Nonlinear phenomena in fluids with temperature-dependent viscosity: an hysteresis model for magma flow in conduits

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Abstract. Magma viscosity is strongly temperature-dependent. When hot magma flows in a conduit, heat is lost through the walls and the temperature decreases along the flow causing a viscosity increase. For particular values of the controlling parameters the steady-flow regime in a conduit shows two stable solutions belonging either to the slow or to the fast branch. As a consequence, this system may show an hysteresis effect, and the transition between the two branches can occur quickly when certain critical points are reached. In this paper we describe a model to study the relation between the pressure at the inlet and the volumetric magma flow rate in a conduit. We apply this model to explain an hysteric jump observed during the dome growth at Soufrière Hills volcano (Montserrat), and described by Melnik and Sparks [1999] using a different model.

1. Introduction

Like other liquids of practical interest (e.g. polymers, oils), magma has a strong temperature-dependent viscosity. It is well known that the flow in conduits of this kind of fluids admits one or more solutions for different ranges of the controlling parameters. The problem of the multiple solutions and their stability was studied by Pearson et al. [1973] and by Aleksanopol’ski and Naidenov [1979]. More recently this phenomenon, applied to magma flows, was investigated with more sophisticated models by Helfrich [1995] and by Wyile and Lister [1995]. In Skul’skiy et al. [1999] a simple method to find the conditions of multiple solutions was adopted and an experimental study to verify the hysteresis phenomenon predicted for this process was performed.

2. The model

In this paper we investigate a simple one-dimensional flow model of a fluid with temperature-dependent viscosity, with the essential physical properties characterizing the phenomenon of the multiple solutions and hysteresis as in Skul’skiy et al. [1999]. The fluid flows in a conduit with constant cross section and constant temperature at the wall boundaries. We assume that the fluid properties are constant with the temperature except for the viscosity, and we neglect the heat conduction along the streamlines, and the viscous heat generation. Moreover, we assume a linear relation between the shear stress and the strain rate (Newtonian rheology). This last assumption is introduced to simplify the model and allows us to demonstrate that the multiple solutions in conduit flows are the direct consequence of the viscosity increase along the conduit induced by cooling (under particular boundary conditions). Under these hypotheses, the equations for momentum and energy balance for the one-dimensional steady flow in a long circular conduit (R ≪ L) at low Reynolds number are:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( \mu(T) r \frac{\partial v}{\partial r} \right) = \frac{dP}{dz} \quad (1)
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( k_r \frac{\partial T}{\partial r} \right) = \rho c_v r \frac{\partial T}{\partial z} \quad (2)
\]

where \( R \) is the conduit radius, \( L \) conduit length, \( r \) radial direction, \( z \) direction along the flow, \( v \) velocity along the flow (we assume \( v = v(r, z) \approx v(r) \)), \( c_v \) specific heat at constant pressure, \( k \) thermal conductivity, \( \rho \) density, \( \mu \) viscosity, and \( P \) pressure, and \( T \) temperature. For magma, the dependence of the viscosity on the temperature is well described by the Arrhenius law:

\[
\mu = \mu_A \exp(B/T) \quad (3)
\]

where \( \mu_A \) is a constant and \( B \) the activation energy. In this paper, for simplicity, we approximate eq. (3) by the Nahme’s exponential law, valid when \((T - T_R)/(T - T_L) \ll 1 \), where \( T_R \) is a reference temperature:

\[
\mu = \mu_R \exp[-\beta(T - T_R)] \quad (4)
\]

with \( \beta = B/T_R^2 \) and \( \mu_R = \mu_A \exp(B/T_R) \). Following Skul’skiy et al. [1999], we introduce two new variables: the volumetric flow rate

\[
Q = 2\pi \int_0^R v(r)rdr \quad (5)
\]

and the convected mean temperature:

\[
T^*(z) = \frac{2\pi}{Q} \int_0^R T(r, z)v(r)rdr \quad (6)
\]

To satisfy the mass conservation, the volumetric flow rate \( Q \) is constant along the flow. Integrating eq. (1) and (2) and expressing the solutions in terms of \( Q \) and \( T^* \), we obtain:

\[
\frac{\pi R^4}{8\mu_w} \exp[\beta(T^* - T_w)] \frac{dP}{dz} = Q \quad (7)
\]

\[
\rho c_v Q \frac{\partial T^*}{\partial z} = 2\pi R \alpha (T_w - T^*) \quad (8)
\]

where \( T_w \) is the wall temperature (taken here as reference, \( T_w = T_R \)), \( \mu_w \) fluid viscosity at \( T = T_w \), and \( \alpha \) the coefficient of heat transfer.
transfer through the walls defined as:

\[ \alpha = \frac{k}{T_w - T^*} \frac{\partial T}{\partial r} \bigg|_{r=R} \]  

(9)

In the averaged model (eq. 7), we have adopted

\[ \mu \approx \mu_R \exp \left( -\beta(T^* - T_R) \right) \]  

(10)

and as pointed out in Helfrich [1995], the value of \( \beta \) in eq. (10), is usually smaller than the actual value of \( \beta \) in the fluid (eq. 4).

Eq. (7) and (8) with the boundary conditions:

\[ P(0) = P_0, \quad P(L) = 0, \quad T^*(0) = T_0 \]  

(11)

give an approximate solution of the problem. These equations are similar to the model of Pearson et al. [1973] (for a plane flow) and were used by Shah and Pearson [1974] to study the viscous heating effects. In the nondimensional form, eq. (7) and (8) are:

\[ \frac{dp}{d\zeta} = -q e^{-\theta} \]  

(12)

\[ q \frac{d\theta}{d\zeta} + \theta = 0 \]

where

\[ q = \frac{\rho c_p Q}{2\pi RL\alpha}, \quad \zeta = \frac{z}{L} \]  

(13)

\[ p = \frac{\rho c_p R^3 P_0}{16\mu_w L^2 \alpha}, \quad \theta = \beta(T^* - T_w) \]

The boundary conditions (11) are rewritten for the new variables:

\[ p(0) = p_0 = \frac{\rho c_p R^3 P_0}{16\mu_w L^2 \alpha}, \quad p(1) = 0, \quad \theta(0) = B \]  

(14)

with \( B = \beta(T_0 - T_w) \). The solutions of eq. (12), satisfying eq. (14) at the boundaries, are:

\[ p(\zeta) - p_0 = q \int_0^\zeta \exp(-B e^{-\zeta/q}) d\zeta \]

\[ \theta(\zeta) = B \exp(-\zeta/q) \]

(15)

and, therefore, the relation between the nondimensional pressure at the conduit inlet \( p_0 \) and the nondimensional flow rate \( q \) is:

\[ p_0 = q \int_0^1 \exp(-B e^{-\zeta/q}) d\zeta \]  

(16)

In Fig. 1 we plot relation (16) obtained numerically. We observe that for values of \( B \) greater than a critical value \( B_c \approx 3 \), there are values of \( p_0 \) which correspond to three different values of \( q \). By using a simpler model, Skul’skiy et al. [1999] found \( B_c = 4 \), whereas Helfrich [1995] found \( B_c \approx 3.03 \), in good agreement with Pearson et al. [1973]. Moreover, Shah and Pearson [1974] showed that when the viscous heat generation is important, the value of \( B_c \) can be lower, but for high values of \( B \), the relation between \( p_0 \) and \( q \) is similar to the case without viscous heat generation.

The stability analysis of the three branches (slow, intermediate and fast) shows that the intermediate branch is never stable. Moreover, one part of the slow branch is stable to two-dimensional perturbations but unstable to the three-dimensional ones, in a way similar to the Saffman-Taylor instability [Wylie and Lister, 1995]. In the case of multiple solutions, an hysteresis phenomenon may occur, as proposed in Wylie and Lister [1995] and verified experimentally in Skul’skiy et al. [1999].

In the experiments of Skul’skiy et al. [1999] a fluid with prescribed temperature and pressure was injected into a capillary tube

**Table 1. Typical Values of the Parameters Used in This Paper**

| Parameter                  | Symbol | Value | Unit   |
|----------------------------|--------|-------|--------|
| Rock density               | \( \rho_r \) | 2600  | Kg m\(^{-3}\) |
| Magma density              | \( \rho_m \) | 2300  | Kg m\(^{-3}\) |
| Conduit radius             | \( R \)  | 15    | m      |
| Conduit length             | \( L \)  | 5000  | m      |
| Magma temperature          | \( T_0 \) | 1123  | K      |
| Magma specific heat        | \( c_p \) | 1000  | J Kg\(^{-1}\) K\(^{-1}\) |
| Magma thermal conductivity | \( \kappa \) | 2     | W m\(^{-1}\) K\(^{-1}\) |
with constant wall temperature and controlled fluid pressure and flow rate. The device is used to show the transition between the two regimes corresponding to the upper and the lower branches.

A comparison between the experimental results (crosses) and theory (full line) is shown in Fig. 2 for the nondimensional variables  and for  = 4.6. The dashed lines indicate the pressure history prescribed in the experiments. The two steady-state regimes corresponding to the slow and to the fast branch were clearly recorded. Starting with a low pressure configuration (point A) and by increasing the pressure, the flow rate increases along the slow branch until it reaches a critical point (point B). Here, a jump to the fast branch occurs (point C). Increasing the pressure further, the flow rate increases along this branch, whereas, by decreasing the pressure, the flow rate decreases moving along the upper branch, until it reaches another critical point (point D) where the jump on the slow branch occurs (point E). On the slow branch the nondimensional flow rate is more than one order of magnitude lower than that on the fast branch.

3. Application

During some basaltic fissure eruptions in Hawaii and in Iceland, the eruption begins with a rapid opening at high flow rate and, after few hours, the flow rate quickly decreases. To explain this phenomenon, Wylie and Lister [1995] and Helfrich [1995] proposed a model similar to the model presented in this paper, based on the hysteretic jump between the fast branch and the slow branch when the pressure driving the eruption decreases.

A similar phenomenon, showing a jump in the mass flow rate, was observed during the dome growth (1995-1999) at Soufrière Hills volcano in Montserrat as described by Melnik and Sparks [1999]. This value is in good agreement with eq. (17); in fact, for  =  and assuming  ≈ 873 K, we have  =  × 10³ Pa s. Moreover, for  =  and  −  ≈ 250 K gives  ≈ 0.014 K⁻¹ (for example, from data of Hess and Dingwell [1996] for a magma with a similar composition and 4% water content, we obtain  ≈ 0.016 K⁻¹). Finally, for the heat transfer coefficient we have:

\[ \alpha \approx \frac{k}{\delta_T} \approx 4 \ \text{W m}^{-2} \text{K}^{-1} \]  

where  is the thermal boundary layer of the flow, while using \( k = 2 \ \text{W m}^{-1} \text{K}^{-1}, \delta_T \approx 50 \ \text{cm} \) (Bruce and Huppert [1989] used \( \delta_T \approx 10 \ \text{cm} \) for dyke length of about one kilometer).

Moreover, we verify the basic assumptions of the model: the assumption of one-dimensional flow is based on the small diameter/length ratio of the conduit (\( R/L \sim 10^{-3} \)), and the small Reynolds

A typical value of the viscosity is \( \mu_0 = \mu(T_0) = 10^7 \ \text{Pa s} \) for Montserrat andesite at 1123 K with 4% water content [Melnik and Sparks, 1999].

The model used by Wylie and Lister [1995] is shown in Fig. 2 for the nondimensional variables  and . The device is used to show the transition between the two regimes corresponding to the upper and the lower branches.

The values of  =  and  =  are chosen by least square best fit.

\[ \alpha_{best} \approx 3.56 \ \text{W m}^{-2} \text{K}^{-1} \]  

\[ \mu_{w, best} = 5 \times 10^8 \ \text{Pa s} \]  

\[ B_{best} = 3.46 \]  

The values obtained by the best fit of the observed data are:

![Figure 3](image3.png)

**Figure 3.** Relation between the discharge rate and the pressure. Full line represents the model prediction; crosses represent observed values at Soufrière Hills from Melnik and Sparks [1999].

![Figure 4](image4.png)

**Figure 4.** Possible pulsating behavior predicted by the model.
number is simply verified:

\[ R_e = \frac{\rho R \bar{v}}{\mu_0} \approx \frac{2300 \times 15 \times 0.003}{10^6} \sim 10^{-5} \]  

(19)

The viscous heating effects can be neglected because the Nahme number based on the shear stress is small:

\[ G = \frac{\beta \left(-\frac{dP}{dx}\right)^2 R^4}{4 \kappa \mu_w} \approx 1 \]  

(20)

The assumption of negligible heat conduction along the streamlines is justified by the high value of the Peclet number (the ratio between the advective and the conductive heat conduction):

\[ P_e = \frac{(\rho c_p \nu R)}{k} \approx 10^3. \]

The existence of the multiple solutions for the steady flow allows the system to show a pulsating behavior between the different solutions. In the case where the initial pressure is greater than the critical pressure corresponding to point E in Fig. 4, the system is on the fast branch of the solution, such as in point A. If the pressure decreases, the system moves along this branch up to the critical point C. In C a jump to the slow branch occurs (point D). If the pressure continues to decrease, the discharge rate tends to zero. Instead, if the pressure increases, the system moves along the slow branch up to the other critical point E. In this point the jump occurs on the fast branch and the system reaches point B.

The overpressure conditions and pulsating activity, typical of dome eruptions, are evident not only at Soufri`ere Hills, but also in Santiaguito (Guatemala), Mount Unzen (Japan), Lascar (Chile), Galeras (Colombia) and Mount St. Helens (USA) [Melnik and Sparks, 1999].

4. Conclusion

Magma flow in conduits shows the existence of multiple solutions, like other fluids with strong temperature dependent viscosity. This is a consequence of the increase of viscosity along the flow due to cooling. For a given pressure drop along the conduit, one or two stable regimes (fast and slow branches) may exist. The transition between the two branches occurs when critical values are reached, and an hysteresis phenomenon is possible. These jumps were evident during the dome growth in the 1995-1999 Soufri`ere Hills (Montserrat) eruption. The pulsating behaviour of the dome growth was previously modeled by Melnik and Sparks [1999] in terms of the nonlinear effects of crystallization and gas loss by permeable magma.

In this paper we propose a model to describe the nonlinear jumps between the two stable solutions as a consequence of the coupling between the momentum and energy equation induced by the strong temperature-dependent viscosity of magma.

However, since the crystal content is a consequence of cooling, the two models, although different, are physically related.

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