PROJECTIVELY GENERATED $d$-ABELIAN CATEGORIES ARE $d$-CLUSTER TILTING

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Abstract. Building on work of Jasso, we prove that any projectively generated $d$-abelian category is equivalent to a $d$-cluster tilting subcategory of an abelian category with enough projectives. This supports the claim that $d$-abelian categories are good axiomatizations of $d$-cluster tilting subcategories.

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1. Introduction

The concept of $d$-cluster tilting subcategories was introduced by Iyama in [I1], and further developed in [I2], [I3]. It is the natural framework for doing higher Auslander-Reiten theory. A $d$-cluster tilting subcategory $\mathcal{M}$ is a contravariantly finite, covariantly finite, and generating-cogenerating subcategory of an abelian category $\mathcal{A}$ satisfying

$$\mathcal{M} = \{ X \in \mathcal{A} \mid \forall i \in \{1, 2, \cdots, d-1\} \ Ext^i_{\mathcal{A}}(X, \mathcal{M}) = 0 \} \quad (1.1)$$

$$\{ X \in \mathcal{A} \mid \forall i \in \{1, 2, \cdots, d-1\} \ Ext^i_{\mathcal{A}}(\mathcal{M}, X) = 0 \} \quad (1.2)$$

Examples of such categories are given in [H1], [H2], [I0]. A problem with this definition is that it is not clear which properties of $\mathcal{M}$ are independent of the embedding into $\mathcal{A}$. To fix this, Jasso introduced in [J] the concept of a $d$-abelian category (see Definition 2.3), which is an axiomatization of $d$-cluster tilting subcategories. He shows that any $d$-cluster tilting subcategory is $d$-abelian. Furthermore, he also shows [J Theorem 3.20] that if $\mathcal{M}$ is a small

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projectively generated $d$-abelian category with category of projective objects denoted by $\mathcal{P}$, such that there exists an exact duality $D: \text{mod} \mathcal{P} \to \text{mod} \mathcal{P}^{\text{op}}$, then the image of the fully faithful functor

$$F: \mathcal{M} \to \text{mod} \mathcal{P} \quad F(X) = \mathcal{M}(-, X)|_{\mathcal{P}}$$

is $d$-cluster tilting in $\text{mod} \mathcal{P}$. Here $\text{mod} \mathcal{P}$ is the category of finitely presented contravariant functors from $\mathcal{P}$ to $\text{Mod} \mathbb{Z}$. In this note we show that the second assumption is unnecessary.

**Theorem 1.3.** Let $\mathcal{M}$ be a small projectively generated $d$-abelian category, let $\mathcal{P}$ be the set of projective objects of $\mathcal{M}$, and let $F: \mathcal{M} \to \text{mod} \mathcal{P}$ be the functor defined by $F(X) := \mathcal{M}(-, X)|_{\mathcal{P}}$. Then the essential image $FM := \{M \in \text{mod} \mathcal{P} \mid \exists X \in \mathcal{M} \text{ such that } F(X) \cong M\}$ is $d$-cluster tilting in $\text{mod} \mathcal{P}$.

We emphasize that almost all of the work towards proving this theorem has been done in [J]. In fact, by Lemma 2.6 the only thing which remains is to show that $FM$ is cogenerating and contravariantly finite, and the proof of these properties are straightforward.

### 2. Preliminaries

We recall the definition of $d$-exact sequences and $d$-abelian categories.

**Definition 2.1 ([J, Definition 2.2]).** Let $\mathcal{M}$ be an additive category and $f^0: X^0 \to X^1$ a morphism in $\mathcal{M}$. A $d$-cokernel of $f^0$ is a sequence of maps

$$(f^1, \ldots, f^d): X^1 \xrightarrow{f^1} X^2 \xrightarrow{f^2} \cdots \xrightarrow{f^{d-1}} X^d \xrightarrow{f^d} X^{d+1}$$

such that the sequence

$$0 \to \mathcal{M}(X^{d+1}, Z) \xrightarrow{- \circ f^d} \mathcal{M}(X^d, Z) \xrightarrow{- \circ f^{d-1}} \cdots$$

$$\cdots \xrightarrow{- \circ f^1} \mathcal{M}(X^1, Z) \xrightarrow{- \circ f^0} \mathcal{M}(X^0, Z)$$

is exact for all $Z \in \mathcal{M}$. Dually, a $d$-kernel of a morphism $g^d: Y^d \to Y^{d+1}$ is a sequence of maps

$$(g^0, \ldots, g^{-d-1}): Y^0 \xrightarrow{g^0} Y^1 \xrightarrow{g^1} \cdots \xrightarrow{g^{-d-2}} Y^{d-1} \xrightarrow{g^{-d-1}} Y^d$$

such that the sequence

$$0 \to \mathcal{M}(Z, Y^0) \xrightarrow{g^0 \circ -} \mathcal{M}(Z, Y^1) \xrightarrow{g^1 \circ -} \cdots$$

$$\cdots \xrightarrow{g^{-d-1} \circ -} \mathcal{M}(Z, Y^{d-1}) \xrightarrow{g^{-d} \circ -} \mathcal{M}(Z, Y^{d+1})$$

is exact for all $Z \in \mathcal{M}$. 

Definition 2.2 ([J, Definition 2.4]). Let \( \mathcal{M} \) be an additive category. A \( d \)-exact sequence is a sequence of maps

\[
X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \cdots \xrightarrow{f^d} X^{d+1}
\]

such that \((f^0, \ldots, f^{d-1})\) is a \( d \)-kernel of \( f^d \), and \((f^1, \ldots, f^d)\) is a \( d \)-cokernel of \( f^0 \).

Recall that \( \mathcal{M} \) is idempotent complete if for any idempotent \( e: X \to X \) in \( \mathcal{M} \) there exists morphisms \( \pi: X \to Y \) and \( i: Y \to X \) such that \( i \circ \pi = e \) and \( \pi \circ i = 1_Y \).

Definition 2.3 ([J, Definition 3.1]). A \( d \)-abelian category is an additive category \( \mathcal{M} \) satisfying the following axioms:

(A0) \( \mathcal{M} \) is idempotent complete.

(A1) Every morphism in \( \mathcal{M} \) has a \( d \)-kernel and a \( d \)-cokernel

(A2) Let \( f^0: X^0 \to X^1 \) be a monomorphism and \((f^1, \ldots, f^d)\) a \( d \)-cokernel of \( f^0 \). Then the sequence

\[
X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} X^2 \xrightarrow{f^2} \cdots \xrightarrow{f^{d-1}} X^d \xrightarrow{f^d} X^{d+1}
\]

is \( d \)-exact.

(A2^op) Let \( g^d: Y^d \to Y^{d+1} \) be an epimorphism and \((g^0, \ldots, g^{d-1})\) a \( d \)-kernel of \( g^d \). Then the sequence

\[
Y^0 \xrightarrow{g^0} Y^1 \xrightarrow{g^1} \cdots \xrightarrow{g^{d-2}} Y^{d-1} \xrightarrow{g^{d-1}} Y^d \xrightarrow{g^d} Y^{d+1}
\]

is \( d \)-exact.

Recall that \( P \in \mathcal{M} \) is projective if for every epimorphism \( f: X \to Y \) in \( \mathcal{M} \) the sequence \( \mathcal{M}(P, X) \xrightarrow{f^0} \mathcal{M}(P, Y) \to 0 \) is exact. The following results holds for projective objects in \( d \)-abelian categories.

Theorem 2.4 ([J, Theorem 3.12]). Let \( \mathcal{M} \) be a \( d \)-abelian category, let \( P \) be a projective object in \( \mathcal{M} \), let \( f^0: X^0 \to X^1 \) be a morphism in \( \mathcal{M} \), and let \((f^1, \ldots, f^d)\) a \( d \)-cokernel of \( f^0 \). Then the sequence

\[
\mathcal{M}(P, X^0) \xrightarrow{f^0} \mathcal{M}(P, X^1) \xrightarrow{f^1} \mathcal{M}(P, X^2) \xrightarrow{f^2} \cdots \xrightarrow{f^{d-1}} \mathcal{M}(P, X^d) \xrightarrow{f^d} \mathcal{M}(P, X^{d+1}) \to 0
\]

is exact.

Definition 2.5 ([J, Definition 3.19]). Let \( \mathcal{M} \) be a \( d \)-abelian category. We say that \( \mathcal{M} \) is projectively generated if for every objects \( X \in \mathcal{M} \) there exists a projective object \( P \in \mathcal{M} \) and an epimorphism \( f: P \to X \).

Let \( \mathcal{M} \) be a projectively generated \( d \)-abelian category, let \( \mathcal{P} \) be the category of projective objects of \( \mathcal{M} \), and let \( F: \mathcal{M} \to \text{mod} \mathcal{P} \) be the functor
Theorem 2.4 tells us that if \((f^1, \cdots, f^d)\) is a \(d\)-cokernel of \(f^0\), then the sequence

\[
F(X^0) \xrightarrow{F(f^0)} F(X^1) \xrightarrow{F(f^1)} F(X^2) \xrightarrow{F(f^2)} \cdots \xrightarrow{F(f^d)} F(X^{d+1}) \rightarrow 0
\]
is exact in \(\mod \mathcal{P}\).

Parts of the proof that a projectively generated \(d\)-abelian category is \(d\)-cluster tilting in \(\mod \mathcal{P}\) follows from the following lemma. Note that there is a typo in [J]; in the lemma they write that \(F\mathcal{M}\) is contravariantly finite, but in the proof they show that it is covariantly finite.

**Lemma 2.6 ([J, Lemma 3.22]).** Let \(\mathcal{M}\) be a small projectively generated \(d\)-abelian category, let \(\mathcal{P}\) the category of projective objects of \(\mathcal{M}\), and let \(F: \mathcal{M} \rightarrow \mod \mathcal{P}\) be the functor defined by \(F(X) = \mathcal{M}(-, X)|_\mathcal{P}\). Also, let \(F\mathcal{M} := \{M \in \mod \mathcal{P} \mid \exists X \in \mathcal{M} \text{ such that } F(X) \cong M\}\) be the essential image of \(F\). Then the following holds:

(i) \(\mod \mathcal{P}\) is abelian;
(ii) \(F\) is fully faithful;
(iii) For all \(k \in \{1, \cdots, d-1\}\) we have \(\text{Ext}^k_{\mod \mathcal{P}}(F\mathcal{M}, F\mathcal{M}) = 0\);
(iv) We have \(F\mathcal{M} = \{M \in \mod \mathcal{P} \mid \forall i \in \{1, 2, \cdots, d-1\} \text{ Ext}^i_{\mod \mathcal{P}}(M, F\mathcal{M}) = 0\}\);
(v) We have \(F\mathcal{M} = \{M \in \mod \mathcal{P} \mid \forall i \in \{1, 2, \cdots, d-1\} \text{ Ext}^i_{\mod \mathcal{P}}(F\mathcal{M}, M) = 0\}\);
(vi) \(F\mathcal{M}\) is covariantly finite in \(\mod \mathcal{P}\).

Since \(F\mathcal{M}\) is obviously generating, it only remains to show that \(F\mathcal{M}\) is cogenerating and contravariantly finite.

### 3. Proof of Theorem 1.3

Throughout this section we fix an integer \(d \geq 2\), a projectively generated \(d\)-abelian category \(\mathcal{M}\), and we let \(\mathcal{P}\) denote the category of projective objects in \(\mathcal{M}\).

**Lemma 3.1.** \(F\mathcal{M}\) is cogenerating in \(\mod \mathcal{P}\).

**Proof.** Let \(G \in \mod \mathcal{P}\) be arbitrary. Since \(G\) is finitely presented, we can find projective objects \(P^0, P^1 \in \mathcal{M}\) and a morphism \(\phi: F(P^0) \rightarrow F(P^1)\) such that \(\text{Cok} \phi \cong G\). Since \(F\) is full, there exists a morphism \(f^0: P^0 \rightarrow P^1\) in \(\mathcal{M}\) such that \(F(f^0) = \phi\). Let

\[
(f^1, \cdots, f^d): P^1 \xrightarrow{f^1} X^2 \xrightarrow{f^2} \cdots \xrightarrow{f^{d-1}} X^d \xrightarrow{f^d} X^{d+1}
\]
be a \(d\)-cokernel of \(f^0\). By Theorem 2.4 we know that the sequence

\[
F(P^0) \xrightarrow{F(f^0)} F(P^1) \xrightarrow{F(f^1)} F(X^2) \xrightarrow{F(f^2)} \cdots \xrightarrow{F(f^d)} F(X^{d+1}) \rightarrow 0
\]
is exact. In particular, we have a monomorphism
\[ G \cong \mathrm{Cok}(F(f^0)) \to F(X^2) \]
This shows that \( F_M \) is cogenerating. \( \square \)

**Lemma 3.2.** \( F_M \) is contravariantly finite in \( \text{mod}\; \mathcal{P} \).

**Proof.** Let \( G \in \text{mod}\; \mathcal{P} \) be arbitrary. By Lemma 3.1 there exist objects \( X^d, X^{d+1} \in \mathcal{M} \) and an exact sequence
\[ 0 \to G \xrightarrow{i} F(X^d) \xrightarrow{\phi} F(X^{d+1}) \]
where \( \phi = F(f^d) \) since \( F \) is full. Let
\[ (f^0, \ldots, f^{d-1}) : X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} \cdots \xrightarrow{f^{d-2}} X^{d-1} \xrightarrow{f^{d-1}} X^d \]
be a \( d \)-kernel of \( f^d \). Since \( F(f^d) \circ F(f^{d-1}) = 0 \), we get an induced morphism \( F(X^{d-1}) \xrightarrow{i} G \). We claim that \( p \) is a right \( F_M \)-approximation of \( G \). Let \( X \in \mathcal{M} \) and let \( F(X) \xrightarrow{\psi} G \) be an arbitrary morphism in \( \text{mod}\; \mathcal{P} \). Since \( F \) is full, the composition \( F(X) \xrightarrow{\psi} G \xrightarrow{i} F(X^d) \) is of the form \( F(f) \) for some morphism \( f : X \to X^d \). Since \( f^d \circ f = 0 \) and
\[ \mathcal{M}(X, X^{d-1}) \xrightarrow{f^{d-1}} \mathcal{M}(X, X^d) \xrightarrow{f^d} \mathcal{M}(X, X^{d+1}) \]
is exact, it follows that \( f = f^{d-1} \circ g \) for some morphism \( g : X \to X^{d-1} \). Applying \( F \) gives
\[ i \circ p \circ F(g) = F(f^{d-1}) \circ F(g) = F(f) = i \circ \psi \]
and since \( i \) is a monomorphism, we get that \( p \circ F(g) = \psi \). This shows that \( p \) is a right \( F_M \)-approximation, and since \( G \) was arbitrary it follows that \( F_M \) is contravariantly finite. \( \square \)

**Remark 3.3.** Let \( \mathcal{M} \) be an injectively cogenerated \( d \)-abelian category, and let \( \mathcal{I} \) be the category of injective objects in \( \mathcal{M} \). Furthermore, let \( G : \mathcal{M} \to (\mathcal{I} \text{mod})^{\text{op}} \) be the functor given by \( G(X) := \mathcal{M}(X, -)_{\mathcal{I}} \). Here \( \mathcal{I} \text{mod} \) denotes the category of finitely presented covariant functors from \( \mathcal{I} \) to \( \text{mod}\; \mathbb{Z} \). The dual of Theorem 1.3 tells us that \( G \) is a fully faithful functor, \( (\mathcal{I} \text{mod})^{\text{op}} \) is an abelian category, and the essential image
\[ G\mathcal{M} := \{ M \in (\mathcal{I} \text{mod})^{\text{op}} \mid \exists X \in \mathcal{M} \text{ such that } G(X) \cong M \} \]
is \( d \)-cluster tilting in \( (\mathcal{I} \text{mod})^{\text{op}} \).

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