Symmetries of cross-ratios and the equation for Möbius structures

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Abstract

We consider orthogonal representations \( \eta_n : S_n \curvearrowright \mathbb{R}^N \) of the symmetry groups \( S_n, \ n \geq 4 \), with \( N = n!/8 \) motivated by symmetries of cross-ratios. For \( n = 5 \) we find the decomposition of \( \eta_5 \) into irreducible components and show that one of the components gives the solution to the equations, which describe Möbius structures in the class of sub-Möbius structures. In this sense, the condition defining Möbius structures is hidden already in symmetries of cross-ratios.

1 Introduction

This note is an extension of [Bu16]. We consider orthogonal representations \( \eta_n : S_n \curvearrowright \mathbb{R}^N \) of the symmetry groups \( S_n, \ n \geq 4 \), with \( N = n!/8 \) motivated by symmetries of cross-ratios. For \( n = 5 \) we find the decomposition of \( \eta_5 \) into irreducible components and show that one of the components gives the solution to the equations obtained in [Bu16], which describe Möbius structures in the class of sub-Möbius structures. In this sense, the condition defining Möbius structures is hidden already in symmetries of cross-ratios.

1.1 Möbius and sub-Möbius structures

Given a set \( X \), we consider ordered 4-tuples \( Q \in X^4 \). Such a \( Q \) is said to be admissible if it contains at most two equal items. Let \( \mathcal{P}_4 = \mathcal{P}_4(X) \) be the set of admissible 4-tuples. A 4-tuple \( Q \in X^4 \) is said to be nondegenerate or regular if all its items are pairwise different. We denote the set nondegenerate 4-tuples as \( \text{reg} \mathcal{P}_4 = \text{reg} \mathcal{P}_4(X) \).

A Möbius structure \( M \) on a set \( X \) is a class of equivalent semi-metrics on \( X \), where two semi-metrics in \( M \) are equivalent if and only if for any nondegenerate 4-tuple \( Q \in X^4 \) cross-ratios of the semi-metrics coincide.

It is convenient to use an alternative description of a Möbius structure \( M \) by three cross-ratios

\[ Q \mapsto \text{cr}_1(Q) = \frac{|x_1x_3||x_2x_4|}{|x_1x_4||x_2x_3|}, \quad \text{cr}_2(Q) = \frac{|x_1x_4||x_2x_3|}{|x_1x_2||x_3x_4|}, \quad \text{cr}_3(Q) = \frac{|x_1x_2||x_3x_4|}{|x_2x_4||x_1x_3|} \]

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for \( Q = (x_1, x_2, x_3, x_4) \in \mathcal{P}_4 \), whose product equals 1, where \(|x_ix_j| = d(x_i, x_j)\) for any semi-metric \( d \in M \). There is a reason to replace cross-ratios \( \text{cr}_k \), \( k = 1, 2, 3 \), by their logarithms. In that way we define a Möbius structure \( M \) on a set \( X \) as the map \( M : \mathcal{P}_4 \rightarrow \mathcal{L}_4 \),

\[
M(Q) = (\ln \text{cr}_1(Q), \ln \text{cr}_2(Q), \ln \text{cr}_3(Q)),
\]

where \( \mathcal{L}_4 \subset \mathbb{R}^3 \) is the 2-plane given by the equation \( a + b + c = 0 \), and \( \mathcal{T}_4 \) is an extension of \( L_4 \) by 3 infinitely remote points \( A = (0, \infty, -\infty), B = (-\infty, 0, \infty), C = (\infty, -\infty, 0) \). The infinitely remote points are added to take into account degenerate 4-tuples \( Q \in \mathcal{P}_4 \).

An important feature of our approach is presence of the symmetry groups \( S_n \) of \( n \) elements for \( n \geq 3 \). The group \( S_4 \) acts on \( \mathcal{P}_4 \) by entries permutations of any \( Q \in \mathcal{P}_4 \). The group \( S_3 \) acts on \( \mathcal{T}_4 \) by signed permutations of coordinates, where a permutation \( \sigma : \mathcal{T}_4 \rightarrow \mathcal{T}_4 \) has the sign “−1" if and only if \( \sigma \) is odd. The map \( M \) is equivariant with respect to the signed cross-ratio homomorphism,

\[
M(\pi(Q)) = \text{sign}(\pi)\varphi(\pi)M(Q)
\]

for every \( Q \in \mathcal{P}_4, \pi \in S_4 \), where \( \varphi : S_4 \rightarrow S_3 \) is the cross-ratio homomorphism, see sect. 2.4.

A sub-Möbius structure on \( X \) is a map \( M : \mathcal{P}_4 \rightarrow \mathcal{L}_4 \) with the basic property (1) and a natural condition for values on degenerate admissible 4-tuples, see sect. 2.2. This notion is introduced in [Bu16], where its importance is demonstrated by showing that on the boundary at infinity \( X \) of every Gromov hyperbolic space \( Y \) there is a canonical sub-Möbius structure \( M \) invariant under isometries of \( Y \) such that the \( M \)-topology on \( X \) coincides with the standard one.

The main result of [Bu16] is a criterion under which a sub-Möbius structure on \( X \) is a Möbius structure, that is, generated by semi-metrics. This criterion is expressed as linear equations (A), (B), see Theorem 2.2 for the codifferential \( \delta M : \mathcal{P}_5 \rightarrow \mathcal{T}_5 \) defined on admissible 5-tuples, see sect. 5.1. In other words, to check whether of a sub-Möbius structure is a Möbius one, one needs to consider its natural extension to admissible 5-tuples. The linearity of the equations (A), (B) is the main reason why we take logarithms in the definition of Möbius structures.

### 1.2 \( S_5 \)-symmetry and its irreducible components

The group \( S_5 \) acts by entries permutations of the space \( \mathcal{P}_5 \subset X^5 \) of admissible 5-tuples. We define a natural representation \( \eta_5 : S_5 \curvearrowright V^5 \) of \( S_5 \) on a 15-dimensional space \( V^5 = \mathbb{R}^5 \otimes V^4 \), where \( V^4 = \mathbb{R}^3 \supset \mathcal{L}_4 \), see sect. 3.1 in a way that \( \eta_5 \) describe the symmetry of any sub-Möbius structure \( M \) regarded on admissible 5-tuples, that is, the codifferential \( \delta M \) is \( \eta_5 \)-equivariant,

\[
\delta M(\pi P) = \eta_5(\pi)\delta M(P)
\]
for every $P \in \mathcal{P}_5$ and every $\pi \in S_5$. In sect. 3.2 we describe the decomposition of $\eta_5$ into irreducible components,

$$\psi = \chi^{32} + \chi^{221} + \chi^{213} + \chi^{15},$$

where $\psi : S_5 \to \mathbb{C}$ is the character of $\eta_5$ and $\chi^{(i)}$ are prime characters of $S_5$.

On the other hand, in sect. 4 we find an $\eta_5$-invariant version of the equations (A), (B), that is, $\eta_5$-invariant subspace $\hat{R} \subset L_5$ of solutions to (A), (B). It turns out that $\hat{R}$ is exactly $\chi^{32}$-subspace $R$ of the $\eta_5$-decomposition into irreducible components, $\hat{R} = R$, see Proposition 5.3. Since $\dim R = 5$, we find in particular that $\dim \hat{R} = 5$. This leads to the main result of this note

**Theorem 1.1.** A sub-Möbius structure $M$ on a set $X$ is Möbius if and only if $\delta M(\text{reg } \mathcal{P}_5) \subset R$, where $R \subset L_5$ is the irreducible component of the canonical representation $\eta_5$ that corresponds to the prime character $\chi^{32}$.

## 2 Sub-Möbius structures

### 2.1 Admissible $n$-tuples and the cross-ratio homomorphism

Here we briefly recall what is a sub-Möbius structure $M$ on a set $X$ and what are conditions under which $M$ is a Möbius structure.

Given an ordered tuple $P = (x_1, \ldots, x_k) \in X^k$ we use notation

$$P_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)$$

for $i = 1, \ldots, k$. We define the set $\mathcal{P}_n$ of the admissible $n$-tuples in $X$, $n \geq 5$, recurrently by asking for every $P \in \mathcal{P}_n$ the $(n-1)$-tuples $P_i$, $i = 1, \ldots, n$, to be admissible. The set sing $\mathcal{P}_n$ of $n$-tuples $P \in \mathcal{P}_n$ having two equal entries is called the singular subset of $\mathcal{P}_n$. Its complement reg $\mathcal{P}_n = \mathcal{P}_n \setminus \text{sing } \mathcal{P}_n$ is called the regular subset of $\mathcal{P}_n$. The set reg $\mathcal{P}_n$ consists of nondegenerate $n$-tuples.

The symmetry group $S_n$ acts on $\mathcal{P}_n$ by permutations of the entries of every $P \in \mathcal{P}_n$. We represent a permutation $\pi \in S_n$ as $\pi = i_1 \ldots i_n$ where $i_k = \pi^{-1}(k)$, $k = 1, \ldots, n$.

The cross-ratio homomorphism $\varphi : S_4 \to S_3$ can be described as follows: a permutation of a tetrahedron ordered vertices $(1, 2, 3, 4)$ gives rise to a permutation of opposite pairs of edges $((12)(34), (13)(24), (14)(23))$ taken in this order. Thus the kernel $K$ of $\varphi$ consists of four elements $1234, 2143, 4321, 3412$, and is isomorphic to the dihedral group $D_4$ of a square automorphisms. The group $D_4$ is also called the Klein group.

We denote by sign : $S_4 \to \{\pm 1\}$ the homomorphism that associates to every odd permutation the sign “−1”.

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2.2 Sub-Möbius structures

We denote by $L_4$ the subspace in $\mathbb{R}^3$ given by $a + b + c = 0$ (subindex 4 is related to 4-tuples rather than to the dimension of $\mathbb{R}^3$). We extend $L_4$ by adding to it the points $A = (0, \infty, -\infty), B = (-\infty, 0, \infty), C = (\infty, -\infty, 0)$, $\mathcal{L}_4 = L_4 \cup \{A, B, C\}$.

A sub-Möbius structure on a set $X$ is given by a map $M : \mathcal{P}_4 \to \mathcal{L}_4$, which satisfies the following conditions

(a) $M$ is equivariant with respect to the signed cross-ratio homomorphism $\varphi$, i.e.

$$M(\pi P) = \operatorname{sign}(\pi) \varphi(\pi) M(P)$$

for every $P \in \mathcal{P}_4$ and every $\pi \in S_4$.

(b) $M(P) \in L_4$ if and only if $P$ is nondegenerate, $P \in \operatorname{reg} \mathcal{P}_4$;

(c) $M(P) = (0, \infty, -\infty)$ for $P = (x_1, x_1, x_3, x_4) \in \mathcal{P}_4$.

Remark 2.1. Note that if $M, M'$ are sub-Möbius structures on $X$, then $M + M'$ and $\alpha M$, $\alpha > 0$, are also sub-Möbius structures on $X$. However, $-M$ is not a sub-Möbius structure in the sense of our definition, because the condition (c) in this case is violated. Therefore, the set $\mathcal{M}$ of sub-Möbius structures on $X$ is a cone.

A function $d : X^2 \to \mathbb{R} = \mathbb{R} \cup \{\infty\}$ is called a semi-metric, if it is symmetric, $d(x, y) = d(y, x)$ for each $x, y \in X$, positive outside of the diagonal, vanishes on the diagonal and there is at most one infinitely remote point $\omega \in X$ for $d$, i.e. such that $d(x, \omega) = \infty$ for some $x \in X \setminus \{\omega\}$. In this case, we require that $d(x, \omega) = \infty$ for all $x \in X \setminus \{\omega\}$. A metric is a semi-metric that satisfies the triangle inequality.

With every semi-metric $d$ on $X$ we associate the Möbius structure $M_d : \mathcal{P}_4 \to \mathcal{L}_4$ given by $M_d(x_1, x_2, x_3, x_4) = (a, b, c)$, where

$$a = \operatorname{cd}(x_1, x_2, x_3, x_4) = (x_1|x_4) + (x_2|x_3) - (x_1|x_3) - (x_2|x_4)$$

$$b = \operatorname{cd}(x_1, x_3, x_4, x_2) = (x_1|x_2) + (x_3|x_4) - (x_1|x_4) - (x_2|x_3)$$

$$c = \operatorname{cd}(x_2, x_3, x_1, x_4) = (x_2|x_4) + (x_1|x_3) - (x_1|x_2) - (x_3|x_4),$$

$(x_i|x_j) = -\ln d(x_i, x_j)$. One easily checks that for every semi-metric $d$ on $X$ the Möbius structure $M_d$ associated with $d$ is a sub-Möbius structure.

A triple $A = (\alpha, \beta, \omega) \in X^3$ of pairwise distinct points is called a scale triple. We use notation $M(P_i) = (a(P_i), b(P_i), c(P_i)), i = 1, \ldots, 5, P \in \mathcal{P}_5$, for a sub-Möbius structure $M$ on $X$. The following theorem has been proved in [Bu16] Theorem 3.4.

Theorem 2.2. A sub-Möbius structure $M$ on $X$ is a Möbius structure if and only if for every scale triple $A = (\alpha, \beta, \omega) \in X^3$ and every admissible 5-tuple $P = (x, y, A) \subset X$ the following conditions (A), (B) are satisfied

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(A) $b(P_1) + b(P_4) = b(P_3) - a(P_1)$ for all $y \in X$, $y \neq \alpha, \beta$;

(B) $b(P_3) = -a(P_4) + b(P_1)$, where $y \neq \alpha, \omega$.

Remark 2.3. An alternative approach to Theorem 2.2 is discussed in [IM17].

3 $V^n$-spaces

The tensor product $\mathbb{R}^m \otimes \mathbb{R}^n$ taken over $\mathbb{R}$ is the space of $m \times n$ real matrices. We use notation $V^4 := \mathbb{R}^3$ and inductively define $V^{n+1} = \mathbb{R}^{n+1} \otimes V^n$ for $n \geq 4$. Then $V^n$ is a real vector space of dimension $\dim V^n = \frac{n!}{2}$ whose elements could be thought as matrices with $n$ rows each of which being an element of $V^{n-1}$, in particular, we have a canonical basis $e^n = e_i \otimes e^{n-1}$ of $V^n$, where $e_i$, $i = 1, \ldots, n$, is the canonical basis of $\mathbb{R}^n$. We also denote by $W^n$, $n \geq 5$, the vector space with $V^n = W^n \otimes V^4$.

Recall that $L_4 \subset \mathbb{R}^3$ is the 2-plane given by the equation $a + b + c = 0$. We put $L_n = W^n \otimes L_4 \subset V^n$, $\overline{L}_n = W^n \otimes \mathcal{L}_4$, $n \geq 5$. Furthermore, $\dim L_n = 2 \dim W^n = \frac{2}{3} \dim V^n = \frac{n!}{12}$.

3.1 The canonical action of $S_n$ on $V^n$

Given $\pi \in S_k$ and $i \in \{1, \ldots, k\}$ we let $\pi_i \in S_{k-1}$ be the permutation of $\{1, \ldots, \hat{i}, \ldots, k\}$ induced by $\pi$, that is, the restriction of the bijection $\pi : \{1, \ldots, k\} \to \{1, \ldots, k\}$ to $\{1, \ldots, \hat{i}, \ldots, k\}$, $\pi_i = \pi|_{\{1, \ldots, \hat{i}, \ldots, k\}} \to \{1, \ldots, \pi(\hat{i}), \ldots, k\}$, where $\hat{i}$ means that the $i$th box is empty. For example, if $\pi$ is a cyclic permutation, $\pi(j) = j + 1 \mod k$, then $\pi_i$ is cyclic for every $1 \leq i \leq k - 1$ while $\pi_k = \text{id}$.

Lemma 3.1. For any $\pi', \pi \in S_k$ we have

$$(\pi' \pi)_i = \pi'_i \circ \pi_i.$$ \hfill \qed

Proof. The right hand side of the equality is well defined because $\pi_i$ is a bijection from $\{1, \ldots, \hat{i}, \ldots, k\}$ to $\{1, \ldots, \pi(\hat{i}), \ldots, k\}$. Furthermore, the both sides are well defined bijections from $\{1, \ldots, \hat{i}, \ldots, k\}$ to $\{1, \ldots, \pi(\hat{i}), \ldots, k\}$. Thus they coincide because $\pi'(\pi(j)) = (\pi' \pi)(j)$ for every $j \in \{1, \ldots, k\}$. \hfill \qed

Let $e^n$ be the canonical basis of $V^n$, $n \geq 4$. Then $e^{n+1} = e_i \otimes e^n$, where $e_i$, $i = 1, \ldots, n + 1$, is the canonical basis of $\mathbb{R}^{n+1}$.

Let $\eta_4 : S_4 \to \text{Aut}(V^4)$ be the signed cross-ratio homomorphism,

$$\eta_4(\pi)(e_j) = \text{sign}(\pi)e_{\varphi(\pi)(j)}, \quad j = 1, 2, 3,$$

where $\pi \in S_4$ and $\varphi : S_4 \to S_3$ is the cross-ratio homomorphism. Now we define an action $\eta_{n+1}$ of $S_{n+1}$ on $V^{n+1}$ inductively by

$$\eta_{n+1}(\pi)(e^{n+1}) = \{e_{\pi(i)} \otimes \eta_n(\pi_i)e^n : i = 1, \ldots, n + 1\}$$

for $\pi \in S_{n+1}$, $n \geq 4$. 

Example 3.2. Let \( \pi = 21345 \in S_5 \). Then \( \pi_1 = \pi_2 = \text{id} \in S_4 \), \( \pi_i = 2134 \in S_4 \) for \( i = 3,4,5 \). For the cross-ratio homomorphism \( \varphi : S_4 \to S_3 \) we have \( \varphi(\pi_1) = \varphi(\pi_2) = \text{id} \in S_3 \), \( \varphi(\pi_i) = 132 \in S_3 \) for \( i = 3,4,5 \). Thus for \( v \in V_5 \),

\[
v = \begin{pmatrix}
a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \\ a_5 & b_5 & c_5
\end{pmatrix},
\]

we have

\[
\eta_5(\pi)(v) = \begin{pmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ -a_3 & -c_3 & -b_3 \\ -a_4 & -c_4 & -b_4 \\ -a_5 & -c_5 & -b_5 \end{pmatrix}.
\]

Lemma 3.3. For every \( n \geq 4 \) the map \( \eta_n : S_n \to \text{Aut}(V^n) \) is a homomorphism, which is a monomorphism for \( n \geq 5 \).

Proof. It suffices to check that \( \eta_n(\pi'\pi) = \eta_n(\pi') \circ \eta_n(\pi) \) for any \( \pi, \pi' \in S_n \). This is true for \( n = 4 \) because \( \eta_4 \) is a homomorphism by definition. By induction we can assume that

\[
\eta_n(\pi') e^n = \eta_n(\pi') \circ \eta_n(\pi) e^n
\]

for \( \pi, \pi' \in S_n \). Using Lemma 3.1 we obtain

\[
(\pi'\pi)_i = \pi'_{\pi(i)} \circ \pi_i
\]

for \( \pi, \pi' \in S_{n+1} \). Then

\[
\eta_{n+1}(\pi'\pi) e^{n+1} = e_{\pi'_{\pi(i)}} \otimes \eta_n(\pi'_{\pi(i)}) \eta_n(\pi_i) e^n = \eta_{n+1}(\pi') (e_{\pi(i)} \otimes \eta_n(\pi_i) e^n).
\]

Hence \( \eta(\pi'\pi) = \eta(\pi') \circ \eta(\pi) \). If \( \eta_n(\pi) = \text{id} \) for \( n \geq 5 \), then \( \pi = \text{id} \) by definition of \( \eta_n \).

The action \( \eta_n \) of \( S_n \) on \( V^n \) is said to be \textit{canonical}. Note that \( \eta_n \) preserves the space \( \mathbb{L}_n \).

3.2 Decomposition of \( \eta_5 \) into irreducible components

In this section, we use elementary facts from the representation theory. The reader may consult e.g. [1].

Let \( \rho \) be a linear representation of a group \( G \) in a vector space \( V \). The function \( \chi_\rho : G \to \mathbb{C}, \chi_\rho(s) = \text{Tr}(\rho(s)) \), where \( \text{Tr}(\rho(s)) \) is the trace of the linear map \( \rho(s) : V \to V \), is called the \textit{character} of \( \rho \). If \( \rho \) is irreducible, then the character \( \chi_\rho \) is \textit{prime}.
The character $\chi_\rho$ is constant on every class of conjugate elements of $G$. For the symmetry group $S_n$ any class of conjugate elements is uniquely determined by decomposition into cycles. We denote by $1^a2^b3^c\ldots$ the class in $S_n$ having $a$ cycles of length one, $b$ cycles of length two, $c$ cycles of length three etc. The number of classes is equal to the number of subdivisions of $n$ into integer positive summands. In its turn, the number of prime characters is equal to the number of classes, and we denote the respective prime character of $S_n$ by $\chi^\lambda$, where $\lambda$ is the respective decomposition of $n$. Here is the table of prime characters of the group $S_5$. The first line is the list of the conjugate classes, the second line lists the orders, i.e. the number of elements in a class. The last line is the character of the canonical representation $\eta_5$, which is easily computed directly. For example, to compute $\psi(21^3)$, we take the transposition $\pi = 21345 \in 21^3$ and note that there are precisely three elements $e_3 \otimes e_1$, $e_4 \otimes e_1$, $e_5 \otimes e_1$ of the canonical basis $e^5$ of $V^5$, which are preserved up to the sign by $\eta_5(\pi)$, moreover, $\eta_5(\pi)(v) = -v$ for all these elements $v = e_3 \otimes e_1$, $e_4 \otimes e_1$, $e_5 \otimes e_1$, see Example 3.2. Hence $\psi(21^3) = -3$. The list of the prime characters can be found in [H].

| cycles  | 1^5 | 21^3 | 22^1 | 31^2 | 32  | 41  | 5   |
|---------|-----|------|------|------|-----|-----|-----|
| order   | 1   | 10   | 15   | 20   | 20  | 30  | 24  |
| $\chi^5$ | 1   | 1    | 1    | 1    | 1   | 1   | 1   |
| $\chi^{41}$ | 4   | 2    | 0    | 1    | -1  | 0   | -1  |
| $\chi^{32}$ | 5   | 1    | 1    | -1   | 1   | -1  | 0   |
| $\chi^{312}$ | 6   | 0    | -2   | 0    | 0   | 0   | 1   |
| $\chi^{221}$ | 5   | -1   | 1    | -1   | -1  | 1   | 0   |
| $\chi^{213}$ | 4   | -2   | 0    | 1    | 1   | 0   | -1  |
| $\chi^{1^5}$ | 1   | -1   | 1    | 1    | -1  | -1  | 1   |
| $\psi$ | 15  | -3   | 3    | 0    | 0   | -1  | 0   |

A function $f : G \to \mathbb{C}$ is central, if it is constant on conjugate classes. The prime characters of a finite group $G$ form an orthonormal basis of the central functions on $G$ with respect to the scalar product

$$\langle \psi, \chi \rangle = \frac{1}{g} \sum_{s \in G} \psi(s)\chi(s)^*,$$

where $g = |G|$ is the order of $G$, $\chi(s)^*$ is the complex conjugate of $\chi(s)$. We have

$$\langle \psi, \chi^\lambda \rangle = \begin{cases} 1, & \lambda = \{32\}, \{2^21\}, \{21^3\}, \{1^5\} \\ 0, & \lambda = \{5\}, \{41\}, \{31^2\} \end{cases},$$

therefore the character $\psi$ of $\eta_5$ has the decomposition

$$\psi = \chi^{32} + \chi^{221} + \chi^{21^3} + \chi^{1^5}$$

(2)
into prime characters of \( S_5 \). The dimensions of prime components of the decomposition can be read off the first column of the list of prime characters, that is

\[
15 = 5 + 5 + 4 + 1.
\]

Thus the respective decomposition of \( \eta_5 \) into irreducible representations can be described as follows. The decomposition \( V^5 = L_5 \oplus L_5^\perp \) is \( \eta_5 \)-invariant. The 5-dimensional subspace \( L_5^\perp \subset V^5 \) orthogonal to \( L_5 \) is decomposed into 1-dimensional and 4-dimensional invariant irreducible subspaces, which correspond to the characters \( \chi_1^{15}, \chi_2^{213} \) respectively. The 1-dimensional \( \chi_1^{15} \)-subspace is spanned by \( w = e_1 - e_2 + e_3 - e_4 + e_5 \), where \( e_i \in V^5 \) has \( i \)-th row \((1, 1, 1)\) and zero the remaining rows. The 10-dimensional \( L_5 \) is decomposed as

\[
L_5 = R \oplus R^\perp
\]

into 5-dimensional irreducible subspaces, where \( R \) is a \( \chi_3^{22} \)-subspace, that is, the character of \( \eta_5 \mid R \) is \( \chi_3^{22} \), and \( R^\perp \) is a \( \chi_2^{21} \)-subspace.

4 Characteristic functions

4.1 An algorithm

Let \( Q = (a, b, c, d) \) be an ordered nondegenerate 4-tuple. The set of the opposite pairs in \( Q \) has a natural order \( \{(ab, cd), (ac, bd), (ad, bc)\} \). We regard an ordered nondegenerate 5-tuple \( P \) as the vertex set of a 4-simplex. Elements of every matrix \( v \in V^5 \) can be labeled by 3-faces of \( P \) as follows. To every 3-face \( P_i = P \setminus i, i \in P \), we associate the \( i \)-th row of \( v \), whose items are labeled by pairs of opposite edges of \( P_i \). The order of the items is induced by the order of \( P_i \) as above, which in its turn is induced by the order of \( P \).

Let \( \Lambda = \Lambda(P) \) be the edges set of \( P \), \( |\Lambda| = 10 \). With each \( \lambda \in \Lambda \) we associate a function \( r_\lambda : V^5 \to \mathbb{R} \) called characteristic as follows. Given \( v \in V^5 \), we consider the rows \( v_i, v_j, v_k \) of \( v \), where \( P \setminus \lambda = \{i, j, k\} \), which are labeled by the 3-faces \( P_i, P_j, P_k \). The function \( r_\lambda \) only depends on items of these three rows. To obtain \( r_\lambda(v) \) we take those items, whose label contains \( \lambda \), and take the sum of the chosen items with signs \((-1)^{p+q+1}\), where \( p \) is the column, \( q \) is the row of the item in the matrix formed by the rows \( v_i, v_j, v_k \).

As an example, we compute \( r_\lambda(v) \) for \( \lambda = 25 \), where

\[
v = \begin{pmatrix}
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2 \\
a_3 & b_3 & c_3 \\
a_4 & b_4 & c_4 \\
a_5 & b_5 & c_5
\end{pmatrix}.
\]

In this case \( i = 1, j = 3, k = 4 \) and
\begin{align*}
v_1 &= (2345, 2435, 2534) \\
v_3 &= (1245, 1425, 1524) \\
v_4 &= (1235, 1325, 1523),
\end{align*}
where we assume that \( P = (1,2,3,4,5) \) and use the cross-ratio labeling. The items containing \( \lambda = 25 \) are \( 2534, 1425, 1325 \). Hence
\[
\begin{align*}
r_{25}(v) &= -c_1 - b_3 + b_4
\end{align*}
\]
Note that Eq. (A) of Theorem 2.2 is equivalent to \( r_{25}(v) = 0 \) because \( a_1 + b_1 + c_1 = 0 \). Similarly, Eq. (B) of Theorem 2.2 is equivalent to \( r_{35}(v) = 0 \).

It this way, we obtain the following list of the characteristic functions
\[
\begin{align*}
r_{25}(v) &= -c_1 - b_3 + b_4 \\
r_{35}(v) &= b_1 - b_2 - a_4 \\
r_{15}(v) &= -c_2 + c_3 - c_4 \\
r_{45}(v) &= -a_1 + a_2 - a_3 \\
r_{12}(v) &= -a_3 + a_4 - a_5 \\
r_{13}(v) &= -a_2 - b_4 + b_5 \\
r_{14}(v) &= b_2 - b_3 - c_5 \\
r_{23}(v) &= -a_1 + c_4 - c_5 \\
r_{24}(v) &= b_1 + c_3 + b_5 \\
r_{34}(v) &= -c_1 + c_2 - a_5.
\end{align*}
\]
Usually, we consider the functions \( r_\lambda \) on the subspace \( L_5 \subset V^5 \), and extend them on \( L_5 \), see sect. 3, with values in \( \mathbb{R} \cup \{ +\infty \} \cup \{ -\infty \} \). In this case we use the agreement \( (a + \infty) - (b + \infty) = 0 \) for any \( a, b \in \mathbb{R} \).

4.2 Action of \( S_5 \) on characteristic functions

The action of \( S_n \) on functions \( f : V^n \to \mathbb{R} \) induced by \( \eta_n \) is defined by
\[
(\pi f)(x) = f(\eta_n(\pi)^{-1}x)
\]
for \( \pi \in S_n, x \in V^n \).

**Proposition 4.1.** The group \( S_5 \) preserves the set \( \mathcal{R} = \{ \pm r_\lambda : \lambda \in \Lambda \} \) and acts on it transitively.

**Proof.** The function \( r_\lambda \) is uniquely determined up to the sign by the choice of \( \lambda \). Thus \( \pi r_\lambda = \pm r_{\pi \lambda} \) for every \( \lambda \in \Lambda, \pi \in S_5 \). One easily checks that the transposition \( \pi_\lambda \in S_5 \) that permutes vertices of \( \lambda \) changes the sign of \( r_\lambda \), \( \pi_\lambda r_\lambda = -r_\lambda \). Since \( S_5 \) acts transitively of the edge set \( \Lambda \), it acts transitively on \( \mathcal{R} \). \( \square \)

**Proposition 4.2.** A sub-Möbius structure \( M \) on \( X \) is a Möbius structure if and only if for every nondegenerate 5-tuple \( P = (x,y,\alpha,\beta,\omega) \) conditions (A), (B) of Theorem 2.2 are satisfied.
Proof. We fix a scale triple \( A = (\alpha, \beta, \omega) \in X^3 \) and consider an admissible 5-tuple \( P = (x, y, A) \). By Theorem 2.2 we only need to show that if \( P \) is degenerate, then (A), (B) are satisfied automatically.

Note that \( r_{25}(M(P)) = 0 \) is equivalent to (A), and \( r_{35}(M(P)) = 0 \) is equivalent to (B). Thus using Proposition 4.1 it suffices to check that if \( x = y \), then \( r_\lambda(M(P)) = 0 \) for every \( \lambda \in \Lambda \). Note that in this case \( x \neq y \not\in A \).

We have \( M(P_1) = M(P_2) = (a, b, c) \in L_4 \), while \( M(P_3) = M(P_4) = (0, \infty, -\infty) \) by definition of a sub-Möbius structure. One directly checks that \( r_\lambda(M(P)) = 0 \) for \( \lambda = 35, 45, 12, 34 \) using that these equations do not involve \( \pm \infty \).

The equation \( r_{25}(v) = 0 \) is equivalent to \( b_1 + b_4 = -a_1 + b_3 \). In the case \( v = M(P) \) the both sides are equal to \( \infty \), thus \( r_{25}(M(P)) = 0 \) according to our agreement. Similar argument shows that \( r_\lambda(M(P)) = 0 \) for the remaining \( \lambda = 15, 13, 14, 23, 24 \), hence \( r_\lambda(M(P)) = 0 \) for all \( \lambda \in \Lambda \).

The functions \( r_\lambda, \lambda \in \Lambda \), on \( V^5 \) are the characteristic functions for Möbius structures. A point \( a \in L_5 \subset V^5 \) is said to be root of \( r_\lambda \) if \( r_\lambda(a) = 0 \). For roots in \( L_5 \) we use notation \( R_\lambda = \{ a \in L_5 : r_\lambda(a) = 0 \} \), and \( \hat{R} = \cap_\lambda R_\lambda \subset L_5 \). By Proposition 4.1, the \( \eta_5 \)-action of \( S_5 \) on \( L_5 \) permutes the sets \( R_\lambda, \lambda \in \Lambda \), and \( \hat{R} \) is \( \eta \)-invariant. We call \( \hat{R} \) the symmetry set.

5 Codifferential of a sub-Möbius structure

5.1 Definition of a codifferential

Recall that \( T_4 = L_4 \cup \{A, B, C\} \), see sect. 2.2. For \( n \geq 5 \) we put \( T_n = W^n \otimes T_4 \), see sect. 3.

Let \( M \) be a sub-Möbius structure on a set \( X \). We define its codifferential as the map \( \delta M : \mathcal{P}_5 \to \mathcal{L}_5 \) given by

\[
\delta M(P)_i = M(P_i) \in \mathcal{L}_4,
\]

where \( \delta M(P)_i \) is the \( i \)th row of \( \delta M(P) \), \( i = 1, \ldots, 5 \). This terminology is related to the fact that \( \delta M \) is computed on a 5-tuple \( P \in \mathcal{P}_5 \) via values of \( M \) on the “boundary” \( dP = P_1 \cup \cdots \cup P_5 \).

Lemma 5.1. Codifferential of any sub-Möbius structure \( M \) on a set \( X \) has the following properties

(i) \( \delta M \) is equivariant with respect to the homomorphism \( \eta_5 \), i.e.

\[
\delta M(\pi P) = \eta_5(\pi) \delta M(P)
\]

for every \( P \in \mathcal{P}_5 \) and every \( \pi \in S_5 \).
(ii) $\delta M(P) \in L_5$ if and only if $P$ is nondegenerate, $P \in \text{reg} \mathcal{P}_5$;

(iii) for every admissible 5-tuple $P = (x, x, \alpha, \beta, \omega)$ the equality

\[
\delta M_i(P) = \begin{cases} (a, b, c) \in L_4, & i = 1, 2 \\
(0, \infty, -\infty), & i = 3, 4, 5 \end{cases}
\]

holds.

**Proof.** By construction, properties of sub-Möbius structures and Lemma 3.3 the map $\delta M$ is equivariant w.r.t. the natural action of $S_5$ on $\mathcal{P}_5$ and the action $\eta_5$ on $L_5$,

\[
\delta M(\pi P) = \eta_5(\pi)\delta M(P)
\]

for every $P \in \mathcal{P}_5$. This is (i).

Property (ii) follows from the similar property for sub-Möbius structures.

For $P = (x, x, \alpha, \beta, \omega) \in \text{sing} \mathcal{P}_5$ we have $P_1 = P_2 = (x, \alpha, \beta, \omega) \in \text{reg}(\mathcal{P}_5)$, and we put $M(P_1) = M(P_2) = (a, b, c) \in L_4$. For $P_3 = (x, x, \beta, \omega)$, $P_4 = (x, x, \alpha, \omega)$, and $P_5 = (x, x, \alpha, \beta)$ we have $M(P_3) = M(P_4) = M(P_5) = (0, \infty, -\infty)$. Thus (iii) holds.

By property (ii) of Lemma 5.1, $\delta M(\text{reg} \mathcal{P}_5) \subset L_5$. Thus we can consider the restriction $\delta M|_{\text{reg} \mathcal{P}_5}$ as a map with values in $L_5$.

**Proposition 5.2.** The following conditions for a sub-Möbius structure $M$ on a set $X$ are equivalent:

(i) $M$ is a Möbius structure;

(ii) $\delta M(\text{reg} \mathcal{P}_5) \subset R_{\lambda}$ for some $\lambda \in \Lambda$;

(iii) $\delta M(\text{reg} \mathcal{P}_5) \subset \hat{R}$.

**Proof.** By Theorem 2.2 we have $\delta M(\text{reg} \mathcal{P}_5) \subset R_{25} \cap R_{35}$ for a Möbius structure $M$. It follows from Proposition 4.1 that $\delta M(\text{reg} \mathcal{P}_5) \subset R_{\lambda}$ for every $\lambda \in \Lambda$, in particular, (i) $\implies$ (ii).

(ii) $\implies$ (iii): Assume $\delta M(\text{reg} \mathcal{P}_5) \subset R_{\lambda}$ for a sub-Möbius structure $M$ and some $\lambda \in \Lambda$. Using $\eta_5$-equivariance of $\delta M$, see Lemma 5.1(i), we obtain that $\delta M(\text{reg} \mathcal{P}_5) \subset R_{\lambda}$ for every $\lambda \in \Lambda$, thus $\delta M(\text{reg} \mathcal{P}_5) \subset \hat{R}$.

(iii) $\implies$ (i): If $\delta M(\text{reg} \mathcal{P}_5) \subset \hat{R}$, then in particular $\delta M(\text{reg} \mathcal{P}_5) \subset R_{25} \cap R_{35}$. Thus $M$ is Möbius by Proposition 4.2.

5.2 Proof of the main theorem

**Proposition 5.3.** The symmetry set $\hat{R} \subset L_5$ coincides with the $\chi^{32}$-subspace $R$ of the decomposition $L_5 = R \oplus R^\perp$ of $\eta_5$ into irreducible components, $\hat{R} = R$. 11
Proof. The set \( \hat{R} \) is the intersection of hyperplanes \( R_\lambda, \lambda \in \Lambda \), thus \( \hat{R} \neq L_5 \). By Proposition 5.2, we have \( \delta M(\text{reg} P_5) \subset \hat{R} \) for any Möbius structure \( M \) on any set \( X \). Taking e.g. the Möbius structure \( M \) of the extended real line \( \hat{R} = \mathbb{R} \cup \{ \infty \} \), we see that \( \delta M(\text{reg} P_5) \neq \{0\} \). This shows that \( \hat{R} \) is a proper subspace of \( L_5 \). Since \( \hat{R} \) is \( \eta_5 \)-invariant, we conclude that \( \hat{R} \) coincides with \( R \) or \( R^\perp \). Note that the characters \( \chi^{32}, \chi^{21} \) differ on the class \( 21^3 \), i.e., on any transposition. To make a choice, we compute \( \text{Tr}(\eta_5(\pi)) \) for \( \pi = 15342 \in 21^3 \) on the orthogonal complement \( R^\perp \) to \( \hat{R} \). It is spanned by the normal vectors \( n_\lambda, \lambda \in \Lambda \), to the hyperplanes \( r_\lambda = 0 \) in \( L_5 \).

For the transposition \( \pi = 15342 \) we have

\[
\eta_5(\pi)(w) = \begin{pmatrix}
-b_1 & -a_1 & -c_1 \\
b_5 & c_5 & a_5 \\
-c_3 & -b_3 & -a_3 \\
-c_4 & -b_4 & -a_4 \\
  c_2 & a_2 & b_2
\end{pmatrix}
\text{ for any } w = \begin{pmatrix}
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2 \\
a_3 & b_3 & c_3 \\
a_4 & b_4 & c_4 \\
a_5 & b_5 & c_5
\end{pmatrix}.
\]

The hyperplane \( r_{25} = 0 \) in \( V^5 \) is given by the equation \(-c_1 - b_3 + b_4 = 0\) with the normal vector

\[
v_{25} = \begin{pmatrix}
0 & 0 & -1 \\
0 & 0 & 0 \\
0 & -1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix} \in V^5.
\]

We take as \( n_{25} \) the projection of \( 3v_{25} \) to \( L_5 \),

\[
n_{25} = \begin{pmatrix}
1 & 1 & -2 \\
 0 & 0 & 0 \\
 1 & -2 & 1 \\
-1 & 2 & -1 \\
 0 & 0 & 0
\end{pmatrix}.
\]

Similarly, we obtain all ten vectors \( n_\lambda \in \hat{R}^\perp, \lambda \in \Lambda \).

Then \( \eta_5(\pi)(n_\lambda) = -n_\lambda \) for \( \lambda = 25, 13, 14, 34 \), and there is no any other \( \lambda \in \Lambda \) such that \( \eta_5(\pi)(n_\lambda) = \pm n_\lambda \). The four vectors \( n_\lambda, \lambda = 25, 13, 14, 34 \), are linearly depended, \( n_{25} + n_{13} = n_{14} + n_{34} \), and they span a 3-dimensional subspace in \( \hat{R}^\perp \) on which \( \eta_5(\pi) \) acts as \(-\text{id}\). Therefore, \( \text{Tr}(\eta_5(\pi))|\hat{R}^\perp \), which must be \( \pm 1 \), cannot be equal to 1, hence \( \text{Tr}(\eta_5(\pi)) = -1 = \chi^{21}(\pi) \). This shows that \( \hat{R}^\perp = R^\perp \) and thus \( \hat{R} = R \). 

Theorem 1.1 now follows from Propositions 5.2 and 5.3.

12
References

[Bu16] S. Buyalo, Möbius and sub-Möbius structures, Algebra i analys, 28 (2016), n.5, 1–20, translation in St. Petersburg Math. J. 29 (2018), no. 5, 715–747.

[H] M. Hamermesh, Group theory and its application to physical problems. Addison-Wesley Series in Physics Addison-Wesley Publishing Co., Inc., Reading, Mass.-London 1962 xv+509 pp. MR0136667

(IM17) M. Incerti-Medici, Geometric structure of Möbius spaces, arX-ive:1706.10166v1 [math.MG], 2017.

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