THE ANALOGUE OF THE FLOQUET-LYAPUNOV THEOREM FOR
THE LINEAR DIFFERENTIAL SYSTEMS OF THE SPECIAL KIND

The analogue of the well known in the theory of the linear differential systems Floquet’s–Lyapunov’s theorem are constructed by the certain conditions for the linear differential system, whose coefficients are represented as an absolutely and uniformly convergent Fourier-series with slowly varying coefficients and frequency.

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Introduction. In the theory of linear systems of differential equations is well known the Floquet–Lyapunov theorem [1]. The fundamental matrix $X(t)$ of the linear homogeneous system

$$\frac{dx}{dt} = A(t)x, \quad t \in \mathbb{R},$$

where $A(t)$ – is a continuous $T$-periodic matrix, has a kind:

$$X(t) = F(t) e^{tK},$$

where $F(t)$ – is a $T$-periodic matrix, and $K$ – is a constant matrix.

There exists many analogues of this theorem for the linear systems of different types, for example, for the systems with quasiperiodic coefficients [2], for the countable systems of differential equations [3], for the differential equations in the Banach spaces [4] and other.

The purpose of this paper is to obtain of analogue of Floquet-Lyapunov theorem for the linear systems of differential equations whose coefficients are represented as an absolutely and uniformly convergent Fourier-series with slowly varying coefficients and frequency. Here we make substantial use of the results of our paper [5].

Notation. Let $G(\varepsilon_0) = \{t, \varepsilon : 0 < \varepsilon < \varepsilon_0, -L\varepsilon^{-1} \leq t \leq L\varepsilon^{-1}, 0 < L < +\infty\}$.

Definition 1. We say, that a function $p(t, \varepsilon)$ belong to class $S(m; \varepsilon_0)$

$$\|p\|_{S(m; \varepsilon_0)} \overset{def}{=} \sum_{k=0}^{m} \sup_{G(\varepsilon_0)} |p_k^*(t, \varepsilon)| < +\infty.$$
**Definition 2.** We say, that a function \( f(t, \varepsilon, \theta(t, \varepsilon)) \) belong to class \( F(m; \varepsilon_0; \theta) \) \((m \in \mathbb{N} \cup \{0\})\), if this function can be represented as:

\[
 f(t, \varepsilon, \theta(t, \varepsilon)) = \sum_{n=-\infty}^{\infty} f_n(t, \varepsilon) \exp (i n \theta(t, \varepsilon)),
\]

and:

1) \( f_n(t, \varepsilon) \in S(m; \varepsilon_0) \);
2) \[
 \|f\|_{F(m; \varepsilon_0; \theta)} \overset{\text{def}}{=} \sum_{n=-\infty}^{\infty} \|f_n\|_{S(m; \varepsilon_0)} < +\infty,
\]
3) \( \theta(t, \varepsilon) = \int_0^t \varphi(\tau, \varepsilon) d\tau, \varphi(t, \varepsilon) \in S(m; \varepsilon_0), \inf_{G(\varepsilon_0)} \varphi(t, \varepsilon) = \varphi_0 > 0. \)

State some properties of functions of classes \( S(m; \varepsilon_0) \), \( F(m; \varepsilon_0; \theta) \) (the proofs are given in [6]). Let \( k = \text{const}, p, q \in S(m; \varepsilon_0), u, v \in F(m; \varepsilon_0; \theta) \). Then \( kp, p \pm q, pq \) belong to class \( S(m; \varepsilon_0) \), \( ku, u \pm v, uv \) belong to class \( F(m; \varepsilon_0; \theta) \), and

1) \( \|kp\|_{S(m; \varepsilon_0)} = |k| \cdot \|p\|_{S(m; \varepsilon_0)} \);
2) \( \|p \pm q\|_{S(m; \varepsilon_0)} \leq \|p\|_{S(m; \varepsilon_0)} + \|q\|_{S(m; \varepsilon_0)} \);
3) \( \|pq\|_{S(m; \varepsilon_0)} \leq 2^m \|p\|_{S(m; \varepsilon_0)} \|q\|_{S(m; \varepsilon_0)} \);
4) \( \|ku\|_{F(m; \varepsilon_0; \theta)} = |k| \cdot \|u\|_{F(m; \varepsilon_0; \theta)} \);
5) \( \|u \pm v\|_{F(m; \varepsilon_0; \theta)} \leq \|u\|_{F(m; \varepsilon_0; \theta)} + \|v\|_{F(m; \varepsilon_0; \theta)} \);
6) \( \|uv\|_{F(m; \varepsilon_0; \theta)} \leq 2^n \|u\|_{F(m; \varepsilon_0; \theta)} \|v\|_{F(m; \varepsilon_0; \theta)} \);
7) let \( u \in F(m; \varepsilon_0; \theta) \), and the function \( f(t, \varepsilon, \theta, u) \) belongs to class \( F(m; \varepsilon_0; \theta) \) with respect \( t, \varepsilon, \theta \) and analytic with respect \( u \), if \( |u| < r \), means

\[
 f(t, \varepsilon, \theta, u) = \sum_{k=0}^{\infty} f_k(t, \varepsilon, \theta) u^k,
\]

where \( f_k(t, \varepsilon, \theta) \in F(m; \varepsilon_0; \theta) \). Then by the condition

\[
 2^n \|u\|_{F(m; \varepsilon_0; \theta)} \leq r_0 < r,
\]

the function \( f(t, \varepsilon, \theta, u) \) belongs to class \( F(m; \varepsilon_0; \theta) \), and

\[
 \|f(t, \varepsilon, \theta, u)\|_{F(m; \varepsilon_0; \theta)} \leq \sum_{k=0}^{\infty} \|f_k(t, \varepsilon, \theta)\|_{F(m; \varepsilon_0; \theta)} r_0^k.
\]

Particularly, all polynomials with respect \( u \) with the coefficients from \( F(m; \varepsilon_0; \theta) \), functions \( \exp u, \sin u, \cos u \) belongs to class \( F(m; \varepsilon_0; \theta) \). For function \( \exp u \) we have:

\[
 \| \exp u \|_{F(m; \varepsilon_0; \theta)} \leq 2^{-m} \exp (2^n \|u\|_{F(m; \varepsilon_0; \theta)}) .
\]

**Statement of the Problem.** We consider the next system of differential equations:

\[
 \frac{dx_j}{dt} = \lambda_j(t, \varepsilon)x_j + \mu \sum_{k=1}^{N} p_{jk}(t, \varepsilon, \theta)x_k, \quad j = 1, N, \tag{3}
\]
The analogue of the Floquet-Lyapunov theorem

where \( \lambda_j(t, \varepsilon) \in S(m; \varepsilon_0), p_{jk}(t, \varepsilon, \theta) \in F(m; \varepsilon_0; \theta) \) \((j, k = 1, N)\), \( \mu \in (0, \mu_0) \subset \mathbb{R}^+ \).

We study the problem about the structure of fundamental system of solutions \( x_{jk}(t, \varepsilon, \mu) \) \((j, k = 1, N)\) of system (3).

**Auxiliary arguments.**

**Lemma 1.** Let the function

\[
f(t, \varepsilon, \theta(t, \varepsilon)) = \sum_{n=-\infty}^{\infty} f_n(t, \varepsilon) \exp(i n \theta(t, \varepsilon))
\]

belongs to class \( F(m-1; \varepsilon_0; \theta) \). Then the function

\[
x(t, \varepsilon, \theta(t, \varepsilon)) = \varepsilon \int_0^t f(\tau, \varepsilon, \theta(\tau, \varepsilon)) d\tau
\]

belongs to class \( F(m-1; \varepsilon_0; \theta) \) also, and there exists \( K_1 \in (0, +\infty) \), that does not depend on the function \( f \), such, that

\[
\|x(t, \varepsilon, \theta)\|_{F(m-1; \varepsilon_0; \theta)} \leq K_1 \|f(t, \varepsilon, \theta)\|_{F(m-1; \varepsilon_0; \theta)}.
\]

The proof are given in the paper [5].

**Lemma 2.** Let we have the linear nonhomogeneous first-order differential equation:

\[
\frac{dx}{dt} = \lambda(t, \varepsilon)x + \varepsilon u(t, \varepsilon, \theta(t, \varepsilon)),
\]

where \( \lambda(t, \varepsilon) \in S(m; \varepsilon_0), u(t, \varepsilon, \theta) \in F(m-1; \varepsilon_0; \theta) \). Let holds the condition \(|\Re(\lambda(t, \varepsilon))| \geq \gamma_0 > 0\). Then equation (4) has a particularly solution \( x(t, \varepsilon, \theta(t, \varepsilon)) \in F(m-1; \varepsilon_0; \theta) \), and there exists \( K_2 \in (0, +\infty) \), that does not depend on the function \( u(t, \varepsilon, \theta) \), such that

\[
\|x(t, \varepsilon, \theta)\|_{F(m-1; \varepsilon_0; \theta)} \leq K_2 \frac{\|u(t, \varepsilon, \theta)\|_{F(m-1; \varepsilon_0; \theta)}}{\gamma_0}.
\]

The proof are given in the paper [7].

**Lemma 3.** Let the system (3) such, that

\[
|\Re(\lambda_j(t, \varepsilon) - \lambda_k(t, \varepsilon))| \geq \gamma_1 > 0, \ (j \neq k).
\]

Then there exists \( \mu_1 \in (0, \mu_0) \), such, that for all \( \mu \in (0, \mu_1) \) there exists the Lyapunov’s transformation of kind

\[ x_j = y_j + \mu \sum_{k=1}^{N} \psi_{jk}(t, \varepsilon, \theta, \mu) y_k, \ j = 1, N, \]

where \( \psi_{jk} \in F(m-1; \varepsilon_0; \theta) \), reducing the system (3) to kind:

\[
\frac{dy_j}{dt} = (\lambda_j(t, \varepsilon) + \mu u_j(t, \varepsilon, \mu)) y_j + \mu \varepsilon \sum_{k=1}^{N} v_{jk}(t, \varepsilon, \theta, \mu) y_k, \ j = 1, N,
\]
where \( u_j \in S(m; \varepsilon_0) \), \( v_{jk} \in F(m - 1; \varepsilon_0; \theta) \) \((j, k = 1, N)\).

The proof are given in the paper [5].

**Lemma 4.** Let holds the condition (6). Then there exists \( \mu_2 \in (0, \mu_1) \) \((\mu_1 \text{ are defined in Lemma 3})\) such, that for all \( \mu \in (0, \mu_2) \) there exists the Lyapunov’s transformaion of kind

\[
y_j = z_j + \mu \sum_{k=1}^{N} q_{jk}(t, \varepsilon, \theta, \mu)z_k, \ j = 1, N,
\]

where \( q_{jk} \in F(m - 1; \varepsilon_0, \theta) \), reducing the system (8) to the pure diagonal form:

\[
\frac{dz_j}{dt} = d_j(t, \varepsilon, \theta, \mu)z_j, \ j = 1, N,
\]

where

\[
d_j = \lambda_j(t, \varepsilon) + \mu u_j(t, \varepsilon, \mu) + \mu \varepsilon v_{jj}(t, \varepsilon, \theta, \mu) +
+ \mu \varepsilon \sum_{k=1}^{N} v_{jk}(t, \varepsilon, \theta, \mu)q_{kj}(t, \varepsilon, \theta, \mu), \ j = 1, N.
\]

**Proof.** We make in the system (8) the substitution (9) and using the condition of diagonality of transformed system. We obtain the next system of differential equations for coefficients \( q_{jk} \):

\[
\frac{dq_{jk}}{dt} = (\lambda_j(t, \varepsilon) - \lambda_k(t, \varepsilon) + \mu (u_j(t, \varepsilon, \mu) - u_k(t, \varepsilon, \mu))) q_{jk} +
+ \mu \varepsilon (v_{jj}(t, \varepsilon, \theta, \mu) - v_{kk}(t, \varepsilon, \theta, \mu)) q_{jk} + \varepsilon v_{jk}(t, \varepsilon, \theta, \mu) +
+ \mu \varepsilon \sum_{s=1}^{N} v_{js}(t, \varepsilon, \theta, \mu)q_{sk} - \mu^2 \varepsilon q_{jk} \sum_{s=1}^{N} v_{ks}(t, \varepsilon, \theta, \mu)q_{sk},
\]

\[
j, k = 1, N, j \neq k.
\]

It is easy to see, that the system (12) is divided into \( N \) independent \((N - 1)\)-order subystems.

Together with the system (12) we consider the linear nonhomogeneous system:

\[
\frac{dq_{jk}^{(0)}}{dt} = (\lambda_j(t, \varepsilon) - \lambda_k(t, \varepsilon) + \mu (u_j(t, \varepsilon, \mu) - u_k(t, \varepsilon, \mu))) q_{jk}^{(0)} +
+ \varepsilon v_{jk}(t, \varepsilon, \theta, \mu), \ j, k = 1, N, j \neq k.
\]

We denote \( u^*(\mu) = \max_{j,k} ||u_j(t, \varepsilon, \mu) - u_k(t, \varepsilon, \mu)||_{S(m; \varepsilon_0)} \). We choose a parameter \( \mu \) so small that \( \mu u^*(\mu) < \gamma_1 \). Then

\[
|\text{Re} (\lambda_j(t, \varepsilon) - \lambda_k(t, \varepsilon) + \mu (u_j(t, \varepsilon, \mu) - u_k(t, \varepsilon, \mu)))| \geq \gamma_1 - \mu u^*(\mu) > 0.
\]

The system (13) is a set of \( N(N - 1) \) independent linear nonhomogeneous equations each of which has form (4). By virtue inequality (14) each of which these
equations is satisfied to conditions of Lemma 2. Therefore the system (13) has a particular solution \( q_{jk}^{(0)}(t, \varepsilon, \theta, \mu) \in F(m - 1; \varepsilon_0; \theta) \), and there exists \( K_3 \in (0, +\infty) \) such that

\[
\|q_{jk}^{(0)}\|_{F(m-1;\varepsilon_0;\theta)} \leq \frac{K_3}{\gamma_1 - \mu u^*(\mu)} \|q_{jk}^{(0)}\|_{F(m-1;\varepsilon_0;\theta)} \quad (j, k = 1, N; \ j \neq k).
\]

We seek the solution from class \( F(m - 1; \varepsilon_0; \theta) \) of system (12) by the method of successive approximations, defining the initial approximation \( q_{jk}^{(0)}(t, \varepsilon, \theta, \mu) \) \( (j, k = 1, N; \ j \neq k) \), and the subsequent approximations defining as solutions from class \( F(m - 1; \varepsilon_0; \theta) \) of the linear nonhomogeneous systems:

\[
\frac{d q_{jk}^{(\nu+1)}}{dt} = (\lambda_j(t, \varepsilon) - \lambda_k(t, \varepsilon) + \mu(u_j(t, \varepsilon, \mu) - u_k(t, \varepsilon, \mu))) q_{jk}^{(\nu+1)} + \\
+ \mu \varepsilon (v_{jk}(t, \varepsilon, \theta, \mu) - v_{kk}(t, \varepsilon, \theta, \mu)) q_{jk}^{(\nu)} + \varepsilon v_{jk}(t, \varepsilon, \theta, \mu) + \\
+ \mu \varepsilon \sum_{s=1}^{\nu} v_{js}(t, \varepsilon, \theta, \mu) q_{sk}^{(\nu)} - \mu^2 \varepsilon q_{jk}^{(\nu)} \sum_{s=1}^{\nu} v_{ks}(t, \varepsilon, \theta, \mu) q_{sk},
\]

\( j, k = 1, N; \ j \neq k; \nu = 0, 1, 2, \ldots . \) \hspace{1cm} (16)

We denote \( V = \max_{j,k} \|v_{jk}\|_{F(m-1;\varepsilon_0;\theta)} \). Then by (15):

\[
\|q_{jk}^{(0)}\|_{F(m-1;\varepsilon_0;\theta)} \leq \frac{K_3 V}{\gamma_1 - \mu u^*(\mu)} \quad (j, k = 1, N; \ j \neq k).
\]

We denote

\[\Omega = \left\{ q_{jk} \in F(m - 1; \varepsilon_0; \theta) : \|q_{jk} - q_{jk}^{(0)}\|_{F(m-1;\varepsilon_0;\theta)} \leq \rho \right\},\]

where \( \rho \in (0, +\infty) \).

Using techniques contraction mapping principle [8] it is easy to show that for sufficiently small values of \( \mu \) all approximations \( q_{jk}^{(\nu)} \ (\nu = 0, 1, 2, \ldots ) \) belongs to \( \Omega \). And process (16) is convergent to solution from class \( F(m - 1; \varepsilon_0; \theta) \) of the system (12).

Lemma 4 are proved.

**Main Results.**

**Theorem.** Let for the system (3) the condition (6) is holds. Then there exists \( \mu_3 \in (0, \mu_0) \) such, that for all \( \mu \in (0, \mu_3) \) the system (3) has a fundamental system of solutions of kind:

\[
x_{jk} = r_{jk}(t, \varepsilon, \theta, \mu) \exp \left( \int_{0}^{t} \sigma_j(s, \varepsilon, \mu) ds \right), \ j, k = 1, N
\]

\( (j - the \ number \ of \ solution, \ k - the \ number \ of \ component) \), where \( r_{jk}(t, \varepsilon, \theta, \mu) \in F(m - 1; \varepsilon_0; \theta), \sigma_j(t, \varepsilon, \mu) \in S(m - 1; \varepsilon_0) \).
Proof. The fundamental system of solutions (FSS) of the system (10) has a kind:

\[ z_{jk} = \delta^k_j \exp \left( \int_0^t d_j(s, \varepsilon, \theta(s, \varepsilon), \mu) ds \right), \quad j, k = 1, N \]  

(19)

\((j - \) the number of solution, \(k - \) the number of component, \(\delta^k_j - \) the symbol of Kronecker). By virtue (9) FSS of system (8) has a kind:

\[ y_{jk} = \bar{q}_{jk}(t, \varepsilon, \theta, \mu) \exp \left( \int_0^t d_j(s, \varepsilon, \theta(s, \varepsilon), \mu) ds \right), \quad j, k = 1, N \]  

(20)

where \(\bar{q}_{jk} = \delta^k_j + (1 - \delta^k_j)\mu q_{jk} \) (\(j - \) the number of solution, \(k - \) the number of component). By virtue (7) FSS of system (3) has a kind:

\[ x_{jk} = \left( \sum_{l=1}^N \bar{\psi}_{kl}(t, \varepsilon, \mu) \bar{q}_{lj}(t, \varepsilon, \theta, \mu) \right) \exp \left( \int_0^t d_j(s, \varepsilon, \theta(s, \varepsilon), \mu) ds \right), \quad j, k = 1, N \]  

(21)

where \(\bar{\psi}_{jk} = \delta^k_j + \mu \psi_{jk} \) (\(\psi_{jk} - \) are defined in Lemma 3).

Consider:

\[ \int_0^t d_j(s, \varepsilon, \theta(s, \varepsilon), \mu) ds = \int_0^t \lambda_j(s, \varepsilon) + \mu_j(s, \varepsilon, \mu) ds + \mu \varepsilon \int_0^t w_j(s, \varepsilon, \theta(s, \varepsilon), \mu) ds, \]

where \(w_j = v_{jj} + \sum_{k=1}^N \psi_{jk} q_{kj} \epsilon \in F(m - 1; \varepsilon_0; \theta)\). We represent the functions \(w_j\) as \(w_j = w_j^*(t, \varepsilon, \mu) + \tilde{w}_j(t, \varepsilon, \theta, \mu)\), where

\[ w_j^*(t, \varepsilon, \mu) = w_j(t, \varepsilon, \theta, \mu) = \frac{1}{2\pi} \int_0^{2\pi} w_j(t, \varepsilon, \theta, \mu) d\theta \in S(m - 1; \varepsilon_0). \]

Accordingly \(\tilde{w}_j \in F(m - 1; \varepsilon_0; \theta)\), and \(\tilde{w}_j(t, \varepsilon, \theta, \mu) = 0\). Then

\[ \exp \left( \int_0^t d_j(s, \varepsilon, \theta(s, \varepsilon), \mu) ds \right) = \exp \left( \int_0^t \left( \lambda_j(s, \varepsilon) + \mu u_j(s, \varepsilon, \mu) + \mu \varepsilon w_j^*(s, \varepsilon, \mu) \right) ds \right) \times \exp \left( \mu \varepsilon \int_0^t \tilde{w}_j(s, \varepsilon, \theta(s, \varepsilon), \mu) ds \right). \]  

(22)

By virtue Lemma 1 we concluding, that

\[ \varepsilon \int_0^t \tilde{w}_j(s, \varepsilon, \theta(s, \varepsilon), \mu) ds \epsilon \in F(m - 1; \varepsilon_0; \theta) \quad (j = 1, N). \]
It follows by virtue the property 7) of functions from class $F(m; \varepsilon_0; \theta)$, that

\[ g_j(t, \varepsilon, \theta, \mu) = \exp \left( \mu \varepsilon \int_0^t \tilde{w}_j(s, \varepsilon, \theta(s, \varepsilon), \mu) ds \right) \in F(m - 1; \varepsilon_0; \theta) \quad (j = 1, N). \quad (23) \]

By virtue (21), (22), (23) we obtain the statement of the Theorem.

Obviously, the formula (18) is an analogue of Floquet-Lyapunov theorem for the systems of kind (3).

**Conclusion.** Thus, the analogue of the Floquet-Lyapunov theorem, well known in the theory of linear homogeneous systems of the differential equations, are obtained for the linear homogeneous systems, whose coefficients are represented as an absolutely and uniformly convergent Fourier-series with slowly varying coefficients and frequency.

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Щоголев С. А.
Аналог теоремы Флоке—Ляпунова для линейных дифференциальных систем специального вида

Резюме
Аналог хорошо известной в теории линейных дифференциальных систем с периодическими коэффициентами теоремы Флоке—Ляпунова построен при определённых условиях для линейной дифференциальной системы, коэффициенты которой представлены абсолютно и равномерно сходящимися рядами Фурье с медленно меняющимися коэффициентами и частотой.

Ключевые слова: линейные дифференциальные системы, ряды Фурье, медленно меняющиеся параметры.