BLOWING-UP SOLUTIONS OF THE TIME-FRACTIONAL DISPERSIVE EQUATIONS

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ABSTRACT. This paper is devoted to the study of initial-boundary value problems for time-fractional analogues of Korteweg-de Vries, Benjamin-Bona-Mahony, Burgers, Rosenau, Camassa-Holm, Degasperis-Procesi, Ostrovsky and time-fractional modified Korteweg-de Vries-Burgers equations on a bounded domain. Sufficient conditions for the blowing-up of solutions in finite time of aforementioned equations are presented. We also discuss the maximum principle and influence of gradient non-linearity on the global solvability of initial-boundary value problems for the time-fractional Burgers equation. The main tool of our study is the Pohožaev nonlinear capacity method. We also provide some illustrative examples.

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1. INTRODUCTION

Nonlinear wave phenomenon is one of the important areas of scientific investigation. Among the mathematical models describing the dynamics of wave equations include...
Korteweg-de Vries equation, Burgers equation, Benjamin-Bona-Mahony equation, Rosenau equation and Ostrovsky equation.

Bateman-Burgers equation or Burgers equation [Bat15, Bur48]
\[ u_t + uu_x = \nu u_{xx}, \quad \nu > 0, \quad (1.1) \]
is a fundamental partial differential equation occurring in various areas of applied mathematics, such as fluid mechanics, nonlinear acoustics, gas dynamics, and traffic flow.

The Korteweg-de Vries equation [KV95] is well known in different fields of science and technology; it reads
\[ u_t + uu_x + u_{xxx} = 0. \quad (1.2) \]

In [BBM72], Benjamin, Bona, and Mahony proposed the following equation to describe long waves on the water surface
\[ u_t - u_{xxx} + uu_x = 0. \quad (1.3) \]
In [Ros86] Rosenau suggested the following equation to describe waves on “shallow” water:
\[ u_t + u_{txxxx} + u_x + uu_x = 0. \quad (1.4) \]
In [Ost78] Ostrovsky derived an equation for weakly nonlinear surface and internal waves in a rotating ocean
\[ u_{tx} + u_{xx} + u_{xxxx} + (uu_x)_x = 0. \quad (1.5) \]

The Camassa-Holm equation
\[ u_t - u_{txx} + 2\kappa u_x + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \quad \kappa > 0, \quad (1.6) \]
was introduced by Camassa and Holm [CH93] as a bi-Hamiltonian model for waves in shallow water and the following Degasperis-Procesi equation, one of the important models of mathematical physics
\[ u_t - u_{txx} + 2\kappa u_x + 4uu_x = 3u_xu_{xx} + uu_{xxx}, \quad \kappa > 0. \quad (1.7) \]

The Korteweg-de Vries-Burgers equation
\[ u_t + uu_x + u_{xxx} = \nu u_{xx}, \quad \nu > 0, \quad (1.8) \]
modified Korteweg-de Vries-Burgers equation
\[ u_t + u^2u_x + u_{xxx} = \nu u_{xx}, \quad \nu > 0, \quad (1.9) \]

Benjamin-Bona-Mahony-Burgers equation
\[ u_t - u_{txx} + uu_x = \nu u_{xx}, \quad (1.10) \]
Korteweg-de Vries-Benjamin-Bona-Mahony equation
\[ u_t - u_{txx} + u_{xxx} + uu_x = 0, \quad (1.11) \]
Rosenau-Burgers equation
\[ u_t + u_{txxxx} + u_x + uu_x = \nu u_{xx}, \quad (1.12) \]
Rosenau-Korteweg-de Vries equation
\[ u_t + u_{txxxx} + u_{xxx} + u_x + uu_x = 0, \quad (1.13) \]
Rosenau-Benjamin-Bona-Mahony equation
\[ u_t - u_{txx} + u_{txxxx} + u_x + uu_x = 0, \quad (1.14) \]
BLOWING-UP SOLUTIONS ...

have important applications in different physical situations such as waves on shallow water, and processes in semiconductors with differential conductivity [BS76, FPS01, Ros89, Shu87, SG69, Zh05].

This paper is devoted to blowing-up solutions of time-fractional analogues of the above equations. The approach to the problem is based on the Pohozaev nonlinear capacity method [MP98, MP01, MP04]; more precisely, on the choice of test functions according to initial and boundary conditions under consideration.

Here, we give a simple case of the analysis of a rough blow-up, i.e., the case where the solution tends to infinity as $t \to T^*$ on $[0, L]$; more exactly, when the integral

$$\int_0^L u(x, t) \phi(x) dx$$

tends to infinity as $t \to T^*$ for the given function $\phi$.

In [Kor12a, Kor12b, KP13, KY14, KY15] Korpusov et al. obtained sufficient conditions for the finite time blow-up of solutions of initial-boundary problems for Burgers, Korteweg-de Vries, Benjamin-Bona-Mahony and Rosenau type equations. We also note that the blow-up of solutions of the initial problems for the Korteweg-de Vries and critical Korteweg-de Vries equations are investigated in [MM02, MM14, MMR14, Po10, Po10a, Po11, Po11a, Po12, Po12a, Po12b]. Blow-up of solutions of the initial problems for the Ostrovsky equation is proved in [LPS10].

Descriptions of some physical applications and numerical simulations of the time-fractional dispersive equations are given in [FH18, HHG19, LZR16, LV18, QTWZ17, SBA18, SB12, XA13, Yok18].

Recently, the study of blowing-up solutions of time-fractional nonlinear partial differential equations received great attention. For example, the authors of this paper obtained results on the blow-up of the solutions of time-fractional Burgers equation [AKT19a, Tor19] and fractional reaction-diffusion equation [AKT19b]. We note that the blow-up of the solution of various nonlinear fractional problems was investigated in [AAAKT15, AAKMA17, CSWSS18, KNS08, Pav18, XX18].

Let us briefly describe the problems investigated in this paper:

- Blowing-up solutions of the time-fractional Rosenau-KdV-BBM-Burgers equation with initial conditions described as follows:

$$\partial_{+0,t}^{\alpha} (u - au_{xx} + bu_{xxxx}) + cu_{xxx} - du_{xx} + u_x + uu_x = 0, \quad 0 < x < L, \quad t > 0,$$

$$u(x, 0) = u_0(x), \quad x \in [0, L],$$

where $a, b, c, d \in \mathbb{R}$ and $u_0$ is a given function.

- Blowing-up solutions of the initial-boundary problem for the time-fractional Camassa-Holm–Degasperis-Procesi equation

$$\partial_{+0,t}^{\alpha} (u - u_{xx}) + au_x + buu_x - cu_x u_{xx} - du_{xxx} = 0, \quad 0 < x < L, \quad t > 0,$$

$$u(x, 0) = u_0(x), \quad x \in [0, L],$$

where $a, b, c, d \in \mathbb{R}$ and $u_0$ is a given function.
Blowing-up solutions of the time-fractional Ostrovsky equation with initial conditions:
\[
\partial_{+0,t}^{\alpha} u_x + a u_{xx} + b u_{xxxx} + (u u_x)_x = 0, \; 0 < x < L, \; t > 0,
\]
\[
u(x, 0) = u_0(x), \; x \in [0, L],
\]
where \( a, b \in \mathbb{R} \) and \( u_0 \) is a given function.

Blowing-up solutions of the initial problem for the time-fractional analogue of the modified Korteweg-de Vries-Burgers equation with dissipation:
\[
\partial_{+0,t}^{\alpha} u + u^2 u_x + a u_{xxx} - b u_{xx} = 0, \; x \in (0, L), \; t > 0,
\]
\[
u(x, 0) = u_0(x), \; x \in [0, L],
\]
where \( a, b \in \mathbb{R} \) and \( u_0 \) is a sufficiently smooth function.

Maximum principle and gradient blow-up in time-fractional Burgers equation
\[
\partial_{+0,t}^{\alpha} u + u u_x = \nu u_{xx}, \; x \in (0, L), \; t > 0,
\]
with an initial condition
\[
u(x, 0) = u_0(x), \; x \in [0, L],
\]
where \( \nu > 0 \) and \( u_0 \) is a sufficiently smooth function.

1.1. Preliminaries.

1.1.1. Fractional operators. Here, we recall definitions and properties of fractional order integral and differential operators [KST06, Nak03, SKM87].

**Definition 1.1.** [KST06] (Riemann-Liouville integral). Let \( f \) be a locally integrable real-valued function on \(-\infty \leq a < t < b \leq +\infty\). The Riemann–Liouville fractional integral \( I_{+a}^{\alpha} \) of order \( \alpha \in \mathbb{R} \) \((\alpha > 0)\) is defined as
\[
I_{+a}^{\alpha} f(t) = (f * K_\alpha)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} f(s)ds,
\]
where \( K_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \), \( \Gamma \) denotes the Euler gamma function.

The convolution here will be understood in the sense of the above definition.

**Definition 1.2.** [KST06] (Riemann-Liouville derivative). Let \( f \in L^1([a, b]), -\infty \leq a < t < b \leq +\infty \) and \( f * K_{m-a}(t) \in W^{m,1}([a, b]), m = [\alpha] + 1, \alpha > 0 \), where \( W^{m,1}([a, b]) \) is the Sobolev space defined as
\[
W^{m,1}([a, b]) = \left\{ f \in L^1([a, b]) : \frac{d^m}{dt^m} f \in L^1([a, b]) \right\}.
\]
The Riemann–Liouville fractional derivative \( D_{+a}^{\alpha} \) of order \( \alpha > 0 \) \((m - 1 < \alpha < m, m \in \mathbb{N})\) is defined as
\[
D_{+a}^{\alpha} f(t) = \frac{d^m}{dt^m} I_{+a}^{m-\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-1-\alpha} f(s)ds.
\]
**Definition 1.3.** [KST06] (Caputo derivative). Let \( f \in L^1([a, b]), -\infty \leq a < t < b \leq +\infty \) and \( f \ast K_{m-\alpha}(t) \in W^{m-1}([a, b]), m = [\alpha], \alpha > 0 \). The Caputo fractional derivative \( \partial_{t+a}^\alpha \) of order \( \alpha \in \mathbb{R} \) \((m-1 < \alpha < m, m \in \mathbb{N})\) is defined as

\[
\partial_{t+a}^\alpha f(t) = D_{t+a}^\alpha \left[ f(t) - f(a) - f'(a) \frac{(t-a)}{1!} - \ldots - f^{(m-1)}(a) \frac{(t-a)^{m-1}}{(m-1)!} \right].
\]

If \( f \in C^m([a, b]) \), then the Caputo fractional derivative \( \partial_{t+a}^\alpha \) of order \( \alpha \in \mathbb{R} \) \((m-1 < \alpha < m, m \in \mathbb{N})\) is defined as

\[
\partial_{t+a}^\alpha f(t) = I_{t+a}^{m-\alpha} f^{(m)}(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds.
\]

**Property 1.4.** [AAK17] Let \( 0 < \alpha \leq 1, f \in C([0, T]), f' \in L^1([0, T]) \) and \( u \) be monotone. Then

\[
2f(t)\partial_{t+0,t}^\alpha f(t) \geq \partial_{t+0,t}^\alpha f^2(t), \quad t \in (0, T].
\] (1.15)

**Property 1.5.** [Lu09] Let \( f \in C^1((0, T)) \cap C([0, T]) \) attain its maximum over the interval \([0, T]\) at \( t_0 \in (0, T) \). Then \( \partial_{t+0,t}^\alpha f(t_0) \geq 0 \).

Let \( f \in C^1((0, T)) \cap C([0, T]) \) attain its minimum over the interval \([0, T]\) at \( t_0 \in (0, T) \). Then \( \partial_{t+0,t}^\alpha f(t_0) \leq 0 \).

1.1.2. **Finite time blow-up of solutions of a fractional differential equation.** We consider the fractional differential equation

\[
\partial_{t+0}^\alpha u(t) = u^2(t), \quad t > 0, \ 0 < \alpha < 1,
\]

\[
u(0) = u_0 \in \mathbb{R}.
\] (1.16)

The blow-up of solutions to (1.16) is assured by the following theorem.

**Theorem 1.6.** [HKL14] If \( u_0 > 0 \), then the solution of problem (1.16) blows-up in a finite time

\[
\left( \frac{\Gamma(\alpha + 1)}{4u_0} \right)^\frac{1}{\alpha} \leq T^* \leq \left( \frac{\Gamma(\alpha + 1)}{u_0} \right)^\frac{1}{\alpha},
\] (1.17)

that is \( \lim_{t \to T^*} u(t) = +\infty \).

2. **Blowing-up solutions of the time-fractional Rosenau-KdV-BBM-Burgers equation**

In this section we consider the time-fractional Rosenau-KdV-BBM-Burgers equation:

\[
\partial_{t+0,t}^\alpha (u - au_{xx} + bu_{xxxx}) + cu_{xxx} - du_{xx} + u_x + uu_x = 0, \ 0 < x < L, \ t > 0,
\]

\[
u(x, 0) = u_0(x), \ x \in [0, L],
\] (2.1)

where \( a, b, c, d \in \mathbb{R} \) and \( u_0 \) is a given function.

The equation (2.1) is called the Rosenau-KdV-BBM-Burgers equation with time-fractional derivative as it is a generalization of the following well-known equations:

- If \( \alpha = 1 \) and \( a = b = c = 0, d > 0 \), then the equation (2.1) coincides with the classical Burgers equation (1.1);
Let the function \( \varphi \) be monotonically nondecreasing:
\[
\varphi'(x) \geq 0 \quad \text{for} \quad x \in [0, L] \tag{2.3}
\]
and satisfy the following properties

\[
\begin{cases}
\theta_1 := \frac{1}{2} \int_0^L \frac{(c\varphi'''(x) + d\varphi''(x) + \varphi'(x))^2}{\varphi'(x)} \, dx < \infty; \\
\theta_2 := 2 \int_0^L \frac{(\varphi(x) - a\varphi''(x) + b\varphi'''(x))^2}{\varphi'(x)} \, dx < \infty.
\end{cases}
\] (2.4)

Then we have

\[
2 \int_0^L u(x,t)(c\varphi'''(x) + d\varphi''(x) + \varphi'(x)) \, dx + \int_0^L u^2(x,t)\varphi'(x) \, dx \\
= \int_0^L v^2(x,t)\varphi'(x) \, dx - \int_0^L \frac{(c\varphi'''(x) + d\varphi''(x) + \varphi'(x))^2}{\varphi'(x)} \, dx,
\]

where

\[v(x,t) = u(x,t) + \frac{c\varphi'''(x) + d\varphi''(x) + \varphi'(x)}{\varphi'(x)}.\]

Using the Hölder inequality, we obtain the following estimate

\[
\left( \int_0^L v(x,t)(\varphi(x) - a\varphi''(x) + b\varphi'''(x)) \, dx \right)^2 \\
\leq \int_0^L v^2(x,t)\varphi'(x) \, dx \int_0^L \frac{(\varphi(x) - a\varphi''(x) + b\varphi'''(x))^2}{\varphi'(x)} \, dx.
\]

Then, expression (2.2) takes the form

\[
\partial_{\tau_0,t}^\alpha F(t) \geq \theta_2^{-1} F^2(t) + \Phi(t) - \theta_1,
\] (2.5)

where

\[F(t) = \int_0^L v(x,t) (\varphi(x) - a\varphi''(x) + b\varphi'''(x)) \, dx\]

and

\[\Phi(t) = \mathcal{B}(u(L,t), \varphi(L)) - \mathcal{B}(u(0,t), \varphi(0)).\]

Then the following theorem holds.

**Theorem 2.1.** Let \(u_0(x) \in L^1([0,L])\) and the solution \(u\) of the equation (2.1) is such that \(u \in C^{1,4}_{t,x}((0,L) \times (0,T))\) and let the function \(\varphi\) satisfy conditions (2.3), (2.4). If \(\Phi(t) - \theta_1 \geq 0\), for all \(t > 0\), and \(F(0) > 0\), then

\[F(t) \to +\infty \text{ for } t \to T^*,\]

where \(T^*\) satisfies estimate (1.17).
Proof. Obviously
\[ \partial_{t}^{\alpha} \tilde{F}(t) \geq \tilde{F}^{2}(t), \]
where \( \tilde{F}(t) = \theta F(t) \).

Since the function \( \tilde{F}(t) \) is an upper solution of equation (1.16), therefore \( \tilde{F}(t) \to +\infty \) for \( t \to T^{*} \), where \( T^{*} \) satisfies estimate (1.17). Whereupon \( F(t) \to +\infty \) for \( t \to T^{*} \). \( \Box \)

Note that the trial function method has great practical convenience.

Example 2.2. (Fractional Korteweg-de Vries equation). Consider the problem (2.1) with \( a = b = d = 0, c = 1 \) on the interval \([0,1]\) equipped with the boundary conditions:
\[
\begin{align*}
    u(0,t) &= 0, \quad t \geq 0, \\
    u(1,t) &= 0, \quad t \geq 0, \\
    u_x(1,t) &= u_x(0,t) + u_{xx}(1,t), \quad t \geq 0.
\end{align*}
\]
Letting \( \varphi(x) = x \), we obtain
\[
\begin{align*}
    \theta_1 &= 0, \quad \theta_2 := \frac{1}{6}
\end{align*}
\]
and
\[
\Phi(t) = \theta_1 = 0, \quad \text{for all } t > 0;
\]
hence it follows from Theorem 2.1 that the solution of problem (2.1) blows up in finite time under the condition
\[
\int_{0}^{1} u_0(x) x dx > 0.
\]

Example 2.3. (Fractional Burgers equation). Let \( a = b = c = 0, d > 0 \) in problem (2.1) on the interval \([0,1]\) and let the solution of problem (2.1) satisfy the Robin type nonlinear boundary conditions:
\[
\begin{align*}
    u(0,t) &= 0, \quad t \geq 0, \\
    du_x(1,t) - du(1,t) - u(1,t) - \frac{1}{2} u^2(1,t) &= 0, \quad t \geq 0.
\end{align*}
\]
Then, if \( \varphi(x) = x \), we obtain
\[
\begin{align*}
    \theta_1 &= 0, \quad \theta_2 := \frac{1}{6}
\end{align*}
\]
and
\[
\Phi(t) = \theta_1 = 0, \quad \text{for all } t > 0;
\]
hence it follows from Theorem 2.1 that the solution of problem (2.1) blows up in finite time under the condition
\[
\int_{0}^{1} u_0(x) x dx > 0.
\]
Example 2.4. (Fractional Benjamin-Bona-Mahony equation). With \( b = c = d = 0 \), \( a = 1 \), consider the problem (2.1) on the interval \([0,1]\) supplemented with final boundary conditions:

\[
\begin{align*}
    u(1, t) &= 0, \quad t \geq 0, \\
    u_x(1, t) &= 0, \quad t \geq 0.
\end{align*}
\]

Taking \( \varphi(x) = x^4 \), we obtain

\[
\begin{align*}
    \theta_1 &:= 0, \quad \theta_2 := \frac{395}{48} \\
    \Phi(t) &= 0, \quad \text{for all } t > 0;
\end{align*}
\]

hence it follows from Theorem 2.1 that the solution of problem (2.1) blows up in finite time under the condition

\[
\int_0^1 u_0(x)x^2(x^2 - 12)dx > 0.
\]

Example 2.5. (Fractional Rosenau equation). Let \( a = c = d = 0 \) and consider problem (2.1) on the interval \([0,1]\) with Dirichlet type boundary conditions

\[
\begin{align*}
    u(0, t) &= 0, \quad t \geq 0, \\
    u(1, t) &= 0, \quad t \geq 0, \\
    u_{xx}(1, t) &= 0, \quad t \geq 0, \\
    \partial_{+0,t}^\alpha u_{xxx}(0, t) - \partial_{+0,t}^\alpha u_{xx}(0, t) &= f(t), \quad t \geq 0.
\end{align*}
\]

Suppose that \( f(t) \geq \frac{1}{2} \), for all \( t > 0 \). Then, if \( \varphi(x) = x - 1 \), we obtain

\[
\begin{align*}
    \theta_1 &:= \frac{1}{2}, \quad \theta_2 := \frac{2}{3} \\
    \Phi(t) - \theta_1 &= f(t) - \frac{1}{2} \geq 0, \quad \text{for all } t > 0;
\end{align*}
\]

hence it follows from Theorem 2.1 that the solution of problem (2.1) blows up in finite time under the condition

\[
\int_0^1 u_0(x)(x - 1)dx > \frac{1}{2}.
\]

Example 2.6. (Fractional Rosenau-Burgers equation). Let \( a = c = 0 \) and consider problem (2.1) on the interval \([0,1]\) with nonlocal dynamical boundary conditions

\[
\begin{align*}
    u(1, t) &= 0, \quad t \geq 0, \\
    u_x(1, t) + u(0, t) &= 0, \quad t \geq 0, \\
    u_{xx}(0, t) &= 0, \quad t \geq 0, \\
    \partial_{+0,t}^\alpha u_{xxx}(1, t) - \partial_{+0,t}^\alpha u_{xx}(1, t) &= \frac{1}{2}, \quad t \geq 0.
\end{align*}
\]
Letting $\varphi(x) = x$, we obtain

$$
\theta_1 := \frac{1}{2}, \quad \theta_2 := \frac{2}{3}
$$

and

$$
\Phi(t) - \theta_1 = 0, \text{ for all } t > 0;
$$

hence it follows from Theorem 2.1 that the solution of problem (2.1) blows up in finite time under the condition

$$
\int_0^1 u_0(x) x dx > -\frac{1}{2}.
$$

### 3. Blowing-up solutions of the time-fractional Camassa-Holm–Degasperis-Procesi equation

In this section we consider the time-fractional Camassa-Holm–Degasperis-Procesi equation:

$$\begin{align*}
\partial_t^{\alpha_{+0,t}} (u - u_{xx}) + au_x + buu_x - cu_x u_{xx} - duu_{xxx} &= 0, \quad 0 < x < L, \ t > 0, \\
u(x,0) &= u_0(x), \quad x \in [0,L],
\end{align*}$$

(3.1)

where $a, b, c, d \in \mathbb{R}$ and $u_0$ is a given function.

The equation (3.1) is called the Camassa-Holm–Degasperis-Procesi equation with time-fractional derivative as it is a generalization of the following well-known equations:

- If $\alpha = 1$ and $a = 2\kappa > 0$, $b = 3$, $c = 2$, $d = 1$, then the equation (3.1) coincides with the classical Camassa-Holm equation (1.6);
- If $\alpha = 1$ and $a = 2\kappa > 0$, $b = 4$, $c = 3$, $d = 1$, then the equation (3.1) coincides with the classical Degasperis-Procesi equation (1.7).

We study the question of the blow-up of a classical solution $u \in C^{1,3}_{t,x}([0,T] \times [0,L])$ of problem (3.1). Let us consider a function $\varphi \in C^3([0,L])$ and suppose that the solution $u \in C^{1,3}_{t,x}([0,T] \times [0,L])$ of problem (3.1) exists. Using the equality

$$(u^2)_{xxx} = 6u_x u_{xx} + 2uu_{xxx},$$

we reduce the equation (3.1) to the equation

$$\partial_t^{\alpha_{+0,t}} (u - u_{xx}) + au_x + buu_x + (3d - c)u_x u_{xx} - \frac{d}{2} (u^2)_{xxx} = 0, \quad 0 < x < L, \ t > 0. \ (3.2)$$
Multiplying equation (3.1) by \(\varphi\) and integrating by parts, we obtain

\[
\frac{\partial^\alpha_{+0,t}}{\partial t^\alpha} \int_0^L u(x,t)(\varphi(x) - \varphi''(x))dx
= a \int_0^L u(x,t)\varphi'(x)dx + \frac{3d-c}{2} \int_0^L u_x^2(x,t)\varphi'(x)dx \\
+ \frac{1}{2} \int_0^L u^2(x,t)(b\varphi'(x) - d\varphi''(x))dx \\
+ B(u(L,t), \varphi(L)) - B(u(0,t), \varphi(0)),
\]

where

\[
B(u(x,t), \varphi(x)) = \frac{\partial^\alpha_{+0,t}}{\partial t^\alpha} u_x(x,t)\varphi(x) - \frac{\partial^\alpha_{+0,t}}{\partial t^\alpha} u(x,t)\varphi'(x) \\
- au(x,t)\varphi(x) - \frac{b}{2} u^2(x,t)\varphi(x) \\
- \frac{d-c}{2} u_x^2(x,t)\varphi(x) + du(x,t)u_{xx}(x,t)\varphi(x) \\
- du(x,t)u_x(x,t)\varphi'(x) + \frac{d}{2} u^2(x,t)\varphi''(x).
\]

Let \(3d-c \geq 0\) and the function \(\varphi(x)\) be monotonically nondecreasing:

\[
\frac{\partial^\alpha_{+0,t}}{\partial t^\alpha} u(x,t)(\varphi(x) - \varphi''(x))dx 
\geq a \int_0^L u(x,t)\varphi'(x)dx \\
+ \frac{1}{2} \int_0^L u^2(x,t)(b\varphi'(x) - d\varphi''(x))dx \\
+ B(u(L,t), \varphi(L)) - B(u(0,t), \varphi(0)).
\]

Let \(\varphi\) satisfy the following properties

\[
\varphi'(x) \geq 0 \text{ for } x \in [0, L],
\]

then from (3.3) we have

\[
\frac{\partial^\alpha_{+0,t}}{\partial t^\alpha} u(x,t)(\varphi(x) - \varphi''(x))dx 
\geq a \int_0^L u(x,t)\varphi'(x)dx \\
+ \frac{1}{2} \int_0^L u^2(x,t)(b\varphi'(x) - d\varphi''(x))dx \\
+ B(u(L,t), \varphi(L)) - B(u(0,t), \varphi(0)).
\]

Let \(\varphi\) satisfy the following properties

\[
\begin{align*}
\theta_1 &:= \frac{1}{2} \int_0^L \frac{a^2\varphi^2(x)}{b\varphi'(x) - d\varphi''(x)} dx < \infty; \\
\theta_2 &:= 2 \int_0^L \frac{(\varphi(x) - \varphi''(x))^2}{b\varphi'(x) - d\varphi''(x)} dx < \infty.
\end{align*}
\]
Then we have
\[
2a \int_0^L u(x,t)\varphi'(x)dx + \int_0^L u^2(x,t) (b\varphi'(x) - d\varphi''(x))dx \\
= \int_0^L v^2(x,t) (b\varphi'(x) - d\varphi''(x))dx - a^2 \int_0^L \varphi^2(x) b\varphi'(x) - d\varphi''(x)dx,
\]
where
\[
v(x,t) = u(x,t) + a \frac{\varphi'(x)}{b\varphi'(x) - d\varphi''(x)}.
\]
Using the Hölder inequality, we obtain the following estimate
\[
\left( \int_0^L v(x,t)(\varphi(x) - \varphi''(x))dx \right)^2 \leq \int_0^L v^2(x,t) (b\varphi'(x) - d\varphi''(x))dx \int_0^L \frac{(\varphi(x) - \varphi''(x))^2}{b\varphi'(x) - d\varphi''(x)}dx.
\]
Then, expression (3.3) takes the form
\[
\partial^{\alpha}_{+0,t} F(t) \geq \theta_2^{-1} F^2(t) + \Phi(t) - \theta_1,
\]
where
\[
F(t) = \int_0^L v(x,t) (\varphi(x) - \varphi''(x))dx
\]
and
\[
\Phi(t) = \mathcal{B}(u(L,t), \varphi(L)) - \mathcal{B}(u(0,t), \varphi(0)).
\]
Then the following theorem holds.

**Theorem 3.1.** Let \(u_0(x) \in L^1([0,L])\) and the solution \(u\) of the equation (3.1) is such that \(u \in C^{1,3}_{t,x}((0,L) \times (0,T))\) and let the function \(\varphi\) satisfy conditions (3.4), (3.6). If \(\Phi(t) - \theta_1 \geq 0\), for all \(t > 0\), and \(F(0) > 0\), then
\[
F(t) \to +\infty \text{ for } t \to T^*,
\]
where \(T^*\) satisfies estimate (1.17).

The Theorem 3.1 can be proved as Theorem 2.1.

Below we give some examples.

**Example 3.2.** (Fractional Camassa-Holm equation). Consider the problem (3.1) with \(a = 2\kappa > 0\), \(b = 3\), \(c = 2\), \(d = 1\), on the interval \([0,1]\) equipped with the dynamical boundary conditions:
\[
\begin{align*}
u(0,t) &= 0, \quad t \geq 0, \\
u(1,t) &= 0, \quad t \geq 0, \\
\partial^{\alpha}_{+0,t} u_x(1,t) + \frac{1}{2} u_x^2(1,t) &= f(t) \geq \frac{2\kappa^2}{3}, \quad t \geq 0.
\end{align*}
\]
Letting $\varphi(x) = x$, we obtain
$$\theta_1 := \frac{2\kappa^2}{3}, \quad \theta_2 := \frac{2}{9}$$
and
$$\Phi(t) - \theta_1 \geq 0, \text{ for all } t > 0;$$
hence it follows from Theorem 3.1 that the solution of problem (3.1) blows up in finite time under the condition
$$\int_0^1 u_0(x)xdx > -\frac{\kappa}{3}.$$

**Example 3.3.** (Fractional Degasperis-Procesi). Let $a = 2\kappa > 0$, $b = 4$, $c = 3$, $d = 1$, in problem (3.1) on the interval $[0, 1]$ and let the solution of problem (3.1) satisfy the nonlinear nonlocal boundary conditions:

$$u(1, t) = 0, \quad t \geq 0,$$
$$u_x(0, t) = 0, \quad t \geq 0,$$
$$\partial_{+0,t}^\alpha u_x(1, t) + \partial_{+0,t}^\alpha u(0, t) + u_x^2(1, t) = g(t) \geq \frac{\kappa^2}{2}, \quad t \geq 0.$$

Then, if $\varphi(x) = x$, we obtain
$$\theta_1 := \frac{\kappa^2}{2}, \quad \theta_2 := \frac{1}{6}$$
and
$$\Phi(t) - \theta_1 \geq 0, \text{ for all } t > 0;$$
hence it follows from Theorem 3.1 that the solution of problem (3.1) blows up in finite time under the condition
$$\int_0^1 u_0(x)xdx > -\frac{\kappa}{4}.$$

4. **Blowing-up solutions of the time-fractional Ostrovsky equation**

We consider the equation
$$\partial_{+0,t}^\alpha u_x + au_{xx} + bu_{xxxx} + (uu_x)_x = 0, \quad x \in (0, L), \quad t > 0,$$  
with Cauchy data
$$u(x, 0) = u_0(x), \quad x \in [0, L],$$
where $a, b \in \mathbb{R}$ and $u_0$ is a sufficiently smooth function.
Multiplying equation (4.1) by a function \( \varphi(x) \in C^4([0, L]) \) and integrating by parts, we obtain

\[
\partial_{t}^{\alpha} \int_{0}^{L} u(x, t) \varphi'(x) dx = \frac{1}{2} \int_{0}^{L} u^2(x, t) \varphi''(x) dx + \int_{0}^{L} u(x, t)(a \varphi''(x) + b \varphi'''(x)) dx + \mathcal{B}(u(x, t), \varphi(x)) \bigg|_{0}^{L},
\]

where

\[
\mathcal{B}(u(x, t), \varphi(x)) = \partial_{t}^{\alpha} u(x, t) \varphi(x) - au(x, t) \varphi(x) + au(x, t) \varphi'(x) - bu_{xxx}(x, t) \varphi(x) - bu_{xx}(x, t) \varphi'(x) + \frac{1}{2} u^2(x, t) \varphi'(x).
\]

Let the function \( \varphi(x) \) satisfy the properties:

\[
\varphi''(x) \geq 0 \text{ for } x \in [0, L]
\]

and

\[
\begin{align*}
\theta_1 := & \frac{1}{2} \int_{0}^{L} \frac{(a \varphi''(x) + b \varphi'''(x))^2}{\varphi''(x)} dx < \infty; \\
\theta_2 := & 2 \int_{0}^{L} \frac{\varphi^2(x)}{\varphi''(x)} dx < \infty.
\end{align*}
\]

Then we have

\[
2 \int_{0}^{L} u(x, t)(a \varphi''(x) + b \varphi'''(x)) dx + \int_{0}^{L} u^2(x, t) \varphi''(x) dx = \int_{0}^{L} v^2(x, t) \varphi''(x) dx - \int_{0}^{L} \frac{(a \varphi''(x) + b \varphi'''(x))^2}{\varphi''(x)} dx,
\]

where

\[
v(x, t) = u(x, t) + \frac{a \varphi''(x) + b \varphi'''(x)}{\varphi''(x)}.
\]

Using the Hölder inequality, we obtain the following estimate

\[
\left( \int_{0}^{L} v(x, t) \varphi'(x) dx \right)^2 \leq \int_{0}^{L} v^2(x, t) \varphi''(x) dx \int_{0}^{L} \frac{\varphi^2(x)}{\varphi''(x)} dx.
\]

Then, the expression (4.3) can be rewritten as

\[
\partial_{t}^{\alpha} F(t) \geq \theta_2^{-1} F^2(t) + \Phi(t) - \theta_1,
\]
where
\[ F(t) = \int_0^L v(x,t)\varphi'(x)dx \]
and
\[ \Phi(t) = B(u(L,t),\varphi(L)) - B(u(0,t),\varphi(0)). \]

Then the following theorem holds.

**Theorem 4.1.** Let \( u_0(x) \in L^1([0,L]) \) and the solution \( u \) of the problem (4.1), (4.2) is such that \( u \in C_{t,x}^{1,4}((0,L) \times (0,T)) \) and let the function \( \varphi \) satisfy conditions (4.4), (4.5). If \( \Phi(t) - \theta_1 \geq 0, \) for all \( t > 0, \) and \( F(0) > 0, \) then
\[ F(t) \to +\infty \text{ for } t \to T^*, \]
where \( T^* \) satisfies estimate (1.17).

The Theorem 4.1 can be proved as Theorem 2.1.

**Example 4.2.** With \( a = 1 \) and \( b = -1 \), consider problem (4.1), (4.2) on the interval \([0,1]\) subject to Dirichlet type boundary conditions
\[
\begin{align*}
  u(0, t) &= 0, \ t \geq 0, \\
  u(1, t) &= 0, \ t \geq 0, \\
  u_x(0, t) &= 0, \ t \geq 0, \\
  u_{xxx}(1, t) - 2u_{xx}(1, t) &= f(t), \ t \geq 0.
\end{align*}
\]

Suppose that \( f(t) \geq 1, \) for all \( t > 0. \) Then, if \( \varphi(x) = x^2, \) we obtain
\[ \theta_1 := 1, \ \theta_2 := \frac{4}{3} \]
and
\[ \Phi(t) - \theta_1 = f(t) - 1 \geq 0, \text{ for all } t > 0; \]
hence it follows by Theorem 4.1 that the solution of problem (4.1), (4.2) blows up in finite time under the condition
\[ \int_0^1 u_0(x)x^2dx > -\frac{1}{3}. \]

5. **Blowing-up solutions of the time-fractional modified KdV-Burgers equation**

Consider the initial value problem for the time-fractional analogue of the well-known modified Korteweg-de Vries-Burgers equation with dissipation:
\[
\begin{align*}
  \partial_{+\alpha,t}^\alpha u + u^2u_x + au_{xxx} - bu_{xx} &= 0, \ x \in (0,L), \ t > 0, \quad (5.1) \\
  u(x, 0) &= u_0(x), \ x \in [0,L], \quad (5.2)
\end{align*}
\]
where \( a, b \in \mathbb{R} \) and \( u_0 \) is a sufficiently smooth function.

Let a function \( \varphi \in C^3([0,L]) \) satisfy the properties:
\[ \varphi(x) \leq 0, \ \varphi'(x) \geq 0 \text{ for } x \in [0,L], \]
and
\[
3a\varphi'(x) + 2b\varphi(x) \leq 0 \text{ for } x \in [0, L],
\] (5.4)

\[
\begin{align*}
\theta_1 &:= 2 \int_0^L \frac{(a\varphi''(x) + b\varphi''(x))^2}{\varphi(x)} \, dx < \infty; \\
\theta_2 &:= \frac{1}{2} \int_0^L \frac{\varphi^2(x)}{\varphi'(x)} \, dx < \infty.
\end{align*}
\] (5.5)

Multiplying equation (5.1) by \( u(x,t)\varphi(x) \) and integrating by parts, we obtain
\[
\begin{align*}
\int_0^L \partial_{+0,t}^{\alpha} u(x,t)u(x,t)\varphi(x) \, dx &\quad = \frac{1}{4} \int_0^L u^4(x,t)\varphi'(x) \, dx + \frac{1}{2} \int_0^L u^2(x,t)(a\varphi'''(x) + b\varphi''(x)) \, dx \\
&\quad - \int_0^L u_x^2(x,t) \left( \frac{3a}{2} \varphi'(x) + b\varphi(x) \right) \, dx + \mathcal{B}(u(x,t), \varphi(x)) |_0^L,
\end{align*}
\] (5.6)

where
\[
\mathcal{B}(u(x,t), \varphi(x)) = -\frac{1}{4} u^4(x,t)\varphi(x) + u(x,t)u_x(x,t)(a\varphi'(x) + b\varphi(x)) \\
- \frac{u^2(x,t)}{2}(a\varphi''(x) + b\varphi'(x)) + \frac{a}{2} u_x^2(x,t)\varphi(x) \\
- au(x,t)u_{xx}(x,t)\varphi(x).
\]

Then we have
\[
\begin{align*}
\frac{1}{2} \int_0^L u^2(x,t)(a\varphi'''(x) + b\varphi''(x)) \, dx &\quad + \frac{1}{4} \int_0^L u^4(x,t)\varphi'(x) \, dx \\
&\quad = \frac{1}{4} \int_0^L v^4(x,t)\varphi'(x) \, dx - \frac{1}{4} \int_0^L \left( \frac{a\varphi'''(x) + b\varphi''(x)}{\varphi'(x)} \right)^2 \, dx,
\end{align*}
\]

where
\[
 v^2(x,t) = u^2(x,t) + \frac{a\varphi'''(x) + b\varphi''(x)}{\varphi'(x)}.
\]

Using the Hölder inequality and inequality (1.15), we obtain
\[
\left( \int_0^L v^2(x,t)\varphi(x) \, dx \right)^2 \leq \int_0^L v^4(x,t)\varphi'(x) \, dx \int_0^L \frac{\varphi^2(x)}{\varphi'(x)} \, dx,
\]
\[
\partial_{+0,t}^{\alpha} u(x,t)u(x,t)\varphi(x) \geq \frac{1}{2} \partial_{+0,t}^{\alpha} \left( u^2(x,t)\varphi(x) \right).
\]
From (5.4), we also get
\[- \int_0^L u_x^2(x, t) \left( \frac{3a}{2} \varphi'(x) + b \varphi(x) \right) dx \geq 0.\]

Then, expression (5.6) takes the form
\[\partial_{+0,t}^\alpha F(t) \geq \theta_2^{-1} F^2(t) + \Phi(t) - \theta_1, \quad (5.7)\]
where
\[F(t) = \int_0^L v^2(x, t) \varphi(x) dx\]
and
\[\Phi(t) = 2\mathcal{B}(u(L, t), \varphi(L)) - 2\mathcal{B}(u(0, t), \varphi(0)).\]

Then the following theorem holds.

**Theorem 5.1.** Let \(u_0(x) \in L^1([0, L])\) and the solution \(u\) of the equation (5.1) is such that \(u \in C_{t,x}^{1,3}((0, L) \times (0, T))\) and let the function \(\varphi\) satisfy conditions (5.3), (5.4) and (5.5). If \(\Phi(t) - \theta_1 \geq 0\), for all \(t > 0\), and \(F(0) > 0\), then
\[F(t) \to +\infty \text{ for } t \to T^*,\]
where \(T^*\) satisfies estimate (1.17).

**Proof.** Since \(\Phi(t) - \theta_1 \geq 0\), for all \(t > 0\), it follows from (5.7) that
\[\partial_{+0,t}^\alpha \tilde{F}(t) \geq \tilde{F}^2(t),\]
where \(\tilde{F}(t) = \theta_2 F(t)\).

As the function \(\tilde{F}(t)\) is an upper solution of equation (1.16), therefore \(\tilde{F}(t) \to +\infty\) for \(t \to T^*\), where \(T^*\) satisfies estimate (1.17). Whereupon \(F(t) \to +\infty\) for \(t \to T^*\). \(\square\)

**Example 5.2.** Consider problem (5.1) with \(a = 2\) and \(b = 3\) on the interval \([0, 1]\), supplemented with Dirichlet type boundary conditions
\[u(0, t) = 0, \ t \geq 0,\]
\[u(1, t) = 0, \ t \geq 0,\]
\[u_x(1, t) = \sqrt{e} u_x(0, t), \ t \geq 0,\]
where \(e = \exp(1)\) is Euler’s number. Then, if
\[\varphi(x) = -\exp(-x),\]
we obtain
\[\theta_1 := 0, \ \theta_2 := \frac{1 - e^{-1}}{2}\]
and
\[\Phi(t) = \theta_1 = 0, \text{ for all } t > 0;\]
hence it follows from Theorem 5.1 that the solution of problem (5.1) blows up in finite time under the condition
\[
\int_0^1 u_0^2(x) \exp(-x) dx < 1 - e^{-1}.
\]

Example 5.3. Let \(a = 0\) and \(b > 0\). Let in problem (5.1) on the interval \([0, 1]\) be given Dirichlet boundary conditions
\[
\begin{align*}
    u(0, t) &= 0, \quad t \geq 0, \\
    u(1, t) &= 0, \quad t \geq 0.
\end{align*}
\]
Then, if \(\varphi(x) = x - 1\), we obtain
\[
\begin{align*}
    \theta_1 &= 0, \quad \theta_2 := \frac{1}{6}
\end{align*}
\]
and
\[
\Phi(t) = \theta_1 = 0, \text{ for all } t > 0;
\]
hence it follows from Theorem 5.1 that the solution of problem (5.1) blows up in finite time under the condition
\[
\int_0^1 u_0^2(x)(x - 1) dx > 0,
\]

6. Maximum principle and gradient blow-up in time-fractional Burgers equation

The purpose of this section is to study time-fractional Burgers equation
\[
\partial_{+0,t}^\alpha u + uu_x = \nu u_{xx}, \quad x \in (0, L), \quad t > 0,
\]
with the initial condition
\[
\begin{align*}
    u(x, 0) &= u_0(x), \quad x \in [0, L],
\end{align*}
\]
where \(\nu > 0\) and \(u_0\) is a sufficiently smooth function.

6.1. Maximum principle. In this subsection, we present a maximum principle for the time-fractional Burgers equation (6.1).

Theorem 6.1. Let \(u(x, t)\) satisfy the time-fractional Burgers equation (6.1) with Cauchy data (6.2). Then
\[
u(x, t) \geq \min_{(x,t)} \{u(0, t), u(L, t), u_0(x)\} \text{ for } (x, t) \in [0, L] \times [0, T].
\]
Proof. Let
\[ m = \min_{(x,t)} \{ u(a,t), u(b,t), u_0(x) \} \]
and
\[ \tilde{u}(x,t) = u(x,t) - m. \]
Then, we have
\[ \tilde{u}(0,t) = u(0,t) - m \geq 0, \ t \in [0,T), \]
\[ \tilde{u}(L,t) = u(L,t) - m \geq 0, \ t \in [0,T), \]
and
\[ \tilde{u}(x,0) = u_0(x) - m \geq 0, \ x \in [0,L]. \]
Since
\[ \partial_{+0,t}^\alpha \tilde{u}(x,t) = \partial_{+0,t}^\alpha u(x,t) \]
and
\[ \tilde{u}_{xx}(x,t) = u_{xx}(x,t), \]
it follows that \( \tilde{u}(x,t) \) satisfies:
\[ \partial_{+0,t}^\alpha \tilde{u}(x,t) + \tilde{u}(x,t)\tilde{u}_x(x,t) + m\tilde{u}_x(x,t) = \nu \tilde{u}_{xx}(x,t), \]
and the initial condition
\[ \tilde{u}(x,0) = u_0(x) - m \geq 0, \ x \in [a,b]. \]
Suppose that there exits some \((x,t) \in [0,L] \times [0,T)\) such that \( \tilde{u}(x,t) \) is negative. Since
\[ \tilde{u}(x,t) \geq 0, \ (x,t) \in \{0\} \times [0,T] \cup \{L\} \times [0,T] \cup [0,L] \times \{0\}, \]
there is \((x_0,t_0) \in (0,L) \times (0,T)\) such that \( \tilde{u}(x_0,t_0) \) is the negative minimum of \( \tilde{u} \) over \((0,L) \times (0,T)\). It follows from Property 1.5 that \( \partial_{+0,t}^\alpha u(x_0,t_0) < 0. \)

Therefore at \((x_0,t_0)\), we get
\[ \partial_{+0,t}^\alpha \tilde{u}(x_0,t_0) < 0, \ \tilde{u}_x(x_0,t_0) = 0 \text{ and } \nu \tilde{u}_{xx}(x_0,t_0) \geq 0. \]
This contradiction shows that \( \tilde{u}(x,t) \geq 0, \) whereupon \( u(x,t) \geq m \) on \([0,L] \times [0,T]\).

A similar result can be obtained for a nonpositive solution \( u(x,t) \) by considering \(-u(x,t)\).

**Theorem 6.2.** Suppose that \( u(x,t) \) satisfies (6.1), (6.2). Then
\[ u(x,t) \leq \max_{(x,t)} \{ u(L,t), u(0,t), u(x,0) \}, \ (x,t) \in [0,L] \times [0,T]. \]
6.2. **Gradient blow-up.** Now suppose that the boundary conditions are set in such a way that the global in time solution of equation (6.1) is bounded. Let there exist a smooth bounded solution \( u \) such that \( |u(x,t)| \leq M \). Differentiating equation (6.1) with respect to \( x \), we obtain

\[
\partial^\alpha_{+0,t}u_x + uu_{xx} + u^2_x - \nu u_{xxx} = 0, \quad x \in (0, L), \quad t > 0.
\]

(6.3)

Substituting the expression for \( u_{xx} \) from (6.1) into (6.3), we obtain

\[
\partial^\alpha_{+0,t}u_x + \frac{1}{\nu}u\partial^\alpha_{+0,t}u + \frac{1}{\nu}u^2 u_x = \nu u_{xxx}, \quad x \in (0, L), \quad t > 0.
\]

(6.4)

Multiply the equation (6.4) by the function \( 0 \leq \varphi(x) \in C^3([0, L]) \) and integrate by parts over the domain \([0, L]\) to get

\[
\int_0^L \left( \partial^\alpha_{+0,t}u_x(x,t) + \frac{1}{\nu}u(x,t)\partial^\alpha_{+0,t}u(x,t) \right) \varphi(x)dx \\
= - \int_0^L \left( u^2(x,t) + \frac{1}{\nu}u^2(x,t)u_x(x,t) \right) \varphi(x)dx - \nu \int_0^L u(x,t)\varphi''(x)dx \\
+ \nu (u_{xx}(x,t)\varphi(x) - u_x(x,t)\varphi'(x) + u(x,t)\varphi''(x)).
\]

(6.5)

Denote \( v = -u_x - \frac{1}{2\nu}u^2 \). Then, using \( |u(x,t)| \leq M \) and inequality (1.15), we obtain

\[
\partial^\alpha_{+0,t} \int_0^L v(x,t)\varphi(x)dx \geq \int_0^L v^2(x,t)\varphi(x)dx \\
- \frac{M^4}{4\nu^2} \int_0^L \varphi(x)dx - M\nu \int_0^L |\varphi''(x)|dx \\
- \nu \left( u_{xx}(x,t)\varphi(x) - u_x(x,t)\varphi'(x) + u(x,t)\varphi''(x) \right)|_0^L.
\]

(6.6)

Let

\[
\theta_1 := \frac{M^4}{4\nu^2} \int_0^L \varphi(x)dx + M\nu \int_0^L |\varphi''(x)|dx < \infty,
\]

(6.7)

\[
\theta_2 := \int_0^L \varphi(x)dx < \infty.
\]

(6.8)

Using Hölder inequality, we can rewrite the expression (6.6) in the form

\[
\partial^\alpha_{+0,t} F(t) \geq \theta_2^{-1} F^2(t) + \Phi(t) - \theta_1,
\]

(6.9)

where

\[
F(t) = \int_0^L v(x,t)\varphi(x)dx
\]
and
\[ \Phi(t) = -\nu (u_{xx}(x, t)\varphi(x) - u_x(x, t)\varphi'(x) + u(x, t)\varphi''(x)) \bigg|_0^L. \]
Then the following theorem holds.

**Theorem 6.3.** Let \( u_0(x) \in C^1([0, L]) \) and the solution \( u \) of the equation (6.1) be such that \( u \in C^{1,3}_{t,x}((0, L) \times (0, T)) \) and let the function \( \varphi \) satisfy conditions (6.7) and (6.8). If \( \Phi(t) - \theta_1 \geq 0 \), for all \( t > 0 \), and

\[ F(0) = -\int_0^L \left( u_0'(x) + \frac{1}{2\nu} u_0^2(x) \right) \varphi(x) dx > 0, \]

then
\[ F(t) \rightarrow +\infty \text{ for } t \rightarrow T^*, \]
where \( T^* \) satisfies estimate (1.17).

We do not provide the proof this theorem as it runs parallel to that of Theorem 5.1.

**Conclusion**

In this article, we have studied blowing-up solutions to some time-fractional non-linear partial differential equations. In precise terms, we have obtained the following results:

- Blowing-up solutions of the time-fractional Rosenau-KdV-BBM-Burgers equation with initial conditions;
- Blowing-up solutions of the time-fractional Camassa-Holm–Degasperis-Procesi equation with initial conditions;
- Blowing-up solutions of the time-fractional Ostrovsky equation with Cauchy data;
- Blowing-up solutions of the initial problem for the time-fractional analogue of the modified Korteweg-de Vries-Burgers equation with dissipation;
- Maximum principle and gradient blow-up in time-fractional Burgers equation.

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