ZEROS OF RELIABILITY POLYNOMIALS AND \( f \)-VECTORS OF MATROIDS

DAVID G. WAGNER

Abstract. For a finite multigraph \( G \), the reliability function of \( G \) is the probability \( R_G(q) \) that if each edge of \( G \) is deleted independently with probability \( q \) then the remaining edges of \( G \) induce a connected spanning subgraph of \( G \); this is a polynomial function of \( q \). In 1992, Brown and Colbourn conjectured that for any connected multigraph \( G \), if \( q \in \mathbb{C} \) is such that \( R_G(q) = 0 \) then \( |q| \leq 1 \). We verify that this conjectured property of \( R_G(q) \) holds if \( G \) is a series-parallel network. The proof is by an application of the Hermite-Biehler Theorem and development of a theory of higher-order interlacing for polynomials with only real nonpositive zeros. We conclude by establishing some new inequalities which are satisfied by the \( f \)-vector of any matroid without coloops, and by discussing some stronger inequalities which would follow (in the cographic case) from the Brown-Colbourn Conjecture, and are hence true for cographic matroids of series-parallel networks.

0. Introduction

Given a finite multigraph \( G = (V, E) \) and \( 0 \leq q \leq 1 \), let \( G(q) \) denote a random spanning subgraph of \( G \) obtained by deleting each edge of \( G \) independently with probability \( q \). The reliability function of \( G \) is the probability \( R_G(q) \) that \( G(q) \) is connected, considered as a function of \( q \). (Trivially, if \( G \) is not connected then \( R_G(q) \equiv 0 \) identically.) In fact (as we see in Section 1) this is a polynomial function of \( q \). In 1992, Brown and Colbourn \( [6] \) made the following fascinating conjecture. A polynomial \( S(q) \) is Schur quasi-stable if every \( q \in \mathbb{C} \) for which \( S(q) = 0 \) is such that \( |q| \leq 1 \); for the relevance of this concept to solutions of linear finite difference equations, see Theorem 3.2 of Barnett \( [4] \).

Conjecture 0.1 (Brown-Colbourn). For any connected multigraph \( G \), the reliability polynomial \( R_G(q) \) is Schur quasi-stable.

In support of this conjecture, Brown and Colbourn verify that this property holds for all simple graphs on up to six vertices, and show that for every multigraph \( G \) there is a multigraph \( G' \) which is obtained from \( G \) by repeatedly subdividing edges, and for which \( R_{G'}(q) \) is Schur quasi-stable. The proofs in \( [6] \) are based on the Eneström-Kakeya Theorem, which gives a sufficient condition for a polynomial with real coefficients to be Schur quasi-stable. As Brown and Colbourn remark, however, there are multigraphs for which the Eneström-Kakeya Theorem fails to
show that $R_G(q)$ is Schur quasi-stable. Some other explanation must be sought if the Brown-Colbourn Conjecture is to be proven.

In this paper we give some indications that the Hermite-Biehler Theorem can provide such an explanation. This theorem is a necessary and sufficient condition for a polynomial $P(u)$ with real coefficients to be such that all its zeros have nonpositive real part; by a fractional linear transformation we can map the unit disc to left half-plane and apply the Hermite-Biehler Theorem to reliability polynomials. Informally, the condition is that if $P(u)$ is expanded into its even and odd parts, $P(u) = P_0(u^2) + uP_1(u^2)$, then the polynomials $P_1(x)$ and $P_0(x)$ have all their zeros on the nonpositive part of the real axis, and these zeros “interlace” (in a sense we make precise in Section 2). This allows the well-developed theory of polynomials with only real zeros to be applied to Conjecture 0.1, but even this is not sufficient. More precisely, this theory must be developed further in order to obtain significant results on reliability polynomials.

The key extension of technique in this paper is the introduction of a useful concept of higher order interlacing for polynomials with only real nonpositive zeros; this involves the definition of “interpolatory hypercubes of polynomials” of any dimension. However, because of the complexity of the relations derived from two-vertex-cut reduction for reliability polynomials, we have applied this theory here for interpolatory cubes only up to dimension four. The limited scope of Theorem 0.2 in relation to the generality of Conjecture 0.1 reflects only this artificial restriction, and should not be interpreted as stemming from some intrinsic limitation of the method.

The class $\mathcal{SP}$ of series-parallel networks is defined recursively as follows. Every multigraph $G$ in $\mathcal{SP}$ has a distinguished unordered pair $\{s, t\}$ of distinct vertices, called the terminals of $G$. If $G$ consists of just one edge connecting its terminals, then $G$ is in $\mathcal{SP}$. Let $G$ and $G'$ be in $\mathcal{SP}$, with terminals $\{s, t\}$ and $\{s', t'\}$, respectively. If $G$ and $G'$ have no edges in common, and only the vertex $t = s'$ in common, then $G \cup G'$ is in $\mathcal{SP}$, and is called a series connection of $G$ and $G'$; its terminals are $\{s, t'\}$. If $G$ and $G'$ have no edges in common, and only the vertices $s = s'$ and $t = t'$ in common, then $G \cup G'$ is in $\mathcal{SP}$, and is called a parallel connection of $G$ and $G'$; its terminals are $\{s, t\}$. Let $\mathcal{SP}'$ denote the class of connected multigraphs every two-connected component of which is in the class $\mathcal{SP}$ (for some choice of terminals).

**Theorem 0.2.** If the multigraph $G$ is in the class $\mathcal{SP}'$ then $R_G(q)$ is Schur quasi-stable.

Since at least the early 1970s there has been some interest in obtaining inequalities valid for the $f$-vectors and/or the $h$-vectors of simplicial complexes belonging to various classes; in part, this developed from similar investigations in the 19th century into the combinatorial geometry of convex polytopes. Sections II.2, II.3, III.1, and III.3 of Stanley [15] provide an excellent overview of these results. Also, Section 5 of Björner [6] considers in detail the case of matroids, and Ball and Provan [3] and Colbourn [12] discuss the application of these ideas to estimation of network reliability. In contrast with the cases of Cohen-Macaulay complexes or simplicial polytopes, the case of matroids is still only rather poorly understood; some recent results in this direction are Brown and Colbourn [8] and Chari [10, 11]. In fact, Conjecture 0.1 implies numerous strong inequalities for the $f$-vector of a cographic matroid, as we shall see in Section 6; thus, by Theorem 0.2, these inequalities
hold for cographic matroids of multigraphs in the class $\mathfrak{P}'$. Moreover, it is an elementary consequence of Chari’s recent work \[10, 11\] that the weakest of these inequalities hold more generally for matroids.

**Theorem 0.3.** Let $M$ be (the set of independent sets of) a matroid of rank $d$, and for $0 \leq i \leq d$ let $f_i$ denote the number of $i$-element sets in $M$. If $M$ has no coloops then for all $0 \leq k \leq d$,

$$0 \leq \sum_{i=k}^d \binom{i}{k} (-2)^{d-i} f_i.$$

These inequalities are violated by some simplicial polytopes and some broken-circuit complexes, and are satisfied with equality for all $0 \leq k < d$ if $M$ is a direct sum of 2-circuits. It thus appears that a better understanding of the phenomena underlying Conjecture 0.1 could lead not only to improved methods for estimating network reliability, but perhaps toward a set of strong necessary conditions on the $f$-vectors of matroids in general.

In Section 1 we review the bare essentials of the combinatorics of reliability polynomials, the deletion/contraction algorithm and two-vertex-cut reduction, and we translate Conjecture 0.1 into a form to which the Hermite-Biehler Theorem applies. (No familiarity with matroid theory is assumed until Section 6.) In Section 2 we review the Hermite-Biehler Theorem and state the lemmas on polynomials with only real zeros which are useful. In Section 3 we sketch how just this amount of theory can be used to verify Conjecture 0.1 for all multigraphs such that the underlying simple graph of every two-connected component is an edge or a cycle. Section 4 contains the new theoretical development of the paper; in Section 5 we apply this technique to prove Theorem 0.2. We conclude in Section 6 with a discussion of reliability polynomials in the more general context of Cohen-Macaulay complexes, for which we assume familiarity with the standard concepts. Readers interested specifically in Theorem 0.3 can skip directly to Section 6.

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## 1. Reliability Polynomials

For a more thorough introduction to reliability polynomials, see Colbourn \[12\]. By a **multigraph** we mean a finite graph which may possess both loops and multiple edges. It is clear that for multigraphs $G$ and $G'$,

$$\text{if } G \simeq G' \text{ then } R_G(q) = R_{G'}(q).$$

(1.1)

If $G$ and $N$ are multigraphs with exactly one vertex in common then

$$R_{G\cup N}(q) = R_G(q)R_N(q),$$

(1.2)

as follows directly from the definition. For a multigraph $G = (V, E)$ and any $e \in E$ let $G \setminus e$ denote $G$ with $e$ deleted and let $G/e$ denote $G$ with $e$ contracted; then

$$R_G(q) = qR_{G\setminus e}(q) + (1 - q)R_{G/e}(q),$$

(1.3)

since the conditional probability that $G(q)$ is connected given that $e$ is deleted is $R_{G\setminus e}(q)$, and the conditional probability that $G(q)$ is connected given that $e$ is not deleted is $R_{G/e}(q)$. If $G'$ is obtained from $G$ by removing all loops, then

$$R_G(q) = R_{G'}(q),$$

(1.4)
since if $e$ is a loop of $G$ then $G \setminus e \simeq G/e$, and we can apply (1.1) and (1.3) and
induction on the number of loops of $G$. Henceforth by a network we shall mean a
finite connected graph which has no loops but may have multiple edges.

For a network $G$ we denote by $G^2$ the underlying simple graph of $G$, which has
the same vertices as $G$ and one edge $u \sim v$ for every pair of vertices \{u, v\} which are
adjacent in $G$. A spindle in a network $G$ is a (nonempty) maximal set of edges in $G$
all of which are incident with the same pair of (distinct) vertices of $G$; if the spindle
has $c$ edges then we say it is a $c$-spindle. There is an obvious natural bijection
between the spindles of $G$ and the edges of $G^2$. If $\sigma$ is a spindle in $G$ then let $G \setminus \sigma$
be obtained from $G$ by deleting all the edges of $\sigma$, and let $G/\sigma$ be obtained from $G$
by contracting all the edges of $\sigma$ (notice that $G/\sigma$ has no loops). If $\sigma$ is a $c$-spindle
in the network $G$ then

\[(1.5) \quad R_G(q) = q^c R_{G \setminus \sigma}(q) + (1 - q^c)R_{G/\sigma}(q),\]

as follows from (1.3) and (1.4) by induction on $c$.

As examples, let $kT_n$ denote a network for which the underlying simple graph is
a tree with $n$ vertices and in which each spindle has $k$ edges, and let $kC_n$ denote
a network for which the underlying simple graph is a cycle with $n$ vertices and in
which each spindle has $k$ edges. Since $R_{kT_n}(q) = 1 - q^k$, it follows by (1.2) and
induction on $n$ that $R_{kT_n}(q) = (1 - q^k)^{n-1}$ for all $k \geq 1$ and $n \geq 2$. From this,
(1.5), and induction on $n$ it follows that $R_{kC_n}(q) = (1 - q^k)^{n-1}(1 + (n - 1)q^k)$
for all $k \geq 1$ and $n \geq 3$. As noted by Brown and Colbourn [6] (Proposition 5.1)
these examples suffice to show that the closure of the set of all zeros of reliability
polynomials contains the whole unit disc \( \{q \in \mathbb{C} : |q| \leq 1\} \).

Generalizing (1.3) and (1.5), let $G$ and $N$ be two networks which intersect in
exactly two vertices $v$ and $w$, and let $G^\bullet$ and $N^\bullet$ be obtained by identifying $v$ and
$w$ in $G$ and in $N$, respectively, and removing any loops thus produced. Then

\[(1.6) \quad R_{G \cup N}(q) = R_G(q)R_{N^\bullet}(q) + R_G^\bullet(q)R_N(q) - R_G(q)R_N(q),\]

since the conditional probability that $G(q) \cup N(q)$ is connected given that $G(q)$ is
connected is $R_{N^\bullet}(q)$, the second term has a similar interpretation, the third term
corrects double counting of the case that both $G(q)$ and $N(q)$ are connected, and
if neither $G(q)$ nor $N(q)$ is connected then $G(q) \cup N(q)$ is not connected.

For a network $G$ we denote by $m_G$ the number of edges of $G$ and by $n_G$ the
number of vertices of $G$, and we let $d_G := m_G - n_G + 1$; we omit the subscript when
no confusion can arise. From (1.5) it follows by induction on $m$ that $R_G(q)$ is a
polynomial in $\mathbb{Z}[q]$ of degree $m$, and the multiplicity of $q = 1$ as a zero of $R_G(q)$ is
at least $n - 1$. In view of this, for each network $G$ we may define the polynomial

\[(1.7) \quad H_G(q) := (1 - q)^{1-n}R_G(q)\]

in $\mathbb{Z}[q]$. The Brown-Colbourn Conjecture is equivalent to the claim that for any
network $G$,

\[(1.8) \quad \text{if } q \in \mathbb{C} \text{ is such that } H_G(q) = 0 \text{ then } |q| \leq 1.\]

It follows from (1.2) and (1.7) that if $G$ and $N$ are networks with exactly one vertex
in common then

\[(1.9) \quad R_{G \cup N}(q) = H_G(q)H_N(q).\]
If $\sigma$ is a $c$-spindle in the network $G$ then

\begin{equation}
H_G(q) = q^d H_G(\cdot, \sigma) + \left( \frac{1 - q^c}{1 - q} \right) H_{G/\sigma}(q),
\end{equation}

as follows from (1.5) and (1.7). Similarly, with notation as in (1.6) we see that

\begin{equation}
H_{G \cup N}(q) = H_G(q) H_N(\cdot, q) + H_G(\cdot, q) H_N(q) - (1 - q) H_G(q) H_N(q).
\end{equation}

It follows from (1.10) by induction on $m$ that $H_G(q)$ is a polynomial of degree $d$ with nonnegative integer coefficients, and the constant term of $H_G(q)$ is 1. (In fact, the coefficients of $H_G(q)$ form the $h$-vector of the cographic matroid of $G$ and have been studied extensively in the context of Cohen-Macaulay simplicial complexes. We shall return to this point in Section 6.)

We make a change of variables $u := (-1 - q)/(1 - q)$ and conversely $q = (1 - u)/(1 - u)$, and define

\begin{equation}
J_G(u) := (u - 1)^d H_G \left( \frac{-1 - u}{1 - u} \right),
\end{equation}

so that

\begin{equation}
H_G(q) = \left( \frac{q - 1}{2} \right)^d J_G \left( \frac{-1 - q}{1 - q} \right).
\end{equation}

From (1.12) and (1.13) it follows that (1.8) is equivalent to

\begin{equation}
\text{if } u \in \mathbb{C} \text{ is such that } J_G(u) = 0 \text{ then } \Re(u) \leq 0.
\end{equation}

From (1.9) and (1.12) it follows that if $G$ and $N$ are networks with exactly one vertex in common then

\begin{equation}
J_{G \cup N}(u) = J_G(u) J_N(u).
\end{equation}

It follows from (1.10) and (1.12) that if $\sigma$ is a $c$-spindle in the network $G$ then

\begin{equation}
J_G(u) = (u + 1)^c J_{G \cdot \sigma}(u) + \left( \frac{(u + 1)^c - (u - 1)^c}{2} \right) J_{G/\sigma}(u).
\end{equation}

Similarly, with notation as in (1.6) and (1.11) we see that

\begin{equation}
J_{G \cup N}(u) = J_G(u) |J_N(u) + J_N(\cdot, u)| + |J_G(u) + J_G(\cdot, u)| J_N(u).
\end{equation}

From (1.16) and induction on $m$ it follows that $J_G(u)$ has nonnegative integer coefficients, but no combinatorial interpretation of these integers is known; we shall return to this point as well in Section 6.

2. The Hermite-Biehler Theorem

The Hermite-Biehler Theorem is a very useful criterion which determines whether a polynomial with real coefficients has all its zeros in the left half-plane. For a nonzero $P(u) \in \mathbb{R}[u]$, if every $u \in \mathbb{C}$ such that $P(u) = 0$ satisfies $\Re(u) \leq 0$ then $P(u)$ is Hurwitz quasi-stable. (For the relevance of this concept to solutions of linear ordinary differential equations, see Theorem 3.1 of Barnett.)

Suppose that $A, B \in \mathbb{R}[x]$ both have only real zeros, that those of $A$ are $\xi_1 \leq \cdots \leq \xi_a$ and that those of $B$ are $\theta_1 \leq \cdots \leq \theta_b$. We say that $A$ interlaces $B$ if $\deg B = 1 + \deg A$ and the zeros of $A$ and $B$ satisfy $\theta_1 \leq \xi_1 \leq \theta_2 \leq \cdots \leq \xi_a \leq \theta_{a+1}$. We also say that $A$ alternates left of $B$ if $\deg A = \deg B$ and the zeros of $A$ and $B$ satisfy $\xi_1 \leq \theta_1 \leq \xi_2 \leq \cdots \leq \xi_a \leq \theta_a$. We use the notation $A \prec B$ for “either $A$ interlaces $B$ or $A$ alternates left of $B$.” (Any polynomial which stands in this
relation a fortiori has only real zeros.) This is a closed condition in the sense that if \( A_n \) and \( B_n \) are sequences of polynomials converging to \( A \) and \( B \), respectively, and if \( A_n \prec B_n \) for all \( n \geq 0 \), then \( A \prec B \). By convention we say that for any polynomial \( A \) with only real zeros, both \( A \prec 0 \) and \( 0 \prec A \) hold. A polynomial is standard when either it is identically zero or its leading coefficient is positive. For brevity, we say that a polynomial has only nonpositive zeros to indicate that either it is identically zero or all of its zeros are real and nonpositive. Henceforth, if we omit the argument of a polynomial then we intend that it is a function of the variable \( x = u^2 \).

**Theorem 2.1** (Hermite-Biehler). Let \( P(u) = P_0(u^2) + uP_1(u^2) \in \mathbb{R}[u] \) be standard. Then \( P(u) \) is Hurwitz quasi-stable if and only if both \( P_0 \) and \( P_1 \) are standard, have only nonpositive zeros, and \( P_1 \prec P_0 \).

The proof of the Hermite-Biehler Theorem in Gantmacher [13] covers only the case of polynomials for which all zeros have strictly negative real part, but the statement given here can be deduced from it easily by a limiting argument. The following lemmas will be useful; Lemmas 2.2 and 2.3 can be proven using the techniques from Section 3 of [16] and Lemma 2.4 can be proven using the techniques from Section 5 of [16].

**Lemma 2.2.** Let \( P_1, \ldots, P_k \) be polynomials in \( \mathbb{R}[x] \), all of which have only real zeros and none of which is identically zero. If \( P_1 \prec P_2 \prec \cdots \prec P_k \) and \( P_1 \prec P_k \) then \( P_i \prec P_j \) for all \( 1 \leq i \leq j \leq k \).

**Lemma 2.3.** Let \( A, P, Q \) be standard polynomials in \( \mathbb{R}[x] \) which have only nonpositive zeros, and assume that \( A \neq 0 \).

(a) \( P \prec Q \) if and only if \( Q \prec xP \).
(b) If \( A \prec P \) and \( A \prec Q \) then \( A \prec P + Q \).
(c) If \( P \prec A \) and \( Q \prec A \) then \( P + Q \prec A \).
(d) If \( P \prec Q \) then \( P \prec P + Q \prec Q \).

**Lemma 2.4.** Let \( P, Q \) be standard polynomials in \( \mathbb{R}[x] \) which have only nonpositive zeros. Then \( P \prec Q \) if and only if for all \( \lambda, \rho > 0 \), both \( \lambda P + \rho Q \) and \( \lambda Q + \rho xP \) have only nonpositive zeros.

3. **Thick Cacti**

Returning to the case of a network \( G \), we define polynomials \( J_G^1 \) and \( J_G^0 \) in \( \mathbb{N}[x] \) by separating \( J_G(u) \) into its odd and even parts, respectively:

\[
J_G(u) = J_0^G(u^2) + uJ_1^G(u^2).
\]

From (3.1) and the Hermite-Biehler Theorem it follows that (1.8) and (1.14) are each equivalent to

\[
J_1^G \prec J_0^G.
\]

From (3.1) and (3.1) it follows that if \( G \) and \( N \) are networks with exactly one vertex in common then

\[
\begin{align*}
J_{G \cup N}^0 &= J_0^G J_0^N + xJ_1^G J_1^N, \\
J_{G \cup N}^1 &= J_0^G J_1^N + J_1^G J_0^N.
\end{align*}
\]
For each natural number $c$ we define $E_c$ and $O_c$ in $\mathbb{N}[x]$ by
\begin{equation}
(u + 1)^c = E_c(u^2) + uO_c(u^2).
\end{equation}

From (1.16), (3.1), and (3.4) it follows that if $\sigma$ is a $c$-spindle in the network $G$ then for $c$ even
\begin{equation}
\begin{aligned}
J^G_0 &= E_cJ^G_0 \sigma + xO_cJ^G_1 \sigma + xO_cJ^G_1 \sigma, \\
J^G_1 &= E_cJ^G_1 \sigma + O_cJ^G_1 \sigma + O_cJ^G_1 \sigma,
\end{aligned}
\end{equation}

and for $c$ odd
\begin{equation}
\begin{aligned}
J^G_0 &= E_cJ^G_0 \sigma + xO_cJ^G_1 \sigma + E_cJ^G_1 \sigma, \\
J^G_1 &= E_cJ^G_1 \sigma + O_cJ^G_1 \sigma + E_cJ^G_1 \sigma.
\end{aligned}
\end{equation}

With $G$, $G^*$, $N$, and $N^*$ as in (1.6), (1.11), and (1.17), let $J(u) := J_G(u)$ and $J_V(u) := J_G(u) + J_G^*(u)$ and $K(u) := J_N(u)$ and $K_V(u) := J_N(u) + J_N^*(u)$; we find that
\begin{equation}
\begin{aligned}
J^G \cup J_V &= J_0 K_0^0 + J_0 K_1^0 + xJ_1 K_0^0 + xJ_1 K_1^0, \\
J^G \cup K_V &= J_0 K_0^0 + J_0 K_1^0 + J_1 K_0^0 + J_1 K_1^0.
\end{aligned}
\end{equation}

**Lemma 3.1.** For any natural numbers $a$ and $b$, $O_{a+b} \prec E_{a+b}$, $E_aO_b \prec E_{a+b}$, $O_aE_b \prec E_{a+b}$, and $O_aO_b \prec O_{a+b} \prec E_aE_b$.

**Proof.** We proceed by induction on $a + b$, the base $a + b = 1$ being evident. If $a = 0$ then the only nontrivial claim is that $O_b \prec E_b$, since $O_0 = 0$ and $E_0 = 1$; we can prove this claim by considering the case $a' := 1$ and $b' := b - 1$ instead. Similarly, we may also dispense with the case $b = 0$, so assume that $a \geq 1$ and $b \geq 1$. From $(u + 1)^{a+b} = (u + 1)^a(u + 1)^b$ it follows that
\begin{equation}
\begin{aligned}
E_{a+b} &= E_aE_b + xO_aO_b, \\
O_{a+b} &= E_aO_b + O_aE_b.
\end{aligned}
\end{equation}

By induction, we have $O_a \prec E_a$ and $O_b \prec E_b$, and hence $O_aO_b \prec E_aO_b \prec E_aE_b$ and $O_aO_b \prec E_aE_b \prec O_aE_b$. Lemma 2.3 now implies the result. \qed

A **cactus** is a connected simple graph in which each edge is contained in at most one cycle. We now consider the special case of networks $G$ such that $G^2$ is a cactus. It is convenient to introduce the notations, for each natural number $c$,
\begin{equation}
\varepsilon(c) := \begin{cases}
1 & \text{if } c \text{ is even}, \\
0 & \text{if } c \text{ is odd},
\end{cases}
\end{equation}
and
\begin{equation}
\delta(c) := 1 - \varepsilon(c),
\end{equation}

and
\begin{equation}
S_c := \begin{cases}
O_c & \text{if } c \text{ is even}, \\
E_c & \text{if } c \text{ is odd}.
\end{cases}
\end{equation}

Denoting a network with two vertices and $m$ edges by $mT_2$ we have for all positive integers $m$,
\begin{equation}
J^{mT_2}_{\varepsilon(m)} = S_m \quad \text{and} \quad J^{mT_2}_{\delta(m)} \equiv 0.
\end{equation}

For a finite list of positive integers $c := (c_1, \ldots, c_{n-1})$, let $T[c]$ denote any network $T$ such that $T^2$ is a tree and the sizes of the spindles of $T$ are given by the list $c$ (so $nT = n$). From (3.3), (3.11), and induction on $n$ it follows that
\begin{equation}
J^{T[c]}_{\varepsilon(n+m)} = x^{\varepsilon(c)} S_{c_1} \cdots S_{c_{n-1}} \quad \text{and} \quad J^{T[c]}_{\delta(n+m)} \equiv 0,
\end{equation}

RELIABILITY POLYNOMIALS 7
Proposition 3.2. If $G$ is a network such that either $J_1^G \equiv 0$ or $J_0^G \equiv 0$ then $G^3$ is a tree.

Proof. If $G^3$ is not a tree then let $\sigma$ be a spindle of $G$ corresponding to an edge of $G^3$ which is not a cut-edge. Since $G \setminus \sigma$ is connected, $R_{G \setminus \sigma}(u) \neq 0$, so $J_{G \setminus \sigma}(u) \neq 0$, so either $J_1^{G \setminus \sigma} \neq 0$ or $J_0^{G \setminus \sigma} \neq 0$. Since all $J$-polynomials of networks have nonnegative coefficients, (3.5) and (3.6) imply that $J_1^G \equiv 0$ and $J_0^G \equiv 0$.

Lemmas 2.3 and 3.1, formulas (3.5), (3.6), and (3.12), and induction on $n$ suffice to prove Theorem 3.3; we omit the details since we obtain the most interesting part of Theorem 3.3 (the condition $J_1^{C^3[c]} \prec J_0^{C^3[c]}$) as a special case of Corollary 5.3. The crucial simplifying feature in the proof of Theorem 3.3 is that, in the notation of (3.12), $J_{\delta(n+m)}^T \equiv 0$.

Theorem 3.3. Let $c := (c_1, \ldots, c_n)$ be a sequence of $n \geq 3$ positive integers, and let $C^3[c]$ denote any network with spindles of sizes $c_1, \ldots, c_n$ such that the underlying simple graph is a cycle. Then $J_{\delta(n+m)} = (n-1)x^{c_1}[S_{c_1} \cdots S_{c_n}]$ and $J_1^{C^3[c]} \prec J_0^{C^3[c]}$, where $m = c_1 + \cdots + c_n$.

Corollary 3.4. Let $G$ be a network for which $G^3$ is a cactus. Then $R_G(q)$ is Schur quasi-stable.

Proof. By (1.8) and (1.9) it suffices to prove the result for two-connected networks. A two-connected network $G$ satisfying the hypothesis is such that $G^3$ is either an edge or a cycle. The result follows from (3.2), (3.11), and Theorem 3.3.

4. Interpolatory Hypercubes of Polynomials

An interpolatory $0$-cube is a standard polynomial $A$ which has only nonpositive zeros. An interpolatory $1$-cube is a pair $(A, B)$ of standard polynomials which have only nonpositive zeros and are such that $A \prec B$. We present the theory next for interpolatory squares and then generalize to higher-dimensional hypercubes. The starting point is an analogue of the “Box Lemma,” Theorem 5.4 of [16].

Proposition 4.1. Let $A, B, P, Q$ be in $\mathbb{R}[x]$, and consider the following two conditions:

$C_1$: For any $\lambda, \rho > 0$ : $\lambda A + \rho B \prec \lambda P + \rho Q$ and $\lambda B + \rho x A \prec \lambda Q + \rho x P$ are interpolatory $1$-cubes.

$C_2$: For any $\kappa, \pi > 0$ : $\kappa A + \pi P \prec \kappa B + \pi Q$ and $\kappa P + \pi x A \prec \kappa Q + \pi x B$ are interpolatory $1$-cubes.

These conditions $C_1$ and $C_2$ are equivalent.

Proof. It follows from Lemma 2.4 that each of the conditions $C_1$ and $C_2$ is equivalent to the condition that for all $\kappa, \lambda, \pi, \rho > 0$, each of the polynomials $\kappa A + \kappa B + \pi \lambda P + \pi \rho Q$, $\kappa \lambda B + \kappa \rho x A + \pi \lambda Q + \pi \rho x P$, $\kappa \lambda P + \kappa \rho Q + \pi \lambda x A + \pi \rho x B$, and $\kappa \lambda Q + \kappa \rho x P + \pi \lambda x B + \pi \rho x^2 A$ have only nonpositive zeros.

Notice that if the conditions of Proposition 4.1 hold then, since $\prec$ is a closed condition, in fact they hold for all $\kappa, \lambda, \pi, \rho \geq 0$. 


In the diagrams which follow it is convenient to use an arrow $A \rightarrow B$ to denote $A \prec B$. If the equivalent conditions of Proposition 4.1 hold for the polynomials $A, B, P, Q$ we say that

$$B \rightarrow Q$$
$$\uparrow \bigoplus \uparrow$$
$$A \rightarrow P$$

is an interpolatory square, and use the notation $\bigoplus$ to indicate this. Notice that if one of the three squares

$$B \rightarrow Q \uparrow \bigoplus \uparrow$$
$$A \rightarrow P \uparrow \bigoplus \uparrow$$

is interpolatory then all three are, by Lemma 2.3.

**Lemma 4.2.** Let $A, B, P, Q$ be in $\mathbb{R}[x]$. If $A \prec B$ and $P \prec Q$ are interpolatory 1-cubes then

$$BP \rightarrow BQ$$
$$\uparrow \bigoplus \uparrow$$
$$AP \rightarrow AQ$$

**Proof.** Condition $C_1$ of Proposition 4.1 is verified easily by using Lemma 2.3.

**Lemma 4.3.** Consider $A, B, P, Q, S, T$ in $\mathbb{R}[x]$, with either $A \not\equiv 0$ or $B \not\equiv 0$.

$$T \leftarrow B \rightarrow Q \quad B \rightarrow Q + T$$
$$S \leftarrow A \rightarrow P \quad A \rightarrow P + S$$

If $\uparrow \bigoplus \uparrow \bigoplus \uparrow$ then $\uparrow \bigoplus \uparrow \bigoplus$.

**Proof.** For any $\lambda, \rho > 0$ we have $\lambda A + \rho B \prec \lambda P + \rho Q$ and $\lambda A + \rho B \prec \lambda S + \rho T$ since the squares are interpolatory. Thus, $\lambda A + \rho B \prec \lambda(P + S) + \rho(Q + T)$ by Lemma 2.3. Also, $\lambda B + \rho xA \prec \lambda Q + \rho xP$ and $\lambda B + \rho xA \prec \lambda T + \rho xS$ since the squares are interpolatory. Thus, $\lambda B + \rho xA \prec \lambda(Q + T) + \rho x(P + S)$ by Lemma 2.3. We have verified condition $C_1$ of Proposition 4.1, and hence the result.

It follows from Lemma 4.3 and (4.1) that, under the hypothesis of Lemma 4.3,

$$T \rightarrow B \leftarrow Q \quad Q + T \rightarrow B$$

if $\uparrow \bigoplus \uparrow \bigoplus \uparrow$ then $\uparrow \bigoplus \uparrow \bigoplus$.

**Lemma 4.4.** Let $A, B$ be in $\mathbb{R}[x]$. If $A \prec B$ is an interpolatory 1-cube then

$$B \rightarrow xA$$
$$\uparrow \bigoplus \uparrow$$
$$A \rightarrow B$$

**Proof.** If either $A \equiv 0$ or $B \equiv 0$ then the result is trivial, so assume that $A \not\equiv 0$ and $B \not\equiv 0$. For any $\lambda, \rho > 0$, Lemma 2.3 implies that $A \prec \lambda A + \rho B \prec B \prec \lambda B + \rho xA \prec xA$, and since $A \prec xA$ it follows from Lemma 2.2 that $\lambda A + \rho B \prec \lambda B + \rho xA$. Also by Lemma 2.3, the condition that $AB + \rho xA \prec \lambda xA + \rho xB$ is equivalent to the condition that $\lambda A + \rho B \prec \lambda B + \rho xA$, which we have just shown, and so condition $C_1$ of Proposition 4.1 is verified.
Using Lemmas 4.2, 4.3, and 4.4, one may adapt the proof of Lemma 3.1 to show that for all \( a \geq 0 \) and \( b \geq 0 \):

\[
\begin{align*}
E_0 O_b & \quad \rightarrow \quad E_{a+b} \quad \rightarrow \quad xO_a E_b \\
\uparrow & \quad \quad \uparrow \quad \quad \uparrow \quad \quad \uparrow \\
O_a O_b & \quad \rightarrow \quad O_{a+b} \quad \rightarrow \quad E_a E_b
\end{align*}
\]

(4.3)

To extend these ideas from squares to hypercubes of any dimension we must first introduce some notation. Fix an integer \( k \geq 0 \), and let \( P : (\mathbb{Z}/2\mathbb{Z})^k \rightarrow \mathbb{R}[x] \). If \( k \geq 1 \) then for \( \lambda, \rho \geq 0 \) we let \( I_\lambda^\rho \) denote the operator which maps \( P \) to the \( 2^{k-1} \) polynomials \( Q : (\mathbb{Z}/2\mathbb{Z})^{k-1} \rightarrow \mathbb{R}[x] \) given by \( Q_\alpha := \lambda P_{1\alpha} + \rho P_{0\alpha} \) for all \( \alpha \in (\mathbb{Z}/2\mathbb{Z})^{k-1} \); for each \( 1 \leq i \leq k \) we define the \( i \)-th interpolation operator \( I_\lambda^\rho \) by a similar interpolation on the \( i \)-th coordinate of \( P \). Given \( k \)-tuples \( \lambda := (\lambda_1, \ldots, \lambda_k) \) and \( \rho := (\rho_1, \ldots, \rho_k) \) of nonnegative real numbers, we define \( I_\lambda^\rho := I_{\lambda_1}^{\rho_1} \cdots I_{\lambda_k}^{\rho_k} \).

For each \( 1 \leq i \leq k \), let \( \eta(i) := 0 \ldots 010 \ldots 0 \) (with the 1 in the \( i \)-th coordinate) be the coordinate vectors of \((\mathbb{Z}/2\mathbb{Z})^k\), and denote by \( \Phi_i \) the \( i \)-th flip operator which associates to \( P : (\mathbb{Z}/2\mathbb{Z})^k \rightarrow \mathbb{R}[x] \) the \( 2^k \) polynomials

\[
(\Phi_i P)_\alpha := \begin{cases} 
  xP_{\alpha + \eta(i)} & \text{if } \alpha_i = 0, \\
  P_{\alpha + \eta(i)} & \text{if } \alpha_i = 1,
\end{cases}
\]

for each \( \alpha \in (\mathbb{Z}/2\mathbb{Z})^k \). For any \( S \subseteq \{1, \ldots, k\} \) we let \( \Phi_S := \prod_{i \in S} \Phi_i \). We say that \( P : (\mathbb{Z}/2\mathbb{Z})^k \rightarrow \mathbb{R}[x] \) is an interpolatory \( k \)-cube of polynomials when the following condition holds: for all \( k \)-tuples \( \lambda := (\lambda_1, \ldots, \lambda_k) \) and \( \rho := (\rho_1, \ldots, \rho_k) \) of nonnegative real numbers, and for all \( S \subseteq \{1, \ldots, k\} \), the polynomial \( I_\lambda^\rho \Phi_S P \) is standard and has only nonpositive zeros.

**Proposition 4.5.** Fix \( k \geq 1 \), and let \( P : (\mathbb{Z}/2\mathbb{Z})^k \rightarrow \mathbb{R}[x] \). Consider the following conditions \( C_i \) for each \( 1 \leq i \leq k \):

\( C_i \): For all \( \lambda, \rho > 0 \): both \( I_\lambda^\rho P \) and \( I_\lambda^\rho \Phi_i P \) are interpolatory \((k-1)\)-cubes.

The conditions \( C_i \) for \( 1 \leq i \leq k \) are each equivalent to the condition that \( P \) is an interpolatory \( k \)-cube.

**Proof.** This follows from Lemma 2.4 (the case \( k = 1 \)) as in the proof of Proposition 4.1 (the case \( k = 2 \)). \( \square \)

**Lemma 4.6.** Fix nonnegative integers \( k \) and \( \ell \), and let \( P : (\mathbb{Z}/2\mathbb{Z})^k \rightarrow \mathbb{R}[x] \) and \( Q : (\mathbb{Z}/2\mathbb{Z})^\ell \rightarrow \mathbb{R}[x] \). Define \( S : (\mathbb{Z}/2\mathbb{Z})^{k+\ell} \rightarrow \mathbb{R}[x] \) by \( S_{\alpha \beta} := P_\alpha Q_\beta \) for all \( \alpha \in (\mathbb{Z}/2\mathbb{Z})^k \) and \( \beta \in (\mathbb{Z}/2\mathbb{Z})^\ell \). If both \( P \) and \( Q \) are interpolatory hypercubes then \( S \) is an interpolatory \((k+\ell)\)-cube.

**Proof.** If \( k = 0 \) then it may be checked directly that \( S \) satisfies the definition of an interpolatory \( \ell \)-cube, and so we proceed by induction on \( k \geq 1 \). For any \( \lambda, \rho > 0 \) we have, by Proposition 4.5, interpolatory \((k-1)\)-cubes \( I_\lambda^\rho P \) and \( I_\lambda^\rho \Phi_i P \). By induction, both \( I_\lambda^\rho S \) and \( I_\lambda^\rho \Phi_i S \) are interpolatory \((k-1+\ell)\)-cubes, and so Proposition 4.5 implies that \( S \) is an interpolatory \((k+\ell)\)-cube. \( \square \)

**Lemma 4.7.** Fix \( k \geq 1 \), and let \( P, Q : (\mathbb{Z}/2\mathbb{Z})^k \rightarrow \mathbb{R}[x] \) be such that \( P_1 \alpha = Q_1 \alpha \) for all \( \alpha \in (\mathbb{Z}/2\mathbb{Z})^{k-1} \), and \( P_1 \alpha \neq 0 \) for at least one \( \alpha \in (\mathbb{Z}/2\mathbb{Z})^{k-1} \). Define \( S : (\mathbb{Z}/2\mathbb{Z})^k \rightarrow \mathbb{R}[x] \) by

\[
S_\alpha := \begin{cases} 
  P_\alpha + Q_\alpha & \text{if } \alpha_1 = 0, \\
  P_\alpha & \text{if } \alpha_1 = 1,
\end{cases}
\]
Lemma 4.9. Fix $k \geq 2$. For any two coordinates $1 \leq i < j \leq k$ of $P$, we may apply Lemma 4.7 to any two coordinates $1 \leq i < j \leq k$. Thus we may apply Lemma 4.7 to $1_{P_1}P_2\Phi_1\Phi_2P$ and $2_{P_1}P_2\Phi_1\Phi_2P$ to find that $3_{P_1}P_2\Phi_1\Phi_2P$ and $3_{P_1}P_2\Phi_1\Phi_2P$ are interpolatory $(k - 1)$-cubes satisfying the hypothesis, and hence $2_{P_1}P_2\Phi_1\Phi_2P$ is an interpolatory $(k - 1)$-cube. This verifies condition $C_2$ of Proposition 4.5 for $S$. □

By an argument analogous to the derivation of (4.2) from Lemma 4.3, we may conjugate everything in Lemma 4.7 by $\Phi_1$ to obtain another similar statement (which is left to the reader).

Lemma 4.8. Fix $k \geq 0$, and let $P : (\mathbb{Z}/2\mathbb{Z})^k \to \mathbb{R}[x]$. Define $Q : (\mathbb{Z}/2\mathbb{Z})^{k+1} \to \mathbb{R}[x]$ by $Q_{10} := P_0$ and $Q_{0\alpha} := (\Phi_1P)_{\alpha}$ for all $\alpha \in (\mathbb{Z}/2\mathbb{Z})^k$. If $P$ is an interpolatory $k$-cube then $Q$ is an interpolatory $(k + 1)$-cube.

Proof. The case $k = 0$ is obvious, and the case $k = 1$ is Lemma 4.4: for $k \geq 2$ we proceed by induction on $k$. Choose $\lambda, \rho > 0$ and apply the induction hypothesis to the interpolatory $(k - 1)$-cubes $2_{P_1}P_2\Phi_1\Phi_2P$ and $2_{P_1}P_2\Phi_1\Phi_2P$ to find that $3_{P_1}P_2\Phi_1\Phi_2P$ and $3_{P_1}P_2\Phi_1\Phi_2P$ are interpolatory $k$-cubes. By Proposition 4.5 it follows that $Q$ is an interpolatory $(k + 1)$-cube.

Lemma 4.9. Fix $k \geq 2$, and let $P : (\mathbb{Z}/2\mathbb{Z})^k \to \mathbb{R}[x]$. Define $Q : (\mathbb{Z}/2\mathbb{Z})^{k-1} \to \mathbb{R}[x]$ by

$$Q_{10} := P_{01}P \quad \text{and} \quad Q_{0\alpha} := P_{10\alpha} + xP_{11\alpha}$$

for all $\alpha \in (\mathbb{Z}/2\mathbb{Z})^{k-2}$. Assume that there is a $\beta \in (\mathbb{Z}/2\mathbb{Z})^{k-2}$ such that either $P_{00\beta} \neq 0$ or $P_{11\beta} \neq 0$, and that there is a $\gamma \in (\mathbb{Z}/2\mathbb{Z})^{k-2}$ such that either $P_{01\gamma} \neq 0$ or $P_{10\gamma} \neq 0$. If $P$ is an interpolatory $k$-cube then $Q$ is an interpolatory $(k - 1)$-cube.

Proof. Since $P$ is an interpolatory $k$-cube, each of $1_{P_1}P_2\Phi_1\Phi_2P$, $2_{P_1}P_2\Phi_1\Phi_2P$, and $2_{P_1}P_2\Phi_1\Phi_2P$ is an interpolatory $(k - 1)$-cube. By exchanging the first and second coordinates if necessary, we may assume that $P$ is such that $P_{10\gamma} \neq 0$ for some $\gamma \in (\mathbb{Z}/2\mathbb{Z})^{k-2}$. Thus we may apply Lemma 4.7 to $1_{P_1}P_2\Phi_1\Phi_2P$ and $2_{P_1}P_2\Phi_1\Phi_2P$; denote the result by $S$. If $P_{01\gamma} \equiv 0$ for all $\gamma \in (\mathbb{Z}/2\mathbb{Z})^{k-2}$ then $S = Q$, which suffices to prove the result. Otherwise, we may also apply Lemma 4.7 to $1_{P_1}P_2\Phi_1\Phi_2P$ and $2_{P_1}P_2\Phi_1\Phi_2P$; denote the result by $T$. Then $1_{P_1}S = 1_{P_1}T = 1_{P_1}Q$, and by the hypothesis there is some $\beta \in (\mathbb{Z}/2\mathbb{Z})^{k-2}$ such that $Q_{0\beta} \neq 0$. Thus we may apply the $\Phi_1$-conjugate form of Lemma 4.7 to $S$ and $T$; the result is $Q$, which proves the result.

Of course, by conjugating with a permutation of indices one may apply Lemma 4.9 to any two coordinates $1 \leq i < j \leq k$ of $P$. In this case the correspondence between the indices of $P$ and the indices of $Q$ will be taken to be $\ell \mapsto \ell$ for $1 \leq \ell < j$, $j \mapsto i$, and $\ell \mapsto \ell - 1$ for $j < \ell \leq k$, generalizing the case of $i = 1$ and $j = 2$ in the statement above.

5. Series-Parallel Networks

After much experimentation one arrives at the following hypothesis. For a network $G$ and distinct vertices $v$ and $w$ of $G$, let $G^-$ be obtained from $G$ by deleting all edges between $v$ and $w$, and let $G^+$ be obtained from $G$ by identifying $v$ and $w$.
and removing any loops thus created. We shall say that \( \{v, w\} \) is *very amicable in* \( G \) if

\[
\begin{align*}
J_0^* & \rightarrow xJ_1^- \\
J_1^* & \rightarrow J_0^* \\
J_0^- & \rightarrow xJ_1^*
\end{align*}
\]  \tag{5.1}

where \( J_-.(u) := J_{G^-}(u) \) and \( J_+(u) := J_{G^+}(u) \). (These conditions are equivalent, by (4.1).) In fact, this condition is too strong, and we shall say that \( \{v, w\} \) is *amicable in* \( G \) if the condition

\[
\begin{align*}
J_0^- + J_1^* & \rightarrow xJ_1^- \\
J_1^- + J_0^* & \rightarrow J_0^*
\end{align*}
\]  \tag{5.2}

is satisfied. Notice that \( J_.(u) \equiv 0 \) if and only if \( v \) and \( w \) are adjacent in \( G \) and \( v \sim w \) is a cut-edge of \( G^k \). In this case, (5.1) and (5.2) are each equivalent to \( J_1^* \sim J_0^-; \) otherwise, from (5.1) we have \( J_1^- \sim J_0^- \), to which we apply Lemma 4.4, and then (4.2) and (5.1) imply (5.2). In either case, if \( \{v, w\} \) is very amicable in \( G \) then \( \{v, w\} \) is amicable in \( G \). Notice that \( J_+(u) \neq 0 \) since \( G \), and hence \( G^k \), is connected.

**Lemma 5.1.** Let \( G \) be a network and let \( \{v, w\} \) be amicable in \( G \). Then \( J_1^G < J_0^G \).

Proof. Let \( G^- \), \( G^+ \), \( J_-.(u) \), and \( J_+(u) \) be as in the above paragraph, and let.

\[
\begin{align*}
O_aJ_0^0 & \rightarrow xO_aJ_1^- \\
O_aJ_1^+ & \rightarrow O_aJ_0^0 \\
E_aJ_0^0 & \rightarrow xE_aJ_1^-
\end{align*}
\]  \tag{5.3}

Index the coordinates of (5.3) by 1, 2, 3 in the order \( \uparrow, \rightarrow, \Rightarrow \). If \( a > 0 \) and \( J_.(u) \equiv 0 \) then we may apply Lemma 4.9 to coordinates 2 and 3 of (5.3), yielding an interpolatory square.

\[
\begin{align*}
E_aJ_0^0 + xO_aJ_1^- & \rightarrow xE_aJ_1^- + xO_aJ_0^0 \\
E_aJ_1^+ + O_aJ_0^0 & \rightarrow E_aJ_0^0 + xO_aJ_1^0
\end{align*}
\]  \tag{5.4}

If \( a = 0 \) then \( O_a = 0 \), and (5.4) is obtained from (5.3) by applying \( 3I_0^0 \); if \( J_.(u) \equiv 0 \) then (5.4) is obtained from (5.3) by applying \( 2I_0^0 \Phi_3 \). Thus, in all cases (5.4) is an interpolatory square. If \( a \) is odd then (3.6) and the left column of (5.4) give \( J_1^G \sim J_0^G \), while if \( a \) is even then (3.5) and the right column of (5.4) give \( J_0^G \sim xJ_1^G \). \( \Box \)

**Theorem 5.2.** Let \( G \) and \( N \) be networks which intersect in exactly one vertex \( v \), let \( w' \neq v \) be a vertex of \( G \), and let \( w'' \neq v \) be a vertex of \( N \). Let \( U := G \cup N \), let \( W \) denote the network obtained from \( U \) by identifying \( w' \) and \( w'' \), and let \( v \) denote the image of \( w' \) and \( w'' \) in \( W \). If \( \{v, w\} \) is amicable in \( G \) and \( \{v, w''\} \) is amicable in \( N \) then:

(a) \( \{v, w'\} \) and \( \{v, w''\} \) are amicable in \( U \), and
(b) \( \{v, w\} \) is amicable in \( W \), and
(c) \( \{w', w''\} \) is very amicable in \( U \).
Proof. Let \( v \) and \( w' \) be joined by exactly \( a \) edges of \( G \), and let \( v \) and \( w'' \) be joined by exactly \( b \) edges of \( N \). Let \( G^- \) denote the network obtained by deleting the \( a \) edges between \( v \) and \( w' \) in \( G \), and let \( G^* \) denote the network obtained by identifying \( v \) and \( w' \) in \( G \) and removing the \( a \) loops thus produced. Let \( N^- \) denote the network obtained by deleting the \( b \) edges between \( v \) and \( w'' \) in \( N \), and let \( N^* \) denote the network obtained by identifying \( v \) and \( w'' \) in \( N \) and removing the \( b \) loops thus produced. Let \( W^- \) denote the network obtained by deleting the \( a+b \) edges between \( v \) and \( w \) in \( W \), and let \( W^* \) denote the network obtained by identifying \( v \) and \( w \) in \( W \) and removing the \( a+b \) loops thus produced. To simplify notation, let \( J(u) := J_G(u) \), \( J_-(u) := J_{G^-}(u) \), and \( J^*(u) := J_{G^*}(u) \), let \( K(u) := J_N(u) \), \( K_-(u) := J_{N^-}(u) \), and \( K^*(u) := J_{N^*}(u) \), and let \( L(u) := J_W(u) \), \( L_-(u) := J_{W^-}(u) \), and \( L^*(u) := J_{W^*}(u) \).

We will also use the notations \( J^*(u) := J_-(u) + J^*(u) \), \( K^*(u) := K_-(u) + K^*(u) \), and \( L^*(u) := L_-(u) + L^*(u) \).

By Lemma 5.1 we see that \( J_1 < J_0 \) and \( K_1 < K_0 \). Now apply Lemma 4.6 to (5.2) (with \( w' \) in place of \( w \)) and \( K_1 < K_0 \) to get an interpolatory 3-cube.

\[
\begin{align*}
J_0^y K_1 & \rightarrow xJ_1^y K_1 \\
J_1^y K_1 & \rightarrow \uparrow \uplus \rightarrow J_0^y K_0 \\
J_0^y K_0 & \rightarrow xJ_1^y K_0 \uplus \uplus \rightarrow \uplus \uplus \rightarrow \uplus \uplus \rightarrow J_0^y K_0
\end{align*}
\]

Index the coordinates of (5.5) as for (5.3). If \( K_1 \equiv 0 \) and \( K_0 \equiv 0 \) then, since \( J^* \equiv 0 \), we may apply Lemma 4.9 to coordinates 1 and 3 of (5.5), yielding the interpolatory square

\[
\begin{align*}
J_0^y K_0 + xJ_1^y K_1 & \rightarrow xJ_0^y K_1 + xJ_1^y K_0 \\
J_1^y K_1 + J_0^y K_0 & \rightarrow J_0^y K_0 + xJ_1^y K_1
\end{align*}
\]

which from (3.3) is seen to be

\[
\begin{align*}
J_0^{-} \cup N + J_1^* \cup N & \rightarrow xJ_1^- \cup N \\
J_1^- \cup N + J_0^* \cup N & \rightarrow J_0^- \cup N
\end{align*}
\]

If \( K_1 \equiv 0 \) then (5.6) is obtained from (5.5) by applying \( \gamma \Phi_1^0 \); if \( K_0 \equiv 0 \) then (5.6) is obtained from (5.5) by applying \( \gamma \Phi_2^0 \). In all cases (5.7) is an interpolatory square, showing that \( \{v, w'\} \) is amicable in \( U \). Since the hypothesis is symmetric in \( G \) and \( N \) we also conclude that \( \{v, w''\} \) is amicable in \( U \), proving part (a).

For part (b), apply Lemma 4.6 to (5.2) and its analogue for \( N \) to get an interpolatory 4-cube.

\[
\begin{align*}
J_0^y K_0 & \rightarrow xJ_1^- K_0^y \\
J_1^y K_0 & \rightarrow J_0^- K_0^y \\
J_0^y K_0 & \rightarrow xJ_1^- K_0^y \\
J_1^y K_0 & \rightarrow J_0^- K_0^y
\end{align*}
\]

Index the coordinates of (5.8) by 1, 2, 3, 4 in the order \( \uparrow, \rightarrow, \uplus, \Rightarrow \). If \( J_-(u) \neq 0 \) and \( K_-(u) \neq 0 \) then we may apply Lemma 4.9 to coordinates 2 and 4 of (5.8) to
obtain an interpolatory 3-cube.

\( xJ_0^0 K_1^- + xJ_1^- K_0^0 \quad \rightarrow \quad xJ_0^0 K_1^- + x^2 J_1^- K_1^- \quad \uparrow \quad \uparrow \)

\( J_0^- K_0^0 + xJ_1^0 K_1^- \quad \rightarrow \quad xJ_0^0 K_1^- + xJ_1^- K_0^0 \quad \uparrow \quad \uparrow \)

\( J_0^0 K_0^- + xJ_1^- K_1^- \quad \rightarrow \quad xJ_0^0 K_1^- + xJ_1^- K_0^- \quad \uparrow \quad \uparrow \)

\( J_0^- K_1^0 + J_1^- K_0^- \quad \rightarrow \quad J_0^0 K_1^- + xJ_1^- K_0^- \quad \uparrow \quad \uparrow \)

(5.10)

If \( J_- (u) \equiv 0 \) then (5.9) is obtained from (5.8) by applying \( 2I^1 \Phi_4 \) (and permuting coordinates); if \( K_- (u) \equiv 0 \) then (5.9) is obtained from (5.8) by applying \( 4I^1 \Phi_2 \). In all cases, (5.9) is an interpolatory 3-cube. If \( J_- (u) \not\equiv 0 \) or \( K_- (u) \not\equiv 0 \) then we may apply Lemma 4.9 to coordinates 1 and 3 of (5.9), yielding

\( xJ_0^0 K_1^- + xJ_1^- K_0^0 \quad \rightarrow \quad xJ_0^0 K_1^- + x^2 J_1^- K_1^- \quad \uparrow \quad \uparrow \)

\( J_0^- K_0^0 + xJ_1^- K_1^- \quad \rightarrow \quad xJ_0^0 K_1^- + xJ_1^- K_0^- \quad \uparrow \quad \uparrow \)

\( J_0^0 K_0^- + xJ_1^- K_1^- \quad \rightarrow \quad xJ_0^0 K_1^- + xJ_1^- K_0^- \quad \uparrow \quad \uparrow \)

(5.10)

If \( J_- (u) \equiv 0 \) and \( K_- (u) \equiv 0 \) then first assume that neither \((G^*)^k\) nor \((N^*)^k\) is a tree. By Proposition 3.2 we may apply Lemma 4.9 to (5.9) to produce (5.10). Otherwise, if \( J^*_u \equiv 0 \) then (5.10) is obtained from (5.9) by applying \( 2I^1 \Phi_3 \), and similarly in case \( K^*_u \equiv 0 \) or \( K^*_u \equiv 0 \). In all cases, (5.10) is an interpolatory square. From (1.17) we have \( L_- (u) = J_- (u)K_0^v(u) + J_v(u)K_- (u) \), and hence \( L_v (u) = L_- (u) + J_v (u)K_0^v(u) = J_- (u)K_- (u) + J_v (u)K_0^v(u) \). From this one sees that (5.10) is

\( xL_1^- \quad \rightarrow \quad xL_0^0 \quad \uparrow \quad \uparrow \quad , \)

\( L_0^- \quad \rightarrow \quad xL_1^0 \)

(5.11)

and from (4.1) it follows that \( \{v, w\} \) is amicable in \( W \), proving part (b).

For part (c) we begin with (5.4) and its analogue for \( N \), that is

\( E_b K_0^0 + xO_b K_1^- \quad \rightarrow \quad xE_b K_1^- + xO_b K_0^0 \quad \uparrow \quad \uparrow \)

\( E_b K_1^- + O_b K_0^- \quad \rightarrow \quad E_b K_0^- + xO_b K_1^- \quad \uparrow \quad \uparrow \)

(5.12)

As in the proof of Lemma 5.1, both (5.4) and (5.12) are interpolatory squares, so that by Lemma 4.6 we obtain an interpolatory 4-cube \( Q \); we index the coordinates of \( Q \) so that 1 and 3 correspond to \( \uparrow \) and \( \rightarrow \) in (5.4) and 2 and 4 correspond to \( \uparrow \) and \( \rightarrow \) in (5.12), respectively. The cases in which either \( a > 0 \) and \( G^3 \) is a tree or \( b > 0 \) and \( N^3 \) is a tree are slightly degenerate; assume first that neither condition holds. Then we can apply Lemma 4.9 to coordinates 1 and 2 of \( Q \) to obtain an interpolatory 3-cube \( T \); the entries of \( T \) are as follows.

\[ T_{000} = (xE_a J_1^- + xO_a J_0^0)(xE_b K_1^- + xO_b K_0^0) + x(E_a J_0^0 + xO_a J_1^-)(E_b K_0^- + xO_b K_1^-) \]

\[ T_{001} = (xE_a J_1^- + xO_a J_0^0)(E_b K_0^0 + xO_b K_1^-) + x(E_a J_0^0 + xO_a J_1^-)(E_b K_1^- + O_b K_0^-) \]
\[
T_{010} = (E_a J_0^q + xO_a J_1^q)(x E_b K_1^q + xO_b K_0^q) \\
\quad + x(E_a J_0^q + O_a J_0^q)(E_b K_0^q + xO_b K_0^q)
\]

\[
T_{011} = (E_a J_0^q + xO_a J_1^q)(E_b K_0^q + xO_b K_1^q) \\
\quad + x(E_a J_0^q + O_a J_0^q)(E_b K_0^q + xO_b K_0^q)
\]

\[
(5.13)
\]

\[
T_{100} = (x E_a J_1^q + xO_a J_0^q)(E_b K_0^q + xO_b K_1^q) \\
\quad + (E_a J_0^q + xO_a J_0^q)(x E_b K_1^q + xO_b K_0^q)
\]

\[
T_{101} = (x E_a J_1^q + xO_a J_0^q)(E_b K_0^q + xO_b K_1^q) \\
\quad + (E_a J_0^q + xO_a J_0^q)(x E_b K_1^q + xO_b K_0^q)
\]

\[
T_{110} = (E_a J_0^q + xO_a J_1^q)(E_b K_0^q + xO_b K_1^q) \\
\quad + (E_a J_0^q + O_a J_0^q)(E_b K_0^q + xO_b K_0^q)
\]

\[
T_{111} = (E_a J_0^q + xO_a J_1^q)(E_b K_0^q + xO_b K_1^q) \\
\quad + (E_a J_0^q + O_a J_0^q)(E_b K_0^q + xO_b K_0^q)
\]

If \( a > 0 \) and \( G^q \) is a tree then \( J_a(u) \equiv 0 \) and either \( J_1^q \equiv 0 \) or \( J_0^q \equiv 0 \); if \( J_1^q \equiv 0 \) then \( T = 10_0^q \Phi_1 \), while if \( J_0^q \equiv 0 \) then \( T = 10_1^q \Phi_1 \). The case when \( b > 0 \) and \( N^q \) is a tree is handled similarly. In all cases \( T \) is an interpolatory 3-cube. Notice that \( J_1(u) \not\equiv 0 \) and if \( a = 0 \) then \( J_a(u) \not\equiv 0 \), and similarly for \( N \); from this it follows that \( T_{00} + T_{10} \not\equiv 0 \) for all \( \alpha \in (\mathbb{Z}/22)^2 \).

If \( a \) and \( b \) are both odd then \( J_1 = E_a J_1^q + O_a J_0^q \) and \( J_0 = E_a J_0^q + xO_a J_1^q \) and \( K_1 = E_b K_1^q + O_b K_0^q \) and \( K_0 = E_b K_0^q + xO_b K_1^q \) and \( L_1 = E_a L_1^q + O_a b L_0^q \) and \( L_0 = E_a b L_0^q + xO_a b L_1^q \). Thus we find that \( T_{011} = J_0 K_1 + J_1 K_0 \) and, by using (3.7) and (3.8), that \( T_{001} + T_{010} = xL_1 \) and \( T_{101} + T_{110} = L_0 \). Applying Lemma 4.3 to \( 21_0^q \Phi_1 \) and \( 31_0^q \Phi_1 \) we see that

\[
\begin{align*}
T_{011} &\rightarrow T_{001} + T_{010} \\
\uparrow &\oplus \uparrow \\
T_{111} &\rightarrow T_{101} + T_{110}
\end{align*}
\]

(5.14)

which shows that \( \{w', w''\} \) is very amicable in \( U \) in this case.

If \( a \) and \( b \) are both even then \( J_1 = E_a J_1^q + O_a J_0^q \) and \( J_0 = E_a J_0^q + xO_a J_1^q \) and \( K_1 = E_b K_1^q + O_b K_0^q \) and \( K_0 = E_b K_0^q + xO_b K_1^q \) and \( L_1 = E_a b L_1^q + O_a b L_0^q \) and \( L_0 = E_a b L_0^q + xO_a b L_1^q \). Thus we find that \( T_{000} = x(J_0 K_0 + J_1 K_1) \) and \( T_{100} = x(J_0 K_1 + J_1 K_0) \) and, by using (3.7) and (3.8), that \( T_{001} + T_{010} = xL_1 \) and \( T_{101} + T_{110} = L_0 \). Applying (4.2) to \( 21_0^q \Phi_1 \) and \( 31_0^q \Phi_1 \) we see that

\[
\begin{align*}
T_{001} + T_{010} &\rightarrow T_{000} \\
\uparrow &\oplus \uparrow \\
T_{101} + T_{110} &\rightarrow T_{100}
\end{align*}
\]

(5.15)

which shows that \( \{w', w''\} \) is very amicable in \( U \) in this case, by (4.1).

In the remaining case, \( a \) and \( b \) have opposite parity; by symmetry we may assume that \( a \) is even and \( b \) is odd. Thus \( J_1 = E_a J_1^q + O_a J_0^q \) and \( J_0 = E_a J_0^q + xO_a J_1^q \) and \( K_1 = E_b K_1^q + O_b K_0^q \) and \( K_0 = E_b K_0^q + xO_b K_1^q \) and \( L_1 = E_a b L_1^q + O_a b L_0^q \) and \( L_0 = E_a b L_0^q + xO_a b L_1^q \). Thus we find that \( T_{001} = x(J_0 K_1 + J_1 K_0) \) and
both with strictly fewer edges than \( G \) provides the induction step. Lemma 5.1 then completes the proof.

5.2(c) provides the induction step; if the connection is parallel then Theorem 5.2(b) series connection or as a parallel connection of series-parallel networks \( G \) amicable in \( G \) is amicable in \( N \). Let \( \mathcal{J} \) be the number of faces \( \Omega \) such that \( \#S = i \); we define the rank-generating function of \( \Omega \) to be

\[
F_{\Omega}(z) = f_0 + f_1 z + f_2 z^2 + \cdots + f_d z^d.
\]

We may factor this polynomial as \( F_{\Omega}(z) = (1 + z)^{d-t} \tilde{F}_{\Omega}(z) \) such that \( \tilde{F}_{\Omega}(-1) \neq 0 \), defining the subdegree \( t_\Omega := \deg \tilde{F}_{\Omega}(z) \) of \( \Omega \), and the coefficients of \( \tilde{F}_{\Omega}(z) = \sum_{i=0}^t \tilde{f}_i z^i \) in the process. Clearly

\[
\tilde{f}_i = \sum_{\ell=0}^{\infty} \binom{d-t+\ell-1}{\ell}(1)^\ell f_{i-\ell} \text{ for all } 0 \leq i \leq t,
\]
and
\[
(6.3) \quad f_i = \sum_{\ell=0}^{d-i} \binom{d-t}{\ell} \tilde{f}_{i-\ell} \quad \text{for all } 0 \leq i \leq d,
\]
with the conventions that \( f_i = \tilde{f}_i = 0 \) if \( i < 0 \) and \( f_i = 0 \) if \( i > d \) and \( \tilde{f}_i = 0 \) if \( i > t \).

The reliability function \( R_{\Omega}(q) \) of \( \Omega \) is the probability that, if each element of \( E \) is selected independently with probability \( 0 \leq q \leq 1 \), then the random subset \( S(q) \subseteq E \) consisting of the selected elements of \( E \) is a face of \( \Omega \). By partitioning the event that \( S(q) \in \Omega \) into its constituent subevents one sees immediately that
\[
(6.4) \quad R_{\Omega}(q) = \sum_{i=0}^{d} f_i q^{d-i}(1-q)^{m-i}.
\]
Thus, we can write \( R_{\Omega}(q) = (1-q)^{m-d} H_{\Omega}(q) \), where the \( H \)-polynomial of \( \Omega \) is defined by
\[
(6.5) \quad H_{\Omega}(q) := (1-q)^d F_{\Omega} \left( \frac{q}{1-q} \right).
\]
A simple calculation shows that \( \deg H_{\Omega}(q) = t_{\Omega} \) and that \( H_{\Omega}(q) = \sum_{i=0}^{t} h_i q^i \) depends only upon \( \tilde{F}_{\Omega}(z) \).

Certain classes of set systems are of special interest with respect to these polynomials. Let \( \mathfrak{S} \) denote the class of simplicial complexes, let \( \mathfrak{M} \) denote the class of (simplicial complexes of independent sets of) matroids, let \( \mathfrak{S}^* \) denote the class of cographic matroids, and let \( \mathfrak{BC} \) denote the class of set systems \( \Omega \) for which \( H_{\Omega}(q) \) is Schur quasi-stable. For a network \( G \), let \( M := M^*(G) \) be the cographic matroid associated with \( G \); then the polynomials \( R_G(q) \), \( H_G(q) \), and \( J_G(u) \) defined in Sections 0 and 1 equal the polynomials \( R_M(q) \), \( H_M(q) \), and \( J_M(u) \) defined in this section, respectively. Thus, the Brown-Colbourn Conjecture is that \( \mathfrak{S}^* \) is a subclass of \( \mathfrak{BC} \).

Lemma 6.1 and Proposition 6.2 were suggested by Theorem 4.3 and the remark on page 585 of Brown and Colbourn. For a positive integer \( k \) and a set system \( \Omega \) on the set \( E \), we define the set system \( k\Omega \) on the set \( E \times \{1, \ldots, k\} \) as follows: \( \{(e_1, i_1), \ldots, (e_r, i_r)\} \subseteq E \times \{1, \ldots, k\} \) is a face of \( k\Omega \) if and only if \( \{e_1, \ldots, e_r\} \) are pairwise distinct elements of \( E \), and this set is a face of \( \Omega \).

**Lemma 6.1.** Let \( \Omega \) be a set system on a set \( E \) of size \( m \) and let \( k \) be a positive integer. Then
\[
R_{k\Omega}(q) = \left( (1-q)^k + kq(1-q)^{k-1} \right)^{m} R_{\Omega} \left( \frac{kq}{1+(k-1)q} \right).
\]

**Proof.** For each of the \( m \) elements \( e \in E \), at most one of the elements \( (e, 1), \ldots, (e, k) \) can be selected if the random subset of selected elements \( S(q) \) is to be a face of \( k\Omega \); these events occur independently, each with probability \( (1-q)^k + kq(1-q)^{k-1} \). Conditioning on the conjunction of these events, the conditional probability that exactly one of \( (e, 1), \ldots, (e, k) \) is selected is
\[
\hat{q} := \frac{kq(1-q)^{k-1}}{(1-q)^k + kq(1-q)^{k-1}} = \frac{kq}{1+(k-1)q}
\]
for each \( e \in E \), and hence the conditional probability that \( S(q) \in \Omega \) is \( R_{\Omega}(\hat{q}) \). \( \square \)
Proposition 6.2. For any set system $\Omega$, there is an integer $K_\Omega$ such that for all $k \geq K_\Omega$, $k\Omega$ is in the class $\mathcal{B}_C$.

Proof. Let $\Omega$ be defined on a set $E$ with $m$ elements. With $q$ defined as in (6.6) we have $R_{k\Omega}(q) = (1-q)^{km-m}(1+(k-1)q)^m R_\Omega(q)$. The zeros of $R_{k\Omega}(q)$ due to the factors $(1-q)^{km-m}(1+(k-1)q)^m$ are inside the unit disc $|q| \leq 1$ for all $k \geq 1$. If $\xi \in \mathbb{C}$ is such that $R_\Omega(\xi) = 0$ then each factor $(q-\xi)$ of $R_{k\Omega}(q)$ contributes a zero of $R_{k\Omega}(q)$ at $q_0 := \xi/(\xi + k(1-\xi))$. If $\xi = 1$ then $q_0 = 1$, and if $\xi \not= 1$ then we can choose $k$ sufficiently large that $|q_0| < 1$. Since $R_\Omega(q)$ has only finitely many zeros, there is some $K_\Omega$ such that $k \geq K_\Omega$ suffices for all factors, proving the result. \qed

In fact, the proof of Brown and Colbourn [3] shows that if $M$ is a matroid then $K_M = d_M + 1$ suffices in Proposition 6.2, although they do not state this explicitly.

Proposition 6.3 provides some weak support for the idea that all matroids are in the class $\mathcal{B}_C$, but at present there is not enough evidence to really justify any opinion on this strengthening of the Brown-Colbourn Conjecture.

Proposition 6.3. For $1 \leq d < m$, let $U^d_m$ denote the uniform matroid of rank $d$ with $m$ elements, let $F^d_m(\nu)$ be its rank-generating function, and construct $H^d_m(q)$ as in (6.5). If $q \in \mathbb{C}$ is such that $H^d_m(q) = 0$ then $(m-d)^{-1} \leq |q| \leq (d-m)^{-1}$.

In particular, $U^d_m$ is in the class $\mathcal{B}_C$.

Proof. For all $1 \leq d < m$ we have $F^d_m(\nu) = \sum_{i=0}^{d} \binom{m}{d} \nu^i$, and so $F^d_m(-1) \neq 0$. For $d = 1$ this gives $F^d_m(\nu) = 1 + m \nu$ and $H^d_m(q) = 1 + (m-d)\nu$, satisfying the statement of the proposition. From the familiar recurrence relations for binomial coefficients it follows that for all $1 < d < m$, $F^d_m(\nu) = z F^d_{m-1}(\nu) + F^d_{m-1}(\nu)$ and $H^d_m(q) = q H^d_{m-1}(q) + H^d_{m-1}(q)$. By induction, one sees that for all $1 \leq d < m$,

$$H^d_m(q) = \sum_{i=0}^{d} \binom{m-d-1+i}{i} q^i.$$

The successive ratios of these coefficients are $\lambda_i := \binom{m-d-1+i}{i} \binom{m-d+i}{i+1} = (i+1)(m-d+i)^{-1}$, which are nondecreasing as $i$ runs from 0 to $d-1$. Thus, by the Eneström-Kakeya Theorem (see Theorem B of Anderson, Saff, and Varga [4]) it follows that all complex zeros of $H^d_m(q)$ satisfy $|q| \leq \lambda_{d-1}$.

The $\bar{f}$-vector $(\bar{f}_0, \ldots, \bar{f}_t)$ of a set system $\Omega$ in the class $\mathcal{B}_C$ must satisfy some strong inequalities, as we now explain; when $t_\Omega = d_\Omega$ this $\bar{f}$-vector agrees with the $f$-vector $(f_0, \ldots, f_d)$ of $\Omega$. We introduce the $J$-polynomial of a set system $\Omega$ by defining

$$J_\Omega(u) := (-2)^t \bar{F}_\Omega \left( \frac{-1-u}{2} \right),$$

so that $\bar{F}_\Omega(z) = J_\Omega(-1 - 2z)$. Here we have the relations $u := -1 - 2z$ and $z = (1-u)/2$. In terms of the coefficients $J_\Omega(u) = \sum_{k=0}^{t} j_k u^k$ this relation is

$$j_k = \sum_{i=k}^{t} \binom{i}{k} (-2)^{t-i} \bar{f}_i \text{ for all } 0 \leq k \leq t,$$

and conversely,

$$\bar{f}_i = 2^{i-t} \sum_{k=i}^{t} \binom{k}{i} (-1)^{t-k} j_k \text{ for all } 0 \leq i \leq t.$$
The relation between \( J_\Omega(u) \) and \( H_\Omega(q) \) is as in (1.12) and (1.13). By reasoning analogous to that showing the equivalence of (1.8) and (1.14), one sees that a set system \( \Omega \) is in the class \( \mathcal{BC} \) if and only if \( J_\Omega(u) \) is Hurwitz quasi-stable. A theorem of Asner [3] (see also Kemperman [14]) states that a polynomial \( J(u) = \sum_{k=0}^{t} j_k u^k \) in \( \mathbb{R}[u] \) with \( j_t > 0 \) is such that all of its zeros have strictly negative real part if and only if every minor of the Hurwitz matrix

\[
H[J(u)] := \begin{bmatrix}
j_0 & 0 & 0 & \cdots & 0 & 0 \\
j_2 & j_1 & j_0 & \cdots & \ddots & 0 \\
j_4 & j_3 & j_2 & \cdots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & j_t & j_t-1 & j_t-2 & j_t-4 \\
0 & 0 & \cdots & 0 & 0 & j_t
\end{bmatrix}
\]

is nonnegative, and \( \det H[J(u)] > 0 \). One direction of this equivalence survives in the limit (the other does not): if \( J(u) \) is Hurwitz quasi-stable then every minor of \( H[J(u)] \) is nonnegative (see [3, 14]). We let \( \mathcal{BC}' \) denote the class of set systems \( \Omega \) such that every minor of \( H[J_\Omega(u)] \) is nonnegative; this class contains \( \mathcal{BC} \) (and hence, by Theorem 0.2, the cographic matroid of each network in \( \mathcal{G} \)). Also, we denote by \( \mathfrak{J}_+ \) the class of set systems \( \Omega \) such that \( j_k(\Omega) \geq 0 \) for all \( 0 \leq k \leq t_\Omega \); this class contains \( \mathcal{BC}' \).

For example, consider the simplicial complex \( I \) consisting of the faces of the icosahedron. We have \( F_I(z) = 1 + 12z + 30z^2 + 20z^3 \) and so \( t_I = d_I = 3 \), and the calculation of \( J_I(u) \) can be illustrated by

\[
\begin{array}{cccc}
1 & -8 & -8 \\
12 & 4 & 48 & 48 \\
30 & -2 & -60 & -120 & -60 \\
20 & 1 & 20 & 60 & 60 & 20 \\
0 & -12 & 0 & 20
\end{array}
\]

so that \( J_I(u) = -12u + 20u^3 \). This is an example of a simplicial polytope which is not in the class \( \mathfrak{J}_+ \). As another example, let \( \Theta \) be the broken-circuit complex (see Brylawski [8]) of (the graphic matroid of) \( K_{2,3} \). Then \( F_\Theta(z) = (1 + z)(1 + 5z + 10z^2 + 7z^3) \), so \( d_\Theta = 4 \) and \( t_\Theta = 3 \) and we calculate that \( J_\Theta(u) = -1 + u + u^2 + 7u^3 \); this \( \Theta \) is a broken-circuit complex which is not in \( \mathfrak{J}_+ \). Simplicial polytopes, broken-circuit complexes, and matroids are each subclasses of the class of Cohen-Macaulay complexes \( \Delta \) in general, see [17].

Our last theorem also supports the possibility that \( \mathfrak{M} \) might be a subclass of \( \mathcal{BC} \).

**Theorem 6.4.** Every matroid is in the class \( \mathfrak{J}_+ \).

**Proof.** Let \( M \) be a matroid of rank \( d \) which has exactly \( c \) coloops, and let \( M' \) be the matroid obtained by deleting all loops and coloops of \( M \). Then \( F_M(z) = (1 + z)^c F_{M'}(z) \), so that \( \bar{F}_M(z) = \bar{F}_{M'}(z) \), and \( M' \) has no loops or coloops. Since \( H_M(q) \) and \( J_M(u) \) depend only upon \( \bar{F}_M(z) \), we may replace \( M \) by \( M' \) and henceforth assume that \( M \) has no loops or coloops.
It is a standard result of matroid theory (see (7.12) of Björner [5], for example) that the Tutte polynomial \( T_M(x, y) \) of a matroid \( M \) may be specialized to yield
\[
T_M(x, 1) = h_0x^d + h_1x^{d-1} + \cdots + h_{d-1}x + h_d = x^dH_M(1/x),
\]
where \( h_i = 0 \) if \( t_M < i \leq d_M \). Another standard result is that if \( M \) has no coloops then \( h_d > 0 \); this follows from Theorem 6.2.13(v) in Brylawski and Oxley [9] (and that fact that \( b_{ij} \geq 0 \) for all \( i, j \) in their notation, see p.127 of [3]). (In other words, \( t_M = d_M \) for a matroid with no coloops.) Chari [10] proves that, since \( M \) has no coloops, there exist integers \( s_i \geq 1 \) for \( 1 \leq i \leq h_d \) and \( 1 \leq \ell \leq s_i \) such that
\[
T_M(x, y) = \sum_{i=1}^{h_d} \prod_{\ell=1}^{s_i} (y + x + x^2 + \cdots + x^{r_i}).
\]
Letting \( a_i := d - \sum_\ell r_{i\ell} \) for \( 1 \leq i \leq h_d \) we obtain from (6.12) and (6.13) that
\[
H_M(q) = \sum_{i=1}^{h_d} q^{a_i} \prod_{\ell=1}^{s_i} (1 + q + q^2 + \cdots + q^{r_i}),
\]
in which each term has degree \( d \). Therefore, applying the relation (1.12) to (6.14) we obtain
\[
J_M(u) = \sum_{i=1}^{h_d} (u + 1)^{a_i} \prod_{\ell=1}^{s_i} \left[ \frac{(u + 1)^{r_{i\ell}} - (u - 1)^{r_{i\ell}}}{2} \right],
\]
which evidently has nonnegative integer coefficients. Therefore \( M \) is in the class \( \mathfrak{J}_+ \).

Theorem 6.4 raises the problem of interpreting the coefficients of the \( J \)-polynomial of a matroid combinatorially; although one can use (6.15) as a guide, a solution to this problem is not presently at hand.

The proof of Theorem 0.3 is now clear. If \( M \) is a matroid with no coloops then \( t_M = d_M \) as in the proof of Theorem 6.4, and thus the \( \tilde{f} \)-vector of \( M \) coincides with the \( f \)-vector of \( M \). By Theorem 6.4, \( M \) is in the class \( \mathfrak{J}_+ \), and hence the conclusion of Theorem 0.3 follows from (6.8).
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DEPARTMENT OF COMBINATORICS AND OPTIMIZATION, UNIVERSITY OF WATERLOO, WATERLOO, ONTARIO, CANADA N2L 3G1

E-mail address: dgwagner@math.uwaterloo.ca