DETERMINANTS OF REPRESENTATIONS OF
COXETER GROUPS

DEBARUN GHOSH AND STEVEN SPALLONE

Abstract. In [APS17], the authors characterize the partitions of \( n \) whose corresponding representations of \( S_n \) have nontrivial determinant. The present paper extends this work to all irreducible finite Coxeter groups \( W \). Namely, given a nontrivial multiplicative character \( \omega \) of \( W \), we give a closed formula for the number of irreducible representations of \( W \) with determinant \( \omega \). For Coxeter groups of type \( B_n \) and \( D_n \), this is accomplished by characterizing the bipartitions associated to such representations.

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1. Introduction

The tools of this paper, and [APS17], have their genesis in the paper [Mac71] of Macdonald, who developed the arithmetic of partitions to give a closed formula for the number \( A(n) \) of odd-dimensional Specht modules (irreducible representations) of \( S_n \). Briefly, if \( n \) is expressed in binary as a sum of powers of 2, then \( A(n) \) is the product of those powers of 2. This formula comes from characterizing the 2-core tower of “odd” partitions; we review this notion in Section 5.1. In [APS17], the authors describe a simple way to read off the determinant of a representation of \( S_n \) from its 2-core tower, and in particular give a closed formula for the number \( B(n) \) of Specht modules whose determinant is the sign character. Since the number \( p(n) \) of partitions of \( n \) itself does not have
a nice closed formula, it is remarkable that such formulas for \(A(n)\) and \(B(n)\) exist.

Most of this paper is devoted to the Coxeter group of type \(B_n\), which we write as \(B_n\) for simplicity. These are the “hyperoctahedral groups”. Here \(B_n\) is the wreath product of \(\mathbb{Z}/2\mathbb{Z}\) by \(S_n\). There are four multiplicative characters of \(B_n\), which we denote by \(\omega = 1, \text{sgn}^0, \varepsilon,\) and \(\text{sgn}^1\), described as follows. The character \(\text{sgn}^0\) is the composition of the projection to \(S_n\) with \(\text{sgn}\), the character \(\varepsilon\) is \((-1)^m\), where \(m\) is the sum of the entries of the \((\mathbb{Z}/2\mathbb{Z})^n\)-factor, and finally \(\text{sgn}^1 = \varepsilon \cdot \text{sgn}^0\).

Irreducible representations of \(B_n\), written \(\rho_{\alpha\beta}\), are parametrized by pairs \((\alpha, \beta)\) of partitions with \(|\alpha| + |\beta| = n\), called bipartitions of \(n\). The representations are induced by certain “Young Subgroups” \(B_a \times B_b \leq B_n\). For each \(\omega\), we characterize the 2-core towers of \(\alpha, \beta\) so that \(\det \rho_{\alpha\beta} = \omega\). For \(\omega \neq 1\), this leads to closed formulas for \(N_\omega(n) = \# \{(\alpha, \beta) : |\alpha| + |\beta| = n, \det \rho_{\alpha\beta} = \omega\}\).

Moreover we prove:

**Theorem 1.** Let \(n \geq 10\).

1. If \(n\) is odd, then \(N_\varepsilon(n) = N_{\text{sgn}^1}(n) < N_{\text{sgn}^0}(n) < N_1(n)\).
2. If \(n\) is even, then \(N_\varepsilon(n) = N_{\text{sgn}^0}(n) < N_{\text{sgn}^1}(n) < N_1(n)\).

In Section 2, after establishing notations, we give formulas for the determinant of a representation of a Coxeter group in terms of the character. In Section 3, for completeness, we indicate the solution to the determinant problem for dihedral groups. Section 4 reviews the work of [Mac71] and [APS17] for \(S_n\), and develops it further for application to the hyperoctahedral case.

Most of the work is in Section 5, where we first compute \(\det \rho_{\alpha\beta}\) in terms of known quantities (‘\(f_\alpha\)’ the degree of the Specht module \(\rho_\alpha\), and ‘\(g_\alpha\)’ the multiplicity of \(-1\) as an eigenvalue of \(\rho_\alpha((12))\), and similarly for \(\beta\)). We use a well-known formula for the determinant of an induced representation. We then provide a table and a logplot illustrating the values of \(N_\omega(n)\) for small \(n\). This is followed by a conceptual proof of the identity \(N_{\text{sgn}^1}(n) = N_\varepsilon(n)\) when \(n \geq 3\) is odd. No such “pure thought” proof is apparent for the other identity \(N_{\text{sgn}^0}(n) = N_\varepsilon(n)\) when \(n\) is even. Next, we calculate each

\[ N_\omega(a, b) = \{(\alpha, \beta) : |\alpha| = a, |\beta| = b, \det \rho_{\alpha\beta} = \omega\}. \]

In Section 5.6, we compute \(N_\omega(n) = \sum_{a+b=n} N_\omega(a, b)\). Considerable work goes into simplifying these sums to arrive at closed formulas. These final formulas take the form

\[ N_\omega(n) = A(2n) \times f_\omega(k), \]
where \( f_\omega(k) \) is a function only depending on \( k = \text{ord}_2(n) \), for \( n \) even, and similarly in terms of \( \text{ord}_2(n - 1) \) when \( n \) is odd.

Next we apply Clifford Theory to the Coxeter group of type \( D_n \). This group is the kernel of \( \varepsilon \) and we denote it by \( \mathbb{D}_n \) (not to be confused with the dihedral groups, of type \( I_2(n) \)). This group has two multiplicative characters, the nontrivial one being the restriction of \( \text{sgn}^0 \), which we denote as ‘\( \text{sgn} \)’. We give a closed formula for \( N'_\text{sgn}(n) \), the number of irreducible representations of \( \mathbb{D}_n \) with determinant \( \text{sgn} \), in terms of our earlier formulas for \( N_\omega(n) \).

We similarly treat the remaining (exceptional) types of finite irreducible Coxeter groups. Given the formulas of Section 2, this comes to a finite calculation with the character tables of \([GP00]\).

**Acknowledgements:** We would like to thank Dipendra Prasad and Amritanshu Prasad for their interest and useful conversations. This research was driven by computer exploration using the open-source mathematical software Sage \([TSD15]\) and its algebraic combinatorics features developed by the Sage-Combinat community \([TSCc08]\).

2. **Notation and Preliminaries**

2.1. **Binary Notation**

The nonnegative integer \( n \) and its binary digits play a substantial role in this paper, so it is convenient to fix notation here. Put

\[
n = \epsilon + 2^{k_1} + \cdots + 2^{k_r},
\]

with \( \epsilon \) either 0 or 1, and \( 1 \leq k_1 < k_2 < \cdots < k_r \). We will sometimes write \( k = k_1 \), and \( n' = n - \epsilon \). Let \( \nu(n) \) denote the number of 1’s in the binary expansion of \( n \); in the above \( \nu(n) = r + \epsilon \).

Write \( \text{bin}(n) \) for the set of powers of 2 in the binary expansion of \( n \); in the above \( \text{bin}(n) = \{k_1, \ldots, k_r\} \), together with 0 if \( n \) is odd.

Given a natural number \( a \), write \( \text{ord}(a) \) for the highest power of 2 dividing \( a \). In the above, \( k = k_1 = \text{ord}(n') \). (It is traditional to write ‘\( \text{ord}_2(n) \)’, but in this paper only ‘2’ matters.) Note for later that if \( 0 \leq j \leq k \), then

\[
\nu(n - 2^j) = \nu(n) + k - j - 1.
\]

**Definition 1.** Let \( a, b \) be nonnegative integers. We say “\( \text{sum}(a, b) \) is neat”, provided that there is no carry when adding \( a \) to \( b \) in binary. We also write ‘\( a + b \doteq n \)” to denote that \( a + b = n \) and the sum is neat. Similarly we say “\( \text{sum}(a, b) \) is messy”, provided that there is a carry.

Thus \( \text{sum}(a, b) \) is neat iff \( \text{bin}(a) \) and \( \text{bin}(b) \) are disjoint. It is a classical observation that the binomial coefficient \( \binom{n}{a,b} = \frac{n!}{a!b!} \) is odd iff
sum\((a, b)\) is neat; we will make extensive use of this without further comment. More generally, \(\text{ord}\left(\binom{n}{a, b}\right)\) is equal to the number of carries when adding \(a\) to \(b\) in binary.

### 2.2. Partition Notation

The notation ‘\(\lambda \vdash n\)’ means that \(\lambda\) is a partition of \(n\). We also write \(|\lambda| = n\) in this case. Write \(\lambda'\) for the conjugate partition; its Young diagram is the transpose of the Young diagram of \(\lambda\). A “hook” is a partition of the form \(\lambda = (a, 1^b)\) for some nonnegative integers \(a, b\).

A bipartition of a number \(n\) is a pair of partitions \((\alpha, \beta)\) with \(|\alpha| + |\beta| = n\). The notation ‘\((\alpha, \beta) \vdash n\)’ means that \((\alpha, \beta)\) is a bipartition of \(n\).

Write \(p(n)\) for the number of partitions of \(n\), and \(p_2(n)\) for the number of bipartitions of \(n\). Write \(\Lambda\) for the set of partitions, \(\Lambda_n\) for the set of partitions of \(n\), and \(\text{Bip}(n)\) for the set of bipartitions of \(n\).

### 2.3. Group Theory Notation

For a group \(G\), write \(D(G)\) for its derived (commutator) subgroup, and \(G_{ab}\) for its abelianization \(G/D(G)\).

There are four infinite families of finite irreducible Coxeter groups, namely ones of type \(A_n\), \(B_n\), \(D_n\), and \(I_2(n)\) (the dihedral groups). The exceptional types are \(E_6\), \(E_7\), \(E_8\), \(F_4\), \(H_3\), \(H_4\). (See [Bou02].)

All representations considered in this paper are finite-dimensional and complex. Write \(\text{Irr}(G)\) for the set of isomorphism classes of irreducible representations of \(G\). If \(\pi\) is a representation, we write \(\chi_{\pi}\) for its character.

By “multiplicative character” we mean a group homomorphism \(W \to \mathbb{C}^\times\), which since \(W\) is generated by involutions, must take values in \(\{\pm 1\}\). For a representation \((\pi, V)\) of a group \(G\), write \(\det \pi\) for the composition of \(\pi : G \to \text{GL}(V)\) with the determinant map. Then \(\det \pi\) is a multiplicative character of \(G\).

If \((\pi_1, V_1)\) and \((\pi_2, V_2)\) are representations of groups \(G_1\) and \(G_2\), write \(\pi_1 \boxtimes \pi_2\) for the external tensor product representation of \(G_1 \times G_2\) on \(V_1 \otimes_{\mathbb{C}} V_2\).

### 2.4. Solomon Principle

Let \(\pi\) be a representation of a Coxeter group \(W\). In this section we show how to infer \(\det \pi\) from its character. So let \((W, S)\) be a Coxeter group \((S\) is a certain set of generators of order 2; see [Bou02]). There is a unique multiplicative character \(\varepsilon_W\) so that \(\varepsilon_W(s) = -1\) for each
s ∈ S, namely
\[ \varepsilon_W(w) = (-1)^{\ell(w)}, \]
where \( \ell(w) \) is the length of \( w \) with respect to \( S \).

For the Coxeter groups of type \( A_n \), \( D_n \), \( E_6 \), \( E_7 \), \( E_8 \), \( H_3 \), \( H_4 \), and \( I_2(p) \) for \( p \) odd, the trivial character and \( \varepsilon_W \) are the only multiplicative characters. This is equivalent to the abelianization \( W_{ab} \) having order 2.

**Proposition 1.** Suppose \(|W_{ab}| = 2\), and let \( s \in S \). If \( \pi \) is a representation of \( W \), then
\[ \det \pi = (\varepsilon_W)^b, \]
where
\[ b = \frac{\dim \pi - \chi_\pi(s)}{2}. \]

**Proof.** Let \( a \) be the multiplicity of 1 as an eigenvalue of \( \pi(s) \), and \( b \) be the multiplicity of \(-1\). Then \( \dim \pi = a + b \) and \( \chi_\pi(s) = a - b \). \( \square \)

We attribute this approach to L. Solomon; see [Sta01], Exercise 7.55.

The abelianization of the other Coxeter groups \((B_n, F_4, \text{and } I_2(p) \text{ for } p \text{ even})\) are Klein 4 groups. For these, fix two non-conjugate simple reflections \( s_1, s_2 \in S \), and multiplicative characters \( \omega_1, \omega_2 \) so that \( \omega_1(s_1) = -1, \omega_1(s_2) = 1, \omega_2(s_1) = 1, \omega_2(s_2) = -1 \). Then \( \varepsilon_W = \omega_1 \cdot \omega_2 \) and the multiplicative characters of \( W \) are \( \{1, \omega_1, \omega_2, \omega_1 \cdot \omega_2\} \).

**Proposition 2.** Suppose \(|W_{ab}| = 4\), and let \( s_1, s_2, \omega_1, \omega_2 \) be as above. If \( \pi \) is a representation of \( W \), then
\[ \det \pi = (\omega_1)^{x_1}(\omega_2)^{x_2}, \]
where
\[ x_1 = \frac{\dim \pi - \chi_\pi(s_1)}{2}, \]
and
\[ x_2 = \frac{\dim \pi - \chi_\pi(s_2)}{2}. \]

**Proof.** Similar to the proof of Proposition 1. \( \square \)

### 3. Type \( I_2(p) \): Dihedral Groups

Let us sketch the case of dihedral groups. Let \( p \geq 1 \) be a positive integer, and let \( W = D_p \) be the dihedral group of order \( 2p \). The irreducible representations of \( W \) are well-known: they are either one-dimensional, or induced from the normal cyclic subgroup of order \( p \), thus two-dimensional. It is elementary to check that the determinant of each two-dimensional irreducible representation is \( \varepsilon_W \).
Case $p$ odd: In this case $|W_{ab}| = 2$, so 1 and $\varepsilon_W$ are the only multiplicative characters. From the above,

$$N_1(p) = \#\{\pi \in \text{Irr}(D_p) \mid \det \pi = 1\} = 1$$

and

$$N_{\varepsilon_W}(p) = \#\{\pi \in \text{Irr}(D_p) \mid \det \pi = \varepsilon_W\} = \frac{p + 1}{2}.$$

Case $p$ even: Now $|W_{ab}| = 4$, so there are four multiplicative characters. We have $N_{\varepsilon_W}(p) = \frac{p}{2}$ and $N_\omega(p) = 1$ for $\omega \neq \varepsilon_W$.

4. Type $A_n$: Symmetric Groups

In this section we review and develop material from [Mac71], [Ols93], and [APS17] for $S_n$, which we will apply to the Type $B_n$ Coxeter groups in the next section. The irreducible representations of $S_n$ are indexed by partitions $\lambda$ of $n$. Given $\lambda \vdash n$, we denote by $(\rho_\lambda, V_\lambda)$ the corresponding representation of $S_n$.

4.1. Cores, Quotients, and Towers

Let us quickly recall the theory of 2-cores and 2-quotients of partitions. Details may be found in [Ols93].

**Definition 2.** We say that a partition $\lambda$ is a 2-core partition, provided that none of its hook lengths are even. The set of 2-core partitions is denoted by $C_2$.

There is assigned to any partition $\lambda$ a partition $\text{core}_2 \lambda \in C_2$ called the 2-core of $\lambda$, and a bipartition $(\alpha, \beta)$, called the 2-quotient of $\lambda$. These have the property that

$$|\lambda| = |\text{core}_2 \lambda| + 2(|\alpha| + |\beta|).$$

Moreover, this assignment is a bijection between $\Lambda$ and $C_2 \times \Lambda \times \Lambda$. In what follows, we iterate the core-quotient procedure to obtain what is called the 2-core tower of $\lambda$. This is an infinite binary tree, with each node labeled with a 2-core partition. For uniformity, let us put $\lambda_0 = \alpha$ and $\lambda_1 = \beta$. The tower is organized into rows, one for each nonnegative integer. Its 0th row is the root of the tree, labeled with $\alpha_0 := \text{core}_2 \lambda$. Its first row comprises two nodes

$$\alpha_0, \alpha_1,$$

where, if $\text{quo}_2 \lambda = (\lambda_0, \lambda_1)$, then $\alpha_i = \text{core}_2 \lambda_i$. Let $\text{quo}_2 \lambda_i = (\lambda_{i0}, \lambda_{i1})$, and define $\alpha_{ij} = \text{core}_2 \lambda_{ij}$. The second row is

$$\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11}.$$
Recursively, having defined partitions $\lambda_x$ for a binary sequence $x$, define the partitions $\lambda_{x0}$ and $\lambda_{x1}$ by

\begin{equation}
\text{quo}_2 \lambda_x = (\lambda_{x0}, \lambda_{x1}),
\end{equation}

and let $\alpha_{x\epsilon} = \text{core}_2 \lambda_{x\epsilon}$ for $\epsilon = 0, 1$. The $i$th row of the 2-core tower of $\lambda$ consists of the partitions $\alpha_x$, where $x$ runs over the set of all $2^i$ binary sequences of length $i$, listed from left to right in lexicographic order.

Thus the 2-core tower is:

This is called the 2-core tower of $\lambda$, which we denote by $T_\lambda$. Note that $T_\lambda$ has nonempty partitions at only finitely many nodes. Moreover $\lambda$ can be reconstructed from $T_\lambda$.

If $w_i(\lambda)$ is the sum of the sizes of the partitions in the $i$th row of $T_\lambda$, then iterating (3) gives

\begin{equation}
|\lambda| = \sum_{i=0}^{\infty} w_i(\lambda)2^i.
\end{equation}

**Example:** Let $\lambda = (12, 2, 1, 1)$. The 2-core of $\lambda$ is empty and its 2-quotient is $((1, 1), (6))$. The 2-core of $(1, 1)$ is also empty, and its 2-quotient is $((1), (\emptyset))$. The partition (6) has empty 2-core, and 2-quotient $(\emptyset, (3))$. Finally, (3) has 2-core (1) and 2-quotient $((1), (\emptyset))$. From this we can write down $T_\lambda$: 
We also define a map $\phi : \Lambda_a \times \Lambda_b \to \Lambda_{2(a+b)}$ so that $\phi(\alpha, \beta)$ has trivial 2-core, and its 2-quotient is $(\alpha, \beta)$. Note $T_{\phi(\alpha, \beta)}$ is obtained simply by joining $T_\alpha$ and $T_\beta$ side-by-side with root node labeled ‘∅’. For example, $(12, 2, 1, 1) = \phi((1, 1), (6))$ and $(6) = \phi(\emptyset, (3))$. All partitions of an even number $n$ with trivial 2-core are uniquely of the form $\phi(\alpha, \beta)$ for some $(\alpha, \beta) \vdash \frac{n}{2}$.

4.2. Review of Macdonald

Definition 3. Write $f_\lambda$ for the dimension of $V_\lambda$. Say that a partition $\lambda$ is odd, provided that $f_\lambda$ is odd. Otherwise say that $\lambda$ is even. Write $\Lambda_{\text{odd}}$ for the set of odd partitions, and $\Lambda_{n, \text{odd}}$ for the set of odd partitions of $n$. Write $A(n) = |\Lambda_{n, \text{odd}}|$.

The main result of [Mac71] is:

**Theorem 2.** A partition is odd precisely when $w_i(\lambda) \leq 1$ for all $i \geq 0$.

In other words, it is odd when its 2-core tower has at most one cell in each row. For example, from the towers we see that the partitions $(12, 2, 1, 1)$ is even, while the partitions $(1, 1)$ and $(6)$ are odd.

By [5], if $\lambda \vdash n$ is an odd partition, then $\bin(n)$ is equal to the set of indices of the nontrivial rows of $T_\lambda$.

**Corollary 1.** For $n$ with binary expansion

$$n = \epsilon + 2^{k_1} + \cdots + 2^{k_r},$$

put

$$\alpha(n) = k_1 + k_2 + \cdots + k_r.$$
Then \( A(n) = \alpha(n) \).

In other words, if we express \( n \) in binary as a sum of powers of 2, then \( A(n) \) is the product of those powers of 2. In particular, if \( n \) is a power of 2, then \( A(n) = n \). In this case the odd partitions of \( n \) are precisely the hooks of length \( n \).

Let us record some properties of \( A(n) \) for later use.

**Lemma 1.**

1. If \( x + y = n \), then \( A(n) = A(x)A(y) \).
2. If \( n \) is odd, then \( A(n - 1) = A(n) \).
3. More generally, if \( 0 \leq j \leq k = \text{ord}(n') \), then
   \[
   A(n - 2^j) = 2^{-k + \binom{k}{2} - \binom{j}{2}}A(n).
   \]
4. For all \( n \) we have
   \[
   A(2n) = 2^{\nu(n)}A(n).
   \]

**Proof.** For the third part, we have
\[
\alpha(n) = \alpha(n - 2^k) + \alpha(2^k)
\]
\[
= \alpha(n - 2^k) + k,
\]
and
\[
\alpha(n - 2^j) = \alpha(n - 2^k) + \alpha(2^k - 2^j).
\]

Now
\[
\alpha(2^k - 2^j) = j + (j + 1) + \cdots + (k - 1)
\]
\[
= \binom{k}{2} - \binom{j}{2}.
\]

Thus
\[
\alpha(n - 2^j) = \alpha(n) - k + \binom{k}{2} - \binom{j}{2}.
\]

The result follows. The other parts are straightforward. Also see Proposition 5 for a bijective proof of the first part. \(\square\)

Here are simple bounds for \( A(n) \):

**Lemma 2.** Let \( n \) be a natural number. We have
\[
\frac{1}{2}n < A(n) \leq n^{\frac{1}{2} \log_2 n + 1}.
\]

**Proof.** We have
\[
\log_2 n - 1 < \lfloor \log_2 n \rfloor \leq \alpha(n) \leq 1 + 2 + \cdots + \lfloor \log_2 n \rfloor
\]
\[
= \frac{\lfloor \log_2 n \rfloor (\lfloor \log_2 n \rfloor + 1)}{2}
\]
\[
\leq \frac{1}{2} \log_2 n (\log_2 n + 1).
\]
Since $A(n) = 2^{a(n)}$, this gives the lemma.

4.3. Review of Ayyer-Prasad-Spallone

Put $s_1 = (12) \in S_n$ for $n \geq 2$.

Definition 4. Given $\lambda \vdash n$, put

$$g_\lambda = \frac{f_\lambda - \chi_\lambda(s_1)}{2}.$$ 

Say that a partition $\lambda$ is chiral, provided that $\det \rho_\lambda = \text{sgn}$. Write $B(n)$ for the number of chiral partitions of $n$.

By Proposition 1, $\lambda$ is chiral iff $g_\lambda$ is odd. Recall the notation ‘ord’, ‘bin’, and $k = \text{ord}(n)$ from Section 2.1.

Theorem 3. [APS17, Theorem 6] Let $\lambda$ be a partition with 2-quotient $(\alpha, \beta)$ with $|\alpha| = a$ and $|\beta| = b$. Then $\lambda$ is chiral if and only if one of the following holds:

1. $\lambda$ is odd, and
   (a) if $n$ is even, then $k - 1 \in \text{bin}(a)$.
   (b) if $n$ is odd, then $k - 1 \in \text{bin}(b)$.
2. $\text{core}_2 \lambda = \emptyset$ or (1), and
   (a) $\alpha$ and $\beta$ are odd,
   (b) $\text{bin}(a) \cap \text{bin}(b) = \{j\}$, with $j = \text{ord}(a) = \text{ord}(b)$.
3. $\text{core}_2 \lambda = (2, 1)$ and $\phi(\alpha, \beta)$ is odd.

Corollary 2. (1) $\#\{\lambda \vdash n \mid f_\lambda \equiv g_\lambda \equiv 1\} = \frac{1}{2} A(n)$.
(2) $\#\{\lambda \vdash n \mid f_\lambda \equiv 1, g_\lambda \equiv 0\} = \frac{1}{2} A(n)$.
(3) $\#\{\lambda \vdash n \mid f_\lambda \equiv 0, g_\lambda \equiv 1\} = B(n) - \frac{1}{2} A(n)$.

Let us give special attention to the chiral partitions of type (2) in Theorem 3. The corresponding towers have at most one cell in each row, except for the $j$th row which has precisely two cells, one in the left half, and another in the right half. Necessarily $j \geq 1$. Moreover, there are no cells in rows 1 through $j - 1$. Its root is labeled $\emptyset$ or (1). Such towers we call “$j$-domino towers”, or “domino towers” if we forget the $j$. The above tower of $\lambda = (12, 2, 1, 1)$ is a 2-domino tower.

Write $\Lambda^D$ for the set of domino towers, and $\Lambda^j$ for the set of $j$-domino towers. Let $D(n) = |\Lambda^D_n|$ and $D_j(n) = |\Lambda^j_n|$. Note that $D(n) = 0$ if $n' \equiv 2 \mod 4$, and $D(n) = \sum_{j=1}^{k-1} D_j(n)$ otherwise.
It is easy to see that
\[ D_j(n) = 2^{2(j-1)}A(n-2^{j+1}) \]
\[ = 2^{2(j-1)-k+\binom{k}{j}-\binom{k+1}{j+1}}A(n) \]
\[ = 2^{\binom{k}{j}-\binom{j}{k}+j+k-2}A(n), \]
using Lemma [1].

**Corollary 3.** \[\text{[APS17, Theorem 1]}\]

For \( n \geq 2 \) and \( k = \text{ord}(n') \), the number of chiral partitions is given by
\[ B(n) = A(n) \left( \frac{1}{2} + \sum_{j=1}^{k-1} 2^{\binom{j}{k} - \binom{k-1}{j} + j+k-2} \right). \]

The sum is understood to be 0 if \( k = 1 \).

**Proof.** The number of partitions satisfying (1) in Theorem 3 is \( \frac{1}{2} A(n) \), and the number satisfying (3) is
\[ \epsilon A(n-3) = \epsilon A(n) 2^{\binom{k}{j}-k}. \]

So
\[ B(n) = \frac{1}{2} A(n) + D(n) + \epsilon A(n) 2^{\binom{k}{j}-k} \]
\[ = \frac{1}{2} A(n) + \left( \sum_{j=1}^{k-1} D_j(n) \right) + \epsilon A(n) 2^{\binom{k}{j}-k}, \]
and the final formula follows from (6). \[\square\]

### 4.4. Case of Trivial 2-core

Consider the case of partitions of the form \( \phi(\alpha, \beta) \vdash 2n \). (See Section 4.1.) Note that \( \phi(\alpha, \beta) \) is chiral iff \( \alpha \) and \( \beta \) are odd and \( \text{sum}(a, b) \) is neat.

**Proposition 3.** The partition \( \lambda = \phi(\alpha, \beta) \) is chiral iff both \( \alpha \) and \( \beta \) are odd, and one of the following holds:

1. \( \text{bin}(a) \cap \text{bin}(b) = \emptyset \) and \( \text{ord}(n) \in \text{bin}(a) \).
2. \( \text{bin}(a) \cap \text{bin}(b) = \{j\} \), with \( j = \text{ord}(a) = \text{ord}(b) \).

The disjunction of the two conditions of the proposition is equivalent to a certain binomial coefficient being odd.

**Proposition 4.** Let \( a + b = n \). The quantity \( \binom{n-1}{a-1, b} \) is odd iff one of the following holds:
(1) $\text{bin}(a) \cap \text{bin}(b) = \emptyset$ and $\text{ord}(n) \in \text{bin}(a)$.
(2) $\text{bin}(a) \cap \text{bin}(b) = \{j\}$, with $j = \text{ord}(a) = \text{ord}(b)$.

The proof is left to the reader. □

**Corollary 4.** Let $(\alpha, \beta) \models n$ with $|\alpha| = a$ and $|\beta| = b$. Then $\lambda = \phi(\alpha, \beta)$ is chiral iff the quantity

$$f_{\alpha f_{\beta}}\left(\frac{n - 1}{a - 1, b}\right)$$

is odd.

### 4.5. Tower Merging

Suppose $\lambda$ and $\mu$ are partitions, with the property that for all $i \geq 0$, either the $i$th row of $T_\lambda$ is empty, or the $i$th row of $T_\mu$ is empty. In this situation, one can define a partition $\nu$ of $|\lambda| + |\mu|$ as follows: For all $i \geq 0$, if the $i$th row of $T_\lambda$ (resp., $T_\mu$) is nonempty, let the $i$th row of $T_\nu$ be equal to the $i$th row of $T_\lambda$ (resp. of $T_\mu$). If the $i$th rows of both $T_\lambda$ and $T_\mu$ are empty, then let the $i$th row of $T_\nu$ be empty. We will refer to this process as “merging” the towers $T_\lambda$ and $T_\mu$.

For example, $\lambda = (3, 3, 2)$, whose 2-core tower is

```
(1)
(1)  (1)
```

can be merged with $\mu = (3, 1, 1, 1, 1, 1)$, whose tower is

```
(1)  (1)  (1)
(1)
```
to yield \( \nu = (11, 3, 2) \), with tower

\[
\begin{array}{ccc}
\emptyset & \emptyset & \emptyset \\
\emptyset & (1) & \emptyset \\
\emptyset & \emptyset & (1) \\
\emptyset & \emptyset & \emptyset \\
\end{array}
\]

Write ‘\( a + b = n \)’ when \( a + b = n \), and \( \text{bin}(a) \cap \text{bin}(b) = \{j\} \), with \( j = \text{ord}(a) = \text{ord}(b) \). Put \( D_{<j}(n) = \sum_{i=1}^{j-1} D_i(n) \).

**Proposition 5.** Let \( a + b = n \) with \( k = \text{ord}(n) \).

1. If \( a + b \not= n \), then \( A(n) = A(a)A(b) \).
2. If \( a + b = n \) with \( k \in \text{ord}(b) \), then \( A(a)D(b) = D(n) \).
3. If \( a+b = n \) with \( \text{ord}(a) = \text{ord}(b) = j \), then \( A(a)D(b) = D_{<j}(n) \).

**Proof.** In each case a bijective proof is given by the merging of towers: In the first case, merging gives a bijection from \( \Lambda^\text{odd}_a \times \Lambda^\text{odd}_b \) to \( \Lambda^\text{odd}_n \). In the second case, it gives a bijection from \( \Lambda^\text{odd}_a \times \Lambda^D_b \) to \( \Lambda^D_n \). In the last case, it gives a bijection from \( \Lambda^\text{odd}_a \times \Lambda^D_b \) to \( \bigcup_{i<j} \Lambda^i_n \). □

5. **Type \( B_n \): Hyperoctahedral Groups**

The Coxeter group of type \( B_n \), which we denote by \( \mathbb{B}_n \), is the wreath product \( (\mathbb{Z}/2\mathbb{Z}) \wr S_n \). It is traditionally called the \( n \)th hyperoctahedral group, being the symmetries of the standard hyperoctahedron (or standard hypercube) in \( \mathbb{R}^n \). For \( a + b = n \), there is a Young subgroup \( \mathbb{B}_a \times \mathbb{B}_b \leq \mathbb{B}_n \) in the evident way.

The four multiplicative characters of \( \mathbb{B}_n \) may be described as follows. Write \( \varepsilon : (\mathbb{Z}/2\mathbb{Z})^n \to \{\pm 1\} \) for the character whose restriction to each factor \( \mathbb{Z}/2\mathbb{Z} \) is nontrivial. As it is \( S_n \)-invariant, it extends to a multiplicative character \( \varepsilon : \mathbb{B}_n \to \{\pm 1\} \). Write \( \text{sgn}^0 \) for the composition of the projection \( \mathbb{B}_n \to S_n \) with the sign character of \( S_n \). Finally write \( \text{sgn}^1 = \varepsilon \cdot \text{sgn}^0 \). The multiplicative characters of \( \mathbb{B}_n \) are then \( 1, \varepsilon, \text{sgn}^0, \text{ and sgn}^1 \).
5.1. Irreducible Representations of $B_n$

Let $\lambda \vdash n$. We consider two extensions of the representation $\rho_\lambda$ of $S_n$ to $B_n$, namely

$$\rho_0^\lambda(x; w) = \rho_\lambda(w) \quad \text{and} \quad \rho_1^\lambda(x; w) = \varepsilon(x)\rho_\lambda(w),$$

for $x \in (\mathbb{Z}/2\mathbb{Z})^n$ and $w \in S_n$. For $(\alpha, \beta) \vdash n$, define

$$\rho_{\alpha\beta} = \text{Ind}^{B_n}_{B_a \times B_b} \rho_0^\alpha \boxtimes \rho_1^\beta.$$

Then

$$\text{Irr}(B_n) = \{ \rho_{\alpha\beta} \mid (\alpha, \beta) \vdash n \}.$$

Put

$$f_{\alpha\beta} = \dim \rho_{\alpha\beta} = \binom{n}{a, b} f_\alpha f_\beta.$$

For a multiplicative character $\omega$ of $B_n$, we define

$$\text{Bip}_\omega(n) = \{ (\alpha, \beta) \vdash n \mid \det \rho_{\alpha\beta} = \omega \},$$

and $N_\omega(n) = |\text{Bip}_\omega(n)|$. Similarly, for $a + b = n$ we put

$$\text{Bip}_\omega(a, b) = \{ (\alpha, \beta) \vdash n \mid \alpha \vdash a, \beta \vdash b, \det \rho_{\alpha\beta} = \omega \},$$

and $N_\omega(a, b) = |\text{Bip}_\omega(a, b)|$.

5.2. The Determinant of $\rho_{\alpha\beta}$

Proposition 29.2 in [BH06] gives the following formula for the determinant of an induced representation:

**Proposition 6.** Let $G$ be a finite group, $H \leq G$ a subgroup, and $\rho$ a representation of $H$. If $\pi = \text{Ind}^G_H \rho$, then

$$\det \pi = \det(\mathbb{C}[G/H])^{\dim \rho} \otimes (\det \rho \circ \text{ver}_{G/H}).$$

Here $\text{ver}_{G/H} : G_{ab} \to H_{ab}$ is the “Verlagerung” map between the abelianizations of $G$ and $H$, and is the main part of our calculation.

Recall the definition of $\text{ver}_{G/H}$: Let $t : G/H \to G$ be a section of the canonical projection. Given $g \in G$, for each $x \in G/H$ we have $gt(x) = t(y)h_{x,g}$ for some $y \in G/H$ and $h_{x,g} \in H$. Note that here $y = g^x$, with $G/H$ considered as a $G$-set. Then

$$\text{ver}_{G/H}(g \mod D(G)) = \prod_{x \in G/H} h_{x,g} \mod D(H).$$

Let us first treat the case where $G = S_n$ and $H$ is the Young subgroup $S_a \times S_b$, with $a + b = n$. Then $G_{ab} = S_n/A_n$, where $A_n$ is the alternating group, and we may identify $H_{ab}$ with $S_a/A_a \times S_b/A_b$. 

Proposition 7. Let $a + b = n$. Then the map

$$\text{ver} = \text{ver}_{S_n/(S_a \times S_b)} : S_n/A_n \to S_a/A_a \times S_b/A_b$$

is given by

$$\text{ver}(\tau_n) = \left( \tau_a^{(n-2)} \tau_b^{(n-2)} \right).$$

Here $\tau_n$ is any transposition in $S_n$, or trivial if $n < 2$.

Proof. Let $J_n = \{1, 2, \ldots, n\}$, and write $\varphi_a(J_n)$ for the set of subsets of $J_n$ of cardinality $a$. For instance $J_a \in \varphi_a(J_n)$. The map $S_n \to \varphi_a(J_n)$ defined by $g \mapsto gJ_a$ descends to an isomorphism of $S_n/(S_a \times S_b)$ with $\varphi_a(J_n)$, as $S_n$-sets. Pick any section $t : \varphi_a(J_n) \to S_n$; thus $t(x)J_a = x$ for all $x \in \varphi_a(J_n)$.

For concreteness, let $\tau_n$ be the transposition $s_1 = (12)$. Since

$$h_{x,s_1} = t(s_1 x)^{-1} s_t(x),$$

we have $h_{x,s_1} h_{s_1 x,s_1} = 1$. Thus

$$\text{ver}(\tau_n) = \prod_{x \in \varphi_a(J_n)} h_{x,s_1} \mod A_n$$

$$= \prod_{x \in \varphi_a(J_n) | s_1 x = x} h_{x,s_1} \mod A_n.$$

Now $s_1 x = x$ iff $\{1, 2\}$ is a subset of $x$, or of the complement of $x$. In the first case, $h_{x,s_1}$ is a transposition $\tau_a \in S_a$, and in the second case, $h_{x,s_1}$ is a transposition $\tau_b \in S_b$. There are $\binom{n-2}{a-2}$ elements of $\varphi_a(J_n)$ containing $\{1, 2\}$, and $\binom{n-2}{b-2}$ elements of $\varphi_a(J_n)$ whose complement contains $\{1, 2\}$. The result follows.

Next we compute the verlagerung for $G = B_n$ and $H$ the Young subgroup $B_a \times B_b$, with $a + b = n$. The derived quotient $(B_n)_{ab} = B_n/D(B_n)$ is a Klein 4 group generated by the images of $\tau_n$ and $e_n$, where $e_n \in (\mathbb{Z}/2\mathbb{Z})^n$ is any vector with $\varepsilon(e_n) = -1$. We identify $H_{ab}$ with $(B_a)_{ab} \times (B_b)_{ab}$.

Proposition 8. The map

$$\text{ver} = \text{ver}_{G/H} : (B_n)_{ab} \to (B_a)_{ab} \times (B_b)_{ab}$$

is given by

$$\text{ver}(\tau_n) = \left( \tau_a^{(n-2)} \tau_b^{(n-2)} \right)$$
and

\[ \text{ver}(e_n) = \left( e_a^{(n-1)_{a-1}}, e_b^{(n-1)_{b-1}} \right). \]

**Proof.** We may identify the quotient \( B_n/(B_a \times B_b) \) with \( S_n/(S_a \times S_b) \), and in particular use a transversal \( S_n/(S_a \times S_b) \to S_n \) to form a transversal \( t : B_n/(B_a \times B_b) \to B_n \) whose image lies in \( S_n \). The following diagram commutes:

\[
\begin{array}{ccc}
(S_n)_{ab} & \xrightarrow{\text{ver}} & (S_a)_{ab} \times (S_b)_{ab} \\
\downarrow & & \downarrow \\
(B_n)_{ab} & \xrightarrow{\text{ver}} & (B_a)_{ab} \times (B_b)_{ab}
\end{array}
\]

Here the vertical maps are the obvious inclusions, and the top horizontal map was computed in the previous proposition. This gives the formula for \( \text{ver}(\tau_n) \).

Now, with notation as in the previous proof, observe that \( t(x)^{-1} x = J_a \), so

\[ t(x)^{-1}(1) \in J_a \iff 1 \in x. \]

We have

\[ h_{x,e_1} = t(x)^{-1} e_1 t(x) = e_i, \]

with \( i = t(x)^{-1}(1) \). Therefore, modulo \( D(H) \), we may write

\[ h_{x,e_1} = \begin{cases} 
  e_a, & \text{if } 1 \in x \\
  e_b, & \text{if } 1 \notin x.
\end{cases} \]

There are \( \binom{n-1}{a-1} \) subsets \( x \) of \( J_n \) with \( 1 \in x \), and \( \binom{n-1}{b-1} \) with \( 1 \notin x \), thus the product of the \( h_{x,e_1} \) is \( \left( e_a^{(n-1)_{a-1}}, e_b^{(n-1)_{b-1}} \right) \), giving the formula for \( \text{ver}(e_n) \).

\[ \square \]

For \( i, j \) we have

\[ \det(\rho^0_{\alpha} (\tau^i_{\alpha}) \boxtimes \rho^1_{\beta}(\tau^j_{\beta})) = (-1)^{g_{\alpha} f_{\beta} i + g_{\beta} f_{\alpha} j}, \]

and so

\[ \det(\rho^0_{\alpha} \boxtimes \rho^1_{\beta})(\text{ver}(s_1)) = (-1)^{g_{\alpha} f_{\beta} \binom{n-2}{a-1} + g_{\beta} f_{\alpha} \binom{n-2}{b-2}}. \]

Similarly,

\[ \det(\rho^0_{\alpha} (e^i_{\alpha}) \boxtimes \rho^1_{\beta}(e^j_{\beta})) = (-1)^{f_{\alpha} f_{\beta} j}. \]
so
\[ \det (\rho_0^1 \boxtimes \rho_1^1)(\text{ver}(e_n)) = (-1)^{f_\alpha f_\beta (n-1)}. \]

The permutation module \( \mathbb{C}[\varphi_a(J_n)] \) coming from the action of \( \mathbb{B}_n \) on \( \varphi_a(J_n) \) factors through the action of \( S_n \). Thus \( e_n \) acts trivially, and \( s_1 \) acts by permuting these sets. The number of doubleton orbits of \( s_1 \) on this set is equal to the number of subsets of \( \varphi_a(J_n) \) which contain 1 but not 2, thus equals \( \binom{n-2}{a-1} \). This gives
\[ \det (\mathbb{C}[G/H])(e_n) = 1 \quad \text{and} \quad \det (\mathbb{C}[G/H])(s_1) = (-1)^{\binom{n-2}{a-1}}, \]
so that
\[ \det (\mathbb{C}[G/H]) = (\text{sgn}^0)^{\binom{n-2}{a-1,b-1}}. \]

Let
\[ x_{\alpha\beta} = f_\alpha f_\beta \left( \frac{n-1}{a,b-1} \right) \in \mathbb{Z}/2\mathbb{Z} \]
and
\[ y_{\alpha\beta} = f_\alpha f_\beta \left( \frac{n-2}{a-1,b-1} \right) + f_\beta g_\alpha \left( \frac{n-2}{a-2,b} \right) + f_\alpha g_\beta \left( \frac{n-2}{a,b-2} \right) \in \mathbb{Z}/2\mathbb{Z}. \]

From the above we deduce:

**Theorem 4.** For a bipartition \((\alpha, \beta)\), we have
\[ \det \rho_{\alpha\beta} = \varepsilon^{x_{\alpha\beta}} \cdot \text{sgn}^0)^{y_{\alpha\beta}}. \]

**Proof.** The preceding shows that
\[ \det (\rho_{\alpha\beta}(e_n)) = (-1)^{x_{\alpha\beta}} \quad \text{and} \quad \det (\rho_{\alpha\beta}(s_1)) = (-1)^{y_{\alpha\beta}}, \]
and this gives the theorem. \( \square \)

An alternate proof of Theorem 4 can be given through Theorem 2, via the Frobenius Character Formula to compute \( \chi_{\alpha\beta}(s_1) \) and \( \chi_{\alpha\beta}(e_1) \). (Here \( \chi_{\alpha\beta} \) denotes the character of \( \rho_{\alpha\beta} \).) In fact,

\[ \chi_{\alpha\beta}(s_1) = \binom{n-2}{a-2,b} f_\beta \chi_\alpha(s_1) + \binom{n-2}{a,b-2} f_\alpha \chi_\beta(s_1) \]
and
\[ \chi_{\alpha\beta}(e_1) = f_\alpha f_\beta \left( 2 \binom{n-1}{a-1,b} - \binom{n}{a,b} \right). \]

Details are left to the reader.
Theorem 5. For \((\alpha, \beta) \models n\), the quantity \(x_{\alpha\beta}\) is odd iff \(\lambda = \phi(\beta, \alpha)\) is chiral. In other words, \(g_{\phi(\beta, \alpha)} \equiv x_{\alpha\beta} \mod 2\).

Proof. This is immediate from Corollary 4.

Corollary 5. For all \(n\),

\[ N_{\varepsilon}(n) + N_{sgu^1}(n) = B(2n). \]

An important case is when \(\alpha = \beta\).

Proposition 9. Let \(a \geq 2\) and \(\alpha \vdash a\). Then

\[
\det \rho_{aa} = \begin{cases} 
\varepsilon, & \text{if } a \text{ is a power of } 2 \text{ and } \alpha \text{ is a hook} \\
1, & \text{otherwise.}
\end{cases}
\]

Proof. In this case

\[ x_{\alpha\alpha} \equiv f_a \left( \binom{2a - 1}{a, a - 1} \right), \text{ and } y_{a\alpha} \equiv 0, \]

since \(\binom{2a - 2}{a-1,a-1}\) is always even. Now \(\text{bin}(a) \cap \text{bin}(a - 1) = \emptyset\) iff \(a\) is a power of 2, and in this case the odd partitions are precisely the hooks. This gives the proposition.

5.3. Values of \(N_\omega(n)\) for small \(n\)

Using Theorem 4, one can compute:
Below is a logplot, base 2, of each $N_{\omega}(n)$ for $2 \leq n \leq 65$. The horizontal axis is $n$. The orange line is $N_1(n)$, the green line is $N_{\text{sgn}^0}(n)$, the red line is $N_{\text{sgn}^1}(n)$, and the blue line is $N_\varepsilon(n)$.

Note the compatibility with Theorem 1.

5.4. An Involution of Bip($n$)

Lemma 3. Mod 2, we have $g_{\lambda} = g_{\lambda} + f_{\lambda}$, and $f_{\phi(\alpha,\beta)} = f_{\alpha\beta}$. 
Proof. For the first part, \( \chi'(s_1) = -\chi(s_1) \), so \( g + g' = f \). The second part is immediate.

Given \((\alpha, \beta) \vdash n\), put \(\sigma(\alpha, \beta) = (\alpha', \beta')\).

Proposition 10. For \((\alpha, \beta) \vdash n\), we have

1. \( x_{\sigma(\alpha, \beta)} = x_{\alpha\beta} \),
2. \( y_{\sigma(\alpha, \beta)} = y_{\alpha\beta} + f_{\alpha\beta} \).

Corollary 6. Let \( n \geq 3 \) be odd. If \( a + b = n \), then \( \sigma \) restricts to a bijection from \( \text{Bip}_{\text{sgn}^1}(a, b) \) to \( \text{Bip}_{\varepsilon}(b, a) \). Thus \( N_{\text{sgn}^1}(b, a) = N_{\varepsilon}(a, b) \) and \( N_{\text{sgn}^1}(n) = N_{\varepsilon}(n) \).

Proof. We must show that if \( x_{\alpha\beta} = y_{\alpha\beta} = 1 \), then \( x_{\sigma(\alpha, \beta)} = 1 \) and \( y_{\sigma(\alpha, \beta)} = 0 \). Now, if \( n \) odd and \( (n - 1)_{a, b - 1} \) is odd, it is elementary to see that \( b \) is odd and \( (n - 1)_{a, b} \) is odd. By the previous proposition we deduce that \( y_{\sigma(\alpha, \beta)} = 0 \). \(\square\)

Corollary 7. Let \( n \geq 3 \) be odd. Then \( N_{\text{sgn}^1}(n) = N_{\varepsilon}(n) = \frac{1}{4} A(2n) \).

Proof. By Corollaries 5 and 6 we have \( N_{\text{sgn}^1}(n) = N_{\varepsilon}(n) = \frac{1}{2} B(2n) \). But by Corollary 3, \( B(2n) = \frac{1}{2} A(2n) \). \(\square\)

5.5. Counting with fixed \( a, b \).

Given \( \omega \in \{1, \varepsilon, \text{sgn}^0, \text{sgn}^1\} \), and \( a + b = n \), put

\[ N_{\omega}(a, b) = \# \{ \alpha \vdash a, \beta \vdash b \mid \det \rho_{\alpha\beta} = \omega \}. \]

We compute \( N_{\varepsilon}(a, b) \) in Proposition 11, \( N_{\text{sgn}^1}(a, b) \) in Proposition 12, and \( N_{\text{sgn}^0}(a, b) \) in Proposition 13. Of course,

\[ p(a)p(b) = N_1(a, b) + N_{\text{sgn}^0}(a, b) + N_{\text{sgn}^1}(a, b) + N_{\varepsilon}(a, b), \]

so one implicitly has an expression for \( N_1(a, b) \) as well.

Recall that

\[ x_{\alpha\beta} = f_\alpha f_\beta \binom{n - 1}{a, b - 1}, \]

and

\[ y_{\alpha\beta} = f_\alpha f_\beta \binom{n - 2}{a - 1, b - 1} + f_\beta g_\alpha \binom{n - 2}{a - 2, b} + f_\alpha g_\beta \binom{n - 2}{a, b - 2}. \]

Lemma 4. Let \( a + b = n \). The following conditions are equivalent:

1. \( \binom{n - 2}{a - 1, b - 1} \) is odd, \( \binom{n - 2}{a, b - 2} \) is even, and \( \binom{n - 2}{a - 2, b} \) is even.

2. \( n \) is even and \( \binom{n - 2}{a - 1, b - 1} \) is odd.

The proof is left to the reader. \(\square\)
Proposition 11. Let $a + b = n$. If $n$ is even, then

$$N_\varepsilon(a, b) = \begin{cases} \frac{1}{2} A(a) A(b) & \text{if } \binom{n-1}{a, b-1} \text{ is odd and } \binom{n-2}{a-1, b-1} \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

If $n$ is odd, then

$$N_\varepsilon(a, b) = \begin{cases} \frac{1}{2} A(a) A(b) & \text{if } \binom{n-1}{a, b-1} \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We must count $\alpha \vdash a, \beta \vdash b$ so that $x_{\alpha \beta} \equiv 1$ and $y_{\alpha \beta} \equiv 0$. For $x_{\alpha \beta}$ to be odd, we need $f_\alpha, f_\beta$ and $\binom{n-1}{a, b-1}$ to be odd.

In this case

$$y_{\alpha, \beta} = \binom{n-2}{a-1, b-1} + g_\alpha \binom{n-2}{a-2, b} + g_\beta \binom{n-2}{a, b-2}.$$  

It is now clear that Bip$_\varepsilon(a, b) = \emptyset$ if either $\binom{n-1}{a, b-1}$ is even, or if $n$ is even and $\binom{n-2}{a-1, b-1}$ is odd (using Lemma 4).

So henceforth assume that $\binom{n-1}{a, b-1}$ is odd, and that the conditions of Lemma 4 do not hold. Since

$$\binom{n-1}{a, b-1} = \binom{n-2}{a-1, b-1} + \binom{n-2}{a, b-2},$$

we consider two cases.

**Case 1:** $\binom{n-2}{a-1, b-1}$ is odd and $\binom{n-2}{a, b-2}$ is even. As we assume the conditions of Lemma 4 do not hold, further $\binom{n-2}{a-2, b}$ is odd. Thus

$$y_{\alpha, \beta} \equiv 1 + g_\alpha \mod 2.$$  

So for such $a, b$ we have

$$\text{Bip}_\varepsilon(a, b) = \{(\alpha, \beta) \mid f_\alpha \text{ is odd, } g_\alpha \text{ is odd, } f_\beta \text{ is odd}\},$$

and by Corollary 2 we compute

$$N_\varepsilon(a, b) = \frac{1}{2} A(a) A(b).$$

**Case 2:** If $\binom{n-2}{a-1, b-1}$ is even and $\binom{n-2}{a, b-2}$ is odd, then

$$y_{\alpha, \beta} \equiv g_\alpha \binom{n-2}{a-2, b} + g_\beta.$$  

For $y_{\alpha, \beta}$ to be even, we need $g_\alpha \binom{n-2}{a-2, b}$ and $g_\beta$ to be both odd or both even. We again have two possibilities.
Case 2a: If \((\binom{n-2}{a-2,b})\) is even, then
\[
\text{Bip}_\varepsilon(a, b) = \{(\alpha, \beta) \mid f_\alpha \text{ is odd, } f_\beta \text{ is odd, } g_\beta \text{ is odd}\},
\]
so \(N_\varepsilon(a, b) = \frac{1}{2} A(a) A(b)\), as claimed.

Case 2b: If \((\binom{n-2}{a-2,b})\) is odd, then
\[
\text{Bip}_\varepsilon(a, b) = \{(\alpha, \beta) \mid f_\alpha \text{ is odd, } g_\alpha \text{ is even, } f_\beta \text{ is odd, } g_\beta \text{ is even}\}
\]
\[\cup\{(\alpha, \beta) \mid f_\alpha \text{ is odd, } g_\alpha \text{ is odd, } f_\beta \text{ is odd, } g_\beta \text{ is odd}\}.
\]
Thus
\[
N_\varepsilon(a, b) = A(a) A(b) - A(a) \frac{1}{2} A(b) - \frac{1}{2} A(a) A(b) + 2 \cdot \frac{1}{2} A(a) \frac{1}{2} A(b)
\]
\[= \frac{1}{2} A(a) A(b).
\]
\[\square\]

**Proposition 12.** For given \(a\) and \(b\), we have
\[
N_{sgn^1}(a, b) = \begin{cases} 0 & \text{if } \binom{n-1}{a,b-1} \text{ is even,} \\
A(a) A(b) & \text{if } n \text{ is even and } \binom{n-2}{a-1,b-1} \text{ is odd,}
\end{cases} \frac{1}{2} A(a) A(b) \text{ otherwise.}
\]

**Proof.** For \(x_{\alpha\beta}\) to be odd, we need \(f_\alpha, f_\beta\) and \(\binom{n-1}{a,b-1}\) to be odd. Again there are two cases to consider.

Case 1: If \((\binom{n-2}{a-1,b-1})\) is odd and \((\binom{n-2}{a,b-2})\) is even, then
\[
y_{\alpha\beta} \equiv 1 + g_\alpha \binom{n-2}{a-2,b} \pmod{2}.
\]

For \(y_{\alpha\beta}\) to be odd, we need \(g_\alpha \binom{n-2}{a-2,b}\) to be even.

Case 1a: If \((\binom{n-2}{a-2,b})\) is even, then \(N_{sgn^1}(a, b) = A(a) A(b)\).

Case 1b: If \((\binom{n-2}{a-2,b})\) is odd, then
\[
N_{sgn^1}(a, b) = \#\{f_\alpha \text{ is odd, } g_\alpha \text{ is even}\} \#\{f_\beta \text{ is odd}\}
\]
\[= \left[\#\{f_\alpha \text{ is odd}\} - \#\{f_\alpha \text{ is odd, } g_\alpha \text{ is odd}\}\right] \#\{f_\beta \text{ is odd}\}
\]
\[= \frac{1}{2} A(a) A(b).
\]

Case 2: If \((\binom{n-2}{a-1,b-1})\) is even and \((\binom{n-2}{a,b-2})\) is odd, then
\[
y_{\alpha\beta} \equiv g_\alpha \binom{n-2}{a-2,b} + g_\beta \pmod{2}.
\]
For \( y_{\alpha \beta} \) to be odd, we have two cases:

Case 2a: If \( g_\alpha \binom{n-2}{a-2,b} \) is even and \( g_\beta \) is odd, then

\[
y_{\alpha \beta} \equiv g_\beta \pmod{2}.
\]

Thus, we have

\[
N_{\text{sgn}}(a, b) = \# \{ f_\beta \text{ is odd}, g_\beta \text{ is odd} \} \# \{ f_\alpha \text{ is odd} \}
= \frac{1}{2} A(b) A(a).
\]

Case 2b: If \( g_\alpha \binom{n-2}{a-2,b} \) is odd and \( g_\beta \) is even, then

\[
y_{\alpha \beta} \equiv g_\beta + g_\alpha \pmod{2}.
\]

Thus, we have

\[
N_{\text{sgn}}(a, b) = \# \{ f_\beta \text{ is odd}, g_\beta \text{ is odd} \} \# \{ f_\alpha \text{ is odd}, g_\alpha \text{ is even} \}
+ \# \{ f_\beta \text{ is odd}, g_\beta \text{ is even} \} \# \{ f_\alpha \text{ is odd}, g_\alpha \text{ is odd} \}
= \frac{1}{2} A(b) A(a).
\]

\[\square\]

Our formulas for \( N_{\text{sgn}}(a, b) \), where \( a + b = n \), depend on the parities of the binomial coefficients \( \binom{n-2}{a-2,b} \), \( \binom{n-2}{a-1,b-1} \), and \( \binom{n-2}{a,b-2} \).

**Proposition 13.** The following table computes \( N_{\text{sgn}}(a, b) \).

| \( \binom{n-2}{a-2,b} \) | \( \binom{n-2}{a-1,b-1} \) | \( \binom{n-2}{a,b-2} \) | \( N_{\text{sgn}}(a, b) \) |
|--------------------------|--------------------------|--------------------------|--------------------------|
| 0                        | 0                        | 0                        | 0                        |
| 0                        | 1                        | 0                        | 0                        |
| 1                        | 0                        | 0                        | \( A(b)B(a) \)            |
| 0                        | 1                        | 1                        | \( A(a)B(b) \)            |
| 1                        | 1                        | 1                        | \( A(a)B(b) + A(b)B(a) - \frac{1}{2} A(a)A(b) \) |
| 1                        | 1                        | 0                        | \( A(b)B(a) - \frac{1}{2} A(a)A(b) \) |
| 0                        | 0                        | 1                        | \( A(a)B(b) - \frac{1}{2} A(a)A(b) \) |
| 1                        | 0                        | 1                        | \( A(a)B(b) + A(b)B(a) - A(a)A(b) \) |

(In the table we read the binomial coefficients mod 2.)

**Proof.** Each of these eight cases we will refer to by the binary “code” coming from the parity of the binomial coefficients. For instance, the first two cases correspond to the codes ‘000’ and ‘010’. Of course, each code further determines the parity of the binomial coefficient \( \binom{n-1}{a,b-1} \).
For the first two cases, suppose that \((n-2)_{a-2,b}\) and \((n-2)_{a,b-2}\) are even, but \(x_{\alpha\beta} = 0\) and
\[
y_{\alpha\beta} = f_{\alpha}f_{\beta}\left(\frac{n-2}{a-1,b-1}\right) = 1.
\]
Then \(f_{\alpha}f_{\beta} = 1\), so it must be that \((n-1)_{a,b-1}\) = 0. But then necessarily \((n-2)_{a-1,b-1}\) = 0, a contradiction. Thus \(\text{Bip}_{sgn^0}(a,b) = \emptyset\) in the ‘000’ and ‘010’ cases.

For the ‘100’ case, we automatically have \(x_{\alpha\beta} = 0\) and \(y_{\alpha\beta} = f_{\beta}g_{\alpha}\). Thus \((\alpha, \beta) \in \text{Bip}_{sgn^0}(a,b)\) iff \(f_{\beta} = g_{\alpha} = 1\). So \(N_{sgn^0}(a,b) = A(b)B(a)\) here.

For the ‘011’ case, again \(x_{\alpha\beta} = 0\) but
\[
y_{\alpha\beta} = f_{\alpha}f_{\beta} + f_{\beta}g_{\alpha} + f_{\alpha}g_{\beta}
\]
must be odd. Note that \(f_{\alpha}\) or \(g_{\alpha}\) must be odd; we consider the three possibilities. For \(f_{\alpha} = g_{\alpha} = 1\), then \(y_{\alpha\beta} = g_{\beta}\); this gives \(\frac{1}{2}A(a)B(b)\).

For \(f_{\alpha} = 0, g_{\alpha} = 1\), then \(y_{\alpha\beta} = f_{\beta}\); this gives \((B(a) - \frac{1}{2}A(a))A(b)\). Next \(f_{\alpha} = 1, g_{\alpha} = 0\) gives \(y_{\alpha\beta} = f_{\beta} + g_{\beta}\); this gives \(\frac{1}{2}A(a)B(b)\). The sum of these three quantities is equal to \(A(a)B(b) + A(b)B(a) - \frac{1}{2}A(a)A(b)\).

For the ‘110’ case, we must have \(f_{\alpha}f_{\beta} = 0\) and so
\[
y_{\alpha\beta} = f_{\beta}g_{\alpha}.
\]
So \((\alpha, \beta) \in \text{Bip}_{sgn^0}\) iff \(f_{\alpha} = 0\) and \(f_{\beta} = g_{\alpha} = 1\). This gives
\[
\left(B(a) - \frac{1}{2}A(a)\right)A(b)
\]
possibilities.

For both ‘001’, we must have \(f_{\beta} = 0\), and \(f_{\alpha} = g_{\beta} = 1\), giving
\[
A(a)\left(B(b) - \frac{1}{2}A(b)\right).
\]

For ‘101’, to solve
\[
y_{\alpha\beta} = f_{\beta}g_{\alpha} + f_{\alpha}g_{\beta} = 1
\]
with \(f_{\alpha}f_{\beta} = 0\), there are two possibilities. Either \(f_{\alpha} = 0, f_{\beta} = g_{\alpha} = 1\), or \(f_{\beta} = 0, f_{\alpha} = g_{\beta} = 1\). The first possibility gives \((B(a) - \frac{1}{2}A(a))A(b)\), and the second gives \((B(b) - \frac{1}{2}A(b))A(a)\). Adding gives the result.

\[\square\]
5.6. Final Count

We compute \( N_\varepsilon(n) \) in Theorem 6, \( N_{\text{sgn}^0}(n) \) in Theorem 7, and \( N_{\text{sgn}^1}(n) \) in Theorem 8. The quantity \( N_1(n) \) can be computed in principle from the formula

\[
p_2(n) = N_1(n) + N_{\text{sgn}^0}(n) + N_{\text{sgn}^1}(n) + N_\varepsilon(n),
\]

where \( p_2(n) \) denotes the number of bipartitions of \( n \).

**Theorem 6.** For \( n \geq 2 \), and \( k = \text{ord}(n) \), we have

\[
N_\varepsilon(n) = \begin{cases} 
\frac{1}{4} A(2n) & \text{if } n \text{ is odd,} \\
\frac{1}{8} A(2n) \left( 2 + \sum_{j=1}^{k-1} 2^{(k_j)} - (j) \right) & \text{if } n \text{ is even.}
\end{cases}
\]

Note that the sum is 0 when \( k = 1 \), and similarly for other sums in this paper.

**Proof.** If \( n \) is odd, this follows from Corollary 7, so let \( n \) be even.

By Proposition 11, we have

\[
N_\varepsilon(n) = \sum_{a+b-1 = n-1 \atop \text{sum(a-1,b-1) messy}} \frac{1}{2} A(a) A(b).
\]

If \( a \) and \( b \) are odd, then \( \text{sum}(a-1,b-1) \) is neat iff \( \text{sum}(a,b-1) \) is neat, so \( N_\varepsilon(a,b) = 0 \). If \( a \) and \( b \) are even, then \( \text{sum}(a-1,b-1) \) is automatically messy. Hence,

\[
N_\varepsilon(n) = \sum_{a+b-1 = n-1 \atop a,b \text{ even}} \frac{1}{2} A(a) A(b)
= \sum_{a+b-1 = n-1 \atop a,b \text{ even}} \frac{1}{2} A(a) A(b-1) 2^{\text{ord}(b) - \left(\frac{\text{ord}(b)}{2}\right)}
= \frac{1}{2} A(n-1) \sum_{a+b-1 = n-1 \atop a,b \text{ even}} 2^{\text{ord}(b) - \left(\frac{\text{ord}(b)}{2}\right)}.
\]

(9)

Now \( \text{sum}(a,b-1) \) is neat iff \( \text{ord}(a) \geq \text{ord}(b) \) and \( \text{sum}(a,b - 2^{\text{ord}(b)}) \) is neat. There are two cases to consider. First if \( \text{ord}(b) = k \), then we must also have \( k < \text{ord}(a) \). The sum of these terms is

\[
\frac{1}{2} A(n-1) 2^{\nu(n-2^k) + k - \left(\frac{k}{2}\right)}.
\]
The other case has $\text{ord}(a) = \text{ord}(b) < k$, the sum over these $a, b$ is

$$\frac{1}{2} A(n - 1) \sum_{j=1}^{k-1} 2^{\nu(n - 2^{j+1}) + 2^{j} - (\text{c})}.$$ 

Further simplification comes by using (2):

$$N_\epsilon(n) = \left( \frac{1}{2} A(n - 1) 2^{\nu(n) + k - 2} \sum_{j=1}^{k-1} 2^{-(\text{c})} \right) + \frac{1}{2} A(n - 1) 2^{\nu(n) + k - 1 - (\text{c})}.$$ 

Now

$$A(n - 1) = A(n) 2^{-k + (\text{c})},$$

so we may rewrite this as

$$N_\epsilon(n) = \left( \frac{1}{2} A(n) 2^{\nu(n) - 2} \sum_{j=1}^{k-1} 2^{(\text{c}) - (\text{c})} \right) + \frac{1}{2} A(n) 2^{\nu(n) - 1}$$

$$= \left( \frac{1}{8} A(2n) \sum_{j=1}^{k-1} 2^{(\text{c}) - (\text{c})} \right) + \frac{1}{4} A(2n).$$

If $k = 1$, then the sum is empty, giving $N_\epsilon(n) = \frac{1}{4} A(2n)$. 

**Theorem 7.** The following computes $N_{\text{sgn}}(n)$ in all cases. Let $k = \text{ord}(n').$

(1) If $n \equiv 1 \mod 4$, then

$$N_{\text{sgn}}(n) = \frac{1}{4} A(2n) \left( 1 + 3 \cdot 2^{k-1} - 2^{k} - k + 1 + \sum_{j=2}^{k-1} \left( 2^{(\text{c}) - (\text{c})} + 2^{(\text{c}) + j} \right) \right).$$

(2) If $n \equiv 3 \mod 4$, then $N_{\text{sgn}}(n) = \frac{1}{2} A(2n)$.

(3) If $n$ is even, then

$$N_{\text{sgn}}(n) = \frac{1}{8} A(2n) \left( 2 + \sum_{j=1}^{k-1} 2^{(\text{c}) - (\text{c})} \right).$$

**Proof.** Case $n \equiv 2 \mod 4$: The codes ‘011’, ‘111’, ‘101’, and ‘110’ cannot occur in the $(n - 2)$nd row of Pascal’s triangle mod 2. Moreover for $a, b$ corresponding to the codes ‘000’ and ‘010’ we have $N_{\text{sgn}}(a, b) = 0$. So we are left with the codes ‘100’ and ‘001’. Let us write
$$N_{\text{sgn}}^\circ(n)^{100} = \sum_{\text{sum}(a-b) \text{ neat}} \sum_{\text{sum}(a-1,b-1) \text{ messy}} \sum_{\text{sum}(a,b-2) \text{ messy}} N_{\text{sgn}}^\circ(a, b),$$

and similarly for other codes. From Proposition 13, $N_{\text{sgn}}^\circ(a, b) = A(b)B(a)$. In this case, since $n - 2 \equiv 0 \mod 4$, the first neatness condition on $a - 2, b$ implies the other two. Moreover it implies that $a \equiv 2 \mod 4$, so that $B(a) = \frac{1}{2}A(a)$ and $A(a - 2) = \frac{1}{2}A(a)$. It follows that

$$N_{\text{sgn}}^\circ(n)^{100} = \sum_{(a-2)+b\equiv n-2} \frac{1}{2}A(a)A(b)$$

$$= \sum_{(a-2)+b\equiv n-2} A(a-2)A(b)$$

$$= 2^{\nu(n-2)}A(n-2)$$

$$= \frac{1}{4}A(2n).$$

Next, the condition that $\text{sum}(a, b-2)$ is neat implies that $b \equiv 2 \mod 4$, so $B(b) = \frac{1}{2}A(b)$. Thus

$$N_{\text{sgn}}^\circ(n)^{001} = \sum_{a+(b-2)\equiv n-2} A(a)B(b) - \frac{1}{2}A(a)A(b)$$

$$= 0.$$

We conclude that $N_{\text{sgn}}^\circ(n) = \frac{1}{4}A(2n)$.

**Case $n \equiv 3 \mod 4$:** This time, the codes ‘111’ and ‘101’ do not occur in the $(n - 2)$nd row of Pascal’s triangle mod 2. It follows that

$$N_{\text{sgn}}^\circ(n) = N_{\text{sgn}}^\circ(n)^{100} + N_{\text{sgn}}^\circ(n)^{011} + N_{\text{sgn}}^\circ(n)^{110} + N_{\text{sgn}}^\circ(n)^{001}.$$

Now

(10)

$$N_{\text{sgn}}^\circ(n)^{100} + N_{\text{sgn}}^\circ(n)^{001} = \sum_{\text{sum}(a-2,b) \text{ neat}} \sum_{\text{sum}(a-1,b-1) \text{ messy}} \sum_{\text{sum}(a,b-2) \text{ messy}} 2A(b)B(a) - \frac{1}{2}A(a)A(b),$$
and

\[(11)\]

\[N_{\text{sgn}^0}(n)^{011} + N_{\text{sgn}^0}(n)^{110} = \sum_{\text{sum}(a-2,b) \text{ messy}} 2A(a)B(b) - \frac{1}{2}A(a)A(b).\]

Next, the conditions on \(a\) and \(b\) for the sum in (10) are equivalent to \((a-2) + b \equiv n-2\), \(a \equiv 3 \mod 4\), and \(b \equiv 0 \mod 4\). But then \(B(a) = A(a)\), and \(A(a) = 2A(a-2)\). So (10) becomes

\[3 \cdot 2^{\nu(n)-3}A(n).\]

Meanwhile, the conditions in (11) are equivalent to \((a-1) + (b-1) \equiv n-2\), \(a \equiv 1 \mod 4\), and \(b \equiv 2 \mod 4\). Now \(B(b) = \frac{1}{2}A(b)\) and \(A(b) = 2A(b-2)\). So (11) becomes

\[2^{\nu(n)-3}A(n).\]

The result follows, since \(A(n)2^{\nu(n)} = A(2n)\).

**Case** \(n \equiv 1 \mod 4\): This time we have

\[N_{\text{sgn}^0}(n) = N_{\text{sgn}^0}(n)^{100} + N_{\text{sgn}^0}(n)^{011} + N_{\text{sgn}^0}(n)^{111} + N_{\text{sgn}^0}(n)^{110} + N_{\text{sgn}^0}(n)^{001}.\]

Again, Equations (10) and (11) hold, but we must add

\[N_{\text{sgn}^0}(n)^{111} = \sum_{\text{sum}(a-1,b-1) \text{ neat}} A(a)B(b) + A(b)B(a) - \frac{1}{2}A(a)A(b).\]

Now

\[N_{\text{sgn}^0}(n)^{111} = N_{\text{sgn}^0}(n)^{111}_0 + N_{\text{sgn}^0}(n)^{111}_1 + N_{\text{sgn}^0}(n)^{111}_2 + N_{\text{sgn}^0}(n)^{111}_3,
\]

where \(N_{\text{sgn}^0}(n)^{111}_i\) takes the sum over \(a \equiv i \mod 4\). But since \(N_{\text{sgn}^0}(a,b) = N_{\text{sgn}^0}(b,a)\) for such \(a, b\), we also have \(N_{\text{sgn}^0}(n)^{111}_3 = N_{\text{sgn}^0}(n)^{111}_2\) and \(N_{\text{sgn}^0}(n)^{111}_0 = N_{\text{sgn}^0}(n)^{111}_1\).

We have

\[N_{\text{sgn}^0}(n)^{111}_2 = \sum_{a \equiv 2 \mod 4} A(a)A(b)
\]

\[= 2A(n-2)2^{\nu(n-5)}
\]

\[= 2^{k-2}A(2n),\]

since \(B(a) = \frac{1}{2}A(a)\), \(B(b) = A(b)\), and \(A(a) = 2A(a-2)\).
On the other hand, putting $k = \text{ord}(n')$,

$$N_{\text{sgn}}^0(n)_0^{111} = \sum_{j=2}^{k-1} \sum_{(a-2)+b=n \atop \text{ord}(a)=j} A(a)B(b) + A(b)A(a) - \frac{1}{2} A(a)A(b).$$

Note that here, $\text{ord}(a) = \text{ord}(b-1) = j$. So $N_{\text{sgn}}^0(n)_0^{111}$ equals

$$\sum_{j=2}^{k-1} \sum_{(a-2)+b=n-2 \atop \text{ord}(a)=j} A(a)a(b)2^\lfloor \frac{j}{2} \rfloor - j + A(a)D(b) + \frac{1}{2} A(a)A(b) + A(b)D(a).$$

Now $a + b \equiv n$ with $\text{ord}(a) = \text{ord}(b) = j$, so by Proposition 5, $A(a)D(b) = A(b)D(a) = D_{<j}(n)$, so this gives

$$\sum_{j=2}^{k-1} \sum_{(a-2)+b=n-2 \atop \text{ord}(a)=j} A(a)A(b) \left( \frac{1}{2} + 2^\lfloor \frac{j}{2} \rfloor - j \right) + 2D_{<j}(n).$$

Since $A(a-2) = 2^{-j+\lfloor \frac{j}{2} \rfloor} A(a)$, we get

$$\sum_{j=2}^{k-1} \sum_{(a-2)+b=n-2 \atop \text{ord}(a)=j} A(n-2)2^{j-\lfloor \frac{j}{2} \rfloor} \left( \frac{1}{2} + 2^\lfloor \frac{j}{2} \rfloor - j \right) + 2D_{<j}(n),$$

which equals

$$\sum_{j=2}^{k-1} 2^{\nu(n)+k-j-3} \left[ A(n-2)2^{j-\lfloor \frac{j}{2} \rfloor} \left( \frac{1}{2} + 2^\lfloor \frac{j}{2} \rfloor - j \right) + 2D_{<j}(n) \right].$$

and then

$$A(n)2^{-k+\lfloor \frac{k}{2} \rfloor} \sum_{j=2}^{k-1} 2^{\nu(n)+k-j-3} \left[ 2^{j-\lfloor \frac{j}{2} \rfloor} \left( \frac{1}{2} + 2^\lfloor \frac{j}{2} \rfloor - j \right) \right] + \sum_{j=2}^{k-1} 2^{\nu(n)+k-j-2} D_{<j}(n).$$

The first sum here is

$$A(2n)2^{\lfloor \frac{k}{2} \rfloor-3} \sum_{j=2}^{k-1} \left( 2^{-\lfloor \frac{j}{2} \rfloor - 1} + 2^{-j} \right),$$

which equals 0 if $k < 3$. 

The second sum is
\[ 2^{\nu(n)-2} \sum_{j=2}^{k-1} 2^{k-j} D_{\leq j}(n) = 2^{\nu(n)-1} \sum_{i=1}^{k-1} (2^{k-i-1} - 1) D_i(n) \]
\[ = \frac{1}{16} A(2n) \left( \sum_{j=1}^{k-1} 2^{(k)-(\frac{j}{2})} \right) - 2^{\nu(n)-1} D(n). \]

Therefore if \( k \geq 3 \) we have
\[ N_{\text{sgn}}(n)_{011} = A(2n) 2^{(k)} \sum_{j=2}^{k-1} \left( 2^{-\left(\frac{j}{2}\right)-4} + 2^{-j-3} \right) + \frac{1}{16} A(2n) \left( \sum_{j=1}^{k-1} 2^{(k)-(\frac{j}{2})} \right) - 2^{\nu(n)-1} D(n); \]
note that it is 0 if \( k = 2 \).

From before,
\[ N_{\text{sgn}}(n)^{111} = 2N_{\text{sgn}}(n)^{111} + 2N_{\text{sgn}}(n)^{111}. \]
If \( k = 2 \), then
\[ N_{\text{sgn}}(n)^{111} = \frac{1}{2} A(2n). \]

If \( k \geq 3 \), then \( N_{\text{sgn}}(n)^{111} \) equals
\[ 2^{(k)-2} A(2n) + A(2n) 2^{(k)} \sum_{j=2}^{k-1} \left( 2^{-\left(\frac{j}{2}\right)-3} + 2^{-j-2} \right) + \frac{1}{8} A(2n) \left( \sum_{j=1}^{k-1} 2^{(k)-(\frac{j}{2})} \right) - 2^{\nu(n)} D(n). \]

Meanwhile,
\[ N_{\text{sgn}}(n)^{011+110} = \sum_{\text{sum}(a,b) \text{ messy}} 2A(a)B(b) - \frac{1}{2} A(a)A(b). \]

The conditions on \( a, b \) imply that \( \text{sum}(a, b) \) is neat, \( a \equiv 1 \mod 4, b \equiv 0 \mod 4 \), and \( k \in \text{bin}(b) \). So
\[ 2A(a)B(b) - \frac{1}{2} A(a)A(b) = 2A(a)(D(b) + \frac{1}{2} A(b)) - \frac{1}{2} A(a)A(b) \]
\[ = 2D(n) + \frac{1}{2} A(n). \]

The number of \( a, b \) satisfying this condition is \( 2^{\nu(n)-2} \), so we have
\[ N_{\text{sgn}}(n)^{011+110} = 2^{\nu(n)-2} \left( 2D(n) + \frac{1}{2} A(n) \right). \]

Next,
\[ \sum_{\text{sum}(a-2, b) \text{ neat}} 2A(b)B(a) - \frac{1}{2} A(a)A(b). \]

The conditions on \( a, b \) again imply that \( \text{sum}(a, b) \) is neat, \( a \equiv 1 \mod 4 \), \( b \equiv 0 \mod 4 \), and \( k \in \text{bin}(a) \). So

\[ 2A(b)B(a) - \frac{1}{2} A(a)A(b) = \frac{1}{2} A(n) + 2\left(\begin{array}{c} k \\ 2 \end{array}\right) A(n) + 2D(n). \]

The number of \( a, b \) satisfying this condition is again \( 2^{\nu(n)-2} \), so we have

\[ N_{\text{sgn}^0}(n)^{100+001} = 2^{\nu(n)-2} \left( \frac{1}{2} A(n) + 2\left(\begin{array}{c} k \\ 2 \end{array}\right) A(n) + 2D(n) \right). \]

Thus

\[ N_{\text{sgn}^0}(n)^{011+110+100+001} = 2^{\nu(n)-2} \left( A(n) + 2\left(\begin{array}{c} k \\ 2 \end{array}\right) A(n) + 4D(n) \right) \]

We obtain \( N_{\text{sgn}^0}(n) = \frac{5}{4} A(2n) \), if \( k = 2 \). For \( k \geq 3 \) we finally obtain

\[ N_{\text{sgn}^0}(n) = A(2n) \left( \frac{1}{4} + 3 \cdot 2\left(\begin{array}{c} k \\ 2 \end{array}\right) - 3 + 2\left(\begin{array}{c} k \\ 2 \end{array}\right) - 1 - \sum_{j=2}^{k-1} 2\left(\begin{array}{c} j \\ 2 \end{array}\right) - (j-2)^2 \right) \).

**Case** \( n \equiv 0 \mod 4 \): Now

\[ N_{\text{sgn}^0}(n) = N_{\text{sgn}^0}(n)^{100} + N_{\text{sgn}^0}(n)^{001} + N_{\text{sgn}^0}(n)^{101}. \]

We have

\[ N_{\text{sgn}^0}(n)^{100} = \sum_{(a-2)+b=n-2 \atop \text{sum}(a,b-2) \text{ messy}} A(b)B(a), \]

\[ N_{\text{sgn}^0}(n)^{001} = \sum_{a+(b-2)=n-2 \atop \text{sum}(a-2,b) \text{ messy}} A(a)D(b). \]

Since \( B(n) = \frac{1}{2} A(n) + D(n) \) for \( n \) even, we may write

\[ N_{\text{sgn}^0}(n)^{101} = \sum_{(a-2)+b=n-2 \atop a+(b-2)=n-2} A(a)D(b) + A(b)D(a). \]

So

\[ N_{\text{sgn}^0}(n)^{100} = \sum_{a+b=n \atop k \in \text{bin}(a)} A(b)B(a), \]
and

\[ N_{\text{sgn}}(n)^{001} = \sum_{a+b|n \atop k \in \text{bin}(b)} A(a)D(b). \]

Thus

\[
N_{\text{sgn}}(n)^{100} + N_{\text{sgn}}(n)^{001} = \sum_{a+b|n \atop k \in \text{bin}(b)} A(a) \left( 2D(b) + \frac{1}{2}A(b) \right)
= \sum_{a+b|n \atop k \in \text{bin}(b)} 2D(n) + \frac{1}{2}A(n)
= 2^{\nu(n)} \left( D(n) + \frac{1}{4}A(n) \right).
\]

We have used Proposition 5 for the equality \( A(a)D(b) = D(n) \).

\[
N_{\text{sgn}}(n)^{101} = 2 \sum_{(a-2)+b|n-2 \atop a+(b-2)|n-2} A(a)D(b)
= 2 \sum_{j=2}^{k-1} \sum_{a+b|n \atop k \in \text{bin}(b) \atop \text{ord}(b) = j} A(a)D(b)
= 2^{\nu(n)-2j+1}D_{\leq j}(n)
= 2^{\nu(n)+k-j-2}D_{\leq j}(n)
= 2^{\nu(n)-1} \sum_{j=2}^{k-1} 2^{k-j}D_{\leq j}(n)
= 2^{\nu(n)} \sum_{i=1}^{k-1} (1 + 2 + \cdots + 2^{k-i-2})D_i(n).
\]
We have again used (2). Therefore

\[ N_{\text{sgn}}(n) = 2^{\nu(n)} \left( \sum_{j=1}^{k-1} (1 + 2 + \cdots + 2^{k-j-2})D_j(n) + \left( \sum_{j=1}^{k-1} D_j(n) \right) + \frac{1}{4} A(n) \right) \]

\[ = 2^{\nu(n)} \left( \frac{1}{4} A(n) + \sum_{j=1}^{k-1} 2^{k-j-1}D_j(n) \right) \]

\[ = 2^{\nu(n)} A(n) \left( \frac{1}{4} + \sum_{j=1}^{k-1} 2^{k-j-3} \right) \]

\[ = \frac{1}{8} A(2n) \left( 2 + \sum_{j=1}^{k-1} 2^{(k-j)-2j} \right). \]

\[ \square \]

**Corollary 8.** When \( n \) is even, we have \( N_\varepsilon(n) = N_{\text{sgn}}(n) \). \( \square \)

**Theorem 8.** For \( n \geq 2 \), and \( k = \text{ord}(n) \), we have

\[ N_{\text{sgn}}(n) = \begin{cases} 
\frac{1}{4} A(2n) & \text{if } n \text{ is odd} \\
\frac{1}{8} A(2n) \left( 2 + 2^k + \sum_{j=1}^{k-1} 2^{(k-j)-2j} \right) & \text{if } n \text{ is even.}
\end{cases} \]

**Proof.** For \( n \) odd, this is Corollary 7. Otherwise this follows from the identity \( N_\varepsilon(n) + N_{\text{sgn}}(n) = B(2n) \), and the formulas for \( N_\varepsilon(n) \) and \( B(2n) \). \( \square \)

### 5.7. Inequalities

In this section, we discuss how to establish, for \( n \geq 10 \):

(1) \( N_\varepsilon(n) = N_{\text{sgn}}(n) < N_{\text{sgn}}(n) < N_1(n) \), for \( n \) odd

(2) \( N_\varepsilon(n) = N_{\text{sgn}}(n) < N_{\text{sgn}}(n) < N_1(n) \), for \( n \) even.

from the formulas in the previous section. We have already established \( N_\varepsilon(n) = N_{\text{sgn}}(n) \) when \( n \) is odd in Corollary 6 and \( N_\varepsilon(n) = N_{\text{sgn}}(n) \) when \( n \) is even in Corollary 8.

If \( n \) is even, we have

\[ N_\varepsilon(n) = \frac{1}{8} A(2n) \left( 2 + \sum_{j=1}^{k-1} 2^{(k-j)-2j} \right) \]

\[ < \frac{1}{8} A(2n) \left( 2 + 2^k + \sum_{j=1}^{k-1} 2^{(k-j)-2j}(2j-1) \right) \]

\[ = N_{\text{sgn}}(n). \]
If \( n \) is odd, then \( N_\varepsilon(n) = \frac{1}{4}A(2n) \), and it is obvious from the formulas that \( N_\varepsilon(n) < N_{\text{sgn}^0}(n) \).

Comparing these formulas to

\[
N_1(n) = p_2(n) - N_\varepsilon(n) - N_{\text{sgn}^0}(n) - N_{\text{sgn}^1}(n)
\]

is considerably more difficult, since the partition function itself does not have an explicit formula. Let us sketch how this can be done. We will take \( n \) even for our sketch; the case \( n \) odd is similar. By the earlier inequalities, we have

\[
\frac{N_1(n)}{N_{\text{sgn}^1}(n)} > \frac{p_2(n)}{N_{\text{sgn}^1}(n)} - 3 \\
> \frac{p_2(n)}{A(2n)(2 + 2^{k+1})} - 3.
\]

So we must estimate \( p_2(n) \) from below and \( A(2n) \) from above; note that \( 2^k \leq n \). It is easy to see that \( p_2(n) \geq 2p(n) \). Lemma 2 yields the (subexponential) upper bound

\[
A(2n) \leq 2n^{\frac{1}{2}(\log_2 n+3)}.
\]

According to Hardy-Ramanujan [HR00], we have

\[
p(n) \sim \frac{1}{4n\sqrt{3}} \exp \left( \frac{\pi \sqrt{2n}}{3} \right)
\]

as \( n \to \infty \). From the above it is straightforward to compute that

\[
\lim_{n \to \infty} \frac{N_1(n)}{N_{\text{sgn}^0}(n)} = \infty.
\]

But the inequality for all \( n \geq 10 \) takes more work. For this, one may start with the classical estimate

\[
p(n) \geq \frac{n^{j-1}}{j!(j-1)!}
\]

for all \( 1 \leq j \leq n \) (see [Pak06]), and use the estimate

\[
k! \leq \sqrt{2\pi k^k + \frac{1}{2}e^{-k}} e^{\frac{k}{\pi}}
\]

of H. Robbins in [Rob55]. With care one can prove from these estimates, and from computation for small \( n \), that

\[
p(n) \geq 5nA(n)
\]

for \( n \geq 64 \). Our desired inequality ultimately follows from this.

This concludes our sketch of how to prove (for \( n \geq 10 \)) that \( N_{\text{sgn}^i}(n) < N_1(n) \) for \( n \) even, and similarly \( N_{\text{sgn}^0}(n) < N_1(n) \) for \( n \) odd.
6. Type $D_n$: Demihyperoctahedral Groups

We denote by $D_n$ the kernel of $\varepsilon : B_n \to \{\pm 1\}$; it is the Weyl group of type $D_n$, called the demihyperoctahedral group. Since $D_1$ is trivial, $D_2$ is the Klein 4-group, and $D_3$ is isomorphic to $S_4$, we may take $n \geq 4$ when convenient.

The group $D_n$, for $n \geq 2$, has two multiplicative characters: 1 and $\text{sgn}$, where $\text{sgn}$ is the restriction of $\text{sgn}^0$ (or of $\text{sgn}^1$) to $D_n$. To avoid confusion with earlier notation, let us write

$$N'_\omega(n) = \#\{\pi \in \text{Irr}(D_n) \mid \det \pi = \omega\},$$

for $\omega = 1, \text{sgn}$.

6.1. Clifford Theory

In this subsection we assume Clifford Theory (especially the index 2 case) as well-known, and tacitly apply it to the subgroup $D_n$ of $B_n$.

Firstly, the restriction of $\rho_{\alpha,\beta}$ from $B_n$ to $D_n$ is irreducible iff $\alpha \neq \beta$. Let us call representations obtained this way “Type I” representations. Moreover $\text{Res}^{B_n}_{D_n} \rho_{\alpha_1,\beta_1} \cong \text{Res}^{B_n}_{D_n} \rho_{\alpha_2,\beta_2}$ iff $(\alpha_2, \beta_2) = (\alpha_1, \beta_1)$ or $(\beta_1, \alpha_1)$.

Next, if $n$ is even and $\alpha \vdash \frac{n}{2}$, then the restriction of $\rho_{\alpha,\alpha}$ to $D_n$ is a direct sum of two non-isomorphic irreducible representations, say $\rho_+^\alpha$ and $\rho_-^\alpha$. Let us call these “Type II” representations. The representation $\rho_-^\alpha$ is the twist of $\rho_+^\alpha$ by $e_1 \in B_n - D_n$. We will write $\chi^\pm_\alpha$ for the character of $\rho^\pm_\alpha$. We have

$$\text{Irr}(D_n) = \{ \text{Res}^{B_n}_{D_n} \rho_{\alpha,\beta} \mid (\alpha, \beta) \models n, \alpha \neq \beta \} \coprod \{ \rho_+^\alpha, \rho_-^\alpha \mid \alpha \vdash \frac{n}{2} \},$$

with the second set nonempty only when $n$ is even. The total number of representations of $D_n$ is therefore equal to $\frac{1}{2}p_2(n)$ when $n$ is odd, and equal to

$$\frac{1}{2}p_2(n) + \frac{3}{2}p\left(\frac{n}{2}\right)$$

when $n$ is even. In particular, there are two multiplicative characters of $D_n$: the trivial character, and the restriction of $\text{sgn}^0$ from $B_n$, which we denote by ‘$\text{sgn}$’.

The determinants of the Type I representations are given by

$$\det \text{Res}^{B_n}_{D_n} \rho_{\alpha,\beta} = (\text{sgn})^{y_\alpha \beta}.$$

So if $n$ is odd, we have

$$N'_1(n) = \frac{1}{2}(N_1(n) + N_\varepsilon(n))$$
and
\[ N'_{\text{sgn}}(n) = \frac{1}{2} (N_{\text{sgn}^0}(n) + N_{\text{sgn}^1}(n)). \]
(The \( N_\omega(n) \) were computed in the previous section.)

6.2. Type II Determinants

Let \( n = 2a \). For \( \alpha \vdash a \), we have \( \det \text{Res}^{D_{n}} \rho_{\alpha,\alpha} = 1 \) by Proposition 9. It follows that \( \det \rho^+_{\alpha} = \det \rho^-_{\alpha} \), so we need only compute \( \det \rho^+_{\alpha} \).

Put
\[ x^+_{\alpha} = \frac{\text{dim } \rho^+_{\alpha} - \chi^+_{\alpha}(s_1)}{2} \in \mathbb{Z}/2\mathbb{Z}. \]
By Proposition 11,
\[ \det \rho^+_{\alpha} = (\text{sgn}) x^+_{\alpha}. \]
If \( n \geq 3 \), it is not hard to see that \( \text{Int}(e_1)(s_1) = e_1 s_1 e_1^{-1} \) is conjugate in \( D_n \) to \( s_1 \); it follows that \( \chi^+_{\alpha}(s_1) = \chi^-_{\alpha}(s_1) \).

Using Equation (7) we compute
\[ \chi^+_{\alpha}(s_1) = \frac{1}{2} \chi_{\rho_{\alpha}}(s_1) = \left( \begin{array}{c} n-2 \\ a, a-2 \end{array} \right) f_{\alpha} \chi_{\alpha}(s_1) = \left( \begin{array}{c} n-2 \\ a, a-2 \end{array} \right) f_{\alpha} (f_{\alpha} - 2g_{\alpha}), \]
and so
\[ x^+_{\alpha} = \frac{1}{4} \left( \begin{array}{c} n \\ a, a \end{array} \right) f_{\alpha}^2 - \frac{1}{2} \left( \begin{array}{c} n-2 \\ a-2, a \end{array} \right) f_{\alpha}^2 + \left( \begin{array}{c} n-2 \\ a, a-2 \end{array} \right) f_{\alpha} g_{\alpha}. \]
If \( f_{\alpha} \) is even, then clearly \( x^+_{\alpha} \) is even. If \( f_{\alpha} \) is odd, then
\[ x^+_{\alpha} = \frac{1}{4} \left( \begin{array}{c} n \\ a, a \end{array} \right) f_{\alpha}^2 - \frac{1}{2} \left( \begin{array}{c} n-2 \\ a-2, a \end{array} \right) + \left( \begin{array}{c} n-2 \\ a, a-2 \end{array} \right) g_{\alpha} \]
\[ = \frac{1}{2} \left( \begin{array}{c} n-2 \\ a-1, a-1 \end{array} \right) + \left( \begin{array}{c} n-2 \\ a, a-2 \end{array} \right) g_{\alpha}. \]

Lemma 5. Let \( n = 2a \geq 6 \) be an even integer.

1. If \( n \) is neither of the form \( 2^k \) nor \( 2^k + 2 \), then \( \left( \begin{array}{c} n-2 \\ a-1, a-1 \end{array} \right) \) is a multiple of 4, and \( \left( \begin{array}{c} n-2 \\ a, a-2 \end{array} \right) \) is even.
2. If \( n \) is of the form \( 2^k \), then \( \left( \begin{array}{c} n-2 \\ a-1, a-1 \end{array} \right) \) is a multiple of 4, and \( \left( \begin{array}{c} n-2 \\ a, a-2 \end{array} \right) \) is odd.
3. If \( n \) is of the form \( 2^k + 2 \), then \( \frac{1}{2} \left( \begin{array}{c} n-2 \\ a-1, a-1 \end{array} \right) \) is odd and \( \left( \begin{array}{c} n-2 \\ a, a-2 \end{array} \right) \) is even.
The proof is left to the reader. □

**Proposition 14.** Let \( n \geq 6 \) be an even integer.

1. If \( n \) is not of the form \( 2^k \) or \( 2^k + 2 \), then all Type II representations have determinant 1.
2. If \( n \) is of the form \( 2^k \), then \( \rho_\alpha^+ , \rho_\alpha^- \) have determinant \( \text{sgn} \) iff \( f_\alpha \) and \( g_\alpha \) are odd.
3. If \( n \) is of the form \( 2^k + 2 \), then \( \rho_\alpha^+ , \rho_\alpha^- \) have determinant \( \text{sgn} \) iff \( f_\alpha \) is odd.

**Theorem 9.** Let \( n \geq 4 \). We have

\[
N'_\text{sng}(n) = \begin{cases} 
\frac{1}{2}(N_{\text{sng}0}(n) + N_{\text{sng}1}(n)) + \frac{1}{2}n & \text{if } n = 2^k \text{ for some } k \geq 1, \\
\frac{1}{2}(N_{\text{sng}0}(n) + N_{\text{sng}1}(n)) + n - 2 & \text{if } n = 2^k + 2 \text{ for some } k \geq 1, \\
\frac{1}{2}(N_{\text{sng}0}(n) + N_{\text{sng}1}(n)) & \text{otherwise.}
\end{cases}
\]

This follows from the above and using \( A(n/2) = \frac{1}{2}n \) when \( n \) is a power of 2, and \( A(n/2) = \frac{1}{2}n - 1 \) when \( n = 2^k + 2 \). (The case \( n = 4 \) is easily done separately, and agrees with the first two cases.)

Note that if \( n \) is even, then

\[
N'_1(n) + N'_\text{sng}(n) = \frac{1}{2}p_2(n) + \frac{3}{2}p(n),
\]

so in principle we have a formula for \( N'_1(n) \) as well.

7. **Exceptional Coxeter Groups**

There are only finitely many irreducible finite Coxeter groups not of type \( A_n, B_n, D_n, \) or \( I_2(n) \), namely the exceptional ones. For completeness, we give \( N_\omega \) for each exceptional group. This is easily done by the propositions in Section 2.4, together with the character tables for these groups, which may be found in [GP00].

For the exceptional Coxeter groups \( W \) with \(|W_{ab}| = 2\), we offer the following table. Below \( N_1 \) is the number of irreducible representations of \( W \) with trivial determinant, and \( N_{\varepsilon_W} \) is the number with \( \varepsilon_W \) as determinant.

| Type | \( N_1 \) | \( N_{\varepsilon_W} \) |
|------|----------|----------------|
| \( H_3 \) | 6 | 4 |
| \( H_4 \) | 19 | 15 |
| \( E_6 \) | 13 | 12 |
| \( E_7 \) | 44 | 16 |
| \( E_8 \) | 63 | 49 |

Finally, for type \( F_4 \), we have \( N_1 = 9 \), \( N_{\varepsilon_W} = 8 \), and \( N_\omega = 4 \) for \( \omega \neq 1, \varepsilon_W \).
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