Non-monotone Submodular Maximization with Nearly Optimal Adaptivity Complexity

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Abstract

As a generalization of many classic problems in combinatorial optimization, submodular optimization has found a wide range of applications in machine learning (e.g., in feature engineering and active learning). For many large-scale optimization problems, we are often concerned with the adaptivity complexity of an algorithm, which quantifies the number of sequential rounds where polynomially-many independent function evaluations can be executed in parallel. While low adaptivity is ideal, it is not sufficient for a distributed algorithm to be efficient, since in many practical applications of submodular optimization the number of function evaluations becomes prohibitively expensive. Motivated by such applications, we study the adaptivity and query complexity of non-monotone submodular optimization. We provide the first constant approximation algorithm for maximizing a non-monotone submodular function with cardinality constraint \(k\) that has nearly-optimal\(^1\) adaptivity complexity \(O(\log(n))\). Furthermore, our algorithm makes only \(O(\log(k))\) calls per element to the function evaluation oracle in expectation.

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\(^1\)By nearly-optimal, we mean up to a \(O(\log \log(n))\) factor.
1 Introduction

Submodular functions have the natural property of diminishing returns, making them prominent in applied fields such as machine learning and data mining. There has been a surge in applying submodular optimization for data summarization [TIWB14, SSS07, SSSJ12], recommendation systems [EAG11], and feature selection for learning models [DK08, KED+17], to name a few applications. There are also numerous recent works that focus on maximizing submodular functions from a theoretical perspective. Depending on the setting where the submodular maximization algorithms are applied, new challenges emerge and hence more practical algorithms have been designed to solve the problem in distributed [MKSK13, MZ15, BENW15a], streaming [BMKK14], and robust [MKK17, MBNF+17, KZK18] optimization frameworks. Most of the existing work assumes access to an oracle that evaluates the submodular function. However, function evaluations (oracle queries) can take a long time to process—for example, the value of a set depends on interactions with the entire input like in Exemplar-based Clustering [DF07] or when the function is computationally hard to evaluate like the log-determinant of sub-matrices [KZK18]. Although distributed algorithms for submodular maximization partition the input into smaller pieces to overcome these problems, each distributed machine may run a sequential algorithm that must wait for the answers of its past queries before making its next query. This motivates the study of the adaptivity complexity of submodular maximization, introduced by Balkanski and Singer [BS18] to study the number of rounds needed to interact with the oracle. As long as we can ask polynomially-many queries in parallel, we can ask them altogether in one round of interaction with the oracle.

To further motivate this adaptive optimization framework, note that in a wide range of machine learning optimization problems, the objective function can only be computed with oracle access to the function. In certain settings, the computation of the oracle is a time-consuming optimization problem that is treated as a black box (e.g., parameter tuning). In these settings, it is desirable to optimize a function with minimal number of rounds of interactions with the oracle. For an example of submodular optimization in this setting, consider the feature selection problem [DK08, KED+17], a critical problem for improving the model accuracy of machine learning models. The accuracy of a machine learning model does not necessarily have a closed-form formula, and in some settings may be computed only by re-evaluating the model with several multi-parameter tuning. It is known that for certain machine learning models, the accuracy of the model satisfies submodularity [DK08, KED+17]. In these cases, we have black-box access to the model accuracy function, which is time-consuming to compute. However, computing the model accuracy for many feature settings can be done in parallel. The adaptive optimization framework [BS18] is a realistic model for these kinds of distributed problems, and the insights from lower bounds and algorithmic techniques developed in this framework have a deep impact on distributed computing for machine learning applications in practice. For more motivations on the importance of round complexity of adaptive optimization framework, we refer the reader to [BS18].

While the number of rounds is an important measure to optimize, the complexity of answering oracle queries also motivates the design of algorithms that are efficient in terms of the total number of oracle queries. Typically we need to make at least a constant number of queries per element in the ground set to have a constant-approximation guarantee. We explore the fundamental question of how many queries per element are needed to achieve optimal approximation guarantees without compromising the minimum number of adaptive rounds. In this paper, we present a constant-factor approximation algorithm for non-monotone submodular maximization with cardinality constraint $k$ that makes $O(\log(k))$ oracle queries per element.
1.1 Our Results: Algorithms and Techniques

Our main result is a simple distributed algorithm for maximizing a non-monotone submodular function with cardinality constraint $k$ that achieves an expected $0.031$-approximation in $O(\log(n) / \varepsilon)$ adaptive rounds and makes $O(n \log(k) / \varepsilon^2)$ queries to the function evaluation oracle in expectation. To the best of our knowledge, this is the first constant approximation algorithm with nearly-optimal adaptive complexity $O(\log(n))$ for the more general problem of maximizing non-monotone submodular functions. We note that due to the known lower bounds in [BS18, MBK+15], the adaptivity complexity of our algorithm is optimal up to $O(\log \log(n))$ factor.

The main building block of our result is the algorithm Threshold-Sampling from [FMZ18] (re-stated in Section 3), which returns a subset of high-valued items from the ground set in $O(\log(n) / \varepsilon)$ adaptive rounds. Specifically, it iteratively adds random subsets of candidate elements to its current solution such that the expected marginal value of each added element is almost as high as a given input threshold. We modify this idea by only adding elements when they are selected from a large enough pool of candidate elements. This ensures that the probability of adding any particular element to the set of selected elements is bounded away from 1, which has recently been shown to be a crucial property for obtaining constant-factor approximations for non-monotone submodular functions [BFNS14]. Our adaptation of Threshold-Sampling guarantees that upon termination only a limited number of unselected items have marginal gain above the threshold. At this point, we run an Unconstrained-Maximization algorithm on this small subset of remaining elements. Unconstrained-Maximization is a simple adaptation of an existing constant-approximation algorithm for unconstrained non-monotone submodular maximization that runs in one adaptive round and also provides a constant-factor approximation for the unconstrained problem. To choose the right input threshold, we bound the range of possible thresholds, try logarithmically many of them, and finally return the best solution over all thresholds.

1.2 Further Related Work

The problem of optimizing query complexity for maximizing a submodular function subject to cardinality constraints has been studied extensively. In fact, for monotone submodular functions a linear-time $(1 - 1/e - \varepsilon)$-approximation algorithm called STOCHASTIC-GREEDY was recently developed [MBK+15]. In this paper, we achieve almost optimal query complexity combined with the nearly-optimal adaptive round complexity of $O(\log(n))$ for non-monotone submodular maximization with cardinality constraint. The applications of efficient algorithms for submodular maximization are widespread due to its numerous applications across machine learning and data mining for very large datasets. Recently, submodular maximization in the streaming and distributed settings has attracted a significant amount of attention [LMSV11, KMVV15, MKSK13, BMKK14, MZ15, BENW15a, BENW15b]. However, the distributed MapReduce and adaptive optimization models differ in that the latter framework does not allow for adaptivity inside each round, while previously studied distributed models allow for adaptivity and completely sequential algorithms for part of the data on the same machine. To demonstrate the fundamental difference between these models, note that Balkanski and Singer [BS18] showed that no constant-factor approximation is achievable in $O(\log(n)/\log \log(n))$ adaptive rounds, yet it is possible to achieve a constant-factor approximation in the MapReduce model in two rounds [MZ15].

Most of the related works focus on monotone submodular functions. Balkanski and Singer [BS18] introduced the adaptive framework model for monotone submodular maximization and showed that
a (1/3)-approximation is achievable in $O(\log(n))$ rounds. They also showed that $\omega(\log(n)/\log(\log(n)))$ rounds are necessary for achieving any constant-factor approximation, but they left the problem of achieving the optimal approximation factor of $1 - 1/e$ open. As a followup, they posted a paper on arXiv achieving a $(1 - (1/e - \varepsilon))$-approximation in $O(\log(n))$ rounds [BRS18] for the monotone submodular functions. However, this algorithm requires $O(nk^2)$ queries. While writing this paper, another recent related work (on arXiv) was brought to our attention [EN18], which achieves the same approximation factor in $O(\log(n))$ adaptive rounds with $O(\text{poly}(\log(n)))$ calls per element. Independently, in [FMZ18] the authors also developed an $O(\log(n))$ adaptive algorithm that achieves a $(1 - 1/e - \varepsilon)$-approximation factor and makes $O(1)$ oracle calls per element in expectation.

Unlike these previous works, we focus on the more general non-monotone submodular functions and provide the first constant-factor approximation algorithm with nearly optimal adaptivity complexity. We also only make $O(\log(k))$ queries per element, as opposed to the polylogarithmic number of oracle queries that are present in existing works.

2 Preliminaries

For a set function $f : 2^N \to \mathbb{R}$ and any $S, T \subseteq N$, let $\Delta(T, S) \overset{\text{def}}{=} f(S \cup T) - f(S)$ be the marginal gain of $f$ at $T$ with respect to $S$. We call $N$ the ground set and let $|N| = n$. A function $f : 2^n \to N$ is submodular if for every $S \subseteq T \subseteq N$ and $x \in N \setminus T$ we have $\Delta(x, S) \geq \Delta(x, T)$, where we overload the marginal gain notation for singletons. A natural class of submodular functions are those which are monotone, meaning that for every $S \subseteq T \subseteq N$ we have $f(S) \leq f(T)$. We call a submodular function non-monotone if it is not monotone. In the arguments of our algorithms, we let $f_S(T) \overset{\text{def}}{=} \Delta(T, S)$ denote a new submodular function with respect to $S$. We also assume that the ground set is global to all algorithms. Let $S^*$ be a solution set to the maximization problem $\max_{S \subseteq N} f(S)$ subject to the cardinality constraint $|S| \leq k$, and let $U(A, t)$ denote the uniform distribution over all subsets of $A$ of size $t$.

Our algorithms take as input an evaluation oracle for $f$, which for any query $S \subseteq N$ returns the value of $f(S)$ in $O(1)$ time. Given an evaluation oracle, we define the adaptivity of an algorithm to be the minimum number of rounds such that in each round the algorithm can make polynomially-many independent queries to the evaluation oracle. We measure the complexity of our distributed algorithms in terms of their query and adaptivity complexity. We also note that in our runtime guarantees we take $1/\delta = \Omega(\text{poly}(n))$ so that the claims hold with high probability.

3 Threshold-Sampling Algorithm

We note that all algorithms and results in this section and Appendix A are restatements of results in [FMZ18]. We are including them in this draft for the sake of completeness.

We start by giving a high-level description of the Threshold-Sampling algorithm. For an input threshold $\tau$, the algorithm iteratively builds a solution $S$ and maintains a set of remaining candidate elements $A$ over $O(\log(n)/\varepsilon)$ adaptive rounds. Initially, the solution is empty and all elements are candidates. In each round, the algorithm first filters out candidate elements whose current marginal gain is less than the threshold. Then the algorithm efficiently finds the largest set size $t^*$ such that for $T \sim U(A, t^*)$ uniformly at random we have $\mathbb{E}[\Delta(T, S)/|T|] \geq (1 - \varepsilon)\tau$. Next, the algorithm samples $T \sim U(A, t^*)$ and updates the current solution to $S \cup T$. This probabilistic guarantee has two beneficial effects. First, it ensures that in expectation the average contribution
of each element in the returned set is at least \((1 - \varepsilon)\tau\). Second, it implies that an expected \(\varepsilon\)-fraction of candidates are filtered out of \(A\) in each round. Therefore, the number of remaining elements that the algorithm considers in each round decreases geometrically in expectation. It follows that \(O(\log(n)/\varepsilon)\) rounds are sufficient to guarantee with high probability that when the algorithm terminates, we have \(|S| = k\) or the marginal gain of all elements is below the threshold.

Before presenting and analyzing Threshold-Sampling, we define the distribution \(D_t\) from which Threshold-Sampling samples when estimating the maximum set size \(t^*\). Note that sampling from this distribution can be simulated with two calls to the evaluation oracle.

**Definition 3.1.** Conditioned on the current state of the algorithm, consider the process where the set \(T \sim U(A, t - 1)\) and then the element \(x \sim A \setminus T\) are drawn uniformly at random. Let \(D_t\) denote the probability distribution over the indicator random variable

\[
I_t = I[\Delta(x, S \cup T) \geq \tau].
\]

It will be useful to think of \(\mathbb{E}[I_t]\) as the probability that the \(t\)-th marginal is at least the threshold \(\tau\) if the candidates in \(A\) are inserted into \(S\) according to a uniformly random permutation.

Now that \(D_t\) is defined, we present the Threshold-Sampling algorithm and its guarantees below. Observe that this algorithm calls the Reduced-Mean subroutine, which detects when the mean of \(D_t\) falls below \(1 - \varepsilon\). We give the exact guarantees of Reduced-Mean in Lemma 3.3. Relating the mean of \(D_t\) to thresholds, this means that after sampling \(T \sim U(A, t^*)\) and adding the elements of \(T\) to \(S\), the expected marginal gain of the remaining candidates is at most \((1 - \varepsilon)\tau\). This is the invariant we want in each iteration for an \(O(\log(n/\delta)/\varepsilon)\) adaptive algorithm. We explain the mechanics of Threshold-Sampling in detail and prove Lemma 3.2 in Appendix A.

**Algorithm 1** Threshold-Sampling [FMZ18]

**Input:** evaluation oracle for \(f : 2^N \rightarrow \mathbb{R}\), constraint \(k\), threshold \(\tau\), error \(\varepsilon\), failure probability \(\delta\)

1: Set smaller error \(\hat{\varepsilon} \leftarrow \varepsilon/3\)
2: Set iteration bounds \(r \leftarrow \lceil \log(1 - \hat{\varepsilon}) - 1 \cdot (2n/\delta) \rceil\), \(m \leftarrow \lceil \log(k)/(\hat{\varepsilon}) \rceil\)
3: Set smaller failure probability \(\hat{\delta} \leftarrow \delta/(2r(m + 1))\)
4: Initialize \(S \leftarrow \emptyset\), \(A \leftarrow N\)
5: for \(r\) rounds do
6:  Filter \(A \leftarrow \{x \in A : \Delta(x, S) \geq \tau\}\)
7:  if \(|A| = 0\) then
8:      break
9:  for \(i = 0\) to \(m\) do
10:     Set \(t \leftarrow \min\{(1 + \hat{\varepsilon})^i, |A|\}\)
11:     if Reduced-Mean\((D_t, \hat{\varepsilon}, \hat{\delta})\) then
12:        break
13:     Sample \(T \sim U(A, \min\{t, k - |S|\})\)
14:     Update \(S \leftarrow S \cup T\)
15:  if \(|S| = k\) then
16:     break
17: return \(S\)
Lemma 3.2 ([FMZ18]). The algorithm Threshold-Sampling outputs \( S \subseteq N \) with \( |S| \leq k \) in \( O(\log(n/\delta)/\varepsilon) \) adaptive rounds such that the following properties hold with probability at least \( 1 - \delta \):

1. There are \( O(n/\varepsilon) \) oracle queries in expectation.
2. The expected average marginal \( \mathbb{E}[f(S)/|S|] \geq (1 - \varepsilon)\tau \).
3. If \( |S| < k \), then \( \Delta(x, S) < \tau \) for all \( x \in N \).

We briefly remark that the Reduced-Mean subroutine is a standard unbiased estimator for the mean of a Bernoulli distribution. Since \( D_t \) is a uniform distribution over indicator random variables, it is in fact a Bernoulli distribution.

Algorithm 2 Reduced-Mean [FMZ18]

Input: access to a Bernoulli distribution \( D \), error \( \varepsilon \), failure probability \( \delta \)

1: Set number of samples \( m \leftarrow 16\lceil \log(2/\delta)/\varepsilon^2 \rceil \)
2: Sample \( X_1, X_2, \ldots, X_m \sim D \)
3: Set \( \bar{\mu} \leftarrow \frac{1}{m} \sum_{i=1}^{m} X_i \)
4: if \( \bar{\mu} \leq 1 - 1.5\varepsilon \) then
5: return true
6: return false

Lemma 3.3 ([FMZ18]). For any Bernoulli distribution \( D \), Reduced-Mean uses \( O(\log(\delta^{-1})/\varepsilon^2) \) samples to correctly report one of the following properties with probability at least \( 1 - \delta \):

1. If the output is true, then the mean of \( D \) is \( \mu \leq 1 - \varepsilon \).
2. If the output is false, then the mean of \( D \) is \( \mu \geq 1 - 2\varepsilon \).

4 Adaptive Non-Monotone Submodular Maximization

In this section we show how to use Threshold-Sampling to give a constant-factor approximation algorithm for non-monotone submodular maximization subject to a cardinality constraint that runs in \( O(\log(n)) \) adaptive rounds. While our approximation factor is only 0.031, it demonstrates the proof of concept that Threshold-Sampling can readily be extended to the non-monotone setting. The corresponding hardness results for adaptive non-monotone submodular maximization show that an exponential number of oracle queries are needed to achieve a 0.491-approximation [GV11], and also that \( \omega(\log(n)/\log\log(n)) \) adaptive rounds are necessary for any constant factor approximation algorithm [BS18].

We start by briefly describing the algorithm Adaptive-Nonmonotone-Maximization and our analysis of its approximation factor. We defer all proofs for results in this section to Appendix B. The following lemmas that are useful for lower bounding our solution in terms of OPT,

Lemma 4.1 ([BFNS14]). Let \( f : 2^N \rightarrow \mathbb{R} \) be a submodular function. Denote by \( A(p) \) a random subset of \( A \) where each element appears with probability at most \( p \) (not necessarily independently). Then \( \mathbb{E}[f(A(p))] \geq (1 - p)f(\emptyset) \).
Lemma 4.2 ([FMV11]). For any nonnegative submodular function $f$, denote the solution to the unconstrained maximization problem by $OPT = \max_{S \subseteq N} f(S)$. If $R$ is a uniformly random subset of $S$, then $\mathbb{E}[f(R)] \geq (1/4)OPT$.

Lemma 4.1 is convenient for non-monotone submodular functions because it allows us to analyze an auxiliary submodular function $g : 2^N \to \mathbb{R}$ such that $g(S) = f(S \cup S^*)$ since $g(\emptyset) = OPT$.

Similar to Exhaustive-Maximization in [FMZ18], Adaptive-Nonmonotone-Maximization algorithm tries different thresholds in parallel, one of which is sufficiently close to $c_1 \cdot OPT/k$, where the constant $c_1 = 100/331$. For each threshold, we run Threshold-Sampling with a smaller cardinality constraint $c_2k$ modified to break if the number of candidates in $A$ falls below $c_3k$, where the constants $c_2 = 0.35, c_3 = 1.65$. If the number of candidates in $A$ is small enough when Threshold-Sampling returns its solution, we also consider a random subset of size $k$ drawn from $A$ in addition to the set $S$ returned by Threshold-Sampling. The constants $c_1, c_2, c_3$ are optimized to balance the expected trade-offs between the two solutions and hence give the best approximation factor. We present the algorithm and its guarantees below.

Algorithm 3 Adaptive-Nonmonotone-Maximization

**Input:** evaluation oracle for $f : 2^N \to \mathbb{R}$, constraint $k$, error $\varepsilon$, failure probability $\delta$

1. Set smaller error $\hat{\varepsilon} \leftarrow \varepsilon/6$
2. Set upper bounds $\Delta^* \leftarrow \max\{f(x) : x \in N\}$, $r \leftarrow \lceil 2\log(k)/\hat{\varepsilon} \rceil$
3. Set smaller failure probability $\hat{\delta} \leftarrow \delta/(2(r + 1))$
4. Set optimized constants $c_1 \leftarrow 100/331, c_2 \leftarrow 0.35, c_3 \leftarrow 1.65$
5. Initialize $R \leftarrow \emptyset$
6. for $i = 0$ to $r$ in parallel do
7.    Set $\tau \leftarrow c_1(1 + \hat{\varepsilon})^i \Delta^*/k$
8.    Set $(S, A) \leftarrow$ Threshold-Sampling($f_S, c_2k, \tau, \hat{\varepsilon}, \hat{\delta}$) modified to break on Line 7 if $|A| < c_3k$
9.    Initialize $U \leftarrow \emptyset, U' \leftarrow \emptyset$
10.  if $|A| < c_3k$ then
11.     Update $U \leftarrow$ Unconstrained-Maximization($f, A, \hat{\varepsilon}, \hat{\delta}$)
12.     if $|U| > k$ then
13.         Sample $D \sim U(U, k)$
14.         Update $U' \leftarrow D$
15.     else
16.         Update $U' \leftarrow U$
17.     Update $R \leftarrow \arg\max_{X \in \{R, S, U\}} f(X)$
18. return $R$

Theorem 4.3. For any nonnegative submodular function $f$, Adaptive-Nonmonotone-Maximization outputs a set $S \subseteq N$ with $|S| \leq k$ in $O(\log(n)/\delta)/\varepsilon^2$ adaptive rounds such that with probability at least $1 - \delta$ it makes $O(n \log(k)/\varepsilon^2)$ oracle queries in expectation and $\mathbb{E}[f(S)] \geq (0.031 - \varepsilon)OPT$.

The algorithm calls Unconstrained-Maximization if the number of candidates in $|A|$ when Threshold-Sampling terminates is less than $c_3k$. By Lemma 4.2, it follows that the output of Unconstrained-Maximization is a constant approximation to the optimal solution of $f$ restricted to $A$ as the ground set. Although this solution might violate the cardinality constraint $k$, it
can only do so by a factor of at most $c_3 = 1.65$. Therefore, by submodularity, we can downsample a large solution while only degrading its approximation factor by $1/c_3$.

**Algorithm 4** UNCONSTRAINED-MAXIMIZATION

**Input:** evaluation oracle for $f : 2^N \to \mathbb{R}$, new ground subset $A \subseteq N$, error $\varepsilon$, failure probability $\delta$

1: Set iteration bound $t \leftarrow \lceil \log(1/\delta) / \log(1 + (4/3)\varepsilon) \rceil$
2: for $i = 1$ to $t$ in parallel do
3: Let $R_i$ be an independent and uniformly random subset of $A$
4: Set $S \leftarrow \arg\max_{X \in \{R_1, \ldots, R_t\}} f(X)$
5: return $S$

**Lemma 4.4.** For any nonnegative submodular function $f$ and allowable ground subset $A \subseteq N$, the algorithm UNCONSTRAINED-MAXIMIZATION outputs a set $S \subseteq A$ in one adaptive round using $O(\log(1/\delta) / \varepsilon)$ oracle queries such that with probability at least $1 - \delta$ we have $f(S) \geq (1/4 - \varepsilon)\text{OPT}_A$, where $\text{OPT}_A = \max_{T \subseteq A} f(T)$ is the solution to the maximization problem without a cardinality constraint but restricted to $A$.

4.1 Analysis of Adaptive-Nonmonotone-Maximization

We start by introducing some notation that is useful for analyzing the states of Threshold-Sampling as a call to this subroutine progresses. Let $T_1, T_2, T_3, \ldots$ be the sequence of randomly generated sets used to build the output set $S$. Similarly, let the corresponding sequence of solutions be $S_1 = \bigcup_{j=1}^{i} T_j$ and the sequence of candidates be $A_0, A_1, A_2, \ldots$. Define $H_i$ to be the distribution over all random choices of the first $i$ sets $T_1, T_2, \ldots, T_i$. In our analysis of the approximation factor of Adaptive-Nonmonotone-Maximization, we start with a fixed threshold $\tau$ and then analyze the resulting sets $S$, $U$, and $U'$. We also use ALG as a convenient alias for the final output set $R$.

**Lemma 4.5.** At any step $i \geq 0$ of Threshold-Sampling, we have $E[f(S_i)] \geq (1 - 2\tilde{\varepsilon})\tau \cdot E[|S_i|]$.

**Proof.** We prove the claim by induction. Assuming $f$ is nonnegative, the base case is clearly true. Therefore, assume as the induction hypothesis that the claim holds for all $i \geq 0$. Conditioning on a history $h \in \text{supp}(H_i)$, we have

$$E[\Delta(T_{i+1}, S_i) \mid h] \geq (1 - 2\tilde{\varepsilon})\tau \cdot E[|T_{i+1}| \mid h]$$

by the argument for Property 2 in the proof of Lemma 3.2, which we provide in Appendix A. Specifically, this equation is a direct consequence of (9). Since $H_i$ partitions the space of possible outcomes, it follows from the total law of expectation and the induction hypothesis that

$$E[f(S_{i+1})] = E[\Delta(T_{i+1}, S_i)] + E[f(S_i)]$$
$$\geq (1 - 2\tilde{\varepsilon})\tau \cdot E[|T_{i+1}|] + E[f(S_i)]$$
$$\geq (1 - 2\tilde{\varepsilon})\tau \cdot E[|T_{i+1}|] + (1 - 2\tilde{\varepsilon})\tau \cdot E[|S_i|]$$
$$= (1 - 2\tilde{\varepsilon})\tau \cdot E[|S_{i+1}|].$$

This completes the proof. \qed
Lemma 4.6. For any element \( x \in N \), we have \( \Pr(x \in S) \leq c_2/c_3 \).

Proof. Let \( X_i \) be an indicator random variable for the event \( x \in T_i \). It follows that
\[
\Pr(x \in S) = \sum_{i=1}^{n} \mathbb{E}[X_i] \leq \sum_{i=1}^{n} \mathbb{E}
\left[ \frac{T_i}{|A_i|} \right] \leq \frac{1}{c_3 k} \sum_{i=1}^{n} \mathbb{E}|T_i| = \frac{1}{c_3 k} \cdot \mathbb{E}|S| \leq \frac{1}{c_3 k} \cdot c_2 k = \frac{c_2}{c_3},
\]
as desired. \qed

Lemma 4.7. For any set \( S \subseteq N \) and \( 0 \leq k \leq |S| \), if \( T \sim \mathcal{U}(S, k) \) then \( \mathbb{E}[f(T)] \geq k/|S| \cdot f(S) \).

Theorem 4.3. For any nonnegative submodular function \( f \), Adaptive-Nonmonotone-Maximization outputs a set \( S \subseteq N \) with \( |S| \leq k \) in \( O(n \log(n/\delta)/\varepsilon^2) \) adaptive rounds such that with probability at least 1 − \( \delta \) it makes \( O(n \log(k)/\varepsilon^2) \) oracle queries in expectation and \( \mathbb{E}[f(S)] \geq (0.031 - \varepsilon)\text{OPT} \).

Proof. We first assume that all subroutines behave as desired with probability at least 1 − \( \delta \) by our choice of \( \delta \) and a union bound. Next, Adaptive-Nonmonotone-Maximization necessarily tries a threshold \( \tau \) such that \( \tau \leq c_1 \cdot \text{OPT} / k \leq \tau(1 + \varepsilon) \), so the analysis that follows considers this particular threshold. We start by proving the approximation factor. Let \( A_{<c_2 k} \) denote the event that the modified Threshold-Sampling subroutine breaks because \( |A| < c_2 k \). Similarly, let \( A_{\geq c_2 k} \) denote the complement (i.e., Threshold-Sampling breaks because \( |S| = c_2 k \)). It follows from Lemma 4.5 that
\[
\mathbb{E}[f(S)] \geq (1 - 2\varepsilon)\tau \cdot \mathbb{E}[|S|] = (1 - 2\varepsilon)\tau \cdot (\mathbb{E}[|S| \mid A_{<c_2 k}] \cdot \Pr(A_{<c_2 k}) + \mathbb{E}[|S| \mid A_{\geq c_2 k}] \cdot \Pr(A_{\geq c_2 k})) \geq (1 - 2\varepsilon)\tau \cdot \mathbb{E}[|S| \mid A_{\geq c_2 k}] \cdot \Pr(A_{\geq c_2 k}) = (1 - 2\varepsilon)\tau \cdot c_2 k \cdot \Pr(A_{\geq c_2 k}).
\]

Therefore, we have
\[
\mathbb{E}[\text{ALG}] \geq \mathbb{E}[f(S)] \geq \frac{1 - 2\varepsilon}{1 + \varepsilon} \cdot c_1 c_2 \cdot \text{OPT} \cdot \Pr(A_{\geq c_2 k}) \geq (1 - \varepsilon) \cdot c_1 c_2 \cdot \text{OPT} \cdot \Pr(A_{\geq c_2 k}). \tag{1}
\]

Next, we show how to lower bound \( \mathbb{E}[\text{ALG}] \) as a function of \( \text{OPT} \cdot \Pr(A_{<c_2 k}) \), which ultimately allows us to achieve a constant approximation to \( \text{OPT} \). For any subsets \( A, S \subseteq N \) returned by Threshold-Sampling, we can partition the optimal set \( S^* \) into
\[
S_1 \overset{\text{def}}{=} S^* \cap A \quad S_2 \overset{\text{def}}{=} S^* \setminus A.
\]

Let \( U_A \) be the output of a call to Unconstrained-Maximization(\( f, A, \varepsilon, \hat{\delta} \)). Assuming that the call succeeds, Lemma 4.4 implies that \( f(S_1^*) \leq (1/4 - \varepsilon)^{-1} \cdot f(U_A) \). Therefore, by subadditivity,
\[
f(S^*) \leq f(S_1^*) + f(S_2^*) = f(S_1^*) + f(S_2^*) - f(S_2^* \cup S) + f(S_2^* \cup S) \leq (1/4 - \varepsilon)^{-1} \cdot f(U_A) + f(S_2^*) - f(S_2^* \cup S) + f(S_2^* \cup S).
\]

8
We claim that each marginal contribution in $f(S_2^* \cup S) \leq f(S) + \sum_{x \in S_2^*} \Delta(x, S) \leq f(S) + k\tau$.

Define $\text{Gap}(A, S) \overset{\text{def}}{=} \max\{f(S_2^*) - f(S_2^* \cup S), 0\}$. It follows from the inequalities above that

$$f(S^*) \leq (1/4 - \hat{\varepsilon})^{-1} \cdot f(U_A) + \text{Gap}(A, S) + f(S) + k\tau. \tag{2}$$

Next, using (2) and the definition of $\tau$, we have

$$f(U_A) \geq (1/4 - \hat{\varepsilon}) \cdot (f(S^*) - \text{Gap}(A, S) - f(S) - k\tau)$$

$$\geq (1/4 - \hat{\varepsilon}) \cdot ((1 - c_1) \cdot \text{OPT} - \text{Gap}(A, S) - f(S)). \tag{3}$$

Our next goal is to upper bound $\text{Gap}(A, S)$ as a function of $S^*$ so that we have an upper bound that is independent of the value of $A$. Specifically, we want to show that for all $A \subseteq N$, we have

$$f(S_2^*) - f(S_2^* \cup S) \leq f(S^*) - f(S^* \cup S).$$

This inequality is true if and only if

$$f(S_2^*) + f(S^* \cup S) \leq f(S^*) + f(S_2^* \cup S).$$

First, observe that by the submodularity definition $f(A \cap B) + f(A \cup B) \leq f(A) + f(B)$, we have

$$f(S^* \cap (S_2^* \cup S)) + f(S^* \cup S) \leq f(S^*) + f(S_2^* \cup S).$$

Therefore, it suffices to show that

$$f(S_2^*) \leq f(S^* \cap (S_2^* \cup S)). \tag{4}$$

We start by rewriting

$$S^* \cap (S_2^* \cup S) = (S^* \cap S_2^*) \cap (S^* \cap S)$$

$$= S_2^* \cup (S_1^* \cap S).$$

Fix an ordering $x_1, x_2, \ldots, x_\ell$ on the elements in $S^*$. Summing the consecutive marginal gains of the elements in the set $S_1^* \cap S$ according to this order gives

$$f(S_2^* \cup (S_1^* \cap S)) = f(S_2^*) + \sum_{x_1, x_2, \ldots, x_{\ell+1} \in S_1^* \cap S} \Delta(x_i, S_2^* \cup \{x_1, \ldots, x_{i-1}\}). \tag{5}$$

We claim that each marginal contribution in (5) is nonnegative. Assume for contradiction that this is not the case. Let $x^* \in S_1^* \cap S$ be the first element violating this property, and let $x_{i-1}^*$ be the previous element according to the ordering. By submodularity, we have

$$0 > \Delta\left(x^*, S_2^* \cup \bigcup_{x_1, \ldots, x_{i-1}^* \in S_1^* \cap S} \{x_i\}\right) \geq \Delta\left(x^*, S_2^* \cup \bigcup_{x_1, \ldots, x_{i-1}^* \in S_1^* \cap S} \{x_i\}\right),$$
which further implies \( f(S^* \setminus \{x^*\}) > f(S^*) = \text{OPT} \), a contradiction. It follows that (4) is true, which gives us a bound for \( \text{Gap}(A, S) \) in terms of \( S^* \). Specifically,

\[
f(S^*_2) - f(S_2 \cup S) \leq f(S^*) - f(S^* \cup S) \implies \text{Gap}(A, S) \leq \text{Gap}(\emptyset, S) \overset{\text{def}}{=} \text{Gap}(S).
\]

Now we give an upper bound for \( \mathbb{E}[\text{Gap}(S)] \), where \( S \) is the output of \text{Threshold-Sampling} conditioned on the subroutine succeeding. Observe that by subadditivity we have

\[
f(S^* \cup S) \leq f(S^*) + f(S) \implies f(S^*) - f(S^* \cup S) + f(S) \geq 0.
\]

Therefore, because \( f \) is nonnegative we have

\[
\text{Gap}(S) = \max\{f(S^*) - f(S^* \cup S), 0\} \leq f(S^*) - f(S^* \cup S) + f(S).
\]

(6)

Next, define a new submodular function \( g : 2^N \rightarrow \mathbb{R} \) such that \( g(S) = f(S^* \cup S) \), and consider the random set \( S \) returned by \text{Threshold-Sampling}. Using Lemma 4.6, each element appears in \( S \) with probability at most \( c_2/c_3 \). Therefore, by Lemma 4.1 applied to \( g \), we have

\[
\mathbb{E}[g(S)] \geq (1 - c_2/c_3)g(\emptyset) \implies \mathbb{E}[f(S \cup S^*)] \geq (1 - c_2/c_3)f(S^*).
\]

It follows that

\[
\begin{align*}
\mathbb{E}[\text{Gap}(S)] & \leq \mathbb{E}[f(S^*) - f(S^* \cup S) + f(S)] \\
& = \text{OPT} - \mathbb{E}[f(S^* \cup S)] + \mathbb{E}[f(S)] \\
& \leq \text{OPT} - (1 - c_2/c_3) \cdot \text{OPT} + \mathbb{E}[f(S)] \\
& = c_2/c_3 \cdot \text{OPT} + \mathbb{E}[f(S)].
\end{align*}
\]

(7)

Now we are prepared to give a second lower bound for \( \mathbb{E}[\text{ALG}] \) in terms of \( \text{OPT} \cdot \Pr(A_{<c_3k}) \). We start by observing that if the algorithm calls \text{Unconstrained-Maximization} (i.e., \( A_{<c_3k} \) occurs), then it follows from (3) and (6) that

\[
\begin{align*}
\mathbb{E}[f(U) \mid A_{<c_3k}] & \geq (1/4 - \hat{\epsilon}) \cdot \mathbb{E}[(1 - c_1) \cdot \text{OPT} - \text{Gap}(A, S) - f(S) \mid A_{<c_3k}] \\
& \geq (1/4 - \hat{\epsilon}) \cdot \mathbb{E}[(1 - c_1) \cdot \text{OPT} - \text{Gap}(S) - f(S) \mid A_{<c_3k}] \\
& = (1/4 - \hat{\epsilon}) \cdot ((1 - c_1) \cdot \text{OPT} - \mathbb{E}[\text{Gap}(S) \mid A_{<c_3k}] - \mathbb{E}[f(S) \mid A_{<c_3k}]).
\end{align*}
\]

It follows from the law of total expectation, the nonnegativity of \( \text{Gap}(S), f(S), f(U) \), and (7) that

\[
\begin{align*}
\mathbb{E}[f(U)] & \geq \mathbb{E}[f(U) \mid A_{<c_3k}] \cdot \Pr(A_{<c_3k}) \\
& \geq (1/4 - \hat{\epsilon}) \cdot \left((1 - c_1) \cdot \text{OPT} \cdot \Pr(A_{<c_3k}) - \mathbb{E}[\text{Gap}(S)] - \mathbb{E}[f(S)]\right) \\
& \geq (1/4 - \hat{\epsilon}) \cdot \left((1 - c_1) \cdot \text{OPT} \cdot \Pr(A_{<c_3k}) - c_2/c_3 \cdot \text{OPT} - 2\mathbb{E}[f(S)]\right).
\end{align*}
\]

It is possible that the unconstrained solution \( U \) exceeds the cardinality constraint \( k \), but by our construction it is of size at most \( c_3k \). When this happens, we uniformly downsample \( U \) to obtain \( U' \) such that \( |U'| = k \). Using Lemma 4.7, it follows that in all cases we have \( \mathbb{E}[f(U')] \geq (1/c_3) \cdot \mathbb{E}[f(U)] \). Therefore, we have the following three inequalities, which allow us to bound \( \mathbb{E}[\text{ALG}] \):
\[ \mathbb{E}[\text{ALG}] \geq \mathbb{E}[f(S)] \]
\[ \mathbb{E}[f(S)] \geq (1 - \varepsilon) \cdot c_1 c_2 \cdot \text{OPT} \cdot \Pr(A_{\geq c_3 k}) \]
\[ \mathbb{E}[\text{ALG}] \geq (1/4 - \hat{\varepsilon})/c_3 \cdot ((1 - c_1) \cdot \text{OPT} \cdot \Pr(A_{< c_3 k}) - c_2/c_3 \cdot \text{OPT} - 2\mathbb{E}[f(S)]) \cdot \text{OPT} \cdot \Pr(A_{\geq c_3 k}). \]

Suppose that \( \mathbb{E}[f(S)] > c_4 \cdot \text{OPT} \) for a new fixed constant \( c_4 \) that we will later optimize. This gives us a \( c_4 \)-approximation to OPT. If this is not the case, then \( \mathbb{E}[f(S)] \leq c_4 \cdot \text{OPT} \), which implies

\[ \mathbb{E}[\text{ALG}] \geq (1 - \varepsilon) \cdot c_1 c_2 \cdot \text{OPT} \cdot \Pr(A_{\geq c_3 k}) \]
\[ \mathbb{E}[\text{ALG}] \geq \frac{1 - \varepsilon}{4c_3} \cdot ((1 - c_1) \cdot \text{OPT} \cdot \Pr(A_{< c_3 k}) - (c_2/c_3 + 2c_4) \cdot \text{OPT}) \cdot \text{OPT}. \]

Averaging these two inequalities gives

\[ \mathbb{E}[\text{ALG}] \geq \frac{1 - \varepsilon}{8c_3} \cdot \left(4c_1 c_2 c_3 \Pr(A_{\geq c_3 k}) + (1 - c_1) \Pr(A_{< c_3 k}) - (c_2/c_3 + 2c_4)\right) \cdot \text{OPT}. \]

Therefore, to optimize our analysis of the approximation factor, we solve

\[ \max \min \left\{ c_4, \frac{1 - \varepsilon}{8c_3} \cdot (1 - (c_1 + c_2/c_3 + 2c_4)) \right\} \]
subject to the constraint

\[ 4c_1 c_2 c_3 = 1 - c_1. \]

In the last part of this proof, we optimize the constants to give a 0.0319-approximation. The equality constraint implies that

\[ c_1 = \frac{1}{1 + 4c_2 c_3}. \]

Second, we can set the two expressions in the max-min problem to be equal since one is increasing in \( c_4 \) and the other is decreasing. This gives us

\[ c_4 = \frac{1 - \varepsilon}{8c_3} \cdot (1 - (c_1 + c_2/c_3 + 2c_4)) \implies c_4 = \frac{(1 - \varepsilon)(1 - c_1 - c_2/c_3)}{2(4c_3 + 1 - \varepsilon)}. \]

Therefore, any values \( c_2, c_3 > 0 \) give a lower bound for the approximation factor of

\[ (1 - \varepsilon) \cdot \frac{1 - \frac{1}{1 + 4c_2 c_3} - \frac{c_2}{c_3}}{2(4c_3 + 1)}. \] (8)

Letting \( c_2 = 0.35 \) and \( c_3 = 1.65 \), we have an approximation factor of \( (1 - \varepsilon) \cdot 0.0319 \geq 0.0319 - \varepsilon \), as desired. These values are close to the numerical optimum and also imply that \( c_1 = 100/331 \).

The proof of the adaptivity and query complexities use the analysis for UNCONSTRAINED-MAXIMIZATION in Lemma 4.4, but is otherwise analogous to the proof of Theorem 4.1 in [FMZ18]. This completes the analysis for the ADAPTIVE-NONMONOTONE-MAXIMIZATION algorithm. \( \square \)
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A Analysis of the Threshold-Sampling Algorithm

In our analysis of the approximation factor of Adaptive-Nonmonotone-Maximization, we use an inequality derived from part of the analysis of Lemma 3.2. We note that all lemmas in this section are restatements borrowed from [FMZ18], and we include them here for the sake of completeness.

**Lemma A.1** ([FMZ18]). In each round of Threshold-Sampling, \( \mathbb{E}[I_1] \geq \mathbb{E}[I_2] \geq \cdots \geq \mathbb{E}[I_A] \).

**Lemma A.2** ([FMZ18]). In each round of Threshold-Sampling, an expected \( \hat{\varepsilon} \)-fraction of \( A \) is filtered with probability at least \( 1 - \hat{\delta} \).

**Lemma A.3** ([FMZ18]). If Threshold-Sampling terminates with \( |S| < k \), then \( |A| = 0 \) with probability at least \( 1 - \delta \).

Using the guarantees for Reduced-Mean and the two lemmas above, we can prove Lemma 3.2.

**Proof of Lemma 3.2** ([FMZ18]). We start by showing the adaptivity complexity of Threshold-Sampling is \( O(\log(n/\delta)/\varepsilon) \). By construction, the number of rounds is \( O(\log(n/\delta)/(\varepsilon^2)) \) and there are polynomially-many queries in each, all of which are independent and rely on the current state of \( S \).

To prove the three properties, we use Lemma A.3 to assume that with probability at least \( 1 - \delta \) all \( O(rm) \) calls to Reduced-Mean yield correct outputs, and also that if the algorithm terminates with \( |S| < k \) then we have \( |A| = 0 \). For the Property 1, the total number of oracle queries incurred by calling Reduced-Mean is \( O(rm \log(\delta^{-1})/\varepsilon^2) = O(\log(n/\delta) \log(k) \log(\delta^{-1})/\varepsilon^4) \) by Lemma 3.3. Note that we can sample from \( D_t \) with two oracle calls. Now we bound the expected number of queries made while filtering over the course of the algorithm. Let \( A_i \) be a random variable for the value of \( A \) in the \( i \)-th round. It follows from the geometric property \( \mathbb{E}[|A_{i+1}|] \leq (1 - \hat{\varepsilon}) \cdot \mathbb{E}[|A_i|] \) in the proof of Lemma A.3 and by linearity that the expected number of queries is bounded by

\[
\mathbb{E} \left[ \sum_{i=0}^{r} |A_i| \right] = \sum_{i=0}^{r} \mathbb{E}[|A_i|] \leq n \sum_{i=0}^{r} (1 - \hat{\varepsilon})^i \leq n/\hat{\varepsilon}.
\]

Since we set \( \delta^{-1} = O(\text{poly}(n)) \), the number of expected queries made when filtering dominates the sum of queries made when calling Reduced-Mean.

For property 2, it suffices to lower bound the expected marginal of every element added to \( S \) if we think of adding each set \( T \) to the output \( S \) one element at a time according to a uniformly random permutation. Let \( t^* = \min\{t, k - |S|\} \) be the size of \( T \) at an arbitrary round. If \( t^* = 1 \) then \( \mathbb{E}[\Delta(T, S)] \geq \tau \) by the definition of \( A \). Otherwise, the candidate size \( t \geq t^*/(1 + \hat{\varepsilon}) \) in the previous iteration has the property that \( \mathbb{E}[I_i] \geq 1 - 2\hat{\varepsilon} \). Since \( T \sim U(A, t^*) \) uniformly at random, we can lower bound the expected marginal \( \mathbb{E}[\Delta(T, S)] \) by the contribution of the first \( t \) elements, giving us

\[
\mathbb{E}[\Delta(T, S)] \geq (\mathbb{E}[I_1] + \mathbb{E}[I_2] + \cdots + \mathbb{E}[I_t]) \tau \\
\geq t(1 - 2\hat{\varepsilon}) \tau \\
\geq \frac{t^*}{1 + \hat{\varepsilon}} \cdot (1 - 2\varepsilon) \tau \\
\geq t^*(1 - \varepsilon) \tau.
\]
The first of the inequalities above uses the definition of \( I_t \) in Definition 3.1 and is analogous to Markov’s inequality, and the second follows from Lemma A.1. Since the expected marginal of any individual element is at least \( (1 - \varepsilon)\tau \), we have Property 2.

To show Property 3, recall that if the algorithm terminates with \(|S| < k\), then we have \(|A| = 0\) with probability at least \(1 - \delta\) by Lemma A.3. Therefore, it follows from the definition of \( A \) and submodularity that \( \Delta(x, S) < \tau \) for all \( x \in N \).

\[\text{Pr}(f(R_i) \leq (1/4 - \varepsilon)\text{OPT}_A) \leq \frac{3}{3 + 4\varepsilon}.\]

By Lemma 4.2 we have \( \mathbb{E}[f(R_i)] \geq (1/4)\text{OPT}_A \). Therefore, using an analog of Markov’s inequality to upper bound \( \mathbb{E}[f(R_i)] \), it follows that

\[
\frac{1}{4} \cdot \text{OPT}_A \leq \mathbb{E}[f(R_i)] \leq p\left(\frac{1}{4} - \varepsilon\right)\text{OPT}_A + (1 - p)\text{OPT}_A \implies p \leq \frac{3}{3 + 4\varepsilon}.
\]

Since the subsets \( R_i \) are chosen independently, our choice of \( t \) gives us a total failure probability of

\[
\text{Pr}(f(S) \leq (1/4 - \varepsilon)\text{OPT}_A) = \prod_{i=1}^{t}\text{Pr}(f(R_i) \leq (1/4 - \varepsilon)\text{OPT}_A) \leq \left(\frac{3}{3 + 4\varepsilon}\right)^t \leq \delta.
\]

This completes the proof that with probability at least \(1 - \delta\) we have \( f(S) \geq (1/4 - \varepsilon)\text{OPT}_A \).

To prove the adaptivity complexity, notice that all subsets \( R_i \) can be generated and evaluated at once in parallel, hence the need for only one adaptive round. For the query complexity, we use the inequality \( \log(1 + (4/3)\varepsilon) \geq 2\varepsilon/3 \), which holds for all errors \( \varepsilon \leq 1/4 \). \(\square\)

### B.2 Downsampling Submodular Sets

**Lemma 4.7.** For any set \( S \subseteq N \) and \( 0 \leq k \leq |S| \), if \( T \sim \mathcal{U}(S, k) \) then \( \mathbb{E}[f(T)] \geq k/|S| \cdot f(S) \).

**Proof.** Fix an ordering \( x_1, x_2, \ldots, x_s \) on the elements in \( S \). Expanding the expected value \( \mathbb{E}[f(T)] \)
and using submodularity, it follows that

\[
\mathbb{E}[f(T)] = \frac{1}{\binom{s}{k}} \sum_{R \in U(S,k)} \sum_{x_i \in R} \Delta(x_i, \{x_1, x_2, \ldots, x_{i-1}\} \cap R)
\geq \frac{1}{\binom{s}{k}} \sum_{R \in U(S,k)} \sum_{x_i \in R} \Delta(x_i, \{x_1, x_2, \ldots, x_{i-1}\})
= \frac{1}{\binom{s}{k}} \sum_{i=1}^{s} \binom{s-1}{k-1} \Delta(x_i, \{x_1, x_2, \ldots, x_{i-1}\})
= \frac{k}{s} \cdot f(S),
\]

which completes the proof. \qed