RAINFALL SETS IN THE INTERSECTION OF TWO MATROIDS

RON AHARONI
Department of Mathematics, Technion, Haifa 32000, Israel

DANIEL KOTLAR
Computer Science Department, Tel-Hai College, Upper Galilee 12210, Israel

RAN ZIV
Computer Science Department, Tel-Hai College, Upper Galilee 12210, Israel

Abstract. Given sets $F_1, \ldots, F_n$, a partial rainbow function is a partial choice function of the sets $F_i$. A partial rainbow set is the range of a partial rainbow function. Aharoni and Berger [2] conjectured that if $M$ and $N$ are matroids on the same ground set, and $F_1, \ldots, F_n$ are pairwise disjoint sets of size $n$ belonging to $M \cap N$, then there exists a rainbow set of size $n - 1$ belonging to $M \cap N$. Following an idea of Woolbright and Brower-de Vries-Wieringa, we prove that there exists such a rainbow set of size at least $n - \sqrt{n}$.

1. Introduction

As in the abstract, a partial rainbow function of a family of sets $\mathcal{F} = (F_1, \ldots, F_n)$ is a partial choice function. A partial rainbow set is the range of a rainbow function, so it is a set consisting of at most one element from each $F_i$, where repeated elements are considered distinct (so, in this terminology a rainbow set is in fact a multiset). A full rainbow set, in which elements are chosen from all $F_i$, is called plainly a rainbow set. Strengthening a conjecture of Brualdi and Stein [4,16], Aharoni and Berger [2] made the following conjecture:

Conjecture 1.1. $n$ matchings of size $n + 1$ in a bipartite graph have a rainbow matching (namely, a rainbow set that is a matching).

This conjecture easily implies:

E-mail addresses: ra@tx.technion.ac.il, dannykot@telhai.ac.il, ranziv@telhai.ac.il.
Conjecture 1.2. \( n \) matchings of size \( n \) in a bipartite graph have a partial rainbow matching of size \( n - 1 \).

The Brualdi-Stein conjecture is that every Latin square of order \( n \) possesses a partial transversal of size \( n - 1 \), namely \( n - 1 \) entries lying in different rows and columns, and containing different symbols. (This is a natural variation on a conjecture of Ryser [14], that an odd Latin square has a full transversal). Each of the \( n \) rows of a Latin square can be considered in a natural way as a matching of size \( n \) between columns and symbols, and applying Conjecture 1.2 to these matchings yields the Brualdi-Stein conjecture.

Lower bounds of order \( n - o(n) \) were proved in both problems. Hatami and Shor [8] proved that in a Latin square of order \( n \) there exists a partial transversal of size at least \( n - 11.053 \log^2 n \). Woolbright [21] and independently Brouwer, de Vries and Wieringa [3] proved (in effect) that \( n \) matchings in a bipartite graph have a partial rainbow matching of size at least \( n - \sqrt{n} \).

Aharoni and Berger [2] extended Conjecture 1.2 to matroids, as follows:

Conjecture 1.3. Let \( M \) and \( N \) be two matroids on the same vertex set. Any \( n \) pairwise disjoint sets of size \( n \), belonging to \( M \cap N \), have a partial rainbow set of size \( n - 1 \) belonging to \( M \cap N \).

Conjecture 1.2 is the special case where both \( M \) and \( N \) are partition matroids. (Here the term partition matroid refers to a direct sum of uniform matroids, each of rank 1.) The aim of this paper is to prove the parallel of the Woolbright-Brower-de Vries-Wieringa result for Conjecture 1.2. We shall prove:

Theorem 1.4. Any \( n \) pairwise disjoint sets of size \( n \) belonging to \( M \cap N \) have a partial rainbow set of size at least \( n - \sqrt{n} \) belonging to \( M \cap N \).

2. Matroid preliminaries

Throughout the paper we shall use the notation \( A + x \) for \( A \cup \{x\} \) and \( A - x \) for \( A \setminus \{x\} \).

Recall that a collection \( M \) of subsets of a set \( S \) is a matroid if it is hereditary and it satisfies an augmentation property: If \( A, B \in M \) and \( |B| > |A| \), then there exists \( x \in B \setminus A \) such that \( A + x \in M \). Sets in \( M \) are called independent and sets not belonging to \( M \) are called dependent. A maximal independent set is called a basis. An element \( x \in S \) is spanned by \( A \) if either \( x \in A \) or \( I + x \notin M \) for some independent set \( I \subseteq A \). The set of elements that are spanned by \( A \) is denoted by \( \text{sp}(A) \), or \( \text{sp}_M(A) \) if the identity of the matroid \( M \) is not clear from the context. A circuit is a minimal dependent set. We shall use some basic facts on matroids, that can be found, for example, in the books of Oxley [13] and Welsh [20].

Fact 2.1. If \( I \) is independent and \( I + x \) is dependent, then there exists a unique minimal subset \( C_M(I, x) \) of \( I \) that spans \( x \).

We shall call \( C_M(I, x) \) the \( M \)-support of \( x \) in \( I \).

Fact 2.2. Let \( A \in M \), \( x \in \text{sp}(A) \), and \( a \in C_M(A, x) \). Then \( A + x - a \in M \) and \( \text{sp}(A + x - a) = \text{sp}(A) \).
Lemma 3.1. If $C_1$ and $C_2$ are circuits with $e \in C_1 \cap C_2$ and $f \in C_1 \setminus C_2$ then there exists a circuit $C_3$ such that $f \in C_3 \subseteq (C_1 \cup C_2) - e$.

The following is an immediate corollary of the augmentation property:

Fact 2.4. Let $I, J$ be independent sets in $\mathcal{M}$. If $|I| < |J|$, then there exists $J_1 \subseteq J \setminus I$ such that $I \cup J_1 \in \mathcal{M}$ and $|I \cup J_1| = |J|$.

Definition 2.5. Let $\mathcal{M}$ and $\mathcal{N}$ be two matroids on the same ground set $S$. We call a set $F \subseteq S$ an independent matching if $F \in \mathcal{M} \cap \mathcal{N}$. A rainbow set for a family $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$ of independent matchings is called a rainbow independent matching if it belongs to $\mathcal{M} \cap \mathcal{N}$.

3. Proof of Theorem 1.4

Let $\mathcal{F} = (F_1, \ldots, F_n)$ be a family of disjoint sets belonging to $\mathcal{M} \cap \mathcal{N}$. Let $R$ be a partial rainbow matching for $\mathcal{F}$ of maximal size. Let $t = |R|$ and $\delta = n - t$. Without loss of generality we may assume that $|R \cap F_i| = 1$ for $i = 1, \ldots, t$.

We shall construct a sequence of sets $(A_1, \ldots, A_\delta)$ such that for all $i = 1, \ldots, \delta$ the following holds:

\begin{align*}
(3.1) & \quad A_i \subseteq F_{t+i}, \\
(3.2) & \quad A_i \subseteq \text{sp}_M(R), \\
(3.3) & \quad |A_i| \geq i\delta.
\end{align*}

Suppose that we succeed in constructing such a sequence. By (3.1) $A_\delta \in \mathcal{M}$ and by (3.2) $A_\delta \subseteq \text{sp}_M(R)$. By (3.3), applied to $i = \delta$, we therefore have $t = |R| \geq |A_\delta| \geq \delta^2$. Clearly, we may assume that $t < n$. Since $\delta = n - t$, it follows that $t > n - \sqrt{n}$, as stated in the theorem.

Construction of the sets $A_i$. We construct the sets $A_i$ inductively, associating with them sets $R_i$, so that $R_1, \ldots, R_\delta$ are disjoint, $R_i \subseteq R$ and $|R_i| \geq \delta$ for all $i = 1, \ldots, \delta$. Since $|F_{t+i}| = n$ and $|R| = t$, there exists, by Fact 2.4, a set $A_1 \subseteq F_{t+i} \setminus R$ such that $|A_1| = \delta$ and $R \cup A_1 \in \mathcal{N}$. By the maximality property of $R$ we have $A_1 \subseteq \text{sp}_M(R)$. Since $|A_1| = \delta$ and $|R| = t$, there exists, again by Fact 2.4, a subset $R' \subseteq R$ of size $t - \delta$ such that $A_1 \cup R' \in \mathcal{M}$ and $|A_1 \cup R'| = t$. Let $R_1 = R \setminus R'$. We have $R \setminus R_1 \cup A_1 \in \mathcal{M}$ and $|R_1| = \delta$.

For the inductive step, assume that $R_1, R_2, \ldots, R_j$ are pairwise disjoint subsets of $R$, each of size at least $\delta$, and $A_1, A_2, \ldots, A_j$ satisfy the conditions (3.1), (3.2) and (3.3), for $i = 1, \ldots, j$. Denote $R^k = R \setminus \cup_{i=1}^{k-1} R_i$ for $k = 2, \ldots$. Notice that $|R^{i+1}| \leq t - j\delta$. Since $F_{t+j+i} \in \mathcal{N}$ and $|F_{t+j+i}| = n$ it follows from Fact 2.4 that there exists $A_{j+1} \subseteq F_{t+j+i} \setminus R$ such that $R^{i+1} \cup A_{j+1} \in \mathcal{N}$ and $|R^{i+1} \cup A_{j+1}| = n$. We have $|A_{j+1}| = n - |R^{i+1}| \geq n - (t - j\delta) = (j+1)\delta$. We see that $A_{j+1}$ satisfies (3.1) and (3.3). The following lemma implies that (3.2) also holds for $A_{j+1}$.

Lemma 3.1. If $j < \delta$ then $A_{j+1} \subseteq \text{sp}_M(R)$.
Before proving Lemma 3.1 let us indicate how it is used to complete the inductive construction of $R_{j+1}$. We use the following observation:

**Observation 3.2.** Let $I$ be an independent set of size $t$ in a matroid $\mathcal{M}$ and suppose $J \subseteq \text{sp}(I)$ has size $n > t$. If $K \subseteq J$ satisfies $J \setminus K \in \mathcal{M}$, then $|K| \geq n - t$.

Assuming Lemma 3.1 we have (*a*) $A_{j+1} \subseteq \text{sp}_\mathcal{M}(R)$. We also have $|R^{j+1} \cup A_{j+1}| = n = |R| + \delta$. Hence $|R^{j+1}| \geq \delta$ (If $|R^{j+1}| < \delta$ then $|A_{j+1}| > |R|$, contradicting (*a*). Let $R_{j+1} \subseteq R^{j+1}$ be of minimal size such that $R^{j+1} \setminus R_{j+1} \cup A_{j+1} \in \mathcal{M}$. By Observation 3.2 we have $|R_{j+1}| \geq \delta$, as required.

The proof of Lemma 3.1 is done by an alternating path argument.

**Definition 3.3.** A **colorful alternating path** (CAP) of length $k$, relative to $R$, consists of

(i) A set $\{b_0, b_1, \ldots, b_k\}$ of distinct elements of the ground set $S$, where each $b_i$ belongs to some $A_j \in \mathcal{A}$ and distinct $b_i$’s belong to distinct $A_j$’s.

(ii) A set of distinct elements $\{r_1, \ldots, r_k\} \subseteq R$ such that

$$\text{sp}_\mathcal{M}(R - r_1 + b_1 - r_2 + b_2 - \cdots - r_k + b_k) = \text{sp}_\mathcal{M}(R).$$

$$\text{sp}_{\mathcal{N}}(R + b_0 - r_1 + b_1 - r_2 - \cdots - r_k + b_k) = \text{sp}_{\mathcal{N}}(R).$$

If, in addition, $R + b_0 - r_1 + b_1 - r_2 + b_2 - \cdots - r_k + b_k \in \mathcal{M} \cap \mathcal{N}$ then the CAP is called augmenting.

Since the $b_i$’s belong to distinct $F_{i+1}$’s we have:

**Observation 3.4.** If $R$ is of maximal size then no augmenting CAP relative to $R$ exists.

In order to extend our alternating path we shall need the following lemma:

**Lemma 3.5.** Let $\mathcal{M}$ be a matroid. Let $I \in \mathcal{M}$ and $X = \{x_1, \ldots, x_k\} \subseteq I$ and $Y = \{y_1, \ldots, y_k\} \subseteq \text{sp}_\mathcal{M}(I) \setminus I$ be such that $\text{sp}_\mathcal{M}((I \setminus X) \cup Y) = \text{sp}_\mathcal{M}(I)$. Suppose $y_{k+1} \in \text{sp}_\mathcal{M}(I) \setminus I$ and $x_{k+1}$ are such that $x_{k+1} \in C(I, y_{k+1}) \setminus X$ and $x_{k+1} \notin C(I, y_i)$ for all $i = 1, \ldots, k$. Then $x_{k+1} \in C(I \setminus X) \cup Y, y_{k+1})$.

**Proof of Lemma 3.5.** Suppose, for contradiction, that $x_{k+1} \notin C(I \setminus X) \cup Y, y_{k+1})$. Let $C' = C(M, y_{k+1}) + y_{k+1}$ and $C'' = C(I, y_{k+1}) + y_{k+1}$. Then, by Fact 2.3 there exits a circuit $C \subseteq C' \cup C''$, such that $x_{k+1} \in C$ and $y_{k+1} \notin C$. Choose such a circuit $C$ with $|C \cap Y|$ minimal. Since $I$ is independent $C$ must contain at least one element $y_j \in Y \cap C''$. Using Fact 2.3 again, since $x_{k+1} \notin C(M, y_j)$, there exists a circuit $\tilde{C} \subseteq C \cup (C(M, y_j) + y_j)$ such that $x_{k+1} \in \tilde{C}$ and $y_j \notin C$. We have $|\tilde{C} \cap Y| < |C \cap Y|$, contradicting the minimality property of $C$. □

**Proof of Lemma 3.7.** We shall show how the existence of some $i$, $1 \leq i \leq \delta$, such that $A_i \not\subseteq \text{sp}_\mathcal{M}(R)$ yields an augmenting CAP relative to $R$. This will contradict the maximality of $R$, by Observation 3.4.

Let $\{A_i\}, \{R_i\}$ and $\{R^i\}$ be defined as above. Recall that for all $i = 1, \ldots, \delta$,

$$R^i = R \setminus \bigcup_{j=1}^{i-1} R_j.$$


(3.5) \( A_i \subseteq F_{i+1} \) satisfies \( R^i \cup A_i \in \mathcal{N} \) and \( |R^i \cup A_i| = n \) and

(3.6) \( R_i \subseteq R^i \) is of minimal size such that \( R^i \setminus R_i \cup A_i \in \mathcal{M} \).

Assume, for contradiction, that \( m, 1 \leq m \leq \delta \), is the minimal index such that \( A_m \not\subseteq \text{sp}_\mathcal{M}(R) \) and let \( a \in A_m \) be such that \( R + a \in \mathcal{M} \). We shall construct a CAP, relative to \( R \), starting from \( a \). Let \( b_0 = a \). We have

(3.7) \( R + b_0 \in \mathcal{M} \)

and, since no augmenting CAP relative to \( R \) exists, we must have \( b_0 \in \text{sp}_\mathcal{N}(R) \). Let \( j \) be the maximal index such that \( b_0 \in \text{sp}_\mathcal{N}(R'^j) \). Since \( b_0 \in A_m \) and, by (3.5), \( R^m \cup A_m \in \mathcal{N} \), we obtain \( b_0 \notin \text{sp}_\mathcal{N}(R^m) \). Thus, \( j < m \). Since \( R_j = R^j \setminus R^{j+1} \), it follows from the maximality of \( j \) that \( C_\mathcal{N}(R^j, b_0) \cap R_j \neq \emptyset \). By Fact 2.2 there exists \( r_1 \in R_j \) such that \( R + b_0 - r_1 \in \mathcal{N} \) and

(3.8) \( \text{sp}_\mathcal{N}(R + b_0 - r_1) = \text{sp}_\mathcal{N}(R) \).

Since \( j < m \), we have, by the minimality of \( m \), that \( A_j \subseteq \text{sp}_\mathcal{M}(R) \). By the minimality of \( R_1 \) (see (3.6)) there exists \( x \in A_j \) such that \( r_1 \in C_\mathcal{M}(R, x) \) (otherwise \( A_j \cup R^{j+1} + r_1 \in \mathcal{M} \)). Let \( l \leq j \) be minimal such that \( A_l \) contains an element \( b_1 \) satisfying \( r_1 \in C_\mathcal{M}(R, b_1) \). By Fact 2.2 we have \( R - r_1 + b_1 \in \mathcal{M} \) and \( \text{sp}_\mathcal{M}(R - r_1 + b_1) = \text{sp}_\mathcal{M}(R) \). This, combined with (3.7), implies that \( R + b_0 - r_1 + b_1 \in \mathcal{M} \). Thus, a CAP of length 1 was created.

Now, suppose that we managed to construct a CAP of length \( k \). We shall show that if the CAP is not augmenting, then it can be extended. Denote \( R_\mathcal{M}(k) = R - r_1 + b_1 - r_2 + b_2 - \cdots - r_k + b_k \) and \( R_\mathcal{N}(k) = R + b_0 - r_1 + b_1 - r_2 + \cdots + b_{k-1} - r_k \).

Note that

(3.9) \( R_\mathcal{M}(k) + b_0 = R_\mathcal{N}(k) + b_k \).

Claim 1. \( b_k \in \text{sp}_\mathcal{N}(R) \).

Proof of Claim 1. By (P.\( R_\mathcal{M} \)), we have \( \text{sp}_\mathcal{M}(R_\mathcal{M}(k)) = \text{sp}_\mathcal{M}(R) \). Hence, from (3.7) we have \( R_\mathcal{M}(k) + b_0 \in \mathcal{M} \). Also, by (P.\( \mathcal{N} \)), we have \( \text{sp}_\mathcal{N}(R_\mathcal{N}(k)) = \text{sp}_\mathcal{N}(R) \). Assume, for contradiction, that \( R + b_k \in \mathcal{N} \). Then, \( R_\mathcal{N}(k) + b_k \in \mathcal{N} \), and by (3.9) we obtain an augmenting CAP, contradicting the maximality property of \( R \).

Assuming Claim 1, let \( p \) be the maximal index such that \( b_k \in \text{sp}_\mathcal{N}(R^p) \). By (3.4), \( p \) is the minimal index such that \( C_\mathcal{N}(R, b_k) \cap R_p \neq \emptyset \). Let \( r_{k+1} \in C_\mathcal{N}(R, b_k) \cap R_p \). By Fact 2.2 \( R + b_k - r_{k+1} \in \mathcal{N} \) and \( \text{sp}_\mathcal{N}(R + b_k - r_{k+1}) = \text{sp}_\mathcal{N}(R) \).

Claim 2. Let \( q \) be the index such that \( b_k \in A_q \). Then, \( p < q \).

Proof of Claim 2. By (3.5), \( R^q \cup A_q \in \mathcal{N} \) and thus, \( b_k \notin \text{sp}_\mathcal{N}(R^q) \). Since \( b_k \in \text{sp}_\mathcal{N}(R^p) \) we conclude that \( R^q \subseteq R^p \), which implies that \( p < q \).

Claim 3. There exists \( x \in A_p \) such that \( r_{k+1} \in C_\mathcal{M}(R, x) \).

Proof of Claim 3. Assume the opposite. Then \( A_p \cup R^{p+1} + r_{k+1} \in \mathcal{M} \). This contradicts the maximality property of \( R_p \) (see (3.6)).
Let $l$ be minimal such that $A_l$ contains an element $b_{k+1}$ satisfying $r_{k+1} \in C_M(R, b_{k+1})$. By Claim 3, $l \leq p$. This, together with Claim 2, yields

\begin{equation}
(3.10) \quad \text{if } b_i \in A_u \text{ and } b_j \in A_v \text{ with } i < j, \text{ then } v < u,
\end{equation}

and

\begin{equation}
(3.11) \quad \text{if } r_i \in R_u \text{ and } r_j \in R_v \text{ with } i < j, \text{ then } v < u.
\end{equation}

Claim 4. $r_{k+1} \notin C_N(R, b_i)$ for all $i = 0, \ldots, k - 1$.

**Proof of Claim 4.** Let $j \in \{1, \ldots, k\}$. In the construction described above, the element $r_j$ was chosen from $R_u$, where $u$ is minimal such that $C_N(R, b_{j-1}) \cap R_u \neq \emptyset$. Recall that $r_{k+1} \in R_p$. Thus, by (3.11), we have $p < u$, and hence $C_N(R, b_{j-1}) \cap R_p = \emptyset$, which implies the claim.

By applying Lemma 3.3 to the sequences $\{r_1, \ldots, r_k, r_{k+1}\}$ and $\{b_0, \ldots, b_{k-1}, b_k\}$, it follows that $r_{k+1} \in C_N(R_N(k), b_k)$. By Fact 2.2, it follows that

\begin{equation}
(3.12) \quad R_N(k) + b_k - r_{k+1} \in N, \text{ and }
sp_N(R_N(k) + b_k - r_{k+1}) = sp_N(R_N(k)) = sp_N(R).
\end{equation}

Claim 5. $r_{k+1} \in C_M(R_M(k), b_{k+1})$.

**Proof of Claim 5.** Let $i \in \{1, \ldots, k\}$. In the construction described above, the element $b_i$ was chosen from $R_u$, where $u$ is minimal such that $A_u$ contains an element $b_i$ such that $r_i \in C_M(R, b_i)$. Recall that $b_{k+1}$ was chosen from $A_i$, and by (3.10), $l < u$. Thus, $r_i \notin C_M(R, b_{k+1})$. Since this is true for any $i \in \{1, \ldots, k\}$, we have $C_M(R, b_{k+1}) \cap \{r_1, \ldots, r_k\} = \emptyset$, and hence, $C_M(R_M(k), b_{k+1}) = C_M(R, b_{k+1})$. Since $b_{k+1}$ was chosen so that $r_{k+1} \in C_M(R, b_{k+1})$, the claim follows.

Assuming Claim 5, by Fact 2.2 we have

\begin{equation}
(3.13) \quad R_M(k) + b_{k+1} - r_{k+1} \in M, \text{ and }
sp_M(R_M(k) + b_{k+1} - r_{k+1}) = sp_M(R_M(k)) = sp_M(R).
\end{equation}

By $\text{(P}_M\text{)}, (\text{P}_N\text{)}, (3.12)$ and (3.13), the CAP was extended to the length of $k + 1$.

By (3.10) and (3.11), the process must end, yielding an augmenting CAP. This completes the proof of Lemma 3.1 and hence of Theorem 1.4.

4. **Independent partial transversals in Matroidal Latin Squares**

Let $\mathcal{M}$ be matroid of rank $n$ defined on a ground set $S$. A *Matroidal Latin Square (MLS)* of degree $n$ over $\mathcal{M}$ was defined in [10] as an $n \times n$ matrix whose entries are from $S$, such that each row and column is a basis of $\mathcal{M}$. (After publication, the authors found out that a similar object had been introduced earlier by Chappell [5].) Note that the notion of MLS generalizes the notion of Latin square, as a Latin square is an MLS over a partition matroid (that is, a direct sum of uniform matroids, each of rank 1). Analogously to Norton’s definition for row Latin square
in [12], we define a row MLS, as an $n \times n$ matrix whose entries are from $S$, such that each row is a basis of $M$. Thus, every MLS is a row MLS.

An independent partial transversal in an MLS, or in a row MLS, $A$, is an independent subset of entries of $A$ where no two of them lie in the same row or column of $A$. It was conjectured in [10] that every MLS of degree $n$ has an independent partial transversal of size $n - 1$. It was shown there that, in general, we cannot expect to find a partial independent transversal of size $n$. The lower bound set in [10] for the size of a partial independent transversal in an MLS was $\lceil 2n/3 \rceil$. Theorem 1.4 yields a significant improvement for that bound:

**Corollary 4.1.** Every row MLS of degree $n$ has an independent partial transversal of size at least $n - \sqrt{n}$.

**Proof.** Let $A$ be a row MLS of degree $n$ over a matroid $M$. The result follows from Theorem 1.4 by taking $N$ as the partition matroid defined by the columns of $A$. □

**Acknowledgments**

The authors thank two anonymous referees for their insightful comments and for their substantial contribution to the clarity of the manuscript.

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