On the method of typical bounded differences

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Abstract

Concentration inequalities are fundamental tools in probabilistic combinatorics and theoretical computer science for proving that random functions are near their means. Of particular importance is the case where \( f(X) \) is a function of independent random variables \( X = (X_1, \ldots, X_n) \). Here the well known bounded differences inequality (also called McDiarmid’s or Hoeffding–Azuma inequality) establishes sharp concentration if the function \( f \) does not depend too much on any of the variables. One attractive feature is that it relies on a very simple Lipschitz condition (L): it suffices to show that \( |f(X) - f(X')| \leq c_k \) whenever \( X, X' \) differ only in \( X_k \). While this is easy to check, the main disadvantage is that it considers worst-case changes \( c_k \), which often makes the resulting bounds too weak to be useful.

In this paper we prove a variant of the bounded differences inequality which can be used to establish concentration of functions \( f(X) \) where (i) the typical changes are small although (ii) the worst case changes might be very large. One key aspect of this inequality is that it relies on a simple condition that (a) is easy to check and (b) coincides with heuristic considerations why concentration should hold. Indeed, given an event \( \Gamma \) that holds with very high probability, we essentially relax the Lipschitz condition (L) to situations where \( \Gamma \) occurs. The point is that the resulting typical changes \( c_k \) are often much smaller than the worst case ones.

To illustrate its application we consider the reverse \( H \)-free process, where \( H \) is 2-balanced. We prove that the final number of edges in this process is concentrated, and also determine its likely value up to constant factors. This answers a question of Bollobás and Erdős.

1 Introduction

In probabilistic combinatorics and theoretical computer science it is often crucial to predict the likely value(s) of a random function. More precisely, in many applications \( f(X) \) is a function of independent random variables \( X = (X_1, \ldots, X_N) \), and we need to prove that it is concentrated in a narrow range around its expected value, i.e., that \( f(X) \) typically is about \( \mu = \mathbb{E} f(X) \). The crux is that the functions of interest are often defined in an indirect or complicated way, so that basic bounds such as Chebychev’s inequality are either hard to evaluate or give error bounds that are too weak in applications. In this work we thus investigate easy-to-check conditions which ensure that the function \( f(X) \) is close to its mean \( \mu \) with very high probability, i.e., that large deviations from \( \mu \) are highly unlikely.

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An important paradigm in this area of research (see e.g. [21]) states that a random function which depends ‘smoothly’ on many independent random variables should be sharply concentrated, meaning that $|f(X) - \mu| = o(\mu)$ holds with probability very close to one. In many applications (e.g. the design of randomized algorithms or random graph theory) each random variable $X_k$ takes values in a set $\Lambda_k$, and in this case a discrete Lipschitz condition for $f : \prod_{j \in [N]} \Lambda_j \to \mathbb{R}$ conveniently ensures that $f(X)$ does not depend too much on any of the variables, where $[N] = \{1, \ldots, N\}$. Perhaps the most famous result in this context is the bounded differences inequality (also called McDiarmid’s or Hoeffding–Azuma inequality), which is nowadays widely used in discrete mathematics and computer science, see e.g. the surveys [28, 29]. Here we only state its one-sided version since the analogous lower tail estimate $\mathbb{P}(f(X) \leq \mu - t)$ follows by considering the function $-f(X)$.

**Theorem 1** (`Bounded differences inequality`). [28] Let $X = (X_1, \ldots, X_N)$ be a family of independent random variables with $X_k$ taking values in a set $\Lambda_k$. Assume that the function $f : \prod_{j \in [N]} \Lambda_j \to \mathbb{R}$ satisfies the following Lipschitz condition:

For any $t \geq 0$ we have
\begin{equation}
\mathbb{P}(f(X) \geq \mu + t) \leq \exp\left(-\frac{2t^2}{\sum_{k \in [N]} c_k^2}\right).
\end{equation}

While the simplicity of (L) makes this inequality very intuitive and easy to apply, its perhaps main drawback is that it considers worst case changes. In particular, the resulting concentration bounds are rather weak (or even trivial) in situations where the worst case $c_k$ are much larger than the typical changes. A standard example is $f(X)$ counting the number of triangles in the binomial random graph $G_{n,p}$: since every pair of vertices has up to $n-2$ common neighbours the worst case is $c_k = \Theta(n)$, which is much larger than we expect from the $\Theta(np^2)$ common neighbours we usually have for $p \geq n^{-1/2+\varepsilon}$. In fact, here Theorem 1 only gives trivial estimates for $p = O(n^{-1/3})$, but it seems plausible that concentration should hold in such applications where the typical changes are much smaller than the worst case ones.

This motivated a line of research [24, 35, 36, 42, 43] which focused on tail inequalities of the form $\mathbb{P}(|f(X) - \mu| \geq t) \leq e^{-\theta(f,X,t)}$ in situations where (intuitively speaking) the average Lipschitz coefficients $c_k$ are small. Pioneered by Kim and Vu [24], such results usually require $f(X)$ to have a special structure (a polynomial of independent random variables of a certain type), reducing their range of applications compared to [2]. Furthermore, the assumptions of such techniques are much more involved (and harder to check) than the simple Lipschitz condition (L).

In contrast, much less research has been devoted to developing easy-to-use tools for proving concentration results in such situations. The Hoeffding–Azuma inequality [26, 3] implies, for example, that (4) essentially remains true if we relax (1) to worst case conditional expected changes:
\begin{equation}
|\mathbb{E}(f(X) \mid X_1, \ldots, X_{k-1}) - \mathbb{E}(f(X) \mid X_1, \ldots, X_{k-1})| \leq c_k.
\end{equation}

While this might be useful in certain textbook examples, it typically has two main drawbacks in involved combinatorial applications: (a) conditional expectations are usually difficult to calculate and (b) it often yields no substantial improvement (for, say, $k \geq N/2$ the worst case in [43] over all
choices of \(X_1, \ldots, X_k\) is often comparable to (11). There are also some approaches which allow \(3\) to be violated occasionally \([13][14][23][29][37]\), but these usually require knowledge about conditional probability distributions, making them particularly difficult to apply when \(f(X)\) is defined in an indirect or complicated way.

### 1.1 Typical bounded differences inequality

In this paper we develop a variant of the bounded differences inequality which can be used to establish concentration of functions \(f(X)\) where (i) the typical changes are small although (ii) the worst case changes might be very large. One key aspect of this inequality is that it relies on a simple and attractive condition that (a) is easy to check and (b) coincides with heuristic considerations why worst case changes might be very large. One key aspect of this inequality is that it may still yield concentration in situations where \(\Gamma\) occurs. More precisely, for the concentration should hold. Indeed, given a ‘good’ event \(\Gamma\) that holds with very high probability, we essentially relax the Lipschitz condition (L) to situations where \(\Gamma\) occurs. More precisely, for the sake of proving concentration the following inequality usually allows us to restrict our attention to such typical changes, which are often much smaller than the worst case ones.

**Theorem 2** (‘Typical bounded differences inequality’). Let \(X = (X_1, \ldots, X_N)\) be a family of independent random variables with \(X_k\) taking values in a set \(\Lambda_k\). Let \(\Gamma \subseteq \prod_{j \in [N]} \Lambda_j\) be an event and assume that the function \(f : \prod_{j \in [N]} \Lambda_j \to \mathbb{R}\) satisfies the following typical Lipschitz condition:

\[
\text{(TL)} \quad \text{There are numbers} \ (c_k)_{k \in [N]} \text{ and} \ (d_k)_{k \in [N]} \text{ with} \ c_k \leq d_k \text{ such that whenever} \ x, \tilde{x} \in \prod_{j \in [N]} \Lambda_j \text{ differ only in the} \ k\text{-th coordinate we have}
\]

\[
|f(x) - f(\tilde{x})| \leq \begin{cases} c_k & \text{if} \ x \in \Gamma, \\ d_k & \text{otherwise}. \end{cases} \tag{4}
\]

For any numbers \((\gamma_k)_{k \in [N]}\) with \(\gamma_k \in (0, 1]\) there is an event \(\mathcal{B} = \mathcal{B}(\Gamma, (\gamma_k)_{k \in [N]})\) satisfying

\[
P(\mathcal{B}) \leq \sum_{k \in [N]} \gamma_k^{-1} \cdot P(X \notin \Gamma) \quad \text{and} \quad \neg \mathcal{B} \subseteq \Gamma, \tag{5}
\]

such that for \(\mu = \mathbb{E}f(X), \ c_k = \gamma_k(d_k - c_k)\) and any \(t \geq 0\) we have

\[
P(f(X) \geq \mu + t \text{ and } \neg \mathcal{B}) \leq \exp \left( -\frac{t^2}{2\sum_{k \in [N]}(c_k + e_k)^2} \right). \tag{6}
\]

**Remark 3.** If each \(X_k\) takes only two values (i.e., when \(|\Lambda_k| = 2\)) the exponent in (6) may be multiplied by factor of 4, analogous to the standard bound (2).

As before, this inequality is only stated for the upper tail since an application to \(-f(X)\) yields the same estimate for \(P(f(X) \leq \mu - t)\). One key property of the ‘bad’ event \(\mathcal{B}\) is that it does not depend on the function \(f(X)\), so that (5) can be used as a tail estimate in union bound arguments. We expect that the typical changes \(c_k\) are usually substantially smaller than the worst case \(d_k\), and the ‘compensation factor’ \(\gamma_k\) is supposed to milder the effects of the \(d_k\) in (5). Indeed, in the typical application \(\gamma_k\) will be very small (this choice is possible if \(\Gamma\) holds with very high probability), so that we can think of \(e_k = \gamma_k(d_k - c_k)\) as a negligible ‘error term’. With this in mind, perhaps the most important aspect of Theorem 2 is that it may still yield concentration in situations where Theorem 1 only gives trivial bounds due to very large worst case Lipschitz coefficients.
To illustrate the ease of application of Theorem 2 consider again the example where \( f(X) \) counts the number of triangles in \( G_{n,p} \). Define \( \Gamma \) as the event that every pair of vertices has at most \( \Delta = \max\{2np^2, n^5\} \) common neighbours, which fails with probability at most \( e^{-\Omega(n^4)} \) by standard Chernoff bounds. It is straightforward to see that in this case (TL) holds with, say, \( c_k = \Delta \) and \( d_k = n \). Setting \( \gamma_k = n^{-1} \) we thus have \( \epsilon_k = o(c_k) \) and \( \mathbb{P}(B) \leq e^{-\Omega(n^4)} \), which means that both terms are negligible for the sake of establishing concentration. It follows that for \( p \geq n^{-2/3+\epsilon} \) the typical bounded differences inequality (Theorem 2) yields tight concentration of the number of triangles (with \( \mathbb{P}(f(X) \notin (1 \pm n^{-\epsilon})\mu) \leq e^{-\Omega(n^4)} \), say), whereas Theorem 1 already fails for \( p = n^{-1/3} \). Note that we picked all parameters in a uniform way, setting \( c_k = C \), \( d_k = D \) and \( \gamma_k = \gamma \); this might be convenient in many applications (where \( \gamma \approx C/D \) should often suffice).

1.1.1 Improvement for Bernoulli random variables

If the underlying probability space is generated by independent Bernoulli random variables we establish much stronger estimates. For example, in the common situation where the success probabilities are all equal to \( p \) (as in \( G_{n,p} \)) the following natural extension of Theorem 2 essentially allows us to multiply the denominator of (6) with an extra factor of \( p \) (on an intuitive level one can perhaps think of this as applying Theorem 2 after conditioning on \( \Theta(N) \) variables being ‘relevant’).

**Theorem 4** (‘Typical bounded differences inequality for 0-1 variables’). Let \( X = (X_1, \ldots, X_N) \) be a family of independent random variables with \( X_k \in \{0,1\} \) and \( p_k = \mathbb{P}(X_k = 1) \). Let \( \Gamma \subseteq \{0,1\}^N \) be an event and assume that the function \( f: \{0,1\}^N \rightarrow \mathbb{R} \) satisfies the typical Lipschitz condition (TL) with \( \Lambda_k = \{0,1\} \). For any numbers \((\gamma_k)_{k \in [N]} \) with \( \gamma_k \in (0,1] \) there is an event \( B = B(\Gamma, (\gamma_k)_{k \in [N]}) \) satisfying (6) such that for \( \mu = \mathbb{E}f(X) \) and any \( t \geq 0 \) we have

\[
\mathbb{P}(f(X) \geq \mu + t \text{ and } \neg B) \leq \exp\left(-\frac{t^2}{2 \sum_{k \in [N]}(1 - p_k)p_k(c_k + \epsilon_k)^2 + 2Ct/3}\right), \tag{7}
\]

where \( \epsilon_k = \gamma_k(d_k - c_k) \) and \( C = \max_{k \in [N]}(c_k + \epsilon_k) \).

**Remark 5.** If \( f(X) \) and \( \Gamma \) are either both monotone increasing or decreasing we have

\[
\mathbb{P}(f(X) \geq \mu + t) \leq \frac{\mathbb{P}(f(X) \geq \mu + t \text{ and } \neg B)}{1 - \mathbb{P}(B)}. \tag{8}
\]

In typical applications of this inequality we hope to be able to ignore the ‘error term’ \( 2Ct/3 \) (and select \( \gamma_k \) such that \( c_k + \epsilon_k \approx c_k \) as before). In this case (7) is close \( e^{-t^2/\left(2 \sum_{k \in [N]}(1 - p_k)p_k(c_k + \epsilon_k)^2\right)} \), which for \( p_k = o(1) \) is a significant improvement of the corresponding \( e^{-2t^2/(\sum \epsilon_k^2)} \) from Remark 3. For example, in the case of triangles in \( G_{n,p} \) this allows us to extend the concentration result of the previous section to edge probabilities satisfying \( p \geq n^{-4/5+\epsilon} \). In fact, the estimates implied by (8) are sometimes comparable to those of Janson’s inequality \cite{21, 32}, see Section 1.2.2.

Ignoring the ‘good’ event \( \Gamma \) in Theorem 4 we also obtain a strengthening of Theorem 1. Since this natural variant of the bounded differences inequality does not seem to be as widely known, we explicitly state it for ease of reference (if each \( (1 - p_k)p_k \) is weakened to \( \max \{1 - p_i, p_i\} \) then (9) follows from Theorem 3.9 in McDiarmid’s survey \cite{29}; Alon, Kim and Spencer \cite{2} also proved a comparable inequality that applies to small values of \( t \) only; for those the contribution of \( Ct \) to the denominator of (9) is negligible).
Corollary 6 (`Bounded differences inequality for 0–1 variables`). Let $X = (X_1, \ldots, X_N)$ be a family of independent random variables with $X_k \in \{0, 1\}$ and $p_k = \mathbb{P}(X_k = 1)$. Assume that the function $f : \{0, 1\}^N \to \mathbb{R}$ satisfies the Lipschitz condition (L) with $\Lambda_k = \{0, 1\}$. Let $\mu = \mathbb{E}f(X)$ and $C = \max_{k \in [N]} c_k$. For any $t \geq 0$ we have

$$
\mathbb{P}(f(X) \geq \mu + t) \leq \exp\left(-\frac{t^2}{2 \sum_{k \in [N]} (1 - p_k) p_k c_k^2 + 2Ct/3}\right).
$$

Proof. Apply Theorem 4 with $\Gamma = \{0, 1\}^N$ and $d_k = c_k$. \qed

This extends Bernstein’s inequality (a strengthening of the Chernoff bounds for small deviations, see e.g. Remark 2.9 in [22]), which applies to sums of independent random variables. One key aspect of (9) is that it is almost tight when $f(X) = \sum_k X_k$, in which case $V = \text{Var } f(X) = \sum_k p_k (1 - p_k)$ and $c_k = 1$. Indeed, the estimate of Corollary 6 is then close to $e^{-t^2/(2V)}$ for $t$ not too large, which is exactly the tail behaviour predicted by the central limit theorem.

Remark 7. Our arguments in fact yield a slightly stronger form of (7) – (9), analogous to Bennet’s sharpening of the Chernoff bounds (see e.g. Remark 2.9 in [22]). Indeed, for $\phi(x) = (1 + x) \log(1 + x) - x$ we can improve terms of the form $e^{-t^2/(2V + 2Ct/3)}$ to $e^{-V/C \phi(Ct/V)}$, where $V$ equals $\sum_k (1 - p_k)p_k(ck + e_k)^2$ and $\sum_k (1 - p_k)p_k c_k^2$ in (7) and (9). For $t = \omega(V/C)$ these refined estimates sharpen the exponents from order $\Theta(t/C)$ to $\Theta(t/C \cdot \log(Ct/V))$, i.e., yield a logarithmic improvement.

Remark 8. Theorem 4 and Corollary 6 extend with minor modifications to the case where each $X_k$ takes values in a set $\Lambda_k$ and satisfies $\max_{\eta \in \Lambda_k} \mathbb{P}(X_k = \eta) \geq 1 - p_k$. Indeed, (7) and (9) both hold after deleting $(1 - p_k)$ and replacing $c_k + e_k$ with $c_k + e_k \cdot (1 - p_k)^{-1}$.

1.1.2 Two-sided Lipschitz conditions

The typical Lipschitz condition (TL) is `one-sided’: $|f(x) - f(\tilde{x})| \leq c_k$ is supposed to hold if $x \in \Gamma$. This keeps the formulas simple, but in many applications it is easier (and perhaps more natural) to verify a `two-sided’ condition where $x, \tilde{x} \in \Gamma$ holds. The following theorem states that we may use a two-sided variant of (TL) at the cost of slightly increasing the ‘error term’ $e_k$.

Theorem 9 (`Two-sided typical Lipschitz condition’). Theorems 4, 5 and Remarks 3, 7, 8 remain valid with $e_k = 2\gamma_k(d_k - c_k)q_k^{-1}$ and $\min_{\eta \in \Lambda_k} \mathbb{P}(X_k = \eta) \geq q_k$ if the Lipschitz condition (4) of (TL) is replaced by the following two-sided variant:

$$
|f(x) - f(\tilde{x})| \leq \begin{cases} 
2\gamma_k d_k & \text{if } x, \tilde{x} \in \Gamma, \\
4\gamma_k c_k & \text{otherwise}.
\end{cases}
$$

Whenever $q_k^{-1}$ is not too big (10) seems the most convenient condition: it is much simpler to check than (4) and does not substantially deteriorate the error bounds. For example, in the random graph $G_{n,p}$ we usually have $q_k^{-1} \leq n^2$, which in the typical application with $\mathbb{P}(X \notin \Gamma) \leq n^{-\omega(1)}$ can be compensated by adapting $\gamma_k$ accordingly (also note that $(1 - p_k)p_kq_k^{-1} \leq 1$ in case of Theorem 4).

Remark 10. As pointed out by Oliver Riordan it is possible to bootstrap (10) from (4) by modifying the good event. Indeed, defining $\Gamma' \subseteq \Gamma$ such that for $x \in \Gamma'$ any single coordinate change results in a sample point satisfying $\tilde{x} \in \Gamma$, it follows that the one-sided condition $x \in \Gamma'$ implies the two-sided condition $x, \tilde{x} \in \Gamma$. Using the bound $\mathbb{P}(X \notin \Gamma') \leq \sum_{k \in [N]} q_k^{-1} \cdot \mathbb{P}(X \notin \Gamma)$ this approach often leads to estimates that are comparable with Theorem 9 (in fact, monotonicity of $\Gamma$ also transfers to $\Gamma'$).
In some applications (a) the $q_k$ are very small and (b) exploiting $x \in \Gamma$ when bounding $|f(x) - f(\tilde{x})|$ is difficult, in which case neither the two-sided \(^\text{[10]}\) nor the one-sided \(^\text{[4]}\) seem to be suitable. In an attempt to deal with such situations we introduce an intermediate variant, which is ‘locally’ two-sided: it only requires the (one-sided) typical Lipschitz condition \(^\text{[4]}\) to hold when each coordinate of both sample points $x, \tilde{x}$ satisfies some local ‘good’ event $x_j, \tilde{x}_j \in \Gamma_j$.

**Theorem 11** (‘Typical bounded differences inequality with truncation’). Let $X = (X_1, \ldots, X_N)$ be a family of independent random variables with $X_k$ taking values in a set $\Lambda_k$. Suppose $(\Gamma_k)_{k \in [N]}$ and $\Gamma \subseteq \prod_{j \in [N]} \Gamma_j$ are events with $\Gamma_k \subseteq \Lambda_k$. Assume that the function $f : \prod_{j \in [N]} \Lambda_j \to \mathbb{R}$ satisfies the Lipschitz condition \(^\text{[4]}\) of (TL) only for all $x, \tilde{x} \in \prod_{j \in [N]} \Lambda_j$ that differ only in the $k$-th coordinate, and that $|f(x) - f(\tilde{x})| \leq s$ for all $x, \tilde{x} \in \prod_{j \in [N]} \Lambda_j$. For any numbers $(\gamma_k)_{k \in [N]}$ with $\gamma_k \in (0, 1]$ there is an event $B = B(\Gamma, (\gamma_k)_{k \in [N]})$ satisfying \(^\text{[5]}\) such that for $\mu = \mathbb{E}[f(X), \Delta = 4\mathbb{P}(X \notin \Gamma)]$, $e_k = \gamma_k(d_k - c_k)$ and any $t \geq 0$ we have

$$
\mathbb{P}(f(X) \geq \mu + t + \Delta \text{ and } \neg B) \leq \exp \left( -\frac{t^2}{2 \sum_{k \in [N]} (c_k + e_k)^2} \right). \tag{11}
$$

**Remark 12.** If $\Gamma = \prod_{j \in [N]} \Gamma_j$ holds we may set $B = \neg \Gamma$, $e_k = 0$ and multiply the exponent of \(^\text{[11]}\) by a factor of 4. If certain monotonicity properties hold we can remove the $\Delta$ term: for example, we can set $\Delta = 0$ if $f(x) \geq f(\tilde{x})$ whenever $x, \tilde{x} \in \prod_{j \in [N]} \Lambda_j$ differ only in their $k$-th coordinates $x_k \in \Lambda_k \setminus \Gamma_k$ and $\tilde{x}_k \in \Gamma_k$.

Theorem \(^\text{[11]}\) seems particularly useful when the underlying random variables are grouped into larger blocks $B_k$, so that each $X_k'$ now takes values in its own product space $\Lambda_k' = \prod_{j \in B_k} \Lambda_j$ (by construction the $X_k'$ are again independent). For example, the so-called ‘vertex exposure’ of $G_{n,p}$ uses $n - 1$ blocks, where $X_k'$ corresponds to the group of edges $E_k = (v_{k+1}, \ldots, v_n)$. In this case $q_k \leq p^{n-k}$ and the ‘good’ event $\Gamma$ of, say, having at most $\Sigma = \max\{2np, n^2\}$ neighbours can dramatically fail after changing the $k$-th coordinate (the degree of $v_k$ can change up to $n-k$). Here we can overcome these issues using the ‘local’ event $\Gamma_k$ that at most $\Sigma$ edges of $E_k$ are present, so that after a one-coordinate change of $x \in \prod_{j \in [N]} \Gamma_j$ from $x_k$ to $\tilde{x}_k \in \Gamma_k$ the degree of every vertex changes by at most $\Sigma$ (if $x \in \Gamma$ then every vertex has at most $2\Sigma$ neighbours). In other words, the local $\Gamma_k$ and global $\Gamma$ can complement each other in order to mitten the large worst case effects (in particular when many variables are associated with each coordinate).

Theorem \(^\text{[11]}\) also allows us to routinely apply certain truncation arguments (without ad-hoc calculations). A typical example is $f(X) = \sum_k X_k$ with $X_k$ having exponential tails, where one often first proves concentration of, say, $\sum_k \min\{X_k, C \log N\}$, and then transfers this result to the original sum, see e.g. \(^\text{[11] [12]}\). Here \(^\text{[11]}\) almost immediately yields concentration of $f(X)$ via the local events $\Gamma_k$ that $X_k \leq C \log N$ occurs (setting $\Gamma = \prod_{j \in [N]} \Gamma_j$, $d_k = c_k = C \log N$ and $\gamma_k = 1$).

### 1.1.3 Dynamic exposure of the variables

The previous inequalities can be refined by exposing the values of the random variables $X_i$ one by one in an *adaptive* order. Intuitively this allows us to exploit that after having learned the values of certain variables, some other $X_j$ may not any more influence the value of $f(X)$. This approach was introduced by Alon, Kim and Spencer \(^\text{[2]}\), and is particularly useful whenever we can determine $f(X)$ without knowing the value of all random variables. More formally, a *strategy* sequentially exposes $X_{q_1}, X_{q_2}, \ldots$, where each index $q_i = q_i(X_{q_1}, \ldots, X_{q_{i-1}})$ may depend on the
previous outcomes and indices (we use the convention that $q_{k+1} = q_k$ if $f(X)$ is determined by $(X_q, \ldots, X_{q_k})$ with $k < N$); every strategy has a natural representation in form of a decision tree. With a fixed strategy in mind, for every possible outcome $X = (X_1, \ldots, X_N)$ we obtain a set of queried indices $Q \subseteq [N]$, and by $Q$ we denote the set of all possible such query sets $Q$. The resulting key improvement is that in most inequalities we essentially may replace $k \in [N]$ with $k \in Q$ for some ‘worst case’ set of indices $Q \in Q$ (note that $\gamma_k = \gamma$ is a typical choice in applications).

**Theorem 13 (‘Dynamic exposure of the variables’).** Suppose that $\gamma_k = \gamma$ for all $k \in [N]$. For any strategy Theorems 4, 7, 10, 11, Corollary 6 and Remarks 3, 6, 12 remain valid with $\sum_{k \in [N]}$ replaced by $\max_{Q \in \cal{Q}} \sum_{k \in Q}$ and $\max_{k \in [N]}$ replaced by $\max_{Q \in \cal{Q}} \max_{k \in Q}$, with the addition that $\cal{B}$ depends on the query strategy.

**Theorem 14 (‘Monotone dynamic exposure of the variables’).** Consider any strategy satisfying $q_{i+1} \geq q_i$ in each step. Then Theorems 4, 7, 10, 11, Corollary 6 and Remarks 3, 6, 12 remain valid with $\sum_{k \in [N]}$ replaced by $\max_{Q \in \cal{Q}} \sum_{k \in Q}$ and $\max_{k \in [N]}$ replaced by $\max_{Q \in \cal{Q}} \max_{k \in Q}$, with the exception that 3 remains unchanged.

Applied to Corollary 6, Theorem 4 and Remark 7 the results tighten and extend an inequality of Alon, Kim and Spencer [2], which is based on the Lipschitz condition (L). In certain applications dynamic exposure yields significant improvements, and for an illustrating example we refer to Claim 2 in [2], where it is crucial to reduce (the order of magnitude of) the number of queried variables. Further refinements are possible by using adaptive Lipschitz bounds $c_k$, which is perhaps most easily exploited by tailoring the arguments of Section 2 to the specific application.

One key feature of Theorem 14 is that the ‘bad’ event $\cal{B}$ does not depend on the strategy used, making it particularly useful in union bound arguments. As an illustration consider the example where $f(X) = f_U(X)$ counts, in $G_n,p$, the number of triangles in a subset $U \subseteq V$ of the vertices. Define $\Gamma$ as in Section 3. Since $f(X)$ depends only on edges in $U$, using Theorems 13 and 14 (sequentially exposing all edges in $U$) we infer for $|U| = u \geq u_0 = u_0(n,p)$ and $\Delta = \min\{\Delta, u\}$ that

$$\mathbb{P}(f(X) \notin (1 \pm \epsilon)u) \mu \text{ and } \mathcal{B} \leq \exp\left(-\Theta\left(\frac{n^{-2\epsilon}u^6p^6}{u^2p\Delta^2 + n^{-2}u^3p^3\Delta}\right)\right) \leq n^{-\omega(u)}.$$  

Taking a union bound the probability that some $U \subseteq V$ with $|U| \geq u_0$ has the ‘wrong’ number of triangles is at most $\mathbb{P}(\mathcal{B}) + n^{-\omega(u_0)}$. Here we crucially exploited that $\mathcal{B}$ is a ‘global’ event not depending on $f(X)$ or $U$, so that $\mathbb{P}(\mathcal{B}) \leq e^{-\Omega(n^\epsilon)}$ does not need to ‘compete’ with the $n^\epsilon$ choices for the subsets (this issue often makes traditional bad events ineffective in union bound arguments).

1.1.4 Weakening the independence assumption

The concentration results discussed so far extend to certain dependent random variables $X = (X_1, \ldots, X_N)$ that are generated by a sequence of ‘nearly’ independent (or uniform) random choices. As we shall see, they e.g. apply to random permutations $\pi \in \Sigma_n$ and uniform random graphs $G_{n,m}$.

To motivate the new (GL) condition below we consider independent random variables, in which case the mapping $\rho_k : \Sigma_a \to \Sigma_b$ that changes the value of the $k$-th coordinate from $a$ to $b$ is a bijection. Here (L) yields $|f(x) - f(\rho_k(x))| \leq c_k$ and independence implies $\mathbb{P}(X = x \mid X \in \Sigma_a) = \mathbb{P}(X = \rho_k(x) \mid X \in \Sigma_b)$. With this in mind 12 can be viewed as a natural analogue of 7 in which the outcomes $x$ and $x = \rho_k(x)$ may differ in more than just one coordinate, and 13 accounts for the fact that the variables are not necessarily independent.
There are numbers \((c_k)_{k \in [N]}\) and \((d_k)_{k \in [N]}\) with \(c_k \leq d_k\) such that the following holds for any two possible sequences of outcomes \(a_1, \ldots, a_{k-1}, a\) and \(a_1, \ldots, a_{k-1}, b\) of \(X_1, \ldots, X_k\). Defining

\[
\Sigma_z = \left\{ x = (a_1, \ldots, a_{k-1}, z, x_{k+1}, \ldots, x_N) \in \prod_{j \in [N]} \Lambda_j : \mathbb{P}(X = x) > 0 \right\},
\]

there is an injection \(\rho_k = \rho_k(\Sigma_a, \Sigma_b) : \Sigma_a \to \Sigma_b\) such that for all \(x \in \Sigma_a\) we have

\[
|f(x) - f(\rho_k(x))| \leq \begin{cases} 
  c_k & \text{if } x \in \Gamma, \\
  d_k & \text{otherwise.}
\end{cases}
\tag{12}
\]

\[
\mathbb{P}(X = x \mid X \in \Sigma_a) \leq \mathbb{P}(X = \rho_k(x) \mid X \in \Sigma_b).
\tag{13}
\]

**Remark 16.** The proof shows that \(\rho_k\) must be a bijection with equality in \(13\). Furthermore, if \(X_k\) takes at most two values conditioned on \(X_1, \ldots, X_{k-1}\), then the exponent in \(10\) may be multiplied by factor of 4. In fact, \(2\) holds if \(\Gamma = \prod_{j \in [N]} \Lambda_j\) (or \(r_k = 0\) below). In addition, for \(6\) to hold with \(c_k = \gamma_{k\Gamma k} \geq 0\) it suffices if we relax (GL) to the average Lipschitz condition

\[
|\mathbb{E}(f(X) \mid X \in \Sigma_a) - \mathbb{E}(f(X) \mid X \in \Sigma_b)| \leq c_k + r_k \mathbb{P}(X \notin \Gamma \mid X \in \Sigma_a).
\tag{14}
\]

To illustrate the application of the (GL) condition we consider uniform permutations \(\pi \in S_n\), which are generated by sequentially choosing each \(\pi(k)\) randomly from \([n] \setminus \{\pi(1), \ldots, \pi(k-1)\}\). Here \(\Sigma_z\) contains all \(\pi\) with \(\pi(k) = z\) and \(\pi(j) = a_j\) for \(1 \leq j < k\). In this case a bijection \(\rho_k : \Sigma_a \to \Sigma_b\) is defined by the transposition of \(a\) and \(b\), so that \(\pi' = \rho_k(\pi)\) satisfies \(\pi'(k) = b\), \(\pi'(\pi^{-1}(b)) = a\) and \(\pi'(i) = \pi(i)\) for \(\pi(i) \notin \{a, b\}\). Using \(|\Sigma_a| = |\Sigma_b| = (n-k)!\) and the uniform measure it is not hard to check that \(13\) holds with equality. We see that for establishing \(12\) it suffices to bound \(|f(\pi) - f(\pi')|\) whenever \(\pi\) and \(\pi'\) are related via a transposition, which is an intuitive and easy to check condition (this may correspond to changing two coordinates).

One key aspect of (GL) is that it often maintains the simplicity of (L) and (TL). Here uniform probability measures are particularly convenient, for which it suffices to first define bijections \(\rho_k : \Sigma_a \to \Sigma_b\) and then check \(12\) only (using \(\mathbb{P}(X = x \mid X \in \Sigma_a) = \mathbb{P}(X = x) / \mathbb{P}(X \in \Sigma_a)\) these must satisfy \(13\) with equality). Indeed, extending the permutations example, for random sequences \(T = (t_1, \ldots, t_m)\) of \(m\) distinct elements from \(W\) it is enough to estimate \(|f(T) - f(T')|\) whenever both sequences are related by changing one coordinate (i.e., \(t_k \neq t'_k\)) or interchanging the order of two coordinates (i.e., \(t_k = t'_j\) and \(t_j = t'_k\)). Note that this example includes the random graph process and various hypergraph processes as special cases. Since every set with \(m\) elements gives rise to \(m!\) ordered sequences, the above result also readily carries over to uniform random subsets \(S \subseteq W\) of size \(|S| = m\): it suffices to bound \(|f(S) - f(S')|\) whenever the sets are minimally different, i.e., satisfy \(|S \cap S'| = m - 1\) (note that for \(m > |W|/2\) better results are obtained by choosing the complement uniformly at random). Here the uniform random graph \(G_{n,m}\) and uniform hypergraphs are special cases. Note that the above construction also extends to multiple (independent) random objects; for example, if \(M\) random subsets \(X = (S_1, \ldots, S_M)\) with \(S_i \in \mathcal{W}_i\) and \(|S_i| = m_i\) are chosen independently it suffices to consider \(|f(X) - f(X')|\) only for the cases where \(X\) and \(X'\) are minimally
Theorem 18 (‘Two-sided general Lipschitz condition’). Let \( X = (X_1, \ldots, X_N) \) be a family of random variables with \( X_k \) taking values in a set \( \Lambda_k \), where \( \max_{\eta \in \Lambda_k} \mathbb{P}(X_k = \eta \mid X_1, \ldots, X_{k-1}) \geq 1 - p_k \) holds. Let \( \Gamma \subseteq \prod_{j \in [N]} \Lambda_j \) be an event and assume that the function \( f: \prod_{j \in [N]} \Lambda_j \to \mathbb{R} \) satisfies the general Lipschitz condition (GL). For any numbers \( (\gamma_k)_{k \in [N]} \) with \( \gamma_k \in (0,1] \) there is an event \( \mathcal{B} = \mathcal{B}(\Gamma, (\gamma_k)_{k \in [N]}) \) satisfying (5) such that for \( \mu = \mathbb{E} f(X) \) and any \( t \geq 0 \) we have

\[
\mathbb{P}(f(X) \geq \mu + t \text{ and } \neg \mathcal{B}) \leq \exp \left( -\frac{t^2}{2 \sum_{k \in [N]} p_k (c_k + \epsilon_k \cdot (1 - p_k)^{-1})^2 + 2Ct/3} \right),
\]

where \( \epsilon_k = \gamma_k (d_k - c_k) \) and \( C = \max_{k \in [N]} (c_k + \epsilon_k) \).

Theorem 17 (‘General bounded differences inequality for asymmetric variables’). Let \( X = (X_1, \ldots, X_N) \) be a family of random variables with \( X_k \) taking values in a set \( \Lambda_k \), where \( \max_{\eta \in \Lambda_k} \mathbb{P}(X_k = \eta \mid X_1, \ldots, X_{k-1}) \geq 1 - p_k \) holds. Let \( \Gamma \subseteq \prod_{j \in [N]} \Lambda_j \) be an event and assume that the function \( f: \prod_{j \in [N]} \Lambda_j \to \mathbb{R} \) satisfies the general Lipschitz condition (GL). For any numbers \( (\gamma_k)_{k \in [N]} \) with \( \gamma_k \in (0,1] \) there is an event \( \mathcal{B} = \mathcal{B}(\Gamma, (\gamma_k)_{k \in [N]}) \) satisfying (5) such that for \( \mu = \mathbb{E} f(X) \) and any \( t \geq 0 \) we have

\[
\mathbb{P}(f(X) \geq \mu + t \text{ and } \neg \mathcal{B}) \leq \exp \left( -\frac{t^2}{2 \sum_{k \in [N]} p_k (c_k + \epsilon_k \cdot (1 - p_k)^{-1})^2 + 2Ct/3} \right),
\]

where \( \epsilon_k = \gamma_k (d_k - c_k) \) and \( C = \max_{k \in [N]} (c_k + \epsilon_k) \).

Theorem 18 (‘Two-sided general Lipschitz condition’). Theorems 15, 17 and Remark 16 remain valid with \( c_k = 2\gamma_k (d_k - c_k) q_k^{-1} \) and \( \min_{\eta \in \Lambda_k} \mathbb{P}(X_k = \eta \mid X_1, \ldots, X_{k-1}) \geq q_k \) if the Lipschitz condition (12) of (GL) is replaced by the following two-sided variant:

\[
|f(x) - f(\rho_k(x))| \leq \begin{cases} c_k & \text{if } x, \rho_k(\tilde{x}) \in \Gamma, \\ d_k & \text{otherwise.} \end{cases}
\]

In addition, \( q_k \leq |\Lambda_k|^{-1} \) suffices when all possible outcomes occur with the same probability.

The sufficient condition \( q_k \leq |\Lambda_k|^{-1} \) often makes the two-sided Lipschitz condition of Theorem 18 easy to apply. For example, in case of random permutations \( \pi \in S_n \) and random graphs \( G_{n,m} \) (or the random graph process) we may take \( q_k = n^{-1} \) and \( q_k = n^{-2} \), respectively.

1.2 Discussion and applications

1.2.1 A wider perspective

As discussed, in probabilistic combinatorics and the analysis of randomized algorithms we frequently need to prove that a random function is not too far from its mean, e.g., that \( f(X) \approx \mu \) or \( f(X) \leq 2\mu \) holds. A common feature of many recent applications is that the functions of interest are only ‘smooth enough’ on a high probability event, whereas their deterministic worst case changes are too large for the standard bounded differences inequality (Theorem 1) to be effective.

In these cases there is no general method, but in the past certain ad-hoc arguments have been successfully used in such situations (see e.g. [7, 10, 25, 20, 33]). Usually the key idea is to construct a random function \( g(X) \) that is a smooth approximation of \( f(X) \), which in particular by definition ensures that the Lipschitz coefficients are always small (approximation usually means that \( f(X) \approx g(X) \) holds with high probability, but often \( \mathbb{E} f(X) \approx \mathbb{E} g(X) \) is also needed). Here smoothness makes it possible to apply concentration inequalities (often the bounded differences inequality) to \( g(X) \), whereas the approximation property ensures that concentration transfers from...
A combination of (8) and (7) now yields

g(X) to f(X). The main disadvantage of this approach is that it relies on ad-hoc arguments (which can be involved); in particular, finding suitable approximation functions may require ingenuity.

One aim of this paper is to provide easy-to-apply tools which can routinely deal with such situations, establishing concentration in a rather simple way. For example, in the frequent case where the good event \( \Gamma \) holds with probability at least \( 1 - N^{-\omega(1)} \) we can typically choose \( \gamma_k = \max |f(X)| \) and then completely ignore the worst case effects, see e.g. the proof of Theorem 28 (this approach also applies, for example, to Lemma 14 in [13] and parts of the martingale-based proof of Theorem 2.2 in [20]). In other words, the crucial advantage of our new inequalities is that they can often remove the need for sometimes difficult ad-hoc arguments using only a minimum amount of calculations (which typically even coincide with heuristic considerations).

### 1.2.2 Comparison with Janson’s inequality

In this section we demonstrate that in certain applications our inequalities give exponential estimates that (i) are tight and (ii) successfully compete with the well known Janson’s inequality. To this end we focus on subgraph counts in the binomial random graph \( G_{n,p} \) since a concrete example seems more illustrative to us. Henceforth we assume that \( H \) is a fixed 2-balanced graph, i.e., where \( H \) has \( e_H \geq 2 \) edges and all its proper subgraphs \( G \subset H \) with \( v_G \geq 3 \) vertices satisfy

\[
\frac{e_G - 1}{v_G - 2} \leq \frac{e_H - 1}{v_H - 2} = d_2(H).
\] (17)

This class of graphs includes, for example, complete graphs and cycles of arbitrary size. Let \( Y_H \) count the number of \( H \) copies in \( G_{n,p} \). For 2-balanced graphs it is well-known (see e.g. [22]) that Janson’s inequality gives

\[
P(Y_H \leq \mu - t) \leq \exp \left( -\Theta \left( \frac{t^2}{\mu^2/(n^2p)} \right) \right)
\] (18)

for \( p \geq n^{-1/d_2(H)} \). Spencer [38] proved that, assuming \( p \geq n^{-1/d_2(H)} (\log n)^b \) with \( 0 < b = b(H) < 2 \), for every \( c > 0 \) the following holds with probability at least \( 1 - n^{-c} \); for \( n \geq n_0(c, H) \); every pair \( xy \) of vertices is contained in at most \( \Delta = O(n^{v_H-2}p^{e_H-1}) \) extensions to copies of \( H \) (for which adding the edge \( xy \) completes a copy of \( H \) containing \( xy \)). The latter event will be our decreasing \( \Gamma \), which allows us to use \( c_k = \Theta(n^{v_H-2}p^{e_H-1}) = \Theta(\mu/(n^2p)) \) as well as \( d_k = n^{v_H} \) and \( \gamma_k = n^{-v_H} \) in our typical bounded differences inequality, so that \( c_k = \gamma_k(d_k - c_k) = o(c_k) \). Applying Spencer’s result with \( c = v_H + 3 \) we have \( \mathbb{P}(B) \leq n^{-1} \) by (3). Note that, since for the lower tail we have \( t = O(\mu) \), it follows that \( \sum_k p c_k^2 + t \max_k c_k = \Theta(\mu^2/(n^2p)) \). For the decreasing function \( f = -Y_H \) a combination of (5) and (7) now yields

\[
P(Y_H \leq \mu - t) \leq \exp \left( -\Theta \left( \frac{t^2}{\mu^2/(n^2p)} \right) \right),
\]

which asymptotically matches (18), i.e., the estimate of Janson’s inequality. In fact, for \( t = \Theta(\mu) \) this bound is best possible (up to constants in the exponent) since \( G_{n,p} \) contains no edges (and thus no copies of \( H \)) with probability \( e^{-\Theta(n^2p)} \).

### 1.2.3 Application: the reverse \( H \)-free process

The following variations of the classical random graph processes were proposed by Bollobás and Erdős at the 1990 Quo Vadis, Graph Theory conference in an attempt to improve Ramsey num-
Theorem 19. Let $H$ be a 2-balanced graph. There are constants $a, A > 0$ such that the final number of edges $M_n$ in the reverse $H$-free process has expectation satisfying $an^{2-1/d_2(H)} \leq \mathbb{E}M_n \leq An^{2-1/d_2(H)}$. Furthermore, for any $c > 0$ we have $|M_n - \mathbb{E}M_n| \leq \sqrt{\mathbb{E}M_n (\log n)}^{e_H}$ with probability at least $1 - n^{-c}$ for $n \geq n_0(c,H)$.

Our arguments partially generalize to arbitrary graphs. Set $d_2(K_2) = 1/2$ and $m_2(H) = \max_{G \subseteq H, e_G \geq 1} d_2(G)$, so that $m_2(H) = d_2(H)$ for 2-balanced graphs $H$. We show that for any graph the expected final number of edges in the reverse $H$-free process is $\Theta(n^{2-1/m_2(H)})$, and prove concentration under certain conditions (satisfied e.g. by a clique $K_r$ with an extra edge hanging off), see Section 3. The proof of Theorem 19 also extends to a finite family of forbidden graphs $\mathcal{H}$, which for the $H$-free process was considered in [11]. Indeed, defining the reverse $\mathcal{H}$-free process in the obvious way (always removing a random edge that is contained in a copy of some $H \in \mathcal{H}$) we obtain, for example, the following generalization.

Theorem 20. Let $\mathcal{H} = \{H_1, \ldots, H_r\}$ be a family of 2-balanced graphs. Define $H, J \in \mathcal{H}$ such that $d_2(H) = \min_{F \in \mathcal{H}} d_2(F)$ and $e_f = \max_{F \in \mathcal{H}} e_F$. There are constants $a, A > 0$ such that the final number of edges $M_n$ in the reverse $\mathcal{H}$-free process has expectation satisfying $an^{2-1/d_2(H)} \leq \mathbb{E}M_n \leq An^{2-1/d_2(H)}$. The $H$-free process, where, starting with an empty graph on $n$ vertices, in each step a new edge is added, chosen uniformly at random from all pairs whose addition does not complete a copy of $H$. The reverse $H$-free process, where, starting with a complete graph on $n$ vertices, in each step an edge is removed, chosen uniformly at random from all edges that are contained in a copy of $H$. The $H$-removal process, where, starting with a complete graph on $n$ vertices, in each step all $e_H$ edges of a copy of $H$ are removed, which is selected uniformly at random from all $H$ copies. All of these processes end with an $H$-free graph, and Bollobás and Erdős asked (among other structural properties) what their typical final number of edges is [8, 11].

These variations have received considerable attention in recent years, in particular the $H$-free process. Its typical final number of edges is nowadays known up to logarithmic factors [5, 34, 39] for the class of strictly 2-balanced graphs $H$, where in (17) the inequality is strict. Matching bounds up to constant factors have only been established for some special forbidden graphs and the class of $C_r$-free processes, see e.g. [41, 44, 45]. The final graph of the $K_s$-free process also yields the best known lower bounds on the Ramsey numbers $R(s,t)$ with $s \geq 4$, see [11, 56]. Recently Makai [27] determined the (asymptotic) final number of edges of the reverse $H$-free process for the class of strictly 2-balanced graphs, but its final graph yields no new estimates for $R(s,t)$. Although the related $H$-removal process has been studied in several papers the final number of edges is known up to multiplicative $n^{o(1)}$ factors only in the special case $H = K_3$, see e.g. [5, 31, 59].

Using our typical bounded differences inequality, in Section 5 we show that the final number of edges in the reverse $H$-free process is sharply concentrated when $H$ is 2-balanced (we do not assume strictly 2-balanced), and also determine the likely number of edges up to constants. This is in contrast to all known results for the widely studied $H$-free and $H$-removal processes. Indeed, in these (a) no sharp concentration results are known, (b) the order of magnitude of the final number of edges is open for most strictly 2-balanced graphs, and (c) no general results apply to the class of 2-balanced graphs. As we shall see, when $H$ is a matching the expected final number of edges in the reverse $H$-free process is $\Theta(1)$. When it comes to concentration we thus restrict our main attention to all other 2-balanced graphs $H$, which in fact satisfy $d_2(H) \geq 1$ (with equality for trees).

Here our next result shows that the reverse $H$-free process typically ends with $\Theta(n^{2-1/d_2(H)})$ edges, answering (up to constant factors) the aforementioned question of Bollobás and Erdős from 1990.
Let \( M_n \) be an \( \mathcal{F}_k \)-measurable variable satisfying \( L_k = M_k - M_{k-1} \leq U_k \). Set \( S_k = \sum_{i \in [k]} (U_i - L_i)^2 \). For every \( t \geq 0 \) and \( S > 0 \) we have
\[
\Pr(M_k \geq M_0 + t \text{ and } S_k \leq S \text{ for some } k \in [N]) \leq e^{-2t^2/S}.
\]

**Remark 23.** Note that \( V_k \) generalizes \( S_k \) since \( \Pr((M_i - M_{i-1})^2 \mid \mathcal{F}_{i-1}) = \mathbb{E}((M_i - M_{i-1})^2 \mid \mathcal{F}_{i-1}) \) holds (it is not hard to check that \( V_k \leq S_k/k \)). In fact, Lemmas 21 and 22 extend with minor modifications to supermartingales: defining \( V_k = \sum_{i \in [k]} \mathbb{E}((M_i - M_{i-1})^2 \mid \mathcal{F}_{i-1}) \) suffices.

Observe that we allow for (accumulative) random bounds on the one-step changes (and other quantities), which in case of Lemma 21 is the main difference to the usual formulation of the classical Hoeffding–Azuma inequality [21, 3]. Lemma 22 also extends the related Theorem 2.2.2 of Kim and Vu [24] (see also Lemma 3.1 in Vu’s survey [13]), which assumes that the underlying probability space is generated by independent random variables (of a special form).

Note that \( L_k, U_k \) are \( \mathcal{F}_{k-1} \)-measurable, whereas \( M_k - M_{k-1} \) is \( \mathcal{F}_k \)-measurable. This difference sometimes causes subtle off-by-one errors. As pointed out by Oliver Riordan, for e.g. the estimate
\[
\Pr(M_N \geq M_0 + t) \leq e^{-t^2/2 \sum_k c_k^2 + \eta}
\]
it does not suffice if \( \sum_k \mathbb{P}(|M_k - M_{k-1}| > c_k) \leq \eta \), as claimed by Theorem 8.4 in [14]. The problem is that, conditional on \( \mathcal{F}_{k-1} \), in the next step it sometimes is always possible for \( |M_k - M_{k-1}| \leq c_k \) to fail (although this might be unlikely). With this in mind, we see that (21) holds e.g. if
\[
\sum_k \mathbb{P}(\text{it is possible, given } M_1, ..., M_{k-1}, \text{ that } |M_k - M_{k-1}| > c_k) \leq \eta.
\]
In fact, assuming that \( |M_k - M_{k-1}| \leq C_k \) always holds, the approach of [13, 37] e.g. implies (21) if
\[
\sum_k (1 + 2C_k/t) \cdot \mathbb{P}(|M_k - M_{k-1}| > c_k/4) \leq \eta.
\]

### 2.1.1 Proof of Lemmas 21 and 22

Our proofs use the following (standard) inequalities due to Hoeffding [20] and Steiger [40]; they follow e.g. from the proofs of Lemmas 2.4, 2.6 and 2.8 in McDiarmid's survey [29].

**Lemma 24.** Let \( X \) be random variable with \( \mathbb{E}(X \mid \mathcal{F}) = 0 \). Let \( L, U \) be \( \mathcal{F} \)-measurable random variables. Set \( g(x) = (e^x - 1 - x)/x^2 \) for \( x \neq 0 \) and \( g(0) = 1/2 \). For any \( \lambda \geq 0 \) the following holds:
\[
\begin{align*}
L \leq X \leq U & \implies \mathbb{E}(e^{\lambda X} \mid \mathcal{F}) \leq e^{\lambda^2 (U - L)/2} \quad \text{and} \\
X \leq U & \implies \mathbb{E}(e^{\lambda X} \mid \mathcal{F}) \leq e^{\lambda^2 g(U)} \mathbb{V} \text{ar}(X \mid \mathcal{F}).
\end{align*}
\]
Furthermore, \( g(x) \) is a non-negative increasing function. \( \square \)

**Lemma 25.** Set \( \phi(x) = (1 + x) \log(1 + x) - x \). For all \( x \geq 0 \) we have \( \phi(x) \geq x^2/(2 + 2x/3) \). \( \square \)

**Proof of Lemmas 21 and 22.** Set
\[
W_k = \sum_{i \in [k]} g(Y_i) \mathbb{V} \text{ar}(M_i - M_{i-1} \mid \mathcal{F}_{i-1}).
\]
The key point is that \( M_k \) and \( L_k, U_k, S_k, W_k \) are \( \mathcal{F}_{k-1} \)-measurable. So, by applying (22) and (23) to \( \mathbb{E}(e^{\lambda(M_k - M_{k-1})} \mid \mathcal{F}_{k-1}) \) we see that
\[
Y_k = e^{\lambda(M_k - M_{k-1}) - \lambda S_k/8} \quad \text{and} \quad Z_k = e^{\lambda(M_k - M_0) - \lambda^2 W_k}
\]
satisfy \( \mathbb{E}(Y_k \mid \mathcal{F}_{k-1}) \leq Y_{k-1} \) and \( \mathbb{E}(Z_k \mid \mathcal{F}_{k-1}) \leq Z_{k-1} \), i.e., are supermartingales. We define the stopping time \( T \) as the minimum of \( N \) and the smallest \( k \in [N] \) with \( M_k - M_0 \geq t \); as usual, we write \( i \wedge T \) as shorthand for \( \min\{i, T\} \). By construction \( (Y_{k \wedge T})_{0 \leq k \leq N} \) and \( (Z_{k \wedge T})_{0 \leq k \leq N} \) are both supermartingales. In particular, we have
\[
\mathbb{E}Y_{N \wedge T} \leq \mathbb{E}Y_0 = 1 \quad \text{and} \quad \mathbb{E}Z_{N \wedge T} \leq \mathbb{E}Z_0 = 1.
\]

Let \( \mathcal{E}_N \) denote the event that \( M_k \geq M_0 + t \) and \( S_k \leq S \) for some \( k \in [N] \). Note that \( \mathcal{E}_N \) implies \( Y_{N \wedge T} = Y_T \geq e^{\lambda - \lambda^2 S/8} \). So, for \( \lambda = 4t/S \) Markov’s inequality gives
\[
\mathbb{P}(\mathcal{E}_N) \leq \mathbb{P}(Y_{N \wedge T} \geq e^{\lambda - \lambda^2 S/8}) \leq e^{\lambda^2 S/8 - \lambda t} = e^{-2t^2/S},
\]
which establishes (19) and thus Lemma 21.

We proceed similarly for \( (Z_{k \wedge T})_{0 \leq k \leq N} \) and let \( \mathcal{E}_N' \) denote the event that \( M_k \geq M_0 + t \) and \( V_k \leq V \) for \( k \geq C \) for some \( k \in [N] \). Using \( V_k \geq 0 \) and monotonicity of \( g(x) \geq 0 \) we see that \( \mathcal{E}_N' \) implies \( Z_{N \wedge T} \geq e^{\lambda - \lambda^2 g(\lambda C)V_k} \geq e^{\lambda^2 - \lambda^2 g(\lambda C)V} \). Recall that \( \phi(x) = (1 + x) \log(1 + x) - x \). For \( \lambda = \log(1 + Ct/V)/C^2 \) Markov’s inequality and Lemma 25 now yield
\[
\mathbb{P}(\mathcal{E}_N') \leq e^{\lambda^2 g(\lambda C)V - \lambda t} = e^{-V/C^2 \cdot \phi(Ct/V)} \leq e^{-t^2/(2V + 2Ct/3)},
\]
which establishes (20) and thus Lemma 22. \( \square \)
2.2 Bounded differences inequalities

The textbook proof of Theorem 11 is based on the Hoeffding–Azuma inequality [20, 3], and essentially uses the ‘worst case’ Lipschitz condition (1) to apply Lemma 21 with $|U_k - L_k| \leq c_k$. We need some modifications to deal with the obstacle that the ‘good’ event $\Gamma$ and thus the ‘typical case’ in (1) does not always hold, and these are partially inspired by the seminal work of Shamir and Spencer [37] from 1987.

When $P(X \notin \Gamma) \leq \eta$ holds one might be tempted to add $\eta$ to the error bound and then always assume that $\Gamma$ holds. The problem is that in the martingale based proof one needs to estimate conditional expected changes as in (3). So, informally speaking, despite $\omega \in \Gamma$ the ‘good’ event can still fail ‘inside’ the corresponding expectations. One can try to overcome this by conditioning on $\Gamma$, but this usually introduces a new technical problem: then the variables are not conditionally independent (in which case Lipschitz conditions comparable to (1) no longer suffice to bound the expected changes). These technicalities seem to cause some confusion in e.g. [15, 18].

We step aside these issues by noting that for good bounds on conditional expected one-step changes it suffices that the conditional probabilities of large changes are small. One key aspect of our approach is that we can always guarantee this via the ‘global’ event $\Gamma$ only, i.e., without having any knowledge about the corresponding conditional distributions.

2.2.1 The general approach

We now introduce the setup used in all subsequent proofs. Let $Y = f(X)$. We consider the increasing sequence of sub-$\sigma$-fields $F_k$ generated by $X_1, \ldots, X_k$. Using Doob’s construction, the sequence $Y_k = \mathbb{E}(Y \mid F_k)$ is a martingale with $Y_0 = \mathbb{E}f(X) = \mu$ and $Y_N = f(X)$. Now we define $F_{k-1}$-measurable events $B_k$, where $\omega \in B_k$ if

$$P(X \notin \Gamma \mid F_{k-1})(\omega) > \gamma_k.$$  \hfill (24)

Let $B = B_0 \cup \bigcup_{k \in [N]} B_{k-1}$. Note that $P(X \notin \Gamma \mid F_0) = P(X \notin \Gamma)$ yields $P(B_0) = 0$ if $\gamma_1 \geq P(X \notin \Gamma)$ and $P(B_0) = 1$ otherwise. Using $\gamma_1 \in (0, 1]$ we infer $P(-\Gamma \cup B_0) \leq \gamma_1^{-1} P(X \notin \Gamma)$. Observing that $P(X \notin \Gamma) = \min_{\Gamma} \mathbb{P}(X \notin \Gamma \mid F_{k-1}) \geq \gamma_k P(B_{k-1})$, the union bound now gives

$$P(B) \leq P(-\Gamma \cup B_0) + \sum_{2 \leq k \leq N} P(B_{k-1}) \leq \sum_{k \in [N]} \gamma_k^{-1} \cdot P(X \notin \Gamma).$$  \hfill (25)

Let the stopping time $T$ be the minimum of $N$ and the smallest $0 \leq k < N$ for which $B_k$ holds (note that $T \leq k - 1$ is $F_{k-1}$-measurable). Setting $M_k = Y_{k+1}$, it follows that the sequence $(M_k)_{0 \leq k \leq N}$ is a martingale with $Y_0 = M_0 = \mu$. Since $T = N$ unless $B$ holds, recalling $Y_N = f(X)$ we see that

$$P(f(X) \geq \mu + t \text{ and } \neg B) = P(Y_N \geq Y_0 + t \text{ and } \neg B) \leq P(M_N \geq M_0 + t).$$  \hfill (26)

It remains to establish suitable tail estimates for $P(M_N \geq M_0 + t)$, and via Lemmas 21 and 22 this reduces to proving (deterministic) upper bounds on the random variables $S_N$, $V_N$, and $C_N$. To this end we consider the martingale difference sequences $\Delta M_k = M_k - M_{k-1}$ and $\Delta Y_k = Y_k - Y_{k-1}$, which satisfy $\mathbb{E}(\Delta M_k \mid F_{k-1}) = 0$ and $\mathbb{E}(\Delta Y_k \mid F_{k-1}) = 0$. Set $e_k = \gamma_k (d_k - c_k)$ and $\Delta_k = c_k + e_k$.

Proof of Theorem 14. It suffices to show $\Delta M_k \in [-\Delta_k, \Delta_k]$ for each $k \in [N]$: then the claim follows by applying Lemma 21 with $S = \sum_{k \in [N]} (2\Delta_k)^2$. The following argument is written with an eye on
the upcoming proofs (where some modifications are needed). Note that $\Delta M_k = 0$ if $T \leq k - 1$ and $\Delta M_k = \Delta Y_k$ if $T \geq k$. So it is enough to prove that $|\Delta Y_k| \leq \Delta_k$ whenever $T \geq k$. For brevity, for $z \in \Lambda_k$ and $y = (y_{k+1}, \ldots, y_N) \in \prod_{k+1 \leq j \leq N} \Lambda_j$ we write $f_y(z)$ for $f(X_1, \ldots, X_{k-1}, z, y_{k+1}, \ldots, y_N)$. Note that

$$E(f(X) \mid F_{k-1}, X_k = a) = \sum_{y_{k+1}, \ldots, y_N} f_y(a) P(X_{k+1} = y_{k+1}, \ldots, X_N = y_N \mid F_{k-1}, X_k = a).$$

Defining $|\Delta Y_k(a, b)|$ via the next equation, since $X_1, \ldots, X_N$ are independent it follows that

$$|\Delta Y_k(a, b)| = |E(f(X) \mid F_{k-1}, X_k = a) - E(f(X) \mid F_{k-1}, X_k = b)| \leq \sum_{y_{k+1}, \ldots, y_N} |f_y(a) - f_y(b)| P(X_{k+1} = y_{k+1}, \ldots, X_N = y_N \mid F_{k-1}, X_k = a).$$  \hspace{1cm} (27)

By distinguishing between $X \in \Gamma$ and $X \not\in \Gamma$, each time applying (24) as appropriate, we infer

$$|\Delta Y_k(a, b)| \leq c_k P(X \in \Gamma \mid F_{k-1}, X_k = a) + d_k P(X \not\in \Gamma \mid F_{k-1}, X_k = a) = c_k + (d_k - c_k)P(X \not\in \Gamma \mid F_{k-1}, X_k = a).$$  \hspace{1cm} (28)

Recall that $B_{k-1}$ fails if $T \geq k$. Using $\sum_{y_k} P(X = y_k \mid F_{k-1}) = 1$ together with (28) and (24), for $T \geq k$ we deduce

$$|\Delta Y_k| = |E(f(X) \mid F_{k-1}) - E(f(X) \mid F_{k})| \leq \sum_{y_k} |\Delta Y_k(y_k, X_k)| P(X = y_k \mid F_{k-1}) \leq c_k + (d_k - c_k)P(X \not\in \Gamma \mid F_{k-1}) \leq c_k + \gamma_k (d_k - c_k) = \Delta_k.$$  \hspace{1cm} (29)

As explained, this completes the proof.

Here we could have used the classical Hoeffding–Azuma inequality \[20\] \[3\] since the proof yields (deterministic) bounds for each individual $\Delta M_k$. We decided to apply Lemma \[21\] since the forthcoming modifications needed for the ‘dynamic exposure’ of Section \[1.1.3\] do use its full strength, i.e., that accumulative estimates of the $\Delta M_k$ suffice.

**Proof of Remark 3** Following the approach of McDiarmid \[28\] we now modify the proof of Theorem 2 whenever $X_k$ takes only two values, say, $\Lambda_k = \{0,1\}$. We focus on the relevant case $T \geq k$, where $\Delta M_k = \Delta Y_k$. Define $L_k$ and $U_k$ as the minimum and maximum of $E(f(X) \mid F_{k-1}, X_k = z) - E(f(X) \mid F_{k})$ for $z \in \{0,1\}$. Clearly $L_k$ and $U_k$ are $F_{k-1}$-measurable and satisfy $L_k \leq \Delta Y_k \leq U_k$. The key observation is that, using $T \geq k$, there exists an $F_{k-1}$-measurable $\alpha \in \{0,1\}$ satisfying

$$\gamma_k \geq P(X \not\in \Gamma \mid F_{k-1}) \geq P(X \not\in \Gamma \mid F_{k-1}, X_k = 0).$$  \hspace{1cm} (30)

So, since $X_k \in \{0,1\}$ takes only two values, using (28) and (30) we infer for $T \geq k$ that

$$|U_k - L_k| \leq |\Delta Y_k(\alpha, 1 - \alpha)| \leq c_k + (d_k - c_k)P(X \not\in \Gamma \mid F_{k-1}, X_k = \alpha) \leq \Delta_k.$$  \hspace{1cm} (31)

This completes the proof (by applying Lemma \[21\] with $S = \sum_{k \in [N]} \Delta_k^2$).

In fact, Theorem 1 follows by a similar modification (here (28) implies $\max_{a,b} |\Delta Y_k(a, b)| \leq c_k$).
Proof of Theorem 4. In the proof of Theorem 2 we established $\Delta M_k \leq \Delta_k$ for every $k \in [N]$. In view of this it suffices to show $\operatorname{Var}(\Delta M_k \mid F_{k-1}) \leq (1 - p_k)p_k \Delta_k^2$ for each $k \in [N]$: then the claim follows by applying Lemma 22 with $V = \sum_{i \in [N]} (1 - p_k)p_k \Delta_i^2$ and $C = \max_{k \in [N]} \Delta_k$. Observe that $E(\Delta M_k \mid F_{k-1}) = 0$ implies $\operatorname{Var}(\Delta M_k \mid F_{k-1}) = E(\Delta M_k^2 \mid F_{k-1})$. Recall that $\Delta M_k = 0$ if $T \leq k - 1$ and $\Delta M_k = \Delta Y_k$ if $T \geq k$. Combining these facts it is enough to prove that $E(\Delta Y_k^2 \mid F_{k-1}) \leq (1 - p_k)p_k \Delta_k^2$ whenever $T \geq k$. Set $D_k = E(Y \mid F_{k-1}, X_k = 1) - E(Y \mid F_{k-1}, X_k = 0)$. Recalling $\Delta Y_k = E(Y \mid F_k) - E(Y \mid F_{k-1})$ we see that

$$|\Delta Y_k| \leq |D_k| \sum_{\beta \in \{0, 1\}} P(X_k = \beta \mid F_{k-1}) \mathbb{I}_{\{X_k = 1 - \beta\}}.$$  \hfill (32)

Arguing as in (30) and (31) we readily obtain $|D_k| \leq \Delta_k$ when $T \geq k$, and thus infer

$$\Delta Y_k^2 \leq \Delta_k^2 \sum_{\beta \in \{0, 1\}} P(X_k = \beta \mid F_{k-1})^2 \mathbb{I}_{\{X_k = 1 - \beta\}}.$$  

Using the independence of $X_1, \ldots, X_N$ it follows that for $T \geq k$ we have

$$E(\Delta Y_k^2 \mid F_{k-1}) \leq \Delta_k^2 \sum_{\beta \in \{0, 1\}} P(X_k = \beta)^2 P(X_k = 1 - \beta) = (1 - p_k)p_k \Delta_k^2,$$  \hfill (33)

where we used $P(X_k = 1) = p_k$ and $(1 - x)^2x + x^2(1 - x) = (1 - x)x$ for the last inequality. \hfill $\square$

Note that (32) implies $\Delta M_k \leq \max\{1 - p_k, p_k\} \cdot \Delta_k$, but the resulting minor improvement of $C$ usually has negligible effect.

Proof of Remark 3. In the more general situation where each $X_k$ takes values in a set $\Lambda_k$ and satisfies $\max_{\eta \in \Lambda_k} P(X_k = \eta) \geq 1 - p_k$, we first show that (7) holds after deleting $(1 - p_k)$ and replacing $c_k + e_k$ with $\tilde{\Delta}_k = c_k + e_k \cdot (1 - p_k)^{-1}$. With the proof of Theorem 4 in mind it suffices to show $\operatorname{Var}(\Delta Y_k \mid F_{k-1}) \leq p_k \tilde{\Delta}_k^2$ whenever $T \geq k$. For $\beta \in \Lambda_k$ satisfying $P(X_k = \beta) \geq 1 - p_k$ set $\tau_k = E(Y \mid F_{k-1}, X_k = \beta)$ and $D_k = E(Y \mid F_k) - \tau_k$. Note that, using the independence of $X_1, \ldots, X_N$, we have $P(D_k \neq 0 \mid F_{k-1}) \leq P(X_k \neq \beta) \leq p_k$. We claim that it suffices to show $|D_k| \leq \tilde{\Delta}_k$. Indeed, since $Y_{k-1}$ and $\tau_k$ are $F_{k-1}$ measurable, we have

$$\operatorname{Var}(\Delta Y_k \mid F_{k-1}) = \operatorname{Var}(D_k \mid F_{k-1}) \leq E(D_k^2 \mid F_{k-1}) \leq \tilde{\Delta}_k^2 P(D_k \neq 0 \mid F_{k-1}) \leq p_k \tilde{\Delta}_k^2.$$  

To bound $|D_k|$, first note that $T \geq k$ and independence of $X_1, \ldots, X_N$ yields

$$\gamma_k \geq P(X \notin \Gamma \mid F_{k-1}) \geq P(X \notin \Gamma \mid F_{k-1}, X_k = \beta) (1 - p_k).$$  \hfill (34)

So, using (28) and (41), for $T \geq k$ we infer

$$|D_k| = |E(Y \mid F_{k-1}, X_k = \beta) - E(Y \mid F_{k})| = |\Delta Y_k(\beta, X_k)| \leq c_k + (d_k - c_k)P(X \notin \Gamma \mid F_{k-1}, X_k = \beta) \leq c_k + \gamma_k (1 - p_k)^{-1} \cdot (d_k - c_k) = \tilde{\Delta}_k,$$  \hfill (35)

establishing the claim.

A similar argument shows that (19) holds after deleting $(1 - p_k)$. The point is that in Corollary 6 there is no ‘good’ event $\Gamma$. Consequently, when invoking (28) in (34) the standard line of reasoning (using (11) instead of (4)) yields $|D_k| \leq c_k$, and the claim follows. \hfill $\square$
Proof of Theorem 11. The crux is that (28) and (29) are at the heart of all previous proofs. In the following we exploit that both can be adapted when (10) instead of (4) holds: it suffices if 
\[ e_k = 2\gamma_k(d_k - c_k)q_k^{-1} \]

is used, where \( \min_{\gamma \in \Lambda_k} \mathbb{P}(X_k = \eta) \geq q_k \).

We start by modifying the proof of Theorem 2. Analogous to (34), if \( T \geq k \) then for all \( b \in \Lambda_k \) we have
\[ \gamma_k \geq \mathbb{P}(X \notin \Gamma | \mathcal{F}_{k-1}) \geq \mathbb{P}(X \notin \Gamma | \mathcal{F}_{k-1}, X_k = b)q_k. \]

The key point of (10) is that \( |f(x) - f(\tilde{x})| \leq c_k \) only holds if \( x, \tilde{x} \in \Gamma \). So, using (10) as appropriate, the corresponding variant of (28) for \( T \geq k \) is
\[ |\Delta Y_k(a, b)| \leq c_k + (d_k - c_k) \left[ \mathbb{P}(X \notin \Gamma | \mathcal{F}_{k-1}, X_k = a) + \mathbb{P}(X \notin \Gamma | \mathcal{F}_{k-1}, X_k = b) \right] \]
\[ \leq c_k + (d_k - c_k) \left[ \mathbb{P}(X \notin \Gamma | \mathcal{F}_{k-1}, X_k = a) + \gamma_k q_k^{-1} \right]. \]

Now, arguing as in (29) and using \( 1 + q_k^{-1} \leq 2q_k^{-1} \), we obtain a natural analogue for \( T \geq k \), namely
\[ |\Delta Y_k| \leq c_k + (d_k - c_k) \left[ \mathbb{P}(X \notin \Gamma | \mathcal{F}_{k-1}) + \gamma_k q_k^{-1} \right] \leq c_k + 2\gamma_k(d_k - c_k)q_k^{-1} = \Delta_k, \]
which establishes the claimed variant of Theorem 2.

In the proofs of Remark 3 and Theorem 4 we only need to adapt (31), and using (36) this follows by straightforward modifications. Similarly, in the proof of Remark 8 it suffices to modify (35), which is standard using (37) together with \( (1 - p_k)^{-1} + q_k^{-1} \leq 2q_k^{-1}(1 - p_k)^{-1}. \)

2.2.2 Some extensions

Proof of Remark 5. Note that \( f(X) \geq \mu + t \) is increasing (decreasing) if \( f(X) \) is increasing (decreasing). Furthermore, in view of (24) it is easy to check that \( \mathcal{B}_{k-1} \) is increasing (decreasing) if \( \Gamma \) is decreasing (increasing). Using the definition \( \mathcal{B} \) and the assumptions of Remark 5, it follows that \( f(X) \geq \mu + t \) and \( \neg \mathcal{B} \) are either both increasing or decreasing. So Harris’ inequality (19) yields
\[ \mathbb{P}(f(X) \geq \mu + t \text{ and } \neg \mathcal{B}) \geq \mathbb{P}(f(X) \geq \mu + t) \cdot \mathbb{P}(\neg \mathcal{B}), \]
which readily establishes (5).

Proof of Theorem 7. The basic idea is to use a truncation that maps every \( x_k \notin \Gamma_k \) to some fixed \( z_k \in \Gamma_k \). As before, we work with the sub-\( \sigma \)-fields \( \mathcal{F}_k \) generated by \( X_1, \ldots, X_k \). Recall that \( \mathcal{B} \) is defined via (24) and satisfies \( \neg \mathcal{B} \subseteq \mathcal{G} \). Since \( \mathbb{P}(f(X) \geq \mu + t \text{ and } \neg \mathcal{B}) \leq \mathbb{P}(X \in \Gamma) \) we may assume that \( \mathbb{P}(X \in \Gamma) > 0 \) and fix some \( z = (z_1, \ldots, z_N) \in \Gamma \subseteq \prod_{j \in [N]} \Gamma_j \). For \( x = (x_1, \ldots, x_N) \) we now define \( x^* = (x_1^*, \ldots, x_N^*) \) via
\[ x_k^* = \begin{cases} x_k & \text{if } x_k \in \Gamma_k, \\ z_k & \text{if } x_k \notin \Gamma_k. \end{cases} \]

The key properties of this construction are (a) that \( x \in \Gamma \) implies \( x^* = x \), and (b) that \( X^* = (X_1^*, \ldots, X_N^*) \) is a family of independent random variables. Set \( \mu^* = \mathbb{E}f(X^*) \). We have \( |\mu - \mu^*| \leq \mathbb{E}|f(X) - f(X^*)| \leq \delta \mathbb{P}(X \notin \Gamma) = \Delta \) (this is not best possible but keeps the formulas simple), and so \( \Gamma \subseteq \neg \mathcal{B} \) yields
\[ \mathbb{P}(f(X) \geq \mu + t + \Delta \text{ and } \neg \mathcal{B}) \leq \mathbb{P}(f(X^*) \geq \mu^* + t \text{ and } \neg \mathcal{B}). \]
Now we estimate the right hand side of (39) for $Y = f(X^*)$ as in the proof of Theorem 2 via (26), and there are only two minor differences. The first is that due to the projection (38) we have $X^* \in \prod_{j \in [N]} \Gamma_j$, so that the ‘refined’ Lipschitz coefficients $c_k$ and $d_k$ of Theorem 11 always apply. The second concerns the case distinction $X^* \in \Gamma$ and $X^* \notin \Gamma$. Here we use that $X^* \notin \Gamma$ implies $X \notin \Gamma$ pointwise, which yields $\mathbb{P}(X^* \notin \Gamma | F_{k-1}) \leq \mathbb{P}(X \notin \Gamma | F_{k-1})$. With this estimate the conclusion of (39) readily carries over, completing the proof.

Here we could have estimated $f(X^*) \geq \mu^* + t$ via Theorem 2 (with $\Lambda_k$ replaced by $\Gamma_k$), using that $\mathbb{P}(X^* \notin \Gamma) \leq \mathbb{P}(X \notin \Gamma)$. The advantage of our more pedestrian approach is that it uses the same ‘bad’ event $\mathcal{B}$ as all other proofs (by applying Theorem 2 it would depend on $z$).

**Proof of Remark 12.** The monotonicity property implies $\mu \geq \mu^*$ (writing $f(X) - f(X^*)$ as a difference sequence of coordinate changes), so $\Delta = 0$ suffices using $\mu + t \geq \mu^* + t$ in (39). Turning to the special case $\Gamma = \prod_{j \in [N]} \Gamma_j$, note that $\mathcal{B} = -\Gamma$ suffices to establish (39). Now, since $X^* = (X^*_1, \ldots, X^*_N)$ is a family of independent random variables with $X^*_k \in \Gamma_k$ satisfying (L), the claimed variant readily follows from Theorem 1.

### 2.2.3 Variants using dynamic exposure

In the following we briefly sketch how to modify the proofs of Sections 2.2.1 and 2.2.2 in case the variables are exposed in a dynamic order, which will eventually establish Theorem 13 and 14 (in contrast to [2] our approach is based on general martingale inequalities). Recall that the strategies introduced in Section 1.2.3 sequentially expose $X_{q_1}, X_{q_2}, \ldots$ with $q_i = q_i(X_{q_1}, \ldots, X_{q_{i-1}})$, where $f(X)$ is determined by $(X_1, \ldots, X_{q_k})$ with $k < N$ if $q_{k+1} = q_k$. For technical reasons we slightly modify these strategies so that (always) all variables are queried. More precisely, for the proof of Theorem 13 we set $\tilde{q}_k = q_k$ until $f(X)$ is determined by $(X_1, \ldots, X_{q_k})$; afterwards $\tilde{q}_{k+1}, \ldots, \tilde{q}_N$ equals the remaining ‘useless’ indices $[N] \setminus \{q_1, \ldots, q_k\}$ in ascending order, say (for the proof of Theorem 14) we simply use the fixed order $\tilde{q}_k = k$ for all $k \in [N])$. We consider an increasing sequence of sub-$\sigma$-fields, where $\mathcal{F}_k$ is generated by $X_{\tilde{q}_1}, \ldots, X_{\tilde{q}_k}$. Note that each index $\tilde{q}_k$ is $\mathcal{F}_{k-1}$-measurable. Furthermore, our modification ensures that the following two key properties hold: $X$ is $\mathcal{F}_N$-measurable (this is needed to apply $\Gamma$ since the value of $f(X)$ must not uniquely determine $X$), and conditional on $\mathcal{F}_{k-1}$ all $X_j$ with $j \notin \{\tilde{q}_1, \ldots, \tilde{q}_{k-1}\}$ are independent random variables. Define $R = \max_{Q \in \mathcal{Q}} |Q|$ and $\mathcal{B} = -\Gamma \cup \bigcup_{k \in [R]} \mathcal{B}_{k-1}$ (in case of the fixed order $\tilde{q}_k = k$ we set $R = N$, so $\mathcal{B}$ remains unchanged). Since the definition of $\mathcal{B}_{k-1}$ via (24) involves $\mathcal{F}_{k-1}$, it follows that $\mathcal{B}$ depends on the query strategy (unless the fixed order $\tilde{q}_k = k$ is used, as in the proof of Theorem 13).

With these changes in mind, all arguments of Sections 2.2.1 and 2.2.2 essentially carry over word by word, the only exception being the proof of Remark 5 (the monotonicity argument needs a fixed order such as $\tilde{q}_k = k$). The crucial observation is that for every variable $X_i$ not queried by the original strategy we know that its value will not change the outcome of $f(X)$. To be more formal, the key point is that whenever such a ‘useless’ variable is queried in step $i$ we have $\Delta M_i = 0$ (note that for $i > R$ this is always the case), i.e., the indices of these variables do not contribute to $S_N, V_N$ or $C_N$. Observe that due to the dynamic exposure we ‘only’ have a connection between $\gamma_k$ and the index $\tilde{q}_k$. We overcome this minor complication using the assumption that $\gamma_k = \gamma$ for all $k \in [N]$ (we may allow for different $\gamma_k$ if $\tilde{q}_k = k$ is used), also ensures that $\max_{Q \in \mathcal{Q}} \sum_{k \in Q} \gamma_k^{-1} = \sum_{k \in [R]} \gamma_k^{-1}$ holds (in case of $\tilde{q}_k = k$ the estimate (25) stays unchanged).

The remaining details for establishing Theorem 13 and 14 are rather straightforward: when invoking the martingale estimates we simply take the ‘worst case’ bounds for $S$, $V$ and $C$ over all
possible sets of queried indices \( Q \in \mathcal{Q} \) (where \( Q \) and \( \mathcal{Q} \) are as defined in Section 1.3); for example, using \( S = \max_{Q \in \mathcal{Q}} \sum_{k \in Q} (2\Delta_k)^2 \) in case of Theorem 2. It is this last step where the accumulative random bounds in Lemmas 21 and 22 are crucial (the behaviour of each individual \( \Delta M_k \) may vary significantly for different sample points due to the dynamic order in which the variables are queried).

### 2.2.4 Variants using the general Lipschitz condition

Finally, we discuss how to modify the proofs in Section 2.2.1 when the independence assumption is replaced by (GL). We first claim that \( \rho_k = \rho_k(\Sigma_a, \Sigma_b) : \Sigma_a \to \Sigma_b \) is injective with equality in (13), i.e., satisfies

\[
\mathbb{P}(X = x \mid X \in \Sigma_a) = \mathbb{P}(X = \rho_k(x) \mid X \in \Sigma_b).
\]

Indeed, using (13) and that \( \rho_k \) is injective it follows that

\[
\sum_{x \in \Sigma_a} \mathbb{P}(X = x \mid X \in \Sigma_a) \leq \sum_{x \in \Sigma_a} \mathbb{P}(X = \rho_k(x) \mid X \in \Sigma_b) \leq \sum_{x \in \Sigma_b} \mathbb{P}(X = x \mid X \in \Sigma_b).
\]

Noting that \( \sum_{x \in \Sigma_b} \mathbb{P}(X = x \mid X \in \Sigma_z) = 1 \) for \( z \in \{a, b\} \) with \( |\Sigma_z| > 0 \) we infer that all inequalities are in fact equalities, which establishes (40). Since every \( x \in \Sigma_z \) satisfies \( \mathbb{P}(X = x) > 0 \) it also follows that \( \rho_k \) must be a bijection, as claimed.

In preparation of our forthcoming arguments we now relate the definitions used in the proofs of Section 2.2.1 with those occurring in (GL). Analogous to \( \Sigma_z \), given any possible sequence of outcomes \( a_1, \ldots, a_{k-1} \) of \( X_1, \ldots, X_{k-1} \) define \( \Sigma \) as the set of all \( x = (a_1, \ldots, a_{k-1}, x_k, \ldots, x_N) \in \prod_{j \in [N]} \Lambda_j \) with \( \mathbb{P}(X = x) > 0 \). Recall that \( \mathcal{F}_k \) is the increasing sequence of sub-\( \sigma \)-fields generated by \( X_1, \ldots, X_k \). The key point is that \( (\mathcal{F}_k)_{0 \leq k \leq N} \) naturally corresponds to an increasing sequence of partitions of the sample space, where two points belong to the same part if and only if they agree on the first \( k \) coordinates. For example, for \( \omega \in \Sigma \) we have

\[
\mathbb{E}(\cdot \mid \mathcal{F}_{k-1}, X_k = z)(\omega) = \mathbb{E}(\cdot \mid X \in \Sigma_z) \quad \text{and} \quad \mathbb{P}(\cdot \mid \mathcal{F}_{k-1}, X_k = z)(\omega) = \mathbb{P}(\cdot \mid X \in \Sigma_z). \quad (41)
\]

**Proof of Theorem 2.** We modify the proof of Theorem 2 where independence is only used to establish (27). Using (13) and that the bijection \( \rho_k : \Sigma_a \to \Sigma_b \) satisfies (40), we obtain

\[
|\Delta Y_k(a, b)| = |\mathbb{E}(f(X) \mid X \in \Sigma_a) - \mathbb{E}(f(X) \mid X \in \Sigma_b)|
\]

\[
= |\sum_{x \in \Sigma_a} f(x)\mathbb{P}(X = x \mid X \in \Sigma_a) - \sum_{x \in \Sigma_b} f(x)\mathbb{P}(X = x \mid X \in \Sigma_b)|
\]

\[
\leq \sum_{x \in \Sigma_a} |f(x) - f(\rho_k(x))|\mathbb{P}(X = x \mid X \in \Sigma_a),
\]

which is the natural analogue of (27). The remainder of the argument carries over with minor modifications. Indeed, proceeding as in (28) (applying (12) instead of (4)) and then appealing to (41), we infer

\[
|\Delta Y_k(a, b)| \leq c_k + (d_k - c_k)\mathbb{P}(X \notin \Gamma \mid X \in \mathcal{F}_{k-1}, X_k = a). \quad (42)
\]

Now, by arguing as in (29), when \( T \geq k \) holds we also have

\[
|\Delta Y_k| \leq c_k + (d_k - c_k)\mathbb{P}(X \notin \Gamma \mid X \in \mathcal{F}_{k-1}) \leq \Delta_k,
\]

completing the proof. □
Remark 10 follows by similar reasoning (noting that the proof of Remark 3 carries over and that (12) equals (14) after replacing $(d_k - c_k)$ with $r_k$).

Proof of Theorem 7. We modify the proof of Remark 3 by picking (some) $\mathcal{F}_{k-1}$-measurable $\beta \in \Lambda_k$ maximizing $\mathbb{P}(X_k = \beta \mid \mathcal{F}_{k-1})$. By assumption we have $\mathbb{P}(X_k = \beta \mid \mathcal{F}_{k-1}) \geq 1 - p_k$, which in turn yields $\mathbb{P}(D_k \neq 0 \mid \mathcal{F}_{k-1}) \leq \mathbb{P}(X_k \neq \beta \mid \mathcal{F}_{k-1}) \leq p_k$. Noting that all remaining applications of independence are already covered by (12) and (13), this completes the proof.  

Proof of Theorem 8. With the above modifications in mind the proof of Theorem 9 carries over word by word, which establishes the first part of the claim. Turning to the second part, our earlier discussion shows that for $\Sigma_2, \Sigma_\eta \subseteq \Sigma$ with $|\Sigma_2|, |\Sigma_\eta| > 0$ there is a bijection $\rho_k : \Sigma_2 \to \Sigma_\eta$. So, since all possible outcomes occur with the same probability, we obtain $\mathbb{P}(X \in \Sigma_2) = \mathbb{P}(X \in \Sigma_\eta)$. For $\eta \in \Lambda_k$ satisfying $|\Sigma_\eta| > 0$ it follows that

$$
\frac{1}{\mathbb{P}(X_k = \eta \mid X \in \Sigma)} = \frac{\mathbb{P}(X \in \Sigma)}{\mathbb{P}(X \in \Sigma_\eta)} = \sum_{z \in \Lambda_k} \frac{\mathbb{P}(X \in \Sigma_z)}{\mathbb{P}(X \in \Sigma_\eta)} \leq |\Lambda_k|.
$$

We deduce that $\min_{\eta \in \Lambda_k} \mathbb{P}(X_k = \eta \mid X_1, \ldots, X_{i-1}) \geq |\Lambda_k|^{-1}$, so $q_k \leq |\Lambda_k|^{-1}$ suffices.  

3 Final number of edges in the reverse $H$-free process

In our analysis of the reverse $H$-free process we use several equivalent definitions (with respect to the final graph). Recall that, starting with the complete graph on vertex set $[n]$, in each step an edge is removed, chosen uniformly at random from all edges contained in a copy of $H$. As in [16, 27], a moment’s thought reveals that we may instead traverse all $\binom{n}{2}$ edges in random order, each time removing the current edge if and only if it is contained in a copy of $H$ in the evolving graph. As observed by Erdős, Suen and Winkler [16], after considering $e_{i,1}, \ldots, e_{i+1}$ the decision whether $e_i$ is removed depends only on the later edges $e_{i-1}, \ldots, e_1$ (all other ‘surviving’ ones are by construction not contained in a copy of $H$). This allows us to consider the edges in reverse order, where $e_i$ is added if and only if it does not complete a copy of $H$ together with $e_1, \ldots, e_{i-1}$ (it does not matter whether these were added or not). Given a random permutation, we denote the corresponding random graph process after $i$ steps by $G_{n,i}(H) \subseteq G_{n,i}$, where $G_{n,i}$ is the uniform random graph with $n$ vertices and $i$ edges.

For technical reasons it will be convenient to also consider a continuous variant of the above process, where each edge is independently assigned a uniform birth time $B_e \in [0,1]$; the edges are then traversed in ascending order of their birth times (which are all distinct with probability one). The resulting process that considers only those edges with $B_e \leq p$ is denoted by $G_{n,p}(H) \subseteq G_{n,p}$. So for $p = 1$ all edges are traversed in random order, and it follows that

$$
G_{n,(\binom{n}{2})}(H) = G_{n,1}(H).
$$

Conditioned on $B_e = q$, the decision whether $e$ is added only depends on the edges $f$ with $B_f \leq q$, which have the same distribution as $G_{n,q}$. As noted by Makai [27], this allows for the use of classical random graph theory when estimating the probability that an edge is added to the evolving graph. Recall that $m_q(H) = d_2(H)$ for 2-balanced graphs $H$. For

$$
m = n^{2-1/m_2(H)}(\log n)^2 \quad \text{and} \quad p = n^{-1/m_2(H)}(\log n)^2
$$

20
the next lemma follows from the results of Spencer \cite{Spencer} mentioned in Section 1.2.2. Note that in $G_{n,p}$ every pair of vertices is expected to have $\Theta((\log n)^{(2e_H-1)})$ ‘extensions’ to copies of $H$.

**Lemma 26.** Let $H$ be a 2-balanced graph. Let $\mathcal{D}$ (and $\mathcal{I}$) denote the event that for every pair $xy$ of vertices the following holds: after adding the edge $xy$ there are at most $\Psi_H = (\log n)^{2e_H}$ copies (is at least one copy) of $H$ containing the edge $xy$. For every $c > 0$ we have $\Pr(G_{n,m} \in \mathcal{D} \cap \mathcal{I}) \geq 1 - n^{-c}$ for $n \geq n_0(c, H)$. \hfill $\square$

The point is that whenever $\mathcal{I}$ holds no further edges are added. This allows us to couple both variants of the reverse $H$-free process such that they agree with very high probability after considering only $m$ edges. So for our purposes they are interchangeable, and we obtain the corresponding formal statement by combining Lemma 26 with \cite{Makai}.

**Lemma 27.** Let $H$ be a 2-balanced graph. There is a coupling such that for every $c > 0$ we have

$$G_{n,m}(H) = G_{n,\frac{m}{n}}(H) = G_{n,1}(H)$$

with probability at least $1 - n^{-c}$ for $n \geq n_0(c, H)$. \hfill $\square$

Turning to the number of edges in $G_{n,m}(H)$, which we denote by $e(G_{n,m}(H))$, recall that each $e_i$ is added if and only if it does not complete a copy of $H$ together with $e_1, \ldots, e_{i-1}$. So one edge can, in the worst case, influence the decisions of up to $O(\min\{m, n^{e_H-2}\})$ edges (whether they are added or not); however, on the ‘typical’ event $\mathcal{D}$ of Lemma 26 this is limited to at most $e_H \cdot \Psi_H = O((\log n)^{2e_H})$ edges. For this reason the standard bounded differences inequality fails to give useful bounds (due to large worst case factors). Our argument is inspired by Makai \cite{Makai}, who proved asymptotically matching bounds for the class of strictly 2-balanced graphs (the case $H = K_3$ is due to Erdős, Suen and Winkler \cite{ErdősSuenWinkler}). In fact, here we determine the correct order of magnitude for all graphs.

**Theorem 28.** Let $H$ be a 2-balanced graph. For every $c > 0$ and $n \geq n_0(c, H)$ we have

$$\Pr(|e(G_{n,m}(H)) - \mathbb{E}e(G_{n,m}(H))| \geq \sqrt{m(\log n)^{3e_H}}) \leq n^{-c}. \quad (45)$$

**Proof.** Lemma 26 implies that $G_{n,m} \in \mathcal{D}$ holds with probability at least $1 - n^{-(2c+6)}$. Note that the random sequence of edges $\xi = (e_1, \ldots, e_m)$ corresponds to the (uniform) random graph process $(G_{n,i})_{0 \leq i \leq m}$ and uniquely determines $f(\xi) = e(G_{n,m}(H))$. The crucial observation is that whenever $G_{n,m}, \tilde{G}_{n,m} \in \mathcal{D}$ have edge sequences $\xi, \tilde{\xi}$ that differ only in one edge (i.e., $e_j \neq \tilde{e}_j$) or the order of two edges (i.e., $e_j = \tilde{e}_k$ and $e_k = \tilde{e}_j$), then our earlier observations imply $|e(G_{n,m}(H)) - e(\tilde{G}_{n,m}(H))| \leq 2e_H(\log n)^{2e_H} = \Delta$. The point is that by the discussion of Section 1.1.4 this is exactly the condition that needs to be checked in order to apply Theorem 13 (with the two-sided Lipschitz condition \cite{ErdősSuenWinkler} of Theorem 13 using $N = m$, the ‘good’ event $\Gamma = \mathcal{D}$, Lipschitz coefficients $c_k = \Delta$, $d_k = n^2$ and the ‘two-sided parameter’ $q_k = n^{-2}$. For the ‘compensation factor’ $\gamma_k = n^{-4}$ we have $e_k \leq 2\gamma_k d_k q_k^{-1} \leq 2$, $c_k = \Theta((\log n)^{2e_H})$ and $\sum_k \gamma_k^{-1} \leq n^2$. So, using (6) and (5) we deduce that the left hand side of (45) is at most $e^{-\Omega((\log n)^{2e_H})} + n^{-2c} \leq n^{-c}$. \hfill $\square$

To establish Theorem 19 it remains to bound the expected final number of edges up to constant factors. Our argument is inspired by Makai \cite{Makai}, who proved asymptotically matching bounds in \cite{ErdősSuenWinkler} for the class of strictly 2-balanced graphs (the case $H = K_3$ is due to Erdős, Suen and Winkler \cite{ErdősSuenWinkler}). In fact, here we determine the correct order of magnitude for all graphs.
Theorem 29. Let $H$ be a graph with $e_H \geq 1$. There are $a, A > 0$ such that
\[ \lceil an^{2-1/m_2(H)} \rceil \leq \mathbb{E}e(G_{n,1}(H)) \leq An^{2-1/m_2(H)} \tag{46} \]
for $n \geq n_0(H)$, where the floor function is only needed when $e_H = 1$.

Proof. When $e_H = 1$ we have $\mathbb{E}e(G_{n,1}(H)) = 0$ and $m_2(H) = 1/2$, so \[ \text{(46)} \]
holds with, say, $a = A = 1/2$. Henceforth we assume $e_H \geq 2$. Define $Z_e = Z_e(H)$ as the event that the edge $e$ is contained $G_{n,1}(H)$. Let $Y_{e,H,q}$ count the number of copies of $H$ in $G_{n,q}$ (the graph obtained by inserting $e$ into $G_{n,q}$ if it is not already present). Recall that, conditioned on $B_e = q$, only edges $f$ with $B_f \leq q$ are relevant for $Z_e$, so $e$ is added if and only if $Y_{e,H,q} = 0$. Hence for $q \in [0,1]$ we have
\[ \mathbb{P}(Z_e \mid B_e = q) = \mathbb{P}(Y_{e,H,q} = 0). \tag{47} \]

For the lower bound in \[ \text{(46)} \] fix $F \subseteq H$ with $d_2(F) = m_2(H)$ that satisfies $e_F \geq 2$ (this choice is possible as $e_H \geq 2$). Given $e$ there are at most $Dn^{v_F-2}$ extensions to $F$ for some $D = D(F) > 0$, so whenever $q \leq n^{-1/m_2(H)}$ holds monotonicity and Harris’ inequality \[ \text{(19)} \] yield
\[ \mathbb{P}(Y_{e,H,q} = 0) \geq \mathbb{P}(Y_{e,F,q} = 0) \geq (1 - q^{e_F-1})Dn^{v_F-2} \geq e^{-2Dn^{v_F-2}q^{e_F-1}} \geq e^{-2D}. \tag{48} \]
Together with \[ \text{(17)} \] we obtain $\mathbb{P}(Z_e) = \mathbb{E}\mathbb{P}(Z_e \mid B_e = q) \geq n^{-1/m_2(H)} \cdot e^{-2D}$, and the lower bound in \[ \text{(46)} \] now follows by linearity of expectation.

Turning to the upper bound in \[ \text{(46)} \], consider $q = \lambda n^{-1/m_2(H)}$ with $1 \leq \lambda \leq n^{-1/m_2(H)}$. We apply Janson’s inequality to $Y_{e,H,q}$, which counts the number of extensions of $e$ to $H$ (viewed as subgraphs these do not contain the edge $e$). Note that $e_H \geq 2$, $\lambda \geq 1$ and $m_2(H) \geq (e_H - 1)/(v_H - 2)$ imply
\[ \mu = \mathbb{E}Y_{e,H,q} = \Theta(n^{v_H-2}q^{e_H-1}) = n^{v_H-2}(e_H-1/m_2(H))\Theta(e_H-1) = \Omega(\lambda). \]
Define $\mathcal{G}$ as the set of all proper subgraphs graphs $G \subset H$ with $e_G \geq 2$. Considering all possible ‘overlaps’ of extensions of $e$ to $H$ (analogous to the textbook proof of the small subgraphs theorem), the $\Delta$ term of Janson’s inequality satisfies
\[ \Delta \leq O(n^{v_H-2}q^{e_H-1}) \cdot \sum_{G \in \mathcal{G}} O(n^{v_H-v_G}q^{e_H-e_G}) = O(\mu^2) \sum_{G \in \mathcal{G}} n^{-(v_G-2)}q^{-e_G-1} = O(\mu^2) \]
where the last inequality follows from $e_G \geq 2$, $\lambda \geq 1$ and $m_2(H) \geq (e_G - 1)/(v_G - 2)$. So, using $\mu/\lambda = \Omega(1)$ we infer $\mu + \Delta = O(\mu^2/\lambda)$ and thus $\mu^2/(\mu + 2\Delta) = \Omega(\lambda) = \Omega(n^{1/m_2(H)}q)$. Applying Janson’s inequality (see e.g. Theorem 2.18 in \[ \text{[22]} \]) we have $\mathbb{P}(Y_{e,H,q} = 0) \leq \exp \left( -Cn^{1/m_2(H)}q \right)$ for $C = C(H) > 0$. Combining this with \[ \text{(17)} \] when $q \geq n^{-1/m_2(H)}$ and the trivial bound $\mathbb{P}(Z_e \mid B_e = q) \leq 1$ otherwise, for $A = 1 + e^{-C}/C$ we obtain
\[ \mathbb{P}(Z_e) = \mathbb{E}\mathbb{P}(Z_e \mid B_e = q) \leq n^{-1/m_2(H)} + \int_{n^{-1/m_2(H)}}^{1} \exp \left( -Cn^{1/m_2(H)}q \right) dq \leq An^{-1/m_2(H)} \tag{49} \]
Linearity of expectation now yields the upper bound in \[ \text{(46)} \].
Our arguments partially generalize to arbitrary graphs, which we shall now briefly discuss. In this case Lemma 26 remains true if we modify $\mathcal{D}$ to at most, say, $\Psi_H = (\log n)n^{v_H-2}2^{r_H-1}$ copies, and so the coupling of Lemma 27 carries over (it only uses $\mathcal{I}$). With (44) in mind, Theorem 28 shows that the expected final number of edges is $\mu = \Theta(n^{2-1/m_2(H)})$. Adjusting the proof of Theorem 28 with $c_F = 2\epsilon_H\Psi_H$, a short calculation shows that we obtain concentration on an interval of length $\mu n^{-\gamma}$ with $\gamma = \gamma(H) > 0$ whenever

$$v_H \geq 4 \text{ and } m_2(H) < (2v_H - 3)/(2v_H - 6) \text{ or } v_H = 3 \text{ and } e_H \geq 2. \tag{50}$$

Perhaps surprisingly, this condition is satisfied by standard examples of ‘unbalanced’ graphs such as a clique $K_r$ with an extra edge hanging off.

The proofs in this section also extend with minor modifications to the more general reverse $\mathcal{H}$-free process considered in Theorem 20. In this case the ‘inverted’ processes $G_{n,m}(\mathcal{H})$ and $G_{n,p}(\mathcal{H})$ are defined in analogous ways, where an edge is added only when it closes no copy of some $F \in \mathcal{H}$. We need to modify $\mathcal{D}$ of Lemma 26 so that for all $F \in \mathcal{H}$ it ensures at most $\Psi_F = \max\{(\log n)n^{v_F-2}2^{r_F-1}, (\log n)^2\}$ copies, whereas the corresponding $\mathcal{I}$ only applies to the distinguished graph $H$ with $m_2(H) = d_2(H) = \min_{F \in \mathcal{H}} d_2(F)$. As before, once $\mathcal{I}$ holds no more edges are added. With this in mind the coupling of Lemma 27 as well as the concentration result of Theorem 28 carry over in a straightforward way (noting that $d_2(H) \leq d_2(F)$ implies $\Psi_F \leq (\log n)^2$ for all $F \in \mathcal{H}$). Turning to the expected final number of edges, for the lower bound of Theorem 29 we avoid all $F \in \mathcal{H}$ simultaneously. The resulting modification of (18) works for $q \leq n^{-1/m_2(H)}$ since $d_2(H) \leq d_2(F)$ implies $n^{v_F-2}q^{r_F-1} \leq 1$. For the upper bound it suffices to just avoid the distinguished 2-balanced graph $H$, so we may reuse the estimates of (49) to establish Theorem 20.

Finally, note that every edge added by $G_{n,m}(\mathcal{H})$ is also added by the $\mathcal{H}$-free process defined in Section 1.2.3 (where $e_i$ is added if and only if it does not complete a copy of $H$ together with the added edges among $e_{1}, \ldots, e_{i-1}$). It follows from Theorem 29 that the expected final number of edges in the $\mathcal{H}$-free process is at least $\Omega(n^{2-1/m_2(H)})$ for any graph $H$, which improves the $\Omega(n^{2-1/d_2(H)})$ bound resulting from the deletion argument of Osthus and Taraz [31]. In fact, if the technical conditions in (50) are satisfied our earlier discussion implies that this lower bound also holds with probability tending to one (not only in expectation), which for ‘unbalanced’ graphs with $m_2(H) > d_2(H)$ does not follow from Theorem 1 in [31].

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