TAP APPROACH FOR MULTI-SPECIES SPHERICAL SPIN GLASSES I: GENERAL THEORY

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Abstract. We develop a generalized TAP approach for the multi-species version of the spherical mixed \( p \)-spin models. In particular, we prove a generalized TAP representation for the free energy at any overlap vector which is multi-samplable in an appropriate sense. Moreover, we show that if a multi-samplable overlap is maximal, then the TAP correction is equal to an analogue of the well-known Onsager reaction term. Finally, in a companion paper we use the results from the current paper to compute the free energy at any temperature for all multi-species pure \( p \)-spin models, assuming the free energy converges.

1. Introduction

In the classical Sherrington-Kirkpatrick (SK) mean-field spin glass model [50], the random interaction coefficients are identically distributed for any two spins. Bipartite and, more generally, multi-species versions of the SK model have been proposed in physics [38, 36, 37, 10, 11, 9]. In those models the spins are divided into groups of different types, and the strength of the interaction between any two spins depends on their types. In this paper we will analyze the spherical mixed \( p \)-spin version of those models defined as follows.

First, consider a finite set of species \( S \), which will be fixed throughout the paper. For each \( N \geq 1 \), suppose that \( \{1, \ldots, N\} = \bigcup_{s \in S} I_s \), for some disjoint \( I_s \). Denoting \( N_s = |I_s| \), we will assume that the proportion of each species converges

\[
\lim_{N \to \infty} \frac{N_s}{N} = \lambda_s \in (0, 1), \quad \text{for all } s \in S.
\]

Let \( S(d) = \{x \in \mathbb{R}^d : ||x|| = \sqrt{d}\} \) be the sphere of radius \( \sqrt{d} \) in dimension \( d \). The configuration space of the spherical multi-species mixed \( p \)-spin model is

\[
S_N = \{(\sigma_1, \ldots, \sigma_N) \in \mathbb{R}^N : \forall s \in S, (\sigma_i)_{i \in I_s} \in S(N_s)\}.
\]

Denoting \( \mathbb{Z}_+ := \{0, 1, \ldots\} \) and \( |p| := \sum_{s \in S} p(s) \) for \( p \in \mathbb{Z}_+^S \), let

\[
P = \{p \in \mathbb{Z}_+^S : |p| \geq 1\}.
\]

Given some nonnegative numbers \( (\Delta_p)_{p \in P} \), define the mixture polynomial in the variables \( x = (x(s))_{s \in S} \in \mathbb{R}^S \),

\[
\xi(x) = \sum_{p \in P} \Delta_p^2 \prod_{s \in S} x(s)^{p(s)}.
\]

We will assume that \( \xi(1+\epsilon) < \infty \) for some \( \epsilon > 0 \), where for \( c \in \mathbb{R} \) we write \( \xi(c) \) for the evaluation of \( \xi \) at the constant function \( x \equiv c \). In analogy to the single-species \( p \)-spin models, (i.e., with \( |S| = 1 \)) we call the models such that \( \Delta_p^2 > 0 \) for exactly one \( p \in P \) multi-species pure \( p \)-spin models.
We define the multi-species mixed p-spin Hamiltonian \( H_N : S_N \to \mathbb{R} \) corresponding to the mixture \( \xi \) by
\[
H_N(\sigma) = \sqrt{N} \sum_{k=1}^{\infty} \sum_{i_1, \ldots, i_k = 1}^N \Delta_{i_1, \ldots, i_k} J_{i_1, \ldots, i_k} \sigma_{i_1} \cdots \sigma_{i_k},
\]
where \( J_{i_1, \ldots, i_k} \) are i.i.d. standard normal variables and if \( \#\{j \leq k : i_j \in I_s\} = p(s) \) for any \( s \in \mathcal{S} \), then \( \Delta_{i_1, \ldots, i_k} \), which depends on \( N \), is defined by
\[
\Delta^2_{i_1, \ldots, i_k} = \Delta^2 \prod_{s \in \mathcal{S}} \frac{p(s)!}{k!} \prod_{s \in \mathcal{S}} N_s^{1-p(s)}.
\]
(1.1)
Here \( \#A \) is the cardinality of a set \( A \). By a straightforward calculation, the covariance function of the random process \( H_N(\sigma) \) is given by
\[
\frac{1}{N} \text{E} H_N(\sigma) H_N(\sigma') = \xi(R(\sigma, \sigma')),
\]
(1.2)
where we define the overlap vector
\[
R(\sigma, \sigma') := (R_s(\sigma, \sigma'))_{s \in \mathcal{S}}, \quad R_s(\sigma, \sigma') := N_s^{-1} \sum_{i \in I_s} \sigma_i \sigma'_i.
\]
Identifying \( S_N \) with the product space \( \prod_{s \in \mathcal{S}} S(N_s) \), let \( \mu \) be the product of the uniform measures on each of the spheres \( S(N_s) \). The free energy \( F_N \) and partition function \( Z_N \) are defined by
\[
F_N := \frac{1}{N} \log Z_N := \frac{1}{N} \log \int_{S_N} e^{H_N(\sigma)} d\mu(\sigma).
\]
(1.3)
The Gibbs measure is the random probability measure on \( S_N \) with density
\[
\frac{dG_N}{d\mu}(\sigma) = Z_N^{-1} e^{H_N(\sigma)}.
\]
In the 70s, Thouless, Anderson and Palmer \[61\] invented their celebrated approach to analyze the Sherrington-Kirkpatrick model. Their approach was further developed in physics, see e.g. \[20, 21, 30, 32, 33, 39, 49\], with the general idea that for large \( N \), \( F_N(\beta) \) is approximated by the free energies associated to the ‘physical’ solutions of the TAP equations. In particular, the single-species spherical pure \( p \)-spin models were analyzed using the TAP approach non-rigorously by Kurchan, Parisi and Virasoro in \[39\] and Crisanti and Sommers in \[30\].

Recently, we introduced in \[53\] a generalized TAP approach for the single-species spherical models. The approach was extended to mixed models with Ising spins by Chen, Panchenko and the author \[20, 27\]. In the current paper we will develop the generalized TAP approach for the multi-species spherical models.

Analogously to the notion of multi-samplable overlaps introduced in \[53\] in the single-species setting, for the multi-species models we use the following definition. Let \( G_N^{\otimes n} \) denote the \( n \)-fold product measure of \( G_N \) with itself.

**Definition 1.** We will say that an overlap vector \( q = (q(s))_{s \in \mathcal{S}} \in [0, 1)^{\mathcal{S}} \) is multi-samplable if and only if for any \( n \geq 1 \) and \( \varepsilon > 0 \),
\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} G_N^{\otimes n} \{ \forall i < j \leq n, \ s \in \mathcal{S} : |R_s(\sigma^i, \sigma^j) - q(s)| < \varepsilon \} = 0.
\]
(1.4)
Suppose that \( q \in [0, 1)^{\mathcal{S}} \) and define
\[
S_N(q) = \{ m \in \mathbb{R}^N : R(m, m) = q \}.
\]
Let \( m \in S_N(q) \) be an arbitrary point. For any \( \sigma \) in
\[
\{ \sigma \in S_N : \forall s \in \mathcal{S}, R_s(\sigma, m) = R_s(m, m) \},
\]
(1.5)
Then, for any two points \( \sigma \), \( \tilde{\sigma} \)

\[
\text{Theorem 2} \quad \text{(Generalized TAP representation)}
\]

We will prove the following representation for the free energy.

\[
\nabla \text{where}
\]

\[
\text{only if, as } N \to \infty,
\]

\[
(1.10)
\]

\[
\frac{1}{N} \mathbb{E} \tilde{H}_N(\tilde{\sigma}) \tilde{H}_N(\tilde{\sigma}^2) = \xi_q(R(\tilde{\sigma}^1, \tilde{\sigma}^2)),
\]

where we define the mixture

\[
\tilde{\xi}_q(x) = \xi((1 - q)x + q) - \xi(q) = \sum_{p \in P} \Delta_{q,p}^2 \prod_{s \in \mathcal{S}} x(s)^p(s),
\]

where

\[
\Delta_{q,p}^2 := \sum_{p' \geq p} \Delta_{p'}^2 \prod_{s \in \mathcal{S}} \begin{pmatrix} p' \end{pmatrix}(s)(1 - q(s))^{p(s)} q(s)^{p(s) - p(s)},
\]

writing \( p' \geq p \) if \( p'(s) \geq p(s) \) for all \( s \in \mathcal{S} \). Here, operations between functions of \( s \) are interpreted in the usual way, namely, \( (1 - q)x + q)(s) = (1 - q(s)x(s) + q(s) \).

Additionally, define the mixture

\[
\xi_q(x) = \xi((1 - q)x + q) - \xi(q) - \sum_{s \in \mathcal{S}} (1 - q)\nabla \xi(q)x(s)
\]

\[
(1.8)
\]

\[
= \sum_{p \in P: |p| \geq 2} \Delta_{q,p}^2 \prod_{s \in \mathcal{S}} x(s)^p(s),
\]

where \( \nabla \xi(q) := \left( \frac{d}{dq(s)} \xi(q) \right)_{s \in \mathcal{S}} \) and the coefficients \( \Delta_{q,p}^2 \) are as above.

With \( H_N^q(\sigma) \) being the Hamiltonian corresponding to \( \xi_q \), define the free energy

\[
F_N(q) = \frac{1}{N} \log \int_{S_N} e^{H_N^q(\sigma)} d\mu(\sigma).
\]

Lastly, define the ground state energy

\[
(1.9)
\]

\[
E_{*,N}(q) := \frac{1}{N} \max_{m \in S_N(q)} H_N(m).
\]

We will prove the following representation for the free energy.

**Theorem 2** (Generalized TAP representation). For any \( q \in [0,1]^\mathcal{S} \), \( q \) is multi-samplable if and only if, as \( N \to \infty \),

\[
(1.10) \quad \mathbb{E} F_N = \mathbb{E} E_{*,N}(q) + \frac{1}{2} \sum_{s \in \mathcal{S}} \lambda_s \log(1 - q(s)) + \mathbb{E} F_N(q) + o(1).
\]

Moreover, for any \( q \in [0,1]^\mathcal{S} \),

\[
(1.11) \quad \mathbb{E} F_N \geq \mathbb{E} E_{*,N}(q) + \frac{1}{2} \sum_{s \in \mathcal{S}} \lambda_s \log(1 - q(s)) + \mathbb{E} F_N(q) + o(1).
\]
Very recently, Bates and Sohn \cite{14,13} proved a Parisi formula \cite{46,47} for the limit of the mean free energy $E_{F_N}$ for multi-species spherical mixed $p$-spin models such that the mixture $\xi(x)$ is a convex function. (For the proof of the formula in the single-species case, see \cite{24,13,58,59}.) The same convexity condition was required in an earlier work of Barra, Contucci, Mingione and Tantari \cite{9}, where they proved for the multi-species Sherrington-Kirkpatrick model that a Parisi type formula is an upper bound for the limiting free energy, using an analogue of Guerra’s interpolation \cite{34}. Panchenko proved the matching lower bound in \cite{44} using an analogue of the Aizenman-Sims-Starr scheme \cite{2} (the proof of the lower bound does not require $\xi(x)$ to be convex). He also proved there that the overlaps $R_s(\sigma, \sigma')$ of independent samples from the Gibbs measure are synchronized for different species. The synchronization mechanism of \cite{44} was also used in the spherical setting in \cite{14}. The combination of \cite{9} and \cite{44} establishes the Parisi formula for the multi-species Sherrington-Kirkpatrick model, assuming the convexity of $\xi(x)$.

Denote the difference of the two sides of the TAP representation by

$$D_N = E_{F_N} - \left( \mathbb{E} E_{\star,N}(q) + \frac{1}{2} \sum_{s \in \mathcal{S}} \log(1 - q(s)) + \mathbb{E} F_N(q) \right).$$

Given a mixture $\xi$, we will assume in the theorem below that

$$\text{(1.12)} \quad \text{for any } q \in [0,1]^\mathcal{S}, \text{ } D_N \text{ converges as } N \to \infty.$$  

If $\xi(x)$ is convex on $[0,1]^\mathcal{S}$, one can check that $\xi_q(x)$ is also convex. In this case, the convergence follows from the results of \cite{14}. For non-convex $\xi(x)$ the convergence was not proved, but is certainly expected.

We will say that a multi-samplable $q \in [0,1]^\mathcal{S}$ is maximal if any $q' \neq q$ such that $q'(s) \geq q(s)$ for all $s \in \mathcal{S}$ is not multi-samplable. For maximal multi-samplable overlaps we prove that the TAP correction is given by the following analogue of the Onsager reaction term (see e.g. \cite{31,39}), assuming the convergence \text{(1.12)}. In the proof of the theorem, we will also show that $\xi_q$ is replica symmetric in the sense that independent samples from the corresponding Gibbs measure typically have overlap approximately zero.

\textbf{Theorem 3 (Onsager correction).} Let $q \in [0,1]^\mathcal{S}$ be a maximal multi-samplable overlap and assume the convergence \text{(1.12)}. Then,

$$\text{(1.13)} \quad \mathbb{E} F_N(q) = \frac{1}{2} \xi_q(1) + o(1).$$

One of the main motivations of the current paper was to develop a method that would allow one to compute the free energy for some class of models which do not satisfy the convexity assumption on $\xi(x)$, that was crucial to the analysis of \cite{9,14,44}. In a companion paper \cite{56} we compute the free energy of the multi-species spherical pure $p$-spin models at any temperature, assuming convergence as in \text{(1.12)}, using the TAP representation \text{(1.10)} and the explicit correction from \text{(1.13)}. One can easily check that for the pure models, $\xi(x)$ is not convex anywhere in $[0,1]^\mathcal{S}$ (assuming that $p(s) \geq 1$ for at least two species).

In the single-species case, the TAP representation from \cite{53} was used in \cite{54} to compute the free energy of the spherical pure $p$-spin models. For the mixed single-species spherical models the free energy is given by the well-known Crisanti-Sommers representation \cite{29} of Parisi formula \cite{48,46}, first prove by Talagrand \cite{58} and extended by Chen \cite{24}. There are several additional earlier results that are connected to the TAP representation in the single-species case. Belius and Kistler \cite{15} established a TAP representation for the free energy for the spherical 2-spin model with an external field. In \cite{52} the free energy was computed for single-species spherical pure $p$-spin models at low enough temperature, building on results about the critical points \cite{43,51,57}, and was shown to be given by the TAP representation. A similar result was proved for mixed models close to pure...
by Ben Arous, Zeitouni and the author [16]. For more recent results on the TAP equations and representation for the single-species models with Ising spins, see [61, 71, 17, 18, 22, 25, 60].

Lastly, we mention earlier results recently proved for the multi-species spherical models. Multi-species models with two species, i.e., \( S = \{s_1, s_2\} \), are called bipartite models. Baik and Lee [8] computed the free energy and the limiting law of its fluctuations for the pure \( p \)-spin spherical bipartite model with \( p(s_1) = p(s_2) = 1 \) using tools from random matrix theory. Auffinger and Chen [5] proved that for mixed spherical bipartite models and in the presence of an external field, if \( \xi(1) \) and the strength of the field are small enough, then the limiting free energy is given by the replica symmetric solution of an analogue of the Crisanti-Sommers formula [29]. Certain bounds on the exponential growth rate of the mean number of minimum points, known as ‘complexity’, were also derived in [5]. McKenna [40] computed the exact asymptotics for the mean complexity of minimum and critical points. Kivimae [35] proved that for large energies, the second moment of the complexity matches its mean squared at exponential scale, assuming \( p(s_1) \) and \( p(s_2) \) are large enough. Combining [40] and [35] yields a variational formula for the ground state energy of the pure bipartite models. An application of the second moment method to compute the critical temperature of some multi-species models has been studied in [55]. As mentioned above, Bates and Sohn proved a Parisi formula for the multi-species spherical mixed \( p \)-spin models in [14], assuming the convexity of \( \xi(x) \). In [13] they showed it admits a representation analogous to the Crisanti-Sommers formula [29].

In the setting of multi-species models with Ising spins, we also mentioned above the results of Barra, Contucci, Mingione and Tantari [9] and Panchenko [44] whose combination proves the Parisi formula for the multi-species SK model, with the same convexity assumption on \( \xi(x) \). Bates, Sloman and Sohn [12] study the replica symmetric phase of the latter formula. Mourrat derived in [41] an upper bound for the free energy of the bipartite SK model with only inter-species interactions, whose mixture is non-convex.

In the next section we state additional results related to the TAP representation and outline the proof of the main results. In Section 3 we prove a concentration for the TAP free energy, stated in Theorem 6 below. Other results which we state in the next section will be proved in Sections 4 and 5. Finally, in Section 6 we will prove Theorems 2 and 3.

## 2. Additional results and outline of the main proofs

Denote the convex hull of \( S_N \) by

\[
M_N = \left\{ (\sigma_1, \ldots, \sigma_N) \in \mathbb{R}^N : \sum_{i \in J_s} \sigma_i^2 < N_s, \forall s \in \mathcal{S} \right\}.
\]

For any \( m \in M_N \), real numbers \( \delta, \rho \geq 0 \) and integer \( n \geq 1 \), define

\[
B(m, \delta) = \left\{ \sigma \in S_N : \forall s \in \mathcal{S}, |R_s(\sigma, m) - R_s(m, m)| \leq \delta \right\};
\]

\[
B(m, n, \delta, \rho) = \left\{ (\sigma^i)_{1 \leq i \leq n} \in B(m, \delta)^n : \forall i \neq j, s \in \mathcal{S}, |R_s(\sigma^i, \sigma^j) - R_s(m, m)| \leq \rho \right\}.
\]

The set \( B(m, \delta) \) is a product of spherical ‘bands’, one for each species. For \( \delta = 0 \), it is equal to the set from \( \mathbf{15} \). The elements of \( B(m, n, \delta, \rho) \) are \( n \)-tuples of points from \( B(m, \delta) \), which are pairwise approximately orthogonal for each species relative to \( m \) for small \( \delta \) and \( \rho \), in the sense that

\[
|R_s(\sigma^i - m, \sigma^j - m)| \leq O(\delta + \rho).
\]
Using those sets, we associate two free energies to any \( m \),
\[
F_N(m, \delta) = \frac{1}{N} \log \int_{B(m, \delta)} \exp \left( H_N(\sigma) - H_N(m) \right) d\mu(\sigma),
\]
(2.2)
\[
F_N(m, n, \delta, \rho) = \frac{1}{N^n} \log \int_{B(m, n, \delta, \rho)} \exp \left( \sum_{i=1}^{n} \left( H_N(\sigma^i) - H_N(m) \right) \right) d\mu(\sigma^1) \cdots d\mu(\sigma^n).
\]
(2.3)

Noting that
\[
F_N(m, n, \delta, \rho) = F_N(m, \delta)
\]
\[+ \frac{1}{N^n} \log G_N^{\otimes n} \{ \left| R_s(\sigma^i, \sigma^j) - R_s(m, m) \right| \leq \rho, \forall s \in \mathcal{S}, i \neq j \mid \sigma^i \in B(m, \delta) \}, \]
the free energy \( F_N(m, n, \delta, \rho) \) defined with \( n \) approximately orthogonal replicas can be thought of as the one with a single replica \( F_N(m, \delta) \), plus a penalty term related to how the conditional Gibbs measure is ‘spread’ on \( B(m, \delta) \).

Remark 4. By definition \( F_N(m, n, \delta, \rho) \) decreases as \( \delta \) and \( \rho \) decrease. Since \( n \rightarrow nF_N(m, n, \delta, \rho) \) is sub-additive, we also have that \( F_N(m, kn, \delta, \rho) \leq F_N(m, n, \delta, \rho) \) for any \( k \geq 1 \).

It is well-known that the free energy \( F_N \) concentrates around its mean, see e.g. \textit{[42, Theorem 1.2]}. For any non-random \( m \), \( F_N(m, \delta) \) and \( F_N(m, n, \delta, \rho) \) are free energies defined on the spaces \( B(m, \delta) \) and \( B(m, n, \delta, \rho) \), and therefore they concentrate around their mean. The concentration of the ground state energy \( E_{*,N}(q) \) around its mean follows from the classical inequality of Borell-TIS \textit{[19, 28]}. We state those results in the proposition below for later use.

**Proposition 5** (Concentration \textit{[19, 28, 42]}). For any \( q \in [0, 1)^{\mathcal{S}} \), \( n \geq 1 \), \( \delta, \rho > 0 \), \( m \in M_N \) and \( t > 0 \), with \( X_N \) being either \( F_N \), \( F_N(m, \delta) \), \( F_N(m, n, \delta, \rho) \) or \( E_{*,N}(q) \),
\[
\mathbb{P}(\left| X_N - \mathbb{E}X_N \right| \geq t) \leq 2 \exp \left( -\frac{Nt^2}{16\xi(1)} \right).
\]

While \( F_N(m, \delta) \) concentrates around its mean for any fixed non-random \( m \), the maximal deviation \( F_N(m, \delta) - \mathbb{E}F_N(m, \delta) \) over \( S_N(q) \) is typically of order \( O(1) \). In contrast to \( F_N(m, \delta) \), the other free energy we defined \( F_N(m, n, \delta, \rho) \) satisfies the following uniform concentration result, crucial to our analysis.

**Theorem 6** (Uniform concentration). For any \( q \in [0, 1)^{\mathcal{S}} \) and \( t, c > 0 \), there exist \( \rho_0, \delta_0 > 0 \) and \( n_0 \geq 1 \) such that for any \( n \geq n_0 \), \( \rho \leq \rho_0 \), and \( \delta \leq \delta_0 \) and any \( N \),
\[
\mathbb{P} \left( \max_{m \in S_N(q)} \left| F_N(m, n, \delta, \rho) - \mathbb{E}F_N(m, n, \delta, \rho) \right| > t \right) < 4e^{-Nc}.
\]
(2.4)

For two sequences of random variables that (may) depend on \( n, \delta \) and \( \rho \), we will write \( A_N(n, \delta, \rho) \approx B_N(n, \delta, \rho) \) if for any \( \epsilon > 0 \),
\[
\lim_{\delta, \rho \rightarrow 0} \lim_{N \rightarrow \infty} \mathbb{P} \left\{ \left| A_N(n, \delta, \rho) - B_N(n, \delta, \rho) \right| < \epsilon \right\} = 1.
\]
(2.5)

We will write \( A_N(n, \delta, \rho) \leq B_N(n, \delta, \rho) \) if the same holds without the absolute value. The same notation will also be used for non-random sequences, in which case it will mean that the corresponding bound holds for large enough \( N \) deterministically.

By definition, for any \( m \in M_N \),
\[
\frac{1}{N} H_N(m) + F_N(m, n, \delta, \rho) \leq \frac{1}{N} H_N(m) + F_N(m, \delta) \leq F_N.
\]
(2.6)
We will see (in Lemma 14) that $q$ is multi-samplable if and only if there exists some (random) $m_* \in S_N(q)$ for which

$$\frac{1}{N} H_N(m_*) + F_N(m_*, n, \delta, \rho) \approx \frac{1}{N} H_N(m_*) + F_N(m_*, \delta) \approx F_N.$$

Combining (2.10) and (2.7) with Proposition 5 and Theorem 6 and the fact that the expectation in (2.4) is constant on $S_N(q)$, one easily sees that

$$\frac{1}{N} H_N(m_*) \approx E_{*,N}(q).$$

Moreover, for any $q \in [0, 1)$,

$$EF_N \approx \mathbb{E}E_{*,N}(q) + EF_N(m, n, \delta, \rho).$$

We will prove the following characterization for multi-samplable overlaps in Section 4. Note that the fact that multi-samplability implies (2.9) follows from (2.7), (2.8) and Theorems 5 and 6.

Proposition 7. An overlap vector $q \in [0, 1)$ is multi-samplable if and only if for arbitrary $m \in S_N(q)$,

$$E_{F_N} \approx \mathbb{E}E_{*,N}(q) + EF_N(m, n, \delta, \rho).$$

Moreover, for any $q \in [0, 1)$,

$$EF_N \geq \mathbb{E}E_{*,N}(q) + EF_N(m, n, \delta, \rho).$$

The proof of Theorem 2 will be completed by the following proposition.

Proposition 8. For any $q \in [0, 1)$,

$$EF_N(m, n, \delta, \rho) \approx \frac{1}{2} \sum_{s \in \mathcal{F}} \lambda_s \log(1 - q(s)) + EF_N(q).$$

Let $H_N^q(\sigma)$ and $\tilde{H}_N^q(\sigma)$ be the Hamiltonians corresponding to the mixtures $\xi_q$ and $\tilde{\xi}_q$ which we defined in the introduction. Define

$$F_N^q(n, \rho) = \frac{1}{N n} \log \int_{B(0, n, \rho)} \exp \left( \sum_{i=1}^n H_N^q(\sigma^i) \right) d\mu(\sigma^1) \cdots d\mu(\sigma^n)$$

and define $\tilde{F}_N^q(n, \rho)$ similarly with $\tilde{H}_N^q(\sigma)$, where $B(0, n, \rho) := B(0, n, \delta, \rho)$ is independent of $\delta$.

The proof of Proposition 8 will consist of the following three lemmas.

Lemma 9. For any $q \in [0, 1)$,

$$\lim_{\rho \to 0} \limsup_{N \to \infty} \mathbb{E}F_N^q(m, n, \delta, \rho) - \left( \mathbb{E}F_N^q(n, \rho) + \frac{1}{2} \sum_{s \in \mathcal{F}} \lambda_s \log(1 - q(s)) \right) = 0.$$

Lemma 10. For any $q \in [0, 1)$,

$$\lim_{\rho \to 0} \limsup_{N \to \infty} |\mathbb{E}F_N^q(n, \rho) - \mathbb{E}\tilde{F}_N^q(n, \rho)| = 0.$$

Lemma 11. For any $q \in [0, 1)$, $\rho > 0$ and $n \geq 1$,

$$\limsup_{N \to \infty} |EF_N(q) - EF_N^q(n, \rho)| = 0.$$
Proof of Lemma 10. Of course, both Hamiltonians $H_N^q(\sigma)$ and $\tilde{H}_N^q(\sigma)$ can be defined on the same probability space such that

\begin{equation}
\tilde{H}_N^q(\sigma) = H_N^q(\sigma) + \sum_{s \in \mathcal{S}} \sqrt{\frac{N}{N_s}} \Delta_{q,p_s} \sum_{i \in I_s} J_i \sigma_i,
\end{equation}

where $J_i$ are i.i.d. standard normal variables independent of $H_N^q(\sigma)$ and $p_s$ is defined by $p_s(s') = 1$ if $s' = s$ and otherwise $p_s(s') = 0$. The point of this representation is that as we let $n \to \infty$, the effect of the extra term (linear in $\sigma$) which is added to $H_N^q(\sigma)$ becomes negligible when computing $\tilde{F}_N^q(n, \rho)$. Precisely, for $(\sigma^1, \ldots, \sigma^n)$, define $\sigma := \sum_{k=1}^n \sigma^k$ and $J := (J_1, \ldots, J_N)$. Then, by Cauchy–Schwarz, uniformly on $B(0, n, \rho)$,

\begin{equation}
\sum_{k=1}^n \sum_{s \in \mathcal{S}} \sqrt{\frac{N}{N_s}} \Delta_{q,p_s} \sum_{i \in I_s} J_i \sigma_i \leq \sum_{s \in \mathcal{S}} \Delta_{q,p_s} \sqrt{N N_s R_s(J, J)} R_s(\sigma, \sigma) \leq \sqrt{N} \|J\|_2 \sum_{s \in \mathcal{S}} \Delta_{q,p_s} R_s(\sigma, \sigma),
\end{equation}

and

$$R_s(\sigma, \sigma) \leq n + n(n-1)\rho.$$ 

Hence, almost surely,

\begin{equation}
|F_N^q(n, \rho) - \tilde{F}_N^q(n, \rho)| \leq \sqrt{\left(\frac{1}{n} + \rho\right) \frac{1}{N} \|J\|_2} \sum_{s \in \mathcal{S}} \Delta_{q,p_s}.
\end{equation}

The proof is completed by Jensen’s inequality since $\mathbb{E}\|J\|_2 \leq (\mathbb{E}\|J\|^2)^{1/2} = \sqrt{N}$. \hfill \square

Lemma 10 allows us to work with the mixture $\xi_q$ instead of $\tilde{\xi}_q$, for which $\Delta_p = 0$ if $|p| = 1$. For such mixtures we will prove the following lemma.

**Lemma 12.** Suppose that $\xi$ is a mixture which does not contain single-spin interactions, namely, such that $\Delta_p = 0$ if $|p| = 1$. Then, $q \equiv 0$ is multi-samplable.

Lemma 11 follows from Lemma 12.

Proof of Lemma 11. The lemma will follow if we show that for a mixture $\xi$ as in Lemma 12 for $m = 0$,

\begin{equation}
\limsup_{N \to \infty} |\mathbb{E}F_N - \mathbb{E}F_N(0, n, \delta, \rho)| = 0,
\end{equation}

since applied to $\xi_q$ this exactly gives (2.12). For such $\xi$, by Lemma 12 and Definition 1, for any fixed $t > 0$, with probability that does not decay exponentially fast as $N \to \infty$,

$$\frac{1}{N} \log G_N \{ \forall i < j \leq n, s \in \mathcal{S} : |R_s(\sigma^i, \sigma^j)| > \rho \} > -t.$$

By (2.23), since $B(0, \delta) = S_N$, for any $t > 0$, with such probability,

$$|F_N - F_N(0, n, \delta, \rho)| < t,$$

and (2.15) follows from the concentration as in Proposition 5. \hfill \square

In Section 5 we will prove Lemmas 9 and 12. This will complete the proof of Proposition 8.
3. Concentration: proof of theorem [6]

Let \( q \in [0, 1)^\mathbb{Z} \) and let \( m \) be an arbitrary point in \( S_N(q) \). For \( \delta, \rho > 0 \) and \( n \geq 1 \), consider the random field \((\sigma^1, \ldots, \sigma^n) \mapsto \sum_{i=1}^n (H_N(\sigma^i) - H_N(m))\) on \( B(m, n, \delta, \rho) \). Its variance at any point satisfies

\[
\frac{1}{N} \sum_{i=1}^n \left( H_N(\sigma^i) - H_N(m) \right)^2 \leq \sum_{i=1}^n \sum_{j=1}^n \left( \xi(R(\sigma^i, \sigma^j)) - \xi(R(\sigma^i, m)) - \xi(R(\sigma^j, m)) + \xi(R(m, m)) \right) \\
\leq n\xi(1) + n^2 \sqrt{|\mathcal{S}|}(2\delta + \rho)\|\nabla \xi(1)\| =: C(n, \rho, \delta),
\]

where the inequality follows since \( R(m, m) = q, \|R(\sigma^i, m) - q\| \leq \sqrt{|\mathcal{S}|}\delta \) and \( \|R(\sigma^j, \sigma^j) - q\| \leq \sqrt{|\mathcal{S}|}\rho \) for \( i \neq j \).

\( \mathcal{F}_N(m, n, \delta, \rho) \) is equal to \( \frac{1}{n} \) times the free energy of the random field above. Hence, from the well-known concentration of the free energy [42 Theorem 1.2], for any fixed \( m \in S_N(q) \),

\[
\Pr \left( \left| \mathcal{F}_N(m, n, \delta, \rho) - \mathbb{E}\mathcal{F}_N(m, n, \delta, \rho) \right| > t \right) < 2e^{-\frac{t^2}{2C(n, \rho, \delta)N}}.
\]

If \( q(s) = 0 \) for any \( s \), this completes the proof. From now on, assume otherwise.

By a union bound, for any \( t, c, a > 0 \) and any subset \( A_N \subset S_N(q) \) of cardinality \( |A| \leq e^{aN} \), if \( n \) is large enough and \( \rho \) and \( \delta \) are small enough,

\[
\Pr \left( \max_{m \in A_N} \left| \mathcal{F}_N(m, n, \delta, \rho) - \mathbb{E}\mathcal{F}_N(m, n, \delta, \rho) \right| > \frac{t}{2} \right) < 2e^{-2cN}.
\]

For any \( m, m' \in S_N(q) \), \( B(m, \delta) \) can be mapped to \( B(m', \delta) \) by a mapping \( \Theta : S_N \to S_N \) acting on the coordinates \((\sigma_i)_{i \in L}\), as a rotation if \( q(s) > 0 \) and as the identity if \( q(s) = 0 \), such that \( \Theta(m) = m' \) and

\[
q(s)R_s(\sigma - \Theta(\sigma), \sigma - \Theta(\sigma)) \leq R_s(m - m', m - m').
\]

In the Appendix we will prove a bound on the Lipschitz constant of the Hamiltonian \( H_N(\sigma) \), see Lemma [25]. From this bound it follows that, for large enough \( L > 0 \),

\[
\Pr \left( \forall m, m' \in S_N(q) : \left| \mathcal{F}_N(m, n, \delta, \rho) - \mathcal{F}_N(m', n, \delta, \rho) \right| \leq L \cdot D(m, m') \right) > 1 - e^{-2cN},
\]

where

\[
D(m, m') := \max_{s \in \mathcal{S}; q(s) > 0} \sqrt{R_s(m - m', m - m')/q(s)}.
\]

Since \( S_N(q) \) is a product of spheres, for any \( x > 0 \) there exists a subset \( A_N \subset S_N(q) \) such that \( |A_N| \leq e^{aN} \) for some large enough \( a = a(x) \) and such that for any \( m' \in S_N(q) \), for some \( m \in A_N \), \( D(m, m') \leq x \). Assuming \( A_N \) is such a subset with \( x = t/2L \), the proof is completed by (3.1) and (3.2), since \( \mathbb{E}\mathcal{F}_N(m, n, \delta, \rho) \) is constant on \( S_N(q) \). \( \square \)

4. First characterization: Proof of Proposition [7]

In this section we prove Proposition [7]. We will need the following lemma.

Lemma 13. For any \( q \in [0, 1)^\mathbb{Z} \), \( q \) is multi-samplable if and only if for any \( n \geq 1 \) and \( \delta, \rho, t > 0 \),

\[
\lim_{N \to \infty} \frac{1}{N} \log \Pr \left\{ \exists m \in S_N(q) : \left| \frac{1}{N} H_N(m) + \mathcal{F}_N(m, n, \delta, \rho) - \mathcal{F}_N \right| < t \right\} = 0.
\]
Proof. Let $q \in [0,1)^\mathcal{S}$ and assume that (4.1) holds for any $n \geq 1$ and $\delta, \rho, t > 0$. We will prove that $q$ is multi-samplable. Note that

$$
\frac{1}{N} H_N(m) + F_N(m, n, \delta, \rho) - F_N = \frac{1}{mN} \log G_N^{\otimes n} \{ (\sigma_1, \ldots, \sigma^n) \in B(m, n, \delta, \rho) \} \leq 0.
$$

(4.2)

Hence, (4.1) is equivalent to

$$
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{P} \left\{ \sup_{m \in S_N(q)} G_N^{\otimes n} \{ (\sigma_1, \ldots, \sigma^n) \in B(m, n, \delta, \rho) \} > e^{-tnN} \right\} = 0.
$$

(4.3)

By definition, if $(\sigma_1, \ldots, \sigma^n) \in B(m, n, \delta, \rho)$, then for any $i < j \leq n$ and $s \in \mathcal{S}$,

$$
|R_s(\sigma_i, \sigma^j) - q(s)| \leq \rho.
$$

Hence,

$$
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} G_N^{\otimes n} \left\{ \forall i < j \leq n, s \in \mathcal{S} : |R_s(\sigma_i, \sigma^j) - q(s)| \leq \rho \right\} \geq \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \sup_{m \in S_N(q)} G_N^{\otimes n} \{ (\sigma_1, \ldots, \sigma^n) \in B(m, n, \delta, \rho) \} = 0,
$$

where the equality follows from (4.3), and thus $q$ is multi-samplable.

Next, we will show that if $q$ is multi-samplable, then (4.1) holds. Let $q \in [0,1)^\mathcal{S}$, $n \geq 1$ and $\epsilon > 0$. Let $\sigma_1, \ldots, \sigma^{2n} \in S_N$ be independent samples from $G_N$. Suppose that for any $i < j \leq 2n$ and $s \in \mathcal{S}$,

$$
|R_s(\sigma^i, \sigma^j) - q(s)| < \epsilon.
$$

(4.4)

Then $m' := \frac{1}{n} \sum_{i=n+1}^{2n} \sigma^i$ satisfies

$$
|R_s(m', m') - q(s)| \leq \frac{1}{n} + \frac{n-1}{n} \epsilon,
$$

and for any $i < j \leq n$ and $s \in \mathcal{S}$,

$$
|R_s(\sigma^i, m') - q(s)| \leq \frac{1}{n} + \frac{n-1}{n} \epsilon
$$

and

$$
|R_s(\sigma^i - m', \sigma^j - m') - q(s)| < \epsilon + 3 \left( \frac{1}{n} + \frac{n-1}{n} \epsilon \right).
$$

If we define $m_* = (m_*, i)_{i \leq N}$ by

$$
m_* := \sqrt{\frac{q(s)}{R_s(m', m')}} m', \text{ if } i \in I_s,
$$

(4.5)

(note that $R_s(m', m') > 0$ a.s.) then $R_s(m_*, m_*) = q(s)$ and by Cauchy–Schwarz,

$$
|R_s(\sigma^*, m_*) - R_s(\sigma^*, m')| \leq \sqrt{R_s(m_* - m', m_* - m')},
$$

$$
|R_s(m_*, m_*) - R_s(m', m')| \leq 2 \sqrt{R_s(m_* - m', m_* - m')},
$$

and

$$
R_s(m_* - m', m_* - m') = \left( \sqrt{R_s(m_*, m_*)} - \sqrt{R_s(m', m')} \right)^2 \leq |R_s(m', m') - q(s)|,
$$

where the equality follows from (4.5).
By combining the above, given some $\delta$ and $\rho$, there exist $n_0 = n_0(\delta, \rho)$ and $\epsilon_0 = \epsilon_0(\delta, \rho)$ such that for any $n \geq n_0$ and $\epsilon \leq \epsilon_0$ the following holds. On the event that $\sigma^1, \ldots, \sigma^{2n} \in S_N$ satisfy (4.4), for $m_\star$ as defined above using the samples $\sigma^{n+1}, \ldots, \sigma^{2n}$, (which are independent of $\sigma^1, \ldots, \sigma^n$)

$$m_\star \in S_N(q) \quad \text{and} \quad (\sigma^1, \ldots, \sigma^n) \in B(m_\star, n, \delta, \rho).$$
Therefore,

$$G_N^{\otimes 2n} \{\forall i < j \leq 2n, s \in \mathcal{S} : |R_s(\sigma^i, \sigma^j) - q(s)| < \epsilon\} \leq G_N^{\otimes 2n} \{\sigma^1, \ldots, \sigma^n \in B(m_\star, n, \delta, \rho)\} \leq \sup_{m \in S_N(q)} G_N^{\otimes n} \{\sigma^1, \ldots, \sigma^n \in B(m, n, \delta, \rho)\}.$$

Now, assume that $q$ is multi-samplable, let $\delta$ and $\rho$ be some positive numbers, and assume that $n \geq n_0(\delta, \rho)$ and $\epsilon \leq \epsilon_0(\delta, \rho)$. Then, for any $t > 0$,

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{P} \left(G_N^{\otimes 2n} \{\forall i < j \leq 2n, s \in \mathcal{S} : |R_s(\sigma^i, \sigma^j) - q(s)| < \epsilon\} > e^{-tN}\right) = 0,$$
and from the inequality above

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{P} \left(\exists m \in S_N(q) : G_N^{\otimes n} \{\sigma^1, \ldots, \sigma^n \in B(m, n, \delta, \rho)\} > e^{-tN}\right) = 0.$$

Combining this with (4.4) implies (4.1), for general $\delta$ and $\rho$ and $n \geq n_0(\delta, \rho)$. By Remark 4 (4.1) holds for general $\delta$, $\rho$ and $n \geq 1$. \hfill \Box

We continue with the proof of Proposition 1. Denote by $\mathcal{E}_N(t, n, \rho, \delta)$ the event

$$\left\{\left|F_N - \mathbb{E}F_N\right| < t, \left|E_{\star,N}(q) - \mathbb{E}E_{\star,N}(q)\right| < t, \sup_{m \in S_N(q)} \left|F_N(m, n, \delta, \rho) - \mathbb{E}F_N(m, n, \delta, \rho)\right| < t\right\}.$$

Recall that by Proposition 5 and Theorem 6 for any $t > 0$ there exist $c_0$, $\rho_0$, and $n_0$, such that if $\delta \leq \delta_0$, $\rho \leq \rho_0$ and $n \geq n_0$, and $N$ is large, then

$$\mathbb{P} \left(\mathcal{E}_N(t, n, \rho, \delta)\right) \geq 1 - e^{-cN}.$$ 

(4.6)

Suppose that $m_\star \in S_N(q)$ is a point such that $\frac{1}{N}H_N(m_\star) = E_{\star,N}(q)$. From (2.6),

$$F_N \geq \frac{1}{N}H_N(m_\star) + F_N(m_\star, n, \delta, \rho) = E_{\star,N}(q) + F_N(m_\star, n, \delta, \rho).$$

Combined with (4.6), this implies that

$$\mathbb{E}F_N \geq \mathbb{E}E_{\star,N}(q) + \mathbb{E}F_N(m, n, \delta, \rho).$$

Suppose that $q$ is multi-samplable. Let $t > 0$ and suppose $c > 0$, $\rho_0$ and $n_0$ are the corresponding constants from (4.4). By Lemma 13 for any $\delta \leq \delta_N$, $\rho \leq \rho_0$ and $n \geq n_0$, for large $N$, with positive probability both $\mathcal{E}_N(t, n, \rho, \delta)$ occurs and there exists $m \in S_N(q)$ such that

$$\left|\frac{1}{N}H_N(m) + F_N(m, n, \delta, \rho) - F_N\right| < t.$$ 

Hence,

$$\mathbb{E}F_N \leq \mathbb{E}E_{\star,N}(q) + \mathbb{E}F_N(m, n, \delta, \rho) + 4t$$
and (2.4) follows, since $t > 0$ is arbitrary.

Conversely, if $q$ satisfies (2.4), then on the event $\mathcal{E}_N(t, n, \rho, \delta)$,

$$\left|\frac{1}{N}H_N(m_\star) + F_N(m_\star, n, \delta, \rho) - F_N\right| < 4t,$$
assuming that \( n \) is large enough and \( \rho \) and \( \delta \) are small enough, that \( N \) is large, and that as before \( m_\ast \in S_N(\rho) \) is a point such that \( \frac{1}{N} H_N(m_\ast) = E_\ast, N(\rho) \). This shows that given \( t > 0 \), for \( n, \rho \) and \( \delta \) as before, for large \( N \),

\[
\Pr \left\{ \exists m \in S_N(\rho) : \left| \frac{1}{N} H_N(m) + F_N(m, n, \delta, \rho) - F_N \right| < 4t \right\} \geq 1 - e^{-cN},
\]

where \( c > 0 \) is a constant as in (4.3). The fact that the same holds for arbitrary \( n \), \( \delta \) and \( \rho \) follows from Remark 3 and (2.6). Thus, by Lemma 13, \( m \) models we used a similar proof in [53].

We finish this section with the following improvement of Lemma 13.

**Lemma 14.** For any \( q \in [0, 1)^\mathcal{Y} \), \( q \) is multi-samplable if and only if for any \( n \geq 1 \) and \( \delta, \rho, t > 0 \),

\[
\limsup_{N \to \infty} \frac{1}{N} \log \Pr \left\{ \forall m \in S_N(q) : \left| \frac{1}{N} H_N(m) + F_N(m, n, \delta, \rho) - F_N \right| > t \right\} < 0.
\]

**Proof.** If for some \( q \in [0, 1)^\mathcal{Y} \) and any \( n \geq 1 \) and \( \delta, \rho, t > 0 \), (4.8) holds, then (4.4) also holds and by Lemma 13 \( q \) is multi-samplable.

Assume that \( q \in [0, 1)^\mathcal{Y} \) is multi-samplable. In the proof above, we have just seen that for such \( q \), (4.7) holds for arbitrary \( n, \delta, \rho \) and \( t \) and large \( N \). From this, (4.8) follows.

5. Computation of the correction: proof of Proposition 8

Proposition 8 follows directly from Lemmas 9, 11 and the triangle inequality. In Section 2 we already proved Lemma 10 and proved that Lemma 11 follows from Lemma 12. In this section we will prove Lemmas 9 and 11 and by doing so we will complete the proof of Proposition 8.

5.1. An auxiliary lemma. For the proof of Lemma 12 we will need the following lemma.

**Lemma 15.** Let \( \theta > 0 \) and suppose that \( \xi \) is a mixture which does not contain single-spin interactions, namely, that \( \Delta_\rho = 0 \) if \( |\rho| = 1 \). If \( \xi(1) < \theta \) and \( N \) is such that \( \max_s N/N_\ast < \theta \), then for any \( t > 0 \),

\[
\Pr(F_N \leq EF_N - t) \leq e^{-CNt^2},
\]

where \( C_N \to \infty \) is a sequence depending only on \( \theta \).

**Proof.** If \( F_N^k \) is the free energy corresponding to the truncated mixture \( \xi^k(x) = \sum_{|\rho| \leq k} \Delta_\rho^2 \prod_{s \in \mathcal{Y}^k} x(s)^{p(s)} \), then, \( F_N^k \to F_N \) in probability and in \( L^1 \) as \( k \to \infty \). Therefore, it will be enough to the lemma assuming that \( \Delta_\rho > 0 \) for infinitely many \( \rho \in \mathcal{P} \).

Note that by Hölder’s inequality, \( F_N \) is a convex function of the disorder variables \( J_{i_1, \ldots, i_k} \). Hence, by the main result of Paouris and Valettas [13],

\[
\Pr(F_N \leq EF_N - t) \leq \exp\left\{ -At^2/\Var(F_N) \right\},
\]

for some absolute constant \( A > 0 \). Therefore, the lemma will follow if we show that

\[
\Var(F_N) \leq \frac{CN}{N},
\]

for some sequence \( c_N \to 0 \) depending only on \( \theta \). In his book [23, Theorem 6.3], Chatterjee proved a bound of this from (specifically, as in [5, 4] below) in the case of the Sherrington-Kirkpatrick model. His method is general and in the rest of the proof we adapt it to our setting. For the usual spherical models we used a similar proof in [32].

For any real (measurable) function \( f((\sigma^i)_{i=1}^n) \) of \( n \) replicas \( \sigma^i \in S_N \) denote by

\[
\langle f((\sigma^i)_{i=1}^n) \rangle = \frac{\int_{S_N^n} f((\sigma^i)_{i=1}^n) e^{\sum_{i=1}^n H_N(\sigma^i)} d\sigma^1 \cdots d\sigma^n}{\int_{S_N^n} e^{\sum_{i=1}^n H_N(\sigma^i)} d\sigma^1 \cdots d\sigma^n}
\]
the average w.r.t. the Gibbs measure.

Note that

\( \frac{\partial}{\partial J_{i_1, \ldots, i_k}} \Delta H_{i_1, \ldots, i_k} = \left\langle \sum_{\alpha=1}^{n} \sqrt{N} \Delta_{i_1, \ldots, i_k} \sigma_{i_1}^\alpha \cdots \sigma_{i_k}^\alpha \right\rangle \).

For any function \( f \) of \( n \) replicas,

\[
\frac{\partial}{\partial J_{i_1, \ldots, i_k}} \left( f(\sigma^\alpha_{a=1}^n) \right) = \left\langle \sum_{\alpha=1}^{n} \sqrt{N} \Delta_{i_1, \ldots, i_k} \sigma_{i_1}^\alpha \cdots \sigma_{i_k}^\alpha f(\sigma^\alpha_{a=1}^n) \right\rangle
- n \left\langle \sum_{\alpha=1}^{n} \sqrt{N} \Delta_{i_1, \ldots, i_k} \sigma_{i_1}^{n+1} \cdots \sigma_{i_k}^{n+1} f(\sigma^\alpha_{a=1}^n) \right\rangle.
\]

Hence, by induction on \( t \in \mathbb{Z}_+ \),

\[
\frac{\partial^{t}}{\partial J_{i_1', \ldots, i_k'} \cdots \partial J_{i_1, \ldots, i_k}} \Delta H_{i_1, \ldots, i_k} = N^t \sum_{(\ell_1, \ldots, \ell_t) \in V(t)} c(\ell_1, \ldots, \ell_t) \left\langle \prod_{n=1}^{t} \Delta_{i_1' \cdots i_k'} \sigma_{i_1}^{\ell_1} \cdots \sigma_{i_k}^{\ell_t} \right\rangle,
\]

where \( V(t) = \{(\ell_1, \ldots, \ell_t) \in \mathbb{Z}_t : 1 \leq \ell_n \leq n\}, c(\ell_1, \ldots, \ell_t) = \prod_{n=1}^{t} a(\ell_n, n) \) and

\[ a(\ell_n, n) = \begin{cases} 1 & \text{if } \ell_n < n, \\ -(n-1) & \text{if } \ell_n = n. \end{cases} \]

In particular, \( |c(\ell_1, \ldots, \ell_t)| \leq (t-1)! \).

Define

\[ D_t := \sum \left( \mathbb{E} \left[ \frac{\partial}{\partial J_{i_1', \ldots, i_k'} \cdots \partial J_{i_1, \ldots, i_k}} \right] \right)^2 \]

where the sum is over all possible choices of \( k_1, \ldots, k_t \geq 1 \) and \( 1 \leq i_1^n, \ldots, i_k^n \leq N \).

Let \( (\hat{\sigma}_i^{(t)})_{i=1}^{k_t} \) be i.i.d. samples from the Gibbs measure corresponding to an independent copy \( \hat{H}_N(\hat{\sigma}) \) of the Hamiltonian \( H_N(\sigma) \). Then,

\[
\left( \sum_{(\ell_1, \ldots, \ell_t) \in V(t)} c(\ell_1, \ldots, \ell_t) \mathbb{E} \left[ \prod_{n=1}^{t} \sigma_{i_1}^{\ell_1} \cdots \sigma_{i_k}^{\ell_t} \right] \right)^2 \leq (t!)^3 \sum_{(\ell_1, \ldots, \ell_t) \in V(t)} \left( \mathbb{E} \left[ \prod_{n=1}^{t} \sigma_{i_1}^{\ell_1} \cdots \sigma_{i_k}^{\ell_t} \right] \right)^2
\]

\[ = (t!)^3 \sum_{(\ell_1, \ldots, \ell_t) \in V(t)} \mathbb{E} \left[ \prod_{n=1}^{t} \sigma_{i_1}^{\epsilon_1} \sigma_{i_2}^{\epsilon_2} \cdots \sigma_{i_k}^{\epsilon_k} \right]. \]

Combining the above, if we denote

\[ I(k_1, \ldots, k_t) := \left\{ (i_n^j)_{1 \leq n \leq t} : i_j^n \in \{1, \ldots, N\}, \forall j, n \right\}, \]

then

\[ D_t \leq N^t(t!)^3 \sum_{k_1, \ldots, k_t} \sum_{(\ell_1, \ldots, \ell_t) \in V(t)} \sum_{I(k_1, \ldots, k_t)} \mathbb{E} \left[ \prod_{n=1}^{t} \Delta_{i_1' \cdots i_k'} \sigma_{i_1}^{\epsilon_1} \cdots \sigma_{i_k}^{\epsilon_k} \right] \]

(5.3)
Define
\[ I(p^1, \ldots, p^t) = \left\{ \left( j^n \right)_{1 \leq n \leq t} : \# \{ j : i^n_j \in I_s \} = p^n(s), \forall n, s \right\}. \]

Then
\[
\sum_{I(k_1, \ldots, k_t)} \mathbb{E} \left\langle \prod_{n=1}^{t} \Delta^2_{\iota_n^{k_n}} \sigma^f_{\iota_n^{k_n}} \cdot \sigma^f_{\iota_n^{k_n}} \right\rangle
= \sum_{T(k_1, \ldots, k_t) \subseteq I(p^1, \ldots, p^t)} \mathbb{E} \left\langle \prod_{n=1}^{t} \Delta^2_{\iota_n^{k_n}} \sigma^f_{\iota_n^{k_n}} \cdot \sigma^f_{\iota_n^{k_n}} \right\rangle.
\]

Since
\[
\prod_{s \in \mathcal{S}} p(s)^! / |p|! \prod_{s \in \mathcal{S}} N_n^{-p(s)} \prod_{(i_j)_{j \leq |p|} \in I(p)} \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_{|p|}} \cdot \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_{|p|}} = \prod_{s \in \mathcal{S}} R_s(\sigma, \hat{\sigma}) p(s),
\]
we have that
\[
\sum_{I(p^1, \ldots, p^t)} \mathbb{E} \left\langle \prod_{n=1}^{t} \Delta^2_{\iota_n^{k_n}} \sigma^f_{\iota_n^{k_n}} \cdot \sigma^f_{\iota_n^{k_n}} \right\rangle
= \prod_{n=1}^{t} \left( \Delta^2_{p^n} / |p^n|! \right) \prod_{s \in \mathcal{S}} N_n^{-p^n(s)} \cdot \mathbb{E} \left\langle \prod_{n=1}^{t} \sum_{(i_j)_{j \leq |p^n|} \in I(p^n)} \sigma^f_{\iota_n^{k_n}} \sigma^f_{\iota_n^{k_n}} \cdots \sigma^f_{\iota_n^{k_n}} \cdot \sigma^f_{\iota_n^{k_n}} \right\rangle
= \left( \prod_{n=1}^{t} \Delta^2_{p^n} \right) \cdot \mathbb{E} \left\langle \prod_{n=1}^{t} \prod_{s \in \mathcal{S}} R_s(\sigma^f, \hat{\sigma}^f) p^n(s) \right\rangle.
\]

And therefore,
\[
\sum_{I(k_1, \ldots, k_t)} \mathbb{E} \left\langle \prod_{n=1}^{t} \Delta^2_{\iota_n^{k_n}} \sigma^f_{\iota_n^{k_n}} \cdot \sigma^f_{\iota_n^{k_n}} \right\rangle
= \mathbb{E} \left\langle \prod_{n=1}^{t} \sum_{p : |p| = k_n} \Delta^2_p R_s(\sigma^f, \hat{\sigma}^f) p^n(s) \right\rangle.
\]

By combining this with \ref{eq:4}, we obtain that
\[
D_t \leq N^t(t)!^3 \sum_{k_1, \ldots, k_t} \sum_{(l_1, \ldots, l_t) \in V(t)} \mathbb{E} \left\langle \prod_{n=1}^{t} \sum_{p : |p| = k_n} \Delta^2_p R_s(\sigma^f, \hat{\sigma}^f) p^n(s) \right\rangle.
\]

By Hölder’s inequality, for \( p^1, \ldots, p^t \in P \) and \( \gamma := \sum_{n \leq t} \sum_{s \in \mathcal{S}} p^n(s) \),
\[
\mathbb{E} \left\langle \prod_{n=1}^{t} \prod_{s \in \mathcal{S}} R_s(\sigma^f, \hat{\sigma}^f) p^n(s) \right\rangle \leq \max_{s \in \mathcal{S}} \mathbb{E} \left\langle |R_s(\sigma, \hat{\sigma})|^\gamma \right\rangle.
\]
Therefore, since that \(|V(t)| = t!\),

\[
D_t \leq N^t(t!)^t \sum_{k_1,\ldots,k_t} \left( \max_{s \in \mathcal{S}} \mathbb{E} \left| R_s(\sigma, \hat{\sigma}) \right|^{\sum_{n=1}^t k_n} \right)^t \prod_{n=1}^t \sum_{p:|p|=k_n} \Delta_p^2
\]

\[
\leq N^t(t!)^t \max_{s \in \mathcal{S}} \mathbb{E} \left| R_s(\sigma, \hat{\sigma}) \right|^{2t} \sum_{k_1,\ldots,k_t} \prod_{n=1}^t \sum_{p:|p|=k_n} \Delta_p^2
\]

\[
= N^t(t!)^t \max_{s \in \mathcal{S}} \mathbb{E} \left| R_s(\sigma, \hat{\sigma}) \right|^{2t},
\]

where in the second inequality we used the fact that \(\Delta_p = 0\) if \(|p| = 1\).

Fix some \(a \in (0,1)\). For large enough \(N_s\), uniformly in \(\gamma \leq N_s^a\), for arbitrary \(i \in I_s\),

\[
\mathbb{E} \left| R_s(\sigma, \hat{\sigma}) \right|^{\gamma} = N_s^{-\frac{\gamma}{2}} \mathbb{E} \left| \sigma_i \right|^{\gamma} = N_s^{-\frac{\gamma}{2}} \frac{1}{\omega_{N_s}^{\gamma-1}} \int_{-\sqrt{N_s}}^{\sqrt{N_s}} |x|^\gamma (1 - x^2/N_s)^{\frac{N_s-3}{2}} \, dx
\]

\[
\leq 2N_s^{-\frac{\gamma}{2}} \int_{-\sqrt{N_s}}^{\sqrt{N_s}} |x|^\gamma e^{-x^2} \, dx \leq 2N_s^{-\frac{\gamma}{2}} \mathbb{E} |X|^\gamma \leq 2N_s^{-\frac{\gamma}{2}} (\gamma - 1)!!,
\]

where \(\omega_M = 2\pi^{M/2}/\Gamma(M/2)\) is the area of the unit sphere in \(\mathbb{R}^M\) and \(X\) is a standard normal variable. Hence, for appropriate constant \(a > 0\), for large \(N\) and any \(t \leq N^a\),

\[
D_t \leq 4t^a \theta^{2t}.
\]

From (6.5.7),

\[
N^2 \mathbb{E} |\nabla F_N|^2 := E \sum_k \sum_{1 \leq i_1,\ldots,i_k \leq N} \left( \frac{\partial}{\partial J_{i_1,\ldots,i_k}} N F_N \right)^2
\]

\[
\leq NE \sum_k \sum_{1 \leq i_1,\ldots,i_k \leq N} \left( \Delta_{i_1,\ldots,i_k} \sigma_{i_1}^2 \cdots \sigma_{i_k}^2 \right) = N \xi(1) \leq N \theta.
\]

By the bound [23, Eq. (6.3)] in Chatterjee's book, for any \(d \geq 1\),

\[
\text{Var}(NF_N) \leq \sum_{t=1}^{d-1} \frac{D_t}{t^4} + \frac{1}{d} N^2 \mathbb{E} |\nabla F_N|^2.
\]

For \(d = \lceil \epsilon \log N / \log \log N \rceil\), with small enough \(\epsilon > 0\) depending on \(\theta\), we obtain that for some large \(C > 0\),

\[
\text{Var}(NF_N) \leq C N \frac{\log \log N}{\log N}.
\]

5.2. Proof of Lemma 12. Let \(n \geq 1\) be an arbitrary integer. Suppose \(X = (x^1,\ldots,x^n)\) is a set of vectors in \(S_N\) such that for any \(i \neq j\) and \(s \in \mathcal{S}\),

\[
R_s(x^i, x^j) = 0.
\]

For \(\delta > 0\), define the set

\[
B(\delta, X) = \{ \sigma \in S_N : |R_s(\sigma, x^i)| \leq \delta, \forall s \in \mathcal{S}, i \leq n \}
\]

and the free energy

\[
F_N(\delta, X) := \int_{B(\delta, X)} e^{H_N(\sigma)} d\mu(\sigma).
\]

We may identify \(B(0, X)\) with a product of \(|\mathcal{S}|\) spheres, one for each \(s \in \mathcal{S}\). Define \(F_N(0, X)\) similarly to the above with \(\mu\) replaced by the product of the uniform measures on those spheres.
Obviously, there is a measure-preserving bijection that maps $B(0, X)$ to
(5.6) $$
\left\{(\sigma_1, \ldots, \sigma_{N-n|\mathcal{F}}) \in \mathbb{R}^{N-n|\mathcal{F}} : \forall s \in \mathcal{F}, (\sigma_i)_{i \in I_s'} \in S(N_s - n) \right\},
$$
where $|I'_s| = |I_s| - n$ and $\cup_{s \in \mathcal{F}} I_s' = \{1, \ldots, N - n|\mathcal{F}|\}$, which also preserves the overlap between any two points (defined on the image space (5.6) using the subsets $I_s'$). Hence, from (1.2), $F_N(0, X)$ is equal to the free energy of a multi-species Hamiltonian on (5.6) with mixture $\xi$ multiplied by $\sqrt{\frac{N}{N - n|\mathcal{F}|}}$. (With this factor accounting for the fact that the variance at any point is $N$. It is easy to see that we may, and therefore will, neglect this factor in our proof.) It follows by Lemma 1.5 that for large $N$ and any fixed $X = (x^1, \ldots, x^n)$,
$$
P \{ F_N(0, X) \leq E F_N(0, X) - t \} \leq e^{-C N t^2},
$$
where $C_N \to \infty$ is a sequence as in the same lemma. Of course, from the concentration of $C$ where we may need to decrease $C_N$.

By a similar argument to the net argument used in the proof of Theorem 6, using the Lipschitz property of Lemma 25 and a union bound, one can see that for any fixed $F$ from the concentration of $C$ where the infimum is over all $N$ satisfying (5.5). Using the Lipschitz property of Lemma 25 again, it is easy to see that for any $t$, $C > 0$, if $\delta > 0$ is small enough then for large $N$,
$$
P \left\{ \inf_{X} (F_N(0, X) - E F_N) \leq -t \right\} \leq e^{-C N},
$$
where the infimum is over all $X$ satisfying (5.5). Using the Lipschitz property of Lemma 25 again, it is easy to see that for any $t$, $C > 0$, if $\delta > 0$ is small enough then for large $N$,
$$
P \left\{ \inf_{X} (F_N(0, X) - E F_N) \leq -t \right\} \leq e^{-C N}.
$$
From the concentration of $F_N$, decreasing the constant $C > 0$ if needed, for large $N$,
(5.7) $$
P \left\{ \inf_{X} (F_N(\delta, X) - F_N) \leq -t \right\} \leq e^{-C N}.
$$
On the complement of the event in (5.7), for any $k \leq n - 1$,
$$
G_N^{\otimes k+1} \left\{ \max_{1 \leq l, s \leq \mathcal{S}} |R_s(\sigma^{l+1}, \sigma^{l})| \leq \delta \left| \sigma^1, \ldots, \sigma^k \right| > e^{-t N}. \right\}
$$
Hence, with probability at least $1 - e^{-C N}$ for large $N$,
$$
G_N^{\otimes n} \left\{ \max_{1 \leq l, s \leq \mathcal{S}} |R_s(\sigma^1, \sigma^j)| \leq \delta \right\} > e^{-t(n-1) N}.
$$
Hence, $q \equiv 0$ is multi-samplable.

5.3. Proof of Lemma 9. We may identify
$$
B(m, 0) = \{ \sigma \in S_N : \forall s \in \mathcal{S}, R_s(\sigma, m) = R_s(m, m) \}
$$
with the product of the spheres
$$
\left\{(\sigma_i)_{i \in I_s} \in S(N_s) : \sum_{i \in I_s} \sigma_i m_i = \sum_{i \in I_s} m_i^2 \right\}.
$$
Endow each of those spheres with the uniform measure, and let $\nu$ be the product measure on $B(m, 0)$. Define
$$
\tilde{F}_N(m, n, \rho) := \frac{1}{N n} \log \int_{B(m, n, 0, \rho)} e^{\sum_{i=1}^n (H_N(\sigma^1) - H_N(m))} d\nu(\sigma^1) \cdots d\nu(\sigma^n).
$$
We will prove the following two lemmas below.

**Lemma 16.** Let $q \in [0, 1)^\mathcal{F}$ and $m \in S_N(q)$ and fix some $t > 0$. Then, for any $\delta, \rho > 0$ there exist $\delta_0 = \delta_0(\delta, \rho) > 0$ and $\rho_0 = \rho_0(\delta, \rho) > 0$ such that if $\rho' < \rho_0$ and $\delta' < \delta_0$, then for any $n \geq 1$ and large $N$,

\[
(5.8) \quad \mathbb{E} F_N(m, n, \delta, \rho) \geq \mathbb{E} \tilde{F}_N(m, n, \rho') + \frac{1}{N} \log \mu(B(m, \delta')) - t,
\]

\[
(5.9) \quad \mathbb{E} F_N(m, n, \delta', \rho') \leq \mathbb{E} F_N(m, n, \rho) + \frac{1}{N} \log \mu(B(m, \delta')) + t.
\]

**Lemma 17.** Let $q \in [0, 1)^\mathcal{F}$ and $m \in S_N(q)$. Then for $n \geq 1$ and $\rho > 0$,

\[
\lim_{N \to \infty} \left| \mathbb{E} F_N(m, n, \rho) - \mathbb{E} \tilde{F}_N^q(m, n, \rho) \right| = 0.
\]

Let $t > 0$ and let $n$ be some large number and $\delta$ and $\rho$ be some small numbers. By Lemma 16, there exist $\delta' < \delta$ and $\rho' < \rho$ such that for large $N$,

\[
\mathbb{E} F_N(m, n, \delta, \rho) \geq \mathbb{E} F_N(m, n, \rho') + \frac{1}{N} \log \mu(B(m, \delta')) - t.
\]

Assuming $\delta$ is small enough,

\[
\limsup_{N \to \infty} \left| \frac{1}{N} \log \mu(B(m, \delta')) - \frac{1}{2} \sum_{s \in \mathcal{F}} \lambda_s \log(1 - q(s)) \right| < t.
\]

Combining this with Lemma 17, we have that for large $N$,

\[
\mathbb{E} F_N(m, n, \delta, \rho) \geq \mathbb{E} \tilde{F}_N^q(m, n, \rho') + \frac{1}{2} \sum_{s \in \mathcal{F}} \lambda_s \log(1 - q(s)) - 3t.
\]

Note that by Lemmas 10 and 11, assuming $\rho' < \rho$ are small enough,

\[
\limsup_{N \to \infty} \left| \mathbb{E} \tilde{F}_N^q(n, \rho') - \mathbb{E} \tilde{F}_N^q(n, \rho) \right| < t.
\]

Hence,

\[
\mathbb{E} F_N(m, n, \delta, \rho) \geq \mathbb{E} \tilde{F}_N^q(n, \rho) + \frac{1}{2} \sum_{s \in \mathcal{F}} \lambda_s \log(1 - q(s)) - 4t.
\]

By a similar argument, for large $n$ and small $\delta$ and $\rho$,

\[
\mathbb{E} F_N(m, n, \delta, \rho) \leq \mathbb{E} \tilde{F}_N^q(n, \rho) + \frac{1}{2} \sum_{s \in \mathcal{F}} \lambda_s \log(1 - q(s)) + 4t.
\]

This completes the proof of Lemma 9. It remains to prove Lemmas 16 and 17.

**5.3.1. Proof of Lemma 17.** The lemma is an easy consequence of the fact that the mapping $\sigma \mapsto \bar{\sigma}$ as defined in (1.6) maps $B(m, 0)$ bijectively to

\[
(5.10) \quad \left\{ \sigma \in S_N : s \in \mathcal{F}, R_s(\sigma, m) = 0 \right\} = \prod_{s \in \mathcal{F}} \left\{ (\sigma_i)_{i \in I_s} : \sum_{i \in I_s} \sigma_i^2 = N_s, \sum_{i \in I_s} \sigma_i m_i = 0 \right\},
\]

and satisfies (1.7). \qed
5.3.2. Proof of Lemma 10. Fix \( q \in (0, 1)^\mathcal{J} \) and \( m \in S_N(q) \), where, for simplicity, we assume for now that \( q(s) > 0 \) for all \( s \in \mathcal{J} \). Define the mapping

\[
\sigma \mapsto \varphi(\sigma) = \varphi(\sigma, m) \in B(m, 0)
\]

by first defining the projection \( \tau = (\tau_i)_{i \leq N} \) by

\[
\tau_i := \sigma_i - \frac{R_s(\sigma, m)}{R_s(m, m)} m_i, \quad \text{if } i \in I_s,
\]

and then defining \( \varphi(\sigma) = \pi = (\pi_i)_{i \leq N} \) by

\[
(5.11) \quad \pi_i := m_i + \sqrt{\frac{1 - R_s(m, m)}{R_s(\tau, \tau)}} \tau_i, \quad \text{if } i \in I_s.
\]

We assume here that \( \sigma \) is such \( R_s(\tau, \tau) > 0 \). Any \( \sigma \in B(m, \delta) \) satisfies this if we assume, as we will from now on, that \( \delta < \min_s (\sqrt{q(s)} - q(s)) \).

Define the free energy

\[
\hat{F}_N(m, n, \delta, \rho) = \frac{1}{Nn} \log \int_{B(m, n, \delta, \rho)} e^{\sum_{i=1}^n (H_N(\sigma^i) - H_N(m))} d\mu(\sigma^1) \cdots d\mu(\sigma^n),
\]

where

\[
B(m, n, \delta, \rho) = \{ (\sigma^i)_{i \leq n} \in B(m, \delta)^n : \forall i \neq j, s \in \mathcal{J}, |R_s(\varphi(\sigma^i), \varphi(\sigma^j)) - R_s(m, m)| \leq \rho \}.
\]

One can check that

\[
(5.12) \quad \sigma \in B(m, \delta) \implies R_s(\varphi(\sigma) - \sigma, \varphi(\sigma) - \sigma) \leq \frac{2\delta}{\sqrt{q(s)}},
\]

and therefore

\[
(5.13) \quad \sigma^1, \sigma^2 \in B(m, \delta) \implies |R_s(\varphi(\sigma^1), \varphi(\sigma^2)) - R_s(\sigma^1, \sigma^2)| \leq \frac{\sqrt{8\delta}}{q(s)^{1/4}}.
\]

Thus, for \( \alpha := \max_{s \in \mathcal{J}} q(s)^{-1/4} \),

\[
B(m, n, \delta, \rho) \subset \hat{B}(m, n, \delta, \rho + \alpha \sqrt{8\delta}),
\]

\[
\hat{B}(m, n, \delta, \rho) \subset B(m, n, \delta, \rho + \alpha \sqrt{8\delta}),
\]

and

\[
\hat{F}_N(m, n, \delta, \rho) \leq \hat{F}_N(m, n, \delta, \rho + \alpha \sqrt{8\delta}),
\]

\[
\hat{F}_N(m, n, \delta, \rho) \leq F_N(m, n, \delta, \rho + \alpha \sqrt{8\delta}).
\]

For any \( t = (t_s) \in (-\delta, \delta)^\mathcal{J} \), define the vector \( m(t) = (m_i(t))_{i \leq N} \) by

\[
m_i(t) := \left( 1 + \frac{t_s}{R_s(m, m)} \right) m_i, \quad \text{if } i \in I_s,
\]

and define \( \varphi(t) = \varphi(\sigma, m(t)) \in B(m(t), 0) \). Note that

\[
\hat{B}(m, n, \delta, \rho) = \{ (\varphi(t^i(\sigma^i)))_{i \leq n} : (\sigma^i)_{i \leq n} \in B(m, n, 0, \rho), t^1, \ldots, t^n \in (-\delta, \delta)^\mathcal{J} \}.
\]

By the co-area formula, we may write

\[
\hat{F}_N(m, n, \delta, \rho) = \frac{1}{Nn} \log \int_{B(m, n, 0, \rho)} e^{\sum_{i=1}^n (H_N(\sigma^i) - H_N(m))} \Theta(\sigma^1, \ldots, \sigma^n) d\nu(\sigma^1) \cdots d\nu(\sigma^n),
\]
where
\[
\Theta(q) = \int_{\mathcal{S}} \cdots \int_{\mathcal{S}} e^{\sum_{i=1}^{N} (H_N(\varphi_i(q)) - H_N(\sigma_i))} D(t^1, \ldots, t^n) dt^1 \cdots dt^n
\]
and \(D(t^1, \ldots, t^n)\) is a Jacobian which integrates to
\[
\int_{\mathcal{S}} \cdots \int_{\mathcal{S}} D(t^1, \ldots, t^n) dt^1 \cdots dt^n = \mu^{\otimes n} (B(m, n, \delta, \rho)) = \mu (B(m, \delta))^{n}.
\]
For \(\sigma^i \in B(m, \delta)\) and \(t^i \in (-\delta, \delta)^{\mathcal{S}}\), \(\varphi_{t^i}(\sigma^i) \in B(m, \delta)\) and \(\varphi(\varphi_{t^i}(\sigma^i)) = \varphi(\sigma^i)\). By (5.12),
\[
R_s(\varphi_{t^i}(\sigma^i) - \sigma^i, \varphi_{t^i}(\sigma^i) - \sigma^i) \\
\leq \left( R_s(\varphi_{t^i}(\sigma^i) - \varphi(\sigma^i), \varphi_{t^i}(\sigma^i) - \varphi(\sigma^i))^{1/2} \\
+ R_s(\varphi(\sigma^i) - \sigma^i, \varphi(\sigma^i) - \sigma^i)^{1/2} \right)^2 \leq \frac{8\delta}{\sqrt{q(s)}}.
\]
Hence, if the event in (6.22) occurs for some \(L\), then for any \((\sigma^1, \ldots, \sigma^n) \in B(m, n, 0, \rho), \)
\[
\left| \sum_{i=1}^{n} (H_N(\varphi_{t^i}(\sigma^i)) - H_N(\sigma^i)) \right| \leq nL \max_{s \in \mathcal{S}} \sqrt{\frac{8\delta}{\sqrt{q(s)}}} = n\sqrt{8\delta} \alpha L
\]
and
\[
\left| \hat{F}_N(m, n, \rho) + \frac{1}{N} \log \mu(B(m, \delta)) - \hat{F}_N(m, n, \delta, \rho) \right| \leq \sqrt{8\delta} \alpha L.
\]
Given \(t, \delta\) and \(\rho\), we may choose some smaller \(\delta'\) and \(\rho'\) such that \(\rho > \rho' + \alpha \sqrt{8\delta'}\) and therefore from the above,
\[
F_N(m, n, \delta, \rho) \geq F_N(m, n, \delta', \rho' + \alpha \sqrt{8\delta'}) \geq \hat{F}_N(m, n, \delta', \rho') \\
\geq \hat{F}_N(m, n, \rho') + \frac{1}{N} \log \mu(B(m, \delta')) - \sqrt{8\delta} \alpha L.
\]
Assume that \(L\) is large enough so that by Lemma 25 the probability in (6.22) goes to 1 with \(N\) and that \(\delta'\) is small enough so that \(\sqrt{8\delta} \alpha L < t/2\). Then, (6.8) follows from the concentration of the free energies \(F_N(m, n, \delta, \rho)\) and \(\hat{F}_N(m, n, \rho')\) around their mean (e.g., by [12] Theorem 1.2). A similar argument gives (5.9).

Lastly, recall that we assumed that \(q(s) > 0\). If \(q(s) = 0\) for some \(s \in \mathcal{S}\), then one only needs to redefine \(\varphi(\sigma) = \pi\) by setting \(\pi_i = \sigma_i\) instead of (5.11) for all \(i \in I_s\) and such \(s\). The inequalities (5.12) and (5.13) then become trivial for such \(s\), and up to obvious modifications the proof remains the same.

6. TAP representation: Proof of Theorems 2 and 3

Theorem 2 follows directly from Propositions 7 and 8. Theorem 3 follows from the following two results.

**Corollary 18.** Let \(q \in [0, 1)^{\mathcal{S}}\) be a maximal multi-samplable overlap of the mixture \(\xi\), and assume \(\xi\) satisfies the convergence condition (6.12). Then, for any \(s \in \mathcal{S}\) and \(\tau > 0,\)
\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \max_{\pi \in \mathcal{S}_N} G_N^q \{ |R_s(\sigma, \pi) | \geq \tau \} < 0,
\]
where \(G_N^q\) is the Gibbs measure corresponding to the Hamiltonian \(H_N^q(\sigma)\) with mixture \(\xi_q(x)\).
Lemma 19. Let $\xi$ be some mixture. If for any $s \in \mathcal{S}$ and $\tau > 0$,
\begin{equation}
\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E} \max_{\pi \in S_N} \{ |R_s(\sigma, \pi)| \geq \tau \} < 0,
\end{equation}
then
\[ \mathbb{E}F_N = \frac{1}{2} \xi(1) + o(1). \]

In the rest of the section we will prove the results above, and the following two results.

Lemma 20. If $q$ is a multi-samplable overlap of $\xi$ and $q'$ is a multi-samplable overlap of $\xi_q$, then
\[ q + (1 - q)q' \]
is also a multi-samplable overlap of $\xi$. Therefore, if $q \in [0, 1]^\mathcal{S}$ is a maximal multi-samplable overlap for some mixture $\xi$, then the only multi-samplable overlap of $\xi_q$ is $q' \equiv 0$.

Lemma 21. If for some $s \in \mathcal{S}$ and $\tau > 0$,
\begin{equation}
\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E} \max_{\pi \in S_N} \{ |R_s(\sigma, \pi)| \geq \tau \} = 0,
\end{equation}
then for some $q \in [0, 1]^\mathcal{S}$ such that $q(s) \geq \tau^2/2$ and some subsequence $N_\ell$, as $\ell \to \infty$,
\begin{equation}
\mathbb{E}F_{N_\ell} = \mathbb{E}E_{*,N}(q) + \frac{1}{2} \sum_{s \in \mathcal{S}} \lambda_s \log(1 - q(s)) + \mathbb{E}F_{N_\ell}(q) + o(1).
\end{equation}

6.1. Proof of Lemma 20. Let $q$ and $q'$ be multi-samplable overlaps as in the lemma, and define $\hat{q}(s) = q(s) + (1 - q(s))q'(s)$. By Theorem \[2 \]
\begin{equation}
\mathbb{E}F_N = \mathbb{E}E_{*,N}(q) + \frac{1}{2} \sum_{s \in \mathcal{S}} \log(1 - q(s)) + \mathbb{E}F_{N}(q) + o(1),
\end{equation}
\begin{equation}
\mathbb{E}F_N(q) = \mathbb{E}E_{*,N}(q') + \frac{1}{2} \sum_{s \in \mathcal{S}} \log(1 - q'(s)) + \mathbb{E}F_{N}(q') + o(1),
\end{equation}
where the terms in the first line are defined using $\xi$, and terms in the second line, with superscript $q$, are defined similarly using the mixture $\xi_q$. Moreover, by the same theorem, to prove the lemma we need to show that
\begin{equation}
\mathbb{E}F_N \leq \mathbb{E}E_{*,N}(\hat{q}) + \frac{1}{2} \sum_{s \in \mathcal{S}} \log(1 - \hat{q}(s)) + \mathbb{E}F_N(\hat{q}) + o(1).
\end{equation}

Since
\[ 1 - \hat{q}(s) = (1 - q(s))(1 - q'(s)), \]
the sum of the two logarithmic terms in (6.5) is equal to the logarithmic term in (6.6). Also, from the definition (1.8) of $\xi_q$, it is straightforward to verify that $F(\hat{q})$ and $F(q')$ are the free energies of the same mixture $\xi_{\hat{q}}$ and are therefore equal. Hence, to prove (6.6) it remains to show that for any $q$, $q'$ and $\hat{q}$ defined as above,
\begin{equation}
\mathbb{E}E_{*,N}(q) + \mathbb{E}E_{*,N}(q') \leq \mathbb{E}E_{*,N}(\hat{q}) + o(1).
\end{equation}

This follows by a straightforward modification of the argument used for the single-species models in the proof of (6.4) in Lemma 31 of \[53 \]. Here, in the multi-species case, one needs to use the decomposition as in (2.13) and the Lipschitz property of Lemma 25 which generalized the analogous results for the single-species used in \[55 \]. (We remark that (6.7) holds for any $q$ and $q'$ in $[0, 1]^\mathcal{S}$, not necessarily multi-samplable overlaps.)
6.2. Proof of Lemma 21. Let $\tau > 0$ and $s' \in \mathcal{S}$. Let $n \geq 1$ and $\epsilon > 0$ such that $1/n \epsilon \leq \tau^2/2$. Denote
\[
Q(\epsilon) = \{ q : \forall s, q(s) \in \{ 0, \epsilon, \ldots, 1 - \epsilon \} \}.
\]
Decompose $[0, 1)^{|\mathcal{S}|}$ into $\epsilon^{-|\mathcal{S}|}$ disjoint sets
\[
[0, 1)^{|\mathcal{S}|} = \bigcup_{q \in Q(\epsilon)} \prod_{s \in \mathcal{S}} \{ q(s), q(s) + \epsilon \}.
\]
By Ramsey’s theorem, if $k = k(n)$ is large enough, then for any choice of $\sigma^1, \ldots, \sigma^k \in S_N$, there is a subset of size $n$ such that the overlaps $R(\sigma^i, \sigma^j)$ of all pairs from it belong to the same subset from the decomposition (6.8), and therefore for some $q \in Q(\epsilon)$,
\[
\max_{s \in \mathcal{S}} |R_s(\sigma^i, \sigma^2) - q(s)| < \epsilon.
\]
Suppose that $\sigma^1, \ldots, \sigma^n \in S_N$ and $q \in Q(\epsilon)$ are as above, i.e. for any $i \neq j$, (6.9) holds, and that for some $\pi \in S_N$ and every $i \leq n$, $R_s(\sigma^i, \pi) > \tau$. Then, for $m = \frac{1}{n} \sum_{i \leq n} \sigma^i$, \[R_s(m, m) = \left(1 + \frac{n - 1}{n} q(s')\right) < \epsilon,\]
and by Cauchy–Schwarz,
\[
R_s(m, \pi) > \tau \implies R_s(m, m) > \tau^2.
\]
Hence, since we assumed that $\frac{1}{n} + \epsilon \leq \tau^2/2$,
\[
q(s') \geq \tau^2/2.
\]
We therefore have that
\[
1\{(\sigma^i, \ldots, \sigma^k) \in A_k(\pi, \tau)\} \leq \sum 1\{(\sigma^i, \ldots, \sigma^j) \in B_n(q, \epsilon)\}
\]
where the sum is over all $q \in Q(\epsilon)$ with $q(s') \geq \tau^2/2$ and $1 \leq i_1 < \cdots < i_n \leq k$ and
\[
A_k(\pi, \tau) = \{ (\sigma^i)_{i \leq k} \in S_N^k : R_s(\sigma^i, \pi) > \tau \},
\]
\[
B_n(q, \epsilon) = \left\{ (\sigma^i)_{i \leq n} \in S_N^n : \max_{s \in \mathcal{S}} |R_s(\sigma^i, \sigma^j) - q(s)| < \epsilon, \forall i \neq j \right\}.
\]
By taking the expectation w.r.t. $G_N^{\otimes k}$ of both sides of (6.11),
\[
\left(\frac{k}{n}\right)^{-|\mathcal{S}|} \max_{q \in Q(\epsilon)} \max_{q(s') \geq \tau^2/2} G_N^{\otimes n} \left\{ \max_{s \in \mathcal{S}, i \neq j} |R_s(\sigma^i, \sigma^j) - q(s)| < \epsilon \right\}
\]
\[
\geq \max_{\pi \in S_N} G_N^{\otimes k} \left\{ \forall i, k, R_s(\sigma^i, \pi) > \tau \right\}^{k} = \left( \max_{\pi \in S_N} G_N \{ R_s(\sigma, \pi) > \tau \} \right)^k.
\]
Now, assume that
\[
\limsup_{N \to \infty} \frac{1}{N} \log E \max_{\pi \in S_N} G_N \{ R_s(\sigma, \pi) > \tau \} = 0.
\]
Since $Q(\epsilon)$ is finite, from (6.12) we obtain that there exists $q = q(n, \epsilon) \in Q(\epsilon)$ independent of $N$, such that $q(s') \geq \tau^2/2$ and
\[
\limsup_{N \to \infty} \frac{1}{N} \log E G_N^{\otimes n} \left\{ \max_{s \in \mathcal{S}, i \neq j} |R_s(\sigma^i, \sigma^j) - q(s)| < \epsilon \right\} = 0.
\]
If $q_*$ is a limit point of $q(n_i, \epsilon_i)$ for some sequences $n_i \to \infty$ and $\epsilon_i \to 0$, then $q_*(s') \geq \tau^2/2$ and for all $n$ and $\epsilon$,
\[
\limsup_{N \to \infty} \frac{1}{N} \log E G_N^{\otimes n} \left\{ \max_{s \in \mathcal{S}, i \neq j} |R_s(\sigma^i, \sigma^j) - q_*(s)| < \epsilon \right\} = 0.
Since the expectation above is increasing in $\epsilon$ and decreasing in $n$, by diagonalization there exists a subsequence $N_\ell$ such that for any $n$ and $\epsilon$,
\[
\lim_{\ell \to \infty} \frac{1}{N_\ell} \log \mathbb{E} G_{N_\ell}^q \left\{ \max_{s \in \mathcal{S}, i < j \leq n} \left| R_s(\sigma^i, \sigma^j) - q_s(s) \right| < \epsilon \right\} = 0.
\]

By going over its proof, one can check that Theorem 2 also holds for subsequences in the sense that an overlap $q \in [0, 1)^{\mathcal{F}}$ satisfies (1.4) for any $n$ and $\epsilon$ on a subsequence $N_\ell$ if and only if (1.10) holds on the same subsequence. Thus, $q_*$ satisfies (6.4).

6.3. **Proof of Corollary 18** Suppose that $q \in (0, 1)^{\mathcal{F}}$ is a maximal multi-samplable overlap of a mixture $\xi$ which satisfies the convergence condition (1.12). Assume towards contradiction that for some $s \in \mathcal{F}$ and $\tau > 0$,
\[
(6.13) \quad \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E} \max_{\pi \in S_N^q} G_N^q \{ |R_s(\sigma, \pi)| \geq \tau \} = 0.
\]
Since
\[
G_N^q \{ |R_s(\sigma, \pi)| \geq \tau \} = G_N^q \{ R_s(\sigma, \pi) \geq \tau \} + G_N^q \{ R_s(\sigma, -\pi) \geq \tau \},
\]
we also have that
\[
\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E} \max_{\pi \in S_N^q} G_N^q \{ R_s(\sigma, \pi) \geq \tau \} = 0.
\]
By Lemma 21 for some $q'$ with $q'(s) \geq \tau^2/2$ and $N_\ell$ as in the lemma, as $\ell \to \infty$,
\[
\mathbb{E} F_{N_\ell}(q) = \mathbb{E} F_{N_\ell}^q = \mathbb{E} F_{N_\ell}^{q'} = \mathbb{E} E_{N_\ell}(q') + \frac{1}{2} \sum_{s \in \mathcal{F}} \lambda_s \log(1 - q'(s)) + \mathbb{E} F_N^q(q') + o(1),
\]
where all the quantities above with superscript $q$ are defined using the Hamiltonian $H_N^q(\sigma)$ with mixture $\xi(q)$.

Since $q$ is multi-samplable, by the same argument we used in the proof of Lemma 20 defining $\hat{q}(s) = q(s) + (1 - q(s)) q'(s)$, on the same subsequence $N_\ell$,
\[
\mathbb{E} F_{N_\ell} = \mathbb{E} E_{N_\ell}(\hat{q}) + \frac{1}{2} \sum_{s \in \mathcal{F}} \log(1 - \hat{q}(s)) + \mathbb{E} F_{N_\ell}(\hat{q}) + o(1).
\]
From (1.12), we have that the same holds without moving to a subsequence. That is,
\[
\mathbb{E} F_N = \mathbb{E} E_N(\hat{q}) + \frac{1}{2} \sum_{s \in \mathcal{F}} \log(1 - \hat{q}(s)) + \mathbb{E} F_N(\hat{q}) + o(1).
\]
By Theorem 2, $\hat{q}$ is therefore multi-samplable. This contradicts the maximality of $q$, and we conclude that (6.13) does not hold for any $s \in \mathcal{F}$ and $\tau > 0$. \(\square\)

6.4. **Proof of Lemma 19** From Jensen’s inequality, $\mathbb{E} F_N \leq \frac{1}{2} \xi(1)$. Hence, to prove the lemma it will be enough to show that
\[
\liminf_{N \to \infty} \mathbb{E} \log Z_{N+1} - \mathbb{E} \log Z_N \geq \frac{1}{2} \xi(1).
\]
We remark that changing the values of $N_\ell$ while keeping the same limiting proportions $\lambda_s$ results in changing the free energy $\mathbb{E} F_N$ by an amount which vanishes as $N \to \infty$.\(^\dagger\) By a similar argument as in the footnote, one can also verify that (6.2) also holds if we change the values of $N_\ell$, as long as the limits $\lambda_s$ are the same. Therefore we may, and will, assume that the values $N_\ell$ are non-decreasing with $N$. In this case, as $N$ increases to $N + 1$, $N_\ell$ increases to $N_\ell + 1$ for exactly one

\(^\dagger\) Note that given $(N_\ell)$ and $(N_\ell^\prime)$ with the same $N \to \infty$ limit, one can relate each of the associated free energies to that of the model corresponding to $\min\{N_\ell, N_\ell^\prime\}$, by integrating over the ‘extra’ coordinates. Using the Lipschitz property in Lemma 24 and the concentration of the free energy, one can then see that the difference of the free energies goes to 0 as $N \to \infty$.\(\square\)
\[s \in \mathcal{S}\) while all other \(N_s\) remain the same. We will denote the species \(s \in \mathcal{S}\) that increases by \(s_\ast = s_\ast(N)\).

Instead of working with \(S_N\) in this proof we will work with
\[T_N = \left\{(\sigma_1, \ldots, \sigma_N) \in \mathbb{R}^N : \forall s \in \mathcal{S}, \sum_{i \in I_s} \sigma_i^2 = 1\right\},\]
by defining for \(\sigma \in T_N\),
\[(6.14)\]
\[h_N(\sigma) = H_N(\sigma) = \sqrt{N} \sum_{k=1}^{\infty} \sum_{i_1, \ldots, i_k=1}^N \bar{\Delta}_{i_1, \ldots, i_k} J_{i_1, \ldots, i_k} \sigma_{i_1} \cdots \sigma_{i_k}\]
where for any \(s \in \mathcal{S}\) and \(i \in I_s, \tilde{\sigma}_i = \sqrt{N_s} \sigma_i\) and if \(\#\{j \leq k : i_j \in I_s\} = p(s)\) for any \(s \in \mathcal{S}\), then \(\bar{\Delta}_{i_1, \ldots, i_k} = \Delta_{i_1, \ldots, i_k}(N)\) is defined by
\[\bar{\Delta}_{i_1, \ldots, i_k} = \Delta_{i_1, \ldots, i_k}(N)\] 
\[\prod_{s \in \mathcal{S}} p(s)! \prod_s \rho_s.\]

Let \(N \geq 1\). When the system size is \(N\), assume WLOG that \(I_{s_\ast} = \{N - N_{s_\ast} + 1, \ldots, N\}\) and when the system size is \(N + 1\) assume that \(I_{s_\ast} = \{N - N_{s_\ast} + 1, \ldots, N + 1\}\). We will use \(\rho\) to denote points from \(T_{N+1}\) and \(\sigma\) for points from \(T_N\), and will assume everywhere that \(\rho, \sigma\) and \(\varepsilon\) are related to each other by
\[(6.15)\]
\[\rho_i = \begin{cases} \sigma_i & \text{if } i \leq N - N_{s_\ast}, \\
\sqrt{1 - \varepsilon^2} \sigma_i & \text{if } N - N_{s_\ast} < i \leq N, \\
\varepsilon & \text{if } i = N + 1. \end{cases}\]
Here \(\varepsilon\) plays the role of the usual cavity coordinate.

Using the same variables \(J_{i_1, \ldots, i_k}\) as in \((6.14)\), define
\[h_{N+1}^{(1)}(\rho) = \sqrt{N} \sum_{k=1}^{\infty} \sum_{i_1, \ldots, i_k=1}^N \bar{\Delta}_{i_1, \ldots, i_k} J_{i_1, \ldots, i_k} \sigma_{i_1} \cdots \sigma_{i_k}(1 - \varepsilon^2)^{\#\{j \leq k : i_j \in I_{s_\ast}\}}.\]
If \(x = (x_i)_{i \leq N}\) is defined by \(x_i = \rho_i\), then
\[\mathbb{E}h_{N+1}^{(1)}(\rho)h_{N+1}^{(1)}(\rho') = N \xi(Q(x, x'))\]
where we define \((Q(x, x'))(s) := \sum_{i \in I_s} x_i x'_i\).

If we write \(h_{N+1}(\rho)\) as a polynomial in \(\rho\) using \((6.14)\), then \(\sqrt{N^{-1}} h_{N+1}^{(1)}(\rho)\) is the sum of all terms which do not contain the last coordinate \(\varepsilon\). Hence, we may assume, as we will, that \(h_{N+1}(\rho), h_{N+1}^{(1)}(\rho)\) and an additional process \(h_{N+1}^{(2)}(\rho)\) independent of \(h_{N+1}^{(1)}(\rho)\) are defined on the same probability space, such that
\[h_{N+1}(\rho) = h_{N+1}^{(1)}(\rho) + h_{N+1}^{(2)}(\rho),\]
and
\[\mathbb{E}\left\{h_{N+1}^{(2)}(\rho)h_{N+1}^{(2)}(\rho')\right\} = \mathbb{E}\left\{h_{N+1}(\rho)h_{N+1}(\rho')\right\} - \mathbb{E}\left\{h_{N+1}^{(1)}(\rho)h_{N+1}^{(1)}(\rho')\right\}.\]
By a Taylor approximation one can check that, if \(\varepsilon, \varepsilon' \in (-t, t),\)
\[(6.16)\]
\[\left|\mathbb{E}\left\{h_{N+1}^{(2)}(\rho)h_{N+1}^{(2)}(\rho')\right\} - \xi(Q(\sigma, \sigma')) - N \varepsilon \varepsilon' \frac{d}{dx(s_\ast)} \xi(Q(\sigma, \sigma'))\right| \leq 4 N t^4 \frac{d^2}{dx(s_\ast)^2} \xi(1) + 4 t^2 \frac{d}{dx(s_\ast)} \xi(1).\]
Fix $C > 0$ and define $B_N := \{ \rho \in T_{N+1} : |\xi| < N^{-1/2}C \}$. Note that
\[
\log Z_{N+1} - \log Z_N = \log \int_{T_{N+1}} e^{h_{N+1}(\rho)} d\mu(\rho) - \log \int_{T_N} e^{h_N(\sigma)} d\mu(\sigma)
\geq \log \int_{B_N} e^{h_{N+1}(\rho)} d\mu(\rho) - \log \int_{T_N} e^{h_N(\sigma)} d\mu(\sigma) =: W_N,
\]
where $\mu = \mu_N$ is the product of the uniform measures on the spheres corresponding to the subsets of coordinates $I_s$. We will show that for any $t > 0$, if $C$ is large enough,
\[
(6.17) \quad \lim_{N \to \infty} P \left( W_N > \frac{1}{2} \xi(1) - t \right) = 1.
\]
Recall (6.16) and our assumption that (6.1) holds. Combining the latter with a straightforward modification of the argument used to prove Lemma 34 in [53] the following lemma follows.

**Lemma 22.** For any $\delta > 0$, with probability going to 1 as $N \to \infty$,
\[
\int_{B_N} e^{h_{N+1}(\rho)} d\rho \geq (1 - \delta/2) \int_{B_N} e^{h_{N+1}(\rho)} + \frac{1}{2} \Var h_{N+1}(\rho) d\rho 
\geq (1 - \delta) \int_{B_N} e^{h_{N+1}(\rho)} + \frac{1}{2} \xi(1) + N \varepsilon^2 \frac{d}{dx(\xi(1))} d\rho.
\]

Define
\[
g_N(\sigma) = \sqrt{N} \sum_{k=1}^{\infty} \sum_{i_1, \ldots, i_k = 1}^{N} p_{i_1, \ldots, i_k}(s_*) \Delta_{i_1, \ldots, i_k} J_{i_1, \ldots, i_k} \sigma_{i_1} \cdots \sigma_{i_k},
\]
\[
\zeta_N(\rho) = \sqrt{N} \sum_{k=1}^{\infty} \sum_{i_1, \ldots, i_k = 1}^{N} \tau_{i_1, \ldots, i_k}(\varepsilon) \Delta_{i_1, \ldots, i_k} J_{i_1, \ldots, i_k} \sigma_{i_1} \cdots \sigma_{i_k},
\]
where $p_{i_1, \ldots, i_k}(s_*) := \# \{ j \leq k : i_j \in I_s \}$ and
\[
\tau_{i_1, \ldots, i_k}(\varepsilon) := (1 - \varepsilon^2) \frac{1}{2} p_{i_1, \ldots, i_k}(s_*) - 1 + \frac{1}{2} p_{i_1, \ldots, i_k}(s_*) \varepsilon^2.
\]
Then,
\[
(6.18) \quad h_{N+1}^{(1)}(\rho) = h_N(\sigma) - \frac{1}{2} \varepsilon^2 g_N(\sigma) + \zeta_N(\rho).
\]

Note that
\[
\frac{1}{2} \frac{d}{dt} \bigg|_{t=0} \mathbb{E} \left( (h_N(\sigma) + tg_N(\sigma))^2 \right) = N \frac{d}{dx(s_*)} \xi(1).
\]
Therefore, by the same argument as in the proof of Lemma 35 in [53], we have the following lemma.

**Lemma 23.** For any $\delta > 0$, with probability going to 1 as $N \to \infty$, there exists a (random) subset $D_N \subset T_N$ such that
\[
\int_{D_N} e^{h_N(\sigma)} d\sigma \geq (1 - \delta) \int_{T_N} e^{h_N(\sigma)} d\sigma,
\]
and
\[
\sup_{\sigma \in D_N} g_N(\sigma) \leq (1 + \delta) N \frac{d}{dx(s_*)} \xi(1).
\]

The last term in (6.18) is negligible by the first bound in the following lemma.
Lemma 24. For some constant $c > 0$ depending only on $\xi$ and $C$, for any large enough $t > 0$, for large $N$,\

$$\P \left( \sup_{\rho \in B_N} |\zeta_N(\rho)| \geq \frac{t}{N} \right) < e^{-Nct^2},$$

$$\P \left( \sup_{\sigma \in T_N} |g_N(\sigma)| \geq Nt \right) < e^{-Nct^2}.$$ 

Proof. The bound for the process $g_N(\sigma)$ follows from Lemma 25, applied with $\pi = 0$. For any $i_1, \ldots, i_k$ define $\kappa_{i_1, \ldots, i_k} := \#\{j \leq k : i_j \in I_s\}$. For $n \geq 1$ define

$$\zeta_{N,n}(\sigma) := \sqrt{N} \sum_{k=1}^{\infty} \sum_{i_1, \ldots, i_k} \bar{\Delta}_{i_1, \ldots, i_k} j_{i_1, \ldots, i_k} \sigma_{i_1} \cdots \sigma_{i_k}.$$ 

By an abuse of notation, write $\zeta_N(\sigma, \varepsilon) = \zeta_N(\rho)$ assuming (6.15). Then,

$$\zeta_N(\sigma, \varepsilon) = \sum_{k \geq 1} \left( (1 - \varepsilon^2)^{\frac{k}{2}} - 1 + \frac{1}{2} k \varepsilon^2 \right) \zeta_{N,k}(\sigma).$$

By the proof of Lemma 25, specifically (6.26),

$$\E \sup_{\sigma \in K_N} \zeta_{N,k}(\sigma) = \E \sup_{\sigma \in K_N} \zeta_{N,k}(\sigma) - \zeta_{N,k}(0) \leq 2N|\mathcal{F}| \Delta_k,$$

where $\Delta_k = \sum_{p \in P : \cdot p(s*) = k} \Delta_p^2 |\mathcal{P}^4|$. (Here we work with $\sigma \in T_N$ instead of $\sigma \in S_N$ as in Lemma 25 which in (6.26) amounts to working with $\Delta_p$ corresponding to $\Delta_{i_1, \ldots, i_k}$ and not the normalized $\bar{\Delta}_{i_1, \ldots, i_k}$.) Since

$$\left| (1 - \varepsilon^2)^{\frac{k}{2}} - 1 + \frac{1}{2} k \varepsilon^2 \right| \leq \frac{1}{4} k (\frac{1}{2} k - 1) \varepsilon^4,$$

we have that

$$\E \sup_{\sigma \in B_N} \zeta_N(\sigma, \varepsilon) \leq \frac{1}{N} C^4 |\mathcal{F}| \sum_{k \geq 1} k (\frac{1}{2} k - 1) \Delta_k.$$ 

Of course,

$$\sup_{\sigma \in B_N} \E \left( \zeta_N(\sigma, \varepsilon)^2 \right) \leq C^8 N^{-3} \sum_{k \geq 1} \left( \frac{1}{4} k (\frac{1}{2} k - 1) \right)^2 \sum_{p \in P : \cdot p(s*) = k} \Delta_p^2,$$

which completes the proof by the Borell-TIS inequality. \qed

Suppose $f(\rho)$ is a smooth function on $T_{N+1}$ and, by an abuse of notation, write $f(\rho) = f(\sigma, \varepsilon)$ assuming (6.15). Then,

$$\int_{B_N} f(\rho) d\rho = \frac{\omega_{N,s-1}}{\omega_{N,s}} \int_{T_N} \int_{C/\sqrt{N}}^{C/\sqrt{N}} (1 - \varepsilon^2)^{\frac{N}{2}\varepsilon^2} f(\sigma, \varepsilon) d\varepsilon d\mu(\sigma)$$

$$= (1 + o_N(1)) \sqrt{\frac{N_s}{2\pi}} \int_{T_N} \int_{C/\sqrt{N}}^{C/\sqrt{N}} e^{-\frac{N}{2\pi} \varepsilon^2} f(\sigma, \varepsilon) d\varepsilon d\mu(\sigma),$$

(6.19)
where \( \omega_{N-1} = 2\pi^{N/2}/\Gamma(N/2) \) is the surface area of \( T_N \) and \( \frac{\omega_{N-1}}{\omega_N} = (1 + o_N(1))\sqrt{\frac{N}{e}} \). Hence, for any \( \delta > 0 \) and with probability going to 1,
\[
\int_{B_N} e^{h_{N+1}(\rho)} d\rho \\
\geq (1 - 2\delta)e^{\xi(1)} \sqrt{\frac{N_s}{2\pi}} \int_{T_N}^{C/\sqrt{N}} e^{-\frac{N_s}{2\pi}r^2 + h_{N+1}^{(1)}(\rho) + \frac{1}{2} N \varepsilon^2 \sum_{i=0}^{d} \varepsilon_i} d\mu(\sigma) \\
\geq (1 - 3\delta)e^{\xi(1)} \sqrt{\frac{N_s}{2\pi}} \int_{D_N}^{C/\sqrt{N}} e^{-\frac{N_s}{2\pi}r^2 + h_N(\sigma) - \frac{1}{2} N \varepsilon^2 \sum_{i=0}^{d} \varepsilon_i} d\mu(\sigma) \\
\geq (1 - 3\delta)e^{\xi(1) - \frac{1}{2} C^2 \varepsilon^2 \ln(\varepsilon) + \xi(1)} \mathbb{P}[|X| \leq C \sqrt{N_s/N}] \int_{T_N} e^{h_N(\sigma)} d\mu(\sigma),
\]
where the first inequality follows from Lemma 24, the second and third inequalities follow from (6.18) and Lemmas 24 and 23 and \( X \) is a standard Gaussian variable. For any \( t \), large enough \( C = C(t) \) and small enough \( \delta = \delta(t, C) \), this yields (6.17). The proof of Lemma 19 will be completed if we show that for fixed \( C, W_N \) is uniformly integrable.

From Lemma 24 and (6.19), since log of the volume of \( B_N \) is bounded from below uniformly in \( N, t \), for large \( t \),
\[
(6.20) \quad \mathbb{P} \left( \left| \log \int_{B_N} e^{h_{N+1}^{(1)}(\rho)} d\mu(\rho) - \log \int_{T_N} e^{h_N(\sigma)} d\mu(\sigma) \right| > a + t \right) \leq e^{-N\delta a^2}
\]
for some constant \( a = a(C) > 0 \). Conditioned on \( h_{N+1}^{(1)}(\rho) \), for any \( \rho \) the variance of \( e^{h_{N+1}(\rho)} \) is equal to the unconditional variance of \( h_{N+1}^{(2)}(\rho) \), which is bounded by some constant \( A = A(C) > 0 \). Hence, by Jensen’s inequality,
\[
0 \leq \mathbb{E} \left( \left| \log \int_{B_N} e^{h_{N+1}(\rho)} d\mu(\rho) \right| h_{N+1}^{(1)}(\rho) \right) - \log \int_{B_N} e^{h_{N+1}^{(1)}(\rho)} d\mu(\rho) \leq \frac{1}{2} A.
\]
Moreover, from the concentration of the free energy (see e.g. [12 Theorem 1.2]) applied conditionally on \( h_{N+1}^{(1)}(\rho) \),
\[
(6.21) \quad \mathbb{P} \left( \left| \log \int_{B_N} e^{h_{N+1}(\rho)} d\mu(\rho) - \log \int_{B_N} e^{h_{N+1}^{(1)}(\rho)} d\mu(\rho) \right| > \frac{1}{2} A + t \right) \leq 2e^{-\frac{t^2}{2}}.
\]
Combining (6.20) and (6.21) we have that
\[
\mathbb{P}[|W_N| > a + \frac{1}{2} A + 2t] \leq e^{-N\delta a^2} + 2e^{-\frac{t^2}{2}}.
\]
Therefore, \( W_N \) is uniformly integrable and proof of Lemma 19 is completed.

APPENDIX: LIPSCHITZ CONTINUITY

Denote the closure of \( M_N \) by
\[
\overline{M}_N = \left\{ (\sigma_1, \ldots, \sigma_N) \in \mathbb{R}^N : \sum_{i \in I_s} \sigma_i^2 \leq N_s, \forall s \in \mathcal{S} \right\}.
\]
The following lemma is proved by an adaptation of the proof of Lemma 6.1 in [26].

Lemma 25 (Lipschitz continuity). For any \( t > 0 \) there exists \( L > 0 \), which depends on \( \xi \) and \( t \) only, such that for any \( N \),
\[
(6.22) \quad \mathbb{P} \left\{ \forall \sigma, \pi \in \overline{M}_N : \frac{1}{N} |H_N(\sigma) - H_N(\pi)| \leq L \max_{s \in \mathcal{S}} \sqrt{R_s(\sigma - \pi, \sigma - \pi)} \right\} > 1 - e^{-tN}.
\]
Proof. Define
\[ G(u, \sigma) := u \cdot \nabla H_N(\sigma) \]
\[ := \sqrt{N} \sum_{k=1}^{\infty} \sum_{i_{1}, \ldots, i_{k} = 1}^{N} \Delta_{i_{1}, \ldots, i_{k}} J_{i_{1}, \ldots, i_{k}}(u_{i_{1}} \cdots \sigma_{i_{k}} + \cdots \sigma_{i_{1}} \cdots u_{i_{k}}). \]
For \( \theta(t) = t\sigma + (1 - t)\pi \), with \( \dot{\theta}(t) = \frac{d\theta}{dt} \theta(t) = \sigma - \pi \),
\[ \frac{1}{N} |H_N(\sigma) - H_N(\pi)| \leq \frac{1}{N} \int_{0}^{1} |\dot{\theta}(t) \cdot \nabla H_N(\theta(t))| dt. \]

Since \( R_{s}(\dot{\theta}(t), \dot{\theta}(t)) = R_{s}(\sigma - \pi, \sigma - \pi) \), for \( a := \max_{s \in S} \sqrt{R_{s}(\sigma - \pi, \sigma - \pi)} \) and some \( u \in \overline{M}_{N} \), \( \dot{\theta}(t) = au \). Thus, the event from (6.22) is contained in the event that
\[ \frac{1}{N} \sup_{\sigma, u \in \overline{M}_{N}} G(u, \sigma) \leq L. \]

By the Borell-TIS inequality, to prove the lemma it will be enough to show that
\[ \frac{1}{N} \mathbb{E} \sup_{\sigma, u \in \overline{M}_{N}} G(u, \sigma) \leq C \]
and that
\[ \frac{1}{N} \sup_{\sigma, u \in \overline{M}_{N}} \mathbb{E}G(u, \sigma)^{2} \leq C, \]
for some \( C > 0 \) independent of \( N \).

Suppose \( \sigma, \pi, u \) and \( v \) are points in \( \overline{M}_{N} \). From the inequality \( (x_{1} + \cdots + x_{k})^{2} \leq k(x_{1}^{2} + \cdots + x_{k}^{2}) \) and symmetry,
\[ \mathbb{E}(G(u, \sigma) - G(v, \pi))^{2} \]
\[ = N \sum_{k=1}^{\infty} \sum_{i_{1}, \ldots, i_{k} = 1}^{N} \Delta_{i_{1}, \ldots, i_{k}}^{2} [(u_{i_{1}} \cdots \sigma_{i_{k}} + \cdots \sigma_{i_{1}} \cdots u_{i_{k}}) - (v_{i_{1}} \cdots \pi_{i_{k}} + \cdots \pi_{i_{1}} \cdots v_{i_{k}})^{2}] \]
\[ \leq N \sum_{k=1}^{\infty} \sum_{i_{1}, \ldots, i_{k} = 1}^{N} \Delta_{i_{1}, \ldots, i_{k}}^{2} k[(u_{i_{1}} \cdots \sigma_{i_{k}} - v_{i_{1}} \cdots \pi_{i_{k}})^{2} + (\sigma_{i_{1}} \cdots u_{i_{k}} - \pi_{i_{1}} \cdots v_{i_{k}})^{2}] \]
\[ = N \sum_{k=1}^{\infty} \sum_{i_{1}, \ldots, i_{k} = 1}^{N} \Delta_{i_{1}, \ldots, i_{k}}^{2} k^{2}(u_{i_{1}} \sigma_{i_{2}} \cdots \sigma_{i_{k}} - v_{i_{1}} \pi_{i_{2}} \cdots \pi_{i_{k}})^{2} \]
\[ \leq N \sum_{k=1}^{\infty} \sum_{i_{1}, \ldots, i_{k} = 1}^{N} \Delta_{i_{1}, \ldots, i_{k}}^{2} k^{3} [(u_{i_{1}} - v_{i_{1}})^{2} \sigma_{i_{2}}^{2} \cdots \sigma_{i_{k}}^{2} \pi_{i_{2}}^{2} \cdots \pi_{i_{k}}^{2}] \]
\[ + \sum_{j=2}^{k} v_{i_{1}}^{2} \sigma_{i_{2}}^{2} \cdots \sigma_{i_{j-1}}^{2} (\sigma_{i_{j}} - \pi_{i_{j}})^{2} \sigma_{i_{j+1}}^{2} \cdots \pi_{i_{k}}^{2}], \]
where for the last inequality we the fact that
\[ u_{i_{1}} \cdots \sigma_{i_{k}} - v_{i_{1}} \cdots \pi_{i_{k}} = (u_{i_{1}} - v_{i_{1}}) \sigma_{i_{2}} \cdots \sigma_{i_{k}} \]
\[ + v_{i_{1}} (\sigma_{i_{2}} - \pi_{i_{2}}) \sigma_{i_{3}} \cdots \sigma_{i_{k}} + \cdots + v_{i_{1}} \pi_{i_{2}} \cdots \pi_{i_{k-1}} (\sigma_{i_{k}} - \pi_{i_{k}}). \]

From this, one can check that
\[ \mathbb{E}(G(u, \sigma) - G(v, \pi))^{2} \leq N \Delta_{s}^{2} \sum_{s \in S} (R_{s}(u - v, u - v) + R_{s}(\sigma - \pi, \sigma - \pi)). \]
for $\Delta_0^2 := \sum_{p \in P} \Delta_p^2 |p|^4 < \infty$.

Note that if we define

$$\tilde{G}(u, \sigma) = \sum_{s \in \mathcal{S}} \sqrt{N/N_s} \sum_{i \in I_s} \Delta_0 J_i \sigma_i + \sum_{s \in \mathcal{S}} \sqrt{N/N_s} \sum_{i \in I_s} \Delta_0 \tilde{J}_i u_i,$$

where $J_i$ and $\tilde{J}_i$ are i.i.d. standard normal variables, then $\mathbb{E}[\tilde{G}(u, \sigma) - \tilde{G}(v, \pi)]^2$ is equal to the right-hand side of (6.25). Hence, from the Sudakov–Fernique inequality, (see e.g. [1, Theorem 2.2.3])

$$\frac{1}{N} \mathbb{E} \sup_{\sigma, u \in \mathcal{M}_N} G(u, \sigma) \leq \frac{1}{N} \mathbb{E} \sup_{\sigma, u \in \mathcal{M}_N} \tilde{G}(u, \sigma) \leq 2\Delta_0 \sum_{s \in \mathcal{S}} \sqrt{N_s/N},$$

which proves (6.24).

A similar computation to the above gives, for any $\sigma, u \in \mathcal{M}_N$,

$$\frac{1}{N} \mathbb{E} G(u, \sigma)^2 \leq \sum_{k=1}^{\infty} \sum_{i_1, \ldots, i_k = 1}^N \Delta_2 \sum_{i_1, \ldots, i_k = 1}^N k^2 \sigma_{i_1}^2 \cdots \sigma_{i_k}^2 \leq \sum_{p \in P} \Delta_p^2 |p|^2,$$

from which (6.24) follows.  \hfill \square

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