PRODUCTS OF MATRICES $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ AND $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ AND THE DISTRIBUTION OF REDUCED QUADRATIC IRRATIONALS

FLORIN P. BOCA

Abstract. Let $\Phi(N)$ denote the number of products of matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ of trace equal to $N$, and $\Psi(N) = \sum_{n=3}^{N} \Phi(n)$ be the number of such products of trace between 3 and $N$. We prove an asymptotic formula of type $\Psi(N) = c_1 N^2 \log N + c_2 N^2 + O(N^{7/4+\epsilon})$ as $N \to \infty$. As a result, the Dirichlet series $\sum_{n=3}^{\infty} \Phi(n)n^{-s}$ has a meromorphic extension in the half-plane $\Re(s) > \frac{7}{4}$ with a single, order two pole at $s = 2$. Our estimate also improves on an asymptotic result of Faivre concerning the distribution of reduced quadratic irrationals, providing an explicit upper bound for the error term.

1. Introduction

Consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and the (free) multiplicative monoid $\mathcal{M}$ they generate. The problem of estimating

$$\Phi(N) = \#\{ C \in \mathcal{M} : \text{Tr}(C) = N \}$$

and

$$\Psi(N) = \sum_{n=3}^{N} \Phi(n) = \#\{ C \in \mathcal{M} : 3 \leq \text{Tr}(C) \leq N \}$$

for large $N$ came across in the study of a number-theoretic spin chain model in statistical mechanics introduced in [10], and further investigated in [5] and [2]. In [9] the estimate

$$\Psi(N) = \frac{N^2 \log N}{\zeta(2)} + O(N^2 \log \log N) \quad (N \to \infty)$$

(1.1)

was proved, using $L$-functions and a result from [4] concerning the distribution of reduced quadratic irrationals.

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In this paper we improve the estimate (1.1). Our approach relies on a result concerning the
distribution of multiplicative inverses, which is a consequence of Weil’s bound on Kloosterman
sums.

**Theorem 1.1.** We have

\[ \Psi(N) = c_1 N^2 \log N + c_2 N^2 + \Psi_0(N), \]  

where

\[ \Psi_0(N) \ll \varepsilon N^{\frac{7}{4} + \varepsilon} \quad (N \to \infty), \]  

and

\[ c_1 = \frac{1}{\zeta(2)}, \quad c_2 = \frac{1}{\zeta(2)} \left( \gamma - \frac{3}{2} - \frac{\zeta'(2)}{\zeta(2)} \right). \]

Using the Mellin transform representation of Dirichlet series and changing \( N \) to \( N - 2 \) in
the right-hand side of (1.2) we obtain for \( \Re(s) > 2 \)

\[
Z(s) = \sum_{n=3}^{\infty} \Phi(n)n^{-s} = s \int_{3}^{\infty} \Psi(x)(x-2)^{-s-1} dx
\]

\[
= s \int_{1}^{\infty} (c_1 x^2 \log x + c_2 x^2) x^{-s-1} dx + s \int_{1}^{\infty} \Psi_0(x)x^{-s-1} dx
\]

\[
= \frac{2c_1}{(s-2)^2} + \frac{c_1 + 2c_2}{s-2} + c_2 + s \int_{1}^{\infty} \Psi_0(x)x^{-s-1} dx.
\]

Since the function \( \Psi_0(x) \) satisfies the growth condition (1.3) the integral above converges
for \( \Re(s) > \frac{7}{4} \), and thus the right-hand side defines an analytic continuation of \( Z(s) \) to the
half-plane \( \Re(s) > \frac{7}{4} \) with the point \( s = 2 \) removed.

The contribution to the main term in the asymptotic formula above only comes from words
of odd length in \( \mathcal{M} \). We estimate the contribution \( \Psi_{ev}(N) \) to \( \Psi(N) \) of words of even length
which begin in \( A \) and end in \( B \), proving

**Proposition 1.2.** \( \Psi_{ev}(N) = \frac{N^2 \log 2}{2\zeta(2)} + O_{\varepsilon}(N^{\frac{7}{4} + \varepsilon}) \quad (N \to \infty). \)

This extends [9, Prop.4.5] which states that

\[ \Psi_{ev}(N) \sim \frac{N^2 \log 2}{2\zeta(2)}, \]

without any estimate on the error term.
Using a transfer operator associated with the Gauss map, Fredholm theory, and Ikehara’s tauberian theorem, Faivre [4] proved the asymptotic formula

$$\sum_{\rho(\omega) \leq X} 1 \sim \frac{e^X \log 2}{2\zeta(2)} \quad (X \to \infty)$$

for the number of reduced quadratic irrationals \( \omega \) of length \( \rho(\omega) \) at most \( X \). Since the final argument relies on a tauberian theorem, no explicit bound was found for the error term. In the last section we use Proposition 1.2 and an explicit identification from [9] between products of matrices \( A \) and \( B \) starting with \( B \) and ending with \( A \), and reduced quadratic irrationals, to prove

**Proposition 1.3.** \( \sum_{\rho(\omega) \leq X} 1 = \frac{e^X \log 2}{2\zeta(2)} + O\left(e^\left(\frac{7}{8} + \epsilon\right)X\right) \quad (X \to \infty). \)

2. **Products of A’s and B’s and continued fractions**

If \( \alpha = [a_1, a_2, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}} \) is a reduced continued fraction with positive integers \( a_i \), the \( k \)th convergent

$$\frac{p_k}{q_k} = [a_1, \ldots, a_k]$$

is given by pairs \((p_n, q_n)\) of relatively prime integers defined recursively as

\[
\begin{align*}
    p_0 &= 0, \quad p_1 = 1, \quad p_n = a_n p_{n-1} + p_{n-2}, \\
    q_0 &= 1, \quad q_1 = a_1, \quad q_n = a_n q_{n-1} + q_{n-2},
\end{align*}
\]

and satisfying \( 0 \leq p_n \leq q_n \) and the equality

$$p_{n-1}q_n - p_nq_{n-1} = (-1)^n.$$

If \((p_n)\) and \((q_n)\) satisfy (2.1) for every \( 0 \leq n \leq k \), then

$$\frac{q_{n-1}}{q_n} = \frac{q_{n-1}}{a_n q_{n-1} + q_{n-2}} = \frac{1}{a_n + \frac{2a_{n-2}}{q_{n-1}}}$$

showing that

$$\frac{q_{k-1}}{q_k} = [a_k, \ldots, a_1]. \quad (2.2)$$

Consider the matrices

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad M(a) = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}. \quad$$
If \((p_n, q_n)\) is as in (2.1) then, as noticed in [9], we have
\[
M(a_1) \cdots M(a_n) = \begin{bmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{bmatrix}.
\] (2.3)

When combined with
\[
B^k A^\ell = M(k)M(\ell), \quad k, \ell \in \mathbb{Z},
\]
equality (2.3) yields
\[
B^{a_1} A^{a_2} \cdots B^{a_{2m-1}} A^{a_{2m}} = M(a_1) \cdots M(a_{2m}) = \begin{bmatrix} q_{2m} & q_{2m-1} \\ p_{2m} & p_{2m-1} \end{bmatrix}.
\] (2.4)

From (2.1), (2.4), and
\[
B = A^T = J A J, \quad A = J B J,
\]
we also infer that
\[
A^{q_1} B^{a_2} \cdots A^{q_{2m-1}} B^{a_{2m}} = J \begin{bmatrix} q_{2m} & q_{2m-1} \\ p_{2m} & p_{2m-1} \end{bmatrix} J = \begin{bmatrix} p_{2m-1} & p_{2m} \\ q_{2m-1} & q_{2m} \end{bmatrix},
\]
\[
B^{a_1} A^{a_2} \cdots A^{a_{2m}} B^{a_{2m+1}} = \begin{bmatrix} q_{2m} & q_{2m-1} \\ p_{2m} & p_{2m-1} \end{bmatrix} \begin{bmatrix} 1 & a_{2m+1} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} q_{2m} & q_{2m+1} \\ p_{2m} & p_{2m+1} \end{bmatrix},
\] (2.5)
\[
A^{q_1} B^{a_2} \cdots B^{a_{2m}} A^{a_{2m+1}} = \begin{bmatrix} p_{2m+1} & p_{2m} \\ q_{2m+1} & q_{2m} \end{bmatrix}.
\]

All matrices in the products from (2.4) and (2.5) have determinant 1. We denote
\[
\mathcal{W}_{ev}(N) = \{ (a_1, \ldots, a_{2m}) \in \mathbb{N}^{2m} : m \geq 1, \quad \text{Tr}(B^{a_1} A^{a_2} \cdots B^{a_{2m-1}} A^{a_{2m}}) \leq N \},
\]
\[
\mathcal{W}_{odd}(N) = \{ (a_1, \ldots, a_{2m+1}) \in \mathbb{N}^{2m+1} : m \geq 1, \quad \text{Tr}(B^{a_1} A^{a_2} \cdots A^{a_{2m}} B^{a_{2m+1}}) \leq N \}.
\]

We consider the sets
\[
\mathcal{J}_{ev}(N) = \left\{ \begin{bmatrix} q' & q \\ p' & p \end{bmatrix} : 0 \leq p \leq q, \quad 0 \leq p' \leq q', \quad q' > q, \quad p + q' \leq N, \quad pq' - p'q = 1 \right\}
\]
and
\[
\mathcal{J}_{odd}(N) = \left\{ \begin{bmatrix} q & q' \\ p & p' \end{bmatrix} : 0 \leq p \leq q, \quad 0 \leq p' \leq q', \quad q' \geq q, \quad p' + q \leq N, \quad p'q - pq' = 1 \right\},
\]
of cardinality \(\Psi_{ev}(N)\) and respectively \(\Psi_{odd}(N)\), and the maps defined as
\[
\beta_{ev}(a_1, \ldots, a_{2m}) = B^{a_1} A^{a_2} \cdots B^{a_{2m-1}} A^{a_{2m}} = M(a_1) \cdots M(a_{2m}),
\]
\[
\beta_{odd}(a_1, \ldots, a_{2m+1}) = B^{a_1} A^{a_2} \cdots A^{a_{2m}} B^{a_{2m+1}} = M(a_1) \cdots M(a_{2m+1}) J,
\]
from \( \bigcup_1^\infty \mathcal{W}_{ev}(N) \) to \( \bigcup_1^\infty \mathcal{S}_{ev}(N) \), and respectively from \( \bigcup_1^\infty \mathcal{W}_{odd}(N) \) to \( \bigcup_1^\infty \mathcal{S}_{odd}(N) \). As a consequence of (2.2), \( \beta_{ev} \) and \( \beta_{odd} \) are injective. It follows from (2.4) and (2.5) that 

\[
\beta_{ev}(\mathcal{W}_{ev}(N)) \subseteq \mathcal{S}_{ev}(N) \quad \text{and} \quad \beta_{odd}(\mathcal{W}_{odd}(N)) \subseteq \mathcal{S}_{odd}(N).
\]

To check \( \mathcal{S}_{ev}(N) \subseteq \beta_{ev}(\mathcal{W}_{ev}(N)) \), let \( \left[ \begin{array}{cc} q' & q \\ p' & p \end{array} \right] \in \mathcal{S}_{ev}(N) \). With \( K = \left[ \begin{array}{cc} \frac{q}{q} \end{array} \right] \) we have \( 0 \leq q' - Kq < q, \ 0 \leq p' - Kp, \) and \((p' - Kp)q - (q' - Kq)p = 1\). Since \( q' > q \) are relatively prime, we also have \( K \leq \frac{q' - 1}{q} \leq \frac{q - p'}{q - p} \), and thus \( p' - Kp \leq q' - Kq \). Since

\[
\left[ \begin{array}{cc} q' & q \\ p' & p \end{array} \right] \left[ \begin{array}{c} 0 \\ 1 - K \end{array} \right] = \left[ \begin{array}{cc} q' & q \\ p' & p \end{array} \right] M(K)^{-1} = \left[ \begin{array}{cc} q & q' - Kq \\ p & p' - Kp \end{array} \right],
\]

it follows, by replacing \((q, q')\) by \((q' - Kq, q)\) and \((p, p')\) by \((p' - Kp, p)\) and performing this process until \(q' - Kq\) becomes equal to 1, that the matrix \( \left[ \begin{array}{cc} q' & q \\ p' & p \end{array} \right] \) is written as a product of \( k \) matrices of form \( M(K) \). Since \( q' > q, k \) ought to be even and therefore \( \left[ \begin{array}{cc} q' & q \\ p' & p \end{array} \right] \in \beta_{ev}(\mathcal{W}_{ev}(N)) \).

One shows in a similar way that \( \mathcal{S}_{odd}(N) \subseteq \beta_{odd}(\mathcal{W}_{odd}(N)) \).

This proves that the elements of \( \mathcal{M} \) are uniquely represented as products of \( A \)'s and \( B \)'s. It also implies that

\[
\Psi(N) = 2\Psi_{ev}(N) + 2\Psi_{odd}(N). \tag{2.6}
\]

3. Estimating \( \Psi_{ev}(N) \)

To estimate \( \Psi_{ev}(N) \), we first keep \( q' \) and \( q \) constant. From \( pq' - p'q = 1 \) and \( q < q' \), it follows that \( q' \) and \( q \) are relatively prime, and that \( p \) is uniquely determined as \( p = \overline{q} \), where \( \overline{q} \) is the unique integer in \( \{1, \ldots, q\} \) for which \( \overline{q}q = 1 \) (mod \( q \)). It is obvious that \( p' = \frac{pq' - 1}{q} \leq q' \) and the map

\[
\left\{ (q, q') : q < q' \leq N, \ (q, q') = 1 \quad q' + \overline{q} \leq N \right\} \ni (q, q') \mapsto \left[ \begin{array}{cc} q' & q \\ p' & p \end{array} \right] \in \mathcal{S}_{ev}(N)
\]

is a one-to-one correspondence. Replacing \( q' \) by \( y \) and \( \overline{q} \) by \( x \) we can write

\[
\Psi_{ev}(N) = \sum_{q < q' \leq N} 1 = \sum_{q < N} \sum_{0 < y \leq N, 0 < x \leq \min\{q, N - y\}, xy = 1 \text{ (mod } q\text{)}} 1. \tag{3.1}
\]

For each \( y \in (0, N] \), there is at most one \( x \in (0, q) \) such that \( xy = 1 \) (mod \( q \)); whence the trivial estimate

\[
\Psi_{ev}(N) \ll N^2.
\]

To give a more precise estimate for \( \Psi_{ev}(N) \), we shall define for \( q > 1 \) integer and \( \Omega \) subset in \( \mathbb{R}^2 \) the number \( \mathcal{A}_q(\Omega) \) of (relatively prime) integers \((x, y) \in \Omega \) such that \( xy = 1 \) (mod \( q \)). It
is known (see for instance [1]) that Weil’s bound on Kloosterman sums yields for any intervals \( I \) and \( J \) of length at most \( q \) the estimate
\[
\mathcal{N}_q(I \times J) = \frac{\varphi(q)}{q^2} |I| |J| + O_\varepsilon(q^{\frac{1}{2}+\varepsilon}).
\]
This immediately extends to intervals of arbitrary size as
\[
\mathcal{N}_q(I \times J) = \frac{\varphi(q)}{q^2} |I| |J| + O_\varepsilon \left( q^{\frac{1}{2}+\varepsilon} \left( 1 + \frac{|I|}{q} \right) \left( 1 + \frac{|J|}{q} \right) \right). \tag{3.2}
\]

Another easy but useful consequence of (3.2) is given next.

**Lemma 3.1.** Suppose that
\[
\Omega = \{(x, y) : \alpha \leq x \leq \beta, \ f_1(x) \leq y \leq f_2(x)\},
\]
with \( C^1 \)-functions \( f_1 \leq f_2 \) on \([\alpha, \beta]\). For every positive integer \( T \) we have
\[
\mathcal{N}_q(\Omega) = \frac{\varphi(q)}{q^2} \text{Area}(\Omega) + \mathcal{E}_q,
\]
with
\[
\mathcal{E}_q \ll \varepsilon \frac{\beta - \alpha}{Tq} \left( V^\beta_\alpha(f_1) + V^\beta_\alpha(f_2) \right) + Tq^{\frac{1}{2}+\varepsilon} \left( 1 + \frac{\beta - \alpha}{Tq} \right) \left( 1 + \frac{\|f_1\|_\infty + \|f_2\|_\infty}{q} \right),
\]
where \( V^\beta_\alpha(f_i) \) denotes the total variation of \( f_i \) on \([\alpha, \beta]\).

**Proof.** One can take without loss of generality \( f_1 = 0 \) and \( f_2 \geq 0 \). Partitioning \([\alpha, \beta]\) into \( T \) intervals \( I_i \), of equal size and denoting by \( M_i \) and \( m_i \) the maximum, respectively the minimum, of \( f_2 \) on \( I_i \), we clearly have
\[
\sum_{i=0}^{T-1} \mathcal{N}_q(I_i \times [0, m_i]) \leq \mathcal{N}_q(\Omega) \leq \sum_{i=0}^{T-1} \mathcal{N}_q(I_i \times [0, M_i]).
\]

By (3.2) we infer that
\[
\mathcal{N}_q(I_i \times [0, m_i]) = \frac{\varphi(q)}{q^2} \cdot \frac{\beta - \alpha}{T} m_i + O_\varepsilon \left( q^{\frac{1}{2}+\varepsilon} \left( 1 + \frac{\beta - \alpha}{Tq} \right) \left( 1 + \frac{\|f_2\|_\infty}{q} \right) \right).
\]
The statement follows from this, the similar estimate for \( \mathcal{N}_q(I_i \times [0, M_i]) \), and from
\[
\int_{\alpha}^{\beta} f_2(x) \, dx = \sum_{i=0}^{T-1} \frac{\beta - \alpha}{T} m_i + O \left( \frac{\beta - \alpha}{T} V^\beta_\alpha(f_2) \right)
\]
\[
= \sum_{i=0}^{T-1} \frac{\beta - \alpha}{T} M_i + O \left( \frac{\beta - \alpha}{T} V^\beta_\alpha(f_2) \right).
\]
Equality (3.1) can also be written as

\[ \Psi_{ev}(N) = \sum_{q < N} \mathcal{N}_q(\Omega_{N,q}), \]  

(3.3)

where \( \Omega_{N,q} = \{(x, y) : q < y \leq N, \ 0 < x \leq \min\{q, N - y\}\} \) coincides with the trapezoid \( \Omega_{N,q}^{(1)} = \{(x, y) : 0 < x \leq q < y \leq N - x\} \) if \( q \leq \frac{N}{2}, \) and with the triangle \( \Omega_{N,q}^{(2)} = \{(x, y) : 0 < x < N - q, \ q < y \leq N - x\} \) if \( q > \frac{N}{2}. \) Employing (3.3) and applying Lemma 3.1 to \( \Omega_{N,q}^{(1)} \) and \( \Omega_{N,q}^{(2)} \) we infer that

\[ \Psi_{ev}(N) = \sum_{q \leq \frac{N}{2}} \mathcal{N}_q(\Omega_{N,q}^{(1)}) + \sum_{\frac{N}{2} < q < N} \mathcal{N}_q(\Omega_{N,q}^{(2)}) \]

\[ = \sum_{q \leq \frac{N}{2}} \frac{\varphi(q)}{q^2} \cdot \frac{q(2N - 3q)}{2} + \sum_{\frac{N}{2} < q < N} \frac{\varphi(q)}{q^2} \cdot \frac{(N - q)^2}{2} + \mathcal{E}_1(N) + \mathcal{E}_2(N), \]

with

\[ \mathcal{E}_1(N) \ll \varepsilon \sum_{q \leq \frac{N}{2}} \left( \frac{q}{Tq} \cdot q + Tq^{\frac{3}{2} + \varepsilon} \cdot \frac{N}{q} \right) \ll \varepsilon \frac{N^2 T}{T} + T N^{\frac{3}{2} + \varepsilon}, \]

\[ \mathcal{E}_2(N) \ll \varepsilon \sum_{\frac{N}{2} < q < N} \left( \frac{N}{Tq} \cdot N + Tq^{\frac{3}{2} + \varepsilon} \left( 1 + \frac{N}{Tq} \right) \frac{N}{q} \right) \ll \varepsilon \frac{N^2 T}{T} + T N^{\frac{3}{2} + \varepsilon}. \]

Taking \( T = \lfloor N^\frac{1}{4} \rfloor \) and using standard summation results (see formulas in Corollary 4.5 and its proof below) this gives

\[ \Psi_{ev}(N) = \frac{N^2 \log 2}{2 \zeta(2)} + O_{\varepsilon}(N^{-\frac{7}{4} + \varepsilon}), \]

which ends the proof of Proposition 1.2.

4. Estimating \( \Psi_{odd}(N) \)

To estimate \( \Psi_{odd}(N), \) we first keep \( q \) and \( p \) fixed. The general solution of \( p'q - pq' = 1 \) is given by

\[ p' = \overline{q} + pt, \quad q' = \frac{p'q - 1}{p} = \frac{q\overline{q} - 1}{p} + qt, \quad t \in \mathbb{Z}, \]

where \( \overline{q} \) is the unique integer in \( \{1, \ldots, p\} \) for which \( \overline{q}q \equiv 1 \pmod{p}. \) Since \( p' > p, \) one has \( t \geq 1. \) The map

\[ \{(q, p, t) : 0 \leq p < q, \ (p, q) = 1 \} \] \[ 1 \leq t \leq \left[ \frac{N - q - \overline{q}}{p} \right] \] \[ \ni (q, p, t) \mapsto \begin{bmatrix} q & q' = \frac{\overline{q} - 1}{p} + qt \\ p & p' = \overline{q} + pt \end{bmatrix} \in \mathcal{S}_{odd}(N) \]
is a bijection. Replacing $p$ by $a$, $q$ by $y$, and $\overline{q}$ by $x$, we can write

$$\Psi_{\text{odd}}(N) = \sum_{q < N} \sum_{p < q \atop q + \overline{q} \leq N} \left[ \frac{N - q - \overline{q}}{p} \right] = \sum_{a < N} \sum_{a < y < N \atop 0 < x \leq \min\{a, N - y\} \atop xy \equiv 1 \pmod{a}} \left[ \frac{N - y - x}{a} \right].$$

When $N - y < a$ we have $N - y - x < a$, and the contribution of such terms to $\Psi_{\text{odd}}(N)$ is null. So we only consider $N - y \geq a$. This gives $a < \frac{N}{2}$ and thus

$$\Psi_{\text{odd}}(N) = \sum_{a < \frac{N}{2}} \sum_{a < y \leq N - a \atop 0 < x \leq a \atop xy \equiv 1 \pmod{a}} \left[ \frac{N - y - x}{a} \right].$$

From

$$\left[ \frac{N - y - x}{a} \right] \leq \frac{N - y - x}{a} < \frac{N - a}{a} = \frac{N}{a} - 1$$

it follows that

$$\left[ \frac{N - y - x}{a} \right] \leq \left[ \frac{N}{a} \right] - 1.$$

The set of points $(x, y) \in (0, a] \times (a, N - a)$ for which $\left[ \frac{N-x-y}{a} \right] = i$ coincides with

$$\Omega_{N,a,i} := \{(x, y) \in (0, a] \times (a, N - a) : N - (i + 1)a < x + y \leq N - ia\}.$$

We can thus write

$$\Psi_{\text{odd}}(N) = \sum_{a < \frac{N}{2}} \sum_{i=1}^{\left[ \frac{N}{a} \right] - 1} i \mathcal{N}_a(\Omega_{N,a,i}) \quad \text{(4.1)}$$

$$= \sum_{a < \frac{N}{2}} \sum_{j=1}^{\left[ \frac{N}{a} \right] - 1} (j \mathcal{N}_a(\Omega_{N,a,j}^{(1)}) + (j - 1) \mathcal{N}_a(\Omega_{N,a,j}^{(2)})},$$

where the sets

$$\Omega_{N,a,j}^{(1)} = \{(x, y) \in (0, a] \times (N - (j + 1)a, N - ja], x + y \leq N - ja\},$$

$$\Omega_{N,a,j}^{(2)} = \{(x, y) \in (0, a] \times (N - (j + 1)a, N - ja], x + y > N - ja\}, \quad 1 \leq j \leq \left[ \frac{N}{a} \right] - 2,$$

$$\Omega_{N,a,\left[\frac{N}{a}\right]-1}^{(1)} = \{(x, y) : 0 < x \leq N - \left[ \frac{N}{a} \right] a, a < y \leq N - (\left[ \frac{N}{a} \right] - 1)a, x + y \leq N - (\left[ \frac{N}{a} \right] - 1)a \},$$

$$\Omega_{N,a,\left[\frac{N}{a}\right]-1}^{(2)} = \{(x, y) : 0 < x \leq a, a < y \leq N - (\left[ \frac{N}{a} \right] - 1)a, x + y > N - (\left[ \frac{N}{a} \right] - 1)a \},$$
give a partition of the trapezoid \( \{(x, y) : 0 < x \leq a < y \leq N - a - x\} \). Since \( \mathcal{N}_a(\Omega) \) does not change when \( \Omega \) is translated by integer multiples of \((a, 0)\), the right-hand side in (4.1) can also be expressed as

\[
\sum_{j=1}^{\left\lceil \frac{N}{a} \right\rceil - 2} \mathcal{N}_a(\mathcal{I}_{N,a,j}) + \mathcal{N}_a(\mathcal{I}_{N,a,\left\lceil \frac{N}{a} \right\rceil - 1}),
\]

where for \(1 \leq j \leq \left\lceil \frac{N}{a} \right\rceil - 2\) we set

\[
\mathcal{I}_{N,a,j} = \{(x, y) : 0 < x \leq N - 2a, a < y \leq N - (\left\lceil \frac{N}{a} \right\rceil - 1)a, x + y \leq N - a\},
\]

and for \(j = \left\lceil \frac{N}{a} \right\rceil - 1\) we set

\[
\mathcal{I}_{N,a,\left\lceil \frac{N}{a} \right\rceil - 1} = \{(x, y) : 0 < x \leq N - 2a, a < y \leq N - \left(\left\lceil \frac{N}{a} \right\rceil - 1\right)a, x + y \leq N - a\}.
\]

The sets \(\mathcal{I}_{N,a,j}, 1 \leq j \leq \left\lceil \frac{N}{a} \right\rceil - 1\), are mutually disjoint and their union is the triangle

\[
\mathcal{I}_{N,a} = \{(x, y) : 0 < x \leq N - 2a, a < y \leq N - a - x\},
\]

and thus we get

\[
\Psi_{\text{odd}}(N) = \sum_{a \leq N} \mathcal{N}_a(\mathcal{I}_{N,a}).
\]

Since \(\mathcal{N}_a(\Omega)\) is invariant under translations by \((0, a)\), this further gives

\[
\Psi_{\text{odd}}(N) = \sum_{a \leq N} \mathcal{N}_a(\widetilde{\mathcal{I}}_{N,a}),
\]

(4.2)

where

\[
\widetilde{\mathcal{I}}_{N,a} = \{(x, y) : 0 < x \leq N - 2a, 0 < y \leq N - 2a - x\}
\]

is the translated triangle \(\mathcal{I}_{N,a} - (0, a)\).

**Lemma 4.1.** For every \(0 < c < 1\) we have

\[
\sum_{a \leq N^c} \mathcal{N}_a(\widetilde{\mathcal{I}}_{N,a}) = \sum_{a \leq N^c} \frac{\varphi(a)}{a^2} \cdot \frac{(N - 2a)^2}{2} + O(N^{1+c}).
\]

**Proof.** Set \(K = \left\lceil \frac{N}{a} \right\rceil - 2 \geq 1\). We partition the triangle \(\widetilde{\mathcal{I}}_{N,a}\) as \(\mathcal{I}_{N,a} \cup \mathcal{R}_{N,a}\), with

\[
\mathcal{I}_{N,a} = \bigcup_{i=1}^{K} (0, ia] \times ((K - i)a, (K - i + 1)a]
\]

and for \(j = \left\lceil \frac{N}{a} \right\rceil - 1\) we set

\[
\mathcal{I}_{N,a,\left\lceil \frac{N}{a} \right\rceil - 1} = \{(x, y) : 0 < x \leq N - 2a, a < y \leq N - \left(\left\lceil \frac{N}{a} \right\rceil - 1\right)a, x + y \leq N - a\}.
\]
and $\mathcal{R}_{N,a} = \mathcal{F}_{N,a} \setminus \mathcal{D}_{N,a}$. As $\mathcal{D}_{N,a}$ is the union of $1 + 2 + \cdots + K = \frac{K(K+1)}{2}$ disjoint squares of size $a$, we have

$$N_a(\mathcal{D}_{N,a}) = \frac{K(K+1)}{2} \frac{\varphi(a)}{a^2} \text{ Area} \mathcal{D}_{N,a}. \quad (4.3)$$

On the other hand it is clear that

$$N_a(\mathcal{R}_{N,a}) \leq (K+1) \frac{\varphi(a)}{a} \leq \frac{N}{a} \cdot a = N,$$

which gives in turn

$$\sum_{a \leq N^c} N_a(\mathcal{R}_{N,a}) \leq N^{1+c}. \quad (4.4)$$

We also have

$$\sum_{a \leq N^c} \frac{\varphi(a)}{a^2} \text{ Area} \mathcal{R}_{N,a} \leq \sum_{a \leq N^c} \frac{1}{a} \text{ Area} \mathcal{R}_{N,a} \leq \sum_{a \leq N^c} \frac{1}{a} \cdot (K+1) \cdot \frac{a^2}{2} \leq \sum_{a \leq N^c} \frac{N}{a} \cdot a \leq N^{1+c}. \quad (4.5)$$

Employing (4.4), (4.3) and (4.5), we gather

$$\sum_{a \leq N^c} N_a(\mathcal{F}_{N,a}) = \sum_{a \leq N^c} N_a(\mathcal{D}_{N,a}) + O(N^{1+c}) = \sum_{a \leq N^c} \frac{\varphi(a)}{a^2} \cdot \text{ Area} \mathcal{D}_{N,a} + O(N^{1+c})$$

$$= \sum_{a \leq N^c} \frac{\varphi(a)}{a^2} \left( \text{ Area} \mathcal{F}_{N,a} - \text{ Area} \mathcal{D}_{N,a} \right) + O(N^{1+c})$$

$$= \sum_{a \leq N^c} \frac{\varphi(a)}{a^2} \cdot \text{ Area} \mathcal{F}_{N,a} + O(N^{1+c})$$

$$= \sum_{a \leq N^c} \frac{\varphi(a)}{a^2} \cdot \frac{(N-2a)^2}{2} + O(N^{1+c}). \quad \square$$

Lemma 4.2. For every $0 < c < 1$ and every integer $T > 1$ we have

$$\sum_{N^c < a < N^c T} N_a(\mathcal{F}_{N,a}) = \sum_{N^c < a < N^c T} \frac{\varphi(a)}{a^2} \cdot \frac{(N-2a)^2}{2} + O\varepsilon \left( \frac{N^2 \log N}{T} + T N^{\frac{1}{2}+\varepsilon} + N^{2+\varepsilon} \right).$$

Proof. Applying Lemma 3.1 we get

$$N_a(\mathcal{F}_{N,a}) = \frac{\varphi(a)}{a^2} \cdot \text{ Area} \mathcal{F}_{N,a} + O\varepsilon \left( \frac{N}{Ta} \cdot N + Ta^{\frac{1}{2}+\varepsilon} \right) \cdot \frac{N}{a}$$

$$= \frac{\varphi(a)}{a^2} \cdot \frac{(N-2a)^2}{2} + O\varepsilon \left( \frac{N^2}{Ta} + T N a^{-\frac{1}{2}+\varepsilon} + N^2 a^{-\frac{3}{2}+\varepsilon} \right),$$
which is summed in the range \( a \in (N^c, \frac{N}{2}) \) to get
\[
\sum_{N^c < a < \frac{N}{2}} \frac{\varphi(a)}{a^2} \cdot \frac{(N - 2a)^2}{2} + O\varepsilon \left( \frac{N^2 \log N}{T} + TN \sum_{a=1}^{N} a^{-\frac{1}{2} + \varepsilon} + N^2 \sum_{a > N^c} a^{-\frac{3}{2} + \varepsilon} \right)
\]
\[\sum_{N^c < a < \frac{N}{2}} \frac{\varphi(a)}{a^2} \cdot \frac{(N - 2a)^2}{2} + O\varepsilon \left( \frac{N^2 \log N}{T} + TN^{\frac{3}{2} + \varepsilon} + N^2(\varepsilon - \frac{1}{2}) \right). \quad \square
\]

Taking \( T = \lfloor N^{\frac{1}{4}} \rfloor \) and (any) \( c \in \left[ \frac{1}{2}, \frac{3}{4} \right] \), the previous two lemmas, together with (4.2), yield

**Corollary 4.3.** \( \Psi_{\text{odd}}(N) = \sum_{a < \frac{N}{2}} \frac{\varphi(a)}{a^2} \cdot \frac{(N - 2a)^2}{2} + O\varepsilon(N^\frac{7}{4} + \varepsilon). \)

**Lemma 4.4.** \( S_N := \sum_{a < \frac{N}{2}} \frac{\varphi(a)(N - 2a)^2}{2a^2} = C_N + O(N), \) where
\[
C_N = \frac{N^2}{2\zeta(2)} \left( \log N + \gamma - \log 2 - \frac{3}{2} - \zeta'(2) \frac{\zeta(2)}{\zeta(2)} \right).
\]

**Proof.** Employing the Dirichlet series
\[
\sum_{a=1}^{\infty} \frac{\varphi(a)}{a^s} = \frac{\zeta(s-1)}{\zeta(s)}, \quad s = \sigma + it, \quad \sigma > 2,
\]
and the Perron integral formula \( (\sigma_0 > 0) \)
\[
\frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \frac{y^s}{s(s+1)(s+2)} ds = \begin{cases} 0 & \text{if } 0 \leq y \leq 1, \\ \frac{1}{2} (1 - \frac{1}{y})^2 & \text{if } y \geq 1, \end{cases}
\]
with \( y = \frac{N}{2a} \) we infer that
\[
S_N = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} g(s) \, ds,
\]
with
\[
g(s) = \frac{N^{s+2}}{2^s(s+1)(s+2)\zeta(s+2)} \frac{\zeta(s+1)}{s} = \frac{N^{s+2}}{2^s(s+1)(s+2)\zeta(s+2)} \left( \frac{1}{s^2} + \frac{\gamma}{s} + O(1) \right) \quad (s \to 0).
\]
In the region \( \Re s > -2 \) the function \( g \) is meromorphic with a removable singularity at \( s = -1 \) and a pole \( C_N = h'(0) \) at \( s = 0 \), where
\[
h(s) = \frac{N^{s+2}(1 + \gamma s)}{2^s(s+1)(s+2)\zeta(s+2)}.
\]
A direct calculation gives
\[
C_N = h'(0) = \frac{N^2}{2\zeta(2)} \left( \log N + \gamma - \log 2 \right) - \frac{3}{2} \frac{\zeta'(2)}{\zeta(2)}.
\]

We seek to change the contour of integration from \( \sigma = \sigma_0 \) to the contour \( \Gamma \) consisting of the five line segments \( s = \sigma_0 \pm it \) \((t \geq T)\), \( s = \sigma \pm iT \) \((-1 \leq \sigma \leq \sigma_0)\), \( s = -1 + it \) \((-T \leq t \leq T)\), getting
\[
\frac{1}{2\pi i} \int_{\sigma_0-i\infty}^{\sigma_0+i\infty} g(s) \, ds = C_N + \frac{1}{2\pi i} \int_{\Gamma} g(s) \, ds.
\] (4.6)

It remains to show that the contribution of the integral on \( \Gamma \) is small. Note first that \( |\zeta(\Sigma)| = |\zeta(s)| \) gives \( |g(\Sigma)| = |g(s)| \). As a result only the case \( \Re s \geq 0 \) will be considered next. Using standard estimates on \( \zeta \) (cf., e.g., [3],[7],[8]) we have
\[
\int_T^\infty |g(\sigma + it)| \, dt \ll \sigma_0 \int_T^\infty \frac{N^{2+\sigma_0} \log t}{2\sigma_0 t^3} \, dt \ll \sigma_0 \varepsilon \mathcal{N}^{2+\sigma_0} T^{2-\varepsilon}.
\] (4.7)

and
\[
\int_{-1}^{\sigma_0} |g(\sigma + iT)| \, d\sigma \ll \int_{-1}^{\sigma_0} \frac{N^{2+\sigma_0}}{T^3} \cdot |\zeta(1+\sigma + iT)| \cdot \frac{1}{|\zeta(2+\sigma + iT)|} \, d\sigma \ll \sigma_0 \int_{-1}^{\sigma_0} \frac{N^{2+\sigma_0}}{T^3} \cdot (T^{1/2} \log T) \cdot \log T \ll \sigma_0 \varepsilon \mathcal{N}^{2+\sigma_0} T^{5/2-\varepsilon}. \] (4.8)

To estimate the contribution of the integrand on the segment \(-1 + it \) \((0 < t \leq T)\) we follow closely the argument from [6], pp. 216-217. The functional equation
\[
\frac{\zeta(it)}{\zeta(1+it)} = \chi(it) \cdot \frac{\zeta(1-it)}{\zeta(1+it)}, \quad \chi(s) = \frac{(2\pi)^s}{2\Gamma(s) \cos \frac{\pi s}{2}},
\]
and the equality
\[
|\Gamma(it)|^2 = \frac{\pi}{\text{tanh } \pi t}
\]
yield
\[
\left| \frac{\zeta(it)}{\zeta(1+it)} \right| = |\chi(it)| = \frac{1}{2\sqrt{\frac{\pi}{\text{tanh } \pi t}} \cdot \cosh \frac{\pi t}{2}} \cdot \sqrt{\frac{t \tanh \frac{\pi t}{2}}{2\pi}}.
\]

Employing also \( \tanh t \leq t \) \((t \geq 0)\), we get (independently of \( T \geq \frac{2}{\pi} \))
\[
\int_0^T |g(-1+it)| \, dt \ll \frac{N}{2} \int_0^T \frac{\zeta(it)}{\zeta(1+it)} \frac{dt}{t(1+t^2)} \ll \frac{N}{2} \int_0^T \frac{dt}{t(1+t^2)} \ll N.
\] (4.9)

The estimates (4.6)-(4.9) with, say, \( T = N^2 \) conclude the proof. \( \square \)

Note also
\[
\text{Corollary 4.5.} \quad \sum_{a \leq N} \frac{\varphi(a)}{a^2} = \frac{1}{\zeta(2)} \left( \log N + \gamma - \frac{\zeta'(2)}{\zeta(2)} \right) + O \left( \frac{\log N}{N} \right).
\]
Proof. This is a consequence of Lemma 4.4 and of the well known formulas

\[ \sum_{a < N} \varphi(a) = \frac{N^2}{2\zeta(2)} + O(N\log N), \quad \sum_{a < N} \frac{\varphi(a)}{a} = \frac{N}{\zeta(2)} + O(\log N). \]

\[ \square \]

Theorem 1.1 now follows from (2.6), Proposition 1.2 and Lemma 4.4.

Remark 4.6. Numerical computations show that the error \( O(N) \) given by Lemma 4.4 on

\[ S_N - C_N = \sum_{a < N} \frac{\varphi(a)(N - 2a)^2}{2a^2} - \frac{N^2}{2\zeta(2)} \left( \log N + \gamma - \frac{3}{2} - \log 2 - \frac{\zeta'(2)}{\zeta(2)} \right) \]

may not be optimal. Moreover, the graph of \( S_N - C_N \) exhibits a surprising regularity (cf. Figure 1). This was brought to our attention by the referee, who also kindly provided the Mathematica notebook. One could hope to improve the theoretical estimate of the error by shifting the segment \( s = -1 + it \) further left to the line \( \Re s = -1 - \delta \). The problem however is that the argument \( 2 + s \) will enter the critical strip \( 0 < \Re s < 1 \) where lower bound estimates for \( \zeta \) are problematic.

5. An application to the distribution of quadratic irrationals

Let \( x \) be an irrational number in \( (0, 1) \) with continued fraction \([a_0(x), a_1(x), \ldots]\) and \( \frac{p_n(x)}{q_n(x)} \) be its \( n^{th} \) convergent. A classical result of P. Lévy states for almost every \( x \) that

\[ \lim_{n \to \infty} \frac{1}{n} \log q_n(x) = \frac{\pi^2}{12 \log 2}. \]
The Gauss map $T : [0, 1) \rightarrow [0, 1)$ defined as $T(0) = 0$ and $T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$ if $x \neq 0$, has the well-known properties

$$T([a_1, a_2, \ldots]) = [a_2, a_3, \ldots],$$

$$xT(x) \cdots T^{n-1}(x) = (-1)^n(xq_n(x) - p_n(x)).$$

When $\omega$ is a quadratic irrational, it is well-known that the limit $\beta(\omega)$ of $\frac{1}{n} \log q_n(\omega)$ exists, and is called the Lévy constant of $\omega$. Let $AX^2 + BX + C$ be the minimal integer polynomial of $\omega$ and $\Delta = B^2 - 4AC$. The length of $\omega$ is defined as $\rho(\omega) = 2 \log \varepsilon_0(\omega)$, where $\varepsilon_0(\omega) = \frac{1}{2}(u_0 + v_0\sqrt{\Delta})$ is the fundamental solution of the Pell equation $u^2 - \Delta v^2 = 4$.

We are interested in the set $R$ of all purely periodic quadratic irrationals, aiming to evaluate

$$\pi_0(X) = \sum_{\omega \in R, \rho(\omega) < X} 1 \quad (X \rightarrow \infty).$$

Following [9], one defines for each such $\omega = [a_1, \ldots, a_n]$ with $n = \text{per}(\omega)$ the quantities

$$\text{eper}(\omega) = \begin{cases} n, & \text{if } n = \text{per}(\omega) \text{ even}, \\ 2n, & \text{if } n = \text{per}(\omega) \text{ odd}, \end{cases}$$

$$M(\omega) = M(a_1) \cdots M(a_n),$$

$$\tilde{M}(\omega) = \Omega^n = \begin{cases} M(\omega), & \text{if } n \text{ even}, \\ M(\omega)^2, & \text{if } n \text{ odd}. \end{cases}$$

According to [4], Proposition 2.2, we have

$$\varepsilon_0(\omega) = \omega T(\omega)T^2(\omega) \cdots T^{\text{eper}(\omega)-1}(\omega),$$

$$\rho(\omega) = 2 \log \varepsilon_0(\omega).$$
Writing $\Delta = f^2 \Delta_0$ for some fundamental discriminant $\Delta_0$ and some positive integer $f$, one considers the group of units of $O_\Delta$

$$E_\Delta = \left\{ \frac{u + v\sqrt{\Delta}}{2} : (u,v) \in \mathbb{Z}^2, \ u^2 - \Delta v^2 = \pm 4 \right\}$$

with fundamental unit $\varepsilon_\Delta > 1$, in the quadratic field $K = \mathbb{Q}(\sqrt{\Delta_0})$ endowed with $\mathbb{Q}$-valued norm $\mathcal{N}$ and trace $\text{tr}$. One also considers the subgroup

$$E_\Delta^+ = \left\{ \frac{u + v\sqrt{\Delta}}{2} : (u,v) \in \mathbb{Z}^2, \ u^2 - \Delta v^2 = 4 \right\}$$

of the totally positive units, which is generated by

$$\varepsilon^+_\Delta = \begin{cases} 
\varepsilon_\Delta & \text{if } \mathcal{N}(\varepsilon_\Delta) = +1, \\
\varepsilon^2_\Delta & \text{if } \mathcal{N}(\varepsilon_\Delta) = -1.
\end{cases}$$

In [9], Section 2, the explicit isomorphism

$$\lambda_\omega : F_\omega \to E_\Delta, \quad \lambda_\omega \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = c\omega + d$$

between the fixed point group

$$F_\omega = \{ g \in G = GL_2(\mathbb{Z})/\pm I : g\omega = \omega \}$$

and $E_\Delta$ was studied. The inverse of $\lambda_\omega$ acts as

$$\lambda^{-1}_\omega \left( \frac{u + v\sqrt{\Delta}}{2} \right) = \begin{bmatrix} \frac{u-Bv}{2} & -Cv \\ Av & \frac{u+Bv}{2} \end{bmatrix}.$$ 

It sends $\varepsilon^+_\Delta = Q\ell\omega + Q\ell^{-1}$ with $\ell = \text{eper}(\omega)$, to $\Omega^+ = \widetilde{M}(\omega)$. Moreover, one has

$$\mathcal{N} \circ \lambda_\omega = \det \quad \text{and} \quad \text{tr} \circ \lambda_\omega = \text{Tr}.$$ 

This implies in particular that $\varepsilon_0(\omega)$ coincides with the spectral radius $R(\widetilde{M}(\omega))$ of $\widetilde{M}(\omega)$, thus

$$\rho(\omega) = 2 \log R(\widetilde{M}(\omega)).$$

Denote by $\mathcal{R}(\Delta)$ the set of reduced quadratic irrationals of discriminant $\Delta$ and consider the set

$$T(N) = \left\{ (k,\omega) : \omega \in \mathcal{R} = \bigcup_{\Delta > 0} \mathcal{R}(\Delta), \ \text{tr}\left( \varepsilon^+_\Delta(\omega)^k \right) \leq N \right\}. $$
As seen in Section 2, the latter has cardinality $\Psi_{ev}(N)$. As shown in [9], Proposition 4.3, the map given by

$$j(a_1, \ldots, a_{2m}) = \left(\frac{2m}{\text{eper}(\omega)}, \omega\right), \quad \text{where} \quad \omega = [a_1, \ldots, a_{2m}],$$

defines a one-to-one correspondence between $\mathcal{W}_{ev}(N)$ and $T(N)$. Denote

$$r(N) = \sum_{\omega \in \mathcal{R}^{+}(\omega) \not< N} 1 = \pi_0(2 \log N).$$

Then the identification between $\mathcal{W}_{ev}(N)$ and $T(N)$ plainly implies as in the proof of [9], Proposition 4.5, the inequalities

$$\sum_{1 \leq k \leq 2 \log N} \frac{r\left(\left(N - \frac{1}{2}\right)^{\frac{1}{k}}\right)}{\Psi_{ev}(N)} < \sum_{1 \leq k < 2 \log N} r\left(N^{\frac{1}{k}}\right). \quad (5.1)$$

From the first inequality we infer

$$r(N) < \Psi_{ev}(N + 1) \ll N^2. \quad (5.2)$$

From (5.1), (5.2), and Proposition 1.2 we derive

$$\Psi_{ev}(N) < \sum_{1 \leq k < 2 \log N} r\left(N^{\frac{1}{k}}\right) < \sum_{1 \leq k < 2 \log N} \Psi_{ev}\left((N + 1)^k\right)$$

$$= \Psi_{ev}(N + 1) + O\left(\sum_{2 \leq k < 2 \log N} N^{\frac{1}{k}}\right) \quad (5.3)$$

$$= \Psi_{ev}(N + 1) + O(N \log N)$$

$$= \frac{N^2 \log 2}{2\zeta(2)} + O_{\varepsilon}(N^{\frac{3}{2}+\varepsilon}).$$

The estimate

$$r(N) = \frac{N^2 \log 2}{2\zeta(2)} + O_{\varepsilon}(N^{\frac{3}{2}+\varepsilon})$$
follows now immediately from (5.2) and (5.3). This completes the proof of Proposition 1.3 by taking $N = e^{\frac{X}{2}}$.

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Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana IL 61801, USA

AND

Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, RO-014700 Bucharest, Romania

E-mail: fboca@math.uiuc.edu