Uniqueness of Self-Similar Asymptotically Friedmann-Robertson-Walker Spacetime in Brans-Dicke theory

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We investigate spherically symmetric self-similar solutions in Brans-Dicke theory. Assuming a perfect fluid with the equation of state \( p = (\gamma - 1)\mu \) \( (1 \leq \gamma < 2) \), we show that there are no non-trivial solutions which approach asymptotically to the flat Friedmann-Robertson-Walker spacetime if the energy density is positive. This result suggests that primordial black holes in Brans-Dicke theory cannot grow at the same rate as the size of the cosmological particle horizon.

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I. INTRODUCTION

It has been pointed out that black holes could be formed in the early universe as a result of the collapse of the initial density perturbations, bubble collisions, or other mechanisms [1,2,3]. They are called primordial black holes (PBHs) and are of special interest since they could be the promising candidates of MACHO or the only ones which could be small enough to be evaporating by the present epoch due to the quantum effects [4]. The evolution of PBHs has been studied by many authors in general relativity [5,6,7,8].

Zeldovich and Novikov pointed out the catastrophic growth of the PBHs in a simple Newtonian argument which implies that the PBHs of which size is comparable to the cosmological particle horizon could continue to grow at the same rate as it throughout the radiation era, while those of which size is much smaller than that could not grow much at all [5]. One may think that the evolution of such PBHs could be described by self-similar solutions in general relativity. Carr and Hawking, however, showed that there is no non-trivial spherically symmetric self-similar solution approaching asymptotically to the flat Friedmann-Robertson-Walker (FRW) spacetime at large radius for a radiation fluid [6]. This argument was extended to the case of a general fluid with \( p = (\gamma - 1)\mu \) \( (1 < \gamma < 2) \) [7]. The result implies that the PBHs must soon become much smaller than the size of the cosmological particle horizon and cannot grow very much at all.

It has also been argued that in the early universe, gravity might obey a scalar-tensor type theory rather than general relativity. A scalar-tensor type theory of gravity arises naturally as a low-energy limit of string theory. In such theories, gravitational “constant” is given by a function of a scalar field which couples non-minimally with gravity so that the effective gravitational “constant” \( G \) generally varies in space or in time. Brans-Dicke theory is the simplest but still very important one among the scalar-tensor theories. Then, the evolution of PBHs in Brans-Dicke theories has been studied by several authors [9,10,11].

Barrow proposed two scenarios for the evolution of the PBHs in Brans-Dicke theory [10]. In his first scenario (A), the Brans-Dicke scalar field evolves everywhere homogeneously so that the gravitational “constant” at the black hole event horizon always the same as that at the cosmological particle horizon. No information about formation epoch remains. In his second scenario (B), however, the gravitational “constant” at the black hole event horizon remains constant with time and is therefore always the same as that at the formation epoch, while it changes at distant scales. Since the gravitational “constant” at the horizon stays an old value when a black hole was formed, the remaining gravitational “constant” near a PBH is called the gravitational memory. Discussing the energy emission due to the Hawking radiation in these scenarios, Barrow and Carr found that there are the significant deviations from that in general relativity in both scenarios [10]. Two alternative scenarios, which are intermediate between the scenario (A) and (B), are proposed by Carr and Goymer [11]. Analyzing perturbations of a scalar field in the Schwarzschild background [12], Jacobson concluded that the gravitational memory is weak. However, this conclusion does not follow when the size of a black hole is comparable to that of the cosmological particle horizon. Using the Tolman-Bondi solution and neglecting back reaction of a scalar field to the background geometry, Harada, Goymer and Carr studied the evolution of a scalar field when a black hole forms from the collapse of dust in the flat FRW spacetime and concluded that there is little gravitational memory [13].

The purpose of this short note is to study the evolution of the PBHs in Brans-Dicke theory by using self-similar solutions which contain a perfect fluid with the equation of state \( p = (\gamma - 1)\mu \) \( (1 \leq \gamma < 2) \). In this paper, we adopt the unit of \( c = 1 \).
II. SPHERICALLY SYMMETRIC SELF-SIMILAR SOLUTION IN BRANS-DICKE THEORY

The field equations in Brans-Dicke theory are given by

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi}{\phi}T_{\mu\nu} + \frac{1}{\phi}(\phi_{;\mu;\nu} - g_{\mu\nu}\Box\phi) \]
\[ + \frac{\omega}{\phi^2}(\phi_{;\mu;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\nu;\mu}), \] (2.1)

\[ T_{\mu\nu} = 0, \] (2.2)
\[ \Box\phi = \frac{8\pi}{3+2\omega}T^{\mu}_{\mu}, \] (2.3)

where the constant \( \omega \) is the Brans-Dicke parameter and \( \phi \) is the Brans-Dicke scalar field, which is related to the gravitational “constant” by \( G = 1/\phi \). \( \omega > -3/2 \) is required for the theory not to contain tachyonic solutions.

We consider a perfect fluid as a matter field \( T_{\mu\nu} = pg_{\mu\nu} + (\mu + p)U_{\mu}U_{\nu} \), where \( U_{\nu} \) is the 4-velocity. The radial coordinate \( r \) is chosen to be comoving. The equation of state is assumed to be \( p = (\gamma - 1)\mu \) where \( 1 \leq \gamma < 2 \). A solution of the Einstein equations is said to be self-similar if it admits a homothetic Killing vector \( \xi \) such that \( \mathcal{L}_\xi g_{\mu\nu} = 2g_{\mu\nu} \), where \( \mathcal{L}_\xi \) denotes Lie derivative along \( \xi \). Cahill and Taub [14] first investigated spherically symmetric self-similar solutions in which the homothetic Killing vector is neither parallel nor orthogonal to the fluid flow vector. They showed that by the suitable coordinate transformations such solutions can be put in a form in which all dimensionless quantities are functions of the dimensionless variable \( z \equiv r/t \). Generalizing their method to the present model, we find the following basic equations. The line element of a spherically symmetric self-similar spacetime is given by

\[ ds^2 = -e^{2\Phi(z)}dt^2 + e^{2\Psi(z)}dz^2 + r^2S^2(z) d\Omega^2, \] (2.4)

where \( d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \). Here we assume the self-similarity of a perfect fluid and the Brans-Dicke scalar field independently, i.e.

\[ \mathcal{L}_\xi p = ap, \quad \mathcal{L}_\xi \mu = b\mu, \quad \mathcal{L}_\xi \phi = \kappa\phi, \] (2.5)

where \( a, b \) and \( \kappa \) are constants. These constants satisfy the relations \( a = b = -(2 - \kappa) \) through the equation of state and the field equations (2.1)-(2.3). Equation (2.3) implies that the quantities \( p, \mu \) and \( \phi \) must be of the form

\[ \mu = \frac{W(z)}{r^{2-\kappa}}, \quad p = \frac{P(z)}{r^{2-\kappa}}, \quad \phi = r^\kappa\psi(z), \] (2.6)

and then equations (2.1)-(2.3) reduce to the ordinary differential equations w.r.t. the self-similar variable \( z \).

The energy-momentum conservation equation is written in self-similar variable as

\[ \Phi' = \frac{(2 - \kappa)P - P'}{W + P} = \frac{\gamma - 1}{\gamma} \left( 2 - \kappa + \frac{W'}{W} \right), \] (2.7)

\[ \Psi' = -\left( \frac{W'}{W + P} + \frac{2S'}{S} \right) = -\left( \frac{1}{\gamma} \frac{W'}{W} + \frac{2S'}{S} \right), \] (2.8)

where prime denotes the derivative with respect to \( \ln z \). These equations can be integrated to give

\[ e^\Phi = c_0(\frac{2(2-\gamma)(\gamma - 1)}{W - 2W'\gamma} + 1), \] (2.9)

\[ e^\Psi = c_1S^{-2W' - \frac{4}{\gamma}}, \] (2.10)

where \( c_0, c_1 \) are integration constants. Equations (2.1) and (2.3) then reduce to the following ordinary differential equations;

\[ \left[ \frac{(2 + \kappa)(\gamma - 1)}{\gamma} \right] \left( \frac{\kappa - 1 + \omega\kappa}{\psi} \right) \]
\[ - \frac{W'}{W} \left[ \frac{(2 + \kappa)(\gamma - 1)}{\gamma} \right] = \frac{8\pi\gamma}{\psi^2}c_1^2S^{-4W'\gamma - \frac{4}{\gamma}}, \] (2.11)

\[ \begin{align*}
2S' &= S + S' S + \frac{\psi'}{\psi} \left( \frac{\kappa - 1 + \omega\kappa}{\psi} \right) + (1 + S'\frac{S'}{S} \]
\[ - \frac{(2 + \kappa)(\gamma - 1)}{\gamma} = \frac{8\pi(3\gamma - 4)}{3 + 2\omega}\psi^2c_1^2S^{-4W'\gamma - \frac{4}{\gamma}}, \] (2.13)

with \( V \equiv \psi = (c_1/c_0)^{\frac{1}{\gamma}(2(2-\gamma)(\gamma - 1)/\gamma}S^{-2W'(\gamma-2)/\gamma} \).

Equations (2.11) and (2.12) are independent of equations (2.7) and (2.8).

III. THE FLAT FRW SOLUTION

The flat FRW solution in Brans-Dicke theory with a perfect fluid obeying the equation of state \( p = (\gamma - 1)\mu \) was found by Nariai [17] and it is a particular solution of the reduced ordinary differential equations, i.e. a self-similar solution. The flat FRW solution in the self-similar variable \((\Phi_{FRW}(z), \Psi_{FRW}(z), S_{FRW}(z), W_{FRW}(z))\) is

\[ e^\Phi_{FRW} = a_0, \quad e^\Psi_{FRW} = b_0z^{-q}, \] (3.1)

\[ S_{FRW} = \frac{b_0}{1 - q}z^{-q}, \quad \psi_{FRW} = \psi_0z^{-2 + 3q}, \] (3.2)

\[ W_{FRW} = \frac{\psi_0(3 + 2\omega/2)(5 - 3\gamma) + (3(2 - \gamma)^2\omega)}{4\omega a_0^2[4 - 3\gamma\omega(\gamma - 2)]^2}z^{3\gamma}, \] (3.3)

\[ \kappa = \frac{2(4 - 3\gamma)}{4 - 3\gamma\omega(\gamma - 2)}, \quad q = \frac{2[1 - \omega(\gamma - 2)]}{4 - 3\gamma\omega(\gamma - 2)}. \] (3.4)
One can put this solution in a more familiar form
\[ ds^2 = -dt^2 + e^{2t}(dr^2 + r^2d\Omega^2), \] (3.5)
by making the coordinate transformation \( t' = a_0 t, r' = a_0^{-1}b_0 r^{-q} / (1-q) \). In the case of radiation (\( \gamma = 4/3 \)), \( q = 1/2 \) and \( \kappa = 0 \) are obtained from equation (3.4). Since \( \kappa = 2 - 3 \gamma, \kappa = 0 \) implies that the Brans-Dicke scalar field \( \phi \) is constant which case is equivalent to that in general relativity. Assuming the energy density is positive (\( W > 0 \)), we find
\[ \omega > 2(3\gamma - 5) / 3(\gamma - 2)^2, \] (3.6)
If \( \gamma = 2 \), any value of \( \omega \) is not possible. On the other hand, in the case of radiation (\( \gamma = 4/3 \)), the energy density is always positive for any value of \( \omega > -3/2 \).

IV. UNIQUENESS OF SELF-SIMILAR ASYMPTOTICALLY FLAT FRW SOLUTION

We are interested in self-similar solutions which are asymptotically flat FRW solution, i.e. in which \( W, S \) and \( \psi \) approach the form \( W = W_{\text{FRW}}(z)e^{A(z)}, S = S_{\text{FRW}}(z)e^{B(z)}, \psi = \psi_{\text{FRW}}(z)e^{C(z)} \) (4.1) and (4.2) imply (through equations (2.7) and (2.8)) that
\[ e^\Phi = e^{\psi_{\text{FRW}}} e^{-2B - \frac{2}{\gamma}}, \] (4.3)
The flat FRW solution is given by \( A = B = C = 0 \) for any \( z \). Since \( V = (b_0/a_0) z^{1-q} \exp(-2B + 2/3), \) the asymptotic form of self-similar solutions which approach the FRW solution for large \( z \) can be found by linearizing equations neglecting the \( V^2 \) term for accelerating expansion case (\( q > 1 \)), while the 1/\( V^2 \) term for deaccelerating expansion case (\( 0 < q < 1 \)).

A. Accelerating expansion case

For accelerating case (\( q > 1 \)), the asymptotic behavior of linearized equations are given by
\[ C'' = -(1-q)C', \] (4.4)
\[ 2B'' = -(1+q)C' + (2q - 2 - k)A' + 6(q-1)B' + (-q + k + \kappa)C', \] (4.5)
\[ A' = \frac{\gamma(k - 1 + \omega\kappa)}{(\gamma - 1)(2 + \kappa)} C'. \] (4.6)
This system of linear differential equations has no solution which vanish as \( z \to \infty \). It means that there are no non-trivial asymptotically flat FRW solutions in accelerating case.

B. Deaccelerating expansion case

For deaccelerating case (\( 0 < q < 1 \)) the asymptotic behavior of linearized equations are given by
\[ -C'' = -\frac{3\gamma - 4}{3 + 2\omega}(3q^2 - \omega/2)A' + (-2\kappa + 1 - 3q)C' - \frac{\kappa(\gamma - 2)}{\gamma} A', \] (4.7)
\[ 2B'' = -C'' + \frac{1}{\gamma} (\kappa \gamma + 2q \gamma - 2 - k)A' + 6(q-1)B' + (-q + k + \kappa)C', \] (4.8)
\[ \frac{2 + \kappa}{\gamma} A' = -\gamma \left( \frac{3q^2 - \omega}{2} - 2q \kappa \right) C' \] (4.9)
which solutions is
\[ C = \frac{2 - \gamma}{\gamma} A + C_0 z^3, \quad 3B + \frac{4}{\gamma} C = C_1 z^3, \quad A = C_2 z^3, \] (4.10)
\[ s \equiv 1 - 3q - \kappa = -\frac{2 - \gamma}{\gamma} \frac{\omega - 2(3\gamma - 5)}{3(\gamma - 2)^2} \] (4.11)
where \( C_0, C_1 \) and \( C_2 \) are constants satisfying one constraint equation
\[ -[3(3\gamma^2 - 8\gamma + 8)] C_0 + \frac{4}{3} [3\omega \gamma (\gamma - 2) + 3\gamma - 8] C_1 + \frac{4}{3\gamma} \] (4.12)
The power \( s \) is positive when
\[ \omega > 2(3\gamma - 5) / 3(\gamma - 2)^2 < 0. \] (4.13)
When \( \gamma = 4/3 \), \( s = -1/2 < 0 \). Since for deaccelerating expansion (\( 0 < q < 1 \)), we find
\[ \omega > \frac{2}{(2 - \gamma)(3\gamma - 2)} > -\frac{4}{3\gamma(2 - \gamma)} \] for \( \gamma > 4/3 \), (4.14)
the inequality (4.13) with (4.14) yields that the energy density of a perfect fluid is negative. Then the positivity of energy density requires \( s < 0 \), i.e., there is no non-trivial solution which approaches the flat FRW universe. We conclude that there are no non-trivial asymptotically flat FRW solutions in deaccelerating case.
V. DISCUSSION AND SUMMARY

We have studied spherically symmetric self-similar solutions in Brans-Dicke theory which contain a perfect fluid with the equation of state \( p = (\gamma - 1)\mu (1 \leq \gamma < 2) \). We have shown that when the energy density is positive, there are no non-trivial solutions which approach to the flat FRW spacetime at large \( z \). It should be noted that although non-trivial solutions do exist, they are unphysical solutions with negative energy density. It is similar to the result in general relativity. In general relativity, non-trivial solutions which approach to the flat FRW spacetime at large \( z \) exist for \( \gamma > 2 \), where the sound speed is faster than the speed of light \([7]\).

The result in this paper suggests that the PBHs does not grow at the same rate as the size of the cosmological particle horizon in Brans-Dicke theory as in the case of general relativity. However, it could be possible to construct a non-trivial solution attached to the flat FRW solution at the point of \( \gamma - 1 \) where the field equations become singular. In general relativity, it was shown that the ingoing Vaidya null fluid solution can be attached to the flat FRW solution at the point of \( \gamma - 1 \) if the equation of state is \( p = \mu \) though this is not a black hole solution \([7]\).

If we allow deficit (surplus) angle in the present self-similar solutions, we can find a one-parameter family of solutions which approaches asymptotically to the “flat FRW spacetime”, as Carr and Hawking found them in the case of general relativity \([7]\).

In this paper, we have imposed the self-similarity on the whole spacetime, however it can be considered that the self-similarity is imposed only on the neighborhood of a black hole and the solution smoothly connects to the non-self-similar spacetime at large distance. Such a solution represents a black hole in the flat FRW universe which grows locally in the self-similar manner. To construct such a solution is an important future work.

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