Norm inequalities for the spectral spread of Hermitian operators

Pedro Massey¹,2 | Demetrio Stojanoff¹,2 | Sebastian Zarate¹,2

¹CMaLP, FCE-UNLP, La Plata, Argentina
²Instituto Argentino de Matemática, CONICET, Buenos Aires, Argentina

Correspondence
Sebastian Zarate, CMaLP, FCE-UNLP, La Plata, Argentina.
Email: seb4.zarate@gmail.com

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Abstract
In this work, we introduce a new measure for the dispersion of the spectral scale of a Hermitian (self-adjoint) operator acting on a separable infinite-dimensional Hilbert space that we call spectral spread. Then, we obtain some submajorization inequalities involving the spectral spread of self-adjoint operators, that are related to Tao’s inequalities for anti-diagonal blocks of positive operators, Kittaneh’s commutator inequalities for positive operators and also related to the arithmetic–geometric mean inequality. In turn, these submajorization relations imply inequalities for unitarily invariant norms (in the compact case).

Keywords
commutator inequalities, spectral spread, Tao’s inequality

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1 | INTRODUCTION

The development of inequalities involving spectral scales and generalized singular values of operators acting on a Hilbert space is a central topic in the operator theory. The literature related to this research area is vast (see [8, 9, 18, 20, 22–24, 30–32] just to mention a few works strictly related to our present research). Therefore obtaining new inequalities, involving new concepts, is of interest in itself.

In the matrix context (i.e., finite-dimensional operator theory), Knyazev and Argentati introduced in [25] an interesting measure for the dispersion of the eigenvalues of an Hermitian matrix called spread. They conjectured that the spread of an Hermitian matrix can be used to bound the so-called absolute variation of its Ritz values. In [26], we obtained some inequalities that correspond to weak versions of Knyazev–Argentati’s conjectures. At that point we realized that although natural and elegant, the spread of Hermitian matrices was not developed in the literature. Hence, in [27] we made a systematic study of this notion. It turns out (see [27]) that the spread of Hermitian matrices is related to several inequalities in terms of a pre-order relation known as submajorization (see [6]). For example, the spread allows us to obtain inequalities that are related to Tao’s inequalities [30] for the singular values of anti-diagonal blocks of positive matrices.

On the other hand, in [27] we showed that the spread of Hermitian matrices is also related to some commutator inequalities for generalized commutators (see also [18, 20, 22–24]). These results were applied in [28] to obtain upper bounds for the absolute variations of Ritz values of Hermitian matrices that partially confirm Knyazev–Argentati’s conjecture (although we point out that the original conjecture remains an open problem at this time).

Motivated by Knyazev–Argentati’s work [25] and our previous works [26–28] we introduce the spectral spread of self-adjoint operators acting on a separable infinite-dimensional Hilbert space $H$. In order to describe the spectral spread of a
self-adjoint operator $A \in B(H)$, we consider its spectral scale $\lambda(A) = (\lambda_i(A))_{i \in \mathbb{Z}_0}$ where $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$ (see Definition 2.1) defined by a “min-max” method. The numbers (entries) in this scale satisfy

$$\lambda_{-i}(A) \leq \lambda_{-i-1}(A) \leq \lambda_{i+1}(A) \leq \lambda_{i}(A) \quad \text{for} \quad i \in \mathbb{N}.$$ 

For example, if $A$ is also compact, then the entries of the sequence $\lambda(A)$ are the eigenvalues of $A$, in such a way that the numbers $\lambda_i(A)$ (for $i \in \mathbb{N}$) are the positive eigenvalues of $A$ counting multiplicities (or zero) arranged in non-increasing order and $\lambda_{-i}(A)$ (for $i \in \mathbb{N}$) are the negative eigenvalues of $A$ counting multiplicities (or zero) arranged in the non-decreasing order. Then, the spectral spread of $A$, noted $\text{Spr}^+(A) \in \ell_\infty(\mathbb{N})$, is the non-negative and non-increasing sequence given by

$$\text{Spr}^+(A) = (\lambda_i(A) - \lambda_{-i}(A))_{i \in \mathbb{N}}.$$ 

After recalling some well-known facts related to spectral scales and singular values (see Section 2), we show some general properties of the spectral spread and its relation with the generalized singular values. Then, we obtain several inequalities for the spectral spread, in terms of submajorization relations between generalized singular values of operators, and therefore for unitarily invariant norms associated with symmetrically normed operator ideals in $B(H)$ (which is a well-known technique, see [16]). Our present results formally extend our previous results for the spread of Hermitian matrices related to Tao’s and Kittaneh’s inequalities (see [22, 30]). Moreover, we obtain some new (stronger) inequalities for commutators of self-adjoint operators in terms of submajorization (see Theorem 4.5).

The fact that the spectral spread of self-adjoint operators is related to Tao’s inequalities for singular values of anti-diagonal blocks suggests that the spectral spread can also be related to Bhatia–Kittaneh’s arithmetic–geometric mean (AGM) inequalities [8, 9] (see [30]) and to Zhan’s inequalities for the difference of positive operators [31, 32]. It turns out that this is the case; indeed, we develop AGM-type inequalities, and Zhan’s type inequalities involving the spread of self-adjoint operators. On the other hand, we show that many of the main inequalities for the spectral spread obtained in the present work are actually equivalent.

The paper is organized as follows. In Section 2, we recall the basic facts about spectral theory for self-adjoint operators in $B(H)$ and submajorization (between self-adjoint operators and between bounded real sequences). In Section 3, we introduce the spectral spread of a self-adjoint operator and obtain some basic results related with this notion. In Section 4, we obtain several submajorization inequalities for the spectral spread. Indeed, in Section 4.1 we obtain an inequality for the generalized singular values of anti-diagonal blocks of a self-adjoint operator that plays a key role throughout our work. In Section 4.2, we obtain inequalities for the generalized singular values of commutators of self-adjoint operators in terms of the spectral spread. In Section 5, we develop several inequalities related to the AGM inequality in terms of submajorization relations and the spectral spread. In turn, these submajorization relations imply (in the compact case) inequalities with respect to unitarily invariant norms. In Section 5.4, we show that many of the main inequalities for the spread obtained in the present work are actually equivalent.

2 | PRELIMINARIES

In this section, we introduce the basic notation and definitions used throughout our work.

2.1 | Notation and terminology

We let $H$ be an infinite-dimensional separable complex Hilbert space and we let $B(H)$ be the algebra of bounded linear operators acting on $H$. In this case, $K(H) \subset B(H)$ denotes the ideal of compact operators and $U(H) \subset B(H)$ denotes the group of unitary operators acting on $H$. In what follows, $K(H)^{sa}$ and $B(H)^{sa}$ denote the real subspaces of self-adjoint and compact and self-adjoint operators, respectively. Also, we denote by $B(H)^+$ the cone of positive operators and $K(H)^+ = B(H)^+ \cap K(H)$.

We write $\mathbb{N} = \{1, 2, \ldots\}$, $I_k = \{1, \ldots, k\} \subset \mathbb{N}$, for $k \in \mathbb{N}$ and $I_\infty = \mathbb{N}$. Also $1$ denotes the constant sequence with all its entries equal to one.
2.2 Spectral scales and submajorization of self-adjoint operators

In what follows, given $0 \leq n$ we let $G(n, H)$ denote the Grassmann manifold of $n$-dimensional subspaces of $H$. Given a subspace $S \subseteq H$, we denote $S_1 = \{ x \in S : \|x\| = 1 \}$ the set of unit vectors in $S$. We also denote $Z_0 = Z \setminus \{0\}$.

**Definition 2.1.** Given $A \in B(H)^{sa}$ we define the spectral scale of $A$ as the sequence $\lambda(A) = (\lambda_i(A))_{i \in Z_0}$ determined by:

for $i \in \mathbb{N}$ we let

$$\lambda_i(A) = \inf_{S \in G(i-1, H)} \sup_{\psi \in S_1} \langle A\psi, \psi \rangle$$

and

$$\lambda_{-i}(A) = \sup_{S \in G(i-1, H)} \inf_{\psi \in S_1^\perp} \langle A\psi, \psi \rangle.$$ 

Note that by construction,

$$\lambda_{-i}(A) \leq \lambda_{-(i+1)}(A) \leq \lambda_{i+1}(A) \leq \lambda_i(A), \quad \text{for } i \in \mathbb{N},$$

(2.1)

and $\lambda_i(-A) = -\lambda_{-i}(A)$ for $i \in Z_0$.

The spectral scale of self-adjoint operators allows to develop the generalized singular values of arbitrary operators in $B(H)$.

**Definition 2.2.** Given $X \in B(H)$ we define the generalized singular values of $X$ as the sequence $s(X) = (s_i(X))_{i \in \mathbb{N}}$ determined by

$$s_i(X) = \lambda_i(|X|)$$

for $i \in \mathbb{N}$, where $|X| = (X^*X)^{1/2} \in B(H)^+$.

We remark that the spectral scale of self-adjoint operators as well as the generalized singular values (s-numbers) of operators have been developed in the more general context of von Neumann algebras endowed with faithful semi-finite normal traces (see [14, 15, 21, 29]).

**Remark 2.3.** Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space and let $\dim \mathcal{H} = d$. Given $A \in B(\mathcal{H})^{sa}$ we denote by $\mu(A) = (R^d)^{\downarrow}$ the vector of eigenvalues of $A$, counting multiplicities and arranged in non-increasing order. In this case, we can consider $G(n, \mathcal{H})$, that is, the Grassmann manifold of $n$-dimensional subspaces of $\mathcal{H}$, for $0 \leq n \leq d$. Then, we can follow Definition 2.1 and set $(\lambda_i(A))_{i \in \mathbb{I}_d}$ and similarly $(\lambda_{-i}(A))_{i \in \mathbb{I}_d}$. Then

$$(\lambda_i(A))_{i \in \mathbb{I}_d} = \mu(A)$$

and

$$(\lambda_{-i}(A))_{i \in \mathbb{I}_d} = \mu(A)^{\downarrow} = (\mu_{d-i+1}(A))_{i \in \mathbb{I}_d}. \quad (2.2)$$

Further, in the Hermitian case the singular values of $A$ can be described as the non-increasing re-arrangement of the vector of eigenvalue modules, that is, $s(A) = (|\lambda_i(A)|)^{\downarrow}$.

We point out that this finite version of $\lambda(A)$ for $A \in B(\mathcal{H})$ does not satisfy Equation (2.1). Nevertheless, we include the definition of $\lambda(A)$ since we need this notion in some cases (e.g., to describe the decreasing rearrangement of eigenvalues of $A_P = PA|_{R(P)} \in B(R(P))^{sa}$, where $P \in B(H)$ is an orthogonal projection onto a finite dimensional subspace of $\mathcal{H}$, appearing in Theorem 2.8). On the other hand, the decreasing rearrangements of the eigenvalues of self-adjoint matrices (i.e. self-adjoint operators acting on finite-dimensional Hilbert spaces) do share some fundamental properties with the spectral scale of self-adjoint operators acting on infinite-dimensional Hilbert spaces. This last fact allows to obtain analogs of our results for self-adjoint matrices, with techniques similar to those included in the present work; we remark that some of these analogues for self-adjoint matrices are new.

**Remark 2.4.** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two separable Hilbert spaces and assume that at least one of them has infinite dimension. Consider $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ a bounded linear transformation. In order to define the generalized singular values for $T$ we consider the following convention:

- If $\mathcal{H}_1$ is infinite dimensional, then we set $s_i(T) = \lambda_i(|T|)$ for $i \in \mathbb{N}$, where $|T| = (T^*T)^{1/2} \in B(\mathcal{H}_1)$;
- If $\dim \mathcal{H}_1 = k \geq 1$ then we set $s_i(T) = \lambda_i(|T|)$ for $i \in \mathbb{I}_k$ (here we use Remark 2.3) and $s_i(T) = 0$, for $i \geq k + 1$. 

Note that in any case, \( s(T) = (s_i(T))_{i \in \mathbb{N}} \) is a non-increasing sequence indexed by \( \mathbb{N} \). Furthermore, with our present convention we always have that \( s(T) = s(T^*) \).

**Remark 2.5.** If \( A \in K(H)^{sa} \) is compact (recall that \( \dim H = \infty \)), it is easy to see that the entries of the sequence \( \lambda (A) = (\lambda_i(A))_{i \in \mathbb{Z}_0} \) are also eigenvalues of \( A \) (or zero), in such a way that the numbers \( \lambda_i(A) \) (for \( i \in \mathbb{N} \)) are the positive eigenvalues of \( A \) counting multiplicities (or zero) arranged in non-increasing order. Similarly, the numbers \( \lambda_{-i}(A) \) (for \( i \in \mathbb{N} \)) are the negative eigenvalues of \( A \) counting multiplicities (or zero) arranged in the non-decreasing order.

Note that if \( T \in K(H) \), then \( |T| \in K(H)^{sa} \). So that, the sequence \( s(T) \) in Definition 2.2 is the usual sequence of singular values of \( T \), considered as the non-increasing rearrangement of the eigenvalues of \( |T| \), counting multiplicities.

**Remark 2.6.** For a general \( A \in B(H)^{sa} \), (see [5, Section 3]) it follows that \( \lim_{i \to \infty} \lambda_{-i}(A) = \min \sigma_e(A) \) and \( \lim_{i \to \infty} \lambda_i(A) = \max \sigma_e(A) \), where \( \sigma_e(A) \) denotes the essential spectrum of \( A \). Hence, we have that

\[
C(A) \overset{\text{def}}{=} \{ \mu \in \mathbb{R} : \lambda_{-i}(A) \leq \mu \leq \lambda_i(A) \text{ for every } i \in \mathbb{N} \} = [\min \sigma_e(A), \max \sigma_e(A)] \neq \emptyset.
\]

Clearly, the elements of \( \Lambda^+(A) = \{ \lambda \in \sigma(A) : \lambda > \max \sigma_e(A) \} \) are eigenvalues with finite multiplicity (since \( A - \lambda I \) is Fredholm). Also, this set is countable and \( \Lambda^+(A) = \{ \lambda_i(A) : i \in \mathbb{N} \} \) or \( |\Lambda^+(A)| = n \) and \( \lambda_i(A) = \max \sigma_e(A) \) for every \( i > n \). A similar phenomenon happens with the sequence \( \lambda_{-i}(A) \) for \( i \in \mathbb{N} \), and \( \min \sigma_e(A) \). A consequence of this characterization of \( \lambda(A) \) is that, for every Hilbert space \( K \) and every \( B \in B(K)^{sa} \),

\[
\sigma(B) \subseteq C(A) \Rightarrow \lambda(A) = \lambda(A \oplus B), \text{ where } A \oplus B \in B(H \oplus K)^{sa}
\]

is the block diagonal operator determined by \( A \) and \( B \). Observe that Equation (2.4) holds for the operators \( A \oplus \mu I_K \) for every \( \mu \in C(A) \). In particular, if we assume that \( A \in K(H)^{sa} \) is compact, then automatically \( C(A) = \{0\} \) and \( \lambda(A) = \lambda(A \oplus 0_K) \).

**Example 2.7.** Let \( a = (a_n)_{n \in \mathbb{N}} \in \ell_\infty^\mathbb{C}(\mathbb{N}) \) and \( A = D_a \in B(\ell_2(\mathbb{N}))^{sa} \) the diagonal multiplication operator as in Equation (2.6). Then, \( C(A) = [\liminf a, \limsup a] \). This fact and Remark 2.6 allow us to compute the sequence \( \lambda(A) \) in this case. For example, if \( a = (1 + 1/1, -1 + 1/1, 1 + 1/2, -1 + 1/2, 1 + 1/3, -1 + 1/3, ...) \),

we see that \( \max \sigma_e(A) = 1, \min \sigma_e(A) = -1, \Lambda^+(A) = \{1 + 1/n : n \geq 1\} \) and \( \Lambda^-(A) = \{\lambda \in \sigma : \lambda < \min \sigma_e(A)\} = \emptyset \). Hence, \( \lambda_i(A) = 1 + 1/i \) while \( \lambda_{-i}(A) = -1 \), for \( i \in \mathbb{N} \).

In what follows, for \( 1 \leq k \leq \infty \), we let

\[
P_k(H) = \{ P \in B(H) : P^2 = P^* = P, \rk(P) = k \},
\]

denote the subset of \( B(H) \) of all the orthogonal projections of rank \( k \). Next, we collect several well-known facts that we will need in the following (for details, see [5, 16]).

**Theorem 2.8.** Let \( A \in B(H)^{sa} \) and \( P \in P_k(H) (1 \leq k \leq \infty) \).

1. **Interlacing inequalities:** denote by \( A_P \overset{\text{def}}{=} P A|_{R(P)} \in B(R(P))^{sa} \). Then

\[
\lambda_j(A) \geq \lambda_j(A_P) \quad \text{and} \quad \lambda_{-j}(A) \leq \lambda_{-j}(A_P) \quad \text{for} \quad j \in I_k.
\]

(In case \( k \in \mathbb{N} \), we have to consider Equation (2.2) in Remark 2.3).
2. For every \( k \in \mathbb{N} \),
\[
\sum_{i=1}^{k} \lambda_i(A) = \sup_{P \in \mathcal{P}_k} \text{tr}(PA) \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i(A) = \inf_{P \in \mathcal{P}_k} \text{tr}(PA).
\]

We can now recall the notion of submajorization and majorization between operators.

**Definition 2.9.** Let \( A, B \in B(H)^{sa} \). We say that \( A \) is submajorized by \( B \), denoted by
\[
A <_w B, \quad \text{if} \quad \sum_{i=1}^{k} \lambda_i(A) \leq \sum_{i=1}^{k} \lambda_i(B) \quad \text{for every} \quad k \in \mathbb{N}.
\]
We say that \( A \) is majorized by \( B \), denoted by \( A \prec B \), if and only if
\[
\sum_{i=1}^{k} \lambda_i(A) \leq \sum_{i=1}^{k} \lambda_i(B) \quad \text{and} \quad \sum_{i=1}^{k} \lambda_{-i}(A) \geq \sum_{i=1}^{k} \lambda_{-i}(B) \quad \text{for every} \quad k \in \mathbb{N}.
\]

We can further consider the notion of (sub)majorization between sequences in \( \ell^\infty(\mathbb{M}) \), where \( \mathbb{M} = \mathbb{N} \) or \( \mathbb{M} = \mathbb{Z}_0 \). In order to do this, we consider the auxiliary Hilbert space \( \ell^2(\mathbb{M}) \). Hence, given \( a = (a_i)_{i \in \mathbb{M}} \in \ell^\infty(\mathbb{M}) \) a bounded real sequence, let \( D_a \in B(\ell^2(\mathbb{M}))^{sa} \) be determined by
\[
D_a((\gamma_i)_{i \in \mathbb{M}}) = (a_i \gamma_i)_{i \in \mathbb{M}}
\]
for \( (\gamma_i)_{i \in \mathbb{M}} \in \ell^2(\mathbb{M}) \).

**Definition 2.10.** Let \( a = (a_i)_{i \in \mathbb{M}_1} \in \ell^\infty(\mathbb{M}_1) \) and \( b = (b_i)_{i \in \mathbb{M}_2} \in \ell^\infty(\mathbb{M}_2) \) be real sequences, with \( \mathbb{M}_1 = \mathbb{N} \) or \( \mathbb{M}_1 = \mathbb{Z}_0 \), for \( i = 1, 2 \).

1. We let \( a^1 = (a^1_i)_{i \geq 1} \in \ell^\infty(\mathbb{N}) \) and \( a^{11} = (a^{11}_i)_{i \in \mathbb{Z}_0} \in \ell^\infty(\mathbb{Z}_0) \) be given by
\[
a^1_i = \lambda_i(D_a) \quad \text{for} \quad i \geq 1 \quad \text{and} \quad a^{11}_i = \lambda_i(D_a) \quad \text{for} \quad i \in \mathbb{Z}_0.
\]

2. We say that \( a \) is submajorized by \( b \), denoted by \( a <_w b \), if
\[
\sum_{i=1}^{k} a^1_i \leq \sum_{i=1}^{k} b^1_i \quad \text{for} \quad k \in \mathbb{N}.
\]

3. We say that \( a \) is majorized by \( b \), denoted as \( a \prec b \), if \( a <_w b \) and \( -a <_w -b \), that is,
\[
\sum_{i=1}^{k} a^{11}_i \leq \sum_{i=1}^{k} b^{11}_i \quad \text{and} \quad \sum_{i=1}^{k} a^{11}_{-i} \geq \sum_{i=1}^{k} b^{11}_{-i} \quad \text{for} \quad k \in \mathbb{N}.
\]

As a consequence of Definition 2.10 and Theorem 2.8, given \( a = (a_n)_{n \in \mathbb{M}} \in \ell^\infty(\mathbb{M}) \), then
\[
\sum_{i=1}^{k} a^{11}_i = \sup \left\{ \sum_{i \in F} a_i : F \subseteq \mathbb{M}, |F| = k \right\} \quad \text{and} \quad \sum_{i=1}^{k} a^{11}_{-i} = \inf \left\{ \sum_{i \in F} a_i : F \subseteq \mathbb{M}, |F| = k \right\} \quad \text{for} \quad k \in \mathbb{N}.
\]

Given sequences \( a = (a_n)_{n \in \mathbb{N}}, \ b = (b_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N}) \) we let \( (a, b) \in \ell^\infty(\mathbb{Z}_0) \) be the sequence determined by
\[
(a, b)_n = \begin{cases} a_{-n} & \text{if} \quad n < 0 \\ b_n & \text{if} \quad n > 0 \end{cases} \quad \text{for} \quad n \in \mathbb{Z}_0.
\]
On the other hand, given \( \mathbf{a} = (a_n)_{n \in \mathbb{M}}, \mathbf{b} = (b_n)_{n \in \mathbb{M}} \in \ell^\infty(\mathbb{M}) \) we let
\[
\mathbf{a} \cdot \mathbf{b} \in \ell^\infty(\mathbb{M}) \quad \text{be given by} \quad (\mathbf{a} \cdot \mathbf{b})_n = a_n b_n, \quad \text{for} \quad n \in \mathbb{M},
\]
(2.8)
where \( \mathbb{M} = \mathbb{N} \) or \( \mathbb{M} = \mathbb{Z}_0 \).

Submajorization relations appear in a natural way in the operator theory. In the following result, we collect some well-known results related to this notion (see [16]).

**Theorem 2.11.** Let \( A, B, X, Y \in B(H) \). Then, the following relations hold:

1. Weyl's inequality for spectral scales: if \( A, B \in B(H)^{sa} \) then \( \lambda(A + B) \prec \lambda(A) + \lambda(B) \).
2. Weyl's inequality for generalized singular values:
   \[
s(A + B) \prec w s(A) + s(B).
   \]
(2.9)
3. \( s_i(XAY) \leq \|X\| \|Y\| s_i(A) \) for \( i \geq 1 \). In particular
   \[
s(XAY) \prec w \|X\| \|Y\| s(A) \quad \text{and} \quad s(UAV) = s(A) \quad \text{for} \quad U, V \in \mathcal{U}(H).
   \]
4. If \( A \in B(H)^{sa} \) and we let \( |\lambda(A)| = (|\lambda_i(A)|)_{i \in \mathbb{Z}_0} \) then \( s(A) = |\lambda(A)|^\frac{1}{4} \).

**Remark 2.12.** Submajorization relations play a central role in the study of symmetrically normed operator ideals (see [16] for a detailed exposition). A symmetrically normed operator ideal \( C \) of \( B(H) \) is a proper two-sided ideal with a symmetric norm \( N(\cdot) \), that is, a norm with the following additional properties:

1. Given \( A \in C \) and \( D, E \in B(H) \), then \( N(DAE) \leq \|D\|N(A)\|E\| \);
2. For any operator \( A \) such that \( \text{rk}(A) = 1 \), \( N(A) = \|A\| = s_1(A) \).

Moreover, any symmetric norm is unitarily invariant as a consequence of item 1; that is, if \( U, V \in \mathcal{U}(H) \) then \( N(UAV) = N(A) \), for every \( A \in C \). Any such norm induces a gauge symmetric function \( g_N \) defined on bounded sequences, such that \( N(A) = g_N(s(A)) \).

In this context, we have that \( C \subset K(H) \). Moreover, given \( B \in C \) and \( A \in K(H) \) then
\[
s(A) \prec w s(B) \Rightarrow A \in C \quad \text{and} \quad N(A) = g_N(s(A)) \leq g_N(s(B)) = N(B).
\]
(2.10)
For the sake of simplicity, in what follows we will call such a norm \( N(\cdot) \) a unitarily invariant norm. As examples of unitarily invariant norms, we mention the Schatten \( p \)-norms, for \( 1 \leq p < \infty \) associated with the Schatten ideals in \( B(H) \).

### 3 | SPECTRAL SPREAD: DEFINITION AND BASIC PROPERTIES

In this section, we introduce and develop the first properties of the spectral spread for self-adjoint operators, which is motivated by the spread of self-adjoint matrices introduced by Knyazev and Argentati in [25].

**Definition 3.1.** Given \( A \in B(H)^{sa} \) we define the **full spectral spread** of \( A \), denoted \( \text{Spr}(A) \in \ell^\infty(\mathbb{Z}_0) \) as the sequence
\[
\text{Spr}(A) \overset{\text{def}}{=} (\lambda_i(A) - \lambda_{-i}(A))_{i \in \mathbb{Z}_0} = \lambda(A) + \lambda(-A).
\]
(3.1)
We also consider the **spectral spread** of \( A \), that is, the non-negative and non-increasing sequence
\[
\text{Spr}^+(A) \overset{\text{def}}{=} (\text{Spr}_i(A))_{i \in \mathbb{N}} \in \ell^\infty(\mathbb{N})
\].
Note that, by Equations (2.2) and (3.1), this definition of spectral spread essentially coincides with the matrix spread defined in [25, 27]. It is clear that the spectral spread of an operator in $\mathcal{B}(H)^{sa}$ is a vector-valued measure of the dispersion of its spectral scale.

In the next result, we collect some basic properties about the spectral spread in $\mathcal{B}(H)^{sa}$.

**Proposition 3.2.** Let $A, B \in \mathcal{B}(H)^{sa}$. The following properties hold:

1. $\text{Spr}(A) \in \ell^\infty(\mathbb{Z}_0)$ is anti-symmetric ($\text{Spr}_{-j}(A) = -\text{Spr}_j(A)$); $\text{Spr}^+(A) \in \ell^\infty(\mathbb{N}) \cap \mathbb{R}_{\geq 0}$.  

2. The spectral spread is invariant under real translations, that is, for every $c \in \mathbb{R}$,
   \[ \text{Spr}(A + c I) = \text{Spr}(A) \quad \text{and} \quad \text{Spr}^+(A + c I) = \text{Spr}^+(A). \]

3. For $c \in \mathbb{R}$, we have that $\text{Spr}(cA) = |c| \text{Spr}(A)$. In particular, $\text{Spr}^+(A) = \text{Spr}^+(-A)$.

4. If $A \in K(H)^{sa}$ is compact, then $\text{Spr}^+(A) = \text{Spr}^+(A \oplus 0_H)$.

**Proof.** Items 1 and 2 are direct consequences of Equation (3.1) in Definition 3.1. Item 3 is a consequence of the following fact: given $A \in \mathcal{B}(H)^{sa}$ then $\lambda(\alpha A) = \alpha \lambda(A)$ if $\alpha \geq 0$ and $\lambda(\alpha A) = (\lambda_{-1}(A))_{i \in \mathbb{Z}_0}$ if $\alpha < 0$. Item 4 is a consequence of Remark 2.6. \hfill \Box

Observe that the equality $\text{Spr}^+(A) = \text{Spr}^+(A \oplus 0_H)$ may be false for general self-adjoint operators (and also for matrices). For example, $\text{Spr}^+(I_H) = 0$ but $\text{Spr}^+(I_H \oplus 0_H) = 1$, the sequence constantly equal to one. The following result describes several relations between the (full) spectral spread and singular values of self-adjoint operators.

**Proposition 3.3.** Let $A, B \in \mathcal{B}(H)^{sa}$.

1. The following entry-wise inequalities hold:
   \[ 0 \leq \text{Spr}^+_i(A) \leq |\lambda_i(A)| + |\lambda_{-i}(A)| \leq 2s_i(A) \quad \text{for every} \quad i \in \mathbb{N}. \tag{3.2} \]
   In the positive case, we have that:
   \[ A \in \mathcal{B}(H)^+ \Rightarrow \text{Spr}^+_i(A) \leq \lambda_i(A) = s_i(A) \quad \text{for every} \quad i \in \mathbb{N}. \tag{3.3} \]

2. Let $A \oplus A \in \mathcal{B}(H \oplus H)^{sa}$ be given by $A \oplus A = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$. Then,
   \[ \text{Spr}^+(A \oplus A) = (\text{Spr}^+(A), \text{Spr}^+(A))^{\dagger} \quad \text{and} \quad \frac{1}{2} \text{Spr}^+(A \oplus A) \prec_w s(A). \tag{3.4} \]

3. If $A < B$ then $\text{Spr}^+(A) \prec_w \text{Spr}^+(B)$.

4. Additive Spread inequality: $\text{Spr}(A + B) \prec \text{Spr}(A) + \text{Spr}(B)$.

**Proof.** Since $s(A) = |\lambda(A)|^{\dagger}$ (see Theorem 2.11), it follows that
   \[ \max\{|\lambda_i(A)|, |\lambda_{-i}(A)|\} \leq s_i(A) \quad \text{for every} \quad i \in \mathbb{N}. \tag{3.5} \]

This proves the first part of item 1. The second part of item 1 follows from the fact that $\lambda_{-i}(A) \geq 0$ for $A \in \mathcal{B}(H)^+$. In order to show item 2 notice that the first claim in Equation (3.4) is straightforward. To show the second claim in Equation (3.4), we note that
   \[ \sum_{i=1}^n \text{Spr}^+_i(A \oplus A) = \begin{cases} \sum_{i=1}^k \text{Spr}^+_i(A) & \text{if} \quad n = 2k \\ 2 \sum_{i=1}^{k+1} \text{Spr}^+_i(A) + \text{Spr}^+_k(A) & \text{if} \quad n = 2k + 1 \end{cases}. \]
Recall that \( \sum_{i=1}^{n} s_i(A) = \sup \left\{ \sum_{i \in F} |\lambda_i(A)| : F \subset \mathbb{Z}_0, |F| = n \right\} \). Then, using that
\[
\text{Spr}^+_{k+1}(A) = \lambda_{k+1}(A) - \lambda_{-(k+1)}(A) \leq 2 \max[|\lambda_{k+1}(A)|, |\lambda_{-(k+1)}(A)|]
\]
and that
\[
2 \sum_{i=1}^{k} \text{Spr}^+_{i}(A) \leq 2 \sum_{i=1}^{k} |\lambda_i(A)| + |\lambda_{-i}(A)|,
\]
we can easily prove the submajorization relation in Equation (3.4).

To show item 3, fix \( k \in \mathbb{N} \). Since \( \lambda(A) \prec \lambda(B) \), then
\[
\sum_{i=1}^{k} \lambda_i(A) \leq \sum_{i=1}^{k} \lambda_i(B) \quad \text{and} \quad \sum_{i=1}^{k} \lambda_{-i}(A) \geq \sum_{i=1}^{k} \lambda_{-i}(B)
\]
\[
\implies \sum_{i=1}^{k} \text{Spr}^+_i(A) = \sum_{i=1}^{k} \lambda_i(A) - \lambda_{-i}(A) \leq \sum_{i=1}^{k} \lambda_i(B) - \lambda_{-i}(B) = \sum_{i=1}^{k} \text{Spr}^+_i(B).
\]

To show item 4, note that \( \text{Spr}(A + B) = \lambda(A + B) + \lambda(-(A + B)) \). Therefore,
\[
\text{Spr}(A + B) = \lambda(A + B) + \lambda(-A - B)
\]
\[
\text{Weyl} < \lambda(A) + \lambda(B) + \lambda(-A) + \lambda(-B) = \text{Spr}(A) + \text{Spr}(B),
\]
where we have used Weyl’s additive inequality for the spectral scale. □

4 | INEQUALITIES FOR THE SPECTRAL SPREAD

In this section, we obtain several submajorization inequalities for the spectral spread of self-adjoint operators. These inequalities show that the spectral spread is a natural measure of dispersion of the spectrum of self-adjoint operators.

4.1 | A key inequality

In [30], Tao showed that given a positive compact operator \( F \in K(H \oplus K) \) represented as a block matrix
\[
F = \begin{bmatrix} F_1 & G \\ G^* & F_2 \end{bmatrix}
\]
then \( 2s_i(G) \leq s_i(F) \), for \( i \in \mathbb{N} \). (4.1)

It is natural to ask whether the inequalities in Equation (4.1) hold in the more general case in which \( F \) is a self-adjoint compact operator. It turns out that these inequalities fail in this more general setting (see [27]).

The next submajorization inequality, which is our first main result of this section, is related to Tao’s inequalities in Equation (4.1); we point out that it will play a key role in the rest of this work. We include the following proof in benefit of the reader; thanks to the definitions given in the previous sections, we can almost reproduce the proof of the matrix case given in [27] for this result:

**Theorem 4.1.** Let \( A \in B(H \oplus K)^{sa} \) be such that \( A = \begin{bmatrix} A_1 & B \\ B^* & A_2 \end{bmatrix} \in \mathcal{H} \) is the block representation for \( A \). Then, \( B \in B(K, H) \) (by construction) and
\[
2s(B) \prec_{s} \text{Spr}^+(A).
\] (4.2)
Proof. Consider \( U = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \in \mathcal{U}(\mathcal{H} \oplus \mathcal{K}) \). Then, \( UA - AU \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K}) \) and

\[
s(UA - AU) = s(A - U^*AU) = |\lambda(A - U^*AU)|^1,\]

where we have used items 3 and 4 in Theorem 2.11. By Weyl’s inequality for spectral scales (see Theorem 2.11), we have that

\[
\lambda(A - U^*A U) < \lambda(A) + \lambda(-U^*A U) = \text{Spr}(A),
\]
since \( \lambda(-U^*A U) = \lambda(-A) \). Using item 2 from Lemma A.1, we deduce that

\[
s(UA - AU) = |\lambda(A - U^*A U)|^1 <_{\omega} |\text{Spr}(A)|^1 = (\text{Spr}^+(A), \text{Spr}^{-}(A))^\perp,
\]

where \( (\text{Spr}^+(A), \text{Spr}^{-}(A)) \in \ell^\infty(\mathbb{Z}_0) \) is constructed as in Equation (2.7). Straightforward computations and Proposition A.3 show that

\[
UA - AU = \begin{bmatrix} 0 & 2B \\ -2B^* & 0 \end{bmatrix} \mathcal{H} \implies s(UA - AU) = 2 (s(B), s(B))^\perp,
\]

and we conclude that

\[
2 (s(B), s(B))^\perp <_{\omega} (\text{Spr}^+(A), \text{Spr}^{-}(A))^\perp \Rightarrow 2s(B) <_{\omega} \text{Spr}^{-}(A). \quad \square
\]

Although simple, the inequality in Equation (4.2) is a useful result. Indeed, it plays a crucial role in the proof of Theorem 4.4. On the other hand, this inequality cannot be improved to an entry-wise inequality in the general case \( A \in \mathcal{B}(\mathcal{H})^{sa} \) (see [27, Remark 2.9]).

**Corollary 4.2.** With the notation of Theorem 4.1, assume further that \( A \in K(\mathcal{H})^{sa} \). Then, for any unitarily invariant norm \( N \) with gauge symmetric function \( g_N \), we have that

\[
2N(B) \leq g_N(\text{Spr}^+(A)).
\]

**Proof.** The inequality follows from Theorem 4.1 and Remark 2.12. \( \square \)

### 4.2 Commutator inequalities

In [22], Kittaneh obtained the following singular value inequalities for commutators of positive compact operators: given \( C, D \in K(\mathcal{H})^+ \) and a bounded operator \( X \in \mathcal{B}(\mathcal{H}) \), then

\[
s_i(CX - XD) \leq \|X\| s_i(C \oplus D), \quad \text{for} \quad i \in \mathbb{N},
\]

where \( \| \cdot \| \) denotes the operator (or spectral) norm. It turns out that Equation (4.3) fails in case \( C \) and \( D \) are arbitrary self-adjoint compact operators (take \( C = X = I \) and \( D = -C \)).

In what follows, we obtain Theorem 4.5 related to Kittaneh’s inequalities above that holds for self-adjoint operators. We begin with the following observation.

**Lemma 4.3.** Let \( A, B \in \mathcal{B}(\mathcal{H})^{sa} \) be such that \( \text{rk}(A) = n \in \mathbb{N} \). Then

\[
\text{tr}(AB) \leq \text{tr} (\lambda(A) \cdot \lambda(B)) \overset{\text{def}}{=} \sum_{k \in \mathbb{Z}_0} \lambda_k(A) \lambda_k(B). \quad (4.4)
\]
Proof. Since $A$ has finite rank, then it is a trace class operator. In particular, $A \in K(H)$ so Remark 2.3 implies that the entries of $\lambda(A)$ are the eigenvalues of $A$ counting multiplicities and zeros; hence, $\text{tr}(A) = \sum_{k \in \mathbb{Z}} \lambda_k(A)$. These last facts show that the inequality (4.4) is invariant if we replace $B$ by $B + \mu I$, for any $\mu \in \mathbb{R}$; so we can assume that $B \in B(H)^+$. We also have that $\lambda_i(A) \geq 0$ and $\lambda_{-i}(A) \leq 0$, for $i \in \mathbb{N}$. Moreover, there is an orthonormal basis for the range of $A$, denoted $R(A)$, say $\{x_i\}_{i=-m, i \neq 0}$ with $m + r = n$, $m, r \geq 0$, such that

$$A = \sum_{i=-m, i \neq 0}^r \lambda_i(A) x_i \otimes x_i \Rightarrow$$

$$\text{tr}(AB) = \text{tr}(BA) = \sum_{i=-m, i \neq 0}^r \lambda_i(A) \text{tr}(B x_i \otimes x_i) = \sum_{i=-m, i \neq 0}^r \lambda_i(A) \langle B x_i , x_i \rangle = \sum_{i=-m, i \neq 0}^r \lambda_i(A) \langle B x_i , x_i \rangle.$$  (4.5)

Let $P$ denote the orthogonal projection onto $\text{Span}\{B\} = R(A)$. Denote by $B_P := P B |_{R(P)} \in B(R(A))^+$ the compression of $B$ to $R(A)$. Using the Schur–Horn theorem (see [6] or [5]) for $B_P$ and its matrix relative to $\{x_i\}$, we get that its “diagonal”

$$d = (\langle B x_m, x_m \rangle, \ldots, \langle B x_{-1}, x_{-1} \rangle, \langle B x_1, x_1 \rangle, \ldots, \langle B x_r, x_r \rangle) \prec (\lambda_i(B_P))_{i \in \mathbb{N}}.$$  

Using Equation (2.2), we can deduce that $d_+ := (\langle B x_i, x_i \rangle)_{i \in \mathbb{I}_r} \prec_w (\lambda_i(B_P))_{i \in \mathbb{I}_r}$ and, since also

$$-d < (-\lambda_i(B_P))_{i \in \mathbb{I}_m} \prec_w (-\lambda_{-i}(B_P))_{i \in \mathbb{I}_m},$$

we now consider the auxiliary vectors $d_+^{(i)} = (a_i)_{i \in \mathbb{I}_r}$ and $d_-^{(i)} = (b_i)_{i \in \mathbb{I}_m}$ that are obtained from $d_+$ and $d_-$ defined above, by rearranging their entries in the non-increasing order. Using that the vector $(\lambda_i(A))_{i \in \mathbb{I}_r}$ has non-negative entries and is arranged in non-increasing order then, by items 5 and 6 in Lemma A.1, we conclude that

$$(\lambda_i(A) \cdot \langle B x_i, x_i \rangle)_{i \in \mathbb{I}_r} \prec_w (\lambda_i(A) \cdot a_i)_{i \in \mathbb{I}_r} \prec_w (\lambda_i(A) \cdot \lambda_i(B_P))_{i \in \mathbb{I}_r},$$

that implies that

$$\sum_{i=1}^r \lambda_i(A) \cdot \langle B x_i, x_i \rangle \leq \sum_{i=1}^r \lambda_i(A) \cdot \lambda_i(B_P) \leq \sum_{i=1}^r \lambda_i(A) \cdot \lambda_i(B),$$  (4.6)

where the last inequality follows from the interlacing inequalities (2.5). Similarly, using the interlacing inequalities, we see that

$$d_-^{(i)} = (b_i)_{i \in \mathbb{I}_m} \prec_w (\lambda_{-i}(B_P))_{i \in \mathbb{I}_m} \prec_w (\lambda_{-i}(B))_{i \in \mathbb{I}_m} \in (\mathbb{R}^m)_+. \quad (2.5)$$

Since $(-\lambda_{-i}(A))_{i \in \mathbb{I}_m} \in (\mathbb{R}_{\geq 0}^m)_+$ then, by item 2 in Lemma A.2 we get that

$$\sum_{i \in \mathbb{I}_m} -\lambda_{-i}(A) \cdot b_i \leq \sum_{i \in \mathbb{I}_m} \lambda_{-i}(A) \cdot \lambda_{-i}(B).$$  (4.7)

On the other hand, since the rearrangement of the vector $(\langle B x_i, x_i \rangle)_{i \in \mathbb{I}_m}$ in the non-decreasing order coincides with $(-b_i)_{i \in \mathbb{I}_m} \in (\mathbb{R}^m)_+$ then, by item 1 in Lemma A.2 we conclude that

$$(\lambda_{-i}(A) b_i)_{i \in \mathbb{I}_m} \prec_w (\lambda_{-i}(A))_{i \in \mathbb{I}_m} \langle B x_i, x_i \rangle_{i \in \mathbb{I}_m}.$$  

The previous submajorization relation and Equation (4.7) imply that

$$\sum_{i \in \mathbb{I}_m} \lambda_{-i}(A) \langle B x_i, x_i \rangle \leq \sum_{i \in \mathbb{I}_m} -\lambda_{-i}(A) b_i \leq \sum_{i \in \mathbb{I}_m} \lambda_{-i}(A) \cdot \lambda_{-i}(B).$$  (4.8)

Using the inequalities in Equations (4.6) and (4.8), we now see that

$$\text{tr}(AB) = \sum_{i=-m, i \neq 0}^r \lambda_i(A) \langle B x_i , x_i \rangle \leq \sum_{i=-m, i \neq 0}^r \lambda_i(A) \lambda_i(B),$$

which completes the proof. \hfill \Box
Theorem 4.4. Let \( A, X \in B(H)^{sa} \). If we let \( i = \sqrt{-1} \), then

\[
\lambda( i(AX-X^A)) <_w \frac{1}{2} \Spr^+(A) \cdot \Spr^+(X). \tag{4.9}
\]

Proof. Let \( \varepsilon > 0 \) and \( k \in \mathbb{N} \); by Theorem 2.8, there exists an orthogonal projection \( P \in B(H) \) with \( k = \text{tr}(P) \) such that

\[
\sum_{j=1}^{k} \lambda_j( i(AX-X^A)) \leq \text{tr}( i(AX-X^A) P ) + \varepsilon.
\]

Moreover, since \( XP-PX \) has finite rank then, by Lemma 4.3, we get that

\[
\text{tr}( i(AX-X^A) P ) = \text{tr}( i(XP-PX) A ) \leq \lambda( i(XP-PX)) \cdot \lambda(A).
\]

Now, consider the block matrix representations induced by \( P \):

\[
X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow i(XP-PX) = i \begin{bmatrix} 0 & -X_{12} \\ X_{12} & 0 \end{bmatrix}.
\]

Denote \( X_{12} = B \in K(H) \); then by Proposition A.3 \( \lambda(i(XP-PX)) = (s(B), -s(B^*)) \). Now, Theorem 4.1 implies that \( s(B) <_w \frac{1}{2} \Spr^+(X) \). The previous fact together with item 6 in Lemma A.1 show that

\[
s(B) \cdot \Spr^+(A) <_w \frac{1}{2} \Spr^+(X) \cdot \Spr^+(A). \tag{4.10}
\]

Moreover, if we let \( k' \leq k \) be the number of non-zero singular values of \( B \) then

\[
\text{tr}( \lambda(i(XP-PX)) \cdot \lambda(A)) = \sum_{j=1}^{k'} s_j(B) \lambda_j(A) - \sum_{j=1}^{k'} s_j(B) \lambda_{-j}(A)
\]

\[
= \sum_{j=1}^{k'} s_j(B) \Spr^+_j(A) \leq \frac{1}{2} \sum_{j=1}^{k'} \Spr^+_j(X) \Spr^+_j(A). \tag{4.10}
\]

Combining the previous arguments it is clear that

\[
\sum_{j=1}^{k} \lambda_j( i(AX-X^A)) \leq \frac{1}{2} \sum_{j=1}^{k'} \Spr^+_j(X) \Spr^+_j(A) + \varepsilon \leq \frac{1}{2} \sum_{j=1}^{k} \Spr^+_j(X) \Spr^+_j(A) + \varepsilon.
\]

Since \( \varepsilon > 0 \) and \( k \in \mathbb{N} \) were arbitrary, we get the submajorization relation in Equation (4.9).

Although Theorem 4.4 contains much information, its statement is rather technical. For example, the submajorization of Equation (4.9) only gives information about the “positive part” of \( \lambda(i(AX-X^A)) \), namely \( (\lambda_j(i(AX-X^A)))_{j \in \mathbb{N}} \). The next result, which is a consequence of Theorem 4.4, is more clear, and it has several direct implications (see Corollary 4.6).

Theorem 4.5. Let \( A, X \in B(H)^{sa} \). Then

\[
s(AX-X^A) <_w \frac{1}{2} \Spr^+(A \oplus A) \cdot \Spr^+(X \oplus X). \tag{4.11}
\]

If we further assume that \( A \) or \( X \in K(H)^{sa} \) then, for any unitarily invariant norm \( N \) with gauge symmetric function \( g_N \),

\[
N(AX-X^A) \leq \frac{1}{2} g_N \left( \Spr^+(A \oplus A) \cdot \Spr^+(X \oplus X) \right). \tag{4.12}
\]

Proof. Using Equation (4.9) applied to \( A, -X \in B(H)^{sa} \),

\[
\lambda(-i(AX-X^A)) <_w \frac{1}{2} \Spr^+(A) \cdot \Spr^+(-X) = \frac{1}{2} \Spr^+(A) \cdot \Spr^+(X),
\]
where \( i = \sqrt{-1} \) and we used that Spr\(^+\)(\(-X\)) = Spr\(^+\)(X). By the comments after Definition 2.1, item 4 in Theorem 2.11 and item 4 in Lemma A.1, we have that

\[
s(AX -XA) = (\lambda_j(i(AX -XA)))_{j \in \mathbb{N}}, (\lambda_j(-i(AX -XA)))_{j \in \mathbb{N}}
\]

\[
\prec_w \frac{1}{2} (\text{Spr}^+(A) \cdot \text{Spr}^+(X), \text{Spr}^+(A) \cdot \text{Spr}^+(X))
\]

\[
= \frac{1}{2} \text{Spr}^+(A \oplus A) \cdot \text{Spr}^+(X \oplus X),
\]

which proves Equation (4.11). Finally, by Equation (2.10) in Remark 2.12 we know that Equation (4.11) \( \Rightarrow \) Equation (4.12).

A statement that is formally analogous to Theorem 4.5 is still valid in the matrix case, with the definition of spread given in [27]. Indeed, the proof of such claim can be obtained by a straightforward adaptation of the proof of Theorem 4.5. The next result, which is formally analogous to [27, Theorem 3.1.] (and played a central role in [28]) is proved here with a new approach, based on Theorem 4.5.

**Corollary 4.6.** Let \( A, B \in B(H)^{sa} \), \( A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in B(H \oplus H)^{sa} \) and \( X \in B(H) \). Then,

\[
s(AX - XB) \prec_w \text{Spr}^+(A \oplus B) \cdot s(X).
\]

**Proof.** First, take \( A = B \) and assume that \( X \in B(H)^{sa} \). By Theorem 4.5, Proposition 3.3 (item 2) and Lemma A.1 (item 6), we have that

\[
s(AX - XA) \prec_w \frac{1}{2} \text{Spr}^+(A \oplus A) \cdot \text{Spr}^+(X \oplus X) \prec_w \text{Spr}^+(A \oplus A) \cdot s(X).
\]

In the general case, let \( C = A \oplus B \in B(H \oplus H)^{sa} \) and \( \hat{X} = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} \in B(H \oplus H)^{sa} \). Then

\[
C\hat{X} - \hat{X}C = \begin{bmatrix} 0 & AX - XB \\ (AX - XB)^* & 0 \end{bmatrix},
\]

and, by the first part of the proof

\[
s(C\hat{X} - \hat{X}C) = (s(AX - XB), s(AX - XB))^\dag \prec_w \text{Spr}^+(C \oplus C) \cdot s(\hat{X}).
\]

Note that Equation (4.13) follows from the previous submajorization relation, since

\[
s(\hat{X}) = (s(X), s(X))^\dag \quad \text{and} \quad \text{Spr}^+(C \oplus C) = (\text{Spr}^+(A \oplus B), \text{Spr}^+(A \oplus B))^\dag.
\]

**Remark 4.7.** Note that inequality (4.13) cannot be improved to an entry-wise inequality, as Kittaneh’s inequality in Equation (4.3) for the positive case. For example, we can consider

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},
\]

all of them embedded on \( K(H)^{sa} \). Then

\[
\lambda(A \oplus B) = ((... , 0, -1), (3, 1, 1, 0, ...)) \quad \text{and} \quad s(AX -XB) = (6, 2, 0, ...)
\]

and \( s(X) = (3, 1, 0, ...) \). Therefore, \( s_2(AX -XB) = 2 > \text{Spr}_2^+(A \oplus B) s_2(X) = 1 \).

The next result gives an upper bound for the generalized singular values of commutators of the form \( AX - XB \), when \( A, B, X \in B(H) \) are arbitrary operators (for related results, see [20]).
Theorem 4.8. Let $A, B$ and $X \in \mathcal{B}(\mathcal{H})$. Consider $A = A_1 + iA_2$, $B = B_1 + iB_2$, where $A_j, B_j \in \mathcal{B}(\mathcal{H})$ for $j = 1, 2$ and $i = \sqrt{-1}$. Then

\[
s(AX - XB) < \omega \left( \text{Spr}^+(A_1 \oplus B_1) + \text{Spr}^+(A_2 \oplus B_2) \right) \cdot s(X).
\]

Proof. Note that

\[
AX - XB = (A_1 + iA_2)X - X(B_1 + iB_2) = A_1X - XB_1 + i(A_2X - XB_2).
\]

Then by Weyl's inequality for generalized singular values (see Equation (2.9)) and Corollary 4.6,

\[
s(AX - XB) < \omega s(A_1X - XB_1) + s(A_2X - XB_2)
\]

\[
< \omega \left( \text{Spr}^+(A_1 \oplus B_1) \cdot s(X) + \text{Spr}^+(A_2 \oplus B_2) \cdot s(X) \right).
\]

\[
(4.14)
\]

Corollary 4.9. Let $A, B \in \mathcal{B}(\mathcal{H})$ and $i = \sqrt{-1}$. Consider $A = A_1 + iA_2$, $B = B_1 + iB_2$, where $A_j, B_j \in \mathcal{B}(\mathcal{H})$ for $j = 1, 2$. Let $a_j, a'_j, b_j, b'_j \in \mathbb{R}$ be such that

\[
\begin{align*}
a_j I &\leq A_j \leq a'_j I, \quad \text{and} \quad b_j I \leq B_j \leq b'_j I \quad \text{for} \quad j = 1, 2. \\
\end{align*}
\]

Then, for $X \in K(\mathcal{H})$ and every unitarily invariant norm $N(\cdot)$ we have that

\[
N(AX - XB) \leq \left( \max\{a'_1, b'_1\} - \min\{a_1, b_1\} + \max\{a'_2, b'_2\} - \min\{a_2, b_2\} \right) N(X).
\]

Proof. Note that since $a_j I \leq A_j \leq a'_j I$ and $b_j I \leq B_j \leq b'_j I$ for $j = 1, 2$, then

\[
\text{Spr}^+(A_j \oplus B_j) \leq \left( \max\{a'_j, b'_j\} - \min\{a_j, b_j\} \right) \mathbb{1} \quad \text{for} \quad j = 1, 2.
\]

From Theorem 4.8, we now see that

\[
s(AX - XB) < \omega \left( \max\{a'_1, b'_1\} - \min\{a_1, b_1\} + \max\{a'_2, b'_2\} - \min\{a_2, b_2\} \right) s(X).
\]

\[
(4.15)
\]

Now, the result follows from Equation (4.15) and Remark 2.12. \qed

Theorem 4.10. Let $A, X \in \mathcal{B}(\mathcal{H})$ and $U = e^{iX} \in \mathcal{U}(\mathcal{H})$, where $i = \sqrt{-1}$. Then

\[
s(A - U^*AU) < \omega \frac{1}{2} \text{Spr}^+(X \oplus X) \cdot \text{Spr}^+(A \oplus A).
\]

Proof. Let $A(\cdot) : [0, 1] \to \mathcal{B}(\mathcal{H})$ be the smooth function given by $A(t) = e^{-itX} A e^{itX}$, for $t \in [0, 1]$. Note that $A(0) = A$ and $A(1) = U^*AU$; using Weyl's inequality in Equation (2.9),

\[
s(A - U^*AU) < \omega \sum_{j=0}^{m-1} s \left( A \left( \frac{j}{m} \right) - A \left( \frac{j+1}{m} \right) \right) \quad \text{for every} \quad m \in \mathbb{N}.
\]

\[
(4.16)
\]

Note that $A(t + h) = e^{-itX} A(h) e^{itX}$ with $e^{itX} \in \mathcal{U}(\mathcal{H})$, for $t, h, t + h \in [0, 1]$. Thus,

\[
s \left( A \left( \frac{j}{m} \right) - A \left( \frac{j+1}{m} \right) \right) = s \left( A - A \left( \frac{1}{m} \right) \right) \quad \text{for} \quad j \in \{0, \ldots, m-1\}.
\]

\[
(4.17)
\]

Since $A - A(0) = 0$ and $\frac{d}{dt} A(t)|_{t=0} = i(AX -XA)$, we get that

\[
s \left( A - A \left( \frac{1}{m} \right) \right) = \frac{1}{m} s(AX -XA) + O(m) \quad \text{with} \quad \lim_{m \to \infty} m O(m) = 0.
\]

\[
(4.18)
\]

Hence, by Theorem 4.5 we have that

\[
s \left( A - A \left( \frac{1}{m} \right) \right) < \omega \frac{1}{2m} \text{Spr}^+(X \oplus X) \cdot \text{Spr}^+(A \oplus A) + O(m).
\]

\[
(4.19)
\]
Therefore, by Equations (4.16) and (4.17) we have that, for sufficiently large \( m \),

\[
s(A - U^*AU) < \omega \frac{1}{2} \text{Spr}^+(X \oplus X) \cdot \text{Spr}^+(A \oplus A) + m O(m).
\]

The statement now follows by taking the limit \( m \to \infty \) in the expression above.

\[\Box\]

5 ARITHMETIC–GEOMETRIC MEAN-TYPE INEQUALITIES AND SPECTRAL SPREAD

Recall the AGM inequality for compact operators: given \( A, B \in K(H) \), Bhatia and Kittaneh showed in [8] that:

\[
2 s_i(A B^*) \leq s_i(A^*A + B^*B) \quad \text{for} \quad i \in \mathbb{N}.
\]  

(5.1)

Then, given \( S, C \in B(H) \) such that \( C^*C + S^*S \leq I \) and \( E \in K(H)^+ \), we get:

\[
2 s_j(SEC^*) \leq s_j(E) \quad \text{for} \quad i \in \mathbb{N},
\]  

(5.2)

by taking \( A = SE^{1/2} \) and \( E^{1/2}C^* = B^* \) in Equation (5.1). Corach et al. motivated by their study of the geometry in the context of operator algebras, obtained in [10] the following inequality with respect to a unitarily invariant norm \( N(\cdot) \)

\[
N(T) \leq \frac{1}{2} N(STS^{-1} + S^{-1}TS),
\]  

(5.3)

where \( T \in K(H) \) is a compact operator and \( S \in B(H) \) is self-adjoint and invertible bounded operator. Later on, Bhatia and Davis showed in [7] the following AGM-type inequality with respect to a unitarily invariant norm

\[
N(A^*XB) \leq \frac{1}{2} N(AA^*X + XBB^*),
\]  

(5.4)

where \( X \in K(H) \) is a compact operator and \( A, B \in B(H) \) are bounded operators. It turns out that inequalities in Equations (5.3) and (5.4) are equivalent (by simple substitutions).

These AGM-type inequalities (both for singular values and for unitarily invariant norms) have been studied and extended in different contexts [1, 3, 4]; it turns out that they are related with deep geometric properties of operators [2, 10–12].

Assume that \( C, S \in B(H) \) and let \( E \in K(H)^+ \). As in Equation (5.2), we have that

\[
s_j(SEC^*) = s_j((SE^{1/2}(CE^{1/2})^*) \leq \frac{1}{2} s_j \left( E^{1/2}(S^*S + C^*C)E^{1/2} \right) \quad \text{for} \quad j \in \mathbb{N}.
\]  

(5.5)

Equations (5.2) and (5.5) were derived in [3], where they were also shown to be stronger than the AGM-type inequalities obtained in [1] (see [17, 19] for related singular values inequalities). In particular, if \( C^*C + S^*S \leq I \), for any unitarily invariant norm \( N \) we have that

\[
N(SEC^*) \leq \frac{1}{2} N(E).
\]  

(5.6)

From Equations (5.5) and (5.6), it is possible to derive more general AGM-type inequalities. Nevertheless, Equation (5.6) fails for arbitrary self-adjoint \( E \in K(H)^{sa} \):

**Example 5.1.** As in previous examples, we shall use that a matrix \( A \in \mathcal{M}_n(\mathbb{C}) \) can be embedded as a finite rank operator, which allow us to build counterexamples using matrices. Now, we see that Equation (5.6) is false if \( E \not\geq 0 \). Consider \( N(X) = \|X\|_2 = (\text{tr}(X^*X))^{1/2} \) the Frobenius norm (which is unitarily invariant). Let

\[
S = \begin{bmatrix}
\sin(\pi/3) & 0 \\
0 & \sin(\pi/5)
\end{bmatrix}, \quad E = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix}
\cos(\pi/3) & 0 \\
0 & \cos(\pi/5)
\end{bmatrix}.
\]

Then, \( C^*C + S^*S = I \) and \( \|SEC^*\|_2 \approx 0.7598 > \sqrt{\frac{\gamma}{2}} = \frac{1}{2} \|E\|_2 \).
5.1 Arithmetic–geometric mean-type inequalities: the general self-adjoint case

In what follows, we obtain a generalization of Equation (5.6) for arbitrary self-adjoint $E \in B(H)^{sa}$; these are new inequalities that involve upper bounds in terms of the spectral spread of self-adjoint operators in $B(H)$. We point out that our results are based on the (weaker) submajorization relations; nevertheless, these results imply inequalities with respect to unitarily invariant norms as in Equation (5.6).

**Proposition 5.2.** Let $C, S \in B(H)$ be such that $C^*C + S^*S = P = P^2$ and let $E \in B(H)^{sa}$. Then,

$$2s(SEC) \prec_w \text{Spr}^+(PEP \oplus 0_H).$$

(5.7)

*Note that if dim ker $P = \infty$, we can rewrite $2s(SEC) \prec_w \text{Spr}^+(PEP).$

**Proof.** By considering the polar decomposition of $S$ and $C$ and Theorem 2.11, we can assume that $S, C \in B(H)^+$. Let $\mathcal{K} = R(P)$ and consider the Hilbert space $H \oplus \mathcal{K}$. We consider an orthogonal decomposition $H \oplus \mathcal{K} = \mathcal{K} \oplus \mathcal{K}^\perp \oplus \mathcal{K}$; this decomposition allows us to represent operators $T \in B(H \oplus \mathcal{K})$ as $3 \times 3$ block matrices. Since the compressions $C^2 + S^2 = I_P \in B(\mathcal{K})$, then the operator $U \in B(H \oplus \mathcal{K})$ whose block representation is given by

$$
\begin{bmatrix}
C_P & 0 & -S_P \\
0 & I & 0 \\
S_P & 0 & C_P 
\end{bmatrix}
$$

is unitary. Furthermore, it is straightforward to check that the block representations of $PEP \oplus 0_{\mathcal{K}}$, $U(PEP \oplus 0_{\mathcal{K}})U^* \in B(H \oplus \mathcal{K})^{sa}$ are given by

$$PEP \oplus 0_{\mathcal{K}} = \begin{bmatrix} E_P & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad U(PEP \oplus 0_{\mathcal{K}})U^* = \begin{bmatrix} C_P E_P C_P & 0 & C_P E_P S_P \\ 0 & 0 & 0 \\ S_P E_P C_P & 0 & S_P E_P S_P \end{bmatrix}.$$ 

where $E_P \in B(\mathcal{K})$ is the compression of $E$ to the subspace $\mathcal{K}$. Since $C = CP = P C P$ and $S = SP = P S P$, we get that $S_P E_P C_P = (SEC)_P$ and hence, the anti-diagonal block in the block representation above

$$[S_P E_P C_P \ 0] = [(SEC)_P \ 0] = SEC \in B(H, \mathcal{K}).$$

Note that in the equality above, we need to restrict the co-domain of $SEC \in B(H)$; yet, this restriction does not affect the generalized singular values (see Remark 2.4). Hence, as a consequence of Theorem 4.1 we get that

$$2s(SEC) \prec_w \text{Spr}^+(U(PEP \oplus 0_{\mathcal{K}})U^*) = \text{Spr}^+(PEP \oplus 0_{\mathcal{K}}).$$

Finally, note that we always get that $\text{Spr}^+(PEP \oplus 0_{\mathcal{K}}) = \text{Spr}^+(PEP \oplus 0_H)$. Indeed, if dim $\mathcal{K} = \text{dim } H$ this is clear; in case dim $\mathcal{K} < \infty$ then $PEP \in K(H)^{sa}$ and therefore, $\lambda(PEP) = \lambda(PEP \oplus 0_{\mathcal{K}}) = \lambda(PEP \oplus 0_H)$ which proves the identity between spectral spreads above.

**Remark 5.3.** The formulation of Proposition 5.2 is sharp in this general case (see Remark 5.8). Nevertheless, if $E \in K(H)^{sa}$ we get a better estimate (see Corollary 5.7).

Proposition 5.4 and Corollary 5.5 complement Proposition 5.2.

**Proposition 5.4.** Let $C, S \in B(H)^+$ be such that $C^2 + S^2 = P = P^2$. If $E_1, E_2 \in B(H)^{sa}$ then

$$s(SE_1C + CE_2S) \prec_w \frac{1}{2} \text{Spr}^+(PE_1P \oplus -PE_2P).$$

(5.8)
Proof. We consider the auxiliary Hilbert space $\mathcal{H} \oplus \mathcal{H}$ together with its orthogonal decomposition

$$\mathcal{H} \oplus \mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp \oplus \mathcal{K} \oplus \mathcal{K}^\perp,$$

where $\mathcal{K} = R(P)$. We also consider $U \in U'(\mathcal{H} \oplus \mathcal{H})$, whose block representation is given by

$$
\begin{bmatrix}
C_P & 0 & -S_P & 0 \\
0 & I & 0 & 0 \\
S_P & 0 & C_P & 0 \\
0 & 0 & 0 & I
\end{bmatrix}
$$

It is straightforward to check that the block representations of $PE_1P \oplus -PE_2P \in B(\mathcal{H} \oplus \mathcal{H})^{sa}$ and $U(PE_1P \oplus -PE_2P)U^*$ are given by

$$
\begin{bmatrix}
(E_1)_P & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -(E_2)_P & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
(CE_1C - SE_2S)_P & 0 & (CE_1S + SE_2C)_P & 0 \\
0 & 0 & 0 & 0 \\
(SE_1C + CE_2S)_P & 0 & (SE_1S - CE_2C)_P & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

Arguing as in the proof of Proposition 5.2, we see that

$$s\left(\begin{bmatrix}
(SE_1C + CE_2S)_P & 0 \\
0 & 0
\end{bmatrix}\right) = s(SE_1C + CE_2S).$$

Hence, as a consequence of Theorem 4.1 we get that

$$2s(SE_1C + CE_2S) \prec_w Spr^+(U(PE_1P \oplus -PE_2P)U^*) = Spr^+(PE_1P \oplus -PE_2P).$$

It is easy to see that, if $E \in B(\mathcal{H})^{sa}$ then

$$Spr^+(E \oplus -E) = (2 |\lambda(E)|)^{1/2} = 2s(E). \quad (5.9)$$

We use this in the following inequality, valid for a general $E \in B(\mathcal{H})^{sa}$:

**Corollary 5.5.** Let $C, S \in B(\mathcal{H})^+$ be such that $C^2 + S^2 = P = P^2$. If $E \in B(\mathcal{H})^{sa}$ then

$$s(\text{Re}(SEC)) \prec_w \frac{1}{2} s(E).$$

**Proof.** Note that $\text{Re}(SEC) = \frac{SEC + CES}{2}$. By Proposition 5.4 with $E_1 = E_2 = E$, we get

$$s(\text{Re}(SEC)) \prec_w \frac{1}{4} Spr^+(PEP \oplus -PEP) \overset{(5.9)}{=} \frac{1}{2} s(PEP) \prec_w \frac{1}{2} s(E). \quad \square$$

### 5.2 Arithmetic-geometric mean-type inequalities: the compact self-adjoint case

We begin by reformulating some facts about spectral scales, submajorization, and spectral spread for compact self-adjoint operators. Recall the definition of the spectral scale of self-adjoint operators given in Definition 2.1. Also recall from Remark 2.5 that if $A \in K(\mathcal{H})^{sa}$ is compact, then the entries of the sequence $\lambda(A) = (\lambda_i(A))_{i \in \mathbb{Z}_0}$ are also eigenvalues of $A$ (or zero), in such a way that the numbers $\lambda_i(A)$, for $i \in \mathbb{N}$, are the positive eigenvalues of $A$ counting multiplicities (or zero) arranged in the non-increasing order. Similarly, the numbers $\lambda_{-i}(A)$, for $i \in \mathbb{N}$, are the negative eigenvalues of $A$ counting multiplicities (or zero) arranged in the non-decreasing order.

Now, we show some properties of the spectral spread in the compact case:
**Proposition 5.6.** Let $A \in K(H)^a$. Then

1. If $P \in P_k(H)$ $(1 \leq k \leq \infty)$ then

$$\lambda_j(A) \geq \lambda_j(PAP) \quad \text{and} \quad \lambda_{-j}(A) \leq \lambda_{-j}(PAP) \quad \text{for} \quad j \in \mathbb{N}.$$ (5.10)

(Note that this is a reformulation of Equation (2.5) for the compact case).

2. For every $i \in \mathbb{N}$,

$$\text{Spr}^+(PAP) \leq \text{Spr}^+(A) \Rightarrow \text{Spr}^+(PAP) <_w \text{Spr}^+(A).$$ (5.11)

**Proof.** Consider $A_P \overset{\text{def}}{=} PA|_{R(P)} \in K(R(P))^a$. Then, $PAP = A_P \oplus 0_{\ker P}$. In case that $k = \dim R(P) = \infty$, we can apply Remark 2.6, so that for every $j \in \mathbb{N}$,

$$\lambda_j(PAP) = \lambda_j(A_P) \geq 0 \quad \text{and} \quad \lambda_{-j}(PAP) = \lambda_{-j}(A_P) \leq 0.$$ If $k < \infty$ then by Remark 2.3, $\lambda_j(PAP) = \max\{\lambda_j(A_P), 0\}$ and $\lambda_{-j}(PAP) = \min\{\lambda_{-j}(A_P), 0\}$ for $j \in \mathbb{N}$. Hence, Equation (5.10) follows in both cases from the interlacing inequalities of Equation (2.5).

In particular, we now see that $\text{Spr}^+(PAP) \leq \text{Spr}^+(A)$, for $j \in \mathbb{N}$. If $k < \infty$, for every $j > k$ we have that $\lambda_j(PAP) = \lambda_{-j}(PAP) = 0$ and then $\text{Spr}^+_j(PAP) = 0 \leq \text{Spr}^+_j(A)$. The submajorization relation $\text{Spr}^+(PAP) <_w \text{Spr}^+(A)$ now follows directly from these facts.

Now, we can reformulate Proposition 5.2 for the compact case:

**Corollary 5.7.** Let $C, S \in B(H)$ be such that $C^* C + S^* S = P$, where $P = P^2$. If $E \in K(H)^a$ then

$$2s(SEC^*) <_w \text{Spr}^+(E).$$ (5.12)

**Proof.** By Proposition 5.2 and item 4 of Proposition 3.2, since $PEP \in K(H)^a$,

$$2s(SEC^*) <_w \text{Spr}^+(PEP \oplus 0_H) = \text{Spr}^+(PEP).$$

Also, by Equation (5.11), $\text{Spr}^+(PEP) <_w \text{Spr}^+(E)$. This shows Equation (5.12).

**Remark 5.8.** The statement of Corollary 5.7 can fail in the non-compact case, since $\text{Spr}^+(PEP) <_w \text{Spr}^+(E)$ does not hold in general (take $P \neq I$ and $E = I$). For example, if $E = I$ we have that $\text{Spr}^+(E) = 0$. But taking $C = S = \frac{1}{\sqrt{2}}I$ we get $s(SEC^*) = \frac{1}{2}$. So that Equation (5.12) fails in this case. Note that even in this extreme case, Equation (5.7) in Proposition 5.2 is still true, because $\text{Spr}^+(I \oplus 0_H) = 1$.

On the other hand, with the notation of Corollary 5.7, if $E \in K(H)^+$ then $\text{Spr}^+(E) = \frac{1}{2} s(E)$. Hence, Corollary 5.7 implies that $N(SEC) < \frac{1}{2} N(E)$ and we recover Equation (5.6).

In case $E \in K(H)^a \setminus K(H)^+$ then it turns out that $s(E) < \text{Spr}^+(E)$, with strict majorization, that is, if $N$ is a strictly convex unitarily invariant norm then

$$N(E) < g_N(\text{Spr}^+(E)).$$

For example, if we consider $N(X) = \|X\|_2 = \left(\text{tr}(X^*X)\right)^{1/2}$, then we have that

$$\|E\|_2^2 = \sum_{j \in \mathbb{Z}_0} \lambda_j(E)^2 < \sum_{j \in \mathbb{N}} (\lambda_j(E) - \lambda_{-j}(E))^2 = g_{\|\cdot\|_2}(\text{Spr}^+(E))^2,$$

for every Hilbert–Schmidt self-adjoint operator $E \notin K(H)^+$. 

**Corollary 5.9.** Let $C, S \in B(H)^+$ be such that $C^2 + S^2 = P = P^2$. If $E_1, E_2 \in K(H)^a$, then

\[ s(SE_1C + CE_2S) \ll_w \frac{1}{2} \text{Spr}^+(E_1 \oplus E_2). \tag{5.13} \]

**Proof.** This is a straightforward consequence of Corollary 5.4, the fact that

\[ PE_1P \oplus PE_2P = (P \oplus P)(E_1 \oplus E_2)(P \oplus P) \in K(H \oplus H)^a \]

and item 2 in Proposition 5.6.

Now, we can state our main result on AGM-type inequalities for unitarily invariant norms.

**Theorem 5.10.** Let $A, B \in B(H)$ and let $E \in K(H)^a$. Then

\[ s(AE^*B) \ll_w \frac{1}{2} \text{Spr}^+ \left( (A^*A + B^*B)^{1/2} E (A^*A + B^*B)^{1/2} \right). \tag{5.14} \]

**Proof.** By Douglas’ theorem (cited as Theorem A.4 in the Appendix), the operator inequality $A^*A \leq A^*A + B^*B$ shows that the (linear) operator equations

\[ A = S (A^*A + B^*B)^{1/2} \quad \text{and} \quad B = C (A^*A + B^*B)^{1/2} \tag{5.15} \]

admit unique solutions $S, C \in B(H)$ also verifying that

\[ \ker (A^*A + B^*B)^{1/2} = R((A^*A + B^*B)^{1/2})^\perp \supseteq R(S^*)^\perp \cap R(C^*)^\perp = \ker S \cap \ker C. \]

Hence, if $z \in R((A^*A + B^*B)^{1/2})$ and $x \in H$ is such that $z = (A^*A + B^*B)^{1/2}x$ then

\[ \langle (S^*S + C^*C)z, z \rangle = \|S(A^*A + B^*B)^{1/2}x\|^2 + \|C(A^*A + B^*B)^{1/2}x\|^2 \]

\[ \overset{(5.15)}{=} \|A x\|^2 + \|B x\|^2 = \langle (A^*A + B^*B)x, x \rangle = \|z\|^2. \]

On the other hand, if $z \in \ker (AA^* + BB^*)^{1/2}$ then $(S^*S + C^*C)z = 0$. Hence, if we let $P$ denote the orthogonal projection onto the closure of $R((A^*A + B^*B)^{1/2})$ then the previous facts show that $\langle (S^*S + C^*C)z, z \rangle = \langle Pz, z \rangle$, for $z \in H$; thus, $S^*S + C^*C = P$. Moreover,

\[ AEB^* = S (A^*A + B^*B)^{1/2} E (A^*A + B^*B)^{1/2} C^*. \]

We now apply Corollary 5.7 and we get Equation (5.14).

**Remark 5.11.** Let $A, B$, and $E$ be as in Theorem 5.10, but assume that $E \geq 0$. Since $s \left( (A^*A + B^*B)^{1/2} E (A^*A + B^*B)^{1/2} \right) = s \left( E^{1/2} (A^*A + B^*B) E^{1/2} \right)$ and $\text{Spr}^+(C) = s(C)$ for $C \in K(H)^+$, we can reformulate Equation (5.14) in the positive compact case as

\[ 2s(AE^*B) \ll_w s \left( E^{1/2} (A^*A + B^*B) E^{1/2} \right). \tag{5.16} \]

Then, Equation (5.1) and its consequence Equation (5.5), being entry-wise inequalities, are stronger than Equation (5.16) in the positive case. As in the previous inequalities of this paper, Equation (5.14) is a substitute of Equation (5.5) in the self-adjoint non-positive case.

Nevertheless, as it happens with Equation (5.4), for the general self-adjoint case the submajorization relation (5.14) cannot be improved to an entry-wise inequality as Equation (5.5). As in previous examples, we shall use that a matrix
A ∈ \mathcal{M}_n(\mathbb{C}) can be embedded as a finite rank operator, which allow us to build counterexamples using matrices. Consider
\[
A = \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
-1 & 0 & 1/2 \\
0 & 1 & 0 \\
1/2 & 0 & 1
\end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 0 \\
2 & 0 & 1
\end{bmatrix}.
\]
In this case, \( F = A^*A + B^*B = \begin{bmatrix}
\frac{13}{4} & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & \frac{13}{4}
\end{bmatrix} \) is such that
\[
\lambda(F^{1/2}EF^{1/2}) = \left(\ldots, 0, \frac{-13}{4}, \left(\frac{39}{4}, 2, 0, \ldots\right)\right) \quad \text{and} \quad \Spr^+(F^{1/2}EF^{1/2}) = (13, 2, 0, \ldots).
\]
Moreover \( s(AE^B) \approx (4, 74, 1, 58, 1, 0 \ldots) \), which shows that
\[
3, 16 \approx 2 s_2(AE^B) > 2 \Spr^+_2(F^{1/2}EF^{1/2}) = 2.
\]

**Remark 5.12.** Using Proposition 5.2, it is not difficult to prove that in the general case we can get a weaker version of Equation (5.14): let \( A, B \in B(H) \) and let \( E \in B(H)^{sa} \). Then
\[
\begin{multline*}
s(AE^B) \approx (3, 16) > 2 \Spr^+_2(F^{1/2}EF^{1/2}) = 2.
\end{multline*}
\]

### 5.3 The matrix case

Note that, by Equation (2.2) and (3.1), the definition of spectral spread of self-adjoint operator given in Definition 3.1 essentially coincides with the matrix spread defined in [25] and considered in [27]; the main difference is that the spread of self-adjoint matrices is a finite vector. Nevertheless, if we embed a self-adjoint matrix \( A \) as the finite rank operator \( A \oplus 0 \) by adding a zero block (as we have done in the examples) we notice that the spread of \( A \) differs from the spectral spread of the compact operator \( A \oplus 0 \). For these reasons, we consider all Hilbert spaces in this paper to have infinite dimension, in order to maintain consistency.

On the other hand, several statements of our present work hold for the finite-dimensional case, by making slight adaptations of the proofs given here. Indeed, Theorem 4.5 is still valid for matrices and it is stronger than every statement about commutator inequalities given in [27]. Also Proposition 5.2, Corollary 5.5 and Equation (5.17) in Remark 5.12 hold in the matrix case, where no results about AGM-type inequalities using spectral spread were known to the best of our knowledge.

Nevertheless, there are some results that do not hold for the finite-dimensional case, which are those statements restricted to compact operators that use the equality \( \Spr^+(A \oplus 0_H) = \Spr^+(A) \); note that this last identity does not hold for self-adjoint matrices \( A \) (see, e.g., Corollary 5.7 and Theorem 5.10). Observe that the counterexamples given in Remark 5.8 also work if \( \dim H < \infty \).

### 5.4 On the equivalence of inequalities for the spectral spread

In this last section, we show that several of the main results in this work are equivalent. It is worth pointing out that each reformulation has a quite different appeal. Indeed, note that the statements involve the key result on the spectral spread (Theorem 4.1), commutator inequalities and AGM-type inequalities, all with respect to submajorization. In the list of equivalent inequalities below, we include a new inequality (item 4) which is a Zhan-type inequality for the singular values of the difference of self-adjoint operators (see [31] and also [32, 33]).

**Theorem 5.13.** The following inequalities are equivalent:

1. \( 2s(P E (I - P)) \approx_{w} \Spr^+(E) \), for every \( E, P \in B(H)^{sa} \), with \( P = P^2 \).
2. \( s(\text{EF} - \text{FE}) \prec_w \frac{1}{2} \text{Spr}^+(\text{E} \oplus \text{E}) - \text{Spr}^+(\text{F} \oplus \text{F}), \) for every \( \text{E}, \text{F} \in B(\mathcal{H})_{\text{sa}}. \)

3. \( s(\text{EX} - \text{XF}) \prec_w \text{Spr}^+(\text{E} \oplus \text{F}) \cdot s(\text{X}), \) for every \( \text{E}, \text{F} \in B(\mathcal{H})_{\text{sa}} \) and \( \text{X} \in B(\mathcal{H}). \)

4. \( s(\text{E} - \text{F}) \prec_w \text{Spr}^+(\text{E} \oplus \text{F}), \) for every \( \text{E}, \text{F} \in B(\mathcal{H})_{\text{sa}}. \)

5. Given \( \text{C}, \text{S} \in B(\mathcal{H}) \) such that \( \text{C}^* \text{C} + \text{S} \cdot \text{S} = \mathcal{I}, \) and \( \text{E} \in B(\mathcal{H})_{\text{sa}}, \) 
   \[ 2s(\text{SEC}^*) \prec_w \text{Spr}^+(\text{E} \oplus \mathcal{I}). \]

**Proof.** By inspection of the proof of Proposition 5.2 we see that 1 \( \rightarrow 5. \) On the other hand, as in Remark 2.6, given \( \text{E} \in B(\mathcal{H})_{\text{sa}} \), there exists \( \lambda \in \mathbb{R} \) such that \( \lambda - i(\text{E}) \leq \lambda \leq \lambda + i(\text{E}) \) for every \( i \in \mathbb{N} \). Denote by \( \text{E}_\lambda = \text{E} - \lambda \mathcal{I}. \) Then, \( \text{Spr}^+(\text{E}) = \text{Spr}^+(\text{E}_\lambda) \). Hence

\[ 2s(\text{PE}(\mathcal{I} - \text{P})) = 2s(\text{PE}(\mathcal{I} - \text{P})) \prec_w \text{Spr}^+(\text{E}_\lambda) \]

That shows that 5 \( \rightarrow 1. \) By inspection of the proofs of Theorem 4.5 and its Corollary 4.6, we see that 1 \( \rightarrow 2 \rightarrow 3. \)

On the other hand, if we let \( \text{X} = \mathcal{I} \) in item 3 we get item 4. Thus, 3 \( \rightarrow 4. \)

To prove 4 \( \rightarrow 1 \) we can assume that \( \mathcal{H} = \mathcal{K} \oplus \mathcal{K} \) and that \( \text{P} = P_{\mathcal{K} \oplus \{0\}}. \) Then

\[ E = \begin{bmatrix} E_1 & B \end{bmatrix}_{\mathcal{K}} \text{ with } s(\text{PE}(\mathcal{I} - \text{P})) = s(B) \in c_0(\mathbb{N})^+. \]

Let \( \text{B} = U |\text{B}| \) be the polar decomposition of \( \text{B}. \) Then \( \text{U} \in B(\mathcal{K}) \) is a partial isometry. If we construct the partial isometry \( \text{W} \in B(\mathcal{H}) = B(\mathcal{K} \oplus \mathcal{K}) \) given by

\[ \text{W} = \begin{bmatrix} U & 0 \\ 0 & I_{\mathcal{K}} \end{bmatrix} \text{ then } \text{W}^* \text{EW} = \begin{bmatrix} U^*E_1U & U^*B \\ B^*U & E_2 \end{bmatrix} = \begin{bmatrix} U^*E_1U & |\text{B}| \\ |\text{B}| & E_2 \end{bmatrix}. \]

Then, by item 4 in Proposition 3.3 we get that \( \text{Spr}^+(\text{W}^* \text{EW}) \prec_w \text{Spr}^+(\text{E}) \) and \( s(B) = s(|\text{B}|). \) Hence, in order to show item 1 we can assume that \( \text{B} \in K(\mathcal{K})_{\text{sa}}. \) In this case, taking the self-adjoint unitary operator \( \text{R} = \begin{bmatrix} 0 & \mathcal{I} \\ \mathcal{I} & 0 \end{bmatrix} \in B(\mathcal{K} \oplus \mathcal{K}), \) we have that

\[ \text{RER} = \begin{bmatrix} E_2 & B \\ B & E_1 \end{bmatrix} \Rightarrow \frac{\text{E} + \text{RER}}{2} = \begin{bmatrix} \frac{E_1 + E_2}{2} & B \\ B & E_1 + E_2 \end{bmatrix}. \]

By item 3 in Theorem 2.8 (Weyl inequality) and item 4 in Proposition 3.3

\[ \lambda \left( \frac{E + \text{RER}}{2} \right) < \lambda(E) + \lambda(\text{RER}) = \lambda(E) \Rightarrow \text{Spr}^+ \left( E + \frac{\text{RER}}{2} \right) \sim_w \text{Spr}^+(E). \]

Therefore, in order to show item 1 we can assume that \( \text{B} \in K(\mathcal{K})_{\text{sa}} \) and \( E_1 = E_2. \)

Take now \( Z = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathcal{I} & \mathcal{I} \\ -\mathcal{I} & \mathcal{I} \end{bmatrix} \in B(\mathcal{K} \oplus \mathcal{K}) \) that is a unitary operator. Since now

\[ E = \begin{bmatrix} E_1 & B \\ B & E_1 \end{bmatrix} \Rightarrow Z^* \text{EZ} = \begin{bmatrix} E_1 - B & 0 \\ 0 & E_1 + B \end{bmatrix}, \text{Spr}^+(Z^* \text{EZ}) = \text{Spr}^+(E). \quad (5.18) \]

Hence, using item 4 we get that

\[ 2s(B) = s([E_1 - B] - [E_1 + B]) \prec_w \text{Spr}^+([E_1 - B] \oplus [E_1 + B]) \overset{\text{(5.18)}}{=} \text{Spr}^+(E). \]

This shows that 4 \( \rightarrow 1 \) and we are done. \( \square \)

In the compact case, we can add another equivalent inequality:

**Proposition 5.14.** The following inequalities are equivalent:
1. For every $E \in K(H)^{sa}$ and $P \in B(H)^{sa}$, with $P = P^2$,
\[ 2s(PE(I-P)) \lesssim_{\omega} S_{pr^+}(E). \]

2. For every $E \in K(H)^{sa}$, and $A, B \in B(H)$,
\[ s(AEB^*) \lesssim_{\omega} \frac{1}{2} S_{pr^+}((A^*A + B^*B)^{1/2} E (A^*A + B^*B)^{1/2}). \]

Proof. By inspection of the proofs of Corollary 5.7 and Theorem 5.10, we see that $1 \rightarrow 2$ On the other hand, if we take $A = P$ and $B = I - P$ in 2, we see that $A^*A + B^*B = I$ and we recover item 1. 

Orcid
Sebastian Zarate https://orcid.org/0000-0002-3219-9128

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APPENDIX A

First, we collect several well-known results about majorization, used throughout our work. For detailed proofs of these results and general references in submajorization theory, see [16]. In what follows, we let $\mathbb{M}$ denote $\mathbb{N}$ or $\mathbb{Z}_0 \overset{\text{def}}{=} \mathbb{Z} \setminus \{0\}.

**Lemma A.1.** Let $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in \ell^\infty(\mathbb{M}) \cap \mathbb{R}^\mathbb{M}$ be real sequences. Then,

1. $\mathbf{x} + \mathbf{y} \prec \mathbf{x}^{\downarrow \downarrow} + \mathbf{y}^{\downarrow \downarrow}$;
2. If $\mathbf{x} \prec \mathbf{y}$ then $|\mathbf{x}| \prec_w |\mathbf{y}|$;
3. If $\mathbf{x} \prec \mathbf{z}, \mathbf{y} \prec \mathbf{w}$ with $\mathbf{z} = \mathbf{z}^{\downarrow}$ and $\mathbf{w} = \mathbf{w}^{\downarrow}$ then, $\mathbf{x} + \mathbf{y} \prec \mathbf{z} + \mathbf{w}$.

If we assume that $\mathbb{M} = \mathbb{N}$, then

4. If $\mathbf{x} \prec_w \mathbf{y}$ and $\mathbf{z} \prec_w \mathbf{w}$ then $(\mathbf{x}, \mathbf{z}) \prec_w (\mathbf{y}, \mathbf{w})$.

If we assume further that $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \ell^\infty(\mathbb{N})$ are non-negative sequences then,

5. $\mathbf{x} \cdot \mathbf{y} \prec_w \mathbf{x}^{\downarrow} \cdot \mathbf{y}^{\downarrow}$;
6. If $\mathbf{x} \prec_w \mathbf{y}$ and $\mathbf{y} = \mathbf{y}^{\downarrow}, \mathbf{z} = \mathbf{z}^{\downarrow}$ then $\mathbf{x} \cdot \mathbf{z} \prec_w \mathbf{y} \cdot \mathbf{z}$.

The following results about finite vectors appear in [6, Chapter II].

**Lemma A.2.** Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$

1. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{\geq 0}$, then $\mathbf{x}^{\downarrow} \cdot \mathbf{y}^{\uparrow} \prec \mathbf{x} \cdot \mathbf{y} \prec \mathbf{x}^{\downarrow} \cdot \mathbf{y}^{\downarrow}$.
2. If $\mathbf{x}, \mathbf{y} \in (\mathbb{R}^n_{\geq 0})^{\downarrow}$ and $\mathbf{z} \in (\mathbb{R}^n_{\geq 0})^{\downarrow}$, then $\mathbf{x} \prec_w \mathbf{y} \Rightarrow \sum_{i \in \mathbb{I}_n} x_i z_i \leq \sum_{i \in \mathbb{I}_n} y_i z_i$.

Next, we consider the following useful fact.

**Proposition A.3.** Let $E \in B(\mathcal{H})^{sa}$ and set $\hat{E} = \begin{bmatrix} 0 & E \\ E^* & 0 \end{bmatrix} \in B(\mathcal{H} \oplus \mathcal{H})^{sa}$. Then,

$$\lambda(\hat{E}) = (s(E), -s(E^*))^{\downarrow 1} = (s(E), -s(E))^{\downarrow 1}.$$

Finally, we state a well-known result of Douglas [13] which contains criteria for the factorization of operators that we will need in the following.

**Theorem A.4.** Let $A, B \in B(\mathcal{H})$. Then, the following conditions are equivalent:

1. $R(A) \subseteq R(B)$.
2. There exists $\lambda \in \mathbb{R}_+$ such that $AA^* \leq \lambda BB^*$.
3. There exists $C \in B(\mathcal{H})$ such that $A = BC$.

In this case, there exists a unique

$$C \in B(\mathcal{H}) \quad \text{such that} \quad A = BC \quad \text{and} \quad R(C) \subseteq R(B^*) = \ker B^\perp. \quad \text{(A.1)}$$