Quantized rotating Taub-bolt instantons

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We argue that previously suggested metrics for rotating Taub-bolt instantons do not satisfy all the necessary regularity conditions, and we present a family of new regular rotating Taub-bolts labelled by an odd integer $k$. There are two types of rotating bolt solutions. The first infinite sequence starts with non-rotating Taub-NUT with positive mass $M_k = N$ for $a = 0$ and goes to $(k - 2)N$ for $|a| \to \infty$ (or $N$ for $k = 1$), where $a$ is the rotation parameter. For the second sequence of rotating Page bolts, the masses $M_k$ go through the value $M_k = 5N/4$ for $a = 0$ and asymptote to $(k + 2)N$ for $|a| \to \infty$.

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Introduction. Gravitational instantons are solutions of the Euclidean signature Einstein equations which are everywhere regular and have finite action \[1, 2\]. They have (Euclidean) time translation symmetry and the corresponding Killing vector in four dimensions can have either zero-dimensional sets of stationary points – NUTs (Newman-Unti-Tamburino), or two-dimensional sets – bolts \[3\], the bolt surfaces being the Euclidean versions of the Lorentzian black hole event horizons. The bolts are therefore very important for developing black hole thermodynamics and holographic applications. Here we revisit the problem of vacuum rotating bolt solutions endowed with a NUT parameter, removing inconsistencies of previous proposals and presenting the correct mass-spectrum of rotating bolts.

As is well-known, the solutions with a NUT parameter $N$ both in the Lorentzian and Euclidean sectors involve a singular Misner string. In the Euclidean regime the Misner string is removed by periodic identification of the Euclidean time by $8\pi|N|$, provided that a coordinate patch is chosen.
so that the space is regular at one of the polar semi-axes, in which case the manifold will be covered by a single coordinate patch. The surface of the bolt is potentially the source of a conical singularity whose removal requires periodicity of the Euclidean time with the inverse Hawking temperature 
\[ T_H = \frac{\kappa}{2\pi}, \]
where \( \kappa \) is the surface gravity of the horizon of the corresponding Lorentzian solution. The identification of the two periods then leads to a constraint on the parameters

\[ |\kappa| = \frac{1}{4|N|}, \]

which defines the masses of the Taub-bolt instantons. Such a construction in the static case was given by Page [3], revealing the relation between the mass and the NUT parameter \( M = 5N/4 \).

**Earlier versions of rotating bolts.** Gibbons and Perry [4] suggested the metric for the rotating bolt by adjusting parameters in the Euclidean version of Carter’s general stationary vacuum solution [5] admitting separation of variables in the Hamilton-Jacobi geodesic equation. The suggested mass-spectrum was subsequently used in a number of further applications and developments such as gravitational thermodynamics [6, 7], generalization for a negative cosmological constant and the associated holographic interpretation [8, 9], dressing by various matter fields [10, 11], the use as Euclidean subspaces of higher dimensional Lorentzian black holes and black rings [12, 13], and others.

It is convenient to start with the Euclidean version of the Kerr-NUT black hole in the following form slightly more general than that used in [4]:

\[ ds^2 = \Sigma \left( \frac{dr^2}{\Delta} + d\theta^2 \right) + \frac{\sin^2 \theta}{\Sigma} d\phi^2 + \frac{\Delta}{\Sigma} d\psi^2, \]

\[ d\phi = P_r(r) \, d\varphi + a \, dt, \quad d\psi = dt + P_\theta(\theta) \, d\varphi, \]

with

\[ P_r(r) = r^2 - N^2 - a^2 + Ca, \]

\[ P_\theta(\theta) = 2N \cos \theta - a \sin^2 \theta + C, \]

\[ \Sigma = P_r - aP_\theta = r^2 - (N + a \cos \theta)^2, \]

\[ \Delta = r^2 - 2Mr + N^2 - a^2, \]

where \( M \) is the mass, \( N \) is the NUT charge, \( a \) is the rotation parameter and we introduced an additional parameter \( C \) which can be added (or removed) by a coordinate transformation

\[ t \rightarrow t - C\varphi. \]
Actually, two choices of $C$ were previously suggested for rotating bolts: one is that of Gibbons and Perry \[4\]:
\[
C = -\frac{aN^2}{N^2 - a^2},
\]
and another is $C = 0$, proposed by Mann \[6\]. We didn’t find in the existing literature a motivation for either of these choices. In fact, by allowing for an arbitrary scale parameter in the general solution given by Carter \[5\], one obtains the metric \( g \) with arbitrary $C$.

**Regularity revisited.** Our point here is that, unless $a = 0$, the above two choices for $C$ lead to different surface gravities $\kappa$, as is clear from the definition
\[
\kappa^2 = \frac{1}{2} (\nabla_\mu \xi_\nu)(\nabla^\mu \xi^\nu),
\]
where $\xi^\nu$ is the time-like Killing vector $\xi^\nu \partial_\nu = \partial_t - \Omega_h \partial_\varphi$ rotating with the velocity of the horizon
\[
\Omega_h = -\frac{a}{P_r(r_+)}.
\]
with $r_+$ the larger root of $\Delta(r)$,
\[
r_\pm = M \pm \sqrt{M^2 - N^2 + a^2}.
\]
The dependence of the angular horizon velocity on $C$, following from \(9\), leads to the dependence on $C$ of the surface gravity:
\[
\kappa = \frac{r_+ - r_-}{2P_r(r_+)} = \frac{r_+ - r_-}{2(r_+^2 - N^2 - a^2 + Ca)},
\]
Consequently, the constraint equation \(11\) leads to different mass spectra for the rotating Taub-bolts with $C$ given by \(9\) \[4\] and $C = 0$ \[6\]. This raises the question of the appropriate choice of $C$.

For this we reconsider the regularity problem for the rotating solutions starting with the reasoning of Bena et al. \[12\]. Potentially dangerous are the two polar semi-axes $r > r_+$, $\theta = 0$, $\pi$, where the circle with $d\psi = dt + P_\theta(\theta) d\varphi = 0$ pinches off. Substituting in the metric for fixed $r$ the corresponding condition
\[
dt = -P_\theta d\varphi,
\]
one finds in view of \(14\)
\[
(\Sigma d\theta^2 + \frac{\sin^2 \theta}{\Sigma} (P_r - aP_\theta)^2 d\varphi^2
\]
\[
= \Sigma [d\theta^2 + \sin^2 \theta d\varphi^2],
\]
\[
(15)\]
which looks perfectly regular. But this is not the end of the story yet. One has to ensure that the condition (14) is consistent with the $2\pi$ periodicity of $\varphi$ and the $8\pi N$ periodicity of $t$. Starting with $\theta = 0$ and integrating (14),

$$t = t_0 - P_\theta(0)\varphi,$$

one finds the periodicity condition

$$t(\varphi + 2\pi) = t(\varphi) - 2\pi P_\theta(0) = t(\varphi), \mod 8\pi |N|,$$

from which follows

$$P_\theta(0) = 4pN, \quad p \in \mathbb{Z}. \quad (18)$$

The similar argument near $\theta = \pi$ leads to

$$P_\theta(\pi) = 4qN, \quad q \in \mathbb{Z}. \quad (19)$$

Using the explicit expression (5) for $P_\theta$, one thus obtains the two conditions:

$$2N + C = 4pN, \quad -2N + C = 4qN. \quad (20)$$

The difference is $p - q = 1$, while the sum expresses $C$ in terms of $N$:

$$C = 2kN, \quad k = 2p - 1. \quad (21)$$

This is obviously neither consistent with (14) (unless a very special relation between $a$ and $N$ is assumed), nor with $C = 0$ of (4). So we conclude that truly regular Taub-bolt instantons were not constructed so far, while actually there exist an infinite countable number of such solutions labelled by an odd integer $k$. We note that quantized rotating Taub-bolt instantons were also obtained from a different approach in [7]. However, contrary to ours, these have the topology of a lens space near infinity.

We find from (1) the following quantization rule

$$M^2 - N(N - ka) = (2\epsilon|N| - M)\sqrt{M^2 - N^2 + a^2}, \quad (22)$$

where $\epsilon = \text{sgn}(P_r(r_+))$. Note that $\epsilon = -1$ corresponds to a negative Hawking temperature, so it is irrelevant for thermodynamic applications, but this still may be used to construct five-dimensional Lorentzian solutions [12], so we keep it here. Without loss of generality we assume $k > 0$ (if $k < 0$ change $a$ to $-a$, i.e. $\varphi$ to $-\varphi$) and $N > 0$ (if $N < 0$ change $\theta$ to $\pi - \theta$).
Another regularity condition which must be imposed on the solution is the absence of true curvature singularities in the physical region \( r > r_+ \). This means that one must have \( \Sigma(r) > 0 \) for all \( r > r_+ \), implying \( Mr_+ \geq N(N + |a|) \). This can be translated, on account of (22) to

\[
\epsilon \sqrt{M^2 - N^2 + a^2} \geq \frac{ka + |a|}{2}.
\]

(23)

For \( a = 0 \) two regular instantons are known: Taub-NUT: \( M = N \) \cite{1}, and Page’s Taub-bolt: \( M = 5N/4 \) \cite{3}. We find two families of rotating solutions which make contact with these.

**NUT-rotons.** From (22) and (23) it is easy to see that regular rotating Taub-NUT solutions exist with \( \epsilon = +1, k = 1 \) with any positive \( a \) and \( \epsilon = -1, k = 3 \) with any negative \( a \). These solutions are of bolt-type with constant masses \( M_{kN}(a) = N \) and bolt-radii

\[
r_+ = N + |a|.
\]

(24)

The Kretschmann scalar for \( M = N \),

\[
K = R_{\mu\nu\lambda\tau}R^{\mu\nu\lambda\tau} = \frac{96N^2}{(N + a\cos\theta + r)^6},
\]

(25)

is obviously non-singular for \( r \geq r_+ \).

Quite unexpectedly, the angular velocity parameter \( \Omega_h \) for the \( k = 1, 3 \) solutions does not depend on \( a \),

\[
\Omega_h = -\frac{1}{4N},
\]

(26)

and thus remains non-zero in the Taub-NUT limit \( a = 0 \), though the metric is non-rotating.

Such instantons, which we call “NUT-rotons”, also exist for higher \( k \geq 5 \) with \( \epsilon = -1, a \leq 0 \). Their masses \( M_{kN}(a) \) become \( a \)-dependent, growing from \( M_{kN}(0) = N \) to the limiting value

\[
M_{kN}(-\infty) = (k - 2)N
\]

(27)

for \( a \rightarrow -\infty \). The functions \( M_{kN} \) are inverse to

\[
a = \frac{M^2 - N^2}{(2N + M)^2 - k^2N^2} \cdot \left( k + \frac{2N + M}{\sqrt{M^2 - N^2}} \sqrt{(k^2 - 5)N^2 - 4MN} \right),
\]

(28)

shown on Fig. 1. These are the **excited** NUT-rotons, with an equidistant mass-spectrum in the limit of infinite rotation. The angular velocity of NUT-rotons with \( k \geq 5 \) depends on \( a \), and in the limit \( a \rightarrow 0 \) also tends to a finite value

\[
\Omega_{hk} = -\frac{1}{2(k - 1)N}.
\]

(29)
FIG. 1: The rotation parameter versus the mass $M_{kN}$ for NUT-rotons with $k = 5, 7, 9, 11$ (down to up). All the curves $M_{kN}$ have vertical asymptotes at $M_{kN} = (k - 2)N$ where $a \to -\infty$ (not shown).

All NUT-rotons have $r_+ \to \infty$ at $|a| \to \infty$.

Rotating Page-bolts. The second family of rotating instantons satisfying the conditions (22) and (23) make contact with Page’s Taub-bolt solution having $M_P = 5N/4$. Consider first the lowest level $k = 1$. In this case the regularity condition (23) can be satisfied only if $\epsilon = +1$. Squaring the Eq. (22) and dividing by $(M - 1)$, we obtain a quadratic equation, whose solutions are

$$M_{1P}^\pm = \frac{1}{8} \left[ (a - N)^2 \pm (a + 3N)\sqrt{(a - N)(a - 9N)} \right].$$

(30)

One has to ensure that i) these are still the solutions of the non-squared equation (22), ii) the regularity condition (23) holds. Both conditions are satisfied for $M_{1P}^+(a)$ on the interval $-\infty < a \leq (\sqrt{2} - 1)N$. This branch of rotating bolts starts at the mass $M = N$ for maximal $a = (\sqrt{2} - 1)N$ (thus intersecting the NUT-roton branch), then with decreasing $a$ goes through the Page value $M = 5N/4$ for $a = 0$, and continues to increase with decreasing $a$, asymptotically approaching the value $M_{1P}^+(\infty) = 3N$ (Fig. 2). The branch $M_{1P}^-(a)$ satisfies i) on the interval $0 \leq a \leq N$, where $M \in [-N, 0]$, but for such $M$ the regularity condition (23) does not hold. Therefore the first level solutions consist of two intersecting branches: Taub-NUT rotons and rotating Page Taub-bolts.

For the second level $k = 3$ one can show that the only branch with negative $\epsilon$ satisfying both conditions (22) and (23) is the Taub-NUT roton described above. For $\epsilon = 1$, squaring the Eq. (22)
FIG. 2: The mass versus the rotation parameter for two rotating Page bolts: $k = 1$ (lower) and $k = 3$ (upper). The curves intersect at $a = 0$, $M = 5N/4$ and tend to $M = 3N$ and $M = 5N$ respectively for $a \to -\infty$ (not shown).

and dividing by $(M + 1)$ one obtains another quadratic equation for $M$ whose solutions are

$$M_{3P}^\pm = \frac{1}{8} \left[ (a - 3N)^2 \pm (a + N) \sqrt{(a - N)^2 + 12a} \right].$$

(31)

One can show that both i) and ii) are satisfied by the branch $M_{3P}^+(a)$ starting at $M/N = 8 - 4\sqrt{3} = 1.0718$ for $a/N = 7 - 4\sqrt{3} = .07179$, passing at $a = 0$ through the Page value $M/N = 5/4$, and then for negative $a$ monotonously going to $M_{3P}^+(\infty) = 5N$. This is therefore an excited rotating Page bolt. The other branch $M_{3P}^-(a)$ fails to satisfy simultaneously i) and ii). For $k \geq 5$ the mass spectrum can be read off from the inverse relation

$$a = \frac{M^2 - N^2}{(2N - M)^2 - k^2N^2} \cdot \left( k + \frac{M - 2N}{\sqrt{M^2 - N^2}} \sqrt{(k^2 - 5)N^2 + 4MN} \right).$$

(32)

One can show that apart from NUT-rotons for each $k$ with $\epsilon = -1$ there exist a family of rotating Page bolts for $\epsilon = +1$ with masses $M_{kP}(a)$ starting at $M = M_0 > N$ for some positive $a_0$, going through the the Page point $a = 0$ at the angle $M'_k = -(2k/3)(k^2 - 3/4)/(k^2 - 9/16)$ (Fig. 3) and tending monotonously for $a \to -\infty$ to the limiting value

$$M_{kP}(\infty) = (k + 2)N.$$  

(33)
FIG. 3: The rotation parameter versus the mass $M_k P$ for rotating Page bolts with $k = 5, 7, 9, 77$ (up to down). All the curves $M_k P$ intersect at $a = 0$, $M = 5N/4$ and have vertical asymptotes at $M_k P = (k + 2)N$ where $a \to -\infty$ (not shown).

Note, that the asymptotic spectra of NUT-rotons and rotating Page bolts coincide for twice neighboring odd $k$.

Conclusions. We have corrected long-standing mistakes regarding spectra of rotating bolt solutions to vacuum Einstein gravity, and shown that the combination of the periodicity conditions eliminating the Misner string singularity and the conical singularities consistently leads (contrary to previous claims) to two new discrete families of rotating solutions labeled by an odd integer $k$. One family makes contact with the non-rotating Taub-NUT instanton converting it into NUT-rotons. The other family represent rotating versions of the Page Taub-bolt solution. The masses of higher $k$ rotating bolts grow with $k$, so these can be regarded as excitations of the lowest mass $k = 1$ solutions.

Our results invalidate previous applications of rotating bolts to thermodynamics of NUTty vacuum black holes, their generalizations including matter fields as well as generalizations to asymptotically AdS solutions with corresponding holographic applications. These topics will be considered in future publications.

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