PERIODIC SOLUTIONS AND HYERS-ULAM STABILITY OF ATMOSPHERIC EKMAN FLOWS

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Abstract. In this paper, we study the classical problem of the wind in the steady atmospheric Ekman layer with constant eddy viscosity. Different from the well-known homogeneous system in [14, 20], we retain the turbulent fluxes and establish a new nonhomogeneous system of first order differential equations involving a term with the horizontal dependent. We present the existence and uniqueness of periodic solutions and show the Hyers-Ulam stability results for the nonhomogeneous systems under the mild conditions via the matrix theory. Further, we consider the nonhomogeneous systems with varying eddy viscosity coefficient and study systems with piecewise constants, systems with small oscillations, systems with rapidly varying coefficients and systems with slowly varying coefficients and give more continued results.

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1. Introduction. Lamina sublayer, surface layer and Ekman layer are three important parts for the atmospheric boundary layer [14, 20]. In particular, the Ekman layer covers ninety percent of the atmospheric boundary layer, which is driven by a three-way balance among frictional effects, pressure gradient and the influence of the coriolis force [9, 14, 28]. Ekman flows are also relevant for the ocean, setting in which recently perturbative approaches provide insight into their dynamics, see [3, 6, 27]. Besides, the Ekman flow (in the atmosphere or in the ocean) is relevant away from the Equator, whereas in equatorial regions the induced deflection from the wind direction does not occur, see [5, 7, 13]. The standard Ekman equations are given by

\[
\begin{align*}
\frac{f}{g}(v - v_g) &= -\frac{\partial}{\partial z}(k \frac{\partial u}{\partial z}), \\
\frac{f}{g}(u - u_g) &= \frac{\partial}{\partial z}(k \frac{\partial v}{\partial z}),
\end{align*}
\]

(1)

where \((u, v)\) represents the horizontal \(z\)-dependent mean wind velocity, with zonal component \(u\) and meridional component \(v\), \(u_g\) and \(v_g\) are the corresponding geostrophic wind component, \(k\) denotes the eddy viscosity, \(f = 2\Omega \sin \theta\) is the Coriolis parameter at the fixed latitude \(\theta\) in the Northern Hemisphere and \(\Omega \approx 7.29 \times 10^{-5} \text{s}^{-1}\) is the angular rotating spherical coordinates [9, 12, 14, 23, 28]. There exists the fundamental contribution to find the explicit formula of the solution to (1) with the constant \(k\) and the classic boundary conditions [12, 23]. Note that \(k\) always varies with the height and it is a challenging work to find the explicit solution for systems with a function \(k(z)\). Thus, there are many continued work to find the approximation solution for the non-constant eddy viscosity \(k\) [11, 19, 21, 22]. Constantin and Johnson [6] studied the homogeneous equation for Ekman flows with variable eddy viscosity \(k(z)\) and they derived the explicit solution through an unclosed form and verified the existence of the solution by transformation and the iterative technique. Following [6], Fečkan et al. [10] derived the existence and uniqueness and smooth results to give the first order approximation of solutions by using the Green’s function approach.

Noting that (1) is formulated by omitting the turbulent fluxes, which has obvious limitations. In this paper, we introduce the following nonhomogeneous model

\[
\begin{align*}
\frac{f}{g}(v - v_g) &= -\frac{\partial}{\partial z}(k \frac{\partial u}{\partial z}) + g_1(z), \\
\frac{f}{g}(u - u_g) &= \frac{\partial}{\partial z}(k \frac{\partial v}{\partial z}) + g_2(z),
\end{align*}
\]

(2)

where \(g_i(z), i = 1, 2\) denote the given turbulent fluxes and they are the functions green of height \(z\). Clearly, (2) is interesting mathematically and might be of physical relevance. By transferring the theory of ordinary differential equations and hyperbolic matrix theory, we obtain the existence and uniqueness of periodic solutions and give a formula involving a projection such that the range of this projection is the stable space of constant coefficient matrix while the kernel of this projection is the unstable space of the constant coefficient matrix. This alternative formula of periodic solutions is useful to find an approximate solution. We also give more transparent computation to characterize the periodic solution by taking Fourier expansions.

The classic definition of asymptotic stability for nonautonomous equations is related to Lyapunov theory. Hyers-Ulam stability is an alternative stability definition for nonautonomous equations from the point view of approximation of solutions, which has been reported in [1, 15, 18]. Jung [16, 17] proved the Hyers-Ulam stability of a system of first order linear differential equations. Hyers-Ulam stability gives a
possible way to find the approximate solution for the given complex systems. Thus, it does provide another way to find the approximation solution, which is different from the Wentzel, Kramers and Brillouin’s method [11, 19, 21, 22]. Based on this investigations, the second aim of this paper is to investigate the Hyers-Ulam stability of (2). We prove the Hyers-Ulam stability by using the matrix theory if the relevant matrix norm and vector norm are appropriate.

Finally, we extend to study some possible generalized systems like piecewise constant systems, systems with small oscillations, systems with rapidly varying coefficients and systems with slowly varying coefficients to enrich the periodic solutions and Hyers-Ulam stability results.

2. Model description. In spherical coordinates, the Ekman layer is governed by the following equations, see [14, 20]

\[
\begin{align*}
\frac{Du}{Dt} &= -\frac{1}{\rho} \frac{\partial P}{\partial x} - 2\Omega \sin \theta \frac{\partial v}{\partial x} - 2\Omega \cos \theta \frac{\partial v}{\partial y} + \frac{u}{a} \frac{\partial \rho}{\partial x} + 2\Omega \sin \theta v - 2\Omega \cos \theta w + \frac{u \sin \theta}{a} \frac{\partial \rho}{\partial x} - \frac{w}{a} + F_{rx}, \\
\frac{Dv}{Dt} &= -\frac{1}{\rho} \frac{\partial P}{\partial y} + 2\Omega \sin \theta \frac{\partial u}{\partial y} + \frac{v}{a} \frac{\partial \rho}{\partial y} - 2\Omega \cos \theta u - \frac{u}{a} \frac{\partial \rho}{\partial y} + F_{ry}, \\
\frac{Dw}{Dt} &= -\frac{1}{\rho} \frac{\partial P}{\partial z} + 2\Omega \cos \theta \frac{\partial u}{\partial z} + \frac{w}{a} \frac{\partial \rho}{\partial z} + F_{rz},
\end{align*}
\]

where

\[
\begin{align*}
F_{rx} &= v \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right], \\
F_{ry} &= v \left[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right], \\
F_{rz} &= v \left[ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right],
\end{align*}
\]

here \( v = \frac{\nu}{\rho} \) is the kinematic viscosity coefficient [14], \( u = u(t, x, y, z) \) and \( v = v(t, x, y, z) \) are the components of the wind in the \( x, y \) and \( z \) directions respectively, \( P \) is the atmospheric pressure, \( \rho \) is the reference density, \( 2\Omega \sin \theta \) is the Coriolis parameter at the fixed latitude \( \theta \).

Using the formula for the transformation of local rectangular coordinate system and spherical coordinate system, we can get the following relationships,

\[
dx = a \cos \phi d\lambda, \quad dy = a d\phi, \quad dz = dr.
\]

In the local cartesian coordinate system, the earth’s surface is approximately regarded as a plane, and the curvature term can be omitted, so we get

\[
\begin{align*}
\frac{Du}{Dt} &= -\frac{1}{\rho} \frac{\partial P}{\partial x} + 2\Omega \sin \phi - 2\Omega \cos \phi + F_{rx}, \\
\frac{Dv}{Dt} &= -\frac{1}{\rho} \frac{\partial P}{\partial y} + 2\Omega \sin \phi u + F_{ry}, \\
\frac{Dw}{Dt} &= -\frac{1}{\rho} \frac{\partial P}{\partial z} - g - 2\Omega \cos \phi u + F_{rz},
\end{align*}
\]

For a wide range of air movements, \( w \ll u, v \) [20], so we assume \( w = 0 \), for the atmosphere below 100km, kinematic viscosity coefficient is negligible except in a thin layer within a few centimeters of the earth’s surface where the vertical shear is very large. So \( F_{rx} = 0, \quad F_{ry} = 0 \) in Ekman layer and from (4), we get

\[
\begin{align*}
\frac{Du}{Dt} &= -\frac{1}{\rho} \frac{\partial P}{\partial x} + 2\Omega \sin \phi = -\frac{1}{\rho} \frac{\partial P}{\partial x} + f v, \\
\frac{Dv}{Dt} &= -\frac{1}{\rho} \frac{\partial P}{\partial y} - 2\Omega u \sin \phi = -\frac{1}{\rho} \frac{\partial P}{\partial y} - f u,
\end{align*}
\]

where \( f = 2\Omega \sin \theta \).

In a turbulent fluid, a field variable such as velocity measured at a point generally fluctuates rapidly in time as eddies of various scales pass the point, so we assume that the field variables can be separated into slowly varying turbulent components, for example, \( u = \bar{u} + u' \), the corresponding means are indicated by overbars and the fluctuating component by primes.
Separating each dependent variable into mean and fluctuating parts, substituting into the chain rule of the differentiation, noting that the mean velocity fields satisfy the following continuity equations

\[ \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0, \]

so we get

\[ \bar{\frac{D\bar{u}}{Dt}} = \frac{\partial \bar{u}}{\partial t} + \frac{\partial}{\partial x}(\bar{u}'w') + \frac{\partial}{\partial y}(\bar{u}'v') + \frac{\partial}{\partial z}(\bar{u}'w'), \]

where

\[ \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} + \bar{v} \frac{\partial}{\partial y} + \bar{w} \frac{\partial}{\partial z} \]

is the rate of change following the mean motion.

Using the above relationships and (5), the mean equations thus have the following form:

\[
\begin{align*}
\frac{\partial \bar{u}}{\partial t} &= -\frac{1}{\rho} \frac{\partial P}{\partial x} + f\bar{v} - \left[ \frac{\partial w'}{\partial x} + \frac{\partial w'}{\partial y} + \frac{\partial w'}{\partial z} \right], \\
\frac{\partial \bar{v}}{\partial t} &= -\frac{1}{\rho} \frac{\partial P}{\partial y} - f\bar{u} - \left[ \frac{\partial u'}{\partial x} + \frac{\partial u'}{\partial y} + \frac{\partial u'}{\partial z} \right].
\end{align*}
\]

(6)

For midlatitude synoptic-scale motions, the inertial acceleration terms (the terms on the left of above equations) can be neglected compared to the Coriolis force and pressure gradient force terms [14], using the geostrophic balance, i.e.,

\[
\begin{align*}
\frac{1}{\rho} \frac{\partial P}{\partial x} &= f\bar{v} - g\bar{u}, \\
\frac{1}{\rho} \frac{\partial P}{\partial y} &= -f\bar{u} - g\bar{v},
\end{align*}
\]

where \( \bar{u} \) and \( \bar{v} \) are the corresponding constant geostrophic wind components, so we obtain

\[
\begin{align*}
f(\bar{v} - \bar{v}_g) - \left[ \frac{\partial w'}{\partial x} + \frac{\partial w'}{\partial y} + \frac{\partial w'}{\partial z} \right] &= 0, \\
-f(\bar{u} - \bar{u}_g) - \left[ \frac{\partial u'}{\partial x} + \frac{\partial u'}{\partial y} + \frac{\partial u'}{\partial z} \right] &= 0.
\end{align*}
\]

By the Flux-Gradient theory, we get

\[
\begin{align*}
\bar{u}'w' &= -k \frac{\partial \bar{u}}{\partial z}, \\
\bar{v}'w' &= -k \frac{\partial \bar{v}}{\partial z},
\end{align*}
\]

where \( k \) is the eddy viscosity coefficient, then we obtain

\[
\begin{align*}
f(\bar{v} - \bar{v}_g) &= -\frac{\partial}{\partial z} (k \frac{\partial \bar{v}}{\partial z}) + \frac{\partial w'}{\partial x} + \frac{\partial w'}{\partial y}, \\
f(\bar{u} - \bar{u}_g) &= \frac{\partial}{\partial z} (k \frac{\partial \bar{u}}{\partial z}) - \frac{\partial u'}{\partial x} - \frac{\partial u'}{\partial y}.
\end{align*}
\]

(7)

The turbulent fluxes are always assuming to be horizontally homogeneous because they are small in comparison to the terms \( \frac{\partial w'}{\partial x}, \frac{\partial w'}{\partial y} \). Thus, the turbulent fluxes terms are really can not be reduced. For simplicity, we assume \( \frac{\partial w'}{\partial x} + \frac{\partial w'}{\partial y} = g_1(z), -\frac{\partial u'}{\partial x} - \frac{\partial u'}{\partial y} = g_2(z) \), then (7) becomes

\[
\begin{align*}
f(\bar{v} - \bar{v}_g) &= -\frac{\partial}{\partial z} (k \frac{\partial \bar{v}}{\partial z}) + g_1(z), \\
f(\bar{u} - \bar{u}_g) &= \frac{\partial}{\partial z} (k \frac{\partial \bar{u}}{\partial z}) + g_2(z).
\end{align*}
\]

Now replacing \( \pi, \nu, \nu_g \) and \( \bar{v}_g \) by \( u, v, u_g \) and \( v_g \), respectively, one can obtain (2).
3. Periodic solutions and Hyers-Ulam stability. If \( k \) reduces to a constant and \( k > 0 \), then (2) reduces to
\[
\begin{align*}
\frac{d^2 u}{dz^2} &= \frac{f(u - u_g)}{k} - \frac{g_1(z)}{k}, \\
\frac{d^3 u}{dz^3} &= -\frac{f(v - v_g)}{k} + \frac{g_2(z)}{k}.
\end{align*}
\] (8)

Let \( U = u - u_g, V = v - v_g \), \( V'(z) = W_1, U'(z) = W_2 \) and \( k = \frac{f}{2k} \), then (8) becomes to
\[
\bar{X}'(z) = A \bar{X}(z) + \bar{F}(z), \quad z \in [0, \infty),
\] (9)
where
\[
\bar{X} = \begin{bmatrix} V \\ U \\ W_1 \\ W_2 \end{bmatrix},
\]
and
\[
A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{g_1(z)}{k} & 0 & 0 \\ -\frac{k}{k} & 0 & 0 & 0 \end{bmatrix}, \quad \bar{F}(z) = \begin{bmatrix} 0 \\ 0 \\ \frac{g_1(z)}{k} \\ \frac{g_2(z)}{k} \end{bmatrix}.
\]

3.1. Existence and uniqueness of periodic solutions.

**Theorem 3.1.** Assume \( \bar{F}(z) \) is a \( T \)-periodic vector function, then (9) has a unique \( T \)-periodic solution.

**Proof.** We know from [4, Corollary 2.119] that a periodic solution has a form
\[
\bar{X}(z) = e^{Az} \bar{X}(0) + \int_0^z e^{A(z-s)} \bar{F}(s) ds
\]
if and only if
\[
(I - e^{AT}) \bar{X}(0) = \int_0^T e^{A(T-s)} \bar{F}(s) ds.
\]
By direct calculation, we get the four eigenvalues of \( A \):
\[
\lambda_1 = \sqrt{\frac{f}{2k}}, \quad \lambda_2 = -\sqrt{\frac{f}{2k}}, \quad \lambda_3 = \sqrt{\frac{f}{2k}}, \quad \lambda_4 = -\sqrt{\frac{f}{2k}},
\]
then the eigenvalues of \( e^{AT} \) are \( e^{\lambda_1}, e^{\lambda_2}, e^{\lambda_3} \) and \( e^{\lambda_4} \), respectively, which show that \( (I - e^{AT})^{-1} \) exists. Thus the periodic solution is given by
\[
\bar{X}(z) = e^{Az} (I - e^{AT})^{-1} \int_0^T e^{A(T-s)} \bar{F}(s) ds + \int_0^z e^{A(z-s)} \bar{F}(s) ds.
\] (10)

On the other hand, we also see that \( A \) is a hyperbolic matrix. So we have a projection \( P : \mathbb{R}^4 \to \mathbb{R}^4 \) such that the range of \( P \) is the stable space of \( A \) while the kernel of \( P \) is the unstable space of \( A \). Then the unique bounded solution of (9) is given by [8]
\[
\bar{X}(z) = \int_0^z e^{A(z-s)} P \bar{F}(s) ds + \int_z^\infty e^{A(z-s)} (I - P) \bar{F}(s) ds,
\] (11)
where \( \bar{F}(z) \) is the \( T \)-periodic extension of \( \bar{F}(0) \) from \([0, \infty) \) to \( \mathbb{R} \). Clearly both (10) and (11) coincide. Note this second approach is expressed in more details in the next Subsection 3.2. The proof is complete. \( \square \)
Remark 1. Theorem 3.1 shows that (9) has a unique $T$-periodic solution given by either (10) or (11), but both formulas are rather messy in general. Now we suggest more transparent computation. We take a Fourier expansions

$$\overrightarrow{X}(z) = \sum_{j \in \mathbb{Z}} e^{\frac{2\pi i}{T} j z} \overrightarrow{x}_j, \quad \overrightarrow{F}(z) = \sum_{j \in \mathbb{Z}} e^{\frac{2\pi i}{T} j z} \overrightarrow{f}_j, \quad \overrightarrow{x}_j, \overrightarrow{f}_j \in \mathbb{R}^4.$$  

Then plugging this into (9), we get

$$\left(\frac{2\pi i}{T} j I - A\right) \overrightarrow{x}_j = \overrightarrow{F}_j,$$

so

$$\overrightarrow{x}_j = \left(\frac{2\pi i}{T} j I - A\right)^{-1} \overrightarrow{F}_j.$$  

Taking the concrete forms of $A$ and $\overrightarrow{F}(z)$ of our paper, we get

$$\overrightarrow{x}_j = \begin{bmatrix} -4\pi^2 j^2 g_{1,j} T^2 + \overline{F}_{2,j} T^4 \\ 16\pi^2 j^2 k + \overline{F}^2 k T^4 \\ 16\pi^2 j^2 T^2 - \overline{F}_{1,j} T^4 \\ 2iT \left(4\pi^2 j g_{1,j} + \overline{F} F_{2,j} T^2\right) \end{bmatrix},$$

where

$$g_{1}(z) = \sum_{j \in \mathbb{Z}} e^{\frac{2\pi i}{T} j z} g_{1,j}, \quad g_{2}(z) = \sum_{j \in \mathbb{Z}} e^{\frac{2\pi i}{T} j z} g_{2,j}, \quad g_{i,j} \in \mathbb{R}.$$  

Consequently, the solution is given by

$$\overrightarrow{X}(z) = \sum_{j \in \mathbb{Z}} e^{\frac{2\pi i}{T} j z} \begin{bmatrix} \frac{-4\pi^2 j^2 g_{1,j} T^2 + \overline{F}_{2,j} T^4}{16\pi^2 j^2 k + \overline{F}^2 k T^4} \\ \frac{16\pi^2 j^2 T^2 - \overline{F}_{1,j} T^4}{2iT \left(4\pi^2 j g_{1,j} + \overline{F} F_{2,j} T^2\right)} \\ \frac{-4\pi^2 j^2 T^2 + \overline{F}_{2,j} T^4}{16\pi^2 j^2 k + \overline{F}^2 k T^4} \\ \frac{16\pi^2 j^2 k + \overline{F}^2 k T^4}{2iT \left(4\pi^2 j g_{1,j} + \overline{F} F_{2,j} T^2\right)} \end{bmatrix}.$$  

3.2. Hyers-Ulam stability. Let $(C^{4 \times 4}, \|\cdot\|_4)$ be a complex normed space and let $C^{4 \times 4}$ be a vector space consisting of all $4 \times 4$ complex matrices. We choose a norm $\|\cdot\|_4$ on $C^{4 \times 4}$ which is compatible with $\|\cdot\|_4$, i.e., both norms obey

$$\|AB\|_{4 \times 4} \leq \|A\|_{4 \times 4} \|B\|_{4 \times 4}, \quad \|A \overrightarrow{F}\|_4 \leq \|A\|_{4 \times 4}\|\overrightarrow{F}\|_4$$  \hspace{1cm} (12)

for all $A, B \in C^{4 \times 4}$ and $\overrightarrow{F} \in C^4$. Furthermore, we assume that $\|\overrightarrow{F}\|_4 < \infty$, and $\|C\|_{4 \times 4} < \infty$ for all $\overrightarrow{F} \in C^4$ and $C \in C^{4 \times 4}$. For any $(u_1, u_2, u_3, u_4)^T, (w_1, w_2, w_3, w_4)^T \in C^4$ with $|u_i| \leq w_i$ ($i = 1, 2, 3, 4$), we assume that

$$|u_i| \leq \|(u_1, u_2, u_3, u_4)^T\|_4 \leq \|(w_1, w_2, w_3, w_4)^T\|_4 \quad \|w_i\|_4 \quad \|(u_1, u_2, u_3, u_4)^T\|_4 \quad \|(w_1, w_2, w_3, w_4)^T\|_4$$  \hspace{1cm} (13)

where $(w_1, w_2, w_3, w_4)^T$ is the transpose of the vector $(w_1, w_2, w_3, w_4)$.

For the matrix $A$, we know that it has 4 distinct eigenvalues $\lambda_\mu$ ($\mu = 1, 2, 3, 4$):

$$\lambda_1 = \sqrt{\frac{f}{2k}} + \sqrt{\frac{f}{2k}}, \quad \lambda_2 = -\sqrt{\frac{f}{2k}} - \sqrt{\frac{f}{2k}}, \quad \lambda_3 = \sqrt{\frac{f}{2k}} - \sqrt{\frac{f}{2k}}, \quad \lambda_4 = -\sqrt{\frac{f}{2k}} + \sqrt{\frac{f}{2k}}.$$
By the direct calculation, it holds that $N^{-1}AN = J$, where
\[
J = \begin{bmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 \\
0 & 0 & 0 & \lambda_4
\end{bmatrix},
\]
and
\[
N = \begin{bmatrix}
-\sqrt{k/27} - i\sqrt{k/27} & \sqrt{k/27} + i\sqrt{k/27} & -\sqrt{k/27} + i\sqrt{k/27} & \sqrt{k/27} - i\sqrt{k/27} \\
\sqrt{k/27} - i\sqrt{k/27} & -\sqrt{k/27} + i\sqrt{k/27} & \sqrt{k/27} + i\sqrt{k/27} & -\sqrt{k/27} - i\sqrt{k/27} \\
-i & 1 & 1 & 1 \\
i & 1 & 1 & 1
\end{bmatrix}.
\] (14)

Furthermore, it holds that $e^{Az} = Ne^{Jz}N^{-1}$ if we set
\[
e^{Jz} = \begin{bmatrix}
e^{\lambda_1z} & 0 & 0 & 0 \\
0 & e^{\lambda_2z} & 0 & 0 \\
0 & 0 & e^{\lambda_3z} & 0 \\
0 & 0 & 0 & e^{\lambda_4z}
\end{bmatrix},
\]
We replace each $e^{\lambda_\mu z}$ in the above matrix by 0, except $e^{\lambda_\mu z}$, and we denote the resulting matrix by $(e^{\lambda_\mu z})$, for example,
\[
(e^{\lambda_1z}) = \begin{bmatrix}
e^{\lambda_1z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]
it obvious holds
\[
e^{Jz} = \sum_{\mu=1}^{4} (e^{\lambda_\mu z}).
\]
Define
\[
(T_1) = \begin{bmatrix}
\frac{1}{Re(\lambda_1)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]
where $Re(\lambda_1)$ is the real part of $\lambda_1$, and similarly, we define $(T_2)$, $(T_3)$ and $(T_4)$, and further define
\[
T = \sum_{\mu=1}^{4} (T_\mu).
\] (15)

**Theorem 3.2.** Assume the $\|\|_4$ and $\|\|_{4\times4}$ satisfy (12) and (13), let $\overrightarrow{F}(z)$ and $\overrightarrow{X}(z)$ be a continuous vector function and a continuously differentiable vector function, respectively. Assume that a continuous vector function $\overrightarrow{v}(z)$ defined by
\[
\overrightarrow{v}(z) = \overrightarrow{X}'(z) - A\overrightarrow{X}(z) - \overrightarrow{F}(z)
\]
satisfies
\[
\|\overrightarrow{v}(z)\|_4 \leq \epsilon
\] (16)
for all $z \in [0, \infty)$ and for some $\epsilon > 0$, then there exists a differentiable vector function $\overrightarrow{X}_0(z)$ such that

$$\overrightarrow{X}_0'(z) = A\overrightarrow{X}_0(z) + \overrightarrow{F}(z)$$

(17)

and

$$\|\overrightarrow{X}(z) - \overrightarrow{X}_0(z)\|_4 \leq \epsilon \|N\|_{4 \times 4}\|N^{-1}\|_{4 \times 4}\|T\|_4,$$

for all $z \in [0, \infty)$, where $\overrightarrow{c} = (1, 1, 1, 1)^T$.

**Proof.** We have

$$\overrightarrow{X}'(z) = A\overrightarrow{X}(z) + \overrightarrow{F}(z) + \overrightarrow{v}(z), \forall z \in [0, \infty),$$

(18)

the solution of (18) is

$$\overrightarrow{X}(z) = e^{At}\overrightarrow{X}(0) + \int_0^z e^{A(z-s)}\overrightarrow{F}(s)ds + \int_0^z e^{A(z-s)}\overrightarrow{v}(s)ds.$$  

(19)

We notice that $N$ is a nonsingular matrix for which $J = N^{-1}AN$, we define vector $\overrightarrow{w}(z) = N^{-1}\overrightarrow{v}(z) = (w_1(z), w_2(z), w_3(z), w_4(z))$, we define the vector $\langle \overrightarrow{w}_\mu(z) \rangle (\mu = 1, 2, 3, 4)$ as:

$$\langle \overrightarrow{w}_\mu(z) \rangle = \begin{cases} w_i(z), & \text{if } i = \mu, \\ 0, & \text{otherwise,} \end{cases}$$

then we have

$$\overrightarrow{w}(z) = \sum_{\mu=1}^4 \langle \overrightarrow{w}_\mu(z) \rangle.$$  

Then

$$e^{-As}\overrightarrow{v}(s) = (Ne^{-Js})(N^{-1}\overrightarrow{v}(s)) = N(e^{-Js}\overrightarrow{w}(s))$$

$$= N\sum_{\mu=1}^4 \langle e^{-\lambda_\mu s}\overrightarrow{w}_\mu(s) \rangle,$$

so

$$\int_0^z e^{-As}\overrightarrow{v}(s)ds = N\sum_{\mu=1}^4 \int_0^z \langle e^{-\lambda_\mu s}\overrightarrow{w}_\mu(s) \rangle ds,$$

where $\int_0^z \langle e^{-\lambda_\mu s}\overrightarrow{w}_\mu(s) \rangle ds$ is a column vector with 4 components. Let us define

$$c_\mu(z) = \int_0^\infty \langle e^{-\lambda_\mu s}\overrightarrow{w}_\mu(s) \rangle ds,$$

and

$$r_\mu(z) = -\int_z^\infty \langle e^{-\lambda_\mu s}\overrightarrow{w}_\mu(s) \rangle ds,$$

the $i$th component of $\langle e^{-\lambda_\mu s}\overrightarrow{w}_\mu(s) \rangle$ is given below:

$$[\langle e^{-\lambda_\mu s}\overrightarrow{w}_\mu(s) \rangle]_i = \begin{cases} e^{-\lambda_\mu s}w_\mu(s), & \text{if } i = \mu, \\ 0, & \text{if } i \neq \mu, \end{cases}$$

(20)

By the direct calculation, we know that $\int_0^\infty [\langle e^{-\lambda_\mu s}\overrightarrow{w}_\mu(s) \rangle]_i ds$ is bounded for $\mu = 1, 2, 3, 4$, we define the vector $\langle c_\mu(z) \rangle (\mu = 1, 2, 3, 4)$ and $\langle r_\mu(z) \rangle (\mu = 1, 2, 3, 4)$ as:

$$\langle c_\mu(z) \rangle = \begin{cases} c_i(z), & \text{if } i = \mu, \\ 0, & \text{otherwise,} \end{cases}$$

$$\langle r_\mu(z) \rangle = \begin{cases} r_i(z), & \text{if } i = \mu, \\ 0, & \text{otherwise}, \end{cases}$$
and

$$\langle \overrightarrow{r}_\mu(z) \rangle = \begin{cases} r_i(z), & \text{if } i = \mu, \\
0, & \text{otherwise.} \end{cases}$$

Because it follows from (12), (13) and (16), we get

$$| w_k(s) | \leq ||N^{-1} \overrightarrow{v}(s)|| \leq \epsilon ||N^{-1}||_{4 \times 4},$$

then we get

$$\int_0^z e^{-A_s} \overrightarrow{v}(s) ds = N \sum_{\mu=1}^4 \left( \langle \overrightarrow{c}_\mu^1 \rangle + \langle \overrightarrow{r}_\mu(s) \rangle \right).$$

If we set

$$\overrightarrow{c}(z) = \sum_{\mu=1}^4 \langle \overrightarrow{c}_\mu^1 \rangle, \quad \overrightarrow{r}(z) = \sum_{\mu=1}^4 \langle \overrightarrow{r}_\mu(z) \rangle,$$

it follows that

$$e^{A_z} \int_0^z e^{-A_s} \overrightarrow{v}(s) ds = N e^{J_z}(\overrightarrow{c}(z) + \overrightarrow{r}(z)).$$

Define

$$\overrightarrow{X}_0(z) = e^{A_z}(\overrightarrow{X}(0) + N \overrightarrow{c}(z)) + e^{A_z} \int_0^z e^{-A_s} \overrightarrow{F}(s) ds,$$

we get

$$||\overrightarrow{X}(z) - \overrightarrow{X}_0(z)||_4 = ||N e^{J_z} \overrightarrow{v}(z)||_4 \leq ||N ||_{4 \times 4} ||e^{J_z} \overrightarrow{v}(z)||_4$$

$$= ||N ||_{4 \times 4} \sum_{\mu=1}^4 \langle e^{\lambda^\mu} \rangle ||\overrightarrow{r}_\mu(z)||_4$$

$$= ||N ||_{4 \times 4} \sum_{\mu=1}^4 \langle e^{\lambda^\mu} \rangle \int_0^{+\infty} \langle e^{-\lambda^\mu} \rangle ||\overrightarrow{w}_\mu(s)||_4 ds.$$ 

It is obvious that $$\langle e^{\lambda^\mu} \rangle \langle e^{-\lambda^\mu} \rangle = \langle e^{\lambda^\mu(z-s)} \rangle$$ for all $$s, z \in [0, \infty)$$, therefore we obtain

$$||\overrightarrow{X}(z) - \overrightarrow{X}_0(z)||_4 = ||N ||_{4 \times 4} \sum_{\mu=1}^4 \int_0^{+\infty} \langle e^{\lambda^\mu(z-s)} \rangle ||\overrightarrow{w}_\mu(s)||_4 ds.$$ 

(21)

Let $$\mu$$ be given, the ith component of $$\langle e^{\lambda^\mu(z-s)} \rangle ||\overrightarrow{w}_\mu(s)||_4$$ is

$$\langle e^{\lambda^\mu(z-s)} \rangle ||\overrightarrow{w}_\mu(s)||_4 = \begin{cases} e^{\lambda^\mu(z-s)} w_\mu(s), & i = \mu, \\
0, & i \neq \mu, \end{cases}$$

hence the i-th component of $$\int_0^{+\infty} \langle e^{\lambda^\mu(z-s)} \rangle ||\overrightarrow{w}_\mu(s)||_4 ds$$ is

$$|\alpha_i| = \begin{cases} \int_0^{+\infty} e^{\lambda^\mu(z-s)} w_\mu(s) ds, & i = \mu, \\
0, & i \neq \mu, \end{cases}$$

so we have

$$|\alpha_i| \leq \int_0^{+\infty} e^{Re(\lambda^\mu)(z-s)} ||\overrightarrow{w}(s)||_4 ds$$

$$\leq \epsilon ||N^{-1}||_{4 \times 4} \frac{1}{|Re(\lambda^\mu)|}, \quad i = \mu,$$

if $$i \neq \mu$$, we get $$|\alpha_i| \leq 0.$$
The $i$th component $[\langle T_\mu \rangle_{-\to}^e]_i$ of $\langle T_\mu \rangle_{-\to}^e$ is

$$[\langle T_\mu \rangle_{-\to}^e]_i = \begin{cases} |\text{Re}(\lambda_\mu)|^{-1}, & i = \mu, \\ 0, & i \neq \mu, \end{cases}$$

where $\overrightarrow{e} = (1, 1, 1, 1)^{tr}$, so

$$|\alpha_i| \leq \epsilon \|N^{-1}\|_{4\times4}[\langle T_\mu \rangle_{-\to}^e]_i, \forall i \in 1, 2, 3, 4.$$  \hspace{1cm} (22)

Consequently, we have

$$\|\overrightarrow{-\to}_X(z) - \overrightarrow{-\to}_X(z)_0\|_4 \leq \|N\|_{4\times4}\|\sum_{\mu=1}^{4} \epsilon \|N^{-1}\|_{4\times4}[T_\mu \rangle_{-\to}^e\|_4 = \epsilon \|N\|_{4\times4}\|N^{-1}\|_{4\times4}\|T_{-\to}^e\|_4.$$  \hspace{1cm} (23)

Remark 2. For more general results on Hyers-Ulam stability of linear systems with exponential splitting, we refer to [26].

We give an example to illustrate the above result. Define $\|A\|_{\infty} = \max_{1 \leq i \leq 4} \sum_{j=1}^{4} |a_{ij}|$, $\|a\|_{\infty} = \max_{1 \leq i \leq 4} |a_i|$, it is easy to verify that both norms satisfy (12) and (13). From (14), we get

$$N^{-1} = \begin{bmatrix} \frac{1}{8} \sqrt{\frac{2}{k}} (-1 + i) & \frac{1}{8} \sqrt{\frac{2}{k}} (1 + i) & \frac{1}{4} i & \frac{1}{4} \\ \frac{1}{8} \sqrt{\frac{2}{k}} (1 - i) & -\frac{1}{8} \sqrt{\frac{2}{k}} (1 + i) & \frac{1}{4} i & \frac{1}{4} \\ -\frac{1}{8} \sqrt{\frac{2}{k}} (1 + i) & \frac{1}{8} \sqrt{\frac{2}{k}} (1 - i) & -\frac{1}{4} i & \frac{1}{4} \\ \frac{1}{8} \sqrt{\frac{2}{k}} (1 + i) & \frac{1}{8} \sqrt{\frac{2}{k}} (-1 + i) & -\frac{1}{4} i & \frac{1}{4} \end{bmatrix}.$$  \hspace{1cm} (23)

From the above definition, we get

$$\|N^{-1}\|_{\infty} = \frac{1}{2} (\sqrt{\frac{2f}{k}} + 1),$$

and

$$\|N\|_{\infty} = \begin{cases} 8 \sqrt{\frac{k}{2f}}, & \text{if } \sqrt{\frac{k}{2f}} \geq \frac{1}{2}, \\ 4, & \text{if } \sqrt{\frac{k}{2f}} < \frac{1}{2}, \end{cases}$$

from (15), we have

$$T = \begin{bmatrix} \sqrt{\frac{2k}{f}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2k}{f}} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{2k}{f}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{2k}{f}} \end{bmatrix}.$$  

If we set $\overrightarrow{\overrightarrow{F}}(z) = \overrightarrow{0}$, assume that a continuously differentiable vector function $\overrightarrow{\alpha}(z)$ satisfies

$$\|\overrightarrow{\alpha}'(z) - A\overrightarrow{\alpha}(z)\|_{\infty} \leq \epsilon$$
for all $z > 0$ and for some $\epsilon \geq 0$, then according to Theorem 3.2, there exists a differentiable vector function $\vec{X}_0$ of the form

$$\vec{X}_0(z) = e^{Az} \vec{k}$$

with

$$\|\vec{X}(z) - \vec{X}_0(z)\|_\infty \leq M\epsilon,$$

where

$$M = \begin{cases} \frac{4k}{T}(\sqrt{\frac{2f}{k}} + 1), & \text{if } \sqrt{\frac{k}{2f}} \geq \frac{1}{2}, \\ 2(\frac{4f}{k} + \sqrt{\frac{2f}{k}}), & \text{if } \sqrt{\frac{k}{2f}} < \frac{1}{2}, \end{cases}$$

and $\vec{k} \in C^4$ is a constant.

4. Further extensions.

4.1. Systems with piecewise constants. Consider (2) with $k$ is piecewise constant, that is, there are $z_0 = 0 < z_1 < \cdots < z_{m-1} < z_m = T$ such that $k(z) = k_i > 0$ for $z \in [z_i, z_{i+1})$, $i = 0, 1, \cdots, m-1$, $k_i > 0$ are constants and $k(z)$ is $T$-periodically extended on $[0, \infty)$. So we have a system

$$\vec{X}'(z) = A(z) \vec{X}(z) + \vec{F}(z)$$

with $A(z)$ and $\vec{F}(z)$ are $T$-periodic and piecewise constant such that $A(z) = A_i$ and $\vec{F}(z) = \vec{F}_j(z)$ on $z \in [z_i, z_{i+1})$ for

$$A_j = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -k_j & 0 & 0 \\ -k_j & 0 & 0 & 0 \end{bmatrix}, \quad \vec{F}_j(z) = \begin{bmatrix} 0 \\ 0 \\ g_1(z) \k_j \\ g_2(z) \k_j \end{bmatrix}$$

and $\k_j = \frac{f}{k_j}$. Let $U(t)$ be the fundamental solution of (24), which exists by a general theory. Then a solution of (24) is

$$X(z) = U(z)X_0 + \int_0^z U(z)U^{-1}(s)F(s)ds.$$  

Note $W(z, s) = U(z)U(s)^{-1}$ is just a Cauchy matrix of (24). We have

$$U(z) = e^{A_i(z-z_i-kT)} \cdots e^{A_0(z_1-z_0)} P_T^k z \text{ for } z \in [z_i + kT, z_{i+1} + kT), \ k \in \mathbb{Z},$$

$$P_T = e^{A(T-z_{m-1})} \cdots e^{A_0(z_1-z_0)}. \quad (26)$$

Here $P_T$ is a monodromy matrix of (24). It is easy to see that $A_j$ are not commutative in this case, since

$$A_jA_i - A_iA_j = (\k_j - k_i) \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

and $\k_i \neq \k_j$, which is supposed. So the above formulas cannot be simplified. Thus we must assume that $P_T$ has no 1 as eigenvalue to get a unique periodic solution of (24). Note each of $A_i$ are hyperbolic, but it is not clear if then $P_T$ is also hyperbolic.
Next, we can find explicitly the formula for (25), but formulas are messy in general. Thus we concentrate on the case $m = 2$. Then

$$U(z) = \begin{cases} 
 e^{A_0(z - kT)} P_k^2 z & \text{for } z \in [kT, z_1 + kT), k \in \mathbb{Z}, \\
 e^{A_1(z - z_1 - kT)} e^{A_0 z_1} P_k^2 z & \text{for } z \in [z_1 + kT, k(T + 1)), k \in \mathbb{Z},
\end{cases}$$

(27)

$$P_T = e^{A_1(T - z_1)} e^{A_0 z_1}.$$  

There is a matrix $\bar{A}$, possible complex, that $P_T = e^{\bar{A}}$. Then the Floquet theory states

$$U(z) = Q(z) e^{z \bar{A}},$$

where $Q(z)$ is $T$-periodic. We can check

$$Q(z) = \begin{cases} 
 e^{A_0(z - kT)} e^{-\bar{A} (z - kT)} & \text{for } z \in [kT, z_1 + kT), k \in \mathbb{Z}, \\
 e^{A_1(z - z_1 - kT)} e^{A_0 z_1} e^{-\bar{A} (z - kT)} & \text{for } z \in [z_1 + kT, k(T + 1)), k \in \mathbb{Z}.
\end{cases}$$

Since $A_0$ and $A_1$ are not commutative when $\bar{F}_0 \neq \bar{F}_1$, which is supposed. Thus a construction of $\bar{A}$ is not easy. On the other hand, we can apply the Baker-Cambell-Hausdorff Theorem for construction $\bar{A}$, when [2]

$$(T - z_1) \norm{A_1} + z_1 \norm{A_0} < \frac{\ln 2}{2},$$

where $\norm{\cdot}$ is a matrix norm. Taking the matrix norm $\norm{\cdot}$ as the maximum absolute column sum of the matrix on $\mathbb{R}^4$, we get

$$(T - z_1) \max\{1, k_1\} + z_1 \max\{1, k_0\} < \frac{\ln 2}{2}.$$  

The second order approximation is

$$\bar{A} \approx A_2 = A_1(T - z_1) + A_0 z_1 + \frac{(T - z_1) z_1}{2} (A_0 A_1 - A_1 A_0) =$$

$$\begin{bmatrix} 
 \frac{z_1}{2} & 0 & \frac{1}{2} z_1 (\bar{F}_1 - \bar{F}_0)(T - z_1) & \frac{T}{2} \\
 0 & 0 & 0 & \frac{T}{2} \\
 \frac{1}{2} z_1 (\bar{F}_1 - \bar{F}_0)(T - z_1) & 0 & 0 & 0 \\
 \frac{z_1}{2} (\bar{F}_1 - \bar{F}_0)(T - z_1) & \frac{T}{2} & 0 & 0 \\
\end{bmatrix}.$$  

By using Mathematica we can check that eigenvalues of $A_2$ are complex, so $\lambda_1$, $\bar{\lambda}_1$, $\lambda_2$, $\bar{\lambda}_2$. Since

$$0 = \text{tr} A_2 = \lambda_1 + \bar{\lambda}_1 + \lambda_2 + \bar{\lambda}_2 = 2 \Re \lambda_1 + 2 \Re \lambda_2,$$

we get $\Re \lambda_1 > 0 > \Re \lambda_2$. Thus $A_2$ is hyperbolic and thus $e^{A_2}$ is also hyperbolic. So we conjecture that $P_T = e^{\bar{A}}$ should be hyperbolic. We cannot prove this but we have the following result:

**Lemma 4.1.** If eigenvalues of $e^{\bar{A}}$ are complex, so $\lambda_1$, $\bar{\lambda}_1$, $\lambda_2$, $\bar{\lambda}_2$. Then either $|\lambda_1| = |\bar{\lambda}_1| = |\lambda_2| = |\bar{\lambda}_2| = 1$ or $|\lambda_1| > 1 > |\lambda_2| = |\bar{\lambda}_2|$.

**Proof.** Since $\text{tr} A_1(T - z_1) = \text{tr} A_0 z_1 = 0$, Liouville’s formula implies that $\det e^{A_1(T - z_1)} = \det e^{A_0 z_1} = 1$, so $\det P = 1$. Thus $|\lambda_1| |\lambda_2| = 1$. This completes the proof.  

So generically $P_T$ is pure hyperbolic for $m = 2$. If so, then (24) is exponentially dichotomic on $[0, \infty)$ and results of Subsections 3.1 and 3.2 are extended to (24), i.e., it holds for any $m$:

**Theorem 4.2.** If $P_T$ is hyperbolic then (24) has a unique $T$-periodic solution. Moreover, (24) is Hyers-Ulam stable on $[0, \infty)$. 

4.2. Systems with small oscillations. Another possibility is to study the case when \( k(z) = k + \epsilon \kappa(z) \) for a \( T \)-periodic continuous function \( \kappa(z) \) and \( \epsilon \) is small. Thus to study the case when \( k(z) \) has a small oscillation. To study (2), we introduce

\[
\begin{align*}
    x &= (k + \epsilon \kappa(z)) \frac{\partial u}{\partial z}, \\
    y &= (k + \epsilon \kappa(z)) \frac{\partial v}{\partial z},
\end{align*}
\]

and (2) is replaced by

\[
\begin{cases}
    \frac{\partial u}{\partial z} = \hat{k} \epsilon \kappa(z) x, \\
    \frac{\partial v}{\partial z} = \hat{k} \epsilon \kappa(z) y, \\
    \frac{\partial x}{\partial z} = -f(u - v) + g_1(z), \\
    \frac{\partial y}{\partial z} = f(u - v) - g_2(z),
\end{cases}
\]

for \( \hat{k}(z) = \frac{1}{k + \epsilon \kappa(z)} \), that is

\[
\vec{X}'(z) = A_\epsilon(z) \vec{X}(z) + \vec{F}(z), \quad z \in [0, \infty),
\]

where

\[
\vec{X} = \begin{bmatrix} u \\ v \\ x \\ y \end{bmatrix}, \quad A_\epsilon(z) = \begin{bmatrix} 0 & 0 & \hat{k}(z) & 0 \\
0 & 0 & 0 & \hat{k}(z) \\
0 & -f & 0 & 0 \\
f & 0 & 0 & 0 \end{bmatrix}, \quad \vec{F}(z) = \begin{bmatrix} 0 \\
0 \\
fv + g_1(z) \\
-fu - g_2(z) \end{bmatrix}.
\]

When \( \epsilon = 0 \), then \( \hat{k}_0(z) = \hat{k} = \frac{1}{k} \) is a constant function and the constant matrix

\[
A_0 = \begin{bmatrix} 0 & 0 & \hat{k} & 0 \\
0 & 0 & 0 & \hat{k} \\
0 & -f & 0 & 0 \\
f & 0 & 0 & 0 \end{bmatrix}
\]

has eigenvalues

\[
\lambda_1 = -\sqrt{\frac{f\hat{k}}{2}(1 + i)}, \quad \lambda_2 = -\sqrt{\frac{f\hat{k}}{2}(1 - i)}, \quad \lambda_3 = \sqrt{\frac{f\hat{k}}{2}(1 - i)}, \quad \lambda_4 = \sqrt{\frac{f\hat{k}}{2}(1 + i)}.
\]

Hence (28) is a small linear periodic perturbation of a hyperbolic linear system of the form

\[
\vec{X}'(z) = (A_0 + \hat{k}_\epsilon(z) B_0) \vec{X}(z) + \vec{F}(z), \quad z \in [0, \infty),
\]

where

\[
B_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{k}_\epsilon(z) = -\frac{\epsilon \kappa(z)}{k + \epsilon \kappa(z)}.
\]

Thus by the roughness of exponential dichotomy or Floquet theory [4, 8], we see that (30) has an exponential dichotomy on \([0, \infty)\) and again, results of Subsections
3.1 and 3.2 are extended to (28), i.e., it has a unique $T$-periodic solution and it is Hyers-Ulam stable on $[0,\infty)$. Note (11) is modified to

$$\begin{align*}
\widehat{X}(z) &= \hat{k}_\varepsilon(z) \left( \int_{-\infty}^{z} e^{A_0(z-s)} P_0 B_0 \widehat{X}(s) ds + \int_{z}^{\infty} e^{A_0(z-s)} (I - P_0) B_0 \widehat{X}(s) ds \right) \\
&\quad + \int_{-\infty}^{z} e^{A_0(z-s)} P_0 \overline{F}(s) ds + \int_{z}^{\infty} e^{A_0(z-s)} (I - P_0) \overline{F}(s) ds,
\end{align*}
$$

(31)

where $P_0 : \mathbb{R}^4 \to \mathbb{R}^4$ is a projection such that the range of $P_0$ is the stable space of $A_0$ while the kernel of $P_0$ is the unstable space of $A_0$. Consider a Banach space $C_b(\mathbb{R}^4)$ of all continuous and bounded function $\widehat{X} : \mathbb{R} \to \mathbb{R}^4$ with the norm $\|\widehat{X}\|_\infty = \sup_{z \in \mathbb{R}} \sum_{j=1}^{4} |X_j(z)|$. Then the corresponding matrix norm $\|\cdot\|_1$ is defined as the maximum absolute column sum of the matrix on $\mathbb{R}^4$. We know from Subsection 3.2 that

$$\|e^{A_0 z} P_0\|_1 \leq K_0 e^{-\alpha z}, \quad z \geq 0, \quad \|e^{A_0 z} (I - P_0)\|_1 \leq K_0 e^{\alpha z}, \quad z \leq 0$$

for some constant $K_0 \geq 1$ and $\alpha = \sqrt{\frac{K_0}{2}}$. Then we get

$$\begin{align*}
\left\| \int_{-\infty}^{z} e^{A_0(z-s)} P_0 \overline{F}(s) ds + \int_{z}^{\infty} e^{A_0(z-s)} (I - P_0) \overline{F}(s) ds \right\|_1 \\
\leq K_0 \left( \int_{-\infty}^{z} e^{-\alpha(z-s)} ds + \int_{z}^{\infty} e^{\alpha(z-s)} ds \right) \|\overline{F}\|_\infty = \frac{2K_0}{\alpha} \|\overline{F}\|_\infty.
\end{align*}$$

Next, introducing a linear operator $\Lambda_\varepsilon : C_b(\mathbb{R}^4) \to C_b(\mathbb{R}^4)$ as

$$(\Lambda_\varepsilon \widehat{X}(\cdot))(z) = \hat{k}_\varepsilon(z) \left( \int_{-\infty}^{z} e^{A_0(z-s)} P_0 B_0 \widehat{X}(s) ds + \int_{z}^{\infty} e^{A_0(z-s)} (I - P_0) B_0 \widehat{X}(s) ds \right),$$

we get like above

$$\|\Lambda_\varepsilon \widehat{X}(\cdot)\|_\infty \leq \frac{2K_0 \|\hat{k}_\varepsilon\|_\infty}{\alpha} \|\widehat{X}\|_\infty.$$ 

So $\Lambda_\varepsilon$ is a continuous linear operator with $\|\Lambda_\varepsilon\| \leq \frac{2K_0 \|\hat{k}_\varepsilon\|_\infty}{\alpha}$. Then (31) has the form

$$\begin{align*}
(I - \Lambda_\varepsilon) \widehat{X}(z) &= \overline{G}(z) = \int_{-\infty}^{z} e^{A_0(z-s)} P_0 \overline{F}(s) ds + \int_{z}^{\infty} e^{A_0(z-s)} (I - P_0) \overline{F}(s) ds.
\end{align*}
$$

(32)

Consequently, if

$$\frac{2K_0 \|\hat{k}_\varepsilon\|_\infty}{\alpha} < 1,$$

(33)

then the Neumann lemma [25] solves (32)

$$\widehat{X} = (I - \Lambda_\varepsilon)^{-1} \overline{G} = (I + \Lambda_\varepsilon + \Lambda_\varepsilon^2 + \Lambda_\varepsilon^3 \cdots) \overline{G}.$$ 

(34)

Finally, introducing a Green function

$$G(z,s) = \begin{cases} 
\hat{k}_\varepsilon(z) e^{A_0(z-s)} P_0 B_0 & \text{for } z \geq s, \\
\hat{k}_\varepsilon(z) e^{A_0(z-s)} (I - P_0) B_0 & \text{for } z \geq s,
\end{cases}$$

we have

$$(\Lambda_\varepsilon \widehat{X}(\cdot))(z) = \int_{-\infty}^{\infty} G(z,s) \widehat{X}(s) ds.$$
Then \[25\]

\[
(A_j \tilde{X}(\cdot))(z) = \int_{-\infty}^{\infty} G_j(z, s) \tilde{X}(s) ds
\]

for \(G_0(z, s) = G(z, s)\) and

\[
G_j(z, s) = \int_{-\infty}^{\infty} G(z, r) G_{j-1}(r, s) dr, \quad j \in \mathbb{N}_0.
\]

Summarizing, we obtain the next result.

**Theorem 4.3.** If (33) holds, then (28) has a unique \(T\)-periodic solution given by (34). Moreover, (28) is Hyers-Ulam stable on \([0, \infty)\) uniformly for \(\epsilon\) small.

4.3. **Systems with rapidly varying coefficients.** We deal with the case when \(k(z) = \kappa(z/\epsilon)\) and \(\tilde{F}(z) = \tilde{H}(z/\epsilon)\) for \(T\)-periodic continuous functions \(\kappa(z) > 0\) and \(\tilde{H}(z)\), and \(\epsilon \neq 0\) is small. So \(k(z)\) and \(\tilde{F}(z)\) are rapidly varying coefficients. Then scaling \(\epsilon z \leftrightarrow z\), like in (29), we get

\[
\tilde{X}'(z) = \epsilon \left( A(z) \tilde{X}(z) + \tilde{H}(z) \right), \quad z \in [0, \infty),
\]

where

\[
A(z) = \begin{bmatrix} 0 & 0 & \dot{k}(z) & 0 \\ 0 & 0 & 0 & \dot{k}(z) \\ 0 & -f & 0 & 0 \\ f & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{H}(z) = \begin{bmatrix} 0 & 0 \\ f \nu + h_1(z) & -f \mu - h_2(z) \end{bmatrix}, \quad \dot{k}(z) = \frac{1}{\kappa(z)}.
\]

We can apply the averaging method [24] to (35). The averaged equation is as follows

\[
\bar{Y}'(z) = \epsilon \left( \bar{A} \bar{Y}(z) + \bar{K} \right), \quad z \in [0, \infty),
\]

where

\[
\bar{A} = \begin{bmatrix} 0 & 0 & \bar{k}_0 & 0 \\ 0 & 0 & 0 & \bar{k}_0 \\ 0 & -f & 0 & 0 \\ f & 0 & 0 & 0 \end{bmatrix}, \quad \bar{K} = \begin{bmatrix} 0 \\ f \nu + h_1 \end{bmatrix}, \quad \bar{k}_0 = \frac{1}{T} \int_0^T \dot{k}(z) dz > 0, \quad \bar{h}_i = \frac{1}{T} \int_0^T h_i(z) dz, \quad i = 1, 2.
\]

We already know that \(\bar{A}\) is hyperbolic, so we obtain the following result.

**Theorem 4.4.** (36) has a unique \(T\)-periodic solution of the form \(\bar{A}^{-1} \bar{K} + O(\epsilon)\). Moreover, (36) is Hyers-Ulam stable on \([0, \infty)\) uniformly for \(\epsilon \neq 0\) small.

4.4. **Systems with slowly varying coefficients.** We deal with the case when \(k(z) = \kappa(z/\epsilon)\) and \(\tilde{F}(z) = \tilde{H}(z/\epsilon)\) for \(T\)-periodic continuously differentiable functions \(\kappa(z) > 0\) and \(\tilde{H}(z)\), and \(\epsilon > 0\) is small. So \(k(z)\) and \(\tilde{F}(z)\) are slowly varying coefficients. Then scaling \(z/\epsilon \leftrightarrow z\), like in (29), we get

\[
\epsilon \bar{X}'(z) = A(z) \bar{X}(z) + \bar{H}(z), \quad z \in [0, \infty),
\]

where \(A(z)\) and \(\bar{H}(z)\) are defined in (35). Setting

\[
\bar{Y}(z) = \bar{X}(z) + \bar{L}(z), \quad \bar{L}(z) = A(z)^{-1} \bar{H}(z),
\]

we derive

\[
\epsilon \bar{Y}'(z) = A(z) \bar{Y}(z) + \epsilon \bar{L}'(z), \quad z \in [0, \infty),
\]
recalling notations of Subsection 3.2, we have $N(z)^{-1}A(z)N(z) = J(z)$. Then we take
\[ \overrightarrow{W}(z) = N(z)^{-1}\overrightarrow{Y}(z), \]
and arrive at
\[ \epsilon\overrightarrow{W}'(z) = \epsilon N(z)^{-1}\overrightarrow{Y}'(z) + \epsilon(N(z)^{-1})'\overrightarrow{Y}(z) \]
where
\[ J(z) = \begin{bmatrix} \lambda_1(z) & 0 & 0 & 0 \\ 0 & \lambda_2(z) & 0 & 0 \\ 0 & 0 & \lambda_3(z) & 0 \\ 0 & 0 & 0 & \lambda_4(z) \end{bmatrix}, \quad \Re\lambda_1(z) = -\Re\lambda_2(z) = \Re\lambda_3(z) = -\Re\lambda_4(z) > 0. \]
Setting
\[ \overrightarrow{Q}(\overrightarrow{W}(z), z) = (N(z)^{-1})'N(z)\overrightarrow{W}(z) + N(z)^{-1}\overrightarrow{L}'(z), \]
(39) becomes
\[ \epsilon\overrightarrow{W}'(z) = J(z)\overrightarrow{W}(z) + \epsilon\overrightarrow{Q}(\overrightarrow{W}(z), z). \]
We prove the following result.

**Lemma 4.5.** For any $\overrightarrow{P}(z) \in C_b(\mathbb{C}^4)$, there is a unique $T$-periodic solution $\overrightarrow{W}(z)$ solving
\[ \epsilon\overrightarrow{W}'(z) = J(z)\overrightarrow{W}(z) + \epsilon\overrightarrow{P}(z). \]
Moreover it holds
\[ \|\overrightarrow{W}\|_\infty \leq \frac{\epsilon}{\lambda_0} \|\overrightarrow{P}\|_\infty, \]
where
\[ \lambda_0 = \min_{z \in \mathbb{R}} \Re\lambda_1(z) > 0. \]
We denote this solution by $\overrightarrow{W}(z) = \gamma_\epsilon(\overrightarrow{P})(z)$.

**Proof.** (41) has the form
\[ W_i'(z) = \frac{\lambda_i(z)}{\epsilon} W_i(z) + P_i(z), \quad i = 1, 2, 3, 4. \]
A unique $T$-periodic solution of (41) is given by
\[ W_1(z) = -\int_z^\infty e^{-\int_z^r \lambda_1(s)ds} P_1(s)ds, \]
\[ W_2(z) = \int_{-\infty}^z e^{\int_z^r \lambda_2(s)ds} P_2(s)ds, \]
\[ W_3(z) = -\int_z^\infty e^{-\int_z^r \lambda_3(s)ds} P_3(s)ds, \]
\[ W_4(z) = \int_{-\infty}^z e^{\int_z^r \lambda_4(s)ds} P_4(s)ds. \]
Next, we have
\[
|W_1(z)| \leq \|\tilde{P}\|_\infty \int_{-\infty}^{\infty} e^{- \frac{f_z z}{\lambda_1(z)} ds} \leq \|\tilde{P}\|_\infty \int_{-\infty}^{\infty} e^{- \lambda_0(z-x) ds} = \frac{\epsilon}{\lambda_0} \|\tilde{P}\|_\infty,
\]
\[
|W_2(z)| \leq \|\tilde{P}\|_\infty \int_{-\infty}^{\infty} e^{\frac{-f_z z}{\lambda_1(z)} ds} \leq \|\tilde{P}\|_\infty \int_{-\infty}^{\infty} e^{- \lambda_0(z-x) ds} = \frac{\epsilon}{\lambda_0} \|\tilde{P}\|_\infty,
\]
\[
|W_3(z)| \leq \|\tilde{P}\|_\infty \int_{-\infty}^{\infty} e^{- \frac{f_z z}{\lambda_1(z)} ds} \leq \|\tilde{P}\|_\infty \int_{-\infty}^{\infty} e^{- \lambda_0(z-x) ds} = \frac{\epsilon}{\lambda_0} \|\tilde{P}\|_\infty,
\]
\[
|W_4(z)| \leq \|\tilde{P}\|_\infty \int_{-\infty}^{\infty} e^{- \frac{f_z z}{\lambda_1(z)} ds} \leq \|\tilde{P}\|_\infty \int_{-\infty}^{\infty} e^{- \lambda_0(z-x) ds} = \frac{\epsilon}{\lambda_0} \|\tilde{P}\|_\infty.
\]
This implies (42). The proof is finished. □

Lemma 4.6. If
\[
\frac{\epsilon}{\lambda_0} \|(N(-1))'N(\cdot)\|_\infty < 1,
\]
then (40) has a unique T-periodic solution satisfying
\[
\|\tilde{W}(\cdot)\|_\infty \leq \frac{\epsilon \|(N(-1))'\tilde{L}(\cdot)\|_\infty}{\lambda_0 - \epsilon \|\|(N(-1))'N(\cdot)\|_\infty\|}.
\]

Proof. Applying Lemma 4.5, T-periodic solution of (40) is given as a fixed point problem
\[
\tilde{W}(z) = \Upsilon(\tilde{Q}(\tilde{W}(\cdot), \cdot))(z).
\]
Next, for any \(\tilde{W}_1(z), \tilde{W}_2(z) \in C_B(\mathbb{C}^4),\) we derive
\[
\|\Upsilon(\tilde{Q}(\tilde{W}_1(\cdot), \cdot) - \Upsilon(\tilde{Q}(\tilde{W}_2(\cdot), \cdot))(z)\| \leq \frac{\epsilon}{\lambda_0} \|	ilde{Q}(\tilde{W}_1(\cdot), \cdot) - \tilde{Q}(\tilde{W}_2(\cdot), \cdot)\|_\infty
\leq \frac{\epsilon}{\lambda_0} \|N(\cdot)\|_\infty \|\tilde{W}_1(\cdot) - \tilde{W}_2(\cdot)\|_\infty \leq \frac{\epsilon}{\lambda_0} \|N(\cdot)\|_\infty \|\tilde{W}_1(\cdot) - \tilde{W}_2(\cdot)\|_\infty,
\]
hence
\[
\|\Upsilon(\tilde{Q}(\tilde{W}_1(\cdot), \cdot)) - \Upsilon(\tilde{Q}(\tilde{W}_2(\cdot), \cdot))\|_\infty \leq \frac{\epsilon}{\lambda_0} \|N(\cdot)\|_\infty \|\tilde{W}_1(\cdot) - \tilde{W}_2(\cdot)\|_\infty.
\]
By (43), we can uniquely solve (45) by the Banach fixed point theorem, and this solution satisfies
\[
\|\tilde{W}(\cdot)\| = \|\Upsilon(\tilde{Q}(\tilde{W}(\cdot), \cdot))\|_\infty
\leq \frac{\epsilon}{\lambda_0} \|\tilde{Q}(\tilde{W}(\cdot), \cdot)\|_\infty
\leq \frac{\epsilon}{\lambda_0} \|N(\cdot)\|_\infty \|\tilde{W}(\cdot)\|_\infty + \|N(\cdot)\|_\infty \|\tilde{W}(\cdot)\|_\infty + \|\tilde{L}(\cdot)\|_\infty,
\]
which gives (44). The proof is ready. □

Applying Lemma 4.6, we obtain

Theorem 4.7. If (43) holds the (37) has a unique T-periodic solution satisfying
\[
\|\tilde{X}(\cdot) - A(\cdot)^{-1}\tilde{L}(\cdot)\|_\infty
\leq \frac{\epsilon \|N(\cdot)\|_\infty \|A(\cdot)^{-1}\|_\infty \|\tilde{L}(\cdot)\|_\infty + \|A(\cdot)^{-1}\|_\infty \|\tilde{L}(\cdot)\|_\infty}{\lambda_0 - \epsilon \|\|(N(\cdot))'N(\cdot)\|_\infty\|}.
\]
Proof. We compute
\[
\|\tilde{X}(z) - \tilde{Y}(z)\| = \|\tilde{Y}(z)\| = \|N(z)\tilde{W}(z)\| \leq \|N(z)\|\|\tilde{W}(z)\|
\]
\[
\leq \frac{\|N(\cdot)\|_{\infty} \|N(\cdot)^{-1}\|_{\infty} \|\tilde{L}'(\cdot)\|_{\infty}}{\lambda_0 - \epsilon \|((N(\cdot)^{-1})'N(\cdot))\|_{\infty}},
\]
which implies (46), since
\[
\|\tilde{L}'(z)\| = \|(A(z)^{-1})'\tilde{H}(z) + A(z)^{-1}\tilde{H}'(z)\|
\]
\[
\leq \|(A(z)^{-1})'\|\|\tilde{H}(z)\| + \|A(z)^{-1}\|\|\tilde{H}'(z)\|.
\]
The proof is finished.

Consequently, the original equation (2) has a unique $Te$-periodic solution $\tilde{X}(\epsilon z)$ which is uniformly $O(\epsilon)$ near to $A(\epsilon z)^{-1}\tilde{H}(\epsilon z)$.

Following the above arguments, we arrive at the following Hyers-Ulam stable result.

**Theorem 4.8.** Assume that there are $C^1$-smooth functions $\tilde{X}(z)$ and $\tilde{v}(z)$ on $[0, \infty)$ such that
\[
\epsilon \tilde{X}'(z) = A(z)\tilde{X}(z) + \tilde{H}(z) + \tilde{v}(z), \ z \in [0, \infty),
\]
\[
\max\{\|\tilde{v}(\cdot)\|_{\infty}, \|\tilde{v}'(\cdot)\|_{\infty}\} \leq \delta.
\]
Then there is a $C^1$-smooth function $\tilde{X}_0(z)$ solving (37) and satisfying
\[
\|\tilde{X}(\cdot) - \tilde{X}_0(\cdot)\|_{\infty}
\]
\[
\leq \left(\|A(\cdot)^{-1}\|_{\infty} + \frac{\epsilon \|N(\cdot)\|_{\infty} \|N(\cdot)^{-1}\|_{\infty} \|((A(\cdot)^{-1})'\|_{\infty} + \|A(\cdot)^{-1}\|_{\infty})}{\lambda_0 - \epsilon \|((N(\cdot)^{-1})'N(\cdot))\|_{\infty}}\right) \delta.
\]

**Proof.** The function
\[
\tilde{M}(z) = \tilde{X}(z) - \tilde{X}_0(z)
\]
solves
\[
\epsilon \tilde{M}'(z) = A(z)\tilde{M}(z) + \tilde{v}(z), \ z \in [0, \infty).
\]
We extend $\tilde{v}(z)$ on $\mathbb{R}$ satisfying the inequality (47). Then Theorem 4.7 ensures the existence of $\tilde{M}(z)$ solving (49) and satisfying
\[
\|\tilde{M}(z)\| \leq \|A(\cdot)^{-1}\|_{\infty} \|\tilde{v}(\cdot)\|_{\infty}
\]
\[
+ \frac{\epsilon \|N(\cdot)\|_{\infty} \|N(\cdot)^{-1}\|_{\infty} \|((A(\cdot)^{-1})'\|_{\infty} \|\tilde{v}(\cdot)\|_{\infty} + \|A(\cdot)^{-1}\|_{\infty} \|\tilde{v}'(\cdot)\|_{\infty})}{\lambda_0 - \epsilon \|((N(\cdot)^{-1})'N(\cdot))\|_{\infty}}
\]
\[
= \left(\|A(\cdot)^{-1}\|_{\infty} + \frac{\epsilon \|N(\cdot)\|_{\infty} \|N(\cdot)^{-1}\|_{\infty} \|((A(\cdot)^{-1})'\|_{\infty} + \|A(\cdot)^{-1}\|_{\infty})}{\lambda_0 - \epsilon \|((N(\cdot)^{-1})'N(\cdot))\|_{\infty}}\right) \delta.
\]
This finishes the proof.
5. **Conclusion.** This paper derives a generalized nonhomogeneous system of first order differential equations involving a term with the horizontal dependent. The existence and uniqueness of periodic solutions and Hyers-Ulam stability results are established for this nonhomogeneous systems with the fixed constant, with the piecewise constants, with the small oscillations, with the rapidly varying coefficients and the slowly varying coefficients, respectively.

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