Determining the surface–density distribution in massive galactic disks with a central black hole

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An efficient method is developed which allows the calculation of distributions of the surface density in the equilibrium disk configurations with an isolated point mass at the center corresponding to known distributions of the angular velocity. We demonstrate the existence of the upper limit for the ratio ‘mass of the galactic disk / mass of the central black hole’ which essentially depends on the form of rotation–curves.

Key words: galaxy disks, rotation curve, black holes

I. INTRODUCTION

The super–massive black holes at the nuclei of galaxies and quasars are able to provide a high luminosity of these objects thanks to the disk accretion (Lynden–Bell 1969). Nowadays there exist serious arguments in favor of that hypothesis for a number of disk–like galaxies (Rees 1998; for the discussion of the observational data providing evidence about the existence of compact nuclei in spirals see Rubin and Graham 1987; Sofue 1996; Ratnam and Salucci 2000), so the problem of finding the mass–density distributions and total masses of flattened galaxies possessing a central black hole from the known distributions of the matter rotation–velocity is of great astrophysical interest. The rotation–curves are normally constructed from the measurements of the Doppler effect for the lines 21cm, \(^{12}H\alpha\), etc. (see, e.g., Schmidt 1957; de Vaucouleurs 1959; Burbidge et al. 1960; Rubin et al. 1980; Carignan and Freeman 1985; Sofue and Rubin 2001; Persic and Salucci 1988). The mathematical problem of reconstruction of the surface mass–density from the rotation–curves in infinite disks without a central body was discussed in the papers by Burbidge et al. 1960; Brandt 1960; Brandt and Belton 1962; Toomre 1963; Binney and Tremaine 1987 in relation with the problem of non-radiating mass in the disk–like galaxies. Mestel 1963 proposed models of flattened galaxies as disks of finite radius with constant angular or linear rotation velocities.

Another important physical aspect of the problem (we do not touch it in this paper) is related to the formation of the exterior massive parts of accretion disks in which the attraction to the disk because of its mass becomes greater or of the order of the vertical component of the attraction force from the part of the black hole. In this case the exterior part of the accretion disk considerably swells, and the proper field of the disk may play a substantial role in the thermal balance, distribution of pressure, etc. (Paczyński 1978; Kozlowsky et al. 1979, Kolykhalov and Sunyaev 1980). For millions years of accretion the mass of the accretion disk can reach 0.001 the mass of the black hole due to transition of the angular momentum into the exterior part of the accretion disk (Kolykhalov and Sunyaev 1980).

In our paper the mass of the accretion disk around a black hole is neglected, and we only take into account the proper gravitational field of the galactic disk. In the latter we neglect the influence of viscosity, pressure and all kind of non–circular movements of matter, as it is done in the classical articles (Schmidt 1957; Burbidge et al. 1960; Brandt 1960) because of a small relative dispersion of velocities. It should be emphasized that the gravitational field of dark matter and that of the non–disk component (bulge) exert the same influence on the rotation–curves as the galactic radiating disk itself. The phenomenological expressions for these potentials are given in the book of Binney and Tremaine 1987 and in the paper by Lovelace et al. 1999, but some researchers are rather sceptic about the methods of separation of those fields (Burstein and Rubin 1985; Persic and Salucci 1988).

The aim of our paper is to solve a new Newtonian potential–theory problem for the gravitational field of a disk extending at some distance from the central point mass. The rotation–curve in the disk is supposed to be known, and it is necessary to find the corresponding mass of the central object and the distribution of the surface mass–density in the disk. We give the general solution of this problem in terms of two successive quadratures (20), (21) and formulas (19), (23) for the masses of the black hole and disk, respectively. As an illustration of the application of general formulas we derive a large class of the asymptotically Keplerian angular–velocity distributions and demonstrate a

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\(^1\)Mention that the disks with an isolated central mass cannot be obtained as limiting cases of confocal spheroidal shells of small eccentricity, unlike the continuous disks previously considered.
strong dependence of the upper limit of the masses of disks on the choice of a class of rotation–curves. In spite of a number of strong assumptions (neglect of a non–disk component, extrapolation of data about the circular motion of matter into the outer parts of galactic disks where there is no radiating gas) our formulation can be considered as a possible approach to the solution of the problem of a hidden mass in the galactic disks with a black hole at the center.

II. THE METHOD

Recall that in the axisymmetric case a thin galactic infinite disk with an attracting central body of mass \( M \) can be described by the following Newtonian potential (in cylindrical coordinates \( \rho, z \))

\[
\phi(\rho, z) = 2G \int_0^\infty \! \int_a^\infty \frac{\sigma(\rho_0)\rho_0 d\rho_0 d\varphi}{\sqrt{\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos \varphi + z^2}} + \frac{GM}{r}, \quad r = \sqrt{\rho^2 + z^2},
\]  

(1)

where \( G \) is Newton’s gravitational constant. The matter of the disk is distributed in the plane \( z = 0 \) in the region exterior to the circumference of radius \( a \). From the condition of balance of the gravitational and centrifugal forces one gets

\[
\omega^2 \rho + \frac{\partial \phi}{\partial \rho} \bigg|_{z \to 0} = 0
\]

(2)

(here we do not consider other forces like pressure of the gas and the radiation pressure).

Substituting the potential (1) into Eq. (2), one easily arrives at the integral equation with the kernel divergent at \( \rho = \rho_0 \):

\[
G \int_a^\infty \sigma(\rho_0)K(\rho, \rho_0) d\rho_0 = \omega^2 \rho - \frac{GM}{\rho^2},
\]

(3)

where

\[
\frac{\rho}{2\rho_0} K(\rho, \rho_0) = \frac{\text{sign}(\rho - \rho_0) E(\tau)}{\rho + \rho_0} + \frac{K(\tau)}{|\rho - \rho_0|}, \quad \tau = -\frac{4\rho_0 \rho}{(\rho - \rho_0)^2}.
\]

(4)

\( E(\tau) \) and \( K(\tau) \) being the complete elliptic integrals.

Eq. (3) is a complicated singular integral equation of the non–Fredholm type. Even the problem of finding \( \omega \) for a given surface density \( \sigma \) would meet difficulties during its numerical resolution. However, from the point of view of the observational astronomy, it is likely to solve the inverse problem, i.e., to determine the mass distribution \( \sigma \) for a known \( \omega \) since from observations it is difficult to establish the matter distribution in the galactic disk because of the problem of a hidden (non–radiating) mass, whereas the rotation–curves are able to provide information about Keplerian frequencies (Schmidt 1957; Burbidge et al. 1960; Brandt 1960; Binney and Tremaine 1987) thanks to the combined attracting field. Viewed in this way (finding \( \sigma \) from \( \omega \)), the straightforward resolution of Eq. (3) looks hopeless even by numerical means.

To circumvent the need of solving the cumbersome Eq. (3), instead of (1) we shall take \( \phi \) in the form

\[
\phi(\rho) = \frac{1}{2\pi} \int_0^\pi \! d\theta \int_a^\infty \ln[(s - \rho \cos \theta)^2 + z^2] \alpha(s) ds + \frac{GM}{r},
\]

(5)

where the real function \( \alpha(s) \) has the meaning of the density of sources distribution.

This potential satisfies Laplace’s equation everywhere except in the plane \( z = 0 \), the sources of the disk being distributed outside the circle \( \rho < a \). The main advantage of such a representation over (1) is that it allows to

\[\text{In practical approaches the surface density is normally reconstructed with the aid of the photometric data (Freeman 1970; for a critical discussion of this approach see Burstein and Rubin 1985; Persic et al. 1996).}\]
introduce into the problem under consideration the powerful theory of analytic functions in the complex plane since the first term in (5) can be cast into the form \[ \int_{\pi}^{0} f(z + i\rho \cos \theta) \, d\theta. \]

Let us show that

\[ \frac{\partial \phi}{\partial z} \bigg|_{z \to +0} = \begin{cases} 
0, & 0 < \rho < a \\
\frac{1}{\pi} \int_{a}^{\rho} \frac{\alpha(s \sigma_{0}) \, ds_{0}}{\sqrt{\rho^{2} - s_{0}^{2}}}, & a \leq \rho. 
\end{cases} \quad (6) \]

Indeed, from (5) we have

\[ \frac{\partial \phi}{\partial z} = \frac{1}{2\pi^{2}} \int_{0}^{\pi} d\theta \int_{a}^{\infty} \left( \frac{-i}{s - \rho \cos \theta - i\rho} + \frac{i}{s - \rho \cos \theta + i\rho} \right) \alpha(s) \, ds - \frac{GM}{r^{3}} z. \quad (7) \]

Tending \( z \to +0 \) for \( \rho \cos \theta = s_{0} > a \) and using the Sokhotsky–Plemelj formula for the Cauchy-type integrals, namely

\[ \lim_{Z \to s_{0}} \int_{L} \frac{\alpha(s) \, ds}{s - Z} = \int_{L} \frac{\alpha(s) \, ds}{s - Z} \pm \pi i \alpha(s_{0}), \quad (8) \]

we get (in our case \( Z = s_{0} \pm iz \))

\[ \frac{\partial \phi}{\partial z} \bigg|_{z \to +0} = \begin{cases} 
0, & 0 < s_{0} < a \\
\frac{1}{\pi} \int_{a}^{\rho} \alpha(s \sigma_{0}) \, ds, & s_{0} \geq a. 
\end{cases} \quad (9) \]

From (9) the formula (6) follows immediately.

Let us consider the dependence of \( \phi \) on \( \rho \) when \( \rho > a \) and \( z \to 0 \). From (8) we obtain

\[ \frac{\partial \phi}{\partial \rho} \bigg|_{z \to 0} = -\frac{GM}{\rho^{2}} + \frac{1}{\pi \rho} \left( \int_{a}^{\infty} \alpha(s) \, ds - \frac{1}{\pi} \int_{a}^{\rho} \alpha(s) \, ds \, ds \, ds \right. \]

\[ \quad \left. \int_{-\rho}^{\rho} \frac{d\rho'}{(s - \rho') \sqrt{\rho^{2} - \rho'^{2}}} \right). \quad (10) \]

Using now the formula

\[ \frac{1}{\pi} \int_{-\rho}^{\rho} \frac{d\rho'}{(s - \rho') \sqrt{\rho^{2} - \rho'^{2}}} = \begin{cases} 
0, & a < s < \rho \\
\frac{1}{\sqrt{s^{2} - \rho^{2}}}, & s > \rho. 
\end{cases} \quad (11) \]

Eq. (10) can be rewritten in the form

\[ \frac{\partial \phi}{\partial \rho} \bigg|_{z \to 0} = -\frac{GM}{\rho^{2}} + \frac{1}{\pi \rho} \left( \int_{a}^{\infty} \alpha(s) \, ds - \int_{\rho}^{\infty} \alpha(s) \, ds \, ds \right. \]

\[ \quad \left. \int_{\rho}^{a} \frac{\alpha(s) \, ds}{\sqrt{s^{2} - \rho^{2}}} \right). \quad (12) \]

Let us show that for a finite mass of the disk, the integral \( \int_{a}^{\infty} \alpha(s) \, ds \) should vanish. Indeed, from (6) we have for \( \rho \geq a \)

\[ -4\pi G \sigma(\rho) = \left[ \frac{\partial \phi}{\partial z} \right]_{z \to 0} = \frac{\partial \phi}{\partial z} \bigg|_{z \to +0} = \frac{2}{\pi} \int_{a}^{\rho} \frac{\alpha(s \sigma_{0}) \, ds_{0}}{\sqrt{\rho^{2} - s_{0}^{2}}}. \quad (13) \]

hence the mass of the disk is equal to

\[ ^{3} \text{Here } s_{0} \text{ is a point on a smooth contour } \mathcal{L}: \ s_{0} \in \mathcal{L}; \ Z \to +s_{0} \text{ means that the point } Z \text{ is approaching the point } s_{0} \text{ from the left side in the sense of a positive running of the curve } \mathcal{L}; \text{ the symbol } \int \text{ denotes the principal value of the respective integral.} \]
\[
M_d = -\frac{1}{4\pi G} \int_0^{2\pi} \int_0^\infty \left( \frac{\partial \phi}{\partial z} \right) \rho \, d\rho = -\frac{1}{G} \int_0^\infty m(\rho) \, d\rho, \quad m(\rho) = \frac{1}{\pi} \int_0^\rho \frac{\alpha(s_0) \rho \, ds_0}{\sqrt{\rho^2 - s_0^2}}. \tag{14}
\]

In order the mass of the disk be finite, it is necessary that
\[
\lim_{\rho \to \infty} m(\rho) = 0 \Rightarrow \int_0^\infty \alpha(s) \, ds = 0. \tag{15}
\]

Substituting now (12) into the equilibrium condition (2), we obtain
\[
\omega^2 \rho - \frac{GM}{\rho^2} = \frac{1}{\pi} \rho \int_0^\rho \alpha(s) \, ds. \tag{16}
\]

After introducing the new variables
\[
t = \frac{a^2}{s^2}, \quad x = \frac{a^2}{\rho^2},
\]
the last equation reduces to the Abel integral equation
\[
\frac{a^2}{2\pi} \int_0^x \frac{\alpha(t) \, dt}{t^{3/2} \sqrt{x-t}} = \frac{a^3 \omega^2}{x^{3/2}} - GM, \tag{17}
\]
and the well-known formula for the solution of this equation yields (see, e.g., Sneddon 1956, p. 318)
\[
\frac{a^2 \alpha(x)}{2x^{3/2}} = \frac{d}{dx} \int_0^x \frac{\omega^2 \, dt}{t^{3/2} \sqrt{x-t}} \left( \frac{a^3 \omega^2}{t^{3/2}} - GM \right). \tag{18}
\]

The substitution of (18) into the condition (15) yields the formula for the mass of the central body
\[
M = \frac{a^3}{2G} \int_0^1 \frac{\omega^2 \, dt}{t^{3/2} \sqrt{1-t}}, \tag{19}
\]
hence the function \( \alpha(x) \) can be expressed through only the known distribution of the angular velocity:
\[
\frac{\alpha(x)}{2ax^{3/2}} = \frac{d}{dx} \int_0^x \frac{\omega^2 \, dt}{t^{3/2} \sqrt{x-t}} - \frac{1}{2\sqrt{x}} \int_0^1 \frac{\omega^2 \, dt}{t^{3/2} \sqrt{1-t}}. \tag{20}
\]

The corresponding distribution of the surface density, obtainable from (13), assumes the form
\[
-4\pi G \sigma(x) = \frac{\sqrt{x}}{\pi} \int_0^1 \frac{\alpha(t) \, dt}{t \sqrt{1-t}}. \tag{21}
\]

From (21), (20) follows that the asymptotics of the surface density near the inner rim of the disk \( \rho = a \) is given by the formula
\[
-4\pi^2 G \sigma(x) \approx A \sqrt{1-x}, \quad A = a \int_0^1 \left[ \left( \frac{\omega^2}{t^{3/2}} \right)_{t=0} - \frac{1}{2} \left( \frac{\omega^2}{t^{3/2}} - \frac{G(M + M_d)}{a^3} \right) \right] \frac{dt}{\sqrt{1-t}}. \tag{22}
\]

It can be seen from (22) that in the presence of a central body possessing a finite mass the surface density near the inner rim of the disk has no finite limit when \( a \) tends to zero. The inequality \( A \leq 0 \) imposes a physical restriction on the angular–velocity distribution.

Therefore, we have finally succeeded in expressing the mass–density distribution in the disk exclusively through the supposedly known distribution of the angular velocity via two quadratures which can be easily taken numerically in the general case, and analytically for large classes of particular distributions.
The total mass of the disk, $M_d = 2\pi \int_a^\infty \rho \sigma d\rho$, is obtainable from (18) and (21):

$$GM_d = \lim_{t \to 0} \frac{a^2 \omega(t)}{2t} = \lim_{t \to 0} \frac{a^3 \omega^2}{t^{3/2}} - GM.$$ (23)

To give an analytical illustration of the results obtained, let us consider the following smooth distribution of the angular velocity possessing the Keplerian asymptotics:

$$\omega^2 \rho^3 = a_0 + a_2 \frac{\omega^2}{\rho^2} + a_3 \frac{\omega^3}{\rho^3} + \ldots + a_n \frac{\omega^n}{\rho^n} \equiv a_0 + \sum_{k=2}^n a_k t^{k/2},$$ (24)

where $a_k$ are some constant coefficients.

Substituting this expression into (18) we arrive at

$$\frac{\alpha(x)}{x^{3/2}} = \frac{2}{\sqrt{x}} (a_0 - GM) + \sum_{k=2}^n (k+1)x^{(k-1)/2} B\left(\frac{1}{2}, 1 + \frac{k}{2}\right) a_k,$$ (25)

$B(x, y)$ denoting the Euler beta–function.

From the condition (19) we obtain the mass of the black hole:

$$2GM = 2a_0 + \sum_{k=2}^n B\left(\frac{1}{2}, 1 + \frac{k}{2}\right) a_k.$$ (26)

Substituting the expression (25) into formula (21) one finds

$$-4\pi^2 G \frac{\sigma(x)}{\sqrt{x}} = 4(a_0 - GM) \sqrt{1 - x} + \sum_{k=2}^n (k+1) B\left(\frac{1}{2}, 1 + \frac{k}{2}\right) a_k J_k, \quad J_k \equiv \int_x^1 \frac{t^{k/2} dt}{\sqrt{t-x}}.$$ (27)

Using integration by parts, for the integrals $J_k$ it is easy to obtain the recurrent formula

$$(k+1) J_k = 2 \sqrt{1-x} + k x J_{k-2},$$ (28)

the use of which in (27) gives the explicit expression for the surface density:

$$\sigma(x) = -\frac{x^{3/2} \sqrt{1-x}}{2\pi^2 G} \sum_{k=2}^n B\left(\frac{1}{2}, 1 + \frac{k}{2}\right) a_k \left\{ \frac{k}{k-1} + x \frac{k(k-2)}{(k-1)(k-3)} + \ldots + x^{[k/2]-1} \right\} .$$ (29)

The inequality $A \leq 0$ (cf. (22)) imposes the following restriction on the parameters $a_k$:

$$\sum_{k=2}^n a_k \left[ \frac{k}{2} B\left(\frac{1}{2}, \frac{k}{2}\right) - \frac{1}{2} B\left(\frac{1}{2}, 1 + \frac{k}{2}\right) \right] \leq 0.$$ (30)

Mention that when the function $\omega^2 \rho^3$ is representable by a polynomial in inverse powers of $\rho^2$, $a_{2k+1} = 0, k = 1, 2, \ldots, N$, then the surface density is given by the formula

$$\sigma(x) = \frac{x^{3/2} \sqrt{1-x}}{\pi^2 a^2 G} \sum_{s=1}^N \beta_s x^{s-1},$$

$$\beta_s = \sum_{k=s}^N \sum_{m=0}^s \frac{2^k (k!)^2 (-1)^{m+1} a_{2k}}{(2k-1)! m! (k-s)! (s-m)! (2k+2m-2s+1)}.$$ (31)
III. SOME EXAMPLES

(A) Let us consider the case when the rotation–curves belong to the family of curves with Keplerian asymptotics of the type

\[ \omega^2 \rho^3 = GM + GM_d (1 + b_1 x + b_2 x^2 + b_3 x^3), \quad a_{2i} = GM_d b_i, \quad x = a^2 / \rho^2. \]  

(32)

From the formula (26) follows that

\[ b_1 = -\frac{3}{2} - \frac{4}{5} b_2 - \frac{24}{35} b_3. \]  

(33)

Taking into account (33), one obtains from (31) the distribution of the surface density corresponding to (30):

\[ \sigma(\rho) = \frac{a M_d \sqrt{1 - a^2 / \rho^2}}{\rho^3} \left[ 2 + \frac{16}{45} b_2 + \frac{64}{175} b_3 - \left( \frac{64}{45} b_2 + \frac{128}{175} b_3 \right) x - \frac{256}{175} b_3 x^2 \right]. \]  

(34)

Formulas (32)–(34) fully describe the two–parameter family of the angular–velocity distribution and the corresponding distribution of the surface density. In Fig. 1 we have plotted the region \( D \) of the plane \((b_2, b_3)\) on which the right–hand side of (34) is a positive quantity, so that (32) has physical sense. The boundary of the region \( D \) consists (i) of two line segments \( AB \) and \( AC \) tangent to the ellipse at the points \( B \) and \( C \), their defining equations are \( 1 + \frac{8}{3} b_2 + \frac{175}{175} b_3 = 0 \) and \( 1 - \frac{2}{5} b_2 - \frac{35}{35} b_3 = 0 \) (ii) of the part of the ellipse between the points \( B \) and \( C \); the equation of the ellipse results by setting to zero the discriminant of the quadratic polynomial at the right–hand side of (34).

Let us call \( \mu(b_2, b_3) \) the minimal value of the cubic polynomial \( f(x) \):

\[ f(x) \equiv 1 - \left( \frac{3}{2} + \frac{4}{5} b_2 + \frac{24}{35} b_3 \right) x + b_2 x^2 + b_3 x^3 \]  

(35)

on the interval \((0,1]\). Then the maximally possible mass of the disk can be calculated from the condition of the non–negativeness of the right–hand side of (32):

\[ M_{d, \text{max}}(b_2, b_3) = -M / \mu(b_2, b_3). \]  

(36)

The polynomial \( f(x) \) can assume its minimal value either inside of the interval \((0,1]\) or at the boundary \( x = 1 \). In Fig. 1 the dashed line separates the region \( D \) into two parts \( D_1 \) and \( D_2 \). Inside of \( D_1 \), where \( f(1) \) is the minimal value of \( f(x) \) on the interval \((0,1]\), the function \( \mu(b_2, b_3) \) has the simple form \( \mu = -\frac{1}{2} + \frac{1}{5} b_2 + \frac{1}{35} b_3 \).

In Fig. 2 the curves \( M_{d, \text{max}} \) are shown as functions of \( b_2 \) for different fixed values of \( b_3 \), the points \((b_2, b_3)\) belonging to \( D \). A simple investigation shows that the maximally possible mass of the disk can reach \( 9M \). It is interesting to point out that the more complicated structure has the angular–velocity distribution in the disk (i.e. the greater is \( n \) in the polynomial (31)), the larger mass the disk can have in principle: for the one–parameter family of the angular–velocity distribution the maximal mass of the disk is equal to \( 5M \), and for the two–parameter family the maximal value elevates up to \( 9M \).

Let us consider three particular cases of the distribution (32).

1) The simplest particular case, \( b_2 = b_3 = 0 \), was found by Lemos and Letelier 1994\footnote{The inequality \( A \leq 0 \) is equivalent to the condition \( 1 - \frac{8}{15} b_2 - \frac{32}{35} b_3 \geq 0 \).}. In this case

\[ \omega^2 \rho^3 = GM + GM_d \left( 1 - \frac{3}{2} \frac{a^2}{\rho^2} \right), \quad \sigma(\rho) = \frac{2a M_d \sqrt{\rho^2 - a^2}}{\pi^2 \rho^4}. \]  

(37)

From (35) and (36) follows that the mass of the disk cannot exceed two masses of the central body.

2) The particular case \( b_2 = 15/8, b_3 = 0 \) corresponds to the distribution of the angular velocity and surface density of the form

\[ \omega^2 \rho^3 = GM + GM_d \left( 1 - \frac{3}{2} \frac{a^2}{\rho^2} + \frac{15}{8} \frac{a^4}{\rho^4} \right), \quad \sigma(\rho) = \frac{8a M_d (\rho^2 - a^2)^{3/2}}{3\pi^2 \rho^5}. \]  

(38)

\footnote{See the discussion of the corresponding gravitational potential in the paper by Semerák and Žáček 2000.}
and the mass of the disk does not exceed $5M$.

3) The choice $b_2 = 45/8$, $b_3 = -35/16$ gives

$$\omega^2 \rho^3 = GM + GM_d \left(1 - \frac{9 a^2}{2 \rho^2} + \frac{45 a^4}{8 \rho^4} - \frac{35 a^6}{16 \rho^6}\right), \quad \sigma(\rho) = \frac{16 a M_d}{5 \pi^2 \rho^3} \left(1 - \frac{a^2}{\rho^2}\right)^{5/2},$$

and the mass of the disk cannot be greater than $6.69M$.

(B) To demonstrate that the mass of the galactic disk can in principle be arbitrarily large compared to the mass of the central black hole, let us consider the following rotation–curves

$$\omega^2 (\rho^2 + c^2)^{3/2} = G(M + M_d), \quad c = \text{const}.$$ (40)

In this case the equation (23) is fulfilled automatically. Let us denote the parameter $c^2/a^2$ as $b$.

Calculating $\alpha(x)$ according to formula (20), we get

$$\frac{\alpha(x)}{2ax^{3/2}} = -\frac{b(-1 + (3 + b)x + bx^2)}{\sqrt{x}(1 + b)(bx + 1)^2} \frac{G(M + M_d)}{a^3}.$$ (41)

Substituting the expression (41) into (21), we obtain the formula for the distribution of the surface density:

$$\sigma(x) = \frac{(M + M_d)\sqrt{bx^3}}{\pi^2 a^2} \left(\frac{\sqrt{b(1 - x)}}{(1 + b)(1 + bx)} + \frac{\text{Arctan} \sqrt{b(1 - x)/(1 + bx)}}{(1 + bx)^{3/2}}\right).$$ (42)

Then from (19) follows a surprisingly simple formula for the mass of the disk:

$$M_d = bM.$$ (43)

The parameter $b$ in formulas (41)–(43) can be an arbitrary positive number. Therefore, the mass of the galactic disk can be by far greater than the mass of the black hole (if the linear velocity $V = \rho \omega$ achieves its maximum at the distances much larger than the accretion part of the disk $b \gg 1$). Mention that when $a \to 0$, the variable $x$ tends to zero too, while the parameter $b$ tends to infinity. From (43) then follows that $M \to 0$, and from formula (42) we obtain Kuzmin–Toomre’s result for the surface–density distribution (40):

$$\sigma(\rho) = \frac{c M_d}{2\pi (\rho^2 + c^2)^{3/2}}.$$ (44)

IV. CONCLUSION

Therefore, we have succeeded in solving the problem of the determination of the surface density in massive galactic disks with a central massive black hole from the known rotation–curves. The problem is reduced to finding two successive quadratures (20) and (21) which can be easily calculated by numerical means in the most general case of smooth distributions of the angular velocity. Besides, we have shown how the knowledge of a rotation–curve permits to calculate the mass of the central body and that of the galactic disk.

The analysis of the formulas obtained gives rise to the following two general observations.

(i) The mass of the disk around a central body in each case should be less than some upper limiting value which depends on the degree of concentration of matter at the inner boundary of the disk. The mass of the disk strongly depends on the form of the rotation–curve and in principle can exceed the mass of the black holes many times.

(ii) For an arbitrary non–degenerated smooth distribution of the angular velocity in the disk, the surface density at the inner boundary tends to zero like $\sqrt{1 - a^2 \rho^{-2}}$. It corresponds to the parabolic profile of thickness of the disk for a finite value of the volume density at the inner boundary of the disk. At infinity the surface density decreases as $\rho^{-3}$.

Lastly, it should be remarked that the method developed in this paper is not restricted to only the case of infinite massive disks with attractive central body, but also permits to treat the cases of finite disks and systems of rings. These last cases are in the stage of analysis and preparation now, and will become a subject of future publications.

7The rotation–curves (40) were first considered by Kuzmin 1956; they enter as a particular case into the class of curves possessing the Keplerian asymptotics at $\rho \to \infty$ proposed by Brandt 1960 for infinite disks without a black hole at the center.
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FIG. 1. The region of positiveness of $\sigma(\rho)$ in the plane $(b_2, b_3)$. 
FIG. 2. The dependence of $M_{d_{\text{max}}}/M$ on $b_{2}$ for different fixed values of $b_{3}$ ($b_{3} = -21, -17, -14, -12, ..., -1, 0, 1, ..., 9$).