On Baxter $Q$-operators

And Their Arithmetic Implications

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Abstract

We consider Baxter $Q$-operators for various versions of quantum affine Toda chain. The interpretation of eigenvalues of the finite Toda chain Baxter operators as local Archimedean $L$-functions proposed recently is generalized to the case of affine Lie algebras. We also introduce a simple generalization of Baxter operators and local $L$-functions compatible with this identification. This gives a connection of the Toda chain Baxter $Q$-operators with an Archimedean version of the Polya-Hilbert operator proposed by Berry-Kitting. We also elucidate the Dorey-Tateo spectral interpretation of eigenvalues of $Q$-operators. Using explicit expressions for eigenfunctions of affine/relativistic Toda chain we obtain an Archimedean analog of Casselman-Shalika-Shintani formula for Whittaker function in terms of characters.

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1 Introduction

Theory of quantum integrable systems provides a reach source of new mathematical concepts. Probably the most famous example is the notion of quantum groups described in terms of $R$-matrices. May be a less widely known is the notion of the $Q$-operator introduced by Baxter [Ba]. $Q$-operators appear as an important tool to solve quantum integrable systems. In [GLO] we introduce $Q$-operators for $\mathfrak{gl}_{\ell+1}$-Toda chains (using previous results of [PG] for periodic Toda chains) and propose an interpretation of Baxter integral $Q$-operators for $\mathfrak{gl}_{\ell+1}$-Toda chains as particular elements of spherical Hecke algebra $\mathcal{H}(GL(\ell+1, \mathbb{R}), SO(\ell+1, \mathbb{R}))$. Surprisingly eigenvalues of the $Q$-operators acting on class one principal series $\mathfrak{gl}_{\ell+1}$-Whittaker functions are given by local Archimedean $L$-factors corresponding to the principal series representations. This implies that the formalism of $Q$-operators might shed light on a hidden structure of the Archimedean ($\infty$-adic arithmetic) geometry (see e.g. [Man1] and references therein).

In this paper we discuss a set of generalizations of the results presented in [GLO]. First we introduce slightly generalized $Q$-operators for finite Toda chains. This construction leads to a natural connection between a representation of local Archimedean $L$-factors (products of $\Gamma$-functions) as appropriately regularized determinants of simple differential operators and the results of Dorey-Tateo [DT1, DT2] on a representation of eigenvalues of $Q$-operators as spectral determinants of auxiliary differential operators. The generalization of [DT1, DT2] to Toda chains was not known before.

Representations of $L$-factors as determinants provide an intriguing relation with the construction of the Weil group $W_{\mathbb{R}}$ for the field $\mathbb{R}$ of real numbers. We demonstrate that the operator arising in our interpretation of $Q$-operator’s eigenvalues for $\mathfrak{gl}_{\ell+1}$-Toda chain is consistent with the structure of the Weil group $W_{\mathbb{R}}$ and with the Archimedean analog of the Polya-Hilbert operator introduced by Berry and Kitting [BK].

Following the interpretation of $Q$-operators for $\mathfrak{gl}_{\ell+1}$-Toda chains given in [GLO] we consider Toda chains for affine Lie algebras and propose to identify the eigenvalues of the corresponding $Q$-operators with the appropriate generalization of local Archimedean $L$-factors for affine Lie algebras. We distinguish affine Toda chains corresponding to generic and critical levels. The first one is known as Relativistic Toda chain [Ru] ($q$-deformed Toda chain in the terminology of [Ei]) and the second one is a standard affine $\hat{\mathfrak{gl}}_{\ell+1}$-Toda chain proper. We construct explicitly $Q$-operators and their eigenvalues for simplest non-trivial cases. Thus in the case of $q$-deformed Toda chain an analog of $L$-factor is given by a Double Sine function [KLS]. To the best of our knowledge these local Archimedean $L$-factors were not considered before.

In this paper we also consider a variant of a Toda chain for affine Lie algebras at generic level with eigenfunctions defined on a lattice. This type of Toda chain naturally arises in the study of $K$-theory version of Gromow-Witten invariants of flag varieties [Gi, GiL]. In the simplest case we derive explicit formulas for lattice analogs of Whittaker functions and eigenvalues of $Q$-operators. Remarkably this provides an interpolation between Archimedean and non-Archimedean Whittaker functions. There is a well-known identification of $p$-adic Whittaker functions for group $G(\mathbb{Q}_p)$ and characters of finite-dimensional representations of
the Langlands dual complex Lie group $L^\mathbb{C}G_0(\mathbb{C})$ due to Casselman-Shalika-Shintani \cite{Sh, CS}. We provide an expression for lattice Whittaker function as a trace over a graded pieces of an infinite-dimensional representation of a Heisenberg algebra. In the appropriate limit this leads to an Archimedean variant of Casselman-Shalika-Shintani formula.

The plan of this paper is as follows. In Section 2 we propose a generalization of the Baxter $Q$-operator for $\mathfrak{gl}_{\ell+1}$-Toda chain and discuss a relations with operators arising in \cite{DT1, DT2} and \cite{BK}. In Sections 3 and 4 we consider various versions of Toda chains related to affine Lie algebras, corresponding $Q$-operators and the eigenfunctions (generalizations of Whittaker functions). Local archimedean $L$-factors of affine Lie algebras are proposed to coincide with the eigenvalues of the $Q$-operators. In Section 5 an interpolation of Archimedean and non-Archimedean Whittaker functions and local $L$-factors is constructed using the a lattice version of the affine Toda eigenfunctions. In Section 6 arithmetical interpretations of the constructions presented in the previous Sections are given. In particular the role of the Weil group $W_\mathbb{R}$ is discussed and an Archimedean analog of the Casselman-Shalika-Shintani formula is proposed.

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2 A Generalization of $\mathfrak{gl}_{\ell+1}$-Toda chain $Q$-operator

In this Section we propose a simple generalization of the Baxter $Q$-operator for $\mathfrak{gl}_{\ell+1}$-Toda chain. The generalized $Q$-operators reveals a connection between $Q$-operator eigenvalues and spectral determinants of ordinary differential operators. This is an analog of a spectral representation of Baxter operator’s eigenvalues in $c < 1$ Conformal Field Theories due to Dorey-Tateo \cite{DT1, DT2}.

We start by recalling the basic constructions in the theory of quantum $\mathfrak{gl}_{\ell+1}$-Toda chain. Quantum $\mathfrak{gl}_{\ell+1}$-Toda chain is a family of mutually commuting quantum Hamiltonians generated by $(\ell + 1)$ elementary Hamiltonians such that the first two are given by

$$\mathcal{H}_1^{\theta_{\ell+1}} = -\ell \sum_{i=1}^{\ell+1} \frac{\partial}{\partial x_i}, \quad (2.1)$$

$$\mathcal{H}_2^{\theta_{\ell+1}} = -\frac{1}{2} \sum_{i=1}^{\ell+1} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{\ell} e^{x_i-x_{i+1}}. \quad (2.2)$$

Common eigenfunctions of the Hamiltonians can be identified with the $\mathfrak{gl}_{\ell+1}$-Whittaker func-
produced. Class one principal series 

gl([Ko1, Ko2]) (see also [Et])

The eigenvalue of the Q-operator for gl_{ℓ+1}-Toda chain was intro-
duced. Class one principal series gl_{ℓ+1}-Whittaker functions are eigenfunctions of this operator with the eigenvalue given by the corresponding local Archimedean L-factor. Let

\[ \Phi^{gl_{ℓ+1}}_{γ_1, ..., γ_ℓ+1}(x_1, ..., x_ℓ+1) = e^{\sum_{j=1}^{ℓ+1} ρ_j x_j} \Psi^{gl_{ℓ+1}}_{γ_1, ..., γ_ℓ+1}(x_1, ..., x_ℓ+1), \]  

where \( ρ \in \mathbb{R}^{ℓ+1} \), with \( ρ_j = \frac{ℓ}{2} + 1 - j \), \( j = 1, ..., ℓ + 1 \).

**Theorem 2.1** Let \( Q^{gl_{ℓ+1}}(λ) \) be an integral operator with the kernel

\[ Q^{gl_{ℓ+1}}(x, y|λ) = 2^{ℓ+1} \exp \left\{ \sum_{j=1}^{ℓ+1} (iλ + ρ_j)(x_j - y_j) - \pi \sum_{k=1}^{ℓ} \left( e^{2(x_k - y_k)} + e^{2(y_k - x_{k+1})} \right) - \pi e^{2(x_{ℓ+1} - y_{ℓ+1})} \right\}. \]  

Let \( \Phi^{gl_{ℓ+1}}(x) \) be a modified gl_{ℓ+1}-Whittaker function \((2.5)\) given by the integral representation

\[ \Phi^{gl_{ℓ+1}}_{γ_1, ..., γ_ℓ+1}(x_1, ..., x_ℓ+1) = e^{\langle ρ, x \rangle} \int_{\mathbb{R}^{ℓ(ℓ+1)/2}} \prod_{k=1}^{ℓ} \prod_{i=1}^{k} dx_{k,i} e^{T^{gl_{ℓ+1}}(x)}, \]  

where

\[ T^{gl_{ℓ+1}}(x) = i \sum_{k=1}^{ℓ+1} γ_k \left( \sum_{i=1}^{k} x_{k,i} - \sum_{i=1}^{k-1} x_{k-1,i} \right) - \pi \sum_{k=1}^{ℓ} \sum_{i=1}^{k} \left( e^{2(x_{k+1,i} - x_{k,i})} + e^{2(x_{k,i} - x_{k+1,i+1})} \right), \]

and \( x_i := x_{ℓ+1,i}, \ i = 1, ..., ℓ + 1 \). The following relation holds

\[ Q^{gl_{ℓ+1}}(λ) \cdot \Phi^{gl_{ℓ+1}}_{γ_1, ..., γ_ℓ+1}(x_1, ..., x_ℓ+1) = \prod_{j=1}^{ℓ+1} \pi^{-iλ - iγ_j} \Gamma\left(\frac{iλ - iγ_j}{2}\right) \Phi^{gl_{ℓ+1}}_{γ_1, ..., γ_ℓ+1}(x_1, ..., x_ℓ+1). \]

The eigenvalue of the Q-operator

\[ Q^{gl_{ℓ+1}}(λ|γ_1, ..., γ_ℓ+1) = \prod_{j=1}^{ℓ+1} \pi^{-iλ - iγ_j} \Gamma\left(\frac{iλ - iγ_j}{2}\right), \]  

(2.8)
is identified with a local Archimedean $L$-factor corresponding to a principal series representation induced from a Borel subgroup of $GL(\ell+1, \mathbb{R})$ with a character parametrized by $(\gamma_1, \ldots, \gamma_{\ell+1})$ (see [GLO] for details). On the other hand the eigenvalue (2.8) is an important ingredient of the Quantum method of separation of variables [Gu, Sk, KL1] for $\mathfrak{sl}_{\ell+1}$-Toda chain. Quantum separation of variables in a quantum mechanical system with $N$ degrees of freedom is a unitary transformation relating Hamiltonian eigenfunctions of the initial system with separated eigenfunctions of a new system given by $N$ mutually non-interacting subsystems. Thus the separated eigenfunctions of the transformed Hamiltonian are given by the products of $N$ elementary eigenfunctions of auxiliary Hamiltonians. Eigenvectors of $\mathfrak{sl}_{\ell+1}$-Toda chain can be obtained from eigenfunctions $\Phi_{\gamma_1, \ldots, \gamma_{\ell+1}}^g(x_1, \ldots, x_{\ell+1})$ by specializing spectral variables $\sum_{j=1}^{\ell+1} \gamma_j = 0$ and satisfy the invariance condition

$$\Phi_{\gamma_1, \ldots, \gamma_{\ell+1}}^g(x_1+a, \ldots, x_{\ell+1}+a) = \Phi_{\gamma_1, \ldots, \gamma_{\ell+1}}^g(x_1, \ldots, x_{\ell+1}).$$

Quantum separation of variables in $\mathfrak{sl}_{\ell+1}$-Toda chain is given by the following integral

$$\Phi_{\gamma_1, \ldots, \gamma_{\ell+1}}^g(x_1, \ldots, x_{\ell+1}) = \int \prod_{j=1}^{\ell} dy_j \tilde{\Phi}_{\gamma_1, \ldots, \gamma_{\ell+1}}^g(y_1, \ldots, y_\ell) R(x_1, \ldots, x_{\ell+1}; y_1, \ldots, y_\ell),$$

where $\tilde{\Phi}_{\gamma_1, \ldots, \gamma_{\ell+1}}^g(y_1, \ldots, y_\ell)$ is a separated eigenfunction

$$\tilde{\Phi}_{\gamma_1, \ldots, \gamma_{\ell+1}}^g(y_1, \ldots, y_\ell) = \prod_{j=1}^{\ell} \phi(y_j; \gamma_1, \ldots, \gamma_{\ell+1}),$$

and

$$R(x_1, \ldots, x_{\ell+1}; y_1, \ldots, y_\ell) = \int \prod_{j=1}^{\ell} d\tilde{\gamma}_j \ e^{-i\sum_{j=1}^{\ell} \tilde{\gamma}_j y_j} \ \Phi^g_{\tilde{\gamma}}(x),$$

$$\Phi^g_{\tilde{\gamma}}(x_1, \ldots, x_{\ell+1}) = e^{-\sum_{j=1}^{\ell} \tilde{\gamma}_j x_{\ell+1}} \Phi^g_{\gamma_1, \ldots, \gamma_{\ell}}(x_1, \ldots, x_{\ell})$$

is a kernel of the integral unitary transformation providing Quantum separation of variables. Elementary eigenfunctions are given by Fourier transform of the eigenvalue (2.8)

$$\phi(y; \gamma_1, \ldots, \gamma_{\ell+1}) = \int_{\mathbb{R}} d\lambda \ e^{i\lambda y} Q^{g_{\ell+1}}(\lambda|\gamma_1, \ldots, \gamma_{\ell+1}).$$

The equations on the separated eigenfunctions

$$\left(\prod_{j=1}^{\ell+1} (\alpha \gamma_j - \partial_y) - e^{-y})\phi(y; \gamma_1, \ldots, \gamma_{\ell+1}) = 0, \right.$$

is an instance of the Baxter equation. The ordinary differential operator in (2.10) can be considered as an oper in the sense of [BD] corresponding to the Whittaker function $\Phi_{\gamma_1, \ldots, \gamma_{\ell+1}}^g(x_1, \ldots, x_{\ell+1})$ (on the relation between quantum separation of variables and opers see [Fr]). Note that this oper has irregular singularities.

Rather surprisingly the integral operator (2.6) is non-trivial even for $\ell = 0$ (the quantum separation of variables is trivial in this case).
Corollary 2.1 The integral operator $Q(\lambda)^{gl_1}$ with the kernel
\begin{equation}
Q^{gl_1}(x, y|\lambda) = 2e^{i\lambda(x-y)-\pi e^{2(x-y)}}, \tag{2.11}
\end{equation}
acts on the $gl_1$-Whittaker function
\begin{equation}
\Phi^{gl_1}_\gamma(x) = e^{i\gamma x}, \tag{2.12}
\end{equation}
as follows
\begin{equation}
Q^{gl_1}(\lambda) \cdot \Phi^{gl_1}_\gamma(x) = \pi^{-\frac{\lambda-\gamma}{2}} \Gamma\left(\frac{\lambda - i\gamma}{2}\right) \Phi^{gl_1}_\gamma(x). \tag{2.13}
\end{equation}
The eigenvalue
\begin{equation}
Q^{gl_1}(\lambda|\gamma) = \pi^{-\frac{\lambda-\gamma}{2}} \Gamma\left(\frac{\lambda - i\gamma}{2}\right), \tag{2.14}
\end{equation}
is an elementary local Archimedean $L$-factor.

Now we introduce a generalized Baxter $Q$-operator for the Toda chain depending on an additional variable. An advantage of the generalized $Q$-operator is in providing a simple relation with [DT1, DT2].

Definition 2.1 Define a generalized Baxter operator by the following explicit formula for its integral kernel
\begin{equation}
Q^{gl_{\ell+1}}(x, y|\lambda, t) = 2^{\ell+1} \exp\left\{\sum_{j=1}^{\ell+1} (i\lambda - i\gamma_j)(x_j - y_j) - \right. \tag{2.15}
\end{equation}
\begin{equation}
- t\pi \sum_{k=1}^\ell \left( e^{2(x_k-y_k)} + e^{2(y_k-x_{k+1})} \right) - t\pi e^{2(x_{\ell+1}-y_{\ell+1})}\}.
\end{equation}

Proposition 2.1 The following relations hold
\begin{equation}
Q^{gl_{\ell+1}}(\lambda, t) \cdot \Phi^{gl_{\ell+1}}_\gamma(x) = Q^{gl_{\ell+1}}(\lambda, t) \Phi^{gl_{\ell+1}}_\gamma(x) = \prod_{j=1}^{\ell+1} (\pi t)^{-\frac{\lambda - i\gamma_j}{2}} \Gamma\left(\frac{\lambda - i\gamma_j}{2}\right) \Phi^{gl_{\ell+1}}_\gamma(x), \tag{2.16}
\end{equation}
\begin{equation}
Q^{gl_{\ell+1}}(\lambda - 2t, t) = \left( \prod_{j=1}^{\ell+1} \frac{i\lambda - i\gamma_j}{2\pi t} \right) Q^{gl_{\ell+1}}(\lambda, t), \tag{2.17}
\end{equation}
\begin{equation}
\left\{ t \frac{\partial}{\partial t} + \frac{1}{2} \sum_{j=1}^{\ell+1} (i\lambda - i\gamma_j) \right\} Q^{gl_{\ell+1}}(\lambda, t) = 0,
\end{equation}
where $\gamma = (\gamma_1, \ldots, \gamma_{\ell+1})$, $x = (x_1, \ldots, x_{\ell+1})$. 6
It is useful to represent \( Q^{t, t+1}(\lambda, t) \) as a determinant of a matrix
\[
Q(t, \lambda) = \det \hat{Q}(t, \lambda), \quad \hat{Q}(t, \lambda) = \Gamma(\frac{i\lambda - i\hat{\gamma}}{2})(t\pi)^{-\frac{i\lambda - i\hat{\gamma}}{2}},
\] (2.18)
where \( \hat{\gamma} = \text{diag}(\gamma_1, \ldots, \gamma_{t+1}) \) is a diagonal matrix. Thus introduced matrix \( \hat{Q} \) satisfies difference-differential equations
\[
(t \frac{\partial}{\partial t} + \frac{i}{2}(\lambda - \hat{\gamma}))\hat{Q}(\lambda, t) = 0, \quad \hat{Q}(\lambda - 2\pi t, t) = \frac{i\lambda - i\hat{\gamma}}{2\pi t} \hat{Q}(\lambda, t).
\] (2.19)

**Remark 2.1** It was argued in [GLO] that eigenvalues of \( Q \)-operators should be identified with local archimedean \( L \)-factors. Given a generalization of \( Q \)-operator introduced above it is natural to consider a corresponding generalization of local non-Archimedean \( L \)-factors
\[
L_p(s, n) = \int_{p^n \mathbb{Z}_p} |x|^s d\mu_p(x) = \frac{p^{-ns}}{1 - p^{-s}}.
\] (2.20)

Then we have
\[
L_p(s, n + 1) = p^{-s} L_p(s, n),
\] (2.21)
and
\[
\frac{L_p(s + 1, n)}{L_p(s, n)} = p^{-n} \frac{L_p(s + 1, 0)}{L_p(s, 0)}, \quad L_p(s, 0) = L_p(s) = \frac{1}{1 - p^{-s}}.
\] (2.22)

The meaning of the matrix operator \( \hat{Q} \) is a fundamental matrix of solutions of the differential equation (2.19) for \( t \in [0, +\infty) \). The value at \( t = 1 \) of the fundamental solution of eigenvalue problem for a linear differential equation \( D(t, \partial_t) = t \frac{\partial}{\partial t} + \frac{i}{2}(\lambda - \hat{\gamma}) \) can be identified with the spectral determinant of \( D(t, \partial_t) \) (see [DT1, DT2] for a general discussion of this relation). Thus we get an identification of the spectral determinant of the operator \( D(t, \partial_t) \) with the Baxter \( Q \)-operator of \( \mathfrak{gl}_{t+1} \)-Toda chain which is in agreement with the results of [DT1, DT2]. Note that this relation is equivalent to the representation (2.29) defined below.

The spectral interpretation of the \( Q \)-operator’s eigenvalues can be supported by the following considerations (closely related with that in [C]). The following integral representation for \( \log \Gamma(z) \) holds
\[
\log \Gamma(\lambda - \gamma) = \int_0^\infty \frac{dt}{t} e^{-t} \left( (\lambda - \gamma - 1) + \frac{e^{-(\lambda - \gamma - 1)t} - 1}{1 - e^{-t}} \right) = \int_0^\infty \frac{dt}{t} \left( \frac{e^{-(\lambda - \gamma)t}}{1 - e^{-t}} + \text{regulators} \right).
\] (2.23)

Consider the operator \( K_t \) of scaling transformations with weight \( \gamma \) acting on the space of functions of one variable
\[
K_t f_\gamma(x) = e^{-i\gamma} f_\gamma(e^{-t}x).
\] (2.24)
It can be represented as an integral operator

\[ K_t = e^{-t(\gamma + x\partial_x)}, \tag{2.25} \]

with the kernel

\[ K_t(x, y) = e^{-t\gamma}\delta(y - e^{-t}x). \tag{2.26} \]

The trace of the operator is given by

\[ \text{Tr} K_t = \int dx e^{-t\gamma}\delta(x - e^{-t}x) = \frac{e^{-t\gamma}}{1 - e^{-t}}. \tag{2.27} \]

Therefore for appropriately regularized determinant (e.g using $\zeta$-function regularization)

\[ \log[\det_{\text{reg}}(\gamma + x\partial_x)]^{-1} = -\text{Tr}_{\text{reg}} \log(\gamma + x\partial_x) = \lim_{\epsilon \to 0} \text{Tr} \left( \int_\epsilon^\infty \frac{dt}{t} K_t - R(\epsilon) \right), \tag{2.28} \]

where $R(\epsilon)$ denotes singular terms such that r.h.s. of (2.28) has a finite limit. Using the integral representation (2.23) we obtain

\[ \Gamma(\lambda - \gamma) = [\det_{\text{reg}}(1 - \gamma\lambda + x\partial_x)]^{-1}. \tag{2.29} \]

Let us finally note that a very close determinant representation for $\Gamma$-functions arises naturally in the representation theory of a (double) of Yangian $Y(\mathfrak{gl}_1)$ \cite{KT}. Consider the following representations of $Y(\mathfrak{gl}_1)$ (for the notations see \cite{KT})

\[ \pi_1(h^-(z)) = 1 - \gamma z, \quad \pi_2(h^+(z)) = 1 - \frac{z}{\lambda}. \tag{2.30} \]

Then for the universal $R$-matrix $\mathcal{R}$ in the representations (2.30) we have

\[ R_{\pi_1 \otimes \pi_2} = (\pi_1(\lambda) \otimes \pi_2(\gamma))\mathcal{R} = \exp(\int dz \log \frac{\Gamma(z - \gamma)}{\Gamma(z)} \otimes \partial_z \log h^+(z)) = \frac{\Gamma(\lambda - \gamma)}{\Gamma(\lambda)} = [\det_{\text{reg}}(1 - \frac{\gamma}{\lambda + x\partial_x})]^{-1}. \tag{2.31} \]

Note that we can consider (2.31) as a continuous version of the fusion. This defines the appropriate version of the representation of $Q$-operator from the universal $R$-matrix for $\mathfrak{g} = \mathfrak{gl}_1$.

## 3 Affine Toda chain: generic level

In this section we generalize some of the results of \cite{GLO} to the case of the Toda chains for affine Lie algebras. The construction is based on the generalization \cite{Et} of the Kostant construction \cite{Ko1, Ko2} to affine Lie algebras. We consider affine Toda chains corresponding
to representations of affine Lie algebras at a generic level and critical level separately. The main objective is the identification of eigenvalues of analogs of Baxter $Q$-operators with the local Archimedean $L$-factors corresponding to affine Lie groups. Therefore, taking into account the discussion in the previous Section we mainly consider the first non-trivial case of $\widehat{\mathfrak{gl}}_1$ and briefly discuss generalizations to $\widehat{\mathfrak{gl}}_{\ell+1}$, $\ell > 0$.

To introduce notations, recall the standard definitions for affine Lie algebra $\widehat{\mathfrak{gl}}_1$. Affine Lie algebra $\widehat{\mathfrak{gl}}_1$ is generated by $K$, $d$, $u_n$, $n \in \mathbb{Z}$ with the relations:

$$[u_n, u_m] = -nK\delta_{n+m, 0}, \quad [K, u_n] = 0, \quad [d, u_n] = nu_n. \quad (3.1)$$

We will consider the following representations of the universal enveloping algebra $\mathcal{U}(\widehat{\mathfrak{gl}}_1)$. A representation $\pi_{\gamma, \kappa}$ is generated by the highest vector $|v_+\rangle$ and satisfies

$$u_n|v_+\rangle = 0, \quad u_0|v_+\rangle = \gamma|v_+\rangle, \quad K|v_+\rangle = \kappa|v_+\rangle. \quad (3.2)$$

This representation can be realized in the space of polynomials of variables $x_n$, $n \in \mathbb{Z}_+$ and exponential functions of $x_0$ as

$$\pi_{\gamma, \kappa} (u_k) = -kx_k, \quad \pi_{\gamma, \kappa} (u_0) = \frac{\partial}{\partial x_0}, \quad \pi_{\gamma, \kappa} (u_k) = \kappa \frac{\partial}{\partial x_k}, \quad (3.3)$$

$$\pi_{\gamma, \kappa} (K) = \kappa, \quad \pi_{\gamma, \kappa} (d) = \left( \mu + \frac{1}{2} \frac{\partial^2}{\partial x_0^2} - \sum_{k=1}^{\infty} kx_k \frac{\partial}{\partial x_k} \right).$$

Another type of representations is a generalization of the evaluation representation to the case of the algebra with the derivative operator $d$

$$\pi_{ev} (u_k) = \tau^k, \quad \pi_{ev} (d) = \tau \frac{\partial}{\partial \tau}, \quad \pi_{ev} (K) = 0, \quad (3.4)$$

acting on the space of functions of $\tau$. Left and right Whittaker vectors $\psi_L$, $\psi_R$ in the representation $\pi_{\gamma, \kappa}$ are defined by the conditions

$$u_{n>0} \psi_R = 0, \quad u_{n<0} \psi_L = 0, \quad u_0 \psi_{R, L} = i\gamma \psi_{R, L}. \quad (3.5)$$

Let us introduce a spherical subalgebra $\mathfrak{f}$ of $\widehat{\mathfrak{gl}}_1$. Define an involution $*: u_k \mapsto -u_{-k}$. The spherical subalgebra $\mathfrak{f} \subset \widehat{\mathfrak{gl}}_1$ is defined as a fixed point subalgebra with respect to the involution $^{[4]} *$. The spherical vector $\psi_K$ is an element of the representation space of $\pi_{\gamma, \mu, \kappa}$ defined by

$$(u_k - u_{-k}) \psi_K = 0, \quad n = 1, 2, \ldots. \quad (3.6)$$

Explicitly Whittaker and spherical vectors can be represented as follows

$$\psi_L = e^{\gamma x_0} \prod_{n=1}^{\infty} \delta(x_n), \quad \psi_R = e^{\gamma x_0}, \quad \psi_K = e^{\gamma x_0} \prod_{n=1}^{\infty} \sqrt{\frac{n}{2\pi \kappa}} e^{-n x_n^2/2\kappa}. \quad (3.7)$$

$^{[4]}$More generally for $\widehat{\mathfrak{gl}}_{\ell+1}$ we should consider $g(\sigma) \rightarrow (g(2\pi - \sigma)^T)^{-1}$. 

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There is a natural non-degenerate pairing between $\mathcal{U}(\mathfrak{gl}_1)$-modules spanned by $\psi_L$ and $\psi_R$, and by $\psi_R$ and $\psi_K$ respectively:

$$\langle \psi_L | \psi_R \rangle = \int_{\mathbb{R}^2} \overline{\psi_L(x)} \psi_R(x) \, dx = 1, \quad (3.8)$$

$$\langle \psi_K | \psi_R \rangle = \int_{\mathbb{R}^2} \overline{\psi_K(x)} \psi_R(x) \, dx = 1, \quad [dx] = \prod_{n=1}^{\infty} dx_n.$$  

Let us parameterize elements of the Cartan subgroup of $\widehat{GL}_1$ as

$$e^u = \exp\left\{ t_0 u_0 + t_+ K + t_- d \right\}. \quad (3.9)$$

One defines Whittaker function as:

$$\Psi(t_0, t_-, t_+) = \langle \psi_L | \pi_{\gamma, \kappa, \mu}(e^{t_0 u_0 + t_+ K + t_- d}) | \psi_R \rangle =$$

$$= \langle \psi_K | \pi_{\gamma, \kappa, \mu}(e^{t_0 u_0 + t_+ K + t_- d}) | \psi_R \rangle = \exp \left\{ \nu \gamma t_0 + t_+ \kappa + t_- (\mu - \frac{\gamma^2}{2}) \right\}. \quad (3.10)$$

The Whittaker function satisfies a set of equations given by the projections of the functions of the generators of $\mathfrak{gl}_1$ that commute with any element of $\mathcal{U}(\mathfrak{gl}_1)$. We consider only the part of equations that govern the dependence on $t_0$. There is a general construction of the central elements in terms of the spectral invariants. In our case it is useful to consider the following element of an appropriate generalization of $\mathcal{U}(\mathfrak{gl}_1)$ (we allow exponential functions of generators, the standard completion of $\mathcal{U}(\mathfrak{gl}_1)$ has a trivial center)

$$\hat{t}(\lambda) = [\det_V \tau \partial_{\tau}]^{-1} \cdot \det_V (\tau \partial_{\tau} + J(\tau) + \lambda), \quad J(\tau) = \sum_{n \in \mathbb{Z}} u_n \tau^n,$$

where $\tau = \exp i\sigma$, $\sigma \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$, $V = L^2(S^1)$ is the space of square-integrable functions on the circle $S^1$, and the determinant $\det_V \tau \partial_{\tau}$ is taken over subspace orthogonal to the constant functions on $S^1$. Here we have used the generalized evaluation representation $[3.2]$. The Whittaker function $\Psi_{\gamma}^{\hat{t}_1}(x)$ satisfies the following equation

$$\hat{t}(\lambda) \Psi_{\gamma}^{\hat{t}_1}(x) = [\det_V \tau \partial_{\tau}]^{-1} \cdot \det_V (\tau \partial_{\tau} + \lambda - \gamma) \Psi_{\gamma}^{\hat{t}_1}(x), \quad (3.11)$$

where $\Psi_{\gamma}^{\hat{t}_1}(x) = \Psi(t_0, t_+, t_-), t_0 = x, t_\pm = 0$. For the eigenvalue of $\hat{t}(\lambda)$ we have the following explicit expression (using the $\zeta$-regularization)

$$t(\lambda) = [\det_V \tau \partial_{\tau}]^{-1} \cdot \det_V (\tau \partial_{\tau} + (\lambda - \gamma)) =$$

$$(\lambda - \gamma) \prod_{n \neq 0} (1 - i \frac{\lambda - \gamma}{n \kappa}) = \frac{\kappa}{\pi} \text{sh} \frac{\lambda - \gamma}{\kappa}. \quad (3.12)$$

Thus the eigenvalue problem $(3.11)$ can be rewritten as follows

$$(q^{\frac{1}{2}(\lambda+i\partial_{x_0})} - q^{-\frac{1}{2}(\lambda+i\partial_{x_0})}) \Psi_{\gamma}^{\hat{t}_1}(x) = (q^{\frac{1}{2}(\lambda-\gamma)} - q^{-\frac{1}{2}(\lambda-\gamma)}) \Psi_{\gamma}^{\hat{t}_1}(x), \quad (3.13)$$
where \( q = \exp(2\pi i/x) \).

Note that (3.13) has infinite number of solutions because the difference operator in r.h.s. commutes with any function of \( x \) periodic with respect to the shift \( x \rightarrow x + 2\pi x^{-1} \). Similarly to the case of the wave-functions for finite Lie algebra Toda chains the choice of the solutions is dictated by the analytic properties of the solutions. As it was argued in [KLS] the appropriate way to impose the right analytic conditions is to use additional “dual” equations

\[
(q^\gamma (\lambda + i\partial_{\gamma_0}) - q^{-\gamma} (\lambda + i\partial_{\gamma_0})) \Psi^\gamma_l(x) = (\bar{q}^\gamma (\lambda - \gamma) - \bar{q}^{-\gamma} (\lambda - \gamma)) \tilde{\Psi}^\gamma_l(x),
\]

where \( \bar{q} = \exp(2\pi i x) \).

Using an analog of the Kostant reduction [Ko1, Ko2] for affine Lie algebra \( \hat{g} \) at the level \( k \), we obtain Hamiltonians for relativistic \((q\text{-deformed})\) Toda chain with \( q = \exp(2\pi i/(k + h^\vee)) \), \( h^\vee \) being the dual Coxter number. This is not surprising taking into account a correspondence between representations of \( \hat{g} \) of level \( k \) and \( \mathcal{U}_q(g) \), \( q = \exp(2\pi i/(k + h^\vee)) \). Thus in the case of \( \hat{g}_{\ell+1} \)-Toda chain the resulting difference equations (analogs of (3.13)) should be eigenfunction equations for generating function of quantum Hamiltonians of \( q \)-Toda chain corresponding to \( g_{\ell+1} \). This theory was considered in the framework of the quantum separation of variables in [KLS]. In the case of the \( sl_{\ell+1} \)-Toda chain we have

\[
\tilde{\Psi}^{\hat{g}_{\ell+1}}_{\gamma_1, \ldots, \gamma_{\ell+1}}(x_1, \ldots, x_{\ell+1}) = \int_{\mathbb{R}^\ell} \prod_{j=1}^{\ell} dy_j \tilde{\Psi}^{\hat{g}_{\ell+1}}_{\gamma_1, \ldots, \gamma_{\ell+1}}(y_1, \ldots, y_{\ell}) R(x_1, \ldots, x_{\ell+1}; y_1, \ldots, y_{\ell}),
\]

where we imply that \( \sum_{j=1}^{\ell+1} \gamma_j = 0 \) and \( \tilde{\Psi}^{\hat{g}_{\ell+1}}_{\gamma_1, \ldots, \gamma_{\ell+1}}(y_1, \ldots, y_{\ell+1}) \) is a separated eigenfunction

\[
\tilde{\Psi}^{\hat{g}_{\ell+1}}_{\gamma_1, \ldots, \gamma_{\ell+1}}(y_1, \ldots, y_{\ell}) = \prod_{j=1}^{\ell} \phi(y_j; \gamma_1, \ldots, \gamma_{\ell+1}),
\]

and

\[
R(x_1, \ldots, x_{\ell+1}; y_1, \ldots, y_{\ell}) = \int \prod_{j=1}^{\ell} d\gamma_j e^{-i\Sigma_{j=1}^{\ell} \gamma_j y_j} \tilde{\Psi}^{\tilde{\sigma}_{\ell+1}}_{\gamma}(x),
\]

is a kernel of the integral unitary transformation providing quantum separation of variables for affine Toda chain. For the explicit form of the integral kernel \( \tilde{\Psi}^{\tilde{\sigma}_{\ell+1}}_{\gamma}(x) \) see [KLS]. Elementary eigenfunctions are given by Fourier transform of the eigenvalue (2.8)

\[
\phi(y; \gamma_1, \ldots, \gamma_{\ell+1}) = \int_{\mathbb{R}} d\lambda e^{i\lambda y} Q^{\hat{g}_{\ell+1}}(\lambda | \gamma_1, \ldots, \gamma_{\ell+1}).
\]

The equations on the separated eigenfunctions

\[
\left( \prod_{j=1}^{\ell+1} \left( q^\gamma (\gamma_j + i\partial_y) - q^{-\gamma} (\gamma_j + i\partial_y) \right) - e^{-y} \right) \phi(y; \gamma_1, \ldots, \gamma_{\ell+1}) = 0,
\]

are equivalent to the Baxter equation on \( Q^{\hat{g}_{\ell+1}}(\lambda | \gamma_1, \ldots, \gamma_{\ell+1}) \). The ordinary difference operator in (3.16) can be considered as a generalization of the oper in the sense of [BD].
corresponding to the Whittaker function $\Psi_{b_{\gamma_1,\ldots,\gamma_{\ell+1}}}(x_1,\ldots,x_{\ell+1})$. Note that this oper has irregular singularities.

By an analogy with the finite Lie algebra case we can expect that the Fourier transform of an elementary wave-function $\phi(y;\gamma_1,\ldots,\gamma_{\ell+1})$ entering the quantum separated problem provides a reasonable candidate for the eigenvalue of the Baxter $Q$-operator for $\mathfrak{gl}_1$. The Fourier transform of the wave-function is given by

$$Q_{b_{\mathfrak{gl}_1}}(i\lambda - i\gamma) = \sqrt{2\pi}(\frac{2\pi}{\kappa})^{\frac{1}{2} - i\lambda + i\gamma} S_2^{-1}(i\lambda - i\gamma), \quad (3.17)$$

where the function $S_2(z)$ is a specialization of the function defined in the Appendix B for $\omega_1 = 1$ and $\omega_2 = \kappa$. The function $S_2^{-1}(z)$ satisfies the following functional relations

$$S_2^{-1}(z + 1) = 2\sin\frac{\pi z}{\kappa} \cdot S_2^{-1}(z), \quad S_2^{-1}(z + \kappa) = 2\sin(\pi z) \cdot S_2^{-1}(z), \quad (3.18)$$

that provide analogs of Baxter equation on the eigenvalue $Q_{b_{\mathfrak{gl}_1}}(i\lambda - i\gamma)$. Note that when $\kappa \to \infty$ the eigenvalues $Q_{b_{\mathfrak{gl}_1}}(i\lambda - i\gamma)$ reduces to the eigenvalues of the Baxter operator for $\mathfrak{gl}_1$-Toda chain

$$\lim_{\kappa \to \infty} Q_{b_{\mathfrak{gl}_1}}(z) = \Gamma(z). \quad (3.19)$$

Introduce the function

$$S(z) = e^{-\frac{4\pi}{\kappa} B_{2,2}(z)} S_2(z), \quad B_{2,2}(z) = \frac{1 + \kappa}{\kappa} z^2 - \frac{1 + \kappa}{\kappa} z + \frac{1 + 3\kappa + \kappa^2}{6\kappa}. \quad (3.19)$$

**Proposition 3.1** Let $Q_{b_{\mathfrak{gl}_1}}(\lambda)$ be an integral operator with the kernel

$$Q_{b_{\mathfrak{gl}_1}}(x,y|\lambda) = 2\pi \sqrt{\kappa} \int_{\mathbb{R} - \kappa} d\rho \, e^{-\frac{\pi}{\kappa} \left( y - x + \ln\frac{2\pi}{\kappa} - \frac{1 + \kappa}{2\kappa} \right)^2} e^{\rho(1 + \kappa)} S^{-1}(-i\frac{\rho\kappa}{2\pi}), \quad (3.20)$$

where $\epsilon > 0$ and $\kappa > 0$. Then the following holds

$$Q_{b_{\mathfrak{gl}_1}}(\lambda) \cdot \Psi_{b_{\mathfrak{gl}_1}}(x) = \int dy \, Q_{b_{\mathfrak{gl}_1}}(x - y|\lambda) \Psi_{b_{\mathfrak{gl}_1}}(y) = Q_{b_{\mathfrak{gl}_1}}(i\lambda - i\gamma) \Psi_{b_{\mathfrak{gl}_1}}(x). \quad (3.21)$$

Thus $Q_{b_{\mathfrak{gl}_1}}(x,y|\lambda)$ is a kernel of the Baxter $Q$-operator for $\mathfrak{gl}_1$. The proposed local Archimedean $L$-factor corresponding to a class of automorphic representations of affine Lie groups is then given by

$$L_{\infty}(s_{\gamma_1,\ldots,\gamma_{\ell+1}}) = \prod_{j=1}^{\ell+1} Q(s - \gamma_j), \quad (3.22)$$

where $Q(s)$ is given by (3.17).
4 Affine Toda chain: critical level

In this Section we shortly comment on the \( \hat{\mathfrak{gl}}_{\ell+1} \)-Toda chain corresponding to representations of affine Lie algebra \( \hat{\mathfrak{gl}}_{\ell+1} \) at the critical level. In this case there is a large center in the universal enveloping algebra \( \mathcal{U}(\hat{\mathfrak{gl}}_{\ell+1})|_{k=-h^\vee} \). Linear and quadratic Hamiltonians of the corresponding Toda chain are given by

\[
\mathcal{H}_1^{\hat{\mathfrak{gl}}_{\ell+1}}(x_i) = -i \sum_{i=1}^{\ell+1} \frac{\partial}{\partial x_i},
\]

\[
\mathcal{H}_2^{\hat{\mathfrak{gl}}_{\ell+1}}(x_i) = -\frac{1}{2} \sum_{i=1}^{\ell+1} \frac{\partial^2}{\partial x_i^2} + e^{x_{\ell+1} - x_1} + \sum_{i=1}^{\ell} e^{x_i - x_{i+1}}.
\]

The corresponding integral Baxter \( \mathcal{Q} \)-operator was introduced in [PG]

\[
\mathcal{Q}_0^{\hat{\mathfrak{gl}}_{\ell+1}}(x_1, \ldots, x_{\ell+1}; y_1, \ldots, y_{\ell+1}|\lambda) = \exp \left\{ \sum_{j=1}^{\ell+1} i\lambda(x_j - y_j) - \sum_{i=1}^{\ell+1} (e^{x_i - y_i} + e^{y_{i+1} - x_i}) \right\},
\]

where \( y_{\ell+2} = y_1 \).

In the case of \( \hat{\mathfrak{gl}}_1 \)-Toda theory one has

\[
\mathcal{H}_1^{\hat{\mathfrak{gl}}_1}(x) = -i \frac{\partial}{\partial x}, \quad \Psi^{\hat{\mathfrak{gl}}_1}_\gamma(x) = e^{x\gamma x}.
\]

Although the Hamiltonian \( \mathcal{H}_1^{\hat{\mathfrak{gl}}_1} \) and \( \hat{\mathfrak{gl}}_1 \)-Whittaker function are very simple the corresponding \( \mathcal{Q} \)-operator is non-trivial

\[
\mathcal{Q}_0^{\hat{\mathfrak{gl}}_1}(x, y|\lambda) = \exp \left\{ i\lambda(x - y) - e^{x-y} - e^{y-x} \right\},
\]

\[
\mathcal{Q}_0^{\hat{\mathfrak{gl}}_1}(\lambda, \gamma) \cdot \Psi^{\hat{\mathfrak{gl}}_1}_\gamma(x) = \int_\mathbb{R} dy \ \mathcal{Q}_0^{\hat{\mathfrak{gl}}_1}(x, y|\lambda) \Psi^{\hat{\mathfrak{gl}}_1}_\gamma(y) = \mathcal{Q}_0^{\hat{\mathfrak{gl}}_1}(\lambda, \gamma) \Psi^{\hat{\mathfrak{gl}}_1}_\gamma(x),
\]

where the eigenvalue \( \mathcal{Q}_0^{\hat{\mathfrak{gl}}_1}(\lambda, \gamma) \) is given by the value of the normalized Macdonald function 2\( K_{i\lambda-r\gamma}(2e^\tau) \) of zero argument \( \tau = 0 \)

\[
\mathcal{Q}_0^{\hat{\mathfrak{gl}}_1}(\lambda, \gamma) = 2K_{i\lambda-r\gamma}(2) = \int_0^\infty \frac{dv}{v} v^{-i(\lambda-\gamma)} e^{-(v+v^{-1})},
\]

where the following integral formula for Macdonald function is used

\[
K_\nu(2e^\tau) = \frac{1}{2} \int_0^\infty v^{-\nu-1} e^{-\tau(v+v^{-1})} dv.
\]
The eigenvalue (4.6) satisfies the Baxter equation
\[ ((\iota \lambda - \iota \gamma) + e^{\iota \theta} - e^{-\iota \theta})Q_0^{\tilde{gl}_1}(\lambda, \gamma) = 0. \] (4.8)

Let us note that this elementary Baxter operator is compatible with the quantum separation of variables in affine Toda chain for \( \hat{\mathfrak{gl}}_{\ell+1} \).

The separated wave function satisfies the Baxter equation
\[ \prod_{j=1}^{\ell+1} (\iota \lambda - \iota \gamma_j) + e^{\iota \theta} - e^{-\iota \theta}Q_0^{\tilde{gl}_{\ell+1}}(\lambda, \gamma_1, \ldots, \gamma_{\ell+1}) = 0. \] (4.9)

In the coordinate representation we have
\[ \prod_{j=1}^{\ell+1} (\partial_y - \iota \gamma_j) + e^{-y} + (1)^{\ell+1}e^{y}\psi(y, \gamma_1, \ldots, \gamma_{\ell+1}) = 0, \] (4.10)

where
\[ \psi(y, \gamma_1, \ldots, \gamma_{\ell+1}) = \int d\lambda e^{\iota \lambda y} Q_0^{\tilde{gl}_{\ell+1}}(\lambda, \gamma_1, \ldots, \gamma_{\ell+1}). \] (4.11)

This can be interpreted as \( \hat{\mathfrak{gl}}_{\ell+1} \)-oper (in terminology of [BD]) corresponding to affine Whittaker function. Let us stress that it has irregular singularities.

Note that (4.10) can be considered as an eigenfunction \( \Psi_{\lambda-\gamma}^{\mathfrak{sl}_2}(e^\tau) \) of \( \mathfrak{sl}_2 \)-Toda chain at \( \tau = 0 \). Thus it is natural to define a generalized \( Q \)-operator for \( \hat{\mathfrak{gl}}_1 \) such that its eigenvalues are given by eigenfunction
\[ \Psi_{\lambda-\gamma}^{\mathfrak{sl}_2}(t) = 2 K_{\iota (\lambda-\gamma)}(2e^\tau), t = e^\tau. \] The eigenvalues \( Q_0^{\tilde{gl}_1}(t, \lambda) \) of the generalized \( Q \)-operator satisfy the relations
\[ Q_0^{\tilde{gl}_1}(t, \lambda + \iota) - Q_0^{\tilde{gl}_1}(t, \lambda - \iota) + \frac{\iota \lambda - \iota \gamma}{t} Q_0^{\tilde{gl}_1}(t, \lambda) = 0, \] (4.12)
\[ (\partial^2_t + \frac{4e^{2\tau}}{t} + (\lambda - \gamma)^2) Q_0^{\tilde{gl}_1}(t, \lambda) = 0, \quad t = e^\tau. \] (4.13)

Considered interpretation of the Baxter \( Q \)-operator as a restriction of the wave function of the \( \mathfrak{sl}_2 \)-Toda chain at \( \tau = 0 \) is consistent with [DT1, DT2], where the same type of relation was established for the \( Q \)-operator describing integrable structure of \( c < 1 \) Conformal Field Theories. The arguments from [DT1, DT2] imply that the eigenvalue of \( Q \)-operator for \( \hat{\mathfrak{gl}}_1 \) is equal to the spectral determinant of the operator (4.13). The generalization of this spectral interpretation for \( \hat{\mathfrak{gl}}_{\ell+1}, \ell \geq 1 \) will be given elsewhere.
5 On a lattice version of affine Whittaker functions

In this subsection we discuss a different definition of the Whittaker function for affine Lie group in the case of generic level. We will use a term $q$-deformed Toda chain for the quantum integrable system such that the affine Whittaker function is a corresponding common eigenvalue (see Et for a detailed description). It is easy to see that a solution of $q$-deformed $\mathfrak{gl}_1$-Toda chain equation (3.11) is unique when the eigenfunction $\Psi^{\hat{t}}_\gamma(x)$ is considered as a function on the lattice $2\pi \mathbf{Z} \ni \mathbf{R}$

$$\Psi^{\hat{t}}_\gamma(2\pi n x^{-1}) = e^{i2\pi n x^{-1} \gamma} = q^{n \gamma}, \quad n \in \mathbf{Z}. \quad (5.1)$$

An analog of the Baxter $Q$-operator in this case can be extracted from the explicit quantum separation of variables for $q$-version of $\mathfrak{gl}_{\ell+1}$-Toda chain. To simplify considerations below we allow complex continuation with respect to $x$. The approach of [KLS] is easily adopted to the case of eigenfunctions on the lattice with the following result for eigenvalues of the Baxter $Q$-operator

$$Q^{\hat{t}_{\ell+1}}_{\text{lattice}}(t|t_1, \ldots, t_{\ell+1}) = \prod_{i=1}^{\ell+1} \Gamma_q(t t_i^{-1}), \quad (5.2)$$

where a $q$-analog of $\Gamma$-functions is given by (see Appendix B)

$$\Gamma_q(t) = \prod_{m=0}^{\infty} \frac{1}{(1 - t q^m)}. \quad (5.3)$$

and we use the notations $t = q^a$, $t_i = q^{\chi_i}$. Eigenvalues (5.2) are reduced to eigenvalues of $Q$-operators for $\mathfrak{gl}_{\ell+1}$ in the limit $t = q^a$, $t_i = q^{\chi_i}$, $q \to 1$ (after simple renormalization of $Q$-operator). There is an analog of the Mellin-Barnes integral representations [KL1] [GKL] for $q$-Whittaker functions in terms of $\Gamma_q$-functions and Jackson integral. In the following we derive another representation of the $q$-version of $\mathfrak{gl}_2$-Whittaker function which has a close relation with Casselman-Shalika-Shintani formula for $p$-adic Whittaker function and will play an important role in the next Section.

The eigenfunction problem for $q$-deformed $\mathfrak{gl}_2$-Whittaker function on the lattice (common eigenfunction of $q$-versions of $\mathfrak{gl}_2$-Toda chain Hamiltonians) can be written in the following form [KLS]

$$t_1^{-1} t_2^{-1} \Psi^{(q)}_{t_1, t_2}(n-1) + (1 - q^{n+1}) \Psi^{(q)}_{t_1, t_2}(n+1) = (t_1^{-1} + t_2^{-1}) \Psi^{(q)}_{t_1, t_2}(n), \quad (5.4)$$

We relate this with the functional equation for $q$-version of Baxter $Q$-operator using the following expansion of the eigenvalue of $Q$-operator (see Appendix B)

$$Q^{\hat{t}_{\ell+1}}_{\text{lattice}}(t|t_1, \ldots, t_{\ell+1}) = \sum_{n_1, \ldots, n_{\ell+1}=0}^{\infty} \frac{\prod_{j=1}^{\ell+1} (t t_j^{-1})^{n_j}}{\prod_{i=1}^{\ell+1} \prod_{j=1}^{n_i} (1 - q^{j_i})} = \sum_{n=0}^{\infty} t^n \left( \sum_{n_1 + \cdots + n_{\ell+1}=n} \frac{\prod_{k=1}^{\ell+1} t_k^{-n_k}}{\prod_{i=1}^{\ell+1} \prod_{j=1}^{n_i} (1 - q^{j_i})} \right) = \sum_{n=0}^{\infty} t^n \chi_n^{(q)}(t_1, \ldots, t_{\ell+1}). \quad (5.5)$$

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where \( t_i = q^{n_i} \) and we introduce the following function\(^5\)

\[
\chi_n^{(q)}(t_1, \ldots, t_{\ell+1}) = \sum_{n_1 + \cdots + n_{\ell+1} = n} \frac{t_1^{-n_1} \cdots t_{\ell+1}^{-n_{\ell+1}}}{(n_1)_q \cdots (n_{\ell+1})_q},
\]

where \((n)_q = (1 - q) \cdots (1 - q^n)\). Consider the eigenvalue of the \( q \)-version of the \( Q \)-operator for \( \mathfrak{gl}_2 \)

\[
Q_{\text{lattice}}^{\hat{y}}(t|t_1, t_2) = \Gamma_q(t_{t_1}^{-1}) \Gamma_q(t_{t_2}^{-1}).
\]

It satisfies the following functional equation

\[
Q_{\text{lattice}}^{\hat{y}}(tq|t_1, t_2) = (1 - t_{t_1}^{-1})(1 - t_{t_2}^{-1}) Q_{\text{lattice}}^{\hat{y}}(t|t_1, t_2),
\]

which leads to the equation for coefficients \( \chi_n^{(q)}(t_1, t_2) \) in \((5.5)\)

\[
t_1^{-1}t_2^{-1} \chi_{n-1}^{(q)}(t_1, t_2) + (1 - q^{n+1}) \chi_n^{(q)}(t_1, t_2) = (t_1^{-1} + t_2^{-1}) \chi_n^{(q)}(t_1, t_2).
\]

Thus using \((5.4)\) we can identify

\[
\Psi_{t_1, t_2}^{(q)}(n) = \chi_n^{(q)}(t_1, t_2).
\]

Note that in the limit \( q \to 0 \) the r.h.s. reduces to the character of the finite-dimensional representation \( V_n = S^n \mathbb{C}^2 \) of \( \mathfrak{gl}_2 \). This is a special case of more general formula expressing Whittaker function in terms of characters. A well-known example of such type of relation is the Casselman-Shalika-Shintani formula for \( p \)-adic Whittaker function for group \( G(\mathbb{Q}_p) \) in terms of the character of the Langlands dual group \( ^L G_0(\mathbb{C}) \) \cite{Sh}, \cite{CS}. Indeed \((5.10)\) in the limit \( q \to 0, t_1 = p^{-y_1}, t_2 = p^{-y_2} \) turns into the Casselman-Shalika-Shintani formula which in the case of \( GL(2, \mathbb{Q}_p) \) is a consequence of a simple identity

\[
\frac{1}{(1 - p^{-y_1})(1 - p^{-y_2})} = \sum_{n=0}^{\infty} \chi_n(p^{-\hat{y}}).
\]

Here \( \hat{y} = \text{diag}(y_1, y_2) \) is a diagonal matrix and \( \chi_n(\hat{y}) \) is a character of the finite-dimensional representation \( S^n \mathbb{C}^2 \) given by \( n \)-th symmetric power of the standard two-dimensional representation of \( \mathfrak{gl}_2 \). In the limit \( t_i = q^{n_i}, t = q^s, q \to 1 \) the \( q \)-deformed Whittaker function \( \Psi_{t_1, t_2}^{(q)}(n) \) reproduces a specialization \( \Psi_{\gamma_1, \gamma_2}^{\hat{y}}(x, 0) \) of the standard \( \mathfrak{gl}_2 \)-Whittaker function and the expansion \((5.5)\) turns into the known relation between local Archimedean \( L \)-factors and \( \mathfrak{gl}_2 \)-Whittaker (see e.g. \cite{Bu})

\[
\Gamma(s + \gamma_1)\Gamma(s + \gamma_2) = \int_{-\infty}^{+\infty} dx \ e^{isx} \Psi_{\gamma_1, \gamma_2}^{\hat{y}}(x, 0).
\]

---

5 In terms of non-commutative variables \{\( t_k \)\} such that \( t_i t_j = q t_j t_i, i < j \) the function \((5.6)\) can be written in the following form

\[
\chi_n^{(q)}(t_1, \ldots, t_{\ell+1}) = \frac{(t_1 + \cdots + t_{\ell+1})^n}{(n)_q^\ell}.
\]
There is a generalization of the integral formulas (5.12) arising naturally in applications of the Rankin-Selberg method to analytic continuations of global $L$-functions

$$\int_{\mathbb{R}^\ell} \prod_{j=1}^\ell dx_j \Psi_{\gamma_1,\ldots,\gamma_\ell}(x_1, \ldots, x_\ell) \Psi_{\lambda_1,\ldots,\lambda_{\ell+1}}^{\ell+1}(x_1, \ldots, x_\ell, 0) = \prod_{i=1}^{\ell+1} \prod_{k=1}^\ell \Gamma(i\lambda_{\ell+1,i} - r\gamma_{i,k}). \quad (5.13)$$

This type of integral representations was considered in [St] (see also [GLO], Lemma 4.2). Its $p$-adic counterpart follows from the Cauchy identity

$$\prod_{i=1}^{\ell+1} \prod_{j=1}^\ell \frac{1}{1-z_i w_j} = \sum_\Lambda s_\Lambda(z) s_\Lambda(w), \quad (5.14)$$

where the sum goes over Young diagrams and $s_\Lambda(z)$ is the Schur polynomial corresponding to a diagram $\Lambda$. Using the expansion of the appropriate product of $\Gamma_q$ and iterative structure of (5.13) one can find an analog of (5.10) for a generic $q$-deformed $\mathfrak{g}_{\ell+1}$-Whittaker function.

An interpretation of the identity (5.10) and its generalization to more general Whittaker functions will be discussed in the next Section.

6 Towards Arithmetic interpretations

In this Section we put the results of the previous Sections in the proper perspective. The natural framework is provided by a geometry of the compactification divisor at infinity (corresponding to the Archimedean valuation) of the arithmetic scheme $\text{Spec}(\mathbb{Z})$. To describe the geometric structures in the vicinity of the compactification divisor it is essential to use a particular generalization of the standard Galois group provided by the notion of the Weil group. In the description of the Weil group of a number field we mainly follow [W], [De], [T] (see also [ABV]).

The Galois group for $\mathbb{R}$ is given by $\text{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}_2$ and a non-trivial generator corresponds to the complex conjugation. By standard arguments of abelian class field theory the Galois group $\text{Gal}(\mathbb{C}/\mathbb{R})$ can be identified with the group of unites $U_\mathbb{R} = \mathbb{R}^\ast / \mathbb{R}_+ = \pm 1$. However this picture is oversimplified. There is a canonical action of the multiplicative group $\mathbb{C}^\ast$ on the complexified cohomology of compact non-singular algebraic varieties over $\mathbb{C}$ which in many respects is analogous to the action of Galois group $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ on the étale cohomology of schemes over $\mathbb{F}_p$ (we do not consider a more general structure of weight filtrations of Hodge structures on non-compact singular algebraic varieties). To take into account this action one should consider Weil groups of $\mathbb{R}$ generalizing the Galois $\text{Gal}(\mathbb{C}/\mathbb{R})$. The Weil group $W_F$ of a number field $F$ should satisfy the following defining properties [T]. First, there should exist a homomorphism with a dense image in the natural topology on $\text{Gal}((\overline{F}/F)$

$$\phi : W_F \rightarrow \text{Gal}(\overline{F}/F).$$

Second, for any Galois extension $E$ of $F$, there should be an inclusion

$$r : E^\ast \rightarrow W_E^{ab},$$
such that the composition
\[ e \circ \phi : E^* \to \text{Gal}^{ab}(\overline{F}/E) \]
is a basic isomorphism of abelian class field theory. Here \( \overline{F} \) is an algebraic closure of \( F \) and \( \Gamma^{ab} = \Gamma/\Gamma, \Gamma \) is the abelianization of \( \Gamma \). Thus for \( F = \mathbb{C} \), \( \text{Gal}(\overline{\mathbb{C}}/\mathbb{C}) \) is trivial, \( W_{\mathbb{C}} = \mathbb{C}^* \), \( \phi \) is trivial and \( r \) is the identity map. For \( F = \mathbb{R} \), one has \( \text{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}_2 \). The Weil group \( W_{\mathbb{R}} = \mathbb{C}^* \cup j\mathbb{C}^* \) is generated by a copy of \( \mathbb{C}^* \) and an element \( j \), subjected to the relations:
\[ jxj^{-1} = x, \quad j^2 = -1 \in \mathbb{C}^*, \]
with the maps
\[ \phi : W_{\mathbb{R}} \to \text{Gal}(\mathbb{C}/\mathbb{R}), \quad \phi(x) = 1, \quad \phi(jx) = -1, \quad x \in \mathbb{C}, \]
\[ r : \mathbb{R}^* \to W_{\mathbb{R}}^{ab}, \quad r(x) = x. \]
Note that \( W_{\mathbb{R}} \) is non-abelian and its abelianization is \( W_{\mathbb{R}}^{ab} = W_{\mathbb{R}}/[W_{\mathbb{R}}, W_{\mathbb{R}}] = \mathbb{R}^* \).

In many problems the Weil group \( W_F \) plays the role of the Galois group of \( F \). Thus in the Langlands correspondence L-packages of automorphic admissible supercaspidal representations of \( G(F) \) are classified by conjugacy classes of continuous homomorphisms from the Weil group into Langlands dual group \( \Lambda G \). In the case of Archimedean fields we have the following explicit description of the spaces of homomorphism (Langlands parameters). Let \( \Lambda G_0(\mathbb{C}) \) be a complex Lie group, Langlands-dual to \( G \) and \( \mathfrak{g}_0 = \text{Lie}(\Lambda G_0) \). Then the set of conjugacy classes of continuous homomorphisms \( \phi \) from \( \mathbb{C}^* \) into \( \Lambda G_0 \) with a semi-simple image can be identified with the set of pairs \( (\lambda_1, \lambda_2) \in \mathfrak{g}_0 \times \mathfrak{g}_0 \) of semi-simple elements, subjected to the following requirements:
\[ [\lambda_1, \lambda_2] = 0, \quad \exp(2\pi i \lambda_1) = \exp(2\pi i \lambda_2). \]
The homomorphism \( \phi \) is then given by:
\[ \phi(e^t e^{i\theta}) = \exp(t(\lambda_1 + \lambda_2)) \exp(i\theta(\lambda_1 - \lambda_2)), \quad \phi(z) = z^\lambda x^{\lambda_2}, \quad z = t e^{i\theta}. \]
Langlands parameters \( \phi \) for \( \Lambda G_0 \), \( F = \mathbb{R} \) can be identified with the set of pairs \( (y, \lambda) \in \Lambda G_0 \times \mathfrak{g}_0 \), \( \lambda \) being semisimple, subjected to the conditions:
\[ [\lambda, \text{Ad}_y(\lambda)] = 0, \quad y^2 = \exp(2\pi i \lambda). \]
In particular, for \( \Lambda G_0(\mathbb{C}) = \text{GL}_1(\mathbb{C}) \) Langlands parameters for \( F = \mathbb{C} \) are given by the quasi-characters \( \chi_{s,m}(z) = |z|^s z^m, s \in \mathbb{C}, m \in \mathbb{Z} \). Similarly for \( F = \mathbb{R} \) the corresponding quasi-characters are given by \( \chi_{s,\epsilon}(z) = |z|^s z^{(1-\epsilon)/2}, s \in \mathbb{C}, \epsilon = \pm 1 \). Langlands parameters classify irreducible representations entering the decomposition of the space of functions on \( \text{GL}_1 \). The correspondence between quasi-characters of the Weil group \( W_{\mathbb{C}} \) and irreducible representations of \( \text{GL}_1(\mathbb{C}) \) can be described as follows. Let \( D = x \frac{\partial}{\partial z} \) be a vector field on a one-dimensional torus \( \mathbb{C}^* \). The following operator
\[ \Pi(z, \bar{z}) = e^{2\pi i D \log z} + 2\pi i D \log \bar{z}, \]

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acts on the space $F$ of functions on $GL_1(\mathbb{C}) = \mathbb{C}^*$. Decompose $F$ as a sum of eigenspaces of $\Pi(z, \bar{z})$. This provides a decomposition on one-dimensional irreducible representations with respect to the action of $GL_1(\mathbb{C})$ by translations. The eigenvalues of $\Pi(z, \bar{z})$ corresponding to a one-dimensional irreducible representation are given by elements of $Hom(\mathbb{C}^*, GL_1)$ in accordance with Langlands correspondence. This construction can be straightforwardly generalized to the case of algebraic tori $G = GL_1 \times \ldots \times GL_1$. Let us remark that a similar construction arises in representation theory of lattice vertex algebras in Conformal Field Theory. In particular as a part of the vertex algebra structure a $A$-graded Frobenius algebras $A = \bigoplus_{\alpha \in A} A_{\Delta_{\alpha}}$, $\Delta_{\alpha} \in \mathbb{R}$, with a semisimple action of $W_\mathbb{R}$ appears. The action of $W_\mathbb{R}$ on one-dimensional $\mathbb{R}$-vector spaces is given by $W_\mathbb{R} : A \to A, \quad t : A_{\Delta_{\alpha}} \to t^{\Delta_{\alpha}} A_{\Delta_{\alpha}}, \quad t \in \mathbb{R}_+.$

Now we are ready to discuss the constructions of the previous Sections from the arithmetic point of view. Note that local non-ramified non-Archimedean $L$-factors over prime numbers $p$ are given by

$$L_p(s) = \frac{1}{1 - g_p p^{-s}} = \exp(- \sum_{n=1}^{\infty} \frac{1}{n} p^{-sn} g_p^n),$$

where $g_p = p^\mu$ is a value of a character of $Gal(\mathbb{F}_p/\mathbb{F}_p)$ on a Frobenius homomorphism $Fr_p$. The close analogy between Weil group $W_\mathbb{R}$ and Galois groups of non-archimedean fields leads to a search of an analog of (6.2) for the Archimedean field $\mathbb{R}$. It seems that (2.29) and (2.23) are good candidates for the proper analogs of representations of $L_p(s)$ in (6.2). This matches with the identification $W_\mathbb{R}^{ab} = \mathbb{R}^*$ discussed above. Indeed the following integral representation (2.23) holds

$$\Gamma(\lambda - \gamma) \sim \exp(\int_{\mathbb{R}_+} d\mu(t) \ e^{-t(\lambda - \gamma)}), \quad d\mu(t) = (1 - e^{-t})^{-1} dt,$$

where $e^{-t(\lambda - \gamma)}$ is a character of the action of an element $t \in W_\mathbb{R}^{ab}$ on a one-dimensional module and the integration goes over elements of $W_\mathbb{R}^{ab}/\{\pm 1\} = \mathbb{R}_+$. Note that the measure $d\mu(t)$ is non-trivial. The integral representation (6.3) of the logarithm of $\Gamma$-function arises naturally in the related context as a contribution at Archimedean places in the “Explicit formulas” in Number theory (see e.g. a related discussion in [C]).

The Weil group $W_\mathbb{R}$ manifest in other representations of local Archimedean $L$-factors. The determinant representation (2.29) for $\Gamma$-function seems to be very close analog of the standard representation of non-Archimedean $L$-factor as an inverse determinant and is compatible with the interpretation of $W_\mathbb{R}^{ab}$ as an abelianization of the generalized Galois group of $\mathbb{R}$. This interpretation should be compared with other closely related proposals. According to Polya and Hilbert it is natural to look for an operator acting on a Hilbert space such that its spectrum is given by non-trivial zeros of Riemann $\zeta$-function. In [BK] Barry and Kitting proposed an Archimedean analog of this hypothetical Polya-Hilbert operator reproducing analytic properties of the local Archimedean factor of the global Riemann $\zeta$-function of $Spec(\mathbb{Z})$. More precisely it was shown that a quantum Hamiltonian operator $\hat{H} = x \partial_x$
should describe a smooth part of the distribution of non-trivial zeroes of $\zeta$-function captured by a local Archimedean $L$-factor. Thus $\hat{H}$ can be interpreted as a local Archimedean version of the Paley-Hilbert operator and is identified with a generator of $W^\mathrm{ab}_\mathbb{R}$. This seems to be in a complete correspondence with the representation of the local Archimedean $L$-factor (expressed in terms of $\Gamma$-function) as an inverse determinant. Let us also remark that the inverse determinant representation (2.29) for $\Gamma$-function is also compatible with Dorey-Tateo type representation of the Baxter $Q$-operator and thus provides a connection between Dorey-Tateo differential operator $D$ and a generator of the Weil group $W^\mathrm{ab}_\mathbb{R}$.

To support the discussed interpretation of various quantum Toda chain constructions in terms of Archimedean geometry we consider Archimedean counterpart of the Casselman-Shalika-Shintani formula [Sh, CS]. One of the manifestations of the Langlands duality over $\mathbb{Q}_p$ is the representation of the $p$-adic Whittaker function for algebraic group $G(\mathbb{Q}_p)$ as a character of a finite-dimensional representation of the Langlands dual group $^L G_0$. It was proposed by Shintani [Sh] for $GL_{\ell+1}$ and by Casselman-Shalika [CS] for all semi-simple Lie groups. Let $G(\mathbb{Q}_p) = GL(\ell+1, \mathbb{Q}_p)$, $\hat{g}_p(\gamma)$ be a semisimple conjugacy class of $\text{diag}(p^{\gamma_1}, \ldots, p^{\gamma_{\ell+1}})$ in $GL(\ell+1, \mathbb{C})$ corresponding to the image of the Frobenius morphism. Let $V_{(n_1,n_2,\ldots,n_{\ell+1})}$ be a finite-dimensional irreducible representation of $GL(\ell+1, \mathbb{C})$ corresponding to a partition $(n_1 \geq n_2 \geq \ldots \geq n_{\ell+1})$. Then for the $p$-adic Whittaker function corresponding to a principal series representation with the character $\chi_{(p^{\gamma_1},\ldots,p^{\gamma_{\ell+1}})}(g)$ of a Borel subgroup $B \subset G$ the following holds

$$W^{(p)}_{(\gamma_1,\ldots,\gamma_{\ell+1})}(\text{diag}(p^{n_1}, \ldots, p^{n_{\ell+1}})) = \text{Tr}_{V_{(n_1,\ldots,n_{\ell+1})}} \hat{g}_p(\gamma).$$ \hspace{1cm} (6.4)

We propose to consider (5.10) and its generalizations as an analog of (6.4). Indeed $q$-deformed Whittaker functions on the lattice defined in the previous Section provide an interpolation between Archimedean and non-Archimedean Whittaker functions. In the non-Archimedean limit $q \to 0$, $t_1 = p^{-y_1}$, $t_2 = p^{-y_2}$ (5.10) reduces to a special case of (6.4). To give an interpretation of Archimedean Whittaker functions as asymptotics of characters it would be enough to represent $q$-deformed Whittaker function on the lattice as a trace of some operator compatible with (5.4) in the limit $q \to 0$. We have for eigenvalues of the Baxter $Q$-operators in the $q$-deformed case

$$Q^\delta_{\ell+1}(t|t_1,\ldots,t_{\ell+1}) = \prod_{i=1}^{\ell+1} \Gamma_q(t_i^{-1}) = \sum_{n_1,\ldots,n_{\ell+1}=0}^{\infty} \prod_{j=1}^{\ell+1} (t_j^{-1})^{n_j} \prod_{i=1}^{\ell+1} \prod_{j=1}^{n_j} (1 - q^{n_j}).$$

Consider an algebra $\mathcal{A}$ with a set of generators $\{a_{n,i}, a_{n,i}^+\}, n \in \mathbb{Z}_+, i = 1, \ldots, \ell + 1$ and relations

$$[a_{n,i}^+, a_{m,j}] = \delta_{n,m} \delta_{i,j}, \quad [a_{n,i}, a_{m,j}^+] = [a_{n,i}, a_{m,j}] = 0, \quad n, m \in \mathbb{Z}_+, \quad i, j = 1, \ldots, \ell + 1.$$ 

Define the following operators

$$L_0 = \sum_{j=1}^{\ell+1} \sum_{n=1}^{\infty} n a_{n,j}^+ a_{n,j}, \quad J_0^{(j)} = \sum_{n=1}^{\infty} a_{n,j}^+ a_{n,j}, \quad I_0 = \sum_{j=1}^{\ell+1} \sum_{n=1}^{\infty} a_{n,j} a_{n,j} = \sum_{j=1}^{\ell+1} J_0^{(j)}.$$ 

\footnote{Let us also remark that there is another representation of $\Gamma$-function as an inverse determinant of the operator with a discrete spectrum [D1, D2] (see also [Man2]).}
Note that one can construct a representation of a version of polynomial loop algebra using the generators \( \{a_{n,i}^+, a_{n,i}^-\} \) with the finite-dimensional subalgebra \( \mathfrak{gl}_{\ell+1} \) generated by \( J_0^{ij} = \sum_{n=1}^{\infty} a_{n,i}^+ a_{n,j}^- \).

A representation \( W_\mathcal{A} \) of \( \mathcal{A} \) can be realized in the space of polynomials of \( \{a_n\}, n = 1, 2, \ldots \) and we have

\[
Q^{\mathfrak{gl}_{\ell+1}}_{\text{lattice}}(t|t_1, \ldots, t_{\ell+1}) = \text{Tr}_{W_\mathcal{A}} q^{-L_0} t_0^{j_0^{(1)}} t_1^{j_0^{(2)}}. 
\]

Therefore one can identify \( \chi^q_n(t_i) \) entering the expansion (5.5) with the characters of \( I_0 \)-graded subspaces \( W_\mathcal{A}^{[n]} = \{v \in W_\mathcal{A} | I_0 v = n v\} \)

\[
\chi^q_n(t_i) = \text{Tr}_{W_\mathcal{A}^{[n]}} q^{-L_0} t_0^{j_0^{(1)}} t_1^{j_0^{(2)}}. 
\]

Note that \( I_0 \) commutes with \( J_0^{ij} \) and thus \( W_\mathcal{A} \) supports the action of \( \mathfrak{gl}_{\ell+1} \). Taking into account the identification of \( \chi^q_n(t_1, t_2) \) with \( q \)-Whittaker functions for \( \mathfrak{g} = \mathfrak{gl}_2 \) we have

\[
\Psi^{(q)}_{t_1,t_2}(n) = \text{Tr}_{W_\mathcal{A}^{[n]}} q^{-L_0} t_1^{j_0^{(1)}} t_2^{j_0^{(2)}}. 
\]

In the asymptotic limit of \( q \to 1 \) one recovers the local Archimedean \( L \)-function. Note that the natural framework of representation theory over archimedean fields should be a kind of asymptotic geometry (also known as a tropical geometry).

7 Conclusions and further directions

In this paper we consider Baxter \( Q \)-operators for various generalization of Toda chains and relate them with local Archimedean \( L \)-factors. The relation between \( Q \)-operators and \( L \)-factors seems provide a consistent approach to construction new candidates for local Archimedean \( L \)-factors. Thus, for instance, in this paper we propose Archimedean \( L \)-factors hypothetically corresponding to generic level automorphic representations of affine Lie algebras. It would be interesting to define its non-Archimedean counterparts and construct the corresponding complete \( \zeta \)-function satisfying certain functional equations (see \([\text{Kap1, Kap2]}\) for a construction of Hecke algebras over two dimensional fields \( \mathbb{Q}_p((t)) \) defined by Parshin \([P]\)). Hopefully further advances in understanding the connection between formalism of Baxter \( Q \)-operators and \( L \)-functions corresponding to automorphic representations will be useful for theories of quantum integrable systems and of automorphic representations.

Let us comment on a possible application of the discussed constructions of affine Lie groups \( Q \)-operators to the study of hidden underlying structures of algebraic geometry over archimedean fields. The very existence of the \( Q \)-operator for finite-dimensional Lie groups introduced in \([\text{GLO}]\) can be traced back to the construction of a spectral curve for the corresponding integrable systems. In contrast to the case of affine Lie algebras the meaning of the spectral curve for finite-dimensional Lie algebras is not so straightforward. The simplest way to uncover the spectral curve and Baxter \( Q \)-operator for finite-dimensional Lie algebras is by
a degeneration of the corresponding constructions for affine Lie algebras. This seems to be an instance of a general phenomena that constructions over real/complex numbers should be understood as degenerations of the corresponding constructions for fields of higher dimensions (see e.g. [Ch] for a proposal to consider Yangian as a hidden symmetry of the representation theory over Archimedean fields). More explicitly the relevance of the higher-dimensional point of view manifests an attempt to find an Archimedean version of Casselman-Shalika-Shintani [Sh, CS] formula. Using a version of the $q$-deformed $\mathfrak{gl}_{t+1}$-Toda chain (related with generic affine Toda chain) we construct generalized Whittaker functions interpolating between non-Archimedean and Archimedean Whittaker functions. These interpolation formulas are closely related with the Macdonald philosophy of symmetric polynomials interpolation [Mac]. One of the consequences of the existence of the Whittaker function interpolation is an Archimedean version of the Casselman-Shalika-Shintani [Sh, CS] relation between principal series $p$-Whittaker function for $G(\mathbb{Q}_p)$ and characters of finite-dimensional representations of the Langlands dual group $L^G_0$. The Archimedean $\mathfrak{gl}_{t+1}$-Whittaker functions are identified with asymptotic of characters of fixed grade components of infinite-dimensional algebras.

The construction of the Archimedean version of the Casselman-Shalika-Shintani formula leads to an interesting opportunity explicit realize $W_\mathbb{R}$-modules and to interpret the $W_\mathbb{R}$-modules using symmetries of the conjectural Archimedean topos $\mathcal{Top}_\mathbb{R}$ responsible for the properties of the algebraic varieties over archimedean fields [Be]. A suitable candidate for appropriate cohomology theory might be found in the papers [GK], [Gi], [GiL] on mirror constructions for flag manifolds of semisimple Lie groups. Note that according to [GK], [Gi], [GiL] one expects that cohomological and $K$-theoretic Gromow-Witten invariants of flag manifolds for a semisimple group $G$ should be given in terms of eigenfunctions of $L^G_0$-Toda chains (i.e. $L^G_0$-Whittaker functions) and its $q$-deformations where $L^G_0 = \text{Lie}(L^G_0)$ and $L^G_0$ is a Langlands dual to $G$. The recent advances in the construction of Whittaker functions for classical Lie groups using a torification of flag varieties [GLO1] provides a direct way to prove this conjecture. Thus it would be quite reasonable to conjecture that the Type $A$ topological strings on $G$-flag varieties should provide a realizations of Weil group modules for Archimedean version of Casselman-Shalika-Shintani formula. The discussion of an explicit realization of this idea obviously deserves a separate publication and we are going to return to this fascinating subject in the future. However taking into account the previous considerations it looks plausible that the hidden underlying structure of algebraic geometry over complex numbers is described by a two-dimensional quantum field theory.

8 Appendix A: Special functions

The following integral representations for logarithm of $\Gamma$-function hold (see e.g. [WW])

$$\log \Gamma(z) = \int_0^\infty \frac{dt}{t} e^{-t} \left( (z - 1) + \frac{e^{(z-1)t} - 1}{1 - e^{-t}} \right),$$

$$\log \Gamma(z) = (z - \frac{1}{2}) \log z - z + \frac{1}{2} \log 2\pi + \int_0^\infty \left( \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-zt}}{t} dt.$$
Let $\omega_1, \omega_2 \in \mathbb{R}_+$. Define functions $S_2(z)$ and $S(z|\omega)$ as

$$S_2(z) = e^{\frac{iz}{2}B_{2,2}(z)} S(z|\omega) = e^{\frac{iz}{2}B_{2,2}(z)} \exp \int_{\mathbb{R}+i0} \frac{e^{zt}}{(e^{\omega_1 t} - 1)(e^{\omega_2 t} - 1)} \frac{dt}{t} = \exp \frac{1}{2\pi i} \int_{C} \frac{\sinh(z - \frac{\omega_1 + \omega_2}{2} t) \ln(-t) dt}{\sinh \frac{\omega_1}{2} t \cdot \sinh \frac{\omega_2}{2} t},$$

(8.1)

where

$$B_{2,2}(z) = \frac{1}{\omega_1 \omega_2} z^2 - \frac{\omega_1 + \omega_2}{\omega_1 \omega_2} z + \frac{\omega_1^2 + 3\omega_1 \omega_2 + \omega_2^2}{6 \omega_1 \omega_2},$$

and the contour $C$ is given in [KLS]. The function $S_2(z)$ satisfies the following functional equations.

$$S_2(z + \omega_1) = \frac{1}{2 \sin \frac{\pi z}{\omega_2}} S_2(z), \quad S_2(z + \omega_2) = \frac{1}{2 \sin \frac{\pi z}{\omega_1}} S_2(z).$$

By a deformation of contours $C$ and $\mathbb{R}+i0$ one can extend the definition of $S_2(z)$ and $S(z|\omega)$ to the complex values of $\omega_1, \omega_2$ with $\text{Im}(\omega_1/\omega_2) > 0$. Then the following holds:

$$S(z|\omega) = \prod_{m=0}^{\infty} \left(1 - q^{2m} e(z/\omega_2)\right) \prod_{m=1}^{\infty} \left(1 - \tilde{q}^{-2m} e(z/\omega_1)\right),$$

where $e(z) = e^{2\pi iz}$, $q = e(\frac{\omega_1}{2 \omega_2})$, $\tilde{q} = e(\frac{\omega_2}{2 \omega_1})$. Fourier transform of the double sine function can be expressed in terms of double sine function as follows

$$\int_{C'} dt \, S(\omega_1 + \omega_2 - it - a|\omega) e^{2\pi i \frac{zt}{\omega_1 \omega_2}} = \sqrt{\omega_1 \omega_2} e^{-\frac{\pi i}{2} B_{2,2}(0)} S^{-1}(tz|\omega) e^{-2\pi \frac{\omega_1 \omega_2}{\omega_1 + \omega_2}},$$

where the contour $C'$ is given e.g. [KLS].

9 Appendix B: q-Math

In this Appendix we follow mainly [CK]. We use the following definition of $q$-numbers

$$[n]_q = \frac{1 - q^n}{1 - q}.$$

Traditionally $q$-Gamma function is defined as

$$\tilde{\Gamma}_q(x) = \frac{(q; q)_{\infty}}{(q^x; q)_{\infty}} (1 - q)^{1-x}, \quad (a; q)_n = \prod_{i=0}^{n} (1 - a q^i).$$

We use the following modified $\Gamma_q$-function

$$\Gamma_q(t) = \frac{1}{(t; q)_{\infty}}, \quad \Gamma_q(t q) = (1 - t) \Gamma_q(t).$$
In the limit \( q \to 0, \ t \to p^{-x} \) we have

\[
\Gamma^{(p)}(x) \equiv \lim_{q \to 0, t = p^{-x}} (q; q)_{\infty} \Gamma_q(t) = \frac{1}{1 - p^{-x}}.
\]

In the limit \( t = q^x, \ q = \exp(2\pi i h), \ h \to 0 \) the \( \Gamma_q \)-function is reduced to the standard \( \Gamma \)-function

\[
\lim_{h \to 0} (q; q)_{\infty} \Gamma_q(q^x) h^{x-1} = \Gamma(x).
\]

The \( q \)-analogs of the exponential function are given by two functions

\[
e_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{(q; q)_n} = \frac{1}{(t, q)_{\infty}}, \quad E_q(t) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2} t^n}{(q; q)_n} = (-t, q)_{\infty},
\]

satisfying the relation \( e_q(t)E_q(-t) = 1 \). We have with our definition of \( \Gamma_q(t) \)

\[
\Gamma_q(t) = e_q(t).
\]

The limits of both \( e_q(t) \) and \( E_q(t) \) for \( t = q^x, \ q \to 1 \) are equal to the standard exponent. In the limit \( q \to 0, \ t \to p^{-x} \) we have

\[
e_p(x) = \frac{1}{1 - p^{-x}}, \quad E_p(x) = (1 - p^x).
\]

Jackson integral is define as

\[
\int_0^{\infty} f(x) d_q x = (1 - q) \sum_{j=-\infty}^{\infty} q^j f(q^j).
\]

For \( q \)-Gamma function there is an analog of the standard integral representation

\[
\Gamma_q(t) = \prod_{m=0}^{\infty} (1 - q^m) \int \chi_t(z) E_q(-qz) z^{-1} d_q z = \frac{1}{\prod_{m=0}^{\infty} (1 - q^m) \prod_{m=0}^{\infty} (1 - tq^m)}
\]

where \( \chi_t(z) = t^{v(z)}, \ v(q^n) = n \). In the limit \( t = p^{-x}, \ q \to 0 \) the integral expression is reduced to following

\[
\Gamma^{(p)}(x) = \sum_{j=0}^{\infty} p^{-xj} = \frac{1}{1 - p^{-x}}.
\]

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