Matrix Models and 2D Critical String Theory
— 2D Black Hole by $c = 1$ Matrix Model —

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In this talk, we first review the possibility of matrix models toward a nonperturbative (critical) string theory. We then discuss whether the $c = 1$ matrix model can describe the black hole solution of 2D critical string theory. We show that there exists a class of integral transformations which send the Virasoro condition for the tachyon field around the 2D black hole to that around the linear dilaton vacuum. In particular, we construct an explicit integral formula which describes a continuous deformation of the linear dilaton vacuum to the black hole background.

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1. Introduction: Basic problems of critical string theory

There are many unsolved questions in string theory. A basic difficulty is that the usual continuum formulation (CFT+integration over moduli) of string theory is intrinsically perturbative with respect to string coupling constant. Namely, we only know the *Feynman rules* without appropriate *derivation* from first principles. This means that we are not yet able to identify the dynamical degrees of freedom and the underlying higher symmetries of the theory. In particular, being interpreted as a finite theory of quantum gravity, we do not know how to describe the (global) *dynamics* of spacetime structure in a stringy language. As a consequence, the most interesting questions of spacetime geometrodynamics, such as background independence, spacetime singularities, black hole evaporation, and so on, have not been appropriately posed within the framework of string theory.

One of traditional approaches to these fundamental questions is the string field theory. In this approach, the fundamental dynamical degrees of freedom are supposed to be the string fields \( \psi[x(\sigma),...] \) as the functionals of a one-dimensional curve in target space. We try to construct (effective) action such that it reproduces the string Feynman rules in perturbation theory. Although we expect that this should be possible, there is no guarantee that the above string fields provide a natural language for formulating the structure of string theory, in particular, its symmetry structure. For a most recent attempt toward a closed string field theory, see ref [2].

Another now popular view is the renormalization-group approach. In this case, the fundamental dynamical degrees of freedom are supposed to be 2D field theories on world sheets and the renormalization-group fixed points are interpreted as the classical equations of motion. This viewpoint seems at least apparently more suitable to incorporate the dual symmetric structure of the theory, especially, modular invariance, than the string field theory. However, it seems much more difficult to go beyond perturbation theory with respect to string coupling constant. For a recent discussion of this approach, see ref [3].

The recent development of matrix models suggests a new possibility towards non-perturbative string theory, at least, in the case of 2D target spacetime. It starts from a completely well-defined quantum mechanics. We can define the matrix model without assuming string coupling constant small. In principle, therefore, it is formulated non-perturbatively and suggests an entirely new possibility concerning the question of the true dynamical degrees of freedom of string theory. Unfortunately, however, it is not at all clear at present whether the matrix models provide a clue to understand the above fundamental questions of geometrodynamics.

In this lecture, after briefly reviewing the connection between the matrix model and the 2D critical string, I would like to make some remarks related to the above problems by focusing to a concrete question, *How to understand the 2D black hole critical string solution (i.e., SL(2,R)/U(1) WZW model) within the framework of c = 1 matrix model?*
2. Interpretation of $c = 1$ matrix model as a 2D critical string theory

2.1 2D critical string

Let us begin from reviewing why the $c = 1$ matrix model may be interpreted as a critical string theory in a 2D target spacetime. The usual continuum world sheet action in 2D spacetime with coordinates $X^\mu = (\phi, x)$, where $\phi$ is spacelike and $x$ is timelike in Minkowski metric, is ($\alpha' = 1$)

$$S = \frac{1}{8\pi} \int d^2 \xi \sqrt{g} (g^{ab} G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu - 2R^{(2)} \Phi(X) + 2T(X)).$$  \hfill (2.1)

The local fields $G_{\mu\nu}(x), \Phi(x)$ and $T(x)$ are background fields corresponding to spacetime metric, dilaton, and tachyon, respectively. This describes a conformally invariant field theory when

$$G_{\mu\nu} = \eta_{\mu\nu},$$  \hfill (2.2)
$$\Phi = -\sqrt{2} \phi,$$  \hfill (2.3)
$$T(X) = 0,$$  \hfill (2.4)

since the 2D energy momentum tensor,

$$\mathcal{T}(z) = \frac{1}{2}((\partial x)^2 - (\partial \phi)^2) - \sqrt{2} \partial^2 \phi,$$  \hfill (2.5)

satisfies the usual OPE with central charge $c = 26$. This solution is called the linear dilaton vacuum.

A problem of this model as a classical solution of critical string theory is that the perturbative treatment of this theory may not be justified because the effective string coupling constant, $g_s^2 \sim e^{-2\sqrt{2} \phi}$, is coordinate dependent owing to the linear dilaton term and becomes infinitely large as $\phi \to -\infty$. This requires to regularize the theory in some way. A natural regularization is to utilize the condensation of the tachyon field $T$.\cite{4,5}

Consider the renormalization-group fixed-point equation (in the low-energy approximation) for the background fields,

$$R_{\mu\nu} - 2\nabla_\mu \nabla_\nu \Phi + \nabla_\mu T \nabla_\nu T = 0,$$  \hfill (2.6)
$$R + 4(\nabla \Phi)^2 - 4\nabla^2 \Phi + (\nabla T)^2 + V(T) = 8,$$  \hfill (2.7)
$$- 2\nabla^2 T + 4\nabla^2 \Phi \nabla T + V'(T) = 0,$$  \hfill (2.8)

where the tachyon potential is

$$V(T) = -2T^2 + O(T^3).$$  \hfill (2.9)

Note that the value 8 in the rhs of the eq. (2.7) due to the dimensionality (2D) of the target spacetime. It is easy to check that the linear dilaton vacuum is an exact solution
of the fixed-point equation, as it should be. The linearized tachyon equation (∼ on shell condition) in this background reduces to

\[-\partial_x^2 + \partial_\phi^2 + 2\sqrt{2}\partial_\phi + 2)T = 0.\]  \hspace{1cm} (2.10)

This allows a static solution

\[T = b(\phi - \phi_0)e^{-\sqrt{2}(\phi - \phi_0)}, \quad (\phi_0, b = \text{constants})\]  \hspace{1cm} (2.11)

Here, \(\phi_0\) is an integration constant. A multiplying constant \(b\) is assumed to be positive. Adding this term to the world sheet action effectively suppresses the region where \(\phi \to -\infty\). This puts an effective cutoff for the coordinate \(\phi\),

\[\phi_0 < \phi.\]  \hspace{1cm} (2.12)

Remarks: (1) This is an infrared cutoff from the point of view of random surface in which the tachyon condensation amounts to nonzero cosmological constant with respect to world sheet; (2) The argument is far from rigorous: there is no known exact solution of the fixed-point equation with nontrivial tachyon field.

Because of the condensation of tachyon, there exists a nonvanishing contribution for the classical vacuum energy, in contrast to the usual critical string solutions in \(D = 26\) or \(D = 10\) spacetimes. Actually the total classical energy (i.e., integrated over \(\phi\)) is divergent for \(\phi \to \infty\). To make the theory completely finite, we then have to introduce another cutoff, say, \(\phi < 0\) by adjusting the integration constant \(\phi_0\) appropriately. This cutoff is ultraviolet in terms of the random surface viewpoint, while is infrared in terms of the target spacetime picture. The introduction of an infrared cutoff is natural from the target-space viewpoint, since the tachyon is massless in the linear dilaton vacuum.

\[-\partial_x^2 + \partial_\phi^2)\tilde{T} = 0, \quad T = e^{-\sqrt{2}\phi}\tilde{T}.\]  \hspace{1cm} (2.13)

We are considering the system in a finite box and then take an infinite-volume limit.

Then, the classical (sphere) free energy, \(F = [\text{range of x direction}] \times f(\Delta)\), turns out to be

\[f(\Delta) = \frac{1}{2\pi g_0^2} \frac{\Delta^2}{\log \Delta} + \cdots = \frac{1}{2\pi g_0^2} \mu^2 \log \mu + \cdots,\]  \hspace{1cm} (2.14)

\[T(0) = b e^{-\sqrt{2}\phi_0} \equiv \Delta, \quad \phi_0 = \frac{1}{\sqrt{2}} \log \mu.\]  \hspace{1cm} (2.15)

This coincides with the famous result of the matrix model if \(\Delta\) is identified with the bare cosmological constant on the world sheet and the bare string coupling constant \(g_0\) is scaled to be proportional to \(1/\beta \sim 1/N\). The double scaling limit is defined as the limit \(N \to \infty\) with \(\beta\mu\) being kept fixed. It is not difficult, using the continuum approach, to estimate the genus-one correction \(f^{(1)}\) to the free energy. The result is, using the standard moduli parameter \(\tau\),

\[f^{(1)} = -\frac{1}{8\pi} \log \mu \int_F d^2 \tau \frac{1}{\tau_2^2} = -\frac{1}{24 \log \mu},\]  \hspace{1cm} (2.16)
and agrees with the matrix model. Note that the integration region with respect to the moduli parameter is the fundamental region $\mathcal{F}$.

It is remarkable that there is no ultraviolet (from the view point of target spacetime) divergence in the genus-one free energy in contrast to local field theory in which case the fundamental region $\mathcal{F}$ is replaced by the usual integration over the proper time $0 < \tau_2 < \infty$. There are also many evidences\textsuperscript{[10]} for the equivalence in the level of correlation functions. From these observations, it seems very natural to regard the matrix model as a toy model which embodies potentially the whole non-perturbative information of 2D critical string theory.

## 2.2 2D black hole solution

Let us next review the 2D black hole solution in string theory. The linear dilaton vacuum as a solution to the fixed point equation is a special case of the following black hole solution\textsuperscript{[12]} (in the conformal gauge for the target spacetime metric $G_{\mu\nu} = e^\sigma \eta_{\mu\nu}$, $(u,v)$=light-like coordinates),

$$e^{-2\Phi} = e^{-\sigma} = uv - C, \quad (C = \text{const.}), \quad T = 0.$$  \hfill (2.17)

This reduces to the linear dilaton vacuum in the limit $\phi \to \infty$ (or equivalently $C \to 0$) upon identification,

$$u = e^{\frac{1}{\sqrt{2}}(\phi+x)}, \quad v = -e^{\frac{1}{\sqrt{2}}(\phi-x)}.$$  \hfill (2.18)

The exact conformal field theory describing the solution is shown by Witten\textsuperscript{[13]} to be the $SL(2,R)/U(1)$ WZW model with $k = \frac{9}{4}$, whose action is given by

$$S = \frac{-k}{8\pi} \int \text{Tr}[(g^{-1} \partial_+ g)(g^{-1} \partial_- g)] - ik \Gamma_{\text{WZ}}$$

$$- \frac{k}{4\pi} \int \text{Tr}(A_- g^{-1} \partial_+ g + A_+ g^{-1} \partial_- g - 2A_+ A_- + A_+ g A_- g^{-1}).$$  \hfill (2.19)

In view of the correspondence of the $c = 1$ matrix model with the linear dilaton vacuum solution, a question arises: Is it possible to represent the black hole solution in the language of the matrix model?

## 2.3 Discrete states of the $c = 1$ model and the black hole

Since for $-uv \to \infty$, the black hole background reduces to the linear dilaton background, let us first study the nature of the deformation of the linear dilaton vacuum to the black hole solution. Consider the asymptotic expansion of the black hole metric using the coordinates (2.18):

$$d^2s = \frac{du dv}{uv-C} = \frac{1}{2}[(d\phi)^2 - (dx)^2 + Ce^{-2\sqrt{2}\phi}((d\phi)^2 - (dx)^2) + O(C^2)].$$  \hfill (2.20)
Note that the parameter $C$ sets the scale of the black hole mass. We can set $C = 1$ by a scale transformation of the light-like coordinates. The first order deformation represented by the term proportional to $C$ is equivalent to adding a term,

$$e^{-2\sqrt{2}\phi((\partial\phi)^2 - (\partial x)^2)},$$  \hspace{1cm} (2.21)

to the world sheet action density. Using the language of CFT, this should be interpreted as a deformation by a marginal operator. In 2D critical string theories, there is no higher excited states than the lowest tachyon modes. However, it is well known\[14\] that there are an countably infinite number of other physical states with special discrete values of x-energy and $\phi$-momentum of the form,

$${\mathcal O}_{r,s} = e^{ipx}e^{\beta(p)\phi} \times (\text{Polynomials of derivatives of } X^\mu).$$ \hspace{1cm} (2.22)

The discrete states may be regarded as the remnants of the physical excitations in higher dimensional critical string theories. The allowed energies $p$ and momenta $\beta$ of the discrete states are

$$p = \frac{r - s}{\sqrt{2}}, \quad \beta_{\pm} = -2 \pm \frac{(r + s)}{\sqrt{2}}$$ \hspace{1cm} (2.23)

at the level $n = rs$ with $(r, s) =$integers. Let us call the discrete operators with $\beta_+$ and $\beta_-$, the positive and negative discrete operators, respectively.

Now we consider in particular the zero-energy $(r = s)$ discrete states which have the following $\phi$-momenta.

| $r$ | $\beta_+$ | $\beta_-$ |
|-----|-----------|-----------|
| 1   | 0         | $-2\sqrt{2}$ |
| 2   | $\sqrt{2}$ | $-3\sqrt{2}$ |
| 3   | $2\sqrt{2}$ | $-4\sqrt{2}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

Comparing with (2.21), we see that the deformation occurring in the black hole solution corresponds to a negative operator with $\beta_- = -2\sqrt{2}$ at level $n = 1$. Another discrete state at level $n = 1$ with $\beta_+ = 0, (\partial\phi)^2 - (\partial x)^2$, can be easily seen to correspond to a global rescaling of the coordinate $(x, \phi) \rightarrow (1 + \epsilon)(x, \phi)$. This deformation should be interpreted as a special case of global gauge transformations whose gauge parameters grow indefinitely at infinity. In general, the positive discrete states with $\beta_+$ are known to generate a closed symmetry algebra $W_{1+\infty}^+$, isomorphic to the Lie algebra of area-preserving diffeomorphisms in two dimensions. It is an important unsolved question to clarify the nature of this infinite symmetry structure in connection with possible higher symmetries underlying the critical string theory. On the other hand, from the view point of Liouville theory, it has been argued\[1\] that the states with $\beta_-$ can not be represented by local operators.
3. Can the matrix model describe the 2D black hole?

Thus, in order to understand the formation of black holes in terms of the matrix model, it is important to identify the negative discrete operators. Before attacking to this question, let us briefly review how to understand the tachyon and the discrete states in the $c = 1$ matrix model.

3.1 Fermi fluid picture to the matrix model

One-dimensional matrix model, after restricting to $U(N)$ singlet states,

$$L(M, \dot{M}) = \frac{1}{2} \text{Tr}[\dot{M}^2 - V(M)]$$

is equivalent to a non-interacting fermion system,

$$S = \beta \int d\lambda dx \psi^\dagger ( -\frac{1}{\beta} \frac{d}{dx} + \frac{1}{2\beta^2} \frac{d^2}{d\lambda^2} + V(\lambda)) \psi,$$

$$\int d\lambda \psi^\dagger \psi = N,$$

$$\{ \psi(\lambda, x), \psi(\lambda', x) \} = \delta(\lambda - \lambda').$$

The constant $\beta^{-1}$ plays the role of $\hbar$. The following table provides a dictionary between the two representations.

| [matrix] | [fermion] |
|----------------|----------------|
| $\int dx e^{ipx} \text{Tr} F(M)$ | $\int dx d\lambda e^{ipx} \psi^\dagger(\lambda, x) \psi(\lambda, x) F(\lambda)$. |

To extract the tachyon field, one of the shortest ways is to utilize the fermi-fluid picture in the semi-classical approximation as first pointed out by Polchinski. This is equivalent with the collective field method. Let $\alpha_{\pm}$ = upper and lower edges of the fermion sea. Then, by the Bohr-Sommerfeld condition, we have the following bosonized representation of the fermion bilinears:

$$\psi^\dagger \psi \Rightarrow \beta \int_\alpha^\alpha \frac{dp}{2\pi} = \frac{\beta}{2\pi} (\alpha_+ - \alpha_-)$$

$$\frac{1}{\beta^2} \frac{\partial^2}{\partial \lambda^2} \psi \Rightarrow \frac{\beta}{2\pi} \int dp p^2 = \frac{\beta}{6\pi} (\alpha_+^3 + - \alpha_-^3)$$
The quantization condition (or more precisely, the Poisson bracket multiplied by $\frac{i}{\beta^2}$ since the semi-classical approximation is assumed) for the fields $\alpha_\pm$ turns out to be

$$[\alpha_\pm(\lambda), \alpha_\pm(\lambda')] = \mp \frac{2\pi i}{\beta^2} \partial_\lambda \delta(\lambda - \lambda'),$$

from the above dictionary. On the basis of the above correspondence, we found that, in terms of the bose fields $\alpha_\pm$, the hamiltonian and the equation of motion are, respectively,

$$H = \beta \int d\lambda \int_{\alpha_{-(\lambda)}}^{\alpha_{+(\lambda)}} dp \, h(p, \lambda),$$

$$\frac{\partial \alpha_\pm}{\partial x} = -\partial_\lambda \frac{\partial H}{\partial \alpha_\pm},$$

where $h(p, \lambda)$ is the one-body hamiltonian

$$h(p, \lambda) = \frac{1}{2} p^2 + V(\lambda).$$

Since the system is symmetric under $p \leftrightarrow -p$, there is a static classical solution such that

$$\alpha_\pm = \pm \alpha_0(\lambda), \quad h(\alpha_0, \lambda) = 0.$$ (3.32)

Then the linear perturbation $\tilde{\alpha} = \alpha_\pm + \alpha_0$ around the classical solution satisfies the massless Klein-Gordon equation,

$$\tilde{\alpha}_\pm \equiv \left(\frac{\partial \lambda}{\partial \sigma}\right)^{-1} \Pi_\zeta + \partial_\sigma \zeta,$$ (3.33)

$$(\partial_x^2 - \partial_\sigma^2)\zeta = 0.$$ (3.34)

$$\frac{\partial \lambda}{\partial \sigma} \equiv \alpha_0(\lambda)$$ (3.35)

This shows that $\zeta$ should be interpreted as the tachyon field $\tilde{T}$. Note that (3.35) defines the spatial coordinate $\sigma$ as a parametrization of the fermi surface.

Now, as is well understood, the scaling limit is governed by the inverse harmonic potential,

$$V(\lambda) = -\frac{1}{2} \lambda^2 + \mu.$$ (3.36)

The one-body hamiltonian describing the scaling limit thus takes a very simple form,

$$h = \frac{1}{2} (p^2 - \lambda^2) + \mu = \frac{1}{4} (A_+ A_- + A_- A_+) + \mu,$$ (3.37)

where

$$A_\pm = p \pm \lambda.$$ (3.38)
satisfying the commutation relations

\[
[A_+, A_-] = i \frac{2}{\beta},
\]

\[
[h, A_\pm] = \mp i \beta A_\pm.
\]

Namely, \(A_\pm\) are energy eigenoperators with pure imaginary (Minkowski) energies (energies become real after Wick rotation to Euclidean metric);

\[A_+^x A_-^x \sim e^{(s-r)x}.
\]

Equivalently, these operators generate an infinite dimensional symmetry of the system

\[
\left[ \frac{i}{\beta} \frac{\partial}{\partial x} - h, e^{-(s-r)x} A_+^x A_-^x \right] = 0
\]

The Poisson brackets among these generators form the \(W_{1+\infty}\) algebra,

\[
\{ H_{J,m}, H_{J',m'} \} = (mJ' - m'J)H_{J+J'-1,m+m'}, \quad (|m| \leq J)
\]

\[
H_{J,m} \equiv A_+^{J+m} A_-^{J-m}.
\]

These properties\cite{16-17-18-19} are similar to the discrete states \cite{20} in the linear dilaton vacuum provided those are identified with the positive discrete states with the correspondence \(J = \frac{r+s}{2}, m = \frac{r-s}{2}\).

Unfortunately, there is no clear indication of negative discrete states in the matrix model. This problem is perhaps related with the fact that the negative discrete states can not be represented by local operators in the language of 2D gravity on the world sheet. However, we have seen that the negative operator is essential for understanding the deformation of the linear dilaton vacuum into the black hole background.

\subsection{3.2 Deformation as canonical transformation}

In the matrix model, the deformations by the positive discrete states can be regarded as canonical transformations. Consider to deform the theory by adding the discrete operator to \(h\),

\[
h \rightarrow h + e^{(r-s)x} H_{J,m}.
\]

The case \(J = 1, m = 0\), for instance, corresponds to a global rescaling of spacetime coordinates which is a marginal deformation in the sense of CFT. as discussed in the subsection 2.3 above. It is important to notice that, since

\[
\frac{d}{dx}(e^{(r-s)x} H_{J,m}) = 0,
\]
the deformation by the discrete operator, in the language of the matrix model, can be regarded as a canonical transformation with

\[ F_{J,m}(x, \partial_x) \equiv xe^{(r-s)x}H_{J,m} \quad (3.46) \]

being the generating function. Note that the presence of a factor \( x \) in the generator (3.46), without which the generating function (3.46) is a symmetry generator. From this simple example, it seems natural to suppose that the deformation by negative discrete states are also generated by more complicated (and possibly singular) canonical transformations.

3.3 Integral transformations of the tachyon field

A GKO analysis\[21\] of the \( SL(2, R)/U(1) \) WZW model shows that the exact on-shell condition for the tachyon in the black hole background takes the following form,

\[ L_0^B T(u, v) = T(u, v), \quad (3.47) \]

with

\[ L_0^B = \frac{1}{k-2} [(1-uv)\partial_u \partial_v - \frac{1}{2} (u\partial_u + v\partial_v)] - \frac{1}{2k(k-2)} (u\partial_u - v\partial_v)^2. \quad (3.48) \]

In the asymptotic limit \( u, -v \to \infty \), this of course reduces to the linear dilaton case. In a previous subsection, we have seen for static background that the scaling limit of the \( c = 1 \) matrix model inevitably gives the linearized massless tachyon equation (3.34) (i.e., Klein-Gordon equation). The assumption of a static classical background for the bosonized field is natural since the classical black hole background (2.17) has a Killing symmetry under the vector field \( u\partial_u - v\partial_v \). Thus the possibility of describing the black hole background within the framework of the matrix model requires, as a necessary condition, that the tachyon operator (3.48) around the black hole background should also be rewritten in the Klein-Gordon form without external fields.

A first step to embed the black hole solution into the matrix model may then be to find a canonical transformation such that it reduces the tachyon equation (3.48) to the Klein-Gordon form. Such a possibility\[1\] has in fact been suggested previously by Martinec and Shatashvili\[22\] from a different context. They argued that the dual transform of the path-integral representation of the model (2.19) is essentially a Liouville theory coupled with the \( c = 1 \) conformal matter. It is indeed possible to construct a transformation which sends (3.48) to the form (3.34), on the basis of their observation. Here we take a different and more direct approach.

The general integral representation\[21\] for the solutions of (3.48) is given by

\[ T(u, v) = \int dx \frac{x}{x} (AB)^{-2i\omega} (AB)^{-1+2i\lambda}, \quad (3.49) \]

\[ \text{§ For other approaches, see } [23, 24].\]
where
\[ A = (\sqrt{1 - uv + \frac{u}{x}})^\frac{1}{2} \equiv \left[ u\left(-\frac{1}{x_2} + \frac{1}{x}\right)\right]^\frac{1}{2}, \]
\[ B = (\sqrt{1 - uv - vx})^\frac{1}{2} \equiv \left[-v(x - x_1)\right]^\frac{1}{2}. \]

The on-shell condition is
\[ \lambda = \pm \omega \frac{3}{3}. \]

There are two independent choices for the integration contours: two among
\[ C_1 = [0, \infty], \quad C_2 = [x_2, 0], \quad C_3 = [x_1, x_2], \quad C_4 = ] - \infty, x_1[. \]

Take, for instance, the contour \(C_1\) and consider the region \(u \equiv \sinh \frac{r}{2} e^t(> 0), \quad v \equiv -\sinh \frac{r}{2} e^{-t}(< 0)\) outside the horizon. We then have
\[ T_{C_1}(u, v) = e^{-2i\omega(-uv)} - \frac{1}{2} - i\lambda B(\nu_+, \nu_-)F(\nu_+, \nu_-; 1 - 2i\lambda; \frac{1}{uv}), \]
with \(\nu_\pm = \frac{1}{2} - i(\lambda \pm \omega)\). The solution (3.53) behaves for large \(r\) as
\[ T_{C_1}(u, v) \sim B(\nu_+, \nu_-)e^{-\frac{r}{2} e^{i\lambda r - 2i\omega t}}. \]

Other choices give similar expressions in terms of the hypergeometric functions.

Now, in the integral representation of \(T_{C_1}\), let us make the following change of integration variables,
\[ x A \quad B \quad \frac{1}{B} \quad = \quad e^{-\tau}, \]
\[ \frac{1}{A} \quad B \quad \frac{1}{B} \quad = \quad e^\sigma. \]

It is easy to see that in terms of new variables, it takes the form
\[ T_{C_1} = \int d\tau d\sigma \delta\left(\frac{-ve^\tau + ue^{-\tau}}{2} - \sinh \sigma\right)e^{-2i\omega\tau - 2i\lambda\sigma}. \]

Thus, \(\delta\left(\frac{-ve^\tau + ue^{-\tau}}{2} - \sinh \sigma\right)\) plays the role of kernel for an integral transformation to the plane-wave solution of the massless Klein-Gordon equation. Similarly, for the solution
\[ T_{C_2}(u, v) = e^{-2i\omega(-uv)} - \frac{1}{2} - i\omega B(\nu_+, \nu_-)F(\nu_+, \nu_-; 1 - 2i\omega; \nu_+), \]

satisfying the boundary condition near horizon \(r \to 0,\)
\[ T_{C_2}(u, v) \sim B(\nu_+, \nu_-)u^{-2i\omega}, \]
we have, for the same region for \(u, v\) as above,
\[ T_{C_2} = \int d\tau d\sigma \delta\left(\frac{-ve^\tau + ue^{-\tau}}{2} - \cosh \sigma\right)e^{-2i\omega\tau - 2i\lambda\sigma}. \]
Denoting these integration kernels by \( M(u, v; \tau, \sigma) \), we can directly check an operator relation

\[
L^B_0 M(u, v; \tau, \sigma) = M(u, v; \tau, \sigma)L^0_0,
\]

(3.61)

\[
L^0_0 \equiv -\frac{1}{4(k - 2)}\partial_\sigma^2 + \frac{1}{4k}\partial_\tau^2 + \frac{1}{4(k - 2)}.
\]

(3.62)

Comparing the rhs of this relation with (2.13), we see that the transformed \( L^0_0 \) operator gives the massless tachyon equation \( L^0_0 = 1 \) in the linear dilaton vacuum if and only if \( k = \frac{9}{4} \) after a trivial renaming of the coordinates.

The above transformations for \( T_{C_1} \) can formally be characterized by the following correspondence using the conventions of (3.61),

\[
\begin{align*}
\partial_u & \rightarrow \frac{1}{2}\partial_\sigma \frac{1}{\cosh \sigma} e^{-\tau}, & (3.63) \\
\partial_v & \rightarrow -\frac{1}{2}\partial_\sigma \frac{1}{\cosh \sigma} e^\tau, & (3.64) \\
u & \rightarrow (\cosh \sigma \partial_\tau + \sinh \sigma \partial_\sigma) \partial_\sigma^{-1} e^\tau, & (3.65) \\
v & \rightarrow (\cosh \sigma \partial_\tau - \sinh \sigma \partial_\sigma) \partial_\sigma^{-1} e^{-\tau}. & (3.66)
\end{align*}
\]

Here, the lhs are the operators for the black hole background and the rhs the ones in the linear dilaton vacuum. For \( T_{C_2} \), \( \cosh \sigma \) and \( \sinh \sigma \) should be interchanged.

These forms (3.63)∼(3.66) may be interpreted as the quantized version of a canonical transformation in the 4-dimensional phase space consisting of the one-particle coordinates and momenta in 2D target spacetime of the matrix model, at least around the fermi surface. The functions \( \sqrt{2|\mu|} \sinh \sigma \) or \( \sqrt{2|\mu|} \cosh \sigma \) appearing in the kernel, are then naturally identified with the coordinate \( \lambda \) representing the matrix eigenvalue whose functional form in terms of \( \sigma \) is defined by (3.33) with negative \( \mu \) or positive \( \mu \), respectively. From this point of view, however, the transformation is unsatisfactory. The reason is that (1) it does not take a form of a unitary transformation in the Hilbert space of one-body problem, namely, as a unitary transformation for the fermion wave functions; (2) the asymptotic limit \( -uv \rightarrow \infty \) of the transformation does not reduce to the identity, even though the black hole background reduces to the linear dilaton vacuum in this limit, as is seen from (3.63)∼(3.66). We conjectured that the black hole background can be treated within the matrix model by making a special canonical transform. This would require the existence of a (at least formally) unitary transformation in the space of fermion single-particle space. On the other hand, our discussion on the deformation of the linear dilaton vacuum into the black hole background in the asymptotic region suggests the existence of a transformation which reduces to the identity as we go far away from the horizon. We will discuss these problems in a forthcoming paper[1]. Below we only briefly discuss the second problem.
3.4 A candidate transformation corresponding to the black hole deformation

We first introduce a one-parameter family of a new integral kernel \( K_a(\tilde{u}, \tilde{v}; \tau, \sigma) \) satisfying the following conditions \((a \neq -1)\),

\[
\begin{align*}
-\frac{1}{2}e^\tau [-e^\sigma + (a + 1 - \partial_\tau) \partial_\sigma^{-1} e^\sigma] K_a &= K_a \tilde{u}, \\
-\frac{1}{2}e^{-\tau} [e^\sigma + (a + 1 + \partial_\tau) \partial_\sigma^{-1} e^\sigma] K_a &= K_a \tilde{v},
\end{align*}
\]

\( (3.67) \)

\( (3.68) \)

\[
\begin{align*}
e^{-\tau - \sigma} \partial_\sigma K_a &= K_a \partial_\tilde{u}, \\
-e^{\tau - \sigma} \partial_\sigma K_a &= K_a \partial_\tilde{v}.
\end{align*}
\]

\( (3.69) \)

\( (3.70) \)

Using the identities \((a \neq -1)\),

\[
\begin{align*}
(ve^\tau - e^\sigma)z^a &= (a + 1 + \partial_\tau) \int dzz^a, \\
(-ue^{-\tau} - e^\sigma)z^a &= (a + 1 - \partial_\tau) \int dzz^a
\end{align*}
\]

\( (3.71) \)

\( (3.72) \)

with \( z \equiv -\frac{ve^\tau + ue^{-\tau}}{2} - e^\sigma \), we can easily check that the solution is, for \( a \neq -1 \),

\[
K_a = (\frac{-\tilde{u}e^\tau + \tilde{v}e^{-\tau}}{2} - e^\sigma)^a.
\]

\( (3.73) \)

For the case \( a = -1 \) in which the identities are violated, the solution is \( \delta(z) \). Let us choose the case \( a = -2 \). Then, the kernel is characterized by

\[
\begin{align*}
\frac{1}{2}[e^\sigma + \partial_\tau \partial_\sigma^{-1} e^\sigma] e^\tau K_{-2} &= K_{-2} \tilde{u}, \\
\frac{1}{2}[e^\sigma - \partial_\tau \partial_\sigma^{-1} e^\sigma] e^{-\tau} K_{-2} &= K_{-2} \tilde{v},
\end{align*}
\]

\( (3.74) \)

\( (3.75) \)

\[
\begin{align*}
e^{-\tau - \sigma} \partial_\sigma K_{-2} &= K_{-2} \partial_\tilde{u}, \\
-e^{\tau - \sigma} \partial_\sigma K_{-2} &= K_{-2} \partial_\tilde{v}.
\end{align*}
\]

\( (3.76) \)

\( (3.77) \)

Note the position of \( e^{\pm \tau} \) in \((3.74)\) and \((3.75)\) comparing with \((3.67)\) and \((3.68)\).

On the other hand, the kernel \( M(u, v; \tau, \sigma) \equiv \delta(\frac{-ue^\tau + ve^{-\tau}}{2} - \sinh \sigma \frac{1}{2}) \) in \((3.57)\) is characterized by \((3.63)\)~\((3.66)\), namely,

\[
\begin{align*}
u M &= M(\cosh \sigma \partial_\tau + \sinh \sigma \partial_\sigma) \partial_\sigma^{-1} e^\tau, \\
v M &= M(\cosh \sigma \partial_\tau - \sinh \sigma \partial_\sigma) \partial_\sigma^{-1} e^{-\tau},
\end{align*}
\]

\( (3.78) \)

\( (3.79) \)

\[
\begin{align*}
\partial_u M &= M \frac{1}{2} \frac{1}{\cosh \sigma} e^{-\tau}, \\
\partial_v M &= -M \frac{1}{2} \frac{1}{\cosh \sigma} e^{\tau}.
\end{align*}
\]

\( (3.80) \)

\( (3.81) \)
We see that the rhs of the asymptotic form of \((3.78) \sim (3.81)\) for large \(\sigma\) coincide with the lhs of \((3.74) \sim (3.77)\) apart from a factor \(e^\sigma\).

Based on this observation, we define a new kernel \(G\) by

\[
G(u, v; \tilde{u}, \tilde{v}) \equiv -\frac{1}{8\pi^2} \int d\tau d\sigma M(u, v; \tau, \sigma) e^\sigma K_{-2}(\tilde{u}, \tilde{v}; \tau, \sigma).
\]  

(3.82)

This satisfies, in the limit of large \(u, -v\) and hence for large \(\tilde{u}, -\tilde{v}\), the following properties

\[
u G \sim G \tilde{u},
\]

(3.83)

\[
v G \sim G \tilde{v},
\]

(3.84)

\[
\partial_u G \sim G \partial_{\tilde{u}},
\]

(3.85)

\[
\partial_v G \sim G \partial_{\tilde{v}},
\]

(3.86)

and

\[
L^B_0(u, v) G = GL_0(\tilde{u}, \tilde{v}),
\]

(3.87)

where

\[
L^B_0(u, v) = \frac{1}{k-2}[(1 - uv)\partial_u \partial_v - \frac{1}{2}(u\partial_u + v\partial_v) - \frac{1}{2k}(u\partial_u - v\partial_v)^2],
\]

(3.88)

\[
L_0(\tilde{u}, \tilde{v}) = \frac{1}{k-2}[-\tilde{u}\tilde{v}\partial_{\tilde{u}} \partial_{\tilde{v}} - \frac{1}{2}(\tilde{u}\partial_{\tilde{u}} + \tilde{v}\partial_{\tilde{v}}) - \frac{1}{2k}(\tilde{u}\partial_{\tilde{u}} - \tilde{v}\partial_{\tilde{v}})^2].
\]

(3.89)

The transformation not only reduces to the identity asymptotically, but also sends the black hole background to the linear dilaton vacuum. This is checked by redefining new \(\tau\) and \(\sigma\) coordinates as

\[
\tilde{u} = e^{\sigma + \tau}, \quad \tilde{v} = -e^{\sigma - \tau}.
\]

(3.90)

Then, we have (compare with (2.13))

\[
L_0(\tilde{u}, \tilde{v}) = e^{-\sigma} L^B_0(\sigma, \tau) e^\sigma.
\]

(3.91)

The new integral transform enable us to identify the first nontrivial negative discrete operator in the bosonized form. More detailed properties of this transformation will be discussed in [1].

Finally, we note that the classical form of the above transformation takes a very simple form. Let the canonical momentum variables, conjugate to \(u, v\) and \(\tilde{u}, \tilde{v}\), be \(\Pi_u, \Pi_v\) and \(\Pi_{\tilde{u}}, \Pi_{\tilde{v}}\), respectively. The classical canonical transformation corresponding to \(G\) is then given as

\[
u = \tilde{u} + \frac{\tilde{v}\Pi^2_v}{(u\Pi_u + v\Pi_v)^2},
\]

(3.92)

\[
v = \tilde{v} + \frac{\tilde{u}\Pi^2_u}{(u\Pi_u + v\Pi_v)^2},
\]

(3.93)

\[
\Pi_{\tilde{u}} = \Pi_u(1 - \frac{\Pi_u\Pi_v}{(u\Pi_u + v\Pi_v)^2}),
\]

(3.94)

\[
\Pi_{\tilde{v}} = \Pi_v(1 - \frac{\Pi_u\Pi_v}{(u\Pi_u + v\Pi_v)^2}).
\]

(3.95)
This of course reduces to the identity transformation in the asymptotic limit. The generating function for this canonical transformation is

\[ F = \tilde{u}\Pi_u + \tilde{v}\Pi_v + \frac{\Pi_u\Pi_v}{\tilde{u}\Pi_u + \tilde{v}\Pi_v} \quad (3.96) \]

\[ \equiv \tilde{u}\Pi_u + \tilde{v}\Pi_v + \tilde{F} \quad (3.97) \]

by which the canonical variables are related as \( \frac{\partial F}{\partial \Pi_u} = u, \frac{\partial F}{\partial \tilde{u}} = \Pi_u, \ldots \). We can easily check that the classical particle hamiltonian satisfies \( (1 - uv)\Pi_u \Pi_v = -\tilde{u}\tilde{v}\Pi_{\tilde{u}} \Pi_{\tilde{v}} \). We may also try to construct the quantum transformation in the single fermion space directly by generalizing the classical form (3.96).

4. Further remarks

There are many questions left. The integral transformations for the linearized tachyon field discussed above indeed describe how to choose the variables in the phase space, at least, near the fermi surface, and suggest the feasibility of our conjecture that the black hole solution can be described by canonical transformations in the matrix model. In particular, our transformation \( G \) may be used to directly obtain the S-matrix around the black hole from the matrix model, since it satisfies the desirable asymptotic properties. The foregoing discussion is not, however, sufficient to decide whether the transformations should be generalized to the whole single-particle phase space of the free fermion. In connection with this, possible relations of our picture to the more conventional approaches as discussed in the first part of this lecture must also be clarified.

Let us finally make a comment related to the question of the correct dynamical degrees of freedom in string theory. The fermi-fluid picture on how the tachyon and the discrete states appear in the matrix model shows that the phase space of the matrix eigenvalue describes both the target space and the field-degrees of freedom in a unified manner. The fermi surface is nothing but a space-like surface of the target spacetime, and the small fluctuation of the surface is interpreted as the propagation of the tachyon field. A canonical transformation connecting different backgrounds, in general, mixes the target space and the field degrees of freedom. In this sense, there is a relativity with respect to the separation between the spacetime and the field degrees of freedom. Clear separation between them is only meaningful in the weak coupling regime where \( \beta\mu \) is large. This suggests that the correct framework for formulating short-distance dynamics of string theory as quantum gravity might be in some enlarged phase space containing target spacetimes as particular sections. As a first step toward the goal, we should be able to unify the case \( c < 1 \) and \( c = 1 \) in such a framework.

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Note added
After the workshop, we found two preprints (in hep.th) [25] [26] in which a similar integral transform as ours, (3.57) or (3.60), is discussed independently.

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