PROJECTIVE BUNDLES AND BLOW-UPS OF PROJECTIVE SPACES.

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Abstract. The aim of this note is to investigate the relation between two types of non-singular projective varieties of Picard rank 2, namely the Projective bundles over Projective spaces and certain Blow-up of Projective spaces.

Keywords: Projective spaces; Projective bundles; Blow-ups.

1. Introduction

Let $X$ be the projective variety obtained from projective space $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$ by blowing up along a linear subspace $\mathbb{P}(W^*)$ of co-dimension $r$, where $\mathbb{C}^{n+1} \to W$ is a quotient vector space of dimension $n-r+1$. If $K$ is the kernel of surjective linear map $\mathbb{C}^{n+1} \to W$ resulting projective variety admits a projective bundle structure over the smaller dimensional projective space $\mathbb{P}(K^*)$. In fact

$$X \simeq \mathbb{P}(W \otimes \mathcal{O}_{\mathbb{P}(K^*)} \oplus \mathcal{O}_{\mathbb{P}(K^*)}(1)).$$

Question: Are there any other examples of blow-ups of a projective space along a sub variety that also have a structure of a projective bundle over a projective space?

Nabanita Ray in [NR] gave examples of non-linear sub varieties in $\mathbb{P}^i, i = 3, 4, 5$ whose blow-ups are projective bundles over $\mathbb{P}^2$.

In this note we provide several other examples of Projective bundles over a Projective space $\mathbb{P}^n$ which can be realised as a blow up of some $\mathbb{P}^m$ along a non-linear sub variety.

Theorem 1.1. Let $n \geq 2$ be an integer and $N = n^2 + n - 1$. Let

$$E = \text{Hom}(\mathcal{O}^n_{\mathbb{P}^n}, T_{\mathbb{P}^n}(-1))$$

be the rank $n^2$ vector bundle on $\mathbb{P}^n$, where $T_{\mathbb{P}^n}(-1)$ is tangent bundle of $\mathbb{P}^n$ twisted by the inverse of the ample bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. The projective bundle $\mathbb{P}(E)$ is isomorphic to a blow-up of $\mathbb{P}^N = \mathbb{P}(\text{Hom}(\mathbb{C}^n, \mathbb{C}^{n+1}))$ along the sub variety of all linear mappings of rank at most $n-1$.

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Theorem 1.2. Let \( n \) be an even integer and \( N = \frac{n(n+1)}{2} - 1 \). Let \( V = \wedge^2(T_{\mathbb{P}^n}(-1)) \) be the rank \( N+1-n = \frac{n(n-1)}{2} \) vector bundle over \( \mathbb{P}^n \). The projective bundle \( \mathbb{P}(V) \) is isomorphic to \( \mathbb{P}^{N} = \mathbb{P}(\text{Alt}(n+1)) \) blown-up along the sub variety consists of all alternating linear mappings of rank at most \( n-1 \), where \( \text{Alt}(n+1) \) denote the space of all alternating linear mappings from \( \mathbb{C}^{n+1} \) to itself.

2. The Projective bundle \( \mathbb{P}(\text{Hom}(O_{\mathbb{P}^n}, T_{\mathbb{P}^n}(-1))) \)

Proof of Theorem(1.1): On \( \mathbb{P}^n \) by applying \( \text{Hom}(O_{\mathbb{P}^n}, -) \) to the standard exact sequence
\[
0 \rightarrow O_{\mathbb{P}^n}(-1) \rightarrow O_{\mathbb{P}^n}^{n+1} \rightarrow T_{\mathbb{P}^n}(-1) \rightarrow 0.
\]
we obtain the exact sequence
\[
0 \rightarrow \text{Hom}(O_{\mathbb{P}^n}, O_{\mathbb{P}^n}(-1)) \rightarrow \text{Hom}(O_{\mathbb{P}^n}, O_{\mathbb{P}^n}^{n+1}) \rightarrow E \rightarrow 0.
\]

Note that \( E \) is a globally generated vector bundle and \( \dim H^0(E) = n(n+1) \). Hence we obtain a morphism from
\[
\phi : \mathbb{P}(E) \rightarrow \mathbb{P}^{n(n+1)-1}.
\]

We claim that this is a surjective morphism of degree one, i.e., \( \phi \) is a bi-rational morphism and hence \( \mathbb{P}(E) \) is a blow-up of \( \mathbb{P}^{n(n+1)-1} \) [See, Chapter II, Theorem 7.17. citeHa] Let \( \xi = O_{\mathbb{P}(E)}(1) \) denote the tautological line bundle on \( \mathbb{P}(E) \). Then there is an exact sequence of vector bundles on \( \mathbb{P}(E) \)
\[
0 \rightarrow \Omega^1_{\mathbb{P}(E)/\mathbb{P}^n} \rightarrow \pi^*(E) \rightarrow \xi \rightarrow 0,
\]
where \( \pi : \mathbb{P}(E) \rightarrow \mathbb{P}^n \) is the natural projection. Since \( \xi^{n(n+1)-1} \) is the degree of the morphism \( \phi \) it is enough to show that \( \xi^{n(n+1)-1} = 1 \). Since \( E \) is vector bundle of rank \( n^2 \) in the cohomology ring of \( \mathbb{P}(E) \) the relation
\[
\xi^{n^2} = \sum_{i=1}^{n} (-1)^{i+1} \pi^*(C_i(E)) \xi^{n^2-i}
\]
holds, where \( C_i(E) \), \( 1 \leq i \leq n \) are the Chern classes of \( E \). Repeated use of the relation 4 gives
\[
\xi^{n^2+n-1} = (-1)^n \pi^*(S_n(E)) \xi^{n^2-1},
\]
where \( S_n(E) \) is the \( n \)th Segre class of \( E \). By equation (3) and the fact that the total Segre class is the inverse of the total Chern class we deduce that \( S_n(E) = (-1)^n h^n \), where \( h \) is the class of the line bundle.
\( \mathcal{O}_{\mathbb{P}^n}(1) \) in \( H^2(\mathbb{P}^n, \mathbb{Z}) \). Hence \( \xi^{n^2+n-1} = 1.\pi^*(h^n)\xi^{n^2-1} \). Since \( \pi^*(h^n)\xi^{n^2-1} \) is the class of a point in \( \mathbb{P}(E) \) it follows that the degree of the map \( \phi \) is one.

**Remark:** The bi-rational morphism \( \phi \) can be described geometrically as follows: let

\[
T = \{(v, \varphi) \in \mathbb{P}^n \times \mathbb{P}(\text{Hom}(\mathbb{C}^n, \mathbb{C}^{n+1})) | (\varphi)^t(v) = 0\},
\]

where \((\varphi)^t : \mathbb{C}^{n+1} \to \mathbb{C}^n\) is the transpose of the map \( \varphi : \mathbb{C}^n \to \mathbb{C}^{n+1} \). Then \( T \simeq \mathbb{P}(E) \) and the bi-rational morphism \( \phi \) is the projection onto the second factor. Let

\[
D = \{ \varphi \in \mathbb{P}(\text{Hom}(\mathbb{C}^n, \mathbb{C}^{n+1})) | \text{rank}(\varphi) \leq n - 1 \},
\]

\( F = \phi^{-1}(D) \) and \( U = \mathbb{P}(E) \setminus F \). The morphism \( \phi \) is one-one on the open set \( U \).

Next we prove that the set \( F \) in the previous Remark is a divisor and we identify its class in the Picard group of \( \mathbb{P}(E) \). Note that Picard group of \( \mathbb{P}(E) \) is equal to \( H^2(\mathbb{P}(E), \mathbb{Z}) = \mathbb{Z}[\pi^*(h)] \oplus \mathbb{Z}[\xi] \).

**Lemma 2.1.** With the notations of previous Remark the set \( F \) is a divisor in \( \mathbb{P}(E) \) and its divisor class is \( n[\xi] - [\pi^*(h)] \).

**Proof:** Rewriting the exact sequence (2) as

\[
0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n}^{(n+1)} \to E \to 0
\]

and taking \( n \)-th symmetric power we obtain a long exact sequence

\[
0 \to \mathcal{O}_{\mathbb{P}^n}(-n) \to \mathcal{O}_{\mathbb{P}^n}^{(n+1)} \otimes \wedge^{n-1}(\mathcal{O}_{\mathbb{P}^n}(-1)) \to \cdots
\]

\[
\cdots \to S^{n-1}(\mathcal{O}_{\mathbb{P}^n}^{(n+1)}) \otimes \mathcal{O}_{\mathbb{P}^n}(-1)^n \to \mathcal{O}_{\mathbb{P}^n}^{n(n+1)} \to S^n(\mathcal{O}_{\mathbb{P}^n}^{n(n+1)}) \to S^n(E) \to 0.
\]

Tensoring the long exact sequence (7) (8) by \( \mathcal{O}_{\mathbb{P}^n}(-1) \) and computing the cohomology we deduce

\[
H^0(\mathbb{P}(E), \xi \otimes \pi^*(\mathcal{O}_{\mathbb{P}^n}(-1))) \simeq H^0(S^n(E)(-1))
\]

\[
\simeq H^n(\mathcal{O}_{\mathbb{P}^n}(-n - 1)) \simeq \mathbb{C}.
\]

This proves that the divisor class \( n[\xi] - [\pi^*(h)] \) contains a unique effective divisor. From the exact sequence (2) we deduce that the Segre class \( S_{n-1}(E) \) is equal to \((-1)^{n-1}nh^{-1}\) and hence \((n[\xi] - [\pi^*(h)]).\xi^{n^2+n-2} = n\pi^*(h^n)\xi^{n^2-1} - (-1)^{n-1}\pi^*(h)\pi^*(S_{n-1}(E))\xi^{n^2-1} = 0 \). For \( v \in \mathbb{P}^n \) the fibre \( E_v \) is isomorphic to \( \text{Hom}(\mathbb{C}^n, \mathbb{C}^{n+1}/\mathbb{C}.v) \). For \( \varphi \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^{n+1}) \) rank of \((\varphi)^t\) is less than or equal to \( n - 1 \) if and only if image of \( \tilde{\varphi} \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^{n+1}/\mathbb{C}.v) \) of \( \varphi \) has rank less than or equal to \( n - 1 \), i.e., \( \tilde{\varphi} \) is not an isomorphism. Since the complement of isomorphisms in
Hom(\mathbb{C}^n, \mathbb{C}^{n+1}/\mathbb{C}.v) is given by vanishing of the homogeneous polynomial of degree in \( n \), i.e., section of \( \mathcal{O}(E_v)(n) \). The set

\[ F_v = \{ \bar{\varphi} \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^{n+1}/\mathbb{C}.v) | \text{rank}(\bar{\varphi}) \leq n - 1 \} \]

is an irreducible divisor given by the vanishing of the restriction to \( \mathbb{P}(E_v) \) of the non-zero section (unique upto non zero scalar) of

\[ H^0(\mathbb{P}(E), \xi \otimes \pi^*(\mathcal{O}_{\mathbb{P}^n}(-1))) \]

The non-zero section \( t \) (unique upto multiplication by a non zero scalar) determines a section of \( \mathcal{O}(\mathbb{P}(E_v))^n \). It is clear that \( F_v \) is the set of zeros of the section \( t|_{\mathbb{P}(E_v)} \). This proves that the set \( F \) is the support of the divisor \( n[\xi] - [\pi^*\xi] \).

\[ \square \]

Remark 1: Theorem (1.1) can be used to obtain for blow up of \( \mathbb{P}^m \), \( (2n - 1 \leq m \leq n(n + 1) - 1) \), along a non linear sub-variety a projective bundle structure over \( \mathbb{P}^n \) for any integer \( n \geq 2 \).

3. Projective bundle \( \mathbb{P}(\wedge^2(T_{\mathbb{P}^n}(-1))) \).

**Proof of Theorem (1.2):**

Let \( n \geq 3 \) be an integer. By taking 2nd exterior power in the exact sequence (1) yields the sequence

\[ 0 \to T_{\mathbb{P}^n}(-2) \to \wedge^2\mathcal{O}_{\mathbb{P}^n}^{n+1} \to \wedge^2(T_{\mathbb{P}^n}(-1)) \to 0. \]

If we identify \( H^0(\mathcal{O}_{\mathbb{P}^n}^{n+1}) \) with \( \mathbb{C}^{n+1} \) then the \( H^0(\wedge^2\mathcal{O}_{\mathbb{P}^n}^{n+1}) \) gets identified with \( \text{Alt}(n + 1) \), where \( \text{Alt}(n + 1) \) denote the set of all alternating linear mappings from \( \mathbb{C}^{n+1} \) to itself. Using the exact sequence (9) we obtain

\[ H^0(\wedge^2(T_{\mathbb{P}^n}(-1))) \simeq \text{Alt}(n + 1). \]

Since \( T_{\mathbb{P}^n}(-1) \) is globally generated \( V = \wedge^2(T_{\mathbb{P}^n}(-1)) \) is generated by \( \text{Alt}(n + 1) \). Therefore we get a morphism

\[ \psi : \mathbb{P}(V) \to \mathbb{P}(\text{Alt}(n + 1)). \]

We claim that \( \psi \) is bi-rational i.e., degree of \( \psi \) is one and hence \( \mathbb{P}(V) \) is a blow-up of \( \mathbb{P}(\text{Alt}(n + 1)) \). [See, Chapter II, Theorem 7.17. citeHa]

Let \( \zeta = \mathcal{O}(V)(1) \) denote the tautological line bundle on \( \mathbb{P}(V) \). Then there is an exact sequence of vector bundles on \( \mathbb{P}(V) \)

\[ 0 \to \Omega_{\mathbb{P}(V)/\mathbb{P}^n}^1 \to \pi^*(V) \to \zeta \to 0, \]

where \( \pi : \mathbb{P}(V) \to \mathbb{P}^n \) is the natural projection. Since \( \zeta^{\left(\frac{n(n+1)}{2} - 1\right)} \) computes the degree of the morphism \( \psi \) it is enough to show that, for \( n \) even,

\[ \zeta^{\left(\frac{n(n+1)}{2} - 1\right)} = \pi^*(h^n)\zeta^{\frac{n(n-1)}{2} - 1}. \]
Since $V$ is vector bundle of rank $n(n-1)/2$ in the cohomology ring of $\mathbb{P}(V)$ the relation
\[
\zeta_n^{n(n-1)/2} = \sum_{i=1}^{n} (-1)^{i+1} \pi^*(C_i(V)) \zeta_n^{n(n+1)/2-i}
\]
holds, where $C_i(V)$, $(1 \leq i \leq n)$ are the Chern classes of $V$. Repeated use of the relation (12) gives
\[
\zeta_n^{n(n+1)/2-1} = (-1)^n \pi^*(S_n(V)) \zeta_n^{n(n+1)/2-1},
\]
where $S_n(V)$ is the $n$th Segre class of $V$. By equation (9) and the fact that the total Segre class of $V$ is the total Chern class $C(T_{\mathbb{P}^n}(-2))$ of $T_{\mathbb{P}^n}(-2)$. Tensoring the equation (2) by $\mathcal{O}_{\mathbb{P}^n}(-1)$ to obtain
\[
C(T_{\mathbb{P}^n}(-2)) = (1 - h)^{n+1}(1 - 2h)^{-1},
\]
where $h$ is the class of the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ in $H^2(\mathbb{P}^n, \mathbb{Z})$. From the equation (14) we deduce that
\[
C_n(T_{\mathbb{P}^n}(-2)) = \left(\sum_{i=0}^{n} (-1)^i \binom{n+1}{i} 2^{n-i}\right) h^n
\]

Thus
\[
S_n(V) = \begin{cases} h^n & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}
\]
Now from equation (13) we deduce that, for $n$ even
\[
\zeta_n^{n(n+1)/2-1} = \pi^*(h^n) \zeta_n^{n(n-1)/2-1}.
\]
Since $\pi^*(h^n) \zeta_n^{n(n-1)/2-1}$ is the class of a point in $\mathbb{P}(V)$ it follows that the degree of the map $\psi$ is one when $n$ is even. \qed

**Remark:** The bi-rational morphism $\psi$ can be described geometrically as follows: let
\[
S = \{(v, \varphi) \in \mathbb{P}^n \times \mathbb{P}(\text{Alt}(n+1)) | (\varphi)(v) = 0 \}.
\]
For $v \in \mathbb{C}^{n+1} \setminus \{0\}$ the linear subspace $\phi \in \text{Alt}(n+1)$ gets identified with fibre $\wedge^2(T_{\mathbb{P}^n}(-1))\nu$ of $\wedge^2(T_{\mathbb{P}^n}(-1))_v$ over the point $[v] \in \mathbb{P}^n$ hence $S \simeq \mathbb{P}(V)$. Under this identification, when $n$ is even, the bi-rational morphism $\psi$ is the projection onto the second factor. From now on we assume $n$ is an even integer say $n = 2k$. Let
\[
D = \{\varphi \in \mathbb{P}(\text{Alt}n+1) | \text{rank}(\varphi) \leq n-1\},
\]
\( F = \phi^{-1}(D) \) and \( U = \mathbb{P}(E) \setminus F \). The morphism \( \psi \) is one-one on the open set \( U \).

Next we prove that the set \( F \) in the previous Remark is a divisor and we identify its class in the Picard group of \( \mathbb{P}(V) \). Note that Picard group of \( \mathbb{P}(V) \) is equal to \( H^2(\mathbb{P}(E), \mathbb{Z}) = \mathbb{Z}[\pi^*(h)] \oplus \mathbb{Z}[\zeta] \).

**Lemma 3.1.** With the notations of previous Remark the set \( F \) is a divisor in \( \mathbb{P}(V) \) and its divisor class is \( k[\zeta] - [\pi^*(h)] \), where \( k = n/2 \).

**Proof:** By equation (9) we obtain a long exact sequence for \( k \)-th symmetric power of \( V \)

\[
0 \to \Lambda^k(T_{\mathbb{P}^n}(-1)) \to \mathcal{O}_{\mathbb{P}^n}^{n(n+1)/2} \otimes \Lambda^{k-1}(T_{\mathbb{P}^n}(-1))) \to \cdots
\]

\[\cdots \to S^{k-1}(\mathcal{O}_{\mathbb{P}^n}^{n(n+1)/2} \otimes (T_{\mathbb{P}^n}(-1)) \to S^k(\mathcal{O}_{\mathbb{P}^n}^{n(n+1)/2}) \to S^k(V) \to 0.\]

Tensoring the long exact sequence (15) (16) by \( \mathcal{O}_{\mathbb{P}^n}(-1) \) and computing the cohomology we deduce

\[
H^0(\mathbb{P}(V), \zeta \otimes \pi^*(\mathcal{O}_{\mathbb{P}^n}(-1))) \simeq H^0(S^k(V)(-1))
\]

\[
\simeq H^k(\Lambda^k(T_{\mathbb{P}^n}(-1))(-1)) \simeq H^k((\Omega_{\mathbb{P}^n}^k)) \simeq \mathbb{C},
\]

where the \( \Omega_{\mathbb{P}^n}^k \) is the bundle \( k \) differential forms and the last isomorphism is the consequence of Bott’s formula [See, Page 8 [CMH]]. This proves that the divisor class \( k[\zeta] - [\pi^*(h)] \) contains a unique effective divisor. From the exact sequence (9) we deduce that the Segre class \( S_{n-1}(V) \) is equal to \((-1)^{n-1}kh^{n-1}\) and hence

\[
(k[\zeta] - [\pi^*(h)]).\zeta^{n(n+1)/2} - 2 = \pi^*(h^n)\zeta^{n(n-1)/2} - 1 - (-1)^{n-1}\pi^*(h)\pi^*(S_{n-1}(V))\zeta^{n(n+1)/2} - 1 = 0.
\]

For \( v \in \mathbb{P}^n \) the fibre \( V_v \) is isomorphic to \( \text{Alt}(\mathbb{C}^n, \mathbb{C}^{n+1}/\mathbb{C}.v) \). For \( \varphi \in Alt^n + 1 \) rank of \( \varphi \) is less than or equal to \( n - 1 \) if and only if image of \( \tilde{\varphi} \in \text{Alt}(\mathbb{C}^n, \mathbb{C}^{n+1}/\mathbb{C}.v) \) of \( \varphi \) has rank less than or equal to \( n - 1 \), i.e., \( \varphi \) is not an isomorphism. Since the complement of isomorphisms in \( \text{Alt}(\mathbb{C}^n, \mathbb{C}^{n+1}/\mathbb{C}.v) \) is given by vanishing of the homogeneous polynomial of degree in \( k \), i.e., section of \( \mathcal{O}_{\mathbb{P}(V_v)}(k) \). The set

\[
F_v = \{ \tilde{\varphi} \in \text{Alt}(\mathbb{C}^n, \mathbb{C}^{n+1}/\mathbb{C}.v)| \text{rank}(\tilde{\varphi}) \leq n - 1 \}
\]

is an irreducible divisor given by the vanishing of the restriction to \( \mathbb{P}(V_v) \) of the non-zero section (unique upto non zero scalar) of \( H^0(\mathbb{P}(V), \zeta \otimes \pi^*(\mathcal{O}_{\mathbb{P}^n}(-1))) \)

The non-zero section \( t \) (unique upto multiplication by a non zero scalar) determines a section of \( \mathcal{O}_{\mathbb{P}(V_v)}(k) \), namely Pfaffian whose square is the determinant of skew symmetric form. It is clear that \( F_v \) is the set of
zeros of the section $t|_{\mathbb{P}(V,v)}$. This proves that the set $F$ is the support of the divisor $k[\zeta] - [\pi^*(h)]$.

**Remark 2:** Theorem (1.2) can be used to obtain for blow up of $\mathbb{P}^n$, $(2n-1 \leq m \leq \frac{n(n-1)}{2}-1)$ along a non linear sub-variety a projective bundle structure over $\mathbb{P}^m$, for even integer $n \geq 4$.

**Remark 3:** The special case of Theorem (1.2) appears first in [EL] and has been used in the context of Quantum Cohomology in [CCGK], [CGKS], [AS1], [AS2], [AS3].

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