Note on Connes spectral distances of qubits

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Abstract

By virtue of the mathematical tools in noncommutative geometry, we study the Connes spectral distances between one- and two-qubit states. We construct a spectral triple corresponding to the 2D fermionic phase spaces, and calculate the Connes spectral distance between one qubits. Based on the Connes spectral distance, we define a coherence measure of quantum states, and calculate the coherence of one-qubit states. We also study some simple cases about two-qubit states, and the corresponding spectral distances satisfy the Pythagoras theorem. We find that the Connes spectral distances are different from quantum trace distances. The Connes spectral distance can be considered as a significant supplement to the trace distances in quantum information sciences. These results are significant for the study of physical relations and geometric structures of qubits and other quantum states.

Keywords: Connes spectral distance; Qubit; Coherence; Noncommutative geometry.

1 Introduction

In quantum mechanics, a physical system is represented by some kind of quantum state. In order to study properties of the physical systems and also the relations between quantum states, or measure the distinguishability between the states, one can define some kinds of abstract distance measures between quantum states. For example, quantum trace distance and quantum fidelity \cite{1}. In quantum information science, one can use quantum trace distance or quantum fidelity to quantify the differences between quantum states.

One usually consider the quantum systems with two levels (namely, qubits) in quantum information and quantum computation. To describe a quantum system with two levels, one can use the Grassmann representation of Fermi operators
in fermionic phase spaces. Since quantum phase spaces are some kinds of non-commutative spaces, one can also use the mathematical tools in noncommutative geometry to study the geometric structures of quantum states in phase spaces [2]. In a noncommutative space, a pure state is the analog of a traditional point in a normal commutative space, and the Connes spectral distance between pure states corresponds to the geodesic distance between points [3]. The Connes spectral distances in some kinds of noncommutative spaces have already been studied in the literatures [4–16]. For example, Dai et al. have studied Connes’ distance in 1D lattices [5]. Cagnache et al. computed Connes spectral distances between the pure states which corresponding to eigenfunctions of the quantum harmonic oscillators in the Moyal plane [6]. Martinetti et al. obtained the spectral distance between coherent states in the so-called double Moyal plane [7]. Scholtz and his collaborators have studied the Connes spectral distances of harmonic oscillator states and also coherent states in Moyal plane and fuzzy space [11–13]. In the present work, we will study the Connes spectral distance between qubits which can be represented by fermionic Fock states in phase spaces.

This paper is organized as follows. In Sec. 2, we consider the 2D fermionic phase space and construct a corresponding spectral triple based on the Hilbert-Schmidt operatorial formulation. In Sec. 3, we review the definition of Connes spectral distance, and derive the explicit expressions of the spectral distances between one qubits with respect to the corresponding Bloch vectors. In Sec. 4, we calculate the Connes spectral distances between some simple cases of two-qubit states. Some discussions and conclusions are given in Sec. 5.

2 2D fermionic phase space and spectral triple

First, let us consider the simplest 2D fermionic phase space \((\theta_1, \theta_2)\), and the coordinate operators \(\hat{\theta}_1, \hat{\theta}_2\) satisfy the following anticommutation relation

\[
\{\hat{\theta}_i, \hat{\theta}_j\} \equiv \hat{\theta}_i\hat{\theta}_j + \hat{\theta}_j\hat{\theta}_i = \delta_{ij}\hbar, \quad i, j = 1, 2.
\] (1)

The Grassmann parity of a function \(f\) is denoted by \(\varepsilon(f)\), for example, \(\varepsilon(\hat{\theta}_i) = 1\) and \(\varepsilon(\hat{\theta}_i\hat{\theta}_j) = 0\). One can define the following annihilation and creation operators,

\[
\hat{f} = \frac{1}{\sqrt{2\hbar}} \left( \hat{\theta}_1 + i\hat{\theta}_2 \right), \quad \hat{f}^\dagger = \frac{1}{\sqrt{2\hbar}} \left( \hat{\theta}_1 - i\hat{\theta}_2 \right).
\] (2)

These operators satisfy the commutation relations \(\{\hat{f}, \hat{f}^\dagger\} = 1\), and \(\{\hat{f}, \hat{f}\} = \{\hat{f}^\dagger, \hat{f}^\dagger\} = 0\). Let |0\rangle be the vacuum state, there are \(\hat{f}|0\rangle = 0, \hat{f}^\dagger|0\rangle = |1\rangle, \hat{f}|1\rangle = |0\rangle, \hat{f}^\dagger|1\rangle = 0\). One can also use the convenient matrix representations, \(|0\rangle = (1, 0)'\), \(|0\rangle = (0, 1)'\), and

\[
\hat{f} = |0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{f}^\dagger = |1\rangle\langle 0| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\] (3)

The fermionic phase space \((\theta_1, \theta_2)\) is some type of noncommutative space. In general, a noncommutative space corresponds to a spectral triple \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) [2]. \(\mathcal{A}\) is an involutive algebra acting on a Hilbert space \(\mathcal{H}\) through a representation \(\pi\), and the Dirac operator \(\mathcal{D}\) is a self-adjoint, densely defined operator on \(\mathcal{H}\) which satisfies:
1. $\mathcal{D}$ can be unbounded operator but $[\mathcal{D}, \pi(a)]$ is bounded;
2. $\mathcal{D}$ has compact resolvent, for $\lambda \in \mathbb{C}/\mathbb{R}$, $(\mathcal{D} - \lambda)^{-1}$ is compact when the algebra $\mathcal{A}$ is unital or $\pi(a)(\mathcal{D} - \lambda)^{-1}$ be compact if it is non-unital.

In the present work, we will use the Hilbert-Schmidt operatorial formulation developed in Refs. [13,17] to construct a spectral triple corresponding to the fermionic phase space. One can define a fermion Fock space $F = \text{span}\{\mid0\rangle, \mid1\rangle\}$ and a quantum Hilbert space $Q = \text{span}\{\midi\rangle\langle j\mid, i,j = 0,1\}$. In the followings, we will also denote the elements $\psi(\hat{\theta}_1, \hat{\theta}_2)$ of the quantum Hilbert space $Q$ by $\mid\psi\rangle$.

A spectral triple $(\mathcal{A}, H, D)$ for the 2D fermionic phase space $(\theta_1, \theta_2)$ can be constructed as follows,

$$\mathcal{A} = Q, \quad H = F \otimes \mathbb{C}^2,$$

and an element $e \in \mathcal{A}$ acts on $\Psi = (\mid\psi_1\rangle, \mid\psi_2\rangle)' \in H$ through the diagonal representation $\pi$ as

$$\pi(e)\Psi = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \begin{pmatrix} \mid\psi_1\rangle \\ \mid\psi_2\rangle \end{pmatrix} = \begin{pmatrix} e\mid\psi_1\rangle \\ e\mid\psi_2\rangle \end{pmatrix}.$$

In order to construct the Dirac operator for the fermionic phase space, one can consider the following extended auxiliary noncommutative space in which the coordinate operators $\hat{\Theta}_i$ and $\hat{\Lambda}_i$ satisfy the following anticommutation relations,

$$\{\hat{\Theta}_i, \hat{\Theta}_j\} = \{\hat{\Lambda}_i, \hat{\Lambda}_j\} = \delta_{ij}\hbar, \quad \{\hat{\Theta}_i, \hat{\Lambda}_j\} = \delta_{ij}\lambda, \quad i,j = 1,2.$$  

(6)

Here $\lambda$ is some real parameter. It is easy to verify that, a unitary representation of the algebra (6) can be obtained by the following actions on the quantum Hilbert space $Q$:

$$\hat{\Theta}_i\mid\phi\rangle = \hat{\theta}_i\phi, \quad \hat{\Lambda}_i\mid\phi\rangle = \frac{\lambda}{\hbar}\hat{\theta}_i\phi + (-1)^{\varepsilon(\phi)}\sqrt{\frac{\lambda^2 - \hbar^2}{\hbar}}\mid\phi\rangle \hat{\theta}_i.$$  

(7)

One can define the following useful operators

$$\hat{B} = \hat{\Lambda}_1 + i\hat{\Lambda}_2, \quad \hat{B}^\dagger = \hat{\Lambda}_1 - i\hat{\Lambda}_2.$$  

(8)

It is easy to verify that,

$$\hat{B}\mid\phi\rangle = \lambda \sqrt{\frac{2}{\hbar}} \mid\hat{f}\phi\rangle - i(-1)^{\varepsilon(\phi)}\sqrt{\frac{2(h^2 - \lambda^2)}{\hbar}}\mid\phi\rangle \hat{f},$$  

$$\hat{B}^\dagger\mid\phi\rangle = \lambda \sqrt{\frac{2}{\hbar}} \mid\hat{f}^\dagger\phi\rangle - i(-1)^{\varepsilon(\phi)}\sqrt{\frac{2(h^2 - \lambda^2)}{\hbar}}\mid\phi\rangle \hat{f}^\dagger.$$  

(9)

By virtue of the result in Ref. [20], one may express the Dirac operator $\mathcal{D}$ as

$$\mathcal{D} = \frac{1}{\lambda} \sum_{i=1,2} \sigma_i \hat{\Lambda}_i,$$  

(10)

where $\sigma_i$’s are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$  

(11)

So the Dirac operator (10) can be written as

$$\mathcal{D} = \frac{1}{\lambda} \begin{pmatrix} 0 & \hat{\Lambda}_1 - i\hat{\Lambda}_2 \\ \hat{\Lambda}_1 + i\hat{\Lambda}_2 & 0 \end{pmatrix} = \frac{1}{\lambda} \begin{pmatrix} 0 & \hat{B}^\dagger \\ \hat{B} & 0 \end{pmatrix}.$$  

(12)
Consider the graded commutator \([D, \pi(a)]_{gr} \equiv D\pi(a) - (-1)^{e(a)}\pi(a)D\) with \(a \in \mathcal{H}\). After some straightforward calculations, one can obtain \([D, \pi(a)]_{gr}\) acting on an element \(\Phi \in Q \otimes \mathbb{C}^2\) as

\[
[D, \pi(a)]_{gr} \Phi = [D, \pi(a)]_{gr} \begin{pmatrix} |\phi_1\rangle \\ |\phi_2\rangle \end{pmatrix} = \frac{1}{\lambda} \begin{pmatrix} 0 & [\hat{B}^\dagger, a]_{gr} \\ [\hat{B}, a]_{gr} & 0 \end{pmatrix} \begin{pmatrix} |\phi_1\rangle \\ |\phi_2\rangle \end{pmatrix} = \sqrt{\frac{2}{\hbar}} \begin{pmatrix} 0 & [\hat{f}^\dagger, a]_{gr} \\ [\hat{f}, a]_{gr} & 0 \end{pmatrix} \begin{pmatrix} |\phi_1\rangle \\ |\phi_2\rangle \end{pmatrix}.
\]

Therefore, one can identify the Dirac operator \(D\) for the 2D fermionic phase space \((\theta_1, \theta_2)\) as

\[
D = \sqrt{\frac{2}{\hbar}} \begin{pmatrix} 0 & \hat{f}^\dagger \\ \hat{f} & 0 \end{pmatrix}.
\]

### 3 Connes spectral distances between one-qubit states

Using the Dirac operator constructed above, one can calculate the Connes spectral distances between quantum states in the fermionic phase space. For the quantum states \(\omega\) which are normal and bounded, they can be represented by density matrices \(\rho\). The action of the state \(\omega\) on an element \(e \in \mathcal{A}\) can be written as

\[
\omega(e) = \text{tr}_F(\rho e),
\]

where \(\text{tr}_F(\cdot)\) denotes the trace over \(F\). Suppose the quantum states \(\omega_1\) and \(\omega_2\) correspond to the density matrices \(\rho_1\) and \(\rho_2\), respectively. The Connes distance between the quantum states \(\omega_1\) and \(\omega_2\) are [3]

\[
d(\omega_1, \omega_2) = \sup_{e \in B} |\text{tr}_F(\rho_1 e) - \text{tr}_F(\rho_2 e)| = \sup_{e \in B} |\text{tr}_F(\Delta \rho e)|,
\]

where \(\Delta \rho = \rho_1 - \rho_2\). The set \(B = \{e \in \mathcal{A} : \|[D, \pi(e)]\|_{op} \leq 1\}\), and \(\|A\|_{op}\) is the operator norm of \(A\),

\[
\|A\|_{op} \equiv \sup_{\psi \in \mathcal{H}, \|\psi\|=1} \|A\psi\|, \quad \|A\|^2 \equiv \text{tr}_F(A^\dagger A).
\]

The inequality \(\|[D, \pi(e)]\|_{op} \leq 1\) is the so-called ball condition.

Since Hermitian elements can give the supremum in the Connes spectral distance functions [18], one only need to consider the element \(e\) being Hermitian. For a Hermitian element \(e \in \mathcal{A}\), using the Dirac operator \(D\) [14], we have

\[
[D, \pi(e)] = \sqrt{\frac{2}{\hbar}} \begin{pmatrix} 0 & [\hat{f}^\dagger, e] \\ [\hat{f}, e] & 0 \end{pmatrix} = \sqrt{\frac{2}{\hbar}} \begin{pmatrix} 0 & -[\hat{f}, e]^\dagger \\ [\hat{f}, e] & 0 \end{pmatrix}.
\]
Any Hermitian element $e \in \mathcal{A}$ can be expressed as the following matrix,

$$
e = \begin{pmatrix} s & w^* \\ w & t \end{pmatrix} = \begin{pmatrix} s & u - iv \\ u + iv & t \end{pmatrix},$$

where $w = u + iv$, and $s, t, u, v$ are real numbers. After some straightforward calculations, one can obtain

$$[\hat{f}, e] = \begin{pmatrix} w & t - s \\ 0 & -w \end{pmatrix}.$$  \hspace{1cm} (20)

Since $\|\mathbf{D}, \pi (e)\|_{\text{op}}$ is just the square root of the largest eigenvalue of the matrix $[\mathbf{D}, \pi (e)]^\dagger [\mathbf{D}, \pi (e)]$, using the ball condition and the above matrix representations (18) and (20), one can obtain

$$2|w|^2 + (s - t)^2 + |s - t| \sqrt{4|w|^2 + (s - t)^2} \leq \hbar.$$ \hspace{1cm} (21)

So there are

$$2|w|^2 \leq \hbar, \quad 2(s - t)^2 \leq \hbar,$$ \hspace{1cm} (22)

or

$$|w| \leq \sqrt{\frac{\hbar}{2}}, \quad |s - t| \leq \sqrt{\frac{\hbar}{2}}.$$ \hspace{1cm} (23)

In general, the density matrice $\rho$ for a qubit can be expressed as [1]

$$\rho = \frac{1 + \mathbf{r} \cdot \mathbf{\Delta}}{2} = \frac{1}{2} \begin{pmatrix} 1 + z & x - i y \\ x + i y & 1 - z \end{pmatrix},$$ \hspace{1cm} (24)

where the real vector $\mathbf{r} = (x, y, z)$ are the so-called Bloch vector, $|\mathbf{r}| \leq 1$, and $\mathbf{\Delta} = (\sigma_1, \sigma_2, \sigma_3)$, $\sigma_i$ are the Pauli matrices. Consider the states $\rho_1$ and $\rho_2$ corresponding to the Bloch vectors $\mathbf{r}_1 = (x_1, y_1, z_1)$ and $\mathbf{r}_2 = (x_2, y_2, z_2)$, respectively. Using the matrix representation (19), one can obtain

$$\text{tr}_F (\Delta \rho e) = \frac{1}{2} (s - t) \Delta z + u \Delta x + v \Delta y,$$ \hspace{1cm} (25)

where $\Delta x = x_1 - x_2$, $\Delta y = y_1 - y_2$, $\Delta z = z_1 - z_2$. So we have

$$d(\rho_1, \rho_2) = \sup_{e \in B} |\text{tr}_F (\Delta \rho e)| = \sup_{e \in B} \left| \frac{1}{2} (s - t) \Delta z + u \Delta x + v \Delta y \right|$$

$$\leq \sup_{e \in B} \left( \frac{1}{2} |(s - t) \Delta z| + |u \Delta x + v \Delta y| \right)$$

$$\leq \sup_{e \in B} \left( \frac{1}{2} |(s - t) \Delta z| + \sqrt{u^2 + v^2} \sqrt{\Delta x^2 + \Delta y^2} \right)$$

$$= \sup_{e \in B} \left( \frac{1}{2} |(s - t) \Delta z| + |w| \sqrt{\Delta x^2 + \Delta y^2} \right).$$ \hspace{1cm} (26)

In the second inequality above, we have used the Cauchy-Schwartz inequality,

$$(a_1 b_1 + a_2 b_2)^2 \leq (a_1^2 + a_2^2) (b_1^2 + b_2^2),$$

where the equality holds if $a_1 b_2 = a_2 b_1$.

So for the given states $\rho_1$ and $\rho_2$, in order to attain the supremum of $|\text{tr}_F (\Delta \rho e)|$, one must choose the Hermitian element $e$ to make $|s - t|$ and $|w|$ as large as possible. So there should be

$$2|w|^2 + (s - t)^2 + |s - t| \sqrt{4|w|^2 + (s - t)^2} = \hbar.$$ \hspace{1cm} (27)
and
\[ |s - t| = \frac{1}{\sqrt{2\hbar}}(\hbar - 2|w|^2). \] (28)

So we have
\[ |\text{tr}_F(\Delta \rho e)| \leq \frac{1}{2}(s - t)\Delta z + |w|\sqrt{\Delta x^2 + \Delta y^2} \]
\[ = \frac{1}{2}\sqrt{\frac{\hbar}{2}}|\Delta z| - \frac{|\Delta z|}{\sqrt{2\hbar}}|w|^2 + |w|\sqrt{\Delta x^2 + \Delta y^2} \]
\[ = \frac{1}{2}\sqrt{\frac{\hbar}{2}}|\Delta r|^2 \left( |w| - \frac{\hbar}{2}\sqrt{\frac{\Delta x^2 + \Delta y^2}{|\Delta z|}} \right)^2 \]
\[ \leq \frac{1}{2}\sqrt{\frac{\hbar}{2}}|\Delta r|^2. \] (29)

and \( \Delta \vec{r} = \vec{r}_1 - \vec{r}_2 = (\Delta x, \Delta y, \Delta z) \), \( |\Delta \vec{r}| = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} \leq 2. \)

From Eq. (23), the ball condition for \( e \in B \) means \( |w| \leq \sqrt{\hbar}/2 \). So when \( \Delta x^2 + \Delta y^2 \leq \Delta z^2 \), one can choose the optimal element \( e \) with
\[ |w| = \frac{\sqrt{\Delta x^2 + \Delta y^2}}{|\Delta z|} \sqrt{\frac{\hbar}{2}}, \quad |s - t| = \frac{\sqrt{\hbar}}{\sqrt{2\hbar}}\Delta z^2 - \frac{\Delta x^2 - \Delta y^2}{\Delta z^2}. \] (30)

In this case, there is
\[ d(\rho_1, \rho_2) = \frac{1}{2}\sqrt{\frac{\hbar}{2}}\Delta x^2 + \frac{\Delta y^2}{|\Delta z|} = \frac{1}{2}\sqrt{\frac{\hbar}{2}}|\Delta \vec{r}|^2. \] (31)

When \( \Delta x^2 + \Delta y^2 > \Delta z^2 \), one can only choose the optimal element \( e \) with
\[ |w| = \frac{\sqrt{\hbar}}{2} < \frac{\sqrt{\Delta x^2 + \Delta y^2}}{|\Delta z|} \sqrt{\frac{\hbar}{2}}, \quad |s - t| = \frac{1}{\sqrt{2\hbar}}(\hbar - 2|w|^2) = 0. \] (32)

In this case, there is
\[ d(\rho_1, \rho_2) = \frac{1}{2}|s - t|\Delta z + |w|\sqrt{\Delta x^2 + \Delta y^2} = \frac{\sqrt{\hbar}}{2}\sqrt{\Delta x^2 + \Delta y^2}. \] (33)

It is convenient to use the spherical coordinates in the Bloch sphere, one can also denote \( \Delta \vec{r} = \vec{r}_1 - \vec{r}_2 = r(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \) \( 0 \leq \theta \leq \pi, \) \( 0 \leq \phi < 2\pi \) and \( 0 \leq r \leq 2. \) The Connes spectral distance between the one-qubit states \( \rho_1 \) and \( \rho_2 \) can be expressed as
\[
\begin{align*}
   d(\rho_1, \rho_2) &= \begin{cases} 
   \frac{\sqrt{\hbar}}{2}\sqrt{\Delta x^2 + \Delta y^2} = r \sin \theta \sqrt{\frac{\hbar}{2}}, & \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}; \\
   \frac{1}{2}\sqrt{\frac{\hbar}{2}}|\Delta r|^2 = \frac{r}{2|\cos \theta|} \sqrt{\frac{\hbar}{2}}, & \text{others.}
   \end{cases}
\end{align*}
\] (34)

These results are similar to those in Refs. [6,9], but the spectral triple constructed in the present work is different from those considered in Refs. [6,9]. The method used in the present work is also different from those used in the literatures.
From the result (34), one can find that these spectral distances are additive when the corresponding points of the states in the Bloch sphere are collinear,

\[ d(\rho_1, \rho_3) = d(\rho_1, \rho_2) + d(\rho_2, \rho_3). \]  

(35)

Furthermore, when the corresponding points of the states in the Bloch sphere are on the same horizontal plane, namely $\Delta z = 0$, the Connes spectral distances between the states are proportional to the Euclidean distances in the Bloch representation with a factor $\sqrt{\hbar/2}$. It is easy to see that, for the diagonal states, namely $x_i = y_i = 0$ in (24), the optimal elements $e$ for the Connes spectral distances can also be diagonal, this is similar to the result in Ref. [19].

Using the formulas (34), one can obtain the Connes spectral distances between some one-qubit states. For example,

\[ d(|0\rangle, |1\rangle) = d(|0\rangle, |x; +\rangle) = d(|0\rangle, |y; +\rangle) = \sqrt{\hbar/2}, \]
\[ d(|x; +\rangle, |y; +\rangle) = \sqrt{\hbar}, \quad d(|x; +\rangle, |x; -\rangle) = \sqrt{2\hbar}, \]
\[ d \left( |x; +\rangle, \frac{I}{2} \right) = d \left( |y; +\rangle, \frac{I}{2} \right) = \sqrt{\hbar/2}, \]
\[ d \left( |0\rangle, \frac{I}{2} \right) = d \left( |1\rangle, \frac{I}{2} \right) = \frac{1}{2} \sqrt{\hbar/2}, \]  

(36)

where the states $|x; \pm\rangle$ and $|y; \pm\rangle$ are

\[ |x; \pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle), \quad |y; \pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm i|1\rangle). \]  

(37)

These spectral distances are depicted in Figure 1.

Figure 1: Connes spectral distances between one-qubit states in the Bloch sphere.

The Connes spectral distance between the mixed states

\[ \rho = p|0\rangle\langle 0| + (1 - p)|1\rangle\langle 1|, \quad \rho' = q|0\rangle\langle 0| + (1 - q)|1\rangle\langle 1| \]  

(38)
is
\[ d(\rho, \rho') = |p - q| \sqrt{\frac{\hbar}{2}}. \] (39)

This is just proportional to the quantum trace distance between the states \( \rho \) and \( \rho' \).

The quantum trace distances between qubits are equal to half the corresponding Euclidean distances in the Bloch representations. In general, the results of Connes spectral distances are quite different from those about the quantum trace distances of one-qubit states. Similar to the trace distance, one can use the Connes spectral distance to analyse the physical properties and relations of qubits, such as quantum discord and coherence.

As an example, let us use the Connes spectral distance to study the coherence of qubits. Quantum coherence is an important resource in quantum information sciences [21]. Similar to Refs. [21,22], one can define the coherence of a state \( \rho \) as follows,
\[ C_{SD}(\rho) = \sqrt{\frac{2}{\hbar}} \min_{\delta \in \mathcal{I}} d(\rho, \delta), \] (40)

where \( \mathcal{I} \) is the set of incoherent states. The elements in \( \mathcal{I} \) are just the diagonal states in a fixed basis. Using the results (34), one can find that, for the one-qubit state \( \rho^{(24)} \),
\[ C_{SD}(\rho) = \sqrt{x^2 + y^2}, \] (41)

and the nearest incoherent state is just \( \rho_{\text{diag}} \) with the Bloch vector \( \vec{r} = (0,0,z) \).

This is just the same as the \( l_1 \) norm of coherence [21] and also the trace norm of coherence defined in Ref. [22].

4 Connes spectral distances between two-qubit states

Next, let us consider the Connes spectral distances between two-qubit states. Similarly, one can construct the following Fock space
\[ \mathcal{F} = \text{span} \{ |m,n\rangle \equiv |m\rangle_1 \otimes |n\rangle_2, \ m,n = 0,1 \}, \] (42)

where \( |1\rangle_i = \hat{f}^\dagger_i |0\rangle_i, \ i = 1,2, \) and \( |0\rangle_i \) is the vacuum state. The creation and annihilation operators \( \hat{f}_1, \hat{f}_2 \) satisfy the commutation relations \( \{ \hat{f}_i, \hat{f}_1^\dagger \} = 1, \ \{ \hat{f}_i, \hat{f}_i \} = 0 \), and \( \{ \hat{f}_1, \hat{f}_2 \} = [\hat{f}_1^\dagger, \hat{f}_2^\dagger] = [\hat{f}_1, \hat{f}_2] = [\hat{f}_1^\dagger, \hat{f}_2^\dagger] = 0 \). We have the following matrix representation with respect to the basis \( \{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \} \),
\[ \hat{f}_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{f}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \] (43)

The corresponding quantum Hilbert space is constructed as follow,
\[ \mathcal{Q} = \text{span} \{ |m_1,n_1\rangle |m_2,n_2\rangle, \ m_1,n_1,m_2,n_2 = 0,1 \}. \] (44)
Using the methods similar to those in the 2D case above, one can construct a spectral triple \((\mathcal{A}', \mathcal{H}', \mathcal{D}')\) as follows,

\[
\mathcal{A}' = \mathcal{Q}, \quad \mathcal{H}' = \mathcal{F} \otimes \mathbb{C}^4, \quad \mathcal{D}' = i \sqrt{\frac{2}{\hbar}} \left( \begin{array}{cccc} 0 & 0 & \hat{f}_2^\dagger & \hat{f}_1^\dagger \\ 0 & 0 & \hat{f}_1 & -\hat{f}_2 \\ -\hat{f}_2 & -\hat{f}_1^\dagger & 0 & 0 \\ -\hat{f}_1 & \hat{f}_1^\dagger & 0 & 0 \end{array} \right). \tag{45}
\]

Now let us study the Connes spectral distances between two-qubit states in this noncommutative space. Usually, the calculations are much more cumbersome and complicated than those in the 2D case. For simplicity, here we only study the spectral distances between the states \(|ij\rangle\), \(i, j = 0, 1\).

First, let us consider the spectral distance between \(|00\rangle\) and \(|10\rangle\),

\[
d(|00\rangle, |10\rangle) = \sup_{e \in B} |\text{tr}_{\mathcal{F}}(\rho_{00} e) - \text{tr}_{\mathcal{F}}(\rho_{10} e)|
= \sup_{e \in B} |\langle 00 | e | 00 \rangle - \langle 10 | e | 10 \rangle|
= \sup_{e \in B} |\langle 00 | \hat{f}_1 | 10 \rangle - \langle 00 | \hat{f}_1 | 10 \rangle|
= \sup_{e \in B} |\langle 00 | [\hat{f}_1, e] | 10 \rangle|
\leq \| [\hat{f}_1, e] \|_{\text{op}}. \tag{46}
\]

Here we have used the Bessel’s inequality \cite{13}: For any operator \(A\) with the matrix elements \(A_{ij}\) in some orthonormal bases, there is

\[
|A_{ij}|^2 \leq \sum_i |A_{ij}|^2 \leq \|A\|_{\text{op}}^2. \tag{47}
\]

For a Hermitian element \(e \in \mathcal{A}'\), using the above Dirac operator \(\mathcal{D}'\), one can calculate the commutator \([\mathcal{D}', \pi(e)]\),

\[
[\mathcal{D}', \pi(e)] = i \sqrt{\frac{2}{\hbar}} \left( \begin{array}{c} M_1 \\ 0 \end{array} \right), \tag{48}
\]

where

\[
M_1 = \left( \begin{array}{cc} -[\hat{f}_2, e]^\dagger & -[\hat{f}_1, e]^\dagger \\ [\hat{f}_1, e] & -[\hat{f}_2, e] \end{array} \right). \tag{49}
\]

Since \(\|M_1^\dagger M_1\|_{\text{op}} = \|M_1 M_1^\dagger\|_{\text{op}}\), for any Hermitian element \(e\), using the ball condition, one can obtain the following inequality,

\[
\| [\mathcal{D}', \pi(e)] \|_{\text{op}}^2 = \| [\mathcal{D}', \pi(e)] [\mathcal{D}', \pi(e)]^\dagger \|_{\text{op}} = \frac{2}{\hbar} \| M_1 M_1^\dagger \|_{\text{op}} \leq 1. \tag{50}
\]

From the above expression \cite{49}, we have

\[
M_1 M_1^\dagger = \left( \begin{array}{c} [\hat{f}_1, e]^\dagger [\hat{f}_1, e] + [\hat{f}_2, e]^\dagger [\hat{f}_2, e] & [\hat{f}_1, e]^\dagger [\hat{f}_2, e]^\dagger - [\hat{f}_2, e]^\dagger [\hat{f}_1, e]^\dagger \\ -[\hat{f}_1, e][\hat{f}_2, e] + [\hat{f}_2, e][\hat{f}_1, e] & [\hat{f}_1, e][\hat{f}_1, e]^\dagger + [\hat{f}_2, e][\hat{f}_2, e]^\dagger \end{array} \right). \tag{51}
\]
By virtue of the Bessel’s inequality (47), one can obtain
\[
\sup_{\phi \in \mathcal{F}, \langle \phi | \phi \rangle = 1} \langle \phi | \hat{f}_1, e \rangle \hat{f}_1, e \rangle + \langle \hat{f}_2, e \rangle \hat{f}_2, e \rangle |\phi \rangle \leq \| M_1 \|_{op} \leq \frac{\hbar}{2}.
\]
(52)

Since \( \langle \phi | \hat{f}_1, e \rangle \hat{f}_1, e \rangle |\phi \rangle \geq 0 \), \( \langle \phi | \hat{f}_2, e \rangle \hat{f}_2, e \rangle |\phi \rangle \geq 0 \), we also have
\[
\sup_{\phi \in \mathcal{F}, \langle \phi | \phi \rangle = 1} \langle \phi | \hat{f}_1, e \rangle \hat{f}_1, e \rangle |\phi \rangle \leq \frac{\hbar}{2}, \quad \sup_{\phi \in \mathcal{F}, \langle \phi | \phi \rangle = 1} \langle \phi | \hat{f}_2, e \rangle \hat{f}_2, e \rangle |\phi \rangle \leq \frac{\hbar}{2},
\]
(53)
or
\[
\| \hat{f}_1, e \|_{op}^2 \leq \frac{\hbar}{2}, \quad \| \hat{f}_2, e \|_{op}^2 \leq \frac{\hbar}{2}.
\]
(54)

Similarly, we also have the following inequalities,
\[
\sup_{\phi \in \mathcal{F}, \langle \phi | \phi \rangle = 1} \langle \phi | \hat{f}_1, e \rangle \hat{f}_1, e \rangle + \langle \hat{f}_2, e \rangle \hat{f}_2, e \rangle |\phi \rangle \leq \frac{\hbar}{2},
\]
(55)

Combine the results (46) and (54), there must be
\[
\langle \langle 00 |, | 10 \rangle \rangle \leq \sqrt{\frac{\hbar}{2}}.
\]
(56)

Now we will find some optimal elements which can saturate the above inequality. Similar to the result in Ref. [19], one can firstly consider the elements \( e_\alpha \) being diagonal,
\[
e_\alpha = \begin{pmatrix} e_1 & 0 & 0 & 0 \\
0 & e_2 & 0 & 0 \\
0 & 0 & e_3 & 0 \\
0 & 0 & 0 & e_4 \end{pmatrix},
\]
(57)
where \( e_i \) are real numbers.

From the relation (55), using the above matrix representations (43), (49) and (57), after some straightforward calculations, one can obtain the following relations,
\[
(e_1 - e_2)^2 \leq \frac{\hbar}{2}, \quad (e_1 - e_3)^2 \leq \frac{\hbar}{2}, \quad (e_2 - e_4)^2 \leq \frac{\hbar}{2}, \quad (e_3 - e_4)^2 \leq \frac{\hbar}{2},
\]
\[
\left( |e_1 - e_2 - e_3 + e_4| + \sqrt{(e_1 - e_4)^2 + (e_2 - e_3)^2} \right)^2 \leq \hbar.
\]
(58)

Now the Connes spectral distance between the states \( |00 \rangle \) and \( |10 \rangle \) can be expressed as
\[
d(|00 \rangle, |10 \rangle) = \sup_{e \in B} |\langle 00 | e |00 \rangle - \langle 10 | e |10 \rangle | = \sup_{e \in B} |e_1 - e_3|.
\]
(59)
Using the relations (58), one can choose the following optimal element

\[
e_o^{(1)} = \begin{pmatrix}
\sqrt{\hbar} & 0 & 0 & 0 \\
0 & \sqrt{\hbar} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = \left( \sqrt{\hbar} \right) \otimes I_2,
\]

(60)

and then obtain

\[
d(|00\rangle, |10\rangle) = |\langle 00|e_o^{(1)}|00\rangle - \langle 10|e_o^{(1)}|10\rangle| = \sqrt{\hbar}
\]

(61)

Similarly, there is

\[
d(|01\rangle, |11\rangle) = \sqrt{\hbar},
\]

(62)

and \(e_o^{(1)}\) can still be the corresponding optimal element.

Using the same method, one can also obtain

\[
d(|00\rangle, |01\rangle) = d(|10\rangle, |11\rangle) = \sqrt{\hbar},
\]

(63)

and the corresponding optimal element can be chosen as

\[
e_o^{(2)} = \begin{pmatrix}
\sqrt{\hbar} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{\hbar} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = I_1 \otimes \left( \sqrt{\hbar} \right).
\]

(64)

It is easy to verify that, there are

\[
[\hat{f}_1, e_o^{(2)}] = 0, \quad [\hat{f}_2, e_o^{(1)}] = 0.
\]

(65)

Next, let us consider the spectral distance between \(|00\rangle\) and \(|11\rangle\),

\[
d(|00\rangle, |11\rangle) = \sup_{\epsilon \in B} |\text{tr}_F(\rho_{00}\epsilon) - \text{tr}_F(\rho_{11}\epsilon)|
\]

\[
= \sup_{\epsilon \in B} |\langle 00|\epsilon|00\rangle - \langle 11|\epsilon|11\rangle|
\]

\[
= \sup_{\epsilon \in B} |\langle 00|\epsilon|00\rangle - \langle 10|\epsilon|10\rangle + \langle 10|\epsilon|10\rangle - \langle 11|\epsilon|11\rangle|
\]

\[
= \sup_{\epsilon \in B} |\langle 00|\epsilon\hat{f}_1|10\rangle - \langle 00|\hat{f}_1\epsilon|10\rangle + \langle 10|\epsilon\hat{f}_2|11\rangle - \langle 10|\hat{f}_2\epsilon|11\rangle|
\]

\[
= \sup_{\epsilon \in B} |\langle 00|[\hat{f}_1, \epsilon]|10\rangle + \langle 10|[\hat{f}_2, \epsilon]|11\rangle|
\]

\[
\leq \sup_{\epsilon \in B} \sqrt{2} \sqrt{|\langle 00|[\hat{f}_1, \epsilon]|10\rangle|^2 + |\langle 10|[\hat{f}_2, \epsilon]|11\rangle|^2}.
\]

(66)

In the inequality above, we have also used the Cauchy-Schwartz inequality.
Since for any states $|\phi\rangle$, $|\varphi\rangle$, there is $|\langle \phi | [\hat{f}_i, e] |\varphi\rangle|^2 \geq 0$. It is easy to see that,

$$
|\langle \phi | [\hat{f}_1, e] |\varphi\rangle|^2 \leq |\langle 00 | [\hat{f}_1, e] |10\rangle|^2 + |\langle 01 | [\hat{f}_1, e] |10\rangle|^2 + |\langle 10 | [\hat{f}_1, e] |10\rangle|^2 + |\langle 11 | [\hat{f}_1, e] |10\rangle|^2
$$

(67)

Here we have used the resolution of the identity,

$$
\sum_{i,j=0,1} |ij\rangle \langle ij| = \mathbb{I}.
$$

(68)

Similarly, there is

$$
|\langle 10 | [\hat{f}_2, e] |11\rangle|^2 \leq |\langle 11 | [\hat{f}_2, e] |10\rangle|^2 \leq |\langle 10 | [\hat{f}_2, e] |11\rangle|.
$$

(69)

So by virtue of the inequalities (55), one can obtain

$$
|\langle 00 | [\hat{f}_1, e] |10\rangle|^2 + |\langle 10 | [\hat{f}_2, e] |11\rangle|^2 \leq |\langle 00 | [\hat{f}_1, e] |10\rangle|^2 + |\langle 01 | [\hat{f}_1, e] |10\rangle|^2 + |\langle 10 | [\hat{f}_1, e] |10\rangle|^2 + |\langle 11 | [\hat{f}_1, e] |10\rangle|^2 = |\langle 10 | [\hat{f}_1, e] |10\rangle|^2.
$$

(66)

$$
\sum_{i,j=0,1} |ij\rangle \langle ij| = \mathbb{I}.
$$

(68)

Similarly, there is

$$
|\langle 00 | [\hat{f}_1, e] |10\rangle|^2 + |\langle 11 | [\hat{f}_2, e] |10\rangle|^2 \leq |\langle 10 | [\hat{f}_2, e] |11\rangle|^2 \leq |\langle 10 | [\hat{f}_2, e] |10\rangle|^2 + |\langle 11 | [\hat{f}_2, e] |10\rangle|^2 + |\langle 10 | [\hat{f}_2, e] |10\rangle|^2 + |\langle 11 | [\hat{f}_2, e] |10\rangle|^2 = |\langle 10 | [\hat{f}_2, e] |10\rangle|^2.
$$

(70)

Combine the results (66) and (70), we have

$$
d(|00\rangle, |11\rangle) \leq \sqrt{\hbar}.
$$

(71)

For example, one can choose the following optimal element

$$
e_o = \begin{pmatrix}
\sqrt{\hbar} & 0 & 0 & 0 \\
0 & \frac{\sqrt{\pi}}{2} & 0 & 0 \\
0 & 0 & \sqrt{\pi} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = \frac{1}{\sqrt{2}} (e_o^{(1)} + e_o^{(2)}),
$$

(72)

and then obtain

$$
d(|00\rangle, |11\rangle) = |\langle 00 | e_o |00\rangle - \langle 11 | e_o |11\rangle| = \sqrt{\hbar}.
$$

(73)

Similarly, there is

$$
d(|01\rangle, |10\rangle) = \sqrt{\hbar},
$$

(74)

and the corresponding optimal element can be chosen as

$$
e_o = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{\sqrt{\pi}}{2} & 0 & 0 \\
0 & 0 & -\frac{\sqrt{\pi}}{2} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = \frac{1}{\sqrt{2}} (e_o^{(1)} - e_o^{(2)}).
$$

(75)

These distances are depicted in Figure 2. From Figure 2 one can see that these spectral distances satisfy the Pythagoras theorem. This is similar to the result in Ref. 9.
As a comparison, let us calculate the quantum trace distances $d_T$ between these states,

$$d_T(\rho_1, \rho_2) = \frac{1}{2} \text{tr}|\rho_1 - \rho_2|,$$

where $|A| := \sqrt{A^\dagger A}$. After some straightforward calculations, one can obtain

$$d_T(|ij\rangle, |kl\rangle) = 1, \quad i \neq k \text{ and/or } j \neq l. \quad (77)$$

We find that the Connes spectral distances between these two-qubit states are quite different from their trace distances. So one can use Connes spectral distances to measure the relations between the qubits. In some special cases, it should give some new results. In this sense, the Connes spectral distance can be considered as a useful supplement to the trace distances in quantum information sciences.

Similarly, one can use the above methods to study Connes spectral distances between $n$-qubit states in higher-dimensional noncommutative spaces.

5 Discussions and conclusions

In this paper, we study the Connes spectral distances between some one- and two-qubit states which can be represented by some fermionic Fock states. We construct a spectral triple corresponding to the $2D$ fermionic phase spaces, and calculate the Connes spectral distance between fermionic Fock states. We also study some simple cases about two-qubit states. In the simple cases, the spectral distances satisfy the Pythagoras theorem. These results are significant for the study of physical relations and geometric structures of qubits and other quantum states.

From the above results of the Connes spectral distances between qubits, one can find that the Connes spectral distances are quite different from quantum trace distances. One can use Connes spectral distances to measure the relations between the qubits. For example, one can use these distances to study the discord and coherence of qubits. In some special cases, it should give some new results. So we
believe that the Connes spectral distance can be used as a significant supplement to the trace distances in quantum information sciences. As an example, we use the Connes spectral distance to calculate the coherence measure of one-qubit states. We find that the Connes spectral distance can obtain the same result as the trace distance.

Furthermore, one can also study the spectral distances between other kinds of pure states and mixed states in higher-dimensional noncommutative spaces. But the calculations will be much more cumbersome and complicated. Our method used in the present work is different from those used in the literatures. We hope that our methods and results can help the studies of mathematical structures of noncommutative spaces and also physical properties of quantum systems.

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