UNIFORMITY OF MULTIPlicative FUNCTIONS AND Partition Regularity of some Quadratic Equations

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Abstract. Since the theorems of Schur and van der Waerden, numerous partition regularity results have been proved for linear equations, but progress has been scarce for non-linear ones, the hardest case being equations in three variables. We prove partition regularity for certain equations involving quadratic forms in three variables, showing for example that the equations $16x^2 + 9y^2 = n^2$ and $x^2 + y^2 - xy = n^2$ are partition regular, where $n$ is allowed to vary freely in $\mathbb{N}$. For each such problem we establish a density analogue that can be formulated in ergodic terms as a recurrence property for actions by dilations on a probability space. Our key tool for establishing such recurrence properties is a decomposition result for multiplicative functions which is of independent interest. Roughly speaking, it states that the arbitrary multiplicative function of modulus 1 can be decomposed into two terms, one that is approximately periodic and another that has small Gowers uniformity norm of degree three.

1. Introduction and main results

1.1. Partition regularity results for quadratic forms. An important question in Ramsey theory is to determine which algebraic equations, or systems of equations, are partition regular over the natural numbers. In this article, we restrict our attention to polynomials in three variables, in which case partition regularity of the equation $p(x, y, z) = 0$ amounts to saying that, for any partition of $\mathbb{N}$ into finitely many cells, some cell contains distinct $x, y, z$ that satisfy the equation.

The case where the polynomial $p$ is linear was completely solved by Rado [24]: For $a, b, c \in \mathbb{N}$ the equation $ax + by = cz$ is partition regular if and only if either $a$, $b$, or $a + b$ is equal to $c$. The situation is much less clear for second or higher degree equations and only scattered results are known. A notorious old question of Erdős and Graham [6] is whether the equation $x^2 + y^2 = z^2$ is partition regular. As Graham remarks in [13] “There is actually very little data (in either direction) to know which way to guess”. More generally, one may ask for which $a, b, c \in \mathbb{N}$ is the equation

$$ax^2 + by^2 = cz^2$$

partition regular. A necessary condition is that at least one of $a, b$, and $a + b$ equals $c$, but currently there are no $a, b, c \in \mathbb{N}$ for which partition regularity of (1) is known.

In this article, we study the partition regularity of equation (1), and other quadratic equations, under the relaxed condition that the variable $z$ is allowed to vary freely in $\mathbb{N}$.

Definition. The equation $p(x, y, n) = 0$ is partition regular in $\mathbb{N}$ if for any partition of $\mathbb{N}$ into finitely many cells, for some $n \in \mathbb{N}$, one of the cells contains distinct $x, y$ that satisfy the equation.

A classical result of Furstenberg-Sárközy [17, 23] is that the equation $x - y = n^2$ is partition regular. Other examples of translation invariant equations are provided by the polynomial van der Waerden Theorem of Bergelson and Leibman [3], but not much is known in the non-translation invariant case. A result of Kalfalah and Szemerédi [21] is
that the equation \( x + y = n^2 \) is partition regular. Again, the situation is much less clear when one considers non-linear polynomials in \( x \) and \( y \), as is the case for the equation \( ax^2 + by^2 = n^2 \) where \( a, b \in \mathbb{N} \). It is one of the main goals of this article to produce the first positive results in this direction. For example, we show that the equations

\[
16x^2 + 9y^2 = n^2 \quad \text{and} \quad x^2 + y^2 - xy = n^2
\]

are partition regular (note that \( 16x^2 + 9y^2 = z^2 \) is not partition regular). In fact we prove a more general result for homogeneous quadratic forms in three variables.

**Theorem 1.1** (The three squares theorem). Let \( p \) be the quadratic form

\[
(2) \quad p(x, y, z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz
\]

where \( a, b \) are positive, \( c \) is negative, and \( d, e, f \) are arbitrary integers. Suppose that all three forms \( p(x, 0, z), p(0, y, z), p(x, x, z) \) have non-zero square discriminants. Then the equation \( p(x, y, n) = 0 \) is partition regular.

The last hypothesis means that the three integers

\[
\Delta_1 := e^2 - 4ac, \quad \Delta_2 := f^2 - 4bc, \quad \Delta_3 := (e + f)^2 - 4c(a + b + d)
\]

are non-zero squares. As a special case, we get the following result:

**Corollary 1.2.** Let \( a, b, \) and \( a + b \) be non-zero squares. Then the equation \( ax^2 + by^2 = n^2 \) is partition regular. More generally, if \( a, b, \) and \( a + b + c \) are non-zero squares, then the equation \( ax^2 + by^2 + cxy = n^2 \) is partition regular.

A partition \( \mathcal{C}_1, \ldots, \mathcal{C}_r \) of \( \mathbb{N} \) induces another partition \( \tilde{\mathcal{C}}_1, \ldots, \tilde{\mathcal{C}}_r \) by the following rule: \( x \in \mathcal{C}_i \) if and only if \( x^2 \in \mathcal{C}_i \). Applying Theorem 1.2 for the induced partition we deduce non-trivial results even for linear equations:

**Corollary 1.3.** Let \( a, b, \) and \( a + b \) be non-zero squares. Then the equation \( ax + by = n^2 \) is partition regular.

Although combinatorial tools, Fourier analysis tools, and the circle method have been used successfully to prove partition regularity of equations that enjoy some linearity features (also for non-linear equations with at least four variables), we have not found such tools adequate for the fully non-linear setup we are interested in. Instead, we found greater utility to the recently developed toolbox of higher order Fourier analysis that relies on inverse theorems for the Gowers uniformity norms and various quantitative equidistribution results on nilmanifolds. We give a summary of our proof strategy in the next subsections.

### 1.2. Parametric reformulation.

In order to prove Theorem 1.1 we exploit some special features of the solution sets of the equations involved given in parametric form. In particular, we have the following result that is proved in Appendix B.

**Proposition 1.4.** Let the quadratic form \( p \) satisfy the hypothesis of Theorem 1.1. Then there exist \( \ell_0, \ell_1 \) positive and \( \ell_2, \ell_3 \) non-negative integers with \( \ell_2 \neq \ell_3 \), such that for every \( k, m, n \in \mathbb{N} \), the integers \( x = k\ell_0 m(m + \ell_1 n) \) and \( y = k\ell_0 (m + \ell_2 n)(m + \ell_3 n) \) satisfy the equation \( p(x, y, z) = 0 \) for some \( z \in \mathbb{N} \).

For example, the equation \( 16x^2 + 9y^2 = z^2 \) is satisfied by the integers \( x = km(m + 3n) \), \( y = k(m + n)(m - 3n) \), \( z = k(5m^2 + 9n^2 + 6mn) \) (replacing \( m \) with \( m + 3n \) leads to coefficients of the announced form), and the equation \( x^2 + y^2 - xy = z^2 \) is satisfied by the integers \( x = km(m + 2n) \), \( y = k(m - n)(m + n) \), \( z = k(m^2 + n^2 + mn) \).

The key properties of the patterns involved in Proposition 1.4 are: (a) they are dilation invariant, which follows from homogeneity, (b) they “factor linearly” which follows from our assumption that the discriminants \( \Delta_1, \Delta_2 \) are squares, and (c) the coefficient of \( m \) in
all forms can be taken to be 1 which follows from our assumption that the discriminant \( \Delta_3 \) is a square.

Using Proposition \([1.4]\) we see that Theorem \([1.1]\) is a consequence of the following result:

**Theorem 1.5** (Parametric reformulation). Let \( \ell_0, \ell_1 \) be positive and \( \ell_2, \ell_3 \) non-negative integers with \( \ell_2 \neq \ell_3 \). Then for every partition of \( \mathbb{N} \) into finitely many cells, there exist \( k, m, n \in \mathbb{N} \) such that the integers \( k\ell_0(m + \ell_1n) \) and \( k\ell_0(m + \ell_2n)(m + \ell_3n) \) are distinct and belong to the same cell.

1.3. From partition regularity to multiplicative functions. Much like the translation invariant case, where partition regularity results can be deduced from corresponding density statements with respect to a translation invariant density, we deduce Theorem \([1.5]\) from the density regularity result of Theorem \([2.1]\) that involves a dilation invariant density (a notion defined in Section \([2.4]\)).

In Section \([2.2]\) we use a multiplicative version of the correspondence principle of Furstenberg to recast Theorem \([2.1]\) as a recurrence property for measure preserving actions of the multiplicative semigroup \( \mathbb{N} \) on a probability space (Theorem \([2.2]\)).

In Section \([2.3]\) we use a corollary of the spectral theorem for unitary operators (see identity \([15]\)) to transform the recurrence result into a positivity property for an integral of averages of products of multiplicative functions (Theorem \([2.3]\)). It is then this positivity property that we seek to prove, and the heavy-lifting is done by a decomposition result for multiplicative functions which is the main number theoretic result of this article (Theorem \([4.0]\)). Assuming this result (we prove it in Sections \([3.0]\)), the proof of the positivity property of Theorem \([2.5]\) is completed in Section \([2.6]\). The reader will also find there a detailed sketch of our proof strategy for this step. We discuss the decomposition result next.

1.4. Multiplicative functions and Gowers uniformity. Our proof of the positivity property mentioned above necessitates that we decompose an arbitrary multiplicative function into two components, one that we can easily control, and another that behaves randomly enough to have a negligible contribution. For our purposes, randomness is measured by the Gowers uniformity norms. Before proceeding to the precise statement of the decomposition result we start with some informal discussion regarding the uniformity norms and the uniformity properties (or lack thereof) of multiplicative functions.

Gowers uniformity. We recall the definition of the \( U^2 \) and \( U^3 \)-Gowers uniformity norm from \([9]\). Here and later, for a function \( f \) defined on a finite set \( A \) we write

\[
E_{x \in A} f(x) := \frac{1}{|A|} \sum_{x \in A} f(x).
\]

**Definition** (Gowers uniformity norms). Given \( N \in \mathbb{N} \) and \( f : \mathbb{Z}_N \to \mathbb{C} \), we define the \( U^2(\mathbb{Z}_N) \)-Gowers norm of \( f \) as follows

\[
\| f \|_{U^2(\mathbb{Z}_N)}^4 = E_{h \in \mathbb{Z}_N} |E_{n \in \mathbb{Z}_N} f(n + h) \cdot \overline{f}(n)|^2;
\]

and the \( U^3(\mathbb{Z}_N) \)-Gowers norm of \( f \) as follows

\[
\| f \|_{U^3(\mathbb{Z}_N)}^8 = E_{h_1, h_2 \in \mathbb{Z}_N} |E_{n \in \mathbb{Z}_N} f(n + h_1 + h_2) \cdot \overline{f}(n + h_1) \cdot \overline{f}(n + h_2) \cdot f(n)|^2.
\]

In an informal way, having a small \( U^2 \)-norm is interpreted as a property of \( U^2 \)-uniformity, and having a small \( U^3 \)-norm as a stronger property of \( U^3 \)-uniformity. Since

\[
\| f \|_{U^3(\mathbb{Z}_N)} \geq \| f \|_{U^2(\mathbb{Z}_N)}^2
\]

for every function \( f \) on \( \mathbb{Z}_N \), \( U^3 \)-uniformity implies \( U^2 \)-uniformity.

We recall that the **Fourier transform** of a function \( f \) on \( \mathbb{Z}_N \) is defined by

\[
\hat{f}(\xi) := E_{n \in \mathbb{Z}_N} f(n) e(-n\xi/N) \quad \text{for} \quad \xi \in \mathbb{Z}_N,
\]
where, as is standard, $e(x) := \exp(2\pi i x)$. A direct computation gives the following identity that links the $U^2$-norm of a function $f$ on $\mathbb{Z}_N$ with its Fourier coefficients:

$$
\|f\|_{U^2(\mathbb{Z}_N)}^4 = \sum_{\xi \in \mathbb{Z}_N} |\hat{f}(\xi)|^4.
$$

Using this and Parseval’s identity, we see that the $U^2$-uniformity of a function can be interpreted as the function having small Fourier coefficients, that is, small correlation with linear phases. We would like to stress though, that this property does not guarantee $U^3$-uniformity; a function bounded by 1 may have small Fourier coefficients, but large $U^3$-norm. In fact, eliminating all possible obstructions to $U^3$-uniformity necessitates the study of correlations with all quadratic phases $e(n^2\alpha + n\beta)$ and the larger class of 2-step nilsequences of bounded complexity (see Theorem 4.2).

**Multiplicative functions.** In this article, we slightly abuse standard terminology\footnote{The functions we call multiplicative are often called completely multiplicative as opposed to functions like the Möbius that are called multiplicative.} and define multiplicative functions as follows:

**Definition (Multiplicative functions).** A multiplicative function is a function $\chi : \mathbb{N} \to \mathbb{C}$ that satisfies $\chi(mn) = \chi(m)\chi(n)$ for all $m, n \in \mathbb{N}$. We denote by $\mathcal{M}$ the family of multiplicative functions of modulus 1.

Note that elements of $\mathcal{M}$ are determined by their values on the primes.

It so happens that several multiplicative functions do not share the strong uniformity properties of the Möbius function established in [18]. The next example illustrates some simple but very important obstructions to uniformity.

**Example (Obstructions to uniformity).** Let $\chi \in \mathcal{M}$ be defined by $\chi(2) = -1$ and $\chi(p) = 1$ for every prime $p \neq 2$. Equivalently, $\chi(2^m(2k + 1)) = (-1)^m$ for all $k, m \geq 0$. Then $\mathbb{E}_{1 \leq n \leq N}|\chi(n)| = 1/3 + o(1)$ and this non-zero mean already gives an obstruction to $U^2$-uniformity. But this is not the only obstruction. Indeed, we have $\mathbb{E}_{1 \leq n \leq N}(-1)^n\chi(n) = -2/3 + o(1)$, and this implies that $\chi - 1/3$ does not have small $U^2$-norm.

Examples similar to the previous one show that normalized multiplicative functions can have significant correlation with periodic phases and thus this is an obstruction to $U^2$-uniformity that we should take into account. However, it is a non-trivial fact that plays a central role in this article, that correlation with periodic phases are, in a sense to be made precise later, the only obstructions not only to $U^2$-uniformity but also to $U^3$-uniformity of multiplicative functions. For $U^2$-uniformity this is already indicated by an old result of Daboussi [3], which states that if $\alpha$ is irrational, then $\sup_{\chi \in \mathcal{M}} |\mathbb{E}_{1 \leq n \leq N}\chi(n)e(n\alpha)| \to 0$ as $N \to \infty$. The proof of this result was later simplified and extended by Kátai [20] (for good quantitative versions of these results see [4, 23]), and it is a simple orthogonality criterion obtained in this article of Kátai (see Lemma 3.2) that is going to be a key number theoretic input for this article.

1.5. $U^3$-Decomposition of multiplicative functions.** We proceed now to the formal statement of our main decomposition result. We are given positive integers $\ell_1, \ell_2, \ell_3$ and these are considered fixed through this article. From this point on we let $\ell := \ell_1 + \ell_2 + \ell_3$.

Also, given $N \in \mathbb{N}$ we let

$$
[N] := \{1, \ldots, N\}.
$$

It is often easier to work on a cyclic group rather than an interval of integers, as this makes Fourier analysis tools more readily available. In order to avoid roundabout issues, we introduce the following notation. Given $N \in \mathbb{N}$, we denote by $\bar{N}$ the smallest prime
that is greater than $10\ell N$. By Bertrand’s postulate we have $\tilde{N} \leq 20\ell N$. For every multiplicative function $\chi \in \mathcal{M}$ and every $N \in \mathbb{N}$, we denote by $\chi_N$ the function on $\mathbb{Z}_N\tilde{N}$, or on $[\tilde{N}]$, defined by

$$\chi_N(n) = \begin{cases} 
\chi(n) & \text{if } n \in [N]; \\
0 & \text{otherwise}.
\end{cases}$$

(4)

The domain of $\chi_N$ will be each time clear from the context. Working with the truncated function $\chi \cdot 1_{[N]}$, rather than the function $\chi$, is a technical maneuver and the reader will not lose much by ignoring the cutoff. We should stress that from this point on, Gowers norms are going to be defined and Fourier analysis is going to happen on the group $\mathbb{Z}_\tilde{N}$ and not on the group $\mathbb{Z}_N$.

We can now state the main decomposition result that will be used below in the proof of the theorems of partition regularity. Its essence is that the restriction of an arbitrary multiplicative function $\chi \in \mathcal{M}$ on a finite interval $[N]$ can be decomposed into three pieces, one that is approximately periodic, one that has small $L^1$-norm, and one that has extremely small $U^3$-norm. In addition, the structured component enjoys some very important features, for example, it is a convolution product of $\chi_N$ with a positive kernel that is independent of $\chi$ and its approximate period is bounded by a constant that does not depend on $\chi$ or on $N$.

**Definition.** By a kernel on $\mathbb{Z}_\tilde{N}$ we mean a non-negative function with average 1.

**Theorem 1.6** (Strong decomposition on average for the $U^3$-norm). For every positive finite measure $\nu$ on the compact group $\mathcal{M}$ of multiplicative functions having modulus 1, every function $F : [N] \times [N] \times \mathbb{R}_+ \to \mathbb{R}_+$, and every $\varepsilon > 0$, and every sufficiently large $N \in \mathbb{N}$, depending only on $F$ and $\varepsilon$, there exist positive integers $Q$ and $R$ that are bounded by a constant which depends only on $F$ and $\varepsilon$, such that, for every $\chi \in \mathcal{M}$, the function $\chi_N$ admits the decomposition

$$\chi_N(n) = \chi_{N,s}(n) + \chi_{N,u}(n) + \chi_{N,e}(n) \quad \text{for every } n \in \mathbb{Z}_\tilde{N},$$

where $\chi_{N,s}$, $\chi_{N,u}$, and $\chi_{N,e}$ satisfy the following properties:

(i) $\chi_{N,s} = \chi_N \ast \psi_{N,1}$ and $\chi_{N,u} = \chi_N \ast \psi_{N,2}$, where $\psi_{N,1}$ and $\psi_{N,2}$ are kernels on $\mathbb{Z}_\tilde{N}$ that do not depend on $\chi$, and the convolution product is defined in $\mathbb{Z}_\tilde{N}$;

(ii) $|\chi_{N,s}(n + Q) - \chi_{N,s}(n)| \leq \frac{R}{N}$ for every $n \in \mathbb{Z}_\tilde{N}$, where $n + Q$ is taken mod $\tilde{N}$;

(iii) $\|\chi_{N,u}\|_{U^3(\mathbb{Z}_\tilde{N})} \leq \frac{1}{F(Q, R, \varepsilon)}$;

(iv) $\mathbb{E}_{n \in \mathbb{Z}_\tilde{N}} \int_{\mathcal{M}} |\chi_{N,e}(n)| \, d\nu(\chi) \leq \varepsilon$.

**Remarks.** (1) For arbitrary bounded sequences, decomposition results with similar flavor have been proved in [10, 11, 16, 26, 27], but working in this generality necessitates the use of structured components that do not satisfy the strong rigidity condition of Property (ii). An additional important feature of our result is that the structured component is defined by a convolution product with a kernel that is independent of $\chi$. All these properties play an important role in the derivation of the combinatorial result in Section 2.

(2) We only plan to use this theorem for the function $F(x, y, z) = cx^2y^2/z^3$ where $c$ is a constant that depends on $\ell$ only. Restricting the statement to this function does not simplify our proof though.

(3) The bound [16] is not uniform in $\chi$ as in part (iii) of Theorem 5.1 below. We do not know if this weakening on the bound is needed or is an artifact of our proof.
(4) It is a consequence of Property 1 that for fixed \(F, N, \epsilon, \nu\), the maps \(\chi \mapsto \chi_s, \chi \mapsto \chi_u, \chi \mapsto \chi_e\) are continuous, and \(\|\chi_s\|_{L^\infty(\mathbb{Z}/N)} \leq 1, \|\chi_u\|_{L^\infty(\mathbb{Z}/N)} \leq 2, \|\chi_e\|_{L^\infty(\mathbb{Z}/N)} \leq 2\).

Most of the work in the proof of Theorem 6.6 goes into verifying the decomposition result of Theorem 6.1 that gives weaker bounds on the uniform component of the decomposition. Two ideas that play a prominent role in its proof, roughly speaking, are:

(a) A multiplicative function that has \(U^2\)-norm bounded away from zero correlates with a linear phase that has frequency close to a rational with small denominator.

(b) A multiplicative function that has \(U^3\)-norm bounded away from zero necessarily has \(U^2\)-norm bounded away from zero.

The proof of (a) uses classical Fourier analysis tools and is given in Section 3. The key number theoretic input is the orthogonality criterion of Kátai stated in Lemma 3.2.

The proof of (b) is much harder and is done in several steps using higher order Fourier analysis machinery. In Section 5 we combine Kátai’s criterion with a quantitative equidistribution result on nilmanifolds of Green and Tao (Theorem 5.1) to study the correlation of multiplicative functions with nilsequences. These results are then combined in Section 6 with modifications of the \(U^3\)-inverse theorem (Theorem 5.2) and the factorization result (Theorem 4.1) of Green and Tao, to conclude the proof of the weak decomposition result of Theorem 6.1. We defer the reader to Section 6.1 for a more detailed sketch of our proof strategy.

Upon proving the weak decomposition result of Theorem 6.1, the proof of Theorem 1.6 consists of a Fourier analysis energy increment argument, and avoids the use of finitary ergodic theory and the use of the Hahn-Banach theorem, tools that are typically used for other decomposition results (see [10, 11, 12, 16, 27]).

1.6. Further directions. Theorem 1.1 establishes that the equation

\[ax^2 + by^2 = cn^2\]

is partition regular provided that all three integers \(ac, bc, (a + b)c\), are non-zero squares. Two interesting cases, not covered by the previous result, are the following:

**Problem 1.** Are the equations \(x^2 + y^2 = n^2\) and \(x^2 + y^2 = 2n^2\) partition regular?\(^2\)

Let us explain why we cannot yet handle these equations using the methods of this article. The equation \(x^2 + y^2 = 2z^2\) has the following solutions: \(x = k(m^2 - n^2 + 2mn), y = k(m^2 - n^2 - 2mn), z = k(m^2 + n^2)\), where \(k, m, n \in \mathbb{Z}\). The values of \(x\) and \(y\) do not factor in linear terms, which leads to the major obstacle of not being able to establish uniformity estimates analogous to the ones stated in Lemma 2.7. The equation \(x^2 + y^2 = z^2\) has the following solutions: \(x = k(m^2 - n^2), y = 2kmn, z = k(m^2 + n^2)\). In this case, it is possible to establish the needed uniformity estimates but we are not able to carry out the argument of Section 2.6 in order to prove the relevant positivity property (see footnote 3 below for more details).

A set \(E \subset \mathbb{N}\) has positive (additive) density if \(\limsup_{N \to \infty} |E \cap [N]|/N > 0\). It turns out that the equations of Corollary 1.2 have non-trivial solutions on every infinite arithmetic progression, making the following statement plausible:

**Problem 2.** Does every set \(E \subset \mathbb{N}\) with positive density contain distinct \(x, y \in \mathbb{N}\) that satisfy the equation \(16x^2 + 9y^2 = n^2\) for some \(n \in \mathbb{N}\)?

We say that the equation \(p(x, y, z) = 0, p \in \mathbb{Z}[x, y, z]\), has no local obstructions if for every infinite arithmetic progression \(P\), there exist distinct \(x, y, z \in P\) that satisfy the

\(^2\)Note that the equation \(x^2 + y^2 = 3n^2\) does not have solutions in \(\mathbb{N}\). Furthermore, the equation \(x^2 + y^2 = 5n^2\) has solutions in \(\mathbb{N}\) but it is not partition regular. Indeed, if we partition the integers in 6 cells according to whether their first non-zero digit in the 7-adic expansion is 1, 2, . . . , 6, it turns out that for every \(n \in \mathbb{N}\) the equation has no solution on any single partition cell.
equation. For example, the equations \( x^2 + y^2 = 2z^2 \) and \( 16x^2 + 9y^2 = 25z^2 \) have no local obstructions.

**Problem 3.** Let \( p \in \mathbb{Z}[x,y,z] \) be a homogeneous quadratic form and suppose that the equation \( p(x,y,z) = 0 \) has no local obstructions. Is it true that every subset of \( \mathbb{N} \) of positive density contains distinct \( x,y,z \) that satisfy the equation?

As we mentioned before, there are no values of \( a, b, c \in \mathbb{N} \) for which the equation
\[
ax^2 + by^2 = cz^2
\]
is known to be partition regular and the condition “at least one of \( a, b, \) and \( a + b \) equals \( c \)” is necessary for partition regularity.

**Problem 4.** Are there \( a, b, c \in \mathbb{N} \) for which equation (6) is partition regular?

Notable examples are the equations \( x^2 + y^2 = z^2 \) and \( x^2 + y^2 = 2z^2 \). In [19] it is conjectured that the second equation is partition regular.

### 1.7. Notation and conventions.

We denote by \( \mathbb{N} \) the set of positive integers. For \( N \in \mathbb{N} \) we denote by \([N]\) the set \( \{1, \ldots, N\} \).

For a function \( f \) defined on a finite set \( A \) we write \( \mathbb{E}_{x \in A} f(x) = \frac{1}{|A|} \sum_{x \in A} f(x) \).

With \( \mathcal{M} \) we denote the set of multiplicative functions \( \chi: \mathbb{N} \to \mathbb{C} \) with modulus 1.

Throughout, we assume that we are given \( \ell_1, \ell_2, \ell_3 \in \mathbb{N} \) and we set \( \ell = \ell_1 + \ell_2 + \ell_3 \).

A kernel on \( \mathbb{Z}_N \) is a non-negative function on \( \mathbb{Z}_N \) with average 1.

For \( N \in \mathbb{N} \) we let \( \tilde{N} \) be the smallest prime that is larger than \( 10\ell N \) (then \( \tilde{N} \leq 20eN \)).

Given \( \chi \in \mathcal{M} \) and \( N \in \mathbb{N} \) we let \( \chi_N: [\tilde{N}] \to \mathbb{C} \) be defined by \( \chi_N = \chi \cdot 1_{[N]} \). The domain of \( \chi_N \) is sometimes thought to be \( \mathbb{Z}_{\tilde{N}} \).

For technical reasons, throughout the article all Fourier analysis happens on \( \mathbb{Z}_{\tilde{N}} \) and all uniformity norms are defined on \( \mathbb{Z}_{\tilde{N}} \).

If \( x \) is a real, \( e(x) \) denotes the number \( \exp(2\pi ix) \), \( \|x\| \) denotes the distance between \( x \) and the nearest integer, \( \lfloor x \rfloor \) the largest integer smaller or equal than \( x \), and \( \lceil x \rceil \) the smallest integer greater or equal than \( x \).

Given \( s \in \mathbb{N} \) we write \( \mathbf{k} = (k_1, \ldots, k_s) \) for a point of \( \mathbb{Z}^s \) and \( \|\mathbf{k}\| = |k_1| + \cdots + |k_s| \). For \( \mathbf{u} = (u_1, \ldots, u_s) \in \mathbb{T}^s \), we write \( \mathbf{k} \cdot \mathbf{u} = k_1u_1 + \cdots + k_su_s \).

Let \( f \) be a function on a metric space \( X \) with distance \( d \). We define
\[
\|f\|_{\text{Lip}(X)} = \sup_{x \in X} |f(x)| + \sup_{x,y \in X, x \neq y} \frac{|f(x) - f(y)|}{d(x,y)}.
\]

There is a proliferation of constants in this article and our general principles are as follows: The constants \( \ell_1, \ell_2, \ell_3 \), are considered as fixed throughout the article, and quantities depending only on these numbers are considered as universal constants. The letters \( \ell_0, \ell, c, c_1, c_2, \ldots \) are reserved for constants of this type independently of whether they represent small or large quantities. Quantities that depend on one or more variables are denoted by Roman capital letters \( C, D, K, \ldots \) if they represent large quantities, and by low case Greek letters \( \gamma, \delta, \varepsilon, \ldots \) if they represent small quantities. It will be very clear from the context when we deviate from these rules.

### 2. Proof of partition regularity assuming the decomposition result.

The goal of this section is to prove the combinatorial Theorem 1.5 assuming the decomposition result of Theorem 1.6 (this is proved in Sections 3-6). We begin by giving three successive reformulations of Theorem 1.5.
2.1. **Reduction to a density regularity result.** We first recast Theorem 1.3 as a density regularity statement for dilation invariant densities on the integers.

**Definition** (Multiplicative Følner sequence). The sequence \((\Phi_N)_{N \in \mathbb{N}}\) of finite subsets of \(\mathbb{N}\) is a multiplicative Følner sequence if for every \(n \in \mathbb{N}\)

\[
\lim_{N \to +\infty} \frac{|n\Phi_N \triangle \Phi_N|}{|\Phi_N|} = 0
\]

where \(n\Phi_N := \{nx : x \in \Phi_N\}\).

The sequence \((\Phi_N)_{N \in \mathbb{N}}\) defined by \(\Phi_N := \{p_1^{k_1} \cdots p_N^{k_N} : 0 \leq k_1, \ldots, k_N \leq N\}\), where \(p_1, p_2, \ldots\) is the sequence of primes, serves as a typical example. To a given multiplicative Følner sequence we associate a notion of multiplicative density as follows:

**Definition** (Multiplicative density). The multiplicative density \(d_{\text{mult}}(E)\) of a subset \(E\) of \(\mathbb{N}\) (relatively to the multiplicative Følner sequence \((\Phi_N)\)) is defined as

\[
d_{\text{mult}}(E) := \limsup_{N \to +\infty} \frac{|E \cap \Phi_N|}{|\Phi_N|}.
\]

We remark that the multiplicative density and the additive density are non-comparable measures of largeness. For instance, the set of odd numbers has zero multiplicative density with respect to any multiplicative Følner sequence, as has any set that omits all multiples of some positive integer. On the other hand, it is not hard to construct sets with multiplicative density \(1\) that have additive density \(0\) (see for instance [1]).

An important property of the multiplicative density, and the reason we work with this notion of largeness, is that for every \(E \subset \mathbb{N}\) and \(n \in \mathbb{N}\), we have

\[
d_{\text{mult}}(nE) = d_{\text{mult}}(E) = d_{\text{mult}}(n^{-1}E), \quad \text{where} \quad n^{-1}E := \{x \in \mathbb{N} : nx \in E\}.
\]

Since a multiplicative density is clearly sub-additive, any finite partition of \(\mathbb{N}\) has at least one cell with positive multiplicative density. Hence, Theorem 1.3 follows from the following stronger result:

**Theorem 2.1** (Density regularity). Let \(\ell_0, \ell_1, \ell_2, \ell_3\) be as in Proposition 1.4. Let \(E \subset \mathbb{N}\) be a set with positive multiplicative density. Then there exist \(k, m, n \in \mathbb{N}\) such that the integers \(k\ell_0(m + \ell_1n)\) and \(k\ell_0(m + \ell_2n)(m + \ell_3n)\) are distinct and belong to \(E\).

In fact, we show that for a set of \((m, n)\) in \(\mathbb{N}^2\) of positive (additive) density the asserted property holds for a set of \(k \in \mathbb{N}\) of positive multiplicative density.

2.2. **Reduction to a recurrence results for actions by dilations.** Our next goal is to reformulate the density statement of Theorem 2.1 as a recurrence statement in ergodic theory.

**Definition.** An action by dilations on a probability space \((X, \mathcal{B}, \mu)\) is a family \((T_n)_{n \in \mathbb{N}}\) of invertible measure preserving transformations of \((X, \mathcal{B}, \mu)\) that satisfy

\[
T_1 := \text{id} \quad \text{and for every} \quad m, n \in \mathbb{N}, \quad T_m \circ T_n = T_{mn}.
\]

We remark that an action by dilations on a probability space \((X, \mathcal{B}, \mu)\) can be extended to a measure preserving action \((T_r)_{r \in \mathbb{Q}^+}\) of the multiplicative group \(\mathbb{Q}^+\) by defining

\[
T_{a/b} := T_a T_b^{-1} \quad \text{for all} \quad a, b \in \mathbb{N}.
\]

We will use a multiplicative version of the (additive) correspondence principle of Furstenberg [8]. Its proof can be found in [1].

**Multiplicative correspondence principle.** Let \(E\) be a subset of \(\mathbb{N}\). Then there exist an action by dilations \((T_n)_{n \in \mathbb{N}}\) on a probability space \((X, \mathcal{B}, \mu)\), and a set \(A \in \mathcal{B}\) with \(\mu(A) = d_{\text{mult}}(E)\), such that for every \(k \in \mathbb{N}\) and for all \(n_1, n_2, \ldots, n_k \in \mathbb{N}\), we have

\[
d_{\text{mult}}(n_1^{-1}E \cap n_2^{-1}E \cap \cdots \cap n_k^{-1}E) \geq \mu(T_{n_1}^{-1}A \cap T_{n_2}^{-1}A \cap \cdots \cap T_{n_k}^{-1}A).
\]
Using this correspondence principle we can recast Theorem 2.1 as a recurrence statement in ergodic theory regarding actions by dilations.

**Theorem 2.2 (Recurrence).** Let \( \ell_1, \ell_2, \ell_3 \) be as in Proposition 1.4. Let \((T_n)_{n \in \mathbb{N}}\) be an action by dilations on a probability space \((X, \mathcal{B}, \mu)\). Then for every \( A \in \mathcal{B} \) with \( \mu(A) > 0 \), there exist \( m, n \in \mathbb{N} \), such that the integers \( m + \ell_1 n \) and \((m + \ell_2 n)(m + \ell_3 n)\) are distinct, and

\[
\mu(T_{m(m + \ell_1 n)}^{-1} A \cap T_{(m + \ell_2 n)(m + \ell_3 n)}^{-1} A) > 0.
\]

**Proof of Theorem 2.2 assuming Theorem 2.3.** Let \( E \subset \mathbb{N} \) have positive multiplicative density. Let the probability space \((X, \mathcal{B}, \mu)\), the action \((T_n)_{n \in \mathbb{N}}, \) and the set \( A \in \mathcal{B} \) be associated to \( E \) by the previous correspondence principle. Let \( \ell_0, \ell_1, \ell_2, \ell_3 \) be as in Proposition 1.4. In order to show that there exist integers \( k, m, n \in \mathbb{N} \) satisfying the conclusions of Theorem 2.1, it suffices to show that there exist \( m, n \in \mathbb{N} \) so that the integers \( \ell_0 m + \ell_1 n \) and \( \ell_0 (m + \ell_2 n)(m + \ell_3 n) \) are distinct, and satisfy

\[
\mu(T_{\ell_0 m + \ell_1 n}^{-1} A \cap T_{\ell_0 (m + \ell_2 n)(m + \ell_3 n)}^{-1} A) > 0.
\]

Since \( \mu \) is \( T_{\ell_0} \)-invariant, the left hand side equals \( \mu(T_{\ell_0 m + \ell_1 n}^{-1} A \cap T_{\ell_0 (m + \ell_2 n)(m + \ell_3 n)}^{-1} A) \), and the existence of \( m, n \in \mathbb{N} \) satisfying the asserted properties follows from Theorem 2.2.

The degenerate case where \( \ell_1 = \ell_2 \) (and similarly if \( \ell_1 = \ell_3 \) or \( \ell_2 \ell_3 = 0 \)) is rather trivial. Indeed, we are then reduced to establishing positivity for \( \mu(T_{m}^{-1} A \cap T_{n}^{-1} A) \)

Letting \( m = \ell_3, n = 2^{n'} - 1 \), we further reduce matters to showing that \( \mu(A \cap T_{n}^{-1} A) > 0 \) for some \( n' \in \mathbb{N} \), and this follows from the Poincaré recurrence theorem applied to \( T_2 \).

Therefore, in the rest of this article, we can and will assume that \( \ell_1, \ell_2, \ell_3 \) are distinct positive integers. In this case, our goal is to show that the set of pairs \((m, n)\) satisfying the conclusion of Theorem 2.2 has positive (additive) density in \( \mathbb{N}^2 \).

**Theorem 2.3 (Recurrence on the average).** Let \( \ell_1, \ell_2, \ell_3 \in \mathbb{N} \) be distinct. Let \((T_n)_{n \in \mathbb{N}}\) be an action by dilations on a probability space \((X, \mathcal{B}, \mu)\). Then for every \( A \in \mathcal{B} \) with \( \mu(A) > 0 \) we have

\[
\liminf_{N \to \infty} \mathbb{E}_{(m, n) \in \Theta_N} \mu(T_{m(m + \ell_1 n)}^{-1} A \cap T_{(m + \ell_2 n)(m + \ell_3 n)}^{-1} A) > 0
\]

where \( \Theta_N = \{(m, n) \in [N] \times [N] : 1 \leq m + \ell_i n \leq N \text{ for } i = 1, 2, 3\} \).

**Remark.** In fact, we prove more: the \( \liminf \) is greater or equal than a positive constant that depends only on the measure of the set \( A \).

**Proof of Theorem 2.2 assuming Theorem 2.3.** It suffices to notice that for \( N \) sufficiently large we have \( |\Theta_N| \geq c_1 N^2 \) and the cardinality of the set of pairs \((m, n) \in \Theta_N\) that satisfy \( m + \ell_1 n = (m + \ell_2 n)(m + \ell_3 n) \) is bounded by \( c_2 N \) for some constants \( c_1 \) and \( c_2 \) that depend only on \( \ell_1, \ell_2, \ell_3 \).

2.3. Reduction to a positivity property for multiplicative functions. Next, we show that Theorem 2.2 is equivalent to a positivity property for multiplicative functions.

Recall that the set \( \mathcal{M} \) consists of all multiplicative functions of modulus 1. When endowed with the topology of pointwise convergence, \( \mathcal{M} \) is a compact (metrizable) Abelian group. If \( \{p_1, p_2, \ldots\} \) denotes the set of primes, then a multiplicative function \( \chi \) is determined by its values on the primes. The map \( \chi \mapsto (\chi(p_n))_{n \in \mathbb{N}} \) is an isomorphism between the groups \( \mathcal{M} \) and \( \mathbb{Q}^\times \). The space of multiplicative functions \( \mathcal{M} \) is the dual group of the multiplicative group \( \mathbb{Q}^+ \), the duality being given by

\[
\chi(m/n) = \chi(m) / \chi(n) \quad \text{for every } \chi \in \mathcal{M} \text{ and every } m, n \in \mathbb{N}.
\]

Recall that an action \((T_n)_{n \in \mathbb{N}}\) by dilations on a probability space \((X, \mathcal{B}, \mu)\) extends to a measure preserving action of the multiplicative group \( \mathbb{Q}^+ \) on the same space. Since \( \mathcal{M} \)
is the dual group of this countable Abelian group, by the spectral theorem for unitary operators, for every function \( f \in L^2(\mu) \) there exists a positive finite measure \( \nu \) on the compact Abelian group \( \mathcal{M} \), called the spectral measure of \( f \), such that, for all \( m, n \in \mathbb{N} \),

\[
\int_X T_m f \cdot T_n \mathcal{T} d\mu = \int T_{m/n} f \cdot \mathcal{T} d\mu = \int_{\mathcal{M}} \chi(m/n) d\nu(\chi) = \int_{\mathcal{M}} \chi(m) \overline{\chi}(n) d\nu(\chi).
\]

Let \( A \in \mathcal{B} \). Letting \( f = 1_A \) in (7) and using the multiplicativity of elements of \( \mathcal{M} \), we get for all \( m, n \in \mathbb{N} \) that

\[
\mu(T_{m+\ell n}^{-1} A \cap T_{m+\ell 2n}^{-1} (m+\ell 3n) A) = \int_{\mathcal{M}} \chi(m+\ell n) \overline{\chi}(m+\ell 2n)(m+\ell 3n) d\nu(\chi) = \int_{\mathcal{M}} \chi(m) \chi(m+\ell 1n) \overline{\chi}(m+\ell 2n) \overline{\chi}(m+\ell 3n) d\nu(\chi).
\]

From this identity we deduce that Theorem 2.3 is equivalent to the following result:

**Theorem 2.4** (Spectral reformulation of recurrence result I). Let \( \ell_1, \ell_2, \ell_3 \in \mathbb{N} \) be distinct and \( (T_n)_{n \in \mathbb{N}} \) be an action by dilations on a probability space \((X, \mathcal{B}, \mu)\). Then for every \( A \in \mathcal{B} \) with \( \mu(A) > 0 \), writing \( \nu \) for the spectral measure of \( 1_A \), we have

\[
\liminf_{N \to \infty} \int_{\mathcal{M}} \mathbb{E}_{m,n} \chi(m) \chi(m+\ell 1n) \overline{\chi}(m+\ell 2n) \overline{\chi}(m+\ell 3n) d\nu(\chi) > 0
\]

where \( \Theta_N \) is as in Theorem 2.3.

**Remark.** An alternate (and arguably more natural) way to try to prove Theorem 2.4 is to replace the additive averages in Theorem 2.3 with multiplicative ones. Upon doing this, one is required to analyze averages of the form

\[
\mathbb{E}_{m,n} \phi_N \chi(m+\ell 1n) \overline{\chi}(m+\ell 2n) \overline{\chi}(m+\ell 3n)
\]

where \((\phi_N)_{N \in \mathbb{N}}\) is a multiplicative Følner sequence in \(\mathbb{N}\) and \(\chi \in \mathcal{M}\). Unfortunately, we were not able to prove anything useful for these multiplicative averages, although one suspects that a positivity property similar to the one in (8) may hold.

Next, for technical reasons we reformulate Theorem 2.4 as a positivity property involving averages over \(\mathbb{Z}_N\). This is going to be the final form of the recurrence statement that we aim to study. Recall that \(\tilde{N}\) was defined in Section 1.3 and the functions \(\chi_N\) on \(\mathbb{Z}_{\tilde{N}}\) were defined by (1) in the same section.

**Theorem 2.5** (Spectral reformulation of recurrence result II). Let \( (T_n)_{n \in \mathbb{N}} \) be an action by dilations on a probability space \((X, \mathcal{B}, \mu)\). Then for every \( A \in \mathcal{B} \) with \( \mu(A) > 0 \), writing \( \nu \) for the spectral measure of \( 1_A \), we have

\[
\liminf_{N \to \infty} \int_{\mathcal{M}} \mathbb{E}_{m,n} 1_{[N]}(n) \chi_N(m) \chi_N(m+\ell 1n) \overline{\chi}_N(m+\ell 2n) \overline{\chi}_N(m+\ell 3n) d\nu(\chi) > 0
\]

where in the above average the expressions \(m + \ell n\) can be considered as elements of \(\mathbb{Z}\) or \(\mathbb{Z}_{\tilde{N}}\) without affecting the value of the average.

We check that Theorems 2.4 and 2.5 are equivalent. Using the definition of the set \(\Theta_N\) given in Theorem 2.3, we can rewrite the averages that appear in the statement of Theorem 2.4 as follows

\[
\mathbb{E}_{m,n} \chi(m) \chi(m+\ell 1n) \overline{\chi}(m+\ell 2n) \overline{\chi}(m+\ell 3n) = \frac{\tilde{N}^2}{|\Theta_N|} \mathbb{E}_{m,n} 1_{[N]}(n) \chi_N(m) \chi_N(m+\ell 1n) \overline{\chi}_N(m+\ell 2n) \overline{\chi}_N(m+\ell 3n).
\]

The value of the last expression remains unchanged when we replace each term \(m + \ell n\) by \(m + \ell n \mod \tilde{N}\). Using this identity and that \(cN^2 \leq |\Theta_N| \leq N^2\) for some positive constant \(c\) that depends only on \(\ell\), we get the asserted equivalence.
2.4. Some estimates involving Gowers norms. Next we establish two elementary estimates that will be used in the sequel. The first one will be used in Section 6.

**Lemma 2.6.** Let $N$ be prime. For every function $a: \mathbb{Z}_N \to \mathbb{C}$ and for every arithmetic progression $P$ contained in the interval $[N]$, we have

$$|E_{n \in [N]} 1_P(n) \cdot a(n)| \leq c_1 \|a\|_{U^2(\mathbb{Z}_N)}$$

for some universal constant $c_1$.

**Proof.** Since $N$ is prime, the $U^2$-norm of a function on $\mathbb{Z}_N$ is invariant under any change of variables of the form $x \mapsto ax + b$, where $a, b \in \mathbb{N}$ and $a \not\equiv 0 \mod N$. By a change of variables of this type, we are reduced to the case that $P$ is an interval $\{0, \ldots, m\}$ with $0 \leq m < N$, considered as a subset of $\mathbb{Z}_N$. A direct computation then shows that

$$|\hat{1}_P(\xi)| \leq \frac{2}{N||\xi/N||} = \frac{2}{\min\{\xi, N - \xi\}} \quad \text{for} \quad \xi = 1, \ldots, N - 1,$$

and as a consequence

$$||1_P(\xi)||_{U^3([N])} \leq c_1$$

for some universal constant $c_1$. Using this estimate, Parseval’s identity, Hölder’s inequality, and identity (3), we deduce that

$$|E_{n \in [N]} 1_P(n) \cdot a(n)| = |\sum_{\xi \in [N]} \hat{1}_P(\xi) \cdot \hat{a}(\xi)| \leq c_1 \cdot \left(\sum_{\xi \in [N]} |\hat{a}(\xi)|^4\right)^{1/4} = c_1 \|a\|_{U^2(\mathbb{Z}_N)}. \quad \square$$

The next estimate is key for the proof of Theorem 2.8. It is the reason we seek for a $U^3$-decomposition result in this article.

**Lemma 2.7 (U^3-uniformity estimates).** Let $a_i$, $i = 0, 1, 2, 3$, be functions on $\mathbb{Z}_N$ with $\|a_i\|_{L^\infty(\mathbb{Z}_N)} \leq 1$ and $\ell_1, \ell_2, \ell_3 \in \mathbb{N}$ be distinct. Then there exists a constant $c_2$, depending only on $\ell$, such that

$$|E_{m,n \in \mathbb{Z}_N} 1_{[N]}(n) \cdot a_0(m) \cdot a_1(m + \ell_1 n) \cdot a_2(m + \ell_2 n) \cdot a_3(m + \ell_3 n)| \leq c_2 \min_{0 \leq j \leq 3} \|a_j\|_{U^3(\mathbb{Z}_N)}^{1/2}.$$

**Proof.** We first reduce matters to estimating a similar average that does not contain the term $1_{[N]}(n)$. Let $r$ be an integer that will be specified later and satisfies $0 < r < N/2$. We define the “trapezoid function” $\phi$ on $\mathbb{Z}_N$ so that $\phi(0) = 0$, $\phi$ increases linearly from 0 to 1 on the interval $[0, r]$, $\phi(n) = 1$ for $r \leq n \leq N - r$, $\phi$ decreases linearly from 1 to 0 on $[N - r, N]$, and $\phi(n) = 0$ for $N < n < N$.

The absolute value of the difference between the average in the statement and

$$E_{m,n \in \mathbb{Z}_N} \phi(n) \cdot a_0(m) \cdot a_1(m + \ell_1 n) \cdot a_2(m + \ell_2 n) \cdot a_3(m + \ell_3 n)$$

is bounded by $2r/N$.

Moreover, it is classical that

$$\sum_{\xi \in \mathbb{Z}_N} |\hat{\phi}(\xi)| \leq \frac{c \sqrt{N}}{r}$$

for some universal constant $c$ that depends only on $\ell$, and thus

$$\left|E_{m,n \in \mathbb{Z}_N} \phi(n) \cdot a_0(m) \cdot a_1(m + \ell_1 n) \cdot a_2(m + \ell_2 n) \cdot a_3(m + \ell_3 n)\right| \leq \frac{c \sqrt{N}}{r} \max_{\xi \in \mathbb{Z}_N} \left|E_{m,n \in \mathbb{Z}_N} e(n\xi/\sqrt{N}) \cdot a_0(m) \cdot a_1(m + \ell_1 n) \cdot a_2(m + \ell_2 n) \cdot a_3(m + \ell_3 n)\right|. $$
Furthermore, notice that upon replacing \( a_0(n) \) with \( a_0(n)e(\ell_1^* n \xi/N) \) and \( a_1(n) \) with \( a_1(n)e(-\ell_1^* n \xi/N) \), where \( \ell_1^* 1 = 1 \mod N \), the \( U_2 \)-norm of all sequences remains unchanged, and the term \( e(n\xi/N) \) disappears. We are thus left with estimating the average
\[
\mathbb{E}_{m,n \in \mathbb{Z}_N} a_0(m) \cdot a_1(m + \ell_1 n) \cdot a_2(m + \ell_2 n) \cdot a_3(m + \ell_3 n),
\]
which is known (see for example [27, Theorem 3.1]) to be bounded by
\[
U := \min_{0 \leq j \leq 3} \|a_j\|_{U^3(\mathbb{Z}_N)}.
\]

Combining the preceding estimates, we get that the average in the statement is bounded by
\[
\frac{2r}{N} + \frac{c\tilde{N}}{r} U.
\]

Assuming that \( U \neq 0 \) and choosing \( r = \lfloor \sqrt{U\tilde{N}/(40\ell)} \rfloor \) (then \( r \leq \tilde{N}/(40\ell) \leq N/2 \)) gives the announced bound. \( \square \)

2.5. A positivity property. We derive now a positivity property that will be used in the proof of Theorem 2.8 in the next subsection. Here we make essential use of the fact that the spectral measure \( \nu \) is associated to a non-negative function on \( X \), and also that the function that defines the convolution product is non-negative.

**Lemma 2.8** (Hidden non-negativity). Let the action by dilations \( (T_n)_{n \in \mathbb{N}} \) on the probability space \( (X, \mathcal{B}, \mu) \), the subset \( A \) of \( X \), and the spectral measure \( \nu \) on \( \mathcal{M} \) be as in Theorem 2.5. Let \( \psi \) be a non-negative function defined on \( \mathbb{Z}_N \). Then
\[
\int_{\mathcal{M}} (\chi_N \ast \psi)(n_1) \cdot (\chi_N \ast \psi)(n_2) \cdot (\chi_N \ast \psi)(n_3) \cdot (\chi_N \ast \psi)(n_4) \ d\nu(\chi) \geq 0
\]
for every \( n_1, n_2, n_3, n_4 \in \mathbb{Z}_N \).

**Proof.** The convolution product \( \chi_N \ast \psi \) is defined on the group \( \mathbb{Z}_N \) by the formula
\[
(\chi_N \ast \psi)(n) = \mathbb{E}_{k \in \mathbb{Z}_N} \psi(n - k) \cdot \chi_N(k).
\]
It follows that for every \( n \in [\tilde{N}] \) there exists a sequence \((a_n(k))_{k \in \mathbb{Z}_N}\) of non-negative numbers that are independent of \( \chi \), such that for every \( \chi \in \mathcal{M} \) we have
\[
(\chi_N \ast \psi)(n) = \sum_{k \in \mathbb{Z}_N} a_n(k) \chi(k).
\]
The left hand side of the expression in the statement is thus equal to
\[
\sum_{k_1, k_2, k_3, k_4 \in \mathbb{Z}_N} \prod_{i=1}^{4} a_{n_i}(k_i) \int_{\mathcal{M}} \chi(k_1) \cdot \chi(k_2) \cdot \chi(k_3) \cdot \chi(k_4) \ d\nu(\chi) = \sum_{k_1, k_2, k_3, k_4 \in \mathbb{Z}_N} \prod_{i=1}^{4} a_{n_i}(k_i) \int_{\mathcal{M}} \chi(k_1 k_2) \cdot \chi(k_3 k_4) \ d\nu(\chi) = \sum_{k_1, k_2, k_3, k_4 \in \mathbb{Z}_N} \prod_{i=1}^{4} a_{n_i}(k_i) \int_{X} T_{k_1 k_2} 1_A \cdot T_{k_3 k_4} 1_A \ d\mu
\]
where the last equality follows from equation (37). This expression is non-negative since the function \( 1_A \) is non-negative, completing the proof. \( \square \)
2.6. Proof of Theorem 2.5 assuming Theorem 1.6 We start with a brief sketch of our proof strategy. Roughly speaking, Theorem 1.6 enables us to decompose the restriction of an arbitrary multiplicative function on a finite interval into three terms, a close to periodic term, a “very uniform” term, and an error term. In the course of the proof of Theorem 2.5, we study these three terms separately. The order of the different steps is important as well as the precise properties of the decomposition. First, we show that the uniform term has a negligible contribution in evaluating the averages in (9). To do this we use the uniformity estimates established in Lemma 2.7. It is for this part of the proof that it is very important to work with patterns that factor into products of linear forms in two variables, otherwise we have no way of controlling the corresponding averages by Gowers uniformity norms. At this point, the error term is shown to have negligible contribution, and thus can be ignored. Lastly, the structured term \( \chi_s \) is dealt with restricting the variable \( n \) to a suitable sub-progression where each function \( \chi_s \) gives approximately the same value to all four linear forms; it then becomes possible to establish the asserted positivity. In fact, the restriction to a sub-progression step is rather delicate, as it has to take place before the component \( \chi_e \) is eliminated (this explains also why we do not restrict both variables \( m \) and \( n \) to a sub-progression), and in addition one has to guarantee that the terms left out are non-negative, a property that follows from Lemma 2.8.

We now enter the main body of the proof. Recall that \( \ell_1, \ell_2, \ell_3 \in \mathbb{N} \) are fixed and distinct and that \( \ell = \ell_1 + \ell_2 + \ell_3 \). We stress also that in this proof the quantities \( m + \ell n \) are computed in \( \mathbb{Z}_N \), that is, modulo \( N \).

Let the action by dilations \((T_n)_{n \in \mathbb{N}}\) on the probability space \((X, B, \mu)\), the set \( A \in B \) with \( \mu(A) > 0 \), and the spectral measure \( \nu \) of \( f = 1_A \), be as in Theorem 2.5. We let

\[
\delta := \mu(A) = \int f \, d\mu ; \\
\varepsilon := c_3 \delta^2 \quad \text{and} \quad F(x, y, z) = c_4 \frac{x^2 y^2}{z^4},
\]

where \( c_3 \) and \( c_4 \) are positive constants that will be specified later, what is important is that they depend only on \( \delta \). Our goal is for all large values of \( N \) (how large will depend only on \( \delta \)) to bound from below the average

\[
A(N) := \int \mathbb{E}_{m, n \in \mathbb{Z}_N} 1_{[N]}(n) \cdot \chi_N(m) \cdot \chi_N(m + \ell_1 n) \cdot \overline{\chi_N}(m + \ell_2 n) \cdot \overline{\chi_N}(m + \ell_3 n) \, d\nu(\chi).
\]

We start by applying the decomposition result of Theorem 1.6 taking as input the spectral measure \( \nu \), the number \( \varepsilon \), and the function \( F \) defined above. Let

\[
Q := Q(F, N, \varepsilon, \nu) = Q(N, \delta, \nu), \quad R := R(F, N, \varepsilon, \nu) = R(N, \delta, \nu)
\]

be the numbers provided by Theorem 1.6. We recall that \( Q \) and \( R \) are bounded by a constant that depends only on \( \delta \). From this point on we assume that \( N \) is sufficiently large, depending only on \( \delta \), so that the conclusions of Theorem 1.6 hold. To ease the notation a bit, we omit the subscript \( N \) when we use the functions \( \chi_{N, s}, \chi_{N, u}, \chi_{N, e} \) provided by Theorem 1.6 and for \( \chi \in \mathcal{M} \), we write

\[
\chi_N(n) = \chi_s(n) + \chi_u(n) + \chi_e(n), \quad n \in \mathbb{Z}_N.
\]

This coincidence of values is very important, not having it is a key technical obstruction that stops us from handling equations like \( x^2 + y^2 = n^2 \). Restricting the range of both variables \( m \) and \( n \) does not seem to help either, as this creates problems with handling the error term in the decomposition.

In a sense, our approach follows the general principles of the circle method. Each multiplicative function is decomposed into two components, with Fourier transform supported on major arc and minor arc frequencies. The contribution of the “major arc component” is further analyzed to deduce the asserted positivity. The “minor arc component” is shown to have negligible contribution, and this step is the hardest, it is done using higher order Fourier analysis tools in the course of proving Theorem 1.6.
for the decomposition that satisfies Properties (ι)–(iv) of Theorem 1.6.

Next, we use the uniformity estimates of Lemma 2.7 in order to eliminate the uniform component $\chi_u$ from the average $A(N)$. We let

$$\chi_{s,e} = \chi_s + \chi_e$$

and

$$A_1(N) := \int_M \mathbb{E}_{m,n \in \mathbb{Z}_N} \mathbbm{1}_{[N]}(n) \cdot \chi_{s,e}(m) \cdot \chi_{s,e}(m + \ell_1 n) \cdot \chi_{s,e}(m + \ell_2 n) \cdot \chi_{s,e}(m + \ell_3 n) \, d\nu(\chi).$$

Using Lemma 2.7 Property (ι) of Theorem 1.6 and the estimates $|\chi_N(n)| \leq 1$, $|\chi_{s,e}(n)| \leq 1$ for every $n \in \mathbb{Z}_N$, we get that

$$|A(N) - A_1(N)| \leq \frac{4c_2}{F(Q, R, \varepsilon)^2}$$

where $c_2$ is the constant provided by Lemma 2.7 and depends only on $\ell$.

Next, we try to eliminate the error term $\chi_e$. But before doing this, it is important to first restrict the range of $n$ to a suitable sub-progression; the utility of this maneuver will be clear on our next step when we estimate the contribution of the leftover term $\chi_s$. We stress that we cannot postpone this restriction on the range of $n$ until after the term $\chi_e$ is eliminated, if we did this the contribution of the term $\chi_e$ would swamp the positive lower bound we get from the term $\chi_s$. We let

$$\eta := \frac{\varepsilon}{QR}.$$ 

By Property (ι) of Theorem 1.6 Lemma 2.8 applies to $\chi_{s,e}$. Note that the integers $Qk$, $1 \leq k \leq \eta N$, are distinct elements of the interval $[N]$. It follows that

$$\sum_{m,n \in \mathbb{Z}_N} \int_M \mathbbm{1}_{[N]}(n) \cdot \chi_{s,e}(m) \cdot \chi_{s,e}(m + \ell_1 n) \cdot \chi_{s,e}(m + \ell_2 n) \cdot \chi_{s,e}(m + \ell_3 n) \, d\nu(\chi) \geq$$

$$\sum_{m \in \mathbb{Z}_N} \sum_{k=1}^{\eta N} \int_M \chi_{s,e}(m) \cdot \chi_{s,e}(m + \ell_1 Qk) \cdot \chi_{s,e}(m + \ell_2 Qk) \cdot \chi_{s,e}(m + \ell_3 Qk) \, d\nu(\chi).$$

Therefore, we have

$$A_1(N) \geq \frac{\eta N}{N} A_2(N) \geq \frac{\eta}{40 \ell} A_2(N) = \varepsilon \frac{1}{40 \ell Q R} A_2(N)$$

where

$$A_2(N) := \int_M \mathbb{E}_{m \in \mathbb{Z}_N} \mathbb{E}_{k \in [\eta N]} \chi_{s,e}(m) \cdot \chi_{s,e}(m + \ell_1 kQ) \cdot \chi_{s,e}(m + \ell_2 kQ) \cdot \chi_{s,e}(m + \ell_3 kQ) \, d\nu(\chi).$$

We let

$$A_3(N) := \int_M \mathbb{E}_{m \in \mathbb{Z}_N} \mathbb{E}_{k \in [\eta N]} \chi_s(m) \cdot \chi_s(m + \ell_1 Qk) \cdot \chi_s(m + \ell_2 Qk) \cdot \chi_s(m + \ell_3 Qk) \, d\nu(\chi).$$

Since for every $n \in \mathbb{Z}_N$ we have $|\chi_s(n)| \leq 1$, and since $|\chi_{s,e}(n)| = |\chi_s(n) + \chi_e(n)| \leq 1$ by Property (ι) of Theorem 1.6 we deduce that

$$|A_2(N) - A_3(N)| \leq 4 \int_M \mathbb{E}_{m \in \mathbb{Z}_N} |\chi_e(m)| \, d\nu(\chi) < 4 \varepsilon$$

where the last estimate follows by Part (iv) of Theorem 1.6.
Next, we study the term \( A_3(N) \). We utilize Property (11) of Theorem 1.6, namely
\[
|\chi_s(n + Q) - \chi_s(n)| \leq \frac{R}{N} \quad \text{for every } n \in \mathbb{Z}_N.
\]
We get for \( m \in \mathbb{Z}_N \), \( 1 \leq k \leq \eta N \), and for \( i = 1, 2, 3 \), that
\[
|\chi_s(m + \ell_i Q k) - \chi_s(m)| \leq \ell_i k \frac{R}{N} \leq \ell \eta N \frac{R}{N} \leq \frac{\varepsilon}{Q}
\]
where the last estimate follows from (11) and the estimate \( \tilde{N} \geq \ell N \). Using this estimate in conjunction with the definition (13) of \( A_3(N) \), we get
\[
A_3(N) \geq \int_{\mathcal{M}} \mathbb{E}_{m \in \mathbb{Z}_N} |\chi_s(m)|^4 \, d\nu(\chi) - \frac{3\varepsilon}{Q}.
\]
We denote by 1 the multiplicative function that is identically equal to 1. We claim that \( \nu(\{1\}) \geq \delta^2 \). Indeed, if \((\Phi_N)_{N \in \mathbb{N}}\) is a multiplicative Følner sequence in \( \mathbb{N} \) we have
\[
\nu(\{1\}) = \lim_{N \to \infty} \int_{\mathcal{M}} \left| \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} \chi(n) \right|^2 \, d\nu(\chi) = \lim_{N \to \infty} \int_{\mathcal{M}} \left| \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} T_n f \right|^2 \, d\mu,
\]
and this is greater or equal than
\[
\lim_{N \to \infty} \int_{\mathcal{M}} \left| \frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} T_n f \right|^2 \, d\mu = \int |f|^2 \, d\mu \geq \delta^2,
\]
proving our claim. Using this we deduce that
\[
\int_{\mathcal{M}} \mathbb{E}_{m \in \mathbb{Z}_N} |\chi_s(m)|^4 \, d\nu(\chi) \geq \nu(\{1\}) \cdot \mathbb{E}_{m \in \mathbb{Z}_N} |1_s(m)|^4 \geq \delta^2 \mathbb{E}_{m \in \mathbb{Z}_N} 1_s(m)^4.
\]
Since \( 1_s = 1_N * \psi \) for some kernel \( \psi \) on \( \mathbb{Z}_N \) we have
\[
\mathbb{E}_{m \in \mathbb{Z}_N} 1_s(m) = \mathbb{E}_{m \in \mathbb{Z}_N} \mathbb{E}_{k \in \mathbb{Z}_N} 1_N(k) \psi(m - k) = \mathbb{E}_{k \in \mathbb{Z}_N} 1_N(k) = \frac{N}{N} \geq \frac{1}{20 \ell}.
\]
Combining the above we get
\[
A_3(N) \geq \frac{\delta^2}{20^4 \ell^4} - \frac{3\varepsilon}{Q}.
\]
Putting (10), (12), (13), and (15) together, we get
\[
A(N) \geq \varepsilon - \frac{1}{40 \ell Q R} \left( \frac{\delta^2}{20^4 \ell^4} - 7\varepsilon \right) - \frac{4c_2}{F(Q, R, \varepsilon)^\frac{1}{2}}.
\]
Recall that \( \varepsilon = c_3 \delta^2 \), for some positive constant \( c_3 \) that we left unspecified until now. We choose \( c_3 < 1 \), depending only on \( \ell \), so that
\[
\frac{1}{40 \ell} \left( \frac{\delta^2}{20^4 \ell^4} - 7\varepsilon \right) \geq c_5 \delta^2
\]
for some positive constant \( c_5 \) that depends only on \( \ell \). Then we have
\[
A(N) \geq \delta^2 \frac{c_5 \varepsilon}{QR} - \frac{4c_2}{F(Q, R, \varepsilon)^\frac{1}{2}}.
\]
Recall that
\[
F(Q, R, \varepsilon) = c_4 Q^2 R^2 \varepsilon^4
\]
where \( c_4 \) was not determined until this point. We choose
\[
c_4 := \frac{8c_2 c_3}{c_5}
\]
and upon recalling that $\varepsilon = c_3 \delta^2$ we get
\[ A(N) \geq \delta^2 \frac{c_5 \varepsilon}{QR} - c_2 \frac{4 \varepsilon^2}{c_4 Q R} = c_3 c_5 \delta^4 \frac{2 \varepsilon^2}{2 Q R} > 0. \]
Recall that $Q$ and $R$ are bounded by a constant that depends only on $\delta$. Hence, $A(N)$ is greater than a positive constant that depends only on $\delta$, and in particular is independent of $N$, provided that $N$ is sufficiently large, depending only on $\delta$, as indicated above. This completes the proof of Theorem [2.7].

3. Fourier analysis of multiplicative functions

In this section we study the Fourier coefficients of multiplicative functions. Our goal is to establish a decomposition
\[ \chi_N = \chi_{N,s} + \chi_{N,u} \]
similar to the one given in Theorem [6.1], but with the $U^2$-norm in place of the $U^3$-norm. We will then use this result in Section [5] as our starting point in the proof of the decomposition result for the $U^3$-norm.

**Convention.** In this section the functions $\chi_N$ are defined on $\mathbb{Z}/N\mathbb{Z}$. In particular, all convolution products are defined on $\mathbb{Z}/N\mathbb{Z}$ and the Fourier coefficients of $\chi_N$ are given by
\[ \hat{\chi}_N(\xi) := \mathbb{E}_{n \in \mathbb{Z}/N\mathbb{Z}} \chi_N(n) e(-n \xi/N) \quad \text{for} \quad \xi \in \mathbb{Z}/N\mathbb{Z}. \]

**Theorem 3.1** (Weak uniform decomposition for the $U^2$-norm). For every $\theta > 0$ there exist positive integers $Q := Q(\theta)$ and $R := R(\theta)$, and for every sufficiently large $N$, depending only on $\theta$, there exists a kernel $\phi_{N,\theta}$ on $\mathbb{Z}/N\mathbb{Z}$ with the following properties:

For every $\chi \in \mathcal{M}$, writing
\[ \chi_{N,s} = \chi_N \ast \phi_{N,\theta} \quad \text{and} \quad \chi_{N,u} = \chi_N - \chi_{N,s}, \]
we have
\begin{enumerate}
  \item \(|\chi_{N,s}(n + Q) - \chi_{N,s}(n)| \leq \frac{R}{N} \quad \text{for every} \quad n \in \mathbb{Z}/N\mathbb{Z}, \quad \text{where} \quad n + Q \quad \text{is taken mod} \quad N; \]
  \item \[\|\chi_{N,u}\|_{U^2(\mathbb{Z}/N\mathbb{Z})} \leq \theta.\]
\end{enumerate}
Moreover, for all $\theta$ and $\theta'$, and every $N$ such that $\phi_{N,\theta}$ and $\phi_{N,\theta'}$ are defined, we have
\[ |\phi_{N,\theta'}(\xi) - \hat{\phi}_{N,\theta}(\xi)| \leq \frac{R}{N} \quad \text{for every} \quad \xi \in \mathbb{Z}/N\mathbb{Z}. \]

The monotonicity property (16) plays a central role in the derivation of Theorem [1.6] from Theorem [6.1] in Section [6.9]. This is one of the reasons why we construct the kernels $\phi_{N,\theta}$ explicitly in Section [3.3].

The values of $Q$ and $R$ given by Theorem [3.1] will be used later in Section [6] and they do not coincide (in fact, they are much smaller) with the values of $Q$ and $R$ in Theorems [6.1] and [1.6].

3.1. Kátai’s orthogonality criterion. We start with the key number theoretic input that we need in this section and which will also be used later in Section [5].

**Lemma 3.2** (Orthogonality criterion [20]). For every $\varepsilon > 0$ and every $K_0 \in \mathbb{N}$ there exists $\delta := \delta(K_0, \varepsilon) > 0$ and $K := K(K_0, \varepsilon) > K_0$ such that the following holds: If $N \geq K$ and $f: [N] \to \mathbb{C}$ is a function with $|f| \leq 1$, and
\[ \max_{\text{primes } p' \text{ primes}} \left| \mathbb{E}_{n \in [N/p']} f(pn) \overline{f}'(p'n) \right| < \delta, \]
then
\[ \sup_{\chi \in \mathcal{M}} \left| \mathbb{E}_{n \in [N]} \chi(n) f(n) \right| < \varepsilon. \]
The dependence of $\delta$ and $K$ on $K_0$ and $\varepsilon$ can be made explicit (for good bounds see \cite{3}) but we do not need such extra information here.

Lemma 3.2 is not strictly speaking contained in the paper \cite{20} that shows only asymptotic results. Moreover, Kátai considers only the case of a function of modulus 1, written as $c(t(n))$, but the estimates are valid without any change for functions of modulus at most 1. For completeness we give the derivation.

**Proof.** Let $K_0 \in \mathbb{N}$ and $\varepsilon > 0$ be given. In \cite{20}, the letter $f$ is used to denote a multiplicative function that we denote here by $\chi$. Let $K \geq K_0$ be a positive integer that will be made explicit below and will depend only on $K_0, \varepsilon$. We write $\mathcal{P}$ for the set of primes $p$ with $K_0 < p < K$ and let

$$A_p := \sum_{p \in \mathcal{P}} \frac{1}{p}.$$ 

Let $N \in \mathbb{N}$, $\chi \in \mathcal{M}$, and $f : [N] \to \mathbb{C}$ be a function with $|f| \leq 1$. We let

$$S(N) := \sum_{n \in [N]} \chi(n)f(n).$$

After correcting some typos in \cite{20} inequality (3.8) of \cite{20} reads as follows

$$(17) \quad \frac{|S(N)|^2 A_P^2}{N^2} \leq c + c' \cdot A_P + c'' \sum_{p \neq p' \in \mathcal{P}} \frac{1}{N} \left| \sum_{n \leq \min(N/p,N/p')} f(pn) f(p'n) \right|$$

for some positive universal constants $c, c', c''$. For $K_0$ given, we choose $K$ sufficiently large so that $c A_P^2 + c' A_P^{-1} \leq c''/2$, and then choose $\delta := \varepsilon^2 A_P^2/(2c''|\mathcal{P}|^2)$. Thus defined, note that $K$ and $\delta$ depend on $K_0$ and $\varepsilon$ only. Assuming that the function $f$ satisfies the hypothesis of the lemma, and inserting the previous bounds in (17) we get the desired estimate. \hfill $\square$

3.2. An application to Fourier coefficients of multiplicative functions. Next, we use the orthogonality criterion of Kátai to prove that the Fourier coefficients of the restriction of a multiplicative function on an interval $[N]$ are small unless the frequency is close to a rational with small denominator. Furthermore, the implicit constants do not depend on the multiplicative function or the integer $N$.

**Corollary 3.3** ($U^2$ non-uniformity). For every $\theta > 0$ there exist positive integers $Q := Q(\theta)$ and $V := V(\theta)$ such that, for every sufficiently large $N$, depending only on $\theta$, for every $\chi \in \mathcal{M}$, and every $\xi \in \mathbb{Z}/\tilde{N}$, we have the following implication

$$(18) \quad \text{if } |\tilde{\chi}_N(\xi)| \geq \theta, \quad \text{then } \left\| \frac{Q\xi}{\tilde{N}} \right\| \leq \frac{QV}{\tilde{N}}.$$ 

**Proof.** Let $\delta := \delta(1, \theta)$ and $K := K(1, \theta)$ be defined by Lemma 3.2 and let $Q = K!$. Suppose that $N > K$. Let $p$ and $p'$ be primes with $p < p' \leq K$ and let $\xi \in \mathbb{Z}/\tilde{N}$. Since $Q$ is a multiple of $p' - p$ we have

$$\|Q\xi/\tilde{N}\| \leq \frac{Q}{p'-p} \|\xi/\tilde{N}\| \leq Q \|\xi/\tilde{N}\|.$$ 

Since $\tilde{N} < 20 \ell N$, we deduce

$$|E_{n \in [N/p']} e(p'n\xi/\tilde{N})e(-pn\xi/\tilde{N})| \leq \frac{Q}{N\|p'/p\xi/\tilde{N}\|} \leq \frac{2KQ}{N\|Q\xi/\tilde{N}\|} \leq \frac{40 \ell KQ}{N\|Q\xi/\tilde{N}\|}.$$ 

$^5$The sign $+$ and the universal constants are missing on the right hand side.
Let $V = 40 \ell K / \delta$. If $\|Q \xi / \tilde{N}\| > QV / \tilde{N}$, then the rightmost term of the last inequality is smaller than $\delta$, and thus, by Lemma 3.2 we have

$$|\tilde{N}(\xi)| = \|E_{n \in [N]} \chi_N(n)e(-n \xi / \tilde{N})\| = \frac{N}{\tilde{N}} \|E_{n \in [N]} \chi(n)e(-n \xi / \tilde{N})\| < \theta$$

contradicting (13). Hence, $\|Q \xi / \tilde{N}\| \leq QV / \tilde{N}$, completing the proof. \hfill \Box

3.3. Some kernels. Next, we make some explicit choices for the constants $Q$ and $V$ of Corollary 3.3. This will enable us to compare the Fourier transforms of the kernels $\phi_{N, \theta}$ defined below for different values of $\theta$ and to establish the monotonicity property (10).

For every $\theta > 0$ we define

$$A(N, \theta) := \left\{ \xi \in \mathbb{Z}_{\tilde{N}} : \exists \chi \in \mathcal{M} \text{ such that } |\tilde{N}(\xi)| \geq \theta^2 \right\};$$

$$W(N, q, \theta) := \max_{\xi \in A(N, \theta)} \frac{\|q \xi / N\|}{\tilde{N}};$$

(19) $$Q(\theta) := \min_{k \in \mathbb{N}} \left\{ k! : \limsup_{N \to \infty} W(N, k!, \theta) < +\infty \right\};$$

(20) $$V(\theta) := 1 + \left[ \frac{1}{Q(\theta)} \limsup_{N \to \infty} W(N, Q(\theta), \theta) \right].$$

It follows from Corollary 3.3 that the set of integers used in the definition of $Q(\theta)$ is non-empty, hence $Q(\theta)$ is well defined. The essence of the preceding definitions is that for every real number $\nu' > V(\theta) - 1$, the implication (18) is valid with $\theta^2$ substituted for $\theta$, $\nu'$ for $V$, $Q(\theta)$ for $Q$, and for every sufficiently large $N$. Furthermore, it follows from these definitions that for $0 < \theta' \leq \theta$, we have $Q(\theta') \geq Q(\theta)$, and thus

(21) for $0 < \theta' \leq \theta$, the integer $Q(\theta')$ is a multiple of $Q(\theta)$.

Moreover, it can be checked that

(22) $V(\theta)$ increases as $\theta$ decreases.

Next, we use the constants just defined to build the kernels $\phi_{N, \theta}$ of Theorem 3.1. We recall that a kernel on $\mathbb{Z}_{\tilde{N}}$ is a non-negative function $\phi$ on $\mathbb{Z}_{\tilde{N}}$ with $E_{n \in \mathbb{Z}_{\tilde{N}}} \phi(n) = 1$. We define the spectrum of a function $\phi$ to be the set

$$\text{Spec}(\phi) := \{ \xi \in \mathbb{Z}_{\tilde{N}} : \hat{\phi}(\xi) \neq 0 \}.$$

For every $m \geq 1$ and $\tilde{N} > 2m$ the “Fejer kernel” $f_{N, m}$ on $\mathbb{Z}_{\tilde{N}}$ is defined by

$$f_{N, m}(x) = \sum_{-m \leq \xi \leq m} \left( 1 - \frac{|\xi|}{m} \right) e(x \xi / \tilde{N}).$$

The spectrum of $f_{N, m}$ is the subset $(-m, m)$ of $\mathbb{Z}_{\tilde{N}}$. Let $Q_N(\theta)^* = Q_N(\theta)$ be the inverse of $Q_N(\theta)$ in $\mathbb{Z}_{\tilde{N}}$, that is, the unique integer in $\{1, \ldots, \tilde{N} - 1\}$ such that $Q_N(\theta)Q_N(\theta)^* = 1 \text{ mod } \tilde{N}$. For $\tilde{N} > 2Q(\theta)V(\theta)[\theta^{-2}]$ we define

(23) $$\phi_{N, \theta}(x) = f_{N, Q_N(\theta)^*/[\theta^{-4}]}(Q_N(\theta)^* x).$$

An equivalent formulation is that $f_{N, Q_N(\theta)^*/[\theta^{-4}]}(x) = \phi_{N, \theta}(Q_N(\theta) x)$. The spectrum of the kernel $\phi_{N, \theta}$ is the set

(24) $$\Xi_{N, \theta} := \{ \xi \in \mathbb{Z}_{\tilde{N}} : \frac{\|Q(\theta)\xi\|}{\tilde{N}} < \frac{Q(\theta)V(\theta)[\theta^{-4}]}{\tilde{N}} \},$$

and we have

(25) $$\phi_{N, \theta}(\xi) = \begin{cases} 1 - \frac{\|Q(\theta)\xi\|}{\tilde{N}} & \text{if } \xi \in \Xi_{N, \theta}; \\ 0 & \text{otherwise.} \end{cases}$$
We remark that the cardinality of $\Xi_{N,\theta}$ depends only on $\theta$.

### 3.4. Proof of Theorem 3.1

First, we claim that property (16) of Theorem 3.1 holds. Indeed, suppose that $\theta \geq \theta' > 0$ and that $N$ is sufficiently large so that $\phi_{N,\theta}$ and $\phi_{N,\theta'}$ are defined. We have to show that $\hat{\phi}_{N,\theta'}(\xi) \geq \hat{\phi}_{N,\theta}(\xi)$ for every $\xi$. Using (21) and (22) and the formula (25) giving the Fourier coefficients of $\phi_{N,\theta}$.

Next, we show the remaining assertions (i) and (ii) of Theorem 3.1 for the decomposition $\chi_{N,\theta} := \phi_{N,\theta} \ast \chi_N$, $\chi_{N,u} := \chi_N - \phi_{N,\theta} \ast \chi_N$.

Let $\theta > 0$, assume that $N$ is sufficiently large depending only on $\theta$, and let $Q = Q(\theta)$, $V = V(\theta)$, $\phi_{N,\theta}$, $\Xi = \Xi_{N,\theta}$, be defined by (19), (20), (23), (24) respectively.

For every $\chi \in M$, if $|\hat{\chi}_N(\xi)| \geq \theta^2$, then by the definition of $Q$ we have $\|Q\xi/\tilde{N}\| \leq QV/\tilde{N}$ and thus $\hat{\phi}_{N,\theta}(\xi) \geq 1 - \theta^4$ by (25). It follows that $|\hat{\chi}_N(\xi) - \phi_{N,\theta} \ast \chi_N(\xi)| \leq \theta^4 \leq \theta^2$.

This last bound is clearly also true when $|\hat{\chi}_N(\xi)| < \theta^2$ and thus using identity (3) we get

$$
\|\chi_N - \phi_{N,\theta} \ast \chi_N\|_{U^2(\mathbb{Z}_{\tilde{N}})}^4 = \sum_{\xi \in \mathbb{Z}_{\tilde{N}}} |\hat{\chi}_N(\xi) - \phi_{N,\theta} \ast \chi_N(\xi)|^4 \leq \theta^4 \sum_{\xi \in \mathbb{Z}_{\tilde{N}}} |\hat{\chi}_N(\xi) - \phi_{N,\theta} \ast \chi_N(\xi)|^2 \leq \theta^4 \sum_{\xi \in \mathbb{Z}_{\tilde{N}}} |\hat{\chi}_N(\xi)|^2 \leq \theta^4,
$$

where the last estimate follows from Parseval’s identity. Hence, $\|\chi_N - \phi_{N,\theta} \ast \chi_N\|_{U^2(\mathbb{Z}_{\tilde{N}})} \leq \theta$, proving Property (i).

Lastly, for $\chi \in M$ and $n \in \mathbb{Z}_{\tilde{N}}$, using the Fourier inversion formula and the estimate $|e(x) - 1| \leq \|x\|$, we get

$$
|\phi_{N,\theta} \ast \chi_N(n + Q) - (\phi_{N,\theta} \ast \chi_N)(n)| \leq \sum_{\xi \in \mathbb{Z}_{\tilde{N}}} |\hat{\phi}_{N,\theta}(\xi)| \cdot \|Q \xi/\tilde{N}\| \leq \frac{|\Xi_{N,\theta}| Q V \theta^{-2}}{\tilde{N}},
$$

where the last estimate follows from (24). As remarked above, $|\Xi_{N,\theta}|$ depends only on $\theta$, thus the last term in this inequality is bounded by $R/\tilde{N}$ for some constant $R$ that depends only on $\theta$. This establishes Property (ii) of Theorem 3.1 and finishes the proof.

### 4. Modifications of the Inverse and Factorization Theorems

In this section, we state and prove some consequences of an inverse theorem of Green and Tao [15] and a factorization theorem by the same authors [17] that are particularly tailored to the applications being pursued in subsequent sections. These results combined, prove that a function that has $U^3$-norm bounded away from zero, either has $U^2$-norm bounded away from zero, or else correlates in a sub-progression with a totally equidistributed 2-step polynomial nilsequence of a very special form.

Essentially all definitions and results of this section extend without important changes to arbitrary nilmanifolds. We restrict to the case of 2-step nilmanifolds as these are the only ones needed in this article and the notation is somewhat lighter in this case.

#### 4.1. Nilmanifolds

We introduce some notions from [17]. We record here only the properties that we need in this section and defer supplementary material to the next section and to Appendix A.

Let $X = G/T$ be a 2-step nilmanifold. Throughout, we assume that $G$ is a connected and simply connected 2-step nilpotent Lie group and $\Gamma$ is a discrete cocompact subgroup of $G$. We view elements of $G/T$ as “points” on the nilmanifold $X$ rather than equivalence classes, and denote them by $x, y, \text{ etc.}$ The nilmanifold $X$ is endowed with a base point $e_X$ which is the projection on $X$ of the unit element of $G$. The action of $G$ on $X$ is
denoted by \((g, x) \mapsto g \cdot x\). The Haar measure \(m_X\) of \(X\) is the unique probability measure on \(X\) that is invariant under this action.

We denote by \(m\) the dimension of \(G\) and by \(r\) the dimension of \(G_2 := [G, G]\), the commutator subgroup of \(G\). We implicitly assume that \(G\) is endowed with a Mal’cev basis \(\mathcal{X}\). \(\mathcal{X}\) is a basis \((\xi_1, \ldots, \xi_m)\) of the Lie algebra \(\mathfrak{g}\) that has the following properties:

(i) The map
\[
\psi: (t_1, \ldots, t_m) \mapsto \exp(t_1 \xi_1) \cdots \exp(t_m \xi_m)
\]
is a homeomorphism from \(\mathbb{R}^m\) onto \(G\);
(ii) \(G_2 = \psi\left(\{0\}^{m-r} \times \mathbb{R}^r\right)\);
(iii) \(\Gamma = \psi(\mathbb{Z}^m)\).

Let \(\mathfrak{g}\) be endowed with the Euclidean structure making \(X\) an orthonormal basis. This induces a Riemannian structure on \(G\) that is invariant under right translations. The group \(G\) is endowed with the associated geodesic distance, which we denote by \(d_G\). This distance is invariant under right translations.

Let the space \(X = G/\Gamma\) be endowed with the quotient metric \(d_X\). Writing \(p: G \to X\) for the quotient map, the metric \(d_X\) is defined by
\[
d_X(x, y) = \inf_{g,h \in G} \{d_G(g, h): p(g) = x, p(h) = y\}.
\]
Since \(\Gamma\) is discrete it follows that the infimum is attained. More precisely, there exists a compact subset \(F_0\) of \(G\), such that
\[
(26) \quad \text{for every } x, x' \in X \text{ there exist } h, h' \in F_0 \text{ with } d_G(h, h') = d_X(x, x').
\]
We frequently use the fact that if \(f\) is a smooth function on \(X\), then \(\|f\|_{\text{Lip}(X)} \leq \|f\|_{C^1(X)}\). We also use the following simple fact:

**Lemma 4.1.** For every bounded subset \(F\) of \(G\) there exists a constant \(H > 0\) such that

(i) \(d_X(g \cdot x, g \cdot x') \leq H d_X(x, x')\) for all \(x, x' \in X\) and every \(g \in F\);
(ii) for every \(f \in C^m(X)\) and every \(g \in F\), writing \(f_g(x) := f(g \cdot x)\), we have
\[
\|f_g\|_{C^m(X)} \leq H \|f\|_{C^m(X)}.
\]

**Proof.** Let \(F_0\) be as in (25). Since the multiplication \(G \times G \to G\) is smooth, its restriction to any compact set is Lipschitz and thus there exists a constant \(H > 0\) with \(d_G(gh, gh') \leq H d_G(h, h')\) for every \(g \in F\) and \(h, h' \in F_0\). The first statement now follows immediately from (26). Since the map \((g, x) \mapsto g \cdot x\), from \(K \times X\) to \(X\), is smooth, the second statement follows as well.

**Definition** (Vertical torus). We keep the same notation as above. The **vertical torus** is the sub-nilmanifold \(G_2/(G_2 \cap \Gamma)\). The Mal’cev basis induces an isometric identification between \(G_2\) and \(\mathbb{R}^r\), and thus of the vertical torus endowed with the quotient metric, with \(\mathbb{T}^r\) endowed with its usual metric. Every \(k \in \mathbb{Z}^r\) induces a character \(u \mapsto k \cdot u\) of the vertical torus. A function \(F\) on \(X\) is a **nilcharacter with frequency** \(k\) if \(F(u \cdot x) = e(k \cdot u)F(x)\) for every \(u \in \mathbb{T}^r = G_2/(G_2\Gamma)\) and every \(x \in X\). The nilcharacter is **non-trivial** if its frequency is non-zero.

**Definition** (Maximal torus and horizontal characters). Let \(X = G/\Gamma\) be a 2-step nilmanifold, let \(m\) and \(r\) be as above, and let \(s := m - r\). The Mal’cev basis induces an isometric identification between the maximal torus \(G/(G_2\Gamma)\), endowed with the quotient metric, and \(\mathbb{T}^s\), endowed with its usual metric. A **horizontal character** is a continuous group homomorphism \(\eta: G \to \mathbb{T}\) with a trivial restriction on \(\Gamma\). A horizontal character \(\eta\) factors through the maximal torus, and we typically abuse notation and think of \(\eta\) as a character of the maximal torus, and identify \(\eta\) with an element \(k\) of \(\mathbb{Z}^s\) by the following rule \(\alpha \mapsto k \cdot \alpha := k_1 \alpha_1 + \cdots + k_s \alpha_s\) for \(\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{T}^s\) and \(k = (k_1, \ldots, k_s) \in \mathbb{Z}^s\).

We define \(\|\eta\| := |k_1| + \cdots + |k_s|\).

Note that a trivial nilcharacter is any function that factors through the maximal torus.
4.2. A corollary of the inverse theorem. We start by stating the inverse theorem of Green and Tao for the $U^3$-norms.

**Theorem 4.2** (The inverse theorem for the $U^3(\mathbb{Z}_N)$-norm [15, Theorem 12.8]). For every $\varepsilon > 0$ there exist $\delta := \delta(\varepsilon) > 0$ and a 2-step nilmanifold $X := X(\varepsilon)$, such that for every sufficiently large $N$, depending only on $\varepsilon$, and for every $f: \mathbb{Z}_N \to \mathbb{C}$ with $|f| \leq 1$ and $\|f\|_{U^3(\mathbb{Z}_N)} \geq \varepsilon$, there exists a function $\Phi: X \to \mathbb{C}$ with $\|\Phi\|_{\text{Lip}(X)} \leq 1$ and an element $g \in G$ such that $|E_{n \in [N]} f(n) \Phi(g^n \cdot e_X)| \geq \delta$.

A sequence $\Phi(g^n \cdot e_X)$ of this form where $\Phi$ is only assumed to be continuous is defined as a 2-step basic nilsequence in [2], if in addition we assume that $\Phi$ is Lipschitz, then we call it a nilsequence of bounded complexity a notion first used in [15].

We state a corollary of this result that is better suited for our purposes:

**Corollary 4.3** (Modified $U^3$-inverse theorem). For every $\varepsilon > 0$ there exist $\delta := \delta(\varepsilon) > 0$, $m := m(\varepsilon)$, and a finite family $\mathcal{H} := \mathcal{H}(\varepsilon)$ of 2-step nilmanifolds, of dimension at most $m$ and having a vertical torus of dimension 1 (identified with $\mathbb{T}$ as explained above), such that: For every sufficiently large $N$, depending only on $\varepsilon$, if $f: \mathbb{Z}_N \to \mathbb{C}$ is a function with $|f| \leq 1$ and $\|f\|_{U^3(\mathbb{Z}_N)} \geq \delta$, then at least one of the following conditions hold

(i) $\|f\|_{U^2(\mathbb{Z}_N)} \geq \delta$;
(ii) there exist a nilmanifold $X$ belonging to the family $\mathcal{H}$, an element $g \in G$, and a nilcharacter $\Psi$ of $X$ with frequency 1, such that $\|\Psi\|_{L^2(X)} \leq 1$ and $|E_{n \in [N]} f(n) \Psi(g^n \cdot e_X)| \geq \delta$.

**Proof.** Let $\varepsilon > 0$. In this proof the constants $\delta, \delta', \delta''$, $C, C_1, \ldots$, depend only on $\varepsilon$.

Let $\tilde{X} = \tilde{G}/\tilde{T}$ and $\delta$ be given by Theorem 4.2. Let $m$ be the dimension of $\tilde{X}$, and $r$ be the dimension of $\tilde{G}_2$. The maximal torus $\tilde{G}/(\tilde{G}_2 \tilde{T})$ has dimension $s := m - r$. As mentioned above, we identify $\tilde{G}_2$ with $\mathbb{R}^r$ and $\tilde{G}_2 \cap \tilde{T}$ with $\mathbb{T}^r$, thus the vertical torus $\tilde{G}_2/(\tilde{G}_2 \cap \tilde{T})$ is identified with $\mathbb{T}^r$.

Let $f: \mathbb{Z}_N \to \mathbb{C}$ be a function with $|f| \leq 1$ and $\|f\|_{U^3} \geq \varepsilon$. Let $\Phi$ and $\tilde{g}$ be given by Theorem 4.2 i.e.

$$|E_{n \in [N]} f(n) \Phi(\tilde{g}^n \cdot e_X)| \geq \delta,$$

and $\|\Phi\|_{\text{Lip}(\tilde{X})} \leq 1$. We can assume that $\|\Phi\|_{L^2(\tilde{X})} \leq 1$. Indeed, there exists a function $\Phi'$ with $\|\Phi - \Phi'\|_{\infty} \leq \delta/2$ and $\|\Phi'\|_{L^2(\tilde{X})} \leq C$ for some constant $C$ depending only on $\delta$ and $\tilde{X}$, and thus only on $\varepsilon$. Up to a change in the constant $\delta$, the conclusion of Theorem 4.2 remains valid with $\Phi'$ substituted for $\Phi$.

We need some preliminary definitions. For $k \in \mathbb{T}^r$, the character $\phi_k$ of $\tilde{G}_2/(\tilde{G}_2 \cap \tilde{T})$ induces a character of $\tilde{G}_2$ given by some linear map $\phi_k: \tilde{G}_2 = \mathbb{R}^r \to \mathbb{R}$. Let $G_k$ be the quotient of $G$ by the subgroup $\ker(\phi_k)$ of $\tilde{G}_2$ and $\Gamma_k$ be the image of $\tilde{T}$ in this quotient. Then $\Gamma_k$ is a discrete and co-compact subgroup of $G_k$ and we let $X_k := G_k/\Gamma_k$. We write $\pi_k: \tilde{X} \to X_k$ for the natural projection and let $e_{X_k} := \pi_k(e_X)$.

If $k$ is non-zero, then $X_k$ is a non-Abelian 2-step nilmanifold. The commutator $G_{k,2}$ of $G_k$ is the quotient of $G_2$ with the kernel of $\phi_k$ and thus has dimension 1 and the vertical torus $G_k/(G_{k,2} \Gamma_k)$ of $X_k$ has dimension 1. If $k$ is the trivial character, then $X_k$ is the maximal torus $G/(G_2 \Gamma)$ of $X$ and thus is a compact Abelian Lie group.

We recall the definition of the vertical Fourier transform. The restriction to $\tilde{G}_2 \cap \tilde{T}$ of the action by translation of $G$ on $\tilde{X}$ is trivial, and thus this action induces an action of the vertical torus on $\tilde{X}$ by $(u, x) \mapsto u \cdot x$ for $u \in \mathbb{T}^r$ and $x \in \tilde{X}$. The vertical Fourier series of the function $\Phi$ is

$$\Phi = \sum_{k \in \mathbb{T}^r} \Phi_k$$

where $\Phi_k(x) = \int_{\mathbb{T}^r} \Phi(u \cdot x) e(-k \cdot u) \, dm_{\mathbb{T}^r}(u)$. 


For every \( k \in \mathbb{Z}' \), the function \( \Phi_k \) is a nilcharacter with frequency \( k \) and thus can be written as

\[
\Phi_k = \Psi_k \circ \pi_k
\]

for some function \( \Psi_k \) on \( X_k \). If \( k \neq 0 \), then \( \Phi_k \) is a nilcharacter of \( X_k \) with frequency equal to 1. Moreover, for every \( k \in \mathbb{Z}' \), since \( \| \Phi_k \|_{C^{2m}(\tilde{X})} \leq 1 \) and \( |\Phi_k(x)| \leq C_1(1 + \|k\|)^{-2m} \) for every \( k \in \mathbb{Z}' \) and every \( x \in \tilde{X} \). Since \( m > r \), there exists a constant \( C_2 \), depending only on \( \tilde{X} \) and \( \delta \), and thus only on \( \varepsilon \), with

\[
\sum_{k: \|k\| > C_2} |\Phi_k(x)| < \delta/2 \quad \text{for every} \quad x \in \tilde{X}.
\]

Replacing \( \Phi \) with its vertical Fourier series in (27), this last bound implies that there exists \( \delta' \), depending only on \( X \) and \( \delta \), and thus only on \( \varepsilon \), such that

\[
|\mathbb{E}_{n \in [N]} f(n) \Psi_k(g_k^n \cdot e_{\tilde{X}})| = |\mathbb{E}_{n \in [N]} f(n) \Phi_k(g_k^n \cdot e_{\tilde{X}})| \geq \delta'
\]

for some \( k \in \mathbb{Z}' \) with \( \|k\| \leq C_2 \), where \( g_k \) is the image of \( \tilde{g} \) in \( G_k \) under the natural projection. Since \( \Phi_k = \Psi_k \circ \pi_k \) and \( \|\Phi_k\|_{C^{2m}(\tilde{X})} \leq 1 \), we have that

\[
\|\Psi_k\|_{C^{2m}(X_k)} \leq C_3 \quad \text{for every} \quad k \in \mathbb{Z}' \quad \text{with} \quad \|k\| \leq C_2.
\]

We define the family \( \mathcal{H} \) of 2-step nilmanifolds as follows

\[
\mathcal{H} := \{X_k: k \neq 0, \|k\| \leq C_2\}.
\]

It remains to show that either Property (1) or Property (11) is satisfied.

Suppose first that the element \( k \in \mathbb{Z}' \) is non-zero. We have \( \|C_3^{-1}\Psi_k\|_{C^m(X_k)} \leq 1 \) and \( \|\mathbb{E}_{n \in [N]} f(n) C_3^{-1}\Psi_k(g_k^n \cdot e_{X_k})\| \geq C_3^{-1}\delta' \), showing that Property (11) is satisfied.

Otherwise, (28) holds for \( k = 0 \), in which case \( X_k \) is the maximal torus \( \tilde{G}/(\tilde{G}_2\tilde{G}) \cong \mathbb{T}^s \) and \( \Psi_0 \) is a function on \( \mathbb{T}^s \) with \( \|\Psi_0\|_{C^{2m}(\mathbb{T}^s)} \leq C_3 \). Let \( \alpha \) be the projection of \( g \) in \( \mathbb{T}^s \). For some constant \( C_4 \), we have

\[
\sum_{\ell \in \mathbb{Z}^s} |\Psi_0(\ell)| \leq C_4;
\]

\[
\delta' \leq \|\mathbb{E}_{n \in [N]} f(n) \Psi_0(n\alpha)| \leq \sum_{\ell \in \mathbb{Z}^s} |\Psi_0(\ell)||\mathbb{E}_{n \in [N]} f(n) e(n \ell \cdot \alpha)|.
\]

Thus, there exists \( \ell \in \mathbb{Z}^s \) with \( \|\mathbb{E}_{n \in [N]} f(n) e(n \ell \cdot \alpha)| \geq \delta'/C_4 \). Since the \( l^{4/3}(\mathbb{Z}_N) \)-norm of the Fourier coefficients of any linear phase is bounded by a universal constant, arguing as in the proof of Lemma 24, we deduce that \( \|f\|_{U^4(\mathbb{Z}_N)} \geq \delta'' \), for some constant \( \delta'' := \delta''(\varepsilon) \), showing that Property (1) is satisfied. This completes the proof. \( \square \)

4.3. A modification of the factorization theorem. We plan to use a decomposition result for polynomial sequences on nilmanifolds from [17]. We keep the notation and the conventions of Section 4.3 and introduce some additional ones next.

**Definition** (Total equidistribution). Let \( X \) be a nilmanifold. We say that the finite sequence \( g: [N] \to G \) is totally \( \varepsilon \)-equidistributed on \( X \) if

\[
|\mathbb{E}_{n \in [N]} 1_{P}(n) F(g(n) \cdot e_X)| \leq \varepsilon,
\]

for all \( F \in \text{Lip}(X) \) with \( \|F\|_{\text{Lip}(X)} \leq 1 \) and \( \int F \, dm_X = 0 \), and all arithmetic progressions \( P \) in \([N]\).

Modulo a change in the constants, our definition of total equidistribution is equivalent to the one given in [17], where the claimed estimate is \( |\mathbb{E}_{n \in P} F(g(n) \cdot e_X)| \leq \varepsilon \) for all \( F \in \text{Lip}(X) \) with \( \|F\|_{\text{Lip}(X)} \leq 1 \) and \( \int F \, dm_X = 0 \), and arithmetic progressions \( P \) in \([N]\) with \( |P| \geq \varepsilon N \).
Notation. If \((g(n))\) is a sequence in \(G\) and \(h \in \mathbb{Z}\), we denote by \((\partial_h(g))\) the sequence defined by \(\partial_h g(n) := g(n + h)g(n)^{-1}\) for \(n \in \mathbb{N}\).

Definition (Polynomial sequences). A polynomial sequence in a nilpotent group \(G\) is a sequence \(g: \mathbb{N} \to G\) that has the form \(g(n) = q_1^{p_1(n)} \cdots q_k^{p_k(n)}\), where \(a_1, \ldots, a_k \in G\) and \(p_1, \ldots, p_k \in \mathbb{Z}[t]\).

An equivalent definition \cite{17} Lemma 6.7 is that there exists an integer \(d\) such that \(\partial_{h+d} \partial_{h+d-1} \cdots \partial_h g(n) = 1_G\) for all \(n \in \mathbb{N}\) and all \(h_1, \ldots, h_{d+1} \in \mathbb{Z}\). The smallest integer \(d\) with this property is the degree of the sequence.

We say that a sequence \((g(n))\) in \(G\) is a degree 2 polynomial sequence with coefficients in the natural filtration if it can be written as

\[
g(n) = g_0 g_1^n g_2^n (n)\]

where \(g_0, g_1 \in G\) and \(g_2 \in G_2\).

Definition (Rational elements). We say that an element \(g \in G\) is \(Q\)-rational for some \(Q \in \mathbb{N}\) if there exists \(m \leq Q\) with \(g^m \in \Gamma\). We say that \(g\) is rational if it is \(Q\)-rational for some \(Q \in \mathbb{N}\).

Rational elements form a countable subgroup of \(G\) \cite{17} Lemma A.12.

Definition (Smooth and rational sequences). Given a nilmanifold \(G/\Gamma\) and \(M, \Omega, N \in \mathbb{N}\) with \(M \leq N\), we say that

- the sequence \(\epsilon: [N] \to G\) is \((M, N)\)-smooth if for every \(n \in [N]\) we have \(d_G(\epsilon(n), 1_G) \leq M\) and \(d_G(\epsilon(n), \epsilon(n-1)) \leq M/N\);
- the sequence \(\gamma: [N] \to G\) is \(M\)-rational if for every \(n \in [N]\), \(\gamma(n)\) is \(M\)-rational.

In \cite{17}, the next result is stated only for functions of the form \(\omega(M) = M^{-A}\), but the same proof works for arbitrary functions \(\omega: \mathbb{N} \to \mathbb{R}^+\). For the notion of a rational subgroup that is used in the next statement we refer the reader to Appendix A.

Theorem 4.4 (Factorization of polynomial sequences \cite{17} Theorem 1.19]). Suppose that \(X := G/\Gamma\) is a 2-step nilmanifold. For every \(M \in \mathbb{N}\) there exists a finite collection \(\mathcal{F}(M)\) of sub-nilmanifolds of \(X\), each of the form \(X' := G'/\Gamma'\), where \(G'\) is a rational subgroup of \(G\) and \(\Gamma' := G' \cap \Gamma\), such that the following holds:

For every function \(\omega: \mathbb{N} \to \mathbb{R}^+\) and every \(M_0 \in \mathbb{N}\), there exists \(M_1 := M_1(M_0, X, \omega)\), such that for every \(N \in \mathbb{N}\), and for every degree 2 polynomial sequence \((g(n))_{n \in [N]}\) in \(G\) with coefficients in the natural filtration of \(G\), there exist \(M \in \mathbb{N}\) with \(M_0 \leq M \leq M_1\), a nilmanifold \(X'\) belonging to the family \(\mathcal{F}(M)\), and a decomposition

\[
g(n) = \epsilon(n) g'(n) \gamma(n), \quad n \in [N],
\]

where \(\epsilon, g', \gamma: [N] \to G\) are degree 2 polynomial sequences with coefficients in the natural filtration of \(G\), that satisfy

(i) \(\epsilon\) is \((M, N)\)-smooth;
(ii) \((g'(n))_{n \in [N]}\) takes values in \(G'\), and the finite sequence \((g'(n), \epsilon X')_{n \in [N]}\) is totally \(\omega(M)\)-equidistributed in \(X'\) with the metric \(d_{X'}\);
(iii) \(\gamma: [N] \to G\) is \(M\)-rational, and \((\gamma(n))_{n \in [N]}\) is periodic with period at most \(M\).

Remark. We emphasize that in the previous result and subsequent corollary, the number \(M_1\) is independent of \(N\) and the family \((\mathcal{F}(M))_{M \in \mathbb{N}}\) is independent of \(\omega\) and \(N\).

The choice of the sub-nilmanifold \(X'\) may depend on \(N\), but it stabilizes for large enough \(N\).

We will use the following corollary of the previous result that gives a more precise factorization for a certain explicit class of polynomial sequences.

Corollary 4.5 (Modified factorization). Let \(X := G/\Gamma\) be a 2-step nilmanifold with vertical torus of dimension 1. For every \(M \in \mathbb{N}\) there exists a finite collection \(\mathcal{F}(M)\) of sub-nilmanifolds of \(X\), each of the form \(X' := G'/\Gamma'\), where \(\Gamma' := G' \cap \Gamma\) and either
(i) \( G' \) is an Abelian rational subgroup of \( G \); or
(ii) \( G' \) is a non-Abelian rational subgroup of \( G \) and \( G'_2/(G'_2 \cap \Gamma') \) has dimension 1, such that the following holds:

For every function \( \omega: \mathbb{N} \to \mathbb{R}^+ \) and every \( M_0 \in \mathbb{N} \), there exists \( M_1 := M_1(M_0, X, \omega) \), such that for every \( N \in \mathbb{N} \) and every \( g \in G \), there exist \( M \in \mathbb{N} \) with \( M_0 \leq M \leq M_1 \), a nilmanifold \( X' \) belonging to the family \( \mathcal{F}(M) \), and a decomposition

\[
g^n = \epsilon(n)g'(n)\gamma(n), \quad n \in [N],
\]

where \( \epsilon, g', \gamma: [N] \to G \) are polynomial sequences that satisfy

(iii) \( \epsilon \) is \((M, N)\)-smooth;
(iv) \((g'(n))_{n \in [N]} \) takes values in \( G' \), has the form

\[
g'(n) = g'_0 g'_1 \cdots g'_d \] where \( g'_0, g'_1, g'_2 \in G' \), and moreover \( g'_2 \in G'_2 \) in case \( \text{iii} \),

and \((g'(n) \cdot e_{X'})_{n \in [N]} \) is totally \( \omega(M) \)-equidistributed in \( X' \) with the metric \( dx' \);
(v) \( \gamma: [N] \to G \) is \( M \)-rational, and \((\gamma(n))_{n \in [N]} \) is periodic with period at most \( M \).

Proof. Let the integers \( M_1, M \) and the nilmanifold \( X' = G'/\Gamma' \) belonging to the family \( \mathcal{F}(M) \) be given by Theorem \( \text{4.3} \). Note that the sequence \((g^n(n))_{n \in [N]} \) is a degree 2 polynomial sequence in \( G \) with coefficients in the natural filtration. Let \((g'(n))_{n \in [N]} \) be the sequence given by the decomposition of Theorem \( \text{4.4} \). This sequence is a polynomial sequence in \( G \) with coefficients in the natural filtration of \( G \) and thus it can be written as \( g'(n) = g'_0 g'_1 \cdots g'_d \) for some \( g'_0, g'_1, g'_2 \in G' \). It remains to show that this sequence has the form \( 29 \), i.e. that \( g'_0, g'_1, g'_2 \in G' \) and furthermore that \( g'_2 \in G'_2 \) in case \( \text{ii} \).

Since \( g'_0 = g'(0) \) we have \( g'_0 \in G' \). Recall that \( \partial_1 g(n) = g(n+1)g(n)^{-1} \). Using the fact that \( G_2 \) is included in the center of \( G \) we obtain

\[
\partial_1 g'(n) = g'_0 g'_1 g'_0^{-1} g'_2 \quad \text{and} \quad \partial_1^2 g'(n) = g'_2.
\]

It follows that \( g'_1 \) and \( g'_2 \) are in \( G' \).

If we are in case \( \text{ii} \), then \( G' \) is Abelian and we are done. Suppose now that we are in case \( \text{ii} \) where \( G' \) is non-Abelian. Then \( G'_2 \) is a non-trivial subgroup of \( G_2 \). Moreover, \( G'_2 \) is closed and connected, and by hypothesis \( G_2 \) is isomorphic to the torus \( T \). It follows that \( G'_2 = G'_2 \). Hence, \( g'_2 \in G'_2 \). This shows that the sequence \((g'(n))_{n \in [N]} \) has the required properties and completes the proof. \( \square \)

5. Correlation of multiplicative functions with nilsequences

The main goal of this section is to establish some correlation estimates needed in the proof of the decomposition results given in the next section. We show that multiplicative functions do not correlate with a class of totally equidistributed 2-step polynomial nilsequences. The precise statements appear in Propositions \( \text{5.3} \) and \( \text{5.4} \).

5.1. Quantitative equidistribution. We start with a quantitative equidistribution result for polynomial sequences on nilmanifolds by Green and Tao \( \text{17} \) that strengthens an earlier qualitative equidistribution result of Leibman \( \text{22} \).

Definition. If \( g: [N] \to T \) is a finite polynomial sequence in \( T \), of the form

\[
g(n) = \alpha_0 + \alpha_1 n + \alpha_2 \binom{n}{2} + \cdots + \alpha_d \binom{n}{d} \quad \text{where} \quad d \in \mathbb{N} \quad \text{and} \quad \alpha_0, \ldots, \alpha_d \in T,
\]

then the smoothness norm of \( g \) is defined by

\[
\|g\|_{C^\infty([N])} := \max_{1 \leq j \leq d} N^j \|\alpha_j\|.
\]
**Theorem 5.1** (Quantitative Leibman Theorem [17]). Let \( X := G/\Gamma \) be a 2-step nilmanifold. Then for every small enough \( \varepsilon > 0 \) there exists \( D := D(X, \varepsilon) > 0 \) with the following property: For every \( N \in \mathbb{N} \), if \( g : [N] \to G \) is a polynomial sequence of degree at most 2 such that the sequence \( (g(n) \cdot e_X)_{n \in [N]} \) is not \( \varepsilon \)-equidistributed in \( X \), then there exists a horizontal character \( \eta := \eta(X, \varepsilon) \) such that

\[
0 < \| \eta \| \leq D \quad \text{and} \quad \| \eta \circ g \|_{C^\infty([N])} \leq D,
\]

where we abuse notation and think of \( \eta \) as a character of the maximal torus and \( g(n) \) as a polynomial sequence on the horizontal torus.

We plan to use a partial converse of this result.

**Lemma 5.2** (A partial converse to Theorem 5.1). Let \( X := G/\Gamma \) be a 2-step nilmanifold. There exists \( C := C(X) > 0 \) with the following property: If \( N \) is sufficiently large, depending only on \( X \), and \( (g(n))_{n \in [N]} \) is a degree 2 polynomial sequence in \( G \) such that there exists a non-trivial horizontal character \( \eta \) of \( X \) with \( \| \eta \| \leq D \) and \( \| \eta \circ g \|_{C^\infty([N])} \leq D \), then the sequence \( (g(n) \cdot e_X)_{n \in [N]} \) is not totally \( CD^{-2} \)-equidistributed in \( X \).

**Proof.** Since \( \| \eta \circ g \|_{C^\infty([N])} \leq D \) we have

\[
\eta(g(n)) = \sum_{0 \leq j \leq 2} \alpha_j \left( \frac{n}{j} \right) \quad \text{for some } \alpha_j \in \mathbb{T} \text{ with } \| \alpha_j \| \leq \frac{D}{Nj} \text{ for } j = 1, 2.
\]

It follows that

\[
|1 - e(\eta(g(n)))| \leq \frac{1}{2} \quad \text{for } 1 \leq n \leq c_1 \frac{N}{D}
\]

where \( c_1 \) is a universal constant. Thus,

\[
\left| \mathbb{E}_{n \leq [c_1 N/D]} e(\eta(g(n))) \right| \geq \frac{1}{2}.
\]

Furthermore, since \( \| \eta \| \leq D \), the function \( x \mapsto e(\eta(x)) \), defined on \( X \), is Lipschitz with constant at most \( C' \) for some \( C' := C'(X) \), and has integral 0 since \( \eta \) is a non-trivial horizontal character. Therefore, the sequence \( (g(n) \cdot e_X)_{n \in [N]} \) is not totally \( (CD^{-2}) \)-equidistributed in \( X \) where \( C = c_1 / C' \). \( \square \)

We can now prove the two main results of this section that give asymptotic orthogonality of multiplicative functions to some totally equidistributed nilsequences. These results will be used later in the proof of Theorem 6.1 to treat each of the two distinct cases arising from an application of Corollary 4.5. Both proofs are based on Kátaı’s orthogonality criterion (Lemma 3.2) and the quantitative Leibman Theorem (Theorem 5.1).

**Proposition 5.3** (Discorrelation estimate 1). Let \( X := G/\Gamma \) be a 2-step nilmanifold and \( \tau > 0 \). There exists \( \sigma := \sigma(X, \tau) > 0 \) with the following property: For every sufficiently large \( N \), depending only on \( X \) and \( \tau \), if \( (g(n))_{n \in [N]} \) is a degree 2 polynomial sequence in \( G \) with coefficients in the natural filtration that is totally \( \sigma \)-equidistributed in \( X \), then

\[
\sup_{m, \chi, \Phi, P} \left| \mathbb{E}_{n \in [N]} 1_P(n) \chi(n) \Phi((g(m + n) \cdot e_X)) \right| < \tau
\]

where the sup is taken over all integers \( m \), multiplicative functions \( \chi \in \mathcal{M} \), functions \( \Phi \in \text{Lip}(X) \) with \( \| \Phi \|_{\text{Lip}(X)} \leq 1 \) and \( \int \Phi \, dm_X = 0 \), and arithmetic progressions \( P \subset [N] \).

**Proof.** Let \( \tau > 0 \) and suppose that

\[
\left| \mathbb{E}_{n \in [N]} 1_P(n) \chi(n) \Phi((g(m + n) \cdot e_X)) \right| \geq \tau
\]

for some \( m \in \mathbb{Z} \), \( \chi \in \mathcal{M} \), \( \Phi \in \text{Lip}(X) \) with \( \| \Phi \|_{\text{Lip}(X)} \leq 1 \) and \( \int \Phi \, dm_X = 0 \), and arithmetic progression \( P \subset [N] \).

We endow the nilmanifold \( X \times X \) with the cartesian product structure and let \( D \) be associated to \( \tau \) (\( \tau \) plays the role of \( \varepsilon \)) and the nilmanifold \( X \times X \) as in Theorem 5.1. We
also let $K$ and $\delta$ be defined as in Lemma 5.2 with $D$ substituted for $K_0$ and $\tau$ substituted for $\varepsilon$. Applying this lemma to the sequence $(1_P(n)\Phi(g(m+n)))_{n \in [N]}$, estimate (30) gives that there exist primes $p, p'$ with $D < p < p' < K$ such that

$$\left| \mathbb{E}_{n \in [[N/p']]} 1_P(n) \Phi(g(m+pn) \cdot e_X) \right| \geq \delta.$$ 

Since $1_P(n)1_P(p'n) = 1_{P_1}(n)$ for some arithmetic progression $P_1 \subset [[N/p']]$ we have

$$\left| \mathbb{E}_{n \in [[N/p']]} 1_{P_1}(n) \Phi(g(m+pn) \cdot e_X) \right| \geq \delta. \tag{31}$$

On the other hand, since $\int \Phi \ dm_X = 0$, we have $\int \Phi \otimes \Phi \ dm_X = 0$. Furthermore, $\|\Phi \otimes \Phi\|_{Lip(X \times X)} \leq 1$, and thus (31) gives that the sequence

$$(g(m+pn) \cdot e_X, g(m+p'n) \cdot e_X), n \in [[N/p']]$$

is not totally $\delta$-equidistributed in $X \times X$. By Theorem 5.1 there exists a non-trivial horizontal character $\eta$ of $X \times X$ with

$$\|\eta\| \leq D, \quad \text{and} \quad \left| \mathbb{E}_{n \in [N/p']} \eta(g(m+pn) \cdot e_X, g(m+p'n) \cdot e_X) \right|_{C^\infty([N/p'])} \leq D. \tag{32}$$

Let $\pi$ is the natural projection on the maximal torus $G/(G_2 \Gamma)$ which we identify with $T^s$ for some $s \in \mathbb{N}$. Recall that $g(n) = g_0 g_1^2 g_2^2$ for some $g_0, g_1 \in G$ and $g_2 \in G_2$. Let $\alpha_0 := \pi(g_0), \alpha_1 := \pi(g_1)$. Since $\pi(g_2) = 0$, we have $\pi(g(n)) = \alpha_0 + n\alpha_1$, and since the horizontal character $\eta$ has the form

$$\eta(x, x') = k \cdot \pi(x) + k' \cdot \pi(x')$$

for some $k, k' \in \mathbb{Z}^s$ with $0 \leq |k| + \|k'\| = \|\eta\| \leq D$, we get that

$$\eta(g(m+pn) \cdot e_X, g(m+p'n) \cdot e_X) = n(pk + p'k') \cdot \alpha_1 + (k + k') \cdot (\alpha_0 + m\alpha_1). \tag{34}$$

Putting together (32) and (33) gives

$$\| (pk + p'k') \cdot \alpha_1 \| \leq \frac{D}{[N/p']} \leq \frac{2DK}{N}.$$ 

Let $\theta$ be the horizontal character of $X$ given by $\theta(x) = (pk + p'k') \cdot \pi(x)$. We have $\theta \circ g(n) = n(pk + p'k') \cdot \alpha_1$ and thus $\|\theta \circ g\|_{C^\infty([N])} \leq 2DK$. Since $p$ and $p'$ are primes and $\|k\|, \|k'\| \leq D < p < p'$, we have $pk + p'k' \neq 0$ and $\theta \neq 0$. Furthermore, $\|\theta\| = \|pk + p'k'\| \leq DK$. It follows from Lemma 5.2 that there exists $\sigma$ depending only on $D, K, X$, and thus only on $X$ and $\sigma$, such that the sequence $(g(n))_{n \in [N]}$ is not totally $\sigma$-equidistributed in $X$. This completes the proof. \hfill \Box

**Proposition 5.4** (Discorrelation estimate II). Let $s \in \mathbb{N}$ and $\tau > 0$. There exists $\sigma := \sigma(s, \tau) > 0$ with the following property: For every sufficiently large $N$, depending only on $s$ and $\tau$, if $(g(n))_{n \in [N]}$ is a polynomial sequence in $T^s$ of the form

$$g(n) = \alpha_0 + \alpha_1 n + \alpha_2 \binom{n}{2}, \quad \alpha_i \in T^s, \quad i = 0, 1, 2,$$

that is totally $\sigma$-equidistributed in $T^s$, then

$$\sup_{m_X \Phi \in P} \left| \mathbb{E}_{n \in [N]} 1_P(n) \chi(n) \Phi(g(m+n)) \right| < \tau$$

where the sup is taken over all integers $m$ with $|m| \leq N$, multiplicative functions $\chi \in \mathcal{M}$, $\Phi \in Lip(X)$ with $\|\Phi\|_{Lip(X)} \leq 1$ and $\int \Phi \ dm_X = 0$, and arithmetic progressions $P \subset [N].$

**Proof.** In this proof $C_1, C_2, \ldots$ are constants that depend only on $s$ and $\tau$.

Without loss of generality, we can assume that $\|\Phi\|_{C^2(X)} \leq 1$. Suppose that

$$\left| \mathbb{E}_{n \in [N]} 1_P(n) \chi(n) \Phi(g(m+n)) \right| \geq \tau$$

then
for some \( \tau > 0 \), integer \( m \) with \( |m| \leq N \), \( \chi \in \mathcal{M} \), \( \Phi \in \operatorname{Lip}(X) \) with \( \|\Phi\|_{\operatorname{Lip}(X)} \leq 1 \) and \( \int \Phi \, dm_X = 0 \), and arithmetic progression \( P \subset [N] \). We have

\[
\tau \leq \left| \mathbb{E}_{n \in [N]} 1_{P}(n) \chi(n) \Phi(m + n) \right| = \left| \sum_{k \in \mathbb{Z}^*} \frac{C_0}{1 + \|k\|^2} \left| \mathbb{E}_{n \in [N]} 1_{P}(n) \chi(n) e(k \cdot g(m + n)) \right| \leq \sum_{k \in \mathbb{Z}^*} \frac{C_0}{1 + \|k\|^2} \left| \mathbb{E}_{n \in [N]} 1_{P}(n) \chi(n) e(k \cdot g(m + n)) \right|
\]

for some constant \( C_0 := C_0(s) \). It follows that there exist constants \( C_1, \theta := \theta(s, \tau) > 0 \), and \( k \in \mathbb{Z}^* \), such that

\[
\|k\| \leq C_1 \quad \text{and} \quad \left| \mathbb{E}_{n \in [N]} 1_{P}(n) \chi(n) e(k \cdot g(m + n)) \right| \geq \theta.
\]

Let \( \delta \) and \( K \) be defined by Lemma \( 3.2 \), with 1 substituted for \( K_0 \) and \( \theta \) substituted for \( \varepsilon \). Note that \( \delta \) and \( K \) depend on \( s \) and \( \tau \) only. There exist primes \( p, p' \) with \( p < p' \leq K \) such that

\[
\left| \mathbb{E}_{n \in [N/p']} 1_{P}(pm) 1_{P}(p'n) e(k \cdot (g(m + pm) - g(m + p'n))) \right| \geq \delta.
\]

Writing \( \beta_1 = k \cdot \alpha_1 \), \( \beta_2 = k \cdot \alpha_2 \), and \( 1_{P}(pm) 1_{P}(p'n) = 1_{P_1}(n) \) where \( P_1 \subset [N/p'] \) is an arithmetic progression, we can rewrite the previous estimate as

\[
\left| \mathbb{E}_{n \in [N/p']} 1_{P_1}(n) e(u(n)) \right| \geq \delta.
\]

where

\[
u(n) = \binom{n}{2} \beta_2(p^2 - p'^2) + n \left( \beta_2 \left( \frac{p}{2} \right) - \left( \frac{p'}{2} \right) \right) + (m\beta_2 + \beta_1)(p - p').
\]

Since \( \lfloor N/p' \rfloor \geq N/2K \), the sequence \((\nu(n))_{n \in [N]}\) is not totally \( \delta/2K \)-equidistributed in the circle. By the Abelian version of Theorem \( 5.1 \), it follows that there exists an integer \( l \) with \( 0 < l \leq D := D(\delta/2K) \) such that

\[
\|l\beta_2(p^2 - p'^2)\| \leq \frac{D}{N^2} \quad \text{and} \quad \|l\beta_2 \left( \frac{p}{2} \right) - \left( \frac{p'}{2} \right) + l(m\beta_2 + \beta_1)(p - p')\| \leq \frac{D}{N}.
\]

We deduce first that \( \beta_2 \) is at a distance \( \leq C_2/N^2 \) of a rational with denominator \( \leq C_3 \), and then that \( \beta_1 \) is at a distance \( \leq C_4/N \) of a rational with denominator \( \leq C_5 \) (here we used that \( |m| \leq N \)). Hence, there exists a non-zero integer \( l' \), bounded by some constant \( C_6 \), such that \( \|l'\beta_2\| \leq C_7/N^2 \) and \( \|l'\beta_1\| \leq C_8/N \). Taking \( k' = l'k \), we have

\[
0 < \|k'\| \leq C_1C_6; \quad \|k' \cdot \alpha_2\| \leq \frac{C_7}{N^2}; \quad \|k' \cdot \alpha_1\| \leq \frac{C_8}{N}.
\]

Using this and Lemma \( 5.2 \) we deduce that the sequence \((g(n))_{n \in [N]}\) is not totally \( \sigma \)-equidistributed for some \( \sigma > 0 \) that depends only on \( s \) and \( \tau \), completing the proof. \( \square \)

We give one more application of Theorem \( 5.1 \) that will be needed in the next section:

**Lemma 5.5 (Shifting the nilmanifold).** Let \( X := G/\Gamma \) be a 2-step nilmanifold, \( G' \) be a rational subgroup of \( G \), \( h \in G \) be a rational element, \( X' := G'/e_X \), \( e_Y := h \cdot e_X \), and \( Y := G' \cdot e_Y \). For every \( \varepsilon > 0 \) there exists \( \delta := \delta(G', X, h, \varepsilon) > 0 \) with the following property: If \((g'(n))_{n \in [N]}\) is a polynomial sequence in \( G' \) of degree at most 2, such that the sequence \((g'(n) \cdot e_X)_{n \in [N]}\) is totally \( \delta \)-equidistributed in \( X' \), then the sequence \((g'(n) \cdot e_Y)_{n \in [N]}\) is totally \( \varepsilon \)-equidistributed in \( Y \).

By Lemma \( 5.2 \) in the Appendix, \( \Gamma \cap G' \) is co-compact in \( G' \) and thus \( X' \) is a closed sub-nilmanifold of \( X \). In a similar fashion, \( (h \Gamma h^{-1}) \cap G' \) is co-compact in \( G' \) and \( Y \) is a closed sub-nilmanifold of \( X \).
Proof. In this proof, $C_1, C_2, \ldots$ are constants that depend only on $G', X, \text{ and } h$.

By Lemma 5.3 in the Appendix, the group $\Gamma \cap h\Gamma^{-1} \cap G'$ has finite index in the two groups $\Gamma \cap G'$ and $h\Gamma^{-1} \cap G'$. We write

$$Z' := G'/G'_2(\Gamma \cap G'), \quad Z_1 := G'/G'_2(\Gamma \cap h\Gamma^{-1} \cap G'), \quad \text{and } Z_2 := G'/G'_2(h\Gamma^{-1} \cap G').$$

Then $Z'$ is the horizontal torus of $X'$, the nilmanifold $Y$ can be identified with $G'/((h\Gamma^{-1} \cap G')$, and thus $Z_2$ is the horizontal torus of $Y$. Let $p: Z_1 \rightarrow Z'$ and $q: Z_1 \rightarrow Z_2$ be the natural projections. These group homomorphisms are finite to one.

Let $\varepsilon > 0$, and suppose that the polynomial sequence $(g(n) \cdot e_Y)_{n \in [N]}$ has degree at most 2 and is not totally $\varepsilon$-equidistributed in $Y$. We denote by $D$ the integer that Theorem 5.1 associates to $\varepsilon$ and $Y$. Then there exists a non-trivial horizontal character $\eta$ of $Y$, with

$$0 \neq \|\eta\| \leq D \quad \text{and} \quad \|\eta(g'(n))\|_{C^\infty[N]} \leq D.$$

Recall that $\eta$ factors to a character of the horizontal torus $Z_2$ of $Y$; we slightly abuse notation and denote it also by $\eta$. We have that $\eta \circ q$ is a character of $Z_1$ and since $q: Z_1 \rightarrow Z_2$ is finite to one, $\|\eta \circ q\| \leq C_1\|\eta\|$ for some constant $C_1$.

Since $\Gamma \cap h\Gamma^{-1} \cap G'$ has finite index in $\Gamma \cap G'$, there exists $\ell \in \mathbb{N}$ such that $\gamma^\ell \in \Gamma \cap h\Gamma^{-1} \cap G'$ for every $\gamma \in \Gamma \cap G'$. Therefore, since the restriction of $\eta \circ q$ to $\Gamma \cap h\Gamma^{-1} \cap G'$ is trivial, for every $\gamma \in \Gamma \cap G'$ we have $\ell\eta \circ q(\gamma) = \eta \circ q(\gamma^\ell) = 1$. Hence, $\ell\eta \circ q$ has a trivial restriction to $G'_2(\Gamma \cap G')$ and so there exists a character $\zeta$ of $Z'$ with $\ell\eta \circ q = \zeta \circ p$.

We have $0 \neq \|\zeta\| \leq C_3\|\ell\eta \circ q\| \leq C_3\|\eta\| \leq C_3D$ for some constants $C_2, C_3$. We consider $\zeta$ as a horizontal character of $X' = G'/(G' \cap \Gamma)$ and thus as a character of $G'$.

For every $n \in \mathbb{N}$ we have $\zeta(g'(n)) = \ell\eta(g'(n))$ and thus, by hypothesis (35), $\|\zeta \circ g'|_{C^\infty[N]} \leq C_4D$ for some constant $C_4$. By Lemma 5.2 the sequence $(g'(n) \cdot e_X)_{n \in [N]}$ is not totally $C_5D^{-1}$-equidistributed in $X'$ for some constant $C_5$. Letting $\delta = C_5D^{-1}$ completes the proof. \hfill \Box

Lastly, we give some uniform discorrelation estimates that serve as a model for the more complicated estimates obtained in the sequel. The argument is based on Kátai’s criterion (Lemma 5.2) and the Abelian version of Theorem 5.1 which is nothing more than a suitable use of Weyl’s estimates.

Proposition 5.6 (Model discorrelation estimates for $\chi_{N,u}$). Let $\varepsilon > 0$. Then there exists $\theta := \theta(\varepsilon)$ such that, for every sufficiently large $N$, depending only on $\varepsilon$, the following holds: If

$$\chi_{N,s} := \chi_N \ast \phi_{N,\theta}, \quad \chi_{N,u} := \chi_N - \chi_{N,s},$$

where $\chi_N = \chi \cdot 1_{[N]}$ and $\phi_{N,\theta}$ is the kernel defined by (25), then

$$\sup_{\chi \in \mathcal{M}, \alpha \in \mathbb{R}} \|E_n \chi_{N,u}(n)e(n^2\alpha)\|_{L^2(Z_N)} \leq \varepsilon.$$

Proof (Sketch). Let $\varepsilon > 0$ and $N$ be sufficiently large depending only on $\varepsilon$ (how large will be determined below).

Let $\theta := \theta(\varepsilon) > 0$ be given by (40) below and for this value of $\theta$ let $\phi_{N,\theta}$ be given by (24). Theorem 5.1 implies that for sufficiently large $N$, depending only on $\varepsilon$, we have

$$\|\chi_{N,u}(n)\|_{L^2(Z_N)} \leq \theta(\varepsilon).$$

We claim that the asserted estimate (36) holds. Arguing by contradiction, suppose that

$$\|E_n \chi_{N,u}(n)e(n^2\alpha)\|_{L^2(Z_N)} > \varepsilon$$

for some $\chi \in \mathcal{M}$ and $\alpha \in \mathbb{R}$. We consider two cases depending on the total equidistribution properties of the sequence $(n^2\alpha)_{n \in [N]}$.
**Minor arcs.** We use Proposition 5.4 with \( s = 1 \) and \( \varepsilon/3 \) in place of \( \tau \). We get that there exists \( \sigma := \sigma(\varepsilon) \), such that for all sufficiently large \( N \), depending only on \( \varepsilon \), if the sequence \( (n^{2}\alpha)_{n\in[N]} \) is totally \( \sigma \)-equidistributed, then

\[
\max_{m\in[-N, N]} |\mathbb{E}_{n\in[N]} \chi(n)e((m + n)^{2}\alpha)| \leq \frac{1}{2}\varepsilon.
\]

Using this and the fact that \( \chi_{N,u} = \chi_{N} \ast (1 - \phi) \) where \( \phi \) is a kernel on \( \mathbb{Z}_{N}^{*} \), we deduce (see Section 6.8 for details) that for all sufficiently large \( N \), depending only on \( \varepsilon \), we have

\[
|\mathbb{E}_{n\in[N]} \chi_{N,u}(n)e(n^{2}\alpha)| \leq \varepsilon
\]

which contradicts (38).

**Major arcs.** Suppose now that the sequence \( (n^{2}\alpha)_{n\in[N]} \) is not totally \( \sigma \)-equidistributed where \( \sigma \) was defined in the minor arc step. Then, as is well known (and also follows by Lemma 5.2), \( \alpha \) has to be close to a rational with a small denominator, more precisely, there exist positive integers \( Q, R \) that depend only on \( \sigma \), and consequently only on \( \varepsilon \), and positive integers \( p, q \leq Q \) such that

\[
|\alpha - \frac{p}{q}| \leq \frac{R}{N^{2}}.
\]

We factor the sequence \( (n^{2}\alpha)_{n\in[N]} \) as follows

\[
n^{2}\alpha = \varepsilon(n) + \gamma(n), \quad \text{where} \quad \varepsilon(n) := n^{2}\left(\alpha - \frac{p}{q}\right), \quad \gamma(n) := n^{2}\frac{p}{q}
\]

Note that \( |\varepsilon(n + 1) - \varepsilon(n)| \leq 2R/N \) for \( n \in [N] \). Furthermore, the sequence \( \gamma(n) \) is periodic with period \( q \). After partitioning the interval \( [N] \) into sub-progressions where \( \varepsilon(n) \) is almost constant and \( \gamma(n) \) is constant, and using the pigeonhole principle, it is not hard to deduce from (38) (see Section 6.5 for details) that

\[
|\mathbb{E}_{n\in[N]} 1_{P(n)} \cdot \chi_{N,u}(n)| > \frac{1}{10QR} \frac{\varepsilon^{2}}{Q^{2}}
\]

for some arithmetic progression \( P \subset [N] \) provided that \( N \) is sufficiently large depending only on \( \varepsilon \). Using (39) and Lemma 2.6 we deduce that

\[
\|\chi_{N,u}(n)\|_{U^{2}(\mathbb{Z}_{N})} > \frac{1}{c_{1}QR} \varepsilon = \theta(\varepsilon)
\]

where \( c_{1} \) is a universal constant. This contradicts (37) and completes the proof. \( \square \)

In the next section we prove a strengthening of the previous result where the place of \( (\varepsilon(n^{2}a)) \) takes any two step nilsequence \( \Phi(a^{0} \cdot e_{X}) \) where \( \Phi \) is a function on a 2-step nilmanifold with Lipschitz norm at most 1. Our proof is much more complicated in this case but the basic strategy remains the same as in the previous argument.

6. Higher order Fourier analysis of multiplicative functions

The goal of this section is to prove the main decomposition result stated in Theorem 1.6. The key ingredient that enters its proof is the following “weaker” decomposition.

**Theorem 6.1** (Weak uniform decomposition for the \( U^{3} \)-norm). For every \( \theta_{0} > 0 \) and \( \varepsilon > 0 \), there exist a positive real \( \theta < \theta_{0} \), and positive integers \( Q := Q(\varepsilon, \theta_{0}) \) and \( R := R(\varepsilon, \theta_{0}) \), with the following properties: For every sufficiently large \( N \), depending only on \( \theta_{0} \) and \( \varepsilon \), and for every \( \chi \in \mathcal{M} \), the function \( \chi_{N} \) admits the decomposition

\[
\chi_{N}(n) = \chi_{N,s}(n) + \chi_{N,u}(n) \quad \text{for every} \quad n \in \mathbb{Z}_{N},
\]

where the functions \( \chi_{N,s} \) and \( \chi_{N,u} \) satisfy:
(i) \( \chi_{N,s} = \chi_N \ast \phi_{N,\theta} \), where \( \phi_{N,\theta} \) is the kernel on \( \mathbb{Z}_N \) defined by \((23)\) and is independent of \( \chi \), and the convolution product is defined in \( \mathbb{Z}_N \);  
(ii) \(|\chi_{N,s}(n + Q) - \chi_{N,s}(n)| \leq \frac{R}{N} \) for every \( n \in \mathbb{Z}_N \), where \( n + Q \) is taken mod \( \tilde{N} \);  
(iii) \( \|\chi_{N,u}\|_{U^3(\mathbb{Z}_N)} \leq \varepsilon \).

The proof of Theorem \(6.1\) takes the largest part of this section. The main disadvantage of this result is that the bound on the uniform component is not strong enough for our applications. In Section \(6.9\) we combine Theorem \(6.1\) with an energy increment argument to prove Theorem \(1.6\) that gives very strong bounds on the uniform component.

6.1. Some preliminary remarks and proof strategy. A substantial part of our proof is consumed in handling correlation estimates of arbitrary multiplicative functions with 2-step nilsequences of bounded complexity. Our proof strategy follows the general ideas of an argument of Green and Tao from \([14, 18]\) where uniformity properties of the Möbius function were studied. In our case, we are faced with a few important additional difficulties stemming from the fact that we are forced to work with all multiplicative functions some of which are not \(U^2\)-uniform (see the example in Section \(1.5\)). Furthermore, we have to establish estimates with implied constants independent of the elements of \( \mathcal{M} \). We give a brief summary of our strategy next.

To compensate for the lack of \(U^2\)-uniformity of a multiplicative function \( \chi \) we subtract from it a suitable “structured component” \( \chi_u \) given by Theorem \(3.1\) so that \( \chi_u = \chi - \chi_s \) has extremely small \(U^2\)-norm. Our goal is then to show that \( \chi_u \) has small \(U^3\)-norm. In view of the \(U^3\)-inverse theorem of Green and Tao (Theorem \(4.2\)), this would follow if we show that \( \chi_u \) has very small correlation with all 2-step nilsequences of bounded complexity. This then becomes our new goal.

The factorization theorem for nilsequences (Theorem \(4.5\)) practically allows us to treat correlation with major arc and minor arc 2-step nilsequences separately. Orthogonality to major arc (approximately periodic) nilsequences is easily implied by the \(U^2\)-uniformity of \( \chi_u \). So our efforts concentrate on the minor arc (totally equidistributed) nilsequences. Combining the orthogonality criterion of Kátai (Lemma \(3.2\)) with some quantitative equidistribution results on nilmanifolds (Theorem \(2.1\)), we deduce that the arbitrary multiplicative function \( \chi \) is asymptotically orthogonal to such sequences (Propositions \(5.3\) and \(5.4\)). The function \( \chi_u \) is not multiplicative though, but this is easily taken care by the fact that \( \chi_u = \chi - \chi_s \) and the fact that \( \chi_s \) can be recovered from \( \chi \) by taking a convolution product with a positive kernel. Using these properties it is an easy matter to transfer estimates from \( \chi \) to \( \chi_u \). Combining the above, we get the needed orthogonality of \( \chi_u \) to all 2-step nilsequences of bounded complexity. Furthermore, a close inspection of the argument shows that all implied constants are independent of \( \chi \). This suffices to complete the proof of the decomposition result.

Although the previous sketch communicates the basic ideas behind the proof of Theorem \(6.1\) the various results needed to implement this plan come with a significant number of parameters that one has to juggle with, making the bookkeeping rather cumbersome. We use the next section to organize some of these data.

6.2. Setting up the stage. In this subsection we define and organize some data that will be used in the proof of Theorem \(6.1\). We take some extra care to do this before the main body of the proof in order to make sure that there is no circularity in the admittedly complicated collection of choices involved. The reader is advised to skip this subsection on a first reading and refer back to it only when necessary.

Essentially all objects defined below will depend on a positive number \( \varepsilon \) and on the integer \( \ell \) defined in Section \(1.5\). We consider these parameters as fixed, and to ease notation we leave the dependence on \( \varepsilon \) and \( \ell \) implicit most of the time.
6.2.3. Objects associated to $H$ family

Furthermore, most objects defined below also depend on a positive integer parameter $M$ that we consider for the moment as a free variable. The explicit choice of $M$ takes place in Section 6.4 and depends on various other choices that will be made subsequently; what is important though is that it is bounded above and below by positive constants that depend only on $\varepsilon$. The parameter $M$ plays a central role in our argument, and to avoid confusion we keep the dependence on $M$ explicit most of the time.

Most objects we define use the families of nilmanifolds $F(M)$ introduced in Corollary 4.5. We would like to stress that these families do not depend on the function $\omega$ that occurs in this statement. This allows us to postpone the definition of $\omega$ until Section 6.2.4.

6.2.1. Objects defined by the inverse theorem. Throughout the argument we let $\varepsilon$ be a fixed positive number. Corollary 4.3 defines the objects $H := H(\varepsilon)$, $\delta := \delta(\varepsilon)$, $m := m(\varepsilon)$ where $H$ is a finite family of nilmanifolds with vertical torus of dimension 1, $\delta$ is a positive number, and $m$ is a positive integer. In the sequel we implicitly assume that $N$ is sufficiently large, depending only on $\varepsilon$, so that the conclusion of Corollary 4.3 holds.

6.2.2. Objects associated to every fixed nilmanifold in $H$. Let $M \in \mathbb{N}$ be fixed and $X := G/H$ be a nilmanifold in $H$. We define below various objects that depend on $M$ and $X$.

By Corollary 4.3, for every $M \in \mathbb{N}$ there exists a finite subset $\Sigma := \Sigma(M, X)$ of $G$, consisting of $M$-rational elements, such that for every $M$-rational element $g \in G$ there exists $h \in \Sigma$ with $h^{-1}g \in \Gamma$, that is, $g \cdot e_X = h \cdot e_X$. We assume that $1_G \in \Sigma$.

Let $F := F(M, X)$ be the family of sub-nilmanifolds of $X$ defined by Corollary 4.5. We define a larger family of nilmanifolds $F' := F'(M, X)$ that have the form

$$Y := G' \cdot e_Y \equiv G'/(h\Gamma h^{-1} \cap G')$$

where

$$X' := G'/\Gamma' \in F', \quad h \in \Sigma, \quad \text{and} \quad e_Y = h \cdot e_X.$$

By Lemma 4.1. there exists a positive constant $H := H(M, X)$ with the following properties:

(i) for every $h \in \Sigma$, every $g \in G$ with $d_G(g, 1_G) \leq M$, and every $x, y \in X$, we have $d_X(gh \cdot x, gh \cdot y) \leq HD_X(x, y)$;

(ii) for every $h \in \Sigma$, every $g \in G$ with $d_G(g, 1_G) \leq M$, and every function $\phi \in \mathcal{C}^{2m}(X)$, we have $\|\phi_{gh}\|_{\mathcal{C}^{2m}(X)} \leq H\|\phi\|_{\mathcal{C}^{2m}(X)}$ where $\phi_{gh}(x) := \phi(gh \cdot x)$.

The distance on a nilmanifold $Y \in F'$ is not the distance induced by the inclusion in $X$. However, the inclusion $Y \subset X$ is smooth and thus we can assume that

(iii) for every nilmanifold $Y \in F'$ and every $x, y \in Y$, we have $d_X(x, y) \leq HD_Y(x, y)$;

(iv) for every $Y \in F'$ and every function $\phi$ on $X$, we have $\|\phi\|_{Y, \mathcal{C}^{2m}(Y)} \leq H\|\phi\|_{\mathcal{C}^{2m}(X)}$.

By Lemma 5.3. for every $X' \in F$, every $\zeta > 0$, and every $h \in \Sigma$, there exists $\rho := \rho(M, X, X', h, \zeta)$ with the following property:

(v) Let $X' = G'/\Gamma' \in F$, $h \in \Sigma$, $e_Y := h \cdot e_X$, and $(g'(n))_{n \in \mathbb{N}}$ polynomial sequence in $G'$ with degree at most 2; if $(g'(n)e_X)_{n \in \mathbb{N}}$ is totally $\rho$-equidistributed in $X'$, then $(g'(n)e_Y)_{n \in \mathbb{N}}$ is totally $\zeta$-equidistributed in $Y := G' \cdot e_Y$.

6.2.3. Objects associated to $H$. We consider now all the nilmanifolds belonging to the family $H$ and define the finite family of nilmanifolds

$$F'(M) = \bigcup_{X \in H} F'(M, X)$$
and the positive numbers
\begin{align}
H(M) &:= \max_{X \in \mathcal{H}} H(M, X) ; \\
\rho(M, \zeta) &:= \min_{X \in \mathcal{H}} \rho(M, X, X', h, \zeta), \tag{41}
\end{align}
where $\rho(M, X, X', h, \zeta)$ was defined by item (vi) above.

We let
\begin{align}
\delta_1(M) &:= \frac{\delta^2}{5M^2} ; \\
\theta(M) &:= \min\left(\frac{\delta}{2}, \frac{\delta_1(M)}{2c_1}\right), \tag{43}
\end{align}
where $\delta$ was defined by Section 6.2 and $c_1$ is the universal constant defined by Lemma 2.6.

To every $\tau > 0$ and every nilmanifold $Y$ in the finite collection $\mathcal{F}'(M)$, either Proposition 5.3 or Proposition 5.4 (applied for $Y$ in place of $X$) associates a positive number $\sigma(Y, \tau)$ (depending on whether $Y$ in non-Abelian or Abelian). We define
\begin{align}
\tilde{\sigma}(M) &:= \min_{Y \in \mathcal{F}'(M)} \sigma(Y, \frac{\delta_1(M)}{10H(M)^2}). \tag{45}
\end{align}

6.2.4. The function $\omega$ and bounds for $M$. We let $\omega: \mathbb{N} \to \mathbb{R}^+$ be the function defined by
\begin{align}
\omega(M) &:= \rho(M, \tilde{\sigma}(M)) \tag{46}
\end{align}
where $\rho$ is defined in (42) and $\tilde{\sigma}$ is defined in (45). We also let
\begin{align}
M_0 &:= \lceil 2/\varepsilon \rceil. \tag{47}
\end{align}

For this choice of $\omega$ and $M_0$, Corollary 4.5 associates to every nilmanifold $X \in \mathcal{H}$ a positive real number $M_1(M_0, X, \omega)$ and we define
\begin{align}
M_1 &:= \max_{X \in \mathcal{H}} M_1(M_0, X, \omega). \tag{48}
\end{align}

We stress that $M_1$ depends only on $\varepsilon$.

6.3. Using the $U^2$-decomposition. After setting up the stage we are now ready to enter the main body of the proof of Theorem 1.6.

Let $\varepsilon > 0$. Let $M_1$ be given by (48) and let
\begin{align}
\theta_1 &:= \theta(M_1), \tag{49}
\end{align}
be given by (44). Note that $\theta_1$ depends on $\varepsilon$ only. We use Theorem 5.1 for $\theta$ substituted with this ‘very small’ value of $\theta_1$. We get that for every sufficiently large $N$, depending only on $\varepsilon$, every $\chi \in \mathcal{M}$ admits the decomposition
\begin{align}
\chi_N = \chi_{N,s} + \chi_{N,u}
\end{align}
where $\chi_{N,s}$ and $\chi_{N,u}$ satisfy the conclusions of Theorem 6.1 in particular
\begin{align}
\|\chi_{N,u}\|_{U^2(\mathbb{Z}_N^2)} &\leq \theta_1. \tag{50}
\end{align}

We claim that $\chi_{N,s}$ and $\chi_{N,u}$ satisfy the conclusion of Theorem 6.1. Theorem 5.1 gives at once that Properties (iii) and (iii) of Theorem 6.1 are satisfied with $Q := Q(M_1)$ and $R := R(M_1)$. It remains to verify Property (iii), namely that
\begin{align}
\|\chi_{N,u}\|_{U^3(\mathbb{Z}_N^2)} &\leq \varepsilon.
\end{align}
We argue by contradiction. We assume that
\begin{align}
\|\chi_{N,u}\|_{U^3(\mathbb{Z}_N^2)} &> \varepsilon, \tag{51}
\end{align}
and in the next subsections we are going to derive a contradiction. To facilitate reading we split the proof into several parts, and to ease notation, we continue to leave the dependence on $\varepsilon, \ell$ implicit.

6.4. **Using the inverse and the factorization theorem.** Suppose that (51) holds. We recall that in Section 6.2 we used Corollary 4.3 to define the positive integer $m$, the positive real number $\delta$, and a finite family $H$ of nilmanifolds with vertical torus of dimension 1. We recall also that these objects depend only on $\varepsilon$ and we assume that $N$ is sufficiently large so that the conclusions of Corollary 4.3 hold.

If the alternative (i) of Corollary 4.3 holds, then

$$\|\chi_{N,u}\|_{U^2(\mathbb{Z}_{\tilde{N}})} \geq \delta,$$

and since by (44) and (49) we have $\theta_1 = \theta(M_1) \leq \delta/2$, this contradicts (50).

As a consequence, alternative (ii) of Corollary 4.3 holds. Our goal is to show that for the particular choices made in the previous subsections we get again a contradiction.

By our assumption, there exists a 2-step nilmanifold $X = G/\Gamma$ belonging to the family $H$, a nilcharacter $\Psi$ on $X$ with frequency 1, and an element $g$ of $G$, such that

$$\|\Psi\|_{C^2_m(X)} \leq 1,$$

and

$$|\mathbb{E}_{n \in [\tilde{N}]} \chi_{N,u}(n)\Psi(g^n \cdot e_X)| \geq \delta$$

where $m$ is the dimension of $X$. Note that $X$, $\Psi$, and $g$, will depend on $\chi$, but this is not going to create problems for us.

Recall that for every $M \in \mathbb{N}$ the finite family $F(M,X)$ of sub-nilmanifolds of $X$ was defined in Section 6.2.2 using Corollary 4.5 of the factorization theorem. Next we apply this corollary for the sequence $(g^n)_{n \in [\tilde{N}]}$ in $G$, the function $\omega: \mathbb{N} \to \mathbb{R}^+$ defined by (46), and the integer $M_0$ defined by (47). For $M_1$ given by (48) we get an integer $M$ with

$$M_0 \leq M \leq M_1,$$

a nilmanifold $X' = G'/\Gamma'$ belonging to the family $F(M,X)$ that satisfies either Property (i) or Property (ii) of Corollary 4.5, and a factorization

$$g^n = \epsilon(n)g'(n)\gamma(n)$$

into polynomial sequences in $G$ that satisfy Properties (iii), (iv), (v) of Corollary 4.5 for this value of $M$. The number $M$ depends on $\chi$, but it belongs on the interval $[M_0, M_1]$, and since the integers $M_0, M_1$ depend only on $\varepsilon$, this suffices for our purposes.

From this point on, we work with this choice of $M$.

6.5. **Eliminating $\epsilon$ and $\gamma$ by passing to a sub-progression.** Our goal on this and the next subsection is to get an estimate of the form (53) with the additional property that the sequence $(g^n \cdot e_X)_{n \in [\tilde{N}]}$ is sufficiently totally equidistributed. To achieve this we pass to an appropriate sub-progression where the sequences $\epsilon$ and $\gamma$ defined by (55) are practically constant and then change the nilmanifold defining the nilsequence to eliminate some extra terms introduced.

By Property (v) of Corollary 4.5, the sequence $(\gamma(n))$ is periodic of period at most $M$; we denote its period by $p$. Let

$$L := \left\lfloor \frac{\delta}{2M^2} \tilde{N} \right\rfloor.$$

We partition $[\tilde{N}]$ into arithmetic progressions of step $p$ and length $L$ and a leftover set that we can ignore since it will introduce error terms bounded by a constant times $\delta/M$ and thus negligible for our purposes (upon replacing $\delta$ with $\delta/2$ below). Using (53) and
the pigeonhole principle, we get that there exists a progression $P$ with step $p$ and length $L$ such that

$$\left| \mathbb{E}_{n \in [\tilde{N}]} \mathbf{1}_p(n) \chi_{N, u}(n) \Psi(g^n \cdot e_X) \right| \geq \frac{\delta L}{N} = \frac{\delta^2}{2M^2} - \frac{\delta}{N}. \tag{57}$$

Let $n_0, n \in P$. We have $\gamma(n) = \gamma(n_0)$ and by Property (iii) of Corollary 4.5, the sequence $(\epsilon(n))$ is $(M, \tilde{N})$-smooth. Using these properties and the right invariance of the metric $d_G$, we get

$$d_G(g^n, \epsilon(n_0)g'(n)\gamma(n_0)) \leq d_G(\epsilon(n), \epsilon(n_0)) \leq |n - n_0| \frac{M}{N} \leq pL \frac{M}{N} \leq \frac{M^2L}{N}. \tag{52}$$

Since by (52) the function $\Psi$ has Lipschitz constant at most 1, it follows that

$$\left| \Psi(g^n \cdot e_X) - \Psi(\epsilon(n_0)g'(n)\gamma(n_0) \cdot e_X) \right| \leq \frac{M^2L}{N}. \tag{58}$$

From this and (50) we deduce that

$$\mathbb{E}_{n \in [\tilde{N}]} \mathbf{1}_p(n) \chi_{N, u}(n) \left| \Psi(g^n \cdot e_X) - \Psi(\epsilon(n_0)g'(n)\gamma(n_0) \cdot e_X) \right| \leq \frac{L}{N} \frac{M^2L}{N} \leq \frac{\delta^2}{4M^2}.$$

Combining this with (57) and the definition of $\delta_1$ given by (43) we get

$$\left| \mathbb{E}_{n \in [\tilde{N}]} \mathbf{1}_p(n) \chi_{N, u}(n) \Psi(\epsilon(n_0)g'(n)\gamma(n_0) \cdot e_X) \right| \geq \frac{\delta^2}{5M^2} = \delta_1(M) \tag{58}$$

provided that $\tilde{N} \geq 20M_2^2/\delta$ (then $\delta/\tilde{N} \leq \delta^2/(20M^2)$).

6.6. Changing the nilmanifold. By Property (v) of Corollary 4.5, $\gamma(n_0)$ is $M$-rational. By the definition of $\Sigma(M, X)$ in Section 6.2.2, we can choose $h_0 \in \Sigma(M, X)$ such that $\gamma(n_0) \cdot e_X = h_0 \cdot e_X$. We define

$$e_Y := h_0 \cdot e_X; \quad Y := G' \cdot e_Y \cong G'/(h_0\Gamma h_0^{-1} \cap G'); \quad \Psi'(x) = \Psi(h_0 \cdot e_X).$$

Note that the nilmanifold $Y$ belongs to the family $\mathcal{F}'(M)$ defined in Section 6.2.3. For every $n \in \mathbb{N}$ we have

$$\Psi(\epsilon(n_0)g'(n)\gamma(n_0) \cdot e_X) = \Psi'(g'(n) \cdot e_Y). \tag{59}$$

Since the sequence $(\epsilon(n))_{n \in [\tilde{N}]}$ is $(M, \tilde{N})$-smooth, we have $d_G(\epsilon(n_0), 1_G) \leq M$. Furthermore, since $\|\Psi\|_{c^{2m}(X)} \leq 1$, by Property (iii) of Section 6.2.2 we have $\|\Psi\|_{c^{2m}(X)} \leq H(M)$ where $H(M)$ was defined by (41). Since the nilmanifold $Y$ belongs to the family $\mathcal{F}'(M)$, by Property (v) of Section 6.2.2 we have

$$\|\Psi'\|_{c^{2m}(Y)} \leq H(M)^2. \tag{60}$$

Recall that the sequence $(g'(n) \cdot e_X)_{n \in [\tilde{N}]}$ arises from the decomposition (55) provided by Corollary 4.5 by Property (iv) of this corollary, the sequence $(g'(n) \cdot e_X)_{n \in [\tilde{N}]}$ is $\omega(M)$-totally equidistributed in $X'$.

Since by the definition of $\omega$ (see (40)) we have $\omega(M) = \rho(M, \tilde{\sigma}(M))$, Property (v) of Section 6.2.2 and (12) give that

$$\text{the sequence } (g'(n) \cdot e_Y)_{n \in [\tilde{N}]} \text{ is totally } \tilde{\sigma}(M)\text{-equidistributed in } Y. \tag{61}$$

Summarizing, we have so far established that

$$\left| \mathbb{E}_{n \in [\tilde{N}]} \mathbf{1}_p(n) \chi_{N, u}(n) \Psi'(g'(n) \cdot e_Y) \right| \geq \delta_1(M) \tag{62}$$

for some arithmetic progression $P \subset [\tilde{N}]$ and properties (50) and (61) are satisfied. In the next subsection we further reduce matters to the case where the function $\Psi'$ has integral zero.
6.7. **Reducing to the zero integral case.** Our goal is to show that upon replacing \( \Psi' \) with \( \Psi' - z \), where \( z = \int_Y \Psi' \, dm_Y \), we get a bound similar to \( (62) \). To do this we make crucial use of the fact that the \( U^2 \)-norm of \( \chi_{N,u} \) is suitably small, in fact, this is the step that determined our choice of the degree of \( U^2 \)-uniformity \( \theta_1 \) of \( \chi_{N,u} \). We write
\[
z = \int_Y \Psi' \, dm_Y \quad \text{and} \quad \Psi'' = \Psi' - z.
\]
Combining Lemma 2.8, equations (44), (19), and estimate (50) we get
\[
\left| \mathbb{E}_{n \in [\tilde{N}]} 1_p(n) \chi_{N,u}(n) \right| \leq c_1 \| \chi_{N,u} \|_{U^2(Z_{\tilde{N}})} \leq c_1 \theta_1 = c_1 \theta(M_1) \leq \frac{1}{2} \theta_1(M_1) \leq \frac{1}{2} \delta_1(M)
\]
where the last estimate follows from (43) and the fact that \( M \leq M_1 \). From this estimate and (62) we deduce that
\[
\left| \mathbb{E}_{n \in [\tilde{N}]} 1_p(n) \chi_{N,u}(n) \Psi''(g'(n) \cdot e_Y) \right| \geq \frac{1}{2} \delta_1(M)
\]
where
\[
\| \Psi'' \|_{C^2(Y)} \leq H(M)^2 \quad \text{and} \quad \int_Y \Psi'' \, dm_Y = 0.
\]
We recall that Property (61) is also satisfied.

6.8. **Proof of the weak \( U^2 \)-decomposition result.** We are now very close to completing the proof of Theorem 6.1. To this end, we are going to combine (63), the total equidistribution of the sequence \( (g'(n) \cdot e_Y)_{n \in [\tilde{N}]} \), and Propositions 5.3 and 5.4 correspondingly to deduce a contradiction.

Recall that \( \chi_{N,s} = \chi_N * \phi \) (the convolution is taken in \( Z_{\tilde{N}} \)) where \( \phi \) is a kernel in \( Z_{\tilde{N}} \), meaning a non-negative function with \( \mathbb{E}_{n \in Z_{\tilde{N}}} \phi(n) = 1 \). Since \( \chi_u = \chi_N - \chi_s \), we can write \( \chi_{N,u} = \chi_N * \phi_s \), where \( \phi_s \) is a function on \( Z_{\tilde{N}} \) with \( \mathbb{E}_{n \in Z_{\tilde{N}}} |\phi_s(n)| \leq 2 \). We deduce from (63) that there exists an integer \( q \) with \( 0 \leq q < \tilde{N} \) such that
\[
\left| \mathbb{E}_{n \in [\tilde{N}]} 1_p(n + q \mod \tilde{N}) \chi_{N}(n) \Psi''(g'(n + q \mod \tilde{N}) \cdot e_Y) \right| \geq \frac{1}{4} \delta_1(M)
\]
where the residue class \( n + q \mod \tilde{N} \) is taken in \( [\tilde{N}] \) instead of the more usual interval \( [0,N] \). It follows that
\[
\left| \mathbb{E}_{n \in [\tilde{N}]} 1_p(n + m) \chi_{N}(n) \Psi''(g'(n + m) \cdot e_Y) \right| \geq \frac{1}{8} \delta_1(M)
\]
where either \( J \) is the interval \( [\tilde{N} - q] \) and \( m = q \), or \( J \) is the interval \( [\tilde{N} - q, \tilde{N}] \) and \( m = q - \tilde{N} \). We remark that in both cases \( 1_p(n + m) \chi_{N}(n) \) is \( \chi_{N,u} \), the convolution is taken in \( Z_{\tilde{N}} \), and thus we have
\[
\left| \mathbb{E}_{n \in [\tilde{N}]} 1_p(n) \chi_{N}(n) \Psi''(g'(n + m) \cdot e_Y) \right| \geq \frac{1}{8} \delta_1(M).
\]
Recall that \( \| \Psi'' \|_{C^2(Y)} \leq H(M)^2 \). By (61), the sequence \( (g'(n) \cdot e_Y)_{n \in [\tilde{N}]} \) is totally \( \tilde{\sigma}(M) \)-equidistributed in \( Y \). We remark that, depending on the case \( [11] \) or \( [10] \) of Corollary 1.3, Proposition 5.3 or Proposition 5.4 correspondingly can be applied to the nilmanifold \( Y \) and the sequence \( (g'(n))_{n \in [\tilde{N}]} \). Since by the definition of \( \tilde{\sigma}(M) \) (see (15)) we have
\[
\tilde{\sigma}(M) \leq \sigma \left( Y, \frac{\delta_1(M)}{10 H(M)^2} \right),
\]
by using one of the two propositions we deduce that
\[ |\mathbb{E}_{n \in [N]} 1_{P_1}(n) \chi(n) \Psi'(g'(n + m) \cdot e_Y)| \leq \]
\[ |\mathbb{E}_{n \in [N]} 1_{P_1}(n) \chi(n) \Psi'(g'(n + m) \cdot e_Y)| \leq \frac{\delta_1(M)}{10 H(M)} \| \Psi''_{1,1} \|_{C^2(Y)} \leq \frac{1}{10} \delta_1(M) \]
which contradicts (65). Hence, (61) fails, giving us that \| \chi_{N,u} \|_{U^3(\mathbb{Z}_N)} \leq \varepsilon. This completes the proof of Theorem [6.1]

6.9. Proof of the strong $U^3$-decomposition on the average. In this subsection we prove Theorem [1.6] Our basic ingredient is the weak decomposition result of Theorem [6.1]. We use it in an iterative way in an argument of energy increment; this is made possible because of the simple structure of \( \chi_{N,s} \) and the particular form of the kernels \( \phi_{N,\theta} \) that occur in Theorem [6.1]. We recall that these kernels were defined by (23) in Section 3.3; the most important property used in the subsequent argument is the monotonicity of their Fourier coefficients:

if \( 0 < \theta' < \theta \), then \( \hat{\phi}_{N,\theta'}(\xi) \geq \hat{\phi}_{N,\theta}(\xi) \geq 0 \) for every \( \xi \in \mathbb{Z}_N \).

We fix a function \( F : \mathbb{N} \times \mathbb{N} \times \mathbb{R}^+ \to \mathbb{R}^+ \), an \( \varepsilon > 0 \), and a finite positive measure \( \nu \) on the compact group \( M \) of multiplicative functions. We can assume without loss of generality that \( \nu \) is a probability measure.

We define inductively a sequence \( (\theta_j) \) of positive reals and sequences \( (N_j), (Q_j), (R_j) \) of positive integers as follows. We let \( \theta_1 = N_1 = Q_1 = R_1 = 1 \). Suppose that \( j \geq 1 \) and that the first \( j \) terms of the sequences are defined. We apply Theorem [6.1] with
\[ \theta_j \text{ substituted for } \theta_0 \text{ and } \frac{1}{F(Q_j, R_j, \varepsilon)} \text{ substituted for } \varepsilon. \]

Let \( N_{j+1} \) be such that the conclusions of Theorem [6.1] hold with these data as input for every \( N \geq N_{j+1} \). We get positive integers \( Q \) and \( R \) that we write respectively as \( Q_{j+1} \) and \( R_{j+1} \), and a real \( \theta_{j+1} \) with \( 0 < \theta_{j+1} < \theta_j \), such that for every \( N \geq N_{j+1} \) and every \( \chi \in M \), the functions
\[ \chi_{j+1,N,s} := \chi_N * \phi_{N,\theta_{j+1}} \quad \text{and} \quad \chi_{j+1,N,u} := \chi_N - \chi_{j+1,N,s} \]
satisfy the Properties (iii) and (iii) of Theorem [6.1] i.e.,
\[ |\chi_{j+1,N,s}(n + Q_{j+1}) - \chi_{j+1,N,s}(n)| \leq \frac{R_{j+1}}{N} \quad \text{for every } n \in \mathbb{Z}_N ; \]
\[ \| \chi_{j+1,N,u} \|_{U^3(\mathbb{Z}_N)} \leq \frac{1}{F(Q_j, R_j, \varepsilon)}. \]
Replacing \( N_{j+1} \) with \( \max_{i \leq j} N_i \) we can assume that the sequence \( (N_j) \) is increasing. By construction, the sequence \( (\theta_j) \) is decreasing.

Let \( J = 1 + [2\varepsilon^{-2}] \) and let \( N_0 = N_{J+1} \). For every \( N \geq N_0 \) we have
\[ \sum_{j=2}^{J} \int_{M} \| \chi_{j+1,N,s} - \chi_{j,N,s} \|_{L^2(\mathbb{Z}_N)}^2 \, d\nu(\chi) = \]
\[ \int_{M} \sum_{\xi \in \mathbb{Z}_N} |\hat{\chi}_N(\xi)|^2 \sum_{j=2}^{J} |\hat{\phi}_{N,\theta_{j+1}}(\xi) - \hat{\phi}_{N,\theta_j}(\xi)|^2 \, d\nu(\chi) \leq \]
\[ 2 \int_{M} \sum_{\xi \in \mathbb{Z}_N} |\hat{\chi}_N(\xi)|^2 \sum_{j=2}^{J} (\hat{\phi}_{N,\theta_{j+1}}(\xi) - \hat{\phi}_{N,\theta_j}(\xi)) \, d\nu(\chi), \]
where to get the last estimate we used that \( \theta_{N,j+1} \leq \theta_{N,j} \) and thus \( \phi_{N,\theta_{j+1}}(\xi) \geq 0 \) for every \( \xi \) by (10). Since \( |\phi_{N,\theta}(\xi)| \leq 1 \), the last quantity in the estimate is at most

\[
2 \int_{\mathcal{M}} \sum_{\xi \in \mathbb{Z}_N} |\hat{X}(\xi)|^2 \, d\nu(\chi) \leq 2.
\]

Therefore, for every \( N \geq N_0 \) there exists \( j_0 := j_0(F,N,\varepsilon,\nu) \) with

\[
(69) \quad 2 \leq j_0 \leq J
\]

such that

\[
(70) \quad \int_{\mathcal{M}} \|\chi_{j_0+1,N,s} - \chi_{j_0,N,s}\|_{L^2(\mathbb{Z}_N)}^2 \, d\nu(\chi) \leq \frac{2}{J - 1} \leq \varepsilon^2.
\]

For \( N \geq N_0 \), we define

\[
\psi_{N,1} := \phi_{N,\theta_{j_0}}; \quad \psi_{N,2} := \phi_{N,\theta_{j_0+1}};
\]

\[
\chi_{N,s} := \chi_N * \psi_{N,1} = \chi_{j_0,N,s}; \quad \chi_{N,u} := \chi_N - \chi_N * \psi_{N,2} = \chi_{j_0+1,N,u};
\]

\[
\chi_{N,e} := \chi_N * (\psi_{N,2} - \psi_{N,1}) = \chi_{j_0+1,N,s} - \chi_{j_0,N,s};
\]

\[
Q := Q_{j_0} \quad \text{and} \quad R := R_{j_0}.
\]

Then we have the decomposition

\[
\chi_N = \chi_{N,s} + \chi_{N,u} + \chi_{N,e},
\]

and furthermore, Property (iii) of Theorem 1.6 follows from (67) (applied for \( j = j_0 - 1 \)), Property (ii) follows from (68) (applied for \( j = j_0 \)), and Property (iv) follows from (70) and the Cauchy-Schwarz estimate. Furthermore, it follows from (69) that the integers \( N_0, Q, R \) are bounded by a constant that depends on \( F \) and \( \varepsilon \) only. Thus, all the announced properties are satisfied, completing the proof of Theorem 1.6. \( \square \)

**Appendix A. Rational elements in a nilmanifold**

In this section, working with 2-step nilmanifolds does not provide any simplification and so the results are stated for general nilmanifolds.

Let \( X = G/\Gamma \) be an \( s \)-step nilmanifold of dimension \( m \). As everywhere in this article we assume that \( G \) is connected and simply connected, and endowed with a Mal’cev basis. Recall that we write \( e_X \) for the image in \( X \) of the unit element \( 1_G \) of \( G \).

From Property (iii) of Mal’cev bases stated in Section 1.1 we immediately deduce:

**Lemma A.1.** \( \Gamma \) is a finitely generated group.

**A.1. Rational elements.** We recall that an element \( g \in G \) is \( Q \)-rational if \( g^m \in \Gamma \) for some \( m \in \mathbb{N} \) with \( m \leq Q \). We collect here some properties of rational elements. We note that all quantities introduced below depend implicitly on the nilmanifold \( X \).

**Lemma A.2** ([17, Lemma A.11]).

(i) For every \( Q \in \mathbb{N} \) there exists \( Q_1 \in \mathbb{N} \) such that the product of any two \( Q \)-rational elements is \( Q_1 \)-rational; it follows that the set of rational elements is a subgroup of \( G \).

(ii) For every \( Q \in \mathbb{N} \) there exists \( q \in \mathbb{N} \) such that the Mal’cev coordinates of any \( Q \)-rational element are rational with denominators at most \( q \); it follows that the set of \( Q \)-rational elements is a discrete subset of \( G \).

(iii) Conversely, for every \( q \in \mathbb{N} \) there exists \( Q \in \mathbb{N} \) such that, if the Mal’cev coordinates of \( g \in G \) are rational with denominators at most \( q \), then \( g \) is \( Q \)-rational.

**Corollary A.3.** For every \( Q \in \mathbb{N} \) there exists a finite set \( \Sigma := \Sigma(Q) \) of \( Q \)-rational elements such that all \( Q \)-rational elements belong to \( \Sigma(Q) \Gamma \).
Proof. Let $K$ be a compact subset of $G$ such that $G = KT$.

Let $Q \in \mathbb{N}$. Let $Q_1$ be associated to $Q$ by Part 3 of Lemma A.2, and let $\Sigma_1$ be the set of $Q_1$-rational elements of $K$. By Part 3 of Lemma A.2, $\Sigma_1$ is finite. Let $g$ be a $Q$-rational element of $G$. There exists $\gamma \in \Gamma$ such that $g\gamma^{-1} \in K$. Since $\gamma$ is obviously $Q$-rational, $g\gamma^{-1}$ is $Q_1$-rational and thus belongs to $\Sigma_1$. For each element $h$ of $\Sigma_1$ obtained this way we choose a $Q$-rational point $g$ such that $h \in g\Gamma$. Let $\Sigma := \Sigma(Q)$ be the set consisting of all elements obtained this way. We have that $\Sigma\Gamma$ contains all $Q$-rational elements. Furthermore, $|\Sigma| \leq |\Sigma_1|$ and so $\Sigma$ is finite, completing the proof. 

A.2. Rational subgroups. We gather here some basic properties of rational subgroups that we use in the main part of the article.

Definition. A rational subgroup $G'$ of $G$ is a closed and connected subgroup of $G$ such that its Lie algebra $g'$ admits a base that has rational coordinates in the Mal’cev basis of $G$.

An equivalent definition is that $\Gamma' := \Gamma \cap G'$ is co-compact in $G$. In this case, $G'/\Gamma'$ is called a sub-nilmanifold of $X$.

Lemma A.4 ([17], Lemma A.13). If $G'$ is a rational subgroup of $G$ and $h$ is a rational element, then $hG'h^{-1}$ is a rational subgroup of $G$.

Proof. The conjugacy map $h \mapsto g^{-1}hg$ is a polynomial map with rational coefficients and thus the linear map $Ad_h$ from $g$ to itself has rational coefficients. Since $g'$ has a base consisting of vectors with rational coefficients, the same property holds for $Ad_h g'$, that is, for the Lie algebra of $hG'h^{-1}$. This proves the claim.

The argument used to deduce Lemma A.4 shows that the group $\Gamma \cap hG'h^{-1}$ is finitely generated.

Lemma A.5. Let $X = G/\Gamma$ be an s-step nilmanifold, $G' \subset G$ be a rational subgroup, and $g \in G$ be a rational element. Then

(i) $\Gamma \cap g^{-1}\Gamma g \cap G'$ is a subgroup of finite index of $\Gamma \cap G'$;
(ii) $\Gamma \cap g^{-1}\Gamma g \cap G'$ is a subgroup of finite index of $g^{-1}\Gamma g \cap G'$.

Proof. By Part 3 of Lemma A.2 all elements of $g\Gamma g^{-1}$ are rational. Hence, if $\gamma \in \Gamma \cap G'$ there exists $k \in \mathbb{N}$ with $(g\gamma g^{-1})^k \in \Gamma$ and so we have $\gamma^k \in g^{-1}\Gamma g \cap \Gamma \cap G'$. Applying Lemma A.4 to $G'$ and $\Gamma \cap G'$ we get that this last group is finitely generated. By induction on $s$ it is easy to deduce that $g^{-1}\Gamma g \cap \Gamma \cap G'$ has finite index in $\Gamma \cap G'$. This proves (ii). Since $g\Gamma g^{-1}$ is co-compact in $G$, substituting this group for $G$ and $g^{-1}$ for $g$ in the preceding statement, we get (i).

Lemma A.6. Let $g \in G$ be a rational element and $G'$ a rational subgroup of $G$. Then $G'g \cdot e_X := \{hg \cdot e_X : h \in G'\}$ is a closed sub-nilmanifold of $X$.

Proof. By Lemma A.4, $g^{-1}G'g$ is a rational subgroup of $G$. Therefore, $\Gamma \cap g^{-1}G'g$ is co-compact in $g^{-1}G'g$ and thus $g\Gamma g^{-1} \cap G'$ is co-compact in $G'$. But $g\Gamma g^{-1} \cap G'$ is the stabilizer $\{h \in G' : h \cdot g \cdot e_X = g \cdot e_X\}$ of $g \cdot e_X$ in $G'$ and thus the orbit $G' \cdot (g \cdot e_X)$ is compact and can be identified with the nilmanifold $G'/(g\Gamma g^{-1} \cap G')$.

Appendix B. Solution sets related to some homogeneous quadratic forms

We give a proof of Proposition 1.4 from the introductory section.

Proposition B.1. Let the quadratic form $p$ satisfy the hypothesis of Theorem 1.4. Then there exist $\ell_0, \ell_1$ positive and $\ell_2, \ell_3$ non-negative integers with $\ell_2 \neq \ell_3$, such that for every $k, m, n \in \mathbb{N}$, the integers $x = k\ell_0(m + \ell_1n)$ and $y = k\ell_0(m + \ell_2n)(m + \ell_3n)$ satisfy the equation $p(x, y, z) = 0$ some $z \in \mathbb{N}$. 

Proof. Let
\[(71) \quad ax^2 + by^2 + cz^2 + dxy + exz + fyz = 0\]
be the equation we are interested in solving. Recall that by assumption \(a, b > 0\) and \(c < 0\), and that all three integers
\[
\Delta_1 := e^2 - 4ac, \quad \Delta_2 := f^2 - 4bc, \quad \Delta_3 := (e + f)^2 - 4c(a + b + d)
\]
are non-zero squares.

A direct computation shows that if \((x_0, y_0, z_0)\) is a solution of (71), then also the following is a solution
\[
x = k(-ax_0 + dy_0 + ez_0)m^2 - (2by_0 + fz_0)mn + bx_0n^2
\]
\[
y = k(ay_0m^2 - (2ax_0 + ez_0)mn - (by_0 + dx_0 + fz_0)n^2)
\]
\[
z = kz_0(am^2 + dmn + bn^2)
\]
where \(k, m, n, \in \mathbb{Z}\).

The discriminant of the quadratic form
\[Q_1(m, n) := -(ax_0 + dy_0 + ez_0)m^2 - (2by_0 + fz_0)mn + bx_0n^2\]
turns out to be \(z_0^2 \Delta_2\) which is a square since by assumption \(\Delta_2\) is a square. Hence, \(Q_1(m, n)\) factors into a product of linear forms with rational coefficients. Similarly, the discriminant of the quadratic form
\[Q_2(m, n) := ay_0m^2 - (2ax_0 + ez_0)mn - (by_0 + dx_0 + fz_0)n^2\]
turns out to be \(z_0^2 \Delta_1\) which is a square since by assumption \(\Delta_1\) is a square. Hence, \(Q_2(m, n)\) factors into a product of linear forms with rational coefficients.

The assumption that \(\Delta_3\) is a square is used to guarantee that a non-trivial choice of \(x_0, y_0, z_0\) can be made so that the coefficients of \(m^2\) in the quadratic forms \(Q_1(m, n)\) and \(Q_2(m, n)\) are equal, i.e.
\[-(ax_0 + dy_0 + ez_0) = ay_0\]
is satisfied. Indeed, if we multiply equation (71) by \(e^2\) and insert \(-(ax_0 + (d + a)y_0)\) in place of \(ez_0\), we lead to the equation
\[a^2ex_0^2 + a(2cd + 2ac - e^2 - ef)x_0y_0 + (be^2 + cd^2 + a^2c + 2acd - def - ae)\]
\[y^2_0 = 0.
\]
A direct computation shows that its discriminant is \(a^2e^2 \Delta_3\), which is square since by assumption \(\Delta_3\) is a square. This leads to the following solution of (71)
\[x_0 = 2ac + 2cd - e^2 - ef + e\sqrt{\Delta_3}
\]
\[y_0 = -2ac
\]
\[z_0 = a(e + f + \sqrt{\Delta_3})
\]
which is non-trivial since \(y_0 > 0\). We work with these choices of \(x_0, y_0, z_0\) next.

Combining the above, we deduce that under the stated assumptions on \(a, b, c\), there exist \(l_1, \ldots, l_8 \in \mathbb{Z}\), with \(l_1l_3 = l_5l_7 \neq 0\), such that for every \(k, m, n \in \mathbb{Z}\) the integers
\[x = k(l_1m + l_2)n(l_3m + l_4n),
\]
\[y = k(l_5m + l_6)n(l_7m + l_8n)
\]
satisfy equation (71) for some \(z := z_{m,n} \in \mathbb{N}\). Inserting \(l_1l_3l_5l_7n\) in place of \(n\), we get that there exist \(l'_1, \ldots, l'_4 \in \mathbb{Z}\), such that for every \(k, m, n \in \mathbb{Z}\) the integers
\[x = k\ell_0(m + l'_1n)(m + l'_2n),
\]
\[y = k\ell_0(m + l'_3n)(m + l'_4n),
\]
where \(\ell_0 = |l_1l_3| = |l_5l_7| \neq 0\), satisfy equation (71) for some \(z' := z'_{m,n} \in \mathbb{N}\).
Since the quadratic forms $Q_1$ and $Q_2$ have non-zero discriminant, we have $l'_i \neq l'_j$ and $l''_3 \neq l''_4$. Without loss of generality we can assume that $l'_1 \leq l'_i$ for $i = 2, 3, 4$. Inserting $m - l'_1n$ in place of $m$, we get that for every $k, m, n \in \mathbb{Z}$ the integers

\[
x = k\ell_0m + (l''_2 - l''_1)n,
\]

\[
y = k\ell_0(m + (l''_3 - l''_1)n)(m + (l''_4 - l''_1)n)
\]
satisfy equation (11) for some $z'' := z''_{m,n} \in \mathbb{N}$. Letting $\ell_1 = l''_2 - l''_1$, $\ell_2 = l''_3 - l''_1$, $\ell_3 = l''_4 - l''_1$, we get the asserted conclusion with $z''_{m,n}$ in place of $z$.

Alternatively, a proof that is free of computations can be given using the fact that the discriminant of the forms $p(x, 0, z)$, $p(0, y, z)$, $p(x, x, z)$ is a non-zero square. We chose a more hands on argument since it determines the integers $\ell_1, \ell_2, \ell_3$ explicitly.

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