Brownian Motion in wedges, last passage time and the second arc-sine law

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14th January 2022

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Abstract

We consider a planar Brownian motion starting from $O$ at time $t=0$ and stopped at $t=1$ and a set $F = \{OI_i; \ i=1,2,\ldots,n\}$ of $n$ semi-infinite straight lines emanating from $O$. Denoting by $g$ the last time when $F$ is reached by the Brownian motion, we compute the probability law of $g$. In particular, we show that, for a symmetric $F$ and even $n$ values, this law can be expressed as a sum of arcsin or $(\text{arcsin})^2$ functions. The original result of Levy is recovered as the special case $n=2$. A relation with the problem of reaction-diffusion of a set of three particles in one dimension is discussed.

The first arc-sine law gives the distribution of the number of positive partial sums in a sequence of independent and identically distributed random variables. It was first discovered by P. Levy in his study of the linear Brownian motion and then discussed a lot for its relevance to the coin-tossing game [1]. The second arc-sine law, also discovered by P. Levy [2], provides an information on the last passage time which can be stated as follows. Consider a linear Brownian motion $B(\tau)$ starting at 0 at time $\tau=0$ and stopped at time $t$ and let $g$ be the last time when 0 is visited. The random variable

$$g = \sup\{\tau < t, B(\tau) = 0\}$$

satisfies

$$P(g < u) = \frac{2}{\pi} \text{arcsin} \sqrt{\frac{u}{t}}$$

with the density

$$P(u) = \frac{1}{\pi} \frac{1}{\sqrt{u(t-u)}}$$

Over the years, this result has been extended in several different directions (see for instance, [3, 4]) and is still a subject of active research in probability [5]. Generalizations of the first arc-sine law have also been considered in different contexts (one-dimensional diffusion in a random medium [6], Brownian motion on graphs [7] and, also, in two dimensions [8]).
The purpose of this Letter is to present a two dimensional generalization of the law (2). As a byproduct of this result we also derive an explicit expression of the first passage time distribution which is relevant for a problem of reaction-diffusion involving three identical particles.

Exit problems for Brownian motion have a rich history and several applications in physics (see for instance [9]). They are in particular related with problems of capture of independent Brownian particles diffusing on the line. This connection was first anticipated by Arratia [10] and then discussed in the mathematics [11, 12] and physics literature [9, 13, 14, 15] mainly in the context of reaction-diffusion models. In the case of three particles, the process \((x_1(t) - x_2(t), x_2(t) - x_3(t))\) defines a certain diffusion in a quadrant of \(\mathbb{R}^2\). By a suitable transformation, this process can be mapped on a diffusion inside a wedge whose angle depends on the diffusion constants. Using this correspondence, it has been shown that the first passage time through the wedge gives the survival probability; a quantity which decays with a power law which only depends on the angle of the wedge \([14, 16]\). At the end of this work, we exploit this correspondence to compute exactly (and not only asymptotically) the first collision time distribution for a three particle problem. Our approach is based on an identity relating the first passage and last passage distribution which has an interesting probabilistic interpretation [17].

To begin with, let us start by considering, as in Figure 1, a wedge of apex \(O\) and angle \(\phi\) with a boundary \(F = \{OI_i ; i = 1, 2\}\) and a two dimensional Brownian motion \(\overrightarrow{r}(t)\) starting from \(O\) at \(t = 0\) and stopped at \(t\) somewhere in the plane. We denote by \(g\) the last time when \(F\) is visited and compute the probability \(P(g < u)\). Due to the scaling property of the Brownian motion, this distribution is a function of the reduced variable \(u/t\). In the following, we will for simplicity set \(t = 1\).

\[ P(g < u) = P^{(1)}(u) + P^{(2)}(u) \quad (4) \]

Expressing the fact that the propagation is free between \(t = 0\) and \(t = u\) and that the particle has not hit the boundary between \(t = u\) and \(t = 1\), we get

\[ P^{(i)}(u) = \int d^2\overrightarrow{r}_0 \int d^2\overrightarrow{r} \frac{1}{2\pi u} e^{-\frac{\|\overrightarrow{r}_0\|^2}{2u}} G^{(i)}(\overrightarrow{r}, 1; \overrightarrow{r}_0, u) \quad ; \quad i = 1, 2 \quad (5) \]

Figure 1: The frontier \(F = \{OI_1, OI_2\}\) of the wedge divides the plane into 2 regions, (1) and (2).

Suppose that the particle reaches some point \(\overrightarrow{r}_0\) at time \(t = u\) (see Figure 1). Clearly, if \(\overrightarrow{r}_0\) belongs to region (1) (resp.(2)), the particle must stay in (1) (resp.(2)) between \(t = u\) and \(t = 1\) in order to satisfy the condition \(g < u\). We can therefore write

\[ P(g < u) = P^{(1)}(u) + P^{(2)}(u) \quad (4) \]

Expressing the fact that the propagation is free between \(t = 0\) and \(t = u\) and that the particle has not hit the boundary between \(t = u\) and \(t = 1\), we get

\[ P^{(i)}(u) = \int_{(i)} d^2\overrightarrow{r}_0 \int_{(i)} d^2\overrightarrow{r} \frac{1}{2\pi u} e^{-\frac{\|\overrightarrow{r}_0\|^2}{2u}} G^{(i)}(\overrightarrow{r}, 1; \overrightarrow{r}_0, u) \quad ; \quad i = 1, 2 \quad (5) \]
The propagator $G^{(1)}$ satisfying the diffusion equation with Dirichlet boundary conditions on $F$ is given by

$$G^{(1)} = \frac{2}{\phi (1-u)} \sum_{m=1}^{\infty} \sin \frac{m\theta \pi}{\phi} \sin \frac{m\theta_{0} \pi}{\phi} e^{-\frac{r^2 + r_{0}^2}{2(1-u)}} I_{m/2} \left( \frac{rr_{0}}{1-u} \right)$$

(6)

where $I_{\nu}$ is a modified Bessel function and the notations are defined on Figure 1.

Performing the spatial integrations in (5), we get \[18\]

$$P^{(1)}(u) = \frac{1}{\pi} \sum_{p,k=0}^{\infty} u^{p+k} \frac{[\Gamma(p/\phi + k + \frac{\pi}{2\phi})]^2}{(2p+k+1)!} = \frac{2}{\pi \arcsin \sqrt{u}}$$

(7)

$P^{(2)}$ is obtained by the change $\phi \to 2\pi - \phi$ in (7). Therefore, for arbitrary values of $\phi$ the law $P(g < u)$ is written in terms of a double series.

As a check, let us first consider the special case $\phi = \pi$. We may write

$$P(g < u) = 2P^{(1)}(u) = \frac{2}{\pi^2} u^{1/2} \sum_{p,k=0}^{\infty} \frac{u^{p+k} [\Gamma(p+k+1/2)]^2}{k! (2p+k+1)!} = \frac{2}{\pi \arcsin \sqrt{u}}$$

(8)

(9)

The fact that one recovers Levy’s second arc-sine law is not surprising since, when $\phi = \pi$, $F$ divides the plane into two half-planes. Therefore the component of the Brownian motion parallel to $F$ factorises and plays no role: we are thus left with a one dimensional problem.

Coming back to general values of $\phi$, we can derive the behavior of the probability density $\mathcal{P}$ ($\equiv \frac{dP}{du}(g < u)$) when $u \to 0$ and $u \to 1$. By using (11) and (7), one gets a power-law behavior when $u \to 0^+$

$$\mathcal{P}(u) \sim \frac{1}{\pi} u^{-1/2} \quad \text{for} \quad \phi = \pi$$

(10)

$$\mathcal{P}(u) \sim C(\mu) u^{\mu-1} \quad \text{for} \quad \phi \neq \pi$$

(11)

with

$$C(\mu) = \frac{\mu^2}{2\pi^2} \frac{[\Gamma(\mu/2)]^2}{\Gamma(\mu+1)}$$

(12)

and

$$\mu = \frac{\pi}{2\pi - \phi} \quad \text{when} \quad 0 < \phi < \pi$$

(13)

$$\mu = \frac{\pi}{\phi} \quad \text{when} \quad \pi < \phi < 2\pi$$

(14)

Now, for the limit $u \to 1^-$, using asymptotic expansions for $\Gamma$ functions and also an equivalence between series and integrals, we get

$$\mathcal{P}(u) \sim \frac{1}{\pi} \frac{1}{\sqrt{1-u}}$$

(15)

i.e. the same behavior as for (11). The expression (15) doesn’t depend on $\phi$ and we have already seen that $\phi = \pi$ gives the Levy’s law. Remark that $u \to 1^-$ corresponds to Brownian curves
that stop close to $F$. Therefore, between $t = u$ and $t = 1$, the Brownian particle only “sees” an infinitesimal part of $F$, i.e. a straight line as for $\phi = \pi$. This is, in our opinion, why the result (15) doesn’t depend on $\phi$. Actually, it only depends on the fact that the plane is divided by $F$ into two regions. We will come back to this point latter on.

Figure 2: $F$ consists in $n$ semi-infinite straight lines starting from $O$.

To go further, let us remark that for $F = \{OI_i ; i = 1, 2, \ldots, n\}$ as in Figure 2, (14) becomes simply:

$$P(g < u) = \sum_{i=1}^{n} P^{(i)}(u)$$

(16)

(Replace $\phi$ by $\phi_i$ in (7) in order to get $P^{(i)}$).

Let us now specialize to the situation when $F$ is symmetric and $n$ is even ($n = 2l$). In that case, $F$ consists in $l$ infinite straight lines crossing at point $O$ and dividing the plane into $2l$ equal angular sectors, each one of angle $\phi = \pi/l$.

Figure 3: $F$ is symmetric. The analytic form of $P(g < u)$ will depend on the parity of $l$. Thus, it will be different for the cases a) and b). For further explanations, see text.
Equation (16) writes
\[
P(g < u) \equiv P_l(u) = \frac{2}{\pi^2} l^2 u^{l/2} \sum_{p=0}^{\infty} u^p \sum_{k=0}^{\infty} \frac{\Gamma(lp + k + l/2)^2 u^k}{(2lp + l + k)!} \frac{1}{k!}
\] (17)

\(l\) being an integer, we can sum the series and, finally, get
\[
P_l(u) = \frac{2l}{\pi} \left( \sum_{k=0}^{l-1} \arcsin \left( \sqrt{u} \cos \frac{2\pi k}{l} \right) \right), \quad l \text{ odd}
\] (18)
\[
P_l(u) = \frac{2l}{\pi^2} \left( \sum_{k=0}^{l-1} (-1)^k \left( \arcsin \left( \sqrt{u} \cos \frac{\pi k}{l} \right) \right)^2 \right), \quad l \text{ even}
\] (19)

which is the central result of this paper.

We remark that the correct small \(u\) behavior for \(P_l(u)\) follows from the two identities
\[
\sum_{k=0}^{l-1} \left( \cos \frac{2\pi k}{l} \right)^m = 0, \quad l \text{ odd, } m = 1, 3, \ldots, l - 2
\] (20)
\[
\sum_{k=0}^{l-1} (-1)^k \left( \cos \frac{\pi k}{l} \right)^m = 0, \quad l \text{ even, } m = 0, 2, 4, \ldots, l - 2
\] (21)

In particular
\[
P_1(u) = \frac{2}{\pi} \arcsin \sqrt{u}
\] (22)
\[
P_2(u) = \frac{4}{\pi^2} (\arcsin \sqrt{u})^2 = P_1^2
\] (23)
\[
P_3(u) = \frac{6}{\pi} \left( \arcsin \sqrt{u} - 2 \arcsin \sqrt{u}/2 \right)
\] (24)
\[
P_4(u) = \frac{8}{\pi^2} \left( (\arcsin \sqrt{u})^2 - 2 \left( \arcsin \sqrt{u}/2 \right)^2 \right)
\] (25)
\[
P_5(u) = \frac{10}{\pi} \left( \arcsin \sqrt{u} - 2 \arcsin \left( \cos \frac{\pi}{5} \sqrt{u} \right) + 2 \arcsin \left( \cos \frac{2\pi}{5} \sqrt{u} \right) \right)
\] (26)

These functions are displayed below in Figure 4.

As expected, the Levy’s second arc-sine law is recovered in (22). Moreover, the result (23) is straightforward since, when \(l = 2\), the 2 components of the Brownian motion factorize. Thus, for \(l = 2\), (\(\arcsin\))\(^2\) functions appear. What is surprising is that they will appear each time \(l\) is even while being absent when \(l\) is odd.

For the probability density, \(P_l \equiv dP_l/du\), with (18) and (19), we obtain:
\[
P_l(u) \sim \frac{l}{\pi} \frac{1}{\sqrt{1 - u}} \quad \text{when} \quad u \to 1^-
\] (27)

This is consistent with (15) that corresponds to \(l = 1\).
Figure 4: The distribution functions $P_l(u)$ for $l = 1, \ldots, 5$.

We now present a formula which relates the first passage and the last passage time distribution. The starting point is (4) and (5) which may be rewritten as

$$P(g < u) = \int Pr(T > 1 - u|r_0) \frac{1}{u} e^{\frac{r_0^2}{2u}} r_0 \, dr_0$$

where $Pr(T > (1 - u)|r_0)$ is the probability distribution of the first passage time $T$ through $F$, given that the process starts at $r_0$. Then, by scaling one has

$$Pr(T > (1 - u)|r_0) = Pr(T > \frac{(1 - u)}{r_0^2}|1)$$

(29)

By a simple change of variables it follows that

$$P(g < \frac{1}{1 + t}) = \int Pr(T > \frac{t}{2x}|1) e^{-x} \, dx$$

(30)

Therefore

$$P(g < \frac{1}{1 + t}) = E(e^{-\frac{T}{2}})$$

(31)

which is a relation between the first passage characteristic function for a process starting at $r_0 = 1$ and the probability distribution of the last passage time. Interestingly enough this formula can also be derived in a more intrinsic fashion using only time inversion and scaling [17]. As an application, let us derive the density of first passage time in a wedge of angle $\phi = \frac{\pi}{3}$. In the context of the capture problem mentioned in the introduction, this corresponds to a set of three identical and independent particles [11]. In this case, the distribution $P(g)$ is given in eq (24). By an inverse Laplace transform [31] gives the density of first passage time:

$$f(T) = \frac{6}{\pi^2 T} e^{-\frac{1}{4T}} \left( \int_0^{\sqrt{\frac{\pi}{8}}} e^{y^2} \, dy - 2 \int_0^{\sqrt{\frac{\pi}{8}}} e^{y^2} \, dy \right)$$

(32)

One can check that this formula is in agreement with (16) of [11] which expresses the first collision time probability for a given set of initial conditions. By averaging this formula over the angle and setting $r = 1$ one recovers [32].
ACKNOWLEDGMENT We are particularly grateful to Prof. Marc Yor for introducing us to the subject and for pointing out (31).

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