The convex real projective orbifolds with radial or totally geodesic ends: The closedness and openness of deformations

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Abstract

A real projective orbifold is an $n$-dimensional orbifold modeled on $\mathbb{R}P^n$ with the group $\text{PGL}(n+1, \mathbb{R})$. We concentrate on an orbifold that contains a compact codimension 0 submanifold whose complement is a union of neighborhoods of ends, diffeomorphic to closed $(n-1)$-dimensional orbifolds times intervals. A real projective orbifold has a radial end if a neighborhood of the end is foliated by projective geodesics that develop into geodesics ending at a common point. It has a totally geodesic end if the end can be completed to have the totally geodesic boundary. The orbifold is said to be convex if any path can be homotoped to a projective geodesic with endpoints fixed.

A real projective structure sometimes admits deformations to parameters of real projective structures. We will prove the local homeomorphism between the deformation space of convex real projective structures on such an orbifold with radial or totally geodesic ends with various conditions with the $\text{PGL}(n+1, \mathbb{R})$-character space of the fundamental group with corresponding conditions. We will use a Hessian argument to show that under a small deformation, a properly (resp. strictly) convex real projective orbifold with generalized admissible ends will remain properly and properly (resp. strictly) convex with generalized admissible ends.

Lastly, we will prove the openness and closedness of the properly (resp. strictly) convex real projective structures on a class of orbifold with generalized admissible ends, where we need the theory of Crampon-Marquis and Cooper, Long and Tillmann on the Margulis lemma for convex real projective manifolds. The theory here partly generalizes that of Benoist on closed real projective orbifolds.
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CHAPTER 1

Introduction

1.1. History and motivations

Recently, there were many research papers on convex real projective structures on manifolds and orbifolds. (See the work of Goldman [46], Choi [19], [20], Benoist [7], Kim [62], Cooper, Long, Thistlethwaite [34], [35] and so on.) One can see them as projectively flat torsion-free connections on manifolds. Topologists will view each of these as a manifold with a structure given by a maximal atlas of charts to $\mathbb{RP}^n$ where transition maps are projective. Hyperbolic and many other geometric structures will induce canonical real projective structures. (See the numerous and beautiful examples in Sullivan-Thurston [77].) Sometimes, these can be deformed to real projective structures not arising from such obvious constructions. In general, the theory of the discrete group representations and their deformations form very much mysterious subjects still. We can use the results in studying linear representations of discrete groups.

Since the examples are more easy to construct, we will be studying orbifolds, natural generalization of manifolds. The deforming a real projective structure on an orbifold to an unbounded situation results in the actions of the fundamental group on affine buildings which hopefully will lead us to some understanding of orbifolds and manifolds in particular in dimension three as indicated by Cooper, Long, and Thistlethwaite.

However, the manifolds studied are usually closed ones so far. (See [11], [23], [34], [35], [31], [32].) We hope to generalize these theories to noncompact orbifolds with conditions on ends. In fact, we are trying the generalize the class of complete hyperbolic manifolds with finite volumes. These are $n$-orbifolds with compact suborbifolds whose complements are diffeomorphic to intervals times closed $(n-1)$-dimensional orbifolds. Such orbifolds are said to be strongly tame orbifolds. An

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{The developing images of convex $\mathbb{RP}^n$-structures on 2-orbifolds deformed from hyperbolic ones: $S^2(3,3,5)$.}
\end{figure}

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**Figure 1.2.** The developing images of convex $\mathbb{R}P^n$-structures on 2-orbifolds deformed from hyperbolic ones: $D^2(2, 7)$.

**Figure 1.3.** The handcuff graph and the construction of 3-orbifold of Tillmann by pasting faces of a hyperbolic tetrahedron.

*end neighborhood* is a component of a complement of a compact subset not contained in any compact subset of the orbifold. An *end* $E$ is an equivalence class of compatible sequences of end neighborhoods. Because of this, we can associate an $(n-1)$-orbifold at each end and we define the *end fundamental group* $\pi_1(E)$ as the fundamental group of the orbifold, a subgroup of the fundamental group $\pi_1(O)$. We also put the condition on end neighborhoods being foliated by radial lines or to have totally geodesic ideal boundary. Of course, this is not the only natural conditions, and we plan to explore the other conditions in some other occasions. (We note that a strongly tame orbifold may have nonempty boundary that is compact.)

We studied some such orbifolds of Coxeter type with ends in [23]. These have convex fundamental polytopes and are easier to understand. This paper generalizes the results there.

S. Tillmann studied a complete hyperbolic 3-orbifold obtained from gluing a complete hyperbolic tetrahedron. The one parameter family of deformations exists and can be solved explicitly. Later, Gye-Seon Lee and I computed more examples starting from hyperbolic Coxeter orbifolds (These are not published results. See Cooper-Long-Tillmann [36], Heusener-Porti [56], and Ballas [3, 4] for some computed 3-manifold examples). However, the convexity of the results was the main question that arose. We will try to answer this.

Also, D. Cooper, D. Long, and S. Tillmann [36] and M. Crampon and L. Marquis [37] are studying these types of orbifolds as quotients of convex domains...
without deforming and hence generalizing the Kleinian group theory for complete hyperbolic manifolds. However, they only study the orbifolds with horospherical types of ends or equivalently finite volume orbifolds. Their work is in a sense dual to this work since we start from orbifolds with real projective structures and deform.

In general, the theory of geometric structures on manifolds with ends is not studied very well. We should try to obtain more results here and find what are the appropriate conditions. This question seems to be also related to how to make sense of the topological structures of ends in many other geometric structures such as ones on modelled on symmetric spaces and so on.

1.2. Some definitions

Given a vector space $V$, we let $P(V)$ denote the space obtained by taking the quotient space of $\mathbb{R}^{n+1} - \{O\}$ under the equivalence relation $v \sim w$ for $v, w \in \mathbb{R}^{n+1} - \{O\}$ iff $v = sw$, for $s \in \mathbb{R} - \{0\}$.

We let $[v]$ denote the equivalence class of $v \in \mathbb{R}^{n+1} - \{O\}$. For a subspace $W$ of $V$, we denote by $P(W)$ the image of $W - \{O\}$ under the quotient map, also said to be a subspace. Recall that the projective linear group $\text{PGL}(\mathbb{R})$ acts on $\mathbb{RP}^n$, i.e., $P(\mathbb{R}^{n+1})$, in a standard manner. Let $O$ be a noncompact strongly tame $n$-orbifold where the orbifold boundary is not necessarily empty.

- A real projective orbifold is an orbifold with a geometric structure modelled on $(\mathbb{RP}^n, \text{PGL}(n+1, \mathbb{R}))$. (See [22] and Chapter 6 of [26].)
- A real projective orbifold also has the notion of projective geodesics as given by local charts and has a universal cover $\tilde{O}$ where a deck transformation group $\pi_1(O)$ acting on.
- The underlying space of $O$ is homeomorphic to the quotient space $\tilde{O}/\pi_1(O)$.
- A real projective structure on $O$ gives us a so-called development pair $(\text{dev}, h)$ where
  - $\text{dev} : \tilde{O} \rightarrow \mathbb{RP}^n$ is an immersion, called the developing map,
  - and $h : \pi_1(O) \rightarrow \text{PGL}(n+1, \mathbb{R})$ is a homomorphism, called a holonomy homomorphism, satisfying
    $\text{dev} \circ \gamma = h(\gamma) \circ \text{dev}$ for $\gamma \in \pi_1(O)$.
  - The pair $(\text{dev}, h)$ is determined only up to the action
    $g(\text{dev}, h(\cdot)) = (g \circ \text{dev}, gh(\cdot)g^{-1})$ for $g \in \text{PGL}(n+1, \mathbb{R})$
    and any chart in the atlas extends to a developing map. (See Section 3.4 of [79].)

Let $\mathbb{R}^{n+1*}$ denote the dual of $\mathbb{R}^{n+1}$. Let $\mathbb{RP}^{n*}$ denote the dual projective space $P(\mathbb{R}^{n+1*})$. $\text{PGL}(n+1, \mathbb{R})$ acts on $\mathbb{RP}^{n*}$ by taking the inverse of the dual transformation. Then $h$ has a dual representation $h^*$ sending elements of $\pi_1(O)$ to the inverse of the dual transformation of $\mathbb{R}^{n+1*}$.

For an element $g \in \text{PGL}(n+1, \mathbb{R})$, we denote

$$g \cdot [w] := [\hat{g}(w)] \quad \text{for } [w] \in \mathbb{RP}^n \text{ or}$$

$$= [(\hat{g}^T)^{-1}(w)] \quad \text{for } [w] \in \mathbb{RP}^{n*}$$

(1.1)

where $\hat{g}$ is any element of $\text{SL}_{\pm}(n+1, \mathbb{R})$ mapping to $g$ and $\hat{g}^T$ the transpose of $\hat{g}$. 
The complement of a codimension-one subspace of $\mathbb{RP}^n$ can be identified with an affine space $\mathbb{R}^n$ where the geodesics are preserved. The group of affine transformations of $\mathbb{R}^n$ are the restriction to $\mathbb{R}^n$ of the group of projective transformations of $\mathbb{RP}^n$ fixing the subspace. We call the complement an affine subspace. It has a geodesic structure of a standard affine space. A convex domain in $\mathbb{RP}^n$ is a convex subset of an affine subspace. A properly convex domain in $\mathbb{RP}^n$ is a convex domain contained in a precompact subset of an affine subspace.

The important class of real projective structures are so-called convex ones where any arc in $\mathcal{O}$ can be homotopied with endpoints fixed to a straight geodesic where $\text{dev}$ is injective to $\mathbb{RP}^n$ except possibly at the end points. If the open orbifold has a convex structure, it is covered by a convex domain $\Omega$ in $\mathbb{RP}^n$. Equivalently, this means that the image of the developing map $\text{dev}(\tilde{\mathcal{O}})$ for the universal cover $\tilde{\mathcal{O}}$ of $\mathcal{O}$ is a convex domain. $\mathcal{O}$ is projectively diffeomorphic to $\text{dev}(\tilde{\mathcal{O}})/h(\pi_1(\mathcal{O}))$. In our discussions, since $\text{dev}$ often is an imbedding, $\tilde{\mathcal{O}}$ will be regarded as an open domain in $\mathbb{RP}^n$ and $\pi_1(\mathcal{O})$ a subgroup of $\text{PGL}(n+1,\mathbb{R})$ in such cases. This simplifies our discussions. (See Chapter 2.)

We will have the following boundary deformability hypothesis for manageability. Otherwise the paper might become too large to handle. Let $\mathcal{O}$ be a strongly tame real projective orbifold. We assume that $\partial \mathcal{O}$ is strictly convex; i.e., each point of $\partial \mathcal{O}$ has a neighborhood mapping to a convex ball with smooth strictly convex boundary under $\text{dev}$. Then each component of $\partial \mathcal{O}$ can be deformed inward to a strictly convex boundary components by arbitrarily small amount since one can find a smooth inward variation of $\partial \mathcal{O}$. The hypersurface remains strictly convex for a short time. Moreover, we observe that if the universal cover $\tilde{\mathcal{O}}$ is a properly convex domain, the deformed $\tilde{\mathcal{O}}$ is one also. (However, we will assume mostly that $\partial \mathcal{O} = \emptyset$ in this paper for convenience and simplicity.)

### 1.2.1. Restricting the ends.

In this case, each end has a neighborhood diffeomorphic to a closed orbifold times an interval. This orbifold is independent of the choice of such a neighborhood, and it said to be the end orbifold associated with the end. The fundamental group of an end is isomorphic to the fundamental group of an end neighborhood.

A lens is a properly convex domain that is bounded by two smooth strictly convex open disks. A lens-orbifold is a compact quotient of a lens by a properly discontinuous action of a projective group.

An end of a real projective orbifold $\mathcal{O}$ is totally geodesic or of type $T$ if

- the end has an end neighborhood that completes to a compact one homeomorphic to a closed $(n - 1)$-dimensional orbifold times a half-open interval and
- each point of the boundary has a neighborhood projectively diffeomorphic to an open set in an affine half-space.

The boundary component is called the totally geodesic ideal boundary (component) of the end. Such an ideal boundary may not be unique as there are two projectively inequivalent ways to add boundary components of elementary annuli (see Section 1.4 of [20]). Two compactified end neighborhoods of an end are equivalent if they contain a common compactified end neighborhood. We also require that
(Lens condition): the ideal boundary be realized as a totally geodesic sub-orbifold in the interior of a lens-orbifold in a cover of a some ambient real projective orbifold of $\mathcal{O}$ corresponding to the end fundamental group.

We also define as follows:

- The equivalence class of the chosen compactified end neighborhood is called a marking of the totally geodesic end.
- We will also call the ideal boundary the end orbifold of the end.

(The reason for the lens condition here is to allow these ends to change to horospherical types.) We will call the totally geodesic ends with the above properties the ends of lens-type.

An end of a real projective orbifold is radial or of type $R$ if

- the end has an end neighborhood foliated by properly imbedded projective geodesics and
- where nearby leaf-geodesics map under a developing map to open geodesics in $\mathbb{R}P^n$ ending at the common point of concurrency.

Two such radial foliations of radial end neighborhoods of an end are compatible if they agree outside some compact subset of the orbifold.

- A radial foliation marking is a compatibility class of radial foliations.
- A real projective orbifold with radial end marks is a strongly tame orbifold with real projective structures and end neighborhoods with radial foliation markings.

The end orbifold has a unique induced real projective structure of one dimension lower since the concurrent lines to a point form $\mathbb{R}P^{n-1}$ and the real projective transformations fixing a point of $\mathbb{R}P^n$ correspond to real projective transformations of $\mathbb{R}P^{n-1}$. To summarize, an end orbifold is a well-defined closed real projective $(n-1)$-dimensional orbifold, which may depends on the choice of radial foliations. (Note that a real projective orbifold could have the same real projective structures and different radial foliation markings as Cooper pointed out. Actually, the totally geodesic ends of lens-type are dual to radial ends and conversely. See Section 4.4.)

The radial foliations and the ideal boundary components compactify $\mathcal{O}$ to a compact orbifold $\mathcal{O}'$, and the universal cover $\tilde{\mathcal{O}}$ is contained in a completion $\tilde{\mathcal{O}}'$ where the developing maps extend, probably not locally injectively. Note that $\mathcal{O}'$ has the unique topology of a tame orbifold by attaching end orbifolds to $\mathcal{O}$ to each ends.

For example, a finite volume hyperbolic $n$-orbifold with cusps and totally geodesic boundary components removed is an example. Let $\mathbb{R}^{n+1}$ have standard coordinates $x_0, x_1, \ldots, x_n$, and let $B$ be the subset in $\mathbb{R}P^n$ corresponding to the cone given by

$$x_0 > \sqrt{x_1^2 + \cdots + x_n^2}.$$

The Klein model gives a hyperbolic space as $B \subset \mathbb{R}P^n$ with the isometry group $\text{PO}(1,n)$ a subgroup of $\text{PGL}(n+1, \mathbb{R})$ acting on $B$. Thus, an above-type hyperbolic orbifold is projectively diffeomorphic to an open submanifold of $B/\Gamma$ for $\Gamma$ in $\text{PO}(1,n)$. (Also, we could allow hyperideal ends by attaching radial ends. See Section 3.1.1 in [27].)

An ellipsoid in $\mathbb{R}P^n = P(\mathbb{R}^{n+1})$ (resp. in $S^n = S(\mathbb{R}^{n+1})$) is the projection $C - \{O\}$ of the null cone $C := \{x \in \mathbb{R}^{n+1} | b(x,x) = 0\}$ for a nondegenerate bilinear
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form \( b : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R} \). Ellipsoids are always equivalent by projective automorphisms. An ellipsoid ball is the closed contractible domain of in \( \mathbb{R}P^n \) (resp. \( S^n \)) bounded by an ellipsoid. A horoball is an ellipsoid ball with a point \( p \) of the boundary removed. An ellipsoid with a point \( p \) on it removed is called a horosphere.

The vertex of the horosphere or the horoball is defined as \( p \).

Define \( \partial A \) for a subset \( A \) of \( \mathbb{R}P^n \) (resp. in \( S^n \)) to be the topological boundary in \( \mathbb{R}P^n \) (resp. in \( S^n \)) and define \( \partial A \) for a manifold or orbifold \( A \) to be the manifold or orbifold boundary and \( A^\circ \) denote the manifold interior. The closure \( \text{Cl}(A) \) of a subset \( A \) of \( \mathbb{R}P^n \) (resp. of \( S^n \)) is the topological closure in \( \mathbb{R}P^n \) (resp. in \( S^n \)).

A real projective orbifold that is real projectively diffeomorphic to an orbifold \( U/\Gamma_p \) for a discrete subgroup \( \Gamma_p \) \( \subset \text{PO}(1,n) \) fixing a point \( p \in \partial B \) and a horoball \( U \) with vertex \( p \) is called a horoball orbifold. A horospherical end is an end with an end neighborhood that is such an orbifold. (In our case, by strong tameness, the group contains an abelian group of maximal rank \( n-1 \) of finite index by Proposition 4.1.)

Given a real projective orbifold \( O \), we add the restriction of the end to be a radial or a totally geodesic type. The end will be either assigned \( \mathcal{R}\text{-type} \) or \( \mathcal{T}\text{-type} \).

- An \( \mathcal{R}\text{-type} \) end is required to be radial.
- A \( \mathcal{T}\text{-type} \) end is required to have totally geodesic properly convex ideal boundary components or be horospherical.

A strongly-tame orbifold will always have such an assignment in this paper, and finite-covering maps will always respect the types.

1.2.2. Definition of the deformation spaces with end marks. An isotopy \( i : O \to O \) is a self-diffeomorphism so that there exists a smooth parameter of self-diffeomorphism \( i_t : O \to O, \ t \in [0,1], \) so that \( i = i_1, i_0 = I_O \).

- Two real projective structures \( \mu_0 \) and \( \mu_1 \) are isotopic if there is an isotopy \( i_0 \) on \( O \) so that \( i^*_0(\mu_1) = \mu_0 \) where \( i^*_0(\mu_1) \) is the induced structure from \( \mu_1 \) by \( i_0 \).
- \( i \) sends the radial end foliation for \( \mu_0 \) from an end neighborhood to the radial end foliation for \( \mu_1 \) in the another end neighborhood, and
- \( i \) extends to the union of totally geodesic ideal boundary components as a diffeomorphism.

We define \( \text{Def}_E(O) \) as the deformation space of real projective structures on \( O \) with end marks; more precisely, this is the quotient space of the real projective structures \( \mu \) on \( O \) satisfying the above conditions for ends of type \( \mathcal{R} \) and \( \mathcal{T} \) under the isotopy equivalence relations. The topology of such a space is defined by the compact open \( C^r \)-topology for the space of developing maps \( \text{dev}, \ r \geq 2 \). We will discuss this more later on. For noncompact orbifolds, these spaces can be very complicated especially if there are no end markings. (see [22], [16] and [47] for more details.)

Remark 1.1. As suggested by Mike Davis, one can look at ends with holonomy groups of end fundamental groups acting on properly convex domains in totally geodesic subspaces of codimension between 2 and \( n-1 \). While they are perfectly reasonable to occur, in particular for Coxeter type orbifolds, we shall avoid these types as they are not understandable yet and we will hopefully study these in other papers. We will only be thinking of ends with holonomy groups of the end...
fundamental groups acting on codimension \( n \) or codimension 1 subspaces. However, we think that the other types of the ends do not change the theory present here in an essential way.

Let \( \{g_1, \ldots, g_m\} \) be the generators of \( \pi_1(O) \). As usual \( \text{Hom}(\pi_1(O), G) \) for a Lie group \( G \) has an \textit{algebraic topology} as a subspace of \( G^m \). This topology is given by the notion of \textit{algebraic convergence}

\[
\{h_i\} \to h \text{ if } h_i(g_j) \to h(g_j) \in G \text{ for each } j, j = 1, \ldots, m.
\]

The \textit{character space} \( \text{rep}(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \) is the quotient space of the homomorphism space

\[
\text{Hom}(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))
\]

where \( \text{PGL}(n+1, \mathbb{R}) \) acts by conjugation

\[
h(\cdot) \mapsto gh(\cdot)g^{-1} \text{ for } g \in \text{PGL}(n+1, \mathbb{R}).
\]

Each element is called a \textit{character} in this paper. A \textit{representation} is an element in the equivalence class of a character. A representation or a character is \textit{stable} if the orbit of it or its representative is closed and the stabilizer is finite under the conjugation action in \( \text{Hom}(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \). (See \cite{60} for more details.) By Theorem 1.1 of \cite{60}, a representation \( \rho \) is stable if and only if any proper parabolic subgroup does not contain the image of \( \rho \). The stability and the irreducibility are open conditions in the Zariski topology. Also, if the image of \( \rho \) is Zariski dense, then \( \rho \) is stable by Lemma 3.5 of \cite{32}.

A representation of a group \( G \) into \( \text{PGL}(n+1, \mathbb{R}) \) or \( \text{SL}_{\pm}(n+1, \mathbb{R}) \) is \textit{strongly irreducible} if the image of every finite index subgroup of \( G \) (resp. every finite index subgroup of \( G \)) is irreducible. Actually, many of the orbifolds have strongly irreducible and stable holonomy homomorphisms by Theorem 4.26.

An \textit{eigen-1-form} of a linear transformation \( \gamma \) is a linear functional \( \alpha \) in \( \mathbb{R}^{n+1} \) so that \( \alpha \circ \gamma = \lambda \alpha \) for some \( \lambda \in \mathbb{R} \).

We define

- \( \text{rep}_E(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \)
- \( \text{rep}_S(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \)
- \( \text{rep}_E^*(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \)
- \( \text{rep}_S^*(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \)

the subspace of stable and irreducible characters, and

- \( \text{rep}_{E,u}(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \)

the subspace of stable and irreducible characters where

- the restricted representation to each radial p-end fundamental group has a unique common eigenspace of dimension 1 and
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– each totally-geodesic end fundamental group has a unique common space of eigen-1-forms of dimension 1 meeting a lens with above properties.

\[ \text{rep}^*_E,\pi_1(\mathcal{O}),\text{PGL}(n+1,\mathbb{R}) \subset \text{rep}_E(\pi_1(\mathcal{O}),\text{PGL}(n+1,\mathbb{R})). \]

- We define

\[ \text{rep}^*_E,\pi_1(\mathcal{O}),\text{PGL}(n+1,\mathbb{R}) := \text{rep}^*(\pi_1(\mathcal{O}),\text{PGL}(n+1,\mathbb{R})) \cap \text{rep}_E,\pi_1(\mathcal{O}),\text{PGL}(n+1,\mathbb{R}). \]

Note that elements of Def\(_E(\mathcal{O})\) have holonomy characters in rep\(_E(\pi_1(\mathcal{O}),\text{PGL}(n+1,\mathbb{R}))\).

Denote by Def\(_E,u(\mathcal{O})\) the subspace of Def\(_E(\mathcal{O})\) of equivalence classes of real projective structures with holonomy characters in rep\(_E(\pi_1(\mathcal{O}),\text{PGL}(n+1,\mathbb{R}))\).

Also, we denote by Def\(_E(\mathcal{O})\) \(\subset\) Def\(_E(\mathcal{O})\) and Def\(_E,u(\mathcal{O})\) \(\subset\) Def\(_E,u(\mathcal{O})\) the subspaces of equivalence classes of real projective structures with stable and irreducible characters.

1.3. The local homeomorphism and homeomorphism theorems

For technical reasons, we will be assuming \(\partial\mathcal{O} = \emptyset\) in most cases. In fact, a proper way to understand the boundary is through understanding the ends as in the hyperbolic manifold theory of Thurston. The following map hol is induced by sending \((\text{dev}, h)\) to the conjugacy class of \(h\) as isotopies preserve \(h\):

**Theorem 1.2.** Let \(\mathcal{O}\) be a noncompact strongly tame real projective \(n\)-orbifold with radial ends or totally-geodesic ends of lens-type with markings and given types \(\mathcal{R}\) or \(\mathcal{T}\). Assume \(\partial\mathcal{O} = \emptyset\). Then the following map is a local homeomorphism:

\[ \text{hol} : \text{Def}_{E,u}(\mathcal{O}) \to \text{rep}_{E,u}(\pi_1(\mathcal{O}),\text{PGL}(n+1,\mathbb{R})). \]

The restrictions of end types are necessary for this theorem to hold. This generalizes results for closed manifolds for many geometric structures starting from the classical results of Weil. (See Goldman \([47]\), Canary-Epstein-Green \([16]\), and \([22]\) for many versions of similar results.)

We will present a more general “section” version Theorem 3.11 in Chapter 3 giving this theorem as a corollary. (Also, see for related work by Cooper and Long \([33]\).)

**Definition 1.3.** An \(\text{SPC-structure}\) or a stable irreducible properly-convex real projective structure on an \(n\)-orbifold \((\text{with radial or totally geodesic end of lens-type})\) is a real projective structure so that the orbifold is projectively diffeomorphic to a quotient orbifold of a properly convex domain in \(\mathbb{R}P^n\) by a discrete group of projective automorphisms that is stable and irreducible.

**Definition 1.4.** Suppose that \(\mathcal{O}\) has an SPC-structure. Let \(\hat{U}\) be the inverse image in \(\hat{\mathcal{O}}\) of the union \(U\) of some choice of a collection of disjoint end neighborhoods of \(\mathcal{O}\). If every straight arc in the boundary of the domain \(\hat{\mathcal{O}}\) and every non-\(C^1\)-point is contained in the closure of a component of \(\hat{U}\) for some choice of \(U\), then \(\mathcal{O}\) is said to be strictly convex with respect to the collection of the ends. And \(\mathcal{O}\) is also said to have a strict \(\text{SPC-structure}\) with respect to the collection of ends.
1.3. THE LOCAL HOMEOMORPHISM AND HOMEOMORPHISM THEOREMS

We will drop the respectiveness when it is obvious. Also \( O \) with its real projective structure is strictly SPC if it the structure is a strict SPC-structure. Later we will show that the word “some” can be replaced by “all” if we restrict the end types. (See Corollary 4.11.)

A segment is a connected arc in a one-dimensional subspace of \( \mathbb{R}P^n \). Given two points or subsets \( A, B \) in an affine subspace \( \mathbb{R}^n \) of \( \mathbb{R}P^n \), we define the join \( A \ast B \) as the union of all segments in \( \mathbb{R}^n \) with end points in \( A \) and \( B \) respectively or its interior. More precisely,

\[
A \ast B := \{ [tv + (1 - t)w] | t \in [0, 1], v \in C_A, w \in C_B \}
\]

where \( C_A \) is the connected cone in \( \mathbb{R}^{n+1} \) mapping to \( A \) and \( C_B \) is one for \( B \) in the open half space \( H \) in \( \mathbb{R}^{n+1} \) corresponding to the affine subspace \( \mathbb{R}^n \). Since \( A \) and \( B \) are usually subsets of a convex domain in an affine space, the join is well-defined subset of the convex domain.

For a topological manifold \( A \), we denote by \( \partial A \) the manifold boundary and by \( A^o \) the manifold interior. A ray is a segment starting from a point \( v \) in \( \mathbb{R}P^n \) (resp. \( S^n \)) that is oriented away from \( v \). Two rays from \( v \) are equivalent if the rays agree in a neighborhood of \( v \). A generalized lens is a properly convex domain bounded by two open disks one of which is smooth and strictly convex and the other boundary component allowed to be not strictly convex. A generalized lens-cone is a cone \( \{ p \} \ast L - \{ p \} \) over a generalized lens \( L \) so that every maximal ray from \( p \) in \( \{ p \} \ast L \) meets each of the two boundary components of \( L \) exactly once and the nonsmooth boundary component must be in the boundary of \( \{ p \} \ast L \). A lens or a generalized lens \( L \) is a strict lens or generalized strict lens if \( L - \partial L \) is nowhere dense in \( \text{bd} L \).

A cone over a lens with vertex \( p \), \( p \not\in \text{Cl}(L) \), is defined as \( \{ p \} \ast L - \{ p \} \) for a lens \( L \) so that every maximal ray from \( p \) in \( \{ p \} \ast L \) meets each of the two boundary components of \( L \) exactly once. Any subdomain of \( \tilde{O} \) projectively diffeomorphic to the interior of the above is called by the same name as if they are in \( \mathbb{R}P^n \).

- An \( R \)-type end is of lens-type if \( \tilde{O} \) contains the lens-cone where the end fundamental group acts on the lens fixing the vertex. An \( R \)-type end is of generalized lens-type if \( \tilde{O} \) contains the interior of a generalized lens-cone where the end fundamental group acts on the generalized lens fixing the vertex.
- An \( R \)-type end of a real projective orbifold is admissible if it is a radial end of lens-type or horospherical type and a \( T \)-type end is admissible if it is totally geodesic (of lens-type) or horospherical type.

We can allow the \( R \)-type ends to be of generalized lens-type. Then we say that the ends are admissible in the generalized sense. We require in both cases that the end fundamental group is a virtually a product of hyperbolic groups and abelian groups. (See Section 2.9 for definitions.) We define

\[
\text{rep}_{E,u,cc}(\pi_1(O), \text{PGL}(n + 1, \mathbb{R}))
\]

to be the subspace of

\[
\text{rep}_{E,u}(\pi_1(O), \text{PGL}(n + 1, \mathbb{R}))
\]
where the holonomy group of each end fundamental group is realized as that of an
admissible end. We define

\[ \text{rep}^*_E(u,ce)(\pi_1(O), \text{PGL}(n+1,\mathbb{R})) := \text{rep}^*_E(u,ce)(\pi_1(O), \text{PGL}(n+1,\mathbb{R})) \cap \text{rep}_E^*(\pi_1(O), \text{PGL}(n+1,\mathbb{R})). \]

(1.3)

We will show that these are semi-algebraic sets or at least open subsets of such sets
(See Section 3.1.)

We are only using “admissible” end as this concept is equivalent to the princi-
pal boundary condition for two-dimensional real projective surfaces [46]. Also, in
[27], we show that they are naturally structurally stable, and definable by natural
eigenvalue conditions. (See Theorems 1.1, 1.2, 8.1, and 8.2 of [27].) We remark
that an example found by Ballas [4] is not lens-type radial end. However, this end
seems to be in the classification [27]. We will attempt to understand these types
of ends at later dates.

We define

- Def_{E,u,ce}(O) to be the subspace Def_E(O) of classes of real projective
structures with generalized admissible ends,
- Def_{E,u,ce}(O) to be the deformation space of classes of real projective
structures with generalized admissible ends and stable and irreducible
holonomy homomorphisms,
- CDef_{E,u,ce}(O) to be the deformation space of SPC-structures with gen-
eralized admissible ends and stable and irreducible holonomy homomor-
phisms, and
- SDef_{E,u,ce}(O) to be that of strict SPC-structures with admissible ends
and stable and irreducible holonomy homomorphisms.

We will allow for these structures that a radial lens-cone end could change to
a horospherical type and vice versa and a totally geodesic lens end could change to
a horospherical one and vice versa. But we will not allow a radial lens-cone end to
change to a totally geodesic lens end.

By an essential annulus A, we mean a map \( f : A \to O \) so that components
of \( \partial A \) are mapped into end neighborhoods and to a homotopy class of infinite
order, and \( f \) is not homotopic into an end neighborhood relative to \( \partial A \). By an
essential torus \( T^2 \), we mean a map \( f : T^2 \to O \) so that the induced homomorphism
\( f_\ast : \pi_1(T^2) \to \pi_1(O) \) is injective to a free abelian group of rank two and where \( f \)
is not freely homotopic into an end of \( O \).

For a strongly tame orbifold \( O \),

(IE) \( O \) or \( \pi_1(O) \) satisfies the infinite-index end fundamental group condition
(IE) if \( [\pi_1(E) : \pi_1(O)] = \infty \) for the end fundamental group \( \pi_1(E) \) of each
end \( E \).

(NA) \( O \) or \( \pi_1(O) \) satisfies the property-NA if \( O \) has no essential annulus and
\( \pi_1(E) \) contains every element \( g \in \pi_1(O) \) normalizing \( \langle h \rangle \) for an infinite
order \( h \in \pi_1(E) \) for an end fundamental group \( \pi_1(E) \) of an end \( E \).

(NA) implies that \( O \) contains no essential torus. These conditions are satisfied
by complete hyperbolic manifolds with cusps, the objects that we are trying to
generalize. These are group theoretical properties with respect to the end groups.

**THEOREM 1.5.** Let \( O \) be a noncompact strongly tame \( n \)-orbifold with generalized
admissible ends. Assume \( \partial O = \emptyset \). Suppose that \( O \) satisfies (IE) and (NA). Then
1.4. OUTLINE

The paper is divided into three parts: The part I is on the local homeomorphism property, i.e., Theorem 1.2.

In Chapter 2, we give elementary definitions of geometric structures, real projective structures, radial ends, totally geodesic ends and so on. We give some well-known reducibility theorems for closed real projective orbifolds due to Koszul, Vey, and Benoist. We discuss the admissible ends and their properties from [27]. Also, the duality of the domains and the actions will be studied. Then we discuss the affine structures. We also study an affine suspension, a method to obtain an affine structure from a real projective structure.

In Chapter 3, we prove the local homeomorphism theorem, i.e., Theorem 1.2; hol send the deformation space to the character space locally homeomorphically. We discuss first the semialgebraic nature of the spaces. We will prove the theorem for the affine structure and change it to be applicable to real projective structures. The methods are similar to what is in [22]. Here, we need to have continuous sections of eigenvectors in the end holonomy groups. Finally, we transfer the theorem to the real projective cases using affine suspensions.

In Part II, we discuss the convexity properties of the orbifolds and related these to the relative hyperbolicity of the fundamental groups of the orbifolds.

In Chapter 4, we discuss convexity and define convex real projective structures on orbifolds. We discuss horospherical ends and lens-shaped ends and their properties and various facts concerning their existence, stability, and examples and so on.

• the subspace of SPC-structures
  \[ \text{CDef}_{E,u,ce}(\mathcal{O}) \subset \text{Def}_{E,u,ce}(\mathcal{O}) \]
  is open.

• Suppose further that every finite-index subgroup of \( \pi_1(\mathcal{O}) \) contains no nontrivial infinite nilpotent normal subgroup and \( \partial \mathcal{O} = \emptyset \). \( \text{hol} \) maps \( \text{CDef}_{E,u,ce}(\mathcal{O}) \) homeomorphically to a union of components of \( \text{rep}_{E,u,ce}(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R})) \).

Theorems 6.1 and 7.1 and Corollary 7.3 prove this and following theorems. (The more general “section versions” will not be proved.)

**Theorem 1.6.** Let \( \mathcal{O} \) be a strict SPC noncompact strongly tame \( n \)-dimensional orbifold with admissible ends and satisfies (IE) and (NA). Assume \( \partial \mathcal{O} = \emptyset \). Then

- \( \pi_1(\mathcal{O}) \) is relatively hyperbolic with respect to its end fundamental group.
- The subspace \( \text{SDef}_{E,u,ce}(\mathcal{O}) \subset \text{Def}^*_E(\mathcal{O}) \), of strict SPC-structures with admissible ends is open.
- Suppose further that every finite-index subgroup of \( \pi_1(\mathcal{O}) \) contains no nontrivial infinite nilpotent normal subgroup and \( \partial \mathcal{O} = \emptyset \). Then \( \text{hol} \) maps the deformation space \( \text{SDef}_{E,u,ce}(\mathcal{O}) \) of strict SPC-structures on \( \mathcal{O} \) with admissible ends homeomorphically to a union of components of 
  \( \text{rep}_{E,u,ce}(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R})) \).

We will also show that an SPC-orbifold \( \mathcal{O} \) with generalized admissible ends is strictly SPC with admissible ends iff \( \pi_1(\mathcal{O}) \) is relatively hyperbolic with respect to its end fundamental groups. (See Theorems 5.7 and 5.9.)
1. INTRODUCTION

We also discuss the duality of $R$-type ends and $T$-type ends. If $O$ satisfies (NA), then we define the boundary of the convex hulls of $p$-ends of $\tilde{O}$. These results are mostly from [27].

In Chapter 5, we define stable irreducible properly convex real projective structures or SPC-structures on orbifolds. We define strict SPC-structures also. We show using Bowditch’s approach that an SPC-orbifold has a relatively hyperbolic fundamental group with respect to its end fundamental groups if and only if the SPC-orbifold is strictly SPC. (Theorems 5.7 and 5.9.)

In Part III, we discuss the openness and the closedness of the deformation spaces of convex real projective structures on orbifolds and finally an example where our theory applies. A deformation means changing the real projective structures so that the developing maps change continuously in the $C^r$-topology, $r \geq 1$, on every compact subset of $\tilde{O}$.

In Chapter 6, we prove that if ends of an orbifold are admissible in a generalized sense, then the deformations of (resp. strictly) SPC-structures will remain (resp. strictly) SPC-structures under irreducibility conditions, i.e., Theorem 1.5. The proof is divided into two: First, we show that there is a Hessian metric and under small perturbations of the real projective structures, we can still find a nearby Hessian metric. Basically, we find that the Koszul-Vinberg functions of the affine suspensions change by very small amounts. Second, the Hessian metric and the boundary orbifold convexity assumption imply convexity.

In Chapter 7, we show that the deformation space $CDef_{E,u,ce}(O)$ or $SDef_{E,u,ce}(O)$ maps homeomorphic to the union of components of

$$\text{rep}_{E,u,ce}(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$$

under appropriate assumptions. We will use the Margulis Lemma of Cooper, Long, and Tillmann for the thin subgroups of the discrete projective automorphism groups acting on properly convex domains. (See [36].)

In Chapter 8, we describe some examples where the theory is applicable. These include the examples of S. Tillmann and a double of a tetrahedron reflection group of all order 3.

1.4.1. Acknowledgment. We thank Yves Benoist, Yves Carrière, Daryle Cooper, Mikael Crampon, Kelly Delp, William Goldman, Ludovic Marquis, Hyam Rubinstein, and Stephan Tillmann for many discussions. This paper began from a discussion the author had with Tillmann on his construction of a parameter of real projective structures on small complete hyperbolic 3-orbifold. We thank D. Fried for helping with the algebraic nature of the character varieties in Subsection 3.1. Many helpful discussions were carried out with C. Hodgson and we hope to publish the resulting examples in another papers. We also thank B. Bowditch, C. Drutu, and M. Kapovich for many technical discussions and the valuable help with the geometric group theory used here. In fact, C. Drutu supplied us with the proof of Theorem 5.8.
Part 1

The local homeomorphisms of the deformation spaces into the spaces of characters
CHAPTER 2

Preliminary

2.1. Orbifolds with ends

Let $H^n$ be the closed half-space of $\mathbb{R}^n$. An $n$-dimensional orbifold is a second-countable Hausdorff space with an orbifold structure. An orbifold structure is given by a fine covering by open sets of form $\phi(U)$ where $(U, G, \phi)$ is a triple of an open subset of $H^n$ and $\phi : U \to \phi(U)$ is a quotient map inducing a homeomorphism $U/G \to \phi$ for a finite group $G$ acting on $U$. An inclusion map of an open set $\phi(U)$ with the model $(U, G, \phi)$ to another $\psi(V)$ with the model $(V, H, \psi)$ induces an inclusion map $U \to V$ equivariant with respect to an injection of the groups $G \to H$ determined up to conjugations. (See [1] for details.) An orbifold structure is a maximal fine covering. We call $(U, G, \phi)$ the model.

An orbifold $O$ often has a simply-connected manifold as a covering space. In this case the orbifold is said to be good. We will assume this always for our orbifolds. (Orbifolds with geometric structures are always good by Thurston. See Chapter 13 of [78] and Chapter 6 of [22].) Such a covering $\tilde{O}$ is unique up to covering equivalences and is said to be the universal cover. There is a discrete group $\pi_1(O)$ acting on the universal cover so that we recover $O$ as a quotient orbifold $\tilde{O}/\pi_1(O)$, where $\pi_1(O)$ is said to be the (orbifold) fundamental group of $O$.

The local group of a point of $\tilde{O}$ is the inverse limit of the group acting on the model neighborhoods ordered by the lifts of the inclusion maps. It is well-defined.

An end neighborhood of an orbifold is a component of the complement of a compact subset of an orbifold. The collection of the end neighborhoods is partially ordered by inclusion maps.

- An end is an equivalence class of sequences of end neighborhoods

$$U_i, i = 1, 2, ..., U_i \supset U_{i+1} \text{ and } \bigcap_{i=1,2,...} \text{Cl}(U_i) = \emptyset.$$  

- Two such sequences $U_i$ and $V_j$ are equivalent if for each $i$, there exist $j, j'$ such that $U_i \supset V_j$ and $V_j \supset U_{j'}$.

Given an end $E$, we can associate the end fundamental group $\pi_1(E)$ since we can always find a sequence of proper end neighborhoods of product type in the end class. That is, $\pi_1(E)$ is defined as the inverse limit of $\{\text{Im}(\pi_1(U_i)) \subset \pi_1(O)\}$ where maps are

$$\text{Im}(\pi_1(U_i)) \to \text{Im}(\pi_1(U_j)) \subset \pi_1(O), i > j.$$  

Given the universal cover of $\tilde{O}$, a proper pseudo-end neighborhood is a component of an inverse image of an end neighborhood. Two proper pseudo-end neighborhoods are equivalent if they intersect such a one. The equivalence class of a system of proper pseudo-end neighborhood is called a pseudo-end in $\tilde{O}$. If we require
that the proper pseudo-end neighborhood is from one end \( E \), there is a one-to-one correspondence
\[
\{ \tilde{E} \in \tilde{O} \text{ a pseudo-end of } \tilde{O} \text{ corresponding to } E \} \leftrightarrow \pi_1(O)/\pi_1(E).
\]
The subgroup of deck transformation groups \( \pi_1(O) \) preserving the class of a pseudo-end \( \tilde{E} \) is called a pseudo-end fundamental group and is denoted by \( \pi_1(\tilde{E}) \), obviously isomorphic to \( \pi_1(E) \). Moreover, the deck transformation group \( \pi_1(O) \) acts on the set of pseudo-ends and each orbit-equivalence class corresponds to an end of \( O \). Hence, there is a natural map from the set of pseudo-ends of \( \tilde{O} \) to the set of ends of \( O \) induced by the covering map.

2.2. Geometric structures on orbifolds

Let \( G \) be a Lie group acting on an \( n \)-dimensional manifold \( X \). For examples, we can let \( X = \mathbb{R}^n \) and \( G = \text{Aff}(\mathbb{R}^n) = \text{GL}(n, \mathbb{R}) \rtimes \mathbb{R}^n \), i.e., the group of transformations of form \( v \mapsto Av + b \) for \( A \in \text{GL}(n, \mathbb{R}) \) and \( b \in \mathbb{R}^n \). Or we can let \( X = \mathbb{R}P^n \) and \( G = \text{PGL}(n + 1, \mathbb{R}) \), the group of projective transformations of \( \mathbb{R}P^n \).

The complement of \( \mathbb{R}P^n \) of a subspace of codimension-one can be identified with an affine subspace. We realize \( \text{Aff}(\mathbb{R}^n) \) as a subgroup of transformations of \( \text{PGL}(n + 1, \mathbb{R}) \) fixing a subspace of codimension-one as there is an isomorphism
\[
(A, b) \mapsto \begin{bmatrix} A & b^T \\ 0 & 1 \end{bmatrix}, A \in \text{GL}(n, \mathbb{R}), b \in \mathbb{R}^n
\]
where \( b^T \) is the transpose of \( b \).

An \((X, G)\)-structure on an orbifold \( O \) is an atlas of charts from open subsets of \( X \) with finite subgroups of \( G \) acting on them, and the inclusions always lift to restrictions of elements of \( G \) in open subsets of \( X \). This is equivalent to saying that the orbifold \( O \) has a simply connected manifold cover \( \tilde{O} \) with an immersion \( D: \tilde{O} \to X \) and the fundamental group \( \pi_1(O) \) acts on \( \tilde{O} \) properly discontinuously so that \( h: \pi_1(O) \to G \) is a homomorphism satisfying \( D \circ \gamma = h(\gamma) \circ D \) for each \( \gamma \in \pi_1(O) \). Here, \( \pi_1(O) \) is allowed to have fixed points. (We shall use this second definition here.) \( (D, h(\cdot)) \) is called a development pair and for a given \((X, G)\)-structure, it is determined only up to an action
\[
(D, h(\cdot)) \mapsto (k \circ D, kh(\cdot)k^{-1}) \text{ for } k \in G.
\]
Conversely, a development pair completely determines the \((X, G)\)-structure.

Thurston showed that an orbifold with an \((X, G)\)-structure is always good, i.e., covered by a manifold with an \((X, G)\)-structure.

An isotopy of an orbifold \( O \) is a map \( f: O \to O \) with a map \( F: O \times I \to O \) so that
- \( F_t: O \to O \) for \( F_t(x) := F(x, t) \) every fixed \( t \) is an orbifold diffeomorphism,
- \( F_0 \) is the identity, and
- \( f = F_1 \).

Given an \((X, G)\)-structure on another orbifold \( O' \), any orbifold diffeomorphism \( f: O \to O' \) induces an \((X, G)\)-structure pulled back from \( O' \) which is given by using the local models of \( O' \) for preimages in \( O \).
Suppose that \( \mathcal{O} \) is compact. We define the isotopy-equivalence space \( \widetilde{\text{Def}}_{X,G}(\mathcal{O}) \) as the space of development pairs \((\text{dev},h)\) quotient by the isotopies of the universal cover \( \tilde{\mathcal{O}} \) of \( \mathcal{O} \). The deformation space \( \text{Def}_{X,G}(\mathcal{O}) \) is given by the quotient of \( \widetilde{\text{Def}}_{X,G}(\mathcal{O}') \) by the action of \( G : g(\text{dev},h(\cdot)) = (g \circ \text{dev}, gh(\cdot)g^{-1}) \). (See [22] for details.) We can also interpret as follows: The deformation space \( \text{Def}_{X,G}(\mathcal{O}) \) of the \((X,G)\)-structures is the space of all \((X,G)\)-structures on \( \mathcal{O} \) quotient by the isotopy pullback actions.

This space can be thought of as the space of pairs \((D,h)\) with compact open \( C^r \)-topology for \( r \geq 1 \) and the equivalence relation generated by the isotopy relation

- \((D,h) \sim (D',h')\) if \( D' = D \circ \iota \) and \( h' = h \) for a lift \( \iota \) of an isotopy and
- \((D,h) \sim (D',h')\) if \( D' = k \circ D \) and \( h(\cdot) = kh(\cdot)k^{-1} \) for \( k \in G \).

(See [22] or Chapter 6 of [26].)

For noncompact orbifolds with end structures, similar definitions hold except that we have to modify the notion of isotopies to preserve the end structures.

### 2.3. Oriented real projective structures

- Given a vector \( v \in \mathbb{R}^{n+1} - \{O\} \), we denote by \([v] \in \mathbb{R}^Pn\) the equivalence class. Let \( \Pi : \mathbb{R}^{n+1} - \{O\} \to \mathbb{R}^Pn \) denote the projection.
- A cone in \( \mathbb{R}^{n+1} \) is a subset \( C \) so that if \( v \in C \), then \( sv \in C \) for all \( s \in \mathbb{R}_+ \).
- Given a connected subset \( A \) of \( \mathbb{R}^Pn \), a cone \( C_A \subset \mathbb{R}^{n+1} \) of \( A \) is given as a cone in \( \mathbb{R}^{n+1} \) mapping onto \( A \) under the projection \( \Pi : \mathbb{R}^{n+1} - \{O\} \to \mathbb{R}^Pn \).
- \( C_A \) is unique up to the antipodal map \( A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) given by \( v \to -v \).

We will be using the standard elliptic metric \( d \) on \( \mathbb{R}^Pn \) (resp. in \( S^n \)) where the set of geodesics coincides with the set of projective geodesics up to parameterizations. An open hemisphere of \( S^n \) is called an affine subspace in \( S^n \). A great segment is a geodesic arc in \( S^n \) with antipodal p-end vertices. A convex segment is an arc contained in a great segment. A convex subset of \( S^n \) is a subset \( A \) where every pair of points of \( A \) connected by a convex segment.

Recall that \( \text{SL}_\pm(n+1,\mathbb{R}) \) is isomorphic to \( \text{GL}(n+1,\mathbb{R})/\mathbb{R}^+ \). Then this group acts on \( S^n \) to be seen as a quotient space of \( \mathbb{R}^{n+1} - \{O\} \) by the equivalence relation

\[ v \sim w, v, w \in \mathbb{R}^{n+1} - \{O\} \text{ if } v = sw \text{ for } s \in \mathbb{R}^+ \]

We let \([v]\) denote the equivalence class of \( v \in \mathbb{R}^{n+1} - \{O\} \). Given a \( V \in \mathbb{R}^{n+1} \), we denote by \( S(V) \) the image of \( V - \{O\} \) under the quotient map. The image is called a subspace. A set of antipodal points is a subspace of dimension 0. There is a double covering map \( S^n \to \mathbb{R}^Pn \) with the deck transformation group generated by \( A \). This gives a projective structure on \( S^n \). The group of projective automorphisms is identified with \( \text{SL}_\pm(n+1,\mathbb{R}) \). The notion of geodesics are defined as in the projective geometry: they correspond to arcs in great circles in \( S^n \).

A collection of subspaces \( P(V_1), \ldots, P(V_n) \) (resp. \( S(V_1), \ldots, S(V_n) \)) are independent if the subspaces \( V_1, \ldots, V_n \) are independent.

An \((S^n,\text{SL}_\pm(n+1,\mathbb{R}))\)-structure on \( \mathcal{O} \) is said to be an oriented real projective structure on \( \mathcal{O} \). We define \( \text{Def}_{S^n}(\mathcal{O}) \) as the deformation space of \((S^n,\text{SL}_\pm(n+1,\mathbb{R}))\)-structures on \( \mathcal{O} \).

The group \( \text{SL}_\pm(n+1,\mathbb{R}) \) of linear transformations of determinant \( \pm 1 \) maps to the projective group \( \text{PGL}(n+1,\mathbb{R}) \) by a double covering homomorphism \( \hat{q} \), and
We now discuss the standard lifting: Given a real projective structure on $\mathcal{O}$, there is a development pair $(\text{dev}, h)$ where $\text{dev} : \tilde{\mathcal{O}} \to \mathbb{RP}^n$ is an immersion and $h : \pi_1(\mathcal{O}) \to \text{PGL}(n + 1, \mathbb{R})$ is a homomorphism. Since $\mathbb{S}^n \to \mathbb{RP}^n$ is a covering map and $\tilde{\mathcal{O}}$ is a simply connected manifold, $\mathcal{O}$ being a good orbifold, there exists a lift $\text{dev}' : \tilde{\mathcal{O}} \to \mathbb{S}^n$ unique up to the action of $\{1, -1\}$. This induces an oriented projective structure on $\tilde{\mathcal{O}}$ and $\text{dev}'$ is a developing map for this geometric structure. Given a deck transformation $\gamma : \tilde{\mathcal{O}} \to \tilde{\mathcal{O}}$, the composition $\text{dev}' \circ \gamma$ is again a developing map for the geometric structure and hence equals $h'(\gamma) \circ \text{dev}'$ for $h'(\gamma) \in \text{SL}_+(n + 1, \mathbb{R})$. We verify that $h' : \pi_1(\mathcal{O}) \to \text{SL}_+(n + 1, \mathbb{R})$ is a homomorphism. Hence, $(\text{dev}', h')$ gives us an oriented real projective structure, which induces the original real projective structure.

Again, we can define the radial end structures and totally geodesic ideal boundary for oriented real projective structures and also horospherical end neighborhoods in obvious ways. They correspond in the direct way in the following theorem also.

**Theorem 2.1.** There is a one-to-one correspondence between the space of real projective structures on an orbifold $\mathcal{O}$ with the space of oriented real projective structures on $\mathcal{O}$. Moreover, a real projective diffeomorphism of real projective orbifolds is an $(\mathbb{S}^n, \text{SL}_+(n + 1, \mathbb{R}))$-diffeomorphism of oriented real projective orbifolds and vice versa.

**Proof.** Straightforward. See p. 143 of Thurston [79].

**Theorem 2.2.** A real projective orbifold $S$ is covered finitely by a real projective manifold $M$ and $S$ is real projectively diffeomorphic to $M/G_1$ for a finite group $G_1$ of real projective automorphisms of $M$. An affine orbifold $S$ is covered finitely by an affine manifold $N$, and $S$ is affinely diffeomorphic to $N/G_2$ for a finite group $G_2$ of affine automorphisms of $N$.

**Proof.** Since $\text{Aff}(\mathbb{R}^n)$ is a subgroup of a general linear group, Selberg’s Lemma shows that there exists a torsion-free subgroup of the deck transformation group. We can choose the group to be a normal subgroup and the second item follows.

A real projective structure induces an $(\mathbb{S}^n, \text{SL}_+(n + 1, \mathbb{R}))$-structure and vice versa. Also a real projective diffeomorphism of orbifolds is an $(\mathbb{S}^n, \text{SL}_+(n + 1, \mathbb{R}))$-diffeomorphism and vice versa. We regard the real projective structures on $S$ and $M$ as $(\mathbb{S}^n, \text{SL}_+(n + 1, \mathbb{R}))$-structures. We are done by Selberg’s lemma that a finitely generated subgroup of a general linear group has a torsion-free subgroup of finite-index.

**2.4. Metrics**

### 2.4.1. The Hausdorff metric.

Recall the standard elliptic metric $d$ on $\mathbb{RP}^n$ (resp. in $\mathbb{S}^n$) where $d(A, B) := \inf\{d(x, y) | x \in A, y \in B\}$.

We can let $A$ or $B$ be points as well obviously.

The Hausdorff distance between two convex subsets $K_1, K_2$ of $\mathbb{RP}^n$ (resp. of $\mathbb{S}^n$) is defined by

$$d^H(K_1, K_2) = \inf\{\varepsilon \geq 0 | \text{Cl}(K_1) \subset N_\varepsilon(\text{Cl}(K_2)), \text{Cl}(K_2) \subset N_\varepsilon(\text{Cl}(K_1))\}$$
where \( N_\epsilon(A) \) is the \( \epsilon \)-neighborhood of \( A \) under the standard metric \( d \) of \( \mathbb{R}^n \) (of \( S^n \)) for \( \epsilon > 0 \). \( d^H \) gives a compact Hausdorff topology on the set of all compact subsets of \( \mathbb{R}^n \) (of \( S^n \)). (See p. 281 of [70].)

We say that a sequence of sets \( \{ K_i \} \) geometrically converges to a set \( K \) if \( d^H(K_i, K) \to 0 \). If \( K \) is assumed to be closed, then the geometric limit is unique.

Suppose that a sequence \( \{ K_i \} \) of compact convex domains geometrically converges to a compact convex domain \( K \) in \( \mathbb{R}^n \) (resp. in \( S^n \)), i.e., \( d^H(K_i, K) \to 0 \). In this case we claim that

\[
(2.1) \quad d^H(\partial K_i, \partial K) \to 0:
\]

For every point \( x \) of \( \partial K \), and an \( \epsilon \)-ball \( B_x \), \( B_x \cap \partial K_i \neq \emptyset \) for sufficiently large \( i \) since \( B_x \) must meet \( K_i \) and be not contained in \( K_i \) for sufficiently large \( i \). There exists some sequence \( x_i \in \partial K_i \) so that \( x_i \to x \). Conversely, every convergence sequence \( \{ x_i \} \), \( x_i \in \partial K_i \), must converge to \( x \in \partial K \).

Given two sets \( A \) and \( B \) of \( \partial \mathcal{O} \) or \( \mathcal{O} \) with a metric \( d \), we define

\[
d(A, B) := \inf\{d(x, y) | x \in A, y \in B \}.
\]

The definition obviously extends to the cases when \( A \) or \( B \) are points.

**2.4.2. The Hilbert metric.** Let \( \Omega \) be a properly convex open domain. A line or a subspace of dimension-one in \( \mathbb{R}^n \) has a two-dimensional homogenous coordinate system. Let \([o, s, q, p]\) denote the cross ratio of four points on a line as defined by

\[
\frac{\bar{o} - \bar{q}}{\bar{s} - \bar{q}} \frac{s - \bar{p}}{s - \bar{o}}
\]

where \( \bar{o}, \bar{p}, \bar{q}, \bar{s} \) denote respectively the first coordinates of the homogeneous coordinates of \( o, p, q, s \) so that the second coordinates equal 1. Define a metric for \( p, q \in \Omega \),

\[
d_\Omega(p, q) = \log |[o, s, q, p]| \quad \text{where } o \ 	ext{and } s \ \text{are endpoints of the maximal segment in } \Omega \ \text{containing } p, q \ \text{where } o, q \ \text{separates } p, s.
\]

The metric is one given by a Finsler metric. (See [63].)

Given an SPC-structure on \( \mathcal{O} \), there is a Hilbert metric which we denote by \( d_\mathcal{O} \) on \( \mathcal{O} \) and hence on \( \mathcal{O} \). Actually, we will make \( \mathcal{O} \) slightly small by inward perturbations of \( \partial \mathcal{O} \) preserving the strict convexity of \( \partial \mathcal{O} \). The Hilbert metric will be defined on original \( \mathcal{O} \). (We call this metric the perturbed Hilbert metric.) This induces a metric on \( \mathcal{O} \), including the boundary now. We will denote the metric by \( d_\mathcal{O} \). More precisely,

Assume that \( K_i \to K \) geometrically for a sequence of properly convex domains \( K_i \) and a properly convex domain \( K \). Suppose that two sequences of points \( \{ x_i | x_i \in K_i^o \} \) and \( \{ y_i | y_i \in K_i^o \} \) converge to \( x, y \in K^o \) respectively. Since the end of a maximal segments always are in \( \partial K_i \) and \( \partial K_i \to \partial K \), the above shows that

\[
(2.2) \quad d_{K^o}(x_i, y_i) \to d_{K^o}(x, y)
\]

holds. We omit the details of the elementary proof.

**2.5. Convexity and convex domains**

An affine manifold is convex if every path can be homotoped to an affine geodesic with endpoints fixed. A complete real line in \( \mathbb{R}^n \) is a 1-dimensional affine subspace of \( \mathbb{R}^n \) with denote it by \( \mathbb{R} \). In \( S^n \), a complete real line is defined as the
interior of a great segment. An affine manifold is \emph{properly convex} if there is no affine map from \( \mathbb{R} \) into it; i.e., there is no complete affine line in its universal cover.

**Proposition 2.3.** An affine manifold is convex if and only if a developing map sends the universal cover to a convex open domain in \( \mathbb{R}^n \). An affine manifold is properly convex if and only if the developing map sends the universal cover to a properly convex open domain in \( \mathbb{R}^n \).

**Proof.** The first part is Theorem 8.1 of Shima [76] or Theorem A.2 of [21]. The second part is Theorem 8.3 of [76] since the hyperbolicity of Kobayashi is equivalent to the proper convexity. (See Kobayashi [63].) \qed

A complete real line in \( \mathbb{R} P^n \) is a 1-dimensional subspace of \( \mathbb{R} P^n \) with one point removed. That is, it is the intersection of a 1-dimensional subspace by an affine subspace. A \emph{convex} projective geodesic is a projective geodesic in a real projective manifold which lifts to a convex segment in \( S^n \). A real projective manifold is convex if every path can be homotopied to a convex projective geodesic with endpoints fixed. It is \emph{properly convex} if there is no projective map from the complete real line \( \mathbb{R} \). (See Chapters 2 and 3 of [21] for more details.)

**Proposition 2.4 (Vey).**

- A strongly tame real projective orbifold is properly convex if and only if the developing map sends the universal cover to a properly convex open domain bounded in an affine subspace of \( \mathbb{R} P^n \).
- If a convex real projective orbifold is not properly convex, then its holonomy homomorphism is virtually reducible.

**Proof.** For the first part, the affine suspension has a developing image to a properly convex subset of an affine subspace. (See Section 2.10.2.) For the final item, see [17]. (See also [54].) \qed

**Lemma 2.5.** Let \( K \) be a compact domain with boundary \( \partial K \neq \emptyset \) with a local homeomorphism \( J \) to \( S^n \) so that each point has a neighborhood in \( J \) imbedding onto a convex domain. Then \( J \) maps \( K \) homeomorphically to a convex domain in \( S^n \) and give \( K \) a convex real projective structure with possibly nonsmooth boundary.

**Proof.** \( J \) induces a real projective structure on \( K \). We can show that \( K \) is now 1-convex. Now, Theorem A.2 of [21] proves this. \qed

**Proposition 2.6 (Corollary 2.13 of Benoist [10]).** Suppose that a discrete subgroup \( \Gamma \) of \( \text{PGL}(n, \mathbb{R}) \) (resp. \( \text{SL}_+^e(n, \mathbb{R}) \)) acts on a properly convex \((n-1)\)-dimensional open domain \( \Omega \) in \( \mathbb{R} P^{n-1} \) (resp. \( S^{n-1} \)) so that \( \Omega/\Gamma \) is compact. Then the following statements are equivalent.

- Every subgroup of finite index of \( \Gamma \) has a finite center.
- Every subgroup of finite index of \( \Gamma \) has a trivial center.
- Every subgroup of finite index of \( \Gamma \) is irreducible in \( \text{PGL}(n, \mathbb{R}) \) (resp. \( \text{SL}_+^e(n, \mathbb{R}) \)). That is, \( \Gamma \) is strongly irreducible.
- The Zariski closure of \( \Gamma \) is semisimple.
- \( \Gamma \) does not contain a normal infinite nilpotent subgroup.
- \( \Gamma \) does not contain a normal infinite abelian subgroup.

The group with properties above is said to be the group with a \emph{virtually center free group} or a \emph{vcf-group}. 


Theorem 2.7 (Theorem 1.1 of Benoist [10]). Let $\Gamma$ be a discrete subgroup of $\text{PGL}(n, \mathbb{R})$ (resp. $\text{SL}_\pm(n, \mathbb{R})$) with a trivial virtual center. Suppose that a discrete subgroup $\Gamma$ of $\text{PGL}(n, \mathbb{R})$ (resp. $\text{SL}_\pm(n, \mathbb{R})$) acts on a properly convex $(n-1)$-dimensional open domain $\Omega$ so that $\Omega/\Gamma$ is a compact orbifold. Then every representation of a component of $\text{Hom}(\Gamma, \text{PGL}(n, \mathbb{R}))$ (resp. $\text{Hom}(\Gamma, \text{SL}_\pm(n, \mathbb{R}))$) containing the inclusion representation also acts on a properly convex $(n-1)$-dimensional open domain cocompactly.

Given subspaces $V_1, \ldots, V_m \subset \mathbb{R}^n$ (resp. $\subset \mathbb{S}^n$) where any two are mutually disjoint, and a subset $C_i \subset V_i$ for each $i$, we define a strict join of $n$ sets $C_1, \ldots, C_m$

$$C_1 \ast \cdots \ast C_m := \left\{ \left[ \sum_{i=1}^m t_i v_i \right] \left\vert \sum_{i=1}^m t_i = 1, t_i \in [0, 1], v_i \in C_{C_i} \right\} ,$$

where $C_{C_i}$ is a cone in $\mathbb{R}^{n+1}$ corresponding to $C_i$. (Of course, this depends on the choices of $C_{C_i}$ up to $A$.)

A point $x$ of a strict join $C_1 \ast \cdots \ast C_j$ for convex sets $C_i$ has join-coordinates $[\lambda_1, \ldots, \lambda_j]$ if $x = [\sum_{i=1}^k \lambda_i \tilde{v}_i]$ for $\tilde{v}_i$ a vector in the cone corresponding to $C_i$.

A cone-over a strictly joined domain is one containing a strictly joined domain $A$ and is a union of segments from a cone-point $\not\in A$ to points of $A$.

Proposition 2.8 (Theorem 1.1 of Benoist [10]). Assume $n \geq 2$. Let $\Sigma$ be a closed $(n-1)$-dimensional properly convex projective orbifold and let $\Omega$ denote its universal cover in $\mathbb{R}P^{n-1}$ (resp. in $\mathbb{S}^{n-1}$). Then

(i) $\Omega$ is projectively diffeomorphic to the interior of a strict join $K_1 \ast \cdots \ast K_{l_0}$ where $K_i$ is a proper convex open domain of dimension $n_i \geq 0$ corresponding to a convex cone $C_i \subset \mathbb{R}^{n_i+1}$.

(ii) $\Omega$ is the image of the interior of $C_1 \oplus \cdots \oplus C_r$.

(iii) The fundamental group $\pi_1(\Sigma)$ is virtually isomorphic to $\mathbb{Z}^{l_0-1} \times \Gamma_1 \times \cdots \times \Gamma_{l_0}$ for $l_0 + \sum n_i = n$. Each $\Gamma_i$ has the property that each finite index subgroup has a trivial center.

(iv) Each $\Gamma_j$ acts on $K_j$ cocompactly and the Zariski closure is a semi-simple Lie group in $\text{PGL}(n_j+1, \mathbb{R})$ (resp. in $\text{SL}_\pm(n_j+1, \mathbb{R})$), and acts trivially on $K_m$ for $m \neq j$.

(v) The subgroup corresponding to $\mathbb{Z}^{l_0-1}$ acts trivially on each $K_j$.

Supposing that $\pi_1(\Sigma)$ is admissible, the Zariski closure of $\Gamma_j$ is one of $O(n_j + 1, 1)$, $\text{PGL}(n_j+1, \mathbb{R})$, $\text{SL}_\pm(n_j+1, \mathbb{R})$, or a union of their components.

A convex hull of a compact subset $A$ of $\mathbb{R}P^n$ is defined as the smallest closed convex subset containing $A$ if $A$ is a bounded subset of an affine subspace of $\mathbb{R}P^n$. This definition is independent of the choice of the affine subspace. A convex hull of a compact subset $A$ of $\mathbb{S}^n$ is defined as the smallest closed convex subset containing $A$ in $\mathbb{S}^n$. (A pair of antipodal points does not have a convex hull.) By the compactness of $\mathbb{R}P^n$ (resp. $\mathbb{S}^n$), a convex hull of a compact set $A$ is a union of the set $S_1$ of 1-simplices with endpoints in the closure of $A$ and the set $S_2$ of 2-simplices with boundary edges in $S_1$ and $S_0$ of $i$-simplices with boundary sides in $S_{i-1}$ for $i = 3, 4, \ldots, n$. We denote it by $\text{CH}(A)$. 

2.5. CONVEXITY AND CONVEX DOMAINS
2.6. Geometric convergence of convex real projective orbifolds

We say that a set $A$ span a subspace $S$ in $\mathbb{R}P^n$ (resp. $S^n$) if $S$ is the smallest subspace containing $A$. Now Proposition A.5 covers the case of Corollary 2.9 when $\Gamma$ is virtually center-free or Benoist since $K_t$ are always properly convex.

**Corollary 2.9.** Suppose that the fundamental group $\Gamma$ of a closed $(n-1)$-orbifold $\Sigma$ is a virtual product of hyperbolic groups and abelian groups. We are given a path $\mu_t$, $t \in [0, 1]$, of convex $\mathbb{R}P^{n-1}$-structures on $\Sigma$ equipped with $C^r$-topology, $r \geq 2$. Suppose that $\mu_0$ is properly convex or complete affine with abelian holonomy. Then we can find a family of developing maps $D_t$ to $\mathbb{R}P^{n-1}$ (resp. in $S^{n-1}$) continuous in the $C^r$-topology and a continuous family of holonomy homomorphisms $h_t : \Gamma \to \Gamma_0$ so that $K_t := \text{Cl}(D_t(\Sigma))$ is a uniformly continuous family of convex domains in $\mathbb{R}P^{n-1}$ (resp. in $S^{n-1}$) under the Hausdorff metric topology of the space of closed subsets of $\mathbb{R}P^{n-1}$ (resp. $S^n$).

- We can find a family of developing maps $D_t$ to $\mathbb{R}P^{n-1}$ (resp. in $S^{n-1}$) continuous in the $C^r$-topology and a continuous family of holonomy homomorphisms $h_t : \Gamma \to \Gamma_0$ so that $K_t := \text{Cl}(D_t(\Sigma))$ is a uniformly continuous family of convex domains in $\mathbb{R}P^{n-1}$ (resp. in $S^{n-1}$) under the Hausdorff metric topology of the space of closed subsets of $\mathbb{R}P^{n-1}$ (resp. $S^n$).

- In other words, given $0 < \epsilon < 1/2$ and $t_0, t_1 \in [0, 1]$, we can find $\delta > 0$ such that if $|t_0 - t_1| < \delta$, then $K_{t_1} \subset N_\epsilon(K_{t_0})$ and $K_{t_0} \subset N_\epsilon(K_{t_1})$.

- Also, given $0 < \epsilon < 1/2$ and $t_0, t_1 \in [0, 1]$, we can find $\delta > 0$ such that if $|t_0 - t_1| < \delta$, then $\partial K_{t_1} \subset N_\epsilon(\partial K_{t_0})$ and $\partial K_{t_0} \subset N_\epsilon(\partial K_{t_1})$.

- Finally, $\mu_t$ is always virtually immediately deformable to a properly convex structure.

Alternatively, these are true whenever we choose $(D_t, h_t)$ so that $h_t$ is chosen to be a continuous path

$$h_t : [0, 1] \to \text{Hom}(\Gamma, \text{PGL}(n, \mathbb{R})) \text{ (resp. } h_t : [0, 1] \to \text{Hom}(\Gamma, \text{SL}_\pm(n, \mathbb{R})).$$

**Proof.** We will prove for $S^{n-1}$. The $\mathbb{R}P^{n-1}$-version follows from this.

We assume first that $\Gamma$ is a virtually-center-free group. First, for any sequence $t_i$, we can choose a subsequence $t_{i_j}$ so that $\{K_{t_{i_j}}\}$ converges to a compact set $K_\infty$ in the Hausdorff metric. $h_{t_0}(\Gamma_0)$ acts on $K_\infty$ as in Choi-Goldman [30]. By Benoist [10], $K_\infty$ is a properly convex domain.

Let us fix $K_\infty$. Now, for any sequence $t_i$, suppose that a convergent subsequence $K_{t_{i_j}}$ converges to $K'_\infty$. Then we claim $K_\infty = K'_\infty$: This follows since the set of attracting and repelling fixed points of elements of $h_{t_0}(\gamma)$ for every $\gamma \neq I$ exist and is in $\partial K'_\infty$ and $\partial K_\infty$ by the $\Gamma$-invariance. They are also dense by Theorem 1.1 of [6] and the density of periodic orbits of Anosov flows and hence $\partial K'_\infty = \partial K_\infty$. This implies $t \mapsto K_t$ is a continuous function by a well-known result in metric topology. This complete the proof in this case.

The join $K_1^t \ast \cdots \ast K_k^t$ is properly convex always by Proposition A.5(i). The subspace $V_i^t$ spanned by $K_i^t$ depends continuously on $t$ since the holonomy of generators of $\Gamma_t$ determines $K_i^t$ uniquely. This completes the proof for the joined cases of this type.

Suppose now that $\Gamma$ is virtually abelian. Then $\Omega_t$ is determined by the generators of the free abelian subgroup $\Gamma'$ of finite index with positive eigenvalues only by Lemma A.3. $\Gamma'$ determines the connected abelian Lie group $\Delta_t$ containing $h_0(\Gamma')$ and $\Omega_t$ is an orbit of $\Delta_t$ by Lemma A.3. Now Lemma A.1 implies the first item.

Now to finish the proof, we suppose that $\Gamma$ has hyperbolic factors. By Proposition A.5(iii), $\Gamma$ acts on $d(\Delta_t) \ast K_1^t \ast \cdots \ast K_k^t$ for $t \in [0, 1]$ where $\Delta_0$ is a subgroup of the Zariski closure of the center of $h_0(\Gamma')$.**
The result for $K_1^p \ast \cdots \ast K_k^p$ is done above. Our proof reduces to the case of $d(\Delta_i)$ and $\Delta_i$ only. This was already accomplished above.

The second item follows from the first one. The third one can be deduced by equation (2.1). The final item follows by Propositions A.5(i),(iii) and A.4.

The final alternative formulation follows: Proposition A.5(iii) determines the image of $D_i$ by the convexity of $\mu_i$. Each of the join part of the image of $D_i$ depends continuously on $h_i$ by Lemma A.3 and by the result proved above in the proof for properly convex domains denoted by as $K_1^p \ast \cdots \ast K_k^p$.

\[ \square \]

2.7. p-ENDS, P-END NEIGHBORHOODS, AND P-END GROUPS

Let $\mathcal{O}$ be a real projective orbifold with the universal cover $\tilde{\mathcal{O}}$. We fix a developing map $\text{dev}$ in this subsection. Given a radial end of $\mathcal{O}$ and an end neighborhood $U$ of a product form $E \times [0,1)$ with a radial foliation, we take a component $U_1$ of $p^{-1}(U)$ and the lift of the radial foliation with leaves whose developing image end at a common point $x$ in $\mathbb{R}^n$.

- We call $x$ the pseudo-end vertex of $\tilde{\mathcal{O}}$. $x$ will be denoted by $\nu_E$ if its neighborhoods corresponds to a $p$-end $\tilde{E}$.
- Generalizing further, we call an open subset $U$ of $\tilde{\mathcal{O}}$ containing a proper pseudo-end neighborhood of $\tilde{E}$, where $\pi_1(\tilde{E})$ acts on a pseudo-end neighborhood. A proper pseudo-end neighborhood is an example. (In the following “pseudo” will be shorted to “$p$”.)
- Let $S_{\nu_E}^{n-1}$ denote the space of equivalence classes of rays from $\nu_E$ diffeomorphic to an $(n-1)$-sphere where $\pi_1(\tilde{E})$ acts as a group of projective automorphisms. Here, $\pi_1(\tilde{E})$ acts on $\nu_E$ and sends leaves to leaves in $U_1$.
- Given a p-end $\tilde{E}$ corresponding to $\nu_E$, we denote by $R_{\nu_E}(\tilde{\mathcal{O}}) = \tilde{S}_E$ the space of directions of developed leaves under $\text{dev}$ oriented away from $\nu_E$ in $\tilde{\mathcal{O}}$. The space develops to $S_{\nu_E}^{n-1}$ by $\text{dev}$ as an immersion.
- Also, for a subset $K$ of $\mathcal{O}$, we denote by $R_{\nu_E}(K)$ the space of directions of developed images of leaves in $\tilde{\mathcal{O}}$ under $\text{dev}$ mapping to rays oriented away from $\nu_E$ ending at $K$. We have $R_{\nu_E}(\tilde{\mathcal{O}}), R_{\nu_E}(K) \subset S_{\nu_E}^{n-1}$ if $\tilde{\mathcal{O}}$ is a convex domain in $\mathbb{R}^n$.
- Recall that $S_{\tilde{E}}/\Gamma_{\tilde{E}}$ is projectively diffeomorphic to the end orbifold to be denoted by $\Sigma_E$.

Given a totally geodesic end of $\mathcal{O}$ and an end neighborhood $U$ of the product from $E \times [0,1)$ with a compactification by a totally geodesic orbifold $E'$, we take a component $U_1$ of $p^{-1}(U)$ and a convex domain $S_{\tilde{E}}$, the ideal boundary component, developing to totally geodesic hypersurface under $\text{dev}$. Here $\tilde{E}$ is the p-end corresponding to $E$ and $U_1$. There exists a subgroup $\Gamma_{\tilde{E}}$ acting on $S_{\tilde{E}}$. Again $S_{\tilde{E}}/\Gamma_{\tilde{E}}$ is projectively diffeomorphic to the end orbifold to be denote by $S_E$ again.

- We call $S_{\tilde{E}}$ the $p$-ideal boundary component of $\tilde{\mathcal{O}}$.
- The group $\Gamma_{\tilde{E}}$ is said to be a $p$-end-fundamental group associated with $\tilde{E}$.

- We call $U_1$ a proper $p$-end neighborhood of $\tilde{E}$.

Generalizing further an open subset $U$ of $\tilde{\mathcal{O}}$ containing a proper $p$-end-neighborhood of $\tilde{E}$, where $\pi_1(\tilde{E})$ acts on, is said to be a $p$-end neighborhood.
2. Preliminary

2.8. The admissible groups

If every subgroup of finite index of a group $\Gamma$ has a finite center, $\Gamma$ is said to be a virtual center-free group or a vcf-group. An admissible group is a finite extension of a finite product of $\mathbb{Z}^l \times \Gamma_1 \times \cdots \times \Gamma_k$ for infinite hyperbolic groups $\Gamma_i$ where $l \geq k - 1$ or $k = 1$ holds and $l + k \leq n$ holds. (See Section 4.1 for details. $l \geq k - 1$ follows from the result of Benoist discussed there. We have $k = 1$ and $l = 0$ if and only if the end fundamental group is hyperbolic. For example, if our orbifold has a complete hyperbolic structure, then end fundamental groups are virtually free abelian.) We will also say that an end is

- **hyperbolic** if the end fundamental group is hyperbolic, i.e., $k = 1, l = 0$ and
- **Benoist** if $l + 1 = k \geq 1$ or $l = k \geq 1$. Benoist ends are said to be permanently properly convex. (See Proposition A.5(i).)

Hyperbolic ends are Benoist. (Of course, these definitions also apply to p-ends.)

2.9. The admissible ends

Let $\tilde{O}$ a convex real projective orbifold with the universal cover $\tilde{O}$.

- A subdomain $K$ of $\mathbb{RP}^n$ (resp. in $S^n$) is said to be horospherical if $\text{Cl}(K)$ is bounded by an ellipsoid and the boundary $\partial K$ is diffeomorphic to $\mathbb{R}^{n-1}$ and $\text{bd}K - \partial K$ is a single point.
- $K$ is lens-shaped if it is a convex domain and the manifold-boundary $\partial K$ is a disjoint union of two smoothly embedded $(n-1)$-cells $A_1$ and $A_2$ not containing any straight segment in them. $K$ is a generalized lens if we allow a component of $\partial K$ to be not necessarily smooth.
- A cone over a point $x$ and a set $A$ in an affine subspace of $\mathbb{RP}^n$ (resp. in $S^n$), $x \notin A$ is the set given by $x * A$ in $\mathbb{RP}^n$ (resp. in $S^n$). (Here no vector in $C_A$ is same or antipodal to $v_x$. Of course, this depends on the choice of $v_x$ and $C_A$ determined up to the inversion $v \rightarrow -v$.)
- A lens-cone is a cone $C := \{x\} * L - \{x\}$ over a lens-domain $L$ and a point $x$, $x \notin \text{Cl}(L)$, so that every maximal segment from $x$ in $C$ ends in one component $\partial_1 L$ of $\partial L$. A generalized lens-cone is a cone over a generalized lens-domain with the same properties where a nonsmooth component has to be in the boundary of the cone. For two components $A_1$ and $A_2$ of $\partial L$, $A_1$ is called a top hypersurface if it faces the exterior of the join and $A_2$ is then called a bottom hypersurface.
- A lens of a lens-cone $C$ is the lens-shaped domain $A$ so that $C = \{x\} * A - \{x\}$ for a point $x \notin \text{Cl}(A)$.
- A totally-geodesic subdomain is a convex domain in a hyperspace.
- A cone-over a totally-geodesic domain $A$ is a cone over a point $x$ not in the hyperspace.

Any subset of $\tilde{O}$ developing diffeomorphic to the above sets under $\text{dev}$ with $x$ being an end vertex is named by the same name. We will also call a real projective orbifold with boundary to be

- a horospherical or
- a lens-cone or
- a lens, provided it is compact,
if it is covered by such domains and is homeomorphic to a closed \((n - 1)\)-orbifold times an interval.

We introduce some relevant adjectives: Let \(\Sigma_E\) be an \((n - 1)\)-dimensional end orbifold corresponding to a \(p\)-end \(\tilde{E}\), and let \(\mu\) be a holonomy homomorphism

\[
\pi_1(\tilde{E}) \to \text{PGL}(n + 1, \mathbb{R}) \quad \text{(resp. } \text{SL}_\pm(n_1, \mathbb{R})\text{)}
\]

restricted from that of \(O\).

- Suppose that \(\mu(\pi_1(\tilde{E}))\) acts on a (generalized) lens-shaped domain \(K\) in \(\mathbb{R}P^n\) (resp. in \(S^n\)) with boundary a union of two open \((n - 1)\)-cells \(A_1\) and \(A_2\) and \(\pi_1(\tilde{E})\) acts properly on \(A_1\) and \(A_2\). Then \(\mu\) is said to be a (generalized) lens-shaped representation for \(\tilde{E}\) with respect to \(x\).
- \(\mu\) is a totally-geodesic representation if \(\mu(\pi_1(\tilde{E}))\) acts on a totally-geodesic subdomain.
- If \(\mu(\pi_1(\tilde{E}))\) acts on a horoball \(K\), then \(\mu\) is said to be a horospherical representation. In this case, \(\text{bd}K - \partial K = \{v_{\tilde{E}}\}\) for the \(p\)-end vertex \(v_{\tilde{E}}\) of \(\tilde{E}\).
- If \(\mu(\pi_1(\tilde{E}))\) acts on a strict joined domain, then \(\mu\) is said to be a strict joined representation.

A concave \(p\)-end-neighborhood is an imbedded \(p\)-end neighborhood of form \(L - C'\) contained in a radial \(p\)-end neighborhood \(L \in O\) with end vertex \(v_{\tilde{E}}\) where \(\text{dev}(L)\) is a generalized lens-cone \(v_{\tilde{E}} \ast \text{dev}(C')\) for a generalized lens \(\text{dev}(C')\).

We redefine the definition given in the introduction. The equivalence is obvious by taking the closures in \(\tilde{O}\) of lens \(p\)-end neighborhoods or concave \(p\)-end neighborhoods where the end-fundamental groups act on.

**Definition 2.10.** (Admissible ends) Let \(O\) be a real projective orbifold with the universal cover \(\tilde{O}\). Let \(E\) be an end of \(O\) and \(\tilde{E}\) be the corresponding \(p\)-end with the admissible \(p\)-end fundamental group \(\pi_1(\tilde{E})\).

- We say that the radial \(p\)-end \(E\) of \(O\) is horospherical if \(\tilde{E}\) has a \(p\)-end neighborhood that is a horoball in \(\tilde{O}\).
- We say that the radial end \(E\) of \(O\) is of lens-type if \(\tilde{E}\) has a \(p\)-end neighborhood that is a lens-cone. \(E\) is of generalized lens-type if \(\tilde{E}\) has a concave \(p\)-end neighborhood. Equivalently, \(\tilde{O}\) has the interior of a generalized lens giving us a \(p\)-end neighborhood of \(\tilde{E}\). in a generalized lens-cone where \(\pi_1(\tilde{E})\) acts on. (We require that the cone-point has to be the end point of the radial lines for the given radial \(p\)-end for the two cases above.)

An end is admissible (resp. admissible in a generalized sense) if it is a radial horospherical or radial lens-type (resp. generalized lens-type) or totally geodesic lens \(p\)-end.

Recall that a totally geodesic \(p\)-end \(E\) of lens-type has the lens-condition that the ideal boundary end orbifold \(S_{\tilde{E}}\) has a lens-neighborhood \(L\) in an ambient orbifold containing \(\tilde{O}\). For a component \(C_1\) of \(L - S_{\tilde{E}}\) inside \(\tilde{O}\), \(C_1 \cup S_{\tilde{E}}\) is said to be the one-sided neighborhood of \(S_{\tilde{E}}\).

(Note that the notion of a totally geodesic radial end can be of lens-type but it is a different concept from that of a totally geodesic lens-type end. However, one can be converted to the other using some geometric operations of cutting and pasting.)
2.10. The duality

A *dilatation* is an affine automorphism $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ of the affine space given by $v \mapsto s(v - w_0) + w_0$ for $s > 0, s \neq 1$ and an arbitrary point $w_0 \in \mathbb{R}^{n+1}$. Here $s$ is the *expansion factor* of the dilatation and is uniquely determined by the dilatation, and $w_0$ is the fixed point.

We start from linear duality. Let us choose the origin $O$ in $\mathbb{R}^{n+1}$. Let $\Gamma$ be a group of linear transformations $\text{GL}(n+1, \mathbb{R})$. Let $\Gamma^*$ be the *affine dual group* defined by $\{g^{-1} | g \in \Gamma\}$ acting on the dual space $\mathbb{R}^{n+1,*}$. Suppose that $\Gamma$ acts on a properly convex cone $C$ in $\mathbb{R}^{n+1}$ with the vertex $O$.

An open convex cone $C^*$ in $\mathbb{R}^{n+1,*}$ is *dual* to an open convex cone $C$ in $\mathbb{R}^{n+1}$ if $C^* \subset \mathbb{R}^{n+1,*}$ is the set of linear transformations taking positive values on $\text{Cl}(C) - \{O\}$. $C^*$ is a cone with vertex as the origin again. Note $(C^*)^* = C$.

Now $\Gamma^*$ will act on $C^*$. Also, if $\Gamma$ acts cocompactly on $C$ if and only if $\Gamma^*$ acts on $C^*$ cocompactly. A *central dilatation extension* $\Gamma'$ of $\Gamma$ by $\mathbb{Z}$ is given by adding a dilatation by a scalar $s \in \mathbb{R}_+ - \{1\}$ with the fixed $O$. The dual of $\Gamma'$ is a central dilatation extension of $\Gamma^*$.

Given a subgroup $\Gamma$ in $\text{PGL}(n+1, \mathbb{R})$, an *affine lift* in $\text{GL}(n+1, \mathbb{R})$ is any subgroup that maps to $\Gamma$ isomorphically under the projection. Given a subgroup $\Gamma$ in $\text{PGL}(n+1, \mathbb{R})$, the dual group $\Gamma^*$ is the image in $\text{PGL}(n+1, \mathbb{R})$ of the dual of any affine lift of $\Gamma$. For $\text{SL}_\pm(n+1, \mathbb{R})$, we define the dual groups as above.

A properly convex open domain $\Omega$ in $P(\mathbb{R}^{n+1})$ (resp. in $S(\mathbb{R}^{n+1})$) is *dual* to a properly convex open domain $\Omega^*$ in $P(\mathbb{R}^{n+1,*})$ (resp. in $S(\mathbb{R}^{n+1,*})$) if $\Omega$ corresponds to an open convex cone $C$ and $\Omega^*$ to its dual $C^*$. We say that $\Omega^*$ is dual to $\Omega$. We also have $(\Omega^*)^* = \Omega$ and $\Omega$ is properly convex if and only if so is $\Omega^*$.

We call $\Gamma$ a *divisible group* if a central dilatational extension acts cocompactly on $C$. $\Gamma$ is divisible if and only if so is $\Gamma^*$. (See [8]).
Note that a hyperspace is an element of $\mathbb{R}P^n$ since it is represented as a 1-form. And an element of $\mathbb{R}P^n$ can be considered as a hyperspace in $\mathbb{R}P^n$. The following definition applies to $\Omega \subset \mathbb{R}P^n$ (resp. $S(\mathbb{R}^{n+1,*})$ and $\Omega^* \subset \mathbb{R}P^n$ (resp. $S(\mathbb{R}^{n+1,*})$). Given a properly convex domain $\Omega$, we define the augmented boundary of $\Omega$

$$\text{bd}^{A\delta}\Omega := \{(x,h)\mid x \in \text{bd}\Omega, h \text{ is a supporting hyperplane of } \Omega, h \ni x\}.$$ 

Note that for each $x \in \text{bd}\Omega$, there exists at least one supporting hyperspace.

**Remark 2.11.** For open properly convex domains $\Omega_1$ and $\Omega_2$, we have

$$\Omega_1 \subset \Omega_2 \text{ if and only if } \Omega_2^* \subset \Omega_1^*.$$ 

We will call the homeomorphism below as the **duality map**.

**Proposition 2.12.** Suppose that $\Omega \subset \mathbb{R}P^n$ (resp. $S(\mathbb{R}^{n+1,*})$) and its dual $\Omega^* \subset \mathbb{R}P^n$ (resp. $S(\mathbb{R}^{n+1,*})$) are properly convex domains. 

(i) There is a proper quotient map $\Pi_{A\delta}: \text{bd}^{A\delta}\Omega \to \text{bd}\Omega$ given by sending $(x,h)$ to $x$.

(ii) Let $\Gamma$ act on properly discontinuously $\Omega$ if and only if so acts $\Gamma^*$ on $\Omega^*$.

(iii) There exists a homeomorphism

$$\mathcal{D}: \text{bd}^{A\delta}\Omega \leftrightarrow \text{bd}^{A\delta}\Omega^*$$

given by sending $(x,h)$ to $(h,x)$.

(iv) Let $A \subset \text{bd}^{A\delta}\Omega$ be a subspace and $A^* \subset \text{bd}^{A\delta}\Omega^*$ be the corresponding dual subspace $\mathcal{D}(A)$. If a group $\Gamma$ acts properly discontinuously on $A$ and only if $\Gamma^*$ so acts on $A^*$.

**Proof.** We will prove for $\mathbb{R}P^n$. (The $S^n$-version has a similar proof.) (i) Each fiber is a closed set of hyperplanes at a point forming a compact set. The set of supporting hyperplanes at a compact subset of $\text{bd}\Omega$ is closed. The closed set of hyperplanes passing a compact subset of $\mathbb{R}P^n$ is compact. Thus, $\Pi_{A\delta}$ is proper. Clearly, $\Pi_{A\delta}$ is continuous since it is induced by a projection.

(ii) Straightforward.

(iii) An element $(x,h)$ is $\text{bd}^{A\delta}\Omega$ if and only if $x \in \text{bd}\Omega$ and $h$ is represented by a 1-form $\alpha_h$ so that $\alpha_h(y) > 0$ for all $y$ in the open cone $C$ corresponding to $\Omega$ and $\alpha_h(v_x) = 0$ for a vector $v_x$ representing $x$.

Since the dual cone $C^*$ consists of all nonzero 1-form $\alpha$ so that $\alpha(y) > 0$ for all $y \in \text{Cl}(C) - \{O\}$. Thus, $\alpha(v_x) > 0$ for all $\alpha \in C^*$ and $\alpha_h(v_x) = 0$. $\alpha_h \notin C^*$ since $v_x \in \text{Cl}(C) - \{O\}$. But $h \in \text{bd}C^*$ as we can perturb $\alpha_h$ so that it is in $C^*$. Thus, $x$ is a supporting hyperspace at $h \in \text{bd}\Omega^*$. Hence we obtain a continuous map $\mathcal{D}: \text{bd}^{A\delta}\Omega \to \text{bd}^{A\delta}\Omega^*$. The inverse map is obtained in a similar way.

(iv) The item follows from (ii) and (iii). 

**Lemma 2.13.** Let $\Omega^*$ be the dual of a properly convex domain $\Omega$. Then

(i) $\text{bd}\Omega$ is $C^1$ and strictly convex if and only $\text{bd}\Omega^*$ is $C^1$ and strictly convex.

(ii) $\Omega$ is a horospherical orbifold if and only if so is $\Omega^*$.

(iii) Let $p \in \text{bd}\Omega$. Then $\mathcal{D}$ sends in a one-to-one and onto manner

$$\{(p,h)|h \text{ is a supporting hyperplane of } \Omega \text{ at } p\}$$

to $\{(h^*,p^*)|h^* \in D = p^* \cap \text{bd}\Omega^*\}$ where $D$ is a properly convex domain disjoint from $\Omega^{**}$. 


(iv) \( \text{bd}\Omega^* \) contains a properly convex domain \( D = P \cap \text{bd}\Omega^* \) open in a totally geodesic hyperplane \( P \) if and only if \( \text{bd}\Omega \) contains a vertex \( p \) with \( R_p(\Omega) \) a properly convex domain. Moreover, \( D \) and \( R_p(\Omega) \) are properly convex and are projectively diffeomorphic to dual domains.

**Proof.** These are straightforward. \( \square \)

### 2.10.1. Affine Orbifolds

An affine orbifold is an orbifold with a geometric structure modelled on \((\mathbb{R}^n, \text{Aff}(\mathbb{R}^n))\). An affine orbifold has a notion of affine geodesics as given by local charts. Recall that a geodesic is **complete** in a direction if the affine geodesic parameter is infinite in the direction.

- An affine orbifold has a **parallel end** if the corresponding end has an end neighborhood foliated by properly imbedded affine geodesics parallel to each other in charts and each leaf is complete in the end direction. We assume that the affine geodesics are leaves assigned as above.
  - We obtain a smooth complete vector field \( X_E \) in a neighborhood of \( E \) for each end following the affine geodesics, which is affinely parallel in the flow; i.e., leaves have parallel tangent vectors. We call this an *end vector field*.
  - We denote by \( X_\mathcal{O} \) the vector field partially defined on \( \mathcal{O} \) by taking a sum of vector fields defined on some mutually disjoint neighborhoods of the ends using the partition of unity.
  - The oriented direction of the parallel end is uniquely determined in the developing image of each p-end neighborhood of the universal cover of \( \mathcal{O} \).
  - Finally, we put a fixed complete Riemannian metric on \( \mathcal{O} \) so that for each end there is an open neighborhood where the metric is invariant under the flow generated by \( X_\mathcal{O} \). Note that such a Riemannian metric always exists.

- An affine orbifold has a **totally geodesic end** \( E \) if each end can be completed by a totally geodesic affine hypersurface. That is, there exists a neighborhood of the end \( E \times [0, 1] \) that compactifies to an orbifold diffeomorphic to \( E \times [0, 1] \), and each point of \( E \times \{1\} \) has a neighborhood affinely diffeomorphic to a neighborhood of a point \( p \) in \( \partial H \) for a half-space \( H \) of an affine space. This implies the fact that the corresponding p-end fundamental group \( \pi_1(\tilde{E}) \) for a p-end \( \tilde{E} \) going to \( E \), \( h(\pi_1(\tilde{E})) \) acts on a totally geodesic hyperplane \( P \) corresponding to \( E \times \{1\} \).

Recall that an orbifold is a topological space stratified by open manifolds. An affine or projective orbifold is **triangulated** if there is a smoothly imbedded \( n \)-cycle consisting of geodesic \( n \)-simplices on the compactified orbifold relative to ends by adding an ideal point to a radial end and an ideal boundary to each totally geodesic end, where the interiors of \( i \)-simplices in the cycle are mutually disjoint and are imbedded in strata of the same or higher dimension.

### 2.10.2. Affine Suspension Constructions

The affine space \( \mathbb{R}^{n+1} \) is a dense open subset of \( \mathbb{R}P^{n+1} \) which is the complement of \((n + 1)\)-dimensional projective space \( \mathbb{R}P^{n+1} \). Thus, an affine transformation is a restriction of a unique projective automorphism acting on \( \mathbb{R}^{n+1} \). The group of affine transformations \( \text{Aff}(\mathbb{R}^{n+1}) \) is
isomorphic to the group of projective automorphisms acting on $\mathbb{R}^{n+1}$ identified this way by the restriction homomorphism.

An affine orbifold $\mathcal{O}'$ is radiant if $h(\pi_1(\mathcal{O}'))$ fixes a point in $\mathbb{R}^n$ for the holonomy homomorphism $h : \pi_1(\mathcal{O}) \to \text{Aff}(\mathbb{R}^{n+1})$. A real projective orbifold $\mathcal{O}$ of dimension $n$ has a developing map $\text{dev}' : \hat{\mathcal{O}} \to \mathbb{S}^n$ and the holonomy homomorphism $h' : \pi_1(\mathcal{O}) \to \text{SL}_+(n+1, \mathbb{R})$. Here, $\mathbb{S}^n$ is imbedded as a unit sphere in $\mathbb{R}^{n+1}$. We obtain a radiant affine $(n+1)$-orbifold by taking $\hat{\mathcal{O}}$ and $\text{dev}'$ and $h'$: Define $D'' : \hat{\mathcal{O}} \times \mathbb{R}^+ \to \mathbb{R}^{n+1}$ by sending $(x, t)$ to $t\text{dev}'(x)$. For each element of $\gamma \in \pi_1(\mathcal{O})$, we define the transformation $\gamma'$ on $\hat{\mathcal{O}} \times \mathbb{R}^+$ by

$$\gamma'(x, t) = (\gamma(x), \theta(\gamma)||h'(\gamma)((t\text{dev}'(x))))|$$

(2.4)

Also, there is a transformation $S_s : \hat{\mathcal{O}} \times \mathbb{R}^+ \to \hat{\mathcal{O}} \times \mathbb{R}^+$ sending $(x, t)$ to $(x, st)$ for $s \in \mathbb{R}^+$. Thus,

$$\hat{\mathcal{O}} \times \mathbb{R}^+/\langle S_\rho, \pi_1(\mathcal{O}) \rangle, \rho \in \mathbb{R}_+, \rho \neq 1$$

is an affine orbifolds with the fundamental group isomorphic to $\pi_1(\mathcal{O}) \times \mathbb{Z}$ with the developing map is given by $D''$ the holonomy homomorphism is given by $h'$ and sending the generator of $\mathbb{Z}$ to $S_\rho$. We call the result the affine suspension of $\mathcal{O}$, which of course is radiant. The representation of $\pi_1(\mathcal{O}) \times \mathbb{Z}$ with the center $\mathbb{Z}$ mapped to a dilatation is called an affine suspension of $h$. A special affine suspension is an affine suspension with $\theta \equiv 1$ identically. (See Sullivan-Thurston [77], Barbot [5] and Choi [24] also.)

**Definition 2.14.** We denote by $C(\hat{\mathcal{O}})$ the manifold $\hat{\mathcal{O}} \times \mathbb{R}$ with the structure given by $D''$, and say that $C(\hat{\mathcal{O}})$ is the affine suspension of $\hat{\mathcal{O}}$.

Let $S_t : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, given by $\vec{v} \to t\vec{v}$, $t \in \mathbb{R}_+$, be a one-parameter family of dilations fixing a common point. A family of self-diffeomorphisms $\Psi_t$ on an affine orbifold $M$ lifting to $\hat{\Psi}_t : \hat{M} \to \hat{M}$ so that $D \circ \hat{\Psi}_t = S_t \circ D$ for $t \in \mathbb{R}_+$ is called a group of radiant flow diffeomorphisms.

**Lemma 2.15.** Let $M$ be a strongly tame $n$-orbifold.

- An affine suspension $\mathcal{O}'$ of a real projective orbifold $\mathcal{O}$ always admits a group of radiant flow diffeomorphisms. Here, $\{\Phi_t\}$ is a circle and all flow lines are closed.
- Conversely, if there exists a group of radiant flow diffeomorphisms with closed orbits on $M \times \mathbb{S}^1$ with an affine structure, then $M \times \mathbb{S}^1$ is affinely diffeomorphic to one obtained by an affine suspension construction from a real projective structure on $M$.

**Proof.** The only the second item is not shown. The generator of $\pi_1(S)$ factor goes to a dilatation. Thus, each closed curve along $* \times \mathbb{S}^1$ gives us a nontrivial homology. The homology direction of the flow equals $[[* \times \mathbb{S}^1]] \in H_1(M \times \mathbb{S}^1)$. By Theorem D of [44], there exists a connected cross-section homologous to $[M \times *] \in H_n(M \times \mathbb{S}^1, V \times \mathbb{S}^1) \cong H^1(M \times \mathbb{S}^1)$ where $V$ is the union of the disjoint end neighborhoods of product form. By Theorem C of [44], any cross-section is isotopic to $M \times *$. The radial flow is transversal to the cross-section isotopic to $M \times *$ and hence $M$ admits a real projective structure. It follows easily now that $M \times \mathbb{S}^1$ is an affine suspension. (See [5] for examples.) $\square$
An affine suspension of a horospherical orbifold is called a \textit{suspended horospherical orbifold}. An end of an affine orbifold with an end neighborhood affinely diffeomorphic to this is said to be of \textit{suspended horospherical type}. This has also a parallel end since the parallel direction is given by the fixed point in the boundary of $\mathbb{R}^n$.

Under the cone-construction, a real projective $n$-orbifold has radial, totally geodesic, or horospherical ends if and only if the affine $(n + 1)$-orbifold affinely suspended from it has parallel, totally geodesic, or suspended horospherical ends.
CHAPTER 3

The local homeomorphism theorems

3.1. The semi-algebraic properties of \( \text{rep}^s(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \) and related spaces

Since \( \mathcal{O} \) is the interior of a compact orbifold, there exists a finite set of generators \( g_1, \ldots, g_m \) with finitely many relators. First, \( \text{Hom}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \) can be identified with an algebraic subset of \( \text{PGL}(n+1, \mathbb{R})^m \) corresponding to the relators.

Let \( \text{Hom}_E(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \) denote the subspace of

\[ \text{Hom}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \]

where the holonomy of each \( p \)-end fundamental group fixes a point of \( \mathbb{R}P^n \) or acts on a subspace \( P \) of codimension-one and on a lens meeting \( P \) satisfying the lens-condition or a horoball tangent to \( P \). Each end of \( \mathcal{O} \) is assigned to be an \( \mathcal{R} \)-type end or a \( \mathcal{T} \)-type end.

Since a set of finitely many elements generates each end fundamental group, and the end fundamental groups are finite up to conjugacy, the conditions of having a common 1-dimensional eigenspace for each of a finite collection of finitely generated subgroups is a semi-algebraic condition.

Let \( E \) denote an end of type \( \mathcal{T} \). Let

\[ \text{Hom}_{E,TL}(\pi(E), \text{PGL}(n+1, \mathbb{R})) \]

denote the space of totally geodesic representations of \( \pi_1(E) \) satisfying the lens-condition, again an open subset of the algebraic set. (This follows by the proof of Theorem 8.1 of [27].) If \( \rho \) is of horospherical type, then \( \pi_1(E) \) is virtually abelian by Theorem 1.1 of [27]. Define \( \text{Hom}_{E,p}(\pi_1(E), \text{PGL}(n+1, \mathbb{R})) \) to be the space of representations where an abelian group of finite index goes into a parabolic subgroup in a copy of \( \text{PO}(n,1) \). By Lemma 3.1, \( \text{Hom}_{E,p}(\pi_1(E), \text{PGL}(n+1, \mathbb{R})) \) is a closed algebraic set.

**Lemma 3.1.** Let \( G \) be a finite extension of a finitely generated free abelian group \( \mathbb{Z}^m \). Then \( \text{Hom}_{E,p}(G, \text{PGL}(n+1, \mathbb{R})) \) is a closed algebraic set.

**Proof.** Let \( P \) be a maximal parabolic subgroup of a copy of \( \text{PO}(n+1, \mathbb{R}) \) that fixes a point \( x \). Then \( \text{Hom}(\mathbb{Z}^m, P) \) is a closed algebraic set.

\[ \text{Hom}_{E,p}(\mathbb{Z}^m, \text{PGL}(n+1, \mathbb{R})) \]

equals a union

\[ \bigcup_{g \in \text{PGL}(n+1, \mathbb{R})} \text{Hom}_p(\mathbb{Z}^m, gPg^{-1}), \]
another closed algebraic set. Now $\text{Hom}_{E,p}(G, \text{PGL}(n + 1, \mathbb{R}))$ is a closed algebraic subset of

$$\text{Hom}_{E,p}(\mathbb{Z}^m, \text{PGL}(n + 1, \mathbb{R})).$$

Therefore we conclude that $\text{Hom}_{E}(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R}))$ is an open subset of an algebraic set.

A parabolic subalgebra $\mathfrak{p}$ is an algebra in a semi-simple Lie algebra $\mathfrak{g}$ whose complexification contains a maximal solvable subalgebra of $\mathfrak{g}$ (p. 279–288 of [80]). A parabolic subgroup $P$ of a semi-simple Lie group $G$ is the full normalizer of a parabolic subalgebra.

Let $\text{Hom}_E^s(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R}))$ denote the subspace of stable irreducible representations. We first note:

• the subset of $\text{Hom}_E(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R}))$ that holonomy groups acting on proper subspaces is a closed algebraic set, the space of proper subspaces in $\mathbb{R}^{n+1}$ is an algebraic set, and
• by Theorem 1.1 of [60], the set of representations not in proper parabolic subgroups is open in $\text{Hom}(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R})).$

Let $E$ be an end orbifold of $\mathcal{O}$. Given $\rho \in \text{Hom}_E(\pi_1(E), \text{PGL}(n + 1, \mathbb{R}))$, we have:

• If $\rho$ is of radial of lens-type, then each element of an open neighborhood is also radial of lens-type by Theorem 3.14 of [27]. Let $\text{Hom}_{E,RL}(\pi(E), \text{PGL}(n + 1, \mathbb{R}))$ denote the space of radial lens-type representations of $\pi_1(E)$. Thus, it is an open subspace of the above algebraic set.
• If $\rho$ is of totally geodesic of lens-type, then each element of an open neighborhood is also totally geodesic of lens-type by Theorem 3.14 of [27]. Let $\text{Hom}_{E,TL}(\pi(E), \text{PGL}(n + 1, \mathbb{R}))$ denote the space of totally geodesic lens-type representations of $\pi_1(E)$. Thus, it is an open subspace of the above algebraic set.

Let $R_{E_i} : \text{Hom}(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R})) \to \text{Hom}(\pi_1(E_i), \text{PGL}(n + 1, \mathbb{R}))$ be the restriction map to the p-end fundamental group $\pi_1(E_i)$ corresponding to the end $E_i$ of $\mathcal{O}$.

Let $\mathcal{R}_\mathcal{O}$ denote the set of radial ends of $\mathcal{O}$, and let $\mathcal{T}_\mathcal{O}$ denote the set of totally geodesic ends of $\mathcal{O}$. We can identify

$$\text{Hom}^s_E(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R}))$$

with the subset

$$\big( \bigwedge_{E_i \in \mathcal{R}_\mathcal{O}} R_{E_i}^{-1}(\text{Hom}_{E_i}(\pi(E_i), \text{PGL}(n + 1, \mathbb{R}))) \big) \cap$$

$$\big( \bigwedge_{E_i \in \mathcal{T}_\mathcal{O}} R_{E_i}^{-1}(\text{Hom}_{E_i,p}(\pi(E_i), \text{PGL}(n + 1, \mathbb{R})) \cup \text{Hom}_{E_i,TL}(\pi(E_i), \text{PGL}(n + 1, \mathbb{R}))) \big).$$
Hence, this is an open subset of a semi-algebraic set.

Let \( \text{Hom}_{E,u}(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \) denote the subspace of \( \text{Hom}_E(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \) where each end of \( \mathcal{R} \)-type fixes a unique point of \( \mathbb{R}P^n \) and each end of \( \mathcal{T} \)-type acts on a unique subspace of codimension-one satisfying the lens-condition or a horosphere tangent to it. We obtain an open subset of a semi-algebraic set since we need to consider finitely many generators of the fundamental groups of the ends as pointed out by D. Fried.

We can identify \( \text{Hom}_{E,u,ce}(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \) to be the subset \( \text{Hom}_{E,u}(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \) ∩ \( (\bigcup_{E_i \in \mathcal{R}O} R_{E_i}^{-1}(\text{Hom}_{E_i,u}(\pi(E_i), \text{PGL}(n+1, \mathbb{R})) \cup \text{Hom}_{E_i,RL}(\pi(E_i), \text{PGL}(n+1, \mathbb{R})))) \) ∩ \( (\bigcup_{E_i \in \mathcal{T}O} R_{E_i}^{-1}(\text{Hom}_{E_i,v}(\pi(E_i), \text{PGL}(n+1, \mathbb{R})) \cup \text{Hom}_{E_i,TL}(\pi(E_i), \text{PGL}(n+1, \mathbb{R})))) \).

Since \( \text{rep}_{E,u}(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \) is the Hausdorff quotient of the above set with the conjugation \( \text{PGL}(n+1, \mathbb{R}) \)-action, this is a topological open subset of a semi-algebraic set. By Proposition 1.1 of [60]. Similarly, so is \( \text{rep}_{E,u,ce}(\pi_1(O), \text{PGL}(n+1, \mathbb{R})). \)

### 3.2. The end condition of affine structures

A suspension of a legion in \( S^n \) in \( \mathbb{R}^{n+1} \) is the inverse image of the region under the projection \( \mathbb{R}^{n+1} - \{O\} \to S^n \). A suspended lens is a quotient affine manifold of a suspension of a lens. A suspended horoball is a quotient affine manifold of a suspension of a horoball.

Given an affine orbifold \( O \) satisfying our end conditions, and each end is given a parallel end type or a totally geodesic lens end type. Each end fundamental group of \( \pi_1(O) \) will have a distinguished infinite cyclic group in the center. Each end of our orbifold \( O \) is given an \( \mathcal{R} \)-type or a \( \mathcal{T} \)-type.

- An \( \mathcal{R} \)-type end is allowed to be parallel always, and
- A \( \mathcal{T} \)-type end is allowed to be totally geodesic with a suspended lens neighborhood in some cover of an ambient affine manifold corresponding to the end fundamental group or be parallel with a suspended horoball neighborhood.

Here the distinguished cyclic central subgroups are required to go to the groups of dilatations preserving the cones corresponding.

Let us make a choice of conjugacy classes of the fundamental group \( \pi_1(E) \) as a subgroup of \( \pi_1(O) \) for every radial end \( E \) as a subgroup of \( \pi_1(O) \).

We define a subspace \( \text{Hom}_E(\pi_1(O), \text{Aff}(\mathbb{R}^{n+1})) \) of \( \text{Hom}(\pi_1(O), \text{Aff}(\mathbb{R}^{n+1})) \) to be the subspace where \( h(\pi_1(E_i)) \) for each end \( E_i \) consists of affine transformations
3. The Local Homeomorphism Theorems

- with linear parts with at least one common eigenvector if \( E_i \) is of \( R \)-type or
- acting on an affine hyperspace \( P \) and properly discontinuously and co-compactly
  - on a suspension \( L \) of a lens meeting \( P \) in its interior
  - or on a suspension of a horoball tangent to \( P \) if \( E_i \) is of \( T \)-type.

(Here we need to fix the generator of the boundary going to a dilatation for each \( T \)-ends.)

Let \( e_1 \) be the number of \( R \)-ends of \( O \). Let \( U \) be an open subspace of

\[ \text{Hom}_E(\pi_1(O), \text{Aff}(\mathbb{R}^{n+1})) \]

invariant under the conjugation action so that one can choose a continuous section \( s_{U}^{(1)} : U \to (\mathbb{R}^n - \{O\})^{e_1} \) sending a holonomy homomorphism \( h \) to a common nonzero eigenvector of \( h(\pi_1(E)) \) for each \( R \)-type end \( E \). Here \( s_{U}^{(1)} \) satisfies

\[ s_{U}^{(1)}(gh(\cdot)g^{-1}) = g(s_{U}^{(1)}(h(\cdot))) \quad \text{for} \quad g \in \text{Aff}(\mathbb{R}^n), h \in U. \]

(The choice of the sections might not be canonical here.) We say that \( s_{U} \) is the eigenvector-section of \( U \).

Let \( AS(\mathbb{R}^{n+1}) \) denote the space of oriented affine hyperplanes in \( \mathbb{R}^{n+1} \). There is a standard action of \( \text{Aff}(\mathbb{R}^{n+1}) \) on \( AS(\mathbb{R}^{n+1}) \). One can choose a continuous section also \( s_{U}^{(2)} : U \to AS(\mathbb{R}^n)^{e_2} \) sending a holonomy homomorphism \( h \) to an invariant hyperplane in \( \mathbb{R}^{n+1} \) of \( h(\pi_1(E)) \) for each totally geodesic end \( E \) of lens-type. Here \( s_{U}^{(2)} \) satisfies

\[ s_{U}^{(2)}(gh(\cdot)g^{-1}) = g(s_{U}^{(2)}(h(\cdot))) \quad \text{for} \quad g \in \text{Aff}(\mathbb{R}^{n+1}), h \in U. \]

(The choice of the sections might not be canonical here.) We say that \( s_{U}^{(2)} \) is the eigen-1-form section of \( U \). We form the eigensection

\[ s_{U} := s_{U}^{(1)} \times s_{U}^{(2)} : U \to (\mathbb{R}^{n+1} - \{O\})^{e_1} \times (AS(\mathbb{R}^{n+1}))^{e_2}. \]

We note that the affine structure with parallel and totally geodesic ends also will determine a point of \( (\mathbb{R}^{n+1} - \{O\})^{e_1} \times (AS(\mathbb{R}^{n+1}))^{e_2}. \)

One can identify \( AS(\mathbb{R}^{n+1}) \) with an open subspace of \( S^{n+1}_{\text{aff}} \) by sending the affine hyperspace to a hyperspace of \( S^{n+1} \) and hence to a point of \( S^{n+1}_{\text{aff}} \) by duality. In fact the open subspace is \( S^{n+1}_{\text{aff}} - \{[\alpha], [-\alpha]\} \) where \( \alpha \) is a 1-form determining \( \mathbb{R}^{n+1} \).

**Remark 3.2** (End fundamental group conditions). There is also an important end fundamental group condition: Let \( P \) be some unspecified condition restricting the holonomy homomorphisms of ends. We say that \( U \) and \( \pi_1(O) \) have the unique fixed direction property with respect to \( P \) for holonomy homomorphisms from the p-end fundamental group \( \pi_1(E) \to \text{Aff}(\mathbb{R}^n) \) arising from \( U \)

- if for each parallel end \( \tilde{E} \), the linear parts of holonomy elements of \( \pi_1(\tilde{E}) \) have a nonzero eigenvector, then it is the nonzero common eigenvector unique up to scalar multiplications for \( U \) under the condition \( P \),
- if for each totally geodesic end \( \tilde{E} \), the holonomy elements of \( \pi_1(\tilde{E}) \) have a common invariant affine hyperplane \( H \), then \( H \) is a unique invariant affine hyperplane under the condition \( P \).
3.3. The End Condition for Real Projective Structures

Of course, \( P \) could be an empty condition.

More precisely, it is not a purely group condition but a geometric condition. In fact, it might be possible that such a condition holds for a component of character space but not for some other subsets of \( \text{Hom}_E^r(\pi_1(\mathcal{O}), \text{Aff}(\mathbb{R}^n)) \). In such cases, our results are valid for the components where the conditions hold.

Finally, we say that the orbifold \( \mathcal{O} \) will have the convex end fundamental group condition if the holonomy group of each of its radial ends has the unique fixed direction and that of each of its totally geodesic end of lens-type has the unique fixed affine hyperplane for every holonomy homomorphism of a convex affine structure on \( \mathcal{O} \) with radial ends or totally geodesic ends in \( \text{Hom}_E^r(\pi_1(\mathcal{O}), \text{Aff}(\mathbb{R}^n)) \).

Example 3.3. For example, if each \( \mathcal{R} \)-type end of \( \mathcal{O} \) has singularity of dimension 1 and there are no \( \mathcal{T} \)-type ends, then the end fundamental group condition holds: If \( \mathcal{O} \) is affine with parallel end, then the singularity line in the universal cover of \( \mathcal{O} \) is in the parallel direction and determines the eigendirection.

3.3. The End Condition for Real Projective Structures

Now, we go over to real projective orbifolds: We are given a real projective orbifold \( \mathcal{O} \) with ends \( E_1, \ldots, E_{\ell_1} \) of \( \mathcal{R} \)-type and \( E_{\ell_1+1}, \ldots, E_{\ell_1+\ell_2} \) of \( \mathcal{T} \)-type. Let us choose representative \( p \)-ends \( \tilde{E}_1, \ldots, \tilde{E}_{\ell_1} \) and \( \tilde{E}_{\ell_1+1}, \ldots, \tilde{E}_{\ell_1+\ell_2} \). Again, \( \ell_1 \) is the number of \( \mathcal{R} \)-type ends, \( \ell_2 \) the number of \( \mathcal{T} \)-type ends of \( \mathcal{O} \).

We define a subspace of \( \text{Hom}_E^r(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \) to be as in Section 3.1. Let \( \mathcal{V} \) be an open subset of

\[
\text{Hom}_E^r(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))
\]

invariant under the conjugation action so that one can choose a continuous section \( s_1^{(1)} : \mathcal{V} \to (\mathbb{R}^n)^{\ell_1} \) sending a holonomy homomorphism to a common fixed point of \( h(\pi_1(\tilde{E}_i)) \) for \( i = 1, \ldots, \ell_1 \) and \( s_1^{(1)} \) satisfies

\[
s_1^{(1)}(gh(\cdot)g^{-1}) = g \cdot s_1^{(1)}(h(\cdot)) \quad \text{for } g \in \text{PGL}(n+1, \mathbb{R}).
\]

There might be more than one choice of a section and the domain of definition. \( s_1^{(1)} \) is said to be a fixed-point section.

Again we assume that for the open subset \( \mathcal{V} \) of

\[
\text{Hom}_E^r(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))
\]

invariant under the conjugation action suppose that one can choose a continuous section \( s_2^{(2)} : \mathcal{V} \to (\mathbb{R}^n)^{\ell_2} \) sending a holonomy homomorphism to a common dual fixed point of \( \pi_1(\tilde{E}_i) \) for \( i = \ell_1 + 1, \ldots, \ell_2 \), and \( s_2^{(2)} \) satisfies \( s_2^{(2)}(gh(\cdot)g^{-1}) = (g^*)^{-1} \circ s_2^{(2)}(h(\cdot)) \) for \( g \in \text{PGL}(n+1, \mathbb{R}) \). There might be more than one choice of section in certain cases. \( s_2^{(2)} \) is said to be a dual fixed-point section.

We define \( s_2^{(1)} : \mathcal{V} \to (\mathbb{R}^n)^{\ell_1} \times (\mathbb{R}^n)^{\ell_2} \) as \( s_2^{(1)} \times s_2^{(2)} \) and call it a fixed-section provided the holonomy group of each \( \mathcal{T} \)-type \( p \)-end fundamental group \( \tilde{E}_i \) acts on a horosphere tangent to \( P \) determined by \( s_2^{(2)} \).

Remark 3.4. Let \( P \) be some condition on holonomies of end fundamental groups. We say that \( \mathcal{V} \) and an end fundamental group \( \pi_1(\tilde{E}) \) have the unique fixed point and dual fixed point property with respect to \( P \) if the holonomy homomorphism of an \( \mathcal{R} \)-type \( p \)-end \( \tilde{E} \) has a common fixed point, then it is the unique fixed point
for \( V \) under the condition \( P \) and if the holonomy homomorphism of the \( T \)-type end \( \tilde{E} \) has a common dual fixed point, then it is the unique dual fixed point for \( V \) under \( P \).

Finally we say that the orbifold \( O \) will have the \textit{end fundamental group condition} if the fundamental group of each of its p-end \( \tilde{E} \) has the uniquely fixed point and dual fixed point property for all representations in \( \text{Hom}^*_{\text{E}}(\pi_1(O), \text{PGL}(n + 1, \mathbb{R})) \) according to the type of \( E \) that arise as holonomy homomorphisms of real projective structures on \( O \). We say that the orbifold \( O \) will have the \textit{convex end fundamental group condition} if the fundamental group of each of its p-end has the unique fixed point and the dual fixed point property for all representations in \( \text{Hom}^*_{\text{E}}(\pi_1(O), \text{PGL}(n + 1, \mathbb{R})) \) according to its type that arise as holonomy homomorphisms of convex real projective structures on \( O \) with radial ends or totally geodesic ends of lens-type.

\textbf{Example 3.5.} If \( O \) is real projective and has some singularity of dimension one in each end-neighborhood of an \( R \)-type end, then the universal cover of \( O \) has more than two lines corresponding to singular loci. The developing image of the lines must meet at a point in \( \mathbb{R}P^n \), which is a fixed point of the holonomy group of an end. If \( O \) has dimension 3, this is equivalent to requiring that the end orbifold has corner-reflectors or cone-points.

\textbf{Example 3.6.} If \( O \) is a real projective \( n \)-orbifolds with \( R \)-type ends and virtually center-free end fundamental groups, then the convex end fundamental group condition holds: Let \( E \) be a radial end. The end fundamental group must be hyperbolic and irreducible. As in the above argument since the end orbifold has a strictly convex real projective structure with irreducible holonomy homomorphism by Benoist \([8]\) and hence cannot preserve a foliation of totally geodesic leaves of any dimension between 1 and \( n - 1 \).

\textbf{Example 3.7 (Cooper).} We do caution the readers that these assumptions are not trivial and exclude some important representations. For example, these spaces exclude some incomplete hyperbolic structures arising in Thurston’s Dehn surgery constructions as they have at least two fixed points for the holonomy homomorphism of the fundamental group of a toroidal end as was pointed out by Cooper.

\section*{3.4. Perturbing horospherical ends}

The following concerns the deformations of \( Z^n \to \text{PGL}(n + 1, \mathbb{R}) \) near horospherical representations. As long as we restrict to deformed representations satisfying the lens-condition, there exist \( n \)-dimensional properly convex domains where the groups act on. (This answers a question of Tillman near 2006. J. Porti also discussed with me on the parabolic representations in 2011.)

Let \( P \) be an oriented hyperspace of \( S^n \) with a dual point \( P^* \in S^n^* \) represented by a 1-form \( \omega_P \). Let \( S^*_{P^*} \) be the space of rays from \( P^* \) corresponding to hyperspaces in \( P \). Then the subspace \( P \) is dual to \( S^*_{P^*} \): each ray in \( S^n^* \) from \( P^* \) defines an oriented hyperspace \( S' \) of \( P \) as the set of common zeros of the 1-forms in the ray. The orientation of \( S' \) is given by the open half-space where the 1-forms near \( \omega_P \) are positive. Conversely, a oriented pencil of hyperplanes determined by a hyperspace of \( P \) is a ray in \( S^*_{P^*} \) from \( P^* \). (The obvious \( \mathbb{R}P^n \)-version is omitted.)
LEMMA 3.8 (Horospherical end perturbation). Let $B$ be a horoball in $\mathbb{R}P^n$ (resp. in $S^n$) and $\Gamma_p$ be a group of projective automorphisms fixing $p$, $p \in b\mathbb{R}B$, (resp. $p \in S^n$) so that $B/\Gamma_p$ is a horospherical orbifold. Let $P$ be a hyperplane in $\mathbb{R}P^n$ (resp. in $P^n$).

- Let $\text{Hom}_{E,p,ce}(\Gamma_p, PGL(n+1, \mathbb{R}))$ denote the space of representations $h$ fixing a common fixed point $p$. Then there exists a sufficiently small neighborhood $K$ of the inclusion homomorphism of $\Gamma_p$ in $\text{Hom}_{E,p,ce}(\Gamma_p, PGL(n+1, \mathbb{R}))$ where for each $h \in K$, $h(\Gamma_p)$ acts on a properly convex domain $B_h$ so that $B_h/h(\Gamma_p)$ is homeomorphic to $B/\Gamma_p$ forming a radial end and fixes $p$.

- Let $\text{Hom}_{E,TL}(\Gamma_p, PGL(n+1, \mathbb{R}))$ denote the space of representations where $h(\Gamma_p)$ for each element $h$ acts on $P$ satisfying the lens-condition. Then there exists a sufficiently small neighborhood $K$ of the inclusion homomorphism of $\Gamma_p$ in $\text{Hom}_{E,TL}(\Gamma_p, PGL(n+1, \mathbb{R}))$ where for each $h \in P$ $h(\Gamma_p)$ acts on a properly convex domain $B_h$ so that $B_h/h(\Gamma_p)$ is homeomorphic to $B/\Gamma_p$ and has a totally geodesic end of lens-type or horospherical end.

PROOF. We will prove for the $S^n$-version, which implies the $\mathbb{R}P^n$-version. Let us choose a smaller horoball $B'$ in $B$. $B'/\Gamma_p$ has a boundary component $S'_E$ so $B'/\Gamma$ is diffeomorphic to $S'_E \times [1,0)$. $S'_E$ is strictly convex and transversal to the radial foliation. There exists a neighborhood $K$ in $\text{Hom}_{E,p,ce}(\Gamma_p, PGL(n+1, \mathbb{R}))$ corresponding to the connection on a fixed compact neighborhood $N$ of $S'_E$ changes only by $\epsilon$ in $C^2$-topology. (See the deformation theorem in [47] which generalize to the compact orbifolds with boundary.) The universal cover $\tilde{S}'_E$ is a strictly convex codimension-one manifold and it deforms to $\tilde{S}'_{E,h}$ that is still convex for sufficiently small $\epsilon$. Here, $\tilde{S}'_{E,h}$ may not be imbedded in $\mathbb{R}P^n$ but is a submanifold of the deformed $n$-manifold $N_h$ from $N$ by the change of connections. Every ray from $p$ meets $\tilde{S}'_{E,h}$ transversally also by the $C^2$-condition.

Let $v_x$ be a vector in direction of $x$ for $x \in \text{Cl}(B')$. We form a cone

$$c(\tilde{S}'_{E,h}) := \{[tv_p + (1-t)v_x]|t \in [0,1], x \in \tilde{S}'_E\}.$$  

Let $\tilde{S}_{E,h}$ denote the space of rays from $p$ ending at $\tilde{S}'_{E,h}$ in $c(\tilde{S}'_{E,h})$. Here $S_h := \tilde{S}_{E,h}/h(\Gamma_p)$ is a compact real projective orbifold of $(n-1)$-dimension.

A map $D_h : \tilde{S}_{E,h} \to S^{n-1}_p$ sends the point $x$ to the image ray in $S^{n-1}_p$. By the assumption on $\text{Hom}_{E,c}(\Gamma_p, PGL(n+1, \mathbb{R}))$, the image is in $S^{n-1}_p$.

It follows that $D_h$ is an imbedding to a domain $\Omega_h$ in $S^{n-1}_p$ where $h(\Gamma_p)$ acts properly discontinuously and cocompactly.

There is a one-to-one correspondence from $\tilde{S}'_{E,h}$ to $\tilde{S}_{E,h} := \Omega_h$. By convexity of $\tilde{S}_{E,h}$ and the strict convexity of $\tilde{S}'_{E,h}$, we obtain that $B_h$ is convex by Lemma 2.5. The proper convexity of $B_h$ follows since $\tilde{S}'_{E,h}$ is strictly convex, and hence $\text{Cl}(B_h)$ cannot contain a pair of antipodal points.

The second item is the dual of the first one. If $h(\Gamma_p)$ acts on a horosphere tangent to $P$ with the vertex in $P$ properly discontinuously, then the dual group $h(\Gamma_p)^*$ acts on a horosphere with a vertex the point $P^*$ dual to $P$. Suppose that $h(\Gamma_p)$ acts on a convex domain $\Omega_p$ in $P$. Then it acts on a convex domain in $S^{n-1}_p$. 


the space of rays from \( P^* \) corresponding to hyperspaces in \( P \) by Proposition 2.12. Therefore, we are reduced to the first item.

An affine space \( \mathbb{R}^{n+1} \) has a great sphere \( \mathbb{S}^n_{\infty} \) as a boundary. We define \( \mathbb{S}^n_{\infty} \) as the dual sphere where the dual group \( G^* \) acts on provided an affine group \( G \) act on \( \mathbb{S}^n_{\infty} \).

Suppose that \( G \) is an extension of an affine group by a cyclic group of dilatations that are in the center. We denote by \( \text{Hom}^S(G, \text{Aff}(\mathbb{R}^{n+1})) \) the representation where the central cyclic group go to the group of dilatations.

**Lemma 3.9 (Affine horospherical end perturbation).** Let \( B \) be a horoball and \( \Gamma_p \) be a group of projective automorphisms fixing \( p \) so that \( B/\Gamma_p \) is a horospherical end orbifold. Let \( C_B \) an affine cone corresponding to \( B \) and \( \Gamma_p \) denote the affine transformation corresponding to \( \Gamma_p \) centrally extended by an infinite cyclic dilatation group acting on \( C_B \).

- Let \( \text{Hom}_{E,p,ce}^S(\Gamma_p', \text{Aff}(\mathbb{R}^{n+1})) \) denote the space of representations \( h \) with linear parts with a common eigenvector \( v_p \) so that \( [v_p] = p \) where the restriction group of \( h(\Gamma_p') \) acts on a lens-cone or a horoball-cone with the end parallel along \( v_p \). Then there exists a sufficiently small neighborhood \( P \) of the inclusion homomorphism of \( \Gamma_p' \) in \( \text{Hom}_{E,p,ce}^S(\Gamma_p', \text{Aff}(\mathbb{R}^{n+1})) \) where for each \( h(\Gamma_p') \) with \( h \in P \) so that \( h(\Gamma_p') \) acts on a properly convex cone \( C_{B_h} \) so that \( C_{B_h}/h(\Gamma_p') \) is homeomorphic to \( C_B/\Gamma_p \) and has a parallel end.

- Let \( \text{Hom}_{E,p,tl}^S(\Gamma_p', \text{Aff}(\mathbb{R}^{n+1})) \) denote the space of representations \( h \) acting on a hyperspace \( P \) in \( \mathbb{R}^{n+1} \) where \( h(\Gamma_p') \) acts on a lens-cone \( L \) properly discontinuously and cocompactly with \( L^o \cap P = L \cap P \neq \emptyset \) or a horoball-cone tangent to \( P \). Then there exists a sufficiently small neighborhood \( K \) of the inclusion homomorphism of \( \Gamma_p' \) in \( \text{Hom}_{E,p,ce}^S(\Gamma_p', \text{Aff}(\mathbb{R}^{n+1})) \) where for each \( h(\Gamma_p') \) with \( h \in K \) so that \( h(\Gamma_p') \) acts on a properly convex cone \( C_{B_h} \) so that \( C_{B_h}/h(\Gamma_p') \) is homeomorphic to \( C_B/\Gamma_p \) and has a totally geodesic or affine horospherical end.

**Proof.** This follows from Lemma 3.8.

### 3.5. Local homeomorphism theorems

Let \( \mathcal{O} \) be a noncompact \((n + 1)\)-orbifold of strongly tame type, and ends are assigned to be of \( \mathcal{R} \)-type or \( \mathcal{T} \)-type as is the convention in this paper.

An affine manifold affinely diffeomorphic to the affine suspension of horospherical end neighborhood is said to be the **affinely suspended horoball neighborhood**. If an end has such a neighborhood, then we call the end **affine horospherical type**. Since the projective automorphism group of a horosphere fixes a point, the fundamental group of the affine horospherical end preserves a direction. Thus, the end of an affine horospherical type is of parallel type.

Again there is a parallel foliation marking for each parallel end of \( \mathcal{O} \) and the ideal boundary components of totally geodesic ends of \( \mathcal{O} \) analogously defined.

We define the end restricted deformation space \( \text{Def}_{A,E}(\mathcal{O}) \) on \( \mathcal{O} \) to be the quotient space of affine structures on \( \mathcal{O} \) where

- each end is parallel if the end is of \( \mathcal{R} \)-type or
- is totally geodesic of suspended lens-type or suspended horospherical type if the end is of \( \mathcal{T} \)-type.
under the action of group of isotopies preserving the markings; i.e., preserves the radial foliation if the end is radial or horospherical or extends to a smooth diffeomorphism if the end is totally geodesic. (As above, each end has a distinguished infinite cyclic group in the center with holonomies in dilatations in $\mathbb{R}^{n+1}$.)

We also define

$$\text{Hom}_E(\pi_1(O), \text{Aff}(\mathbb{R}^{n+1}))$$

as the subspace of

$$\text{Hom}(\pi_1(O), \text{Aff}(\mathbb{R}^{n+1}))$$

of elements $h$ where

- $h(\pi_1(E))$ for each end $E$ has dilatations as images of the distinguished infinite cyclic groups.
- the elements of the representations $h(\pi_1(E))$ of the fundamental group of each end $E$ has a common eigenvector if the end is of $R$-type or
- $h(\pi_1(E))$ acts on a totally geodesic hyperspace $P$ with $C_L \cap P = C_H \cap P \neq \emptyset$ for a lens-cone $C_L$ or tangent to a horoball-cone $C_H$ where $C_L/h(\pi_1(E))$ or $C_H/h(\pi_1(E))$ is a compact orbifold. if the end is of $T$-type.

We define $\text{Def}_{A,E,U,s_U}(O)$ to be the subspace of $\text{Def}_{A,E}(O)$ with the corresponding holonomy homomorphism in the open subset $U$ of

$$\text{Hom}_E(\pi_1(O), \text{Aff}(\mathbb{R}^{n+1}))$$

invariant under the conjugation action and with affine structures so that the end direction is given by

$$s_U : \mathcal{U} \rightarrow (\mathbb{R}^{n+1} - \{O\})^{c_1} \times (\text{AS}(\mathbb{R}^{n+1}))^{c_2}$$

where $\mathcal{U}$ is a conjugation-invariant subset of $\text{Hom}(\pi_1(O), \text{Aff}(\mathbb{R}^{n+1}))$ and

$$s_U(gh(\cdot)g^{-1}) = g \cdot s_U(h(\cdot))$$

for $g \in \text{Aff}(\mathbb{R}^{n+1})$.

(See Section 3.2.)

We define the isotopy-equivalence space $\widetilde{\text{Def}}_{A,E,U,s_U}(O)$ as the quotient space of all development pairs $\text{dev} : \tilde{O} \rightarrow \mathbb{R}^{n+1}$ equivariant with holonomy homomorphisms $h : \pi_1(O) \rightarrow \text{Aff}(\mathbb{R}^{n+1})$ corresponding to the elements of $\text{Def}_{A,E,U,s_U}(O)$ under the isotopies of form $i : \tilde{O} \rightarrow \tilde{O}$ preserving the parallel structures and the totally geodesic ideal boundary. The space has the compact open $C^1$-topology. Here $\text{Def}_{A,E,U,s_U}(O)$ is the quotient space of $\widetilde{\text{Def}}_{A,E,U,s_U}(O)$ under $\text{Aff}(\mathbb{R}^{n+1})$ acting by

$$g(\text{dev}, h(\cdot)) = (g \circ \text{dev}, gh(\cdot)g^{-1}), g \in \text{Aff}(\mathbb{R}^{n+1})$$

(See [22] for details.)

Similarly, we define $\overline{\text{Def}}_{A,E,u}(O)$ as the quotient space of development pairs corresponding to the elements of $\text{Def}_{A,E,u}(O)$ under the isotopies of $\tilde{O}$ preserving the end structures. We also note that

$$\text{Def}_{A,E,u}(O) = \overline{\text{Def}}_{A,E,u}(O)/\text{Aff}(\mathbb{R}^{n+1})$$

The rest of the proof of the first part of Theorem 3.10 is similar to [22]. We cover $O$ by open sets covering a codimension-0 compact orbifold $O'$ and open sets which are end-parallel.
Let $O$ be a noncompact strongly tame affine $(n+1)$-orbifold with parallel ends and totally geodesic ends of lens-type where the types of ends are assigned. Assume $\partial O = \emptyset$. Let $U$ be a conjugation-invariant open subset of $\text{Hom}_E(\pi_1(O), \text{Aff}(\mathbb{R}^{n+1}))$ with an eigensection $s_U$. The map

$$\text{hol} : \tilde{\text{Def}}_{A,E,U,s_U}(O) \to \text{Hom}_E(\pi_1(O), \text{Aff}(\mathbb{R}^{n+1}))$$

sending affine structures determined by the eigensection $s_U$ to the conjugacy classes of holonomy homomorphisms is a local homeomorphism on an open subset of $U'$. Again $\text{Def}_{E,U,s_U}(O)$ is defined to be the subspace of $\text{Def}_E(O)$ with the stable irreducible holonomy homomorphisms in $U$ and the end determined by $s_U$, i.e.,

- each $\mathcal{R}$-type $p$-end has a $p$-end neighborhood foliated by geodesic leaves that go to rays from the fixed points as given by $s_U$ under the developing map, or
- each $\mathcal{T}$-type $p$-end is totally geodesic of lens-type satisfying the lens-condition or horospherical with hyperspace determined by $s_U$. (See Section 3.3.)

**Theorem 3.11.** Let $O$ be a noncompact strongly tame real projective $n$-orbifold with radial ends or totally geodesic ends of lens-type with types assigned and $V$ a conjugation-invariant open subset of

$$\text{rep}_E^s(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$$

and $V'$ the image in

$$\text{rep}_E^s(\pi_1(O), \text{PGL}(n+1, \mathbb{R})).$$

Assume $\partial O = \emptyset$. Let $s_V$ be the fixed-point section defined on $V$ with images in $(\mathbb{R}P^n)^{r_1} \times (\mathbb{R}P^{n*})^{r_2}$. Then the map

$$\text{hol} : \text{Def}_{E,V,s_V}(O) \to \text{rep}_E^s(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$$

sending the real projective structures with ends compatible with $s_V$ to their conjugacy classes of holonomy homomorphisms is a local homeomorphism to an open subset of $V'$.

Define

$$\text{rep}_E^s(\pi_1(O), \text{Aff}(\mathbb{R}^n))$$

as the subspace of stable irreducible holonomy where each $\mathcal{R}$-type end holonomy group has a unique eigenvector and each $\mathcal{T}$-type end holonomy group has a unique eigen-1-form. Define $\text{Def}_{A,E,u}(O)$ as the subspace of $\text{Def}_{A,E}(O)$ mapping to the subspace under $\text{hol}$.

The last part is proved by using affine suspension. This will prove Theorem 1.2 since the uniqueness of the fixed points of the end holonomy groups gives us the section $s_V$ for $V$ equal to the space of representations corresponding to

$$\text{rep}_E^s(\pi_1(O), \text{PGL}(n+1, \mathbb{R})).$$

**Corollary 3.12.** Suppose that $O$ is a noncompact strongly tame $n$-orbifold. Assume $\partial O = \emptyset$. Then the map

$$\text{hol} : \text{Def}_{A,E,u}(O) \to \text{Hom}_{E,u}(\pi_1(O), \text{Aff}(\mathbb{R}^n))$$

sending affine structures to the conjugacy classes of their holonomy homomorphisms is a local homeomorphism. So is the map

$$\text{hol} : \text{Def}_{E,u}(O) \to \text{rep}_E^s(\pi_1(O), \text{PGL}(n+1, \mathbb{R})).$$
3.6. The proof of Theorem 3.10.

We wish to now prove Theorem 3.10 following the proof of Theorem 1 in Section 5 of [22]. Then we will prove Theorem 3.11 in Section 3.7 using this chapter.

Let $\mathcal{O}$ be an affine orbifold with the universal covering orbifold $\tilde{\mathcal{O}}$ with the covering map $p_\mathcal{O} : \tilde{\mathcal{O}} \to \mathcal{O}$ and let the fundamental group $\pi_1(\mathcal{O})$ act on it as an automorphism group.

Let $\mathcal{U}$ and $s_\mathcal{U}$ be as above. We will now define a map

$$\text{hol} : \text{Def}_{A,E,H,s_\mathcal{U}}(\mathcal{O}) \to \text{Hom}_E(\pi_1(\mathcal{O}), \text{Aff}(\mathbb{R}^n))$$

by sending the affine structure to the pair $(\text{dev}, h)$ and to the conjugacy class of $h$ finally. There is a codimension-0 compact submanifold $\mathcal{O}'$ of $\mathcal{O}$ so that $\pi_1(\mathcal{O}') \to \pi_1(\mathcal{O})$ is an isomorphism. The holonomy homomorphism is determined on $\mathcal{O}'$. Since the deformation space has $C^r$-topology, $r \geq 1$, induced by $\text{dev}$, it follows that small changes of $\text{dev}$ on compact domains in the $C^r$-topology imply sufficiently small changes in $h(g'_i)$ for generators $g'_i$ of $\pi_1(\mathcal{O}')$ and hence sufficiently small change of $h(g_i)$ for generators $g_i$ of $\pi_1(\mathcal{O})$. Therefore, $\text{hol}$ is continuous. (Actually for the continuity, we do not need any condition on ends.)

For the purpose of this paper, we use $r \geq 2$. We will use this fact a number of times.

A $v$-parallel set is a subset of $\mathbb{R}^n$ which is invariant under the translation along positive multiples of a fixed nonzero vector $v$. That is, it should be a union of the images under translations along positive multiples of a nonzero vector.

An end-parallel subset of $\tilde{\mathcal{O}}$ or $\mathbb{R}^n$ is a $v$-parallel set where $v$ is the eigenvector of the linear parts of the corresponding p-end.

To show the local homeomorphism property, we take an affine structure $(\text{dev}, h)$ on $\mathcal{O}$ and the associated holonomy map $h$. We cover $\mathcal{O}'$ by small precompact open sets as in Section 5 of [22]. We cover $\mathcal{O} = \mathcal{O}'$ by end-parallel open sets. Consider Lemmas 3, 4, and 5 in [22]. We can generalize these to include the $v$-parallel sets for an invariant direction of $v$ of the finite group $G_B$ where $v$ is an eigenvector of $G_B$ is finite. We repeat them below. The proofs are very similar and use the commutativity of translation by eigenvectors with the action of $G_B$.

**Lemma 3.13.** Let $G_B$ be a finite subgroup of $\text{Aff}(\mathbb{R}^n)$ acting on a $v_0$-parallel open subset $B$ of $\mathbb{R}^n$ for an eigenvector $v_0$ of the linear part of $G_B$. Let $h_t : G_B \to \text{Aff}(\mathbb{R}^n)$, $t \in [0, \epsilon)$, $\epsilon > 0$, be an analytic parameter of representations of $G_B$ so that $h_0$ is the inclusion map. Let $v_t$ be a nonzero eigenvector of $h_t(G_B)$ for each $t$ and we assume that $t \to v_t$ is continuous. Then for $0 \leq t \leq \epsilon$, there exists a continuous family of diffeomorphisms $f_t : B \to B_t$ to a $v_t$-parallel open set $B_t$ in $X$ so that $f_t$ conjugates the $h_t(G_B)$-action to the $h_t(G_B)$-action; i.e., $f_t h_t(g) f_t^{-1} = h_0(g)$ for each $g \in G_B$ and $t \in [0, \epsilon]$.

**Proof.** We find the diffeomorphism $f_t$ in a transversal section of $B$ meeting every lines in $B$ parallel to $v_t$ and extend $f_t$ using the lines. □
Here \( v_h, v_{h,t} \) and \( v_{h,t} \) below are of course determined by \( s_h \).

Since \( \text{Hom}(G_B, \text{Aff}(\mathbb{R}^n)) \) is a semialgebraic set, we obtain that each point has a cone-neighborhood, i.e., a topological neighborhood parameterized by \( I \times S / \sim \) where \( S \) is a semialgebraic set and \( \sim \) is given by \( (0, x) \sim (0, y), x, y \in S \).

**Lemma 3.14.** Let \( G_B \) be a finite subgroup of \( \text{Aff}(\mathbb{R}^n) \) acting on a \( v_0 \)-parallel open subset \( B \) of \( \mathbb{R}^n \) for an eigenvector \( v_0 \) of the linear part of \( G_B \). Suppose that \( h \) is a point of an algebraic set \( V \subset \text{Hom}(G_B, \text{Aff}(\mathbb{R}^n)) \) for a finite group, and let \( C \) be a cone neighborhood of \( h \). Suppose that \( v_h \) is the eigenvector of the linear part of \( h'(G_B) \) for each \( h' \in C \) and \( h' \mapsto v_{h'} \) forms a continuous function \( C \to \mathbb{R}^n \). Then for each \( h' \in C \), there is a corresponding diffeomorphism

\[
    f_{h'} : B \to B_{h'}, B_{h'} = f_{h'}(B)
\]

so that \( f_{h'} \) conjugates the \( h(G_B) \)-action on \( B \) to the \( h'(G_B) \)-action on \( B_{h'} \); i.e.,

\[
    f_{h'}^{-1} h'(g) f_{h'} = h(g)
\]

for each \( g \in G_B \) where \( B_{h'} \) is a \( v_{h'} \)-parallel open set. Moreover, the map \( h' \mapsto f_{h'} \) is continuous from \( C \) to the space \( C^{\infty}(B, X) \) of smooth functions from \( B \) to \( X \).

Continuing to use the notation of Lemma 3.14, we define a parameterization \( l : S \times [0, \epsilon] \to C \) for a cone-neighborhood which is injective except at \( S \times \{0\} \) mapping to \( h \). (We fix \( l \) although \( C \) may become smaller and smaller.) For \( h' \in S \), we denote by \( l(h') : [0, \epsilon] \to C \) be a ray in \( C \) so that \( l(h')(0) = h \) and \( l(h')\epsilon = h' \).

Let the finite group \( G_B \) act on a \( v_h \)-parallel relatively compact submanifold \( F \) of a \( v_h \)-parallel open set \( B \) for an eigenvector \( v_h \) of \( h(G_B) \). Let \( v_{h,t} \) be a nonzero eigenvector of \( l(h')(t) \) for \( h' \in S \) and \( t \in [0, \epsilon] \) and we suppose that \( S \times [0, \epsilon] \to \mathbb{R}^n \) given by \( (h', t) \to v_{h',t} \) is continuous.

A \( G_B \)-equivariant isotopy \( H : F \times [0, \epsilon] \to \mathbb{R}^n \) is a map so that \( H_t \) is an imbedding for each \( t \in [0, \epsilon'] \), with \( 0 < \epsilon' \leq \epsilon \), conjugating the \( G_B \)-action on \( F \) to the \( l(h')(t)(G_B) \)-action on \( \mathbb{R}^n \). Here \( H_0 \) is an inclusion map \( F \to \mathbb{R}^n \) where the image \( H(F, t) \) is a \( v_{h,t} \)-parallel set for each \( t \). Lemma 3.14 above says that for each \( h' \in C \), there exists a \( G_B \)-equivariant isotopy \( H : B \times [0, \epsilon] \to \mathbb{R}^n \) so that the image \( H(B, t) \) is a \( v_{h,t} \)-parallel open set for each \( t \). We will denote by \( H_{h',t} : B \to \mathbb{R}^n \) the map obtained from \( H \) for \( h' \) and \( t = \epsilon' \). Note also by the similar proof, for each \( h' \in S \), there exists a \( G_B \)-equivariant isotopy \( H : F \times [0, \epsilon''] \to \mathbb{R}^n \).

**Lemma 3.15.** Let \( F \) be a \( v_h \)-parallel relatively compact submanifold of a \( v_h \)-parallel open set \( B \) for an eigenvector \( v_h \) of \( h(G_B) \). Let \( H : F \times [0, \epsilon] \times S \to \mathbb{R}^n \) be a map so that \( H(h') : F \times [0, \epsilon'] \times S \to \mathbb{R}^n \) is a \( G_B \)-equivariant isotopy of \( F \) for each \( h' \in S \) where \( 0 < \epsilon' \leq \epsilon \) for some \( \epsilon > 0 \). Then for a neighborhood \( B' \) of \( F \) in \( B \), it follows that \( H \) can be extended to \( \hat{H} : B' \times [0, \epsilon''] \times S \to \mathbb{R}^n \) so that

\[
    \hat{H}(h') : B' \times [0, \epsilon''] \to \mathbb{R}^n, 0 < \epsilon'' \leq \epsilon
\]

is a \( G_B \)-equivariant isotopy of \( B' \) for each \( h' \in S \). The image \( \hat{H}(h')(t)(B') \) is a \( v_{h',t} \)-parallel open set for each \( h', t \).

**Proof of Theorem 3.10.** To finish the proof, we define the local inverse map from a neighborhood in \( \mathcal{U} \) of the image point. Let \( h \) be a representation corresponding to an element \( h \) of it coming from an affine orbifold \( \mathcal{O} \) with radial or totally geodesic boundary. The task is to reassemble \( \mathcal{O} \) with new holonomy homomorphisms as we vary \( h \) as in [22] following Thurston’s approaches.
• For an $R$-type end, this is accomplished as in [22] for precompact open covering sets and for end-parallel open covering sets we use the above lemmas since we are working with finitely many open sets.

• For a $T$-type end that has the totally geodesic ideal boundary, we first complete it with an open subset of a totally geodesic hyperspace. There exists an open subset where the corresponding p-end has totally geodesic hyperspace invariant under each holonomy group of the pseudo-end and is not horospherical. For a sufficiently small open set in $\mathcal{U}$, we can change each open neighborhood in the manner described in [22]. The totally geodesic ideal boundary does not present any difficulty here.

• For a $T$-type end $\tilde{E}$ that is a suspended horospherical end, we take an affinely suspended horospherical neighborhood projectively isomorphic to $C_B/\Gamma_p$ where $\Gamma_p$ is the affine suspension group extended by a central infinite cyclic group generated by a dilatation. Lemma 3.9 shows us how to obtain a totally geodesic end under small deformations of holonomy homomorphisms.

Since we can construct the end neighborhoods as above, we obtain the affine structures for points of $\mathcal{U}$ by using partition of unity and pasting the results as in [22]. To show that the local inverse is a continuous map for the compact open $C^r$-topology we only need to consider compact suborbifolds in $\mathcal{O}$, and we use the fact that the conjugating maps of above Lemmas 3.13, 3.14, and 3.15 depend continuously on $\mathcal{U}$.

Also, finally, we need to prove the local injectivity of $\text{hol}$ as in the last step of the proof of Theorem 3.10. Given two structures $\mu_0$ and $\mu_1$ in a neighborhood of the deformation space, we show that if their holonomy homomorphisms are the same, then we can isotopy one in the neighborhood to the other using vector fields as in [22].

Because of the section $s_{\mathcal{O}}$ defined on $\mathcal{U}$, given a holonomy $h: \pi_1(\mathcal{O}) \to \text{Aff}(\mathbb{R}^n)$, we have a direction of the parallel end that is unique for the holonomy homomorphism.

First assume that $\mathcal{O}$ has only $R$-type ends. Recall the compact suborbifold $\mathcal{O}'$ so that $\mathcal{O} - \mathcal{O}'$ is homeomorphic to $E_i \times (0, 1)$ for each end orbifold $E_i$. Now, $\mathcal{O}$ has a Riemannian metric that is invariant under the flows generated by the end vector fields in the union of its end neighborhoods. On each compact suborbifold $\mathcal{O}'$ of $\mathcal{O}$ with $\partial \mathcal{O}'$ transversal to the vector fields in the end neighborhood, these end vector fields will be uniformly $C^r$-bounded by a small uniform constant depending on how close the two structures $\mu_0$ and $\mu_1$ are in the $C^r$-topology in $\mathcal{O}'$ of the universal cover.

Let $\text{dev}_i$ be the developing map of $\mu_i$ for $i = 1, 2$. Then the $C^r$-norm distance of $\text{dev}_0$ and $\text{dev}_1$ is bounded on each compact set $K \subset \mathcal{O}$. Hence, we can isotopy $\mu_0$ to $\mu_1$ on a neighborhood of $K$ with some $C^r$-bounds $\epsilon > 0$. We can do this for some $\epsilon$ and $K$ mapping onto a suborbifold $\mathcal{O}'$ where $\mathcal{O} - \mathcal{O}'$ is a product of intervals with closed orbifolds. We extend the isotopies using the parallel line extension parametrized by the Riemannian metric. Since the end-orbifolds are determined by their boundary orbifolds in $\mathcal{O}$, we obtain an isotopy from $\mu_0$ to $\mu_1$ in an open neighborhood of the identity map. (Here, we need to only check for compact suborbifolds since we define neighborhoods of the functions using the compact open $C^r$-topology.)
Suppose now that $\mathcal{O}$ has some $T$-type ends. Suppose that $\mu_0$ and $\mu_1$ have totally geodesic ideal boundary corresponding to an end of $\mathcal{O}$. We attach the totally geodesic ideal boundary component for each end, and then we can argue as in [22].

Suppose that $\mu_0$ and $\mu_1$ have horospherical end neighborhoods corresponding to an end of $\mathcal{O}$. Then these are radial ends and the same argument as the above one for $\mathcal{R}$-type ends will apply to show the injectivity. Finally, we cannot have the situation that $\mu_0$ have totally geodesic ideal boundary corresponding to an end while $\mu_1$ have a horoball end neighborhood for the same end. This follows since the end holonomy group acts on a properly convex domain in a totally geodesic hyperspace and as such the end holonomy group elements have some norms of eigenvalues $> 1$. (See Proposition 1.1 of [12] for example.)

3.7. The proof of Theorem 3.11.

Suppose now that $\mathcal{O}$ is a real projective orbifold of dimension $n$. We assume that $\mathcal{O}$ have end that are assigned to be $T$-type or $\mathcal{R}$-type ones. Let $\mathcal{O}' = \mathcal{O} \times S^1$ be the affine suspension. $\pi_1(\mathcal{O}')$ is isomorphic to $\pi_1(\mathcal{O}) \times \mathbb{Z}$. Each end has distinguished infinite cyclic group in the center given by the factor $\mathbb{Z}$. $\mathcal{O}'$ has a parallel end with the end direction determined by the radial ends of $\mathcal{O}$ and totally geodesic ends of lens type determined by that of $\mathcal{O}$. Define $\text{Hom}_E^S(\pi_1(\mathcal{O}'), \text{Aff}(\mathbb{R}^{n+1}))$ to be the subspaces of the representation space $\text{Hom}_E(\pi_1(\mathcal{O}'), \text{Aff}(\mathbb{R}^{n+1}))$ where

- the $\mathbb{Z}$-factor of $\pi_1(\mathcal{O}') = \pi_1(\mathcal{O}) \times \mathbb{Z}$ always maps to a group of dilatations and
- each of whose element $h$ has the stable irreducible linear part $L(h)|\pi_1(\mathcal{O})$.

We define as $\text{rep}^S_E(\pi_1(\mathcal{O}'), \text{Aff}(\mathbb{R}^{n+1}))$ the corresponding subspace of the character variety $\text{rep}_E(\pi_1(\mathcal{O}'), \text{Aff}(\mathbb{R}^{n+1}))$. Let $\mathcal{U}$ be the conjugation invariant subspace of $\text{Hom}_E^S(\pi_1(\mathcal{O}'), \text{Aff}(\mathbb{R}^{n+1}))$ and we are given the fixed section

$s_\mathcal{U} : \mathcal{U} \rightarrow (\mathbb{R}^{n+1} - \{\mathcal{O}\})^c_1 \times (\text{AS}(\mathbb{R}^{n+1}))^c_2$.

For any element $\mu$ of $\text{Def}_{A,E,H,s_\mathcal{U}}(\mathcal{O}')$, $\mathcal{O}'$ with $\mu$ a developing map pulls back a radiant vector field $\sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}$ on $\mathbb{R}^{n+1}$. This gives us a radial flow on $\mathcal{O}'$ with $\mu$. Each point $p$ of $\mathcal{O}'$ has a neighborhood foliated by radial lines. Furthermore, the radial lines are always closed since a dilation from the central elements acts on each radial line giving us a closed orbit always. By Lemma 2.15, $\mathcal{O}'$ with $\mu$ is an affine suspension from $\mathcal{O}$. Since $\mathcal{O}$ can be imbedded transversal to the radial flow, it follows that $\mathcal{O}'$ with $\mu$ gives us an $(\mathbb{S}^n, \text{SL}_+(n+1, \mathbb{R}))$-structure on $\mathcal{O}$.

We define $\text{rep}_E(\pi_1(\mathcal{O}), \text{GL}(n+1, \mathbb{R}))$ and $\text{rep}_E(\pi_1(\mathcal{O}), \text{SL}_+(n+1, \mathbb{R}))$ as the respective subsets where the the holonomy groups of the end fundamental groups of $\mathcal{O}$ have common eigenvectors. By sending dilatations to the expansion factors, we obtain that

$\text{rep}^S_E(\pi_1(\mathcal{O}'), \text{Aff}(\mathbb{R}^{n+1}))$

is identical with

$\text{rep}^S_E(\pi_1(\mathcal{O}), \text{GL}(n+1, \mathbb{R})) \times \mathbb{R}_+$

which is the subspace of

$\text{rep}(\pi_1(\mathcal{O}), \text{GL}(n+1, \mathbb{R})) \times \mathbb{R}_+$.
where the holonomy group of each p-end has an eigendirection or an eigen-1-form.

\[ \text{rep}_E^s(\pi_1(O), GL(n + 1, \mathbb{R})) \times \mathbb{R}_+ \]

can be identified with

\[ \text{rep}_E^s(\pi_1(O), SL_\pm(n + 1, \mathbb{R})) \times H^1(\pi_1(O), \mathbb{R}) \times \mathbb{R} \]

by using the isomorphism

\[ \text{GL}(n + 1, \mathbb{R}) \rightarrow SL_\pm(n + 1, \mathbb{R}) \times \mathbb{R} \]

which is given by sending a matrix \( L \) to \((L/|\det(L)|, \log(|\det(L)|))\). Let

\[ q_S : \text{rep}_E^S(\pi_1(O'), \text{Aff}(\mathbb{R}^{n+1})) \rightarrow \text{rep}_E^s(\pi_1(O), SL_\pm(n + 1, \mathbb{R})) \]

denote the obvious projection.

Let \( \mathcal{U} \) denote a conjugation invariant subset of \( \text{Hom}_E^S(\pi_1(O'), \text{Aff}(\mathbb{R}^{n+1})) \) with a section

\[ s_U : \mathcal{U} \rightarrow (\mathbb{R}^{n+1} - \{O\})^{e_1} \times (\text{AS}(\mathbb{R}^{n+1}))^{e_2}. \]

By Theorem 3.10, and taking the Hausdorff quotients

\[ (3.1) \quad \text{hol} : \text{Def}^S_{\mathcal{U} \cdot \mathcal{U} \cdot s_U}(O') \rightarrow \text{rep}_E^s(\pi_1(O'), \text{Aff}(\mathbb{R}^{n+1})) \]

is a local homeomorphism to its image since the former space is simply the inverse image of the second space.

We define \( \text{Def}_S(O) \) as the deformation space of \( (S^n, SL_\pm(n + 1, \mathbb{R}))-\text{structures} \) on \( O \) and \( \text{Def}_{S^*,E}(O) \) as the quotient space of the space of \( (S^n, SL_\pm(n + 1, \mathbb{R}))-\text{structures} \) on \( O \) with radial ends with radial marks and totally geodesic ends of lens-type with ideal boundary marks. Here the equivalence relation \( \sim \) as before is given by the action of the group of isotopies preserving the radial end structures for radial ends and extending to totally geodesic ideal boundary for totally geodesic ends. Let \( \mathcal{U}' \) denote a conjugation invariant subset of

\[ \text{Hom}_E^S(\pi_1(O), SL_\pm(n + 1, \mathbb{R}))) \]

with a section

\[ s'_U : \mathcal{U}' \rightarrow (S^n)^{e_1} \times (S^n)^{e_2}. \]

We define \( \text{Def}_{S^n, E, \mathcal{U}', s'_U}(O) \) as the subspace of structures whose holonomy characters are in \( \mathcal{U}' \) and the ends are compatible with \( s'_U \), i.e., an end neighborhood of each structure is foliated by concurrent geodesics or by the totally geodesic hyperspace determined by \( s'_U \).

**Proposition 3.16.** Let \( O \) be a noncompact strongly tame \( n \)-orbifold where the types of ends are assigned. Let \( \mathcal{U}' \) be a conjugation-invariant open subset of

\[ \text{Hom}_E^S(\pi_1(O), SL_\pm(n + 1, \mathbb{R}))) \]

with the section

\[ s'_U : \mathcal{U}' \rightarrow (S^n)^{e_1} \times (S^n)^{e_2}. \]

The map

\[ \text{hol} : \text{Def}_{S^n, E, \mathcal{U}', s'_U}(O) \rightarrow \text{rep}_E^s(\pi_1(O), SL_\pm(n + 1, \mathbb{R})) \]

sending \( (S^n, SL_\pm(n + 1, \mathbb{R}))-\text{structures} \) determined by the eigensection \( s'_U \) to the conjugacy classes of holonomy homomorphisms is a local homeomorphism to an open subset of \( \mathcal{U}' \).
PROOF. Let \( \mathcal{O}' \) be the product \( \mathcal{O} \times S^1 \) as above, and this gives each end a distinguished central cyclic group.

Let \( \mathcal{U} \) be the inverse image under \( q_S \) of \( \mathcal{U}' \) in
\[
\text{Hom}^S_{\mathcal{E}}(\pi_1(\mathcal{O}'), \text{Aff}(\mathbb{R}^{n+1})).
\]

Let
\[
q : (\mathbb{R}^{n+1} - \{O\})^{c_1} \times (AS(\mathbb{R}^{n+1}))^{c_2} \rightarrow (S^n)^{c_1} \times (S^{n*)}^{c_2}
\]
be the obvious projections. Let the section \( s_{\mathcal{U}} : \mathcal{U} \rightarrow (\mathbb{R}^{n+1} - \{O\})^{c_1} \times (AS(\mathbb{R}^{n+1}))^{c_2} \)
be the one lifting \( s_{\mathcal{U}'} \). (Here, each hyperplane in \( S^n \) lifts to a hyperplane in \( \mathbb{R}^{n+1} \)
through the fixed points of the holonomy groups of the center.)

By Lemma 2.15, an element of \( \text{Def}_{\mathcal{A},k,E,\mathcal{U},s_{\mathcal{U}}}(\mathcal{O}') \) gives us an element of \( \text{Def}_{\mathcal{S}^n, E, \mathcal{U}', s_{\mathcal{U}}}(\mathcal{O}') \). We can cover \( \mathcal{O}' \) by radial cones with vertex at the origin and project to \( S^n \). Each gluing of open radial cones becomes an element of \( \text{SL}_\pm(n+1, \mathbb{R}) \)
acting on \( S^n \) with positive scalar factors forgotten. The parallel end structures and
totally geodesic ideal boundary components for ends of \( \mathcal{O}' \) go to the radial end structures and
the totally geodesic ideal boundary components of \( \mathcal{O} \). The
isotopies in \( \mathcal{O} \) will give rise to isotopies in \( \mathcal{O}' \) suspending the vector fields on cross-
sections preserving the parallel vector fields and the totally geodesic ideal boundary components.

Therefore, the following map \( P \) is defined:

\[
\text{Def}_{\mathcal{A},k,E,\mathcal{U},s_{\mathcal{U}}}(\mathcal{O}') \xrightarrow{\text{hol}} \text{Def}_{\mathcal{S}^n, E, \mathcal{U}', q_S s_{\mathcal{U}}}(\mathcal{O}) \times H^1(\mathcal{O}, \mathbb{R}) \times (\mathbb{R}^+ - \{1\})
\]

(3.2)

\[
\text{rep}_E^S(\pi_1(\mathcal{O}'), \text{Aff}(\mathbb{R}^{n+1})) \rightarrow \text{rep}_E^S(\pi_1(\mathcal{O}), \text{SL}_\pm(n+1, \mathbb{R})) \times H^1(\mathcal{O}, \mathbb{R}) \times (\mathbb{R}^+ - \{1\}).
\]

A section to \( P \) is defined by taking an affine suspension by the data in \( H^1(\mathcal{O}, \mathbb{R}) \times
(\mathbb{R}^+ - \{1\}) \) and the \( (S^n, \text{SL}_\pm(n+1, \mathbb{R})) \)-structures on \( \mathcal{O} \) using the methods of Section
2.10.2. From this, we deduce that the horizontal maps are local homeomorphisms
in the commutative diagram. Since the left downarrow is a local homeomorphism,
the result is proved.

\[ \square \]

The homomorphism \( q : \text{SL}_\pm(n+1, \mathbb{R}) \rightarrow \text{PGL}(n+1, \mathbb{R}) \) induces a continuous map

\[
\hat{q} : \text{rep}_E^S(\pi_1(\mathcal{O}), \text{SL}_\pm(n+1, \mathbb{R})) \rightarrow \text{rep}_E^S(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})).
\]

Let \( \mathcal{U} \) be a conjugation invariant subset of \( \text{Hom}^S_{\mathcal{E}}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \). \( s_{\mathcal{U}} \)
is an arbitrary fixed section defined on \( \mathcal{U} \). Let \( \mathcal{U}' \) denote the inverse image of \( \mathcal{U} \)
der under \( \hat{q} \). We define \( s_{\mathcal{U}'} : \mathcal{U}' \rightarrow (S^n)^{c_1} \times (S^{n*)}^{c_2} \) be a continuous lift of \( s_{\mathcal{U}} \). The
section is determined up to the action of \( \{e_1, e_2\} \) on each of \( (c_1 + c_2) \)-factors. This
gives us a section \( \tilde{s} : \text{Def}_{\mathcal{E},k,E,\mathcal{U},s_{\mathcal{U}}}(\mathcal{O}) \rightarrow \text{Def}_{\mathcal{S}^n, E, \mathcal{U}', s_{\mathcal{U}}}(\mathcal{O}) \) up to a choice of \( s_{\mathcal{U}'} \),
by Theorem 3.19. The choice here is determined by the lifting of the development
pair \( (\text{dev}, h) \). (For the lifting ideas, see p. 143 of Thurston [79].)

The map \( \tilde{q} : \text{Def}_{\mathcal{S}^n, E, \mathcal{U}', s_{\mathcal{U}}}(\mathcal{O}) \rightarrow \text{Def}_{\mathcal{E},k,E,\mathcal{U},s_{\mathcal{U}}}(\mathcal{O}) \) is induced by the action

\[
(\text{dev}', h') \rightarrow (q \circ \text{dev}', \tilde{q} \circ h').
\]

It is easy to see that the section \( \tilde{s} \) to \( \tilde{q} \) is well-defined since the lifting \( (\text{dev}, h) \)
give us development pairs that are equivalent up to -I \( \in \text{SL}_\pm(n+1, \mathbb{R}) \). The

map $\tilde{s}$ is continuous since for a fixed compact subset of $\tilde{O}$ the $C^r$-closeness of the developing map to $\mathbb{R}P^n$ means the $C^r$-closeness of the lifts for $r \geq 1$.

Thus, we showed that

**Theorem 3.17.** Assume as in the above paragraphs. We obtain a homeomorphism

$$\tilde{q}: \text{Def}^e_{S^m, E, U', s', u'}(O) \to \text{Def}^e_{E, U, s_u}(O).$$

\qed

**Corollary 3.18.** $\tilde{q}: \text{Def}^e_{S^m, E, u}(O) \to \text{Def}^e_{E, u}(O)$ is a homeomorphism.

**Proof.** In the unique eigenvector or eigen-1-form cases, the existence and the continuity of the sections are clear. \qed

**Proof of Theorem 3.11.** We have a commutative diagram:

$$\begin{array}{ccc}
\text{Def}^e_{S^m, E, U, s_u}(O) & \tilde{q} & \text{Def}^e_{E, U, s_u}(O) \\
\downarrow \text{hol} & & \downarrow \text{hol} \\
\text{rep}^e_E(\pi_1(O), \text{SL}_{\pm}(n+1, \mathbb{R})) & \tilde{q} & \text{rep}^e_E(\pi_1(O), \text{PGL}(n+1, \mathbb{R})).
\end{array}$$

(3.3)

First, we remark first that $\tilde{q}$ maps onto the union of components with the associated Stiefel-Whitney number 0. Since $\text{SL}_{\pm}(n+1, \mathbb{R}) \to \text{PGL}(n+1, \mathbb{R})$ is a covering map, so is $\text{Hom}^e_E(\pi_1(O), \text{SL}_{\pm}(n+1, \mathbb{R})) \to \text{Hom}^e_E(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$ to the union of its image components (i.e., corresponding to ones with the corresponding Stiefel-Whitney classes equal to zero.) Each fiber is in one to one correspondence with $\text{Hom}(\pi_1(O), \{-1\})$. The induced map $\tilde{q}$ is a local homeomorphism since the conjugation by $\text{SL}_{\pm}(n+1, \mathbb{R})$ on the first space is equivalent to one by $\text{PGL}(n+1, \mathbb{R})$ since $\{-1\}$ acts trivially.

Since the left hol is locally onto and $\tilde{q}$ is locally onto, so is the right hol by Theorem 3.17.

Given a neighborhood $V'$ in $\text{rep}^e_E(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$ that is in the image of the left hol, we can find a local section to $\tilde{q}$ as $\tilde{q}$ is a local homeomorphism. Since the left hol is a local homeomorphism, and $\tilde{q}$ is a local homeomorphism, there is a local section to the right hol by Theorem 3.17. \qed

### 3.8. A comment on lifting real projective structures

Let $\text{SL}_{-}(n+1, \mathbb{R})$ denote the component of $\text{SL}_{\pm}(n+1, \mathbb{R})$ not containing I. A projective automorphism $g$ of $S^n$ is orientation preserving if and only if $g$ has a matrix in $\text{SL}(n+1, \mathbb{R})$. For even $n$, the quotient map $\text{SL}(n+1, \mathbb{R}) \to \text{PGL}(n+1, \mathbb{R})$ is an isomorphism and so is the map $\text{SL}_{-}(n+1, \mathbb{R}) \to \text{PGL}(n+1, \mathbb{R})$ for the component of $\text{SL}_{\pm}(n+1, \mathbb{R})$ with determinants equal to $-1$. For odd $n$, the quotient map $\text{SL}(n+1, \mathbb{R}) \to \text{PGL}(n+1, \mathbb{R})$ is a 2 to 1 covering map onto its image component with deck transformations given by $A \to \pm A$. Also, so is the map $\text{SL}_{-}(n+1, \mathbb{R}) \to \text{PGL}(n+1, \mathbb{R})$. 


Theorem 3.19. Let $M$ be a strongly tame orbifold. Suppose that $h : \pi_1(M) \to \text{PGL}(n+1, \mathbb{R})$ is a holonomy homomorphism of real projective structure on $M$ with radial or totally geodesic ends of lens-type. Then the image of $h$ for $\text{Def}_E(O)$ in \[ \text{rep}^r(\pi_1(M), \text{PGL}(n+1, \mathbb{R})) \] is homeomorphic to that of $h$ for $\text{Def}_{S^n, E}(O)$ in \[ \text{rep}^r(\pi_1(M), \text{SL}(n+1, \mathbb{R})). \]

$h'$ is unique if $n$ is even.

- Suppose that $M$ is orientable. We can lift to a homomorphism $h' : \pi_1(M) \to \text{SL}(n+1, \mathbb{R})$, which is a holonomy homomorphism of the $(S^n, \text{SL}_\pm(n+1, \mathbb{R}))$-structure lifting the real projective structure.
- Suppose that $M$ is not orientable. Then we can lift $h$ to a homomorphism $h' : \pi_1(M) \to \text{SL}_\pm(n+1, \mathbb{R})$ that is the holonomy homomorphism of the $(S^n, \text{SL}_\pm(n+1, \mathbb{R}))$-structure lifting the real projective structure so that a deck transformation goes to a negative determinant matrix if and only if it reverses orientations.

Proof. Recall $\text{SL}(n+1, \mathbb{R})$ is the group of orientation-preserving linear automorphisms of $\mathbb{R}^{n+1}$ and hence is precisely the group of orientation-preserving projective automorphisms of $S^n$. Since the deck transformations of the universal cover $\tilde{M}$ of the lifted $(S^n, \text{SL}_\pm(n+1, \mathbb{R}))$-orbifold are orientation-preserving, the holonomy of the lift are in $\text{SL}(n+1, \mathbb{R})$. We use as $h'$ the holonomy homomorphism of the lifted structure. For even $n$, the uniqueness of $h'$ follows from the fact that $\text{SL}(n+1, \mathbb{R}) \to \text{PGL}(n+1, \mathbb{R})$ is a homeomorphism.

For the second part, we can double cover $M$ by an orientable orbifold $M'$ with an orientation-reversing $\mathbb{Z}_2$-action of the projective automorphism group generated by $\phi : M' \to M'$. $\phi$ lifts to $\tilde{\phi} : \tilde{M}' \to \tilde{M}'$ for the universal covering manifold $\tilde{M}' = \tilde{M}$ and hence $h(\tilde{\phi}) \circ \text{dev} = \text{dev} \circ \tilde{\phi}$ for the developing map $\text{dev}$ and the holonomy $h(\tilde{\phi}) \in \text{SL}_-(n+1, \mathbb{R})$.

Then it follows from the first item since $\text{dev}$ preserves orientations for a given orientation of $\tilde{M}$. For even $n$, the uniqueness is the consequence of the uniqueness of the lift $h'$ in the orientable case and the fact that $\text{SL}_-(n+1, \mathbb{R}) \to \text{PGL}(n+1, \mathbb{R})$ is a one-to-one homeomorphism also. (See p. 143 of Thurston [79].) \qed

In general, this proposition is used commonly but not written anywhere.
Part 2

Convexity of the orbifolds and the relative hyperbolic fundamental groups
CHAPTER 4

Convexity

4.1. Properties of ends

We will restate the results of [27] here for general understanding need for what follows.

4.1.1. Properties of horospherical ends. In [27], the horospherical ends were defined more generally but the definition was shown to be equivalent to ours. We will not repeat the definition.

Proposition 4.1 (Proposition 5.1 [27]). Let $\mathcal{O}$ be a properly convex real projective $n$-orbifold with radial ends or totally geodesic ends of lens-type. Let $E$ be a horospherical $p$-end of its universal cover $\tilde{\mathcal{O}}$ and $\Gamma_E$ denote the $p$-end fundamental group. Then the following statements hold:

(i) The space $S_E := R_{v_E}(\tilde{\mathcal{O}})$ of rays from the corresponding $p$-end point $v_E$ forms a complete affine subspace of dimension $n - 1$.
(ii) The norms of eigenvalues of $g \in \Gamma_E$ are all 1.
(iii) A $p$-end point of a horospherical $p$-end cannot be on a segment in $\text{bd} \tilde{\mathcal{O}}$.
(iv) For any compact set $K'$ inside a horospherical end-neighborhood, there exists a smooth convex smooth horospherical end-neighborhood disjoint from $K'$.
(v) $\pi_1(\tilde{E})$ is virtually abelian and a finite index subgroup is in a conjugate of a parabolic subgroup of $\text{PO}(n,1)$ of rank $n - 1$ in $\text{PGL}(n+1,\mathbb{R})$ (resp. $\text{SL}_\pm(n+1,\mathbb{R})$) that acts on an ellipsoid in $\text{Cl}(\tilde{\mathcal{O}}) \subset \mathbb{R}P^n$ (resp. $\subset S^n$).

The converse result is the following.

Theorem 4.2 (Theorem 5.2 [27]). Let $\mathcal{O}$ be a properly convex $n$-orbifold with radial ends or totally geodesic ends of lens-type. Suppose that $E$ is a radial $p$-end of its universal cover $\tilde{\mathcal{O}}$. Let $v_E \in \mathbb{R}P^n$ be the $p$-end point and $\pi_1(\tilde{E})$ be the $p$-end fundamental group corresponding to $E$. Suppose that the space $S_E := R_{v_E}(\tilde{\mathcal{O}})$ of rays from the corresponding $p$-end point $v_E$ forms a complete affine subspace of dimension $n - 1$. Then the following statements hold:

(i) The eigenvalues of elements of $h(\pi_1(\tilde{E}))$ have unit norms only.
(ii) A finite index subgroup of $h(\pi_1(\tilde{E}))$ is contained in a unipotent group fixing $v_E$.
(iii) $E$ is horospherical.

4.1.2. The properties of lens-shaped ends. A trivial one-dimensional cone is an open half space in $\mathbb{R}^1$ given by $x > 0$ or $x < 0$.

Recall that if $\pi_1(E)$ is an admissible group, then $\pi_1(E)$ has a finite index subgroup isomorphic to $\mathbb{Z}^l \times \Gamma_1 \times \cdots \times \Gamma_k$ for some $l$ and $k$ for $l \geq k - 1$ where each
\( \Gamma_i \) is hyperbolic. Here, we can identify \( \hat{\mathcal{O}} \) as a convex domain in \( \mathbb{R}P^n \) (resp. in \( \mathbb{S}^n \)) for convenience.

Let us consider \( E \) as a real projective \((n-1)\)-orbifold and consider \( \hat{E} \) as a domain in \( \mathbb{S}^{n-1} \) and \( h(\pi_1(E)) \) induces \( h' : \pi_1(E) \to \text{SL}_{\pm}(n, \mathbb{R}) \) acting on \( \hat{E} \).

**Theorem 4.3** (Proposition 6.7 [27]). Let \( \mathcal{O} \) be a noncompact strongly tame properly convex real projective \( n \)-orbifold with radial ends or totally geodesic ends of lens-type and satisfies (IE) and (NA). Let \( \hat{E} \) be a generalized lens-shaped radial \( p \)-end of \( \hat{\mathcal{O}} \) in \( \mathbb{R}P^n \) (resp. in \( \mathbb{S}^n \)) associated with a \( p \)-end vertex \( v_\xi \). Assume that \( \pi_1(\hat{E}) \) is hyperbolic. Then the following statements hold:

(i) \( D \) is strictly generalized lens-shaped. Moreover, each element \( g \in \Gamma_\xi \) has an attracting fixed point in \( \text{bdCl}(D) \) in the ray from \( v_\xi \) in the direction of \( \text{bd}\Omega_\xi \). The set of attracting fixed points is dense in \( \text{bdCl}(D) - A - B \) for the top and the bottom hypersurfaces \( A \) and \( B \) forming the boundary of the lens \( D \).

(ii) The closure in in \( \mathbb{R}P^n \) (resp. in \( \mathbb{S}^n \)) of a concave \( p \)-end-neighborhood of \( v_\xi \) contains every segment \( l \) in \( \text{bd}\hat{\mathcal{O}} \) meeting the closure of a concave \( p \)-end neighborhood of \( v_\xi \) in \( l^0 \). The set \( S(v_\xi) \) of maximal segments from \( v_\xi \) in the closure of a \( p \)-end-neighborhood of \( v_\xi \) is independent of the \( p \)-end-neighborhood, and \( \bigcup S(v_\xi) \) equals the closure of any \( p \)-end neighborhood of \( v_\xi \) intersected with \( \text{bd}\hat{\mathcal{O}} \).

(iii) Any concave \( p \)-end neighborhood \( U \) of \( v_\xi \) under the covering map \( p_\mathcal{O} \) covers the \( p \)-end neighborhood of \( E \) of form \( U/\pi_1(E) \). That is, a concave \( p \)-end-neighborhood is a proper \( p \)-end neighborhood.

(iv) \( \bigcup S(g(v_\xi)) = g(\bigcup S(v_\xi)) \) for \( g \in \pi_1(E) \). Assume that \( w \) is the \( p \)-end vertex of an irreducible hyperbolic \( p \)-end. \( \bigcup S(v_\xi) \) is an \((n-1)\)-ball. Then \( \bigcup S(v_\xi)^{o} \cap \bigcup S(w) = \emptyset \) or \( v_\xi = w \) for \( p \)-end vertices \( v_\xi \) and \( w \).

Now we go to the cases when admissible \( \pi_1(E) \) has more than two nontrivial abelian or hyperbolic factors. The following theorem shows that each lens-shaped end is totally geodesic and has well-defined \( S(v) \) in this case. The author obtained the proof of (i-3) from Benoist.

**Theorem 4.4** (Theorem 6.9 [27]). Let \( \mathcal{O} \) be a noncompact strongly tame properly convex real projective \( n \)-orbifold with radial ends or totally geodesic ends of lens-type and satisfies (IE) and (NA). Suppose that the holonomy of \( \mathcal{O} \) is strongly irreducible. Let \( \hat{E} \) be a generalized lens-shaped radial \( p \)-end of \( \hat{\mathcal{O}} \) in \( \mathbb{R}P^n \) (resp. in \( \mathbb{S}^n \)) associated with a \( p \)-end vertex \( v_\xi \). Let \( \pi_1(\hat{E}) \) be the \( p \)-end fundamental group corresponding to \( \hat{E} \) containing a finite index abelian subgroup isomorphic to \( \mathbb{Z}^l \times \Gamma_1 \times \cdots \times \Gamma_k \). Assume \( l \geq 1 \). Then the following statements hold:

(i) For \( S^{n-1}_\xi \), we obtain

(i-1) Under the action of the induced group \( \hat{h}(\pi_1(E)) \) of the holonomy group \( h(\pi_1(E)) \), \( \mathbb{R}^n \) splits into \( V_1 \oplus \cdots \oplus V_{l_0} \) and \( S_\xi \) is the quotient of the sum \( C_1 + \cdots + C_{l_0} \) for properly convex or trivial one-dimensional cones \( C_i \subset V_i \) for \( i = 1, \ldots, l_0 \).

(i-2) The Zariski closure of a finite index subgroup of \( \hat{h}(\pi_1(E)) \) is isomorphic to the product \( G = G_1 \times \cdots \times G_{l_0} \times \mathbb{R}^{l_0-1} \) where \( G_i \) is a reductive subgroup of \( \text{SL}_{\pm}(V_i) \).
Let $D_i$ denote the image of $C_i$ in $S^{n-1}_{v_E}$. The number of hyperbolic group factors of $\pi_1(E)$ is $\leq l_0$ and each hyperbolic group factor of $\pi_1(E)$ divides exactly one $D_i$ and acts on other factors trivially.

(i-4) A finite-index subgroup of $\pi_1(\tilde{E})$ has a rank $l_0-1$ free abelian group center corresponding to $\mathbb{Z}^{l_0-1}$ in $\mathbb{R}^{l_0-1}$.

(ii) The p-end is totally geodesic radial p-end of lens-type. $D_i$ corresponds to totally geodesic convex $(n-1)$-ball $D'_i$ disjoint from $v_E$.

(iii) $g$ in the center is diagonalizable with positive eigenvalues. For a non-identity element $g$ in the center, the eigenvalue $\lambda_{v_E}$ of $g$ at $v_E$ is strictly between its largest and smallest eigenvalues.

(iv) The p-end is strictly lens-shaped and each $C_i$ corresponds to a cone $C_i^*$ over a totally geodesic $(n-1)$-dimensional domain $D'_i$ with $v_E$. $C_i^*$ contains a concave open invariant set $\tilde{U}_i$. The p-end has a p-end neighborhood that is a strict join of $D'_i, \ldots, D'_l$ with $v_E$ where the strict join $D$ of $D'_i, \ldots, D'_l$ forms the boundary. They are in a lens part of $\tilde{E}$ for any lens-type p-end neighborhood, and the top and the bottom hypersurfaces of the lens part have the boundary in the boundary of $D'$.

(v) $\bigcup S(v_E)$ is equal to the union of maximal segments with vertex $v_E$ in the union $\bigcup_{i=1}^l v_E \ast D'_1 \ast \cdots \ast D'_{i-1} \ast D'_{i+1} \ast \cdots \ast D'_l$.

(vi) A concave p-end neighborhood of $\tilde{E}$ is a proper end neighborhood. Also the statement in this case for (iv) of Theorem 4.3 holds.

**Theorem 4.5** (Theorem 8.1 [27]). Let $O$ be a strongly tame properly convex real projective manifold with radial ends or totally geodesic ends of lens-type and satisfies (IE) and (NA). Let the holonomy $h(\pi_1(O))$ be strongly irreducible. Let $\tilde{E}$ be a properly convex radial p-end of $\tilde{O}$ in $\mathbb{R}P^n$ (resp. in $\mathbb{S}^n$) associated with a p-end vertex $v_E$. Let 

$$\text{Hom}^\ast_E(\pi_1(\tilde{E}), \text{PGL}(n+1, \mathbb{R})) \text{ (resp. } \text{Hom}^\ast_E(\pi_1(\tilde{E}), \text{SL}_\pm(n+1, \mathbb{R})))$$

be the space of representations of the fundamental group of an $(n-1)$-orbifold $\Sigma_E$ with an admissible fundamental group. Then

(i) the generalized lens-shapedness of an end is equivalent to the strict generalized lens-shapedness of the end, and

(ii) the subspace of generalized lens-shaped representations in the above space is open.

**Lemma 4.6.** Given two concave p-end neighborhoods $U$ and $V$, either they have the same p-end and $U \cap V$ is another concave p-end neighborhood or $U \cap V = \emptyset$ when they have distinct p-ends.

**Proof.** Let $\hat{E}_1$ and $\hat{E}_2$ be the p-end associated with $U$ and $V$ respectively. If $U \cap V \neq \emptyset$, then $S^o(v_{E_1})$ intersect $S^o(v_{E_2})$ since the lens for $U$ is supported by a totally geodesic hyperspace and so is $V$. Thus, the conclusion follows by Theorems 4.3(iv) and 4.4(vi).

**4.2. Expansion and shrinking of admissible p-end neighborhoods**

**Definition 4.7.** Let $\Lambda$ denote $\text{bd} L - \partial L$ for a generalized lens of a radial p-end or a lens of a totally geodesic ends. We call this set the *limit set* of the p-end.
Obviously the limit set of a p-end is independent of the choice of lens by Corollary 8.5 of [27].

**Lemma 4.8 (Lemma 8.6 [27])**. Let $\mathcal{O}$ have a noncompact strongly tame SPC-structure $\mu$ with admissible ends. Let $U_1$ be an admissible p-end neighborhood of a lens-type radial p-end with the vertex $v$ in $\bar{\mathcal{O}}$ that is foliated by segments from $v$ or a totally geodesic p-end $\bar{E}$.

- $\mathcal{O}$ contains a sequence of convex open neighborhoods $U_i$ of $\bar{E}$ so that $(U_i - U_j)/\Gamma_v$ for a fixed $j$ and $i > j$ is homeomorphic to a product of an open interval with the end orbifold.
- Given a compact subset $K$ of $\mathcal{O}$, there exists an integer $i_0$ such that $U_i$ for $i > i_0$ contains $K$.
- We can choose $U_i$ so that $\partial U_i$ is smoothly imbedded and strictly convex with $\partial Cl(\partial U_i) \subset \Lambda$ where $\Lambda$ is the limit set contained in $\bigcup S(v)$ if $v$ is the p-end vertex when $\bar{E}$ is radial and in $\partial S_{\bar{E}}$ if $\bar{E}$ is total geodesic.
- The Hausdorff distance between $U_i$ and $\mathcal{O}$ can be made as small as possible.

See the definition of convex hull of an end in Section 4.4.1.

**Lemma 4.9 (Lemma 8.7 [27])**. Suppose that $\mathcal{O}$ is a strongly tame properly convex real projective orbifold with radial or totally geodesic ends of lens-type and let $\hat{\mathcal{O}}$ is a properly convex domain in $\mathbb{R}P^n$ (resp. in $\mathbb{S}^n$) covering $\mathcal{O}$. Assume that the holonomy group of $\pi_1(\mathcal{O})$ is strongly irreducible.

(i) If $\bar{E}$ is a horospherical p-end, any p-end neighborhood of $\bar{E}$ contains a horospherical p-end neighborhood.

(ii) If $\bar{E}$ is a generalized lens-shaped p-end, any p-end neighborhood whose closure covers a compact end neighborhood and containing the convex hull of the end contains a lens-shaped p-end neighborhood.

(iii) If $\bar{E}$ is a generalized lens-shaped p-end, any p-end neighborhood of $\bar{E}$ contains a concave p-end neighborhood.

(iv) If $\bar{E}$ is totally geodesic p-end of lens type, any end neighborhood contains a one-sided lens p-end neighborhood with strictly convex boundary in $\hat{\mathcal{O}}$.

Proposition 6.7 and Theorem 6.9 of [27] imply the following:

**Corollary 4.10**. Let $\mathcal{O}$ be a noncompact strongly tame $n$-orbifold with radial or totally geodesic ends of lens-type and satisfies (IE) and (NA). Assume that the holonomy group of $\pi_1(\mathcal{O})$ is strongly irreducible. Let $\hat{\mathcal{O}}$ is a properly convex domain in $\mathbb{R}P^n$ (resp. in $\mathbb{S}^n$) covering $\mathcal{O}$, and let $\bar{E}$ be a generalized lens-shaped radial p-end of $\hat{\mathcal{O}}$ associated with a p-end vertex $v_{\bar{E}}$, or $\bar{E}$ is a totally geodesic p-end of lens-type or a horospherical end. Let $U$ be a p-end neighborhood of $\bar{E}$. Then $Cl(U) \cap bd\hat{\mathcal{O}}$ is independent of the choice of $U$.

**Corollary 4.11**. Suppose that $\mathcal{O}$ is a noncompact strongly tame strictly SPC-orbifold with generalized admissible ends. Let $\hat{\mathcal{O}}$ is a properly convex domain in $\mathbb{R}P^n$ (resp. in $\mathbb{S}^n$) covering $\mathcal{O}$. Choose any disjoint collection of end neighborhoods in $\mathcal{O}$. Let $U$ denote their union. Let $p_\mathcal{O} : \hat{\mathcal{O}} \to \mathcal{O}$ denote the universal cover. Then any segment or a non-$C^1$-point of $bd\hat{\mathcal{O}}$ is contained in the closure of a component of $p_\mathcal{O}^{-1}(U)$ for any choice of $U$. 
4.4. TOTALLY GEODESIC ENDS AND DUALITY

Proof. By the definition of a strictly SPC-orbifold, any segment or a non-$C^1$-point has to be in the closure of a p-end neighborhood. Corollary 4.10 proves the claim. □

4.3. Duality of ends

The totally geodesic ends of a properly convex real projective orbifolds are properly convex necessarily.

Theorem 4.12 (Proposition 6.4 [27]). Let $\mathcal{O}$ be a noncompact strongly tame properly convex real projective orbifold with horospherical or properly convex radial or totally geodesics ends. Let $\tilde{\mathcal{O}}$ be the convex domain in $\mathbb{R}P^n$ (resp. $S^n$) that covers $\mathcal{O}$ and $\Gamma$ the projective deck transformation group in $\text{PGL}(n + 1, \mathbb{R})$ (resp. $\text{SL}_\pm(n + 1, \mathbb{R})$). Let $\tilde{\mathcal{O}}^*$ be the dual convex domain and $\Gamma^*$ the dual group to $\Gamma$. Then $\mathcal{O}^* := \tilde{\mathcal{O}}^*/\Gamma^*$ is a noncompact strongly tame properly convex real projective orbifold with radial or totally geodesics ends.

- The set of p-end fundamental groups $\pi_1(\tilde{E})$ of $\tilde{\mathcal{O}}$ corresponds to the set of p-end fundamental groups $\pi_1(\tilde{E})^*$ of $\tilde{\mathcal{O}}^*$.
- There is a one-to-one correspondence

\[
\{ \tilde{E} | \tilde{E} \text{ is a radial properly convex p-end of } \tilde{\mathcal{O}} \} \Leftrightarrow \{ \tilde{E} | \tilde{E} \text{ is a totally geodesic p-end of } \tilde{\mathcal{O}}^* \},
\]

(4.1)

another one

\[
\{ \tilde{E} | \tilde{E} \text{ is a totally geodesic p-end of } \tilde{\mathcal{O}} \} \Leftrightarrow \{ \tilde{E} | \tilde{E} \text{ is a radial properly convex p-end of } \tilde{\mathcal{O}}^* \},
\]

(4.2)

- and one for the set of horospherical ends of $\tilde{\mathcal{O}}$ with and the set of horospherical ends of $\tilde{\mathcal{O}}^*$.

For correspondences of admissible ends, see Lemma 4.17.

4.4. Totally geodesic ends and duality

We discuss somewhat more on totally geodesic ends. For totally geodesic ends, by the lens condition, we only consider the ones that have lens neighborhoods in some ambient orbifolds, i.e., admissible ones. First, we discuss the extension to bounded orbifolds.

Theorem 4.13. Suppose that $\mathcal{O}$ is a noncompact strongly tame properly convex real projective orbifold with generalized admissible ends. Assume that the holonomy group of $\pi_1(\mathcal{O})$ is strongly irreducible. Let $E$ be a lens-shaped totally geodesic end, and let $\Sigma_E$ be a totally geodesic hypersurface that is the ideal boundary corresponding to $E$. Let $L$ be a lens-shaped end neighborhood of $\Sigma_E$ in an ambient real projective orbifold containing $\mathcal{O}$. Then $L \cup \mathcal{O}$ is a properly convex real projective orbifold and has a strictly convex boundary component corresponding to $E$. Furthermore if $\mathcal{O}$ is strictly SPC and $E$ is a hyperbolic end, then so is $L \cup \mathcal{O}$ which now has one more boundary component and one less totally geodesic ends.

Proof. It is sufficient to prove for $S^n$ cases here. Let $\tilde{\mathcal{O}}$ be the universal cover of $\mathcal{O}$ which we can identify with a properly convex bounded domain in an affine subspace. Then $\Sigma_E$ corresponds to a p-end $\tilde{E}$ and to a totally geodesic surface
$S = S_E$. The lens $L$ is covered by a lens $\hat{L}$ containing $S$. The $p$-end fundamental group $\pi_1(\hat{E})$ acts on $\hat{O}$ and $\hat{L}_1$ and $\hat{L}_2$ the two components of $\hat{L} - S_E$ in $\hat{O}$ and outside $\hat{O}$ respectively.

**Definition 4.14.** Let $\mathbb{R}^n$ denote the affine subspace in $\mathbb{S}^n$ with boundary $\mathbb{S}^{n-1}$. Suppose that $\Omega$ is a strictly convex open domain in $\mathbb{S}^{n-1}$. Given a convex open domain $\Omega_1$ with $\text{bd}\Omega_1 \supset \text{Cl}(\Omega)$ in $\mathbb{R}^n$, the supporting hyperplanes at $p \in \Lambda = \text{Cl}(\Omega) - \Omega$ contains the unique hyperplane of codimension-two supporting $\Omega$. Let $A_p := \{H | H$ is a supporting hyperspace of $\Omega_1$ at $p$ in $\mathbb{R}^n\}$ and hence the space $A_p$ of such hyperspaces is homeomorphic to an arc. An asymptotic supporting hyperplane at a point $p$ of $\Lambda$ is a supporting hyperplane at $p$ so that there exists no other element of $A_p$ closer to $\Omega_1$ from a point of $\text{bd}\Omega_1 - \text{Cl}(\Omega)$ (using minimal distance between a point and a set).

**Lemma 4.15.** Suppose that $S_E$ is the totally geodesic ideal boundary of a lens-type totally geodesic end $\hat{E}$ of a strongly tame real projective orbifold $\hat{O}$ and $\pi_1(\hat{E})$ is hyperbolic.

- Given a $\pi_1(\hat{E})$-invariant convex open domain $\Omega_1$ containing $S_E$ in the boundary, at each point of $\Lambda$, there exists a unique asymptotic supporting hyperplane.
- The hyperspace supporting any $\pi_1(\hat{E})$-invariant convex open set $\Omega$ containing $S_E$ at each point of $\Lambda$ is unique.
- Given two $\pi_1(\hat{E})$-invariant convex open domains $\Omega_1$ containing $S_E$ in the boundary and $\Omega_2$ containing $S$ in the boundary from the other side, $\Omega_1 \cup \Omega_2$ is a convex domain and $\text{Cl}(\Omega_1) \cap \text{Cl}(\Omega_2) = \text{Cl}(S_E)$ and their asymptotic supporting hyperplanes at each point of $\Lambda$ coincide.

**Proof.** Let $A$ denote the affine subspace that is the complement of $\mathbb{S}^n$ of the subspace containing $\hat{O}$. Because $\pi_1(\hat{E})$ acts on a lens-type domain, the dual group of $h(\pi_1(\hat{E}))$ is the holonomy group of lens-type radial $p$-end. By Theorem 7.9 of [27], $h(\pi_1(\hat{E}))$ satisfies the uniform middle eigenvalue condition of [27].

If $\Omega_1$ has an asymptotic supporting half-space $H(x)$ for each $x \in \Lambda$ containing $\Omega_1$, $H(x)$ is uniquely determined by $\pi_1(\hat{E})$ and $x$ by Lemmas A.7 and A.8 of [27]. The uniqueness is in Lemma A.8 of [27].

The third item follows since the asymptotically supporting hyperplane at each point of $\text{Cl}(S_E) - S_E$ to $\Omega_1$ and $\Omega_2$ have to agree by Lemma A.7 of [27]. The convexity follows easily from this.

Suppose that $\pi_1(\hat{E})$ is hyperbolic. By Lemma 4.15, $\hat{L}_2 \cup S \cup \hat{O}$ is a convex domain. If $\hat{L}_2 \cup \hat{O}$ is not properly convex, then it is a union of two cones over $S_E$ from a point $x \in A$ but made by two distinct choices of $\pm v_x \in \mathbb{R}^{n+1}, [v_x] = x$ and the same cone over $\hat{O}$. This means that $\hat{O}$ has to be a cone contradicting the irreducibility of $h(\pi_1(O))$. Hence, it follows that $\hat{L}_2 \cup \hat{O}$ is properly convex.

Suppose that $O$ is strictly SPC and $\pi_1(\hat{E})$ is hyperbolic. Then every segment in $\text{bd}\hat{O}$ or a non-$C^1$-point in $\text{bd}\hat{O}$ is in the closure of one of the $p$-end neighborhood. $\text{bd}\hat{L}_2 - \text{Cl}(S_E)$ does not contain any segment in it or a non-$C^1$-point. $\text{bd}\hat{O} - \text{Cl}(S_E)$ does not contain any segment or a non-$C^1$-point outside the union of the closures.
of p-end neighborhoods. By Corollary 4.15, at each point of \( \Lambda := \text{Cl}(S_E) - S_E \), 
\( \tilde{O} \cup \tilde{L}_2 \cup \tilde{S}_E \) is \( C^1 \) and \( \Lambda \) does not contain a segment. This follows because \( S_E \) is 
strictly convex for \( \pi_1(\tilde{E}) \) is a hyperbolic group. (See Theorem 1.1 of \cite{8}.) Therefore, 
\( L_2 \cup \tilde{O} \) is strictly convex relative to the ends.

Suppose now that \( \pi_1(\tilde{E}) \) is a product of hyperbolic and abelian groups. Then 
the dual of the totally geodesic p-end is a radial p-end. By Theorem 6.9 of \cite{27}, 
the dual radial p-end has a neighborhood that is contained in a strict join with a 
vertex \( x \) with a properly convex open domain \( K \) in a hyperplane \( V \). \( \text{Cl}(K) \) is 
a strict join \( C_1 \ast \cdots \ast C_k \) for properly compact convex domains \( C_i \), for \( i = 1, \ldots, k \) by 
Theorem 4.4. Since \( \tilde{O} \) contains a one-sided convex p-end neighborhood \( D \) of \( S_E \).

By equation (2.3), the dual \( D^* \) of \( D \) contains the dual \( \tilde{O}^* \) of \( \tilde{O} \). Since \( D^* \) is the 
interior of a lens-cone by Lemma 2.13, \( D^* \) is contained in the union \( U \) of two strict 
joins \( x \ast K \cup x \ast K \) for the point \( x \) dual to the hyperplane containing ideal boundary 
component \( S_E \) and its antipode \( x \ast K \). Thus, \( \tilde{O}^* \subset x \ast K \cup x \ast K \). The set of supporting hyperspaces at the vertex \( x \) is projectively isomorphic to the 
dual \( K \) of \( \text{Cl}(S_E) \) by Lemma 2.13(iii). Therefore, by Lemma 2.13, the dual \( \tilde{O} \) of 
\( \tilde{O}^* \) is contained in the the cone \( \text{Cl}(S_E) \ast a \) for some point \( a \) dual to the hyperplane \( V \).

Now, \( \tilde{L}_2 \) is a subset of \( \text{Cl}(S_E) \ast a \) sharing boundary \( \text{Cl}(S_E) \) with \( \tilde{O} \) since 
we can treat \( \tilde{L}_2 \) as \( \tilde{O} \) in the above arguments. Since both share \( S_E \) and are in 
\( S_E \ast a \cup S_E \ast a \), the convexity of the union \( \tilde{L}_2 \cup \tilde{O} \) follows. The proper convexity 
follows also as above.

**Corollary 4.16.** Suppose that \( O \) is a noncompact strongly tame properly convex 
real projective orbifold with generalized admissible ends and \( \pi_1(\tilde{E}) \) is hyperbolic. 
Let \( \tilde{E} \) be a lens-type radial p-end. Let \( L \) be a lens in the p-end neighborhood. Define 
\( \Lambda := \text{bd}L - \partial L \). Then each point of \( \Lambda \) has a unique supporting hyperplane of \( L \).

**Proof.** This follows from Lemma 4.15 and the duality Proposition 2.12. \( \Lambda \) and 
the supporting hyperplanes goes to the boundary of a strictly convex domain and 
the supporting hyperplane to \( L^* \) under the duality map. That is, points and the 
supporting hyperplanes change roles here. Then \( L^* \) has strictly convex boundary 
and \( \text{bd}L^* - \partial L^* \) is strictly convex by Lemma 4.15 since \( \pi_1(E) \) is hyperbolic. Thus, 
each hyperplane can meet \( \text{Cl}(L^*) \) at a unique point.

We sharpen Proposition 2.12.

**Lemma 4.17.** Given an end \( E \) of a strongly tame orbifold \( O \) and the corre-
sponding end \( E^* \) of the dual orbifold \( \tilde{O}^* \), \( E \) is a radial end of generalized lens-type 
if and only if \( E^* \) is a totally geodesic end of lens type.

**Proof.** This is given as Remark 2 in \cite{27}.

**4.4.1. The convex hulls of ends.** Here we will be working on \( \mathbb{R}P^n \) exclusively from now on. One can associate a convex hull of a p-end \( E \) of \( \tilde{O} \) as follows:

- For horospherical p-ends, the convex hull of each is defined to be the set of 
the end vertex actually.
• The convex hull of a totally geodesic end $\tilde{E}$ of lens-type is the closure $\text{Cl}(\tilde{S}_E)$ the totally geodesic ideal boundary component $S_E$ corresponding to $\tilde{E}$.

They equal $\text{Cl}(U) \cap \text{bd}\tilde{O}$ for any p-end neighborhood $U$ of $S_E$ by the following proposition 4.18.

**Proposition 4.18.** Let $\mathcal{O}$ be a strongly tame properly convex real projective orbifold with radial ends or totally geodesic ends of lens-type and satisfies (IE) and (NA). Assume that the holonomy group is strongly irreducible. For a horospherical p-end, the closure of a p-end neighborhood meets $\text{bd}\tilde{O}$ at the the p-end only. For a totally geodesic end $\tilde{E}$ of lens-type, a closed p-end neighborhood $L$ of $\tilde{E}$ contains the closure of a proper p-end neighborhood of lens-type, and $\text{Cl}(L) \cap \text{bd}\tilde{O} = \text{Cl}(S_E)$ for the ideal boundary component $S_E$.

**Proof.** For a horospherical end, there exists a finite index free abelian group $\mathbb{Z}^{n-1}$ acting on a compact ellipsoid $E$ with a unique fixed point $x \in \partial E$. Then $(\partial E - \{x\})/\mathbb{Z}^{n-1}$ is compact. We take an open neighborhood $N$ of the fundamental domain in $\tilde{\mathcal{O}}$. Then $\bigcup_{g \in \mathbb{Z}^{n-1}} g(N)$ is an open neighborhood of $\partial E - \{x\}$ in $\tilde{\mathcal{O}}$. Thus, $\partial E \cap \text{bd}\tilde{O} = \{x\}$ since $x$ is a fixed point.

Let $\tilde{E}$ be a lens-type totally geodesic end. By Corollary 8.7(iv) of [27], $S_E$ has a strict lens p-end-neighborhood $L_1 \subset \tilde{\mathcal{O}}$ so that for its boundary component $A$ we have $\text{Cl}(A) - A \subset \text{Cl}(S_E) - S_E$. $\Gamma_E$ acts cocompactly in $A$. As above, we can find an invariant open neighborhood of $A$ in $\tilde{\mathcal{O}}$. Since $\text{Cl}(A) \cup \text{Cl}(S_E)$ is homeomorphic to a $(n - 1)$-sphere, it follows that $\text{bd}\tilde{L}_1 = \text{Cl}(A) \cup \text{Cl}(S_E)$, and $\text{Cl}(L_1) = L_1 \cup \text{Cl}(A) \cup \text{Cl}(S_E)$. Since $A \subset \tilde{\mathcal{O}}$, we obtain $\text{Cl}(L_1) \cap \text{bd}\tilde{O} = \text{Cl}(S_E)$. Since $S_E$ is the boundary of any p-end neighborhood, the result follows. 

For a lens-shaped p-end $E$ with a p-end vertex $v$, the convex hull $I$ is $\text{CH}(\bigcup S(v)) \cap \tilde{\mathcal{O}}$. We can also characterize it as the intersection of every $\text{CH}(\text{Cl}(U_1)) \cap \tilde{\mathcal{O}}$ for a p-end neighborhood $U_1$ of $v$ by (iv) and (v) of Proposition 4.19.

Following Propositions 4.19 and 4.18 imply that the convex hull of an end is a well-defined and is independent of neighborhoods.

**Proposition 4.19.** Let $\mathcal{O}$ be a strongly tame properly convex real projective orbifold with radial ends or totally geodesic ends of lens-type and satisfies (IE) and (NA). Assume that the holonomy group of $\pi_1(\mathcal{O})$ is strongly irreducible. Let $\tilde{E}$ be a radial lens-shaped p-end and $v$ an associated p-end vertex.

(i) A segment in the boundary of $\tilde{\mathcal{O}}$ is always contained in the closure of a convex hull $\text{CH}(\text{Cl}(U_1)) \cap \tilde{\mathcal{O}}$ for a p-end neighborhood $U_1$ of $v$ and more precisely it is in the union $\bigcup S(v)$ of the maximal segments in $\text{bd}\tilde{O}$ ending at $v$ for the corresponding $v$. Thus, the segment is contained in the closure of any p-end neighborhood of $v$.

(ii) $I$ is contained in $\text{CH}(\text{Cl}(U_1)) \cap \tilde{\mathcal{O}}$ for any p-end neighborhood $U_1$ of $v$.

(iii) Any segment in $\bigcup S(v)$ corresponds one-to-one manner to a point of the boundary of the open convex domain $R_v(\tilde{\mathcal{O}}) = S_E$ in $\mathbb{S}_v^{n-1}$.

(iv) $\text{bd}I \cap \tilde{\mathcal{O}}$ is contained in the union of a lens part of a lens-shaped p-end neighborhood.

(v) $I$ contains any concave p-end-neighborhood of $E$ and $I \cap \tilde{\mathcal{O}} = \text{CH}(\text{Cl}(U)) \cap \tilde{\mathcal{O}}$
for a concave $p$-end neighborhood $U$ of $v$. Thus, $I$ has a nonempty interior.

(vi) Each segment from $v$ maximal in $\partial E$ meets the set $\partial I \cap \partial E$ exactly once and $\partial I \cap \partial E / \Gamma_v$ is an orbifold isotopic to $E$ for the end fundamental group $\Gamma_v$ of $v$.

(vii) There exists a nonempty-interior of the convex hull $I$ of a neighborhood of the $p$-end vertex $v$ of $E$ of $\partial E$ and where $\Gamma_v$ acts so that $I \cap \partial E / \Gamma_v$ is diffeomorphic to the end orbifold times an interval.

(viii) $I \cap \partial E$ has a boundary restricting to the covering map is an immersed compact orbifold homotopic to the associated end orbifold.

**Proof.** (i) A segment in $\partial \partial E$ is contained in a closure of a $p$-end neighborhood by the strictness of the SPC-structure. Since it meets the interior of $\bigcup S(v)$, the segment must be in $\bigcup S(v)$ as in the proof of Theorem 4.3(ii) and Theorem 4.4 in [27].

By Theorems 4.3 and 4.4, the set $\bigcup S(v)$ is always contained in the closure of any $p$-end neighborhood of $v$. Thus (ii) follows.

(iii) A segment $s$ from $v$ in $\partial \partial E$ either ends in a lens-shaped domain or is in $\partial \partial E$. In the second case, $s$ is in $\partial \partial E_\sigma(\partial E)$ clearly.

(iv) We define $S_1$ as the set of 1-simplices with endpoints in segments in $\bigcup S(v)$ and we inductively define $S_i$ to be the set of $i$-simplices with faces in $S_{i-1}$. Then $I$ is a union $\bigcup_{\sigma \in S_i} \bigcup_{i \in S_{i-1}} \sigma$. Notice that $\partial I$ is the union $\bigcup_{\sigma \in S_i} \bigcup_{i \in S_{i-1}} \sigma \cap \partial I$ since each point of $\partial I$ is contained in the interior of a simplex which lies in $\partial I$ by the convexity of $I$. If $\sigma \in S_1$ with $\sigma \subset \partial I$, then its end point must be in an endpoint of a segment in $\bigcup S(v)$. If an interior point of $\sigma$ is in a segment in $\bigcup S(v)$, then the vertices of $\sigma$ are in $\bigcup S(v)$ by the convexity of $\partial \partial E_\sigma(\partial E)$. Hence, if $\sigma^o$ meets $\partial E$, then $\sigma^o$ is the lens-shaped domain. Now by induction on $S_i$, $i > 1$, we can verify (iv) since any simplex with boundary in the union of subsimplices in the lens-domain is in the lens-domain by convexity.

(v) Since $I$ contains the segments in $\bigcup S(x)$ and is convex, and so does a concave $p$-end neighborhood $U$, we obtain $\partial U \subset I$. Otherwise, let $x$ be a point of $\partial U \cap \partial I$ where some neighborhood in $\partial U$ is not in $I$. Then a supporting hyperspace at $x$ of the convex set $I$, meets a segment in $\bigcup S(x)$ in its interior. This is a contradiction since $I$ contains the segments entirely. Thus, $U \subset I$.

(vi) $\partial I \cap \partial E$ is a subset of a lens part of a $p$-end neighborhood by (iii). Each point of it meets a maximal segment from $v$ in the end but not in $\bigcup S(x)$ at exactly one point since a maximal segment must leave the lens cone eventually. Thus $\partial I \cap \partial E$ is homeomorphic to an $(n-1)$-cell and the result follows.

(vii) This follows from (v) since we can use rays from $x$ meeting $\partial I \cap \partial E$ at unique points and use them as leaves of a fibration.

(viii) This again follows from (vi).

\[ \square \]

**Definition 4.20.** Let $\tilde{E}$ be a lens-type radial end. Let $L$ be the lens-cone neighborhood of $\tilde{E}$, and let $\Lambda = \partial \partial D - \partial D$ for the lens domain $D$ of $L$, i.e., the limit set of $\pi_1(\tilde{E})$. Let $CH(\Lambda)$ denote the convex hull. We define a maximal concave $p$-end neighborhood or mc-p-end-neighborhood $U$ to be the component of $U' - CH(\Lambda)$ containing a p-end neighborhood of $\tilde{E}$ for any choice of a p-end neighborhood $U'$ of $\tilde{E}$ containing $CH(\Lambda) \cap \partial E$. The closed maximal concave $p$-end neighborhood is
Cl(U) ∩ ˜O. An ϵ-d_3-neighborhood U’ of a maximal concave p-end neighborhood is called an ϵ-mc-p-end-neighborhood U’.

**Lemma 4.21.** Let D be an i-dimensional totally geodesic compact convex domain, i ≥ 1. Let ˜E be a generalized lens-type p-end with the p-end vertex v_˜E. Suppose ∂D ⊂ ∪ S(v_˜E). Then D ⊂ V for a maximal concave p-end neighborhood V, and for sufficiently small ϵ > 0, an ϵ-d_3-neighborhood of D ⊂ V’ for any ϵ-mc-p-end neighborhood V’.

**Proof.** Assume that U is a generalized lens-cone of v_˜E. Then Λ is the set of end points of segments in S_{v_˜E} with v_˜E removed. Let P be the spanning subspace of D and v_˜E. Since ∂D ∩ P ⊂ ∪ S(v_˜E) ∩ P, and ∂D ∩ P is closer than Λ ∩ P from v_˜E, it follows that P ∩ Cl(U) ⊂ D has a component C_1 containing v_˜E and a component C_2 contains Λ ∩ P. Hence Cl(C_2) ⊃ CH(Λ) ∩ P by the convexity of Cl(C_2). This implies that D is disjoint from CH(Λ) or D contains CH(Λ) ∩ P. Let V be an mc-p-end-neighborhood of U. Since Cl(V) contains the closure of the component of U − CH(Λ) whose closure contains v_˜E, it follows that Cl(V) contains D.

Since D is in the mc-p-end neighborhood V, the boundary bdCl(V’) ∩ ˜O of the ϵ-mc-p-end-neighborhood V’ do not meet D. Hence D’ ⊂ V’.

**Corollary 4.22.** Let O be a properly convex real projective manifold with generalized admissible ends and satisfies (IE) and (NA). Assume that the holonomy group of π_1(O) is strongly irreducible. Let ˜E be a generalized lens-type radial end. Then

(i) Also, a concave p-end neighborhood of ˜E is always a subset of an mc-p-end-neighborhood of the same p-end.
(ii) An mc-p-end-neighborhood is a union of concave end neighborhoods.
(iii) Each mc-p-end-neighborhood of ˜E is a proper p-end neighborhood, and covers an end-neighborhood with compact boundary in O.
(iv) An $\epsilon$-mc-p-end-neighborhood of $\hat{E}$ for sufficiently small $\epsilon > 0$ is a proper p-end neighborhood.
(v) We can choose $\epsilon$-mc-p-end neighborhoods of p-ends so that their image end-neighborhoods in $O$ are mutually disjoint.

**Proof.** (i) Since any generalized lens $L$ in a generalized lens-cone p-end neighborhood $U$ of $\hat{E}$ contains $CH(\Lambda) \cap \tilde{O}$ for the limit set $\Lambda$ of $\hat{E}$ by Corollary 8.5 of [27]. Hence, a concave end neighborhood is contained in an mc-p-end-neighborhood.

(ii) Let $V$ be an mc-p-end neighborhood of $\hat{E}$. Then define $S$ to be the set of end points in $\overline{\text{Cl}(\tilde{O})}$ of maximal segments in $V$ from $v_{E}$ in directions of $S_{E}$. That is $\text{Cl}(V) \cap \tilde{O} = V \cup S$, and $S$ is homeomorphic to $S_{E}$. Thus, $S/\pi_{1}(\hat{E})$ is a compact set since $S$ is contractible and $S_{E}/\pi_{1}(\hat{E})$ is a $K(\pi_{1}(\hat{E}))$-space.

We can $d_{2}$-approximate $S$ by the smooth boundary component $S_{e}$ outwards of a generalized lens using the proof of Proposition 7.6 of [27]. For sufficiently small $\epsilon > 0$, $S_{e}$ is strictly convex by the continuity of the Hessian matrices. A component $U - S$, or $S_{e}$ is a concave p-end neighborhood.

(iii) Since a concave p-end neighborhood is a proper p-end neighborhood, we obtain $g(V) \cap V = \emptyset$ or $g(V) = V$ for $g \in \pi_{1}(O)$ by the first item.

Suppose that $g(\text{Cl}(V) \cap \tilde{O}) \cap \text{Cl}(V) \neq \emptyset$. Then $g(V) = V$ and $g \in \pi_{1}(\hat{E})$.

Otherwise, $g(V) \cap V = \emptyset$, and $g(\text{Cl}(V) \cap \tilde{O})$ meets $\text{Cl}(V)$ in a totally geodesic hypersurface $S$ equal to $CH(\Lambda)_{o}$ by the concavity of $V$. Hence for every $g \in \pi_{1}(O)$, $g(S) = S$, and $g(V) \cup S \cup V = \tilde{O}$ since these are subsets of a properly convex domain $\tilde{O}$. Then $\pi_{1}(O)$ acts on $S$ and $S/G$ is homotopy equivalent to $\tilde{O}/G$ for a finite index torsion free subgroup $G$ of $\pi_{1}(O)$ by Selberg’s lemma. This cannot be true since the quotients are manifolds with different dimensions.

Now suppose that $S \cap \bd\tilde{O} \neq \emptyset$. Let $S'$ be a maximal totally geodesic domain in $\text{Cl}(V)$ supporting $S$. Then $S' \subset \bd\tilde{O}$ by convexity by Lemma 7.5 of [27], meaning that $S' = S \subset \bd\tilde{O}$. In this case, $\tilde{O}$ is a cone over $S$ and the end vertex $v_{E}$ of $\hat{E}$. For each $g \in \pi_{1}(O)$, $g(V) \cap V \neq \emptyset$ meaning $g(V) = V$ since $g(v_{E})$ is on $\text{Cl}(S)$. Thus, $\pi_{1}(O) = \pi_{1}(\hat{E})$. This contradicts the infinite index condition of $\pi_{1}(\hat{E})$.

We showed that $\text{Cl}(V) \cap \tilde{O} = V \cup S$. Thus, an mc-p-end-neighborhood $\text{Cl}(V) \cap \tilde{O}$ is a proper end neighborhood of $\hat{E}$ with compact imbedded boundary $S/\pi_{1}(\hat{E})$. Therefore we can choose positive $\epsilon$ so that an $\epsilon$-mc-p-end-neighborhood is a proper p-end neighborhood also. This proves (iv).

(v) For two mc-p-end neighborhoods $U$ and $V$ for different p-ends, we have $U \cap V = \emptyset$ by (iii).

We showed that $\text{Cl}(V) \cap \tilde{O}$ for an mc-p-end-neighborhood $V$ covers an end neighborhood in $O$. Suppose that $U$ is another mc-p-end neighborhood different from $V$ and $\text{Cl}(U) \cap \text{Cl}(V) \cap \tilde{O} \neq \emptyset$. Since $U \cap V = \emptyset$, we have $\text{Cl}(U) \cap \text{Cl}(V)$ are in the boundary of $U$ and $V$ in a properly convex domain $\tilde{O}$, and $\bd\text{Cl}(U) \cap \tilde{O}$ and $\bd\text{Cl}(V) \cap \tilde{O}$ equal a tangent maximal hyperspace $CH(\Lambda)_{o}$ in $\tilde{O}$ and hence they are equal. As above, this is a contradiction. Hence $\text{Cl}(U) \cap \text{Cl}(V) \cap \tilde{O} = \emptyset$.

Since the closures of mc-p-end neighborhoods with different p-ends are disjoint, the final item follows. □

For the following, we need a stronger condition of lens-type ends.
4. CONVEXITY

COROLLARY 4.23. Let $\mathcal{O}$ be a properly convex real projective manifold with admissible ends and satisfies (IE) and (NA). Assume that the holonomy group of $\pi_1(\mathcal{O})$ is strongly irreducible. Let $\mathcal{U}$ be the collection of the components of the inverse image in $\tilde{\mathcal{O}}$ of the union of disjoint collection of end neighborhoods of $\mathcal{O}$ for all radial or totally geodesic ends of lens-type. Now replace each of the p-end neighborhoods of radial lens-type of collection $\mathcal{U}$ by a concave p-end neighborhood by Lemma 4.9 (iii). Then the following statements hold:

(i) Given concave or one-sided lens p-end-neighborhoods $U_1$ and $U_2$ contained in $\bigcup \mathcal{U}$, we have $U_1 \cap U_2 = \emptyset$ or $U_1 = U_2$.

(ii) Let $U_1$ and $U_2$ be in $\mathcal{U}$. Then $\text{Cl}(U_1) \cap \text{Cl}(U_2) \cap \text{bd}\tilde{\mathcal{O}} = \emptyset$ or $U_1 = U_2$ holds.

PROOF. Let $\tilde{E}$ be a p-end of $\tilde{\mathcal{O}}$. Since $L/\pi_1(\tilde{E})$ is compact for a lens of a lens cone of a p-end neighborhood of $\tilde{E}$, each lens-cone p-end neighborhood is a proper p-end neighborhood if we take a finite index subgroup of $\pi_1(\mathcal{O})$. We assume without loss of generality that the lens-cones are proper p-end neighborhoods from now on.

(i) Suppose that $U_1$ and $U_2$ are p-end neighborhoods of radial ends. Let $U'_1$ be the interior of the associated generalized lens-cone of $U_1$ in $\text{Cl}(\mathcal{O})$ and $U'_2$ be that of $U_2$. Let $U''_i$ be the concave p-end-neighborhood of $U'_i$ for $i = 1, 2$ that covers an end neighborhood in $\mathcal{O}$ by Lemma 4.9 (iii). Since the neighborhoods in $\mathcal{U}$ are mutually disjoint,

- $\text{Cl}(U''_1) \cap \text{Cl}(U''_2) \cap \tilde{\mathcal{O}} = \emptyset$ or
- $U''_1 = U''_2$

since we can choose these to cover disjoint or identical p-end neighborhoods in $\mathcal{O}$.

(ii) Assume that $U''_i \in \mathcal{U}$, $i = 1, 2$, and $U''_1 \neq U''_2$. Suppose that the closures of $U''_1$ and $U''_2$ intersect in $\text{bd}\tilde{\mathcal{O}}$. Suppose that they are both radial p-end neighborhoods. Then the respective convex hulls $I_1$ and $I_2$ as obtained by Proposition 4.19 intersect as well. Take a point $z \in \text{Cl}(U''_1) \cap \text{Cl}(U''_2) \cap \text{bd}\tilde{\mathcal{O}}$. Let $p_1$ and $p_2$ be the respective p-end vertices of $U'_1$ and $U'_2$. Then $\overline{p_1z} \in S(p_1)$ and $\overline{p_2z} \in S(p_2)$ and these segments are maximal since otherwise $U''_1 \cap U''_2 \neq \emptyset$. The segments intersect transversally at $z$ since otherwise we violated the maximality in Theorems 4.3 and 4.4. We obtain a triangle $\triangle(p_1 p_2 z)$ in $\text{Cl}(\mathcal{O})$ with vertices $p_1, p_2, z$. We assume that $p_1 p_2 \subset \mathcal{O}$.

If this is not true, we need to perturb $p_1$ and $p_2$ by a small amount. We may not have the geodesic triangle but will have a disk bounded by three arcs. However, the disk has an angle $< \pi$ at $z$ since $z$ is not a $C^1$-point of $\text{bd}\tilde{\mathcal{O}}$. We will denote the disk by $\triangle(p_1 p_2 z)$ still.

We define a convex curve $\alpha_i := \triangle(p_1 p_2 z) \cap \text{bd}I_i$ with an end point $z$ for each $i$, $i = 1, 2$. Let $\tilde{E}_i$ denote the p-end corresponding to $p_i$. Since $\alpha_i$ maps to a geodesic in $R_{p_i}(\mathcal{O})$, there exists a foliation $T$ of $\triangle(p_1 p_2 z)$ by maximal segments from the vertex $p_1$. There is a natural parametrization of the space of leaves by $\mathbb{R}$ as the space is projectively equivalent to an open interval using the Hilbert metric of the interval. We parameterize $\alpha_i$ by these parameters as $\alpha_i$ intersected with a leaf is a unique point. They give the geodesic length parameterizations under the Hilbert metric of $R_{p_i}(\mathcal{O})$ for $i = 1, 2$. 
We now show that an infinite-order element of $\pi_1(\tilde{E}_2)$ is the same as one in $\pi_1(\tilde{E}_1)$: By convexity, either $\alpha_2$ goes into $I_1$ and not leave again or $\alpha_2$ is disjoint from $I_1$. Suppose that $\alpha_2$ goes into $I_1$ and not leave it again. Since $I_2/\pi_1(\tilde{E}_2)$ is compact, there is a sequence $t_i$ so that the image of $\alpha_2(t_i)$ converges to a point of $I_1/\pi_1(\tilde{E}_1)$. Hence, by taking a short path between $\alpha_2(t_i)$s, there exists an essential closed curve $c_2$ in $I_2/\pi_1(\tilde{E}_2)$ homotopic to an element of $\pi_1(\tilde{E}_1)$. In fact $c_2$ is in a lens-cone end neighborhood of the end corresponding to $\tilde{E}_1$. This contradicts (NA). (The element is of infinite order since we can take a finite cover of $O$ so that $\pi_1(O)$ is torsion-free by Selbert’s lemma.)

Suppose now that $\alpha_2$ is disjoint from $I_1$. Then since $\alpha_1$ and $\alpha_2$ have the same end point $z$ and by the convexity of $O$, we parameterize $\alpha_i$ so that $\alpha_1(t)$ and $\alpha_2(t)$ is on a line segment in the triangle with end points in $\mathbb{RP}^1$ and $\mathbb{RP}^2$. We obtain $d_O(\alpha_2(t), \alpha_1(t)) \leq C$ for a uniform constant $C$ since one can project to the space of lines through $z$, a one dimensional projective space where the end points are fixed and the the image of $\beta(t)$ are so that the image of $\beta(t')$ is contained in that of $\beta(t)$ if $t < t'$. And the Hilbert-metric length of the segment $\beta(t) := \overline{\alpha_2(t)\alpha_1(t)}$ is bounded above by the uniform constant.

We have a sequence $t_i \to \infty$ so that

$$p_O \circ \alpha_2(t_i) \to x, d_O(p_O \circ \alpha_2(t_i), p_O \circ \alpha_2(t_{i+1})) \to 0, x \in O.$$ 

So we obtain a closed curve $c_{2,i}$ in $O$ obtained by taking a short path between the two points. By taking a subsequence, the image of $\beta(t_i)$ in $O$ geometrically converges to a segment of Hilbert-length $\leq C$. As $i \to \infty$, we have $d_O(p_O \circ \alpha_1(t_i), p_O \circ \alpha_1(t_{i+1})) \to 0$ by extracting a subsequence. There exists a closed curve $c_{1,i}$ in $O$ again by taking a short path. We see that $c_{1,i}$ and $c_{2,i}$ are homotopic in $O$ since we can use the image of the disk in the quadrilateral bounded by $\alpha_2(t_i)\alpha_2(t_{i+1}), \alpha_1(t_i)\alpha_1(t_{i+1}), \beta(t_i), \beta(t_{i+1})$ and the connecting thin strips between the images of $\beta(t_i)$ and $\beta(t_{i+1})$ in $O$. This again contradicts (NA).

Now, consider when $U_1$ is a one-sided lens-neighborhood of a totally geodesic p-end and let $U_2$ be a concave p-end neighborhood of a radial p-end of $\tilde{O}$. Let $z$
be the intersection point in $\text{Cl}(U_1) \cap \text{Cl}(U_2)$. We can use the same reasoning as above by choosing any $p_1$ in $S_{E_i}$ so that $\overline{p_1z}$ passes the interior of $\tilde{E}_i$. Let $p_2$ be the p-end vertex of $U_2$. Now we obtain the triangle with vertices $p_1, p_2$, and $z$ as above. Then the arguments are analogous and obtain an infinite order elements in $\pi_1(\tilde{E}_1) \cap \pi_1(\tilde{E}_2)$.

Finally, consider when $U_1$ and $U_2$ are one-sided lens-neighborhoods of totally geodesic p-ends respectively. Using the intersection point $z$ of $\text{Cl}(U_1) \cap \text{Cl}(U_2) \cap \tilde{O}$ and we choose $p_i$ in $\text{bd} \tilde{E}_i$ so that $\overline{z p_i}$ passes the interior of $S_{E_i}$ for $i = 1, 2$. Again, we obtain a triangle with vertex $p_1, p_2$, and $z$, and find a contradiction as above.

We fully extend the above result.

**Corollary 4.24.** Let $\mathcal{O}$ be a properly convex real projective manifold with admissible ends and satisfies (IE) and (NA). Assume that the holonomy group of $\pi_1(\mathcal{O})$ is strongly irreducible. Let $\mathcal{U}$ be the collection of components of the inverse image in $\tilde{\mathcal{O}}$ of the union of disjoint collection of p-end neighborhoods of $\mathcal{O}$ for all ends. Then for every $U_1, U_2 \in \mathcal{U}$, $\text{Cl}(U_1) \cap \text{Cl}(U_2) = \emptyset$ or $U_1 = U_2$.

**Proof.** We now consider horospherical p-ends. Since $\text{Cl}(U) \cap \text{bd}\tilde{\mathcal{O}}$ is a unique point, (iii) of Proposition 4.1 implies the result.

4.5. The strong irreducibility and stability of the holonomy group of properly convex strongly tame orbifolds.

First, we modify Theorem 6.9 of [27] by replacing some conditions.

**Lemma 4.25.** Let $\mathcal{O}$ be a strongly tame properly convex real projective manifold with generalized admissible ends and satisfies (IE) and (NA). Let $\tilde{E}$ be a reducible p-end of $\tilde{\mathcal{O}}$ of generalized lens-type. Then there exists a totally geodesic hyperspace $P$ where $h(\pi_1(\tilde{E}))$ acts on and $S_{\tilde{E}} := P \cap \text{Cl}(\tilde{\mathcal{O}})$ is a properly convex domain and $S_{\tilde{E}}^{o} \subset \tilde{\mathcal{O}}$, and $S_{\tilde{E}}/\pi_1(\tilde{E})$ is a compact orbifold. Also, each element of $g \in \pi_1(\tilde{E})$ acts as nonidentity on a subspace properly containing $v$.

**Proof.** The proof of Theorem 6.9 of [27] shows that $\tilde{\mathcal{O}}$ is either a join or the conclusion of Theorem 6.9 of [27] holds and $\pi_1(\tilde{E})$ acts on a totally geodesic convex compact domain $D$ of codimension 1 that is the intersection of $P_{\tilde{E}} \cap \text{Cl}(\tilde{\mathcal{O}})$ for a $\pi_1(\tilde{E})$-invariant subspace $P_{\tilde{E}}$. In the former case, we can show that $\text{Cl}(\tilde{\mathcal{O}})$ is the join $v_{\tilde{E}} * D$ for a compact convex domain $D \subset \text{bdCl}(\tilde{\mathcal{O}})$ of codimension 1.

By (IE), there exists a deck transformation $h$ so that $v_2 := h(v_{\tilde{E}}) \neq v_{\tilde{E}}$. By geometry of the join, $h(v_{\tilde{E}})$ is in $D$. This implies that there exists another totally geodesic domain $D_1 \subset D$ of lower-codimension so that $\text{Cl}(\tilde{\mathcal{O}}) = v_{\tilde{E}} * v_2 * D_2$. Then by induction, we see that such step must terminate in finite steps. This contradicts (IE).

Now, we go to the alternative case. Let $S_{\tilde{E}} = D^o$. Then $D^o \subset \tilde{\mathcal{O}}$. The last part follows again from the proof of Theorem 6.9 (ii).

Because of the following, we no longer need the assumptions of strong irreducibility of the holonomy group of $\pi_1(\mathcal{O})$ in this article.
THEOREM 4.26. Let $\mathcal{O}$ be a noncompact strongly tame properly convex real projective manifold with horospherical, generalized admissible ends and satisfies (IE) and (NA). Assume $\partial \mathcal{O} = \emptyset$. Then any finite-index subgroup of the holonomy group is strongly irreducible and is not contained in a proper parabolic subgroup of $\text{PGL}(n + 1, \mathbb{R})$ (resp. $\text{SL}_\pm(n + 1, \mathbb{R})$).

PROOF. We need to prove for $\text{PGL}(n + 1, \mathbb{R})$ only by Theorem 3.19. Let $h : \pi_1(\mathcal{O}) \to \text{PGL}(n + 1, \mathbb{R})$ be the holonomy homomorphism. Suppose that $h(\pi_1(\mathcal{O}))$ is virtually reducible. Then we can choose a finite cover $\mathcal{O}_1$ so that $h(\pi_1(\mathcal{O}_1))$ is reducible.

We denote $\mathcal{O}_1$ by $\mathcal{O}$. Let $S$ denote a proper subspace where $\pi_1(\mathcal{O})$ acts on. Suppose that $S$ meets $\mathcal{O}$. Then $\pi_1(\mathcal{E})$ acts on a properly convex open domain $S \cap \mathcal{O}$ for each p-end $\mathcal{E}$. Thus, $(S \cap \mathcal{O})/\pi_1(\mathcal{E})$ is a compact orbifold homotopy equivalent to one of the end orbifolds. However, $S \cap \mathcal{O}$ is $\pi_1(\mathcal{E})$-invariant for each end neighborhood $\mathcal{E}$. This contradicts (IE). Therefore, $S \cap \text{Cl}(\mathcal{O}) \subset \text{bd}\mathcal{O}$.

We show that $\text{Cl}(\mathcal{O}) \cap S \neq \emptyset$. Let $\mathcal{E}$ be a p-end. If $\mathcal{E}$ is horospherical, $\pi(\mathcal{E})$ acts on a great sphere $\bar{S}$ tangent to an end vertex. $S$ has to be a subspace in $S$ containing the end vertex by Proposition 5.1(iii) of [27].

Suppose that $\mathcal{E}$ is a radial end of generalized lens-type. Then either $S$ passes the end vertex $v_\mathcal{E}$ or there exists a subspace $S'$ containing $S$ and $v_\mathcal{E}$ where $\pi_1(\mathcal{E})$-invariant. Hence $S'$ correspond to a proper-invariant subspace in $S_{v_\mathcal{E}}^{n-1}$ or $S$ is a hyperspace of dimension $n - 1$ disjoint from $v_\mathcal{E}$. By considering a hyperbolic factor, it follows that there exists some attracting fixed points in the limit sets of $\pi_1(\mathcal{E})$. Considering when $\pi_1(\mathcal{E})$ has nontrivial diagonalizable elements, we obtain $S \cap \text{Cl}(L) \neq \emptyset$ for a lens $L$, $L \subset \mathcal{O}$. The existence of the attracting fixed points of some elements of $\pi_1(\mathcal{E})$ implies that $S \cap \text{Cl}(L) \neq \emptyset$ for a lens $L$, $L \subset \mathcal{O}$. (This follows from Theorem 7.9 of [27] and Proposition 1.1 of [12] and the uniform middle eigenvalue condition.)

If $\mathcal{E}$ is totally geodesic of lens-type, we can apply a similar argument using the attracting fixed points. Therefore, $S \cap \text{Cl}(\mathcal{O})$ is a subset $K$ of $\text{bd}\mathcal{O}$ of dim $\geq 0$. In fact, we showed that the closure of each p-end neighborhood meets $K$.

By taking dual orbifold if necessary, we assume without loss of generality that there exists a radial end of generalized lens-type with a radial p-end vertex $v$.

(I) Suppose that a p-end vertex $v$ is in $K$. $v$ cannot be a horospherical p-end vertex by Proposition 4.1(iii).

Now $v$ is a p-end vertex of a generalized lens-shaped end $\mathcal{E}$. Let $V_v$ be the concave p-end neighborhood of $\mathcal{O}$ of $v$ from a system of mutually disjoint end neighborhood of $\mathcal{O}$. Then $h(\pi_1(\mathcal{E})) \cap h(\pi_1(\mathcal{O}_1))$ is reducible.

We obtain a totally geodesic hypersurface $S_{\mathcal{E}} \subset \mathcal{O}$ by Lemma 4.25. Let $P_{\mathcal{E}}$ denote the spanning subspace of $S_{\mathcal{E}}$. (We are only using the part of the proof where the strong irreducibility of $h(\mathcal{O})$ is not used yet there.) Choose $x \in \mathcal{O}$ in a direction of $S_{\mathcal{E}}$ from $v$. Since $S_{\mathcal{E}}/\pi_1(\mathcal{E})$ is compact, we choose a sequence $g_i \in \pi_1(\mathcal{E})$ of central elements so that $g_i(x)$ converges to a point of $\text{Cl}(V_v) \cap K$ as $i \to \infty$. We can choose unbounded $g_i$ so that $\{g_i|P_{\mathcal{E}} \cap K\} \to \text{I}_{P_{\mathcal{E}} \cap K}$ on $P \cap K$ by Lemma 4.27 applied to $S_{\mathcal{E}}^{n-1}$. Let $\lambda_i^\prime$ denote the $(n_K := \dim P_{\mathcal{E}} \cap K)$-th root of the norm of the determinant of $g_i|P_{\mathcal{E}} \cap K$, and let $\lambda_{v,i}$ denote the eigenvalue of $g_i$ associated with $v$. Since there exists a generalized $g_i$-invariant lens whose closure is disjoint from
for some choice of signs above $g_i$ to a cone $v * (P_E \cap K)$ or alternatively $v * (P_E \cap K) \cup (P_E \cap K) * v_-$, a nonproperly convex set in $\text{Cl}(\hat{O})$ as $g_i|P_E \cap K \to 1_{P_E \cap K}$ by applying Lemma 6.11 of [27] to the closures of the both components of $K - \hat{K} \cap P_E$. This is a contradiction. Only the first case, $K = v * (P_E \cap K)$ holds.

By apply $g_i^n$ to $v$, we obtain a subset $U_v$ equal to $v*(P_E \cap K)^\circ$ with nonempty interior. $U_v$ is in the relative interior of $\text{Cl}(V_v)$ in $\text{Cl}(\hat{O})$ considering segments in $V_v$ in directions of $S^E_E$ and the action of $g_i^n$.

Since $\pi_1(\hat{E})$ is of infinite index in $\pi_1(O)$, there is an element $h \notin \pi_1(\hat{E})$ with $h(v) \neq v$. Since $h(v) \in K$ and $h(v)$ has an end neighborhood $h(U_v)$ in $K$ that is also a neighborhood of $v$ in $K$ with nonempty interior. Since $v \in U_v$, we obtain $h(U_v) \cap U_v \neq \emptyset$. Thus, $h(U_v) \cap U_v \neq \emptyset$ holds and hence $h(U_v) = U_v$. This is a contradiction to $h \notin \pi_1(\hat{E})$.

(ii) Suppose that

$$C < \left| \frac{\lambda_i}{\lambda_{v,i}} \right| < C'$$

for some constants $C, C' > 0$.

Then there exists then a sequence of mutually distinct elements $g_i \in \pi_1(\hat{E})$ so that $\{g_i|K\} \to 1_K$.

By apply $\{g_i\}$ to $v$, we obtain subsets $U_v \subset v * (P_E \cap K)^\circ$ with nonempty interior inside the relative interior of $\text{Cl}(V)$ in $\text{Cl}(\hat{O})$ as above.

Since $\pi_1(\hat{E})$ is of infinite index in $\pi_1(O)$, it follows that there is an element $h \in \pi_1(\hat{E})$ with $h(v), h(v) \neq v$. Since $h(v) \in K$ and $h(v)$ has an end neighborhood $h(U_v)$.

For a sufficiently large $i$, $g_i(h(U_v)) \cap h(U_v) \neq \emptyset$. Since a point of $U_v^\circ$ has a neighborhood $V_v$, this implies $g_i(h(V_v)) \cap h(V_v) \neq \emptyset$, and we obtain $g_i(h(V_v)) = h(V_v)$ and $g_i \in h^{-1}\pi_1(\hat{E})h$. Hence, this implies that there exists an essential annulus, a contradiction to (NA).

(II) Suppose that for every radial p-end $\hat{E}$, the p-end vertex $v$ is not in $K$. Now $v$ cannot be horospherical since the horospherical action fixing $v$ does not preserve a properly convex set except $v$ or horoballs of dimension $n$. Hence, there exists a radial p-end $\hat{E}$ of generalized lens-type as above. Let $v$ be its vertex.

Suppose that $K$ is of dimension $n - 1$. Then $\hat{O}$ is a strict join $v * K$, as we can deduce by the $\pi_1(\hat{E})$-invariance of $K$ and the fact $K \subset \text{bd}\hat{O}$. By (IE), $h(v) \neq v$ for $h \in \pi_1(O)$. We obtain $h(v) \in K$ by the proper convexity of $S^E_E$. This contradicts the premise of (II).

Suppose that $\text{dim } K \leq n - 2$. Then each radial p-end $\hat{E}$ is reducible.

We obtain a totally geodesic hypersurface $S_{E_1} \subset \hat{O}$ by Lemma 4.25. Let $P_{E_1}$ denote the spanning subspace of $S_{E_1}$. Since $K$ is disjoint from $v_{E_1}$, let $P_K$ be the subspace containing $K$ and $v_{E_1}$. Choose $x \in S_{E_1}$. Since $S_{E_1}/\pi_1(\hat{E}_1)$ is compact, we choose a sequence $g_i \in \pi_1(\hat{E}_1)$ of central elements so that $g_i(x)$ converges to a point of $C_K \cap P_E$ as $i \to \infty$. We can choose unbounded $g_i$ so that $\{g_i|C_K \cap P_E\} \to 1_{C_K \cap P_E}$ by Lemma 4.27 applied to $S_{E_1}^{n-1}$. Let $k$ be the dimension of $C_K \cap P_E$. The $(k+1)$-th root $\lambda_i$ of the norm of the determinant of the submatrix of the unit-determinant
matrix $g_i$ corresponding to $C_K \cap P_{\widetilde{E}}$. Then $\hat{\lambda}_i > \lambda(g_i)(v_{\widetilde{E}_1})$ for the eigenvalue $\lambda(g_i)(v_{\widetilde{E}_1})$ of $g_i$ at $v_{\widetilde{E}_1}$ by Theorem 7.9 of [27] since there exists a lens whose closure is disjoint from $\{v_{\widetilde{E}_1}\}$. Since $K$ is $h(g_i)$-invariant, we obtain $K \subset P_{\widetilde{E}}$.

Since $S_{\widetilde{E}_1} \subset \tilde{O}$ and $S_{\widetilde{E}_1}/\pi_1(\tilde{E}_1)$ is a compact surface immersed in $O$, we can assume without loss of generality that $S_{\widetilde{E}_1}$ covers a totally geodesic closed orbifold $S_1$ by taking a finite cover of $O$. Moreover, $K \subset Cl(S_{\widetilde{E}_1})$ by the existence of the sequence $g_i$ acting on $S_{\widetilde{E}_1}$ as above.

By condition (IE), choose $h \in \pi_1(O)$ so that $v_{\widetilde{E}_1} \neq v_{\widetilde{E}_2} = h(v_{\widetilde{E}_1})$. Also, by the same argument as above, we have $Cl(h(S_{\widetilde{E}_1})) \supset K$.

Then as in the proof of Corollary 4.23, we obtain that $d_O(S_{\widetilde{E}_1}, h(S_{\widetilde{E}_1})) \leq C$ for a constant $C > 0$, and there is an infinite order element of $\pi_1(E_1) \cap h\pi_1(\tilde{E}_1)h^{-1}$. This is a contradiction to (NA).

Thus, we have shown that $\pi_1(O)$ is irreducible. Since the argument works for every finite cover of $O$, $\pi_1(O)$ is strongly irreducible.

Since a parabolic group acts on a nontrivial flag in $\mathbb{R}^{n+1}$ by definition, a parabolic group is always reducible. This shows that $\pi_1(O)$ is not parabolic.

\end{proof}

For a matrix $g$ in $\text{GL}(n+1, \mathbb{R})$, we denote the induced projective automorphism by $S(g) : S^n \to S^n$. We state the elementary lemma to clarify.

**Lemma 4.27.** Let $Z^l$ be in the center of the holonomy group of a properly convex closed real projective $(n-1)$-orbifold with admissible fundamental group. Let $r : Z^l \to D$ be the inclusion homomorphism to a group of diagonal matrices on $\mathbb{R}^n = \oplus_{i=1}^{m} V_i$ where $S(r(g))|S(V_i) = I_{S(V_i)}$ for all $g \in Z^l$. For any given $V' := V_{i_1} \oplus \cdots \oplus V_{i_j}$ for a proper set $\{i_1, \ldots, i_j\}$, we can find a sequence of elements $g_i \in Z^l$ so that $\lambda_i/\lambda'_i \to \infty$ for the largest norm $\lambda_i$ of the eigenvalue of $r(g_i)$ on $V'$ and $\lambda'_i$ on the complement of $V'$ and $S(r(g_i))|S(V') \to I_{S(V')}$.\end{lemma}

\begin{proof}
This follows by Theorem 1.1 of [10].\end{proof}
CHAPTER 5

The strict SPC-structures and relative hyperbolicity

In this section, we will be working with $\mathbb{R}P^n$ exclusively.

From now on, we will assume that properly convex strongly tame real projective orbifolds with generalized admissible ends have strongly irreducible holonomy groups by Theorem 4.26.

5.1. The Hilbert metric on $O$.

A Hilbert metric on an orbifold with an SPC-structure is defined as a distance metric given by cross ratios. (We do not assume strictness here.)

Given an open properly convex domain $\Omega$, we note that given any two points $x, y$ in $\Omega$, there is a geodesic arc $xy$ with endpoints $x, y$ so that its interior is in $\Omega$.

**Proposition 5.1.** Let $\Omega$ be a properly convex open domain. Let $P$ be a subspace meeting $\Omega$, and let $x$ be a point of $\Omega - P$:

(i) There exists a shortest path $m$ from $x$ to $P \cap \Omega$ that is a line segment.

(ii) The set of shortest paths have end points in a connected compact subset $K$ of $P \cap \Omega$.

(iii) For any line $m'$ containing $m$ and $y \in m'$, the segment in $m'$ from $y$ to the point of $P \cap \Omega$ is one of the shortest segments.

(iv) When $P$ is a complete geodesic in $\Omega$, outside the compact set $K$, the distance function from $P - K$ to $x$ is strictly increasing or strictly decreasing.

**Proof.** The distance function $f : P \cap \Omega \to \mathbb{R}$ defined by $f(y) = d(x, y)$ is a proper function where $f(x) \to \infty$ as $x \to z$ for any boundary point $z$ of $P \cap \Omega$ in $P$. Hence, there exists a shortest segment with an endpoint $x_0$ in $P \cap \Omega$. (i) follows.

Let $\gamma$ be any geodesic in $P \cap \Omega$ passing $x_0$. We need to consider the 2-dimensional subspace $Q$ containing $\gamma$ and $x$. The set of end points of shortest segments of $\Omega$ in $Q$ is a connected compact subset containing $x_0$ by Proposition 1.4. of [25]. Hence, by considering all geodesics in $P \cap \Omega$ passing $x_0$, (ii) follows.

(iii) Suppose that there exists $y \in m'$, so that the the shorted geodesic $m''$ to $P \cap \Omega$ is not in $m'$. Consider the 2-dimensional subspace $Q$ containing $m'$ and $m''$. Then this is a contradiction by Corollary 1.5 of [25].

(iv) Again follows by considering a 2-dimensional subspace containing $P$ and $m$.

An endpoint in $P$ of a shortest segment is called a foot of the perpendicular from $x$ to $\gamma$. 

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5.2. Strict SPC-structures and the group actions

An elliptic element of $g$ is a nonidentity element of $\pi_1(\mathcal{O})$ fixing an interior point of $\tilde{O}$. Since $\pi_1(\mathcal{O})$ acts discretely on the space $\tilde{O}$ with a metric, an elliptic element has to be of finite order.

**Lemma 5.2.** Let $\mathcal{O}$ be a strongly tame strict SPC-orbifold with admissible ends. Let $\tilde{E}$ be a p-end of $\mathcal{O}$.

(i) Suppose that $\tilde{E}$ is a horospherical p-end. Let $B$ be a horoball at a p-end vertex $p$ corresponding to $\tilde{E}$. There exists a homeomorphism $\Phi_{\tilde{E}} : \partial \tilde{E} \to \partial B$ given by sending a point $x$ to the end point of maximal convex segment containing $x$ and $p$ in $\partial \mathcal{O}$.

(ii) Suppose that $\tilde{E}$ is a radial p-end of lens-type. Let $U$ be a lens-shaped radial p-end neighborhood with the p-end vertex $p$ corresponding to $\tilde{E}$. There exists a homeomorphism $\Phi_{\tilde{E}} : \partial U \cap \mathcal{O} \to \partial \mathcal{O}$ given by sending a point $x$ to the other end point of the maximal convex segment containing $x$ and $p$ in $\partial \mathcal{O}$.

Moreover, each of the maps denoted by $\Phi_{\tilde{E}}$ commutes with elements of $h(\pi_1(\tilde{E}))$.

**Proof.** (i) By Proposition 5.1 (i) of [27], $\Phi_B$ is well-defined. The same proposition implies that $\partial B$ is smooth at $p$ and $\partial \mathcal{O}$ has a unique supporting hyperplane. Therefore the map is onto.

(ii) The second item follows from Theorems 4.3 and 4.4 since they imply that the segments in $S(p)$ are maximal ones in $\partial \mathcal{O}$ from $p$.

We now study the fixed points in $\partial \mathcal{O}$ of elements of $\pi_1(\mathcal{O})$. A great segment is a geodesic arc in $S^n$ with antipodal p-end vertices. It is not properly convex.

**Lemma 5.3.** Let $\mathcal{O}$ be a strict SPC-orbifold with admissible ends. Let $g$ be an infinite order element of a p-end fundamental group. Then every fixed point of $g$ in $\partial \mathcal{O}$ is in the closure of a p-end-neighborhood.

**Proof.** Suppose that the radial p-end $\tilde{E}$ is lens-shaped. The direction of each segment in the interior of the lens cone with an endpoint $v_{\tilde{E}}$ is fixed by only the identity element of $\pi_1(\tilde{E})$ since $\pi_1(\tilde{E})$ acts properly discontinuously on $S_{\tilde{E}}$. Thus, the fixed points are on the rays in the direction of the boundary of $\tilde{E}$. They are in one of $S(v_{\tilde{E}})$ for the p-end vertex $v_{\tilde{E}}$ corresponding to $\tilde{E}$. Hence, the fixed points of the holonomy homomorphism of $\pi_1(\tilde{E})$ is in the closure of the lens-cone with end vertex $v_{\tilde{E}}$ and nowhere else in $\partial \mathcal{O}$.

If $\tilde{E}$ is a horospherical, then the p-end vertex $v_{\tilde{E}}$ is not contained in any segment $s$ in $\partial \mathcal{O}$ by Proposition 4.1. Hence $v_{\tilde{E}}$ is the only point $S \cap \partial \mathcal{O}$ of any invariant subset $S$ of $\pi_1(\tilde{E})$ by Lemma 5.2. Thus, the only fixed point of $\pi_1(E)$ in $\partial \mathcal{O}$ is $v_{\tilde{E}}$.

Suppose that $E$ is a totally geodesic p-end of lens-type, and a fixed point $s \in \partial \mathcal{O}$. Since $\tilde{E}$ is a properly convex real projective orbifold that is closed, we obtain an attracting fixed point $a$ and a repelling fixed point $r$ of $g|\partial \mathcal{O}$ by [9]. Then $a$ and $r$ are attracting and repelling fixed points of $g|\partial \mathcal{O}$ by the existence of the lens neighborhood and Theorem 7.9 in [27].
We claim that $\overline{rs}$ and $\overline{rs}$ are in $\partial \tilde{O}$. Let $P$ denote the two-dimensional subspace containing $r, s, a$. Suppose that the segment intersects $\tilde{O}$ in $x$. Then we take an open convex ball-neighborhood of $x$ in $P \cap \tilde{O}$. Suppose that $x \in \overline{rs}^\circ$. Then using the sequence $g^n(B)$, we obtain a great segment in $\text{Cl}(\tilde{O})$ by choosing $n \to \infty$ by Theorem 7.11 of [27]. This is a contradiction. If $x \in \overline{rs}^\circ$, we can use $g^{-n}(B)$ as $n \to \infty$, again giving us a contradiction.

Since $\tilde{E}$ has a lens type one-sided neighborhood $U$, $\text{Cl}(U) \cap \partial \tilde{O}$ is in $\text{Cl}(S_\tilde{E})$ by Proposition 4.18. By the strict convexity of $\tilde{O}$, we see that $a, r, s$ have to be in $\text{Cl}(S_\tilde{E})$.

See Crampon and Marquis [37] and Cooper-Long-Tillman [36] for similar work to the following.

**Proposition 5.4.** Suppose that $\mathcal{O}$ is a noncompact strongly tame strict SPC-orbifold with admissible ends. Then each nonidentity and infinite-order element $g$ of $\pi_1(\mathcal{O})$ has three mutually exclusive possibilities:

- $g|\text{Cl}(\tilde{O})$ has exactly two fixed points in $\partial \tilde{O}$ none of which is in the closures of the p-end neighborhoods,
- $g$ is in a p-end fundamental group, and $g|\text{Cl}(\tilde{O})$
  - has all fixed points in $\partial \tilde{O}$ in the closure of a concave p-end neighborhood of a lens-shaped radial p-end,
  - has all fixed points in $\partial \tilde{O}$ in $\text{Cl}(S_\tilde{E})$ for the ideal boundary component $S_\tilde{E}$ of a totally geodesic p-end $\tilde{E}$ of lens-type, or
  - has a unique fixed point in $\partial \tilde{O}$ at the horospherical p-end vertex.

**Proof.** Let $g \in \pi_1(\mathcal{O})$. Suppose that $g$ has a fixed point at a horospherical p-end vertex $v$ for a p-end $\tilde{E}$. We can choose the horoball $U$ at $v$ that maps into an end-neighborhood of $\mathcal{O}$. Since $g(U) \cap U \neq \emptyset$ by the geometry of a horoball having a smooth boundary at $v$, $g$ must act on the horoball since the horoball is either sent to a disjoint one or sent to the identical one, and hence $g$ is in the p-end fundamental group: A horoball $U$ has a unique hyperspace that also supports $\tilde{O}$. Thus, $g(U) \cap U \neq \emptyset$ for any horoball p-end neighborhood $U$. Thus, $g(U) = U$ for a horoball p-end neighborhood. Since $\partial U - \partial U$ is a unique point and $\partial U \subset \partial \tilde{O}$ where $g$ acts freely, the p-end vertex is the unique fixed point of $g$ in $\partial \tilde{O}$ by Lemma 5.3.

Similarly, suppose that $g \in \pi_1(\mathcal{O})$ fixes a point of the closure $\tilde{U}$ of a concave p-end neighborhood of a p-end vertex $v$ of lens-type. $g(\text{Cl}(U))$ and $\text{Cl}(U)$ meet at a point. By Corollary 4.23, $g(\text{Cl}(U))$ and $\text{Cl}(U)$ share the p-end vertex and hence $g(U) = U$ as $g$ is a deck transformation. Therefore, $g$ is in the p-end fundamental group of the p-end of $v$. Lemma 5.3 implies the result.

Suppose that $g \in \pi_1(\mathcal{O})$ fixes a point of $\text{Cl}(S_\tilde{E})$ for a totally geodesic ideal boundary $S_\tilde{E}$ corresponding to a p-end $\tilde{E}$. Again by Corollary 4.23 and Lemma 5.3 imply the result for this case.

Suppose that an element $g$ of $\pi_1(\mathcal{O})$ is not homotopic to any element of a p-end fundamental subgroup. Then by above, $g$ does not fix any of the above types of points. We show that $g$ has exactly two fixed points in $\partial \tilde{O}$.

Suppose that $g \in \pi_1(\mathcal{O})$ fixes a unique point $x$ in the closure of $\partial \tilde{O}$ and $x$ is not in the closure of p-end neighborhoods as above. Then $x$ is a $C^1$-point by the strict convexity. Suppose that we have two eigenvalues with largest absolute values $> 1$ and the smallest one with $< 1$. If the eigenvalue is not positive real, $\text{Cl}(\tilde{O})$
contains a nonproperly convex subset as we can see by an action of $g^n$ on a generic point of $\tilde{O}$. Thus, we obtain attracting and repelling subspaces easily with these and there are at least two fixed point. This is a contradiction. Therefore, $g$ has only eigenvalues of unit norms.

Take a line $l(t)$ converging to $x$ as $t \to 0$ where $l(t)$ is a projective function of $t \in \mathbb{R}$. Then a two-dimensional subset $P$ contains $l(t)$ and $g(l(t))$ for all $t$.

We may assume that $O$ is a precompact subset of an affine subspace $H$ and $x$ is the origin for some affine coordinate system of $H$. Find a sequence of dilatation $s_r$ fixing $x$ and acting on $H$, i.e., sending vectors $v$ to $rv$ for $r > 0$ in the vector space $H$ with the origin $O$ identified as $x$. We now use a projective coordinate function $x_1$ on $H$ so that $x_1(l(t)) = t$, $g$ acts on a hyperspace $L$ supporting $\text{Cl}(\tilde{O})$ at $x$.

We choose an affine coordinate on $H$ where $x$ is the origin and $L$ has value 0 for a coordinate function. Then $s_r$ acts on $P$, and two points $s_r(l(1/r))$ and $s_r, g(l(1/r)) = s_r, g s_r^{-1}(s_r(l(1/r)))$ is connected by line $l_r$. Since $s_r, g s_r^{-1}$ converges to a linear map in the compact open topology of $H$, the differential of $g$, with eigenvalues of norm 1 only, the slope of $l_r$ converges to 0 as $r \to \infty$. (One can use an easy estimation.)

Since $x$ is a $C^1$-point, $s_r(\tilde{O})$ converges to the half space $H$ as $r \to \infty$. Since $P \cap \text{Cl}(\tilde{O})$ is a convex subset, and $s_r(P \cap \text{bd}(\tilde{O}))$ geometrically converges to $H$, $s_r,l(1/r)$ and $s_r, g l(1/r)$ are converging to an interior of $H$, it follows that the Hilbert distance

$$d_{\tilde{O}}(l(1/r), g(l(1/r))) = d_{s_r(\tilde{O})}(s_r,l(1/r), s_r, g l(1/r)) \to 0 \text{ as } r \to \infty$$

by equation (2.2).

Then drawing a segment $s(t)$ between $l(t)$ and $g(l(t))$, we obtain a closed circle in $O$ in the homotopy class corresponding to $g$. That is, $s(t)$ maps closed curves $c(t_i)$. Since the Hilbert length of $c(t_i)$ as $i \to \infty$ goes to zero, and there is a uniform lower bound on the non-nullhomotopic closed curve lengths in the complement of the union of end neighborhoods, $c(t_i)$ has to be inside an end neighborhood of $O$ for sufficiently large $i$. In this case $g$ is in a p-end fundamental group, $x$ is in the closure of a p-end neighborhood. This is a contradiction.

We conclude that $g \in \pi_1(O)$ fixes at least two points $a$ and $r$ in $\text{bd}O$. We choose the two fixed points to have the positive real eigenvalues that are largest and smallest absolute values of the eigenvalues of $g$. (As above, the largest and smallest norm eigenvalues must be positive for $\tilde{O}$ to be properly convex.)

No fixed point of $g$ in $\text{bd}O$ is in the closures of p-end neighborhoods. By strict convexity, the interior of $\tilde{O}$ contains an open line segment $l$ connecting $a$ and $r$.

Let $S$ denote the subspace spanned by $a, r, t$. Suppose that there is a third fixed point $t$ in $\text{bd}O$. It is not in the closures of p-end neighborhoods as we assumed that $g$ is not in the p-end fundamental group. Then the line segment connecting it to the $a$ or $r$ must be in $\text{bd}O$: otherwise, we can form a segment $s$ in $\tilde{O} \cap S$ transversal to the segment. Then $\{g^k(s)\}$ geometrically converges to a segment in $\text{bd}O$ containing $a$ or $r$ as $k \to \infty$ or $k \to -\infty$ by the properness of the action. Thus, the existence of $t$ contradicts the strict SPC-property.

Hence, there are exactly two fixed points of $g$ in $\text{bd}O$ of the positive real eigenvalues that are largest and smallest absolute values of the eigenvalues of $g$. \qed
Proposition 5.5. Suppose that $O$ is a noncompact strongly tame strict SPC-orbifold with generalized admissible ends. Let $\tilde{E}$ be an end. Then for a $p$-end $E$, $(\text{bd} \tilde{O} - K)/\pi_1(\tilde{E})$ is compact where $K = \bigcup S(\tilde{E})$ for radial $p$-end $\tilde{E}$ of lens-type, $K = \text{Cl}(S_E)$ for totally geodesic $p$-end $\tilde{E}$, or $K = \{v_E\}$ for horospherical $p$-end $\tilde{E}$.

Proof. Suppose that $\tilde{E}$ is radial type of lens or horospherical type. By Lemma 5.2, the homeomorphism $\Phi_E : S_E \to \text{bd} \tilde{O} - K$ gives us the result.

Suppose that $\tilde{E}$ is a totally geodesic $p$-end of lens-type. Let $\tilde{O}^*$ denote the dual domain. Then there exists a dual radial $p$-end $\tilde{E}^*$ corresponding to $\tilde{E}$. Hence, $(\text{bd} \tilde{O}^* - K')/\pi_1(\tilde{E}^*)$ is compact for $K'$ equal to the closure of $p$-end neighborhoods of $\tilde{E}^*$ in the radial case or the vertex in the horospherical case.

Recall Proposition 2.12. Let $\text{bd}^{Ag} \tilde{O}$ be the augmented boundary with the fibration $\Pi_{Ag}$, and let $\text{bd}^{Ag} \tilde{O}^*$ be the augmented boundary with the fibration map $\Pi_{Ag}^*$. Let $K'' := \Pi_{Ag}^{-1}(K)$ and $K''' := \Pi_{Ag}^{-1}(K')$. There is a duality homeomorphism by Proposition 2.12

$$D : \text{bd}^{Ag} \tilde{O} - K'' \to \text{bd}^{Ag} \tilde{O}^* - K''' .$$

Now $(\text{bd}^{Ag} \tilde{O}^* - K''')/\pi_1(\tilde{E}^*)$ is compact since $\text{bd} \tilde{O}^* - K'$ has a compact fundamental domain and the space is the inverse image in $\text{bd}^{Ag} \tilde{O}^*$. By (iv) of Proposition 2.12, $(\text{bd}^{Ag} \tilde{O} - K'')/\pi_1(\tilde{E})$ is compact also. Since the image of this set under the map induced by a proper map $\Pi_{Ag}$ is $(\text{bd} \tilde{O} - K)/\pi_1(\tilde{E})$, it is is compact. \qed

We call the following construction shaving the ends.

Proposition 5.6. Given a noncompact strongly tame SPC-orbifold $O$ and its universal cover $\tilde{O}$, there exists a collection of mutually disjoint open concave $p$-end neighborhoods for $p$-ends of lens-type. We remove a finite union of concave end-neighborhoods of some radial ends. Then

- we obtain a convex domain as the universal cover of a strongly tame orbifold $O_1$ with additional strictly convex smooth boundary components that are closed $(n-1)$-dimensional orbifolds.
- Furthermore, if $O$ is strictly SPC with respect to all of its ends, and we remove only hyperbolic ends, then $O_1$ is strictly SPC with respect to the remaining ends.

Proof. If $O_1$ is not convex, then there is a triangle $T$ in $\tilde{O}_1$ with three segments $s_0, s_1, s_2$ so that $T - s_0^0 \subset \tilde{O}_1$ but $s_0^0 - \tilde{O}_1 \neq \emptyset$. (See Theorem 2.2 of [21] for details.) Since $\tilde{O}$ is an open manifold, $s_0^0 - \tilde{O}$ is a closed subset of $s_0^0$. Then a boundary point of $x \in s_0^0 - \tilde{O}_1$ is in the boundary of one of the removed concave-open neighborhoods or is in $\text{bd} \tilde{O}$ itself. The second possibility implies that $O$ is not convex as $\tilde{O}_1 \subset \tilde{O}$. These are contradictions. The first possibility implies that there exists a segment in the interior of a concave $p$-end neighborhood $U$ with endpoints in $\text{bd} U \cap \tilde{O}$. This is geometrically not possible. Also, since $\tilde{O}_1 \subset \tilde{O}$, we have the convexity. Since $\tilde{O}$ is properly convex, so is $\tilde{O}_1$.

Now we go to the second part. We suppose that $O$ is strictly SPC. Let $H$ denote the set of $p$-end vertices with hyperbolic $p$-end fundamental groups that were removed in the equivariant manner. For each $p \in H$, denote by $U_p$ the concave neighborhood removed.
Any segment in the boundary of the developing image of \( \mathcal{O} \) is a subset of the closure of a p-end neighborhood of a p-end vertex. For the p-end-vertex \( p \) of a p-end \( \tilde{E} \), the domain \( R_p(\mathcal{O}) \subset S^{n-1}_p \) is strictly convex if \( \pi_1(\tilde{E}) \) is hyperbolic. Since \( \partial R_p(\mathcal{O}) \) contains no straight segment by hyperbolicity in [8], only straight segments in \( \Cl(U) \cap \partial \tilde{O} \) for the concave p-end neighborhood \( U \) of \( \tilde{E} \) are in the segment in \( S(p) \). Thus, their interiors are disjoint from \( \partial \tilde{O} \), and hence \( \partial \tilde{O} \) contains no geodesic segment in \( \bigcup_{p \in \mathcal{H}} \Cl(U_p) \cap \partial \tilde{O} \).

Since we removed concave end neighborhoods of the lens-type ends with the hyperbolic end fundamental groups, any straight segment in \( \partial \tilde{O} \) lies in the closure of a p-end neighborhood of a remaining p-end vertex.

A non-\( C^1 \)-point of \( \partial \tilde{O} \) is not on the boundary of the concave p-end neighborhood \( U \) for a hyperbolic p-end \( \tilde{E} \) nor in \( \partial \tilde{O} - \Cl(U) \). However, \( \Cl(U) \cap \partial \tilde{O} \) contains the limit set \( \Lambda = L - \partial L \) for the lens part \( L \) in a lens-neighborhood. \( \tilde{O} \) has the same set of supporting hyperplanes as \( L \) at points of \( \Lambda \) since they are both \( \pi_1(\tilde{E}) \)-invariant convex domains by Corollary 4.16. However, the supporting hyperplanes at \( \Lambda \) of \( L \) are also supporting ones for \( \tilde{O} \) by Corollary 4.16 since we removed the outside component \( U \) of \( \tilde{O} - L \). Thus, \( \tilde{O} \) is \( C^1 \) at points of \( \Lambda \). Since these are true for all removed concave p-end neighborhood \( U \), \( \tilde{O} \) is strictly SPC. 

\[ \square \]

5.2.1. Bowditch’s method. There are results proved by Cooper, Long, and Tillman [36] and Crampon and Marquis [37] similar to below. However, the ends have to be horospherical in their work. We will use Bowditch’s result [15] to show

**Theorem 5.7.** Let \( \mathcal{O} \) be a noncompact strongly tame strict SPC-orbifold with generalized admissible ends \( E_1, \ldots, E_k \) and satisfies (IE) and (NA). Assume \( \partial \mathcal{O} = \emptyset \). Let \( \tilde{U}_i \) be the inverse image \( U_i \) in \( \mathcal{O} \) for a mutually disjoint collection of neighborhoods \( U_i \) of the ends \( E_i \) for each \( i = 1, \ldots, k \). Then

- \( \pi_1(\mathcal{O}) \) is relatively hyperbolic with respect to the end fundamental groups \( \pi_1(E_1), \ldots, \pi_1(E_k) \).

Hence \( \mathcal{O} \) is relatively hyperbolic with respect to \( U_1 \cup \cdots \cup U_k \).

- If \( \pi_1(E_{l+1}), \ldots, \pi_1(E_k) \) are hyperbolic for some \( 1 \leq l \leq k \) (possibly some of the hyperbolic ones), then \( \pi_1(\mathcal{O}) \) is relatively hyperbolic with respect to the end fundamental group \( \pi_1(E_1), \ldots, \pi_1(E_l) \).

**Proof.** We show that \( \pi_1(\mathcal{O}) \) is relatively hyperbolic with respect to the end fundamental groups \( \pi_1(E_1), \ldots, \pi_1(E_k) \). By Proposition 5.6, we have the second statement.

Choose a collection of \( \pi_1(\mathcal{O}) \)-invariant concave p-end neighborhoods of \( \tilde{O} \) whose union covers \( \bigcup_{i=l+1}^k U_i \) for concave end neighborhoods \( U_{l+1}, \ldots, U_k \). We use \( \tilde{O} \) and remove the union of a collection of \( \pi_1(\mathcal{O}) \)-invariant concave p-end neighborhoods of \( \tilde{O} \) by Proposition 5.6. Now, \( \tilde{O}_1 \) covers a strict SPC-orbifold \( O_1 \) with admissible end.

Thus, we obtain \( \partial \tilde{O}_1 \).

- We now collapse each set of form \( \Cl(U_i) \cap \partial \tilde{O}_1 = S_{\tilde{E}} \) for a concave p-end neighborhood \( U_i \) to a point and
- collapse \( \Cl(S_{\tilde{E}}) \) for each totally geodesic end \( \tilde{E} \) of lens-type to a point.
By Corollary 4.23, these sets are mutually disjoint balls. Let $C_B$ denote the collection, and let $C_B := \bigcup C_B$.

We claim that for each closed set $J$ in $\partial \mathcal{O}_1$, the union of $C_J$ of elements of $C_B$ meeting $J$ is also closed: Let us choose a sequence $\{x_i\}$ for $x_i \in C_i$, $C_i \cap J \neq \emptyset$, $C_i \in C_B$. Suppose that $x_i \to x$. Let $y_i \in C_i \cap J$. Let $v_i$ be the end vertex of $C_i$ if it is radial. Then define $s_i := \overline{x_i v_i} \cup \overline{v_i y_i} \subset C_i$ if $C_i$ is radial or else $s_i := \overline{x_i y_i} \subset C_i$. Choose a subsequence so that $\{s_i\}$ geometrically converges to a limit containing $x$. The limit $s_\infty$ is a singleton, a segment or a union of two segments. By the strict convexity of $\mathcal{O}$, we obtain $s_\infty$ is a subset of an element of $C_B$ and $s_\infty$ meets $J$. Thus, $x \in s_\infty \subset C_i$ for $C_i \cap J$, and $C_J$ is closed.

We denote this quotient space $\partial \mathcal{O}_1 / \sim$ by $B$. By Proposition B.1, $B$ is a metrizable space.

We show that $\pi_1(O)$ acts on the metrizable space $B$ as a geometrically finite convergence group. By Theorem 0.1 of Yaman [85] following Bowditch [15], this shows that $\pi_1(O)$ is relatively hyperbolic with respect to $\pi_1(E_1), \ldots, \pi_1(E_k)$. The definition of conical limit points and so on are from the article.

(I) We first show that the group acts properly discontinuously on the set of triples in $B = \partial \mathcal{O}_1 / \sim$. Suppose not. Then there exists a sequence of nondegenerate triples $\{(p_i, q_i, r_i)\}$ of points in $\partial \mathcal{O}_1$ converging to a distinct triple $\{(p, q, r)\}$ so that $p_i = \gamma_i(p_0), q_i = \gamma_i(q_0), \text{ and } r_i = \gamma_i(r_0)$ where $\gamma_i$ is a sequence of mutually distinct elements of $\pi_1(O)$. We assume here that $p_0, q_0, r_0$ are representatives of distinct points of $B$ and so are $p, q, r$. By multiplying by some uniformly bounded element $R_i$ in $\mathrm{PGL}(n + 1, \mathbb{R})$, we obtain that $R_i \circ \gamma_i$ for each $i$ fixes $p_0, q_0, r_0$ and restricts to a diagonal matrix with entries $\lambda_i, \delta_i, \mu_i$ on the plane with coordinates so that $p_0 = e_1, q_0 = e_2, r_0 = e_3$.

Then we can assume that

$$\lambda_i \delta_i \mu_i = 1, \lambda_i \geq \delta_i \geq \mu_i > 0$$

by restricting to the plane and up to choosing subsequences and renaming. Thus $\lambda_i \to \infty$ and $\mu_i \to 0$ since otherwise these two sequences are bounded and this contradicts the discreteness of the holonomy homomorphism.

Let $P_0$ denote the 2-dimensional subspace containing $p_0, q_0, r_0$. By strictness of convexity, as we collapsed each of the p-end balls, the interiors of the segments $\overline{p_0 q_0}, \overline{q_0 r_0}, \text{ and } \overline{r_0 p_0}$ are in the interior of $\mathcal{O}_1$.

We claim that one of the sequence $\lambda_i / \delta_i$ or the sequence $\delta_i / \mu_i$ are bounded: Suppose not. Then $\lambda_i / \delta_i \to \infty$ and $\delta_i / \mu_i \to \infty$. We choose generic segments $s_0$ and $t_0$ in $\mathcal{O}_1$ with a common end point $q_0$ and the respective other end point $\hat{s}_0$ and $\hat{t}_0$ so that

$$d((s_0, q_0), (t_0, q_0)) \geq \delta \text{ for a uniform } \delta > 0.$$  

We choose $s_0$ and $t_0$ so that their directions from $q_0$ differ from that of $\overline{p_0 q_0}$ and $\overline{q_0 r_0}$ at least by $\delta' > 0$. Then the sequence $R_i \circ \gamma_i(s_0 \cup t_0)$ converges to the segment with end point $p_0$ passing $q_0$ in the middle geometrically. The segment is a great segment. (See Section 2.4.) Since $R_i$ is bounded, this implies that there exists such a segment in $\mathrm{Cl}(\mathcal{O}_1)$. This is a contradiction to the proper convexity of $\mathcal{O}_1$.

Suppose now that the sequence $\lambda_i / \delta_i$ is bounded: Now the sequence of segments $\overline{p_0 q_0}$ converges to $\overline{p q}$ whose interior is in $\mathcal{O}_1$. Then we see that $\overline{p q}$ must be in the boundary as these points must be the limit points of points of sequence of $\gamma_i(s)$ for some compact subsegments $s \subset \overline{p_0 q_0}$ by the boundedness of the above ratio and
the proper-discontinuity of the action. This contradicts the strict convexity as we assumed that \( p, q, \) and \( r \) represent distinct points in \( B \). If we assume that \( \delta_i/\mu_i \) is bounded, then we obtain a contradiction similarly.

This proves the proper discontinuity of the action on the space of distinct triples.

(II) By Propositions 5.4 and 5.5, each group of form \( \Gamma_x \) for a point \( x \) of \( B = \tilde{\mathcal{O}}_1/\sim \) is a bounded parabolic subgroup in the sense of Bowditch [85].

(III) Let \( p \in \text{bd}\tilde{\mathcal{O}} \) be a point that is not in a horospherical endpoint or a singleton corresponding an lens-shaped \( p \)-end of radial or totally geodesic type of \( B \). We show that \( p \) is a conical limit point. This will complete our proof by Theorem 0.1 of [85].

We find a sequence of holonomy transformations \( \gamma_i \) and distinct points \( a, b \in \partial B \) so that \( \gamma_i(p) \to a \) and \( \gamma_i(q) \to b \) locally uniformly for \( q \in \partial B \setminus \{p\} \). To do this, we draw a line \( l(t) \) in \( \tilde{\mathcal{O}} \) from a point of the fundamental domain to \( p \) where as \( t \to \infty \), \( l(t) \to p \) in the compactification. Since \( l(t) \) is not eventually in a \( p \)-end neighborhood, there is a sequence \( \{t_i\} \) going to \( \infty \) so that \( l(t_i) \) is not in any of the \( p \)-end neighborhoods in \( \tilde{U}_1 \cup \cdots \cup \tilde{U}_k \). Let \( p' \) be the other endpoint of the complete extension of \( l(t) \) in \( \tilde{\mathcal{O}} \). We can assume without generality that \( p' \) is not in the closure of any \( p \)-end neighborhood by choosing the line \( l(t) \) differently if necessary.

Since \( (\tilde{\mathcal{O}}_1 - \tilde{U}_1 - \cdots - \tilde{U}_k)/\Gamma \) is compact, we have a compact fundamental domain \( F \) of \( \tilde{\mathcal{O}}_1 - \tilde{U}_1 - \cdots - \tilde{U}_k \) with respect to \( \Gamma \). Note that \( d(F, \text{bd}\tilde{\mathcal{O}}) > C_0 \) for some constant \( C_0 > 0 \). (We remark that this is not true for \( \text{bd}\tilde{\mathcal{O}}_1 \).)

Now we will look at the convex open domain \( \mathcal{O} \) and use the Hilbert metric \( d_\mathcal{O} \). We find points \( z_i \in F \) so that \( \gamma_i(l(t_i)) = z_i \) for a deck transformation \( \gamma_i \). Then by taking subsequences, we assume without loss of generality that \( \gamma_i^{-1}(p) \to a \) for a point \( a \) and \( \gamma_i^{-1}(p') \) also converges to another point \( b, b \neq a \). Take a point \( q \in X - \{p, p'\} \) and find a geodesic \( m \) from \( q \) to \( l \) with the property that every point on \( m \) a shortest geodesic from the point to \( l \) lies in \( m \) by Proposition 5.1. Let \( q' \) be the intersection of \( m \) to \( l \). Then \( \gamma_i^{-1}(q') \) converges to \( b \) by a Hilbert metric consideration.

The sequence of segments \( \gamma_i^{-1}(q'q) \) has the property that every point on it is a shortest geodesic from the point to \( \gamma_i^{-1}(l) \) lies in \( \gamma_i^{-1}(qq') \). (See Figure 5.1.) Because of this property, for any sequence of points \( x_i \in \gamma_i^{-1}(qq') \), we have that the shortest geodesic from \( x_i \) to \( l_i := \gamma_i^{-1}(l) \) lie in \( \gamma_i^{-1}(qq') \).

We show that the sequence \( \gamma_i^{-1}(qq') \) is exiting; that is, for every compact subset \( K \) of \( \tilde{\mathcal{O}}_1 \), there exists \( I_1 \) such that \( \gamma_i^{-1}(qq') \cap K = \emptyset \) for \( i > I_1 \). Suppose not. Then we can choose \( x_i \in \gamma_i^{-1}(qq') \) and \( x_i \to x \in \mathcal{O} \) so that the corresponding sequence of \( d_\mathcal{O} \)-distances in \( \gamma_i^{-1}(qq') \) is converging to \( \infty \). However, as \( x_i \to x \in \mathcal{O} \), the sequence \( d_\mathcal{O}(x_i, l_i) \) is bounded since \( l_i \) passes \( F \) for all \( i \). Since \( d_\mathcal{O}(x_i, l_i) \) is the arclength from \( x_i \) to the point of \( l_i \) in \( \gamma_i^{-1}(qq') \), we have \( d_\mathcal{O}(x_i, l_i) \to \infty \). This is contradiction. (By choosing a continuous parameters of shortest geodesics from a small neighborhood of \( q \), we obtain a local uniform convergence.)

By choosing a subsequence, the sequence \( \{\gamma_i^{-1}(qq')\} \) converges to a point or a segment in the boundary \( \text{bd}\tilde{\mathcal{O}}_1 \). If the limit is a segment, then it is contained in one of the collapsed subsets containing \( b \) since \( \mathcal{O} \) is strictly convex relative to \( p \)-ends. This shows that \( p \) is a conical limit point.

\( \square \)
5.2.2. The Theorem of Drutu. The author obtained a proof of the following theorem from Drutu. See [38] for more details.

**Theorem 5.8 (Drutu).** Let $\mathcal{O}$ be a noncompact strongly tame orbifold with admissible and satisfies (IE) and (NA). Let $\pi_1(E_1), ..., \pi_1(E_m)$ be end fundamental groups where $\pi_1(E_{n+1}), ..., \pi_1(E_m)$ for $n \leq m$ are hyperbolic groups. Then $\pi_1(\mathcal{O})$ is a relatively hyperbolic group with respect to $\pi_1(E_1), ..., \pi_1(E_m)$ if and only if $\pi_1(\mathcal{O})$ is one with respect to $\pi_1(E_1), ..., \pi_1(E_n)$.

**Proof.** With the terminology in the paper [38], $\pi_1(\mathcal{O})$ is a relatively hyperbolic group with respect to the admissible end fundamental groups $\pi_1(E_1), ..., \pi_1(E_m)$ if and only if $\pi_1(\mathcal{O})$ with a word metric is asymptotically tree graded (ATG) with respect to all the left cosets $g\pi_1(E_i)$ for $g \in \pi_1(\mathcal{O})$ and $i = 1, \ldots, m$.

We claimed that $\pi_1(\mathcal{O})$ with a word metric is asymptotically tree graded (ATG) with respect to all the left cosets $g\pi_1(E_i)$ for $g \in \pi_1(\mathcal{O})$ and $i = 1, \ldots, m$ if and only if $\pi_1(\mathcal{O})$ with a word metric is asymptotically tree graded with respect to all the left cosets $g\pi_1(E_i)$ for $g \in \pi_1(\mathcal{O})$ and $i = 1, \ldots, n$.

Conditions $(\alpha_1)$ and $(\alpha_2)$ of Theorem 4.9 in [38] are satisfied still when we drop end fundamental groups $\pi_1(E_{n+1}), ..., \pi_1(E_m)$ or add them. (See also Theorem 4.22 in [38].)

For the condition $(\alpha_3)$ of Theorem 4.9 of [38], it is sufficient to consider only hexagons. According to Proposition 4.24 of [38] one can take the fatness constants as large as one wants, in particular $\theta$ (measuring how fat the hexagon is) much larger than $\chi$ prescribing how close the fat hexagon is from a left coset.

If $\theta$ is very large, left cosets containing such hexagons in their neighborhoods can never be cosets of hyperbolic subgroups since hyperbolic groups do not contain fat hexagons. So the condition $(\alpha_3)$ is satisfied too whether one adds $\pi_1(E_{n+1}), ..., \pi_1(E_m)$ or drop them.

\[\square\]
5.2.3. Converse. We will prove the converse to Theorem 5.7:

For each totally geodesic end of lens-type, we add the totally geodesic ideal boundary component and outside open lens-neighborhood to form \( \mathcal{O}' \) by Theorem 4.13. We need this for a technical reasons.

Note here that we do not assume the “full” admissibility of the ends. The group property will imply that the ends have to be admissible as well.

**Theorem 5.9.** Let \( \mathcal{O} \) be a noncompact strongly tame properly convex real projective orbifold with generalized admissible ends and satisfies (IE) and (NA). Assume \( \partial \mathcal{O} = \emptyset \). Suppose that \( \pi_1(\mathcal{O}) \) is a relatively hyperbolic group with respect to the admissible end groups \( \pi_1(E_1), \ldots, \pi_1(E_k) \) where \( E_i \) are horospherical for \( i = 1, \ldots, m \) and generalized lens-shaped for \( i = m + 1, \ldots, k \) for \( 0 \leq m \leq k \). Then \( \mathcal{O} \) is strictly SPC with respect to the admissible ends \( E_1, \ldots, E_k \).

**Proof.** Since an \( \epsilon \)-mc-p-end-neighborhood is always proper by Corollary 4.22, for any \( i \), we can choose the end neighborhood \( U_i \) of any generalized lens-type end \( \tilde{E}_i \) to be the image of an \( \epsilon \)-mc-p-end-neighborhood. We can choose all such neighborhoods to be mutually disjoint by Corollary 4.22. Let \( \tilde{U} \) denote the union of the inverse images of end neighborhoods \( U_1, \ldots, U_k \).

Suppose that \( \mathcal{O} \) is not strictly convex. We divide into two cases: First, we assume that there exists a segment in \( \partial \mathcal{O} \) not contained in the closure of a p-end neighborhood. Second, we assume that there exists a non-C\(^1\)-point in \( \partial \mathcal{O} \) not contained in the closure of a p-end neighborhood.

(1) We assume the first case now. We will obtain a triangle with boundary in \( \partial \mathcal{O} \) and not contained in the convex hull of p-ends: Let \( l \) be a nontrivial maximal segment in \( \partial \mathcal{O} \) not contained in the closure of a p-end neighborhood. First, \( l \) does not meet the closure of a horospherical p-end neighborhood by Proposition 4.1. By Theorems 4.3 and 4.4 if \( l^o \) meets the closure of a lens-shaped p-end neighborhood, then \( l^o \) is in the closure. Also, suppose that \( l^o \) meets \( S_{\tilde{E}} \) for a totally geodesic p-end \( \tilde{E} \). Then \( l^o \cap \partial S_{\tilde{E}} \neq \emptyset \). Then applying \( \pi_1(\tilde{E}) \), we obtain a great segment in \( \partial \mathcal{O} \) since \( S_{\tilde{E}}/\pi_1(\tilde{E}) \) is a compact properly convex orbifold. (See [8]!) Therefore, \( l \) meets the closures of p-end neighborhoods possibly only at its endpoints.

Let \( P \) be a 2-dimensional subspace containing \( l \) and meeting \( \partial \mathcal{O} \) outside \( \tilde{U} \). By above, \( l^o \) is in the boundary of \( P \cap \partial \mathcal{O} \). Draw two segments \( s_1 \) and \( s_2 \) in \( P \cap \partial \mathcal{O} \) from the end point of \( l \) meeting at a vertex \( v \) in the interior of \( \partial \mathcal{O} \). Since \( l^o \) is not contained in the closure of a single component of \( \tilde{U} \), \( (\partial \mathcal{O} - \tilde{U}) \cap P \) has a sequence of points \( x_i \) converging to a point \( x \) of \( l^o \). Then \( d(x_i, s_1 \cup s_2) \to \infty \) by consideration in the Hilbert metric by looking at all straight segment from \( x_i \) to a point of \( s_1 \) or \( s_2 \) and the maximality of \( l \) in \( \partial \mathcal{O} \).

Recall that there is a compact fundamental domain \( F \) of \( \partial \mathcal{O} - \tilde{U} \) under the action of \( \pi_1(E) \). Now, we can take \( x_i \) to the fundamental domain \( F \) by \( x_i \). We choose \( g_i \) to be a sequence of mutually distinct elements of \( \pi_1(\mathcal{O}) \). We choose a subsequence so that we assume without loss of generality that \( \{ g_i(T) \} \) geometrically converges to a convex set, which could be a point or a segment or a nondegenerate triangle. Since \( g_i(T) \cap F \neq \emptyset \), and the sequence \( \partial g_i(T) \) exists any compact subsets of \( \mathcal{O} \) always while \( d(\partial g_i(T), \partial \mathcal{O}) \to \infty \) and \( g_i(T) \) passes \( F \), we see that a subsequence of \( g_i(T) \) converges to a nondegenerate triangle, say \( T_\infty \).

By following Lemma 5.10, \( T_\infty \) is so that \( \partial T_\infty \) is in \( \bigcup S(v_{\tilde{E}}) \) for a radial generalized lens-type p-end \( \tilde{E} \).
Now, $T_\infty$ is so that $\partial T_\infty \subset \text{Cl}(U_1)$ for a p-end neighborhood $U_1$ of a generalized lens-type end $\tilde{E}$. Then for sufficiently small $\epsilon > 0$, the $\epsilon$-d-$\tilde{O}$-neighborhood of $T_\infty \cap \tilde{O}$ is a subset of $U_1$ as $U_1$ was chosen to be an $\epsilon$-mc-p-end-neighborhood (see Lemma 4.21). However as $g_i(T) \to T_\infty$ geometrically, for any compact subset $K$ of $\tilde{O}$, $g_i(T) \cap K$ is a subset of $U_1$ for sufficiently large $i$. But $g_i(T) \cap F \neq \emptyset$ for all $i$ and the compact fundamental domain $F$ of $\tilde{O} - \tilde{U}$, disjoint from $U_1$. This is a contradiction.

(II) Now we suppose that $\partial \tilde{O}$ has a non-$C^1$-point $x$ outside the closures of p-end neighborhoods. Then we go to the dual $O^*$ and the dual group $\Gamma^*$ where $O^*/\Gamma^*$ is a strongly tame properly convex orbifold with horospherical ends, totally geodesic ends of lens-type or radial ends of generalized lens-type by Theorem 4.12 and Lemma 4.17.

Then we have a one-to-one correspondence of the set of p-ends of $\tilde{O}$ to the set of p-ends of $O^*$, and we obtain that $x$ corresponds to a convex subset of dim $\geq 1$ in $\partial \tilde{O}$ containing a segment $l$ not contained in the closure of p-end neighborhoods using the map $\mathcal{D}$ in Proposition 2.12. Thus, the proof reduces to the case (I).

(III) Finally, we show that the ends are admissible. Given a totally geodesic p-end $\tilde{E}$, it is of lens-type by assumption. A horospherical p-end $\tilde{E}$ is admissible.

We will now study a generalized lens-type radial end $\tilde{E}$ and show that it is of lens-type as well: Given a radial end $\tilde{E}$, $h(\pi_1(\tilde{E}))$ satisfies the uniform middle eigenvalue condition by generalized lens-type condition on the holonomy of the end by the assumption and Theorem 7.9 of [27]. Since the concave-end neighborhood exists by Corollary 8.7 in [27], we know that $\bigcup S(v_{\tilde{E}}) \subset \partial \tilde{O}$. We obtain the compact convex hull $CH(\bigcup S(v_{\tilde{E}})) \subset \text{Cl}(\tilde{O})$.

If $I := \partial CH(\bigcup S(v_{\tilde{E}})) - \bigcup S(v_{\tilde{E}})$ is a subset of $\tilde{O}$, then Proposition 7.6 of [27] shows that if the lens neighborhood of $I$ is in $\tilde{O}$, then we are done.

Suppose not. Then there exists an $i$-dimensional simplex $\sigma$, $\sigma^o \subset I$ for $i \geq 1$ so that $\partial \tilde{O} \cap \sigma^o \neq \emptyset$. Then it must be that $\sigma \subset \partial \tilde{O}$ by Lemma 7.5 of [27]. Hence, there exists a segment $k$ in $I \cap \partial \tilde{O}$ and in $\sigma$ with end points in $\bigcup S(v_{\tilde{E}})$. The two segments $s_1, s_2 \in S(v_{\tilde{E}})$ with endpoints equal to the endpoints of $k$, we obtain a triangle $T$ with $\partial T \subset \partial \tilde{O}$ by the convexity of $\text{Cl}(\tilde{O})$.

By Lemma 5.10, $\partial T \subset \bigcup S(v_{\tilde{E}})$. However, by geometry, this implies $k, T \subset \bigcup S(v_{\tilde{E}})$. This contradicts $I$ being disjoint from $\bigcup S(v_{\tilde{E}})$.

By Theorem 4.26, we obtain that our orbifold is strictly SPC. \hfill $\square$

Now, we come to a lemma with a very long proof.

**Lemma 5.10.** Assume the premise of Theorem 5.9. Then for every triangle $T$ in $\tilde{O}$ with $\partial T \subset \partial \tilde{O}$, we obtain $\partial T \subset \bigcup S(\tilde{E})$ for a p-end $\tilde{E}$ of radial type.

**Proof.** We expand $O$ to $O^*$ by adding lens neighborhoods to totally geodesic ideal boundary components by Theorem 4.13. We will use the Hilbert metric of $O^*$ restricted to $O_1$. From now on, we will denote by $O$ the extended orbifold and $\tilde{O}$ will denote $O^*$ also. Now $S_\tilde{E}$ for totally geodesic p-end $\tilde{E}$ is in $\tilde{O}$.

Suppose that some of the ends $E_i$, $i = 1, \ldots, k$, are hyperbolic. Then remove the concave end neighborhoods for these ends to obtain $O_1$. The universal cover $\tilde{O}_1$ is an open domain in $\tilde{O}$. We have that $\pi_1(O)$ is still relatively hyperbolic with respect
to the rest of the end fundamental groups by Theorem 5.8. Let $E_i, i = 1, \ldots, m$ denote the remaining ends, nonhyperbolic ones.

Let $T'$ be a triangle with $\partial T' \subset \text{bd} \tilde{O}$ and $\partial T'$ is not a subset of $\bigcup S(v_{\tilde{E}})$ of a radial type end or $\text{Cl}(\bigcup S_E)$ of a totally geodesic end $\tilde{E}$ of lens-type. Clearly $T'$ is in $\text{Cl}(\tilde{O}_1)$ and $\partial T' \subset \text{bd} \tilde{O}_1$ since if the interior of a segment meets $\bigcup S(v_{\tilde{E}})$, then the segment must be in $\bigcup S(v_{\tilde{E}})$ by Theorems 4.3 and 4.4.

Let $U$ be the union of all concave $p$-end neighborhoods for radial $p$-ends and lens $p$-end neighborhoods for totally geodesic $p$-ends and horospherical $p$-ends mutually disjoint from one another and covering a union of disjoint end neighborhoods of $\tilde{O}$.

In this case, $S_E$ for the totally geodesic end $\tilde{E}$ is a subset of $\tilde{O}$.

Now we will consider various possibilities for the triangle: By assumption, one of the component $T' - \tilde{U}'$ is not compact. (Otherwise, we are done.) Denote by $L$ a noncompact component of $T' - \tilde{U}'$.

(I) Suppose that there is at least one $g \neq 1$ so that $g(L) = L$. Then clearly $g(T') = T'$ as well. Let $v$ be a vertex of $T'$. Then $L/(g)$ corresponds to an annulus mapping into $\tilde{O}_1$. Let $l$ be a maximal geodesic in $L_o$ so that $l$ and $g(l)$ bound a fundamental domain of the annulus. Then notice that $T_o$ contains a geodesic $\tilde{\alpha}_0$ connecting a point $v$ of $l$ to $g(v$ of $g(l)$ mapping to a closed curve $\alpha_0$. By a similarity based at $v$, we form a parameter of closed curves $\alpha_t$ for $v(t) \in l$ and $t \in \mathbb{R}$. The vertex of $\alpha_t$ is the image of $v(t)$.

Then the $d_3$-lengths of $\alpha_t$ are uniformly bounded above since $g$ acts on a triangle with a diagonalizable matrix and we can compute the $d_3$-lengths of $\alpha_t$ by its vertex $v(t)$. Assume that $\alpha_t$ is periodic with fundamental interval $I \subset \mathbb{R}$ always.

Either

(i) $\alpha_t(I) \subset U$ for an end neighborhood $U$ of $\tilde{E}$ and $t \geq c$ and $t \leq c'$ for some $c$ and $c'$.

(ii) $\alpha_t(I) \subset U$ for an end neighborhood $U$ for some $t$ and $\alpha_t(I) \not\subset U$ for $t \geq c'$ or $t \leq c''$ for some $c', c''$.

(iii) $\alpha_t(I) \not\subset U$ for an end neighborhood $U$ for all $t$.

In the first case, $T'$ must be in a $p$-end neighborhood $U'$ of $\tilde{E}$ so that $U' - U$ covers a compact set in $\mathcal{O}$. Since $\partial T' \subset \text{bd} \tilde{O}$, and $\text{Cl}(U') \cap \text{bd} \tilde{O} = \bigcup S(\tilde{E})$, we obtain $\partial T' \subset \bigcup S(\tilde{E})$. We are finished in this case.

In the second case, $g$ is freely homotopic to the end fundamental group. We can assume that a closed curve freely homotopic to $\alpha_t(I)$ cannot be a subset of another $p$-end neighborhood since otherwise we obtain an essential annulus.

Using the residual finiteness of the linear group $\pi_1(\tilde{O})$, we take a finite index subgroup of $\pi_1(\mathcal{O})$ and a power of $g$ so that we can assume that $g$ is a generator of $\langle g \rangle$. Also, assume that $\pi_1(\mathcal{O})$ is torsion-free by taking a finite cover using the Selberg lemma.

Assume that $\alpha_t$ is not in an end neighborhood entirely for $t > c$. In this case, there is a subsequence $t_i$ so that $\alpha_{t_i}$ converges to a closed curve not contained in any end neighborhood. We assume that $\tilde{\alpha}_{t_0}$ is a subset of an end neighborhood. We can assume that $\bigcup \alpha_{t_i}$ has a convex hull containing $L$ since the sequence is of bounded diameter ones bounding a region with $\tilde{\alpha}_{t_0}$ covering an annuli and hence eventually containing any compact subset of $L$.

We see that $\alpha_{t_i}$ and $\alpha_{t_j}$ are homotopic for $i, j, i > j$ sufficiently large. Let $\tilde{\alpha}_{t_i} \colon \mathbb{R} \to \tilde{O}$ denote the lift of $\alpha_{t_i}$ in $T_o$ where $g$ acts on. The $d_3$-length of $\alpha_{t_i}$ is
uniformly bounded above since in $T^o$ the $d_{\tilde{\Theta}}$-lengths are the $d_{T^o}$-lengths and can be estimated by the action on the projective link of the vertex of $T$.

- the minimum distance $d_{\tilde{\Theta}}(\hat{\alpha}_{t_i}, \hat{\alpha}_{t_j}) \geq 2M + 1$ for infinitely many pairs $i, j$.
- Hence, for every $\epsilon, 0 < \epsilon < 1/2$, there exists infinitely many pair $i, j$ and a deck transformation $h_{i,j}$ for each $i, j$ so that for every $s \in \mathbb{R}$, there exists $s' \in \mathbb{R}$ so that $d_{\tilde{\Theta}}(h_{i,j}(\hat{\alpha}_{t_i}(s')), \hat{\alpha}_{t_j}(s)) \leq \epsilon$, and conversely for every $s' \in \mathbb{R}$, there exists $s \in \mathbb{R}$ satisfying this equation.

Since they have bounded lengths, $[\alpha_{t_i}]$ and $[\alpha_{t_j}]$ have the same homotopy class for infinitely many pairs $i, j$. Thus, $g$ and $h_{i,j}$ commute with each other. Since $h_{i,j}$ sends $\hat{\alpha}_{t_i}$ to points of minimal distance $\geq 2M$, it follows that $h_{i,j}$ is not in $(g)$. By (NA), $h_{i,j}, g \in \pi_1(\tilde{E})$ for a p-end $\tilde{E}$.

We take a sufficiently large lens or lens-cone p-end neighborhood $D$ of $\tilde{E}$ by Lemma 4.8. Since $\alpha_{t_i}$ converges to a closed curve as $i \to \infty$, we may assume that $h_{i,j}(\hat{\alpha}_{t_i}) \subset D$ for $i > I_0$ where we assume $I_0 > i_0$. $h_{i,j} \in \pi_1(\tilde{E})$ implies that $\hat{\alpha}_{t_i} \subset D$ since $D$ is $\pi_1(\tilde{E})$-invariant. $\partial T$ is a subset of the closure of $\bigcup_{k \in \mathbb{Z}} h_k(\hat{\alpha}_{t_i})$. Thus, $\tilde{E}$ is a radial or totally geodesic p-end of lens-type. Since $D$ is a lens-cone or a lens, we obtain

$$T \subset D \text{ and } \partial T \subset D \cap \text{bd} \tilde{O} = \bigcup S(v_E).$$

Also, $\tilde{E}$ is not totally geodesic since otherwise $T^o \cap \tilde{O} = \emptyset$. In this case, we are done.

(II) Now, we suppose that we are in case (iii) or there exists no $g \neq 1$ so that $g(L) = L$ from now on for each component $L$ of $T' - \tilde{U}'$. Now, we obtain a triangle in the asymptotic limit of $\tilde{O}$.

We will obtain an asymptotic limit of $T$ in an asymptotic limit of $\tilde{O}$ and show that we cannot have such an object unless $\partial T'$ is in $\bigcup S(v_E)$ for a radial p-end $\tilde{E}$.

Suppose that $L$ meets infinitely many horospherical p-ends and the $d_{\tilde{\Theta}}$-diameters of $L$ intersected with these are not bounded. Then we can show that $L$ or a leaf $L'$ in its closure of $\bigcup_{j \in \pi_1(O)} g(L)$ meet a horoball and its vertex in its closure. However, in the first case, this gives an arc in $L$ or $L'$ with one horospherical p-endpoint equal to an interior point of an edge or a vertex of a triangle containing $L$ or $L'$. Both cases are ruled out by Proposition 4.1.

Thus, $d_{\tilde{\Theta}}$-diameters of horospherical p-end neighborhoods intersected with $L$ are bounded above uniformly. Therefore, by choosing a horospherical end neighborhood sufficiently far inside each horospherical end neighborhood, we may assume that $L$ does not meet any horospherical p-end neighborhoods. That is we choose a horoball $V'$ inside a one $V$ so that

$$d_{\tilde{\Theta}}(V', \partial V) \geq \frac{1}{2} \sup d_{\tilde{\Theta}}\text{-diam}\{V \cap T\}_{V \in \mathcal{V}, T' \in \mathcal{T}}$$

where $\mathcal{V}$ is the collection of horospherical neighborhoods that we were given in the beginning and $\mathcal{T}$ is the collection of all triangles $T'$ with boundary in $\text{bd} \tilde{O} - (*)$.

We will use the theory of tree-graded spaces and asymptotic cones [39] and [73]. We remove neighborhoods of sufficiently small horospherical p-end neighborhoods from $O_1$ and call the result $\tilde{O}'$. We will be using the restricted path-metric $d_{\tilde{\Theta}}$ on $\tilde{O}'$ restricted from infinitesimal Finsler metric associated with $d_{\tilde{\Theta}}$. (See [63].) Then $\tilde{O}'$ is quasi-isometric with $\pi_1(O)$: This follows since $\tilde{O}'$ has a compact fundamental
domain and hence there is a map from it to \( \pi_1(\mathcal{O}) \) decreasing distances up to a positive constant. Conversely, there is a map from \( \pi_1(\mathcal{O}) \) to \( \mathcal{O}' \) with same property.

Let us recall definitions in Section 3.1 of Drutu-Sapir [39]. An ultrafilter \( \omega \) is a finite additive measure on \( P(\mathbb{N}) \) of \( \mathbb{N} \) so that each subset has either measure 0 or 1 and all finite sets have measure 0. If a property \( P(n) \) holds for all \( n \) from a set with measure 1, we say that \( P(n) \) holds \( \omega \)-almost surely.

Let \( (X, d_X) \) be a metric space. Let \( \omega \) be an ultrafilter over the set \( \mathbb{N} \) of natural numbers. For a sequence \( (x_i)_{i \in \mathbb{N}} \) of points of \( X \), its \( \omega \)-limit is \( x \in X \) if for every neighborhood \( U \) of \( x \) the property that \( x_i \in U \) holds \( \omega \)-almost surely. If \( X \) is Hausdorff, the limit is unique and if \( X \) is compact, every sequence has a convergent sequence.

An ultraproduct \( \prod X_n/\omega \) of a sequence of sets \( (X_n)_{n \in \mathbb{N}} \) is the set of the equivalence classes of sequences \( (x_n) \) where \( (x_n) \sim (y_n) \) if \( x_n = y_n \) holds for \( \omega \)-almost surely.

Given a sequence of metric spaces \( (X_n, d_n) \), consider the ultraproduct \( \prod X_n \) and an observation point \( e = (e_n) \). Let \( D(x, y) = \lim_\omega d_n(x_n, y_n) \). Let \( \prod X_n/\omega \) denote the set of equivalence classes of finite distances from \( e \). The \( \omega \)-limit \( \lim^\omega (X_n) e \) is the metric space obtained from \( \prod X_n/\omega \) by identifying all pair of points \( x, y \) with \( D(x, y) = 0 \).

Given an ultrafilter \( \omega \) over the set \( \mathbb{N} \) of natural numbers, an observation point \( e = (e_n) \), and sequence of numbers \( \delta = (\delta_n)_{n \in \mathbb{N}} \) satisfying \( \lim_\omega \delta_n = \infty \), the \( \omega \)-limit \( \lim^\omega (X, d_X/\delta_n) e \) is called the asymptotic cone of \( X \). (See [52], [53] and Definitions 3.3 to 3.8 in [39].) We denote it by \( \text{Con}^\omega(X, e, \delta) \).

For a sequence \( (A_n) \) of subsets \( A_n \) of \( X \), we denote by \( \text{lim}^\omega (A_n) \) the subset of \( \text{Con}^\omega(X, e, \delta) \) that consists of all elements \( (x_n) \) where \( x_n \in A_n \) \( \omega \)-almost surely. The asymptotic cone is always complete and \( \lim^\omega (A_n) \) is closed.

We will fix an ultrafilter \( \omega \) in \( \mathbb{N} \) and \( \delta \) from now on. We set an observation point \( e \) to be a constant sequence \( e \in \mathcal{O}' \). By Theorem 9.1 of Osin and Sapir [73] and Theorem 5.1 of Drutu and Sapir [39] and Theorem 5.7, \( \pi_1(\mathcal{O}) \) is asymptotically tree-graded with respect to the p-end fundamental groups \( \pi_1(E_i), \ i = 1, \ldots, m \). Thus \( \mathcal{O}' \) is asymptotically tree graded with respect to the p-end neighborhoods. Choose an ultrafilter \( \omega \), and let \( \mathcal{O}_\infty \) denote the asymptotic cone of \( \mathcal{O}' \) with the \( \omega \)-limit of the p-end neighborhoods as pieces. Here a piece is a closed subset satisfying certain properties in [39]. Let \( \mathcal{P} \) denote the set of pieces.

The basic heuristic strategy is to show that a triangle in an asymptotically tree-graded space must be inside one of the p-end neighborhoods.

But the existence of a triangle \( T' \) in \( \text{Cl}(\mathcal{O}) \) gives us a subspace \( T \) isometric with \( T^{\omega_0} \) in the asymptotic limit \( \mathcal{O}_\infty \). We obtain this by considering all sequences \( (x_i)_{i \in \mathbb{Z}_+} \) for \( x_i \in T^{\omega_0} \). The geodesics here are precisely the straight lines since the Hilbert metric \( d_T \) on \( T^{\omega_0} \) scaled by a constant \( d_T/\delta_i \) is isometric with \( d_T \) by an isometry \( f_i \). Hence \( T^{\omega_0} \) with \( d_T/\delta_i \) has \( \omega \)-limit \( T^{\omega_0} \) with \( d_T \). (\( d_T \) is called a hex metric [55]. This fact was first observed by Cooper, Delp and so on, as we understand.)

Each point of \( T' \) has a p-end neighborhood of uniformly bounded \( d_{\mathcal{O}} \)-distance from it as \( \mathcal{O} - U \) is compact by the action of \( \pi_1(\mathcal{O}) \). Hence, each point of \( T \) is in an element of a piece. (See Definition 3.9 of [39].)

(III) We will now show that by the asymptotic tree graded property, \( \partial T' \) must be contained in the closure of a p-end-neighborhood.
For $i$ in the index set of p-ends, we define $L_{1,i}$ to be the subset of $\widehat{U}$

- $\text{Cl}(U(v_{E})) \cap \widehat{O}_{1}$ where $U(v_{E})$ is the concave p-end neighborhood of $E$ when $E$ is a radial p-end of lens-type,
- The closure of the outer lens $L$ of $\tilde{S}_{E}$ if $E$ is a totally geodesic end of lens type, or
- a horoball $U_{E}$ for a horospherical end $\tilde{E}$.

Here $i$ denotes the labeling of the end vertices. We call it the lower boundary component of a lens of the end $E$. Let us choose $\hat{L}_{1,j}$ for $j = 1, \ldots, m$ from each representative in $L_{1,i}$ equivalent under $\pi_{1}(\tilde{O})$.

By Proposition 7.26 in [39], each piece is a $\omega$-limit of $(g_{i_{n}}, \hat{L}_{1,j_{i}})$ where $(g_{i_{n}})$ has the $\omega$-limit in the $\omega$-limit $\pi_{1}(O)^{\omega}$ of $\pi_{1}(O)$ and

$$\lim_{\omega} \frac{d_{\tilde{O}}(e, g_{i_{n}}, \hat{L}_{1,j_{i}})}{\delta_{i}} < \infty.$$
We can assume that $T'$ does not intersect with horoballs as above for $L$ as in (*). Hence, $T' \cap L_{1,i}^0 = \emptyset$ since $T$ is obtained as the $\omega$-limit of triangles outside the concave p-end neighborhoods.

Suppose that the asymptotic limit $T$ of $T'$ is contained in a piece. Since $T$ has a sequence of compact domains $(K_i)$ exhausting it, we obtain by taking sequences $(K_{i,j} \subset T')$ of compact sets converging to these and taking a diagonal sequences the following:

(i) There is a sequence of compact domains $K_i$ of points in $T^{\infty}$

$$d_{\hat{\Omega}}(y_0, \partial K_i) = \lambda_i \to \infty \text{ for } y_i \in K_i$$

so that $\lambda_i/\delta_i \to \infty$ and

(ii) Since every point of $T'$ becomes very close to a the lower boundary component $L_{1,i}$ of a lens of a lens-shaped p-end neighborhood $U_{i,1}$ under the normalization by $1/\delta_i$, we have

$$\max \{d_{\hat{\Omega}}(x, L_{1,i})|x \in K_i\} \leq \mu_i$$

for the closure of $L_{1,i}$, and $\mu_i/\delta_i \to 0$.

This implies also

$$\lambda_i/\mu_i \to \infty. \quad (5.2)$$

(Here $L_{1,i}$ is of form $g_n, \hat{L}_{1,n, i}$ above for some sequence $g_n \in \pi_1(O_i).$)

For any sequence of segments $y_0 \in m_i = m \cap K_i$ with $\partial m_i \subset \partial K_i$ in $T$ and the distance $\lambda_i$ of $m_i$ from $y_0$ to the end points of $m_i$, we have $\lambda_i/\delta_i \to \infty$.

We need a hypothesis:

(H): Suppose that the sequence $\{L_{1,i}\}$ is $d$-bounded away from one of the end points of $m$ in $\partial T$.

Let $s_i \in m_i$ be a point of $\partial m_i \subset \partial K_i$. Let $t_i$ be the end point of the extending line $m_i$ further away from $s_i$ from $y_0$ and let $s_i'$ be the other end point of $m_i$.

We take a supporting hyperspace $H_i$ at the point $t_i$. Then we take a subspace $P_i$ of $H_i$ of codimension 2 in $\mathbb{R}P^n$ disjoint from $\hat{O}$, which exists by the proper convexity of $H_i \cap Cl(\hat{O})$. Then we take hyperspaces containing $P_i$ and points $s_i', t_i, s_i, y_0$ and take the logarithms of the cross ratios to find the distance $d_{\hat{\Omega}}(y_0, s_i) = \lambda_i$ (See Figure 5.2.)

We also take a geodesic $\hat{l}_i$ from $s_i$ to $L_{1,i}$ of length $\mu_i$ and denote by $f_i$ the end point of the extension of $s_i$ not in the boundary of $L_{1,i}$. Then we take the logarithm of the cross ratio of $f_i, s_i$ and the two other points in the closure of $L_{1,i}$ to obtain the distance $\mu_i$. We can replace $f_i$ with $f'_i$ the intersection of the line containing $\hat{l}_i$ with $H_i$. The corresponding logarithm $\mu'_i$ is smaller than or equal to $\mu_i$.

Here to compute $\mu_i$, we need a point $\hat{l}_{1,i}$ on $L_{1,i}$ and one $\hat{m}_i$ in $\bigcup S(v_i)$ for the pseudo-convex p-end vertex $v_i$ corresponding to $L_{1,i}$ if $L_{1,i}$ from a radial p-end of lens-type. We need a point $\hat{l}_i$ on the totally geodesic ideal boundary and one $\hat{m}_i$ in the boundary of the outer part lens if $L_{1,i}$ is of totally geodesic type. For horospherical case, $\hat{l}_i$ is in the boundary of the horosphere and $\hat{m}_i$ is in bd$O'$.

(H1): For now, assume that $O$ has no horospherical ends.

So, now we can just use $d_{\hat{\Omega}}$ instead of $d_{\hat{\Omega}}$.

We will fix $P_i$ sufficiently far way from $\hat{O}$. Let $A$ be an affine space containing $\hat{O}$ containing $P_i$. (We will see things from $P_i$, and the half-hyperspaces for two
points for $\text{Cl}(L_i)$ are bounded away from those of two points $y_0, s'$ uniformly as $L_{1,i}$ is bounded away from $t_i$.)

We have
\[
\lambda_i = |\log[s', t_i, s_i, y_0]|, \mu_i = |\log[\tilde{m}_i, f_i, s_i, \tilde{l}_i]| \geq \mu'_i := |\log[\tilde{m}_i, f'_i, s_i, \tilde{l}_i]|.
\]

We define $P(v)$ to be the unique half-hyperspace in $A$ containing in the boundary $P$ and $v \notin P$. Let $H_i'$ be the half-hyperspace in $A$ containing $P$ and $t_i$. Then we have
\[
\mu'_i := \log[P(\tilde{m}_i), P(f'_i), P(s_i), P(\tilde{l}_i)].
\]

There exists a half-hyperplane $P'$ containing $P$ and furthermore among $P(\tilde{m}_i)$ away from $H_i$ from the view point of $P$. There exists a half-hyperplane $P''$ containing $P$ that $P(\tilde{m}_i)$ can be realized as closest to $H_i$ since $L_{1,i}$ is bounded away from $t_i$ uniformly. Then
\[
\mu'_i \geq |\log[P', H_i, P(s_i), P(\tilde{m}_i)]| \geq |\log[P', H_i, P(s_i), P'']|\]

since we chose the minimal possibility in the right term by taking the half-planes to extreme possibility to lower values first by $P'$ and then by $P''$ second. (Here for sufficiently large $i$, $P(s_i)$ is closer to $H_i$ than $P''$ as $s_i \to t_i$ since $L_{1,i}$ is bounded away from $t_i$.) We have
\[
\lambda_i = \log[P(s'), H_i, P(s_i), P(y_0)].
\]

Now parameterize $s_1$ as a linear function $s_1(t)$ of new variable so that $s_1(t) \to t_i$ as $t \to 1$. Now define functions of $t$ as
\[
f_1(t) := \log[P', H_i, P(s_i(t)), P''] \text{ and } f_2(t) := \log[P(s'), H_i, P(s_i(t)), P(y_0)].
\]

Here $s'$ and $y_0$ are fixed. Then we see easily that only constant terms are different as rational functions of $t$ with a pole of same order at 1. We obtain $f_2(t) \leq C f_1(t)$ for $1 - \epsilon < t < 1$.

Thus, we obtain $\lambda_i \leq C \mu'_i \leq C \mu_i$ for sufficiently large $i$ for some fixed positive constant $C$. Therefore, considering all directions of $m$, we obtain that $K_i$ has a diameter bounded by $C d_{\partial} (x, L_{1,i})$ for a positive constant $C'$. This is a contradiction to equation (5.2).

Now suppose that we have some horospherical $p$-ends. That is, we drop the hypothesis (H1). We follow the same argument as above and we show that a triangle $T'$ in $\text{Cl}(\tilde{O}, T^{\infty} \subset \partial \tilde{O})$, must satisfy $\partial T' \subset \bigcup S(v_E)$ for a radial end $E$. We obtain a sequence $\{L_{1,i}\}$ as above. Since there are only three types, we assume that each $L_{1,i}$ share a common type for each sequence.

If $L_{1,i}$ are from still radial or totally geodesic $p$-ends, then in the metric $d_{\partial'}$, we can do the same arguments where we obtain lower bound in terms of $d_{\partial'}$. We have $d_{\partial} \leq d_{\partial'}$. Since on totally geodesic $T'$ the metrics $d_{\partial'}$ and $d_{\partial}$ are the same, we again obtain that $K_i$ has a diameter bounded by $C d_{\partial'} (x, L_{1,i})$ for a positive constant $C'$.

If each $L_{1,i}$ is the boundary of a horosphere, then we use $d_{\partial}$ and the larger $d_{\partial'}$ to obtain the same proof. (In this case, the hypothesis (H) is always true. The sequence of the $d$-diameters of $\{\text{Cl}(L_{1,i})\}$ goes to zero since the components of the inverse images of the end neighborhoods are locally finite.) Hence, $T$ cannot be in the limit of a sequence $\{L_{1,i}\}$ of horospherical type.

Now we drop the hypothesis (H). Suppose now that the sequence $\{L_{1,i}\}$ is not $d$-bounded away from the both ends of $m$ in $\partial T'$.\]
Suppose that there exists a sequence \((L_{1,i_k})\) so that each end point \(\delta_j m, j = 1, 2,\) of \(m\) has a sequence \(\{x_{i_k}^j\}, j = 1, 2,\) in \((L_{1,i_k})\) where \(\{x_{i_k}^j\} \to \delta_j m\) as \(k \to \infty\) in the \(d\)-metric. Suppose that \(L_{1,i_k}\) is from a radial \(p\)-end of lens-type. We take the union of the two segments in \(\bigcup \{L_{1,i_k}\}\) to form an arc \(s_{i_k}\) with \(\partial s_{i_k} = \{x_{i_k}^1, x_{i_k}^2\}\).

Suppose that \(L_{1,i_k}\) is totally geodesic of lens-type. Then there is an arc \(m_{i_k}\) in a lens obtained by taking the intersection of \(m\) with the lens neighborhood in \(L_{1,i_k}\). Then we take a maximal segment \(s_{i_k}\) in \(\text{Cl}(S_{E_{i_k}})\) near \(m_{i_k}\) with endpoints \(x_{i,k}^j, j = 1, 2\) and \(\{x_{i,k}^j\} \to \delta_j m, j = 1, 2\) in the \(d\)-metric.

We take a subsequence so that \(\{s_{i_k}\}\) geometrically converges to a union \(s\) of one or two segments. By strictness of the SPC-structure on \(O\), it follows \(s\) is a subset of \(\bigcup S(v_E)\) or \(\text{Cl}(S_E)\). Since the end point of \(m\) is in \(s\), we have \(\delta_1 m, \delta_2 m \in S(v_E)\) for a radial end \(E\). \((E\) cannot be totally geodesic.\)

Now, an edge of \(T'\) meets \(\bigcup S(v_i)\) for the corresponding \(p\)-end vertex \(v_i\). By Theorems 4.3 (ii) and 4.4, the edge is \(S(v_j)\). By changing the directions of the maximal segment \(m\) in \(T'\), it follows that \(\partial T' \subset S(v_i)\) since the edges where \(m\) ends have to be in \(S(v_j)\) for the same \(j\) by the above reasoning. This contradicts the assumption.

Therefore, we conclude that the asymptotic limit triangle \(T\) in \(\tilde{O}\) cannot be contained in one piece in the asymptotic limit.

It is not possible for exactly two components \(P'\) contain \(T\): Suppose \(T = C_1 \cup C_2\) for closed \(C_1, C_2\) and \(C_1 \cap C_2\) is a single point. However, removing a point cannot separate \(T\) into two components.

As a consequence, let \(p_1, p_2\) be two points of the interior of \(T\) not in a single piece. Then there exists a point \(p_3\) in general position so that no two of \(p_1, p_2, p_3\) are contained in a common piece.

By taking a geodesics between two of \(p_1, p_2, p_3\), we obtain a simple triangle \(\Delta'\). This contradicts the definition of tree-graded spaces. (See Definition 1.10 of \([39]\).)

We conclude that there exists no triangle such as \(T'\). □

Remark 5.11. We think that there is a proof for \(n = 3\) using trees as Benoist have done in closed 3-dimensional cases in \([11]\).

We recapitulate the results:

**Corollary 5.12.** Assume that \(O\) is a noncompact strongly tame SPC-orbifold with generalized admissible ends and satisfies \((\text{IE})\) and \((\text{NA})\). Let

\[
E_1, \ldots, E_n, E_{n+1}, \ldots, E_k
\]

be the ends of \(O\) where \(E_{n+1}, \ldots, E_k\) are some or all of the hyperbolic ends. Assume \(\partial O = \emptyset\). Then \(\pi_1(O)\) is a relatively hyperbolic group with respect to the admissible end groups \(\pi_1(E_1), \ldots, \pi_1(E_n)\) if and only if \(O_1\) is strictly SPC with respect to admissible ends \(E_1, \ldots, E_n\).

**Proof.** If \(\pi_1(O)\) is a relatively hyperbolic group with respect to the admissible end groups \(\pi_1(E_1), \ldots, \pi_1(E_n)\), then \(\pi_1(O)\) is a relatively hyperbolic group with respect to the admissible end groups \(\pi_1(E_1), \ldots, \pi_1(E_k)\) by Theorem 5.8. By Theorem 5.9, it follows that \(O\) is strictly SPC with respect to the ends \(E_1, \ldots, E_k\). By Proposition 5.6, we obtain that \(O_1\) is strictly SPC with respect to \(E_1, \ldots, E_n\).
For converse, if $O_1$ is strictly SPC with respect to $E_1, \ldots, E_n$, then $O_1$ is strictly SPC with respect $E_1, \ldots, E_k$. By Theorem 5.7, $\pi_1(O)$ is a relatively hyperbolic group with respect to the admissible end groups $\pi_1(E_1), \ldots, \pi_1(E_k)$. The conclusion follows by Theorem 5.8.

\[\square\]

5.2.4. Strict SPC-structures deform to strict SPC-structures.

**Theorem 5.13.** Let $O$ denote a noncompact strongly tame SPC-orbifold with admissible ends and satisfies (IE) and (NA). Assume $\partial O = \emptyset$. Let $E_1, \ldots, E_n, E_{n+1}, \ldots, E_k$ be the ends of $O$ where $E_{n+1}, \ldots, E_k$ are some or all of the hyperbolic ends.

- Given a deformation through SPC-structures with generalized admissible ends of a strict SPC-orbifold with respect to admissible ends $E_1, \ldots, E_k$ to an SPC-structure, the SPC-structure remains strictly SPC with respect to $E_1, \ldots, E_k$.
- Given a deformation through SPC-structures with generalized admissible ends of a strict SPC-orbifold with respect to $E_1, \ldots, E_n$ to an SPC-structure with generalized admissible end, the SPC-structure remains strictly SPC with respect to admissible ends $E_1, \ldots, E_n$.

**Proof.** For the second item, $O_1$ being strictly SPC with respect to $E_1, \ldots, E_n$ implies that $\pi_1(O)$ is relatively hyperbolic with respect to the end fundamental groups $\pi_1(E_1), \ldots, \pi_1(E_n)$ by Corollary 5.12. Then the small deformation does not change the group property. Thus, after deformation $O$ is strictly SPC with respect to $E_1, \ldots, E_n$ by the fourth item by Corollary 5.12.

The first item is simpler to show by Theorem 5.7 and Theorem 5.9. 

\[\square\]
Part 3

The openness and the closedness of the deformations of convex real projective structures
The openness of the convex structures

In this section also, we will only need $\mathbb{R}P^n$ versions. Given a real projective orbifold with radial ends or totally geodesic ends of lens-type, each end has an orbifold structure of dimension $n-1$ and inherits a real projective structure.

Let $\mathcal{U}$ and $s_U : \mathcal{U} \to (\mathbb{R}P^n)^{e_1} \times (\mathbb{R}P^n)^{e_2}$ be as in Section 3.3.

- We define $\text{Def}_{E,u,ce}^r(O)$ to be the subspace of $\text{Def}_{E,u,ce}(O)$ of real projective structures with generalized admissible ends determined by $s_U$, and stable irreducible holonomy homomorphisms in $\mathcal{U}$.
- We define $\text{CDef}_{E,sU,ce}(O)$ to be the subspace consisting of SPC-structures with generalized admissible ends in $\text{Def}_{E,sU,ce}(O)$.
- We define $\text{SDEF}_{E,sU,ce}(O)$ to be the subspace of $\text{Def}_{E,sU,ce}(O)$ consisting of SPC-structures with admissible ends.
- We define $\text{SDef}_{E,sU,ce}(O)$ to be the subspace of $\text{Def}_{E,sU,ce}(O)$ consisting of strict SPC-structures with admissible ends.
- We define $\text{Def}_{E,u,ce}^r(O)$ to be the subspace of $\text{Def}_{E,u,ce}(O)$ consisting of SPC-structures with admissible ends.

**Theorem 6.1.** Let $O$ be a noncompact strongly tame real projective $n$-orbifold with generalized admissible ends and satisfies (IE) and (NA). Assume $\partial O = \emptyset$. In $\text{Def}_{E,u,ce}^r(O)$, the subspace $\text{CDef}_{E,u,ce}(O)$ of SPC-structures with generalized admissible ends is open, and so is $\text{SDef}_{E,u,ce}(O)$.

**Theorem 6.2.** Let $O$ be a noncompact strongly tame real projective $n$-orbifold with generalized admissible ends and satisfies (IE) and (NA). Assume $\partial O = \emptyset$. For an open $\text{PGL}(n+1,\mathbb{R})$-conjugation invariant

$$\mathcal{U} \subset \text{Hom}^E_E(\pi_1(O), \text{PGL}(n+1,\mathbb{R})),$$

and a $\text{PGL}(n+1,\mathbb{R})$-equivariant section $s_U : \mathcal{U} \to (\mathbb{R}P^n)^{e_1} \times (\mathbb{R}P^n)^{e_2}$, $\text{CDef}_{E,sU,ce}(O)$ is open in $\text{Def}_{E,sU,ce}(O)$, and so is $\text{SDef}_{E,sU,ce}(O)$.

For orbifolds such as these the deformation space of convex structures may only be a proper subset of space of the characters.

By Theorem 6.1 and Theorem 1.2, we obtain:

**Corollary 6.3.** Let $O$ be a noncompact strongly tame real projective $n$-orbifold with generalized admissible ends and satisfies (IE) and (NA). Assume $\partial O = \emptyset$. Then

$$\text{hol} : \text{CDef}_{E,u,ce}(O) \to \text{rep}^E_{E,u,ce}(\pi_1(O), \text{PGL}(n+1,\mathbb{R}))$$

is a local homeomorphism. Furthermore, if $O$ has a strict SPC-structure with admissible ends and and satisfies (IE) and (NA), then so is

$$\text{hol} : \text{SDef}_{E,u,ce}(O) \to \text{rep}^E_{E,u,ce}(\pi_1(O), \text{PGL}(n+1,\mathbb{R})).$$
Corollary 6.4. Let \( \mathcal{O} \) be a noncompact strongly tame real projective \( n \)-orbifold with generalized admissible ends and satisfies (IE) and (NA). Assume \( \partial \mathcal{O} = \emptyset \). Let \( \mathcal{U} \) and \( s_{\mathcal{U}} \) be as above. Suppose that \( \mathcal{U} \) has its image \( \mathcal{U}' \) in \( \text{rep}_{E,\mathcal{U},\mathcal{C}}(\pi_1(\mathcal{O}),\text{PGL}(n+1,\mathbb{R})) \). Then

\[
\text{hol} : \text{CDef}_{E,\mathcal{U},\mathcal{C}}(\mathcal{O}) \to \mathcal{U}'
\]

is a local homeomorphism, and so is

\[
\text{hol} : \text{SDef}_{E,\mathcal{U},\mathcal{C}}(\mathcal{O}) \to \mathcal{U}'.
\]

Here, in fact, one needs to prove for every possible continuous section.

Koszul [64] proved these facts for closed affine manifolds and expanded by Goldman [46] for the closed real projective manifolds. See [22] and also Benoist [10].

6.1. The proof of the convexity theorem

Recall that a convex open cone \( V \) is a convex cone of \( \mathbb{R}^{n+1} \) containing the origin \( O \) in the boundary. Recall that a properly convex open cone is a convex cone so that its closure does not contain a pair of \( v, -v \) for a nonzero vector in \( \mathbb{R}^{n+1} \). Equivalently, it does not contain a complete affine line in its interior.

A dual convex cone \( V^* \) to a convex open cone is a subset of \( \mathbb{R}^{n+1*} \) given by the condition \( \phi \in V^* \) if and only if \( \phi(v) > 0 \) for all \( v \in \text{Cl}(V) - \{O\} \).

Recall that \( V \) is a properly convex open cone if and only if so is \( V^* \) and \((V^*)^* = V\) under the identification \((\mathbb{R}^{n+1*})^* = \mathbb{R}^{n+1}\). Also, if \( V \subset W \) for a properly convex open cone, then \( V^* \subset W^* \).

For properly convex open subset \( \Omega \) of \( \mathbb{RP}^n \), its dual \( \Omega^* \) in \( \mathbb{RP}^{n*} \) is given by taking a cone \( V \) in \( \mathbb{R}^{n+1} \) corresponding to \( \Omega \) and taking the dual \( V^* \) and projecting it to \( \mathbb{RP}^{n*} \). The dual \( \Omega^* \) is a properly convex open domain if so was \( \Omega \).

Recall the Koszul-Vinberg function for a properly convex cone \( V \) and the dual properly convex cone \( V^* \)

\[
(6.1) \quad f_{V^*} : V \to \mathbb{R}_+ \text{ defined by } x \in V \mapsto f_{V^*}(x) = \int_{V^*} e^{-\phi(x)} d\phi
\]

where the integral is over the euclidean measure in \( \mathbb{R}^{n+1*} \). This function is strictly convex if \( V \) is properly convex. \( f \) is homogeneous of degree \(-n+1\). Writing \( D \) as the affine connection, we will write the Hessian \( Dd\log(f) \). The hessian is positive definite and norms of unit vectors are strictly bounded below in a compact subset \( K \) of \( V - \{O\} \). (See Chapter 6 of [48].) The metric \( Dd\log(f) \) is invariant under the group \( \text{Aff}(V) \) of affine transformation acting on \( V \). (See Theorem 6.4 of [48].) In particular, it is invariant under dilatation maps \( x \mapsto sx \) for \( s > 0 \).

A Hessian metric on an open subset \( V \) of an affine space is a metric of form \( \partial^2 f / \partial x_i \partial x_j \) for affine coordinates \( x_i \) and a function \( f : V \to \mathbb{R} \) with a positive definite Hessian defined on \( V \). A Riemannian metric on an affine manifold is a Hessian metric if the manifold is affinely covered by a cone and the metric lifts to a Hessian metric of the cone.

Let \( \mathcal{O} \) have an SPC-structure \( \mu \) with admissible ends. Clearly \( \tilde{\mathcal{O}} \) is a properly convex open domain. Then an affine suspension of \( \mathcal{O} \) has an affine Hessian metric defined by \( Dd\phi \) for a function \( \phi \) defined on the cone in \( \mathbb{R}^{n+1} \) corresponding to \( \tilde{\mathcal{O}} \) by above.

For a subset \( K \) of \( S^n \) or \( \mathbb{RP}^n \), we denote by \( C(K) \) the cone in \( \mathbb{R}^{n+1} - \{O\} \) the inverse image of \( K \) under the projection. Recall that a parameter of real projective
structures $\mu_t, t \in [0, 1]$ on a strongly tame orbifold $O$. is a collection so that the restriction $\mu_t|K$ to each compact suborbifold $K$ is continuous parameter; In other words, the associated developing map $\text{dev}_t|K$ for every compact subset $K$ of $\tilde{O}$ is a family in the $C^\infty$-topology continuous for the variable $t$. (See [22] and Canary [16].)

**Proposition 6.5.** Let $O$ be a strongly tame orbifold with ends and satisfies (IE) and (NA). Suppose that $O$ has an SPC structures $\mu_0$ with generalized admissible ends and the suspension of $O$ with $\mu_0$ has a Hessian metric. The ends of $O$ are given $\mathcal{R}$-type or $\mathcal{T}$-types. If

- $\mu_0$ is SPC, and a parameter of real projective structure $\mu_{0,t}, t \in [0,1]$, with generalized admissible ends and $\mu_{0,0} = \mu_0$ where the $\mathcal{R}$-types or $\mathcal{T}$-types of ends are preserved,

then for sufficiently small $t$, the affine suspension $C(\tilde{O})$ for $\tilde{O}$ with $\mu_t$ also has a Hessian metric invariant under dilations and the affine suspensions of the holonomy homomorphism for $\mu_{0,t}$.

**Proof.** We will keep $\tilde{O}$ fixed and only change the structures on it. Note that the subsets here remain fixed and the only changes are on the real projective structures, i.e., the atlas of charts to $\mathbb{R}P^n$.

Let $\tilde{O}$ in $\mathbb{R}P^n$ denote the universal covering domain corresponding to $\mu_0$. Again $\text{dev}_0$ being an embedding identifies the first with subsets of $\mathbb{R}P^n$ but $\text{dev}_t$ is not known to be so. We shall prove this below.

We will make a simplifying assumption:

(H) For $\mu_0$, every radial end of generalized lens-type is an end of radial type.

(A) The first step is to understand the deformations of the end-neighborhoods:

Let $\tilde{E}'$ be a p-end of $\tilde{O}$ and it corresponds to a p-end of $\tilde{O}'$ as well. There exists a $C^r$-parameter of real projective structures $\mu_{0,t}, t \in [0,1]$ with radial or totally geodesic ends of lens-type so that $\mu_{0,0} = \mu_0$. We can also find a parameter of developing maps $\text{dev}_t$ associated with $\mu_{0,t}$ where $\text{dev}_t|K$ is a continuous with respect to $t$ for each compact $K \subset \tilde{O}$. To begin with, we assume that $\tilde{E}'$ keeps being of a radial p-end of lens type or horospherical type.

Here, we do not allow $\mathcal{R}$-type ends to change to $\mathcal{T}$-type ends and vice versa as this will make us to violated the local injectivity property from the deformation space to a space of characters. (See Theorem 1.2.) Thus, we need to consider only three cases to prove openness:

(I) A radial p-end changes to a radial p-end in the cases:

- A radial p-end of lens-type becoming a radial p-end of generalized lens-type.
- A horospherical p-end becoming a radial p-end of generalized lens-type or horospherical type.

(II) A totally geodesic p-end of lens type deforms to a totally geodesic p-end of lens type.

(III) A horospherical p-end deforms to a horospherical p-end or to a totally geodesic p-end.

Here, the premise assumes that these hold for the corresponding holonomy homomorphisms of the fundamental groups of ends. We will show that the above happens in actuality as well.
We will now work on one end at a time: Let us fix a p-end \( \tilde{E} \) of R-type of \( \tilde{\mathcal{O}} \). Let \( v \) be the p-end vertex of \( \tilde{E} \) for \( \mu_0 \) and \( v' \) that for \( \mu_1 \). We denote by \( v = v_0 \) and \( v' = v_1 \). Assume that \( v_1 \) is the p-end vertex of \( \tilde{E} \) for \( \mu_1 \). Let \( \text{dev}_t \) and \( h_t \) denote the developing map and the holonomy homomorphism of \( \mu_t \). Assume first that the corresponding p-end for \( \mu_t \) is of radial or horospherical type. By post-composing the developing map by a transformation near the identity, we assume that the perturbed vertex \( v_t \) of the corresponding p-end \( \tilde{E} \) is mapped to \( v_0 \), i.e., \( v = \text{dev}_t(v_t) \).

(1) If \( \tilde{E} \) is of radial p-end of horospherical or lens-type for \( \mu_0 \), then \( \tilde{E} \) is always a radial p-end of horospherical a generalized lens-type for \( \mu_t \).

Let \( \Lambda \) denote the limit set in the tube of the radial p-end \( \tilde{E} \) for \( \tilde{\mathcal{O}} \) if \( \tilde{E} \) is of lens-type radial p-end, or \( \{v_E = v\} \) if \( \tilde{E} \) is a horospherical type for \( \mu_0 \).

- Let \( R_{v,t}(\tilde{\mathcal{O}}) \) denote the space of rays in \( \tilde{\mathcal{O}} \) mapping to ones from \( v \) under \( \text{dev}_t \).
- \( r_{v,t}(A_t) \) denote the union of segments of \( \Omega_{s_0,t} \) in \( R_{v,t}(\tilde{\mathcal{O}}) \) passing through the set \( A_t \subset \tilde{\mathcal{O}} \) mapping to ones from the p-end vertex \( v \) under \( \text{dev}_t \).

Then for \( \mu_0 \), a smooth and strictly convex hypersurfaces \( \partial \Omega_{s_0} \subset \tilde{\mathcal{O}} \), \( s_0 \in \mathbb{R}_+ \), as obtained by Lemma 4.8 with

\[
\text{Cl}(\partial \Omega_{s_0}) - \partial \Omega_{s_0} \subset \Lambda.
\]

Also, each radial geodesic is transversal to \( \partial \Omega_{s_0} \). \( \partial \Omega_{s_0} \) bounds a properly convex domain \( \Omega_{s_0} \). Here \( \bigcup_{s_0 \in S} \Omega_{s_0} = \tilde{\mathcal{O}} \) where \( S \) is an infinite index set.

For a sufficiently small \( t \) in \( \mu_0,t \), we obtain a domain \( U_t \subset \tilde{\mathcal{O}} \) that is a concave neighborhood or a horospherical one with \( U_0 \subset \Omega_{s_0} \). Now \( U_t \) can be compactified to \( \hat{U}_t \) so that \( \text{dev}_t|U_t \) for the developing map \( \text{dev}_t \) extends to an imbedding \( \text{dev}_t|\hat{U}_t \) to a concave end neighborhood or a horoball. In the first two cases, there exists a point \( v' \in \hat{U}_t \). Let \( S(v)_t \) denote the set of segments from \( v \) in \( \hat{U}_t \) in the corresponding to the boundary of \( S_E \) of \( \mu_t \). Again \( bdU_t \) is assumed to be strictly concave for all sufficiently small \( t \) if \( U_0 \) was a concave p-end neighborhood.

Let the tube \( B_t \) be determined by \( \text{dev}_t(U_t) \); i.e., \( B_t \) is the union of great segments with end points in \( v, v' \) in the direction of \( \text{dev}_t(U_t) \).

Define \( \Lambda_t \) be the limit set in \( \bigcup S(v)_t \) for generalized radial p-end cases and \( \Lambda_t = \{v\} \) for the horospherical case.

We will denote by \( S_{\tilde{E},t} \) the universal cover of the end orbifold corresponding to \( \tilde{E} \) for \( \mu_0,t \). Since \( \tilde{E} \) is a radial p-end and \( S_E \) is properly convex or complete affine for \( \mu_0 \), the admissible holonomy condition on \( \mu_t \) implies that \( \tilde{E} \) is a generalized lens-type end or \( S_{\tilde{E},t} \) is a complete affine space. (See [10] for properly convex cases.) The surface \( S_{\tilde{E},t} \) is always a convex real projective \((n - 1)\)-orbifold.

We assume that the \( C^r \)-change \( r \geq 2 \) of \( \mu_0,t \) from \( \mu_0 \) be sufficiently small so that we obtain a region \( \Omega_{s_0,t} \subset \tilde{\mathcal{O}} \) with \( \partial \Omega_{s_0,t} \) strictly convex and transversal to radial rays under \( \text{dev}_t \). Here, \( \Omega_{s_0,0} = \Omega_{s_0} \). (The strict convexity follows since the change of affine connections are small as the argument of Koszul [64].) Choose a compact domain \( F \) in \( \partial \Omega_{s_0} \). Let \( F_t \) denote the corresponding deformed set in \( \partial \Omega_{s_0,t} \). For sufficiently small \( t \), \( 0 < t < 1 \), \( \text{dev}_t(F_t) \) is a subset of the tube \( B_t \) determined by \( \text{dev}_t(U_t) \) since \( B_t \) and \( \text{dev}_t(F_t) \) depend continuously on \( t \). By transversality to the segments mapping to ones from \( v \) under \( \text{dev}_t \), it follows that \( \text{dev}_t|\partial \Omega_{s_0,t} \) gives
us a smooth immersion to a convex domain $S_{E,t}$ that equals the space of maximal segments $B_t$ with vertices $v$ and $v_\infty$. In this case, $\text{dev}_t|_{\partial\Omega_{s_0,t}}$ is a diffeomorphism to $S_{E,t}$ by [63] if $S_{E,t}$ is properly convex and by Proposition 4.1(i) if $h_t(\pi_1(\tilde{E}))$ is horospherical. Since $\pi_1(\tilde{E})$ is of generalized lens type or horospherical type, it follows that

$$\text{dev}_t((Cl(\partial\Omega_{s_0,t}) - \partial\Omega_{s_0,t})) \subset \text{dev}_t(\Lambda_t)$$

by Corollary 8.5 of [27].

Suppose that $E$ is of radial lens-type. Since $\partial\Omega_{s_0,t}$ is convex, each point of $\partial\Omega_{s_0,t} \cup S(v)_t$ has a neighborhood that maps under the completion $\text{dev}_t$ to a convex open ball. Thus, $\partial\Omega_{s_0,t} \cup S(v)_t$ bounds a compact ball $\Omega_{s_0,t} \cup \bigcup S(v)_t$ by Lemma 2.5.

Suppose that $\tilde{E}'$ is horospherical type. $\partial\Omega_{s_0,t} \cup \{v\}$ bounds a convex domain $\Omega''_{s_0,t}$ by the local convexity of the boundary set $\partial\Omega_{s_0,t} \cup \{v\}$ and Lemma 2.5.

We showed that the $E$ is still a radial p-end of lens-type or horospherical here for $\mu_1$.

Now, we will show how these regions change.

- Let $K$ be a compact convex subset of $\Omega_{s_0,0}$ with smooth boundary, which we can choose to be sufficiently large, and $K_t$ the perturbed one in $\Omega_{s_0,t}$ and $\tilde{E}$ be the corresponding p-end. We can form a compact set inside $\Omega_{s_0,t}$ consisting of segments from the p-end vertex to $K$ in the set of radial segments. For $\mu_{0,t}$ from $\mu_0$ changed by a sufficiently small manner, a compact subset $R_v(K) \subset R_v(\tilde{O})$ is changed to a compact convex domain $R_{v,t}(K_t) \subset R_{v,t}(\tilde{O})$.

The p-end $\tilde{E}$ has either a concave p-end neighborhood or a horospherical p-end neighborhood. If $E$ has a concave p-end neighborhood, then since $\Omega'_{s_0,t}$ is strictly convex, we can obtain a lens. Thus, $\tilde{E}$ is admissible.

Let $K_t$ be a large compact convex domain in $\tilde{O}$ with $\mu_1$. For sufficiently small $t$, $\Omega_{s_0,t} \cap r_v(K_t)$ is a convex domain since $\partial\Omega_{s_0,t}$ is strictly convex and transversal to segments from $v$ and hence embeds to a convex domain under $\text{dev}_t$. We may assume that $\text{dev}_t(\Omega_{s_0,t} \cap r_v(K_t))$ is sufficiently close to $\text{dev}_0(\Omega_{s_0} \cap r_v(K))$ as we changed the real projective structures sufficiently small in the $C^2$-sense and hence the holonomy of the generators of the p-end fundamental group $\pi_1(\tilde{E})$ is changed by a small amount if the change from $\mu_0$ to $\mu_{0,t}$ is sufficiently small.

An $\epsilon$-thin space is a space which is an $\epsilon$-neighborhood of its boundary for small $\epsilon > 0$. By Corollary 2.9, we may assume that $\text{Cl}(R_v(\tilde{O}))$ and $\text{Cl}(R_{v,t}(\tilde{O}))$ as subsets of $S^{-1}_v$ are $\epsilon$-close convex domains in the Hausdorff sense for sufficiently small $t$. Thus, $\text{dev}_t(r_v(\tilde{O}) - r_v(K))$ is an $\epsilon$-thin space and so is $\text{dev}_0(r_v(\tilde{O}) - r_v(K))$ for sufficiently small changes of $\tilde{O}$ and $K$. Given an $\epsilon > 0$, we can choose $K$ and $K_t$ and small deformation of the real projective structures so that

$$\text{Cl}(\text{dev}_t(\Omega_{s_0} \cap (r_v(\tilde{O}) - r_v(K)))) \subset N_{\epsilon}(\text{Cl}(\text{dev}_0(\Omega_{s_0} \cap r_v(K))))$$

(6.2) $$\text{Cl}(\text{dev}_t(\Omega_{s_0,t} \cap (r_v(\tilde{O}) - r_v(K)))) \subset N_{\epsilon}(\text{Cl}(\text{dev}_0(\Omega_{s_0,t} \cap r_v(K))))$$

The reason is that the supporting hyperplanes of $\text{Cl}(\Omega_{s_0})$ at points of $\partial r_v(K) \cap \text{Cl}(\Omega_{s_0})$ are in arbitrarily small acute angles from geodesics from $v$ and similarly for those of $\text{Cl}(\Omega_{s_0,t})$ for sufficiently small $t$. Therefore the Hausdorff distance between
(I) Now suppose that $\tilde{E}$ is totally geodesic p-end, and we suppose that $\tilde{E}$ is totally geodesic for $\mu_t$. We can take dual domains of corresponding p-end neighborhoods and obtain a radial lens-type p-end or a horospherical p-end by Theorem 4.12. We take $\Omega_{s_0}$ be the convex domain obtained as in Lemma 4.8. Then $\partial \Omega_{s_0}/\pi_1(\tilde{E})$ is a strictly convex compact $(n-1)$-orbifold. Suppose that $\mu_{0,t}$ is sufficiently close to $\mu_0$. Then $\partial \Omega_{s_0,t}$ is also cocompact under the $\pi_1(\tilde{E})$-action associated with $\mu_1$ and strictly convex. We have $\partial \text{Cl}(\partial \Omega_{s_0,t}) \subseteq \partial \text{Cl}(S_{\tilde{E},t})$ for a totally geodesic ideal boundary component $S_{\tilde{E},t}$ by Theorem 8.2 of [27] since $h_t(\tilde{E})$ is of lens-type and hence satisfies the uniform middle eigenvalue condition. Therefore the union of $\partial \Omega_{s_0,t}$ and $\text{Cl}(S_{\tilde{E},t})$ bounds a properly convex compact $n$-ball in $\mathbb{R}P^n$. Hence, we obtain a lens-type end or horospherical end for $\tilde{E}$ and $\mu_t$ by Lemma 2.5.

Moreover, we may assume without loss of generality that $\bigcup_{s_0 \in I} \Omega_{s_0,0} = \tilde{O}$ for some infinite index $I$.

Let us choose a sufficiently small $\epsilon > 0$. Let $B$ be a compact $(n-1)$-ball in $\partial \Omega_{s_0}$ so that
\[ d^H(\text{dev}_0(\partial \Omega_{s_0} - B), \text{dev}_0(\partial \text{Cl}(S_{\tilde{E}}))) < \epsilon. \]

Given a supporting hyperplane $W_x$ of a point $x$ of $\text{dev}_0(\partial \Omega_t)$, there exists a supporting closed half-sphere $H_x$ containing it as the the boundary. Let $V_{\tilde{E}}$ as the hyperplane containing $\text{dev}_0(S_{\tilde{E}})$. We define the shadow $S$ of $\partial B$ as the set
\[ \bigcap_{x \in \partial B} H_x \cap V_{\tilde{E}}. \]

Then we can choose sufficiently large $B$ so that $d^H(S, \text{dev}_0(\text{Cl}(S_{\tilde{E}}))) \leq \epsilon$. We can also assure that $W_x$ meets $V_{\tilde{E}}$ in angles in $(\delta, \pi - \delta)$ for some $\delta > 0$ by compactness of $\partial B$ and the continuity of map $x \mapsto W_x$.

Suppose that we change the structure from $\mu_0$ to $\mu_t$ with a small $C^2$-distance. Then $B$ will change to $B'_t$ with $W_x$ change by small amount. The new shadow $S'_t$ will have the property $d^H(S'_t, \text{Cl}(S_{\tilde{E},t})) \leq \epsilon$ for a sufficiently small $C^2$-change of $\mu_t$ from $\mu_0$. Hence, we obtain
\[ d^H(\text{dev}_t(\partial \Omega_{s_0,t} - B'_t), \text{dev}_t(\partial \text{Cl}(S_{\tilde{E},t}))) < \epsilon \]
for a sufficiently small $C^2$-change of $\mu_t$ from $\mu_0$. Therefore by Corollary 2.9 the Hausdorff distance between $\text{Cl}(\Omega_{s_0})$ and $\text{Cl}(\Omega_{s_0,t})$ can be made as small as desired as long as we choose $\mu_{0,t}$ sufficiently close to $\mu_0$. (Note that we can have a change to a horospherical end here.)

Here, the admissibility of the ends of orbifolds follows since by construction we obtain a lens shaped one-sided neighborhood for the totally geodesic p-end $\tilde{E}$.

(II) Lemma 3.8 studies this case. Here, similarly to the case (II), we can obtain that $\text{Cl}(\Omega_{s_0})$ and $\text{Cl}(\Omega_{s_0,t})$ can be made as small as one desires.

(B) We change the Hessian function on the cone associated with the universal covers. We need to obtain one for the deformed end neighborhoods and another one the outside of the union of end neighborhoods and patch the two together.

With $\tilde{O}$ with $\mu_t$, we obtain a special affine suspension on $O \times S^1$ with the affine structure $\tilde{\mu}_t$. Let $C(\tilde{O})$ be the cone over $\tilde{O}$. Then this covers the special affine suspension. Let $\tilde{\mu}_t$ denote the affine structure on $C(\tilde{O})$ corresponding to $\tilde{\mu}_t$. For
each $\mu_t$, it has an affine structure $\tilde{\mu}_t$, different from the induced one from $\mathbb{R}^{n+1}$ as for $t = 0$. We require that scalar multiplication

$$s \cdot v = sv, \quad v \in C(\tilde{\mathcal{O}}), \quad s \in \mathbb{R}$$

for any affine structure $\tilde{\mu}_t$. Also, given a subset $K$ of $\tilde{\mathcal{O}}$, we denote by $C(K)$ the corresponding set in $C(\tilde{\mathcal{O}})$. This set is independent of $\tilde{\mu}_t$ but will have different affine structures nearby.

For $\mu_0$, $\text{dev}_0(\tilde{\mathcal{O}}) = \tilde{\mathcal{O}}$ is a domain in $\mathbf{S}^n$. Recall the Koszul-Vinberg function $f : C(\tilde{\mathcal{O}}) \to \mathbb{R}_+$ homogeneous of degree $-n - 1$. (See Lemma 6.6.) By Lemma 4.8, the Hausdorff distance between $C(\tilde{\Omega}_\mu)$ and $\tilde{\mathcal{O}}$ can be made as small as desired given $s_0$. By the third item of Lemma 6.7, the Hessian functions $f'_1$ defined by equation (6.1) on $C(\tilde{\Omega}_{s_0,t})^\circ$ is very close to the original Hessian function $f$ in compact subsets of $C(\tilde{\Omega}_{s_0})$ in the $C^2$ topology by Lemma 6.6. By construction, $f'_1$ is homogeneous of degree $-n - 1$.

The holonomy groups $\h(\pi_1(\tilde{\mathcal{O}}))$ and $\h(\pi_1(\tilde{E}))$ being in $GL(n + 1, \mathbb{R})$ preserve $f$ and $f'$ under deck transformations respectively.

Now do this for all $p$-ends and we obtain functions $f'_t$ on $C(U_t)$ of the $\pi_1(\tilde{\mathcal{O}})$-invariant mutually disjoint union $U_t$ of $p$-end neighborhoods of $p$-ends of $\tilde{\mathcal{O}}$ for $\mu_t$ and sufficiently small $t_0$.

Let $U$ be the corresponding $\pi_1(\tilde{\mathcal{O}})$-invariant union of proper $p$-end neighborhoods of $\tilde{\mathcal{O}}$ for $\mu_0$. For each component $U_i$ of $U$, we construct $f'_i$ on $C(U_i)$ using $\Omega_{s_0}$ so that $f'_i$ satisfies the above properties. We call $f'_i$ the union of these functions.

Let $V$ be a $\pi_1(\tilde{\mathcal{O}})$-invariant neighborhood of the complement of $U$ in $\tilde{\mathcal{O}}$. Given $\epsilon > 0$, there exists $\delta > 0$ so that we have an $\epsilon$-$C^2$-map close to the identity map on a compact fundamental domain of the set $V$ to $V := \tilde{\mathcal{O}} - U$ since a developing map of $\mu_t$ is $\delta$-close to that of $\mu_0$ in the $C^2$-sense. We obtain a diffeomorphism $k_t : C(V) \to C(V)$ close to the identity on a compact set in the $C^2$-sense so that $k_t(sv) = sk_t(v)$ for $s > 0$ and $v \in V$ holds for the structure $\mu_0$ and $\mu_t$. (Here the scalar product depends on $\mu_0$ and $\mu_t$.) We transfer $f$ to $C(V)$ by this map. Denote the result by $f''_t := f \circ k_t$ where $f''_t(sv) = s^{-n-1}f''_t(v)$ for $s > 0$ and $v \in V'$. (For example, this can be done by deforming the function $f$ only on a section of the radial flow and extending.)

- Let $\partial_s V$ be a copy of $\partial V \times \{s\}$ inside the regular neighborhood of $\partial V$ in $U_t$ parameterized as $\partial V \times [-1, 1] \times \{s\}$ for $s \in [-1, 1]$.
- We assign $\partial V = \partial_0 V$.
- Let $\partial_{[s_1,s_2]} V$ be the image of $\partial V \times \{[s_1,s_2]\}$ inside the regular neighborhood of $\partial V$ in $V \cap U'$ for a neighborhood $U'$ of $C(U') \cap \tilde{\mathcal{O}}$.

We find a $C^\infty$ map $\phi_t : C(U') \cap C(V) \to \mathbb{R}_+$ so that $\phi_t(sv) = \phi_t(v)$ for every $s > 0$ and $f''_t(v) = \phi_t(v)f''_t(v)$ and $\phi_t$ is very close to the constant value 1 function. By making $f''_t$ near 1 and the derivatives of $f''_t$ up to two near 0 as possible, we obtain $\phi_t$ that has derivatives up to order to two as close to 0 in a compact subset as we wish: This is accomplished by taking a partition of unity functions $p_{1,t}, p_{2,t}$ so that

- $p_{1,t} = 1$ on $C(W)$ for

$$W := \partial_{[0,s_1]} V \cup (U' - V)$$

for $s_1 < 1$,
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• $p_{1,t} = 0$ on $C(\widetilde{O} - N)$ for a neighborhood $N$ of $W$ in $\partial_{(-1,1)} V_t \cup (U' - V)$, and

• $p_{1,t} + p_{2,t} = 1$ identically.

We assume that

$$1 - \epsilon < f'_t / f''_t < 1 + \epsilon$$

in $C(U' \cap V)$, and $f'_t / f''_t$ has derivatives up to order two sufficiently close to 0 by taking $f'_t$ and $f''_t$ sufficiently close in $C(U' \cap C(V)$ by taking sufficiently small $t$. We define

$$\phi_t = (f'_t / f''_t - (1 - \epsilon))p_{1,t} + \epsilon p_{2,t} + (1 - \epsilon)$$

as $f'_t$ and $f''_t$ are homogeneous of degree $-n - 1$. Then $1 - \epsilon < \phi_t < 1 + \epsilon$ and derivatives of $\phi_t$ up to order two are sufficiently close to 0 as we can see easily from computations. Thus, using $\phi_t$ we obtain a Hessian function $f'''_t$ obtained from $f'_t$ and $\phi_t f''_t$ on $C(W)$ and extending them smoothly. We can check the welded function from $f'_t$ and $\phi_t f''_t$ has the desired Hessian properties since the derivatives of $\phi_t$ up to order two can be made sufficiently close to zero. Now we do this for every p-end of $\tilde{O}$.

Finally, suppose that we only required the radial p-ends to be of generalized lens type. Then the arguments above changes for only the case (I) of a generalized radial p-end where a radial p-end of generalized lens-type become a radial p-end of generalized lens-type. Then at $\mu_0$, we obtain $\Omega_{s_0}$ that contains $\tilde{O}$ and $\Omega_{s_0}$ in an $\epsilon$-neighborhood of $\tilde{O}$. (Here we do not need $\Omega_{s_0}$ to be a subset of $\tilde{O}$.) We can obtain such a neighborhood by the methods of Section 7.2 (cf. Lemma 7.8) of [27]. As above, by the third item of Lemma 6.7, the Hessian functions $f'_t$ defined by equation (6.1) on $C(\Omega_{s_0,t})$ deformed from $C(\Omega_{s_0,t})$ using the above methods in (I) is very close to the original Hessian function $f_t$ in compact subsets of $C(\Omega_{s_0,t} \cap \tilde{O})$ in the $C^2$ topology by Lemma 6.6. Now we operate as above using possibly not necessarily convex neighborhoods of the radial p-ends to obtain the Hessian metric for $C(\tilde{O})$.

The $-(n + 1)$-homogeneity gives us the invariance of the Hessian metric under the dilatations and the affine lifts of the holonomy groups. (See Chapter 6 of [48].)

\[ \square \]

Figure 6.1. The diagram for Lemma 6.7.
6.1. The Proof of the Convexity Theorem

Lemma 6.6. Let $V$ be a properly convex cone and let $V^*$ be a dual cone. Suppose that a Koszul-Vinberg function $f_{V^*}(x)$ is defined on a compact neighborhood $B$ of $x$ contained in a convex cone $V$. Let $V_1$ be another properly convex cone containing the same neighborhood. Let $V^*$ has projectivization $Ω$ and the dual $V_1^*$ of $V_1$ has a projectivization $Ω_1$. For given any integer $s \geq 1$ and $δ > 0$, there exists $δ > 0$ so that if the Hausdorff distance between $Ω$ and $Ω_1$ is $δ$-close, then $f_{V^*}(x)$ and $f_{V_1^*}(x)$ are $ε$-close in $B$ in the $C^s$-topology.

Proof. By Lemma 6.7, we have
\[ Ω^* \subset Ω_1^*, \quad Ω_1^* \subset N_δ(Ω^*), \]
\[ (Ω - N_δ(∂Ω))^* \subset Ω_1^*, \quad \text{and} \]
\[ (Ω - N_δ(∂Ω_1))^* \subset Ω^*. \]

We choose sufficiently small $δ > 0$ so that
\[ B \subset Ω - N_δ(∂Ω), \quad Ω_1 - N_δ(∂Ω_1). \]

Since the integral is computable from an affine hyperspace meeting $V^*$ and $V_1^*$ in bounded precompact convex sets and $e^{-φ(x)}$ and its derivatives for $φ$ in the domains are uniformly bounded, the integrals and their derivatives are estimable from each other, the result follows by taking the Hausdorff distances of $Ω^*$ and $Ω_1^*$ sufficiently small. (See the proof of Theorem 6.4 of [48].)

Recall the standard elliptic metric $d$ of $\mathbb{R}P^n$. We also have the elliptic metric $d$ on $\mathbb{R}P^{n*}$, denoted by the same letter. Define the thickness of a properly convex domain $Δ$ is given as
\[ \min\{\max\{d(x, bdΔ)|x \in Δ\}, \max\{d(y, bdΔ^*)|y \in Δ^*\}\} \]

for the dual $Δ^*$ of $Δ$.

Lemma 6.7. Let $Δ$ be a properly convex open domain in $\mathbb{R}P^n$ and $Δ^*$ its dual in $\mathbb{R}P^{n*}$. Let $ε$ be a positive number less than the thickness of $Δ$. Then the following hold:

- $N_ε(Δ) \subset (Δ^* - Cl(N_ε(bdΔ^*))^*$.
- If two properly convex open domains $Δ_1$ and $Δ_2$ are of Hausdorff distance $< ε$ for $ε$ less than the thickness of each $Δ_1$ and $Δ_2$, then $Δ_1^*$ and $Δ_2^*$ are of Hausdorff distance $< ε$.
- Furthermore, if $Δ_2 \subset N_ε′(Δ_1)$ and $Δ_1 \subset N_ε′(Δ_2)$ for $0 < ε' < ε$, then we have $Δ_2^* \supset \Delta_1^* - Cl(N_ε(∂Δ_1^*))$ and $Δ_1^* \supset Δ_2^* - Cl(N_ε(∂Δ_2^*))$.

Proof. Using the double covering map $S^n \to \mathbb{R}P^n$ and $S^{n*} \to \mathbb{R}P^{n*}$ of unit spheres in $\mathbb{R}n+1$ and $\mathbb{R}n+1*$, we take components of $Δ$ and $Δ^*$. It is easy to show that the result for properly convex open domains in $S^n$ and $S^{n*}$ is sufficient.

For elements $φ \in S^{n*}$ and $x \in S^n, S^n$, we say $φ(x) < 0$ if $f(v) < 0$ for $φ = [f], x = [v]$ for $f \in \mathbb{R}n+1, v \in \mathbb{R}n+1$.

For the first item, let $y \in N_ε(Δ)$. Suppose that $φ(y) < 0$ for
\[ φ \in Cl((Δ^* - Cl(N_ε(bdΔ^*)) \neq \emptyset. \]

Since $φ \in Δ^*$, the set of positive valued points of $S^n$ under $φ$ is an open hemisphere $H$ containing $Δ$ but not containing $y$. The boundary $bdH$ of $H$ has a closest point $z \in bdΔ$ of distance $≤ ε$. The closest point $z'$ on $bdH$ is in $N_ε(Δ)$ since $y$ is in $N_ε(Δ) - H$ and $z'$ is closest to $bdΔ$. The great circle $S^1$ containing $z$ and $z'$ are
perpendicular to bd$H$ since $\overline{z\overline{z}}$ is minimizing lengths. Hence $S^1$ passes the center of the hemisphere. One can push the center of the hemisphere on $S^1$ until it becomes a supporting hemisphere to $\Delta$. The corresponding $\phi'$ is in bd$\Delta^*$ and the distance between $\phi$ and $\phi'$ is less than $\epsilon$. This is a contradiction. Thus, the first item holds (See Figure 6.1.)

For the final item, we have that

$$\Delta_2 \subset N_{\epsilon'}(\Delta_1), \Delta_1 \subset N_{\epsilon'}(\Delta_2) \text{ for } 0 < \epsilon' < \epsilon.$$ 

Hence, $\Delta_2 \subset (\Delta_1^* - \text{Cl}(N_{\epsilon'}(\text{bd}\Delta_1^*)))^*$. Thus, $\Delta_2^* \supset \Delta_1^* - \text{Cl}(N_{\epsilon'}(\text{bd}\Delta_1^*))$, which proves the third item, and so $N_{\epsilon}(\Delta_2^*) \supset \Delta_1^*$ and conversely. The second item follows. □

**Proof.** [The proof of Theorem 6.1] Suppose that $\mathcal{O}$ has an SPC-structure $\mu$ with generalized admissible ends. We will show that a sufficiently close structure $\mu_s$ that has generalized admissible ends is also SPC. Let $h^t : \pi_1(\mathcal{O}) \to \text{SL}_+(n + 1, \mathbb{R})$ be the lift of the holonomy homomorphism corresponding to $\mu_s$.

Let $\mathcal{O} := C(\mathcal{O}) / h_s(\pi_1(\mathcal{O}))$ with $C(\mathcal{O})$ as the universal cover. Let $\tilde{\mathcal{O}}_s$ denote $\tilde{\mathcal{O}}$ with $\mu_s$. One applies special affine suspension to obtain an affine orbifold $\mathcal{O} \times S^1$. (See Section 2.10.2.) The universal cover is still $C(\tilde{\mathcal{O}})$ and has a corresponding affine structure $\tilde{\mu}_s$. We denote $C(\tilde{\mathcal{O}})$ with the lifted affine structure of $\tilde{\mu}_s$ by $C(\tilde{\mathcal{O}})_s$. Recall the projective completion $\tilde{C}(\tilde{\mathcal{O}})_s$ of $C(\tilde{\mathcal{O}})_s$. This is a completion of $C(\tilde{\mathcal{O}})_s$ the path metric induced from the pull-back of the standard Riemannian metric on $S^{n+1}$ by the developing map $D_s$ of $\tilde{\mu}_s$. (Here the image is in $\mathbb{R}^{n+1}$ as an affine subspace of $S^{n+1}$.) The developing maps always extend to one on $C(\tilde{\mathcal{O}})_s$ which we denote by $D_s$ again. (See [19] and [21] for details.)

By Proposition 6.5, an affine suspension $\tilde{\mu}_s$ of $\mu_s$ also have a Hessian function $\phi$. The Hessian metric $D\phi$ is invariant under affine automorphism groups of $C(\mathcal{O})$ by construction. We prove that $\tilde{\mu}_s$ is properly convex, which will show $\mu_s$ is properly convex:

Suppose that $\tilde{\mu}_s$ is not convex. Then there exists a triangle imbedded in $\tilde{C}(\tilde{\mathcal{O}})_s$ with points in the interior of an edge in the limit set $\Lambda_s := \tilde{C}(\tilde{\mathcal{O}})_s - C(\tilde{\mathcal{O}})_s$. We can move the triangle so that the interior of an edge $l$ has a point $x_{\infty}$ in $\Lambda_s$ and $D_s(l)$ does not pass the origin. We form a parameter of geodesics $l, t \in [0, \epsilon]$ in the triangle so that $l_0 = l$ and $l_t \subset C(\tilde{\mathcal{O}})_s$ is close to $l$ in the triangle. (See Theorem A.2 of [21] for details.)

Let $p, q$ be the endpoints of $l$. Then the Hessian metric is $D^*d\phi$ for a function $\phi$ defined on $C(\mathcal{O})_s$. And $d\phi p$ and $d\phi q$ are bounded, where $D^*$ is the affine connection of $\mu_s$. This should be true for $p_t$ and $q_t$ for sufficiently small $t$ uniformly. Let $u, u \in [0, 1]$, be the affine parameter of $l_t$, i.e., $l_t(s)$ is a constant speed line in $\mathbb{R}^{n+1}$ when developed. We assume that $u \in (\epsilon_t, 1 - \epsilon_t)$ parameterize $l_t$ for sufficiently small $t$ where $\epsilon_t \to 0$ as $t \to 0$ and $dl_t / ds = v$ for a parallel vector $v$. The function $D^*_t d_v \phi(l_t(u))$ is uniformly bounded since its integral $d_v \phi(l_t(u))$ is strictly increasing by the strict convexity and converges to certain values as $u \to \epsilon_t, 1 - \epsilon_t$.

Since

$$\int_{\epsilon_t}^{1 - \epsilon_t} D^*_t d_v \phi(l_t(s)) du = d\phi(p_t)(v) - d\phi(q_t)(v),$$

the function $\sqrt{D^*_v d_v \phi(l_t(u))}$ is also integrable and have a bounded integral by Jensen’s inequality. This means that the length of $l_t$ is bounded.
Let $U$ be a union of disjoint end-neighborhoods of $\mathcal{O}$; $U$ correspond to an inverse image $\tilde{U}$ in $\tilde{\mathcal{O}}$ and to $C(\tilde{U})$ the inverse image in $C(\tilde{\mathcal{O}})$. The minimum distance between components of $U$ is bounded below since the metric is invariant under dilatations $x \mapsto tx$ in $C(\tilde{\mathcal{O}})$. If $l$ meets infinitely many components of $C(\tilde{U})$, then the length is infinite because of this.

As $t \to 0$, the number is thus bounded, $l$ can be divided into finite subsections, each of which meets one component of $C(\tilde{U})$. Any subarc of each with end points in the boundary of a component $C_1$ of $C(\tilde{U})$ is homotopic into a component $C_1$ with endpoints fixed.

Let $\hat{l}$ be the subsegment of $l$ in $C(\tilde{\mathcal{O}})$ containing $x_\infty$ in the ideal boundary and meeting only one component $C(U_1)$ of $C(\tilde{U})$ with $\partial C(U_1)$ in $x_\infty$ is on a line corresponding to the p-end vertex of the radial lines of $U_1$. We project to $\mathbb{S}^n$ from by the projection $\Pi : \mathbb{R}^{n+1} - \{O\} \to \mathbb{R}P^n$. Now suppose that $\Pi(x_\infty)$ is in the middle of the radial line. Then the interior of the triangle is transversal to the radial lines. Since our end orbifold is convex, there cannot be such a line with a single interior point in the ideal set.

If $C(U_1)$ is the inverse image in $C(\tilde{\mathcal{O}})$ of a p-end neighborhood $U_1$ of radial of lens-type or horospherical type in $\tilde{\mathcal{O}}$, and $x_\infty$ is on a line corresponding to the p-end vertex of the radial lines of $U_1$. We project to $\mathbb{S}^n$ from by the projection $\Pi : \mathbb{R}^{n+1} - \{O\} \to \mathbb{R}P^n$. Now suppose that $\Pi(x_\infty)$ is in the middle of the radial line. Then the interior of the triangle is transversal to the radial lines. Since our end orbifold is convex, there cannot be such a line with a single interior point in the ideal set.

This is again a contradiction. Therefore, $\mathcal{O}$ is convex.

Finally, for sufficiently small deformations, the convex real projective structures are properly convex. If not, then there are sufficiently small deformed convex real projective structures which are not properly convex and hence their holonomy homomorphism is reducible. By taking limits, the original one has to be reducible as well. However, we assumed that it was irreducible. Since the subspace of reducible representation is closed, we see that there is an open set of irreducible properly convex projective structures near the initial one $\mu$.

Suppose now that $\mathcal{O}$ with $\mu$ is strictly SPC with admissible ends. The relative hyperbolicity of $\mathcal{O}$ with respect to the p-ends is stable under small deformations since it is a metric property invariant under quasi-isometries by Theorem 5.13.

The irreducibility and the stability follow since these are open conditions. Also, the ends are admissible.

Theorem 6.2 also follows similarly. Hence Corollary 6.3 and 6.4 follow by Theorem 1.2.
CHAPTER 7

The closedness of convex real projective structures

We recall \( \text{rep}_E^s(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \) the subspace of stable irreducible characters of \( \text{rep}_E(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \) which is shown to be an open subset of a semialgebraic set in Section 3.1, and denote by \( \text{rep}_{E,\text{u,ce}}^s(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \) the subspace of stable irreducible characters of \( \text{rep}_{E,\text{u,ce}}(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \), an open subset of a semialgebraic set.

In this section, we will need to discuss \( S^n \) but only inside a proof.

**Theorem 7.1.** Let \( O \) be a noncompact strongly tame SPC \( n \)-orbifold with generalized admissible ends and satisfies (IE) and (NA). Assume \( \partial O = \emptyset \). Assume that every finite index subgroup of \( \pi_1(O) \) has no nontrivial nilpotent normal subgroup. Then the following hold:

- The deformation space \( C\text{Def}_{E,\text{u,ce}}(O) \) of SPC-structures on \( O \) with generalized admissible ends maps under \( \text{hol} \) homeomorphically to a union of components of \( \text{rep}_{E,\text{u,ce}}^s(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \).
- The deformation space \( S\text{Def}_{E,\text{u,ce}}(O) \) of SPC-structures on \( O \) with admissible ends maps under \( \text{hol} \) homeomorphically to the union of components of \( \text{rep}_{E,\text{u,ce}}^s(\pi_1(O), \text{PGL}(n+1, \mathbb{R})) \).

**Proof.** Define \( \widetilde{C\text{Def}}_{E,\text{u,ce}}(O) \) to be the inverse image of \( C\text{Def}_{E,\text{u,ce}}(O) \) in \( \widetilde{\text{Def}}_{E,\text{u,ce}}(O) \). We show that

\[
\text{hol} : \widetilde{C\text{Def}}_{E,\text{u,ce}}(O) \to \text{Hom}_{E,\text{u,ce}}^s(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))
\]

is a homeomorphism onto a union of components. This will imply the results.

Suppose that the map is not injective. Then there exists a homomorphism \( h : \pi_1(O) \to \text{PGL}(n+1, \mathbb{R}) \) and properly convex open domains \( \Omega_1 \) and \( \Omega_2 \) where \( h(\pi_1(O)) \) acts properly on so that \( \Omega_1/h(\pi_1(O)) \) and \( \Omega_2/h(\pi_1(O)) \) are both diffeomorphic to \( O \) by diffeomorphisms inducing \( h \).

Suppose that \( \Omega_1 \) and \( \Omega_2 \) are distinct in \( \mathbb{R}P^n \). We claim that \( \Omega_1 \) and \( \Omega_2 \) are disjoint: Suppose not. Then let \( \Omega' \) be the intersection \( \Omega_1 \cap \Omega_2 \) where \( \Gamma := h(\pi_1(O)) \) acts. Each p-end fundamental group also acts on \( \Omega' \) also. We can form a topological space \( \Omega'/\Gamma \) with end neighborhood system. Since \( \Omega_1, \Omega_2, \) and \( \Omega' \) are all \( n \)-cells, the set of p-ends of \( \Omega_1 \), the set of those of \( \Omega_2 \), and the set of those of \( \Omega' \) are in one-to-one correspondences since the end groups uniquely determines the end vertex and ideal totally geodesic boundary inside \( \Omega_1, \Omega_2, \) and \( \Omega' \) respectively by the uniqueness condition for each end holonomy group. (We need to see the orbits of points of \( \Omega' \) under the end fundamental group.) Also, using concave p-end neighborhoods for radial p-ends, lens p-end neighborhoods for totally geodesic p-ends, and horoball p-end neighborhoods of p-ends, we verify easily that a p-end neighborhood of \( \Omega_1 \) exists if and only if a p-end neighborhood of \( \Omega_2 \) exists and their intersection is
a p-end neighborhood of $\Omega'$. By taking a torsion-free finite-index subgroup $\Gamma'$ of $\Gamma$ using Selberg’s Lemma, $\Omega'/\Gamma'$ is a closed submanifold in $\Omega_1/\Gamma'$ and in $\Omega_2/\Gamma'$. Thus, $\Omega_1/\Gamma'$, $\Omega_2/\Gamma'$, and $\Omega'/\Gamma'$ are all homotopy equivalent relative to the union of disjoint end-neighborhoods. The map has to be onto in order for the map to be a homotopy equivalence as we can show using relative homology theories, and hence, $\Omega' = \Omega_1 = \Omega_2$.

Suppose now that $\Omega_1$ and $\Omega_2$ are disjoint. Each corresponding pair of the p-end neighborhoods share p-end vertices or have antipodal p-end vertices. Since $\Omega_1$ and $\Omega_2$ are disjoint but each pair of the p-ends have same p-end holonomy groups. Now $\text{Cl}(\Omega_1) \cap \text{Cl}(\Omega_2)$ or $\text{Cl}(\Omega_1) \cap A(\text{Cl}(\Omega_2))$ is a compact properly convex subset $K$ of dimension $< n$ and is not empty since the fixed points of the p-ends are in it. The minimal hyperspace containing $K$ is a proper subspace and is invariant under $h(\pi_1(\mathcal{O}))$. This contradicts the irreducibility.

Hence, this proves that hol is injective. hol is an open map by Theorem 6.1 and Theorem A. Actually, this show that a strongly tame SPC-orbifold of given end types is uniquely determined by each holonomy group.

To show that the image is of hol is closed, the subset of

$$\text{Hom}^*_{E,u,ce}(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R}))$$

corresponding to elements in $\widetilde{\text{ClDef}}_{E,u,ce}(\mathcal{O})$ is closed. Let $(\text{dev}, h_i)$ be a sequence of development pairs so that we have $h_i \to h$ algebraically. Let $\Omega_i = \text{dev}(\mathcal{O})$ denote the properly convex domains. The limit $h$ is a discrete representation by Lemma 1.1 of Goldman-Millson [51]. The sequence $\text{Cl}(\Omega_i)$ also converges to a compact convex set $\Omega$ up to choosing a subsequence where $h(\pi_1(\mathcal{O}))$ acts on as in [30]. If $\Omega$ is not properly convex or have the empty interior, $h$ is reducible. Thus, $\Omega^o$ is not empty and is properly convex. As in [30], since $\Omega^o$ has a Hilbert metric, $h(\pi_1(\mathcal{O}))$ acts on $\Omega^o$ properly discontinuously.

The condition of the generalized lens-shapedness is a closed condition in the

$$\text{Hom}^*_{E,u,ce}(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R}))$$

as we defined above. The ends of the orbifold $\mathcal{O}' := \Omega'/h(\pi_1(\mathcal{O}))$ are generalized admissible since the holonomy conditions of the ends insure this by Theorems 7.9 and 7.11 of [27]. We can deform $\mathcal{O}'$ using the openness of hol by Theorem 6.1. We can find a deformed orbifold $\mathcal{O}''$ that has a holonomy $h_i$ for some large $i$. Hence, $\mathcal{O}''$ is diffeomorphic to $\mathcal{O}$ since they share the same open domain as universal cover by the uniqueness above for each holonomy group. By openness of hol for $\mathcal{O}'$, $\mathcal{O}''$ is diffeomorphic to $\mathcal{O}'$. Hence, $\mathcal{O}'$ is diffeomorphic to $\mathcal{O}$.

Therefore, we conclude that $\widetilde{\text{ClDef}}_{E,u,ce}(\mathcal{O})$ goes to a closed subset of

$$\text{Hom}^*_{E,u,ce}(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R})).$$

These imply the first item.

Now, we go to the second item. Define $\widetilde{\text{SDef}}_{E,ce}(\mathcal{O})$ to be the inverse image of $\text{SDef}_{E,ce}(\mathcal{O})$ in $\widetilde{\text{ClDef}}_{E,ce}(\mathcal{O})$. We show that

$$\text{hol} : \widetilde{\text{SDef}}_{E,ce}(\mathcal{O}) \to \text{Hom}^*_{E,u,ce}(\pi_1(\mathcal{O}), \text{PGL}(n + 1, \mathbb{R}))$$

is a homeomorphism onto a union of components. Theorem 6.1 shows that hol is a local homeomorphism to an open set. The injectivity of hol follows the same way as in the above item.
We now show the closedness. By Theorem 5.7, $\pi_1(\mathcal{O})$ is relatively hyperbolic with respect to the admissible end fundamental groups. Let $h$ be the limit of a sequence of holonomy representations $h_i : \pi_1(\mathcal{O}) \to \text{PGL}(n+1, \mathbb{R})$. As above we obtain $\Omega$ as the limit of $\text{Cl}(\Omega_i)$ where $\Omega_i$ is the image of the developing map associated with $h_i$. $\Omega$ is properly convex and $\Omega^o$ is not empty. Since $h$ is irreducible and acts on $\Omega^o$ properly discontinuously, it follows that $\Omega^o/h(\pi_1(\mathcal{O}))$ is an orbifold $\mathcal{O}'$ homotopy equivalent to $\mathcal{O}$ and with generalized admissible ends as above. By Theorem 5.9, $\mathcal{O}'$ is a strict SPC-orbifold with admissible ends. The rest is the same as above.

□

Remark 7.2 (Thurston’s example). We remark that without the end controls we have, there might be counter-examples as we can see from the examples of geometric limits differing from algebraic limits for sequences of hyperbolic 3-manifolds. (See Anderson-Canary [2].)

Recall that an affine subspace of $\mathbb{S}^n$ is an open subspace mapping to an affine subspace of $\mathbb{R}P^n$. Let us choose an affine subspace $A$ and give a coordinate system on $A$. Let $B_R$ denote the ball of radius $R$ with the center at the origin.

Recall the dual sphere $\mathbb{S}^n$ as the space $\mathbb{R}^{n+1} - \{0\}/\sim$ where two vectors are equivalent iff they are positive scalar multiples of each other. Given a properly convex open domain $\Omega \subset \mathbb{S}^n$, the dual domain $\Omega^*$ is the set

$$\{[f] | f \in \mathbb{R}^{n+1}, f(x) > 0 \text{ for all } x \in \text{Cl}(\mathcal{C} \Omega) - \{0\}\}.$$ $\Omega^*$ is properly convex and open also.

We can drop the superscript $s$ from the above space. Hence, the components consist of stable irreducible characters. This is a stronger result.

Corollary 7.3. Let $\mathcal{O}$ be a noncompact strongly tame SPC $n$-dimensional orbifold with admissible ends and satisfies (IE) and (NA). Assume $\partial \mathcal{O} = \emptyset$. Assume that no finite-index subgroups $\pi_1(\mathcal{O})$ has a nontrivial nilpotent normal subgroup. Then hol maps the deformation space $\text{CDef}_{E,u,ce}(\mathcal{O})$ of SPC-structures on $\mathcal{O}$ homeomorphic to a union of components of $\text{rep}_{E,u,ce}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$.

The same can be said for $\text{SDef}_{E,u,ce}(\mathcal{O})$.

Proof. We will show that the image of $\widehat{\text{CDef}}_{E,u,ce}(\mathcal{O})$ under hol in $\text{Hom}^{*}_{E,u,ce}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R}))$ is closed and consists of stable irreducible characters.

It is sufficient to show the closedness of the subspace of $\text{Hom}_{E,u,ce}(\pi_1(\mathcal{O}), \text{SL}_{\pm}(n+1, \mathbb{R}))$ of holonomy homomorphisms of elements of $\text{CDef}_{E,u,ce}(\mathcal{O})$. This again follows from the closedness of the subspace of $\text{Hom}_{E,u,ce}(\pi_1(\mathcal{O}), \text{SL}_{\pm}(n+1, \mathbb{R}))$ of lifted holonomy homomorphisms of elements of $\text{CDef}_{E,u,ce}(\mathcal{O})$.

Using Theorem 3.19, let $h_i : \pi_1(\mathcal{O}) \to \text{SL}_{\pm}(n+1, \mathbb{R})$ be a sequence of holonomy homomorphisms of real projective structures corresponding to liftings of elements of $\text{CDef}_{E,u,ce}(\mathcal{O})$. Let $\Omega_i$ be the sequence of associated properly convex domains in $\mathbb{S}^n$.
and $\Omega_i/h_i(\pi_1(\mathcal{O}))$ is diffeomorphic to $\mathcal{O}$ and has the structure that lifts an element of $\text{CDef}_{E,u,ce}(\mathcal{O})$. We assume that $h_i \rightarrow h$ algebraically, i.e., for a fixed set of generators $g_1, \ldots, g_m$ of $\pi_1(\mathcal{O})$, $h_i(g_j) \rightarrow h(g_j) \in \text{SL}_+ (n+1, \mathbb{R})$ as $i \rightarrow \infty$. We will show that $h$ is a lifted holonomy homomorphism of an element of $\text{CDef}_{E,u,ce}(\mathcal{O})$, and hence $h$ is stable and strongly irreducible.

We take a dual domain $\Omega_i^* \subset \mathbb{S}^{n*}$ Then the sequence $\{\text{Cl}(\Omega_i^*)\}$ also converges geometrically to convex compact set $K^*$. If $K$ has an empty interior and properly convex, then we can show easily that $K^*$ has a nonempty interior since for any 1-form $\alpha$ positive on the cone $C_K$, any sufficiently close 1-form is still positive on $C_K$. Also, if $K^*$ has an empty interior and properly convex, $K$ has a nonempty interior.

(I) The first step is to show that at least one of $K, K^*$ has nonempty interior.

Suppose that both $K$ and $K^*$ are not properly convex and have empty interior. If there exists a radial $p$-end for $\Omega_i$ and the type does not become horospherical, then the mc-p-end neighborhood must be in $K$ since this is true for all structures in $\Omega_i$ and holonomy homomorphisms in

$$\text{Hom}_{E,u,ce}(\pi_1(\mathcal{O}), \text{SL}_+ (n+1, \mathbb{R}))$$

where for each p-end $\tilde{E}$, there exists a distorted compact set $L$ away from $v_{\tilde{E}}$, and the lens-cone $v_{\tilde{E}} * L - \{v_{\tilde{E}}\} \subset K$ has a nonempty interior. If there is a totally geodesic end of lens-type for $\mathcal{O}$ and the type does not become horospherical, then the dual $\Omega_i^*$ and $K^*$ have nonempty interiors. These are contradictions.

Suppose now that $\Omega_i$ has only horospherical ends or the sequence of the end holonomy groups converge to horospherical ones. Put by choosing $\Omega_i^*$ if necessary, we can assume that there is a p-end vertex $v_i \in \text{bd}\Omega_i$ where $\pi_1(\tilde{E})$ is the fixed associated p-end fundamental group. Moreover $v_i = v$ for a fixed vertex $v$ by conjugating by a bounded sequence of projective automorphisms. The generators $g_j \in \pi_1(\tilde{E})$ for $j = 1, \ldots, m$ for some $m$ are convergent. There exists a fundamental domain $F_i$ in $S_{\tilde{E}} = R_{v_i}(\Omega_i)$. Since $\{h_i(g_j)\}$ is convergent for each $j$, we can choose $F_i$ so that $\{F_i \subset \mathbb{S}^{n-1}_v\}$ is geometrically convergent. We choose a great segment $s_i$ with vertex $v_i$ in a direction of $F_i$.

(7.1) For $l_i := \Omega_i \cap s_i$, $\text{d-length}(l_i) \rightarrow 0$.

If not, then the convex hull $C(h_i(\pi_1(\tilde{E}))(s_i)) \subset \Omega_i$ contains balls of fixed radius since $h_i(g_j)$ are convergent to an element of a parabolic group in a copy $P$ of $\text{PO}(n,1)$ for each $j$ as $i \rightarrow \infty$.

Let $h(\pi_1(\tilde{E}))$ be the algebraic limit $h_i(\pi_1(\tilde{E}))$. Then $P \cap h(\pi_1(\tilde{E}))$ is a lattice in $P$.

**Lemma 7.4.** Let $v$ be a fixed point of $P$ and let $L$ be a lattice in $P$. Let $H$ be a $P$-invariant hemisphere with $v$ in the boundary, and let $l$ be the maximal perpendicular line with endpoints $v$ and $v_\perp$. Then there exists a finite subset $F$ of $L$ so that for any point $x \in l$ and a $d$-perpendicular hyperspace at $x$ bounding a closed hemisphere $H_x$, $I_x := \bigcap_{g \in F} g(H_x) \cap H$ is a properly convex domain, and as $x \rightarrow v$, $I_x$ geometrically converges to $\{v\}$.

**Proof.** If $F$ is large enough, then $\{g(H_x), H\}$ is in a general position. The last fact follows by considering the set of outer normal vectors of $\{g(\partial H_x)\}$ in an affine space where $H$ is a half-space. \[\square\]
Let $H$ denote the $P$-invariant hemisphere containing $K$. We assume that $\Omega_i \subset H$ and recall that radial p-end vertices are fixed to be $v$. We assume that the direction of $l$ for $h(\pi_1(E))$ is in $F_i$ always.

Let $\epsilon_i$ be the maximum $d$-length of a maximal segment $s_i'$ in $\Omega_i$ from $v_i$ in direction of $F_i$ for $i \geq I$. Let $F_i'$ denote the set of endpoints of the maximal segments in $\Omega_i$ in direction of $F_i$. Then $\epsilon_i \to 0$ by the above argument. A hyperplane perpendicular to $l$ at $x_i' \in l$ bounds a closed hemisphere $H_i'$ containing $F_i'$. $d(v, x_i) = \delta_i$ satisfy $\{\delta_i\} \to 0$ since otherwise equation (7.1) does not hold. By Lemma 7.4, there is a finite set $F \subset \pi_1(E)$ so that $K_i := \bigcap_{g \in F} h_j(g)(H_i') \cap H$ is properly convex for sufficiently large $j$ since $h_j(g) \to h(g), g \in F$. This set contains $\text{Cl}(\Omega_i)$ since $H_i' \supset \text{Cl}(\Omega_i)$. As $x_i' \to 0$, it follows that $K_i \to \{v\}$ since $x_i' \to v$. (We just need to show that the normal vectors of $h_i(g)(\partial H_i')$, $g \in F$ being sufficiently larger different from that of $H$. Since $h(g)(\partial H_i')$, $g \in F$ is very close to these for sufficiently large $i$, we are done.)

Therefore, we conclude that $K$ is a singleton.

In case, $K$ is a singleton, $K^*$ must be a hemisphere by duality of $\Omega_i$ and $\Omega_i'$. We now conclude that $K$ or its dual $K^*$ has a nonempty interior.

Thus, by choosing $h_i^*$ and $h^*$ if necessary, we may assume without loss of generality that $K$ has nonempty interior. We will show that $K$ is a properly convex domain and this implies that so is $K^*$.

(II) The second step is to show $K$ is properly convex.

Assume that $h(\pi_1(O))$ acts on a convex open domain $K^\circ$. We may assume that $K^\circ \subset A$ for an affine subspace $A$ and $\Omega_i \subset A$ as well by acting by an orthogonal $k_i \in \text{SL}_+(n+1, \mathbb{R})$ converging to $I$. We can accomplish this by moving $\Omega_i$ into $A$.

Also, we may assume that $A$ contains a unit ball $B_1 \subset \Omega_i$ for all $i$. Choose $x_0 \in B_1$ as the origin in the affine coordinates.

Let $g_1, \ldots, g_m$ denote the set of generator of $\pi_1(O)$. Then by extracting subsequences, we may assume without loss of generality that $h_i(g_j) \to h(g_j)$ for each $j = 1, \ldots, m$.

First,

\begin{equation}
(7.2) \quad d(h_i(g_j)(x_0), \text{bd}\Omega_i) \geq C_0 \text{ for a uniform constant } C_0 \, .
\end{equation}

If not, then there is a sequence of a $d$-length constant segment $s_i$ with an origin $x_0$ is sent to the segment $h_i(g_j)(s_i)$ in $\Omega_i$ with end point $h_i(g_j)(x_0)$ and lying on the shortest $d$-length segment from $h_i(g_j)(x_0)$ to $\text{bd}\Omega_i$. Thus, the sequence of the $d$-segment $h_i(g_j)(s_i)$ is going to zero. This implies that $h_i(g_j)$ is not in a compact subset of $\text{SL}_+(n+1, \mathbb{R})$, a contradiction.

By estimation from equation (7.2), a uniform constant $C$ satisfies

\begin{equation}
(7.3) \quad d_{\bar{\Omega}}(x_0, h_i(g_j)(x_0)) < C.
\end{equation}

By Benécri (see C.24 of Goldman [48]), there exists a constant $R_B > 1$ and $\tau_i \in \text{SL}_+(n+1, \mathbb{R})$ so that

\begin{equation}
B_1 \subset \tau_i(\Omega_i) \subset B_R.
\end{equation}

Now, $\tau_i h_i(\pi_1(O)) \tau_i^{-1}$ acts on $\tau_i(\Omega_i)$. Then as in the proof of Theorem 7.1 of Cooper-Long-Tillman [36], we obtain that $\tau_i h_i(g_j) \tau_i^{-1}$ for $j = 1, \ldots, n$ are in a compact subset of $\text{SL}_+(n+1, \mathbb{R})$ independent of $i$. (Theorem 7.1 of [36] is not enough but their proof is sufficient to show this.)
Therefore, up to choosing subsequences, we have \( \tau_\iota(\Omega_\iota) \) geometrically converges to a properly convex domain \( \tilde{K} \) in \( \mathcal{B}_R \) containing \( B_1 \) and \( \tau_\iota h(g_\iota) \tau_\iota^{-1} \) converges to a holonomy homomorphism \( h' : \pi_1(\mathcal{O}) \rightarrow \text{SL}_\pm(n+1, \mathbb{R}) \). And the image of \( h' \) acts on a properly convex domain \( \tilde{K} \).

Suppose that the sequence \( \{ \tau_\iota \} \) is not bounded. Then \( \tau_\iota = \kappa_i d_i k_i' \) where \( d_i \) is diagonal with respect to a standard basis of \( \mathbb{R}^{n+1} \) and \( k_i, k_i' \in \text{O}(n+1, \mathbb{R}) \) by the KTK-decomposition of \( \text{SL}_\pm(n+1, \mathbb{R}) \). Then the sequence of the maximum modulus of the eigenvalues of \( d_i \) are not bounded above. We assume without loss of generality \( k_i \rightarrow k \) and \( k_i' \rightarrow k' \) in \( \text{O}(n+1, \mathbb{R}) \). Thus, \( k_i' h(g_j) k_i'^{-1} \) converges to \( k' h(g_j) k'^{-1} \) for \( k' \in \text{O}(n+1, \mathbb{R}) \). Since \( k_i d_i k_i' h(g_j) k_i'^{-1} d_i^{-1} k_i^{-1} \) is convergent, and the sequence of the norms of \( d_i \) is divergent, \( \{ d_i k_i' h(\pi_1(\mathcal{O})) k_i'^{-1} d_i^{-1} \} \) converges to a reducible group acting on \( k^{-1}(K) \). (Because the sequences of some of the entries must become zero under the conjugation by a sequence of unbounded diagonal matrices.) By following Lemma 7.5, and Theorem 4.26, the limit of \( \{ d_i k_i' h(\pi_1(\mathcal{O})) k_i'^{-1} d_i^{-1} \} \) cannot be reducible. Therefore the sequence of the norms of \( d_i \) is uniformly bounded.

Thus the norm of \( d_i \) is uniformly bounded. We may assume without loss of generality that \( \tau_\iota \) converges to an element \( \tau \in \text{SL}_\pm(n+1, \mathbb{R}) \). We assumed above that \( h_\iota \rightarrow h \). By the above, \( \text{Cl}(\Omega_\iota) \rightarrow \tau(\tilde{K}) \) and \( h(\pi_1(\mathcal{O})) \) acts on \( \tau(\tilde{K}) \).

By the following Lemma 7.5, we obtain that \( \tau(\tilde{K}/h(\pi_1(\mathcal{O}))) \) is a strongly tame SPC-orbifold with generalized admissible ends. This completes the proof for \( \text{CDef}_{\mathcal{R},u,ce}(\mathcal{O}) \).

By the condition on admissibility of the ends, we see that

**Lemma 7.5.** Assume that no finite index subgroup of \( \pi_1(\mathcal{O}) \) contains a normal infinite nilpotent subgroup. Let \( h_\iota \in \text{Hom}(\pi_1(\mathcal{O})), \text{SL}_\pm(n+1, \mathbb{R}) \), \( \Omega_\iota \) a properly convex open domain in \( S^n \), and let \( \Omega_\iota/h_\iota(\pi_1(\mathcal{O})) \) be an \( n \)-dimensional noncompact strongly tame SPC-orbifold with admissible ends and satisfies (IE) and (NA) for each \( \iota \). Assume that the end fundamental groups of \( h_\iota(\pi_1(\mathcal{O})) \) have fixed types. Suppose that \( h_\iota \rightarrow h \) algebraically and \( \text{Cl}(\Omega_\iota) \rightarrow K \) for a compact properly convex domain \( K \subset S^n, K \neq \emptyset \). Then

- \( K^\circ/h(\pi_1(\mathcal{O})) \) is an SPC-orbifold with generalized admissible ends to be denoted \( \mathcal{O}_h \) diffeomorphic to \( \mathcal{O} \).
- For each \( p \)-end \( \tilde{E} \) of the universal cover \( \tilde{O}_h \) of \( \mathcal{O}_h \), \( K^\circ \) has a subgroup \( h(\pi_1(\tilde{E})) \) and \( h(\pi_1(\tilde{E})) \)-invariant open set \( U_{\tilde{E}} \) corresponding fixed vertex \( v_{\tilde{E}} \) or a totally geodesic domain \( S_{\tilde{E}} \).
- \( U_{\tilde{E}}/h(\pi_1(\tilde{E})) \) is real projectively diffeomorphic to an end neighborhood, is either horospherical or of lens type totally geodesic end neighborhood, or else is projectively isomorphic to a concave end neighborhood.
- If \( \pi_1(\mathcal{O}) \) is relatively hyperbolic, then \( K^\circ/h(\pi_1(\mathcal{O})) \) is a strongly tame strict SPC-orbifold with admissible ends.
- Also, the \( \mathcal{R} \)- or \( T \)-types of ends are preserved.

**Proof.** By Goldman–Millson [51], \( h(\pi_1(\mathcal{O})) \) is discrete since each finite index subgroup of it has no normal infinite nilpotent subgroup. Hence, \( K^\circ/h(\pi_1(\mathcal{O})) \) is an orbifold to be denoted \( \mathcal{O}_h \).

For \( h \), let \( [h] \) denote the corresponding homomorphism to \( \text{PGL}(n+1, \mathbb{R}) \). Since \( [h] \in \text{rep}_{\mathcal{R},u,ce}(\pi_1(\mathcal{O}), \text{PGL}(n+1, \mathbb{R})) \) holds, each \( p \)-end fundamental group \( h(\pi_1(\tilde{E})) \) acts on a horoball \( H \subset S^n \), a generalized lens-cone, or a totally geodesic hypersurface \( S_{\tilde{E}} \) with a lens-neighborhood \( L \). In the first case, we can choose a
sufficiently small horoball inside $K^o$ and in $H$ since the supporting hyperplanes at the vertex of $H$ must coincide by the invariance under $h(\pi_1(\tilde{E}))$. In the third case, we can find a one-sided lens-neighborhood of $S_{\tilde{E}}$ in $K^o$ since $S_{\tilde{E}} \subset \text{bd}K$ as the corresponding sets are always in $\text{bd}K_i$ for each $i$.

The types do not change for $h_i$. The limiting types are not changed.

Suppose that $h(\pi_1(\tilde{E}))$ acts on a generalized lens-cone $L$. Then $h(\pi_1(\tilde{E}))$ has a unique fixed point $v$ in $K$ since $h \in \text{rep}_{E,u,ce}(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$. Then $v \notin K^o$ since otherwise the elements fixing it has to be elliptic. Then it satisfies the uniform middle eigenvalue condition. By Theorems 7.9 and 7.10 of [27], the action is distanced and we can find a concave p-end neighborhood. By Theorem 4.26, $O_h$ is a noncompact strongly tame SPC-orbifold with generalized admissible ends.

Near $[h]$ there is an open neighborhood in $\text{rep}_{E,u,ce}(\pi_1(O), \text{PGL}(n+1, \mathbb{R}))$ where $h_i$ in it is realized by a strongly tame SPC-orbifold with admissible ends diffeomorphic to $O_h$ by Theorem 6.1 and lifting by Theorem 3.19. Hence, $\Omega_i/h_i(\pi_1(O))$ is diffeomorphic to $O_h$ for sufficiently large $i$. Hence, $O_h$ is diffeomorphic to $O$.

By Theorem 4.26, $h$ is stable and strongly irreducible.

The final item follows by Theorem 5.9.

□

To prove for $\text{SDef}_{E,u,ce}(O)$, we need additionally the second last item of Lemma 7.5. This completes the proof of Corollary 7.3. □
Examples

For dimension 2, the surfaces with principal boundary component can be made into surfaces with ends since we can select the fixed points for each boundary components and produce radial ends. These are rather trivial examples for our theory.

The complete hyperbolic structure on an orbifold gives a strict SPC-structure on the orbifold with horospherical ends. Sometimes these deform to ones not from the complete hyperbolic ones and with different types of ends.

Let $P$ be a properly convex polytope in an affine subspace of $\mathbb{R}P^n$ with faces $F_i$ where each side $F_i \cap F_j$ of codimension two is given an integer $n_{ij} \geq 2$. A reflection orbifold or Coxeter orbifold based on $P$ is given by a group of reflections $R_i$ associated with faces satisfying relations $(R_i R_j)^{n_{ij}} = I$ for every pair $i,j$ with $F_i \cap F_j$ is a side of codimension 2. Vinberg showed that $R_i$ acts on a convex domain in $\mathbb{R}P^n$ with a fundamental domain $P$ and thus $P$ with a number of vertices removed can be given a structure of an orbifold with interior of faces $F_i$ silvered and the interior of edge is given dihedral group structure and so on.

These orbifolds have radial end always. If $P$ is a hyperbolic polytope, the orbifold is relatively hyperbolic with respect to ends and the ends are virtually reducible having virtually free abelian fundamental group.

Then the theories of this paper are applicable to such orbifolds and the deformation theorems obviously hold. The proper convexity also holds during deformations according to Vinberg’s work also (see [22]). When $P$ is a cube with all edge orders 3, then we obtain a complete hyperbolic orbifold with horospherical ends. Computations done by G. Lee show that there are nontrivial deformations from the hyperbolic structure to real projective structures where ends deform from horospherical ends to totally geodesic radial ends. (See also [31].)

The example of S. Tillmann is an orbifold on a 3-sphere with singularity consisting of two unknotted circles linking each other only once under a projection to a 2-plane and a segment connecting the circles (looking like a linked handcuff) with vertices removed and all arcs given as local groups the cyclic groups of order three. (See Figure 1.3.) This is one of the simplest hyperbolic orbifolds in Heard, Hodgson, Martelli, and Petronio [57] labelled 2h1.1. The orbifold admits a complete hyperbolic structure since we can start from a complete hyperbolic tetrahedron with four dihedral angles equal to $\pi/6$ and two equal to $2\pi/3$ at a pair of opposite edge $e_1$ and $e_2$. Then we glue two faces adjacent to $e_i$ by an isometry fixing $e_i$ for $i = 1, 2$. The end orbifolds are two 2-spheres with three cone points of orders equal to 3 respectively. These end orbifolds always have induced convex real projective structures in dimension 2, and real projective structures on them have to be convex. Each of these is either the quotient of a properly convex open triangle or a complete affine plane as we saw in Proposition 3.3 of [27].
Tillman found a one-dimensional solution set from the complete hyperbolic structure by explicit computations. His main questions are the preservation of convexity and realizability as convex real projective structures on the orbifold.

The another main example can be obtained by doubling a complete hyperbolic Coxeter orbifold based on a convex polytopes. We take a double $DT$ of the reflection orbifold based on a convex tetrahedron with orders all equal to 3. This also admits a complete hyperbolic structure since we can take the two tetrahedra to be the regular complete hyperbolic tetrahedra and glue them by hyperbolic isometries. The end orbifolds are four 2-spheres with three singular points of orders 3. Topologically, this is a 3-sphere with four points removed and six edges connecting them all given order 3 cyclic groups as local groups.

**Theorem 8.1.** Let $O$ denote the hyperbolic 3-orbifolds $2h_{1,1}$ or $DT$. We assign the $R$-type to each end. Then $SDef_E(O)$ equals $SDef_{E,u,ce}(O)$ and hol maps $SDef_E(O)$ as an onto-map to a component of characters 

$$ \text{rep}_E(\pi_1(O), \text{PGL}(4, \mathbb{R}))$$

containing a hyperbolic representation which is also a component of 

$$ \text{rep}_{E,u,ce}(\pi_1(O), \text{PGL}(4, \mathbb{R})).$$

In this case, the components are cells of dimension 1 and dimension 4 respectively for $2h_{1,1}$ and the double $DT$.

**Proof.** The end orbifolds have Euler characteristics equal to zero and all the singularities are of order 3. Since the singularity is in the end neighborhood, it follows that $SDef_E(O)$ equals $SDef_{E,u}(O)$. Each of the ends has to be either horospherical or radial of lens and totally geodesic type by Proposition 3.3 of [27]. Let $\partial O$ denote the union of end orbifolds of $O$.

In [18], we showed that the real projective structures on the ends determined the real projective structure on $O$. First, there is a map $SDef_E(O) \rightarrow \text{CDef}(\partial O)$ given by sending the real projective structures on $O$ to the real projective structures of the ends. (Here if $\partial O$ has many components, then $\text{CDef}(\partial O)$ is the product space of the deformation space of all components.) Let $J$ be the image.

Let $\mu$ be an element of $SDef_E(O)$. The universal cover $\tilde{O}$ is a properly convex domain in $S^3$. The singular geodesic arcs in $\tilde{O}$ connect one p-end vertex to the other. The developing image of $\tilde{O}$ is a convex open domain and the developing map is a diffeomorphism. Their developing images form geodesics meeting at vertices transversally. A choice of six edges will map to a 1-skeleton of a convex tetrahedron in $S^3$. The geometry of the situation forces us that in the universal cover $\tilde{O}$, there exists two convex tetrahedra $T_1'$ and $T_2'$ with vertices removed in $\tilde{O}$. They are adjacent and their images under $\pi_1(O)$ tessellate $\tilde{O}$.

The end orbifold is so that if given an element of the deformation space, then the geodesic triangulation is uniquely obtained. Hence, there is a proper map from $SDef_E(O)$ to the space of invariants of the triangulations as in [18], i.e, the product space of cross-ratios and Goldman-invariant spaces. (The projective structures are bounded if and only if the projective invariants are bounded.) Thus by the result of [18], there is an inverse to the above map $s : J \rightarrow SDef_E(O)$ that is a homeomorphism.

For $2h_{1,1}$, $J$ is connected by Tillmann’s computations.
Now consider when \( O \) is the orbifold obtained from doubling a tetrahedron with edge orders 3, 3, 3. We consider an element of \( \text{SDef}_E(O) \). Since it is convex, we triangulate \( O \) into two tetrahedra and this gives a triangulation for each end orbifold diffeomorphic to \( S_{3,3,3} \), each of which gives us triangulations into two triangles. We can derive from the result of Goldman [46] and Choi-Goldman [30] that given projective invariants \( \rho_1, \rho_2, \rho_3, \sigma_1, \sigma_2 \) for each of the two triangles satisfying \( \rho_1 \rho_2 \rho_3 = \sigma_1 \sigma_2 \), we can determine the structure on \( S_{3,3,3} \) completely.

For \( S_{3,3,3} \) with a convex real projective structure and divided into two geodesic triangles, we compute these invariants \( \rho_1, \rho_2, \rho_3, \sigma_1, \sigma_2 \) for one of the triangles

\[
s^2 + s \tau_1 + 1, s^2 + s \tau_2 + 1, s^2 + s \tau_3 + 1,
\]

(8.1)

and for the other triangle the corresponding invariants are

\[
\frac{1}{s^2} (s^2 + s \tau_1 + 1), \frac{1}{s^2} (s^2 + s \tau_2 + 1), \frac{1}{s^2} (s^2 + s \tau_3 + 1),
\]

(8.2)

where \( s, t \) are Goldman parameters and \( \tau_i = 2 \cos 2\pi/p_i \) for the order \( p_i; p_i = 3 \).

The set \( J \) is given by projective invariants of the \((3,3,3)\) boundary orbifolds satisfying some equations that the cross ratio of an edge are same from one boundary orbifold to the other and that the products of Goldman \( \sigma \)-invariants equal 1 for some quadruples of Goldman \( \sigma \)-invariants. By the method of [18] developed by the author, we obtain the equations that \( J \) satisfies. The equation is solvable:

\[
s_1 = s_2 = s_3 = s_4 = s, t_1 t_2 t_3 t_4 = C(s) \text{ for a constant } C(s) > 0 \text{ depending ons.}
\]

(See the Mathematica file [28].) Thus \( J \) is homeomorphic to a 4-dimensional cell. (The dimension is one higher than that of the deformation space of the reflection 3-orbifold based on the tetrahedron. Thus we have examples not arising from reflection ones here as well.)

Conversely, we can assign invariants at each edge of the tetrahedron and the Goldman \( \sigma \)-invariants at the vertices if the invariants satisfy the equations. This is given by starting from the first convex tetrahedron and gluing one by one using the projective invariants (see [18] and [29]): Let the first one by always be the standard tetrahedron with vertices

\[
[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], \text{ and } [0, 0, 0, 1]
\]

and we let \( T_2 \) a fixed adjacent tetrahedron with vertices

\[
[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0] \text{ and } [2, 2, 2, -1].
\]

Then projective invariants will determine all other tetrahedron triangulating \( \hat{O} \). Given any deck transformation \( \gamma \), \( T_1 \) and \( \gamma(T_1) \) will be connected by a sequence of tetrahedrons related by adjacency and their pasting maps are completely determined by the projective invariants, where cross-ratios do not equal 0. Therefore, as long as the projective invariants are bounded, the holonomy transformations of the generators are bounded. (This method was spoken about in our talk in Melbourne, May 18, 2009 [29].)

From this, we see that \( \text{SDef}_E(O) \) is connected. The results now follow from Theorem 6.1 and Corollary 6.3.
Figure 8.1. A convex developing image example of a tetrahedral orbifold of orders 3, 3, 3, 3, 3, 3.

We remark that the above theorem can be generalized to orders $\geq 3$ with hyperideal ends with similar computations. See [28] for examples to modify orders and so on.
APPENDIX A

Projective abelian group actions on convex domains

**Lemma A.1.** Let $t_0 \in I$ for an interval $I$. Suppose that we have a parameter of compact convex domains $\Delta_t \subset S^{n-1}$ for $t < t_0$, $t \in I$, and a compact convex domain $\Delta_{t_0}$ in $S^{n-1}$ and a transitive group action $\Phi_t : L \times \Delta_{t_0} \to \Delta_t$, $t \in I$ by a connected abelian group $L$ of rank $n$ for each $t \in I$. Suppose that $\Phi_t$ depends continuously on $t$ and $\Phi_t$ is given by a continuous parameter of homomorphisms $h_t : L \to \text{SL}_\pm(n, \mathbb{R})$. Then $\Delta_t \to \Delta_{t_0}$ geometrically.

**Proof.** We may assume without loss of generality that $\bigcap_{t \in I} \Delta_t \neq \emptyset$ by taking a smaller $I$. Choose a generic point $x_0 \in \bigcap_{t \in I} \Delta_t$. Any point $x \in \Delta_{t_0}$ equals $\Phi_{t_0}(g, x_0)$ for $g \in L$. Therefore, $\Phi_t(g, x_0) \in \Delta_t \to \Phi_{t_0}(g, x_0)$ as $t \to t_0$. Hence, every point of $\Delta_{t_0}$ is the limit of a path $\gamma(t) \in \Delta_t$ for $t < t_0$.

Conversely, given any parameter of points $x(t) \in \Delta_t$ for $t \in I$, we obtain that $x(t) = \Phi_t(g_t, x_0)$ for $g_t \in L$. Let $L \cong \mathbb{R}^{n-1}$ have coordinates $(a_1, \ldots, a_{n-1})$. We define $x_t := \Phi_{t_0}(g_t, x_0)$. $\Phi_t(g, \cdot) : S^{n-1} \to S^{n-1}$ is represented as a matrix

$$h_t(g) = \exp(H_t(\sum_{i=1}^{n-1} a_i(g)e_i))$$

where $\{H_t : \mathbb{R}^{n-1} \to \mathfrak{sl}(n, \mathbb{R})\}$ is a uniformly bounded collection of linear maps.

We claim that $\Phi_t$ is an equicontinuous family of functions: Let $v$ be a generic vector $\mathbb{R}^{n-1}$ of norm 1. By dividing by the largest norm matrix entries we obtain a matrix $m_t(g)$ with entries $\leq 1$. Since $\mathbb{R}P^{n-1}$ is bounded, a computation shows that the family of functions $\{m_t| t \in I\} : \mathbb{R}^{n-1} \times \mathbb{R}^n \to \mathbb{R}^n$ have derivatives uniformly bounded above as the entries are rational functions of exponential functions with bounded coefficients and these rational functions are bounded above by 1.

Hence $x(t)$ and $x_t$ have the same set of limit points as $t \to t_0$. Since $x_t \in \Delta_{t_0}$, $x(t)$ has limit points in $\Delta_{t_0}$ only.

The Hausdorff convergence topology and these two facts give us that $\Delta_t \to \Delta_{t_0}$ geometrically, which is an elementary fact.

For a matrix $A$, we denote by $|A|$ the maximum of the norms of entries of $A$.

**Lemma A.2.** Let $h : \mathbb{Z}^l \to \text{SL}_\pm(n, \mathbb{R})$ be a representation to unipotent elements. Let $g_1, \ldots, g_l$ denote the generators. Then given $\epsilon > 0$ there exists a positive diagonalizable representation $h' : \mathbb{Z}^l \to \text{SL}_\pm(n, \mathbb{R})$ with matrices satisfying $|h'(g_i) - h(g_i)| < \epsilon$, $i = 1, \ldots, l$.

**Proof.** First assume that every $h(g_i)$, $i = 1, \ldots, l$, has matrices that upper triangular matrices with diagonal elements equal to 1 since the Zariski closure is in a nilpotent Lie group.
Let $\epsilon > 0$ be given. We will inductively prove that we can find $h'$ as above with eigenvalues of $h'(g_i)$ are all positive and mutually distinct. For $n = 2$, we can simply change the diagonal elements to positive numbers not equal to 1. Then the group imbeds in $\text{Aff}(\mathbb{R}^1)$. We choose positive constant $a_i$ so that $|a_i - 1| < \epsilon$. Let $g_i$ be given as $x \mapsto a_i x + b_i$. The commutativity reduces to equations $a_i b_j = a_j b_i$ for all $i, j$. Then the solution are given by $b_i = a_i^{-1} a_j b_j$.

Suppose that the conclusion is true for dimension $k - 1$. We will now consider a unipotent homomorphism $h : \mathbb{Z}^l \to \text{SL}_+(k, \mathbb{R})$. Since $h(g_i)$ is upper triangular, let $h_1(g_i)$ denote the upper-left $(k-1) \times (k-1)$-matrix. We find a homomorphism $h_1' : \mathbb{Z}^l \to \text{SL}_+(k-1, \mathbb{R})$ a positive diagonalizable representation and the eigenvalues of $h_1'(g)$ are positive and mutually distinct. Also assume $|h_1'(g_i) - h_1(g_i)| < \epsilon/2$ for $i = 1, \ldots, l$. We change

$$h(g_i) = \begin{pmatrix} h_1(g_i) & b(g_i) \\ 0 & 1 \end{pmatrix}$$

to $h'(g_i) = \begin{pmatrix} 1 & h_1'(g_i) \\ \lambda'(g_i) & 0 \end{pmatrix}$ for some choice of $h_1'(g), b'(g), \lambda'(g_i) > 0$ for $i = 1, \ldots, l$. For commutativity, we need to solve for $b'(g_i), i = 1, \ldots, l$,

$$\frac{1}{\lambda'(g_i)} h_1'(g_i) b'(g_j) = \frac{1}{\lambda'(g_j)} h_1'(g_j) b'(g_i), 1 \leq i, j \leq l.$$

The solution is given by

$$b'(g_i) = \frac{\lambda'(g_i)}{\lambda'(g_j)} h_1'(g_i)^{-1} h_1'(g_i) b'(g_i), i = 2, \ldots, l$$

We choose $b'(g_1)$ arbitrarily near $b(g_1)$. Here, $\lambda(g_1)$ has to be chosen generically to make all the eigenvalues distinct and sufficiently near 1 so that $|h'(g_i) - h(g_i)| < \epsilon$, $i = 1, \ldots, l$. We can check the solution by the commutativity. Hence, we complete the induction steps.

Given a connected abelian group $A$ with positive real eigenvalues only, we can form for each $g \in A$, the Jordan block decomposition of $\mathbb{R}^{n+1}$ into subspaces with the same real eigenvalues. We direct-sum all the elementary Jordan blocks that have the same eigenvalue under every $g \in A$: For each $g$, we let $\lambda_i(g)$ denote the eigenvalue of $g$ associated with $V_i, i = 1, \ldots, l$, i.e., $g - \lambda_i(g) 1_{V_i}$ is nilpotent. Two elementary Jordan block subspaces $V_i$ and $V_j$ are equivalent if $\lambda_i(g) = \lambda_j(g)$ for all $g \in A$. We direct sum the Jordan block subspaces in a Jordan decomposition that are equivalent to one $V_i$. We call this subspace a common Jordan block space.

Since $h$ can be connected to the identity, $h \in A$ cannot switch common Jordan blocks of $g$. A scalar group is a group acting by $s I$ for $s \in \mathbb{R}$ and $s > 0$. A scalar unipotent group is a subgroup of the product of a scalar group with a unipotent group.

A join of two convex real projective $m$-dimensional and $l$-dimensional orbifolds $S_1$ and $S_2$ is obtained as follows: we take the convex open domains $\Omega_1$ and $\Omega_2$ covering $S_1$ and $S_2$ respectively. We projectively embed the two in two affine subspaces of the complementary subspaces $U_1$ and $U_2$ of $S^{n-1}$ (resp in $\mathbb{R}P^{n-1}$), for $n = m + 1 + 1$ and obtain the interior of the join $\Omega_1 \ast \Omega_2$. We take the quotient space of $(\Omega_1 \ast \Omega_2)^o$ by direct summing two holonomy groups and adding a diagonalizable projective automorphism fixing all points of $U_1$ and $U_2$. We can of course generalize.
exists a great sphere $S$ is a unipotent radical there. So the eigenvalues are all 1.

One can think of the following lemma as a classification of convex real projective orbifolds with abelian fundamental groups. Benoist [13] investigated thoroughly in this topic also.

**Lemma A.3.** Let $\Gamma$ be an abelian group acting on a convex domain $\Omega$ of $\mathbf{S}^{n-1}$ (resp. $\mathbb{R}P^{n-1}$) cocompactly and properly discontinuously. Then the Zariski closure $L$ of a finite index subgroup $\Gamma'$ of $\Gamma$ is so that $L/\Gamma'$ is compact and with positive eigenvalues. Furthermore, $\Omega$ is an orbit of the abelian Lie group $L$. $\Omega/\Gamma'$ is a join of a closed properly convex manifold and a finite number of closed complete affine manifolds.

**Proof.** We will prove for the case $\Omega \subset \mathbf{S}^{n-1}$. The other case is implied by this. If $\Omega$ is properly convex, then Proposition 2.8 gives us a diagonal matrix group $L$ acting on a simplex.

Assume that $\Omega$ is not properly convex. We assume that $\Gamma$ is torsion-free using Selberg’s Lemma. Since $\Gamma$ is abelian, the syndetic closure $L'$ of $\Gamma$ is an abelian Lie group and $L'/\Gamma$ is compact. We take a connected component $L$ of $L'$ and let $\Gamma' = L \cap \Gamma$. $L/\Gamma'$ is a manifold and $\Omega/\Gamma'$ is a closed manifold. Since they are both $K(\Gamma',1)$-spaces, it follows that $L$ and $\Omega$ have the same dimension and $L/\Gamma'$ is compact also. (see [86].)

If $\Omega$ is a complete affine space, then Proposition T of [58] proves our result. $L$ is a unipotent radical there. So the eigenvalues are all 1.

Suppose that $\Omega$ is not complete affine but not properly convex. Then there exists a great sphere $\mathbf{S}^{i-1}$ in the boundary of $\Omega$ where $L$ acts on and is the common boundary of $i$-dimensional affine spaces foliating $\Omega$. (see [17].) There is a projective projection $\Pi_{S^{i-1}} : \mathbf{S}^{n-1} - \mathbf{S}^{i-1} \to \mathbf{S}^{n-i-1}$. Then the image of $\Omega_1$ of $\Omega$ is properly convex. Since $L$ acts on it, $\Omega_1$ is an $(n-i-1)$-dimensional simplex by Proposition 2.8. Thus, $\Omega$ is the inverse image $\Pi_{S^{i-1}}^{-1}(\Omega_1)$. Since $\Gamma$ acts on $\Omega$, it follows that $L$ acts on $\Omega$. Since $\dim L = \dim \Omega$ and $\Gamma$ acts properly with compact fundamental domain, $L$ acts properly on $\Omega$. (See Section 3.5 of [79].)

Let $N$ denote the kernel of $L$ going to a Lie group $L_1$ acting on $\Omega_1$.

$$1 \to N \to L \to L_1 \to 1.$$  

Since $L_1$ is diagonalizable by Proposition 2.8, $L_1$ acts simply transitively on $\Omega_1$. dim $L_1 = n-i$. Thus, dim $N = i$ and the abelian group $N$ acts on each $i$-dimensional affine space $A_i$ that is a leaf. Since the action of $N$ is proper, $N$ acts on $A_i$ simply transitively and $N$ is also unipotent by Proposition T of Hirsch-Goldman [58]. Hence, we deduce that each element of $L$ has only positive eigenvalues and so do $\Gamma'$. $L$ is also the Zariski closure of $\Gamma'$ by Saito [75]. (See [86].)

By finding the common Jordan block subspaces, we decompose $L$ into subspaces $V_1 \oplus V'_1 \oplus \cdots \oplus V'_k$ where $L$ acts on $V_1$ as a diagonalizable linear group and $L$ acts on $V'_j$ as elements of an abelian scalar unipotent group for $j = 1, \ldots, k$. That is, $g \in L$ is a direct sum of matrices of form

$$\begin{pmatrix}
\lambda_j(g) & 0 & 0 & 0 \\
* & \lambda_j(g) & 0 & 0 \\
* & * & \lambda_j(g) & 0 \\
* & * & * & \lambda_j(g)
\end{pmatrix}$$
where \{\lambda_1, \ldots, \lambda_k\} is a set of mutually distinct homomorphisms \(L \to \mathbb{R}_+\) by the Lie-Kolchin theorem.

A scalar unipotent linear group of a vector space always acts on a half-space of a vector space. The \(L\)-action on \(S(V_j')\) can be seen as direct sum of affine groups on affine subspaces in some affine coordinates of an affine subspace of \(S(V_j')\). It is a unipotent action since \(L\) becomes 1 in the affine coordinate form. Thus, \(L\) is in a nilpotent group. Also, the \(L\)-action on invariant subsets of \(S(V_i)\) can be conjugated to an affine translation group using exponential maps.

Since \(S(V_1)\) and \(S(V_j')\) are \(\Gamma'\)-invariant subspaces of \(S^{n-1}\), we can show that \(\Omega\) is contained in the strict join of the interiors of convex domains \(K_1 \subset S(V_1)\) and \(K_j' \subset S(V_j')\) that are images of the projections of \(\text{Cl}(\Omega)\). (Note \(\dim K_j' = \dim S(V_j') \geq 1\) since \(\Omega\) is open.) By the description of the \(L\)-action and \(\Gamma\)-action on \(S(V_j')\), it follows that \(L\) acts on \(K_1\) and \(K_j'\) also. The induced action of \(\Gamma'\) on each of \(K_j'\) and \(K_j''\) is cocompact since otherwise \(\Omega/\Gamma'\) cannot be compact by the join-coordinate description of the join. \(L\) acts as a unipotent group on \(S(V_j')\) for each \(j\) and hence \(L\) acts on an affine subspace \(A\) meeting \(K_j''\) as a group of affine transformations. Then \(K_j'' / \Gamma\) is on \(A\) since there is a a homotopy equivalence \(K_j'' / \Gamma' \to A / \Gamma'\). In fact each \(K_j''\) is an open hemisphere in \(S(V_j')\) and is an orbit of \(\Gamma\) by Theorem 4.1 of [50] since \(\Gamma'\) acts cocompactly.

Since \(L\) is diagonal on \(K_1\), it acts transitively on \(K_1\) also. \(K_1\) is an open properly convex simplex since \(\Gamma'\) acts as a diagonalizable group. Thus, the orbit of \(L\) on each of \(K_1\) equals \(K_1\) since otherwise we can check that \(K_j'' / L\) is not compact by affine coordinate description above.

We can introduce a coordinate system on \(K_1 * K_1' * \cdots * K_k'\) so that \(L\) acts as unipotent affine transformation group since we have such coordinates for \(K_1\) and \(K_j'\) and we can take logarithms of the join coordinates. Again, by Theorem 4.1 of [50], \(L\) acts transitively on the interior of the strict join \(K_1 * K_1' * \cdots * K_k'\). Thus, it follows that \(L\) has an open orbit in \(\Omega\) and \(\Omega\) equals the interior of \(K_1 * (K_1' * \cdots * K_k')\). This shows that \(\Omega\) is an orbit of \(\Gamma\) as well. (See also [13].) Finally, \(\Omega / \Gamma'\) is a join of \(K_1 / \Gamma'\) and \(K_i / \Gamma'\) for \(i = 1, \ldots, k\). \(\square\)

A convex real projective structure \(\mu_0\) on an orbifold \(\Sigma\) is virtually immediately deformable to a properly convex real structure if there exists a parameter \(\mu_t\) of real projective structures on a finite cover \(\tilde{\Sigma}\) of \(\Sigma\) so that \(\tilde{\Sigma}\) with induced structures \(\tilde{\mu}_t\) is properly convex for \(t > 0\).

**Proposition A.4.** Assume that \(M\) is covered by a cell. Then a convex real projective structure on closed \((n-1)\)-orbifold \(M\) with infinite free abelian holonomy is always virtually immediately deformable to a properly convex real projective. Furthermore, any join of such orbifolds with properly convex orbifolds are also virtually immediately deformable to a properly convex real projective orbifold.

**Proof.** Let \(\mathbb{Z}^l\) denote the fundamental group of a finite cover \(M'\) of \(M\). Let \(h \in \text{Hom}(\mathbb{Z}^l, \text{SL}_+(n, \mathbb{R}))\) be the restriction of the holonomy homomorphism to \(\mathbb{Z}^l\). Nearby every \(h\), there exists a positively diagonalizable holonomy \(h' : \mathbb{Z}^l \to \text{SL}_+(n, \mathbb{R})\) by Lemmas A.2 and A.3. By the deformation theory of [22], \(h''\) is realized as a holonomy of a real projective manifold \(M''\) diffeomorphic to \(M'\). Also, the universal cover of \(M'\) is an orbit of an abelian Lie group \(L\) where \(h'((\mathbb{Z}^l))\) is a lattice by Lemma A.3. Now \(h''((\mathbb{Z}^l))\) is a lattice in a positively diagonalizable abelian Lie
group $L'$. Since $L/h'(Z')$ is diffeomorphic to $L'/h''(Z')$ and they have real projective structures by \cite{13}. $L'/h''(Z')$ with the real projective structure is projectively diffeomorphic to $M''$. From this, we deduce that the universal cover $M''$ is an orbit of $L'$. Hence, $M''$ is properly convex since $L'$ is a positive real diagonal group.

The final part just follows by deforming the first factor orbifold of the join. □

**Proposition A.5.** Let $\Sigma$ be a closed $(n - 1)$-dimensional convex projective orbifold with the real projective structure $\mu$, and let $\Omega$ in $\mathbb{R}P^{n-1}$ (resp. in $\mathbb{S}^{n-1}$) denote a universal cover of $\Sigma$. Let $\mu_t$, $t \in [0,1]$ be a parameter of convex real projective structures on $\Sigma$ where $\mu_0$ is properly convex or complete affine and $\mu = \mu_1$. Let $\pi_1(\Sigma)$ be isomorphic to a finite extension of $\mathbb{Z}^l \times \Gamma_1 \times \cdots \times \Gamma_k$ for a hyperbolic group $\Gamma_i$ for each $i$ and $1 \leq k - 1 \leq l$. Let $h$ denote the holonomy homomorphism of $\pi_1(\Sigma)$ to $\text{PGL}(n, \mathbb{R})$ (resp. $\text{SL}(n, \mathbb{R})$) corresponding to the original real projective structure. Then

(i) If $k \geq 1$ and $k - 1 \leq l \leq k$, then $\Omega$ is a properly convex domain.

(ii) If $k = 0$, $l \geq 1$, then $\Omega$ is a convex domain $d(\Delta)$ that is an orbit of a connected abelian Lie group $\Delta$ in the Zariski closure of a finite index abelian subgroup of $h(\pi_1(\Sigma))$.

(iii) If $l \geq k$, $k \geq 1$, then $\Omega$ is real projectively diffeomorphic to the interior of the strict join of

- an orbit $d(\Delta)$ for a connected abelian Lie group $\Delta$ that is in the Zariski closure of the center of a finite-index subgroup of $h(\pi_1(\Sigma))$ and

- a strict join $K_1 \ast \cdots \ast K_k$ where a hyperbolic factor $\Gamma_i$ of $\Gamma$ acts discretely on a properly convex domain $K_i^\circ$.

(iv) Suppose that $h(\pi_1(\Sigma))$ acts on a properly convex domain in $\mathbb{R}P^{n-1}$ (resp. in $\mathbb{S}^{n-1}$). Then $\Omega$ is properly convex.

**Proof.** We will prove for the case $\Omega \subset \mathbb{S}^{n-1}$. The $\mathbb{R}P^{n-1}$-version follows from this. If $\mu_0$ is complete affine, then we deform for $t < 0$ and assume $\mu_t$ is properly convex by Proposition A.4 and shift our parameter.

We can choose a developing map $D_t : \tilde{\Sigma} \to \mathbb{S}^{n-1}$ and a holonomy homomorphism $h_t$ as above for $\mu_t$ and $h_t$ is a continuous family of representations $\Gamma \to \text{SL}(n, \mathbb{R})$. This follows for each element $g \in \Gamma$, $h_t(g)$ is determined by developing maps in the space of $C^r$-topology. (See Chapter 6 of \cite{26}.) Let $\Omega_t$ denote the image of $D_t$, a convex open domain.

(i) Let $\Gamma$ denote the torsion-free subgroup of $\pi_1(\Sigma)$ isomorphic to $\mathbb{Z}^l \times \Gamma_1 \times \cdots \times \Gamma_k$. We will denote the corresponding subgroups of $\Gamma$ by the same notations in $\mathbb{Z}^l \times \Gamma_1 \times \cdots \times \Gamma_k$.

By Koszul \cite{64}, the set $A_C$ of $t \in [0,1]$ where $\Omega_t$ is properly convex is an open subset of $I$. We will now show that the set is closed. Let $t_0$ be the supremum of a component of $I$ containing 0. Suppose that there exists a parameter of holonomy representations $h_t$, $t \in [0,1]$ acting on a convex domain $\Omega_t$ so that $\Omega_t$ is properly convex for $t < t_0$. Then for $t < t_0$, $h_t(\Gamma)$ acts on a properly convex domain $K_{i,t}$ for $i = 1, \ldots, k$ in $\text{Cl}(\Omega_t)$ and there are $l - k + 1$ number of discrete points $k_{j,t}$ in $\text{Cl}(\Omega_t)$, for $j = 1, \ldots, l - k + 1$, by Proposition 2.8 fixed by $\Gamma$ (up to choosing $\Gamma$ even smaller).

Choose a subsequence $t_i \in A_C$, $t_i \to t_0$, $t_i < t_0$ as $l \to \infty$. By choosing a subsequence of $t_{k,i}$, we may assume without loss of generality that $K_{i,t_i} \to K_{i,t_0}$ for
Let \( k_{L,t_i} \) denote the convex hull of \( \{k_{j,t_i} | j \in L\} \) for any subcollection \( L \) of \( \{1, \ldots , l - k + 1\} \) and

- \( K_{M,t_i} \) denote the convex hull of \( \{K_{j,t_i} | j \in M\} \) for a subcollection \( M \) of \( \{1, \ldots , k\} \).

By Lemma A.1 and \([10]\), we may assume without loss of generality that \( k_{L,t_i} \rightarrow k_{L,t_0} \) as \( l \rightarrow \infty \) and \( K_{i,t_i} \rightarrow K_{i,t_0} \) geometrically as \( l \rightarrow \infty \) for a convex compact sets \( k_{L,t_0} \) and \( K_{i,t_0} \).

By Corollary 1.2 of \([10]\), \( \Gamma_1 \) divides the properly convex \( K_{i,t_0} \) irreducibly for each \( i = 1, \ldots , k \).

Let \( P_{M,t_0} \) denote the subspace spanned by \( K_{M,t_0} \), and let \( Q_{L,1} \) the subspace spanned by \( k_{L,t_0} \). If \( l = 0 \) and \( k = 1 \), then (i) follows from Corollary 1.2 of \([10]\). Now suppose \( l = 1 \) and \( k = 2 \). Then the subspaces \( P_{1,t_0} \) and \( P_{2,t_0} \) are disjoint. Otherwise, \( P_{1,t_0} = P_{2,t_0} \) by the irreducibility of \( \Gamma_1 \) and \( \Gamma_2 \). Since \( \Gamma_1 \) acts trivially on \( P_{2,t_0} \) by the limit argument, this is a contradiction. Now suppose \( l = k - 1 \) and \( k \geq 2 \). Suppose that the subspaces \( P_{M,t_0} \) and \( P_{M',t_0} \) containing \( k_{M,t_0} \) and \( K_{M',t_0} \) meet and \( M \) and \( M' \) are minimal disjoint pair of such sets. We may assume that the collection of subspaces \( \{P_{i,t_0} | i \in M\} \) in \( P_{M,t_0} \) are independent and so are \( \{P_{j,t_0} | j \in M'\} \) in \( P_{M',t_0} \). Since \( P_{M,t_0} \cap P_{M',t_0} \neq \emptyset \), \( \Gamma \) acts reducibly on \( P_{M,t_0} \). Since for the irreducible factors \( P_{i,t_0} \cap P_{j,t_0} = \emptyset \) for \( i \neq j \) and \( P_{i,t_0} \in P_{M,t_0} \) are the only irreducible subspaces, it follows that \( P_{M,t_0} \cap P_{M',t_0} \) is a join of a number of them.

This contradicts the minimality unless \( P_{M,t_0} = P_{M',t_0} \). This is a contradiction as above. Therefore, we showed that \( \{P_1, \ldots , P_k\} \) are independent subspaces of \( S^{n-1} \).

Since \( Cl(\Omega_{t_0}) = K_{1,t_0} \ast \cdots \ast K_{k,t_0} \), we obtain that \( \Omega_{t_0} \) is properly convex.

Now suppose that \( l \geq 1 \) and \( k \geq 1 \) and \( k - 1 \leq l \leq k \). Suppose that \( l = k \).

Then there exists at most one vertex \( k_{1,t_i} \) for each \( i \). Let \( k_{1,t_0} \) denote the limit. Then again we show as above \( \{Q_1,t_0,P_{1,t_0}, \ldots , P_{k,t_0}\} \) are independent. As above, \( \Omega_{t_0} \) is properly convex. This completes the proof of the closedness of the set of \( t \) where \( \Omega_t \) is properly convex. Thus, we showed that \( \Omega_t \) is always properly convex for \( t \in [0,1] \).

**Lemma A.6.** Let \( t \in [0,1] \). We let \( \mu_t \) be convex real projective structures on \( \Sigma \) virtually immediately deformable to properly convex ones. Suppose that \( M \neq \emptyset \).

We assume that the following (*) is true for \( t = 0 \).

(*): The subspace \( Q_{L,t} \) spanned by \( k_{L,t} \) is disjoint from the subspace \( P_{M,t} \) spanned by \( K_{M,t} \).

Then the above (*) is true for all \( t \in [0,1] \).

**Proof.** The set \( I \) of \( t \in [0,1] \) where \( Q_{L,t} \cap P_{M,t} = \emptyset \) is open with \( 0 \in I \).

Now let \( I \) be a set so that \( Q_{L,t} \) is disjoint from \( P_{M,t} \). Let \( t_0 \) be a supremum of a component of \( I \) containing 0. First, we need to only consider a fixed finite index subgroup by the compactness of intervals. \( K_{M,t} \) is properly convex as shown in the proof above. By Lemma A.1 \( k_{M,t} \) is an orbit of a Lie group \( A \) containing the center of holonomy of \( \Gamma \). If \( P_{M,t_0} \cap Q_{L,t_0} \neq \emptyset \), \( P_{M,t_0} \cap Q_{L,t_0} \) equals the join of partial sets \( P_{M',t_0} = Q_{L',t_0} \), \( M' \subset M, L' \subset L \) by the reducibility of \( \Gamma \). Then \( P_i,t_0 \) for \( i \in M' \)
metas $Q_{L,t_0}$. Again, this violates the irreducibility of $\Gamma_i$. The open and closedness argument proves the result. \hfill\Box

(ii) Here, $\Gamma$ is an abelian group. $h_t(\Gamma)$ are diagonalizable with positive eigenvalues for $t < 1$. Thus, the eigenvalues of $h_t(\Gamma)$ are real positive for all $t$. Lemma A.3 implies the result.

(iii) Let $B_C$ denote the set of $t \in [0, 1]$ where the conclusion (iii) holds. We show that $B_C$ is an open set: Let $t \in B_C$. We have $\text{Cl}(\Omega_t) = d(L)_t \ast K_{1,t} \ast \cdots \ast K_{k,t}$. Here, $d(L)$ is characterized as an orbit of $L$ and a $L$-invariant complement to a subspace $P_t$ containing $K_{1,t} \ast \cdots \ast K_{k,t}$. Since $L$ acts semisimply on $P_t$, there exists $\epsilon > 0$ such that for $|t' - t| < \epsilon$, there exists a complement $S_{t'}$ to $P_{t'}$. The subspace $S_{t'}$ and $T_{t'}$ containing $K_{1,t'} \ast \cdots \ast K_{k,t'}$ are disjoint. Then considering the accumulation points of orbits of points under $\Gamma$ that includes $K_{1,t}$ and $d(L)_t$ shows that $\Omega_t$ contains the interior of $d(L)_t \ast K_{1,t} \ast \cdots \ast K_{k,t}$. $\Omega_t$ is a subset of the join since we can project the fundamental to the factors and act by $\Gamma$. By the dimension consideration, $S_{t'}$ is spanned by an orbit $d(L)_{t'}$ by Lemma A.3.

Now we show the closedness of $B_C$. Suppose that there exists a parameter of holonomy representations $h_t$, $t \in I$ acting on convex domain $\Omega_t$. Let $t_0$ be the supremum of the connected component of $I$ containing $0$. and $\Omega_{t_0} = \Omega$. We use the notation of (i).

As in (i), there can only be a pair of antipodal points in $k_{L,t_0}$ for a subcollection $L = \{1, \ldots, l - k + 1\}$. Then $K_{M,t_0}$ is properly convex by (i). Let $S_1$ denote the minimal great sphere containing $k_{L,t_0}$ and $S_2$ the one containing $K_{M,t_0}$ for $M = \{1, \ldots, k\}$. Now $S_1$ and $S_2$ does not intersect by Lemma A.6. $\text{Cl}(\Omega)$ is a subset of a strict join of $k_{L,t_0}$ and $K_{M,t_0}$ again using limit sets and the fundamental domain as above. Thus, $\dim k_{L,t_0} + \dim K_{M,t_0} + 1 = n$ since otherwise $\text{Cl}(\Omega)$ would not be a domain in $S^{n-1}$. Since $S_1$ and $S_2$ are disjoint and are complimentary, there exists a natural projection $S^{n-1} - S_1 \rightarrow S_2$. Then restricting $\Gamma'$ to $k_{L,t_0}'$, we obtain a cocompact action since $k_{L,t_0}'$ is the image of the projection of $\Omega$ and $\Omega/\Gamma'$ is compact. For each $l$, the interior of $k_{L,t_0}'$ is an orbit of a connected abelian group that is the Zariski closure of the subgroup corresponding to $\mathbb{Z}^l$. Hence, $k_{L,t_0}'$ is an orbit of a connected abelian Lie group and is convex as in (ii). Thus $\text{Cl}(\Omega)$ is the strict join of these two sets.

(iv) Suppose that $\pi_1(\Sigma)$ acts on a properly convex domain in $S^{n-1}$. $d(\Delta)$ acts on the domain. Therefore, the previous three items imply the result. \hfill\Box

An immediate corollary is:

**Corollary A.7.** Suppose that a real projective orbifold $\Sigma$ is a closed $(n - 1)$-orbifold that is convex with admissible fundamental group and with the structure deformable to properly convex structures. Suppose that the holonomy group $h'(\pi_1(\Sigma))$ in $\text{PGL}(n, \mathbb{R})$ (resp. $\text{SL}_{\pm}(n, \mathbb{R})$) acts on a properly convex domain or a complete affine space $D$ and the associated developing map maps into $D$. Then $\Sigma$ is also properly convex or complete affine with developing map that is a diffeomorphism to $D$. 
APPENDIX B

A topological result

PROPOSITION B.1. Let $X$ be a compact metrizable space. Let $C_X$ be a countable collection of compact sets. The collection has the property that $C_K := \bigcup_{C \in C_X, C \cap K \neq \emptyset} C$ is closed for any closed set $K$. We define the quotient space $X/\sim$ as follows $x \sim y$ iff $x, y \in C$ for an element $C \in C_X$. Then $X/\sim$ is metrizable.

PROOF. We show that $X/\sim$ is Hausdorff, 2-nd countable, and regular and use the Urysohn metrization theorem. We define a countable collection $B$ of open sets of $X$ as follows: We take an open subset $L$ of $X$ that is an $\epsilon$-neighborhood of an element of $C_X$ or a point of a dense countable set $Y$ in $X \setminus \bigcup C_X$ for $\epsilon \in \mathbb{Q}, \epsilon > 0$. We form $L - \bigcup_{C \cap \text{bd} L \neq \emptyset, C \in C_X} C$ for all such $L$ containing an element of $C_X$ or $Y$. This is an open set since bd$L$ is closed and by the premise, and are neighborhoods of elements of $C_X$ and $Y$. Furthermore, each element of $B$ is a saturated open set under the quotient map. Now, the rest is straightforward. □
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