On stabilization of small solutions in the nonlinear Dirac equation with a trapping potential

Scipio Cuccagna and Mirko Tarulli

August 26, 2014

Abstract

We consider a Dirac operator with short range potential and with eigenvalues. We add a nonlinear term and we show that the small standing waves of the corresponding nonlinear Dirac equation (NLD) are attractors for small solutions of the NLD. This extends to the NLD results already known for the Nonlinear Schrödinger Equation (NLS).

Keywords: Nonlinear Dirac equation, standing waves.

1 Introduction

We consider

\[ \begin{cases} iu_t - Hu + g(u \overline{u}) \beta u = 0 \quad \text{with} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3 \\ u(0, x) = u_0(x) \end{cases} \]  \tag{1.1}

where for \( M > 0 \) we have for a potential \( V(x) \)

\[ H = D_{\#} + V \]  \tag{1.2}

where \( D_{\#} = -i \sum_{j=1}^3 \alpha_j \overline{\partial_j} + M \beta, \) with for \( j = 1, 2, 3 \)

\[ \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_{C^2} & 0 \\ 0 & -I_{C^2} \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

The unknown \( u \) is \( \mathbb{C}^4 \)-valued. Given two vectors of \( \mathbb{C}^4, uv := u \cdot v \) is the inner product in \( \mathbb{C}^4, v^* \) is the complex conjugate, \( u \cdot v^* \) is the hermitian product in \( \mathbb{C}^4, \) which we write as \( uv^* = u \cdot v^* \). We set \( \overline{u} := \beta u^*, \) so that \( u \overline{u} = u \cdot \beta u^*. \)

We introduce the Japanese bracket \( \langle x \rangle := \sqrt{1 + |x|^2} \) and the spaces defined by the following norms:

\[ L^{p,s}(\mathbb{R}^3, \mathbb{C}^4) \] defined with \( \|u\|_{L^p, \#(\mathbb{R}^3, \mathbb{C}^4)} := \|\langle x \rangle^s u\|_{L^p(\mathbb{R}^3, \mathbb{C}^4)}; \]

\[ H^k(\mathbb{R}^3, \mathbb{C}^4) \] defined with \( \|u\|_{H^k(\mathbb{R}^3, \mathbb{C}^4)} := \|\langle x \rangle^k \hat{u}\|_{L^2(\mathbb{R}^3, \mathbb{C}^4)}, \) where \( \hat{u} \) is the Fourier transform;

\[ H^{k,s}(\mathbb{R}^3, \mathbb{C}^4) \] defined with \( \|u\|_{H^{k,s}(\mathbb{R}^3, \mathbb{C}^4)} := \|\langle x \rangle^s u\|_{H^k(\mathbb{R}^3, \mathbb{C}^4)}; \]

\[ \Sigma_k := \bigcap L^{2,k}(\mathbb{R}^3, \mathbb{C}^4) \cap H^k(\mathbb{R}^3, \mathbb{C}^4) \] with \( \|u\|_{\Sigma_k}^2 = \|u\|^2_{\Sigma_k} = \|u\|^2_{H^k(\mathbb{R}^3, \mathbb{C}^4)} + \|u\|^2_{L^{2,k}(\mathbb{R}^3, \mathbb{C}^4)}. \]
For \( f, g \in L^2(\mathbb{R}^3, \mathbb{C}^4) \) consider the bilinear map
\[
\langle f, g \rangle = \int_{\mathbb{R}^3} f(x)g(x)dx = \int_{\mathbb{R}^3} f(x) \cdot g(x)dx.
\] (1.3)

We assume the following.

(H1) \( g(0) = 0, \quad g \in C^\infty(\mathbb{R}, \mathbb{R}) \).

(H2) \( V \in \mathcal{S}(\mathbb{R}^3, S_4(\mathbb{C})) \) with \( S_4(\mathbb{C}) \) the set of self-adjoint \( 4 \times 4 \) matrices and \( \mathcal{S}(\mathbb{R}^3, \mathcal{E}) \) the space of Schwartz functions from \( \mathbb{R}^3 \) to \( \mathcal{E} \), with the latter a Banach space on \( \mathbb{C} \).

(H3) \( \sigma_p(H) = \{ e_1 < e_2 < e_3 \cdots < e_n \} \subset (-\mathcal{M}, \mathcal{M}) \). Here we assume that all the eigenvalues have multiplicity 1. Each point \( \tau = \pm \mathcal{M} \) is neither an eigenvalue nor a resonance (that is, if \( (D_{\mathcal{M}} + V)u = \tau u \) with \( u \in C^\infty \) and \( |u(x)| \leq C|x|^{-1} \) for a fixed \( C \), then \( u = 0 \)).

(H4) There is an \( N \in \mathbb{N} \) with \( N > \mathcal{M}(\min\{e_i - e_j : i > j\})^{-1} \) s.t. if \( \mu \cdot e := \mu_1 e_1 + \cdots + \mu_n e_n \) then
\[
\mu \in \mathbb{Z}^n \text{ with } |\mu| \leq 4N + 6 \Rightarrow |\mu \cdot e| = 0
\] (1.4)
\[
(\mu - \nu) \cdot e = 0 \text{ and } |\mu| = |\nu| \leq 2N + 3 \Rightarrow \mu = \nu.
\] (1.5)

(H5) Consider the set \( M_{\min} \) defined in \([2,5]\) and for any \((\mu, \nu) \in M_{\min} \) we consider the function
\[
G_{\mu\nu}(x) \quad (\text{see the proof of Lemma } [6,11]), \quad \hat{G}_{\mu\nu}(\xi) \quad \text{the Fourier transform associated to } H \text{ of } G_{\mu\nu}(x), \quad \text{the sphere } \mathcal{S}_{\mu\nu} = \{ \xi \in \mathbb{R}^3 : |\xi|^2 + \mathcal{M}^2 = |(\nu - \mu) \cdot e|^2 \}. \]

Then we assume that for any \((\mu, \nu) \in M_{\min} \) the restriction of \( \hat{G}_{\mu\nu} \) on the sphere \( \mathcal{S}_{\mu\nu} \) is \( \hat{G}_{\mu\nu}|_{\mathcal{S}_{\mu\nu}} \neq 0 \).

To each \( e_j \) we associate an eigenfunction \( \phi_j \). We choose them s.t. \( \Re\langle \phi_j, \phi_k^* \rangle = \delta_{jk} \). To each \( \phi_j \) we associate nonlinear bound states.

**Proposition 1.1** (Bound states). Fix \( j \in \{1, \cdots, n\} \). Then \( \exists a_0 > 0 \text{ s.t. } \forall z_j \in B(0, a_0) \), there is a unique \( Q_{jz_j} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4) := \cap_{t \geq 0} \Sigma_t(\mathbb{R}^3, \mathbb{C}^4) \), s.t.
\[
HQ_{jz_j} + g(Q_{jz_j}, \overline{Q_{jz_j}})\beta Q_{jz_j} = E_{jz_j}Q_{jz_j},
\]
\[
Q_{jz_j} = z_j\phi_j + q_{jz_j}, \quad \langle q_{jz_j}, \phi_j^* \rangle = 0,
\] (1.6)
and s.t. we have for any \( r \in \mathbb{N} \):

1. \( (q_{jz_j}, E_{jz_j}) \in C^\infty(B(0, a_0), \Sigma_r \times \mathbb{R}) \); we have \( q_{jz_j} = z_j\tilde{q}_{jz_j}(|z_j|^2) \), with \( \tilde{q}_{jz}(t^2) = t^2\tilde{q}_j(t^2) \), \( \tilde{q}_j(t) \in C^\infty((-a_0^2, a_0^2), \Sigma_r(\mathbb{R}^3, \mathbb{C}^4)) \) and \( E_{jz_j} = E_j(|z_j|^2) \) with \( E_j(t) \in C^\infty((-a_0^2, a_0^2), \mathbb{R}) \);

2. \( \exists C > 0 \text{ s.t. } \|q_{jz_j}\|_{\Sigma_r} \leq C|z_j|^3, \quad |E_{jz_j} - e_j| < C|z_j|^2 \).

For the \( \Sigma_r \) see sect \([2,4]\) The only non elementary point in Prop. \([1,1]\) is the independence of \( a_0 \) with respect of \( r \) (which strictly speaking is not necessary in this paper), which can be proved with routine arguments as in the Appendix of \([1,2]\).

**Definition 1.2.** Let \( b_0 > 0 \) be sufficiently small so that for \( z_j \in B(0, b_0) \), \( Q_{jz_j} \) exists for all \( j \in \{1, \cdots, n\} \). Set \( z_j = z_j R + iz_j I \) for \( z_j R, z_j I \in \mathbb{R} \) and \( D_{jA} := \frac{\partial}{\partial z_j A} \) for \( A = R, I \). We set
\[
\mathcal{H}_c[z] = \mathcal{H}_c[z_1, \cdots, z_n] \text{ where } z := (z_1, \cdots, z_n) := \{ \eta \in L^2(\mathbb{R}^3, \mathbb{C}^4) : \Re \langle i\eta^*, D_{jR}Q_{jz_j} \rangle = \Re \langle i\eta^*, D_{jI}Q_{jz_j} \rangle = 0 \text{ for all } j \}.
\] (1.7)

In particular we have
\[
\mathcal{H}_c[0] = \{ \eta \in L^2(\mathbb{R}^3, \mathbb{C}^4) : \langle \eta^*, \phi_j \rangle = 0 \text{ for all } j \}.
\] (1.8)

We denote by \( P_c \) the orthogonal projection of \( L^2(\mathbb{R}^3, \mathbb{C}^4) \) onto \( \mathcal{H}_c[0] \).
We will prove the following theorem.

**Theorem 1.3.** Assume (H1)–(H5). Then there exist \( \epsilon_0 > 0 \) and \( C > 0 \) such that for \( \epsilon = \| u(0) \|_{H^4} < \epsilon_0 \) then the solution \( u(t) \) of (1.1) can be written uniquely for all times as

\[
    u(t) = \sum_{j=1}^{n} Q_{j} z_j(t) + \eta(t) \quad \text{with} \quad \eta(t) \in \mathcal{H}_e[z(t)]
\]

such that there exist a unique \( j_0 \), a \( \rho_+ \in [0, \infty)^n \) with \( \rho_{+j} = 0 \) for \( j \neq j_0 \), such that \( |\rho_+| \leq C \| u(0) \|_{H^4} \) and an \( \eta_+ \in H^4 \) with \( \| \eta_+ \|_{L^\infty} \leq C \| u(0) \|_{H^4} \) such that we have

\[
    \lim_{t \to +\infty} \| \eta(t, x) - e^{-itD_x} \eta_+(x) \|_{L_x^\infty(R^3, \mathcal{C}^1)} = 0, \\
    \lim_{t \to +\infty} |z_j(t)| = \rho_{+j}.
\]

Furthermore we have \( \eta = \tilde{\eta} + A(t, x) \) such that

\[
    \text{for preassigned } p_0 > 2, \text{ for all } p \geq p_0 \text{ and for } \frac{2}{p} = \frac{3}{2} \left(1 - \frac{2}{q}\right)
\]

we have

\[
    \| z \|_{L_x^r(R^3)} + \| \tilde{\eta} \|_{L_t^r([0, \infty), B_{q, \infty}^{r/2}(R^3, \mathcal{C}^1)} \leq C \| u(0) \|_{H^4(R^3, \mathcal{C}^1)},
\]

\[
    \| \dot{z}_j + i\epsilon_j \dot{z}_j \|_{L_x^r(R^3)} \leq C \| u(0) \|_{L_t^2(H^4(R^3, \mathcal{C}^1))}^2
\]

(for the Besov spaces \( B_{p,q}^r \) see Sect. 2.1) and s.t. \( A(t, \cdot) \in \Sigma_4 \) for all \( t \geq 0 \) with

\[
    \lim_{t \to +\infty} \| A(t, \cdot) \|_{\Sigma_4} = 0.
\]

Theorem 1.3 is a version for the NLD of the result proved for the NLS in [12]. We recall that the dynamical properties of small solutions of the NLS have been investigated in a long list of papers. In particular, after initial work on orbital stability in [20], the asymptotic convergence to standing waves (obtained here when \( \rho_+ \neq 0 \) in Theorem 1.3) has been studied in [22]–[25], [19], [28]–[31], [16], [14]. We refer to [12, 9] for more references.

Theorem 1.3 features a rather peculiar property of nonlinear systems. In the case of the linear Dirac equation \( \dot{u}_t = Hu \) we know that the topological disks of standing waves of Proposition 1.1 are in fact planes, that the standing waves \( e^{it\epsilon_j}Q_{j} z_j(x) \) decouple from each other and from the continuous spectrum components, and in particular that any linear combination of standing waves is a solution.

After adding to the Dirac equation a nonlinearity such as in (1.1), the sets of standing waves still persist, only in a more complicated form. In particular at small energies there are manifolds, topological disks in our case, of standing waves.

When there is only one such topological disk, that is when \( n = 1 \) in (H3), there is not a substantial difference between the evolution under the NLD with or without nonlinearity, as shown in the case of the NLS in [16], and as shown here for the NLD. However, when there are two or more disks, that is \( n \geq 2 \) in (H3), then the literature consistently suggests, and we prove in Theorem 1.3 under a certain set of hypotheses, that linear combinations of two or more standing waves, or specifically nonlinear quasi-periodic versions of linear combinations of standing waves, should not be solutions of the NLD even in an asymptotic sense, and that instead the union of the disks of standing waves is an attractor for the system.
Small solutions of the NLD and of the NLS are asymptotically equal to exactly one standing wave (possibly the vacuum), up to radiation which scatters and up to a phase factor, because of nonlinear interactions between the various components of the solutions, in particular because of friction induced by the radiation on the discrete modes. This phenomenon is discussed at length in [7], in [24, 25], in [28–31], in [2, 9, 12] and in the references therein.

As mentioned Theorem 1.3 is totally consistent to the results in [12] and in previous literature. In fact in the present paper we show how to partially extend the result in [12] to the case of the NLD. Most of work, at the level of the search of appropriate sets of coordinates, is the same of [12], since in terms of the Hamiltonian formalism used in [12] NLD is the same as NLS.

The difference of course is that the NLD requires its own set of technical machinery about linear dispersion theory, Strichartz and smoothing estimates, which is not the same of the NLS and which is trickier to prove. Nonetheless, all the linear theory we need here has been already developed in the literature, in particular in [4, 5, 6]. Notice that these latter references all deal with problems of asymptotic stability of standing waves of the NLD. In particular [4, 5] discusses partial extensions to the NLS of the results in [22–25, 19, 28–31, 16].

Since the main ideas come from [12], we will refer to therein for an extensive discussion of the of the main issues and differences with the previous literature. Here we will only mention succinctly few points. First of all there is a model system where the result can be proved rather easily, and this is represented by the NLD in the coordinate system which is the domain of the map (4.17). Using the linear dispersion theory of Dirac operators developed mostly in [4, 5] and some work in [6], it is rather standard to prove our result in this model problem. In [12] there is a long discussion about the lack of sufficient uncoupling between standing waves inside the energy, reflected in the fact that in the expansion of the energy in (3.3) in the second line the summation on \( l \) starts from 0. However, after moving to Darboux coordinates, the same expansion has the summation on \( l \) starting from 1, see (3.3). This is crucial in the proof, but is solved in [12] and is not discussed here. After that, by the Birkhoff normal form argument we reduce to the model case. This part is proved in [12], so that here we only state the result in Theorem 4.6. The rest of the proof consists in proving dispersion of the continuous component of the solution (and here we use the technology of Dirac operators in [4, 5]) and the so called Nonlinear Fermi Golden Rule which shows that all the discrete modes, except at most one, converge to 0. This part of the proof is similar to [12] (but with some slight modification in the spirit of [11] because of the possible presence of pairs of eigenvalues with different signs). We point out that the fact that here \( u_0 \in H^4 \) is required in the proof of Lemma 6.11, in particular in (5.39).

Here as in [12] we do not prove, as done in [2] in an easier setting, that hypothesis (H5) holds for generic pairs \( (V, g(u\mathcal{M})) \). While in [12] what was missing to repeat the argument in [2] was a meaningful mass term, here we have mass \( \mathcal{M} \) but the dependence of the linear operator \( H \) on \( \mathcal{M} \) requires new ideas.

Furthermore, while in Theorem 1.4 [12] we prove whether a given standing wave is orbitally stable or unstable, we do not give here a similar result.

2 Further notation and coordinates

2.1 Notation

For \( k \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \), the Besov space \( B_{p,q}^k(\mathbb{R}^3, \mathbb{C}^d) \) is the space of all tempered distributions \( f \in \mathcal{S}'(\mathbb{R}^3, \mathbb{C}^d) \) such that

\[
\| f \|_{B_{p,q}^k} = \left( \sum_{j \in \mathbb{N}} 2^{jkq} \| \varphi_j * f \|_p^q \right)^{\frac{1}{q}} < +\infty
\]
with $\hat{\varphi} \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ such that $\sum_{j \in \mathbb{Z}} \hat{\varphi}(2^{-j} \xi) = 1$ for all $\xi \in \mathbb{R}^3 \setminus \{0\}$, $\hat{\varphi}_j(\xi) = \hat{\varphi}(2^{-j} \xi)$ for all $j \in \mathbb{N}^*$ and for all $\xi \in \mathbb{R}^3$, and $\hat{\varphi}_0 = 1 - \sum_{j \in \mathbb{N}^*} \hat{\varphi}_j$. It is endowed with the norm $\|f\|_{B_p^k}^k$.

- We denote by $\mathbb{N} = \{1, 2, \ldots\}$ the set of natural numbers and set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- Given a Banach space $X$, $v \in X$ and $\delta > 0$ we set $B_X(v, \delta) := \{x \in X \mid \|v - x\|_X < \delta\}$.
- We denote $z = (z_1, \ldots, z_n)$, $|z| := \sqrt{\sum_{j=1}^{n} |z_j|^2}$.
- We set $\partial_i := \partial_{z_i}$ and $\partial_{\bar{z}} := \partial_{\bar{z}_i}$. Here as customary $\partial_{z_i} = \frac{1}{2}(D_{IR} - iD_{II})$ and $\partial_{\bar{z}_i} = \frac{1}{2}(D_{IR} + iD_{II})$.
- Occasionally we use a single index $\ell = j, \bar{j}$. To define $\bar{\ell}$ we use the convention $\bar{\bar{j}} = j$. We will also write $z_{\ell} = z_j$.
- We will consider vectors $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and for vectors $\mu, \nu \in (\mathbb{N} \cup \{0\})^n$ we set $z^\mu \nu := z_1^{\mu_1} \cdots z_n^{\mu_n} \overline{z_1}^{\nu_1} \cdots \overline{z_n}^{\nu_n}$. We will set $|\mu| = \sum_{j} \mu_j$.
- We have $dz_j = dz_jR + idz_{jI}$, $d\bar{z}_j = dz_jR - idz_{jI}$.
- We consider the vector $e = (e_1, \ldots, e_n)$ whose entries are the eigenvalues of $H$.

**Remark 2.1.** We draw the attention of the reader to the fact that the complex conjugate of $v \in \mathbb{C}^4$ is $v^*$ with $\overline{v} = \beta v^*$ while for $\zeta \in \mathbb{C}$ the complex conjugate is $\overline{\zeta}$ which is more convenient notation in some later formulas than writing $\zeta^*$.

### 2.2 Coordinates

The first thing we need is an ansatz, see Lemma 2.6 [12].

**Lemma 2.2.** There exists $c_0 > 0$ s.t. there exists a $C > 0$ s.t. for all $u \in H^1$ with $\|u\|_{H^1} < c_0$, there exists a unique pair $(z, \Theta) \in \mathbb{C}^n \times (H^1 \cap H_c[z])$ s.t.

$$u = \sum_{j=1}^{n} Q_j z_j + \Theta \text{ with } |z| + \|\Theta\|_{H^1} \leq C\|u\|_{H^1}. \tag{2.1}$$

Finally, the map $u \to (z, \Theta)$ is $C^\infty(B_{H^1}(0, c_0), \mathbb{C}^n \times H^4)$ and satisfies the gauge property

$$z(e^{i\varphi} u) = e^{i\varphi} z(u) \text{ and } \Theta(e^{i\varphi} u) = e^{i\varphi} \Theta(u). \tag{2.2}$$

We now recall from [12] the following definitions.

**Definition 2.3.** Given $z \in \mathbb{C}^n$, we denote by $\hat{Z}$ the vector with entries $(z_i \overline{z}_j)$ with $i, j \in [1, n]$ ordered in lexicographic order. We denote by $Z$ the vector with entries $(z_i \overline{z}_j)$ with $i, j \in [1, n]$ ordered in lexicographic order but only with pairs of indexes with $i \neq j$. Here $Z \in L$ with $L$ the subspace of $C_0^\infty = \{(a_{i,j})_{i,j=1,\ldots,n} : i \neq j\}$ where $n_0 = n(n-1)$, with $(a_{i,j}) \in L$ iff $a_{i,j} = \overline{z}_j z_i$ for all $i, j$. For a multi index $m = \{m_{ij} \in \mathbb{N}_0 : i \neq j\}$ we set $Z^m = \prod (z_i \overline{z}_j)^{m_{ij}}$ and $|m| := \sum_{i,j} m_{ij}$.
**Definition 2.4.** Consider the set of multiindexes $\mathbf{m}$ as in Def. Consider for any $k \in \{1, \ldots, n\}$ the set

\[
\mathcal{M}_k(r) = \{ \mathbf{m} : \left| \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij}(e_i - e_j) - e_k \right| > \mathcal{M} \text{ and } |\mathbf{m}| \leq r \}
\]

\[
\mathcal{M}_0(r) = \{ \mathbf{m} : \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij}(e_i - e_j) = 0 \text{ and } |\mathbf{m}| \leq r \}.
\]

Set now

\[
M_k(r) = \{ (\mu, \nu) \in \mathbb{N}_0^n \times \mathbb{N}_0^n : \exists \mathbf{m} \in \mathcal{M}_k(r) \text{ s.t. } z^{\mu \nu} = \mathbb{Z}_k \mathbf{Z}^\mathbf{m} \},
\]

\[
M(r) = \bigcup_{k=1}^{n} M_k(r)
\]

We also set $M = M(2N + 4)$ and

\[
M_{\min} = \{ (\mu, \nu) \in M : (\alpha, \beta) \in M \text{ with } \alpha_j \leq \mu_j \text{ and } \beta_j \leq \nu_j \forall j \Rightarrow (\alpha, \beta) = (\mu, \nu) \}.
\]

**Lemma 2.5.** The following facts hold.

1. If $(\mu, \nu) \in M_{\min}$ then for any $j$ we have $\mu_j \nu_j = 0$.

2. Suppose that $(\mu, \nu) \in M_{\min}$, $(\alpha, \beta) \in M_{\min}$ and $(\mu - \nu) \cdot \mathbf{e} = (\alpha - \beta) \cdot \mathbf{e}$. Then $(\mu, \nu) = (\alpha, \beta)$.

**Proof.** First of all it is easy to show that $(\mu, \nu) \in M(r)$ if and only if $|\nu| = |\mu| + 1$, $|\mu| \leq r$ and $|\mu - \nu| \cdot |\mathbf{e}| > \mathcal{M}$.

Suppose that $\mu_j \geq 1$ and $\nu_j \geq 1$. Then $(\mu, \nu) \notin M_{\min}$ because $(\alpha, \beta) \in M$ with

\[
\alpha_k = \begin{cases} 
\mu_k \text{ for } k \neq j, \\
\mu_j - 1 \text{ for } k = j 
\end{cases}
\]

\[
\beta_k = \begin{cases} 
\nu_k \text{ for } k \neq j, \\
\nu_j - 1 \text{ for } k = j.
\end{cases}
\]

This proves claim (1).

By $(\mu - \nu) \cdot \mathbf{e} = (\alpha - \beta) \cdot \mathbf{e}$ and (H3) we obtain $\mu - \nu = \alpha - \beta$, which by claim (1) yields $(\mu, \nu) = (\alpha, \beta)$.

By an use of the same argument of [12] we obtain

**Lemma 2.6.** Assuming (H3) then the following properties are fulfilled.

1. For $\mathbf{Z}^\mathbf{m} = z^{\mu \nu}$, then $\mathbf{m} \in \mathcal{M}_0(2N + 4)$ implies $\mu = \nu$. In particular $\mathbf{m} \in \mathcal{M}_0(2N + 4)$ implies $\mathbf{Z}^\mathbf{m} = |z_1|^{l_1} \ldots |z_n|^{l_n}$ for some $(l_1, \ldots, l_n) \in \mathbb{N}_0^n$.

2. For $|\mathbf{m}| \leq 2N + 3$ and any $j$ we have $\sum_{a,b} (e_a - e_b) m_{ab} - e_j \neq 0$.

The following elementary lemma is very similar to Lemma 2.4 [12].

**Lemma 2.7.** We have the following facts.

1. Consider a vector $\mathbf{m} = (m_{ij}) \in \mathbb{N}_0^n$ s.t. $\sum_{i < j} m_{ij} > N$ for $N > \mathcal{M}(\min\{e_j - e_i : j > i\})^{-1}$, see (H3). Then for any eigenvalue $e_k$ we have

\[
\sum_{i < j} m_{ij}(e_i - e_j) - e_k < -\mathcal{M}.
\]
Proof. The proof is very similar to Lemma 2.4 in [12]. For example, (2.6) follows immediately from the latter inequality due to the definition of Lemma 2.8.

Furthermore, for $m$ regular,

(3) Consider $m \in \mathbb{N}_0^n$ and the monomial $z_j \cdot Z^m$. Suppose $|m| \geq 2N + 3$. Then there are $a, b \in \mathbb{N}_0^n$ such that we have

$$\sum_{i<j} a_{ij} = N + 1 = \sum_{i<j} b_{ij},$$

$$a_{ij} = b_{ij} = 0 \text{ for all } i > j,$$

$$a_{ij} + b_{ij} \leq m_{ij} + m_{ji} \text{ for all } (i, j)$$

and moreover there are two indexes $(k, l)$ s.t.

$$\sum_{i<j} a_{ij}(e_i - e_j) - e_k < -M \quad \text{and} \quad \sum_{i<j} b_{ij}(e_i - e_j) - e_l < -M$$

and such that for $|z| \leq 1$

$$|z_j \cdot Z^m| \leq |z_j| |z_k Z^a| |z_l Z^b|.$$ 

(2.9) For $m$ with $|m| \geq 2N + 3$ there exist $(k, l)$ and $a \in \mathcal{M}_k$ and $b \in \mathcal{M}_l$ s.t. (2.9) holds.

Proof. The proof is very similar to Lemma 2.4 in [12]. For example, (2.6) follows immediately from

$$\sum_{i<j} m_{ij}(e_i - e_j) - e_k \leq -\min\{e_j - e_i : i > j\} N - e_1 < -M,$$

with the latter inequality due to the definition of $N$. All the other claims can be proved like the rest of Lemma 2.4 in [12].

Since $(z, \Theta)$ in (2.1) are not a system of independent coordinates we need the following, see Lemma 2.5 [12].

Lemma 2.8. There exists $d_0 > 0$ such that for all $z \in \mathbb{C}$ with $|z| < d_0$ there exists $R[z] : \mathcal{H}[0] \to \mathcal{H}[z]$ such that $P_c|_{\mathcal{H}[z]} = R[z]^{-1}$, with $P_c$ the orthogonal projection of $L^2$ onto $\mathcal{H}[0]$, see Def. 1.2. Furthermore, for $|z| < d_0$ and $\eta \in \mathcal{H}[0]$, we have the following properties.

(1) $R[z] \in C^\infty(B(\Phi_r, \Phi_r))$ for any $r \in \mathbb{R}$.

(2) For any $r > 0$, we have $||R[z] - 1||_{\Sigma_r} \leq c_r |z|^2 |\eta|_{\Sigma_{-r}}$ for a fixed $c_r$.

(3) We have the covariance property $R[e^{i\theta} z] = e^{i\theta} R[z] e^{-i\theta}$.

(4) We have, summing on repeated indexes,

$$R[z] = \eta - (\alpha_j[z] \eta) \phi_j \text{ with } \alpha_j[z] \eta = \langle B_j(z), \eta \rangle + \langle C_j(z), \eta^* \rangle$$

where, for $\tilde{Z}$ as in Def. 2.5, we have $B_j(z) = \tilde{B}_j(\tilde{Z})$ and $C_j(z) = z_i z_{i,j} \tilde{C}_{i,j}(\tilde{Z})$, for $\tilde{B}$ and $\tilde{C}_{i,j}$ smooth in $\tilde{Z}$.

(5) We have for $z \in \mathbb{R}$ with $Z$ as in Definition 2.5

$$||B_j(z) + \partial_z q_{z,j} ||_{\Sigma_r} + ||C_j(z) - \partial_z q_{z,j} ||_{\Sigma_r} \leq c_r |Z|^2.$$
Then Lemma 2.6 gives us a system of coordinates near the origin in \( H^4 \). The simple proof is the same of Lemma 2.6 \([12]\).

**Lemma 2.9.** For the \( d_0 > 0 \) of Lemma 2.8 the map \((z, \eta) \to u \) defined by

\[
u = \sum_{j=1}^{n} Q_j z_j + R[z] \eta \text{ for } (z, \eta) \in B_{C^n}(0, d_0) \times (H^4 \cap \mathcal{H}_c[0])
\]  

(2.12)
is with values in \( H^4 \) and is \( C^\infty \). Furthermore, there is a \( d_1 > 0 \) such that for \((z, \eta) \in B_{C^n}(0, d_1) \times (B_{H^4}(0, d_1) \cap \mathcal{H}_c[0]) \) the above map is a diffeomorphism and

\[
|z| + \|\eta\|_{H^4} \sim \|u\|_{H^4}.
\]  

(2.13)

Finally, we have the gauge properties \( u(e^{i\theta}z, e^{i\theta} \eta) = e^{i\theta} u(z, \eta) \) and and

\[
z(e^{i\theta}u) = e^{i\theta} z(u) \text{ and } \eta(e^{i\theta}u) = e^{i\theta} \eta(u).
\]  

(2.14)

We end this section exploiting the notation introduced in claim (5) of Lemma 2.8 to introduce two classes of functions. First of all notice that the linear maps \( \eta \to \langle \eta, \phi_j^* \rangle \) extend into bounded linear maps \( \Sigma_r \to \mathbb{R} \) for any \( r \in \mathbb{R} \). We set

\[
\Sigma_r^c := \{ \eta \in \Sigma_r : \langle \eta, \phi_j^* \rangle = 0, \ j = 1, \ldots, n \}.
\]  

(2.15)
The following two classes of functions will be used in the rest of the paper. Recall that in Def. 2.3 we introduced \( Z \in L \) with \( \dim L = n(n-1) \).

**Definition 2.10.** We will say that \( F(t, z, Z, \eta) \in C^M(I \times A, \mathbb{R}) \), with \( I \) a neighborhood of 0 in \( \mathbb{R} \) and \( A \) a neighborhood of 0 in \( \mathbb{C}^n \times L \times \Sigma_{-K}^c \) is \( F = \mathcal{R}_{K,M}^{i,j}(t, z, Z, \eta) \), if there exists a \( C > 0 \) and a smaller neighborhood \( A' \) of 0 s.t.

\[
|F(t, z, Z, \eta)| \leq C(\|\eta\|_{\Sigma_{-K}} + |Z|)^j(\|\eta\|_{\Sigma_{-K}} + |Z|) + |z|^l \text{ in } I \times A'.
\]  

(2.16)

We will specify \( F = \mathcal{R}_{K,M}^{i,j}(t, z, Z) \) if

\[
|F(t, z, Z, \eta)| \leq C|Z|^j|z|^l
\]  

(2.17)
and \( F = \mathcal{R}_{K,M}^{i,j}(t, z, \eta) \) if

\[
|F(t, z, Z, \eta)| \leq C\|\eta\|^j_{\Sigma_{-K}}(\|\eta\|_{\Sigma_{-K}} + |Z|)^i + |z|^l.
\]  

(2.18)

We will omit \( t \) if there is no dependence on such variables.

We write \( F = \mathcal{R}_{K,\infty}^{i,j} \) if \( F = \mathcal{R}_{K,m}^{i,j} \) for all \( m \geq M \). We write \( F = \mathcal{R}_{K,\infty}^{i,j} \) if for all \( k \geq K \) the above \( F \) is the restriction of an \( F(t, z, \eta) \in C^M(I \times A_k, \mathbb{R}) \) with \( A_k \) a neighborhood of 0 in \( \mathbb{C}^n \times L \times \Sigma_{-k} \) and which is \( F = \mathcal{R}_{k,M}^{i,j} \). Finally we write \( F = \mathcal{R}_{\infty,\infty}^{i,j} \) if \( F = \mathcal{R}_{k,\infty}^{i,j} \) for all \( k \).

**Definition 2.11.** We will say that an \( T(t, z, \eta) \in C^M(I \times A, \Sigma_K(\mathbb{R}^3, \mathbb{C})) \), with the above notation, is \( T = S_{K,M}^{i,j}(t, z, \eta) \), if there exists a \( C > 0 \) and a smaller neighborhood \( A' \) of 0 s.t.

\[
\|T(t, z, Z, \eta)\|_{\Sigma_K} \leq C(\|\eta\|_{\Sigma_{-K}} + |Z|)^j(\|\eta\|_{\Sigma_{-K}} + |Z|) + |z|^i \text{ in } I \times A'.
\]  

(2.19)
We use notations \( S_{K,M}^{i,j}(t, z, Z) \), \( S_{K,M}^{i,j}(t, z, \eta) \) etc. as above.
Remark 2.12. For given functions \(F(t, z, \eta)\) and \(T(t, z, \eta)\) we write \(F(t, z, \eta) = R(t, z, \eta)\) and \(T(t, z, \eta) = S(t, z, \eta)\) when they are restrictions to the set of vectors \(Z \in \{(z_i)_{i=1}^n : i \neq j\}\) of functions satisfying the two above definitions. Furthermore, when we write \(R(t, z, \eta)\) and \(S(t, z, \eta)\), we mean \(R(t, z, \eta)\) and \(S(t, z, \eta)\).

Notice that \(F = R(t, z, \eta)\) or \(S = S(t, z, \eta)\) do not mean independence by the variable \(\eta\).

3 Invariants

Equation \((3.1)\) admits the energy and mass invariants, defined as follows for \(G(0) = 0\) and \(G'(s) = \phi(s)\):

\[
E(u) := E_K(u) + E_P(u), \quad \text{where } E_K(u) := \langle D_u u, u^* \rangle \quad \text{and} \quad E_P(u) = \frac{1}{2} \int_{\mathbb{R}^3} G(u \overline{v}) dx;
\]

\[
Q(u) := \langle u, u^* \rangle. \tag{3.1}
\]

We have \(E \in C^\infty(H^1(\mathbb{R}^3, \mathbb{C}), \mathbb{R})\) and \(Q \in C^\infty(L^2(\mathbb{R}^3, \mathbb{C}), \mathbb{R})\). We denote by \(dE\) the Frechét derivative of \(E\). We define \(\nabla E \in C^\infty(H^1(\mathbb{R}^3, \mathbb{C}), H^1(\mathbb{R}^3, \mathbb{C}))\) by \(dE = \text{Re}(\nabla E, X^*)\) for any \(X \in H^1\). We define also \(\nabla_u E\) by

\[
dE \times y = \langle \nabla u, X \rangle + \langle \nabla u^*, X^* \rangle \text{ that is } \nabla u \times y = 2^{-1}(\nabla E)^* \text{ and } \nabla u^* \times y = 2^{-1}\nabla E.
\]

Notice that \(\nabla E = 2Hu + 2Q(u \overline{v}) \beta u\). Then equation \((3.1)\) can be interpreted as

\[
iu = \nabla_u E(u). \tag{3.2}
\]

Proposition 3.1. We have the following expansion of the energy for arbitrary \(N\):

\[
E(u) = \sum_{j=1}^n E(Q_{jz}) + \langle H\eta, \eta^* \rangle + \mathcal{R}_{r_0, 0}^{1, 2}(z, \eta) + \mathcal{R}_{r_0, 0}^{0, 2N+5}(z, \mathbb{Z})
\]

\[
+ \sum_{j=1}^n \sum_{l=0}^{2N+3} \sum\limits_{|m|=l+1} Z_m a_{jm} |z_j|^2 + \sum_{j,k=1}^{2N+3} (z_j z_m \langle G_{jkm}, |z|^2 \rangle, \eta) + c.c.
\]

\[
+ \text{Re}(S_{r_0, 0}^{0, 2N+4}(z, \mathbb{Z}), \eta^*) + \sum_{i+j=2} \sum\limits_{|m|=1} Z_m \langle G_{2mij}, \eta \otimes i \otimes (\eta^*) \otimes j \rangle
\]

\[
+ \sum_{d=2}^{3} \sum_{i+j=3} \mathcal{R}_{r_0, 0}^{z_d - 3d}(z, \eta) \int_{\mathbb{R}^3} G_{di}(x, z, \eta(x)) \eta^* i (x) \otimes (\eta^* (x))^{(d-i)} dx + E_P(\eta) \tag{3.3}
\]

where:

(1) \((a_{jm}, G_{jkm}, G_{2mij}) \in C^\infty(B_{C}(0, d_0), \mathbb{C} \times \Sigma_{r_0}(\mathbb{R}^3, \mathbb{C}) \times \Sigma_{r_0}(B_{\mathbb{C}^3}(\mathbb{R}^4, \mathbb{C})))\);

(2) \(G_{di}(\cdot, z, \eta, \zeta) \in C^\infty(B_{C}(0, d_0), \Sigma_{r_0}(\mathbb{R}^3, \mathbb{C}) \times \mathbb{C}^4, \Sigma_{r_0}(\mathbb{R}^3, B_{\mathbb{C}^4}(\mathbb{C})))\);

(3) for \(|m| = 0\) we have \(G_{2mij}(0) = 0\) and

\[
\sum_{i+j=2} \langle G_{2mij}(z, \eta) \eta^* i (\eta^*) \otimes j \rangle = 2^{-1} \sum_{j=1}^n \langle g(Q_{jz}, \overline{Q_{jz}}) \eta, \eta^* \rangle
\]

\[
+ \sum_{j=1}^n \text{Re}(g'(Q_{jz}, \overline{Q_{jz}}) \text{Re}(Q_{jz}, \eta^*) Q_{jz}, \eta^*). \tag{3.4}
\]
In order to prove Proposition 3.1, we set

\[ K(z, \eta) := E \sum_{j=1}^{n} Q_{jz_j} + R[z] \eta = K_K(z, \eta) + K_P(z, \eta) \]

with

\[ K_K(z, \eta) := E_K \sum_{j=1}^{n} Q_{jz_j} + R[z] \eta \]

and

\[ K_P(z, \eta) := E_P \sum_{j=1}^{n} Q_{jz_j} + R[z] \eta. \]

By Taylor expansion, we write

\[ K(z, \eta) = K(z, 0) + \text{Re} \langle \partial_{\eta} K(z, 0), \eta^* \rangle + \frac{1}{2} \text{Re} \langle \partial_{\eta}^2 K(z, 0) \eta, \eta^* \rangle + K_3(z, \eta) \]

(3.5)

\[ K_3(z, \eta) := \frac{1}{2} \int_0^1 (1 - t)^2 \text{Re} \langle \partial_{\eta}^3 K(z, t \eta) \eta^2, \eta^* \rangle dt. \]

(3.6)

We expand

\[ K_3(z, \eta) = K_3(0, \eta) + R_P(z, \eta) \]

where \( K_3(0, \eta) = K_P(0, \eta) = E_P(\eta) \)

(3.7)

and

\[ R_P(z, \eta) = \int_0^1 \partial_{t} K_3(tz, \eta) dt := \int_0^1 \sum_{j=1}^{n} \sum_{A=R,I} D_{jA} K_3(tz, \eta) z_{jA} dt \]

(3.8)

To prove Proposition 3.1, we compute the terms of

\[ K(z, \eta) = K(z, 0) + \text{Re} \langle \partial_{\eta} K(z, 0), \eta^* \rangle + \frac{1}{2} \text{Re} \langle \partial_{\eta}^2 K(z, 0) \eta, \eta^* \rangle + E_P(\eta) + R_P(z, \eta), \]

(3.9)

starting with \( \partial_{\eta} K, \partial_{\eta}^2 K \) and \( \partial_{\eta}^3 K \).

**Lemma 3.2.** Set \( u = u(z, \eta) = \sum_{j=1}^{n} Q_{jz_j} + R[z] \eta. \) We have the following equalities:

\[ \partial_{\eta} K(z, \eta) = 2R[z]^* Hu, \]

\[ \partial_{\eta}^2 K(z, \eta) = 2R[z]^* HR[z], \]

\[ \partial_{\eta}^3 K(z, \eta) = 0; \]

\[ \partial_{\eta} K_P(z, \eta) = R[z]^* (g(u \bar{\eta}) \beta u), \]

\[ \partial_{\eta}^2 K_P(z, \eta) \nu = R[z]^* g(u \bar{\eta}) \beta R[z] \nu + 2R[z]^* g'(u \bar{\eta}) \text{Re} \left( u \overline{R[z] \nu} \right) \beta u, \]

\[ (\partial_{\eta}^3 K_P(z, \eta) \nu) \nu = 6R[z]^* g'(u \bar{\eta}) \text{Re} \left( u \overline{R[z] \nu} \right) \beta R[z] \nu + 4R[z]^* g''(u \bar{\eta}) \left( \text{Re} \left( u \overline{R[z] \nu} \right) \right)^2 \beta u. \]

(3.10)
Consider the first two terms in the r.h.s. of Lemma 3.3. In particular we have

\[ \partial_\eta K_K(z,0) = 2R[z]^* H \sum_{j=1}^n Q_{jzj}, \quad \partial_\eta^2 K_K(z,0) = 2R[z]^* H R[z], \]

\[ \partial_\eta K_P(z,0) = g(\sum_{j,k} Q_{jzj} \overline{Q}_{kzj}) R[z]^* \beta \sum_{j=1}^n Q_{jzj}, \]

\[ \partial_\eta^2 K_P(z,0) = g(\sum_{j,k} Q_{jzj} \overline{Q}_{kzj}) R[z]^* \beta R[z] \]

\[ + 2g^2(\sum_{j,k} Q_{jzj} \overline{Q}_{kzj}) \text{Re} \left( \sum_{j=1}^n Q_{jzj} R[z]^* \right) R[z]^* \beta \sum_{j=1}^n Q_{jzj}. \]

**Proof.** We get (3.10) by

\[ K_K(z,\eta + \varepsilon \nu) = \text{Re} \langle H(u(z,\eta) + \varepsilon R[z]\nu), (u(z,\eta) + \varepsilon R[z]\nu)^* \rangle \]

\[ = K_K(z,\eta) + 2\varepsilon \text{Re} \langle H(u(z,\eta), (R[z]\nu)^* \rangle + \varepsilon^2 \text{Re} \langle H R[z]\nu, (R[z]\nu)^* \rangle. \]

We get (3.11) by

\[ K_P(z,\eta + \varepsilon \nu) = 2^{-1} \int G((u(z,\eta) + \varepsilon R[z]\nu)(u(z,\eta) + \varepsilon R[z]\nu)) \, dx, \]

and by

\[ G((u(z,\eta) + \varepsilon R[z]\nu)(u(z,\eta) + \varepsilon R[z]\nu)) \]

\[ = G \left( u(z,\eta) u(z,\eta) + 2\varepsilon \text{Re} \left( u(z,\eta) R[z] \nu \right) + \varepsilon^2 R[z]^* \nu R[z] \nu \right) \]

\[ = o(\varepsilon^3) + G(u(z,\eta) u(z,\eta)) + 2\varepsilon g(u(z,\eta) u(z,\eta)) \text{Re} \left( \beta u(z,\eta) (R[z] \nu)^* \right) \]

\[ + \varepsilon^2 \left[ g(u(z,\eta) u(z,\eta)) R[z]^* \nu R[z] \nu + 2g^2(u(z,\eta) u(z,\eta)) \left( \text{Re} \left( u(z,\eta) R[z] \nu \right) \right)^2 \right] + \]

\[ \varepsilon^3 \left[ 2g^2(u(z,\eta) u(z,\eta)) \text{Re} \left( u(z,\eta) R[z]^* \right) R[z]^* \nu R[z] \nu + \frac{4}{3} g^2(u(z,\eta) u(z,\eta)) \text{Re} \left( u(z,\eta) R[z] \nu \right)^3 \right]. \]

We now examine the r.h.s. of (3.9).

**Lemma 3.3.** Consider the first two terms in the r.h.s. of (3.5). We then have

\[ K(z,0) = \sum_{j=1}^n E(Q_{jzj}) + \sum_{j=1}^n \sum_{l=0}^{2N+3} \sum_{|m|=l+1} Z^{|m|} a_{jm}(|z_j|^2) + R_{0,2N+5}^0(z, Z), \]

\[ \text{Re} \langle \partial_\eta K(z,0), \eta^* \rangle = \sum_{j,k=1}^n \sum_{l=0}^{2N+3} \sum_{|m|=l} (\tau_j Z^{|m|} G_{jkm}(|z_k|^2), \eta) + \text{c.c.} \rangle + \text{Re} \langle S_{0,2N+4}^0(z, Z), \eta^* \rangle \]

where the coefficients in the r.h.s.’s have the properties listed in claim (1) in Prop. 3.7.
Proof. First of all, both l.h.s.’s of (3.12)–(3.13) are gauge invariant. Then (3.13) is an immediate consequence of claim (3) of Lemma 3.4 below. We have

\[ K(z, 0) = E(\sum_{j=1}^{n} Q_{jz}) = E(Q_{1z1}) + E(\sum_{j>1} Q_{jz}) + \int_{[0,1]^2} \frac{\partial^2}{\partial s \partial t} E(sQ_{1z1} + t \sum_{j>1} Q_{jz}) dt ds \]

\[ = \sum_{k=1}^{n} E(Q_{kz_k}) + \alpha_k(z) \text{ with } \alpha_k(z) := \sum_{k=1}^{n} \int_{[0,1]^2} \frac{\partial^2}{\partial s \partial t} E(sQ_{kz_k} + t \sum_{j>k} Q_{jz}) dt ds. \]

The \( \alpha_k(z) \) are gauge invariant, so that we can apply to them claim (2) of Lemma 3.4 below. Furthermore, since \( \alpha_k(z) = O(|Z|) \) we conclude that in the expansion (3.14) for \( \alpha_k(z) \) we have equalities \( b_\alpha(|z_j|^2) = 0 \).

\[ \square \]

Lemma 3.4. The following facts hold:

1. For \( a(\zeta) \) smooth from \( B_C(0, \delta) \) to \( \mathbb{R} \) s.t. \( a(e^{i\theta} \zeta) = a(\zeta) \) for any \( \theta \in \mathbb{R} \) there exists \( \alpha \in C^\infty([0, \delta^2); \mathbb{R}) \) s.t. \( a(|\zeta|^2) = a(\zeta) \).

2. Let \( a \in C^\infty(B_C(0, \delta), \mathbb{R}) \) satisfy \( a(e^{i\theta} z_1, \cdots, e^{i\theta} z_n) = a(z_1, \cdots, z_n) \) for all \( \theta \in \mathbb{R} \) and \( a(0) = 0 \). Then for any \( M > 0 \) there exist smooth \( b_{j_0} \) s.t. \( b_{j_0}(|z_j|^2) = a(0, \cdots, 0, z_j, 0, \cdots, 0) \) and

\[ a(z_1, \cdots, z_n) = \sum_{m} Z^m b_{j_m}(|z_j|^2) + R_{0,M}^{0,M}(z, Z). \] (3.14)

3. Let \( a \in C^\infty(B_C(0, \delta), \Sigma_r) \) \( \forall r \in \mathbb{R} \) s.t. \( a(e^{i\theta} z_1, \cdots, e^{i\theta} z_n) = e^{i\theta} a(z_1, \cdots, z_n) \). Then for any \( M > 0 \) \( \exists G_{j_m} \) s.t. \( G_{j_m} \in C^\infty(B_C(0, \delta), \Sigma_r) \) \( \forall r \) \( z_j G_{j_0}(|z_j|^2) = a(0, \cdots, 0, z_j, 0, \cdots, 0) \) and

\[ a(z_1, \cdots, z_n) = \sum_{j=1}^{n} \sum_{m} z_j Z^m G_{j_m}(|z_j|^2) + G^{1,M}_{j,M}(z, Z). \] (3.15)

Proof. This elementary lemma is proved in the course of the proof of Lemma 3.1 [12]. \( \square \)

Lemma 3.5. Assuming (H3) we have the following facts.

1. For \( Z^m = z^\mu \overline{z}^\nu \), then \( m \in M_0(2N + 4) \) implies \( \mu = \nu \). In particular \( m \in M_0(2N + 4) \) implies \( Z^m = |z_1|^{2l_1} \cdots |z_n|^{2l_n} \) for some \( (l_1, \cdots, l_n) \in \mathbb{N}_0^n \).

2. For \( |m| \leq 2N + 3 \) and any \( j \) we have \( \sum_{a,b} (e_a - e_b)m_{ab} - e_j \neq 0 \).

The 3rd term in the r.h.s. of (3.9) is dealt by the following lemma.

Lemma 3.6. There exist \( G_{2m}(z) \) as in the statement of Prop. 5.7 s.t. (3.14) holds and

\[ \text{Re} \langle \partial_\eta \partial_\eta^* K(z, 0) \eta, \eta^* \rangle = \langle H \eta, \eta^* \rangle + R_{1,2}^{1,2} \langle z, \eta \rangle + \sum_{i=0}^{2} \sum_{|m| \leq 1} Z^m (G_{2mi}(z), \eta^{\otimes i} \otimes (2-i)). \] (3.16)
Proof. By Lemma 3.2 and by claim (2) in Lemma 2.8 we have
\[
\frac{1}{2} \text{Re} \langle \partial_{\eta}^2 K(z, 0)\eta, \eta \rangle = \langle HR[z]\eta, (R[z]\eta)^* \rangle + \langle g(\sum_{j,k} Q_{jz_j} \overline{Q_{jz_j}}) R[z]\eta, (R[z]\eta)^* \rangle
\]
\[
+ 2 \text{Re} \langle g'(\sum_{j,k} Q_{jz_j} \overline{Q_{jz_j}}) \text{Re} \left( \sum_{j=1}^{n} Q_{jz_j} \overline{R[z]\eta} \right) \sum_{j=1}^{n} Q_{jz_j}, (R[z]\eta)^* \rangle
\]
\[
= \langle H\eta, \eta^* \rangle + R_{1,\infty}(z, \eta)
\]
\[
+ \langle g(\sum_{j,k} Q_{jz_j} \overline{Q_{jz_j}}) \eta, \eta^* \rangle + 2 \text{Re} \langle g'(\sum_{j,k} Q_{jz_j} \overline{Q_{jz_j}}) \text{Re} \left( \sum_{j=1}^{n} Q_{jz_j} \eta \right) \sum_{j=1}^{n} Q_{jz_j}, \eta^* \rangle.
\]

The last line yields the last term of (3.16) and it is straightforward to see that it has the required properties.

To finish the discussion of the r.h.s. of (3.9) we need to compute \( R_P \). We consider first a preparatory lemma.

Lemma 3.7. Let \( l = 1, \cdots, n \) and \( A = R, I \). Then we have \( (D_{1A} R[z])\eta = \sum_{j=1}^{n} R_{1,\infty}(z, \eta)\phi_j \).

Proof. \( R[z]\eta = \eta + \sum_{j=1}^{n} (\alpha_j[z]\eta)\phi_j \)  So, \( D_{1A} R[z]\eta = \sum_{j=1}^{n} ((D_{1A} \alpha_j[z])\eta) \phi_j \). Now, \( D_{1A} \alpha_j[z]\eta = \int D_{1A} B_j(z)\eta \ dx + \int D_{1A} C_j(z)\eta^* \ dx \), where \( B \) and \( C \) is given in Lemma 2.8 and we have \( D_{1A} B_j(z) = S_{1,\infty}^{0,0} \) and \( D_{1A} C_j(z) = S_{1,\infty}^{1,0} \). This yields the lemma.

We set \( u := \sum_{j=1}^{n} Q_{jz_j} + R[z]\eta \). Then, we have \( D_{1A} u = D_{1A} Q_{jz_j} + (D_{1A} R[z])\eta \). For \( l = 1, \cdots, n \) and \( A = R, I \) we have
\[
D_{1A} \text{Re} \left( (\partial_{\eta}^2 K_P(z, \eta)\eta, \eta^* \right) = 6 \mathcal{A} + 4 \mathcal{B}
\]
\[
\mathcal{A} := D_{1A} \text{Re} \langle g'(u\overline{R[z]\eta}) \beta R[z]\eta, (R[z]\eta)^* \rangle,
\]
\[
\mathcal{B} := D_{1A} \text{Re} \langle g''(u\overline{R[z]\eta}) \left( \text{Re} \left( u\overline{R[z]\eta} \right) \right)^2 \beta u, (R[z]\eta)^* \rangle.
\]

We have
\[
\mathcal{A} = 2 \langle g''(u\overline{R[z]\eta}) \text{Re} \left( D_{1A} u \right) \beta R[z]\eta, (R[z]\eta)^* \rangle
\]
\[
+ \langle g'(u\overline{R[z]\eta}) D_{1A} \left( u \overline{R[z]\eta} \right) \beta R[z]\eta, (R[z]\eta)^* \rangle + \langle g'(u\overline{R[z]\eta}) \beta R[z]\eta, (R[z]\eta)^* \rangle
\]
\[
+ 2 \langle g'(u\overline{R[z]\eta}) \beta D_{1A} R[z]\eta, (R[z]\eta)^* \rangle,
\]
\[
\mathcal{B} = 2 \langle g'''(u\overline{R[z]\eta}) \left( \text{Re} \left( u\overline{R[z]\eta} \right) \right)^2 \beta u, (R[z]\eta)^* \rangle
\]
\[
+ 2 \langle g''(u\overline{R[z]\eta}) \text{Re} \left( D_{1A} u \overline{R[z]\eta} \right) \beta u, (R[z]\eta)^* \rangle
\]
\[
+ 2 \langle g''(u\overline{R[z]\eta}) \text{Re} \left( u \overline{D_{1A} R[z]\eta} \right) \beta u, (R[z]\eta)^* \rangle
\]
\[
+ \langle g''(u\overline{R[z]\eta}) \beta D_{1A} u, R[z]\eta)^* \rangle + \langle g''(u\overline{R[z]\eta}) \beta u, (D_{1A} R[z]\eta)^* \rangle
\]
All the terms in the formulas for \( \mathcal{A} \) and \( \mathcal{B} \) can be expressed as
\[
\sum_{d=2,3} \sum_{i+j=d} (G_{dij}(z, \eta), \eta^{\circ i} \otimes (\eta^*)^{\circ j}) \mathcal{R}^{2,3-d}(z, \eta) + \mathcal{R}^{2,2}(z, \eta).
\] (3.17)

Therefore \( R_P \) admits an expansion of the form (3.17), which is absorbed in terms of the r.h.s. of (3.9).

**Proof of Proposition 4.2.** We have just seen that \( R_P \) is absorbed in the r.h.s. of (3.3). The other terms of the r.h.s. of (3.9) are treated by Lemma 3.6 and 3.8. \( \square \)

### 4 Effective Hamiltonian

In this section we apply the theory of Sections 4 and 5 in [12] which yield an effective Hamiltonian in an appropriate coordinate system, which in turn will be used to prove Theorem 5.1 which yields Theorem 1.3.

The first observation is that system (3.2) is Hamiltonian with respect to the symplectic form
\[
\Omega(X, Y) := -2 \text{Im}(X, Y^*).
\] (4.1)

Notice that the coordinates in Lemma 2.9 do not form a system of Darboux coordinates for (4.1), see Sect. 4 [12].

We consider the symplectic form
\[
\Omega_0 := i \sum_{j=1}^n (1 + \gamma_j(|z_j|^2)) dz_j \wedge d\zeta_j + i \langle d\eta, d\eta^* \rangle - i \langle d\eta^*, d\eta \rangle
\] (4.2)

where
\[
\gamma_j(|z_j|^2) := -\langle \hat{q}_j(|z_j|^2), \hat{q}_j^*(|z_j|^2) \rangle + 2|z_j|^2 \text{Re} \langle \hat{q}_j^*|z_j|^2, \hat{q}_j^*|z_j|^2 \rangle
\]

with \( \hat{q}_j^*(t) = \frac{d}{dt} \hat{q}_j(t) \). By Proposition 1.1 and Definition 2.10 we have \( \gamma_j(|z_j|^2) = \mathcal{R}^{2,0}_{\infty, \infty}(|z_j|^2) \).

**Remark 4.1.** Notice that \( \Omega_0 \) is the same local model symplectic form of (1.4). As explained in [12] we do not know if Proposition 1.2 below holds when choosing \( \gamma_j \equiv 0 \).

In Sect. 4 [12] the following proposition is proved.

**Proposition 4.2 (Darboux Theorem).** There exists a \( \delta_0 \in (0, d_0) \) s.t. the following facts hold.

1. There exists a gauge invariant 1-form \( \Gamma = \Gamma_{\mathcal{A}} \mathcal{A} z \wedge \mathcal{A} \eta + \langle \mathcal{A} \eta, d\eta \rangle + \langle \mathcal{A} \eta^*, d\eta^* \rangle \) with
\[
\Gamma_{\mathcal{A}} = \mathcal{R}^{1,1}_{\infty, \infty}(z, \mathcal{Z}, \mathcal{\eta}) \text{ and } \Gamma_{\mathcal{\eta}} = \mathcal{S}^{1,1}_{\infty, \infty}(z, \mathcal{Z}, \mathcal{\eta}) \text{ for } \xi = \eta, \eta^*
\] (4.3)
s.t. \( d\Gamma = \Omega - \Omega_0 \).

2. For any preassigned \( r \in \mathbb{N} \) and for any \((t, z, \eta) \in (-4, 4) \times B_{\infty}(0, \delta_0) \times B_{\infty}(0, \delta_0) \) there exists exactly one solution \( \mathcal{X}_t(z, \eta) \in L^2 \) of the equation \( i_{\mathcal{X}_t} \Omega_t = -\Gamma \). Furthermore, \( \mathcal{X}_t(z, \eta) \) is gauge invariant, \( \mathcal{X}_t(z, \eta) \in \Sigma_r \) and if we set \( \mathcal{X}_{\mathcal{A}}(z, \eta) = d\mathcal{A} \mathcal{X}_t(z, \eta) \) and \( \mathcal{X}_t(z, \eta) = d\eta \mathcal{X}_t(z, \eta) \), we have \( \mathcal{X}_{\mathcal{A}}(z, \eta) = \mathcal{R}^{1,1}_{r, \infty}(t, z, \mathcal{Z}, \mathcal{\eta}) \) and \( \mathcal{X}_t(z, \eta) = \mathcal{S}^{1,1}_{r, \infty}(t, z, \mathcal{Z}, \mathcal{\eta}). \)

3. Consider the following system in \((t, z, \eta) \in (-4, 4) \times B_{\infty}(0, \delta_0) \times B_{\infty}(0, \delta_0) \) for all \( k \in \mathbb{Z} \cap [-r, r] \):
\[
\dot{z}_j = \mathcal{X}_t(z, \eta) \text{ and } \dot{\eta} = \mathcal{X}_t(z, \eta).
\] (4.4)

Then the following facts hold.
(3.1) For $\delta_1 \in (0, \delta_0)$ sufficiently small we have
\begin{equation}
Z^\delta \in C^\infty((-2, 2) \times B_{C^n}(0, \delta_1) \times B_{S^k}(0, \delta_1), B_{C^n}(0, \delta_0) \times B_{S^k}(0, \delta_0)) \quad \text{for all } k \in \mathbb{Z} \cap [-r, r].
\end{equation}
\begin{equation}
X^\delta \in C^\infty((-2, 2) \times B_{C^n}(0, \delta_1) \times B_{H^1 \cap H_c[0]}(0, \delta_1), B_{C^n}(0, \delta_0) \times B_{H^1 \cap H_c[0]}(0, \delta_0)).
\end{equation}

In particular we have
\begin{equation}
z^\delta_i = z_j + S_j(t, z, \eta) \quad \text{and} \quad \eta^i = \eta + S_\eta(t, z, \eta)
\end{equation}
with $S_j(t, z, \eta) = R_{r, \infty}^{1,1}(t, z, Z, \eta)$ and $S_\eta(t, z, \eta) = S_{r, \infty}^{1,1}(t, z, Z, \eta)$.

(3.2) The map $Z^\delta$ is a local diffeomorphism of $H^1$ into itself near the origin, and we have
$Z^\delta: \Omega \to \Omega$.

(3.3) We have $S_j(t, e^{i\theta} z, e^{i\theta} \eta) = e^{i\theta} S_j(t, z, \eta), \quad S_\eta(t, e^{i\theta} z, e^{i\theta} \eta) = e^{i\theta} S_\eta(t, z, \eta)$.

We now consider the pullback $K := E \circ Z^\delta$.

Lemma 4.3. Consider the $\delta_1 > 0$ and $\delta_0 > 0$ of Prop. 4.2 and set $r = r_0$ with $r_0$ the index in Prop. 4.2. Then we have
\begin{equation}
\mathcal{F}(B_{C^n}(0, \delta_1) \times (B_{H^1}(0, \delta_1) \cap H_c[0])) \subset B_{C^n}(0, \delta_0) \times (B_{H^1}(0, \delta_0) \cap H_c[0])
\end{equation}
and $\mathcal{F}|_{B_{C^n}(0, \delta_1) \times (B_{H^1}(0, \delta_1) \cap H_c[0])}$ is a diffeomorphism between domain and an open neighborhood of the origin in $C^n \times (H^1 \cap H_c[0])$ and furthermore the functional $K$ admits an expansion for $r_1 = r_0 - 2$
\begin{equation}
K(z, \eta) = H_2(z, \eta) + \sum_{j=1}^{2N+4} \lambda_j(|z_j|^2)
+ \sum_{i=1}^{2N+4} \sum_{|m|=l+1} \mathbb{Z}^m \alpha^{(1)}_m(|z_1|^2, |z_n|^2) + \sum_{j=1}^{2N+3} \sum_{|m|=l} \mathbb{Z}^m \alpha(J^{(1)}_m(|z_j|^2), \eta) + c.c.
+ \mathbb{R}_{r, \infty}^{1,2}(z, \eta) + \mathbb{R}_{r, \infty}^{0,2N+5}(z, Z, \eta) + \operatorname{Re}(S_{r, \infty}^{2N+4}(z, Z, \eta), \eta^*)
+ \sum_{d=1}^{3} \sum_{i=1}^{d} \mathbb{R}_{r, \infty}^{0,3-d}(z, \eta) \int \mathbb{G}^{(1)}_{d+i}(x, z, \eta, \eta(x)) \eta^{(i)}(x) \otimes (\eta^*(x))^{(d-i)} dx
+ \sum_{i+j=2} \mathbb{Z}^m \mathbb{G}^{(1)}_{2m,ij}(z, \eta^{(i)} \otimes (\eta^*)^{(j)} + E_P(\eta), (\eta, \eta^*)
\end{equation}
where
\begin{equation}
H_2(z, \eta) = \sum_{j=1}^{n} e_j |z_j|^2 + (H, \eta, \eta^*)
\end{equation}
and where: $G^{(1)}_{jm}, G^{(1)}_{2m,ij}$ are $S_{r, \infty}^{0,0}, \alpha^{(1)}_m(|z_1|^2, |z_n|^2) = \mathbb{R}_{r, \infty}^{0,0}(z); \ c.c. \ means \ complex \ conjugate; \lambda_j(|z_j|^2) = \mathbb{R}_{r, \infty}^{2,0}(z); \mathbb{G}^{(1)}_{d+i}(\cdot, z, \eta, \zeta) \in C^\infty(B_{C^n}(0, \delta_1) \times \Sigma_{-r_1}(\mathbb{R}^3, \mathbb{C}) \times \mathbb{C}^4, \Sigma_{r_1}(\mathbb{R}^3, B^d(\mathbb{C}^4, \mathbb{C}))).$

For $|m| = 0, G^{(1)}_{2m,ij}(z, \eta) = G^{(1)}_{2m,ij}(z)$ is the same of $\mathbb{G}^{(1)}_{d+i}(\cdot, z, \eta, \zeta)$. Finally, we have the invariance $\mathbb{R}_{r, \infty}^{1,2}(e^{i\theta} z, e^{i\theta} \eta) \equiv \mathbb{R}_{r, \infty}^{1,2}(z, \eta)$.
Proof. The proof of the above statement with possibly nonzero terms also corresponding to \( l = 0 \) in both summations in the 2nd line of (1.8) is elementary, see Lemma 4.10 in [12] and Lemma 4.3 [11]. The key fact that in the 2nd line of (1.8) both summations start from \( l = 1 \) and there are no \( l = 0 \) is proved in the Cancelation Lemma 4.11 in [12].

Consider now the symplectic form \( \Omega_0 \) in (1.2). We introduce an index \( \ell = j, \ell, \eta \) for \( \ell = j \) with \( j = 1, \ldots, n \). We write \( \partial_j = \partial z_j \) and \( \partial_\ell = \partial z_\ell = \partial_\eta \). Given \( F \in C^1(\mathcal{U}, \mathbb{C}) \) with \( \mathcal{U} \) an open subset of \( \mathbb{C}^n \times \Sigma_\ell \), its Hamiltonian vector field \( X_F \) is defined by \( i_{X_F} \Omega_0 = dF \). We have summing on \( j \)

\[
i_{X_F} \Omega_0 = i(1 + \gamma_j(|z_j|^2))((X_F)_j dz_j - (X_F)_j^* dz_j^*) + i((X_F)_\eta, d\eta) - \frac{i((X_F)_\eta, d\eta)}{1 + \gamma_j(|z_j|^2)}
\]

where (4.9) is used also to define \( \nabla_\xi F \) for \( \xi = \eta, \eta^* \).

Comparing the components of the two sides of (4.9) we get for \( 1 + \omega_j(|z_j|^2) = (1 + \gamma_j(|z_j|^2))^{-1} \)

\[
(X_F)_j = -i(1 + \omega_j(|z_j|^2))\partial_j F, \quad (X_F)_\eta = i(1 + \omega_j(|z_j|^2))\partial_\eta F
\]

(4.10)

Given \( G \in C^1(\mathcal{U}, \mathbb{C}) \) and \( F \in C^1(\mathcal{U}, \mathbb{E}) \) with \( \mathbb{E} \) a Banach space, we set \( \{ F, G \} := dF X_G \).

Definition 4.4 (Normal Forms). Recall Def. 2.4 and (2.3). Fix \( r \in \mathbb{N}_0 \). A real valued function \( Z(z, \eta) \) is in normal form if \( Z = Z_0 + Z_1 \) with \( Z_0 \) and \( Z_1 \) finite sums of the following type for \( l \geq 1 \); for \( G_{jm}(|z_j|^2) = S_{r, \eta}(|z_j|^2) \), c.c. the complex conjugate and \( a_{m}(|z_1|^2, \ldots, |z_n|^2) = R_{r, \infty}(|z_1|^2, \ldots, |z_n|^2) \),

\[
Z_1(z, \eta) = \sum_{|m|=1}^{n} \sum_{m \in M_1(l)} \langle \tau_j Z^m(G_{jm}(|z_j|^2), \eta) + c.c. \rangle,
\]

(4.11)

\[
Z_0(z, \eta) = \sum_{|m|=0}^{n} \sum_{m \in M_0(l+1)} Z^m a_m(|z_1|^2, \ldots, |z_n|^2).
\]

(4.12)

Remark 4.5. By Hypothesis (H4), in particular by (1.8), for any \( m \in M_0(2N + 4) \) we have \( Z^m = |z_1|^{2m_1} \ldots |z_n|^{2m_n} \) for an \( m \in \mathbb{N}_0^n \) with \( 2|m| = |m| \). Similarly by (H4), in particular by (1.4), for \( |m| \leq 2N + 4 \) we have \( |a_{m}(e_a - e_b) m_ah - e_j| \neq M \).

For \( 0 \leq N + 4 \) we will consider flows associated to Hamiltonian vector fields \( X_\chi \) with real valued functions \( \chi \) of the following form, with \( b_m = R_{r, \eta}(|z_1|^2, \ldots, |z_n|^2) \) and \( B_{jm} = S_{r, \eta}(|z_j|^2) \) for some \( r \in \mathbb{N} \) defined in \( B_{\mathbb{R}^m}(0, d) \) for some \( d > 0 \):

\[
\chi = \sum_{|m|=1}^{n} \sum_{m \in M_1(l+1)} Z^m b_m(|z_1|^2, \ldots, |z_n|^2) + \sum_{|m|=1}^{n} \sum_{m \in M_1(l+1)} \langle \tau_j Z^m(B_{jm}(|z_j|^2), \eta) + c.c. \rangle.
\]

(4.13)

The following result is proved in [12].
Theorem 4.6 (Birkhoff normal forms). For any $\iota \in \mathbb{N} \cap [2, 2N + 4]$ there are a $\delta_0 > 0$, a polynomial $\chi_i$ as in (4.18) with $l = \iota$, $d = \delta_0$, and $r = r_0 = 2(\iota + 1)$ s.t. for all $k \in \mathbb{Z} \cap [-r(\iota), r(\iota)]$ we have for each $\chi_i$ a flow (for $\delta > 0$ the constant in Lemma 4.3)

$$\phi^i_t \in C^\infty((-2, 2) \times B_{C^\infty}(0, \delta_0) \times B_{\Sigma_2^t}(0, \delta_0), B_{C^\infty}(0, \delta_0 - 1) \times B_{\Sigma_2^t}(0, \delta_0 - 1)) \quad \text{(4.14)}$$

and s.t., if we set $\tilde{\phi}^{(i)} := \phi_2 \circ \ldots \circ \phi_1$, with $\phi_i$ the transformation in Prop. 4.2 and the $\phi_j = \phi^j$, then for $(z, \eta) \in B_{C^\infty}(0, \delta_0) \times (B_{H^1}(0, \delta_0) \cap H_c[0])$ we have the following expansion

$$H^{(i)}(z, \eta) := E \circ \tilde{\phi}^{(i)}(z, \eta) = H_2(z, \eta) + \sum_{j=1}^n \lambda_j(|z|^2) + Z^{(i)}(z, Z, \eta)$$

$$+ \sum_{l=1}^{2N+3} \sum_{|m| = l+1} Z^m a^l_m(|z|^2, \ldots, |z|^2) + \sum_{j=1}^n \sum_{|m| = j} (\tau_j Z^m(G^{(j)}_j|z|^2, \eta) + \text{c.c.})$$

$$+ R_{r_0, \infty}^{1, 2}(z, \eta) + R_{r_0, \infty}^{0, 2N+5}(z, Z, \eta) + \Re(S_{r_0, \infty}^{1, 2N+4}(z, Z, \eta, \eta^*)) +$$

$$\sum_{j=1}^3 \sum_{|m| = j} R_{r_0, \infty}^{0, 3-d}(z, \eta) \int_{\mathbb{R}^3} G^{(j)}_j(x, z, \eta(x)) \eta^{*\otimes j}(x) \otimes (\eta^*(x))^{(d-j)} \, dx$$

$$\sum_{j=1}^3 \sum_{|m| = j} R_{r_0, \infty}^{1, 2}(z, \eta)^{*\otimes j} \otimes \eta^*^{(d-j)} + E_P(\eta)$$

where, for coefficients like in Def. 4.4 for $(r, m) = (r_0, \infty)$,

$$Z^{(i)} = \sum_{m \in M_0(i)} Z^m a^l_m(|z|^2, \ldots, |z|^2) + \sum_{j=1}^n (\sum_{m \in M_j(i-1)} Z^m(G^{(j)}_j|z|^2, \eta) + \text{c.c.}). \quad (4.16)$$

We have $R_{r_0, \infty}^{1, 2} = R_{r_0, \infty}^{2, 1}$ and $R_{r_0, \infty}^{1, 2}(e^{i\delta z}, e^{i\delta \eta}) = R_{r_0, \infty}^{1, 2}(z, \eta)$.

In particular we have for $\delta_0 := \delta_{2N+4}$ and for the $\delta_0$ in Prop. 4.2

$$\mathcal{F}(2N+4)(B_{C^\infty}(0, \delta_0) \times (B_{H^1}(0, \delta_0) \cap H_c[0])) \subset B_{C^\infty}(0, \delta_0) \times (B_{H^1}(0, \delta_0) \cap H_c[0]) \quad (4.17)$$

with $\mathcal{F}|_{B_{C^\infty}(0, \delta_0) \times (B_{H^1}(0, \delta_0) \cap H_c[0])}$ a diffeomorphism between its domain and an open neighborhood of the origin in $C^\infty \times (H^1 \cap H_c[0])$.

Furthermore, for $r = r_0 - 4N - 10$ there is a pair $R_{r_0, \infty}^{1, 1}$ and $S_{r_0, \infty}^{1, 1}$ s.t. for $(z', \eta') = \mathcal{F}^{(2N+4)}(z, \eta)$ we have

$$z' = z + R_{r_0, \infty}^{1, 1}(z, Z, \eta) \quad \eta' = \eta + S_{r_0, \infty}^{1, 1}(z, Z, \eta). \quad (4.18)$$

Furthermore, by taking all the $\delta_0 > 0$ sufficiently small, we can assume that all the symbols in the proof, i.e. the symbols in (4.18) and the symbols in the expansions (4.14), satisfy the estimates of Definitions 2.10 and 2.11 for $|z| < \delta_0$ and $||\eta||_{\Sigma_\iota} < \delta_0$ for their respective $\iota$’s.

\[ \square \]

5 Dispersion

We apply Theorem 4.6, set $\mathcal{H} = H^{(2N+4)}$ so that for some $r \in \mathbb{N}$ which we can take arbitrarily large,

$$\mathcal{H}(z, \eta) = H_2(z, \eta) + \sum_{j=1}^n \lambda_j(|z|^2) + Z(z, Z, \eta) + \mathcal{R} \quad (5.1)$$
with \( Z(z, Z, \eta) = Z^{(2N+4)}(z, Z, \eta) \) and

\[
R = R^{1,2}_{\rho, \infty}(z, \eta) + R^{0,2N+5}(z, Z, \eta) + \text{Re}(S^{0,2N+4}(z, Z, \eta), \eta^*) + \\
+ \sum_{d=2}^{3} \sum_{i=1}^{d} R^{3-d}_{\rho, \infty}(z, \eta) \int_{\mathbb{R}^3} G^{(i)}_{d}(x, z, \eta, \eta(x)) \eta^{\otimes i}(x) \otimes (\eta^*)^{\otimes (d-i)} \, dx \\
+ \sum_{i+j=2}^{m} Z^{m}(G^{(i)}_{2mi}, \eta) \eta^{\otimes i} \otimes (\eta^*)^{\otimes j} + E_{P}(\eta). \tag{5.2}
\]

Our ambient space is \( H^4(\mathbb{R}, C^4) \). So under (H1) the functional \( u \to g(u, \beta)u \) is locally Lipschitz and (1.11), (5.2) and the equivalent system with Hamiltonian \( \mathcal{H}(z, \eta) \) and symplectic form \( \Omega \), are locally well posed, see pp. 293–294 volume III [26].

By standard arguments, see [12], Theorem 5.1 below implies Theorem 1.3.

**Theorem 5.1 (Main Estimates).** Consider the constants \( 0 < \epsilon < \epsilon_0 \) of Theorem 1.3 Then if \( \epsilon_0 \) is sufficiently small there is a fixed \( C_0 > 0 \) such that for \( I = [0, \infty) \) we have:

\[
\| \eta \|_{L^1_{t}(I, L^{4/3}(\mathbb{R}^3, C)} \leq C \epsilon \text{ for all the pairs } (p, q) \text{ as of } (1.11) \tag{5.3}
\]

\[
\| \eta \|_{L^{1}_{t}(I, H^{4-10}(\mathbb{R}^3, C)} \leq C \epsilon \tag{5.4}
\]

\[
\| \eta \|_{L^{1}_{t}(I, L^{\infty}(\mathbb{R}^3, C)} \leq C \epsilon \tag{5.5}
\]

\[
\| z_j Z^{m} \|_{L^{1}_{t}(I)} \leq C \epsilon \text{ for all } (j, m) \text{ with } m \in M_j(2N+4), \tag{5.6}
\]

\[
\| z_j \|_{W^{1, \infty}_t(I)} \leq C \epsilon \text{ for all } j \in \{1, \ldots, n\}. \tag{5.7}
\]

Furthermore, there exists \( \rho_+ \in [0, \infty)^n \) s.t. there exist a \( j_0 \) with \( \rho_{+j} = 0 \) for \( j \neq j_0 \) and there exists \( \eta_+ \in L^\infty \) s.t. \( |\rho_+| \leq C \epsilon \) and \( \| \eta_+ \|_{L^\infty} \leq C \epsilon \) s.t.

\[
\lim_{t \to +\infty} \| \eta(t, x) - e^{-itD_{x}} \eta_+(x) \|_{L^\infty(\mathbb{R}^3, C)} = 0, \tag{5.8}
\]

\[
\lim_{t \to +\infty} |z_j(t)| = \rho_{+j}. \tag{5.9}
\]

By an elementary continuation argument, the estimates \( (5.3) - (5.7) \) for \( I = [0, \infty) \) are a consequence of the following Proposition.

**Proposition 5.2.** There exist a constant \( \epsilon_0 > 0 \) such that for any \( C_0 > \epsilon_0 \) there is a value \( \epsilon_0 = \epsilon_0(C_0) \) such that if the inequalities \( (5.3) - (5.7) \) hold for \( I = [0, T] \) for some \( T > 0 \), for \( C = C_0 \) and for \( 0 < \epsilon < \epsilon_0 \), then in fact for \( I = [0, T] \) the inequalities \( (5.3) - (5.7) \) hold for \( C = C_0/2 \).

### 5.1 Proof of Proposition 5.2

We start this section by itemizing some result known in the literature, then we prove some ancillary tools required for the proof of Proposition 5.2. The following theorem is Theorem 1.1 [4].

**Theorem 5.3.** Under hypotheses (H2)–(H3) for \( s > 5/2 \) and any \( k \in \mathbb{R} \) we have

\[
\| e^{iH_{t}} P_{k} u_{0} \|_{H^{k-\epsilon}(\mathbb{R}^3, C)} \leq C_{s,k}(t)^{-\frac{s}{2}} \| P_{k} u_{0} \|_{H^{k-\epsilon}(\mathbb{R}^3, C)}. \tag{5.10}
\]

The subsequent is Theorem 1.1 in [5].
Theorem 5.4 (Smoothness estimates). For any $\tau > 1$ and $k \in \mathbb{R}$ $\exists C$ s.t.

\begin{align*}
\|e^{-itH}P_c\psi\|_{L_t^2(B(0,k^{-\tau};\mathbb{R}^3,\mathbb{C}))} & \leq C\|P_c\psi\|_{H^k(\mathbb{R}^3,\mathbb{C})}, \\
\int_{\mathbb{R}} e^{itH}P_cF(t)\,dt\|_{H^k} & \leq C\|P_cF\|_{L_t^2(B(0,k^{-\tau};\mathbb{R}^3,\mathbb{C}))}, \\
\int_{t' < t} e^{-i(t-t')H}P_cF(t')\,dt'\|_{L_t^2(B(0,k^{-\tau};\mathbb{R}^3,\mathbb{C}))} & \leq C\|P_cF\|_{L_t^2(B(0,k^{-\tau};\mathbb{R}^3,\mathbb{C}))}.
\end{align*}

(5.11) (5.12) (5.13)

The following is Theorem 3.1 in [4].

Theorem 5.5. For $p \in [1,2]$, $\theta \in [0,1]$, with $k - k' \geq (2 + \theta)(\frac{2}{p} - 1)$ and $q \in [1,\infty]$ there is a constant $C$ s.t. for $p' = \frac{p}{p-1}$

\[\|e^{itD_{x,y}}\|_{B^{l,q}_{p',q} \to B^{l',q}_{p',q}} \leq C(K(t))^{\frac{1}{p}-1}\] where $K(t) := \begin{cases} |t|^{-1+\theta/2} & \text{if } |t| \leq 1, \\ |t|^{-1-\theta/2} & \text{if } |t| \geq 1. \end{cases}$

The following is Theorem 1.2 in [5].

Theorem 5.6 (Strichartz estimates). For any $2 \leq p, q \leq \infty$, $\theta \in [0,1]$, with $(1 - \frac{2}{q})(1 + \frac{\theta}{2}) = \frac{2}{p}$ and $(p,\theta) \neq (2,0)$, and for any reals $k, k'$ with $k' - k \geq \alpha(q)$, where $\alpha(q) = (1 + \frac{\theta}{2})(1 - \frac{2}{q})$, there exists a positive constant $C$ such that

\begin{align*}
\|e^{-itH}P_c\psi\|_{L_t^p(B(0,k^{-\tau};\mathbb{R}^3,\mathbb{C}))} & \leq C\|P_c\psi\|_{H^k(\mathbb{R}^3,\mathbb{C})}, \\
\int_{\mathbb{R}} e^{itH}P_cF(t)\,dt\|_{H^k(\mathbb{R}^3,\mathbb{C})} & \leq C\|P_cF\|_{L_t^p(B(0,k^{-\tau};\mathbb{R}^3,\mathbb{C}))}, \\
\int_{t' < t} e^{-i(t-t')H}P_cF(t')\,dt'\|_{L_t^p(B(0,k^{-\tau};\mathbb{R}^3,\mathbb{C}))} & \leq C\|P_cF\|_{L_t^p(B(0,k^{-\tau};\mathbb{R}^3,\mathbb{C}))},
\end{align*}

for any $(a,b)$ chosen like $(p,q)$, and $h - k \geq \alpha(q) + \alpha(b)$.

We have the following facts concerning the resolvent of the operator $D_{x,y}$, see [4,5,6].

Lemma 5.7. The facts are true.

1. For $z \notin \sigma(D_{x,y})$ for the integral kernel we have $R_{D_{x,y}}(x,y,z) = R_{D_{x,y}}(x - y, z)$ with

\[R_{D_{x,y}}(x,z) = \left( \frac{(z + \mathbb{M})I_2}{i\sqrt{\mathbb{M}^2 - z^2} \cdot \hat{x}} \right) e^{-\sqrt{\mathbb{M}^2 - z^2}|x|} \left( \frac{I_2}{z - \mathbb{M}} \right) \frac{e^{-\sqrt{h^2 - z^2}|x|}}{4\pi|x|} + \frac{1}{4\pi|x|^2} e^{-\sqrt{\mathbb{M}^2 - z^2}|x|} \]

where $\hat{x} = x/|x|$ and where for $\zeta = e^{i\varphi}r$ with $r \geq 0$ and $\varphi \in (-\pi, \pi)$ we set $\sqrt{\zeta} = e^{i\varphi/2} \sqrt{r}$.

2. For any $\tau > 1$ there exists $C$ s.t. $\|R_{D_{x,y}}(z)\psi\|_{L_t^{2,-\tau}(\mathbb{R}^3,\mathbb{C})} \leq C\|\psi\|_{L_t^{2,-\tau}(\mathbb{R}^3,\mathbb{C})}$ for all $z \notin \mathbb{R}$.

3. For any $\tau > 1$ the following limits exist in $B(H^{1,\tau}(\mathbb{R}^3,\mathbb{C}^4), L^{2,-\tau}(\mathbb{R}^3,\mathbb{C}^4))$

\[R_{D_{x,y}}^*(\lambda) = \lim_{\varepsilon \searrow 0} R_{D_{x,y}}(\lambda \pm i\varepsilon) \quad \text{for } \lambda \in \mathbb{R}\setminus(-\mathbb{M},\mathbb{M})
\]

and the convergence is uniform for $\lambda$ in compact subsets of $\mathbb{R}\setminus(-\mathbb{M},\mathbb{M})$.

19
Lemma 5.8. For any preassigned \( \tau > 1 \) the following facts hold.

1. The limits \( R_H^\pm(\lambda) = R_H(\lambda \pm i0) := \lim_{\varepsilon \searrow 0} R_H(\lambda \pm i\varepsilon) \) for \( \lambda \in (-\infty, -\mathcal{M}) \cup (\mathcal{M}, \infty) \) exist in \( B(L^{2,\tau}, L^{2,-\tau}) \) and the convergence is uniform in compact subsets of \((-\infty, -\mathcal{M}) \cup (\mathcal{M}, \infty)\).

2. There exists a constant \( C_1 = C_1(\tau) \) s.t. for any \( u_0 \in L^2(\mathbb{R}^3, \mathbb{C}^4) \) and any \( \varepsilon \geq 0 \) we have

\[
\| (x)^{-\tau} R_H(\lambda \pm i\varepsilon) P_c u_0 \|_{L^2_{\chi}(\mathbb{R}, L^2_{\chi}(\mathbb{R}))} \leq C_1 \| P_c u_0 \|_{L^2(\mathbb{R}^3)}. \tag{5.20}
\]

3. Let \( \Lambda \) be a compact subset of \((-\infty, -\mathcal{M}) \cup (\mathcal{M}, \infty)\). There exists a constant \( C_1 = C_1(\tau, \Lambda) \) s.t.

\[
\| (x)^{-\tau} R_H^\pm(\lambda) P_c u_0 \|_{L^\infty_{\chi}(\Lambda, L^2_{\chi}(\mathbb{R}))} \leq C_1 \| P_c u_0 \|_{L^2(\mathbb{R}^3)}. \tag{5.21}
\]

**Proof.** Claim (1) is an immediate consequence of Theorem 5.3 in the case \( \tau > 5/2 \), as observed on p. 783. The extension to the case \( \tau > 1 \) follows by the proof of Proposition 3.10. (5.20) is equivalent to (5.14) for \( k = 0 \).

We prove now (5.21). Let \( u_0 = P_c u_0 \), \( A(x) = (x)^{-\tau} \) and \( B(x) \in \mathcal{S}(\mathbb{R}^3, \mathcal{S}_1(\mathbb{C})) \) s.t. \( B^* A = V \). Then

\[
A R_H(z) u_0 = (1 + A R_{D,\#}(z) B^*)^{-1} A R_{D,\#}(z) u_0 \text{ for } z \in \mathbb{C} \setminus \mathbb{R}.
\]

This equality continues to hold on \( \mathbb{R} \pm i0 \) by Lemmas 5.7 and 5.8. We then have

\[
\| R_H^\pm(\lambda) P_c \|_{B(L^{2,\tau}_{\chi}, L^{2,-\tau}_{\chi})} \leq \| (1 + A R_{D,\#}^+(\lambda) B^*)^{-1} \|_{B(L^{2,\tau}_{\chi}, L^{2,-\tau}_{\chi})} \| R_{D,\#}^+(\lambda) \|_{B(L^{2,\tau}_{\chi}, L^{2,-\tau}_{\chi})}. \tag{5.22}
\]

By (1) there is a \( C'(\tau) > 0 \) s.t. for all \( \lambda \in \mathbb{R} \)

\[
\| A R_{-\Delta}^+(\lambda^2) \|_{B(L^{2,\tau}_{\chi}, L^{2,-\tau}_{\chi})} + \| \nabla R_{-\Delta}^+(\lambda^2) \|_{B(L^{2,\tau}_{\chi}, L^{2,-\tau}_{\chi})} \leq C'(\tau).
\]

Then by (5.19) we have

\[
\| R_{D,\#}^+(\lambda) \|_{B(L^{2,\tau}_{\chi}, L^{2,-\tau}_{\chi})} \leq C(\tau) \text{ for all } \lambda \in \mathbb{R}.
\]

We obtain (5.23) from

\[
\sup_{\lambda \in \Lambda} \| (1 + A R_{D,\#}^+(\lambda) B^*)^{-1} \|_{B(L^{2,\tau}_{\chi}, L^{2,-\tau}_{\chi})} < \infty, \tag{5.23}
\]

which follows from the analytic Fredholm alternative.

**Remark 5.9.** Notice that (5.23) is in fact true for \( \Lambda = \mathbb{R} \) by (H3) and, for large \( \lambda \), by [13], see Appendix A [15].

The next lemma is proved by an argument of [17] reviewed in Lemma 5.7 [16].

20
Lemma 5.10. Consider pairs \((p, q)\) as in Theorem 5.7 with \(p > 2\), \(k \in \mathbb{R}\) arbitrary and \(k' - k \geq \alpha(q)\). Then for any \(\tau > 1\) there is a constant \(C_0 = C_0(\tau, k, p, q)\) such that

\[
\left\| \int_0^t e^{iH(t-t')} P_c F(t') dt' \right\|_{L^p_t B^k_{q,2}} \leq C_0 \| P_c F \|_{L^\infty_t H^{k',\tau}}. \tag{5.24}
\]

Proof. For \(F(t, x) \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)\) set

\[ T F(t) := \int_0^{+\infty} e^{i(t-t')H} P_c F(t') dt', \quad f := \int_0^{+\infty} e^{i't'H} P_c F(t') dt'. \]

Theorem 5.6 implies \(\|TF\|_{L^p_t B^k_{q,2}} \leq \|f\|_{H^{k'}}\) for \(k' - k = \alpha(q)\). By Theorem 5.4 we have \(\|f\|_{H^{k'}} \leq C \|F\|_{L^\infty_t H^{k',\tau}}\). By \(p > 2\) a lemma by Christ and Kiselev [8], see Lemma 3.1 [21], yields Lemma 5.10. \(\square\)

Lemma 5.11. Assume the hypotheses of Prop. 5.2 and recall the definition of \(M\) Def. 2.4. Let \(\tau_0 > 1\). Then there is a fixed \(c\) such that for all admissible pairs \((p, q)\)

\[
\|\eta\|_{L^p_t([0,T],B^{k-\tau_0}_{q,2}) \cap L^\infty_t(\mathbb{R}^3)} \leq c \epsilon + c \sum_{(\mu, \nu) \in M} |z^{\mu,\nu}|_{L^2_t(0,T)}. \tag{5.25}
\]

Proof. By picking \(\epsilon_0 > 0\) sufficiently small and \(\epsilon = \|u(0)\|_{H^4} < \epsilon_0\), for a fixed \(c_1 > 0\) for the final coordinates \((z(0), \eta(0))\) of \(u(0)\) we have

\[
|z(0)| + \|\eta(0)\|_{H^4} \leq c_1 \epsilon, \tag{5.26}
\]

We have for \(G_{jm}^* = G_{jm}^*(0)\)

\[
i\dot{\eta} = i\{\eta, H\} = H\eta + \sum_{j=1}^n \sum_{l=1}^{2N+3} \sum_{|m|=l} z_j \overline{z}_m G_{jm}^* + A \quad \text{where}
\]

\[ A := \sum_{j=1}^n \sum_{l=1}^{2N+3} \sum_{|m|=l} z_j \overline{z}_m [G_{jm}^*(|z_j|^2) - G_{jm}^*] + \nabla \eta \cdot R. \tag{5.27}
\]

We rewrite

\[
\sum_{j=1}^n \sum_{l=1}^{2N+3} \sum_{|m|=l} z_j \overline{z}_m G_{jm}^* = \sum_{(\mu, \nu) \in M} \mathbb{Z}^{\mu,\nu} G_{jm}^*. \tag{5.28}
\]

Notice that (5.6) is the same as

\[
\|z^{\mu,\nu}\|_{L^2_t(0,T)} \leq C\epsilon \text{ for all } (\mu, \nu) \in M. \tag{5.29}
\]

We have the following lemma.

Lemma 5.12. For \(I_T := [0, T]\) and for \(S \in \mathbb{R}\) and \(\epsilon_0 > 0\) small enough then for a constant \(C(S, C_0)\) independent from \(T\) and \(\epsilon\) we have

\[
\|A\|_{L^2(I_T, H^4, S) + L^1(I_T, H^4)} \leq C(S, C_0)\epsilon^2. \tag{5.30}
\]
Proof. The proof is standard. For example we have

\[ \|z_j Z_m^n|G_{jm}^n(\|z_j\|^2) - G_{jm}^n||_{L^2(\Omega_T,H^s)} \leq \|z_j Z_m^n||_{L^2(\Omega_T,H^s)}\|G_{jm}^n(\|z_j\|^2) - G_{jm}^n\|_{L^\infty(\Omega_T,H^s)} \leq C_0 \sup\{\|G_{jm}^n(\|z_j\|^2)\|_{L^\infty(\Omega_T,H^s)} \leq C C_0 c^3. \]  

(5.31)

We have for a fixed \( c_1 > 0 \)

\[ \|\nabla \eta E_p(\eta)\|_{L^1(\Omega_T,H^s)} = 2\|g(\eta,\eta)\|_{L^1(\Omega_T,H^s)} \leq c_1 \|\eta\|_{L^\infty(\Omega_T,H^s)} \|\|_{L^2(\Omega_T,H^s)} \leq c_1 C_0 c^3. \]  

(5.32)

The rest of Lemma 5.12 follows by the fact that for arbitrarily preassigned \( S > 2 \)

\[ \|R_1\|_{L^2(\Omega_T,H^s)} \leq C(S,C_0)c^2 \]  

(5.33)

\( R_1 \) is a sum of various terms obtained from the expansion (5.2). The following estimates are proved in [12]:

\[ \|\nabla \eta R^1_{\Omega_T}\|_{L^2(\Omega_T,H^s)} + \|\nabla \eta R^0_{\Omega_T}\|_{L^2(\Omega_T,H^s)} \leq C(S,C_0)c^2. \]  

(5.34)

Other terms in \( R_1 \) can be bounded with similarly elementary arguments, yielding (5.33). Then (5.31), (5.32) and (5.33) imply (5.30).

Thanks to Lemma 5.12 then Lemma 5.11 is a consequence of Lemmas 5.13 and 5.14 below.

Lemma 5.13. Consider \( \psi \) \(-\psi - H\psi = F \) where \( P_c \) and \( \psi = P_c \psi \). Let \( k \in \mathbb{Z} \) with \( k \geq 0 \) and \( \tau_0 > 1 \). Then for \( (p,q) \) as in (5.1) and \( \tau_0 > 1 \) for a constant \( C = C(p,q,k,\tau_0) \) we have

\[ \|\psi\|_{L^p_T([0,T],B_{p,2}^{k-\frac{2}{p}}) \cap L^2_T([0,T],H^k_{-\tau_0})} \leq C\|\psi(0)\|_{H^k} + C\|\psi(0)\|_{H^k} + L^2_T([0,T],H^k_{-\tau_0}) \]  

(5.35)

Proof. We split \( F = F_1 + F_2 \) with \( F_1 \in L^1([0,T],H^k) \) and \( F_2 \in L^2([0,T],H^k_{-\tau_0}) \) and we write

\[ \psi(t) = e^{-itH}\psi(0) - i \sum_{j=1}^2 \int_0^t e^{-i(t-s)H} F_j(s) ds. \]  

(5.36)

Estimate (5.35) in the special case \( F = 0 \) is a consequence of (5.14) and (5.11). The case \( \psi_0 = 0 \), \( F_2 = 0 \) follows by (5.10) and (5.11). Finally, the case \( \psi_0 = 0 \), \( F_1 = 0 \) follows by (5.24) and (5.13).

Lemma 5.14. Using the notation of Lemma 5.13, but this time picking \( \tau_0 > 3/2 \), we have

\[ \|\psi\|_{L^2_T([0,T],H^k_{-\tau_0})} \leq C\|\psi(0)\|_{H^k} + C\|\psi(0)\|_{H^k} + L^2_T([0,T],H^k_{-\tau_0}) \]  

(5.37)

Proof. The argument is the same of Lemma 10.5 [5]. We consider

\[ \psi(t) = e^{-itD,\#}\psi(0) + i \int_0^t e^{-i(t-s)D,\#} V\psi(s) ds - i \sum_{j=1}^2 \int_0^t e^{-i(t-s)D,\#} F_j(s) ds. \]  

(5.38)

We have for \( k \in (1/2,3] \)

\[ \|e^{-itD,\#}\psi(0)\|_{L^2_TH^k_{-\tau_0}} \leq C\|\psi(0)\|_{H^k} \leq C'\|\psi(0)\|_{H^k} \leq C'\|\psi(0)\|_{H^k}. \]

(5.39)
by the flat version of (5.14), which holds by [4]. Similarly we have
\[ \| \int_0^t e^{iD}\phi(t')dt' \|_L^2L^\infty \leq C \| \int_0^t e^{iD}\phi(t')dt' \|_{L^2B^k_{\infty,2}} \]
\[ \leq C' \| F_1 \|_{L^1_tH^{k+1}} \leq C' \| F_1 \|_{L^1_tH^4}. \]

Using \( B^k_{\infty,2} \subseteq L^\infty \) for \( k > 1/2 \) and picking \( k < 1 \), by Theorem 5.3 we have
\[ \| \int_0^t e^{iD}\phi(t')dt' \|_L^2L^\infty \leq C \| \int_0^t \min\{|t-t'|^{-1/2},|t-t'|^{-1/2}\} \| F_2(t') \|_{B^1_{1,2}}^2 dt' \]
\[ \leq C' \| F_2 \|_{L^1_tB^1_{1,2}} \leq C' \| (x)^{-\tau_0}F_2 \|_{L^2_{1,2}} = C'' \| F_2 \|_{L^2_{1,2}}H^{\tau_0}, \]

where we have used \( \| \varphi_j * F_2 \|_{L^1_2} \leq \| (x)^{-\tau_0}F_2 \|_{L^2_2} \| (x)^{\tau_0} \varphi_j \|_{L^2_2} \leq C''' \| \varphi_j * (\cdot)^{\tau_0}F_2 \|_{L^2_2} \)

for fixed \( C'' > 0 \) and fixed \( \tau_0 > 3/2 \). With \( F_2 \) replaced by \( V\psi \) we get a similar estimate. This yields inequality (5.37).

We then have
\[ \| \varphi_j * F_2 \|_{L^1_2} \leq \| (x)^{-\tau_0}F_2 \|_{L^2_2} \| (x)^{\tau_0} \varphi_j \|_{L^2_2} \leq C''' \| \varphi_j * (\cdot)^{\tau_0}F_2 \|_{L^2_2} \]

for fixed \( C'' > 0 \) and fixed \( \tau_0 > 3/2 \). With \( F_2 \) replaced by \( V\psi \) we get a similar estimate. This yields inequality (5.37).

Setting \( M = M(2N + 4) \), see Def. 2.4 we now introduce a new variable \( g \) setting
\[ g = \eta + Y \] with \( Y := \sum_{(\alpha,\beta) \in M} z^\alpha \bar{z}^\beta R^s_H(e \cdot (\beta - \alpha))G_{\alpha\beta}. \]

The following lemma is an easier version of Lemma 10.7 [9] and so we give the proof in few lines.

**Lemma 5.15.** Assume the hypotheses of Prop. 5.2 and fix \( S > 9/2 \). Then there is a \( c_1(S) > 0 \) s.t. for any \( C_0 \) there is a \( c_0 = c_0(C_0, S) > 0 \) such that for \( \epsilon \in (0, c_0) \) in Theor [7] we have
\[ \| g \|_{L^2([0,T],L^2_{-S})} \leq c_1(S)\epsilon. \]  \( (5.41) \)

**Proof.** We have
\[ ig = Hg + \lambda + T \]
where
\[ T := \sum_{j} [\partial_{\bar{z}_j}Y(i\bar{z}_j - c_jz_j) + \partial_{z_j}Y(z_j + i\bar{z}_j)]. \]

We then have
\[ g(t) = e^{-iHt}\eta(0) + e^{-iHt}Y(0) - i \int_0^t e^{-iH(t-s)}(\lambda(s) + T(s))ds. \]

We have \( \| e^{-iHt}\eta(0) \|_{L^2(R,L^2_{-S})} \leq c \| \eta(0) \|_{L^2} \leq c\epsilon \) by (5.11). The rest of the proof of Lemma 5.15 follows by Lemma 5.16 below, which is the analogue of Lemma 6.5 [12], and by exactly the same proof of Lemma 6.4 [12].

We get, following the proof of Lemma 5.8 [5].

**Lemma 5.16.** Let \( \Lambda \) be a compact subset of \((-\infty, -M) \cup (M, \infty) \) and let \( S > 7/2 \). Then there exists a fixed \( c(S, \Lambda) \) s.t. for every \( t \geq 0 \) and \( \lambda \in \Lambda \)
\[ \| e^{-iHt}R^+_{\Lambda}(\lambda)Pc\|_{L^2_{-S}(R^3)} \leq c(S, \Lambda)(t)^{-\frac{1}{2}} \| PcV_0 \|_{L^2_{-S}(R^3)} \]
for all \( V_0 \in L^{2S}(R^3). \]

(5.44)
Proof. We expand $R^t_H(\lambda) = R^t_{D,\sigma}(\lambda) - R^t_{D,\sigma}^+(\lambda)V R^t_{H}(\lambda)$. For $\tau_1 > 5/2$, by Theorem \ref{thm5.3} and by \[2\] Theorem 2 we have
\[
\|e^{-itD,\sigma} R^t_{D,\sigma}(\lambda)\psi_0\|_{L^2, -\tau_1(\mathbb{R}^3)} \leq C(t)^{-\frac{7}{2}} \|R^t_{D,\sigma}(\lambda)\psi_0\|_{L^2, -\tau_1(\mathbb{R}^3)} \leq C_1(t)^{-\frac{7}{2}} \|\psi_0\|_{L^2, -\tau_1(\mathbb{R}^3)},
\]
with $C_1$ locally bounded in $\lambda$ and $\tau_1$. Hence, by the rapid decay of $V$ and by Lemma 5.8
\[
\|e^{-itD,\sigma} R^t_{D,\sigma}(\lambda)V R^t_{H}(\lambda) \psi_0\|_{L^2, -\tau_1} \leq C_1(t)^{-\frac{7}{2}} \|V\|_{B(L^2, -\tau_1, L^2, -\tau_1)} \|R^t_{H}(\lambda)\psi_0\|_{L^2, -\tau_1} \leq C'(t)^{-\frac{7}{2}}.
\]

\[\square\]

5.2 The analysis of the discrete modes

We turn now to the analysis of the Fermi Golden Rule (FGR). Consider as in Lemma 5.10 the set
\[
\Lambda := \{ (\nu - \mu) \cdot e : (\mu, \nu) \in M \}. \tag{5.45}
\]
We have
\[
i \hat{z}_j = (1 + \varpi_j(|z|^2))(e_j z_j + \partial_{z_j} Z_0(|z_1|^2, ..., |z_n|^2) + \partial_{z_j} R) + (1 + \varpi_j(|z|^2)) \sum_{(\mu, \nu) \in M} \nu_j \frac{z^{\mu + \beta + \alpha}}{z_j} \langle \eta, G_{\mu \nu} \rangle + \sum_{(\mu', \nu') \in M} \mu'_j \frac{z^{\mu'+\nu'+\beta'}}{z_j} \langle \eta^*, G^{*\mu'\nu'} \rangle + (1 + \varpi_j(|z|^2)) \sum_{m \in M_j(2N+3)} |z_j|^2 Z^m \langle G^{*}_{jm}, \eta \rangle + z_j^2 \overline{Z}^m \langle G^{*}_{jm}, \eta^* \rangle.
\]

We use \[5.40\] to substitute an appropriately $\eta$ in the equations above getting, for $M_{\text{min}}$ defined as in \[23\],
\[
i \hat{z}_j = (1 + \varpi_j(|z|^2))(e_j z_j + \partial_{z_j} Z_0(|z_1|^2, ..., |z_n|^2) + \mathcal{F}_j + X_j(M_{\text{min}}) \tag{5.47}
\]
where for $\widehat{M} \subset M$ we set
\[
X_j(\widehat{M}) := \sum_{(\mu, \nu) \in \widehat{M}} \nu_j \frac{z^{\mu + \beta + \alpha}}{z_j} \langle R^+_H(e \cdot (\beta - \alpha))G^*_{\alpha \beta}, G_{\mu \nu} \rangle - \sum_{(\mu', \nu') \in \widehat{M}} \mu'_j \frac{z^{\mu'+\nu'+\beta'}}{z_j} \langle R_H(e \cdot (\beta' - \alpha'))G_{\alpha' \beta'}, G^{*\mu'\nu'} \rangle \tag{5.48}
\]
and where $\mathcal{F}_j = \mathcal{A}_j + \mathcal{B}_j$ with
\[
\mathcal{A}_j = (1 + \varpi_j(|z|^2))\partial_{\hat{z}_j} R + \varpi_j(|z|^2) \sum_{(\mu, \nu) \in M} \nu_j \frac{z^{\mu + \beta + \alpha}}{z_j} \langle g, G_{\mu \nu} \rangle + \sum_{(\mu', \nu') \in M} \mu'_j \frac{z^{\mu'+\nu'+\beta'}}{z_j} \langle g^*, G^{*\mu'\nu'} \rangle
\]
\[
+ (1 + \varpi_j(|z|^2)) \sum_{m \in M_j(2N+3)} |z_j|^2 Z^m \langle g_{jm}, \eta \rangle + z_j^2 \overline{Z}^m \langle g^{*}_{jm}, \eta^* \rangle \tag{5.49}
\]
Lemma 2.5 we get

Proof.
The proof the bounds (5.52) is the same of [12] (see also reference therein) so we skip. By
We introduce the variable \( \zeta \) defined by

\[
\begin{align*}
\zeta_j &= - \sum_{(\mu, \nu) \in M_{\min}} \frac{\nu_j z^{\mu + \beta} z^{\nu + \alpha}}{((\mu - \nu) \cdot e - (\alpha - \beta) \cdot e) \zeta_j} (R^+_{H}(e \cdot (\beta - \alpha)) G^*_{\alpha \beta}, G_{\mu \nu}) \\
- \sum_{(\mu', \nu') \in M_{\min}} \frac{\mu_j' z^{\nu' + \alpha'} z^{\mu' + \beta'}}{((\alpha' - \beta') \cdot e - (\mu' - \nu') \cdot e) \zeta_j} (R^-_{H}(e \cdot (\beta' - \alpha')) G_{\alpha' \beta', \mu' \nu'})
\end{align*}
\]

where the summation is performed over the pairs where the formula makes sense, that is \((\mu - \nu) \cdot e \neq (\alpha - \beta) \cdot e\). We have the following lemma, (see also [12]).

Lemma 5.17. We have

\[
\|\zeta - z\|_{L^2(0,T)} \leq c(N, C_0) c^2 \quad \text{and} \quad \|\zeta - z\|_{L^\infty(0,T)} \leq c(N, C_0) c^2
\]

and \( \zeta \) equations

\[
i \dot{\zeta}_j = (1 + \sigma(z_j^2))(c_j \zeta_j + \lambda_j(z_j^2) \zeta_j + \partial_{\frac{\lambda_j}{\nu_j}} \zeta_j) z_0(\zeta_j)
- \sum_{(\mu, \nu) \in M_{\min}} \frac{\nu_j z^{\mu + \beta} z^{\nu + \alpha}}{\zeta_j} (R^+_{H}(e \cdot (\beta - \alpha)) G^*_{\alpha \beta}, G_{\mu \nu})
- \sum_{(\mu', \nu') \in M_{\min}} \frac{\mu_j' z^{\nu' + \alpha'} z^{\mu' + \beta'}}{\zeta_j} (R^-_{H}(e \cdot (\beta' - \alpha')) G_{\alpha' \beta', \mu' \nu'}) + G_j,
\]

where there are fixed \( c_4 \) and \( \epsilon_0 > 0 \) such that for \( T > 0 \) and \( \epsilon \in (0, \epsilon_0) \) we have

\[
\|G_j \cdot \zeta_j\|_{L^\infty(0,T)} \leq (1 + C_0) c_4 c^2.
\]

Proof. The proof the bounds (5.52) is the same of [12] (see also reference therein) so we skip. By Lemma 2.5 we get

\[
i \dot{\zeta}_j = (1 + \sigma(z_j^2))(c_j \zeta_j + \lambda_j(z_j^2) \zeta_j + \partial_{\frac{\lambda_j}{\nu_j}} \zeta_j) z_0(\zeta_j)
- \sum_{(\mu, \nu) \in M_{\min}} \frac{\nu_j z^{\mu + \beta} z^{\nu + \alpha}}{\zeta_j} (R^+_{H}(e \cdot (\nu - \mu)) G^*_{\mu \nu}, G_{\mu \nu})
- \sum_{(\mu', \nu') \in M_{\min}} \frac{\mu_j' z^{\nu' + \alpha'} z^{\mu' + \beta'}}{\zeta_j} (R^-_{H}(e \cdot (\nu - \mu)) G_{\mu \nu}, G^*_{\mu \nu}) + G_j,
\]

and

\[
B_j = X_j(M) - X_j(M_{\min}).
\]
where for some $A_{\alpha\beta\mu\nu}, B_{\alpha\beta\mu\nu}$ we have
\[ G_j = F_j + (1 + \varpi(|z_j|^2))|\partial_j Z_0(|z_1|^2, ..., |z_n|^2) - \partial_j Z_0(|\zeta_1|^2, ..., |\zeta_n|^2)| 
- e_j \varpi(|z_j|^2) \sum_{(\mu,\nu) \in M_{\text{min}}} \frac{\nu_j z^{\mu+\beta} z^{\nu+\alpha}}{(\mu - \nu) \cdot e - (\alpha - \beta) \cdot e} \zeta_j(R^+_H(e \cdot (\beta - \alpha)G^*_{\alpha\beta}, G_{\mu\nu}) 
+ \sum_{(\mu,\nu) \in M_{\text{min}}} \frac{\mu_j z^{\nu} z^{\mu} + \alpha z^{\alpha} + \beta}{(\mu' - \nu') \cdot e - (\mu - \nu) \cdot e} \zeta_j(R^-_H(e \cdot (\beta' - \alpha')G_{\alpha'\beta'}, G^*_{\mu'\nu'})) \]
\[ + \sum_{l \in M_{\text{min}}} \left[(i \partial_l - e_l z_l) \frac{z^{\mu+\beta} z^{\nu+\alpha}}{z_j} A_{\alpha\beta\mu\nu} + (i \partial_l - e_l z_l) \frac{z^{\mu+\beta} z^{\nu+\alpha}}{z_j} B_{\alpha\beta\mu\nu}. \right] \]

The proof of the inequality (5.57) is similar to the analogous one in [12] (see also reference therein) with the minor addition that we have to estimate the additional terms $\mathcal{B}_j$ given by (5.56). Hence we need to check only the estimate
\[ \|B_j \zeta_j\|_{L^1_t} \leq \|B_j z_j\|_{L^1_t} + \|B_j\|_{L^2_t} \|z_j - \zeta_j\|_{L^2_t} \leq C(C_0) \varepsilon^3. \] (5.57)

To prove (5.57) we use
\[ \|B_j z_j\|_{L^1_t} \lesssim \sum_{(\mu,\nu) \in M \setminus M_{\text{min}}} \|z^{\mu+\nu} z^{\nu} z^{\mu} z^{\alpha} z^{\beta}\|_{L^2_t} \leq C(C_0) \varepsilon^3, \]
by the definition of $M_{\text{min}}$. In an analogous way it is possible to prove the remaining estimates needed for the second term on the r.h.s. of (5.57).

By multiplying the identity (5.55) above by $\zeta_j$ and summing over the index $j$ we achieve, like in [12],
\[ 2^{-1} \sum_j \frac{d}{dt} |\zeta_j|^2 = - \sum_j \text{Im} \left[ \sum_{(\mu,\nu) \in M_{\text{min}}} \nu_j |\zeta_j|^2 (R^+_H(\nu - \mu) \cdot e) G^*_{\mu\nu}, G_{\mu\nu} \right] \]
\[ + \sum_{(\mu,\nu) \in M_{\text{min}}} \mu_j |\zeta_j|^2 (R^-_H(\nu - \mu) \cdot e) G_{\mu\nu}, G^*_{\mu\nu}) + \sum_j \text{Im}[\zeta_j]. \] (5.58)
Thus by using the substitution, for any $(\nu - \mu) \cdot e \in \Lambda$ (see the formula (A.7) in the Lemma A.1 of the Appendix [A]),
\[ R^+_H((\nu - \mu) \cdot e) = P.V. \frac{1}{H - (\nu - \mu) \cdot e} \pm i \pi \delta(H - (\nu - \mu) \cdot e), \]
we can state the following lemma.

**Lemma 5.18.** For any $(\mu, \nu) \in M_{\text{min}}$ we have
\[ \sum_j \text{Im} \left[ \sum_{(\mu,\nu) \in M_{\text{min}}} \nu_j |\zeta_j|^2 (P.V. \frac{1}{H - (\nu - \mu) \cdot e}) G^*_{\mu\nu}, G_{\mu\nu}) \right] + \sum_{(\mu,\nu) \in M_{\text{min}}} \mu_j |\zeta_j|^2 (P.V. \frac{1}{H - (\nu - \mu) \cdot e}) G_{\mu\nu}, G^*_{\mu\nu}) = 0. \] (5.59)
Proof. By formula (A.9) below the terms $\langle P.V. \frac{1}{(\nu - \mu) \cdot e} f, f^* \rangle$, for $f = G_{\mu \nu}, G^*_{\mu \nu}$, are real valued. 

We have also the lemma below.

**Lemma 5.19.** For any $(\mu, \nu) \in M_{\min}$ we have

$$\pi \sum_j \text{Im} \left[ i \sum_{(\mu, \nu) \in M_{\min}} \nu_j |\zeta^\nu|^2 \langle \delta(H - (\nu - \mu) \cdot e) G_{\mu \nu}^*, G_{\mu \nu} \rangle \right] - i \sum_{(\mu, \nu) \in M_{\min}} \mu_j |\zeta^\mu|^2 \langle \delta(H - (\nu - \mu) \cdot e) G_{\mu \nu}^*, G_{\mu \nu} \rangle$$

$$= \pi \sum_{(\mu, \nu) \in M_{\min}} |\zeta^\mu|^2 \langle \delta(H - (\nu - \mu) \cdot e) G_{\mu \nu}^*, G_{\mu \nu} \rangle \geq 0. \tag{5.60}$$

Proof. By (A.8) we have $\langle \delta(H - (\nu - \mu) \cdot e) G_{\mu \nu}, G^*_{\mu \nu} \rangle = \langle \delta(H - (\nu - \mu) \cdot e) G_{\mu \nu}^*, G_{\mu \nu} \rangle \geq 0$. Then, commuting the summations and by $|\nu| - |\mu| = 1$ we get

$$\text{l.h.s. (5.60)} = \pi \sum_j \text{Re} \left[ \sum_{(\mu, \nu) \in M_{\min}} (\nu_j - \mu_j) |\zeta^\nu|^2 \langle \delta(H - (\nu - \mu) \cdot e) G_{\mu \nu}^*, G_{\mu \nu} \rangle \right]$$

$$= \pi \sum_{(\mu, \nu) \in M_{\min}} \langle \delta(H - (\nu - \mu) \cdot e) G_{\mu \nu}^*, G_{\mu \nu} \rangle \geq 0.$$

By an application of Lemmas 5.18 and 5.19 to the identity (5.58) we arrive as in [12] at the following.

**Corollary 5.20.** We have

$$2^{-1} \frac{d}{dt} \sum_j |z_j|^2 = \sum_j \text{Im}[\bar{G}_j z_j] - \pi \sum_{(\mu, \nu) \in M_{\min}} |\zeta^\nu|^2 \langle \delta(H - L) G_{\mu \nu}^*, G_{\mu \nu} \rangle. \tag{5.61}$$

By (H5) and Lemma [A.1] in the Appendix [A] we can assume that there exists a $\Gamma > 0$ s.t.

$$\langle \delta(H - (\nu - \mu) \cdot e) G_{\mu \nu}^*, G_{\mu \nu} \rangle > \Gamma > 0 \text{ for all } (\mu, \nu) \in M_{\min}. \tag{5.62}$$

Now we complete the proof of Proposition 5.2. By the inequalities (5.3) – (5.7) and (5.54) in Lemma 5.17 integrating (5.61) and using (5.62) we get

$$\sum_j |z_j(t)|^2 + \sum_{(\mu, \nu) \in M} \|z^{\mu + \nu}\|^2_{L^2(0,t)} \lesssim \epsilon^2 + C_0 \epsilon^2. \tag{5.63}$$

The bound (5.63) above tells us that $\|z^{\mu + \nu}\|^2_{L^2(0,t)} \lesssim C_0 \epsilon^2$ implies $\|z^{\mu + \nu}\|^2_{L^2(0,t)} \lesssim \epsilon^2 + C_0 \epsilon^2$ for all $(\mu, \nu) \in M$. This means that we can take $C_0 \sim 1$. This completes the proof of Prop. 5.2.

27
5.3 Proof of the asymptotics [5.8]

We write (5.27) in the form i\dot{\eta} = D_\# \eta + V\eta + B with, see (5.27),

$$B = \sum_{(\mu, \nu) \in M} \bar{\psi}^{\mu} z^{\nu} G_{\mu \nu}^* \hat{A}.$$ 

Then \( \partial_t (e^{iD_\# t} \eta) = -ie^{iD_\# t}(V\eta + B) \) and so

$$e^{iD_\# t_2 \eta(t)} - e^{iD_\# t_1 \eta(t)} = -i \int_{t_1}^{t_2} e^{iD_\# t}(V\eta(t) + B(t))dt$$

for \( t_1 < t_2 \).

Then for a fixed \( c_2 \) by Lemma 5.14 we have

$$\|e^{iHt_2 \eta(t)} - e^{iHt_1 \eta(t)}\|_{L^\infty} \leq c_2 \|V\eta(t) + B(t)\|_{L^1([t_1, t_2], H^s) + L^2([t_1, t_2], H^{s+1})}.$$  \hspace{1cm} (5.64)

By (5.3), valid now in \([0, \infty)\) and for a fixed \( C \), we have

$$\|V\eta(t)\|_{L^1([t_1, t_2], H^{s+1})} \leq c_1 \|\eta\|_{L^1([t_1, t_2], H^{s+1})} \leq C\epsilon.$$ 

By Theorem 4.6 and (5.29) we similarly have

$$\| \sum_{(\mu, \nu) \in M} \bar{\psi}^{\mu} z^{\nu} G_{\mu \nu}^* \|_{L^2([0, \infty), H^{s+1})} \leq C' \sum_{(\mu, \nu) \in M} \|\bar{\psi}^{\mu} z^{\nu}\|_{L^2([0, \infty), C)} \leq C\epsilon.$$ 

We also have (5.30) for \( I_T = [0, \infty) \). We then conclude that there exists an \( \eta_+ \in L^\infty \cap H_c[0] \) with

$$\lim_{t \to \infty} e^{iD_\# t} \eta(t) = \eta_+ \text{ in } L^\infty \text{ and with } \|\eta_+\|_{L^\infty(\mathbb{R}^3)} \leq C\epsilon.$$ 

Now we prove the existence of \( z_+ \) and the facts about it in Theorem 5.1. First of all we have

$$\frac{1}{2} \sum_j \frac{d}{dt} |z_j|^2 = \sum_j \text{Im} \left[ \partial_j \mathcal{R} z_j + \sum_{(\mu, \nu) \in M} \nu_j z^\nu \bar{\psi}^{\mu}(\eta, G_{\mu \nu}) + \sum_{(\mu', \nu') \in M} \mu'_j z^{\mu'} \bar{\psi}^{\mu}(\eta, G_{\mu' \nu'}) \right].$$ 

Since the r.h.s. has \( L^1(0, \infty) \) norm bounded by \( C\epsilon^2 \) for a fixed \( C \), we conclude that the limit

$$\lim_{t \to \infty} (|z_1(t)|, ..., |z_n(t)|) = (\rho_{1+}, ..., \rho_{n+})$$

exists, with \( |\rho_+| \leq C\|u(0)\|_{H^s} \). By \( \lim_{t \to \infty} Z(t) = 0 \) we can conclude that all but at most one of the \( \rho_{j+} \) are equal to 0.

A Appendix: proof of the formula [A.7]

This section is devoted to prove the Plemeljji formula associated to the resolvent of the operator [12]. With this aim we need to rely on the following facts. Borrowed by [4] we first introduce the matrix functions \( \psi_0(x, \xi) \in M_4(\mathbb{C}) \) with vector column given by

$$\psi^j_0(x, \xi) = e^{ix \cdot \xi} v(\xi) e_j, \quad j = 1, ..., 4, \quad (A.1)$$

28
with \( \lambda(\xi) = \sqrt{|\xi|^2 + \mathcal{M}^2} \),

\[
v(\xi) = \frac{(\lambda(\xi) + \mathcal{M}) I - \beta \alpha \cdot \xi}{\sqrt{2 \lambda(\xi)((\lambda(\xi) + \mathcal{M}))}}
\]  

(A.2)
is a unitary matrix and \( e_j \) are vectors of the canonical basis of \( \mathbb{C}^4 \) (for more details see \[27\], Section 1). We recall that the transformation \( v(\xi) \mathcal{F}(u)(\xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^3} \psi_0(x, \xi)^* u(x) dx \)diagonalizes the free Dirac operator \( D, \mathcal{M} \) as follows \( v^*(\xi) \mathcal{F} D, \mathcal{M} \mathcal{F}^* v(\xi) = \lambda(\xi) \beta \) (here \( \mathcal{F} \) denotes the classical Fourier transform with inverse \( \mathcal{F}^* \)). Consequently we can define the distorted plane wave functions \( \psi_V^\pm(x, \xi) \in \mathcal{M}_4(\mathbb{C}) \) associated to the continuous spectrum of \( H \) (see for instance \[1\] and \[2\] for the cases of Schrödinger and Klein-Gordon) as follows,

\[
\psi_V^j(\pm)(x, \xi) = \psi_0^j(\pm)(x, \xi) - \Lambda_V^{j, \pm}(x, \xi), \quad j = 1, \ldots, 4,
\]  

(A.3)
where

\[
\Lambda_V^{j, \pm}(x, \xi) = \left\{ \begin{array}{ll}
\lim_{\epsilon \to 0} R_H(\lambda(\xi) \pm i \epsilon) V \psi_0^j(x, \xi) & \text{for } j \in \{1, 2\}, \\
\lim_{\epsilon \to 0} R_H(-\lambda(\xi) \pm i \epsilon) V \psi_0^j(x, \xi) & \text{for } j \in \{3, 4\}.
\end{array} \right.
\]  

(A.4)
We recall also that the distorted plane wave associated to the perturbed Dirac operator \( D, \mathcal{M} \), here denoted by \( \psi_V^+(x, \xi) \), satisfies the equation

\[
H \psi_V^+(x, \xi) = \pm \lambda(\xi) \psi_V^+(x, \xi),
\]  

(A.5)
(the same equation holds for \( \psi_V^-(x, \xi) \)). By this, for any \( g \in S(\mathbb{R}^3, \mathbb{C}^4) \), we have the distorted Fourier transform associated to \( H \) (in the sense of Section 3.3 in \[4\]),

\[
\mathcal{F}_V^+(g)(\xi) = \frac{1}{(2\pi)^{2}} \int_{\mathbb{R}^3} \psi_V^+(x, \xi)^* g(x) dx,
\]  

(A.6)
is a bounded linear operator from \( \mathcal{H}_{\mathcal{M}}[0] \) in \( L^2(\mathbb{R}^3, \mathbb{C}^4) \) and with inverse \( \mathcal{F}_V^* \) defined as in Theorem 3.2 of \[4\]. Notice that the we have also the relation \( \mathcal{F}(g)(\xi) = v^*(\xi) \mathcal{F}_0^+(g)(\xi) \). Motivated by this we state the following lemma

**Lemma A.1** (Plemelji formula for \( R_H^+(L) \)). Let be \( L \in \mathbb{R}(\mathbb{R}, \mathcal{M}) \), then we have the following representation of the resolvent of the perturbed Dirac operator \( H \) given as in Lemma 5.8,

\[
R_H^+(L) = PV \frac{1}{H - L} \pm i\pi \delta(H - L),
\]  

(A.7)
characterized by

\[
\pi i \delta(H - L) f, f^* = \frac{1}{2} \langle R_H^+(L) - R_H^-(L) f, f^* \rangle
\]

\[
= \frac{\pi i |L|}{\sqrt{L^2 - \mathcal{M}}^2} \int_{|\xi| = \sqrt{L^2 - \mathcal{M}}} |\mathcal{F}_V^+(f)|^2 d\xi,
\]  

(A.8)
and

\[
\langle PV \frac{1}{H-L} f, f^* \rangle = \frac{1}{2} \langle R_H^+(L) + R_H^-(L)f, f^* \rangle + \lim_{\varepsilon \to 0} \int_{||\xi| - \sqrt{L^2 - M^2}|| > \varepsilon} \frac{\lambda(\xi)\beta + LI_{C^4}}{||\xi||^2 + M^2 - L^2} |F_{V,+}(f)|^2 d\xi;
\]

\[\text{(A.9)}\]

for any function \(f \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4) \cap \mathcal{H}[0].\)

**Proof.** We deal with the proof of the formulas \(\text{(A.8)}\) and \(\text{(A.9)}\) for \(R_H^+(L)\) because the one for \(R_H^-(L)\) is similar. Select a \(f \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)\), then by transposing the arguments of \(\text{(1)}\) and following \(\text{(32)}\) one can see that, for any \(z \in \mathbb{C} \setminus \sigma(H),\) the following identity is fulfilled

\[
\mathcal{F}(R_H(z)f)(\xi) = \mathcal{F} \left( \frac{1}{H - z} f \right)(\xi) = v(\xi) \frac{\lambda(\xi)\beta + zI_{C^4}}{||\xi||^2 + M^2 - z^2} f(\xi, z),
\]

\[\text{(A.10)}\]

where we set

\[
\frac{(\lambda(\xi)\beta + zI_{C^4})}{||\xi||^2 + M^2 - z^2} = \begin{pmatrix}
\frac{\lambda(\xi) - z}{||\xi||^2} & 0 \\
0 & \frac{1}{\lambda(\xi) + z}I_{C^2}
\end{pmatrix},
\]

(see once again \(\text{(27)}\)) and with the vector valued function \(f(\xi, z)\) having the form

\[
f(\xi, z) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} (\psi_0(x, \xi) - R_H(z^*)V\psi_0(x, \xi))^* f(x) dx.
\]

\[\text{(A.11)}\]

Moreover by an use of \(R_H(z) - R_H(z^*) = 2iR_H(z)R_H(z^*) \text{Im} z\) and \(R_H(z^*) = (R_H(z))^*\) for any \(z \in \mathbb{C} \setminus \mathbb{R},\) in connection with the Parseval identity and with the fact that \(v(\xi)v^*(\xi) = v^*(\xi)v(\xi) = I_{C^4},\) one gets

\[
\frac{1}{2} \langle [R_H(z) - R_H(z^*)] f, f^* \rangle = i \int_{\mathbb{R}^3} \text{Im} \left( \frac{(\lambda(\xi)\beta + zI_{C^4})}{||\xi||^2 + M^2 - z^2} f(\xi, z) f(\xi, z)^* d\xi,
\]

\[\text{(A.12)}\]

with the matrix

\[
\text{Im} \left( \frac{(\lambda(\xi)\beta + zI_{C^4})}{||\xi||^2 + M^2 - z^2} \right) = \begin{pmatrix}
\frac{\lambda(\xi) - z}{||\xi||^2} & 0 \\
0 & \frac{1}{\lambda(\xi) + z}I_{C^2}
\end{pmatrix}.
\]

Pick now \(z = L + i\varepsilon,\) then we are allowed by the trace Lemma \(\text{(5.8)}\) to take the limit \(\varepsilon \searrow 0\) (see also \(\text{(1)}\)). Combining this step with an application of the Plemelj formula \(\frac{1}{x - \delta} = PV \frac{1}{x} \pm i\pi\delta(x),\) we obtain from \(\text{(A.12)}\) the identity

\[
\frac{1}{2} \langle [R_H^+(L) - R_H^-(L)] f, f^* \rangle = \pi i \int_{\mathbb{R}^3} \Xi(\lambda(\xi), L) F_{V,+}(f) F_{V,+}(f)^* d\xi,
\]

\[\text{(A.13)}\]

with

\[
\Xi(\lambda(\xi), L) = \begin{pmatrix}
\delta(\lambda(\xi) - L)I_{C^2} & 0 \\
0 & \delta(-\lambda(\xi) - L)I_{C^2}
\end{pmatrix}.
\]

At this point, an application of the identities (in \(\mathcal{S}'(\mathbb{R}^3, \mathbb{C}^4)\), see for example \(\text{(14)}\), Chap. II, Sec. 2.5 or Chap III for a more general theory)

\[
\delta(\lambda(\xi) - L) = \frac{L}{\sqrt{L^2 - M^2}} \delta(||\xi|| - \sqrt{L^2 - M^2}), \quad \text{for } L > M,
\]

\[\text{(A.14)}\]

\[
\delta(-\lambda(\xi) - L) = \frac{-L}{\sqrt{L^2 - M^2}} \delta(||\xi|| - \sqrt{L^2 - M^2}), \quad \text{for } L < -M,
\]

\[\text{(A.15)}\]
shows that the r.h.s of identity (A.13) is equal to
\[
\begin{cases}
\frac{\pi L}{\sqrt{L^2 - \mathcal{M}^2}} \int_{|\xi| = \sqrt{L^2 - \mathcal{M}^2}} 2 \sum_{i=1}^{2} |F_{V,+}(f)|^2 d\xi, & \text{for } L > \mathcal{M}, \\
\frac{-\pi L}{\sqrt{L^2 - \mathcal{M}^2}} \int_{|\xi| = \sqrt{L^2 - \mathcal{M}^2}} 4 \sum_{i=3}^{4} |F_{V,+}(f)|^2 d\xi, & \text{for } L < -\mathcal{M},
\end{cases}
\tag{A.16a}
\]
which in turn implies the identity (A.8). A similar discussion (actually easier) yields
\[
\frac{1}{2} \langle R_H^+(L) + R_H^-(L)f, f^* \rangle = PV \int \frac{\lambda(\xi) \beta + LI_{C^4}}{|\xi|^2 + \mathcal{M}^2 - L^2} |F_{V,+}(f)|^2 d\xi,
\]
that is the identity (A.9). This completes the proof of the lemma.

\textbf{Remark A.2.} All the convergence arguments are well defined because the Lemma 5.8. Moreover in the proof we used functions in \(f \in S(\mathbb{R}^3, \mathbb{C}^4)\). One can easily extend to \(f \in H^{k,s}(\mathbb{R}^3, \mathbb{C}^4) \cap \mathcal{H}[0]\) for \(k \geq 0\) and \(s > 1/2\) (which implies \(F_{V,\pm} \in H_{loc}^{s}(\mathbb{R}^3, \mathbb{C}^4)\) so that the restriction to spheres makes sense) by a density argument.

\section*{Acknowledgments}

The authors were funded by the grant FIRB 2012 (Dinamiche Dispersive) from MIUR, the Italian Ministry of Education, University and Research. S.C. was supported also by a grant FIRB 2013 from the University of Trieste.

The authors wish to thank Professor Masaya Maeda for help with the proof of Proposition 3.1.

\section*{References}

[1] S.Agmon, \textit{Spectral properties of Schrödinger operators and scattering theory}, An. Sc. N. Pisa 2 (1975),151–218.

[2] D.Bambusi, S.Cuccagna, \textit{On dispersion of small energy solutions of the nonlinear Klein Gordon equation with a potential}, Amer. Math. Jour. 133 (2011), 1421–1468.

[3] A.M.Berthier, V.Georgescu, \textit{On the point spectrum of Dirac operators}, J. Fun. Anal. 71 (2011), 309–338.

[4] N.Boussaid, \textit{Stable directions for small nonlinear Dirac standing waves}, Com. Math. Phys. 268 (2006), 757–817.

[5] N.Boussaid, \textit{On the asymptotic stability of small nonlinear Dirac standing waves in a resonant case}, SIAM J. Math. Anal., 40 (2008), 1621–1670.

[6] N.Boussaid, S.Cuccagna, \textit{On stability of standing waves of nonlinear Dirac equations}, Comm. in Partial Diff. Eq. 37 (2012), 1001–1056.
[7] V.Buslaev, G.Perelman, *On the stability of solitary waves for nonlinear Schrödinger equations*, Nonlinear evolution equations, editor N.N. Uraltseva, Transl. Ser. 2, 164, Amer. Math. Soc., pp. 75–98, Amer. Math. Soc., Providence (1995).

[8] M.Christ, A.Kiselev, *Maximal functions associated to filtrations*, J. Fun. Anal. **179** (2001), 409–425.

[9] S. Cuccagna, *The Hamiltonian structure of the nonlinear Schrödinger equation and the asymptotic stability of its ground states*, Comm. Math. Physics **305** (2011), 279–331.

[10] S. Cuccagna, *On scattering of small energy solutions of non autonomous hamiltonian nonlinear Schrödinger equations*, J. Differential Equations **250** (2011), no. 5, 2347–2371.

[11] S.Cuccagna, *On the Darboux and Birkhoff steps in the asymptotic stability of solitons*, Rend. Istit. Mat. Univ. Trieste **44** (2012), 197–257.

[12] S.Cuccagna, M.Maeda, *On small energy stabilization in the NLS with a trapping potential*, arXiv:1309.4878

[13] M.B.Erdogan, M. Goldberg, W.Schlag, *Strichartz and smoothing estimates for Schrödinger operators with almost critical magnetic potentials in three and higher dimensions*, Forum Math (2009) **21**, 687–722.

[14] A. Gelfand, G. Shilov *Generalized Functions*, Vol.1, Academic Press, 1964, pp 423.

[15] S.Gustafson, T.V.Phan, *Stable directions for degenerate excited states of nonlinear Schrödinger equations*, SIAM J. Math. Anal. **43** (2011), 1716–1758.

[16] S.Gustafson, K.Nakanishi, T.P.Tsai, *Asymptotic stability and completeness in the energy space for nonlinear Schrödinger equations with small solitary waves*, Int. Math. Res. Not. 2004 (2004) no. **66**, 3559–3584

[17] T.Mizumachi, *Asymptotic stability of small solitons to 1D NLS with potential*, Jour. of Math. Kyoto University, **48** (2008), 471–497.

[18] K.Nakanishi, T.V.Phan, T.P.Tsai, *Small solutions of nonlinear Schrödinger equations near first excited states*, Jour. Funct. Analysis **263** (2012), 703–781.

[19] C.A.Pillet, C.E.Wayne, *Invariant manifolds for a class of dispersive, Hamiltonian partial differential equations* J. Diff. Eq. 141 (1997), pp. 310–326.

[20] H.A.Rose, M.I.Weinstein, *On the bound states of the nonlinear Schrödinger equation with a linear potential*, Physica D, 30 (1988), pp. 207–218

[21] H.Smith, C.Sogge, *Strichartz estimates for nontrapping perturbations of the Laplacian*, Com. Part. Diff. Eq. **25** (2000),2171–2183.

[22] A.Soffer, M.I.Weinstein, *Multichannel nonlinear scattering for nonintegrable equations*, Comm. Math. Phys., 133 (1990), pp. 116–146

[23] A.Soffer, M.I.Weinstein, *Multichannel nonlinear scattering II. The case of anisotropic potentials and data*, J. Diff. Eq., 98 (1992), pp. 376–390.
[24] A.Soffer, M.I.Weinstein, Selection of the ground state for nonlinear Schrödinger equations, Rev. Math. Phys. 16 (2004), 977–1071.

[25] A.Soffer, M.I.Weinstein, Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations, Invent. Math. 136 (1999), 9–74.

[26] M.E. Taylor, Partial Differential Equations, volumes 115–117 of App. Math. Sci., Springer, New York (1996).

[27] B.Thaller. The Dirac equation. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1992.

[28] T.P.Tsai, H.T.Yau, Asymptotic dynamics of nonlinear Schrödinger equations: resonance dominated and radiation dominated solutions, Comm. Pure Appl. Math. 55 (2002), 153–216.

[29] T.P.Tsai, H.T.Yau, Relaxation of excited states in nonlinear Schrödinger equations, Int. Math. Res. Not. 31 (2002), 1629–1673.

[30] T.P.Tsai, H.T.Yau, Classification of asymptotic profiles for nonlinear Schrödinger equations with small initial data, Adv. Theor. Math. Phys. 6 (2002), 107–139.

[31] T.P.Tsai, H.T.Yau, Stable directions for excited states of nonlinear Schrödinger equations, Comm. P.D.E. 27 (2002), 2363–2402.

[32] O. Yamada, On the principle of limiting absorption for the Dirac operators, Publ. Res. Inst. Math. Sci. 8 (1972/73), 557-577.

Department of Mathematics and Geosciences, University of Trieste, via Valerio 12/1, Trieste 34127, Italy
E-mail Address: scuccagna@units.it

Department of Mathematics, University of Pisa, Largo Bruno Pontecorvo 5, Pisa 56127, Italy
E-mail Address: tarulli@mail.dm.unipi.it