Long Alternating Paths Exist

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The Problem

Given: 2n points, convex, n red, n blue

Want: (noncrossing) alternating path: alternate between red and blue, every point used at most once, no crossings

No!
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Question: What is the longest alternating path? algorithmically easy (dynamic programming)
The Problem

Given: 2n points, convex, n red, n blue
Want: (noncrossing) alternating path: alternate between red and blue, every point used at most once, no crossings
Question: What is the longest alternating path as a function of n, alt(n)? (min over all colorings)
Easy Lower Bound (Erdős, 1980s)

Take any halving line. One side has $\geq n/2$ red points. Other side has $\geq n/2$ blue points. Connect into an alternating path with $n$ points.

Thus: $\text{alt}(n) \geq n$
Better Lower Bounds

run: maximal sequence of consecutive points of the same color

Theorem [Kynčl, Pach and Tóth ‘08]: $\text{alt}(n) \geq n + \text{#runs}/2 - 1$

Theorem [Mészáros‘11]: $\text{alt}(n) \geq n + \lceil (n - 1) / \text{#runs} \rceil$

Corollary: $\text{alt}(n) \geq n + \Omega(\sqrt{n})$
Our Result

Theorem: \( \exists \epsilon > 0: \text{alt}(n) \geq (1+\epsilon)n \)

Remark: also for monochromatic matchings can also interpreted as a statement about (anti)palindromic subsequences in circular words.
More Background: Upper Bounds

[Erdős, 1980s]

\[ \text{alt}(n) \leq 1.5n + 2 \]
More Background: Upper Bounds

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Assume \( \text{alt}(n) > 1.5n + 2 \)

\( \leq 0.5n \text{ red points.} \)
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Assume \( \text{alt}(n) > 1.5n + 2 \)

\[ \leq 0.25n \text{ blue points} \]

\[ 0.5n \]

\[ 0.75n \leq 0.5n \text{ red points} \]

\[ 0.5n \]

\[ 0.25n \]
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[Abellanas, Garcia, Hurtado, and Tejel ‘03; Kynčl, Pach and Tóth ’08; Mészáros ’11]
\[ \text{alt}(n) \leq \frac{4n}{3} \approx 1.33n \]

[Csóka, Blázsik, Király and Lenger ’20]
\[ \text{alt}(n) \leq (4 - 2\sqrt{2})n \approx 1.17n \]
Our Approach – Chunks

k-chunk: k points of one color and <k points of other color
k-configuration: partition into k-chunks
index (chunk): #points minority color/#points majority color
index (configuration): average index over all chunks
Our Approach – Configurations

Suppose: For every $k$, we can find a canonical $k$-configuration $\Gamma_k$ on $P$.

Observation 1: If $\Gamma_{1000}$ has index $\geq 0.1$, a long alternating path exists.

Reason: There must be many runs.
Our Approach – Configurations

Suppose: For every $k$, we can find a canonical $k$-configuration $\Gamma_k$ on $P$

Observation 2: If $\Gamma_{n/1000}$ has index $<0.1$, a long alternating path exists.

Reason: There must be a large unbalanced chunk.

→ Kynčl, Pach and Tóth
Our Approach – Configurations

Suppose: For every $k$, we can find a canonical $k$-configuration $\Gamma_k$ on $P$

Thus: We can focus on a canonical $3k$-configuration $\Gamma_{3k}$ with $1000 < 3k < n/1000$ and index $\approx 0.1$
Our Approach – Separated Matchings

We now look at separated matchings.

**separated matching:** plane bichromatic matching, all segments intersected by one line

**Obvious:** separated matching with $k$ edges $\rightarrow$ alternating path with $2k$ points
Our Approach – Separated Matchings

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**separated matching:** plane bichromatic matching, all segments intersected by one line

**Obvious:** separated matching with $k$ edges $\rightarrow$ alternating path with $2k$ points

**We show:** $\exists \, \varepsilon > 0 \Rightarrow \forall$ suitable $\Gamma_{3k} \; \exists$ sep. matching of $(1/2 + \varepsilon)n$ edges
Our Approach – Chunk Matchings

chunk matching: match $3k$-chunks in $\Gamma_{3k}$ along a chunk-halving-line
random chunk  pick chunk-halving-line uniformly at random
matching
Our Approach – Chunk Matchings

Observation: chunk matching \(\rightarrow\) separated matching
Our Approach – Chunk Matchings

Observation: chunk matching $\rightarrow$ separated matching
Our Approach – Chunk Matchings

**Observation:** chunk matching → separated matching

**Fact:** A random chunk matching yields a separated matching of expected size $n/2$ (# edges).

**Proof:** Brute-force calculation.

**Crucial:** bound $\max\{r_1, r_2\} \geq (r_1 + r_2)/2$
Our Approach – Proof Strategy

Suppose: $3k$-configuration $\Gamma_{3k}$ of index $\approx 0.1$ is at hand

Consider: random chunk matching in $\Gamma_{3k}$

Lemma: If the individual chunk indices in $\Gamma_{3k}$ have “large variance”, we get a separated matching with $(1/2 + \varepsilon)n$ edges in expectation.

\[ \max\{b_1, b_2\} >> (b_1 + b_2)/2 \text{ edges} \]
Our Approach – Proof Strategy

Suppose: \(3k\)-configuration \(\Gamma_{3k}\) of index \(\approx 0.1\) is at hand

Consider: random chunk matching in \(\Gamma_{3k}\)

Lemma: If \(\Gamma_{3k}\) has “large variance”, we get a separated matching with \((1/2 + \varepsilon)n\) edges in expectation.

Otherwise: Consider refined \(k\)-configuration \(\Gamma_k\) for \(\Gamma_{3k}\) (it exists).

Lemma: If \(\Gamma_k\) has “large variance”, we get a separated matching with \((1/2 + \varepsilon)n\) edges in expectation.

3 red \(k\)-chunks

3 red \(k\)-chunks
Our Approach – Proof Strategy

Remains: 3k-configuration $\Gamma_{3k}$ and refined $k$-configuration $\Gamma_k$ with “uniform” chunks.

Main trick: gain when matching two 3k-chunks of the same color!

\[ \max\{b_1, b_2\} \text{ edges} \]
Our Approach – Proof Strategy

Remains: $3k$-configuration $\Gamma_{3k}$ and refined $k$-configuration $\Gamma_k$ with “uniform” chunks.

Main trick: gain when matching two $3k$-chunks of the same color!

$\approx (4/3) \max\{b_1, b_2\} \text{ edges}$
Conclusion

very technical

very small $\varepsilon$

What is the right bound for $\text{alt}(n)$?

Questions?