Abstract. Wang, Jiang and Cao have obtained a generalized version of the Jørgensen inequality in Proc. Indian Acad. Sci. Math. Sci., 123(2):245–251, 2013, for two generator subgroups of $\text{SL}(2, \mathbb{C})$ where one of the generators is loxodromic. We prove that their inequality is strict.

1. Introduction

The Jørgensen inequality is a classical result that provides necessary condition of discreteness for a two generator subgroup of $\text{SL}(2, \mathbb{C})$, see [4]. Extremality of the Jørgensen inequality has been investigated in [5]. In the literature there are several generalizations of the Jørgensen inequality, and extremalities of some of those inequalities have also been investigated, e.g. [2], [3], [6], [7]. In this note we investigate extremality of one such generalized Jørgensen inequality.

In [8], Wang, Jiang and Cao have obtained a generalized version of the Jørgensen inequality for two generator subgroups of $\text{SL}(2, \mathbb{C})$ where one of the generators is loxodromic. We recall their result.

**Theorem WJC.** [8] Let $g, h$ are elements in $\text{SL}(2, \mathbb{C})$ such that $g$ is loxodromic. Suppose that $g, h$ are of the form: for $|\lambda| > 1$,

\begin{equation}
(1.1) \quad g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\end{equation}

Let $g$ be such that $M_g < 1$, where

\[ M_g = |\lambda - 1| + |\lambda^{-1} - 1|.
\]

If $\langle g, h \rangle$ is discrete and non-elementary, then

\begin{equation}
(1.2) \quad |abcd|^{\frac{1}{2}} \geq \frac{1 - M_g}{M_g^2}.
\end{equation}

We prove that the above inequality is strict, see Theorem 2.1 below. Further, we note down a few generalized Jørgensen type inequalities which are also strict.

2. Proof of Strictness of the Inequality

**Theorem 2.1.** Under the hypothesis of the above theorem, equality does not hold in (1.2).

Date: September 20, 2018.

2000 Mathematics Subject Classification. Primary 20H10; Secondary 51M10.

Key words and phrases. Jørgensen inequality, discreteness.

Gongopadhyay acknowledges partial support from SERB MATRICS grant MTR/2017/000355. Tiwari is supported by NBHM-SRF.
Proof. If possible suppose equality holds in (1.2). Let $h_1 = hgh^{-1}$. Then

$$h_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \begin{pmatrix} ad\lambda - bc\lambda^{-1} & -(\lambda - \lambda^{-1})ab \\ (\lambda - \lambda^{-1})cd & ad\lambda^{-1} - bc\lambda \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}.$$

Note that

$$b_1c_1 = -(\lambda - \lambda^{-1})^2abcd = -(\lambda - \lambda^{-1})^2bc(1 + bc).$$

$$|a_1d_1| = |1 + (\lambda - \lambda^{-1})^2abcd|$$

$$\leq 1 + |\lambda - 1 + 1 - \lambda^{-1}||abcd|$$

$$\leq 1 + \frac{(1 - M_g)^2}{M_g^2}$$

$$\leq \frac{(M_g + 1 - M_g)^2}{M_g^2}.$$

Thus we have

$$(2.1) \quad |a_1d_1|^\frac{1}{2} \leq \frac{1}{M_g}.$$  

Also, we have $|b_1| \leq M_g|ab|$, $|c_1| \leq M_g|cd|$. Hence

$$(2.2) \quad |b_1c_1|^\frac{1}{2} \leq \frac{1 - M_g}{M_g}.$$  

In particular,

$$(2.3) \quad M_g(1 + |b_1c_1|^\frac{1}{2}) < 1.$$  

Now note that $\langle g, h_1 \rangle$ is discrete and non-elementary. Discreteness of $\langle g, h_1 \rangle$ is obvious, and if it was elementary, that would have implied that $g$ and $h$ had a common fixed point and hence, would have contradicted the assumption that $\langle g, h \rangle$ is non-elementary. So, applying Theorem WJC to $\langle g, h_1 \rangle$, we have

$$\frac{1 - M_g}{M_g^2} \leq |a_1b_1c_1d_1|^{\frac{1}{2}} \leq |a_1d_1|^\frac{1}{2}|b_1c_1|^\frac{1}{2}$$

$$\leq \frac{1 - M_g}{M_g^2}, \quad \text{by (2.1) and (2.2)}.$$  

This implies,

$$|a_1b_1c_1d_1|^\frac{1}{2} = \frac{1 - M_g}{M_g^2}.$$
Next we observe that
\[
\left( \frac{1 - M_g}{M_g^2} \right)^2 \leq |a_1 b_1 c_1 d_1| \\
\leq |1 + b_1 c_1||b_1 c_1| \\
\leq |1 + |b_1 c_1|((\lambda - \lambda^{-1})^2|b_0 c_0||a_0 d_0| \\
\leq (\lambda - \lambda^{-1})^2(1 + |b_1 c_1|)\left( \frac{1 - M_g}{M_g^2} \right)^2 \\
\leq M_g^2(1 + |b_1 c_1| + 2|b_1 c_1|^{1/2})\left( \frac{1 - M_g}{M_g^2} \right)^2 \\
\leq (M_g(1 + |b_1 c_1|^{1/2})^2\left( \frac{1 - M_g}{M_g^2} \right)^2 \\
\leq \left( \frac{1 - M_g}{M_g^2} \right)^2, \text{ by } (2.3).
\]

Hence we have
\[
(\lambda - \lambda^{-1})^2(1 + |b_1 c_1|) = 1.
\]

Noting that
\[
tr^2(g) - 4 = (\lambda - \lambda^{-1})^2, \text{ and } tr[g, h_1] - 2 = -(\lambda - \lambda^{-1})^2b_1 c_1,
\]
the above equality implies that \((g, h_1)\) satisfies equality in the classical Jørgensen inequality. By a theorem of Jørgensen and Kiikka, see [5, Theorem 2], this implies that \(g\) is either elliptic or parabolic, which is a contradiction. Hence equality can not hold in (1.2).

Combining Theorem WJC and Theorem 2.1, we can rephrase the generalized Jørgensen inequality as follows.

**Theorem 2.2.** Let \(g, h\) are elements in \(SL(2, \mathbb{C})\) such that \(g\) is loxodromic. Suppose that \(g, h\) are of the form (1.1). Let \(g\) be such that \(M_g < 1\). If
\[
(\lambda - \lambda^{-1})^2(1 + |b_1 c_1|) = 1.
\]
then \((g, h)\) is either elementary or non-discrete.

2.1. Some more inequalities. The main idea in [8] was to embed \(SL(2, \mathbb{C})\) into the isometry group \(Sp(1, 1)\) of the one dimensional quaternionic hyperbolic space \(H^1_{\mathbb{H}}\), and then use quaternionic Jørgensen inequality of Cao and Parker, see [1, Theorem 1.1]. In view of the above theorem, following arguments as used in the proof of [1, Corollary 1.2], we note the following.

**Corollary 2.3.** Let \(g, h\) are elements in \(SL(2, \mathbb{C})\) such that \(g\) is loxodromic. Suppose that \(g, h\) are of the form (1.1). Let \(g\) be such that \(M_g < 1\). If \((g, h)\) is discrete and non-elementary, then each of the following strict inequalities holds.
\[
(2.6) \quad |bc|^{1/2} > \frac{1 - M_g}{M_g}.
\]
\[
(2.7) \quad |1 + bc|^{1/2} > \frac{1 - M_g}{M_g}.
\]
(2.8) \[ |1 + bc| + |bc| > \frac{2(1 - M_g)}{M_g^2}. \]

**Proof.** If possible, suppose \( |bc| \leq \frac{1 - M_g}{M_g} \). Then

\[ |ad| \leq (1 + |bc|)|bc| \leq \frac{1 - M_g}{M_g^2}. \]

Using Theorem 2.2, \( \langle g, h \rangle \) is either discrete or non-elementary.

If possible suppose \( |1 + bc| \leq \frac{1 - M_g}{M_g} \). Then it follows similarly as above noting that \( |bc| \leq 1 + |ad| \) and \( ad - bc = 1 \).

Finally, if \( |1 + bc| + |bc| \leq \frac{2(1 - M_g)}{M_g^2} \), then

\[ |ad| \leq \frac{1 - M_g}{M_g^2}, \]

and the result follows from Theorem 2.2.

This completes the proof. \( \square \)

In view of the results noted in this communication, the following question is natural to ask.

**Question 1.** What are sharp bounds for the inequalities (1.2) and (2.6) – (2.8)?

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