THE BOUNDARY OF THE FREE FACTOR GRAPH AND THE FREE SPLITTING GRAPH

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Abstract. We show that the Gromov boundary of the free factor graph for the free group $F_n$ with $n \geq 3$ generators is the space of equivalence classes of minimal very small indecomposable projective $F_n$-trees without point stabilizer containing a free factor equipped with a quotient topology. Here two such trees are equivalent if the union of their metric completions with their Gromov boundaries are $F_n$-equivariantly homeomorphic with respect to the observer’s topology. The boundary of the cyclic splitting graph is the space of equivalence classes of trees which either are indecomposable or split as very large graph of actions. The boundary of the free splitting graph is the space of equivalence classes of trees which either are indecomposable or which split as large graphs of actions.

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1. Introduction

A free splitting of the free group $F_n$ with $n \geq 3$ generators is a one-edge graph of groups decomposition of $F_n$ with trivial edge group. Thus it either is a decomposition as a free product $F_n = A \ast B$ where $A, B$ is a proper subgroups of $F_n$, or as an HNN-extension $F_n = A \ast$. The subgroups $A, B$ are free factors of $F_n$.

A vertex of the free splitting graph is a conjugacy class of a free splitting of $F_n$. Two such free splittings are connected by an edge of length one if up to conjugation, they have a common refinement. The free splitting graph is hyperbolic in the sense of Gromov [HM13b, HiHo12]. The outer automorphism group $\text{Out}(F_n)$ of $F_n$ acts on the free splitting graph as a group of simplicial isometries.

A cyclic splitting of $F_n$ is a one-edge graph of groups decomposition of $F_n$ with infinite cyclic edge group. Thus it either is a decomposition as an amalga mated product $F_n = A \ast \langle w \rangle B$ or as an HNN-extension $F_n = A \ast \langle w \rangle$ where $\langle w \rangle$ is an infinite cyclic subgroup of $F_n$. A vertex of the cyclic splitting graph is a conjugacy class of a free or of a cyclic splitting of $F_n$. Two such splittings are connected by an edge of length one if up to conjugation, they have a common refinement. The cyclic splitting graph is hyperbolic in the sense of Gromov [Mn12]. The outer automorphism group $\text{Out}(F_n)$ of $F_n$ acts on the cyclic splitting graph as a group of simplicial isometries. The free splitting graph is a coarsely dense $\text{Out}(F_n)$-invariant subgraph of the cyclic splitting graph.

The free factor graph is the graph $FF$ whose vertices are conjugacy classes of free factors of $F_n$. Two free factors $A, B$ are connected by an edge of length one if up to conjugation, either $A < B$ or $B < A$. The free factor graph is hyperbolic in the sense of Gromov [BF14a]. The outer automorphism group $\text{Out}(F_n)$ of $F_n$ acts on the free factor graph as a group of simplicial isometries.

The goal of this article is to determine the Gromov boundaries of these three graphs.

Let $cv(F_n)$ be the unprojectivized Outer space of minimal free simplicial $F_n$-trees, with its boundary $\partial cv(F_n)$ of minimal very small $F_n$-trees [CL95, BF92] which either are not simplicial or which are not free. Write $cv(F_n) = cv(F_n) \cup \partial cv(F_n)$.

A tree $T \in \partial cv(F_n)$ is called indecomposable if for any finite, non-degenerate arcs $I, J \subset T$, there are elements $g_1, \ldots, g_r \in F_n$ so that

$$J \subset g_1 I \cup \cdots \cup g_r I$$

and such that $g_i I \cap g_{i+1} I$ is non-degenerate for $i \leq r - 1$ (Definition 1.17 of [G08]). The $F_n$-orbits on an indecomposable tree are dense (Lemma 1.18 of [G08]). Being indecomposable is invariant under scaling and hence it is defined for projective trees.

A graph of actions is a minimal $F_n$-tree $G$ which consists of

(1) a simplicial tree $S$, called the skeleton, equipped with an action of $F_n$
(2) for each vertex $v$ of $S$ an $\mathbb{R}$-tree $Y_v$, called a vertex tree, and
(3) for each oriented edge $e$ of $S$ with terminal vertex $v$ a point $p_e \in Y_v$, called an attaching point.

It is required that the projection $Y_v \to p_e$ is equivariant, that for $g \in F_n$ one has $g p_e = p_{g e}$ and that each vertex tree is a minimal very small tree for the action of its stabilizer in $F_n$.

Associated to a graph of actions $G$ is a canonical action of $F_n$ on an $\mathbb{R}$-tree $T_G$ which is called the dual of the graph of actions [L94]. Define a pseudo-metric $d$ on $\bigcup_{v \in V(S)} Y_v$ as follows. If $x \in Y_{v_0}, y \in Y_{v_k}$ let $e_1 \ldots e_k$ be the reduced edge-path from $v_0$ to $v_k$ in $S$ and define

$$d(x, y) = d_{Y_{e_1}}(x, p_{e_1}) + \cdots + d_{Y_{e_k}}(p_{e_k}, y).$$

Making this pseudo-metric Hausdorff gives an $\mathbb{R}$-tree $T_G$. The vertex trees of the graph of actions $G$ are isometrically embedded in $T_G$ and will be called the vertex trees of $T_G$.

An minimal very small $F_n$-tree $T$ splits as a graph of actions if there is a graph of actions $G$ with dual tree $T_G$, and there is an equivariant isometry $T \to T_G$. We call $G$ a structure of a graph of action for $T$. We also say that the projectivization $[T]$ of an $F_n$-tree $T$ splits as a graph of actions if $T$ splits as a graph of actions.

A tree $T \in \partial cv(F_n)$ with dense orbits splits as a large graph of actions if $T$ splits as a graph of actions with the following properties.

1. There is a single $F_n$-orbit of non-degenerate vertex trees.
2. The stabilizer of a non-degenerate vertex tree is not contained in any proper free factor of $F_n$.
3. A non-degenerate vertex tree is indecomposable for its stabilizer.

Note that by the second requirement, the stabilizer of an edge of the skeleton of a large graph of actions is non-trivial.

We say that a tree $T \in \partial cv(F_n)$ with dense orbits splits as a very large graph of actions if $T$ splits as a large graph of actions and if the stabilizer of an edge of the skeleton of the is not infinite cyclic.

If $T \in \partial cv(F_n)$ is a tree with a dense action of $F_n$ then the union $\hat{T}$ of the metric completion $\overline{T}$ of $T$ with the Gromov boundary $\partial T$ can be equipped with an observer’s topology [CHL07]. This topology only depends on the projective class of $T$.

Denote by $\partial CV(F_n)$ the boundary of projectivized Outer space, i.e. the space of projective classes of trees in $\partial cv(F_n)$. Let $ST \subset \partial CV(F_n)$ be the $\text{Out}(F_n)$-invariant subspace of projective trees which either are indecomposable or split as large graph of actions. Let $\sim$ be the equivalence relation on $ST$ which is defined as follows. The equivalence class of the projective class $[T]$ of a tree $T$ consists of all projective classes of trees $S$ so that the trees $\hat{S}$ and $\hat{T}$ equipped with the observer’s topology are $F_n$-equivariantly homeomorphic. We equip $ST/\sim$ with the quotient topology.
Theorem 1. The Gromov boundary of the free splitting graph is the space $\mathcal{ST} / \sim$ of equivalence classes of projective very small trees which either are indecomposable or which split as large graph of actions.

Let $\mathcal{CT} \subset \mathcal{ST}$ be the $\text{Out}(F_n)$-invariant subset of equivalence classes of projective trees which either are indecomposable or split as very large graph of actions. This subspace is saturated for the equivalence relation $\sim$.

Theorem 2. The Gromov boundary of the cyclic splitting graph is the space $\mathcal{CT} / \sim$ of equivalence classes of projective trees which either are indecomposable or which split as very large graph of actions.

An indecomposable projective tree $[T] \in \mathcal{ST}$ is called arational if no nontrivial point stabilizer of $[T]$ contains a free factor. The space of arational trees is saturated for the equivalence relation $\sim$.

Theorem 3. The Gromov boundary of the free factor graph is the space $\mathcal{FT} / \sim$ of equivalence classes of arational trees equipped with the quotient topology.

The strategy of proof for the above theorems builds on the strategy of Klarreich [K99] who determined the Gromov boundary of the curve graph of a non-exceptional surface of finite type.

As in [K99], we begin with describing in Section 2 two electrifications of Outer space which are quasi-isometric to the free factor graph and to the free splitting graph, respectively.

In Section 3 we introduce folding paths and collect those of their properties which are used later on.

Section 4 establishes a technical result on folding paths and their relation to the geometry of the free splitting graph. Section 5 gives some information on trees in $\partial \text{cv}(F_n)$ for which the orbits of the action of $F_n$ are not dense.

In Section 6 we show that the boundary of each of the above three $\text{Out}(F_n)$-graphs is the image of a $\text{Out}(F_n)$-invariant subspace of the boundary $\partial \text{CV}(F_n)$ of projectivized Outer space under a continuous map. The main technical tool to this end is a detailed analysis on properties of folding paths.

In Section 7 we investigate the structure of indecomposable trees, and Section 8 contains some information on trees with point stabilizers containing a free factor. In Section 9 we show that only indecomposable projective trees can give rise to points in the boundary of the free factor graph.

The proof of Theorem 3 is contained in Section 10, and the proofs of Theorem 1 and Theorem 2 are completed in Section 11.

Theorem 3 was independently and at the same time obtained by Mladen Bestvina and Patrick Reynolds [BR12]. Very recently Horbez [Ho14] extended Theorem 2 to a more general class of groups, with a somewhat different proof.
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2. Geometric models

As in the introduction, we consider a free group $F_n$ of rank $n \geq 3$. Our goal is to introduce geometric models for the free factor graph and the free splitting graph. We also construct $n - 3$ additional $\text{Out}(F_n)$-graphs which geometrically lie between the free factor graph and the free splitting graph; they are analogs of the graphs considered in [H11]. Although these graphs are not used for the proofs of the theorems from the introduction, they shed some light on the geometry of the free splitting graph and the structure of its boundary. They will be useful in another context.

The free splitting graph [KL09] is defined to be the graph whose vertices are one-edge graph of groups decompositions of $F_n$ with trivial edge group. Two such vertices are connected by an edge if up to conjugation, they have a common refinement. It is more convenient for our purpose to use instead the first barycentric subdivision $FS$ of the free splitting graph. Its vertices are graph of groups decompositions of $F_n$ with trivial edge groups. Two vertices $\Gamma, \Gamma'$ are connected by an edge if $\Gamma$ is a collapse or blow-up of $\Gamma'$. The outer automorphism group $\text{Out}(F_n)$ of $F_n$ acts on $FS$ as a group of simplicial isometries.

The cyclic splitting graph [Mn12] is the graph whose vertices are one-edge graph of groups decompositions of $F_n$ with trivial or infinite cyclic edge group. Two such vertices are connected by an edge if up to conjugation, they have a common refinement. The vertices of the first barycentric subdivision $CS$ of the cyclic splitting graph are graph of groups decompositions of $F_n$ with trivial or infinite cyclic edge groups. Two vertices $\Gamma, \Gamma'$ are connected by an edge if $\Gamma$ is a collapse or a blow-up of $\Gamma'$. The graph $FS$ is an $\text{Out}(F_n)$-invariant subgraph of $CS$. The inclusion is an $\text{Out}(F_n)$-equivariant one-Lipschitz embedding

$$\Psi : FS \to CS.$$ 

Let $cv(F_n)$ be the unprojectivized Outer space of all simplicial trees with minimal free isometric actions of $F_n$, equipped with the equivariant Gromov-Hausdorff topology. We refer to [P88] for detailed information on this topology. The quotient $T/F_n$ of each tree $T \in cv_0(F_n)$ defines a graph of groups decomposition of $F_n$ with trivial edge groups and hence a vertex $\Upsilon(T)$ in $FS$. This graph of groups decomposition is invariant under scaling the metric of $T$.

An $F_n$-tree $T$ is called very small if $T$ is minimal and if moreover the following holds.
(1) Stabilizers of non-degenerate segments are at most cyclic.
(2) If $g^n$ stabilizes a non-degenerate segment $e$ for some $n \geq 1$ then so does $g$.
(3) Fix($g$) contains no tripod for $g \neq 1$.

Here a tripod is a compact subset of $T$ which is homeomorphic to a cone over three points.

The equivariant Gromov Hausdorff topology extends to the space $\partial \text{cv}(F_n)$ of minimal very small $F_n$-trees [BF92, CL95] which either are not simplicial or which are not free. The subspace $\text{cv}(F_n)$ is dense in $\text{cv}(F_n) = \text{cv}(F_n) \cup \partial \text{cv}(F_n)$.

Let
\[ \text{cv}(F_n)^+ \subset \text{cv}(F_n) \]
be the $\text{Out}(F_n)$-invariant subset of all minimal very small simplicial $F_n$-trees. Then $\text{cv}(F_n)^+ - \text{cv}(F_n)$ consists of simplicial $F_n$-trees so that the action of $F_n$ is not free.

For each $T \in \text{cv}(F_n)^+$ the quotient graph $T/F_n$ defines a graph of groups decomposition of $F_n$ with at most cyclic edge groups (we refer to [CL95] for a detailed discussion). Thus there is a natural $\text{Out}(F_n)$-equivariant map
\[ \Upsilon_C : \text{cv}(F_n)^+ \rightarrow \mathcal{CS}. \]
Its image is the vertex set of $\mathcal{CS}$.

Let
\[ \text{cv}(F_n)^0 \subset \text{cv}(F_n) \]
be the $\text{Out}(F_n)$-invariant subspace of simplicial trees with at least one $F_n$-orbit of edges with trivial edge stabilizer. To each tree $T \in \text{cv}(F_n)^+$ we can associate the graph of groups decomposition $\Upsilon(T)$ with trivial edge groups obtained by collapsing all edges in $T/F_n$ with non-trivial edge groups to a point. Thus there is an $\text{Out}(F_n)$-equivariant coarsely surjective map
\[ \Upsilon : \text{cv}(F_n)^+ \rightarrow \mathcal{FS}. \]
Note that for $T \in \text{cv}(F_n)^+$ the distance in $\mathcal{CS}$ between $\Upsilon(T) \in \mathcal{FS} \subset \mathcal{CS}$ and $\Upsilon_C(T)$ is at most one.

The free factor graph $\mathcal{FF}$ is the graph whose vertices are free factors of $F_n$ and where two such vertices $A, B$ are connected by an edge of length one if and only if up to conjugation, either $A < B$ or $B < A$. We next observe that there is a (coarsely) $\text{Out}(F_n)$-equivariant (coarsely) consistent way to associate to a point in $\text{cv}(F_n)^+$ a vertex in the free factor graph.

To this end say that a map $f : X \rightarrow Y$ between metric spaces $X,Y$ equipped with an isometric action of a group $\Gamma$ is coarsely $\Gamma$-equivariant if there is a number $C > 0$ such that
\[ d(g(f(x)), f(gx)) \leq C \]
for all $x \in X$, all $g \in \Gamma$. The map $f$ is called coarsely $L$-Lipschitz if
\[ d(f(x), f(y)) \leq Ld(x, y) + L \]
Lemma 2.1. There is a number $k > 1$, and there is a coarsely $k$-Lipschitz coarsely $\text{Out}(F_n)$-equivariant map $\Omega : \mathcal{CS} \to \mathcal{FF}$.

Proof. There is a second description of the cyclic splitting graph as follows. Namely, let $ZF_n$ be the graph whose vertex set is the set of one edge free splittings of $F_n$. Two such splittings $X, Y$ are connected by an edge if either

(a) they are connected in the free splitting graph by an edge or
(b) there exists a $Z$-splitting $T$ and equivariant edge folds $X \to T, Y \to T$.

Roughly speaking, an edge fold of a free splitting $A \ast B$ is a splitting of the form $A \ast C(B, C)$ where $C$ is any maximal cyclic subgroup of $A$. We refer to [St65, BF92, Mn12] for details of this construction.

By Proposition 2 of [Mn12], the vertex inclusion of the set of vertices of $ZF_n$ into the set of vertices of the cyclic splitting graph extends to an $\text{Out}(F_n)$-equivariant quasi-isometry.

Define a map $R$ from the vertex set of $ZF_n$ into the vertex set of the free factor graph by associating to a vertex $X$ of $ZF_n$ a vertex group of the corresponding free splitting.

Since for every free splitting of $F_n$ of the form $F_n = A \ast B$ the distance in $\mathcal{FF}$ between the free factors $A, B$ is at most three, the map $R$ coarsely does not depend on choices and hence it is coarsely $\text{Out}(F_n)$-equivariant. We claim that it extends to a coarsely Lipschitz map $ZF_n \to \mathcal{FF}$.

To this end let $d_F$ be the distance in $\mathcal{FF}$. Since the metrics on the graphs $\mathcal{CS}, \mathcal{FF}$ are geodesic, by an iterated application of the triangle inequality it suffices to show that there is a number $L > 1$ so that $d_F(R(X), R(Y)) \leq L$ whenever $X, Y \in ZF_n$ are vertices connected by an edge.

Consider first the case that $X, Y$ are connected by an edge in the free splitting graph. Assume furthermore that $X, Y$ are one edge two vertex splittings, i.e. that these splittings are of the form $X = A \ast B$ and $Y = C \ast D$. Then up to conjugation, these splitting have a common refinement, i.e. up to exchanging $A, B$ and $C, D$ there is a free factor $E$ of $F_n$ which is a subgroup of both $A, C$.

Then $d_F(E, A) \leq 1, d_F(E, C) \leq 1$, moreover the distance in $\mathcal{FF}$ between $A$ and $B$ and between $C$ and $D$ is at most three. Thus the distance between $R(X)$ and $R(Y)$ is at most 8. The case that one or both of the splittings $X, Y$ is a one-loop splitting follows in the same way and will be omitted (see [KR14] for details).

Now assume that $X, Y$ are connected by an edge of type (b) above. Following Lemma 1 of [Mn12] and the discussion in the proof of Theorem 5 of [Mn12], let $\langle w \rangle$ be the edge group of the $Z$-splitting to which $X, Y$ fold and let $A$ be the smallest free factor of $F_n$ which contains $\langle w \rangle$. Then $A$ is a subgroup of a vertex group of both $X$ and $Y$ and hence the distance in the free factor graph between $R(X), R(Y)$ is at most 8. \qed
By Lemma 2.1 the map
\[
\Upsilon_F = \Omega \circ \Upsilon_C : \overline{cv(F_n)} \to \mathcal{F}\mathcal{F}
\]
is coarsely $\text{Out}(F_n)$-equivariant and coarsely surjective.

For a number $\ell > 0$ call a free basis $e_1, \ldots, e_n$ of $F_n$ $\ell$-short for a tree $T \in \overline{cv(F_n)}$ if there is a vertex $v$ of $T$ so that for each $i$ the distance in $T$ between $v$ and $e_iv$ is at most $\ell$. If $T \in \overline{cv(F_n)}$ then this is equivalent to stating that there is a marked rose $R$ with $n$ petals of equal length one representing each one of the basis elements $e_i$, and there is an $\ell$-Lipschitz map $u : R \to T/F_n$ which maps the vertex of $R$ to a vertex of $T/F_n$, and which maps each petal marked with $e_i$ to a path in $T/F_n$ representing $e_i$.

Let $\overline{cv_0(F_n)} \subset \overline{cv(F_n)}$ be the subspace of all simplicial trees $T$ with quotient graph $T/F_n$ of volume one. The group $\text{Out}(F_n)$ acts on $\overline{cv_0(F_n)}$ by precomposition of marking. Let $\overline{cv_0(F_n)}^+ = \overline{cv(F_n)}^+ \cap \overline{cv_0(F_n)}^+$ and define
\[
\overline{cv_0(F_n)}^{++} \subset \overline{cv_0(F_n)}^+
\]
to be the set of all simplicial very small $F_n$-trees with volume one quotient and no non-trivial edge stabilizer.

**Lemma 2.2.** Every tree $T \in \overline{cv_0(F_n)}^{++}$ admits a 3-short basis.

**Proof.** If $T \in \overline{cv_0(F_n)}$ then choose any vertex $v \in G = T/F_n$. Collapse a maximal tree in $G$ to a point. The resulting graph is a rose $R$ with $n$ petals which determines a free basis $e_1, \ldots, e_n$ of $F_n$. In the graph $G$, each basis element $e_i$ is represented by a loop based at $v$ which passes through every edge of $G$ at most twice. Since the volume of $G$ equals one, the basis $e_1, \ldots, e_n$ is 2-short for $T$.

If $T$ has non-trivial vertex stabilizers then each of these stabilizers is a free factor of $F_n$. This means that there is a free simplicial $F_n$-tree $T' \in cv(F_n)$ with quotient of volume smaller than $3/2$, and there is a one-Lipschitz equivariant map $T' \to T$ obtained by collapsing the minimal subtrees of $T'$ which are invariant under the vertex stabilizers of $T$ to points. The tree $T'$ is the universal covering of a graph $G$ obtained from $T/F_n$ by attaching a marked rose with $m$ petals of very small length to the projection of a vertex in $T$ with stabilizer of rank $m$. This construction is carried out in detail in the Combination Lemma 8.6 of [CL95].

As $T'$ has a 3-short basis by the discussion in the first paragraph of this proof, the same holds true for $T$. The lemma follows.

The following lemma shows that the statement of Lemma 2.2 can be extended to a class of trees with non-trivial edge stabilizers. It serves as an illustration for the results established later on, but it will not be used directly in the sequel. We also refer to Lemma 8.6 of [CL95] for related and more general constructions.
Lemma 2.3. Let $T \in \text{cv}_0(F_n)^+$ be a simplicial $F_n$-tree such that the graph of groups decomposition $T/F_n$ has a single edge with non-trivial edge group, and this edge is separating. Then $T$ admits a 3-short basis.

Proof. Let $T \in \text{cv}_0(F_n)^+$ be a tree as in the lemma. There is a single separating edge $s$ in $T/F_n = G$ with non-trivial edge group. The edge $s$ defines a one-edge two vertex cyclic splitting of $F_n$. Let $S < F_n$ be the cyclic edge group.

By assumption, the edge $s$ is separating in $G$. Let $v_1, v_2 \in G$ be the two vertices on which $s$ is incident and let $H(v_1), H(v_2)$ be the vertex groups. The infinite cyclic edge group $S$ is a free factor in at least one of the vertex groups $H(v_i)$, say in the group $H(v_2)$ [St65]. Then $H(v_1)$ is a free factor of $F_n$ (this is not true in general for the vertex group $H(v_2)$).

Let $\tilde{v}_1$ be a preimage of $v_1$ in $T$. As $H(v_1)$ is a free factor of $F_n$ containing $S$ as a subgroup, the subset $A$ of $F_n$ of all elements $g$ so that the segment in $T$ connecting $\tilde{v}_1$ to $g\tilde{v}_1$ does not cross through a preimage of $s$ is a free factor of $F_n$. The minimal $A$-invariant subtree $T_A$ of $T$ does not have edges with non-trivial stabilizer. Its quotient graph $T_A/A$ is an embedded subgraph of $G$ and hence its volume is smaller than one.

Choose the vertex $\tilde{v}_1$ as a base point. By Lemma 2.2 and the above discussion, there is a 3-short free basis $e_1, \ldots, e_k$ for the free factor $A$ of $F_n$. Extend this basis of $A$ to a basis of $F_n$ as follows. First add a free basis $e_{k+1}, \ldots, e_u$ for the free factor $B$ of $F_n$ generated by all based loops at $v_2$ in $G$ which do not cross through $s$. We require that each such based loop passes through every edge of $G$ at most twice. Up to conjugation of $B$, an element $e_i$ ($i \geq k + 1$) translates the vertex $\tilde{v}_1$ a distance which equals the sum of twice the length of $s$ with the length of the defining loop in $G$. In particular, the translation length of each of the elements $e_i$ is at most three. This partial basis can be extended to a 3-short free basis of $F_n$ by adding some elements in the point stabilizer of $v_2$. \hfill \Box

Lemma 2.3 does not seem to hold for all points in $\text{cv}_0(F_n)^+$. However, there is a weaker statement which holds true.

For its formulation, define a pure cyclic splitting to consist of a graph of groups decomposition for $F_n$ with all edge groups cyclic. Define a free refinement of a pure cyclic splitting $s$ to be a refinement $s'$ of $s$ so that each edge group of an edge in $s' - s$ is trivial. We have

Lemma 2.4. For every pure cyclic splitting $s$ of $F_n$ there is a number $\ell(s) > 0$ with the following property. Every simplicial tree $T \in \text{cv}_0(F_n)^+$ so that $\Upsilon_C(T)$ is a free refinement of $s$ admits an $\ell(s)$-short basis.

Proof. Let $s$ be a pure cyclic splitting of $F_n$. For some $\epsilon << 1/3n - 4$ let $B \subset \text{cv}_0(F_n)^+$ be the set of all trees $T$ with the following properties.

1. $\Upsilon_C(T) \in CS$ is a free refinement of $s$.
2. The length of each edge of $T$ is at least $\epsilon$. 
Note that \( B \) is a closed subset of \( \overline{cv_0(F_n)}^s \) (this standard fact is discussed in detail in [CL95]).

As there are only finitely many topological types of quotient graphs \( T/F_n \) for trees \( T \in B \), the stabilizer in \( \text{Out}(F_n) \) of the pure cyclic splitting \( s \) acts on \( B \) properly and cocompactly. On the other hand, by the definition of the equivariant Gromov Hausdorff topology [P89], if \( k > 0 \) and if \( A \) is a \( k \)-short free basis for a tree \( T \in \overline{cv_0(F_n)}^s \) then there is a neighborhood \( U \) of \( T \) in \( \overline{cv_0(F_n)}^s \) such that \( A \) is a \( k+1 \)-short free basis for every tree \( S \in U \). Thus by invariance under the action of \( \text{Out}(F_n) \) and cocompactness, there is a number \( q(s) > 0 \) so that any tree \( T \in B \) admits a \( q(s) \)-short basis.

Now there is a number \( b > 0 \) depending on \( s \) and \( \epsilon \), and for every tree \( T \in \overline{cv_0(F_n)}^s \) such that \( \Upsilon_c(T) \) is a free refinement of \( s \) there is a tree \( T' \in B \) and there is an equivariant \( b \)-Lipschitz map \( T' \to T \). This map preserves the topological type of the quotient graph and expands or decreases the lengths of the edges of \( T' \). As \( T' \) admits \( q(s) \)-short basis, the tree \( T \) admits a \( bq(s) \)-short basis. This shows the lemma. \( \square \)

Fix a number \( k \geq 3 \) and a number \( \ell \leq n-1 \). Let \( R_\ell \) be the rose with \( \ell \) petals of equal length one and let \( \tilde{R}_\ell \) be its universal covering. Let \( A \) be a free factor of \( F_n \) and identify \( A \) with the fundamental group of \( R_\ell \). The free factor \( A \) is called \( k \)-\emph{short} for \( T \in \overline{cv_0(F_n)}^s \) if there is an \( A \)-equivariant \( k \)-Lipschitz map

\[
F : \tilde{R}_\ell \to T
\]

which maps vertices to vertices. If \( T \in \overline{cv_0(F_n)}^s \) then this is equivalent to the requirement that the quotient map \( f : R_\ell \to T/F_n \) of \( F \) is a \( k \)-Lipschitz map which maps the vertex \( v \) of \( R_\ell \) to a vertex of \( T/F_n \) and so that \( f_\ast(\pi_1(R_\ell,v)) \) is conjugate to \( A \). By Lemma 2.22 each \( T \in \overline{cv_0(F_n)}^s \) admits 3-short free factors of any rank \( \ell \leq n-1 \).

The following observation is a version of Lemma 3.2 of [BF14a] and Lemma A.3 of [BF14c] (see also [HM13a]). For its formulation, note that a corank one free factor \( A < F_n \) determines a one-loop graph of groups decomposition of \( F_n \) with vertex group \( A \) (see Section 4.1 of [HM13a] for details). In the sequel we often view a corank one free factor of \( F_n \) as a point in \( \mathcal{FS} \) without further notice.

\begin{lemma}
For every \( k \geq 3 \) there is a number \( c = c(k) > 0 \) with the following property. For every \( T \in \overline{cv_0(F_n)}^{++} \), the distance in \( \mathcal{FS} \) between \( \Upsilon(T) \) and any corank one free factor which is \( k \)-short for \( T \) is at most \( c \).
\end{lemma}

\textbf{Proof.} Since the volume of \( T/F_n \) equals one and since \( T/F_n \) has at most \( 3n-4 \) edges, there is an edge \( e \) in \( T/F_n \) of length at least \( 1/(3n-4) \).

Assume first that \( e \) is non-separating. Collapsing a maximal forest in \( T/F_n \) not containing \( e \) yields an \( F_n \)-tree \( V \) whose quotient \( V/F_n \) is a rose and an equivariant one-Lipschitz map

\[
F : T \to V.
\]
The induced quotient map $f : T/F_n \to V/F_n$ maps the edge $e$ isometrically onto a petal $e_0$ of $V/F_n$. We refer again to [CL95] for details of this construction.

Let $v$ be a vertex of $V$ and let $A < F_n$ be the corank one free factor of all elements $g$ with the property that the segment connecting $v$ to $gv$ does not cross through a preimage of $e_0$. The free factor $A$ defines a one-loop free splitting of $F_n$ which is obtained from $V/F_n$ by collapsing the complement of $e_0$ to a point. In particular, this splitting is a collapse of $\mathcal{T}(T)$. Therefore it suffices to show that the distance in $\mathcal{FS}$ between $A$ and any corank one free factor of $F_n$ which is $k$-short for $T$ is bounded from above by a number only depending on $k$.

Let as before $R_{n-1}$ be a rose with $n - 1$ petals of equal length one and universal covering $\tilde{R}_{n-1}$. Let $Q : \tilde{R}_{n-1} \to T$ be a $k$-Lipschitz map which maps the vertices of $\tilde{R}_{n-1}$ to vertices in $T$ and which is equivariant with respect to the action of a corank free factor $C$ of $F_n$, viewed as the fundamental group of $\tilde{R}_{n-1}$. Then $Q_0 = F \circ Q : \tilde{R}_{n-1} \to V$ is an equivariant $k$-Lipschitz map. Its quotient $q_0 : R_{n-1} \to V/F_n$ maps the vertex of $R_{n-1}$ to the vertex of $V/F_n$. The image under $q_0$ of a petal of $R_{n-1}$ passes through the loop $e_0$ at most $k(3n - 4)$ times. The claim of the lemma now follows from Lemma A.3 and Remark A.8 of [BF14c].

If the edge $e$ is separating then the same reasoning applies. Let $v_1, v_2$ be the two vertices in $T/F_n$ on which $e$ is incident. Collapsing maximal trees in the two components of $T/F_n - e$ yields an $F_n$-tree $V$ whose quotient $V/F_n$ consists of two roses connected at their vertices by the edge $e$. Lemma A.3 and Remark A.8 of [BF14c] are valid in this situation as well and yield the lemma. □

**Corollary 2.6.**

(1) There is a number $m = m(n) > 0$, and for every tree $T \in \overline{cv_0(F_n)^{++}}$ there is a neighborhood $U$ of $T$ in $\overline{cv_0(F_n)^{++}}$ such that

$$\text{diam}(\mathcal{T}(U)) \leq m.$$ 

(2) For every simplicial tree $T \in \overline{cv_0(F_n)^{}}$ there is a neighborhood $U$ of $T$ in $\overline{cv_0(F_n)^{++}}$ such that the diameter of $\mathcal{T}(U \cap \overline{cv_0(F_n)^{++}})$ is finite.

**Proof.** By Lemma 2.2, every tree $T \in \overline{cv_0(F_n)^{++}}$ admits a 3-short basis. Let $e_1, \ldots, e_n$ be such a basis. By the definition of the equivariant Gromov Hausdorff topology, there is a neighborhood $V$ of $T$ in $\overline{cv_0(F_n)^{}}$, and for every $S \subseteq V$ there is a vertex $v \in S$ so that for each $i$ the distance in $S$ between $v$ and $e_i v$ is at most 4. Thus the corank one free factor with basis $e_1, \ldots, e_{n-1}$ is 4-short for each $S \subseteq V$. The first part of the corollary for $U = V \cap \overline{cv_0(F_n)^{++}}$ is now immediate from Lemma 2.5.

To show the second part of the corollary, let $T \in \overline{cv_0(F_n)^{}}$. Choose any free basis $e_1, \ldots, e_n$ for $F_n$. Let $v \in T$ be any vertex and let $\ell = \max \{ \text{dist}(v, e_i v) \}$. Then the corank one free factor with basis $e_1, \ldots, e_{n-1}$ is $\ell$-short for $T$. As in the first part of this proof, this factor is $\ell + 1$-short for every tree in some neighborhood of $T$ in $\overline{cv_0(F_n)^{}}$. The second part of the corollary now follows from Lemma 2.5. □
Remark 2.7. By the main result of [G00], any neighborhood in $cv(F_n)$ of a simplicial tree $T \in cv_0(F_n)$ contains trees with dense orbits. Moreover, it is unclear whether all neighborhoods of $T$ contain points $S \neq T$ in $cv_0(F_n)$.

Corollary 2.8. Let $T \in cv_0(F_n)$ be a simplicial tree and let $T_i \subset cv_0(F_n)^{++}$ be a sequence converging to $T$. Then $\text{diam}(\Upsilon\{T_i | i\}) < \infty$.

Fix once and for all a number $k \geq 4$. Quantitative versions of all statements in the sequel will depend on this choice of $k$, but for simplicity we will drop this dependence in our notations.

Let $cv_0(F_n)$ be the space of all free simplicial $F_n$-trees with volume one quotient. For a number $\ell \leq n-1$ call two trees $T, T' \in cv_0(F_n)$ $\ell$-tied if there is a free factor $A$ of $F_n$ of rank $\ell$ which is $k$-short for both $T, T'$. For trees $T \neq T' \in cv_0(F_n)$ let $d^\ell_{ng}(T,T')$ be the minimum of all numbers $s \geq 1$ with the following property. There is a sequence $T = T_0, \ldots, T_s = T' \subset cv_0(F_n)$ so that for all $i$ the trees $T_i, T_{i+1}$ are $\ell$-tied. Define moreover $d^\ell_{ng}(T,T) = 0$ for all $T \in cv_0(F_n)$.

Proposition 2.9. For all $1 \leq \ell \leq n-1$, the function

$$d^{\ell}_{ng} : cv_0(F_n) \times cv_0(F_n) \to \mathbb{N}$$

is a distance on $cv_0(F_n)$.

Proof. Symmetry of $d^\ell_{ng}$ is immediate from the definition, as well as the fact that $d^\ell_{ng}(T,T') = 0$ if and only if $T = T'$. The triangle inequality is built into the construction, so all we have to show is that $d^\ell_{ng}(T,T') < \infty$ for all $T, T' \in cv_0(F_n)$. For $\ell = n-1$, this is a consequence of Lemma 2.10 below and its proof, and the case $\ell \leq n-2$ is established in Lemma 2.13 $\blacksquare$

Lemma 2.10. The map $\Upsilon : (cv_0(F_n), d^{n-1}_{ng}) \to \mathcal{FS}$ is a coarsely Out$(F_n)$-equivariant quasi-isometry.

Proof. Recall that the map $\Upsilon$ is coarsely surjective. We begin with showing that $\Upsilon$ is coarsely $L$-Lipschitz for some $L = L(k) \geq 1$.

Let $d_{FS}$ be the distance in $\mathcal{FS}$. By definition, if $T, T' \in cv_0(F_n)$ are $(n-1)$-tied then there is a free factor $A$ of $F_n$ of rank $n-1$ which is $k$-short for both $T, T'$. By Lemma 2.8

$$d_{FS}(A, \Upsilon(T)) \leq c(k), \quad d_{FS}(A, \Upsilon(T')) \leq c(k)$$

and hence $d_{FS}(\Upsilon(T), \Upsilon(T')) \leq 2c(k)$. Thus $\Upsilon$ is $2c(k)$-Lipschitz by an iterated application of the triangle inequality (remember that $d^{n-1}_{ng}$ only assumes integral values).

We are left with showing that the map $\Upsilon$ coarsely decreases distances by at most a fixed positive multiplicative constant. Choose for every free factor $A$ of $F_n$ of corank one a marked rose $R(A)$ with $n$ petals of length $1/n$ each which represents this factor $A$, i.e. such that $n-1$ petals of $R(A)$ generate $A$. We denote by $\tilde{R}(A) \in cv_0(F_n)$ the universal covering of $R(A)$. By construction, $d_{FS}(A, \Upsilon(\tilde{R}(A))) = 1$. 

Let $\hat{FS}$ be the complete subgraph of the free splitting graph (not of its first barycentric subdivision) whose vertex set consists of one-loop free splittings of $F_n$. It is well known that the vertex inclusion extends to a coarsely $\text{Out}(F_n)$-equivariantly quasi-isometry $\hat{FS} \to FS$ (see e.g. [H14b] for a detailed proof in the case of the curve graph which carries over word by word).

Since $\hat{FS}$ is a metric graph and, in particular, a geodesic metric space, and since for all corank one free factors $A$ the distance between $A$ and $\Upsilon(\tilde{R}(A))$ equals one, it now suffices to show the following. Whenever two one-loop graph of groups decompositions $A, C$ of $F_n$ are of distance one in $\hat{FS}$, i.e. if they have a common refinement, then $d_{ng}^{-1}(\tilde{R}(A), \tilde{R}(C)) \leq 2$.

Thus let $A < F_n, C < F_n$ be corank one free factors defining one-loop graph of groups decompositions with a common refinement. Then up to replacing one of these free factors by a conjugate, the intersection $B = A \cap C$ is a free factor of rank $n - 2$, and there is a free splitting $F_n = U \ast B \ast D$ with $U \ast B = A$ and $B \ast D = C$.

Let $G$ be a metric rose with $n$ petals of length $1/n$ which represents this splitting, with universal covering $\tilde{G} \in \text{cv}_0(F_n)$. Then both corank one free factors $A, C$ of $F_n$ are $2$-short for $\tilde{G}$. This implies that the distance with respect to the metric $d_{ng}^{-1}$ between $\tilde{R}(A)$ and $\tilde{G}$ is at most one, and the same holds true for the distance between $\tilde{G}$ and $\tilde{R}(C)$. Therefore we have $d_{ng}^{-1}(\tilde{R}(A), \tilde{R}(C)) \leq 2$ which is what we wanted to show. The lemma is proven. □

The final goal of this section is to give a geometric interpretation of the free factor graph. We will take a slightly more general viewpoint and introduce $n - 3$ additional intermediate $\text{Out}(F_n)$-graphs.

Fix a number $\ell \leq n - 2$. Define a graph $FF_\ell$ as follows. Vertices of $FF_\ell$ are free factors of rank $n - 1$. Two such vertices $A, B$ are connected by an edge of length one if and only if up to conjugation, the intersection $A \cap B$ contains a free factor of rank $\ell$. Note that the graphs $FF_\ell$ all have the same set of vertices, and for each $\ell \leq n - 2$ the graph $FF_\ell$ can be obtained from the graph $FF_{\ell-1}$ by deleting some edges. In particular, the vertex inclusion extends to a one-Lipschitz map $FF_\ell \to FF_{\ell-1}$. The group $\text{Out}(F_n)$ acts as a group of simplicial automorphisms on each of the graphs $FF_\ell$. We have

**Lemma 2.11.** The graph $FF_1$ is coarsely $\text{Out}(F_n)$-equivariantly quasi-isometric to the free factor graph.

**Proof.** Associate to a free factor $A$ in $F_n$ a free factor $h(A) > A$ of corank one. If the free factors $A, B$ are connected by an edge in the free factor graph, then up to conjugation and exchanging $A$ and $B$ we have $A < B$. But this just means that up to conjugation, $h(A)$ and $h(B)$ intersect in the free factor $A$ and hence $h(A)$ and $h(B)$ are connected by an edge in $FF_1$. In other words, the map $h$ which maps the vertex set of the free factor graph into the vertex set of the graph $FF_1$ is one-Lipschitz with respect to the metric of $FF$ and the metric of $FF_1$. Moreover, $h$ is coarsely $\text{Out}(F_n)$-equivariant.
Now if $A, B$ are free factors of rank $n - 1$ which are connected by an edge in $\mathcal{FF}_1$ then up to conjugation, they intersect in some free factor $C$. Then the distance between $A, C$ and between $B, C$ in $\mathcal{FF}$ equals one and hence the distance between $A, B$ in $\mathcal{FF}$ is at most two. The lemma follows. \hfill \Box

**Remark 2.12.** The dual of a graph $G$ is a graph $G'$ whose vertex set is the set of edges of $G$ and where two distinct vertices of $G'$ are connected by an edge if the corresponding edges of $G$ are incident on the same vertex. For $\ell \leq n - 2$, the dual $\mathcal{FF}'_\ell$ of $\mathcal{FF}_\ell$ has a simple description. Its vertex set consists of free factors of rank $\ell$, and two such free factors are connected by an edge if up to conjugation, they are subgroups of the same free factor of rank $n - 1$. Although we do not use these dual graphs directly, they are more closely related to the geometrical description of the graphs $\mathcal{FF}_\ell$ given in Lemma 2.13 below.

For each $\ell \leq n - 2$ let

$$\Upsilon_\ell : \text{cv}_0(F_n) \to \mathcal{FF}_\ell$$

be a map which associates to a tree $T$ a $k$-short corank one free factor— in fact we can take the same map for every $\ell$ and view its images as vertices in any one of the graphs $\mathcal{FF}_\ell$.

The statement of the following lemma is only needed in the case $\ell = 1$.

**Lemma 2.13.** For $\ell \leq n - 2$ the map $\Upsilon_\ell : (\text{cv}_0(F_n), d_{ng}^\ell) \to \mathcal{FF}_\ell$ is a coarsely $\text{Out}(F_n)$-equivariant quasi-isometry.

**Proof.** The proof of the lemma is similar to the proof of Lemma 2.10. Note first that the map $\Upsilon_\ell$ is coarsely surjective.

Let $\ell \leq n - 2$ and let $A < F_n$ be a free factor of rank $\ell$. Let $T \in \text{cv}_0(F_n)$ be such that $A$ is $k$-short for $T$. Recall from 5 the definition of the map $\Upsilon : \text{cv}_0(F_n) \to \mathcal{FS}$. We claim that there is a free factor $B > A$ of rank $n - 1$ whose distance to $\Upsilon(T)$ in the free splitting graph is bounded from above by a number only depending on $k$.

Let as before $R_\ell$ be a rose with $\ell$ petals of equal length one and let $f : R_\ell \to T/F_n$ be a $k$-Lipschitz map such that $f_*(\pi_1(R_\ell))$ is conjugate to $A$. Let $e \subset T/F_n$ be an edge of length at least $1/(3n - 4)$. Then $e$ is covered by $f(R_\ell)$ at most $k(3n - 4)\ell$ times.

As in the proof of Lemma 2.10 assume first that the edge $e$ of $T/F_n$ is non-separating. Collapse a maximal forest in $T/F_n$ not containing $e$ to a point and let $V$ be the resulting rose. Let $e_0 \subset V$ be the image of $e$ under the collapsing map.

Embed $V$ into the manifold $M = \sharp_2(S^2 \times S^1)$ to define a marked isomorphism of fundamental groups. Let $S_0 \subset M$ be an embedded sphere which intersects the interior of the petal $e_0$ of $V \subset M$ in a single point and has no other intersection with $V$. Then $S_0$ defines a one-loop free splitting of $F_n$ obtained by collapsing the complement of $e_0$ in $V$ to a point. Its vertex group is up to conjugation the group of all homotopy classes of loops in $M$ based at a point $x \in M - S_0$ which do not intersect $S_0$. The distance in the graph $\mathcal{FS}$ between $\Upsilon(T)$ and the one-loop free splitting defined by $S_0$ equals one.
Let $G \subset M$ be an embedded rose with $\ell$ petals, with vertex $x \in M - S_0$ and with fundamental group the free factor $A$. We may assume that the rose $G$ intersects the sphere $S_0$ in at most $k(3n - 4)\ell$ points. Let $\Sigma \subset M$ be a sphere which is disjoint from the rose $G$. Such a sphere exists since the rank of $A$ equals $\ell \leq n - 2$. The free factor $A$ is a subgroup of the corank one free factor of $F_n$ which is the vertex group $B$ of the splitting defined by $\Sigma$.

If $\Sigma$ is disjoint from $S_0$ then $S_0, \Sigma$ define free splittings of $F_n$ of distance two in $FS$. The distance in the free splitting graph between $B$ and $\Upsilon(T)$ is at most four, so the above claim follows in this case.

If $\Sigma$ intersects $S_0$ then use $\Sigma$ to surger $S_0$ so that the intersection number with the rose $G$ is decreased as follows. Choose a disc component $D$ of $\Sigma - S_0$. Its boundary divides $S_0$ into two discs $D_1, D_2$. Assume that $D_1$ is the disc with fewer intersections with $G$. Then $D \cup D_1 = S_1$ is a sphere disjoint from $S_0$ with at most $k(3n - 4)\ell/2$ intersections with $G$. The sphere $S_1$ defines a free splitting of $F_n$ whose distance in $FS$ to the splitting defined by $S_0$ equals two (we refer to [HiHo12] for details of this well known construction).

Repeat this construction with $S_1$ and $\Sigma$. After at most $\log_2 k(3n - 4)\ell$ such surgeries the resulting sphere $Q$ is disjoint from $G$. The sphere $Q$ determines a corank one free factor of $F_n$ containing $A$ whose distance to $\Upsilon(T)$ in $FS$ is at most $1 + \log_2 k(3n - 4)\ell$.

As a consequence, if $e$ is non-separating then in the free splitting graph, $\Upsilon(T)$ is at uniformly bounded distance from a one-loop graph of groups decomposition $F_n = C \ast$ with trivial edge group and vertex group $C > A$. The same reasoning also applies if the edge $e$ is separating- in this case the sphere $S_0$ is chosen to be separating as well.

To summarize, if the free factor $A < F_n$ is $k$-short for $T \in cv_0(F_n)$ then the distance in $FS$ between $\Upsilon(T)$ and some corank one free factor $B > A$ is uniformly bounded. By the definition of the metric $d_{ng}$ and of the graph $FF$, this implies that whenever $T, T' \in cv_0(F_n)$ are such that $d_{ng}(T, T') = 1$ then the distance in $FF$ between $\Upsilon(T)$ and $\Upsilon(T')$ is uniformly bounded. By an iterative application of the triangle inequality, this shows that the map $\Upsilon : (cv_0(F_n), d_{ng}) \to FF$ is coarsely Lipschitz.

To show that the map $\Upsilon$ coarsely decreases distances at most by a fixed positive multiplicative constant, note that if two corank one free factors $A, A'$ of $F_n$ are connected by an edge in $FF$ then there are free bases $e_1, \ldots, e_n$ and $e'_1, \ldots, e'_n$ of $F_n$ such that $e_i = e'_i$ for $1 \leq i \leq \ell$ and that $A = \langle e_1, \ldots, e_{n-1} \rangle, A' = \langle e'_1, \ldots, e'_{n-1} \rangle$ up to conjugation. The universal coverings $R, R' \in cv_0(T)$ of the marked roses with petals of length $1/n$ determined by these bases are $\ell$-tied. As in the proof of Lemma 2.10 this yields the lemma.

**Remark 2.14.** Purists may find the simultaneous use of the sphere graph and of Outer space as a model for the free splitting graph unsatisfactory. However we felt that using spheres makes the proof of Lemma 2.13 more transparent. We invite...
the reader to find another proof along the lines of the proofs of Lemma A.3 and Lemma 4.1 of [BF14c].

3. Skora paths

The goal of this section is to introduce folding paths and establish some of their properties needed later on.

A morphism between $F_n$-trees $S, T$ is an equivariant map $\varphi : S \to T$ such that every segment of $S$ can be subdivided into finitely many subintervals on which $\varphi$ is an isometric embedding. If the tree $S$ is simplicial then we require that the restriction of $\varphi$ to any edge of $S$ is an isometric embedding.

The following (well known) construction is taken from Section 2 of [BF14a]. Let for the moment $U$ be an arbitrary $F_n$-tree. A direction at a point $x \in U$ is a germ of non-degenerate segments $[x, y]$ with $y \neq x$. At each interior point of an edge of $U$ there are exactly two directions. A collection of directions at $x$ is called a gate at $x$. A turn at $x$ is an unordered pair of distinct directions at $x$. It is called illegal if the directions belong to the same gate, and it is called legal otherwise. A train track structure on $U$ is an $F_n$-invariant family of gates at the points of $U$ so that at each $x \in U$ there are at least two gates.

A morphism $\varphi : S \to T$ determines a collection of gates as follows. Define a turn in $S$ to be illegal if it is given by two directions which are identified by the morphism $\varphi$. Otherwise the turn is defined to be legal. Two directions $d, d'$ at the same point belong to the same gate if either $d = d'$ or if the turn $d, d'$ is illegal. If these gates determine a train track structure on $S$ and if moreover there is a train track structure on $T$ so that legal turns are sent to legal turns, then $\varphi$ is called a train track map (see p.110 of [BF14a]).

Let as before $\text{cv}(F_n)$ be unprojectivized Outer space, with its boundary $\partial \text{cv}(F_n)$ of minimal very small actions of $F_n$ on $\mathbb{R}$-trees which are either not simplicial or which are not free. As in Section 2 denote by $\text{cv}_0(F_n) \subset \text{cv}(F_n)$ the subspace of trees with quotient of volume one. Let $\text{CV}(F_n)$ be the projectivization of $\text{cv}(F_n)$, with its boundary $\partial \text{CV}(F_n)$. The union

$$\text{CV}(F_n) = \text{CV}(F_n) \cup \partial \text{CV}(F_n)$$

is a compact space. The outer automorphism group $\text{Out}(F_n)$ of $F_n$ acts on $\text{CV}(F_n)$ as a group of homeomorphisms. The projection map restricts to an $\text{Out}(F_n)$-equivariant homeomorphism $\text{cv}_0(F_n) \to \text{CV}(F_n)$.

There is a natural bijection between conjugacy classes of free bases of $F_n$ and roses (=marked metric roses with $n$ petals of length $1/n$ each). Define the standard simplex of a free basis of $F_n$ to consist of all simplicial trees

$$U \in \text{cv}_0(F_n)^{++} \subset \text{cv}_0(F_n)^{+} \subset \text{cv}(F_n) = \text{cv}(F_n) \cup \partial \text{cv}(F_n)$$

with quotient of volume one which are universal coverings of graphs obtained from the rose $R$ corresponding to the basis by changing the lengths of the edges. We allow that $U$ is contained in the boundary of unprojectivized Outer space, i.e. that
the rank of the fundamental group of the graph $U/F_n$ is strictly smaller than $n$. A standard simplex is a compact subset of the space $cv_0(F_n)^{++}$ of simplicial $F_n$-trees with trivial edge stabilizers and quotient of volume one.

In the proof of Lemma 3.2 below and several times later on we will use the main result of [P88] which we report here for easy reference. We do not define the equivariant Gromov Hausdorff topology, but we note that this is the topology used for $cv(F_n)$ (see [P89] for a precise statement).

**Theorem 3.1.** Let $(X_i)$ be a sequence of complete $\mathbb{R}$-trees. Let $\Gamma$ be a countable group acting by isometries on all $X_i$. Suppose that for each $i$ there exists a point $x_i \in X_i$ such that the following holds true. For every finite subset $P$ of $\Gamma$, the closed convex hulls of the images of $x_i$ under $P$ admit for all $\epsilon > 0$ a covering by balls of radius $\epsilon$ of uniformly bounded cardinality. Then there exists a subsequence converging in the equivariant Gromov Hausdorff topology to a $\mathbb{R}$-tree.

The construction in Lemma 3.2 below is discussed in detail in Section 2 of [BF14a]. We also refer to this paper for references to earlier works where this construction is introduced. In the rest of the paper, we always denote by $[T] \in CV(F_n)$ the projectivization of a tree $T \in cv(F_n)$.

**Lemma 3.2.** For every $[T] \in CV(F_n)$ and every standard simplex $\Delta \subset cv(F_n)^{++}$ there is a tree $U \in \Delta$ and a train track map $f : U \to T$ where $T$ is some representative of $[T]$.

**Proof.** In the case that $[T] \in CV(F_n)$ is free simplicial a detailed argument is given in the proof of Proposition 2.5 of [BF14a] (see also [FM11]). A limiting argument then yields the result for trees $[T] \in \partial CV(F_n)$.

Let $\Sigma \subset cv(F_n)$ be the set of all trees $U \in cv(F_n)$ such that the minimum over all trees $S \in \Delta$ of the smallest Lipschitz constant for equivariant maps $S \to U$ equals one. Let $[T_i] \subset CV(F_n)$ be a sequence of free simplicial projective $F_n$-trees which converge in $CV(F_n)$ to a projective $F_n$-tree $[T]$. By Proposition 2.5 of [BF14a], for each $i$ there is a point $S_i \in \Delta$, a representative $T_i \in \Sigma$ of $[T_i]$ and a train track map $f_i : S_i \to T_i$ of Lipschitz constant one.

Since $\Delta \subset cv_0(F_n)^{++}$ is compact, the set $\Sigma$ is compact with respect to the equivariant Gromov Hausdorff topology $[P88]$ $[P89]$ and hence $\mathcal{P} = \{U \times V \mid U \in \Delta, V \in \Sigma\}$ is compact in the equivariant Gromov Hausdorff topology for the diagonal action of $F_n$. Then $S_i \times T_i \in \mathcal{P}$ for each $i$ and therefore up to passing to a subsequence, we may assume that $S_i \times T_i$ converges in the equivariant Gromov Hausdorff topology to $S \times T \in \mathcal{P}$ where $S \in \Delta$ and where $T \in \Sigma$ is a representative of $[T]$ (see Theorem 3.1 and [P88]).

For each $i$ the graph $A_i$ of $f_i$ is a closed $F_n$-invariant subset of $S_i \times T_i$. By passing to another subsequence we may assume that the sequence $A_i$ converges in
the equivariant Gromov Hausdorff topology to a closed $F_n$-invariant subset $A$ of $S \times T$ (see once more Theorem 3.1 and [PSS]).

Let $d_S, d_T$ be the distance on $S, T$. For each $i$ the set $A_i \subset S_i \times T_i$ is the graph of a one-Lipschitz map $S_i \to T_i$. Thus by continuity, for $(x_1, y_1), (x_2, y_2) \in A \subset S \times T$ we have $d_T(y_1, y_2) \leq d_S(x_1, x_2)$. In particular, for each $x \in S$ there is a unique point $f(x) \in T$ so that $(x, f(x)) \in A$, and the assignment $x \mapsto f(x)$ is an equivariant map $S \to T$ with Lipschitz constant one. Now $T \in \Sigma$ and therefore the Lipschitz constant one is optimal. The map $f$ is an isometry on edges since this is the case for each of the maps $f_i$. Thus $f$ is a morphism which induces a collection of gates on $S$. Since the Lipschitz constant one for equivariant maps $S \to T$ is optimal, there are at least two gates at each preimage of the unique vertex of $S/F_n$ (compare the end of the proof of Proposition 2.5 of [BF14a] for details). In other words, $f$ has the required properties.

**Remark 3.3.** The train track map $f : U \to T$ constructed in the proof of Lemma 3.2 minimizes the Lipschitz constant among all equivariant maps from points in the standard simplex $\Delta$ to $T$. We call such a map *optimal*.

**Remark 3.4.** The proof of Lemma 2.3 shows the following special case of Lemma A.3 of [BF14c]. Let $x, y \in cv(F_n)$ and assume that there is a train track map $f : x \to y$ which maps a non-degenerate segment $c \subset x$ isometrically to a non-degenerate segment $f(c)$ so that $f^{-1}(f(c)) = c$; then the distance in $\mathcal{FS}$ between $\Upsilon(x)$ and $\Upsilon(y)$ is uniformly bounded.

Let $S \in \overline{cv(F_n)}$ be a simplicial tree, let $T \in \overline{cv(F_n)}$ and let $f : S \to T$ be a train track map. By the definition of a train track map, $f$ isometrically embeds every edge.

Let $\epsilon > 0$ be half of the smallest length of an edge of $S$. Let $e, e'$ be edges with the same initial vertex $v$ which define an illegal turn. Assume that $e, e'$ are parametrized by arc length on compact intervals $[0, a], [0, a']$ where $e(0) = e'(0) = v$. Then there is some $t \in (0, \epsilon]$ so that $f(e[0, t]) = f(e'[0, t])$. For $s \in [0, t]$ let $S_s$ be the quotient of $S$ by the equivalence relation $\sim_s$ which is defined by $u \sim_s v$ if and only if $u = ge(r)$ and $v = ge'(r)$ for some $r \leq s$ and some $g \in F_n$. The tree $S_s$ is called a *fold* of $S$ obtained by folding the illegal turn defined by $e, e'$ (once again, compare the discussion in Section 2 of [BF14a]). Note that for $s > 0$ the volume of the graph $S_s/F_n$ is strictly smaller than the volume of $S/F_n$. There also is an obvious notion of a maximal fold at the illegal turn defined by $e, e'$.

Using the terminology of the previous paragraph, the assignment $s \to S_s$ ($s \in [0, t]$) is a path in $\overline{cv(F_n)}$ through $S_0 = S$ which is called a folding path. We call the train track map $f : S \to T$ used to construct the path a *guide* for the path. The semigroup property holds for folding paths. For each $s \in [0, t]$ the train track map $f : S \to T$ decomposes as

$$f = f_s \circ \varphi_s$$

where $f_s : S_s \to T$ is a train track map and $\varphi_s : S \to S_s$ is a train track map for the train track structures on $S, S_s$ defined by $f$ and $f_s$. We refer to [HM11] and to Section 2 of [BF14a] for details of this construction. We insist that we view the initial train track map $f : S \to T$ as part of the data defining a folding path.
Repeat this construction with $S_t$ and a perhaps different pair of edges. The path constructed in this way by successive foldings terminates if $T$ is free simplicial (Proposition 2.2 of [BF14a]).

We next observe that with perhaps the exception of the endpoint, such folding paths are in fact contained in the subspace $\overline{cv(F_n)}^{++}$ of all simplicial trees with at least one orbit of edges with trivial stabilizer.

**Lemma 3.5.** Let $U \in \overline{cv(F_n)}^{++}$ be a simplicial tree and assume that there is a non-trivial folding path $(x_i) \subset \overline{cv(F_n)}$ connecting $U = x_0$ to a tree $U \neq T \in \overline{cv(F_n)}$. Then $U \in \overline{cv(F_n)}^{++}$. If all stabilizers of non-degenerate segments in $T$ are trivial then $U \in \overline{cv(F_n)}^{++}$.

**Proof.** Let $f : U \to T$ be a train track map which guides the folding path $(x_i)$. Let $e_1, e_2$ be two edges in $U$ incident on the same vertex $v$ with the property that the map $f$ identifies two non-degenerate initial proper subsegments $e_1', e_2'$ of $e_1, e_2$. Assume that the stabilizers $A_1, A_2 < F_n$ of both edges $e_1, e_2$ are non-trivial. Since $U$ is very small, the groups $A_1, A_2$ are maximal cyclic. By equivariance, the non-degenerate segments $f(e_i) \subset T$ are stabilized by $A_i$ $(i = 1, 2)$ and hence $f(e_1') = f(e_2') \subset f(e_1) \cap f(e_2)$ is stabilized by the subgroup of $F_n$ generated by $A_1$ and $A_2$. Since $T$ is very small and hence stabilizers of non-degenerate segments in $T$ are maximal cyclic, we conclude that $A_1 = A_2 = A$.

Fold the tree $U$ and equivariantly identify $ge_1'$ and $ge_2'$ for all $g \in F_n$. The resulting tree $U'$ contains a tripod which consists of the images of the identified segments $e_1', e_2'$ and the images of the segments $e_i - e_i' (i = 1, 2)$. As both $e_1, e_2$ are stabilized by $A$, by equivariance this tripod is stabilized by $A$ as well. But $U'$ is contained in a folding path connecting $U$ to $T$ and hence $U'$ is very small. In particular, stabilizers of tripods are trivial. This is a contradiction which implies that the stabilizer of at least one of the edges $e_1, e_2$ is trivial. The first part of the lemma follows.

To show the second part, note that if there is some $t \geq 0$ so that $x_t$ has a non-trivial edge stabilizer then by equivariance and the fact that the train track map $x_t \to T$ maps edges isometrically, there is a non-degenerate segment in $T$ with non-trivial edge stabilizer. Thus indeed $(x_i) \subset \overline{cv(F_n)}^{++}$ if all stabilizers of non-degenerate segments in $T$ are trivial.\hfill $\square$

**Remark 3.6.** Lemma 3.5 shows in particular that the map $\Upsilon$ as defined in (14) of Section 2 is defined on every folding path, perhaps with the exception of its endpoint.

We can also fold all illegal turns with unit speed at once [BF14a]. The resulting path is unique (Proposition 2.2 of [BF14a]). In the same vein we can rescale all trees along the path to have volume one quotient and rescale the endpoint tree $T$ accordingly while folding with unit speed all illegal turns at once. Using Proposition 2.2 and Proposition 2.5 of [BF14a] (compare [BF14a] for references), a path constructed in this way from a train track map $f : S \to T$ is unique and will be called a Skora path in the sequel (however, the path depends on the guiding train
track map \( f : S \to T \). If \( T \in \text{cv}(F_n) \) then this path has finite length, otherwise its length may be infinite. By convention, even if the path has finite length we do not consider its endpoint to be a point on the path.

We say that a (normalized or unnormalized) folding path or Skora path \((x_t)_{0 \leq t < \xi} \) (\( \xi \in [0, \infty] \)) converges to a projective tree \([T] \in \overline{\text{CV}}(F_n) \) if as \( t \to \xi \) the projective trees \([x_t] \) converge in \( \text{CV}(F_n) \) to \([T] \). If \((x_t)\) is a Skora path then this is equivalent to stating that \((x_t)\) is guided by a train track map \( x_0 \to T \). We summarize the discussion as follows (see also Proposition 2.5 of \([BF14a]\)).

**Lemma 3.7.** For every standard simplex \( \Delta \) and every tree \([T] \in \text{CV}(F_n) \) there is a Skora path \((x_t) \subset \text{cv}_0(F_n)^+ \) with \( x_0 \in \Delta \) which converges to \([T] \).

**Proof.** By Lemma 3.2, there is a representative \( T \) of \([T] \), a tree \( U \in \Delta \) and a train track map \( f : U \to T \). This train track map then determines a unique Skora path \((x_t)\) issuing from \( U \). The projectivizations \([x_t] \) of the trees \( x_t \) converge as \( t \to \infty \) in \( \text{CV}(F_n) \) to the projectivization \([T] \) of \( T \) (see \([FM11, BF14a]\)). \( \square \)

In the sequel we will often use volume renormalization to define a Skora path \((x_t) \subset \text{cv}_0(F_n)^+ \). However, most of the time we consider unnormalized Skora paths, i.e. we scale the trees along the path \((x_t)\) in such a way that the train track maps along the path are edge isometries onto a fixed endpoint tree \( T \). Note that our parametrization of such unnormalized Skora path does not coincide with the parametrization used in \([BF14a]\) although the paths coincide as sets.

### 4. Alignment preserving maps

For a number \( L > 1 \), an \( L\)-quasi-geodesic in a metric space \((X,d)\) is a map \( \rho : J \subset \mathbb{R} \to X \) such that \(|s - t|/L - L \leq d(\rho(s), \rho(t)) \leq L|s - t| + L\) for all \( s, t \in J \). The path \( \rho \) is called a reparametrized \( L\)-quasi-geodesic if there is a homeomorphism \( \psi : I \to J \) such that \( \rho \circ \psi : I \to X \) is an \( L\)-quasi-geodesic.

Recall from \([5, 3, 6]\) of Section 2 the definition of the maps \( \Upsilon, \Upsilon_C, \Upsilon_F \). A liberal folding path is a folding path where folding occurs with any speed, and there may also be rest intervals. Building on the work \([HM13, BF14c]\), it was observed in \([BF14a]\) that there is a number \( L > 1 \) such that the image under \( \Upsilon \) of any liberal folding path in \( \text{cv}(F_n)^{++} \) is an \( L\)-unparametrized quasi-geodesic, and the same holds true for the maps \( \Upsilon_C, \Upsilon_F \). Moreover, following the construction in the proof of Proposition 2.2 of \([BF14a]\), if \((x_t)\) is a liberal folding path converging to a projective tree \([T] \), i.e. such that \([x_t] \to [T] \) in \( \overline{\text{CV}}(F_n) \), then there is a Skora path \((y_t)\) converging to \([T] \) such that the Hausdorff distance between \( \Upsilon(x_t) \) and \( \Upsilon(y_t) \) is uniformly bounded.

By Lemma 3.5, all folding paths (with perhaps the exception of their endpoints) are contained in \( \text{cv}(F_n)^{++} \) and therefore the map \( \Upsilon \) is defined on any folding path. Our next goal is to show that the image under \( \Upsilon \) of any folding path is a...
reparametrized $L$-quasi-geodesic and that the same holds true for the maps $\Upsilon_c, \Upsilon_r$.

The main result of this section (Proposition 4.2 below) will be used several times again in later sections.

An alignment preserving map between two $F_n$-trees $T, T' \in \overline{cv}(F_n)$ is an equivariant map $\rho : T \to T'$ with the property that $x \in [y, z]$ implies $\rho(x) \in [\rho(y), \rho(z)]$. An equivariant map $\rho : T \to T'$ is alignment preserving if and only if the preimage of every point in $T'$ is convex ([G00] and Def. 10.7 of [R10]). The map $\rho$ is then continuous on segments.

An example of an alignment preserving map can be obtained as follows. Let $G$ be a finite metric graph with fundamental group $F_n$ and without univalent vertices and let $G'$ be obtained from $G$ by collapsing a forest. The collapsing map $G \to G'$ lifts to a one-Lipschitz alignment preserving map $T \to T'$ where $T, T'$ is the universal covering of $G, G'$. An alignment preserving morphism is an equivariant isometry.

In the next lemma, $d$ denotes the distance in $FS$. Versions of the lemma can be found in [HM13b] and [BF14c].

Lemma 4.1. Let $S, S' \in \overline{cv}(F_n)^+$ and assume that there is an alignment preserving map $\rho : S \to S'$; then $d(\Upsilon(S), \Upsilon(S')) \leq 2$.

Proof. By equivariance, an edge in $S$ with non-trivial stabilizer is mapped by $\rho$ to a (perhaps degenerate) edge in $S'$ with non-trivial stabilizer. Thus $\rho$ induces an alignment preserving map $\rho_0 : S_0 \to S'_0$ where $S_0, S'_0$ is obtained from $S, S'$ by collapsing all edges with non-trivial stabilizers to points. The map $\rho_0$ projects to a quotient map $\hat{\rho}_0 : S_0/F_n \to S'_0/F_n$.

Let $e'$ be an edge in $S'_0/F_n$ and let $V$ be the graph obtained from $S'_0/F_n$ by collapsing the complement of $e'$ to a point. Let $\zeta : S'_0/F_0 \to V$ be the collapsing map. Then $\zeta \circ \hat{\rho}_0 : S_0/F_n \to V$ collapses $S_0/F_n$ to $V$. As $S_0/F_n$ is the graph of groups decomposition defining $\Upsilon(S)$ and $S'_0/F_n$ is the graph of groups decomposition defining $\Upsilon(S')$, this shows that $\Upsilon(S)$ and $\Upsilon(S')$ collapse to the same vertex in $FS$. Thus the distance in $FS$ between $\Upsilon(S)$ and $\Upsilon(S')$ is at most two. □

Proposition 4.2. There is a number $\chi > 0$ with the following property. Let $T, T' \in \partial cv(F_n)$ and assume that there is a one-Lipschitz alignment preserving map $\rho : T \to T'$. Let $\Delta \subset \overline{cv}(F_n)^{++}$ be a standard simplex and let $(x_t)$ be any folding path connecting a point in $\Delta$ to $T$. Then there is a liberal folding path $(y_t)$ connecting a point in $\Delta$ to $T$ such that for all $t$ we have

$$d(\Upsilon(x_t), \Upsilon(y_t)) \leq \chi.$$  

Proof. Let $T, T' \in \partial cv(F_n)$ and assume that there is a one-Lipschitz alignment preserving map

$$\rho : T \to T'.$$
As this property is invariant under scaling the metric on both trees by the same constant, given a standard simple \( \Delta \subset \text{cv}(F_n) \) we may assume that there is a point \( S \in \Delta \) and there is a train track map

\[
\varphi : S \to T.
\]

Then the map \( \rho \circ \varphi : S \to T' \) is equivariant and one-Lipschitz.

We claim that there is a tree \( S' \in \text{cv}(F_n) \) which can be obtained from \( S \) by decreasing the lengths of some edges of \( S \), there is a one-Lipschitz alignment preserving map \( \alpha : S \to S' \) and there is a morphism \( \varphi' : S' \to T' \) such that

\[
\varphi' \circ \alpha = \rho \circ \varphi.
\]

The map \( \alpha \) may collapse some edges of \( S \) to points.

The tree \( S' \) and the map \( \alpha \) are constructed as follows. Let \( e \) be an edge in \( S \). By definition of a train track map, the restriction of \( \varphi \) to \( e \) is an isometric embedding and hence \( \varphi(e) \) is a segment in \( T \) whose length equals the length \( \ell(e) \) of \( e \). The alignment preserving map \( \rho \) maps \( \varphi(e) \) to a segment \( \rho(\varphi(e)) \) of length \( \ell'(e) \leq \ell(e) \).

To construct \( S' \) we reduce the length of the edge \( e \) to \( \ell'(e) \geq 0 \). The natural map \( \alpha : S \to S' \) is one-Lipschitz, equivariant and simplicial, and it associates to \( e \) the (possibly degenerate) edge in \( S' \) obtained in this way (compare with the discussion in [G00, HM13b, BF14c]).

Let \( a, b \subset S \) be subsegments of edges incident on the same vertex \( v \) which are identified by the map \( \varphi \). Then the segments \( a, b \) are also identified by \( \rho \circ \varphi \). This means the following. Let \( U \) be the simplicial tree obtained from \( S \) by equivariantly folding the \( F_n \)-translates of the segments \( a, b \). Then there is a morphism \( \chi : U \to T \) which maps the identified segments \( a, b \) to a segment in \( T \), and this segment is mapped by \( \rho \) to a segment in \( T' \). By equivariance of the map \( \rho \), there is a tree \( U' \) which is be obtained from \( S' \) by equivariantly identifying the \( F_n \)-translates of \( \alpha(a) \) and \( \alpha(b) \), i.e. by a (perhaps trivial) fold, and there is a one-Lipschitz alignment preserving map \( \beta : U \to U' \) and a morphism \( \chi' : U' \to T' \) such that

\[
\chi' \circ \beta = \rho \circ \chi.
\]

Now let \( (\hat{x}_t) \) be an (unnormalized) Skora path connecting \( S \) to \( T \). Following the discussion in Proposition 2.2 of [BF14a] and its proof, there is an (unnormalized) folding path \( (x_t) \) so that only a single fold is performed at the time, and there are sequences \( t_i \to \infty, s_i \to \infty \) such that \( \hat{x}_{s_i} = x_{t_i} \). In particular, we have \( (x_{t_i}) \to T \) in the equivariant Gromov Hausdorff topology. For each \( t \) there is a train track map \( h_t : x_t \to T \). The Hausdorff distance between the path \( \Upsilon(\hat{x}_t) \) and the path \( \Upsilon(x_t) \) is at most \( \chi \) for a number \( \chi > 0 \) not depending on the paths.

The discussion in the beginning of this proof shows that there is a (suitably parametrized) liberal folding path \( (y_t) \) connecting \( S' \) to \( T' \) with the following property. For each \( t \), there is a one-Lipschitz alignment preserving map \( \zeta_t : x_t \to y_t \),
and there is a morphism $h' : y_t \to T'$ such that the diagram

$$
\begin{array}{ccc}
x_t & \xrightarrow{h_t} & T \\
\downarrow & & \downarrow \rho \\
y_t & \xrightarrow{h'_t} & T'
\end{array}
$$

commutes. As $t \to \infty$, $y_t$ converges in the equivariant Gromov Hausdorff topology to $T'$. We refer to [HM13b, BF14c] for a detailed proof of this fact in the case that $T \in cv(F_n)$.

By Lemma 4.1, for each $t$ the distance in $FS$ between $\Upsilon(x_t)$ and $\Upsilon(y_t)$ is at most two. Consequently for each $t$ there is some $s(t)$ so that the distance between $\Upsilon(\hat{x}_t)$ and $\Upsilon(y_{s(t)})$ is at most $\chi + 2$. This shows the proposition. \hfill $\square$

Recall the definitions of the maps $\Upsilon : cv(F_n)^+ \to FS$ and $\Upsilon_C : cv(F_n)^s \to CS$ and $\Upsilon_F = \Omega \circ \Upsilon_C : cv(F_n) \to FF$.

**Corollary 4.3.** There is a number $L > 1$ such that the image under $\Upsilon$ (or under $\Upsilon_C, \Upsilon_F$) of every folding path is a reparametrized $L$-quasi-geodesic in $FS$ (or $CS, FF$).

**Proof.** It suffices to show the corollary for bounded subsegments of folding paths. Now if $S, T \in cv(F_n)^+$ and if there is a folding path $(x_t)$ connecting $S$ to $T$, then Proposition 4.2 shows that there are trees $S', T' \in cv(F_n)^+$ with $\Upsilon(S) = \Upsilon(S')$, $\Upsilon(T) = \Upsilon(T')$ and there is a liberal folding path $(y_t)$ connecting $S'$ to $T'$ such that $d(\Upsilon(x_t), \Upsilon(y_t)) \leq \chi$ for all $t$. Then $(y_t) \subset cv(F_n)^+$ by Lemma 3.5. The corollary now follows from the fact that the image under $\Upsilon$ of a liberal folding path in $cv(F_n)^+$ is an reparametrized $m$-quasi-geodesic for a universal number $m > 1$ [HM13b]. \hfill $\square$

5. **Trees in $\partial CV(F_n)$ without dense orbits**

A tree $T \in \partial cv(F_n)$ decomposes canonically into two disjoint $F_n$-invariant subsets $T_d$ and $T_c$. Here $T_d$ is the set of all points $p$ such that the orbit $F_n p$ is discrete, and $T_c = T - T_d$. The set $T_c \subset T$ is closed. Each of its connected components is a subtree $T'$ of $T$. The stabilizer of $T'$ acts on $T'$ with dense orbits. We have $T_d = \emptyset$ if and only if the group $F_n$ acts on $T$ with dense orbits. This property is invariant under scaling and hence it is defined for projective trees.

Let $T \in \partial cv(F_n)$ be a very small $F_n$-tree with $T_d \neq \emptyset$. The quotient $T/F_n$ admits a natural pseudo-metric. Let $\overline{T/F_n}$ be the associated metric space.

Since $T$ is very small, by Theorem 1 of [L94], the space $\overline{T/F_n}$ is a finite graph. Edges correspond to orbits of the action of $F_n$ on $\pi_0(T - \overline{B})$ where $B \subset T$ is the set of branch points of $T$. The graph $\overline{T/F_n}$ defines a graph of groups decomposition for $F_n$, with at most cyclic edge groups.
Denote as before by $FS$ and $CS$ the first barycentric subdivision of the free splitting graph and the cyclic splitting graph. The proof of the following result uses an argument which was shown to me by Vincent Guirardel.

**Proposition 5.1.** Let $[T] \in \partial\text{CV}(F_n)$ be such that $T \neq \emptyset$. Let $(x_t) \subset \text{cv}(F_n)^+$ be a Skora path converging to $[T]$.

1. $\text{diam}(\Upsilon_c(x_t)) < \infty$.
2. If $T$ contains an edge with trivial edge stabilizer then $\text{diam}(\Upsilon(x_t)) < \infty$.

**Proof.** Let $[T] \in \partial\text{CV}(F_n)$ be such that $T \neq \emptyset$. Let $(x_t)$ be an unnormalized Skora path connecting a point $x_0$ in a standard simplex $\Delta$ to a representative $T$ of $[T]$. By this we mean that there is a folding path $(y_t) \subset \text{cv}_0(F_n)$ guided by a train track map $f : x_0 = y_0 \to T$ such that for each $t$ we have $x_t = u_t y_t$. Here $u_t > 0$ is such that for $s < t$ there is a morphism $g_{st} : x_s \to x_t$ and a train track map $f_t : x_t \to T$ with $f_s = f_t \circ g_{st}$.

Let $a > 0$ be the smallest length of an edge in $\overline{T/F_n}$ and let $c \geq a$ be the volume of $T/F_n$. The volume of $x_t/F_n$ is a decreasing function in $t$ which converges as $t \to \infty$ to some $b \geq c$. Thus there is a number $t_0 > 0$ such that for each $t \geq t_0$ the volume of $x_t/F_n$ is smaller than $b + a/8$.

Let $e_0$ be an edge of $T$. The length of $e_0$ is at least $a$. By equivariance and the fact that for each $t$ the map $f_t$ is an edge isometry, there is a non-degenerate subarc $\rho$ of $e_0$ of length $a/2$ (the central subarc of length $a/2$) such that for each $t \geq t_0$ the preimage of $\rho$ under $f_t$ consists of $k(t) \geq 1$ pairwise disjoint subarcs of edges of $x_t$ (i.e. the preimage does not contain any vertex). Namely, since $f_t$ is an edge isometry, otherwise there are two segments in $x_0$ of length at least $a/4$ which are identified by the unnormalized Skora path connecting $x_t$ to $T$. By equivariance, this implies that there is some $u > t$ such that the volume of $x_u/F_n$ is strictly smaller than the volume of $x_t/F_n$ minus $a/8$. This violates the assumption on the volume of $x_t/F_n$.

We claim that the number $k(t)$ of preimages of $\rho$ in $x_t$ does not depend on $t \geq t_0$. Namely, by equivariance and the definition of a Skora path, otherwise there is some $t > t_0$ and a vertex of $x_t$ which is mapped by $f_t$ into $\rho$. By the discussion in the previous paragraph, this is impossible.

Write $k = k(t_0)$. (It is not hard to see that $k = 1$, however this information is not needed in the sequel). For each $u \geq t_0$ the preimage of $\rho$ in $x_u$ consists of exactly $k$ subsegments of edges of $x_u$. The morphism $g_{tu} : x_t/F_n \to x_u/F_n$ maps the $k$ components of the preimage of $\rho$ in $x_t$ isometrically onto the $k$ components of the preimage of $\rho$ in $x_u$. Thus by equivariance, there is an edge $h_{tu}$ of $x_t/F_n$, and there is a subsegment $\rho_{tu}$ of $h_{tu}$ of length $a/2$ which is mapped by the quotient map $\tilde{g}_{tu} : x_t/F_n \to x_u/F_n$ of the train track map $g_{tu}$ isometrically onto a subsegment $\rho_u$ of an edge $h_u$ in $x_u/F_n$, and $\tilde{g}_{tu}^{-1}(\rho_u) = \rho_{tu}$. 


By Lemma \[A.1\] (see also Lemma A.3 of \[BFI14c\]), this implies that the distance in \(\mathcal{C}S\) between \(\Upsilon_C(x_u)\) and \(\Upsilon_C(x_u)\) is uniformly bounded, independent of \(u \geq t_0\). As a consequence, we have \(\text{diam}(\Upsilon_C(x_i)) < \infty\) as claimed.

Now if the stabilizer of the edge \(e_0\) of \(T\) is trivial then by equivariance, the same holds true for the stabilizer of \(h_u\) for all \(u \geq t_0\). Then Lemma A.3 of \[BFI14c\] shows that \(\text{diam}(\Upsilon(x_i)) < \infty\).

From Corollary \[6.10\] and Proposition \[5.1\] we obtain as an immediate consequence

**Corollary 5.2.** If \([T] \in \partial \mathcal{CV}(F_n)\) is such that \(T_d \neq \emptyset\) and if \((x_i)\) is a Skora path converging to \([T]\) then \(\text{diam}(\Upsilon(x_i)) < \infty\).

For the map \(\Upsilon\), the analog of the first part of Proposition \[6.1\] may not hold. To obtain information in this case as well we formulate first a technical observation which will be used again in Section 6.

**Lemma 5.3.** Let \((x_i)\) be an unnormalized Skora path connecting \(x_0 \in \mathcal{CV}(F_n)\) to a tree \([T] \in \partial \mathcal{CV}(F_n)\). For each \(t \geq 0\) let \(f_t : x_t \rightarrow T\) be the corresponding train track map. If there is some \(t > 0\) such that \(x_t\) has an edge \(e\) with non-trivial stabilizer then \(f_t(e)\) is contained in the closure of the discrete set \(T_d \subset T\).

**Proof.** For some \(t > 0\) let \(e\) be an edge in \(x_t\) with non-trivial stabilizer \(A < F_n\). By equivariance, a train track map \(f_t : x_t \rightarrow T\) maps \(e\) isometrically onto a segment \(f(e)\) in \(T\) stabilized by \(A\).

For \(s > t\) let \(g_{ts} : x_t \rightarrow x_s\) be the train track map induced by the Skora path, i.e. such that \(f_s = f_t \circ g_{ts}\) for all \(s > t\). The image \(g_{ts}(e)\) of \(e\) is isometrically embedded in the fixed point set of \(A\) in \(x_s\). Since the tree \(x_s\) is very small, this fixed point set is a line segment \(h_s\) in \(x_s\) whose length is not smaller than the length of \(e\).

We claim that the projection of \(h_s\) into \(x_s/F_n\) is an immersed arc with only finitely many double points. Namely, otherwise there is an element \(g \in F_n - A\) such that \(gh_s \cap h_s\) contains a non-degenerate segment \(a\). Now the group \(A\) fixes \(h_s\), and by equivariance, the group \(Ag^{-1}\) fixes \(gh_s\). Then \(a\) is fixed by \(\langle A, Ag^{-1}\rangle\). Since \(a\) is non-degenerate and \(x_s\) is very small, this implies that \(g\) normalizes \(A\) and hence \(g \in A\). This contradiction shows the claim.

As a consequence, for all \(s > t\) the volume of \(x_s/F_n\) is not smaller than the length of \(e\). In particular, we have \(T_d \neq \emptyset\).

We next observe that the segment \(f(e)\) is contained in the closure \(\overline{T_d}\) of \(T_d\). Namely, otherwise \(f(e) \cap \overline{T_d}\) is a non-trivial subsegment of \(f(e)\). Let \(T'\) be the tree obtained from \(T\) by equivariantly collapsing the segments in the \(F_n\)-orbit of \(f(e) \cap \overline{T_d}\) to points and let \(\rho : T \rightarrow \hat{T}\) be the collapsing map. By the construction in the proof of Proposition \[4.2\] there is a tree \(\hat{x}_t\), there is an alignment preserving map \(\alpha : x_t \rightarrow \hat{x}_t\) and there is a train track map \(\varphi : \hat{x}_t \rightarrow T'\) such that \(\varphi \circ \alpha = \rho \circ f\).
Since $f(e) \not\subset T_d$, the segment $\alpha(e) \subset \tilde{x}_t$ is non-degenerate, and by equivariance, it is stabilized by $A$. The above discussion then shows that $\varphi(\xi(e))$ intersects $T'_d$ in a non-degenerate segment which is impossible. This contradiction yields the lemma. \hfill \Box

Corollary 5.4.  

(1) Let $[T] \in \partial CV(F_n)$ be such that $T_d = \emptyset$. Then any Skora path $(x_t)$ converging to $[T]$ is contained in $cv(F_n)^{++}$. 

(2) For every tree $T \in \partial CV(F_n)$ there is a number $k([T]) > 0$ with the following property. Let $(x_t) \subset cv_0(F_n)^+$ be a folding path converging to $[T]$. Then for all sufficiently large $t$ there is a neighborhood $W$ of $x_t$ in $cv_0(F_n)^+$ such that $\text{diam} \left( \Upsilon((W \cap cv_0(F_n)^{++}) \cup x_t) \right) \leq k([T])$.

Proof. The first part of the corollary is immediate from Lemma 5.3. To show the second part, let $T$ be such that $T_d \neq \emptyset$ and let $(x_t) \subset cv_0(F_n)^+$ be any normalized folding path which converges to $[T]$. For all $t$ let $f_t : x_t \to T$ be the corresponding train track map.

As in the beginning of this Section, let $T/F_n$ be the graph of groups decomposition of $F_n$ defined by $T$. Lemma 5.3 shows that for every $t$ and every edge $e$ of $x_t$ with non-trivial stabilizer $A$, the segment $f_t(e) \subset T$ is contained in an edge $h$ of $T_d$ with stabilizer $A$. The edge $h$ defines a cyclic splitting $e$ of $F_n$ which is a collapse of $T/F_n$, and this splitting coincides with the splitting defined by the edge $e$ of $x_t$, i.e. which is determined by the tree obtained by equivariantly collapsing all edges in $x_t/F_n$ which are not contained in the $F_n$-orbit of $e$ to points.

As $T/F_n$ only has finitely many edges, we deduce that there are only finitely many possibilities for the pure cyclic splitting defined by the union of all edges of $x_t$ with non-trivial stabilizer. Lemma 2.4 shows that for every $t$ there is a neighborhood $W$ of $x_t$ in $cv_0(F_n)^+$ so that the diameter of $\Upsilon((W \cap cv_0(F_n)^{++}) \cup x_t)$ is uniformly bounded, independent of $t$. The construction in the proof of Lemma 2.4 also yields that $\Upsilon((W \cap cv_0(F_n)^{++}) \cup x_t)$ is uniformly bounded in diameter as well. \hfill \Box

We use Lemma 5.3 to show

Lemma 5.5. Let $(x_t)$ be a Skora path connecting a point $x_0 \in cv_0(F_n)^{++}$ to a simplicial tree $[T]$. Then $\text{diam} \left( \Upsilon(x_t) \right) < \infty$.

Proof. Proposition 5.1 yields the lemma for projective trees $[T]$ with at least one edge with trivial stabilizer. Thus assume that $[T] \in \partial CV(F_n)$ is simplicial, and that $T/F_n$ is a graph of groups with each edge group maximal cyclic.

Let $(x_t)$ be an unnormalized Skora path connecting $x_0 \in cv_0(F_n)^{++}$ to a representative $T$ of $[T]$. Since $(x_t) \subset cv(F_n)^+$, for each $t$ we can consider the tree $y_t$ obtained from $x_t$ by equivariantly collapsing all edges with non-trivial stabilizers to points. By definition, we have $\Upsilon(x_t) = \Upsilon(y_t)$ for all $t$. 

As \( T \) is simplicial, the quotient \( T/F_n \) is a finite graph of positive volume. The volumes of the quotient graphs \( x_t/F_n \) are decreasing. The volumes of the graphs \( y_t/F_n \) are decreasing as well. We distinguish two cases.

**Case 1:** The volumes \( \beta_t \) of the graphs \( y_t/F_n \) are bounded from below by a number \( b > 0 \).

Let \( e_1, \ldots, e_n \) be any free basis of \( F_n \). Choose a vertex \( v \in T \) and let \( k = \sup \{ d(v, e_i, v) \mid i \} \). As \( x_t \to T \), for sufficiently large \( t \) the basis \( e_1, \ldots, e_n \) is \((k+1)\)-short for \( x_t \) and hence it is \((k+1)\)-short for \( y_t \). Then \( e_1, \ldots, e_n \) is \((k+1)/a\)-short for \( y_t/\beta_t \in c_{0}(F_n)^{++} \). By Lemma 2.5, this means that \( \Upsilon(x_t) = \Upsilon(y_t/\beta_t) \) is contained in a uniformly bounded neighborhood of the corank one free factor of \( F_n \) with basis \( e_1, \ldots, e_{n-1} \). The lemma follows in this case.

**Case 2:** The volumes of \( y_t/F_n \) converge to zero.

Let \( c > 0 \) be the smallest length of an edge in \( T \). Choose \( t > 0 \) sufficiently small that the volume of \( y_t/F_n \) is smaller than \( c/10 \). Let \( f_t : x_t \to T \) be the guiding train track map. Then for each edge \( e \) of \( T \), there is a subsegment \( \rho \) of \( e \) (the middle subsegment of length at least \( c/2 \)) so that \( f_t^{-1}(\rho) \) consists of a finite number \( k \geq 1 \) of subarcs of edges (compare the proof of Proposition 5.1).

In fact, we have \( k = 1 \). Namely, otherwise there is an embedded edge path \( \alpha \subset x_t \) with endpoints in the center of distinct edges with non-trivial stabilizer which is mapped to a loop in \( T \). As \( T \) is simplicial, this means that \( f(\alpha) \) is a compact subtree of \( T \). If \( v \in x_t \) is such that \( f_t(v) \) is a leaf of this subtree then there are two proper subsegments of edges incident on \( v \) which are indentified by the map \( f_t \). Then \( x_t \) can be folded in such a way that these segments are identified. In finitely many such folding steps, we obtain a new tree \( \hat{x}_t \) and train track maps \( g : x_t \to \hat{x}_t, h : \hat{x}_t \to T \) so that \( f = h \circ g \) and that \( g(\alpha) \) consists of a single point. Then the volume of \( \hat{x}_t \) is smaller than the volume of \( x_t \) minus the length of an edge with non-trivial stabilizer. This violates the assumption on the volume of \( x_t \).

Let \( z_t \) the tree obtained from \( x_t \) by collapsing all edges with trivial stabilizers to points. By equivariance and the above discussion, the graph of groups decomposition \( \Upsilon_C(z_t/F_n) \) coincides with \( \Upsilon_C(T) \). This means that there is a simplicial tree \( S \) with an edge with non-trivial stabilizer, and there is a one-Lipschitz alignment preserving map \( S \to T \). The claim of the lemma now follows from Proposition 4.2 and Proposition 5.1.

### 6. Boundaries and the boundary of Outer space

This section contains the main geometric results of this work. We establish some properties which are valid for all three graphs \( FS, CS, FF \). To ease notation we will prove the results we need only for the graph \( FS \). It will be clear that the argument is also valid for any graph \( G \) which can be obtained from \( FS \) by a surjective coarsely \( \text{Out}(F_n) \)-equivariant Lipschitz map \( \Psi \) so that the image under \( \Psi \circ \Upsilon \) of any folding path is a uniform reparametrized quasi-geodesic. Recall that by Corollary 4.3...
the image under $\Upsilon$ of any folding path is a reparametrized $L$-quasi-geodesic for a universal number $L > 1$.

Let $\partial FS$ be the Gromov boundary of $FS$ and let $\Theta$ be an isolated point. For every standard simplex $\Delta$ define a map $\varphi_\Delta : \partial CV(F_n) \to \partial FS \cup \Theta$ as follows. For a projective tree $[T] \in \partial CV(F_n)$ choose a Skora path $(x_t)$ connecting a point in $\Delta$ to $[T]$. If the reparametrized quasi-geodesic $t \to \Upsilon(x_t) \in FS$ has finite diameter then define $\varphi_\Delta([T]) = \Theta$. If the diameter of the reparametrized quasi-geodesic $\Upsilon(x_t)$ is infinite then define $\varphi_\Delta([T])$ to be the unique endpoint in $\partial FS$ of the path $\Upsilon(x_t)$.

Note that a priori, the map $\varphi_\Delta$ depends on choices since a Skora path connecting a tree $S$ in $\Delta$ to a representative $T$ of $[T]$ depends on the choice of a train track map $S \to T$. The next lemma shows that the map $\varphi_\Delta$ does not depend on $\Delta$ nor on any choices made. It is related to a construction of Handel and Mosher [HM13b].

For the purpose of the proof, for a number $c > 0$ we say that a path $\alpha : [0, \infty) \to FS$ is a $c$-fellow traveler of a path $\beta : [0, \infty) \to FS$ if there is a nondecreasing function $\tau : [0, \infty) \to [0, \infty)$ such that for all $t \geq 0$ we have $d(\alpha(t), \beta(\tau(t))) \leq c$. We allow that the function $\tau$ has bounded image.

**Proposition 6.1.** The map $\varphi_\Delta : \partial CV(F_n) \to \partial FS \cup \Theta$ does not depend on the choice of Skora paths, nor does it depend on the standard simplex $\Delta$. Moreover, it is equivariant with respect to the action of $Out(F_n)$ on $\partial CV(F_n)$ and on $\partial FS$.

**Proof.** If $[T] \in \partial CV(F_n)$ is simplicial then Lemma 5.5 shows that the diameter of the image under $\Upsilon$ of any Skora path converging to $[T]$ is finite. Thus it suffices to show the proposition for trees $[T] \in \partial CV(F_n)$ with $T_c \neq \emptyset$.

Let $T$ be such a tree. As $T$ is not simplicial, the tree $T'$ obtained from $T$ by equivariantly collapsing all edges of $T$ to points is not trivial, and there is an one-Lipschitz alignment preserving map $T \to T'$. By Proposition 1.12 for every Skora path $(x_t)$ connecting a point $x_0$ in a standard simplex $\Delta$ to $T$ there is a Skora path $(y_t)$ connecting a point $y_0 \in \Delta$ to $T'$ so that the Hausdorff distance between $\Upsilon(x_t)$ and $\Upsilon(y_t)$ is uniformly bounded. Thus it suffices to consider trees $[T]$ with $T_c = \emptyset$.

Let $[T] \in \partial CV(F_n)$ with $T_d = \emptyset$ and let $(x_t)_{t \geq 0} \subset \overline{cv_0(F_n)}$ be a Skora path guided by an optimal train track map $x_0 \to T$ where $x_0 \in \Delta$ and where $\hat{T}$ is a representative of $[T]$ (see Remark 4.3 for the existence of such a map). By Lemma 5.3 with perhaps the exception of its endpoint, the path $(x_t)$ is contained in $\overline{cv_0(F_n)}^{++}$.

Choose another (not necessarily distinct) standard simplex $\Lambda$ and a Skora path $(y_t)$ guided by an optimal train track map $y_0 \to T$ where $y_0 \in \Lambda$ and where $T$ is a representative of $[T]$ (in general we expect that $T \neq \hat{T}$). If the diameters of both $\Upsilon(x_t)$ and $\Upsilon(y_t)$ are finite then there is nothing to show, so by perhaps exchanging $(x_t)$ and $(y_t)$ we may assume that the diameter of $\Upsilon(y_t)$ is infinite.

Let $f : y_0 \to T$ be an optimal train track map which guides the folding path $(y_t)$. We claim that there is a sequence $t_i \to \infty$, a sequence $S_i \subset \Delta$ of points in
the standard simplex $\Lambda$, a sequence $a_{t_i} > 0$ such that $a_{t_i}x_{t_i} \to T$ in $\text{cv}(F_n)$ and a sequence of train track maps $f_i : S_i \to a_{t_i}x_{t_i}$ with the following properties.

(1) For every $U \in \Lambda$ the Lipschitz constant of any equivariant map $U \to a_{t_i}x_{t_i}$ is at least $1/2$.

(2) The maps $f_i$ converge as $i \to \infty$ to the map $f$ in the following sense: The graphs of $f_i$ in $S_i \times a_{t_i}x_{t_i}$ converge in the equivariant Gromov Hausdorff topology to the graph of $f$.

Let $B \subset \text{cv}(F_n)$ be the set of all trees $V$ such that the minimum of the smallest Lipschitz constant for equivariant maps $U \to V$ where $U$ runs through the points in the standard simplex $\Lambda$ equals one. Then $B$ is a compact subset of $\text{cv}(F_n)$, and $T \in B$. For $t > 0$ let $b_t > 0$ be such that $b_t x_t \in B$. Then we have $b_t x_t \to T$ in $\text{cv}(F_n)$ ($t \to \infty$) with respect to the equivariant Gromov Hausdorff topology.

Let $\tilde{v} \in S$ be a preimage of the unique vertex $v$ of the graph $S/F_n$. Let $\{g_1, \ldots, g_n\}$ be a free basis of $F_n$ which determines the simplex $\Lambda$ and such that for each $i$ the axis of $g_i$ passes through $\tilde{v}$ (in particular, if $g_i$ has a fixed point in $S$ then this fixed point equals $\tilde{v}$). Let $K$ be the compact subtree of $T$ which is the convex hull of the points $f(\tilde{v}), g_j f(\tilde{v}), g_j^{-1} f(\tilde{v})$ ($j = 1, \ldots, n$). Then for each $t$ there is a point $v_t \in b_t x_t$, and there is a compact subtree $K_t$ of $b_t x_t$ which is the convex hull of the points $v_t, g_j v_t, g_j^{-1} v_t$ ($j = 1, \ldots, n$) and such that the trees $K_t$ converge to $K$ in the usual Gromov Hausdorff topology (see [P89] for details).

Since for each $U \in \Lambda$ the quotient graph $U/F_n$ has a single vertex, for each $t$ there is a tree $U_t \in \Lambda$, there is a number $a_t \leq b_t$ and a morphism $U_t \to a_t x_t$ which maps a preimage of a vertex of $U_t/F_n$ to $v_t$. Note that in general we have $a_t \neq b_t$, however by construction, $a_t/b_t \to 1$ ($t \to \infty$). In particular, for sufficiently large $t$ the trees $a_t x_t$ have property (1) above.

Using again the fact that $U_t$ has a single vertex, we may in fact assume that the morphism $U_t \to a_t x_t$ is a train track map, however it may not be optimal. Namely, the elements $g_i$ ($1 \leq i \leq n$) generate $F_n$ and therefore if there was only one gate for the morphism at a vertex of $U_t$ then the tree $a_t x_t$ can not be minimal. By Theorem 3.1 applied to the products $U_t \times a_t x_t$ (see [P88] for more details), there is a sequence $t_i \to \infty$ so that the sequence

$$f_i : S_i = U_{t_i} \to a_{t_i} x_{t_i}$$

of train track maps has property (2) above as well.

The subspace $\Sigma \subset \text{cv}(F_n)$ of all trees $x$ for which the minimum of the smallest Lipschitz constant for equivariant maps from trees $U \in \Lambda$ to $x$ is contained in the interval $[1/2, 1]$ is compact, and it contains $B$. For each $i$ connect $S_i$ to $a_{t_i} x_{t_i}$ by a Skora path $(y'_k)$ guided by the train track map $f_i$. For sufficiently large $i$ the corresponding unnormalized Skora paths are entirely contained in $\Sigma$.

Normalized Skora paths are geodesics for the one-sided Lipschitz metric on the locally compact space $\text{cv}_0(F_n)$ [EM11] (note however that the one-sided Lipschitz distance between an ordered pair of points $(x, y) \in \text{cv}_0(F_n)$ may be infinite). Thus we can apply to these paths the Arzela Ascoli theorem. As a consequence, up to
passing to a subsequence, the paths \((y_i')\) converge as \(i \to \infty\) to a Skora path \((y_s)\). By the discussion in the previous paragraph and construction, this path is guided by the train track map \(f\), and it connects \(S\) to \([T]\). The image under the map \(\Upsilon\) of the family of paths \((y_i')\), \((y_s)\) is a family of reparametrized \(L\)-quasi-geodesics in the hyperbolic graph \(FS\). Moreover, since \(T_d = \emptyset\), Corollary 5.4 shows that the path \((y_s)\) is contained in \(cv_0(F_n)^{++}\), and the same holds true for each of the paths \((y_i')\).

The diameter of \(\Upsilon(\Lambda) \subset FS\) is bounded independent of the standard simplex \(\Lambda\). Let \(b \geq 0\) be such that \(\Upsilon(x_0)\) is a coarsely well defined shortest distance projection of \(\Upsilon(\Lambda)\) into the reparametrized \(L\)-quasi-geodesic \(\Upsilon(x_i)\).

By hyperbolicity of \(FS\), for every \(u > b\) and every \(v > u\), the image under \(\Upsilon\) of any Skora path connecting a point in \(\Lambda\) to \(x_u\) passes through a uniformly bounded neighborhood of \(\Upsilon(x_u)\). Thus for all \(u > b\) and all \(i\) such that \(t_i > u\), the reparametrized \(L\)-quasi-geodesic \(\Upsilon(y_i')\) in \(FS\) passes through a uniformly bounded neighborhood of \(\Upsilon(x_u)\).

We claim that the same holds true for the reparametrized \(L\)-quasi-geodesic \(\Upsilon(y_s)\). To this end recall that we assumed that the diameter of \(\Upsilon(y_s)\) is infinite. Let \(\tau > 0\) be arbitrary. Since \((y_s) \subset cv_0(F_n)^{++}\), Corollary 2.6 shows that for all sufficiently large \(i\) the reparametrized quasi-geodesic \(\Upsilon(y_i')\) connecting a point in \(\Upsilon(\Lambda)\) to \(\Upsilon(x_{t_i})\) passes through the \(L\)-neighborhood of \(\Upsilon(y_\tau)\).

On the other hand, it also passes through a uniformly bounded neighborhood of \(\Upsilon(x_u)\). Since the diameter of \(\Upsilon(y_s)\) is infinite, for sufficiently large \(\tau\) the point \(\Upsilon(x_u)\) coarsely lies between \(\Upsilon(y_0)\) and \(\Upsilon(y_\tau)\). This shows that the reparametrized quasi-geodesic \(\Upsilon(y_s)\) passes through a uniformly bounded neighborhood of \(\Upsilon(x_u)\) as claimed above.

We use this fact to show that the diameter of \(\Upsilon(x_i)\) is infinite. Namely, assume to the contrary that the diameter of \(\Upsilon(x_i)\) is finite. Then by construction, for each \(i\) the diameter of \(\Upsilon(y_i')\) is bounded from above by a number \(C > 0\) not depending on \(i\). Choose a number \(t > 0\) so that the distance between \(\Upsilon(\Lambda)\) and \(\Upsilon(y_i)\) is at least \(C + 10m\) where \(m > 0\) is as in Corollary 2.6. For sufficiently large \(i\) the path \(\Upsilon(y_i')\) passes through the \(m\)-neighborhood of \(\Upsilon(y_\tau)\) which is a contradiction.

By symmetry, we conclude that the diameter of \(\Upsilon(x_i)\) is infinite if and only if the diameter of \(\Upsilon(y_i)\) is infinite. Moreover, if this holds true then there is a number \(p > 0\) only depending on \(\Lambda, \Delta\) such that \(\Upsilon(y_s)\) is a \(p\)-fellow traveller of \(\Upsilon(x_i)\). As a consequence, the map \(\varphi_{\Delta}\) indeed does not depend on the choice of \(\Delta\) or on the choice of Skora paths. This then implies that the map \(\varphi_{\Delta}\) is moreover \(\text{Out}(F_n)\)-equivariant.

Remark 6.2. The proof of Proposition 6.1 shows more generally the following. Let \(\Delta\) be a standard simplex and let \((x_i) \subset cv_0(F_n)^{++}\) be any normalized folding path. If \([x_i] \to [T] \in \partial CV(F_n)\) in the equivariant Gromov Hausdorff topology then the reparametrized quasi-geodesic \(\Upsilon(x_i)\) in \(FS\) is of infinite diameter if and only if \(\varphi_{\Delta}([T]) \in \partial FS\), and in this case we have \(\Upsilon(x_i) \to \varphi_{\Delta}([T])\) in \(FS \cup \partial FS\).
Remark 6.3. The proof of Proposition 6.1 also gives some information on Skora paths whose images under $\Upsilon$ have finite diameter. Namely, if $\varphi_\Delta([T]) = \Theta$ then there is a point $\xi \in FS$ and there is a number $r > 0$ so that for every Skora path $(x_t) \subset S0(F_n) + \partial CV(F_n)$ and for all large enough $t$ the point $\Upsilon(x_t)$ is contained in the $r$-neighborhood of $\xi$ in $FS$.

By Proposition 6.1 we can define
$$\psi = \varphi_\Delta : \partial CV(F_n) \to \partial FS \cup \Theta$$
for some (and hence every) standard simplex $\Delta$. Similarly we define
$$\psi_C : \partial CV(F_n) \to \partial CS \cup \Theta, \psi_F : \partial CV(F_n) \to \partial FF \cup \Theta.$$
The maps $\psi, \psi_C, \psi_F$ do not depend on choices, and they are $\text{Out}(F_n)$-equivariant.

Our next goal is to show that the map $\psi$ is onto $\partial FS$. As a main preparation we establish the following lemma which is motivated by the work of Klarreich (Proposition 6.4 of [K99]). For convenience of notation we extend the map $\psi$ to $CV(F_n)$ by defining $\psi([T]) = \Theta$ for every simplicial free projective $F_n$-tree $[T] \in CV(F_n)$.

Lemma 6.4. Let $(T_i) \subset S0(F_n)$ be a sequence whose projectivization converges in $CV(F_n) \cup \partial CV(F_n)$ to a point $[T] \in CV(F_n)$. If $\psi([T]) = \Theta$ then $(\Upsilon(T_i))$ does not converge to a point in $\partial FS$.

Proof. We follow the reasoning in the proof of Proposition 6.4 of [K99]. The case that $[T] \in CV(F_n)$ is immediate from Corollary 2.6 so let $(|[T_i]|) \subset CV(F_n)$ be a sequence which converges to a point $[T] \in \partial CV(F_n)$ with $\psi([T]) = \Theta$.

For each $i$ let $T_i \in S0(F_n)$ be a representative of $[T_i]$. We argue by contradiction and we assume that the sequence $(\Upsilon(T_i))$ converges to a point in the Gromov boundary of $FS$.

For each $i$ there is a standard simplex $\Delta_i \subset S0(F_n)^{++}$ so that the distance in $FS$ between $\Upsilon(T_i)$ and $\Upsilon(\Delta_i)$ is at most $k$ where $k > 0$ is a universal constant. Such a simplex can be constructed as follows. Collapse all edges of $T_i$ with non-trivial stabilizer to a point. Let $S$ be the resulting tree. By definition, we have $\Upsilon(T_i) = \Upsilon(S)$.

Collapse all but one edge in $S/F_n$ to a point. If the resulting one-edge graph of groups decomposition $G$ of $F_n$ is a two-vertex decomposition then the vertex groups are free factors of $F_n$. Replace each such vertex by a marked rose representing a basis of the free factor. The resulting graph of groups decomposition collapses to both $G$ and a decomposition defined by a rose of rank $n$. This rose determines a standard simplex as required. The case that $G$ is a one-loop decomposition follows in the same way.

Let $\Sigma_i \subset S(F_n)$ be the set of all trees for which the minimum of the optimal Lipschitz constants for equivariant maps from points in $\Delta_i$ equals one.
For $j > i$ let $T^j_i = b_i T_j \in \Sigma_i$ be a representative of $[T_j]$. Choose an optimal train track map $f_j : r^i_j \to T^j_i$ where $r^i_j \in \Delta_i$ and let $(r^i_j)$ be a Skora path connecting $r^i_0$ to $T^j_i$ which is guided by $f_j$. As the endpoints of the paths are contained in $cv(F_n)_i^{+}$, the same holds true for the entire paths.

The initial points of the paths $(r^i_j) \ (j > i)$ are contained in the compact subset $\Delta_i$ of $cv_0(F_n)_i^{++}$. Hence by the Arzela Ascoli theorem, applied to the geodesics $r^i_j$ for the one-sided Lipschitz metric, up to passing to a subsequence we may assume that the paths $(r^i_j)$ converge as $j \to \infty$ locally uniformly to a Skora path $t \to r^i_t$ issuing from a point $r^i_0 \in \Delta_i$. By the reasoning in the proof of Lemma 5.2 since $[T_j] \to [T]$ we may assume that the path $(r^i_t)$ is defined by a train track map $r^i_0 \to T$ where $T$ is a representative of $[T]$ and hence it connects $r^i_0$ to $[T]$.

By Remark 6.3, there is a point $\eta \in \mathcal{FS}$, and there is a number $b > 0$ only depending on $T$ with the following property. For any Skora path $(x_i) \subset cv_0(F_n)_i^{++}$ guided by some train track map $x_0 \to T$ where $T$ is a representative of $[T]$ and for all large enough $t$, the point $\Upsilon(x_t)$ is contained in the $b$-neighborhood of $\eta$ in $\mathcal{FS}$. In particular, for large enough $t$ the point $\Upsilon(r^i_t)$ is contained in the $b$-neighborhood of $\eta$. Fix a number $t_0$ with this property.

By the second part of Corollary 5.3 for large enough $j$ the distance between $\Upsilon(r^i_{t_0})$ and $\Upsilon(r^j_{t_0})$ is bounded by $k([T])$. As a consequence, if $(\xi)_\xi$ is the Gromov product based at $\eta \in \mathcal{HG}$ then we have

$$((\Upsilon(T_i)) \ | \ \Upsilon(T_j))_\eta \leq q$$

for infinitely many $i, j$ where $q > 0$ is a constant depending on $T$ but not on $i, j$. This is a contradiction to the assumption that the sequence $(\Upsilon(T_i))$ converges to a point in the Gromov boundary of $\mathcal{FS}$. □

**Remark 6.5.** Lemma 6.4 does not state that if $(x_i) \subset cv_0(F_n)$ is any sequence converging to a point $[T] \in \partial CV(F_n)$ with $\psi([T]) = \Theta$ then $\Upsilon(x_i)$ is bounded. In fact, we believe that there should be sequences for which this is not true.

Next we have

**Lemma 6.6.** If $[T_i] \subset CV(F_n)$ is any sequence which converges to $[T] \in \partial CV(F_n)$ with $\psi([T]) = \Theta$ then $\Upsilon(T_i)$ converges to $\psi([T])$ in $\mathcal{FS} \cup \partial \mathcal{FS}$.

**Proof.** Let $\Delta$ be a standard simplex. For each $i$ let $(r^i_t)$ be a Skora path connecting a point in $\Delta$ to a tree $T_i \in cv_0(F_n)$ representing the class $[T_i]$. By the Arzela Ascoli theorem, up to passing to a subsequence we may assume that the paths $(r^i_t)$ converge as $i \to \infty$ to a Skora path $(x_i)$ connecting a point in $\Delta$ to a representative of $[T]$.

Since the paths $\Upsilon(r^i_t)$ are uniform reparametrized quasi-geodesics in $\mathcal{FS}$ and since the path $(x_i)$ connects a point in $\Delta$ to $[T]$, the path $\Upsilon(x_i)$ is a (reparametrized) quasi-geodesic ray connecting $\Upsilon(x_0)$ to $\psi([T])$. The lemma now follows from the second part of Corollary 5.3 and hyperbolicity of $\mathcal{FS}$. □
As a corollary we obtain

**Corollary 6.7.** $\psi(\partial CV(F_n)) \supset \partial FS$.

**Proof.** Since the map $\Upsilon : cv_0(F_n) \to FS$ is coarsely surjective, for every $\eta \in \partial FS$ there is a sequence $(T_i) \subset cv_0(F_n)$ so that $\Upsilon(T_i) \to \eta$.

By compactness of $CV(F_n)$, by passing to a subsequence we may assume that $[T_i] \to [T]$ for some $[T] \in CV(F_n)$. By Lemma 6.3 if $\psi([T]) = \Theta$ then the sequence $\Upsilon(T_i)$ does not converge to $\eta$. Thus $\psi([T]) = \chi \in \partial FS$, and it follows from Lemma 6.6 that $\chi = \xi$. \hfill $\square$

Let now

$\mathcal{F}T = \psi^{-1}(\partial FS) \subset \partial CV(F_n)$.

By $Out(F_n)$-equivariance of the map $\psi$, the set $\mathcal{F}T$ is an $Out(F_n)$-invariant subset of $\partial CV(F_n)$. We showed above that the restriction of $\psi$ maps $\mathcal{F}T$ onto $\partial FS$. We also have

**Lemma 6.8.** The restriction of the map $\psi$ to $\mathcal{F}T$ is continuous and closed.

**Proof.** To show that the restriction to $FS$ of the map $\psi$ is continuous, note that if $[T_i] \to [T] \in \mathcal{F}T$ then there is a sequence $(r^i_t)$ of Skora paths starting at a point in a standard simplex $\Delta$ so that $r^i_t \to [T_i] (t \to \infty)$ and $r^i \to r$ locally uniformly where $r$ is a Skora path connecting $\Delta$ to $[T]$. The images under $\Upsilon$ of these paths are reparametrized quasi-geodesics in $FS$. Continuity now follows from hyperbolicity of $FS$.

Since the Gromov topology on $\partial FS$ is metrizable, to show that the restriction of $\psi$ is closed it suffices to show the following. If $[T_i] \subset \mathcal{F}T$ is any sequence and if $\psi([T_i]) \to \eta \in \partial FS$ then up to passing to a subsequence, we have $[T_i] \to [U]$ where $\psi([U]) = \eta$.

Assume to the contrary that this is not the case. By compactness of $\partial CV(F_n)$ there is then a sequence $[T_i] \subset \mathcal{F}T$ so that $\psi([T_i]) \to \eta$ and such that $[T_i] \to [S] \in \partial CV(F_n)$ where either $[S] \notin \mathcal{F}T$ or $\psi([S]) \neq \eta$.

Now if $[S] \in \mathcal{F}T$ then by continuity of the map $\psi$, we have

$\psi([S]) = \lim_{i \to \infty} \psi([T_i]) = \eta$.

Since we assumed that $\psi([S]) \neq \eta$ this is impossible.

Thus $[S] \notin \mathcal{F}T$. However, it follows from the hypotheses that there is a sequence $(S_i) \subset cv_0(F_n)$ with $[S_i] \to [S]$ and such that $\Upsilon(S_i) \to \eta$ in $FS \cup \partial FS$. Here $S_i$ can be chosen to be a point on the Skora path $r^i_t$ which is such that $\Upsilon(S_i)$ is sufficiently close to $\Upsilon(T_i)$ in $FS \cup \partial FS$. This violates Lemma 6.3. The corollary follows. \hfill $\square$

**Remark 6.9.** The results in this section do not use any specific property of Outer space and can easily be formulated in a more abstract setting which is valid for example for Teichmüller space and the curve graph as in \[K99\] and for various disc graphs in a handlebody \[H11\].
Recall the definition of the hyperbolic $\Out(F_n)$-graphs $(\mathcal{CS}, \mathcal{T}_C)$ and $(\mathcal{FF}, \mathcal{T}_F)$ and of the maps

$$\psi_C : \partial \CV(F_n) \to \partial \mathcal{CS} \cup \Theta \text{ and } \psi_F : \partial \CV(F_n) \to \partial \mathcal{FF} \cup \Theta.$$  

Since for any tree $T \in cvo(F_n)^+$ the distance in $\mathcal{CS}$ between $\Upsilon(T) \in \mathcal{FS} \subset \mathcal{CS}$ and $\Upsilon_C(T)$ is at most one and since $\Upsilon_F = \Omega \circ \Upsilon$ for a coarsely Lipschitz map $\Omega : \mathcal{CS} \to \mathcal{FF}$, we obtain as an immediate consequence

**Corollary 6.10.**  
(1) If $\psi([T]) = \Theta$ then $\psi_C([T]) = \Theta$.  
(2) If $\psi_C([T]) = \Theta$ then $\psi_F([T]) = \Theta$.  

As a consequence of the results in this section, the boundaries of $\mathcal{FS}, \mathcal{CS}$ and $\mathcal{FF}$ can be identified as a set with the quotient of an $\Out(F_n)$-invariant subspace of $\partial \CV(F_n)$ by an equivalence relation defined by the map $\psi$ and the map $\psi_C$ and the map $\psi_F$.

We complete the section with some first easy information on the fibres of $\psi$. From now on we only consider trees $T \in \partial \cv(F_n)$ (or projective trees $[T] \in \partial \CV(F_n)$) with dense orbits. For simplicity we call such a tree dense.

The following definition is due to Paulin (see [G00]).

**Definition 6.11.** A length measure $\mu$ on $T$ is an $F_n$-invariant collection

$$\mu = \{ \mu_I \}_{I \subset T}$$

of locally finite Borel measures on the finite arcs $I \subset T$; it is required that for $J \subset I$ we have $\mu_J = (\mu_I)|J$.

The Lebesgue measure $\lambda$ defining the metric on $T$ is an example of a length measure on $T$ with full support.

Denote by $M_0(T)$ the set of all non-atomic length measures on $T$. By Corollary 5.4 of [G00], $M_0(T)$ is a finite dimensional convex set which is projectively compact. Up to homothety, there are at most $3n - 4$ non-atomic ergodic length measures. In particular, $M_0(T)$ is a cone over a compact convex polyhedron with finitely many vertices. Each non-atomic length measure $\mu \in M_0(T)$ defines an $F_n$-tree $T_\mu \in \partial \cv(F_n)$ as follows [G00]. Define a pseudo-metric $d_\mu$ on $T$ by $d_\mu(x, y) = \mu([x, y])$. Making this pseudo-metric Haussdorf gives an $\mathbb{R}$-tree $T_\mu$.

**Corollary 6.12.** Let $T \in \partial \cv(F_n)$ be a tree with dense orbits; then $\psi([T_\mu]) = \psi([T])$ for all $\mu \in M_0(T)$.

**Proof.** Let $\nu$ be a length measure on $T$ which is contained in the interior of the convex set $M_0(T)$. Let $\zeta$ be a point in $M_0(T)$ which projects to a vertex in the projectivization of $M_0(T)$; this is an ergodic measure in $M_0(T)$. Up to rescaling, there is a one-Lipschitz alignment preserving map $T_\nu \to T_\zeta$. Proposition 4.2 shows that $\psi([T_\nu]) = \psi([T_\zeta])$.

Now if $\xi \in M_0(T)$ is arbitrary then there is an ergodic measure $\beta \in M_0(T)$, and there is a one-Lipschitz alignment preserving map $T_\xi \to T_\beta$. Using once more Proposition 4.2 we deduce that $\psi([T_\xi]) = \psi([T_\beta]) = \psi([T_\nu])$. Since the Lebesgue measure on $T$ defines a point in $M_0(T)$ this shows the corollary. \(\square\)
We use Proposition 4.2 and Corollary 6.12 to show

**Corollary 6.13.** Let \( T, T' \in \partial cv(F_n) \) be trees with dense orbits and let \( \rho : T \to T' \) be alignment preserving. Then \( \psi([T]) = \psi([T']) \).

**Proof.** Let \( T, T' \in \partial cv(F_n) \) be trees with dense orbits and assume that there is an alignment preserving map \( \rho : T \to T' \).

As is shown in [G00], if \( \mu' \) is a non-atomic length measure on \( T' \) then there is a length measure \( \mu \) on \( T \) such that \( \rho_* \mu = \mu' \). This means that for every segment \( I \subset T \) we have \( \mu(I) = \mu'(\rho I) \). As a consequence, there is a one-Lipschitz alignment preserving map \( \hat{\rho} : T \to T' \). Proposition 4.2 shows that \( \psi([T]) = \psi([T']) \). The corollary now follows from Corollary 6.12. \( \square \)

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7. Indecomposable trees

The goal of this section is to obtain some information on the structure of indecomposable trees which is needed for the proof of the theorems from the introduction.

Let \( \Delta \subset \partial F_n \times \partial F_n \) be the diagonal. The zero lamination \( L^2(T) \) of an \( \mathbb{R} \)-tree \( T \) with an isometric action of \( F_n \) is the closed \( F_n \)-invariant subset of \( \partial F_n \times \partial F_n - \Delta \) which is the set of all accumulation points of pairs of fixed points of any family of conjugacy classes with translation length on \( T \) that tends to 0. The zero lamination of a tree \( T \in cv(F_n) \) is empty. For \( T \in \partial cv(F_n) \), it only depends on the projective class \( [T] \in \partial CV(F_n) \) of \( T \).

In [CHL07] the following topological interpretation of the zero lamination of an \( \mathbb{R} \)-tree \( T \in \partial cv(F_n) \) is given.

The union \( \hat{T} = T \cup \partial T \) of the metric completion \( \overline{T} \) of \( T \) with the Gromov boundary \( \partial T \) of \( T \) can be equipped with an observer’s topology. With respect to this topology, \( \hat{T} \) is a compact \( F_n \)-space, and the inclusion \( T \to \hat{T} \) is continuous [CHL07]. Isometries of \( T \) induce homeomorphisms of \( \hat{T} \) (see p.903 of [CHL09]).

There is an explicit description of \( \hat{T} \) as follows. Namely, let again \( L^2(T) \) be the zero lamination of \( T \). There is an \( F_n \)-equivariant [LL03] continuous (Proposition 2.3 of [CHL07]) map

\[
Q : \partial F_n \to \hat{T}
\]

such that \( L^2(T) = \{ (\xi, \zeta) \mid Q(\xi) = Q(\zeta) \} \). This map determines an equivariant homeomorphism

\[
\partial F_n/L^2(T) \to \hat{T}
\]

(Corollary 2.6 of [CHL07]), i.e. the tree \( \hat{T} \) is the quotient of \( \partial F_n \) by the equivalence relation obtained by identifying all points \( \xi, \xi' \in \partial F_n \) with \( Q(\xi) = Q(\xi') \), and a pair of points \( (\xi, \xi') \) is identified if and only if it is contained in \( L^2(T) \).

The limit set of \( \overline{T} \) is defined to be the set

\[
\Omega = Q^2(L^2(T)) \subset \overline{T} \subset \hat{T}
\]
where the notation $Q^2$ refers to applying the map $Q$ to a pair of points with the same image.

A finitely generated subgroup $H < F_n$ is free, and its boundary $\partial H$ is naturally a closed subset of the boundary $\partial F_n$ of $F_n$. If $H$ fixes a point in an $\mathbb{R}$-tree $\Gamma$ then the set $\partial H \times \partial H - \Delta$ of pairs of distinct points in $\partial H$, viewed as a subset of $\partial F_n \times \partial F_n - \Delta$, is contained in $L^2(\Gamma)$. We say that a leaf $\ell \in L^2(\Gamma)$ is carried by a finitely generated subgroup $H$ of $F_n$ if it is a point in $\partial H \times \partial H - \Delta$.

Define a leaf $\ell \in L^2(\Gamma)$ to be regular if $\ell$ is not carried by the stabilizer of a point in $\Gamma$ (this makes sense since point stabilizers of $\Gamma$ are finitely generated) and if moreover there exists a sequence $\ell_n \in L^2(\Gamma)$ of leaves converging to $\ell$ such that the $x_n = Q^2(\ell_n)$ are distinct. The set of regular leaves of $L^2(\Gamma)$ is the regular sublamination $L_r(\Gamma)$. It is $F_n$-invariant but in general not closed.

The space $\mathcal{ML}$ of measured laminations for $F_n$ is a closed subspace of the space of all locally finite $F_n$-invariant Borel measures on $\partial F_n \times \partial F_n - \Delta$, equipped with the weak-"-topology. Dirac measures on pairs of fixed points of all elements in some primitive conjugacy class of $F_n$ are dense in $\mathcal{ML}$ $[\text{Ma95}]$. There is a continuous length pairing $[\text{KL09}]$

$$\langle \cdot, \cdot \rangle : \mathcal{ML} \times \mathcal{ML} \to [0, \infty).$$

For $T \in \mathcal{ML}$ and $\mu \in \mathcal{ML}$ we have $\langle T, \mu \rangle = 0$ if and only if $\mu$ is supported in $L^2(\Gamma)$ $[\text{KL09}]$.

The following is the main result of this section.

**Proposition 7.1.** Let $[T] \in \partial \mathcal{CV}(F_n)$ be indecomposable and let $\mu$ be a measured lamination with $\langle T, \mu \rangle = 0$ and $\mu(L^2(T) - L_r(T)) = 0$. If $[S] \in \partial \mathcal{CV}(F_n)$ is indecomposable and if $\langle S, \mu \rangle = 0$ then $\mathcal{H}([S], [T])$ is homeomorphic.

Denote by $\mathcal{IT} \subset \partial \mathcal{CV}(F_n)$ the Out($F_n$)-invariant subset of indecomposable projective trees. For an indecomposable projective tree $[T] \in \mathcal{IT} \subset \partial \mathcal{CV}(F_n)$ define $\mathcal{H}([T]) \subset \mathcal{IT}$ to be the set of all indecomposable projective trees $[S]$ so that $S$ is $F_n$-equivariantly homeomorphic to $T$.

**Corollary 7.2.** Let $[T] \in \mathcal{IT}$ be such that there is a measured lamination $\mu$ with $\langle T, \mu \rangle = 0$ and $\mu(L^2(T) - L_r(T)) = 0$. Then $\mathcal{H}([T])$ is a closed subset of $\mathcal{IT}$.

**Proof.** If $\hat{S}, \hat{T}$ are $F_n$-equivariantly homeomorphic then for every measured lamination $\mu$ we have $\langle T, \mu \rangle = 0$ if and only if $\langle S, \mu \rangle = 0$ $[\text{CHL07}, \text{KL09}]$. Thus by Proposition 7.1 if $[T] \in \mathcal{IT}$ and if there is a measured lamination $\mu$ with $\mu(L^2(T) - L_r(T)) = 0$ then we have

$$\mathcal{H}([T]) = \{[S] \in \mathcal{IT} \mid \langle S, \mu \rangle = 0\}.$$

The corollary now follows from continuity of the length pairing $\langle \cdot, \cdot \rangle$ $[\text{KL09}]$. □

**Remark 7.3.** We do not know whether every indecomposable tree $[T] \in \mathcal{IT}$ admits a measured lamination $\mu$ with $\langle T, \mu \rangle = 0$ and $\mu(L^2(T) - L_r(T)) = 0$. 


Recall that a minimal subset of an $F_n$-space is an invariant set with each orbit dense. In the case that $T$ is an indecomposable tree with a free action of $F_n$, Proposition 5.14 of [CH14] together with Proposition 3.7 and Lemma 4.10 of [CHR11] show that the regular sublamination $L_r(T)$ is minimal. The main technical tool for the proof of Proposition 7.1 is the following extension of this result to indecomposable trees which are not necessarily free.

**Lemma 7.4.** Let $[T] \in \partial \text{CV}(F_n)$ be indecomposable; then the regular sublamination $L_r(T)$ of $L^2(T)$ is minimal.

**Proof.** Choose a basis $A$ for $F_n$. Elements of $F_n$ are reduced finite words in the alphabet $A^\pm$. A point $X$ in the boundary $\partial F_n$ of $F_n$ is an infinite reduced word in $A^\pm$ whose first letter will be denoted by $X_1$. The cylinder

$$C_A(1) = \{(X,Y) \in (\partial F_n)^2 \mid X_1 \neq Y_1\}$$

is compact, and the same holds true for the relative limit set

$$\Omega_A = Q^2(L^2(T) \cap C_A(1)) \subset T.$$

The compact heart $K_A$ of $T$ is the convex hull of $\Omega_A$. It is a compact subtree of $T$ (see Section 5.3 of [CH14]).

The tree $T$ is called of **Levitt type** if the limit set $\Omega$ is a totally disconnected subset of $T$. That this definition is equivalent to earlier definitions in the literature is Theorem 5.11 of [CH14]. We first show the lemma for indecomposable trees $T$ of Levitt type. For such trees the relative limit set $\Omega_A$ is a Cantor set. For any element $a \in A^\pm$ consider the partial isometry defined by the restriction of the action of $a^{-1}$.

$$K_A \cap aK_A \to K_A \cap a^{-1}K_A, \ x \to a^{-1}x.$$  

The thus defined system of partial isometries determines a directed labeled graph $\Gamma$ whose vertex set is the set of components of $K_A$ (the compact heart is connected, but we will use this construction below also for compact subsets of $K_A$ which may be disconnected) and where an edge labeled with $a \in A$ is a partial isometry connecting a component of $K_A$ which intersects the domain of definition of $a$ to the component containing its image. As $K_A$ is connected, the graph $\Gamma$ is a rose.

Apply the **Rips machine** (see Section 3 of [CH14]) to the system of isometries $(K_A,A^\pm)$. Its first step consists in choosing $K_A(1)$ to be the subset of $K_A$ of all points which are in the domain of at least two distinct partial isometries from $A^\pm$ and restricting the partial isometries from $A^\pm$ to $K_A(1)$. This determines a new directed labeled graph $\Gamma_1$. There is a natural morphism $\tau_1$ from the directed graph $\Gamma_1$ into the directed graph $\Gamma$ (see Section 3 of [CH14] for details).

Since $T$ is of Levitt type, the Rips machine never stops. The output is a sequence $\Gamma_i$ ($i > 0$) of graphs without vertices of valence zero or one (Proposition 5.6 of [CH14]) so that every leaf of $L^2(T)$ can be represented by a path in each of the $\Gamma_i$. For each $i$ there is a homotopy equivalence $\tau_{i+1} : \Gamma_{i+1} \to \Gamma_i$. 

The remainder of this section is devoted to the proof of Proposition 7.1.
Define the size of an element of $F_n$ to be its length with respect to the generating set $A^n$. There are only finitely many $F_n$-orbits of points in $T$ with nontrivial point stabilizers [CH14], and each point stabilizer for $T$ is finitely generated [GL95]. This implies that there are numbers $i_0 > 0$, $p > 0$, $u \geq 0$ with the following properties. For each $i \geq i_0$ the graph $\Gamma_i$ contains precisely $u \geq 0$ labeled loops $\alpha_1^i, \ldots, \alpha_u^i$ which define elements of $F_n$ of size at most $p$. Each of these loops corresponds to a conjugacy class in a point stabilizer of $T$. The point stabilizers of $T$ are generated by elements in the conjugacy classes of these loops, and the homotopy equivalence $\tau_i : \Gamma_{i+1} \rightarrow \Gamma_i$ [CH14] maps the loop $\alpha_j^{i+1}$ to the loop $\alpha_j^i$.

By Theorem 5.3 of [CH14], there are only finitely many $F_n$-orbits of points in $Q^2(L^2(T))$ whose preimage under the map $Q$ have cardinality at least three. These points contain the points with non-trivial stabilizer. Thus the union of all $F_n$-orbits of points in the limit set $\Omega = Q^2(L^2(T))$ whose preimage under $Q$ have cardinality at least three is a countable subset of $\Omega$.

We follow the proof of Lemma 3.6 of [CHR11] and we assume to the contrary that there is a closed proper subset $L_0 \subset L_r(T)$. Since $\Omega_A$ is uncountable and since $L_r(T)$ does not have isolated points, by the previous paragraph there is a leaf $\ell \in L_r(T) - L_0$ such that $X = Q^2(\ell) \in \Omega_A$ and that moreover the preimage of $X$ under $Q$ consists precisely of the two endpoints of $\ell$.

The leaf $\ell$ is not contained in the closure of $L_0$ and therefore there is a compact subarc $\rho = \ell[-k,k]$ of $\ell$ which is not shared by any leaf in $L_0$. We may assume that the size $2k$ of the arc is at least $4p$ and that none of its prefixes coincides with any of the words defining a loop $\alpha_j^i$ or its inverse. Note that this makes sense since the loops $\alpha_j^i$ define only finitely many conjugacy classes in $F_n$.

For sufficiently large $i$, every (non-oriented) loop of size at most $p$ in the graph $\Gamma_i$ is one of the loops $\alpha_j^i$ (see the proof of Lemma 3.6 of [CHR11]). As a consequence, the initial segment of the subarc $\rho$ of $\ell$ is distinct from any of these loops.

Choose now a sequence $\ell_j \subset L_r(T)$ of pairwise distinct leaves which converge to $\ell$. We may assume that $\ell_j[-k,k] = \rho$ for all $j$. Since $\Omega$ is a Cantor set, for every $u$ and for sufficiently large $m$ the leaves $\ell_j$ ($0 \leq j \leq u$) are contained in distinct components of $K_A(m)$. On the other hand, there exists a number $i(k) > i_0$ such that for $i > i(k)$ the size of any loop in $\Gamma_i$ which is distinct from one of the loops $\alpha_j^i$ is at least $2k$. This implies that there are $n > 0$, $m > 0$ such that $\ell_n[-k,k]$ does not cross through any vertex of $\Gamma_m$ of valence strictly bigger than two (compare the proof of Lemma 3.6 of [CHR11] for details).

As in the proof of Lemma 3.6 of [CHR11] we conclude that there is an edge of $\Gamma_m$ which contains a subsegment of the arc $\ell_n[-k,k]$ and which is missed by any leaf of $L_0$. Hence every leaf of $L_0$ is contained in some free factor of $F_n$. Since $T$ is indecomposable, by Theorem 4.5 of [R11], for every free factor $H$ of $F_n$, the minimal $H$-invariant subtree of $T$ is discrete and therefore a leaf of $L^2(T)$ contained in $H$ is contained in a point stabilizer of $T$. Then $L_0 \not\subset L_r(T)$ which is a contradiction. This shows the lemma in the case that the tree $T$ is of Levitt type.
By definition, a tree $T$ is of surface type if the Rips machine stops. By Proposition 5.14 of [CH14], an indecomposable tree $T$ either is of Levitt type or of surface type. Thus we are left with showing the lemma for indecomposable trees of surface type.

Let $(\mathcal{K}, \mathcal{A})$ be a system of isometries which is not modified further. The set $\mathcal{K}$ is a compact forest, in particular it has finitely many connected components. For each $a \in \mathcal{A}$ let $\mathcal{K}(a) \subset \mathcal{K}$ be the union of those components which intersect the domain of definition for $a$. We claim that there is some $a \in \mathcal{A}$ and some component $K \in \mathcal{K}(a)$ such that the domain of definition for $a$ intersects $K$ in a proper subtree.

Namely, otherwise each $a \in \mathcal{A}$ maps each component $K$ of $\mathcal{K}(a)$ isometrically onto a component $a(K)$ of $\mathcal{K}$. Since $\mathcal{K}$ has only finitely many components, this implies that there is a component $K$ of $\mathcal{K}$, and there is a finite cyclically reduced nontrivial word $w = a_1 \ldots a_n$ in $\mathcal{A}^\pm$ such that $w(K) = K$. Then $w$ is an isometry of the compact tree $K$. Since $K$ is compact, $w$ has a fixed point $p \in K$ and permutes the components of $K - \{p\}$. Using again compactness of $K$ we conclude that there is some $k \geq 1$ such that $w^k$ stabilizes a non-degenerate segment in $K$. Since $K \subset \overline{T}$ the tree $\overline{T}$ has a segment with non-trivial stabilizer. This contradicts the assumption that $T$ is indecomposable.

As a consequence, there is some $a \in \mathcal{A}^\pm$, and there is a component $K \in \mathcal{K}(a)$ containing an extreme point $x$ of the domain of definition of $a$. By definition, such an extreme point $x$ is contained in the domain of definition of $a$. Moreover, this domain of definition contains a segment abutting at $x$, the complement $K - x$ of $x$ has at least two connected components, and there is at least one segment abutting at $x$ whose interior is not contained in the domain of $a$.

Following the proof of Lemma 4.10 of [CHR11], we can now split the suspension surface by cutting $K_x$ at $x$ and cutting the suspension intervals containing $x$ accordingly. We obtain a new system of isometries. Repeat this construction. Since by the above discussion the splitting process does not terminate, we obtain the conclusion of the lemma as in the proof of Lemma 4.10 of [CHR11].

Let again $|T| \in \mathcal{IT} \subset \partial\text{CV}(\mathcal{F}_n)$ be indecomposable and let $\mu$ be an ergodic measured lamination with support $\text{Supp}(\mu) \subset L^2(T)$ and $\mu(L^2(T) - L_\tau(T)) = 0$. We call such a measured lamination regular. By [KL09], a measured lamination $\mu$ is regular if and only if $(T, \mu) = 0$ and $\mu(L^2(T) - L_\tau(T)) = 0$.

Define an equivalence relation $\sim_\mu$ on $\partial\mathcal{F}_n$ as the smallest equivalence relation with the following property. The equivalence class of a point $\xi$ contains all points $\xi'$ with $(\xi, \xi') \in \text{Supp}(\mu)$. Let $\sim$ be the closure of $\sim_\mu$. By invariance of $\text{Supp}(\mu)$ under the action of $\mathcal{F}_n$, the equivalence relation $\sim$ is $\mathcal{F}_n$-invariant, and its quotient $\partial\mathcal{F}_n / \sim$ is a compact $\mathcal{F}_n$-space. Note that it is unclear whether $\partial\mathcal{F}_n / \sim$ is a tree. The action of $\mathcal{F}_n$ on $\partial\mathcal{F}_n / \sim$ is minimal and dense since this holds true for the action of $\mathcal{F}_n$ on $\partial\mathcal{F}_n$.

By Corollary 2.6 of [CHL07], the tree $\hat{T}$ with the observer’s topology is $\mathcal{F}_n$-equivariantly homeomorphic to $\partial\mathcal{F}_n / L^2(T)$. Now the zero lamination $L^2(T)$ contains the support $\text{Supp}(\mu)$ of $\mu$ and hence the compact $\mathcal{F}_n$-space $\partial\mathcal{F}_n / \sim$ admits an $\mathcal{F}_n$-equivariant continuous surjection onto the tree $\hat{T}$. 

The following observation is also used in Section 11 and completes the proof of Proposition 7.1 under the assumption that $\partial F_n/\sim$ is a tree. In its statement, we do not assume that the tree $S$ is very small.

**Lemma 7.5.** Let $S$ be a minimal dense $F_n$-tree which admits an alignment preserving map onto indecomposable trees $T, T'$. Then $\hat{T}$ is $F_n$-equivariantly homeomorphic to $\hat{T}'$.

**Proof.** Let $S$ be a minimal dense $F_n$-tree, let $T \in \partial cv(F_n)$ be indecomposable and assume that there is an alignment preserving map $f: S \to T$.

A *transverse family* for an $F_n$-tree $U$ with dense orbits is an $F_n$-invariant family $\{Y_v\}$ of non-degenerate subtrees $Y_v \subset U$ with the property that if $Y_v \neq Y_{v'}$ then $Y_v \cap Y_{v'}$ contains at most one point.

Since $S$ is dense we can apply Theorem 12.12 of [R10]. It shows that there is an ergodic length measure $\nu$ for $S$ so that with the notations from Section 5 we have $T = S_\nu$. Moreover, there is a transverse family $\mathcal{F}$ for $S$ with the following properties.

Each component of $\mathcal{F}$ is a subtree $Y$ of $S$. Its stabilizer is a vertex group of a very small splitting of $F_n$, and it acts on $Y$ with dense orbits (see the explicit construction in Section 10 of [R10] for this fact which is attributed to Guirardel and Levitt). The tree $T$ is obtained by collapsing each component of the transverse family $\mathcal{F}$ to a point. In particular, the family $\mathcal{F}$ is trivial (i.e. it is empty or its components are points) if and only if $\hat{S}$ is equivariantly homeomorphic to $\hat{T}$.

Let now $T'$ be a second indecomposable tree for which there is an alignment preserving map $f': S \to T'$. Then the tree $T'$ is obtained from $S$ by collapsing each tree from a second transverse family $\mathcal{F}'$ to a point. If $\hat{T}$ is not $F_n$-equivariantly homeomorphic to $\hat{T}'$ then the transverse families $\mathcal{F}, \mathcal{F}'$ do not coincide. In particular, up to exchanging $T$ and $T'$ we may assume that there is a component $Y'$ of $\mathcal{F}'$ which is not mapped to a point by the alignment preserving map $f: S \to T$.

As the stabilizer $H$ of $Y'$ is a vertex group of a very small splitting of $F_n$, it is a finitely generated subgroup of $F_n$ [GL95] of infinite index, and it acts on $Y'$ with dense orbits. By equivariance and the assumption that $f$ is alignment preserving, the image under $f$ of the minimal $H$-invariant subtree $Y''$ of $S$ equals the minimal $H$-invariant subtree $Z$ of $T$. Since $Y'$ is not mapped to a point, the tree $Z$ is non-trivial, and by equivariance, $H$ acts on $Z$ with dense orbits. However, by the main result of [R11], since $T$ is indecomposable and $H$ is a finitely generated subgroup of $F_n$ of infinite index, the minimal $H$-invariant subtree of $T$ is simplicial. This is a contradiction which shows the lemma.  

**Proof of Proposition 7.1.** Let $T \in \partial cv(F_n)$ be indecomposable and let $\mu \in ML$ be a regular measured lamination for $T$, i.e. such that $\langle T, \mu \rangle = 0$ and $\mu(L^2(T) - L_r(T)) = 0$. 

Recall that the zero lamination $L_2(T)$ for $T$ defines a closed equivalence relation on $\partial F_n$, and $\hat{T} = \partial F_n/L_2(T)$ (Corollary 2.6 of [CHL07]). Let $\sim$ be the closure of the equivalence relation on $\partial F_n$ defined by $\text{Supp}(\mu) \subset L_2(T)$. There is a natural $F_n$-equivariant continuous surjection $G : \partial F_n/\sim \to \hat{T}$.

A leaf $\ell$ of $L_2(T)$ is diagonal over a sublamination $L_0$ if $\ell = (x_1, x_n)$ and if there are points $x_2, \ldots, x_{n-1} \in \partial F_n$ so that $(x_1, x_2), \ldots, (x_{n-1}, x_n) \in L_0$. A leaf of $L_2(T)$ which is diagonal over $\text{Supp}(\mu)$ is contained in the equivalence relation $\sim$. If the tree $T$ is free then $L_2(T) - \text{Supp}(\mu)$ consists of finitely many $F_n$-orbits of diagonal leaves [CHR11] and therefore $G$ is an equivariant homeomorphism. This implies the proposition in the case that the tree $T$ is free.

Now let $T$ be an arbitrary indecomposable tree, let $\mu$ be a regular measured lamination for $T$ and let $S \in \partial \text{cv}(F_n)$ be another indecomposable tree with $\langle S, \mu \rangle = 0$. We claim that a subgroup $H$ of $F_n$ stabilizes a point in $T$ if and only if $H$ stabilizes a point in $S$.

To this end recall that the stabilizer $H$ of a point $x \in T$ is finitely generated [GL95] and therefore the Gromov boundary $\partial H$ of $H$ is a closed $H$-invariant subset of $\partial F_n$. By equivariance and continuity, the preimage in $\partial F_n/\sim$ of the fixed point $x$ of $H$ under the surjection $G$ is a compact $H$-invariant subset $A$ of $\partial F_n/\sim$. This set is just the projection of $\partial H \subset \partial F_n$ to $\partial F_n/\sim$. The union $\bigcup_{gH \in F_n/H} gA$ is disjoint and defines a transverse family $\mathcal{F}$ of compact spaces. Each component of $\mathcal{F}$ is invariant under a conjugate of $H$ (these spaces may be reduced to points or may not be trees).

There also is an equivariant projection $Q : \partial F_n/\sim \to \hat{S}$. If $H$ does not fix a point in $S$ then the transverse family $\mathcal{F}$ of $\partial F_n/\sim$ is not collapsed in $\hat{S}$ to a countable set of points. However, as in the proof of Lemma 7.5, since the interior of $\hat{S}$ is $F_n$-equivariantly homeomorphic to the tree $S$ [CHL07], this implies that the minimal $H$-invariant subtree of $S$ is not reduced to a point, and $H$ acts on it with dense orbits. As in the proof of Lemma 7.5 we conclude that this is impossible.

By the definition of the regular lamination $L_r(T)$, the zero lamination $L^2(T)$ is the union of $L_r(T)$, the sets $\partial H \times \partial H - \Delta$ where $H$ runs through the point stabilizers of $T$ and some isolated diagonal leaves.

As diagonal leaves are contained in the closure of the equivalence relation defined by $\text{Supp}(\mu)$ and the point stabilizers of $T$, the zero lamination $L^2(T)$ is determined uniquely by $\text{Supp}(\mu)$ and the point stabilizers of $T$. The above discussion then shows that indeed $L^2(T) = L^2(S)$ as claimed. This completes the proof of Proposition 7.1. □
8. Trees with point stabilizers containing a free factor

The goal of this section is to analyze trees in $\partial \text{CV}(F_n)$ with dense orbits which have some point stabilizers containing a free factor.

Let $A < F_n$ be a free factor of rank $r \geq 1$. A free basis extension of $A$ is a free basis $e_1, \ldots, e_{n-r}, e_{n-r+1}, \ldots, e_n$ of $F_n$ such that $e_{n-r+1}, \ldots, e_n$ is a free basis of $A$. Define the standard simplex relative to $A$ of a free basis extension of $A$ to be the set

$$\Delta(A) \subset \text{cv}_0(F_n)^{++}$$

of all trees $T$ with volume one quotient with the following property. Either $T$ is contained in the boundary of the standard simplex $\Delta$ for the basis and admits $A$ as a point stabilizer, or $T/F_n$ can be obtained as follows.

Let $R_0, R_{n-r+1}, \ldots, R_n$ be roses of rank $n - r$ and 1, respectively, with petals of length $1/(n+r)$ which are marked by the subgroup of $F_n$ generated by $e_1, \ldots, e_{n-r}$ and by $e_j$ ($n - r \leq j \leq n$). Connect the roses $R_j$ in an arbitrary way by $r$ edges of length $1/(n+r)$ so that these edges form a forest in the resulting connected graph $G_1$. For $j \geq n - r + 1$ collapse each rose $R_j$ in $G_1$ to a point and let $G_2$ be the resulting graph. The graph $T/F_n$ is obtained from $G_2$ by changing the lengths of the remaining edges, allowing some of the edges to shrink to zero length.

If $A < F_n$ is a free factor which fixes a point in a tree $[T] \in \partial \text{CV}(F_n)$ then we say that the action of $A$ on $[T]$ is elliptic.

The next observation is a relative version of Lemma 3.2.

Lemma 8.1. Let $A < F_n$ be a free factor and let $[T] \in \partial \text{CV}(F_n)$ be such that the action of $A$ on $[T]$ is elliptic. Then for every standard simplex $\Delta(A)$ relative to $A$ there is a tree $U \in \Delta(A)$ and a train track map $f : U \to T$ where $T$ is some representative of $[T]$.

Proof. Let $A < F_n$ be a free factor of rank $r \leq n - 1$ and let $[T] \in \partial \text{CV}(F_n)$ be a tree containing a point $x$ which is stabilized by $A$. Let $e_1, \ldots, e_n$ be a free basis extension of $A$ and let $\Delta \subset \text{cv}_0(F_n)^{++}$ be the standard simplex for this basis.

By Lemma 3.2 there is some $S \in \Delta$, and there is a train track map $f : S \to T$ where $T$ is a representative of $[T]$. The map $f$ is determined by the image of a point $\tilde{v} \in S$ in the preimage of the unique vertex $v$ of $S/F_n$.

There are now two possibilities. In the first case, $f(\tilde{v})$ is stabilized by a conjugate of $A$. Now up to conjugation, for $i \geq n - r + 1$ a train track map $f : S \to T$ maps the edge of $S$ with endpoints $\tilde{v}, e_i \tilde{v}$ isometrically onto the segment in $T$ connecting $f(\tilde{v})$ to $e_i f(\tilde{v})$. Since $f(\tilde{v}) = e_i f(\tilde{v})$ and since $f$ is an edge isometry, the tree $S$ is contained in $\text{cv}_0(F_n) - \text{cv}_0(F_n)$, and the vertex $\tilde{v}$ is fixed by a conjugate of $A$. As a consequence, we have $S \in \Delta(A)$ and we are done.

If $f(\tilde{v})$ is not stabilized by a conjugate of $A$ then by equivariance, the vertex $\tilde{v}$ of $S$ is not stabilized by a conjugate of $A$. Thus there is a petal of $S/F_n$ which represents an element $a$ of $A$. This petal lifts to a line in $S$ through $\tilde{v}$ which is stabilized by
a conjugate of $a$ (but not pointwise). Let us suppose that this conjugate is just $a$. The edge in $S$ with endpoints $\hat{v}$ and $a\hat{v}$ is mapped by $f$ isometrically to a segment in $T$.

By our assumption on $T$, there is a fixed point for $a$ on $T$. Since $T$ is very small, the fixed point set $\text{Fix}(a)$ of $a$ is a point or a closed segment in $T$. Let $\hat{x}$ be the point of smallest distance to $f(\hat{v})$ in $\text{Fix}(a)$.

Let $s$ be the unique segment in $T$ connecting $f(\hat{v})$ to $\hat{x}$. The segment meets $\text{Fix}(a)$ only in $\hat{x}$. Then $a(s)$ is the segment in $T$ connecting $af(\hat{v})$ to $a(\hat{x})$, and the segment $s \cap a(s)$ is fixed by $a$. Since $s$ meets the fixed point set of $a$ only in $\hat{x}$, we conclude that $s \cap a(s) = \hat{x}$. Thus the geodesic segment in $T$ connecting $f(\hat{v})$ to $af(\hat{v})$ passes through $\hat{x}$, and the geodesic segment connecting $f(\hat{v})$ to $a^{-1}f(\hat{v})$ passes through $\hat{x}$ as well.

As a consequence, for the train track structure on $S$ defined by $f$, the turn at $\hat{v}$ containing the two directions of the axis of $a$ is illegal. Folding this illegal turn collapses the petal in $S/F_n$ defining $a$ to a single segment without changing the rest of $S/F_n$. After volume renormalization, the resulting tree $S_1$ is contained in $\overline{cv_0(F_n)}$. Its quotient $S_1/F_n$ contains a segment which is a collapse of the petal of $S/F_n$ defining $a$. There is a train track map $f_1 : S_1 \to T_1$ where $T_1$ is a rescaling of $T$.

The graph $S_1/F_n$ has precisely two vertices, and one of these vertices is univalent. There are at most $r - 1$ loops in $S_1/F_n$ defining elements of $A$. Repeat this construction with a petal of $S_1/F_n$ defining an element of the free factor $A$. After a total of $r$ such steps we obtain a simplicial tree $S_r \in \overline{cv_0(F_n)}$ and a train track map $f_r : S_r \to T_r$ where $T_r$ is a rescaling of $T$. The graph $S_r/F_n$ consists of a rose with $n - r$ petals (some of them may be degenerate to points) with a collection of edges attached. By construction, we have $S_r \in \Delta(A)$ as claimed. \hfill $\square$

Recall from Section 2 the definition of the map $\Upsilon_F : \overline{cv_0(F_n)}^+ \to \mathcal{FF}$. Recall also from Section 4 the definition of the map $\psi_F : \partial CV(F_n) \to \partial \mathcal{FF} \cup \Theta$. We use Lemma 8.1 to show

**Corollary 8.2.** Let $A < F_n$ be a free factor of rank $\ell \geq 1$ and let $[T] \in \partial CV(F_n)$ be a tree such that the action of $A$ on $[T]$ is elliptic. Then $\psi_F([T]) = \Theta$.

**Proof.** Let $\Delta(A)$ be a standard simplex relative to $A$. By Lemma 8.1 there is a tree $S \in \Delta(A)$ and a train track map $f : S \to T$ where $T$ is a representative of $[T]$.

Any point on a Skora path $(x_t)$ guided by $f$ contains a point stabilized by a fixed free factor contained in $A$. By Lemma 2.13 this implies that for every $t > 0$ the free factor $\Upsilon_F(x_t)$ is contained in a uniformly bounded neighborhood of $A$. In particular, the diameter of $\Upsilon_F(x_t)$ is finite. The claim of the lemma now follows from Proposition 6.1. \hfill $\square$
9. Trees which split as graphs of actions

In this section we investigate the structure of trees in $\partial CV(F_n)$ with dense orbits which resemble trees $T$ with $T_d \neq \emptyset$. The description of such trees is as follows [G08, L94].

**Definition 9.1.** A graph of actions is a minimal $F_n$-tree which consists of

(1) a simplicial tree $S$, called the skeleton, equipped with an action of $F_n$
(2) for each vertex $v$ of $S$ an $R$-tree $Y_v$, called a vertex tree, and
(3) for each oriented edge $e$ of $S$ with terminal vertex $v$ a point $p_e \in Y_v$, called an attaching point.

It is required that the projection $Y_v \rightarrow p_e$ is equivariant, that for $g \in F_n$ one has $gp_e = p_ge$ and that each vertex tree is a minimal very small tree for its stabilizer.

Associated to a graph of actions $G$ is a canonical action of $F_n$ on an $R$-tree $T_G$ which is called the dual of the graph of actions [L94]. Define a pseudo-metric $d$ on $\bigsqcup_{v \in V(S)} Y_v$ as follows. If $x \in Y_{v_0}, y \in Y_{v_k}$ let $e_1 \ldots e_k$ be the reduced edge-path from $v_0$ to $v_k$ in $S$ and define

$$d(x, y) = d_{Y_{e_1}}(x, p_{e_1}) + \cdots + d_{Y_{e_k}}(p_{e_k}, y).$$

Making this pseudo-metric Hausdorff gives an $R$-tree $T_G$. Informally, the tree $T_G$ is obtained by first inserting the vertex trees into the skeleton $S$ and then collapsing each edge of the skeleton to a point.

We say that an $F_n$-tree $T$ splits as a graph of actions if either $T_d \neq \emptyset$ or if there is a graph of actions $G$ with dual tree $T_G$, and there is an equivariant isometry $T \rightarrow T_G$. We also say that the projectivization $[T]$ of an $F_n$-tree $T$ splits as a graph of actions if $T$ splits as a graph of actions.

A transverse family for an $F_n$-tree $T$ with dense orbits is an $F_n$-invariant family $\{Y_v\}$ of non-degenerate subtrees $Y_v \subseteq T$ with the property that if $Y_v \neq Y_{v'}$ then $Y_v \cap Y_{v'}$ contains at most one point. The transverse family is a transverse covering if any finite segment $I \subseteq T$ is contained in a finite union $Y_{v_1} \cup \cdots \cup Y_{v_r}$ of components from the family. The vertex trees of a graph of actions with dual tree $T_G$ define a transverse covering of $T_G$. More precisely, by Lemma 4.7 of [G04], a minimal very small $F_n$-tree $T$ admits a transverse covering if and only if $T$ splits as a graph of actions.

A dense tree $T$ splits as a large graph of actions if it is equivariantly isometric to the dual tree of a graph of actions $G$ with the following additional properties.

(1) There is a single $F_n$-orbit of vertex trees.
(2) A stabilizer of a vertex tree is not contained in a proper free factor of $F_n$.
(3) A vertex tree is indecomposable for its stabilizer.
(4) The skeleton of $G$ does not have an edge with trivial stabilizer.
Note that there is some redundancy in the above definition.

The tree \( T \) splits as a \textit{very large graph of actions} if it splits as a large graph of actions and if moreover the skeleton does not have an edge with infinite cyclic stabilizer.

The following result is Corollary 11.2 of [R10].

**Proposition 9.2.** Let \( T \in \text{cv}(F_n) \) have dense orbits, and assume that \( T \) is neither indecomposable nor splits as a graph of actions. Then there is an alignment preserving map \( f : T \to T' \) such that either \( T' \) is indecomposable or \( T' \) splits as a graph of actions.

We use Proposition 9.2 to show

**Proposition 9.3.**

1. Let \([T] \in \partial\text{CV}(F_n)\) be such that \( \psi([T]) \in \partial\text{FS} \). Then \( T \) admits an alignment preserving map onto a tree which either is indecomposable or splits as a large graph of actions.
2. Let \([T] \in \partial\text{CV}(F_n)\) be such that \( \psi_{\text{C}}([T]) \in \partial\text{CS} \). Then \( T \) admits an alignment preserving map onto a tree which either is indecomposable or splits as a very large graph of actions.

**Proof.** Let \([T] \in \partial\text{CV}(F_n)\) be such that \( \psi([T]) \in \partial\text{FS} \) (or \( \psi_{\text{C}}([T]) \in \partial\text{CS} \)). By Corollary 5.5 \([T]\) is not simplicial. A tree which is neither dense nor simplicial admits a one-Lipschitz alignment preserving map onto a dense tree. By Proposition 4.2 we may therefore assume without loss of generality that \( T \) is dense. By Proposition 9.2 and Corollary 6.13, it now suffices to show the following. If \( T \) splits as a graph of actions, then \( T \) admits an alignment preserving map onto a tree which splits as a large (or very large) graph of actions.

Thus assume that \( T \) is the dual tree of a graph of actions, with skeleton \( S \). Let \( U \) be the tree obtained from \( S \) by insertion of the vertex trees. The tree \( U \) may not be very small, but it admits an alignment preserving map onto \( T \) obtained by equivariantly collapsing each edge of \( U \) to a point.

If there is an edge in \( S \) with trivial (or either trivial or infinite cyclic) stabilizer then there is an edge \( e \) in \( U \) with trivial (or infinite cyclic) stabilizer. The tree \( V \) obtained from \( U \) by equivariantly collapsing those edges of \( U \) to points which are not contained in the orbit of \( e \) is very small, and \( V_d \neq \emptyset \). There is an edge in \( V \) with trivial (or either trivial or infinite cyclic) stabilizer. Moreover, \( V \) admits an alignment preserving map onto \( T \). We then have \( \psi([T]) = \Theta \) (or \( \psi_{\text{C}}([T]) = \Theta \)) by Corollary 6.13 and Proposition 5.1.

If there is no edge \( e \) in the skeleton \( S \) with trivial (or either trivial or infinite cyclic) stabilizer then choose an arbitrary edge \( e \) in \( U \) and equivariantly collapse all edges of \( U \) to a point which are not contained in the \( F_n \)-orbit of \( e \). This defines the structure of a graph of actions for \( T \) whose skeleton projects to a one-edge graph of groups decomposition with edge group which is not trivial (or neither trivial nor infinite cyclic).
If there is more than one $F_n$-orbit of vertex trees for this graph of actions then we can collapse the vertex trees in all but one of these orbits to points. We obtain a tree $W$ which is dual to a graph of actions with a single orbit of vertex trees. The edge stabilizer of each edge of the skeleton is not trivial (or neither trivial nor infinite cyclic).

Let $T_v$ be one of the vertex trees of the $F_n$-tree $W$. Its stabilizer is a finitely generated subgroup $H$ of $F_n$ (and hence a finitely generated free group) which acts on $T_v$ with dense orbits. By Proposition 9.2, the $H$-tree $T_v$ either admits an alignment preserving map onto an indecomposable tree or onto a tree which splits as a graph of actions.

Assume first that $T_v$ admits an alignment preserving map onto an indecomposable $H$-tree. Then $T$ admits an alignment preserving map onto a tree $Z$ which splits as a graph of actions, with a single orbit of indecomposable vertex trees. Moreover, $Z$ is dual to a graph of actions with a single orbit of edges whose stabilizers are not trivial (or neither trivial not infinitely cyclic). In particular, $Z$ splits as a large (or as a very large) graph of actions provided that the group $H$ is not contained in a proper free factor of $F_n$.

To see that the latter property indeed holds true consider the one-edge splitting of $F_n$ defined by the $F_n$-quotient of the skeleton of $Z$. Assume first that this splitting is a one-edge two-vertex splitting. Then this splitting is of the form $H \ast C B$ where $C$ is a nontrivial free subgroup of $H$ (or a free subgroup of $H$ of rank at least two).

Let $A < F_n$ be the smallest free factor containing $H$. Assume to the contrary that $A$ is a proper subgroup of $F_n$. Then there is a free splitting $F_n = A \ast B_1$ where $B_1 < B$, and there is a refinement $(H \ast C B_0) \ast B_1$ of $H \ast C B$. Since there is a unique orbit of indecomposable vertex trees for $Z$ whose stabilizers are conjugate to $H$, the group $B$ fixes a vertex in $Z$, and the tree $Z$ can be represented as a graph of actions with an edge with trivial edge group. Proposition 5.1 now shows that $\psi([T]) = \Theta$ which contradicts the assumption on $T$. Thus we have $A = F_n$ and $Z$ splits as a large (or as a very large) graph of actions.

If the splitting of $F_n$ defined by an edge in the skeleton of $Z$ is a one-edge one-vertex splitting and if the vertex group $H$ is contained in a proper free factor $A$ of $F_n$ then the vertex group is a free factor of rank $n - 1$ (since $F_n$ is generated by $A$ and a single primitive element) and the splitting is a one-loop free splitting of $F_n$ which violates once more the assumption that $\psi([T]) \in \partial FS$ (or $\psi_c([T]) \in \partial CS$). This completes the proof of the proposition in the case that $T_v$ admits an alignment preserving map onto an indecomposable tree.

We are left with the case that a vertex tree $T_v$ of the graph of actions with dual tree $W$ admits an alignment preserving map onto a tree $Y_v$ which splits as a graph of actions for the stabilizer of $T_v$. Then the tree $W$ admits an alignment preserving map onto the tree $Y$ which is obtained by collapsing each tree in the orbit of $T_v$ to a tree in the orbit of $Y_v$.

Iterating the above argument, in the case that $Y_v$ has more than one orbit of vertex trees then all but one of these orbits can be collapsed to points. Extending
the collapsing map equivariantly yields an alignment preserving map of $Y$ onto a tree $Y'$ which splits as a graph of actions, with a single orbit of vertex trees. The sums of the ranks of conjugacy classes of point stabilizers in $Y'$ is strictly bigger than those of $Y$.

Since the sum of the ranks of conjugacy classes of point stabilizers in a very small $F_n$-tree is uniformly bounded [GL95], we find in finitely many such steps an $F_n$-tree $Q$ which splits as a graph of actions with a single $F_n$-orbit of vertex trees, and each such vertex tree is indecomposable for its stabilizer. Moreover, there is an alignment preserving map $T \to Q$. The skeleton for the structure of a graph of actions for $Q$ is an $F_n$-tree with finite quotient. Stabilizers of edges are not trivial (or neither trivial nor cyclic). In other words, $Q$ splits as a large (or as a very large) graph of actions. This completes then proof of the proposition.

The following example was shown to me by Camille Horbez.

**Example:** Let $T$ be an indecomposable $F_n$-tree with a point stabilizer which contains a free factor $H$ of $F_n$ of rank $k \geq 3$. Let $B$ be a free group of rank $\ell < k$ and choose a finite index subgroup $H'$ of $B$ which is a free group of rank $k$. Identifying $H$ and $H'$ defines a one-edge two vertex graph of groups decomposition $F_n *_H B$ for a free group $F_m$ where $m = n + \ell - k < n$. Let the group $B$ act trivially on a point and define a graph of actions for $F_m$ with a single orbit of edges in its skeleton containing $T$ as a vertex tree. This graph of actions is very large.

We can also iterate this construction by beginning with an indecomposable $F_n$-tree $T$ with more than one $F_n$-orbit of points with point stabilizer containing a free factor of rank $k \geq 3$ and constructing from $T$ a very large graph of actions defined by a tree whose skeleton has more than one connected component.

The final goal of this section is to show

**Proposition 9.4.** If $[T] \in \partial CV(F_n)$ splits as a graph of actions then $\psi_F([T]) = \Theta$.

**Proof.** By Corollary 6.10, Corollary 6.13 and Proposition 9.3, it suffices to show that $\psi_F([T]) = \Theta$ for every projective tree $[T]$ which splits as a very large graph of actions.

By Corollary 8.2, this indeed holds true if a tree which splits as a very large graph of actions admits a point stabilizer which contains a free factor of $F_n$. That this is indeed the case is due to Reynolds (Proposition 10.3 of [R12]).

Proposition 6.11, Corollary 9.4, Proposition 9.2 and Corollary 6.13 immediately imply

**Corollary 9.5.** Let $[T] \in \partial CV(F_n)$ be such that $\psi_F([T]) \in \partial FF$. Then $T$ admits an alignment preserving map onto an indecomposable tree.
10. The boundary of the free factor graph

In this section we complete the proof of Theorem 3 from the introduction.

An indecomposable tree $T$ is called arational if no point stabilizer for $T$ contains a free factor. We begin with collecting some additional information on arational trees. Part of what we need is covered by the main result of [R12] which shows that an arational tree $T$ either is free or dual to a measured lamination of an oriented surface with a single boundary component. We will not use this information.

A closed $F_n$-invariant subset $C$ of $\partial F_n \times \partial F_n - \Delta$ intersects a free factor if there is a proper free factor $H$ of $F_n$ so that $C \cap \partial H \times \partial H - \Delta \neq \emptyset$.

The following lemma is a simple consequence of Theorem 4.5 of [R11].

**Lemma 10.1.** Let $T \in \partial \text{cv}(F_n)$ be indecomposable. If the zero lamination $L^2(T)$ of $T$ intersects a proper free factor of $F_n$ then there is a point stabilizer for the action of $F_n$ on $T$ which contains a proper free factor.

**Proof.** The intersection of $L^2(T)$ with a proper free factor $H < F_n$ is contained in the zero lamination of the minimal $H$-invariant subtree $T_H$ of $T$. By Theorem 4.5 of [R11], since $T$ is indecomposable the action on $T$ of any proper free factor $H$ is discrete and hence $T_H$ is simplicial. Thus the intersection of $L^2(T)$ with $H$ is non-empty if and only if there is a leaf of $L^2(T)$ which is carried by a point stabilizer for the action of $H$ on $T_H$.

Since $T$ is indecomposable, the stabilizer of any non-degenerate segment of $T$ is trivial. As a consequence, the tree $T_H$ is very small simplicial, with trivial edge stabilizers. Therefore $T_H/H$ defines a graph of groups decomposition of $H$ with trivial edge groups, i.e. it defines a free splitting of $H$. Each vertex group is a free factor of $H$. Hence if the action of $H$ on $T_H$ is not free, then there is a point stabilizer of $T_H$ which is a proper free factor of $H$ and hence of $F_n$. The lemma follows. \qed

We use Lemma 10.1 to show that only arational trees can give rise to points in the boundary of the free factor graph $\mathcal{FF}$.

Say that a measured lamination $\mu$ is supported in a finitely generated subgroup $H$ of $F_n$ if $\text{Supp}(\mu) = F_n(\text{Supp}(\mu) \cap \partial H \times \partial H - \Delta)$. Our next goal is to understand measured laminations whose supports are contained in the point stabilizer of an indecomposable tree. For this we use the following consequence of Theorem 49 of [Ma95].

**Lemma 10.2.** Let $H < F_n$ be a finitely generated subgroup of infinite index which does not intersect a free factor. Then $\partial H \times \partial H - \Delta$ does not support a measured lamination for $F_n$. 

Proof. Assume to the contrary that there is a measured lamination $\nu$ for $F_n$ supported in $\partial H \times \partial H - \Delta$. Let $\ell \in \partial H \times \partial H - \Delta$ be a density point for $\nu$. Then there is no proper free factor $A$ of $F_n$ so that $\ell \in \partial A \times \partial A - \Delta$.

Choose a free basis $A$ for $F_n$ and represent $\ell$ by a biinfinite word $(w_i)$ in that basis. Since $H < F_n$ is finitely generated of infinite index, $H$ is quasiconvex and hence we may assume that there is a sequence $n_i \to \infty$ such that each of the prefixes $(w_{n_i})$ of the word $(w_i)$ represents an element of $H$.

By Theorem 49 of [Ma95] (which is attributed to Bestvina), for each $i$ there is a free basis $B_i$ for $F_n$ such that the Whitehead graph of $w_{n_i}$ with respect to $B_i$ is connected and does not have a cut vertex. Since $\ell$ is a density point for $\nu$, the Whitehead graph of $\nu$ for the basis $B_i$ contains the Whitehead graph of $w_{n_i}$, in particular it does not have a cut vertex and is connected. On the other hand, since $\nu$ is a measured lamination, Proposition 21 of [Ma95] shows that the Whitehead graph of $\nu$ with respect to $B_i$ has a cut vertex or is disconnected. This is a contradiction and shows the lemma. □

In Section 7 we defined a measured lamination $\mu \in \mathcal{ML}$ to be regular for an indecomposable tree $T$ if $\langle T, \mu \rangle = 0$ and $\mu(L^2(T) - L_r(T)) = 0$. By Lemma 10.1 and Lemma 10.2, if $T$ is arational then every measured lamination $\mu$ with $\langle T, \mu \rangle = 0$ is regular for $T$.

Example: Let $n = 2g \geq 4$ and let $S$ be an oriented surface of genus $g$ with non-empty connected boundary. The fundamental group of $S$ is the group $F_n$. Every measured lamination $\mu$ on $S$ is dual to an $F_n$-tree $T$. If the support of $\mu$ is minimal and decomposes $S$ into ideal polygons, then this tree is arational. The free homotopy class of the boundary circle defines the unique conjugacy class in $F_n$ whose elements have fixed points in $T$. The point stabilizers of $T$ do not support a measured lamination.

Denote as before by $\mathcal{IT} \subset \partial \mathit{CV}(F_n)$ the set of indecomposable projective trees. Let $\mathcal{FT} \subset \mathcal{IT} \subset \partial \mathit{CV}(F_n)$ be the set of arational trees. Corollary 9.5 Corollary 8.2 and Lemma 10.1 show that $\psi(\mathcal{FT}) \supset \partial \mathcal{FF}$. Our next task is to show that $\mathcal{FT} \subset \psi^{-1}(\partial \mathcal{FF})$. To this end recall that each conjugacy class of a primitive element $g \in F_n$ determines a measured lamination which is the set of all Dirac masses on the pairs of fixed points of the elements in the class. The measured lamination is called dual to the conjugacy class.

A primitive conjugacy class $\alpha$ in $F_n$ is short for a tree $T \in \mathit{cv}_0(F_n)$ if it can be represented by a loop on $T/F_n$ of length at most 4. Recall that the length pairing $\langle \cdot, \cdot \rangle$ on $\mathit{cv}(F_n) \times \mathcal{ML}$ is continuous.

We have
Lemma 10.3. Let \([T_i] \subset \text{CV}(F_n)\) be a sequence converging to \([T] \in \partial \text{CV}(F_n)\). For each \(i\) let \(T_i \in \text{cv}_0(F_n)\) be a representative of \([T_i]\) and let \(\alpha_i\) be a primitive short conjugacy class on \(T_i\) with dual measured lamination \(\mu_i\). If \([T]\) is dense then up to passing to a subsequence, there is a sequence \(b_i \in (0,1]\) such that the measured laminations \(b_i \mu_i\) converge weakly to a measured lamination \(\mu\) with \(\langle T, \mu \rangle = 0\). Moreover, either up to scaling \(\mu\) is dual to a primitive conjugacy class, or \(b_i \to 0\).

Proof. For each \(i\) let \(T_i \in \text{cv}_0(F_n)\) be a representative of \([T_i]\), let \(T\) be a representative of \([T]\) and let \(a_i \in (0,\infty)\) be such that \(a_i T_i \to T\). Since the \(F_n\)-orbits on \(T\) are dense, we have \(a_i \to 0\) \((i \to \infty)\).

Fix some tree \(S \in \text{cv}_0(F_n)\). Then the set

\[ \Sigma = \{ \zeta \in \mathcal{ML} \mid \langle S, \zeta \rangle = 1 \} \]

defines a continuous section of the projection \(\mathcal{ML} \to \mathcal{PML}\) for the weak*-topology. In particular, the space \(\Sigma\) is compact.

There is a number \(\epsilon > 0\) so that \(\langle S, \zeta \rangle \geq \epsilon\) whenever \(\zeta\) is dual to any primitive conjugacy class. Let \(\mu_i\) be the measured lamination dual to a primitive short conjugacy class \(\alpha_i\) on \(T_i\). If \(b_i > 0\) is such that \(b_i \mu_i \in \Sigma\) then the sequence \((b_i)\) is bounded. Since \(\Sigma\) is compact, by passing to a subsequence we may assume that \(b_i \mu_i \to \mu\) for some measured lamination \(\mu \in \Sigma\).

Now \(\langle a_i T_i, \mu_i \rangle \leq k a_i\) where \(k \geq 2\) is as in the definition of a short conjugacy class (see Section 2) and hence since \(a_i \to 0\) \((i \to \infty)\) and since the sequence \((b_i)\) is bounded, we have

\[ \langle a_i T_i, b_i \mu_i \rangle \to 0 \ (i \to \infty). \]

The first part of the lemma now follows from continuity of the length pairing. Moreover, either \(b_i \to 0\) or the length on \(S/F_n\) of the conjugacy classes \(\alpha_i\) is uniformly bounded. However, there are only finitely many conjugacy classes of primitive elements which can be represented by a loop on \(S/F_n\) of uniformly bounded length. Thus either \(b_i \to 0\), or the sequence \((\alpha_i)\) contains only finitely many elements and hence there is some primitive conjugacy class \(\alpha\) so that \(\alpha_i = \alpha\) for infinitely many \(i\). Then clearly \(\mu\) is a multiple of the dual of \(\alpha\). \(\Box\)

The following proposition is the main remaining step towards the proof of Theorem\[\text{[10.3]}\] For its formulation, define two trees \([S], [T] \in \mathcal{IT}\) to be equivalent if the trees \(\hat{S}, \hat{T}\) are \(F_n\)-equivariantly homeomorphic with respect to the observer’s topology.

Proposition 10.4. (1) Let \([T] \in \mathcal{IT}\); then \(\psi_F([T]) \in \partial \mathcal{FF}\) if and only if \([T] \in \mathcal{FF}\).

(2) If \([T], [T'] \in \mathcal{FF}\) then \(\psi_F([T]) = \psi_F([T'])\) if and only if \([T], [T']\) are equivalent.

Proof. Let \(\Upsilon_F = \Omega \circ \Upsilon : \text{cv}_0(F_n) \to \mathcal{FF}\) be the map constructed in Section 2. By the discussion preceding Lemma\[\text{[10.3]}\] it suffices to show that \(\psi([T]) \neq \Theta\) for every \([T] \in \mathcal{FF}\). Since Skora paths map to uniform unparametrized quasi-geodesics in \(\mathcal{FF}\), this holds true if for every \([T] \in \mathcal{FF}\) the image under \(\Upsilon_F\) of a Skora path converging to \([T]\) is unbounded.
We show more generally the following. If \( [T_i] \subset CV(F_n) \) is any sequence which converges to \( [T] \in \mathcal{FT} \) and if \( T_i \in cv_0(F_n) \) is a representative of \( [T_i] \) then the sequence \( \Upsilon_F(T_i) \subset \mathcal{FF} \) is unbounded. For this we use a variant of an argument of Luo as explained in [MM99].

We argue by contradiction and we assume that after passing to a subsequence, the sequence \( \Upsilon_F(T_i) \) remains in a bounded subset in \( \mathcal{FF} \).

Since by Lemma 2.13, the map \( \Upsilon_F \) is a quasi-isometry for the metric \( d_{ng} = d^1_{ng} \) on \( cv_0(F_n) \) which only assumes integral values, after passing to another subsequence we may assume that for all \( i \geq 1 \) the distance between \( T_i \) and \( T_0 \) in \( (cv_0(F_n), d_{ng}) \) equals \( m \) for some \( m \geq 0 \) which does not depend on \( i \).

By the definition of the metric \( d_{ng} \), this implies that for all \( i \geq 1 \) there is a sequence \( (T_{j,i})_{0 \leq j \leq m} \subset cv_0(F_n) \) with \( T_{0,i} = T_0 \) and \( T_{m,i} = T_i \) so that for all \( j \) the trees \( T_{j,i} \) and \( T_{(j+1),i} \) are one-tied. In particular, for each \( j < m \) there is a primitive conjugacy class \( \alpha_{j,i} \) which can be represented by a curve of length at most \( 4 \) on both \( T_{j,i}/F_n \) and \( T_{(j+1),i}/F_n \). Let \( \mu_{j,i} \) be the measured lamination which is dual to \( \alpha_{j,i} \).

By assumption, we have \( [T_{m,i}] \to [T] (i \to \infty) \) in \( CV(F_n) \). Since \( T \) is dense, Lemma 10.3 implies that up to passing to a subsequence, there is a bounded sequence \( (b_i) \) such that the measured laminations \( b_i \mu_{m-1,i} \) converge as \( i \to \infty \) to a measured lamination \( \nu_{m-1} \) supported in the zero lamination of \( T \). Since \( [T] \in \mathcal{FT} \), by Lemma 10.1 the support of \( \nu_{m-1} \) does not intersect a free factor and hence \( b_i \to 0 \) by Lemma 10.3.

By passing to another subsequence, we may assume that the projective trees \( [T_{(m-1),i}] \) converge as \( i \to \infty \) to a projective tree \( [U_{m-1}] \). Choose a representative \( U_{m-1} \) of \( [U_{m-1}] \). Since \( b_i \mu_{m-1,i} \to \nu_{m-1} \) for a sequence \( b_i \to 0 \) and since \( \langle U_{m-1}, \mu_{m-1,i} \rangle \leq 4 \) for all \( i \). Lemma 10.3 shows that \( \langle U_{m-1}, \nu_{m-1} \rangle = 0 \). In particular, \( \nu_{m-1} \) is supported in the zero lamination of \( U_{m-1} \). Proposition 7.1 now shows that \( U_{m-1} \) admits an alignment preserving map onto \( T \). Moreover, there is a subsequence of the sequence \( \{ \mu_{m-2,i} \} \) which converges as \( i \to \infty \) to a projective measured lamination supported in the zero lamination of \( [U_{m-1}] \) and hence of \( [T] \). (Since \( [T] \) is arational, it can easily be seen that \( [U_{m-1}] \in \mathcal{FT} \), a fact which is immediate from [KT2].)

Repeat this argument with the sequence \( [T_{(m-2),i}] \) and the tree \( [U_{m-1}] \). After \( m \) steps we conclude that \( T_0 \) admits an alignment preserving map onto \( T \) which is impossible. The first part of the proposition is proven.

The second part of the proposition is shown in the same way. By Lemma 2.13 we only have to show that if \( [T], [T'] \in \mathcal{FT} \) are such that \( \psi_F([T]) = \psi_F([T']) \) then \( [T], [T'] \) are equivalent. Let \( (x_t), (y_t) \) be Skora paths connecting a point in a standard simplex \( \Delta \) to \( [T], [T'] \). Since the paths \( \Upsilon_F(x_t), \Upsilon_F(y_t) \) are uniform reparametrized quasi-geodesics in \( \mathcal{FF} \), by hyperbolicity there is a number \( m \geq 0 \) and for each \( t > 0 \) there is a number \( s(t) > 0 \) so that \( d_{ng}(\Upsilon_F(x_t), \Upsilon_F(y_{s(t)})) \leq m \).

As above, by passing to a subsequence (and perhaps changing the constant \( m \)) we may assume that there is a sequence \( t_i \to \infty \) so that \( d_{ng}(\Upsilon_F(x_{t_i}), \Upsilon_F(y_{s(t_i)})) = \)
m for all i. For each i let \( u_i \in cv_0(F_n) \) be such that \( d_{ng}(x_1, u_i) = 1 \) and \( d_{ng}(u_i, y_{i_0(t)}) = m - 1 \). By the above discussion, up to passing to a subsequence the sequence \([u_i]\) converges to a point \([U]\) \( \in \partial CV(F_n) \) so that \( U \) admits an alignment preserving map onto \( T \). Repeat with the sequence \((u_i)\). After \( m \) steps we conclude that \([T']\) is equivalent to \([T]\) which is what we wanted to show. \( \square \)

**Remark 10.5.** The proof of Proposition \( 10.4 \) shows that for every \([T]\) \( \in FT \) there is a measured lamination \( \mu \) with \( \langle T, \mu \rangle = 0 \). Thus Corollary \( 7.2 \) applies to arational trees.

We showed so far that the map \( Y = \psi|FT \) is a continuous closed surjection of \( FT \) onto \( \partial FF \). Each fibre consists of the closed set of all trees in a fixed equivalence class for the relation \( \sim \). Thus the Gromov boundary of \( FF \) is homomorphic to \( FT/\sim \). Theorem \( 3 \) is proven.

We complete this section with an easy consequence which will be useful in other context. For its formulation, following \( [H14a] \) we call a pair \((\mu, \nu) \in ML \times ML\) positive if for every tree \( T \in cv(F_n) \) we have \( \langle T, \mu \rangle + \langle T, \nu \rangle > 0 \).

**Corollary 10.6.** Let \( \mu, \nu \in ML \) be measured laminations which are supported in the zero lamination of trees \([T], [S] \in FT \). If \( Y([T]) \neq Y([S]) \) then \((\mu, \nu)\) is a positive pair.

**Remark 10.7.** The topology for the boundary of the free factor graph can also be described as a measure forgetful topology in the following sense. The projection to \( \partial FF \) of a sequence \([T_i] \subset FT \) converges to the projection of \([T]\) if and only if the following holds true. For each i let \( \beta_i \) be a measured lamination with \( \langle T_i, \beta_i \rangle = 0 \). Assume that the measured laminations converge to a measured lamination \( \beta \); then \( \langle T, \beta \rangle = 0 \).

11. THE BOUNDARIES OF THE FREE AND THE CYCLIC SPLITTING GRAPH

In this section we identify the Gromov boundary of the free and of the cyclic splitting graph.

We continue to use all assumptions and notations from the previous sections. In particular, we use the maps

\[ \psi : \partial CV(F_n) \to \partial FS \cup \Theta \text{ and } \psi_C : \partial CV(F_n) \to \partial CS \cup \Theta \]

defined in Section \( 6 \).

As in the introduction, let \( ST \subset \partial CV(F_n) \) be the \( \text{Out}(F_n) \)-invariant set of projective dense trees which either are indecomposable or split as large graphs of actions. It contains the \( \text{Out}(F_n) \)-invariant subspace \( CT \) of projective trees which either are indecomposable or split as very large graph of actions. By Corollary \( 6.7 \), Corollary \( 6.14 \) and Proposition \( 9.3 \) we have

\[ \psi(ST) \supset \partial FS \text{ and } \psi(C(T)) \supset \partial CS. \]

We have to show that \( \psi^{-1}(\partial FS) = ST \) and \( \psi^{-1}(\partial CS) = CT \). We begin with an extension of Lemma \( 7.5 \). To this end we call as before two dense trees \( T, T' \in FT \)
\[ \partial \text{cv}(F_n) \] equivalent if the unions \( \hat{T}, \hat{T}' \) of their metric completions with their Gromov boundaries are equivariantly homomorphic.

**Lemma 11.1.** Let \( [T] \in ST \), let \( T' \in \partial \text{cv}(F_n) \) be dense and assume that there is a tree \( S \in \partial \text{cv}(F_n) \) which admits an alignment preserving map onto both \( T, T' \). Then \( T' \) admits an alignment preserving map onto a tree which is equivalent to \( T \).

**Proof.** By Proposition 9.2 we may assume that \( T' \) is either indecomposable or splits as a graph of actions.

We argue as in the proof of Lemma 7.5. Namely, if \( T \) splits as a large graph of actions then let \( V \subset T \) be a vertex tree of the transverse covering of \( T \) defined by the structure of a large graph of actions. If \( T \) is indecomposable then we let \( V = T \). Let \( H < F_n \) be the stabilizer of \( V \). Then \( H \) is finitely generated, and \( V \) is indecomposable for the action of \( H \). Let \( S_H \) be the minimal \( H \)-invariant subtree of \( S \). By equivariance, there is a one-Lipschitz alignment preserving map \( S_H \rightarrow V \), and there also is an alignment preserving map \( S_H \rightarrow T' \) where \( T' \) is the minimal \( H \)-invariant subtree of \( T' \).

Using again Proposition 9.2 we may assume that \( T' \) either is indecomposable or splits as a graph of actions. Now \( S_H \) is a minimal very small \( H \)-tree and therefore by the results in [R11], there is no alignment preserving map from \( S_H \) onto both an indecomposable tree and a tree which splits as a graph of actions. Since \( V \) is indecomposable, this shows that \( T' \) is indecomposable.

An application of Lemma 7.5 to \( S_H \) and \( T_H, T'_H \) now shows that \( \hat{T}'_H \) is equivariantly homeomorphic to \( V \). Since there is a single \( F_n \)-orbit of vertex trees for the structure of a large graph of actions on \( T \) we conclude that \( T' \) admits an alignment preserving map onto a tree which is equivalent to \( T \). This is what we wanted to show. \( \square \)

The next proposition is the main remaining step for the proof of Theorem 11 and Theorem 2. It entirely relies on the work of Handel and Mosher [HM13a] and its variation due to Bestvina and Feighn [BF14c]. Recall that two trees \( T, T' \) with dense orbits are equivalent if the union of their metric completions with their Gromov boundaries are equivariantly homeomorphic.

**Proposition 11.2.** Let \( \Delta \subset \text{cv}(F_n) \) be a standard simplex, let \( [T] \in ST \) and let \( (x_t) \) be a Skora path connecting a point \( x_0 \in \Delta \) to \([T] \).

1. \( \Upsilon(x_t) \subset FS \) is unbounded.
2. If \([T] \in CT \) then \( \Upsilon_C(x_t) \subset CS \) is unbounded.

**Proof.** Let \( (x_t) \subset \text{cv}(F_n) \) be an unnormalized Skora path connecting a point \( x_0 \in \Delta \) to a representative \( T \) of \([T] \).

For the proof of the first part of the proposition we argue by contradiction and we assume that \( \Upsilon(x_t) \subset FS \) is bounded. Let \( d \) be the distance in \( FS \). Since the
distance between two vertices in $FS$ assumes only integral values, there is some $k \geq 0$ and a sequence $t_i \to \infty$ such that
\[
d(\Upsilon(x_0), \Upsilon(x_{t_i})) = k
\]
for all $i$.

For each $i$ let $y_i(k)$ be the tree obtained from $x_{t_i}$ by equivariantly rescaling edges in such a way that the lengths of all edges of the quotient $y_i(k)/F_n$ coincide. This can be arranged in such a way that there is a one-Lipschitz alignment preserving map $y_i(k) \to x_{t_i}$ and that the volume of $y_i(k)/F_n$ is bounded from above by a uniform multiple of the volume of $x_{t_i}/F_n$. Thus after passing to a subsequence, we may assume that the trees $y_i(k)$ converge to a tree $T_k \in cv(F_n)$. By Proposition 5.5 of [G00], there is an alignment preserving map $T_k \to T$.

For each $i$ let $\Gamma_i(j)$ ($0 \leq j \leq k$) be a geodesic in $FS$ connecting $\Upsilon(x_0)$ to $\Upsilon(x_{t_i})$. Then for all $i$, the vertex $\Gamma_i(k-1)$ can be obtained from $x_{t_i}$ by either an expansion or a collapse. By passing to a subsequence, assume first that for all $i$, $\Gamma_i(k-1)$ is obtained from $\Gamma_i(k)$ by a collapse. Let $y_i(k-1) \in cv(F_n)$ be a simplicial tree whose quotient $y_i(k-1)/F_n$ has edges of equal length and defines the splitting $\Gamma_i(k-1)$. As $\Gamma_i(k-1)$ is obtained from $\Gamma_i(k)$ by a collapse, we may assume that the volume of $y_i(k-1)$ is bounded from below by a uniform multiple of the volume of $y_i(k)$ and that there is a one-Lipschitz alignment preserving map $y_i(k) \to y_i(k-1)$. By passing to a subsequence, we may moreover assume that the trees $y_i(k-1)$ converge to a tree $T_{k-1} \in cv(F_n)$. Since $T_k$ is dense, by equivariance the same holds true for $T_{k-1}$. By Proposition 5.5 of [G00], there is an alignment preserving map $T_k \to T_{k-1}$. Lemma 11.1 then shows that $T_{k-1}$ admits an alignment preserving map onto a tree $T'_{k-1}$ which is equivalent to $T$.

In the case that $\Gamma_i(k-1)$ is obtained from $\Gamma_i(k)$ by an expansion for all but finitely many $i$ we construct the trees $y_i(k-1)$ as before, and we find by passing to a subsequence that $y_i(k-1)$ converges to a tree $T_{k-1}$ which admits an alignment preserving map onto $T_k$ and hence onto a tree equivalent to $T$.

Repeat this construction with the sequence $y_i(k-1)$ and trees $y_i(k-2)$ which define the splitting $\Gamma_i(k-2)$. In finitely many steps we conclude that $x_0$ admits an alignment preserving map onto a tree which is equivalent to $T$. This is a contradiction which shows the first part of the proposition.

The argument for the second part of the proposition is identical and will be omitted. $\square$

As an immediate consequence of Lemma 6.4, Proposition 11.2 and Lemma 6.8 we obtain

**Corollary 11.3.** The map $\psi$ (or $\psi_C$) restricts to a continuous closed $Out(F_n)$-equivariant surjection $ST \to \partial FS$ (or a surjection $CT \to \partial CS$).
We are left with calculating the fibres of the map. This is done in the next lemma.

**Lemma 11.4.**  
(1) $\psi([T]) = \psi([S])$ for trees $[S], [T] \in ST$ if and only if $[T]$ and $[S]$ are equivalent. 
(2) $\psi_C([T]) = \psi_C([S])$ for trees $[S], [T] \in CT$ if and only if $[T]$ and $[S]$ are equivalent.

**Proof.** If $\hat{S}, \hat{T}$ are $F_n$-equivariantly homeomorphic then there exists an alignment preserving map $T \to S$ \cite{CHL07} and therefore $\psi([S]) = \psi([T])$ by Corollary 6.13.

Now let $[S], [T] \in ST$ be such that $\psi([S]) = \psi([T])$. Let $\Delta$ be a standard simplex and let $(x_t), (y_t)$ be Skora paths connecting a point in $\Delta$ to $[S], [T]$. By Corollary 11.3 the paths $\Upsilon(x_t), \Upsilon(y_t)$ are uniform unparametrized quasi-geodesics in $\mathcal{FS}$ of infinite diameter with the same endpoint. The distance between their starting points is uniformly bounded.

By hyperbolicity, there is a number $m > 0$ such that for every $t \geq 0$ the free splitting $\Upsilon(x_t)$ is contained in the $m$-neighborhood of the quasi-geodesic ray $(\Upsilon(y_t))_{t \geq 0}$ in $\mathcal{FS}$. As a consequence, for each $t > 0$ there is some $s(t) > 0$ such that the distance between $\Upsilon(x_t)$ and $\Upsilon(y_{s(t)})$ is at most $m$.

For each $i$ and each $k = 0, \ldots, m$ let $\beta_i(k)$ be a graph of groups decomposition for $F_n$ with trivial edge group with $\beta_i(0) = x_i/F_n, \beta_i(m) = y_{s(i)}/F_n$ such that for each $j \leq m/2$, the splitting $\beta_i(2j)$ collapses to both $\beta_i(2j-1)$ and $\beta_i(2j + 1)$. For $1 \leq k \leq m-1$ the splitting $\beta_i(k)$ defines a simplicial tree $\tilde{\beta}_i(k) \in CV(F_n)$ which is unique if we require that all edges have the same length.

Choose a sequence $(i_t)_{t \geq 0}$ so that for each $k \leq m$ the projectivizations $[\tilde{\beta}_{i_t}(k)]$ of the trees $\tilde{\beta}_{i_t}(k)$ converge in $CV(F_n)$ to a tree $[U_k]$. Apply Proposition 5.5 of \cite{G00} and conclude that for each $j$ there is an alignment preserving map $U_{2j} \to U_{2j-1}$ and $U_{2j} \to U_{2j+1}$.

As in the proof of Proposition 11.2 we can now successively apply Lemma 11.4 and deduce that $\hat{S}$ and $\hat{T}$ are homeomorphic which is what we wanted to show. $\square$

**Example:** We give an example which shows that the Gromov boundary of the free splitting graph does not coincide with the Gromov boundary of the cyclic splitting graph.

Namely, let $S$ be a compact surface of genus $g \geq 2$ with connected boundary $\partial S$. The *arc graph* of $S$ is defined as follows. Vertices are embedded arcs in $S$ with both endpoints on $\partial S$. Two such arcs are connected by an edge of length one if they are disjoint. The arc graph of $S$ is quasi-isometrically embedded in the free splitting graph of the free group $\pi_1(S) = F_{2g}$ \cite{HH11}. The Gromov boundary of the arc graph has been determined in \cite{HH11}.

Let $c \subset S$ be a non-separating simple closed curve and let $\varphi$ be a pseudo-Anosov mapping class of $S - c$. It follows from the results in \cite{MS13, HH11} and \cite{HH11} that $\varphi$ acts as a hyperbolic isometry on the free splitting graph of $F_{2g}$. However,
ψ preserves the cyclic splitting $A^\infty(\psi)$ and therefore it acts as an elliptic element on the cyclic splitting graph.

We conclude this section with some remarks and open questions.

**Question 1:** An element $\varphi \in \text{Out}(F_n)$ acts on the free factor graph as a hyperbolic isometry if and only if it is irreducible with irreducible powers. An irreducible element with irreducible powers $\varphi$ acts on $\partial CV(F_n)$ with north-south dynamics [LL03], fixing precisely two points. It was shown in [BFH97] (see also [KL11]) that the stabilizer in $\text{Out}(F_n)$ of such a fixed point of $\varphi$ is virtually cyclic.

A fixed point $[T]$ of $\varphi$ defines a point in the boundary of the free factor graph and hence is an arational tree. In analogy of the mapping class group action on the Thurston boundary of Teichmüller space, it can be shown that $[T]$ is uniquely ergodic, i.e. there is a unique projective measured lamination $\mu$ with $\langle T, \mu \rangle = 0$ [HH14].

More generally, in [H14a] we defined an $F_n$-invariant set $UT$ of projective trees in $\partial CV(F_n)$ as follows. If $[T] \in UT$ and if $\mu \in \mathcal{ML}$ is such that $\langle T, \mu \rangle = 0$ then the projective class of $\mu$ is unique, and $[T]$ is the unique projective tree with $\langle T, \mu \rangle = 0$. It follows immediately from this work that $UT \subset FS$ (see also [R12, HH14]).

Define the $\epsilon$-thick part $\text{Thick}_\epsilon(F_n)$ of $cv_0(F_n)$ to consist of simplicial trees with quotient of volume one which do not admit any essential loop of length smaller than $\epsilon$. In analogy to properties of the curve graph and Teichmüller space, we conjecture that whenever $(r_t)$ is a normalized Skora path in $cv_0(F_n)$ with the property that $r_{t_i} \in \text{Thick}_\epsilon(F_n)$ for a sequence $t_i \to \infty$ and some fixed number $\epsilon > 0$ then $[r_t]$ converges as $t \to \infty$ to a tree $[T] \in UT$ (see [Ho12] and [HH14] for results in this direction). Moreover, if $(r_t) \subset \text{Thick}_\epsilon(F_n)$ for all $t$ then the path $t \to \Upsilon(r_t)$ is a parametrized $L$-quasi-geodesic in $FS$ for a number $L > 1$ only depending on $\epsilon$.

**Question 2:** Let $\varphi \in \text{Out}(F_n)$ be a reducible outer automorphism which acts as a hyperbolic isometry on the free splitting graph $FS$. It follows from Theorem [H] that $\varphi$ fixes a tree $[T] \in ST$. It is true that up to scale there is a unique measured lamination $\mu$ with $\langle T, \mu \rangle = 0$ and $\mu(L^2(T) - L_r(T)) = 0$? What can we say about the subgroup of $\text{Out}(F_n)$ which fixes $[T]$? Recent work of Handel and Mosher [HM13c] indicates that this group may be quite large (i.e. it may not be infinitely cyclic).

Vice versa, given a map $\varphi \in \text{Out}(F_n)$, it is known that $\varphi$ can be represented by a relative train track map. It was observed by Handel and Mosher that $\varphi$ defines a hyperbolic automorphism of the free splitting complex if the support of the top stratum of the relative train track is all of $F_n$. Theorem [H] shows that the converse is also true: If the action of $\varphi$ on $FS$ is hyperbolic, then the support of the top stratum of a relative train track for $\varphi$ is all of $F_n$.

Note that Handel and Mosher showed [HM13c] that there are non-torsion elements $\varphi \in \text{Out}(F_n)$ which act with bounded orbits on the free splitting graph, but for which there is no $k \geq 1$ so that $\varphi^k$ fixes a point in $FS$. 
Appendix A. Splitting control

In this appendix we collect some information from the work of Handel and Mosher \cite{HM13b}.

Consider a free group $F_n$ of rank $n \geq 3$. Denote by $\mathcal{FS}$ and $\mathcal{CS}$ the first barycentric subdivision of the free splitting graph and the cyclic splitting graph of $F_n$, respectively. Vertices of $\mathcal{FS}$ are graphs of groups decompositions of $F_n$ with trivial edge groups. Such a graph of groups decomposition is a finite graph $G$ whose vertices are labeled with (possibly trivial) free factors of $F_n$. Two vertices $G, G'$ in $\mathcal{FS}$ are connected by an edge if $G'$ is either a collapse or an expansion of $G$. Vertices of $\mathcal{CS}$ are graph of groups decompositions of $F_n$ with at most cyclic edge groups. Two such vertices $G, G'$ are connected by an edge if $G'$ is either a collapse or an expansion of $G$.

Throughout this appendix we follow the appendix of \cite{BF14c}.

A simplicial $F_n$-tree $T \in \text{cv}(F_n)$ with at least one trivial edge stabilizer projects to a finite metric graph $T/F_n$ which defines a graph of groups decomposition for $F_n$ with trivial edge groups, i.e. it defines a vertex in $\mathcal{FS}$. Namely, an arbitrary simplicial tree $T \in \text{cv}(F_n)$ defines a vertex in $\mathcal{CS}$.

A finite folding path is a path $(x_t) \subset \text{cv}(F_n) (t \in [0, L])$ which is guided by an optimal morphism $f : x_0 \to x_L$ as introduced in Section 3. The points along the path are obtained from $x_0$ by identifying directions which are mapped to the same direction in $x_L$. We allow folding at any speed, and we also allow rest intervals. Such paths are called liberal folding paths in \cite{BF14c}.

An optimal morphism projects to a map of the quotient graphs which minimizes the Lipschitz constant in its homotopy class. There are induced optimal maps

$$f_t : x_t \to x_L$$

for all $t \in [0, L]$.

Let $G_t \subset \text{cv}(F_n) (0 \leq t \leq L)$ be any folding path and let $F_L \subset G_L$ be a proper equivariant forest. Then for each $t$ the preimage $F_t \subset G_t$ of $F_L$ under the map $f_t : G_t \to G_L$ is defined, and $G'_t = G_t/F_t$ is an $F_n$-tree. By Proposition 4.4 of \cite{HM13b} (see also Lemma A.1 of \cite{BF14c}), the path $t \to G'_t$ is a liberal folding path, and we obtain a commutative diagram

$$\begin{array}{ccc}
G_o & \longrightarrow & G'_L \\
\bullet & \longrightarrow & \bullet \\
\uparrow & & \uparrow \\
G_0 & \longrightarrow & G_L
\end{array}$$
Similarly, if $G_t \ (0 \leq t \leq L)$ is a folding path and if $G'_L \rightarrow G_L$ is a collapse map then there is a folding path $G'_t$ so that the diagram

$$
\begin{array}{ccc}
G'_0 & \longrightarrow & G'_L \\
\downarrow & & \downarrow \\
G_0 & \longrightarrow & G_L
\end{array}
$$

commutes (Lemma A.2 of [BF14c]). Here one may have to insert rest intervals into the folding path $G_t$.

The following lemma is a version of Lemma A.3 of [BF14c].

**Lemma A.1.** There is a number $M > 0$ with the following property. Let $G, G' \in \text{cv}(F_n)$ be such that $G/F_n, G'/F_n$ are finite metric graphs defining a graph of groups decomposition for $F_n$. Let $f : G \rightarrow G'$ be an optimal morphism. Assume that there is a point $y \in G'$ such that the cardinality of $f^{-1}(y)$ is at most $p$. Then the distance in $\text{CS}$ between $G, G'$ is at most $Mp$. If the edge groups of $G/F_n, G'/F_n$ are trivial then the same holds true for the distance in $\mathcal{FS}$.

**Proof.** The proof of Lemma 4.1 of [BF14c] is valid without modification. A variation of the argument is used in Section 11, so we provide a sketch.

Connect $G$ to $G'$ by a folding path $G_t$ guided by $f$, with $G_0 = G$. For each $t$ there are optimal morphisms $\varphi_t : G \rightarrow G_t$ and $f_t : G_t \rightarrow G'$ so that $f = f_t \circ \varphi_t$. The number of preimages of $y$ under $f_t$ decreases with $t$.

Let $t > 0$ be such that this number coincides with the number of preimages of $y$ in $G$. Then there is point $z_t \in G_t/F_n$ whose preimage under the quotient of $\varphi_t$ consists of a single point $z$. If $z_t$ is a vertex then the preimage of a nearby interior point of an adjacent edge consists of a single point as well, so may assume that the points $z, z_t$ are contained in the interior of edges $e, e_t$. A loop in $G/F_n$ not passing through the projection of $z$ is mapped by the quotient of $\varphi_t$ to a loop in $G_t/F_n$ not passing through $z_t$. Thus collapsing the complement of the edges $e, e_t$ in $G/F_n, G_t/F_n$ to a single point yields two identical graph of groups decompositions containing $A$ as a vertex group. In particular, $G/F_n$ and $G_t/F_n$ collapse to the same vertex in $\text{CS}(A)$, and if the edge group of $e$ is trivial then $G(F_n)$ and $G_t/F_n$ collapse to the same point in $\mathcal{FS}$.

Let $t_0 > 0$ be the first point so that the number of preimages of $y$ in $G_{t_0}$ is strictly smaller than the number of preimages of $y$ in $G_0$. For $t < t_0$ sufficiently close to $t_0$, the graphs $G_t/F_n, G_{t_0}/F_n$ collapse to the same graph of groups decomposition of $F_n$. Hence the lemma now follows by induction. \[\square\]

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