Z_2 \times Z_2\text{-symmetric spaces}

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Abstract

The notion of a Γ-symmetric space is a generalization of the classical notion of a symmetric space, where a general finite abelian group Γ replaces the group $Z_2$. The case $Γ = Z_k$ has also been studied, from the algebraic point of view by V. Kac [14] and from the point of view of the differential geometry by Ledger, Obata [17], Kowalski [16] or Wolf-Gray [20] in terms of $k$-symmetric spaces. In this case, a $k$-manifold is an homogeneous reductive space and the classification of these varieties is given by the corresponding classification of graded Lie algebras. The general notion of a Γ-symmetric space was introduced by R. Lutz in [18]. We approach the classification of such spaces in the case $Γ = Z_2 \times Z_2$ using recent results (see [2]) on the classification of complex $Z_2 \times Z_2$-graded simple Lie algebras.

1 Introduction

A symmetric space is a homogeneous space $M = G/H$ where $G$ is a connected Lie group with an involutive automorphism $σ$ and $H$ a closed subgroup which lies between the subgroup of all fixed points of $σ$ and its identity component. This automorphism $σ$ induces an involutive diffeomorphism $σ_0$ of $M$ such that $σ_0(π(x)) = π(σ(x))$ for every $x \in G$ where $π : G \to G/H$ is the canonical projection. It also induces an automorphism $γ$ on the Lie algebra $g$ of $G$. This automorphism satisfies $γ^2 = Id$, hence is diagonalizable and the Lie algebra $g$ of $G$ admits a $Z_2$-grading, $g = g_0 \oplus g_1$ where $g_0$ and $g_1$ are the eigenspaces of $σ$ corresponding to the eigenvalues 1 and $-1$. Conversely, every $Z_2$-grading $g = g_0 \oplus g_1$ on a Lie algebra $g$ makes it into a symmetric Lie algebra, that is a triple $(g, g_0, g_1)$ where $γ$ is an involutive automorphism of $g$ such that $γ(X) = X$ if and only if $X \in g_0$ and $γ(X) = -X$ for all $X \in g_1$. If $G$ is a connected simply connected Lie group with Lie algebra $g$, then $γ$ induces an automorphism $σ$ of $G$ and for any subgroup $H$ lying between $G^σ = \{ x \in G, σ(x) = x \}$ and the identity component of $G^σ$, $(G/H, σ)$ is a symmetric space. In the Riemannian case, $H$ is compact and $g$ admits an orthogonal symmetric decomposition, that is the Lie group of linear transformations of $g$ generated by $ad_H$ is compact. As a result, the study is reduced to the effective irreducible case and $g$ is semi-simple.

In this paper we will look at more general Γ-symmetric homogeneous spaces. They were first introduced by R. Lutz in [18], and A. Tsagas at a workshop in Bucharest. We propose here to develop the corresponding algebraic structures and to give, using the results on complex simple Lie algebras graded by finite abelian groups [2], [5], [4], (see also [11], [12]), the algebraic classification of $Z_2 \times Z_2$-symmetric spaces $G/H$ whose associated Lie algebra $g$ is classical simple.
2 Group graded Lie algebras

2.1 Definition

Definition 1 Let \( P \) be a group with identity element \( 1 \). A Lie algebra \( \mathfrak{g} \) over a field \( F \) is graded by \( P \) if
\[
\mathfrak{g} = \bigoplus_{p \in P} \mathfrak{g}_p
\]
with
\[
[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{pq}
\]
for all \( p, q \in P \).

Definition 2 Given two \( P \)-gradings
\[
\mathfrak{g} = \bigoplus_{p \in P} \mathfrak{g}_p \text{ and } \mathfrak{g}' = \bigoplus_{p \in P} \mathfrak{g}'_p
\]
of an algebra \( \mathfrak{g} \) by a group \( P \) we call them equivalent if there exists an automorphism \( \theta \) of \( \mathfrak{g} \) such that \( \mathfrak{g}'_p = \theta(\mathfrak{g}_p) \), for all \( p \in P \).

An important subset of the grading group is defined by the following.

Definition 3 Given a grading as above, the set
\[
\text{Supp } \mathfrak{g} = \{ p \in P \mid \mathfrak{g}_p \neq \{0\} \}
\]
is called the support of the grading.

It has been established in \([19]\) (see also \([4]\)) that if \( \mathfrak{g} \) is complex simple then any two elements in the support of the grading commute. So one can always restrict oneself to the case of abelian groups. In this paper we restrict ourselves to finite abelian grading groups \( P \).

2.2 Action of the dual group

Let \( \Gamma = \hat{P} \) be the dual group associated to \( P \), that is, the group of characters
\[
\gamma : P \rightarrow \mathbb{C}^*
\]
of \( P \). If we assume that a Lie algebra \( \mathfrak{g} \) is \( P \)-graded then we obtain a natural action of \( \Gamma \) by linear transformations on \( \mathfrak{g} \otimes \mathbb{C} \) if for any homogeneous elements \( X \in \mathfrak{g}_p \) we set \( \gamma(X) = \gamma(p)X \). Since for \( X \in \mathfrak{g}_p \) and \( Y \in \mathfrak{g}_q \) we have \([X,Y] \in \mathfrak{g}_{pq}\), it follows that
\[
\gamma([X,Y]) = \gamma(pq)[X,Y] = [\gamma(p)X, \gamma(q)Y] = [\gamma(X), \gamma(Y)],
\]
that is, \( \Gamma \) acts by Lie automorphisms on \( \mathfrak{g} \). In this case there is a canonical homomorphism
\[
\alpha : \Gamma \rightarrow \text{Aut}(\mathfrak{g} \otimes \mathbb{C}) \text{ given by } \alpha(\gamma)(X) = \gamma(X).
\]
If for any \( p \in \text{Supp } \mathfrak{g} \) we have \( p^2 = 1 \) then the action is defined even on \( \mathfrak{g} \) itself and the above homomorphism maps \( \Gamma \) onto a subgroup of \( \text{Aut } \mathfrak{g} \).

Conversely, suppose there is a homomorphism \( \alpha : \Gamma \rightarrow \text{Aut } \mathfrak{g} \), for a finite abelian group \( \Gamma \). Then \( \Gamma \) acts on \( \mathfrak{g} \), hence on \( \mathfrak{g} \otimes \mathbb{C} \) by automorphisms if one sets \( \gamma(X) = \alpha(\gamma)(X) \). This
action extends to $\mathfrak{g} \otimes \mathbb{C}$ and yields a $P$-grading, $\Gamma = \widehat{P}$, of $\mathfrak{g} \otimes \mathbb{C}$ by subspaces $(\mathfrak{g} \otimes \mathbb{C})_p$, for each $p \in P$, defined as follows:

$$(\mathfrak{g} \otimes \mathbb{C})_p = \{ X \mid \gamma(X) = \gamma(p)X \}.$$ 

That $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{pq}$ easily follows from (1). Now the vector space decomposition $\mathfrak{g} = \bigoplus_{p \in P} \mathfrak{g}_p$ is just a standard weight decomposition under the action of an abelian semisimple group of linear transformations over an algebraically closed field. Again, if we have $\alpha(\gamma)^2 = 1$ for any $\gamma \in \Gamma$, then we have a $P$-grading on $\mathfrak{g}$ itself.

Explicitly, let us assume that $\mathfrak{g}$ is a complex Lie algebra and $K$ a finite abelian subgroup of $\text{Aut}(\mathfrak{g})$. One can write $K$ as $K = K_1 \times \ldots \times K_p$, where $K_i$ is a cyclic group of order $r_i$. Let $\kappa_i$ be a generator of $K_i$. The automorphisms $\kappa_i$ satisfy:

$$\begin{cases} 
\kappa_i^r = \text{Id}, \\
\kappa_i \circ \kappa_j = \kappa_j \circ \kappa_i,
\end{cases}$$

for all $i, j = 1, \ldots, p$. These automorphisms are simultaneously diagonalizable. If $\xi_i$ is a primitive root of order $r_i$ of the unity, then the eigenspaces

$$\mathfrak{g}_{s_1, \ldots, s_p} = \{ X \in \mathfrak{g} \text{ such that } \kappa_i(X) = \xi_i^{s_i}X, \ i = 1, \ldots, p \}$$

give the following grading of $\mathfrak{g}$ by $P = \mathbb{Z}_{r_1} \times \ldots \times \mathbb{Z}_{r_p}$:

$$\mathfrak{g} = \bigoplus_{(s_1, \ldots, s_p) \in \mathbb{Z}_{r_1} \times \ldots \times \mathbb{Z}_{r_p}} \mathfrak{g}_{s_1, \ldots, s_p}.$$ 

We can summarize some of what was said above as follows.

**Proposition 4** Let $P$ be a finite abelian group and $\Gamma = \widehat{P}$, the group of complex characters of $P$.

(a) A complex Lie algebra $\mathfrak{g}$ is $P$-graded if and only if the dual group $\Gamma$ maps homomorphically onto a finite abelian subgroup of $\text{Aut}(\mathfrak{g})$, by a canonical homomorphism $\alpha$ described in (E).

(b) A real Lie algebra $\mathfrak{g}$ is $P$-graded, with $p^2 = 1$ for each $p \in \text{Supp} \mathfrak{g}$, if and only if there is a homomorphism $\alpha : \Gamma \to \text{Aut}(\mathfrak{g})$ such that $\alpha(\gamma)^2 = \text{id}_{\mathfrak{g}}$ for any $\gamma \in \Gamma$.

(c) In both cases above, $\text{Supp} \mathfrak{g}$ generates $P$ if and only if the canonical mapping $\alpha$ has trivial kernel, that is, $\Gamma$ is isomorphic to a (finite abelian) subgroup of $\text{Aut}(\mathfrak{g})$.

**Proof.** We need only to comment on (c). If $\Lambda \subset \Gamma$ then by $\Lambda^\perp$ we denote the set of all $p \in P$ such that $\lambda(p) = 1$ for all $\lambda \in \Lambda$. Similarly we define $Q^\perp$ for any $Q \subset P$. We have that $\Lambda^\perp$ and $Q^\perp$ are always subgroups in $P$ and $\Gamma$, respectively. If $\Lambda$ and $Q$ are subgroups then $|\Lambda| \cdot |\Lambda^\perp| = |Q| \cdot |Q^\perp| = |\Gamma| = |P|$. We claim that if $\Lambda = \text{Ker} \alpha$ then the subgroup generated by $\text{Supp} \mathfrak{g}$ is $P = \Lambda^\perp$. This follows because for any $p \notin Q$ there is $\lambda \in \Lambda$ such that $\lambda(p) \neq 1$. If $\mathfrak{g}_p \neq \{0\}$ and $0 \neq X \in \mathfrak{g}_p$ then $\lambda(X) = \lambda(p)X \neq X$ and $\lambda \notin \text{Ker} \alpha$. Conversely, let $\text{Supp} \mathfrak{g} \subset T$, where $T$ is a proper subgroup of $Q = \text{Ker} \alpha^\perp$. Then $T^\perp$ properly contains $\Lambda$ and for any $\gamma \in T^\perp \setminus \Lambda$ and any $X \in \mathfrak{g}_t$, $t \in T$, we have $\gamma(X) = \gamma(t)X = X$. Since all such $X$ span $\mathfrak{g}$, we have that $\gamma \in \text{Ker} \alpha = \Lambda$, a contradiction. Thus $\text{Supp} \mathfrak{g}$ must generate the whole of $Q$, proving (c).
2.3 Examples

1. The gradings of classical simple complex Lie algebras by finite abelian groups have been described in [1], [2], [4], and [9] (see also [11], [12]). We will use this classification in the case $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ to obtain some classification-type results in the theory of $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces.

2. In the non simple case the study of gradings is more complicated. Consider, for example, the nilpotent case. In contrast to the simple case, there is no classification of these Lie algebras, except in the dimensions up to 7, [10]. Even then one has to distinguish between two classes of complex nilpotent Lie algebras:

(a) The non characteristically nilpotent Lie algebras. These Lie algebras admit a nontrivial abelian subalgebra of the Lie algebra $\text{Der}(\mathfrak{g})$ of derivations such that the elements are semisimple. In this case $\mathfrak{g}$ is graded by the roots.

(b) The characteristically nilpotent Lie algebras. Every derivation is nilpotent and we do not have root decompositions. Nevertheless, these nilpotent Lie algebras can be graded by groups. For example, the following nilpotent Lie algebra, denoted by $(n^3_7)$ in [10] and given by

$$\begin{align*}
[X_1, X_i] &= X_{i+1}, \ i = 2, ..., 6 \\
[X_2, X_3] &= X_5 + X_7 \\
[X_2, X_4] &= X_6 \\
[X_2, X_5] &= X_7
\end{align*}$$

is characteristically nilpotent and admits the following grading

$$n^3_7 = g_0 \oplus g_1$$

where $g_0$ is the nilpotent subalgebra generated by $\{X_2, X_4, X_6\}$ and $g_1$ is the $g_0$-module generated by $\{X_1, X_5, X_5, X_7\}$. On the other hand, the nilpotent Lie algebras $n^2_7$ and $n^5_7$ do not admit nontrivial group gradings.

3 $\Gamma$-symmetric spaces

3.1 Definition

**Definition 5** Let $\Gamma$ be a finite abelian group. A homogeneous space $M = G/H$ is called $\Gamma$-symmetric if

1. The Lie group $G$ is connected

2. The group $G$ is effective on $G/H$ (i.e. the Lie algebra $\mathfrak{h}$ of $H$ does not contain a nonzero proper ideal of the Lie algebra $\mathfrak{g}$ of $G$)

3. There is an injective homomorphism

$$\rho : \Gamma \to \text{Aut } G$$

such that if $G^{\Gamma}$ is the closed subgroup of all elements of $G$ fixed by $\rho(\Gamma)$ and $(G^{\Gamma})_c$ the identity component of $G^{\Gamma}$ then

$$(G^{\Gamma})_c \subset H \subset G^{\Gamma}.$$
Obviously, in the case of $\Gamma = \mathbb{Z}_2$ we obtain ordinary symmetric spaces.

We denote by $\rho_\gamma$ the automorphism $\rho(\gamma)$ for any $\gamma \in \Gamma$. If $H$ is connected, we have

\[
\begin{align*}
\rho_{\gamma_1} \circ \rho_{\gamma_2} &= \rho_{\gamma_1 \gamma_2}, \\
\rho_\varepsilon &= \text{Id} \\
\rho_\gamma(g) &= g, \forall \gamma \in \Gamma \iff g \in H.
\end{align*}
\]

where $\varepsilon$ is the identity element of $\Gamma$. If $\Gamma = \mathbb{Z}_2$ then a $\mathbb{Z}_2$-symmetric space is a symmetric space, if $\Gamma = \mathbb{Z}_p$ we find again the $p$-manifolds in the sense of Ledger-Obata \cite{17}.

### 3.2 $\hat{\Gamma}$-grading of the Lie algebra of $G$

Let $M = G/H$ be a $\Gamma$-symmetric space. Each automorphism $\rho_\gamma$ of $G$, $\gamma \in \Gamma$, induces an automorphism of $\mathfrak{g}$, denoted by $\tau_\gamma$ and given by $\tau_\gamma = (T\rho_\gamma)_e$ where $(Tf)_x$ is the tangent map of $f$ at the point $x$.

**Lemma 6** The map $\tau : \Gamma \rightarrow \text{Aut}(\mathfrak{g})$ given by

\[\tau(\gamma) = (T\rho_\gamma)_e\]

is an injective homomorphism.

**Proof.** Let $\gamma_1, \gamma_2$ be in $\Gamma$. Then $\rho_{\gamma_1} \circ \rho_{\gamma_2} = \rho_{\gamma_1 \gamma_2}$. It follows that $(T\rho_{\gamma_1})_e \circ (T\rho_{\gamma_2})_e = (T\rho_{\gamma_1 \gamma_2})_e$, that is, $\tau(\gamma_1 \gamma_2) = \tau(\gamma_1) \tau(\gamma_2)$. Now let us assume that $\tau(\gamma) = \text{Id}_g$. Then $(T\rho_\gamma)_e = \text{Id} = (T\rho_e)_e$. But $\rho_\gamma$ is uniquely determined by the corresponding tangent automorphism of $\mathfrak{g}$. Then $\rho_\gamma = \rho_e$ and $\gamma = \varepsilon$. \qed

From this lemma we derive the following.

**Proposition 7** If $M = G/H$ is a $\Gamma$-symmetric space then the complex Lie algebra $\mathfrak{g}_C = \mathfrak{g} \otimes \mathbb{C}$ where $\mathfrak{g}$ is the Lie algebra of $G$ is $\hat{\Gamma}$-graded and if $\Gamma = \mathbb{Z}_2^k$ then the real Lie algebra $\mathfrak{g}$ of $G$ is $\hat{\Gamma}$-graded. The subgroup of $\hat{\Gamma}$ generated by the support of the grading is $\hat{\Gamma}$ itself.

**Proof.** Indeed, by Lemma 6 $\alpha : \Gamma \rightarrow \text{Aut}(\mathfrak{g})$ is an injective homomorphism, so all our claims follow by Proposition 4. \qed

For convenience, recall that if a finite abelian group $P$ is such that $\hat{\Gamma} = P$ and $\alpha$ is a canonical homomorphism introduced in (2) then the components of the grading are given by the following equation.

\[\mathfrak{g}_p = \{X \in \mathfrak{g}, | \alpha(\gamma)(X) = \gamma(p)X, \forall \gamma \in \Gamma\}. \tag{3}\]

### 3.3 $\Gamma$-symmetric spaces and graded Lie algebras

To study $\Gamma$-symmetric spaces, we need to start with the study of $P$-graded Lie algebras where $\Gamma = \hat{P}$. But in a general case if $G$ is a connected Lie group corresponding to $\mathfrak{g}$, the $P$-grading of $\mathfrak{g}$ or $\mathfrak{g}_C$ does not necessarily give a $\Gamma$-symmetric space $G/H$. Some examples are given in \cite{8}, even in the symmetric case. Still, if $G$ is simply connected, $\text{Aut}(G)$ is a Lie group isomorphic to $\text{Aut}(\mathfrak{g})$.

**Proposition 8** Let $P = \mathbb{Z}_2^k$, with the identity element $\varepsilon$, $\hat{\Gamma} = P$, and $\mathfrak{g}$ a real $P$-graded Lie algebra such that the subgroup generated by $\text{Supp} \mathfrak{g}$ equals $P$ and the identity component $\mathfrak{h} = \mathfrak{g}_\varepsilon$ of the grading does not contain a nonzero ideal of $\mathfrak{g}$. If $G$ is a connected simply connected Lie group with the Lie algebra $\mathfrak{g}$ and $H$ a Lie subgroup associated with $\mathfrak{h}$, then the homogeneous space $M = G/H$ is a $\Gamma$-symmetric space.
Proof. By Proposition 9 there is an injective homomorphism \( \alpha : \Gamma \to \text{Aut} \mathfrak{g} \) defined by this grading. The subgroup \( \alpha(\Gamma) \) of \( \text{Aut} \mathfrak{g} \) is isomorphic to \( \Gamma \). Choosing, for each \( \alpha(\gamma) \), a unique automorphism \( \rho(\gamma) \) of \( \mathfrak{g} \) such that \( (T(\rho(\gamma)))_e = \tau(\gamma) \) we obtain an injective homomorphism \( \rho : \Gamma \to \text{Aut} G \), making \( G/H \) into a \( \Gamma \)-symmetric space. \( \blacksquare \)

Motivated by Propositions 4, 7 and 8, we introduce the following.

Definition 9 Given a real or complex Lie algebra \( \mathfrak{g} \), a subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) and a finite abelian subgroup \( \Gamma \subset \text{Aut} \mathfrak{g} \), we say that \( (\mathfrak{g}/\mathfrak{h}, \Gamma) \) is a local \( \Gamma \)-symmetric space if and only if \( \mathfrak{h} = \mathfrak{g}^\Gamma \), the set of all fixed points of \( \mathfrak{g} \) under the action of \( \Gamma \). We call \( \mathfrak{h} \) the isotropy subalgebra of \( (\mathfrak{g}/\mathfrak{h}, \Gamma) \).

Any \( \Gamma \)-symmetric space gives rise to a local \( \Gamma \)-symmetric space and, in the case of connected simply connected groups, the converse is also true. If \( \Gamma = \mathbb{Z}_2^k \) then \( (\mathfrak{g}/\mathfrak{h}, \Gamma) \) is a local \( \Gamma \)-symmetric space if and only if \( \mathfrak{g} \) is \( P \)-graded, where \( P = \hat{\Gamma} \), and the isotropy subalgebra \( \mathfrak{h} \) is the identity component of the grading. If \( \Gamma \) is a more general finite abelian group then the grading by \( \Gamma = \hat{\Gamma} \) arises only on the complexification \( \mathfrak{g} \otimes \mathbb{C} \) and still \( \mathfrak{h} \otimes \mathbb{C} \) is both the set of fixed points of \( \Gamma \) and the identity component of the grading. Again, the study of local complex \( \Gamma \)-symmetric spaces amounts to the study of \( P \)-graded Lie algebras, where \( P = \hat{\Gamma} \).

3.4 \( \Gamma \)-symmetries on the homogeneous space \( M = G/H \)

Given a \( \Gamma \)-symmetric space \( (G/H, \Gamma) \) it is easy to construct, for each point \( x \) of the homogeneous space \( M = G/H \), a subgroup of the group \( \text{Diff}(M) \) of diffeomorphisms of \( M \), isomorphic to \( \Gamma \), which has \( x \) as an isolated fixed point. We denote by \( \bar{\mathfrak{g}} \) the class of \( g \in G \) in \( M \). If \( e \) is the identity of \( G \), \( \gamma \in \Gamma \), we set

\[
s(\gamma,e)(\bar{g}) = \rho_\gamma(g).
\]

If \( \bar{g} \) satisfies \( s(\gamma,e)(\bar{g}) = \bar{g} \) then \( \rho_\gamma(g) = \bar{g} \) that is \( \rho_\gamma(g) = gh_\gamma \) for \( h_\gamma \in H \). Thus \( h_\gamma = g^{-1}\rho_\gamma(g) \). But \( \Gamma \cong \hat{\Gamma} \) is a finite abelian group. If \( p_\gamma \) is the order of \( \gamma \) then \( p_\gamma p_\gamma = \text{Id} \). Then

\[
h_\gamma^2 = g^{-1}\rho_\gamma(g)\rho_\gamma(g^{-1})\rho_\gamma^2(g) = g^{-1}\rho_\gamma^2(g).
\]

Applying induction, and considering \( h^m \in H \) for any \( m \), we have

\[
(h_\gamma)^m = g^{-1}\rho_\gamma^m(g).
\]

For \( m = p_\gamma \) we obtain

\[
(h_\gamma)^{p_\gamma} = e.
\]

If \( g \) is near the identity element of \( G \), then \( h_\gamma \) is also close to the identity and \( h_\gamma^{p_\gamma} = e \) implies \( h_\gamma = e \). Then \( \rho_\gamma(g) = g \). This is true for all \( \gamma \in \Gamma \) and thus \( g \in H \). It follows that \( \bar{g} = \bar{e} \) and that the only fixed point of \( s(\gamma,e) \) is \( \bar{e} \). In conclusion, the family \( \{s(\gamma,e)\}_{\gamma \in \Gamma} \) of diffeomorphisms of \( M \) satisfy

\[
\left\{
\begin{array}{l}
 s(\gamma_1,e) \circ s(\gamma_2,e) = s(\gamma_1 \gamma_2,e) \\
 s(\gamma,e)(\bar{g}) = \bar{g}, \forall \gamma \in \Gamma \Rightarrow \bar{g} = \bar{e}.
\end{array}
\right.
\]

Thus,

\[
\Gamma_{\bar{e}} = \{s(\gamma,e), \gamma \in \Gamma\}
\]

is a finite abelian subgroup of \( \text{Diff}(M) \) isomorphic to \( \Gamma \), for which \( \bar{e} \) is an isolated fixed point.
In another point $\bar{g}_0$ of $M$ we put
\[ s(\gamma, \bar{g}_0)(\bar{g}) = g_0(s(\gamma, \bar{g}_0^{-1})(\bar{g})). \]

As above, we can see that
\[ \left\{ \begin{array}{l}
s(\gamma_1, \bar{g}_0) \circ s(\gamma_2, \bar{g}_0) = s(\gamma_1 \gamma_2, \bar{g}_0) \\
s(\gamma, \bar{g}_0)(\bar{g}) = \bar{g}, \forall \gamma \in \Gamma \Rightarrow \bar{g} = \bar{g}_0. \end{array} \right. \]

and
\[ \Gamma_{\bar{g}_0} = \{ s(\gamma, \bar{g}_0), \ \gamma \in \Gamma \} \]
is a finite abelian subgroup of $\text{Diff}(M)$ isomorphic to $\Gamma$, for which $\bar{g}_0$ is an isolated fixed point.

Thus for each $\bar{g} \in M$ we have a finite abelian subgroup $\Gamma_{\bar{g}}$ of $\text{Diff}(M)$ isomorphic to $\Gamma$, for which $\bar{g}$ is an isolated fixed point.

**Definition 10** Let $(G/H, \Gamma)$ be a $\Gamma$-symmetric space. For any point $x \in M = G/H$ the subgroup $\Gamma_x \subset \text{Diff}(M)$ is called the group of symmetries of $M$ at $x$.

Since for every $x \in M$ and $\gamma \in \Gamma$, the map $s(\gamma, x)$ is a diffeomorphism of $M$ such that $s(\gamma, x)(x) = x$, the tangent linear map $(Ts(\gamma, x))_x$ is in $\text{GL}(T_x M)$. For every $x \in M$, we obtain a linear representation
\[ S_x : \Gamma \rightarrow \text{GL}(T_x M) \]
defined by
\[ S_x(\gamma) = (Ts(\gamma, x))_x. \]

Thus for every $\gamma \in \Gamma$ the map
\[ S(\gamma) : M \rightarrow T(M) \]
defined by $S(\gamma)(x) = S_x(\gamma)$ is a $(1,1)$-tensor on $M$ which satisfies:

1. the map $S(\gamma)$ is of class $C^\infty$,
2. for every $x \in M$,
\[ \{ X_x \in T_x(M) \mid S_x(\gamma)(X_x) = X_x, \forall \gamma \in \Gamma \} = \{0\}. \]

In fact, this last remark is a consequence of the property : $s(\gamma, x)(y) = y$ for every $\gamma$ implies $y = x$.

**Definition 11** Let $M$ be a real differential manifold and $\Gamma$ a finite abelian group. We note by $T_x M$ the tangent space to $M$ at the point $x$.

A $\Gamma$-symmetric structure on $M$ is given for all $x \in M$ by a linear representation of $\Gamma$ on the vector space $T_x M$
\[ \rho_x : \Gamma \rightarrow \text{GL}(T_x M) \]
such that

1. For every $\gamma \in \Gamma$, the map $x \in M \rightarrow \rho_x(\gamma)$ is of class $C^\infty$,
2. For every $x \in M$, \( \{ X_x \in T_x(M) \mid \rho_x(\gamma)(X_x) = X_x, \forall \gamma \} = \{0\}. \)

**Proposition 12** If $(G/H, \Gamma)$ is a $\Gamma$-symmetric space, the family $\{ S_x \}_{x \in M}$ is a $\Gamma$-symmetric structure on the homogeneous space $M = G/H$. 

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3.5 Canonical connections on the homogeneous space $G/H$

Let $(G/H, \Gamma)$ be a $\Gamma$-symmetric space. As we learned, the complexified Lie algebra $g \otimes \mathbb{C}$ of $G$ is then $P$-graded, $P = \tilde{P}$, $g \otimes \mathbb{C} = \oplus_{p \in P} \ (g \otimes \mathbb{C})_p$. If $e$ is the identity element of $P$ then the component $h = (g \otimes \mathbb{C})_1$ is a Lie subalgebra of $g \otimes \mathbb{C}$ of the form $h \otimes \mathbb{C}$ where $h = g^\Gamma$, the set of fixed points of the action of $\Gamma$ on $g$ and also the Lie algebra of the subgroup $H$.

Let us consider the subspace $g'$ of $g$:

$$g' = \oplus_{p \neq 1} \ g_p.$$ 

For every $1 \neq p \in P$, if $p^2 = 1$ then $(g \otimes \mathbb{C})_p = g_p \otimes \mathbb{C}$ where $g_p$ is given by (3), and if not, then $(g \otimes \mathbb{C})_p \oplus (g \otimes \mathbb{C})_{p-1} = \bar{g}_p \otimes \mathbb{C}$ where $\bar{g}_p$ is the subspace of $g$ spanned by the real and imaginary parts of the vectors in $(g \otimes \mathbb{C})_p$.

This simple claim follows because we have $\gamma(u + vi) = \gamma(u + vi)$ where the action of $\Gamma$ on $g \otimes \mathbb{C}$ is given by $\gamma(u + vi) = \gamma(u) + \gamma(v)i$. Thus if $p^2 = 1$ and $u + vi \in (g \otimes \mathbb{C})_p$, then $\gamma(u + vi) = \gamma(p)(u - vi)$ and $\gamma(u + vi) = \gamma(u) - \gamma(v)i$. Since $\gamma(p)$ is real, $\gamma(u) = \gamma(p)(u)$ and $\gamma(v) = \gamma(p)(v)$, proving that $u, v \in g_p$ and $(g \otimes \mathbb{C})_p = g_p \otimes \mathbb{C}$. But if $p^2 \neq 1$ and again $u + vi \in (g \otimes \mathbb{C})_p$, then

$$\gamma(u + vi) = \gamma(p)(u - vi) = \gamma(p^{-1})u + vi = \gamma(u - vi)$$

showing that the complex conjugation leaves invariant $(g \otimes \mathbb{C})_p \oplus (g \otimes \mathbb{C})_{p-1}$. Then, of course, our claim follows.

If we set $m$ the sum of all $g_p$ and $\bar{g}_p$ if $p \neq 1$ then $g \otimes \mathbb{C} = g_1 \otimes \mathbb{C} \oplus m \otimes \mathbb{C}$ and $g = g_1 \oplus m$. We also have

$$[g_1, m] \subset m$$

so that $m$ is an ad $g_1$-invariant subspace. If $H$ is a connected Lie group then $[g_1, m] \subset m$ is equivalent to $\text{(ad } H)(m) \subset m$, that is, $m$ is an ad $H$-invariant subspace. This property is true without any conditions on $H$.

Lemma 13 Any $\Gamma$-symmetric space $(G/H, \Gamma)$ is reductive.

**Proof.** Let us consider the associated local $\Gamma$-symmetric space $(g/h, \Gamma)$. We need to find a decomposition $g = h \oplus m$ such that $m$ is invariant under the adjoint action of the isotropy subgroup $H$ or, which is the same, under the action of the isotropy subalgebra $h$. Now since $h \otimes \mathbb{C} = (g \otimes \mathbb{C})_1$ and $m \otimes \mathbb{C} = \oplus_{p \neq 1} (g \otimes \mathbb{C})_p$, we have that $[h \otimes \mathbb{C}, m \otimes \mathbb{C}] \subset m \otimes \mathbb{C}$. Then, of course, also $[h, m] \subset m$, and the proof is complete. 

We now deduce from [13], Chapter X, that $M = G/H$ admits two $G$-invariant canonical connections denoted by $\nabla$ and $\bar{\nabla}$. The first canonical connection, $\nabla$, satisfies

$$\nabla(X, Y) = -\text{ad} \left( [X, Y]_h \right), \quad T(X, Y) = -[X, Y]_m, \quad T(X, Y) \in m,$$

where $T$ and $R$ are the torsion and the curvature tensors of $\nabla$. The tensor $T$ is trivial if and only if $[X, Y]_m = 0$ for all $X, Y \in m$. This means that $[X, Y] \in h$ that is $[m, m] \subset h$. Then the Lie algebra $g$ is $\mathbb{Z}_2$-graded and the homogeneous space $G/H$ is symmetric. If the grading of $g$ is given by $\Gamma$ where $\Gamma$ is not isomorphic to $\mathbb{Z}_2$, then $[m, m]$ need not be a subset of $\text{span} h$ and then the torsion $T$ need not vanish. In this case another connection, $\bar{\nabla}$, will be defined if one sets $\bar{\nabla}_X Y = \nabla_X Y - T(X, Y)$. This is an affine invariant torsion free connection on $G/H$ which has the same geodesics as $\nabla$. This connection is called the second canonical
connection or the torsion-free canonical connection. For example, if $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ then these connections can be distinct, as one can see from a number of examples in Section 4.

Remark. Actually, there is another way of writing the canonical affine connection of a $\Gamma$-symmetric space, without any reference to Lie algebras. This is done by an intrinsic construction of $\Gamma$-symmetric spaces proposed by Lutz in [18].

4 Classification of local $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric complex spaces

We have seen that the classification of $\Gamma$-symmetric spaces $(G/H, \Gamma)$, when $G$ is connected and simply connected, corresponds to the classification of Lie algebras graded by $P = \hat{\Gamma}$. Below we establish the classification of local $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces $(\mathfrak{g}, \Gamma)$ in the case where the corresponding Lie algebra $\mathfrak{g}$ is simple complex and classical.

We recall Definition 2 that given two $P$-gradings $\mathfrak{g} = \bigoplus_{p \in P} \mathfrak{g}_p$ and $\mathfrak{g} = \bigoplus_{p \in P} \mathfrak{g}_p^\prime$ of an algebra $\mathfrak{g}$ by a group $P$ we call them equivalent if there exists an automorphism $\alpha$ of $\mathfrak{g}$ such that $\mathfrak{g}_p^\prime = \alpha(\mathfrak{g}_p)$. To make the classification even more compact we will use another equivalence relation on the gradings. We will call two $P$-gradings $\mathfrak{g} = \bigoplus_{p \in P} \mathfrak{g}_p$ and $\mathfrak{g} = \bigoplus_{p \in P} \mathfrak{g}_p^\prime$ of an algebra $\mathfrak{g}$ by a group $P$ weakly equivalent if there exists an automorphism $\pi$ of $\mathfrak{g}$ and an automorphism $\omega$ of $P$ such that $\mathfrak{g}_p^\prime = \pi(\mathfrak{g}_{\omega(p)})$. So the classification we are about to produce will be up to the weak equivalence.

4.1 Introductory remarks about the $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings

In this section $P = \{e, a, b, c\}$ is the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ with identity $e$ and $a^2 = b^2 = c^2 = e$, $ab = c$. We will consider the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading on a complex simple Lie algebra $\mathfrak{g}$ one if the types $A_l$, $l \geq 1$, $B_l$, $l \geq 2$, $C_l$, $l \geq 3$ and $D_l$, $l \geq 4$. We are going to use some results of [2] and [5]. That is why in the remainder of the paper we denote by $e$ the identity of the grading group $P$.

Note that in those papers we do not consider the case of $\mathfrak{so}(8)$. All fine group gradings on this algebra have been described in [9]. We are grateful to the referee for pointing out that one can use this classification to construct all $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings on $D_4$ by Proposition 2 of [8]. At the same time, we note that if one considers the $P$-gradings of $\mathfrak{g} = \mathfrak{so}(8)$ where $P$ is an elementary abelian 2-group, then each grading is equivalent to a grading induced from a grading of the matrix algebra $M_8$. This quickly follows from the description of the outer automorphisms of order 2. If we fix a canonical realization of $D_4$ in $M_8$ and a basis of the root system for $D_4$ then one of the three diagram automorphisms of order 2 can be given as the conjugation by an appropriate nonsingular matrix in $M_8$ while the others are the conjugates of this fixed one by the diagram automorphisms of order 3 (see, for example, [13 Chapter III]).

4.1.1 According to [2] any $P$-grading of a simple Lie algebra $\mathfrak{g} = \mathfrak{so}(2l+1)$, $l \geq 2$, $\mathfrak{g} = \mathfrak{so}(2l)$, $l > 4$ and $\mathfrak{sp}(2l)$, $l \geq 3$ is induced from an $P$-grading of the respective associative matrix algebra $R = M_{2l+1}$ in the first case, or $M_{2l}$ in the second and the third case. As we just explained, this is also true for $\mathfrak{so}(8)$ provided that $P = \mathbb{Z}_2 \times \mathbb{Z}_2$. Two kinds of $P$-grading on the associative matrix algebra $M_8 = R = \bigoplus_{p \in P} R_p$ are of special importance:

1) Elementary gradings. Each elementary grading is defined by an $n$-tuple $(p_1, ..., p_n)$ of elements of $P$ in such a way that all matrix units $E_{ij}$ are homogeneous with $E_{ij} \in R_p$ if and only if $p = p_i^{-1} p_j$. 

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2) Fine gradings. The characteristic property of such gradings is that for every \( p \in \text{Supp}(R) \), \( \dim R_p = 1 \) where \( \text{Supp}(R) = \{ p \in P, \dim R_p \neq 0 \} \). In the case of \( P = \mathbb{Z}_2 \times \mathbb{Z}_2 \), each fine grading is either trivial or weakly equivalent to the grading on \( R \cong M_2 \) given by the Pauli matrices

\[
X_e = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_a = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X_c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

in such a way that the graded component of degree 0 is spanned by \( X_p, p = e, a, b, c \). Notice \( [1] \) that the support of a fine grading of a simple associative algebra is always a subgroup of \( P \).

According to \( [1] \) and \( [3] \) any \( P \)-grading of \( R = M_n \) can be written as the tensor product of two graded matrix subalgebras \( A = R \otimes B \), where \( A \cong M_k \) and its grading is (equivalent to) elementary, and the grading of \( B = M_l \) is fine with \( \text{Supp} A \cap \text{Supp} B = \{ e \} \), \( kl = n \). Thus the only cases possible, when \( P = \mathbb{Z}_2 \times \mathbb{Z}_2 \), are

1) \( B = \mathbb{C} \) and \( R = A \otimes \mathbb{C} = A \)
2) \( B = M_2 \) and the grading on \( A \) is trivial.

If \( R \) is graded by \( P \) as above, then an involution \( * \) of \( R = M_n \) is called graded if \( (R_p)^* = R_p \) for any \( p \in P \). In the case of such involution, the spaces \( K(R, *) = \{ X \in R, X^* = -X \} \) of skew-symmetric elements under \( * \) and \( H(R, *) = \{ X \in R, X^* = X \} \) of symmetric elements under \( * \) are graded and the first is a simple Lie algebra of one of the types \( B, C, D \).

According to \( [1] \) and \( [3] \), any \( r \)-grading of \( R \) is trivial.

Now any involution has the form \( * : X \to X^* = \Phi^{-1}X^t\Phi \), for a nonsingular matrix \( \Phi \), which is either symmetric in the orthogonal case and skew-symmetric in the symplectic case. Since the elementary and fine components are invariant under the involution, we have that \( \Phi = \Phi_1 \otimes \Phi_2 \) where \( \Phi_1 \) defines a graded involution on \( A \) and \( \Phi_2 \) on \( B \).

First we recall the description of the graded involutions for the elementary gradings. Given an element \( p \) of a group \( P \) and a natural number \( k \) we denote by \( p^{(n)} \) the \( n \)-tuple \( p^{(n)} = (p_1 \cdots p_n) \). Using the argument of \( [5] \), one may assume that this grading is given by an \( n \)-tuple \( \nu = (p_1^{(k_1)}, p_2^{(k_2)}, \ldots, p_s^{(k_s+2t)}) \), \( n = k_1 + \ldots + k_s + 2t \), where \( p_i \in \Gamma \) are pairwise different, \( k_{s+1} = k_{s+2}, \ldots, k_{s+2t-1} = k_{s+2t} \), and there is \( p_0 \) such that \( p_0 = p_1^2 = \ldots = p_s^{2} = p_{s+1}p_{s+2} = \ldots = p_{s+2t-1}p_{s+2t} \). However, since \( p^2 = e \) for the elements of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), we have \( p_0 = e \) and if \( st \neq 0 \) we have \( p_{s+1}p_{s+2} = e \) this implies \( p_{s+1} = p_{s+2} \) which gives a contradiction. Then in this case either \( t = 0 \) and the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-elementary grading corresponds to \( e = a^2 = b^2 = c^2 \) and is given by the \( p \)-tuple

\[
\nu = (e^{(k_1)}, a^{(k_2)}, b^{(k_3)}, c^{(k_4)})
\]

or \( s = 0 \) and the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-grading corresponds to \( a = ae = bc \) with \( k_1 = k_2 \) and \( k_3 = k_4 \).

In the general case the matrix \( \Phi \) defining the involution has the form

\[
\Phi = \text{diag} \left\{ I_{k_1}, \ldots, I_{k_r}, \begin{pmatrix} 0 & I_{k_{s+1}} \\ I_{k_{s+1}} & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & I_{k_{s+2t-1}} \\ I_{k_{s+2t-1}} & 0 \end{pmatrix} \right\}
\]

if \( * \) is a transpose involution, i.e. \( \Phi \) is symmetric, or

\[
\Phi = \text{diag} \left\{ S_{k_1}, \ldots, S_{k_r}, \begin{pmatrix} 0 & I_{k_{p+1}} \\ -I_{k_{p+1}} & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & I_{k_{p+2t-1}} \\ -I_{k_{p+2t-1}} & 0 \end{pmatrix} \right\}.
\]
in the case of a skew-symmetric $\Phi$, where we denote by $I_k$ the identity matrix of order $k$ and by $S_{2t}$ the skew-symmetric matrix $\begin{pmatrix} 0 & I_t \\ -I_t & 0 \end{pmatrix}$. If $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$, when we consider the case $t = 0$, the matrices of $\Phi$ are the identity in the symmetric case and

$$\Phi = \text{diag}\{S_{k_1}, \ldots, S_{k_s}\}$$

in the skew-symmetric case and if $s = 0$ then

$$\Phi = \text{diag}\left\{ \begin{pmatrix} 0 & I_{k_1} \\ I_{k_1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_{k_3} \\ I_{k_3} & 0 \end{pmatrix} \right\}$$

if $\Phi$ is symmetric and

$$\Phi = \text{diag}\left\{ \begin{pmatrix} 0 & I_{k_1} \\ -I_{k_1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_{k_3} \\ -I_{k_3} & 0 \end{pmatrix} \right\}$$

if $\Phi$ is skew-symmetric.

If $R = A \otimes B$ and $B \neq C$, then $\text{Supp}(A) = \{e\}$. We have $\Phi = \Phi_1 \otimes \Phi_2$ and the involution on $R$ defines involutions on $A$ and $B$. It follows that $\Phi$ is symmetric if and only if either $\Phi_1$ and $\Phi_2$ are both symmetric or they both are skew-symmetric. Similarly, $\Phi$ is skew-symmetric if one of $\Phi_1, \Phi_2$ is symmetric and the other is skew-symmetric. It was proved in \cite{2} that $M_2$ with graded involution is isomorphic to $M_2$ with $\Gamma$-graded basis

$$\left\{ X_e = I_2, X_a = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, X_b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, X_c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

and the graded involution is given by one of $\Phi = X_e, X_a, X_b, X_c$.

### 4.1.2

Now let $\mathfrak{g}$ be a simple Lie algebra of type $A_l$. We view $\mathfrak{g}$ as the set $\mathfrak{g} = \text{sl}(n)$ of all matrices of trace zero in the matrix algebra $R = M_n(\mathbb{C})$ where $n = l + 1$. In this case any $P$-grading of $\mathfrak{g}$ belongs to one of the following two classes (see \cite{3}).

- For Class I gradings, any grading of $\mathfrak{g}$ is induced from a $P$-grading of $R = \bigoplus_{p \in P} R_p$ and one simply has to set $\mathfrak{g}_p = R_p$ for $p \neq e$ and $\mathfrak{g}_e = R_e \cap \mathfrak{g}$, otherwise. For $P = \mathbb{Z}_2 \times \mathbb{Z}_2$ we still have to distinguish between the cases $R = A \otimes C$ with an elementary grading on $A = M_n$ or $R = A \otimes B$ with trivial grading on $A$ and fine $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading on $B = M_2$.

- For Class II gradings, we have to fix an element $q$ of order 2 in $P$ and an involution $P$-grading $R = \bigoplus_{p \in P} R_p$. Then for any $p \in P$ one has

$$\mathfrak{g}_p = K(R_p, *) \oplus H(R_{pq}, *) \cap \mathfrak{g}.$$ 

The involution grading on $M_n$ have been discussed just before in 4.1.1. It should be noted that in the case where $B \neq C$ in $R = A \otimes B$ we have

$$(*) \quad K(R_p, *) = K(A, *) \otimes H(B_p, *) \oplus H(A, *) \otimes K(B_p, *),$$

$$(***) \quad H(R_p, *) = H(A, *) \otimes H(B_p, *) \oplus K(A, *) \otimes K(B_p, *).$$

As noted above, $\Phi = \Phi_1 \otimes \Phi_2$ with $\Phi_2 = X_p$ for $p = e, a, b, c$. If $X_p$ is symmetric with respect to $\Phi_2$ then $(*)$ and $(***)$ become

$$K(R_p, *) = K(A, *) \otimes X_p,$$
\[ H(R_p, *) = H(A, *) \otimes X_p. \]

If \( X_p \) is skew-symmetric with respect to \( \Phi_2 \) then
\[ K(R_p, *) = H(A, *) \otimes X_p, \]
\[ H(R_p, *) = K(A, *) \otimes X_p. \]

All these remarks allow to determine up to the weak equivalence the pairs \((g, g_e)\) inside the respective matrix algebra \( M_n \). This gives a local classification of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-symmetric homogeneous spaces \( G/H \) where \( G \) is simple classical connected Lie group.

### 4.2 Classification : \( B_l, C_l, D_l \) cases

We have that \( g = K(M_n, \Phi) \), \( \Phi \) being symmetric for the cases \( B_l \) and \( D_l \) and \( \Phi \) skew-symmetric for \( C_l \). In the case of \( B_l \) we have \( n = 2l + 1 \) and in the case of \( C_l \) and \( D_l \) we have \( n = 2l \).

#### 4.2.1 Lie gradings corresponding to the elementary grading of \( M_n \)

Since we are interested in the gradings only up to the weak equivalence, it is sufficient to consider the following tuples defining the elementary gradings:

\[
\begin{align*}
\nu_1 &= (e^{(k_1)}, a^{(k_2)}) \\
\nu_2 &= (e^{(k_2)}, a^{(k_3)}, b^{(k_3)}) \\
\nu_3 &= (e^{(k_1)}, a^{(k_2)}, b^{(k_3)}, c^{(k_4)}).
\end{align*}
\]

Notice that the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-gradings corresponding to \( \nu_1 \) coincide with \( \mathbb{Z}_2 \)-gradings and thus the corresponding homogeneous spaces are symmetric in the classical sense. The matrices defining the graded transpose involution in the case of \( \nu_1 \) are

\[ \Phi_1 = \text{diag} \{ I_{k_1}, I_{k_2} \} \quad \text{and} \quad \Phi'_1 = \begin{pmatrix} 0 & I_{k_1} \\ I_{k_1} & 0 \end{pmatrix}. \]

In the case of \( \nu_2 \) we have

\[ \Phi_2 = \text{diag} \{ I_{k_1}, I_{k_2}, I_{k_3} \}. \]

Finally, in the case of \( \nu_3 \) we have

\[ \Phi_3 = \text{diag} \{ I_{k_1}, I_{k_2}, I_{k_3}, I_{k_4} \} \quad \text{or} \quad \Phi'_3 = \text{diag} \left\{ \begin{pmatrix} 0 & I_{k_1} \\ I_{k_1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_{k_3} \\ I_{k_3} & 0 \end{pmatrix} \right\}. \]

If the involution is symplectic, then the respective matrices, in the case of \( \nu_1 \), are as follows:

\[ \overline{\Phi}_1 = \text{diag} \{ S_{k_1}, S_{k_2} \} \quad \text{and} \quad \overline{\Phi}_1 = \begin{pmatrix} 0 & I_{k_1} \\ -I_{k_1} & 0 \end{pmatrix}. \]

In the case of \( \nu_2 \):

\[ \overline{\Phi}_2 = \text{diag} \{ S_{k_1}, S_{k_2}, S_{k_3} \}. \]

and in the case of \( \nu_3 \):

\[ \overline{\Phi}_3 = \text{diag} \{ S_{k_1}, S_{k_2}, S_{k_3}, S_{k_4} \} \quad \text{or} \quad \overline{\Phi}_3 = \text{diag} \left\{ \begin{pmatrix} 0 & I_{k_1} \\ -I_{k_1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_{k_3} \\ -I_{k_3} & 0 \end{pmatrix} \right\}. \]

In the cases \((\nu_1, \Phi_1)\) and \((\nu_2, \overline{\Phi}_1)\) we have \( g_e = \text{so}(k_1) \oplus \text{so}(k_2) \) and \( g_e = \text{sp}(k_1) \oplus \text{sp}(k_2) \), respectively.
In the cases \((\nu_1, \Phi'_1)\) and \((\nu_3, \Phi'_3)\) we have that
\[
\mathfrak{g}_e = \left\{ \begin{pmatrix} U_1 & 0 \\ 0 & -U_1' \end{pmatrix} \right\} | U_1 \in M_{k_1}.
\]
So for both \(D_{k_1}\) and \(C_{k_1}\) cases we will have \(\mathfrak{g}_e = \mathfrak{gl}(k_1)\).

In the cases \((\nu_2, \Phi_2)\) and \((\nu_2, \overline{\Phi}_2)\) we have \(\mathfrak{g}_e = \mathfrak{so}(k_1) \oplus \mathfrak{so}(k_2) \oplus \mathfrak{so}(k_3)\) and \(\mathfrak{g}_e = \mathfrak{sp}(k_1) \oplus \mathfrak{sp}(k_2) \oplus \mathfrak{sp}(k_3)\), respectively.

In the cases \((\nu_3, \Phi_3)\) and \((\nu_3, \overline{\Phi}_3)\) we have \(\mathfrak{g}_e = \mathfrak{so}(k_1) \oplus \mathfrak{so}(k_2) \oplus \mathfrak{so}(k_3) \oplus \mathfrak{so}(k_4)\) and \(\mathfrak{g}_e = \mathfrak{sp}(k_1) \oplus \mathfrak{sp}(k_2) \oplus \mathfrak{sp}(k_3) \oplus \mathfrak{sp}(k_4)\), respectively.

In the cases \((\nu_3, \Phi'_3)\) and \((\nu_3, \overline{\Phi}'_3)\) we have that
\[
\mathfrak{g}_e = \text{diag} \left\{ \begin{pmatrix} U_1 & 0 \\ 0 & -U_1' \end{pmatrix}, \begin{pmatrix} U_2 & 0 \\ 0 & -U_2' \end{pmatrix} \right\}, U_1 \in M_{k_1} \text{ and } U_2 \in M_{k_2}.
\]
So for both \(D_{k_1+k_3}\) and \(C_{k_1+k_3}\) cases we have \(\mathfrak{g}_e = \mathfrak{gl}(k_1) \oplus \mathfrak{gl}(k_3)\).

| $\mathfrak{g}$       | $\mathfrak{g}_e$       |
|----------------------|------------------------|
| \(\mathfrak{so}(k_1 + k_2)\) | \(\mathfrak{so}(k_1) \oplus \mathfrak{so}(k_2)\) |
| \(\mathfrak{so}(k_1 + k_2 + k_3)\) | \(\mathfrak{so}(k_1) \oplus \mathfrak{so}(k_2) \oplus \mathfrak{so}(k_3)\) |
| \(\mathfrak{so}(k_1 + k_2 + k_3 + k_4)\) | \(\mathfrak{so}(k_1) \oplus \mathfrak{so}(k_2) \oplus \mathfrak{so}(k_3) \oplus \mathfrak{so}(k_4)\) |
| \(\mathfrak{sp}(k_1 + k_2)\) | \(\mathfrak{sp}(k_1) \oplus \mathfrak{sp}(k_2)\) |
| \(\mathfrak{sp}(k_1 + k_2 + k_3)\) | \(\mathfrak{sp}(k_1) \oplus \mathfrak{sp}(k_2) \oplus \mathfrak{sp}(k_3)\) |
| \(\mathfrak{sp}(k_1 + k_2 + k_3 + k_4)\) | \(\mathfrak{sp}(k_1) \oplus \mathfrak{sp}(k_2) \oplus \mathfrak{sp}(k_3) \oplus \mathfrak{sp}(k_4)\) |
| \(\mathfrak{so}(2m)\) | \(\mathfrak{gl}(m)\) |
| \(\mathfrak{so}(2(k_1 + k_2))\) | \(\mathfrak{gl}(k_1) \oplus \mathfrak{gl}(k_2)\) |
| \(\mathfrak{sp}(2m)\) | \(\mathfrak{gl}(m)\) |
| \(\mathfrak{sp}(2(k_1 + k_2))\) | \(\mathfrak{gl}(k_1) \oplus \mathfrak{gl}(k_2)\) |

*Table 1*
Conclusion. The $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces corresponding to the pairs $(g, g_e)$ considered so far are given in the Table 1.

In all the above cases the $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradings of $g$ are very easy to compute, using explicit matrices of the involutions. For example, the gradings of $\text{so}(n)$ are given in [7].

4.2.2 Gradings corresponding to $R = A \otimes B$, with $B$ nontrivial

The algebra $B$ is endowed with a fine grading given by the Pauli matrices

$$X_e = I_2, \quad X_a = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X_c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $A = M_m, \Phi = \Phi_1 \otimes \Phi_2$. We have that

$$g = K(R, \Phi) = K(A, \Phi_1) \otimes H(B, \Phi_2) \oplus H(A, \Phi_1) \otimes K(B, \Phi_2).$$

In particular, $g_e = K(R_e, \Phi_1) \otimes I_2$. If $\Phi_1$ is symmetric then $g_e = \text{so}(m)$, and if $\Phi_1$ is skew symmetric, $g_e = \text{sp}(m)$.

Conclusion. We obtain the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces corresponding to the pairs $(g, g_e)$ given in the following table:

| $g$   | $g_e$  |
|-------|--------|
| $\text{so}(2m)$ | $\text{so}(m)$ |
| $\text{so}(4m)$ | $\text{sp}(2m)$ |
| $\text{sp}(2m)$ | $\text{so}(m)$ |

Table 2

In these cases we will describe the components of the gradings explicitly.

If $g = \text{so}(2m)$ then $\Phi$ is symmetric and is of one of the form

$$\Psi_1 = I_m \otimes I_2, \quad \Psi_2 = I_m \otimes X_a, \quad \Psi_3 = I_m \otimes X_b, \quad \Psi_4 = S_m \otimes X_c.$$ 

For $\Psi_1$ we have

$$g = g(\Psi_1) = K(A, I_m) \otimes I \oplus K(A, I_m) \otimes X_a \oplus K(A, I_m) \otimes X_b \oplus H(A, I_m) \otimes X_c,$$

for $\Psi_2$ we have

$$g = g(\Psi_2) = K(A, I_m) \otimes I \oplus K(A, I_m) \otimes X_a \oplus H(A, I_m) \otimes X_b \oplus K(A, I_m) \otimes X_c,$$
for \( \Psi_3 \) we have
\[
g = g(\Psi_3) = K(A, I_m) \otimes I \oplus H(A, I_m) \otimes X_a \oplus K(A, I_m) \otimes X_b \oplus K(A, I_m) \otimes X_c,
\]
and for \( \Psi_4 \) we have
\[
g = g(\Psi_4) = K(A, S_m) \otimes I \oplus H(A, S_m) \otimes X_a \oplus H(A, S_m) \otimes X_b \oplus H(A, S_m) \otimes X_c.
\]
The conjugation by \( I \otimes \left( \begin{smallmatrix} i & 0 \\ 0 & 1 \end{smallmatrix} \right) \) maps \( g(\Psi_1) \) to \( g(\Psi_2) \) while mapping \( K(A, I_m) \otimes I \) and \( K(A, I_m) \otimes X_a \) into themselves, \( K(A, I_m) \otimes X_b \) into \( K(A, I_m) \otimes X_c \) and \( H(A, I_m) \otimes X_c \) into \( H(A, I_m) \otimes X_b \). If we apply an automorphism of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) changing places of \( b \) and \( c \) we will see that the first and the second gradings are weakly equivalent. Quite similarly, the conjugation by \( I \otimes \frac{1}{\sqrt{2}} \left( \begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix} \right) \) maps \( g(\Psi_1) \) to \( g(\Psi_3) \) while mapping \( K(A, I_m) \otimes I \) into itself, \( K(A, I_m) \otimes X_a \) into \( K(A, I_m) \otimes X_b \), \( K(A, I_m) \otimes X_b \) into \( K(A, I_m) \otimes X_c \) and \( H(A, I_m) \otimes X_c \) into \( H(A, I_m) \otimes X_a \). It remains to apply an automorphism of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) mapping \( a \) to \( c \), \( c \) to \( b \) and \( b \) to \( a \) to make sure that the first and the third gradings are weakly equivalent. Thus, all the first three gradings are weakly equivalent. None of them is weakly equivalent to the fourth one because in these cases \( g_c \cong so(m) \) while in the fourth case we have \( g_c \cong sp(m) \).

The matrix form of the first and the fourth gradings are given below.

For \( \Psi_1 \) we have
\[
g = \left\{ \begin{pmatrix} U_1 - U_2 & U_3 - V \\ U_3 + V & U_1 + U_2 \end{pmatrix} \right\}, \quad U_1, U_2, U_3 \in so(m), \quad ^tV = V.
\]
The components are:
\[
g_c \oplus g_a \oplus g_b \oplus g_c = \left\{ \begin{pmatrix} U_1 & 0 \\ 0 & U_1 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} -U_2 & 0 \\ 0 & U_2 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & U_3 \\ U_3 & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & -V \\ V & 0 \end{pmatrix} \right\}.
\]

For \( \Psi_4 \) we have
\[
g = \left\{ \begin{pmatrix} P - Q_1 & Q_2 - Q_3 \\ Q_2 - Q_3 & P + Q_1 \end{pmatrix} \right\}, \quad P \in sp(m), \quad Q_1, Q_2, Q_3 \in H(M_m, S_m).
\]
The components are:
\[
g_c \oplus g_a \oplus g_b \oplus g_c = \left\{ \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} -Q_1 & 0 \\ 0 & Q_1 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & Q_2 \\ Q_2 & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & -Q_3 \\ -Q_3 & 0 \end{pmatrix} \right\}.
\]

If \( g = sp(2m) \) then \( \Phi \) is skew-symmetric and is of one of the form
\[
\overline{V}_1 = S_m \otimes I_2, \quad \overline{V}_2 = S_m \otimes X_a, \quad \overline{V}_3 = S_m \otimes X_b, \quad \overline{V}_4 = I_m \otimes X_c.
\]

For \( \overline{V}_1 \) we have
\[
g = g(\overline{V}_1) = K(A, S_m) \otimes I \oplus K(A, S_m) \otimes X_a \oplus K(A, S_m) \otimes X_b \oplus H(A, S_m) \otimes X_c,
\]
for \( \overline{V}_2 \) we have
\[
g = g(\overline{V}_2) = K(A, S_m) \otimes I \oplus K(A, S_m) \otimes X_a \oplus H(A, S_m) \otimes X_b \oplus K(A, S_m) \otimes X_c,
\]
for \( \overline{V}_3 \) we have
\[
g = g(\overline{V}_3) = K(A, S_m) \otimes I \oplus H(A, I_m) \otimes X_a \oplus K(A, I_m) \otimes X_b \oplus K(A, I_m) \otimes X_c,
\]

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and for $\Psi_4$ we have

$$g = g(\Psi_4) = K(A, S_m) \otimes I \oplus H(A, S_m) \otimes X_a \oplus H(A, S_m) \otimes X_b \oplus H(A, S_m) \otimes X_c.$$ 

The same argument as before shows that the first three gradings are weakly equivalent and none of them is weakly equivalent to the fourth one. In the first three cases we have $g_c \cong sp(m)$ while in the fourth case we have $g_c \cong so(m)$.

Again we give the matrix form of the first and the fourth gradings. If $\Phi$ is skew symmetric then $g = sp(2m)$ and $\Phi$ is of one of the form

$$\overline{\Psi}_1 = S_m \otimes I_2, \overline{\Psi}_2 = S_m \otimes X_a, \overline{\Psi}_3 = S_m \otimes X_b, \overline{\Psi}_4 = I_m \otimes X_c.$$ 

For $\overline{\Psi}_1$ we have

$$g = \left\{ \begin{pmatrix} U_1 - U_2 & U_3 - V \\ U_3 + V & U_1 + U_2 \end{pmatrix}, U_1, U_2, U_3 \in sp(m), V \in H(A, S_m) \right\}.$$ 

The components are:

$$g_c \oplus g_a \oplus g_b \oplus g_c = \left\{ \begin{pmatrix} U_1 & 0 \\ 0 & U_1 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} -U_2 & 0 \\ 0 & U_2 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & U_3 \\ U_3 & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & -V \\ V & 0 \end{pmatrix} \right\}.$$ 

For $\overline{\Psi}_4$ we have

$$g = \left\{ \begin{pmatrix} P - Q_1 & Q_2 - Q_3 \\ Q_2 - Q_3 & P + Q_1 \end{pmatrix}, P \in so(m), Q_1, Q_2, Q_3 \in H(M_m, I_m) \right\}.$$ 

The components are:

$$g_c \oplus g_a \oplus g_b \oplus g_c = \left\{ \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} -Q_1 & 0 \\ 0 & Q_1 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & Q_2 \\ Q_2 & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & -Q_3 \\ Q_3 & 0 \end{pmatrix} \right\}.$$ 

### 4.3 Classification of Class I gradings on $A_1$-type Lie algebras

If no fine component is present in $R = M_n \supset g = sl(n)$, $n = l + 1$, then all is defined by the $n$-tuples

$$\nu_1 = (e^{(k_2)}, a^{(k_2)}), \nu_2 = (e^{(k_2)}, a^{(k_2)}, b^{(k_2)}) , \nu_3 = (e^{(k_3)}, a^{(k_2)}, b^{(k_2)}, c^{(k_2)}).$$ 

In the case of $\nu_1$, the grading correspond to a symmetric decomposition (in fact we obtain the symmetric pair

$$(sl(n), sl(k_1) \oplus sl(k_2) \oplus \mathbb{C})$$

(or $\mathbb{R}$ if we are in the real case.

In the case of $\nu_2$ the

$$g_c = \text{diag} \{ X, Y, Z, | X \in M_{k_1}, Y \in M_{k_2}, Z \in M_{k_3}, tr(X + Y + Z) = 0 \}$$

and $g_c = sl(k_1) \oplus sl(k_2) \oplus sl(k_3) \oplus \mathbb{C}^2$.

In the case of $\nu_3$ the

$$g_c = \text{diag} \{ X, Y, Z, T, | X \in M_{k_1}, Y \in M_{k_2}, Z \in M_{k_3}, T \in M_{k_4}, tr(X + Y + Z + T) = 0 \}$$

and $g_c = sl(k_1) \oplus sl(k_2) \oplus sl(k_3) \oplus sl(k_4) \oplus \mathbb{C}^3$. 

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In all these case the grading is obvious. If $R = A \otimes B = M_n$, $A = M_m$, $n = 2m$, with a
trivial grading on $A$, then

$$g_e = \left\{ \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \bigg| X \in sl(m) \right\}$$

and the grading is given by

$$g = g_e \oplus g_a \oplus g_b \oplus g_c = g_e \oplus \left\{ \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} \bigg| X \in M_m \right\} \oplus \left\{ \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \bigg| X \in M_m \right\} \oplus \left\{ \begin{pmatrix} 0 & -X \\ X & 0 \end{pmatrix} \bigg| X \in M_m \right\}.$$ 

**Conclusion.** We obtain the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces corresponding to the pairs $(g, g_e)$
given in the following table:

| $g$          | $g_e$          |
|-------------|---------------|
| $sl(2n)$    | $sl(n)$       |
| $sl(k_1 + k_2)$ | $sl(k_1) \oplus sl(k_2) \oplus \mathbb{C}$ |
| $sl(k_1 + k_2 + k_3)$ | $sl(k_1) \oplus sl(k_2) \oplus sl(k_3) \oplus \mathbb{C}^2$ |
| $sl(k_1 + k_2 + k_3 + k_4)$ | $sl(k_1) \oplus sl(k_2) \oplus sl(k_3) \oplus sl(k_4) \oplus \mathbb{C}^3$ |

**Table 3**

### 4.4 Classification of Class II gradings on $A_l$-type Lie algebras

The general approach described in [4.1.2] enables one to classify the Class II gradings on
$\mathfrak{g} = sl(n)$, for any $n \geq 2$ and any grading group $P$. However, in the case $P = \mathbb{Z}_2 \times \mathbb{Z}_2$ the
amount of work can be significantly reduced if one uses the results of [4] and [5]. In the
former paper, in the case of outer gradings of $sl(n)$, the authors showed that the dual $\Gamma$ of the
grading group $P$ decomposes as the direct product $\langle \varphi \rangle \times \Lambda$ where $\varphi$ is an antiautomorphism
of order $2^n$ and $\Lambda$ acts by inner matrix automorphisms. If $H = \Lambda^\perp$ then $H$ is a subgroup of
order 2 and the induced $G/H$-grading of $sl(n)$ is induced from $M_n$. To obtain the $G$-grading
of $\mathfrak{g}$ one has to refine the $G/H$-grading by intersecting them with the eigenspaces of $\varphi$. In
the case $P = \mathbb{Z}_2 \times \mathbb{Z}_2$, $\varphi$ is (a negative to) a graded involution of $M_n$, $\Lambda$ is generated by an
automorphism $\lambda$ of order 2, the generators of $P$ are $a$ and $b$ such that $\lambda(a) = -1$, $\lambda(b) = 1$,
$\varphi(a) = 1$, and $\varphi(b) = -1$. In the latter paper the authors described all graded involutions on
graded matrix algebras. In our particular case the gradings by $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $sl(n)$ correspond
to $\mathbb{Z}_2$-graded eigenspaces of a (negative to) a graded involution $\varphi$ on a $\mathbb{Z}_2$-graded associative
algebra \( R = M_n \). Any \( \mathbb{Z}_2 \)-grading on \( R \) is elementary, given by a tuple \( \nu_1 = (e^{(k_1)}, a^{(k_2)}) \) where \( a \) is the generator of \( \mathbb{Z}_2 \) and \( k_1 + k_2 = n \). Now, as described in [5, Theorem 3], any graded involution is graded equivalent to \( \omega(X) = \Phi^{-1}X^t\Phi \) where \( \Phi \) is of one the following types

\[
\Phi_1 = \text{diag}\{I_{k_1}, I_{k_2}\} \quad \text{(4)}
\]
\[
\Phi_1' = \begin{pmatrix} 0 & I_{k_1} \\ I_{k_1} & 0 \end{pmatrix} \quad \text{(5)}
\]
\[
\Phi_1'' = \text{diag}\{S_{k_1}, S_{k_2}\} \quad \text{(6)}
\]
\[
\Phi_1''' = \begin{pmatrix} 0 & I_{k_1} \\ -I_{k_1} & 0 \end{pmatrix}. \quad \text{(7)}
\]

Now it remains to apply [4, Corollary 5.6], where \( K = \langle a \rangle \) to obtain that all Class II gradings of \( g \) have the following form

\[
\begin{aligned}
g_e &= K(R_e, \Phi) \\
g_a &= K(R_a, \Phi) \\
g_b &= H(R_e, \Phi) \\
g_c &= H(R_a, \Phi)
\end{aligned}
\]

In all four cases with, depending on the choice of \( \Phi \), we have

\[
R = \left\{ \begin{pmatrix} U & V \\ W & T \end{pmatrix} \bigg| U \in M_{k_1}, T \in M_{k_2} \right\}, \quad \text{and} \quad g = \left\{ \begin{pmatrix} U & V \\ W & T \end{pmatrix} \bigg| \text{tr}U + \text{tr}T = 0 \right\}
\]

and in the last two cases, we additionally have that \( k_1 = k_2 \). It easily follows that for the \( \mathbb{Z}_2 \)-grading we have \( R_e = \left\{ \begin{pmatrix} U & 0 \\ 0 & T \end{pmatrix} \right\} \subset R \) and \( R_a = \left\{ \begin{pmatrix} 0 & V \\ W & 0 \end{pmatrix} \right\} \subset R \).

Now we can explicitly write all the gradings in this case. In what follows we keep the following notation. For any \( X \) of size \( c \times d \), we denote by \( X^* \) the ordinary transpose of \( X \), except in the case of \( \Phi_1'' \) where it means \( S_{p^{-1}}X^tS_q \). By \( U, U_1, \ldots \) we will denote the matrices with \( X^* = -X, U, U_1, \ldots \) those with \( X^* = X \), while by \( W, W_1, \ldots \) we will mean any matrices of appropriate sizes. All matrices must be in \( g \).

In both cases \( \Phi = \Phi_1, \Phi_1'' \) then we will have

\[
g_e \oplus g_a \oplus g_b \oplus g_c = \left\{ \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & W \quad 0 \\ -W^* & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & W \quad 0 \\ W^* & 0 \end{pmatrix} \right\}.
\]

However, because we have the transpose involution in the first case and the symplectic in the second, we obtain two inequivalent local symmetric spaces

\[
\text{sl}(k_1 + k_2)/(\text{so}(k_1) \oplus \text{so}(k_2))
\]

and

\[
\text{sl}(k_1 + k_2)/(\text{sp}(k_1) \oplus \text{sp}(k_2)).
\]

It should be noted that \( k_1 \) is always nonzero while we could have \( k_2 = 0 \). In this case we actually have a \( \mathbb{Z}_2 \)-grading rather than a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-grading. The respective local symmetric spaces are well-known to be

\[
\text{sl}(n)/\text{so}(n)
\]

and

\[
\text{sl}(2m)/\text{sp}(2m).
\]
In the case of $\Phi = \Phi'_1$ we will have
\[
g_e \oplus g_a \oplus g_b \oplus g_c = \left\{ \begin{pmatrix} W & 0 \\ 0 & -W^t \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & U_1 \\ U_2 & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & W \\ W^t & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & V_1 \\ V_2 & 0 \end{pmatrix} \right\}.
\]

Finally, in the case of $\Phi = \Phi'_1$ we will have
\[
g_e \oplus g_a \oplus g_b \oplus g_c = \left\{ \begin{pmatrix} W & 0 \\ 0 & -W^t \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & V_1 \\ V_2 & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & W \\ W^t & 0 \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} 0 & U_1 \\ U_2 & 0 \end{pmatrix} \right\}.
\]

Obviously, these two latter grading are weakly equivalent, and the weak equivalence is achieved by an automorphism of $\mathcal{P}$ which changes places of $a$ and $c$. So, we obtain the third local symmetric space as follows
\[
\text{sl}(2k)/\text{gl}(k).
\]

**Conclusion.** We obtain the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces corresponding to the pairs $(\mathfrak{g}, \mathfrak{g}_e)$ given in the following table:

| $\mathfrak{g}$   | $\mathfrak{g}_e$   |
|-----------------|-------------------|
| sl($2k$)        | gl($k$)           |
| sl($n$)         | so($n$)           |
| sl($2m$)        | sp($2m$)          |
| sl($k_1 + k_2$) | so($k_1$) $\oplus$ so($k_2$) |
| sl($2(k_1 + k_2)$) | sp($2k_1$) $\oplus$ sp($2k_2$) |

**Table 4**

We summarize the result obtained in Section 4 as follows.

**Theorem 14** All local complex $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces in cases $A_l$, $l \geq 1$, $B_l$, $l \geq 2$, $C_l$, $l \geq 3$ or $D_l$, $l \geq 4$ are given in Tables 1, 2, 3, and 4. Each space $\mathfrak{g}/\mathfrak{g}_e$ is uniquely, up to a weak equivalence of respective $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading, defined by $\mathfrak{g}$ and $\mathfrak{g}_e$.

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References

[1] Bahturin, Y., Sehgal, S, and M. Zaicev, *Group Gradings on Associative Algebras*, J. Algebra **241** (2001), 677–698.

[2] Bahturin, Yuri; Shestakov, Ivan; Zaicev, Mikhail, *Gradings on simple Jordan and Lie algebras*, J. Algebra **283**(2005), 849 - 868.

[3] Bahturin, Yuri; Zaicev, Mikhail, *Graded algebras and graded identities*, Polynomial identities and combinatorial methods (Pantelleria, 2001), 101-139, Lecture Notes in Pure and Appl. Math., **235**, Dekker, New York, 2003

[4] Bahturin, Yuri; Zaicev, Mikhail, *Group gradings on simple Lie algebras of type “A”*, J. Lie Theory **16**(2006), 719–742.

[5] Bahturin, Yuri; Zaicev, Mikhail, *Involutions on graded matrix algebras*, J. Algebra **315**(2007), 527 - 540.

[6] Berger, M., *Les espaces symétriques non compacts*, Ann.E.N.S. **74**, 2, (1957), 85-177.

[7] Bouyakoub, Abdelkader; Goze, Michel; Remm, Elisabeth; *On Riemannian $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces and flag manifolds* arXiv math. DG/0609790. Preprint Mulhouse 2006.

[8] Draper, Cristina; Martín, Candido., *Gradings on $g_2$*, Linear Algebra Appl. **418** (2006), 85 - 111.

[9] Draper, C; Viruel, A., *Gradings on o(8, C)*, arXiv: 0709.0194

[10] Goze, M; Remm, E., *Classifications of nilpotent Lie algebras*, www.math.uha.fr/ algebre/.

[11] Havlíček, Miloslav; Jiri Patera; Edita Pelantova, *On Lie gradings, II*, Linear Algebra Appl. **277** (1998), 97 - 125.

[12] Havlíček, Miloslav; Jiri Patera; Edita Pelantova, *On Lie gradings, III. Gradings of the real forms of classical Lie algebras*, Linear Algebra Appl. **314** (2000), 87 - 159.

[13] Jacobson, N., *Lie Algebras*, Interscience Tracts on Pure and Applied Mathematics, no. 10. Interscience Publishers, New York, 1962.

[14] Kac, Victor, *Graded algebras and symmetric spaces*, Funct. Anal. Pril. 2 (1968), 93 - 94.

[15] Kobayashi, S., Nomizu, K., *Foundations of Differential Geometry*, Vol II. John Wiley and sons, 1969.

[16] Kowalski O., *Generalized symmetric spaces*, Lecture Notes in Mathematics, 805. Springer-Verlag, Berlin-New-York, 1980.

[17] Ledger, A.J.; Obata, M., *Affine and Riemannian s-manifolds*, J. Differential Geometry **2**, (1968), 451-459.

[18] Lutz, Robert *Sur la géométrie des espaces $\Gamma$-symétriques* C. R. Acad. Sci. Paris Sér. I Math. **293** (1981), no. 1, 55–58.

[19] Patera, Jiri; Zassenhaus, Hans, *On Lie gradings, I*, Linear Algebra Appl. **112** (1989), 87 - 159.

[20] Wolf, Joseph; Gray, Alfred, *Homogeneous spaces defined by Lie group automorphisms. I, II*, J. Differential Geometry **2**(1968), 77 - 114.