Igusa’s Modular Form and the Classification of Siegel Modular Threefolds

Klaus Hulek

0 Introduction

For an integer $d \geq 1$ let

$$E_d = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \quad \Lambda_d = \begin{pmatrix} 0 & E_d \\ -E_d & 0 \end{pmatrix}.$$  

We consider the symplectic group

$$\tilde{\Gamma}_{1,d} := \text{Sp}(\Lambda_d, \mathbb{Z}).$$

For $d = 1$ this is the usual integer symplectic group $\text{Sp}(4, \mathbb{Z})$. The group $\tilde{\Gamma}_{1,d}$ operates on the Siegel space of genus 2

$$\mathbb{H}_2 = \left\{ \tau = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \text{Mat}(2 \times 2, \mathbb{C}); \text{Im} \tau > 0 \right\}$$

by

$$\tilde{M} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}: \tau \mapsto (\tilde{A}\tau + \tilde{B}E_d)(\tilde{C}\tau + \tilde{D}E_d)^{-1}E_d.$$  

The quotient

$$A_{1,d} = \tilde{\Gamma}_{1,d} \backslash \mathbb{H}_2$$

is the moduli space of $(1, d)$-polarized abelian surfaces. Alternatively we can consider the following subgroup of the usual rational symplectic group $\text{Sp}(4, \mathbb{Q})$. Let

$$R_d = \text{diag}(1, 1, 1, d)$$

and set

$$\Gamma_{1,d} := R_d^{-1}\tilde{\Gamma}_{1,d}R_d \subset \text{Sp}(4, \mathbb{Q}).$$

Then $\Gamma_{1,d}$ acts in the usual way on $\mathbb{H}_2$ by

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}: \tau \mapsto (A\tau + B)(C\tau + D)^{-1}$$

and

$$A_{1,d} = \tilde{\Gamma}_{1,d} \backslash \mathbb{H}_2 = \Gamma_{1,d} \backslash \mathbb{H}_2.$$  

Let $L = \mathbb{Z}^4$ be the lattice on which $\Lambda_d$ defines a symplectic form and let $L^\vee$ be the dual lattice of $L$. We consider the following subgroups of $\tilde{\Gamma}_{1,d}$ defined by

$$\tilde{\Gamma}_{1,d}^{\text{lev}} : = \left\{ M \in \tilde{\Gamma}_{1,d}; \ M|_{L^\vee/L} = \text{id} \right\}$$

$$\tilde{\Gamma}_{1,d}(n) : = \left\{ M \in \tilde{\Gamma}_{1,d}; \ M \equiv 1 \mod n \right\} \quad (n \geq 1)$$

$$\tilde{\Gamma}_{1,d}^{\text{lev}}(n) : = \tilde{\Gamma}_{1,d}^{\text{lev}} \cap \tilde{\Gamma}_{1,d}(n).$$  

This gives rise to subgroups of Sp(4,\mathbb{Q}):

\[ \begin{align*}
\Gamma_{1,d}^{\text{lev}} & := R_d^{-1}\tilde{\Gamma}_{1,d} R_d \\
\Gamma_{1,d}(n) & := R_d^{-1}\tilde{\Gamma}_{1,d}(n) R_d \\
\Gamma_{1,d}^{\text{lev}}(n) & := R_d^{-1}\tilde{\Gamma}_{1,d}^{\text{lev}}(n) R_d = \Gamma_{1,d}^{\text{lev}} \cap \Gamma_{1,d}(n),
\end{align*} \]

resp. to the moduli spaces

\[ \begin{align*}
\mathcal{A}_{1,d}^{\text{lev}} & = \tilde{\Gamma}_{1,d}^{\text{lev}} \backslash \mathbb{H}_2 = \Gamma_{1,d}^{\text{lev}} \backslash \mathbb{H}_2 \\
\mathcal{A}_{1,d}(n) & = \tilde{\Gamma}_{1,d}(n) \backslash \mathbb{H}_2 = \Gamma_{1,d}(n) \backslash \mathbb{H}_2 \\
\mathcal{A}_{1,d}^{\text{lev}}(n) & = \tilde{\Gamma}_{1,d}^{\text{lev}}(n) \backslash \mathbb{H}_2 = \Gamma_{1,d}^{\text{lev}}(n) \backslash \mathbb{H}_2.
\end{align*} \]

The geometric meaning of these moduli spaces is the following:

\[ \mathcal{A}_{1,d}^{\text{lev}} = \{(A, H, \alpha); (A, H) \text{ is a (1,}d\text{-polarized abelian surface,} \\
\quad \alpha \text{ is a canonical level-structure}\}. \]

Here a canonical level-structure is a symplectic basis of the kernel of the map \(\lambda_H : A \to \hat{A} = \text{Pic}^0 A\). (Note that this kernel is (non-canonically) isomorphic to \(\mathbb{Z}/d \times \mathbb{Z}/d\).) Similarly

\[ \mathcal{A}_{1,d}(n) = \{(A, H, \beta); (A, H) \text{ is a (1,}d\text{-polarized abelian surface,} \\
\quad \beta \text{ is a full level-}\text{ }n\text{-structure}\}. \]

Here a full level-n structure is a symplectic basis of the group \(A^{(n)}\) of \(n\)-torsion points of \(A\). Finally

\[ \mathcal{A}_{1,d}^{\text{lev}}(n) = \{(A, H, \alpha, \beta); (A, H) \text{ is a (1,}d\text{-polarized abelian surface,} \\
\quad \alpha \text{ is a canonical level structure,} \beta \text{ is a full level-}\text{ }n\text{-structure}\}. \]

Note that

\[ \tilde{\Gamma}_{1,d}^{\text{lev}} \tilde{\Gamma}_{1,d} \cong \Gamma_{1,d}^{\text{lev}} / \Gamma_{1,d} \cong \text{SL}(2, \mathbb{Z}/d) \]

and that we have, therefore, a Galois covering \(\mathcal{A}_{1,d}^{\text{lev}} \to \mathcal{A}_{1,d}\) with Galois group \(\text{PSL}(2, \mathbb{Z}/d)\).

The aim of this short note is to prove two results about the classification of these Siegel modular varieties.

**Theorem 0.1** \(\mathcal{A}_{1,d}(n)\) is of general type if \((d, n) = 1\) and \(n \geq 4\).

Since \(\mathcal{A}_{1,1}(n)\) is rational for \(n \leq 3\) this is the best result which one can hope for if one considers all \(d\) simultaneously. The space \(\mathcal{A}_{1,3}(2)\) has a Calabi-Yau model ([BN], [GH]) and hence Kodaira dimension 0, whereas \(\mathcal{A}_{1,3}(3)\) is of general type ([GH, Theorem 3.1]). For prime numbers \(p\) Sankaran [S] has proved that \(\mathcal{A}_{1,p}\) is of general type for \(p \geq 173\). A similar result for \(\mathcal{A}_{1,d}\), where \(d\) is not necessarily prime, is, as far as I know, not known. Borisov [Bo] has shown that, up to conjugation, there are only finitely many subgroups \(\Gamma\) of \(\text{Sp}(4, \mathbb{Z})\) such that \(\mathcal{A}(\Gamma) = \Gamma \backslash \mathbb{H}_2\) is not of
general type. Recall however, that the groups $\Gamma_{1,d}(n)$ are not subgroups of $\text{Sp}(4,\mathbb{Z})$ unless $d$ divides $n$ and that, in general, they are also not conjugate to subgroups of $\text{Sp}(4,\mathbb{Z})$. An essential ingredient in the proof of the theorem is Igusa's modular form $\Delta_{10}$.

The above theorem is a result about the birational classification of these varieties. If one wants to ask more precise questions, such as whether $K$ is ample, then one has to specify the compactification with which one wants to work.

**Theorem 0.2** The Voronoi compactification $(\mathcal{A}_{1,p}^{\text{lev}}(n))^*$ for a prime number $p$ with $(p,n) = 1$ is smooth and has ample canonical bundle (i.e. is a canonical model in the sense of Mori theory) if and only if $n \geq 5$.

Here a few words are in order: By Voronoi compactification we mean the compactification defined by the second Voronoi decomposition. We choose this compactification because Alexeev [11] has shown that it appears naturally when one wants to construct a toroidal compactification which represents a geometrically meaningful functor. The addition of a canonical level structure has two reasons: The spaces $(\mathcal{A}_{1,p}(n))^*$ have non-canonical singularities for infinitely many $p$ and $n$. These singularities come from the toroidal construction, not from fixed points of the group which is neat for $n \geq 3$. Moreover, it is necessary to introduce at least some kind of level structure to obtain a functorial description of the compactifications. A canonical level structure will be sufficient for this [A2]. Finally the restriction to prime numbers $p$ in Theorem 0.2 is done to keep the technical difficulties to an acceptable level. I believe that this restriction is not essential. This result also supports a conjecture made in [11] for principal polarizations. This is in so far interesting as the case treated here cannot, as it could be in the case $p = 1$, be easily derived from known results on $\mathcal{M}_2$.

**Acknowledgement:** I am grateful to M. Friedland and G.K. Sankaran for useful discussions.

1 General type

In this section we want to prove

**Theorem 1.1** If $(d, n) = 1$ and $n \geq 4$ then $\mathcal{A}_{1,d}(n)$ is of general type.

We shall work with the Voronoi compactification $\mathcal{A}_{1,d}^*(n)$ of $\mathcal{A}_{1,d}(n)$. Before we can prove the theorem we need to know something about the coordinates of $\mathcal{A}_{1,d}^*(n)$ near a cusp. Recall that the codimension 1 cusps are given by lines $l \subset \mathbb{Q}^4$ up to the action of the group $\Gamma_{1,d}(n)$ and that the codimension 2 cusps are similarly given by isotropic planes $h \subset \mathbb{Q}^4$. For any
such \(l\), resp. \(h\) and any group \(\Gamma\) we denote the lattice part of the stabilizer of \(l\), resp. \(h\) in \(\Gamma\) by \(P'_{\Gamma}(l)\), resp. \(P'_{\Gamma}(h)\). These lattices have rank 1, resp. 3.

**Lemma 1.2**  
(i) For every line \(l \subset \mathbb{Q}^4\) there is an inclusion \(P'_{\text{Sp}(4,\mathbb{Z})}(l) \subset P'_{\Gamma_{1,d}}(l)\) with cokernel \(\mathbb{Z}/d_1\) where \(d_1|d\).

(ii) For every isotropic plane \(h \subset \mathbb{Q}^4\) there is an inclusion \(P'_{\text{Sp}(4,\mathbb{Z})}(h) \subset P'_{\Gamma_{1,d}}(h)\) with cokernel \(\mathbb{Z}/d_1 \times \mathbb{Z}/d_2 \times \mathbb{Z}/d_3\) where \(d_i|d\) for \(i = 1, 2, 3\).

**Proof.** (i) Let \(l\) be a line which corresponds to a given cusp of \(\Gamma_{1,d}\). By [FS, Satz 2.1] we may assume that

\[
(l = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} l_0, \quad \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} =: M \in \text{Sp}(4,\mathbb{Z})
\]

where \(l_0 = e_3 = (0, 0, 1, 0)\). The group \(Q'(l_0) = M^{-1}P'_{\Gamma_{1,d}}(l_0)M\) is a rank 1 lattice which fixes \(l_0\). We want to compare this to the rank 1 lattice

\[
P'_{\text{Sp}(4,\mathbb{Z})}(l_0) = \left\{ \begin{pmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; s \in \mathbb{Z} \right\} \subset \text{Sp}(4,\mathbb{Z}).
\]

Recall from [HKW, Proposition I.1.16] that every element \(g\) in \(\Gamma_{1,d}\) fulfills the following congruences

\[
g - 1 \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \frac{1}{d_1} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}
\]

Hence

\[
\begin{pmatrix} tD & 0 \\ -tC & tA \end{pmatrix} \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} = \begin{pmatrix} * & S \\ * & * \end{pmatrix}, \quad S \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}.
\]

In particular \(P'_{\text{Sp}(4,\mathbb{Z})}(l_0)\) is contained in \(Q'(l_0)\). Hence \(P'_{\Gamma_{1,d}}(l)/P'_{\text{Sp}(4,\mathbb{Z})}(l) \cong \mathbb{Z}/d_1\) for some \(d_1\). The claim \(d_1|d\) follows since

\[
M \begin{pmatrix} 1 & 0 & d & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} M^{-1} \in P'_{\Gamma_{1,d}}(l).
\]

(ii) Again we can choose an element \(M \in \text{Sp}(4,\mathbb{Z})\) such that \(h = M(h_0)\) where \(h_0 = e_3 \wedge e_4\). Then

\[
Q'(h_0) = M^{-1}P'_{\Gamma_{1,d}}(h)M
\]
consists of elements of the form
\[
\begin{pmatrix}
1 & 0 & d_1Z & d_2Z \\
0 & 1 & d_2Z & d_3Z \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

By (i) we can conclude that \(d_1, d_3 \in \mathbb{N}\). We claim that also \(d_2 \in \mathbb{N}\). To prove this recall that there is a sublattice \(L_0 \subset L = \mathbb{Z}^4\) with \(L/L_0 \cong \mathbb{Z}/d\) such that \(\Gamma_{1,d}(L_0) \subset L\). (This is simply the lattice spanned by \(e_1, e_2, e_3, de_4\). Hence the same statement must be true for \(\Gamma_{1,d}(h)M\), but this implies that \(d_2 \in \mathbb{N}\). The assertions \(d_1 | d\) and \(d_3 | d\) follow from (i) and \(d_2 | d\) follows again since

\[
M \begin{pmatrix}
1 & 0 & 0 & d \\
0 & 1 & d & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} M^{-1} \in \mathcal{P}_{1,d}(h).
\]

\(\square\)

**Proof of the theorem.** We consider the following maps of moduli spaces

\[
\begin{array}{ccc}
\mathcal{A}_{1,d}^{\text{lev}} & \rightarrow & \mathcal{A}_{1,d}(n) \\
\downarrow & & \downarrow \\
\mathcal{A}_{1,1} & \rightarrow & \mathcal{A}_{1,d}
\end{array}
\]

The map \(\mathcal{A}_{1,d}^{\text{lev}} \rightarrow \mathcal{A}_{1,1}\) comes from the inclusion \(\Gamma_{1,d}^{\text{lev}} \subset \text{Sp}(4, \mathbb{Z})\). (The argument given in \([\text{HKW}], \text{Proposition I.1.20}\) for \(d\) prime goes through unchanged for all \(d\).) Note that \(\Gamma_{1,d}^{\text{lev}}\) is not normal in \(\text{Sp}(4, \mathbb{Z})\) and hence \(\mathcal{A}_{1,d}^{\text{lev}} \rightarrow \mathcal{A}_{1,1}\) is not Galois. The other maps \(\mathcal{A}_{1,d}^{\text{lev}} \rightarrow \mathcal{A}_{1,d}\) and \(\mathcal{A}_{1,d}(n) \rightarrow \mathcal{A}_{1,d}\) are Galois covers.

An essential ingredient in the proof is Igusa’s modular form

\[
\Delta_{10} = \prod_{m \text{ even}} \Theta^2_m(\tau)
\]

given by the product of the squares of all even theta null values. This is a cusp form of weight 10 with respect to \(\text{Sp}(4, \mathbb{Z})\). In fact it is, up to scalar, the unique weight 10 cusp form with respect to \(\text{Sp}(4, \mathbb{Z})\). Recall that it vanishes exactly along the \(\text{Sp}(4, \mathbb{Z})\)-translates of the diagonal

\[
\mathbb{H} \times \mathbb{H} = \left\{ \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix}; \text{Im} \tau_1, \text{Im} \tau_3 > 0 \right\} \subset \mathbb{H}^2
\]

where it vanishes of order 2. Since \(\Gamma_{1,d}^{\text{lev}}\) is a subgroup of \(\text{Sp}(4, \mathbb{Z})\) we can also consider \(\Delta_{10}\) as a cusp form with respect to \(\Gamma_{1,d}^{\text{lev}}\). Recall that for any
modular form $G$ and a matrix $M$ the slash-operator is defined by

$$G|_k M := \det(C\tau + D)^{-k} G(M\tau) \quad (M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}).$$

We consider the multiplicative symmetrization

$$F_0 := \prod_{M \in \text{PSL}(2,\mathbb{Z}/d)} \Delta_{10}|_{\text{codim}1} M.$$

It is straightforward to check that $F_0$ is a cusp form with respect to $\Gamma_1,d$ of weight $10\mu(d)$ where

$$\mu(d) = \frac{1}{2}|\text{SL}(2,\mathbb{Z}/d)| = \frac{1}{2}d^3 \prod_{p|d} \left(1 - \frac{1}{p^2}\right) \quad (d \geq 3),$$

resp. $\mu(2) = 6$. Clearly we can also consider $F_0$ as a cusp form with respect to the smaller group $\Gamma_1,d(n)$. Let $L$ be the $(\mathbb{Q})$-line bundle of modular forms of weight 1. By abuse of notation we shall use the same notation for whatever moduli space we are considering.

**Claim 1:** For every point $P$ on the boundary of $\mathcal{A}_{1,d}^+(n)$ the modular form $F_0$ defines an element in $m_P^{\mu(d)} L^{10\mu(d)}$.

For points on the codimension 1 cusps this follows immediately from Lemma 1.2 (i) and $(n, d) = 1$. To prove it in general we consider an isotropic plane $h$ and the lattices $N := P'_{\Gamma_1,d}(h)$ and $N' := P'_{\Gamma_1,d(n)}(h)$. Let $\sigma \in \Sigma_{\text{vor}}$ be a 3-dimensional cone and let $T_{\sigma}(N)$, resp. $T_{\sigma}(N')$ be the corresponding affine parts in the toric variety $T_{\Sigma_{\text{vor}}}(N)$, resp. $T_{\Sigma_{\text{vor}}}(N')$. We claim that $\Delta_{10}$ defines a function on the closure of the image of $P_{\Gamma_1,d}(h) \setminus \mathbb{H}_2$ in $T_{\sigma}(N)$.

First of all $\Delta_{10}$ is a function on $P_{\Gamma_1,d}(h) \setminus \mathbb{H}_2$ by Lemma 1.2 (ii). Since $T_{\sigma}(N)$ is normal, it is enough to show that this function extends to the codimension 1 boundary components. This follows from Lemma 1.2 (i). Since $\Delta_{10}$ is a cusp form it follows that $\Delta_{10} \in m_P$ for every point $P$ on the boundary. By construction of $F_0$ this gives claim 1 in the case $n = 1$. Since $(d, n) = 1$ we have $N' = nN$ and this gives the claim for general $n$.

Let $\mathcal{A}_{1,d}^+(n)$ be the Voronoi compactification of $\mathcal{A}_{1,d}(n)$, i.e. the toroidal compactification given by the second Voronoi decomposition of the cone of semi-positive definite symmetric real $(2 \times 2)$-matrices (in $\mathbb{R}^2$ this was also called Legendre decomposition). If $n \geq 3$ the group $\Gamma_1,d(n)$ is neat (this follows from a general result of Serre which says that every algebraic integer which is a unit and which is congruent to 1 mod $n(n \geq 3)$ is equal to 1). In particular $\mathcal{A}_{1,d}(n)$ is smooth. The toroidal compactification $\mathcal{A}_{1,d}^+(n)$ will, in general, however have singularities. These arise because the fan given by the Voronoi decomposition is not always basic, i.e. there may be cones which are not spanned by elements of a basis of the lattice. We can always choose
a suitable subdivision of the fan given by the Voronoi decomposition and in this way construct a smooth resolution \( \psi : \tilde{A}_{1, d}(n) \to A^*_{1, d}(n) \) such that the exceptional divisor is a normal crossing divisor.

Let \( \omega = d\tau_1 \wedge d\tau_2 \wedge d\tau_3 \). It is well known that, if \( G \) is a weight 3k cusp form which vanishes of order \( \geq k \) along all 1-codimensional cusps, then \( G\omega^k \) defines a k-fold canonical form on the smooth part of \( A^*_{1, d}(n) \).

**Claim 2:** The space of k-fold canonical forms which extends to the smooth part of \( A^*_{1, d}(n) \) grows (at least for sufficiently divisible k) as \( ck^3 \) for some positive constant \( c \).

To prove this claim recall that \( K = 3L - D \) on the smooth part of \( A^*_{1, d}(n) \), where \( L \) is the (\( \mathbb{Q}^+ \)) line bundle of modular forms of weight 1 and \( D \) is the boundary, i.e. the union of all 1-codimensional cusps. By claim 1 the form \( F_0 \) gives the equality

\[
10\mu(d) L = n\mu(d) D + D_{\text{eff}}
\]

for some effective divisor \( D_{\text{eff}} \) on the smooth part of \( A^*_{1, d}(n) \). From this we obtain

\[
-D = -\frac{10}{n} L + \frac{1}{n\mu(d)} D_{\text{eff}}.
\]

Combining this equality with the expression for \( K \) gives us

\[
K = \left( 3 - \frac{10}{n} \right) L + \frac{1}{n\mu(d)} D_{\text{eff}}.
\]

For \( n \geq 4 \) the factor in front of \( L \) is positive and the claim follows since \( h^0(L^k) \) grows as \( ck^3 \).

**Claim 3:** If \( F_{3k} \omega^k \) defines a k-fold canonical form on the smooth part of \( A^*_{1, d}(n) \) then \( (F_{3k} \omega^{10\mu(d)})^k (F_{3k} \omega^k) \) extends to \( \tilde{A}_{1, d}(n) \). We first notice that, since \( n \geq 4 \), the form \( F_{3k} \omega^{10\mu(d)} \) extends to the smooth part of \( A^*_{1, d}(n) \). The following part of the argument follows closely the proof of [S, Theorem 6.3].

It is enough to prove that the form in question extends to the generic point of each component of the exceptional divisor. Let \( E \) be a component of the exceptional divisor of the resolution \( \tilde{A}_{1, d}(n) \to A^*_{1, d}(n) \). It is enough to consider points which lie on only one boundary component. We can choose local analytic coordinates \( z_1, z_2, z_3 \) on an open set \( U \) such that \( E = \{ z_1 = 0 \} \). Recall that \( U \) is an open set in some toroidal variety \( T_{\Sigma}(N') \) where \( N' = \mathbb{P}^{\mu_{1, d}(n)}(h) \) for some isotropic plane \( h \) and \( \Sigma \) is a refinement of the fan \( \Sigma_{\text{Vor}} \) defined by the Voronoi decomposition. Moreover the coordinates of the torus are of the form \( t_i = e^{2\pi i a_i \tau_i} \) for some rational numbers \( a_i \). A local equation for \( E \) is given by \( t_1^{b_1} t_2^{b_2} t_3^{b_3} \) for suitable \( b_i \) and hence we can set \( z_1 = t_1^{b_1} t_2^{b_2} t_3^{b_3} \). Since \( \partial z_1 / \partial \tau_j = 2\pi i a_j b_j z_1 \) we can conclude that the order of
\[ J = \det (\partial \tau_i / \partial z_j) \text{ along } E \text{ is } v_E(J) \geq -1. \] It follows again from claim 1 that
\[ v_E \left( F_0^3 J^{10 \mu(d)} \right) \geq (3n - 10) \mu(d). \]

Therefore \( (F_0^3 \omega^{10 \mu(d)}) \) defines a section of \( \mu(d)(10K - (3n - 10)E) \). By assumption \( F_{3k} \omega^k \) defines a section of \( \psi^* \left( kK_{A_1^{(3d)}}(n) \right) = k(K - \alpha E) \) where \( \alpha \) is the discrepancy of \( E \). Altogether \( (F_0^3 \omega^{10 \mu(d)}) \) defines a section of
\[ (10 \mu(d)kK - k\mu(d)(3n - 10)E) + k(K - \alpha E) = k \left[ (10 \mu(d) + 1) K - (\mu(d)(3n - 10) + \alpha) E \right]. \]

All singularities here are cyclic quotient singularities. This follows from Lemma 1.2 and the fact that \( T_{\Sigma, \psi}(P_{Sp(4, Z)}(h)) \) is smooth. Hence the singularities are log-terminal, i.e. \( \alpha > -1 \). This implies that \( \mu(d)(3n - 10) + \alpha > 0 \) for \( n \geq 4 \) and thus the claim follows.

The theorem now follows easily from by combining claim 2 and claim 3. \( \square \)

### 2 Ample canonical bundle

It is the aim of this section to prove the following

**Theorem 2.1** Let \( p \) be an odd prime number and assume that \( (n, p) = 1 \). The Voronoi compactification \( (A_{lev}^{(n)})^* \) is smooth and has ample canonical bundle if and only if \( n \geq 5 \).

We remark that this result is also known to be true for \( p = 1 \) (cf. [Bd], [H]). Before we prove this theorem we recall the geometry of the spaces \( (A_{lev}^{(p)})^* \) which was described in detail in [HKW]. The Tits building of the group \( \Gamma_{lev}^{(p)} \) consists of \( 1 + (p^2 - 1)/2 \) lines and \( p+1 \) isotropic planes. The lines consist of one so-called central line and \( (p^2 - 1)/2 \) peripheral lines. If \( D(l_0) \) is the closed boundary surface which belongs to the central line, then there is a map \( K(p) \rightarrow D(l_0) \) which is an immersion, but not an embedding if \( p > 3 \). Here \( K(p) \) is the Kummer modular surface of level \( p \), i.e. the quotient of Shioda’s modular surface \( S(p) \) by the involution which acts by \( x \mapsto -x \) on every fibre. For each peripheral boundary component \( D(l) \) there exists an isomorphism \( K(1) \cong D(l) \) where \( K(1) \) is the Kummer modular surface of level 1.

If we add a level-\( n \) structure clearly the number of inequivalent cusps will increase. We shall, however, still speak about central or peripheral cusps with respect to \( \Gamma_{lev}^{(p)}(n) \) depending on whether this defines a central or peripheral cusp with respect to \( \Gamma_{lev}^{(p)} \). Now assume \( n \geq 3 \). Then one shows exactly as in the proof of [HKW, Theorem I.3.151] that there are immersions \( S(np) \rightarrow D(l_c) \) if \( l_c \) is a central cusp, resp. \( S(n) \rightarrow D(l_p) \) if \( l_p \)
is a peripheral cusp. The reason why we have Shioda modular surfaces here instead of Kummer modular surfaces is that for $n \geq 3$ the matrix $-\mathbf{1}$ is not contained in $\Gamma_{1,k}^{lev}(n)$. It will be immaterial for us whether these maps are immersions or embeddings.

We shall write the boundary as

$$D = \sum_{i \in I} D_i^c + \sum_{j \in J} D_j^p$$

where $D_i^c$ are the central and $D_j^p$ the peripheral boundary components.

We recall the following well known facts about Shioda modular surfaces. For $k \geq 3$ the surface $S(k) \to X(k)$ is the universal elliptic curve with a level-$k$ structure. The base curve $X(k)$ is the modular curve of level $k$. It has $t(k) = 1 + (k-6)t(k)/12$ cusps and the fibre of $S(k)$ over the cusps are singular of type $I_k$, i.e. a $k$-gon of $(-2)$-curves. The genus of $X(k)$ equals $1 + (k-6)t(k)/12$ and the line bundle $L_{X(k)}$ of modular forms of weight 1 has degree $kt(k)/12$. The elliptic fibration $\pi : S(k) \to X(k)$ has $k^2$ sections $L_{ij}$ which form a group $\mathbb{Z}/k \times \mathbb{Z}/k$. By $F$ we denote a general fibre of $S(k)$.

It is well known (cf.[BH, pp.78-80]) that

$$K_{S(k)} = \frac{k-4}{4}t(k)F,$$

$$L_{ij}^2 = \frac{k}{12}t(k) = -\deg L_{X(k)}.$$  

Let $f_c : S(np) \to D_c$, resp. $f_p : S(n) \to D_p$ be the map from $S(np)$ to a central, resp. from $S(n)$ to a peripheral boundary component. Since these maps are immersions we can consider the normal bundles $N_c$, resp. $N_p$ of these maps.

**Proposition 2.2**  
(i) $N_c \equiv -\frac{2}{np}L_{X(np)} - \frac{2}{np} \sum_{i,j \in \mathbb{Z}/np \times \mathbb{Z}/np} L_{ij}$

(ii) $N_p \equiv -\frac{2}{n}L_{X(n)} - \frac{2}{n} \sum_{i,j \in \mathbb{Z}/n \times \mathbb{Z}/n} L_{ij}.$

**Proof.** We shall give the proof for the central boundary components and indicate how it has to be adopted to the peripheral boundary components. There is a natural action of the group $\sum L_{ij} = \mathbb{Z}/np \times \mathbb{Z}/np$ on $S(np)$. It is an easy calculation to check that this action is induced by elements of $\Gamma_{1,k}^{lev}(n)/\Gamma_{1,p}$. It follows that $N_c$ is invariant under the group $\mathbb{Z}/np \times \mathbb{Z}/np$ and hence

$$N_c \equiv aF + b \sum L_{ij}$$

for some $a, b \in \mathbb{Q}$ (cf. [BH]). To determine $a, b$ we have to compute the degree of the normal bundle $N_c$ on a general fibre of $S(np)$ and on a section, e.g.
the zero section $L_{00}$.
As a representative for a central cusp we can take the line $l_0 = (0, 0, 1, 0)$.
Assume $(n, p) = 1$. We set

$$\Gamma_1'(np) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}); a, d \equiv 1 \mod np, c \equiv 0 \mod n, \\
                            b \equiv 0 \mod np^2 \right\}. $$

Note that by conjugation with $E = \text{diag}(1, p)$ the group $\Gamma_1'(np)$ is conjugate to the principal subgroup $\Gamma_1(np)$. Then by [HKW, Proposition I.3.98] the stabilizer subgroup $P(l_0)$ of $\Gamma_{l_0}^{\text{lev}}(np)$ is given by

$$P(l_0) = \left\{ \begin{pmatrix} 1 & k & s & m \\ 0 & a & m & b \\ 0 & 0 & 1 & 0 \\ 0 & c & -k & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1'(np), k, s \in n\mathbb{Z}, m \in pn\mathbb{Z} \right\}. $$

The action of

$$\begin{pmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mapsto \begin{pmatrix} \tau_1 + s & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix},$$

gives rise to the partial quotient

$$\mathbb{H}_2 \rightarrow \mathbb{C}^* \times \mathbb{C} \times \mathbb{H}_1,$$

$$\begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mapsto (t_1 = e^{2\pi i \tau_1/n}, \tau_2, \tau_3).$$

The other elements of $P(l_0)$ act as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix} : \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mapsto \begin{pmatrix} \tau_1 - \tau_2(c\tau_3 + d)^{-1}c\tau_2 \\ \tau_2(c\tau_3 + d)^{-1}(a\tau_3 + b)(c\tau_3 + d)^{-1} \end{pmatrix}^*,$$

$$\begin{pmatrix} 1 & k & 0 & m \\ 0 & 1 & m & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -k & 1 \end{pmatrix} : \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mapsto \begin{pmatrix} \tau_1' \\ \tau_2 + k\tau_3 + m & \tau_3 \end{pmatrix},$$

$$\tau_1' = \tau_1 + k^2\tau_3 + 2k\tau_2 + km.$$
defines an action on $\mathbb{C} \times \mathbb{H}_1$ by

$$
\begin{pmatrix}
1 & k & m \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} : (\tau_2, \tau_3) \mapsto (\tau_2 + k\tau_3 + m, \tau_3)
$$

$$
\begin{pmatrix}
1 & 0 & 0 \\
a & b & 0 \\
c & d & 1
\end{pmatrix} : (\tau_2, \tau_3) \mapsto (\tau_2(c\tau_3 + d)^{-1}, (a\tau_3 + b)(c\tau_3 + d)^{-1}).
$$

Then $D^0(l_0) = P''(l_0) \setminus \{0\} \times \mathbb{C} \times \mathbb{H}_1$ is the open boundary surface associated to $l_0$ and conjugation with $E = \text{diag}(1, p)$ shows that $D^0(l^0) \cong S^0(np)$, the open part of $S(np)$ which does not lie over the cusps.

A local equation of $D^0(l_0)$ in $\mathbb{C} \times \mathbb{C} \times \mathbb{H}_1$ is given by $t_1 = 0$ and hence $t_1/t_1^2$ is a local section of the conormal bundle. Under the action of the group $P(l_0)$ this transforms as follows:

$$
t_1/t_1^2 \mapsto t_1/t_1^2 e^{2\pi i [k^2\tau_3 - 2k\tau_2]/n},
$$

$$
t_1/t_1^2 \mapsto t_1/t_1^2 e^{2\pi i \left(-\frac{c\tau_3^2}{c\tau_3 + d}\right)/n}.
$$

We can use the formulae (1) and (2) to determine the coefficients $a$ and $b$. We first determine the degree of $N_c$ on a general fibre $F$. Since $k \in n\mathbb{Z}$, $m \in pn\mathbb{Z}$ the fibre of $S(np)$ over the point $[\tau_3] \in X(np)$ is given by $E_{[\tau_3]} = \mathbb{C}/(\mathbb{Z}n\tau_3 + \mathbb{Z}np)$. The standard theta function $\vartheta(\tau_3, \tau_2)$ defines a line bundle of degree $n^2p$ on $E_{[\tau_3]}$ and transforms as follows

$$
\vartheta(\tau_3, \tau_2 + k\tau_3 + m) = e^{2\pi i [\frac{1}{2}k^2\tau_3 - k\tau_2]}.
$$

Comparing formulae (1) and (3) we find that the degree of $N_c$ on $F$ equals $-2np$. Since we have $n^2p^2$ sections $L_{ij}$ it follows that $b = -2/np$. To determine the coefficient $a$ we have to compute the degree of $N_c$ on the zero section $L_{00}$. Since $L_{00}^2 = -\deg L_{X(np)}$ we must show that this degree is 0. There are two ways to see this. The first is to use formula (2) and specialise it to $\tau_2 = 0$. One then has to show that this description extends over the cusps which is an easy local calculation. Alternatively one can proceed as follows:

The section $L_{00}$ is the transversal intersection of $D_c$ with the closure of the image of the diagonal $\mathbb{H}_1 \times \mathbb{H}_1 \subset \mathbb{H}_2$ which parametrizes products. This closure is isomorphic to $X(n) \times X(np)$ and $L_{00}$ is equal to $\{\text{cusp}\} \times X(np)$. Hence the normal bundle of $L_{00}$ in $X(n) \times X(np)$ is trivial and by adjunction

$$
K_{L_{00}} = K|_{L_{00}} + L_{00}|_{L_{00}}
$$

where $K$ is the canonical bundle of $(A^1_{\text{lev}}(n))^*$. Using the fact that $K = 3L - D$ and pulling this back to $S(np)$ we obtain

$$
K_{L_{00}} = (3L_{X(np)} - t(np)F - N_c + L_{00})|_{L_{00}}.
$$
Since \( \text{deg } K_{L_{00}} = t(np)(np - 6)/6 \) a straightforward calculation shows that the degree of \( N_{c}|_{L_{00}} \) is equal to 0.

The calculation for \( N_p \) is essentially the same. The only differences are that \( t_3 = e^{2\pi i \tau_3/np^2} \) and that the fibre of \( S(n) \) over \( [\tau_1] \in X(n) \) is equal to \( E_{[\tau,3]} = \mathbb{C}/(\mathbb{Z} np\tau_1 + np\mathbb{Z}) \).

\( \square \)

**Proof of the theorem:** We first remark that \( (A_{\text{lev}}(n))^* \) is smooth under the assumptions made. Since \( n \geq 4 \) is the group \( \Gamma_{\text{lev}}^p(n) \) is neat. Therefore it is enough to show that for a given cusp \( h \) the toroidal variety \( T_{\Sigma_{\text{vor}}}(P'_{\Gamma_{\text{lev}}^p(n)}(h)) \) is smooth. If \( n \) and \( p \) are coprime, the lattice \( P'_{\Gamma_{\text{lev}}^p(n)}(h) \) is simply \( n \) times the corresponding lattice in \( \Gamma_{\text{lev}}^p(n) \). Hence every \( \sigma \in \Sigma_{\text{vor}} \) is spanned over \( \mathbb{R} \) by a basis of the lattice and this implies that \( T_{\Sigma} \) is smooth.

The next observation is that the condition \( n \geq 5 \) is necessary. We have already remarked that the closure of the diagonal \( \mathbb{H}_1 \times \mathbb{H}_1 \subset \mathbb{H}_2 \) parametrizing split abelian surfaces is isomorphic to \( X(n) \times X(np) \). Consider a curve \( C = X(n) \times \{ \text{point} \} \). Then

\[
K|_C = (3L - D)|_C = 3L_{X(n)} - X_{\infty}(n)
\]

where \( X_{\infty}(n) \) is the divisor of cusps on \( X(n) \). Hence

\[
KC = \frac{n}{4} t(n) - t(n)
\]

and this is positive if and only if \( n \geq 5 \).

We shall now assume \( n \geq 5 \). Let \( C \) be an irreducible curve which is not entirely contained in the boundary, i.e. \( C \cap A_{\text{lev}}^p(n) \neq \emptyset \) and consider a point \( [\tau] \in C \). Choose some \( \varepsilon > 0 \) with \( \varepsilon < 3n/5 \). By Weissauer’s result [We, p. 220] we can find a cusp form \( F \) with respect to \( \text{Sp}(4, \mathbb{Z}) \) such that \( F(\tau) \neq 0 \) and \( \text{ord}(F) \geq 1/(12 + \varepsilon) \). Here \( \text{ord}(F) \) is the order of \( F \), i.e. the vanishing order of \( F \) divided by the weight of \( F \). Let \( m \) and \( k \) be the vanishing order, resp. the weight of \( F \). Since \( \Gamma_{\text{lev}}^p(n) \subset \text{Sp}(4, \mathbb{Z}) \) the form \( F \) is also a modular form with respect to \( \Gamma_{\text{lev}}^p(n) \). In terms of divisors this gives us

\[
kL = mnD + D_{\text{eff}}, \quad C \not\subseteq D_{\text{eff}}.
\]

Here \( D_{\text{eff}} \) contains in particular multiples of peripheral boundary components since the vanishing order of \( F \) along these boundary components is at least \( np^2 \). From the above formula we find that

\[
\left( \frac{k}{mn} L - D \right) . C = \frac{1}{mn} D_{\text{eff}} . C \geq 0.
\]

Since \( L.C > 0 \) we find that \( K.C > 0 \) provided \( 3 > k/mn \). But this follows immediately from the inequalities \( m/k \geq 1/(12 + \varepsilon) \) and \( \varepsilon < 3n/5 \). It remains to prove that the restriction of \( K \) to every boundary component
is ample. Let $D_0$ be a boundary component and set $D'_0 = D - D_0$. We have already observed that there is an immersion $f : \tilde{D} \to D_0$ which is the normalization. The surface $\tilde{D}$ is either isomorphic to $S(np)$ or to $S(n)$ depending on whether we have a central or a peripheral boundary component. The map $f$ embeds every component of a singular fibre. The image of such a component in $(A_{1,p}^{\text{lev}}(n))^*$ is a $\mathbb{P}^1$. Away from $\{0, \infty\}$ this line is either the intersection of 2 different boundary components or 2 branches of $D_0$ intersecting transversally. In either case we have the

$$f^*K = f^*(3L - D'_0 - D_0) = 3L_X - F_\infty - N_f.$$

Here $L_X$ is either $L_{X(np)}$ or $L_{X(n)}$ depending on the type of the boundary component, the divisor $F_\infty$ is the union of the singular fibres and $N_f$ is the normal bundle of the immersion $f$. Let $k = np$ or $n$. Then

$$\deg(3L_X - F_\infty) = \frac{1}{4}kt(k) - t(k) > 0$$

for $n > 4$. Hence $(3L_X - F_\infty).C \geq 0$ for every curve $C$ and $(3L_X - F_\infty).C > 0$ unless $C$ is contained in a union of fibres. It follows immediately from our proposition that $-N_f.C > 0$ for every curve $C$ which does not contain a section $L_{ij}$. Since $-N_f.L_{ij} = 0$ we can conclude that $f^*K.C > 0$ for every curve $C$.

**Remark** The above proof can also be adapted to show that $K$ is nef for $n = 4$. We had already seen that $K$ is not ample in this case. In other words $(A_{1,p}^{\text{lev}}(4))^*$ is a minimal, but not a canonical model for $p = 1$ or $p \geq 3$ prime.

**References**

[A1] V. Alexeev, Complete moduli in the presence of semiabelian group action. math AG/9905103.

[A2] V. Alexeev, Private communication.

[Bo] L. Borisov, A finiteness theorem for $\text{Sp}(4, \mathbb{Z})$. [alg-geom/9510002](alg-geom/9510002).

[Br] H.J. Brasch, Modulräume abelscher Flächen. Thesis, Erlangen 1994.

[BH] W. Barth and K. Hulek, Projective models of Shioda modular surfaces. manuscr. math. 50 (1985), 73–132.

[BN] W. Barth and I. Nieto, Abelian surfaces of type $(1,3)$ and quartic surfaces with 16 skew lines. J. Algebr. Geom. 3 (1994), 173–222.

[FS] M. Friedland and G.K. Sankaran, Das Titsgebäude einiger arithmetischer Gruppen. Preprint 1999.
[GH] V. Gritsenko and K. Hulek, The modular form of the Barth-Nieto quintic. Intern. Mathematics Research Notices 17 (1999), 915–937.

[H] K. Hulek, Nef devisors on moduli spaces of abelian varieties. To appear: Volume dedicated to M. Schneider, de Gruyter, 1999.

[HKW] K. Hulek, C. Kahn, S. H. Weintraub, Moduli spaces of abelian surfaces: Compactification, degenerations and theta functions. de Gruyter 1993.

[S] G.K. Sankaran, Moduli of polarized abelian surfaces. Math. Nach. 188 (1997), 321–340.

[We] R. Weissauer, Untervarietäten der Siegelschen Modulmannigfaltigkeiten von allgemeinem Typ. Math. Ann. 275 (1986), 207–220.

Klaus Hulek
Universität Hannover
Institut für Mathematik
Postfach 6009
D-30060 Hannover
Germany
hulek@math.uni-hannover.de