YANG–MILLS CONNECTIONS ON COMPACT COMPLEX TORI

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Abstract. Let $G$ be a connected reductive complex affine algebraic group and $K \subset G$ a maximal compact subgroup. Let $M$ be a compact complex torus equipped with a flat Kähler structure and $(E_G, \theta)$ a polystable Higgs $G$–bundle on $M$. Take any $C^\infty$ reduction of structure group $E_K \subset E_G$ to the subgroup $K$ that solves the Yang–Mills equation for $(E_G, \theta)$. We prove that the principal $G$–bundle $E_G$ is polystable and the above reduction $E_K$ solves the Einstein–Hermitian equation for $E_G$. We also prove that for a semistable (respectively, polystable) Higgs $G$–bundle $(E_G, \theta)$ on a compact connected Calabi–Yau manifold, the underlying principal $G$– bundle $E_G$ is semistable (respectively, polystable).

1. Introduction

Let $X$ be a compact connected Kähler manifold equipped with a Kähler form $\tilde{\omega}$. Let $(E, \theta)$ be a Higgs vector bundle on $X$. Given a Hermitian structure $h$ on $E$, the curvature of the corresponding Chern connection on $E$ will be denoted by $\mathcal{K}_h$. A Hermitian structure $h$ is said to satisfy the Yang–Mills equation for $(E, \theta)$ if there is $c \in \mathbb{R}$ such that
\[ \Lambda_{\tilde{\omega}}(\mathcal{K}_h + \theta \wedge \theta^*) = c \sqrt{-1} \cdot \text{Id}_E, \]
where $\Lambda_{\tilde{\omega}}$ is the adjoint of multiplication of forms by $\tilde{\omega}$, and $\theta^*$ is the adjoint of $\theta$ with respect to $h$. A Higgs bundle admits a Hermitian structure satisfying the Yang–Mills equation if and only if it is polystable [Si1, Hi]. If $\theta = 0$, then the above Yang–Mills equation is also known as the Einstein–Hermitian equation. A holomorphic vector bundle $E$ admits an Einstein–Hermitian metric if and only if $E$ is polystable [UY], [Do].

More generally, let $G$ be a connected reductive affine algebraic group over $\mathbb{C}$. Fix a maximal compact subgroup $K \subset G$. The center of the Lie algebra of $K$ will be denoted by $\mathfrak{z}(k)$. Let $(E_G, \theta)$ be a Higgs $G$–bundle on $X$ (its definition is recalled in Section 4.1). A $C^\infty$ reduction of structure group of $E_G$ to $K$
\[ E_K \subset E_G \]
is said to satisfy the Yang–Mills equation for $(E, \theta)$ if there is an element $c \in \mathfrak{z}(\mathfrak{k})$ such that
\[ \Lambda_{\tilde{\omega}}(\mathcal{K} + \theta \wedge \theta^*) = c, \]
where $\mathcal{K}$ is the curvature of the Chern connection associated to the reduction $E_K$ and $\theta^*$ is the adjoint of $\theta$ constructed using $E_K$ (see [At, p. 191–192, Proposition 5] for Chern connections on principal bundles). A Higgs $G$–bundle admits a Yang–Mills connection if

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and only if it is polystable [Si2, BS]. As mentioned before, if $\theta = 0$, then the Yang–Mills equation is also called the Einstein–Hermitian equation.

We consider Higgs $G$–bundles over a torus $M$ equipped with a flat Kähler form $\tilde{\omega}$. If $(E_G, \theta)$ is a polystable Higgs $G$–bundle on $M$, we prove that the principal $G$–bundle $E_G$ is polystable. If a reduction to $K$ 

$$E_K \subset E_G$$

satisfies the Yang–Mills equation for $(E, \theta)$, we show that $E_K$ also satisfies the Einstein–Hermitian equation for $E_G$.

In the last section we observe some properties of Higgs bundles on Kähler manifolds with nonnegative tangent bundle.

## 2. Higgs vector bundles on a torus

### 2.1. Semistable and polystable Higgs bundles.

Let $M$ be a compact complex torus of complex dimension $d$. Fix a Kähler class

$$\omega \in H^{1,1}(M) \cap H^2(M, \mathbb{R}).$$

The degree of any torsionfree coherent analytic sheaf $F$ on $M$ is defined to be

$$\text{degree}(F) := (c_1(F) \cap \omega^{d-1}) \cap [M] \in \mathbb{R}.$$ 

If $F$ is of positive rank, then

$$\mu(F) := \frac{\text{degree}(F)}{\text{rank}(F)} \in \mathbb{R}$$

is called the slope of $F$.

Let $E$ be a holomorphic vector bundle on $M$. A Higgs field on $E$ is a holomorphic section

$$\theta \in H^0(M, \text{End}(E) \otimes \Omega^1_M)$$

such that section

$$\theta \wedge \theta \in H^0(M, \text{End}(E) \otimes \Omega^2_M)$$

vanishes identically. A Higgs bundle is a holomorphic vector bundle equipped with a Higgs field.

The following lemma is well-known (see [BF, FGN1]).

**Lemma 2.1.** Let $(E, \theta)$ be a semistable Higgs bundle on $M$. Then the holomorphic vector bundle $E$ is semistable.

**Proof.** Assume that $E$ is not semistable. Let

$$(2.1)\quad E_1 \subset E_2 \subset \cdots \subset E_n = E$$

be the Harder–Narasimhan filtration of $E$. We have

$$H^0(M, \text{End}(E_1)) = H^0(M, \text{Hom}(E_1, E))$$

because $H^0(M, \text{Hom}(E_1, E_i/E_{i-1})) = 0$ for every $i \in \{2, \cdots, n\}$. Since $\Omega^1_M$ is the trivial vector bundle of rank $d$, this implies that

$$\theta(E_1) \subset E_1 \otimes \Omega^1_M.$$
Therefore, \( E_1 \) contradicts the given condition that \((E, \theta)\) is semistable. Hence the vector bundle \( E \) is semistable. \( \square \)

We note that the following lemma is a consequence of Corollary 2.2 of [Bi, p. 73] (see also [FGN1]).

**Lemma 2.2.** Let \((E, \theta)\) be a polystable Higgs bundle on \( M \). Then the holomorphic vector bundle \( E \) is polystable.

**Proof.** From Lemma 2.1 we know that \( E \) is semistable. Let \( V \subset E \) be the coherent analytic subsheaf generated by all polystable subsheaves of \( E \) with slope \( \mu(E) \). This \( V \) is a polystable subsheaf with slope \( \mu(E) \) [HL, page 23, Lemma 1.5.5]. We will show that

\[
\theta(V) \subset V \otimes \Omega^1_M.
\]

To show (2.2), fix a holomorphic trivialization of \( \Omega^1_M \). Using this trivialization, the homomorphism \( \theta \) is written as

\[
\theta = (\theta_1, \ldots, \theta_d),
\]

where \( \theta_i \in H^0(M, \text{End}(E)) \) for every \( i \). Since \( E \) is semistable, and \( V \) is polystable with \( \mu(V) = \mu(E) \), it follows that \( \theta_i(V) \) is polystable with \( \mu(\theta_i(V)) = \mu(E) \). Therefore, (2.2) holds.

Assume that \( E \) is not polystable. So \( V \neq E \). Since the Higgs bundle \((E, \theta)\) is polystable, from (2.2) and the fact that \( \mu(V) = \mu(E) \) we conclude that there is a coherent analytic subsheaf

\[
V' \subset E
\]
such that \( \mu(V') = \mu(E) \) and \( V \cap V' = 0 \). Let \( V'' \subset V' \) be a polystable subsheaf such that \( \mu(V'') = \mu(V') \). From the definition of \( V \) it follows that \( V'' \subset V \). But this contradicts the condition that \( V \cap V'' \subset V \cap V' = 0 \). So \( E \) is polystable. \( \square \)

### 2.2. Higgs fields on a polystable vector bundle.

Let \( E \to M \) be a polystable vector bundle. Our aim in this subsection is to describe all Higgs fields \( \theta \) on \( E \) such that the Higgs bundle \((E, \theta)\) is polystable.

Since \( E \) is polystable, we can write

\[
E = \bigoplus_{j=1}^\ell E_j \otimes \mathbb{C}^{n_j},
\]

where

- each \( E_j \) is a stable vector bundle with \( \mu(E_j) = \mu(E) \),
- \( E_j \) is not isomorphic to \( E_{j'} \) if \( j \neq j' \), and
- \( n_j > 0 \) for every \( j \).

From the first two conditions it follows immediately that \( H^0(M, \text{Hom}(E_j, E_{j'})) = 0 \) if \( j \neq j' \). Therefore, we have

\[
H^0(M, \text{Hom}(E_j, E_{j'}) \otimes \Omega^1_M) = 0 \quad \text{if} \quad j \neq j'.
\]
Since $E_j$ is stable, we also have
\begin{equation}
H^0(M, End(E_j)) = \mathbb{C}.
\end{equation}
In view of (2.4) and (2.5), any $\beta \in H^0(M, End(E))$ can be written as
\begin{equation}
\beta = \bigoplus_{j=1}^\ell \text{Id}_{E_j} \otimes T_j
\end{equation}
in terms of the isomorphism in (2.3), where
\[ T_j \in M(n_j, \mathbb{C}) = \text{End}_\mathbb{C}(\mathbb{C}^{n_j}). \]

As before, fix a holomorphic trivialization of $\Omega^1_M$. Using this trivialization, any $\theta \in H^0(M, End(E) \otimes \Omega^1_M)$ can be written as
\[ \theta = (\theta_1, \cdots, \theta_d), \]
where $\theta_i \in H^0(M, End(E))$.

Take any
\begin{equation}
\theta \in H^0(M, End(E) \otimes \Omega^1_M).
\end{equation}
Write
\[ \theta = (\theta_1, \cdots, \theta_d), \]
as above. Let
\begin{equation}
\theta_i = \bigoplus_{j=1}^\ell \text{Id}_{E_j} \otimes T^i_j,
\end{equation}
where $T^i_j \in M(n_j, \mathbb{C})$.

**Proposition 2.3.** The pair $(E, \theta)$ in (2.6) is a polystable Higgs bundle if and only if
\begin{enumerate}
\item $T^i_j T^k_j = T^k_j T^i_j$ (see (2.7)) for all $i, k \in \{1, \cdots, d\}$ and all $j$, and
\item each $T^i_j$ is semisimple.
\end{enumerate}

**Proof.** First assume that the two conditions in the proposition are satisfied. The first condition implies that $\theta \wedge \theta = 0$. The second condition implies that $(E, \theta)$ can be expressed as a direct sum of stable Higgs bundles of same slope. Therefore, $(E, \theta)$ is polystable.

Now assume that $(E, \theta)$ is a polystable Higgs bundle. Since $\theta \wedge \theta = 0$, the first condition in the proposition holds. The Higgs bundle $(E, \theta)$ is a direct sum of stable Higgs bundles of same slope. From this it follows that the second condition in the proposition is satisfied.

**Remark 2.4.** A sum of commuting semisimple matrices is again semisimple. Therefore, the two conditions in Proposition 2.3 are independent of the choice of the trivialization of $\Omega^1_M$. 

3. Yang–Mills Hermitian metric on polystable Higgs bundles

Let $\text{Aut}^0(M)$ denote the connected component, containing the identity element, of the group of holomorphic automorphisms of $M$. The complex manifold $\text{Aut}^0(M)$ is isomorphic to $M$. If we consider $M$ as a complex abelian Lie group, then $\text{Aut}^0(M)$ coincides with the group of translations of $M$.

There is a unique Kähler form $\tilde{\omega}$ on $M$ such that

- the cohomology class of $\tilde{\omega}$ coincides with $\omega$, and
- the form $\tilde{\omega}$ is preserved by the action of $\text{Aut}^0(M)$ on $M$.

The Kähler structure on $M$ given by $\tilde{\omega}$ is flat. Fix the Kähler form $\tilde{\omega}$ on $M$.

Proposition 3.1. Let $(E, \theta)$ be a polystable Higgs bundle on $M$. There is a Yang–Mills Hermitian metric $h$ on $E$ for the Higgs field $\theta$ such that $h$ satisfies the Einstein–Hermitian equation for the polystable vector bundle $E$.

Proof. Fix a trivialization of $\Omega^1_M$ using holomorphic sections of $\Omega^1_M$ that are pointwise orthonormal. Such a trivialization exists because the connection on $\Omega^1_M$ corresponding to $\tilde{\omega}$ is flat with trivial monodromy. Take the endomorphisms $T^i_j$ in Proposition 2.3. Take any $j \in \{1, \cdots, \ell\}$. Since $T^j_i T^k_j = T^k_i T^j_i$ for all $i, k \in \{1, \cdots, d\}$, we have a simultaneous eigenspace decomposition of $\mathbb{C}^n_j$ for the eigenvalues of $T^i_j$, $i \in \{1, \cdots, d\}$. Fix an inner product $h_j$ on $\mathbb{C}^n_j$ such that the above decomposition of $\mathbb{C}^n_j$ given by the eigenspaces of $\{T^i_j\}_{i=1}^d$ is orthogonal.

Fix an Einstein–Hermitian structure $h'_j$ on the stable vector bundle $E_j$ in (2.3). The Hermitian structures $h_j$ and $h'_j$ together produce a Hermitian structure on the vector bundle $E_j \otimes \mathbb{C}^n_j$ in (2.3). These together in turn define a Hermitian structure $h$ on $E$ using the isomorphism in (2.3) after imposing the condition that the subbundles $E_j \otimes \mathbb{C}^n_j$ in (2.3) are orthogonal.

The above Hermitian structure $h$ on $E$ clearly satisfies the Einstein–Hermitian equation for the polystable vector bundle $E$.

Let $\theta^* \in C^\infty(M; \text{End}(E) \otimes \Omega^0_{M^1})$ be the adjoint of $\theta$. From the construction of $h$ it follows that $\theta \wedge \theta^* = 0$. Using this it is straightforward to check that $h$ satisfies the Yang–Mills equation for the Higgs bundle $(E, \theta)$. □

Theorem 3.2. Let $(E, \theta)$ be a polystable Higgs bundle on $M$. Let $h'$ be a Yang–Mills Hermitian metric on $E$ for the Higgs field $\theta$. Then $h'$ satisfies the Einstein–Hermitian equation for the polystable vector bundle $E$.

Proof. Consider the Yang–Mills Hermitian metric $h$ on $E$ constructed in Proposition 3.1. The two Hermitian structures $h$ and $h'$ differ by a holomorphic automorphism of $E$. In other words, there is a holomorphic automorphism

$$T : E \longrightarrow E$$

such that

$$(3.1) \quad h'(v, w) = h(T(v), T(w))$$

for all $v, w \in E_x$ and all $x \in M$. From this we will derive that $h'$ satisfies the Einstein–Hermitian equation for the polystable vector bundle $E$. 

Consider the holomorphic vector bundle $\operatorname{End}(E) = E \otimes E^\ast$. The Hermitian structure $h$ on $E$ produces a Hermitian structure on $\operatorname{End}(E)$. The corresponding Chern connection $\nabla$ on $\operatorname{End}(E)$ is Einstein–Hermitian, because $h$ satisfies the Einstein–Hermitian equation. Note that $c_1(\operatorname{End}(E)) = 0$. Therefore, the mean curvature of the Einstein–Hermitian connection $\nabla$ on $\operatorname{End}(E)$ vanishes identically (see [Ko, p. 51] for mean curvature). In particular, the section $T$ in (3.1) is flat with respect to $\nabla$.

Since $h$ satisfies the Einstein–Hermitian equation for $E$, and $T$ is flat with respect to the connection $\nabla$ given by $h$, it follows that $h'$ defined by (3.1) also satisfies the Einstein–Hermitian equation for $E$. □

4. Higgs $G$–bundles on $M$

4.1. Semistable and polystable Higgs $G$–bundles. Let $G$ be a connected reductive affine algebraic group defined over $\mathbb{C}$. The Lie algebra of $G$ will be denoted by $\mathfrak{g}$. For a holomorphic principal $G$–bundle $E_G$ on $M$, let $\operatorname{ad}(E_G) := E_G \times^G \mathfrak{g}$ be the adjoint bundle. A section

$$\theta \in H^0(M, \operatorname{ad}(E_G) \otimes \Omega^1_M)$$

is called a Higgs field on $E_G$ if $\theta \wedge \theta = 0$. A Higgs $G$–bundle is a holomorphic principal $G$–bundle equipped with a Higgs field.

The proof of the following lemma is very similar to the proof of Lemma 2.1.

**Lemma 4.1.** Let $(E_G, \theta)$ be a semistable Higgs $G$–bundle on $M$. Then the principal $G$–bundle $E_G$ is semistable.

**Proof.** Assume that the principal $G$–bundle $E_G$ is not semistable. Let

$$E_P \subset E_G$$

be the Harder–Narasimhan reduction of $E_G$ over the dense open subset $U$ associated to $E_G$. We have

$$H^0(U, \operatorname{ad}(E_G)/\operatorname{ad}(E_P)) = 0$$

[AAB, p. 705, Corollary 1]. Since the vector bundle $\Omega^1_M$ is trivial, this implies that the image of $\theta$ in $H^0(U, (\operatorname{ad}(E_G)/\operatorname{ad}(E_P))) \otimes \Omega^1_M$ vanishes identically. In other words,

$$\theta \in H^0(U, \operatorname{ad}(E_P) \otimes \Omega^1_M).$$

Therefore, the above reduction $E_P$ contradicts the given condition that the Higgs $G$–bundle $(E_G, \theta)$ is semistable. Consequently, the principal $G$–bundle $E_G$ is semistable. □

Lemma 4.1 and Lemma 4.2 are proved in [FGN2] under the assumption that $M$ is an elliptic curve.

**Lemma 4.2.** Let $(E_G, \theta)$ be a polystable Higgs $G$–bundle on $M$. Then the principal $G$–bundle $E_G$ is polystable.

**Proof.** Since $(E_G, \theta)$ is polystable, it admits a Yang–Mills connection $\nabla$ [BS, p. 554, Theorem 4.6]. Let $\operatorname{ad}(\theta)$ be the Higgs field on the vector bundle $\operatorname{ad}(E_G)$ induced by $\theta$. The connection on $\operatorname{ad}(E_G)$ induced by $\nabla$ satisfies Yang–Mills equation for the Higgs bundle $(\operatorname{ad}(E_G), \operatorname{ad}(\theta))$. Therefore, $(\operatorname{ad}(E_G), \operatorname{ad}(\theta))$ is polystable. Hence the vector bundle
ad\((E_G)\) is polystable by Lemma 2.2. This implies that the principal \(G\)-bundle \(E_G\) is polystable [AB, p. 224, Corollary 3.8]. □

4.2. A Levi reduction associated to a semisimple section. Let \(E_G\) be a holomorphic principal \(G\)-bundle over \(M\). Let
\[
(4.1) \quad \eta \in H^0(M, \text{ad}(E_G))
\]
be a section such that \(\eta(x) \in \text{ad}(E_G)_x\) is semisimple for every \(x \in M\). Since the Lie algebra \(\text{ad}(E_G)_x\) is identified with the Lie algebra \(g\) of \(G\) up to an inner automorphism, the element \(\eta(x) \in \text{ad}(E_G)_x\) defines a conjugacy class in \(g\). Let
\[
C_x \subset g
\]
denote this orbit of \(G\) in \(g\) given by \(\eta(x)\).

**Proposition 4.3.** The above conjugacy class \(C_x\) is independent of the point \(x\).

**Proof.** Fix a maximal torus \(T \subset G\). Let \(W := N(T)/T\) be the corresponding Weyl group, where \(N(T) \subset G\) is the normalizer of \(T\). The Lie algebra of \(T\) will be denoted by \(t\). The space of semisimple conjugacy classes in \(g\) is identified with the quotient \(t/W\).

Since \(t/W\) is an affine variety, and \(M\) is a compact connected complex manifold, there are no nonconstant holomorphic maps from \(M\) to \(t/W\). This immediately implies that \(C_x\) is independent of the point \(x\). □

Fix an element
\[
(4.2) \quad \eta' \in C_x \subset g.
\]
Let
\[
(4.3) \quad L = C(\eta') \subset G
\]
be the centralizer of \(\eta'\). It is known that \(L\) is a Levi subgroup of \(G\) [DM, p. 26, Proposition 1.22]; we recall that a Levi subgroup of \(G\) is a maximal connected reductive subgroup of some parabolic subgroup of \(G\).

**Proposition 4.4.** Given \(\eta\) and \(\eta'\) as above, the principal \(G\)-bundle \(E_G\) has a natural holomorphic reduction of structure group to the subgroup \(L\) defined in (4.3).

**Proof.** For any \(g \in G\), let
\[
\text{Ad}(g) : g \rightarrow g
\]
be the Lie algebra automorphism corresponding to the automorphism of the group \(G\) defined by \(z \mapsto g^{-1}zg\). We recall that \(\text{ad}(E_G)\) is the quotient of \(E_G \times g\) where two point \((y_1, v_1), (y_2, v_2) \in E_G \times g\) are identified if there is an element \(g \in G\) such \(y_2 = y_1 g\) and \(v_2 = \text{Ad}(g)(v_1)\). Let
\[
q : E_G \times g \rightarrow \text{ad}(E_G)
\]
be the quotient map.

Let
\[
p_1 : E_G \times g \rightarrow E_G
\]
be the projection to the first factor. Define
\[
(4.4) \quad \mathcal{Z} := p_1((q^{-1}(\eta(M))) \cap (E_G \times \eta')) \subset E_G.
\]
It is straightforward to check that $Z$ is a holomorphic reduction of structure group of $E_G$ to the subgroup $L$. □

If $\eta'$ in (4.2) is replaced by $\text{Ad}(g)(\eta')$ for some $g \in G$, then the subgroup $L$ in (4.3) gets replaced by $g^{-1}Lg$.

**Corollary 4.5.** If $\eta'$ in (4.2) is replaced by $\text{Ad}(g)(\eta')$ for some $g \in G$, then $Z$ in (4.4) gets replaced by $Zg \subset E_G$.

**Proof.** This follows immediately from the construction in (4.4). □

Let $E_L \subset E_G$ be the reduction of structure group to $L$ constructed in Proposition 4.4. Let $\text{ad}(E_L)$ be the adjoint vector bundle for $E_L$.

**Corollary 4.6.** The subbundle $\text{ad}(E_L) \subset \text{ad}(E_G)$ is independent of the choice of the element $\eta'$ in (4.2).

**Proof.** This follows immediately from Corollary 4.5. □

**Corollary 4.7.** For any $x \in M$, the subalgebra $\text{ad}(E_L)_x \subset \text{ad}(E_G)_x$ coincides with the centralizer of $\eta(x) \in \text{ad}(E_G)_x$.

**Proof.** This follows from the construction of $E_L$ in (4.4). □

4.3. A Levi reduction associated to a semisimple Higgs field. Let $(E_G, \theta)$ be a Higgs $G$–bundle on $M$. For any $\alpha \in H^0(M, TM)$, we have

$$\theta(\alpha) \in H^0(M, \text{ad}(E_G)).$$

Take a basis $\{\alpha_1, \cdots, \alpha_d\}$ of $H^0(M, TM)$.

**Lemma 4.8.** If $\theta(\alpha_i)$ is pointwise semisimple for every $i \in \{1, \cdots, d\}$, then for any $\alpha \in H^0(M, TM)$, the section $\theta(\alpha)$ is pointwise semisimple.

**Proof.** Since $\theta$ is a Higgs field, we have $[\theta(\alpha_i), \theta(\alpha_j)] = 0$ for every $i, j \in \{1, \cdots, d\}$. The lemma follows from the fact that a sum of commuting semisimple elements of $g$ is again semisimple. □

Assume that $\theta(\alpha_i)$ is pointwise semisimple for every $i \in \{1, \cdots, d\}$. Let $L_1 \subset G$ be the Levi subgroup constructed as in (4.3) for the section $\theta(\alpha_1)$. Let

$$E_{L_1} \subset E_G$$

be the reduction constructed as in Proposition 4.3 for $\theta(\alpha_1)$. Since $[\theta(\alpha_1), \theta(\alpha_2)] = 0$, from Corollary 4.7 we know that

$$\theta(\alpha_2) \subset H^0(M, \text{ad}(E_{L_1})) \subset H^0(M, \text{ad}(E_G)).$$

Therefore, proceeding inductively, we get from the Higgs field $\theta$

- a Levi subgroup $L \subset G$,
• a holomorphic reduction of structure group

\[(4.5) \quad E_L \subset E_G\]

to \(L\).

The subgroup \(L\) is unique up to a conjugation. If \(L\) is replaced by \(g^{-1}Lg\) for some \(g \in G\), then \(E_L\) gets replaced by \(E_{Lg}\). Consequently, the subbundle

\[\text{ad}(E_L) \subset \text{ad}(E_G)\]

is uniquely determined by \(\theta\). Also, note that

\[(4.6) \quad \theta \in H^0(M, \text{ad}(E_L) \otimes \Omega^1_M) \subset H^0(M, \text{ad}(E_G) \otimes \Omega^1_M).\]

From Corollary 4.7 it follows that for every \(i \in \{1, \cdots, d\}\) and \(x \in M\), the subalgebra \(\text{ad}(E_L)_x \subset \text{ad}(E_G)_x\) is contained in the centralizer of \(\theta(\alpha_i)(x)\). More precisely, \(\text{ad}(E_L)_x\) is the centralizer of the subset \(\{\theta(\alpha_1)(x), \cdots, \theta(\alpha_d)(x)\} \subset \text{ad}(E_G)_x\).

5. Yang–Mills structure on polystable Higgs \(G\)–bundles on \(M\)

Let \((E_G, \theta)\) be a polystable Higgs \(G\)–bundle on \(M\).

**Proposition 5.1.** For any \(i \in \{1, \cdots, d\}\) and \(x \in M\), the element \(\theta(\alpha_i)(x) \in \text{ad}(E_G)_x\) is semisimple.

**Proof.** Let

\[(5.1) \quad Z(G) \subset G\]

be the connected component of the center of \(G\) containing the identity element. Take a finite dimensional holomorphic representation

\[\rho : G \longrightarrow \text{GL}(V)\]

such that \(\rho(Z(G))\) is contained in the center of \(\text{GL}(V)\). Let

\[E_V := E_G \times^G V \longrightarrow M\]

be the vector bundle associated to \(E_G\) for this \(G\)–module \(V\). The Higgs field \(\theta\) induces a Higgs field on \(E_V\). This induced Higgs field on \(E_V\) will be denoted by \(\theta_V\). The connection on \(E_V\) induced by a Yang–Mills connection for \((E_G, \theta)\) satisfies the Yang–Mills equation for the Higgs bundle \((E_V, \theta_V)\). This implies that \((E_V, \theta_V)\) is polystable.

Now from Proposition 2.3 we conclude that \(\theta_V(\alpha_i)(x) \in \text{End}(E_V)_x\) is semisimple for every \(i \in \{1, \cdots, d\}\) and \(x \in M\). Since \(\rho\) is an arbitrary holomorphic representation such that \(\rho(Z(G))\) is contained in the center of \(\text{GL}(V)\), this implies that \(\theta(\alpha_i)(x)\) is semisimple. \(\square\)

Consider the Higgs \(L\)–bundle \((E_L, \theta)\) constructed from the given polystable Higgs \(G\)–bundle \((E_G, \theta)\) (see \((4.5), (4.6))\). We note that \((E_L, \theta)\) is polystable because \((E_G, \theta)\) is so. From Lemma 4.2 we know that the principal \(L\)–bundle \(E_L\) is polystable.

As in Section 3 fix the Kähler form \(\tilde{\omega}\) on \(M\). Fix a maximal compact subgroup

\[K_L \subset L.\]

Let

\[E_{K_L} \subset E_L\]
be a $C^\infty$ reduction of structure group to $K_L$ that solves the Yang–Mills equation for $(E_L, \theta)$ (see [BS, p. 554, Theorem 4.6]).

**Proposition 5.2.** The above reduction

$$E_{K_L} \subset E_L$$

solves the Einstein–Hermitian equation for the polystable principal $L$–bundle $E_L$.

**Proof.** We observed earlier that for every $i \in \{1, \cdots, d\}$ and $x \in M$, the subalgebra $\text{ad}(E_L)_x \subset \text{ad}(E_G)_x$ is contained in the centralizer of $\theta(\alpha_i)(x)$. Therefore, for every $i \in \{1, \cdots, d\}$ and $x \in M$, the element $\theta^*(\overline{\alpha_i})(x) \in \text{ad}(E_L)_x$ also is contained in the center of $\text{ad}(E_L)_x$. Consequently, we have $\theta \wedge \theta^* = 0$. This immediately implies that the reduction

$$E_{K_L} \subset E_L$$

solves the Einstein–Hermitian equation for the polystable principal $L$–bundle $E_L$. \qed

Fix a maximal compact subgroup

$$K \subset G$$

such that $K \cap L = K_L$.

**Theorem 5.3.** Let $(E_G, \theta)$ be a polystable Higgs $G$–bundle on $M$. Let

$$E_K \subset E_G$$

be a $C^\infty$ reduction of structure group to $K$ that solves the Yang–Mills equation for $(E_G, \theta)$. Then the reduction $E_K \subset E_G$ solves the Einstein–Hermitian equation for the polystable principal $G$–bundle $E_G$.

**Proof.** As before, let $(E_L, \theta)$ be the Higgs $L$–bundle constructed from the polystable Higgs $G$–bundle $(E_G, \theta)$ (see (4.5), (4.6)). Take a $C^\infty$ reduction

$$E_{K_L} \subset E_L$$

that solves the Yang–Mills equation for $(E_L, \theta)$. Let

$$E'_K := E_{K_L}(K) \longrightarrow M$$

be the principal $K$–bundle obtained by extending the structure group of $E_{K_L}$ using the inclusion of $K_L$ in $K$. We note that $E'_K$ is a reduction of structure group of $E_G$ to $K$ because $E_{K_L}$ is a reduction of structure group of $E_G$ to $K_L$. The above reduction

$$E'_K \subset E_G$$

solves the Yang–Mills equation for $(E_G, \theta)$ because the reduction $E_{K_L} \subset E_L$ solves the Yang–Mills equation for $(E_L, \theta)$.

Therefore, there is a holomorphic automorphism $T$ of $E_G$ such that $E'_K = T(E'_K)$.

Let $\text{Ad}(E_G) = E_G \times^G G \longrightarrow M$ be the holomorphic fiber bundle associated to $E_G$ for the adjoint action of $G$ on itself. It can be shown that the holomorphic sections of $\text{Ad}(E_G)$ are flat with respect to the connection on $\text{Ad}(E_G)$ induced by the connection on $E_G$ given by the reduction $E'_K$. To prove this, take any finite dimensional holomorphic $G$–module

$$\rho : G \longrightarrow \text{GL}(V)$$
such that $\rho(Z(G))$ (see [5,1]) is contained in the center of $GL(V)$. Let

$$E_V := E_G \times^G V \rightarrow M$$

be the associated vector bundle. The Einstein–Hermitian connection on $E_G$ given by the reduction $E'_K$ produces an Einstein–Hermitian connection on $\text{End}(E_V)$; this Einstein–Hermitian connection on $\text{End}(E_V)$ will be denoted by $\nabla'$.

Given any holomorphic section $T'$ of $\text{Ad}(E_G)$, let $T''$ be the automorphism of $E_V$ given by $T'$. As done in the proof of Theorem 3.2, using [Ko, p. 52, Theorem 1.9] we conclude that the section $T''$ of $\text{End}(E)$ is flat with respect to $\nabla'$. From this it follows that the automorphism $T'$ of $E_G$ is flat with respect to the connection on $\text{Ad}(E_G)$ induced by the connection on $E_G$ given by the reduction $E'_K$.

In particular, the earlier automorphism $T$ is flat with respect to the connection on $\text{Ad}(E_G)$ corresponding to the reduction $E_K$. From this it follows that the reduction $E_K \subset E_G$ solves the Einstein–Hermitian equation for the polystable principal $G$–bundle $E_G$.

6. Higgs bundles on Kähler manifolds with nonnegative tangent bundle

Let $X$ be a compact connected Kähler manifold equipped with a Kähler class $\omega$. Let

$$W_1 \subset \cdots \subset W_m = \Omega^1_X$$

be the Harder–Narasimhan filtration of $\Omega^1_X$.

**Lemma 6.1.** Assume that $\mu_{\text{max}}(\Omega^1_X) := \mu(W_1) < 0$. Let $(E, \theta)$ be a semistable Higgs bundle on $X$. Then $\theta = 0$.

**Proof.** To prove that $E$ is semistable, consider $E_1$ in (2.1). We have

$$H^0(X, \text{Hom}(E_1, (E/E_1) \otimes \Omega^1_X)) = 0$$

because $\mu_{\text{max}}((E/E_1) \otimes \Omega^1_X) = \mu_{\text{max}}(E/E_1) + \mu_{\text{max}}(\Omega^1_X) < \mu(E_1) + 0 = \mu(E_1)$. Form this it follows that $\theta(E_1) \subset E_1 \otimes \Omega^1_X$. Since $(E, \theta)$ is semistable, this implies that $E = E_1$. So $E$ is semistable.

Since $E$ is semistable,

$$\mu_{\text{max}}(E \otimes \Omega^1_X) = \mu(E) + \mu_{\text{max}}(\Omega^1_X) < \mu(E).$$

Hence $H^0(X, \text{End}(E) \otimes \Omega^1_X) = 0$. In particular, $\theta = 0$. \qed

Combining the proofs of Lemma 4.1 and Lemma 6.1 it is easy to deduce that Lemma 6.1 remains valid for Higgs $G$–bundles on $X$. The only point to note is that [AAB, p. 705, Corollary 1] (which is used in the proof of Lemma 4.1) is proved by showing that $\mu_{\text{max}}(\text{ad}(E_G)/\text{ad}(E_P)) < 0$.

6.1. Higgs bundles on Calabi–Yau manifolds. Let $X$ be a compact connected Kähler manifold such that $c_1(TX) \in H^2(X, \mathbb{Q})$ is zero. These are known as Calabi–Yau manifolds. Fix a Kähler class $\omega$ on $X$. A celebrated theorem of Yau says that there is a Kähler form $\tilde{\omega}$ in the class $\omega$ such that the Ricci curvature for $\tilde{\omega}$ vanishes identically [Ya] (this was conjectured earlier by Calabi). In particular, $\tilde{\omega}$ is an Einstein–Hermitian structure on $\Omega^1_X$. This implies that the vector bundle $\Omega^1_X$ is polystable.
**Lemma 6.2.** Let \((E, \theta)\) be a semistable Higgs bundle on \(X\). Then the vector bundle \(E\) is semistable. This is also true for Higgs \(G\)–bundles, meaning if \((E_G, \theta)\) is a semistable Higgs \(G\)–bundle on \(X\), then the underlying principal \(G\)–bundle \(E_G\) is semistable.

*Proof.* Since \(\Omega^1_X\) is polystable of slope zero, the proof of it given in Lemma 6.1 remains valid. To prove for Higgs \(G\)–bundles, just note that for \(\text{ad}(E_P)\) in the proof of Lemma 4.1 we have \(\mu_{\text{max}}(\text{ad}(E_G)/\text{ad}(E_P)) < 0\) (see the proof of Corollary 1 in [AAB, p. 705]). \(\square\)

**Lemma 6.3.** Let \((E, \theta)\) be a polystable Higgs bundle on \(X\). Then the vector bundle \(E\) is polystable.

*Proof.* From Lemma 6.2 we know that \(E\) is semistable. As in the proof of Lemma 2.2, \(V \subset E\) is the coherent analytic subsheaf generated by all polystable subsheaves of \(E\) with slope \(\mu(E)\). Let \(\theta' : TX \otimes E \longrightarrow E\) be the following composition homomorphism

\[
TX \otimes E \xrightarrow{\text{Id}_{TX} \otimes \theta} TX \otimes \Omega^1_X \otimes E \xrightarrow{\text{trace} \otimes \text{Id}_E} E.
\]

Since both \(TX\) and \(V\) are polystable, it follows that \(TX \otimes V\) is polystable. Also, note that \(\mu(TX \otimes V) = \mu(V) = \mu(E)\). Therefore, the image

\[
\theta'(TX \otimes V) \subset E
\]

is polystable with \(\mu(\theta'(TX \otimes V)) = \mu(E)\). Hence, we have

\[
\theta'(TX \otimes V) \subset V.
\]

This implies that \(\theta(V) \subset V \otimes \Omega^1_X\). Now the last part of the proof of Lemma 2.2 shows that \(E\) is polystable. \(\square\)

**Lemma 6.4.** Let \((E_G, \theta)\) be a polystable Higgs \(G\)–bundle on \(X\). Then the principal \(G\)–bundle \(E_G\) is polystable.

*Proof.* The proof is identical to the proof of Lemma 4.2. \(\square\)

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