POLARIZED ENDOMORPHISMS OF COMPLEX NORMAL VARIETIES

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ABSTRACT. It is shown that a complex normal projective variety has non-positive Kodaira dimension if it admits a non-isomorphic quasi-polarized endomorphism. The geometric structure of the variety is described by methods of equivariant lifting and fibrations.

1. Introduction

We work over the complex number field \( \mathbb{C} \). Much progress has been recently made in the study of endomorphisms of smooth projective varieties from the algebro-geometric viewpoint. Especially, the following cases of varieties are well studied: projective surfaces ([35], [13]), homogeneous manifolds ([40], [9]), Fano manifolds ([1], [3], [20]), projective bundles ([2]), and projective threefolds with non-negative Kodaira dimension ([12], [14]). Additionally, étale endomorphisms are investigated in [39] from the viewpoint of the birational classification of algebraic varieties. However, there is neither a classification of endomorphisms of singular varieties even when they are of dimension two (except for [38]), nor any reasonably fine classification of non-étale endomorphisms of smooth threefolds, which are then necessarily uniruled.

Let \( V \) be a normal projective variety of dimension \( n \). An endomorphism \( f: V \to V \) is called polarized if there is an ample divisor \( H \) such that \( f^*H \) is linearly equivalent to \( qH \) (\( f^*H \sim qH \)) for a positive number \( q \). In this case, \( f \) is a finite surjective morphism, \( q \) is an integer, and \( \deg f = q^n \) (cf. Lemma 2.1 below). A surjective endomorphism of a variety of Picard number one is always polarized. Polarized endomorphisms of smooth projective varieties are studied in papers [11] and [45]. In the present article, we shall study the polarized endomorphisms of normal projective varieties (not only smooth ones). The following Theorems 1.1 and 1.3 are our main results.

Theorem 1.1. Let \( f: X \to X \) be a non-isomorphic polarized endomorphism of a normal projective variety \( X \). Then there exist a finite morphism \( \tau: V \to X \) from a normal projective variety \( V \), a dominant rational map \( \pi: V \to A \times S \) for an abelian variety \( A \) and an integer \( q \) such that \( f^*H \sim qH \).
and a weak Calabi–Yau variety \( S \) (cf. Definition 2.9 below), and polarized endomorphisms \( f_V : V \to V \), \( f_A : A \to A \) and \( f_S : S \to S \) satisfying the following conditions:

1. \( \tau \circ f_V = f \circ \tau \), \( \pi \circ f_V = (f_A \times f_S) \circ \pi \), i.e., the diagram below is commutative:

\[
\begin{array}{ccc}
A \times S & \xleftarrow{\pi} & V \xrightarrow{f} X \\
\downarrow f_A \times f_S & & \downarrow f_V & \downarrow f \\
A \times S & \xleftarrow{\pi} & V \xrightarrow{\tau} X.
\end{array}
\]

2. \( \tau \) is étale in codimension one.

3. If \( X \) is not uniruled, then the Kodaira dimension \( \kappa(X) = 0 \) and \( \pi \) is an isomorphism.

4. If \( X \) is uniruled, then, for the graph \( \Gamma_\pi \subset V \times A \times S \) of \( \pi \), the projection \( \Gamma_\pi \to A \times S \) is an equi-dimensional morphism birational to the maximal rationally connected fibration (MRC fibration in the sense of [29], cf. [7], [17]) of a smooth model of \( V \).

5. If \( \dim S > 0 \), then \( \dim S \geq 4 \) and \( S \) contains a non-quotient singular point.

In case \( X \) is smooth and \( \kappa(X) \geq 0 \), Theorem 1.1 with \( S \) being a point is proved in [11], Theorem 4.2. For uniruled \( X \), there is a discussion related to Theorem 1.1 on endomorphisms and maximal rationally connected fibrations in [15], Section 2.2, especially in Proposition 2.2.4 (cf. Remark 4.2 below). We expect that \( \dim S = 0 \) for the variety \( S \) in Theorem 1.1. To be precise, we propose:

**Conjecture 1.2.** A non-uniruled normal projective variety admitting a non-isomorphic polarized endomorphism is \( Q \)-abelian (cf. Definition 2.13 below), i.e., there is a finite surjective morphism étale in codimension one from an abelian variety onto the variety.

The conjecture has been proved affirmatively in [11], Theorem 4.2, for the case of smooth varieties with non-negative Kodaira dimension. In Theorem 3.4 below, we confirm the conjecture for a non-uniruled variety \( X \) such that \( \dim X \leq 3 \) or that \( X \) has only quotient singularities.

Applying Theorem 1.1 and more, we have the following classification result, where \( q^\tau(X, f) \) denotes the supremum of irregularities \( q(X') \) of a smooth model \( X' \) of \( X \) for all the finite coverings \( \tau : X' \to X \) étale in codimension one and admitting an endomorphism \( f' : X' \to X' \) with \( \tau \circ f' = f \circ \tau \); we also define a similar notion \( q^\tau(X) \) (independent of \( f \)) so that \( q^\tau(X, f) \leq q^\tau(X) \) in general, with equality holds when \( X \) is non-uniruled (cf. Definition 2.6, Proposition 3.5, Theorem 3.2); see also Lemmas 4.5 and 2.8.

**Theorem 1.3.** Let \( f : X \to X \) be a non-isomorphic polarized endomorphism of a normal projective variety \( X \) of dimension \( n \). Then \( \kappa(X) \leq 0 \) and \( q^\tau(X, f) \leq n \). Furthermore, \( X \) is described as follows:
(1) Assume that \( q^\natural(X, f) = 0 \). If \( n \leq 3 \), or more generally, if Conjecture 1.2 is true for varieties of dimension at most \( n \), then \( X \) is rationally connected.

(2) \( q^\natural(X, f) = n \) if and only if \( X \) is \( Q \)-abelian (cf. Definition 2.13 below).

(3) Assume that \( q^\natural(X, f) \geq n - 3 \), or more generally, that Conjecture 1.2 is true for varieties of dimension at most \( n - q^\natural(X, f) \). Then there exist a finite covering \( \tau: V \to X \) étale in codimension one, a birational morphism \( \rho: Z \to V \) of normal projective varieties, and a flat surjective morphism \( \varpi: Z \to A \) onto an abelian variety \( A \) of dimension \( q^\natural(X, f) \), and polarized endomorphisms \( f_V: V \to V \), \( f_Z: Z \to Z \), \( f_A: A \to A \) such that

- every fiber of \( \varpi \) is irreducible, normal, and rationally connected,
- \( \tau \circ f_V = f \circ \tau \), \( \rho \circ f_Z = f_V \circ \rho \), and \( \varpi \circ f_Z = f_A \circ \varpi \), i.e., the diagram below is commutative:

\[
\begin{array}{ccc}
A & \xrightarrow{\varpi} & Z \\
\downarrow f_A & & \downarrow f_Z \\
A & \xrightarrow{\varpi} & Z
\end{array}
\begin{array}{ccc}
\xrightarrow{\rho} & V & \xrightarrow{\tau} & X \\
\downarrow f_V & & \downarrow f \\
\xrightarrow{\rho} & V & \xrightarrow{\tau} & X.
\end{array}
\]

Moreover, the fundamental group \( \pi_1(X) \) has a finite-index subgroup which is a finitely generated abelian group of rank at most \( 2q^\natural(X, f) \).

(4) If \( q^\natural(X, f) = n - 1 \), then there is a finite covering \( \tau: V \to X \) étale in codimension one from a normal projective variety \( V \) admitting an endomorphism \( f_V: V \to V \) with \( \tau \circ f_V = f \circ \tau \) such that one of the following conditions is satisfied:

(a) \( V \) is a \( \mathbb{P}^1 \)-bundle over an abelian variety.

(b) There exist a \( \mathbb{P}^1 \)-bundle \( Z \) over an abelian variety and a birational morphism \( Z \to V \) whose exceptional locus is a section of the \( \mathbb{P}^1 \)-bundle.

Notation and Conventions. The readers may refer to the standard references such as [25] and [28] for things related to the birational classification theory of algebraic varieties and the minimal model theory of projective varieties, e.g., log-terminal singularity, canonical singularity, etc.

For a normal variety \( X \), the canonical divisor, denoted by \( K_X \), is defined as the natural extension of the canonical divisor of the smooth locus of \( X \). Details on a relation between the canonical divisor and the dualizing sheaf \( \omega_X \) and details on Weil divisors on normal varieties are explained in [41], Appendix to §1. The notion of canonical singularity is introduced in the same paper [41].

The Kodaira dimension \( \kappa(M) \) of a smooth projective variety \( M \) is a birational invariant. The Kodaira dimension \( \kappa(X) \) of a singular projective variety \( X \) is defined as the Kodaira dimension \( \kappa(M) \) of a smooth model \( M \) of \( X \), i.e., a smooth projective variety \( M \) birational to \( X \).
The linear equivalence relation of divisors is denoted by the symbol $\sim$, the $\mathbb{Q}$-linear equivalence relation by $\sim_{\mathbb{Q}}$, and the numerical equivalence relation by $\sim_{\mathbb{N}}$.

For a projective variety $Z$, the singular locus is denoted by $\text{Sing } Z$ and the smooth locus $Z \setminus \text{Sing } Z$ by $Z_{\text{reg}}$.

Let $f: Z' \to Z$ be a finite surjective morphism of normal varieties. We denote by $R_f$ the ramification divisor of $f$, which is just the natural extension of the ramification divisor of the restriction $Z'_{\text{reg}} \cap f^{-1}(Z_{\text{reg}}) \to Z_{\text{reg}}$ of $f$, where the closed subset $Z' \setminus (Z'_{\text{reg}} \cap f^{-1}(Z_{\text{reg}}))$ has codimension at least two; in other words, for a prime divisor $\Gamma$ on $Z'$, $\text{mult}_\Gamma(R_f) = m - 1$ if and only if $\text{mult}_\Gamma(f^*(f(\Gamma))) = m$, where $\text{mult}_\Gamma(D)$ denotes the multiplicity of a divisor $D$ along $\Gamma$. As usual, we have the ramification formula: $K_{Z'} = f^*(K_Z) + R_f$. The finite surjective morphism is called étale in codimension one if $R_f = 0$ or equivalently if $f$ is étale over $Z \setminus \Sigma$ for a closed subset $\Sigma$ with $\text{codim}(\Sigma) \geq 2$.

Remark 1.4. If $Z$ is smooth and if $\Sigma$ is a closed subset with $\text{codim}(\Sigma) \geq 2$, then the natural homomorphism $\pi_1(Z \setminus \Sigma) \to \pi_1(Z)$ of the fundamental groups is isomorphic. This property implies the birational invariance of the fundamental group of a smooth projective variety. Moreover, the same property implies that if $Z$ is smooth and if $Z' \to Z$ is a finite surjective morphism étale in codimension one from a normal variety $Z'$, then, it is actually étale. Therefore, for an arbitrary normal variety $V$, a finite surjective morphism $V' \to V$ étale in codimension one from a normal variety $V'$ is determined uniquely up to isomorphism over $V$ by a finite index subgroup of $\pi_1(V_{\text{reg}})$.

The irregularity $q(X)$ of a normal projective variety $X$ is defined as $\dim H^1(X, \mathcal{O}_X)$. In Definition 2.6 below, we define $q^\circ(X)$ to be the supremum of $q(X')$ for all the normal projective varieties $X'$ with finite surjective morphisms $X' \to X$ étale in codimension one. More variants of the irregularity $q(X)$ are defined in Definition 2.6.

A normal projective variety $Y$ with only canonical singularities is called a weak Calabi–Yau variety if $K_Y \sim 0$ and $q^\circ(Y) = 0$ (cf. Definition 2.9 below and its remark).

A normal projective variety $W$ is called $Q$-abelian if there is a finite surjective morphism $A \to W$ étale in codimension one from an abelian variety $A$ (cf. Definition 2.13 below and its remark). A similar notion “$Q$-torus” is introduced in [34], which is a Kähler version and is restricted to étale coverings.

An endomorphism $f: X \to X$ is called polarized (resp. quasi-polarized) if $f^*H \sim qH$ for an ample divisor (resp. a nef and big divisor) $H$ for some positive integer $q$ (cf. Lemma 2.1 below).

Acknowledgement. The second author would like to express his gratitude to Professors Frédéric Campana and Nessim Sibony for the valuable comments, and to Research Institute for Mathematical Sciences, Kyoto University for the support and warm hospitality.
during the visit in the second half of 2007. He would also like to thank the following institutes for the support and hospitality: University of Tokyo, Nagoya University, and Osaka University.

2. Some basic properties

A surjective endomorphism of a normal projective variety is a finite morphism by the same argument as in [12], Lemma 2.3. In fact, such an endomorphism \( f: X \to X \) induces an automorphism \( f^*: N^1(X) \to N^1(X) \) of the real vector space \( N^1(X) := \text{NS}(X) \otimes \mathbb{R} \) for the Néron–Severi group \( \text{NS}(X) \), so the pullback of an ample divisor is ample, which implies the finiteness of \( f \).

**Lemma 2.1.** Let \( f: X \to X \) be an endomorphism of an \( n \)-dimensional normal projective variety \( X \) such that \( f^*H \sim qH \) for a positive number \( q \) and for a nef and big divisor \( H \). Then \( q \) is a positive integer and \( \deg f = q^n \). Moreover, the absolute value of any eigenvalue of \( f^*: N^1(X) \to N^1(X) \) is \( q \).

**Proof.** Comparing the self-intersection numbers \( (f^*H)^n \) and \( H^n \), we have \( \deg f = q^n \). In particular, \( q \) is an algebraic integer. Since the numerical equivalence classes of \( f^*H \) and \( H \) in \( N^1(X) \) belong to the image of Néron–Severi group \( \text{NS}(X) \), \( q \) is a rational number. Hence, \( q \) is an integer. Let \( \lambda \) be the spectral radius of \( f^*: N^1(X) \to N^1(X) \), i.e., the maximum of the absolute values of eigenvalues of \( f^* \). Then there is a nef \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisor \( D \) such that \( D \not\sim 0 \) and \( f^*D \sim \lambda D \), by a version of the Perron–Frobenius theorem (cf. [5]). Suppose that \( \lambda \neq q \). Then \( D^{n-1} = 0 \) by the equalities

\[
\lambda q^{n-1}DH^{n-1} = f^*D(f^*H)^{n-1} = (\deg f)DH^{n-1} = q^nDH^{n-1}.
\]

Thus, \( D \sim 0 \) by Lemma 2.2 below. This is a contradiction. Therefore, \( \lambda = q \). For the spectral radius \( \lambda' \) of \( (f^*)^{-1} \), \( \lambda'^{-1} \) is the minimum of the absolute values of eigenvalues of \( f^* \). By the same version of the Perron–Frobenius theorem, we also have a nef \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisor \( D' \) such that \( D' \not\sim 0 \) and \( f^*D' = \lambda'^{-1}D' \). Then, \( \lambda' = q^{-1} \) by the same reason as above. Hence, the absolute value of any eigenvalue of \( f^* \) is \( q \). \( \square \)

**Corollary.** The degree of a quasi-polarized endomorphism of a normal projective variety of dimension \( n \) is the \( n \)-th power \( q^n \) of a positive integer \( q \).

The lemma below is regarded as a generalization of a part of the Hodge index theorem and it is used in the proofs of Lemma 2.1 and Theorem 3.2.

**Lemma 2.2.** Let \( X \) be a smooth projective variety of dimension \( n \geq 2 \). Suppose that an \( \mathbb{R} \)-divisor \( D \) satisfies the following two conditions:
Then the following assertions hold:

Lemma 2.3. Let \( M \) be a projective variety such that \( a \) is a positive integer and \( b \) is an effective \( R \)-divisor such that \( a + b \) is holomorphic. Then, we have a commutative diagram:

\[
\begin{array}{ccc}
H^1(X, O_X) & \xrightarrow{f^*} & H^1(X, O_X) \\
\mu^* & & \nu^* \\
H^1(M, O_M) & \xrightarrow{h^*} & H^1(Z, O_Z) & \xrightarrow{\psi^*} & H^1(M, O_M).
\end{array}
\]

Proof. Let \( A \) be an ample divisor on \( X \). Then, there exist a rational number \( a \) and an effective \( R \)-divisor \( E \) such that \( aH_1 \cong E + A \), since \( H_1 \) is big. Thus,

\[ 0 \leq DAH_2 \cdots H_{n-1} = -DEH_2 \cdots H_{n-1} \leq 0 \]

by (1) and (2). Hence, we may assume \( H_1 = A \). Applying the same argument to \( H_i \) for \( i \geq 2 \), we have \( DA^{n-1} = 0 \). Hence, \( D^2A^n \leq 0 \) in which the equality holds if and only if \( D \cong 0 \), by the hard Lefschetz theorem. Thus, it suffices to show: \( D^2A^n \geq 0 \). There is a positive integer \( b \) such that \( D + bA \) is ample. In particular, \( D + bA \cong \Delta \) for an effective \( R \)-divisor \( \Delta \). Hence, we have \( D^2A^n \geq 0 \) by (1), since

\[ 0 \leq D\Delta A^n = D(D + bA)A^n = D^2A^n. \]

The endomorphism in Lemma 2.1 is shown to be quasi-polarized by the following:

Lemma 2.3. Let \( f: X \to X \) be an endomorphism of an \( n \)-dimensional normal projective variety \( X \) such that \( f^*H \cong qH \) for a positive number \( q \) and for a nef and big divisor \( H \). Then the following assertions hold:

1. The absolute value of any eigenvalue of \( f^* \): \( H^1(X, O_X) \to H^1(X, O_X) \) is \( \sqrt{q} \).
2. There is a nef and big divisor \( H' \) such that \( H' \cong H \) and \( f^*H' \sim Q qH' \).

In particular, \( f \) is quasi-polarized by \( H' \).

Proof. (1): There exist birational morphisms \( \mu: M \to X \) and \( \nu: Z \to X \) from smooth projective varieties \( M \) and \( Z \), and a generically finite surjective morphism \( h: Z \to M \) such that \( \mu \circ h = f \circ \nu \). We may assume that the birational map \( \psi := \mu^{-1} \circ \nu: Z \to M \) is holomorphic. Then, we have a commutative diagram:

\[
\begin{array}{ccc}
H^1(X, O_X) & \xrightarrow{f^*} & H^1(X, O_X) \\
\mu^* & & \nu^* \\
H^1(M, O_M) & \xrightarrow{h^*} & H^1(Z, O_Z) & \xrightarrow{\psi^*} & H^1(M, O_M).
\end{array}
\]

Let \( \phi(x) \) be the image of \( x \in H^1(X, O_X) \) by the composition

\[ H^1(X, O_X) \xrightarrow{\mu^*} H^1(M, O_M) \xrightarrow{\cong} H^{0,1}(M) \subset H^1(M, C), \]

where \( H^{0,1}(M) \) is the \((0,1)\)-part of the Hodge decomposition of \( H^1(M, C) \). Then, for \( x \in H^1(X, O_X) \), we have \( \psi^*\phi(f^*(x)) = h^*\phi(x) \) by the diagram above. We consider the
following Hermitian form on $H^1(X, \mathcal{O}_X)$:

$$\langle x, y \rangle = -\sqrt{-1} \int_M \phi(x) \cup \overline{\phi(y)} \cup (\mu^* c_1(H))^{n-1} \in \mathbb{C}.$$ 

This is positive definite by Lemma 2.4 below applied to $L = \mu^*(H)$. We have the equality

$$\langle f^*(x), f^*(y) \rangle = q \langle x, y \rangle$$

for $x, y \in H^1(X, \mathcal{O}_X)$ by the calculation

$$(\deg h) \langle x, y \rangle = -\sqrt{-1} \int_Z h^* \phi(x) \cup \overline{h^* \phi(y)} \cup (h^* \mu^* c_1(H))^{n-1}$$

$$= -\sqrt{-1} \int_Z \psi^* \phi(f^*(x)) \cup \overline{\psi^* \phi(f^*(y))} \cup (\nu^* f^* c_1(H))^{n-1}$$

$$= -\sqrt{-1} \int_M \phi(f^*(x)) \cup \overline{\phi(f^*(y))} \cup (\mu^* f^* c_1(H))^{n-1}$$

$$= q^{n-1} \langle f^*(x), f^*(y) \rangle,$$

where $\deg h = \deg f = q^n$. Therefore, $q^{-1/2} f^*$ is a unitary transformation with respect to $\langle \ , \ \rangle$. Thus, the absolute value of any eigenvalue of $q^{-1/2} f^*$ is 1.

(2): Let $m$ be the order of $c_1(f^*H - qH)$ in $H^2(X, \mathbb{Z})$. By the exponential exact sequence

$$H^1(X, \mathcal{O}_X) \xrightarrow{\epsilon} H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$$

we can find an element $x \in H^1(X, \mathcal{O}_X)$ with $\mathcal{O}_X(m(f^*H - qH)) = m \epsilon(x)$. There is an element $y \in H^1(X, \mathcal{O}_X)$ such that $f^*(y) - qy = x$ by (1). Let $H'$ be a divisor such that $\mathcal{O}_X(H - H') = \epsilon(y)$. Then $m(f^*H' - qH') \sim 0$. Thus, we are done.

\[\square\]

**Remark.** The proof of Lemma 2.3 is similar to that of [43], Theorem 1.1.2, where $X$ is assumed to be smooth.

In the proof of Lemma 2.3 we used the result below:

**Lemma 2.4.** Let $M$ be an $n$-dimensional smooth projective variety and $L$ a nef and big divisor. Then the Hermitian form $\langle \ , \ \rangle$ on $H^{0,1}(M)$ defined by

$$\langle \xi, \eta \rangle = -\sqrt{-1} \int_M \xi \cup \overline{\eta} \cup c_1(L)^{n-1}$$

is positive definite.

**Proof.** We may assume that $n \geq 2$, since it is well-known to be positive definite in case $n = 1$. If $L$ is ample, then the bilinear form is positive definite by the hard Lefschetz theorem. Thus, the bilinear form is positive semi-definite even if we replace $L$ with a nef divisor. Let $W$ be a prime divisor of $M$. Then

$$-\sqrt{-1} \int_M \xi \cup \overline{\xi} \cup c_1(L)^{n-2} \cup c_1(W)$$
is non-negative for any $\xi \in H^{0,1}(M)$. In fact, it is equal to

$$-\sqrt{-1}\int_W \varphi^*(\xi) \cup \overline{\varphi^*(\xi)} \cup c_1(\varphi^*L)^{n-2}$$

for a resolution of singularities $\varphi: \tilde{W} \to W$, and it is non-negative by the reason above.

There exist a positive integer $m$, a smooth ample divisor $A$, and an effective divisor $E = \sum e_iE_i$ such that $mL \sim A + E$. Then

$$m\langle \xi, \xi \rangle = -\sqrt{-1}\int_M \xi \cup \overline{\xi} \cup mc_1(L)^{n-1}$$

$$= -\sqrt{-1}\int_A \xi|_A \cup \overline{\xi|_A} \cup c_1(L|_A)^{n-2} + \sum e_i(-\sqrt{-1})\int_{E_i} \xi|_{E_i} \cup \overline{\xi|_{E_i}} \cup c_1(L|_{E_i})^{n-2}.$$  

Hence, if $\langle \xi, \xi \rangle = 0$, then

$$-\sqrt{-1}\int_A \xi|_A \cup \overline{\xi|_A} \cup c_1(L|_A)^{n-2} = 0.$$  

Since $L|_A$ is nef and big, we can consider the induction on $\dim M$. Then, we have $\xi|_A = 0$ as an element of $H^{0,1}(A)$ by induction. Therefore, $\xi = 0$, since $H^1(M, \mathcal{O}_M(-A)) = 0$ by the Kodaira vanishing theorem for $n \geq 2$ and hence $H^1(M, \mathcal{O}_M) \to H^1(A, \mathcal{O}_A)$ is injective. Thus, we are done. $\blacksquare$

We borrow the following property of Galois closures of powers $f^k = f \circ \cdots \circ f$ from [38]:

**Lemma 2.5.** Let $f: X \to X$ be a non-isomorphic surjective endomorphism of a normal projective variety $X$. Let $\theta_k: V_k \to X$ be the Galois closure of $f^k: X \to X$ for $k \geq 1$ and let $\tau_k: V_k \to X$ be the induced finite Galois covering such that $\theta_k = f^k \circ \tau_k$. Then there exist finite Galois morphisms $g_k, h_k: V_{k+1} \to V_k$ such that $\tau_k \circ g_k = \tau_{k+1}$ and $\tau_k \circ h_k = f \circ \tau_{k+1}$.

**Proof.** The composite $f^k \circ \tau_{k+1}: V_{k+1} \to X$ is Galois, since so is $f^{k+1} \circ \tau_{k+1} = \theta_{k+1}$. Hence, $f^k \circ \tau_{k+1}$ factors through the Galois closure $\theta_k$ of $f^k$. Thus, $\tau_{k+1} = \tau_k \circ g_k$ for a morphism $g_k: V_{k+1} \to V_k$. Let $H_i$ be the Galois group of $f^i \circ \tau_{k+1}: V_{k+1} \to X$ for $0 \leq i \leq k + 1$. Then $V_k$ is regarded as the Galois closure of $V_{k+1}/H_1 \to V_{k+1}/H_{k+1}$, thus $V_k \simeq V_{k+1}/H$ for the maximal normal subgroup $H$ of $H_{k+1}$ contained in $H_1$. Hence, we have a morphism $h_k: V_{k+1} \to V_k$ with $\tau_k \circ h_k = f \circ \tau_{k+1}$. $\blacksquare$

**Definition 2.6.** Let $X$ be a normal projective variety. The irregularity $q(X)$ is defined as $\dim H^1(X, \mathcal{O}_X)$. We define the following variants of $q(X)$:

1. $\bar{q}(X) := q(\bar{X})$ for a smooth model $\bar{X}$ of $X$ (This is well-defined).
(2) $q^\circ(X)$ (resp. $q^\natural(X)$) is defined to be the supremum of $q(X')$ (resp. $\tilde{q}(X')$) for a normal projective variety $X'$ with a finite surjective morphism $\tau: X' \to X$ étale in codimension one. Namely, 

$$
q^\circ(X) := \sup\{q(X') | X' \to X \text{ is finite, surjective, and étale in codimension one}\},
$$

$$
q^\natural(X) := \sup\{\tilde{q}(X') | X' \to X \text{ is finite, surjective, and étale in codimension one}\}.
$$

(3) Suppose that $X$ admits a surjective endomorphism $f: X \to X$. Then we define $q^\circ(X, f)$ (resp. $q^\natural(X, f)$) to be the supremum of $q(X')$ (resp. $\tilde{q}(X')$) for a normal projective variety $X'$ with a finite surjective morphism $\tau: X' \to X$ étale in codimension one and with an endomorphism $f': X' \to X'$ such that $\tau \circ f' = f \circ \tau$.

Remark. If $X$ is a smooth projective variety, then $q^\circ(X)$ equals $q^{\max}(X)$ defined in [39].

Remark 2.7. Let $\tau: X' \to X$ be a finite surjective morphism of normal varieties étale in codimension one. Then:

1. $X$ has only log-terminal singularities if and only if so does $X'$.
2. If $X$ has only canonical singularities, then so does $X'$.

These well-known properties are derived from [41], Proposition (1.7) and [22], Proposition 1.7, as follows. The assertion (1) is proved just by the same argument as in the proof of [22], Proposition 1.7. If we replace the logarithmic ramification formula in the proof with the usual ramification formula, then we can prove (2); this was already done in [41], Proposition (1.7), (I). Another proof of these properties is found in [28], Proposition 5.20, but it is essentially the same as above.

Lemma 2.8. Let $X$ be a normal projective variety with only log-terminal singularities. Then $q^\circ(X) = q^\natural(X)$. If $f$ is a surjective endomorphism of $X$, then $q^\circ(X) \geq q^\circ(X, f) = q^\natural(X, f)$.

Proof. If $X'$ is a normal variety with a finite covering $X' \to X$ étale in codimension one, then $X'$ is also log-terminal (cf. Remark 2.7). In particular, $X'$ has only rational singularities, and hence $q(X') = \tilde{q}(X')$. Thus, $q^\circ(X) = q^\natural(X)$. Considering the special case where $X'$ admits an endomorphism compatible with $f$, we have $q^\circ(X, f) = q^\natural(X, f)$. We also have $q^\circ(X) \geq q^\circ(X, f)$ by definition. □

Definition 2.9. A normal projective variety $Y$ with only canonical singularities is called a weak Calabi–Yau variety if $K_Y \sim 0$ and $q^\circ(Y) = 0$.

Remark. The notion of weak Calabi–Yau variety is slightly different from that in [39] in which only finite étale coverings were taken into consideration. A weak Calabi–Yau variety has dimension at least two. A two-dimensional weak Calabi–Yau variety is nothing
but a normal projective surface such that the minimal resolution of singularities is a K3 surface and that there is no finite surjective morphism from any abelian surface.

**Proposition 2.10.** Let $X$ be a normal projective variety with only log-terminal singularities such that $K_X \sim Q 0$. Then:

1. $q^o(X) \leq \dim X$. In particular, there is a finite Galois covering $X' \to X$ étale in codimension one such that $q(X') = q^o(X)$.
2. $q(X) = \dim X$ if and only if $X$ is an abelian variety.
3. There exists a finite covering $A \times S \to X$ étale in codimension one for an abelian variety $A$ of dimension $q^o(X)$ and a weak Calabi–Yau variety $S$.

**Proof.** Let $r$ be the smallest positive integer such that $rK_X \sim 0$. Then, there is a cyclic covering $\widetilde{X} \to X$ of degree $r$ étale in codimension one from a normal projective variety $\widetilde{X}$ such that $K_{\widetilde{X}} \sim 0$. The covering is unique up to isomorphism over $X$ and is called the *global index-one covering* (or the canonical cover in [22]). Then, $\widetilde{X}$ has only canonical singularities by [22], Proposition 1.7. Let $Y \to \widetilde{X}$ be a finite covering étale in codimension one from a normal projective variety $Y$. Then, $K_Y \sim 0$ and $Y$ has only canonical singularities by [22], Proposition (1.7). Let $\alpha : Y \to A := \text{Alb}(Y)$ be the Albanese map; this is holomorphic, since $Y$ has only rational singularities (cf. [23], Lemma 8.1). Then, $\alpha$ is an étale fiber bundle by [23], Theorem 8.3, i.e., there is a finite étale covering $A' \to A$ from an abelian variety $A'$ such that $Y \times_A A' \simeq F \times A'$ over $A'$ for a fiber $F$ of $\alpha$. In particular, $q(Y) = \dim A \leq \dim X$. As a consequence, we have $q^o(X) \leq \dim X$, since any finite covering $X' \to X$ étale in codimension one is dominated by such a variety $Y$. By the boundedness of $q^o(X)$, we have a finite covering $X' \to X$ étale in codimension one such that $q^o(X) = q(X')$. The Galois closure $X'' \to X$ of $X' \to X$ is also étale in codimension one and $q^o(X) = q(X') \leq q(X'') \leq q^o(X)$. Thus, (1) has been proved.

In order to prove the other assertions (2) and (3), we may assume that $q^o(X) = q(Y)$ and that the composite $Y \to \widetilde{X} \to X$ is Galois. Let $G$ be the Galois group of $Y \to X$.

Assume that $q(X) = \dim X$. Then, $q(X) = q(Y) = \dim Y$ and $\alpha : Y \to A$ is an isomorphism. Since the natural pullback homomorphism $H^1(X, \mathcal{O}_X) \to H^1(Y, \mathcal{O}_Y)$ is an isomorphism, the action of $G$ on $H^1(Y, \mathcal{O}_Y)$ is trivial. Therefore, every element of $G$ acts on $A$ as a translation. Hence, the quotient variety $X \simeq G \backslash A$ is also an abelian variety. Conversely, if $X$ is an abelian variety, then $q(X) = \dim X$. Thus, (2) has been proved.

We shall prove the remaining assertion (3): If $q^o(X) = \dim X$, then $Y$ is an abelian variety by the argument above. Thus, we may assume that $q^o(X) < \dim X$. Then the fiber $F$ of $\alpha$ is positive-dimensional and has only canonical singularities with $K_F \sim 0$. If $q^o(F) > 0$, then applying the same argument above to $F$, we have a finite covering...
$A_0 \times F_0 \to F$ étale in codimension one for a positive-dimensional abelian variety $A_0$ and a normal projective variety $F_0$. Thus, we have a finite covering $A' \times A_0 \times F_0 \to X$ étale in codimension one and a contradiction by

$$q^0(X) = q(Y) = \dim A' < \dim A' + \dim A_0 \leq q(A' \times A_0 \times F_0) \leq q^0(X).$$

Therefore, $F$ is a weak Calabi–Yau variety. Hence, the covering $F \times A' \simeq Y \times_A A' \to Y \to X$ satisfies the required condition of (3). Thus, we are done.

**Corollary 2.11.** Let $S$ be a weak Calabi–Yau variety and $A$ an abelian variety. Then

$q^0(A \times S) = q(A) + q(S) = \dim A$.\[2\]

**Proof.** We have $q^0(A \times S) \geq q(A \times S) = q(A) + q(S) = \dim A$. Assume that $q^0(A \times S) > \dim A$. By Proposition 2.10, there is a finite surjective morphism $\tau: A' \times S' \to A \times S$ étale in codimension one for an abelian variety $A'$ of dimension $q^0(A \times S)$ and a weak Calabi–Yau variety $S'$. Since the first projections $p_1: A \times S \to A$ and $p'_1: A' \times S' \to A'$ are the Albanese maps, we have a surjective morphism $\varphi: A' \to A$ such that $p_1 \circ \tau = \varphi \circ p'_1$. Let $B$ be a connected component of the fiber $\varphi^{-1}(P)$ for a general point $P \in A$. Then, $B$ is a positive-dimensional abelian variety. By restricting $\tau$, we have a finite covering $B \times S' \to \{P\} \times S$ étale in codimension one. This contradicts that $q^0(S) = 0$. Thus, $q^0(A \times S) = \dim A$.\[2\]

We have the following variant of [39], Proposition 4.3, which treats only étale coverings and varieties with only canonical singularities:

**Lemma 2.12.** Let $V$ be a normal projective variety with only log-terminal singularities such that $K_V \sim_Q 0$. Then there exists a finite morphism $\tau: V^\sim \to V$ satisfying the following conditions, uniquely up to isomorphism over $V$:

1. $\tau$ is étale in codimension one.
2. $q^0(V) = q(V^\sim)$.
3. $\tau$ is Galois.
4. If $\tau': V' \to V$ satisfies the conditions (1), (2), then there exists a finite surjective morphism $\sigma: V' \to V^\sim$ étale in codimension one such that $\tau' = \tau \circ \sigma$.

We call $\tau$ the Albanese closure of $V$ in codimension one.

**Proof.** The same argument as in the proof of [39], Proposition 4.3 works as follows: We may assume that $q^0(V) > 0$. There is a Galois covering $W \to V$ étale in codimension one with $q(W) = q^0(V)$ by Proposition 2.10. Then $K_W \sim_Q 0$ and $W$ has only log-terminal singularities (cf. Remark 2.7). Let $W \to \Alb(W)$ be the Albanese map of $W$ and $\Gal(W/V)$ the Galois group of $W \to V$. Then we have a natural homomorphism
Gal(W/V) → Aut(H₁(Alb(W), Z)). Let G₀ be the kernel and let W₀ be the quotient variety G₀\W of W by the action of G₀. Then q(W₀) = q(W), since the quotient variety of Alb(W) by G₀ is an abelian variety, as in the proof of Proposition 2.10. Therefore, the Galois covering W₀ → V satisfies the conditions (1)–(3). Let W' → V be an arbitrary covering satisfying the conditions (1) and (2). Then there exist finite morphisms W'' → W and W'' → W' over V such that the composite W'' → V is Galois and étale in codimension one. Thus, W₀'' ≃ W₀ for the quotient variety W₀'' for W'' obtained by the same procedure as in defining W₀ from W, and there is a morphism W' → W₀ over V. Hence, V~ := W₀ satisfies all the required conditions (1)–(4), and V~ → V is unique up to non-canonical isomorphism over V.

**Definition 2.13.** A normal projective variety W is called Q-abelian if there are an abelian variety A and a finite surjective morphism A → W which is étale in codimension one.

**Remark.** By Proposition 2.10, a Q-abelian variety is characterized as a normal projective variety X with only log-terminal singularities such that K_X ∼ Q 0 and q^c(X) = dim X. The Albanese closure of a Q-abelian variety is abelian, by Proposition 2.10 (2).

A surjective endomorphism of the direct product of certain varieties is split. The following gives an example:

**Lemma 2.14.** Let A be an abelian variety and S a normal projective variety with only rational singularities. Suppose that q(S) = 0 and that S is not uniruled. Let f : S × A → S × A be a surjective morphism. Then f = f_S × f_A for suitable endomorphisms f_S and f_A of S and A, respectively.

**Proof.** By the universality of the Albanese map, f induces a surjective endomorphism f_A of A = Alb(S × A). We can write the endomorphism f as S × A ⊃ (s, a) ↦ f(s, a) = (ρ_a(s), f_A(a)), where ρ : A → Sur(S), a ↦ ρ_a, is a morphism into

Sur(S) := \{g : S → S | g is a surjective morphism\}.

By [19], Theorem 3.1, the compact subvariety Im(ρ) is contained in the orbit of some f_S ∈ Sur(S) by the action of Aut^0(S). For a smooth model S' of S, the birational automorphism group Bir(S') contains Aut^0(S) as a subgroup. By [18], Theorem (2.1), Bir(S') is a disjoint union of abelian varieties of dimension equal to q(S') = q(S) = 0. Thus Im(ρ) consists of a single element, say \{f_S\}. Then f = f_S × f_A. □
3. The non-uniruled case

In this section, we shall study non-isomorphic quasi-polarized endomorphisms of non-uniruled normal projective varieties. The following gives examples of such polarized endomorphisms:

**Example 3.1.** Let $A$ be an abelian variety of dimension $n \geq 2$ and let $H$ be a symmetric ample divisor, i.e., $H$ is ample and $\iota^* H \sim H$ for the involution $\iota: x \mapsto -x$. Then the multiplication map $\mu_m: A \ni x \mapsto mx = x + \cdots + x \in A$ by an integer $m \neq 0$ is polarized by $H$ as $\mu_m^* H \sim m^2 H$ (cf. [33], Chapter II, §6, Corollary 3). Let $X = A/\iota$ be the quotient variety by the involution $\iota$. Then $\mu_m$ descends to a polarized endomorphism $f_m$ of $X$ of degree $\deg(\mu_m) = m^2 n$. If $n = 2$, then $X$ has only 16 rational double points of type $A_1$ as singularities and its minimal resolution of singularities is a K3 surface, called the Kummer surface of $A$; in particular, $f_m$ for $m > 1$ is not nearly étale in the sense of [39], Definition 3.2 (cf. [39], Example 3.14). If $n \geq 3$, then $X$ has only $2^{2n}$ terminal cyclic quotient singular points of type $(1,1,\ldots,1)$ as singularities, and $2K_X \sim 0$. Thus, $X$ is not uniruled and $f_m$ is a non-isomorphic polarized endomorphism for $m > 1$.

In the examples above, $X$ has only canonical singularities and $K_X \sim_Q 0$. These properties hold in general by the following fundamental result:

**Theorem 3.2.** Let $f: V \to V$ be a surjective endomorphism of a normal projective variety $V$ and let $H$ be a nef and big Cartier divisor on $V$ such that $f^* H \sim qH$ for a positive integer $q > 1$. Suppose that $V$ is not uniruled. Then, there exist a projective birational morphism $\sigma: V \to X$ onto a normal projective variety $X$, an endomorphism $f_X$ of $X$, and an ample divisor $A$ on $X$ such that

1. $X$ has only canonical singularities with $K_X \sim_Q 0$,
2. $f_X^* A \sim qA$,
3. $f_X \circ \sigma = \sigma \circ f$,
4. $H \sim \sigma^* A$, and
5. $f_X$ is étale in codimension one.

In particular, if $H$ is ample, then $V$ has only canonical singularities, $K_V \sim_Q 0$, and $f$ is étale in codimension one.

**Proof.** We may assume that $V$ is of dimension $n \geq 2$. Taking intersection numbers with $(f^*(H))^{n-1} = f^*(H) \cdots f^*(H)$ of the both sides of the ramification formula: $K_V = f^*(K_V) + R_f$, we obtain

$$(q - 1)K_VH^{n-1} + R_f H^{n-1} = 0.$$
Thus, \( K_Y H^{n-1} \leq 0. \) Let \( \mu : Y \to V \) be a birational morphism from a smooth projective variety \( Y. \) Since \( Y \) is not uniruled, \( K_Y (\mu^* H)^{n-1} \geq 0 \) by [31]. Thus, \( K_Y (\mu^* H)^{n-1} = K_Y H^{n-1} = R_f H^{n-1} = 0. \) Moreover, \( K_Y \) is pseudo-effective by [6] (cf. [30], §11.4.C). Thus, we have the \( \sigma \)-decomposition \( K_Y = P_\sigma (K_Y) + N_\sigma (K_Y) \) in the sense of [32]: \( N_\sigma (K_Y) \) is an effective \( \mathbb{R} \)-divisor determined by the following property: \( P_\sigma (K_Y) = K_Y - N_\sigma (K_Y) \) is movable, and if \( B \) is an effective \( \mathbb{R} \)-divisor such that \( K_Y - B \) is movable, then \( N_\sigma (K_Y) \leq B. \)

Here, an \( \mathbb{R} \)-divisor \( D \) is called \textit{movable} if: for any \( \varepsilon > 0, \) any ample divisor \( H' \) and any prime divisor \( \Gamma, \) there is an effective \( \mathbb{R} \)-negative part is \( \text{(cf. [36], Chapter III, §1.b).} \) In particular, \( P_\sigma (K_Y) \) satisfies the condition (1) of Lemma 2.2. Furthermore,

\[ 0 \leq P_\sigma (K_Y) (\mu^* H)^{n-1} = (K_Y - N_\sigma (K_Y)) (\mu^* H)^{n-1} = -N_\sigma (K_Y) (\mu^* H)^{n-1} \leq 0. \]

Therefore, \( P_\sigma (K_Y) \approx 0 \) and \( K_Y \approx N_\sigma (K_Y) \) by Lemma 2.2. This implies that the numerical Kodaira dimension \( \kappa_\sigma (Y) \) of \( Y \) in the sense of [36], Chapter V, is zero. Namely, for any ample divisor \( H' \) on \( Y, \) the function \( m \mapsto \dim H^0 (Y, \mathcal{O}_Y (mK_Y + H')) \) is bounded (cf. [36], Chapter V, Corollary 1.12). By Theorem 4.8 of [36], Chapter V, which is the abundance theorem for \( \kappa_\sigma = 0, \) we have \( \kappa (Y) = 0. \) In particular, \( K_Y \sim_\mathbb{Q} E \) for an effective \( \mathbb{Q} \)-divisor \( E \) such that \( E (\mu^* H)^{n-1} = 0. \) Therefore, \( K_Y + \mu^* H \) has a Zariski-decomposition whose negative part is \( E \) and whose positive part is \( \mathbb{Q} \)-linearly equivalent to \( \mu^* H \) by [36], Chapter III, Proposition 3.7, i.e., \( N_\sigma (K_Y + \mu^* H) = E, \) and \( P_\sigma (K_Y + \mu^* H) = \mu^* H \) is nef. The Zariski-decomposition above in the sense of [36] coincides with the Zariski-decomposition in the sense of Cutkosky–Kawamata–Moriwaki ([10], [24], [32]) or that of Fujita [15], since the divisor \( K_Y + \mu^* H \) is big (cf. [36], Chapter III, Remark 1.17). Thus, the positive part \( P_\sigma (K_Y + \mu^* H) = \mu^* H \) is semi-ample, and furthermore, \( Bs |m \mu^* H| = \emptyset \) for \( m \gg 0, \) by a version of the base point free theorem (cf. [15], (A.5); [24], Theorem 1; [32], Theorem 0). In particular, \( Bs |m H| = \emptyset \) for \( m \gg 0. \) Let \( \sigma : V \to X \) be the birational morphism onto a normal projective variety \( X \) defined by the free linear system \( |m H| \) for \( m \gg 0. \) Then \( H \sim \sigma^* A \) for an ample divisor \( A \) on \( X. \) Since \( (\mu_* E) H^{n-1} = R_f H^{n-1} = 0, \) \( \mu_* E \) and \( R_f \) are \( \sigma \)-exceptional. Therefore, \( K_X = \sigma_* (K_Y) \sim_\mathbb{Q} \sigma_* (\mu_* (E)) = 0 \) and \( X \) has only canonical singularities, since \( K_Y - \mu^* \sigma^* (K_X) = E \geq 0. \) By considering the Stein factorization of the composite \( \sigma \circ f : V \to X, \) we have an endomorphism \( f_X \) of \( X \) such that \( f_X \circ \sigma = \sigma \circ f \) and \( f_X^* A \sim q A. \) Moreover, \( R_{f_X} = \sigma_* (R_f) = 0. \) Hence, \( f_X \) is étale in codimension one. The last assertion follows immediately, since \( \sigma \) is isomorphic if \( H \) is ample. Thus, we are done.

The following gives a sufficient condition for a normal projective variety admitting polarized endomorphisms to be \( \mathbb{Q} \)-abelian (cf. Definition 2.13):
Theorem 3.3. Let $f : X \to X$ be a non-isomorphic polarized endomorphism of an $n$-dimensional normal projective variety $X$. Assume that $f$ is étale in codimension one and that, for any point $P \in \text{Sing } X$, there is a connected analytic open neighborhood $U$ of $P$ such that the algebraic fundamental group $\pi_1^{\text{alg}}(U_{\text{reg}})$ is finite. Then $X$ is a $Q$-abelian variety.

Proof. For a positive integer $k$, let $\theta_k : V_k \to X$ be the Galois closure of $f^k$, and let $\tau_k, \theta_k, g_k, h_k$ be as in Lemma 2.5, which are all étale in codimension one (cf. Remark 1.4). Then, $g_k$ and $h_k$ are both étale for $k \gg 0$ by the claim below applied to the cases $(\alpha_k, \gamma_k) = (\theta_k, h_k)$ and $(\alpha_k, \gamma_k) = (\tau_k, g_k)$.

Claim. For the $X$ above, let $\alpha_k : V_k \to X$ be finite Galois coverings and let $\gamma_k : V_{k+1} \to V_k$ be finite surjective morphisms defined for $k \geq 1$ such that $\alpha_k$ and $\gamma_k$ are étale in codimension one and $\alpha_{k+1} = \alpha_k \circ \gamma_k$ for $k \geq 1$. Then, $\gamma_k$ is étale for $k \gg 0$.

Proof. For a point $P \in \text{Sing } X$, let $U \subset X$ be a connected analytic open neighborhood such that $\pi_1^{\text{alg}}(U_{\text{reg}})$ is finite. For a point $Q \in \alpha_k^{-1}(P)$, let $V$ (depending on $Q$ and $k$) be the connected component of $\alpha_k^{-1}(U)$ containing $Q$. We set

$$\Pi(U; k) := \pi_1^{\text{alg}}(V \setminus \alpha_k^{-1}(\text{Sing } X)).$$

Note that $\Pi(U; k) = \pi_1^{\text{alg}}(V_{\text{reg}})$ by Remark 1.4. Since $\alpha_k$ is Galois, $\Pi(U; k)$ is independent of the choice of $Q \in \alpha_k^{-1}(P)$ and is a normal subgroup of $\pi_1^{\text{alg}}(U_{\text{reg}})$. By the finiteness assumption of $\pi_1^{\text{alg}}(U_{\text{reg}})$, we have a positive integer $k_P$ such that the injection $\gamma_k : \Pi(U; k + 1) \to \Pi(U; k)$ is isomorphic for any $k \geq k_P$. As a consequence, we infer that $\gamma_k : V_{k+1} \to V_k$ is étale along $\alpha_k^{-1}(P)$ for any $k \geq k_P$. Since Sing $X$ is compact, we can find a positive integer $k_0$ such that $\gamma_k : V_{k+1} \to V_k$ is étale for any $k \geq k_0$. □

Proof of Theorem 3.3 continued. We fix a large positive integer $k$ such that $g_k$ and $h_k$ are both étale. We shall show that $V_k$ is smooth. Assume the contrary that $\text{Sing } V_k \neq \emptyset$. We set $d := \dim \text{Sing } V_k$. Then $0 \leq d \leq n - 2$. Since $g_k^{-1}(\text{Sing } V_k) = h_k^{-1}(\text{Sing } V_k) = \text{Sing } V_{k+1}$, the mapping degrees of $g_k : \text{Sing } V_{k+1} \to \text{Sing } V_k$ and $h_k : \text{Sing } V_{k+1} \to \text{Sing } V_k$ are $\deg g_k$ and $\deg h_k$, respectively. Then $d > 0$; otherwise, we have a contradiction by $\sharp \text{Sing } V_{k+1} = (\deg g_k)\sharp \text{Sing } V_k = (\deg h_k)\sharp \text{Sing } V_k$ and $\deg h_k > \deg g_k$. Let $A$ be an ample divisor on $X$ such that $f^*A \sim qA$ for an integer $q > 1$. We set $A_l$ to be the ample divisor $\tau_l^*A$ for any $l \geq 1$. Then $g_k^*A_k \sim A_{k+1}$ and $h_k^*A_k \sim qA_{k+1}$. Hence, we have the equalities

$$(\text{Sing } V_{k+1})A_{k+1}^d = (\text{Sing } V_{k+1})g_k^*(A_k)^d = (\deg g_k)(\text{Sing } V_k)A_k^d = q^{-d}(\deg h_k)(\text{Sing } V_k)A_k^d$$
of intersection numbers. Thus, \((\text{Sing } V_k) A_k^d = 0\) by \(\deg h_k = (\deg f)(\deg g_k) = q^n \deg g_k\) and \(d < n\); this is absurd, since \(A_k\) is ample. Consequently, \(V_k\) is smooth. Since \(g_k, h_k: V_{k+1} \to V_k\) are étale morphisms with \(\deg h_k > \deg g_k\), we have
\[ c_1(V_k) A_k^{n-1} = c_1(V_k)^2 A_k^{n-2} = c_2(V_k) A_k^{n-2} = 0 \]
by a similar calculation of intersection numbers as above. Then \(c_1(V_k)\) is numerically trivial by the hard Lefschetz theorem. Moreover, the vanishing of \(c_2(V_k) A_k^{n-2}\) implies that an étale covering of \(V_k\) is an abelian variety by \([44]\) (cf. \([4]\)). Therefore, \(X\) is a Q-abelian variety.

\[ \square \]

Remark. An argument on the Galois closure in the proof of Theorem 3.3 is borrowed from \([38]\). A result of Campana \([8]\), Corollary 6.3, gives another proof of Theorem 3.3 in the case where \(K_X \sim Q 0\) and \(X\) has only quotient singularities.

Applying Theorems 3.2 and 3.3, we have the following partial answer to Conjecture 1.2.

**Theorem 3.4.** Let \(X\) be a non-uniruled normal projective variety such that \(\dim X \leq 3\) or that \(X\) has only quotient singularities. If \(X\) admits a non-isomorphic polarized endomorphism, then \(X\) is Q-abelian.

**Proof.** By Theorem 3.3, it is enough to show that any singular point has a connected analytic open neighborhood \(U\) such that \(\pi_1^{\text{alg}}(\mathcal{U}_{\text{reg}})\) is finite. If \(X\) has only quotient singularities, then this is true. We know that \(X\) has only canonical singularities by Theorem 3.2. If \(\dim X \leq 2\), then \(X\) has only quotient singularities. If \(\dim X = 3\), then the finiteness of \(\pi_1^{\text{alg}}(\mathcal{U}_{\text{reg}})\) is proved in \([42]\), Theorem 3.6. Thus, we are done. \[ \square \]

Even though Conjecture 1.2 is still open, we have the following:

**Proposition 3.5.** Let \(X\) be a normal projective variety and \(f: X \to X\) a polarized endomorphism of \(X\) such that \(\deg f = q^{\dim X}\) for an integer \(q \geq 1\). Assume that \(X\) has only log-terminal singularities and \(K_X \sim Q 0\). Then there exist a finite covering \(\tau: A \times S \to X\) étale in codimension one for an abelian variety \(A\) and a weak Calabi-Yau variety \(S\), and polarized endomorphisms \(f_A: A \to A, f_S: S \to S\) such that \(\deg f_A = q^{\dim A}, \deg f_S = q^{\dim S}\), and \(\tau \circ (f_A \times f_S) = f \circ \tau:\)

\[
\begin{array}{ccc}
A \times S & \xrightarrow{f_A \times f_S} & A \times S \\
\tau \downarrow & & \tau \downarrow \\
X & \xrightarrow{f} & X.
\end{array}
\]

In particular, \(q^\circ(X) = q^\circ(X) = q^\circ(X, f) = q^\circ(X, f) = q(A) = \dim A\).
Proof. By Lemma 2.1 and its corollary, there is an ample divisor $H$ on $X$ such that $f^*(H) \sim qH$.

Step 1. Reduction to the case where $X$ has only canonical singularities with $K_X \sim 0$:
Let $\nu: \tilde{X} \to X$ be the minimal cyclic covering satisfying $K_{\tilde{X}} \sim 0$ (cf. the proof of Proposition 2.10). Then $\tilde{X}$ has only canonical singularities by [22], Proposition 1.7. By the uniqueness of the global-index-one cover, there is an endomorphism $\tilde{\rho}: \tilde{X} \to \tilde{X}$ such that $\nu \circ \tilde{\rho} = \rho \circ \nu$. This is shown as follows: For the normalization $X^\circ$ of the fiber product $\tilde{X} \times_X X$ of $\nu$ and $f$ over $X$, let $p_1^\circ: X^\circ \to \tilde{X}$ and $p_2^\circ: X^\circ \to X$ be the morphisms induced from the first and second projections, respectively. Then, the restriction $X^\circ_i \to \tilde{X}$ of $p_1^\circ$ to any connected component $X^\circ_i$ of $X^\circ$ is finite and surjective, since so is $f$. Thus, pulling back a nowhere vanishing section of $\mathcal{O}_{\tilde{X}}(K_{\tilde{X}})$, we have a holomorphic section of $\mathcal{O}_{X^\circ}(K_{X^\circ})$, which is not zero on each connected component of $X^\circ$. On the other hand, $p_2^\circ$ is étale in codimension one, since so is $\nu$. Hence, $K_{X^\circ} \sim 0$. Noting that $\deg p_2^\circ = \deg \nu$ and by the minimality and the uniqueness of the global-index one covering, we infer that $X^\circ$ is irreducible and that $p_2^\circ: X^\circ \to X$ is isomorphic to the global index-one covering $\nu: \tilde{X} \to X$ over $X$. Thus, $p_1^\circ$ produces an endomorphism $\tilde{\rho}: \tilde{X} \to \tilde{X}$ satisfying $\nu \circ \tilde{\rho} = \rho \circ \nu$. Then, $\tilde{\rho}$ is a polarized endomorphism with $f^*(\nu^*(H)) \sim q\nu^*(H)$, since $f^*(H) \sim qH$. Therefore, we may assume that $X$ has only canonical singularities with $K_X \sim 0$ by replacing $(X, f)$ with $(\tilde{X}, \tilde{f})$. Note that the replacement does not affect the last equalities of Proposition 3.5 by Lemma 2.8.

Step 2. Reduction to the case where $q^\circ(X) = q(X)$: Let $\lambda: X^\sim \to X$ be the Albanese closure of $X$ in codimension one defined in Lemma 2.12. By the uniqueness of $\lambda$, $X^\sim$ admits an endomorphism $\tilde{\rho}$ such that $\lambda \circ \tilde{\rho} = \rho \circ \lambda$. This is shown as follows: For the normalization $X^\sharp$ of the fiber product $X^\sim \times_X X$ of $\lambda$ and $f$, let $p_1^\sharp: X^\sharp \to X^\sim$ and $p_2^\sharp: X^\sharp \to X$ be the morphisms induced from the first and second projections, respectively. Then the restriction $X^\sharp_i \to X^\sim$ of the morphism $p_1^\sharp$ to any connected component $X^\sharp_i$ of $X^\sharp$ is a finite surjective morphism, since so is $f$. Thus, $q(X^\sharp_i) \geq q(X^\sim) = q^\circ(X)$. On the other hand, $p_2^\sharp$ is étale in codimension one, since so is $X^\sim \to X$. By Lemma 2.12, the restriction $X^\sharp_i \to X$ of $p_2^\sharp$ to $X^\sharp_i$ factors through $\lambda: X^\sim \to X$. Thus, $\deg(X^\sharp_i/X) \geq \deg(X^\sim/X)$. Since $\deg(X^\sim/X) = \deg(X^\sharp/X) = \sum_i \deg(X^\sharp_i/X)$, we infer that $X^\sharp$ is irreducible and $p_2^\sharp: X^\sharp \to X$ is isomorphic to the Albanese closure $\lambda: X^\sim \to X$ in codimension one over $X$. Thus, $p_1^\sharp: X^\sharp \to X^\sim$ produces an endomorphism $\tilde{\rho}: X^\sim \to X^\sim$ satisfying $\lambda \circ \tilde{\rho} = \rho \circ \lambda$. Then, $\tilde{\rho}$ is a polarized endomorphism with $f^*(\lambda^*(H)) \sim \lambda f^*(H)$, since $f^*(H) \sim qH$. Therefore, we may assume that $q^\circ(X) = q(X)$ by replacing $(X, f)$ with $(X^\sim, \tilde{\rho})$. Note again that the replacement does not affect the last equalities of Proposition 3.5 by Lemma 2.8.
Step 3. The final step: We may assume that \( X \) has only canonical singularities, \( K_X \sim 0 \), and \( q^o(X) = q(X) \), by the previous steps. If \( q(X) = 0 \), then \( X \) is weak Calabi–Yau, and Proposition 3.5 holds in this case. Thus, we may assume that \( q(X) > 0 \). Let \( \alpha: X \to A := \text{Alb}(X) \) be the Albanese map. Then, there is an endomorphism \( f_A': A \to A \) such that \( \alpha \circ f = f_A' \circ \alpha \) by the universality of the Albanese map. By [23], Theorem 8.3, we can find an étale covering \( \theta: T \to A \) such that \( X \times A T \simeq S \times T \) over \( T \) for a fiber \( S \) of \( \alpha \). Here, \( S \) is weak Calabi–Yau by the definition of \( q^o(X) \). Taking a further étale covering, we may assume that \( T \simeq A \) and \( \theta: T \to A \) is just the multiplication map by a positive integer \( m \) for a certain group structure of \( A \). There is an endomorphism \( f_A \) of \( A \) such that \( \theta \circ f_A = f_A' \circ \theta \) by [39], Lemma 4.9. Let \( W \) be the fiber product \( X \times_A A \) of \( \alpha: X \to A \) and \( \theta: A \to A \) over \( A \), and let \( \varphi: W \to X \) be the finite étale covering induced from the first projection. Then \( W \simeq S \times A \) over \( A \) as above, and \( f \times f_A: X \times A \to X \times A \) induces an endomorphism \( f_W \) of \( W \subset X \times A \) such that \( \varphi \circ f_W = f \circ \varphi \). In particular, \( f_W \) is a polarized endomorphism with \( f_W^*(\varphi^*(H)) \sim q\varphi^*(H) \) for the ample divisor \( \varphi^*(H) \).

We have an endomorphism \( f_S: S \to S \) such that \( f_W = f_S \times f_A \) by Lemma 2.14. Then, \( f_S \) and \( f_A \) are polarized endomorphisms with \( \deg f_A = q^\dim A \) and \( \deg f_S = q^\dim S \) by [37], Proposition 4.17. It remains to show the last equalities. We have \( \hat{q}(X, f) = q^o(X, f) \) by Lemma 2.8 and \( q^o(A \times S) = q(A) = \dim A \) by Corollary 2.11. In view of the covering \( A \times S \to X \) étale in codimension one, we have

\[
q(A) \leq q^o(A \times S, f_A \times f_S) = q^o(X, f) \leq q^o(X) = q^o(A \times S) = q(A).
\]

Thus, the expected equalities also hold. \( \square \)

Theorem 1.1 for non-uniruled \( X \) is a consequence of Theorems 3.2 and 3.4 and Proposition 3.5.

4. The Proof of Theorems 1.1 and 1.3

The following result gives a descent property of polarized endomorphisms by maximal rationally connected fibrations, which is proved in [37], Section 4.3.

Lemma 4.1 ([37], Corollary 4.20). Let \( f: X \to X \) be a quasi-polarized endomorphism of a normal projective variety \( X \). Suppose that \( X \) is uniruled. Then there exist a birational morphism \( \sigma: W \to X \), an equi-dimensional surjective morphism \( p: W \to Y \), and quasi-polarized endomorphisms \( f_W: W \to W \), \( f_Y: Y \to Y \) satisfying the following conditions:

1. \( W \) and \( Y \) are normal projective varieties, and \( \dim Y < \dim X \).
2. \( Y \) is not uniruled if \( \dim Y > 0 \).
3. A general fiber of \( p \) is rationally connected.
4. \( \deg f_Y = (\deg f)^\dim Y / \dim X \).


\(\sigma \circ f_W = f \circ \sigma\) and \(p \circ f_W = f_Y \circ p\), i.e., the diagram below is commutative:

\[
\begin{array}{ccc}
Y & \xrightarrow{p} & W \\
\downarrow{f_Y} & & \downarrow{f_W} \\
Y & \xleftarrow{p} & W \\
\end{array}
\]

(5) If \(f\) is polarized, then both \(f_W\) and \(f_Y\) are polarized.

An outline of the proof of Lemma 4.1 is as follows: We take a dominant rational map \(X \to Y\) which is birational to the maximal rationally connected fibration \(\tilde{X} \to \tilde{Y}\) (cf. [7], [29], [17]) of a smooth model \(\tilde{X}\) of \(X\). It is determined uniquely up to birational equivalence by the property that \(Y\) is not uniruled (when \(\dim Y > 0\)) and a general ‘fiber’ of \(X \to Y\) is rationally connected. Among the choices of the rational maps \(X \to Y\), we can select a unique one up to isomorphism by the following two properties:

- The graph \(\Gamma_{X/Y}\) of \(X \to Y\) is equi-dimensional over \(Y\).
- If \(\nu: Y' \to Y\) is a birational map from another normal projective variety \(Y'\) such that the graph \(\Gamma_{X/Y'}\) of the composite \(X \to Y' \to Y''\) of rational maps is equi-dimensional over \(Y'\), then \(\nu\) is holomorphic.

The existence and the uniqueness of \(X \to Y\) is proved in [37], Proposition 4.14 (cf. [37], Theorem 4.18). The proof uses the notion of intersection sheaves, which we do not explain here. The variety \(W\) is just the normalization of \(\Gamma_{X/Y}\). The endomorphism \(f\) descends to an endomorphism \(f_Y\) of \(Y\) by [37], Theorem 4.19. If \(f\) is polarized (resp. quasi-polarized) then so is \(f_Y\) by [37], Corollary 4.20; more precisely, if \(f^*(H) \sim qH\) for an ample (resp. a nef and big) divisor \(H\) on \(X\), where \(q = (\deg f)^{1/\dim X}\), then \(f_Y^*(H_Y) \sim qH_Y\) for an ample (resp. a nef and big) divisor \(H_Y\) on \(Y\). The proofs of two assertions also use the notion of intersection sheaves. The endomorphism \(f_W\) of \(W\) is induced from \(f \times f_Y\). Since \(f^*(H) \sim qH\), we have \(f_W^*(H_W) \sim qH_W\) for the ample (resp. nef and big) divisor \(H_W = \sigma^*(H) + p^*(H_Y)\) for the induced morphisms \(\sigma: W \to X\) and \(p: W \to Y\). Thus, \(f_W\) is also polarized (resp. quasi-polarized).

Remark 4.2. The same assertion as in Lemma 4.1 for polarized endomorphisms is stated in [45], Proposition 2.2.4. However, the argument there is valid only when the maximal rationally connected fibration is flat, which is not a priori available. The study of intersection sheaves in [37] renders the flatness requirement redundant, and consequently the expected assertion is proved in [37], Section 4.3.

Remark 4.3. In the situation of Lemma 4.1 assume that \(\deg f > 1\). Then, there exist a birational morphism \(Y \to Y''\) onto a normal projective variety \(Y''\) with only canonical singularities such that \(K_{Y''} \sim_{\mathbb{Q}} 0\) and a polarized endomorphism \(f_{Y''}: Y'' \to Y''\) compatible
with \( f_Y \) by Theorem 3.2. Applying [11], Theorem 5.1 to \( f_{Y'} \), we infer that the set \( \mathcal{Y} \) of periodic points of \( f_Y \) is Zariski dense in \( Y \). Here, \( y \in \mathcal{Y} \) if and only if \( f_Y^r(y) = y \) for a positive integer \( r = r(y) \). Thus, if \( y \in \mathcal{Y} \) is general, then a multiple of \( f \) induces a non-isomorphic quasi-polarized endomorphism of the rationally connected variety \( p^{-1}(y) \).

Hence the study of quasi-polarized endomorphisms on uniruled varieties is reduced, to some extent, to that on rationally connected varieties.

**Lemma 4.4.** In the situation of Lemma [4.1], assume that \( f \) is polarized. Let \( \theta: Y' \to Y \) be a finite covering étale in codimension one from a normal variety \( Y' \) and let \( f_{Y'}: Y' \to Y' \) be a polarized endomorphism such that \( \theta \circ f_{Y'} = f_Y \circ \theta \). Then there exist normal projective varieties \( X', W' \), finite coverings \( \tau: X' \to X \) and \( \delta: W' \to W \) both étale in codimension one, a birational morphism \( \sigma': W' \to X' \), a fibration \( p': W' \to Y' \) whose general fiber is rationally connected, and polarized endomorphisms \( f': X' \to Y' \), \( f_{W'}: W' \to W' \) such that \( \tau \circ f' = f \circ \tau \), \( \sigma' \circ f_{W'} = f' \circ \sigma' \), \( \delta \circ f_{W'} = f_W \circ \delta \), and \( p' \circ f_{W'} = f_{Y'} \circ p' \); hence, the diagram below is commutative and all the varieties admit mutually compatible polarized endomorphisms:

\[
\begin{array}{ccc}
Y' & \xleftarrow{f'} & W' \\
\downarrow{\theta} & & \downarrow{\delta} \\
Y & \xleftarrow{p} & W \\
\downarrow{\tau} & & \downarrow{\tau} \\
X' & \xrightarrow{f} & X.
\end{array}
\]

In particular, \( \sigma', p', f_{W'} \), and \( f_{Y'} \) satisfy the same conditions as in Lemma 4.1 for \( f': X' \to X' \).

**Proof.** Let \( W' \) be the normalization of \( W \times_Y Y' \). Let \( \delta: W' \to W \) and \( p': W' \to Y' \) be the morphisms induced from the first and second projections, respectively. Then, a general fiber of \( p' \) is also a rationally connected variety. In particular, \( W' \) is connected; thus \( W' \) is a normal projective variety. Since \( p \) is equi-dimensional, \( p' \) is also equi-dimensional and the finite morphism \( \delta: W' \to W \) is étale in codimension one. As the Stein factorization of the composite \( \sigma \circ \delta: W' \to W \to X \), we have a birational morphism \( \sigma': W' \to X' \) and a finite morphism \( \tau: X' \to X \) for a normal projective variety \( X' \) such that \( \tau \circ \sigma' = \sigma \circ \delta \). Let \( U \subset X \) be the domain of \( \sigma^{-1}: X \to W \). Then, \( \operatorname{codim}(X \setminus U) \geq 2 \), and the restriction \( \tau^{-1}(U) \to U \) of \( \tau \) is étale in codimension one, since so is \( \delta \). Therefore, \( \tau: X' \to X \) is étale in codimension one.

A polarized endomorphism \( f_{W'}: W' \to W' \) is induced from \( f_W \times f_{Y'} \) of \( W \times Y' \). It satisfies \( \delta \circ f_{W'} = f_W \circ \delta \) and \( p' \circ f_{W'} = f_{Y'} \circ p' \). Moreover, we have relations

\[(\tau \circ \sigma') \circ f_{W'} = (\sigma \circ \delta) \circ f_{W'} = \sigma \circ f_W \circ \delta = f \circ (\sigma \circ \delta) = f \circ (\tau \circ \sigma').\]

Thus, the Stein factorization of \( (\tau \circ \sigma') \circ f_{W'}: W' \to X \) is given by the birational morphism \( \sigma': W' \to X' \) and the finite morphism \( f \circ \tau: X' \to X \). Since \( \tau \) is finite, the Stein
factorization of $\sigma' \circ f_{W'}: W' \to X'$ is also given by the same birational morphism $\sigma': W' \to X'$. Therefore, we have an endomorphism $f': X' \to X'$ such that $\sigma' \circ f_{W'} = f' \circ \sigma'$. We have also $\tau \circ f' = f \circ \tau$ by the surjectivity of $\sigma'$ and by the relation $(\tau \circ \sigma') \circ f_{W'} = f \circ (\tau \circ \sigma')$ above. The endomorphism $f'$ is polarized by the pullback of an ample divisor on $X$ polarizing $f$. Thus, we are done.

\[ \square \]

**Lemma 4.5.** In the situation of Lemma 4.4, assume that $f$ is non-isomorphic and polarized. Then:

1. $Y$ has only canonical singularities with $K_Y \sim_{Q} 0$.
2. $q(X) \leq q(Y) \leq q^0(Y) = q^0(Y, f_Y) = q^2(Y, f_Y) \leq \dim Y$ and $q^2(Y, f_Y) \leq q^2(X, f)$.
3. The homomorphism $p_*: \pi_1(W) \to \pi_1(Y)$ of the fundamental groups is isomorphic and $\sigma_*: \pi_1(W) \to \pi_1(X)$ is surjective.

**Proof.** The assertion (1) follows from Theorem 3.2 and Lemma 4.4. In particular, $q(Y) \leq q^0(Y) = q(Y, f_Y) = q^2(Y, f_Y) \leq \dim Y$ by Proposition 3.3. We can take birational morphisms $\mu_W: \bar{W} \to W$ and $\mu_Y: \bar{Y} \to Y$ from smooth projective varieties $\bar{W}$ and $\bar{Y}$, respectively, such that the induced rational map $\tilde{p}: \bar{W} \to \bar{Y}$ is holomorphic and smooth over the complement of a normal crossing divisor on $\bar{Y}$. Then, $q(\bar{W}) \geq q(W) \geq q(X)$ by the injections $\sigma^*: H^1(X, O_X) \to H^1(W, O_W)$ and $\mu^*_W: H^1(W, O_W) \to H^1(\bar{W}, O_{\bar{W}})$. On the other hand, $q(\bar{Y}) = q(Y) = q(\bar{Y})$, since $Y$ has only rational singularities. We have $R^1 \tilde{p}_* O_{\bar{W}} = 0$ by Kollár’s torsion free theorem [20], since a general fiber of $\tilde{p}$ is rationally connected. Hence, $q(\bar{W}) = q(\bar{Y})$, consequently, $q(Y) = \tilde{q}(X) \geq q(X)$.

For the proof of (2), it remains to show the inequality: $q^2(Y, f_Y) \leq q^2(X, f)$. We can take a normal projective variety $Y'$, a finite covering $\theta: Y' \to Y$ étale in codimension one, and an endomorphism $f_{Y'}: Y' \to Y'$ such that $\theta \circ f_{Y'} = f_Y \circ \theta$ and $q(Y') = q^2(Y, f_Y)$. By Lemma 4.4, we have a normal projective variety $X'$, a finite covering $\tau: X' \to X$ étale in codimension one, and an endomorphism $f': X' \to X'$ such that $\tau \circ f' = f \circ \tau$ and $f_{Y'}$ is obtained from $f'$ as in Lemma 4.4. Hence, $q^2(X, f) \geq \tilde{q}(X') = q(Y')$ by the argument above. Thus, the assertion (2) has been proved.

Next, we shall prove (3). Let $U \subset X$ be a Zariski open dense subset of $X$ such that $\mu^{-1}_W \sigma^{-1}(U) \simeq \sigma^{-1}(U) \simeq U$ for the birational morphisms $\sigma$ and $\mu_W$. Note that the homomorphism $\pi_1(U) \to \pi_1(X)$ associated with the open immersion $U \hookrightarrow X$ is surjective, since $X$ is normal. Similarly, $\pi_1(\sigma^{-1}(U)) \to \pi_1(W)$ and $\pi_1(\mu^{-1}_W \sigma^{-1}(U)) \to \pi_1(\bar{W})$ are surjective. Thus, $\mu_{W*}: \pi_1(\bar{W}) \to \pi_1(W)$ and $\sigma_*: \pi_1(W) \to \pi_1(X)$ are surjective. On the other hand, by [43], $\mu_{Y*}: \pi_1(\bar{Y}) \to \pi_1(Y)$ is an isomorphism, since $Y$ has only canonical singularities. Moreover, $\tilde{p}_*: \pi_1(\bar{W}) \to \pi_1(\bar{Y})$ is an isomorphism, by [27], Theorem 5.2 (cf.
is $Q$-abelian, by Proposition 3.5. Therefore, we have only to show (3), Lemma 5.3). Hence,

$$\pi \reg_1(W) \simeq \pi_1(Y) \simeq \pi_1(Y).$$

Thus the assertion (3) has been proved. \hfill \Box

Corollary 4.6. Let $X$ be an $n$-dimensional normal projective variety admitting a non-isomorphic polarized endomorphism $f: X \to X$. Then:

(1) The inequality $q^*(X, f) \leq n$ holds, in which the equality holds if and only if $X$ is $Q$-abelian.

(2) If Conjecture [1.2] is true for the varieties of dimension at most $n - q^*(X, f)$, then $\pi_1(X)$ contains a finite-index subgroup which is a finitely generated abelian group of rank at most $2q^*(X, f)$.

Proof. (1): Suppose that $X$ is not uniruled. Then, $X$ has only canonical singularities and $K_X \sim_Q 0$ by Theorem 3.2. Hence, $q^*(X, f) \leq n$ in which the equality holds if and only if $X$ is $Q$-abelian, by Proposition 3.5. Therefore, we have only to show $q^*(X, f) < n$ assuming that $X$ is uniruled. By replacing $X$ with a finite covering $\tilde{X} \to X$ étale in codimension one and by replacing $f$ with an endomorphism of $\tilde{X}$ compatible with the original $f$, we may assume that $q^*(X, f) = \tilde{q}(X)$. Then, for the morphisms $\sigma: W \to X$ and $p: W \to Y$ in Lemma 4.1, we have $\tilde{q}(X) = q(Y) \leq \dim Y < \dim X = n$ by Lemma 4.5. (2). Thus, the assertion (1) has been proved.

(2): Let $\tilde{\tau}: \tilde{X} \to X$ be a finite covering étale in codimension one from a normal projective variety $\tilde{X}$ and $\tilde{f}: \tilde{X} \to \tilde{X}$ an endomorphism such that $\tilde{\tau} \circ \tilde{f} = f \circ \tilde{\tau}$. Then, $\tau_*: \pi_1(\tilde{\tau}^{-1}(X_{\reg})) \to \pi_1(X_{\reg})$ is injective and its image is a finite-index subgroup (cf. Remark 1.4). Since the natural inclusion $X_{\reg} \hookrightarrow X$ induces a surjection $\pi_1(X_{\reg}) \to \pi_1(X)$, the image of $\tau_*: \pi_1(\tilde{X}) \to \pi_1(X)$ is also a finite-index subgroup. Thus, we may replace $(X, f)$ with $(\tilde{X}, \tilde{f})$. Therefore, we can assume that $q^*(X, f) = \tilde{q}(X)$. Let $q$ be the positive integer defined by $q^n = \deg f$.

Assume first that $X$ is not uniruled. Then, $X$ has only canonical singularities and $K_X \sim_Q 0$ by Theorem 3.2. By Proposition 3.5 we may assume that $X = A \times S$ and $f = f_A \times f_S$ for an abelian variety $A$, a weak Calabi–Yau variety $S$, and polarized endomorphisms $f_A: A \to A$ and $f_S: S \to S$ with $\deg f_A = q^{\dim A}$ and $\deg f_S = q^{\dim S}$, respectively. Since $\dim S = n - \dim A = n - q^*(X, f)$, we infer that $S$ is a point by our assumption on Conjecture 1.2. Thus, $n = q^*(X, f)$, $X = A$, and $\pi_1(X)$ is a free abelian group of rank $2n = 2q^*(X, f)$. Therefore, the assertion (2) is true when $X$ is not uniruled.

Assume next that $X$ is uniruled. Let $\sigma: W \to X$ and $p: W \to Y$ be as in Lemma 4.1. Then, we have a surjection $\pi_1(Y) \to \pi_1(X)$ by Lemma 4.5. (3). In particular, if $Y$ is a point, then $\pi_1(X)$ is trivial. Thus, we may assume that $\dim Y > 0$. Then, $Y$ is not uniruled and it admits a polarized endomorphism $f_Y: Y \to Y$ of degree $q^{\dim Y} > 1$ by Lemma 4.1. We have $q^*(Y, f_Y) = q^*(Y, f_Y) = \tilde{q}(X) = q^*(X, f)$ by Lemma 4.5. (2).
Since \( \dim Y - q^i(Y, f_Y) < n - q^i(X, f) \), we can apply the previous argument to the non-uniruled variety \( Y \). Thus, \( \pi_1(Y) \) contains a finite-index subgroup which is a finitely generated abelian group of rank at most \( 2q^i(Y, f_Y) = 2q^i(X, f) \). Hence, \( \pi_1(X) \) has the same property, since we have the surjection \( \pi_1(Y) \rightarrow \pi_1(X) \). Therefore, the assertion (2) has been proved.

Now, we are ready to prove Theorem 1.1, which is a consequence of Theorems 3.2 and 3.4, Proposition 3.5, and Lemmas 4.1 and 4.4.

**Proof of Theorem 1.1.** If \( X \) is not uniruled, then it is proved in Theorems 3.2 and 3.4, and Proposition 3.5. Thus, we may assume that \( X \) is uniruled. We apply Lemma 4.1 to the polarized endomorphism \( f: X \rightarrow X \). Let \( \sigma: W \rightarrow Y \), \( p: W \rightarrow Y \), \( f_W: W \rightarrow W \), and \( f_Y: Y \rightarrow Y \) be the same objects as in Lemma 4.1. Then \( Y \) has only canonical singularities and \( K_Y \sim_\mathbb{Q} 0 \) by Theorem 3.2. Moreover, by Proposition 3.5, there exist a finite surjective morphism \( A \times S \rightarrow Y \) étale in codimension one from the direct product \( A \times S \) for an abelian variety \( A \) and a weak Calabi–Yau variety \( S \), and polarized endomorphisms \( f_A, f_S \) such that \( f_A \times f_S: A \times S \rightarrow A \times S \) is compatible with \( f_Y \). Here, if \( \dim S > 0 \), then \( \dim S \geq 4 \) and \( S \) has a non-quotient singularity by Theorem 3.4.

Let \( Z \) be the normalization of the fiber product of \( p: W \rightarrow Y \) and \( A \times S \rightarrow Y \). Let \( \varpi: Z \rightarrow A \times S \) be the morphism induced from the second projection and let \( Z \xrightarrow{\rho} V \xrightarrow{\tau} X \) be the Stein factorization of \( Z \rightarrow W \xrightarrow{\sigma} X \). Then, we have a commutative diagram:

\[
\begin{array}{ccc}
A \times S & \xleftarrow{\varpi} & Z \\
\downarrow & & \downarrow \\
Y & \xleftarrow{p} & W \\
\downarrow & & \downarrow \\
& \xrightarrow{\sigma} & X.
\end{array}
\]

By Lemma 4.4, the following hold:

- \( Z \) is irreducible.
- The fibration \( Z \rightarrow A \times S \) is equi-dimensional and is birational to the maximal rationally connected fibration of a smooth model of \( Z \).
- The finite surjective morphisms \( Z \rightarrow W \) and \( V \rightarrow X \) are étale in codimension one.
- There exist polarized endomorphisms \( f_Z: Z \rightarrow Z \) and \( f_V: V \rightarrow V \) such that \( f_V \) is compatible with \( f \) and that \( f_Z \) is compatible with \( f_W, f_V \) and \( f_A \times f_S \).

Let \( \pi: V \rightarrow A \times S \) be the rational map \( \varpi \circ \rho^{-1} \). Then \( Z \) is just the normalization of the graph \( \Gamma_\pi \) of \( \pi \). In particular, \( \Gamma_\pi \rightarrow A \times S \) is equi-dimensional. Thus, the conditions required in Theorem 1.1 are all satisfied.

The proof of Theorem 1.3 uses the following:
Lemma 4.7. Let $Z$ be the normalization of the graph of $\pi : V \to A \times S$ in Theorem 1.1 and let $\varpi : Z \to A \times S$ be the induced equi-dimensional morphism. Suppose that $\dim S = 0$. Then $\varpi$ is flat, and any fiber of $\varpi$ is irreducible, normal, and rationally connected. If $\dim Z = \dim A + 1$, then $\varpi$ is a holomorphic $\mathbb{P}^1$-bundle.

Proof. Let $V \to X$, $Z \to V$, $Z \to A$, and $A \to Y$ be as in the proof of Theorem 1.1 where $S$ is a point. We may assume that $\dim A > 0$. Let $Z_1$ be the fiber product of $\varpi : Z \to A$ and $f_A : A \to A$. Then the other endomorphism $f_Z : Z \to Z$ induces a commutative diagram
\[
\begin{array}{ccc}
Z & \xrightarrow{\psi} & Z_1 \\
\downarrow & & \downarrow p_1 \\
A & \overset{f_A}{\longrightarrow} & A
\end{array}
\]
where $p_1$ and $p_2$ denote the first and second projections, and $f_Z = p_1 \circ \psi$. Note that $p_1$ is étale, since so is $f_A$. Thus, $Z_1$ is also a normal projective variety and $\psi$ is a finite surjective morphism.

Step 1. We shall prove: If a subset $\Sigma \subset A$ is not Zariski dense and $f_A^{-1}(\Sigma) \subset \Sigma$, then $\Sigma = \emptyset$.

We shall derive a contradiction by assuming $\Sigma \neq \emptyset$. First, we note that $f_A^{-1}(\Sigma) \subset \Sigma$ for the Zariski-closure $\Sigma$ of $\Sigma$. In fact, by assumption, we have $f_A^{-1}(A \setminus \Sigma) \supset A \setminus \Sigma \supset A \setminus \Sigma \neq \emptyset$. Thus, $f_A(A \setminus \Sigma)$ is a Zariski-open subset contained in $A \setminus \Sigma$, since $f_A$ is an open map. Hence $f_A(A \setminus \Sigma) \subset A \setminus \Sigma$, and equivalently, $f_A^{-1}(\Sigma) \subset \Sigma$. Therefore, replacing $\Sigma$ with $\Sigma$, we may assume that $\Sigma$ is Zariski-closed. There is a positive integer $l$ such that $f_A^{-l}(\Sigma) = f_A^{-l-1}(\Sigma)$ by the Noetherian condition for Zariski-closed subsets. Hence, $f_A^{-1}(\Sigma) = \Sigma$. Replacing $f$ with a power $f^k$, we may assume that $f_A^{-1}$ preserves every irreducible component of $\Sigma$. Thus, we may assume that $\Sigma$ is irreducible. Let $f_{\Sigma} : \Sigma \to \Sigma$ be the polarized endomorphism of $\Sigma$ induced from $f_A$. Then $\deg f_{\Sigma} = q^{\dim A}$ and $\deg f_{\Sigma} = q^{\dim \Sigma}$ for some $q > 1$ by Lemma 2.1. On the other hand, $\deg f_{\Sigma} = \deg f_A$, since $f_A$ is étale. Thus, $\dim \Sigma = \dim A$. This is a contradiction.

Step 2. We shall prove: Any fiber of $\varpi$ is irreducible.

Let $\Sigma$ be the set of points $y \in A$ such that $\varpi^{-1}(y)$ is reducible. Then $\varpi^{-1}(y')$ is reducible for any $y' \in f_A^{-1}(y)$, since $\psi$ in the diagram (*) is surjective. Thus, $f_A^{-1}(\Sigma) \subset \Sigma$. Since a general fiber of $\varpi$ is irreducible, we have $\Sigma = \emptyset$ by Step 1.

Step 3. We shall prove: $\varpi$ is flat.

Let $L$ be an ample divisor on $Z$ such that $f_Z^* L \sim qL$. Since $\mathcal{O}_Z$, is a direct summand of $\psi_* \mathcal{O}_Z$ (cf. the first part of the proof of Lemma 4.8 below), we infer that $\varpi_* \mathcal{O}_Z(f_Z^* L) \sim \mathcal{O}_Z.$
\(\varpi_* \mathcal{O}_Z(qL)\) contains
\[p_{2*} \mathcal{O}_{Z_1}(p_1^* L) \simeq f_A^* (\varpi_* \mathcal{O}_Z(L))\]
as a direct summand. In particular, if \(\varpi_* \mathcal{O}_Z(qL)\) is locally free at a point \(y \in A\), then so is \(\varpi_* \mathcal{O}_Z(L)\) at \(f_A(y)\). Let \(U\) be the set of points \(y \in A\) such that \(\varpi\) is flat along \(\varpi^{-1}(y)\). Then \(U\) is a Zariski open dense subset. The argument above says that \(f_A(U) \subset U\), since \(y \in U\) if and only if \(\varpi_* \mathcal{O}_Z(mL)\) is locally free at \(y\) for \(m \gg 0\) (cf. [16], Proposition 7.9.14).

Thus, for the complement \(\Sigma\) of \(U\) in \(A\), we have \(f_A^{-1}(\Sigma) \subset \Sigma\). Then \(\Sigma = \emptyset\) by Step 1, and hence \(\varpi\) is flat.

**Step 4.** We shall prove: Any fiber of \(\varpi\) is normal.

Let \(\Sigma\) be the set of points \(y \in A\) such that the fiber \(F_y := \varpi^{-1}(y)\) is not normal. We fix a point \(y \in \Sigma\) and a non-normal point \(x\) of \(F_y\). For a point \(y' \in f_A^{-1}(y)\), let \(x'\) be a point of \(F_{y'}\) such that \(f(x') = x\). Then, \(x_1 := \psi(x')\) is a point of \(Z_1\) such that \(\{x_1\} = p_1^{-1}(x) \cap p_2^{-1}(y')\). Note that \(x_1\) is a non-normal point of \(p_2^{-1}(y')\), since \(p_2^{-1}(y') \simeq F_y\).

Assume that \(y' \notin \Sigma\), i.e., \(F_{y'}\) is normal. We have affine open neighborhoods \(U' \subset Z\) and \(U_1 \subset Z_1\) of \(x'\) and \(x_1\), respectively, such that \(U' = \psi^{-1}(U_1)\). Thus, \(U' = \text{Spec} \ R'\) and \(U_1 = \text{Spec} \ R_1\) for finitely generated \(\mathbb{C}\)-algebras \(R'\) and \(R_1\) such that \(R'\) and \(R_1\) are normal domains, \(R_1\) is a subalgebra of \(R'\) and that \(R'\) is a finite \(R_1\)-module. Let \(I\) be the ideal of \(R_1\) defining the closed subscheme \(U_1 \cap p_2^{-1}(y') \simeq U_1 \times_A \{y'\}\). Then, \(R'/IR'\) is normal, since \(U' \cap F_{y'} \simeq U' \times_A \{y'\}\) is normal. Thus, \(R_1/I\) is normal by Lemma 4.8 below. Therefore, \(p_2^{-1}(y')\) is normal at \(x_1\). This is a contradiction. Thus, \(y' \in \Sigma\). Hence, \(f_A^{-1}(\Sigma) \subset \Sigma\). Since a general fiber of \(\varpi\) is normal, we have \(\Sigma = \emptyset\) by Step 1.

**Step 5.** We shall prove: Any fiber of \(\varpi\) is rationally connected.

A general fiber of \(\varpi\) is rationally connected by the construction of \(\varpi\) in the proof of Theorem 1.1. Let \(\Sigma\) be the set of points \(y \in A\) such that the fiber \(F_y = \varpi^{-1}(y)\) is not rationally connected. If \(F_{y'}\) is rationally connected for a point \(y' \in f_A^{-1}(y)\), then \(F_y\) is rationally connected, since \(F_{y'} \to F_y\) is surjective. Thus, \(f_A^{-1}(\Sigma) \subset \Sigma\). Therefore, \(\Sigma = \emptyset\) by Step 1.

**Step 6.** End of the proof:

Finally, we consider the case where \(\dim \ Z/A = 1\). Then, \(\varpi\) is flat and any fiber of \(\varpi\) is \(\mathbb{P}^1\) by Steps 2–5. In particular, \(\varpi\) is smooth and is a holomorphic \(\mathbb{P}^1\)-bundle. Thus, we are done.

The lemma below on commutative algebra is used in Step 4 of the proof of Lemma 4.7.

**Lemma 4.8.** Let \(R_0\) and \(R_1\) be commutative algebras finitely generated over a field \(k\) of characteristic zero. Assume that \(R_0\) and \(R_1\) are normal integral domains, \(R_0\) is a
\textbf{k-subalgebra of }R_1\text{ and that } R_1\text{ is a finite } R_0\text{-module. Let } I\text{ be an ideal of } R_0\text{ such that } R_1/I R_1\text{ is normal. Then, } R_0/I \text{ is also normal.}

\textbf{Proof.} Since the characteristic of } k \text{ is zero, } R_0 \text{ is a direct summand of the } R_0\text{-module } R_1. \text{ This is shown as follows: Let } K_i \text{ be the field of fractions of } R_i \text{ for } i = 0, 1. \text{ Then, } K_1 \text{ is a finite extension of } K_0. \text{ Let } t: K_1 \rightarrow K_0 \text{ be the trace map of the extension } K_1/K_0: \text{ for } a \in K_1, t(a) \text{ is the trace of the multiplication map } \mu(a): K_1 \rightarrow K_1 \text{ by } a. \text{ The composite } K_0 \rightarrow K_1 \xrightarrow{t} K_0 \text{ with the canonical inclusion } K_0 \hookrightarrow K_1 \text{ is just the multiplication map by } \deg(K_1/K_0) = \dim_{K_0} K_1. \text{ We have } t(R_1) \subset R_0, \text{ since the eigenvalues of } \mu(a) \text{ for } a \in R_1 \text{ are integral over } R_0 \text{ and since } R_0 \text{ is integrally closed in } K_0. \text{ Moreover, the map } R_1 \rightarrow R_0 \text{ induced by } t, \text{ is } R_0\text{-linear. Thus, } R_0 \text{ is a direct summand of the } R_0\text{-module } R_1, \text{ since } \deg(K_1/K_0) \neq 0 \text{ in } k.

Therefore, the natural homomorphism

\[ R_0/I \rightarrow R_1 \otimes_{R_0} (R_0/I) \simeq R_1/I R_1 \]

is injective and } R_0/I \text{ is regarded as a direct summand of } R_1/I R_1. \text{ Since } R_1/I R_1 \text{ is an integral domain, so is } R_0/I. \text{ Let } \mathcal{R} \text{ be the normalization of } R_0/I. \text{ Then, } R_0/I \subset \mathcal{R} \subset R_1/I R_1, \text{ since } R_1/I R_1 \text{ is normal. Thus, } R_0/I \text{ is a direct summand of the } R_0/I\text{-module } \mathcal{R}. \text{ Let } \mathcal{R} \twoheadrightarrow R_0/I \text{ be a projection to the direct summand and let } M \text{ be the kernel of } \mathcal{R} \twoheadrightarrow R_0/I. \text{ Then, } M \otimes_{R/I_0} \mathcal{R} = 0 \text{ for the field } \mathcal{R} \text{ of fractions of } R/I_0. \text{ Since } \mathcal{R} \twoheadrightarrow \mathcal{R} \otimes_{R/I_0} \mathcal{R} \simeq \mathcal{R} \text{ is injective, we have } M = 0. \text{ Therefore, } R/I_0 \simeq \mathcal{R}, \text{ i.e., } R/I_0 \text{ is normal.} \qed

A holomorphic } \mathbb{P}^1\text{-bundle is not necessarily associated with a locally free sheaf of rank two. But we have the following result on holomorphic } \mathbb{P}^1\text{-bundles over abelian varieties:

\textbf{Lemma 4.9.} Let } Z \rightarrow A \text{ be a holomorphic } \mathbb{P}^1\text{-bundle over an abelian variety } A. \text{ For the multiplication map } \nu_2: A \rightarrow A \text{ by } 2, \text{ let } Z' \rightarrow A \text{ be the } \mathbb{P}^1\text{-bundle obtained by the pullback of } Z \rightarrow A \text{ by } \nu_2. \text{ Then, there exists a locally free sheaf } \mathcal{E} \text{ of rank two on } A \text{ such that } Z' \simeq \mathbb{P}^1_A(\mathcal{E}) \text{ and det } \mathcal{E} \simeq \mathcal{O}_A.

\textbf{Proof.} We have an exact sequence

\[ 1 \rightarrow \mu_2, A \rightarrow SL(2, \mathcal{O}_A) \rightarrow PGL(2, \mathcal{O}_A) \rightarrow 1 \]

of sheaves of non-commutative groups on } A, \text{ where } SL(2, \mathcal{O}_A) \text{ (resp. } PGL(2, \mathcal{O}_A)) \text{ is the sheaf of germs of SL(2, } \mathbb{C})\text{-valued (resp. PGL(2, } \mathbb{C})\text{-valued) holomorphic functions on } A, \text{ and } \mu_2, A \simeq (\mathbb{Z}/2\mathbb{Z})_A \text{ is the constant sheaf of } \mu_2 = \{ \pm 1 \} \subset \mathbb{C}^*. \text{ Since } \mu_2 \text{ is the center of } SL(2, \mathbb{C}), \text{ we have an associated exact sequence}

\[ H^1(A, SL(2, \mathcal{O}_A)) \rightarrow H^1(A, PGL(2, \mathcal{O}_A)) \rightarrow H^2(A, \mu_2, A) \]
of the Čech cohomology sets. The holomorphic $\mathbb{P}^1$-bundle $Z/A$ is associated with an element of $\eta$ of $\check{H}^1(A, PGL(2, \mathcal{O}_A))$. If the image $\eta$ in $H^2(A, \mu_2)$ is zero, then $Z \simeq \mathbb{P}_A(\mathcal{E})$ for a locally free sheaf $\mathcal{E}$ of rank two with $\det \mathcal{E} \simeq \mathcal{O}_A$ which is associated with an element of $\check{H}^1(A, SL(2, \mathcal{O}_A))$. Thus, it is enough to show that $\eta$ is mapped to zero by the homomorphism $\nu_2^*: H^2(A, \mathbb{Z}/2\mathbb{Z}) \to H^2(A, \mathbb{Z}/2\mathbb{Z})$. The pullback homomorphism $\nu_2^*: H^1(A, \mathbb{Z}) \to H^1(A, \mathbb{Z})$ is just the multiplication map by 2. Since $H^3(A, \mathbb{Z}) \simeq \wedge^3 H^1(A, \mathbb{Z})$ is torsion free, the natural sequence

$$H^2(A, \mathbb{Z}) \xrightarrow{2*} H^2(A, \mathbb{Z}) \to H^2(A, \mathbb{Z}/2\mathbb{Z}) \to 0$$

is exact; equivalently, $H^2(A, \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} \simeq H^2(A, \mathbb{Z}/2\mathbb{Z})$. Hence, $\nu_2^*: H^2(A, \mathbb{Z}/2\mathbb{Z}) \to H^2(A, \mathbb{Z}/2\mathbb{Z})$ is zero; in particular, $\eta$ is mapped to zero by $\nu_2^*$. Thus, we are done. \hfill \Box

Now we are ready to prove Theorem 1.3 which follows essentially from Theorems 1.1, 3.2, and 3.4, Lemmas 4.1, 4.5, and 4.7, and Corollary 4.6.

**Proof of Theorem 1.3.** We may assume that $n = \dim X > 0$. We have $\kappa(X) \leq 0$ by Theorem 1.1. The inequality $q^2(X, f) \leq n$ and the assertion (2) are proved in Corollary 4.6 (1).

We shall prove (1): Assume that $q^2(X, f) = 0$ and Conjecture 1.2 is true for varieties of dimension at most $n = \dim X$. If $X$ is not uniruled, then $X$ is Q-abelian by the conjecture; thus $n = q^2(X, f) = 0$ by Corollary 4.6 (1). Hence $X$ is uniruled. Let $\sigma: W \to X$ and $\rho: W \to Y$ be as in Lemma 4.1. Since $\dim Y \leq n$, $Y$ is Q-abelian by the conjecture. On the other hand, $q^2(Y) = 0$ by Lemma 4.5 (2). Thus, $Y$ is a point. This means that $X$ is rationally connected.

Next, we shall prove (3) and (4): Suppose that Conjecture 1.2 is true for varieties of dimension at most $n - q^2(X, f)$. Let $\tau: V \to X$, $\rho: Z \to V$, $\sigma: Z \to A \times S$, and $A \times S \to Y$ be as in the proof of Theorem 1.1. By the proof of Corollary 4.6 (2), we infer that $S$ is a point and $q^2(X, f) = \dim A$. Then $\sigma: Z \to A$ is a flat morphism whose fibers are all irreducible, normal, and rationally connected by Lemma 4.7. We have polarized endomorphisms $f_V: V \to V$, $f_Z: Z \to Z$, and $f_A: A \to A$ satisfying the compatibility conditions in (3) by the proof of Theorem 1.1. Thus the assertion (3) follows.

Suppose that $q^2(X, f) = n - 1$. Then $\sigma: Z \to A$ is a holomorphic $\mathbb{P}^1$-bundle by Lemma 4.7. Assume that $\rho: Z \to V$ is an isomorphism. Then, this corresponds to the case (1a) in a weak sense. However, by replacing $V$ by a finite étale covering, we can prove that $V$ is a $\mathbb{P}^1$-bundle associated with a locally free sheaf of rank two, as follows. By Lemma 4.9, the fiber product $Z \times_{A, \nu_2} A$ is a $\mathbb{P}^1$-bundle associated with a locally free sheaf of rank two for the multiplication map $\nu_2: A \to A$ by 2 with respect to a certain group structure of $A$. There is an endomorphism $f_A': A \to A$ such that $\nu_2 \circ f_A' = f_A \circ \nu_2$, etc.
by [39], Lemma 4.9. Thus, we may replace \( Z \simeq V \) with the étale covering \( Z \times_{A,\nu^2} A \) by Lemma 4.4 (cf. the proof of Theorem 1.1).

Assume next that \( \psi: Z \to V \) is not isomorphic. Let \( E \subset Z \) be the exceptional locus. Then \( f^{-1}_Z(E) = E \), since \( f_Z \) is compatible with \( f_V \). Moreover, \( f^{-1}_A(\varpi(E)) = \varpi(E) \), since \( \psi: Z \to Z_1 \) is surjective. Thus, \( \varpi(E) = A \) by Step 1 in the proof of Lemma 4.7. Let \( \Sigma \subset A \) be the set of points \( y \in A \) such that \( \varpi^{-1}(y) \subset E \). Then \( f^{-1}_A(\Sigma) \subset \Sigma \). Hence, \( \Sigma = \emptyset \) by Step 1 in the proof of Lemma 4.7. Therefore, \( \varpi|_E: E \to A \) is a finite surjective morphism. It is enough to show that \( E \) is a section of \( \varpi \), which is equivalent to that \( \varpi|_E: E \to A \) is bijective, since \( A \) is normal. If \( E \) is a section of \( \varpi \), then \( \varpi \) is a \( \mathbb{P}^1 \)-bundle associated with the locally free sheaf \( \varpi_*\mathcal{O}_Z(E) \) of rank two.

Let \( P \in A \) be an arbitrary point. Then, there exists a positive-dimensional fiber \( \Gamma \) of \( E \subset Z \to V \) such that \( P \in \varpi(\Gamma) \). Let \( C \) be the normalization of an irreducible curve in \( \varpi(\Gamma) \) passing through \( P \) and let \( \nu: C \to A \) be the induced finite morphism. Then, \( Z \times_A C \) is a \( \mathbb{P}^1 \)-bundle over \( C \). It suffices to prove that the support of \( E \times_A C \) is a section of the \( \mathbb{P}^1 \)-bundle. Note that \( Z \times_A C \to Z \to V \) is generically injective and it contracts any irreducible component \( \gamma \) of \( E \times_A C \) to a point. Since \( Z \times_A C \) is a \( \mathbb{P}^1 \)-bundle over \( C \), the irreducible component \( \gamma \) is a unique curve of \( Z \times_A C \) with negative self-intersection number and it is a section of the \( \mathbb{P}^1 \)-bundle. Hence the support of \( E \times_A C \) is just the section \( \gamma \). Therefore, \( E \) is a section of \( \varpi \), and the condition (4b) is satisfied. Thus, we are done.

\( \square \)

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