Some Notes on Pythagorean Triples in Diophantine Equations

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Abstract. There exist solutions of algebraic equation in the study of the early mature conclusion. Within the scope of the complex numbers sets, the 5 times following algebraic equation has formula solution. However, when we put the solution of the limit in the range of natural numbers or within the scope of rational number, the equation becomes Diophantine equation. These solutions are not easily found. In this paper, all kinds of solution of the Pythagorean equation were studied. The derivation of using chord method got a formula of rational number solution. This formula can be used to easily get countless set of integer solutions.

1. Introduction

Solving equations is the traditional goal of algebra[1], and particular parts of algebra have been developed to analyze particular methods of solution. Solution by radicals is one branch of the tradition, typified by the ancient formula (1), for the solution of real number.

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \] (1)

In this paper. Equations whose integer solutions are sought are called Diophantine [2-4], even though it is not really the equations that are "Diophantine equations", but the solutions. Nevertheless, certain equations stand out as “Diophantine” because their integer solutions are of exceptional interest.

2. The Pythagorean Equation

The Pythagorean equation \( x^2 + y^2 = z^2 \), whose natural number solutions \((x, y, z)\) are known as Pythagorean triples [5-6].

Table 1: Plimpton 322.

| \( y \) | \( z \) |
|-------|-------|
| 119   | 169   |
| 3367  | 4825  |
| 4601  | 6649  |
| 12709 | 18541 |
| 65    | 97    |
| 319   | 481   |
| 2291  | 3541  |
| 799   | 1249  |
| 481   | 769   |
| 4961  | 8161  |
| 45    | 75    |
| 1679  | 2929  |
| 161   | 289   |
| 1771  | 3229  |
| 56    | 106   |
The Pell equation \( x^2 - ny^2 = 1 \) for any non-square natural number \( n \).

The Bachet equation \( y^2 = x^2 + n \) for any natural number \( n \).

The Fermat equation \( x^n + y^n = z^n \) for any integer \( n > 2 \).

The Pythagorean equation is the oldest known mathematical problem, being the subject of a Babylonian clay tablet from around 1800 BCE known as Plimpton 322 (from its museum catalogue number). The tablet contains the two columns of natural numbers, \( y \) and \( z \) shown in Table 1. The left part of the table is missing, but it is surely a column of values of \( x \), because each value of \( z^2 - y^2 \) is an integer square \( x^2 \), and so the table is essentially a list of Pythagorean triples.

This means that Pythagorean triples were known long before Pythagoras (who lived around 500 BCE), and the Babylonians apparently had sophisticated means of producing them. Notice that Plimpton 322 does not contain any well known Pythagorean triples, such as (3, 4, 5), (5, 12, 13) or (8, 15, 17). It does, however, contain triples derived from these, mostly in nontrivial ways.

Around 300 BCE, Euclid showed that all natural number solutions of \( x^2 + y^2 = z^2 \) can be produced by the formulas (2) - (4).

\[
\begin{align*}
x &= (u^2 - v^2)w \\
y &= 2uvw \\
z &= (u^2 + v^2)w
\end{align*}
\]

It is easily checked that these formulas give in formula (5)

\[
x^2 + y^2 = z^2
\]

Another approach, using rational numbers, was found by Diophantus around 200 CE. Diophantus specialized in solving equations in rationals, so his solutions are not properly "Diophantine" in our sense, but in this case rational and integer solutions are essentially equivalent.

3. The Diophantus chord method

An integer solution \( (x, y, z) = (a, b, c) \) of \( x^2 + y^2 = z^2 \) implies:

\[
\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1
\]

So \( X = \frac{a}{c} \), \( Y = \frac{b}{c} \) is a rational solution of the equation (7):

\[
X^2 + Y^2 = 1
\]

In other words, a rational point on the unit circle. Admittedly, any multiple of the triple, \( (ma, mb, mc) \), corresponds to the same point, but we can easily insert multiples once we have found \( a, b, \) and \( c \) from \( X \) and \( Y \).

Diophantus found rational points on \( X^2 + Y^2 = 1 \) by an algebraic method, which has the geometric interpretation shown in Figure 1.

![Figure 1. The chord method for rational points.](image-url)
If we draw the chord connecting an arbitrary rational point $R$ to the point $Q = (-1, 0)$, we get a line with rational slope, because the coordinates of $R$ and $Q$ are rational. If the slope is $t$, the equation of this line is:

$$Y = t(X + 1)$$  \hspace{1cm} (8)$$

Conversely, any line of this form, with rational slope, meets the circle at a rational point $R$. This can be seen by computing the coordinates of $R$. We do this by substituting equation (8) in equation (7), obtaining:

$$X^2 + t^2(X + 1)^2 = 1$$  \hspace{1cm} (9)$$

Which is the following quadratic equation for $X$:

$$X^2(1 + t^2) + 2t^2X + t^2 - 1 = 0$$  \hspace{1cm} (10)$$

The quadratic formula gives the solutions $(X = -1, \frac{1-t^2}{1+t^2})$. The solution $X = -1$ corresponds to the point $Q$, so the $X$ coordinate at $R$ is $\frac{1-t^2}{1+t^2}$, hence the $Y$ coordinate is:

$$Y = t\left(\frac{1-t^2}{1+t^2} + 1\right) = \frac{2t}{1+t^2}$$  \hspace{1cm} (11)$$

To sum up: an arbitrary rational point on the unit circle $X^2 + Y^2 = 1$ has coordinates $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$, for arbitrary rational $t$.

Now we can recover Euclid's formulas.

An arbitrary rational $t$ can be written $t = \frac{v}{u}$, where $u, v \in \mathbb{Z}$, and the rational point $R$ then becomes:

$$\left(\frac{1-v^2}{u^2}, \frac{2v}{u} \cdot \frac{2uv}{u^2 + v^2}, \frac{2uv}{u^2 + v^2}\right)$$  \hspace{1cm} (12)$$

Thus if this is $\left(\frac{x}{z}, \frac{y}{z}\right)$ for some $x, y, z \in \mathbb{Z}$, we must have $\frac{x}{z} = \frac{u^2 - v^2}{u^2 + v^2}, \frac{y}{z} = \frac{2uv}{u^2 + v^2}$, for some $u, v \in \mathbb{Z}$.

Euclid's formulas for $x, y, z$ also give these formulas for $\frac{x}{z}$ and $\frac{y}{z}$, so the results of Euclid and Diophantus are essentially the same.

There is little difference between rational and integer solutions of the equation $x^2 + y^2 = z^2$ because it is homogeneous in $x, y$ and $z$, hence any rational solution can be multiplied through to give an integer solution.

4. Conclusion

As we have seen above, the integer solutions of Pythagorean equation is not easy to find, but they are all included in the rational solution. And all rational numbers have been find solution by us. Not only that, but we also got all the rational solution of the above formula. Pythagorean equation integer solutions are included in the rational solution. As long as we give enough $t$ value, from the above formula can we find enough of the $x, y, z$. When the $x, y, z$ are integers group, we have plenty of integer solutions are constructed.

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