Ghost-free and Modular Invariant Spectra
of a String in
$SL(2, R)$ and Three Dimensional Black Hole Geometry

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Abstract

Spectra of a string in $SL(2, R)$ and three dimensional (BTZ) black hole geometry are discussed. We consider a free field realization of $\hat{sl}(2, R)$ different from the standard ones in treatment of zero-modes. Applying this to the string model in $SL(2, R)$, we show that the spectrum is ghost-free. The essence of the argument is the same as Bars’ resolution to the ghost problem, but there are differences; for example, the currents do not contain logarithmic cuts. Moreover, we obtain a modular invariant partition function. This realization is also applicable to the analysis of the string in the three dimensional black hole geometry, the model of which is described by an orbifold of the $SL(2, R)$ WZW model. We obtain ghost-free and modular invariant spectra for the black hole theory as well. These spectra provide examples of few sensible spectra of a string in non-trivial backgrounds with curved time and, in particular, in a black hole background with an infinite number of propagating modes.

PACS codes: 04.70.Dy, 11.25.-w, 11.25.Hf, 11.40 Ex
Keywords: $SL(2, R)$ WZW model, free field realization, ghost problem, modular invariance, BTZ black holes, orbifold

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1 Introduction

A string in $SL(2, R)$ provides one of the simplest models of strings in non-trivial backgrounds with curved space-time. This model is described by the $SL(2, R)$ WZW model and $SL(2, R)$ becomes an exact string background. In contrast to well-studied curved space cases, we have only a few consistent string models with curved time \[1, 2\]. Therefore the investigation of the string in $SL(2, R)$ \[3-10\] is important as a step to understand strings in non-trivial backgrounds.

The three dimensional (BTZ; Bañados-Teitelboim-Zanelli) black hole geometry \[11\] is another exact string background with curved time. The corresponding string model is described by an orbifold of the $SL(2, R)$ WZW model \[12, 13\]. This model is interesting also as a model of quantum black holes in string theory which has an infinite number of propagating modes.

However, the spectrum of the string in $SL(2, R)$ is known to contain ghosts \[3-7\]. In the orbifold model, the original ghosts in $SL(2, R)$ disappear, but a different type of ghosts appears from the twisted sectors \[14\]. These analyses about ghosts are based on the standard argument about current algebras.

The appearance of the ghosts means that these string models are not sensible as they stand. However, there are arguments which support the existence of a sensible string model in the $SL(2, R)$ (and hence possibly in the BTZ black hole) background. First, let us consider the effective action for the S-matrix of the bosonic string theory. In addition to flat space, this has an extremal point at nearly flat three dimensional anti-de Sitter space ($AdS_3$), or equivalently $SL(2, R)$, and is shown to be unitary at one-loop order \[15\]. Second, a $D = 5$ black hole solution in type IIB theory has the $SL(2, R) \times SU(2)$ WZW model structure near the horizon \[16\]. So, there is a possibility of ghosts. Nevertheless, this black hole is mapped to a bound state of D-string and D5-brane. Since the resultant model is regarded as unitary, we expect that the original model is also unitary\[16\]. Third, Bars has proposed a ghost-free spectrum of the $SL(2, R)$ model \[7\]. By making use of ‘modified’ currents, namely, a non-standard free field realization of the $\hat{sl}(2, R)$ currents, he showed that the ghosts in the standard argument disappear. Analyses of the classical string propagation indicate no pathologies either \[3, 7\].

The standard argument about current algebras is well established for compact group manifolds but it leads to the ghosts for non-compact $SL(2, R)$. Then one might think that it cannot be applied to non-compact cases. In addition, although we expect to get the three dimensional flat string theory in the flat limit of the $SL(2, R)$ theory, this is impossi-

\[1\] These two arguments are due to A.A. Tseytlin. The author thanks him for comments on these issues.
ble in the standard argument. To see this, let us first remember that $SL(2, R)(AdS_3)$ has a constant scalar curvature, which we denote by $-6l^{-2}$. In the context of string theory, this is given by $l^2 = (k - 2)\alpha'$ \cite{12, 14} where $(2\pi\alpha')^{-1}$ is the string tension, $k$ is the level of the WZW model and $-2$ is the second Casimir of $sl(2, R)$. Thus, the flat limit is achieved by $l$ or $k \to \infty$. Next, recall that the $sl(2, R)$ algebra is given by

$$ [J^a_0, J^b_0] = i e^{abc} \eta_{cd} J^d_0, \tag{1.1} $$

where $a-d = 0, 1, 2$ and $\eta_{ab} = \text{diag} (-1, 1, 1)$. $J^{1, 2}_0$ correspond to the non-compact direction of $SL(2, R)$ and $J^0_0$ to the compact direction. The primary states of the standard $SL(2, R)$ current algebra are represented by the states of the global $sl(2, R)$ representations $|j, J\rangle$. Here, $-j(j + 1)$ is the Casimir of $sl(2, R)$; $J$ is the eigenvalue of $J^0_0$, $J^2_0$ or $J^+_0 \equiv J^0_0 + J^1_0$. According to the choice of $J$, the representations are called elliptic, hyperbolic or parabolic respectively. Note that the primary states are labeled by two parameters (zero-modes). On the other hand, the base (primary) states of the flat three dimensional theory are labeled by three zero-modes (momenta), for instance, as $|p^0, p^1, p^2\rangle$. Therefore, the standard $SL(2, R)$ model cannot get to the flat theory. Also, it is impossible to observe how the ghosts disappear in the flat limit.

What is wrong in applying the standard argument to the $SL(2, R)$ model? The above argument about the flat limit indicates the importance of the careful treatment of zero-modes. In fact, this is one of the points of Bars’ argument. The importance of the zero-modes in a non-trivial background is also pointed out in \cite{2}.

Along this line of thought, we first consider a free field realization of $\hat{sl}(2, R)$ in the following. This is different from the standard ones in treatment of zero-modes. Applying this realization to the string theory in $SL(2, R)$, we show that the spectrum is ghost-free. The essence of the argument is the same as Bars’. However, our realization is different: Bars’ modified currents are in the parabolic basis and contain logarithmic cuts but ours are in the hyperbolic basis and contain no logarithmic cuts. The treatment of zero-modes is also different. Moreover, we obtain a modular invariant partition function. Our realization turns out to be useful for the analysis of the string in the three dimensional (and the two dimensional $SL(2, R)/U(1)$ \cite{17}) black hole geometry. For the three dimensional black hole theory, we obtain ghost-free and modular invariant spectra as well. We see that these spectra are consistent with some self-dual T-duality and closed time-like curves \cite{11, 12} are removed in constructing the spectrum invariant under this T-dual transformation. This work resolves the ghost problem of the string theories in the $SL(2, R)$ and the BTZ black hole geometry and provides examples of few sensible spectra of a string in non-trivial backgrounds with curved time. In particular, this is the first time that such a
spectrum is obtained for a string in a black hole background with an infinite number of propagating modes. Summary and brief discussion about future problems are given in the final section.

2 Spectrum of a string in $SL(2, R)$

2.1 Free field realizations of $\hat{sl}(2, R)$

We begin our discussion with the Wakimoto construction of $\hat{sl}(2, R)$ [13]. It is given by a free boson $\phi$ and the bosonic $\beta$-\(\gamma\) ghosts:

\[
\begin{align*}
  iJ^+(z) &= \beta(z), \\
  iJ^-(z) &= \gamma^2\beta(z) + \sqrt{2k'} \gamma\partial\phi(z) + k\partial\gamma(z), \\
  iJ^2(z) &= \gamma\beta(z) + \sqrt{k'/2} \partial\phi(z),
\end{align*}
\]

where $J^\pm = J^0 \pm J^1$, $k' \equiv k - 2$ and

\[
\begin{align*}
  \beta(z)\gamma(w) &= -\gamma(z)\beta(w) \sim \frac{1}{z-w}, \\
  \phi(z)\phi(w) &\sim -\ln(z-w).
\end{align*}
\]

It is easy to check the algebra

\[
\begin{align*}
  J^+(z)J^-(w) &\sim \frac{-k}{(z-w)^2} + \frac{-2iJ^2(w)}{z-w}, \\
  J^2(z)J^\pm(w) &\sim \frac{\pm iJ^\pm(w)}{z-w}, \\
  J^2(z)J^2(w) &\sim \frac{k/2}{(z-w)^2}.
\end{align*}
\]

In terms of the modes, this is written as

\[
\left[J^a_n, J^b_m\right] = i\epsilon^{ab} c^n_{n+m} + \frac{k}{2} \eta^{ab} \delta_{m+n}.
\]

In addition, the energy-momentum tensor is given by

\[
T(z) = \frac{1}{k-2} \eta_{ab} : J^a J^b : \\
= -\frac{1}{2} (\partial\phi)^2 - \frac{1}{\sqrt{2k'}} \partial^2\phi + \beta\partial\gamma,
\]

and the central charge is

\[
c = \left(1 + 12 \left(1/\sqrt{2k'}\right)^2\right) + 2 = \frac{3k}{k-2}.
\]
At the critical value \( c = 26 \), we have \( k = 52/23 \). Notice that the level of the corresponding WZW model \( k \) need not be an integer since \( \pi_3(SL(2, R)) = 0 \).

In the above construction, a generator to the non-compact direction \( J_0^2 \) is the Cartan subalgebra of \( SL(2, R) \) and \( J_0^\pm \) are the step operators. Another choice of the basis is \( I^0 \equiv J^0 \) and \( I^\pm \equiv J^1 \pm iJ^2 \) in which \( I_0^0 \) is the Cartan subalgebra and \( I_0^\pm \) are the step operators. The algebra in the latter basis is obtained from (2.3) formally by \( J^\pm \to -iI^\pm \) and \( J^2 \to iI^0 \) \([19]\). The similarity to \( \hat{su}(2) \) becomes clearer in the latter.

The base states on which the operators act are labeled by two zero-modes; the Casimir and, e.g., the eigenvalue of \( J_0^2 \). Because of the deficiency of the zero-modes, the string model based on this standard realization cannot have the flat limit as discussed in the introduction. It may be natural on physical grounds for a sensible \( SL(2, R) \) model to have such a limit. Thus, we will try to make the \( \hat{sl}(2, R) \) realization which has the flat limit by incorporating an additional zero-mode.

For this purpose, we first bozonize the \( \beta-\gamma \) ghosts. In principle, various bozonizations are possible because the current algebra is maintained as long as the OPE (operator product expansion) of the \( \beta-\gamma \) ghosts are preserved.

One simple bozonization is given by two free bosons \( \phi_{0,1} : \)

\[
\beta = \frac{1}{\sqrt{2}} \partial \phi_+, \quad \gamma = \frac{1}{\sqrt{2}} \phi_-, \tag{2.7}
\]

where \( \phi_\pm = \phi_0 \pm \phi_1, \phi_i(z)\phi_j(w) \sim -\eta_{ij} \ln(z-w) \) and \( \eta_{ij} = \text{diag} (-1, +1) \). We can readily check the current algebra in this realization. In fact, the resultant currents are nothing but Bars’ modified currents \([7]\). He found these currents by trials and errors, but we got them in a simple way. By a careful treatment of zero-modes, he showed that the spectrum of the \( SL(2, R) \) model using these currents is ghost-free. Notice that substituting (2.7) into (2.1) yields logarithmic cuts in the currents because \( \phi_- \) has the mode expansion \( \phi_- = q^{-} - i\alpha_0 ^{-} \ln z + \cdots \). The currents are ‘modified’ by this logarithmic term. We have to require that the logarithmic cuts of the currents and primary fields have no effects in the physical sector. As a result, only particular combinations of the left and the right sector are allowed in the physical sector of the full theory. Also, \( J_0^+ \) is diagonal on the base states \( |P^i \rangle \), where \( P^i \) are the momenta of \( \phi_\pm \) and \( \phi \). This means that the representations are parabolic.

It turns out that the representations in the hyperbolic (i.e., \( J_0^2 \)-diagonal) basis are necessary to analyze the string theory in the three dimensional (and the two dimensional \( SL(2, R)/U(1) \)) black hole geometry. Since only certain states are allowed in Bars’ realization, we cannot make the change from the parabolic to the hyperbolic basis. We are interested in the application to the black hole physics. So, we will consider a different
realization using a different bosonization of the $\beta$-$\gamma$ ghosts in what follows. We will get a ghost-free and, moreover, modular invariant spectrum of the $SL(2, R)$ model using this realization.

The bosonization we then take is the standard one in [20]. First, we represent the $\beta$-$\gamma$ ghosts by

$$\beta(z) = -e^{-\varphi_1(z)} \partial \xi(z), \quad \gamma(z) = e^{\varphi_1(z)} \eta(z),$$

where $\varphi_1(z)$ is a free boson, $\eta(z)$ and $\zeta(z)$ are a dimension-1 and a dimension-0 fermionic field, respectively. The fermionic fields have the OPE

$$\xi(z)\eta(w) \sim \eta(z)\xi(w) \sim + \frac{1}{z - w}. \quad (2.9)$$

We further bosonize the fermionic fields by another free boson;

$$\xi(z) = e^{-\varphi_0(z)}, \quad \eta(z) = e^{\varphi_0(z)}.$$

(2.10)

$\varphi_i(z)$ ($i = 0, 1$) have the OPE's

$$\varphi_i(z)\varphi_j(w) \sim -\eta_{ij} \ln(z - w), \quad (2.11)$$

where $\eta_{ij} = \text{diag}(-1, 1)$. Note that the zero-mode of $\xi$ never appears in the $\beta$-$\gamma$ ghosts.

In order to get representations in the hyperbolic basis, we make a change of variables

$$X_0 = \sqrt{k'/k} \phi + \sqrt{k'/2} \varphi_0 + \sqrt{2k} \varphi_1,$$

$$X_1 = \sqrt{k'/k} \phi - \sqrt{2/k} \varphi_1,$$

$$X_2 = \phi + \sqrt{k'/2} (\varphi_0 + \varphi_1). \quad (2.12)$$

The new fields have the OPE's $X_i(z)X_j(w) \sim -\eta_{ij} \ln(z - w)$ with $\eta_{ij} = \text{diag}(-1, 1, 1)$. A similar transformation has been discussed, e.g., in [21] for $\hat{su}(N)$. Consequently, the currents and the energy-momentum tensor are written as

$$iJ^\pm = e^{\mp \sqrt{2/k} X_\mp} \partial \left( \sqrt{k'/2} X_0 \mp \sqrt{k'/2} X_2 \right),$$

$$iJ^2 = \sqrt{k'/2} \partial X_1,$$

$$T = \frac{1}{2} \partial X_+ \partial X_- - \frac{1}{2} (\partial X_2)^2 - \frac{1}{\sqrt{2k'}} \partial^2 X_2, \quad (2.13)$$

where $X_\pm = X_0 \pm X_1$. We remark that (i) the currents have no logarithmic cuts, (ii) $J_0^2$ is diagonal on the base states $|p^-, p^0, p^2\rangle$ as desired, where $p^i$ are the momenta of $X_i$.
and (iii) the energy-momentum tensor is expressed by flat light-cone free fields $X_{\pm}$ plus a space-like free field of a Liouville type $X_2$.

Using these expressions, we find that

$$V_j^j(z) \equiv \exp \left[ i J \sqrt{\frac{2}{k}} X_-(z) + j \sqrt{\frac{2}{k'}} X_2(z) \right]$$  \hspace{1cm} (2.14)

are the primary fields of the current algebra. Actually, it follows that

$$\text{dim. } V_j^j(z) = \frac{-j(j+1)}{k-2},$$  \hspace{1cm} (2.15)

$$J^2(z)V_j^j(w) \sim \frac{J}{z-w} V_j^j(w), \quad J^\pm(z)V_j^j(w) \sim \frac{J \mp ij}{(z-w)} V_j^j(w).$$

Also, we find the three screening operators which are dimension-1 Virasoro primary fields and have no non-trivial OPE’s with the currents;

$$\eta(z) = \exp \left[ \sqrt{k/2} X_0(z) - \sqrt{k'/2} X_2(z) \right],$$

$$\tilde{\eta}(z) = \exp \left[ -\sqrt{k/2} X_0(z) - \sqrt{k'/2} X_2(z) \right],$$

$$S(z) = \partial X_0(z) e^{-\sqrt{2/k'} X_2(z)}.$$  \hspace{1cm} (2.16)

These screening operators are determined up to factors and total derivative terms. In particular, $S(z)$ is expressed also as $i J^\pm e^{\pm \sqrt{2/k' X_2(z)} - \sqrt{2/k} X_2(z)}$.

### 2.2 Realization in an extended space

We have made a free field realization of $\hat{sl}(2,R)$ in the basis $J^\pm$ and $J^2$. This is a simple application of the literature about $\hat{s}u(2)$, e.g. [18, 21, 22], and $\hat{sl}(2,R)$ in a different basis [9]. In the standard argument, the representation space is constructed as follows. First, we take the vacuum $|0\rangle$ satisfying $J_n^a|0\rangle = 0$ for $n \geq 0$. The primary states are given by $|j;J\rangle \equiv \lim_{z \to 0} V_j^j(z)|0\rangle$. In addition, the current module is obtained by acting $J_n^a$ on the primary states. The Virasoro weight is written as

$$L_0 = -\frac{j(j+1)}{k-2} + N,$$  \hspace{1cm} (2.17)

where $N$ is the total grade.

Let us rephrase this in terms of the free bosons. First, we note that $X_2$ has the background charge $iQ = -\frac{1}{\sqrt{2k'}}$. Because of $Q$, the mode expansion of $X_2$ and the corresponding Virasoro operators are written as

$$X_2(z) = q^2 - i(\alpha_0^2 - iQ) \ln z + i \sum_{n\neq 0} \frac{\alpha_n^2}{n} z^{-n},$$

$$L_n^2 = \frac{1}{2} \sum_m \alpha_m^2 \alpha_{n-m}^2 + i Q n \alpha_n^2 + \frac{1}{2} Q^2 \delta_n.$$  \hspace{1cm} (2.18)
The total Virasoro operators are then given by
\[ L_n = L_n^\pm + L_n^2, \]
where
\[ L_n^\pm = -\frac{1}{2} \sum_m \alpha_m^\pm \alpha_{-m}^- , \tag{2.19} \]
and \( \alpha_n^\pm = \alpha_n^0 \pm \alpha_n^1. \) From the expression of \( L_n, \) we find that \( |p^\pm = 0, p^2 = iQ\rangle \) is the \( sl_2\)-invariant vacuum. This is nothing but the vacuum in the standard argument \(|0\rangle\). Also, from (2.14), we have
\[ |j; J\rangle = |p^\mp = 0, p^\pm = \sqrt{2/k'} J, p^2 = i(Q - \sqrt{2/k'} j)\rangle . \tag{2.20} \]
This shows the relation between the Casimir, \(-j(j+1),\) and the eigenvalue of \( \alpha_0^2, p^2; \)
\[ j = -\frac{1}{2} + i\sqrt{k'/2} p^2. \tag{2.21} \]
Substituting \( p^i \) in (2.20) into \( L_0 \) reproduces (2.17). We remark that, for \( p^2 \in \mathbb{R}, \) the above \( j\)-values are precisely those of the principal continuous series of the unitary \( SL(2,R) \) representations. (For details, see, e.g., [23, 14].)

On the entire module \( p^- \) is fixed to be zero because the currents shift \( p^\pm \) only (see (2.13)). \( J^\pm \) shift \( J \) by \( \pm i. \) This might be curious but this is one of the characteristic features of the representations of \( SL(2,R) \) in the hyperbolic basis [19, 23, 14]. First, since \( J_0^2 \) corresponds to the non-compact direction of \( SL(2,R), \) the spectrum of \( J \) is not discrete but continuous. So, an element (a state) of the global representation space is given by a “wave packet”
\[ |\Psi\rangle = \int_{-\infty}^{\infty} dJ \Psi(J) |j; J\rangle . \tag{2.22} \]
This is analogous to a state in a field theory using a plane wave basis in infinite space. The action of the operators is given by
\[ J_0^2 |\Psi\rangle = \int_{-\infty}^{\infty} dJ J \Psi(J) |j; J\rangle , \]
\[ J_0^+ |\Psi\rangle = \int_{-\infty}^{\infty} dJ f_+(J) \Psi(J - i) |j; J\rangle , \tag{2.23} \]
\[ J_0^- |\Psi\rangle = \int_{-\infty}^{\infty} dJ f_-(J + i) \Psi(J + i) |j; J\rangle , \]
where \( f_\pm(J) \) play the role of the matrix elements of \( J_0^\pm. \) The above shift of \( J \) should be understood in this way as that of the argument of \( \Psi(J). \)

So far, nothing is special. Here, we will make a change from the standard argument. Namely, we take all \( |p^\pm, p^2 \in \mathbb{R}\rangle \) as base states. This means that (i) we extend the representation space so that \( p^- \neq 0 \) are allowed and (ii) we concentrate on the principal
continuous series as the global $SL(2, R)$ representations and take all the representations of this type. By this prescription, we incorporate the deficient zero-mode in the standard argument.

2.3 Ghost-free spectrum

Now, we are ready to show that the spectrum is ghost-free. First, following the arguments for the no-ghost theorem in flat space-time [25], we can prove that the module

$$\prod_{l=1}^{\infty} (\alpha_{l}^+) a_l \prod_{m=1}^{\infty} (\alpha_{-m}^-) b_m \prod_{n=1}^{\infty} (\alpha_{-n}^2) c_n |p^-, p^+, p^2\rangle$$

contains no ghosts, where $p^+, p^2 \in \mathbb{R}$ and $a_l, b_m$ and $c_n$ are non-negative integers. Namely, the physical state conditions $(L_n - \delta_n) |\psi\rangle = 0$ $(n \geq 0)$ remove all the negative-norm states in this module. We do not repeat the detailed argument. Instead, it is sufficient to check that (i) the central charge of $L_n$ is equal to 26, (ii) the energy-momentum tensor is hermitian, i.e., $(L_n)^\dagger = L_{-n}$ and (iii) the transverse space is positive definite (please remember that we have the flat light-cone ($X^\pm$) plus transverse ($X^2$) space). (i) follows from setting $k$ to be the critical value $k = 52/23$. Once we define the hermiticity of the modes by $(\alpha_{n}^i)^\dagger = \alpha_{-n}^{i}$, (ii) is also satisfied since $Q$ is real. The oscillator part of $X^2$ causes no trouble to show (iii) because $X^2$ is space-like. But we still need to give the precise prescription of the inner product of the zero-mode part. For this purpose, we first note that the conjugate of the $sl_2$-invariant vacuum is given by $|p^- = 0, p^2 = -iQ\rangle$. Then we define the conjugates of the in-states $|p^2 \in \mathbb{R}\rangle = \lim_{z \to 0} e^{i \sqrt{2/k} X_2(z)} |iQ\rangle$ by $|p^2| = \lim_{z \to \infty} \langle -iQ | e^{i \sqrt{2/k} X_2(z)} z^{2L^2_0}. Notice that the imaginary momentum in the $sl_2$-invariant vacuum is precisely canceled by acting the primary fields $V_j^i$ and that the complex conjugation $j \rightarrow \bar{j} = -j - 1$ does not change the Casimir and the Virasoro weight. Next, we define the inner product by $\langle p^2' | p^2\rangle = 2\pi \delta(p^2' - p^2)$. This assures the positivity of the transverse space.

From the viewpoint of the original $SL(2, R)$ model, the above statement means that the spectrum of the $SL(2, R)$ model is ghost-free. Also, by the limit $k \rightarrow \infty$, we obtain a flat theory (although it should be a part of the critical theory).

It may be instructive to see the difference from the standard argument in detail. First, in order to obtain the base states $|p^\pm, p^2 \in \mathbb{R}\rangle$ by acting the primary fields, we have to prepare $|p^- \neq 0, 0, iQ\rangle$ as ‘vacua’ as well as $|0\rangle = |p^\pm = 0, p^2 = iQ\rangle$. On these

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2 The degrees of freedom of the fixed zero-mode such as $p^-$ play roles also in different contexts. For example, vacua $|P^1 = i n\rangle$, where $P^1$ is the momentum of $\varphi_1$ in [23] and $n \in \mathbb{Z}$, are the picture-changed vacua with respect to the $\beta\gamma$ system [20]. In addition, the operators $\exp(n \sqrt{k/2} q^1)$, where $n \in \mathbb{Z}$, generate the twists of the current algebra by integers [24].
base states, \( J_0^0 | p^- \neq 0, 0, iQ \rangle \neq 0 \) in general. This is similar to spontaneous symmetry breaking as Bars mentioned in his argument. Moreover, since \( p^- \neq 0 \), which never appeared in the standard argument, are allowed, the on-shell condition is changed as

\[
L_0 = -\frac{1}{2} p^+ p^- - \frac{j(j+1)}{k-2} + N = 1.
\]

(2.25)

If \( p^- = 0 \), the \( j \)-value satisfying this condition is real for \( N \geq 1 \) and corresponds to the discrete series of the unitary \( SL(2, R) \) representations. In the standard argument, the ghosts arise from the discrete series for \( N \gg 1 \). However, real \( j \)-values imply purely imaginary \( p^2 \) and such states are removed from our module. Instead, because of non-zero \( p^- \), states with real \( p^2 \) can be physical. They correspond to the principal continuous series. We remark that the positivity of the transverse space becomes invalid for purely imaginary \( p^2 \). It is easy to find examples of negative norm states in this case from

\[
\left\{ (p^2)^2 + 1/(2k') \right\} \langle p^2 | p^2 \rangle = \langle p^2 | 2L_0^2 | p^2 \rangle = \langle p^2 | L_1^2 L_2^1 | p^2 \rangle.
\]

(2.26)

Arguments similar to ours do not hold for compact groups since the principal continuous series and representations in the hyperbolic basis are characteristic of non-compact groups. We reduced our \( SL(2, R) \) theory almost to that of three free bosons. It is the above characteristics that make this possible. Because we use the hyperbolic basis, the eigenvalue of \( J_0^0 \) is continuous and has the one-to-one correspondence with the momentum of \( X_1 \). Since we take the principal continuous series at the base, the character of the \( SL(2, R)/SO(1,1) \) module (the coset module by \( J_0^0 \)) coincides with that of two free bosons. Namely, we have no non-trivial null states [8]. Thus, the sum of the characters over all \( p^i \) for our \( SL(2, R) \) model coincides with that of the three free bosons with the same central charge. Also, for the \( j \)-values of the principal continuous series, we find another close connection between two theories: on \( | p^i = 0 \rangle \), \( V_j \) look like primary fields of the flat theory;

\[
\lim_{z \to 0} V_j^i(z) | 0 \rangle = \lim_{z \to 0} e^{ip^+ X_-} e^{ip^2 X_2} | p^i = 0 \rangle.
\]

(2.27)

Moreover, we can construct the spectral generating operators like DDF (Del Giudice-Di Vecchia-Fubuni) operators in the 26-dimensional flat theory [25, 27]. The difference arises from the fact that \( \partial X_2 \) is not a primary field because of the background charge. To compensate this, we use the \( b\text{-}c \) ghosts or the logarithmic operators \( \ln \partial X_\pm \) [27]. By making use of them, the spectral generating operators are written as

\[
A_n^\pm = \frac{1}{\sqrt{1 - 4Q^2/9}} \int \frac{dz}{2\pi i} e^{i n X^/p^\pm} i \left( \partial X_2 - \frac{2Q}{3} \partial_c \right),
\]

or

\[
B_n^\pm = \int \frac{dz}{2\pi i} e^{i n X^/p^\pm} i (\partial X_2 - Q \partial \ln \partial X_\pm),
\]

(2.28)
with \( n \in \mathbb{Z} \). In fact, since the integrands are dimension-1 primary fields, \( A_n^\pm \) and \( B_n^\pm \) commute with all the Virasoro operators and create physical states. The non-triviality of these operators is easily checked. In addition, each set of operators has the same commutation relations as free field oscillators;

\[
[A_m^\pm, A_n^\pm] = [B_m^\pm, B_n^\pm] = m\delta_{m+n}.
\]

(2.29)

Finally, a comment may be in order on the hermiticity of the currents and the spectral generating operators. Because of the background charge, \( X_2 \) is not hermitian:

\[
(X_2(1/z))^{\dagger} = X_2(z) + 2Q\ln z.
\]

However, we need to take into account the prescription of the conjugation. Then we have

\[
(X_2(z) | p^2 = iQ) = \langle p^2 = -iQ | X_2(1/z) \rangle \text{ for the } sl_2-\text{invariant vacuum.}
\]

The hermiticity of the currents and the spectrum generating operators should be understood in this sense.\footnote{The contributions from the background charge in \( A_n^\pm \) and \( B_n^\pm \) are spurious in any case.}

The essence of our argument is the same as Bars’\footnote{}; the special treatment concerning zero-modes leads to the change of the Virasoro condition and hence the principal continuous series become relevant instead of the discrete series. Nevertheless, our currents contain no logarithmic cuts and the treatment of zero-modes is different. Furthermore, using our realization, we can argue for the modular invariance of the spectrum and the application becomes possible to the three (and two) dimensional black holes. These are the subjects of the following sections.

2.4 Modular invariance

So far, we have focused on the left sector of the string model in \( SL(2,R) \). In this subsection, we introduce the right sector and discuss the modular invariance of the spectrum. In what follows, \( L(R) \) implies the quantities in the left (right) sector and tildes refer to the quantities in the right sector.

We discussed that the sum of the characters for the chiral sector of the \( SL(2,R) \) model was the same as that of the three free bosons. Therefore, we immediately obtain the modular invariant partition function (1-loop vacuum amplitude) by identifying the left and the right momenta as

\[
p^i_L = \pm p^i_R.
\]

This means that \( j_L = j_R \) or \( -j_R - 1 \) and \( J_L = \pm J_R \) where \( j_{L(R)} \) and \( J_{L(R)} \) are the \( j \)-values and the eigenvalue of \( J_0^2(J_0^2) \), respectively. Explicitly, we find that

\[
Z = \int \frac{d^2 \tau}{\text{Im} \tau} Z_{bc}(\tau) \text{ Tr } e^{i\tau(L_0-cX/24)} e^{-i\tau(\tilde{L}_0-cX/24)},
\]

(2.30)

is modular invariant where \( \tau \) is the modular parameter; \( \text{Tr} \) is the trace over the entire module of \( X_i \); \( Z_{bc} \) is the contribution from the \( b-c \) ghosts; \( c_X = 3k/(k-2) = 26 \). We
remark that it is necessary to Wick-rotate $p^0$ as in the flat theory.

In the above construction, the combined primary fields take the form

$$V^{ij}_{j;\pm, j}(z, \bar{z}) \equiv V^{ij}_j(z) \tilde{V}^{ij}_{\pm, j}(\bar{z}), \quad (2.31)$$

where $\tilde{V}^{ij}_{\pm, j}$ is defined similarly to (2.14). For the $j$-values, we have set $j_L = j_R$ as usual in the WZW models. This is possible because $j$ and $-j - 1$ represent the same Casimir and hence the same representation. To take $-j - 1 = \tilde{j}$ means just to take the conjugate.

Suppose that the combined primary fields are expressed by the matrix elements of the $SL(2, \mathbb{R})$ representations which have the same transformation properties. Then they are represented by the hypergeometric functions [23, 28, 14]. The above condition $J_L = \pm J_R$ leads to the degenerate cases, and the hypergeometric functions have singular behavior near certain points of $SL(2, \mathbb{R})$. These points correspond to the origin or the horizon for the two dimensional $SL(2, \mathbb{R})/U(1)$ black holes [28], whereas they correspond to the inner or the outer horizon of the three dimensional black holes [14].

3 Spectrum of a string in three dimensional black hole geometry

In this section, we turn to the application to the string theory in the three dimensional black hole geometry. Actually, this was important part of our motivation to investigate the $SL(2, \mathbb{R})$ theory. The application is straightforward and we obtain ghost-free and modular invariant spectra again.

3.1 Three dimensional black holes

Let us begin with a brief review of the three dimensional black holes [11] in the context of string theory [12]–[14].

The three dimensional black holes are solutions to the vacuum Einstein equations with the cosmological constant $-l^{-2}$. The simplest way to obtain the black hole geometry is to start from the three dimensional anti-de Sitter space ($AdS_3$), or equivalently, the $SL(2, \mathbb{R})$

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4 A modular invariant of $\hat{sl}(2, \mathbb{R})$ has been discussed in [1] by including the degrees of freedom of zero-modes which are absent in the standard argument. The additional degrees of freedom are related to the twists of $\hat{sl}(2, \mathbb{R})$ by integers, the Weyl reflection of $\hat{sl}(2, \mathbb{R})$ or the winding around the compact direction of $SL(2, \mathbb{R})$.

5 Harmonic analysis on $SL(2, \mathbb{R})$ shows that normalizable functions on $SL(2, \mathbb{R})$ are expanded by the matrix elements of the discrete and the principal continuous series [2].
In a parametrization, the metric of (a part of) $AdS_3$ is written as
\[ ds^2 = -\left(\frac{r^2}{l^2} - M_{BH}\right) dt^2 - J_{BH} dt d\varphi + r^2 d\varphi^2 + \left(\frac{r^2}{l^2} - M_{BH} + \frac{J_{BH}^2}{4r^2}\right)^{-1} dr^2, \]
(3.1)

where $-\infty < t, \varphi < \infty$, $0 \leq r < \infty$; $M_{BH}$ and $J_{BH}$ are some parameters. The black hole geometry is obtained by identifying $\varphi + 2\pi$ with $\varphi$. In the following, we denote this identification by $Z_{\varphi}$. The above two parameters, $M_{BH}$ and $J_{BH}$ represent the mass and the angular momentum of the black hole. If we express them as $l^2 M_{BH} = r_+^2 + r_-^2$ and $l J_{BH} = 2r_+ r_-$ for $r_+ \geq r_- \geq 0$, then $r = r_\pm$ correspond to the location of the outer and the inner horizon, respectively. The variables $t$, $r$ and $\varphi$ represent the time, the radial and the angle coordinate.

The string theory in this geometry is described by the $SL(2,R)/Z_{\varphi}$ orbifold of the $SL(2,R)$ WZW model [12, 13]. Thus, the black hole geometry provides an exact and simple string background with curved time. The model is interesting both as a string model in non-trivial backgrounds and as a model of quantum black holes in string theory. However, a detailed analysis based on the standard argument about the current algebras showed that the model contains ghosts as mentioned in the introduction [14]. The ghosts in the black hole model are different from those in the untwisted model and originate from the twisted sectors.

### 3.2 Application of the new realization

Now we discuss the application of the new realization. In order to analyze the spectrum of the orbifold model by $Z_{\varphi}$, we need to construct twist (winding) operators. Although our realization is different from the standard ones, the basic strategy is the same. Thus, we follow the argument in [29] and [14]. First, we recall that the coordinates $\varphi$ and $t$ are expressed by analogs of the Euler angles $\theta_L$ and $\theta_R$ as [14]
\[ \varphi = \frac{1}{2} \left( \frac{\theta_L}{\Delta_-} + \frac{\theta_R}{\Delta_+} \right), \quad t/l = \frac{1}{2} \left( \frac{\theta_L}{\Delta_-} - \frac{\theta_R}{\Delta_+} \right), \]  
(3.2)

where $\Delta_\pm = \hat{r}_\pm \pm \hat{r}_- \equiv (r_+ \pm r_-)/l$. The untwisted primary fields $V_{j_L}^j(z) \left( \tilde{V}_{j_R}^j(\tilde{z}) \right)$ should have the $\theta_L (\theta_R)$ dependence as $e^{-iJ_L \theta_L} \cdot (e^{-iJ_R \theta_R})$ [14]. Next, we decompose $\theta_{L,R}$ into the free field parts and the non-free field parts as
\[ \theta_L = \theta_L^F(z) + \theta_L^{NF}(z, \tilde{z}), \quad \theta_R = \theta_R^F(\tilde{z}) + \theta_R^{NF}(z, \tilde{z}). \]  
(3.3)

In this section, we consider the universal covering group (space) of $SL(2,R) (AdS_3)$. The argument in Section 2 holds without change.
Note that $\theta^F_L$ is holomorphic and $\theta^F_R$ is anti-holomorphic. In terms of these free fields, $J^2(z)$ and $\tilde{J}^2(\bar{z})$ are written as $iJ^2 = (k/2)\partial \theta^F_L$, $i\tilde{J}^2 = (k/2)\partial \theta^F_R$. Comparing these expressions with $J^2$ in (2.13) yields

$$X_1(z) = \sqrt{k/2} \theta^F_L(z), \quad \bar{X}_1(\bar{z}) = \sqrt{k/2} \theta^F_R(\bar{z}).$$

Then, we find that the twist operator representing $n$-fold winding is given by

$$W(z, \bar{z}; n_w) = \exp \left\{ i \left( \mu_L X_1(z) + \mu_R \bar{X}_1(\bar{z}) \right) \right\},$$

where $\mu_L = n_w \Delta_- \sqrt{k/2}, \mu_R = -n_w \Delta_+ \sqrt{k/2}$. In fact, this operator has the OPE's

$$X_1(z)W(0, 0; n_w) \sim -in_w \Delta_- \sqrt{k/2} \ln z \ W(0, 0; n_w),$$
$$\bar{X}_1(\bar{z})W(0, 0; n_w) \sim in_w \Delta_+ \sqrt{k/2} \ln \bar{z} \ W(0, 0; n_w),$$

and this means that $\delta \varphi = 2\pi n_w$ and $\delta t = 0$ under $\sigma \to \sigma + 2\pi$ where $z = e^{\tau + i\sigma}$. A general primary field is obtained by combining an untwisted primary field and $W(z, \bar{z}; n_w)$. As a result, the eigenvalue of $\alpha^0(\alpha^1_0)$ of the twisted primary field becomes $p^v_{L(R)} = p^1_{L(R)} + \mu_{L(R)}$ where $p^1_{L(R)} = -\sqrt{2/k}J_{L(R)}$ is that of the untwisted part.

Next, we solve the level matching condition $L_0 - \tilde{L}_0 \in \mathbb{Z}$. Here we will set the momenta of the untwisted part to be $p^j_L = \pm p^j_R$ as in the previous section because the untwisted model was sensible for these cases. Also, it turns out that we obtain the sensible orbifold model starting from this untwisted part. We will see that the twist with respect to $Z_\varphi$ becomes similar to the toroidal compactification.

The difference from the analysis in [14] arises because (i) the Virasoro weights $L_0$ and $\tilde{L}_0$ take the forms as (2.25), e.g., $2L_0 = -(p^0)^2 + (p^1)^2 - 2j(j + 1)/k' + 2N$ and (ii) the left and right zero-modes of the untwisted part are connected by $p^1_L = \pm p^j_R$. According to $p^1_L = \pm p^1_R$, we have two cases. First, we consider $p^1_L = p^1_R \equiv p^1$ case. Then, the level matching leads to the condition $2\tilde{r}_+ n_w \left( \sqrt{k/2} p^1 - (k/2)\tilde{r}_- n_w \right) = m_J$, where $m_J$ is an integer. Furthermore, this plus the closure of the OPE give

$$\sqrt{k/2} p^1 = \frac{1}{2} \left( \frac{m_J}{\tilde{r}_+} + k\tilde{r}_- n_w \right),$$

in other words,

$$\sqrt{k/2} p^v_L = \frac{1}{2} \left( \frac{m_J}{\tilde{r}_+} + k\tilde{r}_+ n_w \right), \quad \sqrt{k/2} p^v_R = \frac{1}{2} \left( \frac{m_J}{\tilde{r}_+} - k\tilde{r}_+ n_w \right).$$

When the above condition is satisfied, the primary field is invariant under

$$\delta \theta^F_L = \delta \theta^F_{L'} = \pi \Delta_-, \quad \delta \theta^F_R = \delta \theta^F_{R'} = \pi \Delta_+.$$
This follows from the $\theta_{L,R}$-dependence of the untwisted part and $\theta_{F,L,R}^F$-dependence of the twist operator, and ensures the single-valuedness under $\delta \varphi = 2\pi$. The non-free parts $\theta_{L,R}^{NF}$ appear in the orbifolding in this way as in [29] and [14]. The states corresponding to the primary fields are obtained in the usual way.

The discussion for $p^1_L = -p^1_R$ case is similar, and we do not repeat it. The results are obtained simply by the exchange $\hat{r}_+ \leftrightarrow \hat{r}_-$ and appropriate changes of signs.

Consequently, the effects of the twist by $Z_\varphi$ are summarized in discretizing the eigenvalue of $\alpha_0^1$ and $\tilde{\alpha}_0^1$ as, e.g., in (3.8). Therefore, we can readily confirm that the spectrum for each case is ghost-free and modular invariant. First, once we restrict ourselves to the left or the right sector, the spectrum of each sector is just a subset of the spectrum of the untwisted model. Thus, we can prove the no-ghost theorem in each sector and hence that of the full theory. It is easy to see that the examples of the ghosts in [14] disappear. Second, $p^1_{L,R}$ are discretized in the same way as the momenta of the toroidally compactified field. So, we find that the replacement of the integral over $p^1 \equiv p^1_L = \pm p^1_R$ in (2.30) by the summation $\sum_{n,m,j \in \mathbb{Z}}$ does not break the modular invariance of the partition function. Also, in order to get the physical states, we can make use of the same spectral generating operators as in the untwisted model.

For the three dimensional black holes, there is a self-dual T-dual transformation (T) [14] which is caused by $\theta_R \rightarrow -\theta_R$ [30]. The flip of the sign of $p^1_R$ corresponds to $T$. This is consistent with the above result since $T$ is interpreted also as the exchanges $\hat{r}_+ \leftrightarrow \hat{r}_-$ and $r^2 \leftrightarrow -r^2 + l^2M_{BH}$. Although there is no rigorous proof that T-dual transformations along non-compact directions are the exact symmetries of CFT’s, it might be natural to suppose that that is the case. Then the partition function should be invariant under $T$. Such an invariant partition function is obtained simply by summing up the two sectors $p^1_L = \pm p^1_R$ [14]. The resultant spectrum is regarded as that of the model further twisted by this $\mathbb{Z}_2(T)$ symmetry. Thus, in this construction the region $r^2 < 0$ where closed time-like curves exists [11, 12] is truncated because of $r^2 \leftrightarrow -r^2 + l^2M_{BH}$ (at the expense of the additional truncated region $0 \leq r^2/l^2 < M_{BH}/2$). This shows a way to remove the closed time-like curves in the three dimensional black holes.

4 Summary and discussion

In this paper, we discussed a free field realization of $\hat{sl}(2, R)$, which was different from the standard ones in the treatment of zero-modes. This gave a resolution to the ghost problem.

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7 In the flat limit $k$ (or $l$) $\rightarrow \infty$, we have $iJ^2 \sim (k/2)\partial(\theta_L - \theta_R)$ and $i\tilde{J}^2 \sim -(k/2)\bar{\partial}(\theta_L - \theta_R)$ [14]. So, this latter case smoothly leads to the flat limit.

8 $M_{BH}$ and $J_{BH}$ are also $T$-invariant.
of the string theory in $SL(2,R)$. The essence of the argument was the same as Bars’ which had given a resolution to this problem. However, our currents did not have logarithmic cuts and the treatment of zero-modes was also different. Moreover, we obtained a modular invariant partition function. Our realization was useful for the analysis of the string in the three dimensional black hole geometry as well. The model was described by an orbifold of the $SL(2,R)$ WZW model. By a simple application, we again obtained ghost-free and modular invariant spectra for the black hole theory. These spectra were consistent with some self-dual T-duality and we saw that closed time-like curves were removed in constructing the spectrum invariant under this T-dual transformation.

Each spectrum obtained here provides one of few sensible spectra of a string in non-trivial backgrounds with curved time (at least so far). Furthermore, this is the first time that such a sensible spectrum is obtained for a string theory in a black hole background with an infinite number of propagating modes.

The argument in this paper may have a wide variety of applications to string models in non-trivial backgrounds containing non-compact group manifolds. In particular, the application to the $SL(2,R)/U(1)$ black holes is straightforward because, for the lorentzian black holes, the model is described by the coset of the $SL(2,R)$ WZW model by $J^2(z) \pm \tilde{J}^2(\bar{z}) (i\partial X_1(z) \pm i\partial \bar{X}_1(\bar{z}))$.

Finally, let us discuss the remaining problems. First, we need to check that the spectrum is closed at loop orders, namely, when interactions are introduced. This is also necessary to assure the unitarity of the model. For this purpose, we have to analyze the fusion rules. This requires careful treatment of screening operators as in other free field systems including background charges. Calculation of correlators is closely related.

One might be able to derive the fusion rules from the modular property of the partition function \cite{31,32}. A naive application of the formula in \cite{31} leads to $\phi_{\vec{p}} \times \phi_{\vec{p}'} = \phi_{\vec{p} + \vec{p}'}$, where $\phi_{\vec{p}}$ are the chiral primary fields with momenta of $X_i$, $\vec{p} - i\vec{Q} = (p^0, p^1, p^2 - iQ)$; $p^i \in \mathbb{R}$. Nevertheless, there is no proof of this formula for CFT’s with an infinite number of primary fields to the author’s knowledge. Second, it is important to consider how to extract physical information. What we should do first is to calculate correlation functions. However, this may not be enough to understand black hole physics such as Hawking radiation and black hole entropy. Recently, these issues are intensively analyzed by using D-branes. It is interesting to approach these problems in a different way using our model.
Acknowledgements

The author would like to thank I. Bars, S. Hirano, M. Natsuume and, especially, M. Kato for useful discussions. He would also like to acknowledge useful comments from A. A. Tseytlin on [14]. This work is supported in part by JSPS Postdoctoral Fellowships for Research Abroad.

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