Compression based bound for non-compressed network: unified generalization error analysis of large compressible deep neural network

Taiji Suzuki
Graduate School of Information Science and Technology, The University of Tokyo
Center for Advanced Intelligence Project, RIKEN
taiji@mist.i.u-tokyo.ac.jp

Abstract

One of the biggest issues in deep learning theory is the generalization ability of networks with huge model size. The classical learning theory suggests that overparameterized models cause overfitting. However, practically used large deep models avoid overfitting, which is not well explained by the classical approaches. To resolve this issue, several attempts have been made. Among them, the compression based bound is one of the promising approaches. However, the compression based bound can be applied only to a compressed network, and it is not applicable to the non-compressed original network. In this paper, we give a unified framework that can convert compression based bounds to those for non-compressed original networks. The bound gives even better rate than the one for the compressed network by improving the bias term. By establishing the unified framework, we can obtain a data dependent generalization error bound which gives a tighter evaluation than the data independent ones.

1 Introduction

Deep learning has shown quite successful results in wide range of machine learning applications, such as image recognition [28], natural language processing [16] and image synthesis tasks [43]. The success of deep learning methods is mainly due to its flexibility, expression power and computational efficiency for large dataset training. Due to its significant importance in wide range of application areas, its theoretical analysis is also getting much important. For example, it has been known that the deep neural network has universal approximation capability [13, 24, 44] and its expressive power grows up in an exponential order against the number of layers [37, 8, 12, 11, 42, 47]. However, theoretical understandings are still lacking in several important issues.

Among several topics of deep learning theories, a generalization error analysis is one of the biggest issues in the machine learning literature. An important property of deep learning is that it generalizes well even though its parameter size is quite large compared with the sample size [41]. This can not be well explained by a classical learning theory which suggests that overparameterized models cause overfitting and thus result in poor generalization ability.

For this purpose, norm based bounds have been extensively studied so far [39, 6, 40, 18]. These bounds are beneficial because the bounds are not explicitly dependent on the number of parameters and thus are useful to explain the generalization error of overparameterized network [41]. However, these bounds are typically exponentially dependent on the number of layers and thus tends to be loose for deep network situations. As a result, [1] reported that a simple VC-dimension bound [32, 23] can still give sharper evaluations than these norm based bounds in some practically used deep

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networks. [56] improved this issue by involving a data dependent Lipschitz constant as performed in [1, 38].

On the other hand, compression based bound is another promising approach for tight generalization error evaluation which can avoid the exponential dependence on the depth. The complexity of deep neural network model is regulated from several aspects. For example, we usually impose explicit regularization such as weight decay [29], dropout [43, 54], batch-normalization [23], and mix-up [59, 53]. [58] reported that such explicit regularization does not have much effect but implicit regularization induced by SGD [20, 19, 26] is important. Through these explicit and implicit regularizations, deep learning tends to produce a simpler model than its full expression ability [50, 53].

To measure how “simple” the trained model is, one of the most promising approaches currently investigated is the compression bounds [1, 7, 48]. These bounds measure how much the network can be compressed and characterize the size of the compressed network as the implicit effective dimensionality. [1] characterized the implicit dimensionality based on so called layer-cushion quantity and suggested to perform random projection to obtain a compressed network. Along with a similar direction, [7] proposed a pruning scheme called Corenet and derived a bound of the size of the compressed network. [48] has developed a spectrum based bound for their compression scheme. Unfortunately, all of these bounds guarantee the generalization error of only the compressed network, not the original network. Hence, it does not give precise explanations about why large network can avoid overfitting.

In this paper, we derive a unified framework to obtain a compression based bound for a non-compressed network. Unlike the existing researches, our bound is valid to evaluate the original network before compression, and thus gives a direct explanation about why deep learning generalizes despite its large network size. The difficulty to apply the compression bound to the original network lies in evaluation of the population $L_2$-bound between the compression network and the original network. A naive evaluation results in the VC-bound which is not preferable. This difficulty is overcome by developing novel data dependent capacity control technique using local Rademacher complexity bounds [55, 4, 27, 17]. Then, the bound is applied to some typical situations where the network is well compressed. Our analysis stands on the implicit bias hypothesis [19, 26] that claims deep learning tends to produce rather simple models. Actually, [19, 26] showed gradient descent results in (near) low rank parameter matrices in each layer in linear network settings. [34] evaluated the eigenvalue decays of the weight matrix through random matrix theories and several numerical experiments. These observations are also supported by the flat minimum analysis [23, 57, 30], that is, the product of the eigenvalues of the Hessian around the SGD solution tends to be small, which means SGD converges to a flat minimum and possess stability against small perturbations leading to good generalization. Based on these observations, we make use of the eigenvalue decay of the weight matrix and the covariance matrix among the nodes in each layer. The eigenvalue decay speed characterizes the redundancy in each layer and thus is directly relevant to compression ability. Our contributions in this paper are summarized as follows:

- We give a unified framework to obtain a compression based bound for non-compressed network which properly explains that a compressible network can generalizes well. The bound can convert several existing compression based bounds to that for non-compressed one in a unifying manner. The bound is applied to near low rank models as concrete examples.
- We develop a data dependent capacity control technique to bound the discrepancy between the original network and compressed network. As a result, we obtain a sharp generalization error bound which is even better than that of the compressed network. All derived bounds are characterized by data dependent quantities.

Other related work Recently, the role of over-parameterization for two layer networks has been extensively studied [41, 2]. These are for the shallow network and the generalization error is essentially given by the norm based bounds. It is not obvious that these bounds also give sharp bounds for deep models.

PAC-Bayes bound is also applied to obtain a non-vacuous compression based bound [60]. However, the bound is still for the compressed (quantized) models and it is not obvious that that bound can be converted to that for the original network.

Relation between compression and learnability was traditionally studied in a different framework as in [33] and minimum description code length [22]. Our bound would share the same spirits
Table 1: Comparison of each generalization error to our bound. $R_F^c$ is the Frobenius norm of the weight matrix, $R_2$ is the operator norm of the weight matrix, $R_{p,q}$ is the $(p, q)$ matrix norm, $L$ is the depth, $m$ is the maximum of the width, $n$ is the sample size. $\hat{R}_n$ and $\hat{R}$ represent the Rademacher complexity and local Rademacher complexity respectively. $\kappa$ is a Lipschitz constant between layers. $\alpha$ represents the eigenvalue drop rate of the weight matrix, and $\beta$ represents that of the covariance matrix among the nodes in each internal layer. $\hat{r}$ is the bias induced by compression. “Original” indicates whether the bound is about the original network or not.

| Authors          | Rate                                                                 | Bound type               | Original |
|------------------|----------------------------------------------------------------------|--------------------------|----------|
| Neyshabur et al. | $2\hat{L}\|\mathbf{F}\|_n$                                       | Norm base               | Yes      |
| Bartlett et al.  | $R_2\sqrt{n}$                                                       | Norm base               | Yes      |
| Wei & Ma [8]     | $R_2\sqrt{n}L^{2/3}m^{2/3}(1+L\kappa^2R_2^2L^{1/3})^{3/2}$         | Norm base               | Yes      |
| Neyshabur et al. | $\hat{R}_F\min\left\{\frac{L\sqrt{mR_2^2}}{m}, \sqrt{3\frac{m}{n}}\right\}$ | Norm base               | Yes      |
| Golowich et al.  | $R_2\sqrt{\frac{n}{m}}L^{2/3}m^{2/3}$                              | VC-dim.                  | Yes      |
| Li et al. [12]   | $R_2\sqrt{\frac{n}{m}}L^{2/3}m^{2/3}$                              | VC-dim.                  | Yes      |
| Harvey et al.    | $R_2\sqrt{\frac{n}{m}}L^{2/3}m^{2/3}$                              | VC-dim.                  | Yes      |
| Arora et al.     | $\hat{r} + \sqrt{\frac{1}{n} \max_{1 \leq i \leq n} |f(x_i)|^2 \sum_{l=1}^L \frac{1}{\mu_i^{2l+1}}}$ | Compression              | No       |
| Suzuki et al.    | $\hat{r} + \sqrt{\frac{1}{n} \max_{1 \leq i \leq n} |f(x_i)|^2 \sum_{l=1}^L \frac{1}{\mu_i^{2l+1}}}$ | Compression              | No       |
| Ours (Thm. [1])  | $\hat{r} + \hat{R}_n(\mathbf{F} - \mathbf{G}) + \hat{R}_n(\mathbf{G})$ | General                  | Yes      |
| Ours (Cor. [1])  | $\sqrt{L(Lm^2L)^{1/4} \frac{\kappa}{n}}$                         | Low rank weight          | Yes      |
| Ours (Thm. [4])  | $\sqrt{\frac{1}{n} \max_{1 \leq i \leq n} \frac{1}{\mu_i^{2l+1}}}$ | Low rank cov.            | Yes      |

with these studies but give a new analysis by incorporating recent observations in deep learning researches.

2 Preliminaries: Problem formulation and notations

In this section, we give the problem setting and notations that will be used in the theoretical analysis. We consider the standard supervised learning formulation where data consists of input $x \in \mathbb{R}^d$ and output (or label) $y \in \mathbb{R}$. We consider a single output setting, i.e., the output $y$ is a 1-dimensional real value, but it is straightforward to generalize the result to a multiple output case. Suppose that we are given $n$ i.i.d. observations $D_n = \{(x_i, y_i)\}_{i=1}^n$ distributed from a probability distribution $P$ the marginal distribution of $x$ is denoted by $P_X$ and the one corresponding to $y$ is denoted by $P_Y$. To measure the performance of a trained function $f$, we use a loss function $\psi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and define a training error and its expected one as

$$\hat{\Psi}(f) := \frac{1}{n} \sum_{i=1}^{n} \psi(y_i, f(x_i)), \quad \Psi(f) := \mathbb{E}[\psi(Y, f(X))],$$

where the expectation is taken with respect to $(X, Y) \sim P$. Basically, we are interested in the generalization error $\hat{\Psi}(\hat{f}) - \hat{\Psi}(f)$ for an estimator $\hat{f}$. We denote the empirical $L_2$-norm by $\|f\|_n := \sqrt{\sum_{i=1}^{n} (f(z_i))^2/n}$ for an empirical observation $z_i = (x_i, y_i)$ ($i = 1, \ldots, n$). The population $L_2$-norm is denoted by $\|f\|_{L_2} := \sqrt{\mathbb{E}_{Z \sim P} [f(Z)^2]}$.

This paper deals with deep neural networks as a model. The activation function is denoted by $\eta$ which will be assumed to be 1-Lipschitz as satisfied by ReLU (Assumption [1]). Let the depth of the network be $L$ and the width of the $\ell$-th internal layer be $m_\ell$ ($\ell = 1, \ldots, L + 1$) where we set
model is moderately large. On top of the last layer is because the clipping operator restricts the change the classification error for binary classification. We express $F_2$ any estimator networks with weight matrices that have bounded norms and are near low rank (Sec. 3.1 or Sec. 3.2). The main difficulty in generalization error analysis of deep learning is that the Rademacher complexity is the typical tool to evaluate the generalization error on a function class $\mathcal{F}$, which is denoted by $\hat{R}_n(\mathcal{F}) := E_{\mathcal{D}_n} \left[ \frac{1}{N} \sum_{i=1}^{N} \epsilon_i f(z_i) \right]$ where $\mathcal{D}_n = (z_i)_{i=1}^n = (x_i, y_i)_{i=1}^n$, and $\epsilon_i$ (i = 1, ..., n) is an i.i.d. Rademacher sequence (P($\epsilon_i = 1$) = P($\epsilon_i = -1$) = 1/2). This is also called conditional Rademacher complexity because the expectation is taken conditioned on fixed $\mathcal{D}_n$. Its expectation with respect to $\mathcal{D}_n$ is denoted by $\hat{R}_n(\mathcal{F}') := E_{\mathcal{D}_n} [\hat{R}_n(\mathcal{F})]$. Roughly speaking the Rademacher complexity measures the size of the model and it gives an upper bound of the generalization error [52, 36].

The main difficulty in generalization error analysis of deep learning is that the Rademacher complexity of the full model $\mathcal{F}$ is quite large. One of the successful approaches to avoid this difficulty is the compression based bound [1, 7, 48] which measures how much the trained network can be regarded as small. To describe it more precisely, suppose that the trained network is included in a submodel: $\hat{f} \in \hat{\mathcal{F}} \subset \mathcal{F}$. For example, $\mathcal{F}$ can be a set of networks with weight matrices that have bounded norms and are near low rank (Sec. 3.1 or Sec. 3.2). We do not assume a specific type of training procedure, but we give a uniform bound valid for any estimator $\hat{f}$ that falls into $\hat{\mathcal{F}}$ and satisfies the following compressibility condition. We suppose that the network $\hat{f}$ is easy to compress, that is, $\hat{f}$ can be compressed to a smaller network $\hat{g}$ which is included in a submodel: $\hat{g} \in \hat{\mathcal{G}}$. For example, $\hat{\mathcal{G}}$ can be a set of networks with a smaller size than $\hat{f}$. How small the trained network $\hat{f}$ can be compressed has been characterized by several notions such as “layer-cushion” [1]. Typical compression based bounds give generalization errors of the compressed model $\hat{g}$, not the original network $\hat{f}$. Our approach converts an error bound of $\hat{g}$ to that of $\hat{f}$ and eventually obtains a tighter evaluation.

The biggest difficulty for transforming the compression bound to that of $\hat{f}$ lies in evaluation of the population $L_2$-norm between $\hat{f}$ and $\hat{g}$. Basically, the compression based bounds are given as $\Psi(\hat{g}) \leq \hat{\Psi}(\hat{f}) + ||\hat{f} - \hat{g}||_n + C \hat{R}_n(\hat{g})$, (1) for a constant $C > 0$ under some assumptions (Table I). The term $||\hat{f} - \hat{g}||_n$ appears to adapt the empirical error of $\hat{f}$ to that of $\hat{g}$, that is called “compression error” which can be seen as a bias term. We see that, in the right hand side, there appears the complexity of $\hat{\mathcal{G}}$ which is assumed to be much smaller than that of the full model $\mathcal{F}$. However, the left hand side is not the expected error of $\hat{f}$ but that of $\hat{g}$. One way to transfer this bound to that of $\hat{f}$ is that we have $||\Psi(\hat{g}) - \Psi(\hat{f})||_L_2 \leq ||\hat{g} - \hat{f}||_L_2$ by assuming Lipschitz continuity of the loss function and then convert the bound to $\Psi(\hat{f}) \leq \hat{\Psi}(\hat{f}) + (||\hat{f} - \hat{g}||_n + ||\hat{f} - \hat{g}||_{L_2}) + \hat{R}_n(\hat{g})$.

1In this paper, $\| \cdot \|$ denotes the Euclidean norm: $\| u \| = \sqrt{\sum_i u_i^2}$. 

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However, to bound the term \( \| \hat{f} - \hat{g} \|_n + \| \tilde{f} - \tilde{g} \|_{L_2} \), there typically appears the complexity of the model \( \tilde{F} \) which is larger than the compressed model \( \tilde{G} \) like \( \| \hat{f} - \hat{g} \|_n \leq \sqrt{\| \tilde{f} - \tilde{g} \|_2^2 + O_p(R(\tilde{F}))} \), which results in slow convergence rate. To overcome this difficulty, we need to carefully control the difference between the training and test error of \( \hat{f} \) and \( \hat{g} \) by utilizing the local Rademacher complexity technique \([35, 4, 27, 17]\). The local Rademacher complexity of a model \( F' \) with radius \( r > 0 \) is defined as

\[
R_r(F') := \tilde{R}_n(\{ f \in F' | \| f \|_{L_2} \leq r \}).
\]

The main difference from the standard Rademacher complexity is that the model is localized to a set of functions satisfying \( \| f \|_{L_2} \leq r \). As a result, we obtain a tighter error bound.

Throughout this paper, we always assume the following assumptions.

**Assumption 1** (Lipschitz continuity of loss and activation functions). The loss function \( \psi \) is 1-Lipschitz continuous with respect to the function output:

\[
|\psi(y, u) - \psi(y, u')| \leq |u - u'| \quad (\forall y \in \text{supp}(P_Y), \; u, u' \in \mathbb{R}).
\]

The activation function \( \eta \) is also 1-Lipschitz continuous: \( \| \eta(u) - \eta(u') \| \leq \| u - u' \| \quad (\forall u \in \mathbb{R}^d) \) where \( d' \) is any positive integer.

**Assumption 2.** The norm of input is bounded by \( B_x > 0: \| x \| \leq B_x \quad (\forall x \in \text{supp}(P_X)) \).

**Assumption 3.** The \( L_\infty \)-norms of all elements in \( \tilde{F} \) and \( \tilde{G} \) are bounded by \( M \geq 1: \| f \|_\infty, \| g \|_\infty \leq M \) for all \( f \in \tilde{F} \) and \( g \in \tilde{G} \).

This assumption can be ensured by applying the clipping operator \( G \) to the output of the functions. In this paper, all the variables \( L, m_L, R_2, R_f, M, B_x \) are supposed to be \( o(n) \). What we will derive in the following is a bound which has mild dependency on the depth \( L \) and depends on the width \( \sum_{L=1}^{L} m_L R_L^2 \) in a sub-linear order by using the compression based approach.

**Existing bounds for no-compressed network** Here we give a brief review of the generalization error bound for non-compressed models. (i) VC-bound: The Rademacher complexity of the full model \( F \) can be bounded by a naive VC-dimension bound \([21]\) which is

\[
\tilde{R}_n(F) = O \left( \sqrt{\frac{L \sum_{l=1}^{L} m_l m_{l+1}}{n} \log(n)} \right).
\]

In this bound, there appears the number of parameters \( \sum_{l=1}^{L} m_l m_{l+1} \) in the numerator. However, the number of parameters is often larger than the sample size \( n \) in practical use. Hence, this bound is not appropriate to evaluate generalization ability of overparameterized networks. (ii) Norm-based bound: \([18]\) showed the norm based bound which is given as \( \tilde{R}_n(F) = O \left( \sqrt{\frac{R_f^2}{n}} \right) \). However, this is exponentially dependent on the depth as \( R_f^2 \) resulting in quite loose bound. \([40]\) showed a norm based bound of \( \tilde{R}_n(F) = O \left( \frac{L^2 (\max_m m_1) R_f^2}{n R_f^2} \right) \) which avoids the exponential dependency. However, there is still dependency on the width, \( \sum_{L} m_L R_L^2 \), which is larger than the linear order of the width since \( R_f^2 \) could be moderately large. \([6]\) showed \( \tilde{R}_n(F) = O \left( \frac{R_f^2}{\sqrt{n} \left( \frac{L R_f^2}{R_f^2} \right)^{3/2}} \right) \). The norm constraint on \( R_{2 \to 1} \) implicitly assumes sparsity on the weight matrix and \( R_{2 \to 1} \) typically depends on the width linearly. \([56]\) improved the exponential dependency \( R_f^2 \) appearing in this bound \([6]\) to obtained a bound \( O \left( \frac{1}{\sqrt{n}} \left( 1 + L \kappa^{2/3} R_{2 \to 1}^{2/3} + L \kappa^{2/3} R_1^{2/3} \right)^{3/2} \right) \) where \( \kappa \) is the Lipschitz continuity between layers. We can see that \( R_{2 \to 1}^2 \) and \( R_1^2 \) can depend on the width linearly and quadratically respectively even though \( R_f^2 \) is bounded.

### 3 Compression bound for noncompressed network

Here, we give a general theoretical tool that converts a compression based bound to that for the original network \( \tilde{F} \). We suppose the model classes \( \tilde{F} \) and \( \tilde{G} \) are fixed independently on each data...
We denote the Minkowski difference of \( \hat{\mathcal{F}} \) and \( \hat{\mathcal{G}} \) by \( \hat{\mathcal{F}} - \hat{\mathcal{G}} := \{ f - g \mid f \in \hat{\mathcal{F}}, g \in \hat{\mathcal{G}} \} \). We assume that the local Rademacher complexity of this set has a concave shape with respect to \( r > 0 \): Suppose that there exists a function \( \phi : [0, \infty) \rightarrow [0, \infty) \) such that
\[
\hat{R}_r (\hat{\mathcal{F}} - \hat{\mathcal{G}}) \leq \phi(r) \quad \text{and} \quad \phi(2r) \leq 2\phi(r) \quad (\forall r > 0).
\]
This condition is not restrictive, and usual bounds for the local Rademacher complexity satisfy this condition \([35,4] \). Using this notation, we define \( r_\ast = r_\ast(t) \) as
\[
r_\ast(t) := \inf \left\{ r > 0 \mid 8\phi (r) r^2 + 1 \leq \frac{1}{2} \right\}.
\]
This is roughly given by the fixed point of a function \( r^2 \rightarrow \phi(r) \), and it is useful to bound the ratio of the empirical \( L_2 \)-norm and the population \( L_2 \)-norm of an element \( h \) in \( \hat{\mathcal{F}} - \hat{\mathcal{G}} \): \( \| h \| L_2^2 / (\| h \|_n^2 + r_r^2) \leq 1/2 \) with high probability. Then, we obtain the following theorem that gives the compression based bound for non-compressed networks.

**Theorem 1.** Suppose that the empirical \( L_2 \)-distance between \( \hat{f} \) and \( \hat{g} \) is bounded by \( \| \hat{f} - \hat{g} \|_n \leq \tilde{r}^2 \) for a fixed \( \tilde{r} > 0 \) almost surely. Let \( \hat{r} := \sqrt{2(\tilde{r}^2 + r_r^2)} \), then, under Assumptions \([1,2,3] \) there exists a universal constant \( C > 0 \) such that
\[
\Psi(\hat{f}) \leq \hat{\Psi}(\hat{f}) + 2\hat{R}_n(\hat{g}) + \begin{cases} M \frac{2\hat{t}}{n} & \text{main term} \\ +C \left[ \hat{R}_r (\hat{\mathcal{F}} - \hat{\mathcal{G}}) \log(n)^{2} + \sqrt{\frac{r}{n}} \frac{1}{n} + \frac{1 + tM}{n} \right] & \text{fast term} \end{cases}
\]
with probability at least \( 1 - 3e^{-t} \) for all \( t \geq 1 \).

The proof is given in Appendix \([A] \). The bound consists of two terms: “main term” and “fast term.” The main term represents the complexity of the compressed model \( \hat{\mathcal{G}} \) which could be much smaller than \( \hat{\mathcal{F}} \). The fast term represents a sample complexity to bridge the original model and the compressed model. If we set \( \hat{r} = o_r(1) \), then it can be faster than the main term which is \( O(1/\sqrt{n}) \). Indeed, the fast term achieves \( o(1/n) \) in a typical situation. The term \( \hat{R}_r (\hat{\mathcal{F}} - \hat{\mathcal{G}}) \log(n)^{2} \) can be refined by directly evaluating the covering number of the model (the poly-log(n) factor can be improved). The refined version is given in Appendix \([A] \). This bound is general, and can be combined with the compression bounds derived so far such as \([1,2,48] \) where the complexity of \( \hat{\mathcal{G}} \) and the bias \( \hat{r} \) are analyzed for their generalization error bounds.

The main difference from the compression bound \([1] \) for \( \hat{g} \) is that the bias term \( \hat{r} = \| \hat{f} - \hat{g} \|_n \) is replaced by \( \frac{1}{\sqrt{n}} \| \hat{f} - \hat{g} \|_n \), which is \( \sqrt{n} \) times smaller. Since \( r_r^2 \) and \( \Phi(\hat{r}) \) are typically \( o(1/\sqrt{n}) \), we may neglect these terms, and then the bound is informally written as
\[
\Psi(\hat{f}) \leq \hat{\Psi}(\hat{f}) + O_p \left( \hat{R}_n(\hat{g}) + \frac{1}{\sqrt{n}} \| \hat{f} - \hat{g} \|_n + \sqrt{1/n} \right).
\]
This allows us to obtain tighter bound than the compression bound for \( \hat{g} \) because the bias term \( \hat{r} / \sqrt{n} \) is much smaller than \( \hat{r} \) and eventually we can let the variance term \( \hat{R}_r(\hat{g}) \) much smaller by taking small compressed model \( \hat{\mathcal{G}} \) when we balance the bias and variance trade-off. This is an advantageouse point of directly bounding the generalization error of \( \hat{f} \) instead of \( \hat{g} \).

Finally, we note that some existing bounds such as \([1,2,56] \) assumes a constant margin so that the bias term can be a sufficiently small constant (which does not need to converge to 0). On the other hand, our bound does not assume it and the bias term should converge to 0 so that the bias is balanced with the variance term, which is a more difficult problem setting.

**Example 1.** In practice, a trained network can be usually compressed to one with sparse weight matrix via pruning techniques \([14,15] \). Based on this observation, \([2] \) derived a compression based bound based on a pruning procedure. In this situation, we may suppose that \( \hat{\mathcal{G}} \) is the set of

\footnote{We can extend the result to data dependent models \( \hat{\mathcal{F}} \) and \( \hat{\mathcal{G}} \) by taking uniform bound for all possible choice of the pair \( \hat{\mathcal{F}} \) and \( \hat{\mathcal{G}} \). However, we omit explicit presentation of this uniform bound for simplicity.}
networks with $S$ non-zero parameters where $S$ is much smaller than the total number of parameters: 
$\tilde{G} = \{ f \in \text{NN}(m, R_2, R_F) \mid \sum_{\ell=1}^{L} \| W^{(\ell)} \|_0 \leq S \}$ where $\| W^{(\ell)} \|_0$ is the number of nonzero parameters of the weight matrix $W^{(\ell)}$. In this situation, its Rademacher complexity is bounded by 
$R(\tilde{G}) \leq CM \sqrt{\frac{L}{n} \log(n)}$ (see Appendix B.2 for the proof). This is much smaller than the VC-dimension bound $\sqrt{\frac{L \sum_{\ell=1}^{L} m_{\ell} m_{\ell+1}}{n}} \log(n)$ if $S \leq \sum_{\ell=1}^{n} m_{\ell} m_{\ell+1}$.

Although our bound can be adopted to several compression based bounds, we are going to demonstrate how small the obtained bound can be through some typical situations in the following.

### 3.1 Compression bound with near low rank weight matrix

Here, we analyze the situation where the trained network has near low rank weight matrices $(W^{(\ell)})_{\ell=1}^{L}$. It has been reported that the trained network tends to have near low rank weight matrices experimentally [19, 26]. This situation has been analyzed in [1] where the low rank property is characterized by their original quantities such as layer cushion. However, we employ a much simpler and intuitive condition to highlight how the low rank property affects the generalization.

**Assumption 4.** Assume that each of weight matrices $W^{(\ell)}$ ($\ell = 1, \ldots, L$) of any $f \in \tilde{F}$ is near low rank, that is, there exists $\alpha > 1/2$ and $V_0 > 0$ such that

$$
\sigma_j(W^{(\ell)}) \leq V_0 j^{-\alpha},
$$

where $\sigma_j(W)$ is the $j$-th largest singular value of a matrix $W$. $\sigma_1(W) \geq \sigma_2(W) \geq \cdots \geq 0$.

In this situation, we can see that for any $1 \leq s \leq \min\{m_\ell, m_{\ell+1}\}$, we can approximately $W^{(\ell)}$ by a rank $s$ matrix $W'$ as $\| W^{(\ell)} - W' \|_2 \leq V_0 s^{-\alpha}$. Let the set of networks with exactly low rank weight matrices be $\text{NN}(m, s, R_2, R_F) := \{ f \in \text{NN}(m, R_2, R_F) \mid \text{the weight matrix } W^{(\ell)} \text{ of } f \text{ has rank } s_{\ell} \}$ for $s = (s_1, \ldots, s_L)$. If we set $\tilde{G} = \text{NN}(m, s, R_2, R_F)$, then we have the following theorem.

**Theorem 2.** The compressed model $\tilde{G} = \text{NN}(m, s, R_2, R_F)$ has the following complexity:

$$
\tilde{R}_n(\tilde{G}) \leq CM \sqrt{\frac{L \sum_{\ell=1}^{L} s_{\ell}(m_\ell + m_{\ell+1})}{n} \log(n)}.
$$

If $\tilde{F}$ satisfies Assumption 4, we can set $\hat{r} = V_0 R_2^{L-1} B_2 \sum_{\ell=1}^{L} s_{\ell}^{-\alpha}$; for any $\hat{f} \in \tilde{F}$, there exists $\hat{g} \in \tilde{G}$ such that $\| \hat{f} - \hat{g} \|_n \leq \hat{r}$. Then, letting $A_1 = L \sum_{\ell=1}^{L} s_{\ell}(m_\ell + m_{\ell+1}) \log(n)$ and 
$A_2 = L \sqrt{\sum_{\ell=1}^{L} m_\ell (2L V_0 R_2^{L-1} B_2)^{1/\alpha}}$, the overall generalization error is bounded by

$$
\Psi(\hat{f}) \leq \tilde{\Psi}(\hat{f}) + C \left[ M A_1 + M^{2\alpha - 1} A_2^{2\alpha - \alpha} + \sqrt{2(1-2\alpha)} A_2 + (\hat{r} + M) \sqrt{A_1} + \frac{1 + tM}{n} \right],
$$

with probability $1 - 3e^{-t}$ for any $t > 1$ where $C > 0$ is a constant depending on $\alpha$.

See Appendix B.3 for the proof. This indicates that, if $\alpha > 1/2$ is large (in other words, each weight matrix is close to rank 1), then we have a better generalization error bound. Note that the rank $s_{\ell}$ can be arbitrary chosen and $\hat{r}$ and $A_1$ are in a trade-off relation. Hence, by selecting the rank appropriately so that this trade-off is balanced, then we obtain the optimal upper bound as in the following corollary.

**Corollary 1.** Under Assumption 4 using the same notation as Theorem 2 it holds that

$$
\Psi(\hat{f}) \leq \tilde{\Psi}(\hat{f}) + C \left[ M^{1-1/2\alpha} \sqrt{L \sum_{\ell=1}^{L} m_\ell (2L V_0 R_2^{L-1} B_2)^{1/\alpha}} \log(n) + M^{2\alpha - 1} A_2^{2\alpha - \alpha} + \frac{1 + tM}{n} \right]
$$

with probability $1 - 3e^{-t}$ for any $t > 1$ where $C$ is a constant depending on $\alpha$.

An important point here is that the bound is $O(\sqrt{L \sum_{\ell=1}^{L} m_\ell})$ which has linear dependency on the width $m_\ell$ in the square root, but the naive VC-dimension bound has quadratic dependency.
$O(\sqrt{L \sum_{l=1}^{L-1} m_l m_{l+1}})$. In other words, the term in the square root has linear dependency to the number of nodes instead of the number of parameters. This is huge gap because the width can be quite large in practice. This result implies that a compressible model achieves much better generalization than the naive VC-bound.

In the generalization error bound, there appears $R^2_F$. Even though $R_2$ can be much smaller than $R_F$, the exponential dependency $R^2_F$ can give loose bound as pointed out in [1]. This is due to a rough evaluation of the Lipschitz continuity between layers, but the practically observed Lipschitz constant is usually much smaller. To fix this issue, we give a refined version of Corollary 1 in Appendix B.4 by using data dependent Lipschitz constants such as interlayer cushion and interlayer smoothness introduced by [1]. The refined bound does not involve the exponential term $R^2_F$, but instead $\kappa^2$ (\(\kappa\): Lipschitz continuity) appears.

### 3.2 Compression bound with near low rank covariance matrix

Strictly speaking, the near low rank condition on the weight matrix in the previous section can be dealt with a standard Rademacher complexity argument. Here, we consider more data dependent bound: We assume the near low rank property of the covariance matrix.

\[\sum_{n=1}^{n} \| \Theta(x_i) \|_i \leq \sum_{n}^{\ell} (m^2) - \beta/2.\]

More precisely, for given \(\hat{r}_\ell > 0 (\ell = 1, \ldots, L)\) which corresponds to the compression error in the \(\ell\)-th layer, let \(\tilde{m}_\ell := \max \{ 1 \leq j \leq m_\ell | \hat{\mu}_j^{(\ell)} \geq \hat{r}_\ell^2 / 4 \}.\) Then, we define \(N_\ell(\hat{r}) = \beta + 1 \tilde{m}_\ell + 8(\sum_{n=1}^{\ell} R^2_\ell R_\ell^2)\), for \(\hat{r} = \{ \hat{r}_1, \ldots, \hat{r}_L \}.\) Correspondingly we set

\[m^2 := 5N_\ell(\hat{r}) \log(80N_\ell(\hat{r})).\]

Then, we obtain the following theorem.

**Theorem 3.** Let \(\hat{r} := \sum_{k=1}^{L} R^2_\ell (L-k) R_\ell \hat{r}_k.\) Then, under Assumption 5 there exists \(\hat{g}\) with width \(m^2 = (m_1, m_2^2, \ldots, m_L^2)\) that satisfies \(\hat{g} \in \mathbb{N}(m^2, \sqrt{2\beta} \max_m m_R 2^{\sqrt{2\beta} m_m R} F)\) and

\[\| \hat{f} - \hat{g} \|_n \leq \hat{r}.\]

In particular, we may set \(\hat{G} = \mathbb{N}(m^2, \sqrt{2\beta} \max_m m_R 2^{\sqrt{2\beta} m_m R}),\) and then it holds that \(\hat{R}_n(\hat{G}) \leq C \sqrt{L \sum_{l=1}^{L} m^2_{l+1} m^2_l} \log(n)\) for a constant \(C > 0.\)

See Appendix B.5 for the proof. Here, we again observe that there appears a trade-off between \(\hat{r}\) and \(m^2\) because as \(\hat{r}\) becomes small, then \(m^2\) becomes large and thus \(m^2\) becomes large. The evaluation given in Theorem 3 can be substituted to the general bound (Theorem 1). If \(\hat{F}\) is the full model \(\mathcal{F},\) then there appears the number \(\sum_{f=1}^{L} m_f m_{f+1}\) of parameters which could be larger than \(n,\) which is unavoidable. This dependency on the number of parameters becomes much milder if both of Assumptions 4 and 5 are satisfied.
Theorem 4. Under Assumptions $4$ and $5$ it holds that

$$
\Psi(f) \leq \hat{\Psi}(\hat{f}) + C \left[ M \beta \frac{1 + \beta}{\log(1 + \beta)} \left( \sum_{\ell=1}^{L} m_{\ell} \right)^{\frac{4/\beta + 2(1-1/2\alpha)}{n}} \log(n)^{3} \right] + M \frac{2^{\alpha+1}}{\beta n} \left( LP_{L} \frac{\sum_{\ell=1}^{L} m_{\ell}}{n} \log(n)^{2} + M \frac{R_{L}^{2} L^{2}}{R_{2}^{2}} \sqrt{\frac{\log(n)^{2}}{n} + 1 + \frac{4}{\beta} n} \right),
$$

for $P_{L} = (2L_{0}R_{2}^{L-1}B_{L})^{1/\alpha}$ and $Q_{L} = \left[ \frac{4L_{0}R_{L}^{2}(1 + R_{L})^{2} \exp\left(\frac{4}{\beta}(2\sqrt{L} - 1)\right)}{(0.25)^{1/R_{2}^{2}}} \right]^{2/\beta}$ with probability $1 - 3e^{-t}$ ($t > 1$), where $C$ is a constant depending on $\alpha$, $\beta$.

If we omit $L$ and $\log(n)$ terms for simplicity of presentation, then the bound can be written as

$$
\hat{O} \left( \sqrt{\left( \sum_{\ell=1}^{L} m_{\ell} \right)^{\frac{4/\beta + 2(1-1/2\alpha)}{n}}} + \left( \sum_{\ell=1}^{L} m_{\ell} \right)^{\frac{2}{\beta n} \frac{2}{n}} \right),
$$

where the $\hat{O}(\cdot)$ symbol hides the poly-log order. This is tighter than that of Corollary $1$ We can see that as $\beta$ and $\alpha$ get large, the bound becomes tighter. Actually, by taking the limit of $\alpha$, $\beta \to \infty$, then the bound goes to $L^{2} \sqrt{\frac{\log(n)}{n}} + L \frac{\sum_{\ell=1}^{L} m_{\ell}}{n}$. Moreover, the term dependent on the width is $O(1/n)$

with respect to the sample size $n$ which is faster than the rate $O(\sqrt{\sum_{\ell=1}^{L} m_{\ell}})$ which was presented in Corollary $1$. Hence, the low rank property of both the covariance matrix and the weight matrix helps to obtain better generalization. Although the bound contains the exponential term $R_{L}^{2}$, we can give a refined version that does not contain the exponential term by assuming interlayer cushion $\hat{1}$. See Appendix $B.3$ for the refined version.

There appears $\exp\left(\frac{4}{\beta}(2\sqrt{L} - 1)\right)$ which is exponentially dependent on $L$. However, this term is moderately small for realistic settings of the depth $L$. Actually, it is $7.27$ for $L = 20$ and $26.7$ for $L = 50$ (we can replace this term in exchange for larger polynomial dependency on $L$). The bound is not optimized with respect to the dependency on the depth $L$. In particular, the term $L^{2} \sqrt{\log(n)/n}$ could be an artifact of the proof technique and the $L^{2}$ term would be improved.

Finally, we compare our bound with the following norm based bounds; $O \left( \frac{n_{L}}{n} \left( \frac{L_{\beta/2}^{2/3}}{R_{2}^{2}} \right)^{3/2} \right)$ by $[6]$ and $O \left( \frac{1}{\sqrt{n}} \left( 1 + L_{\beta/2}^{2/3} R_{2\to1}^{2/3} + \kappa^{\beta/2} L_{\beta/2\to1} R_{2\to1}^{2/3} \right)^{3/2} \right)$ by $[5]$. Since our bound and their bounds are derived from different conditions, we cannot tell which is better. Here, we consider a special case where $m_{\ell} = m$ ($\forall \ell$) and $W^{(\ell)} = \frac{1}{m} \mathbf{1} \mathbf{1}^{\top} \in \mathbb{R}^{m \times m}$ which is an extreme case of low rank settings (note that $W^{(\ell)}$ has rank 1). Then, $R_{2} = 1$, $R_{F} = 1$, $R_{2\to1} = \sqrt{m}$ and $R_{1\to1} = m$, and thus their bounds are $O(\sqrt{m/n})$ and $O(\sqrt{(m + m^{2})/n})$ respectively. However, $\beta$ and $\alpha$ in our bound $\sqrt{m_{\ell}^{\beta} + 2(1-1/2\alpha)/n} = \sqrt{m_{\ell}^{\beta} + 2(1-1/2\alpha)/n}$ can be arbitrary large in this situation, so that our bound has much milder dependency on the width $m$. On the other hand, if the weight matrix has small norm and has no spectral decay (corresponding to small $\alpha$ and $\beta$), then our bound can be looser than theirs. Combining compression based bounds and norm based bounds would be interesting future work.

4 Conclusion

In this paper, we derived a compression based error bound for non-compressed network. The bound is general and it can be adopted to several compression based bound derived so far. The main difficulty lies in evaluating the population $L_{2}$-norm between the original network and the compressed network, but it can be overcome by utilizing the data dependent bound by the local Rademacher complexity technique. We have applied the derived bound to a situation where low rank properties of the weight matrices and the covariance matrices are assumed. The obtained bound gives much better dependency on the parameter size than ever obtained compression based ones.
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References

[1] S. Arora, R. Ge, B. Neyshabur, and Y. Zhang. Stronger generalization bounds for deep nets via a compression approach. In J. Dy and A. Krause (eds.), Proceedings of the 35th International Conference on Machine Learning, volume 80 of Proceedings of Machine Learning Research, pp. 254–263, Stockholmsmassan, Stockholm Sweden, 10–15 Jul 2018. PMLR.

[2] S. Arora, S. S. Du, W. Hu, Z. Li, and R. Wang. Fine-Grained Analysis of Optimization and Generalization for Overparameterized Two-Layer Neural Networks. arXiv e-prints, art. arXiv:1901.08584, Jan 2019.

[3] F. Bach. On the equivalence between kernel quadrature rules and random feature expansions. Journal of Machine Learning Research, 18(21):1–38, 2017.

[4] P. Bartlett, O. Bousquet, and S. Mendelson. Local Rademacher complexities. The Annals of Statistics, 33:1487–1537, 2005.

[5] P. Bartlett, D. J. Foster, and M. Telgarsky. Spectrally-normalized margin bounds for neural networks. arXiv preprint arXiv:1706.08498, 2017.

[6] P. L. Bartlett, D. J. Foster, and M. J. Telgarsky. Spectrally-normalized margin bounds for neural networks. In Advances in Neural Information Processing Systems, pp. 6241–6250, 2017.

[7] C. Baykal, L. Liebenwein, I. Gilitschenski, D. Feldman, and D. Rus. Data-dependent coresets for compressing neural networks with applications to generalization bounds. In International Conference on Learning Representations, 2019.

[8] M. Bianchini and F. Scarselli. On the complexity of neural network classifiers: A comparison between shallow and deep architectures. IEEE transactions on neural networks and learning systems, 25(8):1553–1565, 2014.

[9] S. Boucheron, G. Lugosi, and P. Massart. Concentration Inequalities: A Nonasymptotic Theory of Independence. OUP Oxford, 2013.

[10] O. Bousquet. A Bennett concentration inequality and its application to suprema of empirical process. C. R. Acad. Sci. Paris Ser. I Math., 334:495–500, 2002.

[11] N. Cohen and A. Shashua. Convolutional rectifier networks as generalized tensor decompositions. In Proceedings of the 33th International Conference on Machine Learning, volume 48 of JMLR Workshop and Conference Proceedings, pp. 955–963, 2016.

[12] N. Cohen, O. Sharir, and A. Shashua. On the expressive power of deep learning: A tensor analysis. In The 29th Annual Conference on Learning Theory, pp. 698–728, 2016.

[13] G. Cybenko. Approximation by superpositions of a sigmoidal function. Mathematics of Control, Signals, and Systems (MCSS), 2(4):303–314, 1989.

[14] M. Denil, B. Shakibi, L. Dinh, M. A. Ranzato, and N. de Freitas. Predicting parameters in deep learning. In C. J. C. Burges, L. Bottou, M. Welling, Z. Ghahramani, and K. Q. Weinberger (eds.), Advances in Neural Information Processing Systems 26, pp. 2148–2156. Curran Associates, Inc., 2013.

[15] E. L. Denton, W. Zaremba, J. Bruna, Y. LeCun, and R. Fergus. Exploiting linear structure within convolutional networks for efficient evaluation. In Z. Ghahramani, M. Welling, C. Cortes, N. D. Lawrence, and K. Q. Weinberger (eds.), Advances in Neural Information Processing Systems 27, pp. 1269–1277. Curran Associates, Inc., 2014.

[16] J. Devlin, M.-W. Chang, K. Lee, and K. Toutanova. BERT: Pre-training of Deep Bidirectional Transformers for Language Understanding. arXiv e-prints, art. arXiv:1810.04805, Oct 2018.
[17] E. Giné and V. Koltchinskii. Concentration inequalities and asymptotic results for ratio type empirical processes. *The Annals of Probability*, 34(3):1143–1216, 2006.

[18] N. Golowich, A. Rakhlin, and O. Shamir. Size-independent sample complexity of neural networks. In S. Bubeck, V. Perchet, and P. Rigollet (eds.), *Proceedings of the 31st Conference On Learning Theory*, volume 75 of *Proceedings of Machine Learning Research*, pp. 297–299. PMLR, 06–09 Jul 2018.

[19] S. Gunasekar, J. D. Lee, D. Soudry, and N. Srebro. Implicit bias of gradient descent on linear convolutional networks. In *Advances in Neural Information Processing Systems*, pp. 9482–9491, 2018.

[20] M. Hardt, B. Recht, and Y. Singer. Train faster, generalize better: Stability of stochastic gradient descent. In M. F. Balcan and K. Q. Weinberger (eds.), *Proceedings of The 33rd International Conference on Machine Learning*, volume 48 of *Proceedings of Machine Learning Research*, pp. 1225–1234, New York, New York, USA, 20–22 Jun 2016. PMLR.

[21] N. Harvey, C. Liaw, and A. Mehrabian. Nearly-tight VC-dimension bounds for piecewise linear neural networks. In S. Kale and O. Shamir (eds.), *Proceedings of the 2017 Conference on Learning Theory*, volume 65 of *Proceedings of Machine Learning Research*, pp. 1064–1068, Amsterdam, Netherlands, 07–10 Jul 2017. PMLR.

[22] G. Hinton and D. Van Camp. Keeping neural networks simple by minimizing the description length of the weights. In *in Proc. of the 6th Ann. ACM Conf. on Computational Learning Theory*. Citeseer, 1993.

[23] S. Hochreiter and J. Schmidhuber. Flat minima. *Neural Computation*, 9(1):1–42, 1997.

[24] K. Hornik. Approximation capabilities of multilayer feedforward networks. *Neural Networks*, 4(2):251–257, 1991.

[25] S. Ioffe and C. Szegedy. Batch normalization: Accelerating deep network training by reducing internal covariate shift. In F. Bach and D. Blei (eds.), *Proceedings of the 32nd International Conference on Machine Learning*, volume 37 of *Proceedings of Machine Learning Research*, pp. 448–456, Lille, France, 07–09 Jul 2015. PMLR.

[26] Z. Ji and M. Telgarsky. Gradient descent aligns the layers of deep linear networks. In *International Conference on Learning Representations*, 2019.

[27] V. Koltchinskii. Local Rademacher complexities and oracle inequalities in risk minimization. *The Annals of Statistics*, 34:2593–2656, 2006.

[28] A. Krizhevsky, I. Sutskever, and G. E. Hinton. Imagenet classification with deep convolutional neural networks. In *Advances in neural information processing systems*, pp. 1097–1105, 2012.

[29] A. Krogh and J. A. Hertz. A simple weight decay can improve generalization. In *Advances in neural information processing systems*, pp. 950–957, 1992.

[30] J. Langford and R. Caruana. (not) bounding the true error. In T. G. Dietterich, S. Becker, and Z. Ghahramani (eds.), *Advances in Neural Information Processing Systems 14*, pp. 809–816. MIT Press, 2002.

[31] M. Ledoux and M. Talagrand. *Probability in Banach Spaces. Isoperimetry and Processes*. Springer, New York, 1991. MR1102015.

[32] X. Li, J. Lu, Z. Wang, J. Haupt, and T. Zhao. On tighter generalization bound for deep neural networks: Cnns, resnets, and beyond. *arXiv preprint arXiv:1806.05159*, 2018.

[33] N. Littlestone and M. K. Warmuth. Relating data compression and learnability. Technical report, University of California, Santa Cruz, 1986.

[34] C. H. Martin and M. W. Mahoney. Implicit self-regularization in deep neural networks: Evidence from random matrix theory and implications for learning. *arXiv preprint arXiv:1810.01075*, 2018.
[35] S. Mendelson. Improving the sample complexity using global data. *IEEE Transactions on Information Theory*, 48:1977–1991, 2002.

[36] M. Mohri, A. Rostamizadeh, and A. Talwalkar. Foundations of machine learning. 2012.

[37] G. F. Montufar, R. Pascanu, K. Cho, and Y. Bengio. On the number of linear regions of deep neural networks. In Z. Ghahramani, M. Welling, C. Cortes, N. Lawrence, and K. Weinberger (eds.), *Advances in Neural Information Processing Systems* 27, pp. 2924–2932. Curran Associates, Inc., 2014.

[38] V. Nagarajan and Z. Kolter. Deterministic PAC-bayesian generalization bounds for deep networks via generalizing noise-resilience. In *International Conference on Learning Representations*, 2019.

[39] B. Neyshabur, R. Tomioka, and N. Srebro. Norm-based capacity control in neural networks. In *Proceedings of The 28th Conference on Learning Theory*, pp. 1376–1401, Montreal Quebec, 2015.

[40] B. Neyshabur, S. Bhojanapalli, D. McAllester, and N. Srebro. A PAC-Bayesian approach to spectrally-normalized margin bounds for neural networks. *arXiv preprint arXiv:1707.09564*, 2017.

[41] B. Neyshabur, Z. Li, S. Bhojanapalli, Y. LeCun, and N. Srebro. The role of overparametrization in generalization of neural networks. In *International Conference on Learning Representations*, 2019.

[42] B. Poole, S. Lahiri, M. Raghu, J. Sohl-Dickstein, and S. Ganguli. Exponential expressivity in deep neural networks through transient chaos. In D. D. Lee, M. Sugiyama, U. V. Luxburg, I. Guyon, and R. Garnett (eds.), *Advances in Neural Information Processing Systems* 29, pp. 3360–3368. Curran Associates, Inc., 2016.

[43] A. Radford, L. Metz, and S. Chintala. Unsupervised Representation Learning with Deep Convolutional Generative Adversarial Networks. *arXiv e-prints*, art. arXiv:1511.06434, Nov 2015.

[44] S. Sonoda and N. Murata. Neural network with unbounded activation functions is universal approximator. *Applied and Computational Harmonic Analysis*, 2015.

[45] N. Srivastava, G. Hinton, A. Krizhevsky, I. Sutskever, and R. Salakhutdinov. Dropout: a simple way to prevent neural networks from overfitting. *The Journal of Machine Learning Research*, 15(1):1929–1958, 2014.

[46] I. Steinwart and A. Christmann. *Support Vector Machines*. Springer, 2008.

[47] T. Suzuki. Adaptivity of deep reLU network for learning in besov and mixed smooth besov spaces: optimal rate and curse of dimensionality. In *International Conference on Learning Representations*, 2019.

[48] T. Suzuki, H. Abe, T. Murata, S. Horiuchi, K. Ito, T. Wachi, S. Hirai, M. Yukishima, and T. Nishimura. Spectral-Pruning: Compressing deep neural network via spectral analysis. *arXiv e-prints*, art. arXiv:1808.08558, Aug 2018.

[49] M. Talagrand. New concentration inequalities in product spaces. *Inventiones Mathematicae*, 126:505–563, 1996.

[50] G. Valle-Perez, C. Q. Camargo, and A. A. Louis. Deep learning generalizes because the parameter-function map is biased towards simple functions. In *International Conference on Learning Representations*, 2019.

[51] A. W. van der Vaart and J. A. Wellner. *Weak Convergence and Empirical Processes: With Applications to Statistics*. Springer, New York, 1996.

[52] V. N. Vapnik. *Statistical Learning Theory*. Wiley, New York, 1998.
Appendix

In the appendix, we give the proofs of the main text. We use the following notation throughout the appendix:

\[ P_n f := \frac{1}{n} \sum_{i=1}^{n} f(z_i), \quad P f = \mathbb{E}[f(Z)]. \]

To evaluate it, the covering number is useful \[51\]. For a metric space \( \tilde{F} \) equipped with a metric \( \tilde{d} \), the \( \epsilon \)-covering number \( N(\tilde{F}, \tilde{d}, \epsilon) \) is defined as the minimum number of balls with radius \( \epsilon \) (measured by the metric \( \tilde{d} \)) to cover the metric space \( \tilde{F} \). Hereafter, \( C \) denotes a constant which will be dependent on the context.

We let \( a \wedge b := \min\{a, b\} \) and \( a \vee b := \max\{a, b\} \) for \( a, b \in \mathbb{R} \).

A Proof of Theorem 1

For \( \hat{\gamma}_n = \hat{\gamma}_n(D_n) := \sup\{\|f - g\|_n \mid \|f - g\|_{L_2} \leq r, f \in \widehat{F}, g \in \widehat{G}\} \), we define

\[
\Phi(r) := \min \left\{ \hat{R}_r(\widehat{F} - \widehat{G}) \sqrt{\log(n) \log(2nM)}, \right. \\
\left. \mathbb{E} \left[ \int_{1/n}^{\hat{r} n} \sqrt{\log(N(\widehat{F}, \| \cdot \|_n, \epsilon/2))} d\epsilon + \int_{1/n}^{\hat{r} n} \sqrt{\log(N(\widehat{G}, \| \cdot \|_n, \epsilon/2))} d\epsilon \right] \right\}.
\]

Here, we restate Theorem 1 in the following in more complete form.

**Theorem 5.** Suppose that the empirical \( L_2 \)-distance between \( \widehat{f} \) and \( \widehat{g} \) is bounded by \( \|\widehat{f} - \widehat{g}\|_n \leq r^2 \) for a fixed \( r > 0 \) almost surely. Let \( \hat{\epsilon} := \sqrt{2(\hat{r}^2 + r_0^2)} \), then, under Assumptions \(7, 8, 9\) there exists a universal constant \( C > 0 \) such that

\[
\Psi(\hat{f}) \leq \hat{\Psi}(\hat{f}) + 2R_n(\widehat{G}) + \sqrt{\frac{2M}{n}} + C \left[ \Phi(\hat{r}) + \hat{r} \sqrt{\frac{\hat{r} n + 1 + tM}{n}} \right].
\]

with probability at least \( 1 - 3e^{-t} \) for all \( t \geq 1 \).
Proof. First, by the standard Rademacher complexity analysis, we have that

\[
\Psi(\hat{g}) - \hat{\Psi}(\hat{g}) = \frac{1}{n} \sum_{i=1}^{n} (\psi(z_i, \hat{g}(x_i)) - \mathbb{E}[\psi(Z, \hat{g}(X))])
\]

\[
\leq \mathbb{E}\left[\left.\sup_{g \in \hat{G}} \frac{1}{n} \sum_{i=1}^{n} (\psi(z_i, g(x_i)) - \mathbb{E}[\psi(Z, g(X))])\right| 1 \leq n\right]
\]

\[
\leq 2\mathbb{E}_{D_n, \epsilon} \left[\sup_{g \in \hat{G}} \frac{1}{n} \sum_{i=1}^{n} \epsilon_i \psi(z_i, g(x_i))\right] + \sqrt{\frac{M 2\epsilon}{n}} \leq 2\tilde{R}_n(\hat{g}) + \sqrt{\frac{M 2\epsilon}{n}}
\]

with probability $1 - e^{-\epsilon}$, where we used the Rademacher concentration inequality (Theorem 3.1 of [56]) in the third line and the contraction inequality (Theorem 11.6 of [9] or Theorem 4.12 of [31] and its proof) in the last line. We let this event be $E_0(t)$.

Next, we observe that

\[
\Psi(\hat{f}) - \hat{\Psi}(\hat{f}) = \Psi(\hat{f}) - \Psi(\hat{g}) + \Psi(\hat{g}) - \hat{\Psi}(\hat{g}) - \hat{\Psi}(\hat{f})
\]

\[
\leq (\Psi(\hat{f}) - \Psi(\hat{g}) - (\hat{\Psi}(\hat{f}) - \hat{\Psi}(\hat{g}))) + \Psi(\hat{g}) - \hat{\Psi}(\hat{g})
\]

\[
\leq [\Psi(\hat{f}) - \Psi(\hat{g}) - (\hat{\Psi}(\hat{f}) - \hat{\Psi}(\hat{g}))] + 2\tilde{R}_n(\hat{g}) + \sqrt{\frac{M 2\epsilon}{n}}
\]

where we used Eq. (3) in the last line. Here, it should be noticed that it is not a good strategy to bound the first term $\Psi(\hat{f}) - \Psi(\hat{g}) - (\hat{\Psi}(\hat{f}) - \hat{\Psi}(\hat{g}))$ by bounding $\Psi(\hat{f}) - \hat{\Psi}(\hat{f})$ and $\Psi(\hat{g}) - \hat{\Psi}(\hat{g})$ independently. Instead, we should bound them simultaneously to obtain tighter bound. This can be accomplished by using the local Rademacher complexity technique.

Since both of $\hat{f}$ and $\hat{g}$ are data dependent random variable, the contraction inequality [31] is not trivially applied. We overcome this difficulty as follows. Let $\hat{\gamma}_n = \hat{\gamma}_n(D_n) := \sup\{\|\hat{f} - \hat{g}\|_n \mid \|\hat{f} - \hat{g}\|_{L_2} \leq r\}$, then the conditional Rademacher complexity of the set $\{\psi(y, f(x)) - \psi(y, g(x)) \mid f \in \hat{F}, g \in \hat{G}, \|f - g\|_{L_2} \leq r\}$ can be bounded by a constant times the following Dudley integral (see Theorem 5.22 of [55] or Lemma A.5 of [5] for example):

\[
\inf_{\alpha > 0} \left[\alpha + \int_{\alpha}^{\tilde{\gamma}_n} \frac{\log(N(\{\psi(f) - \psi(g) \mid f \in \hat{F}, g \in \hat{G}, \|f - g\|_{L_2} \leq r\}, \|\cdot\|_n, \epsilon))}{n} \right] d\epsilon
\]

\[
\leq \frac{1}{n} + \int_{1/n}^{\tilde{\gamma}_n} \frac{\log(N(\{f - g \mid f \in \hat{F}, g \in \hat{G}, \|f - g\|_{L_2} \leq r\}, \|\cdot\|_n, \epsilon))}{n} d\epsilon
\]

\[
\leq \left[\frac{1}{n} + \int_{1/n}^{\tilde{\gamma}_n} \frac{\log(N(\hat{F}, \|\cdot\|_n, \epsilon/2)) + \log(N(\hat{G}, \|\cdot\|_n, \epsilon/2))}{n} d\epsilon\right]
\]

\[
\leq \left[\frac{1}{n} + \int_{1/n}^{\tilde{\gamma}_n} \frac{\log(N(\hat{F}, \|\cdot\|_n, \epsilon/2)) + \log(N(\hat{G}, \|\cdot\|_n, \epsilon/2))}{n} d\epsilon\right]
\]

\[
\leq \left[\frac{1}{n} + \int_{1/n}^{\tilde{\gamma}_n} \frac{\log(N(\hat{F}, \|\cdot\|_n, \epsilon/2))}{n} d\epsilon + \int_{1/n}^{\tilde{\gamma}_n} \frac{\log(N(\hat{G}, \|\cdot\|_n, \epsilon/2))}{n} d\epsilon\right],
\]

where we used 1-Lipschitz continuity of the loss function $\psi$ in the first inequality, $\|f - g - (f' - g')\|_n \leq \|f - f'\|_n + \|g - g'\|_n \leq \epsilon$ for $f, f' \in \hat{F}$ and $g, g' \in \hat{G}$ with $\|f - f'\|_n \leq \epsilon/2$ and $\|g - g'\|_n \leq \epsilon/2$ in the third line.
On the other hand, the Sudakov’s minoration (Corollary 4.14 of [31]) gives an upper bound of the right hand side of Eq. 5:

\[
\int_{1/n}^{\gamma_n} \frac{\log(\mathcal{N}(\{ \hat{f} - \tilde{g} \mid \hat{f} \in \tilde{F}, \tilde{g} \in \tilde{G}, \|\hat{f} - \tilde{g}\|_{L_2} \leq r \}, \| \cdot \|_{n, \epsilon})}{n} \, d\epsilon \\
\leq \int_{1/n}^{\gamma_n} \hat{R}_{n,r}(\hat{F} - \tilde{G}) \sqrt{\log(n)} \frac{1}{\epsilon} \, d\epsilon \leq \hat{R}_{n,r}(\hat{F} - \tilde{G}) \sqrt{\log(n)} \log(nr\gamma_n).
\]

Since \( \gamma_n \leq 2M \), the expectation of the right hand side with respect to \( D_n \) is \( \hat{R}_{n,r}(\hat{F} - \tilde{G}) \sqrt{\log(n)} \log(2nM) \). Finally, we note that \( \hat{R}_{n,r}(\hat{F} - \tilde{G}) \) is again bounded by the Dudley integral as

\[
\hat{R}_{n,r}(\hat{F} - \tilde{G}) \leq C + C_E D_n \left[ \int_{1/n}^{\gamma_n} \frac{\log(\mathcal{N}(\{ \hat{f} - \tilde{g} \mid \hat{f} \in \tilde{F}, \tilde{g} \in \tilde{G}, \|\hat{f} - \tilde{g}\|_{L_2} \leq r \}, \| \cdot \|_{n, \epsilon})}{n} \, d\epsilon \right] \\
\leq C + C_E D_n \left[ \int_{1/n}^{\gamma_n} \frac{\log(\mathcal{N}(\hat{F}, \| \cdot \|_{n, \epsilon}/2))}{n} \, d\epsilon + \int_{1/n}^{\gamma_n} \frac{\log(\mathcal{N}(\tilde{G}, \| \cdot \|_{n, \epsilon}/2))}{n} \, d\epsilon \right]
\]

where the last line is by Eq. (8) and \( C > 0 \) is a universal constant.

Next, we bound the population \( L_2 \) norm of \( \hat{f} - \tilde{g} \) given \( \|\hat{f} - \tilde{g}\|_n \leq \hat{r} \). This is done by the local Rademacher complexity argument. Suppose that there exists a function \( \phi : [0, \infty) \rightarrow [0, \infty) \) such that the following conditions are satisfied:

\[
\hat{R}_{r}(\hat{F} - \tilde{G}) \leq \phi(r)
\]

and

\[
\phi(2r) \leq 2\phi(r).
\]

Then, by the so-called *peeling device*, we can show that for any \( r > 0 \),

\[
P \left( \sup_{h \in \hat{F} - \tilde{G}} \frac{(P - P_n)(h^2)}{P h^2 + r^2} \geq 8 \frac{\phi(r)}{r^2} + M \sqrt{\frac{4t}{r^2 n} + M^2 \frac{2t}{r^2 n}} \right) \leq e^{-t}
\]

for all \( t > 0 \) (Theorem 7.7 and Eq. (7.17) of [40]). Hence, if we choose \( r_s = r_s(t) \) so that

\[
8 \frac{\phi(r_s)}{r_s^2} + M \sqrt{\frac{4t}{r_s^2 n} + M^2 \frac{2t}{r_s^2 n}} \leq \frac{1}{2},
\]

then it holds that

\[
P(h^2) \leq 2P_n(h^2) + 2r_s^2
\]

with probability greater than \( 1 - e^{-t} \). We let this event as \( \mathcal{E}(t) \). In this event, if \( \|\hat{f} - \tilde{g}\|_n^2 \leq r^2 \), then it holds that

\[
\|\hat{f} - \tilde{g}\|_{L_2}^2 \leq 2(r^2 + r_s^2).
\]

Next, we bound \( \Psi(\hat{f}) - \Psi(\tilde{g}) - (\Psi(\hat{f}) - \Psi(\tilde{g})) \). To bound this term, we apply the Talagrand’s concentration inequality (Proposition 2 and [49] [10]). To apply it, we should bound the variance and \( L_\infty \)-norm of \( \psi(y, f(x)) - \psi(y, g(x)) - (E[\psi(Y, f(X))] - E[\psi(Y, g(X))]) \) for any \( f \in \hat{F}, g \in \tilde{G} \) with \( \|f - g\|_{L_2} \leq r \) (where \( r \) will be set \( 2(r^2 + r_s^2) \)). Due to the Lipschitz continuity of \( \psi \), we have that

\[
\text{Var}[\psi(Y, f(X)) - \psi(Y, g(X))] \leq \text{Var}[f(X) - g(X)] \leq \|f - g\|_{L_2} \leq r^2.
\]

Similarly, it holds that

\[
|\psi(y, f(x)) - \psi(y, g(x)) - (E[\psi(Y, f(X))] - E[\psi(Y, g(X))])| \leq |\psi(y, f(x)) - E[\psi(Y, f(X))]| + |\psi(y, g(x)) - E[\psi(Y, g(X))]| \leq 2M.
\]
Hence, by the Talagrand’s concentration inequality (Proposition 2 and [49, 10]), it holds that
\[
\sup_{f,g \in \mathbb{R}} \mathbb{E} \left[ \left( P - P_n \right) \left( \psi(Y, f(X)) - \psi(Y, g(X)) \right) \right] \\
\leq 2 \mathbb{E} \left[ \sup_{f,g \in \mathbb{R}} \left( P - P_n \right) \left( \psi(Y, f(X)) - \psi(Y, g(X)) \right) \right] + r \sqrt{\frac{2t}{n} + \frac{4tM}{n}},
\]
with probability at least \(1 - e^{-t}\) for any \(t > 0\). The first term in the right hand side can be bounded as
\[
\mathbb{E} \left[ \sup_{f,g \in \mathbb{R}} \left( P - P_n \right) \left( \psi(Y, f(X)) - \psi(Y, g(X)) \right) \right] \\
\leq 2 \mathbb{E}_{D_n,e} \left[ \sup_{f,g \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} \epsilon_i \left( \psi(z_i, f(x_i)) - \psi(z_i, g(x_i)) \right) \right],
\]
where we used the standard symmetrization argument (see Lemma 11.4 of [2] for example). The right hand side is further bounded by
\[
C \mathbb{E} \left[ \inf_{\alpha > 0} \left\{ \alpha + \int_{\alpha}^{\sqrt{n}} \log \mathcal{N} \left( \|f - g\|_{L_2} \leq r, \|f - g\|_{L_2} \leq r, \|\cdot\|_{L_2}, \epsilon \right) \right\} \right] \\
\leq C \left( \frac{1}{n} + \Phi(r) \right),
\]
for a universal constant \(C > 0\) (see Eq. (5)). Combining these inequalities, it holds that
\[
\sup_{f,g \in \mathbb{R}} \left( P - P_n \right) \left( \psi(Y, f(X)) - \psi(Y, g(X)) \right) \leq C \left( \Phi(r) + r \sqrt{\frac{t}{n} + \frac{1 + M}{n}} \right),
\]
for a universal constant \(C > 0\) with probability at least \(1 - e^{-t}\) for all \(t > 0\). We denote by this event as \(\mathcal{E}_2(t, r)\).

We define an event \(\mathcal{E}_3(t) = \mathcal{E}_1(t) \cap \mathcal{E}_2(t, \sqrt{2(t^2 + r^2)})\). Then \(P(\mathcal{E}_3(t)) \geq 1 - 2e^{-t}\) for all \(t > 0\). In this event, it holds that
\[
\Psi(\hat{f}) - \Psi(\hat{g}) - (\hat{\Psi}(\hat{f}) - \hat{\Psi}(\hat{g})) \leq C \left[ \Phi(\sqrt{2(t^2 + r^2)}) + \sqrt{2(t^2 + r^2)} \sqrt{\frac{t}{n} + \frac{1 + M}{n}} \right],
\]
for a universal constant \(C > 0\). Combining this and Eq. (4), we obtain the assertion on the event \(\mathcal{E}_0(t) \cap \mathcal{E}_3(t)\).

Hereafter, we derive some upper bounds of the (local) Rademacher complexities under some covering number conditions.

**Lemma 1.**
\[
\mathbb{E} \left[ \sup \left\{ P_n h^2 \mid h \in \hat{F} - \hat{G} : \|h\|_{L_2} \leq r \right\} \right] \leq r^2 + 2M \phi(r).
\]

**Proof.** By the contraction inequality of the Rademacher complexity (Theorem 4.12 of [31] and its proof), we have
\[
\mathbb{E} \left[ \sup \left\{ P_n h^2 \mid h \in \hat{F} - \hat{G} : \|h\|_{L_2} \leq r \right\} \right] \\
\leq \mathbb{E} \left[ \sup \left\{ (P_n - P) h^2 \mid h \in \hat{F} - \hat{G} : \|h\|_{L_2} \leq r \right\} \right] + r^2 \\
\leq 2 \mathbb{E} \left[ \sup \left\{ \frac{1}{n} \sum_{i=1}^{n} \epsilon_i h(x_i)^2 \mid h \in \hat{F} - \hat{G} : \|h\|_{L_2} \leq r \right\} \right] + r^2
\]
(symmetrization; Lemma 11.4 of [3])

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holds that

\[ \sup_{D_n} \log(\mathcal{N}(\widehat{\mathcal{F}}, \| \cdot \|_n, \epsilon/2)) + \sup_{D_n} \log(\mathcal{N}(\widehat{\mathcal{G}}, \| \cdot , \epsilon/2)) \leq S_1 + S_2 \log(1/\epsilon) + S_3 \epsilon^{-2q} \]

for \( q < 1 \). Then, for a universal constant \( C > 0 \) and a constant \( C_q > 0 \) which depends on \( q < 1 \), it holds that

\[ \Phi(r) \leq C \max \left\{ \frac{1}{n} + M \frac{S_1 + S_2 \log(n)}{n} + r^{1+q} \sqrt{S_1 + S_2 \log(n)}, C_q \left[ \frac{1}{n} + \left( \frac{M^{1-q}S_3}{n} \right)^{\frac{1}{1-q}} + r^{1-q} \sqrt{S_3} \right] \right\} . \]

In particular,

\[ r^*_2 \leq C \left[ M \frac{S_1 + S_2 \log(n)}{n} + \left( \frac{M^{1-q}S_3}{n} \right)^{\frac{1}{1-q}} + 1 + Mt \right] . \]

**Proof.** Under the assumption, we may set

\[ \phi(r) = C \left( \frac{1}{n} + \frac{1}{\sqrt{n}} E \left[ \int_{1/n}^{\gamma_n} \sqrt{S_1 + S_2 \log(\epsilon^{-1}) + S_3 \epsilon^{-2q}} \, d\epsilon \right] \right) \]

for a universal constant \( C > 0 \). Then, for \( \gamma_n := \sup \{ \| f - g \|_n : \| f - g \|_{L_2} \leq r, f, g \in \widehat{\mathcal{F}}, g \in \widehat{\mathcal{G}} \} \), it holds that

\[ E \left[ \int_{1/n}^{\gamma_n} \sqrt{S_1 + S_2 \log(\epsilon^{-1}) + S_3 \epsilon^{-2q}} \, d\epsilon \right] \leq E \left[ \gamma_n \sqrt{S_1 + S_2 \log(n)} \right] + \frac{1}{1-q} \sqrt{S_3} E [\gamma_n^{1-q}] \]

\[ \leq \sqrt{E \{ P_n h^2 | h \in \widehat{\mathcal{F}} - \widehat{\mathcal{G}}, \| h \|_{L_2} \leq r \} } \sqrt{S_1 + S_2 \log(n)} + \frac{\sqrt{S_3}}{1-q} E \{ P_n h^2 | h \in \widehat{\mathcal{F}} - \widehat{\mathcal{G}}, \| h \|_{L_2} \leq r \} \]

\[ \leq \sqrt{r^2 + 2M \phi(r) \sqrt{S_1 + S_2 \log(n)} + \frac{\sqrt{S_3}}{1-q} (r^2 + 2M \phi(r))} \]

(7)

where we used Lemma[1] Hence, if the first term is larger than the second term, we have that

\[ \phi(r) \leq C \left( \frac{1}{n} + \sqrt{\frac{S_1 + S_2 \log(n)}{n}} \sqrt{r^2 + 2M \phi(r)} \right) \]

\[ \leq C \frac{1}{n} + C^2 M \frac{S_1 + S_2 \log(n)}{n} + Cr \sqrt{\frac{S_1 + S_2 \log(n)}{n}} + \frac{\phi(r)}{2} . \]

Therefore, we obtain that

\[ \phi(r) \leq \frac{2C}{n} + 2C^2 M \frac{S_1 + S_2 \log(n)}{n} + 2Cr \sqrt{\frac{S_1 + S_2 \log(n)}{n}} \]

(8)

On the other hand, if the second term in Eq. (7) is larger than the first one, then Young’s inequality gives that

\[ \phi(r) \leq C \left( \frac{1}{n} + \frac{S_3}{n(1-q)^2} (r^2 + 2M \phi(r))^{\frac{1}{1-q}} \right) \]
\[ \leq \frac{C}{n} + C \left[ q \left( \frac{c'^{1-q} C^2 S_3}{n(1-q)^2} \right)^{1/q} + (1-q) \frac{2M \phi(r)}{c'} + \sqrt{\frac{S_3 n}{n(1-q)^2 r^{2(1-q)}}} \right]. \]

where \( c' > 0 \) is any positive real. Thus taking \( c' \) sufficiently large (which depends on \( M, q \)), we conclude that

\[ \phi(r) \leq C_q \left[ \frac{1}{n} + \left( \frac{M^{1-q} S_3}{n} \right)^{1/q} + \sqrt{\frac{S_3 n r^{2(1-q)}}{r^{2}}} \right], \]

where \( C_q \) is a constant depending only on \( q < 1 \). These two inequalities (Eq. (8) and Eq. (9)) give the first assertion. By noticing the assumption \( M \geq 1 \), \( r_* \) can be derived from a simple calculation.

\[ \tag{9} \]

B Derivation of compression based bound for non-compressed networks

B.1 Full model bound

Here, we assume that the model \( \hat{F} \) of the trained network is the full model \( \hat{F} = F = \text{NN}(m, R_2, R_F) \) and \( \hat{G} \) is included in \( \text{NN}(m, \hat{R}_2, \hat{R}_F) \). Then, their covering entropy is bounded by

\[ \log(N(\hat{F}, \| \cdot \|_\infty, \epsilon)) \leq \left( \sum_{\ell=1}^{L} m_\ell m_{\ell+1} \right) \log(\epsilon^{-1}) + L \left( \sum_{\ell=1}^{L} m_\ell m_{\ell+1} \right) \log(L(R_2 \lor 1)(\max_\ell m_\ell + 1)), \]

\[ \log(N(\hat{G}, \| \cdot \|_\infty, \epsilon)) \leq \left( \sum_{\ell=1}^{L} m_\ell m_{\ell+1} \right) \log(\epsilon^{-1}) + L \left( \sum_{\ell=1}^{L} m_\ell m_{\ell+1} \right) \log(L(\hat{R}_2 \lor 1)(\max_\ell m_\ell + 1)). \]

Hence, the condition in Lemma 2 holds for \( S_1 = \sum_{\ell=1}^{L} m_\ell m_{\ell+1}, S_2 = L S_1 \log(L(R_2 \lor 1)(\max_\ell m_\ell + 1)) \) and \( S_3 = 0 \). In this case, we can set

\[ r_*^2 = C (M + 1)(S_1 + 1 + S_2 \log(n)) + M \frac{n}{n}, \]

for a constant \( C > 0 \).

B.2 Complexity of a sparse model (Proof of Example 1)

Suppose that \( \hat{G} \) is the model with sparse weight matrices given in Example 1. Let \( m = \max_\ell m_\ell \) and \( B = R_2 \), then we can see that

\[ \hat{G} \subset \Phi(L, m, S, B) \]

where the definition of \( \Phi(L, m, S, B) \) is given in Appendix C.1. Therefore, its covering number is bounded by

\[ \log(N(\hat{G}, \| \cdot \|_\infty, \epsilon)) \leq S \log(\epsilon^{-1}) + L S \log(L(R_2 \lor 1)(\max_\ell m_\ell + 1)) \leq O(S L \log(n) + S \log(\epsilon^{-1})), \]

by Lemma 3. Hence, the Rademacher complexity is bounded as

\[ \hat{G} \leq C M \sqrt{L \frac{S}{n} \log(n L(R_2 \lor 1)(\max_\ell m_\ell + 1))} = O \left( M \sqrt{L \frac{S}{n} \log(n)} \right), \]

by the Dudley integral, \( R(\hat{G}) \leq \int_0^M \sqrt{\frac{\log(N(\hat{G}, \| \cdot \|_\infty, \epsilon))}{n}} d\epsilon, \) where \( C \) is a universal constant.
B.3 Near low rank condition on the weight matrix (Proof of Theorem 2 and Corollary 1)

Here, we give proofs of Theorem 2 and Corollary 1 which give a generalization error bound when the trained network has near low rank weight matrices \((W^{(l)})_{l=1}^{L}\) (Assumption 4).

Under Assumption 4, we can see that for any \(1 \leq s \leq \min\{m_{\ell}, m_{\ell+1}\}\), we can approximate \(W^{(l)}\) by a rank \(s\) matrix \(W'\) as

\[
\|W^{(l)} - W'\|_2 \leq V_0 s^{-\alpha}, \quad \|W'\|_2 \leq \|W^{(l)}\|_2
\]

\[
\|W^{(l)} - W'\|_F \leq \frac{1}{\sqrt{2\alpha - 1}} V_0 (s - 1)^{(1-\alpha)/2}.
\]

This can be checked by discarding the singular vectors corresponding to the singular values smaller than the \(s\)-th largest one. This ensures that, for any \(f \in \hat{F}\), there exists \(f' \in F\) such that it has width \(s = (s_1, \ldots, s_L)\), weight matrix \(W^{(l)}\) with \(\|W^{(l)}\|_2 \leq R_{2L}/\|W\|_F \leq R_F\) and

\[
\|f - f'\|_\infty \leq \sum_{\ell=1}^{L} V_0 R_{2\ell-1} s_\ell^{-\alpha} B_\ell.
\]

This can be proved as follows. Let \(f'(x) = G \circ (W^{(L)}(\eta(\cdot))) \circ \cdots \circ (W^{(1)}(\eta(\cdot)))\) where \(W^{(l)}\) is a rank \(s_l\) matrix that satisfies Eqs. (10) and (11) for \(W' = W^{(l)}\). Let \(f_{\ell}(x) = G \circ (W^{(L)}(\eta(\cdot))) \circ \cdots \circ (W^{(l+1)}(\eta(\cdot))) \circ (W^{(l)}(\eta(\cdot))) \circ \cdots \circ (W^{(1)}(\eta(\cdot)))\) and \(f_0(x) = f'\). Then, \(\|f - f'\|_\infty \leq \sum_{\ell=1}^{L} \|f_\ell - f_{\ell-1}\|_\infty\). We can see that \(\|W^{(l)}(\eta(\cdot)) \circ \cdots \circ (W^{(1)}(\eta(\cdot)))\| \leq \prod_{k=1}^{l} \|W^{(k)}(\eta(\cdot))\|_2 \leq R_{2\ell} B_\ell\),

\[
\|W^{(l)}(\eta(\cdot)) \circ (W^{(l-1)}(\eta(\cdot))) \circ \cdots \circ (W^{(1)}(\eta(\cdot))) - (W^{(l)}(\eta(\cdot))) \circ \cdots \circ (W^{(1)}(\eta(\cdot)))\| \leq \|W^{(l)} - W^{(l-1)}\|_2 \|W^{(l-1)}(\eta(\cdot))\|_2 \cdots \|W^{(1)}(\eta(\cdot))\|_2 \leq V_0 s_\ell^{-\alpha} R_{2\ell-1} B_\ell.
\]

This gives \(\|f_\ell - f_{\ell-1}\|_\infty \leq \|W^{(l)}(\eta(\cdot)) \circ (W^{(l-1)}(\eta(\cdot))) \circ \cdots \circ (W^{(1)}(\eta(\cdot)))\| \leq V_0 s_\ell^{-\alpha} R_{2\ell-1} B_\ell\). Finally, summing up this from \(\ell = 1\) to \(L\), we obtain Eq. (12).

In particular, for any \(\epsilon > 0\), by setting \(s_\ell' = s_\ell(\epsilon) = \min\{[(\epsilon/(L V_0 R_{2\ell-1} B_\ell)^{1/\alpha}), m_\ell \wedge m_{\ell+1}\}\) for all \(\ell\), then \(\|f - f'\|_\infty \leq \epsilon\). This indicates that, by Lemma 4, the covering entropy of \(\hat{F}\) is bounded by

\[
\log(N(\hat{F}, \|\cdot\|_\infty, \epsilon)) \leq \left(\sum_{\ell=1}^{L} (m_\ell + m_{\ell+1}) s_\ell'(\epsilon)/2\right) \log(\epsilon^{-1}) + 2L \log(2L(R_2 \vee 1)(\max_{\ell} m_\ell + 1))
\]

\[
\leq 2\left(\sum_{\ell=1}^{L} m_\ell (2L V_0 R_{2\ell-1} B_\ell)^{1/\alpha} \epsilon^{-1/\alpha} \log(\epsilon^{-1}) + 2L \log(2L(R_2 \vee 1)(\max_{\ell} m_\ell + 1))\right).
\]

As the compressed network \(\hat{G}\), we may choose \(\hat{G} = \text{NN}(m, s, R_2, R_F)\) for \(s = (s_1, \ldots, s_L)\) so that, for all \(f \in \hat{F}\), there exists \(g \in \hat{G}\) satisfying

\[
\|f - g\|_\infty \leq (V_0 R_{2L-1} B_\ell) \sum_{\ell=1}^{L} s_\ell^{-\alpha}.
\]

Hence, we may set \(r^2 = [(V_0 R_{2L-1} B_\ell) \sum_{\ell=1}^{L} s_\ell^{-\alpha}]^2\). In this case, the covering number of \(\hat{G}\) is bounded as (13) by replacing \(s_\ell'\) with \(s_\ell\).

Therefore, Lemma 2 gives that

\[
r^* = C(M + 1)(S_1 + 1 + S_2 \log(n)) + Mt \sqrt{\frac{2a+1}{n}} \leq M^{2a+1} \left(\frac{S_3}{n}\right)^{2a/2a+1},
\]

where

\[
S_1 = \sum_{\ell=1}^{L} s_\ell(m_\ell + m_{\ell+1}),
\]

\[
S_2 = LS_1 \log(L(R_2 \vee 1)(\max_{\ell} m_\ell + 1)),
\]

\[
S_3 = L^{2a+1} \left(\frac{S_3}{n}\right)^{2a/2a+1}.
\]
Finally, we observe that for the trained network (Lipschitz continuity between layers: Interlayer cushion, interlayer smoothness [1]) in [1]. We improve this exponential dependency by assuming the following condition.

\[ \kappa \]

Proof of Corollary 2.

\( (\cdots \circ \ell) \) transformation from the then we show that Eq. (12) can be replaced by

In the generalization error bound of Theorem 2 and Corollary 1, there appears \( s \) where

\[ \hat{\Psi}(\hat{f}) \leq \hat{\Psi}(\hat{f}) + C \left[ M^{1-1/2\alpha} \sqrt{L \left( \sum_{\ell=1}^{L} m_{\ell}\right)(2LV_0 R_2^{L-1} B_{x})^{1/\alpha}} \log(n) + M^{\frac{1}{2\alpha+1}} A_2^{\frac{1}{2\alpha+1}} + \frac{1 + tM}{n} \right] \]

for \( A_2 = L \left( \sum_{\ell=1}^{L} m_{\ell}\right)(2LV_0 R_2^{L-1} B_{x})^{1/\alpha} \) with probability \( 1 - 3e^{-t} \) for any \( t > 1 \) where \( C \) is a constant depending on \( \alpha \).

This is almost same as Corollary 1 but the exponential dependency on \( R_2^{L} \) is replaced by the Lipschitz continuity \( \kappa^2 \).

Proof of Corollary 2 Suppose that

\[ s_{\ell} \geq \min \left\{ m_{\ell}, m_{\ell+1}, \left[ (4\kappa V_0 L)^{1/\alpha} \right] \right\}. \]

then we show that Eq. (12) can be replaced by

\[ \| f - f' \|_n \leq 4 \sum_{\ell=1}^{L} V_0 \kappa^2 s_{\ell}^{-\alpha} B_{x}. \]
where if $s_\ell = \min\{m_\ell, m_{\ell-1}\}$, then $s^{-\alpha}_\ell$ term can be replaced by 0 (which means no-compression in the layer $\ell$). Once we obtain this evaluations, then the following argument is same as the proof of Theorem 2 and Corollary 1 (Sec. B.3).

Let $\phi(x) = \eta \circ (W^{(\ell)}\eta(\cdot)) \circ \cdots \circ (W^{(1)}x)$ and $\phi^\sharp(x) = \eta \circ (W^{(\ell)}\eta(\cdot)) \circ \cdots \circ (W^{(1)}x)$ for $\ell = 1, \ldots, L - 1$, and let $\phi_L(x) = G \circ (W^{(L)}\eta(\cdot))$\) for $\ell = 1, \ldots, L$. Let $C_B := 2\kappa B_x$. We will show that

$$\|\phi_k - \phi^\sharp_k\|_n \leq 2\kappa V_0 C_B \left( \sum_{j=1}^k s^{-\alpha}_j \right), \quad \|\phi^\sharp_k\|_n \leq C_B,$$

for all $k = 1, \ldots, L$. We show this by inductive reasoning. To do so, we assume that, for $k = 1, \ldots, \ell - 1$, this is satisfied, and then we show this for $k = \ell$. Note that, for all $k$ with $k < \ell$, it holds that, for any $\ell' > k$,

$$\|M_{k, \ell'} \circ \phi^\sharp_k - M_{k-1, \ell'} \circ \phi^\sharp_{k-1}\|_n \leq \kappa (\|\phi^\sharp_k - \eta(W^{(k)}\phi^\sharp_{k-1})\|_n + \tau \|\eta(W^{(k)}\phi^\sharp_{k-1}) - \phi_k\|_n)
\leq \kappa (V_0 s^{-\alpha}_{k-1}) \|\phi^\sharp_{k-1}\|_n + \tau \kappa \|\phi^\sharp_{k-1} - \phi_{k-1}\|_n
\leq \kappa V_0 C_B \left( s^{-\alpha}_{k-1} \right) \left( \sum_{j=k}^\ell \sum_{j=1}^\tau s^{-\alpha}_j \right) \quad \text{(by induction)}
\leq \kappa V_0 C_B \left( s^{-\alpha}_{k-1} \right) \left( \sum_{j=k}^\ell \sum_{j=1}^\tau s^{-\alpha}_j \right) \quad \text{(by the assumption of $\tau$)}.$$

Note that the term $s^{-\alpha}_k$ can be replaced by 0 if $s_k = \min\{m_k, m_{k+1}\}$ which corresponds to the full rank setting ($W^{(k)} = W^{(k)}$). Therefore, we have that

$$\|\phi_\ell - \phi^\sharp_\ell\|_n \leq \sum_{j=1}^{\ell} \|M_{j, \ell} \circ \phi^\sharp_j - M_{j-1, \ell} \circ \phi^\sharp_{j-1}\|_n \leq 2\kappa V_0 C_B \left( \sum_{k=1}^\ell s^{-\alpha}_k \right).$$

Under the setting (14), this gives that

$$\|\phi_\ell - \phi^\sharp_\ell\|_n \leq C_B / 2.$$

Finally, noting that $\|\phi_\ell\|_n \leq C_B / 2$, we have

$$\|\phi^\sharp_\ell\|_n \leq C_B.$$

This concludes the inductive reasoning.

Finally, noting that $f = \phi_L$ and $f' = \phi^\sharp_L$, we have Eq. (15). \hfill \box

### B.5 Near low rank condition on the covariance matrix (Proof of Theorem 3 and Theorem 4)

Under Assumption 5, $\hat{f}$ can be compressed as follows. Suppose that the network is compressed to smaller one up to the $\ell - 1$-th layer and the weight matrix of the compressed one is denoted by $(W^{(\ell)}\hat{f})_{k=1}^{\ell-1}$ where each $W^{(k)}\hat{f}$ has size $m_{k+1}^2 \times m_k^2$ (here, $m_k^2 \leq m_k$ is assumed), and, in the $\ell - 1$-th layer, $W^{(\ell-1)}\hat{f}$ has size $m_{\ell} \times m_{\ell-1}^2$. The input to the $\ell$-th layer of the compressed network is denoted by $\phi^\sharp_{\ell}(x) = \eta(W^{(\ell-1)}\eta(\cdots W^{(1)}x)) \cdots$. Let $r^2 = \|\phi_\ell - \phi^\sharp_\ell\|_n^2$ and $\Sigma_{(\ell)}^\sharp := \frac{1}{n} \sum_{i=1}^n \phi^\sharp_{\ell}(x_i)(\phi^\sharp_{\ell}(x_i))^\top$.

For a given matrix $\Sigma$ and a precision $r^2 > 0$, the degrees of freedom $N_\ell$ are defined as

$$N_\ell(r^2, \Sigma) := \sum_{j=1}^{m_{\ell}} \frac{\sigma_j(\Sigma)}{\sigma_j(\Sigma) + r^2}.$$

The definition is not dependent on $\ell$, but to make it clear that we are dealing with the $\ell$-th layer, we use the notation $N_\ell$. 

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Since the degrees of freedom are monotonically increasing with respect to each \( \sigma_j(\Sigma) \), we can see that \( N_\ell(r^2, \Sigma) \geq N_\ell(r^2, \Sigma') \) if \( \Sigma \geq \Sigma' \). Let

\[
m_\ell^4 = \lfloor 5N_\ell(r^2, \Sigma_{(\ell)}^*) \log(80N_\ell(r^2, \Sigma_{(\ell)}^*)) \rfloor,
\]

then Proposition\(^4\) tells that there exits a matrix \( \hat{A}_\ell \in m_\ell \times m_\ell^2 \) and \( J_\ell \subset \{1, \ldots, m_\ell\}^{m_4} \) such that

\[
\| w^T \phi_\ell^2 - w^T \hat{A}_\ell \phi_\ell^2, J_\ell \|_2^2 \leq 4r^2 w^T \Sigma_{(\ell)}^*(\Sigma_{(\ell)}^* + r^2I)^{-1} w \leq 4r^2 \| w \|_2^2,
\]

for any \( w \in \mathbb{R}^{m_\ell} \) and the norm of \( \hat{A}_\ell \) is bounded as

\[
\| \hat{A}_\ell \|_2 \leq \sqrt{\frac{20}{3} m_\ell}.
\]

Next, we evaluate the degrees of freedom of \( \Sigma_{(\ell)}^2 \). We bound this by using the degrees of freedom of \( \hat{\Sigma}_{(\ell)} \). First note that

\[
r_\ell^2 = \| \phi_\ell - \hat{\phi}_\ell^2 \|_2^2.
\]

Let \( s \leq m_\ell \). For any matrix \( U \in \mathbb{R}^{m_\ell \times s} \) such that \( U^T U = I_s \), \( \text{Tr}[U^T \Sigma_{(\ell)}^2 U] = P_n[\phi_\ell^T UU^T \phi_\ell] \leq 2\{P_n[\phi_\ell^T UU^T \phi_\ell] + P_n[(\phi_\ell - \hat{\phi}_\ell^2)^T UU^T (\phi_\ell - \hat{\phi}_\ell^2)] \} \) by the Cauchy-Schwarz inequality. Here, let \( U \) be the matrix that gives \( P_n[\phi_\ell^T UU^T \phi_\ell] = \sum_{j=m_\ell-s+1}^{m_\ell} \sigma_j(\hat{\Sigma}_{(\ell)}) = \inf_{U^T U = I_s} P_n[\phi_\ell^T UU^T \phi_\ell] \), then by noticing \( P_n[(\phi_\ell - \hat{\phi}_\ell^2)^T UU^T (\phi_\ell - \hat{\phi}_\ell^2)] \leq P_n\| \phi_\ell - \hat{\phi}_\ell^2 \|_2^2 \leq r_\ell^2 \), we obtain that \( P_n[\phi_\ell^T UU^T \phi_\ell] \leq 2\sum_{j=m_\ell-s+1}^{m_\ell} \sigma_j(\hat{\Sigma}_{(\ell)}) + r_\ell^2 \). Finally, by minimizing the left hand side with respect to \( U \), we obtain that

\[
\sum_{j=m_\ell-s+1}^{m_\ell} \sigma_j(\Sigma_{(\ell)}^2) \leq 2 \left( \sum_{j=m_\ell-s+1}^{m_\ell} \sigma_j(\hat{\Sigma}_{(\ell)}) + r_\ell^2 \right).
\]

By setting \( s = m_\ell - m + 1 \) for \( 1 \leq m \leq m_\ell \), this indicates that

\[
\sum_{j=m}^{m_\ell} \sigma_j(\Sigma_{(\ell)}^2) \leq 2 \left( \sum_{j=m}^{m_\ell} \sigma_j(\hat{\Sigma}_{(\ell)}) + r_\ell^2 \right) \leq 2 \left( \sum_{j=m}^{m_\ell} \mu_{j}^{(\ell)} + r_\ell^2 \right).
\]

Now, let \( \hat{m}_\ell := \min\{j \in \{1, \ldots, m_\ell\} \mid \mu_{j}^{(\ell)} \leq r_\ell^2\} \) (if \( \mu_{m_\ell}^{(\ell)} > r_\ell^2 \), then we set \( \hat{m}_\ell = m_\ell \)). Then,

\[
N_\ell(r^2, \Sigma_{(\ell)}^2) = \sum_{j=1}^{m_\ell} \frac{\sigma_j(\Sigma_{(\ell)}^2)}{\sigma_j(\Sigma_{(\ell)}^2) + r^2} \leq \hat{m}_\ell + \sum_{j=\hat{m}_\ell}^{m_\ell} \frac{\sigma_j(\Sigma_{(\ell)}^2)}{\sigma_j(\Sigma_{(\ell)}^2) + r^2} \leq \hat{m}_\ell + \frac{2}{r^2} \left( r_\ell^2 + \sum_{j=\hat{m}_\ell}^{m_\ell} \sigma_j(\hat{\Sigma}_{(\ell)}) \right) \leq \hat{m}_\ell + \frac{2}{r^2} \left( r_\ell^2 + \frac{\hat{m}_\ell^{1-\beta}}{\beta - 1} \right) \leq \hat{m}_\ell + \frac{2}{r^2} \left( r_\ell^2 + \frac{\hat{m}_\ell r^2}{\beta - 1} \right) = \frac{\beta + 1}{\beta - 1} \hat{m}_\ell + \frac{2}{r^2} \frac{r^2}{r^2}.
\]

Now, let

\[
r^2 = \frac{1}{4} r_\ell^2,
\]

then

\[
N_\ell(r^2, \Sigma_{(\ell)}^2) \leq \frac{\beta + 1}{\beta - 1} \hat{m}_\ell + \frac{8}{r_\ell^2}.
\]

\(^4[x] \) is the smallest integer that is not less than \( x \in \mathbb{R} \).

\(^5\)For a vector \( x \in \mathbb{R}^m \) and index set \( J \subset \{1, \ldots, m\} \), \( x_J \) is the vector corresponding to the index set \( J \), that is, \( x_J = (x_j)_{j \in J} \).
We define the right hand side as $m_{\ell}^r$:
\[
m_{\ell}^r := \frac{\beta + 1}{\beta - 1} \bar{m}_{\ell} + 8 \frac{r_{\ell}^2}{r_{\ell}^*}.
\]

We have, by Eq. (17),
\[
\sum_{j=1}^{m_{\ell+1}^r} ||\eta(W_{j;}: \phi_{\ell}^j) - \eta(W_{j;}: \hat{A}_T \phi_{\ell,J_{\ell}}^j)||_n^2 \leq 4 \sum_{j=1}^{m_{\ell+1}^r} ||W_{j;}: \phi_{\ell}^j||_n^2 \leq 4 R_{\ell}^2 \times \frac{1}{4} r_{\ell}^2 = R_{\ell}^2 r_{\ell}^2.
\]

By the induction assumption, we also have that
\[
||\eta(W_{(\ell)} \phi_{\ell+1}^j) - \eta(W_{(\ell)} \phi_{\ell+1})||_n^2 = ||\eta(W_{(\ell)} \phi_{\ell}) - \eta(W_{(\ell)} \phi_{\ell}))||_n^2 \leq ||W_{(\ell)}||_n^2 r_{\ell+1}^2 \leq R_{(\ell)}^2 r_{\ell+1}^2.
\]

Combining these inequalities, if we define
\[
\phi_{\ell+1}^j = \eta(W_{(\ell)} \hat{A}_T \phi_{\ell+1,J_{\ell}}^j(x)),
\]
and set $W_{(\ell)} = W_{(\ell)} \hat{A}_T$ and reset $W_{(\ell-1)}$ ← $W_{(\ell-1)}$, then it holds that
\[
||\phi_{\ell+1} - \phi_{\ell+1}||_n \leq r_{\ell+1}
\]
where we let
\[
r_{\ell+1} = R_2 r_{\ell} + R_2 \tilde{r}_{\ell}.
\]
Letting $r_0 = 0$, by an induction argument, we obtain
\[
r_{\ell+1} \leq \sum_{k=1}^{\ell} R_{2(\ell-k)} R_{2} \tilde{r}_{k}.
\]

Finally, we obtain
\[
||\hat{f} - f^*||_n^2 \leq \left[ \sum_{k=1}^{L} R_{2(L-k)} R_{2} \tilde{r}_{k} \right]^2,
\]
for a compressed network $f^*$ that has width $m^2 = (m_{1}^\ell, \ldots, m_{L}^\ell)$ with parameters $W_{(\ell)} = W_{(\ell)} \hat{A}_T$. Note that
\[
||W_{(\ell)}||_n \leq ||W_{(\ell)} \hat{A}_T||_n = R_2 \sqrt{\frac{20}{3} m_{\ell}}, \quad ||W_{(\ell)}||_F \leq ||W_{(\ell)} \hat{A}_T||_F \leq R_2 \sqrt{\frac{20}{3} m_{\ell}}.
\]
Therefore, if we set $\hat{G} = \text{NN}(m^2, \sqrt{\frac{20}{3} m_{\ell}}, R_2, \sqrt{\frac{20}{3} m_{\ell}} R_{FP})$, then there exists $\hat{g} \in \hat{G}$ such that
\[
||\hat{f} - \hat{g}||_n \leq \hat{r}
\]
where
\[
\hat{r}^2 = r_{L}^2.
\]
Moreover, applying Lemma 4 to $\hat{G}$ and the Dudley integral yields
\[
\bar{R}(\hat{G}) \leq C_M \sqrt{L \sum_{\ell=1}^{L} m_{\ell}^2 m_{\ell+1}^{\ell}}} \log(nL(R_2 \vee 1)(\max_{\ell} m_{\ell} + 1)^2).
\]
This gives the assertion of Theorem 3.

Here, we consider a situation where $R_2^2 r_{\ell}^2 = c_0^2 \frac{r_{\ell}^2}{r_{\ell}^*}$ for some constant $c_0 > 0$. Then it holds that
\[
r_{\ell+1} = R_2 \left( 1 + \sqrt{\frac{c_0^2}{c_{k}}} \right) r_{\ell} = R_2 \prod_{k=1}^{\ell} \left( 1 + \sqrt{\frac{c_0^2}{c_{k}}} \right) r_{1} \leq R_2^2 \exp \left( c_0(2\sqrt{\ell} - 1) \right) r_{1}.
\]
Therefore, by setting $C_L := (1 \vee R_2) \exp \left( c_0 (2 \sqrt{L} - 1) \right)$, it holds that $r_\ell \leq C_L r_1$ for $\ell = 1, \ldots, L$, in particular, we have

$$\hat{r} \leq C_L r_1.$$ 

In this situation, the degrees of freedom are bounded by

$$N_\ell(r^2, \Sigma^\ell_{(\ell)}) \leq m^\ell = \frac{\beta + 1}{\beta - 1} \hat{m}_\ell + 8 \ell \frac{R^2_0}{c_0^2 R^2_2}.$$ 

Next, we bound $\hat{m}_\ell$. To do so, we should bound $\hat{r}_\ell$ from below. Note that

$$\hat{r}_\ell = \frac{c_0 R_2}{R_F} \frac{1}{\sqrt{\ell}} r_\ell = \frac{c_0 R_2}{R_F} \frac{1}{\sqrt{\ell}} R_\ell^{\ell=1} \prod_{k=1}^{\ell-1} \left( 1 + \sqrt{\frac{c_0^2 k}{k}} \right) r_1 \geq \frac{c_0 R_2}{R_F} \frac{1}{\sqrt{\ell}} \prod_{k=1}^{\ell-1} \left( 1 + \sqrt{\frac{c_0^2 k}{k}} \right) r_1 \geq \frac{c_0 R_2}{R_F} \left( 1 + \sum_{k=1}^{\ell-1} \sqrt{\frac{c_0^2 k}{k}} \right) r_1 \geq \frac{c_0 R_2}{R_F} \left( 1 + 2 \sqrt{c_0^2 (\sqrt{\ell} - 1)} + 2 \sqrt{c_0^2 (\sqrt{\ell} - 1)} \right) r_1.$$ 

Hence,

$$\hat{m}_\ell \leq (\hat{r}_\ell^2 / (4U_0))^{-1/\beta} \leq (4U_0)^{1/\beta} \left[ \frac{c_0 R_2}{R_F} \left( \frac{1}{\sqrt{\ell}} + 2 \sqrt{c_0^2 (\sqrt{\ell} - 1)} \right) r_1 \right]^{-2/\beta} \leq (4U_0)^{1/\beta} \left[ \frac{R_F}{c_0 R_2} \right]^{2/\beta} \left( \frac{1}{2} \wedge c_0 \right)^{-2/\beta} r_1^{-2/\beta} \leq \left[ \frac{4U_0 R_2^2}{0.5 \wedge c_0^2 c_0^2 R_2^2} \right]^{1/\beta} r_1^{-2/\beta}.$$ 

By Lemma 2 we can evaluate $r_\ell^2$ for $\hat{F}$ satisfying Assumption 4 as

$$r_\ell^2 \leq C \frac{(M + 1)(S_1 + 1 + S_2 \log(n)) + Mt}{n} \vee M \frac{2a - 1}{n} \left( S_3 \right)^{2a/3 + 2a/3},$$

where

$$S_1 = \sum_{\ell=1}^L m^\ell m^\ell_{\ell+1},$$

$$S_2 = LS_1 \log(L(R_2 \vee 1)(\max m^{\ell+1} + 1)^2) = O \left( L \sum_{\ell=1}^L m^\ell m^\ell_{\ell+1} \log(n) \right),$$

$$S_3 = (\sum_{\ell=1}^L m^\ell)(2LV_0 R_2^{L-1} B_2)^{1/\alpha} \left[ \log(n) + 2L \log(2L(R_2 \vee 1)(\max m^\ell + 1)^2) \right]$$

$$= O \left( L \sum_{\ell=1}^L m^\ell)(2LV_0 R_2^{L-1} B_2)^{1/\alpha} \log(n) \right),$$

Then, the overall generalization error is upper bounded by

$$\Psi(\hat{f}) \leq \hat{\Psi}(\hat{f}) + C \left[ r^2 + \sqrt{\frac{S_3}{n}} 2^{(1-1/2a)} + \frac{\left( M^2 + \hat{r}^2 \right) L \sum_{\ell=1}^L m^\ell m^\ell_{\ell+1} \log(n) + 1 + Mt}{n} \right],$$

with probability $1 - 3e^{-t}$ for all $t \geq 1$. By letting $Q'_{L,\alpha,\alpha} := L (2LV_0 R_2^{L-1} B_2)^{1/\alpha} \log(n)$ and assuming $\hat{r} \leq 1$, the second and third terms in $C[\cdot]$ is bounded by

$$\sqrt{Q'_{L,\alpha,\alpha} \left( \sum_{\ell=1}^L m^\ell \right)(C_L r_1)^{2(1-1/2a)} + C' M \sqrt{L \sum_{\ell=1}^L m^\ell + \ell R^2_0 / (c_0^2 R_2^2)^2} \log(n)^3}.$$
Then, through a cumbersome calculation, we have that
Corollary 3.
Hence, by setting $C = \frac{\sum_{\ell=1}^{L} m_{\ell}}{L} \frac{1}{\sqrt{\lambda + 2(1/2)\alpha}}$, which balances the first and the second terms, then $\hat{r} \leq C L r_{1} \leq 1$ and the right hand side is bounded by
\[
\sqrt{Q'_{L,\alpha}(\sum_{\ell=1}^{L} m_{\ell})^{2}} \text{exp} \left( c_{0} (2\sqrt{L} - 1) \right) \leq \left[ \frac{4U_{0}R_{F}^{2}(1 \vee R_{2})^{L} \exp \left( \frac{U}{2}(2\sqrt{L} - 1) \right)}{(0.25)^{4}(1 \wedge R_{2})^{2L}} \right]^{2/\beta}.
\]
This gives
Theorem 4.

B.6 Improved bound of Theorem 4 with Lipschitz continuity constraint

Here, we again note that there appears $R_{F}^{2}$ in $P_{L}$ and $Q_{L}$ in the bound of Theorem 4. This is due to a rough evaluation of the interlayer Lipschitz continuity. We can reduce this exponential dependency under Assumption 6.

Corollary 3. Assume Assumption 6 in addition to Assumptions 4 and 5, then the bound in Theorem 4 holds for the following redefined $P_{L}$ and $Q_{L}$:
\[
P_{L} = (2LV_{0}\kappa^{2} B_{x})^{1/\alpha}, \quad Q_{L} = \left[ \frac{4U_{0}R_{F}^{2}(1 \vee R_{2})^{L} \exp \left( \frac{U}{2}(2\sqrt{L} - 1) \right)}{(0.25)^{4}(1 \wedge R_{2})^{2L}} \right]^{2/\beta},
\]
except that the term $M^{2}L^{2}R_{F}^{2}L^{2} \log(n)^{3}/n$ is replaced by $M^{2}L^{2}R_{F}^{2}L^{2} \sqrt{\log(n)^{3}/n}$:
\[
\Psi(\hat{f}) \leq \widetilde{\Psi}(\hat{f}) + C \left[ M^{2} \left( \frac{P_{L} \vee Q_{L}}{n} \right)^{1/\alpha} \left( \frac{4U_{0}R_{F}^{2}(1 \vee R_{2})^{L} \exp \left( \frac{U}{2}(2\sqrt{L} - 1) \right)}{(0.25)^{4}(1 \wedge R_{2})^{2L}} \right)^{2/\beta} \log(n)^{3}/n \right]
\]
\[
+ M^{\frac{2\alpha + 2}{\alpha}} \left( L P_{L} \frac{\sum_{\ell=1}^{L} m_{\ell}}{n} \log(n) \right)^{\frac{2\alpha + 2}{\alpha}} + M^{\frac{2\alpha + 2}{\alpha}} \left( L P_{L} \frac{\sum_{\ell=1}^{L} m_{\ell}}{n} \log(n) \right)^{\frac{2\alpha + 2}{\alpha}} + M^{\frac{2\alpha + 2}{\alpha}} \left( L P_{L} \frac{\sum_{\ell=1}^{L} m_{\ell}}{n} \log(n) \right)^{\frac{2\alpha + 2}{\alpha}} + M^{\frac{2\alpha + 2}{\alpha}} \left( L P_{L} \frac{\sum_{\ell=1}^{L} m_{\ell}}{n} \log(n) \right)^{\frac{2\alpha + 2}{\alpha}}.
\]

Proof of Corollary 3. To show Corollary 3 we set
\[
\hat{r}_{\ell} = \left[ \frac{1}{\sqrt{\ell}} \prod_{k=1}^{\ell-1} \left( 1 + \sqrt{\frac{c_{0}}{k}} \right) \right] \frac{r_{1}}{R_{F}},
\]
where $c_{0}$ is a constant, and by the same argument as in the proof of Corollary 2 we can show that
\[
r_{\ell} \leq 2\kappa \sum_{k=1}^{\ell} \frac{1}{\sqrt{k}} \prod_{j=1}^{k-1} \left( 1 + \sqrt{\frac{c_{0}}{j}} \right) r_{1}.
\]
Then, through a cumbersome calculation, we have that
\[
\frac{r_{\ell}}{\hat{r}_{\ell}} \leq C \frac{\kappa R_{F}}{c_{0}} \sqrt{\ell},
\]
for a universal constant $C$. Moreover, we can show that $r_L$ can be bounded as

$$r_L \leq C' \frac{K}{c_0} \exp(c_0(2\sqrt{L} - 1)) r_1.$$ 

This also gives

$$r_L \leq C' \frac{K}{c_0} \exp(c_0(2\sqrt{L} - 1)) r_1,$$

for a universal constant $C'$. Then, redefining $C_L = \kappa \exp(c_0(2\sqrt{L} - 1))$, we can apply the same argument as in the proof of Corollary 2. Indeed, we can show

$$\hat{m}_\ell \lesssim \frac{4U_0 R_2^2}{(0.5 \cdot c_0)^2 c_0^2 \kappa^2} \right)^{1/\beta}, \quad m_\ell^* = \frac{\beta + 1}{\beta - 1} \hat{m}_\ell + C^2 \ell r_1^2 - 2 \beta.$$

From the above argument, if we set $\hat{G} = \text{NN}(m^2, \sqrt{20/3} \max_\ell m_\ell R_2, \sqrt{\frac{20}{3} \max_\ell m_\ell R_F})$, then there exists $\hat{f} \in \hat{G}$ such that

$$\| \hat{f} - \hat{g} \|_n \leq \hat{r},$$

where

$$\hat{r}^2 = r_L^2.$$

Moreover, we can show

$$r_L^2 \leq C (M + 1)(S_1 + 1 + S_2 \log(n)) + Mt n^{2a-1} \left( \frac{S_3}{n} \right)^{\frac{2a}{a+1}},$$

where

$$S_1 = \sum_{\ell=1}^L m_\ell^2 m_{\ell+1}^2,$$

$$S_2 = LS_1 \log(L(R_2 \lor 1)(\max_\ell m_\ell + 1)^2) = O \left( L \sum_{\ell=1}^L m_\ell^2 m_{\ell+1}^2 \log(n) \right),$$

$$S_3 = \sum_{\ell=1}^L m_\ell (2LV_0 \kappa^2 B_x)^{1/\alpha} [\log(n) + 2L \log(2L(R_2 \lor 1)(\max_\ell m_\ell + 1)^2)]$$

$$= O \left( L \sum_{\ell=1}^L m_\ell (2LV_0 \kappa^2 B_x)^{1/\alpha} \log(n) \right).$$

Here, to evaluate $S_3$, we used the argument in Sec. [B.4](proof of Corollary 2) of Theorem 3 and Theorem 4 (Sec. [B.5]).

## C Auxiliary lemmas

In this section, we give several auxiliary lemmas that are used in the proof of the theorems. These results are not new at all, but we explicitly present them for completeness.

### C.1 Covering number of deep network models

Define the neural network with height $L$, width $m$, sparsity constraint $S$ and norm constraint $B$ as

$$\Phi(L, m, S, B) := \{ G \circ (W^{(L)} \eta) \circ (b^{(L)}) \circ \cdots \circ (W^{(1)} \eta) \circ (b^{(1)}) \mid W^{(L)} \in \mathbb{R}^{1 \times m}, b^{(L)} \in \mathbb{R},$$

$$W^{(1)} \in \mathbb{R}^{m \times d}, b^{(1)} \in \mathbb{R}^m, W^{(\ell)} \in \mathbb{R}^{m \times m}, b^{(\ell)} \in \mathbb{R}^m (1 < \ell < L),$$

$$\sum_{\ell=1}^L (\| W^{(\ell)} \|_0 + \| b^{(\ell)} \|_0) \leq S, \max_\ell \| W^{(\ell)} \|_\infty \lor \| b^{(\ell)} \|_\infty \leq B \},$$

where $\| \cdot \|_0$ is the $\ell_0$-norm of the matrix (the number of non-zero elements of the matrix) and $\| \cdot \|_\infty$ is the $\ell_\infty$-norm of the matrix (maximum of the absolute values of the elements).

The following evaluation of the covering number of the model $\Phi(L, m, S, B)$ is shown by [47].
Lemma 3 (Covering number evaluation [47]). The covering number of $\Phi(L, m, S, B)$ can be bounded by

$$\log N(\Phi(L, m, S, B), \| \cdot \|_\infty, \delta) \leq S \log(\delta^{-1} L(B \lor 1)^{L-1}(m + 1)^2 L)$$

$$\leq 2SL \log((B \lor 1)(m + 1)) + S \log(\delta^{-1} L).$$

Proof of Lemma 3 Given a network $f \in \Phi(L, m, S, B)$ expressed as

$$f(x) = G \circ (W^{(L)} \eta(\cdot) + b^{(L)}) \circ \cdots \circ (W^{(1)} x + b^{(1)}),$$

let

$$A_k(f)(x) = \eta \circ (W^{(k-1)} \eta(\cdot) + b^{(k-1)}) \circ \cdots \circ (W^{(1)} x + b^{(1)}),$$

and

$$B_k(f)(x) = G \circ (W^{(L)} \eta(\cdot) + b^{(L)}) \circ \cdots \circ (W^{(k)} b^{(k)}),$$

for $k = 2, \ldots, L$. Corresponding to the last and first layer, we define $B_{L+1}(f)(x) = x$ and $A_1(f)(x) = x$. Then, it is easy to see that $f(x) = B_{k+1}(f) \circ (W^{(k)} \cdot + b^{(k)}) \circ A_k(f)(x)$. Now, suppose that a pair of different two networks $f, g \in \Phi(L, m, S, B)$ given by

$$f(x) = G \circ (W^{(L)} \eta(\cdot) + b^{(L)}) \circ \cdots \circ (W^{(1)} x + b^{(1)}),$$

$$g(x) = G \circ (W^{(L)}' \eta(\cdot) + b^{(L)'}) \circ \cdots \circ (W^{(1)}' x + b^{(1)'})',$

has a parameters with distance $\delta$: $\| W^{(\ell)} - W^{(\ell)'} \|_{\infty} \leq \delta$ and $\| b^{(\ell)} - b^{(\ell)'} \|_{\infty} \leq \delta$. Now, not that $\| A_k(f) \|_{\infty} \leq \max_j \| W^{(k-1)}_j \|_{1} \| A_{k-1}(f) \|_{\infty} + \| b^{(k-1)} \|_{\infty} \leq mB \| A_{k-1}(f) \|_{\infty} + B \leq (B \lor 1)(m + 1) \| A_{k-1}(f) \|_{\infty}$, and similarly the Lipschitz continuity of $B_k(f)$ with respect to $\| \cdot \|_{\infty}$-norm is bounded as $(Bm)^{L-k}$. Then, it holds that

$$| f(x) - g(x) |$$

$$= \left| \sum_{k=1}^{L} B_{k+1}(g) \circ (W^{(k)} \cdot + b^{(k)}) \circ A_k(f)(x) - B_{k+1}(g) \circ (W^{(k)'} \cdot + b^{(k)'}) \circ A_k(f)(x) \right|$$

$$\leq \sum_{k=1}^{L} (Bm)^{L-k} \| (W^{(k)} \cdot + b^{(k)}) \circ A_k(f)(x) - (W^{(k)'} \cdot + b^{(k)'}) \circ A_k(f)(x) \|_{\infty}$$

$$\leq \sum_{k=1}^{L} (Bm)^{L-k} \delta (mB \lor 1)^{k-1} m_k^{k-1} + 1$$

$$\leq \sum_{k=1}^{L} (Bm)^{L-k} \delta (B \lor 1)^{k-1} m_k^{k-1} \leq \delta L(B \lor 1)^{L-1}(m + 1)^L.$$

Thus, for a fixed sparsity pattern (the locations of non-zero parameters), the covering number is bounded by $(\delta / L(B \lor 1)^{L-1}(m + 1)^L)^{-S}$. There are the number of configurations of the sparsity pattern is bounded by $(m + 1)^LS$. Thus, the covering number of the whole space $\Phi$ is bounded as

$$(m + 1)^LS \left( \delta / \left[ L(B \lor 1)^{L-1}(m + 1)^L \right] \right)^{-S} = [\delta^{-1} L(B \lor 1)^{L-1}(m + 1)^2L]^{S},$$

which gives the assertion.

Lemma 4 (Covering number evaluation). Let NN(m, R_2, R_F) be the set of neural networks with depth $m(m_1, \ldots, m_{L})$, $\| W^{(\ell)} \|_{\infty} \leq R_2$ and $\| W^{(\ell)} \|_{F} \leq R_F$. The covering number of $NN(m, R_2, R_F)$ can be bounded by

$$\log N(NN(m, R_2, R_F), \| \cdot \|_{\infty}, \delta)$$

$$\leq \left( \sum_{\ell=1}^{L} m_{\ell}m_{\ell+1} \right) \log(\delta^{-1} L(R_2 \lor 1)^{L-1}(\max m_{\ell} + 1)^{L})$$
\[
\leq \left( \sum_{\ell=1}^{L} m_{\ell} m_{\ell+1} \right) \log(\delta^{-1}) + L \left( \sum_{\ell=1}^{L} m_{\ell} m_{\ell+1} \right) \log(L(R_2 \lor 1)(\max m_{\ell} + 1)).
\]

Moreover, the set of networks with low rank weight matrices, \( \text{NN}(m, s, R_2, R_F) \), has the following covering number bound:
\[
\log \mathcal{N}(\text{NN}(m, s, R_2, R_F), \| \cdot \|_\infty, \delta) \\
\leq \sum_{\ell=1}^{L} s_\ell (m_\ell + m_{\ell+1}) \log(\delta^{-1}) L(R_2 \lor 1)^{2L-1}(\max m_\ell + 1)^{2L}.
\]

Proof of Lemma 4 Let \( B = R_2, m = \max_\ell m_\ell, \) and \( S = \sum_{\ell=1}^{L} m_\ell m_{\ell+1}, \) then we can see that \( \text{NN}(m, R_2, R_F) \) is a subset of \( \Phi(L, m, S, B) \) because \( \| W \|_\infty \leq \| W \|_2. \) Hence Lemma 3 gives the first assertion. As for the second one, we can easily check that the covering number of \( \text{NN}(m, s, R_2, R_F) \) can be bounded by the one given in Lemma 4 for \( \Phi(2L, m, S, B) \) with \( S = \sum_{\ell=1}^{L} s_\ell (m_\ell + m_{\ell+1}). \) Then, we obtain the second assertion.

\[\square\]

C.2 Compression error bound for one layer

The following proposition was shown by [3,48]. Let \( \tilde{\Sigma}_{I,I'} \in \mathbb{R}^{K \times H} \) for integers \( K, H \in \mathbb{N} \) and a matrix \( \tilde{\Sigma}_{I,I'} \in \mathbb{R}^{K \times H} \) be a matrix \( \tilde{\Sigma}_{j,j} \in I, I' \) for the index sets \( I \in [m]^K \) and \( I' \in [m]^H. \) Let \( F = \{1, \ldots, m_\ell\} \) be the full index set. Let the degrees of freedom corresponding to \( \tilde{\Sigma} \) be \( \hat{N}(\lambda) := N_\ell(\lambda, \Sigma) \) (see Eq. (16)) for \( \lambda > 0. \)

**Proposition 1.** Suppose that
\[
\tau_j' = \frac{1}{\hat{N}(\lambda)} \sum_{\ell=1}^{m_\ell} \mu_\ell^{(\ell)} / \mu_\ell^{(\ell)} + \lambda = \frac{1}{\hat{N}(\lambda)} [\hat{\Sigma}(\hat{\Sigma} + \lambda I)^{-1}]_{j,j} \quad (j \in \{1, \ldots, m_\ell\}),
\]
where \( U = (U_{j,j})_{j,j} \) is the orthogonal matrix that diagonalizes \( \hat{\Sigma} \) that is, \( \hat{\Sigma} = U \text{diag}(\hat{\mu}_1, \ldots, \hat{\mu}_{m_\ell}) U^\top. \) For \( \lambda > 0, \) if
\[
m \geq 5\hat{N}(\lambda) \log(80\hat{N}(\lambda)),
\]
then there exist \( v_1, \ldots, v_m \in \{1, \ldots, m_\ell\} \) such that, for every \( \alpha \in \mathbb{R}^{m_\ell}, \)
\[
\inf_{\beta \in \mathbb{R}^m} \left\{ \left\| \alpha^\top \eta(F_{\ell-1}(\cdot)) - \sum_{j=1}^{m_\ell} \beta_j \eta(F_{\ell-1}(\cdot)) v_j \right\|_2^2 + m\lambda\|\beta\|_2^2 \right\} \leq 4\lambda \alpha^\top \hat{\Sigma}(\hat{\Sigma} + \lambda I)^{-1} \alpha,
\]
and \( \sum_{j=1}^{m_\ell} \tau_j'^{-1} \leq \frac{2}{3} m \times m_\ell, \) where \( \|\beta\|_2^2 := \sum_{j=1}^{m_\ell} \beta_j^2 \tau_j'^2. \) Let \( \tau := m\lambda \tau' \) and \( I_\tau = \text{diag}(\tau). \) Then, \( \hat{A} := \hat{\Sigma}_{F,j}(\hat{\Sigma}_{j,j} + I_\tau)^{-1} \) for \( J = \{v_1, \ldots, v_m\} \) satisfies
\[
\|\hat{A}\|_2 \leq \sqrt{\frac{20}{3} m_\ell},
\]
and the optimal \( \beta \) that achieves the infimum is given by \( \hat{\beta} = \hat{A}^\top \alpha \) for any \( \alpha \in \mathbb{R}^{m_\ell}. \)

C.3 Concentration inequality

**Proposition 2** (Talagrand’s Concentration Inequality [49,10]). Let \( \mathcal{G} \) be a function class on \( X \) that is separable with respect to \( \infty \)-norm, and \( \{x_i\}_{i=1}^n \) be i.i.d. random variables with values in \( X. \) Furthermore, let \( \mathcal{B} \geq 0 \) and \( U \geq 0 \) be \( \mathcal{B} := \sup_{g \in \mathcal{G}} E[(g - E[g])^2] \) and \( U := \sup_{g \in \mathcal{G}} \| g \|_\infty, \) then for \( Z := \sup_{g \in \mathcal{G}} \frac{1}{n} \sum_{i=1}^n g(x_i) - E[g], \) we have
\[
P \left( Z \geq 2E[Z] + \sqrt{\frac{2B}{n} + \frac{2Ut}{n}} \right) \leq e^{-t},
\]
for all \( t > 0. \)