HIGHER HOMOTOPIES AND MAURER-CARTAN ALGEBRAS: QUASI-LIE-RINEHART, GERSTENHABER, AND BATALIN-VILKOVISKY ALGEBRAS

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Dedicated to Alan Weinstein on his 60th birthday

Abstract. Higher homotopy generalizations of Lie-Rinehart algebras, Gerstenhaber, and Batalin-Vilkovisky algebras are explored. These are defined in terms of various antisymmetric bilinear operations satisfying weakened versions of the Jacobi identity, as well as in terms of operations involving more than two variables of the Lie triple systems kind. A basic tool is the Maurer-Cartan algebra—the algebra of alternating forms on a vector space so that Lie brackets correspond to square zero derivations of this algebra—and multialgebra generalizations thereof. The higher homotopies are phrased in terms of these multialgebras. Applications to foliations are discussed: objects which serve as replacements for the Lie algebra of vector fields on the “space of leaves” and for the algebra of multivector fields are developed, and the spectral sequence of a foliation is shown to arise as a special case of a more general spectral sequence including as well the Hodge-de Rham spectral sequence.

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Introduction

In this paper we will explore, in the framework of Lie-Rinehart algebras and suitable higher homotopy generalizations thereof, various antisymmetric bilinear operations satisfying weakened versions of the Jacobi identity, as well as similar operations involving more than two variables; such operations have recently arisen in algebra, differential geometry, and mathematical physics but are lurking already behind a number of classical developments. Our aim is to somewhat unify these structures by means of the relationship between Lie-Rinehart, Gerstenhaber, and Batalin-Vilkovisky algebras which we first observed in our paper [19]. This will be, perhaps, a first step towards taming the bracket zoo that arose recently in topological field theory, cf. what we wrote in the introduction to [19]. The notion of Lie-Rinehart algebra and its generalization are likely to provide a good conceptual framework for that purpose. It will also relate new notions like those of Gerstenhaber and Batalin-Vilkovisky algebra, and generalizations thereof, with classical ones like those of connection, curvature, and torsion, as well as with less classical ones like Yamaguti’s triple product [62] and operations of the kind introduced in [35]; it will connect new developments with old results due to E. Cartan [9] and Nomizu [49] describing the geometry of Lie groups and of reductive homogeneous spaces and, more generally, with more recent results in the geometry of Lie loops [34,55]. We will see that the new structures have incarnations in mathematical nature, e.g. in the theory of foliations. The higher homotopies which are exploited below are of a special kind, though, where only the first of an (in general) infinite family is non-zero.

Let \( R \) be a commutative ring with 1. A Lie-Rinehart algebra \((A,L)\) consists of a commutative \( R \)-algebra \( A \), an \( R \)-Lie algebra \( L \), an \( A \)-module structure on \( L \), and an action \( L \otimes_R A \to A \) of \( L \) on \( A \) by derivations. These are required to satisfy suitable compatibility conditions which arise by abstraction from the pair \((A,L) = (C^\infty(M),\text{Vect}(M))\) consisting of the smooth functions \( C^\infty(M) \) and smooth vector fields \( \text{Vect}(M) \) on a smooth manifold \( M \). In a series of papers [16–21], we studied these objects and variants thereof and used them to solve various problems in algebra and geometry. See [23] for a survey and leisurely introduction. In differential geometry, a special case of a Lie-Rinehart algebra arises from the space of sections of a Lie algebroid.

In [19,21,22] we have shown that certain Gerstenhaber and Batalin-Vilkovisky algebras admit natural interpretations in terms of Lie-Rinehart algebras. The starting point was the following observation: It is nowadays well understood that a skew-symmetric bracket on a vector space \( g \) is a Lie-bracket (i.e. satisfies the Jacobi identity) if and only if the coderivation \( \partial \) on the graded exterior coalgebra \( \Lambda'[sg] \) corresponding to the bracket on \( g \) has square zero, i.e. is a differential; this coderivation is then the ordinary Lie algebra homology operator. This kind of characterization is not available for a general Lie-Rinehart algebra: Given a commutative algebra \( A \) and an \( A \)-module \( L \), a Lie-Rinehart structure on \((A,L)\) cannot be characterized in terms of a coderivation on \( \Lambda_A[sL] \) with reference to a suitable coalgebra structure on \( \Lambda_A[sL] \) (unless the \( L \)-action on \( A \) is trivial); in fact, in the Lie-Rinehart context, a certain dichotomy between \( A \)-modules and chain complexes which are merely defined over \( R \) persists throughout; cf. e.g. the Remark 2.5.2 below. On the other hand, Lie-Rinehart algebra structures on \((A,L)\) correspond to Gerstenhaber algebra structures on the exterior \( A \)-algebra \( \Lambda_A[sL] \); cf. e.g. [38].
particular, when $A$ is the ground ring and $L$ just an ordinary Lie algebra $\mathfrak{g}$, under the obvious identification of $\Lambda[\mathfrak{s}g]$ and $\Lambda'[\mathfrak{s}g]$ as graded $R$-modules, the (uniquely determined) generator of the Gerstenhaber bracket on $\Lambda[\mathfrak{s}g]$ is exactly the Lie algebra homology operator on $\Lambda'[\mathfrak{s}g]$. Given a general commutative algebra $A$ and an $A$-module $L$, the interpretation of Lie-Rinehart algebra structures on $(A,L)$ in terms of Gerstenhaber algebra structures on $\Lambda_A[\mathfrak{s}L]$ provides, among other things, a link between Gerstenhaber’s and Rinehart’s papers [13] and [52] (which seems to have been completely missed in the literature). In the present paper, we will extend this link to suitable higher homotopy notions which we refer to by the attribute “quasi”; we will introduce Lie-Rinehart triples, quasi-Lie-Rinehart algebras, and certain quasi-Gerstenhaber algebras and quasi-Batalin-Vilkovisky algebras, and we will explore the various relationships between these notions. Below we will comment on the relationship with notions of quasi-Gerstenhaber and quasi-Batalin-Vilkovisky algebras already in the literature.

When a Lie algebra $\mathfrak{g}$ over a field $k$ is “resolved” by an object, which we here somewhat vaguely refer to as a “resolution” (free, or projective, or variants thereof) having the given structure as its zero-th homology, on the resolution, the algebraic structure is in general defined only up to higher homotopies; likewise, an $A_\infty$ structure is defined in terms of a bar construction or variants thereof, cf. e.g. [31], [32] and the references there. Exploiting higher homotopies of this kind, in a series of articles [27-30] we constructed small free resolutions for certain classes of groups from which we then were able to do explicit calculations in group cohomology which until today still cannot be done by other methods. A historical overview related with $A_\infty$-structures may be found in the Addendum to [33]; cf. also [24] and [31] for more historical comments.

In the present paper, we will explore a certain higher homotopy related with Lie-Rinehart algebras and variants thereof. A Lie algebra up to higher homotopies (equivalently: $L_\infty$-algebra) on an $R$-chain complex $\mathfrak{h}$ may be defined in terms of a coalgebra perturbation of the differential on the graded symmetric coalgebra on the suspension of $\mathfrak{h}$; alternatively, it may be defined in terms of a suitable Maurer-Cartan algebra (see below). Since a genuine Lie-Rinehart structure on $(A,L)$ cannot be characterized in terms of a coderivation on $\Lambda_A[\mathfrak{s}L]$, the first alternative breaks down for a general Lie-Rinehart algebra. The higher homotopies we will explore in the present paper do not live on an object close to a resolution of the above kind or close to a symmetric coalgebra; they may conveniently be phrased in terms of an object of a rather different nature which, extending terminology introduced by van Est [60], we refer to as a Maurer-Cartan algebra. A special case thereof arises in the following fashion: Given a finite dimensional vector space $\mathfrak{g}$ over a field $k$, skew symmetric brackets on $\mathfrak{g}$ correspond bijectively to degree $-1$ derivations of the graded algebra of alternating forms on $\mathfrak{g}$ (with reference to multiplication of forms), and those brackets which satisfy the Jacobi identity correspond to square zero derivations, i.e. differentials. This observation generalizes to Lie-Rinehart algebras of the kind $(A,L)$ under the assumption that $L$ be a finitely generated projective $A$-module; see Theorem 2.2.16 below. For an ordinary Lie algebra $\mathfrak{g}$ over a field $k$, in [60], the resulting differential graded algebra $\text{Alt}(\mathfrak{g},k)$ (which calculates the cohomology of $\mathfrak{g}$) has been called Maurer-Cartan algebra. The main point of this paper is that higher homotopy variants of the notion of Maurer-Cartan algebra
provide the correct framework to phrase certain higher homotopy versions of Lie-Rinehart-, Gerstenhaber, and Batalin-Vilkovisky algebras to which we will refer as quasi-Lie-Rinehart-, quasi-Gerstenhaber, and quasi-Batalin-Vilkovisky algebras.

The differential graded algebra of alternating forms on a Lie algebra occurs, at least implicitly, in [10] and has a long history of use since then, cf. [41], and once I learnt in a talk by van Est that this algebra has been used by E. Cartan in the 1930’s to characterize the structure of Lie groups and Lie algebras.

For the reader’s convenience, we will explain briefly and somewhat informally a special case of a quasi-Lie-Rinehart algebra at the present stage: Let \((M,\mathcal{F})\) be a foliated manifold, the foliation being written as \(\mathcal{F}\), let \(\tau_\mathcal{F}\) be the tangent bundle of the foliation \(\mathcal{F}\), and choose a complement \(\zeta\) of \(\tau_\mathcal{F}\) so that the tangent bundle \(\tau_M\) of \(M\) may be written as \(\tau_M = \tau_\mathcal{F} \oplus \zeta\). Let \(L_\mathcal{F} \subseteq \text{Vect}(M)\) be the Lie algebra of smooth vector fields tangent to the foliation \(\mathcal{F}\), and let \(Q\) be the \(C^\infty(M)\)-module \(\Gamma(\zeta)\) of smooth sections of \(\zeta\). The Lie bracket in \(\text{Vect}(M)\) induces a left \(L_\mathcal{F}\)-module structure on \(Q\)—the Bott connection—and the space \(Q_{L_\mathcal{F}}\) of invariants, that is, of vector fields on \(M\) which are horizontal (with respect to the decomposition \(\tau_M = \tau_\mathcal{F} \oplus \zeta\)) and constant on the leaves inherits a Lie bracket. The standard complex \(A\) arising from a fine resolution of the sheaf of germs of functions on \(M\) which are constant on the leaves acquires a differential graded algebra structure and has \(H^0(A)\) equal to the algebra of functions on \(M\) which are constant on the leaves, and the Lie algebra \(Q_{L_\mathcal{F}}\) of invariants arises as \(H^0(Q)\) where \(Q\) is the complex coming from a fine resolution of the sheaf \(\mathcal{V}_Q\) of germs of vector fields on \(M\) which are horizontal (with respect to the decomposition \(\Gamma(\tau_M) = L_\mathcal{F} \oplus Q\)) and constant on the leaves. In a sense, \(Q_{L_\mathcal{F}}\) is the Lie algebra of vector fields on the “space of leaves”, that is, the space of sections of a certain geometric object which may be seen as a replacement for the in general non-existant tangent bundle of the “space of leaves”.

Within our approach, this philosophy is pushed further in the following fashion: The pair \((A, Q)\) acquires what we will call a quasi-Lie-Rinehart structure in an obvious fashion; see (4.12) and (4.15) below for the details. We view \(A\) as the algebra of generalized functions and \(Q\) as the generalized Lie algebra of vector fields for the foliation. The pair \((H^0(A), H^0(Q))\) is necessarily a Lie-Rinehart algebra, and the entire cohomology \((H^*(A), H^*(Q))\) acquires a graded Lie-Rinehart algebra structure. As a side remark, we note that here the resolution of the sheaf \(\mathcal{V}_Q\) is by no means a projective one; indeed, it is a fine resolution of that sheaf, the bracket on \(Q\) is not an ordinary Lie(-Rinehart) bracket, in particular, does not satisfy the Jacobi identity, and the entire additional structure is encapsulated in certain homotopies which may conveniently be phrased in terms of a suitable Maurer-Cartan algebra which here arises from the de Rham algebra of \(M\). When the foliation does not come from a fiber bundle, the structure of the graded Lie-Rinehart algebra \((H^*(A), H^*(Q))\) will in general be more complicated than that for the case when the foliation comes from a fiber bundle. Thus the cohomology of a quasi-Lie-Rinehart algebra involves an ordinary Lie-Rinehart algebra in degree zero but in general contains considerably more information. In particular, in the case of a foliation it contains more than just “functions and vector fields on the space of leaves”; the additional information partly includes the history of the “space of leaves”, that is, it includes information as to how this space arises from the foliation, how the leaves sit inside the ambient space, about singularities, etc. In Section 6 we will show that, when the foliation
is transversely orientable with a basic transverse volume form $\omega$, a corresponding quasi-Batalin-Vilkovisky algebra isolated in Theorem 6.10 below has an underlying quasi-Gerstenhaber algebra which, in turn, yields a kind of generalized Schouten algebra (generalized algebra of multivector fields) for the foliation; the cohomology of this quasi-Gerstenhaber algebra may then be viewed as the Schouten algebra for the “space of leaves”. See (6.15) below for details.

Thus our approach will provide new insight, for example, into the geometry of foliations; see in particular (1.12), (2.10), (4.15), (6.15) below. The formal structure behind foliations which we will phrase in terms of quasi-Lie-Rinehart algebras and its offspring does not seem to have been noticed in the literature before—indeed, it involves, among a number of other things, a suitable grading which seems unfamiliar in the literature on quasi-Gerstenhaber and quasi-Batalin-Vilkovisky algebras, cf. (6.17) below—, nor the formal connections with Yamaguti’s triple product and with Lie loops.

A simplified version of the question we will examine is this: Given a Lie algebra $\mathfrak{g}$ with a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ where $\mathfrak{h}$ is a Lie subalgebra, what kind of structure does $\mathfrak{q}$ then inherit? Variants of this question and possible answers may be found at a number of places in the literature, cf. e. g. [9, 49] where, in particular, in a global situation, an answer is given for reductive homogeneous spaces. In the framework of Lie-Rinehart algebras, this issue does not seem to have been raised yet, not even for the special case of Lie algebroids.

As a byproduct, we find a certain formal relationship between Yamaguti’s triple product and certain forms $\Phi^*_3$ which may be found in [41]. In particular, the failure of a quasi-Gerstenhaber bracket to satisfy the Jacobi identity is measured by an additional piece of structure which we refer to as an $h$-Jacobiator; an $h$-Jacobiator, in turn, is defined in terms of Koszul’s forms $\Phi^*_3$. Likewise the quadruple and quintuple products studied in Section 3 below are related with Koszul’s forms, and these, in turn, are related with certain higher order operations which may be found e. g. in [55]. We do not pursue this here; we hope to eventually come back to it in another article.

A Courant algebroid has been shown in [54] to acquire an $L_\infty$-structure, that is, a Lie algebra structure up to higher homotopies. The present paper paves, perhaps, the way towards finding a higher homotopy Lie-Rinehart or higher homotopy Lie algebroid structure on a Courant algebroid incorporating the Courant algebroid structure.

Graded quasi-Batalin-Vilkovisky algebras have been explored already in [14]. Our notions of quasi-Gerstenhaber and quasi-Batalin-Vilkovisky algebra, while closely related, do not coincide with those in [4], [5], [14], [40], [53]. In particular, our algebras are bigraded while those in the quoted references are ordinary graded algebras; the appropriate totalization (forced, as noted above, by our application of the newly developed algebraic structure to foliations and written in Section 6 below as the functor Tot) of our bigraded objects leads to differential graded objects which are not equivalent to those in the quoted references. See Remark 6.17 below for more details on the relationship between the various notions. Also the approaches differ in motivation; the guiding idea behind [14] and [40] seems to be Drinfeld’s quasi-Hopf algebras. Our motivation, as indicated above, comes from foliations and the search for appropriate algebraic notions encapsulating the infinitesimal structure of
the “space of leaves” and its history, as well as the search for a corresponding Lie-Rinehart generalization of the operations on a reductive homogenous space isolated by Nomizu and elaborated upon by Yamaguti (mentioned earlier) and taken up again by M. Kinyon and A. Weinstein in [35]. Indeed, the present paper was prompted by the preprint versions of [35] and [61]. It is a pleasure to dedicate it to Alan Weinstein. Throughout this work I have been stimulated by M. Kinyon via some e-mail correspondence at an early stage of the project as well as by M. Bangoura, P. Michor, D. Roytenberg and Y. Kosmann-Schwarzbach. I am indebted to J. Stasheff and to the referees for a number of comments on a draft of the manuscript which helped improve the exposition.

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1. Lie-Rinehart triples

Let $R$ be a commutative ring with 1, not necessarily a field; $R$ could be, for example, the algebra of smooth functions on a smooth manifold, cf. [20]. The problem we wish to explore is this:

**Question 1.1.** Given a Lie-Rinehart algebra $(A,L)$ and an $A$-module direct sum decomposition $L = H \oplus Q$ inducing an $(R,A)$-Lie algebra structure on $H$, what kind of structure does then $Q$ inherit, and by what additional structure are $H$ and $Q$ related?

**Question 1.2.** Given an $(R,A)$-Lie algebra structure on $H$ and the (new) structure (which we will isolate below) on $Q$, what kind of additional structure turns the $A$-module direct sum $L = H \oplus Q$ into an $(R,A)$-Lie algebra in such a way that the latter induces the given structure on $H$ and $Q$?

**Example 1.3.1.** Let $g$ be an ordinary $R$-Lie algebra with a decomposition $g = h \oplus q$ where $h$ is a Lie subalgebra. Recall that the decomposition of $g$ is said to be reductive [49] provided $[h,q] \subseteq q$. Such a reductive decomposition arises from a reductive homogeneous space [9, 34, 49, 55, 62]. For example, every homogeneous space of a compact Lie group or, more generally, of a reductive Lie group, is reductive. Nomizu has shown that, on such a reductive homogeneous space, the torsion and curvature of the “canonical affine connection of the second kind” (affine connection with parallel torsion and curvature) yield a bilinear and a ternary operation which, at the identity, come down to a certain bilinear and ternary operation on the constituent $q$ [49], and Yamaguti gave an algebraic characterization of pairs of such operations [62].

**Example 1.3.2.** A quasi-Lie bialgebra $(h,q)$, cf. [37], consists of a (real or complex) Lie algebra $h$ and a (real or complex) vector space $q$ with suitable additional structure where $q = h^*$, so that $g = h \oplus h^*$ is an ordinary Lie algebra; the pair $(g,h)$ is occasionally referred to in the literature as a Manin pair. Quasi-Lie bialgebras arise as classical limits of quasi-Hopf algebras; these, in turn, were introduced by Drinfeld [11].

**Example 1.4.1.** Let $R$ be the field $\mathbb{R}$ of real numbers, let $(M,\mathcal{F})$ be a foliated manifold, let $\tau_{\mathcal{F}}$ be the tangent bundle of the foliation $\mathcal{F}$, and choose a complement $\zeta$ of $\tau_{\mathcal{F}}$ so that the tangent bundle $\tau_M$ of $M$ may be written as $\tau_M = \tau_{\mathcal{F}} \oplus \zeta$. 
Thus, as a vector bundle, \( \zeta \) is canonically isomorphic to the normal bundle of the foliation. Let \((A,L)\) be the Lie-Rinehart algebra \((C^\infty(M),\text{Vect}(M))\), let \(L_\mathcal{F} \subseteq L\) be the \((\mathbb{R},A)\)-Lie algebra of smooth vector fields tangent to the foliation \(\mathcal{F}\), and let \(Q\) be the \(A\)-module \(\Gamma(\zeta)\) of smooth sections of \(\zeta\). Then \(L = L_\mathcal{F} \oplus Q\) is a \(A\)-module direct sum decomposition of the \((\mathbb{R},A)\)-Lie algebra \(L\), and the question arises what kind of Lie structure \(Q\) carries. This question, in turn, may be subsumed under the more general question to what extent the “space of leaves” can be viewed as a smooth manifold. This more general question is not only of academic interest since, for example, in interesting physical situations, the true classical state space of a constrained system is the “space of leaves” of a foliation which is in general not fibrating, and the Noether theorems are conveniently phrased in the framework of foliations.

**Example 1.4.2.** Let \(R\) be the field \(\mathbb{C}\) of complex numbers, \(M\) a smooth complex manifold, \(A\) the algebra of smooth complex functions on \(M\), \(L\) the \((\mathbb{C},A)\)-Lie algebra of smooth complexified vector fields, and let \(L'\) and \(L''\) be the spaces of smooth sections of the holomorphic and antiholomorphic tangent bundle of \(M\), respectively. Then \(L'\) and \(L''\) are \((\mathbb{C},A)\)-Lie algebras, and \((A,L',L'')\) is a twilled Lie-Rinehart algebra in the sense of [21,22]. Adjusting the notation to that in (1.4.1), let \(H = L'\) and \(Q = L''\). Thus, in this particular case, \(Q = L''\) is in fact an ordinary \((R,A)\)-Lie algebra, and the additional structure relating \(H\) and \(Q\) is encapsulated in the notion of twilled Lie-Rinehart algebra. The integrability condition for an almost complex structure may be phrased in term of the twilled Lie-Rinehart axioms; see [21,22] for details.

The situation of Example 1.4.1 is somewhat more general than that of Example 1.4.2 since in Example 1.4.1 the constituent \(Q\) carries a structure which is more general than that of an ordinary \((R,A)\)-Lie algebra. Another example for a decomposition of the kind spelled out in Questions 1.1 and 1.2 above arises from combining the situations of Example 1.4.1 and of Example 1.4.2, that is, from a smooth manifold foliated by holomorphic manifolds, and yet another example arises from a holomorphic foliation. Abstracting from these examples, we isolate the notion of Lie-Rinehart triple. For ease of exposition, we also introduce the weaker concepts of almost pre-Lie-Rinehart triple and pre-Lie-Rinehart triple. Distinguishing between these three notions may appear pedantic but will clarify the statement of Theorem 2.7 below. See also Remark 2.8.4 below. As for the terminology we note that our notion of triple is not consistent with the usage of Manin triple in the literature. However, a Lie-Rinehart algebra involves a pair consisting of an algebra and a Lie algebra, and in this context, it is also common in the literature to refer to this structure as a pair which, in turn, is not consistent with the notion of Manin pair. We therefore prefer to use our terminology Lie-Rinehart triple etc.

Let \(A\) be a commutative \(R\)-algebra. Consider two \(A\)-modules \(H\) and \(Q\), together with

- skew-symmetric \(R\)-bilinear brackets of the kind (1.5.1.\(H\)) and (1.5.1.\(Q\)) below, not necessarily Lie brackets;
- \(R\)-bilinear operations of the kind (1.5.2.\(H\)), (1.5.2.\(Q\)), (1.5.3), (1.5.4) below; and
— a skew-symmetric $A$-bilinear pairing $\delta$ of the kind (1.5.5) below:

$$(1.5.1.H) \quad [\cdot,\cdot]_H: H \otimes_R H \to H,$$

$$(1.5.1.Q) \quad [\cdot,\cdot]_Q: Q \otimes_R Q \to Q,$$

$$(1.5.2.H) \quad H \otimes_R A \to A, \quad x \otimes_R a \mapsto x(a), \quad x \in H, \ a \in A,$$

$$(1.5.2.Q) \quad Q \otimes_R A \to A, \quad \xi \otimes_R a \mapsto \xi(a), \quad \xi \in Q, \ a \in A,$$

$$(1.5.3) \quad \cdot: H \otimes_R Q \to Q,$$

$$(1.5.4) \quad \cdot: Q \otimes_R H \to H,$$

$$(1.5.5) \quad \delta: Q \otimes_A Q \to H.$$

We will say that the data $(A,H,Q)$ constitute an almost pre-Lie-Rinehart triple provided they satisfy (i), (ii), and (iii) below.

(i) The values of the adjoints $H \to \text{End}_R(A)$ and $Q \to \text{End}_R(A)$ of (1.5.2.H) and (1.5.2.Q) respectively lie in $\text{Der}_R(A)$;

(ii) (1.5.1.H), (1.5.2.H) and the $A$-module structure on $H$ and, likewise, (1.5.1.Q), (1.5.2.Q) and the $A$-module structure on $Q$, satisfy the following Lie-Rinehart axioms (1.5.6.H), (1.5.7.H), (1.5.6.Q), (1.5.7.Q):

$$(1.5.6.H) \quad (ax)(b) = a(x(b)), \quad a,b \in A, \ x \in H,$$

$$(1.5.7.H) \quad [x,ay]_H = x(a)y + a[x,y]_H, \quad a \in A, \ x,y \in H,$$

$$(1.5.6.Q) \quad (a\xi)(b) = a(\xi(b)), \quad a,b \in A, \ \xi \in Q,$$

$$(1.5.7.Q) \quad [\xi,a\eta]_Q = \xi(a)\eta + a[\xi,\eta]_Q, \quad a \in A, \ \xi,\eta \in Q;$$

(iii) (1.5.3) and (1.5.4) behave like connections, that is, for $a \in A, \ x \in H, \ \xi \in Q,$ the identities

$$(1.5.8) \quad x \cdot (a\xi) = (x(a))\xi + a(x \cdot \xi),$$

$$(1.5.9) \quad (ax) \cdot \xi = a(x \cdot \xi),$$

$$(1.5.10) \quad \xi \cdot (ax) = (\xi(a))x + a(\xi \cdot x),$$

$$(1.5.11) \quad (a\xi) \cdot x = a(\xi \cdot x),$$

are required to hold.

We will say that an almost pre-Lie-Rinehart triple $(A,H,Q)$ is a pre-Lie-Rinehart triple provided that (i) $(A,H)$, endowed with the operations (1.5.1.H) and (1.5.2.H), is a Lie-Rinehart algebra—equivalently, the bracket (1.5.1.H) satisfies the Jacobi identity—, and that (ii) the operation (1.5.3) turns $Q$ into a left $(A,H)$-module, that is, the “connection” given by this operation is “flat”, i.e. satisfies the identity

$$(1.5.12) \quad [x,y]_H \cdot \xi = x \cdot (y \cdot \xi) - y \cdot (x \cdot \xi), \quad x,y \in H, \ \xi \in Q.$$

$$(1.5.13) \quad \text{Thus a pre-Lie-Rinehart triple (A,H,Q) consists of a Lie-Rinehart algebra (A,H) (the structure of which is given by (1.5.1.H), (1.5.2.H)) and a left (A,H)-module Q (given by the operation (1.5.3) which, in turn, is required to satisfy the}$$
axioms (1.5.8) and (1.5.9)) together with the additional structure (1.5.1.\(Q\)), (1.5.2.\(Q\)), (1.5.4), (1.5.5) subject to the axioms (1.5.6.\(Q\)), (1.5.7.\(Q\)), (1.5.10), (1.5.11).

Given an almost pre-Lie-Rinehart triple \((A,H,Q)\), let \(L = H \oplus Q\) be the \(A\)-module direct sum, and define an \(R\)-bilinear skew-symmetric bracket

\[
(1.6.1) \quad [\cdot,\cdot]: L \otimes_R L \rightarrow L
\]

by means of the formula

\[
(1.6.2) \quad [(x,\xi),(y,\eta)] = [x,y]_H + [\xi,\eta]_Q + \delta(\xi,\eta) + x \cdot \eta - \eta \cdot x + \xi \cdot y - y \cdot \xi
\]

and, furthermore, an operation

\[
(1.6.3) \quad L \otimes_R A \rightarrow A
\]

in the obvious way, that is, by means of the association

\[
(1.6.4) \quad (\xi,x) \otimes_R a \mapsto \xi(a) + x(a), \quad x \in H, \xi \in Q, a \in A.
\]

By construction, the values of the adjoint of (1.6.3) then lie in \(\text{Der}_R(A)\), that is, this adjoint is then of the form

\[
(1.6.5) \quad L = H \oplus Q \rightarrow \text{Der}_R(A).
\]

An almost pre-Lie-Rinehart triple \((A,H,Q)\) will be said to be a Lie-Rinehart triple if (1.6.1) and (1.6.3) turn \((A,L)\) where \(L = H \oplus Q\) into a Lie-Rinehart algebra. A Lie-Rinehart triple \((A,H,Q)\) where \(\delta\) is zero is a twilled Lie-Rinehart algebra \([21, 22]\). Thus Lie-Rinehart triples generalize twilled Lie-Rinehart algebras.

A direct sum decomposition \(L = H \oplus Q\) of an \((R,A)\)-Lie algebra \(L\) such that \((A,H)\) inherits a Lie-Rinehart structure yields a Lie-Rinehart triple \((A,H,Q)\) in an obvious fashion: The brackets (1.5.1.\(H\)) and (1.5.1.\(Q\)) result from restriction and projection; the operations (1.5.2.\(H\)) and (1.5.2.\(Q\)) are obtained by restriction as well; further, the requisite operations (1.5.3) and (1.5.4) are given by the composites

\[
(1.7.1) \quad \cdot: H \otimes_R Q \xrightarrow{[\cdot,\cdot]_{H \otimes_R Q}} H \oplus Q \xrightarrow{\text{pr}_Q} Q
\]

and

\[
(1.7.2) \quad \cdot: Q \otimes_R H \xrightarrow{[\cdot,\cdot]_{Q \otimes_R H}} H \oplus Q \xrightarrow{\text{pr}_H} H
\]

where, for \(M = H \otimes_R Q\) and \(M = Q \otimes_R H\), \([\cdot,\cdot]_M\) denotes the restriction of the Lie bracket to \(M\). The pairing (1.5.5) is the composite

\[
(1.7.3) \quad \delta: Q \otimes_A Q \xrightarrow{[\cdot,\cdot]_{Q \otimes_R Q}} L = H \oplus Q \xrightarrow{\text{pr}_H} H;\]

at first it is only \(R\)-bilinear but is readily seen to be \(A\)-bilinear. The formula (1.6.2) is then merely a decomposition of the initially given bracket on \(L\) into components.
according to the direct sum decomposition of $L$ into $H$ and $Q$, and (1.6.3) is accordingly a decomposition of the $L$-action on $A$. Furthermore, given $x, y \in H$ and $\xi \in Q$, in $L$ we have the identity

$$[x, y] \cdot \xi - \xi \cdot [x, y] = [[x, y], \xi] = [x, [y, \xi]] - [y, [x, \xi]]$$

$$= x \cdot (y \cdot \xi) - (y \cdot \xi) \cdot x - [x, \xi \cdot y]$$

$$- y \cdot (x \cdot \xi) + (x \cdot \xi) \cdot y - [\xi \cdot x, y]$$

which at once implies (1.5.12).

**Remark 1.8.1.** Thus we see that, in particular, if an almost pre-Lie-Rinehart triple $(A, H, Q)$ is a Lie-Rinehart triple, it is necessarily a pre-Lie-Rinehart triple, cf. (1.5.13).

**Remark 1.8.2.** In the situation of Example 1.3.2, when $g$ arises from a quasi-Lie bialgebra (so that $q = h^*$), in the literature, the piece of structure $\delta$ is often written as an element of $A^3 h$.

**Theorem 1.9.** A pre-Lie-Rinehart triple $(A, H, Q)$ is a genuine Lie-Rinehart triple, that is, the bracket $[\cdot, \cdot]$, cf. (1.6.1), and the operation (1.6.3) turn $(A, L)$ where $L = H \oplus Q$ into a Lie-Rinehart algebra, if and only if the brackets $[\cdot, \cdot]_H$ and $[\cdot, \cdot]_Q$ on $H$ and $Q$, respectively, and the operations (1.5.3), (1.5.4), and (1.5.5), are related by

(1.9.1) \[ \xi(x(a)) - x(\xi(a)) = (\xi \cdot x)(a) - (x \cdot \xi)(a) \]

(1.9.2) \[ x \cdot [\xi, \eta]_Q = [x \cdot \xi, \eta]_Q + [\xi, x \cdot \eta]_Q - (\xi \cdot x) \cdot \eta + (\eta \cdot x) \cdot \xi \]

(1.9.3) \[ \xi \cdot [x, y]_H = [\xi \cdot x, y]_H + [x, \xi \cdot y]_H - (x \cdot \xi) \cdot y + (y \cdot \xi) \cdot x, \]

(1.9.4) \[ \xi(\eta(a)) - (\eta(a)) = [\eta, \xi]_Q(a) + (\delta(\xi, \eta))(a) \]

(1.9.5) \[ [\xi, \eta]_Q \cdot x = \xi \cdot (\eta \cdot x) - \eta \cdot (\xi \cdot x) - \delta(x \cdot \xi, \eta) - \delta(\xi, x \cdot \eta) + [x, \delta(\xi, \eta)]_H \]

(1.9.6) \[ \sum_{(\xi, \eta, \vartheta) \text{ cyclic}} (\xi, \eta, \vartheta)_Q = \sum_{(\xi, \eta, \vartheta) \text{ cyclic}} \xi \cdot \delta(\eta, \vartheta) \]

where $a \in A$, $x, y \in H$, $\xi, \eta, \vartheta \in Q$.

Recall [18] that, given a commutative algebra $A$ and Lie-Rinehart algebras $(A, L')$, $(A, L)$ and $(A, L'')$ where $L'$ is an ordinary $A$-Lie algebra, an extension of Lie-Rinehart algebras

$$0 \rightarrow L' \rightarrow L \rightarrow L'' \rightarrow 0$$

is an extension of $A$-modules which is also an extension of ordinary Lie algebras so that the projection from $L$ to $L''$ is a morphism of Lie-Rinehart algebras. Theorem 1.9 entails at once the following.

**Corollary 1.9.8.** Given a Lie-Rinehart triple $(A, H, Q)$, the left $(A, H)$-module structures (1.5.2.1) on $A$ and (1.5.3) on $Q$ are trivial if and only if $(A, Q)$ is a Lie-Rinehart algebra in such a way that the projection from $E = H \oplus Q$ to $Q$ fits into an extension

$$0 \rightarrow H \rightarrow E \rightarrow Q \rightarrow 0$$
Thus Lie-Rinehart triples \((A, H, Q)\) having trivial left \((A, H)\)-module structures on \(A\) and \(Q\) and extensions of Lie-Rinehart algebras of the kind \((A, L)\) together with an \(A\)-module section of the projection map are equivalent notions.

**Proof of Theorem** (1.9). The bracket (1.6.1) is plainly skew-symmetric. Hence the proof comes down to relating the Jacobi identity in \(L\) and the Lie-Rinehart compatibility properties with (1.9.1)–(1.9.7).

Thus, suppose that the bracket \([\cdot, \cdot]\) on \(L = H \oplus Q\) given by (1.6.1) and the operation \(L \otimes_A A \to A\) given by (1.6.3) turn \((A, L)\) into a Lie-Rinehart algebra. Given \(\xi \in Q\) and \(x \in H\), we have \([\xi, x] = \xi \cdot x - x \cdot \xi\); since \(L\) acts on \(A\) by derivations, for \(a \in A\), we conclude

\[
\xi(x(a)) - x(\xi(a)) = [\xi, x](a) = (\xi \cdot x)(a) - (x \cdot \xi)(a),
\]

that is, (1.9.1) holds. Likewise, given \(\xi, \eta \in Q\), \([\xi, \eta] = [\xi, \eta]_Q + \delta(\xi, \eta) \in L\) whence, for \(a \in A\),

\[
\xi(\eta(a)) - \eta(\xi(a)) = [\xi, \eta](a) = [\xi, \eta]_Q(a) + (\delta(\xi, \eta))(a),
\]

that is, (1.9.4) holds. Next, since \(L\) is a Lie algebra, its bracket satisfies the Jacobi identity. Hence, given \(x \in H\) and \(\xi, \eta \in Q\),

\[
x \cdot [\xi, \eta]_Q - [\xi, \eta]_Q \cdot x = [x, [\xi, \eta]_Q] = [x, [\xi, \eta]] - [x, \delta(\xi, \eta)]
\]

\[
= [[x, \xi], \eta] + [\xi, [x, \eta]] - [x, \delta(\xi, \eta)]
\]

\[
= [x \cdot \xi - \xi \cdot x, \eta] + [\xi, x \cdot \eta - \eta \cdot x] - [x, \delta(\xi, \eta)]_H
\]

\[
= [x \cdot \xi, \eta] + [\xi, x \cdot \eta] - [\xi, \eta \cdot x] - [x, \delta(\xi, \eta)]_H
\]

\[
= [x \cdot \xi, \eta] + [\xi, x \cdot \eta] - (\xi \cdot x) \cdot \eta + \eta \cdot (\xi \cdot x) + (\eta \cdot x) \cdot \xi - \xi \cdot (\eta \cdot x)
\]

\[
- [x, \delta(\xi, \eta)]_H
\]

whence, comparing components in \(H\) and \(Q\), we conclude

\[
x \cdot [\xi, \eta]_Q = [x \cdot \xi, \eta]_Q + [\xi, x \cdot \eta]_Q - (\xi \cdot x) \cdot \eta + (\eta \cdot x) \cdot \xi
\]

\[
[\xi, \eta]_Q \cdot x = \xi \cdot (\eta \cdot x) - \eta \cdot (\xi \cdot x) - \delta(x \cdot \xi, \eta) - \delta(\xi, x \cdot \eta) + [x, \delta(\xi, \eta)]_H
\]

that is, (1.9.2) and (1.9.5) hold.
Likewise given $\xi \in Q$ and $x, y \in H$,
$$
\xi \cdot [x, y]_H - [x, y]_H \cdot \xi = [\xi, [x, y]_H]
$$
$$
= [[\xi, x], y] + [x, [\xi, y]]
$$
$$
= [\xi \cdot x - x \cdot \xi, y] + [x, \xi \cdot y - y \cdot \xi]
$$
$$
= [\xi \cdot x, y] - [x \cdot \xi, y] + [x, \xi \cdot y] - [x, y \cdot \xi]
$$
$$
= [\xi \cdot x, y] + [x, \xi \cdot y]
$$
whence, comparing components in $Q$ and $H$, we conclude
$$
\xi \cdot [x, y]_H = [\xi \cdot x, y]_H + [x, \xi \cdot y]_H - (x \cdot \xi) \cdot y + (y \cdot \xi) \cdot x
$$
$$
[x, y]_H \cdot \xi = x \cdot (y \cdot \xi) - y \cdot (x \cdot \xi)
$$
that is, (1.9.3) and (1.5.12) hold; notice that (1.5.12) holds already by assumption.

Next, given $\xi, \eta, \vartheta \in Q$,
$$
[[\xi, \eta], \vartheta] = [[\xi, \eta]_H, \vartheta] + [[\xi, \eta]_Q, \vartheta]
$$
$$
= [\delta(\xi, \eta), \vartheta] + [[\xi, \eta]_Q, \vartheta]
$$
$$
= [\delta(\xi, \eta), \vartheta]_H + [\delta(\xi, \eta), \vartheta]_Q + [[\xi, \eta]_Q, \vartheta]_H + [[\xi, \eta]_Q, \vartheta]_Q
$$
$$
= (\delta(\xi, \eta)) \cdot \vartheta - \vartheta \cdot \delta(\xi, \eta) + \delta([\xi, \eta]_Q, \vartheta) + [[\xi, \eta]_Q, \vartheta]_Q
$$
Hence
$$
[[\xi, \eta], \vartheta] + [[\eta, \vartheta], \xi] + [[\vartheta, \xi], \eta] = (\delta(\xi, \eta)) \cdot \vartheta + (\delta(\eta, \vartheta)) \cdot \xi + (\delta(\vartheta, \xi)) \cdot \eta
$$
$$
+ [[\xi, \eta]_Q, \vartheta]_Q + [[\eta, \vartheta]_Q, \xi]_Q + [[\vartheta, \xi]_Q, \eta]_Q
$$
$$
- \xi \cdot \delta(\eta, \vartheta) - \eta \cdot \delta(\vartheta, \xi) - \vartheta \cdot \delta(\xi, \eta)
$$
$$
+ \delta([\xi, \eta]_Q, \vartheta) + \delta([\eta, \vartheta]_Q, \xi) + \delta([\vartheta, \xi]_Q, \eta)
$$
Thus the Jacobi identity implies
$$
[[\xi, \eta]_Q, \vartheta]_Q + [[\eta, \vartheta]_Q, \xi]_Q + [[\vartheta, \xi]_Q, \eta]_Q + (\delta(\xi, \eta)) \cdot \vartheta + (\delta(\eta, \vartheta)) \cdot \xi + (\delta(\vartheta, \xi)) \cdot \eta = 0
$$
$$
\delta([\xi, \eta]_Q, \vartheta) + \delta([\eta, \vartheta]_Q, \xi) + \delta([\vartheta, \xi]_Q, \eta) - \xi \cdot \delta(\eta, \vartheta) - \eta \cdot \delta(\vartheta, \xi) - \vartheta \cdot \delta(\xi, \eta) = 0,
$$
that is, (1.9.6) and (1.9.7) are satisfied.

Conversely, suppose that the brackets $[\cdot, \cdot]_H$ and $[\cdot, \cdot]_Q$ on $H$ and $Q$, respectively, and the operations (1.5.3), (1.5.4), and (1.5.5), are related by (1.9.1)–(1.9.7). We can then read the above calculations backwards and conclude that the bracket (1.6.1) on $L$ satisfies the Jacobi identity and that the operation (1.6.3) yields a Lie algebra action of $L$ on $A$ by derivations. The remaining Lie-Rinehart algebra axioms hold by assumption. Thus $(A, L)$ is then a Lie-Rinehart algebra. □

**Remark 1.10.** Under the circumstances of Example 1.3, the requirements (1.5.6.Q), (1.5.6.H), (1.5.7.Q), (1.5.7.H), (1.5.8)–(1.5.11) are vacuous, and so are (1.9.1) and (1.9.4) as well.

Given an $(R, A)$ Lie algebra $L$ and an $(R, A)$ Lie subalgebra $H$, the invariants $A^H \subseteq A$ constitute a subalgebra of $A$; we will then denote the normalizer of $H$ in $L$ in the sense of Lie algebras by $L_H$, that is, $L_H$ consists of all $\alpha \in L$ having the property that $[\alpha, \beta] \in H$ whenever $\beta \in H$. 

Corollary 1.11. Given a Lie-Rinehart triple \((A,H,Q)\), the corresponding \((R,A)\)-Lie algebra being written as \(L = H \oplus Q\), the intersection \(Q \cap L_H\) coincides with the invariants \(Q_H^H\) under the \(H\)-action on \(Q\) (given by the corresponding operation (1.5.3)), the pair \((A^H,Q^H)\) acquires a Lie-Rinehart algebra structure, and the projection from \(L_H\) to \(Q^H\) fits into an extension

\[
0 \rightarrow H \rightarrow L_H \rightarrow Q^H \rightarrow 0
\]

of \((R,A^H)\)-Lie algebras. Furthermore, the restriction of \(\delta\) to \(Q^H\) is a cocycle for this extension, that is, it yields the curvature of the connection for the extension determined by the \(A^H\)-module direct sum decomposition \(L_H = H \oplus Q^H\).

Notice that \(H\) is here viewed as an ordinary \(A^H\)-Lie algebra, the \(H\)-action on \(A^H\) being trivial by construction.

**Proof.** Indeed, given \(\alpha \in Q\) and \(\beta \in H\),

\[
[\alpha, \beta] = \alpha \cdot \beta - \beta \cdot \alpha \in L
\]

whence \([\alpha, \beta] \in H\) for every \(\beta \in H\) if and only if \(\beta \cdot \alpha = 0 \in Q\) for every \(\beta \in H\), that is, if and only if \(\alpha\) is invariant under the \(H\)-action on \(Q\). The rest of the claim is an immediate consequence of Theorem 1.9. \(\square\)

1.12. ILLUSTRATION. Under the circumstances of Example 1.4.1, Corollary 1.11 obtains, with \(H = L_F\). Now \(A^H = A^{L_F} \subseteq A\) is the algebra of smooth functions which are constant on the leaves, that is, the algebra of functions on the “space of leaves”, and \(L_H\) consists of the vector fields which “project” to the “space of leaves”. Indeed, given a function \(f\) which is constant on the leaves and vector fields \(X \in L_H\) and \(Y \in L_F\), necessarily \(Y(Xf) = [Y,X]f + X(Yf) = 0\) whence \(Xf\) is constant on the leaves as well. Thus we may view \(Q^H\) as the Lie algebra of vector fields on the “space of leaves”, that is, as the space of sections of a certain geometric object which serves as a replacement for the in general non-existant tangent bundle of the “space of leaves”.

**Remark 1.13.** In analogy to the deformation theory of complex manifolds, given a Lie-Rinehart triple \((A,H,Q)\), we may view \(H\) and \(Q\) as what corresponds to the antiholomorphic and holomorphic tangent bundle, respectively, and accordingly study deformations of the Lie-Rinehart triple via morphisms \(\vartheta: H \rightarrow Q\) and spell out the resulting infinitesimal obstructions. This will include a theory of deformations of foliations. Details will be given elsewhere.
2. Lie-Rinehart triples and Maurer-Cartan algebras

In this section we will explore the relationship between Lie-Rinehart triples and suitably defined Maurer-Cartan algebras. In particular, we will show that, under an additional assumption, the two notions are equivalent; see Theorem 2.8.3 below for details. As an application we will explain how the spectral sequence of a foliation and the Hodge-de Rham spectral sequence arise as special cases of a single conceptually simple construction. More applications will be given in subsequent sections.

2.1. Maurer-Cartan algebras. Given an $A$-module $L$ and an $R$-derivation $d$ of degree $-1$ on the graded $A$-algebra $\text{Alt}_A(L,A)$, we will refer to $(\text{Alt}_A(L,A),d)$ as a Maurer-Cartan algebra (over $L$) provided $d$ has square zero, i.e. is a differential.

Recall that a multicomplex (over $R$) is a bigraded $R$-module $\{M^{p,q}\}_{p,q}$ together with an operator $d_j:M^{p,q} \to M^{p+j,q-j+1}$ for every $j \geq 0$ such that the sum $d = d_0 + d_1 + \ldots$ is a differential, i.e. $dd = 0$, cf. [43], [44]. The idea of multicomplex occurs already in [15] and was exploited at various places in the literature including [29], [30]. We note that an infinite sequence of the kind $(d_2, d_3, \ldots)$ is a system of higher homotopies. We will refer to a multicomplex $(M; d_0, d_1, d_2, \ldots)$ whose underlying bigraded object $M$ is endowed with a bigraded algebra structure such that the operators $d_j$ are derivations with respect to this algebra structure as a multi $R$-algebra.

Given $A$-modules $H$ and $Q$, consider the bigraded $A$-algebra $(\text{Alt}_A(Q, \text{Alt}_A(H,A)))$; we will refer to a multi $R$-algebra structure (beware: not multi $A$-algebra structure) on this bigraded $A$-algebra having at most $d_0, d_1, d_2$ non-zero as a Maurer-Cartan algebra structure. The resulting multi $R$-algebra will then be written as

\[(\text{Alt}_A(Q, \text{Alt}_A(H,A)); d_0, d_1, d_2)\]

and referred to as a (multi) Maurer-Cartan algebra (over $(Q,H)$). Usually we will discard “multi” and more simply refer to a Maurer-Cartan algebra. We note that, for degree reasons, when (2.1.1) is a Maurer-Cartan algebra, the operator $d_2$ is necessarily an $A$-derivation (since $d_2(a) = 0$ for every $a \in A \cong \text{Alt}_A^0(Q, \text{Alt}_A^0(H,A)))$.

Remark 2.1.2. In this definition, we could allow for non-zero derivations of the kind $d_j$ for $j \geq 3$ as well. This would lead to a more general notion of multi Maurer-Cartan algebra not studied here. The presence of a non-zero operator at most of the kind $d_2$ is an instance of a higher homotopy of a special kind which suffices to explain the “quasi” structures explored later in the paper.

Remark 2.1.3. Given a (multi) Maurer-Cartan algebra of the kind (2.1.1), the sum $d = d_0 + d_1 + d_2$ turns $\text{Alt}_A(Q \oplus H,A)$ into a Maurer-Cartan algebra. However, not every Maurer-Cartan structure on $\text{Alt}_A(Q \oplus H,A)$ arises in this fashion, that is, a multi Maurer-Cartan algebra structure captures additional structure of interaction between $A$, $Q$, and $H$, indeed, it captures essentially a Lie-Rinehart triple structure. The purpose of the present section is to make this precise.

For later reference, we spell out the following, the proof of which is immediate.

Proposition 2.1.4. Given the three derivations $d_0, d_1, d_2$,

\[(\text{Alt}_A^*(Q, \text{Alt}_A^*(H,A)); d_0, d_1, d_2)\]
is a (multi) Maurer-Cartan algebra if and only if the following identities are satisfied.

(2.1.4.1) \quad d_0 d_0 = 0
(2.1.4.2) \quad d_0 d_1 + d_1 d_0 = 0
(2.1.4.3) \quad d_0 d_2 + d_1 d_1 + d_2 d_0 = 0
(2.1.4.4) \quad d_1 d_2 + d_2 d_1 = 0
(2.1.4.5) \quad d_2 d_2 = 0. \quad \Box

2.2. Lie-Rinehart and Maurer-Cartan algebras. Let $A$ be a commutative $R$-algebra and $L$ an $A$-module, together with a skew-symmetric $R$-bilinear bracket

(2.2.1) \quad [\cdot,\cdot]_L : L \otimes_R L \to L

and an operation

(2.2.2) \quad L \otimes_R A \to A, \quad x \otimes_R a \mapsto x(a), \quad x \in H, \quad a \in A

such that the values of the adjoint $L \to \text{End}_R(A)$ lie in $\text{Der}_R(A)$ and that (2.2.1), (2.2.2) and the $A$-module structure on $L$ satisfy the Lie-Rinehart axioms (2.2.3) and (2.2.4) below:

(2.2.3) \quad (ax)(b) = a(x(b)), \quad a, b \in A, \quad x \in L,
(2.2.4) \quad [x, ay]_L = x(a)y + a[x, y]_L, \quad a \in A, \quad x, y \in L.

Let $M$ be a graded $A$-module, together with an operation

(2.2.5) \quad L \otimes_R M \to M, \quad x \otimes m \mapsto x(m), \quad x \in L, \quad m \in m

subject to the following requirement: For $\alpha \in L$, $a \in A$, $m \in M$

(2.2.6) \quad (a \alpha)(m) = a(\alpha(m)),
(2.2.7) \quad \alpha(a m) = a \alpha(m) + \alpha(a) m.

We refer to an operation of the kind (2.2.5) as a generalized $L$-connection on $M$. Under these circumstances, the ordinary CARTAN-CHEVALLEY-EILENBERG (CCE) operator $d$ is defined, at first on the bigraded object $\text{Alt}_R(L, M)$ of $M$-valued $R$-multilinear alternating forms on $L$. Indeed, given an $R$-multilinear alternating function $f$ on $L$ of $n-1$ variables which is homogeneous, i.e. the values of $f$ lie in a homogeneous constituent of $M$, the Cartan-Chevalley-Eilenberg (CCE) formula yields

\begin{equation}
(-1)^{|f|+1}(df)(\alpha_1, \ldots, \alpha_n) = \sum_{i=1}^n (-1)^{(i-1)} \alpha_i (f(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_n)) + \sum_{1 \leq j < k \leq n} (-1)^{(j+k)} f([\alpha_j, \alpha_k], \alpha_1, \ldots, \hat{\alpha}_j \ldots, \hat{\alpha}_k \ldots, \alpha_n),
\end{equation}

(2.2.8)
where \(\alpha_1, \ldots, \alpha_n \in L\) and where as usual \(\cdots\) indicates omission of the corresponding term. We note that, when the values of the homogeneous alternating function \(f\) on \(L\) of \(n - 1\) variables lie in \(M_q, |f| = q - n + 1\). Here and below our convention is that, given graded objects \(N\) and \(M\), a homogeneous morphism \(h: N_p \to M_q\) has degree \(|h| = q - p\). This is the standard grading on the Hom-functor for graded objects. The requirements (2.2.3), (2.2.4), (2.2.6), and (2.2.7) entail that the operator \(d\) of (2.2.9) entails that the operator \(d\) on \(\text{Alt}_R(L, M)\) passes to an \(R\)-linear operator on the (bi)graded \(A\)-submodule \(\text{Alt}_A(L, M)\) of \(A\)-multilinear functions, written here and henceforth as

\[
d: \text{Alt}_A(L, M) \to \text{Alt}_A(L, M)
\]

as well. The sign \((-1)^{|f|+1}\) in (2.2.8) is the appropriate one according to the customary Eilenberg-Koszul convention in differential homological algebra since (2.2.9) involves graded objects. For \(M = A\), the operator \(d\) is plainly a derivation on \(\text{Alt}_A(L, A)\).

Let \(M_1\) and \(M_2\) be graded \(A\)-modules endowed with generalized \(L\)-connections of the kind (2.2.5), and let

\[
\langle \cdot, \cdot \rangle: M_1 \otimes_A M_2 \to M
\]

be an \(A\)-module pairing which is compatible with the generalized \(L\)-connections in the sense that

\[
x(\langle m_1, m_2 \rangle) = (x(m_1), m_2) + (m_1, x(m_2)), \quad x \in L, \quad m_1 \in M_1, \quad m_2 \in M_2.
\]

This pairing induces a (bi)graded pairing

\[
(2.2.10) \quad \text{Alt}_A(L, M_1) \otimes_R \text{Alt}_A(L, M_2) \to \text{Alt}_A(L, M)
\]

which is compatible with the generalized CCE operators.

An \(A\)-module \(M\) will be said to have property \(P\) provided for \(x \in M, \phi(x) = 0\) for every \(\phi: M \to A\) implies that \(x\) is zero. For example, a projective \(A\)-module has property \(P\), or a reflexive \(A\)-module has this property as well or, more generally, any \(A\)-module \(M\) such that the canonical map from \(M\) into its double \(A\)-dual is injective. On the other hand, for example, for a smooth manifold \(X\), the \(C^\infty(X)\)-module \(D\) of formal (\(=\) Kähler) differentials does not have property \(P\): On the real line, with coordinate \(x\), consider the functions \(f(x) = \sin x\) and \(g(x) = \cos x\). The formal differential \(df - gdx\) is non-zero in \(D\); however, the \(C^\infty(X)\)-linear maps from \(D\) to \(C^\infty(X)\) are the smooth vector fields, whence every such \(C^\infty(X)\)-linear map annihilates the formal differential \(df - gdx\).

**Lemma 2.2.11.** When \(L\) has the property \(P\), the pair \((A, L)\), endowed with the bracket \([\cdot, \cdot]_L\) (cf. (2.2.1)) and operation (2.2.2) is a Lie-Rinehart algebra, that is, the bracket \([x, y]_L\) satisfies the Jacobi identity and the adjoint of (2.2.2) is a morphism of \(R\)-Lie algebras, if and only if \((\text{Alt}_A(L, A), d)\) is a Maurer-Cartan algebra.

**Proof.** A familiar calculation shows that \(d\) is a differential if and only if the bracket \([x, y]_L\) satisfies the Jacobi identity and if the adjoint of (2.2.2) is a morphism of \(R\)-Lie algebras. Cf. also 2.8.5(i) below. \(\square\)

**Example 2.2.12.** The Lie algebra \(L\) of derivations of a polynomial algebra \(A\) in infinitely many indeterminates (over a field) has property \(P\) as an \(A\)-module but
is not a projective $A$-module. To include this kind of example and others, it is necessary to build up the theory for modules having property P rather than just projective ones or even finitely generated projective modules.

Let now $(A, L)$ be an (ungraded) Lie-Rinehart algebra, and let $(\text{Alt}_A(L, A), d)$ be the corresponding Maurer-Cartan algebra; notice that the operator $d$ is not $A$-linear unless $L$ acts trivially on $A$. For reasons explained in [23] we will refer to this operator as Lie-Rinehart differential. We will say that the graded $A$-module $M$, endowed with the operation (2.2.5), is a graded (left) $(A, L)$-module provided this operation is an ordinary Lie algebra action on $M$. When $M$ is concentrated in degree zero, we simply refer to $M$ as a (left) $(A, L)$-module. In particular, with the obvious $L$-module structure, the algebra $A$ itself is a (left) $(A, L)$-module. The proof of the following is straightforward and left to the reader.

**Lemma 2.2.13.** When $(A, L)$ is a Lie-Rinehart algebra and when $M$ has the property $P$, the operation (2.2.5) turns $M$ into a left $(A, L)$-module if and only if the operator $d$ on $\text{Alt}_A(L, M)$ turns $(\text{Alt}_A(L, M), d)$ into a differential graded $(\text{Alt}_A(L, A), d)$-module via (2.2.10) (with $M_1 = A$ and $M_2 = M$). □

Given a graded $(A, L)$-module $M$, we will refer to the resulting (co)chain complex

$$\text{(2.2.14)} \quad (\text{Alt}_A(L, M), d)$$

as the Rinehart complex of $M$-valued forms on $L$; often we write this complex more simply in the form $\text{Alt}_A(L, M)$. It inherits a differential graded $\text{Alt}_A(L, A)$-module structure via (2.2.10).

We now spell out the passage from Maurer-Cartan algebras to Lie-Rinehart algebras.

**Lemma 2.2.15.** Let $L$ be a finitely generated projective $A$-module. Then an $R$-derivation $d$ on the graded $A$-algebra $\text{Alt}_A(L, A)$ determines a skew-symmetric $R$-bilinear bracket $[\cdot, \cdot]_L$ on $L$ of the kind (2.2.1) and an operation $L \otimes R A \to A$ of the kind (2.2.2) such that the identities (2.2.3) and (2.2.4) are satisfied and that the corresponding CCE operator (2.2.8) (for $M = A$) coincides with $d$. Furthermore, $\text{Alt}_A(L, A)$ is then a Maurer-Cartan algebra if and only if $(A, L)$ is a Lie-Rinehart algebra.

**Proof.** The operator

$$d: \text{Alt}^q_A(L, A) \to \text{Alt}^{q+1}_A(L, A) \quad (q \geq 0)$$

induces, for $q = 0$, an operation $L \otimes_R A \to A$ of the kind (2.2.2) and, for $q = 1$, a skew-symmetric $R$-bilinear bracket $[\cdot, \cdot]_L$ on $L$ of the kind (2.2.1). More precisely: Given $x \in L$ and $a \in A$, let

$$x(a) = -(d(a))(x).$$

This yields an operation of the kind (2.2.2). Given $x, y \in L$, using the hypothesis that $L$ is a finitely generated projective $A$-module, identify $x$ and $y$ with their images in the double $A$-dual $L^{**}$ and define the value $[x, y]_L$ by

$$[x, y]_L(\alpha) = x(\alpha(y)) - y(\alpha(x)) - d\alpha(x, y)$$

...
where \( \alpha \in L^* = \text{Hom}_A(L, A) \). This yields a bracket of the kind (2.2.1), that is, an \( R \)-bilinear (beware: not \( A \)-bilinear) skew-symmetric bracket on \( L \). Notice that, at this stage, the operation of the kind (2.2.2) is already defined whence the definition of the bracket makes sense. Since, by assumption, \( d \) is a derivation on \( \text{Alt}_A(L, A) \), the identities (2.2.3) and (2.2.4) are satisfied. By construction, the resulting CCE operator coincides with \( d \) in degree 0 and in degree \(-1\) whence the two operators coincide. Since a finitely generated projective \( A \)-module has property P, Lemma 2.2.11 completes the proof. \( \square \)

Combining Lemma 2.2.13 and 2.2.15, we arrive at the following.

**Theorem 2.2.16.** Given a finitely generated projective \( A \)-module \( L \), Lie-Rinehart algebra structures on \((A, L)\) and Maurer-Cartan algebra structures on \( \text{Alt}_A(L, A) \) are equivalent notions. \( \square \)

2.3. **Connections.** Let \((A, L)\) be a Lie-Rinehart algebra. Given a graded \( A \)-module \( M \), a degree zero operation \( L \otimes_R M \to M \), not necessarily a graded left \( L \)-module structure but still satisfying (2.2.6) and (2.2.7), is referred to as an \((A, L)\)-connection, cf. [16, 19] or, somewhat more precisely, as a graded left \((A, L)\)-connection; in this language, a (graded) \((A, L)\)-module structure is a (graded) flat \((A, L)\)-connection. Given a graded \( A \)-module \( M \), together with a graded \((A, L)\)-connection, we extend the definition of the Lie-Rinehart operator to an operator

\[
(2.3.1) \quad d : \text{Alt}_A(L, M) \to \text{Alt}_A(L, M)
\]

by means of the formula (2.2.8). The resulting operator \( d \) is well defined; it is a differential if and only if the \((A, L)\)-connection on \( M \) is flat, i.e. an ordinary \((A, L)\)-module structure.

2.4. **From Lie-Rinehart Triples to Maurer-Cartan Algebras.** Let \((A, H, Q)\) be an almost pre-Lie-Rinehart triple. Consider the bigraded \( A \)-module

\[
(2.4.1) \quad \text{Alt}_A^{*, *}(Q \oplus H, A) \cong \text{Alt}_A^*(Q, \text{Alt}_A^*(H, A)).
\]

Henceforth we spell out a particular homogeneous constituent of bidegree \((p, q)\) (according to the conventions used below, such a homogenous constituent will be of bidegree \((-p, -q)\) but for the moment this usage of negative degrees is of no account) in the form

\[
(2.4.2) \quad \text{Alt}_A^p(Q, \text{Alt}_A^q(H, A)).
\]

The operations (1.5.3) and (1.5.4) induce degree zero operations

\[
(2.4.3) \quad H \otimes_R \text{Alt}_A^*(Q, A) \to \text{Alt}_A^*(Q, A)
\]

\[
(2.4.4) \quad Q \otimes_R \text{Alt}_A^*(H, A) \to \text{Alt}_A^*(H, A)
\]

on \( \text{Alt}_A^*(Q, A) \) and \( \text{Alt}_A^*(H, A) \), respectively, when (1.5.3) and (1.5.4) are treated like connections. By evaluation of the expression given on the right-hand side of (2.2.8),
with (1.5.1.H) and (1.5.1.Q) instead of (2.2.1), and with (2.5.1) and (2.5.2) instead of (2.2.5), these operations, in turn, induce two operators

\[(2.4.5) \quad d_0: \text{Alt}_A^p(Q, \text{Alt}_A^q(H, A)) \to \text{Alt}_A^p(Q, \text{Alt}_A^{q+1}(H, A))\]

\[(2.4.6) \quad d_1: \text{Alt}_A^p(Q, \text{Alt}_A^q(H, A)) \to \text{Alt}_A^{p+1}(Q, \text{Alt}_A^q(H, A)).\]

A little thought reveals that, in view of (1.5.6), these operations, in turn, induce two operators, which are at first defined only on the \(R\)-multilinear alternating functions, in fact pass to operators on \(A\)-multilinear alternating functions. Furthermore, the skew-symmetric \(A\)-bilinear pairing \(\delta\), cf. (1.5.5), induces an operator

\[(2.4.7) \quad d_2: \text{Alt}_A^p(Q, \text{Alt}_A^q(H, A)) \to \text{Alt}_A^{p+2}(Q, \text{Alt}_A^{q-1}(H, A)).\]

Hence, when \((A, H, Q)\) is a Lie-Rinehart triple, that is, when (1.6.1) and (1.6.3) turn \((A, H \oplus Q)\) into a Lie-Rinehart algebra, \((\text{Alt}_A(Q, \text{Alt}_A(H, A)); d_0, d_1, d_2)\) is a (multi) Maurer-Cartan algebra.

**2.5. Explicit description of the operators \(d_0, d_1, d_2\):** Let \(f\) be an alternating \(A\)-multilinear function on \(Q\) of \(p\) variables with values in \(\text{Alt}_A^q(H, A)\), so that \(|f| = -q - p\) and \((-1)^{|f|+1} = (-1)^{p+q+1}\). Let \(\xi_1, \ldots, \xi_{p+2} \in Q\) and \(x_1, \ldots, x_{q+1} \in H\).

The operator \(d_0\):

\[
(-1)^{p+q+1} ((d_0 f)(\xi_1, \ldots, \xi_p))(x_1, \ldots, x_{q+1}) = \\
\sum_{j=1}^{q+1} (-1)^{p+j-1} x_j ((f(\xi_1, \ldots, \xi_p))(x_1, \ldots, \hat{x}_j, \ldots, x_{q+1})) \\
+ \sum_{1 \leq j < k \leq q+1} (-1)^{p+j+k} (f(\xi_1, \ldots, \xi_p))( [x_j, x_k]_H, x_1, \ldots, \hat{x}_j \ldots \hat{x}_k, \ldots, x_{q+1}) \\
+ \sum_{j=1}^{p} \sum_{k=1}^{q+1} (-1)^{j+k+p+1} \left( f(x_k \cdot \xi_j, \xi_1, \ldots, \hat{\xi}_j, \ldots, \xi_p) \right)(x_1, \ldots, \hat{x}_k, \ldots, x_{q+1})
\]

The last term involving the double summation necessarily appears since, for \(1 \leq j \leq p\) and 
\(1 \leq k \leq q+1\), the bracket \([x_k, \xi_j]\) in \(Q \oplus H\), cf. (1.6.2), is given by

\([x_k, \xi_j] = x_k \cdot \xi_j - \xi_j \cdot x_k\).

**Remark 2.5.2.** A crucial observation is this: The operator \(d_0\) may be written as the sum

\[d_0 = d_H + d_Q\]

of certain operators \(d_H\) and \(d_Q\) defined on \(\text{Alt}_R(Q, \text{Alt}_R(H, A))\) by

\[
(-1)^{p+q+1} ((d_H f)(\xi_1, \ldots, \xi_p))(x_1, \ldots, x_{q+1}) = \\
\sum_{j=1}^{q+1} (-1)^{p+j-1} x_j ((f(\xi_1, \ldots, \xi_p))(x_1, \ldots, \hat{x}_j, \ldots, x_{q+1})) \\
+ \sum_{1 \leq j < k \leq q+1} (-1)^{j+k} (f(\xi_1, \ldots, \xi_p))( [x_j, x_k]_H, x_1, \ldots, \hat{x}_j \ldots \hat{x}_k, \ldots, x_{q+1})
\]

\[
(-1)^{p+q+1} ((d_Q f)(\xi_1, \ldots, \xi_p))(x_1, \ldots, x_{q+1}) = \\
\sum_{1 \leq j \leq p, 1 \leq k \leq q+1} (-1)^{j+k+p+1} \left( f(x_k \cdot \xi_j, \xi_1, \ldots, \hat{\xi}_j, \ldots, \xi_p) \right)(x_1, \ldots, \hat{x}_k, \ldots, x_{q+1}).
\]
However, even when \((A, H, Q)\) is a (pre-)Lie-Rinehart triple, the individual operators \(d_H\) and \(d_Q\) are well defined merely on \(\text{Alt}_R(Q, \text{Alt}_R(H, A))\); only their sum is well defined on \(\text{Alt}_A(Q, \text{Alt}_A(H, A))\).

The operator \(d_1\):

\[
(-1)^{p+q+1} ((d_1 f)(\xi_1, \ldots, \xi_{p+1}))(x_1, \ldots, x_q) = \\
\sum_{j=1}^{p+1} (-1)^{j-1} \xi_j \left( f(\xi_1, \ldots, \xi_j, \ldots, \xi_{p+1}) \right)(x_1, \ldots, x_q) \\
\sum_{1 \leq j < k \leq p} (-1)^{j+k} f([\xi_j, \xi_k]Q, \xi_1 \ldots \hat{\xi}_j \ldots \hat{\xi}_k \ldots \xi_{p+1})(x_1, \ldots, x_q) \\
\sum_{j=1}^{p+1} \sum_{k=1}^{q} (-1)^{j+k+1} f(\xi_1, \ldots, \hat{\xi}_j, \ldots, \xi_{p+1})(\xi_j, x_k, x_1, \ldots, \hat{x}_k, \ldots, x_q)
\]

(2.5.3)

The last term involving the double summation necessarily appears in view of (1.6.4).

With the generalized operation of Lie-derivative

\[ (\xi, \alpha) \mapsto \xi(\alpha), \quad \xi \in Q, \quad \alpha \in \text{Alt}_A^q(H, A) \quad (q \geq 0) \]

which, for \(x_1, \ldots, x_q \in H\), is given by

\[ (\xi(\alpha))(x_1, \ldots, x_q) = \xi(\alpha(x_1, \ldots, x_q)) - \sum_{k=1}^{q} \alpha(x_1, \ldots, x_{k-1}, \xi \cdot x_k, x_{k+1}, \ldots, x_q), \]

the identity (2.5.3) may be written as

\[
(-1)^{p+q+1} (d_1 f)(\xi_1, \ldots, \xi_{p+1}) = (-1)^{|f|+1} (d_1 f)(\xi_1, \ldots, \xi_{p+1}) \\
= \sum_{j=1}^{p+1} (-1)^{j-1} \xi_j \left( f(\xi_1, \ldots, \hat{\xi}_j, \ldots, \xi_{p+1}) \right) \\
+ \sum_{1 \leq j < k \leq p} (-1)^{j+k} f([\xi_j, \xi_k]Q, \xi_1 \ldots \hat{\xi}_j \ldots \hat{\xi}_k \ldots \xi_{p+1}).
\]

(2.5.3’)

The operator \(d_2\):

\[
(-1)^{p+q+1} ((d_2 f)(\xi_1, \ldots, \xi_{p+2}))(x_1, \ldots, x_{q-1}) = \\
\sum_{1 \leq j < k \leq p+2} (-1)^{j+k+p} f(\xi_1, \ldots, \hat{\xi}_j \ldots \hat{\xi}_k \ldots \xi_{p+2})(\delta(\xi_j, \xi_k), x_1, \ldots, x_{q-1})
\]

(2.5.4)

**Remark 2.5.5.** The operator \(d_2\) does not involve the pieces of structure \((1.5.1.H), (1.5.1.Q), (1.5.2.H), (1.5.2.Q), (1.5.3), (1.5.4)\). Hence, for an arbitrary \(A\)-module \(M\), the formula (2.5.4) given above yields an operator

\[
d_2 : \text{Alt}_A^p(Q, \text{Alt}_A^q(H, M)) \to \text{Alt}_A^{p+2}(Q, \text{Alt}_A^{q-1}(H, M)) \quad (p \geq 0, \quad q \geq 1).
\]

(2.4.7’)

We will use this observation in (5.8.7) and (5.8.8) below.

**Remark 2.6.** Given an almost pre-Lie-Rinehart triple \((A, H, Q)\), the vanishing of \(d_2d_2\) is automatic, for the following reason: View \(H\) and \(Q\) as abelian \(A\)-Lie algebras and \(H\) as being endowed with the trivial \(Q\)-module structure. Since \(\delta\) is a skew-symmetric \(A\)-bilinear pairing, we may use it to endow the \(A\)-module direct sum \(L = H \oplus Q\) with a nilpotent \(A\)-Lie algebra structure (of class two) by setting

\[
[(x, \xi), (y, \eta)] = (\delta(\xi, \eta), 0), \quad \xi, \eta \in Q, \quad x, y \in H.
\]

We write \(L_{\text{nil}}\) for this nilpotent \(A\)-Lie algebra. The ordinary CCE complex for calculating the Lie algebra cohomology \(H^*(L_{\text{nil}}, A)\) (with trivial \(L_{\text{nil}}\)-action on \(A\)) is just \((\text{Alt}_A(L, A), d_2)\). Thus the vanishing of \(d_2d_2\) is automatic.

**Theorem 2.7.** An almost pre-Lie-Rinehart triple \((A, H, Q)\) such that \(H\) and \(Q\) have property \(P\) is a Lie-Rinehart triple, that is, (1.6.1) and (1.6.3) then turn \((A, H \oplus Q)\) into a Lie-Rinehart algebra, if and only if \((\text{Alt}_A(Q, \text{Alt}_A(H, A)); d_0, d_1, d_2)\) is a (multi) Maurer-Cartan algebra.

**Proof.** The direct \(A\)-module sum \(L = Q \oplus H\) has the property \(P\). The sum \(d = d_0 + d_1 + d_2\) is an \(R\)-derivation on \(\text{Alt}_A(L, A)\). Hence the claim is an immediate consequence of Lemma 2.2.11. \(\square\)

### 2.8. From Maurer-Cartan Algebras to Lie-Rinehart Triples

Let \(H\) and \(Q\) be finitely generated projective \(A\)-modules, and let \(d_0, d_1, d_2\) be homogeneous \(R\)-derivations of the bigraded \(A\)-algebra \(\text{Alt}_A(Q, \text{Alt}_A(H, A))\) of the kind

\[
d_j: \text{Alt}_A^p(Q, \text{Alt}_A^q(H, A)) \to \text{Alt}_A^{p+j}(Q, \text{Alt}_A^{q-j+1}(H, A)).
\]

**Proposition 2.8.1.** The operators \(d_0, d_1, d_2\) induce an almost pre-Lie-Rinehart triple structure on \((A, H, Q)\).

**Proof.** Write \(L = Q \oplus H\). The sum \(d = d_0 + d_1 + d_2\) is a derivation on \(\text{Alt}_A(L, A)\). By Lemma 2.2.15, \(d\) induces a bracket \([\cdot, \cdot]_L\) on \(L\) (of the kind (2.2.1)) and an operation \(L \otimes_R A \to A\) of the kind (2.2.2). Taking homogeneous components with reference to the direct sum decomposition \(L = Q \oplus H\), we obtain an almost pre-Lie-Rinehart triple structure of the kind (1.5.1.1), (1.5.2.1), (1.5.1.2), (1.5.2.2), (1.5.3), (1.5.4), (1.5.5) on \((A, H, Q)\). The three almost pre-Lie-Rinehart triple axioms are implied by the fact that the operators \(d_0, d_1, d_2\) are derivations of the bigraded algebra \(\text{Alt}_A(Q, \text{Alt}_A(H, A))\). \(\square\)

**Theorem 2.8.2.** The triple \((A, H, Q)\), endowed with the induced operations of the kind (1.5.1.1), (1.5.2.1), (1.5.1.2), (1.5.2.2), (1.5.3), (1.5.4), (1.5.5) given in (2.8.1) above, is a pre-Lie-Rinehart triple if and only if \(d_0\) is a differential; \((A, H, Q)\) is a Lie-Rinehart triple if and only if \((\text{Alt}_A(Q, \text{Alt}_A(H, A)); d_0, d_1, d_2)\) is a Maurer-Cartan algebra.

**Proof.** This is a consequence of Lemmata 2.2.13 and 2.2.15. \(\square\)

Combining Theorem 2.7 and Theorem 2.8.2, we arrive at the following.
Theorem 2.8.3. Given finitely generated projective $A$-modules $H$ and $Q$, Lie-Rinehart triple structures on $(A,H,Q)$ and (multi) Maurer-Cartan algebra structures on $\text{Alt}_A(Q,\text{Alt}_A(H,A))$ are equivalent notions. □

Remark 2.8.4. Concerning the hypotheses and hence the range of applications, cf. e.g. Example 2.2.12 above, Theorem 2.7 is somewhat more general than Theorem 2.8.3. This justifies, hopefully, the terminology “almost-” and “pre-Lie-Rinehart triple”, admittedly a bit cumbersome. In fact, it would be interesting and important to establish the statement of Theorem 2.8.3 for $A$-modules more general than finitely generated and projective.

2.8.5. Direct verification of the Lie-Rinehart triple structure. Let $(A,H,Q)$ be an almost pre-Lie-Rinehart triple such that $H$ and $Q$ have property P, and suppose that $(\text{Alt}_A(Q,\text{Alt}_A(H,A));d_0,d_1,d_2)$ is a (multi) Maurer-Cartan algebra. It is then instructive to deduce directly that $(A,H,Q)$ is a Lie-Rinehart triple.

(i) Consider the operator

$$d_0d_0 : \text{Alt}^j_A(H,A) \to \text{Alt}^{j+2}_A(H,A)$$

for $j = 0$ and $j = 1$. Notice that $\text{Alt}^j_A(H,A)$ equals $\text{Alt}^j_A(H,\text{Alt}^0_A(Q,A))$ and that $\text{Alt}^{j+2}_A(H,A)$ equals $\text{Alt}^{j+2}_A(H,\text{Alt}^0_A(Q,A))$. For $j = 1$, given $x,y,z \in H$ and $\phi \in \text{Hom}_A(H,A) = \text{Alt}^j_A(H,A)$, we find

$$(d_0d_0\phi)(x,y,z) = \phi([[x,y]_H,z]_H + [[y,z]_H,x]_H + [[z,x]_H,y]_H).$$

Since $H$ has property P, we conclude that the bracket on $H$ satisfies the Jacobi identity, that is, $H$ is an $R$-Lie algebra. Likewise, for $j = 0$, given $x,y \in H$ and $a \in A$, we find

$$(d_0d_0a)(x,y) = x(y(a)) - y(x(a)) - [x,y](a).$$

Consequently the adjoint $H \to \text{Der}_R(A)$ of (1.5.2.$H$) is a morphism of $R$-Lie algebras. In view of (1.5.6.$H$) and (1.5.7.$H$), we conclude that (1.5.1.$H$) and (1.5.2.$H$) turn $(A,H)$ into a Lie-Rinehart algebra.

(ii) Next, consider the operator

$$d_0d_0 : \text{Alt}^1_A(Q,\text{Alt}^0_A(H,A)) \to \text{Alt}^1_A(Q,\text{Alt}^2_A(H,A)).$$

We note that $\text{Alt}^1_A(Q,\text{Alt}^0_A(H,A)) = \text{Alt}^1_A(Q,A) = \text{Hom}_A(Q,A)$. Let $\xi \in Q$, $x,y \in H$, and $\phi \in \text{Hom}_A(H,A)$. A straightforward calculation gives

$$(d_0d_0\phi)(\xi)(x,y) = \phi(y \cdot (x \cdot \xi) - x \cdot (y \cdot \xi) + [x,y]_H \cdot \xi).$$

Since $H$ is assumed to have property P, we conclude that, for every $\xi \in Q$, $x,y \in H$,

$$[x,y]_H \cdot \xi = x \cdot (y \cdot \xi) - y \cdot (x \cdot \xi),$$

that is, (1.5.3) is a left $(A,H)$-module structure on $Q$.

(iii) Pursuing the same kind of reasoning, consider the operator

$$d_0d_1 + d_1d_0 : A = \text{Alt}^0_A(Q,\text{Alt}^0_A(H,A)) \to \text{Alt}^1_A(Q,\text{Alt}^1_A(H,A)).$$
Let \( a \in A, \xi \in Q, x \in H \). Again a calculation shows that

\[
((d_0d_1 + d_1d_0)a)(\xi)(x) = x(\xi(a)) - \xi(x(a)) - ((x \cdot \xi)(a) - (\xi \cdot x)(a))
\]

whence the vanishing of \( d_0d_1 + d_1d_0 \) in bidegree \((0,0)\) entails the compatibility property (1.9.1). Likewise consider the operator

\[
d_0d_1 + d_1d_0: \text{Hom}_A(H, A) = \text{Alt}^0_A(Q, \text{Alt}^1_A(H, A)) \to \text{Alt}^1_A(Q, \text{Alt}^2_A(H, A)).
\]

Again a calculation shows that, for \( \xi \in Q, x, y \in H, \phi \in \text{Hom}_A(H, A) \),

\[
((d_0d_1 + d_1d_0)\phi)(\xi)(x, y) = \phi([x, y]_H - (x \cdot \xi) \cdot y + (y \cdot \xi) \cdot x)
\]

whence the vanishing of \( d_0d_1 + d_1d_0 \) in bidegree \((0,1)\) entails the compatibility property (1.9.3). Likewise, the vanishing of the operator \( d_0d_1 + d_1d_0 \) in bidegree \((1,0)\), that is,

\[
d_0d_1 + d_1d_0: \text{Alt}^1_A(Q, \text{Alt}^0_A(H, A)) \to \text{Alt}^2_A(Q, \text{Alt}^1_A(H, A)),
\]

tests the compatibility property (1.9.2).

In the same vein:
(iv) The vanishing of the operator

\[
d_1d_1 + d_2d_0 = d_0d_2 + d_1d_1 + d_2d_0: \text{Alt}^0_A(Q, \text{Alt}^0_A(H, A)) \to \text{Alt}^2_A(Q, \text{Alt}^0_A(H, A))
\]

entails the compatibility property (1.9.4).
(v) The vanishing of the operator

\[
d_1d_1 + d_2d_0 = d_0d_2 + d_1d_1 + d_2d_0: \text{Alt}^1_A(Q, \text{Alt}^0_A(H, A)) \to \text{Alt}^3_A(Q, \text{Alt}^0_A(H, A)),
\]

together with (1.9.4), entails the compatibility property (1.9.6), the “generalized Jacobi identity for the bracket \([\cdot, \cdot]_Q\)”.

For intelligibility and later reference (cf. (4.10) and (6.11) below), we sketch the argument: Let \( \alpha \in \text{Alt}^1_A(Q, \text{Alt}^0_A(H, A)) \) and \( \xi, \eta, \vartheta \in Q \). A straightforward calculation yields

\[
(d_1d_1)\alpha(\xi, \eta, \vartheta) = - \sum_{(\xi, \eta, \vartheta) \text{ cyclic}} \alpha([[\xi, \eta]_Q, \vartheta]_Q)
\]

\[
- \sum_{(\xi, \eta, \vartheta) \text{ cyclic}} (\xi(\eta(\alpha(\vartheta))) - \eta(\xi(\alpha(\vartheta))) - [\xi, \eta]_H(\alpha(\vartheta)))
\]

Using (1.9.4), we substitute \((\delta(\xi, \eta))(\alpha(\vartheta))\) for \( \xi(\eta(\alpha(\vartheta))) - \eta(\xi(\alpha(\vartheta))) - [\xi, \eta]_H(\alpha(\vartheta)) \) and obtain

\[
(d_1d_1)\alpha(\xi, \eta, \vartheta) = - \sum_{(\xi, \eta, \vartheta) \text{ cyclic}} \alpha([[\xi, \eta]_Q, \vartheta]_Q) - \sum_{(\xi, \eta, \vartheta) \text{ cyclic}} (\delta(\xi, \eta))(\alpha(\vartheta))
\]

Likewise, a calculation gives

\[
(d_2d_0)\alpha(\xi, \eta, \vartheta) = \sum_{(\xi, \eta, \vartheta) \text{ cyclic}} (\delta(\xi, \eta))(\alpha(\vartheta)) - \sum_{(\xi, \eta, \vartheta) \text{ cyclic}} \alpha(\delta(\xi, \eta) \cdot \vartheta)
\]
whence the vanishing of the operator $d_1d_1 + d_2d_0$ on $\text{Alt}_A^1(Q, \text{Alt}_A^0(H, A))$ implies
\[
\sum_{(\xi, \eta, \vartheta) \text{ cyclic}} \alpha([\xi, \eta]_Q, \vartheta)_Q + (\delta(\xi, \eta)) \cdot \vartheta = 0.
\]
For later reference we note that
\[
(d_2d_0\alpha(\vartheta))(\xi, \eta) = (\delta(\xi, \eta))(\alpha(\vartheta))
\]
whence
\[
(2.8.6) \quad \sum_{(\xi, \eta, \vartheta) \text{ cyclic}} \alpha([\xi, \eta]_Q, \vartheta)_Q = (d_2d_0\alpha)(\xi, \eta, \vartheta) + \sum_{(\xi, \eta, \vartheta) \text{ cyclic}} (d_2d_0\alpha(\vartheta))(\xi, \eta)
\]
(vi) The vanishing of the operator
\[
d_0d_2 + d_1d_1 + d_2d_0: \text{Alt}_A^0(Q, \text{Alt}_A^1(H, A)) \to \text{Alt}_A^2(Q, \text{Alt}_A^1(H, A))
\]
entails the compatibility property (1.9.5), the “generalized $Q$-module structure on $H$”.
(vii) The vanishing of the operator
\[
d_1d_2 + d_2d_1: \text{Alt}_A^0(Q, \text{Alt}_A^1(H, A)) \to \text{Alt}_A^3(Q, \text{Alt}_A^0(H, A))
\]
entails the compatibility property (1.9.7). Indeed, given $\xi, \eta, \vartheta \in Q$ and $\alpha: H \to A$,
\[
((d_1d_2 + d_2d_1)\alpha)(\xi, \eta, \vartheta) = \sum_{(\xi, \eta, \vartheta) \text{ cyclic}} \alpha(\delta([\xi, \eta]_Q, \vartheta) - \xi \cdot \delta(\eta, \vartheta)).
\]
2.9. THE SPECTRAL SEQUENCE. Let $(A, H, Q)$ be a Lie-Rinehart triple. The filtration of $\text{Alt}_A(Q, \text{Alt}_A(H, A))$ by $Q$-degree leads to a spectral sequence
\[
(2.9.1) \quad (E_r^{*, *}, d_r)
\]
having
\[
(2.9.2) \quad (E_0, d_0) = (\text{Alt}_A(Q, \text{Alt}_A(H, A)), d_0)
\]
whence $E_1^{p,q}$ amounts to the Lie-Rinehart cohomology $H^q(H, \text{Alt}_A^p(Q, A))$ of $H$ with values in the left $(A, H)$-module $\text{Alt}_A^p(Q, A)$. There is a slight conflict of notation here but it will always be clear from the context whether $d_j \ (j \geq 0)$ refers to the differentials of a spectral sequence or to a system of multicomplex operators. The spectral sequence (2.9.1) is an invariant of the Lie-Rinehart triple structure. In particular, $E_1^{0,0} = A^H$ and $E_1^{1,0} = \text{Hom}(Q, A)^H$, and $H^*(H, A)$ inherits an $(A^H, Q^H)$-module structure, with reference to the Lie-Rinehart structure on $(A^H, Q^H)$, cf. Corollary 1.11. Thus the Rinehart complex $(\text{Alt}_A^\mu(Q^H, H^*(H, A)), d)$ is defined.
2.10. ILLUSTRATION. The spectral sequence (2.9.1) includes as special cases that of a foliation and the Hodge-de Rham spectral sequence. This provides a conceptually
simple approach to these spectral sequences and subsumes them under a single more general construction. We will now make this precise.

(i) Consider a foliated manifold $M$, the foliation being written as $\mathcal{F}$. Recall that a $p$-form $\omega$ on $M$ is called \textit{horizontal} (with reference to the foliation $\mathcal{F}$) provided $\omega(X_1, \ldots, X_p) = 0$ if some $X_j$ is vertical, i.e., tangent to the foliation, or, equivalently, $i_X \omega = 0$ whenever $X$ is vertical; a horizontal $p$-form $\omega$ is said to be \textit{basic} provided it is constant on the leaves (i.e., $\lambda_X \omega = 0$ whenever $X$ is vertical). The sheaf of germs of basic $p$-forms is in general not fine and hence gives rise to in general non-trivial cohomology in non-zero degrees, cf. [51]. Thus, under the circumstances of germs of basic forms $\omega_i$ of the Example 1.4.1, and those of (1.12) as well, so that

(ii) Suppose that the foliation $\mathcal{F}$ of germs of basic forms $\omega_i$ is an invariant of the foliation. The cohomology $E_p^f,0$ of a foliation, studied already in the literature, cf. [51, 56, 57]; this spectral sequence corresponding spectral sequence (2.9.1) comes down to the ordinary spectral sequence (2.10.1) 0 → $L'$ → $Q^H$ → $\text{Vect}(W) → 0$. The corresponding spectral sequence (2.9.1) comes down to the ordinary spectral sequence of a foliation, studied already in the literature, cf. [51, 56, 57]; this spectral sequence is an invariant of the foliation. The cohomology $E_p^f,0$ is sometimes called “basic cohomology”, since it may be viewed as the cohomology of the “space of leaves”.

(iii) Returning to (i) above, suppose in particular that the foliation is transversely complete [2]. Then the closures of the leaves constitute a smooth fiber bundle $M → W$, the algebra $A^H$ is isomorphic to that of smooth functions on $W$ in an obvious fashion, and the obvious map from $Q^H$ to $\text{Vect}(W) ≅ \text{Der}(A^H)$ which is part of the Lie-Rinehart structure of $(A^H, Q^H)$ is surjective [47] and hence fits into an extension of $(\mathbb{R}, A^H)$-Lie algebras of the kind

\begin{equation}
0 → L' → Q^H → \text{Vect}(W) → 0.
\end{equation}

Here $L'$ is the space of sections of a Lie algebra bundle on $W$, and the underlying extension of Lie algebroids on $W$ is referred to as the \textit{Atiyah sequence} of the (transversely complete) foliation $\mathcal{F}$ [47]. Thus we see that the interpretation of $Q^H$ as the space of vector fields on the “space of leaves” requires, perhaps, some care, since $L'$ will then consist of the “vector fields on the “space of leaves” which act trivially on every function".
To get a concrete example, let $M = \text{SU}(2) \times \text{SU}(2)$, and let $\mathcal{F}$ be the foliation defined by a dense one-parameter subgroup in a maximal torus $S^1 \times S^1$ in $\text{SU}(2) \times \text{SU}(2)$. Then the space $W$ is $S^2 \times S^2$, and $L'$ is the space of sections of a real line bundle on $S^2 \times S^2$, necessarily trivial. One easily chooses a vector bundle $\zeta$ on $\text{SU}(2) \times \text{SU}(2)$ which is complementary to $\tau_{\mathcal{F}}$, and the Lie-Rinehart triple structure is defined on $(\mathcal{C}^\infty(M), L_{\mathcal{F}}, \Gamma(\zeta))$. In particular, the operation $\delta$ is non-zero. We note that the Chern-Weil construction in [18] yields a characteristic class in $H^2_{\text{deRham}}(S^2 \times S^2, \mathbb{R})$ for the extension (2.10.1), and this class may be viewed as an irrational Chern class [18] (Section 4). The non-triviality of this class entails that the differential $d_2$ of the spectral sequence (2.9.1) is non-trivial. We also note that, in view of a result of Almeida and Molino [2], the transitive Lie algebroid corresponding to (2.10.1) does not integrate to a principal bundle; in fact, Mackenzie's integrability obstruction [45] is non-zero.

(iv) Under the circumstances of the Example 1.4.2, the cohomology $H^*(H, \text{Alt}^*(Q, A))$ is the Hodge cohomology of the smooth complex manifold $M$, i.e. $H^*(H, \text{Alt}^p(Q, A))$ is the cohomology of $M$ with values in the sheaf of germs of holomorphic $p$-forms, and the spectral sequence (2.9.1) is the Hodge-de Rham spectral sequence, sometimes referred to as the Frölicher spectral sequence in the literature.

(v) Under the circumstances of Corollary 1.9.8, so that $(A, Q, H)$ is a Lie-Rinehart triple with trivial $(A, H)$-module structures on $A$ and $Q$, the spectral sequence (2.9.1) is the ordinary spectral sequence for the corresponding extension of Lie-Rinehart algebras. If, furthermore, $A$ is the ground ring so that $Q$ and $H$ are ordinary Lie algebras, this comes down to the Hochschild-Serre spectral sequence of the Lie algebra extension.

3. The additional structure on $Q$

Let $(A, H, Q)$ be a Lie-Rinehart triple. Theorem 1.9 gives a possible answer to Question 1.2 as well as to Question 1.1. What is missing is an intrinsic description of the structure induced on the constituent $(A, Q)$ which, in turn, should then in particular encapsulate the Lie-Rinehart triple structure on $(A, H, Q)$. We now proceed towards finding such an intrinsic description. To this end, we will introduce, on the constituent $Q$, certain operations similar to those introduced by Nomizu on the constituent $\mathfrak{q}$ of a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ of a Lie algebra [49]; the operations in [49] come from the curvature and torsion of an affine connection of the second kind. We note that the naive generalization to Lie-Rinehart algebras of the notion of reductive decomposition of a Lie algebra is not consistent with the Lie-Rinehart axioms. Given a Lie-Rinehart algebra $L$ and an $A$-module decomposition $L = H \oplus Q$ where $(A, H)$ inherits a Lie-Rinehart structure, since for $x \in H$, $\xi \in Q$, and $a \in A$, necessarily

$$[x, a\xi] = a[x, \xi] - \xi(a)x,$$

the defining property $[H, Q] \subset Q$ of a reductive decomposition cannot be satisfied unless the constituent $Q$ acts trivially on $A$.

Let $(A, H, Q)$ be an almost pre-Lie-Rinehart triple. We will now define triple-,
quadruple-, and quintuple products of the kind

\[(3.1) \quad \{\cdot,\cdot,\cdot\}: Q \otimes_R Q \otimes_R A \to A\]

\[(3.2) \quad \{\cdot,\cdot,\cdot\}: Q \otimes_R Q \otimes_R Q \otimes_R A \to A\]

\[(3.3) \quad \{\cdot,\cdot,\cdot,\cdot\}: Q \otimes_R Q \otimes_R Q \otimes_R Q \otimes_R A \to A\]

\[(3.4) \quad \{\cdot,\cdot\}: Q \otimes_R Q \rightarrow Q\]

\[(3.5) \quad \{\cdot,\cdot\}: Q \otimes_R Q \otimes_R Q \otimes_R Q \rightarrow Q\]

\[(3.6) \quad \{\cdot,\cdot\}: Q \otimes_R Q \otimes_R Q \otimes_R Q \otimes_R Q \rightarrow Q.\]

To this end, pick \(\alpha, \beta, \gamma, \xi, \eta, \vartheta, \kappa \in Q\) and \(a \in A\). For \(1 \leq j \leq 6\), we will spell out an explicit description of each of the operations \((3,j)\) and label it as \((3,j')\), as follows.

\[(3.1') \quad \{\xi, \eta; a\} = (\delta(\xi, \eta))(a)\]

\[(3.2') \quad \{\alpha; \xi, \eta; a\} = (\alpha \cdot \delta(\xi, \eta))(a)\]

\[(3.3') \quad \{\alpha; \beta; \xi, \eta; a\} = (\alpha \cdot (\beta \cdot \delta(\xi, \eta)))(a)\]

\[(3.4') \quad \{\xi, \eta; \vartheta\} = (\delta(\xi, \eta)) \cdot \vartheta\]

\[(3.5') \quad \{\alpha; \xi, \eta; \kappa\} = (\alpha \cdot \delta(\xi, \eta)) \cdot \kappa\]

\[(3.6') \quad \{\alpha; \beta; \xi, \eta; \gamma\} = (\alpha \cdot (\beta \cdot \delta(\xi, \eta))) \cdot \gamma.\]

**Proposition 3.7.** Suppose that \((A, H, Q)\) is a pre-Lie-Rinehart triple.

(i) The operations \(\{\xi, \eta; \cdot\}: A \to A, \quad \{\alpha; \xi, \eta; \cdot\}: A \to A, \quad \{\alpha; \beta; \xi, \eta; \cdot\}: A \to A\) are derivations.

(ii) The operations \(\{\xi, \eta; \cdot\}: A \to A, \quad \{\alpha; \xi, \eta; \cdot\}: A \to A, \quad \{\alpha; \beta; \xi, \eta; \cdot\}: A \to A\) are skew in the variables \(\xi\) and \(\eta\).

(iii) The operations \(\{\xi, \eta; \cdot\}\) (on \(A\) as well as on \(Q\)) are \(A\)-linear in the variables \(\xi\) and \(\eta\), and the operations (on \(A\) as well as on \(Q\)) \(\{\alpha; \xi, \eta; \cdot\}\) and \(\{\alpha; \beta; \xi, \eta; \cdot\}\) are \(A\)-linear in the variable \(\alpha\).

(iv) The triple, quadruple, and quintuple products \(\{\xi, \eta; \vartheta\}, \{\alpha; \xi, \eta; \kappa\}, \{\alpha; \beta; \xi, \eta; \gamma\}\) are skew in the variables \(\xi\) and \(\eta\).

(v) Furthermore, these operations are related by the following identities.

\[
\{\xi, \eta; a \vartheta\} = a\{\xi, \eta; \vartheta\} + \{\xi, \eta; a\} \vartheta
\]

\[
\{\alpha; \alpha \xi, \eta; b\} = \{\alpha; \alpha \eta; b\} = a\{\alpha; \xi, \eta; b\} + \alpha(a)\{\xi, \eta; b\}
\]

\[
\{\alpha; \alpha \xi, \eta; \kappa\} = \{\alpha; \alpha \kappa; \eta\} = a\{\alpha; \xi, \eta; \kappa\} + \alpha(a)\{\xi, \eta; \kappa\}
\]

\[
\{\alpha; \xi, \eta; a \kappa\} = a\{\alpha; \xi, \eta; a\} + \{\alpha; \xi, \eta; a\} \kappa
\]

\[
\{\alpha; \alpha \beta; \xi, \eta; b\} = a\{\alpha; \beta; \xi, \eta; b\} + \alpha(a)\{\beta; \xi, \eta; b\}
\]

\[
\{\alpha; \beta; \alpha \xi, \eta; b\} = \{\alpha; \beta; \xi, \eta; a\} \vartheta
\]

\[
= a\{\alpha; \beta; \xi, \eta; b\} + \alpha(\alpha)\{\xi, \eta; b\} + \beta(a)\{\alpha; \xi, \eta; b\} + \alpha(a)\{\beta; \xi, \eta; b\}
\]

\[
\{\alpha; a \beta; \xi, \eta; \gamma\} = a\{\alpha; \beta; \xi, \eta; \gamma\} + a(\alpha)\{\beta; \xi, \eta; \gamma\}
\]

\[
\{\alpha; \beta; \alpha \xi, \eta; \gamma\} = \{\alpha; \beta; \xi, \eta; a\} \gamma
\]

\[
= a\{\alpha; \beta; \xi, \eta; \gamma\} + a(\alpha)\{\xi, \eta; \gamma\} + \beta(a)\{\alpha; \xi, \eta; \gamma\} + \alpha(a)\{\beta; \xi, \eta; \gamma\}
\]

\[
\{\alpha; \beta; \xi, \eta; a \gamma\} = a\{\alpha; \beta; \xi, \eta; a\} \gamma + \{\alpha; \beta; \xi, \eta; a\} \gamma
\]

**Proof.** These assertions are immediate consequences of the pre-Lie-Rinehart triple properties of \((A, H, Q)\).
Proposition 3.8. Suppose that \((A,H,Q)\) is a pre-Lie-Rinehart triple, and let \(\alpha, \beta, \gamma, \zeta, \xi, \eta, \vartheta, \kappa \in Q\) and \(a \in A\). With the notation \(x = \delta(\alpha, \beta)\) and \(y = \delta(\gamma, \zeta)\), the compatibility properties \((1.9.1)-(1.9.7)\) take the following form.

\[
\begin{align*}
(3.8.1) \quad & \xi \{\alpha, \beta; a\} - \{\alpha, \beta; \xi(a)\} = \{\xi; \alpha, \beta; a\} - \{\alpha, \beta; \xi\}(a) \\
& \{\alpha, \beta; [\xi, \eta]_Q\} = \{[\alpha, \beta; \xi], \eta\}_Q + [\xi, \{\alpha, \beta; \eta\}]_Q \\
(3.8.2) \quad & (\xi \cdot [\delta(\alpha, \beta), \delta(\gamma, \zeta)]_H) \cdot \kappa = \{\xi; \alpha, \beta; \gamma, \zeta; \kappa\} - \{\gamma, \zeta; \xi; \alpha, \beta; \kappa\} \\
& + \{\alpha, \beta; \{\gamma, \zeta; \kappa\}\} - \{\xi; \gamma, \zeta; \alpha, \beta; \kappa\} \\
(3.8.3) \quad & - \{\{\alpha, \beta; \xi\}; \gamma, \zeta; \kappa\} + \{\gamma, \zeta; \alpha, \beta; \kappa\} \\
(3.8.4) \quad & \xi(\eta(a)) - \eta(\xi(a)) = [\xi, \eta]_Q(a) + \{\xi, \eta; a\} \\
& \{[\xi, \eta]_Q; \alpha, \beta; \gamma\} = \{\xi; \eta; \alpha, \beta; \gamma\} - \{\eta; \xi; \alpha, \beta; \gamma\} \\
& - \{\{\alpha, \beta; \xi\}; \eta; \gamma\} - \{\xi; \{\alpha, \beta; \eta\}; \gamma\} \\
& + \{\alpha, \beta; \{\xi, \eta; \gamma\}\} - \{\xi, \eta; \alpha, \beta; \gamma\} \\
(3.8.5) \quad & \sum_{(\xi, \eta, \vartheta) \text{ cyclic}} ([\xi, \eta]_Q, \vartheta)_Q + \{\xi, \eta; \vartheta\} = 0 \\
(3.8.7) \quad & \sum_{(\xi, \eta, \vartheta) \text{ cyclic}} \{[\xi, \eta]_Q; \vartheta; \kappa\} = \sum_{(\xi, \eta, \vartheta) \text{ cyclic}} \{\xi; \eta, \vartheta; \kappa\} \\

Furthermore, the compatibility property \((1.5.12)\) takes the form

\[
(3.8.8) \quad \lfloor \delta(\alpha, \beta), \delta(\xi, \eta) \rfloor_H \cdot \xi = \{\alpha, \beta; \xi; \eta; \xi\} - \{\xi, \eta; \alpha, \beta; \xi\} \\

Proof. This is an immediate consequence of Theorem 1.9. We leave the details to the reader. \(\square\)

We note that \((3.8.5)\) is equivalent to

\[
\begin{align*}
\{\alpha, \beta; \{\xi, \eta; \gamma\}\} - \{\xi, \eta; \alpha, \beta; \gamma\} &= \{\alpha, \beta; \xi, \eta\} + \{\xi, \alpha, \beta; \eta\}; \gamma\} \\
& + \{[\xi, \eta]_Q; \alpha, \beta; \gamma\} \\
& - \{\xi; \eta; \alpha, \beta; \gamma\} + \{\eta; \xi; \alpha, \beta; \gamma\} \\
\end{align*}
\]

Somewhat more explicitly, \((3.8.7)\) reads

\[
\begin{align*}
\{[\xi, \eta]_Q, \vartheta, \kappa\} + \{\vartheta, [\xi, \eta]_Q, \kappa\} + \{\vartheta, [\xi, \eta]_Q, \kappa\} \\
= (\xi \cdot \delta(\eta, \vartheta)) \cdot \kappa + (\eta \cdot \delta(\vartheta, \xi)) \cdot \kappa + (\vartheta \cdot \delta(\xi, \eta)) \cdot \kappa \\

Moreover, with the notation \(x = \delta(\xi, \eta)\), \((3.8.2)\) comes down to

\[
x \cdot [\vartheta, \kappa]_Q = [x \cdot \vartheta, \kappa]_Q + [\vartheta, x \cdot \kappa]_Q - (\vartheta \cdot x) \cdot \kappa + (\kappa \cdot x) \cdot \vartheta, \\
\]

which is just \((1.9.2)\), and \((3.8.5)\) reads

\[
[x, \delta(\alpha, \beta)]_H = \delta(x \cdot \alpha, \beta) + \delta(\alpha, x \cdot \beta) \\
+ [\alpha, \beta]_Q \cdot x - \alpha \cdot (\beta \cdot x) + \beta \cdot (\alpha \cdot x),
\]
which is (1.9.5).

**Remark 3.9.** The description of the structure on \((A,Q)\) given in Propositions 3.7 and 3.8 is nearly intrinsic: Only the left-hand side \((\xi \cdot [\delta(\alpha,\beta),\delta(\gamma,\zeta)]_H)(\kappa)\) of the equation (3.8.3) and the left-hand side \([\delta(\alpha,\beta),\delta(\xi,\eta)]_H \cdot \xi\) of (3.8.8) involve the Lie-Rinehart bracket \([\cdot,\cdot]_H\) on \(H\) explicitly, and this bracket is not covered by the structure on \((A,Q)\). The Lie-Rinehart structure of \((A,H)\) encapsulates a whole bunch of additional compatibility conditions which the triple-, quadruple-, quintuple products necessarily satisfy.

### 3.10. Reconstruction of the Lie-Rinehart triple structure.

Starting from \((A,Q)\), endowed with the pieces of structure (1.5.1) and (1.5.2) which are supposed to satisfy (1.5.6) and (1.5.7) and, furthermore, with the triple-, quadruple-, quintuple products (3.1)–(3.6), to reconstruct an \((R,A)\)-Lie algebra complement \(H\) such that \(E = H \oplus Q\) inherits an \((R,A)\)-Lie algebra structure which, in turn, then determines the given structure on \((A,Q)\), we might proceed as follows, where we pursue a reasoning similar to that in the proof of Theorem 18.1 in [49] and that of Theorem 7.1 in [34]: Suppose that those compatibility properties spelled out in (3.7) and (3.8) which are merely phrased in terms of \(Q\) and, in particular, do not involve the bracket \([\cdot,\cdot]_H\) on \(H\) explicitly, are satisfied. Given \(\xi,\eta \in Q\), define \(\delta(\xi,\eta) \in \text{End}_R(Q)\) by

\[
\delta(\xi,\eta)(\vartheta) = \{\xi,\eta;\vartheta\}
\]

and let \(H \subseteq \text{End}_R(Q)\) be the \(A\)-linear span of the \(\delta(\xi,\eta)\)’s in \(\text{End}_R(Q)\) \((\xi,\eta \in Q)\); notice that, by assumption, \(Q\) comes with an \(A\)-module structure whence it makes sense to take the \(A\)-linear span of the \(\delta(\xi,\eta)\)’s in \(\text{End}_R(Q)\) \((\xi,\eta \in Q)\). The restriction of the evaluation pairing \(\text{End}_R(Q) \otimes_R Q \to Q\) to \(H\) yields the pairing (1.5.3), to be written as the association

\[
(\delta(\xi,\eta), \vartheta) \mapsto \delta(\xi,\eta) \cdot \vartheta, \quad \xi,\eta,\vartheta \in Q,
\]

and the requisite bilinear pairing (1.5.5) is just \(\delta\), viewed as a function from \(Q \otimes_A Q\) to \(H\). Since the triple product (3.4) is \(A\)-bilinear, the pairing (1.5.3) will then satisfy (1.5.9), and \(\delta\) is well defined on \(Q \otimes_A Q\). Next, define a pairing

\[
H \otimes_R A \to A, \quad (x,a) \mapsto x(a),
\]

by means of

\[
\delta(\xi,\eta)(a) = \{\xi,\eta;a\}, \quad \xi,\eta \in Q, \ a \in A.
\]

This yields the requisite pairing (1.5.2). Since the triple product (3.1) is \(A\)-bilinear, (1.5.6) will hold. Thereafter, define a pairing

\[
:: Q \otimes_R H \to H
\]

by setting

\[
(\alpha \cdot \delta(\xi,\eta))(\kappa) = \{\alpha;\xi,\eta;\kappa\}, \quad \alpha,\xi,\eta,\kappa \in Q.
\]

This yields the requisite pairing (1.5.4). Since the quadruple product is \(A\)-linear in \(\alpha\), (1.5.11) will hold. The compatibility properties in (3.7) and (3.8) imply that the pairings (1.5.3) and (1.5.4) will satisfy (1.5.8) and (1.5.10).
To complete the construction, we must require that the ordinary commutator bracket on \( \text{End}_R(Q) \) descend to a bracket \([\cdot, \cdot]_H\) on \( H \) in such a way that \((A, H)\), with this bracket and the pairing \((1.5.2.H)\) (which we reconstructed from the triple product \(3.4)\), be a Lie-Rinehart algebra in such a way that \((3.8.3)\) and \((3.8.8)\) are satisfied. The remaining compatibility properties in order for \((A, H, Q)\) to be a Lie-Rinehart triple will then be implied by the structure isolated in \((3.7)\) and \((3.8)\).

4. Quasi-Lie-Rinehart algebras

Let \((A, H, Q)\) be a Lie-Rinehart triple. Thus \((A, H)\) is a Lie-Rinehart algebra whence the Rinehart complex \( A = (\text{Alt}_A(H, A), d) \) inherits a differential graded \( R \)-algebra structure and \( Q \) is, in particular, an \((A, H)\)-module whence the Rinehart complex \( Q = (\text{Alt}_A(H, Q), d) \) is a differential graded \( A \)-module in an obvious fashion. For the special case where \((A, H, Q)\) is a twilled Lie-Rinehart algebra (i.e. the operation \(\delta: Q \otimes_A Q \to H\), cf. \((1.5.5)\), is zero), we have shown in [21] \((3.2)\) that the pair \((A, Q)\) acquires a differential graded Lie-Rinehart structure and that the twilled Lie-Rinehart algebra compatibility conditions can be characterized in terms of this differential graded Lie-Rinehart structure. We will now show that, for a general Lie-Rinehart triple \((A, H, Q)\) (i.e. with in general non-zero \(\delta\)), the pair \((A, Q)\) inherits a higher homotopy version of a differential graded Lie-Rinehart algebra structure; abstracting from the structure which thus emerges, we isolate the notion of quasi-Lie-Rinehart algebra. This structure provides a complete solution of the problem of describing the structure on the constituent of a Lie-Rinehart triple written as \(Q\) and hence yields a complete answer to Question 1.1.

We begin by describing the requisite pieces of structure, independently of any given (pre-)Lie-Rinehart triple, in the following fashion: Let \(A\) be a graded commutative algebra concentrated in non-negative degrees \((A^q = 0\) for \(q < 0)\), at this stage not a differential graded commutative algebra, and let \(Q\) be a graded (left) \(A\)-module which we suppose to be an induced graded \(A\)-module of the kind \(Q = A \otimes_A Q\) where \(A = A^0\) and where \(Q\) is concentrated in degree zero; the notation \((A, Q)\) will refer to this kind of structure throughout, perhaps endowed with additional structure. A homogeneous \(A\)-multilinear function \(\phi\) on \(Q\) in \(\ell\) variables with values in a graded \(A\)-module \(M\) is said to be \(A\)-graded multilinear if, for every \(\alpha_1, \ldots, \alpha_\ell \in A\) and every \(\xi_1, \ldots, \xi_\ell \in Q\),

\[
\phi(\xi_1, \ldots, \xi_{j-1}, \alpha_j \xi_j, \xi_{j+1}, \ldots, \xi_\ell)
= (-1)^{|\phi| + |\xi_1| + \cdots + |\xi_{j-1}| + |\alpha_j| |\alpha_j|} \phi(\xi_1, \ldots, \xi_{j-1}, \xi_j, \xi_{j+1}, \ldots, \xi_\ell);
\]

it is called graded alternating if, for every \(\xi_1, \ldots, \xi_\ell \in Q\),

\[
\phi(\xi_1, \ldots, \xi_j, \xi_{j+1}, \ldots, \xi_\ell) = (-1)^{|\xi_j| |\xi_{j+1}|} \phi(\xi_1, \ldots, \xi_{j+1}, \xi_j, \ldots, \xi_\ell).
\]

A pairing is graded skew-symmetric provided it is graded alternating as a graded bilinear function.

With these preparations out of the way suppose that, in addition, \((A, Q)\) carries — a graded skew-symmetric \(R\)-bilinear pairing of degree zero

\[
(4.1) \quad [\cdot, \cdot]_Q: Q \otimes_R Q \to Q,
\]
— an $R$-bilinear pairing of degree zero

\[(4.2) \quad Q \otimes_R A \to A, \ (\xi, \alpha) \mapsto \xi(\alpha),\]

— an $A$-trilinear operation of degree $-1$

\[(4.3.Q) \quad \langle \cdot, \cdot; \cdot \rangle_Q: Q \otimes_A Q \otimes_A A \to A\]

which is graded skew-symmetric in the first two variables (i.e. in the $Q$-variables).

We will say that the pair $(A, Q)$ constitutes a pre-quasi-Lie-Rinehart algebra provided it satisfies (i) and (ii) below.

(i) The values of the adjoints $Q \to \text{End}_R(A)$ and $Q \otimes_A Q \to \text{End}_R(A)$ of (4.2) and (4.3.Q) respectively, lie in $\text{Der}_R(A)$ so that, in particular, given $\xi, \eta \in Q$ and homogeneous $\alpha, \beta \in A$,

\[\langle \xi, \eta; \beta \alpha \rangle_Q = \langle \xi, \eta; \beta \rangle_Q \alpha + (-1)^{|\beta|} \beta \langle \xi, \eta; \alpha \rangle_Q;\]

(ii) the bracket (4.1), the operation (4.2), and the graded $A$-module structure on $Q$ satisfy the following graded Lie-Rinehart axioms (4.4) and (4.5):

\[(4.4) \quad (a\xi)(b) = a(\xi(b)), \ a, b \in A, \ \xi \in Q,\]
\[(4.5) \quad [\xi, a\eta]_Q = \xi(a)\eta + a[\xi, \eta]_Q, \ a \in A, \ \xi, \eta \in Q.\]

The graded Lie-Rinehart algebra axioms (4.4) and (4.5) imply that (4.1) and (4.2) are determined by their restrictions

\[(4.1.Q) \quad [\cdot; \cdot]_Q: Q \otimes_R Q \to Q\]
\[(4.2.Q) \quad Q \otimes_R A \to A, \ (\xi, \alpha) \mapsto \xi(\alpha)\]

Here the values of (4.1.Q) necessarily lie in $Q$ since $[\cdot, \cdot]_Q$ is supposed to be of degree zero; in particular, (4.1.Q) is skew-symmetric in the usual sense. We note that, when $A$ is concentrated in degree zero, the operation (4.3.Q) is necessarily zero.

Given a pre-quasi-Lie-Rinehart algebra $(A, Q)$, consider the bigraded algebra

\[(4.6) \quad \text{Alt}_A(Q, A) \cong \text{Alt}_A(Q, A),\]

of $A$-valued $A$-multilinear alternating functions on $Q$ and define the operators

\[(4.7.1) \quad d_1: \text{Alt}_A^p(Q, A^q) \to \text{Alt}_A^{p+1}(Q, A^q) \ (p, q \geq 0)\]

and

\[(4.8.1) \quad d_2: \text{Alt}_A^p(Q, A^q) \to \text{Alt}_A^{p+2}(Q, A^{q-1}) \ (p, q \geq 0)\]
by
\[(4.7.2)\]
\[
(-1)^{|f|+1}(d_1 f)(\xi_1, \ldots, \xi_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j-1} \xi_j (f(\xi_1, \ldots, \hat{\xi}_j, \ldots, \xi_{p+1})) \\
+ \sum_{1 \leq j < k \leq p+1} (-1)^{j+k} f([\xi_j, \xi_k]_Q, \xi_1, \ldots, \hat{\xi}_j \ldots \hat{\xi}_k \ldots, \xi_{p+1})
\]

(the graded CCE formula)
\[(4.8.2)\]
\[
(-1)^{|f|+1}(d_2 f)(\xi_1, \ldots, \xi_{p+2}) = (-1)^p \sum_{1 \leq j < k \leq p+2} \langle \xi_j, \xi_k; f(\xi_1, \ldots, \hat{\xi}_j \ldots \hat{\xi}_k \ldots, \xi_{p+2}) \rangle_Q
\]

where \(\xi_1, \ldots, \xi_{p+2} \in Q\). The graded Lie-Rinehart axioms (4.4) and (4.5) imply that the operator \(d_1\) is well defined on \(\mathrm{Alt}_A(Q, \mathcal{A})\) as an \(R\)-linear (beware, not \(A\)-linear) operator. The usual argument shows that \(d_1\) is a derivation on the bigraded \(A\)-algebra \(\mathrm{Alt}_A(Q, \mathcal{A})\). Since the operation \(\langle \cdot, \cdot; \cdot \rangle_Q\), cf. (4.3.Q), is \(A\)-trilinear, the operator \(d_2\) is well defined on \(\mathcal{A}\)-valued \(A\)-multilinear functions on \(Q\). Since (4.3.Q) is skew-symmetric in the first two variables, the operator \(d_2\) automatically has square zero, i.e. is a differential.

**Lemma 4.8.3.** The operator \(d_2\) is an \(A\)-linear derivation on the bigraded \(A\)-algebra \(\mathrm{Alt}_A(Q, \mathcal{A})\).

**Proof.** Since, as a graded \(\mathcal{A}\)-module, \(Q\) is an induced graded \(\mathcal{A}\)-module, the bigraded algebra \(\mathrm{Alt}_A(Q, \mathcal{A})\) may be written as the bigraded tensor product \(\mathrm{Alt}_A(Q, \mathcal{A}) \cong \mathrm{Alt}_A(Q, \mathcal{A}) \otimes \mathcal{A}\), and it suffices to consider forms which may be written as \(\beta \alpha\) where \(\beta \in \mathrm{Alt}_A(Q, \mathcal{A})\) and \(\alpha \in \mathcal{A}\); the formula (4.8.2) yields
\[
d_2(\beta) = 0, \quad d_2(\beta \alpha) = (-1)^{|\beta|} \beta d_2(\alpha)
\]
and, since for \(\xi, \eta \in Q\), the operation \(\langle \xi, \eta; \cdot \rangle_Q\) is a derivation of \(\mathcal{A}\), we conclude that the operator \(d_2\) is an \(R\)-linear derivation on \(\mathrm{Alt}_A(Q, \mathcal{A})\). Furthermore, since for \(a \in A = \mathcal{A}^0\), for degree reasons, \(d_2(a)\) is necessarily zero the operator \(d_2\) is plainly well defined on \(\mathcal{A}\)-valued \(A\)-multilinear functions on \(Q\) and in fact an \(A\)-linear derivation on \(\mathrm{Alt}_A(Q, \mathcal{A})\) as asserted. \(\square\)

**Remark 4.8.4.** On the formal level, the notion of quasi-Lie-Rinehart algebra isolated above is somewhat unsatisfactory since the definition involves the structure of \(Q\) as an induced \(\mathcal{A}\)-module. The operator \(d_1\) may be written out as an operator on the bigraded \(A\)-module \(\mathrm{Alt}_A(Q, \mathcal{A})\) of \(A\)-graded multilinear alternating forms on \(Q\) directly in terms of the operations (4.1) and (4.2), that is, in terms of the arguments of these operations, without explicit reference to the induced \(A\)-module structure. Indeed, given an \(n\)-tuple \(\eta = (\eta_1, \ldots, \eta_n)\) of homogeneous elements of \(Q\), write \(|\eta| = |\eta_1| + \ldots + |\eta_n|\) and \(|\eta|^{(j)} = |\eta_1| + \ldots + |\eta_j|\), for \(1 \leq j \leq n\), and define the operators
\[
d_{\langle \cdot, \cdot \rangle}: \mathrm{Alt}_R^p(Q, \mathcal{A}^q) \to \mathrm{Alt}_R^{p+1}(Q, \mathcal{A}^q), \quad d_{[\cdot, \cdot]}: \mathrm{Alt}_R^p(Q, \mathcal{A}^q) \to \mathrm{Alt}_R^{p+1}(Q, \mathcal{A}^q),
\]
by means of

\[-1\right)^{|f|+1+|\eta|}(d_{(,\ldots,)}(f))(\eta_1,\ldots,\eta_{p+1})

\[=

\sum_{j=1}^{p+1} (-1)^{j-1+(|\eta|^{(j-1)}+|f|)|\eta_j| f(\eta_1,\ldots,\widehat{\eta_j}\ldots,\eta_{p+1})

\[=

\sum_{1\leq j< k \leq p+1} (-1)^{j+k+|\eta|^{(j-1)}|\eta_j|+(|\eta|^{(k-1)}-|\eta_j|)|\eta_k| f(\eta_j,\eta_k,\eta_1,\ldots,\widehat{\eta_j}\ldots,\widehat{\eta_k}\ldots,\eta_{p+1})

where \(\eta_1,\ldots,\eta_{p+1}\) are homogeneous elements of \(Q\). Then the sum \(d_{(,\ldots,)}+d_{[\ldots,]}\) descends to an operator on \(\text{Alt}_A(Q,A)\) which, in turn, coincides with \(d_1\). In this fashion, \(d_1\) appears as being given by the CCE formula (2.2.8) with respect to (4.1) and (4.2).

We were so far unable to give a similar description of the operator appears as being given by the CCE formula (2.2.8) with respect to (4.1) and (4.2). We were so far unable to give a similar description of the operator \(d_2\), though, in terms of a suitable extension of (4.3) to an operation of the kind \(Q\bigotimes_A Q\bigotimes_A A \to A\).

4.9. Definition. Let \((A,Q)\) be a pre-quasi-Lie-Rinehart algebra so that, in particular, \(A\) is a differential graded commutative algebra and \(Q\) a differential graded \(A\)-module. Consider the bigraded \(A\)-algebra

\[\text{Alt}_A(Q,A) \cong \text{Alt}_A(Q,A) \subseteq \text{Mult}_R(Q,A),\]

cf. (4.6) above, where \(\text{Mult}_R(Q,A)\) refers to the bigraded algebra of \(A\)-valued \(R\)-multilinear forms on \(Q\). The differentials on \(Q\) and \(A\) (both written as \(d\), with an abuse of notation,) induce a differential \(D\) on \(\text{Mult}_R(Q,A)\) in the usual way, that is, given an \(R\)-multilinear \(A\)-valued form \(f\) on \(Q\),

\[Df = df + (-1)^{|f|+1}fd\]

where, with a further abuse of notation, the “\(d\)” in the constituent \(df\) signifies the induced operator on any of the tensor powers \(Q\bigotimes_R \ell\) \((\ell \geq 1)\). We will say that the pre-quasi-Lie-Rinehart algebra \((A,Q)\) is a quasi-Lie-Rinehart algebra provided it satisfies the requirements (4.9.1)–(4.9.6) below where \(d_1\) and \(d_2\) are the operators (4.7.1) and (4.8.1), respectively.

(4.9.1) The differential \(D\) descends to an operator on \(\text{Alt}_A(Q,A)\), necessarily a differential, which we then write as \(d_0\).

(4.9.2) The differential on \(Q\) is a derivation for the bracket (4.1).

(4.9.3) The pairing (4.2) is compatible with the differentials on \(A\) and \(Q\).

(4.9.4) For every \(\xi, \eta \in Q\) and \(\alpha \in A\),

\[\xi(\eta(\alpha)) - \eta(\xi(\alpha)) - [\xi, \eta]_Q(\alpha) = ((d_0d_2 + d_2d_0)(\alpha))(\xi, \eta)\]

(4.9.5) For every \(\xi, \eta, \vartheta \in Q\) and \(\alpha \in \text{Alt}_A^1(Q,A^0) = \text{Hom}_A(Q,A),\)

\[\sum_{(\xi, \eta, \vartheta) \text{ cyclic}} \alpha([\xi, \eta]_Q, \vartheta)_Q = (d_2d_0\alpha)(\xi, \eta, \vartheta) + \sum_{(\xi, \eta, \vartheta) \text{ cyclic}} (d_2d_0\alpha(\vartheta))(\xi, \eta)\]
In (4.9.6), it suffices to require the vanishing of the operator \(d_1d_2 + d_2d_1\) on \(\text{Alt}^0_A(Q, A^1)\). We leave it to the reader to spell out a description of this requirement directly in terms of the structure (4.1)–(4.3); this description would be less concise than the requirement given as (4.9.6).

**Theorem 4.10.** Let \((A, Q)\) be a pre-quasi-Lie-Rinehart algebra, consider the bigraded \(A\)-algebra

\[
\text{Alt}_A(Q, A) \cong \text{Alt}_A(Q, A) \subseteq \text{Mult}_R(Q, A),
\]

suppose that the operator \(D\) on \(\text{Mult}_R(Q, A)\) descends to an operator \(d_0\) on \(\text{Alt}_A(Q, A)\), and let \(d_1\) and \(d_2\) be the operators on \(\text{Alt}_A(Q, A)\) given by (4.7.1) and (4.8.1), respectively. Then \((A, Q)\) is a quasi-Lie-Rinehart algebra if and only if \((\text{Alt}_A(Q, A), d_0, d_1, d_2)\) is a multialgebra.

**Proof.** (i) The identity \(0 = d_0d_1 + d_1d_0\) on \(\text{Alt}^1_A(Q, A^0)\) is equivalent to (4.9.2), that is, to the differential on \(Q\) being a derivation for the bracket \([\cdot, \cdot]_Q\), cf (4.1). See also (2.8.5(iii)).

(ii) The identity \(0 = d_0d_1 + d_1d_0\) on \(\text{Alt}^0_A(Q, A^1)\) is equivalent to (4.9.3), that is, to the differentials on \(A\) and \(Q\) being compatible with the pairing (4.2).

(iii) The identity \(0 = d_0d_2 + d_1d_1 + d_2d_0\) on \(\text{Alt}^0_A(Q, A^0)\) is equivalent to the special case of (4.9.4) where \(\alpha \in A = A^0\). Cf. (2.8.5(iv)).

(iv) Once (4.9.4) holds, the identity \(0 = d_0d_2 + d_1d_1 + d_2d_0\) on \(\text{Alt}^1_A(Q, A^0)\) is equivalent to (4.9.5). Cf. (2.8.5(v)).

(v) The identity \(0 = d_0d_2 + d_1d_1 + d_2d_0\) on \(\text{Alt}^0_A(Q, A^1)\) is equivalent to the special case of (4.9.4) where \(\alpha \in A^1\). Cf. (2.8.5(vi)). \(\square\)

Under the circumstances of Theorem 4.10, we will refer to the multialgebra

\[
(\text{Alt}_A(Q, A), d_0, d_1, d_2)
\]
as the *Maurer-Cartan algebra* for the quasi-Lie-Rinehart algebra structure on \((A, Q)\).

**4.11. Relationship with Almost Pre-Lie-Rinehart Triples.** Our goal is to show how a Lie-Rinehart triple determines a quasi-Lie-Rinehart algebra. Here we explain the first step, that is, how a structure of the kind (4.1.Q)–(4.3.Q) that underlies a pre-quasi-Lie-Rinehart algebra arises: Let \((A, Q, H)\) be an almost pre-Lie-Rinehart triple, and let \(A = \text{Alt}_A(H, A)\) and \(Q = \text{Alt}_A(H, Q)\). Then \(A = \text{Alt}_A(H, A)\) is a graded commutative algebra (beware, not necessarily a *differential* graded commutative algebra) and \(Q = \text{Alt}_A(H, Q)\) is a graded \(A\)-module (not necessarily a *differential* graded module). The pairings (1.5.2.Q) and (1.5.4) induce a pairing \(Q \otimes_R A \to A\) of the kind (4.2.Q) by means of the association

\[
(4.11.1) \quad \xi \otimes \alpha \mapsto \xi(\alpha), \quad \xi \in Q, \quad \alpha \in A = \text{Alt}_A(H, A)
\]

where

\[
(4.11.2) \quad (\xi(\alpha))(x_1, \ldots, x_n) = \xi(\alpha(x_1, \ldots, x_n)) - \sum_{j=1}^n \alpha(x_1, \ldots, \xi \cdot x_j, \ldots x_n).
\]
The corresponding induced pairing of the kind (4.2) has the form

\[(4.11.3)\]  
\[Q \otimes_R A \to A, \ (\xi, \alpha) \mapsto \xi(\alpha), \ \xi \in Q, \ \alpha \in A.\]

Furthermore, the bracket \([\cdot, \cdot]_Q\) is exactly of the kind (4.1). It extends to a graded skew-symmetric bracket

\[(4.11.4)\]  
\[[\cdot, \cdot]_Q: Q \otimes_R Q \to Q\]

of the kind (4.1). To get an explicit formula for this bracket we suppose, for simplicity, that the canonical map from \(A \otimes_A Q\) to \(Q = \text{Alt}_A(H, Q)\) is an isomorphism of graded \(A\)-modules so that \(Q\) is indeed an induced graded \(A\)-module of the kind considered above. This will be the case, for example, when \(H\) is finitely generated and projective as an \(A\)-module or when \(Q\) is projective as an \(A\)-module. Under these circumstances, given homogeneous elements \(\alpha, \beta \in A\) and \(\xi, \eta \in Q\), the value \([\alpha \otimes \xi, \beta \otimes \eta]_Q\) of the bracket (4.11.4) is given by

\[(4.11.5)\]  
\[[\alpha \otimes \xi, \beta \otimes \eta]_Q = (\alpha \xi(\beta)) \otimes \eta - (\beta \eta(\alpha)) \otimes \xi + (\alpha \beta) \otimes [\xi, \eta]_Q.\]

Furthermore, setting

\[(4.11.6)\]  
\[\langle \xi, \eta; \alpha \rangle_Q = i_\delta(\xi, \eta)\alpha, \ \alpha \in A, \ \xi, \eta \in Q,\]

where, for \(x \in H\), \(i_x\) refers to the operation of contraction, that is,

\[(4.11.7)\]  
\[\langle \xi, \eta; \alpha \rangle_Q(x_1, \ldots, x_{q-1}) = \alpha(\delta(\xi, \eta), x_1, \ldots, x_{q-1}), \ x_1, \ldots, x_{q-1} \in H,\]

we obtain a pairing of the kind (4.3). Thus, summing up, we conclude that, on \((A, Q)\), the operations (4.11.1), (4.11.4), and (4.11.6) which, in turn, come from the almost pre-Lie-Rinehart triple structure on \((A, Q, H)\) determine a structure of the kind (4.1)–(4.3) which underlies that of a pre-quasi-Lie-Rinehart algebra. Indeed, the structure on \((A, Q)\) given by (4.11.1), (4.11.4), and (4.11.6) is essentially a rewrite of the almost pre-Lie-Rinehart triple structure on \((A, Q, H)\); the two structures are equivalent when \(H\) is finitely generated projective as an \(A\)-module and when \(Q\) has property P. At this stage we do not make any claim as to whether or not the structure given by (4.11.1), (4.11.4), and (4.11.6) turns \((A, Q)\) into a pre-quasi-Lie-Rinehart algebra.

4.12. Lie-Rinehart Triples and Quasi-Lie-Rinehart Algebras. Suppose now that \((A, Q, H)\) is a pre-Lie-Rinehart triple; with reference to the Lie-Rinehart structure on \((A, H)\) and the left \((A, H)\)-module structure on \(Q\), the Lie-Rinehart differentials then turn \(A = \text{Alt}_A(H, A)\) into a differential graded commutative algebra and \(Q = \text{Alt}_A(H, Q)\) into a differential graded (left) \(A\)-module; cf. (1.5.13). Furthermore, the bigraded algebra \(\text{Alt}_A(Q, A)\) of alternating \(A\)-multilinear \(A\)-valued forms on \(Q\) may be rewritten in the form \(\text{Alt}_A(H, \text{Alt}_A(Q, A))\); equivalently, the algebra \(\text{Alt}_A(Q, A)\) may be viewed as the bigraded algebra \(\text{Alt}_A(Q, A)\) of alternating \(A\)-multilinear \(A\)-valued forms on \(Q\). We write the resulting operator \(d_0\), cf. (2.4.5) and (2.5.1), as

\[(4.12.1)\]  
\[d_0: \text{Alt}_A^p(Q, A^q) \to \text{Alt}_A^p(Q, A^{q+1}) \quad (p, q \geq 0).\]
Consider the operators \( d_1 \) and \( d_2 \) on \( \text{Alt}_A(Q,A) \) given as (4.7.1) and (4.8.1) above, respectively. These operators now come down to the operators (2.4.6) and (2.4.7), respectively. By Theorem 2.7, when \((A,Q,H)\) is a genuine Lie-Rinehart triple,

\[
(4.12.2) \quad (\text{Alt}_A(Q,A), d_0, d_1, d_2) = (\text{Alt}_A(Q, \text{Alt}_A(H,A)), d_0, d_1, d_2)
\]
is a Maurer-Cartan algebra, that is, \( d = d_0 + d_1 + d_2 \) turns \( \text{Alt}_A(Q,A) \) into a differential graded algebra. Furthermore, still by Theorem 2.7, under the assumption that \(H\) and \(Q\) both have property \(P\), the converse holds, i.e. when (4.12.2) is a Maurer-Cartan algebra, \((A,Q,H)\) is a genuine Lie-Rinehart triple. In view of Theorem 4.10 we conclude the following.

**Theorem 4.13.** Let \((A,H,Q)\) be a pre-Lie-Rinehart triple and suppose that both \(H\) and \(Q\) have property \(P\), (e.g. \(H\) and \(Q\) are both projective as \(A\)-modules). Then \((A,H,Q)\) is a genuine Lie-Rinehart triple if and only if

\[
(A, Q) = (\text{Alt}_A(H,A), \text{Alt}_A(H,Q)),
\]

endowed with the pairing (4.11.1), the bracket \([\cdot, \cdot]_Q\), cf. (4.11.4), and the operation \(\langle \xi, \eta; \alpha \rangle_Q\), cf. (4.11.6), is a quasi-Lie-Rinehart algebra. \(\square\)

The proof of the following is straightforward and left to the reader:

**Proposition 4.14.** The homology \((H^*(A), H^*(Q))\) of a quasi-Lie-Rinehart algebra \((A, Q)\) inherits a graded Lie-Rinehart algebra structure. \(\square\)

Given a Lie-Rinehart triple \((A,H,Q)\), the graded Lie-Rinehart algebra \((H^*(A), H^*(Q))\) of the corresponding quasi-Lie-Rinehart algebra

\[
(A, Q) = (\text{Alt}_A(H,A), \text{Alt}_A(H,Q))
\]
contains more information than the Lie-Rinehart algebra \((A^H, Q^H) = (H^0(A), H^0(Q))\) spelled out in Corollary 1.11.

**Illustration 4.15.** Let \((M, \mathcal{F})\) be a foliated manifold, maintain the notation established earlier in (1.4.1), (1.12), and (2.11), let \((A,H,Q) = (C^\infty(M), L_\mathcal{F}, Q)\), the corresponding Lie-Rinehart triple, and consider the resulting quasi-Lie-Rinehart algebra \((A, Q) = (\text{Alt}_A(H,A), \text{Alt}_A(H,Q))\). We may view \(A\) as the algebra of generalized functions and \(Q\) as the generalized Lie algebra of vector fields for the foliation. Thus \(A\) is the standard complex arising from a fine resolution of the sheaf of germs of functions on \(M\) which are constant on the leaves. Likewise, the constituent \(Q^H\) of the Lie-Rinehart algebra \((A^H, Q^H)\) (discussed earlier), cf. (1.12) and (2.10) (iii), amounts to the space of global sections of the sheaf \(V_Q\) of germs of vector fields on \(M\) which are horizontal (with respect to the decomposition \(\Gamma(\tau_M) = L_\mathcal{F} \oplus Q\)) and constant on the leaves, and \(Q\) is the standard complex arising from a fine resolution of this sheaf. Thus \(H^*(A)\) is the cohomology of \(M\) with values in the sheaf of germs of functions which are constant on the leaves, and \(H^*(Q)\) is the cohomology of \(M\) with values in the sheaf \(V_Q\).

Under the circumstances of (2.10(ii)), so that the foliation \(\mathcal{F}\) comes from a fiber bundle and the space of leaves coincides with the base \(B\) of the corresponding
fibration, from the graded commutative $\mathbb{R}$-algebra structure of $H^*(F,\mathbb{R})$, the space $\Gamma(\zeta^*)$ of sections of the induced graded vector bundle

$$\zeta^*: P \times_G H^*(F,\mathbb{R}) \to B$$

inherits a graded $C^\infty(B)$-algebra structure and, as a graded $C^\infty(B)$-algebra, $H^*(A)$ coincides with the graded commutative algebra $\Gamma(\zeta^*)$ of sections of $\zeta^*$; in particular, $H^0(A) = C^\infty(B)$. Furthermore, $H^0(Q)$ is the $(\mathbb{R}, C^\infty(B))$-Lie algebra $\text{Vect}(B)$ of smooth vector fields on the base $B$ and, as a graded $(\mathbb{R}, H^*(A))$-Lie algebra, $H^*(Q)$ is the graded crossed product

$$(4.15.1) \quad H^*(Q) = H^*(A) \otimes_{C^\infty(B)} \text{Vect}(B)$$

(cf. [21] for the notion of graded crossed product Lie-Rinehart algebra).

Under the circumstances of (2.10(i)), when the foliation does not come from a fiber bundle, the structure of the graded Lie-Rinehart algebra $(H^*(A), H^*(Q))$ will in general be more complicated than that for the case when the foliation comes from a fiber bundle. The significance of this more complicated structure has been commented on already in the introduction.

**Remark 4.16.** We are indebted to P. Michor for having pointed out to us a possible connection of the notion of quasi-Lie-Rinehart bracket with that of Frölicher-Nijenhuis bracket [12], [48]. Given a smooth manifold $M$, the Frölicher-Nijenhuis bracket is defined on the graded vector space of forms on $M$ with values in the tangent bundle $\tau_M$ of $M$ and endows this graded vector space with a graded Lie algebra structure which in degree zero amounts to the ordinary Lie bracket of vector fields on $M$. Given a Lie-Rinehart algebra $(A, L)$, an obvious generalization of the Frölicher-Nijenhuis bracket endows the graded $A$-module $\text{Alt}_A(L, L)$ with a graded $R$-Lie algebra structure. Given a Lie-Rinehart triple $(A, H, Q)$, with corresponding Lie-Rinehart algebra $(A, L)$ where $L = H \oplus Q$, the induced quasi-Lie-Rinehart bracket $(4.11.4)$ is defined on $\text{Alt}_A(H, Q)$, and the obvious question arises how this quasi-Lie-Rinehart bracket is related with the Frölicher-Nijenhuis bracket on $\text{Alt}_A(L, L)$.

**5. Quasi-Gerstenhaber algebras**

The notion of Gerstenhaber algebra has recently been isolated in the literature but implicitly occurs already in Gerstenhaber’s paper [13]; see [19] for details and more references. In this section we will introduce a notion of quasi-Gerstenhaber algebra which generalizes that of strict differential bigraded Gerstenhaber algebra isolated in [21, 22] (where the attribute “strict” refers to the requirement that the differential be a derivation for the Gerstenhaber bracket). The generalization consists in admitting a bracket which does not necessarily satisfy the graded Jacobi identity and incorporating an additional piece of structure which measures the deviation from the graded Jacobi identity.

For intelligibility, we recall the notion of graded Lie algebra, tailored to our purposes. As before, $R$ denotes a commutative ring with $1$. A graded $R$-module $\mathfrak{g}$, endowed with a graded skew-symmetric degree zero bracket $[\cdot, \cdot, \cdot]: \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, is called a graded Lie algebra provided the bracket satisfies the graded Jacobi identity

$$\sum_{(a,b,c) \text{ cyclic}} (-1)^{|a||c|}[a, [b, c]] = 0,$$
for every triple \((a, b, c)\) of homogeneous elements of \(\mathcal{G}\).

Given a graded commutative algebra \(\mathcal{A}\), an (ordered) \(m\)-tuple \(a = (a_1, \ldots, a_m)\) of homogeneous elements thereof, and a permutation \(\sigma\) of \(m\) objects, we denote by \(\varepsilon(a, \sigma)\) the sign defined by

\[
a_1 \cdot \ldots \cdot a_m = \varepsilon(a, \sigma)a_{\sigma 1} \cdot \ldots \cdot a_{\sigma m}
\]

according to the Eilenberg-Koszul convention.

We will consider bigraded \(R\)-algebras. Such a bigraded algebra is said to be bigraded commutative provided it is commutative in the bigraded sense, that is, graded commutative with respect to the total degree. Given such a bigraded commutative algebra \(\mathcal{G}\), for bookkeeping purposes, we will write its homogeneous components in the form \(\mathcal{G}^p_0\), the superscript being viewed as a cohomology degree and the subscript as a homology degree; the total degree \(|\alpha|\) of an element \(\alpha\) of \(\mathcal{G}^p_0\) is, then, \(|\alpha| = p - q\).

We will explore differential operators in the bigraded context. We recall the requisite notions from [41] (Section 1), cf. also [1]. Let \(\mathcal{G}\) be a bigraded commutative \(R\)-algebra with 1, and let \(r \geq 1\). A (homogeneous) differential operator on \(\mathcal{G}\) of order \(\leq r\) is a homogeneous \(R\)-endomorphism \(D\) of \(\mathcal{G}\) such that a certain \(\mathcal{G}\)-valued \((r + 1)\)-form \(\Phi^{r+1}_D\) on \(\mathcal{G}\) (the definition of which for general \(r\) we do not reproduce here) vanishes. For our purposes, it suffices to recall explicit descriptions of these forms in low degrees. Thus, given the homogeneous \(R\)-endomorphism \(D\) of \(\mathcal{G}\), for homogeneous \(\xi, \eta, \vartheta\),

\[
\Phi^1_D(\xi) = D(\xi) - D(1)\xi \\
\Phi^2_D(\xi, \eta) = D(\xi\eta) - D(\xi)\eta - (-1)^{|\xi||\eta|}D(\eta)\xi + D(1)\xi\eta \\
\Phi^3_D(\xi, \eta, \vartheta) = D(\xi\eta\vartheta) \\
- D(\xi\eta)\vartheta - (-1)^{|\xi|(|\eta|+|\vartheta|)}D(\eta)\vartheta\xi - (-1)^{|\vartheta|(|\xi|+|\eta|)}D(\vartheta)\xi\eta \\
+ D(\xi)\eta\vartheta + (-1)^{|\xi|(|\eta|+|\vartheta|)}D(\eta)\vartheta\xi + (-1)^{|\vartheta|(|\xi|+|\eta|)}D(\vartheta)\xi\eta \\
- D(1)\xi\eta\vartheta.
\]

In the literature, a (homogeneous) differential operator \(D\) of order \(\leq r\) with \(D(1) = 0\) is also referred to as a (homogeneous) derivation of order \(\leq r\). In particular, a homogeneous derivation \(d\) of (total) degree 1 and order 1 is precisely a differential turning \(\mathcal{G}\) into a differential graded \(R\)-algebra.

With these preparations out of the way, consider a bigraded commutative \(R\)-algebra \(\mathcal{G}\) with 1, with \(\mathcal{G}^0_0\) zero when \(q < 0\) or \(p < 0\), together with

— a homogeneous bracket \([\cdot, \cdot]: \mathcal{G} \otimes_R \mathcal{G} \to \mathcal{G}\) of bidegree \((0, -1)\), where “bidegree \((0, -1)\)” means that, in given bidegrees \((q_1, p_1)\) and \((q_2, p_2)\), the bracket takes the form

\[
[\cdot, \cdot]: \mathcal{G}^{q_1}_{p_1} \otimes \mathcal{G}^{q_2}_{p_2} \to \mathcal{G}^{q_1+q_2}_{p_1+p_2-1};
\]

— a differential \(d: \mathcal{G}_*^{\cdot} \to \mathcal{G}_*^{\cdot+1}\) of bidegree \((1, 0)\) which endows \(\mathcal{G}\) (with respect to the total degree) with a differential graded \(R\)-algebra structure, and
— a homogeneous differential operator \( \Psi: \mathcal{G} \to \mathcal{G} \) of order \( \leq 3 \) with \( \Psi(1) = 0 \) which is \( \mathcal{G}^0_o \)-linear and of bidegree \((-1, -2)\), i.e. in bidegree \((q, p)\), \( \Psi \) may be depicted as

\[
(5.1) \quad \Psi: \mathcal{G}^q_p \to \mathcal{G}^{q-1}_{p-2} \quad (q \geq 1, p \geq 2).
\]

In particular, \( \Psi \) is zero on \( \mathcal{G}^0_o, \mathcal{G}^*_0, \mathcal{G}^*_1 \). Notice that \( d \) and \( \Psi \) both lower total degree by 1, that is, are homogeneous operators on \( \mathcal{G} \) of degree -1.

We will refer to the bracket \([\cdot, \cdot]\) as a quasi-Gerstenhaber bracket and to \( \Psi \) as an \( h\)-Jacobiator for the bracket \([\cdot, \cdot]\) provided \([\cdot, \cdot]\) and \( \Psi \) satisfy (5.i)–(5.vi) below.

(5.i) The bracket \([\cdot, \cdot]\) is graded skew-symmetric when the total degree of \( \mathcal{G} \) is regraded down by one, that is, for homogeneous \( \alpha, \beta \in \mathcal{G} \),

\[
(5.2) \quad [\alpha, \beta] = -(-1)^{|\alpha|(|\beta|-1)}[\beta, \alpha];
\]

(5.ii) for each homogeneous element \( \alpha \) of \( \mathcal{G} \) of bidegree \((q, p)\), the operation \([\alpha, \cdot]\) is a derivation of \( \mathcal{G} \) of bidegree \((q-1, p)\) for the multiplicative structure on \( \mathcal{G} \); that is to say, \([\alpha, \cdot]\) may be depicted as

\[ [\alpha, \cdot]: \mathcal{G}^*_q \to \mathcal{G}^{*+q-1}_{q+p} \]

and, for homogeneous \( \beta, \gamma \in \mathcal{G} \),

\[
(5.3) \quad [\alpha, \beta \gamma] = [\alpha, \beta] \gamma + (-1)^{|\alpha||\beta|}[\beta, \gamma] \alpha.
\]

(5.iii) The differential \( d \) behaves as a derivation for the bracket \([\cdot, \cdot]\), that is, for homogeneous \( x, y \in \mathcal{G} \),

\[
(5.4) \quad d[x, y] = [dx, y] - (-1)^{|x|}[x, dy].
\]

(5.iv) Given homogeneous elements \( \xi, \eta, \vartheta \) of \( \mathcal{G} \),

\[
(5.5) \quad \sum_{(\xi, \eta, \vartheta) \text{ cyclic}} (-1)^{|\xi|(|\eta|-1)(|\vartheta|-1)}[\xi, [\eta, \vartheta]] = (-1)^{|\xi|+|\eta|+|\vartheta|} \Phi^3_{d\Psi+\Psi d}(\xi, \eta, \vartheta).
\]

(5.v) the differential operator \( \Psi \) has square zero and

(5.vi) the bracket \([\cdot, \cdot]\) and \( \Psi \) are related by the following requirement: For every ordered quadruple \( \mathbf{a} = (a_1, a_2, a_3, a_4) \) of homogeneous elements of \( \mathcal{G} \),

\[
(5.6) \quad \sum_{\sigma} \varepsilon(\sigma)\varepsilon(\mathbf{a}, \sigma)\Phi^3_{\Psi}(a_{\sigma_1}, a_{\sigma_2}, a_{\sigma_3}, a_{\sigma_4}) = \sum_{\tau} \varepsilon(\tau)\varepsilon(\mathbf{a}, \tau)\Phi^3_{\Psi}(a_{\tau_1}, a_{\tau_2}, a_{\tau_3}, a_{\tau_4})
\]

where \( \sigma \) runs through \((3,1)\)-shuffles and \( \tau \) through \((2,2)\)-shuffles and where \( \varepsilon(\sigma) \) and \( \varepsilon(\tau) \) are the signs of the permutations \( \sigma \) and \( \tau \). The data \((\mathcal{G}; d, [\cdot, \cdot], \Psi)\) will then be referred to as a quasi-Gerstenhaber algebra. Notice that (5.3) implies that \([\alpha, 1] = 0\) for every homogeneous element \( \alpha \) of \( \mathcal{G} \).

We note that, given an \( L_{\infty} \)-algebra \( \mathfrak{h} \) with only two-variable and three-variable brackets \([\cdot, \cdot]\) and \([\cdot, \cdot, \cdot]\), respectively (and no non-zero higher order bracket operation), the compatibility condition which relates \([\cdot, \cdot]\) and \([\cdot, \cdot, \cdot]\) is exactly an identity of the kind (5.6), when \([\cdot, \cdot, \cdot]\) is substituted for \( \Phi^3_{\Psi} \).

A quasi-Gerstenhaber algebra having \( \Psi \) zero is just an ordinary strict differential bigraded Gerstenhaber algebra. Indeed, in the general (quasi-) case, in view of the requirement (5.iv), the operation \( \Psi \) measures the failure of the quasi-Gerstenhaber bracket \([\cdot, \cdot]\) to satisfy the graded Jacobi identity in a coherent fashion. A strict differential bigraded Gerstenhaber algebra having zero differential is called a bigraded Gerstenhaber algebra [21, 22]. Given a quasi-Gerstenhaber algebra \((\mathcal{G}; d, [\cdot, \cdot], \Psi)\), we denote its \( d \)-homology by \( H^*_d(\mathcal{G}) \). The following is straightforward.
**Proposition 5.7.** Given a quasi-Gerstenhaber algebra \((G; d, [\cdot, \cdot], \Psi)\), the quasi-Gerstenhaber bracket \([\cdot, \cdot]\) induces a bracket

\[
[\cdot, \cdot]: H^q_{p_1}(G) \otimes H^q_{p_2}(G) \to H^{q_1+q_2}_{p_1+p_2-1}(G)
\]

on the \(d\)-homology \(H^*_d(G)\) which turns \(H^*_d(G)\) into an ordinary bigraded Gerstenhaber algebra. \(\square\)

5.8. Relationship with Lie-Rinehart triples. We will now explain how quasi-Gerstenhaber algebras arise from Lie-Rinehart triples. To this end, we recall that, given an ordinary Lie-Rinehart algebra \((A, L)\), Gerstenhaber algebras arise from Lie-Rinehart triples. To this end, we recall that, given an ordinary Lie-Rinehart algebra \((A, L)\), the Lie bracket on \(L\) and the \(L\)-action on \(A\) determine a Gerstenhaber bracket on the exterior \(A\)-algebra \(\Lambda_A L\) on \(L\); for \(\alpha_1, \ldots, \alpha_n \in L\), the bracket \([u, v]\) in \(\Lambda_A L\) of \(u = \alpha_1 \wedge \ldots \wedge \alpha_\ell\) and \(v = \alpha_{\ell+1} \wedge \ldots \wedge \alpha_n\) is given by the expression

\[
[u, v] = (-1)^\ell \sum_{1 \leq j \leq \ell < k \leq n} (-1)^{j+k}[\alpha_j, \alpha_k] \wedge \alpha_1 \wedge \ldots \widehat{\alpha_j} \ldots \widehat{\alpha_k} \ldots \wedge \alpha_n,
\]

where \(\ell = |u|\) is the degree of \(u\), cf. [19] (1.1). In fact, given the \(R\)-algebra \(A\) and the \(A\)-module \(L\), a bracket of the kind (5.8.1) yields a bijective correspondence between Lie-Rinehart structures on \((A, L)\) and Gerstenhaber algebra structures on \(\Lambda_A L\). Our goal, which will be achieved in the next section, is now to extend this observation to a relationship between Lie-Rinehart triples, quasi-Lie-Rinehart algebras, and quasi-Gerstenhaber algebras.

Thus, let \((A, Q, H)\) be a pre-Lie-Rinehart triple. Consider the graded exterior \(A\)-algebra \(\Lambda_A Q\), and let \(G = \text{Alt}_A(H, \Lambda_A Q)\), with the bigrading \(G^p_q = \text{Alt}^p_q(A(H, \Lambda_A Q)\) \((p, q \geq 0)\). Suppose for the moment that \((A, Q, H)\) is merely an almost pre-Lie-Rinehart triple. Recall that the almost pre-Lie-Rinehart triple structure induces operations of the kind (4.11.3), (4.11.4), and (4.11.6) on the pair

\[
(A, Q) = (\text{Alt}_A(H, A), \text{Alt}_A(H, Q))
\]

but, at the present stage, this pair is not necessarily a quasi-Lie-Rinehart algebra. Consider the bigraded algebra \(\text{Alt}_A(H, \Lambda_A Q)\); at times we will view it as the exterior \(A\)-algebra on \(Q\), and we will accordingly write

\[
\Lambda_A Q = \text{Alt}_A(H, \Lambda_A Q).
\]

The graded skew-symmetric bracket (4.11.4) on \(Q\) \((= \text{Alt}_A(H, Q))\) extends to a (bigraded) bracket

\[
[\cdot, \cdot]: \Lambda_A Q \otimes_R \Lambda_A Q \to \Lambda_A Q
\]

on \(\Lambda_A Q = \text{Alt}_A(H, \Lambda_A Q)\). Indeed, with reference to the graded bracket \([\cdot, \cdot]\) on \(Q\) spelled out as (4.11.4) (and written there as \([\cdot, \cdot]_Q\) and the pairing (4.11.1), the bigraded bracket (5.8.3) on \(\Lambda_A Q = \text{Alt}_A(H, \Lambda_A Q)\) is determined by the formulas

\[
[\alpha, \beta, \gamma] = \alpha[\beta, \gamma] + (-1)^{|\alpha||\beta|}[\beta, \alpha, \gamma],
\]

\[
[\xi, \alpha] = \xi(\alpha),
\]

\[
[\alpha, \beta] = -(-1)^{(|\alpha|-1)(|\beta|-1)}[\beta, \alpha]
\]

(5.8.4)
where \( \alpha, \beta, \gamma \) are homogeneous elements of \( \Lambda_A Q = \text{Alt}_A(H, \Lambda_A Q) \), and where \( \xi \in Q \) and \( a \in A \).

We now construct an operation \( \Psi \) of the kind (5.1) from the operation \( \langle \cdot, \cdot, \cdot \rangle_Q \), that is, one which formally looks like an \( h \)-Jacobiator for (5.8.3). To this end we suppose that, as an \( A \)-module, at least one of \( H \) or \( Q \) is finitely generated and projective; then the canonical \( A \)-linear morphism from \( \text{Alt}_A(H, A) \otimes \Lambda_A Q \) to \( \text{Alt}_A(H, \Lambda_A Q) \) is an isomorphism of bigraded \( A \)-algebras. Let \( \xi_1, \ldots, \xi_p \in Q \). Now, given a homogeneous element \( \beta \) of \( \text{Alt}_A(H, A) \), with reference to the operation \( \langle \cdot, \cdot, \cdot \rangle_Q \) induced by \( \delta \), cf. (4.11.6), let

\[
\tag{5.8.5}
\Psi(\beta \xi_1 \wedge \ldots \wedge \xi_p) = \sum_{1 \leq j < k \leq p} (-1)^{j+k} \langle \xi_j, \xi_k; \beta \rangle_Q \xi_1 \wedge \ldots \hat{\xi}_j \ldots \hat{\xi}_k \ldots \wedge \xi_p;
\]

we will write \( \Psi_\beta \) rather than just \( \Psi \) whenever appropriate. As an operator on the graded \( A \)-algebra \( \text{Alt}_A(H, \Lambda_A Q) \), \( \Psi \) may be written as a finite sum of operators which are three consecutive contractions each; since an operator which consists of three consecutive contractions is a differential operator of order \( \leq 3 \), the operator \( \Psi \) is a differential operator of order \( \leq 3 \). Furthermore, since for \( \xi, \eta \in Q \), the operation \( \langle \xi, \eta; \cdot \rangle_Q \) is a derivation of the graded \( A \)-algebra \( \text{Alt}_A(H, A) \), given homogeneous elements \( \beta_1 \) and \( \beta_2 \) of \( \text{Alt}_A(H, A) \),

\[
\tag{5.8.6}
\Psi(\beta_1 \beta_2 \xi_1 \wedge \ldots \wedge \xi_p) = (-1)^{|\beta_1|} \beta_1 \Psi(\beta_2 \xi_1 \wedge \ldots \wedge \xi_p) + (-1)^{|\beta_1|+1} |\beta_2| \beta_2 \Psi(\beta_1 \xi_1 \wedge \ldots \wedge \xi_p)
\]

A somewhat more intrinsic description of \( \Psi \) results from the observation that the operation

\[
\Psi: \text{Alt}_A^1(H, \Lambda_A^2 Q) = \text{Hom}_A(H, \Lambda_A^2 Q) \rightarrow A \cong \text{Alt}_A^0(H, \Lambda_A^0 Q)
\]

is simply given by the assignment to \( \chi: H \rightarrow \Lambda_A^2 Q \) of the trace of the \( A \)-module endomorphism \( \delta \circ \chi \) of \( H \) when \( H \) is finitely generated and projective as an \( A \)-module, and of the trace of the \( A \)-module endomorphism \( \chi \circ \delta \) of \( \Lambda_A^2 Q \) when \( Q \) is finitely generated and projective as an \( A \)-module.

We now give another description of \( \Psi \), cf. (5.8.11) below, under an additional hypothesis: Suppose that, as an \( A \)-module, \( Q \) is finitely generated and projective of constant rank \( n \). Then the canonical \( A \)-module isomorphism

\[
\phi: \Lambda_A^* Q \rightarrow \text{Alt}_A^{n-*}(Q, \Lambda_A^n Q)
\]

extends to an isomorphism

\[
\tag{5.8.7}
\phi: \text{Alt}_A^*(H, \Lambda_A^* Q) \rightarrow \text{Alt}_A^*(H, \text{Alt}_A^{n-*}(Q, \Lambda_A^n Q))
\]

of graded \( A \)-modules. In this fashion, \( \text{Alt}_A^*(H, \Lambda_A^* Q) \) acquires a bigraded \( \text{Alt}_A^*(H, \text{Alt}_A^{n-*}(Q, A)) \)-module structure, induced from the graded \( A \)-module \( \Lambda_A^n Q \). Further, the skew-symmetric \( A \)-bilinear pairing (1.5.5) induces an operator

\[
\tag{5.8.8}
d_2: \text{Alt}_A^*(H, \text{Alt}_A^{n-*}(Q, \Lambda_A^n Q)) \rightarrow \text{Alt}_A^{*-1}(H, \text{Alt}_A^{n-(n-2)}(Q, \Lambda_A^n Q)).
\]
This is just the operator \((2.4.7')\), suitably rewritten, with \(M = \Lambda_A^Q\), where the degree of the latter \(A\)-module forces the correct sign: The \(A\)-module \(\Lambda_A^n Q\) is concentrated in degree \(n\), and a form in \(\text{Alt}^{n-p}_A(Q, \Lambda_A^n Q)\) has degree \(p\). In bidegree \((q, p)\), given
\[
\psi \in \text{Alt}^q_A(H, \text{Alt}^{n-p}_A(Q, \Lambda_A^n Q)),
\]
the value
\[
d_2(\psi) \in \text{Alt}^{q-1}_A(H, \text{Alt}^{n-p+2}_A(Q, \Lambda_A^n Q))
\]
of the operator \((5.8.8)\) is given by the formula
\[
(-1)^{|\psi|+1} ((d_2 \psi)(x_1, \ldots, x_{q-1}))(\xi_{p-1}, \ldots, \xi_n)
= \sum_{p-1 \leq j < k \leq n} (-1)^{j+k} (\psi(\delta(\xi_j, \xi_k), x_1, \ldots, x_{q-1})) (\xi_{p-1}, \ldots, \xi_j \hat{\ldots} \hat{\xi}_k \ldots, \xi_n),
\]
where \(x_1, \ldots, x_{q-1} \in H\) and \(\xi_{p-1}, \ldots, \xi_n \in Q\) and, with \(|\psi| = q + p\) (the correct degree would be \(|\psi| = q - p\) but modulo 2 this makes no difference), this simplifies to
\[
(-1)^p ((d_2 \psi)(x_1, \ldots, x_{q-1}))(\xi_{p-1}, \ldots, \xi_n)
= \sum_{p-1 \leq j < k \leq n} (-1)^{j+k} (\psi(x_1, \ldots, x_{q-1}, \delta(\xi_j, \xi_k))) (\xi_{p-1}, \ldots, \xi_j \hat{\ldots} \hat{\xi}_k \ldots, \xi_n);
\]
cf. \((2.5.4)\).

**Lemma 5.8.10.** The operator \(\Psi\) makes the diagram
\[
\begin{array}{ccc}
\text{Alt}^*_A(H, \Lambda_A^Q) & \xrightarrow{\Psi} & \text{Alt}^{*}_A(H, \Lambda_A^{*-2}Q) \\
\phi \downarrow & & \downarrow \phi \\
\text{Alt}^*_A(H, \text{Alt}^{*-p}_A(Q, \Lambda_A^n Q)) & \xrightarrow{d_2} & \text{Alt}^{*}_A(H, \text{Alt}^{*-2}_A(Q, \Lambda_A^n Q))
\end{array}
\]
commutative.

Thus, under the isomorphism \((5.8.7)\), the operator \(\Psi\) is induced by the operator \(d_2\) (on the right-hand side of \((5.8.7)\)).

**Proof.** In a given bidegree \((q, p)\), the isomorphism \((5.8.7)\) sends \(\alpha \in \text{Alt}^q_A(H, \Lambda_A^n Q)\) to
\[
\phi_\alpha \in \text{Alt}^q_A(H, \text{Alt}^{n-p}_A(Q, \Lambda_A^n Q))
\]
determined by the identity
\[
(\phi_\alpha(x_1, \ldots, x_q))(\xi_{p+1}, \ldots, \xi_n) = (\alpha(x_1, \ldots, x_q)) \wedge \xi_{p+1} \wedge \ldots \wedge \xi_n,
\]
for arbitrary \(x_1, \ldots, x_q \in H\) and \(\xi_{p+1}, \ldots, \xi_n \in Q\). Under the isomorphism \((5.8.7)\), the operator \(d_2\) (on the right-hand side of \((5.8.7)\)) induces an operator
\[
(5.8.11) \quad \Theta = \Theta_\delta : \text{Alt}^q_A(H, \Lambda_A^n Q) \to \text{Alt}^{q-1}_A(H, \Lambda_A^{p-2} Q)
\]
of the kind (5.1) on the left-hand side of (5.8.7); by construction, for $x_1, \ldots, x_{q-1} \in H$ and $\xi_{p-1}, \ldots, \xi_n \in Q$, in view of (5.8.9),

$$(-1)^p ((\Theta\alpha)(x_1, \ldots, x_{q-1})) \wedge \xi_{p-1} \wedge \cdots \wedge \xi_n = \sum_{p-1 \leq j < k \leq n} (-1)^{j+k} (\alpha(x_1, \ldots, x_{q-1}, \delta(\xi_j, \xi_k))) \wedge \xi_{p-1} \wedge \cdots \xi_j \wedge \xi_k \wedge \xi_n.$$ 

Let $\beta \in \text{Alt}^q_A(H, A), \eta_1, \ldots, \eta_p \in Q$, and $\alpha = (\eta_1 \wedge \cdots \wedge \eta_p)\beta$; then, for $p-1 \leq j < k \leq n$,

$$\alpha(x_1, \ldots, x_{q-1}, \delta(\xi_j, \xi_k)) = (\eta_1 \wedge \cdots \wedge \eta_p)\beta(x_1, \ldots, x_{q-1}, \delta(\xi_j, \xi_k)) = \beta(x_1, \ldots, x_{q-1}, \delta(\xi_j, \xi_k))\eta_1 \wedge \cdots \wedge \eta_p = (-1)^{q-1}\beta(\delta(\xi_j, \xi_k), x_1, \ldots, x_{q-1})\eta_1 \wedge \cdots \wedge \eta_p = (-1)^{q-1}(\xi_j, \xi_k; \beta)Q(x_1, \ldots, x_{q-1})\eta_1 \wedge \cdots \wedge \eta_p = (-1)^{q-1+p(q-1)}((\xi_j, \xi_k; \beta)Q)\eta_1 \wedge \cdots \wedge \eta_p) (x_1, \ldots, x_{q-1})$$

whence

$$(-1)^{p+q-1} ((\Theta\alpha)(x_1, \ldots, x_{q-1})) \wedge \xi_{p-1} \wedge \cdots \wedge \xi_n = \sum_{p-1 \leq j < k \leq n} (-1)^{j+k} (\xi_j, \xi_k; \beta)Q(x_1, \ldots, x_{q-1})\eta_1 \wedge \cdots \wedge \eta_p \wedge \xi_{p-1} \wedge \cdots \xi_j \wedge \xi_k \wedge \xi_n.$$ 

Let $(\eta_1, \ldots, \eta_p) = (\xi_1, \ldots, \xi_p)$. With $j = p-1$ and $k = p$, this yields

$$(-1)^{p+q-1} ((\Theta\alpha)(x_1, \ldots, x_{q-1})) \wedge \xi_{p-1} \wedge \cdots \wedge \xi_n = - (\xi_{p-1}, \xi_p; \beta)Q(x_1, \ldots, x_{q-1})\xi_1 \wedge \cdots \wedge \xi_p \wedge \xi_{p+1} \wedge \cdots \wedge \xi_n$$

or, equivalently, since $|x_1| + \cdots + |x_{q-1}| = q-1$ and $|\xi_1 \wedge \cdots \wedge \xi_p| = p$,

$$(-1)^{pq} ((\Theta\alpha)(x_1, \ldots, x_{q-1})) \wedge \xi_{p-1} \wedge \cdots \wedge \xi_n = - (\xi_{p-1}, \xi_p; \beta)Q\xi_1 \wedge \cdots \wedge \xi_{p-2} \wedge \cdots \wedge \xi_{p-2} \wedge \xi_{p-1} \wedge \cdots \wedge \xi_n$$

Hence

$$(-1)^{pq}(\Theta\alpha) = - (\xi_{p-1}, \xi_p; \beta)Q\xi_1 \wedge \cdots \wedge \xi_{p-2} \pm \cdots$$

or, equivalently,

$$\Theta(\beta\xi_1 \wedge \cdots \wedge \xi_p) = - (\xi_{p-1}, \xi_p; \beta)Q\xi_1 \wedge \cdots \wedge \xi_{p-2} \pm \cdots$$

where $\cdots$ stands for terms involving $\xi_1 \wedge \cdots \xi_j \cdots \xi_k \wedge \xi_{p-2}$ with $(j, k) \neq (p-1, p)$. Consequently

$$\Theta(\beta\xi_1 \wedge \cdots \wedge \xi_p) = \sum_{1 \leq j < k \leq p} (-1)^{j+k} (\xi_j, \xi_k; \beta)Q\xi_1 \wedge \cdots \xi_j \wedge \xi_k \wedge \xi_{p}.$$ 

However, this is exactly the definition (5.8.5) of $\Psi$. □
In view of Remark 2.6, the operator Ψ thus calculates essentially the Lie algebra cohomology $H^*(L_{nil}, \Lambda^n_A Q)$ of the (nilpotent) $A$-Lie algebra $L_{nil}$ (= $H \oplus Q$ as an $A$-module) with values in the $A$-module $\Lambda^n_A Q$, viewed as a trivial $L_{nil}$-module. In particular, Ψ is $A$-linear.

Suppose finally that $(A, Q, H)$ is a genuine Lie-Rinehart triple, not just an almost pre-Lie-Rinehart triple. By Proposition 4.13, $(A, Q)$ then acquires a quasi-Lie-Rinehart structure. Our ultimate goal is now to prove that, likewise, $\Lambda^n_A Q$ endowed with the bigraded bracket (5.8.3) and the operation Ψ, cf. (5.8.5), (which formally looks like an $h$-Jacobiator) acquires a quasi-Gerstenhaber structure. The verification of the requirements (5.2)–(5.4) does not present any difficulty at this stage, and the vanishing of ΨΨ is immediate. However we were so far unable to establish (5.5) and (5.6) without an additional piece of structure, that of a generator of a (quasi-Gerstenhaber) bracket. The next section is devoted to the notion of generator and the consequences it entails. A precise statement is given as Corollary 6.10.4 below.

6. Quasi-Batalin-Vilkovisky algebras and quasi-Gerstenhaber algebras

Let $G = G^*_\ast$ be a bigraded commutative $R$-algebra, endowed with a bigraded bracket $[\cdot, \cdot] : G \otimes_R G \to G$ of bidegree $(0, -1)$ which is graded skew-symmetric when the total degree is regraded down by 1. Extending terminology due to Koszul, cf. the definition of $[\cdot, \cdot]_D$ on p. 260 of [41], we will say that an $R$-linear operator $\Delta$ on $G$ of bidegree $(0, -1)$ generates the bracket $[\cdot, \cdot]$ provided, for every homogeneous $a, b \in G$,

$$[a, b] = (-1)^{|a|} \left( \Delta(ab) - (\Delta a)b - (-1)^{|a|}a(\Delta b) \right) \quad (= (-1)^{|a|}\Phi_\Delta^2(a, b));$$

we then refer to the operator $\Delta$ as a generator.

In particular, let $(G; d, [\cdot, \cdot], \Psi)$ be a quasi-Gerstenhaber algebra over $R$. In view of the identity (1.4) on p. 260 of [41], a generator $\Delta$ is then necessarily a differential operator on $G$ of order $\leq 2$. Indeed, given a differential operator $D$, this identity reads

$$\Phi_D^3(a, b, c) = \Phi_D^2(a, bc) - \Phi_D^2(a, b)c - (-1)^{|b||c|}\Phi_D^2(a, c)b.$$

Hence, when a differential operator $\Delta$ generates a quasi-Gerstenhaber bracket $[\cdot, \cdot]$,

$$\Phi_\Delta^3(a, b, c) = (-1)^{|a|} \left( [a, bc] - [a, b]c - (-1)^{|b||c|}[a, c]b \right).$$

However, by virtue of (5.3), the right-hand side of this identity is zero, whence $\Delta$ is necessarily of order $\leq 2$.

A generator $\Delta$ of a quasi-Gerstenhaber bracket satisfies $\Delta(1)a = 0$ for every $a \in G$ since, with respect to the multiplication map on $G$, the quasi-Gerstenhaber bracket behaves as a derivation of the appropriate degree in each variable of the bracket, cf. (5.3). We will say that a generator $\Delta$ is strict provided $\Delta(1) = 0$ and

(6.2) $d\Delta + \Delta d = 0$,
(6.3) $d\Psi + \Delta\Delta + \Psi d = 0$;
(6.4) $\Delta\Psi + \Psi\Delta = 0$;
Let $G$ be a bigraded commutative algebra, with differential operators

$$d: G^* \to G^{*+1}, \quad \Delta: G^* \to G^{*-1}, \quad \Psi: G^* \to G^{*-1},$$

having orders, respectively, $\leq 1$, $\leq 2$, $\leq 3$, and having the properties $d(1) = 0$, $\Delta(1) = 0$, $\Psi(1) = 0$. We will say that $(G; d, \Delta, \Psi)$ is a quasi-Batalin-Vilkovisky algebra provided $dd = 0$ and the operators $d, \Delta, \Psi$ satisfy the identities (6.2)–(6.4) as well as the identity

$$\Psi \Psi = 0.$$

Thus $(G; d, \Delta, \Psi)$ is a quasi-Batalin-Vilkovisky algebra if and only if, on the totalization, the operator $D = d + \Delta + \Psi$ has square zero.

### 6.6. From quasi-Batalin-Vilkovisky algebras to quasi-Gerstenhaber algebras.

**Theorem 6.6.1.** Given a quasi-Batalin-Vilkovisky algebra $(G; d, \partial, \Psi)$ with $G^p = 0$ for $q < 0$ and $p < 0$, let $[\cdot, \cdot]$ be the bracket on $G$ generated by $\partial$. Then $(G; d, [\cdot, \cdot], \Psi)$ is a quasi-Gerstenhaber algebra provided, as a bigraded $G^0$-algebra, $G$ is generated by its homogeneous constituents $G^1_0$ and $G^1_1$.

To prepare for the proof, we need the following.

**Lemma 6.6.2.** Let $G = \{G^*_q\}$ be a bigraded commutative $R$-algebra with $G^p = 0$ for $q < 0$ and $p < 0$, let $\Delta: G^* \to G^{*-1}$ and $\Psi: G^* \to G^{*-1}$ be differential operators of orders $\leq 2$ and $\leq 3$, respectively, and let $[\cdot, \cdot]: G^* \to G^{* - 1}$ be the bracket (6.1) generated by $\Delta$. Let $A = G^1_0$, and suppose that, as a bigraded $A$-algebra, $G$ is generated by its homogeneous constituents $G^1_0$ and $G^1_1$ and that $\Psi$ is $A$-linear. Then

$$\Delta \Psi + \Psi \Delta = 0$$

if and only if, for every ordered quadruple $a = (a_1, a_2, a_3, a_4)$ of homogeneous elements of $G$,

$$\sum_{\sigma} \varepsilon(\sigma) \varepsilon(a, \sigma) \Phi^3_{\Psi}(a_{\sigma_1}, a_{\sigma_2}, a_{\sigma_3}, a_{\sigma_4}) = \sum_{\tau} \varepsilon(\tau) \varepsilon(a, \tau) \Phi^3_{\Psi}([a_{\tau_1}, a_{\tau_2}], a_{\tau_3}, a_{\tau_4})$$

where $\sigma$ runs through $(3,1)$-shuffles and $\tau$ through $(2,2)$-shuffles and where $\varepsilon(\sigma)$ and $\varepsilon(\tau)$ are the signs of the permutations $\sigma$ and $\tau$. 

We note the identity (6.6.4) is formally the same as (5.6), but the circumstances are now more general. We also note that the hypotheses of the Lemma imply that $\Delta(1) = 0$ and $\Psi(1) = 0$.

**Remark 6.6.5.** For a graded commutative (not bigraded) algebra, multiplicatively generated by its homogeneous degree 1 constituent and endowed with a suitable Batalin-Vilkovisky structure, formally the same identity as (5.6) has been derived in Theorem 3.2 of [5]. Our totalization Tot yields a notion of Batalin-Vilkovisky algebra not equivalent to that explored in [5], though; see Remark 6.17 below for details. The distinction between the ground ring $R$ and the $R$-algebra $A$, crucial
for our approach (involving in particular Lie-Rinehart algebras and variants thereof), complicates the situation further. We therefore give a complete proof of the Lemma.

Proof of Lemma 6.6.2. We start by exploring the operator

\[ \Delta \Psi + \Psi \Delta : G^1_3 \rightarrow G^0_0 = A. \]

Let \( \alpha \in G^1_0 \) and \( \xi_1, \xi_2, \xi_3 \in G^1_1 \); then \( \alpha \xi_1 \xi_2 \xi_3 \in G^1_3 \). Since \( \Psi \) is of order 3,

\[ \Phi^4_\Psi(\alpha, \xi_1, \xi_2, \xi_3) = 0 \]

and, for degree reasons, this identity boils down to

\[ \Psi(\alpha \xi_1 \xi_2 \xi_3) = \Psi(\alpha \xi_1 \xi_2) \xi_3 + \Psi(\alpha \xi_2 \xi_3) \xi_1 + \Psi(\alpha \xi_3 \xi_1) \xi_2. \]

In view of the definition (6.1) of the bracket \([\cdot, \cdot]\),

\[ [\Psi(\alpha \xi_1 \xi_2), \xi_3] = \Delta(\Psi(\alpha \xi_1 \xi_2) \xi_3) - \Psi(\alpha \xi_1 \xi_2) \Delta(\xi_3) \]

whence

\[ \Delta \Psi(\alpha \xi_1 \xi_2 \xi_3) = [\Psi(\alpha \xi_1 \xi_2), \xi_3] + [\Psi(\alpha \xi_2 \xi_3), \xi_1] + [\Psi(\alpha \xi_3 \xi_1), \xi_2] + \Psi(\alpha \xi_1 \xi_2) \Delta(\xi_3) + \Psi(\alpha \xi_2 \xi_3) \Delta(\xi_1) + \Psi(\alpha \xi_3 \xi_1) \Delta(\xi_2). \]

On the other hand,

\[ \Delta(\xi_1 \xi_2 \xi_3) = \Delta(\xi_1) \xi_2 \xi_3 + \Delta(\xi_2) \xi_3 \xi_1 + \Delta(\xi_3) \xi_1 \xi_2 - [\xi_1, \xi_2] \xi_3 - [\xi_2, \xi_3] \xi_1 - [\xi_3, \xi_1] \xi_2 \]

\[ [\alpha, \xi_1 \xi_2 \xi_3] = - (\Delta(\alpha \xi_1 \xi_2 \xi_3) + \alpha \Delta(\xi_1 \xi_2 \xi_3)). \]

Hence

\[ \Psi \Delta(\alpha \xi_1 \xi_2 \xi_3) = - \Psi ([\alpha, \xi_1 \xi_2 \xi_3] + \alpha \Delta(\xi_1 \xi_2 \xi_3)) \]

\[ = - \Psi [\alpha, \xi_1 \xi_2 \xi_3] - \Psi(\alpha \Delta(\xi_1) \xi_2 \xi_3) - \Psi(\alpha \Delta(\xi_2) \xi_3 \xi_1) - \Psi(\alpha \Delta(\xi_3) \xi_1 \xi_2) + \Psi(\alpha [\xi_1, \xi_2] \xi_3) + \Psi(\alpha [\xi_2, \xi_3] \xi_1) + \Psi(\alpha [\xi_3, \xi_1] \xi_2) \]

that is,

\[ \Psi \Delta(\alpha \xi_1 \xi_2 \xi_3) = - \Psi [\alpha, \xi_1 \xi_2 \xi_3] - \Psi(\alpha \xi_2 \xi_3) \Delta(\xi_1) - \Psi(\alpha \xi_3 \xi_1) \Delta(\xi_2) - \Psi(\alpha \xi_1 \xi_2) \Delta(\xi_3) + \Psi(\alpha [\xi_1, \xi_2] \xi_3) + \Psi(\alpha [\xi_2, \xi_3] \xi_1) + \Psi(\alpha [\xi_3, \xi_1] \xi_2) \]

since \( \Psi \) is \( A \)-linear. Exploiting the identity

\[ \Psi [\alpha, \xi_1 \xi_2 \xi_3] = \Psi [\alpha, \xi_1] \xi_2 \xi_3 + \Psi [\alpha, \xi_2] \xi_3 \xi_1 + \Psi [\alpha, \xi_3] \xi_1 \xi_2 \]

we conclude

\[(\Delta \Psi + \Psi \Delta)(\alpha \xi_1 \xi_2 \xi_3) = [\Psi(\alpha \xi_1 \xi_2), \xi_3] + [\Psi(\alpha \xi_2 \xi_3), \xi_1] + [\Psi(\alpha \xi_3 \xi_1), \xi_2]
+ \Psi(\alpha [\xi_1, \xi_2] \xi_3) + \Psi(\alpha [\xi_2, \xi_3] \xi_1) + \Psi(\alpha [\xi_3, \xi_1] \xi_2)
- \Psi([\alpha, \xi_1] \xi_2 \xi_3) - \Psi([\alpha, \xi_2] \xi_3 \xi_1) - \Psi([\alpha, \xi_3] \xi_1 \xi_2)\]

Thus the graded commutator \(\Delta \Psi + \Psi \Delta\) vanishes on \(G_3^1\) if and only if, for every \(\alpha \in G_0^1\) and \(\xi_1, \xi_2, \xi_3 \in G_0^0\),

\[
[\Psi(\alpha \xi_1 \xi_2), \xi_3] + [\Psi(\alpha \xi_2 \xi_3), \xi_1] + [\Psi(\alpha \xi_3 \xi_1), \xi_2] =
\Psi([\alpha, \xi_1] \xi_2 \xi_3) + \Psi([\alpha, \xi_2] \xi_3 \xi_1) + \Psi([\alpha, \xi_3] \xi_1 \xi_2)
- \Psi([\alpha, \xi_1] \xi_2 \xi_3) - \Psi([\alpha, \xi_2] \xi_3 \xi_1) - \Psi([\alpha, \xi_3] \xi_1 \xi_2);
\]

since \(\Psi(\xi_1 \xi_2 \xi_3) = 0\), the latter identity is equivalent to

\[
[\Psi(\alpha \xi_1 \xi_2), \xi_3] - [\Psi(\xi_1 \xi_2 \xi_3), \alpha] + [\Psi(\alpha \xi_2 \xi_3 \alpha), \xi_1] - [\Psi(\xi_3 \alpha \xi_1), \xi_2] =
\Psi([\alpha, \xi_1] \xi_2 \xi_3) + \Psi([\alpha, \xi_2] \xi_3 \xi_1) + \Psi([\alpha, \xi_3] \xi_1 \xi_2)
+ \Psi([\xi_1, \xi_2] \alpha \xi_3) + \Psi([\xi_2, \xi_3] \alpha \xi_1) + \Psi([\xi_3, \xi_1] \alpha \xi_2).
\]

With the more neutral notation \((a_1, a_2, a_3, a_4) = (\alpha, \xi_1, \xi_2, \xi_3)\) since, for degree reasons,

\[
\Phi_3^\Psi(a_1, a_2, a_3) = -\Psi(a_1 a_2 a_3), \quad \Phi_3^\Psi([a_1, a_2], a_3, a_4) = -\Psi([a_1, a_2] a_3 a_4)
\]

e.t., the identity takes the form

\[
[\Phi_3^\Psi(a_1, a_2, a_3), a_4] - [\Phi_3^\Psi(a_2, a_3, a_4), a_1] + [\Phi_3^\Psi(a_3, a_4, a_1), a_2] - [\Phi_3^\Psi(a_4, a_1, a_2), a_3] =
\Phi_3^\Psi([a_1, a_2], a_3, a_4) - \Phi_3^\Psi([a_1, a_3], a_2, a_4) + \Phi_3^\Psi([a_1, a_4], a_2, a_3)
+ \Phi_3^\Psi([a_2, a_3], a_1, a_4) - \Phi_3^\Psi([a_2, a_4], a_1, a_3) + \Phi_3^\Psi([a_3, a_4], a_1, a_2).
\]

This is the identity (6.6.4) for the special case where the elements \(a_1, a_2, a_3, a_4\) are from \(G_0^1 \cup G_0^0\). The operator \(\Delta\) being of order \(\leq 2\) means precisely that the bracket \([\cdot, \cdot] = \pm \Phi_3^\Delta\) (generated by it) behaves as a derivation in each argument and, accordingly, the operator \(\Psi\) being of order \(\leq 3\) means that the operation \(\Phi_3^\Psi\) is a derivation in each of its three arguments. The equivalence between the identities (6.6.3) and (5.7.1) for arbitrary arguments is now established by induction on the degrees of the arguments. \(\square\)

**Proof of Theorem 6.6.1.** The quasi-Lie-Rinehart bracket \([\cdot, \cdot]\) on \(G\) is that generated by \(\Delta = \partial\) via (6.1). This bracket is plainly graded skew-symmetric in the correct sense, and the reasoning in Section 1 of [41] shows that this bracket satisfies the identities (5.3)–(5.5). In particular, the identity (5.5) is a consequence of the identity (6.3): This identity may be rewritten as

\[(6.3')\]  

\[d\Psi + \Psi d = -\Delta \Delta.\]

Hence, given homogeneous elements \(\xi, \eta, \vartheta\) of \(G\), the identity (5.5) takes the form

\[(5.5')\]  

\[\sum_{(\xi, \eta, \vartheta) \text{ cyclic}} (-1)^{|\xi|+|\eta|+|\vartheta|} [\xi, [\eta, \vartheta]] = -(-1)^{|\xi|+|\eta|+|\vartheta|} \Phi_3^\Delta(\xi, \eta, \vartheta).\]
This is exactly the identity in line -5 on p. 260 of [41], which measures the failure of the bracket \([\cdot, \cdot]\) to satisfy the graded Jacobi identity in terms of the square \(\Delta^2\) of the generating operator \(\Delta\). The identity (5.6) holds by virtue of Lemma 6.6.2. □

An observation due to Koszul [41] (p. 261) extends to the present case in the following fashion: For any quasi-Batalin-Vilkovisky algebra \((G; d, \partial, \Psi)\), the operator \(\partial\) (which is strict by assumption) behaves as a derivation for the quasi-Gerstenhaber bracket \([\cdot, \cdot]\), up to a suitable correction term which we now determine: The identity in line 6 on p. 261 of [41] implies that, for homogeneous \(a, b \in G\),

\[
\partial[a, b] - ([\partial a, b] - (-1)^{|a|}[a, \partial b]) = (-1)^{|a|} \Phi^2_{d\Psi + \Psi d}(a, b).
\]

Since, by virtue of (6.3), \(\partial\partial + d\Psi + \Psi d = 0\), we conclude

\[
\partial[a, b] - ([\partial a, b] - (-1)^{|a|}[a, \partial b]) = (-1)^{|a|} - 1 \Phi^2_{d\Psi + \Psi d}(a, b).
\]

The correction term \(\Phi^2_{d\Psi + \Psi d}(a, b)\) is plainly an instance of the occurrence of a homotopy. We also note that, in view of (6.1), a generator, even if strict, behaves as a derivation for the multiplication of \(G\) only if the quasi-Gerstenhaber bracket \([\cdot, \cdot]\) is zero.

A quasi-Batalin-Vilkovisky algebra having \(\Psi\) zero is just an ordinary differential bigraded Batalin-Vilkovisky algebra, and a differential bigraded Batalin-Vilkovisky algebra having zero differential is called a bigraded Batalin-Vilkovisky algebra [21, 22]. Maintaining notation introduced in the previous section, given a quasi-Batalin-Vilkovisky algebra \((G; d, \partial, \Psi)\), we denote its \(d\)-homology by \(H^*_d(G)\); Proposition 5.7 above says that \(H^*_d(G)\) inherits a bigraded Gerstenhaber bracket. Plainly, under the present circumstances this homology inherits more structure; indeed, the proof of the following is straightforward and left to the reader.

**Proposition 6.7.** Given a quasi-Batalin-Vilkovisky algebra \((G; d, \partial, \Psi)\), the strict operator \(\partial\) induces a generator

\[
\partial: H^*_d(G) \to H^*_{d-1}(G)
\]

for the bigraded Gerstenhaber bracket on its \(d\)-homology \(H^*_d(G)\) and hence turns the latter into a bigraded Batalin-Vilkovisky algebra. □

A quasi-Batalin-Vilkovisky algebra has an invariant which is finer than just ordinary homology, though: Let \((G; d, \partial, \Psi)\) be a quasi-Batalin-Vilkovisky algebra, and consider the following \(\text{Tot}G\) of \(G\) given by

\[
(\text{Tot}G)_n = \sum_{q-p=n} G^q_p = G^0_0 \oplus G^1_1 \oplus \ldots \oplus G^n_k \oplus \ldots
\]

This totalization is forced by the isomorphism (6.8) and by Theorem 6.10 below. In a given bidegree \((q, p)\), the operators \(d, \partial, \Psi\) may be depicted as

\[
d: G^q_p \to G^{q+1}_p, \quad \partial: G^q_p \to G^{q}_{p-1}, \quad \Psi: G^q_p \to G^{q-1}_{p-2},
\]
and the defining properties (6.2)–(6.5) say that the sum

\[ D = d + \partial + \Psi \]

is a square zero operator on \( \text{Tot} \mathcal{G} \), i.e., a differential. Consider the ascending filtration \( \{ F_r \}_{r \geq 0} \) of \( \text{Tot} \mathcal{G} \) given by

\[ F_r(\text{Tot} \mathcal{G})_n = \sum_{q-p=n, p \leq r} \mathcal{G}_p^q = \mathcal{G}_0^n \oplus \mathcal{G}_1^{n+1} \oplus \ldots \oplus \mathcal{G}_r^{n+r} \]

This filtration gives rise to a spectral sequence

\[ (E^q(r), d(r)), \quad d(r): E^q_p(r) \to E^q_{p-r+1}(r) \]

having

\[ (E(0), d(0)) = (\mathcal{G}, d) \]

whence

\[ (E(1), d(1)) = (H^*_d(\mathcal{G}, \partial), \partial) \]

which is the bigraded homology Batalin-Vilkovisky algebra spelled out in Proposition 6.7 above. This spectral sequence is an invariant for the quasi-Batalin-Vilkovisky algebra \( \mathcal{G} \) which is finer than just the bigraded homology Batalin-Vilkovisky algebra \( (H^*_d(\mathcal{G}), \partial) \).

We will now take up and extend the discussion in (5.8) and describe how quasi-Gerstenhaber and quasi-Batalin-Vilkovisky algebras arise from Lie-Rinehart triples. To this end, let \( (A, H, Q) \) be a pre-Lie-Rinehart triple and suppose that, as an \( A \)-module, \( Q \) is finitely generated and projective, of constant rank \( n \). Consider the graded exterior \( A \)-algebra \( \Lambda A_Q \), and let \( \mathcal{G} = \text{Alt}_A(H, \Lambda A_Q) \), with \( \mathcal{G}_n = \text{Alt}_A^n(H, \Lambda^n A_Q) \); this is a bigraded commutative \( A \)-algebra. The Lie-Rinehart differential \( d \), with respect to the canonical graded \( (A, H) \)-module structure on \( \Lambda A_Q \), turns \( \mathcal{G} \) into a differential graded \( R \)-algebra. Our aim is to determine when \( (A, Q, H) \) is a genuine Lie-Rinehart triple in terms of conditions on \( \mathcal{G} \).

The graded \( A \)-module \( \text{Alt}_A^*(Q, \Lambda^n A_Q) \) acquires a canonical graded \( (A, H) \)-module structure. Further, since \( (A, H, Q) \) is a pre-Lie-Rinehart triple (not just an almost pre-Lie-Rinehart triple), the canonical bigraded \( A \)-module isomorphism (5.8.7) is now an isomorphism

\[ \phi: (\text{Alt}_A^*(H, \Lambda^*_A Q), d) \to (\text{Alt}_A^*(H, \text{Alt}_A^{n-*}(Q, \Lambda^n A_Q)), d) \]

of Rinehart complexes, with reference to the graded \( (A, H) \)-module structures on \( \Lambda^*_A Q \) and \( \text{Alt}_A^{n-*}(Q, \Lambda^n A_Q) \). We will say that \( (A, H, Q) \) is weakly orientable if \( \Lambda^n A_Q \) is a free \( A \)-module, that is, if there is an \( A \)-module isomorphism \( \omega: \Lambda^n A_Q \to A \), and \( \omega \) will then be referred to as a weak orientation form. Under the circumstances of Example 1.4.1, this notion of weak orientability means that the foliation \( \mathcal{F} \) is transversely orientable, with transverse volume form \( \omega \). For a general pre-Lie-Rinehart triple \( (A, H, Q) \), we
will say that a weak orientation form \( \omega \) is invariant provided it is invariant under the \( H \)-action; we will then refer to \( \omega \) as an orientation form, and we will say that \((A,H,Q)\) is orientable. In the situation of Example 1.4.1, with a grain of salt, an orientation form in this sense amounts to an orientation for the “space of leaves”, that is, with reference to the spectral sequence (2.9.1), the class in the top basic cohomology group \( E_{2}^{n,0} \) (cf. 2.10(i)) of such a form is non-zero and generates this cohomology group. Likewise, in the situation of Example 1.4.2, an orientation form is a holomorphic volume form, and the requirement that an (invariant) orientation form exist is precisely the Calabi-Yau condition.

Let \((A,H,Q)\) be a general orientable pre-Lie-Rinehart triple, and let \( \omega \) be an invariant orientation form. Then \( \omega \) induces an isomorphism \( \text{Alt}_{A}^{*}(Q,\Lambda_{A}^{*}Q) \rightarrow \text{Alt}_{A}^{*}(Q,A) \) of graded \((A,H)\)-modules and hence an isomorphism

\[
(6.9) \quad \phi_{\omega}^{*} : (\text{Alt}_{A}^{*}(H,\Lambda_{A}^{*}Q),d) \rightarrow (\text{Alt}_{A}^{*}(H,\text{Alt}_{A}^{-*}^{*}(Q,A)),d_{0})
\]

of Rinehart complexes. Here, on the right-hand side of (6.9), the operator \( d_{0} \) is that given earlier as (2.4.5), with the orders of \( H \) and \( Q \) interchanged. On the right-hand side of (6.9), we have as well the operator \( d_{1} \) given as (2.4.6) and the operator \( d_{2} \) given as (2.4.7) (the order of \( H \) and \( Q \) being interchanged), cf. also (5.8.8). The operator \( d_{1} \) induces an operator

\[
(6.10.1) \quad \Delta_{\omega} : \text{Alt}_{A}^{*}(H,\Lambda_{A}^{*}Q) \rightarrow \text{Alt}_{A}^{*}(H,\Lambda_{A}^{-1}^{*}Q)
\]

on the left-hand side of (6.9) by means of the the relationship

\[
\phi_{\Delta_{\omega}}^{*}(\alpha) = (-1)^{n+1}d_{1}(\phi_{\omega}^{*}(\alpha)), \quad \alpha \in \Lambda_{A}^{*}Q.
\]

By Lemma 5.8.10, the operator \( d_{2} \) on the right-hand side of (6.9) corresponds to the operator \( \Psi_{\delta} \) on the left-hand side of (6.9) given as (5.8.5) above. Notice that \( \Delta_{\omega} \) is an \( R \)-linear operator on \( G_{*}^{*} = \text{Alt}_{A}^{*}(H,\Lambda_{A}^{*}Q) \) of bidegree \((0,-1)\) which looks like a generator for the corresponding bracket (5.8.3). We will now describe the circumstances where \( \Delta_{\omega} \) is a generator.

**Theorem 6.10.** Let \((A,H,Q)\) be an orientable pre-Lie-Rinehart triple, with invariant orientation form \( \omega \). If \((A,H,Q)\) is a genuine Lie-Rinehart triple, then \((\text{Alt}_{A}^{*}(H,\Lambda_{A}^{*}Q);d,\Delta_{\omega},\Psi_{\delta})\) is a quasi-Batalin-Vilkovisky algebra, and \( \Delta_{\omega} \) is a strict generator for the bracket \([\cdot,\cdot]\) given by (5.8.3). Conversely, under the additional hypothesis that \( H \) satisfy the property \( P \), if \((\text{Alt}_{A}^{*}(H,\Lambda_{A}^{*}Q),d,\Delta_{\omega},\Psi_{\delta})\) is a quasi-Batalin-Vilkovisky algebra, then \((A,H,Q)\) is a genuine Lie-Rinehart triple.

**Proof.** We note first that, when \((A,d_{0},d_{1},d_{2})\) is a multialgebra, so is \((A,d_{0},-d_{1},d_{2})\). Furthermore, when \((\text{Alt}_{A}^{*}(H,\text{Alt}_{A}^{*}(Q,A)),d_{0},-d_{1},d_{2})\) is a multialgebra,

\[
(\text{Alt}_{A}^{*}(H,\text{Alt}_{A}^{*}(Q,\Lambda_{A}^{*}Q)),d_{0},-d_{1},d_{2})
\]

is a multicomplex, the operators \( d_{j} \) \((0 \leq j \leq 2)\) (where the notation \( d_{j} \) is abused somewhat) being the induced ones, with the correct sign, that is,

\[
\omega_{*}(d_{j}(\cdot)) = (-1)^{n}d_{j}\omega_{*}(\cdot)
\]
where $\omega_*$ is the induced bigraded morphism of degree $n$. Hence the equivalence between the Lie-Rinehart triple and quasi-Batalin-Vilkovisky properties is straightforward, in view of Theorem 2.7 and Theorem 6.6.1. In particular, the identities (2.1.4.2)–(2.1.4.5) correspond to the identities (6.2)–(6.5) which characterize $(\text{Alt}^*(H, \Lambda^n_A Q); d, \Delta_\omega, \Psi_\delta)$ being a quasi-Batalin-Vilkovisky algebra. It remains to show that, when $(A, H, Q)$ is a genuine Lie-Rinehart triple, the operator $\Delta_\omega$ (given by (6.10.1)) is indeed a strict generator for the bigraded bracket (5.8.3) to which the rest of the proof is devoted.

6.10.2. Verification of the generating property. We note first that, in view of the derivation properties of a quasi-Gerstenhaber bracket, it suffices to establish the generating property (6.1) on $\Lambda^n_A Q$, viewed as the bidegree $(0, *)$-constituent of $\text{Alt}^*(H, \Lambda^n_A Q) = \Lambda^n_A Q$. To make the operator $\Delta_\omega$ somewhat more explicit, we note that the pairing (1.5.2.Q) and the choice of $\omega$ determine a generalized $Q$-connection

$\nabla : Q \otimes \Lambda^n_A Q \to \Lambda^n_A Q$

on $\Lambda^n_A Q$ determined by requiring that the diagram

$\begin{align*}
Q \otimes_R \Lambda^n_A Q & \xrightarrow{\nabla} \Lambda^n_A Q \\
\text{Id} \otimes \omega & \downarrow \omega \\
Q \otimes_R A & \xrightarrow{(1.5.2.Q)} A
\end{align*}$

be commutative, and the multialgebra compatibility property $d_0 d_1 + d_1 d_0 = 0$ (cf. (2.1.4.2)) is equivalent to this generalized $Q$-connection being compatible with the $H$-module structures. In the situation of Example 1.4.2, such a generalized $Q$-connection on $\Lambda^n_A Q$ amounts to a flat holomorphic connection on the highest exterior power of the holomorphic tangent bundle. In the present general case, the operator $d_1$ on $\text{Alt}_A(H, \text{Alt}_A(Q, A))$ (given by (2.5.3) then corresponds to an operator

$d_1^{\nabla} : \text{Alt}_A(H, \text{Alt}_A^p(Q, \Lambda^n_A Q)) \to \text{Alt}_A(H, \text{Alt}_A^{p+1}(Q, \Lambda^n_A Q)) \quad (p \geq 0)$

determined by the commutativity of the diagram

$\begin{align*}
\text{Alt}_A(H, \text{Alt}_A^p(Q, \Lambda^n_A Q)) & \xrightarrow{d_1^{\nabla}} \text{Alt}_A(H, \text{Alt}_A^{p+1}(Q, \Lambda^n_A Q)) \\
\downarrow & \\
\text{Alt}_A(H, \text{Alt}_A^p(Q, A)) & \xrightarrow{(-1)^n d_1} \text{Alt}_A(H, \text{Alt}_A^{p+1}(Q, A))
\end{align*}$

whose vertical arrows are induced by $\omega$. Consider, then, the operator $D$ determined by the requirement that the diagram

$\begin{align*}
\text{Alt}_A(H, \Lambda^n_A Q) & \xrightarrow{\phi} \text{Alt}_A(H, \text{Alt}_A^{n-p}(Q, \Lambda^n_A Q)) \\
D & \\
\downarrow & \\
\text{Alt}_A(H, \Lambda^{n-1}_A Q) & \xrightarrow{\phi} \text{Alt}_A(H, \text{Alt}_A^{n-(p-1)}(Q, \Lambda^n_A Q))
\end{align*}$

be commutative. This operator coincides with the operator $\Delta_\omega$ but we prefer to use a neutral notation. In view of the derivation properties of a quasi-Gerstenhaber bracket, to establish the generating property, it will suffice to study the restriction

$$D: \Lambda^p_A Q \to \Lambda^{p-1}_A Q \quad (1 \leq p \leq n)$$

of this operator.

Given $\alpha \in \Lambda^p_A Q$, we will write $\phi_\alpha \in \text{Alt}^{n-p}_A(Q, \Lambda^n_A Q)$ for the image under $\phi$ so that, for $\xi_{p+1}, \ldots, \xi_n$,

$$\phi_\alpha(\xi_{p+1}, \ldots, \xi_n) = \alpha \land \xi_{p+1} \land \ldots \land \xi_n.$$ Let $\alpha_1$ and $\alpha_2$ be homogeneous elements of $\Lambda^*_A Q$. We will now establish the generating property

$$(6.10.3) \quad (-1)^{|\alpha_1|}[\alpha_1, \alpha_2] = D(\alpha_1\alpha_2) - (D\alpha_1)\alpha_2 - (-1)^{|\alpha_1|}\alpha_1(D\alpha_2).$$

Let $\beta \in \Lambda^{n+1-|\alpha_1|-|\alpha_2|}_A Q$; it will suffice to study the expression

$$(D(\alpha_1\alpha_2)) \land \beta - ((D\alpha_1)\alpha_2) \land \beta - (-1)^{|\alpha_1|}(\alpha_1(D\alpha_2)) \land \beta - (-1)^{|\alpha_1|}[\alpha_1, \alpha_2] \land \beta \in \Lambda^n_A Q$$
or, equivalently, the expression

$$\phi_{D(\alpha_1\alpha_2)}(\beta) - \phi_{(D\alpha_1)\alpha_2}(\beta) - (-1)^{|\alpha_1|}\phi_{\alpha_1 D\alpha_2}(\beta) - (-1)^{|\alpha_1|}\phi_{[\alpha_1, \alpha_2]}(\beta) \in \Lambda^n_A Q.$$ To this end, we note first that

$$-\phi_{D(\alpha_1\alpha_2)}(\beta) = (d^\nabla \phi_{\alpha_1\alpha_2})(\beta)$$
$$-\phi_{(D\alpha_1)\alpha_2}(\beta) = -(D\alpha_1) \land \alpha_2 \land \beta = (d^\nabla \phi_{D\alpha_1})(\alpha_2 \land \beta)$$
$$-\phi_{\alpha_1(D\alpha_2)}(\beta) = -\alpha_1 \land (D\alpha_2) \land \beta = -(-1)^{|\alpha_1|(|\alpha_2|-1})(D\alpha_2) \land \alpha_1 \land \beta$$
$$\quad = (-1)^{|\alpha_1|(|\alpha_2|-1)}(d^\nabla \phi_{D\alpha_2})(\alpha_1 \land \beta).$$

Let $\vartheta_1, \vartheta_2 \in Q$ and $\xi_2, \ldots, \xi_n \in Q$. Letting $\xi_1 = \vartheta_2$ we obtain

$$(d^\nabla \phi_{\vartheta_1})(\vartheta_2, \xi_2, \ldots, \xi_n) = (d^\nabla \phi_{\vartheta_1})(\xi_1, \xi_2, \ldots, \xi_n)$$
$$= \sum_{1 \leq j \leq n} (-1)^{j-1}\nabla_{\xi_j}(\vartheta_1 \land \xi_1 \land \ldots \hat{\xi}_j \ldots \land \xi_n)$$
$$+ \sum_{1 \leq j < k \leq n} (-1)^{j+k}\vartheta_1 \land [\xi_j, \xi_k] \land \xi_1 \land \ldots \hat{\xi}_j \ldots \hat{\xi}_k \ldots \land \xi_n$$

and a straightforward calculation gives

$$(d^\nabla \phi_{\vartheta_1})(\xi_1, \xi_2, \ldots, \xi_n) = \nabla_{\vartheta_2}(\vartheta_1 \land \xi_2 \land \ldots \land \xi_n)$$
$$+ \sum_{2 \leq j \leq n} (-1)^{j-1}\nabla_{\xi_j}(\vartheta_1 \land \vartheta_2 \land \xi_2 \land \ldots \hat{\xi}_j \ldots \land \xi_n)$$
$$+ \sum_{1 < k \leq n} (-1)^{1+k}\vartheta_1 \land [\vartheta_2, \xi_k] \land \xi_2 \land \ldots \hat{\xi}_k \ldots \land \xi_n$$
$$+ \sum_{2 \leq j < k \leq n} (-1)^{j+k}\vartheta_1 \land [\xi_j, \xi_k] \land \vartheta_2 \land \xi_2 \land \ldots \hat{\xi}_j \ldots \hat{\xi}_k \ldots \land \xi_n.$$
Likewise letting $\xi_1 = \vartheta_1$ we obtain

$$(d^\nabla \varphi_{\vartheta_2})(\vartheta_1, \xi_2, \ldots, \xi_n) = (d^\nabla \varphi_{\vartheta_2})(\xi_1, \xi_2, \ldots, \xi_n)$$

$$= \sum_{1 \leq j \leq n} (-1)^{j-1}\nabla_{\xi_j}(\vartheta_2 \wedge \xi_1 \wedge \ldots \hat{\xi}_j \ldots \wedge \xi_n)$$

$$+ \sum_{1 \leq j < k \leq n} (-1)^{j+k}\vartheta_2 \wedge [\xi_j, \xi_k] \wedge \xi_1 \wedge \ldots \hat{\xi}_j \ldots \hat{\xi}_k \ldots \wedge \xi_n$$

and again a calculation yields

$$(d^\nabla \varphi_{\vartheta_2})(\xi_1, \xi_2, \ldots, \xi_n) = \nabla_{\vartheta_1}(\vartheta_2 \wedge \xi_2 \wedge \ldots \wedge \xi_n)$$

$$+ \sum_{2 \leq j \leq n} (-1)^{j-1}\nabla_{\xi_j}(\vartheta_2 \wedge \vartheta_1 \wedge \xi_2 \wedge \ldots \hat{\xi}_j \ldots \wedge \xi_n)$$

$$+ \sum_{1 \leq k \leq n} (-1)^{1+k}\vartheta_2 \wedge [\vartheta_1, \xi_k] \wedge \xi_2 \wedge \ldots \hat{\xi}_k \ldots \wedge \xi_n$$

$$+ \sum_{2 \leq j < k \leq n} (-1)^{j+k}\vartheta_2 \wedge [\xi_j, \xi_k] \wedge \vartheta_1 \wedge \xi_2 \wedge \ldots \hat{\xi}_j \ldots \hat{\xi}_k \ldots \wedge \xi_n.$$
whereas, by a calculation the details of which are not given here,

\[(d^\nabla \phi_{\theta_2})(\theta_1, \xi_2, \ldots, \xi_n) = \nabla_{\xi_2}(\xi_1 \land \xi_2 \land \xi_3 \land \ldots \land \xi_n) + [\xi_1, \xi_2] \land \xi_3 \land \ldots \land \xi_n + \sum_{2<k \leq n} (-1)^k \xi_1 \land \xi_2 \land [\xi_2, \xi_k] \land \xi_3 \land \ldots \hat{\xi}_k \ldots \land \xi_n.\]

Consequently

\[\begin{align*}
(d^\nabla \phi_{\theta_1 \land \theta_2})(\xi_2, \ldots, \xi_n) - (d^\nabla \phi_{\theta_1})(\theta_2, \xi_2, \ldots, \xi_n) + (d^\nabla \phi_{\theta_2})(\theta_1, \xi_2, \ldots, \xi_n)
&= \sum_{2<k \leq n} (-1)^{k+1} \xi_1 \land \xi_2 \land [\xi_2, \xi_k] \land \xi_3 \land \ldots \hat{\xi}_k \ldots \land \xi_n + [\xi_1, \xi_2] \land \xi_3 \land \ldots \land \xi_n + \sum_{2<k \leq n} (-1)^k \xi_1 \land \xi_2 \land [\xi_2, \xi_k] \land \xi_3 \land \ldots \hat{\xi}_k \ldots \land \xi_n \\
&= [\xi_1, \xi_2] \land \xi_3 \land \ldots \land \xi_n
\end{align*}\]

that is, with \(\beta = \xi_2 \land \xi_3 \ldots \land \xi_n\),

\[(D(\theta_1 \land \theta_2)) \land \beta - ((D\theta_1)\theta_2) \land \beta - (\theta_1(D\theta_2)) \land \beta = -[\theta_1, \theta_2] \land \beta \in \Lambda^n A Q.\]

This establishes the generating property (6.10.3) for \(\alpha_1\) and \(\alpha_2\) homogeneous of degree 1 since, as an \(A\)-module, \(Q\) is finitely generated and projective of constant rank \(n\). Since, as an \(A\)-algebra, \(\Lambda_A Q\) is generated by its elements of degree 1, a straightforward induction completes the proof of Theorem 6.10. \(\Box\)

For the special case where \(\delta\) and hence \(\Psi_\delta\) is zero, the statement of the theorem is a consequence of Theorem 5.4.4 in [21].

**Corollary 6.10.4.** Let \((A, H, Q)\) be an orientable Lie-Rinehart triple, and let \(G = \text{Alt}_A(H, \Lambda_A Q)\) be endowed with the Lie-Rinehart differential \(d\), the bigraded bracket (5.8.3), and Jacobiator (5.8.5). Then \((G, d, [\cdot, \cdot], \Psi)\) is a quasi-Gerstenhaber algebra.

Indeed, the identity (5.6) then corresponds to (1.9.7); cf. also (4.9.6) and (6.11)(vii) below.

**Remark 6.11.** It is instructive to spell out the relationship between the quasi-Batalin-Vilkovisky compatibility conditions (6.2)–(6.4) and the Lie-Rinehart triple axioms (1.9.1)–(1.9.7); cf. (2.8.5) above. As before, write \(G = \text{Alt}_A(H, \Lambda_A Q)\), and recall that \(n\) is the rank of \(Q\) as a projective \(A\)-module.

(i) The vanishing of \(d \Delta + \Delta d: G^0_n \rightarrow G^1_{n-1}\) (a special case of (6.2)) corresponds to (1.9.1).

(ii) The vanishing of the operator \(d \Delta + \Delta d: G^1_n \rightarrow G^2_{n-1}\) (a special case of (6.2), too) corresponds to (1.9.2).

(iii) The vanishing of \(d \Delta + \Delta d: G^0_{n-1} \rightarrow G^1_{n-2}\) (still a special case of (6.2)) corresponds to (1.9.3).

(iv) The vanishing of \(\Delta \Delta + \Psi d = d \Psi + \Delta \Delta + \Psi d: G^0_n \rightarrow G^0_{n-2}\) (a special case of (6.3)) corresponds to (1.9.4).
(v) The vanishing of $\Delta \Delta + \Psi d = d \Psi + \Delta \Delta + \Psi d: G^0_{n-1} \to G^0_{n-3}$ (a special case of (6.3), too) corresponds to (1.9.5).

(vi) The vanishing of $d \Psi + \Delta \Delta + \Psi d: G^1_n \to G^1_{n-2}$ (still a special case of (6.3)) corresponds to (1.9.6).

(vii) The vanishing of $\Delta \Psi + \Psi \Delta: G^1_n \to G^0_{n-3}$ (a special case of (6.4)) corresponds to (1.9.7). Cf. also (4.9.6) above.

When $(A, H, Q)$ is an orientable Lie-Rinehart triple, with orientation form $\omega$, pursuing the philosophy developed in Section 7 of [21] (cf. in particular (7.14)), we may view $(\text{Alt}_A^*(H, \Lambda_A^* Q), d, \Delta, \Psi_\delta)$ as an object the category of $A$-modules calculating the “quasi-Lie-Rinehart homology $H_*^q(Q, A_\Delta)$ of the quasi-Lie-Rinehart algebra $(A, Q)$, with values in the right $(A, Q)$-module $A_\Delta$”, the right $(A, Q)$-module structure being induced by $\Delta$. The isomorphism

\begin{equation}
(6.12) \quad (\text{Alt}_A^*(H, \Lambda_A^* Q); d, \Delta, \Psi_\delta) \to (\text{Alt}_A^*(H, \text{Alt}_A^{n-*}(Q, A)); d_0, -d_1, d_2)
\end{equation}

is then a kind of “duality isomorphism” of chain complexes inducing a “duality isomorphism” which, in bidegree $(q, p)$, is of the kind

\begin{equation}
(6.13) \quad H_p^q(Q, A_\Delta) \to H^{q,n-p}(Q, A) \cong H^{q,n-p}(L, A)
\end{equation}

where $L = H \oplus Q$ is the $(R, A)$-Lie algebra which corresponds to the given Lie-Rinehart triple $(A, H, Q)$. Proposition 7.14 in [21] makes this precise for the special case where $(A, Q, H)$ is a twilled Lie-Rinehart algebra. In our case, pushing further, consider the filtrations of $\text{Alt}_A^*(H, \Lambda_A^* Q)$ and $\text{Alt}_A^*(H, \text{Alt}_A^{n-*}(Q, \Lambda_A^* Q))$ by $Q$-degree. In view of what was said above, the corresponding spectral sequence (6.7.6), which we now write in the form

\begin{equation}
(6.14.1) \quad (E^*_r(r), d(r)), \quad d(r): E^*_r(r) \to E^*_{p-r+1}(r),
\end{equation}

has

\begin{equation}
(6.14.2) \quad (E(0), d(0)) = (\text{Alt}_A^*(H, \Lambda_A^* Q), d)
\end{equation}

whence

\begin{equation}
(6.14.3) \quad (E(1), d(1)) = (H_*^r(G)_d, \partial);
\end{equation}

this is the bigraded homology Batalin-Vilkovisky algebra spelled out in Proposition 6.7 above, for the quasi-Batalin-Vilkovisky algebra $G_*^r = (\text{Alt}_A^*(H, \Lambda_A^* Q); d, \Delta, \Psi_\delta)$. The isomorphism (6.8) is compatible with these filtrations. Hence it identifies the corresponding spectral sequence (2.9.1) with (6.14.1).

**Illustration 6.15.** Return to the situation of (1.4.1), and maintain the notation established there as well as in (2.10), cf. also (4.15). Thus $(M, \mathcal{F})$ is a foliated manifold, $(A, H, Q) = (C^\infty(M), L_F, Q)$ is the corresponding Lie-Rinehart triple, and $(A, Q) = \text{Alt}_A^*(H, A), \text{Alt}_A^*(H, Q)$ is the corresponding quasi-Lie-Rinehart algebra. We now push further the interpretation, advertised already in (4.15) above, of $A$ as the algebra of generalized functions and of $Q$ as the generalized Lie algebra of vector
fields for the foliation. This interpretation relies crucially on the totalization spelled out as (6.7.2) above; with the more familiar totalization \( \text{Tot}' \mathcal{G} \) given by

\[
(\text{Tot}' \mathcal{G})^n = \sum_{p+q=n} \mathcal{G}_{p+q}^q,
\]

such an interpretation is not visible.

Thus, consider the bigraded algebra \( \mathcal{G}_*^* = \text{Alt}_A^*(H, \Lambda_A^*Q) = \Lambda_A^*Q \), where as before \( \mathcal{A} = \text{Alt}_A(H, A) \) and \( \mathcal{Q} = \text{Alt}_A^*(H, Q) \). Suppose that the foliation is transversely orientable with a basic transverse volume form \( \omega \), and consider the resulting quasi-Batalin-Vilkovisky algebra \( (\text{Alt}_A^*(H, \Lambda_A^*Q); d, \Delta, \Psi) \), cf. Theorem 6.10. In particular, \( \mathcal{G}_*^* \) is then a quasi-Gerstenhaber algebra. This quasi-Gerstenhaber algebra yields a kind of generalized Schouten algebra (algebra of multivector fields) for the foliation; the cohomology \( H^*_0(\mathcal{G}) \) may be viewed as the Schouten algebra for the “space of leaves”. However the entire cohomology contains more information about the foliation than just \( H^*_0(\mathcal{G}) \).

Under the circumstances of (2.10(ii)), where the foliation comes from a fiber bundle, cf. also (4.15), let \( B \) denote the “space of leaves” or, equivalently, the base of the corresponding bundle; an orientation \( \omega \) in our sense is now essentially equivalent to a volume form \( \omega_B \) for the base \( B \). Let \( L_B = \text{Vect}(B) \). The volume form \( \omega_B \) induces an exact generator \( \partial_{\omega_B} \) for the ordinary Gerstenhaber algebra \( G_* = \Lambda_{C^\infty(B)}^\ast(L_B, \partial_{\omega_B}) \), and the corresponding bigraded homology Batalin-Vilkovisky algebra \( (H_*^*(\text{Alt}_A(H, \Lambda_A^*Q))_d, \partial_{\omega}) \) coming into play in Theorem 6.10 may then be written as the bigraded crossed product

\[
(6.15.1) \quad (H_*^*(\text{Alt}_A(H, \Lambda_A^*Q))_d, \partial_{\omega}) = H_*^*(\mathcal{A}) \otimes_{C^\infty(B)} (G_*^*, \partial_{\omega_B})
\]

of \( H_*^*(\mathcal{A}) \) with the ordinary Batalin-Vilkovisky algebra \( (G_*^*, \partial) = (\Lambda_{C^\infty(B)}^\ast(L_B, \partial_{\omega_B})) \) (cf. [21] for the notion of bigraded crossed product Batalin-Vilkovisky algebra); here \( \mathcal{A} = (\text{Alt}_A(H, A), d) \) which, cf. (2.10(ii)), computes the cohomology of \( M \) with values in the sheaf of germs of functions which are constant on the leaves, i.e. fibers.

Under the circumstances of (2.10(ii)), when the foliation does not come from a fiber bundle, the structure of the bigraded homology Batalin-Vilkovisky algebra \( H_*^*(\text{Alt}_A(H, \Lambda_A^*Q))_d \) may be more intricate.

ILLUSTRATION 6.16. For a (finite dimensional) quasi-Lie bialgebra \( (\mathfrak{g}, \mathfrak{h}^*) \) [37], with Manin pair \( (\mathfrak{g}, \mathfrak{h}) \), where \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^* \), the resulting quasi-Batalin-Vilkovisky algebra has the form

\[
\text{Alt}(\mathfrak{h}, \Lambda_{\mathfrak{h}^*}) \cong \Lambda_{\mathfrak{h}^*}^* \otimes \Lambda_{\mathfrak{h}^*} \cong \Lambda(\mathfrak{h}^* \oplus \mathfrak{h}^*).
\]

REMARK 6.17. Given the bigraded commutative algebra \( \mathcal{G}_*^* \), consider the totalization \( \text{Tot}' \mathcal{G} \) spelled out above. Suppose there be given operators \( d, \partial, \Psi \) which endow \( \mathcal{G} \) with a quasi-Batalin-Vilkovisky algebra structure in our sense. These operators induce operators

\[
d; (\text{Tot}' \mathcal{G})^* \to (\text{Tot}' \mathcal{G})^* + 1, \quad \partial; (\text{Tot}' \mathcal{G})^* \to (\text{Tot}' \mathcal{G})^* - 1, \quad \Psi; (\text{Tot}' \mathcal{G})^* \to (\text{Tot}' \mathcal{G})^* - 3
\]

such that \( \mathcal{L} = d\partial + \partial d = 0 \), \( d\Psi + \partial \partial + \Psi d = 0 \), \( \partial \Psi + \Psi \partial = 0 \), \( \Psi \Psi = 0 \), whence, endowed with these operators, \( \text{Tot}' \mathcal{G} \) is precisely a quasi-Batalin-Vilkovisky algebra.
in the sense of [14] with zero Laplacian $\mathcal{L}$. This notion of quasi-Batalin-Vilkovisky algebra extends that of differential GBV-algebra in [46] (III.9.5) (which corresponds to the structure under discussion with $\Psi = 0$, with reference to the totalization $\text{Tot}^{\prime} \mathcal{G}$) and is a special case of a more general notion of generalized BV-algebra explored in [42]. In [5] (Definition 3.2), a corresponding notion of quasi-Gerstenhaber algebra has been isolated. When $(\mathcal{G}, d, [\cdot, \cdot], \Psi)$ is a quasi-Gerstenhaber algebra in our sense, the operations $d, [\cdot, \cdot], \Psi$ induce as well corresponding pieces of structure $d, [\cdot, \cdot], \Psi$ on $\text{Tot}^{\prime} \mathcal{G}$ and, in view of Lemma 2.2 in [5], the requirement (5.5) above (which makes precise how under our circumstances the $h$-Jacobiator $\Psi$ controls the failure of the strict Jacobi identity) entails the requirement (3.7) in [5] which, in turn, describes the failure of the strict Jacobi identity under the circumstances of [5]. Moreover, our requirements (5.i)–(5.iii) in Section 5 above now amount to the corresponding requirements (3.6)–(3.8) in [5]. Likewise the requirement (5.6) corresponds to the requirement (3.9) in [5]. These observations make precise the relationship between our notions of quasi-Gerstenhaber and of quasi-Batalin-Vilkovisky algebra and that of quasi-Gerstenhaber algebra in [5] and those of quasi-Batalin-Vilkovisky algebra (with zero Laplacian) explored in [5] and [14]. However the notion of Laplacian does not seem to have a meaning for the totalization $\text{Tot}$ which we use in this paper, in particular, does not have an interpretation (at least not an obvious one) in terms of foliations.

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