Self-embeddings of models of arithmetic; fixed points, small submodels, and extendability

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Abstract

In this paper we will show that for every cut $I$ of any countable nonstandard model $\mathcal{M}$ of $I\Sigma_1$, each $I$-small $\Sigma_1$-elementary submodel of $\mathcal{M}$ is of the form of the set of fixed points of some proper initial self-embedding of $\mathcal{M}$ iff $I$ is a strong cut of $\mathcal{M}$. Especially, this feature will provide us with some equivalent conditions with the strongness of the standard cut in a given countable model $\mathcal{M}$ of $\Sigma_1$. In addition, we will find some criteria for extendability of initial self-embeddings of countable nonstandard models of $\Sigma_1$ to larger models.

1 Introduction

In 1973, Harvey Friedman proved a striking result for countable nonstandard models of finite set theory, and consequently for countable models of Peano arithmetic, PA, stating that every countable nonstandard model of PA carries a proper initial self-embedding; here an initial self-embedding is a self-embedding whose image is an initial segment of the ground model [5]. Afterwards, many versions of Friedman’s style Theorem appeared in the literature of model theory of arithmetic (e.g. see [3] or [16]). In [1], it is shown that some results on the set of fixed points of automorphism of countably saturated models of PA can be generalized for initial self-embeddings of countable nonstandard models of $\Sigma_1$ (see Theorem 2.4 below). In this paper, inspired by results about automorphisms of models of PA, we will investigate some more properties of countable models of $\Sigma_1$ through initial self-embeddings.

In [4], Enayat generalized the notion of a small submodel from [15], to $I$-small for a given cut $I$ of a model of PA (see Definition 1 below), and proved that:

\footnote{In his paper [4], Enayat called such submodels $I$-coded. The name $I$-small is borrowed from Kossak-Schmerl’s book [15].}
Theorem 1.1 (Enayat). Suppose $\mathcal{M} \models PA$ is countable, recursively saturated, and $I$ is a strong cut of $\mathcal{M}$. Moreover, let $\mathcal{M}_0$ be an $I$-small elementary submodel of $\mathcal{M}$. Then there exists some automorphism $j$ of $\mathcal{M}$ such that $\mathcal{M}_0$ is equal to the set of fixed points of $j$.

In section 3 of this paper, after investigating some basic properties of $I$-small $\Sigma_1$-elementary submodels of a countable model $\mathcal{M}$ of $I\Sigma_1$ for some cut $I$ of $\mathcal{M}$, we will refine the above theorem for initial self-embeddings; i.e we will show that $I$ is strong in $\mathcal{M}$ iff every $I$-small $\Sigma_1$-elementary submodel of $\mathcal{M}$ is equal to the set of fixed points of some proper initial self-embedding of $\mathcal{M}$. This result also generalizes one of the main theorems of [1] (see Corollary 4.3 below).

Section 4 of this paper, is devoted to the investigation of equivalent conditions to strongness of the standard cut, denoted by $\mathbb{N}$, in a countable model of $I\Sigma_1$, through the set of fixed points of initial self-embeddings. In [13], it is shown that:

Theorem 1.2 (Kossak-Schmerl). Suppose $\mathcal{M}$ is a countable recursively saturated model of PA. If $\mathbb{N}$ is not strong in $\mathcal{M}$, then for every automorphism $j$ of $\mathcal{M}$ the set of fixed points of $j$ is isomorphic to $\mathcal{M}$.

In Corollary 4.2, we will show that for every countable nonstandard model $\mathcal{M}$ of $I\Sigma_1$, if $\mathbb{N}$ is not strong in $\mathcal{M}$, then the set of fixed points of any initial self-embedding $j$ of $\mathcal{M}$ is either a model of $\neg B\Sigma_1$, or is isomorphic to some proper initial segment of $\mathcal{M}$. Then, we conclude that $\mathbb{N}$ is strong in a countable recursively saturated model $\mathcal{M}$ of PA iff there exists some proper initial self-embedding $j$ of $\mathcal{M}$ such that the set of fixed points of $j$ is small in $\mathcal{M}$ and consequently it is not isomorphic to any proper initial segment of $\mathcal{M}$.

In section 5, we will study the extendability of initial embeddings of models of $I\Sigma_1$ to larger models. In particular, we will prove that any isomorphism between two $\Sigma_1$-elementary initial segment of a countable nonstandard model $\mathcal{M}$ of $I\Sigma_1$ is extendable to some initial self-embedding of $\mathcal{M}$ iff it preserves coded subsets (for the case of automorphisms of countable recursively saturated models of PA this condition is only a necessary condition for extendability to larger models [11]).

2 Preliminaries

In this section we will review some definitions and results which are used through this paper. All unexplained notions can be found in [6] and [7].

• Through this paper, we will work in the language of arithmetic $L_A := \{+,,<,0,1\}$. For a given class $\Gamma$ of $L$-formulas (where $L \supseteq L_A$), $\Pi^\Gamma$ is the fragment of $PA^\ast := PA(L)$ with the induction scheme limited to formulas of $\Gamma$. The $\Gamma$-Collection scheme, denoted by $B\Gamma$, consists of the formulas of the following form for every $\varphi \in \Gamma$:
\[
\forall z, u \left( \left( \forall x < u \ \exists y \ \varphi(x, y, z) \right) \rightarrow \exists v \left( \forall x < u \ \exists y < v \ \varphi(x, y, z) \right) \right).
\]
Moreover, the strong $\Gamma$-Collection scheme, denoted by $B^+\Gamma$, consists of the formulas of the following form for every $\varphi \in \Gamma$:

$$\forall \vec{z}, u \exists v \forall x < u (\exists y \varphi(x, y, \vec{z}) \to \exists y < v \varphi(x, y, \vec{z}))$$

It is folklore that $I\Sigma_{n+1} \vdash B^+\Sigma_{n+1} \vdash B\Sigma_{n+1}$ for all $n \in \omega$; moreover, for every $n \in \omega$, neither $I\Sigma_n \nvdash B\Sigma_n$, nor $I\Sigma_n \nvdash \neg B\Sigma_{n+1}$ (see [6, Ch. I]).

- Within $I\Delta_0 + \text{Exp}$, the $\Delta_0$-formula $x Ey$ denotes the Ackermann’s membership relation, asserting that "the $x$-th bit of the binary expansion of $y$ is 1". For every $\mathcal{M} \models I\Delta_0 + \text{Exp}$ and each $a \in \mathcal{M}$, $a_{\vec{z}}$ denotes the set of E-members of $a$ in $\mathcal{M}$. Moreover, the $\Delta_0$-formulas $\text{Card}(x) = y$, $\langle \vec{x} \rangle = y$, $\text{Len}(x) = y$, $(x)_y = z$, and $x \mid y = z$ respectively express that "there exists some bijection between $y$ and the set coded by $x$", "the sequence number of $\vec{x}$ is $y$", "length of the sequence coded by $x$ is $y$", "the $y$-th element of the sequence number $x$ is $z$", and "the restriction of the sequence number $x$ to $y$ is $z$". In addition, for every formula $\varphi(x, y)$, by the formula $y = \mu_x \varphi(x)$ we mean "$y$ is the least element such that $\varphi(y)$ holds”.

Furthermore, for every $n \in \omega$ there exist $\mathcal{L}_A$-formulas $\text{Sat}_{\Sigma_n}$ and $\text{Sat}_{\Pi_n}$ which define the satisfaction predicate for $\Sigma_n$-formulas and $\Pi_n$-formulas respectively, in an ambient model. For every natural number $n > 0$, it can be shown that $\text{Sat}_{\Sigma_n}$ and $\text{Sat}_{\Pi_n}$ are $\Sigma_n$ and $\Pi_n$ respectively in $I\Sigma_1$. Moreover, $\text{Sat}_{\Delta_0} \in \Delta^1_{\text{II}}$ ([6, ch. I, Thm. 1.75]). If $\mathcal{M}$ is a nonstandard model of $I\Sigma_n$, the aforementioned feature along with $\Sigma_n$-Overspill in $\mathcal{M}$ imply that every coded $\Sigma_n$-type and every coded bounded $\Pi_n$-type is realized in $\mathcal{M}$.

- $\Sigma_n$-Pigeonhole Principle. For every $n > 0$, if $\mathcal{M} \models I\Sigma_n$, $a \in \mathcal{M}$, and $\varphi$ is a $\Sigma_1$-formula which defines a function from $a + 1$ into $a$ in $\mathcal{M}$, then $\varphi$ is not one-to-one.

- Given $\mathcal{L}_A$-structure $\mathcal{M}$ and subset $X$ of $\mathcal{M}$, for every $n > 0$, we define:

- $K^n(\mathcal{M}; X) := \text{the set of all } \Sigma_n\text{-definable element of } \mathcal{M} \text{ with parameters from } X$;
- $\Gamma^n(\mathcal{M}; X) := \{x : x \leq a \text{ for some } a \in K^n(\mathcal{M}; X)\}$;
- $H^n(\mathcal{M}; X) := \bigcup_{k \in \omega} H^n_k(\mathcal{M}; X)$, where:

$$H^n_0(\mathcal{M}; X) := \Gamma^n(\mathcal{M}; X) \text{, and } H^n_{k+1}(\mathcal{M}; X) := \Gamma^n(\mathcal{M}; H^n_k(\mathcal{M}; X)).$$

- $K(\mathcal{M}; X) := \bigcup_{n \in \omega} K^n(\mathcal{M}; X)$.

(When $X = \emptyset$, we omit $X$ from the notations.) Clearly, $\Gamma^n(\mathcal{M}; X)$ and $H^n(\mathcal{M}; X)$ are initial segments of $\mathcal{M}$. The following properties of these submodels of $\mathcal{M}$ are well-known (e.g. see [6, Ch. IV, Thm. 1.33]):

**Theorem 2.1.** Suppose $n > 0$, and $\mathcal{M} \models I\Sigma_n$ and $X \subseteq M$, then the following hold:
A given structure $\mathcal{M}$ is called \textit{recursively saturated} if it realizes every recursive type with finite parameters in $\mathcal{M}$. In [2], Barwise and Shilipf showed that \textit{any countable model $\mathcal{M}$ of PA is recursively saturated iff it carries an inductive satisfaction class}; here an \textit{inductive satisfaction class} $S$ of $\mathcal{M}$ is a subset of $\mathcal{M}$ which contains $\langle \varphi, a \rangle$ such that (1) $\mathcal{M} \models \text{Form}(\varphi)$, (2) $(\mathcal{M}; S) \models \text{PA}^*$, and (3) $(\mathcal{M}; S)$ satisfies Tarski’s inductive conditions for satisfaction (for a more precise definition see [1]). It is folklore that for every countable recursively saturated model $\mathcal{M}$ of PA there exists some inductive satisfaction class $S$ such that $(\mathcal{M}; S)$ is also recursively saturated (e.g. see [11]).

For every cut $I$ of $\mathcal{M}$ the I-\textit{Standard System} of $\mathcal{M}$, denoted by $SSy_I(\mathcal{M})$, is the family of subsets of $I$ of the form $I \cap a_E$ for some $a \in \mathcal{M}$. By $SSy(\mathcal{M})$ we mean $SSy_{\mathbb{N}}(\mathcal{M})$. It is well-known that for every model $\mathcal{M}$ of $I\Sigma_n$ (for $n > 0$), $SSy_I(\mathcal{M})$ is equal to the family of subsets of $I$ which are $\Sigma_n$-definable (with parameters) in $\mathcal{M}$ (see [6, Ch. I]). Moreover, it is easy to check that if $\mathcal{N}$ is an initial segment and a submodel of $\mathcal{M}$ containing $I$, then $SSy_I(\mathcal{M}) = SSy_I(\mathcal{N})$ (see [7]).

A given model $\mathcal{M}$ of $I\Delta_0$ is called \textit{1-tall} if $K^1(\mathcal{M}; a)$ is cofinal in $\mathcal{M}$ for no $a \in \mathcal{M}$; and it is called \textit{1-extendable} if it possesses some end extension $\mathcal{N} \models I\Delta_0$ such that $\text{Th}_{\Sigma_1}(\mathcal{M}) = \text{Th}_{\Sigma_1}(\mathcal{N})$. Dimitracopoulos and Paris, in [3] showed that:

\textbf{Theorem 2.2} (Dimitracopoulos-Paris). (1) \textit{For any two countable and nonstandard models $\mathcal{M}$ and $\mathcal{N}$ of $I\Delta_0 + \text{Exp}$ such that $\mathcal{M}$ is 1-extendable and $\mathcal{N}$ is 1-tall, there exists a proper initial embedding from $\mathcal{M}$ into $\mathcal{N}$ iff $SSy(\mathcal{M}) = SSy(\mathcal{N})$ and $\text{Th}_{\Sigma_1}(\mathcal{M}) \subseteq \text{Th}_{\Sigma_1}(\mathcal{N})$.}

(2) \textit{Any 1-tall countable model $\mathcal{M}$ of $\Sigma_1 + \text{Exp}$ in which $\mathbb{N}$ is not $\Pi_1$-definable (without parameters), is 1-extendable.}

A given cut $I$ of a model $\mathcal{M}$ is called \textit{strong} if for every coded function $f$ of $\mathcal{M}$ whose domain contains $I$, there exists some $e > I$ such that $f(i) \in I$ iff $f(i) < e$ for all $i \in I$. Paris and Kirby, in [9], proved that $I$ is a strong cut of a model $\mathcal{M}$ of $I\Delta_0 + \text{Exp}$ iff $(I, SSy_I(\mathcal{M})) \models \text{ACA}_0$ (here $\text{ACA}_0$ is the subsystem of second order arithmetic with the comprehension scheme restricted to formulas with no second order quantifier).

For given $L_A$-structures $\mathcal{M}$ and $\mathcal{N}$, an (a proper) \textit{initial embedding} $j$ is an embedding from $\mathcal{M}$ into $\mathcal{N}$ whose image is an (a proper) initial segment of $\mathcal{N}$. To every self-embedding $j$ of $\mathcal{M}$, we associate two subsets of $\mathcal{M}$:

$$I_{\text{fix}}(j) := \{ m \in M : \forall x \leq m \ j(x) = x \},$$

and

$$\text{Fix}(j) := \{ m \in M : j(m) = m \}.$$
In [1], it is shown that for every model \( M \) of \( \text{I} \Sigma_1 \), and any self-embedding \( j \) of \( M \), it holds that \( K^1(M) \prec \text{I} \Sigma_1 \text{Fix}(j) \prec \text{I} \Sigma_1 M \). Consequently, \( \text{Fix}(j) \models \text{I} \Delta_0 + \text{Exp} \). The following results on the set of fixed points of initial self-embeddings were also proved in [1]:

**Theorem 2.3** (B-Enayat). Let \( M \) and \( N \) be countable nonstandard models of \( \text{I} \Sigma_1 \), \( c \in M \) and \( d, b \in N \), and \( I \) be a proper cut shared by \( M \) and \( N \) which is closed under exponentiation. Then the following are equivalent:

1. There exists some proper initial embedding \( j \) from \( M \) into \( N \) such that \( I \subseteq \text{I} \text{fix}(j) \), \( j(M) < b \), and \( j(c) = d \).
2. \( \text{SSy}_I(M) = \text{SSy}_I(N) \), and for every \( \Delta_0 \)-formula \( \delta(z, x, y) \) and every \( i \in I \) it holds that:
   \[ M \models \exists z \delta(z, c, i) \Rightarrow N \models \exists z < b \delta(z, d, i). \]

**Remark 1.** With the above assumptions, suppose \( a \in M \cap N \) such that for all \( \Delta_0 \)-formula \( \delta \) and for every \( i \in I \) it holds that:

\[ M \models \exists z \delta(z, c, (a)_i) \Rightarrow N \models \exists z < b \delta(z, d, (a)_i). \]

Then, by an appropriate modification in the proof of Theorem 2.3, we can manage to construct the above proper initial embedding \( j \) with the additional feature that \( j((a)_i) = (a)_i \) for every \( i \in I \).

**Theorem 2.4** (B-Enayat). Suppose \( M \models \text{I} \Sigma_1 \) is countable and nonstandard and \( I \) is a cut of \( M \). Then the following hold:

1. \( I \) is closed under exponentiation iff there exists some proper initial self-embedding \( j \) of \( M \) such that \( \text{I} \text{fix}(j) = I \).
2. \( I \) is strong in \( M \) and \( I \prec \text{I} \Sigma_1 M \), iff there exists some proper initial self-embedding \( j \) of \( M \) such that \( \text{Fix}(j) = I \).
3. \( N \) is strong in \( M \) iff there exists some proper initial self-embedding \( j \) of \( M \) such that \( \text{Fix}(j) = K^1(M) \).

- The following lemma from [1] will be useful in section 4 of this paper:

**Lemma 2.5.** Suppose \( M \models \text{I} \Delta_0 + \text{Exp} \) in which \( \mathbb{N} \) is not a strong cut, then for any self-embedding \( j \) of \( M \), the following hold:

1. The nonstandard fixed points of \( j \) are downward cofinal in the nonstandard part of \( M \).
2. For every element \( a \in M \), and \( m \in \text{Fix}(j) \) there exists an element \( b \in \text{Fix}(j) \) such that:
   \[ \text{Th}_{\Sigma_1}(M; a, m) \subseteq \text{Th}_{\Sigma_1}(M; b, m). \]
• **Convention.** Suppose $M \models \Sigma_1$ and $\langle \delta_r : r \in M \rangle$ is a canonical enumeration of $\Delta_0$-formulas in $M$. For every $r \in M$:

- $f_r(\langle \rangle) = \Diamond$ denotes the following partial $\Sigma_1$-function in $M$:
  $$\exists \bar{z}(\bar{z}_0 = \Diamond \land z = \mu_y \mathop{\text{Sat}}_{\Delta_0}(\delta_r(\langle \rangle, (y)_0, (y)_1))).$$

- The notation $f_r(\bar{x}) \downarrow$ denotes the $\Sigma_1$-formula $\exists z, y \mathop{\text{Sat}}_{\Delta_0}(\delta_r(\bar{x}, y, z))$, and $f_r(\bar{x}) \downarrow^w$ stands for the formula $\exists z, y < w \mathop{\text{Sat}}_{\Delta_0}(\delta_r(\bar{x}, y, z))$.

Finally, we put $F(M)$ to be the collection of all $\emptyset$-definable partial $\Sigma_1$-functions in $M$. As noted in [II], if $M$ and $N$ are two models of $I\Delta_0$ such that $\mathop{\text{Th}}_{\Sigma_1}(M) = \mathop{\text{Th}}_{\Sigma_1}(N)$, then $F(M) = F(N) = F := \{f_n : n \in \mathbb{N}\}$. Moreover, in [I] it is shown that:

$$K^1(M; a) = \{f(a) : f \in F \text{ and } M \models [f(a) \downarrow]\}.$$

### 3 I-small $\Sigma_1$-elementary submodels

In [13], Lascar introduced a class of submodels of models of arithmetic, namely small submodels, which resemble those submodels of a model of set theory whose cardinality is less than the cardinality of the ground model. Then, Enayat inspired by a result of Schmerl (stated without proof as Theorem 5.7 in [8]), generalized this notion in [I]. In this section we will prove some results about these submodels.

**Definition 1.** For a given proper cut $I$ of a model $M$ of $I\Delta_0 + \text{Exp}$, subset $X$ of $M$ is called $I$-small in $M$ if there exists some $a \in M$ such that $X = \{(a)_i : i \in I\}$, and $(a)_i \neq (a)_j$ for all distinct $i, j \in I$. When $I = \mathbb{N}$, we simply use small for $\mathbb{N}$-small.

It is easy to see that for every model $M$ of $\Sigma_1$, each proper cut $I$ of $M$ is $I$-small. Moreover, for every $a \in M$, $K^1(M; a)$ is small in $M$. In [13], it is shown that every recursively saturated model $M$ of PA possesses some small submodel which is not finitely generated. This result can be generalized for $I$-small submodels, when $I$ is a strong cut of $M$ (see Theorem 3.2 below). Furthermore, By using compactness arguments, for every model $M$ of $\Sigma_1$, we can find some elementary extension of $M$ in which it is small. And finally, in [12] it is shown that every nonstandard small submodel is a mixed submodel (i.e. neither cofinal, nor initial segment). In a similar manner, for every cut $I$ of a model $M$ of $\Sigma_1$, and each $I$-small submodel $M_0$ of $M$, if $I \subseteq M_0$ then $M_0$ is mixed in $M$ (since if $M_0 := \{(a)_i : i \in I\}$, and $A := \{i \in I : M \models \neg \epsilon(a)_i\}$, then $A \in \text{SSy}_I(M) \setminus \text{SSy}_I(M_0)$. So $M_0$ cannot be an initial segment of $M$).

In the following lemma we will show that in the definition of $I$-small, if $I$ is a strong cut or it is equal to $\mathbb{N}$, then the condition $(a)_i \neq (a)_j$ for all distinct $i, j \in I$, can be eliminated:

**Lemma 3.1.** Suppose $M \models \Sigma_1$ is nonstandard, $I \subseteq M$, $M_0$ is a submodel of $M$ such that $M_0 = \{(a)_i : i \in I\}$ for some $a \in M$. Then the following hold:

---
(1) If $I = \mathbb{N}$, then $\mathcal{M}_0$ is small.

(2) If $I$ is strong in $\mathcal{M}$, then $\mathcal{M}_0$ is $I$-small.

**Proof.** First, we will inductively define the following $\Delta_0$-function (with parameters) in $\mathcal{M}$:

$$
g(0) := (a)_0,$$

and

$$
g(x + 1) := y \text{ iff } y = (a)_r \land g(x) = (a)_u \land \forall w < z ((a)_w \neq (a)_z \land \exists v \leq u((a)_w = (a)_v)) \right).$$

Note that by the way we defined $g$, its domain is an initial segment of $\mathcal{M}$, and $\text{Dom}(g) \leq \text{Len}(a)$. Moreover, since $I$ and $\mathcal{M}_0$ are not $\Delta_0$-definable in $\mathcal{M}$, then $I \subset \text{Dom}(g)$. So by $\Sigma_1$-induction in $\mathcal{M}$, we can find some $d \in M$ such that $(d)_i = g(i)$ for every $i \in I$. Clearly, $(d)_i \neq (d)_j$ for every distinct $i, j \in I$, and $\mathcal{M}_0 \subseteq \{(d)_i : i \in I\}$. Now, in each case of the statement of theorem we will prove that $\{(d)_i : i \in I\} \subseteq \mathcal{M}_0$:

(1) Suppose $I = \mathbb{N}$. If $\{(d)_n : n \in \mathbb{N}\} \not\subseteq \mathcal{M}_0$, then there exists the least number $n \in \mathbb{N}$ such that $(d)_n \notin \mathcal{M}_0$. So by the definition of $g$, there exist some $m \in \mathbb{N}$ and some $r \in M \setminus \mathbb{N}$ such that $(d)_{n-1} = (a)_m$ and $(d)_n = (a)_r$. Therefore, by the definition of $g$, it holds that $\mathcal{M}_0 = \{(a)_0, \ldots, (a)_m\}$, which is a contradiction.

(2) In the general case with the extra assumption that $I$ is strong in $\mathcal{M}$, consider the following partial $\Delta_0$-function in $\mathcal{M}$:

$$\forall x \in M : \mathcal{M} \models (a)_r = (a)_i.$$ 

Since $I$ is strong and $I \subseteq \text{dom}(h)$ (because $g$ is well-defined on $I$), there exists some $e \in M$ such that $h(i) \in I$ iff $h(i) < e$, for all $i \in I$. Moreover, by the definition of $d, g$ and $h$, for every $i \in I$ it holds that $(d)_i = (a)_{h(i)}$. So it suffices to prove that $h(i) < e$ for every $i \in I$. Suppose not; so there exists some $i_0 \in I$ which is the least element of $M$ such that $h(i_0) > e$. Now, by the way we defined $g$ and $h$, it holds that:

$$\mathcal{M} \models \forall i < h(i_0) ((a)_i \neq (a)_{h(i_0)} \land \exists j \leq h(i_0 - 1)((a)_i = (a)_j)).$$

Therefore, $\mathcal{M}_0 = \{x \in M : \mathcal{M} \models \exists i \leq h(i_0 - 1) x = (a)_i\}$. So $\mathcal{M}_0$ is $\Delta_0$-definable in $\mathcal{M}$, which is a contradiction.

In the following theorem, we will show that when $I$ is strong, the basic properties which hold for small submodels, also hold for $I$-small ones.
Theorem 3.2. Let $\mathcal{M} \models \Sigma_1$ be nonstandard, and $I$ be a strong cut of $\mathcal{M}$. Then:

1. For every $a \in M$, $K^1(\mathcal{M}; I \cup \{a\})$ is $I$-small.

2. If $\mathcal{M}_0$ is an $I$-small submodel of $\mathcal{M}$, then $I \subseteq M_0$.

3. If $\mathcal{M} \models \text{PA}$ is countable and recursively saturated, then there exists some $I$-small elementary submodel of $\mathcal{M}$ which is not of the form of $K(\mathcal{M}; I \cup \{a\})$ for any $a \in M$.

Proof. (1) First fix some arbitrary $s > I$. So by using strong $\Sigma_1$-Collection in $\mathcal{M}$ for the formula $\text{Sat}_{\Delta_0}(\delta_r(i, a, z))$, we will find some $b \in M$ such that:

\[
\mathcal{M} \models \forall(r, i) < s ( [f_r(i, a) \downarrow ] \to [f_r(i, a) \downarrow ] < b).
\]

Then, by using $\Sigma_1$-induction we observe that $\mathcal{M} \models \exists y \forall(r, i) < s \varphi(y, r, i, a, b)$, in which $\varphi(r, i, a, b)$ is the following $\Delta_0$-formula:

\[
( [f_r(i, a) \downarrow ] < b \to (y)_{r, i} = f_r(i, a)) \land (\neg[f_r(i, a) \downarrow ] < b \to (y)_{r, i} = 0).
\]

As a result, if $d \in M$ is such that $\mathcal{M} \models \forall(r, i) < s \varphi(d, r, i, a, b)$, then:

\[
K^1(\mathcal{M}; I \cup \{a\}) = \{(d)_i : i \in I\}.
\]

So by Lemma 3.1, $K^1(\mathcal{M}; I \cup \{a\})$ is $I$-small in $\mathcal{M}$.

(2) The exact argument used in [4] Thm. 4.5.1 works here: let $M_0 = \{(a)_i : i \in I\}$ for some $a \in M$ such that $(a)_i \neq (a)_j$ for all distinct $i, j \in I$. Then put:

\[
Z := \{(y, z) \in M : \mathcal{M} \models (a)_y = z\}.
\]

Since $Z$ is $\Delta_0$-definable in $\mathcal{M}$, then $X := I \cap Z \in \text{SSy}_f(\mathcal{M})$. As a result, because $I$ is strong in $\mathcal{M}$, $(I, X) \models \text{PA}^\ast$. Now, suppose $I \nsubseteq M_0$. So $(I, X) \models \exists x (\forall y \langle y, x \rangle \notin X)$. Let $(I, X) \models x_0 := \mu_x(\forall y \langle y, x \rangle \notin X)$. Therefore, $x_0 \notin M_0$. So since $x_0 \neq 0$, and by the definition of $x_0$, we conclude that $x_0 - I \in M_0$, which contradicts the fact that $\mathcal{M}_0$ is a submodel of $\mathcal{M}$.

(3) We will generalize the method used in [13] Pro. 2.10]: let $S$ be a nonstandard inductive satisfaction class for $\mathcal{M}$ such that $(\mathcal{M}; S)$ is recursively saturated. Put $\mathcal{M}_s := (\mathcal{M}; S)$, and $N := K(\mathcal{M}_s; I \cup \{s\})$ for some $s > I$. First note that $N$ is $I$-small in $\mathcal{M}$: since $\mathcal{M}_s$ is a countable recursively saturated model of $\text{PA}^\ast$, so it also possesses an inductive satisfaction class. Moreover, $I$ is also strong in $\mathcal{M}_s$. Therefore, by repeating the argument used in the proof of part (1) of this theorem, and Lemma 3.1(2), we can show that $N$ is $I$-small in $\mathcal{M}$.

Moreover, on one hand, it is easy to see that $S \cap N$ is a nonstandard satisfaction class for the $\mathcal{L}_s$-structure $N$. So $N$ is also a recursively saturated model of $\text{PA}$. On the other hand, $I$ is a proper initial segment of $N$ (because $s > I$). Therefore, $N$ is of the form of $K(\mathcal{M}; I \cup \{a\})$ for no $a \in M$. 

\[
\square
\]
The following lemma will be useful in the proof of the main theorem of this section:

**Lemma 3.3.** Suppose \( M \models \Sigma_1 \), \( I \) is a strong cut of \( M \), and \( a \in M \setminus I \) such that \( (a)_i \neq (a)_j \) for all distinct \( i, j \in I \). Moreover, let \( M_0 = \{(a)_i : i \in I\} \) be a \( \Sigma_1 \)-elementary submodel of \( M \), \( X \subseteq M_0 \) be coded in \( M \), and \( i_0 \in I \) such that \( i < i_0 \) for all \( (a)_i \in X \). Then \( X \) is coded in \( M_0 \).

**Proof.** Suppose \( \alpha \in M \) codes \( X \) in \( M \). So \( M \models \alpha = \sum_{i \leq i_0} \delta^{(i)} \), in which \( i_1 = \text{Card}(X) \leq i_0 \) and \( \sigma := \langle x : xE_\alpha \rangle \) (so \( \text{Len}(\sigma) = i_1 \)). Since \( \delta(x, y, z) \) is a \( \Delta_0 \)-formula and \( M_0 \prec_{\Sigma_1} M \), it suffices to prove that \( \sigma \in M_0 \). For this purpose let \( Y := \{i < i_0 : M \models (a)_iE_\alpha\} \). Then there exists some \( \gamma \in I \) which codes \( Y \).

Now, we define:

\[
h(z) := \begin{cases} \mu_u((a)_x : xEz) = (a)_u \land u < \text{Len}(a) & \text{if } M \models \exists u < \text{Len}(a) \langle (a)_x : xEz \rangle = (a)_u; \\ 0 & \text{otherwise} \end{cases}
\]

Since \( I \) is strong in \( M \), there exists some \( e \) such that \( h(i) > e \) if \( h(i) > I \), for all \( i \in I \). We claim that \( M \models \forall x \varphi(x, a, \gamma, e) \), where \( \varphi(x, a, \gamma, e) \) is the following \( \Delta_0 \)-formula:

\[
\forall y < \text{Len}(x) \exists zE_\gamma \langle (x)_y = (a)_z \rangle \rightarrow \exists w < \min\{e, \text{Len}(a)\} \langle x = (a)_w \rangle.
\]

Therefore, \( M \models \varphi(\sigma, a, \gamma, e) \), which implies that \( \sigma = (a)_c \) for some \( c < \min\{e, \text{Len}(a)\} \). So \( \sigma = (a)_{h(\gamma)} \) and \( h(\gamma) < e \), which implies that \( \sigma \in M_0 \).

In order to prove the above claim, we will use \( \Delta_0 \)-induction inside \( M \): let \( x \in M \) such that \( M \models \varphi(w, a, \gamma, e) \) for every \( w < x \), and \( M \models \forall y < \text{Len}(x) \exists zE_\gamma \langle (x)_y = (a)_z \rangle \). So by induction hypothesis \( M \models x \models \exists z < \text{Len}(x) \exists zE_\gamma \langle (x)_y = (a)_z \rangle \). Then, we put \( Z := \{i < \gamma : M \models \exists z < \text{Len}(x) \exists zE_\gamma \langle (x)_y = (a)_z \rangle \} \). So by induction hypothesis \( Z \subseteq M_0 \). Then, we put \( z_0 \in I \) such that \( M \models \exists z < \text{Len}(x) \exists zE_\gamma \langle (x)_y = (a)_z \rangle \). As a result, \( h(z_0) \leq z < \min\{e, \text{Len}(a)\} \), which implies that \( x \models x \models \exists z \in I \models \exists z < \text{Len}(x) \exists zE_\gamma \langle (x)_y = (a)_z \rangle \). So since \( M_0 \prec_{\Sigma_1} M \), then \( x \) is in \( M_0 \). Therefore, \( x = (a)_i \) for some \( i \in I \) < \min\{e, \text{Len}(a)\} \).

Now we are ready to prove the main theorem and corollary of this section. The method we use for proving Theorem 3.4 is a combination of the back-and-forth method used in [1 Thm. 6.1] and [2 Thm. 5.6].

**Theorem 3.4.** Assume \( N \models \Sigma_1 \) is countable and nonstandard, \( I \) is a strong cut of \( N \), and \( N_0 \) is an \( I \)-small \( \Sigma_1 \)-elementary submodel of \( N \) such that \( I \neq N_0 \). Then there exists some proper initial self-embedding \( j \) of \( H^1(N; N_0) \) such that \( N_0 = \text{Fix}(j) \).

**Proof.** Put \( M := H^1(N; N_0) \). So by Theorem 2.1, \( M \) is a \( \Sigma_1 \)-elementary initial segment of \( N \) such that \( M \models \Sigma_1 \), and it is easy to see that \( I \) is also strong in \( M \). Moreover, since \( N_0 \neq I \), by using \( \Sigma_1 \)-Overspill in \( M \) we can find some \( a \in M \) such that \( N_0 = \{(a)_i : i \in I\} \) and \( (a)_i \neq (a)_j \) for distinct \( i, j \in I \). In order to construct \( j \), first by using strong \( \Sigma_1 \)-Collection in \( M \), we will find some \( b \in M \) such that:

...
\[ \mathcal{M} \models [f((a)_i) \downarrow] \rightarrow [f((a)_i) \downarrow]^<_{b}, \text{ for all } f \in \mathcal{F} \text{ and all } i \in I. \]

Then, by using back-and-forth method we will inductively build finite functions \( \bar{u} \mapsto \bar{v} \) such that \( \bar{u}, \bar{v} \in M \), and \( \mathcal{M} \models (\bar{v} < b \land P(\bar{u}, \bar{v}) \land Q(\bar{u}, \bar{v})) \), in which:

\[
P(\bar{u}, \bar{v}) \equiv [f(\bar{u}, (a)_i) \downarrow] \rightarrow [f(\bar{v}, (a)_i) \downarrow]^<_{b}, \text{ for all } f \in \mathcal{F} \text{ and } i \in I;\]

\[
Q(\bar{u}, \bar{v}) \equiv \left( [f(\bar{u}, (a)_i) \downarrow] \land [f(\bar{v}, (a)_i) \downarrow]^<_{b} \land f(\bar{u}, (a)_i) \notin N_0 \right) \Rightarrow f(\bar{u}, (a)_i) \neq f(\bar{v}, (a)_i), \text{ for all } f \in \mathcal{F} \text{ and all } i \in I.\]

Through the ‘forth’ stages of back-and-forth we shall make the domain of \( j \) to be equal to \( M \), and ‘back’ stages are for making the range of \( j \) to be an initial segment of \( M \). For the first step of induction, we will choose \( 0 \mapsto 0 \). Then, suppose \( \bar{u} \mapsto \bar{v} \) is built such that \( \mathcal{M} \models (\bar{v} < b \land P(\bar{u}, \bar{v}) \land Q(\bar{u}, \bar{v})) \).

**‘Forth’ stages:** Let \( m \in M \setminus \{\bar{u}\} \). By the definition of \( \mathcal{M} \), without loss of generality, we can assume that \( m \leq t(\bar{u}, (a)_i) \) for some \( t \in \mathcal{F} \) and \( i \in I \). In order to find some image for \( m \), first note that since \( P(\bar{u}, \bar{v}) \) holds in \( \mathcal{M} \), Theorem 2.3 and Remark 1 imply that:

\[
(1) : \quad \text{There exists some initial self-embedding } j_0 \text{ of } \mathcal{M} \text{ such that } j_0(M) < b, \ j_0(\bar{u}) = \bar{v}, \text{ and } N_0 \subseteq \text{Fix}(j_0).\]

Then, we define:

\[ C := \{ \langle r, i \rangle \in I : \mathcal{M} \models [f_r(\bar{u}, m, (a)_i) \downarrow] \text{ and } f_r(\bar{u}, m, (a)_i) \notin K^1(\mathcal{M}; N_0 \cup \{\bar{u}\}) \}. \]

We claim that \( C \in \text{SSy}_I(\mathcal{M}) \); so there exists some \( \alpha \in M \) such that \( C = I \cap \alpha_E \). To prove this claim, let:

\[ R := \left\{ \langle \langle r, i \rangle, k, t \rangle \in I : \mathcal{M} \models \left( [f_r(\bar{u}, m, (a)_i) \downarrow] \land [f_t(\bar{u}, (a)_k) \downarrow] \rightarrow f_r(\bar{u}, m, (a)_i) = f_t(\bar{u}, (a)_k) \right) \right\}. \]

On one hand, since \( R \) is \( \Pi^1_1 \)-definable in \( \mathcal{M} \), then \( R \in \text{SSy}_I(\mathcal{M}) \). On the other hand, by Lemma 3.2(2), it holds that:

\[
I \setminus C \models \exists k, t \langle \langle r, i \rangle, k, t \rangle \in R. \]

Since \( I \) is strong in \( \mathcal{M} \), which implies that \( (I, \text{SSy}_I(\mathcal{M})) \models \text{ACA}_0 \), and because \( B \) is arithmetical in \( R \) and \( R \in \text{SSy}_I(\mathcal{M}) \), we may deduce that \( B \in \text{SSy}_I(\mathcal{M}) \), and consequently \( C \in \text{SSy}_I(\mathcal{M}) \).
Now, for every $s \in M$, we define:
\[
    p_s(y) := \{ y \leq t(\bar{v}, (a)_1) \} \cup p_{s1}(y) \cup p_{s2}(y); \text{ where:}
\]
\[
    p_{s1}(y) := \{ \forall i < s([f(\bar{u}, m, (a)_1) \downarrow] \rightarrow [f(\bar{v}, y, (a)_i) \downarrow]^{<b} : f \in \mathcal{F}) \};
\]
\[
    p_{s2}(y) := \{ \forall i < s \left( \left( [f_n(\bar{v}, y, (a)_i) \downarrow]^{<b} \land \langle n, i \rangle \varepsilon \alpha \right) \rightarrow f_n(\bar{u}, m, (a)_1) \neq f_n(\bar{v}, y, (a)_i) \right) : n \in \mathbb{N} \}.
\]

We shall show that there is some $s > I$ such that $p_s$ is finitely satisfiable; then since $p_s$ is $\Pi_1$, bounded and recursive, there exists some $m'$ which realises $p_s$ in $M$. Therefore, $m'$ serves as the image of $m$, and this finishes the ‘fortth’ stage.

In order to find such $s$, we claim that for every $k \in \mathbb{N}$ it holds that:
\[
    \left( \text{For every } f \in \mathcal{F}, \text{ every } z \in N_0, \text{ and any nonempty finite set } \{ f_{n_0}, \ldots, f_{n_k} \} \text{ of elements of } \mathcal{F}, \right.
\]
\[
    \text{there exists some } s > I \text{ such that } M \models \Psi(f, f_{n_0}, \ldots, f_{n_k}, \bar{u}, m, \bar{v}, b, a, s, \alpha, z, (a)_1),
\]
\[
    \text{where } \Psi(f, f_{n_0}, \ldots, f_{n_k}, \bar{u}, m, \bar{v}, b, a, s, \alpha, z, (a)_1) \text{ is the following } \Pi_1\text{-formula:}
\]
\[
    \left. \exists y \leq t(\bar{v}, (a)_1) \left( \forall i < s([f(\bar{u}, m, (a)_1), z) \downarrow] \rightarrow [f(\bar{v}, y, (a)_i) \downarrow]^{<b} \land \right. \right.
\]
\[
    \left. \left. \left( [f_n(\bar{v}, y, (a)_i) \downarrow]^{<b} \land \langle n, i \rangle \varepsilon \alpha \right) \rightarrow f_n(\bar{u}, m, (a)_1) \neq f_n(\bar{v}, y, (a)_i) \right) \right). \right)
\]

This claim completes the proof in the following way:

Let $d > I$ be an arbitrary and fixed element of $M$. Suppose $i, s \in M$, and $\Theta(s, i, \bar{u}, m, \bar{v}, b, a, \alpha, \beta, (a)_1)$ is the following $\Delta_0$-formula:
\[
    \forall r < i \exists y \leq t(\bar{v}, (a)_1) \left( \forall w < s(\langle r, w \rangle \varepsilon \beta \rightarrow [f_r(\bar{v}, y, (a)_w) \downarrow]^{<b} \land \right.
\]
\[
    \left. \left. \forall w < s \forall r' < i \left( (\{f_r(\bar{v}, y, (a)_w) \downarrow]^{<b} \land \langle r', w \rangle \varepsilon \alpha \right) \rightarrow f_r(\bar{u}, m, (a)_w) \neq f_r(\bar{v}, y, (a)_w) \right) \right); \right)
\]

where $\beta$ is the code of the following $\Sigma_1$-definable set in $M$:
\[
L := \{ (r, w) < d : M \models [f_r(\bar{u}, m, (a)_w) \downarrow] \}.
\]

Now, for every $i \in M$, we define:
\[
    g(i) := \max\{ x < d : M \models \Theta(x, i, \bar{u}, m, \bar{v}, b, a, \alpha, \beta, (a)_1) \}.
\]

Clearly $g$ is $\Delta_0$-definable function in $M$, and $I \subseteq \operatorname{Dom}(g)$ (we assume $\max(\emptyset) = 0$). Therefore, since $I$ is strong, there exists some $e > I$ such that for all $i \in I$, $g(i) > I$ iff $g(i) > e$.

We will show that $p_e(y)$ is a finitely satisfiable type. First, note that by statement (1), $p_{e1}(y)$ is closed under conjunctions. So let $f_n, f_{n_0}, \ldots, f_{n_k}$ be some finite number of elements
of $\mathcal{F}$, and let $n^* = \max\{n, n_0, \ldots, n_k\}$. Then, use $(*)_{n^*}$, $(n^* + 2)$-many times; i.e for every $t = 0, \ldots, n^* + 1$ consider $f_t$ instead of $f$ in the assertion of $(*)_{n^*}$, $0 \in M_0$ instead of $z$, and $f_1, \ldots, f_{n^*}$. So by statement $(*)_{n^*}$, for every $t = 0, \ldots, n^* + 1$ there exists some $s_t > I$ such that $\mathcal{M} \models \Psi(f_t, f_0, \ldots, f_{n^* + 1}, \bar{u}, m, \bar{v}, b, a, s_t, \alpha, 0, (a)_1)$. Then, let $s^* := \min\{s_t : t < n^* + 1\}$. Therefore, $\mathcal{M} \models \Theta(s^*, n^*, \bar{u}, m, \bar{v}, b, a, \alpha, \beta, (a)_1)$. It is easy to see that if $d \leq s^*$ then $g(n^*) = d - 1$, and if $s^* < d$ then $s^* \leq g(n^*)$; so in both cases $g(n^*) > I$ and consequently $g(n^*) > e$. So $\mathcal{M} \models \Theta(e, n^*, \bar{u}, m, \bar{v}, b, a, \alpha, \beta, (a)_1)$; this proves that $p_e$ is finitely satisfiable.

**Proof of the claim $(*)_k$ for every $k \in \mathbb{N}$:** Suppose the claim is not true; i.e there is some $k \in \mathbb{N}$ for which there exists some nonempty finite set $\{f, f_{n_0}, \ldots, f_{n_k}\}$ of elements of $\mathcal{F}$, and some $z \in N_0$ such that for all $s > I$ it holds that:

$$\mathcal{M} \models \neg\Psi(f, f_{n_0}, \ldots, f_{n_k}, \bar{u}, m, \bar{v}, b, a, s, \alpha, z, (a)_1).$$

Therefore, by $\Sigma_1$-Underspill in $\mathcal{M}$, there exists some $s \in I$ such that:

$$\mathcal{M} \models \neg\Psi(f, f_{n_0}, \ldots, f_{n_k}, \bar{u}, m, \bar{v}, b, a, s, \alpha, z, (a)_1).$$

(2) : Let $k_0 \in \mathbb{N}$ be the least natural number, for which there exists a set $\{f, f_{n_0}, \ldots, f_{n_{k_0}}\}$ of elements of $\mathcal{F}$, some $z_0 \in N_0$, and some $s_0 \in I$ such that:

$$\mathcal{M} \models \neg\Psi(f, f_{n_0}, \ldots, f_{n_{k_0}}, \bar{u}, m, \bar{v}, b, a, s_0, \alpha, z_0, (a)_1).$$

Put:

$$X := \{x \in M : \mathcal{M} \models \exists i < s_0(x = (a)_i \land [f(\bar{u}, m, (a)_i, z_0) \downarrow]\}$$

and

$$X' := \{\langle n, x \rangle \in M : \mathcal{M} \models \exists i < s_0(x = (a)_i \land \bigvee_{t=0}^{k_0} n = n_t \land \langle n, i \rangle \in \alpha(\bar{v})\}.$$

By Lemma 3.3, there exist $(a)_{\xi} \in N_0$ and $(a)_{\xi} \in N_0$ which code $X$ and $X'$ respectively. So we can restate statement (2) in the following form:

(3) : Let $k_0 \in \mathbb{N}$ be the least natural number, for which there exists a set $\{f, f_{n_0}, \ldots, f_{n_{k_0}}\}$ of elements of $\mathcal{F}$, some $z_0, (a)_{\xi}, (a)_{\xi} \in N_0$ such that:

$$\mathcal{M} \models \forall b \leq t(\bar{v}, (a)_1) \left( \forall \epsilon < (a)_{\xi}(\epsilon E(a)_{\xi} \rightarrow [f(\bar{v}, y, \delta, z_0) \downarrow] < b \rightarrow \exists \epsilon < E(a)_{\xi} \bigvee_{t=0}^{k_0} \langle n_t, \epsilon \rangle E(a)_{\xi} \land [f_{n_t}(\bar{v}, m, \epsilon) \downarrow] < b \land f_{n_t}(\bar{v}, m, \epsilon) = f_{n_t}(\bar{v}, m, \epsilon) \right) \right).$$

Now, by considering the sequence number of $(f_{n_t}(\bar{v}, m, \epsilon)) : (n_t, \epsilon) E(a)_{\xi}$ in $\mathcal{M}$, we may quantify out $f_{n_t}(\bar{u}, m, \epsilon)$s from the formula in statement (3), and deduce that:
(4): $M \models \exists x \forall y \leq t(\bar{v}, (a)\downarrow) \theta(y, b, \bar{v}, x, (a)\xi, (a)\zeta, z_0)$, where $\theta(y, b, \bar{v}, x, (a)\xi, (a)\zeta, z_0)$ is the following $\Delta_0$-formula:

$$
\forall \epsilon < (a)\xi (\epsilon E(a) \zeta \rightarrow [f(\bar{v}, y, \epsilon, z_0) \downarrow]^{<b}) \rightarrow (\exists (n_t, \epsilon) E(a) \zeta (\partial n_t(\bar{v}, y, \epsilon) \downarrow^{<b} \land (x)_{(n_t, \epsilon)} = n_t(\bar{v}, y, \epsilon) ) ) .
$$

Then, we will define $\Sigma_1$-definable partial functions $b(\bar{v}, y, (a)\xi, (a)\zeta, z_0)$ and $s(\bar{v}, (a)\xi, (a)\zeta, z_0, (a)\downarrow)$, as follows (we omit the parameters $(a)\xi, (a)\zeta, (a)\downarrow$, and $z_0$ in the presentations of these functions for the sake of simplicity):

- $b(\bar{v}, y) := \min \left\{ w : \left( \forall \epsilon (\epsilon E(a) \zeta \rightarrow [f(\bar{v}, y, \epsilon, z_0) \downarrow]^{<w}) \land \exists (n_t, \epsilon) E(a) \zeta (\partial n_t(\bar{v}, y, \epsilon) \downarrow^{<w}) \right) \right\} .

- $s(\bar{v}) := x$ iff $\exists z = \mu w \forall y \leq t(\bar{v}, (a)\downarrow) \left( \left\{ \begin{array}{l}
\langle z \rangle_0 = x \\
[b(\bar{v}, y) \downarrow]^{<w(\epsilon)} \rightarrow (\exists (n_t, \epsilon) E(a) \zeta (\partial n_t(\bar{v}, y, \epsilon) \downarrow^{<b(\bar{v}, y)}) \land \theta(y, b(\bar{v}, y), \bar{v}, (a)\downarrow, (a)\xi, (a)\zeta, z_0) ) ) ;
\end{array} \right\} ;$

and $s_t(\bar{v}, \epsilon) := (s(\bar{v})_{(n_t, \epsilon)})$, for every $(n_t, \epsilon) E(a) \zeta (\partial n_t(\bar{v}, y, \epsilon) \downarrow^{<w})$.

From the definition of $s_t(\bar{v}, \epsilon)$s and statement (4) we may infer that:

(5): $M \models \forall y \leq t(\bar{v}, (a)\downarrow) \left( \left\{ \begin{array}{l}
[b(\bar{v}, y) \downarrow]^{<b} \land \forall \epsilon < (a)\xi (\epsilon E(a) \zeta \rightarrow [f(\bar{v}, y, \epsilon, z_0) \downarrow]^{<b(\bar{v}, y)}) ) \rightarrow (\exists (n_t, \epsilon) E(a) \zeta (\partial n_t(\bar{v}, y, \epsilon) \downarrow^{<b(\bar{v}, y)}) \land \theta(y, b(\bar{v}, y), \bar{v}, (a)\downarrow, (a)\xi, (a)\zeta, z_0) ) ) ;
\end{array} \right\} .

It is not difficult to express the formula in the statement (5) in the form of $\forall z < b \delta(\bar{v}, (a)\xi, (a)\zeta, z_0)$ for some $\Delta_0$-formula $\delta$. Therefore, by the property $P(\bar{u}, \bar{v})$, the definition of function $s$, and statement (5) we deduce that:

(6): $M \models \forall y \leq t(\bar{u}, (a)\downarrow) \left( \left\{ \begin{array}{l}
[b(\bar{u}, y) \downarrow]^{<b} \land \forall \epsilon < (a)\xi (\epsilon E(a) \zeta \rightarrow [f(\bar{u}, y, \epsilon, z_0) \downarrow]^{<b(\bar{u}, y)}) ) \rightarrow (\exists (n_t, \epsilon) E(a) \zeta (\partial n_t(\bar{u}, y, \epsilon) \downarrow^{<b(\bar{u}, y)}) \land \theta(y, b(\bar{u}, y), \bar{u}, (a)\downarrow, (a)\xi, (a)\zeta, z_0) ) ) ;
\end{array} \right\} .

Now, we will simultaneously define two more $\Sigma_1$-definable functions in $M$:

$$
\langle o(\bar{v}, y), h(\bar{v}, y) \rangle := \min \left\{ \langle n_t, \epsilon \rangle E(a) \zeta : \left( \partial n_t(\bar{v}, y, \epsilon) \downarrow^{<b(\bar{v}, y)}) \land \theta(y, b(\bar{v}, y), \bar{v}, (a)\downarrow, (a)\xi, (a)\zeta, z_0) ) \rightarrow (\exists (n_t, \epsilon) E(a) \zeta (\partial n_t(\bar{v}, y, \epsilon) \downarrow^{<b(\bar{v}, y)}) \land \theta(y, b(\bar{v}, y), \bar{v}, (a)\downarrow, (a)\xi, (a)\zeta, z_0) ) ) ;
\end{array} \right\} .
$$

(Note that, similar to the way we defined function $s$, we can express the above definition by a $\Sigma_1$-formula.) Then, by statement (5) it holds that:

(7): $M \models \forall y \leq t(\bar{v}, (a)\downarrow) \left( \left\{ \begin{array}{l}
[b(\bar{v}, y) \downarrow]^{<b} \land \forall \epsilon < (a)\xi (\epsilon E(a) \zeta \rightarrow [f(\bar{v}, y, \epsilon, z_0) \downarrow]^{<b}) ) \rightarrow (\exists (n_t, \epsilon) E(a) \zeta (\partial n_t(\bar{v}, y, \epsilon) \downarrow^{<b}) ) .
\end{array} \right\} .
$$
Similarly, from statement (6) we may deduce that:

$$(8): \mathcal{M} \models \forall y \leq t(\bar{u}, (a)_{i_0}) \left( \left[ [b(\bar{u}, y) \downarrow] \land \forall \varepsilon < (a)_{\xi}(\varepsilon E(a)_{\xi} \rightarrow [f(\bar{u}, y, \varepsilon, z_0) \downarrow]^{<b(\bar{u}, y)}) \right] \rightarrow \right)$$

Finally, we obtain a contradiction by dividing $k_0$ into two cases in the following way:

- If $k_0 = 1$, we inductively define the following $\Sigma_1$-function in $\mathcal{M}$:

$$w(\hat{\diamond}, 0) := \min \left\{ y \leq t(\hat{\diamond}, (a)_{i_0}) : \left( \forall \varepsilon < (a)_{\xi}(\varepsilon E(a)_{\xi} \rightarrow [f(\hat{\diamond}, y, \varepsilon, z_0) \downarrow]^{<b(\hat{\diamond}, y)}) \right) \right\},$$

and

$$w(\hat{\diamond}, i+1) := \min \left\{ y \leq t(\hat{\diamond}, (a)_{i_0}) : \varphi(\hat{\diamond}, i, y, (a)_{\xi}, (a)_{\xi}, z_0) \right\},$$

where $\varphi(\hat{\diamond}, i, y, (a)_{\xi}, (a)_{\xi}, z_0)$ is the following formula:

$$\forall x \leq i \left( \begin{array}{c}
[b(\hat{\diamond}, y) \downarrow] \land [b(\hat{\diamond}, y) \downarrow]^{<b(\hat{\diamond}, y)} \land \\
\forall \varepsilon < (a)_{\xi}(\varepsilon E(a)_{\xi} \rightarrow [f(\hat{\diamond}, y, \varepsilon, z_0) \downarrow]^{<b(\hat{\diamond}, y)}) \land \\
\left( \begin{array}{c}
[h(\hat{\diamond}, w(\hat{\diamond}, x)) \downarrow]^{<b(\hat{\diamond}, y)} \\
[f_n(\hat{\diamond}, w(\hat{\diamond}, x)) \downarrow]^{<b(\hat{\diamond}, w(\hat{\diamond}, x))} \\
f_n(\hat{\diamond}, w(\hat{\diamond}, x)) \neq f_n(\hat{\diamond}, w(\hat{\diamond}, x))
\end{array} \right) \\
\end{array} \right) \rightarrow \).$$

First, we will show that $\mathcal{M} \models [w(\bar{u}, i) \downarrow]$ for all $i \in I$. Otherwise, there exists the least $0 < i_0 \in I$ such that:

$$(9): \mathcal{M} \models \forall y \leq t(\bar{u}, (a)_{i_0}) \neg \varphi(\bar{u}, i_0, y, (a)_{\xi}, (a)_{\xi}, z_0).$$

Note that by the definition of $(a)_{\xi}$ and $(a)_{\xi}$ it holds that:

$$(10): \mathcal{M} \models ([b(\bar{u}, m) \downarrow] \land \forall \varepsilon < (a)_{\xi}(\varepsilon E(a)_{\xi} \rightarrow [f(\bar{u}, m, \varepsilon, z_0) \downarrow]^{b(\bar{u}, m)}).$$

So by statements (8), (9) and (10), there exists some $i_1 < i_0$ such that:

$$(11): \mathcal{M} \models \left( \begin{array}{c}
[h(\bar{u}, w(\bar{u}, i_1)) \downarrow] \land [f_n(\bar{u}, m, h(\bar{u}, w(\bar{u}, i_1))) \downarrow] \\
[f_n(\bar{u}, w(\bar{u}, i_1), h(\bar{u}, w(\bar{u}, i_1))) \downarrow] \\
f_n(\bar{u}, m, h(\bar{u}, w(\bar{u}, i_1))) = f_n(\bar{u}, w(\bar{u}, i_1))
\end{array} \right).$$

Clearly, $f_n(\bar{u}, w(\bar{u}, i_1), h(\bar{u}, w(\bar{u}, i_1))) \in K^1(M; N_0 \cup \{\bar{u}\})$. So by statement (11), $f_n(\bar{u}, m, h(\bar{u}, w(\bar{u}, i_1))) \in K^1(M; M_0 \cup \{\bar{u}\})$. So $\mathcal{M} \models \neg (\exists n_0, h(\bar{u}, w(\bar{u}, i_1)) E(a)_{\xi}$ (by the definition of $(a)_{\xi}$), which is in contradiction with the definition of the function $h$. 

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As a result, by the definition of \(w(\bar{u}, i)\) and statement (8), the function \(i \mapsto h(\bar{u}, w(\bar{u}, i))\) from \(\{i : i \leq s_0 + 1\}\) into \((\langle a \rangle \zeta)_E\) is well-defined and coded in \(\mathcal{M}\). So, since the cardinality of \((a)\zeta\) is less than \(s_0 + 1\), by \(\Sigma_1\)-Pigeonhole Principle in \(\mathcal{M}\), there exists some distinct \(i_0 < i_1 \leq s_0 + 1\) such that:

\[
(12) : \quad \mathcal{M} \models h(\bar{u}, w(\bar{u}, i_0)) = h(\bar{u}, w(\bar{u}, i_1)).
\]

Therefore, by statement (12) and the definition of \(h\) we conclude that:

\[
(13) : \quad \mathcal{M} \models \left( [s_0(\bar{u}, h(\bar{u}, w(\bar{u}, i_0))) \downarrow \wedge s_0(\bar{u}, h(\bar{u}, w(\bar{u}, i_1))) \downarrow \right) \land
\]

Moreover, by the definition of \(h(\bar{u}, w(\bar{u}, i))\), for \(i = i_0, i_1\) it holds that:

\[
(14) : \quad \mathcal{M} \models \left( [f_{n_0}(\bar{u}, w(\bar{u}, i), h(\bar{u}, w(\bar{u}, i))) \downarrow \wedge s_0(\bar{u}, h(\bar{u}, w(\bar{u}))) \downarrow \right) \land
\]

So statements (12), (13) and (14) imply that:

\[
(15) : \quad \mathcal{M} \models f_{n_0}(\bar{u}, w(\bar{u}, i_1), h(\bar{u}, w(\bar{u}, i_0))) = f_{n_0}(\bar{u}, w(\bar{u}, i_0), h(\bar{u}, w(\bar{u}, i_0))).
\]

But statement (15) is in contradiction with the definition of \(w(\bar{u}, i_1)\).

- If \(k_0 > 1\), by using Lemma 3.3, let \((a)\rho \in N_0\) be the code of the following subset of \(N_0\):

\[
A := \left\{ \langle o(\bar{v}, y), h(\bar{v}, y) \rangle : \mathcal{M} \models \left( \begin{array}{l}
y \leq t(\bar{v}, (a)_1) \wedge [(a(\bar{v}, y), h(\bar{v}, y)) \downarrow]^{<b} \wedge \\
\forall \varepsilon < (a)\zeta (E(\langle a \rangle \zeta) \rightarrow [f(\bar{v}, y, \varepsilon, z_0) \downarrow]^{<b} \wedge \\
\exists \varepsilon < (a)\zeta (E(\langle a \rangle \zeta) \land [f_{n_0}(\bar{v}, y, \varepsilon) \downarrow]^{<b} \wedge \\
f_{n_0}(\bar{v}, y, \varepsilon) = f_{n_0}(\bar{u}, m, \varepsilon)) \end{array} \right) \right\}
\]

So, by statements (3), (7), and the definition of \((a)\rho\), we conclude that:

\[
(16) : \quad \mathcal{M} \models \forall y \leq t(\bar{v}, (a)_1) \left( \begin{array}{l}
\forall \varepsilon < (a)\zeta (E(\langle a \rangle \zeta) \rightarrow [f(\bar{v}, y, \varepsilon, z_0) \downarrow]^{<b} \wedge \\
[(a(\bar{v}, y), h(\bar{v}, y)) \downarrow]^{<b} \wedge \\
\langle n_1, \varepsilon \rangle E(\langle a \rangle \zeta) \land [f_{n_1}(\bar{v}, y, \varepsilon) \downarrow]^{<b} \wedge \\
f_{n_1}(\bar{v}, y, \varepsilon) = f_{n_1}(\bar{u}, m, \varepsilon)) \end{array} \right).
\]

Let \(f' \in \mathcal{F}\) such that:

\[
f'(\Diamond, y, \varepsilon, \langle z_0, (a)\rho, (a)\zeta, (a)\zeta \rangle) = x
\]
iff
\[ x = f(\Diamond, y, \epsilon, z_0) \land [o(\Diamond, y), h(\Diamond, y)] \downarrow \land \neg(o(\Diamond, y), h(\Diamond, y))E(a) \rho. \]

So by considering \( f' \) instead of \( f \) in statement (3), statement (16) leads to contradiction with the minimality of \( k_0 \).

‘Back’ stages: Let \( m' \in M \setminus \{\bar{v}\} \) such that \( m' < v_0 := \max\{\bar{v}\} \), and \( u_0 := \max\{\bar{u}\} \). In order to find some element of \( M \) whose image is \( m' \), we modify the proof of the ‘forth’ stage in the following way:

- Let \( \alpha' \) be the code of the following set in \( \mathcal{M} \):

\[ C' := \{\langle r, i \rangle \in I : \mathcal{M} \models [f_r(\bar{v}, m', (a)_i) \downarrow]^{<b} \text{ and } f_r(\bar{v}, m', (a)_i) \notin K^1(\mathcal{M}; N_0 \cup \{\bar{v}\})\}. \]

- Replace \( p_s(y) \) by:

\[
q_s(x) := \{x < u_0\} \cup q_{s1}(x) \cup q_{s2}(x); \quad \text{where:}
\]

\[
q_{s1}(x) := \{\forall i < s(-[f(\bar{v}, m', (a)_i) \downarrow]^{<b} \rightarrow -[f(\bar{u}, x, (a)_i) \downarrow]) : f \in \mathcal{F}\}; \quad \text{and}
\]

\[
q_{s2}(x) := \{\forall i < s \left( \begin{array}{c}
[n_i, i]Ea' \wedge
\{f_n(\bar{u}, x, (a)_i) \downarrow \rightarrow f_n(\bar{v}, m', (a)_i) \}
\end{array} \right) : n \in \mathbb{N}\}. \]

- Let:

\[
{(s'_{k})_k} : \quad \left( \begin{array}{c}
\forall r < i \exists x < u_0
\forall i < s(-[f(\bar{v}, m', (a)_i) \downarrow]^{<b} \rightarrow -[f(\bar{u}, x, (a)_i) \downarrow]) \wedge
\forall i < s \forall z \leq k \left( \langle n_i, i \rangle E\text{a} \wedge [f_n(\bar{u}, x, (a)_i) \downarrow] \rightarrow f_n(\bar{u}, x, (a)_i) \neq f_n(\bar{v}, m', (a)_i) \right) \end{array} \right).
\]

- Replace \( \Theta(s, i, \bar{u}, m, \bar{v}, b, a, \alpha, \beta) \) with \( \Theta'(s, i, \bar{u}, \bar{v}, m', b, a, \alpha', \beta') \):

\[
\forall r < i \exists x < u_0 \left( \begin{array}{c}
\forall w < s(\langle x, r, w \rangle E\beta' \rightarrow [f_r(\bar{v}, m', (a)_w) \downarrow]^{<b}) \wedge
\forall w < s \forall r' > i \left( \langle r', w \rangle E\alpha' \wedge \langle x, r', (a)_w \rangle \right) \rightarrow f_r(\bar{u}, m', (a)_w) \neq f_r(\bar{v}, y, (a)_w) \end{array} \right); \]

where \( \beta' \) is the code of the following \( \Sigma_1 \)-definable set in \( \mathcal{M} \):

\[ L' := \{\langle x, r, w \rangle : \mathcal{M} \models (x < u_0 \land w < d \land r < d \land [f_r(\bar{a}, x, (a)_w) \downarrow])\}. \]
Between statements (3) and (4) we need to use $\Sigma_1$-Collection to deduce:

\[(3') : \mathcal{M} \models \exists w \forall x < u_0 \left( \forall \varepsilon < (a)_\lambda \left( [f(\bar{u}, x, \varepsilon, z_\lambda^0) \downarrow]^{< w} \rightarrow \varepsilon \mathcal{E}(a)_\lambda \rightarrow \exists \varepsilon < (a)_\eta \bigvee_{t \leq k_t} \left( (n_t, \varepsilon) \mathcal{E}(a)_\eta \land [f_{n_t}(\bar{u}, x, \varepsilon) \downarrow]^{< \omega} \land \nabla_{n_t}(\bar{v}, m', \varepsilon) \right) \right) \right) ;\]

in which $(a)_\lambda \in N_0$ and $(a)_\eta \in N_0$ code the following $Y$ and $Y'$ respectively:

\[Y := \{ x \in M : \mathcal{M} \models \exists i < s_0(x = (a)_i \land [f(\bar{v}, m', (a)_i, z_\lambda^0) \downarrow]^{< b}) \}, \quad \text{and} \]
\[Y' := \{ (n, x) \in M : \mathcal{M} \models \exists i < s_0(x = (a)_i \land \bigvee_{t=0}^{k_\lambda} n = n_t \land \langle n, i \rangle \alpha \rangle \}. \]

The rest of the argument goes smoothly by modifying the ‘forth’ stage according to the above changes, and this completes the proof. \(\Box\)

**Corollary 3.5.** Assume $\mathcal{M} \models \Pi_1$ is countable and nonstandard, $I$ is a proper cut of $\mathcal{M}$, and $\mathcal{M}_0$ is an $I$-small $\Sigma_1$-elementary submodel of $\mathcal{M}$. Then the following are equivalent:

1) $I$ is strong in $\mathcal{M}$.

2) There exists some proper initial self-embedding $j$ of $\mathcal{M}$ such that $M_0 = \text{Fix}(j)$.

**Proof.** Suppose $M_0 = \{(a)_i : i \in I\}$, for some $a \in M$ such that $(a)_i \neq (a)_j$ for all distinct $i, j \in I$.

(1) $\implies$ (2): If $M_0 = I$, then by Theorem 2.4(2), we are done. So suppose $I \subsetneq M_0$. First, by using Theorem 3.4 let $h$ be some proper initial self-embedding of $H^1(\mathcal{M}; M_0)$ such that $\text{Fix}(h) = M_0$. Moreover, fix some $b \in H^1(\mathcal{M}; M_0) \setminus M_0$ such that $h(H^1(\mathcal{M}; M_0)) < b$. Now, by using strong $\Sigma_1$-Collection in $H^1(\mathcal{M}; M_0)$, and since $H^1(\mathcal{M}; M_0) \prec \Sigma_1 \mathcal{M}$, we can find some $d \in H^1(\mathcal{M}; M_0)$ such that:

$\mathcal{M} \models [f((a)_i, b) \downarrow] \rightarrow [f((a)_i, b) \downarrow]^{< d}$, for all $f \in F$ and all $i \in I$.

Therefore, by Theorems 2.1 and 2.3 and Remark 1, there exists some proper embedding $k : \mathcal{M} \hookrightarrow H^1(\mathcal{M}; M_0)$ such that $M_0 \subseteq \text{Fix}(k)$, $k(M) < d$ and $b \in k(M)$ (note that since $H^1(\mathcal{M}; M_0)$ is an initial segment of $\mathcal{M}$, then $\text{SSy}_I(\mathcal{M}) = \text{SSy}_I(H^1(\mathcal{M}; M_0)))$. Finally, we put $j := k^{-1}hk$. It is easy to check that $j$ is a well-defined proper initial self-embedding of $\mathcal{M}$ such that $\text{Fix}(j) = M_0$.

(2) $\implies$ (1): We combine the methods used in the proof of Theorem 5.1 and 6.1 of \([1]\). Suppose $I$ is not strong; i.e. there exists some coded function $f$ in $\mathcal{M}$ such that $I \subseteq \text{Dom}(f)$, and the set $D := \{f(i) : i \in I \land I < f(i)\}$ is downward cofinal in $M \setminus I$.

Let $b \in M \setminus M_0$ and $g := j(f)$. For every $k \in M$, we put:

$A_k := \{(r, y) < k : \mathcal{M} \models \text{Sat}_{\Delta_0}(\delta_r((a)_y, b))\}$. 

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Since $A_k$ is bounded and $\Delta_1$-definable, it is coded by some $s_k$ in $M$. Moreover, the function $k \mapsto s_k$ is $\Delta_1$-definable in $M$. Now, we define:

$$h(k) := \mu_x (\forall (r, y) < k \langle \langle r, y \rangle \rangle E s_k \rightarrow \text{Sat}_{\Delta_0}(\delta_r((a)_y, x))).$$

So note that:

(I) For every $k > I$, we have $\text{Th}_{\Delta_0}(M; b, \{(a)_i\}_{i \in I}) \subseteq \text{Th}_{\Delta_0}(M; h(k), \{(a)_i\}_{i \in I})$.

(II) For every $i \in I$, $h(i)$ is well-defined and inside $M_0 = \text{Fix}(j)$; the reason behind this statement is that for every $i \in I$ we consider the following set:

$$B_i := \{(r, \epsilon) : M \models \exists y < i((a)_y = \epsilon \land \langle r, y \rangle E s_i)\}.$$

Then, by Lemma 3.3, $B_i$ is coded by some $\alpha_i \in M_0 = \text{Fix}(j)$). So it holds that:

$$M \models h(i) = \mu_x (\forall (r, \epsilon) E \alpha_i \text{ (Sat}_{\Delta_0}(\delta_r(\epsilon, x))).$$

As a result, since $M_0 \prec \Sigma_1 M$, statement (II) holds.

Now, let $h' := j(h)$. So for all $i \in I$, and all $u < i$ such that $f(u) < i$, statement (II) implies that:

$$h'(g(u)) = j(h)(j(f)(u)) = j(h)(j(f)(j(u))) = j(h(f(u))) = h(f(u)).$$

Therefore, for all $i \in I$, $M \models \varphi(i, f, g, h, h')$, where $\varphi(i, f, g, h, h')$ is the following $\Delta_1$-formula:

$$\forall u < i \ (f(u) < i \rightarrow h(f(u)) = h'(g(u))).$$

So by $\Sigma_1$-Overspill in $M$, there exists some $s > I$ such that:

$$\langle \circ \rangle : \forall u < s \ (f(u) < s \rightarrow h(f(u)) = h'(g(u))).$$

Since $D$ is downward cofinal in $M \setminus I$, there is some $i_0 \in I$ such that $I < f(i_0) < s$. Let $c := h(f(i_0))$. On one hand, by (I), $\text{Th}_{\Delta_0}(M; b, \{(a)_i\}_{i \in I}) \subseteq \text{Th}_{\Delta_0}(M; c, \{(a)_i\}_{i \in I})$. As a result, because $b \notin M_0$, we have $c \notin M_0 = \text{Fix}(j)$. On the other hand $\langle \circ \rangle$ implies that:

$$j(c) = j(h(f(i_0))) = j(h)(j(f)(j(i_0))) = h'(g(i_0)) = h(f(i_0)) = c.$$

As a result, $I$ has to be strong in $M$. 

\[\Box\]
4 Strongness of the standard cut and fixed points

In this section, we will show some properties of Fix(j), when \( N \) is not strong in \( \mathcal{M} \). Then we will conclude some criteria for strongness of \( N \) in a countable nonstandard model of \( \text{I} \Sigma_1 \) through the set of fixed points of its initial self-embeddings.

**Lemma 4.1.** Suppose \( \mathcal{M} \) is a nonstandard model of \( \text{I} \Sigma_1 \) in which \( N \) is not strong. Then for any self-embedding \( j \) of \( \mathcal{M} \) the following hold:

1. \( \text{Fix}(j) \) is 1-tall.
2. If \( \text{Fix}(j) \) is a countable model of \( \text{B} \Sigma_1 \), then it is 1-extendable.

**Proof.**

1. Let \( a \in \text{Fix}(j) \) be arbitrary and fixed. Since \( \text{Fix}(j) \not\preceq \Sigma_1 \mathcal{M} \), it suffices to prove that \( K^1(\mathcal{M};a) \) is not cofinal in \( \text{Fix}(j) \). Since \( \mathcal{M} \models \text{B}^+ \Sigma_1 \), there exists some \( t_0 \in M \) such that \( K^1(\mathcal{M};a) < t_0 \). Moreover, by Lemma 2.5(2) there exists some \( t_{00} \in \text{Fix}(j) \) such that \( \text{Th}_{\Sigma_1}(\mathcal{M};t_{00},a) \subseteq \text{Th}_{\Sigma_1}(\mathcal{M};t_{00},a) \). Therefore, \( K^1(\mathcal{M};a) < t_{00} \).

2. By Theorem 2.2(2), and part (1) of this lemma, it suffices to prove that \( N \) is not \( \Pi_1 \)-definable in \( \text{Fix}(j) \). Suppose not; i.e. \( N \) is definable in \( \text{Fix}(j) \) by some \( \Pi_1 \)-formula \( \pi(x) \). By Lemma 2.5(1), \( \text{Fix}(j) \cap M \setminus N \) is downward cofinal in \( M \setminus N \). So by \( \Sigma_1 \)-Underspill in \( \mathcal{M} \), there exists some \( n \in N \) such that \( \mathcal{M} \models \neg \pi(n) \), and consequently since \( \text{Fix}(j) \not\preceq \Sigma_1 \mathcal{M} \), \( \text{Fix}(j) \models \neg \pi(n) \), which is a contradiction.

The following corollary generalizes Theorem 1.2:

**Corollary 4.2.** Let \( \mathcal{M} \models \text{I} \Sigma_1 \) be countable and nonstandard in which \( N \) is not strong, and \( j \) is an initial self-embedding of \( \mathcal{M} \) such that \( \text{Fix}(j) \models \text{B} \Sigma_1 \). Then \( \text{Fix}(j) \) is isomorphic to a proper cut of \( \mathcal{M} \).

**Proof.** By Theorem 2.2(1) and the previous lemma, it is enough to prove that \( \text{SSy}(\text{Fix}(j)) = \text{SSy}(\mathcal{M}) \).

We conclude this section with a generalization of a similar result about automorphisms of countable recursively saturated models of \( \text{PA} \) in \( \text{I} \Sigma_1 \). Moreover, the following corollary refines Theorem 2.4(3).

**Corollary 4.3.** Let \( \mathcal{M} \models \text{I} \Sigma_1 \) be countable and nonstandard. Then the following are equivalent:

1. \( N \) is strong in \( \mathcal{M} \).
2) There exists some proper initial self-embedding $j$ of $\mathcal{M}$ such that $\text{Fix}(j) = K^1(\mathcal{M})$.

3) There exists some proper initial self-embedding $j$ of $\mathcal{M}$, and some small $\mathcal{M}_0 \prec \Sigma_1 \mathcal{M}$, such that $\text{Fix}(j) = \mathcal{M}_0$.

4) For every small $\mathcal{M}_0 \prec \Sigma_1 \mathcal{M}$ there exists some proper initial self-embedding $j$ of $\mathcal{M}$ such that $\text{Fix}(j) = \mathcal{M}_0$.

5) There exists some proper initial self-embedding $j$ of $\mathcal{M}$ such that $\text{Fix}(j) \subseteq I^1(\mathcal{M})$.

If $\mathcal{M} \models \text{PA}$ and it is recursively saturated, then the above statements are equivalent to the following:

6) There exists some proper initial self-embedding $j$ of $\mathcal{M}$ such that $\text{Fix}(j) \models \text{B}\Sigma_1$ and it is isomorphic to no proper initial segments of $\mathcal{M}$.

Proof. The equivalences of statements (1) to (5) is a straightforward implication of Corollary 3.5 and Theorem 4.1(1). Moreover, (6) ⇒ (1) holds by Corollary 4.2. In order to prove (4) ⇒ (6), similar to the proof of Theorem 3.2(3), we will find some small recursively saturated elementary submodel $\mathcal{M}_0$ of $\mathcal{M}$. So statement (4) will provide us with a proper initial self-embedding $j$ of $\mathcal{M}$ such that $\text{Fix}(j) = \mathcal{M}_0$. Clearly $\text{Fix}(j) \models \text{B}\Sigma_1$. Moreover, as we mentioned in the beginning of Section 3, $\text{SSy}(\mathcal{M}_0) \neq \text{SSy}(\mathcal{M})$. As a result, $\text{Fix}(j)$ is isomorphic to no proper initial segment of $\mathcal{M}$. \qed

5 Extendability

In this section, we will study the extendability of initial embeddings. Most of the theorems of this section are generalizations of results about automorphisms of countable recursively saturated models of PA obtained in [11] and [12].

Definition 2. Suppose $\mathcal{M}$ and $\mathcal{N}$ are models of $\Sigma_1$, $\mathcal{M}_0$ and $\mathcal{N}_0$ are bounded submodels (or proper cuts) of $\mathcal{M}$ and $\mathcal{N}$ respectively. We call an initial embedding $j : \mathcal{M}_0 \rightarrow \mathcal{N}_0$ an initial $(\mathcal{M}, \mathcal{N})$-embedding if for every $A \subseteq \mathcal{M}_0$ it holds that:

$$A \in \text{SSy}_I(\mathcal{M}) \iff j(A) \in \text{SSy}_J(\mathcal{N}),$$

where $I := I^1(\mathcal{M}; \mathcal{M}_0)$, and $J := I^1(\mathcal{N}; j(\mathcal{M}_0))$.

If $\mathcal{M} = \mathcal{N}$, we call such $j$ an initial $\mathcal{M}$-embedding.

First, in the next lemma we will show that the condition in the above definition, i.e. preserving coded subsets, is a necessary condition for extendability of an initial embedding:
Lemma 5.1. Suppose $\mathcal{M}$ and $\mathcal{N}$ are models of $I\Sigma_1$, $\mathcal{M}_0 \subseteq \mathcal{M}$ and $\mathcal{N}_0 \subseteq \mathcal{N}$ are bounded submodels (or proper cuts), and $j : \mathcal{M}_0 \rightarrow \mathcal{N}_0$ is an initial embedding. If $j$ is extendable to some initial embedding $\hat{j} : \mathcal{M} \rightarrow \mathcal{N}$, then $j$ is an initial $(\mathcal{M}, \mathcal{N})$-embedding.

Proof. Put $I := I^1(\mathcal{M}; \mathcal{M}_0), J := I^1(\mathcal{N}; j(\mathcal{M}_0)),$ and let $A \subseteq \mathcal{M}_0$ be arbitrary. If $A = I \cap (\alpha E)^{\mathcal{M}}$ for some $\alpha$ in $\mathcal{M}$, then clearly $j(A) = J \cap ((\hat{j}(\alpha))E)^{\mathcal{N}}$. Conversely, suppose $j(A) \in SSy_{J}(\hat{j}(\mathcal{M}))$. Since $\mathcal{M}_0$ is bounded in $\mathcal{M}$, we have $J \subseteq \hat{j}(\mathcal{M})$. As a result, $j(A) \in SSy_{J}(\hat{j}(\mathcal{M}))$, which implies that $A \in SSy_I(\mathcal{M})$. □

Converse of the above lemma holds, when $\mathcal{M}_0$ and $j(\mathcal{M}_0)$ are $\Sigma_1$-elementary initial segments of $\mathcal{M}$ and $\mathcal{N}$:

Theorem 5.2. Suppose $\mathcal{M}$ and $\mathcal{N}$ are countable and nonstandard models of $I\Sigma_1$, and $I$ and $J$ are $\Sigma_1$-elementary initial segments of $\mathcal{M}$ and $\mathcal{N}$, respectively. Then for any isomorphism $j : I \rightarrow J$ which is an initial $(\mathcal{M}, \mathcal{N})$-embedding and each $b > J$, there exists some proper initial embedding $\hat{j} : \mathcal{M} \rightarrow \mathcal{N}$ such that $\hat{j} |_I = j$ and $\hat{j}(\mathcal{M}) < b$.

Sketch of proof. The proof is conducted by a back-and-forth argument similar to the one used in the proof of [1, Thm. 3.3]; we will build finite partial functions $\bar{u} \mapsto \bar{v}$ such that the following induction hypothesis holds:

If $\mathcal{M} \models [f(\bar{u}, i) \downarrow]$, then $\mathcal{N} \models [f(\bar{v}, j(i)) \downarrow] < b$,

for every $f \in F$ and $i \in I$.

For the ‘forth’ steps, if $\bar{u} \mapsto \bar{v}$ is built, for given $m \in \mathcal{M}$ we define:

$H := \{ (r, i) \in I : \mathcal{M} \models [f_r(\bar{u}, m, i) \downarrow] \}$.

Then, let $h \in \mathcal{M}$ such that $H = I \cap h E$. Since $j$ is an initial $(\mathcal{M}, \mathcal{N})$-embedding, there exists some $h' \in \mathcal{N}$ such that $j(H) = J \cap h'_E$. Therefore, by induction hypothesis for every $s \in I$ it holds that:

(1) : $\mathcal{N} \models \exists x, w < b \forall (r, i) < j(s) \left( (\langle r, i \rangle E h' \rightarrow [f_r(\bar{v}, x, i) \downarrow] < w) \right)$.

Since $j$ is onto, statement (1) implies that for every $t \in J$ it hold that:

(2) : $\mathcal{N} \models \exists x, w < b \forall (r, i) < t \left( (\langle r, i \rangle E h' \rightarrow [f_r(\bar{v}, x, i) \downarrow] < w) \right)$.

Therefore, by using $\Sigma_1$-Overspill in $\mathcal{N}$, we will find some image for $m$, for which induction hypothesis holds. The ‘back’ stages can be done similarly. □
The proof of the above theorem can also be modified for $I$-small submodels:

**Theorem 5.3.** Suppose $M \models I \Sigma_1$ is countable and nonstandard, $I$ is a strong cut of $M$, $M_0$ is an $I$-small $\Sigma_1$-elementary submodel of $M$ such that $M_0 := \{(a)_i : i \in I\}$, and $j$ is an initial embedding of $M_0$ such that $j(I) \subseteq M$. Then the following are equivalent:

1. $j \upharpoonright I$ is an initial $M$-embedding, and there exists some $b \in M$ such that $M \models j((a)_i) = (b)_{j(i)}$ for all $i \in I$.
2. $j$ extends to some proper initial self-embedding of $M$.

**Sketch of proof.** $(2) \Rightarrow (1)$ holds by Lemma 5.1. In order to prove $(1) \Rightarrow (2)$, we will use a similar argument to the proof of [1, Thm. 3.3] to obtain an extension $\hat{j}$ of $j$. For this purpose, first we will fix some $d \in M$ which is an upper bound for $M_0$. Then, we will build finite partial functions $\bar{u} \mapsto \bar{v}$ such that the following induction hypothesis holds:

$$M \models \exists x < \max\{\bar{u}\} \forall \langle r, i \rangle < s \langle \langle r, i \rangle E \bar{l}' \rangle \rightarrow \neg [f_r(\bar{u}, x, (a)_i) \downarrow].$$

Here, we outline the proof for the ‘back’ steps and the proof of ‘forth’ steps is left to the reader. Suppose $\bar{u} \mapsto \bar{v}$ is built, and $m < \max\{\bar{v}\}$ is given. We define:

$$L := \{\langle r, i \rangle \in j(I) : M \models \neg [f_r(\bar{v}, m, (b)_i)]^{< d}\}.$$

Then, let $l \in M$ such that $L = j(I) \cap l_E$. Since $j \upharpoonright I$ is an initial $M$-embedding, then there exists some $l' \in M$ such that $j^{-1}(L) = I \cap l'_E$. Moreover, by using Lemma 3.3, for every $s \in I$ there exists some $(a)_i \in M_0$ which codes of the following subset of $M_0$:

$$A := \{\langle r, (a)_i \rangle : M \models (\langle r, i \rangle < s \land (r, i) E \bar{l}')\}.$$

By $\Pi_1$-Overspill, it suffices to prove that for every $s \in I$ it holds that:

\[\forall x < \max\{\bar{u}\} \exists \langle r, i \rangle < s (\langle r, i \rangle E \bar{l}' \rightarrow \neg [f_r(\bar{u}, x, (a)_i) \downarrow]).\]

Suppose not; i.e. there exists some $s \in I$ which for statement $(\ast)$ does not hold. So we have:

\[\forall x < \max\{\bar{u}\} \exists \langle r, i \rangle < s (\langle r, i \rangle E \bar{l}' \land [f_r(\bar{u}, x, (a)_i) \downarrow]).\]

As a result, by using $\Sigma_1$-Collection in $M$, from statement $(i)$, induction hypothesis, and the way we chose $(a)_i$, we may conclude that:
(ii) : \( M \models \forall x < \max \{ \bar{v} \} \exists \langle r, \epsilon \rangle E(b_{j(i)}) ([f_r(\bar{v}, x, \epsilon) \downarrow]^{<d}). \)

So by statement (ii), there exists some \( \langle r, i \rangle < s \) such that:

(iii) : \( M \models \langle \langle r, i \rangle E l' \land [f_{j(r)}(\bar{v}, m, (b_{j(i)}) \downarrow]^{<d}. \)

But statement (iii) is in direct contradiction with the way we chose \( l' \).

\[ \square \]

In the last theorem, we investigate whether we can control the set of fixed points, while extending an isomorphism to an initial self-embeddings with larger domain:

**Theorem 5.4.** Suppose \( M \models \Sigma_1 \) is countable and nonstandard, \( I \) is a strong \( \Sigma_1 \)-elementary initial segment of \( M \), and \( j : I \to I \) is an isomorphism and an initial \( M \)-embedding. Then there exists some proper initial self-embedding \( \hat{j} \) of \( M \) such that \( \hat{j} \lvert_I = j \), and \( \text{Fix}(\hat{j}) = \text{Fix}(j) \).

**Sketch of proof.** First, we will fix some arbitrary \( a > I \). Since \( I \) is strong in \( M \), there exists some \( b > I \) such that:

\[ \langle \star \rangle : \text{if } M \models [f(a, i) \downarrow] \text{ and } f(a, i) > I \text{ then } f(a, i) > b, \text{ for all } f \in \mathcal{F} \text{ and } i \in I. \]

So by Theorem 5.2, there exists some proper initial self-embedding \( \bar{j} \) of \( M \) such that \( \bar{j} \lvert_I = j \) and \( \bar{j}(M) < b \). If \( \text{Fix}(\bar{j}) = \text{Fix}(j) \), then we are done. Otherwise we will build \( \hat{j} \) in the following way:

- **By using a similar argument to the proof of Theorem 3.4, we construct some proper initial self-embedding \( h \) of \( N := H^1(M; a) \) such that \( h \lvert_I = j \), \( \text{Fix}(h) = \text{Fix}(j) \), and \( h(N) < b \). In order to construct such \( h \), we will inductively construct finite functions \( \bar{u} \mapsto \bar{v} \) such that:

\[
\begin{align*}
P(\bar{u}, \bar{v}) &\equiv [f(\bar{u}, i) \downarrow] \rightarrow [f(\bar{v}, j(i)) \downarrow]^{<b}, \text{ for all } f \in \mathcal{F} \text{ and } i \in I; \text{ and} \\
Q(\bar{u}, \bar{v}) &\equiv \left( [f(\bar{u}, i) \downarrow] \land [f(\bar{v}, j(i)) \downarrow]^{<b} \land \\
f(\bar{u}, i) \notin I \right) \Rightarrow f(\bar{u}, i) \neq f(\bar{v}, j(i)), \text{ for all } f \in \mathcal{F} \text{ and all } i \in I.
\end{align*}
\]

- For the first step of induction, we will take \( a \mapsto \bar{j}(a) \); clearly \( P(a, \bar{j}(a)) \) holds in \( M \). Moreover, by statement \( (\star) \) and since \( \text{Fix}(\bar{j}) < b \), the property \( Q(a, \bar{j}(a)) \) also holds in \( M \).

- Then suppose \( \bar{u} \mapsto \bar{v} \) is built. We will just mention the changes that should be made in the ‘forth’ steps of Theorem 3.4, and ‘back’ steps should be modified similarly:
* Suppose \( m \in N \setminus \{ \bar{u} \} \) is given. By the definition of \( \mathcal{N} \), without loss of generality, we may assume that \( m \leq t(\bar{u}, a) \) for some \( t \in F \). Put:

\[
C := \{ \langle r, i \rangle \in I : \mathcal{N} \models [f_r(\bar{u}, m, i) \downarrow] \wedge f_r(\bar{u}, m, i) \notin K^1(\mathcal{N} ; I \cup \{ \bar{u} \}) \}. 
\]

Let \( \alpha, \alpha' \in N \) such that \( C = I \cap \alpha_E \) and \( j(C) = I \cap \alpha'_E \) (note that since \( \mathcal{N} \) is a \( \Sigma_1 \)-elementary initial segment of \( \mathcal{M} \) containing \( I, j \) is an initial \( \mathcal{N} \)-embedding):

* Let \( L := \{ \langle r, i \rangle \in I : \mathcal{N} \models [f_r(\bar{u}, m, i) \downarrow] \}, L = I \cap \beta_E \), and \( j(L) = I \cap \beta'_E \) for \( \beta, \beta' \in N \).

* For every \( s \in \bar{j}(N) \) such that \( \bar{j}(s') = s \) for some \( s' \in N \), let:

\[
p_s(y) := \{ y < t(\bar{v}, \bar{j}(a)) \} \cup p_s(y) \cup p_s(y); 
\]

where:

\[
p_s(y) := \forall i < s(\langle n, i \rangle E \beta' \rightarrow [f_n(\bar{v}, y, i) \downarrow < b] : n \in N \}; 
\]

and:

\[
p_s(y) := \forall w < s' \forall i < s \left( \forall w < s \forall w' < s' \forall r' < i \left( \forall r < i \left( \forall \langle r, w \rangle E \beta' \rightarrow [f_r(\bar{v}, y, w) \downarrow < b] \wedge \langle r', w \rangle E \alpha' \rightarrow [f_r(\bar{v}, y, w) \downarrow < b] \wedge [f_r(\bar{u}, m, w) \neq f_r(\bar{v}, y, w)] \right) \right) \right) ; n \in N \right \} \right) . 
\]

* In order to find some \( s > I \) such that \( s \in \bar{j}(N) \) and \( p_s(y) \) is finitely satisfiable, we will adapt the rest of the proof of Theorem 3.4 accordingly; for instance, we will mention two of these adaptations:

(1) Let \( d' \in N \) such that \( d' > I \) and \( d := \bar{j}(d') \). Moreover, for every \( i, s, s' \in N \), let \( \Theta(s, s', i, \bar{u}, m, \bar{v}, b, j(a), \alpha, \alpha', \beta') \) be the following \( \Delta_0 \)-formula:

\[
\forall r < i \exists y \leq t(\bar{v}, j(a)) \left( \forall w < s(\langle r, w \rangle E \beta' \rightarrow [f_r(\bar{v}, y, w) \downarrow < b] \wedge \langle r', w \rangle E \alpha' \rightarrow [f_r(\bar{v}, y, w) \downarrow < b] \wedge \langle r', w \rangle E \alpha' \rightarrow [f_r(\bar{v}, y, w) \downarrow < b] \wedge \langle r', w \rangle E \alpha' \rightarrow [f_r(\bar{v}, y, w) \downarrow < b] \wedge [f_r(\bar{u}, m, w) \neq f_r(\bar{v}, y, w)] \right) . 
\]

Then, for every \( i \in M \), we define:

\[
g(i) := \max \{ w < d' : \mathcal{M} \models \exists x \leq d \Theta(x, w, i, \bar{u}, m, \bar{v}, b, j(a), \alpha, \alpha', \beta') \} . 
\]

Since \( I \) is strong, there exists some \( e' > I \) such that \( e' \leq d' \), and for all \( i \in I \), \( g(i) > I \) iff \( g(i) > e' \). Then, for every \( i \in M \) put:

\[
l(i) := \max \{ x < j(e') : \mathcal{M} \models \exists x \leq j(e') \left( [g(i) \downarrow < d'] \wedge \Theta(x, g(i), i, \bar{u}, m, \bar{v}, b, j(a), \alpha, \alpha', \beta') \right) \} . 
\]

Again, since \( I \) is strong, there exists some \( e > I \) such that \( e \leq d \), and for all \( i \in I \), \( l(i) > I \) iff \( l(i) > e \). Then \( p_e(y) \) is a finitely satisfiable type.

(2) Instead of the function \( \langle o(\Diamond, y), h(\Diamond, y) \rangle \) we need to define the following function:
\[
\langle o(\Diamond, y), h(\Diamond, y), h'(\Diamond, y) \rangle := \min \left\{ \langle n_t, i, w \rangle \mathcal{E} \alpha s_0 : \begin{array}{l}
[b(\Diamond, y) \downarrow \wedge [f_{n_t}(\Diamond, y, i) \downarrow <b(\Diamond, y) \wedge \mathcal{S}_t(\Diamond, w) \downarrow <b(\Diamond, y) \wedge \\
\mathcal{S}_l(\Diamond, w) \downarrow b(\Diamond, y) \wedge \mathcal{S}_t(\Diamond, w) = f_{n_t}(\Diamond, y, i)] \end{array} \right\};
\]

where \( \alpha s_0 \in I \) is the code of the following subset of \( I \):

\[
\{ \langle n, i, w \rangle : \mathcal{M} \models i < s_0 \wedge w < j^{-1}(s_0) \wedge \langle n, w \rangle \mathcal{E} \alpha \wedge \langle n, i \rangle \mathcal{E} \alpha' \}
\]

The rest of the adaptations should be made similar to statements (1) and (2) in order to construct \( h \).

- If \( \mathcal{M} = \mathcal{N} \), then we are done. Otherwise, by using Theorem 2.3 we shall find some proper initial embedding \( k : \mathcal{M} \hookrightarrow \mathcal{N} \) such that \( I \subseteq \mathcal{I}_{\text{fix}}(k) \) and \( b \in k(M) \).
- Finally, we put \( \hat{j} := k^{-1}hk \).

\[\square\]

**Remark 2.** If we let \( j \) be the trivial automorphism of \( I \), then Theorem 5.4 implies Theorem 2.4(2).

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