ON THE PARTIAL SUMS OF VILENKIN-FOURIER SERIES

GEORGE TEPHNADZE

Abstract. The main aim of this paper is to investigate weighted maximal operators of partial sums of Vilenkin-Fourier series. We also use our results to prove approximation and strong convergence theorems on the martingale Hardy spaces $H_p$, when $0 < p \leq 1$.

2010 Mathematics Subject Classification. 42C10.
Key words and phrases: Vilenkin system, partial sums, martingale Hardy space, modulus of continuity, convergence.

1. Introduction

It is well-known that Vilenkin system forms not basis in the space $L_1(G_m)$. Moreover, there is a function in the martingale Hardy space $H_1(G_m)$, such that the partial sums of $f$ are not bounded in $L_1(G_m)$-norm, but partial sums $S_n$ of the Vilenkin-Fourier series of a function $f \in L_1(G_m)$ convergence in measure [12].

Uniform convergence and some approximation properties of partial sums in $L_1(G_m)$ norms was investigate by Goginava [8] (see also [9]). Fine [3] has obtained sufficient conditions for the uniform convergence which are in complete analogy with the Dini-Lipschitz conditions. Guliev [13] has estimated the rate of uniform convergence of a Walsh-Fourier series using Lebesgue constants and modulus of continuity. Uniform convergence of subsequence of partial sums was investigate also in [7]. This problem has been considered for Vilenkin group $G_n$ by Fridli [4], Blahota [2] and Gát [6].

It is also known that subsequence $S_{n_k}$ is bounded from $L_1(G_m)$ to $L_1(G_m)$ if and only if $n_k$ has uniformly bounded variation and subsequence of partial sums $S_{M_n}$ is bounded from the martingale Hardy space $H_p(G_m)$ to the Lebesgue space $L_p(G_m)$, for all $p > 0$. In this paper we shall prove very unexpected fact:

There exists a martingale $f \in H_p(G_m)$ ($0 < p < 1$), such that

$$\sup_{n \in \mathbb{N}} \|S_{M_n+1} f\|_{L_p, \infty} = \infty.$$ 

The reason of divergence of $S_{M_n+1} f$ is that when $0 < p < 1$ the Fourier coefficients of $f \in H_p(G_m)$ are not bounded (See [17]).

\[\text{The research was supported by Shota Rustaveli National Science Foundation grant no.13/06 (Geometry of function spaces, interpolation and embedding theorems).}\]
In Gát [3] the following strong convergence result was obtained for all \( f \in H_1(G_m) \):
\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \|S_k f - f\|_1 = 0,
\]
where \( S_k f \) denotes the \( k \)-th partial sum of the Vilenkin-Fourier series of \( f \). (For the trigonometric analogue see Smith [16], for the Walsh system see Simon [14]). For the Vilenkin system Simon [15] proved that there is an absolute constant \( c_p \), depending only on \( p \), such that
\[
\sum_{k=1}^\infty \frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p,
\]
for all \( f \in H_p(G_m) \), where \( 0 < p < 1 \). The author [18] proved that for any nondecreasing function \( \Phi : \mathbb{N} \to [1, \infty) \), satisfying the condition \( \lim_{n \to \infty} \Phi(n) = +\infty \), there exists a martingale \( f \in H_p(G_m) \), such that
\[
\sum_{k=1}^\infty \frac{\|S_k f\|_{L_p, \infty}^p \Phi(k)}{k^{2-p}} = \infty, \text{ for } 0 < p < 1.
\]

Strong convergence theorems of two-dimensional partial sums was investigate by Weisz [23], Goginava [10], Gogoladze [11], Tepnadze [19].

The main aim of this paper is to investigate weighted maximal operators of partial sums of Vilenkin-Fourier series. We also use this results to prove some approximation and strong convergence theorems on the martingale Hardy spaces \( H_p(G_m) \), when \( 0 < p \leq 1 \).

2. Definitions and Notations

Let \( \mathbb{N}_+ \) denote the set of the positive integers, \( \mathbb{N} := \mathbb{N}_+ \cup \{0\} \).

Let \( m := (m_0, m_1, ...) \) denote a sequence of the positive integers not less than 2.

Denote by
\[
Z_{m_k} := \{0, 1, ..., m_k - 1\}
\]
the additive group of integers modulo \( m_k \).

Define the group \( G_m \) as the complete direct product of the group \( Z_{m_j} \) with the product of the discrete topologies of \( Z_{m_j} \)'s.

The direct product \( \mu \) of the measures \( \mu_k(\{j\}) := 1/m_k \) is the Haar measure on \( G_m \) with \( \mu(G_m) = 1 \).

If the sequence \( m := (m_0, m_1, ...) \) is bounded than \( G_m \) is called a bounded Vilenkin group, else it is called an unbounded one.

The elements of \( G_m \) represented by sequences
\[
x := (x_0, x_1, ..., x_j, ...), \quad (x_k \in Z_{m_k}).
\]
It is easy to give a base for the neighborhood of $G_m$

\[ I_0(x) := G_m, \]

\[ I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \quad (x \in G_m, \ n \in \mathbb{N}). \]

Denote $I_n := I_n(0)$ for $n \in \mathbb{N}$ and $\overline{I_n} := G_m \setminus I_n$.

It is evident

\[ (3) \quad \overline{I_N} = \bigcup_{s=0}^{N-1} I_s \setminus I_{s+1}. \]

If we define the so-called generalized number system based on $m$ in the following way

\[ M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in \mathbb{N}) \]

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_j M_j$, where $n_j \in \mathbb{Z}_{m_j}$ ($j \in \mathbb{N}$) and only a finite number of $n_j$’s differ from zero. Let $|n| := \max \{j \in \mathbb{N}, n_j \neq 0\}$.

Denote by $L_1(G_m)$ the usual (one dimensional) Lebesgue space.

Next, we introduce on $G_m$ an orthonormal system which is called the Vilenkin system.

At first define the complex valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions as

\[ r_k(x) := \exp \left( \frac{2\pi i x k}{m_k} \right) \quad (i^2 = -1, \ x \in G_m, \ k \in \mathbb{N}). \]

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on $G_m$ as:

\[ \psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}). \]

Specifically, we call this system the Walsh-Paley one if $m \equiv 2$.

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ [11, 20].

Now we introduce analogues of the usual definitions in Fourier-analysis.

If $f \in L_1(G_m)$ we can establish the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to the Vilenkin system $\psi$ in the usual manner:

\[ \hat{f}(k) := \int_{G_m} f \psi_k d\mu, \quad (k \in \mathbb{N}), \]

\[ S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k, \quad (n \in \mathbb{N}_+, \ S_0 f := 0), \]

\[ D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}_+). \]
Recall that (see [1])

\[ D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n \\ 0, & \text{if } x \notin I_n \end{cases} \]

and

\[ D_n(x) = \psi_n(x) \left( \sum_{j=0}^{\infty} D_{M_j}(x) \sum_{u=m_j-n_j}^{m_j-1} r^u_j(x) \right). \]

The norm (or quasinorm) of the space \( L^p(G_m) \) is defined by

\[ \|f\|_p := \left( \int_{G_m} |f|^p d\mu \right)^{1/p} \quad (0 < p < \infty). \]

The space \( L_{p,\infty}(G_m) \) consists of all measurable functions \( f \) for which

\[ \|f\|_{L_{p,\infty}} := \sup_{\lambda > 0} \lambda \mu(f > \lambda)^{1/p} < +\infty. \]

The \( \sigma \)-algebra generated by the intervals \( \{I_n(x) : x \in G_m\} \) will be denoted by \( F_n(n \in \mathbb{N}) \). Denote by \( f = (f_n, n \in \mathbb{N}) \) a martingale with respect to \( F_n(n \in \mathbb{N}) \). (for details see e.g. [21]). The maximal function of a martingale \( f \) is defined by

\[ f^* = \sup_{n \in \mathbb{N}} \left| f^{(n)} \right|. \]

In case \( f \in L_1(G_m) \), the maximal functions are also given by

\[ f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u) d\mu(u) \right|. \]

For \( 0 < p < \infty \) the Hardy martingale spaces \( H^p(G_m) \) consist of all martingales, for which

\[ \|f\|_{H^p} := \|f^*\|_p < \infty. \]

The dyadic Hardy martingale spaces \( H^p(G_m) \) for \( 0 < p \leq 1 \) have an atomic characterization. Namely the following theorem is true (see [24]):

**Theorem W**: A martingale \( f = (f_n, n \in \mathbb{N}) \) is in \( H^p(G_m) \) \((0 < p \leq 1)\) if and only if there exists a sequence \((a_k, k \in \mathbb{N})\) of \( p \)-atoms and a sequence \((\mu_k, k \in \mathbb{N})\) of real numbers such that for every \( n \in \mathbb{N} \)

\[ \sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f_n \]

and

\[ \sum_{k=0}^{\infty} |\mu_k|^p < \infty. \]
Moreover, \( \|f\|_{H^p} \sim \inf (\sum_{k=0}^{\infty} |\mu_k|^p)^{1/p} \), where the infimum is taken over all decomposition of \( f \) of the form (6).

Let \( X = X(G_m) \) denote either the space \( L_1(G_m) \), or the space of continuous functions \( C(G_m) \). The corresponding norm is denoted by \( \|\cdot\|_X \). The modulus of continuity, when \( X = C(G_m) \) and the integrated modulus of continuity, where \( X = L_1(G_m) \) are defined by

\[
\omega(1/M_n, f)_X = \sup_{h \in I_n} \|f(\cdot + h) - f(\cdot)\|_X.
\]

The concept of modulus of continuity in \( H^p(G_m) \) \((0 < p \leq 1)\) can be defined in following way

\[
\omega(1/M_n, f)_{H^p(G_m)} := \|f - S_{M_n} f\|_{H^p(G_m)}.
\]

If \( f \in L_1(G_m) \), then it is easy to show that the sequence \( (S_{M_n}(f) : n \in \mathbb{N}) \) is a martingale.

If \( f = (f_n, n \in \mathbb{N}) \) is martingale then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

\[
\hat{f}(i) := \lim_{k \to \infty} \int_{G_m} f_k(x) \Psi_i(x) d\mu(x).
\]

The Vilenkin-Fourier coefficients of \( f \in L_1(G_m) \) are the same as the martingale \( (S_{M_n}(f) : n \in \mathbb{N}) \) obtained from \( f \).

For the martingale \( f \) we consider maximal operators

\[
S^* f := \sup_{n \in \mathbb{N}} |S_n f|,
\]

\[
S_p^* f := \sup_{n \in \mathbb{N}} \frac{|S_n f|}{(n+1)^{1/p-1} \log^p (n+1)}, \quad 0 < p \leq 1,
\]

where \( [p] \) denotes integer part of \( p \).

A bounded measurable function \( a \) is \( p \)-atom, if there exist a dyadic interval \( I \), such that

\[
\int_I ad\mu = 0, \quad \|a\|_{\infty} \leq \mu(I)^{-1/p}, \quad \text{supp} \ (a) \subset I.
\]

3. Formulation of Main Results

**Theorem 1.** a) Let \( 0 < p \leq 1 \). Then the maximal operator \( S_p^* \) is bounded from the Hardy space \( H^p(G_m) \) to the space \( L_p(G_m) \).

b) Let \( 0 < p \leq 1 \) and \( \varphi : \mathbb{N}_+ \to [1, \infty) \) be a nondecreasing function satisfying the condition

\[
\lim_{n \to \infty} \frac{(n+1)^{1/p-1} \log^p(n+1)}{\varphi(n)} = +\infty.
\]
Then
\[
\sup_{n \in \mathbb{N}} \left\| \frac{S_n f}{\varphi(n)} \right\|_{L^p,\infty(G_m)} = \infty, \text{ for } 0 < p < 1
\]
and
\[
\sup_{n \in \mathbb{N}} \left\| \frac{S_n f}{\varphi(n)} \right\|_1 = \infty.
\]

**Corollary 1.** (Simon [15]) Let \( 0 < p < 1 \) and \( f \in H_p(G_m) \). Then there is an absolute constant \( c_p \), depends only \( p \), such that
\[
\sum_{k=1}^{\infty} \frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p.
\]

**Theorem 2.** Let \( 0 < p \leq 1 \), \( f \in H_p(G_m) \) and \( M_k < n \leq M_{k+1} \). Then there is an absolute constant \( c_p \), depends only \( p \), such that
\[
\|S_n (f) - f\|_{H_p(G_m)} \leq c_p n^{1/p-1} \lg n \omega\left(\frac{1}{M_k}, f\right)_{H_p(G_m)}.
\]

**Theorem 3.** a) Let \( 0 < p < 1 \), \( f \in H_p(G_m) \) and
\[
\omega\left(\frac{1}{M_n}, f\right)_{H_p(G_m)} = o\left(\frac{1}{M_n^{1/p-1}}\right), \text{ as } n \to \infty.
\]
Then
\[
\|S_k (f) - f\|_{L^p,\infty(G_m)} \to 0, \text{ when } k \to \infty.
\]
b) For every \( p \in (0,1) \) there exists martingale \( f \in H_p(G_m) \), for which
\[
\omega\left(\frac{1}{M_{2n}}, f\right)_{H_p(G_m)} = O\left(\frac{1}{M_{2n}^{1/p-1}}\right), \text{ as } n \to \infty
\]
and
\[
\|S_k (f) - f\|_{L^p,\infty(G_m)} \to 0, \text{ when } k \to \infty.
\]

**Theorem 4.** Let \( f \in H_1(G_m) \) and
\[
\omega\left(\frac{1}{M_n}, f\right)_{H_1(G_m)} = o\left(\frac{1}{n}\right), \text{ as } n \to \infty.
\]
Then
\[
\|S_k (f) - f\|_1 \to 0, \text{ when } k \to \infty.
\]
b) There exists martingale \( f \in H_1(G_m) \) for which
\[
\omega\left(\frac{1}{M_{2M_n}}, f\right)_{H_1(G_m)} = O\left(\frac{1}{M_n}\right), \text{ as } n \to \infty
\]
and
\[
\|S_k (f) - f\|_1 \to 0 \text{ when } k \to \infty.
\]
4. Auxiliary propositions

Lemma 1. Suppose that an operator $T$ is sublinear and for some $0 < p \leq 1$

$$\int |T a|^p d\mu \leq c_p < \infty,$$

for every $p$-atom $a$, where $I$ denote the support of the atom. If $T$ is bounded from $L_\infty$ to $L_\infty$. Then

$$\|T f\|_p \leq c_p \|f\|_{H_p(G_m)}.$$

Lemma 2. Let $n \in \mathbb{N}$ and $x \in I_s \setminus I_{s+1}$, $0 \leq s \leq N - 1$. Then

$$\int_{I_N} |D_n (x - t)| d\mu (t) \leq \frac{cM_s}{M_N}.$$

5. Proof of the Theorems

Proof of Theorem 1. Since $\tilde{S}_p$ is bounded from $L_\infty(G_m)$ to $L_\infty(G_m)$ by Lemma 1 we obtain that the proof of theorem 1 will be complete, if we show that

$$\int_{I_N} \left| S_p a (x) \right|^p d\mu (x) \leq c < \infty, \text{ when } 0 < p \leq 1,$$

for every $p$-atom $a$, where $I$ denotes the support of the atom.

Let $a$ be an arbitrary $p$-atom with support $I$ and $\mu (I) = M_N$. We may assume that $I = I_N$. It is easy to see that $S_n (a) = 0$ when $n \leq M_N$. Therefore we can suppose that $n > M_N$.

Since $\|a\|_\infty \leq M_N^{1/p}$ we can write

$$|S_n (a)| \leq \int_{I_N} |a (t)||D_n (x - t)| d\mu (t)$$

$$\leq \|a\|_\infty \int_{I_N} |D_n (x - t)| d\mu (t) \leq M_N^{1/p} \int_{I_N} |D_n (x - t)| d\mu (t).$$

Let $0 < p < 1$ and $x \in I_s \setminus I_{s+1}$. From Lemma 2 we get

$$\frac{|S_n a (x)|}{\log^p (n + 1) (n + 1)^{1/p-1}} \leq \frac{cM_N^{1/p-1} M_s}{\log^p (n + 1) (n + 1)^{1/p-1}}.$$
Combining (3) and (9) we obtain

\[(10) \int_{I_N} \left| \sum_{s=0}^{N-1} S_p^s a(x) \right|^p d\mu(x) = \sum_{s=0}^{N-1} \int_{I_k \setminus I_{s+1}} \left| S_p^s a(x) \right|^p d\mu(x) \leq \frac{cM_N^{1-p}}{\log |p|^p (n+1)^{1-p}} \sum_{s=0}^{N-1} M_s^{p} \leq \frac{cM_N^{1-p} N^{|p|}}{\log |p|^p (n+1)^{1-p}} < c_p < \infty. \]

Let 0 < p < 1. Applying (S), (10) and Theorem W we have

\[(11) \sum_{k=M_N}^{\infty} \frac{\|S_k a\|_p^p}{k^{2-p}} \leq \sum_{k=M_N}^{\infty} \frac{1}{k} \int_{I_N} \left| S_k a(x) \right|^p \frac{1}{k^{1/p-1}} d\mu(x) + \sum_{k=M_N}^{\infty} \frac{M_N^{p}}{k^{2-p}} \int_{I_N} \left( \int_{I_N} |D_k(x-t)| d\mu(t) \right)^p d\mu(x) \leq c_p M_N^{1-p} \sum_{k=M_N}^{\infty} \frac{1}{k^{2-p}} + c_p M_N^{1-p} \sum_{k=M_N}^{\infty} \frac{\log^p k}{k^{2-p}} \leq c_p < \infty. \]

Which complete the proof of corollary 1. Let prove second part of Theorem 1. Let

\[f_{n_k}(x) = D_{M_2n_k+1}(x) - D_{M_{2n_k}}(x).\]

It is evident

\[\widehat{f_{n_k}}(i) = \begin{cases} 1, & \text{if } i = M_{2n_k}, \ldots, M_{2n_k+1} - 1 \\ 0, & \text{otherwise}. \end{cases} \]

Then we can write

\[(12) S_i f_{n_k}(x) = \begin{cases} D_{i}(x) - D_{M_{2n_k}}(x), & \text{if } i = M_{2n_k} + 1, \ldots, M_{2n_k+1} - 1, \\ f_{n_k}(x), & \text{if } i \geq M_{2n_k+1}, \\ 0, & \text{otherwise}. \end{cases} \]

From (11) we get

\[(13) \|f_{n_k}\|_{H_p(G_m)} = \|S_{M_n}(f_{n_k})\|_p = \left\|D_{M_{2n_k+1}} - D_{M_{2n_k}}\right\|_p \leq c_p M_{2n_k}^{1-1/p}. \]

Let 0 < p < 1. Under condition (7) there exists positive integers n_k such that

\[\lim_{k \to \infty} \frac{(M_{2n_k} + 2)^{1/p-1}}{\varphi(M_{2n_k} + 2)} = \infty, \quad 0 < p < 1.\]

Applying (4), (5) and (12) we can write

\[
\frac{|S_{M_{2n_k+1}} f_{n_k}|}{\varphi(M_{2n_k} + 2)} = \frac{|D_{M_{2n_k+1}} - D_{M_{2n_k}}|}{\varphi(M_{2n_k} + 2)} = \frac{|w_{M_{2n_k}}|}{\varphi(M_{2n_k} + 2)} = \frac{1}{\varphi(M_{2n_k} + 2)}. \]

Hence we can write:
(14) \[ \mu \left\{ x \in G_m : \left| \frac{S_{M2n_k+1}f_{n_k}(x)}{\varphi(M_{2n_k}+2)} \right| \geq \frac{1}{\varphi(M_{2n_k}+2)} \right\} = 1. \]

Combining (13) and (14) we have

\[ \frac{1}{\varphi(M_{2n_k}+2)} \left( \mu \left\{ x \in G_m : \left| \frac{S_{M2n_k+1}f_{n_k}(x)}{\varphi(M_{2n_k}+2)} \right| \geq \frac{1}{\varphi(M_{2n_k}+2)} \right\} \right)^{1/p} \leq \frac{1}{\varphi(M_{2n_k}+2)M_{2n_k}^{1-1/p}} \to \infty, \text{ when } k \to \infty. \]

Now consider the case when \( p = 1 \). Under condition (7) there exists \( \{n_k : k \geq 1\} \), such that

\[ \lim_{k \to \infty} \frac{\log q_{n_k}}{\varphi(q_{n_k})} = \infty. \]

Let \( q_{n_k} = M_{2n_k} + M_{2n_k-2} + M_2 + M_0 \) and \( x \in I_{2s} \setminus I_{2s+1}, s = 0, \ldots, n_k \).

Combining (14) and (5) we have

\[ D_{q_{n_k}}(x) \geq |D_{M_{2s}}(x)| - \sum_{l=0}^{s-2} r_{2l+1}^{m_{2l}} D_{M_2}(x) \geq M_{2s} - \sum_{l=0}^{s-2} M_{2l} \geq M_{2s} - M_{2s-1} \geq M_{2s}/2. \]

Hence

(15) \[ \int_{G_m} \left| D_{q_{n_k}}(x) \right| d\mu(x) \geq \frac{1}{2} \sum_{s=0}^{n_k} \int_{I_{2s} \setminus I_{2s+1}} M_{2s} d\mu(x) \geq c \sum_{s=0}^{n_k} 1 \geq cn_k. \]

From (12), (13) and (15) we have

\[ \frac{1}{\|f_{n_k}(x)\|_{H_1(G_m)}} \int_{G_m} \left| S_{q_{n_k}} f_{n_k}(x) \right| d\mu(x) \geq \frac{1}{\|f_{n_k}(x)\|_{H_1(G_m)}} \left( \int_{G_m} \left| D_{q_{n_k}}(x) \right| d\mu(x) - \int_{G_m} \left| D_{M_{2n_k}}(x) \right| d\mu(x) \right) \geq \frac{c}{\varphi(q_{n_k})} (\log q_{n_k} - 1) \geq \frac{c\log q_{n_k}}{\varphi(q_{n_k})} \to \infty, \text{ when } k \to \infty. \]

Which complete the proof of theorem 1.
Proof of Theorem 2. Let $0 < p \leq 1$ and $M_k < n \leq M_{k+1}$. Using Theorem 1 we have

$$\|S_n f\|_p \leq c_p n^{1/p - 1} \log^p[ n ] \|f\|_{H_p(G_m)}.$$ 

Hence

$$\|S_n f - f\|_p^p \leq \|S_n f - S_{M_k} f\|_p^p + \|S_{M_k} f - f\|_p^p = \|S_n (S_{M_k} f - f)\|_p^p$$

$$+ \|S_{M_k} f - f\|_p^p \leq c_p (n^{1-p} + 1) \log^p[ n ] \omega^p \left( \frac{1}{M_k}, f \right)_{H_p(G_m)}$$

and

$$\|S_n f - f\|_p \leq c_p n^{1/p - 1} \log^p[ n ] \omega \left( \frac{1}{M_k}, f \right)_{H_p(G_m)}.$$ 

Proof of Theorem 3. Let $0 < p < 1$, $f \in H_p(G_m)$ and

$$\omega \left( \frac{1}{M_{2n}}, f \right)_{H_p(G_m)} = o \left( \frac{1}{M_{2n}^{1/p - 1}} \right), \text{ as } n \to \infty.$$ 

Using (16) we immediately get

$$\|S_n f - f\|_p \to \infty, \text{ when } n \to \infty.$$ 

Let proof of second part of theorem 3. We set

$$a_k (x) = \frac{M_{2k}^{1/p - 1}}{\lambda} \left( D_{M_{2k+1}} (x) - D_{M_{2k}} (x) \right),$$

where $\lambda = \sup_{n \in \mathbb{N}} m_n$ and

$$f_A (x) = \sum_{i=0}^{A} \frac{\lambda}{M_{2i}^{1/p - 1}} a_i (x).$$

Since

$$S_{M_A} a_k (x) = \begin{cases} a_k (x), & 2k \leq A, \\ 0, & 2k > A, \end{cases}$$

and

$$\text{supp}(a_k) = I_{2k}, \quad \int_{I_{2k}} a_k \, d\mu = 0, \quad \|a_k\|_\infty \leq M_{2k}^{1/p - 1} \cdot M_{2k} = M_{2k}^{1/p} = (\text{supp } a_k)^{-1/p},$$

if we apply Theorem W we conclude that $f \in H_p$. 

It is easy to show that

\[(18) \quad f - S_{M_n}f = \left( f^{(1)} - S_{M_n}f^{(1)}, \ldots, f^{(n)} - S_{M_n}f^{(n)}, \ldots, f^{(n+k)} - S_{M_n}f^{(n+k)} \right)\]

\[= \left( 0, \ldots, 0, f^{(n+1)} - f^{(n)}, \ldots, f^{(n+k)} - f^{(n)}, \ldots \right)\]

\[= \left( 0, \ldots, 0, \sum_{i=n}^{k} \frac{a_i(x)}{M_i^{1/p-1}}, \ldots \right), \quad k \in \mathbb{N}_+\]

is martingale. Using (18) we get

\[\omega(\frac{1}{M_n}, f)_{H_p} \leq \sum_{i=\lfloor n/2 \rfloor + 1}^{\infty} \frac{1}{M_{2i}^{1/p-1}} = O \left( \frac{1}{M_n^{1/p-1}} \right).\]

where \([n/2]\) denotes integer part of \(n/2\). It is easy to show that

\[(19) \quad \hat{f}(j) = \begin{cases} 1, & \text{if } j \in \{M_{2i}, \ldots, M_{2i+1} - 1\}, \quad i = 0, 1, \ldots \\ 0, & \text{if } j \notin \bigcup_{i=0}^{\infty} \{M_{2i}, \ldots, M_{2i+1} - 1\}. \end{cases}\]

Using (19) we have

\[
\limsup_{k \to \infty} \|f - S_{M_{2k+1}}(f)\|_{L_p, \infty(G_m)} \\
\geq \limsup_{k \to \infty} \left( \|w_{M_{2k+1}}\|_{L_p, \infty(G_m)} - \sum_{i=k+1}^{\infty} (D_{M_{2i+1}} - D_{M_{2i}}) \|L_p, \infty(G_m)\right) \\
\geq \limsup_{k \to \infty} \left( 1 - c/M_{2k}^{1/p-1} \right) c > 0.
\]

Which complete the proof of Theorem 3.

Proof of Theorem 4. Analogously we can prove first part of Theorem 4. Let proof it’s second part. We set

\[a_i(x) = D_{M_{2i}+1}(x) - D_{M_{2i}}(x)\]

and

\[f_A(x) = \sum_{i=1}^{A} \frac{a_i(x)}{M_i}.\]

Since

\[(20) \quad S_M a_k(x) = \begin{cases} a_k(x), & 2M_k \leq A, \\ 0, & 2M_k > A, \end{cases}\]

and

\[\text{supp}(a_k) = I_{2M_k}, \quad \int_{I_{2M_k}} a_k d\mu = 0, \quad \|a_k\|_{\infty} \leq M_{2M_k} = \mu(\text{supp } a_k),\]
if we apply Theorem W we conclude that $f \in H_1$.

It is easy to show that

$$(21) \quad \omega\left(\frac{1}{M_n}, f\right)_{H_1(G_m)} \leq \sum_{i=\lfloor \lg n/2 \rfloor}^{\infty} \frac{1}{M_i} = O\left(\frac{1}{n}\right),$$

where $\lfloor \lg n/2 \rfloor$ denotes integer part of $\lg n/2$. By simple calculation we get

$$(22) \quad \hat{f}(j) = \begin{cases} \frac{1}{M_{2i}}, & \text{if } j \in \{M_{2M_i}, \ldots, M_{2M_i+1} - 1\}, \; i = 0, 1, \ldots \\ 0, & \text{if } j \notin \bigcup_{i=0}^{\infty} \{M_{2M_i}, \ldots, M_{2M_i+1} - 1\}. \end{cases}$$

Combining (15) and (22) we have

$$\limsup_{k \to \infty} \|f - S_{QM_k}(f)\|_1 \geq \limsup_{k \to \infty} \left( \frac{1}{M_{2k}} \|D_{QM_k}\|_1 - \frac{1}{M_{2k}} \|D_{2M_{M_k}}\|_1 - \sum_{i=k+1}^{\infty} \frac{D_{M_{2M_k+1}} - D_{M_{2M_i}}}{M_{2i}} \right)$$

$$\geq \limsup_{k \to \infty} \left( c - \sum_{i=k+1}^{\infty} \frac{1}{M_{2i}} - \frac{1}{M_{2k}} \right) \geq c > 0.$$

Theorem 4 is proved.

REFERENCES

[1] G. N. Agaev, N. Ya. Vilenkin, G. M. Dzafary and A. I. Rubinshtein, Multiplicative systems of functions and harmonic analysis on zero-dimensional groups, Baku, Ehim, 1981 (in Russian).

[2] I. Blahota, Acta Acad. Paed. Nyiregyhaziensis, Approximation by Vilenkin-Fourier sums in $L_p(G_m)$. 13(1992), 35-39.

[3] N. I. Fine, On Walsh function, Trans. Amer. Math. Soc. 65 (1949), 372-414.

[4] S. Fridli, Approximation by Vilenkin-Fourier series, Acta Math. Hungarica 47 (1-2), (1986), 33-44.

[5] G. Gát, Investigations of certain operators with respect to the Vilenkin sistem, Acta Math. Hung., 61 (1993), 131-149.

[6] G. Gát, Best approximation by Vilenkin-Like systems, Acta Acad. Paed. Nyiregyhaziensis, 17(2001), 161-169.

[7] U. Goginava, G. Tkebuchava, Convergence of subsequence of partial sums and logarithmic means of Walsh-Fourier series, Acta Sci. Math (Szeged) 72 (2006), 159-177.

[8] U. Goginava, On Uniform convergence of Walsh-Fourier series, Acta Math. Hungar. 93 (1-2) (2001), 59-70.

[9] U. Goginava, On approximation properties of partial sums of Walsh-Fourier series, Acta Sci. Math. (Szeged), 72 (2006), 569-579.

[10] U. Goginava, L. D. Gogoladze, Strong Convergence of Cubic Partial Sums of Two-Dimensional Walsh-Fourier series, Constructive Theory of Functions, Sozopol 2010: In memory of Borislav Bojanov. Prof. Marin Drinov Academic Publishing House, Sofia, 2012, pp. 108-117.

[11] L. D. Gogoladze, On the strong summability of Fourier series, Bull of Acad. Scie. Georgian SSR, 52, 2 (1968), 287-292.
[12] B. I. Golubov, A. V. Efimov and V. A. Skvortsov, Walsh series and transforms. (Russian) Nauka, Moscow, 1987, English transl, Mathematics and its Applications (Soviet Series), 64. Kluwer Academic Publishers Group, Dordrecht, 1991.

[13] N. V. Gulićev, Approximation to continuous functions by Walsh-Fourier series, Analysis Math. 6(1980), 269-280.

[14] P. Simon, Strong convergence of certain means with respect to the Walsh-Fourier series, Acta Math. Hung., 49 (1-2) (1987), 425-431.

[15] P. Simon, Strong convergence Theorem for Vilenkin-Fourier Series. Journal of Mathematical Analysis and Applications, 245, (2000), pp. 52-68.

[16] B. Smith, A strong convergence theorem for $H_1(T)$, in Lecture Notes in Math., 995, Springer, Berlin, (1994), 169-173.

[17] G. Tephnadze, A note on the Vilenkin-Fourier coefficients, Georgian Mathematical Journal, (to appear).

[18] G. Tephnadze, A note on the Fourier coefficients and partial sums of Vilenkin-Fourier series, Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis (AMAPN), (to appear).

[19] G. Tephnadze, Strong convergence of two-dimensional Walsh-Fourier series, Ukrainian Mathematical Journal (UMJ), (to appear).

[20] N. Ya. Vilenkin, A class of complete orthonormal systems, Izv. Akad. Nauk. U.S.S.R., Ser. Mat., 11 (1947), 363-400.

[21] F. Weisz, Martingale Hardy spaces and their applications in Fourier Analysis, Springer, Berlin-Heidelberg-New York, 1994.

[22] F. Weisz, Hardy spaces and Cesáro means of two-dimensional Fourier series, Bolyai Soc. math. Studies, (1996), 353-367.

[23] F. Weisz, Strong convergence theorems for two-parameter Walsh-Fourier and trigonometric-Fourier series. (English) Stud. Math. 117, No.2, (1996), 173-194.

[24] F. Weisz, Hardy spaces and Cesáro means of two-dimensional Fourier series, Bolyai Soc. math. Studies, (1996), 353-367.

G. Tephnadze, Department of Mathematics, Faculty of Exact and Natural Sciences, Tbilisi State University, Chavchavadze str. 1, Tbilisi 0128, Georgia

E-mail address: giorgitephnadze@gmail.com