Monte Carlo and Renormalization Group
Effective Potentials in Scalar Field Theories

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We study constraint effective potentials for various strongly interacting $\phi^4$ theories. Renormalization group (RG) equations for these quantities are discussed and a heuristic development of a commonly used RG approximation is presented which stresses the relationships among the loop expansion, the Schwinger-Dyson method and the renormalization group approach. We extend the standard RG treatment to account explicitly for finite lattice effects. Constraint effective potentials are then evaluated using Monte Carlo (MC) techniques and careful comparisons are made with RG calculations. Explicit treatment of finite lattice effects is found to be essential in achieving quantitative agreement with the MC effective potentials. Excellent agreement is demonstrated for $d = 3$ and $d = 4$, $O(1)$ and $O(2)$ cases in both symmetric and broken phases.
I. INTRODUCTION

Strong-coupled relativistic quantum field theories (RQFT’s) play an essential role in any modern understanding of subatomic phenomena from the Higgs mechanism to fundamental theories of nucleon and nuclear phenomena. There are important theoretical issues, such as spontaneous symmetry breaking, which can be studied with relatively simple scalar field theories. Unfortunately, even the simplest interacting relativistic quantum field theories are notoriously difficult to solve. A classic example is the issue of triviality in 4-dimensional $\phi^4$ scalar field theory which has been tackled over the years using nearly the full complement of weapons in the theorists’ arsenal (loop expansion, strong coupling- and $\epsilon$-expansions, truncated systems of Schwinger-Dyson (SD) equations, renormalization group (RG) techniques and Monte Carlo studies), without yielding a definitive answer. (See, e.g., Ref. [1].) It is self-evident that accurate and relatively simple approximate methods for solving RQFT’s would be of enormous value.

In the present paper we discuss approximations to $\phi^4$ theories emphasizing throughout comparison with lattice Monte Carlo calculations which, in a sense to be made explicit below, we consider exact. In particular we will focus on the uniform properties which can be expressed in terms of “effective potentials” (or more precisely, “constraint effective potentials” as defined, e.g., in Ref. [2] ) from which, for example, zero-momentum $n$-point couplings may be extracted. The present results may be summarized most succinctly by stating that the renormalization group (RG) effective potential – evaluated so as to account for lattice artifacts – provides a spectacularly accurate description of its lattice counterpart with little calculational complexity for a wide variety of scalar field theories including symmetric and broken phases in 2, 3, and 4-D with O(1) and O(2) symmetries. In contrast, other methods such as the use of truncated SD equations [3] yield only qualitative agreement at best. Moreover, as will be emphasized below, the simple RG effective potential used here is known to be the first term in a momentum expansion meaning that systematic improvements in the approximation are possible. Perhaps even more important is the evidence that – in contrast to, e.g., the hierarchy of SD equations – this expansion appears to be strongly convergent for cases with strong coupling. Indeed, estimates of next-to-leading order correction to be made below are quite small but consistently improve the already extraordinary agreement with lattice results. Finally, as will also be addressed below, the RG approach is ideally suited to the construction of effective low energy theories which can economically account for the long-wavelength behavior of the full theories. We will also discuss some anticipated exploitations of this possibility.

In section II we review briefly the history of the version of the RG effective potential used in the present work providing highlights and references to several other recent studies. In section III we provide a heuristic derivation of the RG equation for the effective potential and compare with truncated Schwinger-Dyson. In section IV we describe the methods we use in extracting the effective potential from Monte Carlo simulations and in solving the RG equations numerically. In section V we present the results of our calculations comparing Monte Carlo and RG descriptions for broken and symmetric phases of O(1) and O(2) $\phi^4$ field theories in 3- and 4-dimensions.
II. REVIEW OF THE RG EFFECTIVE POTENTIAL

We now trace the development of the form of the RG equations used in the present study. The original work by Wilson on the renormalization group [4] was in essence a discussion of how to obtain the effective action at long wavelengths from a bare action at short wavelengths. He and his coworkers discussed a variety of approximations to the full (and intractable) RG equations. Shortly thereafter, Wegner and Houghton [7] derived an exact RG equation for the effective potential by integrating over thin shells in momentum space, a technique referred to as the “sharp cutoff method.” (See Ref. [8] for an early discussion of this and alternative “smooth cutoff” methods. Also see Ref. [5] for early discussions of many RG concepts. Ref. [6] provides a later discussion of some of these ideas in the course of constructing a novel and intuitively appealing RG-based proof of the renormalizability of $\phi^4$ field theories.) A seminal early application of the Wegner and Houghton method to $\phi^4$ theories was presented by Hasenfratz and Hasenfratz [9]. (See also Ref. [10] for an illuminating comparison of analyses of the cutoff dependence of the Higgs mass using the Wilson-based and Wegner/Houghton-based RG methods.) These authors obtained a tractable approximation to the exact Wegner and Houghton equation by projecting the full effective action onto an effective potential for which all $n$-point couplings are momentum independent. They used the resulting RG equation to extract information about RG flows in the vicinity of Wilson and Gaussian fixed points in $d = 3$ and to establish the absence of a Wilson fixed point in $d = 4$. They also extracted various critical exponents for the $d = 3$ case and found them to be in good agreement with values determined by a variety of other techniques.

There has been a recent resurgence of interest in RG effective potentials. Wetterich and coworkers [11], for example, have extended the approach of Hasenfratz and Hasenfratz using the so-called “smooth cutoff” method in deriving their RG equations. Their technique permits evaluation of, e.g., the anomalous dimension, $\eta$, which is implicitly zero in the Hasenfratz and Hasenfratz approximation and yields values of critical exponents which are in impressive agreement with those determined by other methods. (See Ref. [12] for an early calculation of $\eta$ using an extension of one of Wilson’s original approximate RG equations.) This increase in calculational power comes, however, at the cost of a substantial increase in numerical complexity. In other work, Morris [13] gives an in-depth discussion of the RG effective potential as derived using both sharp and smooth cutoffs. He also [14] develops a version of the systematic momentum expansion alluded to above and presents an analysis of $\phi^4$ in $d = 3$ using the first correction to what is in effect the Hasenfratz and Hasenfratz method (which appears as the zero-momentum limit in both the sharp- and smooth-cutoff derivations). Without attempting to be comprehensive, we mention other related papers which appear here as Refs. [15], [16] and [17].

III. A HEURISTIC DEVELOPMENT OF THE RG EQUATION FOR THE EFFECTIVE POTENTIAL

Derivations of RG equations for the effective action or their reduction to expressions governing the effective potential appear in Refs. [4,7,6,9,11,13,17] and will not be reproduced here. Instead we provide a heuristic and, we hope, intuitive development of the RG equation.
in the zero-momentum limit which has as its point of departure the familiar expression for the one-loop contribution to the effective action of a uniform system.

In $d$ Euclidean dimensions, the bare action for a one-component $\phi^4$ theory is given by

$$ S_0 = \int d^4x \left[ \frac{1}{2} (\partial_\mu \phi)^2 + U_0(\phi) \right] $$

where

$$ U_0(\phi) = \frac{1}{2} m_0^2 \phi^2 + \frac{1}{4} \lambda_0 \phi^4. $$

The one-loop perturbative correction to the bare potential for a uniform system corresponds to the familiar expression (primes denote differentiation with respect to $\phi$)

$$ U_1(\phi) = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \ln[k^2 + U''_0(\phi)] $$

which can be interpreted physically as the sum of zero-point energies of vacuum modes for a free scalar meson theory with meson mass-squared $m^2 = U''_0(\phi)$. This quantity is divergent and we regularize it by adopting a UV cutoff $\Lambda_0$:

$$ U_1(\phi) = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \theta(\Lambda_0 - |\vec{k}|) \ln[k^2 + U''_0(\phi)] $$

We note that the lowest order truncated Schwinger-Dyson equation can be obtained from the following expression for what we will call the Schwinger-Dyson effective potential, $U_{SD}$, in which the 1-loop contribution $U_1(\phi)$ as given above is self-consistent in the following sense:

$$ U_{SD}(\phi) \approx U_0(\phi) + \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \theta(\Lambda_0 - |\vec{k}|) \ln[k^2 + U''_0(\phi)] $$

The RG equation for the effective potential $U_{RG}$ may be obtained heuristically by starting with the loop expansion of the effective action, the first term of which is given in Eq. 4. Assume we know $U(\phi, \Lambda)$, the effective potential at momentum scale $\Lambda$. We then have the following expression for $U(\phi, \Lambda - d\Lambda)$, the effective potential at a slightly smaller momentum scale:

$$ U(\phi, \Lambda - d\Lambda) = U(\phi, \Lambda) + \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \theta(\Lambda - |\vec{k}|) \theta(|\vec{k}| - \Lambda + d\Lambda) \ln[k^2 + U''_0(\phi)] $$

$$ = U(\phi, \Lambda) + \frac{A_d}{2} \Lambda^{d-1} \ln[\Lambda^2 + U''(\phi, \Lambda)] \ d\Lambda + O(d\Lambda^2) $$

where $A_d = \int d\Omega_d/(2\pi)^d = 1/(2^{d-1} \pi^{d/2} \Gamma(d/2))$ and the higher order loops contribute at $O(d\Lambda^2)$. Taylor expanding the left hand side and taking the limit $d\Lambda \to 0$ gives the RG equation:

$$ \frac{dU_{RG}(\phi, \Lambda)}{d\Lambda} = -\frac{A_d}{2} \Lambda^{d-1} \ln[\Lambda^2 + U''_{RG}(\phi, \Lambda)]. $$

The essential point is that where the Eq. 5 requires $U_{SD}$ be made self-consistent across all momentum scales at once, Eq. 6 ensures that $U_{RG}$ is self-consistent at every scale. Put
another way, the evolution (or “flow”) of $U_{RG}$ at some scale $\Lambda$ is determined self-consistently by $U_{RG}$ at that same scale. We make think of $U_{SD}$ as the result of the crudest possible discretization of the flow which yields $U_{RG}$, namely one in which the integration of the differential equation in Eq. 7 is carried out in a single step.

The extension to $O(N)$ is straight-forward. Defining $\rho \equiv \sqrt{\phi^2}$ and with primes denoting differentiation with respect to $\rho$, we find

$$\frac{dU_{RG}(\rho, \Lambda)}{d\Lambda} = -\frac{A_d}{2} \Lambda^{d-1} \left\{ \ln[\Lambda^2 + U''_{RG}(\rho, \Lambda)] + (N - 1) \ln[\Lambda^2 + U'_{RG}(\rho, \Lambda)/\rho] \right\}. \quad (8)$$

To make contact with, e.g., Hasenfratz and Hasenfratz [9], we scale all dimensionful quantities by the appropriate power of $\Lambda$ so as to render them dimensionless. For example, $\phi \rightarrow x \equiv \phi/\Lambda^{d/2-1}$ (where in this approximation the anomalous dimension, $\eta$, vanishes).

Designating the rescaled potential as

$$\tilde{U}_{RG}(x, t) \equiv U_{RG}(\phi, \Lambda)/\Lambda^d, \quad (9)$$

where $t \equiv \ln(\Lambda_0/\Lambda)$, the one-component equation (Eq. 7) becomes, after dropping irrelevant constants, becomes

$$-\Lambda \frac{d\tilde{U}_{RG}(x, \Lambda)}{d\Lambda} = \frac{d\tilde{U}_{RG}(x, t)}{dt} = \frac{A_d}{2} \ln[1 + \tilde{U}''_{RG}(x, t)] + (1 - d/2) x \tilde{U}'_{RG}(x, t) + d \tilde{U}_{RG}(x, t) \quad (10)$$

where primes now denote differentiation with respect to $x$. This is Eq. 22 of Ref. [9] (with $\eta = 0$). Finally, defining $f(x, t) \equiv \tilde{U}'_{RG}(x, t)$, Eq. [10] becomes

$$\dot{f} = \frac{A_d}{2} \frac{f''}{1 + f'} + (1 - d/2) x f' + (1 + d/2) f \quad (11)$$

where the “dot” means differentiation with respect to $t$. This is Eq. 9 of Ref. [9] and is a form of the RG equation for the effective potential frequently encountered in the literature.

These RG equations may be integrated to find the effective potential at momentum scale $\Lambda$ subject to the boundary condition that the bare and effective potentials coincide at the UV cutoff scale $\Lambda_0$. In so doing, quantum corrections associated with single meson modes with momenta between $\Lambda$ and $\Lambda_0$ are (approximately) included in the effective potential while contributions from modes at lower frequencies must still be treated explicitly. As touched upon above, the power of the RG approach derives from its self-consistency at each momentum scale. This feature is evident in Eq. [7] where it is clear that the evolution of $U_{RG}$ at momentum scale $\Lambda$ depends on a quantum correction which is consistent with the potential itself at that particular momentum scale.

Another important and perhaps surprising aspect of the RG equations is that – while they have the form of a truncated loop expansion – no such truncation has been made. As originally shown by Wegner and Houghton [7] higher-order loops are suppressed by powers of the differential $d\Lambda$ and their equation therefore has the form of a one-loop result but is in fact exact. Thus the RG equation for the effective potential, obtained by Hasenfratz and Hasenfratz as an approximation to the Wegner and Houghton expression has nothing to do
with explicitly dropping high order quantum corrections but instead depends on projecting
the full effective action onto a form with independent $n$-particle couplings. (This is what
is meant by the “zero-momentum” RG effective potential.) Of course, just as is the case
when solving truncated SD equations [3], quantum corrections of arbitrarily high order in a
perturbative loop expansion are automatically generated in the course of integrating the RG
equation. (The relationship between the RG equation and the loop expansion was touched
upon by Polchinski [6] and addressed in more detail by Morris [13] who also made a direct
comparisons with the SD expression given below in Eq. (15).

Once the effective potential is obtained, the renormalized mass and 4-point coupling are
given by:

$$m^2 = \frac{d^2U_{eff}}{d\phi^2} \bigg|_{\phi \to \phi_0}$$

$$\lambda = \frac{1}{6} \frac{d^4U_{eff}}{d\phi^4} \bigg|_{\phi \to \phi_0}$$

where $\phi_0$ is the value of the field at the minimum of $U_{eff}$. For example in the case of the
Schwinger-Dyson equation, Eq. (5), in the symmetric phase where $\phi_0 = 0$, we have

$$m_{SD}^2 = m_0^2 + 3\lambda_{SD} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m_{SD}^2}$$

where we have used $U_{SD}''(\phi = 0) \leftrightarrow m_{SD}^2$ and $U_{SD}'''(\phi = 0) \leftrightarrow 6\lambda_{SD}$. Approximating $\lambda_{SD}$
by $\lambda_0$ yields the familiar (see, e.g., Eq. 2.17 of Ref. [3]) SD self-consistency relation for the
“dressed” mass:

$$m_{SD}^2 = m_0^2 + 3\lambda_0 \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m_{SD}^2}$$

where the self-consistent nature of the SD method is apparent. (We note that the approxi-
mation defined by this equation for $m_{SD}$ is also known [13] as the “cactus” approximation.)
Likewise, within the same approximation, the 4-point is

$$\lambda_{SD} = \frac{\lambda_0}{1 + 9\lambda_0 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + m_{SD}^2)^2}}$$

Examination of RG renormalized masses and 4-point couplings following from Eq. (4) (or
the equivalent expressions given subsequently) requires finding numerical solutions to the
non-linear partial differential RG equations. In the next section we discuss our numerical
methods for solving the RG equations and for determining the “exact” effective potentials
from our Monte Carlo calculations.

**IV. NUMERICAL METHODS**

We have carried out RG calculations for the effective constraint potential in two com-
plementary ways. In the first, appropriate to studying the continuum limit, we treat $\Lambda$ as
a continuous variable. In the second approach, appropriate to direct comparison with our Monte Carlo calculations which of course refer to a finite lattice, we treat the lattice normal modes explicitly. We now give detailed descriptions of both methods.

In the first approach, we solve Eq. 7 by treating \( \Lambda \) as a continuous variable. (We have also evaluated the equivalent expressions, Eq. 10 or Eq. 11. Although one version is often better suited to a given calculation than the others, we found them to be entirely equivalent to one another as expected.) We begin by expanding \( U_{RG}(\phi, \Lambda) \) in a power series in \( \phi \) truncated at \( \phi^{2M} \):

\[
U_{RG}(\phi, \Lambda) = \sum_{i=1}^{M} \frac{1}{2^{i}} U_{2i}(\Lambda) \phi^{2i}.
\]

The boundary condition referred to in Section III is imposed by setting

\[
U_{2}(\Lambda_0) = m_0^2, \quad U_{4}(\Lambda_0) = \lambda_0, \quad \text{and} \quad U_{2i}(\Lambda_0) = 0 \quad \text{for} \quad 2i \geq 6.
\]

The range of the \( \Lambda \) integration is divided into \( N \) intervals of equal length \( \Delta \Lambda \):

\[
\Lambda_n = \Lambda_0 - n \Delta \Lambda
\]

with

\[
\Delta \Lambda = (\Lambda_0 - \Lambda_{IR})/N,
\]

where \( \Lambda_{IR} = 2\pi/N_{\text{lat}} \) is the infrared scale appropriate to a lattice of \( N_{\text{lat}}^d \) sites and \( \Lambda_0 \) is chosen to yield the same phase space volume as the finite sum-over-modes.

A similar decomposition is made of a range of \( \phi \), the other independent variable:

\[
\phi_0 < \ldots \phi_j < \phi_{j+1} < \ldots < \phi_J.
\]

Each step in the integration is accomplished by first making a least squares fit of the polynomial on the r.h.s of Eq. 17 to \( U_{RG}(\phi, \Lambda_n) \) in order to determine the \( U_{2i}(\Lambda_n) \) and then computing

\[
U_{RG}(\phi_j, \Lambda_{n+1}) = U_{RG}(\phi_j, \Lambda_n) + \Delta \Lambda \cdot \Lambda_n^{d-1} \frac{A_d}{2} \ln[\Lambda_n^2 + U''_{RG}(\phi_j, \Lambda_n)]
\]

for each \( j \) where \( U''_{RG}(\phi_j, \Lambda_n) \) is computed analytically from the polynomial fit. Starting with \( n = 0 \), this procedure is iterated until ultimately the \( J + 1 \) values of \( U_{RG}(\phi_j, \Lambda_{IR}) \) are found. We refer to this as the “continuum RG effective potential” (C-RG). Obviously this method yields directly the \( m \)-point couplings, \( U_m(\Lambda_{IR}) \), as well. The calculation can sometimes be unstable in the vicinity of the \( \phi \)-endpoint, \( \phi_J \), and some care is required in choosing values of \( M, N \) and \( J \) to ensure stable integration. In the calculations to be presented below, we typically used

\[
M \approx 10, \quad N \approx 4000, \quad \text{and} \quad J \approx 50.
\]

In our second approach, an alternate RG calculation was formulated to facilitate comparison with the lattice results to be described below. The derivative terms in the action
(Eq. 4) are approximated on the lattice by a finite difference formula. For a uniform system on a $N_{\text{lat}}^d$ lattice with periodic boundary conditions, the vacuum modes are momentum eigenstates with wavenumbers in each direction given by $\kappa_n \equiv 2n\pi/N_{\text{lat}}$, $n = 1, 2, \ldots N_{\text{lat}}$ where we take the lattice constant to be unity, $a = 1$. We make the following identification:

$$\int d^d k (2\pi)^d f(k^2) \rightarrow \frac{1}{N_{\text{lat}}^d} \sum_{n_1=1}^{N_{\text{lat}}} \sum_{n_2=1}^{N_{\text{lat}}} \ldots \sum_{n_d=1}^{N_{\text{lat}}} f(k_{n_1}^2 + k_{n_2}^2 + \ldots + k_{n_d}^2)$$

where $k_n^2 \equiv 4 \sin^2(\kappa_n/2)$. Now define the $N_{\text{lat}}^d$ scalar quantities

$$k_{n_1 \ldots n_d}^2 \equiv k_{n_1}^2 + k_{n_2}^2 + \ldots + k_{n_d}^2,$$

then sort them by magnitude and relabel:

$$k_1^2 \geq \ldots k_N^2 \geq k_{N+1}^2 \geq \ldots \geq k_{N_{\text{lat}}^d}^2$$

where $k_1^2 = 4d$ and $k_{N_{\text{lat}}^d}^2 = 0$. Our “latticized” version of the RG equation for the effective potential, Eq. 22, is

$$U_{\text{RG}}(\phi_i, k_{N+1}^2) = U_{\text{RG}}(\phi_i, k_N^2) + \frac{1}{N_{\text{lat}}^d} \frac{A_d}{2} \ln[k_N^2 + U''_{\text{RG}}(\phi_i, k_N^2)]$$

with the boundary condition

$$U_{\text{RG}}(\phi_i, k_1^2) = U_0(\phi_i).$$

We will refer to this as the “lattice RG effective potential” (L-RG).

The lattice Monte Carlo calculations to be presented below employed $d$-dimensional cubic lattices with periodic boundary conditions in conjunction with the Metropolis algorithm to sample the $\phi$-configurations with weights given by the exponentiated action. The Monte Carlo observable of primary interest here is the “constraint effective potential” [2] which we refer to as the “MC effective potential” and designate by $U_{\text{eff}}$. Formally, this quantity is defined via

$$\exp[-N_{\text{lat}}^d U_{\text{eff}}(\bar{\phi})] = \int \mathcal{D}\{\phi\} \delta(\bar{\phi} - \sum_i \phi_i/N_{\text{lat}}^d) \exp[-S[\phi]]$$

where $\sum_i$ means a sum over all lattice sites and $\bar{\phi}$ is the average field on the lattice for a given field configuration in the path integral.

Since the Monte Carlo method generates sequences of field configurations, $[\phi]_n$, weighted by $\exp[-S[\phi]_n]$, $U_{\text{eff}}(\bar{\phi})$ is extracted from the lattice Monte Carlo calculation from a histogram of the number of configurations generated $\text{vs}$ the average value of the field for those configurations. Let $d\mathcal{N}(\bar{\phi})$ be the number of configurations with average field values in an interval $d\bar{\phi}$ about $\bar{\phi}$. Then

$$d\mathcal{N}(\bar{\phi}) \propto \exp[-N_{\text{lat}}^d U_{\text{eff}}(\bar{\phi})] \, d\bar{\phi}$$

and, to within an overall constant,
\[
U_{\text{eff}}(\bar{\phi}) = -\frac{1}{N_{\text{lat}}} \ln \frac{dN(\bar{\phi})}{d\bar{\phi}}. \tag{31}
\]

For \(O(N)\) theories this expression is modified slightly. Let \(\bar{\rho}\) be the average value of \(\rho \equiv \sqrt{\langle \bar{\phi}^2 \rangle}\) on the lattice and let \(dN(\bar{\rho})\) be the number of configurations in an interval \(d\bar{\rho}\) about \(\bar{\rho}\). Then, since

\[
dN(\bar{\rho}) \propto \bar{\rho}^{N-1} \exp[-N_{\text{lat}} U_{\text{eff}}(\bar{\rho})] d\bar{\rho}, \tag{32}
\]

we have

\[
U_{\text{eff}}(\bar{\rho}) = -\frac{1}{N_{\text{lat}}} \ln \left[ \frac{1}{\bar{\rho}^{N-1}} \frac{dN(\bar{\rho})}{d\bar{\rho}} \right]. \tag{33}
\]

Of course the most probable average field values are near the minimum of \(U_{\text{eff}}\). In order to determine \(U_{\text{eff}}\) away from its minimum; i.e., to sample a range of relatively improbable values of \(\bar{\phi}\) (or \(\bar{\rho}\)), it is often necessary to add a constraint to the updating algorithm which rejects “moves” which take the average field value outside a preset “window” \([18]\). By matching \(U_{\text{eff}}\)’s for a number of adjacent windows, it is possible to determine the lattice effective potential over a wide range of average field values. (See Ref. \([19]\) for an interesting alternative method.)

Finally we comment on the “convexity” of \(U_{\text{eff}}\). It can be shown by very general arguments \([2]\) that the condition \(U_{\text{eff}}''(\bar{\phi}) \geq 0\) (primes denote differentiation with respect to \(\bar{\phi}\)) must be satisfied for all \(\bar{\phi}\). Presently, we will show \(U_{\text{eff}}\)’s for “broken” symmetry phases of \(\phi^4\) theories which \(do\ not\) satisfy the convexity condition. The apparent contradiction is resolved by noting that convexity is required only in the thermodynamic limit; i.e., as \(N_{\text{lat}} \to \infty\). For the finite lattice results to be presented below, convexity is not a fundamental property of \(U_{\text{eff}}\).

V. COMPARISON OF MC AND RG EFFECTIVE POTENTIALS

We now present MC effective potentials as defined in Eqs. \([31]\) or \([33]\) for a variety of cases. We also show the corresponding continuum and lattice RG effective potentials of Eqs. \([22]\) and \([27]\), respectively. In computing the former RG potential, we always set the IR cutoff; i.e., \(\Lambda_N\) in Eq. \([19]\) to \(2\pi/N_{\text{lat}}\). Fixing the UV cutoff, \(\Lambda_0\), is ambiguous. We adopt the prescription that \(\Lambda_0\) should be chosen so that the phase space volumes integrated over should be the same for continuum and lattice calculations. This correspondence is made quantitative by setting \(f(k^2) = 1\) in Eq. \([24]\) which implies, in effect,

\[
A_d \int_0^{\Lambda_0} k^{d-1} dk = A_d \frac{\Lambda_0^d}{d} = 1. \tag{34}
\]

Then, e.g.,

\[
\begin{align*}
\Lambda_0(d = 2) &= (4\pi)^{1/2} \\
\Lambda_0(d = 3) &= (6\pi^2)^{1/3} \\
\Lambda_0(d = 4) &= (32\pi^4)^{1/4}.
\end{align*} \tag{35}
\]
In the calculations shown in this work the bare coupling was fixed at $\lambda_0 = 10$ while the bare mass $m_0$ was adjusted to give a desired renormalized mass. The MC effective potentials were fit with a polynomial of the form

$$U_{\text{eff}}(\phi) = \frac{1}{2}m^2_{\text{fit}}\phi^2 + \frac{1}{4}\lambda_{\text{fit}}\phi^4$$

(36)

to determine the renormalized MC masses and 4-point couplings.

In our RG calculations we also kept the 4-point coupling fixed at $\lambda_0 = 10$. We now describe our procedure for determining the bare mass in the RG calculations. For symmetric phases, the RG bare masses were adjusted to yield $\partial^2 U_{\text{RG}}/\partial \phi^2 |_{\phi=0} = m^2_{\text{fit}}$. For broken phases, the RG bare masses were adjusted so as to give the best fit “by eye” to $U_{\text{eff}}$ in the vicinity of its minimum (see discussion below). Since the MC and RG effective potentials in the symmetric phase behave identically at small $\phi$ by construction, we typically compare $-$ for both symmetric and broken phases – MC and RG results for the subtracted quantity $U - \frac{1}{2}m^2_{\text{fit}}\phi^2$ rather than results for the $U$’s themselves. Table 1 summarizes results of $O(1)$ and $O(2)$ calculations in both symmetric (S) and broken (B) phases in $d = 3$ and $d = 4$. For the RG results, $m \equiv \sqrt{U}$ and $\lambda \equiv U/6$. The Table shows that our bare masses squared are typically $m^2_0 \sim -2^2$ while the renormalized masses squared are $m^2 \sim \pm 0.2^2$ to $\pm 0.3^2$ with renormalized couplings of $\lambda \sim 1$ to 4.

Clearly quantum corrections are large and simple perturbation theory (e.g., the loop expansion) cannot account for them. For comparison purposes we examine results from the simplest Schwinger-Dyson (SD) equations [8]. Consider, for example, a “latticized” $d = 3$, $O(1)$ version of Eq. 13:

$$m^2_{SD} = m^2_0 + 3\lambda_0 \frac{1}{N^3_{\text{lat}}} \sum_{N=1}^{N^3_{\text{lat}}} \frac{1}{k_N^2 + m^2_{SD}}$$

(37)

where the $k_N^2$’s are defined in Eq. 26. In this expression, a value of $m^2_0 = -2.61^2$ is required to give $m_{SD} = 0.285$ for $\lambda_0 = 10$ and $N_{\text{lat}} = 16$ while the corresponding MC calculation yielding $m_{\text{fit}} = 0.285$ requires $m^2_0 = -2.16^2$. The latticized expression for the SD 4-point coupling, Eq. 16, is

$$\lambda_{SD} = \lambda_0/\left[1 + 9\lambda_0 \frac{1}{N^3_{\text{lat}}} \sum_{N=1}^{N^3_{\text{lat}}} \frac{1}{(k_N^2 + m^2_{SD})^2}\right]$$

(38)

and gives $\lambda_{SD} = 0.65$ as compared to the MC value of 1.49. As another example, we find that, for $d = 4$, $O(1)$, the MC-determined renormalized mass is $m_{\text{fit}} = 0.276$ using $m^2_0 = -1.88^2$ compared to the latticized SD value $m^2_0 = -2.13^2$ needed to yield the same renormalized mass. Similarly, the latticized SD equation for the 4-point coupling gives $\lambda_{SD} = 1.28$ vs $\lambda_{MC} = 3.42$. Not surprisingly, these simple SD equations do not in general provide a quantitatively accurate picture of the quantum effects. Although we do not display SD results for the effective potential, it is apparent from the 4-point couplings that the SD-derived effective potential will yield at best only a qualitative description of the MC effective potential.

We turn now to the Monte Carlo calculations of the effective potential and comparisons with the two RG effective potentials, C-RG and L-RG, discussed in the previous section.
As mentioned previously, since we adjust the bare mass in the RG calculations to give the MC fit mass, \( m_{fit} \), in most of the figures that follow we display the subtracted quantity, \( U - \frac{1}{2} m^2_{fit} \phi^2 \), to emphasize the predictive power of our calculations.

Figure 1 displays the subtracted (i.e., \( U - \frac{1}{2} m^2_{fit} \phi^2 \)) MC effective potential for \( d = 3 \), \( O(1) \) in the symmetric phase on a 16\(^3\) lattice. A MC bare mass squared of \(-2.16^2\) yields \( m_{fit} = 0.285 \) and \( \lambda_{fit} = 1.49 \), as mentioned above and shown in Table 1. Note that we have used the “windowing” technique described in Section IV to extend the range of \( \phi \) at which the MC effective potential is determined from a maximum value of about 0.1 which is feasible without windowing to a maximum value of 0.22. (The relative normalizations of adjacent windows were determined by matching intermediate values well determined in both windows.) However, the values of \( m_{fit} \) and \( \lambda_{fit} \) were determined by fits to the MC values of \( U_{eff} \) for \( \phi \leq 0.1 \) only. To reproduce this value of \( m_{fit} \), a C-RG bare mass squared of \(-1.91^2\) was required. Of course this numerical value depends on our prescription for \( \Lambda_0 \) which was discussed above. (However, the C-RG effective potential is only weakly dependent on \( \Lambda_0 \) provided the RG bare mass is adjusted to give a particular value of the renormalized mass, \( m \).) The C-RG value for the effective 4-point coupling is \( \lambda = 1.89 \) and it is evident from Figure 1 that the C-RG calculation (solid line) does not accurately reproduce the quartic and higher \( \phi \)-dependence of the MC effective potential. By comparison, a bare mass squared of \(-2.14^2\) was required in the L-RG calculation to give \( m = m_{fit} \). That the L-RG bare mass differs from the MC value by less than 1% is especially gratifying; in contrast, the C-RG bare mass differs by about 12% while the SD difference is also about 12% but in the opposite direction. The L-RG value of \( \lambda = 1.43 \) is also quite close to \( \lambda_{fit} = 1.49 \). However, inspection of Figure 1 — where the agreement between MC and L-RG results over the wide range of \( \phi \) spanned using the windowing technique is spectacular — suggests that even this small discrepancy is probably due to the fact that the lattice \( U_{RG} \) contains terms of order \( \phi^6 \) and higher while our fit to \( U_{eff} \) does not. We note that our limited MC “data” are not good enough to extract the \( \phi^6 \) or higher terms reliably. Also appearing in Figure 1 as the dashed line is the L-RG result omitting all contributions of order \( \phi^6 \) and higher. Comparison of this curve with the MC results show that these high order contributions are important for large values of \( \phi \) and that the L-RG accounts for them quite well. To summarize, the L-RG calculation which includes lattice effects in the integration of the differential equation for \( U_{RG} \) yields excellent quantitative agreement with MC effective potentials even on a relatively small lattice provided a very small adjustment in the RG bare mass is permitted. For the C-RG calculation, comparison with the MC results is not straightforward because of the ambiguity in choosing \( \Lambda_0 \) which then results in ambiguities in the bare mass. Furthermore, even when the bare mass is allowed to vary freely, agreement is poorer than for the L-RG.

Unsubtracted effective potentials for the \( d = 3 \), \( O(1) \) case in the broken phase appear in Figure 2. As mentioned above, the bare masses were adjusted in the RG calculations to yield the correct location of the minimum. From Figure 2 we see again that the description of the effective potential provided by the L-RG calculation is excellent. The C-RG calculation, however, is incapable of simultaneously reproducing the location in \( \phi \) and the depth of the minimum in the MC effective potential. The fit shown is a (non-unique) compromise. The subtracted potentials for this case are shown in Figure 3 and again demonstrate the superior description of \( \phi^6 \) and higher contributions to the effective potential provided by the L-RG. Indeed, the description of the MC results by the L-RG is again spectacular. The bare mass...
squared used in this calculation is $-2.23^2$ which differs from the MC value of $-2.24^2$ by less than 1%. (The high quality of this fit again leads us to attribute the differences between $\lambda_{fit}$ and $\lambda$ from the L-RG calculation — 1.48 and 0.52, respectively — to the absence of $\phi^6$ and higher terms in the fitting function rather than to shortcomings of the L-RG method.)

Similar findings were obtained for $d = 4$, $O(1)$ as well as $O(2)$ cases in $d = 3$ and $d = 4$ in both symmetric and broken phases. Values of the bare and renormalized masses and 4-point couplings are reported in Table I. The L-RG description of the subtracted MC effective potential is nearly perfect in all cases and is distinctly superior to that provided by the C-RG. We note that this is true even when the correlation length (i.e., $m^{-1}$) is large compared to the overall lattice size. For our $d = 4$, $O(2)$ case, a small MC mass of 0.06 was found (i.e., the correlation length was $\sim 15$) on an $8^4$ lattice (!) and, yet, as shown in Figure 4, the L-RG calculation agrees almost perfectly with the subtracted MC effective potential. Furthermore, the L-RG bare masses never differ by more than 1.5% from the MC values and are typically much closer. Overall, we conclude that the L-RG method provides an extraordinarily accurate, semianalytic description of lattice MC calculations of the effective potential at a tiny fraction of the computational cost.

VI. SUMMARY AND CONCLUSIONS

We have presented a discussion of the renormalization group (RG) effective potential for $\phi^4$ scalar field theories which stresses its relationship to the loop expansion and the Schwinger-Dyson (SD) method. We show in a simple way that the unique feature of the RG method – which is presumably the origin of its ability to describe the full theory as well as it does – is its self-consistency at all momentum scales. We have formulated a version of the RG equation for the effective potential which explicitly treats finite lattice effects so that RG calculations may be compared directly with lattice Monte Carlo (MC) calculations without extrapolations of the latter to infinite lattice size and zero lattice spacing (See Ref. [19] and references contained therein for some examples of such extrapolations). We compare such “latticized” or L-RG potentials along with standard continuum or C-RG potentials [9] to constraint effective potentials obtained via MC methods. The L-RG effective potentials are in excellent quantitative agreement with the MC results for $d = 3$ and $d = 4$, $O(1)$ and $O(2)$ cases in both symmetric and broken phases. More specifically, we find that, upon adjusting the L-RG bare masses so as to yield renormalized masses equal to those found in the MC calculations, the L-RG bare masses never differ by more than 1.5% from the MC values and are typically much closer. The renormalized 4-point couplings found by solving the L-RG equations are also typically in close agreement with the MC values. In at least one case for which the MC effective potential was determined over a wide range of $\phi$ using windowing techniques, there is strong evidence that $\phi^6$ and higher contributions are also well accounted for in the L-RG approach. In another case, we showed that the L-RG approach works very well even when the correlation length is appreciably larger than the overall lattice size ($m N_{lat} \simeq 0.5$). Comparison of C-RG calculations with the MC results was found to be ambiguous; however, even when ambiguous parameters were tuned to give optimal agreement with the MC quantities, this accord was found to be inferior to that achieved using the L-RG without tuning. Finally, latticized SD equations were seen to yield only a rough qualitative description of of MC masses and 4-point couplings.
We conclude that the L-RG method provides an very accurate, relatively simple description of lattice MC calculations. It may prove to be a useful guide in extrapolating finite lattice results to the continuum limit since the relation between the L-RG and the C-RG is simple and well-defined. We may also be able to apply it to some specific problems of current interest such as accounting for the “Universal Effective Potential” for $\phi^4$ field theories in $d = 3$ recently obtained via MC techniques by Tsypin [19]. The success of the method encourages us to make various extensions of it. For example, we plan to go beyond the “zero-momentum” approximation so as to be able to compute wavefunction renormalizations [12,20]. We have also begun to examine including Yukawa-coupled fermions in the approach which would be a crucial step toward our long term goal of constructing effective theories of low-energy nuclear phenomena whose field theoretic structure is fully understood and for which we have an accurate and relatively simple approximation scheme of proven reliability.

VII. ACKNOWLEDGEMENTS

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REFERENCES

1. See, e.g., R. Fernández, J. Fröhlich and A.D. Sokal, Random Walks, Critical Phenomena and Triviality in Quantum field Theory, Springer-Verlag, Berlin, 1992; also see M. Consoli and P.M. Stevenson, Z. Phys. C63, 427 (1994) and hep-ph/9310333.

2. See, e.g., L. O’Raifeartaigh, A. Wipf, H. Yoneyama, Nucl.Phys.B271,653 (1986).

3. R. J. Rivers, Path Integral Methods in Quantum Field Theory, Cambridge University Press, Cambridge (1987).

4. K. G. Wilson, Phys. Rev. D6, 419 (1971).

5. S. Weinberg, Erice Lectures on “Critical Phenomena for Field Theorists,” 1 (1976).

6. J. Polchinski, Nucl. Phys. B231, 269 (1984).

7. F. J. Wegner and A. Houghton, Phys. Rev. A8, 401 (1973)

8. F.J. Wegner in Phase Transitions and Critical Phenomena, C. Domb and M.S. Green, eds., Academic Press, London (1976).

9. A. Hasenfratz and P. Hasenfratz, Nucl. Phys. B270, 687 (1986)

10. P. Hasenfratz and J. Nager, Z. Phys. C37, 477 (1988)

11. N. Tetradis and C. Wetterich, Nucl. Phys. B422, 541 (1994) and hep-ph/9305214 and references therein.

12. G.R. Golner, Physical Review B8, 339 (1973).

13. T. R. Morris, Int. J. Mod. Phys. A9, 2411 (1994) and hep-ph/9308265.

14. T. R. Morris, Phys. Lett. B329, 241 (1994) and hep-ph/9403346.

15. J. Kuti, L. Lin and Y. Shen, in Lattice Higgs Workshop, B. Berg, G. Bhanot, M. Burkbank, M. Creutz and J. Owens, eds., World Scientific, Singapore, 1988 and references contained therein.

16. See, e.g., M. Alford, Phys. Lett. B336, 237 (1994) and hep-ph/9403324, T.R. Morris, Phys. Lett. B334, 355 (1994) and hep-th/9405190, P.E. Haagensen et al., Phys. Lett. B323, 330 (1994) and hep-th/9310032.

17. S.-B. Liao and C. Gong, Duke preprint TH-94-63 and hep-th/9404086.

18. D. Chandler, Introduction to Modern Statistical Mechanics, Oxford University Press, Oxford, 1987.

19. M.M. Tsypin, Phys. Rev. Lett. 73, 2105 (1994) and hep-lat/9401034.

20. T.R. Morris, Southampton preprint SHEP 94/95-10 and hep-ph/9411053.
FIGURES

FIG. 1. Subtracted effective potentials for the $d = 3$, $O(1)$ case in the symmetric phase. The solid dots are the MC results while the solid and dotted lines correspond to the C-RG and L-RG calculations, respectively. Statistical uncertainties for the MC effective potential are not shown but are typically smaller than the size of the dots. The dashed curve is the L-RG result with $\phi^6$ and higher contributions omitted. Input and output quantities are discussed in the text and are presented in Table I.

FIG. 2. Effective potentials for the $d = 3$, $O(1)$ case in the broken phase. The solid dots are the MC results while the solid and dotted lines correspond to the C-RG and L-RG calculations, respectively. Input and output quantities are discussed in the text and are presented in Table I.

FIG. 3. Same as Figure 2 but for the subtracted effective potentials.

FIG. 4. Same as for Figure 3 but for $d = 4$, $O(2)$ in the symmetric phase.
TABLE I. Input and output quantities for Monte Carlo (MC) as well as continuum (C-RG) and lattice (L-RG) renormalization group calculations for $d = 3$ and $d = 4$, $O(1)$ and $O(2)$ cases in both symmetric (S) and broken (B) phases. The bare masses used are denoted by $m_0$ while the MC renormalized masses and 4-point couplings are $m_{fit}$ and $\lambda_{fit}$, respectively, as defined in the text and their RG counterparts are $m$ and $\lambda$. The MC and L-RG calculations use the same value of $N_{lat}$.

| Case, Phase | MC | C-RG | L-RG |
|-------------|------------------|------------------|------------------|
| $d = 3, O(1)$ | | | |
| S | 16 | 2.16 | 0.285 | 1.49 | 1.91i | 0.285 | 1.89 | 2.14i | 0.285 | 1.43 |
| B | 16 | 2.24i | 0.264i | 1.48 | 2.00i | 0.258i | 1.10 | 2.23i | 0.277i | 0.52 |
| $d = 4, O(1)$ | | | |
| S | 8 | 1.88i | 0.276 | 3.42 | 1.62i | 0.277 | 4.49 | 1.86i | 0.276 | 3.47 |
| B | 8 | 1.97i | 0.388i | 3.98 | 1.74i | 0.409i | 4.40 | 1.94i | 0.365i | 3.00 |
| $d = 3, O(2)$ | | | |
| S | 16 | 2.55i | 0.251 | 1.34 | 2.26i | 0.251 | 1.67 | 2.55i | 0.250 | 1.22 |
| B | 16 | 2.62i | 0.221i | 1.17 | 2.34i | 0.229i | 1.07 | 2.63i | 0.199i | 0.59 |
| $d = 4, O(2)$ | | | |
| S | 8 | 2.22i | 0.060 | 3.28 | 1.92i | 0.060 | 4.09 | 2.20i | 0.060 | 3.10 |
| B | 8 | 2.25i | 0.256i | 3.01 | 1.96i | 0.291i | 4.32 | 2.23i | 0.258i | 2.93 |
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$d=3$, $O(1)$ in Symmetric Phase
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$d=3$, $O(1)$ in Broken Phase

$U_{\text{eff}}(\phi)$ vs $\phi$
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$d=3$, $O(1)$ in Broken Phase
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$d=4, \ O(2) \ in \ Symmetric\ Phase$