Which is the least complex explanation?
Abduction and complexity∗

Fernando Soler-Toscano
Grupo de Lógica, Lenguaje e Información
University of Seville
fsoler@us.es

1 How an abductive problem arises?

What is abductive reasoning and when is it used? First, it is a kind of inference. In a broad sense, a logical inference is any operation that, starting with some information, allows us to obtain some other information. People are continually doing inference. For example, I cannot remember which of the two keys in my pocket opens the door of my house. I try with the first one but it does not open. So I conclude that it should be the other. In this case, the inference starts with some data (premises): one of the two keys opens the door, but the first I tried did not open. I reach some new information (conclusion): it is the second key.

Inference (or reasoning) does not always follow the same way. In the example, if I know that one of the keys opens my house and I cannot open with the first key, it is necessary that the second opens. When this is the case (that is, the conclusion follows necessarily from the premises), then we are facing a deductive inference. Given that the conclusion is a necessary consequence of the premises, there is no doubt that the conclusion is true, whenever premises are all true: the door must be opened with the second key.

But there are many contexts in which we cannot apply deductive reasoning. Sometimes, we use it but we become surprised by the outcome. What happens if finally the second key does not open the door? This kind of surprise was studied by the philosopher Charles S. Peirce as the starting point of abductive reasoning:

∗Draft of the paper published in Max A. Freund, Max Fernandez de Castro and Marco Ruffino (eds.), Logic and Philosophy of Logic. Recent Trends in Latin America and Spain. College Publicacions. Studies in Logic, Vol. 78, 2018, pp. 100–116.
The surprising fact, \( C \), is observed;
But if \( A \) were true, \( C \) would be a matter of course,
Hence, there is reason to suspect that \( A \) is true. (CP 5.189, 1903).

Peirce mentions a surprising fact, \( C \), that in our example is that none of the keys opens the door, despite we strongly believed that one of them was the right one. This is an abductive problem: a surprising fact that we cannot explain with our current knowledge. Then, we search for a solution, an explanation \( A \) that would stop \( C \) from being surprising. To discover this \( A \) we put into play our knowledge about how things usually happen. For example, we may realise that maybe someone locked the door from the inside. Also, if someone usually locks the door from the inside, then the explanation becomes stronger and, as Peirce says, there is reason to suspect that it is true.

Not all abductive problems are identical. A common distinction is between novel and anomalous abductive problems [3]. A novel abductive problem is produced when the surprise produced by \( C \) is coherent with our previous information. Contrary, in an anomalous abductive problem we previously thought that \( C \) could not be the case, and the surprise contradicts our previous belief. The example of the key that does not open the door is a case of anomalous abductive problem.

To clarify the notions, we now offer some informal definitions of the concepts that are commonly used in the logical study of abductive reasoning [13]. Suppose that the symbol \( \vdash \) represents our reasoning ability, so that \( A, B \vdash C \) means that from premises \( A \) and \( B \) it is possible to infer \( C \) by a necessary inference (deduction, as explained above). The negated symbol, as in \( A, B \nvdash C \), means that the conclusion \( C \) cannot be obtained from premises \( A \) and \( B \). Also, consider that \( \Theta \) is a set of sentences (logical propositions) representing our knowledge (that I have two keys, one of them is the right one, etc.) and \( \varphi \) is the surprising fact (none of the keys opens the door). Then, in a novel abductive problem \((\Theta, \varphi)\) the following holds:

1. \( \Theta \nvdash \varphi \)
2. \( \Theta \nvdash \neg \varphi \)

The first condition is necessary for \( \varphi \) to be surprising: it does not follow from our previous knowledge. The second condition is specific for a novel abductive problem: the negation (the opposite) of \( \varphi \), represented by \( \neg \varphi \), does not follow from our knowledge \( \Theta \). So, in a novel abductive problem our previous knowledge was not useful to predict either the surprising fact \( \varphi \) or the contrary \( \neg \varphi \).

In an anomalous abductive problem \((\Theta, \varphi)\), the conditions that are satisfied are the following:

1. \( \Theta \nvdash \varphi \)
2. \( \Theta \vdash \neg \varphi \)

Now, although the first condition is the same, the second is different: our previous knowledge \( \Theta \) predicted \( \neg \varphi \), the negation of the surprising fact \( \varphi \).

## 2 How an abductive problem is solved?

Logicians say that in deductive reasoning the conclusion is contained in the premises. This means that the information given by the conclusion is implied by the information in the premises. For example, the information that one of my keys open the door but the first does not open contains the information that the second key will open. But this does not happen in abductive reasoning: the information that the door is locked from the inside is not implied by the information of my keys not opening the door. So, abductive reasoning raises conclusions that introduce new information not present in the premises. Because of this, abduction requires a dose of creativity to propose the solutions. Moreover, there are frequently several different solutions, and the ability to select the best of them is required. We will return later to this issue.

We have distinguished two kinds of abductive problems. Now we will comment the kinds of abductive solutions that are usually considered. We denoted above by \((\Theta, \varphi)\) an abductive problem that arises when our knowledge is represented by \( \Theta \) and the surprising fact is \( \varphi \). The solution to this problem is given by some information \( \alpha \) such that, together with the previous knowledge we had, allows us to infer \( \varphi \), logically represented by

\[
\Theta, \alpha \vdash \varphi
\]

This is the minimal condition for an abductive solution \( \alpha \) to solve the problem \((\Theta, \varphi)\). Atocha Aliseda \[3\] calls *plain* to those abductive solutions satisfying this requirement.

There are other very interesting kinds of abductive solutions. For example, *consistent* solutions satisfy the additional condition of being coherent with our previous knowledge. It is formally represented by

\[
\Theta, \alpha \not\vdash \bot,
\]

where the symbol \( \bot \) represents any contradiction. It is important that our abductive solutions are consistent. Possibly, we will not know whether the abductive solution \( \alpha \) is true, but usually, if it is inconsistent with our previous knowledge, we have reason to discard it.

Finally, *explanatory* abductive solutions are those satisfying

\[
\alpha \not\vdash \varphi,
\]
that is, the surprising fact \( \varphi \) cannot be inferred with \( \alpha \) alone without using the knowledge given by \( \Theta \). This is to avoid self-contained explanations: the key idea behind this criterion is that a good abductive explanation offers the missing piece to solve a certain puzzle, but all the other pieces were previously given.

When an abductive solution satisfies the three conditions above, we call it a consistent explanatory solution. To avoid useless or trivial solutions (the key does not open the door because it does not open it), it is frequent to focus on consistent explanatory abduction.

The logical study of abductive reasoning has been receiving a notable attention for several years, and many calculi have been proposed for abduction in different logical systems \[6, 13, 16\]. Now, we are not interested in offering a specific calculus for a particular logic, but in looking to an old problem in abductive reasoning: the selection of the best hypothesis. Which is the best abductive solution? First, we will proceed conceptually, by introducing some notions from information theory. It will be in Section \[4\] when, as an example, we will apply the introduced idea in the context of epistemic logic.

3 Which is the least complex explanation?

It may happen that for a certain abductive problem there are several possible explanations, not all of them mutually compatible. For example, to explain why the key does not open the door, we have proposed that someone locked it from the inside. But it could also happen that the key or the door lock are broken, or that someone changed the lock while we were outside, or made a joke, etc. It is necessary to select one of the many possible explanations, because it cannot be that case that all of them happened, it is enough just one of them to explain that we cannot open the door with our key. What explanation is selected and which criteria are used to select it? This is the well-known problem of the selection of abductive hypotheses \[17\].

Moreover, different to deductive reasoning, abductive conclusions (selected solutions) are not necessary true. It is easy to observe that, despite we think that someone locked the door from the inside, it may have not been the case, and that in fact the lock is broken. So, we usually have to replace an explanation with another one, when we come to know that the originally chosen is false.

Several criteria have been proposed to solve the problem of the selection of abductive hypotheses. A common one is minimality, that prefers explanations assuming fewer pieces of new information. So, if I can solve a certain abductive problem both assuming \( \alpha_1 \) or \( \alpha_2 \), it is possible that I can also solve it by simultaneously assuming \( \alpha_1 \) and \( \alpha_2 \), or maybe \( \alpha_1 \) and a certain \( \beta \), but we will usually discard those options because they are not the simplest possible ones. In logical terms, if \( A \vdash B \) and both \( A \) and \( B \) can solve a certain abductive
problem, we prefer $B$, given that $A$ is at least equally strong than $B$, and maybe stronger, in the sense of assuming more information.

Frequently, the minimality criterion is not enough to select the best explanation. Which one is simpler: to think that someone locked the door from the inside, or that they spent a joke by changing the door lock?

Are there criteria that can help us to select the simplest explanation in a broad spectrum of abductive problems? To give an (affirmative) answer to this question we will move to a field in theoretical computer science: Algorithmic Information Theory (AIT), which is due to the works of Ray Solomonoff [14], [20], Andrés Kolmogorov [7], Leonid Levin [9] and Gregory Chaitin [5].

A central notion in AIT is the measure known as Kolmogorov complexity, or *algorithmic complexity*. To understand it, let us compare these two sequences of 0s and 1s:

```
0101010101010101010101010101010101010101
0001101000100110111101010010111011100100
```

If we were asked which one of them is simpler, we will answer that the first one. Why? It is built up from 20 repetitions of the pattern 01. The second sequence is a random string. The difference between the regularity of the first sequence and the randomness of the second one is related with one property: the first sequence has a much shorter description than the second. The first sequence can be described as ‘twenty repetitions of 01’, while the second one can be hardly described with a description shorter than itself.

The idea behind the notion of Kolmogorov complexity is that if some object $O$ can be fully described with $n$ bits (*bit: binary digit, information unit*), then $O$ does not contain more information. So the shorter description of the object $O$ indicates how much information is contained in $O$. We would like to measure in this way the complexity of abductive solutions, and introduce an informational minimality criterion: we select the least complex explanation, that is, the least informative one. But we will look at how this complexity measure is quantified.

Kolmogorov uses the concept of universal Turing machine [21]. An universal Turing machine (UTM) $M$ is a programmable device capable of implementing any algorithm. The important point for us now is that the machine $M$, similar to our computers, takes a program $p$, runs it and eventually (if the computation stops) produces a certain output $o$. To indicate that $o$ is the output produced by UTM $M$ with program $p$ we write $M(p) = o$. Then, for a certain string of characters $s$, we define its Kolmogorov complexity, $K_M(s)$, as

$$K_M(s) = \min \{ l(p) \mid M(p) = s, \ p \text{ is a program} \}$$

where $l(p)$ is the length in bits of the program $p$. That is, $K_M(s)$ is equal to the size of the shorter program producing $s$ in the UTM $M$. The subindex $M$ in $K_M(s)$ means that its value depends on the choice of UTM, because
not all of them interpret the programs in the same way, despite all having the same computational power. If we choose another machine \( M' \) instead of \( M \), it can happen that \( K_{M'}(s) \) is pretty different to \( K_M(s) \). However, these bad news are only relative, given that the Invariance Theorem guarantees that the difference between \( K_M(s) \) and \( K_{M'}(s) \) is always lower than a certain constant not depending on \( s \), but on \( M \) and \( M' \). So, as we face more and more complex strings, it is less relevant the choice of UTM. Then, we can simply write \( K(s) \) to denote the Kolmogórov complexity of \( s \).

The use of Turing machines and programs allows us to set an encoding to describe any computable (that is, that can be produced by some algorithm) object \( O \). Then, for the first binary sequence above, the shortest description will not be ‘twenty repetitions of 01’ (26 characters) but the shortest program producing that string.

To approach the relation between algorithmic complexity and abductive reasoning, we can look at the work of Ray Solomonoff, that conceives algorithmic complexity as a tool to create a model that explains all the regularities in the observed universe (see [19], Section 3.2). The idea of Solomonoff is ambitious, but it is in line with the common postulates of the inference to the best explanation [11]. A theory can be conceived as a set of laws (axioms, hypotheses, etc., depending on the kind of theory) trying to give account of a set of observations in a given context. The laws in the theory try to explain the regularities in those observations. So, what is the best theory? From Solomonoff’s point of view, the best theory is the most compact one, that describing the highest number of observations (the most general one) with the fewest number of postulates (the most elegant from a logical point of view). It is the condition for the lowest algorithmic complexity. The best theory is then conceived as the shortest program generating the observations that we want to explain. Such generation consist of the inferential mechanism underlying the postulates of the theory. If it were a set of logical rules, the execution of the program given by the theory is equivalent to what logicians denote by the deductive closure of the theory: the set of all consequences that can be deduced from the theory axioms. Such execution does not always finish in a finite number of steps, because logical closures frequently (always in classical logic) are infinite sets, but usually there are procedures (in decidable logical systems) that, in a finite number of steps, check whether a certain formula belongs to such closure.

As we can see, the algorithmic complexity measure \( K(s) \) can be used to determine which is the best theory within those explaining a set of observations. However, the problem with \( K(s) \) is its uncomputability: there is no algorithm such that, given the object \( s \) (a binary string or a set of observations) returns, in a finite number of steps, the value \( K(s) \) (the size of the shortest program producing \( s \), or the smallest theory explaining our observations). Therefore, if we can not generally know the value of \( K(s) \), we cannot know which is the
shortest program (or theory) generating the string $s$ (explaining our observations). The most interesting consequence of the above explanation is that, in general, the problem of determining which is the best explanation for a given set of observations is uncomputable (if we understand the best as the most compact).

However, despite the uncomputability of $K(s)$, there are good approximations that allow to measure the algorithmic complexity of an object. One of the most used approximations is based on lossless compression algorithms. These algorithms are frequently used in our computers to compress documents. A very common compression algorithm is Lempel-Ziv, on which the ZIP compression format is based. It allows to define a computable complexity measure that approximates $K(s)$ \[8\]. If we have some file $s$ and the output of the compression algorithm is $c(s)$ (it is important to use a lossless compression algorithm so that when decompressed it produces exactly the original file $s$), we can understand $c(s)$ as a program that, when is run in certain computer (the decompressor program) produces $s$. So, the length of the compressed file $c(s)$ is an approximation to $K(s)$. It is not necessary that $c(s)$ is the shortest possible description of $s$, as we can consider it an approximation. In fact, many applications based on $c(s)$ to measure complexity are used in different disciplines line physics, cryptography or medicine \[10\].

How can we approximate $K(s)$ to compare the complexity of several abductive explanations and choose the best one? Compression-based approximations to $K(s)$ are frequently good when $s$ is a character string. It also happens with other approximations based on the notion of algorithmic probability \[18\]. However, abductive explanations are usually produced in the context of theories with a structure that can be missed when treated as character strings. However, we can use several tricks to reproduce some aspects of the structure of the theories into the structure of the strings. For example, in classical propositional logic, both sets of formulas $A = \{p \to q\}$ and $B = \{\neg q \to \neg p, \neg p \lor q\}$ are equivalent, but if we understand $A$ and $B$ as character sequences and we compress them, $B$ will probably seem more complex than $A$. We can avoid this problem by converting both sets into a normal form, for example the minimal clausal form—sets (conjunctions) of sets (disjunctions) of literals (propositional variables or their negations)—which in both cases is $\{\{\neg p, q\}\}$.

Another important point to be considered is that the complexity of an abductive explanation $\alpha$ should be measured related to the context in which it is proposed: the theory $\Theta$. Hence, $K(\alpha)$ may not be a good approximation to the complexity of $\alpha$ as an abductive solution to a certain abductive problem $(\Theta, \varphi)$. Because of that, in certain cases it is more reasonable to use the notion of conditional algorithmic complexity $K(s | x)$ measuring the length of the shortest program that produces $s$ with input $x$. So, $K(\alpha | \Theta)$ would be a better approximation to the complexity of the abductive solution $\alpha$ (within the theory...
Θ) than just $K(\alpha)$. Using lossless compression, if $c(f)$ represents the compression of $f$ and $|c(f)|$ is the length in bits of $c(f)$, a common approximation to $K(s \mid x)$ is given by $|c(xs)| - |c(x)|$, where $xs$ represents the concatenation of $x$ and $s$. It can be observed that, in general, this approximation gives different values for $K(y \mid x)$ and $K(x \mid y)$, and the value of $K(x \mid x)$ approaches 0 for an ideal compressor, given that the size of the compression of $xx$ is almost equal to the compression of $x$, only one instruction to repeat all the output has to be included.

As we can see, the notions of algorithmic complexity make sense to approach the problem of the complexity of abductive solutions and to tackle with computational tools the problem of the selection of the best explanation. However, good choices have to be made about the way to represent the theories (for example, in clausal form) and which approach to $K(s) \circ K(s \mid x)$ is to be used. We presented above a very simple example on propositional logic where clausal form can be fine. But other options are also possible. For example, Kripke frames can be used to represent relations between theories [15]. That way, each world $w$ represents a possible theory $\Theta_w$, and the accessibility relation indicates which modifications can be done to the theories. Then, if world $w$ can access to $u$, then $\Theta_w$ can be modified to become $\Theta_u$. An abductive problem appears when we are in a certain world $w$ and there is a certain formula $\varphi$ which does not follow from theory $\Theta_w$. Then, we solve the abductive problem by moving to another accessible world $u$ (we modify our theory $\Theta_w$ to get $\Theta_u$) such that $\varphi$ is a consequence of $\Theta_u$. This way we give account of modifications in theories that go beyond adding new formulas. That is, if does not necessary happen $\Theta_w \subset \Theta_u$, because the change of theory can entail deeper modifications, for example in the structural properties of the logical consequence relation. Then, it may be possible to pass, for example, from $\Theta_w$ with a monotonous reasoning system, to a non-monotonic reasoning in $\Theta_u$. However, within all accessible theories from $w$ that explain $\varphi$, the problem of determining the least complex explanation still remains. The complexity measure that should be used here is $K(\Theta_u \mid \Theta_w)$ and, among all accessible theories from $w$ explaining $\varphi$, the one which minimises this complexity measure should be chosen.

Despite offering resources to compare different abductive solutions and to choose the simplest one, algorithmic complexity notions have two problems: (1) to determine a good representation for theories (or formulas) and (2) to choose a computable approximation to $K(s)$ or $K(s \mid x)$, through lossless compression or by other means. In the next section we present an example, based on epistemic logic, illustrating how we can do this in a specific case.
4 A proposal using epistemic logic

In this section we introduce an application of $K(s)$ to the selection of the best abductive explanation, in the context of dynamic epistemic logic (DEL). The presentation is based on previous papers where we use the same logical tools [14, 17, 12], but the selection criteria are now different. Here, an approximation to $K(s)$ is applied to choose among several abductive explanations.

One of the possible ways to model the knowledge and belief of an agent is offered by plausibility models [4]. We start by presenting the semantic notions that will be later used to propose and solve abductive problems.

Definition 1 (Language $L$) Given a set of atomic propositions $P$, formulas $\varphi$ of the language $L$ are given by

$$
\varphi ::= p \mid \neg \varphi \mid \varphi \lor \varphi \mid \langle \leq \rangle \varphi \mid \langle \sim \rangle \varphi
$$

where $p \in P$. Formulas of the form $\langle \leq \rangle \varphi$ are read as “there is a world at least as plausible as the current one where $\varphi$ holds”, and those of the form $\langle \sim \rangle \varphi$ are read as “there is a world epistemically indistinguishable from the current one where $\varphi$ holds”. Other Boolean connectives ($\land$, $\rightarrow$, $\leftrightarrow$) as well as the universal modalities, $[\leq]$ and $[\sim]$, are defined as usual ($[\leq] \varphi ::= \neg\langle\neg\rangle \neg\varphi$ and $[\sim] \varphi ::= \neg\langle\sim\rangle \neg\varphi$ for the latter).

It can be observed that the language $L$ is like propositional logic with two new modal connectives, $\langle \leq \rangle$ and $\langle \sim \rangle$, that will allow to define the notions of belief and knowledge. These notions will depend on a plausibility order that the agent sets among the worlds in the model. We now see how these models are built.

Definition 2 (Plausibility model) Let $P$ be a set of atomic propositions. A plausibility model is a tuple $M = (W, \leq, V)$, where:

- $W$ is a non-empty set of possible worlds
- $\leq \subseteq (W \times W)$ is a locally connected and conversely well-founded preorder, the agent’s plausibility relation, representing the plausibility order of the worlds from her point of view ($w \leq u$ is read as “$u$ is at least as plausible as $w$”)
- $V : W \rightarrow \wp(P)$ is an atomic valuation function, indicating the atoms in $P$ that are true at each possible world.

\[1\] A relation $R \subseteq (W \times W)$ is locally connected when every two elements that are $R$-comparable to a third are also $R$-comparable. It is conversely well-founded when there is no infinite $R$-ascending chain of elements in $W$, where $R$, the strict version of $R$, is defined as $Rwu$ iff $Rwu$ and not $Ruw$. Finally, it is a preorder when it is reflexive and transitive.
A pointed plausibility model \((M, w)\) is a plausibility model with a distinguished world \(w \in W\).

The key idea behind plausibility models is that an agent’s beliefs can be defined as what is true in the most plausible worlds from the agent’s perspective, and modalities for the plausibility relation \(\leq\) will allow this definition to be formed. In order to define the agent’s knowledge, the approach is to assume that two worlds are epistemically indistinguishable for the agent if and only if she considers one of them at least as plausible as the other (i.e., if and only if they are comparable via \(\leq\)). The epistemic indistinguishability relation \(\sim\) can therefore be defined as the union of \(\leq\) and its converse, that is, as \(\sim := \leq \cup \geq\). Thus, \(\sim\) is the symmetric closure of \(\leq\) and hence \(\leq \subseteq \sim\). Moreover, since \(\leq\) is reflexive and transitive, \(\sim\) is an equivalence relation. This epistemic indistinguishability relation \(\sim\) should not be confused with the equal plausibility relation, denoted by \(\equiv\), and defined as the intersection of \(\leq\) and \(\geq\), that is, \(\equiv := \leq \cap \geq\).

Now we can see how a formula is evaluated in a plausibility model. Modalities \(\langle \leq \rangle\) and \(\langle \sim \rangle\) are interpreted in the standard way, using their respective relations.

**Definition 3 (Semantic interpretations)** Let \((M, w)\) be a plausibility model \(M = \langle W, \leq, V \rangle\) with distinguished world \(w \in W\). By \((M, w) \vDash \psi\) we indicate that the formula \(\psi \in \mathcal{L}\) is true in the world \(w \in W\). Formally,

\[
\begin{align*}
(M, w) \vDash p & \iff p \in V(w), \text{ for every } p \in \mathbf{P} \\
(M, w) \vDash \neg \varphi & \iff (M, w) \not\vDash \varphi \\
(M, w) \vDash \varphi \land \psi & \iff (M, w) \vDash \varphi \text{ and } (M, w) \vDash \psi \\
(M, w) \vDash \langle \leq \rangle \varphi & \iff \text{there exists } u \in W \text{ such that } w \leq u \text{ and } (M, u) \vDash \varphi \\
(M, w) \vDash \langle \sim \rangle \varphi & \iff \text{there exists } u \in W \text{ such that } w \sim u \text{ and } (M, u) \vDash \varphi
\end{align*}
\]

In plausibility models, knowledge is defined using the indistinguishability relation. So, an agent knows \(\varphi\) in some world \(w\) iff \(\varphi\) is true in all worlds that cannot be distinguished from \(w\) by her, that is, all worlds considered epistemically possible for her. However, within those worlds there is a plausibility order, not all of them are equally plausible for the agent. This is relevant for the notion of belief: agent believes \(\varphi\) in a certain world \(w\) iff \(\varphi\) is true in the most plausible worlds that are reachable from \(w\). Due to the properties of the plausibility relation, \(\varphi\) is true in the most plausible worlds iff by following the plausibility order, from some stage we only reach \(\varphi\)-worlds [4]. We can express this idea with modalities \(\langle \leq \rangle\) and \(\leq\). Formally [4].

---

[2] Operator \(K\) is commonly used for knowledge. It should not be confused with Kolmogórov complexity \(K\(s\)) also usually represented by \(K\).
Agent knows $\varphi$ \quad $K\varphi := [\neg] \varphi$

Agent believes $\varphi$ \quad $B\varphi := \langle \leq \rangle [\leq] \varphi$

Figure 1: Example of a plausibility model

Fig. 1 shows a plausibility model example $M$. Plausibility relation $\leq$ is represented by arrows between worlds. For the agent, world $w_2$ is more plausible than $w_1$. In this case, $p$ is true in both worlds, while $q$ is true only in $w_2$ ($\neg q$ represents that $q$ is false). So, agent knows $p$ in $w_1$ but does not know $q$, that is, $(M, w_1) \models Kp \land \neg Kq$. However, agent believes $q$, $(M, w_1) \models Bq$. Indeed, she also believes $p$, $(M, w_1) \models Bp$.

If a formula $\varphi$ is true in all the states of a certain model $M$, then $\varphi$ is valid in $M$, represented as $M \models \varphi$. In the example, $M \models Kp \land \neg Kq \land Bq$.

In the logical literature about abduction and belief revision in general [2], logical operations adding or removing information of the theory are frequently considered. In the same way, in the context of plausibility models, agents can perform epistemic actions modifying the agent’s information. We now present two of the main actions that agents in plausibility models can perform. One of the actions modifies the knowledge and the other the belief. For more details about the properties of these actions, see [4].

The first operation, observation, modifies the agent’s knowledge. It is defined in a very natural way: it consists of removing all worlds where the observed $\psi$ is not satisfied, so that the domain of the model is reduced.

**Definition 4 (Observation)** Let $M = \langle W, \leq, V \rangle$ be a plausibility model. The observation of $\psi$ produces the model $M_{\psi!} = \langle W', \leq', V' \rangle$ where

\[
\begin{align*}
W' &:= \{ w \in W \mid (M, w) \models \psi \} \\
\leq' &:= \leq \cap (W' \times W') \\
V'(w) &:= V(w), \text{ for each } w \in W'
\end{align*}
\]

This operation removes worlds of $W$, keeping only those that satisfy (before the observation) the observed $\psi$. The plausibility relation is restricted to the conserved worlds.

Another operation that agents can do is to modify just the plausibility relation. It can be done in several ways. The operation we call conjecture is also known as radical upgrade in the literature.
Definition 5 (Conjecture) Let $M = \langle W, \preceq, V \rangle$ be a plausibility model and $\psi$ a formula. The conjecture of $\psi$ produces the model $M_{\uppsi} = \langle W, \preceq', V \rangle$, that differs from $M$ only in the plausibility relation, which is now,
\[
\preceq' := \{(w, u) \mid w \preceq u \quad \text{and} \quad (M, u) \Vdash \psi\} \cup \\
\{(w, u) \mid w \preceq u \quad \text{and} \quad (M, w) \Vdash \neg \psi\} \cup \\
\{(w, u) \mid w \sim u \quad \text{and} \quad (M, w) \Vdash \neg \psi \quad \text{and} \quad (M, u) \Vdash \psi\}
\]

The new plausibility relation indicates that, after the conjecture of $\psi$, all $\psi$-worlds (before the conjecture) are more plausible than all $\neg \psi$-worlds. The previous order between $\psi$-worlds or between $\neg \psi$-worlds does not change [22]. This operation preserves the properties of the plausibility relation, as shown in [23].

We now discuss how an abductive problem can appear and be solved within the plausibility models formalism. In the classical definition of abductive problem, formula $\varphi$ is an abductive problem because it is not entailed by the theory $\Theta$. But, where does $\varphi$ come from? For Peirce, $\varphi$ is an observation, that is, it comes from an agent’s epistemic action. As we have seen in Def. 4 the action of observing $\varphi$ can be modelled in DEL. What does it mean, then, that $\varphi$ is an abductive problem? After observing $\varphi$, if it is a propositional formula, the agent knows $\varphi$, so we cannot affirm that an abductive problem arises when the agent does not know $\varphi$. However, we can go back to the moment before observing $\varphi$; if the agent did not know $\varphi$, then after the observation it becomes an abductive problem. Formally,
\[
\varphi \quad \text{is an abductive problem in } (M_{\upvarphi}, w) \quad \text{iff} \quad (M, w) \not\Vdash K\varphi \quad (1)
\]

This definition of abductive problem within plausibility models is in line with Peirce’s idea that an abductive problem appears when the agent observes $\varphi$.

The notion of abductive problem in (1) has been defined in terms of knowledge. If could have been defined in terms of belief too, considering that $\varphi$ is an abductive problem in $(M_{\upvarphi}, w)$ iff $(M, w) \not\Vdash B\varphi$. Then, the condition for $\varphi$ to be an abductive problem becomes stronger than in (1), because $\neg B\varphi$ implies $\neg K\varphi$.

Now the notion of abductive solution can also be interpreted in the plausibility models semantics. According to Peirce’s idea, the agent knows that if $\psi$ were true, then the truth of the surprising fact $\varphi$ would be obvious. It is expressed in DEL by requiring that the agent knows $\psi \rightarrow \varphi$, that is, $K(\psi \rightarrow \varphi)$. Then, when the agent faces an abductive problem $\varphi$ and knows $\psi \rightarrow \varphi$, how does she solve it? Again, Peirce says that there is reason to suspect that $\psi$ is true. We now discuss what does ‘to suspect’ $\psi$ mean in DEL and how can it be modelled as an epistemic action.
Something not usually considered within logical approaches to abductive reasoning is how to integrate the solution. Maybe because in classical logic there is no way to suspect a formula. But in epistemic logic there are beliefs. It is an information kind weaker than knowledge, as we have seen. So, belief seems the most natural candidate to model the suspicion. In line with Peirce, we then distinguish the agent’s knowledge of $ψ → ϕ$ from her belief in $ψ$.

But for a reasonable suspicion, as Peirce requires, it is necessary that the agent knows $ψ → ϕ$. Joining all the presented ideas, given abductive problem $ϕ$ in $(M, w)$ (see (1)), formula $ψ$ is a solution for it iff

$$(M, w) \models K(ψ → ϕ)$$

(2)

Condition $(M, w) \not\models K(ψ → ϕ)$ cannot be required because it is trivially verified in all cases in which $ϕ$ is a propositional formula, as it becomes known after being observed, so the agent knows also $ψ → ϕ$ for every $ψ$.

What does the agent do to suspect $ψ$? The most adequate abductive action to integrate $ψ$ into the agent’s information is to conjecture $ψ$ (def. 5). In this way, $ψ$ is integrated into the agent’s information as a belief.

![Figure 2: Solving an abductive problem](image)

Fig. 2 shows an example of the whole explained process. In the model of the left, $K(p → q)$ is verified, but also $¬K q$. In the central model, after observing $q$ agent knows $q$, and of course she continues knowing $p → q$, that is, $K q ∧ K(p → q)$. Then, $q$ is an abductive problem, given that the agent knows it after the observation but not before. A possible solution is then $p$. After the abductive action of conjecturing $p$, the model on the right shows that the agent believes $p$, $B p$, given that $p$ is true in the most plausible world.

Briefly, (1) shows the condition for $ϕ$ being an abductive problem and (2) for $ψ$ being a solution for it. As we have proposed, is it reasonable to conjecture the abductive solution as a belief (def. 5).

We are now ready to use Kolmogórov complexity as a selection criterion within solutions of an abductive problem. First, observe that a binary relation $R$ over a set $\{l_1, l_2, \ldots, l_n\}$ can be represented as a binary matrix $A$ with dimension $n \times n$. In such matrix, each cell $a_{i,j}$ is equal to 1 iff $(l_i, l_j) \in R$ and is 0.
otherwise. For example, consider relation \( R = \{(1, 1), (1, 2), (2, 1), (2, 3), (3, 1)\} \) over the set \( \{1, 2, 3\} \). The matrix representing \( R \) is

\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\]

Given a matrix \( A \), its Kolmogórov complexity \( K(A) \) can be approximated for example using lossless compression. The least complex relations are equally the empty relation (all the matrix is filled with 0s) and the total relation (all filled with 1s). Those are the most compressible matrices. Other very regular relations, as that containing only pairs \((e, e)\) for each element \( e \) (diagonal matrix), are also quite compressible. The more random a relation is, the more random is the obtained matrix and the less compressible it is.

In addition to approximate the value of \( K(A) \) for a given matrix \( A \), compression allows us to approximate \( K(B \mid A) \), given matrices \( A \) and \( B \). To do that, the concatenation of \( A \) and \( B \) (in this order) is first compressed and then \( A \) alone is also compressed. The difference of the size of both files approximates \( K(B \mid A) \). This approximation to \( K(B \mid A) \) can be used to determine which is the best abductive solution in the context of plausibility models.

Consider an abductive problem \( \varphi \) in \((M_{\varphi!}, w)\), and two competing explanations \( \psi_1 \) and \( \psi_2 \) for it. The argument below can be extended to any number of competing explanations. Which of both explanations will be chosen? It was explained below that the way to integrate an abductive explanation \( \psi_i \) is to conjecture \( \psi_i \) in the model \((M_{\psi!}, w)\) (def. 5). The effect of conjecturing \( \psi_i \) is only to modify the plausibility relation of \( M_{\psi!} \). Then, which is the best explanation? The answer offered by the notion of algorithmic complexity if that the best explanation is the one modifying the model (the plausibility relation) in the least complex way: this is not the one making the smallest modification, but the modification with the shortest description given the initial model. If \( A_{\psi!} \) is the matrix representing the plausibility relation of \( M_{\psi!} \) and \( A_{\psi!} \) the matrix for the plausibility relation of \((M_{\psi!})_{\psi_1, \psi_2}\), then the best explanation is the one minimising the value of

\[
K(A_{\psi_1} \mid A_{\psi!})
\]

The explanation minimising (3) is the one having the shortest description starting at the plausibility relation for the agent after observing \( \varphi \).

---

3Now \( K \) is used for algorithmic complexity and not for knowledge as in previous paragraphs.

4Lossless compression approximations are not good for small matrices as the one in the example, because habitual compressors cannot detect regularities in very short binary sequences. For small matrices, it is more convenient to use the tools presented in [18, 24] also available at The Online Algorithmic Complexity Calculator: http://www.complexitycalculator.com/.

5Formally, the chosen solution \( \psi \) is the one, among all possible solutions, that minimises
This methodology cannot be applied to small epistemic models, as those in the previous examples, because compression is not a good approximation to the complexity of small matrices. But it can be applied to models of medium and large size appearing in applications of DEL to multi-agent systems. For small models there are other methods that can be applied [24], only changing in the way to approximate $K(s)$.

5 Discussion

Ray Solomonoff, one of the drivers of algorithmic information theory, was convinced that the notion of algorithmic complexity can be used to define a system explaining all the observed regularities in the Universe. But there are strong limitations that make impossible to build such a system, mainly the uncomputability of $K(s)$. However, by using computable approximations, though not being able to build a system explaining all the regularities in the Universe, it is possible, in a specific context (plausibility models for us), to establish criteria based on $K(s)$ (and its conditional version) that allow to select the best explanation among the possible ones.

There are still many issues to explore. For example, it would be interesting to study the relevance of the notion of facticity introduced by Pieter Adriaans [1]. He considers $K(s)$ the sum of two terms, one is the structural information of $s$ and the other the ad hoc information in $s$. Then the best explanation could be selected by specially looking at the contained structural information. Also, algorithmic complexity measures can be combined with other common selection criteria that avoid triviality. Ideally, the least complex solution should be selected among all possible consistent and explanatory ones.

References

[1] Pieter Adriaans. Facticity as the amount of self-descriptive information in a data set. CoRR, abs/1203.2245, 2012.

[2] C.E. Alchourrón, P. Gärdenfors, and D. Makinson. On the logic of theory change: partial meet contraction and revision functions. Journal of Symbolic Logic, 50(2):510–530, 1985.

[3] Atocha Aliseda. Abductive Reasoning: Logical Investigations into Discovery and Explanation, volume 330 of Synthese Library. Springer, 2006.
[4] Alexandru Baltag and Sonja Smets. A qualitative theory of dynamic interactive belief revision. In Giacomo Bonanno, Wiebe van der Hoek, and Michael Wooldridge, editors, Logic and the Foundations of Game and Decision Theory (LOFT7), volume 3 of Texts in Logic and Games, pages 13–60. Amsterdam University Press, 2008.

[5] Gregory J. Chaitin. A theory of program size formally identical to information theory. J. ACM, 22(3):329–340, July 1975.

[6] Marta Cialdea Mayer and Fiora Pirri. First order abduction via tableau and sequent calculi. Bulletin of the IGPL, 1:99–117, 1993.

[7] A. N. Kolmogorov. Three approaches to the quantitative definition of information. Problems of Information Transmission, 1(1):1–7, 1965.

[8] Abraham Lempel and Jacob Ziv. On the complexity of finite sequences. IEEE Transactions on Information Theory, 22(1):75–81, 1976.

[9] Leonid A Levin. On the notion of a random sequence. Soviet Math. Dokl, 14(5):1413–1416, 1973.

[10] Ming Li and Paul M. B. Vitányi. An Introduction to Kolmogorov Complexity and Its Applications, Third Edition. Texts in Computer Science. Springer, 2008.

[11] Peter Lipton. Inference to the Best Explanation. Routledge, New York, 1991.

[12] Ángel Nepomuceno-Fernández, Fernando Soler-Toscano, and Fernando R. Velázquez-Quesada. An epistemic and dynamic approach to abductive reasoning: selecting the best explanation. Logic Journal of IGPL, 21(6):943–961, 2013.

[13] Fernando Soler-Toscano. Razonamiento abductivo en lógica clásica. Colege Publications, 2012.

[14] Fernando Soler-Toscano. El giro dinámico en la epistemología formal: el caso del razonamiento explicativo. THEORIA, 29(2):181–199, 2014.

[15] Fernando Soler-Toscano, David Fernández-Duque, and Ángel Nepomuceno-Fernández. A modal framework for modelling abductive reasoning. Logic Journal of IGPL, 20(2):438–444, 2012.

[16] Fernando Soler-Toscano, Ángel Nepomuceno-Fernández, and Atocha Aliseda-Llera. Abduction via c-tableaux and δ-resolution. Journal of Applied Non-Classical Logics, 19(2):211–225, 2009.
[17] Fernando Soler-Toscano and Fernando R. Velázquez-Quesada. Generation and selection of abductive explanations for non-omniscient agents. *Journal of Logic, Language and Information*, 23(2):141–168, 2014.

[18] Fernando Soler-Toscano, Hector Zenil, Jean-Paul Delahaye, and Nicolas Gauvrit. Calculating Kolmogorov complexity from the output frequency distributions of small turing machines. *PLoS ONE*, 9(5):e96223, 05 2014.

[19] R.J. Solomonoff. A formal theory of inductive inference. Part I. *Information and Control*, 7(1):1–22, 1964.

[20] R.J. Solomonoff. A formal theory of inductive inference. Part II. *Information and Control*, 7(2):224–254, 1964.

[21] Alan M. Turing. On computable numbers, with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*, 2(42):230–265, 1936.

[22] Johan van Benthem. Dynamic logic for belief revision. *Journal of Applied Non-Classical Logics*, 17(2):129–155, 2007.

[23] Fernando R. Velázquez-Quesada. Dynamic epistemic logic for implicit and explicit beliefs. *Journal of Logic, Language and Information*, 23(2):107–140, 2014.

[24] Hector Zenil, Fernando Soler-Toscano, Kamaludin Dingle, and Ard A. Louis. Correlation of automorphism group size and topological properties with program-size complexity evaluations of graphs and complex networks. *Physica A: Statistical Mechanics and its Applications*, 404(0):341 – 358, 2014.