Isometry group orbit quantization of spinning strings in AdS$_3 \times$ S$^3$

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Received 27 October 2014
Accepted for publication 28 January 2015
Published 6 March 2015

Abstract

Describing the bosonic AdS$_3 \times$ S$^3$ particle and string in SU(1,1) × SU(2) group variables, we provide a Hamiltonian treatment of the isometry group orbits of solutions via analysis of the pre-symplectic form. For the particle we obtain a one-parameter family of orbits parameterized by creation–annihilation variables, which leads to the Holstein–Primakoff realization of the isometry group generators. The scheme is then applied to spinning string solutions characterized by one winding number in AdS$_3$ and two winding numbers in S$^3$. We find a two-parameter family of orbits, where quantization again provides the Holstein–Primakoff realization of the symmetry generators with an oscillator-type energy spectrum. Analyzing the minimal energy at strong coupling, we verify the spectrum of short strings at special values of winding numbers.

Keywords: AdS/CFT correspondence, AdS$_3 \times$ S$^3$, co-adjoint orbit method, string solutions, energy spectrum

1. Introduction and conclusion

Finding the energy spectrum of string excitations in AdS × S backgrounds is one of the major goals in the study of the AdS/CFT correspondence [1–3]. For the cases of AdS$_5 \times$ S$^5$ and AdS$_3 \times$ CP$^1$, a solution of the spectral problem has been proposed recently in terms of the so-

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called quantum spectral curve \cite{4–5}, though it heavily relies on the conjectured quantum integrability (for a review see \cite{6}). Calculation of the string spectrum by first principles appears intricate, and since the pioneering works \cite{7–9}, the main considerations were restricted to semiclassical analysis around solutions of string dynamics (see reviews \cite{10–12}).

In addition, these studies require that some of the \( \text{psu}(2, 2|4) \) charges diverge in the 't Hooft coupling as \( \lambda \gg 1 \), usually corresponding to the classical string solutions becoming long, whereas for finite charges, the short string regime, the analysis formally breaks down.

As observed in \cite{13}, this subtlety seems to be connected to the particular role played by the string zero-modes, which obtain a mass-term determined by the non-zero-mode excitations and which for short strings scale differently in \( \lambda \) than the non-zero-modes. Therefore, working in the bosonic subsector and using a static gauge \cite{14}, in \cite{15} a generalization of the pulsating string \cite{16, 17} was constructed, which explicitly allowed for unconstrained \( \text{AdS}_3 \times S^5 \) zero-modes. This so-called single-mode solution showed classical integrability and invariance under the isometries, even at the quantum level. Heuristically taking into account supersymmetric corrections, indeed the energy of the lowest excited state dual to a member of the Konishi multiplet was recovered up to order \( \lambda^{-14} \).

The present work should certainly be seen in this context. Note that as the single-mode solution \cite{15} is invariant under the isometries, it is nothing but the \( \text{SO}(2, 4) \times \text{SO}(6) \) group orbit of the pulsating string solution \cite{16, 17} constructed in \( \text{AdS}_3 \). Therefore, to devise similar systems one can consider the isometry group orbits of different well-known string solutions. To find the supersymmetric generalization one should construct the orbits of the full symmetry group \( \text{PSU}(2, 2|4) \) in the case of \( \text{AdS}_3 \times S^5 \).

The Kirillov–Kostant–Souriau theory of co-adjoint orbit quantization is a powerful tool (see, for example, the seminal work \cite{18}), and there is yet another reason to be interested in this method. As the bosonic string zero-modes become massive, the work \cite{15} benefited immensely from thorough understanding of the massive bosonic particle in \( \text{AdS} \times S \) \cite{19, 20}. Hence, for a generalization to the full superstring one should also expect that at least some of the fermionic zero-modes obtain a mass and that knowledge of the massive \( \text{AdS} \times S \) superparticle will be advantageous. However, even for the massless case, our understanding of this system seems unsatisfactory, where for the case of \( \text{AdS}_3 \times S^5 \) \cite{21} progress has been made in \cite{22–25}.

In the present paper we describe the dynamics of the bosonic \( \text{AdS}_3 \times S^5 \) particle and string in \( \text{SU}(1, 1) \times \text{SU}(2) \) group variables. After fixing our notation, we construct the isometry group orbits of a point particle sitting in the center of \( \text{AdS}_3 \) and rotating in \( S^5 \) and devise a Hamiltonian treatment by analyzing the corresponding pre-symplectic 1-form. We find a one-parameter family of orbits naturally parametrized in creation–annihilation variables, which yields a Holstein–Primakoff realization of the isometry algebra \cite{26, 27} and results in an oscillator-type energy spectrum. Hence, with relative ease we acquire exact quantization of the \( \text{AdS}_3 \times S^5 \) particle, which shows consistency with previous results \cite{28}. By this, it seems plausible that quantization of the bosonic particle in other \( \text{AdS} \times S \) spaces could be achieved by similar means. More interestingly, however, quantization of the \( \text{AdS} \times S \) superparticle, massless or massive, should be feasible by investigation of the supergroup orbits.

Next, in the spirit of the single-mode string \cite{15}, we apply the orbit method to the spinning string solutions introduced in \cite{29} (see also \cite{30, 31} for more details). Following essentially the same steps as for the particle, the isometry group orbits are characterized by two parameters and the winding numbers of the spinning string. Investigation of the pre-symplectic 1-form again prompts a description in creation–annihilation variables, giving an
oscillator-type realization of the symmetry generators and a corresponding spectrum. This yields exact quantization of the system, where in comparison to the particle, one has more freedom in the Casimir numbers.

However, the exact formula for the minimal energy $E_0$ turns out to be rather involved. Moreover, we expect our findings to match the result for the full superstring at leading order in strong coupling, $\lambda \gg 1$, only. Therefore, we conclude by investigating the minimal energy in this limit. As a check of our method, we study the different possibilities for the winding numbers and consistently identify long and short string solutions, characterized by their typical scaling behavior in 't Hooft coupling, $E_0 \propto \lambda^{1/2}$ and $E_0 \propto \lambda^{1/4}$, respectively.

The main goal of this work is to demonstrate the applicability of the quantization scheme utilizing the isometry group co-adjoint orbits for a well-known problem of current interest, namely, quantization of particles and classical string solutions in $\text{AdS} \times \text{S}$ spaces. We are looking forward to extending our analysis by exploring the orbit method for supergroups, which hopefully give new insights on the spectral problem, especially in the limit of short strings.

In particular, the $\text{AdS}_3/\text{CFT}_3$ duality has recently sparked extensive studies in related models. The machinery developed for $\text{AdS}_3 \times \text{S}^3$ is currently adapted to less supersymmetric spaces [32, 33], viz., the superstring theory in $\text{AdS}_3 \times \text{S}^3 \times \mathbb{T}^6$ [34, 35] and $\text{AdS}_7 \times \text{M}_4$ [1, 36–38]. Especially in [39], the $\text{AdS}_3 \times \text{S}^3$ spinning string studied in this work has been investigated in the presence of an NS–NS flux.

Another prevailing topic is the investigation of the $\mathbb{q}$-deformed $\text{AdS}_3 \times \text{S}^5$ superstring, which was first discovered by tracing its integrability structure [40, 41]. Only recently the corresponding space–time has been understood [42–44], and the generalization to other $\text{AdS} \times \text{S}$ spaces has been discussed in [45]. However, the dual-field theory is still unknown, and instead of a conformal boundary, the space–time shows a singularity, which seems to repel long string solutions [46, 47].

We are eager to see whether the presented method proves to be useful in both of these contexts.

## 2. Notation and conventions

Let us denote coordinates of $\mathbb{R}^{2,2}$ and $\mathbb{R}^4$ by $(X^0, X^1, X^2)$ and $(Y^1, Y^2, Y^3, Y^4)$, respectively. The $\text{AdS}_3$ and $\text{S}^3$ spaces are defined by the embedding conditions

\[
X \cdot X = (X^1)^2 + (X^2)^2 - (X^0)^2 - (X^0)^2 = -1,
\]

\[
Y \cdot Y = (Y^1)^2 + (Y^2)^2 + (Y^3)^2 + (Y^4)^2 = 1.
\]

One identifies $\text{AdS}_3$ with $\text{SU}(1, 1)$ and $\text{S}^3$ with $\text{SU}(2)$ by defining the group elements

\[
g = \begin{pmatrix} X^0 + iX^0 & X^1 - iX^2 \\ X^1 + iX^2 & X^0 - iX^0 \end{pmatrix}, \quad \tilde{g} = \begin{pmatrix} Y^0 + iY^3 & Y^2 + iY^4 \\ -Y^2 + iY^1 & Y^4 - iY^3 \end{pmatrix},
\]

where generally, due to their similarity, quantities corresponding to $\text{SU}(2)$ are denoted as the ones of $\text{SU}(1, 1)$, just with tildes.

We use the following basis of the $\mathfrak{su}(1, 1)$ algebra

\[
t_0 = i\sigma_3, \quad t_1 = \sigma_1, \quad t_2 = \sigma_2,
\]
with \( \{ \sigma_1, \sigma_2, \sigma_3 \} \) the Pauli matrices, such that the generators \( t_a \) satisfy the relations
\[
t_a \, t_b = \eta_{ab} I - \epsilon_{abc} t_c, \quad \text{for} \quad a, b, c = 0, 1, 2. \quad (4)
\]
Here \( I \) is the unit matrix, \( \eta_{ab} = \text{diag}(-1, 1, 1) \), and \( \epsilon_{abc} \) is the Levi-Civita tensor, with \( \epsilon_{012} = 1 \). The inner product defined by \( \langle t_a \, t_b \rangle = \frac{1}{2} \text{tr} (t_a \, t_b) = \eta_{ab} \) provides the isometry between \( su(1, 1) \) and the three-dimensional (3D) Minkowski space, since for \( u = u^a \, t_a \) one gets \( \langle u \, u \rangle = u^a \, u_a \). Then \( u \) can be timelike, spacelike, or lightlike as the corresponding 3D vector \( (u^0, u^1, u^2) \).

A standard basis in \( su(2) \) is given by \( \tilde{t}_j = i \tilde{\sigma}_j \), and one has
\[
\tilde{t}_i \, \tilde{t}_j = -\delta_{ij} I - \epsilon_{ijk} \, \tilde{t}_k, \quad \text{for} \quad i, j, k = 1, 2, 3. \quad (5)
\]
Hence, \( su(2) \), with inner product \( \langle \tilde{t}_i \, \tilde{t}_j \rangle = -\frac{1}{2} \text{tr} (\tilde{t}_i \, \tilde{t}_j) = \delta_{ij} \), is isometric to \( \mathbb{R}^3 \); i.e., \( \langle \tilde{u} \, \tilde{u} \rangle = \tilde{u}_j \, \tilde{u}_j \), where \( \tilde{u}_j = \langle \tilde{t}_j \, \tilde{u} \rangle \).

The matrices \( g \) and \( \tilde{g} \) in (2) and their inverse group elements can be written as
\[
g = X^a \, I + X^a \, t_a, \quad \tilde{g} = Y^j \, I + Y^j \, \tilde{t}_j, \quad g^{-1} = X^a \, I - X^a \, t_a, \quad \tilde{g}^{-1} = Y^j \, I - Y^j \, \tilde{t}_j, \quad (6)
\]
and from (4) and (5) one obtains the following relations between the length elements
\[
\langle g^{-1} d g \, g^{-1} d \tilde{g} \rangle = dX \cdot dX, \quad \langle g^{-1} d g \, \tilde{g}^{-1} d \tilde{g} \rangle = dY \cdot dY. \quad (7)
\]
The isometry transformations are therefore given by the left–right multiplications
\[
g \mapsto g_t \, g \, g_r, \quad \tilde{g} \mapsto \tilde{g}_t \, \tilde{g} \, \tilde{g}_r. \quad (8)
\]

### 3. The particle in \( SU(1,1) \times SU(2) \)

The dynamics of a particle in \( SU(1, 1) \times SU(2) \) is described by the action
\[
S = \int_0^\tau d\xi \left( \frac{1}{2 \xi} \left( \langle g^{-1} \tilde{g} \, g^{-1} \tilde{g} \rangle + \langle g^{-1} \tilde{g} \, g^{-1} \tilde{g} \rangle \right) - \frac{\xi \mu_0^2}{2} \right), \quad (9)
\]
where \( \xi \) plays the role of the world-line einbein and \( \mu_0 \) is the particle mass. In the first-order formalism, this action is equivalent to
\[
S = \int d\xi \left( \langle R g^{-1} \tilde{g} \rangle + \langle \tilde{R} \, \tilde{g}^{-1} \tilde{g} \rangle - \frac{\xi}{2} \left( \langle RR \rangle + \langle \tilde{R} \tilde{R} \rangle + \mu_0^2 \right) \right), \quad (10)
\]
where \( R \) and \( \tilde{R} \) are Lie algebra-valued phase–space variables, \( \xi \) becomes a Lagrange multiplier, and its variation defines the mass–shell condition with timelike \( R \)
\[
\langle RR \rangle + \langle \tilde{R} \tilde{R} \rangle + \mu_0^2 = 0. \quad (11)
\]
The Hamilton equations obtained from (10),
\[
g^{-1} \tilde{g} = \xi R, \quad \tilde{g}^{-1} \tilde{g} = \xi \tilde{R}, \quad R = 0, \quad \tilde{R} = 0, \quad (12)
\]
provide the conservation of \( R \) and \( \tilde{R} \) as well as of their ‘left’ counterparts
\[
L = g \, R g^{-1}, \quad \bar{L} = g \, \tilde{R} \tilde{g}^{-1}. \quad (13)
\]
The dynamical integrals \( L, \bar{L}, R, \) and \( \tilde{R} \) are the Noether charges related to the invariance of the action (9) with respect to the isometry transformations (8).
The first-order action (10) defines the pre-symplectic form of the system
\[ \Theta = \{ R g^{-1} dg \} + \{ \tilde{R} g^{-1} d\tilde{g} \}, \]  
which leads to the following Poisson brackets
\[
\begin{align*}
\{ L_a, L_b \} & = 2 \varepsilon_{a b c} L_c, \\
\{ R_a, R_b \} & = -2 \varepsilon_{a b c} R_c, \\
\{ \tilde{L}_a, \tilde{L}_b \} & = 2 \varepsilon_{a b c} \tilde{L}_c, \\
\{ \tilde{R}_a, \tilde{R}_b \} & = -2 \varepsilon_{a b c} \tilde{R}_c, \\
\{ \tilde{L}_a, \tilde{R}_b \} & = 0,
\end{align*}
\]
where \( L_a, \tilde{L}_j, R_a, \tilde{R}_j \) are the components of the charges in the bases (4) and (5)
\[
L_a = \langle t_a L \rangle, \quad \tilde{L}_j = \langle \tilde{t}_j \tilde{L} \rangle, \quad R_a = \langle t_a R \rangle, \quad \tilde{R}_j = \langle \tilde{t}_j \tilde{R} \rangle.
\]  
Since \( R = R^a t_a \) and \( \tilde{R} = \tilde{R}_j \tilde{t}_j \), the mass–shell condition (11) can be written as
\[
R_a R^a + \tilde{R}_j \tilde{R}^j + \mu^2 = 0,
\]
and it obviously has vanishing Poisson brackets with components (16). Hence, the components are gauge invariant and, therefore, the Poisson brackets algebra (15) will be preserved after gauge fixing.

Let us choose the gauge \( \xi = 1 \) and consider a solution of (12) in the SU(1, 1) part
\[
\begin{align*}
g & = e^{\mu t_a} g^a, \\
R & = \mu t_a,
\end{align*}
\]
which corresponds to the AdS_3 particle of mass \( \mu \geq 0 \) in the rest frame. The isometry transformations of (18) provide a class of solutions parameterized by \( \mu \) and the group variables
\[
\begin{align*}
g & = g_i e^{\alpha t_u} g^u, \\
R & = g^-\mu t_0 g^r.
\end{align*}
\]
To find the Poisson bracket structure on the space of parameters, we calculate the SU(1, 1) part of the pre-symplectic form (14). For fixed \( \tau \) this calculation yields
\[
\Theta = \{ R g^{-1} dg \} = \mu \{ t_0 g^{-1} dg \} + \mu \{ t_0 d g \} - \tau d\mu,
\]
and we can neglect the exact form \( -\tau d\mu \). With help of the Cartan decomposition,
\[
\begin{align*}
g_i & = e^{t_0 t_1} g^{(1)} e^{t_1} g^1, \\
g_r & = e^{t_0 t_1} g^{(1)} e^{t_1} g^{(1)},
\end{align*}
\]
as elaborated in the appendix, the remaining terms in (20) reduces to a canonical 1-form
\[
\Theta = \mu d\varphi + H_i d\phi_i + H_r d\phi_r,
\]
where we defined the following quantities,
\[
\varphi = -\left( \alpha_i + \beta_i + \alpha_r + \beta_r \right), \quad \phi_i = \frac{\pi}{2} - 2\alpha_i, \quad \phi_r = \pi - 2\alpha_r,
\]
\[
H_i = \frac{\mu}{2} \left( \cosh \left( 2\rho_i \right) - 1 \right), \quad H_r = \frac{\mu}{2} \left( \cosh \left( 2\rho_r \right) - 1 \right)
\]
The conserved Noether charges constructed from (19) and (21) read
\[
L = \mu e^{\alpha t_1} e^{\gamma t_1} t_0 e^{-\beta t_1} e^{-\eta t_0}, \quad R = \mu e^{-\alpha t_0} e^{-\gamma t_1} t_0 e^{\beta t_1} e^{\eta t_0},
\]
and by (23) and (24) their components become
\[
\begin{align*}
L_0 & = \mu + 2H_i, \\
R_0 & = \mu + 2H_r, \\
L_\pm & = \sqrt{\mu H_i + H_i^2} e^{\pm i\phi}, \\
R_\pm & = \sqrt{\mu H_r + H_r^2} e^{\pm i\phi},
\end{align*}
\]
where \( L_\pm = \frac{1}{2}(L_1 \pm iL_2) \) and \( R_\pm = \frac{1}{2}(R_2 \pm iR_1) \).
Similarly, for SU(2) we consider the isometry group orbit of the solution $\tilde{g} = e^{\mu \tilde{t}_3}$, with $\mu \geq \tilde{\nu}$. Repeating the same steps we obtain the canonical 1-form

$$\tilde{\theta} = (\tilde{R} \tilde{g}^{-1} \tilde{d}\tilde{g}) = \tilde{\mu} d\tilde{\phi}, \quad \tilde{H}_i d\phi_i + \tilde{H}_r d\tilde{\phi}_r. \quad (27)$$

The canonical coordinates (89) and (90) given in the appendix parameterize the Noether charges $\tilde{L}_3, \tilde{L}_z = \frac{1}{2}(\tilde{L}_1 \pm i\tilde{L}_2)$ and $\tilde{R}_3, \tilde{R}_z = \frac{1}{2}(\tilde{R}_2 \pm i\tilde{R}_1)$ as follows

$$\tilde{L}_3 = \tilde{\mu} - 2\tilde{H}_i, \quad \tilde{R}_3 = \tilde{\mu} - 2\tilde{H}_r,$$

$$\tilde{L}_z = \sqrt{\tilde{\mu} \tilde{H}_i - \tilde{H}_r^2} e^{\pm i\tilde{\phi}}, \quad \tilde{R}_z = \sqrt{\tilde{\mu} \tilde{H}_r - \tilde{H}_i^2} e^{\pm i\tilde{\phi}}. \quad (28)$$

From the canonical variables $H \geq 0$ and $\phi \in S^1$ one naturally defines creation–annihilation variables as

$$a^+ = \sqrt{H} e^{i\tilde{\phi}}, \quad a = \sqrt{H} e^{-i\tilde{\phi}}. \quad (29)$$

The form of the functions (26) and (28) then dictates the realization of the isometry group generators in terms of creation–annihilation operators, which is known as the Holstein–Primakoff transformation [26, 27]. Thus, we have

$$L^0 = \mu + 2a^+_i a_i, \quad R^0 = \mu + 2a^+_r a_r,$$

$$L_+ = a^+_i \sqrt{|\mu + a^+_i a_i|}, \quad R_+ = a^+_r \sqrt{|\mu + a^+_r a_r|},$$

$$L_- = \sqrt{|\mu + a^+_i a_i|}, \quad R_- = \sqrt{|\mu + a^+_r a_r|}. \quad (30)$$

These yield a representation of $su(1, 1) \oplus su(1, 1) \oplus su(2) \oplus su(2)$ with basis vectors

$$\{\mu, \tilde{\mu}, k_i, k_i, \tilde{k}_i, \tilde{k}_i\} = \{\mu, k_i\} \{\tilde{\mu}, \tilde{k}_i\}. \quad (32)$$

where $k_{i,r}, \tilde{k}_{i,r}$ are non-negative integers and, furthermore, $\tilde{k}_{i,r} \leq \tilde{\mu}$.

The representation is characterized by the Casimir numbers

$$C_{AdS} = -L_a L^a = -R_a R^a = \mu (\mu - 2),$$

$$C_k = \tilde{L}_j \tilde{L}_j = \tilde{R}_j \tilde{R}_j = \tilde{\mu} (\tilde{\mu} + 2), \quad (33)$$

which are related through the mass–shell condition (17)

$$C_{AdS} = C_k + \mu_0^2. \quad (34)$$

and we find

$$\mu = 1 + \sqrt{\mu_0^2 + (\tilde{\mu} + 1)^2}. \quad (35)$$

Since translations along the AdS3 time direction correspond to rotations in the $(X^0, X^0')$ plane, the energy operator is given by

$$E = \frac{1}{2} (L^0 + R^0). \quad (36)$$
and from (30) we obtain the energy spectrum

$$E = \mu + k_i + k_r.$$  \hfill (37)

Here, $\mu$ is defined by (35) and corresponds to the lowest energy level for a given total angular momentum $\mu$ on $S^3$. Equations (35) and (37) reproduce the result obtained in the covariant quantization or in the static gauge approach [20].

In the following section we use a similar scheme to calculate the energy spectrum of SU(1, 1) × SU(2) string solutions.

### 4. The spinning string in SU(1, 1) × SU(2)

The Polyakov action for the SU(1, 1) × SU(2) string is given by

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int \sqrt{-g} \sqrt{-\hat{g}} \left( \langle g^{-1}\partial_\sigma g \langle g^{-1}\partial_\rho g \rangle + \langle g^{-1}\partial_\sigma \hat{g} \langle g^{-1}\partial_\rho \hat{g} \rangle \rangle \right).$$ \hfill (38)

Here, $\lambda$ is a dimensionless coupling constant, which in the context of the AdS/CFT correspondence plays the role of the 't Hooft coupling. In analogy to the case of the particle, for the closed string this action is equivalent to

$$S = \int \sqrt{-\hat{h}} \sqrt{-\hat{h}} \left( \langle R g^{-1}\hat{g} \rangle + \langle \dot{R} \hat{g}^{-1}\dot{\hat{g}} \rangle - \xi_1 \left( \langle R g^{-1}\dot{g} \rangle + \langle \dot{R} \hat{g}^{-1}\dot{\hat{g}} \rangle \right) \right) - \frac{\xi_2}{2\sqrt{-\hat{h}}} \left( \langle R R \rangle + \langle \dot{R} \dot{R} \rangle + \lambda \left( \langle g^{-1}\dot{g} \rangle^2 + \langle \hat{g}^{-1}\dot{\hat{g}} \rangle^2 \right) \right).$$ \hfill (39)

The Lagrange multipliers $\xi_1$ and $\xi_2$ are related to the worldsheet metric by

$$\xi_1 = -\sqrt{-\hat{h}} \frac{1}{\sqrt{-\hat{h}}}, \quad \xi_2 = -\frac{\lambda}{\sqrt{\hat{h}}},$$ \hfill (40)

and their variations provide the Virasoro constraints

$$\langle RR \rangle + \langle \dot{R} \dot{R} \rangle + \lambda \left( \langle g^{-1}\dot{g} \rangle^2 + \langle \hat{g}^{-1}\dot{\hat{g}} \rangle^2 \right) = 0,$n\langle R g^{-1}\dot{g} \rangle + \langle \dot{R} \hat{g}^{-1}\dot{\hat{g}} \rangle = 0.$$ \hfill (41)

The conformal gauge corresponds to $\xi_1 = 1$ and $\xi_2 = 0$. In this case the equations of motion obtained from (39) become

$$\sqrt{\lambda} g^{-1}\dot{g} = \mathcal{R}, \quad \mathcal{R} = \sqrt{\lambda} (g^{-1}\dot{g}),$$

$$\sqrt{\lambda} \hat{g}^{-1}\dot{\hat{g}} = \mathcal{\dot{R}}, \quad \mathcal{\dot{R}} = \sqrt{\lambda} (\hat{g}^{-1}\dot{\hat{g}}),$$ \hfill (42)

and they are equivalent to

$$\partial_\sigma (g^{-1}\dot{g}) = \partial_\sigma (g^{-1}\dot{g}), \quad \partial_\sigma (\hat{g}^{-1}\dot{\hat{g}}) = \partial_\sigma (\hat{g}^{-1}\dot{\hat{g}}).$$ \hfill (43)

We now consider the following solution of these equation [48]

$$g = \begin{pmatrix} \cosh \theta e^{i(\sigma + \pi n)} & \sinh \theta e^{i(\sigma + \pi n)} \\ \sinh \theta e^{-i(\sigma + \pi n)} & \cosh \theta e^{-i(\sigma + \pi n)} \end{pmatrix}. \hfill (44)$$
\[
\tilde{g} = \begin{pmatrix}
\cos \tilde{\theta} e^{i(\tilde{\tau} + i\sigma)} & i \sin \tilde{\theta} e^{i(\tilde{\tau} + i\sigma)} \\
i \sin \tilde{\theta} e^{-i(\tilde{\tau} + i\sigma)} & \cos \tilde{\theta} e^{-i(\tilde{\tau} + i\sigma)}
\end{pmatrix},
\]
(45)
with the parameters fulfilling
\[
p^2 - \tilde{p}^2 = n^2 - \tilde{n}^2, \quad \rho^2 - \tilde{\rho}^2 = \tilde{m}^2.
\]
(46)
which turns out to be the renowned spinning string solution [29–30].

In the appendix we present the matrices \( g^{-1} \tilde{g}, g^{-1} g', \tilde{g}^{-1} \tilde{g}, \tilde{g}^{-1} g' \) calculated from (44) and (45). The corresponding equations (91) and (92) show that the conditions (46) indeed provide (43). The matrices \( \mathcal{R} \) and \( \mathcal{R} \) are defined by the Hamilton equations (42) and the Virasoro constraints (41) then lead to the additional conditions
\[
(e^2 + m^2) \cosh^2 \theta - (p^2 + n^2) \sinh^2 \theta = (\tilde{e}^2 + \tilde{m}^2) \cos^2 \tilde{\theta} + (\tilde{p}^2 + \tilde{n}^2) \sin^2 \tilde{\theta},
\]
(47)
which are obtained from the induced metric (94) given in the appendix.

Note that the components of the induced metric tensor are constants on both the SU(1, 1) and the SU(2) projections. The scheme of the Pohlmeyer reduction [49, 50] for a flat induced metric yields a linear system with constant coefficients, which is simply integrated in the exponential form, like in (44)(45), [51]. This is a typical feature of these so-called homogeneous solutions [29–31].

Since we consider a closed string in SU(1, 1) \( \times \) SU(2), the parameters \( m, n, \tilde{m}, \tilde{n} \) have to be integers. However, if we unwrap the time coordinate, the polar angle in the \((X^0, X^0)\) plane, it has to be periodic in \( \sigma \) itself. This is obviously achieved for \( m = 0 \) only, which is assumed below.

Thus, our solutions are parameterized by three winding numbers and six continuous variables, which satisfy the four conditions in (46) and (47). Hence, for given winding numbers, we have a two-parameter family of solutions\(^5\).

Similarly to the particle dynamics, we consider the isometry group orbits of the solutions with the aim to find their Hamiltonian description and quantization. For this purpose we analyze the pre-symplectic form defined by (39),
\[
\Theta = \int_{0}^{2\pi} \frac{d\sigma}{2\pi} \left( (\mathcal{R} g^{-1} dg) + (\tilde{\mathcal{R}} \tilde{g}^{-1} d\tilde{g}) \right).
\]
(48)
To calculate this 1-form on the space of orbits, one has to make the replacements
\[
\mathcal{R} \mapsto \sqrt{\lambda} \ g^{-1}_r g^{-1}_r \ g_r, \quad g \mapsto g_r g_r, \quad g^{-1} \mapsto g^{-1} g^{-1}_r g^{-1}_r,
\]
(49)
similarly for the SU(2) term, and then identify \( g \) with (44) and \( \tilde{g} \) with (45), respectively. For the SU(1, 1) part this yields
\[
\theta = \langle L \ g^{-1}_r dg \rangle + \langle R \ dg_r g^{-1}_r \rangle + \sqrt{\lambda} \int_{0}^{2\pi} \frac{d\sigma}{2\pi} \langle g^{-1} g^{-1} g^{-1}dg \rangle,
\]
(50)
where \( L \) and \( R \) are the Noether charges related to the isometries (8) as
\[
L = \sqrt{\lambda} \int_{0}^{2\pi} \frac{d\sigma}{2\pi} \tilde{g} \tilde{g}^{-1}, \quad R = \sqrt{\lambda} \int_{0}^{2\pi} \frac{d\sigma}{2\pi} \tilde{g}^{-1} \tilde{g},
\]
(51)
\(^5\) Note that the particle solutions in SU(1, 1) \( \times \) SU(2) were parameterized by one variable \( \tilde{\mu} \).
and the differential of $g$ in the last term of (50) is taken with respect to the parameters of the solution (44). The calculations given by (95) in the appendix show that the last term in (50) is an exact form and can be neglected. The SU(2) part is computed in a similar way, and altogether we find the 1-form

$$\Theta = \langle L \, g^{-1} \, dg_L \rangle + \langle R \, dg_r, g_r^{-1} \rangle + \langle \tilde{L} \, \tilde{g}^{-1} \, d\tilde{g} \rangle + \langle \tilde{R} \, d\tilde{g}, \tilde{g}^{-1} \rangle,$$  

(52)

where $L$ and $R$ are the Noether charges similar to (51),

$$L = \sqrt{x} \int_0^{2\pi} \frac{d\sigma}{2\pi} \bar{g} \bar{g}^{-1}, \quad R = \sqrt{x} \int_0^{2\pi} \frac{d\sigma}{2\pi} \bar{g}^{-1} \bar{g}.$$  

(53)

These charges are easily calculable by the currents given in the appendix. However, their matrix form depends on the winding numbers, and one has to distinguish between the cases $n \neq 0$ and $n = 0$ for SU(1, 1), as well as $\tilde{m}^2 \neq \tilde{n}^2$ and $\tilde{m}^2 = \tilde{n}^2$ for SU(2).

Let us consider the case $n \neq 0$ and $\tilde{m}^2 \neq \tilde{n}^2$. The integration of the off-diagonal terms of the currents (91)–(93) vanish, and we obtain

$$L = \mu_1 t_0, \quad \tilde{L} = \tilde{\mu}_1 \check{t}_3, \quad R = \mu_r t_0, \quad \tilde{R} = \tilde{\mu}_r \check{t}_3,$$  

(54)

where

$$\mu_i = \sqrt{x} \left( e \cosh^2 \theta - p \sinh^2 \theta \right), \quad \mu_r = \sqrt{x} \left( e \cosh^2 \theta + p \sinh^2 \theta \right),$$  

(55)

$$\tilde{\mu}_i = \sqrt{x} \left( \tilde{e} \cos^2 \hat{\theta} - \tilde{p} \sin^2 \hat{\theta} \right), \quad \tilde{\mu}_r = \sqrt{x} \left( \tilde{e} \cos^2 \hat{\theta} + \tilde{p} \sin^2 \hat{\theta} \right).$$  

(56)

We can assume that the numbers $\mu_i, \mu_r, \tilde{\mu}_i, \tilde{\mu}_r$ are non-negative.

Similarly to the particle case, the 1-form (52) then becomes

$$\Theta = \mu_1 \langle t_0 \, g^{-1} \, dg_L \rangle + \mu_r \langle t_0 \, dg_r, g_r^{-1} \rangle + \tilde{\mu}_1 \langle t_3 \bar{g}^{-1} \, d\bar{g} \rangle + \tilde{\mu}_r \langle t_3 \, d\bar{g}, \bar{g}^{-1} \rangle,$$  

(57)

and the same parametrization as in (21) leads to the canonical 1-form

$$\Theta = \mu_1 \, d\phi_1 + H_1 \, d\phi_1 + H_2 \, d\phi_2 + \tilde{\mu}_1 \, d\tilde{\phi}_1 + \tilde{H}_1 \, d\tilde{\phi}_1 + \tilde{\mu}_r \, d\tilde{\phi}_r + \tilde{H}_r \, d\tilde{\phi}_r.$$  

(58)

The components of the symmetry generators have the same form as in (26) and (28)

$$L^0 = \mu_i + 2H_i, \quad R^0 = \mu_r + 2H_r,$$

(59)

$$L_+ = \sqrt{\mu_1 H_1 + H_1^2} \, e^{\pm i\phi_1}, \quad R_+ = \sqrt{\mu_r H_r + H_r^2} \, e^{\pm i\phi_r},$$

$$L_3 = \mu_1 - 2H_1, \quad R_3 = \mu_r - 2H_r,$$

$$L_+ = \sqrt{\mu_r H_r - H_r^2} \, e^{\pm i\phi_r}, \quad R_+ = \sqrt{\mu_1 H_1 - H_1^2} \, e^{\pm i\phi_1}.$$  

(60)

Here, now the Casimir numbers $\mu_i$ and $\mu_r$ are independent, whereas $\tilde{\mu}_i$ and $\tilde{\mu}_r$ are integers of the same parity, which ensures that the total angular momentum $\frac{1}{2}(\tilde{\mu}_i + \tilde{\mu}_r)$ on $S^3$ takes integer values.

Hence, as in (30) and (31), the Holstein–Primakoff transformation provides a realization of the isometry group generators, and the energy given by (36) is now obtained from (59) and (55), having the spectrum

$$E = E_0 + k_1 + k_r,$$  

(61)

where $k_1$ and $k_r$ are non-negative integers and $E_0 = \sqrt{x} \, e \cosh^2 \theta$ corresponds to the minimal energy for given $\tilde{\mu}_i, \tilde{\mu}_r$. To find the dependence of this term on $\tilde{\mu}_i, \tilde{\mu}_r$ and the winding numbers, one has to use (56) and the constraints (46) and (47). Hence, we get
\[ \tilde{\varepsilon} = \frac{1}{\sqrt{\lambda}} \left( \mu_i + \mu_j \right), \quad \tilde{\rho} = \frac{1}{\sqrt{\lambda}} \left( \mu_i - \mu_j \right). \]  

(62)

Inserting them in (46), one gets a fourth-order equation for \( \cos 2\tilde{\theta} \)

\[
\left( \tilde{\mu}_i + \tilde{\mu}_j \right)^2 \left( 1 - \cos 2\tilde{\theta} \right)^2 - \left( \tilde{\mu}_i - \tilde{\mu}_j \right)^2 \left( 1 + \cos 2\tilde{\theta} \right)^2 = \lambda \left( \tilde{m}^2 - \tilde{n}^2 \right) \left( 1 - \cos^2 2\tilde{\theta} \right)^2.
\]  

(63)

The solution of this equation and (62) define the right-hand sides of (47) as a function of the coupling constant and four integers \( m, n \). Solving (47) for \( e^2 \) and \( \sinh \tilde{\theta} \), one obtains a third-order equation, which can be solved in a standard way. Hence, we acquired exact quantization of the spinning string solution at hand, where, however, the final answer for \( E_0 \) takes a rather complicated form.

Furthermore, in analogy to the discussion in [15], we expect that the obtained spectrum concurs with the one of corresponding states of the full superstring theory only at the leading order in strong coupling, \( \lambda \gg 1 \). Therefore, let us present the analysis only in this limit, which corresponds to the near-flat-space regime.

First we consider the case when both \( \tilde{m} \) and \( \tilde{n} \) are non-zero and assume \( 0 < \tilde{m}^2 < \tilde{n}^2 \). Using (46) and (56), the system (47) can be written as

\[
e^2 - 2n^2 \sinh^2 \theta = \tilde{e}^2 + 2n^2 \sin^2 \tilde{\theta} + \tilde{m}^2 \cos 2\tilde{\theta},
\]

\[
|n| \sqrt{e^2 + n^2 \sin^2 \theta} = \left| m \left( \tilde{\mu}_i + \tilde{\mu}_j \right) + \tilde{n} \left( \tilde{\mu}_i - \tilde{\mu}_j \right) \right| \lambda^{-1/2}.
\]  

(64)

At large \( \lambda \), from (63) and (62) we find

\[
\cos 2\tilde{\theta} = 1 - \frac{\vert \tilde{\mu}_i - \tilde{\mu}_j \vert}{\sqrt{\tilde{n}^2 - \tilde{m}^2}} \lambda^{-1/2} + \mathcal{O}(\lambda^{-1}), \quad \tilde{\varepsilon} = \frac{\tilde{\rho}}{2} \lambda^{-1/2} + \mathcal{O}(\lambda^{-1}).
\]  

(65)

and then (64) yields \( \sinh^2 \theta = \mathcal{O}(\lambda^{-1/2}) \), \( e = |\tilde{m}| + \mathcal{O}(\lambda^{-1/2}) \), and \( E_0 = |\tilde{m}| \lambda^{1/2} + \mathcal{O}(\lambda^0) \).

The case \( 0 < \tilde{n}^2 < \tilde{m}^2 \) is analyzed similarly. Its large \( \lambda \) behavior is governed by

\[
\cos 2\tilde{\theta} = -1 + \frac{\tilde{\mu}_i + \tilde{\mu}_j}{\sqrt{\tilde{m}^2 - \tilde{n}^2}} \lambda^{1/2} + \mathcal{O}(\lambda^{-1}), \quad \tilde{\rho} = \frac{\tilde{\mu}_i - \tilde{\mu}_j}{2} \lambda^{-1/2} + \mathcal{O}(\lambda^{-1}).
\]  

(66)

which again follows from (63) and (62). Writing now the first equation of (47) as

\[
e^2 - 2n^2 \sinh^2 \theta = \tilde{\rho}^2 + 2\tilde{m}^2 \cos^2 \tilde{\theta} - \tilde{n}^2 \cos 2\tilde{\theta},
\]  

(67)

we find \( \sinh^2 \theta = \mathcal{O}(\lambda^{-1/2}) \), \( e = |\tilde{n}| + \mathcal{O}(\lambda^{-1/2}) \), and \( E_0 = |\tilde{n}| \lambda^{1/2} + \mathcal{O}(\lambda^0) \).

The analysis of the case \( |\tilde{m}| = |\tilde{n}| \) is the simplest, and it leads to the same answer. Thus, if \( \tilde{m} \neq 0 \) and \( \tilde{n} \neq 0 \), the leading-order behavior of \( E_0 \) is given by

\[
E_0 = \min \left( |\tilde{m}|, |\tilde{n}| \right) \lambda^{1/2} + \mathcal{O}(\lambda^0).
\]  

(68)

Note that for \( \tilde{m} = 0 = \tilde{n} \), from (47) one has \( n = 0 \). The solution then becomes \( \sigma \), independent, and it describes the massless particle in AdS3 × S3.

It remains to analyze the two cases \( \tilde{m} = 0, \tilde{n} \neq 0, \) and \( \tilde{m} \neq 0, \tilde{n} = 0 \). In the first case the system (47) reduces to

\[
e^2 - 2n^2 \sinh^2 \theta = \tilde{e}^2 + 2n^2 \sin^2 \tilde{\theta}, \quad \sqrt{n^2 \tilde{e}^2 + n^4 \sin^2 \theta} = \sqrt{n^2 \tilde{e}^2 + \tilde{n}^2 \sin^2 \tilde{\theta}}.
\]  

(69)
Here, one has to use the same large \( \lambda \) behavior as in (65)

\[
\cos 2\hat{\theta} = 1 - \frac{[\tilde{\mu}_i - \tilde{\mu}_f]}{|\tilde{m}|} \lambda^{-1/2} + \mathcal{O}\left(\lambda^{-1}\right), \quad \tilde{e} = \frac{\tilde{\mu}_i + \tilde{\mu}_f}{2} \lambda^{-1/2} + \mathcal{O}\left(\lambda^{-1}\right).
\]

(70)

From (69) we then find

\[
sinh^2 \theta = \frac{[\tilde{n}(\tilde{\mu}_i - \tilde{\mu}_f)]}{2n^2} \lambda^{-1/2} + \mathcal{O}\left(\lambda^{-1}\right), \quad e^2 = 2 [\tilde{n}(\tilde{\mu}_i - \tilde{\mu}_f)] \lambda^{-1/2} + \mathcal{O}\left(\lambda^{-1}\right),
\]

(71)

\[
E_0 = \sqrt{2 [\tilde{n}(\tilde{\mu}_i - \tilde{\mu}_f)]} \lambda^{1/4} + \mathcal{O}\left(\lambda^{-1/4}\right).
\]

(72)

Note that for \( \mu = \tilde{\mu} \), the exact solution of the system takes the following simple form

\[
\cos 2\hat{\theta} = 1, \quad \tilde{e} = \tilde{\mu}_f \lambda^{1/2} = e, \quad \sinh^2 \theta = 0, \quad E_0 = \bar{\mu}_f,
\]

(73)

and it corresponds to a particle solution in (44) and (45).

In the second case, \( \tilde{n} = 0 \), the system (47) can be written in the form

\[
e^2 - 2n^2 \sinh^2 \theta = \bar{\rho}^2 + 2\bar{m}^2 \cos^2 \bar{\theta}, \quad \sqrt{n^2 e^2 + n^4 \sinh^2 \theta} = \sqrt{\bar{n}^2 \bar{\rho}^2 + \bar{m}^2 \cos^2 \bar{\theta}}.
\]

(74)

The solutions of (63) and (62) at large \( \lambda \) now are

\[
\cos 2\hat{\theta} = -1 + \frac{[\tilde{\mu}_i + \tilde{\mu}_f]}{|\tilde{m}|} \lambda^{-1/2} + \mathcal{O}\left(\lambda^{-1}\right), \quad \bar{\rho} = \frac{\tilde{\mu}_i - \tilde{\mu}_f}{2} \lambda^{-1/2} + \mathcal{O}\left(\lambda^{-1}\right),
\]

(75)

and (74) leads to

\[
sinh^2 \theta = \frac{[\tilde{n}(\tilde{\mu}_i + \tilde{\mu}_f)]}{2n^2} \lambda^{-1/2} + \mathcal{O}\left(\lambda^{-1}\right), \quad e^2 = 2 [\tilde{n}(\tilde{\mu}_i + \tilde{\mu}_f)] \lambda^{-1/2} + \mathcal{O}\left(\lambda^{-1}\right),
\]

(76)

\[
E_0 = \sqrt{2 [\tilde{n}(\tilde{\mu}_i + \tilde{\mu}_f)]} \lambda^{1/4} + \mathcal{O}\left(\lambda^{-1/4}\right).
\]

(77)

The case \( \mu = \mu \) is again special, giving the simple solution in the SU(2) part

\[
\bar{\rho} = 0, \quad \tilde{e} = |\tilde{m}|, \quad \cos^2 \bar{\theta} = \frac{\tilde{\mu}_i}{|\tilde{m}|} \lambda^{-1/2},
\]

(78)

and the corresponding minimal energy

\[
E_0 = 2 \sqrt{|\tilde{m}|} \lambda^{1/4} + \mathcal{O}\left(\lambda^{-1/4}\right).
\]

(79)

Note that (69)–(72) become (74)–(77) by substituting

\[
\{ \tilde{n}, \tilde{e}, \tilde{\theta}, \tilde{\mu}_i + \tilde{\mu}_f \} \leftrightarrow \{ \tilde{m}, \bar{\rho}, \bar{\theta} + \pi/2, \tilde{\mu}_i - \tilde{\mu}_f \},
\]

(80)

which corresponds to interchanging the \((Y^1, Y^2)\) with the \((Y^3, Y^4)\) plane\(^6\).

We can now compare the results qualitatively. For \( m, n > 0 \) the scaling of the minimal energy (68), \( E_0 \propto \lambda^{1/2} \), suggests that the corresponding strings are long. In contrast, for \( m = 0 \) or \( n = 0 \) we found \( E_0 \propto \lambda^{1/4} \) (see (72) and (77)), which is the typical scaling behavior of

\(^6\) This symmetry could have been made manifest by also allowing for negative \( su(2) \) Casimir numbers, \( \bar{\mu}_{\alpha_i} \in \mathbb{Z} \) with \( \frac{1}{2}(\bar{\mu}_i + \bar{\mu}_f) \in \mathbb{Z} \), which, however, would have complicated most formulas.
short strings. Indeed, in the first case the string wraps both circles, the one in the \((Y^1, Y^2)\) and the one in the \((Y^3, Y^4)\) plane, and hence cannot become small, while for the latter cases this is possible, as the string now only wraps one circle.

As we are particularly interested in the short-string regime, recall that the \(su(2)\) Casimir numbers \(\tilde{\mu}_l\) and \(\tilde{\mu}_r\) have the same parity. Hence, (72) and (77) both yield the minimal energy of the form

\[
E_0 = 2\sqrt{N\lambda^{-1/4}} + \mathcal{O}\left(\lambda^{1/4}\right), \quad \text{with } N \in \mathbb{N}
\]

which is nothing but the renowned result by Gubser, Klebanov, and Polyakov [9] for the near-flat-space limit. The lowest excited states, \(N = 1\), ought to be dual to some members of the Konishi multiplet.

Acknowledgments

We thank Harald Dorn, Sergei Frolov, Ben Hoare, and Jan Plefka for useful discussions, and Stijn van Tongeren for valuable comments on the manuscript. We also thank Chrysostomos Kalousios and Zurab Kepuladze for collaboration at an initial stage of the work. G J thanks the Humboldt University of Berlin and the Max-Planck Institute for Gravitational Physics in Potsdam for kind hospitality. M H thanks Nordita in Stockholm for kind hospitality. The research leading to these results has received funding from the Volkswagen Foundation, WFS, Rustaveli GNSF, the International Max Planck Research School for Geometric Analysis, Gravitation and String Theory, and a DFG grant in the framework of the SFB 647.

Appendix

The commutation relations of the basis vectors (3) provide the following adjoint transformation properties

\[
e^{\gamma_1} t_0 e^{-\gamma_1} = \cosh (2\gamma) t_0 + \sinh (2\gamma) t_2, \quad e^{\alpha t_0} t_2 e^{-\alpha t_0} = \cos (2\alpha) t_2 + \sin (2\alpha) t_1. \quad (82)
\]

The conserved charges (25) then can be written as

\[
L = \mu \left( \cosh (2\gamma) t_0 + \sinh (2\gamma)(\cos (2\alpha) t_2 + \sin (2\alpha) t_1) \right),
\]

\[
R = \mu \left( \cosh (2\gamma) t_0 - \sinh (2\gamma)(\cos (2\alpha) t_2 - \sin (2\alpha) t_1) \right). \quad (83)
\]

From these equations it follows that the angle variables \(\phi_l\) and \(\phi_r\) defined in (23) correspond to the phases of \(L_1 + iL_2\) and \(R_2 + iR_1\), respectively, as in (26).

By (21), the ‘left’ term of the 1-form (20) becomes

\[
\mu \langle t_0 s_l^{-1} d t_l \rangle = \mu \left( \langle e^{\gamma_1} t_0 e^{-\gamma_1} t_0 \rangle d\gamma_1 - d\beta_l \right) \quad (84)
\]

with the coefficient of \(d\gamma_l\) being \(\langle t_0 t_l \rangle = 0\). Similarly, the ‘right’ term in (20) reads

\[
\mu \langle t_0 s_r^{-1} d t_r \rangle = \mu \left( \langle e^{\gamma_1} t_0 e^{-\gamma_1} t_0 \rangle d\gamma_1 - d\beta_r \right). \quad (85)
\]

Taking into account then (82) and \(\langle t_0 t_0 \rangle = \eta_{ab}\), we arrive at (22)–(24) and (26). Note that in (24) we substracted 1 from \(\cosh (2\gamma)\) to have \(H > 0\).
For \( \text{SU}(2) \) one has transformations similar to (82),
\[
e^{ij_1} t_1 e^{-i j_1} = \cos (2\gamma) t_1 + \sin (2\gamma) t_2,
\]
\[
e^{i j_2} t_2 e^{-i j_2} = \cos (2\tilde{\gamma}) t_2 + \sin (2\tilde{\gamma}) t_1.
\]
Repeating the same steps as for \( \text{SU}(1, 1) \), one obtains equations similar to (83)–(85),
where one has to substitute untilded with tilded parameters along with the replacements
\[
cosh (2\gamma) \mapsto \cos (2\tilde{\gamma}), \quad \sinh (2\gamma) \mapsto \sin (2\tilde{\gamma}), \quad t_0 \mapsto \tilde{t}_3.
\]
This procedure yields the following 1-form
\[
\tilde{\vartheta} = \tilde{\mu}(d\tilde{\varphi} + d\tilde{\bar{\varphi}}) + \tilde{\bar{\mu}} \cos (2\tilde{\gamma}) d\tilde{a}_i + \tilde{\mu} \cos (2\tilde{\gamma}) d\tilde{\bar{a}}_i,
\]
which takes the canonical form (27) with
\[
\tilde{\varphi} = \tilde{\alpha}_l + \tilde{\beta}_l + \tilde{\bar{\alpha}}_r + \tilde{\bar{\beta}}_r, \quad \tilde{\varphi}_l = \tilde{\varphi}_l = \pi = 2\tilde{\alpha}_i, \quad \tilde{\varphi}_r = \pi - 2\tilde{\bar{\alpha}}_i.
\]
(88)

The \( \text{SU}(2) \) version of (83) then provides (89).

To check that (44) and (45), together with (46), satisfy equation (43), we calculate the left-invariant currents and find
\[
g^{-1} g = \frac{1}{2} \begin{pmatrix}
(e - p) + (e + p) \cosh 2\theta & (e + p) e^{i\omega \sigma} \sinh 2\theta \\
-e(e + p) e^{i\omega \sigma} \sinh 2\theta & -(e - p) - (e + p) \cosh 2\theta
\end{pmatrix},
\]
\[
g^{-1} g' = \frac{1}{2} \begin{pmatrix}
(m - n) + (m + n) \cosh 2\theta & (m + n) e^{i\omega \sigma} \sinh 2\theta \\
-(m + n) e^{i\omega \sigma} \sinh 2\theta & -(m - n) - (m + n) \cosh 2\theta
\end{pmatrix},
\]
(91)

with the abbreviations \( \omega \sigma = (e \pm p) \tau + (m \pm n) \sigma \) and \( \tilde{\omega} \sigma = (\tilde{e} \pm \tilde{p}) \tau + (\tilde{m} \pm \tilde{n}) \sigma \).

Since the diagonal components of these matrices are constants, one has to check (43) for the off-diagonal entries only, giving the conditions (46).

Similar calculations for the right-invariant currents yield
\[
g g^{-1} = \frac{1}{2} \begin{pmatrix}
(e + p) + (e - p) \cosh 2\tilde{\theta} & -(e - p) e^{i\omega \sigma} \sinh 2\tilde{\theta} \\
(e - p) e^{i\omega \sigma} \sinh 2\tilde{\theta} & -(e + p) - (e - p) \cosh 2\tilde{\theta}
\end{pmatrix},
\]
\[
\tilde{g} g^{-1} = \frac{1}{2} \begin{pmatrix}
(e + \tilde{p}) + (e - \tilde{p}) \cosh 2\tilde{\bar{\theta}} & -i(e - \tilde{p}) e^{i\omega \sigma} \sinh 2\tilde{\bar{\theta}} \\
i(e - \tilde{p}) e^{i\omega \sigma} \sinh 2\tilde{\bar{\theta}} & -i(e + \tilde{p}) - (e - \tilde{p}) \cosh 2\tilde{\bar{\theta}}
\end{pmatrix}.
\]
(93)
The induced metric tensor components obtained from (91) and (92) read

\[
\begin{align*}
\left( g^{-1} \tilde{g} \right)^{\theta \theta} &= p^2 \sinh^2 \theta c^2 \cosh^2 \theta, \\
\left( g^{-1} \tilde{g} \right)^{\theta \dot{\theta}} &= \tilde{p}^2 \sin^2 \tilde{\theta} c^2 \cos^2 \tilde{\theta}, \\
\left( g^{-1} \tilde{g} \right)^{\theta \rho} &= n^2 \sinh^2 \theta m^2 \cosh^2 \theta, \\
\left( g^{-1} \tilde{g} \right)^{\theta \tilde{\rho}} &= \tilde{n}^2 \sin^2 \tilde{\theta} m^2 \cos^2 \tilde{\theta}, \\
\langle g^{-1} \tilde{g} g^{-1} g' \rangle &= m \tilde{n} \sinh \cosh \theta c \cosh \tilde{\theta}, \\
\langle \tilde{g}^{-1} \tilde{g} \tilde{g}^{-1} \tilde{g}' \rangle &= m \tilde{n} \sin \cosh \theta \tilde{c} \cosh \tilde{\theta}.
\end{align*}
\]

The Virasoro constraints (41) are expressed through these components, and one gets additional conditions on the parameters given by (47).

Finally, we present formulas useful for calculations of the pre-symplectic form

\[
\begin{align*}
\langle g^{-1} \tilde{g} \partial_n g \rangle &= 0, \\
\langle g^{-1} \tilde{g} \left( p g^{-1} \partial_p g + e g^{-1} \partial_p \tilde{g} \right) \rangle &= -epr, \\
\langle \tilde{g}^{-1} \tilde{g} \partial_n \tilde{g} \rangle &= 0, \\
\langle \tilde{g}^{-1} \tilde{g} \left( \tilde{p} \tilde{g}^{-1} \partial_{\tilde{p}} \tilde{g} + \tilde{e} \tilde{g}^{-1} \partial_{\tilde{p}} \tilde{g} \right) \rangle &= \tilde{e} \tilde{p} \tau.
\end{align*}
\]

Here \( g \) and \( \tilde{g} \) are given again by (44)–(45), and the calculation is straightforward. Taking into account then the constraint (46) between the parameters \( e \) and \( \rho \), the last term in (50) becomes the exact form \( - \text{red} e \). Obviously, the same is valid for the SU(2) part.

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