YET ANOTHER NOTION OF IRREGULARITY THROUGH SMALL BALL ESTIMATES

MARCO ROMITO AND LEONARDO TOLOMEO

ABSTRACT. We introduce a new notion of irregularity of paths, in terms of control of growth of the size of small balls by means of the occupation measure of the path. This notion ensures Besov regularity of the occupation measure and thus extends the analysis of Catellier and Gubinelli [CG16] to general Besov spaces. On stochastic processes this notion is granted by suitable properties of local non-determinism.

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1. INTRODUCTION

The seminal paper [CG16] introduces the idea of noiseless regularization by noise, further developed in [GG22b, GG20], that considers regularization by noise from the point of view of generic perturbations, without a prescribed
probabilistic distribution on the driving function. The motivating problem is to solve the stochastic differential equation,

\begin{align}
\begin{cases}
\dot{X}_t = b(t, X_t) + \dot{\omega}_t, \\
X_0 = x,
\end{cases}
\end{align}

(1.1)

in the case of singular drift $b$. Regularization by noise phenomena concern the effects of perturbation, mainly due to irregularity, to improve results of existence/uniqueness/stability of solutions of the above (1.1).

Regularization by noise has a long history, starting from [Zvo74, Ver80], and is mainly devoted to Brownian perturbation, in finite (see for instance [KR05, FGP10, FIR17, CdR17, CdR18, BFGM19, CG19]) and infinite dimension (for instance [DPF10, DPFRV16, BM19]). A recent review of probabilistic regularization by noise is [Fla11]. Regularization by noise has then found application for a wider class of driving processes (mainly fractional Brownian motion, or Lévy stable processes), see for instance [NO02, NO03, HN07, Pri12, AbnP20, ABM20, HL20, GH20, KP22, Ger22, BH22] and the forthcoming [GG22a], or for distribution dependent SDEs [GHM21, GHM22].

Davie in [Dav07] (see also [Fla11, Sha16]) introduced the notion of “path-by-path” uniqueness, and with it a path-wise strategy to regularization by noise. It is [CG16] that has initiated an approach to regularization by noise based on Young’s integral and on a notion of irregularity of paths that does not depend on an underlying probabilistic structure. The approach has been further developed in [GG22b, GG20] (see also [GH20, HP20]). [CG16] introduces the notion of $(\rho, \gamma)$-irregularity, which corresponds to space-time regularity of the Fourier transform of the occupation measure

\begin{align}
\mu^\omega_{s,t}(\cdot) = \int_s^t \mathbb{1}_{\omega_r}(\cdot) \, dr,
\end{align}

(1.2)

of the path $\omega$. Indeed, with the position $\theta_t = X_t - \omega_t$, (1.1) reads as $\dot{\theta}_t = b(t, \omega_t + \theta_t)$, or more precisely as

$$\theta_t = x + \int_0^t b(s, \omega_s + \theta_s) \, ds.$$ 

The above integral can be understood in terms of the occupation measure by means of a suitable Young’s integral. In this respect, [CG16] proves that the occupation measure of a $(\rho, \gamma)$-irregular path is Hölder in time with values in Fourier-Lebesgue spaces.

Noiseless regularization by noise can be developed from different points of view. In the perspective of (1.1), regularization by noise can be derived for a given drift $b$ and generic perturbations $\omega$, or for generic drifts and a given perturbation $\omega$. A key feature of this second approach is that, for a given irregular perturbation with regular occupation measure, regularization
results hold for large classes of singular drifts, and is particularly suited for application to PDEs [GG21]. The basic tool of this analysis is the *averaging* operator $T^\omega$, that can be recast as a convolution with the occupation measure,

$$T^\omega_{s,t} b(x) = \int_s^t b(x + \omega_r) \, dr = b \ast \mu^\omega_{s,t}.$$  

Mapping properties of the averaging operator can be achieved by means of regularity of the occupation measure.

A fundamental review on occupation measure is [GH80]. A recent review on nonlinear Young integration can be found in [Gal21].

In this paper we introduce a new notion of irregularity, based on control of the growth of the size of small balls measured through the occupation measure. Our new notion (briefly denoted by $\mathbb{SBE}$, a contraction for *small ball estimate*) is comparable with the $(\rho, \gamma)$-irregularity of [CG16], and presents a series of features: the $\mathbb{SBE}$ property is invariant by re-parametrization by suitable bi-Lipschitz maps, moreover is stable by regular perturbation. Above all, $\mathbb{SBE}$ regularity is essentially equivalent to Besov regularity of the occupation measure, and this allows us to extend the validity of [CG16] for (1.1) to drift in Besov spaces, instead of Fourier-Lebesgue spaces as in [CG16]. When we turn to stochastic processes, the $\mathbb{SBE}$ property is granted by simple conditions on the law of increments of the perturbation $\omega$, which are known in the literature as *local non-determinism* [Ber74, GG20]. The connection of small ball estimates with irregularity of paths is not unexpected. For instance, small ball estimates of stochastic processes yield their $\theta$-roughness in the sense of [HP13, FH14] (and in some way also vice versa, see [HRT22]). Likewise, Esséen concentration inequality (see for instance [TV06]) provides a bound of small balls by means of the Fourier transform of the occupation measure, a quantity of interest in the $(\rho, \gamma)$-irregularity of [CG16].

The paper is organised as follows. In Section 2 we introduce our new notion, by means of a regularity property on measures. We then prove that, at least on compactly supported measures, this new notion is comparable to the scale of Besov spaces. Section 3 contains the proofs that the new notion, when applied to occupation measures, is invariant by re-parametrization by suitable bi-Lipschitz maps, and stable by perturbations. In Section 4 we turn to stochastic processes, and prove $\mathbb{SBE}$ regularity of the occupation measure of a process under a very simple and slick condition on the density of increments of the process. This condition, once read on Gaussian processes, becomes even simpler and very elementary. In Section 5 we obtain a higher time regularity, possibly at the price of a lower space regularity, under a stronger assumption on the joint laws of time increments. This condition is very close to Berman’s definition of local non-determinism for general stochastic processes [Ber83].
Even though we could rely on the general theory of non-linear Young’s integration developed in [CG16] (see also [Gal21]) for Hölder continuous occupation measures, we give a limited version of the theory, targeted at equations with additive perturbation, that works in spaces of variations. Finally in Section 7 we apply the theory developed so far to solutions to one-dimensional equations driven by Brownian motion, and to equations driven by fractional Brownian motion.

1.1. Notation. If $E$ is a normed space, we shall always denote by $\| \cdot \|_E$ its norm. We denote by $S(\mathbb{R}^d)$ the Schwarz space, by $B_{p,q}^s(\mathbb{R}^d)$ the standard Besov spaces on $\mathbb{R}^d$, by $L^p(A, \mu)$ the space of functions on $A$ whose $p$-power is integrable with respect to the measure $\mu$. If $E$ is normed space, the space $V^p(a,b; E)$ is the set of functions $f : [a,b] \to E$ such that

$$\|f\|_{V^p(a,b; E)} := \sup_{a = t_0 < t_1 < \cdots < t_n = b} \sum_{h=1}^n \|f(t_n) - f(t_{n-1})\|_E^p < +\infty,$$

where the supremum is extended over all finite partitions of $[a, b]$. The space $C^{p-\text{var}}$ is the space of all continuous functions that are in $V^p(a, b; E)$, with semi-norm $\| \cdot \|_{C^{p-\text{var}}(a,b; E)} = \| \cdot \|_{V^p(a,b; E)}$. Notice that $\| \cdot \|_{C^{p-\text{var}}(a,b; E)}$ is a semi-norm, and it can be made a norm by adding $\| \cdot \|_\infty$ or the value at one single point. It becomes a norm on the subspace of $C^{p-\text{var}}(a, b; E)$ of functions with given value at one single point.

We denote by $M(\mathbb{R}^d)$ the space of finite measures on $\mathbb{R}^d$, and by $\delta_x$ the delta measure on $x$.

Finally, $\ast$ denotes convolution, $S^d$ is the $d$-sphere, $| \cdot |$ is the $\mathbb{R}^d$-norm or the Lebesgue measure (use follows obviously from the context), Law$(X)$ is the law of the random object $X$. We will sometimes write $x_{1:n}$ to denote the vector $(x_1, x_2, \ldots, x_n)$. Likewise, in integrals, $dx_{1:n}$ is the differential $dx_1 dx_2 \ldots dx_n$. We shall use the symbol $\lesssim$ for inequalities up to numerical factors, which may change from line to line and that do not depend on the main quantities of the problem.

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2. Yet another notion of irregularity

For a measure $\mu \in M(\mathbb{R}^d)$, let $F_\mu : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ be given by

$$F_\mu(t, y) := \mu(\{x \in \mathbb{R}^d : |x - y| \leq t\}).$$
Notice that \( F_\mu(r, y) \) is simply the measure of the ball with radius \( r \) and centre in \( y \). For a function \( F : \mathbb{R} \to \mathbb{R} \), define recursively the operators \( \Delta_k \) by the formula

\[
\begin{align*}
\Delta_0 F(r) &= F(r) - F(r/2) \\
\Delta_{k+1} F(r) &= \Delta_k F(r) - 2^{k+1} \Delta_k F(r/2).
\end{align*}
\]

It is easy to check that, if \( p_k(r) \) is a polynomial of degree \( \leq k \), then

\[
\Delta_k p_k(r) = 0.
\]

Define finally, for \( y \in \mathbb{R}^d \), the left translation \( \tau_y \) by

\[
\tau_y F(x) = F(x - y).
\]

**Definition 2.1.** Given \( \alpha > 0, 1 \leq p, q \leq \infty \), we say that a compactly supported measure \( \mu \in \mathcal{M}(\mathbb{R}^d) \) satisfies a small ball estimate of regularity \( \alpha \), integrability \( p \), and values in \( L^q \), in short \( \mathcal{S}^\alpha_{q,p} \), if for \( k = \lceil \alpha + d - 1 \rceil \), the quantity

\[
\| \mu \|_{\mathcal{S}^\alpha_{q,p}} := \| r^{-\alpha - d}(\Delta_k)_r F_\mu(r, y) \|_{L^q_{d\mu}([\mathbb{R}^d \times \mathbb{R}^+, dy \otimes dr])} < +\infty
\]

is finite.

The small ball estimate condition we have introduced is comparable to the scale of Besov spaces. The proof will be given at page 8.

**Theorem 2.2.** Let \( \alpha > 0, 1 \leq p < \infty \), and \( 1 \leq q \leq \infty \). If \( \mu \in \mathcal{S}^\alpha_{q,p} \), then \( \mu \) belongs to the Besov space \( B^\alpha_{q,\infty} \). More precisely,

\[
\| \mu \|_{B^\alpha_{q,\infty}} \lesssim \| \mu \|_{\mathcal{S}^\alpha_{q,p}}.
\]

A converse is also available (the proof is detailed at page 11).

**Proposition 2.3.** Let \( \mu = fdx \) be a compactly supported finite measure such that for some \( \alpha > 0, \alpha \notin \mathbb{N} \),

\[
f \in B^\alpha_{q,1}.
\]

Then, for every \( 1 \leq p \leq \infty \), we have that

\[
\| \mu \|_{\mathcal{S}^\alpha_{q,p}} \lesssim \| \mu \|_{B^\alpha_{q,1}}.
\]

### 2.1. Proofs of Theorem 2.2 and Proposition 2.3.

Before moving to the proofs, we will need some preparatory lemmas. Let \( \Delta_k^* \) be the adjoint of \( \Delta_k \) on \( L^2(\mathbb{R}^+, \frac{ds}{s}) \), then

\[
\begin{align*}
\Delta_0^* F(r) &= F(x) - F(2r), \\
\Delta_{k+1}^* F(r) &= \Delta_k^* F(r) - 2^{k+1} \Delta_k^* F(2r).
\end{align*}
\]
Lemma 2.4. Let $\varphi : \mathbb{R}^+ \to \mathbb{C}$ be a Schwartz function. Then for every $k \geq 0$ and $r \in \mathbb{R}^+$,

$$\varphi(r) = \sum_{h=0}^{+\infty} c_{h,k} \Delta^*_k \varphi(2^h r),$$

where

$$c_{h,k} = \sum_{h_0, \ldots, h_k = h} \prod_{j=1}^k 2^{j h_j} \leq 2^k \cdot 2^{h k}.$$  

Proof. We start by proving the inequality $c_{h,k} \leq C(k) 2^{hk}$. We clearly have that $c_{h,0} = 1$, so $C(0) = 1$. By proceeding inductively, we have

$$c_{h,k+1} = \sum_{h_0, \ldots, h_{k+1} = h} \prod_{j=0}^{h+1} 2^{j h_j}$$

$$= \sum_{h_0=0}^{h} \left( 2^{h-h_0} \sum_{h_1, \ldots, h_{k+1} = h-h_0} \prod_{j=0}^{k+1} 2^{(j-1) h_j} \right)$$

$$= \sum_{h_0=0}^{h} 2^{h-h_0} c_{h-h_0,k}$$

thus, by the induction assumption,

$$\leq C(k) \sum_{h_0=0}^{h} 2^{h-h_0} 2^{(h-h_0)k}$$

$$\leq C(k+1) 2^{h(k+1)},$$

if we set

$$C(k+1) = C(k) \sum_{h_0=0}^{h} 2^{-h_0(k+1)}.$$

Since the latter sum is bounded from above by 2, the inequality follows. Therefore, in order to obtain (2.6), it is enough to take

$$C(k+1) = C(k) \sup_{h \in \mathbb{N}} \sum_{h_0=0}^{h} 2^{-h_0(k+1)} \leq 2C(k) < +\infty.$$

We now move to the proof of formula (2.5). We first notice that since $\varphi$ is a Schwartz function, then $\Delta^*_k \varphi$ is a Schwartz function as well for every $k$. Hence the summation in (2.5) converges absolutely, in view of the estimate
Moreover, \( 2^{(k+1)h} \Delta_k^* \varphi(2^h r) \to 0 \) as \( h \to \infty \), so by a telescoping sum argument,

\[
\varphi(r) = \sum_{h=0}^{+\infty} \Delta_h^* \varphi(2^h r),
\]

which is (2.5) in the case \( k = 0 \), and similarly

\[
\Delta_k^* \varphi(r) = \sum_{h=0}^{+\infty} 2^{(k+1)h} \Delta_{k+1}^* \varphi(2^h r).
\]

Therefore, proceeding inductively,

\[
\varphi(r) = \sum_{h=0}^{+\infty} c_{h,k} \Delta_k^* \varphi(2^h r)
\]

\[
= \sum_{h=0}^{+\infty} c_{h,k} \sum_{h_k+1=0}^{+\infty} 2^{(k+1)h_k+1} \Delta_{k+1}^* \varphi(2^{h+h_k+1} r)
\]

\[
= \sum_{h=0}^{+\infty} \sum_{h_0+\cdots+h_k=h_k+1=0}^{+\infty} \prod_{j=0}^{k} 2^{h_j} 2^{(k+1)h_k+1} \Delta_{k+1}^* \varphi(2^{h+h_k+1} r)
\]

\[
= \sum_{h'=0}^{+\infty} \sum_{h_k+1=0}^{+\infty} \prod_{j=0}^{k+1} 2^{h_j} \Delta_{k+1}^* \varphi(2^{h'} r)
\]

\[
= \sum_{h'=0}^{+\infty} c_{h',k+1} \Delta_{k+1}^* \varphi(2^{h'} r),
\]

which shows (2.5). \( \square \)

**Lemma 2.5.** Let \( \varphi \) be a Schwartz function, \( N \geq 1, k \geq 0 \), and let \( c_{h,k} \) be as in (2.6). Then for every \( \alpha > \min(k, 1) \) and \( 1 \leq p' \leq \infty \),

\[
(2.7) \quad \sum_{h=0}^{+\infty} c_{h,k} N \| \varphi(2^h N r) \|_{L^p(\mathbb{R}^d)} \| r^\alpha \|_{L^{p'}(\mathbb{R}^d)} \lesssim N^{-\alpha}.
\]

**Proof.** Since \( \varphi \) is a Schwartz function and \( \alpha > 1 \), then \( |r|^{\alpha} \varphi \in L^{p'}(\mathbb{R}^d) \). Hence, by a simple change of variable,

\[
\| \varphi(2^h N r) \|_{L^p(\mathbb{R}^d)} \sim 2^{-h \alpha} N^{-\alpha}.
\]
Therefore,
\[
\left\| \sum_{h=0}^{+\infty} c_{h,k} N \varphi(2^h N r)|r|^\alpha \right\|_{L_p'} \leq \sum_{h=0}^{+\infty} c_{h,k} N \left\| \varphi(2^h N r)|r|^\alpha \right\|_{L_p'} \lesssim \sum_{h=0}^{+\infty} N^{-\alpha} 2^{-h(\alpha-k)} \lesssim N^{-\alpha},
\]
by (2.6).

\[\square\]

**Proof of Theorem 2.2.** Let \( \varphi_N = N^d \varphi(N \cdot) \) be the Littlewood-Paley projector on frequencies \( \sim N \). The estimate (2.4) is equivalent to
\[
(2.8) \quad \left\| \int_{\mathbb{R}^d} \varphi_N(x) \mu(x - y) \, dx \right\|_{L_q^p} \lesssim N^{-\alpha} \|\mu\|_{SBE^{\alpha,p}}.
\]

We will first show the above inequality in the case \( \mu \in C^\infty(\mathbb{R}^d) \), with a pre-factor independent from \( \mu \).

Recall that \( \varphi \) is a radial function. Let \( \partial_r \varphi \) be the derivative of \( \varphi \) in the radial direction. With a slight abuse of notation, we denote \( \partial_r \varphi \big( r \big) := \partial_r \varphi \big( x \big) \), for any \( x \) with \( |x| = r \). Since \( \varphi \) is a Schwartz function, we have that
\[
\varphi(x) = - \int_0^\infty \partial_r \varphi(r) \mathbb{1}_{[0,r]}(|x|) \, dr.
\]

Therefore,
\[
\int_{\mathbb{R}^d} \varphi_N(x) \mu(x - y) \, dx = \int_{\mathbb{R}^d} \varphi_N(x - y) \mu(x) \, dx = - \int_0^\infty \int_{\mathbb{R}^d} \partial_r \varphi_N(r) \mathbb{1}_{[0,r]}(|x - y|) \mu(x) \, dx \, dr
\]
\[
= - \int_0^\infty \partial_r \varphi_N(r) F_\mu(r, y) \, dr
\]
\[
= - \int_0^\infty N^d \cdot N r (\partial_r \varphi)(N r) F_\mu(r, y) \frac{dr}{r}
\]

In order to compensate the divergence at \( r = 0 \) that we will find when estimating formula (2.5) (see (2.11) below), we notice that, since \( \int \varphi_N(r) r^n \, dr = 0 \) for every \( n \in \mathbb{N} \), by similar computations,
\[
\int_0^\infty N^d \cdot N r (\partial_r \varphi)(N r) r^{n+1} \frac{dr}{r} = 0
\]
for every \( n \in \mathbb{N} \). Therefore,
\[
(2.9) \quad \int \varphi_N(x) \mu(x - y) \, dx = - \int_0^\infty N^d \cdot N r (\partial_r \varphi)(N r) \left( F_\mu(r, y) - \sum_{j=0}^k \alpha_j r^j \right) \frac{dr}{r},
\]
and we choose \( a_j = a_j(y) = \frac{1}{j!} F_\mu(\cdot, y)^{(j)}(0) \), so that

\[
|F_\mu(r, y) - \sum_{j=d}^{k} a_j r^j| \lesssim r^{k+1}.
\]

(2.10)

Set \( \psi(r) = r \partial_r \), so that \( N^d \cdot N r(\partial_r \psi)(N r) = N^d \psi(N r) =: \psi_N(r) \). By (2.5),

\[
\psi_N(r) = \sum_{h=0}^{+\infty} c_{h,k} \Delta_k^h \psi_N(2^h r).
\]

Since \( \psi_N \) is a Schwartz function, and thus so is \( \Delta_k^h \psi_N \), from (2.6) we have that

\[
\sum_{h=0}^{+\infty} c_{h,k} |\Delta_k^h \psi_N(2^h r)| \lesssim \frac{r^{-k}}{1 + r^2}.
\]

(2.11)

Therefore, by dominated convergence, (2.9), (2.10), (2.3), Hölder’s inequality (we denote by \( p' \) the Hölder conjugate exponent of \( p \)) and (2.7),

\[
\left| \int \varphi_N(x) \mu(x - y) \, dx \right| = \left| \sum_{h=0}^{+\infty} c_{h,k} \int \Delta_k^h \psi_N(2^h r) \left( F_\mu(r, y) - \sum_{j=d}^{k} a_j r^j \right) \frac{dr}{r} \right|
\]

\[
= \left| \sum_{h=0}^{+\infty} c_{h,k} \int \psi_N(2^h r) \Delta_k F_\mu(r, y) \frac{dr}{r} \right|
\]

\[
= \sum_{h=0}^{+\infty} c_{h,k} \int r^{\alpha+d} \psi_N(2^h r) r^{-\alpha-d} \Delta_k F_\mu(r, y) \frac{dr}{r}
\]

\[
\leq \sum_{h=0}^{+\infty} c_{h,k} ||r^{\alpha+d} \psi_N(2^h r)||_{L_p'} ||r^{-\alpha-d} \Delta_k F_\mu(r, y)||_{L_p}
\]

\[
\lesssim N^{-\alpha} ||r^{-\alpha-d} \Delta_k F_\mu(r, y)||_{L_p},
\]

from which (2.8) follows easily.

We now move to the case of \( \mu \) being not necessarily smooth. Let \( \rho \geq 0 \) be a smooth function with compact support and \( \int \rho = 1 \), and for \( \epsilon > 0 \), let

\[
\rho_\epsilon := \epsilon^{-1} \rho(\epsilon^{-1} x).
\]

Fix \( N \gg 1 \), and let \( \mu_\epsilon := \mu \ast \rho_\epsilon \). Recalling that \( \mu \) has compact support and \( \varphi_N \) is a Schwartz function, we have that

\[
\int_{\mathbb{R}^d} \varphi_N(x-y)(d\mu(x) - \mu_\epsilon(x)) \, dx = \int_{\mathbb{R}^d} (\varphi_N - \varphi_N \ast \rho_\epsilon)(x-y) \, d\mu(x)
\]

\[
\lesssim C(N, \mu) \epsilon(1 + |y|)^{-(d+1)},
\]
so \( \| \int \varphi_N(x-y)(d\mu(x) - \mu_\varepsilon(x)dx) \|_{L^q_y} \to 0 \) as \( \varepsilon \to 0 \). Moreover,

\[
\| \mu_\varepsilon \|_{\text{SBF}_{q}^{\alpha,p}} = \| r^{-\alpha-d} \Delta_k F_{\mu_\varepsilon}(r,y) \|_{L^q_{y}L^p_r} \\
= \| r^{-\alpha-d}(\Delta_k)_r(F_\mu \ast \rho_\varepsilon)(r,y) \|_{L^q_{y}L^p_r} \\
\leq \left\| \int \| r^{-\alpha-d} \Delta_k F_\mu (r,y-y_0) \|_{L^p_r} \rho_\varepsilon(y_0) dy_0 \right\|_{L^q_y} \\
\leq \left\| \int \| r^{-\alpha-d} \Delta_k F_\mu (r,y-y_0) \|_{L^q_{y}L^p_r} \rho_\varepsilon(y_0) dy_0 \right\|_{L^q_y} \\
= \int \| r^{-\alpha-d} \Delta_k F_\mu (r,y) \|_{L^q_{y}L^p_r} \rho_\varepsilon(y_0) dy_0 \\
= \| \mu \|_{\text{SBF}_{q}^{\alpha,p}}.
\]

Therefore, by choosing \( \varepsilon \) small enough,

\[
\left\| \int \varphi_N(x-y)d\mu(x) \right\|_{L^q_y} \\
\leq \left\| \int \varphi_N(x-y)(d\mu(x) - \mu_\varepsilon(x)dx) \right\|_{L^q_y} + \left\| \int \varphi_N(x-y) d\mu_\varepsilon(x) \right\|_{L^q_y} \\
\lesssim N^{-\alpha} \| \mu \|_{\text{SBF}_{q}^{\alpha,p}},
\]

which yields (2.8). \( \square \)

**Lemma 2.6.** Let \( \varphi \in S(\mathbb{R}^d) \). Then for every \( \alpha > 0 \), \( \alpha \notin \mathbb{N} \), and for every \( 1 \leq p, q \leq \infty \),

\[ \| \varphi \|_{\text{SBF}_{q}^{\alpha,p}} < \infty. \]

**Proof.** We begin by estimating \( \Delta_k F_\varphi(r,y) \). For some coefficients \( \{ a_n = a_n(y) \}_{n \leq k} \) to be chosen later, define

\[ g(r,y) := F_\varphi(r,y) - \sum_{n \leq k} a_n r^{n+d}. \]

From (2.3), we have that \( (\Delta_k)_r g(r,y) = \Delta_k F_\varphi(r,y) \). Moreover, since \( \varphi \in C^\infty(\mathbb{R}^d) \), from the formula

\[ F_\varphi(r,y) = \int_{|x-y| \leq r} \varphi(x) dx = \int_0^r \int_{S^{d-1}} \mu(r \cdot \theta + y) d\sigma(\theta) r^{d-1} dr, \]

we have that \( F_\varphi(\cdot,y) \in C^\infty([0, \infty)) \) as well, and moreover, for every \( h \in \mathbb{N} \),

(2.12) \[ |F_\varphi^{(h)}(\cdot,y)(r)| \lesssim \| \varphi \|_{C^h(B(y,r))}, \]

and for every \( h \leq d \),

(2.13) \[ |F_\varphi^{(h)}(\cdot,y)(0)| = 0. \]
Therefore, choosing
\[ a_n(y) := \frac{f_{\varphi}^{(n+d)}(\cdot, y)(0)}{(n+d)!}, \]
by (2.12) and (2.13), we obtain that
\[ |g(r, y)| \lesssim r^{d+k+1} \| \varphi \|_{C^{k+1}(B(y, r))}, \]
and so we also have the same estimate for \((\Delta_k)_r g(r, y) = \Delta_k f_{\varphi}(r, y)\),
\[ |\Delta_k f_{\varphi}(r, y)| \lesssim \| \varphi \|_{C^{k+1}(B(y, r))}. \]
Together with the simple estimate
\[ |\Delta_k f_{\varphi}(r, y)| \lesssim \| \varphi \|_{L^1}, \]
by splitting the domain of integration in \{r \leq |y|/2\} and \{r > |y|/2\}, we obtain that
\[ \| r^{-\alpha-d}(\Delta_k)_r f_{\mu}(r, y) \|_{L^p(\frac{dy}{r^{d-\alpha}})} \lesssim (|y| + 1)^{k+1-\alpha} \| \varphi \|_{C^{k+1}(B(y, |y|/2))} + (|y| + 1)^{-d-\alpha} \]
Recalling that \( \varphi \in S(\mathbb{R}^d) \), it follows easily that \( \| \varphi \|_{\mathcal{S}^\alpha,p} < \infty. \)

**Proof of Proposition 2.3.** For \( N \geq 1 \), let \( \varphi_N(x) := N^d \varphi(Nx) \) be the Littlewood-Paley projector, and let \( \varphi_0 = \sum_{N \leq 0} N^d \varphi(N \cdot) \). Since \( f \in B_{q,1}^\alpha \), we can write
\[ f = \sum_{N \geq 0} \text{dyadic } f_N \ast \varphi_N, \]
with
\[ \sum_N N^\alpha \| f_N \|_{L^q} \lesssim \| f \|_{B_{q,1}^\alpha}. \]
By change of variables,
\[ f_{\varphi_N}(r, y) = \int_{B_1} r^d \varphi_N(y + rx) \, dx = \int_{B_1} r^d N^d \varphi(Ny + Nrx) \, dx = F_{\varphi}(Nr, Ny). \]
Therefore, we also have \((\Delta_k)_r f_{\varphi_N}(r, y) = ((\Delta_k)_r f_{\varphi})(Nr, Ny)\). Thus,
\[ \| \varphi_N \|_{\mathcal{S}^\alpha,p} = \| r^{-\alpha-d}(\Delta_k)_r f_{\varphi_N}(r, y) \|_{L^p_y L^p(\mathbb{R}^d \times \mathbb{R}^d, dy \otimes \frac{dy}{r^{d-\alpha}})} \]
\[ = \| r^{-\alpha-d}((\Delta_k)_r f_{\varphi})(Nr, Ny) \|_{L^p_y L^p(\mathbb{R}^d \times \mathbb{R}^d, dy \otimes \frac{dy}{r^{d-\alpha}})} \]
\[ = \| N^{\alpha+d}((\Delta_k)_r f_{\varphi})(\cdot, Ny) \|_{L^p_y(\frac{dy}{r^{d-\alpha}})} \|_{L^p_y} \]
\[ = N^\alpha \| \varphi \|_{\mathcal{S}^\alpha,p}. \]
which is finite by Lemma 2.6. Similarly, again by Lemma 2.6, we have that 
\[ \|\varphi_0\|_{SGE^{\alpha,p}_q} \lesssim 1. \] Therefore, by Young’s inequality,

\[
\|f\|_{SGE^{\alpha,p}_q} \lesssim \sum_N \|f_N \ast \varphi_N\|_{SGE^{\alpha,p}_q}
\]

\[
= \sum_N \|r^{-\alpha-d}(\Delta_k)_r F_{f_N \ast \varphi_N}(r, y)\|_{L^q_\delta L^p([R^d \times R^+, dy \otimes dr])}
\]

\[
= \sum_N \|r^{-\alpha-d}(f_N \ast \varphi_N)(\Delta_k)_r F\varphi_N(r, y)\|_{L^q_\delta L^p([R^d \times R^+, dy \otimes dr])}
\]

\[
\lesssim \sum_N \|f_N\|_{L^q} \|r^{-\alpha-d}(\Delta_k)_r F\varphi_N(r, y)\|_{L^q_\delta L^p([R^d \times R^+, dy \otimes dr])}
\]

\[
\lesssim \sum_N \|f_N\|_{L^q} N^\alpha \lesssim \|f\|_{B^{\alpha,1}_q}. \]

\[ \square \]

3. $SGE$ Regularity for Occupation Measures

We wish to use this definition for the occupation measure (defined in (1.2)), to measure the irregularity of paths. First, we give a connection with $\rho, \gamma$ irregularity.

**Proposition 3.1.** Let $\alpha > 0$, $1 \leq p, q \leq \infty$, and $\gamma \in (0,1)$. Given a path $\omega : [a, b] \rightarrow \mathbb{R}^d$, if the occupation measure $(\mu_{a,t})_{t \in [a, b]}$ is $\gamma$-Hölder with values in $SGE^{\alpha,p}_q$, then $\omega$ is $(\alpha, \gamma)$-irregular.

**Proof.** We recall that by a simple remark given in [HRT22], a path $\omega$ is $(\rho, \gamma)$-irregular if and only if $t \mapsto \mu_{a,t}^\omega$ is $\gamma$-Hölder in time with values in $B^{d,\infty}_{1,1}$. Hence, for every $\rho \in C^\infty_\gamma$ such that $\rho \equiv 1$ on supp$(\mu_{a,t}^\omega)$, we have that $\mu_{a,t}^\omega = \rho \mu_{a,t}^\omega$ for every $\rho \in C^\infty_\gamma$, and

\[
\|\rho\|_{B^{d,\infty}_{1,1}} \lesssim \|\mu_{a,t}^\omega\|_{B^{\alpha,\infty}_1} \lesssim \|\mu_{a,t}^\omega\|_{SGE^{\alpha,p}_q}. \]

In view of the negative result presented in [HRT22] about the non-existence for Hölder continuous maps of a condition that ensures $(\rho, \gamma)$ irregularity and is invariant by bi-Lipschitz reparametrisation, we prove that our notion of irregularity of paths is invariant by bi-Lipschitz reparametrisations, under an additional $V^p$ condition on the derivative of the reparametrisation. To this end we recall the definition of spaces of variations $V^p(a, b; E)$ defined in Section 1.1.
**Proposition 3.2.** Let \( \omega : [a, b] \to \mathbb{R}^d \), and let \( r > 1 \). Suppose that the occupation measure \( \mu \) of \( \omega \) satisfies
\[
\mu_{a, r} \in V^r(\mathbb{S}^d_{E_q^a, p}).
\]
Let \( \varphi : [a', b'] \to [a, b] \) be a bi-Lipschitz map with \( \varphi' \in V^{r' - \epsilon} \) for some \( \epsilon > 0 \), where \( r' \) is the Hölder conjugate of \( r \). Then the process \( \omega \circ \varphi \) has occupation measure \( \mu_{\varphi} \) that satisfies
\[
\mu_{\varphi, a', r'} \in V^{r'}(\mathbb{S}^d_{E_q^{a'}, p}),
\]
as well.

We will prove the above proposition at page 13. We additionally give a property of stability with respect to (suitable) smooth perturbations of paths, proved at page 15.

**Proposition 3.3.** Let \( \omega : [a, b] \to \mathbb{R}^d \). Suppose that for some \( r > 1 \), \( 1 < p < \infty \), \( 1 \leq q \leq \infty \), the occupation measure \( \mu \) of \( \omega \) satisfies
\[
\mu_{a, r} \in V^r(B_{p, q}^\alpha).
\]
Let \( f : [a, b] \to \mathbb{R}^d \) be a function with \( f \in V^{r_1} \) for some \( r_1 \geq 1 \), let \( \omega_s^f := \omega_s + f_s \), and let \( \mu^f \) be its occupation measure. Then, for every \( \frac{1}{r_1} < \gamma \leq 1 \),
\[
\mu_{a, r}^f \in V^r(B_{p, q}^{\alpha - \gamma}),
\]
where \( r' \) is the Hölder conjugate of \( r \). Moreover, if \( f, g \in V^{r_1} \), then for every \( \frac{1}{r_1} < \gamma \leq 1 \),
\[
\| \mu_{a, r}^f - \mu_{a, r}^g \|_{V^r(B_{p, q}^{\alpha - \gamma})} \lesssim \| \mu_{a, r} \|_{V^r(B_{p, q}^\alpha)} \| f - g \|_{V^{r_1}} + (1 + \|g\|_{V^{r_1}}) \| f - g \|_{L^\infty}.
\]

### 3.1. Proofs of reparametrisation and perturbation.

**Proof of Proposition 3.2.** We preliminarily state the following estimate. Let \( g : [a, b] \to \mathbb{R} \) be a bounded function in \( V^{r' - \epsilon} \), and set
\[
g \cdot \mu_{s, t} := \int_s^t g(\tau) \delta_{\omega_{\tau}} \, d\tau.
\]
Then
\[
\| g \cdot \mu_{a, r} \|_{V^r(\mathbb{S}^d_{E_q^a, p})} \lesssim \| \mu_{a, r} \|_{V^r(\mathbb{S}^d_{E_q^a, p})} \left( \|g\|_{V^{r' - \epsilon}} + \|g\|_{L^\infty} \right).
\]

We postpone the proof of the estimate and use it to prove the proposition. By a change of variables, we have that
\[
\mu_{\varphi, s, t} = \int_s^t \delta_{\omega_{\varphi(\tau)}} \, d\tau = \int_{\varphi(s)}^{\varphi(t)} \frac{\delta_{\omega_{\tau}}}{\varphi'(\tau^{-1})} \, d\tau.
\]
Let \( g(\tau) := \frac{1}{\varphi'(\varphi^{-1}(\tau))} \). Since \( \varphi' \in V^{r'-e} \), and \( \varphi \) is bi-Lipschitz, then we also have \( g \in V^{r-e} \). Therefore, by (3.2),

\[
\|\mu_{a_{r'}}\|_{V^r(g \mathcal{E} A_\rho)} \lesssim \|\mu_{a_{r'}}\|_{V^r(S g \mathcal{E} A_\rho)} \left( \|\frac{1}{\varphi'} \|_{V^{r'-e}} + \|\frac{1}{\varphi'}\|_{L^\infty} \right) < +\infty,
\]

which concludes the proof of the main statement.

We turn to the proof of (3.2). We first assume that \( g \) is smooth. If \( g \) is constant, this inequality is trivial. Therefore, we can assume that \( g \) is not constant. Let \( N \) be a dyadic number. Define the partitions \( s = t_0^N < t_1^N < \cdots < t_N^N = t \) so that

\[
(3.3) \quad \begin{cases} \|g\|_{V^{r'-e}(\mathbb{I}_{[t_j^N, t_{j+1}^N]})} = \|g\|_{V^{r'-e}(\mathbb{I}_{[t_{j+1}^N, t_{j+2}^N]})} \\ t_{j+1}^N = t_j^N. \end{cases}
\]

Since

\[
\|g\|_{V^{r'-e}(\mathbb{I}_{[t_j^N, t_{j+1}^N]})} + \|g\|_{V^{r'-e}(\mathbb{I}_{[t_{j+1}^N, t_{j+2}^N]})} \leq \|g\|_{V^{r'-e}(\mathbb{I}_{[t_j^N, t_{j+2}^N]})} = \|g\|_{V^{r'-e}(\mathbb{I}_{[t_j^N, t_{j+1}^N]})}
\]

proceeding inductively we have that

\[
(3.4) \quad \|g\|_{V^{r'-e}(\mathbb{I}_{[t_j^N, t_{j+1}^N]})} \leq \frac{1}{N} \|g\|_{V^{r'-e}(\mathbb{I}_{[t_j^N, t_{j+1}^N]})}
\]

Moreover, we have that

\[
g \cdot \mu_{s,t} = \lim_{N \to \infty} (g \cdot \mu_{s,t})^{(N)}
\]

in total variation norm, where

\[
(g \cdot \mu_{s,t})^{(N)} := \sum_{j=0}^{N-1} \mu_{t_j^N, t_{j+1}^N, g(t_j^N)}.
\]

By (3.4), we have that

\[
\sum_{j=0}^{N-1} \|g(t_{j+1}^{2N}) - g(t_j^{2N})\|_{V^{r'-e}} \leq \sum_{j=0}^{N-1} \|g\|_{V^{r'-e}(\mathbb{I}_{[t_j^{2N}, t_j^{2N+1}]} + g(t_j^{2N}) - g(t_{j+1}^{2N}))\|_{V^{r'-e}}
\]

\[
\leq N^{-r'-e} \|g\|_{V^{r'-e}(s,t)} \sum_{j=0}^{N-1} \|g(t_{j+1}^{2N}) - g(t_j^{2N})\|_{V^{r'-e}}
\]

\[
\leq N^{-r'-e} \|g\|_{V^{r'-e}(s,t)}
\]
thus,
\[
\| (g \cdot \mu_{s,t})^{(2N)} - (g \cdot \mu_{s,t})^{(N)} \|_{SB^{\alpha,p}_q}
\]
\[
= \left\| \sum_{j=0}^{N-1} \mu_{t_{2j+1}, t_{2(j+1)}}^{(2N)} (g(t_{2j+1}^{2N}) - g(t_{2j}^{2N})) \right\|_{SB^{\alpha,p}_q}
\]
\[
\leq \sum_{j=0}^{N-1} \| \mu_{t_{2j+1}, t_{2(j+1)}}^{(2N)} \|_{SB^{\alpha,p}_q} |g(t_{2j+1}^{2N}) - g(t_{2j}^{2N})|
\]
\[
\leq \left( \sum_{j=0}^{N-1} \| \mu_{t_{2j+1}, t_{2(j+1)}}^{(2N)} \|_{SB^{\alpha,p}_q} \right)^{\frac{1}{r}} \left( \sum_{j=0}^{N-1} |g(t_{2j+1}^{2N}) - g(t_{2j}^{2N})|^{r'} \right)^{\frac{1}{r'}}
\]
\[
\leq N^{-\frac{1}{r}(r' - \epsilon)} \| \mu \|_{V^{r'}(s,t), SB^{\alpha,p}_q} \| g \|_{V^{r'} - \epsilon}.
\]
Therefore, by telescopic summation,
\[
\| g \cdot \mu_{s,t} \|_{SB^{\alpha,p}_q} \lesssim \| g \cdot \mu_{s,t} \|^{(1)}_{SB^{\alpha,p}_q} + \| \mu \|_{V^{r'}(s,t), SB^{\alpha,p}_q} \| g \|_{V^{r'} - \epsilon}
\]
\[
\lesssim \| \mu_{s,t} \|_{SB^{\alpha,p}_q} \| g \|_{L^\infty} + \| \mu \|_{V^{r'}(s,t), SB^{\alpha,p}_q} \| g \|_{V^{r'} - \epsilon},
\]
from which (3.2) follows when \( g \) is smooth.

Assume now that \( g \in V^{r'} - \epsilon \) is bounded. It is well known that smooth functions are not dense in spaces of variations, see for instance [FV10, Theorem 5.31], and moreover our \( g \) is not continuous. So we consider the following regularization \( g_\delta \) of \( g \) defined as
\[
g * \rho_\delta(t) = \int_R g(t - s) \rho_\delta(s) \, ds,
\]
with \( g \) continuously extended as constant outside \([a,b]\), and where \( \rho_\delta(t) = \delta^{-1} \rho(\delta^{-1} t) \) is a smooth mollifier. It is elementary to see that \( \| g_\delta \|_{V^{r'} - \epsilon} \leq \| g \|_{V^{r'} - \epsilon} \) and \( \| g_\delta \|_{L^\infty} \leq \| g \|_{L^\infty} \). Moreover,
\[
\| F_{g \cdot \mu_{s,t}} - F_{g_\delta \cdot \mu_{s,t}} \|_{L^1} \leq \| g - g_\delta \|_{L^1},
\]
therefore by Fatou’s lemma,
\[
\| g \cdot \mu_{a,r} \|_{V^r(SB^{\alpha,p}_q)} \leq \liminf_{\delta \to 0} \| g_\delta \cdot \mu_{a,r} \|_{V^r(SB^{\alpha,p}_q)},
\]
and from this (3.2) follows. \( \square \)

**Proof of Proposition 3.3.** The proof follows closely the proof of Proposition 3.2. We want to show that
\[
(3.5) \quad \| \mu_{a,r} \|_{V^r(B^{\alpha - \gamma}_{p,q})} \lesssim \| \mu_{a,r} \|_{V^r(B^{\alpha}_{p,q})} \| f \|_{V^r} + \| \mu_{a,r} \|_{V^{r}(B^{\alpha - \gamma}_{p,q})},
\]
If $f$ is constant, the inequality is trivial, so we can assume that $f$ is not constant. Define the partitions $s = t_0^N < t_1^N < \cdots < t_N^N = t$ as in (3.3) (with $r' - \epsilon$ replaced by $r_1$), so that for every $x, y \in [t_j^N, t_{j+1}^N]$,

\begin{equation}
|f(x) - f(y)|^{r_1} \leq \frac{1}{N} ||f||_{V^r([s,t])}.
\end{equation}

Recall that $t_j^N = t_j$. Moreover, we have that

$$\mu_s^f = \lim_{N \to \infty} \mu_{s,t}^{f,N}$$

in (any) Wasserstein norm, where

$$\mu_{s,t}^{f,N} := \sum_{j=0}^{N-1} \tau_{t_j} \mu_{t_j^N, t_{j+1}^N}.$$ 

For any function $u$ such that its Fourier transform has support in $[-10N, 10N]$, we have the inequality

$$\|\tau_y u - u\|_{L^p} \lesssim |y|N \|u\|_{L^p}.$$ 

By interpolation with $\|\tau_y u - u\|_{L^p} \lesssim \|u\|_{L^p}$, this implies

$$\|\tau_y u - u\|_{L^p} \lesssim |y|N^\gamma \|u\|_{L^p}.$$ 

Therefore, for any $x, y \in \mathbb{R}^d$, we have

\begin{equation}
||\tau_y u - \tau_x u||_{B_{p,q}^{\alpha-\gamma}} \lesssim |x - y|^\gamma \|u\|_{B_{p,q}^\alpha}.
\end{equation}

Set $\epsilon = \gamma r' - r_1$, then by (3.6),

\begin{equation}
\sum_{j=0}^{N-1} |f(t_{2j+1}^N) - f(t_{2j}^N)|^{r_1} \leq \sum_{j=0}^{N-1} (N - \epsilon)^{r_1} \|f\|_{V^r([s,t])} \|f(t_{2j}^N) - f(t_{2j+1}^N)\|^{r_1}
\end{equation}

\begin{equation}
\leq N^{\gamma r' - r_1} \|f\|_{V^r([s,t])}.
\end{equation}

The above estimate and (3.7) yield,

\begin{align*}
\|\mu_{s,t}^{f,2N} - \mu_{s,t}^{f,N}\|_{B_{p,q}^{\alpha-\gamma}} &= \left\| \sum_{j=0}^{N-1} (\tau_{f(t_{2j+1}^N)} - \tau_{f(t_{2j}^N)}) \mu_{t_{2j+1}^N, t_{2j}^N} \right\|_{B_{p,q}^{\alpha-\gamma}} \\
&\leq \sum_{j=0}^{N-1} \left\| \mu_{t_{2j+1}^N, t_{2j}^N} \right\|_{B_{p,q}^\alpha} \left\| f(t_{2j+1}^N) - f(t_{2j}^N) \right\|^{\gamma} \\
&\leq \left( \sum_{j=0}^{N-1} \left\| \mu_{t_{2j+1}^N, t_{2j}^N} \right\|_{B_{p,q}^\alpha} \right)^{\frac{r_1}{r}} \left( \sum_{j=0}^{N-1} \left\| f(t_{2j}^N) - f(t_{2j+1}^N) \right\|^{r_1} \right)^{\frac{r_1}{r}} \\
&\leq N^{\gamma r' - r_1} \|\mu\|_{V^r([s,t], B_{p,q}^\alpha)} \|f\|_{V^r([s,t])}.
\end{align*}
By telescopic summation, we obtain

\[ \|\mu_{s,t}^f\|_{B^{\alpha-\gamma}_{p,q}} \leq \|\mu_{s,t}\|_{B^{\alpha-\gamma}_{p,q}} + \sum_{N} \|\mu_{s,t}^{f_{2N}} - \mu_{s,t}^{f_{N}}\|_{B^{\alpha-\gamma}_{p,q}} \]

\[ \lesssim \|\mu_{s,t}\|_{B^{\alpha-\gamma}_{p,q}} + \|\mu\|_{V^r_{\gamma}(t,s,t]} \|f\|_{V^r_{\gamma}}, \]

from which (3.5) follows easily.

Now we move to proving (3.1). We proceed analogously, and fix partitions

\[ s = t_0^N \leq t_1^N \leq \cdots \leq t_N^N = t \]

as in (3.3) (with \( r' - \epsilon \) replaced by \( r_1 \)), so that for every \( x, y \in [t_j^N, t_{j+1}^N] \),

\[ |f(x) - f(y)|_{r_1} \leq 2 \frac{N}{|f|}_{V^r_1([s,t])}, \]

\[ |(f - g)(x) - (f - g)(y)|_{r_1} \leq 2 \frac{N}{|f - g|}_{V^r_1([s,t])}. \]

In order to realise such a partition, we can construct \( s = t_0^N(f) < t_1^N(f) < \cdots < t_N^N(f) = t \) and \( s = t_0^N(f - g) < t_1^N(f - g) < \cdots < t_N^N(f - g) = t \) as in (3.3), and then take \( \{t_0^{2N}, \ldots, t_{2N}^N\} = \{t_j^N(f)\}_{j \leq N} \cup \{t_j^N(f - g)\}_{j \leq N} \), appropriately rearranged so they are in increasing order.

For any function \( u \) such that its Fourier transform has support in \([-10N, 10N]\), we have the inequality

\[ \|\tau_{y_2} - \tau_{x_2}\| u - (\tau_{y_1} - \tau_{x_1}) u\|_{L^p} \leq \|\tau_{y_2} - \tau_{x_2}\| u - (\tau_{y_2 - (x_2 - x_1)} - \tau_{x_2 - (x_2 - x_1)}) u\|_{L^p} + \|\tau_{y_2 - (x_2 - x_1)} - \tau_{y_1} u\| \]

\[ \lesssim N^2 \|u\|_{L^p} |x_2 - x_1| |y_2 - x_2| + N \|u\|_{L^p} (y_2 - x_2) - (y_1 - x_1), \]

and similarly,

\[ \|\tau_{y_2} - \tau_{x_2}\| u - (\tau_{y_1} - \tau_{x_1}) u\|_{L^p} \lesssim N \|u\|_{L^p} (y_2 - x_2) + (y_2 - x_2) - (y_1 - x_1)). \]

Interpolating between the two, we obtain

\[ \|\tau_{y_2} - \tau_{x_2}\| u - (\tau_{y_1} - \tau_{x_1}) u\|_{L^p} \lesssim |x_2 - x_1| |y_2 - x_2| N^{1/2} \|u\|_{L^p} + (y_2 - x_2) - (y_1 - x_1) N \|u\|_{L^p}. \]

Therefore, for any \( x_1, y_1, x_2, y_2 \in \mathbb{R}^d \), we have

\[ \|\tau_{y_2} - \tau_{x_2}\| u - (\tau_{y_1} - \tau_{x_1}) u\|_{B_{p,q}^{\alpha-\gamma-1}} \]

\[ \lesssim (|x_2 - x_1|^r |y_2 - x_2| + (y_2 - x_2) - (y_1 - x_1)) \|u\|_{B_{p,q}^\alpha}. \]

\[ (3.10) \]
Notice that (3.8) still holds for $g$ and $f - g$, due to the choice of the partition (3.9). Therefore, by (3.10) and (3.8), we obtain that
\[
\| (\mu_{s,t}^g - \mu_{s,t}^f) - (\mu_{s,t}^g - \mu_{s,t}^f) \|_{B_{p,q}^{\alpha - \gamma - 1}}
\]
\[
= \left\| \sum_{j=0}^{N-1} \left( (\tau_{f(t_{2j+1}^{2N})} - \tau_{g(t_{2j+1}^{2N})}) - (\tau_{f(t_{2j}^{2N})} - \tau_{g(t_{2j}^{2N})}) \right) \mu_{t_{2j+1}^{2N}, t_{2j+1}^{2N}} \right\|_{B_{p,q}^{\alpha - \gamma - 1}}
\]
\[
\leq \sum_{j=0}^{N-1} \| \mu_{t_{2j+1}^{2N}, t_{2j+1}^{2N}} \|_{B_{p,q}^{\alpha - \gamma - 1}} \cdot \left( (g(t_{2j+1}^{2N}) - g(t_{2j}^{2N})) \| f - g \|_{L^\infty([s,t])} + |(f - g)(t_{2j+1}^{2N}) - (f - g)(t_{2j}^{2N})| \right)
\]
\[
\leq \left( \sum_{j=0}^{N-1} \| \mu_{t_{2j+1}^{2N}, t_{2j+1}^{2N}} \|_{B_{p,q}^{\alpha - \gamma - 1}} \right) \frac{1}{N} \left( \sum_{j=0}^{N-1} |g(t_{2j+1}^{2N}) - g(t_{2j}^{2N})| \right)^{\gamma} \| f - g \|_{L^\infty([s,t])} + \left( \sum_{j=0}^{N-1} |(f - g)(t_{2j+1}^{2N}) - (f - g)(t_{2j}^{2N})| \right)^{\gamma}
\]
\[
\leq \| \mu \|_{V^r([s,t], B_{p,q}^{\alpha - \gamma - 1})} \left( N^{-\frac{\gamma}{r' - \gamma}} \| g \|_{V^{r'}} \| f - g \|_{L^\infty} + N^{\frac{\gamma}{r' - \gamma}} \| f - g \|_{V^{r'}} \right)
\]

By telescopic summation, we obtain that
\[
\| \mu_{s,t}^f - \mu_{s,t}^g \|_{B_{p,q}^{\alpha - \gamma - 1}} \leq \| f - g \|_{L^\infty} \| \mu_{s,t} \|_{B_{p,q}^{\alpha - \gamma - 1}}
\]
\[
+ \| \mu \|_{V^r([s,t], B_{p,q}^{\alpha - \gamma - 1})} \left( \| g \|_{V^{r'}} \| f - g \|_{L^\infty} + \| f - g \|_{V^{r'}} \right),
\]
from which (3.1) follows easily. 

\[\square\]

4. \textbf{SpE regularity for stochastic processes}

We wish to discuss here SpE regularity for paths of stochastic processes. It turns out that it is sufficient to control the density of the increments of paths to get SpE regularity.

\textbf{Theorem 4.1.} Let $\beta > 0$ and let $\omega : [a, b] \to \mathbb{R}^d$ be a stochastic process with continuous paths. For $a \leq s, t \leq b$, let $\nu_{st} = \text{Law}(\omega_{st})$, and suppose that for every interval $J \subseteq [a, b]$,
\[
(4.1) \quad \int \int_{J \times J} \| \nu_{st} \|_{C^\beta} \ ds \ dt \leq C \nu |J|.
\]

Then for every $\alpha < \frac{\beta}{2}$ with $2\alpha \not\in \mathbb{N}$, the occupation measure satisfies
\[
\mu_{a, t} \in V^2(\text{SpE}^{\alpha, 2}_2).
\]
More precisely, for every $\delta \in (0, 1)$, and for every $1 \leq q < 2$, we have that

$$E \left[ \| \mu_{a,} \|_{V^q(\mathcal{SBE}_{2}^\gamma)}^q \left( 1 + \sup_s |\omega_s| \right)^{-\frac{d}{2}} \right] \lesssim 1 + C_{\nu}^{(1-\delta)\frac{d}{2}}.$$

The proof of the theorem can be found at page 27. A straightforward application of Kolmogorov’s continuity theorem yields the following result (see page 32 for the proof).

**Corollary 4.2.** Given a path $\omega : [a, b] \to \mathbb{R}^d$, under the same assumptions of Theorem 4.1, the occupation measure $\mu_a$, of $\omega$ is $\gamma$-Hölder continuous with values in $\mathcal{SBE}_{2}^{\alpha,2}$, for every $\gamma < \frac{\beta - 2\alpha}{2(\beta + 2d)}$.

**Remark 4.3.** Notice that the Hölder exponent in the above corollary is, no matter what is the value of parameters, smaller than $\frac{1}{2}$. This apparently prevents the use of the Young integration theory developed in [CG16]. We will overcome the issue in two ways. On the one hand we shall directly develop Young’s integration theory in spaces of finite variation in Section 6. On the other hand the Hölder exponent can be improved in two ways. First, (4.1) is apparently a bit restrictive and we could assume instead, without changing significantly our proofs, that

$$\int J \times J \| \nu_{st} \|_{C^\beta} \leq C_{\nu} |J|^{2\eta},$$

with $\eta \in [0, 1]$. Under this assumption, Corollary 4.2 yields $\gamma$-Hölder continuity with values in $\mathcal{SBE}_{2}^{\alpha,2}$ for $\gamma < \frac{1}{2} - (1 - \eta)(1 - \delta_0)$, and $\delta_0$ is as in (4.14). Unfortunately, the Hölder exponent is still too small for our purposes. A second possibility is a trade-off between time and space regularity. Unfortunately this will require much stronger assumptions and will not follow from the slick condition (4.1) of Theorem 4.1. We postpone to Section 5 this discussion.

The condition of Theorem 4.1 simplifies dramatically when dealing with a Gaussian process.

**Corollary 4.4.** Let $0 < H < 1$, let $\omega : [a, b] \to \mathbb{R}^d$ be a Gaussian process with continuous paths. Suppose that for $a \leq s, t \leq b$,

$$\text{Cov}(\omega_{st}) \gtrsim |t-s|^{2H}\text{Id.}$$

Then for every $\alpha < \frac{1}{2H} - \frac{d}{2}$,

$$\mu_{a,} \in V^{2}(\mathcal{SBE}_{2}^{\alpha,2}).$$

**Proof.** It is enough to check that $\omega$ satisfies the assumptions of Theorem 4.1 for any $\beta < \frac{1}{H} - d$. Since $\omega$ is Gaussian, we have that

$$\nu_{st}(x) = \frac{1}{(2\pi)^{\frac{d}{2}} \sqrt{\det(\text{Cov}(\omega_{st}))}} \exp \left( -\frac{1}{2} (x - \mathbb{E}[\omega_{st}])^T \text{Cov}(\omega_{st})^{-1} (x - \mathbb{E}[\omega_{st}]) \right),$$
therefore, we have that
\[ \|\mathcal{V}_{st}\|_{C^\beta} \lesssim |t-s|^{-H(\beta+d)}. \]
Let \( J = [a', b'] \subseteq [a, b] \) be an interval. For \( \beta < \frac{1}{H} - d \), we have
\[ \int_{J \times J} \|\mathcal{V}_{st}\|_{C^\beta} d\sigma dt \lesssim \int_{J \times J} |t-s|^{-(\beta+d)H} d\sigma dt \lesssim \int_{2a'}^{2b'} \int_{2a'}^{2b'} |s'|^{-(\beta+d)H} ds' dt' \lesssim |J|. \]

Remark 4.5. The condition above in Corollary 4.4 can be reinterpreted as a property of local non-determinism. Indeed, if \( \mathcal{F}_t = \sigma(\omega_s : a \leq s \leq t) \), for \( t \in [a, b] \), is the filtration generated by the process \((\omega_t)_{t \in [a,b]}\), and if the assumption
\[ \text{Cov}(\omega_t | \mathcal{F}_s) \gtrsim |t-s|^{-2H} \text{Id}, \]
then (4.3) holds. More on this will be considered in Section 5, see Remark 5.4.

4.1. Proof of Theorem 4.1. We start with some preliminary results.

Lemma 4.6. Let \( k \geq 0 \). We have for \( x, y > 0 \) that
\[ \Delta_k \mathbb{1}_{[y, \infty)}(x) = \Delta_k^* \mathbb{1}_{[0,x]}(y). \]

Proof. For \( k = 0 \), we have that
\[ \Delta_0 \mathbb{1}_{[y, \infty)}(x) = \mathbb{1}_{[y, \infty)}(x) - \mathbb{1}_{[y, \infty)}(x/2) = \mathbb{1}_{[0,x]}(y) - \mathbb{1}_{[0,x/2]}(y) = \mathbb{1}_{[0,x]}(y) - \mathbb{1}_{[0,x]}(2y) = \Delta_0^* \mathbb{1}_{[0,x]}(y). \]

Therefore, by proceeding inductively, for \( k \geq 1 \),
\[ \Delta_k \mathbb{1}_{[y, \infty)}(x) = \Delta_{k-1} \mathbb{1}_{[y, \infty)}(x) - 2^k \Delta_{k-1} \mathbb{1}_{[y, \infty)}(x/2) = \Delta_k^* \mathbb{1}_{[0,x]}(y) - 2^k \Delta_k^* \mathbb{1}_{[0,x/2]}(y) = \Delta_k \mathbb{1}_{[0,x]}(y) - 2^k \Delta_k^* \mathbb{1}_{[0,x]}(2y) = \Delta_k^* \mathbb{1}_{[0,x]}(y). \]

Lemma 4.7. Fix \( \beta \geq 0 \), \( \alpha_0 \geq 0 \), and let \( k = \lceil \beta - 1 \rceil \). Suppose that for some \( R < \infty \), \( f : \mathbb{R} \to \mathbb{R} \) satisfies \( \|f\|_{C^\beta + \alpha_0 - k(B_R)} < \infty \). Then
\[ \|\Delta_k f\|_{C^\alpha_0(B_R)} \lesssim R^\beta \|f^{(k)}\|_{C^{\beta + \alpha_0 - k(B_R)}}. \]

Proof. By (2.3), we have that \( \Delta_k x^h = 0 \) as long as \( h \leq k \). Consider the function
\[ g(x) := f(x) - \sum_{h=k}^{\infty} \frac{f^{(h)}(0)}{h!} x^h. \]
We then have that
\[ \Delta_k g = \Delta_k f. \]
Moreover, by Taylor’s remainder theorem,

\[ \| g \|_{C^\alpha_0(B_R)} \lesssim R^\beta \| f^{(k)} \|_{C^{\beta + \alpha_0 - k}}. \]

By the definition of \( \Delta_k \), we then have that

\[ \| \Delta_k g \|_{C^\alpha_0(B_R)} \lesssim \| g \|_{C^\alpha_0(B_R)} \lesssim R^\beta \| f^{(k)} \|_{C^{\beta + \alpha_0 - k}}. \]

**Corollary 4.8.** Let \( m \in \mathbb{N} \), \( \beta > 0 \), \( \beta \not\in \mathbb{N} \), and let \( k = \lfloor \beta \rfloor \). Suppose that \( f : \mathbb{R}^m \to \mathbb{R} \) satisfies

\[ \| | \partial_{x_1} |^\beta \ldots | \partial_{x_m} |^\beta f \|_{L^\infty} < \infty, \]

where \( | \partial_x | \) is defined as \( | \partial_x | = \sqrt{-\partial_x^2} \). Then for every \( R > 0 \),

\[ \| (\Delta_k)_{x_1} \ldots (\Delta_k)_{x_m} f \|_{L^\infty(B_R^m)} \lesssim R^m \| | \partial_{x_1} |^\beta \ldots | \partial_{x_m} |^\beta f \|_{L^\infty}. \]

Moreover, if \( f \in C^{m\beta}(B_R^m) \),

\[ \| (\Delta_k)_{x_1} \ldots (\Delta_k)_{x_m} f \|_{L^\infty(B_R^m)} \lesssim R^m \| f \|_{C^{m\beta}(B_R^m)} \]

**Proof.** We proceed inductively. For \( m = 1 \), (4.4) follows directly from Lemma 4.7, together with the usual equivalence (for \( \beta \not\in \mathbb{N} \)) of \( C^\beta \) and \( B^\beta_{\infty,\infty} \). If \( m > 1 \), by induction,

\[ \| (\Delta_k)_{x_1} \ldots (\Delta_k)_{x_m} f \|_{L^\infty(B_R^m)} \lesssim R^{(m-1)\beta} \| | \partial_{x_1} |^\beta \ldots | \partial_{x_{m-1}} |^\beta (\Delta_k)_{x_m} f \|_{L^\infty(R^{m-1} \times B_R)} \]

\[ = R^{(m-1)\beta} \| (\Delta_k)_{x_1} \ldots (\Delta_k)_{x_m} f \|_{L^\infty} \]

\[ \lesssim R^m \| | \partial_{x_{m-1}} |^\beta \ldots | \partial_{x_1} |^\beta f \|_{L^\infty} \]

\[ = R^m \| f \|_\infty, \]

which is (4.4). To obtain (4.5), we proceed similarly. If \( m = 1 \), (4.5) follows directly from Lemma 4.7. For \( m > 1 \), proceeding inductively and using Lemma
Proof. We proceed inductively. By expanding the definition (2.2) of \( \Delta_k \), we have that

\[
\Delta_d - 1 (r^d F (r)) = \sum_{h=0}^{d} c_{h,d} r^d F \left( \frac{r}{2^n} \right)
\]

for some coefficients \( c_{h,d} \). Therefore, by definition of \( \Delta_k \),

\[
\Delta_d (r^d F (r)) = \sum_{h=0}^{d} c_{h,d} \left( r^d F \left( \frac{r}{2^n} \right) - 2^d r^d F \left( \frac{r}{2^{n+1}} \right) \right)
\]

\[
= \sum_{h=0}^{d} c_{h,d} r^d \left( F \left( \frac{r}{2^n} \right) - F \left( \frac{r}{2^{n+1}} \right) \right)
\]

\[
= \sum_{h=0}^{d} c_{h,d} r^d \Delta_0 F \left( \frac{r}{2^n} \right),
\]

for some coefficients \( c_{h,d} \). Therefore, by definition of \( \Delta_0 \),
so (4.6) holds for $k = d$. Moreover, if (4.6) holds for some $k \geq d$,

$$
\Delta_{k+1}(r^d F(r)) = \sum_{h=0}^{d} c_{h,d} (r^d \Delta_{k-d} F(\frac{r}{2^h}) - 2^{k+1} \frac{r^d}{2^a} \Delta_{k-d} F(\frac{r}{2^{h+1}}))
$$

$$
= \sum_{h=0}^{d} c_{h,d} r^d (\Delta_{k-d} F(\frac{r}{2^h}) - 2^{k+1-d} \Delta_{k-d} F(\frac{r}{2^{h+1}}))
$$

$$
= \sum_{h=0}^{d} c_{h,d} r^d \Delta_{k+1-d} F(\frac{r}{2^h}),
$$

which is (4.6) for $k+1$. \qed

We finally give a finite $p$-variation criterion for stochastic processes under assumptions similar to those of Kolmogorov’s continuity theorem.

**Lemma 4.10.** Let $E$ be a normed space, and let $f : [0, 1] \to E$. Then for $q \geq 1$ and $\epsilon > 0$,

$$
\|f\|_{V_q([0,1];E)}^q \lesssim \sum_{N \text{ dyadic}} \left( N^\epsilon \sum_{k=0}^{N-1} \|f(t_{k+1}^N) - f(t_k^N)\|_E^q \right),
$$

with a constant that depends only on $q$, $\epsilon$.

**Proof.** Let $\pi = \{0 = s_0 < s_1 < \cdots < s_n = 1\}$ be a partition of $[0, 1]$, and for a dyadic number $N$ and an integer $0 \leq k \leq N$, let $t_k^N := \frac{k}{N}$. Define recursively the set of intervals $J_h^N$ by

$$
\begin{align*}
J_h^1 &= \{I = [t_k^1, t_{k+1}^1] : I \subseteq [s_{h}, s_{h+1}]\}, \\
J_h^N &= \{I = [t_k^N, t_{k+1}^N] : I \subseteq [s_{h}, s_{h+1}], \ I \not\subseteq \bigcup J \forall J \in \bigcup M < N J_h^M\},
\end{align*}
$$

and let $J_h = \bigcup M J_h^M$. Informally, $J_h^N$ is the collection of all dyadic intervals of size $\sim N^{-1}$ that are not contained into any interval chosen previously (for a smaller value of $N$). Note that, by definition, $J_h^N$ has at most two elements: if not, there would be contiguous intervals that should have been chosen before.
Therefore, by the Hölder inequality,

\[
\|f(s_{h+1}) - f(s_h)\|_E^q = \left\| \sum_{I \in J_h} f(\max I) - f(\min I) \right\|_E^q \\
\leq \left( \sum_{N \text{ dyadic}} N^{-\frac{q}{p}} N^{\frac{q}{p}} \sum_{I \in J_h^N} \|f(\max I) - f(\min I)\|_E \right)^q \\
\leq \left( \sum_{N \text{ dyadic}} N^{-\frac{cq}{p-q}} \right)^{q-1} \left( \sum_{N \text{ dyadic}} N^{\epsilon} \left( \sum_{I \in J_h^N} \|f(\max I) - f(\min I)\|_E \right)^q \right) \\
\lesssim \sum_{N \text{ dyadic}} N^{\epsilon} \sum_{I \in J_h^N} \|f(\max I) - f(\min I)\|_E^q.
\]

So, by summing up over the partition,

\[
\sum_{h=0}^{n-1} \|f(s_{h+1}) - f(s_h)\|_E^q \lesssim \sum_{N \text{ dyadic}} N^{\epsilon} \sum_{h=0}^{n-1} \sum_{I \in J_h^N} \|f(\max I) - f(\min I)\|_E^q \\
\leq \sum_{N \text{ dyadic}} N^{\epsilon} \sum_{k=0}^{N-1} \|f(t_{k+1}^N) - f(t_k^N)\|_E^q.
\]

Since the right-hand side is independent from the partition, the same inequality holds for the p-variation. \(\square\)

**Proposition 4.11.** Let \((X_t)_{t \in [0,1]}\) be a stochastic process with values in a normed space \(E\). Assume that given \(M > 0\) there are \(p > 1\), \(\alpha > 0\) and \(C_M > 0\) such that

\[
\mathbb{E}[\|X_t - X_s\|_E^p \mathbb{1}_{G(M)}] \leq C_M |t - s|^{1+\alpha}, \quad s, t \in [0,1],
\]

for any event \(G(M) \subset \{\sup_{s \in [0,1]} \|X_s\|_E \leq M\}\). Then for every \(q\) such that \(\frac{p}{1+\alpha} < q \leq p\),

\[
\mathbb{E}[\|X\|_{V^q(0,1;E)}^p \mathbb{1}_{G(M)}] \lesssim C_M.
\]
Proof. By the assumption, for \( \varepsilon > 0 \),

\[
E \left[ \sup_N \left( N^{a-\varepsilon} \sum_{k=0}^{N-1} \| X_{t_{k+1}^N} \|_E \right) \right] \leq E \left[ 1_{G(M)} \sum_N N^{a-\varepsilon} \sum_{k=0}^{N-1} \| X_{t_{k+1}^N} \|_E \right] \]

\[
\leq C_M \sum_N N^{-\varepsilon} \leq C_M
\]

By Lemma 4.10 and the Hölder inequality,

\[
\| X \|_{V_q}^q \leq \sum_N N^\varepsilon \sum_{k=0}^{N-1} \| X_{t_{k+1}^N} \|_E^q
\]

\[
\leq \sum_N N^\varepsilon \left( N^{a-\varepsilon} \sum_{k=0}^{N-1} \| X_{t_{k+1}^N} \|_E^p \right)^{\frac{q}{p}}
\]

\[
\leq \left( \sup_N N^{a-\varepsilon} \sum_{k=0}^{N-1} \| X_{t_{k+1}^N} \|_E^p \right)^{\frac{q}{p}},
\]

with \( \varepsilon = 1 + \left( 1 + \frac{a}{p} \right) \varepsilon - \frac{a}{p} (1 + a) < 0 \) by assumption, for \( \varepsilon \) small enough. By multiplying by \( 1_{G(M)} \) the above inequality and by using (4.7), the conclusion follows. \( \square \)

We are ready to turn to the proof of Theorem 4.1. We start with the following proposition, which yields the same conclusions under an assumption on the 2-points joint density of the path.

**Proposition 4.12.** Given \( \beta > 0 \), let \( \omega : [a, b] \to \mathbb{R}^d \) be a stochastic process with continuous paths. Suppose that

- for a.e. \((s, t)\), the joint density \( \rho_{s,t} := \text{Law}(\omega_s, \omega_t) \) belongs to \( C^\beta(B_R \times B_R) \) for every ball \( B_R = \{ x \in \mathbb{R}^d : |x| \leq R \} \),
- for every \( R > 0 \) there is \( C_p(R) > 0 \) such that for every \( J \subseteq [a, b] \),

\[
\int_{J \times J} \| \rho_{s,t} \|_{C^\beta(B_R \times B_R)} \, ds \, dt \leq C_p(R)|J|.
\]

Then for every \( \alpha < \frac{\beta}{2} \) with \( 2\alpha \notin \mathbb{N} \), the occupation measure satisfies

\[
\mu_{a, *} \in V^2(SBE_2^{\alpha, 2}).
\]
More precisely, there is $\delta_0(\alpha, \beta) \in (0, 1)$ such that for every $0 < \delta < \delta_0(\alpha, \beta)$, if

$$E(M) = \{ \sup_{a \leq s \leq b} |\omega_s| \leq M \}, \quad F(M_0) = \{ \sup_{a \leq s, t \leq b} |\omega_{st}| \leq M_0 \},$$

then

$$E \left[ \mathbb{E}(M) \mathbb{E}(F(M_0)) \| \mu_{s,t} \|^2_{\mathbb{S}^{2\alpha,2}} \right] \lesssim M_0^2 (M^d C \rho^2 (2M))^{1-\delta} + M_0.$$  \hfill (4.9)

**Proof.** Notice that up to making $\beta$ smaller, we can assume that $2\lceil \alpha \rceil \geq \beta$, and that $\beta \not\in \mathbb{N}$.

Fix $s, t \in [a, b]$ and $M, M_0 > 0$. We start by estimating

$$E \left[ \mathbb{E}(M) \mathbb{E}(F(M_0)) \| \mu_{s,t} \|^2_{\mathbb{S}^{2\alpha,2}} \right].$$

Notice that $F_{\mu_{s,t}}(\cdot, \cdot) = \int_s^t F_{\delta \omega, r}(\cdot, r) \, dr$. Moreover, for $r \geq 0$, by Lemma 4.6

$$(\Delta_k)^r F_{\delta \omega, r}(r, y) = (\Delta_k)^r \mathbb{I}_{[0, r]}(|y - y_0|) = (\Delta_k)^r \mathbb{I}_{[|y - y_0|, \infty)}(r) = (\Delta_k)^r \mathbb{I}_{[0, r]}(|y - y_0|).$$

Let $k = \lceil \alpha + d - 1 \rceil = \lceil \frac{d}{2} + d - 1 \rceil$, and let $\chi \in C_c^\infty(\mathbb{R})$ be such that $\chi(r) \equiv 1$ for $r \leq 1$, supp$(\chi) \subseteq [0, 2]$, and $\|\chi\|_{C^\beta} \lesssim 1$. For $r \leq 1$ and $|y| \leq M + 1$, we have that

$$E \left[ |\Delta_k F_{\mu_{s,t}}(\cdot, \cdot)(r)|^2 \right]$$

$$= E \left[ \int_{[s, t]^2} \int \Delta_k F_{\delta \omega_s, r}(\cdot, y) \Delta_k F_{\delta \omega_t, r}(\cdot, y) \, ds \, dt' \right]$$

$$= \int_{[s, t]^2} \int \mathbb{E} \left[ (\Delta_k^{s}\mathbb{I}_{[0, r]}(|y - \omega_s|))(\Delta_k^{t}\mathbb{I}_{[0, r]}(|y - \omega_t|)) \right] \, ds \, dt'$$

$$= \int_{[s, t]^2} \int (\Delta_k^{s}\mathbb{I}_{[0, r]}(|y - y_1|))(\Delta_k^{t}\mathbb{I}_{[0, r]}(|y - y_2|)) \rho_{s', t'}(y_1, y_2) \, dy_1 \, dy_2 \, ds \, dt'$$

and since $\Delta_k^{s}\mathbb{I}_{[0, r]}(|y|) = 0$ if $|y| \geq 1$,

$$= \int_{[s, t]^2} \int (\Delta_k^{s}\mathbb{I}_{[0, r]}(|y_1|))(\Delta_k^{t}\mathbb{I}_{[0, r]}(|y_2|)) \chi(|y_1|) \chi(|y_2|) \times \rho_{s', t'}(y + y_1, y + y_2) \, dy_1 \, dy_2 \, ds \, dt'$$

$$= \int_{[s, t]^2} \int (\Delta_k^{s}\mathbb{I}_{[0, r]})(r_1)(\Delta_k^{t}\mathbb{I}_{[0, r]})(r_2) r_1^d r_2^d g_{s', t'}(r_1, r_2) \, \frac{dr_1}{r_1} \frac{dr_2}{r_2} \, ds \, dt',$$

where

$$g_{s', t'}(r_1, r_2) := \int_{(S_1 - 1)^2} \chi(r_1) \chi(r_2) \rho_{s', t'}(y + r_1 \theta_1, y + r_2 \theta_2) \, d\sigma(\theta_1), \, d\sigma(\theta_2).$$
We clearly have that \( \|g_{s',t'}\|_{C^\beta} \lesssim \|\rho_{s',t'}\|_{C^\beta(2M)} \). Therefore, by (4.6) and (4.5),
\[
\mathbb{E}[|\Delta_{k}\mathbb{F}_{\mu_{s,t}}(\cdot, y)(r)|^2] = \int_{[s,t]^2} \int_{[0,r]} \mathbb{I}_{[0,r]}(r_1)\mathbb{I}_{[0,r]}(r_2)(\Delta_{k})r_1(\Delta_{k})r_2(r_1^4 r_2^4 s_{s',t'}) \frac{dr_1 dr_2 ds'dt'}{r_1 r_2}
\]
\[
\lesssim \int_{[s,t]^2} \int_{R^2} \mathbb{I}_{[0,r]}(r_1)\mathbb{I}_{[0,r]}(r_2)r_1^4 r_2^4 (r_1^4 + r_2^4) \|g_{s',t'}\|_{C^\beta} \frac{dr_1 dr_2 ds'dt'}{r_1 r_2}
\]
\[
\lesssim \int_{[s,t]^2} r^{\beta+2d} \|\rho_{s',t'}\|_{C^\beta(2M)} ds'dt'
\]
\[
\lesssim C_\rho(2M)(r^{2d} \wedge r^{\beta+2d})|t-s|,
\]
from which we obtain, for \( r \leq 1 \),
\[(4.10) \quad \mathbb{E}\left[ \mathbb{I}_{E(M)} \int |\Delta_{k}\mathbb{F}_{\mu_{s,t}}(\cdot, y)(r)|^2 dy \right] \lesssim C_\rho(2M)M^d r^{\beta+2d}|t-s|.
\]
In the following, for simplicity of notation, we write \( C_\rho = C_\rho(2M) \). We have the simple estimate
\[(4.11) \quad \mathbb{E}\left[ \mathbb{I}_{F(M_0)} |\Delta_{k}\mathbb{F}_{\mu_{s,t}}(\cdot, y)(r)| \right] \lesssim \|\mathbb{F}_{\mu_{s,t}}\|_{L^\infty} \mathbb{I}_{(|y-\omega_m-M_0|, + \leq r \leq 2^k(|y-\omega_m+M_0|)}
\]
\[(4.12) \quad \mathbb{E}\left[ \mathbb{I}_{F(M_0)} \int |\Delta_{k}\mathbb{F}_{\mu_{s,t}}(\cdot, y)(r)|^2 dy \right] \lesssim (M_0^d + r^d)|t-s|^2.
\]
Therefore, from (4.10), (4.12), we have for \( \delta \in [0, 1] \),
\[(4.13) \quad \mathbb{E}\left[ |\Delta_{k}\mathbb{F}_{\mu_{s,t}}(\cdot, y)(r)|^2 \mathbb{I}_{E(M)} \mathbb{I}_{F(M_0)} \right] \lesssim \lesssim (C_\rho M^d)^{1-\delta} M_0^{\delta} \rho^{(\beta+2d)(1-\delta)}|t-s|^{1+\delta} \mathbb{I}_{\{|r| \leq 1\}} + (M_0^d + r^d)|t-s|^2 \mathbb{I}_{\{|r| > 1\}}.
\]
Therefore, if \( \delta < \delta_0 \), and \( \delta_0 \) is such that
\[(4.14) \quad (\beta + 2d)(1 - \delta_0) = 2(\alpha + d),
\]
then
\[(4.15) \quad \mathbb{E}\left[ \|\mu_{s,t}\|^2 \mathbb{I}_{E(M)} \mathbb{I}_{F(M_0)} \right] \lesssim (C_\rho M^d)^{1-\delta} M_0^{\delta} |t-s|^{1+\delta} + M_0^\delta |t-s|^2.
\]
Inequality (4.9) now follows from formula above and Proposition 4.11. \( \square \)

**Proof of Theorem 4.1.** We first consider the case in which, for the joint density \( \rho_{s,t} \) of \((\omega_s, \omega_t)\), for every interval \( J \subseteq [a, b] \),
\[
\int_{J \times J} \sup_{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d} (1 + \|x\|^n + \|y\|^n)(\|d^n \rho_{s,t}(x, y)\|_\infty + \|\rho_{s,t}(x, y)\|_\infty) \leq C_p'|J|,
\]
where $d^n$ is the total differential, and $n$ is an integer with $n > \beta + d$. Let $X$ be a $d$-dimensional standard Gaussian random variable, independent from $\omega$, and for $\lambda \in \mathbb{R}$, consider the path $\omega_{s}^{\lambda} := \omega_{s} + \lambda X$, and let $\mu_{s}^{\lambda}$ its associated occupation measure. We have that

\begin{equation}
(4.17) \quad \mu_{s,t}^{\lambda} = \tau_{\lambda X} \mu_{s,t},
\end{equation}

so in particular, for every choice of the parameters $\alpha, p, q$,

\begin{equation}
(4.18) \quad \| \mu_{s,t}^{\lambda} \|_{\mathcal{S}B_{E,q}^{\alpha,p}} = \| \mu_{s,t} \|_{\mathcal{S}B_{E,q}^{\alpha,p}}.
\end{equation}

Moreover, since $X$ is independent from $\omega$, if $\rho_{s,t}^{\lambda} = \text{Law}(\omega_{s}^{\lambda}, \omega_{t}^{\lambda})$, we have

$$
\rho_{s,t}^{\lambda}(x, y) = \frac{\lambda^{-d}}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \rho_{s,t}(x - z, y - z) \exp\left(-\frac{|z|^2}{2\lambda^2}\right) dz.
$$

From the identity

$$
\nu_{s,t}(x - y) := \int_{\mathbb{R}^d} \rho_{s,t}(x - z, y - z) dz,
$$

and (4.16), we obtain for $\lambda \geq 1$,

$$
\left\| (2\pi\lambda^2)^\frac{d}{2} \rho_{s,t}^{\lambda}(x, y) - \exp\left(-\frac{|x|^2}{2\lambda^2}\right) \nu_{s,t}(x - y) \right\|_{C^\beta((\mathbb{R}^d)^2)} \\
\lesssim \left\| \int_{\mathbb{R}^d} \rho_{s,t}(x - z, y - z) \left[ \exp\left(-\frac{|z|^2}{2\lambda^2}\right) - \exp\left(-\frac{|x|^2}{2\lambda^2}\right) \right] dz \right\|_{C^\beta((\mathbb{R}^d)^2)} \\
\lesssim \| d^n \int_{\mathbb{R}^d} \rho_{s,t}(x - z, y - z) \left[ \exp\left(-\frac{|z|^2}{2\lambda^2}\right) - \exp\left(-\frac{|x|^2}{2\lambda^2}\right) \right] dz \right\|_{L^\infty((\mathbb{R}^d)^2)} \\
+ \| \int_{\mathbb{R}^d} \rho_{s,t}(x - z, y - z) \left[ \exp\left(-\frac{|z|^2}{2\lambda^2}\right) - \exp\left(-\frac{|x|^2}{2\lambda^2}\right) \right] dz \right\|_{L^\infty((\mathbb{R}^d)^2)} \\
\lesssim \left\| \left(\frac{|x - z|}{\lambda} + \frac{1}{\lambda^2}\right)(\| d^n \rho_{s,t}(x - z, y - z) \| + \| \rho_{s,t}(x - z, y - z) \|) \right\|_{L^\infty((\mathbb{R}^d)^2)}
$$

Therefore, for $J \subseteq [a, b]$, by (4.1) and (4.16), we have

\begin{equation}
(4.19) \quad \int_{J \times J} \| \rho_{s,t}^{\lambda} \|_{C^\beta(B_R \times B_R)} ds dt \\
\lesssim \frac{\lambda^{-d}}{\lambda^2} \left( \int_{J \times J} \left\| (2\pi\lambda^2)^\frac{d}{2} \rho_{s,t}^{\lambda}(x, y) - \exp\left(-\frac{|x|^2}{2\lambda^2}\right) \nu_{s,t}(x - y) \right\|_{C^\beta(B_R \times B_R)} ds dt \\
+ \int_{J \times J} \| \exp\left(-\frac{|x|^2}{2\lambda^2}\right) \nu_{s,t}(x - y) \|_{C^\beta} ds dt \right) \\
\lesssim \lambda^{-d-1} C_{\rho} |J| + \lambda^{-d} C_{\nu} |J| \\
\lesssim \lambda^{-d} C_{\nu} |J|
\end{equation}
for $\lambda = \lambda(C_{\rho'}, C_\nu)$ big enough. Now, consider the events $E_\lambda(M) = \{ \sup_{a \leq s \leq b} |\omega^\lambda_s| \leq M \}$ and $F_\lambda(M_0) = \{ \sup_{a \leq s \leq b} |\omega^\lambda_s| \leq M_0 \} = F(M_0)$. We have that

$$E_\lambda(M) = \{ \sup_{a \leq s \leq b} |\omega^\lambda_s| - \lambda |X| \leq M \}$$

$$\subseteq \{ \sup_{a \leq s \leq b} |\omega^\lambda_s| \leq M + \lambda |X| \} \cap F_\lambda(2M)$$

$$\subseteq F_\lambda(2M) \cap \bigcup_{N \text{ dyadic}} \{ \sup_{a \leq s \leq b} |\omega^\lambda_s| \leq M + 4\lambda N \} \cap \{ |X| \sim N \}$$

$$F_\lambda(2M) \cap \bigcup_{N \text{ dyadic}} E_\lambda(M + 4\lambda N) \cap \{ |X| \sim N \}.$$

Therefore, by applying Proposition 4.12 to the path $\omega^\lambda$, and by (4.19), we have that, for any $q < 2$,

$$\mathbb{E} \left[ \| \mu_{a,s} \|_{V^q_2(S_{B^{\alpha^2}})}^q \mathbb{I}_{E(M)} \right] \leq \sum_{N \text{ dyadic}} \mathbb{E} \left[ \| \mu_{a,s} \|_{V^q_2(S_{B^{\alpha^2}})}^q \right] \mathbb{P} \left[ |X| \sim N \right] \frac{2-q}{2}$$

where we have used the simple fact that $\| \mu_{a,s} \|_{V^q_2(S_{B^{\alpha^2}})} = \| \mu_{a,s} \|_{V^q_2(S_{B^{\alpha^2}})}$. By taking limits as $\lambda \to \infty$, we obtain that

$$\mathbb{E} \left[ \| \mu_{a,s} \|_{V^q_2(S_{B^{\alpha^2}})}^q \mathbb{I}_{E(M)} \right] \leq C_{q,\delta} \frac{(1-\delta)q}{2} M^{d\delta q} + M^{d\frac{q}{2}},$$

so in particular, we obtain

$$\mathbb{E} \left[ \| \mu_{a,s} \|_{V^q_2(S_{B^{\alpha^2}})}^q \left( 1 + \sup_s |\omega_s| \right)^{-d} \right] \lesssim 1 + C_{\nu} \frac{(1-\delta)q}{2}.$$

We now move to the case in which $\rho_{s,t}$ does not necessarily satisfy (4.16). Let $\varepsilon \ll 1$, $M_1 \gg 1$ to be determined later. Let $Y_s : [a, b] \to \mathbb{R}$ be a process, independent from $\omega$, such that its density satisfies (4.16), and let $\sigma_{s,t}$ be the joint density of $(Y_s, Y_t)$. Finally, let

$$\omega_{s,M_1}^{c,M_1} := H_{M_1} (\omega_s) + \varepsilon Y_s,$$

\footnote{For instance, one can take a fractional Brownian motion with Hurst index $H \ll \frac{1}{p}$, starting at $Y_0 \sim N(0,1)$.}
where $H_{M_1}$ is defined as

$$H_{M_1}(x) = \begin{cases} \frac{x}{M_1} & \text{if } |x| \leq M_1, \\ \frac{M_1 x}{|x|} & \text{if } x \geq M_1. \end{cases}$$

In particular, if $\rho_{s,t}^{M_1} = \text{Law}(H_{M_1}(\omega_s), H_{M_1}(\omega_t))$, one has that

$$(4.23) \quad \rho_{s,t}^{M_1}(x, y) = \rho_{s,t}(x, y) \mathbb{1}_{|x|,|y| < M_1} + \tilde{\rho}_{s,t}^{M_1}(x, y) = \rho_{s,t}(x, y) + \tilde{\rho}_{s,t}^{M_1}(x, y),$$

where $\tilde{\rho}_{s,t}^{M_1}$ is a measure supported on

$$(4.24) \quad \text{supp}(\tilde{\rho}_{s,t}^{M_1}) \subseteq \{|x| = M_1\} \cup \{|y| = M_1\},$$

and

$$\|\tilde{\rho}_{s,t}^{M_1}\|_{TV} = \mathbb{P}[^{\max(|\omega_s|, |\omega_t|) \geq M_1}] \leq \mathbb{P}[E(M_1/2)^c],$$

and similarly,

$$(4.25) \quad \|\tilde{\rho}_{s,t}^{M_1}\|_{TV} \leq 2\mathbb{P}[E(M_1/2)^c].$$

By definition of $\omega_s^{e,M_1}$, recalling that $Y_s$ is independent from the path $\omega$, for $\rho_{s,t}^{e,M_1} = \text{Law}(\omega_{e,M_1}^{s,t}, \omega_{e,M_1}^{e,M_1})$, by (4.23), we have that

$$(4.26) \quad \rho_{s,t}^{e,M_1} = \rho_{s,t}^{M_1} \ast e^{-2}\sigma_{s,t}(e^{-1} \cdot) = \rho_{s,t} \ast e^{-2}\sigma_{s,t}(e^{-1} \cdot) + \tilde{\rho}_{s,t}^{M_1} \ast e^{-2}\sigma_{s,t}(e^{-1} \cdot).$$

Let

$$\nu_{s,t}^{e,M_1}(x - y) := d \text{ Law}(\omega_{st}^{e,M_1})(x - y) = \int \rho_{s,t}^{e,M_1}(x - z, y - z) dz,$$

and

$$\nu_{s,t}^{\sigma,e}(x - y) := e^{-2} \int \sigma_{s,t}(e^{-1}(x - z), e^{-1}(y - z)) dz.$$

From (4.26), the properties of convolutions, and (4.25), we have

$$\|\nu_{s,t}^{e,M_1}\|_{C^\beta} \leq \|\nu_{s,t} \ast e^{-2}\sigma_{s,t}(e^{-1} \cdot)\|_{C^\beta} + \|\tilde{\rho}_{s,t}^{M_1} \ast \nu_{s,t}^{\sigma,e}\|_{C^\beta} \leq \|\nu_{s,t}\|_{C^\beta} + \|\tilde{\rho}_{s,t}^{M_1}\|_{TV} \|\nu_{s,t}^{\sigma,e}\|_{C^\beta} \leq \|\nu_{s,t}\|_{C^\beta} + 2\mathbb{P}[E(M_1/2)^c] e^{-\beta-d} \|\nu_{s,t}\|_{C^\beta}.$$

Therefore, if we choose $M_1 = M_1(\epsilon)$ such that $\mathbb{P}[E(M_1/2)^c] \ll \epsilon^{\beta+d+1}$, we obtain

$$\int \int_{|x|} \|\nu_{s,t}^{e,M_1}\|_{C^\beta} \leq (\mathcal{C}_Y + \epsilon) |J|.$$

By (4.23), (4.24), $\rho_{s,t}^{M_1}$ has compact support. Recalling that $\sigma_{s,t}$ satisfies (4.16), then $\rho_{s,t}^{e,M_1}$ satisfies (4.16) as well. Therefore, we can apply (4.20) to $\omega_{s}^{e,M_1}$, and we obtain

$$\mathbb{E} \left[ \|H_{M_1}(\epsilon)\|^q \mathbb{V}_{2}(\mathbb{S}^2_{e,M_1}) \left(1 + \sup_{s} |\omega_{s}^{e,M_1}(\epsilon)|^{-d} \right) \right] \lesssim 1 + (\mathcal{C}_Y + \epsilon)^{(1-\delta)\frac{d}{2}}.$$
From the definition of $\Delta_k$ and $F_{\mu}$, it is not hard to check that $\Delta_kF_{\mu}$ is a continuous function of $\omega \in C([a, b]; \mathbb{R}^d)$, see Lemma 4.13 below. As a consequence, we have that $\|\mu\|_{V^2(S\mathbb{B}_{2}^{a,p})}$ is a l.s.c. function of $\omega \in C([a, b]; \mathbb{R}^d)$. Noticing that $\omega_{\varepsilon,M_1(e)} \to \omega$ in $C([a, b]; \mathbb{R}^d)$ a.s. as $\varepsilon \to 0$,

$$
\mathbb{E} \left[ \|\mu_{\varepsilon,M_1(e)}\|_{V^2(S\mathbb{B}_{2}^{a,p})}^q (1 + \sup_s |\omega_s|)^{-d} \right] \leq \\
\leq \liminf_{\varepsilon \to 0} \mathbb{E} \left[ \|\mu_{\varepsilon,M_1(e)}\|_{V^2(S\mathbb{B}_{2}^{a,p})}^q (1 + \sup_s |\omega_s^{e,M_1(e)}|)^{-d} \right] \lesssim \\
\lesssim C_\varepsilon (1-8)^{\frac{q}{p}},
$$

by Fatou’s lemma.

**Lemma 4.13.** Let $\omega^1, \omega^2 : [a, b] \to \mathbb{R}^d$ be two continuous paths, and set $F_1(t, r, y) = F_{\omega^1}(r, y)$. Then for every $y \in \mathbb{R}^d$ and $1 \leq p < \infty$,

$$
\| \sup_{t \in [a, b]} |F_1(t, \cdot, y) - F_2(t, \cdot, y)| \|_{L^p(\mathbb{R}^+)} \leq (b - a)\|\omega^1 - \omega^2\|_{L^p}^{\frac{1}{p}}
$$

In particular, the map $\omega \mapsto \mu_{\omega}$ is lower semi-continuous from $C([a, b])$ to $S\mathbb{B}_{q}^{a,p}$, for all $\alpha > 0$, $1 \leq p$, $q < \infty$.

Finally, the map that, given $\omega \in C([a, b]; \mathbb{R}^d)$, returns $t \mapsto \mu_{\omega,t}$ is lower semi-continuous with values in $V^e(S\mathbb{B}_{q}^{a,p})$, for all $e > 0$.

**Proof.** We have

$$
|F_1(t, r, y) - F_2(t, r, y)| \leq \int_a^b |1_{[0, r]}(|y - \omega^1_s|) - 1_{[0, r]}(|y - \omega^2_s|)| \, ds \\
= \int_a^b (1_{[|y - \omega^1_s| < r < |y - \omega^2_s|]} + 1_{[|y - \omega^2_s| < r < |y - \omega^1_s|]}) \, ds,
$$

thus, by the Hölder inequality,

$$
\int_0^\infty \|F_1(\cdot, r, y) - F_2(\cdot, r, y)\|_\infty^p \, dr \leq \\
\leq \int_0^\infty \left( \int_a^b (1_{[|y - \omega^1_s| < r < |y - \omega^2_s|]} + 1_{[|y - \omega^2_s| < r < |y - \omega^1_s|]}) \, ds \right)^p \, dr \\
\leq (b - a)^{p-1} \int_a^b \int_0^\infty (1_{[|y - \omega^1_s| < r < |y - \omega^2_s|]} + 1_{[|y - \omega^2_s| < r < |y - \omega^1_s|]}) \, dr \, ds \\
\leq (b - a)^{p-1} \int_a^b \|y - \omega^1_s - y - \omega^2_s\|_\infty \, ds \\
\leq (b - a)^p \|\omega^1 - \omega^2\|_{L^p}.
$$
This proves the first claim. The second and third claims follow by semi-continuity of the integral, and the fact that the estimate proved above is uniform in time. □

Proof of Corollary 4.2. Under the assumptions of Proposition 4.12, (4.15) holds. Since $E(M) \subset F(2M)$, (4.15) reads, for every $M > 0$ as

$$E[||\mu_{a,t} - \mu_{a,s}||^2_{\mathbb{S}E^\alpha_2}] \leq C(M)|t - s|^{1+\delta},$$

with $C(M)$ a number depending on $M$, and $\delta < \delta_0 := \frac{\beta - 2\alpha}{\beta + 2\alpha}$, where $\delta_0$ is defined in (4.14). From here, it is not difficult to prove that $\mu_{a,\cdot}$ is a.s. $\gamma$-Hölder continuous, for every $\gamma < 1/2\delta_0$ with values in $\mathbb{S}E^\alpha_2$ on dyadic times. For instance, if

$$U_\lambda = \sum_{n=0}^{\infty} \sum_{0 < |t-s| \leq 2^{-un}} \frac{||\mu_{a,t} - \mu_{a,s}||^2_{\mathbb{S}E^\alpha_2}}{|t - s|^{\gamma}},$$

then $\mathbb{E}[U_\lambda 1_{E(M)}] < \infty$, and the events $(E(M))_{M \geq 1}$ fill in the probability space. Then any standard proof of Kolmogorov’s continuity theorem (for instance, see [Dur10]) yields Hölder continuity on dyadic times with probability 1. Finally, by Lemma 4.13, the map $t \mapsto \mu_{a,t}$ is lower semi-continuous in $\mathbb{S}E^\alpha_2$ and thus, by density of dyadic times, it is Hölder continuous on all times. In order to extend the result to the assumptions of Theorem 4.1, notice that from (4.18) and Proposition 4.12 applied to the path $\omega^\lambda$, we know that (4.15) holds for the occupation measure $\mu_{a,\cdot}^\lambda$ defined in (4.17), if (4.16) holds. On the other hand, (4.16) holds for the density $\rho^{c,M_1}$ of (4.21), thus (4.15) holds for $\mu_{a,\cdot}^{c,M_1}$. By Lemma 4.13, (4.15) holds for $\mu_{a,\cdot}$, and therefore the same proof given above above yields Hölder regularity under the assumptions of Theorem 4.1. □

5. Space-time regularity trade-off

In view of Remark 4.3 and the fact that there is a gap between the regularity in variation given by Theorem 4.1 and the lower Hölder regularity of Corollary 4.2, we discuss how to fill in this gap, under a stronger assumption over regularity of joint laws of increments.

**Theorem 5.1.** Given $\beta > 0$, with $\beta \notin \mathbb{N}$, and an integer $m \geq 1$, let $\eta > 0$ be such that $\frac{1}{2m} \leq \eta \leq 1$. Let $\omega : [a,b] \to \mathbb{R}^d$ be a stochastic process with continuous paths, and denote by $\nu_{s_1 \ldots s_{2m}}$ the joint density of $(\omega_{s_2} - \omega_{s_1}, \ldots, \omega_{s_{2m}} - \omega_{s_{2m-1}})$, with $s_1, \ldots, s_{2m} \in [a,b]$.

Assume that there is $C_\nu > 0$ such that for every interval $J \subseteq [a, b]$ and for every family $\beta_1, \ldots, \beta_{2m-1} \in \mathbb{R}$ satisfying

$$\beta_j \in [0, \beta, 2\beta), \quad \beta_1 + \cdots + \beta_{2m-1} = 2m\beta,$$
we have that

$$\int_{s_1 < s_2 < \cdots < s_{2m}, s_j \in [j]} \| (1 - \Delta)_{x_j}^{\beta_1} \cdots (1 - \Delta)_{x_{2m-1}}^{\beta_{2m-1}} \nu_{s_{1,m}} \|_{1/2} ds_{1,m} \leq C \nu[J]^{2m\eta}. $$

Then for every $\alpha < \beta$, with $\alpha \notin \mathbb{N}$, the occupation measure satisfies

$$\mu_{a,\nu} \in V^{1/\eta}(SBE_{2m}^{2m}).$$

More precisely, there is $\delta_0 = \delta_0(\alpha, \beta) \in (0, 1)$ such that for every $\delta < \delta_0$, and for every $1 \leq q < 2m$,

$$E \left[ \| \mu_{a,\nu}^{\nu} \|_{V^{1/q}(SBE_{2m}^{2m})} (1 + \sup_s |\omega_s|)^{-d} \right] \lesssim 1 + C \nu^{(1-\delta)}2^{m\eta}. $$

These results immediately yield H"older regularity in time of the occupation measure.

**Corollary 5.2.** Given a path $\omega : [a, b] \to \mathbb{R}^d$, under the same assumptions of Theorem 5.1, the occupation measure $\mu_{a,\nu}$ of $\omega$ is $\gamma$-H"older continuous with values in $SBE_{2m}^{2m}$, for every $\gamma < \eta + (1 - \eta)\delta_0 - \frac{1}{2m}$.

**Proof.** The result follows by the same argument of Corollary 4.2, using (5.14). \qed

We finally consider the case of Gaussian processes.

**Corollary 5.3.** Let $0 < H < \frac{1}{d}$, and let $\omega_t : [a, b] \to \mathbb{R}^d$ be a Gaussian process with continuous paths. Suppose moreover that $\omega$ is locally non-deterministic with parameter $H$, that is, for every integer $n \geq 2$, there exist constants $c_n$ and $\delta_n$ such that for every sequence of times $a \leq s_1 < s_2 < \cdots < s_n \leq b$, and every vector $(x_1, \ldots, x_{n-1}) \in (\mathbb{R}^d)^{n-1}$,

$$\text{Var} \left( \sum_{k=1}^{n-1} x_k \cdot (\omega_{s_k,s_{k+1}}) \right) \geq 2c_n \sum_{k=1}^{n-1} \|x_k\|^2 |s_{k+1} - s_k|^{2H}$$

for every $s_1, \ldots, s_n$ with $|s_n - s_1| \leq \delta_n$. Let $\alpha \geq 0$, $1 \leq p \leq q \leq \infty$ be such that

$$\frac{1}{p} + \left( \alpha - \frac{d}{q'} \right) H < 1 - dH, \quad \alpha < \left( \frac{1}{H} - d \right) \min \left( \frac{1}{2}, \frac{1}{q'} \right),$$

where $q'$ is the H"older conjugate of $q$, i.e. $\frac{1}{q'} = 1 - \frac{1}{q}$. Then we have that

$$\mu_{a,\nu} \in C^{\alpha-p}([a, b]; B_{q,\infty}^\alpha).$$

**Remark 5.4** (Local non determinism). Condition (5.3) says that the process is locally $\phi$-non deterministic, with $\phi(r) = r^{2H}$, according to [Cuz78]. Local non determinism was first introduced by Berman in [Ber74]. A recent review on the subject can be found in [Xia06].
A notion of local non determinism for general processes (that is to say, non Gaussian), was given in [Ber83]. A process \( \omega : [a, b] \to \mathbb{R}^d \) is \textit{locally} \textit{g non deterministic} if for every \( n \geq 2 \) there is \( c_n \) such that for all \( s_1 < s_2 < \cdots < s_n \),

\[
\nu_{s_1:n}(0, \ldots, 0) \geq c_n g(s_2 - s_1)g(s_3 - s_2)\ldots g(s_n - s_{n-1}),
\]

where \( \nu_{s_1:n} \) is the joint density of the increments \( \omega_{s_2s_1}, \ldots, \omega_{s_{n-1}s_n} \).

**Remark 5.5.** Condition (5.1) could be slightly weakened by adding an extra combinatorial condition to the sequence of \( \beta_j \). Indeed, the requirement is that, given the sequence of \( \beta_j \), then between each pair of values \( 2\beta \) there must be at least one \( 0 \). Likewise, between each pair of values \( 0 \) there must be at least a value \( 2\beta \). A short way to write the condition is that for all \( 1 \leq n_1 \leq n_2 \leq 2m - 1 \),

\[
\sum_{j=n_1}^{n_2} \beta_j \leq (n_2 - n_1 + 2)\beta.
\]

This condition though does not seem to be more useful or easier to check than (5.1).

**5.1. Proof of Theorem 5.1 and Corollary 5.3.** As in Proposition 4.12, we first prove the final result under assumptions on the joint density of the path, instead of the joint density of increments.

**Proposition 5.6.** Given \( \beta > 0 \), an integer \( m \geq 1 \), \( \frac{1}{2m} \leq \eta \leq 1 \), let \( \omega : [a, b] \to \mathbb{R}^d \) be a stochastic process with continuous paths. For \( (s_1, s_2, \ldots, s_{2m}) \in [a, b] \), let \( \nu_{s_1:s_{2m}} \) be the joint density of \( (\omega_{s_1}, \ldots, \omega_{s_{2m}}) \). Suppose that for every \( \beta \in [a, b] \),

\[
\int_{\substack{s_1 < s_2 < \cdots < s_{2m} \\{s_j \in [a, b]\}}} \|1 - \Delta\frac{\beta}{s_{2m}}s_1 \cdots (1 - \Delta\frac{\beta}{s_{2m}}s_{2m})
\]

\[
\nu_{s_{2m}:s_1} \rho_{s_{2m}:s_1} \|_{L^\infty} ds_{1:2m} \leq C_\beta |J|^{2m\eta}.
\]

Then for every \( \alpha < \beta \), with \( \alpha \notin \mathbb{N} \), the occupation measure satisfies

\[
\mu_{\alpha, \beta} \in V^{1/\eta}([a, b]; \mathbb{R}^{\alpha, 2m}).
\]

More precisely, there is \( \delta_0 = \delta_0(\alpha, \beta) \in (0, 1) \) such that for every \( \delta < \delta_0 \),

\[
\mathbb{E}[\mathbb{I}_{E(M)} \mathbb{I}_{F(M_0)} \|\mu_{\alpha, \beta}\|_{V^{1/\eta}(\mathbb{R}^{\alpha, 2m})}^{2m}] \lesssim M_0^{d\delta} (M^d C_\beta)^{1-\delta} + M_0^d.
\]

**Proof.** We follow the lines of the proof of Proposition 4.12. It is not restrictive to assume that \( \lceil \alpha \rceil \geq \beta \). First, for \( s, t \in [a, b] \) and \( M, M_0 > 0 \), we estimate

\[
\mathbb{E}[\mathbb{I}_{E(M)} \mathbb{I}_{F(M_0)} \|\mu_{s,t}\|_{\mathbb{R}^{\alpha, 2m}}^{2m}] .
\]

Let \( k = \lceil \alpha + d - 1 \rceil = \lceil \beta + d - 1 \rceil \), and, for simplicity of notation, set

\[
D_{2m}(s, t) := \{s_1 < \cdots < s_{2m} : s_j \in [s, t] \text{ for all } 1 \leq j \leq 2m\}.
\]
For $r \leq 1$ and $|y| \leq M + 1$, by Lemma 4.6,

$$
\mathbb{E} \left[ |\Delta_k F_{\mu_{s,t}}(\cdot, y)(r)|^{2m} \right] 
= \mathbb{E} \left[ \int_{[s,t]^{2m}} \Delta_k F_{\Delta \omega_{s_1}}(\cdot, y)(r) \ldots \Delta_k F_{\Delta \omega_{s_{2m}}}(\cdot, y)(r) \, ds_{1:2m} \right] 
= (2m)! \mathbb{E} \left[ \int_{D_{2m}(s,t)} \Delta_k F_{\Delta \omega_{s_1}}(\cdot, y)(r) \ldots \Delta_k F_{\Delta \omega_{s_{2m}}}(\cdot, y)(r) \, ds_{1:2m} \right] 
= (2m)! \int_{D_{2m}(s,t)} \left( \Delta_k \mathbb{1}_{[0,r]}(y) \right) \ldots \left( \Delta_k \mathbb{1}_{[0,r]}(r_2) \right) \rho_{s_{1:2m}} \, dy \, ds_{1:2m} 
= (2m)! \int_{D_{2m}(s,t)} \left( \Delta_k \mathbb{1}_{[0,r]}(r_1) \right) \ldots \left( \Delta_k \mathbb{1}_{[0,r]}(r_{2m}) \right) \rho_{s_{1:2m}} \, dy \, ds_{1:2m} 
\times \frac{dr_{1:2m}}{r_1 \ldots r_{2m}} \, ds_{1:2m},
$$

where

$$
g_{s_{1:2m}}(r_{1:2m}) := \int_{(S^{d-1})^{2m}} \rho_{s_{1:2m}}(y + r_1 \theta_1, \ldots, y + r_{2m} \theta_{2m}) \, d\sigma(\theta_{1:2m}).
$$

We have that for every $\epsilon > 0$, the operator $|\partial_\theta|^{\beta - \epsilon} (1 - \Delta)^{\frac{\beta}{2}}$ is bounded on $L^\infty(\mathbb{R}^d)$, with its norm independent from $\theta \in S^{d-1}$. Therefore,

$$
\| \partial_\theta^{\beta - \epsilon} g_{s_{1:2m}} \|_{L^\infty} \leq \left\| \int_{(S^{d-1})^{2m}} \prod_{j=1}^{2m} \partial_\theta^{\beta - \epsilon} \rho_{s_{1:2m}}(y + x_1, \ldots, y + x_{2m}) \, d\sigma(\theta_{1:2m}) \right\|_{L^\infty} 
\lesssim \| (1 - \Delta)^{\frac{\beta}{2}} \rho_{s_{1:2m}} \|_{L^\infty}.
$$

Therefore, by (4.6) and (4.4),

$$
\mathbb{E} \left[ |\Delta_k F_{\mu_{s,t}}(\cdot, y)(r)|^{2m} \right] 
= (2m)! \int_{D_{2m}(s,t)} \left( \Delta_k r_1 \ldots \Delta_k r_{2m} \right) \rho_{s_{1:2m}} \, ds_{1:2m} 
\lesssim C_p r^{2m(\beta - \epsilon) + 2md} |t - s|^{2m\eta},
$$

and thus,

$$
\mathbb{E} \left[ 1_{E(M)} \left\| \Delta_k F_{\mu_{s,t}}(\cdot, y)(r) \right\|_{L^2}^{2m} \right] \lesssim C_p M^d r^{2m(\beta - \epsilon) + 2md} |t - s|^{2m\eta}.
$$
By interpolating the above estimate with the simple estimate (4.11), and by choosing \( \varepsilon \) so that \( \alpha + \varepsilon < \beta \),

\[
\mathbb{E} \left[ \mathbb{I}_{E(M)} \mathbb{I}_{F(M)} \| \mu_{s,t} \|_{S^{2m,2m}_p}^2 \right] \lesssim \\
\lesssim (C_p M^{d})^{1-\delta} M_0^d |t - s|^\delta |\alpha(1-\delta) + 2m| + M_0^d |t - s|^{2m},
\]

for \( \delta \in (0, \delta_0) \), with

\[
(\beta + d)(1 - \delta_0) = (\alpha + d).
\]

Inequality (5.6) follows from Proposition 4.11, since \( \eta(1-\delta) + \delta > \eta \). \( \square \)

**Proof of Theorem 5.1.** For \( J = [s,t] \), let \( D_{2m}(J) := D_{2m}(s,t) \) be the set defined in (5.7). Assume first that for every interval \( J \subset [a,b] \),

\[
\int_{D_{2m}(J)} \sup_{x_i \in \mathbb{R}^d} (1 + |x_1|^n + \cdots + |x_{2m}|^n) \left( \| \rho_{1;2m}(x_{1;2m}) \| + \| d^n \rho_{1;2m}(x_{1;2m}) \| \right) ds_{1;2m} \leq C J^{2m \eta},
\]

where \( \rho_{1;2m} \) is the density of \( (\omega_{1;1}, \ldots, \omega_{1;2m}) \), \( d^n \) is the total differential, and \( n \) is an integer with \( n > 2m \beta + d \).

For a \( d \)-dimensional standard Gaussian random variable \( X \), independent from \( \omega \), and for \( \lambda \in \mathbb{R} \), define the path \( \omega^\lambda_s := \omega_s + \lambda X \), and let \( \mu^\lambda_s \) be its associated occupation measure, so that \( \| \mu^\lambda_s \|_{S^{\alpha,p}_q} = \| \mu_{s,t} \|_{S^{\alpha,p}_q} \) for all \( \alpha, p, q \). We have the identity

\[
\nu_{1;2m}(x_2 - x_1, x_3 - x_2, \ldots, x_{2m} - x_{2m-1}) = \\
\frac{1}{|J|} \rho_{1;2m}(x_1 - z, \ldots, x_{2m} - z) dz.
\]

By (5.11) and (5.10), for \( \lambda \geq 1 \), by Sobolev embeddings we have

\[
\left\| \partial_\beta \left( (2\pi \lambda^2)^{-\frac{d}{2}} \rho^\lambda_{1;2m}(x_{1;2m}) - e^{-\frac{|x_1|^2}{2\lambda^2}} \nu_{1;2m}(x_2 - x_1, \ldots, x_{2m} - x_{2m-1}) \right) \right\|_{L^\infty(\mathbb{R}^{2md})} \\
= \left\| \partial_\beta \int_{\mathbb{R}^d} \left( e^{-\frac{|x_1|^2}{2\lambda^2}} - e^{-\frac{|z|^2}{2\lambda^2}} \right) \rho_{1;2m}(x_1 - z, \ldots, x_{2m} - z) dz \right\|_{L^\infty(\mathbb{R}^{2md})} \\
\lesssim \left\| d^n \int_{\mathbb{R}^d} \left( e^{-\frac{|x_1|^2}{2\lambda^2}} - e^{-\frac{|z|^2}{2\lambda^2}} \right) \rho_{1;2m}(x_1 - z, \ldots, x_{2m} - z) dz \right\|_{L^\infty(\mathbb{R}^{2md})} \\
+ \left\| \int_{\mathbb{R}^d} \left( e^{-\frac{|x_1|^2}{2\lambda^2}} - e^{-\frac{|z|^2}{2\lambda^2}} \right) \rho_{1;2m}(x_1 - z, \ldots, x_{2m} - z) dz \right\|_{L^\infty(\mathbb{R}^{2md})} \\
\lesssim \left\| \frac{|x_1 - z|}{\lambda} + \frac{1}{\lambda^d} \right\|_{L^\infty(\mathbb{R}^{2md})} \left\| d^n \rho_{s,t}(x_1 - z, \ldots, x_{2m} - z) + \rho_{s,t}(x_1 - z, \ldots, x_{2m} - z) \right\|_{L^\infty(\mathbb{R}^{2md})}
\]
where \( \mathcal{D}_\beta = (1 - \Delta)^{\beta \over 2} \mathcal{F}_x \ldots (1 - \Delta)^{\beta \over 2m} \) and \( \rho^{\lambda}_{s_{1:2m}} \) is the density of \( (\omega^{\lambda}_{s_1}, \ldots, \omega^{\lambda}_{s_{2m}}) \). Moreover, we notice that for \( \lambda \geq 1 \), for every \( \epsilon > 0 \), by (5.2)

\[
\int_{D_{2m}(f)} \|\mathcal{D}_\beta - e^{i |x| \over 2} \mathcal{F}_{s_{1:2m}}(x_1 - x_2, \ldots, x_{2m-1} - x_{2m})\|_{L^\infty(\mathbb{R}^{2m} \cdot d)} \, ds_{1:2m}
\]

\[
\lesssim \int_{D_{2m}(f)} \|\mathcal{D}_\beta - e^{i |x| \over 2} \mathcal{F}_{s_{1:2m}}(x_1 - x_2, \ldots, x_{2m-1} - x_{2m})\|_{L^\infty(\mathbb{R}^{2m} \cdot d)} \, ds_{1:2m}
\]

(5.13)

\[
\lesssim \int_{D_{2m}(f)} \left\| \left( \prod_{j=1}^{2m-1} ((1 - \Delta_{x_j})^{\beta \over 2} + (1 - \Delta_{x_{j+1}})^{\beta \over 2}) \mathcal{F}_{s_{1:2m}}(x_1 - x_2, \ldots, x_{2m-1} - x_{2m}) \right) \right\|_{L^\infty(\mathbb{R}^{2m} \cdot d)} \, ds_{1:2m}
\]

\[
\lesssim C_{\epsilon} |J|^{2m \eta}
\]

Therefore, combining (5.12) and (5.13), we obtain

\[
\int_{D_{2m}(f)} \|\mathcal{D}_\beta - e^{i |x| \over 2} \mathcal{F}_{s_{1:2m}}(x_1 - x_2, \ldots, x_{2m-1} - x_{2m})\|_{L^\infty(\mathbb{R}^{2m} \cdot d)} \, ds_{1:2m}
\]

\[
\lesssim \frac{1}{\lambda^d} \int_{D_{2m}(f)} \left\| (2\pi \lambda^2)^{\beta \over 2} \rho^{\lambda}_{s_{1:2m}}(x_1 - x_2, \ldots, x_{2m-1} - x_{2m}) \right\|_{L^\infty(\mathbb{R}^{2m} \cdot d)} \, ds_{1:2m}
\]

\[
= \int_{D_{2m}(f)} \left\| \mathcal{D}_\beta - e^{i |x| \over 2} \mathcal{F}_{s_{1:2m}}(x_1 - x_2, \ldots, x_{2m-1} - x_{2m}) \right\|_{L^\infty(\mathbb{R}^{2m} \cdot d)} \, ds_{1:2m}
\]

\[
\lesssim \lambda^{-d} C_{\epsilon} |J|^{2m \eta}
\]

for \( \lambda = \lambda(C'_\rho, C_{\epsilon}) \) big enough.

Fixing \( 2 \epsilon < \beta - \alpha \), we can use the above estimate, (5.8) applied to \( \omega^{\lambda} \), and the fact \( \|\mu_{s,t}\|_{S_{E_{2m}}^{\alpha,2m}} = \|\mu_{s,t}\|_{S_{E_{2m}}^{\alpha,2m}} \), to obtain that for every \( 1 \leq q < 2m \) and \( \delta < \delta_0 \) (defined as in (5.9)),

\[
\mathbb{E} \left[ \|\mu_{s,t}\|^q_{S_{E_{2m}}^{\alpha,2m}} \mathbb{I}_{E(M)} \right] \leq \sum_{N \text{ dyadic}} \mathbb{E} \left[ \|\mu_{s,t}\|^q_{S_{E_{2m}}^{\alpha,2m}} \mathbb{I}_{F_{2m}(2M)} \mathbb{I}_{E_{\lambda}(M+4\lambda N)} \mathbb{I}_{|X|-N} \right]
\]

\[
\leq \sum_{N \text{ dyadic}} \left( \mathbb{E} \left[ \|\mu_{s,t}\|_{S_{E_{2m}}^{\alpha,2m}} \mathbb{I}_{F_{2m}(2M)} \mathbb{I}_{E_{\lambda}(M+4\lambda N)} \right] \right)^{q \over 2m} \mathbb{P}(|X|-N)^{2m-q \over 2m}
\]

\[
\lesssim \sum_{N \text{ dyadic}} \left( (\lambda^{-d} C_{\epsilon})(M + 4\lambda N)^{d} \right)^{1-\delta} (M^{d \delta} + M^{d})^{q \over 2m} (t-s)^{q(\delta+\eta(1-\delta))} \mathbb{E} \left[ \left( \frac{CN^2}{2m} (2m-q) \right)^{q \delta \eta \over 2m} \right]
\]
and by taking the limit $\lambda \to \infty$,

$$\mathbb{E} \left[ \|\mu_{s,t}\|_{S_{q}^{2m,\alpha}}^q \right] \lesssim \left( C_{\nu}^{1-\delta} M^{d\delta} + M^d \right)^{\frac{a_q}{2m}} |t - s|^2 (\delta + \eta(1-\delta)).$$

Likewise, by (5.6) for $\omega^\lambda$ and the fact $\|\mu_a,\|_{\mathcal{V}_{p}(S_{q}^{2m,\alpha})} = \|\mu_a,\|_{\mathcal{V}_{p}(S_{q}^{2m,\alpha})^j}$, we obtain for $1 < q < 2m$, $\delta < \delta_0$, and $p(\delta + \eta(1-\delta)) > 1$,

$$\mathbb{E} \left[ \|\mu_{a,s}\|_{\mathcal{V}_{p}(S_{q}^{2m,\alpha})} \right] \lesssim \left( C_{\nu}^{1-\delta} M^{d\delta} + M^d \right)^{\frac{a_q}{2m}},$$

and in particular,

$$\mathbb{E} \left[ \|\mu_{a,s}\|_{\mathcal{V}_{p}(S_{q}^{2m,\alpha})} \left( 1 + \sup_s |\omega_s| \right)^{-d} \right] \lesssim 1 + C_{\nu}^{1-\delta} \frac{d}{2m}.$$

To get rid of assumption (5.10), we proceed as in the proof of Theorem 4.1. Define $\omega^{\epsilon,M_1} := H_{M_1}(\omega) + \epsilon Y$, where $H_{M_1}$ is given in (4.22), and $Y$ is a process, independent from $\omega$, that satisfies (5.10). Given $s_1, s_2, \ldots, s_{2m}$, we can write (as in the proof of Theorem 4.1) $\rho_{s_1:s_2}^{M_1} = \rho_{s_1:s_2} + \rho_{s_1:s_2}^{M_1}$, where $\rho_{s_1:s_2}^{M_1}$ is the density of $(H_{M_1}(\omega_{s_1}), \ldots, H_{M_1}(\omega_{s_{2m}}))$ and

$$\|\rho_{s_1:s_2}^{M_1}\|_{TV} \leq 2\mathbb{P}[E(M_{1}/2)^c].$$

Let now $\rho_{s_1:s_2}^{\epsilon,M_1}$ be the density of $(\omega_{s_1}^{\epsilon,M_1}, \ldots, \omega_{s_{2m}}^{\epsilon,M_1})$, let $\sigma_{s_1:s_2}^{\epsilon}$ be the density of $(Y_{s_1}, \ldots, Y_{s_{2m}})$ and $\sigma_{s_1:s_2}^{\epsilon} = \frac{1}{\epsilon^{2m}} \sigma_{s_1:s_2}^{\epsilon}(\cdot/\epsilon)$ the density of $\epsilon(Y_{s_1}, \ldots, Y_{s_{2m}})$, then

$$\rho_{s_1:s_2}^{\epsilon,M_1} = \rho_{s_1:s_2}^{\epsilon,\sigma_{s_1:s_2}^{\epsilon}} + \rho_{s_1:s_2}^{M_1} \cdot \sigma_{s_1:s_2}^{\epsilon}.$$

We wish to apply the conclusions of the first part of this proof to $\omega^{\epsilon,M_1}$. First, since the joint density of $(Y_{s_1}, \ldots, Y_{s_{2m}})$ satisfies (5.10), the same is true for $\rho_{s_1:s_2}^{\epsilon,M_1}$. It remains to verify that the joint density $\nu_{s_1:s_2}^{\epsilon,M_1}$ of $(\omega_{s_1}^{\epsilon,M_1} - \omega_{s_1}^{\epsilon,M_1}, \ldots, \omega_{s_{2m}}^{\epsilon,M_1} - \omega_{s_{2m-1}}^{\epsilon,M_1})$ satisfies (5.2). Let $\nu_{s_1:s_2}^{\epsilon}$ be the joint density of $\epsilon(Y_{s_2} - Y_{s_1}, \ldots, Y_{s_{2m}} - Y_{s_{2m-1}})$, then by (5.11), (5.16) and (5.17),

$$\|D\nu_{s_1:s_2}^{\epsilon,M_1}\|_{L^\infty} \lesssim \|D\nu_{s_1:s_2}^{\epsilon,\sigma_{s_1:s_2}^{\epsilon}}\|_{L^\infty} + \|\rho_{s_1:s_2}^{M_1} \cdot \nu_{s_1:s_2}^{\epsilon}\|_{C^\beta} \lesssim \|D\nu_{s_1:s_2}^{\epsilon}\|_{L^\infty} + \epsilon^{-(2m\beta + d)} \|\nu_{s_1:s_2}^{\epsilon}\|_{C^\beta} \mathbb{P}[E(M_{1}/2)^c],$$

with $D = (1 - \Delta_{x_1})^{\beta_1/2} \cdots (1 - \Delta_{x_{2m-1}})^{\beta_{2m-1}/2}$ for brevity. If we choose $M_1 = M_1(\epsilon)$ so that $\mathbb{P}[E(M_{1}/2)^c] \leq e^{2m\beta - d} C_{\nu}$, in conclusion $\nu_{s_1:s_2}^{\epsilon,M_1}$ satisfies (5.2), thus (5.15) yields,

$$\mathbb{E} \left[ \|\mu_{a,s}\|_{\mathcal{V}_{p}(S_{q}^{2m,\alpha})} \left( 1 + \sup_s |\omega_s^{\epsilon,M_1(\epsilon)}| \right)^{-d} \right] \lesssim 1 + (C_{\nu})^{1-\delta} \frac{d}{2m},$$

for $p(\delta + \eta(1-\delta)) > 1$, $1 \leq q < 2m$ and $\delta < \delta_0$, as well as (5.14). The conclusion now follows again by semi-continuity, thanks to Lemma 4.13. \qed
Proof of Corollary 5.3. First of all, we can assume without loss of generality that $\delta_n = |b - a|$. If this is not the case, we can apply the result with $\delta_n = |b - a|$ to the intervals $[a, a + \delta_n], [a + \delta_n, a + 2\delta_n], [a + 2\delta_n, a + 3\delta_n]$ and so on, and obtain that $\mu_{a+k\delta_n, a+(k+1)\delta_n} \in C^{q-\var}B^\gamma_{p,\infty}$. By patching the intervals back together, we obtain that $\mu_{a, b} \in C^{q-\var}B^\gamma_{p,\infty}$ as well.

We first check the statement in the case $q = 2m$, with $m \in \mathbb{N}$. By Sobolev embeddings and (2.4), it is enough to prove that for every $\alpha < \frac{1}{2H} - \frac{d}{2}$ and $p$ such that $\frac{1}{p} < 1 - dH - \frac{dH}{2m} + \alpha H$, $\mu_{a, b} \in C^{p-\var}([a, b]; S^{\alpha, 2m})$.

By Theorem 5.1, it is enough to check that (5.2) holds for $\beta < \frac{1}{2H} - \frac{d}{2}$ and $\eta = 1 - (d + \beta)H - \frac{dH}{2m}$. Indeed, if this holds, we can choose $\beta$ in such a way that

$$\alpha < \beta < \frac{1}{2H} - \frac{d}{2}, \quad \frac{1}{p} < \eta = 1 - (d + \beta)H + \frac{dH}{2m}.$$ 

Recall that if a Gaussian random variable $X$ has covariance matrix $C$ and mean $\bar{x}$, then

$$E[e^{-i\xi \cdot X}] = \exp \left( - i\xi \cdot \bar{x} - \frac{1}{2}(C\xi, \xi) \right).$$

For simplicity of notation, let $D_{2m}(J) := \{s_1 < s_2 < \cdots < s_{2m} : \forall j, s_j \in J\}$. We have that

$$\int_{D_{2m}(J)} \left\| (1 - \Delta_{x_1})^{\frac{\beta_1}{2}} \cdots (1 - \Delta_{x_{2m-1}})^{\frac{\beta_{2m-1}}{2}} \nu_{s_1, 2m} \right\|_{L^\infty} ds_{1, 2m}$$

$$\leq \int_{D_{2m}(J)} \left\| \prod_{j=1}^{2m-1} (1 + |\xi_j|^2)^{\frac{\beta_j}{2}} v_{s_1, 2m}(\xi_{1:2m-1}) \right\|_{L^1(\mathbb{R}^d)^{2m-1}} ds_{1, 2m}$$

$$= \int_{D_{2m}(J)} \left\| \prod_{j=1}^{2m-1} (1 + |\xi_j|^2)^{\frac{\beta_j}{2}} \exp \left[ -\frac{1}{2} \text{Var} \left( \sum_{k=1}^{n} \xi_k \cdot (\omega_{s_k, s_{k+1}}) \right) \right] \right\|_{L^1(\mathbb{R}^d)^{2m-1}} ds_{1, 2m}$$

$$\leq \int_{D_{2m}(J)} \left\| \prod_{j=1}^{2m-1} (1 + |\xi_j|^2)^{\frac{\beta_j}{2}} \exp \left[ - c_n \sum_{k=1}^{n-1} \|\xi_k\|^2 |s_{k+1} - s_k|^2H \right] \right\|_{L^1(\mathbb{R}^d)^{2m-1}} ds_{1, 2m}$$

$$\lesssim \int_{D_{2m}(J)} \prod_{j=1}^{2m-1} |s_{k+1} - s_k|^{-(d + \beta_j)H} ds_{1, 2m}$$

$$\lesssim \prod_{j=1}^{2m-1} |J|^{1-(d + \beta_j)H}$$

$$\lesssim |J|^{2m(1-dH-\beta H + \frac{d}{2m})}$$

$$\lesssim |J|^{2m\eta},$$
where we used the condition \( \beta_j \leq 2\beta < \frac{1}{H} - d \) in order to integrate over \( D_{2m}(J) \), and the condition \( \sum_j \beta_j = 2m\beta \) in order to evaluate the last product.

We now move to the general case. By the usual interpolation inequalities between variation spaces and Besov spaces, we have that for every \( 1 \leq p_1, q_1, p_2, q_2 \leq \infty, \alpha_2 \) and for every \( 0 \leq \theta \leq 1 \), by defining

\[
\frac{1}{p_0} := \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q_0} := \frac{\theta}{q_1} + \frac{1-\theta}{q_2}, \quad \alpha_0 := \theta\alpha_1 + (1-\theta)\alpha_2,
\]

we have that

\[
\|\mu\|_{C_{p_0}^{\theta-\varrho B_{q_0}^{\alpha_0}}} \lesssim \|\mu\|_{C_{p_1}^{\theta-1}B_{q_1}^{\alpha_1}} \|\mu\|_{C_{p_2}^{\theta-1}B_{q_2}^{\alpha_2}} \|
\]

Therefore, by interpolating between even integers, we obtain that for every \( 2 \leq q < \infty \), \( \mu_{s,t} \in C^{p-\varrho B_{q}^{\alpha}} \) for every \( 2 \leq q < \infty \), \( \alpha < \frac{1}{2}(\frac{1}{H} - d) \), \( \frac{1}{p} < 1 - (d + \alpha)H + \frac{dH}{q} \). By Sobolev embeddings, we can extend the result to \( q = \infty \). Therefore, it only remains to check the result for \( 1 \leq q < 2 \). We observe that in the case \( p = q = 1 \), \( \alpha = 0 \), we have that

\[
\|\mu^u_{s,t}\|_{B_{1,\infty}^{0}} \leq \int_s^t \|\delta_{\omega_{\tau}}\|_{B_{1,\infty}^{0}} d\tau \lesssim |t-s|,
\]

therefore \( \mu_{a,b} \in C^{1-\varrho B_{1,\infty}^{0}} \). Finally, the case where \( 1 \leq q < 2 \) follows by interpolation between the case \( q = 2 \), \( \alpha < \frac{1}{2}(\frac{1}{H} - d) \), \( \frac{1}{p} < 1 - \frac{dH}{2} - \alpha H \), and the case \( p = q = 1, \alpha = 0 \).

6. Integration and ODEs with \( SBE \) paths

We briefly show how to solve ODEs driven by \( SBE \) paths. We first recall a few notions and results from [FV10]. Given a normed space \( E \) and \( T > 0 \), we denote by \( C^{r-\varrho}(]-0, T[; E) \) the space of continuous paths of finite \( r \)-variation with values on \( E \). Likewise, if \( \Delta_T = \{ (s, t) : 0 \leq s \leq t \leq T \} \) we define \( C^{r-\varrho}(\Delta_T; E) \). We also recall a sewing lemma for finite variation paths (see for instance [FZ18, Theorem 2.22]).

**Lemma 6.1.** Let \( \chi : \Delta_2 \rightarrow E \) be continuous and assume there are two controls \( \rho, \sigma \) and two numbers \( a, b > 0 \) with \( a + b > 1 \) such that

\[
\|\delta X_{st}\|_{\mathcal{T}} \leq \rho_{su}^a \sigma_{ut}^b, \quad s \leq u \leq t,
\]

where \( \delta X_{st} = X_{st} - X_{su} - X_{ut} \). Then there is a unique \( \mathcal{J} \chi \in C([-0, T]; E) \) such that \( \mathcal{J} X_0 = 0 \) and

\[
(6.1) \quad \|\mathcal{J} X_{st} - X_{st}\|_{\mathcal{T}} \leq \rho_{st}^a \sigma_{st}^b.
\]

Moreover, if \( \chi \in C^{r-\varrho}(\Delta_T; E) \), then \( \mathcal{J} \chi \in C^{r-\varrho}(0, T; E) \).
We shall use the above lemma to define the integral

$$\int_0^T f(s, \theta_s - \omega_s) \, ds$$

for suitable paths \((\theta_s)_{s \in [0,T]}, (\omega_s)_{s \in [0,T]}\) on \(\mathbb{R}^d\), and non-smooth \(f : \mathbb{R}^d \to \mathbb{R}^d\). To this end we recall a standard result for convolutions (see for instance [KL21]).

**Theorem 6.2.** Let \(\alpha_1, \alpha_2 \in \mathbb{R}, q, q_1, q_2 \in (0, \infty), p, p_1, p_2 \in [1, \infty)\) be such that

$$\frac{1}{q} \leq \frac{1}{q_1} + \frac{1}{q_2}, \quad 1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}. $$

If \(f \in B_{p_1,q_1}^{\alpha_1} \) and \(g \in B_{p_2,q_2}^{\alpha_2} \), then \(f \ast g \in B_{p,q}^{\alpha_1+\alpha_2} \) and

$$\|f \ast g\|_{B_{p,q}^{\alpha_1+\alpha_2}} \lesssim \|f\|_{B_{p_1,q_1}^{\alpha_1}} \|g\|_{B_{p_2,q_2}^{\alpha_2}}. $$

By collecting the above results, we can prove existence and properties of a Young integral.

**Proposition 6.3.** Let \(p_1, q_1, p_2 \in [1, \infty), q_2 \in [1, \infty], \alpha_1 > 0, \alpha_2 \in \mathbb{R}, \) and \(r_1, r_2, r_3 > 0\) be such that

$$1 < \frac{1}{q_1} + \frac{1}{p_2} < 1 + \frac{\alpha_1 + \alpha_2}{d}, \quad 1 + \frac{1}{r_1} + \frac{1}{r_2} > 1, \quad \frac{1}{r_1} + \frac{\gamma}{r_3} > 1,$$

for every \(\gamma \in (0, 1), \gamma \leq \gamma_0, \) with \(\gamma_0 = \alpha_1 + \alpha_2 - d\left(\frac{1}{q_1} + \frac{1}{p_2} - 1\right)\).

Let \(\omega : [0, T] \to \mathbb{R}^d\) be a continuous path such that its occupation measure \(\mu_{\omega}^w \in C^{r_1-\var\var}([0, T]; \mathbb{B}_{q_1}^{p_1,p_1})\), let \(f \in C^{r_2-\var\var}([0, T]; B_{p_2,q_2}^{\alpha_2} (\mathbb{R}^d)) \) and \(\theta \in C^{r_3-\var\var}([0, T]; \mathbb{R}^d)\).

Then

$$t \mapsto \int_0^t f(s, \theta_s - \omega_s) \, ds$$

is well defined in \(C^{r_1-\var\var}([0, T]; \mathbb{R}^d)\), and

$$\left\| \int_0^t f(s, \theta_s - \omega_s) \, ds \right\|_{r_1-\var\var} \lesssim \left(\|f\|_{C^{r_2-\var\var}([0, T]; B_{p_2,q_2}^{\alpha_2})} + (1 + \|\theta\|_{L^{\infty}(0, T; B_{p_2,q_2}^{\alpha_2})}) \|\mu_{\omega}^w\|_{C^{r_1-\var\var}([0, T]; \mathbb{B}_{q_1}^{p_1,p_1})} \right).$$

Moreover, if \(\theta^1, \theta^2 \in C^{r_3-\var\var}([0, T]; \mathbb{R}^d)\), and if \(\lambda \in (0, 1)\) is such that \(\frac{1}{r_1} + \frac{\gamma(1-\lambda)}{r_3} > 1\), then

$$\left\| \int_0^t f(s, \theta^1_s - \omega_s) \, ds - \int_0^t f(s, \theta^2_s - \omega_s) \, ds \right\|_{r_1-\var\var} \lesssim$$

\[
(\|\theta^1 - \theta^2\|_{L^{\infty}} + \|\theta^1\|_{C^{r_3-\var\var}([0, T]; \mathbb{R}^d)} + \|\theta^2\|_{C^{r_3-\var\var}([0, T]; \mathbb{R}^d)})^{\gamma(1-\lambda)}.
\]
Finally, if $\gamma_0 > 1$, then the map
\[
\theta \to \mathcal{J}(\theta) = \int_0^1 f(s, \theta_s - \omega_s) \, ds
\]
is Fréchet differentiable and, for $\gamma \in (0, 1)$, with $\gamma \leq \gamma_0 - 1$,
(6.3)
\[
\|D \mathcal{J}(\theta)\| \lesssim \|\mu^w_{\omega}\|_{C^{1-\gamma_0}(\Omega; C^{\alpha_1,p_1,1})} \cdot \left(\|f\|_{L^\infty(0,T; B^\alpha_{p,2,q_2})} + \|f\|_{C^{2-\gamma_0}(0,T; B^\alpha_{p,2,q_2})} + \|\theta\|_{C^1(\Omega; \mathbb{R}^d)}\right).
\]

Proof. Set
(6.4)
\[
X_{st} = \int_s^t f(s, \theta_s - \omega_r) \, dr = f(s, \cdot) \ast \mu^w_{st}(\theta_s),
\]
thus for $s \leq u \leq t$,
\[
\delta X_{sut} = (f(s, \cdot) - f(u, \cdot)) \ast \mu^w_{ut}(\theta_s) + f(u, \cdot) \ast \mu^w_{ut}(\theta_s) - f(u, \cdot) \ast \mu^w_{ut}(\theta_u).
\]
Notice preliminary that by Theorem 2.2, $\mu^w_{ut} \in B^\alpha_{q_1,\infty}$, therefore by Theorem 6.2 $f(u, \cdot) \ast \mu^w_{ut} \in B^{\alpha_1+\alpha_2}_{p,2}$, with $1 + \frac{1}{p} = \frac{1}{q_1} + \frac{1}{p_2}$, By Sobolev's embeddings, $f(u, \cdot) \ast \mu^w_{ut} \in B^\alpha_{\infty,q_2}$ with $\gamma_0 = \alpha_1 + \alpha_2 - \frac{d}{p}$ and
\[
\|f(u, \cdot) \ast \mu^w_{ut}\|_{B^{\alpha_1}_{\infty,q_2}} \lesssim \|f(u, \cdot) \ast \mu^w_{ut}\|_{B^{\alpha_1}_{p,2}} \lesssim \|f(u, \cdot)\|_{B^{\alpha_1}_{p,2}} \lesssim \|f(u, \cdot)\|_{B^{\alpha_1}_{q_1,\infty}} \lesssim \|f(u, \cdot)\|_{B^{\alpha_1}_{q_1,\infty}}.
\]
A similar estimate clearly holds for $(f(s, \cdot) - f(u, \cdot)) \ast \mu^w_{ut}$.

Fix $\gamma \in (0, 1)$, with $\gamma \leq \gamma_0$. We use the above inequalities to estimate $|\delta X_{sut}|$,
\[
|\delta X_{sut}| \leq \|(f(s, \cdot) - f(u, \cdot)) \ast \mu^w_{ut}(\theta_s) - f(u, \cdot) \ast \mu^w_{ut}(\theta_u)| \leq \|(f(s, \cdot) - f(u, \cdot)) \ast \mu^w_{ut}\|_{B^{\alpha_{\infty,q_2}}} + \|f(u, \cdot) \ast \mu^w_{ut}\|_{B^{\alpha_{\infty,q_2}}|\theta_s - \theta_u|}
\leq \rho_{st} \sigma_{st} |
\]

where $F_T = \|f\|_{L^\infty(0,T; B^{\alpha_{\infty,q_2}})}$, $r_0 = \max(r_2, r_3/\gamma)$, and
\[
\sigma_{st} = \|\mu^w_{\omega}\|_{C^1-\gamma_0(\Omega; C^{\alpha_1,p_1,1})} \cdot \|f\|_{C^{1-\gamma_0}(0,T; B^{\alpha_{p,2,q_2}})} + \|\theta\|_{C^1(\Omega; \mathbb{R}^d)}.
\]

are controls, and by assumption $\frac{1}{r_1} + \frac{1}{r_0} > 1$. Moreover,
\[
|X_{st}| \lesssim \|f\|_{L^\infty(0,T; B^{\alpha_{p,2,q_2}})} \mu^w_{st} \lesssim F_T \sigma_{st}^{-1/r_1}
\]
therefore by Lemma 6.1 the conclusion follows.
We turn to the proof of (6.2). Given $\theta^1, \theta^2 \in C^{r_3-var}(\mathbb{R}^d)$, it is not difficult to prove through Lemma 6.1 that

$$\int_0^t \left( f(s, \theta^1_s - \omega_s) ds - \int_0^t f(s, \theta^2_s - \omega_s) ds \right) = \int_0^t \left( f(s, \theta^1_s - \omega_s) - f(s, \theta^2_s - \omega_s) \right) ds,$$

and that the integral on the right hand side can be obtained by Lemma 6.1 with the local description

$$\chi^0_{st} = f(s, \cdot) \ast \mu^\omega_{st}(\theta^1_s) - f(s, \cdot) \ast \mu^\omega_{st}(\theta^2_s).$$

Moreover,

$$\delta \chi^0_{st} = (f(s, \cdot) - f(u, \cdot)) \ast \mu^\omega_{st}(\theta^1_s) - (f(s, \cdot) - f(u, \cdot)) \ast \mu^\omega_{st}(\theta^2_s) + f(u, \cdot) \ast \mu^\omega_{st}(\theta^1_s) - f(u, \cdot) \ast \mu^\omega_{st}(\theta^2_s) - f(u, \cdot) \ast \mu^\omega_{st}(\theta^1_u) + f(u, \cdot) \ast \mu^\omega_{st}(\theta^2_u)$$

can be estimated as

$$|\delta \chi^0_{st}| \leq \| (f(s, \cdot) - f(u, \cdot)) \ast \mu^\omega_{st} \|_{B^{\gamma_1}_{\infty,q_2}} |\theta^1_s - \theta^2_s|^{\gamma} + \| f(u, \cdot) \ast \mu^\omega_{st} \|_{B^{\gamma_1}_{\infty,q_2}} (|\theta^1_s - \theta^2_s|^{|\gamma|} + |\theta^1_s - \theta^2_s|^{\gamma}) \leq \sigma^{1/r_1} \left( (\rho^f_{su})^{1/r_2} |\theta^1_s - \theta^2_s|^{\gamma} + F_T |\theta^1_s - \theta^2_s|^{\gamma} \right),$$

and as

$$|\delta \chi^0_{st}| \leq (\rho^f_{su})^{1/r_2} \sigma^{1/r_1} \| \theta^1_s - \theta^2_s \|_{\infty} + \| f(u, \cdot) \ast \mu^\omega_{st} \|_{B^{\gamma_1}_{\infty,q_2}} (|\theta^1_s - \theta^2_s|^{\gamma} + |\theta^1_s - \theta^2_s|^{\gamma}) \leq \sigma^{1/r_1} \left( (\rho^f_{su})^{1/r_2} \| \theta^1_s - \theta^2_s \|_{\infty} + F_T |\theta^1_s - \theta^2_s|^{\gamma} \right).$$

where $\rho^f_{st} = \| f \|^{r_2}_{r_2-var,[s,t]}$ and similarly for $\rho^{\theta^i}_{st}$, $i = 1, 2$ in $C^{r_3-var}$. So for $\lambda \in (0, 1)$,

$$|\delta \chi^0_{st}| \leq \sigma^{1/r_1} \left( (\rho^f_{su})^{1/r_2} \| \theta^1_s - \theta^2_s \|_{\infty} + F_T |\theta^1_s - \theta^2_s|^{\gamma} \right).$$

Take this time $r_0 = \max(r_2, r_3/(\gamma(1 - \lambda)))$, and $\lambda$ small enough that $1/r_1 + 1/r_0 > 1$ still holds. Since we also have

$$|\chi^0_{st}| \leq \| f(s, \cdot) \ast \mu^\omega_{st} \|_{B^{\gamma_1}_{\infty,q_2}} \| \theta^1_s - \theta^2_s \|_{\infty} \leq F_T \sigma^{1/r_1} \| \theta^1_s - \theta^2_s \|_{\infty},$$

(6.2) follows.

We finally prove differentiability of $\mathscr{S}$. Assume $\gamma_0 > 1$, and let now $\gamma \in (0, 1)$ with $\gamma \leq \gamma_0 - 1$. First notice that if $0, \eta \in C^{r_3-var}(0, T; \mathbb{R}^d)$, the function $\chi^\prime_{st} = \eta_s \cdot \nabla (f(s, \cdot) \ast \mu_{st})(\theta_s)$ meets the assumption of Lemma 6.1, and thus defines an integral $\mathscr{S}(\theta)_{\eta}$, which is moreover linear in $\eta$. This follows as for the local description defined in (6.4). Indeed, $|\chi^\prime_{st}| \leq \| \eta \|_\infty \| f \|_{L^{\infty}[0, T; B^{\gamma_1}_{\infty,q_2}]} \| \mu^\omega_{st} \|_{B^{\gamma_1}_{\infty,q_2}^{\gamma_1}},$

and

$$\delta \chi^\prime_{st} = \eta_s \cdot \nabla \left( (f(s, \cdot) - f(u, \cdot)) \ast \mu^\omega_{st}(\theta_s) + (\eta_s - \eta_u) \cdot \nabla (f(u, \cdot) \ast \mu^\omega_{st})(\theta_s) + \eta_u \cdot \left( \nabla (f(u, \cdot) \ast \mu^\omega_{st})(\theta_s) - \nabla (f(u, \cdot) \ast \mu^\omega_{st})(\theta_u) \right) \right).$$
thus

\[ |\delta \chi^s_u| \lesssim \sigma^s_{ut} \left( \|\eta\|_{\infty} (\rho^s_{su})^{1/r_2} + F_T(\rho^\eta_{su})^{1/r_3} + \|\eta\|_{\infty} F_T(\rho^\theta_{su})^{\gamma/r_3} \right) \]

where \( \sigma, F_T, \rho^f, \rho^0 \) are as before, and \( \rho^\eta \) is defined as \( \rho^0 \). Let us prove that \( \mathcal{I}(\theta) \) is differentiable and \( D\mathcal{I}(\theta) = \mathcal{I}'(\theta) \). By our previous considerations, given \( \theta, \eta \in C^{r_3-\text{var}}([0, T]; \mathbb{R}^d) \) and \( \epsilon > 0 \), the integral

\[ \frac{1}{\epsilon} (\mathcal{I}(\theta + \epsilon \eta) - \mathcal{I}(\theta)) - \mathcal{I}(\theta) \eta \]

has the local description

\[ \chi^s_{st} = \frac{1}{\epsilon} \left( (f(s, \cdot) \ast \mu^w_{st})(\theta_s + \epsilon \eta_s) - (f(s, \cdot) \ast \mu^w_{st})(\theta_s) \right) - \eta_s \cdot \nabla (f(s, \cdot) \ast \mu^w_{st})(\theta_s) \]

\[ = \frac{1}{\epsilon} \int_0^\epsilon \eta_s \cdot \left( \nabla (f(s, \cdot) \ast \mu^w_{st})(\theta_s + \alpha \eta_s) - \nabla (f(s, \cdot) \ast \mu^w_{st})(\theta_s) \right) \, d\alpha \]

Now,

\[ |\chi^s_{st}| \lesssim \frac{1}{\epsilon} \int_0^\epsilon \alpha^r \|\eta_s\|^r + \|f(s, \cdot) \ast \mu^w_{st}\|_{L^{r+1}} \, d\alpha \lesssim \epsilon \|\eta\| \|f\| \|\mu^w_{st}\|_{L^{r+1}} + \|\eta\| \|f\| \|\mu^w_{st}\|_{L^{r+1}} \]

and \( \delta \chi^s_{st} = \mathcal{O} + \mathcal{O} + \mathcal{C} \), where

\[ |\mathcal{O}| = \frac{1}{\epsilon} \int_0^\epsilon (\eta_s - \eta_u) \cdot \left( \nabla (f(s, \cdot) \ast \mu^w_{st})(\theta_s + \alpha \eta_s) - \nabla (f(s, \cdot) \ast \mu^w_{st})(\theta_s) \right) \, d\alpha \]

\[ \lesssim \epsilon \|\eta\| \|f\| \|\mu^w_{st}\|_{L^{r+1}} (\rho^\eta_{su})^{1/r_3}, \]

\[ |\mathcal{O}| = \frac{1}{\epsilon} \int_0^\epsilon \eta_u \cdot \left( \nabla ((f(s, \cdot) - f(u, \cdot)) \ast \mu^w_{ut})(\theta_s + \alpha \eta_s) - \nabla ((f(s, \cdot) - f(u, \cdot)) \ast \mu^w_{ut})(\theta_s) \right) \, d\alpha \]

\[ \lesssim \epsilon \|\eta\| \|f\| \|\mu^w_{st}\|_{L^{r+1}} (\rho^\eta_{su})^{1/r_3}, \]

and finally,

\[ |\mathcal{C}| = \frac{1}{\epsilon} \int_0^\epsilon \eta_u \cdot \left( \nabla (f(u, \cdot) \ast \mu^w_{ut})(\theta_u + \alpha \eta_u) - \nabla (f(u, \cdot) \ast \mu^w_{ut})(\theta_u) \right) \, d\alpha \]

\[ \lesssim \epsilon \|\eta\| \|f\| \|\mu^w_{st}\|_{L^{r+1}} (\rho^\theta_{su} + \rho^\eta_{su})^{\gamma/r_3}, \]

so that for \( \lambda \in (0, 1) \),

\[ |\mathcal{O}| \lesssim \epsilon^\lambda \|\eta\| \|f\| \|\mu^w_{st}\|_{L^{r+1}} (\rho^\theta_{su} + \rho^\eta_{su})^{\gamma/r_3(1-\lambda)}. \]

In conclusion,

\[ \left\| \frac{1}{\epsilon} (\mathcal{I}(\theta + \epsilon \eta) - \mathcal{I}(\theta)) - \mathcal{I}(\theta) \eta \right\|_{C^{r_3-\text{var}}} \lesssim \epsilon^\lambda. \]
Moreover, the above estimate is uniform for \( \eta \) in a bounded ball centred at 0. This proves Fréchet differentiability of \( \mathcal{I}(\theta) \).

We turn to ODEs driven by SBE-paths. Following [GG22b], we shall use the integral as given by the previous proposition to give a meaning, for general drifts, to differential equations.

**Definition 6.4.** Given \( x_0 \in \mathbb{R}^d, r_2 \geq 1, \alpha_2 \in \mathbb{R}, p_2 \in [1, \infty), q_2 \in [1, \infty], \) and \( f \in C^{r_2-\text{var}}([a, b]; B_{p_2,q_2}^{\alpha_2}(\mathbb{R}^d)), \) a map \( x \in C([a, b]; \mathbb{R}^d) \) is a solution of the ODE

\[
x_t = x_0 - \omega_t + \int_a^t f(s, x_s) \, ds, \quad t \in [a, b],
\]

if there are \( p_1, q_1 \in [1, \infty), \alpha_1 > 0 \) and \( r_1 \geq 1 \) such that

\[
1 - \frac{1}{q_1} + \frac{1}{p_2} < 1 + \frac{\alpha_1 + \alpha_2}{d}, \quad \frac{1}{r_1} + \frac{1}{r_2} > 1, \quad r_1 < 1 + \gamma,
\]

for some \( \gamma \in (0, 1) \) and \( \gamma < \gamma_0, \) with \( \gamma_0 = \alpha_1 + \alpha_2 - d\left(\frac{1}{q_1} + \frac{1}{p_2} - 1\right), \) and \( \theta = x + \omega \) has occupation measure \( \mu_\alpha^\omega \in C^{r_1-\text{var}}([a, b]; \mathcal{S}^{\alpha_1,p_1}), \) and if

\[
\theta_t = x_0 + \int_0^t f(s, \theta_s - \omega_s) \, ds, \quad t \in [a, b].
\]

**Theorem 6.5.** Let \( p_1, q_1, p_2 \in [1, \infty), q_2 \in [1, \infty], \alpha_1 > 0, \alpha_2 \in \mathbb{R}, \) and \( r_1, r_2 > 0 \) be such that (6.5) holds for some \( \gamma \in (0, 1) \) and \( \gamma \leq \gamma_0, \) with \( \gamma_0 = \alpha_1 + \alpha_2 - d\left(\frac{1}{q_1} + \frac{1}{p_2} - 1\right). \)

Let \( x_0 \in \mathbb{R}^d, \omega : [0, T] \to \mathbb{R}^d \) be a continuous path such that its occupation measure \( \mu_\alpha^\omega \in C^{r_1-\text{var}}([0, T]; \mathcal{S}^{\alpha_1,p_1}), \) and let \( f \in C^{r_2-\text{var}}([0, T]; B_{p_2,q_2}^{\alpha_2}(\mathbb{R}^d)). \)

Then there is \( x \in C^{r_1-\text{var}}([0, T]; \mathbb{R}^d) \) such that

\[
x_t = x_0 - \omega_t + \int_0^t f(s, x_s) \, ds,
\]

for \( 0 \leq t \leq T, \) in the sense of Definition 6.4.

Moreover, if \( \gamma_0 > 1 \) then the solution is unique.

**Proof.** We look for a solution \( x_t = \theta_t - \omega_t, \) with \( \theta \) a fixed point of the map

\[
\mathcal{I} : \mathcal{K}_R \to C^{r_3-\text{var}}([0, T_*]; \mathbb{R}^d),
\]

where

\[
(\mathcal{I}\theta)_t = x_0 + \int_0^t f(s, \theta_s - \omega_s) \, ds,
\]

\( \mathcal{K}_R = \{ \theta \in C^{r_3-\text{var}}([0, T_*]; \mathbb{R}^d) : \|\theta\|_{r_3-\text{var}} \leq R \}, \)

\( r_3 > r_1 \) such that \( \frac{1}{r_1} + \frac{\gamma}{r_3} > 1, \) and the integral is defined by Proposition 6.3.

We use Schauder’s fixed point. To this end, we shall prove that

- \( \mathcal{I} \) maps \( \mathcal{K}_R \) onto and is continuous,
- \( \mathcal{I}(\mathcal{K}_R) \) is relatively sequentially compact.
First, from Proposition 6.3, for \( r_3 > r'_1 > r_1 \) but such that \( \frac{1}{r'_1} + \frac{T}{r_3} > 1 \) and (6.5) holds with \( r_1 \) replaced by \( r'_1 \),

\[
\| T0 \|_{r_3 - \text{var}} \leq \| T0 \|_{r_1 - \text{var}} \\
\leq (\| f \|_{C^{r_2 - \text{var}}([0,T];B^s_{p,2,q_2})} + (1 + \| \theta \|_{r_3 - \text{var}}) \| f \|_{L^\infty(0,T,B^s_{p,2,q_2})}) \| \mu_0 \|_{C^{r'_1 - \text{var}}([0,T_x],S^s_{q_1})} \\
\]

and by assumption \( \| \mu_0 \|_{C^{r'_1 - \text{var}}([0,T_x],S^s_{q_1})} \rightarrow 0 \) as \( T_x \rightarrow 0 \). Therefore it is possible to find, given \( R > 0 \), a time \( T_s \in [0, T) \) that depends only on \( f \) and \( \mu_0 \), but not on \( x_0 \), such that \( T \) maps \( K_R \) onto. Moreover, the same estimate proves that \( TK \) is a subset of \( C^{r_1 - \text{var}}([0, T], \mathbb{R}^d) \).

To prove sequential compactness, it suffices by \([\text{FV}10, \text{Proposition 5.28}]\) to prove equicontinuity. Indeed, if \( \theta \in K_R \) and \( r'_1 > r_1 \) is as above,

\[
(\langle T0 \rangle)_t - (\langle T0 \rangle)_s = \int_s^t f(u, \theta_u - \omega_u) \, du \lesssim \| \mu_0 \|_{C^{r'_1 - \text{var}}([s, t], S^s_{q_1})} \\
\]

where the constant depends only in \( R \) and \( f \). Therefore elements in \( T(K_R) \) have the same modulus of continuity and equicontinuity holds. Finally, continuity of \( T \) follows from (6.2).

Finally, we construct a solution on the whole interval \([0, T]\). As noticed before, the existence time \( T_s \), we have found depends on \( R \), \( \| f \|_{C^{r_2 - \text{var}}([0,T];B^s_{p,2,q_2})} \) and \( \| \mu_0 \|_{C^{r'_1 - \text{var}}([0, T_x], S^s_{q_1})} \)' with \( r'_1 > r_1 \), sufficiently close to \( r_1 \). So the global solution is obtained by gluing the solution \( x^1 \) in \([0, T_s]\), with \( T_s = T_x \), started at \( x_0 \), with the solution \( x^2 \) on \([T_s, T]\) with initial condition \( x^2(T_s) = x^1(T_s) \), etc. If \( T_s = 0 \), the existence times \( (T_k)_k \) can be chosen so that

\[
(\| f \|_{C^{r_2 - \text{var}}([0,T];B^s_{p,2,q_2})} + (1 + R_k) \| f \|_{L^\infty(0,T,B^s_{p,2,q_2})}) \| \mu_0 \|_{C^{r'_1 - \text{var}}([T_k - 1, T_k], S^s_{q_1})} \geq R_k, \\
\]

for \( R_k > 0 \) chosen in the proof. Let \( R_k = 1 \), then in a finite number of steps the whole \([0, T]\) is covered (that is \( T_k^* = T \) for some \( k \)). Indeed, we have that \( \| \mu_0 \|_{C^{r'_1 - \text{var}}([T_k - 1, T_k], S^s_{q_1})} \geq C_f \), for a constant depending only on \( f \), thus

\[
\sum_{k=1}^N C_{T_k} \leq \sum_{k=1}^N \| \mu_0 \|_{C^{r'_1 - \text{var}}([T_k - 1, T_k], S^s_{q_1})} \leq \| \mu_0 \|_{C^{r'_1 - \text{var}}([0,T], S^s_{q_1})}, \\
\]

which concludes the proof of global existence.

Finally, uniqueness holds by using the Banach fixed point theorem and the estimate on the differential of the integral given in Proposition 6.3.

**Remark 6.6.** The global control on \( f \) yields a global solution. If on the other hand \( f \in C^{r_2 - \text{var}}([0,T];B_{p,2,q_2}(D)) \) for a domain \( D \), or \( f \in C^{r_2 - \text{var}}([0,T];B_{p,2,q_2,loc}) \),
the previous theorem provides a local solution, namely there is a time \( T_s \in (0, T] \) and a solution \( x \) defined on \([0, T]\). Indeed if \( \eta \in C^\infty \) is a cut-off function such that \( \eta \equiv 1 \) in a neighbourhood of \( x_0 \), then in both cases \( \eta f \in C^{r_2-\var}([0, T]; B^{s_2}_{p_2, \alpha_2}) \), and the previous theorem applies.

Recall that \( \phi : [0 \leq s \leq t \leq T] \times \mathbb{R}^d \to \mathbb{R}^d \) is a (continuous) flow if

- \( \phi(t, t, x_0) = x_0 \), for all \( t, x_0 \),
- \( \phi(u, t, \phi(s, u, x_0)) = \phi(s, t, x_0) \) for all \( s \leq u \leq t \) and \( x_0 \),
- \( \phi(s, t, \cdot) \) is continuous with continuous inverse, for all \( s, t \).

**Corollary 6.7.** Under the same assumptions of Theorem 6.5, if \( \gamma_0 > 1 \), then there is a flow \( \phi \) of diffeomorphisms (namely, \( \phi(s, t, \cdot) \) is continuously differentiable) such that \( \phi(s, t, x_0) \in C^{r_1-\var}(s, T; \mathbb{R}^d) \) for all \( s, x_0 \), and

\[
\phi(s, t, x_0) = x_0 - (\omega_t - \omega_s) + \int_s^t f(r, \phi(s, r, x_0)) \, dr,
\]

for all \( s \leq t \).

**Proof.** Define \( \phi \) according to (6.6), using the previous theorem. The first property of flows is obvious, the second follows by uniqueness. The fact that \( \phi(s, t, x_0) \in C^{r_1-\var}(s, T; \mathbb{R}^d) \) follows by definition.

To prove continuity, recall that if \( \theta(s, t, x_0) = \phi(s, t, x_0) + (\omega_t - \omega_s) \), then

\[
\theta(s, t, x_0) = x_0 + \int_s^t f(r, \theta(s, r, x_0)) \, dr,
\]

therefore, by (6.2), for \( u \in [s, t] \),

\[
\|\theta(s, t, x_0) - \theta(s, t, x'_0)\|_{C^{r_1-\var}([s, u])} \leq \|x_0 - x'_0\| + c_u \|\theta(s, t, x_0) - \theta(s, t, x'_0)\|_{C^{r_1-\var}([s, u])}^a,
\]

where \( a \in (0, 1) \) and

\[
c_u \leq \|\mu_u\|_{C^{r_1-\var}([s, t])} \|f\|_{C^{r_2-\var}([s, t])} \cdot (\|\theta(s, t, x_0)\|_{C^{r_1-\var}([s, t])} + \|\theta(s, t, x'_0)\|_{C^{r_1-\var}([s, t])})^b
\]

with \( r_1' > r_1 \) and \( b < 1 \). Since \( c_u \to 0 \) as \( u \to s \), we can find a value \( u_1 \) such that \( \|\theta(s, t, x_0) - \theta(s, t, x'_0)\|_{C^{r_1-\var}([s, u_1])} \leq 2|x_0 - x'_0| \). Since the estimate on \( c_u \) depends only on \( \mu^u \), we can replicate the estimate on \( a \) at most finite number of intervals (as in the proof of Theorem 6.5) and get continuity on \([s, t]\).

To prove that \( \phi \) is invertible with continuous inverse, notice that, for fixed \( s, t \), if we define the time-reversed flow

\[
\psi(s, r, y_0) = y_0 + (\omega_{t+s-r} - \omega_1) - \int_s^r f(t + s - u, \psi(s, u, y_0)) \, du,
\]
with \( r \in [s, t] \), then \( \psi(s, t, \phi(s, t, x_0)) = x_0 \). The flow \( \psi \) is also defined according to \textbf{Theorem 6.5}, and it is immediate to check that the reversed perturbation \( r \mapsto \omega_{1+s-r} \) is in \( C^{1-\text{var}}([s, t]; \mathbb{SBE}^{\alpha_1, \beta_1}) \). Finally, differentiability follows by an argument similar to continuity, using this time (6.3).

\[ \square \]

7. Examples

Finally, we discuss a few examples of processes with occupation measure in \( \mathbb{SBE} \)-type spaces. First, using either (especially) \textbf{Corollary 4.4} or \textbf{Corollary 5.3}, it is not difficult, in general, to establish that Gaussian processes have occupation measure in \( \mathbb{SBE} \) spaces. We provide a pair of examples of non-Gaussian processes with occupation measure in \( \mathbb{SBE} \) spaces. The two examples are solutions of stochastic differential equations driven either by Brownian motion or fractional Brownian motion. Albeit this is not surprising, since it is expected that short time asymptotics of densities of solutions of SDEs look like the rough driving process, it remains non obvious to prove similar statements (namely that solutions of equation with \( \mathbb{SBE} \)-driven input have \( \mathbb{SBE} \) occupation measures) in the general framework of Young differential equations (Section 6).

7.1. Markov processes and one dimensional SDEs. Consider a Markov process \( (X_t)_{t \in [0, T]} \) on \( \mathbb{R}^d \) with transition density function \( p(s, t, x, y) \), and assume \( X_0 = x_0 \in \mathbb{R}^d \). Given \( n \geq 1 \) and \( s_0 = 0 < s_1 < s_2 < \cdots < s_n \), the joint density of \( (X_{s_1}, X_{s_2}, \ldots, X_{s_n}) \) is

\[
\rho_{s_1:n}(x_{1:n}) = \prod_{j=1}^{n} p(s_{j-1}, s_j, x_{j-1}, x_j)
\]

and the density of an increment is

\[
\nu_{s_1s_2}(y) = \int_{\mathbb{R}^d} \rho_{s_1s_2}(x, x+y) \, dx = \int_{\mathbb{R}^d} p(0, s_1, x_0, x)p(s_1, s_2, x, x+y) \, dx.
\]

Therefore,

\[
\|\nu_{s_1s_2}\|_{C^\beta} \leq \int_{\mathbb{R}^d} \|p(s_1, s_2, x, \cdot)\|_{C^\beta} p(0, s_1, x_0, x) \, dx,
\]

and, for instance, \textbf{Theorem 5.1} applies with \( m = 1 \) if there are \( C_\nu > 0 \) and \( \eta \in (0, 1) \) such that for all \( 0 \leq a < b \leq T \),

\[
\int_{s_1}^{b} \sup_{s \in [s_1, x \in \mathbb{R}^d] \mathbb{E} \|p(s, s_2, x, \cdot)\|_{C^\beta} ds_2 \leq C_\nu (b-a)^{2\eta}.
\]

If on the other hand we wish to use \textbf{Theorem 5.1} with \( m \geq 2 \), we notice that, to use for instance (5.5), in view of (7.1) it is sufficient to get a bound of

\[
\|(1 - \Delta_x)^{\beta} (1 - \Delta_y)^{\beta} p(s, t, x, y)\|_{L^\infty}.
\]
As an example for this, consider a stochastic equation in dimension 1,
\[
\begin{aligned}
\frac{dX_t}{dt} &= b(t, X_t) dt + \sigma(t, X_t) dB_t, \\
X_0 &= x_0,
\end{aligned}
\]
and let \( p(s, t, x, \cdot) \) be the density of \( X_t \) subject to \( X_s = x \). A simple method for the existence and regularity of the density of the solution can be found in [Rom18]. Short time asymptotics are classical, see for instance [Var67b, Var67a, Mol75]. Assume for simplicity \( \sigma \equiv 1 \) (otherwise, one gets the same asymptotics by [Var67a]) and \( b \) bounded measurable, then
\[
p(s, t, x, y) = q_{t-s}(x-y) + \int_s^t \delta_y q_{t-r} * (b(r, \cdot) p(s, r, x, \cdot)) \, dr,
\]
where \( q_t \) is the heat kernel. Then
\[
\|p(s, t, x, \cdot)\|_{C^\beta} \leq \|q_{t-s}\|_{C^\beta} + \|b\|_{L^\infty} \int_s^t \|q_{t-r}\|_{W^{1+\beta, \alpha}} \|p(s, r, x, \cdot)\|_{C^\beta} \, dr,
\]
so that by the heat kernel asymptotics \( \|q_t\|_{C^\beta} \sim t^{-\frac{\beta}{2}(1+\beta)} \), \( \|q_t\|_{W^{\beta,1}} \sim t^{-\frac{\beta}{2}} \) and a simple Gronwall’s argument (using for instance [MW17, Lemma 3.3]) yields
\[
\|p(s, t, x, \cdot)\|_{C^\beta} \lesssim (t-s)^{-\frac{\beta}{2}(1+\beta)},
\]
with a pre-factor independent from \( x \). Thus Theorem 5.1 applies with \( m = 1 \) and \( \eta = \frac{1}{2}(1-\beta) \).

Likewise, if we wish to use Theorem 5.1 with \( m \geq 2 \), we get an estimate of (7.2) notice that to use, for instance, (5.5), in view of (7.1) it is sufficient to bound
\[
\|(1-\Delta_x)^{\frac{\beta}{2}} (1-\Delta_y)^{\frac{\beta}{2}} p(s, t, \cdot, \cdot)\|_{\infty}
\]
and with similar computations and Schauder estimates we get
\[
\|(1-\Delta_x)^{\frac{\beta}{2}} (1-\Delta_y)^{\frac{\beta}{2}} p(s, t, \cdot, \cdot)\|_{\infty} \lesssim (t-s)^{-\frac{\beta}{2}(1+2\beta)},
\]
for \( \beta \), so that
\[
\|(1-\Delta_{x_1})^{\frac{\beta}{2}} \ldots (1-\Delta_{x_{2m}})^{\frac{\beta}{2}} \rho s_{1:2m}\|_{\infty} \lesssim s_1^{-\frac{\beta}{2}(1+\beta)} \Pi_{l=1}^{2m-1} (s_{i+1} - s_i)^{-\frac{\beta}{2}(1+2\beta)},
\]
and the occupation measure of \( (X_t)_{t \in [0,T]} \) is in \( C^{p-\text{var}}([0, T]; B^{\alpha,2m}_{2m-1}) \) for \( \alpha < 1 \) and \( p > \frac{4m}{(2m-1)(1-\alpha)} \). A smaller value of \( p \) can be obtained on intervals away from zero, or using Theorem 5.1 instead of Proposition 5.6.
7.2. Fractional Brownian motion. From Corollary 4.4 and Theorem 6.5 we can immediately deduce the following result.

**Corollary 7.1.** Let \((B^H_t)_{t \geq 0}\) be a \(d\)-dimensional fractional Brownian motion with Hurst index \(H \in (0, 1)\), with \(H < \frac{1}{d}\), and \(x_0 \in \mathbb{R}^d\). Then if \(\alpha_1 < \frac{1}{2H} - \frac{d}{2}\) and \(f \in C^{r_2-\text{var}}([0, T]; \mathbb{B}^\alpha_{p_1, q_2})\), with \(r_2 < 2\), \(p_2 < 2\), and \(\alpha_1 + \alpha_2 > 1 + \frac{d}{p_2} - \frac{d}{2}\), then there is \(x \in C([0, T]; \mathbb{R}^d)\) such that

\[
x_t = x_0 + B^H_t + \int_0^t f(s, x_s) \, ds, \quad t \leq T.
\]

From [Pit78, Lemma 7.1] we see that fractional Brownian motion with Hurst index \(H\) is strongly local \(\phi\) non deterministic, with \(\phi(r) = r^{2H}\). As noticed by [Xiao06], by modifying the proof of [Pit78, Proposition 7.2], this yields (5.3) on time intervals that are bounded away from 0. Thus the occupation measure of fractional Brownian motion is in \(C^{p-\text{var}}([a, b]; \mathbb{B}^\alpha_{q, \infty})\) for \(0 < a < b\) and \(\alpha, p, q\) as in (5.4).

We turn to regularity of the occupation measure of solutions of equations driven by a fractional Brownian motion. We consider the setting of [BNOT16], namely we consider

\[
X_t = x + \int_0^t V_0(X_s) \, ds + \sum_{i=1}^d \int_0^t V_i(X_s) \, dB^i_s,
\]

where \(x \in \mathbb{R}^m\), \((B^i, \ldots, B^d)\) is a \(d\)-dimensional fractional Brownian motion with Hurst parameter \(H > \frac{1}{2}\), and \(V_0, V_1, \ldots, V_d\) are smooth vector fields on \(\mathbb{R}^m\), bounded with all derivatives bounded. Moreover, \((V_1, \ldots, V_d)\) are elliptic and non-degenerate (see [BNOT16, Hypotheses 1.2,1.3]). Existence of the above equation has been established for instance in [FV10], while existence and regularity of the density has been proved for instance in [BH07, HNo7, NS09] \((H > \frac{1}{2})\) and [CLL13, CHLT15] \((H < \frac{1}{2})\).

**Proposition 7.2.** Let \(d \leq 3\) and \(H \in (\frac{1}{4}, 1)\) with \(H < \frac{1}{d}\). Under the above assumptions, let \((X_t)_{t \in [0, T]}\) be a solution of (7.3), and \(\mu^X_s\) its occupation measure. Then for every \(a > 0\), \(\alpha > 0\), with \(\alpha < \frac{1}{2H} - \frac{d}{2}\), \(\mu^X_{a, \alpha} \in V_p([a, T]; \mathbb{S_{\mathbb{B}^\alpha_{q, \infty}}^2})\) for all \(p > 1 - \frac{1}{2}H(d + 2\alpha)\).

**Proof.** Fix \(\varepsilon \leq s < t\), and let \(\nu_{s,t}\) be the density of \(X_t - X_s\). For an integer \(k \geq 1\), let \(\alpha' \in \{1, \ldots, d\}^k\) be a multi-index, and we write \(\partial^{\alpha'} = \partial_{x_{\alpha'1}} \cdots \partial_{x_{\alpha'k}}\). Let \(\alpha = \alpha' \cup \{1, 2, \ldots, k\}\), so that \(\partial_{\alpha}\) computes \(k + d\) derivatives. Let \(\varphi : \mathbb{R}^d \to \mathbb{R}\) be a smooth function, bounded by 1, then by [Nua06, Proposition 2.1.4], integration by parts holds and there is a random variable \(H_{\alpha,s,t}\) such that

\[
\mathbb{E}[\partial^{\alpha}\varphi(X_t - X_s)] = \mathbb{E}[\varphi(X_t - X_s)H_{\alpha,s,t}].
\]
Moreover,
\[ \mathbb{E}[H^2_{\alpha,s,t}]^{1/2} \lesssim \| \Gamma_{s,t}^{-1} D(X_t - X_s) \|_{k+d,2k+d+1}, \]
where \( D(X_t - X_s) \) is the Malliavin derivative, \( \Gamma_{s,t} \) is the Malliavin covariance matrix of \( X_t - X_s \), and \( \| \cdot \|_{k,p} \) is the Malliavin-Sobolev norm (see [Nua06, Section 2.1]). We thus have
\[ \| \mathbb{E}[\partial_\alpha \varphi(X_t - X_s)] \| \lesssim \| \Gamma_{s,t}^{-1} \|_{k+d,2k+d+2} \| D(X_t - X_s) \|_{k+d,2k+d+2}. \]
As in [BNOT16, Proposition 5.9] (the proposition proves only the case \( k = 0 \), conditional to \( s \), but the case \( k > 0 \) can be carried on as in Lemma 4.1, Lemma 4.2 of the same paper),
\[ \| D(X_t - X_s) \|_{k+d,2k+d+2} \lesssim (t - s)^{1/2}, \]
\[ \| \Gamma_{s,t}^{-1} \|_{k+d,2k+d+2} \lesssim (t - s)^{-2H} \]
with constants depending on \( k \) and \( \epsilon \). In conclusion, \( \| \mathbb{E}[\partial_\alpha \varphi(X_t - X_s)] \| \lesssim (t - s)^{-H(k+d)}. \) To prove an estimate of the \( k \)th derivatives of \( \nu_{s,t} \), we use a sequence of smooth functions that converge, monotonically increasing, to suitable Heaviside functions. This yields \( \| \nu_{s,t} \|_{k} \lesssim (t - s)^{-H(k+d)}. \)

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Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, I–56127 Pisa, Italia

*Email address:* marco.romito@unipi.it

*URL:* http://people.dm.unipi.it/romito

Matematichesches Institut, Universität Bonn, Endenicher Allee 60, D–53115 Bonn, Nordrhein-Westfalen, Deutschland

*Email address:* tolomeo@math.uni-bonn.de

*URL:* https://www.math.uni-bonn.de/~tolomeo