Relativistic Generalization and Extension to the Non-Abelian Gauge Theory of Feynman’s Proof of the Maxwell Equations

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Abstract

R.P. Feynman showed F.J. Dyson a proof of the Lorentz force law and the homogeneous Maxwell equations, which he obtained starting from Newton’s law of motion and the commutation relations between position and velocity for a single nonrelativistic particle. We formulate both a special relativistic and a general relativistic versions of Feynman’s derivation. Especially in the general relativistic version we prove that the only possible fields that can consistently act on a quantum mechanical particle are scalar, gauge and gravitational fields. We also extend Feynman’s scheme to the case of non-Abelian gauge theory in the special relativistic context.
1 Introduction

In 1990 F.J. Dyson published a paper [2] about one of R.P. Feynman’s works, the proof of the Lorentz force law and the homogeneous Maxwell equations. According to Dyson [1], Feynman showed Dyson the proof in 1948. Feynman started with the commutation relations between position and velocity of a single nonrelativistic particle obeying Newton’s law of motion and deduced the existence of electric and magnetic fields satisfying the equations of Maxwell. However he had never published his proof. After Feynman’s death, Dyson published it with some editorial comments. Thanks to Dyson, Feynman’s work is now available to us.

Feynman’s proof is mathematically rigorous. However relativistic covariance is not manifest in his proof. In this paper we propose both a special and a general relativistic generalization of it. Especially in the general relativistic version we show that the only possible fields that can consistently act on a quantum mechanical particle are scalar, gauge and gravitational fields. We also extend Feynman’s scheme to the case of non-Abelian gauge theory. We add some remarks to each case.

2 Review of Feynman’s proof

First we review Feynman’s proof of the Maxwell equations following Dyson [2].

Their assumptions:

(i) A particle is moving in a 3-dimensional Euclidean space with position \(x_i(t)\) \((i = 1, 2, 3)\), where \(t\) is time.

(ii) Its position and velocity \(\dot{x}_i(t)\) satisfy the commutation relations

\[
\begin{align*}
[x_i, x_j] &= 0, \\
m[x_i, \dot{x}_j] &= i\hbar \delta_{ij}. 
\end{align*}
\]

(iii) It obeys the equation of motion

\[
m\ddot{x}_i = F_i(x, \dot{x}, t).
\]

Their results:
(i) The force $F_i(x, \dot{x}, t)$ can be written as
\[ F_i(x, \dot{x}, t) = E_i(x, t) + \epsilon_{ijk} \langle \dot{x}_j B_k(x, t) \rangle, \] (2.4)
where the symbol $\langle \cdots \rangle$ refers to the Weyl-ordering prescription. Later we shall give explanation on this prescription.

(ii) The fields $E_i(x, t)$ and $B_i(x, t)$ satisfy the homogeneous Maxwell equations
\[ \text{div } B = 0, \] (2.5)
\[ \frac{\partial B}{\partial t} + \text{rot } E = 0, \] (2.6)
which implies that there exist a scalar potential $\phi(x, t)$ and a vector potential $A_i(x, t)$ such that
\[ B = \text{rot } A, \] (2.7)
\[ E = - \text{grad } \phi - \text{rot } A. \] (2.8)

From these results they identified $E$ and $B$ with electric and magnetic field respectively. Dyson appreciated the point that the proof shows that the only possible fields that can consistently act on a quantum mechanical particle are gauge fields.

Their proof: Differentiating equation (2.2) with respect to time and using (2.3), one obtains
\[ m [ \dot{x}_i, \dot{x}_j ] + [ x_i, F_j ] = 0. \] (2.9)
This allows one to write
\[ m [ \dot{x}_i, \dot{x}_j ] = - [ x_i, F_j ] = \frac{i\hbar}{m} \epsilon_{ijk} B_k. \] (2.10)
One may consider this equation as the definition of the field $B$. Equation (2.10) may be written as
\[ B_k = \frac{m^2}{2i\hbar} \epsilon_{klm} [ \dot{x}_l, \dot{x}_m ]. \] (2.11)
The field $B$ would depend on $x, \dot{x}$ and $t$. But the Jacobi identity and Eq. (2.2) imply
\[ [ x_i, B_k ] = \frac{m^2}{2i\hbar} \epsilon_{klm} [ x_i, [ \dot{x}_l, \dot{x}_m ] ] \]
\[ = \frac{m^2}{2i\hbar} \epsilon_{klm} \left( [ [ x_i, \dot{x}_l ], \dot{x}_m ] + [ \dot{x}_l, [ x_i, \dot{x}_m ] ] \right) \]
\[ = 0, \] (2.12)
which means that $B$ is a function of $x$ and $t$ only. For a function $f(x,t)$ one has the formula

$$[\hat{x}_k, f(x,t)] = -\frac{i\hbar}{m} \frac{\partial f}{\partial x_k}, \quad (2.13)$$

which is easily verified by Eqs. (2.1), (2.2). One can use Eq. (2.13) to obtain

$$[\hat{x}_k, B_k] = -\frac{i\hbar}{m} \frac{\partial B_k}{\partial x_k}. \quad (2.14)$$

On the other hand Eq. (2.11) and the Jacobi identity give

$$[\hat{x}_k, B_k] = \frac{m^2}{2i\hbar} \epsilon_{klm} [\hat{x}_k, [\hat{x}_l, \hat{x}_m]] = 0. \quad (2.15)$$

These prove (2.5).

Next one takes Eq. (2.4) as the definition of the field $E$. Here it is necessary to explain the Weyl-ordering. It refers to complete symmetrization of operator-products, for instance,

$$\langle x_i \hat{x}_j \rangle = \frac{1}{2} (x_i \hat{x}_j + \hat{x}_j x_i),$$

$$\langle x_i \hat{x}_j \hat{x}_k \rangle = \frac{1}{6} (x_i \hat{x}_j \hat{x}_k + x_i \hat{x}_k \hat{x}_j + \hat{x}_j x_i \hat{x}_k + \hat{x}_j \hat{x}_k x_i + \hat{x}_k x_i \hat{x}_j + \hat{x}_k \hat{x}_j x_i)$$

and so on. Again, $E$ would depend on $x, \dot{x}$ and $t$ in general, but using Eqs. (2.4), (2.10), (2.2) and (2.12) in this order, one obtains

$$[x_l, E_i] = [x_l, F_i] - \epsilon_{ijk} \langle [x_l, \dot{x}_j] B_k \rangle \hat{x}_j [x_l, B_k] = -\frac{i\hbar}{m} \epsilon_{lik} B_k - \epsilon_{ijk} \frac{i\hbar}{m} \delta_{lj} B_k = 0, \quad (2.16)$$

which says that $E$ is a function of $x$ and $t$ only.

It remains to prove the second Maxwell equation (2.6). One takes the total time-derivative of (2.11) and obtains

$$\frac{\partial B_k}{\partial t} + \langle \hat{x}_j \frac{\partial B_k}{\partial x_j} \rangle = \frac{m^2}{i\hbar} \epsilon_{klm} [\hat{x}_l, \hat{x}_m]. \quad (2.17)$$

Now by (2.3), (2.4), (2.10), (2.13) and (2.7), the right side of (2.17) becomes

$$\frac{m}{i\hbar} \epsilon_{klm} [\hat{x}_l, E_m + \epsilon_{mij} \langle \hat{x}_i B_j \rangle] = \frac{m}{i\hbar} \left( \epsilon_{klm} [\hat{x}_l, E_m] + (\delta_{ik} \delta_{jl} - \delta_{ij} \delta_{lk}) \langle [\hat{x}_l, \dot{x}_i] B_j + \dot{x}_i [\hat{x}_l, B_j] \rangle \right)$$
\begin{align*}
&= -\epsilon_{klm} \frac{\partial E_m}{\partial x_l} + (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) \left( \frac{1}{m} \epsilon_{lin} B_n B_j - \dot{x}_i \frac{\partial B_j}{\partial x_l} \right) \\
&= -\epsilon_{klm} \frac{\partial E_m}{\partial x_l} + \left( \frac{1}{m} \epsilon_{jkn} B_n B_j - \dot{x}_k \frac{\partial B_j}{\partial x_l} + \dot{x}_l \frac{\partial B_k}{\partial x_i} \right) \\
&= -\epsilon_{klm} \frac{\partial E_m}{\partial x_l} + \langle \dot{x}_i \frac{\partial B_k}{\partial x_i} \rangle. \tag{2.18}
\end{align*}

Eqs. (2.17) and (2.18) give (2.6). End of proof.

Remark: We observe the properties of the commutator [ , ] used in their argument. They are

(i) bilinearity

\[ [\lambda A + \mu B, C] = \lambda [A, C] + \mu [B, C], \tag{2.19} \]
\[ [A, \lambda B + \mu C] = \lambda [A, B] + \mu [A, C], \tag{2.20} \]

(ii) antisymmetry

\[ [A, B] = -[B, A], \tag{2.21} \]

(iii) Jacobi identity

\[ [A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0, \tag{2.22} \]

(iv) Leibniz rule I

\[ [A, BC] = [A, B] C + B [A, C], \tag{2.23} \]

(v) Leibniz rule II

\[ \frac{d}{dt} [A, B] = [\frac{dA}{dt}, B] + [A, \frac{dB}{dt}]. \tag{2.24} \]

The Poisson bracket \{ , \} also possesses the properties from (i) to (iv) as automatic consequence of its definition. However the property (v) is not trivial for the Poisson bracket. Unless the canonical equation of motion is given, the Poisson bracket does not satisfy the property (v). We shall show what could be the matter.

We treat a classical mechanical system of one degree of freedom for simplicity. \(A(q, p)\) and \(B(q, p)\) are observables which do not depend on time explicitly. The Poisson bracket is defined by

\[ \{A, B\} = \frac{\partial A}{\partial q} \frac{\partial B}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial B}{\partial q}. \tag{2.25} \]
Take the total time-derivative of Eq. (2.25) through time-dependence of \( q(t) \) and \( p(t) \). We obtain
\[
\frac{d}{dt}\{A, B\} = \{\frac{dA}{dt}, B\} + A, \frac{dB}{dt}\} - \{A, B\}\left(\frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p}\right).
\] (2.26)

Without the equation of motion \( \dot{q} = f(q, p) \), \( \dot{p} = g(q, p) \), we can say nothing about the last term of Eq. (2.26). If we have the Hamiltonian \( H(q, p) \) and use the canonical equation of motion
\[
\dot{q} = \frac{\partial H}{\partial p},
\]
\[
\dot{p} = -\frac{\partial H}{\partial q},
\] (2.27)
the last term of Eq. (2.26) vanishes and the Leibniz rule II is satisfied. It is one of virtues of Feynman’s proof that there is no need of a priori existence of Hamiltonian, Lagrangian, canonical equation or Heisenberg equation.

## 3 Special relativistic version

It is a weak point of Feynman’s derivation that Lorentz covariance is not manifest \[3\] \[4\]. We propose a special relativistic version of it.

Assumptions:

(i) A particle is moving in \( d \)-dimensional Minkowski space-time with coordinate \( x^\mu(\tau) \) \( (\mu = 0, 1, \cdots, d-1) \), where \( \tau \) is a parameter.

(ii) Its coordinate and velocity \( \dot{x}^\mu(\tau) \) satisfy the commutation relations
\[
[ x^\mu, x^\nu ] = 0,
\]
\[
m[ x^\mu, \dot{x}^\nu ] = -i\hbar \eta^{\mu\nu},
\] (3.1) (3.2)
where the dot refers to the derivative with respect to \( \tau \), and \( \eta^{\mu\nu} \) is Minkowskian metric \( \eta = \text{diag}(+1, -1, \cdots, -1) \).

(iii) It obeys the equation of motion
\[
m \ddot{x}^\mu = F^\mu(x, \dot{x}).
\] (3.3)
Results:

(i) The force \( F^\mu(x, \dot{x}) \) can be written as

\[
F^\mu(x, \dot{x}) = G^\mu(x) + \langle F^\mu_\nu(x) \dot{x}^\nu \rangle,
\]

where the symbol \( \langle \cdots \rangle \) also refers to the Weyl-ordering prescription.

(ii) The fields \( G^\mu(x), F^\mu_\nu(x) \) satisfy

\[
\partial_\mu G_\nu - \partial_\nu G_\mu = 0, \tag{3.5}
\]

\[
\partial_\mu F^\nu_\rho + \partial_\nu F^\rho_\mu + \partial_\rho F^\mu_\nu = 0, \tag{3.6}
\]

which implies that there exist a scalar field \( \phi(x) \) and a vector field \( A_\mu(x) \) such that

\[
G_\mu = \partial_\mu \phi, \tag{3.7}
\]

\[
F^\mu_\nu = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.8}
\]

Proof: Differentiating Eq. (3.2) with respect to \( \tau \) and using (3.3), we obtain

\[
m [ \dot{x}^\mu, \dot{x}^\nu ] + [ x^\mu, F^\nu ] = 0. \tag{3.9}
\]

We define \( F^{\mu\nu} \) by

\[
m [ \dot{x}^\mu, \dot{x}^\nu ] = - [ x^\mu, F^\nu ] = - \frac{i\hbar}{m} F^{\mu\nu}. \tag{3.10}
\]

By definition, \( F^{\mu\nu} = - F^{\nu\mu} \). \( F^{\mu\nu} \) would depend on \( x \) and \( \dot{x} \). But using the Jacobi identity and Eq. (3.2) we get

\[
[x^\lambda, F^{\mu\nu}] = - \frac{m^2}{\hbar} [x^\lambda, [\dot{x}^\mu, \dot{x}^\nu]]
\]

\[
= - \frac{m^2}{\hbar} \left( [ [x^\lambda, \dot{x}^\mu], \dot{x}^\nu ] + [ \dot{x}^\mu, [x^\lambda, \dot{x}^\nu] ] \right)
\]

\[
= 0,
\]

which means that \( F^{\mu\nu} \) is a function of \( x \) only.

Here we should pay attention to the fact that raising and lowering of tensor-indices by \( \eta^{\mu\nu} \) and \( \eta_{\mu\nu} \) are compatible with operator-product. For example if we define \( \dot{x}_\mu \) and \( F_{\nu\rho} \) as

\[
\dot{x}_\mu = \eta_{\mu\alpha} \dot{x}^\alpha, \tag{3.12}
\]

\[
F_{\nu\rho} = \eta_{\nu\alpha} \eta_{\rho\beta} F^{\alpha\beta}, \tag{3.13}
\]
respectively, it is justified to write

\[ F_{\nu\rho} = -\frac{m^2}{i\hbar} [\dot{x}_\nu, \dot{x}_\rho]. \quad (3.14) \]

Eqs. (3.12), (3.12) imply

\[ m[ x^\mu, \dot{x}_\nu ] = -i\hbar \delta^\mu_\nu, \quad (3.15) \]

from which we derive a useful formula

\[ [\dot{x}_\nu, f(x)] = \frac{i\hbar}{m} \frac{\partial f}{\partial x^\nu}, \quad (3.16) \]

for a function \( f(x) \).

The Jacobi identity with (3.14) and (3.16) implies

\[
0 = [\dot{x}_\mu, [\dot{x}_\nu, \dot{x}_\rho]] + [\dot{x}_\nu, [\dot{x}_\rho, \dot{x}_\mu]] + [\dot{x}_\rho, [\dot{x}_\mu, \dot{x}_\nu]]
- \frac{i\hbar}{m^2} \left( [\dot{x}_\mu, F_{\nu\rho}] + [\dot{x}_\nu, F_{\rho\mu}] + [\dot{x}_\rho, F_{\mu\nu}] \right)
- \frac{\hbar^2}{m^3} \left( \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} \right),
\]

which is nothing but Eq. (3.7).

Next we take Eq. (3.4) as the definition of \( G_\mu \), that is,

\[ G_\mu = F_\mu(x, \dot{x}) - \langle F^{\mu\nu}(x) \dot{x}_\nu \rangle. \quad (3.18) \]

Again, \( G_\mu \) might depend on \( x \) and \( \dot{x} \), but using Eqs. (3.10), (3.11) and (3.15) we get

\[
[x^\lambda, G_\mu] = [x^\lambda, F_\mu] - \langle F^{\mu\nu}[x^\lambda, \dot{x}_\nu] \rangle
= \frac{i\hbar}{m} F^{\lambda\mu} + \frac{i\hbar}{m} F^{\nu\mu} \delta^\lambda_\nu
= 0,
\]

which says that \( G_\mu \) is also a function of \( x \) only.

It completes the proof to show equation (3.5). Eqs. (3.18), (3.3), (3.16) and (3.10) imply

\[
[x^\lambda, G_\nu] = [x^\lambda, F_\nu] - \langle [\dot{x}_\mu, F_{\nu\rho}] \dot{x}_\rho \rangle - \langle F_{\nu\rho}[\dot{x}_\mu, \dot{x}_\rho] \rangle
= m[\dot{x}_\mu, \dot{x}_\nu] - \frac{i\hbar}{m} (\partial_\mu F_{\nu\rho} \dot{x}_\rho) + \frac{i\hbar}{m^2} F_{\nu\rho} F_{\mu\rho},
\]
which leads to

\[
\left[ \dot{x}_\mu, G_{\nu} \right] - \left[ \dot{x}_\nu, G_\mu \right] = m\left[ \dot{x}_\mu, \ddot{x}_\nu \right] - m\left[ \dot{x}_\nu, \ddot{x}_\mu \right] - \frac{i\hbar}{m} \left( (\partial_\mu F_{\nu\rho} - \partial_\nu F_{\mu\rho}) \dot{x}^\rho \right) + \frac{i\hbar}{m^2} (F_{\nu\rho} F_{\mu}^\rho - F_{\mu\rho} F_{\nu}^\rho)
\]

\[
= m \frac{d}{d\tau} \left[ \dot{x}_\mu, \dot{x}_\nu \right] - \frac{i\hbar}{m} \left( (\partial_\mu F_{\nu\rho} - \partial_\nu F_{\mu\rho}) \dot{x}^\rho \right)
\]

\[
= -\frac{i\hbar}{m} \frac{d}{d\tau} F_{\mu\nu} - \frac{i\hbar}{m} \left( (\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu}) \dot{x}^\rho \right)
\]

\[
= -\frac{i\hbar}{m} \left( (\partial_\mu F_{\rho\nu} + \partial_\nu F_{\rho\mu}) \dot{x}^\rho \right).
\]

(3.20)

Eqs. (3.6), (3.16) and (3.20) give (3.5). End of proof.

Remarks: The properties of the commutator which we use above are same as those used by Feynman. We also use the Leibniz rule II (2.24).

The dimension of the space-time is irrelevant to our proof. This point forms a contrast to Feynman’s case. In his proof he used the complete antisymmetric tensor \( \epsilon_{ijk} \), which depends on the dimension of space(-time). The signature of the metric \( \eta_{\mu\nu} \) is also irrelevant. It is sufficient that the metric is symmetric and regular.

The parameter \( \tau \) is introduced as just a parameter. What could it be? It may be identified with the proper-time but that is wrong. First we do not use the condition

\[
\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 1,
\]

(3.21)

which claims \( \tau \) is the proper-time. Secondly this condition is contradictory to the assumption (3.2). Eq. (3.21) implies

\[
\left[ x^\lambda, \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right] = 0.
\]

(3.22)

However Eq. (3.2) implies

\[
\left[ x^\lambda, \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right] = -\frac{2i\hbar}{m} \dot{x}^\lambda.
\]

(3.23)

Apparently Eqs. (3.22), (3.23) are inconsistent.

Our argument is not invariant under arbitrary reparametrization. If we reparametrize \( \tau \) by another parameter \( \tau' \) as \( \tau = f(\tau') \), Eq. (3.2) is transformed to

\[
m \left[ x^\mu, x'^\nu \right] = -i\hbar \dot{f}' \eta'^{\mu\nu},
\]

(3.24)
where the dot and prime refer to differentiation with respect to $\tau'$. If the right-hand side of Eq. (3.24) is not constant, the derivation of Eq. (3.9) cannot be justified. Only permissible reparametrization is affine transformation, $\tau = a\tau' + b$. In this case, metric, force and other quantities are transformed as follows:

$$
\dot{x}^\mu = a \dot{x}^\mu \\
\ddot{x}^\mu = a^2 \ddot{x}^\mu \\
\eta'^{\mu\nu} = a \eta^{\mu\nu} \\
F'^\mu = a^2 F^\mu \\
G'^\mu = a^2 G^\mu \\
F'^{\mu\nu} = a^2 F^{\mu\nu}.
$$

4 General relativistic version

Absence of gravity is one of the unsatisfactory points of Feynman’s argument and our previous one. However there is gravitation in our world! How can we harmonize it with our framework?

Here we propose one way to introduce it. We take notice of the commutation relation (3.2). It seems natural to replace the Minkowskian metric $\eta_{\mu\nu}$ by an arbitrary metric $g_{\mu\nu}(x)$ to derive gravity. We have found that this assumption leads to an anticipated result.

Assumptions:

(i) A particle is moving in $d$-dimensional space-time with coordinate $x^\mu(\tau) \ (\mu = 0, 1, \cdots, d - 1)$, where $\tau$ is a parameter.

(ii) Its coordinate and velocity $\dot{x}^\mu(\tau)$ satisfy the commutation relations

$$
[x^\mu, x^\nu] = 0,
$$

$$
m [x^\mu, \dot{x}^\nu] = -i\hbar g^{\mu\nu}(x),
$$

where the dot refers to the derivative with respect to $\tau$, and $g^{\mu\nu}(x)$ is a metric of the space-time.

(iii) It obeys the equation of motion

$$
m \ddot{x}^\mu = F^\mu(x, \dot{x}).$$
Results:

(i) The force $F^\mu(x, \dot{x})$ can be written as

$$F^\mu(x, \dot{x}) = G^\mu(x) + \langle F^\mu_\nu(x) \dot{x}^\nu \rangle - m \langle \Gamma^\mu_\nu_\rho \dot{x}^\nu \dot{x}^\rho \rangle.$$  \hspace{1cm} (4.4)

(ii) The fields $G^\mu(x), F^\mu_\nu(x)$ satisfy

$$\partial_\mu G_\nu - \partial_\nu G_\mu = 0,$$ \hspace{1cm} (4.5)

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0.$$ \hspace{1cm} (4.6)

(iii) $\Gamma^\mu_\nu_\rho(x)$ is a Levi-Civita connection, which is defined by

$$\Gamma^\mu_\nu_\rho = \frac{1}{2}(\partial_\rho g^\mu_\nu + \partial_\nu g^\mu_\rho - \partial_\mu g^\nu_\rho).$$ \hspace{1cm} (4.7)

Speaking after Dyson's fashion, these results say that the only possible fields that can consistently act on a quantum mechanical particle are scalar, gauge and gravitational fields.

Proof: The tactics are almost the same as in the preceding section but there is a problem concerning compatibility between raising and lowering of tensor-indices and ordering of operator-product. The metric $g_{\mu\nu}(x)$ is not a commutative operator. Here we define lowering of the index of $\dot{x}^\mu$ as

$$\dot{x}_\mu = \langle g_{\mu\nu}(x) \dot{x}^\nu \rangle,$$ \hspace{1cm} (4.8)

using Weyl ordering $\langle \cdots \rangle$. We write derivatives of a function $f(x)$ as

$$\partial_\nu f = \frac{\partial f}{\partial x^\nu},$$ \hspace{1cm} (4.9)

$$\partial^\nu f = g^{\nu\mu} \partial_\mu f.$$  

These definitions and Eqs. (4.4), (4.5) give useful formulas

$$[\dot{x}_\nu, f(x)] = \frac{i\hbar}{m} \partial_\nu f,$$ \hspace{1cm} (4.10)

$$[\dot{x}^\nu, f(x)] = \frac{i\hbar}{m} \partial^\nu f.$$
For a general tensor field $T^{\alpha\beta}(x)$, we define lowering of its indices as usual,

$$T_{\alpha\beta} = g_{\alpha\mu} g_{\beta\nu} T^{\mu\nu}. \quad (4.11)$$

Now we begin the proof. Differentiating Eq. (4.2) with respect to $\tau$ and using (4.3), we obtain

$$m \left[ \dot{x}^\mu, \dot{x}^\nu \right] + \left[ x^\mu, F^\nu \right] = -i\hbar \langle \partial^\rho g^{\mu\nu} \dot{x}^\rho \rangle. \quad (4.12)$$

We define $W^{\mu\nu}$ as

$$W^{\mu\nu} = -\frac{m^2}{i\hbar} \left[ \dot{x}^\mu, \dot{x}^\nu \right]. \quad (4.13)$$

The Jacobi identity and Eqs. (4.2), (4.10) imply

$$[x^\lambda, W^{\mu\nu}] = -\frac{m^2}{i\hbar} \left[ x^\lambda, [\dot{x}^\mu, \dot{x}^\nu] \right]$$

$$= -\frac{m^2}{i\hbar} \left( \left[ \left[ x^\lambda, \dot{x}^\mu \right], \dot{x}^\nu \right] + \left[ \dot{x}^\mu, \left[ x^\lambda, \dot{x}^\nu \right] \right] \right)$$

$$= m \left( [ g^{\lambda\mu}, \dot{x}^\nu ] + [ \dot{x}^\mu, g^{\lambda\nu} ] \right)$$

$$= -i\hbar (\partial^\nu g^{\lambda\mu} - \partial^\rho g^{\lambda\nu}). \quad (4.14)$$

Therefore if we put

$$F^{\mu\nu} = W^{\mu\nu} - m \langle (\partial^\nu g^{\lambda\mu} - \partial^\rho g^{\lambda\nu}) \dot{x}_\lambda \rangle, \quad (4.15)$$

we obtain

$$[x^\lambda, F^{\mu\nu}] = 0, \quad (4.16)$$

which means that $F^{\mu\nu}$ is a function of $x$ only. Eqs. (4.12), (4.13) and (4.15) imply

$$[x^\mu, F^\nu] = \frac{i\hbar}{m} F^{\mu\nu} + \frac{2i\hbar}{m} \langle \Gamma^{\nu\lambda\mu} \dot{x}_\lambda \rangle,$$

where we define $\Gamma^{\nu\lambda\mu}$ as

$$\Gamma^{\nu\lambda\mu} = -\frac{1}{2}(\partial^\mu g^{\lambda\nu} + \partial^\lambda g^{\mu\nu} - \partial^\nu g^{\lambda\mu}), \quad (4.18)$$

which is nothing but the Levi-Civita connection.

Next we want to prove equation (4.6). For that purpose we should lower the indices of $F^{\mu\nu}$. It is easily seen that

$$-\frac{m^2}{i\hbar} \langle \left[ \dot{x}_\alpha, \dot{x}_\beta \right] \rangle = -\frac{m^2}{i\hbar} \langle \left[ \langle g_{\alpha\mu} \dot{x}^\mu \rangle, \langle g_{\beta\nu} \dot{x}^\nu \rangle \right] \rangle.$$
\[ \begin{align*}
&= -\frac{m^2}{i\hbar} \left( g_{\alpha\mu} g_{\beta\nu} [\dot{x}^\mu, \dot{x}^\nu] + g_{\alpha\mu} [\dot{x}^\mu, g_{\beta\nu} \dot{x}^\nu] \\
&\quad + g_{\beta\nu} [g_{\alpha\mu}, \dot{x}^\nu] \dot{x}^\mu \right) \\
&= \langle g_{\alpha\mu} g_{\beta\nu} W^{\mu\nu} \rangle - m \langle \partial_\alpha g_{\beta\nu} \dot{x}^\nu - \partial_\beta g_{\alpha\mu} \dot{x}^\mu \rangle. \tag{4.19}
\end{align*} \]

Eqs. (4.11), (4.15) and (4.19) imply
\[ F_{\alpha\beta} = g_{\alpha\mu} g_{\beta\nu} F^{\mu\nu} = \langle g_{\alpha\mu} g_{\beta\nu} W^{\mu\nu} \rangle - m \langle \partial_\alpha g_{\beta\nu} \dot{x}^\nu - \partial_\beta g_{\alpha\mu} \dot{x}^\mu \rangle = -\frac{m^2}{i\hbar} \langle [\dot{x}_\alpha, \dot{x}_\beta] \rangle, \tag{4.20} \]

when the second line turns to the third, the equation \( 0 = \partial_\beta (g_{\alpha\mu} g^{\lambda\mu}) = (\partial_\beta g_{\alpha\mu}) g^{\lambda\mu} + g_{\alpha\mu} \partial_\beta g^{\lambda\mu} \) is used. The Jacobi identity Eqs. (4.20) and (4.10) give
\[ 0 = \langle [\dot{x}_\mu, [\dot{x}_\nu, \dot{x}_\rho]] + [\dot{x}_\nu, [\dot{x}_\rho, \dot{x}_\mu]] + [\dot{x}_\rho, [\dot{x}_\mu, \dot{x}_\nu]] \rangle = -\frac{m^2}{i\hbar} \langle [\dot{x}_\mu, [\dot{x}_\nu, \dot{x}_\rho]] \rangle = \frac{\hbar^2}{m^3} (\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu}), \tag{4.21} \]

which is just Eq. (4.6).

As in the previous proofs, we take Eq. (4.4) as the definition of \( G^\mu \), that is
\[ G^\mu = F^\mu(x, \dot{x}) - \langle F^{\mu\nu}(x) \dot{x}_\nu \rangle + m \langle \Gamma^{\mu\rho\lambda} \dot{x}_\nu \dot{x}_\rho \rangle. \tag{4.22} \]

Using Eqs. (4.22), (4.17) and (4.10) in this order, we obtain
\[ [x^\lambda, G^\mu] = [x^\lambda, F^\mu] - \langle F^{\mu\nu}[x^\lambda, \dot{x}_\nu] \rangle + m \langle \Gamma^{\mu\rho\lambda} [x^\lambda, \dot{x}_\rho] \rangle + \frac{i\hbar}{m} F^{\lambda\mu} + 2 i\hbar \langle \Gamma^{\lambda\rho\lambda} \dot{x}_\mu \rangle + \frac{i\hbar}{m} F^{\mu\nu} \delta^\lambda_\nu \\
- i\hbar (\langle \Gamma^{\lambda\rho\lambda} \dot{x}_\mu \rangle + \langle \Gamma^{\mu\rho\lambda} \dot{x}_\nu \delta^\lambda_\rho \rangle) = 0, \tag{4.23} \]

which says that \( G^\mu \) is a function of \( x \) only.

The remaining task is to show equation (4.5). It takes tedious and long calculation, so a reader who is not interested in the detail may skip the following part of
the proof. Lowering of the index of Eq. (4.22) gives

\[ G_\nu = \langle g_{\nu\alpha} F^{\alpha} \rangle - \langle g_{\nu\alpha} F^{\alpha\beta} \dot{x}_\beta \rangle + m \langle g_{\nu\alpha} \Gamma^{\alpha\beta\gamma} \dot{x}_\beta \dot{x}_\gamma \rangle. \] (4.24)

The first term of the right-hand side can be rewritten as

\[ \langle g_{\nu\alpha} F^{\alpha} \rangle = m \langle g_{\nu\alpha} \dot{x}^{\alpha} \rangle = m \langle g_{\nu\alpha} \frac{d}{dt} \langle g^{\alpha\beta} \dot{x}_\beta \rangle \rangle = m \langle g_{\nu\alpha} g^{\alpha\beta} \dot{x}_\beta + g_{\nu\alpha} \partial^\gamma g^{\alpha\beta} \dot{x}_\beta \dot{x}_\gamma \rangle = m \langle \dot{x}_\nu \rangle + m \langle g_{\nu\alpha} \partial^\gamma g^{\alpha\beta} \dot{x}_\beta \dot{x}_\gamma \rangle. \] (4.25)

In the second term of the last line, the indices \( \beta \) and \( \gamma \) of \( \partial^\gamma g^{\alpha\beta} \) are symmetrized, so it is rewritten using Eq. (4.18) as

\[ \frac{1}{2} (\partial^\gamma g^{\alpha\beta} + \partial^\beta g^{\alpha\gamma}) = -\Gamma^{\alpha\beta\gamma} + \frac{1}{2} \partial^\alpha g^{\beta\gamma}. \] (4.26)

Eqs. (4.24), (4.25) and (4.26) imply

\[ G_\nu = m \langle \dot{x}_\nu \rangle + \frac{1}{2} m \langle g_{\nu\alpha} \partial^\alpha g^{\beta\gamma} \dot{x}_\beta \dot{x}_\gamma \rangle - \langle g_{\nu\alpha} F^{\alpha\beta} \dot{x}_\beta \rangle = m \langle \dot{x}_\nu \rangle + \frac{1}{2} m \langle \partial^\alpha g^{\alpha\beta} \dot{x}_\alpha \dot{x}_\beta \rangle - \langle F_{\nu\alpha} g^{\alpha\beta} \dot{x}_\beta \rangle. \] (4.27)

Using Eqs. (4.27), (4.10) and (4.20), we obtain

\[
\begin{align*}
[\dot{x}_\mu, G_\nu] &= m \langle [\dot{x}_\mu, \dot{x}_\nu] \rangle + \frac{1}{2} \langle \partial^\alpha \partial^\beta g^{\alpha\beta} \dot{x}_\alpha \dot{x}_\beta \rangle - \frac{i\hbar}{2m} \langle \partial^\alpha g^{\alpha\beta} (F_{\mu\alpha} \dot{x}_\beta + \dot{x}_\alpha F_{\mu\beta}) \rangle \\
&\quad - \frac{i\hbar}{m} \langle \partial^\beta g^{\beta\gamma} \dot{x}_\beta \rangle + \frac{i\hbar}{m^2} \langle \partial^\gamma F_{\nu\alpha} g^{\alpha\beta} \dot{x}_\beta F_{\mu\beta} \rangle \\
&= m \langle [\dot{x}_\mu, \dot{x}_\nu] \rangle + \frac{1}{2} \langle i\hbar \partial^\beta g^{\alpha\beta} \dot{x}_\alpha \dot{x}_\beta \rangle - \frac{i\hbar}{m} \langle \partial^\beta g^{\alpha\beta} \cdot F_{\nu\alpha} \dot{x}_\beta \rangle \\
&\quad - \frac{i\hbar}{m} \langle \partial^\beta g^{\alpha\beta} \cdot F_{\nu\alpha} \dot{x}_\beta \rangle - \frac{i\hbar}{m^2} \langle \partial^\gamma F_{\nu\alpha} g^{\alpha\beta} \dot{x}_\beta \rangle + \frac{i\hbar}{m^2} \langle \partial^\gamma F_{\nu\alpha} g^{\alpha\beta} \dot{x}_\beta \rangle + \frac{i\hbar}{m^2} \langle \partial^\gamma F_{\nu\alpha} g^{\alpha\beta} \dot{x}_\beta \rangle.
\end{align*}
\] (4.28)

Finally antisymmetrization with respect to the indices \( \mu \) and \( \nu \) gives

\[
\begin{align*}
[\dot{x}_\mu, G_\nu] - [\dot{x}_\nu, G_\mu] &= m \langle [\dot{x}_\mu, \dot{x}_\nu] - [\dot{x}_\nu, \dot{x}_\mu] \rangle - \frac{i\hbar}{m} \langle (\partial^\beta F_{\nu\alpha} - \partial^\beta F_{\mu\alpha}) g^{\alpha\beta} \dot{x}_\beta \rangle \\
&= m \langle \frac{d}{dt} [\dot{x}_\mu, \dot{x}_\nu] \rangle - \frac{i\hbar}{m} \langle (\partial^\beta F_{\nu\alpha} + \partial^\beta F_{\alpha\nu}) \dot{x}_\alpha \rangle \\
&= -\frac{i\hbar}{m} \langle \frac{d}{dt} F_{\mu\nu} \rangle - \frac{i\hbar}{m} \langle (\partial^\rho F_{\nu\rho} + \partial^\rho F_{\rho\nu}) \dot{x}^\rho \rangle \\
&= -\frac{i\hbar}{m} \langle (\partial^\rho F_{\mu\nu} + \partial^\rho F_{\nu\mu} + \partial^\rho F_{\mu\nu}) \dot{x}^\rho \rangle.
\end{align*}
\] (4.29)
Eqs. (4.6), (4.10) and (4.29) imply (4.5). End of proof.

Remark : Unfortunately we have not yet found the way to make our formulation reparametrization-invariant.

5 Non-Abelian gauge theory

We can also bring the non-Abelian gauge field into our scheme. However for this purpose we should admit more assumptions. Our argument here is a special relativistic reconstruction of C.R. Lee’s work [4]. He generalized Feynman’s proof to the case of non-Abelian gauge theory in the nonrelativistic context.

Assumptions :

(i) A particle is moving in \(d\)-dimensional Minkowski space-time with coordinate \(x^\mu(\tau) \ (\mu = 0, 1, \cdots, d - 1)\). And the particle carries isospin (or color) \(I^a(\tau) \ (a = 1, \cdots, n)\). \(I^a\)'s are linearly independent operators.

(ii) Its coordinate and velocity \(\dot{x}^\mu(\tau)\) and isospin satisfy the commutation relations

\[
\begin{align*}
[x^\mu, x^\nu] &= 0, \\
m[x^\mu, \dot{x}^\nu] &= -i\hbar \eta^{\mu\nu}, \\
[I^a, I^b] &= i\hbar f^{abc} I^c, \\
[x^\mu, I^a] &= 0,
\end{align*}
\]

where \(f^{abc}\)'s are the structure constants of the gauge group.

(iii) Its coordinate obeys the equation of motion

\[
m \ddot{x}^\mu = F^\mu(x, \dot{x}, I),
\]

where the force \(F^\mu\) is linear with respect to \(I^a\), that is,

\[
F^\mu(x, \dot{x}, I) = \langle F^\mu_a(x, \dot{x}) I^a \rangle.
\]

And its isospin obeys the equation

\[
\dot{I}^a - f^{ab}_c \langle A_{\mu b}(x) \dot{x}^\mu I^c \rangle = 0,
\]

where \(A_{\alpha\mu}(x)\) is a gauge field.
Results:

(i) The force \( F^\mu(x, \dot{x}, I) \) can be written as

\[
F^\mu(x, \dot{x}, I) = G^\mu_a(x) I^a + \langle F^\mu_{a\nu}(x) I^a \dot{x}^\nu \rangle.
\]

(ii) The fields \( G^\mu_a(x), F^\mu_{a\nu}(x) \) satisfy

\[
(D_\mu G_\nu - D_\nu G_\mu)_a = 0,
\]

\[
(D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu})_a = 0,
\]

where \( D \) denotes covariant derivative with the gauge field \( A_{a\mu} \), for instance,

\[
(D_\mu F_{\nu\rho})_a = \partial_\mu F_{a\nu\rho} - f^{bc}_a A_{b\mu} F_{c\nu\rho}.
\]

(iii) The field \( F^{a\mu\nu} \) is related to the gauge potential \( A_{a\mu} \) by

\[
f^{ab}_c \left( F^{a\mu\nu} - (\partial_\mu A_{a\nu} - \partial_\nu A_{a\mu} - f^{de}_a A_{d\mu} A_{e\nu}) \right) = 0.
\]

If we could remove \( f^{ab}_c \)'s from Eq. (5.12), we would identify \( F^{a\mu\nu} \) with the field strength of the gauge potential \( A_{a\mu} \).

The set of Eqs. (5.7) and (5.8) is known as Wong’s equation [6], which describes the motion of a particle carrying isospin. In other words, it is non-Abelian extension of the Lorentz law. Eq. (5.7) just says that the isospin is parallel-transported along the trajectory of the particle under the influence of the gauge field.

Proof: The tactics are almost the same as in the previous sections. Differentiating Eq. (5.2) with respect to \( \tau \) and using (5.5), we obtain

\[
m \left[ \dot{x}^\mu, \dot{x}^\nu \right] + \left[ x^\mu, F^{\nu} \right] = 0.
\]

We define \( F^{\mu\nu} \) by

\[
F^{\mu\nu} = -\frac{m^2}{\hbar} \left[ \dot{x}^\mu, \dot{x}^\nu \right] = \frac{m}{\hbar} \left[ x^\mu, F^{\nu} \right] = \frac{m}{\hbar} \left( \left[ x^\mu, F^{\nu}_{a} \right] I^a \right),
\]

where we put

\[
F^{\mu\nu}_{a} = \frac{m}{\hbar} \left[ x^\mu, F^{\nu}_{a}(x, \dot{x}) \right],
\]
therefore \( F_{\mu\nu} = \langle F_{a\mu\nu} I^a \rangle \). The Jacobi identity and Eq. (5.2) imply

\[
[x^\lambda, F^{\mu\nu}] = -\frac{m^2}{\hbar} [x^\lambda, [\dot{x}^\mu, \dot{x}^\nu]]
= -\frac{m^2}{\hbar} \left( \left( [x^\lambda, \dot{x}^\mu], \dot{x}^\nu \right) + \left( [\dot{x}^\mu, x^\lambda], \dot{x}^\nu \right) \right)
= 0,
\]

(5.16)

which means that \( F^{\mu\nu} \) is independent of \( \dot{x} \) and \( F_{a\mu\nu} \) is a function of \( x \) only.

Differentiating Eq. (5.4) with respect to \( \tau \) and substituting (5.7) and (5.2), we obtain

\[
0 = \left[ \dot{x}^\mu, I^a \right] + \left[ x^\mu, \dot{I}^a \right]
= \left[ \dot{x}^\mu, I^a \right] + \frac{i\hbar}{m} f^{ab}_c A_{b\mu} \eta^c_{\nu} I^c,
\]

which is rewritten as

\[
\left[ \dot{x}^\mu, I^a \right] - \frac{i\hbar}{m} f^{ab}_c A_{b\mu} I^c = 0
\]

(5.17)

by lowering the index \( \mu \). From Eqs. (5.2) and (5.17), we derive a useful formula

\[
\left[ \dot{x}^\mu, \phi_a(x) I^a \right] = \frac{i\hbar}{m} \left( \frac{\partial \phi_a}{\partial x^\mu} - f^{bc}_a A_{b\mu} \phi_c \right) I^a
= \frac{i\hbar}{m} \left( D^\mu \phi \right)_a I^a
\]

(5.18)

for functions \( \phi_a(x) \) \( (a = 1, \ldots, n) \).

The Jacobi identity with (5.14) and (5.18) implies

\[
0 = \left[ \dot{x}_\mu, [\dot{x}_\nu, \dot{x}_\rho] \right] + \left[ \dot{x}_\nu, [\dot{x}_\rho, \dot{x}_\mu] \right] + \left[ \dot{x}_\rho, [\dot{x}_\mu, \dot{x}_\nu] \right]
= -\frac{i\hbar}{m^2} \left( \left[ \dot{x}_\mu, F_{\nu\rho} \right] + \left[ \dot{x}_\nu, F_{\rho\mu} \right] + \left[ \dot{x}_\rho, F_{\mu\nu} \right] \right)
= \frac{\hbar^2}{m^3} \left( D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} \right)_a I^a,
\]

(5.19)

and hence Eq. (5.10) follows from the linear independence of \( I^a \)'s.

Next we show equation (5.12). Using Eqs. (5.17), (5.14), (5.2) and (5.3), we obtain

\[
0 = -\frac{m^2}{\hbar} \left( [\dot{x}_\nu, [\dot{x}_\mu, I^a]] - \frac{i\hbar}{m} f^{ab}_c [\dot{x}_\nu, A_{b\mu} I^c] \right)
\]
\[
-m^2 \left( \left[ [\dot{x}_\nu, \dot{x}_\mu], I^a \right] + [\dot{x}_\mu, [\dot{x}_\nu, I^a]] \right)
+m f^{ab}_c \left( [\dot{x}_\nu, A_{b\mu}] I^c + A_{b\mu} [\dot{x}_\nu, I^c] \right)
\]

\[
= [F_{\nu\mu}, I^a] - m [\dot{x}_\nu, f^{ab}_c A_{b\mu} I^c]
+i\hbar f^{ab}_c (\partial_\nu A_{b\mu} I^c + A_{b\mu} f_{ced} A_{d\nu} I^e)
\]

\[
= i\hbar f^{ab}_c I^c(F_{b\mu
u} - \partial_\mu A_{b\nu} + \partial_\nu A_{b\mu})
-ih \left( f^{ab}_c f_{ed} - f^{ad}_c f_{eb} \right) I^e A_{b\nu} A_{d\mu}.
\] (5.20)

Concerning the last term, the Jacobi identity implies

\[
-\hbar^2 (f^{ab}_c f_{ed} - f^{ad}_c f_{eb}) I^c = \left[ \left[ I^a, I^b \right], I^d \right] - \left[ \left[ I^a, I^d \right], I^b \right]
= \left[ \left[ I^a, I^b \right], I^d \right] + \left[ I^b, \left[ I^a, I^d \right] \right]
= [I^a, [I^b, I^d]]
= -\hbar^2 f^{ae}_c f^{bd}_e I^c.
\] (5.21)

Therefore the last term of Eq. (5.20) becomes

\[
(f^{ab}_c f_{ed} - f^{ad}_c f_{eb}) I^c A_{b\nu} A_{d\mu} = f^{ae}_c f^{bd}_e I^c A_{b\nu} A_{d\mu}
= -f^{ab}_c f^{de}_b I^c A_{e\nu} A_{d\mu}.
\] (5.22)

Equations (5.20) and (5.22) give (5.12).

As before, we take Eq. (5.8) as the definition of \(G^\mu\), that is,

\[
G^\mu = G^\mu_a I^a = F^\mu(x, \dot{x}) - (F^\mu_{a\nu}(x) \dot{x}_\nu I^a).
\] (5.23)

Again, \(G^\mu_a\) might depend on \(x\) and \(\dot{x}\), but using Eqs. (3.14), (5.1), (5.2) and (5.4), we get

\[
[x^\lambda, G^\mu] = [x^\lambda, F^\mu] - (F^\mu_{a\nu}[x^\lambda, \dot{x}_\nu]).
\]
\[
\begin{align*}
    &\frac{i\hbar}{m} F^{\lambda\mu} + \frac{i\hbar}{m} F^{\mu\nu} \delta^\lambda_{\nu} \\
    &= 0,
\end{align*}
\]

which says that \( G^\mu_a \) is also a function of \( x \) only.

The remaining task is to show equation (5.9). Eqs. (5.18) and (5.14) imply

\[
\begin{align*}
    [\dot{x}_\mu, G_\nu] \\
    &= [\dot{x}_\mu, F_\nu] - \langle [\dot{x}_\mu, F_{\nu\rho}] \dot{x}^\rho \rangle - \langle F_{\nu\rho} [\dot{x}_\mu, \dot{x}^\rho] \rangle \\
    &= m [\dot{x}_\mu, \ddot{x}_\nu] - \frac{i\hbar}{m} \langle D_\mu F_{\nu\rho} \dot{x}^\rho \rangle + \frac{i\hbar}{m^2} \langle F_{\nu\rho} F_\mu^\rho \rangle,
\end{align*}
\]

which leads to

\[
\begin{align*}
    [\dot{x}_\mu, G_\nu] - [\ddot{x}_\nu, G_\mu] \\
    &= m [\dot{x}_\mu, \ddot{x}_\nu] - m [\dot{x}_\nu, \ddot{x}_\mu] - \frac{i\hbar}{m} \langle (D_\mu F_{\nu\rho} - D_\nu F_{\mu\rho}) \dot{x}^\rho \rangle \\
    &\quad + \frac{i\hbar}{m^2} \langle F_{\nu\rho} F_\mu^\rho - F_\mu^\rho F_{\nu\rho} \rangle \\
    &= m \frac{d}{d\tau} [\dot{x}_\mu, \ddot{x}_\nu] - \frac{i\hbar}{m} \langle (D_\mu F_{\nu\rho} - D_\nu F_{\mu\rho}) \dot{x}^\rho \rangle \\
    &= -\frac{i\hbar}{m} \langle (D_\mu F_{\mu\nu} + D_\nu F_{\nu\mu}) \dot{x}^\rho \rangle \\
    &= -\frac{i\hbar}{m} \langle (D_\mu F_{\mu\nu} + D_\nu F_{\nu\mu}) \dot{x}^\rho \rangle,
\end{align*}
\]

with the aid of

\[
\begin{align*}
    \frac{d}{d\tau} (F_{\alpha\mu} I^a) &= \langle \partial_\mu F_{\alpha\mu}, \dot{x}_\rho I^a + F_{\alpha\mu} f^{abc}_{\rho} A_{\rho} \dot{x}_\mu I^c \rangle \\
    &= \langle (\partial_\mu F_{\alpha\mu} - f^{abc}_{\rho} A_{\rho} F_{\mu\nu}) \dot{x}_\rho I^a \rangle.
\end{align*}
\]

Eqs. (5.18), (5.25) and (5.26) give (5.9). End of proof.

Remarks: In our proof we should assume the existence of the gauge potential \( A_{\alpha\mu} \) in contrast with the Abelian case, in which its existence is not an assumption but a result. And we need the explicit form of the equation for the isospin, that is Eq. (5.4).

We have not yet found the way to remove \( f^{ab}_c \)'s from Eq. (5.12).

It may be possible to extend our argument to general relativistic formulation.

### 6 Concluding remarks

We have reviewed Feynman’s proof and proposed both a special relativistic and a general relativistic version and extension to the non-Abelian gauge theory of his
consideration. However some questions remain.

What on earth do these proofs show? They may show to what degree commutation relations restrict allowable form of the force. However we have *not* used the noncommutativity of operators, in other words, we have *not* used the very definition $[A, B] = AB - BA$. We have used only the properties of $[,]$, that is, equations from (2.19) to (2.24).

If we persist in quantum mechanical context, we should face the fact that there is no satisfactory quantum theory of a single relativistic particle. The commutation relation

$$[x^\mu, m\dot{x}^\nu] = -i\hbar \eta^{\mu\nu}$$

implies that the eigenvalue of $m\dot{x}^\mu$ runs over the whole real numbers. If we interpret $m\dot{x}^\mu$ as energy-momentum $p^\mu$ of the particle, the problem of negative energy annoys us. Even the on-shell condition $p^\mu p_\mu = m^2$ is not satisfied.

If we take classical mechanical context, we can make our consideration rather trivial. In the Lagrangian formalism, the assumption (4.2) may be replaced by the statement that the Lagrangian is a quadratic function with respect to the velocity, that is to say, the action is

$$S = \int \left( \frac{1}{2} m g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + A_\mu(x) \dot{x}^\mu + \phi(x) \right) d\tau.$$  

This assumption directly gives the results from (4.4) to (4.7). But the above consideration is less attractive because it demands much assumption, say, existence of the potentials $A_\mu, \phi$. In our proof using the commutation relations, their existence is not an assumption but a result.

How seriously do we have to accept our result, “the only possible fields that can consistently act on a quantum mechanical particle are scalar, gauge and gravitational fields”? If we want other fields, there may be methods to introduce them. A probable easy method is to replace the right-hand side of Eq. (4.2) by functions of both $x$ and $\dot{x}$. In general, we can put

$$m [x^\mu, \dot{x}^\nu] = -i\hbar \left\{ g^{\mu\nu}(x) + h^\rho_\rho(x) \dot{x}^\rho + k^{\mu\rho\sigma}(x) \dot{x}^\rho \dot{x}^\sigma + \cdots \right\}.$$  

We have not yet pursued extension by this method.

A final question is this; can we involve the spin in our argument? The force which the electromagnetic and gravitational fields exert over a particle with spin has
different form from Eq. (4.4). According to J.W. van Holten [7] [8], it is

\[ m \ddot{x}^\mu = e g^{\mu\nu} F_{\nu\rho} \dot{x}^\rho - m \Gamma_{\nu\rho}^{\mu} \dot{x}^\nu \dot{x}^\rho + \frac{e}{2m} g^{\mu\nu} (\nabla_\nu F_{\rho\sigma}) S^{\rho\sigma} + \frac{1}{2} R^{\mu}_{\nu\rho\sigma} \dot{x}^\nu S^{\rho\sigma}, \] (6.4)

where \( e \) is electric charge of the particle, which has been absorbed in the definition of \( F_{\mu\nu} \) in the previous sections; \( \nabla \) denotes covariant derivative with the Levi-Civita connection \( \Gamma_{\nu\rho}^{\mu} \); \( R^{\mu}_{\nu\rho\sigma} \) is the Riemann curvature tensor; \( S^{\rho\sigma} \) is intrinsic angular momentum tensor, that is, spin. We naively expect the Lie algebra of the (local) Lorentz group

\[ [S_{\mu\nu}, S_{\rho\sigma}] = -i\hbar (g_{\mu\rho} S_{\nu\sigma} - g_{\nu\rho} S_{\mu\sigma} - g_{\mu\sigma} S_{\nu\rho} + g_{\nu\sigma} S_{\mu\rho}) \] (6.5)

to lead to Eq. (6.4). Here we indicate the ‘local’ Lorentz group parenthetically, because the metric \( g_{\mu\nu}(x) \) depends on \( x \). The derivation of Eq. (6.4) is left for a future work.

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