Explicit Solutions of Integrable Variable-coefficient Cylindrical Toda Equations

Ting Su¹, Jia Wang² and Quan Zhen Huang³*

¹College of Science, Henan University of Engineering, China.
²School of Mathematics and Statistics, Zhengzhou University, Zhengzhou, China.
³School of Electrical Information Engineering, Henan University of Engineering, China.

Authors’ contributions

This work was carried out in collaboration among all authors. Each author has done his or her share of the work in the manuscript. Author TS presented the operators $M_1, M_2$ and constructed $N_1, N_2$ and the solution formula. Author TS was a major contributor in writing the manuscript. Author JW was responsible for the derivation of equations. Author QZH derived one soliton solution and two soliton solution. All authors read and approved the final manuscript.

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Abstract

Integrable cylindrical Toda lattice equations are proposed by utilizing a generalized version of the dressing method. A compatibility condition is given which insures that these equations are integrable. Further, soliton solutions for new type equations are shown in explicit forms, including one soliton solution and two soliton solutions, respectively.

Keywords: Cylindrical Toda equation; the generalized dressing method; two soliton solutions.

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*Corresponding author: E-mail: huangquanzhen666@126.com, huang2004_susu@126.com;
Abbreviations

KDV: Korteweg-Devries
NLS: Nonlinear Schrödinger
1D: 1 Dimensional
IST: Inverse scattering transform
GLM: Gelfand-Levitan-Marchenko

1 Introduction

It is well known that there are lots of methods to establish the equations and derive their explicit solutions. Recently, numerical solutions of equations have been hot topic. There are a lot of researchers to study the topic by using different methods [1-10]. Dressing method [11,12] is presented by Zaharov and Shabat. Later, most scholars studied and developed the method in [13,14,15]. However, Dai and Jeffery [16,17] authors extended the dressing method [11,12] to obtain integrable variable-coefficient equations. The generalization provided a procedure for construction of integrable variable-coefficient equations and gave their explicit solutions. By utilizing the generalization, we have studied integrable variable-coefficient coupled Hirota equation in [18]. In [19,20], integrable variable-coefficient Manakov model and cylindrical NLS equation are discussed in detailed, respectively. In [21], we developed the generalized dressing method to discrete system and integrable coefficient Toda equation is researched.

The well-known 1D Toda lattice equation

\[ x_{n,tt} + e^{x_{n} - x_{n+1}} - e^{x_{n+1} - x_{n}} = 0, \tag{1} \]

models the interaction of a chain of particles with the nearest neighbors via exponential potential in a nonlinear one-dimensional crystal [22]. A number of authors investigated the Toda lattice equation (1). In [23], the scattering operator associated with (1) first presented by Toda. Flaschka [24] and Manakov [25] developed the inverse scattering transform (IST) scheme for (1) and obtained its N-soliton solutions. Its algebro-geometric solution is derived by Tanaka and Date in [26]. Manakov[27] and Flaschka [24] proved its integrability for initial conditions rapidly decreasing at infinity. In the present work, we shall use the generalized dressing method developed in [21] to produce this type, which takes the form

\[ \Delta e^{w_{n} - w_{n+1}} - \frac{1}{r} w_{n,r} - w_{n,rr} = 0, \tag{2} \]

We shall also construct its one soliton and two soliton solutions of the above interesting integrable variable-coefficient Toda equation.

The outline of the letter is as follows. In section 2, a new integrable 2D variable-coefficient Toda lattice equation is obtained and its reductions are cylindrical Toda equations. In section 3, the exact solutions of the obtained Toda equations are given, including one soliton solution and two soliton solutions.

2 Integrable Cylindrical Toda Equations

In this section, we extend the generalized version of the dressing method to discrete systems and derive different integrable cylindrical Toda equations by choosing different operators.
First, we consider three linear difference operators [21]

\[
F(n, m, x, y)\psi_n = \sum_{m=-\infty}^{\infty} F(n, m, x, y)\psi_m,
\]

\[
K_-(n, m, x, y)\psi_n = \sum_{m=-\infty}^{\infty} K_-(n, m, x, y)\psi_m, \quad (3)
\]

\[
K_+(n, m, x, y)\psi_n = \sum_{m=-\infty}^{\infty} K_+(n, m, x, y)\psi_m.
\]

Similar to the generalized dressing method application to continuous system, we introduce the triangular factorization about the operator \(F^n\)

\[
I + F = (I + K_+)^{-1}(I + K_-), \quad (4)
\]

where \(I\) is the identity operator, \(K_+(n, m, x, y) = 0\) for \(m < n\) and \(K_-(n, m, x, y) = 0\) for \(m > n\). It is assumed that

\[
\sup_{m=n_0} \sum_{m=-\infty}^{\infty} |K_+(n, m, x, y)|\psi_m < \infty, \quad \sup_{m=n_0} \sum_{m=-\infty}^{\infty} |F(n, m, x, y)|\psi_m < \infty,
\]

for all \(n_0 > -\infty\). For convenience, we denote \(F(n, m, x, y) = F(n, m), K_+(n, m, x, y) = K_+(n, m)\).

The discrete Gelfand-Levitan-Marchenko (GLM) equation can be obtained from (4), which reads [11]

\[
F(n, m) + K_+(n, m) + \sum_{s=m}^{\infty} K_+(n, s)F(s, m) = 0, \quad m > n. \quad (5)
\]

In what follows, we introduce two differential-difference operators \(P_1\) and \(P_2\) defined by

\[
P_1 = a_1\partial_x + a_2\partial_y + a_3E, \quad (6)
\]

\[
P_2 = b_1\partial_x + b_2\partial_y + b_3E^{-1}, \quad (7)
\]

where \(E\) is the shift operator of the discrete variable \(n\), defined by \(E^k f(n) = f(n + k), k \in \mathbb{Z}, x\) and \(y\) are continuous variables, \(a_i, b_i (i = 1, 2, 3)\) are functions of \(x\) and \(y\).

The dressing operators \(Q_1\) and \(Q_2\) can be derived from the relations

\[
Q_1(I + K_+(n, m)) - (I + K_+(n, m))P_1 = 0, \quad (8)
\]

\[
Q_2(I + K_+(n, m)) - (I + K_+(n, m))P_2 = 0. \quad (9)
\]

Similar to a theorem [21], it can be proved that \(Q_1\) and \(Q_2\) are differential-difference operators. For convenience, we denote \(K_+(n, m) = K(n, m)\).

We write

\[
Q_1 = P_1 + D_1, \quad (10)
\]

\[
Q_2 = P_2 + d_1E^{-1}. \quad (11)
\]

Acting on function \(\psi_n\) for (8) and with aid of (10), after some calculations, we find that

\[
P_1K(n, m)\psi_n + D_1K(n, m)\psi_n - D_1\psi_n - K(n, m)P_1\psi_n
\]

\[
= a_1 \sum_{m=n}^{\infty} K_x(n, m)\psi_m + a_2 \sum_{m=n}^{\infty} K_y(n, m)\psi_m + a_3 \sum_{m=n+1}^{\infty} K(n + 1, m)\psi_m
\]

\[
+ D_1 \sum_{m=n}^{\infty} K(n, m)\psi_m + D_1\psi_n - a_3 \sum_{m=n}^{\infty} K(n, m - 1)\psi_m,
\]
from which, comparing coefficient of \( \psi_n \), we have
\[
a_1 K_x(n, n) + a_2 K_y(n, n) + D_1 K(n, n) + D_1 - a_3 K(n, n - 1) = 0,
\]
thus, we obtain
\[
D_1 = \frac{a_3 K(n, n - 1) - a_1 K_x(n, n) - a_2 K_y(n, n)}{1 + K(n - 1, n - 1)}.
\]
Similarly, acting on function \( \psi_n \) on (9) and with aid of (11), the following equation is read
\[
P_2 K(n, m) \psi_n + d_1 E^{-1} \psi_n + d_1 E^{-1} K(n, m) \psi_n - K(n, m) P_2 \psi_n = 0,
\]
that is
\[
(b_1 \partial_x + b_2 \partial_y + b_3 E^{-1}) \sum_{m=n}^{\infty} K(n, m) \psi_m + d_1 \psi_{n-1}
\]
\[
+ d_1 E^{-1} \sum_{m=n}^{\infty} K(n, m) \psi_m - \sum_{m=n}^{\infty} K(n, m) K(n, n)(b_1 \partial_x + b_2 \partial_y + b_3 E^{-1}) \psi_m
\]
\[
= b_1 \sum_{m=n}^{\infty} K_x(n, m) \psi_m + b_2 \sum_{m=n}^{\infty} K_y(n, m) \psi_m + b_3 \sum_{m=n-1}^{\infty} K(n - 1, m) \psi_m
\]
\[
+ d_1 \psi_{n-1} + d_1 \sum_{m=n-1}^{\infty} K(n - 1, m) \psi_m - b_3 \sum_{m=n}^{\infty} K(n, m) \psi_{m-1} = 0,
\]
comparing the coefficient of \( \psi_{n-1} \), we obtain that
\[
b_3 K(n - 1, n - 1) + d_1 K(n - 1, n - 1) + d_1 - b_3 K(n, n) = 0,
\]
from which, we derive that
\[
d_1 = b_3 K(n, n) - K(n - 1, n - 1) \\
1 + K(n - 1, n - 1)
\]
\[
(15)
\]
The following theorem in [21] is an extension of original dressing method, which can yield a wide range of integrable variable-coefficient nonlinear evolution equations.

**Theorem:** If the operators \( P_1 \) and \( P_2 \) satisfy a relation
\[
[P_1, P_2] = \rho_1 P_1 + \rho_2 P_2,
\]
where \( \rho_1(y, x) \) and \( \rho_2(y, x) \) are arbitrary functions, then their corresponding dressing operators will satisfy the relation
\[
[Q_1, Q_2] = \rho_1 Q_1 + \rho_2 Q_2.
\]
Actually, variable-coefficient Toda equations are obtained from (17).

In fact, from (16), \( a_i, b_i (i = 1, 2, 3), \rho_1 \) and \( \rho_2 \) satisfy the relations
\[
-b_1 a_{1x} - b_2 a_{1y} = \rho_1 a_1, \quad a_1 b_{1x} + a_2 b_{1y} = \rho_2 b_1,
\]
\[
-b_1 a_{2x} - b_2 a_{2y} = \rho_1 a_2, \quad a_1 b_{2x} + a_2 b_{2y} = \rho_2 b_2,
\]
\[
-b_1 a_{3x} - b_2 a_{3y} = \rho_1 a_3, \quad a_1 b_{3x} + a_2 b_{3y} = \rho_2 b_3.
\]
Acting function \( \psi_n \) on (17), which is reduced to
\[
[a_1 d_{1;x} + a_2 d_{1;y} + (b_3 + d_1)(D_1 - E^{-1} D_1)] \psi_{n-1}
\]
\[
+ (a_3 \Delta d_1 - b_1 D_{1;x} - b_2 D_{1,y}) \psi_n = \rho_1 D_1 \psi_n + \rho_2 d_{1} \psi_{n-1},
\]
comparing the coefficient $\psi_{n-1}$ and $\psi_n$, we have
\begin{align}
a_1d_{1,x} + a_2d_{1,y} + (1 - E^{-1})D_1 &= \rho_2d_1, \\
a_2\Delta d_1 - b_1D_{1,x} - b_2D_{1,y} &= \rho_1D_1.
\end{align}
(19)

Let
\begin{equation}
u_n - 1 = \frac{K(n, n) - K(n - 1, n - 1)}{1 + K(n - 1, n - 1)}, \quad D_1 = v_n.
\end{equation}
(20)

Utilizing (18) and (20), (19) turns out to be
\begin{align}a_1u_{n,x} + a_2u_{n,y} + u_n(v_n - v_{n-1}) &= 0, \\
a_3b_3\Delta u_n - b_1v_{n,x} - b_2v_{n,y} &= \rho_1v_n.
\end{align}
(21)

Assuming that
\begin{equation}u_n = e^{-n-1}-w_n, \quad v_n = a_1w_{n,x} + a_2w_{n,y},
\end{equation}
(22)

thus, we obtain integrable two dimensional variable-coefficient Toda equation
\begin{equation}a_3b_3(e^{-n-1}-w_{n+1}) - (b_1a_2 + b_2a_1)w_{n,xy} - (b_1a_1w_{n,xx} + b_2a_2w_{n,yy}) = 0.
\end{equation}
(23)

In what follows, we consider two special cases:

Case 1: $b_1 = ib_2, a_1 = -ia_2, s = \sqrt{x^2 + y^2}$, Eq.(23) is reduced to the following integrable variable-coefficient cylindrical Toda equation
\begin{align}e^{-n-1}-w_{n+1} - e^{-n-1}-w_n &= \frac{a_1b_1}{a_3b_3}(w_{n,xx} + w_{n,yy}) = 0, \\
(e^{-n-1}-w_{n+1} - e^{-n-1}-w_n) &= \frac{a_1b_1}{a_3b_3}(w_{n,xx} + \frac{1}{s}w_{n,s}) = 0.
\end{align}
(24) (25)

Eq.(24) is discussed by Nakamura in [28] for $a_1b_1 = a_3b_3 = constant$, which has spherical Bessel function solution and ripple solution under the transformation.

Case 2: $a_1b_1 = a_2b_2, r = \sqrt{x^2 - y^2}$, Eq.(23) is reduced to the following integrable variable-coefficient cylindrical Toda equation
\begin{equation}a_3b_3(e^{-n-1}-w_{n+1} - e^{-n-1}-w_n) - (b_1a_2 + b_2a_1)w_{n,xy} - a_1b_1(w_{n,rr} + \frac{1}{r}w_{n,r}) = 0.
\end{equation}
(26)

In what follows, we shall research the solutions of the above derived equations.

3 The Explicit Solutions of Integrable Cylindrical Toda Equations

Suppose that the operator $F$ commute with $P_1$ and $P_2$, i.e.
\begin{align}[P_1, F] &= P_1F - FP_1 = 0, \\
[P_2, F] &= P_2F - FP_2 = 0.
\end{align}
(27) (28)

From (6) and (27), we have an equation for $F$:
\begin{equation}a_1F_x(n, m) + a_2F_y(n, m) + a_3F(n + 1, m) - a_3F(n, m - 1) = 0.
\end{equation}
(29)
Similarly, from (7) and (28), we obtain another equation for $F$:

$$b_1 F_x(n,m) + b_2 F_y(n,m) + b_3 F(n-1,m) - F(n,m+1)b_3 = 0. \quad (30)$$

In what follows, we consider the solution of separation variable for (23), furthermore, we obtain the solution of (24-26). For the $N$-soliton solution of the integrable equation (23), we let $F$ be

$$F(n,m) = \sum_{j=1}^{N} f_j(x,y,n)g_j(x,y,m), \quad (31)$$

where $f_j(x,y,n)$ and $g_j(x,y,m)$ are some $l \times l$ matrices. Moreover, we suppose that

$$K(n,m) = \sum_{j=1}^{N} k_j(x,y,n)g_j(x,y,m). \quad (32)$$

Substituting (31) and (32) into discrete GLM equation (5) gives

$$K(n,n) = \sum_{j=1}^{N} k_j(x,y,n)g_j(x,y,n) = -(f_1, f_2, \ldots, f_N)L^{-1}(g_1, g_2, \ldots, g_N)^T, \quad (33)$$

where $L$ is defined by

$$L_{jl} = \delta_{jl} + \sum_{s=n}^{\infty} g_j(x,y,s)f_l(x,y,s)g_j(x,y,s), \quad 1 \leq j,l \leq N.$$

The $N$-soliton solution for the integrable 2D variable-coefficient Toda equation (23) can be obtained from (33).

**Case 1. one-soliton solution**

We have $N = 1$ in (31) and (33). We assume that

$$F(n,m) = f_1g_1 = e^{p(x)+q(y)+\mu_1n+\mu_2m}, \quad K(n,m) = k_1g_1 = k_1e^{\mu_2m}. \quad (34)$$

Substituting (34) into (29-30), we have

$$q = \int \frac{b_1a_3(e^{\mu_1} - e^{-\mu_2}) + a_1b_3(e^{\mu_2} - e^{-\mu_1})}{b_1a_2 - a_1b_2}dy,$$

$$p = \int \frac{b_2a_3(e^{\mu_1} - e^{-\mu_2}) + a_2b_3(e^{\mu_2} - e^{-\mu_1})}{b_2a_1 - a_2b_1}dx.$$

$$u_n = \frac{(1 - e^{\mu_1+\mu_2} + e^{p(x)+q(y)+\mu_1n}(1 - e^{\mu_1+\mu_2} + e^{p(x)+q(y)+\mu_1n+\mu_2})^2)}{(1 - e^{\mu_1+\mu_2} + e^{p(x)+q(y)+\mu_1n+\mu_2})^2}, \quad (35)$$

by using (20).

Under transformation $u_n = e^{w_{n-1} - w_n}$, we derive the solution of (23)

$$w_n = w_0 - \ln(u_1u_2\ldots u_n). \quad (36)$$

Through transformation, we may obtain the explicit solutions of (24-26).
Case 2. two-soliton solution

We have \( N = 2 \) in (31), from (33), then

\[
K(n, n) = -\frac{1}{|L|} [g_1(f_1 L_{22} - f_2 L_{21}) + g_2(f_2 L_{11} - f_1 L_{12})],
\]

with

\[
L_{11} = 1 + \sum_{s=n}^{\infty} g_1(s)f_1(s), \quad L_{12} = \sum_{s=n}^{\infty} g_1(s)f_2(s), \quad L_{21} = \sum_{s=n}^{\infty} g_2(s)f_1(s), \quad L_{22} = 1 + \sum_{s=n}^{\infty} g_2(s)f_2(s),
\]

\(|L| = L_{11} L_{22} - L_{12} L_{21}.\)

We assume that

\[
f_1(n) = e^{p_1(x) + q_1(y) + \mu_1 n + \eta_0}, \quad g_1 = e^{\mu_2 n},
\]

\[
f_2(n) = e^{p_2(x) + q_2(y) + \mu_3 n + \eta_0}, \quad g_2 = e^{\mu_4 n},
\]

from which, we have

\[
L_{11} = 1 + \frac{e^{p_1(x) + q_1(y) + (\mu_1 + \mu_2)n + \eta_0}}{1 - e^{\mu_1 + \mu_2}}, \quad L_{12} = \frac{e^{p_2(x) + q_2(y) + (\mu_2 + \mu_3)n + \eta_0}}{1 - e^{\mu_2 + \mu_3}},
\]

\[
L_{21} = \frac{e^{p_1(x) + q_1(y) + (\mu_1 + \mu_4)n + \eta_0}}{1 - e^{\mu_1 + \mu_4}}, \quad L_{22} = 1 + \frac{e^{p_2(x) + q_2(y) + (\mu_3 + \mu_4)n + \eta_0}}{1 - e^{\mu_3 + \mu_4}}.
\]

with

\[|L| = (1 + \frac{e^{p_1(x) + q_1(y) + (\mu_1 + \mu_2)n + \eta_0}}{1 - e^{\mu_1 + \mu_2}})(1 + \frac{e^{p_2(x) + q_2(y) + (\mu_2 + \mu_3)n + \eta_0}}{1 - e^{\mu_2 + \mu_3}})\]

\[-\frac{e^{p_1(x) + q_1(y) + (\mu_1 + \mu_4)n + \eta_0}}{1 - e^{\mu_1 + \mu_4}} - \frac{e^{p_2(x) + q_2(y) + (\mu_3 + \mu_4)n + \eta_0}}{1 - e^{\mu_3 + \mu_4}}.\]

From (37), we derive

\[
1 + K(n, n) = 1 - \frac{1}{|L|} [e^{p_1(x) + q_1(y) + (\mu_1 + \mu_2)n + \eta_0} + e^{p_2(x) + q_2(y) + (\mu_2 + \mu_3)n + \eta_0} + e^{p_1(x) + q_1(y) + (\mu_1 + \mu_4)n + \eta_0} + e^{p_2(x) + q_2(y) + (\mu_3 + \mu_4)n + \eta_0} + \sum_{i=1}^{4} \mu_i n + \eta_0 + \eta_0]
\]

\[
+ e^{p_1(x) + q_1(y) + p_2(x) + q_2(y) + \sum_{i=1}^{4} \mu_i n + \eta_0 + \eta_0} \left( \frac{1}{1 - e^{\mu_3 + \mu_4}} - \frac{1}{1 - e^{\mu_1 + \mu_3}} + \frac{1}{1 - e^{\mu_1 + \mu_4}} - \frac{1}{1 - e^{\mu_2 + \mu_4}} \right),
\]

where

\[
q_1 = \int \frac{b_1 a_3 (e^{\mu_1} - e^{-\mu_2}) + a_1 b_3 (e^{\mu_2} - e^{-\mu_1})}{b_1 a_2 - a_1 b_2} dy,
\]

\[
p_1 = \int \frac{b_3 a_2 (e^{\mu_1} - e^{-\mu_2}) + a_3 b_2 (e^{\mu_2} - e^{-\mu_1})}{b_2 a_1 - a_2 b_1} dx,
\]

\[
q_2 = \int \frac{b_1 a_3 (e^{\mu_3} - e^{-\mu_4}) + a_1 b_3 (e^{\mu_4} - e^{-\mu_3})}{b_1 a_2 - a_1 b_2} dy,
\]

\[
p_2 = \int \frac{b_3 a_2 (e^{\mu_3} - e^{-\mu_4}) + a_3 b_2 (e^{\mu_4} - e^{-\mu_3})}{b_2 a_1 - a_2 b_1} dx.
\]

According to \( u_n = \frac{1 + K(n, n)}{1 - K(n, n)} \) and (35), we have two soliton solution of (23). Further, two soliton solutions of eqs.(24-26) will be given.
Availability of Data and Material

All the data and material during the study are available from the corresponding author.

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Competing Interests

Authors have declared that no competing interests exist.

References

[1] Shah K, Ali A, Khan RA. Numerical solutions of fractional order system of bagley-torvik equation using operational matrices. Evaluation and Assessment in Educational Information Technology; 2015.
[2] Kamal Shah, Rahmat Ali Khan. The applications of natural transform to the analytical solutions of some fractional order ordinary differential Equations. Sindh University Research Journal (Science Series). 2015;47(4):683-686.
[3] Shah, Kamal, Khalil, Hammad, Khan, Rahmat Ali. Analytical solutions of fractional order diffusion equations by natural transform method. Iranian Journal of Science and Technology Transactions A Science. 2016;42(3):1479-1490.
[4] Deconinck B, Trogdon T, Yang X. Numerical inverse scattering for the sine-Gordon equation. Physica D. 2019;399:159-172.
[5] Amjad Ali, Kamal Shah, Rahmat Ali Khan. Numerical treatment for traveling wave solutions of fractional Whitham-Broer-Kaup equations. Alexandria Engineering Journal. 2018;57(3):1991-1998.
[6] Sajjad Ali, Samia Bushnaq, Kamal Shah. Numerical treatment of fractional order Cauchy reaction diffusion equations. Chaos, Solitons and Fractals. 2017;103:578-587.
[7] Kamal S, Mohammad A. Numerical treatment of non-integer order partial differential equations by omitting discretization of data. Computational and Applied Mathematics. 2018;37(5):6700-6718.
[8] Shah K, Wang JR. A numerical scheme based on non-discretization of data for boundary value problems of fractional order differential equations. RACSAM. 2018;2018:1-18.
[9] Shah K, Wang JR. A numerical scheme based on non-discretization of data for boundary value problems of fractional order differential equations. Revista de la Real Academia de Ciencias Exactas. Fisicas y Naturales. Serie A. Matematicas; 2018.
[10] Trogdon T, Olver S, Deconinck B. Numerical inverse scattering for the Korteweg-Cde Vries and modified Korteweg-Cde Vries equations. Physica D Nonlinear Phenomena. 2012;241(11):1003-1025.
[11] Zakharov E, Shabat AB. A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem I. Funt. Anal.Appl. 1974;8:226-238.

[12] Zakharov E, Shabat AB. A scheme for integrating the nonlinear equations of mathematical physics by the method of the inverse scattering problem II. Funt. Anal. Appl. 1979;8:13-16.

[13] Chowdhury AR, Basak S. On the complete solution of the Hirota-Satsuma system through the ’dressing’ operator technique. J. Phys. A. 1984;17(16)L863.

[14] Dye JM, Parker A. An inverse scattering scheme for the regularized long-wave equation. J. Math. Phys. 2000;41(5):2889.

[15] Parker A. A reformulation of the ‘dressing method’ for the Sawada-Kotera equation. Inverse Problems. 2001;17:885-895.

[16] Dai HH, Jeffery A. The inverse scattering transforms for certain types of Variable coefficient KdV equations. Phys. Lett. A. 1989;139:369.

[17] Jeffery A, Dai HH. On the application of a generalized version of the dressing method to the integration of variable- coefficient KdV equation. Rendiconti. Di Mathematica, Serie VII. 1990;10:439.

[18] Su T, Dai HH, Geng XG. On the application of a generalized dressing method to the integration of variable-coefficient coupled Hirota equations. J. Math. Phys. 2009;50:226.

[19] Su T, Dai HH, Geng XG. A variable-coefficient manakov model and its explicit solutions through the generalized dressing method. Chin. Phys. Letts. 2013;30:060201.

[20] Su T, Ding GH, Fang JY. Integrable variable-coefficient coupled cylindrical NLS equations and their explicit solutions. Acta Mathematicae Applicatae Sinica, English Series. 2014;30(4):1017-1024.

[21] Dai HH, Su T. The generalized dressing method with applications to the integration of variable-coefficient Toda equations. Proceedings of the Estonian Academy of Sciences. 2010;59:293-298.

[22] Toda M. Theory of nonlinear lattices, 2nd. Springer, Berlin; 1989.

[23] Toda M. Variational of a chain with nonlinear interaction. J. Phys. Soc. Jpn. 1967;22:431.

[24] Flaschka H. The toda lattice. I: Existence of integrals. Phys. Rev. 1974;B9:1924.

[25] Manakov SV. Complete integrability and stochastization of discrete dynamical systems. Sov. Phys. JETP. 1975;40:269.

[26] Date E, Tanaka S. Analogue of inverse scattering theory for the discrete Hill’s equation and exact solutions for the periodic Toda lattice. Prog. Theoret. Phys. 1976;56:457-465.

[27] Manakov SV. Complete integrability and stochastization of discrete dynamical systems. Zhurnal Ekperimentalnoi I Teoreticheskoi Fiziki. 1974;67(2):269-274.

[28] Nakamura A. Exact bessel type solution of the two-dimensional toda lattice equation. J. Phys. Soc. Jpn. 1983;52(2):380-387.

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