BERNSTEIN POLYNOMIALS ON SIMPLEX

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Abstract. We prove two identities for multivariate Bernstein polynomials on simplex, which are considered on a pointwise. In this paper, we study good approximations of Bernstein polynomials for every continuous function on simplex and the higher dimensional $q$-analogue of Bernstein polynomials on simplex.

1. Introduction and motivation

Recently many mathematicians study on the theory of multivariate Bernstein polynomials on simplex. This theory has many applications in different areas in mathematics and physics, see [1–17]. Throughout this paper we set $I = [0,1]$ and $k \in \mathbb{N}$. Taking a $k$-dimensional simplex $\Delta_k$:

$$\Delta_k = \{ \vec{x} = (x_1, \ldots, x_k) \in I^k : x_1 + \ldots + x_k \leq 1 \}.$$

As well-known, Bernstein polynomials (1) are the most important and interesting concrete operators on a space of continuous functions (see [15,16]). The purpose of this paper is to study their generalization to $k$-dimensional simplex.

Definition 1. The $n$-th degree ordinary Bernstein polynomial $B_{\nu,n} : I \rightarrow \mathbb{R}$ is given by

$$B_{\nu,n}(x) = \binom{n}{\nu} x^\nu (1 - x)^{n - \nu},$$

$\nu = 0, 1, \ldots, n$. We extend this to

$$B_{\vec{\nu},n} : \Delta_k \rightarrow \mathbb{R}$$

by taking $\vec{\nu}$ to be a multi-index, $\vec{\nu} = (\nu_1, \ldots, \nu_k) \in \mathbb{N}_0^k$, define

$$|\vec{\nu}| := \nu_1 + \ldots + \nu_k \in \{0, 1, \ldots, n\}$$

and setting

$$B_{\vec{\nu},n}(\vec{x}) = \binom{n}{\vec{\nu}} \vec{x}^{\vec{\nu}} (1 - |\vec{x}|)^{n - |\vec{\nu}|}.$$
where
\[ \vec{x} = (x_1, \ldots, x_k) \in \Delta_k, \quad \vec{x}^v = \prod_{i=1}^{k} x_i^{v_i}, \quad \vec{v}! = v_1! \cdots v_k! \quad \text{and} \quad (n^v) = \frac{n!}{v!(n-|\vec{v}|)!}. \]

For every \( f \) defined on \( \Delta_k \), we write
\[ \mathbb{B}_n(f | \vec{x}) = \sum_{|\vec{v}| \leq n} f(\vec{v}/n) B_{\vec{v}, n}(\vec{x}). \]

Here we have the convergence

**Proposition 1.** If \( f : \Delta_k \to \mathbb{R} \) is continuous, then \( \mathbb{B}_n(f | \cdot) \to f \) uniformly on \( \Delta_k \) as \( n \to \infty \).

**Proof.** The proof of this proposition is quite simple and is based on the partition property of the \( B_{\vec{v}, n}(\vec{x}) \) that
\[ \sum_{|\vec{v}| \leq n} B_{\vec{v}, n}(\vec{x}) = 1. \]

This proof is similar to that Theorem 1.1.1 in [15, p.5–8]. Since \( f \) is uniformly continuous on \( \Delta_k \), for given \( \epsilon > 0 \), there exists \( \delta > 0 \) with the property that if \( \vec{x} = (x_1, \ldots, x_k) \), \( \vec{y} = (y_1, \ldots, y_k) \), and \( |x_i - y_i| < \delta \) for all \( i \), then \( |f(\vec{x}) - f(\vec{y})| < \epsilon \). We define the distance on \( \Delta_k \) by
\[ d(\vec{x}, \vec{y}) = \max \{ |x_1 - y_1|, \ldots, |x_k - y_k| \}. \]

We suppose that \( n \) is sufficiently large that \( \frac{1}{4n\delta^2} < \epsilon \). We have
\[ |\mathbb{B}_n(f | \vec{x}) - f(\vec{x})| \leq \sum_{d(\vec{v}/n, \vec{x}) < \delta} |f(\vec{v}/n) - f(\vec{x})| B_{\vec{v}, n}(\vec{x}) + \sum_{d(\vec{v}/n, \vec{x}) \geq \delta} |f(\vec{v}/n) - f(\vec{x})| B_{\vec{v}, n}(\vec{x}) \]

where \( \vec{v} = (v_1, \ldots, v_k) \), a multi-index. The first sum satisfies
\[ \sum_{d(\vec{v}/n, \vec{x}) < \delta} |f(\vec{v}/n) - f(\vec{x})| B_{\vec{v}, n}(\vec{x}) < \epsilon \]
by uniform continuity of \( f \). Now we study the second sum. Let \( M = \max f \). Then
\[ \sum_{d(\vec{v}/n, \vec{x}) \geq \delta} |f(\vec{v}/n) - f(\vec{x})| B_{\vec{v}, n}(\vec{x}) \leq 2M \sum_{d(\vec{v}/n, \vec{x}) \geq \delta} B_{\vec{v}, n}(\vec{x}). \]

We see that
\[ \sum_{d(\vec{v}/n, \vec{x}) \geq \delta} B_{\vec{v}, n}(\vec{x}) < n\epsilon. \]
Thus \( |\mathbb{B}_n(f | \vec{x}) - f(\vec{x})| < \epsilon + 2Mn\epsilon \), and we are done. \( \square \)
The generating functions of the $k$-dimensional Bernstein polynomials are as follows:

**Proposition 2.**

\[
\sum_{n \geq |\vec{v}|} B_{\vec{v},n} (\vec{x}) \frac{t^n}{n!} = \left(\frac{t}{\vec{x}}\right)^{\vec{v}} e^{t(1-|\vec{x}|)}.
\]

**Proof.** Writing

\[
\sum_{n \geq |\vec{v}|} B_{\vec{v},n} (\vec{x}) \frac{t^n}{n!} = \sum_{n \geq |\vec{v}|} \left(\frac{n}{\vec{v}}\right) \vec{x}^{\vec{v}} (1 - |\vec{x}|)^{n-|\vec{v}|} \frac{t^n}{n!}
\]

\[
= \sum_{n \geq |\vec{v}|} \left(\frac{t}{\vec{x}}\vec{v}\right)! \frac{(1 - |\vec{x}|)^{n-|\vec{v}|} t^n}{(n - |\vec{v}|)!}
\]

\[
= \left(\frac{t}{\vec{x}}\vec{v}\right)! \sum_{m \geq 0} \frac{(1 - |\vec{x}|)^{m+t}}{m!}.
\]

This yields the equality

\[
\sum_{n \geq |\vec{v}|} B_{\vec{v},n} (\vec{x}) \frac{t^n}{n!} = \left(\frac{t}{\vec{x}}\vec{v}\right)! e^{t(1-|\vec{x}|)}.
\]

\[
\square
\]

2. Main results and proofs

This section contains the main results of this paper. The first main result can be stated as follows.

**Theorem 1.** For $n \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $\vec{v} \in \mathbb{N}_0^k$ such that $m \leq \min \left( |\vec{v}|, n \right)$. Then we have the following identity

\[
\sum_{\vec{u} \leq \vec{v}, |\vec{u}| \leq m} \frac{\vec{u}!(m - |\vec{u}|)!}{m!} B_{\vec{u},n}(\vec{x}) B_{\vec{v},n-m}(\vec{x}) = B_{\vec{v},n}(\vec{x}),
\]

where $\vec{u} \leq \vec{v}$ means that $0 \leq u_i \leq v_i$ for all $i = 1, \cdots, k$.

The formula (9) can be viewed as a pointwise recurrence or orthogonality formula for the Bernstein polynomials in $k$-dimensional simplex.

**Remark 1.** For $m = 1$, we obtain from Theorem 1 the following recurrence formula

\[
(1 - |\vec{x}|) B_{\vec{v},n-1}(\vec{x}) + \sum_{\vec{u} \leq \vec{v}, |\vec{u}| = n-1} \vec{x}^{\vec{u}} B_{\vec{v},n-1}(\vec{x}) = B_{\vec{v},n}(\vec{x})
\]
Proof. We prove this theorem by induction on \( n \) and \( m \). For \( n = 0, 1 \) the statement is trivial. Let \( n \geq 2 \). Taking \( m = 1 \). The sum

\[
\sum_{\substack{u \leq v \leq \tilde{v} \leq \tilde{u} \leq m \\mid u \leq m}} \frac{u!(m - |u|)!}{m!} B_{\tilde{u}, m}(\tilde{x}) B_{\tilde{v} - \tilde{u}, n-m}(\tilde{x})
\]

\[
= B_{\tilde{u}, 1}(\tilde{x}) B_{\tilde{v}, n-1}(\tilde{x}) + \sum_{\substack{\tilde{u} \leq \tilde{v} \leq \tilde{x} \\mid \tilde{u} \leq 1}} \tilde{x}^\tilde{u} B_{\tilde{v} - \tilde{u}, n-1}(\tilde{x})
\]

\[
= (1 - |\tilde{x}|) B_{\tilde{v}, n-1}(\tilde{x}) + \sum_{\substack{\tilde{u} \leq \tilde{v} \leq \tilde{x} \\mid \tilde{u} \leq 1}} \tilde{x}^\tilde{u} B_{\tilde{v} - \tilde{u}, n-1}(\tilde{x}).
\]

By using the following fact \( |\tilde{u}| = 1 \) if and only if one index of \( \tilde{u} \) is 1 and all the others are zero, after simple manipulation, we obtain the relation

\[
(1 - |\tilde{x}|) B_{\tilde{v}, n-1}(\tilde{x}) + \sum_{\substack{\tilde{u} \leq \tilde{v} \leq \tilde{x} \\mid \tilde{u} \leq 1}} \tilde{x}^\tilde{u} B_{\tilde{v} - \tilde{u}, n-1}(\tilde{x}) = B_{\tilde{v}, n}(\tilde{x}).
\]

Then the Theorem 1 is valid for any \( n \) and \( m = 1 \). Now we suppose the theorem holds up to \( n \geq m \geq 1 \). We can write for \( n+1 \) and \( m = 1 \) the following

\[
B_{\tilde{v}, n+1}(\tilde{x}) = \sum_{\substack{\tilde{u} \leq \tilde{v} \leq \tilde{x} \\mid \tilde{u} \leq 1}} \tilde{u}!(1 - |\tilde{u}|)! B_{\tilde{u}, 1}(\tilde{x}) B_{\tilde{v} - \tilde{u}, n}(\tilde{x})
\]

Then we get

\[
(11) \quad B_{\tilde{v}, n+1}(\tilde{x}) = \sum_{\substack{\tilde{u} \leq \tilde{v} \leq \tilde{x} \\mid \tilde{u} \leq 1}} B_{\tilde{u}, 1}(\tilde{x}) B_{\tilde{v} - \tilde{u}, n}(\tilde{x}),
\]

by using the recurrence hypothesis, from (11), we obtain

\[
B_{\tilde{v}, n+1}(\tilde{x}) = \sum_{\substack{\tilde{u} \leq \tilde{v} \leq \tilde{x} \\mid \tilde{u} \leq 1}} B_{\tilde{u}, 1}(\tilde{x}) \sum_{\substack{\tilde{u}' \leq \tilde{v} - \tilde{u} \leq m \\mid \tilde{u}' \leq 1, \tilde{u}' \leq m}} \tilde{u}'!(m - |\tilde{u}'|)! B_{\tilde{u}', m}(\tilde{x}) B_{\tilde{v} - \tilde{u}' - \tilde{u}, n-m}(\tilde{x})
\]

\[
= \sum_{\substack{\tilde{u} \leq \tilde{v} \leq \tilde{x} \\mid \tilde{u} \leq 1, \tilde{u} \leq m}} \tilde{u}!(m - |\tilde{u}|)! B_{\tilde{u}, 1}(\tilde{x}) B_{\tilde{u}', m}(\tilde{x}) B_{\tilde{v} - \tilde{u}', n-m}(\tilde{x}).
\]
Setting $\vec{w} = \vec{u} + \vec{u'}$. From the relation (2), we deduce the identity

$$B_{\vec{v},n+1}(\vec{x}) = \sum_{\substack{\vec{w} \leq \vec{v} \leq \vec{m}+1 \atop |\vec{w}| \leq m+1}} \frac{\vec{w}!(m+1-|\vec{w}|)!}{(m+1)!} B_{\vec{w},m+1}(\vec{x}) B_{\vec{v}-\vec{w},n-m}(\vec{x}).$$

This completes the proof of the theorem. \(\square\)

For every $j, m, 1 \leq j \leq k, m \geq 1$, we define the affine transformations $T_{j,m}$ by

$$(12)\quad T_{j,m}(\vec{x}) = (x_1, \cdots, x_{j-1}, m-|\vec{x}|, x_{j+1}, \cdots, x_k).$$

and for every $\sigma \in S_k$ permutation of the set $\{1, \cdots, k\}$ we put

$$(13)\quad \sigma(\vec{x}) = (x_{\sigma(1)}, \cdots, x_{\sigma(k)}).$$

We state now the second main result of this paper.

**Theorem 2.** For $n \in \mathbb{N}$ and $\vec{v} \in \mathbb{N}_0^k$. Then we have the following identities

$$(14)\quad B_{\vec{v},n}(T_{j,1}(\vec{x})) = B_{T_{j,1}(\vec{v}),n}(\vec{x}),$$

and

$$(15)\quad B_{\vec{v},n}(\sigma(\vec{x})) = B_{\sigma^{-1}(\vec{v}),n}(\vec{x}).$$

The relation (14) is a multivariate symmetry formula for the Bernstein polynomials.

**Remark 2.** Taking $j = 1$ and $\sigma = (12)$ we get from the Theorem 2 the symmetries relations

$$(16)\quad B_{\vec{v},n}(1-|\vec{x}|, x_2, \cdots, x_k) = B_{(n-|\vec{v}|, v_2, \cdots, v_k),n}(\vec{x}),$$

and

$$(17)\quad B_{\vec{v},n}(x_2, x_1, x_3, \cdots, x_k) = B_{(v_2, v_1, v_3, \cdots, v_k),n}(\vec{x}).$$

**Proof.** By using the equalities (2) and (12) we have

$$B_{\vec{v},n}(T_{j,1}(\vec{x})) = B_{\vec{v},n}(x_1, \cdots, x_{j-1}, \cdots, 1-|\vec{x}|, x_{j+1}, \cdots, x_k)$$

$$= \left(\frac{n}{\vec{v}}\right) x_1^{v_1} \cdots x_{j-1}^{v_{j-1}} (1-|\vec{x}|)^{v_j} x_{j+1} \cdots x_k^{v_k} (1-|\vec{x}|)$$

$$= \left(\frac{n}{\vec{v}}\right) x_1^{v_1} \cdots x_{j-1}^{v_{j-1}} x_j^{n-|\vec{v}|} x_{j+1} \cdots x_k^{v_k} (1-|\vec{x}|)^{v_j}.$$

This implies, by using the relation (12), the identity

$$B_{\vec{v},n}(T_{j,1}(\vec{x})) = B_{T_{j,1}(\vec{v}),n}(\vec{x}).$$

The equality (15) of the Theorem 2 can be obtained in a similar way of (14). \(\square\)
3. \(q\)-extension of Bernstein polynomials on simplex

When one talks of \(q\)-extension, \(q\) is variously considered as an indeterminate, a complex number \(q \in \mathbb{C}\), or \(p\)-adic number \(q \in \mathbb{C}_p\). If \(q \in \mathbb{C}\), then we always assume that \(|q| < 1\). If \(q \in \mathbb{C}_p\), we usually assume that \(|1 - q|_p < 1\). Here, the symbol \(|\cdot|_p\) stands for the \(p\)-adic absolute value on \(\mathbb{C}_p\) with \(|p|_p \leq 1/p\).

For each \(x\), the \(q\)-basic numbers are defined by

\[ [x]_q = \frac{1 - q^x}{1 - q}. \]

We extend this by

\[ [\vec{x}]_q = ([x_1]_q, \ldots, [x_k]_q) \]

and the \(q\)-extension of Bernstein polynomials on \(\Delta_k\) is defined by

\begin{equation}
B_{\vec{v},n}(\vec{x} | q) = \binom{n}{\vec{v}}[\vec{x}]_q[1 - \vec{x}]_q^{n-|\vec{v}|}. \tag{18}
\end{equation}

Here again we have the \(q\)-extensions of Theorem 1 and Theorem 2.

**Theorem 3.** For \(n \in \mathbb{N}\), \(m \in \mathbb{N}_0\) and \(\vec{v} \in \mathbb{N}_0^k\) such that \(m \leq \min \left( |\vec{v}|, n \right)\). Then we have the following identity

\[ \sum_{\underline{\vec{u}} \leq \vec{v} \atop |\underline{\vec{u}}| \leq m} \frac{\underline{\vec{u}}!(m - |\underline{\vec{u}}|)!}{m!} B_{\underline{\vec{u}},m}(\vec{x} | q)B_{\vec{v} - \underline{\vec{u}},n-m}(\vec{x} | q) = B_{\vec{v},n}(\vec{x} | q), \]

where \(\underline{\vec{u}} \leq \vec{v}\) means that \(0 \leq u_i \leq v_i\) for all \(i = 1, \ldots, k\).

**Theorem 4.** For \(n \in \mathbb{N}\) and \(\vec{v} \in \mathbb{N}_0^k\). Then we have the following identities

\begin{equation}
B_{\vec{v},n}(T_{j,1}(\vec{x}) | q) = B_{T_{j,1}(\vec{v}),n}(\vec{x} | q), \tag{19}
\end{equation}

and

\begin{equation}
B_{\vec{v},n}(\sigma(\vec{x}) | q) = B_{\sigma^{-1}(\vec{v}),n}(\vec{x} | q). \tag{20}
\end{equation}

The proofs of these theorems are quite similar to those of Theorems 1 and 2. Then we omit them.

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