On Borsuk–Ulam theorems and convex sets

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Abstract
The Intermediate Value Theorem is used to give an elementary proof of a Borsuk–Ulam theorem of Adams, Bush and Frick [1] that if \( f : S^1 \to \mathbb{R}^{2k+1} \) is a continuous function on the unit circle \( S^1 \) in \( \mathbb{C} \) such that \( f(-z) = -f(z) \) for all \( z \in S^1 \), then there is a finite subset \( X \) of \( S^1 \) of diameter at most \( \pi - \pi/(2k+1) \) (in the standard metric in which the circle has circumference of length \( 2\pi \)) such the convex hull of \( f(X) \) contains 0 \( \in \mathbb{R}^{2k+1} \).

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1 INTRODUCTION

We shall use the Intermediate Value Theorem to give an elementary proof of the following Borsuk–Ulam theorem of Adams, Bush and Frick in which \( k \geq 1 \) is a natural number, \( \zeta = e^{2\pi i/(2k+1)} \in \mathbb{C} \) and the metric \( d \) on the unit circle, \( S(\mathbb{C}) \), in \( \mathbb{C} \) is given by \( d(z, e^{i\theta}z) = |\theta| \) if \( |\theta| \leq \pi \).

**Theorem 1.1** [1, Theorems 1 and 5]. Let \( f : S^1 = S(\mathbb{C}) \to \mathbb{R}^{2k+1} \) be a continuous map such that \( f(-z) = -f(z) \) for all \( z \in S(\mathbb{C}) \). Then there exist \( e_i \in \{\pm 1\}, i = 0, \ldots, 2k \), and \( z \in S(\mathbb{C}) \) such that 0 lies in the convex hull of the image \( f(X) \) of the finite set \( X = \{e_i\zeta^i z \mid i = 0, \ldots, 2k\} \), which is a subset of \( S(\mathbb{C}) \) with diameter at most \( \pi - \pi/(2k+1) \).

Moreover, there is an example of such a map \( f \) with the property that any finite subset \( X \subseteq S(\mathbb{C}) \) such that 0 lies in the convex hull of \( f(X) \) has diameter greater than or equal to \( \pi - \pi/(2k+1) \).

Consider, more generally, a continuous map \( f : S(\mathbb{R}^n) \to \mathbb{R}^{m+n-1} \), where \( m \geq 0 \) and \( n > 1 \), such that \( f(-v) = -f(v) \) for all vectors \( v \) on the unit sphere \( S(\mathbb{R}^n) \) in \( \mathbb{R}^n \). The classical Borsuk–Ulam theorem asserts, in one form, that if \( m = 0 \), there is a point \( x \in S(\mathbb{R}^n) \) such that \( f(x) = 0 \). If \( n = 2 \), the proof is an elementary exercise using the Intermediate Value Theorem, while for
general $n$ a proof (one of many) can be given using the $F_2$-cohomology of real projective space. If $m > 0$, the theorem clearly fails, in the sense that there exist maps $f$ having no zero, for example, the inclusion $S(\mathbb{R}^n) \subseteq \mathbb{R}^n \hookrightarrow \mathbb{R}^{m+n-1}$. But the Adams–Bush–Frick theorems of [1] show that there is a finite subset $X \subseteq S(\mathbb{R}^n)$ with the property that zero is a convex linear combination of the values $f(x)$, $x \in X$, and, to make the assertion non-trivial (because $\frac{1}{2}f(x) + \frac{1}{2}f(-x) = 0$ for any $x$), satisfying the condition that $X$ does not contain any pair of antipodal points. And this condition is refined, for given $m$ and $n$, by bounding the diameter of $X$. A precise statement is given in Corollary 2.4.

More technical geometric arguments can be used to establish a Borsuk–Ulam theorem for higher dimensional spheres strengthening another of the Adams–Bush–Frick theorems [1, Theorem 3]. The unit sphere $S(\mathbb{R}^n)$ in $\mathbb{R}^n$ is equipped with the standard metric $d$: $\cos(d(u,v)) = \langle u, v \rangle$, $0 \leq d(u,v) \leq \pi$.

**Theorem 1.2.** Let $m, n \geq 1$ be positive integers such that $m \leq 2r \leq n < 2r+1$, where $r \geq 0$ is a non-negative integer. Suppose that $f: S^{n-1} = S(\mathbb{R}^n) \to \mathbb{R}^{m+n-1}$ is a continuous map such that $f(-v) = -f(v)$ for all $v \in S(\mathbb{R}^n)$. Then there exists a finite subset $X \subseteq S(\mathbb{R}^n)$ with cardinality at most $m + n$ and diameter at most $\pi - \arccos(1/n)$ such that 0 lies in the convex hull of $f(X)$ in $\mathbb{R}^{m+n-1}$.

Moreover, there is an example of such a map $f$ with the property that any finite subset $X \subseteq S(\mathbb{R}^n)$ such that 0 lies in the convex hull of $f(X)$ has diameter greater than or equal to $\pi - \arccos(1/n)$.

The proof of Theorem 1.2, requiring methods from Algebraic Topology (although not much more than the calculation of the $F_2$-cohomology of real projective spaces and the properties of Stiefel–Whitney classes of vector bundles), is beyond the scope of this elementary note, but can be found, with additional material, in [3].

For the wider context of these results the reader is referred to [1, 2].

### 2 | The Intermediate Value Theorem

**Lemma 2.1.** Let $f: S(\mathbb{C}) \to \mathbb{R}^{2k+1}$ be a continuous map such that $f(-z) = -f(z)$. Suppose that $w_0, \ldots, w_{2k}$ are any $2k + 1$ points in $S(\mathbb{C})$. Then there exist $e_i \in \{\pm 1\}$, $\lambda_i \geq 0$, for $i = 0, \ldots, 2k$, with $\sum \lambda_i = 1$, and $z \in S(\mathbb{C})$, such that $\sum_{i=0}^{2k} \lambda_i f(e_i z w_i) = 0$.

If $f(w_0), \ldots, f(w_{2k})$ lie in a $2k$-dimensional subspace of $\mathbb{R}^{2k+1}$, we can require that $z = 1$.

**Proof.** Consider the determinant map

$$\varphi: S(\mathbb{C}) \to \Lambda^{2k+1} \mathbb{R}^{2k+1}, z \mapsto f(z w_0) \wedge \cdots \wedge f(z w_{2k}).$$

(Thus, $\varphi(z) \in \mathbb{R}$ is the determinant of the $(2k + 1) \times (2k + 1)$-matrix with columns the vectors $f(z w_0), \ldots, f(z w_{2k})$.) Then $\varphi(-z) = (-1)^{2k+1} \varphi(z) = -\varphi(z)$. So, by the Intermediate Value Theorem, $\varphi$ has a zero.

If $\varphi(z) = 0$, the vectors $f(z w_i)$ in $\mathbb{R}^{2k+1}$ are linearly dependent and there exist $\mu_i \in \mathbb{R}$, not all zero such that $\sum \mu_i f(w_i) = 0$. We may assume, by scaling, that $\sum |\mu_i| = 1$. Choose $\lambda_i \geq 0$ and $e_i = \pm 1$ so that $\lambda_i e_i = \mu_i$.

**Lemma 2.2.** For $z \in S(\mathbb{C})$ and $e_i = \pm 1$, $i = 0, \ldots, 2k$, the distance $d(e_i z^i, e_j z^j)$ is less than or equal to $\pi - \pi / (2k + 1)$. 


Proof. Indeed, the $2(2k + 1)$ points $\pm z^i\zeta$ on the unit circle lie at the vertices of a regular polygon.

These two lemmas already prove the first part of Theorem 1.1.

Example 2.3 [1, Theorem 5]. Let $P$ be the $k$-dimensional complex vector space of complex polynomials $p(z)$ of degree $\leq 2k - 1$ such that $p(-z) = -p(z)$. Let $g : S(\mathbb{C}) \to P^*$ be the evaluation map to the dual $P^* = \text{Hom}_\mathbb{C}(P, \mathbb{C})$. Suppose $w_0, \ldots, w_{2k}$ are points of $S(\mathbb{C})$ such that $0$ lies in the convex hull of the $g(w_i)$. Then $d(w_i, w_j) \geq \pi - \pi/(2k + 1)$ for some $i, j$.

For the sake of completeness, we include a concise version of the proof in [1].

Proof. Assume first that the $2k + 1$ points $w_i^2$ are distinct. By relabelling, we may arrange that $w_1 = e^{i\theta}w_0$, where $\theta$ is the minimum of the distances $d(w_i, w_j), i \neq j$.

Suppose that $\sum \lambda_i g(w_i) = 0$, where the $\lambda_i \in \mathbb{R}$ are not all equal to zero. This means that $\sum \lambda_i p(w_i) = 0$ for each $p \in P$, and hence, since $\lambda_i \in \mathbb{R}$ and $w_i^{-1} = \overline{w_i}$, that $\sum \lambda_i p(w_i^{-1}) = 0$ too (because $\sum \lambda_i p(w_i)$ can be written as the complex conjugate of $\sum \lambda_i \overline{p(w_i)}$ with $\overline{p} \in P$). For $r \neq s$, $0 \leq r, s \leq 2k$, we may write

$$z^{-2k+1} \prod_{j \neq r, j \neq s} (z^2 - w_j^2) = p_+(z) + p_-(z^{-1}),$$

for unique polynomials $p_+, p_- \in P$. Then we find, because $\sum \lambda_i p_+(w_i) = 0$ and $\sum \lambda_i p_-(w_i^{-1}) = 0$, that

$$\sum \lambda_i w_i^{-2k+1} \prod_{j \neq r, j \neq s} (w_i^2 - w_j^2) = \sum \lambda_i (p_+(w_i) + p_-(w_i^{-1})) = 0,$$

that is,

$$\lambda_r w_r^{-2k+1} \prod_{j \neq r, j \neq s} (w_r^2 - w_j^2) + \lambda_s w_s^{-2k+1} \prod_{j \neq r, j \neq s} (w_s^2 - w_j^2) = 0,$$

or

$$\lambda_r \prod_{j \neq r} (w_r w_j^{-1} - w_j^{-1} w_r) = \lambda_s \prod_{j \neq s} (w_s w_j^{-1} - w_s^{-1} w_j).$$

It follows that, for some non-zero $c \in \mathbb{R}$,

$$\lambda_i \delta_i = c,$$

where $\delta_i = \prod_{j \neq i} (w_i w_j^{-1} - w_i^{-1} w_j)$.

for all $i$. (Notice that $\delta_i$, being the product of the $2k$ purely imaginary numbers $w_i \overline{w_j} - \overline{w_i} w_j$, is real.) In particular, all the $\lambda_i$ are nonzero.

Given that $0$ lies in the convex hull of the points $g(w_i)$, we can now assume further that all $\lambda_i$ are non-negative, and so, because they are non-zero, strictly positive. Then the $\delta_i \in \mathbb{R}$ all have the same sign. We show that there is some $i$ such that $w_i = -e^{it\theta}w_0$ for $0 < t < 1$. 
Indeed, write $\psi(t) = \prod_{1 < j \leq 2k} (e^{i\theta} w_0 w_{-1}^j - e^{-i\theta} w_0^{-1} w_j) \in i\mathbb{R}$, for $0 \leq t \leq 1$. Then $\delta_0 = (w_0 w_{-1}^1 - w_{-1}^1 w_1) \psi(0)$ and $\delta_1 = (w_1 w_0^{-1} - w_0^{-1} w_1) \psi(1)$. So, by the Intermediate Value theorem again, $\psi(t) = 0$ for some $t$, and then $w_i^2 = (e^{i\theta} w_0)^2$ for some $i$, $1 < i \leq 2k$. But $w_i \neq e^{i\theta} w_0$, by the minimality of $\theta$. So $w_i = -e^{i\theta} w_0$. Now $d(w_1, w_0) = \pi - t\theta$ and $d(w_i, w_1) = \pi - (1-t)\theta$. But clearly $\theta \leq 2\pi/(2k+1)$ and either $t \geq 1/2$ or $1-t \geq 1/2$.

This completes the proof if all the points $w_i^2$ are distinct. In general, we can apply the result to $2k+1$ vectors $w'_0, \ldots, w'_{2k}$ with the $w_i^2$ distinct such that $\{w_0^2, \ldots, w_{2k}^2\} \subseteq \{w'_0, \ldots, w'_{2k}\}$. For any $\delta > 0$, we can choose the $w_i'$ in such a way that $d(w_i, w_i') < \delta$, and then conclude that, for some $i, j \geq 0$, we have $d(w_i, w_j) + 2\delta > \pi - \pi/(2k+1)$.

\[ \square \]

Lemmas 2.1 and 2.2, together with Example 2.3, establish Theorem 1.1, using the standard fact\footnote{We have $\lambda_y > 0$, $y \in Y$, such that $\sum \lambda_y y = 0$. Suppose that $\mu_y \in \mathbb{R}$, $y \in Y$, satisfy $\sum \mu_y y = 0$ and $\sum \mu_y = 0$. Then, for any $t \in \mathbb{R}$ such that $\lambda_y > |t\mu_y|$ for all $y$, $\sum (\lambda_y + t\mu_y) y = 0$, $\sum (\lambda_y - t\mu_y) y = 0$, $\lambda_y \pm t\mu_y \geq 0$, and so $\lambda_y > \pm t\mu_y$, that is, $\lambda_y > |t\mu_y|$, for all $y$. Hence $\mu_y = 0$ for all $y$.} that, if $Y$ is a finite subset of a real vector space such that the convex hull of $Y$ contains 0 but no proper subset of $Y$ has this property, then the set $Y$ is affinely independent.

\textbf{Corollary 2.4.} Let $m \geq 0$ and $n > 1$ be integers. Then there is a non-negative real number $\delta < 1$ with the property that, for any continuous map $f : S(\mathbb{R}^n) \to \mathbb{R}^{m+n-1}$ such that $f(-v) = -f(v)$ for all $v \in S(\mathbb{R}^n)$, there is a finite subset $X$ of $S(\mathbb{R}^n)$ of cardinality at most $m+n$ and diameter at most $\pi - \arccos(\delta) < \pi$ such that 0 lies in the convex hull of $f(X)$.

The classical Borsuk–Ulam theorem deals with the case $m = 0$; we may take $\delta = 0$ so that $X$ consists of a single point.

\textbf{Proof.} By restricting $f$ to $S(\mathbb{R}^2) \subseteq S(\mathbb{R}^n)$ and including $\mathbb{R}^{m+n-1}$ in $\mathbb{R}^{2k+1}$ for the smallest $k \geq 1$ such that $m + n - 1 \leq 2k + 1$, we see from Theorem 1.1 that the assertion is true with $\delta$ equal to $\cos(\pi/(2k+1))$.

\[ \square \]

There is an easy extension of Lemma 2.1 to higher dimensions.

\textbf{Lemma 2.5.} For integers $n, k \geq 1$, suppose that $f : S(\mathbb{C}^n) \to \mathbb{R}^{2k+1}$ is a continuous map such that $f(-v) = -f(v)$ for all $v \in S(\mathbb{C}^n)$. Then, for any $2k + 1$ vectors $w_0, \ldots, w_{2k}$ in $S(\mathbb{C}^n)$, there exist $e_i \in \{\pm 1\}$, $\lambda_i \geq 0$, for $i = 0, \ldots, 2k$, with $\sum \lambda_i = 1$, and $z \in S(\mathbb{C})$ such that $\sum_{i=0}^{2k} \lambda_i f(e_i z w_i) = 0$.

\textbf{Proof.} This can be established, using the same arguments as in the proof of Lemma 2.1, by looking at the function $\varphi : S(\mathbb{C}) \to \Lambda^{2k+1} \mathbb{R}^{2k+1}$ defined by $\varphi(z) = f(z w_0) \wedge \cdots \wedge f(z w_{2k})$.

\[ \square \]

\textbf{Example 2.6.} For $n > 1$, write $e_1, \ldots, e_n$ for the standard orthonormal $\mathbb{C}$-basis of $\mathbb{C}^n$, let $\eta = e^{2\pi i/(2l+1)}$ where $l \geq 1$ is a positive integer, and fix an integer $r$ in the range $1 \leq r \leq n$.

If $k$ satisfies $2k + 1 \leq (2l + 1)^r \binom{n}{r}$, we can choose distinct vectors $w_0, \ldots, w_{2k}$ from the set

$$\{(\sum_{s=1}^{r} \eta^{a_s} e_{i_s})/\sqrt{r} \mid 1 \leq i_1 < i_2 \cdots < i_r \leq n, a_s = 0, \ldots, 2l\} \subseteq S(\mathbb{C}^n).$$

Then \(|\langle w_i, w_j \rangle| \leq \delta = 1 - (1 - \cos(\pi/(2l+1))/r \quad \text{for} \quad i \neq j\). Hence $d(e_i z w_i, e_j z w_j) \leq \pi - \arccos(\delta)$. \[ \square \]
Application of Lemma 2.5 to the case \( r = 1 \), for which \( \arccos(\delta) = \pi/(2l + 1) \), gives a result which is close to [1, Theorem 2], but slightly weaker. For general \( r \), if \( l \) is large, \( \arccos(\delta) \) is close to \((\pi/(2l + 1))/\sqrt{r}\). In particular, the case \( r = n \) gives a much stronger result than that provided by [1, Theorem 2] when \( k \) is sufficiently large. (A similar observation is made in [2, Remark 4.2].)

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