Rigorous Mathematical Thinking: Why are Cognitive Levels Important in Three-Dimensional Learning for Pre-service teachers?

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Abstract. Learning material three dimensions is closely related to processing high-level abstract thinking skills. The ability in question is Rigorous Mathematical Thinking (RMT). Some research results do not focus on the relationship of RMT with cognitive levels. The cognitive relationship between RMT is essential because students must have abstraction skills to be able to teach according to the concept. The study aimed to determine the ability of RMT mathematics teacher pre-service based on cognitive levels in a three-dimensional material. The research method is qualitative descriptive because of the use of test sheets and interviews as instruments. The research sample consisted of three students, each with low, medium and high cognitive abilities. Research has resulted in the ability of prospective RMT mathematics teachers to have a close relationship with their cognitive level. High-level RMT students can link the three-dimensional concepts of the Pythagoras theorem, great line propositions, angular definitions, the definition of squared angles, and several trigonometric principles. Medium RMT, able to sort diagonal concepts until the trigonometric proposition is systematic. A low RMT can label a right triangle and cube side. The results of this study are the basis for giving appropriate treatment to students before carrying out teaching practices.

1. Introduction
Mathematical thinking is one of the ability to achieve the most important goals of the mathematics education process which has the role of improving conceptual learning [1,2]. Conceptual learning is concerned with the ability to connect between new and old understandings using the main elements of mathematics [3]. In this case, it means three main things of mathematics namely to live, justify, and show [4]. Therefore, people who think can be helpful in many ways in their mathematics [5]. Abstraction ability leads one to optimise his cognitive ability.

There are three levels of cognitive ability that is low, medium, and high [6]. The three levels of cognitive are different based on the value obtained when given the test or the final value obtained by a person from a material. The process of understanding learning material influences the cognitive of students. The need for thoroughness in synthesising and improving cognitive processes to enhance abstraction skills needs to have Rigorous Mathematical Thinking (RMT) [7,8]. In a strict paradigm of thought, the cognitive process builds a strict conceptual understanding as well. It can position the students to create a higher level of patterns and relationships in mathematics [9].
RMT (Rigorous mathematical thinking) is a learning approach that emphasises the interaction between students and teachers that produces good transformations and concepts by linking ideas [8]. RMT mentions the mental processes that students pass through based on their cognitive function, their ability to learn from cognitive functions to high levels of abstraction. Through RMT, we can use the values used by the students, because RMT with three dimensions of cognitive function is qualitative thinking, quantitative thinking, and abstract relational thinking [7,9].

Qualitative thinking suggests that the spontaneous concepts encountered in the student's daily life have led to cognitive functioning but are not systematic. Cognitive functions for qualitative thinking are labelling, visualising, comparing, systematically searching information for collecting and completing the information, using more than one information, encoding-decoding. Levels of quantitative thinking with meticulousness, linking the conceptual bases with general cognitive function. The cognitive function is to maintain firmness, measure space and spatial relationships, measure time and temporal relationships, analyse-integrate, generalise, and precisely. For the third stage of abstract relational thinking, students can link logic and high-level understanding in solving math problems. The stages of its cognitive function are to activate mathematically related knowledge before deciphering mathematical activity through cognitive categories.

The three levels of RMT cognitive functioning related to three levels of cognitive abilities when learning mathematics. Each representing each cognitive level has a distinct RMT cognitive function. To more clearly consider the relationship is selected mathematical material that is appropriate three-dimensional geometry. In this material the student's abstraction level is visible because students will visualise, interpret, and calculate the physics of a solid object [10]. The three-dimensional material presents geometric knowledge with conceptual difficulties but can develop students' strict visual abstraction skills [11].

The three-dimensional material learning will be properly conveyed to the student if the teacher who teaches the concept understands, has adequate cognitive skills, and a strict abstraction. Having these three abilities and understanding becomes mandatory for pre-service mathematics teachers [12]. Romano explained in his research that the pre-service mathematics teachers must have a very strong ability with the material [13]. Lowrie and Jorgensen have done research that took a sample of pre-service mathematics teachers. The theory used to measure the abilities and abstractions of mathematician candidates on the geometric material is quite high, this may be a provision in later teaching times [14]. From both studies, it takes into account the cognitive abilities of pre-service mathematics teachers because they have to master basic skills related to all materials.

Based on the above exposure, the researchers conducted a study on the relevance of mathematical thinking rigor with the importance of the cognitive abilities of pre-service mathematics teachers on the three-dimensional material. Discussion and research on RMT are still so rare that it raises ideas and empowers researchers to investigate further. The results of this study expect teachers of pre-service mathematics teachers to find out early on cognitive abilities associated with rigorous mathematical thinking students. So that lecturers can give appropriate treatment during lecturing.

2. Methods
Researchers use the qualitative descriptive method because the research instrument is in the form of a test sheet and interview. Both instruments are by the statement of Coloravi and Evans that the data obtained for qualitative descriptive research can be interview, observation, document or artefact [15]. The study focuses on describing the cognitive abilities of pre-service teachers from the results of the material test questions in three subsections of the calculation of the angle of space; then the interview results reinforce the answer. This research is collecting data in the form of the results of the pre-service teacher's test and individual interviews to be analysed, but not the new theory [16].

The research begins by giving pre-test to 42 students of mathematics teacher candidate contracting Kapita Seleksa Mathematics lectures of three-dimensional materials with different cognitive abilities. Then the pre-test results are used as a benchmark to select the study sample. Researchers selected three pre-service teachers as a sample, in which one person represented a low cognitive level, one was
moderate, and the rest were high using random sampling techniques. Choose the technique because the three cognitive levels have the same ratio to be sampled [17, 18]. Then the researchers gave further tests to the six samples at the same time. The next session is an interview with each pre-service teachers. Researchers conduct tests and interviews also at the same time. The final step of the researcher analyses and describes the results of the three-dimensional material test of three corners in the third dimension space and interview

3. Results and Discussion
Researchers use the qualitative descriptive method because the research instrument The results of the pre-test found that there are three cognitive levels of pre-service mathematics teachers reasoning ability is low, medium, and high. Gaining all three cognitive levels needs to pay attention to the final score. From a total score of 100, if obtained 0 - 49 includes a low cognitive level, 50 - 69 medium, and 70 to 100 high [19]. Based on the assessment guide, each person is chosen to represent each cognitive level. The three mathematician candidates are working on a matter of determining the angle on the building of the three-dimensional space.

To select advanced test questions requires precision as it should reflect the rigorous mathematical thinking ability. Here's a look at the answers to all three samples and their analysis in order. Figure 1 is the answer view of the pre-service mathematics teachers at a high cognitive level.

![Figure 1](image_url)

**Figure 1.** Pre-service Mathematics Teacher Answer with High Cognitive Level.

Mathematics teacher candidates who are in high cognitive abilities enter the RMT level of abstract relational thinking. This level is the highest level on the RMT. The mathematics teacher candidate can connect the three-dimensional material with the prerequisite material, namely real number root operations, the Pythagoras theorem on right triangles, high line propositions, angular definitions, coincident angle definitions, and several principles in trigonometry. The four materials are linked to produce the correct answer. He compiled the proofs in orderly, detailed, and clear. Starting from
determining the simplest aspect of the problem is the diagonal of the base plane, then draws lines that are perpendicular to the needs of the three fields. The end of the right answer to the points of the cosine is different and three angles in one cube.

This pre-service mathematics teacher can connect the concept of lines perpendicular to the triangle, then determine the length of the line in the triangle using the Pythagoras formula equation of an arbitrary triangle. So the conclusion is the formulation of trigonometry for right triangles that are part of the arbitrary triangle. Appears here the mathematical relational thinking ability has been fulfilled to be able to analyse their result and obtained the correct answer.

Next will analyse the results of pre-service mathematics teacher answers to the level of moderate cognitive. Figure 2 shows the answer.

![Figure 2. Pre-service Mathematics Teacher Answer with Medium Cognitive Level.](image)

The mathematics teacher candidate in Figure 2 can identify the concepts that need to be present in solving the problem. These concepts include building space, real number root operations, Pythagoras theorems on right triangles, heavy line propositions, and several principles in trigonometry. He also has been able to solve the initial problem using reference examples of questions that have ever existed so that the systematics of answers is similar. He can observe the properties of cubes, equilateral triangles, and right angles, but errors occur when starting to refer to more specific concepts. It can be seen from the answer when it has been able to show the correct concept of a right triangle but entering into a special concept of high lines shows a concept incompatibility.

The pre-service mathematics teacher shows good accuracy when performing the number operations. He also sequenced the material systematically from the initial concept of field diagonals to trigonometry.
This obedience shows that at a moderate cognitive level, students have been able to demonstrate quantitative thinking ability with thoroughness. At RMT these capabilities go at level 2.

The last level indicates the low-level cognitive abilities found in Figure 3. The pre-service mathematics teacher demonstrates their ability to answer the problem by focusing on their questions. Here's the look of the test answers from the pre-service teachers of low-level cognitive math.

![Pre-service Mathematics Teacher Answer with Low Cognitive Level](image)

**Figure 3.** Pre-service Mathematics Teacher Answer with Low Cognitive Level.

Based on the results of the above answers, obtained the conclusion that the pre-service mathematics teacher can label the right triangle symbol, and the same side that formed in the cube. He has been able to translate the matter into the picture correctly with a neat visual. He has been able to complete the information contained in the picture for purposes in the next stage of reply. E.g. calculates the length of the diagonals of the plane, and which angle to represent is formed from the line and the plane of three. A pre-service teacher can interpret the angle symbols, the shadow of a point, and lines of mutual perpendicular. From these statements obtained the conclusion that pre-service mathematics teachers who are at a low cognitive level including RMT level of qualitative thinking.

The above description obtained confirmation from the interview result of the researcher with the three samples. The results of the interviews indicate that pre-service mathematics teachers who are at a high cognitive level do not have significant difficulties when working on the problem. He was able to work on the problem by connecting the old and new concepts, and not to the example given by the lecturer. He states that when he receives a question then reads it he immediately visualises it in the form of an image, then searches for what elements are needed to answer the question. While for pre-service teachers at the level of cognitive is stating that he first identifies all the concepts used to answer the problem and then connect it thoroughly. It is just that he wrongly specifies formulas for special or profound concepts. For pre-service teachers at a low cognitive level, he or she understands how to
construct an image of a given problem, then collects whatever information is necessary for the next step. However, have not been able to visualise the picture and solve the problem correctly.

The mathematical thinking abilities of the three samples indicate differences that should be subject to different treatments. This treatment becomes an important task of lecturers who can abstract the subject matter to give appropriate treatment to the candidate's cognitive ability based on the level. They must work more deeply on how to explore the rigorous mathematical thinking abilities of all mathematics teacher. The reason for the pre-service mathematics teacher at the time of teaching later is not overwhelmed when meeting abstract materials, and convey the correct concept to the side. Kinard writes in his writings that rigorous mathematical thinking is the dynamics of constructing a logical framework in a structured way through the discovery, definition, and pattern of a real object [20]. Therefore every pre-service mathematics teachers must be able to optimise the ability of mathematical thinking rigor so that his mindset in learning all mathematical concepts arranged.

4. Conclusion
The ability of rigor mathematical thinking to prospective mathematics teachers has a close relationship with the cognitive level. Rigor high-level students with abstract relational skills can associate three-dimensional concepts from real number root operations, Pythagoras theorems, high line propositions, angle definitions, coincident angle definitions, and trigonometric principles. For rigor medium-level students with the ability to think quantitatively with precision, they can systematically sort material from diagonal field concepts to trigonometric principles. Whereas for low-level rigor with the ability to think qualitatively, it can label the right triangle and sides of the cube symbol. When the lecturer knows the weaknesses in mathematical thinking of prospective teachers, then the appropriate treatment will be easily given at the beginning.

5. References
[1] Zeynivandnezhad F, Ismail Z, and Yusuf Y M 2013 Mathematical Thinking in Differential Equations Among Pre-Service Teachers Jurnal Teknologi (Social Science) 63 51
[2] Yorulmaz A, Altintas S, and Sidekli S 2017 Investigation of the Effects of Mathematical Thinking States of Form Teachers on Their Mathematics Teaching Anxieties European Journal of Educational Research 6 485
[3] Lambrer D and Meaney T 2016 Preschool Children Learning Mathematical Thinking on Interactive Tables POEM2 Conference 235
[4] Sherman M 2014 The Role of Technology in Supporting Students’ Mathematical Thinking: Extending the Metaphors of Amplifier and Reorganizer Contemporary Issues in Technology and Teacher Education 14 220
[5] Breen S and O’Shea A 2011 The Use of Mathematical Tasks to Develop Mathematical Thinking Skills in Undergraduate Calculus Courses – a Pilot Study Proceedings of the British Society for Research into Learning Mathematics (BSRLM) 31 43
[6] Williamson J R, Quatieri T F, Smalt C J, Perricone J, Helfer B J, Nolan M A, Eddy M, and Moran J 2016 Using EEG to Discriminate Cognitive Workload and Performance Based on Neural Activation and Connectivity (USA: MIT Lincoln Laboratory Lexington United States)
[7] Fitriyani H and Khasanah U 2017 Student’s rigorous mathematical thinking based on cognitive style In Journal of Physics: Conference Series 943 1 012055).
[8] Hidayat D, Nurlaelah E, and Dahlan J A 2017 Rigorous Mathematical Thinking Approach to Enhance Students Mathematical Creative and Critical Thinking Abilities J. Phys.: Conf. Ser. 895 1 012087
[9] Kinard J T and Kozulin A 2008 Rigorous Mathematical Thinking Conceptual Formation in the Mathematics Classroom (New York: Cambridge University Press)
[10] Friansah D, Zulkardi, and Somakim 2015 The Development of Cabri 3D Based Learning Material Three Dimensional at Class X High School Proceeding The 3rd SEA-DR Sriwijaya University 94
[11] Mammana M F, Micale B, and Pennisi M 2012 Analogy and Dynamic Geometry System Used to Introduce Three-Dimensional Geometry. *International Journal of Mathematical Education in Science and Technology* 43 818

[12] Kamina P and Iyer N N 2009 From Concrete to Abstract: Teaching for Transfer of Learning when Using Manipulatives. *Northeastern Educational Research Association (NERA) Annual Conference Proceedings* 1

[13] Romano D I 2010 Prospective B&H Elementary School Teachers’ Understanding of Processes with Basic Geometric Concepts. *Open Mathematical Education Notes* IMVI Journal 7 29

[14] Lowrie T and Jorgensen R 2015 Pre-Service Teachers’ Mathematics Content Knowledge: Implications for How Mathematics is Taught in Higher Education. *Teaching Mathematics and Its Applications* 35 4 202-215

[15] Colorafi K J and Evans B 2016 Qualitative Descriptive Methods in Health Science Research. *Health Environments Research & Design Journal* 9 4 16-25.

[16] Lambert V A and Lambert C E 2012 Qualitative Descriptive Research: an Acceptable Design. *Journal of Nursing Research* 16 255

[17] Etikan I and Bala K 2017 Sampling and Sampling Methods. *Biometrics & Biostatistics International Journal* 5 1

[18] Kadane B S 2011 Probability Sampling in Litigation. *Connecticut Insurance Law Journal* 18 311

[19] Abdulwahab N, Oyelekan OS, and Olorundare AS 2016 Effects of Cooperative Instructional Strategy on Senior School Students’ Achievement in Electrochemistry. *Eurasian Journal of Physics & Chemistry Education* 8 37

[20] Kinard J 2006 Creating rigorous mathematical thinking: a dynamic that drives mathematics and science conceptual development. *Transsyllvanian Journal of Psychology-Erdély Pszichológiai Szemle, Special issue Vol. Supplement 2006,* 2 251-266

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