A formulation of the relaxation phenomenon for lane changing dynamics in an arbitrary car following model

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Abstract

This paper develops a model of lane changing dynamics which can be applied to an arbitrary car following model. Without the lane changing dynamics, car following models often create unrealistic trajectories because they react too strongly to the changes in space headway caused by lane changing maneuvers. With the lane changing dynamics added, car following models avoid these unrealistic behaviors and achieve better fits when calibrated to empirical data. The lane changing dynamics can be applied using a single parameter with a physical meaning, and can describe multiple types of lane changes. Validation is performed using the NGSim trajectory data and three different car following models.

Keywords: lane changing dynamics, car following, relaxation phenomenon, calibration
1 Introduction

Lane changing is a vital component of traffic dynamics which plays a role in many observed macroscopic phenomena. Recent works have confirmed that shockwaves on the highway are often initiated by lane changing maneuvers. [1] and [2] both examined the same traffic data from Interstate I-80 in the U.S., with the former concluding that all shockwaves observed in that data originated from disturbances caused by lane changing. The latter concluded that 16 out of 18 traffic oscillations in the data were caused by lane changing. [2] also looked at another study site, and found 12 out of 35 traffic oscillations were caused by lane changing. These results indicate that understanding lane changing dynamics is necessary for understanding congestion. Another example of the importance of lane changing is [3], where incorporating lane changing dynamics into a kinematic wave model was found to explain the drop in the discharge rate frequently observed at the onset of congestion.

Despite the evidence that lane changing is often responsible for the creation of shockwaves/disturbances, there has been a limited amount of literature which attempts to study lane changing dynamics [4]. Rather, it seems that most literature concentrates on lane changing decisions (the motivations/decision process behind lane changing), rather than lane changing dynamics (the effect that lane changes have on traffic flow). For the works that have focused on lane changing dynamics, many have focused on macroscopic models, for example [5], [6], [7]. These works are only interpretable in the setting of kinematic wave theory/the LWR model. For microscopic traffic models, specifically car following models, works have developed formulations for lane changing dynamics which are only interpretable for specific models. [8], [9], [10], [11], [12] are all such examples. As identified in the literature review [4], one significant literature gap is a general model of lane changing dynamics which can be combined with an arbitrary car following/microscopic model. The present work addresses this gap.

One well known feature of lane changing dynamics is the so-called relaxation phenomenon. This refers to the observation that drivers are willing to accept abnormally short spacings at the onset of lane changes, and that these short spacings gradually transition (“relax”) back to a normal spacing. Car following models react to this short spacing by decelerating far too strongly, causing unrealistic behavior. Of the previous works mentioned, [9], [10], [11], [12] all incorporated the relaxation phenomenon by changing some parameter in the model in order to let the model temporarily accept the short spacing. The present work differs because instead of changing model parameters, the input to the model (specifically, the headway) is altered. This creates a formulation of the relaxation phenomenon which can be applied to a general car following model. The formulation was found to be applicable to wide range of lane changing maneuvers, and not just those which involve shorter headways.

The rest of the paper is organized as follows. Section 2 shows some empirical examples of the relaxation phenomenon. Section 3 formulates the model for lane changing dynamics. Section 4 gives the calibration/validation procedure. The remainder of the paper is a case study which shows the efficacy of the lane changing dynamics applied to three different car following models (the optimal velocity model, intelligent driver model, and Newell model).

2 Empirical Evidence for Incorporating the Relaxation Phenomenon

It can be observed that car following models produce unrealistic description of driving in the presence of lane changing. Fig. 1 shows a motivating example of this, using the reconstructed [13] NGSim trajectory data [14] with the optimal velocity model [15]. In the top left panel, a lane changing vehicle causes the headway for the following vehicle to change from 100 ft. to 50 ft. The shorter headway of 50 ft. gradually increases back to 100 ft. over approximately 15 seconds. The bottom left panel shows the same behavior for different vehicles, where the lane change causes the follower’s headway to change from 50 ft. to 10 ft., and the short headway is relaxed back to 50 ft. after about 10 seconds. This empirical behavior of a short spacing immediately after a lane change gradually reverting back to an equilibrium spacing is known as the relaxation phenomenon. The middle panels show the typical behavior of a car following model in those situations, where the model reacts to the suddenly shorter headway by decelerating very strongly. The model predicts a pronounced dip in the trajectory due to the sudden deceleration. The result is that the calibrated trajectory describes the empirical behavior poorly. On the right panels, the proposed formulation for the relaxation phenomenon is added to the car following model, which removes the unrealistic behavior previously observed and allows a better fit to be achieved by the calibration.
Figure 1: Left panels show measured vehicle trajectories for a single lane. In both cases, a vehicle inserts itself which causes a significantly shorter headway (black dashed lines). This temporarily short headway is gradually reduced back to the full headway (only partially shown). Middle panel shows the calibrated trajectory for a vanilla car following model with no lane changing dynamics. The abnormally short headway induced by the lane change causes the car following model to predict strong decelerations. Right panel shows the same car following model with a single parameter added to model lane changing dynamics.

Table 1 gives some more empirical evidence for the importance of lane changing dynamics by examining the reconstructed NGSim data. Lane changes were categorized according to whether the headway increased or decreased after the lane change. The 3 seconds (30 observations) before and after each lane change were averaged to get the headway and speed before and after that particular lane change. In the reconstructed NGSim data, there are 1020 vehicles which experience lane changing, and among those, there are 1114 instances of lane changes where the headway decreases, and 743 where the headway increased.

Table 1: Change in headway and velocity for lane changes resulting in shorter headway

|          | Avg. Headway (ft) | Avg. Velocity (ft/s) |
|----------|-------------------|----------------------|
| Before LC| 78.0              | 25.0                 |
| After LC | 36.9              | 24.8                 |

Table 2: Change in headway and velocity for lane changes resulting in longer headway

|          | Avg. Headway (ft) | Avg. Velocity (ft/s) |
|----------|-------------------|----------------------|
| Before LC| 43.7              | 26.1                 |
| After LC | 91.4              | 28.3                 |

This simple analysis shows the fundamental problem lane changing causes to car-following models. After a lane change, the headway for all vehicles involved can change drastically, effectively halving or doubling itself. In every car following model the authors are aware of, this sudden change in headway will cause a drastic change in the speed/acceleration the model predicts. But the data tells a very different story. After a sudden decrease in headway (due to a LC) vehicles hardly decelerate; after a large increase in headway, vehicles accelerate only slightly.
3 Formulation of Lane Changing Dynamics Model

Simply put, the issue with lane changing is that the space headway a following vehicle “sees” suddenly jumps a large amount. A way to avoid this sudden jump is to gradually adjust the headway over time.

Consider following vehicle \( x_i(t) \) which has the lead trajectory \( x_{i-1}(t) \). Assuming that \( x_i(t) \) experiences the effects of lane changing, then \( x_{i-1}(t) \) necessarily refers to at least two different vehicles. Suppose that \( x_{i-1}(t) \) is defined on some discrete set of times \( \{t_n\} \) that are evenly spaced with timestep \( \Delta t \). At some time \( t_j \), a lane change occurs (not necessarily one which is initiated by the following vehicle) which causes \( x_{i-1}(t_j+1) \) to refer to a different vehicle than \( x_{i-1}(t_j) \).

Define the space headway \( s(t) = x_{i-1}(t) - l_{i-1}(t) - x_i(t) \) where \( l_{i-1}(t) \) is the length of the lead vehicle. Define \( r(t) \) and \( \gamma \)

\[
\begin{align*}
  r(t) &= \begin{cases} 
    -\frac{1}{c}(t - t_j) + 1 & t_j < t < t_j + c \\
    0 & \text{otherwise}
  \end{cases} \\
  \gamma &= s(t_j) - s(t_{j+1})
\end{align*}
\]

where \( c \) is a new non-negative parameter (to be calibrated) which has the same units as \( t \). Now consider the car-following model \( h \) posed as a second order ODE with some parameters \( p \)

\[
\dot{x}_i(t) = h(x_{i-1}(t), \ldots, p)
\]

where the \( \ldots \) denotes that \( h \) may depend on other quantities \( (x_i, \dot{x}_{i-1}, \text{etc.}) \) The model with lane changing dynamics is defined as

\[
\dot{x}_i(t) = h(x_{i-1}(t) + r(t)\gamma, \ldots, p) \tag{2}
\]

In other words, use the modified lead trajectory \( x_{i-1}(t) + r(t)\gamma \) instead of simply \( x_{i-1}(t) \) (this is equivalent to adding \( r(t)\gamma \) to the headway \( s(t) \)). Normally, just after \( t = t_j \), the lane change will cause the space headway \( s(t) \) to change suddenly, but the addition of \( \gamma \) will prevent this. \( r(t) \) then slowly relaxes \( \gamma \) to 0, at which point the normal car following dynamics resume. The modification reduces to the original model in the case where \( c = 0 \) (in continuous time) or \( c \leq \Delta t \) (in discrete time). Note in discrete time that \( r(t) \) will be nonzero on \( \{t_j + \Delta t, t_j + 2\Delta t, \ldots, t_j + \lfloor \frac{\Delta t}{\Delta t} \rfloor \Delta t \} \) where \( \lfloor \cdot \rfloor \) is the floor function.

This modification is not limited to class of car-following models posed as second order ODEs. For example, for some car-following model posed as a DDE (delay differential equation) with time delay \( \tau \)

\[
\dot{x}_i(t) = h(x_{i-1}(t - \tau), \ldots, p) \quad \rightarrow \quad \dot{x}_i(t) = h(x_{i-1}(t - \tau) + r(t - \tau)\gamma, \ldots, p) \tag{3}
\]

or for a trajectory translation model with time shift \( \tau \) and space shift \( p \)

\[
x_i(t) = x_{i-1}(t - \tau) - p \quad \rightarrow \quad x_i(t) = x_{i-1}(t - \tau) + r(t - \tau)\gamma - p \tag{4}
\]

The authors are not aware of any car-following model to which the lane changing dynamics cannot be applied. The physical interpretation of the lane changing dynamics model is that when a following vehicle’s leader changes, the follower allows some time \( c \) to adjust to the new headway caused by the lane change. During this time, the follower is willing to accept shorter headways when the new leader is nearer (or longer headways when the new leader is further); the follower behaves as if there was \( r(t)\gamma \) extra (less) distance between them and the lead vehicle. An example of this rule is shown below.
Figure 2: Left shows the headway for vehicle "93" as recorded in the data. The gap in headway is caused by a lane changing maneuver. On the right shows the headway $+ r(t) \gamma$ with $c = 15$ seconds. $\gamma > 0$ since the lane change induces a shorter headway.

The lane changing dynamics model can be applied for instances of many lane changes in a short amount of time. Suppose vehicle $x_i(t)$ undergoes a series of lane changes $J$ at times $t_j, j \in J$. Define $r_j(t)\gamma_j$ as the relaxation for the lane change at time $t_j$. Then,

$$r(t)\gamma = \sum_{j \in J} r_j(t)\gamma_j$$

or in other words, add up the relaxation amounts for the individual lane changes. This ensures that the headway will always be adjusted smoothly, and some examples of this are shown below.

Figure 3: Left shows the headway for vehicle "373" as recorded in the data. The gaps in headway are caused by two different lane changing maneuvers. On the right shows the headway $+ r(t)\gamma$ with $c = 15$ seconds. The first lane change has $\gamma < 0$ and the second $\gamma > 0$. Since $c = 15$ the vehicle is still being relaxed from the first lane change when the second lane change occurs.
Figure 4: Left shows the headway for vehicle “320” as recorded in the data. Right shows headway \( +r(t)\gamma \). In this example the following vehicle’s initial leader merged to the left, creating a large space headway. Then 3 additional vehicles merged from the right, each inserting themselves in front of the following vehicle. There were 5 unique leaders in a span of only 30 seconds.

One can see that even in fig. 4 even when there are 5 unique leaders in a span of 30 seconds, the lane changing dynamics model still will keep the headway varying smoothly. This ensures that a car-following model will not behave strangely around a lane change; namely the car-following model will not predict sudden jumps in position/speed due to the sudden change in headway.

It should be stressed that the lane changing dynamics model does not alter the positions of the lead vehicles. The relaxation \( r(t)\gamma \) is only “felt” by the following vehicle \( x_i \).

3.1 Discussion of the relaxation constants \( \gamma \)

\( \gamma \) is not a parameter, it is defined by the change in headway after a lane change. Thus, it will be defined by whatever model one chooses to use. In the context of a calibration problem, \( \gamma \) can either be calculated during each run of the simulation or it can be calculated once from the measurements. This distinction is discussed further in appendix B.

Figure 5: The three possible situations for a lane changing maneuver (from the perspective of following vehicle \( x_i \)). Rectangles represent vehicles, and the dashed rectangle indicates a lane changing maneuver. Note that if the current leader initiates a lane change, it will always induce a larger headway (\( \gamma < 0 \)), and if the new leader initiates the lane change, it will always induce a shorter headway (\( \gamma > 0 \)). The follower changing lanes can induce either a larger or smaller headway (\( \gamma > 0 \) pictured).

As far as the authors are aware, past studies of the relaxation phenomenon [11], [5], [6], [10], [9],[8] have been focused on specific models, and have only considered lane changes which induce shorter headways (\( \gamma > 0 \)). In the current formulation, we have proposed a far more general framework for an arbitrary microsimulation model. Lane changes can induce larger or smaller headways, the maneuvers might arise due to mandatory or discretionary lane changes, and there is no distinction made between which vehicle \( (x_i(t), x_{i-1}(t_j), \) or \( x_{i-1}(t_{j+1})) \) initiates the maneuver. All of
these different scenarios have, thus far, been grouped in the general category of “lane changing”, with no distinction between them.

Clearly, one can make the lane changing dynamics model more restrictive if desired. For example, consider only allowing $\gamma \neq 0$ if the lane change is initiated by $x_i(t)$: this would mean the lane changing dynamics are not applied to either of the maneuvers depicted in the middle and right panels of figure 4. Similarly, one could use different parametrizations of $r(t)$ based on the situation. For example, one could use $c = 25$ for lane changes when $\gamma > 0$ and $c = 15$ when $\gamma < 0$. This would mean drivers react differently whether a lane change induces a larger or shorter headway (they would need more time to adjust to a shorter headway). There are two important questions regarding the present formulation of the relaxation phenomenon. First, given all the possible different maneuvers/situations we have just discussed, how well does the lane changing dynamics model perform for each one? And second, is the proposed formulation really able to be applied to some arbitrary car-following rule? The remainder of the paper is devoted to a case study to investigate these two questions.

4 Calibration procedure

We will validate the formulation by comparing the fit achieved with the lane changing dynamics added compared to the fit achieved with no model of lane changing dynamics (i.e. the vanilla car following model). Then, for each model considered, it will need to be calibrated to the data. This necessitates a reliable calibration procedure that can be used to compare the model results.

The calibration follows the methodology described in [16] and is briefly described here. The reconstructed NGSim data is used as the source of trajectory data. Each vehicle is treated separately, so each vehicle has its own individual parameters to be calibrated. For each vehicle, the longest period of having a continuous leader is the period over which the model is calibrated. For the majority of vehicles, this corresponds to the time the vehicle enters the road section up until the time when the lead vehicle leaves the road section (a lead vehicle is required to use the model). For some vehicles though, they may be missing leaders in certain time periods, so they will be calibrated for the longest possible continuous time period.

The calibration problem consists of solving the following optimization problem

$$\min_p F = \sum_{t_i}^{T_{i-1}} f(x_i, \hat{x}_i, t) dt$$

s.t.  
$$\dot{x}_i(t) - h_i(x_i(t), x_{L(i)}(t) + r_i(t)\gamma, p_i) = 0, \quad t \in [t_i, T_{i-1}]$$
$$x_i(t_i) - \hat{x}_i(t_i) = 0$$
$$b_{low} \leq p \leq b_{high}$$

where $x_i(t)$ is the (follower) trajectory of vehicle $i$ at time $t$, $\hat{x}_i(t)$ is the measured trajectory of vehicle $i$, $x_{L(i)}$ is the lead trajectory (which refers to several different vehicles if lane changing is present), $r_i(t)\gamma$ is the relaxation amount which is defined by Eqs. 4 and 5. The follower is to be calibrated between times $[t_i, T_{i-1}]$, and its measurements are known up to time $T_i$. $f(x_i, \hat{x}_i, t)$ is a loss function which represents the goodness-of-fit at time $t$, and $F$ is the loss over the entire calibrated trajectory. $h_i(x_i(t), x_{L(i)}(t) + r_i(t)\gamma, p_i)$ represents the functional form of the model, with given initial conditions $x_i(t_i) = \hat{x}_i(t_i)$ and parameters $p$. The optimization is performed with respect to the parameters $p$.

In this paper we use a loss function of squared error in distance

$$f(x_i, \hat{x}_i, t) = (x_i(t) - \hat{x}_i(t))^2$$

$$F = \sum_{t_i}^{T_{i-1}} f(x_i, \hat{x}_i, t)$$

which is equivalent to minimizing the root mean square error (RMSE).

$$\text{RMSE} = \sqrt{\frac{F}{T_{i-1} - t_i}}$$
The optimization problem (6) is a nonlinear and nonconvex optimization problem which is solved numerically. [10] compared several different algorithms for solving the optimization problem. It was found in that paper that gradient-based optimization algorithms and the adjoint method can be used to find globally optimal solutions to the calibration problem in significantly less time than the more commonly used gradient free methods. Algorithms were evaluated based on their overall RMSE achieved, ability to find the global optimum, and the average time needed to solve the calibration problem. Based on those 3 metrics, 3 algorithms were identified as being on the pareto front: genetic algorithm, limited memory bfgs for bound constraints (l-bfgs-b), and truncated newton conjugate (tnc). The genetic algorithm is a method for global optimization, and l-bfgs-b and tnc are quasi-newton methods. It was found that all three algorithms gave very similar performance with respect to accuracy (although the genetic algorithm gave slightly worse RMSE), but either of the quasi-newton methods were significantly faster (5-75 times faster). In this paper the calibration problem is solved with the l-bfgs-b algorithm, using the adjoint method to calculate the gradient.

5 Results

This section validates the proposed formulation of lane changing dynamics by analyzing its effects on goodness-of-fit for the optimal velocity model (OVM). The following section applies the same analysis to the intelligent driver model (IDM) and Newell car following model to validate that the lane changing dynamics can be applied to an arbitrary car-following model. The functional form of the OVM is shown below [17, 15]

\[ \ddot{x}_n(t) = c_4(V(s(t)) - \dot{x}_n(t)) \]  (3)

\[ V(s) = c_1[\tanh(c_2 s - c_3) - \tanh(-c_3)] \]  (4)

where \( c_1, \ldots, c_5 \) are parameters to be calibrated. The OVM with lane changing dynamics is simply (3) with \( s(t) \) replaced by \( s(t) + r(t)\gamma \). In addition to the vanilla OVM (with no lane changing dynamics), we consider the following four variants of the lane changing dynamics model

1. **Baseline**: no lane changing dynamics
2. **Relax \( \gamma > 0 \)**: lane changing dynamics are only applied when shorter headways are induced
3. **Relax \( \gamma < 0 \)**: lane changing dynamics are only applied when longer headways are induced
4. **Relax**: vanilla formulation where any lane change uses the same relaxation constant \( c \)

5. **2 Parameter Relax**: Different parameters for \( \gamma > 0 \) and \( \gamma < 0 \)

For each of the five variants of OVM, each vehicle in the reconstructed NGSim data [13] is calibrated following the procedure outlined in section [4]. Out of the 2033 vehicles in the reconstructed NGSim data, 1020 vehicles experienced at least a single lane change. Among those, there were 1857 total lane changing maneuvers: 386 instances of a follower initiating a change with \( \gamma > 0 \), 213 instances of a follower initiating a change with \( \gamma < 0 \), 530 instances of a leader initiating a change with \( \gamma < 0 \), and 728 instances of a leader initiating a change with \( \gamma > 0 \).

We consider the model’s performance for lane changing vehicles based on several different measures. Table 3 shows overall RMSE (over entire vehicle trajectories), averaged across all vehicles which experience lane changes. Table 4 considers the RMSE of calibrated trajectories in the 10 seconds immediately following a lane change. That table also categorizes lane changes depending on whether they are initiated by the leader or follower, and whether \( \gamma \) is positive or negative. Table 5 looks at overall RMSE with vehicles categorized according to how many lane changing events they experience.

Table 3: Goodness of fit for the four variants of the lane changing dynamics model compared against the baseline of the OVM model with no lane changing dynamics. For reference, the goodness of fit of the model for vehicles which do not experience lane changes is included.

|                      | Baseline | Relax \( \gamma > 0 \) | Relax \( \gamma < 0 \) | Relax \( \gamma < 0 \) | Relax \( \gamma < 0 \) |
|----------------------|----------|------------------------|------------------------|------------------------|------------------------|
| Overall RMSE for LC (ft) | 8.18     | 6.58                   | 7.74                   | 6.37                   | 6.03                   |
| Overall* RMSE for LC (ft) | 9.07     | 6.99                   | 8.24                   | 6.92                   | 5.90                   |
| Overall RMSE for no LC (ft) | 5.49     | -                      | -                      | -                      | -                      |
Table 4: Goodness of fit for the four variants of the lane changing dynamics model compared against the baseline of the OVM model with no lane changing dynamics. Instead of showing the overall fit, this table only shows the fit in the time immediately following a lane change. The lane changes are split up according to the four possibilities (see fig. 5). The average relaxation times are also reported.

|                              | Baseline | Relax $\gamma > 0$ | Relax $\gamma < 0$ | Relax 2 Parameter Relax | 2 Parameter Relax |
|------------------------------|----------|---------------------|---------------------|-------------------------|-------------------|
| RMSE (ft), Follower initiates, $\gamma > 0$ | 10.86    | 7.40                | 10.57               | 7.64                    | 7.04              |
| RMSE (ft), Follower initiates, $\gamma < 0$ | 9.48     | 8.41                | 9.38                | 8.75                    | 8.08              |
| RMSE (ft), Leader initiates, $\gamma < 0$ | 9.64     | 8.04                | 7.62                | 6.57                    | 5.76              |
| RMSE (ft), Leader initiates, $\gamma > 0$ | 10.43    | 6.78                | 10.23               | 6.87                    | 6.36              |
| Relaxation time (s)          | -        | 26.3                | 13.0                | 22.7                    | 27.9 ($> 0$), 17.8 ($< 0$) |

Table 5: Overall RMSE where vehicles are categorized based on how many lane changing events they experience.

| Overall RMSE (ft), # LC     | Baseline | Relax | 2 Parameter Relax |
|-----------------------------|----------|-------|--------------------|
| Overall RMSE (ft), 1 LC     | 7.25     | 5.73  | 5.73               |
| Overall RMSE (ft), 2 LC     | 8.26     | 6.54  | 6.08               |
| Overall RMSE (ft), 3 LC     | 9.92     | 7.47  | 6.47               |
| Overall RMSE (ft), 4+ LC    | 11.10    | 8.08  | 7.06               |

Table 4 shows that the overall RMSE for vehicles which do not experience lane changing is 5.49 ft (note that this excludes vehicles in the HOV lane because those vehicles give meaningless calibration results; those vehicles simply move at free flow speeds the entire observation time [18]). The RMSE for vehicles which do experience lane changing is 8.18 ft (with no lane changing dynamics), significantly worse. When adding only a single parameter in the Relax scheme, the RMSE for lane changing vehicles reaches 6.37 ft, or 6.03 ft when 2 parameters are used. The 2 parameter formulation is equivalent to the 1 parameter formulation if a vehicle experiences only one type of lane change (either $\gamma > 0$ or $\gamma < 0$), but helps significantly for describing vehicles which experience both types. We also see from that table that the Relax $\gamma < 0$ formulation does not result in much improvement to goodness-of-fit, contrary to the $\gamma > 0$ formulation. This result is consistent with the intuition that lane changing dynamics are more important in the $\gamma > 0$ case (indeed, this is the case that previous studies have focused on).

In table 4, RMSEs are computed for the 10 seconds immediately after every lane changing event, and categorized according to the type of lane changing maneuver. This allows insight into the model’s efficacy for different types of maneuvers. It shows that the relaxation phenomenon better describes lane changes initiated than a leader than it does for lane changes initiated by the follower. This is likely due to followers anticipation of the imminent lane change (for example, by accelerating/decelerating before the change to create a more favorable headway after the change); these anticipation effects are not part of the current formulation, since the lane changing dynamics are only applied after the change. We also see that in general, $\gamma > 0$ LC have a higher RMSE than $\gamma < 0$ LC. The calibrated relaxation times are consistent with the values reported throughout the literature. The relaxation for $\gamma < 0$ can be observed as being shorter than for $\gamma > 0$, meaning vehicles need more time to adjust to a shorter headway.

Table 5 categorizes lane changing vehicles according to how many lane changing events they experience. Either the 1 parameter or 2 parameter formulations of Relax give a significant improvement for any number of LC events. They both give the same result for vehicles with only a single LC since the two formulations are equivalent in that case. For only a single lane change, the accuracy (5.73 ft) is almost equivalent to the accuracy for vehicles with no lane changes (5.49 ft). As the number of LC events increases, the RMSE of any of the model variants increase, albeit it increases at the slowest rate for the 2 parameter formulation.

Overall these tables all show that the simple addition of the relaxation phenomenon enables the OVM to describe lane changing vehicles with a high accuracy which is comparable to its ability to describe car following behavior. In particular, we saw that it is easiest to model vehicles which have a lane change initiated by its leader, or vehicles which experience a small number of lane changing events. The calibrated reaction times (ranging from 15-25 sec.) are consistent with the values reported in the literature.
Figure 6: Histogram showing the distribution of RMSE for 1 parameter relax, 2 parameter relax, and the baseline OVM. In this figure the four different lane changing scenarios are aggregated. Some outliers with RMSE > 20 are not shown.

Fig. 6 shows the distributions of the RMSEs immediately after LC for the different Relax variants compared to baseline. Looking at the distribution for the baseline model, it is centered around the 6.5-8.5 ft RMSE bin, with a very heavy tail corresponding to the large amount of lane changes which are described very poorly by the calibrated model. After adding the relaxation phenomenon, the distribution peaks around the 2.5-4.5 ft bin. Although there is still a tail, it decays much faster: then it can be concluded that the number of lane changes which are described poorly decreases substantially. Notably, the differences between the 1 parameter and 2 parameter formulations are not big when compared to the baseline.

Figures 7 and 8 show some examples of how the final calibrated trajectory varies between the baseline OVM and the 1 parameter relax OVM. The stars on those plots indicate the times of lane changes. These plots show the need of the incorporation of lane changing dynamics into the car following model. After a lane change, the model reacts very strongly and unrealistically, with the speed suddenly changing. This behavior is to be expected when considering that the lane change causes a large jump in headway. This large jump in headway will of course be associated with a large jump in the output of the car following model. After adding the relaxation phenomenon, the calibrated trajectory behaves in a manner consistent with how actual drivers behave.

Figure 8 is a more extreme example of the problems lane changes can cause to car following models. In that figure, the following vehicle is travelling in the rightmost lane, and experiences 6 different lane changes in a short period of time due to merging vehicles. Similar to the vehicle in Fig. 7, these lane changes can cause spikes in acceleration. However, they also cause more serious problems in this case, since the unrealistic driving behaviors are present throughout the whole trajectory instead of just around the lane changing times. In order to compensate for the numerous sudden changes in headway, the model assumes an essentially constant speed, which only changes after the 5th and 6th lane change both cause a significantly shorter headway. Compare this to the model parameters on the right plot, which describe a realistic behavior. Then it is clear that although the overall goodness of fit is not much different between these two results (8.7 ft on the left compared to 8.3 ft on the right), the parameters for the relaxed OVM are far better, since those correspond to a realistic behavior, while the parameters for the baseline OVM correspond to a type of overfitting, where the model parameters become unrealistic in order to account for the lane changing.

Overall, these two example trajectories illustrate something not captured in the previous tables and histogram. The lane changing dynamics not only increase the goodness of fit achieved, but they also allow the calibration to find more realistic parameters. This is an important point, seeing as parameters found by minimizing some error metric...
are hardly meaningful if those parameters correspond to an unrealistic driving behavior.

Figure 7: The calibration results for vehicle 31, which experiences two different lane changes. On the left, the calibration results for the baseline OVM. On the right, calibration results for the OVM with the 1 parameter relaxation. Stars indicate the times of the lane changes.

Figure 8: Calibration results for vehicle 603. The RMSE is not not much different between the two trajectories, although the lack of lane changing dynamics on the left cause the trajectory to become excessively unrealistic.

5.1 The relaxation phenomenon for mergers

The typical merging maneuver is different than a lane change because there is usually a period of time for which there is no leader for the merging vehicle (i.e. there are no vehicles on the on-ramp ahead of the merger). This is not the case when a vehicle on the on-ramp merges onto the highway before it's lead vehicle has merged, so that there is always some lead vehicle for the merger; regard this as scenario b and the former as scenario a. In the reconstructed NGSim data, there are 156 instances of scenario a and 25 of scenario b. Scenario b is not much different than a
normal lane change and the relaxation can be applied as such, so the remainder of the section is concerned with scenario a, which is henceforth regarded as simply a merger.

The relaxation constant $\gamma$ can not be calculated for a merger since there is no leader before the merge. To apply the relaxation the $\gamma$ constants were calculated as

$$\gamma = \bar{s} - s(t_{j+1})$$

where $\bar{s}$ is the average headway for the vehicle. To prevent the shorter/longer headways observed immediately after a lane change from causing an erroneous value of $\bar{s}$, only observations which were at least 10 seconds after a lane change were used. After $\gamma$ is calculated for a merging vehicle, the relaxation phenomenon can be applied as in the case of a normal lane change.

For the 156 merging vehicles in the reconstructed NGSim data, the above strategy was applied to the OVM with a single relaxation parameter. For each vehicle, the trajectory was calibrated with and without the merger dynamics, and the RMSE was measured in the 10 seconds following the merging maneuver. Additionally, each merger was categorized into either having $\gamma > 0$ (126 vehicles) or $\gamma < 0$ (30 vehicles). The results are shown in the histogram below and summarized in the table.

Table 6: RMSE for the calibrated trajectory in the vicinity of a merger for the 1 parameter relaxation phenomenon + OVM compared to the unmodified OVM. Averaged over both categories, the overall fit was 5.82 ft for OVM and 3.66 ft for relaxed OVM.

| Category          | Baseline | Relax |
|-------------------|----------|-------|
| $\gamma > 0$, RMSE (ft) | 6.24     | 3.62  |
| $\gamma < 0$, RMSE (ft) | 4.08     | 3.85  |
| Relaxation time (s)| -        | 26.7  |

Figure 9: RMSE for the calibrated trajectory in the vicinity of a merger for the 1 parameter relaxation phenomenon + OVM compared to the unmodified OVM.

Similar to the case of a normal lane change, the RMSE was higher for $\gamma > 0$ than it is for $\gamma < 0$. This matches the intuition that it is more difficult to describe lane changing when the maneuver causes a short gap. The overall improvement in ft ($\approx 2-3$ feet) from adding a single parameter is similar as well. It can be concluded that adding a single parameter is once again adequate for a significant increase to the fit of the model.

The fit for merging vehicles is overall much better than it is for lane changing vehicles. Part of this is explained by the fact that most merging vehicles do not change lanes after entering the highway, and are therefore easier to calibrate compared to vehicles which change lanes several times. The rest is explained by the rightmost lane in the NGSim data being the most congested. In general it can be observed that describing unstable/stop-and-go traffic (which has the most variation) is more difficult than describing either free flow or highly congested traffic.
6 Validation with two other car following models

6.1 Intelligent Driver Model

Two additional car-following models commonly used in the literature were tested to verify that the lane changing dynamics can be applied to an arbitrary model. The first one was the Intelligent Driver Model (IDM) \cite{19}, \cite{20} which has the following parametrization

\begin{align*}
\ddot{x}_i(t) &= c_4 \left(1 - \left(\frac{\dot{x}_i(t)}{c_1}\right)^4 - \left(\frac{s^*}{s(t)}\right)^2\right) \\

s^* &= c_3 + c_2 \dot{x}_i(t) + \frac{\dot{x}_i(t)(\dot{x}_i(t) - \dot{x}_{i-1}(t))}{2\sqrt{c_4 c_5}}
\end{align*}

where $c_1, \ldots c_5$ are parameters and $s$ is the space headway. Although the parametrization is quite different than the OVM, the overall logic behind the model is similar. Whereas the OVM relies on the optimal velocity function $V(s)$ to determine whether the vehicle accelerates/decelerates, the IDM combines an acceleration strategy $1 - (\dot{x}_i/c_1)^4$ with deceleration strategy $-(s^*/s)^2$.

Similar to OVM, the IDM is a relatively recent car-following model that can be categorized as a stimulus response model. It does not include a time-delay, and most works on calibration comparing IDM to OVM have found that the IDM is more accurate. One major difference it has compared to OVM is that it uses the speed and position of the lead vehicle whereas OVM uses only the lead vehicle’s position.

The IDM is well known as being a “collision free” model because it will accelerate arbitrarily strongly to avoid a collision when the headway becomes small. However, we found that these arbitrarily strong accelerations can cause problems in the context of lane changing, where short headways induced by lane changing can cause accelerations strong enough to force the vehicle to assume negative speeds (in particular, if $\dot{x}_i < -c_1$ the model will quickly decelerate to negative infinity and overflow). To circumvent this problem, the velocity of the model was capped at 0. This will still allow the model to accelerate arbitrarily strongly, but prevents negative speeds. In a discrete setting with timestep $\Delta t$ this is expressed as

\[
\text{acceleration} = \begin{cases} 
\dot{x}_i(t) & \dot{x}_i(t) + \Delta t \ddot{x}_i(t) > 0 \\
-\dot{x}_i(t)/\Delta t & \text{otherwise}
\end{cases}
\]

The results for IDM are shown below, in an identical format to how they were shown for OVM.

Table 7: Goodness of fit for the four variants of the lane changing dynamics model compared against the baseline of the IDM model with no lane changing dynamics. For reference, the goodness of fit of the model for vehicles which do not experience lane changes is included.

| Overview | Baseline | Relax $\gamma > 0$ | Relax $\gamma < 0$ | Relax | 2 Parameter Relax |
|----------|----------|-------------------|-------------------|-------|-------------------|
| Overall RMSE for LC (ft) | 6.05 | 5.23 | 5.81 | 5.18 | 4.96 |
| Overall RMSE for no LC (ft) | 4.20 | - | - | - | - |

Table 8: Goodness of fit for the four variants of the lane changing dynamics model compared against the baseline of the OVM model with no lane changing dynamics. The fit in the time immediately following the lane change is shown. The lane changes are split up according to the four possibilities.

| Overview | Baseline | Relax $\gamma > 0$ | Relax $\gamma < 0$ | Relax | 2 Parameter Relax |
|----------|----------|-------------------|-------------------|-------|-------------------|
| RMSE (ft), Follower initiates, $\gamma > 0$ | 7.84 | 5.91 | 7.54 | 6.09 | 5.69 |
| RMSE (ft), Follower initiates, $\gamma < 0$ | 9.15 | 8.53 | 8.47 | 8.07 | 7.76 |
| RMSE (ft), Leader initiates, $\gamma < 0$ | 6.22 | 5.35 | 5.87 | 5.33 | 4.77 |
| RMSE (ft), Leader initiates, $\gamma > 0$ | 7.03 | 5.75 | 6.87 | 5.81 | 5.41 |
| Relaxation time (s) | - | 18.7 | 10.2 | 16.6 | 19.8 ($>0$), 12.8 ($<0$) |
It can be observed that the IDM model is more accurate to begin with, and the % improvement from adding the relaxation is similar albeit smaller. The overall RMSE for the 1 parameter relax is a 28% improvement for OVM and a 16% improvement for IDM. It can be observed that both models show the same shortcomings, with describing a follower-initiated change resulting in a lower fit, $\gamma > 0$ being more difficult to describe than $\gamma < 0$, and less accuracy for trajectories with a large number of LC events.

Also, the relaxation time is observed to be shorter for IDM than it was for OVM, even though the times are in the same ratio. The way we interpret this result is as follows. The relaxation phenomenon can be thought as a period of time in which the car-following behavior is modified in order to prevent excessively strong accelerations. If a model requires a shorter relaxation time, then that implies the model is less prone to predicting those strong accelerations. Consider the thought experiment where we add the relaxation phenomenon to a car-following model which already includes some mechanism for describing lane changing dynamics. In that scenario, then the car-following model would already describe the lane changes well, the relaxation phenomenon would offer no improvement, and so the calibrated relaxation time would always be 0 (equivalent to no relaxation). Overall, this implies that a model that describes lane changes better will have shorter relaxation times. This also explains why the relaxation times for $\gamma < 0$ changes were found as shorter than those for $\gamma > 0$ for both the IDM and OVM. The $\gamma < 0$ changes (those which induce longer headways) are better described by car following, so they do not require as long of relaxation times.

![Figure 10: Histogram showing the distribution of RMSEs around a lane change, aggregated between the four different lane changing scenarios.](image)

Figs. 11 and 12 show some examples of the difference between the baseline IDM and relaxed IDM with a single relaxation parameter. Very similar to the OVM, excessively strong accelerations are predicted in reaction to the headway jump caused by lane changes. After adding the relaxation, a more realistic car following behavior is achieved. Fig. 12 corresponds to the same vehicle shown in 8. Again, we see that the baseline calibrated trajectory for that vehicle gives a reasonable accuracy but has a very unrealistic behavior. After adding the relaxation, the fit is somewhat improved and the unrealistic acceleration spikes are no longer present.

Overall, these differences from the results from OVM show that the IDM is a more accurate model which, in its
unmodified state, is also better at describing lane changing dynamics. However, adding the relaxation still gave a considerable improvement to accuracy, as well as prevented instances of unrealistic acceleration.

![Figure 11: The calibration results for vehicle 31, which experiences two different lane changes. On the left, the calibration results for the baseline IDM. On the right, calibration results for the IDM with the 1 parameter relaxation. Stars indicate the times of the lane changes.](image1)

![Figure 12: Calibration results for vehicle 603. Compare to Fig. 8, which is the same vehicle but for the OVM](image2)

6.2 Newell model

The Newell car-following model [21], also referred to as a trajectory translation model, is widely used for several reasons. First, it only has two parameters, and trajectories are uncoupled (i.e. they can be calibrated in any order) which makes the model easy to calibrate. Additionally, since it is not a differential equation, it can be solved exceedingly quickly. Most importantly, the model is “parsimonious” in that it preserves the shockwave behavior of the LWR macroscopic model [22]. Overall, it’s tractability and correspondence with macroscopic theory makes the
model both appealing and unique. However, results in the calibration literature have shown that the Newell model results in a poor fit compared to other, more complicated models. The (baseline) model is formulated as follows

\[ x_i(t) = x_{i-1}(t - c_1) - l_{i-1}(t - c_1) - c_2 \]

Where \( c_1 \) is the translation over time and \( c_2 \) the translation over space. The formulation Eq. (1) applied to the Newell model reduces to a special case where the simulation’s speed is shifted by a constant amount during the relaxation time. Again the same analysis is carried out on the Newell model to further verify that the formulation of lane changing dynamics can be applied to an arbitrary car following model.

Table 10: Goodness of fit for the four variants of the lane changing dynamics model compared against the baseline of the Newell model with no lane changing dynamics. For reference, the goodness of fit of the model for vehicles which do not experience lane changes is included.

| Overall RMSE for LC (ft) | Baseline | Relax γ > 0 | Relax γ < 0 | Relax | 2 Parameter Relax |
|--------------------------|----------|--------------|--------------|-------|-------------------|
| Overall RMSE for LC (ft) | 18.13    | 14.68        | 14.34        | 11.14 | 10.59             |
| Overall* RMSE for LC (ft)| 20.73    | 17.03        | 14.37        | 11.41 | 9.80              |
| Overall RMSE for no LC (ft)| 8.77    | -            | -            | -     | -                 |

Table 11: Goodness of fit for the four variants of the lane changing dynamics model compared against the baseline of the OVM model with no lane changing dynamics. The fit in the time immediately following the lane change is shown. The lane changes are split up according to the four possibilities.

| RMSE (ft), Follower initiates, γ > 0 | Baseline | Relax γ > 0 | Relax γ < 0 | Relax | 2 Parameter Relax |
|------------------------------------|----------|--------------|--------------|-------|-------------------|
| RMSE (ft), Follower initiates, γ < 0 | 18.98    | 18.41        | 11.95        | 11.66 | 10.94             |
| RMSE (ft), Leader initiates, γ < 0  | 29.00    | 26.44        | 11.90        | 9.97  | 8.79              |
| RMSE (ft), Leader initiates, γ > 0  | 24.70    | 14.26        | 21.89        | 12.55 | 11.74             |
| Relaxation time (s)               | -        | 36.8         | 13.6         | 29.1  | 38.0 (> 0), 17.3 (< 0) |

Table 12: Overall RMSE where vehicles are categorized based on how many lane changing events they experience.

| Overall RMSE (ft), 1 LC | Baseline | Relax | 2 Parameter Relax |
|-------------------------|----------|-------|-------------------|
| Overall RMSE (ft), 2 LC | 15.41    | 10.27 | 10.27             |
| Overall RMSE (ft), 3 LC | 19.33    | 11.04 | 10.34             |
| Overall RMSE (ft), 4+ LC| 21.80    | 13.00 | 11.41             |
| Overall RMSE (ft), 4+ LC| 25.03    | 14.01 | 12.42             |

The Newell model has a higher RMSE compared to either the OVM or IDM, which is expected since it has fewer parameters. Despite the fact that the Newell model is a trajectory translation model, and not a differential equation, the lane changing dynamics still apply, and the results are very similar to those obtained for the OVM and IDM.
Figure 13: Distribution of RMSEs immediately after lane changing, aggregated over all types of lane changing events and compared between the baseline Newell model and 1 and 2 parameter relax variants.

Contrary to the IDM and OVM, which will predict acceleration spikes in reaction to lane changing, the Newell model will predict jumps in position. Clearly this is extremely unphysical, as the Newell model formulation was not intended to be applied to trajectories with lane changing. Regardless, the lane changing dynamics can be applied to extend the model to be interpretable in the case of lane changing. Instead of showing the speed-time plots to demonstrate the difference between the baseline Newell and relaxed Newell, the space-time plots are shown instead.

The method shown in (1) of choosing the relaxation amounts $\gamma$ based on the change in headway will eliminate the jumps in position, but may still cause unrealistic spikes in acceleration/speed. Appendix C gives an example of a modified version of choosing the $\gamma$ constants for the Newell model that will prevent those spikes by enforcing a smoothness condition.

Overall, the Newell model had the largest difference in RMSE between non-lane-changing and lane changing vehicles; it also had the largest improvement from the addition of the lane changing dynamics. This is in line with the results from comparing the IDM to the OVM. We can conclude that the poorer a model is at describing lane changing, the bigger the benefit will be from adding the relaxation. The relaxation times are longer as well, which is consistent with our interpretation of longer relaxation times as being an indicator of the model being poor at describing lane changes. Looking at the histogram, it is clear that the Newell model’s distribution of RMSE has a very heavy tail. This is not unexpected when considering the behavior of the model. Since the following vehicle will always assume some fixed translation in space, if a lane change causes the headway to change drastically, the simulated trajectory can change drastically as well, and become far removed from the actual trajectory. The space-time plots show example of this; for example in time 1930 for fig. [14] the calibrated simulation is extremely far away from the measurements.
Figure 14: Newell trajectory for vehicle 603 (the same as Figs. 7 and 11 but shown in a space-time plot instead of speed-time)

Figure 15: Newell trajectory for vehicle 522.

7 Discussion and Conclusion

A novel formulation of the relaxation phenomenon is proposed. The resulting lane changing dynamics model can be applied to an arbitrary car following rule, and was tested on the IDM, OVM, and Newell car following models using the reconstructed NGSim data as a testbed. The formulation is shown to significantly improve the ability of the car following models to describe lane changing dynamics, evident by the increases in goodness of fit seen for every type of lane change, as well as mergers. The relaxation phenomenon also prevents calibrated trajectories from becoming unrealistic.

The main strengths of the lane changing dynamics model are its ability to be applied to an arbitrary microsimulation model, as well as its simple formulation which uses only 1 or 2 parameters with physical meanings, and has
no “hidden” parameters which need to be tuned. The main drawback of the model is that it is not as accurate for describing lane changing initiated by the following vehicle.

Even after the calibration, lane changing vehicles do not have as good as a fit as those vehicles which do not change lanes (although the fit is very close for vehicles which have only 1 or 2 lane changing events). We suggest three main reasons for this

1. Our implementation of lane changing dynamics assumes that the headway is adjusted at a constant rate (i.e. linear), controlled only by a single parameter (which controls the rate). Other similar strategies could be tested, such as the rate being logistic, or the rate depending on the headway and/or speed of the vehicle.

2. There are many other effects on lane changing dynamics other than drivers adjusting the new headways. For example, drivers cooperating to allow other vehicles to enter, or drivers becoming more aggressive to pursue a lane change.

3. Anticipation and post relaxation dynamics are not included in the current formulation.

Some lesser concerns are as follows. It could be the case that some vehicles are still being relaxed by the time they enter the simulation, but since we cannot observe this we have no way of knowing in what state the relaxation is in. Another ambiguity comes from how the data is recorded for vehicles changing lanes. The number of lanes are discrete; thus there has to be some cutoff point when vehicles are switched from 1 lane to another. Then this can lead to some ambiguity in the lane changing dynamics: when should a vehicle be regarded as having stopped following its previous leader and started following its new leader?

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Appendix A - Adjoint calculation for the relaxation

Assume that γ is fixed (i.e. it comes from the measurements) and one desires to calculate the gradient of \( F \) using the adjoint method. Let \( p_i^* = [p_i, c_i] \), where \( p_i \) are the parameters of the car-following model and \( c_i \) is the parameter associated with the relaxation. When \( \gamma \) is fixed, the adjoint system is unchanged, and is given by Eq. (22) in [16] \((n = 1)\). Eq. (23) in the same paper gives \( dF/dp_i \) (gradient with respect to the base car following parameters), and \( dF/dc_i \) (new component of the gradient due to the new relaxation parameter) is given below

\[
\frac{dF}{dc_i} = - \int_{t_i}^{T_{i-1}} \lambda^T_i(t) \frac{\partial h}{\partial (r_i \gamma)} \gamma \frac{\partial r_i(t)}{\partial c_i} dt
\]

Because of the linearity of differentiation, this can easily be extended to handle all the different extensions/variants mentioned in section 3.1, some of which were tested in section 5. For example, suppose vehicle \( i \) experiences several lane changes indexed by \( j \in J \), so that

\[
r_i(t) \gamma = \sum_{j \in J} r_{i,j}(t) \gamma_j
\]

Suppose further that some of those lane changes used relaxation constant \( c^1_i \) and others used relaxation constant \( c^2_i \). Then

\[
\frac{dF}{dc^1_i} = - \int_{t_i}^{T_{i-1}} \lambda^T_i(t) \frac{\partial h}{\partial (r_i \gamma)} \sum_j \gamma_j \frac{\partial r_{i,j}(t)}{\partial c^1_i} dt
\]

and similarly for \( c^2_i \).
Appendix B - Non-fixed relaxation amount for the calibration of a platoon of vehicles

Define $s_i(t)$ as the simulated headway of vehicle $i$ at time $t$, and $s_i^*(t)$ as the measured headway. Defining $\gamma$ through $s_i^*$ has the advantage of causing $\gamma$ to be fixed; this is convenient when using the adjoint method because the adjoint system is unchanged. Whether one chooses to define $\gamma$ through $s_i^*$ or $s_i$ makes virtually no difference for the calibration of a single vehicle. This is because the change in headway is caused almost exclusively by the different lead trajectories, and the lead trajectories will be fixed if one solves the calibration problem on individual vehicles.

In the scenario where one is calibrating several vehicles together, it may be beneficial to calculate $\gamma$ from the simulation. This non-trivially alters the adjoint system. In this situation the optimization problem (6) still applies, with the difference being that instead of having a single vehicle $i$, there are $n$ vehicles to be calibrated total, so $i \in [1, n]$. The objective $F$ is then summed over all $i$, and the constraints apply for all $i$.

Define $L(i, t)$ as returning the index to vehicle $i$ at time $t$, (so $x_{L(i)}(t)$ is the lead trajectory for vehicle $i$), and define $G(i, t)$ similarly as returning the index of the follower of vehicle $i$ at time $t$. If there is no follower at time $t$, let $G(i, t)$ return 0.

Define $J_{G(i)}$ as the set of times that lane changes occur for the follower trajectory $x_{G(i)}$, $J_i$ as the set of times of lane changes for vehicle $i$, and $e_1$ as the vector $[1, 0]$. The adjoint system of calibration problem is

$$\frac{\partial f}{\partial x_i} - \lambda_i^T - \Pi(t \leq T_i - 1)\lambda_i^T \frac{\partial h_i}{\partial x_i} - \Pi(G(i) \neq 0)\lambda_i^T \frac{\partial h_{G(i)}}{\partial x_i} + \sum_{j \in J_i} (\Pi(t = t_j) - \Pi(t = t_{j+1})) \xi(i, i, j)
\begin{align*}
    &- \sum_{j \in J_{G(i)}} (\Pi(t = t_j) - \Pi(t = t_{j+1})) \xi(i, G(i), j) = 0, \quad t \in [T_i, t_i], \forall i
\end{align*}$$

where

$$\xi(z, i, j) = \int_{t_j}^{\min(T_{z-1}, t_j + c_i)} \lambda_i^T(s) \frac{\partial h_i(s)}{\partial r_i \gamma} r_{i, j}(s)e_1 ds$$

The way to interpret $\xi(z, i, j)$ is that it is the effect vehicle $z$ has on vehicle $i$ through the lane change $j$. After the adjoint system is computed, the gradient can be computed as usual. All quantities are assumed to be dependent on $t$ unless explicitly stated otherwise.

Appendix C - Example of a modified formulation for Newell model

Consider a vehicle which obeys the Newell model. Suppose that at time $t_j$ the vehicle experiences some lane changing maneuver, causing it to have a new leader at time $t_{j+1}$. The original choice of $\gamma = s(t_j) - s(t_{j+1})$ can result in unrealistic spikes in speed for the Newell model. Consider the below speed-time plots for the persistent example of vehicle 603 (space-time plot in Fig. 14 right panel).
Figure 16: On the left, the original choice for $\gamma$. On the right, $\gamma$ was chosen to satisfy the smoothness condition.

Now, compared to the baseline model (where vehicles can move several hundred feet in a single timestep, e.g. Figs. 14, 15), the left panel of the above figure is a considerable improvement. But the relaxation has clearly left some artifacts, evident by the dips in speed at the times of lane changes. It can be shown that for the Newell model,

$$\dot{x}_i(t_{j-1} + p_1) = \dot{x}_{i-1}(t_{j-1})$$
$$\dot{x}_i(t_{j+1} + p_1) = \dot{x}_{i-1}(t_{j+1}) - \frac{\gamma}{c}$$

where $c$ is the relaxation time, $p_1$ is the time delay $t_j$ is the time of the lane change and we assume the vehicle is not relaxed prior to $t_j$. Then instead of the original choice of $\gamma = s(t_j) - s(t_{j+1})$ we will choose $\gamma$ such that

$$\dot{x}_i(t_{j} + p_1) = \frac{1}{2} (\dot{x}_{i-1}(t_{j-1}) + \dot{x}_{i-1}(t_{j+1}))$$

or in other words, choose the relaxation constant so as to ensure the smoothest possible transition for the model. This results in

$$\gamma = \frac{2c}{2c - \Delta t} \left( \frac{h}{2} (\dot{x}_{i-1}(t_{j-1}) + \dot{x}_{i-1}(t_{j+1})) - x_{i-1}(t_{j+1}) + l_{i-1}(t_{j+1}) + x_{i-1}(t_{j}) - l_{i-1}(t_{j}) \right)$$

Choosing $\gamma$ in this fashion results in the right panel of Fig. 16

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