Improving fidelity of continuous-variable teleportation via local operations

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We study the Braunstein-Kimble setup for teleportation of quantum state of a single mode of optical field. We assume that the sender and receiver share a two-mode Gaussian state and we identify optimum local Gaussian operations that maximize the teleportation fidelity. We consider fidelity of teleportation of pure Gaussian states and we also introduce fidelity of the teleportation transformation. We show on an explicit example that in some cases the optimum local operation is not a simple unitary symplectic transformation but some more general completely positive map.

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I. INTRODUCTION

Quantum state teleportation is undoubtedly one of the most exciting developments in the rapidly growing field of Quantum Information Processing. In quantum teleportation, the information about the teleported quantum state is transferred from the sender, Alice, to the receiver, Bob, via dual classical and quantum EPR channels [1]. The latter is established via an entangled state shared by Alice and Bob. The teleportation protocol goes as follows: Alice carries out a Bell-type measurement on the state she wants to teleport and her part of the shared entangled state. She sends the result of her measurement via classical channel to Bob, who applies to his part of entangled state a transformation which depends on the classical information received from Alice.

The teleportation is perfect and Bob recovers an exact copy of the state teleported to him by Alice only if the quantum channel is ideal maximally entangled state. If we deal with qubits represented by polarization states of photons, then we can employ pair of polarization-entangled photons generated by means of spontaneous parametric down-conversion, where the entanglement is almost perfect [2, 3]. However, in case of continuous quantum variables [4, 5], an ideal EPR channel is an unphysical infinitely squeezed state. In quantum optics, the available resource is a two-mode squeezed vacuum state with some finite degree of squeezing [5, 6]. Moreover, the parts of the entangled state must be distributed among Alice and Bob, e.g., through optical fibers. This transmission inevitably introduces losses and noise and the entangled state shared by Alice and Bob will be some mixed state in general.

An important question is whether one can somehow improve the quality of the teleportation by means of local operations on the parts of the shared entangled state. Recently, this problem has been studied for teleportation of qubits and it was shown that local transformations may indeed be helpful [6]. Moreover, it was demonstrated that the optimum local transformation that maximizes the average teleportation fidelity need not be simple unitary transformation, but some completely positive (CP) map [7, 8]. In other words, it may be advantageous to let the parts of the shared quantum state interact with local ancillas.

In this paper, we investigate how to improve the fidelity of teleportation of continuous quantum variables by means of local operations on sender’s and receiver’s side. The first steps in this direction were already taken. Bowen et al. showed that in certain cases the fidelity of teleportation of coherent or squeezed states may be improved when Alice and Bob locally apply squeezing transformations to their parts of the shared quantum state [7]. Kim and Lee considered an asymmetric mixed quantum channel and showed that in that case the fidelity of teleportation of coherent states may be enhanced when a local transformation accompanied by decoherence is applied to one part of the quantum channel [8].

To make the problem tractable, we restrict ourselves to the class of trace-preserving Gaussian CP maps [12]. These maps preserve the Gaussian shape of the Wigner function of the transformed state. The restriction to Gaussian CP maps is very reasonable from the experimental point of view, because these maps can be implemented in the laboratory as a unitary symplectic transformation (linear canonical transformation of quadrature operators) on the signal mode and auxiliary modes initially prepared in some Gaussian states. In quantum optical setups this can be done with the help of phase shifters, beam splitters and squeezers. Gaussian CP maps were recently applied to description of cloning of continuous quantum variables [13]. Another recent paper discussed the conditions under which a given two-mode shared Gaussian state can be transformed into another Gaussian state by means of local Gaussian CP maps [14].

The paper is organized as follows. In Sec. II we will briefly describe the Braunstein-Kimble teleportation setup and we will derive compact formulas for fidelities of teleportation of any pure Gaussian state. We shall also introduce a fidelity for the teleportation operation itself. It will turn out that this latter fidelity can be interpreted as a fidelity of entanglement swapping. In Sec. III we will briefly review the properties of Gaussian CP maps and we will derive optimum local Gaussian CP map which maximizes a chosen teleportation fidelity. We shall consider two scenarios: in the first case the transformation is applied only on one side, in the second case both Alice and Bob may locally apply some CP maps. In Sec.
IV we present an example of our optimization procedure. Finally, Sec. V contains conclusions.

II. Fidelities

We shall consider the Braunstein-Kimble setup for teleportation of a single mode of optical field [1]. The quantum channel between Alice and Bob is established via two-mode entangled state \( \rho_{AB} \) fully described by its Wigner function \( W_{AB}(x_A,p_A,x_B,p_B) \). Alice mixes the mode whose state she wants to teleport with her part of entangled state on balanced beam splitter and she carries out a homodyne detection on each output mode thereby measuring two commuting quadratures \( X_+ = (x_+ + x_A)\sqrt{2} \) and \( P_- = (p_- - p_A)/\sqrt{2} \). After receiving the measured values of \( X_+ \) and \( P_- \) from Alice, Bob displaces his part of entangled state as follows: \( x_B \to x_B + \sqrt{2}X_+ \), \( p_B \to p_B + \sqrt{2}P_- \). We assume ideal homodyne detectors on Alice’s side and a zero coherent component of the entangled state \( \rho_{AB} \) (mean values of all quadratures \( x_{A,B}, p_{A,B} \) vanish). Under these conditions the resulting state on Bob’s side possesses the same coherent component as the original state teleported to him by Alice and the teleportation is invariant under displacement transformation.

Ide et al. [13] showed that the fidelity of continuous variable (CV) teleportation can be improved by optimizing the gain \( g \) in the modulation of the output field whose quadratures are displaced by the amount \( gX_+ \) and \( gP_- \). However, in this case the teleportation is not in general invariant under displacement transformation and, for instance, the fidelity of teleportation of coherent state \( |\alpha\rangle \) depends on the intensity \( |\alpha|^2 \). Here we keep the gain \( g \) fixed and improve the teleportation fidelity by suitable local transformations of the shared entangled state. Assuming fixed gain \( g = \sqrt{2} \), the relation between input and output Wigner functions of the teleported state is given by convolution [14]

\[
W_{out}(x_2,p_2) = \int_{-\infty}^{\infty} K(x_2-x_1,p_2-p_1)W_{in}(x_1,p_1)dx_1dp_1. \tag{1}
\]

In order to express the kernel \( K \) it is convenient to rewrite \( W_{AB} \) as a function of the variables \( x_{\pm} = x_A \pm x_B \) and \( p_{\pm} = p_A \pm p_B \),

\[
W_{AB}(x_A,p_A,x_B,p_B) = W_{AB}(x_+,p_+,x_-,p_-). \tag{2}
\]

With the help of \( W_{AB} \) we can write

\[
K(x_+,p_-) = \frac{1}{4} \int_{-\infty}^{\infty} W_{AB}(x_+,p_+,x_-,p_-)dx_-dp_+. \tag{3}
\]

In what follows we shall assume that the shared quantum state \( \rho_{AB} \) is two-mode Gaussian state. This is reasonable assumption since this class of states can be prepared in the lab. It is computationally convenient to deal with characteristic function of this state, defined as Fourier transform of the Wigner function,

\[
W_{AB}(r) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} w_{AB}(q) \exp(iq \cdot r) \, dq, \tag{4}
\]

where \( r = (x_A,p_A,x_B,p_B) \) and \( q = (\xi_A,\eta_A,\xi_B,\eta_B) \) are real vectors. For Gaussian state with vanishing coherent component we have [17]

\[
w_{AB}(q) = \exp\left[-\frac{1}{4}q^{\dagger} \Gamma_{AB} q\right]. \tag{5}
\]

The elements of the covariance matrix \( \Gamma_{AB} \) are given by

\[
\Gamma_{AB,ij} = \langle \Delta r_i \Delta r_j \rangle + \langle \Delta r_i \Delta r_i \rangle, \tag{6}
\]

where \( \Delta r_j = r_j - \langle r_j \rangle \) (note that if the coherent component of the state vanishes then \( \langle r_j \rangle = 0 \)). We can express the real covariance matrix \( \Gamma_{AB} \) in terms of three \( 2 \times 2 \) matrices \( A, B, \) and \( C, \)

\[
\Gamma_{AB} = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}. \tag{7}
\]

Here \( A \) and \( B \) are covariance matrices of the single modes on Alice’s and Bob’s side, respectively, and \( C \) contains the inter-modal correlations.

Now consider teleportation of a pure single-mode Gaussian state with covariance matrix \( \Gamma \). Since the teleportation is invariant under displacement transformation, all states with the same covariance matrix but different coherent components are teleported with the same fidelity. It thus suffices to consider state with vanishing coherent amplitude, whose characteristic function reads

\[
w_{in}(q) = \exp\left[-\frac{1}{4}q^{\dagger} \Gamma_{in} q\right], \tag{8}
\]

where \( q_{in} = (\xi_{in},\eta_{in}) \). Fidelity of teleportation of a pure state can be calculated as an overlap integral of input and output Wigner functions over the whole phase space

\[
F = 2\pi \int_{-\infty}^{\infty} W_{in}(x,p)W_{out}(x,p)dxdp. \tag{9}
\]

After making use of the formulas [11] and [12], expressing all Wigner functions as Fourier transforms of the characteristic functions and carrying out all integrals, we arrive at a compact formula for the teleportation fidelity,

\[
F = \frac{2}{\sqrt{\det E}}, \tag{10}
\]

where the matrix \( E \) reads

\[
E = 2D + RAR^T + RC + C^T R^T + B \tag{11}
\]

and

\[
R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{12}
\]
Besides the fidelity of teleportation of certain class of Gaussian states, one can introduce the fidelity of the teleportation process itself. How this can be accomplished becomes clear when one notices that the teleportation transformation is a trace-preserving CP map. Any CP map can be represented by positive semidefinite operators on a Hilbert space which is a tensor product of Hilbert space of input states and Hilbert space of output states. This representation is not only mathematical, the state \( \chi \) can be actually prepared in the lab if we first prepare a maximally entangled state on Hilbert space \( \mathcal{H} \otimes \mathcal{K} \) and then apply the CP map to one part of the entangled state. In case of CV teleportation, the maximally entangled state is the EPR state

\[
W_{\text{EPR}} = \frac{1}{2\pi} \delta(x_1 - x_2)\delta(p_1 + p_2)
\]

and the teleportation of one part of that state can be interpreted as an entanglement swapping. Hence the fidelity we obtain in this way is the fidelity of entanglement swapping of the EPR state.

Formally, the CP map that transforms input density matrix \( \rho_{\text{in}} \) onto output density matrix \( \rho_{\text{out}} \) can be written as a partial trace over the input Hilbert space,

\[
\rho_{\text{out}} = \text{Tr}_\mathcal{K}[\chi \rho_{\text{in}}^T \otimes \mathbb{1}_{\mathcal{K}}].
\]

In our case, the Wigner function \( W_{\text{tel}} \) of the teleportation CP map \( \chi_{\text{tel}} \) is closely related to the kernel \( K \) because the convolution is essentially the partial trace rewritten in terms of Wigner functions,

\[
W_{\text{tel}} = \frac{1}{2\pi} K(x_2 - x_1, p_2 + p_1).
\]

Notice the change of sign in front of \( p_1 \) which reflects the transposition in Eq. (13). The ideal teleportation is an identity map represented by the EPR state. Now since the CP maps are represented by positive semidefinite operators and since the ideal transformation is represented by a pure state, we can calculate the fidelity between the ideal and actual teleportation as fidelity of these two states. Thus we can write

\[
F_\chi = 4\pi^2 \int_{-\infty}^{\infty} W_{\text{EPR}}(x_1, p_1, x_2, p_2) \times W_{\text{tel}}(x_1, p_1, x_2, p_2) dx_1 dp_1 dx_2 dp_2.
\]

On inserting the explicit formulas (13) and (13) into Eq. (16) we obtain

\[
F_\chi = K(0, 0) \int_{-\infty}^{\infty} dx dp.
\]

We can see that \( F_\chi \) is infinite, as could have been expected since we work in infinite dimensional Hilbert space. Nevertheless, the fidelity \( F_\chi \) can be renormalized. If we drop an infinite constant proportional to Dirac delta function and multiply by \( 2\pi \), then we obtain

\[
F = 2\pi K(0, 0).
\]

If we insert the explicit formula (3) for kernel \( K \) into Eq. (18), then we find that

\[
F = 2\pi \int_{-\infty}^{\infty} W_{AB}(x, p, -x, p) dx dp.
\]

For Gaussian quantum channels, this formula simplifies to

\[
F = \frac{2}{\sqrt{\det \mathbf{E}'}}
\]

where the matrix \( \mathbf{E}' \) reads

\[
\mathbf{E}' = \mathbf{R} \mathbf{A} \mathbf{R}^T + \mathbf{R} \mathbf{C} + \mathbf{C}^T \mathbf{R}^T + \mathbf{B}.
\]

The expression for the fidelity \( F \) is a rather special case of the formula for the fidelity of teleportation of pure Gaussian states where we set the covariance matrix \( \mathbf{D} \) equal to zero. Of course, this means that \( F \) is unbounded. Nevertheless, \( F \) is a good measure of the quality of teleportation. For instance it can be shown that \( F > 1 \) only if the state \( \rho_{AB} \) is entangled. In particular, if \( \rho_{AB} \) is two-mode squeezed vacuum state parametrized by squeezing constant \( r \) then one gets

\[
F = \exp(2r),
\]

hence the fidelity monotonically exponentially grows with the squeezing.

### III. OPTIMUM LOCAL GAUSSIAN CP MAP

Our task is to maximize the fidelity of teleportation (either \( F \) or \( F' \)) by means of local Gaussian trace-preserving CP maps on Alice’s and Bob’s side. We shall consider two scenarios: in the first, simpler scenario the CP map is applied only on Bob’s side while in the second case both Alice and Bob may locally apply some CP maps. Gaussian CP maps are those maps for which the Wigner function of the corresponding operator \( \chi \) has a Gaussian form. The teleportation with Gaussian quantum channel is an example of Gaussian CP map. The Wigner function representing single-mode trace-preserving Gaussian CP map reads

\[
W_\chi = \frac{1}{2\pi^2 \sqrt{\det \mathbf{G}}} \exp(-\Delta \mathbf{r}^\mathsf{T} \mathbf{G}^{-1} \Delta \mathbf{r}),
\]

where \( \mathbf{S} \) and \( \mathbf{G} \) are real \( 2 \times 2 \) matrices, moreover, \( \mathbf{G} \) is symmetric positive semidefinite matrix,

\[
\Delta \mathbf{r} = \mathbf{r}_{\text{out}} - \mathbf{S} \mathbf{r}_{\text{in}}^*,
\]

and

\[
\mathbf{r}_{\text{out}} = \begin{pmatrix} x_{\text{out}} \\ p_{\text{out}} \end{pmatrix}, \quad \mathbf{r}_{\text{in}}^* = \begin{pmatrix} x_{\text{in}} \\ -p_{\text{in}} \end{pmatrix}
\]

are column vectors of output and input quadratures, respectively. Since we deal with Gaussian states whose
form is invariant under Gaussian CP maps, it suffices to provide rule for transformation of the covariance matrix $\Gamma$. The relation between input and output single-mode covariance matrices $\Gamma_{\text{in}}$ and $\Gamma_{\text{out}}$ is given by a simple linear map $\mathbb{C}^3$

$$\Gamma_{\text{out}} = S\Gamma_{\text{in}}S^T + G.$$  

(25)

The map $\mathbb{C}^3$ is completely positive if and only if $S$ and $G$ satisfy an inequality $\mathbb{C}^3$

$$G + i\Sigma - iS\Sigma S^T \geq 0,$$  

(26)

where

$$\Sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  

(27)

The condition $\mathbb{C}^3$ can be derived as follows: the Wigner function $\mathbb{C}^3$ must represent a positive semidefinite operator, which imposes constraint on the covariance matrix $G$ $\mathbb{C}^3$. Namely, the matrix

$$M_{ij} = G_{ij} + [\Delta r_i, \Delta r_j],$$  

(28)

must be positive semidefinite, where $[\cdot, \cdot]$ stands for commutator. Making use of canonical commutation relations for the quadratures $x_{\text{in}}$, $p_{\text{in}}$ and $x_{\text{out}}$, $p_{\text{out}}$ one arrives after some algebra at the inequality $\mathbb{C}^3$.

Assume now that a Gaussian CP map $\mathbb{C}^3$ is applied to Bob’s part of shared two-mode state $\rho_{\text{AB}}$. This modifies the covariance matrix $\Gamma_{\text{AB}}$

$$\Gamma_{\text{AB}} = \begin{pmatrix} A & CS^T \\ SC^T & SBS^T + G \end{pmatrix},$$  

(29)

The maximization of the fidelity then amounts to the minimization of the determinant

$$D = \det[2D + RAR^T + RCS^T + SC^T R^T + SBS^T + G]$$  

(30)

under the constraints $\mathbb{C}^3$. Recall that on setting $D = 0$ we obtain as a special case the fidelity of entanglement swapping $\mathbb{C}^3$.

We divide the optimization of the CP map into two steps. In the first step we find optimum $G$ for a given matrix $S$ and then we shall optimize over all possible matrices $S$. Since the matrices $S$ and $G$ have altogether seven independent elements,

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}, \quad S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix},$$  

(31)

we have to find a global minimum of a function of seven real variables under the constraint $\mathbb{C}^3$, which can be equivalently expressed as

$$g_{11} \geq 0, \quad g_{22} \geq 0$$  

(32)

and

$$g_{11}g_{22} - g_{12}^2 - (1 - s)^2 \geq 0,$$  

(33)

where $s = s_{11}s_{22} - s_{12}s_{21}$. We introduce a short-hand notation for the elements of matrix

$$2D + RAR^T + RCS^T + SC^T R^T + SBS^T = \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix}.$$  

(34)

Notice that this matrix is, by definition, positive semidefinite, and its elements $\alpha$, $\beta$, $\gamma$ are functions of $s_{ij}$. Thus we can write the determinant $\mathbb{C}^3$ in a compact form,

$$D = (\alpha + g_{11})(\beta + g_{22}) - (g_{12} + \gamma)^2.$$  

(35)

It is always optimal to choose “extremal” matrix $G$ that satisfies the inequality $\mathbb{C}^3$ as an equality. Indeed, if a sharp inequality holds in $\mathbb{C}^3$, then we can reduce the value of diagonal elements $g_{11}$ and $g_{22}$ until the equality is reached in $\mathbb{C}^3$ and this would obviously reduce also the value of $D$. Hence we can write

$$g_{12} = \pm \sqrt{g_{11}g_{22} - (1 - s)^2}$$  

(36)

and insert into $\mathbb{C}^3$. Furthermore, we can see that it is optimal to choose the sign of $g_{12}$ the same as the sign of $\gamma$ and we have,

$$D = (\alpha + g_{11})(\beta + g_{22}) - (\sqrt{g_{11}g_{22} - (1 - s)^2} + |\gamma|)^2.$$  

(37)

Upon solving the set of two nonlinear extremal equations

$$\frac{\partial D}{\partial g_{11}} = 0, \quad \frac{\partial D}{\partial g_{22}} = 0,$$  

(38)

we find that the optimum matrix $G$ is proportional to the matrix $\mathbb{C}^3$,

$$G = \frac{1 - s}{\sqrt{\alpha\beta - \gamma^2}} \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix}.$$  

(39)

On inserting the elements of the optimum $G$ back into Eq. $\mathbb{C}^3$ we finally obtain

$$D = \left(1 - s + \sqrt{\alpha\beta - \gamma^2}\right)^2.$$  

(40)

Now $D$ is a function of four variables $s_{11}$, $s_{22}$, $s_{12}$ and $s_{21}$ and we have to find its global minimum. In general, such optimization is a hard task and can be solved only numerically. However, we shall see that when making some assumptions we will be able to solve this problem analytically.

It is well known that by means of local symplectic transformations it is possible to bring any two-mode covariance matrix $\Gamma_{\text{AB}}$ into tridiagonal form $\mathbb{C}^3$.

$$\Gamma_{\text{AB}} = \begin{pmatrix} a & 0 & c_1 & 0 \\ 0 & a & 0 & c_2 \\ c_1 & 0 & b & 0 \\ 0 & c_2 & 0 & b \end{pmatrix}.$$  

(41)

It suffices to consider Gaussian quantum channels for which all the matrices $A$, $B$ and $C$ in $\mathbb{C}^3$ are diagonal.
Further assume that also the covariance matrix $D$ of the teleported state is diagonal, $D = \text{diag}(d_{11}, d_{22})$. Note that this assumption is not a serious restriction, because any $D$ can be diagonalized by means of reversible symplectic transformation. In this case it can be shown that the necessary conditions on extremum

$$\frac{\partial D}{\partial s_{12}} = 0, \quad \frac{\partial D}{\partial s_{21}} = 0,$$

are satisfied when $s_{12} = s_{21} = 0$. One has to be a bit careful here because there is an absolute value in Eq. (41) and three cases must be distinguished: (i) $1 - s > 0$, (ii) $1 - s < 0$, and (iii) $s = 1$. In all cases, the conditions on extremum are satisfied when $s_{12} = s_{21} = 0$.

We are thus lead to make the hypothesis that if the matrices $A, B, C, D$ are diagonal, then the optimum $G$ and $S$ are also diagonal. When $S$ is diagonal, then $\gamma = 0$, and the matrix elements $\alpha$ and $\beta$ become quadratic functions of $s_{11}$ and $s_{22}$, respectively,

$$\alpha(s_{11}) = 2d_{11} + a + 2c_1 s_{11} + b s_{11}^2, \quad \beta(s_{22}) = 2d_{22} + a - 2c_2 s_{22} + b s_{22}^2.$$  

(43)

For the sake of notational simplicity we define $x = s_{11}$ and $y = s_{22}$ and we must minimize the function

$$f(x, y) = |1 - xy| + \sqrt{\alpha(x)\beta(y)}.$$

(44)

The extremal equations are obtained by setting the partial derivatives of $f(x, y)$ equal to zero,

$$x = \pm \sqrt{\frac{\alpha(x)}{\beta(y)}(by - c_2)}, \quad y = \pm \sqrt{\frac{\beta(y)}{\alpha(x)}(bx + c_1)},$$  

(45)

where the signs $+$ and $-$ correspond to the cases when $1 - xy > 0$ and $1 - xy < 0$, respectively. From the product of the formulas for $x$ and $y$, we can express $y$ in terms of $x$,

$$y = \frac{c_2(bx + c_1)}{x(b^2 - 1) + bc_1}. $$

(46)

Substituting this formula back into the second Eq. (45) and squaring that equation, we arrive at

$$c_2^2 \alpha(x) = [x(b^2 - 1) + bc_1]^2 \beta \left( \frac{c_2(bx + c_1)}{x(b^2 - 1) + bc_1} \right).$$

(47)

This is a quadratic equation for $x$ and can be solved analytically. In this way we identify all potential minima outside the boundary $xy = 1$. It remains to localize minima on the boundary where $y = 1/x$ and we must minimize the function $\alpha(x)\beta(1/x)$. The condition on extremum

$$\frac{d}{dx} \left[ \alpha(x)\beta \left( \frac{1}{x} \right) \right] = 0$$

(48)

reduces to quartic equation for $x$. Upon solving this equation we get positions of all possible minima on the boundary, i.e., we determine all potentially optimum symplectic transformations.

Let us now consider a more general protocol, where both Alice and Bob are allowed to apply some local Gaussian CP maps. To make the problem tractable, we do not assume any communication between Alice and Bob at this stage, hence they both apply their local operations independently. Furthermore, we shall assume that all relevant matrices are diagonal, hence we shall seek the optimum two-mode CP map in the form

$$S = \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & y \end{pmatrix},$$

(49)

$$G = \begin{pmatrix} g_{A,11} & 0 & 0 & 0 \\ 0 & g_{A,22} & 0 & 0 \\ 0 & 0 & g_{B,11} & 0 \\ 0 & 0 & 0 & g_{B,22} \end{pmatrix}.$$  

(50)

The covariance matrix $\Gamma_{AB}$ transforms according to

$$\Gamma_{AB} \rightarrow S \Gamma_{AB} S^T + G.$$  

(51)

From Eq. (52) where the equality should hold and where $g_{12} = 0$, we obtain the following relations between the elements of the optimum matrix $G$,

$$g_{A,11} g_{A,22} = (1 - uv)^2, \quad g_{B,11} g_{B,22} = (1 - xy)^2.$$  

(52)

The determinant $D$ can be expressed as

$$D = (\alpha + g_{A,11} + g_{B,11}) \left( \beta + \frac{(1 - uv)^2}{g_{A,11}} + \frac{(1 - xy)^2}{g_{B,11}} \right).$$

(53)

This function attains its global minimum when

$$g_{A,11} = |1 - uv| \sqrt{\frac{\alpha}{\beta}}, \quad g_{B,11} = |1 - xy| \sqrt{\frac{\alpha}{\beta}}.$$  

(54)

On inserting these expressions back into Eq. (52), we get

$$D = (|1 - xy| + |1 - uv| + \sqrt{\alpha\beta})^2,$$

(55)

where $\alpha$ and $\beta$ are functions of four real variables $u, v, x, y$, the elements of matrix $S$. In general, the minimum of the function (52) must be found numerically. In what follows we shall focus on the fidelity of entanglement swapping and we shall see that in this case one can find the global minimum analytically. The square root of the determinant (55) that we must minimize reads in this case ($d_{11} = d_{22} = 0$)

$$f(u, v, x, y) = |1 - uv| + |1 - xy|$$

$$+ (u^2 + 2uvx + x^2 b) (v^2 a - 2vyc + y^2 b)^{1/2}.$$  

(56)
The function \( f(w, x, y) = |1 - w| + |1 - xy| + |w|[(a + 2qc_1 + q^2b)(a - 2yc_2 + y^2b)]^{1/2} \). \hfill (58)

After another substitution \( x = qw \) the function \( f(w, q, y) = |1 - w| + |1 - wqy| + |w|[(a + 2qc_1 + q^2b)(a - 2yc_2 + y^2b)]^{1/2} \). \hfill (59)

From the linearity of \( f \) it is clear that the extrema are localized at points, where one absolute value is equal to zero. Hence we have to consider three different possibilities:

(i) \( w = 1 \), no operation is applied on Alice’s side and a CP map is applied on Bob’s side.

(ii) \( wqy = 1 \), a symplectic transformation is applied on Bob’s side. However, this symplectic transformation can be in our case “absorbed” into CP map on Alice’s side, hence another possibly optimum strategy is to do nothing on Bob’s side and to apply a CP map on Alice’s side.

(iii) \( w = 0 \), this means that both Alice and Bob throw away their parts of shared quantum state and replace them with vacuum states. Clearly, this strategy is optimum if the quantum channel is not in entangled state, because with vacuum state at both sides one gets maximum fidelity \( F \) obtainable without the aid of entanglement, \( F_{\text{max, class}} = 1 \).

One may object that the substitution \( x = wq \) is problematic when \( w = 0 \) and \( x \neq 0 \). However, a detailed analysis reveals that if one of the four parameters \( u, v, x, y \) is set equal to zero and the three remaining parameters are optimized, then we once again arrive at the above listed alternatives (i)–(iii).

The strategies (i) and (ii) represent a CP map on only one side, while nothing is performed on the other side. It was shown above that these optimum one-sided CP maps can be found analytically. The search for optimum CP map would thus consist of three parts: find one-sided optimum Gaussian CP maps on Alice’s side, on Bob’s side and also consider replacement of the shared quantum state with vacuum state and choose the optimum alternative that yields maximum fidelity.

IV. EXAMPLE OF OPTIMIZATION

To illustrate how the optimization works in practice, let us assume that the covariance matrix \( \Gamma_{AB} \) has the tridiagonal structure given by Eq. \( (41) \) and the nonzero elements read

\[ a = 1 + 2 \sinh^2 r, \quad c_1 = -\sinh(2r), \]
\[ b = 1 + 2 \sinh^2 r + b_0, \quad c_2 = \sinh(2r). \] \hfill (60)

For \( b_0 = 0 \) we recover the covariance matrix of pure two-mode squeezed vacuum state and the quantum channel is in a mixed state for any \( b_0 > 0 \).

Let us analyse how the fidelity of teleportation of coherent state can be improved by means of local transformations on Bob’s side. Since \( d_{11} = d_{22} = 1 \) and also \( c_1 = -c_2 \) [c.f. Eq. \( (61) \)], the solution of Eq. \( (47) \) simplifies considerably because the functions \( \alpha \) and \( \beta \) are identical. The optimum \( x \) and \( y \) are equal, \( x = y \), and the two roots of Eq. \( (47) \) read

\[ x_1 = \frac{c_2}{b - 1}, \quad x_2 = \frac{c_2}{b + 1}. \] \hfill (61)

Furthermore, the quartic equation \( (48) \) for optimum symplectic transformation splits into two quadratic equations that have only two real roots \( x = \pm 1 \).

We must evaluate the fidelity of teleportation of coherent state for all these potentially optimum transformations on Bob’s side, and choose the maximum value. The resulting fidelity is plotted in Fig. 1 for \( b_0 = 1/2 \) and a variable degree of squeezing \( r \). It turns out that if the squeezing is lower than certain threshold \( r_{\text{th}} = -[\ln(1 - b_0)]/2 \), then the optimum transformation on Bob’s side is a CP map with \( x \) given by \( x_1 \) in Eq. \( (47) \). The parameter \( x_1 \) grows from zero for \( r = 0 \) to the value \( x_1 = 1 \) that is attained when \( r = r_{\text{th}} \). For higher squeezing, the best strategy is to do nothing, i.e., the optimum operation is a symplectic transformation with \( x = y = 1 \). The optimum CP map for \( r < r_{\text{th}} \) is a simple damping process which can be implemented with the help of a beam splitter with amplitude transmittance.
t = c_2/(b - 1) whose two input ports are fed with Bob’s part of entangled state and a vacuum state, respectively. This transformation reduces the noise represented by b_0 in Eq. (60), which in turn improves the teleportation fidelity.

FIG. 2: The same as Fig. 1 but the fidelity of entanglement swapping F is plotted.

With the help of CP map on Bob’s side, the fidelity of teleportation of coherent state is always larger than the maximum fidelity 1/2 achievable without the aid of entanglement. On the other hand, if we allow only for unitary symplectic transformations on Bob’s part of the state, then there is a region of squeezing where the maximum achievable fidelity is lower than 1/2. This example clearly illustrates that in certain cases it is advantageous to couple the shared state to the local environment [7, 9].

Similar results are obtained for the fidelity of entanglement swapping F. In this case we can optimize over all local Gaussian CP maps on both Alice’s and Bob’s side, because the problem reduces to the optimization of a Gaussian CP map on only one side, as discussed in the previous Section. For our specific example, the optimum CP map is actually the same as that for the fidelity of teleportation of coherent state. Also the dependence of the fidelity F on r is qualitatively similar to that shown in Fig. 1, see Fig. 2. In particular, there is a region where F > 1 if Bob applies the optimum CP map, but a restriction to local symplectic transformation results in F < 1.

V. CONCLUSIONS

In this paper we have shown that one can improve the fidelity of teleportation of continuous quantum variables by means of local operations on the sender’s and receiver’s parts of the shared entangled state ρ_{AB} (quantum channel). We have considered the fidelity of teleportation of pure Gaussian states and we have also introduced a fidelity measure for the teleportation transformation. The latter fidelity was interpreted as a fidelity of entanglement swapping of infinitely squeezed EPR state.

We have restricted ourselves to the class of local trace-preserving Gaussian completely positive maps and we have shown that in this case the optimization problem can be solved analytically. We have demonstrated on a simple example that the optimum local operation need not be a unitary symplectic transformation but some more general CP map.

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APPENDIX A

Here we prove that the inequality F ≤ 1 holds if the shared quantum state W_{AB}(x_A, p_A, x_B, p_B) is separable. Our starting point is the the formula (19):

\[ F = 2\pi \int_{-\infty}^{\infty} W_{AB}(x, p, -x, p) dx dp. \] (A1)

Density matrix ρ_{AB} of any separable state can be written as a convex mixture of product states,

\[ ρ_{AB} = \sum_j p_j ρ_{A,j} \otimes ρ_{B,j}, \] (A2)

where p_j > 0 and \( \sum_j p_j = 1 \). Formula (A2) implies that

\[ W_{AB}(x_A, p_A, x_B, p_B) = \sum_j p_j W_{A,j}(x_A, p_A) W_{B,j}(x_B, p_B). \] (A3)

If \( W_{B,j}(x_B, p_B) \) is Wigner function of the quantum state \( ρ_{B,j} \), then \( W_{B,j}(-x_B, p_B) \) is a Wigner function of the quantum state \( ρ_{B,j}^T \), because the transformation \( x \rightarrow -x \) and \( p \rightarrow p \) is the transposition. For separable state (A2), the formula (A1) thus reduces to

\[ F = \sum_j p_j \text{Tr}[ρ_{A,j} ρ_{B,j}^T], \] (A4)

where we used that

\[ \text{Tr}[ρ_{AB}] = 2\pi \int_{-\infty}^{\infty} W_A(x, p) W_B(x, p) dx dp. \] (A5)

The Schwarz inequality implies that \( \text{Tr}[ρ_{A,j} ρ_{B,j}^T] \leq 1 \) and with the help of normalization of \( p_j \) we finally obtain

\[ F \leq 1. \] (A6)
The fidelity $\mathcal{F} = 1$ forms a boundary between classical information transfer and quantum teleportation. The state $\rho_{AB}$ must be entangled in order to achieve $\mathcal{F} > 1$. This illustrates the essential and central role of the entanglement in the teleportation.

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