On a Microscopic Representation of Space-Time VII
– On Spin

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Abstract. We recall some basic aspects of line and line Complex representations, of symplectic
symmetry emerging in bilinear point transformations as well as of Lie transfer of lines to spheres.
Here, we identify SU(2) spin in terms of (classical) projective geometry and obtain spinorial
representations from lines, i.e. we find a natural non-local geometrical description associated to
spin. We discuss the construction of a Lagrangean in terms of line/Complex invariants. We
discuss the edges of the fundamental tetrahedron which allows to associate the most real form
SU(4) with its various related real forms covering SO(n,m), n + m = 6.

1. Introduction
At this stage (see [5] and references), we’ve established the ties from our original Lie group
based approach of hadronic (and quantum) degrees of freedom in terms of SU(4) and SU∗(4),
respectively, to (classical and advanced) projective geometry of real 3-space. Now, in parts VII
and VIII of this series, we want to focus more detailed on two of the included major aspects or
building blocks – spin and relativity. So departing from a description common to both parts
and based on elementary line geometry, we are going to present a more detailed treatment of
each spin and relativity related to classical line and projective geometry. This, however, at the
same time almost immediately and inherently comprises a necessary branch of the two topics
here and in upcoming papers.

It is the task of this paper to recall briefly some (very) elementary geometrical properties and
features, and to relate this approach to spin and associated algebraical concepts and physical
background. As such, we’ll rearrange coordinates and relate this rep1 to nowadays common
spin (and ’quantum’) aspects and some of their reps. Moreover, we use this approach to discuss
further aspects as mentioned in the abstract so that inherently we obtain a relation between the
various reps and, moreover, between their respective interpretations and related concepts. Here,
the crucial point with respect to spin discussions is the application of Lie’s transfer principle
[7] and a special complexification of the real (inhomogeneous) coordinates. On the other hand,
this is also the branching point mentioned above so that, based on the same Ansatz but on real
coordinates, in part VIII we’ll concentrate on the original real (and the line geometric) aspects,
and there we’ll continue this second branch to discuss special linear Complexes and relativity.
We feel that the next few pages grant to state the relation of the usually disconnected ’quantum’

1 As in the papers before, we use the shorthand notation ’rep’ to denote both representations or realizations as
long as the respective context is unambiguous. ’PG’ subsequently denotes ’projective geometry’.
world to classical geometrical concepts, however, the wealth of possibilities comprised in Lie’s mechanism [7], e.g. with respect to higher order (or class) objects or using further choices of the transfer functions $f_i$, are still far from being applied or even exhausted. The ‘degenerate cases’ mentioned in Lie’s paper are related to very special choices and cannot serve to comprise the full power of this Ansatz based on Plücker’s algebraic concept of a general (higher dimensional) description of ordinary space and Lie’s extension thereof (see [7], especially page 151 above/end of §2 with respect to Lie’s understanding of Plücker’s original algebraical Ansatz [9]).

So although at this time we’ll include general aspects only, we feel it is the time to split our original series on a microscopic rep of space-time at this point into a spin (using complex numbers and elliptic geometry) and a relativistic (hyperbolic) part. On one hand, it is worth to investigate Lie transfer in more detail and to relate Clifford’s and Study’s work on elliptic geometry as well as especially Study’s further work on (different) line geometries and on kinematics (see e.g. [11] and later publications). To our opinion, in conjunction with (quaternary) invariant and form theory, this should be sufficient to exhaust and comprise what is nowadays called ‘quantum theory’. On the other hand, it is worth to separate a discussion focussing on real aspects and hyperbolic geometry in order to keep in parallel the track to relativity - special relativity using (special) linear Complexes as well as the full geometry of regular linear Complexes in $P^5$ and/or higher order Complexes and nonlinear transformations. Stepping back to line and Complex geometry as the basic and underlying framework of Lie’s approach [8], we feel to escape the various restrictions of early use of (linear) rep theory of groups and algebras as well as concepts of differential geometry, and we think it’s sufficient when asking for integral and/or nonlinear representations only afterwards.

As such, in the upcoming sections, we use Plücker’s line rep (in inhomogeneous coordinates) and feel free to rearrange and reinterpret this rep in a special form so that we can write the coordinates and transformations in matrix notation. This helps to identify elements and relate them to quantum reps, e.g. in Lagrangean or current algebra notation, and to ongoing discussions. Please note, too, that the next section is useful both for spin and relativity, especially after switching to homogeneous coordinates and departing from affine to non-euclidean geometry, and that it includes nothing but elementary PG. Afterwards, in section 3, we branch from real projective geometry by Lie’s transfer principle [7] to the derivation of (complex) spin reps and the known algebraic formalism. Throughout this section, we feel free to recall some historic contexts and include them in the text, not to write some science-history related paper but to put some known facts in an understandable context right from the original development. The last section is devoted to a brief outlook in order to sketch the wealth of opening possibilities which appear by Lie transfer and sphere geometry.

2. Line representation and coordinates

The reason to cite some (very elementary) issues from [10] is mainly to connect the reasoning of the 19th century view of projective geometry to the dynamic picture used nowadays in terms of ‘time’ and ‘velocity’ coordinates which emerge only after we tag points in the respective

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2 According to current discussions and the attribution of achievements, we feel the urgent necessity to reconsider the real historical development, however, personally, we neither have the competency nor the time to evaluate and honor those achievements scientifically and relate them in detail to the original authors. So we focus here on few spin-related aspects only which to our opinion need (among other authors) to be attributed mainly to S. Lie and especially E. Study already back in the late 19th and first decade of 20th century instead of E. Cartan in 1938 [1] (or even 1913, as he claimed there, see his introduction). However, we cannot rise a fundamental and historically complete priority discussion here.
geometry and 'observe' them moving around. So in choosing [10] the (euclidean) rep

\[ x = r z + \rho, \quad y = s z + \sigma, \tag{1} \]

we obtain first of all a (cartesian) line rep of 3-space which is 'globally' defined by two linear functions in terms of the third cartesian coordinate z. In each of the planes (x, z), i.e. y = 0, and (y, z), i.e. x = 0, we find linear line projections, too, which we may map to separate line reps as well. So far, we do not have tagged special points on the line like the origin as a reference or a unit point or two (fundamental) points to declare anharmonic ratios or three points to declare a coordinate (or measure) on the line itself. Moreover, in the same picture, we do not yet apply the vector calculus of real 3-space common today by pointing with a vector to the line and defining a 'time' parameter and a velocity on the line (or at least its direction).

2.1. Coordinates

So the following arguments are concerned with the (real) parameters r, \( \rho \), s and \( \sigma \) from eq. (1), i.e. what enters these equations are position and orientation (i.e. 'the geometry') of the line itself and as a 'whole' object in the 3-dim euclidean coordinate system x, y, z. Now Plücker founds his definition of (4-dim) line coordinates and their necessary extension by a fifth line coordinate \( \eta = r \sigma - s \rho \) on this basis, and in [10] he shows how to define and use sets of lines in terms of Complex, Congruences and Configurations as foundation of a space description, his so-called 'new geometry', in distinction to standard (projective) geometry which uses the point as base element and planes to complete the point-based framework with respect to duality (see e.g. his contemporary summary [9] or later text books on PG also by other authors).

Suppressing PG details here, we want to prepare our discussion of both parts VII and VIII, i.e. spin and relativity, by rearranging the (euclidean) coordinates x, y, as still a function of z only, with respect to the line projection onto the (x, y)-plane at z = 0. Our main concern and driving force having been inclusion and understanding of special Lorentz transformations, however, it turned out that if we transform (in the point picture) along the z-axis, and if we require as usual the x- and y-coordinates to remain invariant, we have remarkable freedom in that this transform concerns only \( z \sim \frac{\eta}{\sigma} \) as (euclidean) normal to (x, y). Moreover, weakening the invariance requirement(s), we can rearrange the variables in the projection plane (x, y) as needed to investigate invariances of products, quotients, etc., too.

So if we arrange e.g. the ratio of the two (cartesian) coordinate projections x and y of the line into a new coordinate \( \zeta \) by \( \zeta = \frac{y}{x} \), using eq. (1) we obtain the relation \( \frac{x}{\sigma} = \zeta = \frac{r z + \rho}{s z + \sigma} \). Now we can interpret the rhs as projective transformation and, of course, map the parameters to a

3 We want to emphasize the difference of the 'pictures' because there are alternative reps of points e.g. by considering points in terms of (inter-)sections of lines in 3-space. This leads almost immediately to a discussion of Congruences when including (moving) observers into the physical description, see also our preparatory discussions in [4], [5]. We have to remark as well the dependent relation of velocities to the respective time coordinates of the individual lines, or axes, or of the (local) coordinate systems which in its simplest form of (linear) motion with uniform velocity read as \( s = v \cdot t \) in those coordinates. Note also, that Plücker (as well as Lie in his approach) uses (inhomogeneous) euclidean coordinates x, y, z and not fourfold/quaternary point coordinates, neither homogeneous nor (affine) Hesse coordinates by introducing the additional t-coordinate common at that time to denote the fourth (affine) point coordinate. This reduces the anharmonic ratio using point reps to a simple quotient of point distances or 'coordinate' projections.

4 We are going to extend this approach by homogeneous coordinates mainly in part VIII and postpone the more general discussion, also with respect to anharmonic ratios and projective transformations of the lines, to upcoming parts.

5 This ensures to preserve the grade during transformations whereas a sixth coordinate is necessary to linearize the transformations of the line coordinates.

6 We discuss more details related to relativity in part VIII.

7 The three (cartesian) projections of the original line onto each of the planes (x, z), (y, z) and (x, y) in the general case are, of course, related to the line by perspective.
(real) matrix formalism by
\[
\frac{x}{y} = \zeta \rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r & \rho \\ s & \sigma \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}.
\]

2.2. Homogeneous Coordinates
Moreover, by formally\(^8\) introducing homogeneous coordinates for \(\zeta \rightarrow \frac{\zeta_1}{\zeta_2}\) and \(z \rightarrow \frac{x_3}{x_0}\), we can rewrite eq. (2) according to
\[
\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} r & \rho \\ s & \sigma \end{pmatrix} \begin{pmatrix} x_3 \\ x_0 \end{pmatrix}
\]
and apply the usual 2 × 2 matrix formalism.

Now, dependent on whether we focus on the rhs of eqns. (2) and (3), or on the lhs of these equations, we steer towards a discussion of spin for the rhs or (special) relativity for the lhs as we’ll discuss soon. Note, however, that throughout our approach \(\frac{x}{y}\) remains ‘invariant’ if we replace the cartesian coordinates \(x\) and \(y\) homogeneously by \(x \rightarrow \frac{x_1}{x_0}\) and \(y \rightarrow \frac{x_2}{x_0}\), i.e. \(\frac{x}{y} = \frac{x_1}{x_2}\), following the affine interpretation by tagging a certain plane within PG.

Plücker’s fifth line coordinate \(\eta = r\sigma - s\rho\) then turns out to represent the determinant of the transformation matrix above, i.e. via the line rep in eq. (1), we have associated (formally) a projective transformation of the (euclidean) \(z\)-axis to a (projective) transformation of the coordinate \(\zeta\). Accordingly, we can relate this ‘new’ coordinate axis of \(\zeta\) to \(z\) by associated projective transformations \(\zeta = f(z)\). More general, the projective relation of \(z\) and \(\zeta\) allows to apply the full framework of point transformations on first order (projective) elements and to investigate binary forms (and the related invariant theory) in terms of mappings of point sets on lines to each other. In addition, our approach suggests the discussion of involutions on projective elements of first kind, so that we may include complex numbers via von Staudt’s or Lüroth’s geometric interpretation of complex numbers naturally. By considering this approach from a more general viewpoint, we have now set up the very foundation of the complete framework of PG, here in terms of point geometry on two lines coordinated by \(z\) and \(\zeta\).

3. Lie Transfer and Spin
Following Lie’s approach here, we are going to discuss the rhs of eq. (2) only. Also, we require the matrix determinant (or Plücker coordinate \(\eta\)) here as nonzero, i.e. the original line doesn’t hit the \(z\)-axis, and the line of the respective projections in the \((x,y)\)-plane is skew versus the \(z\)-axis. In other words, \((0,0)\) being the (cartesian) projection of the \(z\)-axis in the \((x,y)\)-plane (and all its parallel planes, i.e. planes hitting the same ’absolute line’ within the spatial affine picture), the non-vanishing determinant ensures considering the case of two skew lines, and moreover eases the introduction of homogeneous coordinates by avoiding singular/special cases. Some aspects related to the general case, including \(\eta = 0\), will be discussed in part VIII, also in the context of using homogeneous coordinates.

3.1. Lie’s Mapping – aka ‘Lie Transfer’
Lie restricted the general cases of his mapping at first by considering lines instead of general curves ([7], #5 and #21), and further by choosing special constants in that the second Complex in the degenerate case ([7], #21) hits the conic section in the absolute plane of the affine geometry. Note, that it is this linear extension of euclidean geometry\(^9\) which introduces the

\(^8\) Here, we may only use the affine (linear) coordinate extension by an absolute plane \(x_0 = 0\) (better by \(\pm \sqrt{x_0^2}\) in the denominator.

\(^9\) . . . which also forces the switch to homogeneous coordinates to describe absolute elements. We refer to Study’s remarks [12] and some of his further work cited here. See also our few remarks in section 3.6.
isotropy discussion (and as such the 'spinor' rep), whereas with respect to relativistic (or non-
euclidean) transformations, leaving a general second order (i.e. quadratic) surface invariant, we
have to switch to 3-space instead of (linear) planar discussions. There will remain the connecting
link that \( 2^{nd} \) order surfaces can be generated by two families of lines, i.e. in each point of the
surface, we can find appropriate members of the two generating families, however, in order
to handle spin and (special) relativity from the viewpoint of the degeneracies discussed here
and in the upcoming part VIII with respect to special linear Complexe, it is obvious that one
should switch to Complexe in general as geometrical base elements as we have worked out and
proposed throughout this paper sequence (see [5] and references there). A strongly supporting,
direct argument for switching to line and Complex geometry of \( P^5 \) would be to refer back to Lie’s
original investigations [7] and by noting that the general theory underlying Lie’s approach and
reasoning is Complex geometry. Indeed, Lie obtained this (bilinear) approach given in eq. (4)
only by further specializing his originally more general mapping to a line Complex hitting the
absolute circle of affine geometry\(^{10}\), or isotropic lines, respectively. For now, we think that Lie’s
Ansatz in general establishes sphere geometry in a separate space besides the line (and the
represented Complex) geometry of ordinary 3-space, whereas concerning 'spin', here we treat
only simple/linear and (multiply) degenerate cases of Lie’s general mapping which 'overlap' or
relate in certain aspects due to the (linear) cases and/or the degeneracies considered.

3.2. Matrix Reps and Spin
Nevertheless, we depart from Lie’s restricted (bilinear) Ansatz (see [7], p. 146, or [7], §8, #24
(1)),

\[
-Zz = x - (X + iY), \quad (X - iY)z = y - Z
\]  
(4)
to relate the (cartesian) coordinates of the spaces \( r \) and \( R \), which we rearrange into

\[
x = -Zz + (X + iY), \quad y = (X - iY)z + Z,
\]  
(5)
and according to our choice\(^{11}\) of the line representation in eq. (1) and the matrix definition in
(2), this allows to identify the transformation matrix directly by

\[
\begin{pmatrix}
r & \rho \\
\sigma & s
\end{pmatrix}
\sim
\begin{pmatrix}
-Z & X + iY \\
X - iY & +Z
\end{pmatrix}.
\]  
(6)

Note here, that on the lhs, we originally describe a \textit{linear} transformation of cartesian coordinate
projections which yields the context for discussion of [3] later on. So the original line coordinates
of space \( r \) are represented (or 'replaced') by (complexified point) coordinate combinations of the
space \( R \), where the rhs of eq. (6), however, may be re-expressed in nowadays common notation
according to

\[
\begin{pmatrix}
-Z & X + iY \\
X - iY & +Z
\end{pmatrix}
= X\sigma_1 - Y\sigma_2 - Z\sigma_3
\]  
(7)
when introducing standard 'Pauli matrices' \( \sigma_i \). So the important point here – derived from the
background of Complex geometry – is the identification of real (point) coordinates of \( R \) in the
\(^{10}\) It is obvious from the Cayley-Klein construction of a metric that this construction is relevant to metrical
properties, too. However, we do not want to discuss general lengths, 'masses' or 'charges' here as this approach
is, first of all, connected to absolute elements of an affine Ansatz only.

\(^{11}\) For detailed analytic consideration and calculations one should recall that at the time of Plücker’s and Lie’s
writing, the authors were used to coordinate systems with different handedness, especially to a different sign of
the \( y \)-coordinate, which sometimes leads to signs especially in linear equations involving \( y \) or \( \sigma \).

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Pauli/spin basis by eq. (7) with (inhomogeneous) line coordinates\(^\text{12}\) of real 3-space, and it is the \text{su}(2) (Lie) algebra which maps the geometries and the respective coordinate interpretation. However, for the general environment and the context of our current discussion, it is necessary to mention and cite three more important aspects given by Lie in his paper [7].

### 3.3. Geometry in \(R\)-space

Casting the equations above in a form to identify coordinates in \(R\)-space, Lie shows ([7], p. 168) that those (still inhomogeneous) coordinates fulfil the equation \(R^2 + S^2 + 1 = 0\), or when cast in differential form\(^\text{13}\), based on [7], §8, eq. (3), by\(^\text{14}\) \(R = \frac{dX}{dz}\), \(S = \frac{dY}{dz}\), which yields \(dX^2 + dY^2 + dZ^2 = 0\) and suggests the considerations of isotropic elements. This led Lie to the statement that the line Complex in \(R\)-space is being built of imaginary lines of zero length\(^\text{15}\). We want to address those aspects later and separately.

Here, to connect to 'Cartan's' spin rep, we want to recall Lie's equations ([7], p. 168) of the line coordinates \(R\) and \(S\),

\[
R = \frac{1}{2} \left( z - \frac{1}{z} \right), \quad S = \frac{1}{2i} \left( z + \frac{1}{z} \right),
\]

and the quadratic equation \(R^2 + S^2 + 1 = 0\) from above. We can now start from either 'side' – using the quadratic equation and introducing homogeneous coordinates, or recalling the inhomogeneous character of \(z\) and introducing homogeneous quaternary coordinates of 3-space in (8). By the second approach, we obtain from eq. (8)

\[
R = \left( \frac{z^2 - 1}{2z} \right) = \left( \frac{x_3^2 - x_0^2}{2x_0x_3} \right), \quad S = \left( \frac{z^2 + 1}{2iz} \right) = \left( \frac{x_3^2 + x_0^2}{2ix_0x_3} \right).
\]

By introducing primed 'homogeneous' variables \(R' = \frac{R'}{T'}\), \(S' = \frac{S'}{T'}\), we can extract the identifications

\[
R' = x_0^2 - x_3^2, \quad S' = i \left( x_0^2 + x_3^2 \right), \quad T' = -2x_0x_3,
\]

where we’ve used the standard relation \(z = \frac{x_3}{x_0}\) from affine geometry with respect to the \(z\)-axis in all places. The equation \(R^2 + S^2 + 1 = 0\) converts to \(R'^2 + S'^2 + T'^2 = 0\), thus featuring (finite) line coordinates and 4\textsuperscript{th} order surfaces in the original (point) coordinates. Eq. (10) yields Cartan's coordinate and 'spinor' 'definition' in [1], \#52.

### 3.4. Lie Transfer and 'Spinors'

Now regarding eqns. (9) and (10), it is easy to trace back Cartan’s miraculous 'definition' of 'a spinor' emerging in [1], \#52 by recalling the relation of inhomogeneous and homogeneous coordinates of (real) 3-space, the definition of the absolute element of affine geometry. What

\(^{12}\)This is especially noteworthy facing the current discussions on non-local aspects of quantum theories if we depart from real parametrizations of Pauli matrices. Note in this contexts, that already special linear Complexe are \textit{sets} of lines in 3-space, and that using Complexe, we also have to take into account regular and higher order Complexe, i.e. the geometry of \(P^3\) as well as quadratic manifolds. Here, we are discussing linear mappings and degenerate cases only, so these are only the very beginnings of a potentially relevant description of observations in nature.

\(^{13}\)We cite the expression as given, not discussing partial derivatives. Please note, that this replacement of coordinates by derivatives is attached to Plicker’s original line rep and to a linear choice of the coordinates. So this Ansatz yields the very foundation of Lie’s contact transformations based on line elements (or five-parameter space reps \((x,y,z,p,q)\)) later on. However, due to the intrinsically linear rep, we may also switch directly to the point rep on lines in terms of binary forms [3] and the related symbolism.

\(^{14}\)We want to mention the relation to three homogeneous (planar) coordinates only.

\(^{15}\)Isotropic lines.
enters in addition, are the absolute circle, by \( x_1^2 + x_2^2 + x_3^2 = 0 \), \( x_0 = 0 \) in homogeneous spatial coordinates, and an identification \( x_0^2 - x_3^2 \) here with \( \xi_0^2 - \xi_3^2 \) in Cartan’s notation. Accordingly, his ‘spinor’ coordinates \( \xi_0 \) and \( \xi_1 \) map to two homogeneous point coordinates \( x_0 \) and \( x_3 \) of real 3-space, whereas by eqns. (9) or (10) the three linear coordinates \( x_1 \) of Cartan map to Lie’s ‘line coordinates’ \( R \) and \( S \), or \( R', S' \) and \( T' \) from above and from the ‘inverse’ Lie mapping, obeying \( R^2 + S^2 + 1 = 0 \) or \( R'^2 + S'^2 + T'^2 = 0 \), respectively.

There is, however, a much easier way to establish the ‘spinor’ interpretation versus Lie’s mapping by starting from [1], #55, eq. (1), and by recalling \( z = \frac{x_2}{x_0} \sim \frac{\xi_2}{\xi_0} \) from above, i.e.

\[
\xi_0 x_3 + \xi_1 (x_1 - ix_2) = 0 \; , \; \xi_0 (x_1 + ix_2) - \xi_1 x_3 = 0 , \tag{11}
\]

Here, we can at first change the order of the equations and divide by \( x_0 \), thus switching back to inhomogeneous, cartesian coordinates \( X, Y \) and \( Z \). We obtain

\[
-\xi_1 Z + \xi_0 (X + iY) = 0 \; , \; \xi_0 Z + \xi_1 (X - iY) = 0 , \tag{12}
\]

\[
0 = -Z\xi_1 + (X + iY)\xi_0 , \; 0 = (X - iY)\xi_1 + Z\xi_0 . \tag{13}
\]

We may put these equations in matrix form, too, by extracting a ‘spinor’ \( (\xi_1, \xi_0)^T \) to the right, and celebrating the \( \text{su}(2) \) (Lie) algebra in terms of Pauli matrices. What is more important, however, is the comparison with eq. (5) if we introduce an inhomogeneous coordinate \( \xi := \frac{\xi_1}{\xi_0} \) (which, as we see above, is Lie’s original \( z \)). Dividing eq. (13) by \( \xi_0 \), we obtain

\[
0 = -Zz + (X + iY) , \; 0 = (X - iY)z + Z , \tag{14}
\]

which recovers a special case of Lie transfer, i.e. instead of Lie’s general coordinate projections \( x \) and \( y \), Cartan’s definition is obviously fixed to the \( z \)-axis at \((0,0)\) in the \((x,y)\)-plane, which from Lie’s approach is related to vanishing \( \eta \) (or determinant).

More important is the missing dependence of \( x \) and \( y \) at all, or according to (14) \( x = 0 \) and \( y = 0 \). So in addition to the restrictions and the resulting degeneracies/singularities applied by Lie, there is a priori no suitable coordinate dependence besides the ‘spinor’ itself describing a line in terms of two homogeneous coordinates \( \xi_i \), or a simple 1-dim projective transformation of the quotient. The quotient may be reversed by the exchange \( \xi_1 \leftrightarrow \xi_0 \), in ‘spinor’ notation induced by \( \sigma_3 \) or \( i\sigma_2 \), which correspond to reciprocal transformations\(^{16}\) \( z \leftrightarrow \pm \frac{1}{2} \). Due to the remarkable situation that (in the usual point picture) special relativity doesn’t transform \( x \) and \( y \) when we choose the transformation along \( z \), Cartan’s approach – being independent of \( x \) and \( y \) – survives and reflects those transformations. One should, however, place question marks to general space-dependent \( \text{SU}(2) \) spinors like \( \psi(x) \) or \( \psi(x^\mu) \) founding on Cartan’s definition.

\[3.5. \text{Further Linear Aspects}\]

The second aspect covers a more general case. Here, Lie’s equations (4) mapping the two spaces \( r \) and \( R \), relate both spaces in a manner that points in \( r \) map to isotropic lines in \( R \), and points in \( R \) map to the linear Complex \( r + s = 0 \) in \( r \). In upcoming papers, we hope to have the possibility to discuss more geometrical details, especially with respect to further details and geometrical relations mentioned by Lie in [7]. For now, we want to close this subsection by reference to Lie’s remarks in §9, p. 171 with respect to the second unnumbered equation (see also eq. (15)) holding

\[\text{By a suitable setup of the system, arranging a (metric) radius along } \hat{z}, \text{ we have } r \leftrightarrow \pm \frac{1}{2}, \text{ and planar duality as well. Conjugation of eq. (14) in } R \text{-space is invariant if, at the same time, we perform } z \leftrightarrow -\frac{1}{2}, \text{ etc. Note also sign behaviour of } R \text{ and } S \text{ in eq. (8) in the } z\text{-cases. Such methods can be applied throughout Cartan’s ‘spinor’ formalism, however, it should at least be founded on Lie’s approach. The more general treatment has to include and respect polar systems.}\]
a quadratic relation. Tracing this back with our matrix rep, it is obvious that the variable $H$ in $R$ corresponds to the unit matrix $\sigma_0$, and it thus relates the unit matrix of the $R$ matrix rep with the linear Complex $r+\sigma$ of $r$-space and it’s special rôle. In Lie’s original work [7], p. 171, the spheres in $R$-space (i.e. using Plücker’s inhomogeneous coordinates$^{17}$ to describe Complex lines) fulfill
\begin{equation}
[X-(s+r)]^2+[Y-i(s-\rho)]^2+[Z-(\sigma-r)]^2=(\sigma+r)^2. \tag{15}
\end{equation}
So the coordinate set$^{18}$ $X'=s+\rho$, $Y'=i(s-\rho)$ and $Z'=\sigma-r$ describes the center of the sphere in $R$-space in terms of line coordinates (or a shift from an original center $(0,0,0)$) whereas $\sigma+r$ takes the rôle of an oriented coordinate or even a ‘radius’ $H=r+\sigma$ in terms of line coordinates of the line Complex in $r$-space. This enhances the mapping $r \leftrightarrow -Z$, $\rho \leftrightarrow X+iY$, $s \leftrightarrow X-iY$ and $\sigma \leftrightarrow +Z$ in eq. (6) into (see [7], §9, eq. (3))
\begin{equation}
r = \frac{1}{2}((\pm H'-Z') \ , \ \rho = \frac{1}{2}((X'+iY') \ , \ s = \frac{1}{2}((X'-iY') \ , \ \sigma = \frac{1}{2}((\pm H'+Z') \tag{16}
\end{equation}
which now corresponds to the expression
\begin{equation}
\frac{1}{2}\begin{pmatrix}
-Z' \pm H' & X' + iY' \\
X' - iY' & Z' \pm H'
\end{pmatrix}
\sim X'\sigma_1 - Y'\sigma_2 - Z'\sigma_3 \pm H'\sigma_0. \tag{17}
\end{equation}
Recalling Plücker’s fifth coordinate $\eta = r\sigma-s\rho$, eq. (16) yields $r\sigma = H'^2-Z'^2$ and $s\rho = X'^2+Y'^2$, so $\eta = H'^2-Z'^2 - X'^2-Y'^2$. This prepares the basis of the 2-spinor formalism and all its applications and analytical forms, in addition, we thus map the quest for ‘relativistic’ invariance to $2 \times 2$-matrix notation (17) or complexified quaternions, and invariance of the determinant $\eta$.

Especially, this gives some insight on how (due to the quadratic nature of the manifold) the unit matrix serves [7] in this rep: A line given in $r$-space determines a sphere in $R$. However, a sphere in $R$ given by $(X,Y,Z,H^2)$ maps to two lines in $r$, both being reciprocal polars with respect to the linear line Complex $\pm H=r+\sigma=0$ in $r$. $H$ in $R$-space plays the rôle of a sphere radius, so choosing $H=0$ amounts to ’point spheres’$^{19}$ mapping uniquely to the Complex $r+\sigma=0$ of $r$-space.

Finally, and pointing towards a third important aspect, it is worth mentioning [7] article #28 which relates the contact of Lie’s spheres to intersection of lines. This emphasizes the relevance of contact interactions in both pictures when constructing and investigating symmetry and invariance principles.

For us, the essence of the reasoning so far – remembering all the various restrictions applied up to this point – justifies according to our current use of the Pauli spin (or the quaternion) algebra in various physical contexts to reverse the viewpoint: If we are to interpret Pauli matrices with real coefficients and including the unit matrix to represent a ‘radius’, we have to recall their origin from transformations in $R$-space and their identification with some (inhomogeneous) line coordinates which themselves describe, according to their direct relation with the underlying linear line Complex, automatically an extended (i.e. ‘nonlocal’) object – a set of lines – in real 3-space $r$.

$^{17}$ Generalizing the coordinates via affine homogeneous coordinate extensions, note that $\rho$ and $\sigma$ are related to the ‘new’ affine coordinate $t$ or $x_0$ and as such to the absolute plane in $r$-space. Recall also, that transformations of the inhomogeneous Plücker coordinates $r, s, \rho, \sigma$ do not preserve grade, i.e. (16) pretends a linear relationship due to the projective rep.

$^{18}$ Note, that $Y'$ is imaginary, or $iY'$ is real, and remember polar relationships in affine geometry, too.

$^{19}$ German: Punktkugeln [7], p. 171
3.6. Some Aspects of Study’s work

Driven by some discussions throughout and following this year’s conferences with respect to Cartan’s spin and to unitary symmetries in QFT, especially when based on one or more su(2) algebras, we’ve searched through literature. So before performing some algebra related to sect. 3.3, it is necessary and important to recall few historical but obviously forgotten aspects which allows to rearrange some aspects back into their original context.

Especially when facing the discussion of spin and unitary symmetries, typically attributed to Pauli, Cartan and su(2) (and certain su(n)) algebras, the origins can be easily traced back at least to Hesse’s transfer principle (1866) and Meyer’s generalization (1883) (for references and an outline see [11], §151) of rational curves of a point \( \lambda = \frac{x_4}{x_2} \) on a line. For \( n = 2 \), we thus obtain (planar) conic sections parametrized by three homogeneous coordinates \( \rho x_0 = \lambda_1^2, \rho x_1 = \lambda_1 \lambda_2 \) and \( \rho x_2 = \lambda_2^2 \), or appropriate linear combinations. Choosing \( n = 3 \) for later use, we obtain \( 3^{rd} \) order curves parametrized by the four coordinates \( \rho x_0 = \lambda_1^3, \rho x_1 = \lambda_1^2 \lambda_2, \rho x_2 = \lambda_1 \lambda_2^2 \) and \( \rho x_3 = \lambda_2^3 \). Now linear transformations of the line, alternatively re-expressed by matrix transformations of the vector/‘spinor’ \( (\lambda_1, \lambda_2)^T \) with non-vanishing determinant, i.e. by a 3-dim group, lead to collineations in \( \mathbb{R}^3 \) which exchange points on the (invariant) curve, for \( n = 2 \) the conic section. For further details see [6] or [3].

In the context of line and Complex geometry – in a certain sense continuing the heritage of Plücker and Lie – one must consider and honor Study’s work (see [11], or [12] and references) which has been definitely known to Cartan. Ad hoc, without enough time to work through and synchronize notation, we found references and work in Münchener and Leipziger Berichte (where Cartan published in german, too), and Mathematische Annalen, besides Study’s remarks on Lie’s transfer principle in [13]. The main task to make the mappings unique will focus on introducing ’orientation’ so that e.g. (projective) lines correspond to two (oriented) ‘Speere’ which – on the other hand – introduces and requires metric (and sometimes euclidean) concepts and objects, see e.g. [11].

Here, mentioning [12], part II, p. 151, eqns. (51,l) and (51,r), Study explains the above ‘spinor’ notation, and as such Cartan’s ‘definition’, in ’elliptic’ context and SU(2)×SU(2) (or quaternionic) transformations, whereas in eq. (52), he resolves the ‘spinor’ components linearly in terms of his ’elliptic’ line coordinates.

From our point of view, this emphasizes once more the use of Complex geometry. However, Study’s part II and his reasoning with respect to ’elliptic’ geometry has to be regarded with time and care in detail, because in eq. (1) on p. 119 he assumes the absolute polar system as \((xy) = x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3\). This allows him to introduce real combinations of (homogeneous) line coordinates in his eq. (3). However, his eq. (4), re-expressing the Plücker condition of the line coordinates \( X_{\alpha \beta} \) and using six line coordinate combinations \( X_i \), can be cast into the Complex expression

\[
X_1^2 - X_2^2 + X_3^2 - X_4^2 + X_5^2 - X_6^2 = 0
\]

while exhibiting SO(3,3) symmetry for a point interpretation of \( X_i \), which directly relates to the fundamental form of Plücker’s line geometry (see also [6], §23, especially p. 98).

However, if we switch Study’s absolute polar system to \((xy) = \frac{-x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3}{i} \) by introducing an \( i \) in the first (point) coordinate \( \{\} \) so that the Plücker coordinates \( X_{0k} \) change to \( iX_{0k} \), the coordinates (up to overall \( i \)’s) in Study’s eq. (3) are Klein’s coordinates \( z \) ([6], §23).

Anyhow, citing Klein’s work on six Complexe in involution and their threefold ±-handedness ([6], §23, p. 99), or facing Study’s discussion of two spheres (or left/right quaternions) above, it is evident that these properties represent PG in terms of line and Complex geometry which reflects in certain known symmetry groups like SU(2), SU(2)×SU(2) or even SU(4) of the

\[\text{German: dreigliedrige Gruppe}\]
\[\text{U(1) is almost always evident by planar projections.}\]
(spinorial) formulation of quantum theories.

3.7. Basic Spin Commutators

Now, having established the above matrix form of Lie transfer, we want to insert a brief section related to practical application and identification.

This time, we start from the matrix reps above in that we define points in R-space right from the beginning and ask for the rôle of the commutator as natural (antisymmetric) 'product' of two such objects\(^{22}\). As such, we define two 'points' \(X \sim \{X,Y,Z\}\) and \(X' \sim \{X',Y',Z'\}\), keeping the notation of the inhomogeneous coordinates from above, but switching to positive signs in eqns. (7) and (17), respectively, by

\[
X := X\sigma_1 + Y\sigma_2 + Z\sigma_3 = \begin{pmatrix} Z & X - iY \\ X + iY & -Z \end{pmatrix}, \quad X' := \begin{pmatrix} Z' & X' - iY' \\ X' + iY' & -Z' \end{pmatrix}.
\]

Defining \(\mathcal{C}\) mainly as commutator by \(\mathcal{C}(X, X') := \frac{1}{2} [X, X']\), straightforward calculation yields

\[
\mathcal{C} = \begin{pmatrix} i(XY' - YX') & ZX' - XZ' + i(YZ' - ZY') \\ ZX' - ZX' + i(YZ' - ZY') & -i(XY' - YX') \end{pmatrix}.
\]  
(19)

In order to get more insight, by using the two points \(X\) and \(X'\), we construct line coordinates in R-space, i.e. we define and use \(P_{12} = XY' - YX', \ P_{23} = YZ' - ZY'\) and \(P_{31} = ZX' - XZ'\). So eq. (19) reads as

\[
\mathcal{C} = \begin{pmatrix} iP_{12} & P_{31} + iP_{23} \\ -P_{31} + iP_{23} & -iP_{12} \end{pmatrix} = i \begin{pmatrix} P_{12} & P_{23} - iP_{31} \\ P_{23} + iP_{31} & P_{12} \end{pmatrix},
\]  
(20)

where the rhs can be re-expressed in terms of Pauli spin matrices as \(\mathcal{C} = i(P_{23}\sigma_1 + P_{31}\sigma_2 + P_{12}\sigma_3)\). Although we are used to having the \(i\) accompanying the commutator of (Pauli) spin matrices, it is easy to transform eq. (20) into a real equation by introducing quaternions, \(q_k = -i\sigma_k\). Eq. (20) changes into the equation

\[
\mathcal{C} = \frac{1}{2}(X\ X' - X'\ X) = i(P_{23}\sigma_1 + P_{31}\sigma_2 + P_{12}\sigma_3).
\]  
(21)

By multiplying both sides with \(-1\), i.e. \((-i)^2\) on the lhs, we obtain \(-\mathcal{C} =: \mathcal{C} = \frac{1}{2}(X\ X' - X'\ X) = P\) where each underlining now denotes a real quaternion, respectively, e.g. \(X = Xq_1 + Yq_2 + Zq_3\), \(P = P_3q_1 + P_1q_2 + P_2q_3\), etc. If we transfer the spin notation above of \(\mathcal{C} = -\mathcal{C}(X, X')\) to real quaternion calculus, eq. (21) reads as \(\mathcal{C}(X, X') = P\), i.e. the product (or commutator) \(\mathcal{C}\) of two real 'point' quaternions \(X\) and \(X'\) yields a real quaternion holding the three spatial Plücker coordinates. We may, in addition, recognize the striking index ordering in eqns. (20) and (21) as well as ask for further commutators of those objects.

3.8. Advanced Spin Commutators

If we denote the components \(P_i\) of the 'Plücker' quaternion \(P\) above through \(P = Pq_i\), by performing some algebra we obtain \(\mathcal{C}(X, P) = (P_{kj}x_j)q_k =: U\). Whereas in QFT, we would now have to consider higher and increasing order of 'vector' products, reducible by quadratic (mass/momentum) constraints and angular momentum algebra, here we can stress the origin of \(P\).

\(^{22}\) There is some parallel background from within classical polar theory of second order surfaces, however, the calculations here are self-contained so we'll discuss those issues in projective geometry at a later time.
By switching to the general context in terms of homogeneous coordinates, we can use the relation of linear Complexes and null systems. As such, in terms of homogeneous coordinates it is $P_{\alpha\beta}$ which relates $x_\beta$ to the dual plane coordinatized by $u_\alpha$. Whereas in the polar case, incidence $u_\alpha x_\alpha = 0$ of point and plane is special, treating null systems, the incidence relation is always satisfied, i.e. point and plane intersect because of antisymmetry of $P_{\alpha\beta}$ in $u_\alpha x_\alpha = P_{\alpha\beta} x_\beta x_\alpha = 0$.

Recalling this background from PG, by considering the real quaternion $\mathcal{U}$ above, its three components realize (almost) the spatial part of a null system in terms of the associated planar coordinates $u_i$.

It is obvious that we have to enhance the mechanism because at this stage we’ve lost contact to the $0$-components of the complete null system, and to the $0$-coordinates of points and planes in homogeneous rep. What remains open here, is the question whether this quaternionic calculus is sufficient to represent inhomogeneous calculations in mechanics and electrodynamics, i.e. comparison to experimental (classical) data.

Possible enhancements are obvious because the $\text{su}(2)$ rep above is related to one line only. By recalling the two skew (and thus independent) lines above which we can identify either as opposite edges of a (fundamental) tetrahedron or as the two lines of a Congruency [4], we can almost automatically enhance the scheme by another independent $\text{su}(2)$ (or quaternion) of the second (skew) line, so that the general construction scheme comprises $\text{SU}(2) \times \text{SU}(2)$ (or $\text{SL}(1,\mathbb{H}) \times \text{SL}(1,\mathbb{H})$) or $\text{SU}(4)$ (or $\text{SL}(2,\mathbb{H}) \cong \text{SU}(4)$), so we’ve closed the loop to our original starting point $\text{SU}(4)$ or Dirac algebra in terms of $\text{SU}^*(4)$ (see [4], [5] and refs there). $\text{SO}(6)$ and the various real forms $\text{SO}(n,m)$, $m+n = 6$, can be related either to their unitary covering groups, or by starting from $\text{SO}(3,3)$ or $\text{SO}(6)$ in line geometry by coordinate complexification, either of lines or points (thus, however, changing the respective absolute element or polar system, and in consequence the related underlying geometry). We postpone further discussions here.

Last not least, while realizing PG aspects by quaternions and their commutators, it is important to note that (in general) $\mathcal{C}(X'',P) \sim [X'',P] \sim [X'',[X,X']]$. This discussion can be extended to Jacobi identities, triple systems, curvature, and further geometric relations.

### 3.9. Lagrangean Construction

So far, after having related ‘spinors’ and $\text{SU}(2)$ spin reps by means of Lie transfer to line and Complex geometry, we have to focus on appropriate Lagrangean constructions in order to reflect Complex invariants and dynamics, and provide the Lagrangean (or Hamiltonian) apparatus.

Here, we’ll firstly focus on the photon rep of the QED Lagrangean, i.e. we’ll postpone the construction of matter fields and the discussion of Dirac or Clifford algebra for a moment and focus on the construction principle using Complexes. This reflects the fact that the photon rep almost automatically corresponds to a special linear Complex [4], [5] while obeying (and respecting) special relativity. Relativistic aspects will be treated in more detail in part VIII.

Above, we’ve shown that the ‘spinor’ discussion is a subset of Complex geometry right from Lie’s paper. So to discuss invariance and construct invariant elements in Complex geometry, we have lots of geometrical possibilities, some of which we’ve already mentioned elsewhere. On the ground of Klein’s Erlanger program, being in the regime of projective transformations, we may stress directly (quadratic) tetrahedral Complexes leaving anharmonic ratios of line intersections with a (fundamental) tetrahedron invariant. With respect to second order surfaces to be discussed in part VIII as absolute elements of non-euclidean geometry, we may as well consider polar or tangential Complexes. And possible degeneracy or Clebsch’s paper on Complex symbolism [2] suggests to consider assemblies or powers of individual (special) linear Complexes as well in the basic construction process.

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23 For details, see [5] sec. III with respect to relativistic symmetry and to symplectic geometry, thus stabilizing a certain linear Complex.
Now the first major step is to identify electromagnetism by means of a special linear Complex. The Plücker condition \( P := p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0 \) suggests to identify the field strengths \( \vec{E} \) and \( \vec{B} \), i.e. we have \( \vec{E} \cdot \vec{B} = 0 \), so initially we can identify \( \vec{E}_i \) and \( \vec{B}_i \) as six Plücker coordinates. Having argued in [5] with Klein’s paper of 1869 and 1871 (see also [6]), here we introduce some linear combinations \( y \) of Plücker’s line coordinates \( p \sim x \) according to the notation [6], p. 98, and we obtain (up to a factor 4)

\[
\vec{E}^2 - \vec{B}^2 = \vec{E}^2 + i\vec{B}^2 =: \Omega,
\]

which Klein defines as an invariant \( \Omega \) of the linear Complex and states \( \Omega = 0 \) as condition for a special linear Complex. Recall that this decomposition historically gave rise to various discussions of \( SU(2) \times SU(2) \) reps, as well as various complexifications thereof, mostly celebrating vector/’spinor’ calculus.

Re-expressed in contemporary notation, we can compare e.g. with

\[
-F_{\mu\nu}F^{\mu\nu} \sim \vec{E}^2 - \vec{B}^2 = \Omega, \quad G_{\mu\nu}F^{\mu\nu} \sim \vec{E} \cdot \vec{B} = P, \quad G_{\mu\nu} = \frac{1}{2} \epsilon_{\alpha\beta\mu\nu} F^{\alpha\beta}
\]

being the hodge dual which we’ve related earlier to conjugation and polar theory of Complexes.

So working with simple invariants of linear special Complexes, only, comprises known QED Lagrangean formulations. From the decomposition of quadratic Complexes or investigating pencils of linear Complexes, we know that we have to treat regular Complexes as well, so besides using the above quadratic expressions to enforce projective invariance or absolute elements, we expect the individual as well as the simultaneous invariance theory of general linear Complexes (special as well as regular) to comprise an appropriate basis of Lagrangean construction. Standard unitary symmetries are included as we’ve shown already above.

4. Summary and Outlook

Essentially, we have shown that \( su(2) \) and the Pauli rep according to Lie’s setup is an artifact emerging from line geometry, and a degenerated case of Complex geometry. Lie transfer puts classical PG and sphere geometry side by side, and we’ve found especially real quaternion calculus formally realizing known geometry based on null systems, as soon as one understands the commutator as a generalized product. However, this needs deeper and more complete research. The general framework superseding this type of algebra is Complex geometry, and we have connected previous work by closing the circle to twofold quaternions and certain unitary symmetries we’ve started with. The outline and development, however, suggests to consider typical spin applications or \( SU(2) \) symmetry in physics from the viewpoint of line or Complex geometry as well.

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