Chowla and Sarnak conjectures for Kloosterman sums

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Abstract

We formulate several analogs of the Chowla and Sarnak conjectures, which are widely known in the setting of the Möbius function, in the setting of Kloosterman sums. We then show that for Kloosterman sums, in some cases, these conjectures can be established unconditionally.

KEYWORDS

Chowla conjecture, Kloosterman sum, Sarnak conjecture

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1 INTRODUCTION

1.1 Background and motivation

Recently, there has been a lot of activity related to the Chowla and Sarnak conjectures for the Möbius function $\mu(n)$. We recall that these conjectures assert noncorrelation between shifted values of $\mu(n)$ and between $\mu(n)$ and low complexity sequences, respectively, see [1–4, 9, 20, 26, 29, 44, 47–49, 51, 53, 54, 61–64] and references therein. Both conjectures are special cases of the general Möbius Randomness Law, formulated, for example, in [31, section 13.1].

Here, we introduce and investigate similar conjectures for Kloosterman sums,

$$K_m(a) = \sum_{\substack{x=1 \\ \gcd(a,m) = 1}}^m e_m(ax + \bar{x}),$$

where $\bar{x}$ is the multiplicative inverse of $x$ modulo an integer $m \geq 1$ and

$$e_m(z) = \exp(2\pi iz/m),$$

see [31, eq. (1.56)]. There are however several important distinctions, which may affect their difficulty (perhaps adversely) compared with their analogs for the Möbius function. These are

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(i) lack of multiplicativity between Kloosterman sums and hence the approach of [9, 13, 32] does not apply;
(ii) a dense set of their values rather than a discrete set \{-1, 0, 1\} as in the case of the Möbius function.

Nevertheless, we establish them in several special cases.

In fact, some special instances of what one may call Kloosterman Randomness Law is already known, see Section 2.1 below. Moreover, in some cases, for Kloosterman sums, we are able to establish results, which are superior to those known for the Möbius function.

\[ \sum_{n=1}^{N} b_n K_m(n) \text{ and } \sum_{n=1}^{N} K_m(n + h_1) \cdots K_m(n + h_s) \]

are small compared to their trivial bound \( Nm^{o(1)} \), which follows from \( (1.1) \), for some “interesting” (bounded) arithmetic sequence \( b_n, n = 1, \ldots, N, \) and some integer shifts \( 0 \leq h_1 < \ldots < h_s < q \); see Lemma 3.3, due to Fouvry, Kowalski, and Michel [22, Corollary 1.6], for one of such results.

It is especially interesting to estimate the above sums with power saving and obtain estimates of the type \( O(M^{1-\eta}) \) and \( O(NM^{-\eta}) \), respectively, for some positive constant \( \eta > 0 \).
1.3 | Notation

Throughout the paper, the notation \( U = O(V) \), \( U \ll V \), and \( V \gg U \) are equivalent to \( |U| \leq c|V| \) for some positive constant \( c \), which throughout the paper may depend on the degree \( d \) and occasionally on the small real positive parameters \( \varepsilon \) and \( \delta \).

For any quantity \( V > 1 \), we write \( V^{o(1)} \) (as \( V \to \infty \)) to indicate a function of \( V \), which does not exceed \( V^\varepsilon \) for any \( \varepsilon > 0 \), provided \( V \) that is large enough. The conjunct notation \( U \ll V^{o(1)} \) is thus subsequently to be interpreted as for any \( \varepsilon \), there is a \( V_0(\varepsilon) > 1 \) and a \( c = c(\varepsilon) > 0 \), depending only on \( \varepsilon \) such that \( U \leq c(\varepsilon)V^\varepsilon \) for all \( V \geq V_0(\varepsilon) \).

More generally, when we say that a certain parameter is \emph{fixed}, this means that we allow all implied constants to depend on this parameter.

As usual, we use \( \mu(m) \) to denote the Möbius function, that is, \( \mu(m) = (-1)^s \) when \( m \) is a product of \( s \) distinct primes and \( \mu(m) = 0 \) otherwise.

We further simply write \( e(z) = \exp(2\pi i z) \) for \( e_1(z) \).

We use \# \( A \) to denote the cardinality of a set \( A \).

Throughout the paper, \( p \) always denotes a prime number.

We always follow the following conventions:

(1) When we write \( \mathcal{K}_m(a) \), we always assume that \( a \) is \emph{fixed} and thus the implied constant in “\( \ll \)” and similar expressions may depend on \( a \) (and hence (1.1) simply implies \( |\mathcal{K}_m(a)| \leq m^{o(1)} \)).

(2) When we write \( \mathcal{K}_m(n) \), or \( \mathcal{K}_m(n + h) \), we assume that \( n \) is a parameter varying in some interval \([1, N]\), where \( N \) may grow as fast as some power of \( m \).

Given a number-theoretic function \( \xi : \mathbb{N} \to \mathbb{C} \), we define the \emph{horizontal} averages

\[
H_{\xi, \mathcal{K}}(M) = \sum_{m=1}^{M} \xi(m) \mathcal{K}_m(a) \tag{1.3}
\]

and the \emph{vertical} averages

\[
V_\xi(m; N) = \sum_{n=1}^{N} \xi(n) \mathcal{K}_m(n). \]

These are our main objects of study.

It is also convenient to define the sums

\[
\overline{H}_\xi(M) = \sum_{m=1}^{M} |\mathcal{K}_m(a)| \quad \text{and} \quad \overline{V}(m; N) = \sum_{n=1}^{N} |\mathcal{K}_m(n)|, \tag{1.4}
\]

which we are going to use as benchmarks for our estimates on \( H_{\xi, \mathcal{K}}(M) \) and \( V_\xi(m; N) \).

It is certainly natural to expect that

\[
\overline{H}_\xi(M) = M^{1+o(1)} \quad \text{and} \quad \overline{V}(m; N) = N^{1+o(1)}
\]

in wide range of parameters. For example, by [23, Theorem 1.2], for any fixed \( r \), we have

\[
M \left( \frac{\log \log M}{\log M} \right)^r \ll \overline{H}_1(M) \ll M \left( \frac{\log \log M}{\log M} \right)^{2-8/(3r)} \left( \frac{\log M}{\log M} \right)^{1-8/(3r)},
\]

and perhaps a similar result also holds for \( \overline{H}_\xi(M) \) for any fixed integer \( a \neq 0 \).
The bound (1.1) further suggests to define
\[ \mathcal{K}_m^r(a) = \frac{|\mathcal{K}_m(a)|}{2^\omega(m)}, \]
where \( \omega(n) \) is the number of distinct prime factors in \( n \). Thus, in particular, for an integer \( a \neq 0 \),
\[ \mathcal{K}_m^r(a) \leq 2^{3/2}. \]

Analogously to (1.3) and (1.4), we define \( H_{a,\xi}^r(M) \) and \( \overline{H}_{a}^r(M) \). According to Fouvry and Michel [23, Theorem 1.1], for any fixed \( r \geq 1 \), we have
\[ M\left(\frac{\log \log M}{\log M}\right)^r \ll H_{1}^r(M) \ll M\left(\frac{\log \log M}{\log M}\right)^{1-4/(3\pi)} . \]

2 MAIN RESULTS

2.1 Previous results

When it comes to horizontal randomness, only a few techniques have been successfully applied. The one that stands out is, of course, the use of Kuznetsov’s trace formula. Kuznetsov [42] developed said formula to prove a strong bound toward the Linnik–Selberg conjecture on sums of Kloosterman sums (see also [31, section 16.1])
\[ H_{a,1}(M) \ll M^{2/3+\omega(1)}, \quad a \geq 1, \text{ fixed}, \]
where \( \mathbf{1} \) indicates the constant weight \( \xi(m) = 1 \). Similar results stemming from more general Kuznetsov formulas were subsequently derived, see, for example, [8, 14, 16, 17, 24, 27, 36, 59]. These cover horizontal averages against sequences of the type \( \xi(m) = 1 \) if \( q \mid m \) and \( \xi(m) = 0 \) otherwise, for a fixed integer \( q; \xi = \chi \), a fixed Dirichlet character, or mixtures thereof. In the direction of vertical randomness, several results and techniques are known, often in the most interesting case of prime \( m = p \).

1 Correlations between shifted values: For a prime \( m = p \), general results of Fouvry, Kowalski, and Michel [22, Corollary 1.6] contain as a special case a bound on the second sum in (1.2). Furthermore, Fouvry, Michel, Rivat, and Sárközy [25, Theorem 1.1] have estimated a variant of the second sum in (1.2) with the product of the sign-functions \( \text{sign} \mathcal{K}_p(n + h_1) \cdots \text{sign} \mathcal{K}_p(n + h_s) \) instead of the sums themselves.

2 Correlations with some arithmetic functions: Fouvry, Kowalski, and Michel [21, Theorem 1.7] and Kowalski, Michel, and Sawin [41, Corollary 1.4] have given bounds
\[ V_{\mu}(m; N) \ll N^{1-\eta} \quad \text{and} \quad V_{\tau}(m; N) \ll N^{1-\eta} \]
with the Möbius \( \mu(n) \) and divisor \( \tau(n) \) functions provided that for some fixed \( \varepsilon > 0 \), we have
\[ N \geq p^{3/4+\varepsilon} \quad \text{and} \quad N \geq p^{2/3+\varepsilon}, \]
respectively, where \( \eta > 0 \) depends only on \( \varepsilon > 0 \). Perhaps the argument of the proof of [7, Theorem 1.8] can be used to improve the dependence \( \eta \) on \( \varepsilon \) for sums with \( \mu(n)\mathcal{K}_p(n) \) and thus improve the bound of [21, Theorem 1.7], however it is not likely to help to extend any of the above ranges. Korolev and Shparlinski [38, Theorems 2.1 and 2.2], using a different method, have obtained nontrivial bounds on both sums already for
\[ N \geq p^{1/2+\varepsilon}, \]
however the saving is only logarithmic. All these methods also apply in broader generality to sums with other arithmetic functions.
It is also quite plausible that the results and methods of [45, 46] are able to produce estimates on similar sums modulo a prime power \( m = p^k \) for a fixed \( p \) and growing \( k \), and in fact it is reasonable to expect that these results are nontrivial starting with very small values of \( N \), for example, for \( N \geq p^\varepsilon \).

(3) **Correlations with some digital functions:** For any integers \( 0 \leq s \leq r \), let \( \chi_{r,s}(n) \) be the characteristic function of the set \( G_s(r) \) of \( r \)-bit integers with only \( s \) nonzero binary digits, thus
\[
\# G_s(r) = \binom{r}{s}.
\]

Korolev and Shparlinski [38, section 9] have shown that if
\[
2^r = p^{1+o(1)} \quad \text{and} \quad r/2 \geq s \geq (\rho_0 + \delta)r,
\]
where \( \rho_0 = 0.11002786 \ldots \) is the root of the equation
\[
H(\vartheta) = 1/2, \quad 0 \leq \vartheta \leq 1/2,
\]
with the binary entropy function
\[
H(\gamma) = \frac{-\gamma \log \gamma - (1-\gamma) \log(1-\gamma)}{\log 2},
\]
then, for any \( \varepsilon > 0 \), there exists \( \eta > 0 \) such that,
\[
\sum_{n=1}^{2^r-1} \mathcal{K}_p(n)\chi_{r,s}(n) \ll \binom{r}{s}^{1-\eta}.
\]

(4) **Correlations with automatic sequences:** Let \( M_b \) be the free monoid with generators \( \{0, 1, \ldots, b-1\} \) acting from the right on a finite set \( \mathcal{X} \). By considering the base \( b \) expansion of an integer, we may embed \( \iota : \mathbb{N} \rightarrow M_b \) as sets. For an automatic sequence \( \xi(n) = F(x_0 \cdot \iota(n)) \), where \( F : \mathcal{X} \rightarrow \mathbb{C} \) and \( x_0 \in \mathcal{X} \), Drappeau and Müllner [18, Remark 3 to Theorem 2 & Proposition 13] have outlined that their approach together with Lemma 3.3 yields a power saving in \( N \) for \( V_\xi(p; N) \) for a prime \( p \) satisfying \( \gcd(p, b) = 1 \), perhaps in the full range \( p^A \geq N \geq p^{1/2+\varepsilon} \) for any fixed positive \( A \) and \( \varepsilon \).

(5) **Bilinear correlations:** The results of [6, 34, 40, 41, 56–58] give various bounds on bilinear sums with Kloosterman sums
\[
\sum_{k=1}^{K} \sum_{n=1}^{N} \alpha_k \mathcal{K}_m(kn), \quad \text{and} \quad \sum_{k=1}^{K} \sum_{n=1}^{N} \alpha_k \beta_n \mathcal{K}_m(kn),
\]
which are known as Type I and Type II sums, with some complex weights. Bounds of such sums, besides being other instances of Kloosterman randomness, are also important for many applications, see [6, 7, 33, 40].

### 2.2 New results in the horizontal aspect

It is natural to assume that both \( \mathcal{K}_m(a) \) and \( \mathcal{K}_m^\ast(a) \) and the Möbius function function \( \mu(m) \) are not correlated.

**Conjecture 2.1.** For any fixed integer \( a \neq 0 \), we have
\[
H_{\tilde{a},\tilde{\psi}}(M) = o\left(\sqrt{H_{\psi}(M)}\right) \quad \text{and} \quad H_{\tilde{a},\tilde{\psi}}^\ast(M) = o\left(\sqrt{H_{\psi}(M)}\right)
\]
as \( M \rightarrow \infty \).
It is quite possible that the convergence to zero in Conjecture 2.1 is quite fast and in fact

$$H_{\alpha, \mu}(M) \ll M^{-\eta} H_\alpha(M)$$

and

$$H^*_{\alpha, \mu}(M) \ll M^{-\eta} H^*_\alpha(M),$$

for some constant $\eta > 0$.

This leads us to formulating the following versions of the Chowla conjecture, introduced in [12] (see also [51, 62] about its importance and links to other problems), for the Kloosterman sums. We note that since we still do not know the exact order of magnitude of $H^*_{\alpha, 1}(M)$, see (1.6), we need more explicit bounds on the assumed rate of decay than in the classical Chowla conjecture.

**Conjecture 2.2.** For any fixed integer $a \neq 0$, and any fixed positive integers $\nu_1, \ldots, \nu_s$ and integers $h_s > \ldots > h_1 \geq 0$, as $M \to \infty$, we have:

1. If $\nu_1, \ldots, \nu_s$ are not all even, then

$$\left| \frac{1}{M} \sum_{m=1}^{M} \mathcal{K}^*_{m+h_1}(a) \cdots \mathcal{K}^*_{m+h_s}(a)^{\nu_s} \right| = o \left( \frac{H^*_\alpha(M)}{M} \right)^{\nu_1 + \ldots + \nu_s};$$

2. If $\nu_1, \ldots, \nu_s$ are all even, then

$$\left| \frac{1}{M} \sum_{m=1}^{M} \mathcal{K}^*_{m+h_1}(a) \cdots \mathcal{K}^*_{m+h_s}(a)^{\nu_s} \right| \leq \left( \frac{A H^*_\alpha(M)}{M} \right)^{\nu_1 + \ldots + \nu_s},$$

with some constant $A \geq 1$, which may depend only upon $a$ and is uniform with respect to all other parameters.

We note that the second part of Conjecture 2.2 is still nontrivial (which makes a remarkable difference between Conjecture 2.2 and the Chowla conjecture for the Möbius function).

Next, under Conjecture 2.2, we show that $H^*_{\alpha, \xi}(M) = o \left( \frac{H^*_\alpha(M)}{M} \right)$ for a natural class of “low complexity” sequences.

First, we introduce the notion of topological entropy following works of Bowen [10, 11] and Dinaburg [15].

For a compact metric set $\mathcal{X}$ and a homeomorphism $T$ on it, we consider a topological metric space $(\mathcal{X}, T)$.

We define the distance on $\mathcal{X}$ at step $n$ by

$$d_n(x, y) = \max_{0 \leq k \leq n-1} d(T^k x, T^k y).$$

We say that a set $S \subseteq \mathcal{X}$ is $(n, \varepsilon)$-separated if for all $x, y \in S$ with $x \neq y$, we have $d_n(x, y) \geq \varepsilon$. Since $\mathcal{X}$ is compact, a separated set cannot be infinite and we can define $s(n, \varepsilon)$ to be the largest cardinality of an $(n, \varepsilon)$-separated set. Similarly, we say that a set $R \subseteq \mathcal{X}$ is an $(n, \varepsilon)$-span set if

$$\mathcal{X} \subseteq \bigcup_{x \in R} B_{d_n}(x, \varepsilon),$$

where $B_{d_n}(x, \varepsilon)$ is the ball of radius $\varepsilon$ with respect to $d_n$:

$$B_{d_n}(x, \varepsilon) = \{ y \in \mathcal{X} : d_n(x, y) \leq \varepsilon \}.$$

By compactness, there are finite $(n, \varepsilon)$-spanning sets. Let $r(n, \varepsilon)$ be the minimum cardinality of the $(n, \varepsilon)$-spanning sets. This number corresponds to the minimal number of points in $\mathcal{X}$ such that each orbit of length $n$ can be $\varepsilon$-approximated by the orbit of one of these points.

It is easy to see that these two numbers are linked by the following inequality for $n \in \mathbb{N}$ and $\varepsilon > 0$:

$$r(n, \varepsilon/2) \leq s(n, \varepsilon) \leq r(n, \varepsilon).$$
Following [10], the topological entropy of $T$ is defined by

$$h_{\text{top}}(T) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \epsilon) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \epsilon).$$

(2.3)

We introduce also the notion of deterministic sequences.

**Definition 2.3.** The function $\xi : \mathbb{N} \to \mathbb{C}$ is said to be deterministic if there exists a dynamical system $(\mathcal{X}, T)$ of topological entropy zero, a continuous function $f : \mathcal{X} \to \mathbb{C}$, and a point $x \in \mathcal{X}$ such that for all $n \in \mathbb{N}$ we have $\xi(n) = f(T^n x)$.

**Theorem 2.4.** Under Conjecture 2.2, for any deterministic function $\xi : \mathbb{N} \to \mathbb{C}$ and a fixed integer $a \neq 0$, we have

$$H_{a, \xi}(M) = o\left( \overline{H}_{a}(M) \right).$$

**Remark 2.5.** For the proof of Theorem 2.4, as well as of Theorem 2.9 below, we follow combinatorial ideas indicated by Sarnak [51] and fully worked out by Tao [62]. We also note that a similar question has been discussed by Kowalski [39]. While Theorems 2.4 and 2.9 are not surprising, we believe they deserve to be recorded in a precise form. More importantly, we use this opportunity to reshape the argument used in [39, 62] into a more canonical form, which has been used for many other problems of analytic number theory, most significantly, in the modern proof of the Burgess bound (see [31, Theorem 12.6]). Furthermore, if explicit quantitative versions of Conjecture 2.2 become available, for functions $\xi(n) = f(T^n x)$ with a known rate of convergence in (2.3), this approach may also lead to stronger versions of Theorems 2.4 and 2.9.

As a further evidence of validity of Conjecture 2.1, we now obtain variants of (2.1) for other weights of arithmetic nature. For example, we give a result about noncorrelation of Kloosterman sums and the characteristic function $\chi_k(m)$ of $k$th-power free numbers (i.e., numbers indivisible by $k$th power of a prime), in particular,

$$\chi_2(m) = |\mu(m)|$$

and the Euler function $\varphi(m)$. In order to state the theorem, we first need to introduce $\theta$ as the currently best bound toward the Selberg eigenvalue conjecture. That is, the cuspidal spectrum of the (negated) hyperbolic Laplacian on congruence quotients $\Gamma(N) \backslash \mathbb{H}$ is lower bounded by $\frac{1}{4} - \theta^2$. Note that we have

$$\theta \leq \frac{7}{64}$$

by a result of Kim and Sarnak [35], while the Selberg eigenvalue conjecture states that one should have $\theta = 0$. We now define

$$\gamma = \max\{1/6, 2\theta\}.$$  

(2.4)

In particular, we see that $\gamma \leq 7/32$.

**Theorem 2.6.** For any fixed integer $a$, and $k \geq 2$, we have

$$H_{a, \chi_k}(M) \leq M^{1/2 + \gamma + o(1)} + M^{1/2 + 1/(2k) + o(1)}.$$

Next we show that Kloosterman sums do not correlate with the Euler function $\varphi(m)$.

**Theorem 2.7.** For any fixed integer $a$, with $\gamma$ given by (2.4), we have

$$H_{a, \varphi}(M) \leq M^{3/2 + \gamma + o(1)}.$$
2.3 New results in the vertical aspect

We start with recalling that the vertical analog of Conjecture 2.2 is established by Fouvry, Kowalski, and Michel [22], see Lemma 3.3.

Next, similarly to [22, Definition 1.3], we say that an integer vector \((h_1, \ldots, h_s) \in \mathbb{Z}^s\) is normal modulo \(p\) if there is some \(h\) such that
\[
\#\{j : 1 \leq j \leq s, h_j \equiv h \pmod{p}\} \equiv 1 \pmod{2}.
\]

**Theorem 2.8.** For a prime \(p\) and integer \(N < p\), uniformly over normal modulo \(p\) vectors \((h_1, \ldots, h_s) \in \mathbb{Z}^s\) and polynomials \(g \in \mathbb{R}[X]\) of degree \(d \geq 0\), we have
\[
\sum_{n=1}^{N} K_p(n + h_1) \cdots K_p(n + h_s) e(g(n)) \ll N^{1-2^{-d}} p^{3^{-d-1}} (\log p)^{3^{-d}}.
\]

We also have the following unconditional analog of Theorem 2.4 (we also recall Remark 2.5 describing our motivation).

**Theorem 2.9.** Let \(\psi(z)\) be a fixed arbitrary real function with \(\psi(z) \to \infty\) as \(z \to \infty\). For any dynamical system \((\mathcal{X}, T)\) with zero topological entropy, for any continuous function \(f : \mathcal{X} \to \mathbb{C}\) and any \(x \in \mathcal{X}\), for a prime \(p\), for an integer \(N\) with \(p > N \geq \sqrt{p\psi(p)}\), we have
\[
\sum_{n=1}^{N} K_p(n) f(T^n x) = o(N)
\]
as \(p \to \infty\).

3 PRELIMINARIES

3.1 Sums of Kloosterman sums in the horizontal aspect

We start by considering the special case of sums of Kloosterman sums along moduli, which are multiples of a given integer \(q \geq 1\). For \(q = 1\), cancellation along such sums was first conjectured by Selberg [55] and Linnik [43] (independently) and later proven by Kuznetsov [42] with the eponymous trace formula. For general \(q\), adaptations of the Kuznetsov trace formula as well as cancellations along these sums have been demonstrated by, for example, Deshouillers and Iwaniec [14]. For more uniform versions, the reader may wish to consult [27, 28, 52, 59] and for more general arithmetic progressions [8, 16, 36].

Here, we slightly improve the \(q\) dependence in the bound for the sums of Kloosterman sums along moduli divisible by \(q\). We achieve this by applying a more optimized treatment of the exceptional spectrum. Otherwise, we follow the arguments of prior works [14, 27, 28, 59] almost verbatim.

It is useful to record the following trivial bound (i.e., when one forgoes any cancellation in the sum over \(m\))
\[
\sum_{m=1}^{M} \frac{1}{m^{1/2}} K_m(a) \ll M^{1/2+o(1)} q^{-1},
\]
(3.1)

implied by a direct application of the Weil bound (1.1).

**Lemma 3.1.** For a fixed nonzero integer \(a\) and a positive integer \(q\), with \(\gamma\) given by (2.4), we have
\[
\sum_{m=1}^{M} \frac{1}{m^{1/2}} K_m(a) \ll (M q^{-2})^{\gamma} (qM)^{o(1)}.
\]
Proof. We observe that the trivial bound (3.1) is stronger if $M \leq q^2$. Hence, for the remainder of the proof, we always assume

$$M > q^2. \quad (3.2)$$

It is sufficient to treat dyadic sums

$$S_{\text{dyad}}(M) = \sum_{M \leq m < 2M \mid q} \frac{1}{m^{1/2}} \mathcal{K}_m(a),$$

which we compare to smooth sums

$$S_{\text{smooth}}(M) = \sum_{m \equiv 0 \pmod{q}} \frac{1}{m^{1/2}} \mathcal{K}_m(a) g \left( \frac{4\pi \sqrt{|a|}}{m} \right), \quad (3.3)$$

where $g$ is a smooth bump function with

1. $g(x) = 1$ for

$$\frac{2\pi \sqrt{|a|}}{M} \leq x \leq \frac{4\pi \sqrt{|a|}}{M},$$

2. $g(x) = 0$ for

$$x \leq \frac{2\pi \sqrt{|a|}}{M + T} \quad \text{or} \quad \frac{4\pi \sqrt{|a|}}{M - T} \geq x,$$

3. with $L_1$-norms satisfying

$$\|g'\|_1 \ll 1 \quad \text{and} \quad \|g''\|_1 \ll \frac{M^2}{\sqrt{|a|}T},$$

for some parameter $1 \leq T \leq M/2$. The Weil bound (1.1) shows that

$$\left| S_{\text{dyad}}(M) - S_{\text{smooth}}(M) \right| \ll M^{-1/2}(1 + T/q)(Mq)^{(1)}$$

see [59, eq. (3.2)].

The smooth sum (3.3), following the analysis [14, pp. 264–268], may be bounded as

$$S_{\text{smooth}}(M) \ll \frac{M^{1/2}}{T^{1/2}} + \sum_{j, \text{exc.}} |\rho_j(a)\rho_j(1)| M^{2|j|},$$

where the sum is over an orthonormal basis of exceptional Maass forms of $\Gamma_0(q)\backslash \mathbb{H}$ with Fourier expansion

$$y^{1/2} \sum_{n \neq 0} \rho_j(n) K_{n, j}(2\pi |n| y) e(nx), \quad x + iy \in \mathbb{H}.$$ 

We refer to [14, 30] for a background and standard notations. This exceptional term may be further bounded using

$$|\rho_j(a)| \ll |a|^{1/2} |\rho_j(1)| \ll |\rho_j(1)|$$
and also a density estimate for exceptional eigenvalues, see [30, eq. (11.25)], along the lines of [24, sections 2.1] and [60, Lemma 2.10]:

\[
\sum_{j, \text{exc.}} |\varphi_j(a)\varphi_j(1)| \ll (Mq^{-2})^{2\delta} \sum_{j, \text{exc.}} |\varphi_j(1)|^2 q^{4|l_j|} \ll (Mq^{-2})^{2\delta} q^{o(1)}.
\]

In conclusion,

\[
S_{\text{dyad}}(M) \ll \left(\frac{T}{qM^{1/2}} + \frac{M^{1/2}}{T^{1/2}} + (Mq^{-2})^{2\delta}\right) (qM)^{o(1)}.
\]

The choice \( T = 0.5(qM)^{2/3} \) (which due to (3.2) ensures that \( 1 \leq T \leq M/2 \), as required) gives the bound

\[
S_{\text{dyad}}(M) \ll (M^{1/6}q^{-1/3} + (Mq^{-2})^{2\delta})(qM)^{o(1)}.
\]

The desired conclusion follows upon recalling the restriction (3.2). \(\square\)

Via partial summation, we further derive from Lemma 3.1.

**Corollary 3.2.** For a fixed nonzero integer \( a \) and a positive integer \( q \), with \( \gamma \) given by (2.4), we have for any fixed \( \alpha \geq -1/2 \),

\[
\sum_{m=1}^{M} m^{\alpha} \mathcal{K}_m(a) \ll M^{\alpha+1/2} (Mq^{-2})^{\gamma} (Mq)^{o(1)}.
\]

### 3.2 | Sums of Kloosterman sums in the vertical aspect

We start with a result, which is a special case of Fouvry, Kowalski, and Michel [22, Corollary 1.6]. We also recall the definition of normal vectors from Section 2.3.

**Lemma 3.3.** For a prime \( p \), for any integer \( b \), uniformly over normal modulo \( p \) vectors \((h_1, \ldots, h_s) \in \mathbb{Z}^s\), we have

\[
\left| \sum_{n=1}^{P} \mathcal{K}_p(n+h_1) \cdots \mathcal{K}_p(n+h_s) e_p(bn) \right| \ll p^{1/2}.
\]

Using the standard completion technique, see [31, section 12.2], we immediately derive from Lemma 3.3 that for a prime \( p \) and integer \( N \geq 1 \), uniformly over normal modulo \( p \) vectors \((h_1, \ldots, h_s) \in \mathbb{Z}^s\), where \( s \) is a fixed integer, we have

\[
\left| \sum_{n=1}^{N} \mathcal{K}_p(n+h_1) \cdots \mathcal{K}_p(n+h_s) \right| \ll p^{1/2} \log p.
\] (3.4)

However, we need a different bound, which is usually weaker than (3.4) but instead is nontrivial for smaller values of \( N \), namely, in the range \( p^{1/2} \leq N \leq p^{1/2} \log p \).

**Corollary 3.4.** For a prime \( p \) and integer \( N < p \), uniformly over normal modulo \( p \) vectors \((h_1, \ldots, h_s) \in \mathbb{Z}^s\), where \( s \) is a fixed integer, we have

\[
\left| \sum_{n=1}^{N} \mathcal{K}_p(n+h_1) \cdots \mathcal{K}_p(n+h_s) \right| \ll N^{1/2} p^{1/4}.
\]
Proof. Clearly for any integer $K \geq 1$, we have

$$
\sum_{n=1}^{N} \mathcal{K}_p(n + h_1) \cdots \mathcal{K}_p(n + h_s) = \frac{1}{K} W + O(K), \quad (3.5)
$$

where

$$
W = \sum_{k=1}^{K} \sum_{n=1}^{N} \mathcal{K}_p(n + k + h_1) \cdots \mathcal{K}_p(n + k + h_s)
$$

$$
= \sum_{n=1}^{N} \sum_{k=1}^{K} \mathcal{K}_p(n + k + h_1) \cdots \mathcal{K}_p(n + k + h_s).
$$

By the Cauchy inequality,

$$
|W|^2 \leq N \sum_{n=1}^{N} \left| \sum_{k=1}^{K} \mathcal{K}_p(n + k + h_1) \cdots \mathcal{K}_p(n + k + h_s) \right|^2
$$

$$
\leq N \sum_{n=1}^{p} \left| \sum_{k=1}^{K} \mathcal{K}_p(n + k + h_1) \cdots \mathcal{K}_p(n + k + h_s) \right|^2
$$

$$
= N \sum_{n=1}^{p} \sum_{k, \ell=1}^{K} \mathcal{K}_p(n + k + h_1) \cdots \mathcal{K}_p(n + k + h_s)
\mathcal{K}_p(n + \ell + h_1) \cdots \mathcal{K}_p(n + \ell + h_s)
$$

$$
= N \sum_{k, \ell=1}^{K} \sum_{n=1}^{p} \mathcal{K}_p(n + k + h_1) \cdots \mathcal{K}_p(n + k + h_s)
\mathcal{K}_p(n + \ell + h_1) \cdots \mathcal{K}_p(n + \ell + h_s).
$$

For $K < p$, there are at most $s^2 K$ pairs $(k, \ell)$ with

$$
\quad k - \ell \equiv h_i - h_j \pmod{p}
$$

for some $1 \leq i, j \leq s$ (including $i = j$). In these occurrences, we estimate the inner sum trivially as $p$. For the remaining pairs $(k, \ell)$, the $2s$ integers $k + h_1, \ldots, k + h_s$, $\ell + h_1, \ldots, \ell + h_s$ are pairwise distinct modulo $p$ and so Lemma 3.3 applies. Hence, we derive

$$
W^2 \ll N(Kp + K^2 p^{1/2}).
$$

We now choose $K = \lceil p^{1/2} \rceil$ for which $W \ll N^{1/2} p^{3/4}$, which after substitution in (3.5) implies

$$
\sum_{n=1}^{N} \mathcal{K}_p(n + h_1) \cdots \mathcal{K}_p(n + h_s) \ll N^{1/2} p^{1/4} + p^{1/2}. \quad (3.6)
$$

Clearly this bound is trivial for $N \leq p^{1/2}$, while for $N > p^{1/2}$, we see that $N^{1/2} p^{1/4} > p^{1/2}$ and hence that term $p^{1/2}$ in (3.6) can be discarded. □
4 | PROOFS OF MAIN RESULTS

4.1 | Proof of Theorem 2.4

Let $\xi(n) = f(T^n x)$ as in Definition 2.3. Since $\mathcal{X}$ is compact, it follows that $f$ is uniformly continuous by [50, Proposition 23]. This means for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\forall x, y \in \mathcal{X}, \ d(x, y) < \delta \implies |f(x) - f(y)| < \varepsilon. \quad (4.1)$$

Let $\varepsilon > 0$ and set $\delta$ as in (4.1).

Next, we observe that for any integer $h \geq 0$, we have

$$\mathcal{H}_{a, \delta}^*(M) = \sum_{m=1}^{M} \mathcal{K}_{m}(a)\xi(m) = \sum_{m=1}^{M} \mathcal{K}_{m+h}(a)\xi(m + h) + O(h),$$

where throughout the proof all implied constants may depend only on $f$ and do not depend on the parameters $\varepsilon$ and $H$ we introduce below.

We now fix some sufficiently large (in terms of $\varepsilon$) integer $H$ and note that by (1.5) we have

$$\mathcal{H}_{a, \delta}^*(M) = \frac{1}{H} \sum_{m=1}^{M} \sum_{h=1}^{H} \mathcal{K}_{m+h}(a)\xi(m + h) + O(H), \quad (4.2)$$

since $\xi(n) = f(T^n x)$ and $f$ is a continuous function on the compact set $\mathcal{X}$.

We now argue as in [62, Section 2] and note that since the topological entropy of $T$ is zero. We also recall the definition (2.2).

Then, for a sufficiently large $H$ (depending on $\varepsilon$), there is a set $\{x_1, \ldots, x_t\} \subseteq \mathcal{X}$ of cardinality

$$t \ll \exp(\varepsilon^3 H), \quad (4.3)$$

which spans $(\mathcal{X}, T)$ by balls $B_{d_{H}}(x_i, \delta), i = 1, \ldots, t$, of radius $\delta$ with respect to the metric induced by $d_{H}$. That is,

$$\mathcal{X} \subseteq \bigcup_{i=1}^{t} B_{d_{H}}(x_i, \delta),$$

for some $x_1, \ldots, x_t \in \mathcal{X}$ with $t \ll \exp(\varepsilon^3 H)$.

In other words, for any $x \in \mathcal{X}$, and $m \geq 1$, there is $i_m \in \{1, \ldots, t\}$ for which we have

$$d(T^{h}x_{i_m}, T^{m+h}x) \leq \delta,$$

for all $1 \leq h \leq H$.

The rest of the argument, while is inspired by the exposition of Tao [62], deviates from that in [62, section 2]. Therefore, recalling (4.1), we see that for all $1 \leq h \leq H$, we also have

$$\frac{1}{H} \sum_{h=1}^{H} \mathcal{K}_{m+h}(a)f(T^{m+h}x) = \frac{1}{H} \sum_{h=1}^{H} \mathcal{K}_{m+h}(a)f(T^{h}x_{i_m}) + O\left(\frac{\varepsilon}{H} \sum_{h=1}^{H} |\mathcal{K}_{m+h}(a)|\right).$$
Hence, together with (4.2), we obtain
\[ H^*_{a,\xi}(M) = W + O(\Delta), \tag{4.4} \]
where
\[ W = \sum_{m=1}^{M} \frac{1}{H} \sum_{h=1}^{H} \mathcal{K}^*_{m+h}(a)f(T^h x_{im}), \]
and
\[ \Delta = \sum_{m=1}^{M} \frac{\varepsilon}{H} \sum_{h=1}^{H} |\mathcal{K}^*_{m+h}(a)| = \frac{\varepsilon}{H} \sum_{h=1}^{H} \sum_{m=1}^{M} |\mathcal{K}^*_{m+h}(a)| \]
\[ \ll \frac{\varepsilon}{H} \sum_{h=1}^{H} \left( \sum_{m=1}^{M} |\mathcal{K}^*_{m}(a)| + O(H) \right) \]
\[ \ll \varepsilon H_0(M) + \varepsilon H \ll \varepsilon H_0(M), \tag{4.5} \]
as \( M \to \infty \).

We fix some integer \( s \geq 1 \) and, applying the Hölder's inequality, derive
\[ |W|^2 \leq H^{-2s} M^{2s-1} \sum_{m=1}^{M} \left| \sum_{h=1}^{H} \mathcal{K}^*_{m+h}(a)f(T^h x_{im}) \right|^2. \]

It is now convenient to denote
\[ \rho(M) = \frac{H_0(M)}{M}. \]

To estimate \( W \), we eliminate the dependence of \( x_{im} \) on \( m \) with the trivial inequality
\[ |W|^2 \leq H^{-2s} M^{2s-1} \sum_{i=1}^{t} \sum_{m=1}^{M} \left| \sum_{h=1}^{H} \mathcal{K}^*_{m+h}(a)f(T^h x_{i}) \right|^2. \]

Since \( \mathcal{X} \) is compact, we can define
\[ F = \sup_{x \in \mathcal{X}} |f(x)| < \infty. \]

We fix some integer \( s \geq 1 \) and, applying Hölder’s inequality, derive
\[ W^{2s} \leq \frac{H^{-2s} M^{2s-1}}{2^s} \sum_{i=1}^{t} \sum_{m=1}^{M} \left| \sum_{h=1}^{H} \mathcal{K}^*_{m+h}(a)f(T^h x_{i}) \right|^{2s} \]
\[ = H^{-2s} M^{2s-1} \sum_{i=1}^{t} \sum_{h_{1}, h_{2}, \ldots, h_{s}=1}^{H} \prod_{j=1}^{s} f(T^{h_j} x_{i}) f(T^{h_{1}+h_{j}} x_{i}) \sum_{m=1}^{M} \prod_{j=1}^{s} \mathcal{K}^*_{m+h_{j}}(a) \mathcal{K}^*_{m+h_{1}+h_{j}}(a) \]
\[ \leq F^{2s} H^{-2s} M^{2s-1} \sum_{i=1}^{t} \sum_{h_{1}, h_{2}, \ldots, h_{s}=1}^{H} \left| \sum_{m=1}^{M} \prod_{j=1}^{s} \mathcal{K}^*_{m+h_{j}}(a) \mathcal{K}^*_{m+h_{1}+h_{j}}(a) \right| \]
\[ = tF^{2s} H^{-2s} M^{2s-1} \sum_{h_{1}, h_{2}, \ldots, h_{s}=1}^{H} \left| \sum_{m=1}^{M} \prod_{j=1}^{s} \mathcal{K}^*_{m+h_{j}}(a) \mathcal{K}^*_{m+h_{1}+h_{j}}(a) \right|. \]
Each tuple \((h_1, \ldots, h_{2s}) \in [1, H]^{2s}\) that does not satisfy the conditions of Conjecture 2.2 can be split into \(s\) pairs \(h_j = h_k, j \neq k\). We thus see that their cardinality may be bounded by

\[
\binom{2s}{s} s! H^s \leq (sH)^s,
\]

Therefore, under Conjecture 2.2, recalling (1.5), we have

\[
W^{2s} \leq tF^{2s}H^{-2s}M^{2s-1}\left((sH)^s M(A\rho(M))^{2s} + o(M\rho(M)^{2s})\right)
\]

\[
= t(AF)^{2s} M^{2s} \rho(M)^{2s}((s/H)^s + o(1))
\]

\[
= (AF)^{2s} M^{2s} \rho(M)^{2s}(t(s/H)^s + o(1))
\]

\[
= (AF)^{2s} t(s/H)^s M^{2s} \rho(M)^{2s},
\]

as \(M \to \infty\) (while \(s\) and \(H\) are fixed). Thus,

\[
W \ll t^{1/2s}(s/H)^{1/2} M\rho(M) = t^{1/3}(s/H)^{1/3} H_{\omega}(M),
\]

(4.6)

where the implies constant only depends on \(a\) and \(f\).

Choosing

\[
s = \lceil \varepsilon^2 H \rceil
\]

provided that \(\varepsilon < 1/2\), assuming that \(H\) is large enough so that \(s \geq 2\), and recalling (4.3) we see that

\[
t^{1/s} \leq \exp(\varepsilon) \leq 2 \quad \text{and} \quad (s/H)^{1/2} \leq 2\varepsilon,
\]

provided that \(\varepsilon < 1/2\). Therefore, with the above choice of \(s\), the bound (4.6) becomes

\[
W \ll H_{\omega}^{s}(M)  (\varepsilon + o(1)) \ll \varepsilon H_{\omega}^{s}(M),
\]

(4.7)

when \(M \to \infty\).

Substituting the bounds (4.5) and (4.7) in (4.4), we obtain

\[
H_{\omega, \varepsilon}(M) \ll \varepsilon H_{\omega}^{s}(M)
\]

and since \(\varepsilon > 0\) is arbitrary, the result follows.

### 4.2 Proof of Theorem 2.6

Using the inclusion–exclusion principle, we write

\[
\sum_{m=1}^{M} \mathcal{K}_m(a) x_k(m) = \sum_{d \leq M^{1/6}} \mu(d) \sum_{m=1}^{M} \mathcal{K}_m(a),
\]

We now choose some parameter \(D \in [1, M]\) and use Corollary 3.2 for \(d \leq D\), while for \(d > D\), we use the trivial bound

\[
\sum_{m=1}^{M} \mathcal{K}_m(a) \ll M^{1+o(1)}/d^k,
\]

where

\[
\sum_{m=1}^{M} \mathcal{K}_m(a) \ll M^{1+o(1)}/d^k,
\]
which is similar to (3.1). Hence

\[ \sum_{m=1}^{M} \mathcal{K}_m(a) \varphi_k(m) \ll \sum_{d \leq D} M^{1/2+o(1)}(Md^{-2k})^\gamma + \sum_{d > D} M^{1+o(1)} d^{k} \]

\[ = M^{1/2+\gamma+o(1)}(D^{(1)} + D^{1-2k\gamma}) + M^{1+o(1)} D^{1-k}. \]

We now define \( D \) by the equation

\[ M^{1/2+\gamma} D^{1-2k\gamma} = MD^{1-k} \]

or

\[ D = M^{1/(2k)} \]

(which in fact does not depend on \( \gamma \)). This implies the bound

\[ H_{a,k}(M) \ll M^{1/2+\gamma+o(1)} + M^{1/2+1/(2k)+o(1)}. \]

### 4.3 Proof of Theorem 2.7

Using the well-known elementary formula (see, for instance, [5, Theorem 2.3])

\[ \varphi(m) = m \sum_{d|m} \frac{\mu(d)}{d}, \]

we write

\[ \sum_{m=1}^{M} \mathcal{K}_m(a) \varphi(m) = \sum_{m=1}^{M} m \mathcal{K}_m(a) \sum_{d|m} \frac{\mu(d)}{d} \]

\[ = \sum_{d=1}^{M} \frac{\mu(d)}{d} \sum_{m=1}^{M} \mathcal{K}_m(a). \]

Hence, by Corollary 3.2,

\[ \sum_{m=1}^{M} \mathcal{K}_m(a) \varphi(m) \ll \sum_{d=1}^{M} M^{3/2+o(1)} d^{1-2\gamma} \]

\[ = M^{3/2+\gamma+o(1)} \sum_{d=1}^{M} d^{-1-2\gamma} \leq M^{3/2+\gamma+o(1)}, \]

which concludes the proof.

### 4.4 Proof of Theorem 2.8

We use a version of the Weyl differencing and establish the result by induction on \( d \geq 0 \).

For \( d = 0 \) that is for a pure sum the result is instant from Corollary 3.4.
Now assume that \( d \geq 1 \). Given integers \( h_1, \ldots, h_s \) and a polynomial \( g(X) \in \mathbb{R}[X] \) we square the sum

\[
S = \sum_{n=1}^{N} \prod_{j=1}^{s} \mathcal{K}_p(n + h_j) e(g(n)).
\]

and obtain

\[
S^2 = \sum_{m,n=1}^{N} \prod_{j=1}^{s} \mathcal{K}_p(m + h_j) \prod_{j=1}^{s} \mathcal{K}_p(n + h_j) e(g(n) - g(m))
= 2W + O(N).
\]

where

\[
W = \sum_{1 \leq m < n \leq N} \prod_{j=1}^{s} \mathcal{K}_p(m + h_j) \prod_{j=1}^{s} \mathcal{K}_p(n + h_j) e(g(n) - g(m)).
\]

Writing \( n = m + \ell \) we obtain

\[
W = \sum_{m=1}^{N-\ell} \sum_{\ell=1}^{N-m} \prod_{j=1}^{s} \mathcal{K}_p(m + h_j) \prod_{j=1}^{s} \mathcal{K}_p(m + \ell + h_j) e(g(m + \ell) - g(m))
= \sum_{\ell=1}^{N-1} \sum_{m=1}^{N-\ell} \prod_{j=1}^{s} \mathcal{K}_p(m + h_j) \prod_{j=1}^{s} \mathcal{K}_p(m + \ell + h_j) e(g(m + \ell) - g(m)).
\]

We now observe that for every \( \ell \) the polynomial \( g(X + \ell) - g(X) \) is a polynomial of degree at most \( d - 1 \).

Further more, if \( (h_1, \ldots, h_s) \in \mathbb{Z}^s \) is normal modulo \( p \) then for all but at most \( s(s - 1)/2 \) values of \( \ell = 1, \ldots, N - 1 \) (avoiding the differences \( |h_i - h_j|, 1 \leq i < j \leq s \)) the vector

\[
(h_1, \ldots, h_s, h_1 + \ell, \ldots, h_s + \ell) \in \mathbb{Z}^{2s}
\]

is also normal modulo \( p \). For these exceptional values of \( \ell \) we estimate the sums over \( m \) trivially as \( N \), and apply the induction assumption for the remaining values of \( \ell \). Hence

\[
W \ll N^{2 - \frac{d+1}{2}} p^{\frac{d}{2} - d} (\log p)^{-d + 1} + N,
\]

which after substitution in (4.8) implies

\[
S \ll N^{1 - \frac{d}{2}} p^{\frac{d}{2} - d-1} (\log p)^{\frac{d}{2} - d} + N^{1/2}.
\]

It remains to note that for \( N \leq p^{1/2} \) the result is trivial and for \( N \geq p^{1/2} \) we have

\[
N^{1 - \frac{d}{2}} p^{\frac{d}{2} - d-1} (\log p)^{\frac{d}{2} - d} \gg N^{1 - \frac{d}{2}} p^{\frac{d}{2} - d-1} \geq N^{1/2},
\]

from which the result follows.

### 4.5 Proof of Theorem 2.9

We start by noticing that

\[
| \mathcal{K}_p(n) | \leq 2, \quad \forall n \in \mathbb{N}.
\]
We proceed also as in the proof Theorem 2.4 by writing

\[ \sum_{n=1}^{N} \kappa_p(n) \xi(n) = \sum_{n=1}^{N} \kappa_p(n + h) \xi(n + h) + O(h). \]

We again fix some \( \varepsilon > 0 \) and let \( \delta \) be as in (4.1). We choose also the parameters \( H \) and \( t \) as in (4.3). Therefore,

\[ \sum_{n=1}^{N} \kappa_p(n) \xi(n) = \frac{1}{H} \sum_{h=1}^{H} \sum_{n=1}^{N} \kappa_p(n + h) \xi(n + h) + O(H), \]

Recalling that \( \xi(n) = f(T^n x) \), as in the proof of Theorem 2.4, we have

\[ \frac{1}{H} \sum_{h=1}^{H} \kappa_p(n + h) f(T^{n+h} x) = \frac{1}{H} \sum_{h=1}^{H} \kappa_p(n + h) f(T^h x_i_n) + O(\varepsilon). \]

Hence, it is now enough to show for some \( H = o(N) \) that

\[ W = \sum_{n=1}^{N} \frac{1}{H} \sum_{h=1}^{H} \kappa_p(n + h) f(T^h x_i_n) \]

satisfies

\[ W \ll \varepsilon N. \] (4.9)

Now, for fix \( s \geq 2 \), we apply Hölder’s inequality and we relax the dependence of \( i_n \) to obtain

\[ |W|^{2s} \leq \left| \sum_{n=1}^{N} \frac{1}{H} \sum_{h=1}^{H} \kappa_p(n + h) f(T^h x_i_n) \right|^{2s} \]

\[ \leq H^{-2s} N^{2s-1} \sum_{i=1}^{s} \sum_{j=1}^{s} \prod_{1 \leq h_1, \ldots, h_{2s} \leq 1} f(T^{h_j} x_i) f(T^{h_{s+j}} x_i) \]

\[ \sum_{n=1}^{N} \prod_{j=1}^{s} \kappa_p(m + h_j) \kappa_p(m + h_{s+j}) \]

\[ \leq t F^{2s} H^{-2s} N^{2s-1} \sum_{h_1, \ldots, h_{2s} = 1}^{H} \left| \sum_{n=1}^{N} \prod_{j=1}^{s} \kappa_p(m + h_j) \kappa_p(m + h_{s+j}) \right|. \]

Each tuple \((h_1, \ldots, h_{2s}) \in [1, H]^{2s}\) that does not satisfy the conditions of Corollary 3.4 can be split into \( s \) pairs \((j, k)\) such that \( h_j \equiv h_k \pmod{p} \). We thus see that their cardinality may be bounded by

\[ \binom{2s}{s} s^H \leq (sH)^s. \]

We estimate these tuples trivially and bound the others using Corollary 3.4. We thus find

\[ |W|^{2s} \ll t F^{2s} H^{-2s} N^{2s-1} \left( (sH)^s N4^s + H^{2s} N^{1/2} p^{1/4} \right) \]

\[ \ll t F^{2s} \left( \frac{8}{H} \right)^{s} N^{2s} + t F^{2s} N^{2s-1/2} p^{1/4} \]
provided that $p$ is sufficiently large in terms of $H$ and $s$. This gives

$$W \ll t^{1/2s} \sqrt{\frac{s}{H} N}.$$

Choose $s = \lfloor \varepsilon^2 H \rfloor$ with $\varepsilon < 1/2$ and assume $H$ is large enough so that $s \geq 2$. For the ease of the reader, we repeat the dependencies of the involved parameters

$$\varepsilon \leftarrow \delta \leftarrow H \leftarrow s \leftarrow p,$$

where each parameter may depend on all of the preceding ones. In conclusion, by (4.3) we see that $t^{1/2s} \leq \sqrt{2}$ and $\sqrt{s/H} \leq 2\varepsilon$. Therefore, we obtain (4.9) and complete the proof.

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