0$ and $\lambda_1$ give trivially zero contributions to the contour integral, since on both $\lambda$'s $dq = 0$ and $dt = 0$.

The precise definition is: suppose there are two histories $\tilde{\gamma}_0(t)$ and $\tilde{\gamma}_1(t)$ in the extended phase space whose projections $\text{Proj}(\tilde{\gamma}_0)$ and $\text{Proj}(\tilde{\gamma}_1)$ on the extended configuration space connect $(q_0, t_0)$ with $(q_1, t_1)$. Then $U(\tilde{\gamma}_0, \tilde{\gamma}_1)$ contains all continuous and oriented surfaces $\Sigma$ in the extended phase space such that

$$\Sigma(t, s = 0) = \tilde{\gamma}_0(t) \quad \text{and} \quad \text{Proj}(\Sigma(t = t_0, s)) = (q_0, t_0)$$

$$\Sigma(t, s = 1) = \tilde{\gamma}_1(t) \quad \text{and} \quad \text{Proj}(\Sigma(t = t_1, s)) = (q_1, t_1).$$

If $\tilde{\gamma}_0 \neq \tilde{\gamma}_1$, the orientation of $\Sigma$ is supposed to be such that $\tilde{\gamma}_1 - \tilde{\gamma}_0$ is a part of the boundary of $\Sigma$. The Projection is simply the momentum forgetting map, or to put it in a more elevated way, the canonical projection from the cotangent bundle to the base manifold. A special case is the set $U(\tilde{\gamma}, \tilde{\gamma})$, comprised of all closed surfaces in the extended phase space containing $\tilde{\gamma}$. In what follows, we consider any history $\tilde{\gamma}$ as a degenerated (shrunken) closed surface, hence $\tilde{\gamma} \in U(\tilde{\gamma}, \tilde{\gamma})$ by definition.

Strictly speaking, the expression for the friction force can be used only for the motion with increasing $q$. The universally applicable expression is $-\kappa \text{sgn}(\dot{q})|\dot{q}|^\alpha$.

The prefix $A$ in the subscript $Acl$, as well as in the subscript $Astr$ introduced later, refers to the power in the expression for the friction force.

The external source in the functional integral $\int d\lambda$ can be viewed as a sort of additional time-dependent potential, therefore one would expect that the Chapman-Kolmogorov equation (3) will be satisfied. However, this argument is not applicable in the present situation. The point is that if we merge two paths $KL$ and $LM$ into a new path $KM$, then the time-dependent potentials $-\kappa p_{\dot{q}} - KL(t)$ and $-\kappa p_{\dot{q}} - LM(t)$ differ, for a general position of $L$, from the potential $-\kappa p_{\dot{q}} - KM(t)$ in the corresponding time (sub)intervals,

$$-\kappa p_{\dot{q}} - KL(t) \neq -\kappa p_{\dot{q}} - LM(t) \neq -\kappa p_{\dot{q}} - KM(t) \quad \text{for} \ t \in [t_K, t_L]$$

Since this differs substantially from the standard case, the argument about the validity of the Chapman-Kolmogorov equation cannot be used.

An exhaustive list of various quantization techniques applicable in the presence of linear dissipation, including those mentioned in the introduction, can be found in [19] and [20].

The ordering problem and its consequences are not discussed here.

Since the traveling wave $|p_0\rangle$ is not a square integrable function, we evolved by $A_{str}(q, t_1|q', t_0)$ a modified (normalized) state $|\Psi_\epsilon\rangle = \int dp e^{-i(p-p_0)^2/\epsilon} |p\rangle$ ($\epsilon > 0$). After the stringy evolution of $|\Psi_\epsilon\rangle$, the expectation value of the momentum was computed and the regularization parameter $\epsilon$ was set to zero.

The situation here strongly resembles that discussed in section [11]. However, there is one substantial difference between the two cases. In the present case, we allow the $\lambda$-curves to vary in both coordinates $q$ and $p$, only the time $t$ must stay unchanged along them. In the case considered in section [11] we were more stringent: the $\lambda$-curves were allowed to vary only with respect to $p$ and had to stay unchanged with respect to both $q$ and $t$.

From now on we are using a new symbol $f$ for all discretized integrals. In the continuum limit $K$ and $L$ approach independently infinity and $f \rightarrow f$. However, we must keep in mind that we must perform simultaneously the limits $\Delta \rightarrow 0$ and $\epsilon \rightarrow 0$ so that the quantities $K_\Delta = t_1 - t_0$ and $L_\epsilon = 1$ stay finite.