FILTERED ENDS, PROPER HOLOMORPHIC MAPPINGS OF KÄHLER MANIFOLDS TO RIEMANN SURFACES, AND KÄHLER GROUPS

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Abstract. The main result of this paper is that a connected bounded geometry complete Kähler manifold which has at least 3 filtered ends admits a proper holomorphic mapping onto a Riemann surface. As an application, it is also proved that any properly ascending HNN extension with finitely generated base group, as well as Thompson’s groups V, T, and F, are not Kähler. The results and techniques also yield a different proof of the theorem of Gromov and Schoen that, for a connected compact Kähler manifold whose fundamental group admits a proper amalgamated product decomposition, some finite unramified cover admits a surjective holomorphic mapping onto a curve of genus at least 2.

This version of this paper contains details not in the version submitted for publication.

Introduction

This is a version of a paper which is similar to another paper of the same title submitted for publication. This version contains details not in the version submitted for publication.

The main goal of this paper is the following (the required definitions appear later in this introduction):

Theorem 0.1. Let \( X \) be a connected complete Kähler manifold satisfying at least one of the following hypotheses:

(i) \( X \) has bounded geometry,
(ii) \( X \) admits a positive Green’s function \( G \) that vanishes at infinity, or
(iii) \( X \) is weakly \( 1 \)-complete.

If \( \tilde{e}(X) \geq 3 \), then \( X \) admits a proper holomorphic mapping onto a Riemann surface.

In particular, this paper provides a unified framework for some of the results of [Gro1], [L], [Gro2], [GroS], [NR1]–[NR4], and [DelG]. This framework relies on the notion of filtered

Date: July 6, 2006.
2000 Mathematics Subject Classification. 32C17.
Key words and phrases. Fundamental groups, potential theory.
*Research partially supported by NSF grant DMS0306441.
ends, first introduced by Kropholler and Roller [KroR] in the group theoretic context and later given a topological interpretation by Geoghegan [Ge] (we work with Geoghegan’s topological notion in this paper). Theorem 0.1 for $X$ satisfying conditions (i) and (ii) together was first proved by Delzant and Gromov [DelG] using harmonic maps into trees (as in [GroS], [KoS1], [KoS2], [Sun]). They applied their result to the problem of determining which hyperbolic groups are Kähler. In this paper, we consider a different approach which yields the more complete result Theorem 0.1 and which is more elementary in the sense that it only uses harmonic functions. Cousin’s example [Co] of a 2-ended weakly 1-complete covering of an Abelian variety which has only constant holomorphic functions demonstrates that one cannot weaken the hypotheses to $\tilde{e}(X) \geq 2$. On the other hand, a slightly stronger version (Theorem 3.1) is obtained in the case in which each end separately (in an appropriate sense) has bounded geometry, is weakly 1-complete, or admits a positive Green’s function $G$ that vanishes at infinity.

Theorem 0.1 and elementary facts from geometric group theory together give the following:

**Theorem 0.2.** Let $X$ be a connected compact Kähler manifold with fundamental group $\Lambda = \pi_1(X)$ satisfying at least one of the following:

(a) (Gromov and Schoen [GroS]) $\Lambda$ admits a proper amalgamated product decomposition (i.e. $\Lambda = \Gamma_1 \ast_\Gamma \Gamma_2$ where the index of $\Gamma$ in $\Gamma_1$ is at least 3 and the index of $\Gamma$ in $\Gamma_2$ is at least 2); or

(b) (See [NR3]) $\Lambda$ is a properly ascending HNN extension.

Then some finite (unramified) covering of $X$ admits a surjective holomorphic mapping onto a curve of genus $g \geq 2$.

**Remarks.**

1. Conversely, if a connected compact manifold $M$ admits a surjective continuous mapping onto a curve $S$ of genus $g \geq 2$, then $\pi_1(M)$ admits a proper amalgamated product decomposition. For such a decomposition exists for $\pi_1(S)$ by Van Kampen’s theorem and one may pull this back to $\pi_1(M)$.

2. By considering the action of $\Lambda$ on the associated tree and applying Theorem 0.1, one gets the theorem in both cases (a) and (b) simultaneously (see the proof of Theorem 5.1). On the other hand, according to a theorem of Baumslag and Shalen (see Theorem 6 of Chapter 4 of [Bau]), a finitely presented group which can be expressed as a properly
ascending HNN extension, but not one with finitely generated base group, is virtually a proper amalgamated product. Thus Theorem 0.2 is actually contained within the theorem of Gromov and Schoen (i.e. the case (a)) together with part (ii) of Theorem 0.3 below and the theorem of Baumslag and Shalen.

Theorem 0.1 and its consequences also lead to new restrictions on Kähler groups (i.e. fundamental groups of compact Kähler manifolds).

**Theorem 0.3.** The following groups are not Kähler:

(i) Thompson’s groups $V$, $T$, and $F$; and

(ii) Any properly ascending HNN extension with finitely generated base group.

The question as to whether or not $F$ is Kähler was first posed by Geoghegan (see [Br]) and the first proof that $F$ is not Kähler appeared in [NR4]. Since $F$ is a properly ascending HNN extension with finitely generated base group, this may now be viewed as a special case of part (ii) of Theorem 0.3. Daniel Farley has independently obtained the result that $V$ and $T$ are not Kähler. For more on Kähler groups, the reader may refer to [Ar] and [ABCKT].

Before sketching the proof of Theorem 0.1, we make some remarks which put these results in context and we recall the required definitions. For $X$ a connected compact Kähler manifold, a natural and much studied problem is to determine when $X$ admits a surjective holomorphic mapping onto a curve of genus $g \geq 2$. According to the classical theorem of Castelnuovo and de Franchis (see [Be], [BarPV]), this is the case if and only if there exists a pair of linearly independent holomorphic 1-forms $\omega_1, \omega_2$ such that $\omega_1 \wedge \omega_2 \equiv 0$. The main point is that the meromorphic function $f \equiv \omega_1/\omega_2$ actually has no points of indeterminacy (this is a general fact about closed holomorphic 1-forms; see, for example, [NR2] for an elementary proof). Stein factorization then gives the required mapping. For some of the many other results in this context, the reader may refer to the work of Beauville (see [Cat1], [Car1], [Sim], [Sim1], [GroS], [JY1], [JY2], [Ar], and [ABCKT]).

For a connected noncompact complete Kähler manifold $(X, g)$, the analogous problem is to determine when $X$ admits a proper holomorphic mapping onto a Riemann surface. We will mainly consider the case in which $X$ has bounded geometry of order $k \geq 2$ (in the sense that there exists a constant $C > 0$ and, for each point $p \in X$, a biholomorphism $\Psi$ of the unit ball $B = B(0; 1) \subset \mathbb{C}^n$ onto a neighborhood $U$ of $p$ in $X$ such that $\Psi(0) = p$.
and, on $B$, $C^{-1}g_{Cn} \leq \Psi^*g \leq Cg_{Cn}$ and $|D^m\Psi^*g| \leq C$ for $m = 0, 1, 2, \ldots, k$; the case in which $X$ is weakly 1-complete (i.e. $X$ admits a continuous plurisubharmonic exhaustion function); or the case in which $X$ admits a positive Green’s function which vanishes at infinity. As shown in [Gro1], [Gro2], [L], and [NR1], if such an $X$ has at least 3 ends, then $X$ admits a proper holomorphic mapping onto a Riemann surface. The main step is to produce pluriharmonic functions $\rho_1, \rho_2$ which have different limits at infinity along the various ends and whose holomorphic differentials $\omega_1 = \partial \rho_1$, $\omega_2 = \partial \rho_2$ satisfy $\omega_1 \wedge \omega_2 \equiv 0$. In particular, $1, \rho_1, \rho_2$, and hence $d\rho_1, d\rho_2$, are linearly independent. If $\omega_1, \omega_2$ are linearly independent, then one Stein factors the holomorphic map

$$f = \frac{\omega_1}{\omega_2} : X \to \mathbb{P}^1.$$ 

Otherwise, one gets a holomorphic function $f = \rho_1 + c\rho_2$, for some constant $c \in \mathbb{C}$, which one may Stein factor. Thus it has been known for some time that the ends structure is relevant to the problem of finding a proper holomorphic mapping onto a Riemann surface.

We now recall the definitions of ends and filtered ends. Depending on the context, by an end of a connected manifold $M$, we will mean either a component $E$ of $M \setminus K$ with noncompact closure, where $K$ is a given compact subset of $M$, or an element of

$$\lim_{\leftarrow} \pi_0(M \setminus K)$$

where the limit is taken as $K$ ranges over the compact subsets of $M$ (or the compact subsets of $M$ whose complement $M \setminus K$ has no relatively compact components). The number of ends of $M$ will be denoted by $e(M)$. For a compact set $K$ such that $M \setminus K$ has no relatively compact components, we get an ends decomposition

$$M \setminus K = E_1 \cup \cdots \cup E_m,$$

where $E_1, \ldots, E_m$ are the distinct components of $M \setminus K$.

As in the work of Geoghegan [Ge], for $\tilde{Y} : \tilde{M} \to M$ the universal covering of $M$, consider the set

$$\lim_{\leftarrow} \pi_0[\tilde{Y}^{-1}(M \setminus K)],$$

where the limit is taken as $K$ ranges over the compact subsets of $M$ (or the compact subsets of $M$ whose complement $M \setminus K$ has no relatively compact components). Following [Ge], we will call elements of the above set filtered ends. The number of filtered ends of $M$ will be denoted by $\tilde{e}(M)$. Clearly, $\tilde{e}(M) \geq e(M)$. In fact, for $k \in \mathbb{N}$, we have $\tilde{e}(M) \geq k$ if
and only if there exists an ends decomposition $M \setminus K = E_1 \cup \cdots \cup E_m$ for $M$ such that, for $\Gamma_j = \text{im } [\pi_1(E_j) \to \pi_1(M)]$ for $j = 1, \ldots, m$, we have
\[ \sum_{j=1}^{m} [\pi_1(M) : \Gamma_j] \geq k. \]
Moreover, if \( \widehat{M} \to M \) is a connected covering space, then $\tilde{e}(\widehat{M}) \leq \tilde{e}(M)$ with equality if the covering is finite.

To illustrate some of the arguments in the proof of Theorem 0.1, let us consider the case in which $e(X) = 2$ and $X$ admits a positive Green’s function $G$ that vanishes at infinity. In this case, $X$ admits an ends decomposition $X \setminus K = E_1 \cup E_2$ such that the image $\Gamma$ of $\pi_1(E_1)$ in $\pi_1(X)$ is a proper subgroup. By standard arguments, there exists a pluriharmonic function $\rho : X \to (0,1)$ with finite energy such that
\[ \lim_{x \to \infty} \rho \mid_{E_1}(x) = 1 \quad \text{and} \quad \lim_{x \to \infty} \rho \mid_{E_2}(x) = 0. \]
In particular, $X$ is weakly 1-complete. Taking $\Upsilon : \widehat{X} \to X$ to be a connected covering space (not the universal covering) with $\Upsilon_*\pi_1(\widehat{X}) = \Gamma$, we see that $\Upsilon$ maps some component $\Omega_1$ of $\Upsilon^{-1}(E_1)$ isomorphically onto $E_1$ and the set $\Omega_2 = \Upsilon^{-1}(E_1) \setminus \Omega_1 \neq \emptyset$. Again, there exists a pluriharmonic function $\rho_2 : \widehat{X} \to (0,1)$ with finite energy such that
\[ \lim_{x \to \infty} \rho_2 \mid_{\Omega_1}(x) = 1 \quad \text{and} \quad \liminf_{x \to \infty} \rho_2 \mid_{\widehat{X} \setminus \Omega_1}(x) = 0. \]
If $\Gamma$ is of finite index, then $e(\widehat{X}) \geq 3$ and hence, by [NR1], $\widehat{X}$ admits a proper holomorphic mapping onto a Riemann surface. Since $\widehat{X} \to X$ is a finite covering in this case, $X$ also admits such a mapping. If $\Gamma$ is of infinite index, then the lift $\rho_1 = \rho \circ \Upsilon$ does not have finite energy and so $d\rho_1$ and $d\rho_2$ must be linearly independent. On the other hand, since $\rho_1$ and $\rho_2$ have compact levels in $\Omega_1$ over values near 1, we must have $\partial \rho_1 \wedge \partial \rho_2 \equiv 0$ (see, for example, [NR3], Lemma 2.1). It follows that some (nonempty) open subset of $\Omega_1 \cong E_1$ admits a proper holomorphic mapping onto a Riemann surface. Standard arguments now imply that this is the case for $X$.

For the general case, we will again pass to the appropriate covering spaces. We will produce suitable pluriharmonic functions $\rho_1$ and $\rho_2$ with prescribed values at infinity along filtered ends by applying the theory of massive sets as in Grigor’yan [Gri]. A version of the cup product lemma (see Lemma 2.7 below) will give $\partial \rho_1 \wedge \partial \rho_2 \equiv 0$ (Gromov [Gro2] was the first to notice the cup product lemma in the context of bounded geometry and
subsequent refinements were formed and used by others). The idea of filtered ends was
applied in some special cases in the context of Lefschetz type theorems in \[NR1\], \[NR2\],
and \[NR3\]. The arguments in this paper formalize this approach.

According to a theorem of Simpson \[Sim2\], if \(\Upsilon: \hat{X} \rightarrow X\) is a connected covering
space of a smooth projective variety \(X\), \(\rho\) is a nonconstant pluriharmonic function on \(\hat{X}\) such
that \(\partial\rho\) descends to a holomorphic 1-form \(\alpha\) on \(X\) and such that, for any \(\zeta \in \mathbb{C}\), the fiber
\(F = \rho^{-1}(\zeta)\) satisfies
\[
\sum_{L \text{ a component of } F} \left[\pi_1(\hat{X}) : \text{im} \left[\pi_1(L) \rightarrow \pi_1(\hat{X})\right]\right] > 1,
\]
then there exists a surjective holomorphic mapping \(\Phi\) of \(X\) onto a curve \(Y\) and a holomorphic
1-form \(\beta\) on \(Y\) with \(\alpha = \Phi^*\beta\) (Simpson also obtains a version for \(\alpha\) the push-forward of
d\(\rho\) for a real-valued pluriharmonic function \(\rho\)). Simpson’s result and Theorem 0.1 appear
to be related, but neither is known to imply the other.

Another application of Theorem 0.1, which will be applied in the proof of part (i) of
Theorem 0.3, is the following (cf. Theorem 3.6):

**Theorem 0.4.** Let \(X\) be a connected noncompact complete Kähler manifold which has
bounded geometry of order 2 or which is weakly 1-complete or which admits a positive
Green’s function \(G\) that vanishes at infinity. Assume that \(\hat{e}(X) \geq 2\) and \(\pi_1(X)\) is infinitely
generated. Then \(X\) admits a proper holomorphic mapping onto a Riemann surface.

This is an immediate consequence of Theorem 0.1 and the following fact:

**Lemma 0.5.** Let \(M\) be a connected \(C^\infty\) manifold such that \(\hat{e}(M) \geq 2\) and \(\Lambda = \pi_1(M)\) is
infinitely generated. Then \(\hat{e}(M) = \infty\).

**Proof.** If \(e(M) = 1\), then the image \(\Theta\) of the fundamental group of some end \(E = M \setminus K\)
is a proper subgroup of \(\Lambda\). If \(\left[\Lambda : \Theta\right] = \infty\), then \(\hat{e}(M) = \infty\) as claimed. If \(\left[\Lambda : \Theta\right] < \infty\),
then \(\Theta\) is infinitely generated and the finite covering \(\hat{M} \rightarrow M\) with \(\text{im} \left[\pi_1(\hat{M}) \rightarrow \Lambda\right] = \Theta\)
satisfies \(e(\hat{M}) \geq 2\) and \(\hat{e}(\hat{M}) = \hat{e}(M)\). Thus we may assume without loss of generality that
\(e(M) \geq 2\).

We may fix a \(C^\infty\) relatively compact domain \(\Omega\) in \(M\) such that \(M \setminus \overline{\Omega}\) has exactly
two components \(E_0'\) and \(E_1'\), each with noncompact closure. Fix a point \(x_0 \in \Omega\), let
\(\Gamma \equiv \text{im} \left[\pi_1(\Omega, x_0) \rightarrow \pi_1(M, x_0)\right]\), and, for \(i = 0, 1\), let \(E_i\) be the end defined by \(E_i = \overline{E_i'} \cup \Omega\)
and let $\Gamma_i \equiv \text{im} \left[ \pi_1(E_i, x_0) \to \pi_1(M, x_0) \right]$. Then $\Gamma_0 \cap \Gamma_1 = \Gamma$ (for example, by Van Kampen’s theorem), so

$$[\Lambda : \Gamma_0] \geq [\Gamma_1 : \Gamma_0 \cap \Gamma_1] = [\Gamma_1 : \Gamma].$$

If $\Gamma_1$ is finitely generated, then $[\Lambda : \Gamma_1] = \infty$ because $\Lambda$ is infinitely generated. Hence $\hat{c}(M) = \infty$ in this case. If $\Gamma_1$ is infinitely generated, then, since $\Gamma$ is finitely generated, $[\Gamma_1 : \Gamma] = \infty$. Hence $[\Lambda : \Gamma_0] = \infty$ by the above inequality and, again, $\hat{c}(M) = \infty$. □

We close this section with a proof that Thompson’s groups $V$ and $T$ are not Kähler (part (i) of Theorem 0.3). The proof that any properly ascending HNN extension with finitely generated base group (for example, Thompson’s group $F$) is not Kähler (part (ii) of Theorem 0.3) will be given in Section 5. The group $V$ is the group of right-continuous bijections $\lambda: [0, 1] \to [0, 1]$ such that $\lambda$ maps dyadic rational numbers to dyadic rational numbers, $\lambda$ is differentiable except at finitely many dyadic rational numbers, and $\lambda$ is affine with derivative a power of 2 on each interval on which it is differentiable. The group $T$ is the subgroup consisting of all $v \in V$ such that $v$ induces a homeomorphism of the circle $[0, 1)/0 \sim 1$, and $F$ is the subgroup consisting of all homeomorphisms in $V$. For the proof that $V$ and $T$ are not Kähler, we will apply the following fact:

**Lemma 0.6.** If $M$ is a connected compact manifold with fundamental group $V$ or $T$, then there exists a connected covering $\hat{M} \to M$ such that $\hat{e}(\hat{M}) = \infty$.

**Proof.** For a subset $S \subset [0, 1]$, let $V_S$ denote the subgroup of $V$ consisting of those elements whose restriction to $S$ is the identity and let $T_S = T \cap V_S$. Applying the work of Farley (see Proposition 6.1 of [F]) and the work of Sageev [S], one sees that $V_{[0,1/2)}$ and $T_{[0,1/2)}$ admit finite index subgroups $G$ and $H$, respectively, such that the group pairs $(V, G)$ and $(T, H)$ are multi-ended (actually, we have $G = V_{[0,1/2)}$, since $V_{[0,1/2)} \cong V$ and $V$ is infinite and simple). Now

$$G \subset V_{[0,1/2)} \subset \Gamma \equiv \bigcup_{n=1}^{\infty} V_{[0,4^{-1}+2^{-n})}.$$

As a proper increasing union of a sequence of groups, $\Gamma$ is infinitely generated. If $\pi_1(M) = V$, then, forming covering spaces $\hat{M} \to \hat{M} \to M$ with $\text{im} \left[ \pi_1(M) \to \pi_1(M) \right] = G$ and $\text{im} \left[ \pi_1(\hat{M}) \to \pi_1(M) \right] = \Gamma$, we get $\hat{e}(\hat{M}) \geq \hat{e}(\hat{M}) \geq e(\hat{M}) \geq 2$. Therefore, by Lemma 0.3 we have $\hat{e}(\hat{M}) = \infty$. A similar proof applies for $\pi_1(M) = T$. □
Proof that $V$ and $T$ are not Kähler (part (i) of Theorem 0.3). If $X$ is a connected compact Kähler manifold with $\Lambda = \pi_1(X)$ equal to $V$ or $T$, then, by Lemma 0.6 and Theorem 4.1, some finite covering $X'$ of $X$ admits a surjective holomorphic mapping with connected fibers onto a curve $S$ of genus $g \geq 2$. In particular, some finite index subgroup $\Lambda'$ of $\Lambda$ admits a surjective homomorphism onto a cocompact Fuchsian group $\Theta = \pi_1(S)$. However, $V$ and $T$ are infinite simple groups, so it follows that $\Lambda = \Lambda' \cong \Theta$. But any cocompact Fuchsian group is not simple, so we have arrived at a contradiction and, therefore, $V$ and $T$ are not Kähler. □

As mentioned above, Farley has independently obtained the result that $V$ and $T$ are not Kähler. In fact, he has recently shown that $e(V, V_{[0,1/2)}) = e(T, T_{[0,1/2)}) = \infty$, so one may obtain the desired proper holomorphic mapping onto a Riemann surface in the proof by applying [NR1] in place of Theorem 0.1 (in his first proof, Farley applied Theorem 0.1 and a theorem of Klein [Kl] according to which, if $G$ is a finitely generated group, $K < H < G$, $[G : H] = [H : K] = \infty$, and $\tilde{e}(G, K) > 1$, then $\tilde{e}(G, H) = \infty$).

Section 1 contains a summary of the required facts from the theory of massive sets [Gr1]. In Section 2 we consider the required versions of the cup product lemma. Section 3 contains the proof of Theorem 0.1. Section 4 contains consequences for compact Kähler manifolds. Finally, Section 5 contains the details of the proof of the theorem of Gromov and Schoen (Theorem 0.2) using Theorem 0.1 as well as a proof that a properly ascending HNN extension with finitely generated base group is not Kähler (part (ii) of Theorem 0.3). Section 6 (which does not appear in the version submitted for publication) provides, for the convenience of the reader, the proof of Sario’s existence theorem of principal functions [RS] and Nakai’s construction of the Evans-Selberg potential [Na1, Na2, SaNo]. These facts were applied in [NR1] (and, therefore, indirectly here). However, it is difficult to find proofs for a general oriented Riemannian manifold in a convenient form in the literature.

Acknowledgement. We would like to thank Misha Kapovich for bringing the work of Delzant and Gromov to our attention and for helping us understand the geometric group theory component. We would also like to thank Matt Brin, Ross Geoghegan, and John Meier for many helpful conversations on filtered ends and Thompson’s groups. We would like to further thank Ross Geoghegan for providing a copy of his forthcoming book.
1. Massive sets

By applying the work of Sario, Nakai, and their collaborators (see [Na1], [Na2], [SaNa], [SaNo], [RS], and also [LT]), one can produce independent harmonic functions by prescribing limiting values along ends. This is the main fact from potential theory applied in [NRI]. In order to produce independent harmonic functions on a manifold with a small number of ends, it is natural to consider massive sets (in place of ends) as studied by Grigor’yan [Gri]. Throughout this section, $(M, g)$ will denote a connected Riemannian manifold.

Definition 1.1. Let $U$ be an open subset of $M$.

(a) A bounded nonnegative continuous subharmonic function $\alpha$ on $M$ such that

$$\alpha \equiv 0 \text{ on } M \setminus U \quad \text{and} \quad \sup_M \alpha = \sup_U \alpha > 0$$

is called an admissible subharmonic function for $U$.

(b) If there exists an admissible subharmonic function for $U$, then $U$ is called massive.

(c) If there exists an admissible subharmonic function for $U$ with finite energy, then $U$ is called $D$-massive.

Remarks. 1. If $M$ is a Kähler manifold, then a plurisubharmonic function is subharmonic. If $\alpha$ is a plurisubharmonic admissible subharmonic function for $U$, then we will simply call $\alpha$ an admissible plurisubharmonic function for $U$ and we will say that $U$ is plurimassive. If, in addition, $\alpha$ has finite energy, then we will say that $U$ is pluri-$D$-massive.

2. To say that a continuous function $\alpha$ has finite energy is to say that $\alpha \in W^{1,2}_{\text{loc}}(M, g)$ and

$$\int_M |d\alpha|^2_g dV_g < \infty.$$  

3. If $M$ contains a proper massive subset $U$, then $M$ is hyperbolic; i.e. $M$ admits a positive Green’s function $G$. Moreover, if $\alpha$ is an admissible subharmonic function for $U$, then $\lim_{j \to \infty} G(x_j, \cdot) = 0$ for any sequence $\{x_j\}$ in $M$ such that $\alpha(x_j) \to \sup_U \alpha$ as $j \to \infty$.

4. An end $E \subset M$ with $E \neq M$ is a hyperbolic end if and only if $E$ is a massive set. In fact, a hyperbolic end $E$ is $D$-massive. For if we fix a $C^\infty$ relatively compact domain $\Omega$ in $M$ such that $\partial E \subset \Omega$ and $M \setminus \Omega$ has no compact components and we let $u : M \setminus \Omega \to [0, 1)$ be the harmonic measure of the ideal boundary of $M$ with respect to $M \setminus \overline{\Omega}$, then, for
0 < \epsilon < 1, the function

\[ \alpha \equiv \begin{cases} 
\max(u - \epsilon, 0) & \text{on } E \setminus \Omega \\
0 & \text{on } (M \setminus E) \cup \Omega 
\end{cases} \]

is a finite energy admissible subharmonic function for \( E \). Observe also that, for a sequence \( \{x_j\} \) in \( E \), we have, as \( j \to \infty \),

\[ G(x_j, \cdot) \to 0 \iff u(x_j) \to 1 \iff \alpha(x_j) \to \sup_M \alpha. \]

**Proposition 1.2** (See [Gri]). Suppose \( U \) is a proper massive subset of \( M \) and \( \alpha : M \to [0,1) \) is an admissible subharmonic function for \( U \) with \( \sup_M \alpha = 1 \). Then there exists a harmonic function \( \rho : M \to (0,1] \) with the following properties:

(i) \( \alpha \leq \rho \leq 1 \) on \( M \);

(ii) If \( M \setminus \overline{U} \) is massive and \( \beta : M \to [0,1) \) is an admissible subharmonic function for \( M \setminus \overline{U} \), then \( 0 < \rho \leq 1 - \beta \) on \( M \).

(iii) If \( \alpha \) has finite energy (hence \( U \) is \( D \)-massive), then \( \rho \) has finite energy. In fact,

\[ \int_M |d\rho|^2_g dV \leq \int_M |d\alpha|^2_g dV. \]

**Remark.** If \( M \setminus \overline{U} \) is massive, then \( \rho \) is nonconstant and the maximum principle gives \( 0 < \rho < 1 \) on \( M \).

**Proof.** We may choose a sequence of \( C^\infty \) domains \( \{\Omega_m\}_{m=1}^\infty \) in \( M \) such that

\[ \Omega_m \Subset \Omega_{m+1} \text{ for } m = 1, 2, 3, \ldots \quad \text{and} \quad \bigcup_{m=1}^\infty \Omega_m = M. \]

For each \( m \), let \( \rho_m : \overline{\Omega_m} \to [0,1) \) be the continuous function satisfying

\[ \Delta \rho_m = 0 \text{ in } \Omega_m \quad \text{and} \quad \rho_m = \alpha \text{ on } \partial \Omega_m. \]

Since \( \alpha \) is subharmonic, we have \( \alpha \leq \rho_m \) on \( \overline{\Omega_m} \) and, in particular, on \( \partial \Omega_m \subset \Omega_{m+1} \), we have \( \rho_m = \alpha \leq \rho_{m+1} \). Thus \( \rho_m \leq \rho_{m+1} \) on \( \Omega_m \) and hence \( \rho_m \not> \rho \) for some harmonic function \( \rho : M \to (0,1] \) with \( \alpha \leq \rho \leq 1 \) on \( M \).

If \( \beta : M \to [0,1) \) is an admissible subharmonic function for \( M \setminus \overline{U} \), then the superharmonic function \( 1 - \beta \) satisfies, for each \( m = 1, 2, 3, \ldots \),

\[ \rho_m = \alpha = 0 \leq 1 - \beta \text{ on } (\partial \Omega_m) \setminus U \quad \text{and} \quad \rho_m = \alpha \leq 1 = 1 - \beta \text{ on } (\partial \Omega_m) \cap U. \]

It follows that \( \rho_m \leq 1 - \beta \) on \( \Omega_m \) and hence \( 0 < \rho \leq 1 - \beta \) on \( M \).
Finally, suppose $\alpha$ has finite energy. Then, since harmonic functions minimize energy, we have, for each $m$,
\[ \int_{\Omega} |d\rho_m|^2 g_dV_g \leq \int_{\Omega} |d\alpha|^2 g_dV_g \leq \int_M |d\alpha|^2 g_dV_g. \]
Therefore, since $d\rho_m \to d\rho$ uniformly on compact sets, we get
\[ \int_M |d\rho|^2 g_dV_g \leq \int_M |d\alpha|^2 g_dV_g. \]
\[ \square \]

2. Special ends and the cup product lemma

It will be convenient to have the terminology contained in the next two definitions.

**Definition 2.1.** Let $S$ be a subset of a complex manifold $X$ of dimension $n$.

(a) For $g$ a Hermitian metric on $X$ and $k$ a nonnegative integer, we will say that $(X, g)$ has bounded geometry of order $k$ along $S$ if, for some constant $C > 0$ and for every point $p \in S$, there is a biholomorphism $\Psi$ of the unit ball $B = B(0; 1) \subset \mathbb{C}^n$ onto a neighborhood of $p$ in $X$ such that $\Psi(0) = p$ and, on $B$,
\[ C^{-1}g_{\mathbb{C}^n} \leq \Psi^*g \leq Cg_{\mathbb{C}^n} \quad \text{and} \quad |D^m\Psi^*g| \leq C \text{ for } m = 0, 1, 2, \ldots, k. \]

(b) We will say that $X$ is weakly 1-complete along $S$ if there exists a continuous plurisubharmonic function $\varphi$ on $X$ such that
\[ \{ x \in S \mid \varphi(x) < a \} \subseteq X \quad \forall a \in \mathbb{R}. \]

**Remark.** Both (a) and (b) hold if $S \subseteq X$.

**Definition 2.2.** We will call an end $E \subset X$ in a connected noncompact complete Kähler manifold $(X, g)$ special if $E$ is of at least one of the following types:

(BG) $(X, g)$ has bounded geometry of order 2 along $E$;

(W) $X$ is weakly 1-complete along $E$;

(RH) $E$ is a hyperbolic end and the Green’s function vanishes at infinity along $E$; or

(SP) $E$ is a parabolic end, the Ricci curvature of $g$ is bounded below on $E$, and there exist positive constants $R$ and $\delta$ such that
\[ \text{vol} \left( B(p; R) \right) > \delta \quad \forall p \in E. \]
An ends decomposition for $X$ in which each of the ends is special will be called a \textit{special ends decomposition}.

\textbf{Remarks.} 1. (BG) stands for “bounded geometry,” (W) for “weakly 1-complete,” (RH) for “regular hyperbolic,” and (SP) for “special parabolic.”  
2. A parabolic end of type (BG) is also of type (SP).  
3. If $E$ and $E'$ are ends with $E' \subset E$ and $E$ is special, then $E'$ is special.  
4. For our purposes, it is generally enough to replace the condition (BG) by the condition that $E$ is a hyperbolic end along which $(X,g)$ has bounded geometry of order 0. However, we then lose the condition described in Remark 3 above and so the existence of a special ends decomposition is no longer determined by the set of ends $\lim_{\leftarrow} \pi_0(X \setminus K)$.  

\textbf{Lemma 2.3.} Suppose $(X,g)$ is a connected complete Kähler manifold with an ends decomposition 

$$X \setminus K = E_1 \cup E_2 \cup \cdots \cup E_m;$$

where $m \geq 2$, $E_1$ is a special end of the type (W), (RH), or (SP) in Definition 2.2 and, for $j = 2,3,\ldots,m$, $E_j$ is a hyperbolic or special end. Then $X$ is weakly 1-complete along $E_1$ (i.e. $E_1$ is of type (W)).

\textit{Proof.} We may arrange $E_2,\ldots,E_m$ so that, for some $k$ with $1 \leq k \leq m$, $E_2,\ldots,E_k$ are ends of type (W) and $E_{k+1},\ldots,E_m$ are ends which are not of type (W). We set $E' = E_2 \cup \cdots \cup E_k$ and $E'' = E_{k+1} \cup \cdots \cup E_m$. We have a continuous plurisubharmonic function $\varphi : X \to \mathbb{R}$ such that $\{ x \in E' \mid \varphi(x) < a \} \Subset X$ for each $a \in \mathbb{R}$ and, by fixing a domain $\Omega$ with $K \subset \Omega \Subset X$ and replacing $\varphi$ with the function

\[ \begin{cases} 
\max(\varphi - \max_{\overline{\Omega}} \varphi - 1,0) & \text{on } E' \setminus \Omega \\
0 & \text{on } \Omega \cup (X \setminus E')
\end{cases} \]

we may assume that $\varphi \geq 0$ on $X$ and $\varphi \equiv 0$ on the connected set 

$$\Omega \cup (X \setminus E') = \Omega \cup E_1 \cup E''.$$ 

By a theorem of [Nk] and [Dem], the component $Y$ of $\{ x \in X \mid \varphi(x) < 1 \}$ containing $\Omega \cup (X \setminus E')$ admits a complete Kähler metric $h$ such that $h = g$ on a neighborhood of $\Omega \cup (X \setminus E')$ (one must modify their proofs slightly since the associated plurisubharmonic function $-\log(1-\varphi)$ exhausts $Y$ at the compact boundary $\partial Y$ but not entirely along $Y$). Thus $E' \cap Y$ is a union of ends of $Y$ of type (RH) and hence we may assume without
loss of generality that $E_2, \ldots, E_k$ are of type (RH) (as well as type (W)). Thus, for each $j = 2, \ldots, m$, $E_j$ is hyperbolic or $E_j$ is of type (SP). If $E_1$ is of type (SP), then Theorem 2.6 of [NR1] (which is contained implicitly in the work of Sario, Nakai, and their collaborators [Na1], [Na2], [SaNa], [SaNo], [RS] together with the work of Sullivan [Sul]) provides a pluriharmonic function $\rho: X \to \mathbb{R}$ such that $\lim_{x \to \infty} \rho |_{E_1}(x) = \infty$. In particular, $E_1$ is of type (W) in this case. Thus we may assume that $E_1$ is of type (RH). Moreover, if, for some $j$ with $k + 1 \leq j \leq m$, $E_j$ were of type (SP), then $X$ would be weakly 1-complete along $E_j$ and hence $E_j$ would have been included in $E'$. Therefore, $E_{k+1}, \ldots, E_m$ must be hyperbolic ends.

Thus we have hyperbolic, and therefore $D$-massive, ends $E_1, \ldots, E_m$ with $m \geq 2$ and $E_1$ is of type (RH). By Remark 4 following Definition 1.1 there is a finite energy admissible subharmonic function $\alpha: X \to [0, 1)$ for $E_1$ such that

$$\lim_{x \to \infty} \alpha |_{E_1}(x) = 1$$

and, applying Proposition 1.2 ($X \setminus E_1 \supset E' \cup E''$ is massive), we get a finite energy harmonic function $\rho: X \to (0, 1)$ such that $\alpha \leq \rho < 1$ on $X$. The Gaffney theorem [Ga] implies that $\rho$ is pluriharmonic and we have

$$1 > \rho(x) \geq \alpha(x) \to 1$$

as $x \to \infty$ in $E_1$.

Therefore $X$ is weakly 1-complete along $E_1$ (with plurisubharmonic function $-\log(1 - \rho)$ exhausting $E_1$) in this case as well.

As described in the introduction, the main goal of this paper is to obtain a filtered ends version of the following:

**Theorem 2.4** ([Gro1], [L], [Gro2], and Theorem 3.4 of [NR1]). If $(X, g)$ is a connected complete Kähler manifold which admits a special ends decomposition and $e(X) \geq 3$, then $X$ admits a proper holomorphic mapping onto a Riemann surface.

The following lemma is well known (see, for example, the proof of Theorem 4.6 of [NR1]):

**Lemma 2.5.** Let $(X, g)$ be a connected complete Kähler manifold which is compact or which admits a special ends decomposition. If some nonempty open subset of $X$ admits a surjective proper holomorphic mapping onto a Riemann surface, then $X$ admits a surjective proper holomorphic mapping onto a Riemann surface.
Lemma 2.6 (See Lemma 2.1 of [NR3]). Let $\rho_1$ and $\rho_2$ be two real-valued pluriharmonic functions on a connected complex manifold $X$. If $\rho_1$ has a nonempty compact fiber, then $\partial \rho_1 \wedge \partial \rho_2 \equiv 0$ on $X$. Furthermore, if the differentials $d\rho_1$ and $d\rho_2$ are (globally) linearly independent for some such pair of functions, then some nonempty open subset of $X$ admits a proper holomorphic mapping onto a Riemann surface.

Remark. Two real-valued pluriharmonic functions $\rho_1$ and $\rho_2$ on a connected complex manifold have linearly dependent differentials $d\rho_1$ and $d\rho_2$ (i.e. 1, $\rho_1$, $\rho_2$ are linearly dependent functions) if and only if $d\rho_1 \wedge d\rho_2 \equiv 0$.

Lemma 2.7. Let $(X,g)$ be a connected complete Kähler manifold and let $\rho_1$ and $\rho_2$ be two real-valued pluriharmonic functions on a domain $Y \subset X$. Assume that, for some constant $a$ with $\inf \rho_1 < a < \sup \rho_1$, some component $\Omega$ of $\{ x \in Y \mid a < \rho_1(x) \}$ has the following properties:

(i) $\overline{\Omega} \subset Y$;
(ii) $|d\rho_1|^g$ is bounded on $\Omega$; and
(iii) $\int_{\Omega} |d\rho_j|^2_g dV_g < \infty$ for $j = 1, 2$.

Then $\partial \rho_1 \wedge \partial \rho_2 \equiv 0$ on $Y$.

Furthermore, if $d\rho_1$ and $d\rho_2$ are linearly independent and $(X,g)$ has bounded geometry along $\Omega$, then $\Omega$ admits a proper holomorphic mapping onto a Riemann surface.

Remark. In this paper, we will only need the fact that some nonempty open subset of $\Omega$ admits a proper holomorphic mapping onto a Riemann surface.

Proof of Lemma 2.7. We may assume without loss of generality that $\rho_1$ and $\rho_2$ are non-constant. We denote the Levi form of a $C^2$ function $\varphi$ by

$$\mathcal{L}(\varphi) = \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j.$$ 

The Hermitian tensor

$$h = g + \mathcal{L}(-\log(\rho_1 - a)) = g + (\rho_1 - a)^{-2} \partial \rho_1 \overline{\partial \rho_1}$$

is a complete Kähler metric on $\Omega$ with $h \geq g$ (see [NK] or [Dem]). Moreover, on $\Omega$, we have

$$|\partial_1|^2_h = \left[1 + (\rho_1 - a)^{-2} |\partial \rho_1|^2_g \right]^{-1} |\partial \rho_1|^2_g, \quad dV_h = \left[1 + (\rho_1 - a)^{-2} |\partial \rho_1|^2_g \right] dV_g,$$

and $|\partial_2|^2_h \leq |\partial \rho_2|^2_g$. 

For \( g = h \) at any point \( p \in \Omega \) where \( (\partial \rho_1)_p = 0 \) while, at any point \( p \in \Omega \) where \( (\partial \rho_1)_p \neq 0 \), one may get the above by writing \( g \) and \( h \) in terms of a \( g \)-orthonormal basis \( e_1, \ldots, e_n \) for \( T_p^{1,0} \Omega \) with dual basis
\[
e_1^* = |\partial \rho_1|^{-1}(\partial \rho_1)_p e^*_2, \ldots, e^*_n.\
\]
In particular, we have
\[
|\partial \rho_1|^2_h dV_h = |\partial \rho_1|^2_g dV_g
\]
and hence \( \rho_1 \mid \Omega \) has finite energy with respect to \( h \) as well as \( g \).

Following [Gro2], we consider the closed holomorphic 1-forms
\[
\alpha_j = \partial \rho_j = \beta_j + i \gamma_j \quad \text{for } j = 1, 2,
\]
where
\[
\beta_j = \frac{1}{2}(\partial \rho_j + \bar{\partial} \rho_j) = \frac{1}{2} d\rho_j \quad \text{and} \quad \gamma_j = \frac{1}{2i}(\partial \rho_j - \bar{\partial} \rho_j) = \frac{1}{2} d^c \rho_j
\]
are closed real 1-forms (since \( \rho_j \) is pluriharmonic) for \( j = 1, 2 \). Thus we get a closed holomorphic 2-form
\[
\alpha_1 \wedge \alpha_2 = \eta + i \theta
\]
on \( Y \) where
\[
\eta = \beta_1 \wedge \beta_2 - \gamma_1 \wedge \gamma_2 \quad \text{and} \quad \theta = \beta_1 \wedge \gamma_2 + \gamma_1 \wedge \beta_2
\]
are closed real 2-forms. Moreover, \( \alpha_1 \wedge \alpha_2 \) is in \( L^2 \) with respect to \( h \) on \( \Omega \) because
\[
|\alpha_1 \wedge \alpha_2|^2_h dV_h \leq |\alpha_1|^2_h |\alpha_2|^2_h dV_h = |\partial \rho_1|^2_h |\rho_2|^2_h dV_h = |\partial \rho_2|^2_h |\rho_1|^2_h dV_g \leq |\partial \rho_2|^2_g |\rho_1|^2_g dV_g
\]
while \( |\partial \rho_1|_g \) is bounded on \( \Omega \) and \( \rho_2 \mid \Omega \) is of finite energy with respect to \( g \). Furthermore, the closed form \( \alpha_1 \wedge \alpha_2 \) is harmonic with respect to the complete Kähler metric \( h \) (and \( g \)) because \( \bar{\partial}^*_h(\alpha_1 \wedge \alpha_2) = 0 \) (since \( \alpha_1 \wedge \alpha_2 \) is of type \( (2,0) \)) and \( \bar{\partial}(\alpha_1 \wedge \alpha_2) = 0 \). Therefore
\[
d(\alpha_1 \wedge \alpha_2) = d\eta = d\theta = 0 \quad \text{and} \quad d^*_h(\alpha_1 \wedge \alpha_2) = d^*_h \eta = d^*_h \theta = 0
\]
\( (L^2 \) harmonic forms are closed and coclosed by the Gaffney theorem [Ga]).

For each \( R > 0 \), let \( \varphi_R : \Omega \to (0,R] \) and \( \psi_R : \Omega \to [-R,R] \) be the bounded locally Lipschitz functions given by, for each \( x \in \Omega \),
\[
\varphi_R(x) = \frac{1}{2} \min(\rho_1(x) - a, R)
\]
and

\[
\psi_R(x) = \begin{cases} 
\frac{1}{2} R & \text{if } \rho_2(x) \leq -R \\
-\frac{1}{2} \rho_2(x) & \text{if } -R < \rho_2(x) < R \\
-\frac{1}{2} R & \text{if } \rho_2(x) \geq R
\end{cases}
\]

Since \( \rho_1 \) and \( \rho_2 \) are nonconstant pluriharmonic functions, the sets \( \{ x \in Y \mid \rho_1(x) = a + R \} \) and \( \{ x \in Y \mid |\rho_2(x)| = R \} \) are sets of measure 0 and we have

\[
d\varphi_R = \begin{cases} 
\frac{1}{2} d\rho_1 & \text{on } \{ x \in \Omega \mid \rho_1(x) < a + R \} \\
0 & \text{on } \{ x \in \Omega \mid \rho_1(x) > a + R \}
\end{cases}
\]

and

\[
d\psi_R = \begin{cases} 
-\frac{1}{2} d\rho_2 & \text{on } \{ x \in \Omega \mid |\rho_2(x)| < R \} \\
0 & \text{on } \{ x \in \Omega \mid |\rho_2(x)| > R \}
\end{cases}
\]

Thus

\[
\beta_1 \wedge \gamma_2 = \frac{1}{2} d\rho_1 \wedge \gamma_2 = d\left[ \frac{1}{2} \rho_1 \gamma_2 \right] = d[\varphi_R \gamma_2] \quad \text{when } \rho_1 < a + R
\]

and

\[
\gamma_1 \wedge \beta_2 = \gamma_1 \wedge \frac{1}{2} d\rho_2 = d\left[ -\frac{1}{2} \rho_2 \gamma_1 \right] = d[\psi_R \gamma_1] \quad \text{when } |\rho_2| < R.
\]

Moreover, \( \varphi_R \gamma_2 \) is in \( L^2 \) with respect to \( h \) on \( \Omega \) because

\[
|\varphi_R \gamma_2|_h^2 dV_h \leq \varphi_R^2 |\partial \rho_2|_h^2 dV_h \leq \varphi_R^2 |\partial \rho_2|_g^2 dV_h = \varphi_R^2 |\partial \rho_2|_g^2 (1 + (\rho_1 - a)^2 |\partial \rho_1|_g^2) dV_g
\]

\[
= (\varphi_R^2 + \varphi_R^2 (\rho_1 - a)^2 |\partial \rho_1|_g^2) |\partial \rho_2|_g^2 dV_g \leq (R^2 + |\partial \rho_1|_g^2) |\partial \rho_2|_g^2 dV_g
\]

while \( |\partial \rho_1|_g \) is bounded on \( \Omega \) and \( \rho_2 \mid_\Omega \) has finite \( g \)-energy. The form \( \psi_R \gamma_1 \) is also in \( L^2 \) with respect to \( h \) because \( |\psi_R| \leq R \) and \( \rho_1 \mid_\Omega \) has finite energy with respect to \( h \) (as well as \( g \)). Thus the form \( \lambda_R = \varphi_R \gamma_2 + \psi_R \gamma_1 \) is in \( L^2 \) with respect to \( h \) on \( \Omega \) and, as \( R \to \infty \), \( \lambda_R \) converges pointwise to \( \lambda = \frac{1}{2} (\rho_1 - a) \gamma_2 - \frac{1}{2} \rho_2 \gamma_1 \).

Moreover, \( d\lambda_R \to d\lambda = \theta \) in \( L^2 \) with respect to \( h \) as \( R \to \infty \). For \( d\lambda_R = \theta \) when we have both \( \rho_1 < a + R \) and \( |\rho_2| < R \), so \( d\lambda_R \to \theta \) pointwise as \( R \to \infty \). We also have (almost everywhere)

\[
|d\lambda_R|_h \leq |d\varphi_R \wedge \gamma_2|_h + |d\psi_R \wedge \gamma_1|_h \leq |d\varphi_R|_h |\gamma_2|_h + |d\psi_R|_h |\gamma_1|_h \leq |\beta_1|_h |\gamma_2|_h + |\beta_2|_h |\gamma_1|_h
\]
and the last expression is in $L^2$ with respect to $h$ (by an argument similar to that showing that $\alpha_1 \wedge \alpha_2$ is in $L^2$ with respect to $h$). Thus the Lebesgue dominated convergence theorem gives the claim.

By the Gaffney theorem [Ga], we get

$$0 = \int_\Omega \langle 0, \lambda_R \rangle_h \, dV_h = \int_\Omega \langle d^* h \theta, \lambda_R \rangle_h \, dV_h = \int_\Omega \langle \theta, d\lambda_R \rangle_h \, dV_h \rightarrow \int_\Omega |\theta|^2 h \, dV_h$$

as $R \rightarrow \infty$. Thus $\theta \equiv 0$ on $\Omega$ and hence, since $\alpha_1 \wedge \alpha_2 = \eta + i\theta$ is a holomorphic 2–form, we get $\alpha_1 \wedge \alpha_2 \equiv 0$ on $\Omega$ and, therefore, on $Y$.

Assuming now that $d\rho_1$ and $d\rho_2$ are linearly independent and that $(X,g)$ has bounded geometry along $\Omega$, the arguments in [Gro2], [ArBR] (see also Chapter 4 of [ABCKT]), together with some easy observations, give the required proper holomorphic mapping of $\Omega$ onto a Riemann surface. For the convenience of the reader, we include a sketch of the arguments.

We may assume $n = \dim X > 1$. Because we have $\alpha_1 \wedge \alpha_2 = \partial \rho_1 \wedge \partial \rho_2 \equiv 0$, the meromorphic function $\frac{\alpha_1}{\alpha_2} : \Omega \rightarrow \mathbb{P}^1$ is actually a holomorphic map (i.e. $\alpha_1/\alpha_2$ has no points of indeterminacy). If this function is equal to a constant $\zeta$, then $\rho_1 - \zeta \rho_2$ is a nonconstant holomorphic function on $\Omega$. In any case, we get a nonconstant holomorphic map $f : \Omega \rightarrow \mathbb{P}^1$ such that $\rho_1$ and $\rho_2$ are constant on each level of $f$. In fact, $f$ is locally constant on the (complex) analytic set

$$A = \{ x \in \Omega \mid (\alpha_1)_x = (\alpha_2)_x = 0 \}$$

and the levels of $f \mid_{\Omega \setminus A}$ are precisely the (smooth) leaves of the holomorphic foliation determined by $\alpha_1$ and $\alpha_2$ in $\Omega \setminus A$ (see, for example, [NR2], pp. 387–388). If $L$ is a level of $f$, then $\rho_1$ is equal to a constant $t$ on $L$ with $a < t$ and $\bar{L} \subset \overline{\Omega} \subset Y$. It follows that $\bar{L} \subset \Omega$ (since $\Omega$ is a component of $\{ x \in Y \mid \rho_1(x) > a \}$) and hence that $L = \bar{L} \cap \Omega = L$. Thus $L$ is closed as an analytic subset of $X$.

The coarea formula for the map $\Phi = (\rho_1, \rho_2) : \Omega \rightarrow \mathbb{R}^2$ gives us

$$\int_{\mathbb{R}^2} \text{vol}_{\Phi^{-1}(t_1,t_2)} (\Phi^{-1}(t_1,t_2)) \, dt_1 \wedge dt_2 = \int_\Omega |d\rho_1 \wedge d\rho_2|_g \, dV_g < \infty.$$ 

Hence $\text{vol} (\Phi^{-1}(t)) < \infty$ for almost every point $t \in \mathbb{R}^2$. Thus we may fix a regular value $t_0$ in the interior of $\Phi(\Omega)$, with $t_0$ in the complement of the countable set $\Phi(A)$, such that $\text{vol} (\Phi^{-1}(t_0)) < \infty$. Since $\Phi$ is constant on each leaf of the foliation in $\Omega \setminus A$ and $\Phi^{-1}(t_0) \subset \Omega \setminus A$ is a $C^\infty$ submanifold of $\Omega$ (not just $\Omega \setminus A$) with $\dim_{\mathbb{R}} \Phi^{-1}(t_0) = 2n - 2$,
we see that a component $L_0$ of $\Phi^{-1}(t_0)$ is a leaf of the foliation in $\Omega \setminus A$, $L_0$ is closed in $X$, and $L_0$ is a level of $f$.

Since $(X, g)$ has bounded geometry along $\Omega$, Lelong’s monotonicity formula (see 15.3 in [Chi]) shows that there is a constant $C > 0$ such that each point $p \in \Omega$ has a neighborhood $U_p$ in $X$ such that $\text{diam}_X U_p < 1$ and $\text{vol} (D \cap U_p) \geq C$ for every complex analytic set $D$ of pure dimension $n - 1$ in $X$ with $p \in D$. Therefore, since $L_0$ has finite volume, $L_0$ must be compact. Thus $f : \Omega \to \mathbb{P}^1$ has a compact level $L_0 \subset \Omega$.

It follows that the set $V = \{ x \in \Omega \mid x \text{ lies in a compact level of } f \}$ is a nonempty open subset of $\Omega$. To show that $V$ is also closed relative to $\Omega$, let $V_0$ be a component of $V$, let $\{x_j\}$ be a sequence in $V_0$ converging to a point $p \in \overline{V_0} \cap \Omega$, and, for each $j$, let $L_j \subset V_0$ be the compact level of $f$ through $x_j$. Stein factoring $f \mid_{V_0}$, we get a proper holomorphic mapping $\Psi : V_0 \to W$ with connected fibers of $V_0$ onto a Riemann surface $W$. We may choose each $x_j$ to lie over a regular value of $f$ and of $\Psi$. Applying Stokes’ theorem as in [Ste], we see that $\text{vol} (L_j)$ is constant in $j$ and so the above volume estimate implies that, for some $R \gg 0$, we have $L_j \subset B(p; R)$ for $j = 1, 2, 3, \ldots$. On the other hand, by [Ste] (see [TW] and Theorem 4.23 in [ABCKT]), a subsequence of $\{L_j\}$ converges to the level $L$ of $f$ through $p$. So we must have $L \subset B(p; R) \cap \Omega$ and hence, since $L$ is a closed analytic subset of $X$, $L$ must be compact. Thus $p \in \overline{V_0} \cap V$ and, therefore, $p \in V_0$. It follows that $V = V_0 = \Omega$. Thus every level of $f$ is compact and we get our proper holomorphic map $\Psi : \Omega \to W$. \hfill \Box

### 3. Filtered ends and mappings to Riemann surfaces

Theorem 0.1 is an immediate consequence of the following theorem which will be proved in this section:

**Theorem 3.1.** If $(X, g)$ is a connected complete Kähler manifold which admits a special ends decomposition and $\bar{e}(X) \geq 3$, then $X$ admits a proper holomorphic mapping onto a Riemann surface.

We first consider two lemmas which will allow us to replace special ends of type (W) with special ends of type (RH).

**Lemma 3.2.** Let $M$ be a connected noncompact $C^\infty$ manifold and let $k \in \mathbb{N}$. 

Proof. For the proof of (a), we fix a point \( x \) at the image \( \alpha \) of \( \U \) components of \( \Omega \) with \( F \) and \( \partial E \) endpoints in \( \Omega \cap \U \). We may now choose a compact set \( D \) in \( \Omega \) such that, if \( x \) belongs to \( \U \) and we may let \( B \) be the finite set of loops in \( \Omega \) given by \( \pi_1(\Omega) : \text{im} [\pi_1(F) \to \pi_1(\Omega)] \geq k \).

(b) If \( \varepsilon(M) \geq k \), then there exists a compact set \( D \subseteq M \) such that, for every domain \( \Omega \) containing \( D \), we have \( \varepsilon(\Omega) \geq k \).

Suppose \( \Omega \) is a domain containing \( D \) and \( \Omega \setminus K = F_1 \cup \cdots \cup F_m \) is an ends decomposition for \( \Omega \) with \( F = F_1 \subseteq E \). The intersection \( \Omega \cap E \) is connected. For a path in \( \Omega \) with endpoints in \( \Omega \cap E \) which leaves \( E \) must meet \( D \supseteq \hat{D} \supseteq \partial E \). Hence the segments between the endpoints and the first and last points in \( D \) together with a path in \( D \cap E \) joining these first and last points yields a path in \( \Omega \cap E \) between the endpoints. Thus \( \Omega \cap E \) is an end of \( \Omega \). In particular, we may fix a point \( y_0 \in F \) and a path \( \lambda \) in \( \Omega \cap E \) from \( x_0 \) to \( y_0 \), and we may let \( B \) be the finite set of loops in \( \Omega \) given by \( B = \{ \lambda^{-1} \ast \alpha \ast \lambda \mid \alpha \in A \} \). If \( \alpha, \beta \in A \) and \( [\lambda^{-1} \ast \beta \ast \lambda]^{-1} \ast [\lambda^{-1} \ast \alpha \ast \lambda] \) is homotopic in \( \Omega \) to a loop in \( F \), then \( \beta^{-1} \ast \alpha \) is homotopic in \( X \) to a loop in \( E \) and hence \( \alpha = \beta \). Thus

\[
[\pi_1(\Omega) : \text{im} [\pi_1(F) \to \pi_1(\Omega)]] \geq \#B = \#A = k.
\]

For the proof of (b), we fix positive integers \( k_1, \ldots, k_m \in \mathbb{N} \) and an ends decomposition \( M \setminus K = E_1 \cup \cdots \cup E_m \) such that \( [\pi_1(M) : \text{im} [\pi_1(E_j) \to \pi_1(M)]] \geq k_j \) for each \( j = 1, \ldots, m \) and \( \sum k_j \geq k \). By (a), we may choose a compact set \( D \subseteq M \) such that, if \( \Omega \) is any domain containing \( D \) and \( 1 \leq j \leq m \), then \( F_j = E_j \cap \Omega \) is an end of \( \Omega \) and \( [\pi_1(\Omega) : \text{im} [\pi_1(F_j) \to \pi_1(\Omega)]] \geq k_j \). The claim now follows. \( \square \)

**Lemma 3.3.** Let \((X,g)\) be a connected complete Kähler manifold, let \( E \) be a special end of type \((W)\) in \( X \), let \( k, l \in \mathbb{N} \) with \( \varepsilon(X) \geq k \) and \( [\pi_1(X) : \text{im} [\pi_1(E) \to \pi_1(X)]] \geq l \), and
let $D$ be a compact subset of $X$. Then there exists a domain $X'$ in $X$, a complete Kähler metric $g'$ on $X'$, and an ends decomposition $X' \setminus K = E_0 \cup E_1 \cup E_2 \cup \cdots \cup E_m$ such that

(i) $(X \setminus E) \cup D \subset E_0$;

(ii) On $E_0$, $g' = g$;

(iii) For each $j = 1, \ldots, m$, $E_j$ is a special end of type (RH) and (W) satisfying 
\[ \pi_1(X) : \text{im} [\pi_1(E_j) \to \pi_1(X)] \geq l; \]

(iv) $\tilde{e}(X') \geq k$.

Proof. By Lemma 3.2, we may assume without loss of generality that $D$ is connected; $\partial E \subset D$; and, if $\Omega$ is any domain in $X$ containing $D$, then $\tilde{e}(\Omega) \geq k$, $\Omega \cap E$ is an end of $\Omega$, and, for any end $F$ of $\Omega$ contained in $E$, we have 
\[ \pi_1(\Omega) : \text{im} [\pi_1(F) \to \pi_1(\Omega)] \geq l. \]

By hypothesis, there exists a continuous plurisubharmonic function $\psi$ on $X$ which exhausts $E$. For constants $r_1$ and $r_2$ with $\max_D \psi < r_1 < r_2$, the component $X'$ of 
\[ \{ x \in E \mid \psi(x) < r_2 \} \cup (X \setminus E) \] containing the connected set $(X \setminus E) \cup D$ admits a complete Kähler metric $g'$ such that $g' = g$ on 
\[ \{ x \in E \mid \psi(x) < r_1 \} \cup (X \setminus E) \cap X' \]
(see [Nk], [Dem]). Since $\psi \to r_2$ at $\partial X'$, the component $E_0$ of 
\[ \{ x \in E \mid \psi(x) < r_1 \} \cup (X \setminus E) \] containing $(X \setminus E) \cup D$ is contained in $X'$ and the set $K \equiv X' \setminus [E_0 \cup \{ x \in E \cap X' \mid \psi(x) > r_1 \}]$ is compact. Furthermore, by the maximum principle, the components $E_1, \ldots, E_m$ of the nonempty set 
\[ \{ x \in E \cap X' \mid \psi(x) > r_1 \} \] are not relatively compact in $X'$. Thus the domain $X'$ and the ends decomposition $X' \setminus K = E_0 \cup \cdots \cup E_m$ have the required properties. $\Box$

Several cases of Theorem 3.1 are contained in the following:

**Lemma 3.4.** Let $(X, g)$ be a connected complete Kähler manifold which contains a special end $E$ and suppose that $X \setminus \overline{E}$ contains two disjoint massive subsets $U_1$ and $U_2$ of $X$ such that $U_1$ is $D$-massive or $U_1$ has an associated $C^\infty$ admissible plurisubharmonic function (i.e. $U_1$ is $C^\infty$ pluri-massive). Then some nonempty open subset of $E$ admits a proper holomorphic mapping onto a Riemann surface.

**Remark.** Lemma 3.4 remains true if we allow the admissible plurisubharmonic function for $U_1$ to be only continuous. However, it is then harder to produce a complete Kähler metric on a sublevel and we will only need the $C^\infty$ case.

**Proof of Lemma 3.4.** We first observe that we may assume without loss of generality that $E$ is a $C^\infty$ domain and $E' = X \setminus \overline{E}$ is connected (i.e. $E'$ is an end). For we may choose a
\( C^\infty \) relatively compact domain \( \Theta \) such that \( \partial E \subset \Theta \) and such that \( X \setminus \Theta \) has no compact components, and we may replace \( E \) with a component of \( E \setminus \overline{\Theta} \). Observe also that \( E' \) is then a massive, and therefore hyperbolic, end.

Next, we observe that we may assume without loss of generality that \( E \) is a hyperbolic end of type (BG) or type (RH). For, if \( E \) is of type (W), then we may apply Lemma 3.3 and work on a suitable subdomain in place of \( X \). If \( E \) is of type (SP) (for example, if \( E \) is parabolic of type (BG)), then, by Lemma 2.3, \( E \) is also of type (W) and the above applies.

Since \( E \) is a hyperbolic end, there exists a finite energy admissible subharmonic function \( \alpha_0 : X \to (0, 1) \) for \( E \) such that \( \alpha_0(x_j) \to 1 \) as \( j \to \infty \) whenever \( \{x_j\} \) is a sequence in \( E \) such that \( G(x_j, \cdot) \to 0 \) as \( j \to \infty \); where \( G \) is the Green’s function on \( X \) (see Remark 4 following Definition 1.1). Applying Proposition 1.2, we get a finite energy harmonic, hence pluriharmonic, function \( \rho_1 : X \to (0, 1) \) such that \( \alpha_0 \leq \rho_1 < 1 \) on \( X \) and, for any admissible subharmonic function \( \beta : X \to [0, 1) \) for \( E' \), \( 0 < \rho_1 \leq 1 - \beta \) on \( X \). In particular, \( \rho_1(x_j) \to 1 = \sup \rho_1 \) whenever \( \{x_j\} \) is a sequence in \( E \) such that \( G(x_j, \cdot) \to 0 \) as \( j \to \infty \). We will produce a second pluriharmonic function and apply Lemma 2.6 and Lemma 2.7.

Toward this end, we fix a constant \( a \) with \( \max_{\partial E} \rho_1 < a < 1 \) and a component \( \Omega \) of \( \{ x \in X \mid a < \rho_1(x) \} \) contained in \( E \).

We have admissible subharmonic functions \( \alpha_1 \) and \( \alpha_2 \) for \( U_1 \) and \( U_2 \), respectively, such that \( \sup \alpha_1 = \sup \alpha_2 = 1 \) and such that \( \alpha_1 \) is of finite energy or \( \alpha_1 \) is \( C^\infty \) plurisubharmonic.

Assuming first that \( \alpha_1 \) has finite energy (\( U_1 \) is \( D \)-massive), we may apply Proposition 1.2 to get a finite energy pluriharmonic function \( \rho_2 : X \to (0, 1) \) such that \( \alpha_1 \leq \rho_2 < 1 \) on \( X \) and, for any admissible subharmonic function \( \beta : X \to [0, 1) \) for \( X \setminus U_1 \), \( 0 < \rho_2 \leq 1 - \beta \) on \( X \). The functions 1, \( \rho_1 \), and \( \rho_2 \) are then linearly independent on \( X \). To see this, suppose \( a_1, a_2, a_3 \in \mathbb{R} \) with \( a_1 \rho_1 + a_2 \rho_2 + a_3 \equiv 0 \). Choosing a sequence \( \{x_j\} \) in \( U_2 \) with \( \alpha_2(x_j) \to 1 \) as \( j \to \infty \), we get

\[
0 < \rho_1(x_j), \rho_2(x_j) \leq 1 - \alpha_2(x_j) \to 0 \text{ as } j \to \infty
\]

and it follows that \( a_3 = 0 \). Taking a sequence \( \{x_j\} \) in \( U_1 \) with \( \alpha_1(x_j) \to 1 \) as \( j \to \infty \), we get

\[
1 > \rho_2(x_j) \geq \alpha_1(x_j) \to 1 \quad \text{and} \quad 0 < \rho_1(x_j) \leq 1 - \alpha_1(x_j) \to 0 \text{ as } j \to \infty
\]

and it follows that \( a_2 = 0 \). Thus \( a_1 \rho_1 \equiv 0 \) and hence \( a_1 = 0 \). In this case, we also set \( Y = X \) and \( h = g \).
If \( \alpha_1 \) is \( C^\infty \) and plurisubharmonic (\( U_1 \) is \( C^\infty \) pluri-massive), then we fix a connected compact set \( H \) such that
\[
\partial E \subset H \quad \text{and} \quad \max_H \alpha_i > 0 \text{ for } i = 0, 1, 2.
\]

Fixing \( b \) with \( \max_H \alpha_1 < b < 1 \), we see that the component \( Y \) of \( \{ x \in X \mid \alpha_1(x) < b \} \) containing \( H \) admits the complete Kähler metric
\[
h = g + \mathcal{L}(−\log(b−\alpha_1)) = g + (b−\alpha_1)^{-1}\mathcal{L}(\alpha_1) + (b−\alpha_1)^{-2}\partial \alpha_1 \bar{\partial} \alpha_1
\]
(see [Nk], [Dem]). We have \( \overline{\Omega} \subset E \subset \overline{E} \subset Y \), \( \sup_Y \alpha_i \geq \sup_H \alpha_i > 0 \text{ for } i = 0, 1, 2 \), and \( h = g \) on \( Y \setminus U_1 \supset E \cup (U_2 \cap Y) \). Thus, with respect to \((Y, h)\), \( \alpha_0 \mid_Y \) is a finite energy admissible subharmonic function for \( E \), \( \alpha_1 \mid_Y \) is a \( C^\infty \) admissible plurisubharmonic function for \( U_1 \cap Y \), and \( \alpha_2 \mid_Y \) is an admissible subharmonic function for \( U_2 \cap Y \). Applying Proposition 1.2, we get a finite \( h \)-energy harmonic, hence pluriharmonic, function \( \rho_2 : Y \to (0, 1) \) such that \( \alpha_0 \leq \rho_2 < 1 \) on \( Y \) and, for any admissible subharmonic function \( \beta : Y \to [0, 1) \) for \( Y \setminus \overline{E} \), \( 0 < \rho_2 \leq 1 − \beta \) on \( Y \). To see that the functions \( 1, \rho_1 \mid_Y \), and \( \rho_2 \) are linearly independent, suppose \( a_1, a_2, a_3 \in \mathbb{R} \) with \( a_1\rho_1 + a_2\rho_2 + a_3 \equiv 0 \). Choosing a sequence \( \{x_j\} \) in \( U_2 \cap Y \) with \( \alpha_2(x_j) \to \sup_Y \alpha_2 \) as \( j \to \infty \), we get
\[
0 < \rho_1(x_j), \rho_2(x_j) \leq 1−(\alpha_2(x_j)/\sup_Y \alpha_2) \to 0 \text{ as } j \to \infty
\]
and it follows that \( a_3 = 0 \). Here, we have used the fact that the function given by
\[
\begin{cases}
\alpha_2 / \sup_Y \alpha_2 & \text{on } U_2 \cap Y \\
0 & \text{on } X \setminus (U_2 \cap Y)
\end{cases}
\]
is an admissible \( g \)-subharmonic function for \( U_2 \cap Y \subset X \setminus \overline{E} \) in \((X, g)\) (which is the case because \( g = h \) on \( U_2 \cap Y \), \( \overline{U_2} \cap \partial Y = \emptyset \), and \( \alpha_2 \equiv 0 \) on \( X \setminus U_2 \)). Taking a sequence \( \{x_j\} \) in \( U_1 \cap Y \) with \( x_j \to x_0 \in \partial Y \) as \( j \to \infty \), we get
\[
0 < \rho_2(x_j) \leq 1−b^{-1}\alpha_1(x_j) \to 0 \quad \text{and} \quad \rho_1(x_j) \to \rho_1(x_0) > 0 \text{ as } j \to \infty
\]
and it follows that \( a_1 = 0 \). Thus \( a_2\rho_2 \equiv 0 \) and hence \( a_2 = 0 \).

If the end \( E \) is of type (RH), then, for any \( c \) with \( a < c < 1 = \sup_\Omega \rho_1 \), the fiber \( \rho_1^{-1}(c) \cap \Omega \) of \( \rho_1 \mid_\Omega \) is compact. Thus, in either of the above cases, Lemma 2.6 implies that some nonempty open subset of \( \Omega \subset E \) admits a proper holomorphic mapping onto a Riemann surface.
Suppose the end \( E \supset \overline{\Omega} \) is of type (BG). We have, in either case, \( h = g \) on \( E \). Thus
\[
\int_{\Omega} |d\rho_1|_g^2 dV_g \leq \int_X |d\rho_1|_g^2 dV_g < \infty \quad \text{and} \quad \int_{\Omega} |d\rho_2|_g^2 dV_g = \int_{\Omega} |d\rho_2|_h^2 dV_h < \infty.
\]
Moreover, the \( L^2/L^\infty \) comparison for holomorphic 1-forms on a bounded geometry Kähler manifold shows that \( |d\rho_1|_g \) is bounded on \( E \supset \Omega \). Lemma 2.7 now gives the lemma in this case as well. \( \square \)

For the proof of Theorem 3.1, we will consider the cases of \( e(X) \geq 2 \) and \( e(X) = 1 \) separately.

**Lemma 3.5.** Let \((X,g)\) be a connected complete Kähler manifold which admits a special ends decomposition. Assume that \( e(X) \geq 2 \) and \( \tilde{e}(X) \geq 3 \). Then \( X \) admits a proper holomorphic mapping onto a Riemann surface.

**Proof.** By Lemma 2.5, it suffices to find a nonempty open subset of \( X \) that admits a proper holomorphic mapping onto a Riemann surface.

If \( e(X) \geq 3 \), then Theorem 2.4 provides the required proper holomorphic mapping to a Riemann surface.

If \( e(X) = 2 \), then there exists a special ends decomposition \( X \setminus K = E_1 \cup E_2 \) where \( E_1 \) and \( E_2 \) are \( C^\infty \) domains with \( K = \partial E_1 = \partial E_2 \) and, for some point \( x_0 \in E_1 \), the image \( \Gamma \) of \( \pi_1(E_1,x_0) \) in \( \pi_1(X,x_0) \) is a proper subgroup. We may assume without loss of generality that, for \( j = 1, 2 \), \( E_j \) is a hyperbolic special end (of type (BG) or (RH)). For if \( E_j \) is of type (W), then we may apply Lemma 3.3 while, if \( E_j \) is of type (SP), then, by Lemma 2.3, \( E_j \) is also of type (W). Finally, a parabolic end of type (BG) is also of type (SP).

Applying Proposition 1.2 and Remark 4 following Definition 1.1 we get a finite energy harmonic, hence pluriharmonic, function \( \rho : X \to (0,1) \) such that \( \limsup_{x \to \infty} \rho \big|_{E_1} \rho = 1 \), \( \liminf_{x \to \infty} \rho \big|_{E_2} \rho = 0 \), and \( \lim_{j \to \infty} \rho(x_j) = 1 \), \( \lim_{j \to \infty} \rho(x_j) = 0 \) for any sequence \( \{x_j\} \) in \( E_1 \) (respectively \( E_2 \)) such that \( G(x_j, \cdot) \to 0 \) as \( j \to \infty \). We may choose a connected covering space \( \Upsilon : \tilde{X} \to X \) such that, for some point \( y_0 \in \tilde{X} \), we have \( \Upsilon \circ \pi_1(\tilde{X}, y_0) = \Gamma \). Thus \( \Upsilon \) maps the component \( E \) of \( \tilde{E}_1 = \Upsilon^{-1}(E_1) \) isomorphically onto \( E_1 \) and, since \( \Gamma \) is a proper subgroup, \( \tilde{E}_1 \setminus E \neq \emptyset \). Since \( E_1 \) is a \( C^\infty \) domain, loops in a small neighborhood of \( \tilde{E}_1 \) homotop into \( E_1 \). So \( \Upsilon \) maps a neighborhood of \( \tilde{E}_1 \) isomorphically onto a neighborhood of \( \tilde{E}_1 \). Thus \( E \) is a hyperbolic special end in \( (\tilde{X}, \tilde{g} = \Upsilon^*g) \). Fix constants \( a \) and \( b \) with
\[
0 < a < \min_{K=\partial E_1=\partial E_2} \rho \leq \max_{K} \rho < b < 1.
\]
In $\hat{X}$,
\[
\alpha_2 \equiv \begin{cases} 
\max(\rho \circ \Upsilon - b, 0) & \text{on } U_2 = \hat{E}_1 \setminus E \\
0 & \text{on } \hat{X} \setminus U_2
\end{cases}
\]
is an admissible plurisubharmonic function for $U_2 = \hat{E}_1 \setminus E$. Choosing a $C^\infty$ function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\chi' \geq 0$, $\chi'' \geq 0$, $\chi(t) = 0$ for $t \leq 1 - a$, and $\chi'(t) > 0$ for $t > 1 - a$, we get a $C^\infty$ admissible plurisubharmonic function
\[
\alpha_1 \equiv \begin{cases} 
\chi(1 - \rho \circ \Upsilon) & \text{on } U_1 = \Upsilon^{-1}(E_2) \\
0 & \text{on } \hat{X} \setminus U_1
\end{cases}
\]
for $U_1 \equiv \Upsilon^{-1}(E_2)$. Applying Lemma 3.4, we get a proper holomorphic mapping of a nonempty open subset of $E \cong E_1$ onto a Riemann surface; as required.

**Proof of Theorem 3.1.** By Lemma 3.5, it remains to consider the case $e(X) = 1$. The manifold $X$ is itself a special end in this case, so we have an ends decomposition $X \setminus K = E_1$ such that $E_1$ is a $C^\infty$ domain and, for a point $x_0 \in E_1$, $\Gamma \equiv \im \pi_1(E_1, x_0) \to \pi_1(X, x_0)$ is of index $\geq 3$. We may fix a connected covering space $\Upsilon : \hat{X} \to X$ and a point $y_0 \in \hat{X}$ such that $\Upsilon \circ \pi_1(\hat{X}, y_0) = \Gamma$. Hence $\Upsilon$ maps the component $E$ of $\hat{E}_1 = \Upsilon^{-1}(E_1)$ containing $y_0$ isomorphically onto $E_1$ and, since $\# \Upsilon^{-1}(x_0) = [\pi_1(X, x_0) : \Gamma] \geq 3$, we have $\hat{E}_1 \setminus E \neq \emptyset$. Again, since $E_1$ is a $C^\infty$ domain, $\Upsilon$ maps a neighborhood of $\overline{E}$ isomorphically onto a neighborhood of $\overline{E}_1$ and hence $E$ is a special end in $(\hat{X}, \hat{g} = \Upsilon^* g)$.

We may again assume without loss of generality that $X$ (and hence any end in $X$) is hyperbolic of type (BG) or (RH). For if $X$ is of type (W), then we may apply Lemma 3.3. If $X$ is of type (SP), then $E$ is a special end of type (SP) in $\hat{X}$ and any other end in $\hat{X}$ is either a hyperbolic end or a special end of type (SP). Therefore, by Lemma 2.3, there exists a continuous plurisubharmonic function $\hat{\psi}$ on $\hat{X}$ which exhausts $\overline{E}$ and hence the function
\[
\psi \equiv \begin{cases} 
\max(\hat{\psi} \circ (\Upsilon |_E)^{-1}, \max_{\partial E} \hat{\psi}) & \text{on } E_1 \\
\max_{\partial E} \hat{\psi} & \text{on } K
\end{cases}
\]
is a continuous plurisubharmonic exhaustion function on $X$ and the above applies. Finally, if $X$ is parabolic of type (BG), then $X$ is of type (SP) and the above applies. Thus we may assume without loss of generality that $X$ is hyperbolic of type (BG) or (RH).

In particular, $E_1$ is $D$-massive and there is a finite energy admissible subharmonic function $\beta : X \to [0, 1)$ for $E_1$ in $X$ such that $\beta(x_j) \to 1$ as $j \to \infty$ for any sequence $\{x_j\}$ in
$E_1$ with $G(x_j, \cdot) \to 0$ as $j \to \infty$. It follows that $E$ is $D$-massive in $\hat{X}$ with finite energy admissible subharmonic function

$$
\begin{cases}
\beta \circ \Upsilon & \text{on } E \\
0 & \text{on } \hat{X} \setminus E
\end{cases}
$$

and $\hat{E}_1 \setminus E$ is massive in $\hat{X}$ with admissible subharmonic function

$$
\begin{cases}
\beta \circ \Upsilon & \text{on } \hat{E}_1 \setminus E \\
0 & \text{on } \hat{X} \setminus (\hat{E}_1 \setminus E)
\end{cases}
$$

Applying Proposition 1.2, we get a finite energy harmonic, hence pluriharmonic, function $\rho_1 : \hat{X} \to (0, 1)$ such that $\beta \circ \Upsilon \leq \rho_1 < 1$ on $E$ and such that, for any admissible subharmonic function $\gamma : \hat{X} \to [0, 1)$ for $\hat{X} \setminus E$, we have $0 < \rho_1 \leq 1 - \gamma$ on $\hat{X}$.

Fix constants $t_0$, $t_1$, and $t_2$ with $\max_{\partial E} \rho_1 < t_2 < t_1 < t_0 < 1$ and a $C^\infty$ function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\chi' \geq 0$, $\chi'' \geq 0$, $\chi(t) = 0$ if $t \leq t_2$, and $\chi(t) = (t - t_1)/(1 - t_1)$ if $t \geq t_0$. Thus

$$
\beta_1 = \begin{cases}
\chi(\rho_1 \circ (\Upsilon |_E)^{-1}) & \text{on } E_1 \\
0 & \text{on } X \setminus E_1
\end{cases}
$$

is a finite energy $C^\infty$ admissible plurisubharmonic function for $E_1$ which is pluriharmonic on $\{ x \in X \mid \beta(x) > t_0 \}$. For we have, on $E_1$,

$$
|d\chi(\rho_1 \circ (\Upsilon |_E)^{-1})|_{\hat{g}} = \chi'(\rho_1 \circ (\Upsilon |_E)^{-1})|d\rho_1|_{\hat{g}} \circ (\Upsilon |_E)^{-1} \leq (1 - t_1)^{-1}|d\rho_1|_{\hat{g}} \circ (\Upsilon |_E)^{-1}
$$

and $\rho_1$ has finite $\hat{g}$-energy, so $\chi(\rho_1 \circ (\Upsilon |_E)^{-1})$ has finite $g$-energy. If $\{x_j\}$ is a sequence in $E_1$ with $G(x_j, \cdot) \to 0$ as $j \to \infty$, then

$$
1 > \chi(\rho_1((\Upsilon |_E)^{-1}(x_j))) \geq \chi(\beta(x_j)) \to \chi(1) = 1,
$$

so $\chi(\rho_1((\Upsilon |_E)^{-1}(x_j))) \to 1$. Finally,

$$
1 > \rho_1 \geq \chi(\rho_1) \geq \chi(\beta \circ \Upsilon) \quad \text{on } E,
$$

so the relation $\rho_1 \geq \beta \circ \Upsilon$ on $E$ is preserved if we replace $\beta$ with $\beta_1$. Thus we may assume that $\beta = \chi(\rho_1 \circ (\Upsilon |_E)^{-1})$ on $E_1$.

If $U$ is any component of $\hat{E}_1 = \Upsilon^{-1}(E_1)$, then the function

$$
\begin{cases}
\beta \circ \Upsilon & \text{on } U \\
0 & \text{on } \hat{X} \setminus U
\end{cases}
$$
is a $C^\infty$ admissible plurisubharmonic function for $U$ (with finite energy if $U \to E_1$ is a finite covering). Therefore, if $\hat{E}_1$ has at least 3 components, then Lemma 3.4 provides an open subset of $E \cong E_1$ which admits a proper holomorphic mapping onto a Riemann surface. If $\hat{E}_1$ has exactly two components $\hat{E}$ and $\hat{E}'$ and $\hat{E}' \to E_1$ is a finite covering, then we see that $\hat{X} \to X$ is a finite covering and $\hat{X}$ has the special ends decomposition $\hat{X} \setminus \hat{K} = E \cup E'$, where $\hat{K} = \Upsilon^{-1}(K)$. Thus $\hat{e}(\hat{X}) = \hat{e}(X) \geq 3$ and $e(\hat{X}) \geq 2$, and therefore, by Lemma 3.5 $\hat{X}$ admits a proper holomorphic mapping $\Psi : \hat{X} \to S$ with connected fibers onto a Riemann surface $S$. The pluriharmonic function $\rho_1$ descends to a pluriharmonic function $\tau$ on $S$. Hence any component of $\{ x \in \hat{X} \mid \rho_1(x) > \max_{\partial E} \rho_1 \}$ contained in $E \cong E_1$ admits a proper holomorphic mapping onto a component of $\{ \zeta \in S \mid \tau(\zeta) > \max_{\partial E} \rho_1 \}$. Thus it remains to consider the case in which $\hat{E}_1 = \Upsilon^{-1}(E_1)$ has exactly two components $E$, $E'$ and $E' \to E_1$ is an infinite covering (i.e. $[\pi_1(X, x_0) : \Gamma] = \infty$). We have the finite energy pluriharmonic function $\rho_1 : \hat{X} \to (0, 1)$ with $$\beta \circ \Upsilon \leq \rho_1 < 1 \text{ on } E \quad \text{and} \quad 0 < \rho_1 \leq 1 - \gamma$$ for every admissible subharmonic function $\gamma : X \to [0, 1)$ for $X \setminus \overline{E}$. By construction, for some constants $t_0$, $t_1$, and $t_2$ with $\max_{\partial E} \rho_1 < t_2 < t_1 < t_0 < 1$, we have on $E$ $$\beta \circ \Upsilon = \begin{cases} 0 & \text{on } \{ x \in E \mid \rho_1(x) \leq t_2 \} \\ \frac{\rho_1 - t_1}{1 - t_1} & \text{on } \{ x \in E \mid \rho_1(x) \geq t_0 \} = \{ x \in E \mid \beta(\Upsilon(x)) \geq r_0 \} \end{cases}$$ where $r_0 = (t_0 - t_1)/(1 - t_1)$. We will produce a second pluriharmonic function and apply Lemma 2.6 and Lemma 2.7. Lifting, we get $C^\infty$ admissible plurisubharmonic functions $$\alpha = \begin{cases} \beta \circ \Upsilon & \text{on } E \\ 0 & \text{on } \hat{X} \setminus E \end{cases} \quad \text{and} \quad \alpha' = \begin{cases} \beta \circ \Upsilon & \text{on } E' \\ 0 & \text{on } \hat{X} \setminus E' \end{cases}$$ for $E$ and $E'$, respectively. Moreover, $\alpha$ has finite energy, $\alpha'$ has infinite energy, and $$\alpha = \frac{\rho_1 - t_1}{1 - t_1} \text{ on } \{ x \in \hat{X} \mid \alpha(x) \geq r_0 \}$$ and $$\alpha' = \frac{\rho_1 \circ (\Upsilon |_{E'})^{-1} \circ \Upsilon - t_1}{1 - t_1} \text{ on } \{ x \in \hat{X} \mid \alpha'(x) \geq r_0 \}.$$ Let $V = \{ x \in X \mid \beta(x) > r_0 \} \subset E_1$ and let $\hat{V} = \Upsilon^{-1}(V) \subset \hat{E}_1$. If $\hat{V} \cap E'$ is not connected, then we get two disjoint $C^\infty$ plurimassive subsets of $\hat{X}$ contained in $\hat{V} \cap E' \subset \hat{X} \setminus \overline{E}$ and we may apply Lemma 3.4 as before. Thus we may assume without loss of
generality that \( \hat{V} \cap E' \) is connected. In particular, \( \hat{V} \cap E' \to V \) is a connected infinite covering. Observe also that \( \alpha \) and \( \alpha' \) are pluriharmonic on \( \hat{V} \).

For each \( r \) with \( r_0 < r < 1 \), the component \( Y_r \) of \( \{ x \in \hat{X} \mid \alpha'(x) < r \} \) containing \( E \) (and \( \partial E \)) admits the complete Kähler metric

\[
    h_r = \hat{g} + \mathcal{L}(-\log(r - \alpha')) = \hat{g} + (r - \alpha')^{-1}\mathcal{L}(\alpha') + (r - \alpha')^{-2}\partial\alpha' \partial\overline{\alpha'}
\]

with \( h_r \geq \hat{g} \) on \( \hat{X} \) and \( h_r = \hat{g} \) at points in \( \{ x \in \hat{X} \mid \alpha'(x) = 0 \} \supset \hat{X} \setminus E' \). Moreover, \( \alpha \mid_{Y_r} \) and \( \alpha' \mid_{Y_r} \) are \( C^\infty \) admissible plurisubharmonic functions for \( E = E \cap Y_r \) and \( E' = E' \cap Y_r \), respectively, and \( \alpha \mid_{Y_r} \) has finite \( h_r \)-energy (since \( \alpha = 0 \) on \( \hat{X} \setminus E \) and \( h_r = \hat{g} \) on \( E \)). Applying Proposition 1.2, we get a finite \( h_r \)-energy harmonic, hence pluriharmonic, function \( \tau_r : Y_r \to (0, 1) \) such that \( \alpha \leq \tau_r < 1 \) on \( Y_r \) and such that, for any admissible \( h_r \)-subharmonic function \( \gamma : Y_r \to [0, 1) \) for \( Y_r \setminus \partial E, 0 < \tau_r \leq 1 - \gamma \) on \( Y_r \).

We will show that the functions 1, \( \rho_1 \mid_{Y_r} \), and \( \tau_r \) are linearly independent for some \( r \) with \( r_0 < r < 1 \). To see this, suppose that, on the contrary, 1, \( \rho_1 \mid_{Y_r} \), and \( \tau_r \) are linearly dependent functions for every \( r \) with \( r_0 < r < 1 \). Then, since \( 0 < \tau_r \leq 1 - r^{-1}\alpha' \to 0 \) at \( \partial Y_r \), we see that \( \rho_1 \) is constant on \( \partial Y_r \subset \hat{V} \cap E' \) for each \( r \in (r_0, 1) \). Fixing a regular value \( r \in (r_0, 1) \) for \( \alpha' \) and a point \( p \in \partial Y_r \), we may choose a holomorphic coordinate neighborhood \( (W, \Phi = (\zeta_1, \ldots, \zeta_n)) \) mapping \( W \) onto a \( 2n \)-dimensional open rectangle \( (r - \epsilon, r + \epsilon) \times (0, 1) \) in \( \mathbb{R}^n \) with \( \Phi(p) = (r, 0, \ldots, 0) \) and \( \alpha' = \text{Re} \zeta_1 = u_1 \) on \( W \). For each \( s \) with \( r < s < r + \epsilon \), \( \{ x \in W \mid \alpha'(x) < s \} \cong (r - \epsilon, s) \times (0, 1) \) is a connected open subset of \( \{ x \in \hat{X} \mid \alpha'(x) < s \} \) meeting \( Y_r \subset Y_s \), so this set is contained in \( Y_s \). Thus \( W \cap Y_s = \{ x \in W \mid \alpha'(x) < s \} \) and so \( \rho_1 \) is constant on \( W \cap \partial Y_s = \{ x \in W \mid \alpha'(x) = s \} \). It follows that \( d\rho_1 \wedge d\alpha' \equiv 0 \) on \( \{ x \in W \mid r \leq \alpha'(x) < r + \epsilon \} \) and, therefore, since \( \rho_1 \) and \( \alpha' \) are pluriharmonic on the connected open set \( \hat{V} \cap E' \), we see that \( d\rho_1 \mid_{(\hat{V} \cap E')} \) and \( d\alpha' \mid_{(\hat{V} \cap E')} \) are linearly dependent forms. On the other hand, \( \hat{V} \cap E' \to V \) is an infinite covering. Thus we have, for some constant \( C > 0 \),

\[
    \infty > \int_{\hat{V} \cap E'} |d\rho_1|^2 dV_{\hat{g}} = C \int_{\hat{V} \cap E'} |d\alpha'|^2 dV_{\hat{g}} = C \int_{\hat{V} \cap E'} |Y^* d\beta|^2 dV_{\hat{g}} = \infty.
\]

Thus we have arrived at a contradiction, so 1, \( \rho_1 \mid_{Y_r} \), and \( \tau_r \) must be linearly independent functions on \( Y_r \) for some \( r \in (r_0, 1) \).

As in the last part of the proof of Lemma 3.4 (applied to \( \rho_1 \) and \( \tau_r \)), for \( \max_{\partial E} \rho_1 < a < 1 \), one gets a proper holomorphic mapping of some open subset of a component of
\{ x \in X \mid a < \rho_1(x) \} \text{ contained in } E \cong E_1 \text{ onto a Riemann surface and the theorem follows.} \]

**Remarks.** 1. If \( M \) is a connected noncompact manifold with \( 3 \leq \tilde{e}(M) < \infty \), then \( M \) admits a finite covering space \( \tilde{M} \) with \( e(\tilde{M}) \geq 3 \). To see this, we fix an ends decomposition \( M \setminus K = E_1 \cup \cdots \cup E_m \) such that the lifting of \( M \setminus K \) to the universal covering \( \Upsilon : \tilde{M} \to M \) has \( \tilde{e}(M) \) components. The action of \( \pi_1(M) \) permutes these components and so we get a homomorphism of \( \pi_1(M) \) into the symmetric group on \( \tilde{e}(M) \) objects. Thus the kernel \( \Gamma \) is a normal subgroup of finite index and hence the quotient \( \hat{M} = \Gamma \setminus \tilde{M} \to M \) is the desired finite covering. This observation together with Theorem 2.4 gives Theorem 3.1 for the case in which \( 3 \leq \tilde{e}(X) < \infty \). For the associated finite covering space admits a proper holomorphic mapping onto a Riemann surface and any normal complex space which is the image of a holomorphically convex complex space under a proper holomorphic mapping is itself holomorphically convex. In particular, in the proof of Theorem 3.1 we could have avoided the argument in the case in which \( e(X) = 1 \), \( \hat{E}_1 = \Upsilon^{-1}(E_1) \) has exactly two components \( E, E' \), and \( E' \to E_1 \) is a finite covering.

2. In general, the number of components of \( \Upsilon^{-1}(E_1) \) is equal to the number of distinct double cosets of the image of the fundamental group of \( E_1 \).

Theorem 3.1 and Lemma 0.5 give the following:

**Theorem 3.6.** Let \( X \) be a connected complete Kähler manifold which admits a special ends decomposition. Assume that \( \tilde{e}(X) \geq 2 \) and \( \pi_1(X) \) is infinitely generated. Then \( X \) admits a proper holomorphic mapping onto a Riemann surface.

4. **Mappings of compact Kähler manifolds to curves**

In this section, we consider the following consequence of Theorem 0.1 (and Theorem 3.1).

**Theorem 4.1.** If \( X \) is a connected compact Kähler manifold for which there is a connected infinite covering space \( \Upsilon : \hat{X} \to X \) with \( \tilde{e}(\hat{X}) \geq 3 \), then some finite covering space \( X' \to X \) admits a surjective holomorphic mapping onto a curve of genus \( g \geq 2 \).

The main point of the proof is the following (cf. Proposition 1.2.11 of [Kol] and 1.2.3, p. 490, of [Cam]):
Proposition 4.2. Let \((X, g)\) be a connected complete Kähler manifold with bounded geometry. Suppose some connected noncompact covering space \(Y : \hat{X} \to X\) admits a proper surjective holomorphic mapping \(\hat{\Phi} : \hat{X} \to \hat{S}\) onto a Riemann surface \(\hat{S}\). Then, for every level \(F\) of \(\hat{\Phi}\) over a regular value \(\zeta\) of \(\hat{\Phi}\), the normalizer \(N\) of \(\text{im}[\pi_1(F) \to \pi_1(X)]\) is of finite index in \(\pi_1(X)\). Furthermore, the associated finite covering space \(X' \to X\) with \(\text{im}[\pi_1(X') \to \pi_1(X)] = N\) admits a surjective proper holomorphic mapping \(\Phi' : X' \to S\) onto a Riemann surface \(S\). In particular, if \(X\) is noncompact, then \(X\) admits a proper holomorphic mapping onto a Riemann surface.

Remark. The last statement is a consequence of the fact that the normal proper holomorphic image of a holomorphically convex complex space is holomorphically convex.

The following elegant proof of Proposition 4.2, which we give in steps, is due to Delzant and Gromov [DelG]. Let \(\Phi : X \to S\) be a proper holomorphic mapping with connected fibers of a connected complex manifold \(X\) onto a Riemann surface \(S\), let \(X_A = \Phi^{-1}(A)\) for each set \(A \subset S\), and let \(X_\zeta = X_{\{\zeta\}}\) for each point \(\zeta \in S\). If \(P\) is the (discrete) set of critical values of \(\Phi\) and, for each point \(p \in P\), \(m_p\) is the greatest common divisor of the multiplicities of the components of the divisor \(\Phi^{-1}(p)\) and \(\gamma_p\) is a simple loop tracing the boundary circle of a coordinate disk \(D_p\) in \(S\) centered at \(p\) with \(D_p \cap P = \{p\}\), then the orbifold fundamental group is given by

\[
\pi_1^{\text{orb}}(\Phi) \equiv \pi_1(S \setminus P)/N
\]

where \(N\) is the normal subgroup of \(\pi_1(S \setminus P)\) generated by the loops \(\{\gamma_p^{m_p}\}_{p \in P}\). The following lemma is well known (see [Cat2], [CatKO], [Sim1]):

Lemma 4.3. For \(\Phi : X \to S \supset P\) as above, we have

(a) The map \(\pi_1(X) \to \pi_1(S)\) is surjective.
(b) For each point \(\zeta \in S \setminus P\), the induced maps give an exact sequence

\[
\pi_1(X_\zeta) \to \pi_1(X) \to \pi_1^{\text{orb}}(\Phi) \to 1.
\]

(c) For each pair of points \(\zeta_1, \zeta_2 \in S \setminus P\), each choice of a point \(x_j \in X_{\zeta_j}\) for \(j = 1, 2\), and each path \(\alpha\) in \(X_{\zeta_j \setminus P}\) from \(x_1\) to \(x_2\), the isomorphism \(\pi_1(X, x_1) \to \pi_1(X, x_2)\) given by \([\gamma] \mapsto [\alpha^{-1} \ast \gamma \ast \alpha]\) restricts to a surjective isomorphism of \(\Gamma_1 \equiv \text{im}[\pi_1(X_{\zeta_1}, x_1) \to \pi_1(X, x_1)]\) onto \(\Gamma_2 \equiv \text{im}[\pi_1(X_{\zeta_2}, x_2) \to \pi_1(X, x_2)]\).
(d) If $\Upsilon : X' \to X$ is a connected covering space and, for some point $\zeta_1 \in S \setminus P$, the group $\Gamma_1 \equiv \text{im}[\pi_1(X_{\zeta_1}) \to \pi_1(X)]$ is contained in (is equal to) the group $\Lambda \equiv \text{im}[\pi_1(X') \to \pi_1(X)]$, then this is the case for every point $\zeta \in S \setminus P$ and we get a commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\Phi'} & S' \\
\downarrow{\Upsilon} & & \downarrow{\theta} \\
X & \xrightarrow{\Phi} & S
\end{array}
\]

where $S'$ is a Riemann surface, $\Phi'$ is a surjective proper holomorphic map with connected fibers, and $\theta$ is a (possibly infinite, but locally finite) holomorphic branched covering map with branch locus $B$ contained in $\{ p \in P \mid m_p > 1 \}$. In particular, if $\Gamma_1 = \Lambda$, then $S' \cong \mathbb{P}^1$, $\mathbb{C}$, or $\Delta$ and we get injective homomorphisms of $\pi_1^{\text{orb}}(\Phi) = \pi_1(X)/\Gamma_1$ into $\text{Aut}(X')$ and into $\text{Aut}(S')$, $\Phi'$ is equivariant with respect to the action of $\pi_1^{\text{orb}}(\Phi)$, and $\Upsilon$ and $\theta$ are the quotient maps given by

$$
\Upsilon : X' \to X = \pi_1^{\text{orb}}(\Phi) \setminus X' \quad \text{and} \quad \theta : S' \to S = \pi_1^{\text{orb}}(\Phi) \setminus S'.
$$

Lemma 4.4. Let $(M, g)$ be a connected complete Riemannian manifold.

(a) For each point $p \in M$ and each constant $L > 0$, the set

$$K(p, L) \equiv \{ [\alpha] \in \pi_1(M, p) \mid \alpha \text{ is a piecewise } C^\infty \text{ loop in } M \text{ of length } < L \}$$

is finite.

(b) Let $A$ be a path connected compact subset of $M$, let $r > 0$ be a lower bound for the injectivity radius at points in $A$, and let $A_1, \ldots, A_m$ be a covering of $A$ by path connected subsets which are relatively open in $A$ and which have diameter $< r$ (with respect to the distance function in $M$). Then, for each point $p \in A$, the group $\Gamma \equiv \text{im}[\pi_1(A, p) \to \pi_1(M, p)]$ is generated by the set

$$\{ [\alpha] \in \Gamma \mid \alpha \text{ is a piecewise } C^\infty \text{ loop in } M \text{ based at } p \text{ of length } < 2(m^2 + 1)r + 1 \}.$$

(c) Let $\Upsilon : \hat{M} \to M$ be a connected covering space, let $\hat{g} = \Upsilon^* g$, and let $\{ A_\lambda \}_{\lambda \in \Lambda}$ be a family of path connected compact subsets of $\hat{M}$. Assume that there exist a positive integer $m$ and a positive constant $r$ such that, for each $\lambda \in \Lambda$, $r$ is a lower bound for the injectivity radius in $\hat{M}$ at each point in $A_\lambda$ and there is a covering $A^\lambda_1, \ldots, A^\lambda_m$
of $A$ by path connected relatively open subsets of $A$ of diameter $< r$ (in $\hat{M}$). Then, for each point $p \in M$, the (possibly empty) collection of subgroups

$$\mathcal{H}_p \equiv \{ \text{im} [\pi_1(A, \hat{p}) \to \pi_1(M, p)] \mid \lambda \in \Lambda, \hat{p} \in \Upsilon^{-1}(p) \cap A_\lambda \}$$

is finite.

**Sketch of the proof.** For the proof of (a), we fix a number $r > 0$ with $3r$ less than the injectivity radius at each point in the set $D \equiv \overline{B(p, L)}$, points $p = p_1, p_2, \ldots, p_k \in D$ such that the balls $B_1 = B(p_1, r), \ldots, B_k = B(p_k, r)$ form a covering for $D$, a Lebesgue number $\delta > 0$ for this covering, and a positive integer $m$ such that $L/m < \delta$. For each pair of indices $i, j$, we let $\lambda_{ij} = \lambda_{ij}^{-1}$ be a minimal geodesic from $p_i$ to $p_j$. Now any piecewise $C^\infty$ loop $\alpha$ of length $< L$ based at $p$ is homotopic to a loop $\alpha_1 \ast \alpha_2 \ast \cdots \ast \alpha_m$ in $D$; where, for each $\nu = 1, \ldots, m$, $\alpha_\nu$ is a piecewise $C^\infty$ path of length $< \delta$ and is, therefore, contained in $B_{\nu}$ for some index $i_\nu$. We may assume that $i_1 = i_m = 1$. Thus $\alpha$ is homotopic to the loop $\lambda_{i_1i_2} \ast \lambda_{i_2i_3} \ast \cdots \ast \lambda_{i_{m-1}i_m}$ and the claim follows.

For the proof of (b), we let $I$ be the set of pairs of indices $(i, j)$ with $1 \leq i, j \leq m$ and $A_i \cap A_j \neq \emptyset$ and, for each $(i, j) \in I$, we fix a point $p_{ij} = p_{ji} \in A_i \cap A_j$. If $(i, j), (j, k) \in I$, then $A_j \subset B_g(p_{ij}, r)$ and we get a unique minimal geodesic $\gamma_{ijk} = \gamma_{kji}^{-1}$ from $p_{ij}$ to $p_{jk}$.

Fix a point $p \in A$. Given a point $q \in A$, we may form a broken geodesic $\lambda$ of the form

$$\lambda = \zeta \ast \gamma_{i_0i_1i_2} \ast \gamma_{i_1i_2i_3} \ast \cdots \ast \gamma_{i_{k-2}i_{k-1}i_k} \ast \eta$$

from $p$ to $q$ where $p \in A_{i_0} \subset B(p_{i_0i_1}, r)$, $q \in A_{i_k} \subset B(p_{i_{k-1}i_k}, r)$, and $\zeta$ and $\eta$ are the unique minimal geodesics from $p$ to $p_{i_0i_1}$ and from $p_{i_{k-1}i_k}$ to $q$, respectively. On the other hand, any broken geodesic $\lambda$ of the above form is homotopic to a path in $A$. If we choose $\lambda$ so that $k$ is minimal, then each pair $(i, j) \in I$ can be equal to $(i_{\nu-1}, i_\nu)$ for at most one $\nu \in \{1, 2, 3, \ldots, k\}$. Thus we must have $k \leq m^2$ and hence $l_g(\lambda) < r + (m^2 - 1)r + r = (m^2 + 1)r$. Thus we see that any point $q \in A$ may be joined to $p$ by a piecewise $C^\infty$ path $\lambda$ in $M$ which has length $< (m^2 + 1)r$ and which is homotopic (in $M$) to a path in $A$.

Now given a loop $\beta$ in $A$ based at $p$, we may choose a subdivision $0 = t_0 < t_1 < \cdots < t_k = 1$ such that, for $\nu = 1, \ldots, k$, $\beta \mid [t_{\nu-1}, t_\nu]$ is homotopic to a $C^\infty$ path of length $< 1$. By the above, for each $\nu = 1, \ldots, k - 1$, we may also choose a piecewise $C^\infty$ path $\lambda_\nu$ from $p$ to $\beta(t_\nu)$ such that $l(\lambda_\nu) < (m^2 + 1)r$ and $\lambda_\nu$ is homotopic to a path in $A$. Thus, in $\Gamma$, we have

$$[\beta] = [\beta_1] \cdots [\beta_k]$$
where $\beta_1 = \beta \mid_{[t_0, t_1]} * \lambda_1^{-1}$, $\beta_\nu = \lambda_{\nu - 1} * \beta \mid_{[t_{\nu - 1}, t_\nu]} * \lambda_\nu^{-1}$ for $\nu = 2, \ldots, k - 1$, and $\beta_k = \lambda_{k - 1} * \beta \mid_{[t_{k - 1}, t_k]}$. Since each of the above loops is homotopic to a piecewise $C^\infty$ loop of length $< (m^2 + 1)r + 1 + (m^2 + 1)r$, the claim (b) follows.

Finally, for the proof of (c), we fix $p \in M$ and we set $L = 2(m^2 + 1)r + 1$. Applying Part (b) in the covering space $(\hat{M}, \hat{g})$, we see that each element $\Gamma \in \mathcal{H}_p$ is generated by the set $K(p, L) \cap \Gamma$. Therefore, since $K(p, L)$ is a finite set by Part (a), $\mathcal{H}_p$ must be finite. \hspace{1cm} \Box

Proof of Proposition 4.2 (Delzant-Gromov \cite{DelG}). Stein factoring, we may assume that $\hat{\Phi}$ has connected fibers. Furthermore, by passing to the appropriate covering space as in part (d) of Lemma 4.3, we may assume that $\pi_1(\hat{X}_\zeta) \rightarrow \pi_1(\hat{X})$ is surjective for every regular value $\zeta$ of $\hat{\Phi}$ and, therefore, for any $\zeta \in \hat{S}$ (since $\pi_1(\hat{X}_\zeta)$ surjects onto $\text{im} \left[ \pi_1(X_U) \rightarrow \pi_1(X) \right] = \pi_1(X)$ for a sufficiently small connected neighborhood $U$ of $\zeta$ in $\hat{S}$). With these additional assumptions, we will show that the normalizer of $\text{im} \left[ \pi_1(X) \rightarrow \pi_1(X) \right]$ is of finite index. Equivalently, the normalizer of $\text{im} \left[ \pi_1(F) \rightarrow \pi_1(X) \right]$ is of finite index for any (possibly singular) fiber $F$ of $\hat{\Phi}$.

Clearly, we may assume without loss of generality that $n = \dim X > 1$ and, since $\hat{X}$ is noncompact, we have $\hat{S} = \mathbb{C}$ or $\Delta$. Fixing a point $\zeta_0 \in \hat{S}$ and a point $\hat{x}_0 \in F_0 \equiv \hat{X}_{\zeta_0} = \hat{\Phi}^{-1}(\zeta_0)$, we get

$$\Gamma_0 = \pi_1(\hat{X}, \hat{x}_0) = \text{im} \left[ \pi_1(F_0, \hat{x}_0) \rightarrow \pi_1(\hat{X}, \hat{x}_0) \right] \cong \Gamma_0 \equiv \mathcal{Y}_* \Gamma_0 \subset \Lambda \equiv \pi_1(X, x_0),$$

where $x_0 = \mathcal{Y}(\hat{x}_0)$. We must show that the normalizer

$$N_0 \equiv \left\{ \lambda \in \Lambda \mid \lambda \Gamma_0 \lambda^{-1} = \Gamma_0 \right\}$$

is of finite index in $\Lambda$. For this, it suffices to show that $\Gamma_0$ has only finitely many distinct conjugates in $\Lambda$.

The collection of conjugates of $\Gamma_0$ in $\Lambda$ is precisely the collection $\mathcal{H}$ of subgroups $\Gamma$ of $\Lambda$ of the form

$$\Gamma = \text{im} \left[ \pi_1(\hat{X}, \hat{x}) \rightarrow \Lambda \right] = \text{im} \left[ \pi_1(F, \hat{x}) \rightarrow \Lambda \right]$$

where $\hat{x} \in \mathcal{Y}^{-1}(x_0)$ and $F$ is the (not necessarily smooth) fiber of $\hat{\Phi}$ containing $\hat{x}$. Let $\hat{g} = \mathcal{Y}^* g$. According to Lemma 4.3, to show that $\mathcal{H}$ is finite, it suffices to find constants $m \in \mathbb{N}$ and $r > 0$ such that, for each fiber $F$ of $\hat{\Phi}$ meeting $\mathcal{Y}^{-1}(x_0)$, $r$ is a lower bound for the injectivity radius in $\hat{X}$ at each point in $F$ and there is a covering $A_1, \ldots, A_m$ of $F$ by connected relatively open subsets of $F$ of diameter $< r$ (in $\hat{X}$).
The covering space \( \hat{X} \) has bounded geometry because \( X \) does. Thus, for some constant \( C > 0 \) and for each point \( p \in \hat{X} \), there is a biholomorphism \( \Psi_p \) of the unit ball \( B_{C^n}(0; 1) \subset \mathbb{C}^n \) onto a neighborhood \( B_{E,p} = B_{E,p}(1) \) of \( p \) in \( \hat{X} \) such that \( \Psi_p(0) = p \) and

\[
C^{-1}\Psi_p^*g \leq g_{\mathbb{C}^n} \leq C\Psi_p^*g \quad \text{on } B_{C^n}(0; 1).
\]

We set \( B_{E,p}(r) = \Psi_p(B_{C^n}(0; r)) \) for each \( r \in (0, 1) \). Thus, for constants \( R_1, R_2, \) and \( R \) with \( 1/4 \gg R_1 > R_1/2 \gg R > R/2 \gg R_2 > 0 \), we get, for each point \( p \in \hat{X} \),

\[
B_{E,p}(1/4) \supset B_{\hat{g}}(p, R_1) \supset B_{\hat{g}}(p, R_1/2) \supset B_{E,p}(R) \supset B_{E,p}(R/2) \supset B_{\hat{g}}(p, R_2).
\]

If \( F \) is a fiber of \( \hat{\Phi} \) meeting \( \Upsilon^{-1}(x_0) \), then we may choose points \( p_1, \ldots, p_s \in F \) such that

\[
 F \subset B_{\hat{g}}(p_1, R_1) \cup \cdots \cup B_{\hat{g}}(p_s, R_1)
\]

but

\[
p_j \notin F \setminus \left[ B_{\hat{g}}(p_1, R_1) \cup \cdots \cup B_{\hat{g}}(p_{j-1}, R_1) \right] \quad \text{for } j = 2, \ldots, s.
\]

In particular, the balls \( B_{\hat{g}}(p_1, R_1/2), \ldots, B_{\hat{g}}(p_s, R_1/2) \) are disjoint and, therefore, the balls \( B_{E,p_1}(R), \ldots, B_{E,p_s}(R) \) are disjoint. According to [Sto], since \( \hat{X} \) is Kähler, the fibers of \( \hat{\Phi} \) have equal volume \( v_0 \) (counting multiplicities). But, by Lelong’s monotonicity formula (see estimate 15.3 in [Ch]), there is a constant \( \delta > 0 \) such that any analytic set \( A \) in \( B(0, 1/2) \) of pure dimension \( n - 1 \) passing through 0 satisfies \( \text{vol}_{C^n}(A \cap B_{C^n}(0, R)) \geq \delta \) (recall that \( R < 1/4 \)). It follows that

\[
 v_0 = \text{vol}_{\hat{g}}(F) \geq \sum_{j=1}^s \text{vol}_{\hat{g}}(F \cap B_{E,p_j}(R)) \geq \sum_{j=1}^s C^{-n}\text{vol}_{C^n}(\Psi_{p_j}^{-1}(F \cap B_{E,p_j}(1/2)) \cap B_{C^n}(0, R)) \geq C^{-n} \cdot s \cdot \delta.
\]

Thus we have the uniform bound \( s \leq C^n v_0 / \delta \) for \( s \). Consequently, we also get the uniform bound \( \text{diam } F \leq 2R_1 C^n v_0 / \delta \) for the diameter (with respect to the distance in \( \hat{X} \)) of \( F \).

It follows that the union \( Z \) of the images in \( X \) of the fibers of \( \hat{\Phi} \) meeting \( \Upsilon^{-1}(x_0) \) is a bounded set. Hence we may choose a constant \( r > 0 \) so that \( r \) is a lower bound for the injectivity radius in \( X \) at each point in \( Z \) and, therefore, for the injectivity radius in \( \hat{X} \) at each point in any fiber of \( \hat{\Phi} \) which meets \( \Upsilon^{-1}(x_0) \). We may also choose each of the neighborhoods \( \{B_{E,p}\}_{p \in \hat{X}} \) so that \( \text{diam } B_{E,p} < r \) for each \( p \in \hat{X} \) (although the above proof of the boundedness of \( Z \) involved the choice of \( \{B_{E,p}\}_{p \in \hat{X}} \), this boundedness property is clearly independent of the choice of \( \{B_{E,p}\}_{p \in \hat{X}} \).
Now suppose that \( F \) is again a fiber of \( \hat{\Phi} \) meeting \( \Upsilon^{-1}(x_0) \) and, in the above notation, \( 1 \leq j \leq s \) and \( A \) is a connected component of \( F \cap B_{E,p_j} \) which meets \( B_{\hat{g}}(p_j; R_1) \). Choosing a point \( a \in A \cap B_{\hat{g}}(p_j; R_1) \) and applying the above volume estimate then gives

\[
\text{vol}_{\hat{g}}(A) \geq C^{-n}\text{vol}_{C^n}((\Psi_{p_j}^{-1}(A) \cap B_{C^n}(\Psi_{p_j}^{-1}(a); 1/2)) \cap B_{C^n}(\Psi_{p_j}^{-1}(a); R)) \geq C^{-n} \cdot \delta.
\]

It follows that \( F \cap B_{E,p_j} \) can have at most \( C_nv_0/\delta \) such components. Combining this uniform bound with the uniform bound for \( s \), we see that, if we fix a positive integer \( m \geq (C_nv_0/\delta)^2 \), then, for any such \( F \), we may choose sets \( A_1, \ldots, A_m \) so that \( F = A_1 \cup \cdots \cup A_m \) (possibly with some repetition) and, for each \( \nu = 1, \ldots, m \), \( A_\nu \) a component of \( F \cap B_{E,p_j} \) (which meets \( B_{\hat{g}}(p_j, R_1) \)) for some \( j \). The finiteness of \( \mathcal{H} \), and hence of \([\Lambda : N_0] \), now follows.

Thus we get a commutative diagram

\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{\Upsilon'} & X' = \Lambda' \setminus \hat{X} \\
\downarrow{\Upsilon} & & \downarrow{\Upsilon}'' \\
\hat{X} & \xrightarrow{\hat{\Phi}} & \hat{S} \\
\end{array}
\]

in which \( \Upsilon' \) is a Galois covering map, \( \Upsilon'' \) is a finite covering map, and \( \Lambda' = N'_0/\Gamma'_0 \); where \( N'_0 \equiv \pi_1(X', x'_0) \xrightarrow{\sim} N_0 \) for \( x'_0 = \Upsilon'(\hat{x}_0) \) and \( \Gamma'_0 = \Upsilon'_0 \pi_1(\hat{X}, \hat{x}_0) \).

It remains to show that \( X' \) admits a proper holomorphic mapping onto a Riemann surface. But each automorphism \( \sigma \in \Lambda' \) maps fibers (of \( \hat{\Phi} \)) to fibers, because \( \hat{\Phi} \) is constant on every connected compact analytic set. Thus \( \sigma \) descends to an automorphism of \( \hat{S} \) and \( \Lambda' \) acts properly discontinuously on \( \hat{S} \) (in other words, the image of the homomorphism \( \Lambda' \to \text{Aut}(\hat{S}) \) acts properly discontinuously and the kernel is finite). We therefore get a commutative diagram

\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{\hat{\Phi}} & \hat{S} \\
\downarrow{\Upsilon'} & & \downarrow{\theta} \\
X' & \xrightarrow{\Phi} & S' \equiv \Lambda' \setminus \hat{S} \\
\end{array}
\]

where \( \theta \) is a branched infinite covering map, \( S' \) is a Riemann surface, and \( \Phi \) is a surjective proper holomorphic map with connected fibers. \( \square \)

**Lemma 4.5.** Let \((X, g)\) be a connected compact Kähler manifold. Suppose some connected covering space \( \Upsilon : \hat{X} \to X \) admits a surjective proper holomorphic mapping \( \hat{\Phi} : \hat{X} \to \hat{S} \)
onto a Riemann surface \( \widehat{S} \) whose universal covering is the unit disk \( \Delta \). Then some finite covering space of \( X \) admits a surjective holomorphic mapping onto a curve of genus \( g \geq 2 \).

**Proof.** Clearly, we may assume that \( \Upsilon : \widehat{X} \to X \) is an infinite covering and, by Stein factorization, we may assume that \( \Phi \) has connected fibers. After passing to the appropriate covering space, we may also assume that \( \pi_1(F) \) surjects onto \( \pi_1(\widehat{X}) \) for every fiber \( F \) of \( \Phi \) and hence that \( \widehat{S} = \Delta \) (in the above, we have used part (d) of Lemma 4.3 and the fact that, for a surjective holomorphic map of Riemann surfaces \( S^* \to \widehat{S} \), we get a lifting \( \widetilde{S}^* \to \Delta \), where \( \widetilde{S}^* \) is the universal covering of \( S^* \), and hence \( \widetilde{S}^* = \Delta \) by Liouville’s theorem). Furthermore, applying (the proof of) Proposition 4.2, we get a commutative diagram of holomorphic mappings

\[
\begin{array}{ccc}
\widehat{X} & \xrightarrow{\Phi} & \widehat{S} = \Delta \\
\downarrow{\Upsilon} & & \downarrow{\theta} \\
\tilde{X} & \xrightarrow{\tilde{\Phi}} & \tilde{S} = \tilde{\Theta} \setminus \Delta \\
\end{array}
\]

where \( \beta : \tilde{X} \to X \) is the finite covering with \( \beta_* \pi_1(X) \) equal to the normalizer of \( \Upsilon_* \pi_1(\tilde{X}) \); \( \alpha \) is the corresponding intermediate covering map; \( \tilde{\Lambda} = \pi_1(\tilde{X})/\tilde{\Gamma} \) is the quotient by the normal subgroup \( \tilde{\Gamma} = \alpha_* \pi_1(\tilde{X}) \); \( \tilde{\Theta} \) is the image of \( \tilde{\Lambda} \) under the homomorphism \( \tilde{\Lambda} \to \text{Aut}(\Delta) \); and \( \theta \) is the corresponding branched covering map of the quotient Riemann surface \( \tilde{S} \). Since \( \tilde{X} \) is compact (and, therefore, \( \tilde{\Lambda} \) is finitely generated), Selberg’s lemma provides a finite index torsion free normal subgroup \( \Theta' \) of \( \tilde{\Theta} \) and hence a commutative diagram of holomorphic mappings

\[
\begin{array}{ccc}
\widehat{X} & \xrightarrow{\Phi} & \widehat{S} = \Delta \\
\downarrow{\Upsilon} & & \downarrow{\theta} \\
\Lambda' \setminus \widehat{X} = X' & \xrightarrow{\Phi'} & \Theta' \setminus \Delta \\
\downarrow{\beta} & & \downarrow{\theta'} \\
\tilde{\Lambda} \setminus \tilde{X} = \tilde{X}' & \xrightarrow{\tilde{\Phi}} & \tilde{\Theta} \setminus \Delta \\
\end{array}
\]

where \( \Lambda' \subset \tilde{\Lambda} \) is the inverse image of \( \Theta' \). Clearly, the map \( \Phi' \) is surjective and the map \( \theta' \) is an unramified covering map, so the proof is complete. \( \square \)
Proof of Theorem 4.1. By Theorem 0.1, \( \hat{X} \) admits a proper holomorphic mapping \( \hat{\Phi} \) with connected fibers onto a (noncompact) Riemann surface \( \hat{S} \). For an ends decomposition \( \hat{X} \setminus K = E_1 \cup \cdots \cup E_m \) and a generic fiber \( F \) of \( \hat{\Phi} \) contained in \( E_1 \), we get a commutative diagram of holomorphic maps

\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{\hat{\Phi}} & \hat{S} = \Delta \text{ or } \mathbb{C} \\
\alpha \downarrow & & \rho \\
\hat{\Gamma} & \xrightarrow{\hat{\Phi}} & \tilde{\hat{S}} = \tilde{\hat{\Gamma}} \setminus \tilde{\hat{S}} \\
\beta & \downarrow & \tau \\
\hat{X} & \xrightarrow{\hat{\Phi}} & \tilde{\hat{S}} = \pi_1^{\text{orb}}(\hat{\Phi}) \setminus \tilde{\hat{S}}
\end{array}
\]

where \( \hat{\Gamma} : \hat{X} \to \hat{X} \) is the connected Galois covering space with

\[ \hat{\Gamma} \ast \pi_1(\hat{X}) = \Gamma \equiv \text{im} \left[ \pi_1(F) \to \pi_1(\hat{X}) \right] = \ker \left[ \pi_1(\hat{X}) \to \pi_1^{\text{orb}}(\hat{\Phi}) \right] ; \]

\( \beta : \hat{X} \to \hat{X} \) is the connected covering space with

\[ \beta \ast \pi_1(\hat{X}) = \tilde{\Gamma} \equiv \text{im} \left[ \pi_1(E_1) \to \pi_1(\hat{X}) \right] \supset \Gamma ; \]

\( \tilde{\hat{\Gamma}} \) is the group given by

\[ \tilde{\hat{\Gamma}} = \tilde{\Gamma} / \Gamma \subset \pi_1(\hat{X}) / \Gamma \cong \pi_1^{\text{orb}}(\hat{\Phi}) ; \]

\( \alpha : \tilde{\hat{X}} \to \hat{X} \) is the connected Galois covering space with

\[ \alpha \ast \pi_1(\tilde{\hat{X}}) = \text{im} \left[ \pi_1(F) \to \pi_1(\hat{X}) \right] = (\beta \ast)^{-1}(\Gamma) \cong \Gamma \]

for a fiber \( \tilde{\hat{F}} \) of \( \tilde{\hat{\Phi}} \) which \( \beta \) maps isomorphically onto \( F \); the maps \( \theta, \rho, \) and \( \tau \) are branched covering maps (with branch locus mapping into the set of critical values of \( \hat{\Phi} \)); and the maps \( \tilde{\hat{\Phi}} \) and \( \hat{\Phi} \) (and \( \hat{\Phi} \)) are surjective proper holomorphic mappings with connected fibers. Here, we identify \( \pi_1^{\text{orb}}(\hat{\Phi}) \) with the corresponding discrete subgroup of \( \text{Aut}(\hat{\Gamma}) \) or \( \text{Aut}(\tilde{\hat{\Gamma}}) \), depending on the context. According to Lemma 4.5, it now suffices to show that \( \tilde{\hat{S}} = \Delta \) (in fact, this will imply that one may get the desired finite covering \( X' \) of \( X \) by applying Lemma 1.5 to \( X \) and a suitable covering \( \tilde{\hat{X}} \) of the given covering \( \hat{X} \)). Observe that this is the case if some intermediate infinite covering space \( X^* \) between \( \tilde{\hat{X}} \) and \( X \) admits a proper holomorphic mapping \( \Phi^* \) with connected fibers onto a Riemann surface \( S^* \) whose universal covering is \( \Delta \). For the fibers of \( \Phi^* \) lift to a union of fibers of \( \hat{\Phi} \) (since \( \Phi^* \) is constant on the image of each fiber of \( \hat{\Phi} \) in \( X^* \)), so the covering map \( \tilde{\hat{X}} \to X^* \) descends to a surjective
holomorphic mapping $\tilde{S} \to S^*$. Lifting to a map $\tilde{S} \to \Delta$, we see that $\tilde{S} \neq \mathbb{C}$ and hence $\tilde{S} = \Delta$.

In particular, it suffices to consider the cases $\hat{S} = \mathbb{C}$ or $\mathbb{C}^\ast$. Since $e(\hat{X}) = e(\tilde{S})$ and $\tilde{e}(\hat{X}) \geq 3$, we may choose the ends decomposition $\hat{X} \setminus K = E_1 \cup \cdots \cup E_m$ so that $m = e(\hat{X}) = 1$ or $2$ and $\hat{\Gamma} \neq \pi_1(\hat{X})$. It follows that $e(\tilde{X}) \geq 2$ and therefore, since $e(\tilde{X}) = e(\tilde{S}) = 1$, $\hat{\lambda} = \hat{\Gamma}/\Gamma$ is infinite (i.e. $\alpha : \hat{X} \to X$ is an infinite covering). On the other hand, we have $[\pi_1^{\text{orb}}(\hat{\Phi}) : \hat{\Lambda}] = [\pi_1(\hat{X}) : \hat{\Gamma}]$. If this index is finite (i.e. $\beta : \hat{X} \to X$ is a finite covering), then $\tilde{e}(\hat{X}) = \tilde{e}(\hat{X}) \geq 3$ and hence, in this case, we may replace $\hat{X}$ by $\tilde{X}$. Thus we may assume without loss of generality that $\hat{\lambda}$ is of infinite index in $\pi_1^{\text{orb}}(\hat{\Phi})$ or $e(\hat{X}) = 2$ (i.e. $\hat{X} = \mathbb{C}^\ast$).

We now consider the possible properties of $\pi_1^{\text{orb}}(\hat{\Phi})$. We first observe that, if $\pi_1^{\text{orb}}(\hat{\Phi})$ contains an infinite cyclic subgroup of finite index, then every infinite subgroup of $\pi_1^{\text{orb}}(\hat{\Phi})$ is of finite index. In particular, we get $e(\hat{X}) = 2$ and $\beta : \hat{X} \to X$ is a finite covering and, therefore, $e(\tilde{X}) \geq 3$. Therefore, $\tilde{S} \neq \mathbb{C}$ or $\mathbb{C}^\ast$ and hence $\tilde{S} = \Delta$ in this case. In general, if $\hat{S} = \mathbb{C}$, then $\pi_1^{\text{orb}}(\hat{\Phi})$ is either trivial or a free product of a countable collection of nontrivial finite cyclic groups while, if $\hat{S} = \mathbb{C}^\ast$, then $\pi_1^{\text{orb}}(\hat{\Phi})$ is either $\mathbb{Z}$ or the free product of $\mathbb{Z}$ and a countable collection of nontrivial finite cyclic groups. The group $\mathbb{Z}_2 \ast \mathbb{Z}_2$ contains the finite index infinite cyclic subgroup $\langle a_1a_2 \rangle$, where $a_1$ and $a_2$ are the generators for the first and second copy of $\mathbb{Z}_2$, respectively. Thus, by the above remarks, we need only consider cases in which the free product representation for $\pi_1^{\text{orb}}(\hat{\Phi})$ has at least three nontrivial factors or at least two nontrivial factors with one of order $> 2$. But in these cases, $\pi_1^{\text{orb}}(\hat{\Phi})$ contains a non-Abelian free group and is therefore Fuchsian. □

**Remark.** Lemma 0.5 and Theorem 4.1 together give part (b) of Theorem 0.2. A proof giving both parts (a) (Gromov and Schoen) and (b) simultaneously appears in the next section.

5. **Amalgamations and mappings to Riemann surfaces**

Theorem 4.1 together with standard facts from geometric group theory (see Proposition 5.2 below) give Theorem 0.2 which is equivalent to the following:

**Theorem 5.1.** Suppose $X$ is a connected compact Kähler manifold whose fundamental group $\Lambda = \pi_1(X)$ induces a minimal action without inversion on a (simplicial) tree $T$
which is not a line or a point. Then some finite covering space $X' \to X$ admits a surjective holomorphic mapping onto a curve of genus $g \geq 2$.

Remarks. 1. Equivalently, if $X$ is a connected compact Kähler manifold whose fundamental group $\Lambda = \pi_1(X)$ is the fundamental group of a minimal reduced graph of groups for which the universal covering tree is not a line or a point, then some finite covering space admits a surjective holomorphic mapping onto a curve of genus $g \geq 2$.
2. We will only consider group actions on simplicial trees.

A brief discussion of the required facts from Bass-Serre Theory [Se] will be provided for the convenience of the reader. For a graph $Y$, we will denote the set of vertices by $\text{Vert}(Y)$ and the set of edges by $\text{Edge}(Y)$, the origin and terminus maps by

$$\alpha : \text{Edge}(Y) \to \text{Vert}(Y) \quad \text{and} \quad \omega : \text{Edge}(Y) \to \text{Vert}(Y),$$

respectively, and the edge inversion by $e \mapsto \bar{e}$. We often identify each edge $e \in \text{Edge}(Y)$ with the corresponding map $e : [0, 1] \to Y$. In a metric space $Z$, for $r > 0$, we denote the ball of radius $r$ centered at a point $z \in Z$ by $B_Z(z; r)$ and the $r$-neighborhood of a subset $A \subset Z$ by $N_Z(A; r)$. In particular, for a connected graph $Y$, a vertex $v \in \text{Vert}(Y)$, and a number $r \in (0, 1]$, we have

$$B_Y(v; r) = \bigcup_{e \in \text{Edge}(Y), \alpha(e) = v} e([0, r)).$$

Suppose $\Lambda$ is a group which acts without inversion on a tree $T$ (i.e. $\lambda e \neq \bar{e}$ for every edge $e \in \text{Edge}(T)$). The action is called minimal if $T$ contains no proper subtree that is invariant under the action. In general, if $\Lambda$ is finitely generated, then one can form a minimal $\Lambda$-invariant subtree $T_{\min}$. For we may take $T_{\min}$ to be a vertex $v$ of $T$ fixed by $\Lambda$ if such a vertex exists (i.e. if the action is elliptic). If no vertex is fixed, then we may take $T_{\min}$ to be the intersection of all $\Lambda$-invariant subtrees.

Observe also that, if the action of $\Lambda$ on $T$ is minimal and $T$ is not a single point (i.e. no vertex is fixed), then the valence of every vertex is at least 2 (i.e. there are no “dead ends”). For the collection of edges having an endpoint of valence 1 in $T$ is invariant and hence, after removing such edges, one gets an invariant subtree which, by minimality, must be equal to $T$. Thus no such edges can exist. Equivalently, each edge of $T$ is contained in a line.

Theorem 5.1 is an immediate consequence of Theorem 4.1 and the following fact:
Proposition 5.2. Suppose $M$ is a connected compact $C^\infty$ manifold whose fundamental group $\Lambda = \pi_1(M)$ induces a minimal action without inversion on a tree $T$ which has a vertex $v_0$ of valence at least 3. Then the connected covering $\hat{M} \to M$ for which $\text{im}[\pi_1(\hat{M}) \to \pi_1(M)]$ is equal to the isotropy subgroup $\Lambda_{v_0}$ at $v_0$ is an infinite covering (i.e. $[\Lambda : \Lambda_{v_0}] = \infty$) and $\hat{e}(\hat{M}) \geq 3$.

Remark. For $\Lambda = \pi_1(X) = \Gamma_1 * \Gamma_2$ as in the theorem of Gromov and Schoen (Theorem 0.2), $\Lambda$ induces a minimal action on a tree $T$ with fundamental domain $e \in \text{Edge}(T)$ such that $\Gamma$ is the stabilizer of $e$, $\Gamma_1 = \Lambda_{\alpha(e)}$, and $\Gamma_2 = \Lambda_{\omega(e)}$, and the index of $\Gamma$ in $\Gamma_1$ and $\Gamma_2$ is equal to the valence of $\alpha(e)$ and $\omega(e)$, respectively.

For the proof of the proposition, we first consider the following standard fact:

Lemma 5.3. Given a minimal cocompact action without inversion of a finitely generated group $\Lambda$ on a tree $T$ which is not a point, let $\hat{\theta} : T \to Y$ be the quotient map to the finite quotient graph $Y = \Lambda \setminus T$, let $v_0$ be a vertex in $T$, let $\theta : T \to \hat{T} \equiv \Lambda_{v_0} \setminus T$ be the quotient by the isotropy subgroup $\Lambda_{v_0}$, let $\hat{\theta} : \hat{T} \to Y$ be the induced map, let $y_0 = \hat{\theta}(v_0)$, let $\hat{v}_0 = \theta(v_0)$, and let $T_0$ be the subtree of $T$ with

$$\text{Edge}(T_0) = \{ e \in \text{Edge}(T) | \alpha(e) = v_0 \text{ or } \omega(e) = v_0 \}$$

and

$$\text{Vert}(T_0) = \{ v_0 \} \cup \{ \omega(e) | e \in \text{Edge}(T) \text{ and } \alpha(e) = v_0 \}.$$ 

Then we have the following:

(a) The image $Y_0 \equiv \hat{\theta}(T_0)$ (i.e. $\text{Vert}(Y_0) = \hat{\theta}(\text{Vert}(T_0))$, $\text{Edge}(Y_0) = \hat{\theta}(\text{Edge}(T_0)))$ is the (finite) subgraph of $Y$ with

$$\text{Edge}(Y_0) = \{ e \in \text{Edge}(Y) | \alpha(e) = y_0 \text{ or } \omega(e) = y_0 \}$$

and

$$\text{Vert}(Y_0) = \{ y_0 \} \cup \{ \omega(e) | e \in \text{Edge}(Y) \text{ and } \alpha(e) = y_0 \}.$$ 

(b) The graph $\hat{T}$ is a tree.

(c) $\Lambda_{v_0}$ acts on $T_0$ and on $T \setminus T_0$, $\hat{T}_0 \equiv \Lambda_{v_0} \setminus T_0$ is the finite subtree of $\hat{T}$ with

$$\text{Edge}(\hat{T}_0) = \{ e \in \text{Edge}(\hat{T}) | \alpha(e) = \hat{v}_0 \text{ or } \omega(e) = \hat{v}_0 \}$$

and

$$\text{Vert}(\hat{T}_0) = \{ \hat{v}_0 \} \cup \{ \omega(e) | e \in \text{Edge}(\hat{T}) \text{ and } \alpha(e) = \hat{v}_0 \},$$
\[ T_0 = \theta^{-1}(\hat{T}_0), \] and there exists a finite subtree \( T_1 \) of \( T_0 \) which \( \theta \) maps isomorphically onto \( \hat{T}_0 \).

(d) If \( R : [0, \infty) \rightarrow T \) is a ray in \( T \) with vertex \( R(0) = v_0 \), then \( \theta \) maps \( R \) isomorphically onto a ray \( \theta(R) \) in \( \hat{T} \).

Proof. For the proof of (a), we observe that if \( e \in \text{Edge}(Y_0) \) with \( \alpha(e) = y_0 = \hat{\theta}(v_0) \), then there is an edge \( f \in \text{Edge}(T) \) with \( \hat{\theta}(f) = e \). Since \( \hat{\theta}(\alpha(f)) = y_0 \), there is an element \( \lambda \in \Lambda \) with \( \alpha(\lambda \cdot f) = \lambda \cdot \alpha(f) = v_0 \). Thus \( \lambda \cdot f \in \text{Edge}(T_0) \) and \( \hat{\theta}(\lambda \cdot f) = e \).

For the proof of (b), suppose \( \hat{T} \) contains a circuit. Then (equivalently) \( \hat{T} \) contains a loop \( \hat{\gamma} = \hat{\gamma}_1 \circ \hat{\gamma}_2 \circ \cdots \circ \hat{\gamma}_k \); where \( \hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_k \) are edges such that \( \alpha(\hat{\gamma}_1) = \omega(\hat{\gamma}_k) = \theta(v_0) = \hat{v}_0 \) and, for some \( i \in \{1, \ldots, k\} \), we have \( \hat{\gamma}_i \neq \hat{\gamma}_j \) and \( \hat{\gamma}_i \neq \hat{\gamma}_j \). We may then lift \( \hat{\gamma} \) to a path \( \gamma = e_1 \circ e_2 \circ \cdots \circ e_k \) with \( \alpha(e_1) = v_0 \). Since \( \omega(e_k) \in \Lambda_{v_0} \cdot v_0 = \{v_0\} \), \( \gamma \) must be a loop. But we also have \( e_i \neq e_j \) and \( e_i \neq e_j \) for \( j = 1, \ldots, i, \ldots, k \), which is impossible since \( T \) is a tree. Thus \( \hat{T} \) is a tree.

For (c), it is clear that \( \Lambda_{v_0} \) acts on \( T_0 \) and on \( T \setminus T_0 \) and the proof of (a) shows that \( \hat{T}_0 \) is the subtree of \( \hat{T} \) as described. We may form the finite subtree \( T_1 \) of \( T_0 \) as follows. For each edge \( e \in \text{Edge}(Y_0) \) with \( \alpha(e) = y_0 \), we may choose an edge \( f_e \in \text{Edge}(T_0) \) with \( \alpha(f_e) = v_0 \) and \( \hat{\theta}(f_e) = e \). If \( e \) is a loop edge, then we may choose an element \( \lambda_e \in \Lambda \) such that \( \omega(\lambda_e f_e) = \lambda_e \cdot \omega(f_e) = v_0 \). Hence \( \alpha(\lambda_e f_e) = v_0 \), \( \hat{\theta}(\lambda_e f_e) = \hat{\theta}(\lambda_e f_e) = \hat{\theta}(f_e) = \hat{e} \), and \( f_e = \lambda_e^{-1}(\lambda_e f_e) \). Thus we may choose \( f_e, f_e, \lambda_e, \) and \( \lambda_e \) so that \( f_e = \lambda_e f_e \) and \( \lambda_e = \lambda_e^{-1} \) for each loop edge \( e \in \text{Edge}(Y_0) \). Note also that \( f_e \neq f_e \) since \( \Lambda \) acts without inversion. We now define the subtree \( T_1 \) by

\[
\text{Edge}(T_1) = \{ f_e \mid e \in \text{Edge}(Y_0) \text{ and } \alpha(e) = y_0 \} \cup \{ \overline{f_e} \mid e \in \text{Edge}(Y_0) \text{ and } \alpha(e) = y_0 \}
\]

and

\[
\text{Vert}(T_1) = \{v_0\} \cup \{ \omega(f_e) \mid e \in \text{Edge}(Y_0) \text{ and } \alpha(e) = y_0 \}.
\]

We have \( \Lambda_{v_0} \cdot T_1 = T_0 \). For, if \( f \in \text{Edge}(T_0) \) with \( \alpha(f) = v_0 \), then the edge \( e = \hat{\theta}(f) \) is an edge in \( Y_0 \) with initial point \( y_0 \) and hence \( \hat{\theta}(f) = e = \hat{\theta}(f_e) \). Thus \( f = \lambda f_e \) for some \( \lambda \in \Lambda \).

In particular, \( \lambda v_0 = \lambda \alpha(f_e) = \alpha(f) = v_0 \), so \( \lambda \in \Lambda_{v_0} \). Thus \( T_0 = \Lambda_{v_0} \cdot T_1 \). On the other hand, if \( f, f' \in \text{Edge}(T_1) \) with \( \lambda f = f' \) for some \( \lambda \in \Lambda_{v_0} \), then \( f = f' \). For we may assume without loss of generality that \( \alpha(f) = \alpha(f') = v_0 \) (otherwise, we replace the pair by \( \bar{f}, \bar{f}' \)). We then have \( \hat{\theta}(f) = \hat{\theta}(f') = e \in \text{Edge}(Y_0) \) and hence \( f = f' = f_e \) by the construction of
Thus $\theta$ maps $T_1$ isomorphically onto $\hat{T}_0$. In particular, $\hat{T}_0 = \Lambda_{v_0}\backslash T_0$ is a finite subtree of the tree $\hat{T} = \Lambda_{v_0}\backslash T$.

Finally, for (d), observe that if $v = R(j)$ and $w = R(k)$ are vertices in $R$ ($j, k \in \mathbb{N}$) and $\lambda \in \Lambda_{v_0}$ with $\lambda v = w$, then $\lambda \cdot [v_0, v] = [v_0, w] \subset R$. Since we then have $j = l([v_0, v]) = l([v_0, w]) = k$, we get $v = w$.

□

Proof of Proposition 5.2. Let $\tilde{\Upsilon} : \tilde{M} \to M$ be the universal covering of $M$ and let $B = \tilde{M} \times_A T \to M$ be the associated bundle. Here, we will consider the left action of $\Lambda$ on $\tilde{M}$ so that

$$B = \tilde{M} \times T/[(x, t) \sim (\lambda \cdot x, \lambda \cdot t) \ \forall \lambda \in \Lambda]$$

(in terms of the right action, $(\lambda \cdot x, \lambda \cdot t) = (x \cdot \lambda^{-1}, \lambda \cdot t) \sim (x, t)$). Since the fiber $T$ is contractible, $B \to M$ admits a continuous section $\sigma$. Thus we get a continuous equivariant map $\tilde{\Psi} : \tilde{M} \to T$ given by

$$\tilde{\Psi}(x) = t \iff [(x, t)] = \sigma(\tilde{\Upsilon}(x)).$$

Furthermore, the minimality of the action implies that $\tilde{\Psi}$ is surjective. To see this, we let $T'$ be the subgraph of $T$ for which $\text{Edge}(T')$ is the set of edges $e \in \text{Edge}(T)$ which lie entirely (including the endpoints) in $\tilde{\Psi}(\tilde{M})$ and $\text{Vert}(T') = \text{Vert}(T) \cap \tilde{\Psi}(\tilde{M})$. Then $T'$ is $\Lambda$-invariant because $\tilde{\Psi}(\tilde{M})$ is $\Lambda$-invariant. Furthermore, $T'$ is connected and, therefore, $T'$ is a subtree. For if $p$ and $q$ are distinct points in $T'$, then we may form a geodesic $\eta$ in $T$ from $p$ to $q$ which is contained in the path connected set $\tilde{\Psi}(\tilde{M})$. Clearly, each vertex in $T$ which $\eta$ meets will lie in $T'$. If $e$ is an edge whose interior meets $\eta$, then either $p$ or $q$ is in the interior or $e$ is a segment of $\eta$. In either case, we get $e \in \text{Edge}(T')$. Thus $\eta$ is contained in $T'$ and hence $T'$ is a subtree. Therefore, by minimality, we have $T' = T$ and hence $\tilde{\Psi}(\tilde{M}) = T$. Thus we get a commutative diagram of surjective continuous mappings.
In particular, the quotient graph $Y$ is compact; that is, finite. Thus we may form $v_0 = \tilde{\theta}(v_0)$, $\hat{v}_0 = \theta(v_0)$, $T \supset T_0 \supset T_1$, $\hat{T}_0 = \Lambda_{v_0} \backslash T_0 \subset \hat{T}$, and $Y_0 = \hat{\theta}(T_0) = \hat{\theta}(\hat{T}_0)$ as in Lemma 5.3.

Let
\[
C_0 \equiv \overline{B_T(v_0; 1/4)} = \bigcup_{e \in \text{Edge}(T), \alpha(e) = v_0} e([0, 1/4])
\]
and
\[
C_1 \equiv \overline{B_{T_1}(v_0; 1/4)} = \bigcup_{e \in \text{Edge}(T_1), \alpha(e) = v_0} e([0, 1/4]).
\]

We then have
\[
C_0 = \bigcup_{e \in \text{Edge}(T_0), \alpha(e) = v_0} e([0, 1/4]) = \overline{B_{T_0}(v_0; 1/4)} = \Lambda_{v_0} \cdot C_1.
\]

Moreover, $\theta$ maps $C_1$ isomorphically onto the connected compact set
\[
\hat{C}_0 \equiv \theta(C_0) = \overline{B_{\hat{T}}(\hat{v}_0; 1/4)} = \overline{B_{\hat{T}_0}(\hat{v}_0; 1/4)}.
\]

We will pull back components of $\hat{T} \setminus \hat{C}_0$ to get ends in $\hat{M}$. For this, we first observe that the set $K_0 \equiv \hat{\Psi}^{-1}(\hat{C}_0)$ is compact. For, given a point
\[
x_0 \in \hat{\Upsilon}(K_0) = \Psi^{-1}(\hat{\theta}(\hat{C}_0)) = \Psi^{-1}(\tilde{\theta}(C_0)) = \Psi^{-1}(\tilde{\theta}(C_1)),
\]
we may choose connected neighborhoods $B$ and $B'$ such that $x_0 \in B \subset B'$, $B'$ is contractible, and diam $\Psi(B') < 1/4$. Suppose $\hat{B}_1$ and $\hat{B}_2$ are two components of $\hat{\Upsilon}^{-1}(B)$ which meet $K_0$. Then, for $i = 1, 2$, we may choose a component $\hat{B}_i$ of $\hat{\Upsilon}^{-1}(\hat{B}_i)$ so that $\hat{\Psi}(\hat{B}_i)$ meets $C_1$. We then have
\[
\tilde{\theta}(\hat{\Psi}(\hat{B}_i)) = \Psi(B) \subset N_Y(\tilde{\theta}(C_1); 1/4) \subset B_Y(y_0; 1/2).
\]
Thus $\hat{\Psi}(\hat{B}_i)$ is a connected subset of $N_T(\Lambda \cdot v_0; 1/2)$ which meets $C_1 \subset B_T(v_0; 1/2)$ and hence $\hat{\Psi}(\hat{B}_i) \subset B_T(v_0; 1/2)$. On the other hand, for some $\lambda \in \Lambda$, $\hat{B}_2 = \lambda \cdot \hat{B}_1$ and so $B_T(v_0; 1/2) \cap B_T(\lambda v_0; 1/2) \neq \emptyset$. Thus $\lambda \in \Lambda_{v_0}$ and hence $\hat{B}_1 = \Upsilon(\hat{B}_1) = \Upsilon(\hat{B}_2) = \hat{B}_2$.

Therefore, a unique component $\hat{B}_1 \in \hat{M}$ of $\hat{\Upsilon}^{-1}(B)$ meets $K_0$. Covering the compact set $\hat{\Upsilon}(K_0) = \Psi^{-1}(\tilde{\theta}(C_1))$ by finitely many such sets $B$, we see that $K_0$ is compact (in fact, we have shown that $\hat{\Upsilon}$ maps $K_0$ homeomorphically onto $\hat{\Upsilon}(K_0)$).

We may now choose an ends decomposition $\hat{M} \setminus K = E_1 \cup \cdots \cup E_m$ in which $K$ is a connected compact set with $K_0 \subset K$ and we may set $C = \hat{\Psi}(K) \supset \hat{C}_0$. If $R : [0, \infty) \to \hat{T}$ is a ray with vertex $R(0) = \hat{v}_0$, then, since $C$ is a compact connected subset of the tree $\hat{T}$ and $\hat{C}_0 \subset C$, we have $C \cap R = R([0, b])$ for some $b \geq 1/4$. Similarly, for each $j = 1, \ldots, m$,
extension with base group $\Gamma$ and stable letter $\tau$

Let $\psi$ be a proper homomorphism of $\Gamma$, and let $\psi(E) \cap R$ be a singleton or a (possibly unbounded) interval whose closure meets $C \cap R$. But then

$$R((b, \infty)) = R \setminus C = R \setminus \psi(K) \subset \psi(M \setminus K) = \psi(E_1) \cup \cdots \cup \psi(E_m)$$

(since $\psi$ is surjective) and hence $R \setminus C \subset \psi(E_j)$ for some $j$. In other words, for each ray $R$ in $\hat{T}$ with vertex $\hat{v}_0$, there is an index $j \in \{1, \ldots, m\}$ such that the unbounded interval $R \setminus C$ is contained in $\psi(E_j)$. Note also that, if $S$ is the component of $\hat{T} \setminus \{v_0\}$ containing $R((0, \infty))$, then $\psi(E_j) \subset S$ because

$$\psi(E_j) \subset \psi(M \setminus K) \subset \psi(M \setminus K_0) = \hat{T} \setminus C_0 \subset \hat{T} \setminus \{v_0\}$$

and $\psi(E_j)$ meets $R((0, \infty))$.

Now, by hypothesis, there exist distinct edges $u_1, u_2, u_3 \in \text{Edge}(T)$ and rays $R_1, R_2, R_3$ with vertex $v_0$ such that, for $\nu = 1, 2, 3$, $\alpha(u_\nu) = v_0$ and $u_\nu = R_\nu \setminus [0,1]$ (see the remarks preceding the statement of the proposition). We may choose (not necessarily distinct) indices $i_1, i_2, i_3 \in \{1, \ldots, m\}$ such that, for $\nu = 1, 2, 3$,

$$\hat{R}_\nu \setminus C \subset \psi(E_{i_\nu}) \subset \hat{S}_\nu,$$

where $\hat{R}_\nu$ is the ray given by $\hat{R}_\nu = \theta(R_\nu)$ and $\hat{S}_\nu$ is the component of $\hat{T} \setminus \{\hat{v}_0\}$ containing $\hat{R}_\nu \setminus \{\hat{v}_0\}$. We then have, for some $b_\nu \geq 1/4$, $\hat{R}_\nu \cap C = \hat{R}_\nu \setminus [0, b_\nu]$, $\hat{R}_\nu \setminus C = \hat{R}_\nu \setminus (b_\nu, \infty)$, $R_\nu \cap \theta^{-1}(C) = R_\nu \setminus [0, b_\nu]$, and $R_\nu \setminus \theta^{-1}(C) = R_\nu \setminus (b_\nu, \infty)$. We may also choose a component $F_\nu$ of $\Psi^{-1}(E_{i_\nu})$ such that $\psi(F_\nu)$ meets $R_\nu \setminus \theta^{-1}(C)$. We have

$$\theta(\psi(F_\nu)) = \psi(\Psi(F_\nu)) = \psi(E_{i_\nu}) \subset \hat{S}_\nu,$$

so $\psi(F_\nu)$ is a connected subset of $\theta^{-1}(\hat{S}_\nu)$ meeting $R_\nu \setminus \{v_0\}$. Thus $\psi(F_\nu)$ is contained in the component $S_\nu$ of $T \setminus \{v_0\}$ containing $R_\nu \setminus \{v_0\}$ (note that $\theta^{-1}(\hat{v}_0) = \{v_0\}$). But $S_1, S_2,$ and $S_3$ are disjoint because the sets $e_\nu((0, 1]) \subset S_\nu$ for $\nu = 1, 2, 3$ are in different components of $T \setminus \{v_0\}$. Therefore, $F_1, F_2,$ and $F_3$ are disjoint and it follows that $e(\hat{M}) \geq 3$. \hfill \Box

We now complete the proof of Theorem 0.3. Suppose $\Lambda$ is a properly ascending HNN extension with base group $\Gamma$ and stable letter $\tau$. In other words, for some isomorphism $\varphi$ of $\Gamma$ onto a proper subgroup of $\Gamma$, we have

$$\Lambda = \langle \Gamma, \tau; \varphi(\gamma) = \tau^{-1} \gamma \tau \text{ for } \gamma \in \Gamma \rangle.$$

Remarks. 1. We have $\Gamma \hookrightarrow \Lambda$ and

$$\cdots \subset \tau^{-2} \Gamma \tau^2 \subset \tau^{-1} \Gamma \tau \subset \Gamma \subset \tau \Gamma \tau^{-1} \subset \Gamma \tau^2 \Gamma \tau^{-2} \subset \cdots.$$
2. The group $\hat{\Lambda} = \bigcup_{m \in \mathbb{Z}} \tau^m \Gamma \tau^{-m}$ is an infinitely generated normal subgroup of $\Lambda$. The quotient group $\Lambda/\hat{\Lambda}$ is infinite cyclic with generator $\tau \hat{\Lambda}$.

3. Clearly, if $\Gamma$ is finitely generated, then $\Lambda$ is finitely generated. According to a theorem of Bieri and Strebel [BiS], if $\Lambda$ is finitely presented, then one express $\Lambda$ as an HNN extension with finitely generated base group. However, it may be impossible to express $\Lambda$ as a properly ascending HNN extension with finitely generated base group.

4. $\Lambda$ induces a minimal left action without inversion on a tree $T$ such that the quotient graph $\Lambda \backslash T$ is a single loop edge. Under the embedding $\Gamma \hookrightarrow N$, we may identify the base group $\Gamma$ with the isotropy subgroup $\Lambda_{v_0}$ for some vertex $v_0$. For some unique edge $e_0$, we have $v_0 = \alpha(e_0)$ and $v_1 \equiv \omega(e_0) = \tau \cdot v_0$. The edges $\tau^m e_0$, $m \in \mathbb{Z}$, form a line $l: \mathbb{R} \to T$, the axis for $\tau$, with $l \mid_{m,m+1} = \tau^m \cdot e_0$ for each $m \in \mathbb{Z}$. For each $m \in \mathbb{Z}$, we set $v_m = \tau^m \cdot v_0 = l(m)$. We then have $\Lambda_{v_m} = \tau^m \Lambda_{v_0} \tau^{-m} \Lambda_{v_m} \supset \Lambda_{v_0}$ if $m \geq 0$ (while $\Lambda_{v_m} \subset \Lambda_{v_0}$ if $m \leq 0$), so $\Lambda_{v_0}$ must fix each point in the ray $l \mid_{[0,\infty)}$. In particular, $\Lambda_{v_0}$ fixes the edge $e_0$ and $\Lambda_{v_0}$ acts transitively on the remaining edges with initial vertex $v_0$. For if $f$ is an edge not equal to $e_0$ or $\tau^{-1} \cdot e_0$ with $\alpha(f) = v_0$, then there exists a $\lambda \in \Lambda$ with $\lambda \tau^{-1} \cdot e_0 = f$ or $\bar{f}$. If the former, then $\lambda \in \Lambda_{v_0}$. If the latter, then $f = \lambda \tau^{-1} \cdot e_0$ with $\lambda \tau^{-1} \in \Lambda_{v_0}$, contradicting the above.

5. Let $m \in \mathbb{Z}_{\geq 0}$ and let $D$ be the end in $T$ which is the component of $T \backslash \{v_{m+1}\}$ containing $v_0$. Then $\Lambda_{v_m} = \tau^m \Gamma \tau^{-m}$ is precisely the set of elements $\lambda \in \hat{\Lambda}$ with $\lambda \cdot v_0 \in D$. For each element of $\Lambda_{v_{m+1}}$ maps $D$ onto a component of $T \backslash \{v_{m+1}\}$. If $\lambda \in \Lambda_{v_m} \subset \Lambda_{v_{m+1}}$, then $\lambda \cdot \tau^m \cdot e_0 = \tau^m \cdot e_0$, and hence $\lambda \cdot D = D$. Conversely, if $\lambda \in \hat{\Lambda} \setminus \Lambda_{v_m}$, then we have $\lambda \in \Lambda_{v_{k+1}}$ for some minimal $k \geq m$. In particular, since $\tau^k \cdot e_0$ has endpoints $v_k$ and $v_{k+1}$, $\lambda$ must map the component $F$ of $T \backslash \{v_{k+1}\}$ containing $\tau^k \cdot e_0$ onto a different component. Hence, since $e_0 \subset D \subset F$, $D$ and $\lambda \cdot D$ must be disjoint.
Lemma 5.4. Let \( M \) be a connected compact \( C^\infty \) manifold whose fundamental group \( \Lambda = \pi_1(M) \) is a properly ascending HNN extension with finitely generated base group \( \Gamma \) and stable letter \( \tau \), let \( \hat{\Lambda} = \bigcup_{m \in \mathbb{Z}} \tau^m \Gamma \tau^{-m} \), and let \( \hat{\Psi} : \hat{M} \to M \) be a connected (Galois) covering space with \( \hat{\Psi}_* \pi_1(\hat{M}) = \hat{\Lambda} \). Then there is an end \( E \) for \( \hat{M} \) such that \( \text{im} \left[ \pi_1(E) \to \pi_1(\hat{M}) \right] \) is finitely generated. In fact, for any \( C^\infty \) relatively compact domain \( \Omega' \) in \( \hat{M} \) containing \( \partial E \), the image of the fundamental group of the end \( E' = E \cup \Omega' \) in \( \pi_1(\hat{M}) \) will be finitely generated.

Proof. As in the proof of Proposition 5.2 for \( \hat{\Psi} : \hat{M} \to M \) the universal covering and \( T \) the universal covering tree as above, we get a surjective continuous equivariant map \( \hat{\Psi} : \hat{M} \to T \) and a commutative diagram

\[
\begin{array}{ccc}
\hat{M} & \xrightarrow{\hat{\Psi}} & T \\
\downarrow{\\hat{\Psi}} & & \downarrow{\theta} \\
\hat{M} & \xleftarrow{(\hat{\Psi})} & \hat{\Lambda} \setminus \hat{M} \equiv \hat{M} \\
\end{array}
\]

The quotient \( \hat{T} \) is a line onto which \( \theta \) maps the line \( l \) isomorphically. We set \( \hat{l} = \theta \circ l \). Furthermore, the map \( \hat{\Psi} \) is proper (since \( \Lambda / \hat{\Lambda} \cong \mathbb{Z} \) acts properly discontinuously on \( \hat{M} \) and \( \hat{T} \)) and surjective and \( e(\hat{M}) = 2 \). For some point \( \hat{x}_0 \in \hat{M} \) with \( \hat{\Psi}(\hat{x}_0) = v_0 \) and for \( \hat{x}_0 = \gamma(\hat{x}_0), \hat{v}_0 = \theta(v_0) = \hat{\Psi}(\hat{x}_0), x_0 = \rho(\hat{\gamma}(\hat{x}_0)) = \hat{\gamma}(\hat{x}_0), \) and \( y_0 = \hat{\theta}(v_0) = \hat{\gamma}(\hat{v}_0) = \hat{\Psi}(x_0) \), we may identify \( \Lambda = \pi_1(M) \) with \( \pi_1(M, x_0) \), \( \pi_1(\hat{M}) \) with \( \pi_1(M, \hat{x}_0) \), and \( \hat{\Lambda} \) with \( \hat{\gamma}_* \pi_1(\hat{M}, \hat{x}_0) \).

By the arguments in the proof of Proposition 5.2, some end of \( \hat{M} \) is a component of \( \hat{\Psi}^{-1}(\hat{l}(\langle -\infty, 1 \rangle)) \). Forming the union of this end with a large relatively compact domain in \( \hat{M} \), we get an end \( E_0 \) of \( \hat{M} \) such that \( \hat{x}_0 \in E_0 \) and \( \hat{\Psi}(E_0) \subset \hat{l}(\langle -\infty, m \rangle) \) for some positive integer \( m \). We may now choose a \( C^\infty \) relatively compact domain \( \Omega \) in \( \hat{M} \) such that \( \hat{x}_0 \in \Omega \), \( \partial E_0 \subset \Omega \), \( \Omega \cap E_0 \) is connected, and \( \text{im} \left[ \pi_1(\Omega, \hat{x}_0) \to \pi_1(M, x_0) \right] \supset \Lambda_{vm} = \tau^m \Gamma \tau^{-m} \) (here we have used the fact that \( \Gamma \) is finitely generated).

We now show that, for \( E = E_0 \cup \Omega \), \( \text{im} \left[ \pi_1(E, \hat{x}_0) \to \pi_1(M, x_0) \right] \) is finitely generated. Since \( \Omega \) is a \( C^\infty \) domain, we may choose a domain \( \Theta \) such that \( \Omega \subset \Theta \subset \hat{M} \) and \( \overline{\Theta} \) is a strong deformation retract of \( \Theta \). Given a loop \( \beta \) in \( E \) based at \( \hat{x}_0 \), we may choose a partition \( 0 = s_0 < t_0 < s_1 < t_1 < \cdots < s_k < t_k = 1 \) such that \( \beta(s_j), \beta(t_j) \in \overline{\Omega} \) and...
β([s_j, t_j]) \subset \Theta$ for $j = 0, \ldots, k$, and $β((t_{j-1}, s_j)) \subset E \setminus \Omega = E_0 \setminus \Omega$ for $j = 1, \ldots, k$. For each $j = 1, \ldots, k$, we may choose a path $δ_j$ in $E_0 \cap \overline{\Omega}$ from $β(t_{j-1})$ to $β(s_j)$ (since these points lie in $E \cap \partial \Omega = E_0 \cap \partial \Omega$ and $E_0 \cap \overline{\Omega}$ is connected), and, for each $j = 0, \ldots, k$, we may choose a path $ε_j$ in $E_0 \cap \overline{\Omega}$ from $x_0$ to $β(t_j)$. The loop $β$ is homotopic to the loop $η_0 * κ_1 * η_1 * κ_2 * \cdots * η_{k-1} * κ_k * η_k$, where, $η_0 = β\mid_{[s_0, t_0]} * ε_0^{-1}$, $κ_j = ε_{j-1} * β\mid_{[t_{j-1}, s_j]} * δ_{j-1}^{-1} * ε_{j-1}^{-1}$ for $j = 1, \ldots, k$, $η_j = ε_{j-1} * δ_j * β\mid_{[s_j, t_j]} * ε_j^{-1}$ for $j = 1, \ldots, k-1$, and $η_k = ε_{k-1} * \delta_k * β\mid_{[s_k, t_k]}$. For each $j = 0, \ldots, k$, $η_j$ is contained in $\Theta$ and, therefore, homotopic to a loop in $Ω$. For each $j = 1, \ldots, k$, $κ_j$ is a loop in $E_0$ and hence the lifting $\tilde{κ}_j$ to $\tilde{M}$ with $\tilde{κ}_j(1) = \tilde{x}_0$ lies in $\tilde{Ψ}^{-1}(D)$; where $D \subset θ^{-1}(\hat{U}((−\infty, m+1)))$ is the component of $T \setminus \{v_{m+1}\}$ containing $v_0$. Therefore, the element $λ_j = [\tilde{Ψ}(κ_j)] \in π_1(M, x_0)$ satisfies $λ_j \cdot v_0 = \tilde{Ψ}(λ_j \cdot \tilde{x}_0) = \tilde{Ψ}(\tilde{κ}_j(0)) \in D$. Thus, by the remarks preceding this proof, we have $λ_j \in Λ_{vm}$ and hence, by construction, $κ_j$ is homotopic to a loop in $Ω$. Since $Ω \subset E$, we get

$$\text{im} \left[ π_1(Ω, \tilde{x}_0) → π_1(M, x_0) \right] \subset \text{im} \left[ π_1(E, \tilde{x}_0) → π_1(M, x_0) \right] \subset \text{im} \left[ π_1(Ω, \tilde{x}_0) → π_1(M, x_0) \right].$$

Thus we get equality of the above finitely generated groups.

Finally, given a relatively compact $C^∞$ domain $Ω'$ in $\tilde{M}$ containing $\partial E$, we may choose a $C^∞$ relatively compact domain $Ω''$ in $\tilde{M}$ such that $Ω \subseteq Ω''$, $E_0 \cap Ω''$ is connected, and $Ω'' \setminus E = Ω' \setminus E$. The above argument applied to the ends $E_0 \subset E' = E \cup Ω' = E_0 \cup Ω''$ gives finite generation of $\text{im} \left[ π_1(E', \tilde{x}_0) → π_1(M, x_0) \right]$. \(\square\)

**Remark.** If $M = S$ is a compact Riemann surface of genus 2 and $Λ = π_1(M)$ is expressed as a properly ascending HNN extension with (infinitely generated) base group $Γ$ as in the remarks preceding the above lemma, then the corresponding covering $\tilde{M}$ is, topologically, an infinite tube with an infinite sequence of handles attached; and $\mathbb{Z}$ acts transitively and freely on this collection of handles. In particular, for any end $E$, the image of $π_1(E)$ is *not* finitely generated. This illustrates how the lemma and the theorem fail if the base group is not assumed to be finitely generated.

**Completion of the proof of Theorem 0.3.** It remains to show that any properly ascending HNN extension $Λ$ with finitely generated base group $Γ$ and stable letter $τ$ is *not* Kähler. For this, we assume that $Λ = π_1(X)$ for some connected compact Kähler manifold $X$ and reason to a contradiction.

Let $\hat{Λ} = ∪_{m \in \mathbb{Z}} \tau^m Γ τ^{-m}$ and let $Υ: \hat{X} → X$ be a connected (Galois) covering space with $Υ∗ π_1(\hat{X}) = \hat{Λ}$. According to Lemma 5.3, $\text{im} \left[ π_1(E) → π_1(\hat{X}) \right]$ is finitely generated.
for some end $E$ of $\hat{X}$. In particular, this image group is of infinite index and hence, by Theorem 0.1 there exists a proper holomorphic mapping $\hat{\Phi}$ with connected fibers of $\hat{X}$ onto a Riemann surface $\hat{S}$, with $e(\hat{S}) = 2$. The action of $\mathbb{Z} \cong \Lambda/\Lambda'$ on $\hat{X}$ descends to a properly discontinuous action of $\mathbb{Z}$ on $\hat{S}$ and, since $\mathbb{Z}$ is torsion-free, the action is free. Thus we get a commutative diagram

$$
\begin{array}{ccc}
\hat{X} & \xrightarrow{\hat{\Phi}} & \hat{S} \\
\Upsilon \downarrow & & \downarrow \theta \\
\mathbb{Z}/\hat{X} = X & \xrightarrow{\Phi} & S \equiv \mathbb{Z}/\hat{S}
\end{array}
$$

where $\theta$ is an (unbranched) infinite covering map, $S$ is a compact Riemann surface, and $\Phi$ is a surjective proper holomorphic map with connected fibers. In particular, the singular fibers of $\hat{\Phi}$ are precisely the liftings of the singular fibers of $\Phi$.

By the second conclusion of Lemma 5.4, we may assume that $E = \hat{\Phi}^{-1}(F)$ for some end $F$ of $\hat{S}$ with smooth boundary. Hence $\text{im} \left[ \pi_1(F) \to \pi_1(\hat{S}) \right]$ is finitely generated and, therefore, the Riemann surface $F$ is of finite type. Thus $F$ contains an end $D$ which is isomorphic to either a punctured disk or an annulus.

Moreover, $\text{im} \left[ \pi_1^{\text{orb}}(\hat{\Phi} \mid_E) \to \pi_1^{\text{orb}}(\hat{\Phi}) \right]$ is finitely generated. If $F$ contains a point $p$ such that the greatest common divisor of the multiplicities of the components of the divisor $\hat{\Phi}^{-1}(p)$ is greater than 1, then $F$ contains infinitely many such points (since the singular fibers are liftings of the singular fibers of $\Phi$). Hence $\pi_1^{\text{orb}}(\hat{\Phi} \mid_E)$ contains an infinite free product of finite cyclic groups which injects into $\pi_1^{\text{orb}}(\hat{\Phi})$. But this contradicts the finite generation property. Thus $F$ cannot contain such points and therefore, applying the action of $\mathbb{Z}$ to move the fibers over points in $\hat{S} \setminus F$ into $F$, we see that, for each fiber of $\hat{\Phi}$ or $\Phi$, the greatest common divisor of the multiplicities of the components is 1. Thus $\pi_1(\hat{S}) = \pi_1^{\text{orb}}(\hat{\Phi})$ is infinitely generated and hence, in particular, $\hat{S} \neq \mathbb{C}^*$. Hence $S$ is a curve of genus $g \geq 2$ and therefore, since a punctured disk end for $\hat{S}$ would imply the existence of a parabolic element in $\pi_1(S)$, the end $D \subset F$ must be isomorphic to an annulus.

On the other hand, $\hat{X}$ admits a proper pluriharmonic function $\alpha: \hat{X} \to \mathbb{R}$ (for example, one may take $\alpha$ to be the integral of the lifting of a closed real harmonic 1-form on $X$ which integrates to 1 on $\tau$, and 0 on $\Lambda'$). The function $\alpha$ descends to a proper (pluri)harmonic function $\beta: \hat{S} \to \mathbb{R}$. This implies that $\hat{S}$ is parabolic. For assuming, as we may, that $\pm 1$
is a regular value for $\beta$, the harmonic function $|\beta|/R$ on the set $\{ x \in \hat{S} \mid 1 < |\beta(x)| < R \}$ vanishes on $\beta^{-1}(\pm 1)$ and is equal to 1 on $\beta^{-1}(\pm R)$. Letting $R \to \infty$, we see that the harmonic measure of the ideal boundary of $\hat{S}$ with respect to $\{ x \in \hat{S} \mid |\beta(x)| < 1 \}$ vanishes. Thus we have again arrived at a contradiction and, therefore, $\Lambda$ cannot be Kähler.

\section{Principal functions and the Evans-Selberg potential}

For the convenience of the reader, the proof of Sario’s existence theorem of principal functions \cite{RS} and Nakai’s construction of the Evans-Selberg potential \cite{Na1}, \cite{Na2}, \cite{SaNo} are provided in this section. These facts were applied in \cite{NR1}. However, it is difficult to find proofs for a general oriented Riemannian manifold in a convenient form in the literature. This section will not appear in the version submitted for publication. Throughout this section, $(M, g)$ will denote a connected noncompact oriented Riemannian manifold of dimension $n > 2$.

\subsection{Principal functions}

Throughout this subsection, we will assume that $(M, g)$ is parabolic and $M_0$ will denote a $C^\infty$ relatively compact domain in $M$. In this subsection, we recall the following theorem of Sario \cite{RS}:

\begin{theorem}[Sario]
If $u$ is a continuous function on $M \setminus M_0$ which is harmonic on $M \setminus \overline{M_0}$ and satisfies the flux condition:
\[ \int_{\partial M_0} \frac{\partial u}{\partial \nu} = 0, \]
then there exists a harmonic function $v$ on $M$ such that $u - v$ is bounded on $M \setminus M_0$.
\end{theorem}

\begin{remarks}
1. This theorem is also true in the hyperbolic case, but we will only need it for the parabolic case.
2. The flux condition gives
\[ \int_{\partial \Omega} \frac{\partial u}{\partial \nu} = 0 \]
for some (hence for every) $C^\infty$ relatively compact domain $\Omega$ containing $\overline{M_0}$.

For the rest of this subsection, we will assume, as we may, that $M \setminus \overline{M_0}$ has no relatively compact connected components.
Lemma 6.2. There exists a linear map

\[ L : C^0(\partial M_0) \to \text{Harm}(M \setminus \overline{M}_0) \cap C^0(M \setminus M_0) \]

such that, for every continuous function \( \alpha \) on \( \partial M_0 \), we have:

(i) \( (L\alpha) \mid_{\partial M_0} = \alpha \),
(ii) \( \min_{\partial M_0} \alpha \leq L\alpha \leq \max_{\partial M_0} \alpha \), and
(iii) \( \int_{\partial M_0} \frac{\partial (L\alpha)}{\partial \nu} = 0 \).

Remark. Such an operator \( L \) is called a normal operator for \( M \setminus M_0 \).

Proof of Lemma 6.2. Let \( \{ M_k \}_{k=1}^{\infty} \) be a fixed exhaustion of \( M \) by \( C^\infty \) relatively compact domains containing \( \overline{M}_0 \), and, for each positive integer \( k \), let \( v_k \) be the harmonic measure of \( \partial M_k \) with respect to \( M_k \setminus \overline{M}_0 \). Given a continuous function \( \alpha \) on \( \partial M_0 \), for each positive integer \( k \), let \( w_k \in \text{Harm}(M_k \setminus \overline{M}_0) \cap C^0(M_k \setminus M_0) \) be the function which vanishes on \( \partial M_k \) and is equal to \( \alpha \) on \( \partial M_0 \). Since \( M \) is parabolic, \( v_k \to 0 \) uniformly on compact subsets of \( M \setminus M_0 \). Since the sequence of nonnegative functions \( \{ w_k - (\min_{\partial M_0} \alpha)(1 - v_k) \} \) is bounded and nondecreasing, Harnack’s principle (Lemma 1.3) implies that the sequence \( \{ w_k \} \) converges uniformly on compact subsets of \( M \setminus M_0 \) to a function \( w \in \text{Harm}(M \setminus \overline{M}_0) \cap C^0(M \setminus M_0) \).

We set \( L\alpha \equiv w \). It remains to verify the properties (i), (ii), and (iii). The property (i) is clear. The property (ii) follows from the minimum principle for superharmonic functions on parabolic manifolds (see Sect. 1). Finally, to verify the property (iii), we observe that

\[
\int_{\partial M_1} \frac{\partial w_k}{\partial \nu} = \int_{\partial M_k} \frac{\partial w_k}{\partial \nu} = \int_{\partial M_k} v_k \frac{\partial w_k}{\partial \nu} = \int_{\partial (M_k \setminus M_1)} v_k \frac{\partial w_k}{\partial \nu} + \int_{\partial M_1} v_k \frac{\partial w_k}{\partial \nu} = \int_{\partial (M_k \setminus M_1)} \frac{\partial v_k}{\partial \nu} w_k + \int_{\partial M_1} v_k \frac{\partial w_k}{\partial \nu} = -\int_{\partial M_1} \frac{\partial v_k}{\partial \nu} w_k + \int_{\partial M_1} v_k \frac{\partial w_k}{\partial \nu} \to 0 \quad \text{as} \quad k \to \infty.
\]

Therefore

\[
\int_{\partial M_0} \frac{\partial w}{\partial \nu} = \int_{\partial M_1} \frac{\partial w}{\partial \nu} = \lim_{k \to \infty} \int_{\partial M_1} \frac{\partial w_k}{\partial \nu} = 0.
\]

Thus \( L \) is a normal operator. \( \square \)

Next we recall the following important consequence of Harnack’s principle:
Lemma 6.3 (q-lemma). Given a compact subset $K$ of an oriented Riemannian manifold $N$, there exists a constant $q \in (0, 1)$ such that

$$q \inf_{N} u \leq u(x) \leq q \sup_{N} u \quad \forall x \in K$$

for every harmonic function $u$ on $N$ which changes sign on $K$.

Proof of Theorem 6.1. Since $L(u \mid_{\partial M_0})$ is bounded, $u - v$ will be bounded for a given function $v$ if and only if $u - L(u \mid_{\partial M_0}) - v$ is bounded. Hence we may assume without loss of generality that $u$ vanishes on $\partial M_0$.

Fix a $C^\infty$ domain $\Omega$ with $M_0 \Subset \Omega \Subset M$ and let $K : C^0(\partial \Omega) \to \text{Harm}(\Omega) \cap C^0(\overline{\Omega})$ be the linear map which associates to each (finitely) continuous function $\alpha$ on $\partial \Omega$ the continuous function on $\overline{\Omega}$ which is harmonic on $\Omega$ and equal to $\alpha$ on $\partial \Omega$. Clearly, it suffices to find a harmonic function $v$ on $M$ such that

$$u - v = -L((K(v \mid_{\partial \Omega})) \mid_{\partial M_0})$$

on $M \setminus M_0$; since the right-hand side is bounded. For this, we need only find a solution $\alpha \in C^0(\partial \Omega)$ to

(1) $$(I-J)\alpha = u \mid_{\partial \Omega}$$

where $J : C^0(\partial \Omega) \to C^0(\partial \Omega)$ is the continuous linear operator defined by

$$J\alpha = (L((K\alpha) \mid \partial M_0)) \mid_{\partial \Omega} \quad \forall \alpha \in C^0(\partial \Omega).$$

For the function $v$ defined by

$$v \mid_{\Omega} \equiv K\alpha \quad \text{and} \quad v \mid_{(M \setminus M_0)} \equiv u + L((K\alpha) \mid \partial M_0)$$

will then have the required properties. For this, we will prove uniform convergence of the series

$$\sum_{m=0}^{\infty} J^m(u \mid_{\partial \Omega}).$$

The sum $\alpha \in C^0(\partial \Omega)$ will then be a solution to equation (1).

We first prove two identities. Let $w$ be the harmonic measure of $\partial \Omega$ with respect to $\Omega \setminus \overline{M_0}$. Then

(a) $\int_{\partial M_0} \beta \frac{\partial w}{\partial \nu} = \int_{\partial \Omega} (L\beta) \frac{\partial w}{\partial \nu} \quad \forall \beta \in C^0(\partial M_0)$; and

(b) $\int_{\partial M_0} (K\beta) \frac{\partial w}{\partial \nu} = \int_{\partial \Omega} \beta \frac{\partial w}{\partial \nu} \quad \forall \beta \in C^0(\partial \Omega)$.
For the proofs, it suffices to consider $C^\infty$ functions since $C^\infty(\partial M_0)$ and $C^\infty(\partial \Omega)$ are dense in $C^0(\partial M_0)$ and $C^0(\partial \Omega)$, respectively. If $\beta \in C^\infty(\partial M_0)$, then $L\beta$ is harmonic on $M \setminus M_0$ with $C^\infty$ boundary data. Therefore $L\beta \in C^\infty(M \setminus M_0)$ and
\[ \int_{\partial M_0} \beta \frac{\partial w}{\partial \nu} = \int_{\partial M_0} (L\beta) \frac{\partial w}{\partial \nu} = \int_{\partial \Omega} (L\beta) \frac{\partial w}{\partial \nu} - \int_{\partial \Omega} \frac{\partial (L\beta)}{\partial \nu} w + \int_{\partial M_0} \frac{\partial (L\beta)}{\partial \nu} w \]
since $w \equiv 0$ on $\partial M_0$, $w \equiv 1$ on $\partial \Omega$, and $L\beta$ satisfies the flux condition (iii) of Lemma 6.2. Thus the identity (a) is proved. The proof of (b) is similar.

Next we show that the function $KJ^m(u \mid \partial \Omega)$ changes sign on $\partial M_0$ for every nonnegative integer $m$ (which will allow us to apply the $q$-lemma). In fact, we show that
\[ \int_{\partial M_0} KJ^m(u \mid \partial \Omega) \frac{\partial w}{\partial \nu} = 0. \]
Since $\frac{\partial w}{\partial \nu} > 0$ on $\partial M_0$, it will follow that $KJ^m(u \mid \partial \Omega)$ changes sign on $\partial M_0$. For $m = 0$, the identity (b) implies that
\[ \int_{\partial M_0} (K(u \mid \partial \Omega)) \frac{\partial w}{\partial \nu} = \int_{\partial \Omega} u \frac{\partial w}{\partial \nu} = \int_{\partial M_0} u \frac{\partial w}{\partial \nu} + \int_{\partial \Omega} \frac{\partial u}{\partial \nu} w - \int_{\partial M_0} \frac{\partial u}{\partial \nu} w = 0 \]
since $u \equiv w \equiv 0$ on $\partial M_0$, $w \equiv 1$ on $\partial \Omega$, and $u$ satisfies the flux condition. If $m > 0$, then, by the identities,
\[ \int_{\partial M_0} (KJ^m(u \mid \partial \Omega)) \frac{\partial w}{\partial \nu} = \int_{\partial \Omega} J^m(u \mid \partial \Omega) \frac{\partial w}{\partial \nu} \]
\[ = \int_{\partial \Omega} L((KJ^{m-1}(u \mid \partial \Omega)) \mid \partial M_0) \frac{\partial w}{\partial \nu} \]
\[ = \int_{\partial M_0} KJ^{m-1}(u \mid \partial \Omega) \frac{\partial w}{\partial \nu}. \]
The claim now follows by induction on $m$. Thus $KJ^m(u \mid \partial \Omega)$ is a harmonic function on $\Omega$ which changes sign on the compact set $\partial M_0$. Therefore, by the $q$-lemma, there exists a number $q \in (0, 1)$ (independent of $m$) such that, for each nonnegative integer $m$,
\[ q \min_{\partial \Omega}(KJ^m(u \mid \partial \Omega)) = q \inf_{\Omega}(KJ^m(u \mid \partial \Omega)) \]
\[ \leq KJ^m(u \mid \partial \Omega) \mid \partial M_0 \]
\[ \leq q \sup_{\Omega}(KJ^m(u \mid \partial \Omega)) \]
\[ = q \max_{\partial \Omega}(KJ^m(u \mid \partial \Omega)). \]
Hence
\[ q \min_{\partial \Omega}(J^m(u \mid_{\partial \Omega})) \leq (KJ^m(u \mid_{\partial \Omega})) \mid_{\partial M_0} \leq q \max_{\partial \Omega}(J^m(u \mid_{\partial \Omega})) \]
since \( KJ^m(u \mid_{\partial \Omega}) = J^m(u \mid_{\partial \Omega}) \) on \( \partial \Omega \). By the property (ii) of the normal operator \( L \) stated in Lemma 6.2, we get
\[
\max_{\partial \Omega}(J^{m+1}(u \mid_{\partial \Omega})) = \max_{\partial \Omega}(L((KJ^m(u \mid_{\partial \Omega})) \mid_{\partial M_0}))
\leq \max_{\partial M_0}(KJ^m(u \mid_{\partial \Omega}))
\leq q \max_{\partial \Omega}(J^m(u \mid_{\partial \Omega})).
\]
Similar inequalities hold for the corresponding minima. Therefore, by induction on \( m \), we get
\[
q^m \min_{\partial \Omega} u \leq \min_{\partial \Omega}(J^m(u \mid_{\partial \Omega})) \leq \max_{\partial \Omega}(J^m(u \mid_{\partial \Omega})) \leq q^m \max_{\partial \Omega} u.
\]
Since \( 0 < q < 1 \), the series
\[
\sum_{m=0}^{\infty} J^m(u \mid_{\partial \Omega})
\]
converges uniformly on \( \partial \Omega \) and the proof is complete.

\[ \square \]

**B. Green’s potentials and the energy principle.** In this subsection, we recall the facts concerning Green’s potentials which are used in Nakai’s construction of the Evans-Selberg potential. We include sketches of some of the proofs. For more details the reader may refer to [An] and [M]. Throughout this subsection, we will assume that \((M, g)\) is hyperbolic with Green’s function \( G \).

A nonconstant nonnegative superharmonic function \( \varphi \) on \( M \) is called a potential if, for every harmonic function \( u \) with \( 0 \leq u \leq \varphi \) on \( M \), we have \( u \equiv 0 \). For example, \( G_x \) is a potential for each point \( x \in M \).

Given a positive regular Borel measure \( \mu \) supported in a compact subset \( K \) of \( M \), the function \( G_\mu : M \to [0, +\infty) \) given by
\[
G_\mu(x) = \int_M G(x, y) \, d\mu(y) \quad \forall x \in M
\]
is a potential called the Green’s potential of \( \mu \). Moreover, \( G_\mu \) is harmonic on an open subset of \( U \) if and only if \( \mu(U) = 0 \).

We will need two well-known facts concerning Green’s potentials. The first is the following:
Lemma 6.4. Suppose \( \varphi \) is a potential on \( M \) which is harmonic on the complement of some compact subset \( K \) of Lebesgue measure zero in \( M \). Then there exists a positive regular Borel measure \( \mu \) on \( M \) (supported in \( K \)) such that \( \varphi = G_\mu \).

Remark. In fact, every potential is a Green’s potential, but we will only need to consider this special case. For the general case, see [An]. The general fact is the key element in the proof that the distributional Laplacian of a subharmonic function is a positive measure. Moreover, one can prove that the measure \( \mu \) is unique.

Proof of Lemma 6.4. We first show that we may assume without loss of generality that \( M \) is a \( C^\infty \) relatively compact domain in an oriented Riemannian manifold \( (M',g') \), \( g = g' \mid_M \), and the restriction of \( \varphi \) to \( M \setminus K \) vanishes smoothly at \( \partial M \). Let \( \{M_k\} \) be an exhaustion of \( M \) by \( C^\infty \) relatively compact domains containing \( K \), and, for each \( k \), let \( G_k \) be the Green’s function on \( M_k \). Given an upper semicontinuous function \( \alpha \) and a \( C^\infty \) domain \( \Omega \) in \( M \) with Green’s function \( \tilde{G} \), we denote by \( \rho_\Omega(\alpha) \) the harmonic function given by

\[
\rho_\Omega(\alpha)(x) = \int_{\partial\Omega} \frac{\partial \tilde{G}_x(y)}{\partial \nu(y)} \alpha(y) d\sigma(y).
\]

Since \( \varphi \) is harmonic and therefore of class \( C^\infty \) on \( M \setminus K \), for each \( k \), the function

\[
\psi_k \equiv \varphi - \rho_{M_k}(\varphi)
\]

is a potential which is \( C^\infty \) on \( M \setminus K \), harmonic on \( M_k \setminus K \), and identically equal to 0 on \( \partial M_k \). Moreover, since

\[
\rho_{M_{k+1}}(\varphi) \leq \varphi = \rho_{M_k}(\varphi)
\]
on \( \partial M_k \) and the functions \( \rho_{M_{k+1}}(\varphi) \) and \( \rho_{M_k}(\varphi) \) are harmonic, we have

\[
0 \leq \rho_{M_{k+1}}(\varphi) \leq \rho_{M_k}(\varphi) \leq \varphi \quad \text{and} \quad \psi_{k+1} \geq \psi_k \geq 0
\]
on \( M_k \). By Harnack’s principle, the sequence of functions \( \{\rho_{M_k}(\varphi)\} \) converges uniformly on compact subsets of \( M \) to some harmonic function \( h \) satisfying \( 0 \leq h \leq \varphi \). Since \( \varphi \) is a potential, \( h \) must vanish identically and therefore \( \varphi - \psi_k \downarrow 0 \) uniformly on compact sets as \( k \to \infty \).

Now suppose that, for each \( k \), \( \psi_k = (G_k)_{\mu_k} \) for some positive regular Borel measure \( \mu_k \). Then \( \mu_k \) is supported in \( K \) since \( \psi_k \) is harmonic on \( M \setminus K \). Fixing any point \( x_0 \in M \setminus K \),
we get
\[ \mu_k(M) = \int_K d\mu_k(y) \leq \left( \min_K (G_k)_{x_0} \right)^{-1} \int_K G_k(x_0, y) d\mu_k(y) = \left( \min_K (G_k)_{x_0} \right)^{-1} \psi_k(x_0) \leq \left( \min_K (G_k)_{x_0} \right)^{-1} \varphi(x_0) < +\infty. \]

Hence the sequence \( \{\mu_k(K)\} \) is bounded. Thus, by passing to a subsequence, we may assume that the sequence of measures \( \{\mu_k\} \) converges \textit{weakly} to some positive regular Borel measure \( \mu \) supported in \( K \); that is,
\[ \int_M \alpha d\mu_k \to \int_M \alpha d\mu \]
for every continuous function \( \alpha \) on \( M \). On the other hand, for every point \( x \in M \setminus K \), \( G_x \) is continuous on \( K \) and \( G - G_k \to 0 \) uniformly on compact subsets of \( M \times M \). Therefore
\[ G_\mu(x) = \int_K G(x, y) d\mu(y) = \lim_{k \to \infty} \int_M G(x, y) d\mu_k(y) = \lim_{k \to \infty} \left[ \int_K (G(x, y) - G_k(x, y)) d\mu_k(y) + \int_K G_k(x, y) d\mu_k(y) \right] = \lim_{k \to \infty} (G_k)_\mu(x) = \lim_{k \to \infty} \psi_k(x) = \varphi(x). \]

Therefore \( G_\mu = \varphi \) on \( M \setminus K \). But \( G_\mu \) and \( \varphi \) are superharmonic and the set \( K \) has Lebesgue measure zero. Therefore \( G_\mu \equiv \varphi \) on \( M \). Thus we may assume that \( M \) is a \( C^\infty \) relatively compact domain in an oriented Riemannian manifold \( (M', g') \), \( g = g' \mid M \), \( \varphi \) is of class \( C^\infty \) on \( \overline{M} \setminus K \), and \( \varphi \equiv 0 \) on \( \partial M \).

Next, we approximate \( \varphi \) by Lipschitz continuous potentials. Let \( \{\Omega_m\} \) be a sequence of \( C^\infty \) relatively compact open sets in \( M \) such that
\[ \Omega_m \ni \Omega_{m+1} \quad \forall m = 1, 2, 3, \ldots \quad \text{and} \quad \bigcap_{m=1}^\infty \Omega_m = K. \]

For each \( m \), let \( \varphi_m \) be the Lipschitz continuous potential defined by
\[ \varphi_m(x) = (P_{\Omega_m}(\varphi))(x) = \begin{cases} \varphi(x) & \text{if } x \in M \setminus \Omega_m \\ (\rho_{\Omega_m}(\varphi))(x) & \text{if } x \in \Omega_m. \end{cases} \]

Then \( \varphi_m \) is harmonic on \( M \setminus \partial \Omega_m \), \( \varphi_m \) is smooth up to the boundary on \( \overline{M} \setminus \Omega_m \) and on \( \Omega_m \), and \( \varphi_m \leq \varphi_{m+1} \leq \varphi \) on \( M \). In particular, the sequence \( \{\varphi_m\} \) converges to a superharmonic function on \( M \). Since this function is equal to \( \varphi \) on \( M \setminus K \) and \( K \) has
Lebesgue measure zero, we get $\varphi_m \not\approx \varphi$ on $M$. We will prove that each function $\varphi_m$ is a Green’s potential and then pass to the limit to obtain a Green’s potential equal to $\varphi$.

For each positive integer $m$, let $u_m = \rho_{\Omega_m}(\varphi) = \varphi_m \mid_{\overline{\Omega}}$. Then

$$\frac{\partial}{\partial \nu}(u_m - \varphi) \geq 0 \quad \text{on } \partial \Omega_m$$

because $u_m - \varphi \leq 0$ on $\overline{\Omega}$ and $u_m - \varphi \equiv 0$ on $\partial \Omega_m$. Let $\tau_{n-1}$ denote the volume of the unit sphere in $\mathbb{R}^n$. It follows that the distributional Laplacian of the function $-\varphi_m/((n-2)\tau_{n-1})$ determines a positive regular Borel measure

$$d\mu_m \equiv \frac{1}{(n-2)\tau_{n-1}} \left[ \frac{\partial u_m}{\partial \nu} - \frac{\partial \varphi}{\partial \nu} \right] d\sigma_m,$$

where $d\sigma_m$ is the volume element on $\partial \Omega_m$. For if $\alpha$ is a $C^\infty$ function on $\overline{M}$ which vanishes on $\partial M$, then, since $u_m \equiv \varphi$ on $\partial \Omega_m$,

$$\int_M (\Delta \alpha) \varphi_m dV = \int_{M \backslash \overline{\Omega}_m} (\Delta \alpha) \varphi dV + \int_{\Omega_m} (\Delta \alpha) u_m dV$$

$$= -\int_{\partial \Omega_m} \frac{\partial \alpha}{\partial \nu} \varphi d\sigma_m + \int_{\partial \Omega_m} \alpha \frac{\partial \varphi}{\partial \nu} d\sigma_m$$

$$+ \int_{\partial \Omega_m} \frac{\partial \alpha}{\partial \nu} u_m d\sigma_m - \int_{\partial \Omega_m} \alpha \frac{\partial u_m}{\partial \nu} d\sigma_m$$

$$= -\int_{\partial \Omega_m} \alpha \left[ \frac{\partial u_m}{\partial \nu} - \frac{\partial \varphi}{\partial \nu} \right] d\sigma_m$$

Moreover, $\varphi_m = G_{\mu_m}$. For if $x \in M \backslash \overline{\Omega}_m$ and $G^*$ is the Green’s function on $M \backslash \overline{\Omega}_m$, then, since $G_x$ is harmonic on $\Omega_m$, $G_x - G^*_x$ and $\varphi$ are harmonic on $M \backslash \overline{\Omega}_m$, $G_x$, $G^*_x$, and $\varphi$ are
equal to zero on \( \partial M \), and \( G^*_x = 0 \) and \( \varphi = u_m \) on \( \partial \Omega_m \), we have

\[
(n - 2) \tau_{n-1} G_{\mu_m}(x) = (n - 2) \tau_{n-1} \int_M G(x, y) \, d\mu_m(y)
\]

\[
= \int_{\partial \Omega_m} G(x, y) \left[ \frac{\partial u_m(y)}{\partial \nu} - \frac{\partial \varphi(y)}{\partial \nu} \right] d\sigma_m(y)
\]

\[
= \int_{\partial \Omega_m} \frac{\partial G_x}{\partial \nu} u_m \, d\sigma_m - \int_{\partial \Omega_m} \frac{\partial \varphi}{\partial \nu} \, d\sigma_m
\]

\[
= \int_{\partial \Omega_m} \frac{\partial (G_x - G^*_x)}{\partial \nu} \varphi \, d\sigma_m - \int_{\partial \Omega_m} (G_x - G^*_x) \frac{\partial \varphi}{\partial \nu} \, d\sigma_m
\]

\[
+ \int_{\partial \Omega_m} \frac{\partial G^*_x}{\partial \nu} \varphi \, d\sigma_m
\]

\[
= \int_{\partial M} \frac{\partial (G_x - G^*_x)}{\partial \nu} \varphi \, d\sigma_m - \int_{\partial M} (G_x - G^*_x) \frac{\partial \varphi}{\partial \nu} \, d\sigma_m
\]

\[
- \int_{\partial (M \setminus \Omega_m)} \frac{\partial G^*_x}{\partial \nu} \varphi \, d\sigma_m
\]

\[
= (n - 2) \tau_{n-1} \varphi(x) = (n - 2) \tau_{n-1} \varphi_m(x).
\]

Therefore \( G_{\mu_m} \equiv \varphi \equiv \varphi_m \) on \( \overline{M \setminus \Omega_m} \). Similarly (by working with the Green’s function of \( \Omega_m \)) one can show that \( G_{\mu_m} \equiv u_m \equiv \varphi_m \) on \( \Omega_m \). Since \( G_{\mu_m} \) and \( \varphi_m \) are superharmonic and \( \partial \Omega_m \) has Lebesgue measure zero, \( G_{\mu_m} \) and \( \varphi_m \) are equal on \( \overline{M} \). In particular, \( G_{\mu_m} = \varphi_m \searrow \varphi \) as \( m \to \infty \).

Since \( u_m \) is harmonic on \( \Omega_m \) and \( \varphi \) is harmonic on \( M \setminus K \equiv \Omega_1 \setminus \Omega_m \), we have

\[
\mu_m(M) = \int_{\partial \Omega_m} d\mu_m = \frac{1}{(n - 2) \tau_{n-1}} \int_{\partial \Omega_m} \left[ \frac{\partial u_m}{\partial \nu} - \frac{\partial \varphi}{\partial \nu} \right] \, d\sigma_m
\]

\[
= \frac{1}{(n - 2) \tau_{n-1}} \int_{\partial \Omega_m} \frac{\partial \varphi}{\partial \nu} \, d\sigma_m
\]

\[
= \frac{1}{(n - 2) \tau_{n-1}} \int_{\partial \Omega_m} \frac{\partial \varphi}{\partial \nu} \, d\sigma_m.
\]

Thus \( \{ \mu_m(M) \} \) is a bounded (in fact, constant) sequence and therefore, by replacing \( \{ \mu_m \} \) by a subsequence (if necessary), we may assume that \( \{ \mu_m \} \) converges weakly to a positive regular Borel measure \( \mu \) on \( M \) supported in \( K = \bigcap_m \Omega_m \). If \( x \in M \setminus K \), then, for \( m_0 \) a
sufficiently large positive integer, the function $G_x$ is finite and continuous on $\overline{\Omega}_{m_0}$ and, for $m > m_0$,

$$\varphi_m(x) = G_{\mu_m}(x) = \int_M G(x, y) d\mu_m(y) = \int_{\Omega_{m_0}} G(x, y) d\mu_m(y).$$

Passing to the limit as $m \to \infty$, we get

$$\varphi(x) = \int_{\Omega_{m_0}} G(x, y) d\mu(y).$$

Thus $\varphi \equiv G_\mu$ on $M \setminus K$. Since $K$ has Lebesgue measure zero and these functions are superharmonic, $\varphi = G_\mu$ on $M$. Thus the lemma is proved. $\square$

The second fact which is needed in Nakai’s construction of the Evans-Selberg potential ([Na1] and [Na2]) is the energy principle, which is an analogue of the Schwarz inequality. Given two compactly supported positive regular Borel measures $\lambda$ and $\mu$ on $M$, the nonnegative numbers

$$\langle \mu, \lambda \rangle \equiv \int_M \int_M G(x, y) d\mu(x) d\lambda(y) = \int_M G_\mu d\lambda \quad \text{and} \quad \|\mu\|^2 \equiv \langle \mu, \mu \rangle$$

are called the mutual energy of $\mu$ and $\lambda$ and the energy of $\mu$, respectively.

**Lemma 6.5 (Energy principle).** For all compactly supported positive regular Borel measures $\mu$ and $\lambda$, we have

$$\langle \mu, \lambda \rangle \leq \|\mu\| \cdot \|\lambda\|.$$  

**Proof.** Recall that $M$ admits a minimal heat kernel; that is, a positive $C^\infty$ function

$$P : M \times M \times (0, +\infty) \to (0, +\infty)$$

such that for all $x, y \in M$,

$$P(x, y, \cdot) = P(y, x, \cdot) \quad \text{and} \quad G(x, y) = \int_0^\infty P(x, y, t) dt.$$

Moreover, for all $t, s > 0$,

$$P(x, y, t + s) = \int_M P(x, z, t) P(z, y, s) dV(z).$$
Therefore
\[
\langle \mu, \lambda \rangle = \int_{M} \int_{M} G(x, y) \, d\mu(x) \, d\lambda(y)
\]
\[
= \int_{0}^{\infty} \int_{M} \int_{M} P(x, y, t) \, d\mu(x) \, d\lambda(y) \, dt
\]
\[
= \int_{0}^{\infty} \int_{M} \left[ \left( \int_{M} P(x, z, t/2) \, d\mu(x) \right) \left( \int_{M} P(y, z, t/2) \, d\lambda(y) \right) \right] \, dV(z) \, dt.
\]
In particular,
\[
\|\mu\|^2 = \int_{0}^{\infty} \int_{M} \left( \int_{M} P(x, z, t/2) \, d\mu(x) \right)^2 \, dV(z) \, dt
\]
and similarly for \( \|\lambda\|^2 \). Therefore, by the Schwarz inequality,
\[
\langle \mu, \lambda \rangle \leq \left[ \int_{0}^{\infty} \int_{M} \left( \int_{M} P(x, z, t/2) \, d\mu(x) \right)^2 \, dV(z) \, dt \right]^{1/2}
\cdot \left[ \int_{0}^{\infty} \int_{M} \left( \int_{M} P(y, z, t/2) \, d\lambda(y) \right)^2 \, dV(z) \, dt \right]^{1/2}
\]
\[
= \|\mu\| \cdot \|\lambda\|.
\]

Remark. For a different proof see [M].

C. Evans-Selberg potential. In this subsection, we recall Nakai’s construction of a harmonic exhaustion function on the closure of an end of a parabolic Riemannian manifold [Na1] and [Na2] (see also [SaNo]). We begin with two useful facts:

Lemma 6.6. Suppose \( M \) is an oriented Riemannian manifold, \( N \) is a topological space, and \( H(x, y) \) is a positive function on \( M \times N \) which is harmonic in \( x \) and continuous in \( y \). Then \( H \) is continuous on \( M \times N \).

Proof. Let \((x_0, y_0)\) be a point in \(M \times N\). By the Harnack inequality, there exists a continuous function \( \delta : M \times M \to [0, +\infty) \) such that \( \delta \) vanishes on the diagonal and
\[
(1 + \delta(x, x_0))^{-1} H(x_0, y) \leq H(x, y) \leq (1 + \delta(x, x_0)) H(x_0, y)
\]
for all points \( x \in M \) sufficiently close to \( x_0 \) and all points \( y \in N \). Hence
\[
|H(x, y) - H(x_0, y)| \leq \delta(x, x_0) (1 + \delta(x, x_0)) H(x_0, y).
\]
Since the function $H(x_0, \cdot)$ is continuous on $N$, $H(x_0, y) \to H(x_0, y_0)$ as $y \to y_0$. It now follows easily that

$$|H(x, y) - H(x_0, y_0)| \leq |H(x, y) - H(x_0, y)| + |H(x_0, y) - H(x_0, y_0)| \to 0$$

as $(x, y) \to (x_0, y_0)$. Thus $H$ is continuous at $(x_0, y_0)$. □

**Lemma 6.7.** Suppose $M$ and $N$ are two oriented noncompact connected Riemannian manifolds and $H(x, y)$ is a positive function on $M \times N$ which is harmonic in each variable. Then $H$ is Harmonic with respect to the product metric on $M \times N$. In particular, $H$ is of class $C^\infty$.

**Proof.** By the previous lemma, $H$ is continuous. Moreover, given a smooth compactly supported function $\alpha$ on $M \times N$, we have

$$\int_{M \times N} H(x, y) \Delta \alpha(x, y) \, dV(x, y) = \int_M \left[ \int_N H(x, y) \Delta \alpha_x(y) \, dV(y) \right] \, dV(x)$$

$$+ \int_N \left[ \int_M H(x, y) \Delta \alpha_y(x) \, dV(x) \right] \, dV(y) = 0;$$

where, in each integral, $\Delta$ and $dV$ denote the appropriate Laplacian and volume element, and, for each $x \in M$ and $y \in N$, $\alpha_x \equiv \alpha(x, \cdot)$ and $\alpha_y \equiv \alpha(\cdot, y)$. Thus $H$ is harmonic on $M \times N$. □

**Stone-Čech compactification.** Let $X$ be a topological space. We will call a continuous map $f : X \to [-\infty, +\infty]$ a continuous function on $X$ and we will denote the space of continuous functions by $C(X)$. If $f(X) \subset (-\infty, +\infty)$, then we will call $f$ a finitely continuous function. We will denote the space of finitely continuous functions by $C^0(X)$.

If $X$ is locally compact Hausdorff, then the Stone-Čech compactification $\tilde{X}$ is the unique Hausdorff compactification of $X$ to which every continuous function on $X$ extends continuously. The set $\Gamma = \tilde{X} \setminus X$ is the Stone-Čech boundary of $X$.

**Green’s function on the Stone-Čech compactification.** For the rest of this subsection, we will assume that $(M, g)$ is parabolic. $\tilde{M}$ will denote the Stone-Čech compactification of $M$, $\Gamma$ will denote the Stone-Čech boundary of $M$, and $M_0$ will denote a fixed (nonempty) $C^\infty$ relatively compact domain in $M$ with connected boundary.

In particular, the manifold $M \setminus \overline{M}_0$ is connected and hyperbolic. For if $M_1$ and $\Omega$ are $C^\infty$ domains in $M$ with

$$M_0 \Subset M_1 \Subset M \quad \text{and} \quad \Omega \Subset M \setminus \overline{M}_1,$$
\( u \) is the harmonic measure of the ideal boundary of \( M \setminus \overline{M_0} \) with respect to the complement of \( \overline{\Omega} \), and \( v \) is the continuous function on \( \overline{M}_1 \setminus M_0 \) which is harmonic on \( M_1 \setminus \overline{M}_0 \), equal to 0 on \( \partial M_1 \), and equal to 1 on \( \partial M_0 \), then \( v \leq u \leq 1 \) on \( M_1 \setminus \overline{M}_0 \subset (M \setminus \overline{M}_0) \setminus \overline{\Omega} \). Hence \( u \not\equiv 0 \).

We will denote the Green’s function on \( M \setminus \overline{M}_0 \) by \( G(x, y) \). In the above, we may extend \( u \) continuously to \( \partial M_0 \) by the constant 1. Thus every sequence in \( M \setminus \overline{M}_0 \) approaching \( \partial M_0 \) is a regular sequence. Hence if we extend \( G \) to a function on \( (M \setminus M_0) \times (M \setminus M_0) \) by setting

\[
G(x, y) = 0 \quad \text{if} \quad x \in \partial M_0 \text{ or } y \in \partial M_0,
\]

then, for each point \( x_0 \in M \setminus M_0 \), the function \( G_{x_0} \) is continuous on \( M \setminus M_0 \). In fact, by Lemma 6.4, \( G \) is continuous on the set

\[
[(M \setminus M_0) \times (M \setminus M_0)] \setminus (\partial M_0 \times \partial M_0).
\]

Nakai’s main observation is that the Green’s function \( G(x, y) \) may be continuously extended, in each variable, to the Stone-Čech boundary. More precisely, we have the following:

**Proposition 6.8 (Nakai).** The Green’s function \( G \) on \( M \setminus \overline{M}_0 \) extends to a function

\[
\tilde{G} : (\tilde{M} \setminus M_0) \times (\tilde{M} \setminus M_0) \to [0, +\infty]
\]

given by the double limit

\[
\tilde{G}(x_0, y_0) \equiv \lim_{x \to x_0} \left( \lim_{y \to y_0} G(x, y) \right) \quad \forall x_0, y_0 \in \tilde{M} \setminus M_0,
\]

where, in the above limits, \( x, y \in M \setminus \overline{M}_0 \). Moreover, this function has the following properties:

(i) \( \tilde{G}(x, y) = \tilde{G}(y, x) \) \( \forall (x, y) \in (M \setminus M_0) \times (\tilde{M} \setminus M_0) \);

(ii) \( \tilde{G} \) is continuous on \( [(M \setminus M_0) \times (\tilde{M} \setminus M_0)] \setminus (\partial M_0 \times \partial M_0) \) and finitely continuous on \( (M \setminus M_0) \times \Gamma \);

(iii) For each point \( y \in \tilde{M} \setminus M_0 \), the function \( \tilde{G}_y \equiv \tilde{G}(\cdot, y) \) is continuous on \( \tilde{M} \setminus M_0 \), harmonic on \( (M \setminus \overline{M}_0) \setminus \{y\} \), and equal to 0 on \( \partial M_0 \);

(iv) For each point \( y \in \tilde{M} \setminus \overline{M}_0 \) (\( y \in \partial M_0 \)), \( \tilde{G}_y > 0 \) (\( \equiv 0 \)) on \( \tilde{M} \setminus \overline{M}_0 \); and
(v) For each point \( y \in \tilde{M} \setminus \overline{M_0} \),
\[
\int_{\partial M_0} \frac{\partial \tilde{G}_y}{\partial \nu}(x) \, d\sigma(x) = (n - 2)\tau_{n-1}.
\]

**Remark.** It is not clear that \( \tilde{G} \) is symmetric on \((\tilde{M} \setminus M_0) \times (\tilde{M} \setminus M_0)\) since, for \( x \in \Gamma \), it is not clear that the function \( y \mapsto \tilde{G}(x, y) \) is continuous on \( \tilde{M} \setminus M_0 \).

**Proof of Proposition 6.8.** For each point \( x \in \{ \} \), the function
\[
G_x = G(x, \cdot) = G(\cdot, x) : M \setminus M_0 \to [0, +\infty]
\]
is continuous and hence extends to a continuous function on \( \tilde{M} \setminus M_0 \). Thus the function \( G^*(x, y) \) defined by
\[
G^*(x, y) = G^*_y(x) \equiv \lim_{z \to y} G(x, z) \quad \forall (x, y) \in (M \setminus M_0) \times (\tilde{M} \setminus M_0),
\]
with \( z \in M \setminus \overline{M_0} \) is an extension of \( G \) which is continuous in \( y \) for each fixed \( x \in M \setminus M_0 \). We show that, for each fixed \( y_0 \in \Gamma \), the function \( G^*_{y_0} \) is continuous on \( M \setminus M_0 \), positive and harmonic on \( M \setminus \overline{M_0} \), and identically equal to 0 on \( \partial M_0 \).

Suppose \( \Omega_1 \) and \( \Omega_2 \) are domains with
\[
\Omega_1 \Subset \Omega_2 \Subset M \setminus \overline{M_0}.
\]
Then, by the Harnack inequality (Lemma 1.2), there exists a continuous function \( \delta : \Omega_2 \times \Omega_2 \to [0, +\infty) \) such that \( \delta \) vanishes on the diagonal and, for every positive harmonic function \( u \) on \( \Omega_2 \),
\[
|u(x) - u(x_0)| \leq \delta(x, x_0) \max(u(x), u(x_0)) \quad \forall x, x_0 \in \Omega_1.
\]
Next, observe that
\[
0 \leq G(x, y) \leq a \equiv \max_{\overline{\Omega}_1 \times \partial \Omega_2} G < +\infty \quad \forall (x, y) \in \overline{\Omega}_1 \times ((M \setminus M_0) \setminus \Omega_2).
\]
For if \( \Omega \) is a \( C^\infty \) domain with
\[
\Omega_2 \Subset \Omega \Subset M \setminus \overline{M_0},
\]
and \( G' \) is the Green’s function of \( \Omega \), then, for each point \( x \in \overline{\Omega}_1 \), \( G'_x \upharpoonright_{(\Omega \setminus \overline{\Omega}_1)} \) is a positive harmonic function which vanishes continuously at \( \partial \Omega \). Hence
\[
G'_x \upharpoonright_{(\Omega \setminus \overline{\Omega}_2)} \leq \max_{\partial \Omega_2} G'_x \leq \max_{\overline{\Omega}_1 \times \partial \Omega_2} G'.
\]
Since $G$ is the pointwise limit of an increasing limit of such Green’s functions, the inequality follows. It also follows that the collection of positive finitely continuous functions $\mathcal{G} \equiv \{ G_y \mid y \in (M \setminus M_0) \setminus \Omega_2 \}$ is precompact in $C^0(\Omega_1)$. For if $y \in (M \setminus M_0) \setminus \Omega_2$, then $G_y$ is bounded by $a$ and harmonic on $\Omega_2$. Hence

$$|G_y(x) - G_y(x_0)| \leq a \delta(x, x_0) \quad \forall x, x_0 \in \Omega_1.$$  

Since $\delta(x, x_0) \to 0$ as $x \to x_0$, $G_y$ is bounded and equicontinuous, hence precompact, by Ascoli’s Theorem.

Now let \{ $y_\alpha$ \} be a net in $(M \setminus M_0) \setminus \Omega_2$ which converges to the given point $y_0 \in \Gamma$. Then, by passing to a subnet, we may assume that \{ $(G_{y_\alpha}) \mid \Omega_1$ \} converges uniformly on $\Omega_1$ to the function $G^*_{y_0} \mid \Omega_1$. In particular, $G^*_{y_0}$ is nonnegative and harmonic on $\Omega_1$.

Finally, for a fixed point $x_0 \in \Omega_1$, the function $G_{x_0}$ is continuous, positive, and superharmonic on $M \setminus M_0$. Hence, if $\Omega_0$ is a $C^\infty$ domain with $M_0 \subseteq \Omega_0 \subseteq M \setminus \Omega_1$, then, by the minimum principle for parabolic manifolds,

$$\inf_{M \setminus \Omega_0} G_{x_0} = \min_{\partial \Omega_0} G_{x_0} \geq \min_{\Omega_1 \times \partial \Omega_0} G(x, y) \equiv b > 0.$$  

Therefore

$$G^*_{y_0}(x_0) = \lim_\alpha G_{y_\alpha}(x_0) = \lim_\alpha G_{x_0}(y_\alpha) \geq b > 0.$$  

Therefore $G^*_{y_0} > 0$ on $\Omega_1$. Since the domain $\Omega_1 \subseteq M \setminus M_0$ is arbitrary, the function $G^*_{y_0}$ is positive and harmonic on $M \setminus M_0$.

Next we verify that $G^*_{y_0}$ is continuous at $\partial M_0$. It is clear from the definition that $G^*_{y_0}$ vanishes on $\partial M_0$. Let $\Omega$ be a $C^\infty$ domain with $M_0 \subseteq \Omega \subseteq M$.

Then $\partial \Omega$ is a compact subset of $M \setminus M_0$. By the above discussion, every net $\{ y_\alpha \}_{\alpha \in A}$ in $M \setminus M_0$ converging to the given point $y_0 \in \Gamma$ admits a subnet $\{ y_\beta \}_{\beta \in B}$ in $M \setminus \Omega$ such that $\{ G_{y_\beta} \}$ converges uniformly on $\partial \Omega$ to $G^*_{y_0}$. For all $\beta, \gamma \in B$ and $x \in \Omega \setminus M_0$, we have

$$|G_{y_\beta}(x) - G_{y_\gamma}(x)| \leq \max_{\Omega \setminus M_0} |G_{y_\beta} - G_{y_\gamma}| \leq \max_{\partial \Omega} |G_{y_\beta} - G_{y_\gamma}|.$$
because $G_{y_0}$ and $G_y$ are harmonic on $\Omega \setminus \overline{M}_0$ and equal to 0 on $\partial M_0$. Since $\{G_{y_0}\}$ converges to $G_{y_0}^*$ pointwise on $\overline{\Omega} \setminus M_0$ and uniformly on $\partial \Omega$, we get
\[
|G_{y_0}(x) - G_{y_0}^*(x)| \leq \max_{\partial \Omega} |G_{y_0} - G_{y_0}^*|.
\]
Hence
\[
\sup_{\Omega \setminus M_0} |G_{y_0} - G_{y_0}^*| \leq \max_{\partial \Omega} |G_{y_0} - G_{y_0}^*| \to 0.
\]
Therefore the continuous functions $\{G_{y_0}\}$ converge uniformly on $\overline{\Omega} \setminus M_0$ to $G_{y_0}^*$ and hence $G_{y_0}^*$ vanishes continuously at $\partial M_0$.

Thus, for each point $y_0 \in \bar{M} \setminus M_0$, the function $G_{y_0}^*$ is continuous on $M \setminus M_0$. Hence $G_{y_0}^*$ extends to a continuous function $\tilde{G}_{y_0}$ on $\bar{M} \setminus M_0$. We may therefore define $\tilde{G}(x_0, y_0)$ for each pair of points $x_0, y_0 \in \bar{M} \setminus M_0$ by
\[
\tilde{G}(x_0, y_0) \equiv \tilde{G}_{y_0}(x_0) \equiv \lim_{y \to y_0} \left( \lim_{x \to x_0} G(x, y) \right) = \lim_{x \to x_0} \left( G_{y_0}^*(x) \right);
\]
where, in the above limits, $x, y \in M \setminus \overline{M}_0$.

We now verify that $\tilde{G}$ has the properties (i)–(v). Given points $x \in M \setminus M_0$ and $y \in \bar{M} \setminus M_0$, we have
\[
\tilde{G}(y, x) = \lim_{z \to y} \left( \lim_{w \to x} G(z, w) \right) = \lim_{z \to y} G(z, x) = \lim_{z \to y} G(x, z)
\]
\[
= G_y^*(x) = \tilde{G}_y \mid_{M \setminus M_0} (x) = \tilde{G}(x, y).
\]
with $w, z \in M \setminus \overline{M}_0$. Thus (i) is proved.

The properties (iii) and (iv) follow by construction.

By the above discussion, given a cpt set $K$ in $M \setminus M_0$ and a point $y_0 \in \Gamma$, every net $\{y_\alpha\}$ in $M \setminus \overline{M}_0$ converging to $y_0$ admits a subnet $\{y_\beta\}$ such that $\{G_{y_\beta}\}$ converges to $G_{y_0}^*$ uniformly on $K$. It follows that $G_y \to G_{y_0}^*$ uniformly on $K$ as $y \in M \setminus \overline{M}_0$ approaches $y_0$. Moreover, we have shown that the functions $G_y$ are uniformly bounded on $K$ for $y \in M \setminus \overline{M}_0$ near $y_0$. Hence $\tilde{G}_{y_0} = G_{y_0}^*$ is bounded on $K$.

Since $\tilde{G} = G$ on $((M \setminus M_0) \times (M \setminus M_0)) \setminus (\partial M_0 \times \partial M_0)$, in order to prove (ii), it remains to show that $\tilde{G}$ is finitely continuous at each point $(x_0, y_0) \in (M \setminus M_0) \times \Gamma$. By the above remarks, $\tilde{G}(x_0, y_0) < +\infty$ and for points $x \in M \setminus M_0$ near $x_0$ and $y \in M \setminus \overline{M}_0$ near $y_0$, we
have
\[
|\tilde{G}(x, y) - \tilde{G}(x_0, y_0)| \leq |G(x, y) - \tilde{G}(x, y_0)| + |\tilde{G}(x, y_0) - \tilde{G}(x_0, y_0)|
\]
\[
= |G_y(x) - G^*_y(x)| + |G^*_y(x) - G^*_y(x_0)| \to 0
\]
as \((x, y) \to (x_0, y_0)\). Thus we need only consider points in \((M \setminus M_0) \times \Gamma\) approaching \((x_0, y_0)\). Moreover, the term \(\tilde{G}(x, y_0) - \tilde{G}(x_0, y_0)\) is independent of \(y\) and approaches 0 as \(x\) approaches \(x_0\). Thus it remains to show that
\[
|\tilde{G}(x, y) - \tilde{G}(x_0, y_0)| \to 0 \quad \text{as} \quad (x, y) \to (x_0, y_0) \quad \text{with} \quad (x, y) \in (M \setminus M_0) \times \Gamma.
\]
If \(z\) is a point in \(M \setminus M_0\), then
\[
|\tilde{G}(x, y) - \tilde{G}(x, y_0)| \leq |\tilde{G}(x, y) - \tilde{G}(x, z)| + |\tilde{G}(x, z) - \tilde{G}(x, y_0)|.
\]
Since \(\tilde{G}_z = G_z\) converges uniformly to \(G^*_y\) on a relatively compact neighborhood of \(x_0\) as \(z \in M \setminus M_0\) approaches \(y_0\), given a positive real number \(\epsilon\) there exists a neighborhood \(U\) of \(y_0\) in \(M\) such that
\[
|\tilde{G}(x, z) - \tilde{G}(x, y_0)| < \frac{\epsilon}{2} \quad \forall \ z \in (M \setminus M_0) \cap U.
\]
Hence, given a point \(y \in U \cap \Gamma\) and a point \(z \in (M \setminus M_0) \cap U\) so close to \(y\) that \(|\tilde{G}(x, y) - \tilde{G}(x, z)| < \epsilon/2\) for all points \(x \in M \setminus M_0\) near \(x_0\), we get
\[
|\tilde{G}(x, y) - \tilde{G}(x_0, y_0)| < \epsilon.
\]
Thus the claim is proved.

For the proof of (v), let \(y_0 \in M \setminus M_0\), let \(\{M_k\}\) be an exhaustion of \(M\) by \(C^\infty\) relatively compact domains containing \(M_0 \cup \{y_0\}\), let \(\Omega\) be a \(C^\infty\) relatively compact domain in \(M_1 \setminus M_0\) containing \(y_0\), and, for each positive integer \(k\), let \(G_k\) be the Green's function on \(M_k \setminus M_0\) and let \(v_k\) be the continuous function on \(M_k \setminus (M_0 \cup \Omega)\) which is harmonic on \(M_k \setminus (M_0 \cup \Omega)\), equal to 1 on \(\partial M_0 \cup \partial \Omega\), and equal to 0 on \(\partial M_k\). Then the function \(1 - v_k\) is nonnegative and not greater than the restriction of the harmonic measure of \(\partial M_k\) with respect to \(M_k \setminus M_0\). Therefore, since \(M\) is parabolic, \(v_k \to 1\) uniformly on compact sets in \(M \setminus (M_0 \cup \Omega)\). Consequently, \(\nabla v_k \to 0\) uniformly on compact subsets of \(M \setminus (M_0 \cup \Omega)\).

The function \(G_{y_0} - (G_k)_{y_0}\) is positive and continuous on \(M_k \setminus M_0\), harmonic on \(M_k \setminus M_0\), and equal to 0 on \(\partial M_0\). Moreover, \(G_{y_0} - (G_k)_{y_0} \to 0\) uniformly on compact subsets of \(M \setminus M_0\). For \(\{(G_k)_{y_0}\}\) dominates a sequence of Greens functions on relatively compact domains exhausting \(M \setminus M_0\) and hence \(G_{y_0} - (G_k)_{y_0} \to 0\) uniformly on compact subsets.
of $M \setminus \overline{M}_0$. By applying the maximum principle and the fact that $G_{y_0}$ and the functions \{(G_k)_{y_0}\} vanish on $\partial M_0$, we get uniform convergence near $\partial M_0$ as well. In particular, 

$$\nabla G_{y_0} - \nabla (G_k)_{y_0} \to 0$$

uniformly on compact subsets of $M \setminus \overline{M}_0$. Therefore

$$D_{M_k \setminus (\overline{M}_0 \cup \overline{\Omega})}((G_k)_{y_0}, v_k) = \int_{M_k \setminus (\overline{M}_0 \cup \overline{\Omega})} \langle \nabla (G_k)_{y_0}, \nabla v_k \rangle$$

$$= \int_{\partial (M_k \setminus (\overline{M}_0 \cup \overline{\Omega}))} \frac{\partial (G_k)_{y_0}}{\partial \nu} v_k$$

$$= \int_{\partial M_k} \frac{\partial (G_k)_{y_0}}{\partial \nu} v_k - \int_{\partial M_0} \frac{\partial (G_k)_{y_0}}{\partial \nu} v_k - \int_{\partial \Omega} \frac{\partial (G_k)_{y_0}}{\partial \nu} v_k$$

$$= - \int_{\partial M_0} \frac{\partial (G_k)_{y_0}}{\partial \nu} - \int_{\partial \Omega} \frac{\partial (G_k)_{y_0}}{\partial \nu}$$

$$= - \int_{\partial M'_0} \frac{\partial (G_k)_{y_0}}{\partial \nu} - \int_{\partial \Omega} \frac{\partial (G_k)_{y_0}}{\partial \nu}$$

where $M'_0$ is any $C^\infty$ relatively compact domain in $M_1 \setminus \overline{\Omega}$ containing $\overline{M}_0$. Hence

$$D_{M_k \setminus (\overline{M}_0 \cup \overline{\Omega})}((G_k)_{y_0}, v_k) \to - \int_{\partial M'_0} \frac{\partial G_{y_0}}{\partial \nu} - \int_{\partial \Omega} \frac{\partial G_{y_0}}{\partial \nu} = - \int_{\partial M_0} \frac{\partial G_{y_0}}{\partial \nu} - \int_{\partial \Omega} \frac{\partial G_{y_0}}{\partial \nu}$$

as $k \to \infty$. On the other hand, we have

$$D_{M_k \setminus (\overline{M}_0 \cup \overline{\Omega})}((G_k)_{y_0}, v_k) = \int_{\partial (M_k \setminus (\overline{M}_0 \cup \overline{\Omega}))} (G_k)_{y_0} \frac{\partial v_k}{\partial \nu}$$

$$= \int_{\partial M_k} (G_k)_{y_0} \frac{\partial v_k}{\partial \nu} - \int_{\partial M_0} (G_k)_{y_0} \frac{\partial v_k}{\partial \nu} - \int_{\partial \Omega} (G_k)_{y_0} \frac{\partial v_k}{\partial \nu}$$

$$= - \int_{\partial \Omega} (G_k)_{y_0} \frac{\partial v_k}{\partial \nu}$$

$$= - \int_{\partial \Omega'} (G_k)_{y_0} \frac{\partial v_k}{\partial \nu} + \int_{\partial \Omega'} \frac{\partial (G_k)_{y_0}}{\partial \nu} v_k - \int_{\partial \Omega} \frac{\partial (G_k)_{y_0}}{\partial \nu} v_k$$

$$\to \int_{\partial \Omega'} \frac{\partial G_{y_0}}{\partial \nu} - \int_{\partial \Omega} \frac{\partial G_{y_0}}{\partial \nu} = 0 \text{ as } k \to \infty;$$

where $\Omega'$ is any $C^\infty$ relatively compact domain in $M_1 \setminus \overline{M}_0$ containing $\overline{\Omega}$. Hence if $\hat{G}$ is the Green’s function on $\Omega$, then

$$\int_{\partial M_0} \frac{\partial G_{y_0}}{\partial \nu} = - \int_{\partial \Omega} \frac{\partial G_{y_0}}{\partial \nu} = - \int_{\partial \Omega} \frac{\partial \hat{G}_{y_0}}{\partial \nu} + \int_{\partial \Omega} \frac{\partial (G_{y_0} - \hat{G}_{y_0})}{\partial \nu}$$

$$= - \int_{\partial \Omega} \frac{\partial \hat{G}_{y_0}}{\partial \nu} \cdot 1 = (n - 2)\tau_{n-1},$$
since \( G_{y_0} - \tilde{G}_{y_0} \) is a \( C^\infty \) function on \( \overline{\Omega} \) which is harmonic on the interior.

Finally, suppose \( y_0 \in \Gamma \) and let \( M_1 \) be a \( C^\infty \) relatively compact domain in \( M \) containing \( \overline{M_0} \). Then, as \( y \to y_0 \) with \( y \in M \setminus \overline{M_1} \), \( G_y \to \tilde{G}_{y_0} \) uniformly on \( \partial M_1 \setminus M_0 \); and therefore

\[
(n - 2) \tau_{n-1} = \int_{\partial M_0} \frac{\partial G_y}{\partial \nu} = \int_{\partial M_1} \frac{\partial \tilde{G}_y}{\partial \nu} \to \int_{\partial M_1} \frac{\partial \tilde{G}_{y_0}}{\partial \nu}.
\]

The claim (v) follows. \( \square \)

Transfinite Diameter and Tchebycheff constant. Let \( K \) be a compact subset of \( \tilde{M} \setminus \overline{M_0} \). Given an integer \( m > 1 \) and points \( x_1, \ldots, x_m \in K \), let \( D_m(K, x_1, \ldots, x_m) = D_m(x_1, \ldots, x_m) \) and \( E_m(K, x_1, \ldots, x_m) \) be the following numbers:

\[
D_m(x_1, \ldots, x_m) \equiv \left( \frac{m}{2} \right)^{-1} \sum_{1 \leq i < j \leq m} \tilde{G}(x_i, x_j)
\]

and

\[
E_m(K, x_1, \ldots, x_m) \equiv \frac{1}{m} \left( \inf_{x \in K} \sum_{i=1}^{m} \tilde{G}(x, x_i) \right).
\]

We then define \( D_m(K) \) and \( E_m(K) \) by

\[
D_m(K) \equiv \inf_{x_1, \ldots, x_m \in K} D_m(x_1, \ldots, x_m)
\]

and

\[
E_m(K) \equiv \sup_{x_1, \ldots, x_m \in K} E_m(K, x_1, \ldots, x_m).
\]

The numbers

\[
D(K) \equiv \sup_{m>0} D_m(K) \quad \text{and} \quad E(K) \equiv \sup_{m>0} E_m(K)
\]

are called the transfinite diameter of \( K \) and the Tchebycheff constant of \( K \), respectively.

Lemma 6.9. For every compact subset \( K \) of \( \tilde{M} \setminus \overline{M_0} \),

(i) \( D_m(K) \not\nearrow D(K) \) and \( E_m(K) \to E(K) \) as \( m \to \infty \); and

(ii) \( 0 \leq D(K) \leq E(K) \leq +\infty \).
Proof. Given a positive integer \( m \) and points \( x_1, \ldots, x_{m+1} \in K \), we have, for each \( k = 1, \ldots, m + 1 \),
\[
\left( \frac{m+1}{2} \right) D_{m+1}(x_1, \ldots, x_{m+1}) = \sum_{1 \leq i < k} \tilde{G}(x_i, x_j) + \sum_{k < j \leq m+1} \tilde{G}(x_i, x_j) + \sum_{1 \leq i < j < k} \tilde{G}(x_i, x_j)
\]
\[
= \sum_{1 \leq i < k} \tilde{G}(x_i, x_j) + \sum_{k < j \leq m+1} \tilde{G}(x_i, x_j) + \left( \frac{m}{2} \right) D_m(x_1, \ldots, \hat{x}_k, \ldots, x_{m+1})
\]
\[
\geq \sum_{1 \leq i < k} \tilde{G}(x_i, x_j) + \sum_{k < j \leq m+1} \tilde{G}(x_i, x_j) + \left( \frac{m}{2} \right) D_m(K),
\]
where \((x_1, \ldots, \hat{x}_k, \ldots, x_{m+1})\) denotes the \( m \)-tuple obtained by removing the \( k \)th term from the \((m+1)\)-tuple \((x_1, \ldots, x_{m+1})\). Summing over \( k \), we get
\[
(m+1) \left( \frac{m+1}{2} \right) D_{m+1}(x_1, \ldots, x_{m+1}) \geq 2 \left( \frac{m+1}{2} \right) D_{m+1}(x_1, \ldots, x_{m+1}) \]
\[
+ (m+1) \left( \frac{m}{2} \right) D_m(K).
\]
It follows that \( D_{m+1}(x_1, \ldots, x_{m+1}) \geq D_m(K) \). Therefore
\[
D_m(K) \leq D_{m+1}(K) \rightarrow D(K) \quad \text{as} \quad m \rightarrow \infty.
\]
Given positive integers \( m \) and \( l \) and points \( x_1, \ldots, x_{m+l} \in K \), we have
\[
(m+l) E_{m+l}(K, x_1, \ldots, x_{m+l}) = \inf_{x \in K} \left( \sum_{i=1}^{m} \tilde{G}(x,x_i) + \sum_{i=m+1}^{m+l} \tilde{G}(x,x_i) \right)
\]
\[
\geq \inf_{x \in K} \left( \sum_{i=1}^{m} \tilde{G}(x,x_i) \right) + \inf_{x \in K} \left( \sum_{i=m+1}^{m+l} \tilde{G}(x,x_i) \right)
\]
\[
= mE_m(K, x_1, \ldots, x_m) + lE_l(K, x_{m+1}, \ldots, x_{m+l}).
\]
Therefore \((m+l) E_{m+l}(K) \geq mE_m(K) + lE_l(K)\). Let \( q \) and \( r \) be integers satisfying
\[
m = ql + r \quad \text{and} \quad 0 \leq r < l.
\]
By the above, we have \( qIE_q(K) \geq (q - 1)IE_{(q-1)}(K) + IE_l(K) \). Proceeding inductively, we get \( qIE_q(K) \geq qIE_l(K) \). Therefore

\[
mE_m(K) = (ql + r)E_{ql+r}(K) \geq qIE_q(K) + rE_r(K) \geq qIE_l(K).
\]

Hence

\[
E(K) \geq E_m(K) \geq \left( \frac{ql}{ql + r} \right) E_l(K).
\]

For fixed \( l \), we have \( q \to +\infty \) as \( m \to +\infty \) and \( 0 \leq r < l \). Thus

\[
E(K) = \sup_{m>0} E_m(K) \geq \limsup_{m \to \infty} E_m(K) \geq \liminf_{m \to \infty} E_m(K) \geq E_l(K)
\]

for every positive integer \( l \) and hence

\[
E(K) = \lim_{m \to \infty} E_m(K).
\]

Thus (i) is proved.

Given a positive integer \( m \), there exist points \( x_1, \ldots, x_m \in K \) such that

\[
(1) \quad iE_i(K, x_{m-i+1}, \ldots, x_m) = \sum_{j=m-i+1}^{m} \tilde{G}(x_{m-i}, x_j)
\]

for \( i = 1, \ldots, m - 1 \). To see this, let \( x_m \in K \) be arbitrary, let \( 1 \leq k < m \), and suppose \( x_{m-k+1}, \ldots, x_m \in K \) have been chosen so that the equality (1) holds for \( i = 1, \ldots, k - 1 \) (the case \( k = 1 \) is vacuously true). By Proposition 6.8 (iii), the function

\[
x \mapsto \sum_{j=m-k+1}^{m} \tilde{G}(x, x_j)
\]

is continuous and therefore assumes its minimum value on \( K \) at some point \( x_{m-k} = x_{m-(k+1)+1} \). The equality (1) then holds for \( i = k \). Thus, proceeding inductively, we obtain points \( x_1, \ldots, x_m \in K \) with the required properties.

From (1) it follows that

\[
iE_i(K) \geq \sum_{j=m-i+1}^{m} \tilde{G}(x_{m-i}, x_j)
\]

for \( i = 1, \ldots, m - 1 \). Summing over \( i \), we get

\[
\sum_{i=1}^{m-1} iE_i(K) \geq \sum_{i=1}^{m-1} \sum_{j=m-i+1}^{m} \tilde{G}(x_{m-i}, x_j) = \sum_{1 \leq i < j \leq m} \tilde{G}(x_i, x_j)
\]

\[
= \binom{m}{2} D_m(x_1, \ldots, x_m) \geq \binom{m}{2} D_m(K).
\]
Clearly, we may assume that $E(K) < +\infty$. Therefore, for each integer $k$ with $0 < k < m - 1$,

$$D_m(K) \leq \left( \frac{m}{2} \right)^{-1} \sum_{i=1}^{m-1} iE_i(K) = E(K) + \left( \frac{m}{2} \right)^{-1} \sum_{i=1}^{m-1} i(E_i(K) - E(K))$$

$$\leq E(K) + \left( \frac{m}{2} \right)^{-1} \sum_{i=1}^{k} i|E_i(K) - E(K)| + \left( \frac{m}{2} \right)^{-1} \sum_{i=k+1}^{m-1} i|E_i(K) - E(K)|.$$

Hence

$$D_m(K) \leq E(K) + \left( \frac{m}{2} \right)^{k+1} \left( \frac{k+1}{2} \right) \max_{1 \leq i \leq k} |E_i(K) - E(K)| + \sup_{i > k} |E_i(K) - E(K)|.$$

By choosing $k$ sufficiently large, we can make the third term on the right-hand side of the above equation arbitrarily small. Moreover, for $k$ fixed, the second term approaches 0 as $m \to \infty$. The claim (ii) now follows.

The main step in Nakai’s construction of the Evans-Selberg potential is a proof that the transfinite diameter of the Čech boundary is infinite. For this, the first step is the following lemma:

**Lemma 6.10.** If $\Omega$ is a $C^\infty$ domain with $M_0 \Subset \Omega \Subset M$, then

$$D(\tilde{M} \setminus \Omega) = D(\partial\Omega).$$

**Proof.** It suffices to show that, for every integer $m > 2$,

$$D_m(\tilde{M} \setminus \Omega) = D_m(\partial\Omega);$$

that is, given points $x_1, \ldots, x_m \in \tilde{M} \setminus \Omega$ we have

$$D_m(x_1, \ldots, x_m) \geq \inf_{y_1, \ldots, y_m \in \partial\Omega} D_m(y_1, \ldots, y_m).$$

We will prove by induction on $k = 0, 1, 2, \ldots, m$ that there exist points $y_1, \ldots, y_k$ in $\partial\Omega$ such that

$$D_m(x_1, \ldots, x_m) \geq D_m(y_1, \ldots, y_k, x_{k+1}, \ldots, x_m).$$

The case $k = 0$ is clear. Suppose $0 \leq k \leq m - 1$ and there exist points $y_1, \ldots, y_k$ in $\partial\Omega$ satisfying the above inequality. By Proposition 6.8 (parts (i) and (iii)), the function $\psi$
defined by
\[ \psi(y) = \binom{m}{2} D_m(y_1, \ldots, y_k, y, x_{k+2}, \ldots, x_m) \]
\[ = \binom{m-1}{2} D_{m-1}(y_1, \ldots, y_k, x_{k+2}, \ldots, x_m) + \sum_{i=1}^{k} \tilde{G}(y, y) + \sum_{i=k+2}^{m} \tilde{G}(y, x_i) \]
is nonnegative and continuous on \( \tilde{M} \setminus M_0 \supset M \setminus \Omega \) and superharmonic on \( M \setminus M_0 \supset M \setminus \Omega \).

Therefore, by the minimum principle for parabolic manifolds (Sect. 1),
\[ \inf_{M \setminus \Omega} \psi = \inf_{M \setminus \Omega} \psi = \min_{\partial \Omega} \psi = \psi(y_{k+1}) \]
for some point \( y_{k+1} \in \partial \Omega \). Therefore
\[ \binom{m}{2} D_m(x_1, \ldots, x_m) \geq \binom{m}{2} D_m(y_1, \ldots, y_k, x_{k+1}, \ldots, x_m) = \psi(x_{k+1}) \]
\[ \geq \psi(y_{k+1}) = \binom{m}{2} D_m(y_1, \ldots, y_k, y_{k+1}, x_{k+2}, \ldots, x_m). \]
Thus the claim and the lemma follow by induction. \( \square \)

**Capacity.** Given a nonempty compact subset \( K \) of \( M \setminus \overline{M_0} \), we denote by \( W(K) \) the positive number \( \inf \| \mu \|^2 \), where the infimum is over all unit positive regular Borel measures supported in \( K \) and, for each such measure \( \mu \),
\[ \| \mu \|^2 = \int_M G_\mu(y) \, d\mu(y) = \int_M \int_M G(x, y) \, d\mu(x) \, d\mu(y) \]
is the energy of \( \mu \). Observe that
\[ W(K) \geq \min_{K \times K} G > 0. \]
The number \( 1/W(K) \) is called the **capacity** of \( K \).

**Lemma 6.11.** Let \( \Omega \) be a \( C^\infty \) domain with \( M_0 \Subset \Omega \Subset M \). Then
\[ W(\partial \Omega) = \frac{(n-2)\tau_{n-1}}{D_{\Omega \setminus \overline{M_0}}(u)}, \]
where \( u \) is the harmonic measure of \( \partial \Omega \) with respect to \( \Omega \setminus \overline{M_0} \).

**Proof.** The nonnegative finitely continuous function \( \varphi \) on \( M \setminus M_0 \) defined by
\[ \varphi(x) = \begin{cases} u(x) & \text{if } x \in \overline{\Omega} \setminus M_0 \\ 1 & \text{if } x \in M \setminus \Omega \end{cases} \]
is a potential on $M \setminus \overline{M_0}$. For $\varphi$ is superharmonic and, if $v$ is a harmonic function on $M \setminus \overline{M_0}$ with $0 \leq v \leq \varphi$, then $v$ vanishes continuously at $\partial M_0$. Hence $1 - v$ is a nonnegative continuous function on $M \setminus M_0$ which is superharmonic on $M \setminus \overline{M_0}$ and equal to 1 on $\partial M_0$. Therefore, by the minimum principle for a superharmonic function on the complement of a $C^\infty$ domain in a parabolic manifold, $1 \leq 1 - v \leq 1$ on $M \setminus M_0$. Therefore $v \equiv 0$ and $\varphi$ is a potential. Moreover, $\varphi$ is harmonic on $(M \setminus \overline{M_0}) \setminus \partial \Omega$. Therefore, by Lemma 6.4 there exists a positive regular Borel measure $\mu$ supported in $\partial \Omega$ such that

$$\varphi(x) = G_\mu(x) = \int_M G(x, y) \, d\mu(y) \quad \forall \, x \in M \setminus \overline{M_0}.$$ 

Since $\varphi > 0$ on $M \setminus \overline{M_0}$, $\mu(\partial \Omega) > 0$.

The infimum $W(\partial \Omega)$ is attained by the energy $\|\hat{\mu}\|^2$ of the unit measure

$$\hat{\mu} \equiv \frac{\mu}{\mu(\partial \Omega)}.$$ 

For

$$\|\mu\|^2 = \int_{\partial \Omega} G_\mu \, d\mu = \int_{\partial \Omega} \varphi \, d\mu = \int_{\partial \Omega} d\mu = \mu(\partial \Omega).$$

Hence

$$\|\hat{\mu}\|^2 = \frac{\|\mu\|^2}{(\mu(\partial \Omega))^2} = \|\mu\|^{-2}.$$ 

If $\lambda$ is a positive regular unit Borel measure $\mu$ supported in $\partial \Omega$, then, by the energy principle (Lemma 6.5),

$$1 = \int_{\partial \Omega} d\lambda = \int_{\partial \Omega} \varphi \, d\lambda = \int_{\partial \Omega} G_\mu \, d\lambda = \langle \mu, \lambda \rangle \leq \|\mu\| \cdot \|\lambda\|.$$ 

Hence $\|\lambda\|^2 \geq \|\mu\|^{-2} = \|\hat{\mu}\|^2$. Therefore

$$W(\partial \Omega) = \|\hat{\mu}\|^2 = 1/\|\mu\|^2 = 1/\mu(\partial \Omega)$$

as claimed.
On the other hand, since $u = 0$ on $\partial M_0$, $u = 1$ on $\partial \Omega$, and $u$ is harmonic on $\Omega \setminus \overline{M}_0$, we have

$$D_{\Omega \setminus \overline{M}_0}(u) = \int_{\Omega \setminus \overline{M}_0} |\nabla u|^2 = \int_{\partial (\Omega \setminus \overline{M}_0)} u(x) \frac{\partial u}{\partial \nu}(x) \, d\sigma(x) = \int_{\partial \Omega} \frac{\partial u}{\partial \nu}(x) \, d\sigma(x)$$

$$= \int_{\partial M_0} \frac{\partial u}{\partial \nu}(x) \, d\sigma(x) = \int_{\partial M_0} \frac{\partial \varphi}{\partial \nu}(x) \, d\sigma(x) = \int_{\partial M_0} \frac{\partial G_\mu}{\partial \nu}(x) \, d\sigma(x)$$

$$= \int_{\partial \Omega} \left[ \int_{\partial M_0} \frac{\partial G_\mu}{\partial \nu}(x) \, d\sigma(x) \right] \, d\mu(y) = \int_{\partial \Omega} \left[ \int_{\partial M_0} \frac{\partial \tilde{G}_\mu}{\partial \nu}(x) \, d\sigma(x) \right] \, d\mu(y)$$

$$= (n - 2) \tau_{n-1} \int_{\partial \Omega} \, d\mu(y) \quad \text{(Proposition 6.8 part (v))}$$

$$= (n - 2) \tau_{n-1} \mu(\partial \Omega)$$

The lemma now follows. \qed

**Lemma 6.12.** If $\Omega$ is a $C^\infty$ domain with $M_0 \Subset \Omega \Subset M$, then

$$D(\partial \Omega) \geq W(\partial \Omega).$$

**Proof.** For each integer $m > 1$, there exist points $z_{m1}, \ldots, z_{mm} \in \partial \Omega$ such that

$$D_m(z_{m1}, \ldots, z_{mm}) \geq D_m(\partial \Omega) \geq D_m(z_{m1}, \ldots, z_{mm}) - \frac{1}{m}.$$  

In particular,

$$D_m(z_{m1}, \ldots, z_{mm}) \to D(\partial \Omega) \quad \text{as} \quad m \to \infty.$$

Let $\mu_m$ be the unit positive regular Borel measure supported in the subset

$$\{z_{m1}, \ldots, z_{mm}\}$$

of $\partial \Omega$ with

$$\mu_m(\{z_{mj}\}) = 1/m \quad \text{for} \quad j = 1, \ldots, m.$$  

Then, since the sequence $\{\mu_m(\partial \Omega)\}$ is constant (hence bounded), there is a subsequence $\{\mu_{m_k}\}$ of $\{\mu_m\}$ converging weakly to a unit positive regular Borel measure $\mu$ supported in $\partial \Omega$. In other words,

$$\int_{\partial \Omega} f \, d\mu_{m_k} \to \int_{\partial \Omega} f \, d\mu \quad \forall f \in C^0(\partial \Omega).$$
It follows that, for every continuous function \( f \) on \( \partial \Omega \times \partial \Omega \),
\[
\int_{\partial \Omega} \int_{\partial \Omega} f(x, y) d\mu_{m_k}(x) d\mu_{m_k}(y) \to \int_{\partial \Omega} \int_{\partial \Omega} f(x, y) d\mu_{m}(x) d\mu_{m}(y).
\]
One can verify this by applying the Stone-Weierstrass theorem to approximate \( f \) by a linear combination of functions of the form \( a(x)b(y) \) with \( a, b \in C^0(\partial \Omega) \).

Given a positive real number \( c \), let \( G_c \equiv \min(G, c) \); a finitely continuous function on \((M \times M) \setminus (\partial M \times \partial M)\). Then, for each \( k \), we have
\[
D_{m_k}(z_{m_k1}, \ldots, z_{m_km_k}) \geq \frac{2}{m_k(m_k - 1)} \sum_{1 \leq i < j \leq m_k} G_c(z_{m_ki}, z_{m_kj}) \\
\geq \frac{1}{m_k^2} \sum_{1 \leq i, j \leq m_k; i \neq j} G_c(z_{m_ki}, z_{m_kj}) \\
= \frac{1}{m_k^2} \sum_{i,j=1}^{m_k} G_c(z_{m_ki}, z_{m_kj}) - \frac{1}{m_k^2} \sum_{i=1}^{m_k} G_c(z_{m_ki}, z_{m_ki}) \\
= \frac{1}{m_k^2} \sum_{i,j=1}^{m_k} G_c(z_{m_ki}, z_{m_kj}) - \frac{c}{m_k} \\
= \int_{\partial \Omega} \int_{\partial \Omega} G_c(x, y) d\mu_{m_k}(x) d\mu_{m_k}(y) - \frac{c}{m_k}.
\]
Thus
\[
D(\partial \Omega) = \lim_{k \to \infty} D_{m_k}(z_{m_k1}, \ldots, z_{m_km_k}) \geq \int_{\partial \Omega} \int_{\partial \Omega} G_c(x, y) d\mu(x) d\mu(y)
\]
for every \( c > 0 \). Therefore, by the monotone convergence theorem,
\[
D(\partial \Omega) \geq \int_{\partial \Omega} \int_{\partial \Omega} G(x, y) d\mu(x) d\mu(y) = \|\mu\|^2 \geq W(\partial \Omega).
\]

We now come to the main point:

**Lemma 6.13.** \( E(\Gamma) = D(\Gamma) = +\infty \).

**Proof.** Let \( \{M_k\} \) be an exhaustion of \( M \) by \( C^\infty \) relatively compact domains containing \( \overline{M}_0 \).
Then, for each positive integer \( k \),
\[
E(\Gamma) \geq D(\Gamma) \geq D(\tilde{M} \setminus M_k) = D(\partial M_k) \geq W(\partial M_k) = \frac{(n-2)\tau_{n-1}}{D_{M_k \setminus \overline{M}_0}(u_k)},
\]
where $u_k$ is the harmonic measure of $\partial M_k$ with respect to $M_k \setminus M_0$. Since $M$ is parabolic, the sequence $\{u_k\}$ converges to 0 uniformly on compact sets. Hence

$$D_{M_k \setminus M_0}(u_k) = \int_{\partial (M_k \setminus M_0)} u_k \frac{\partial u_k}{\partial \nu} = \int_{\partial M_k} \frac{\partial u_k}{\partial \nu} = \int_{\partial M_0} \frac{\partial u_k}{\partial \nu} \to 0 \quad \text{as} \quad k \to \infty.$$ 

The lemma now follows. □

**Theorem 6.14.** Let $(M,g)$ be a connected noncompact oriented parabolic Riemannian manifold of dimension $n > 2$ and $M_0$ a relatively compact domain with $C^\infty$ connected boundary. Then there exists a continuous exhaustion function $\varphi$ on $M \setminus M_0$ such that $\varphi$ is harmonic on $M \setminus M_0$ and equal to 0 on $\partial M_0$.

**Remark.** Of course, this theorem holds for $n = 2$ as well.

**Proof of Theorem 6.14.** In the notation of this section, we have $E_m(\Gamma) \to E(\Gamma) = +\infty$ as $m \to \infty$.

Hence there is an increasing sequence of integers $\{m_k\}$ such that, for each $k$, $m_k > 1$ and $E_{m_k}(\Gamma) > 2^k$. Hence there exist points $x_{k1}, \ldots, x_{km_k}$ in $\Gamma$ such that $E_{m_k}(\Gamma, x_{k1}, \ldots, x_{km_k}) > 2^k$. Let $\varphi_k$ be the nonnegative continuous function defined by

$$\varphi_k(x) \equiv \frac{1}{m_k2^k} \sum_{i=1}^{m_k} \tilde{G}(x, x_{ki}) \quad \forall x \in \mathring{M} \setminus M_0.$$ 

Then, by Proposition 6.8, $\varphi_k$ is positive and harmonic on $M \setminus M_0$ and equal to 0 on $\partial M_0$. On $\Gamma$, we have

$$\varphi_k \geq \frac{1}{2^k} E_{m_k}(\Gamma, x_{k1}, \ldots, x_{km_k}) > 1.$$ 

Hence there exists an exhaustion $\{\Omega_k\}$ of $M$ by domains such that $\varphi_k > 1$ on $\mathring{M} \setminus \Omega_k$ for each $k$. If $K$ is a compact subset of $M \setminus M_0$, then $\tilde{G}$ is finitely continuous, hence bounded, on $K \times \Gamma$. Therefore, for some positive constant $a$ (depending on $K$), we have

$$0 \leq \varphi_k \leq a \sum_{i=1}^{m_k} \frac{1}{m_k2^k} = a2^{-k}$$

for point $x \in K$ and integer $k > 1$. Hence $\sum \varphi_k$ converges uniformly on compact subsets of $M \setminus M_0$ to a continuous function $\varphi$ which is positive and harmonic on $M \setminus M_0$ and equal
to 0 on $\partial M_0$. Moreover, for $x \in M \setminus \Omega_k$, we have

$$\varphi(x) \geq \sum_{j=1}^{k} \varphi_j(x) \geq \sum_{j=1}^{k} 1 = k \to +\infty \quad \text{as} \quad k \to \infty.$$ 

Hence $\varphi$ exhausts $M \setminus M_0$.

The corresponding result for parabolic ends is applied in Sect. 2 of [NR1].

**Corollary 6.15.** Let $(M, g)$ be a connected noncompact oriented Riemannian manifold, let $\Omega$ be a $C^\infty$ relatively compact domain in $M$, and let $E$ be a connected component of $M \setminus \overline{\Omega}$ which is not relatively compact in $M$. Assume that $E$ is a parabolic end. Then there exists a continuous exhaustion function $\psi$ on $\overline{E}$ which is harmonic on $E$ and equal to 0 on $\partial E$.

**Proof.** Clearly, we may assume that $M$ is the double of $E$; a parabolic manifold. Let $M_0$ be a $C^\infty$ relatively compact domain in $\Omega$ such that $M \setminus M_0$ is connected, let $\varphi$ be as in Theorem 6.14, and let $\{M_k\}$ be an exhaustion of $M$ by $C^\infty$ relatively compact domains containing $\Omega$, and for each positive integer $k$, let $v_k$ be the continuous function on $\overline{M_k} \setminus \Omega$ which is harmonic on $M_k \setminus \Omega$, equal to 0 on $\partial M_k$, and equal to $\varphi$ on $\partial \Omega$. Then $0 \leq v_k \leq a \equiv \max_{\partial \Omega} \varphi$. Moreover, the sequence $\{v_k\}$ is nondecreasing and therefore converges uniformly on compact subsets of $M \setminus \Omega$ to a nonnegative continuous function $v$ such that $v$ is harmonic on $M \setminus \overline{\Omega}$, $v \equiv \varphi$ on $\partial \Omega$, and $0 \leq v \leq a$ on $M \setminus \Omega$. Hence the function $\psi \equiv \varphi - v$ has the desired properties. \hfill $\square$

**Remark.** Given a point $x_0 \in M_0$, by applying Sario’s existence theorem of principal functions (Theorem A.1) to the function $\varphi$ of Theorem 6.14 one can construct a unique potential $p$ on $M$ such that $p$ is harmonic on $M \setminus \{x_0\}$; $p - G'_{x_0}$ is harmonic on $M_0$, where $G'$ is the Green’s function on $M_0$, and $p$ exhausts $M \setminus M_0$. The function $p$ is called the **Evans-Selberg potential** of $M$ with pole at $x_0$. The existence of the Evans-Selberg potential is equivalent to parabolicity. For the details, the reader may refer to [SaNo].

**References**

[ABCKT] J. Amorós, M. Burger, K. Corlette, D. Kotschick, D. Toledo, *Fundamental groups of compact Kähler manifolds*, Math. Surveys and Monographs, 44, American Mathematical Society, Providence, RI, 1996.

[An] A. Ancona, *Théorie du potentiel sur les graphes et les variétés*, École d’Été de Probabilités de Saint-Flour XVIII 1988, Lect. Notes in Math., vol. 1427, pp. 1–112, Springer, Berlin, Heidelberg, New York, 1990.
[Ar] D. Arapura, *Fundamental groups of smooth projective varieties*, Current Topics in Complex Algebraic Geometry (Berkeley, CA, 1992/93), 1–16, Math. Sci. Res. Inst. Publ., 28, Cambridge Univ. Press, Cambridge, 1995.

[ArBR] D. Arapura, P. Bressler, and M. Ramachandran, *On the fundamental group of a compact Kähler manifold*, Duke Math. J. 64 (1992), 477–488.

[BarPV] W. Barth, C. Peters, A. Van de Ven, *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1984.

[Bau] G. Baumslag, *Topics in combinatorial group theory*, Birkhäuser, Basel, Boston, Berlin, 1993.

[Be] A. Beauville, *Complex algebraic surfaces*, London Math. Soc. Lect. Note Series (68), Cambridge Univ. Press, Cambridge, 1983.

[BiS] R. Bieri, R. Strebel, *Almost finitely presented soluble groups*, Comment. Math. Helvetici 53 (1978), 258–278.

[Br] K. Brown, *The homology of Richard Thompson’s group F*, to appear in Proceedings of the Conference on Probabilistic and Asymptotic Aspects of Group Theory and Topological Aspects of Group Theory (Nashville, 2004), Contemporary Mathematics, American Mathematical Society, Providence.

[Cam] F. Campana, Frédéric, *Connexité abélienne des variétés kähleriennes compactes*, Bull. Soc. Math. France 126 (1998), no. 4, 483–506.

[Cat1] F. Catanese, *Moduli and classification of irregular Kaehler manifolds (and algebraic varieties) with Albanese general type fibrations*, Invent. Math. 104 (1991), no. 2, 263–289.

[Cat2] F. Catanese, *Fibred Kähler and quasi-projective groups*, Special issue dedicated to Adriano Barlotti, Adv. Geom. 2003, suppl., S13–S27.

[CatKO] F. Catanese, J. Keum, K. Oguiso, *Some remarks on the universal cover of an open K3 surface*, Math. Ann. 325 (2003), no. 2, 279–286.

[Cha] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, Orlando, 1984.

[Chi] E. M. Chirka, *Complex Analytic Sets*, Translated from the Russian by R. A. M. Hoksbergen, Mathematics and its Applications (Soviet Series), 46, Kluwer Academic Publishers Group, Dordrecht, 1989.

[Co] P. Cousin, *Sur les fonctions triplement périodiques de deux variables*, Acta Math. 33 (1910), 105–232.

[DelG] T. Delzant, M. Gromov, *Cuts in Kaehler groups*, to appear in Proceedings of the conference in honor of R. Grigorchuk, 2004.

[Dem] J.-P. Demailly, *Estimations $L^2$ pour l’opérateur $\bar{\partial}$ d’un fibré vectoriel holomorphe semi-positif au-dessus d’une variété Kahlerienne complète*, Ann. Sci. École Norm. Sup. 15 (1982), 457–511.

[F] D. Farley, *Actions of picture groups on CAT(0) cubical complexes*, preprint.

[Ga] M. Gaffney, *A special Stokes theorem for Riemannian manifolds*, Ann. of Math. 60 (1954), 140–145.

[Ge] R. Geoghegan, *Topological Methods in Group Theory*, forthcoming book.

[Gri] A. Grigor’yan, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds*, Bull. Amer. Math. Soc. 36 (1999), no. 2, 135–249.

[Gro] M. Gromov, *Sur le groupe fondamental d’une variété kähleriennne*, C. R. Acad. Sci. Paris 308, (1989), no. 3, 67–70.

[Gro2] M. Gromov, *Kähler hyperbolicity and $L^2$-Hodge theory*, J. Differential Geom. 33 (1991), 263–292.

[GroS] M. Gromov, R. Schoen, *Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one*, Publ. Math. I.H.E.S., No. 76, (1992), 165–246.

[JY1] J. Jost, S.-T. Yau, *Applications of quasilinear PDE to algebraic geometry and arithmetic lattices*, Algebraic geometry and related topics (Inchon, 1992), 169–193, Conf. Proc. Lect. Notes Alg. Geom., I, Internat. Press, Cambridge, MA, 1993.
[JY2] J. Jost, S.-T. Yau, Harmonic mappings and algebraic varieties over function fields, Amer. J. Math. 115 (1993), no. 6, 1197–1227.

[Kl] T. Klein, Filtered ends of infinite covers and groups, preprint.

[Kol] J. Kollár, Shafarevich maps and automorphic forms, M. B. Porter Lectures, Princeton Univ. Press, Princeton, 1995.

[KoS1] N. Korevaar, R. Schoen, Sobolev spaces and harmonic maps for metric space targets, Comm. Anal. Geom. 1 (1993), no. 3-4, 561–659.

[KoS2] N. Korevaar, R. Schoen, Global existence theorems for harmonic maps to non-locally compact spaces, Comm. Anal. Geom. 5 (1997), no. 2, 333–387.

[KroR] P. Kropholler, M. Roller, Relative ends and duality groups, J. Pure Appl. Algebra 61 (1989), no. 2, 197–210.

[L] P. Li, On the structure of complete Kähler manifolds with nonnegative curvature near infinity, Invent. Math. 99 (1990), 579–600.

[LT] P. Li, L.-F. Tam, Harmonic functions and the structure of complete manifolds, J. Diff. Geom. 35 (1992), 359–383.

[M] F.-Y. Maeda, Dirichlet Integrals on Harmonic Spaces, Lect. Notes in Math., vol. 803, Springer, Berlin, Heidelberg, New York, 1980.

[Na1] M. Nakai, Infinite boundary value problems for second order elliptic partial differential equations, J. Fac. Sci. Univ. Tokyo, Sect. I 17 (1970), 101–121.

[Na2] M. Nakai, On Evans potential, Proc. Japan. Acad. 38 (1962), 624–629.

[Nk] S. Nakano, Vanishing theorems for weakly 1-complete manifolds II, Publ. R.I.M.S. Kyoto 10 (1974), 101–110.

[NR1] T. Napier, M. Ramachandran, Structure theorems for complete Kähler manifolds and applications to Lefschetz type theorems,Geom. Funct. Anal. 5 (1995), 809–851.

[NR2] T. Napier, M. Ramachandran, The Bochner-Hartogs dichotomy for weakly 1-complete Kähler manifolds, Ann. Inst. Fourier (Grenoble) 47 (1997), 1345–1365.

[NR3] T. Napier, M. Ramachandran, Hyperbolic Kähler manifolds and proper holomorphic mappings to Riemann surfaces, Geom. Funct. Anal. 11 (2001), 382–406.

[NR4] T. Napier, M. Ramachandran, Thompson’s Group F is not Kähler, to appear in Proceedings of the Conference on Probabilistic and Asymptotic Aspects of Group Theory (Nashville, 2004), Contemporary Mathematics, American Mathematical Society, Providence.

[RS] B. Rodin, L. Sario, Principal Functions, Van Nostrand, Princeton, 1968.

[S] M. Sageev, Ends of group pairs and non-positively curved cube complexes, Proc. London Math. Soc. 71, no. 3, (1995), 585–617.

[SaNa] L. Sario, M. Nakai, Classification Theory of Riemann Surfaces, Springer, Berlin-Heidelberg-New York, 1970.

[SaNo] L. Sario, K. Noshiro, Value Distribution Theory, Van Nostrand, Princeton, 1966.

[Se] J.-P. Serre, Trees, Translated from the French original by John Stillwell, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

[Sim1] C. Simpson, The ubiquity of variations of Hodge structure, Complex geometry and Lie theory (Sundance, 1989), 329–348, Proc. Sympos. Pure Math., 53, Amer. Math. Soc., Providence, RI, 1991.

[Sim2] C. Simpson, Lefschetz theorems for the integral leaves of a holomorphic one-form, Comp. Math. 87 (1993), 99–113.

[Siu] Y. T. Siu, Strong rigidity for Kähler manifolds and the construction of bounded holomorphic functions, Discrete groups in geometry and analysis (New Haven, Conn., 1984), 124–151, Progr. Math., 67, Birkhäuser Boston, Boston, MA, 1987.

[Ste] K. Stein, Maximale holomorphe und meromorphe Abbildungen, I, Amer. J. Math. 85 (1963), 298–315.

[Sto] W. Stoll, The fiber integral is constant, Math. Z. 104 (1968), 65–73.
[Sul] D. Sullivan, *Growth of positive harmonic functions and Kleinian group limit sets of zero planar measure and Hausdorff dimension two*, in Geometry Symposium (Utrecht 1980), Lect. Notes in Math. 894, 127–144, Springer, Berlin-Heidelberg-New York, 1981.

[Sun] X. Sun, *Regularity of harmonic maps to trees*, Amer. J. Math. 125 (2003), no. 4, 737–771.

[TW] P. Tworzewski, T. Winiarski, *Continuity of intersection of analytic sets*, Ann. Polon. Math. 42 (1983), 387–393.

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