SINGULAR SPECTRUM OF LEBESGUE MEASURE ZERO FOR ONE-DIMENSIONAL QUASICRYSTALS

DANIEL LENVZ

1 Institute of Mathematics, The Hebrew University, Jerusalem 91904, Israel
2 Fachbereich Mathematik, Johann Wolfgang Goethe-Universität, 60054 Frankfurt, Germany
E-mail: dlenz@math.uni-frankfurt.de

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ABSTRACT. The spectrum of one-dimensional discrete Schrödinger operators associated to strictly ergodic dynamical systems is shown to coincide with the set of zeros of the Lyapunov exponent if and only if the Lyapunov exponent exists uniformly. This is used to obtain Cantor spectrum of zero Lebesgue measure for all aperiodic subshifts with uniform positive weights. This covers, in particular, all aperiodic subshifts arising from primitive substitutions including new examples as e.g. the Rudin-Shapiro substitution.

Our investigation is not based on trace maps. Instead it relies on an Oseledectype theorem due to A. Furman and a uniform ergodic theorem due to the author.

1. Introduction

This article is concerned with discrete random Schrödinger operators associated to minimal subshifts over a finite alphabet. This means we consider a family \((H_\omega)_{\omega \in \Omega}\) of operators acting on \(l^2(\mathbb{Z})\) by

\[
(H_\omega u)(n) \equiv u(n + 1) + u(n - 1) + \omega(n)u(n),
\]

where \(\omega \in \Omega\) and \((\Omega, T)\) is a subshift over the finite set \(A \subset \mathbb{R}\). Recall that \((\Omega, T)\) is called a subshift (over \(A\)) if \(\Omega\) is a closed subset of \(A^\mathbb{Z}\), invariant under the shift operator \(T : A^\mathbb{Z} \to A^\mathbb{Z}\) given by \((Ta)(n) \equiv a(n + 1)\). Here, \(A\) carries the discrete topology and \(A^\mathbb{Z}\) is given the product topology. A subshift is called minimal if every orbit is dense. For minimal subshifts \((\Omega, T)\), there exists a set \(\Sigma \subset \mathbb{R}\) s.t.

\[
\sigma(H_\omega) = \Sigma, \quad \text{for all } \omega \in \Omega,
\]

where we denote the spectrum of the operator \(H\) by \(\sigma(H)\) (cf. [1, 38]).

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Operators of this type arise in the quantum mechanical treatment of quasicrystals (cf. [4, 11] for background on quasicrystals). Various examples of such operators have been studied in recent years. The main examples can be divided in two classes. These classes are given by primitive substitution operators (cf. e.g. [4, 7, 11, 41, 42]) and Sturmian operators respectively more generally circle map operators (cf. e.g. [6, 12, 15, 16, 27, 28, 31]). A recent survey can be found in [14].

For these classes and in fact for arbitrary operators of type (1) satisfying suitable ergodicity and aperiodicity conditions, one expects the following features:

(S) Purely singular spectrum; (A) absence of eigenvalues; (Z) Cantor spectrum of Lebesgue measure zero.

Note that (S) combined with (A) implies purely singular continuous spectrum and note also that (S) is a consequence of (Z). Let us mention that (S) is by now completely established for all relevant subshifts due to recent results of Last/Simon [35] in combination with earlier results of Kotani [33]. For discussion of (A) and further details we refer the reader to the cited literature.

The aim of this article is to investigate (Z) and to relate it to ergodic properties of the underlying subshifts.

The property (Z) has been investigated for several models by a number of authors: For the period-doubling substitution it was shown by Bellissard/Bovier/Ghez in [7]. There, it is also shown for the Thue-Morse substitution (cf. earlier work of Bellissard [4] as well). The most general result for primitive substitutions so far has been obtained by Bovier/Ghez [7]. They can treat a rather large class of primitive substitutions (given by an algorithmically accessible condition) including the examples mentioned above as well as new examples as e.g. the binary non-Pisot substitution. For Sturmian operators (Z) has been established by Sütő in the golden mean case (= Fibonacci substitution) [41, 42] and, extending this work, by Bellissard/Iochum/Scoppola/Testard in the general case [6]. A different approach has been developed in [19] by Damanik and the author. In [19], this approach is used to recover (Z) in the Sturmian case. A suitably modified version of this approach can also be used to establish the result for a certain class of substitutions as shown by Damanik [13].

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While this class it not as big as the class investigated in [7], it contains many prominent examples including those mentioned above.

All these results rely on the technique of trace maps (cf. [6, 7] as well for study of trace maps). Most of the cited works tackle not only (Z) but also (A) (cf. Section 5 for further discussion).

A canonical starting point in the investigation of (Z) is the fundamental result of Kotani [33] that the set \( \{ E \in \mathbb{R} : \gamma(E) = 0 \} \) has Lebesgue measure zero if \((\Omega, T)\) is aperiodic. Here, \( \gamma \) denotes the Lyapunov exponent (precise definition given below). This reduces the problem (Z) to establishing the equality

\[
\Sigma = \{ E \in \mathbb{R} : \gamma(E) = 0 \}.
\]

As do all other investigations of (Z) so far, our approach starts from (3). Unlike the earlier treatments mentioned above our approach does not rely on trace maps. Instead, we present a new method, the cornerstones of which are the following:

- A strong type of Oseledec theorem by A. Furman [22].
- A uniform ergodic theorem for a large class of subshifts by the author [37].

This ergodic setting allows us to show that equation (3) may not only be seen as a consequence of hierarchical structures as the trace map methods suggest but
can rather be considered to be a consequence of certain ergodic properties of the system. In fact, we are able to even characterize validity of (3) by an ergodic type condition, namely uniform existence of the Lyapunov exponent.

On the conceptual level, this provides a new prospective on equality (3). On the practical level it provides a soft argument for (Z) for a large class of examples. This class contains all linearly repetitive subshifts and therefore, in particular, all primitive substitutions. Thus, it gives information on the Rudin-Shapiro substitution which could not be derived earlier (cf. discussion in [9]). To summarize, our aims are

(*) to characterize validity of (3) by an essentially ergodic property of the subshift viz by uniform existence of the Lyapunov exponent (Theorem 1),
(*** to present a large class of subshifts satisfying this property (Theorem 3).

Here, (*) gives the new conceptual point of view of our treatment and (**) gives a large class of examples.

Let us also point out that our characterization of validity of (3) is not confined to subshifts over finite alphabets but applies to arbitrary strictly ergodic systems.

The paper is organized as follows. In Section 2 we present the subshifts we will be interested in, introduce some notation and state our results. In Section 3, we recall results of Furman [22] and of the author [37] and adopt them to our setting. Section 4 is devoted to a proof of our results. Finally, in Section 5, we provide some further comments.

Note added. After this work was completed, we learned about the very recent preprint “Measure Zero Spectrum of a Class of Schrödinger Operators” by Liu/Tan/Wen/Wu (mp-arc 01-189). They present a detailed and thorough analysis of trace maps for primitive substitutions. Based on this analysis, they establish (Z) for all primitive substitutions thereby extending the approach developed in [5, 7, 9, 41].

2. Notation and Results

In this section we discuss basic material concerning subshifts and the associated operators and state our results.

We begin with a short discussion of subshifts. We will consider the elements of \((\Omega, T)\) as double sided infinite words and use notation and concepts from the theory of words. To \(\Omega\) we associate the set \(\mathcal{W}\) of words associated to \(\Omega\) consisting of all finite subwords of elements of \(\Omega\). The length \(|x|\) of a word \(x = x_1 \ldots x_n\) with \(x_j \in A, j = 1, \ldots, n\), is defined by \(|x| \equiv n\). The number of occurrences of \(v \in \mathcal{W}\) in \(x \in \mathcal{W}\) is denoted by \(\sharp_v(x)\). A subshift is called uniquely ergodic if there exist only one normalized invariant measure on \(\Omega\). It is said to be strictly ergodic (SE), if (SE) the subshift is both uniquely ergodic and minimal.

A minimal subshift is called aperiodic if there does not exist an \(n \in \mathbb{Z}, n \neq 0\), and \(\omega \in \Omega\) with \(T^n \omega = \omega\).

Virtually all models for quasicrystals in one dimension considered so far are based on strictly ergodic systems. This applies in particular for the two classes of Sturmian and substitution subshifts mentioned above.

Recently, a further class of strictly ergodic subshifts has received attention, viz linearly repetitive ones. In fact, this class (and its higher dimensional analog) is put
forward in a recent paper by Lagarias and Pleasants [34] as models for “perfectly ordered quasicrystals”. In the one-dimensional case it has been investigated by Durand from a different point of view including a characterization in terms of primitive \( S \)-adic systems [20] (cf. [21] as well). It contains all subshifts arising from primitive substitutions as well as all those Sturmian systems whose rotation number has bounded continued fraction as has e.g. the Fibonacci system [34, 38].

This class is particularly attractive as it is not given by a generating procedure but by a combinatorial condition. More precisely, a subshift is said to satisfy linear repetitivity (LR), if the following holds:

\[
(LR) \text{ There exists a } \kappa \in \mathbb{R} \text{ with } \# v(x) \geq 1 \text{ whenever } |x| \geq \kappa |v| \text{ for } x, v \in \mathcal{W}.
\]

It is true but not immediate from the definition that (LR) implies strict ergodicity [20, 34, 37] (s. below as well).

We can now introduce the class of subshifts we will be dealing with. They are those satisfying uniform positivity of weights (PW) given as follows:

\[
(PW) \text{ There exists a } C > 0 \text{ with } \lim \inf_{|x| \to \infty} \frac{|v(x)|}{|x|} |v| \geq C \text{ for every } v \in \mathcal{W}.
\]

One might think of (PW) as a strong type of minimality condition. Indeed, minimality can easily be seen to be equivalent to \( \lim \inf_{|x| \to \infty} |x|^{-1} z_v(x) |v| > 0 \) for every \( v \in \mathcal{W} \) [39]. To further put this condition in perspective, we mention that the following holds

\[
(LR) \implies (PW) \implies (SE).
\]

The first implication is clear from the definitions (take \( C \equiv \kappa^{-1} \)). The second implication follows from results of the author [37]. In our setting the class of subshifts satisfying (PW) appears naturally as it is exactly the class of subshifts admitting a strong form of uniform ergodic theorem [37]. Such a theorem in turn is needed to apply Furman’s results (s. below for details).

After this discussion of background from dynamical systems let us now get back to spectral theoretic issues. An important tool in spectral theoretic considerations are transfer matrices and Lyapunov exponents. These quantities will be introduced next. To have the appropriate setting for the discussion in Section 3, we will actually choose a rather general approach.

Let \( GL(2, \mathbb{R}) \) be the group of invertible \( 2 \times 2 \)-matrices over \( \mathbb{R} \) and let \( SL(2, \mathbb{R}) \) be the subgroup of \( GL(2, \mathbb{R}) \) consisting of matrices with determinant equal to one. The operator norm \( \| \cdot \| \) on the set of \( 2 \times 2 \)-matrices induces a topology on \( GL(2, \mathbb{R}) \) and \( SL(2, \mathbb{R}) \). For a continuous function \( A : \Omega \to GL(2, \mathbb{R}) \), \( \omega \in \Omega \), and \( n \in \mathbb{Z} \), we define the cocycle \( A(n, \omega) \) by

\[
A(n, \omega) = \begin{cases} 
A(T^{n-1} \omega) \cdots A(\omega) & : n > 0 \\
Id & : n = 0 \\
A^{-1}(T^n \omega) \cdots A^{-1}(T^{-1} \omega) & : n < 0
\end{cases}
\]

By Kingman’s subadditive ergodic theorem (cf. e.g. [22]), there exists \( \Lambda(A) \in \mathbb{R} \) with

\[
\Lambda(A) = \lim_{|n| \to \infty} \frac{1}{|n|} \log \| A(n, \omega) \|
\]

for \( \mu \text{ a. e. } \omega \in \Omega \) if \( (\Omega, T) \) is uniquely ergodic with invariant probability measure \( \mu \). Following [24], we introduce the following definition.
\textbf{Definition 1.} Let \((\Omega, T)\) be strictly ergodic. The continuous function \(A : (\Omega, T) \rightarrow \mathbb{GL}(2, \mathbb{R})\) is called uniform if the limit \(\Lambda(A) = \lim_{|n| \to \infty} \frac{1}{|n|} \log \|A(n, \omega)\|\) exists for all \(\omega \in \Omega\) and the convergence is uniform on \(\Omega\).

\textbf{Remark 1.} It is possible to show that uniform existence of the limit in the definition already implies uniform convergence. The author learned this from Furstenberg and Weiss [24]. They actually have a more general result. Namely, they consider a topological minimal dynamical system \((\Omega, T)\) with compact metric space \(\Omega\) and a continuous subadditive cocycle \((f_n)\) (i.e. \(f_n\) are continuous real-valued functions on \(\Omega\) with \(f_{n+m}(\omega) \leq f_n(\omega) + f_m(T^n \omega)\) for all \(n, m \in \mathbb{N}\) and \(\omega \in \Omega\)). Their result then gives that existence of \(\phi(\omega) = \lim_{n \to \infty} n^{-1} f_n(\omega)\) for all \(\omega \in \Omega\) implies constancy of \(\phi\) as well as uniform convergence.

The proof proceeds by exhibiting a dense \(G_\delta\)-set in \(\Omega\) on which both \(\phi(\omega) \leq \liminf_{n \to \infty}(\min \{n^{-1} f_n(\omega) : \omega \in \Omega\})\) and \(\phi(\omega) \geq \limsup_{n \to \infty}(\max \{n^{-1} f_n(\omega) : \omega \in \Omega\})\) hold.

For spectral theoretic investigations a special type of \(SL(2, \mathbb{R})\)-valued function is relevant. Namely, for \(E \in \mathbb{R}\), we define the continuous function \(M^E : \Omega \to SL(2, \mathbb{R})\) by
\[
(4) \quad M^E(\omega) \equiv \begin{pmatrix} E - \omega(1) & -1 \\ 1 & 0 \end{pmatrix}.
\]
It is easy to see that a sequence \(u\) is a solution of the difference equation
\[
(5) \quad u(n + 1) + u(n - 1) + (\omega(n) - E)u(n) = 0
\]
if and only if
\[
(6) \quad \begin{pmatrix} u(n + 1) \\ u(n) \end{pmatrix} = M^E(n, \omega) \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}, n \in \mathbb{Z}.
\]
By the above considerations, \(M^E\) gives rise to the average \(\gamma(E) \equiv \Lambda(M^E)\). This average is called the Lyapunov exponent for the energy \(E\). It measures the rate of exponential growth of solutions of (5).

Now, we are in a position to state our main result.

\textbf{Theorem 1.} Let \((\Omega, T)\) be a strictly ergodic subshift over a finite alphabet. Then the following are equivalent:
(i) The function \(M^E\) is uniform for every \(E \in \mathbb{R}\).
(ii) \(\Sigma = \{E \in \mathbb{R} : \gamma(E) = 0\}\).
In this case the Lyapunov exponent \(\gamma : \mathbb{R} \to [0, \infty)\) is continuous. If in this case furthermore \((\Omega, T)\) is aperiodic, then the spectrum \(\Sigma\) is a Cantor set of Lebesgue measure zero.

\textbf{Remark 2.} (a) This theorem links (ii) to ergodic properties of the subshift, thereby giving the new prospective on (ii) discussed in the introduction.
(b) As will be seen later on, \(M^E\) is always uniform for \(E\) with \(\gamma(E) = 0\) and for \(E \in \mathbb{R} \setminus \Sigma\). From this point of view, the theorem essentially states that \(M^E\) can not be uniform for \(E \in \Sigma\) with \(\gamma(E) > 0\).
(c) Continuity of the Lyapunov exponent can easily be inferred from (ii) (though this does not seem to be in the literature). More precisely, continuity of \(\gamma\) on \(\{E \in \mathbb{R} : \gamma(E) = 0\}\) is a consequence of subharmonicity. Continuity of \(\gamma\) on \(\mathbb{R} \setminus \Sigma\)
follows from the Thouless formula (see e. g. [10] for discussion of subharmonicity and the Thouless formula). Below, we will show that continuity of $\gamma$ follows from (i) and this will be crucial in our proof of (i) $\implies$ (ii).

(d) The theorem (except its last statement) and the proof given below remain valid for arbitrary strictly ergodic dynamical systems. Here, a dynamical system $(\Omega, T)$ is called strictly ergodic if $\Omega$ is a compact metric space and $T$ is a homeomorphism of $\Omega$ s.t. every orbit is dense (minimality) and there is only one normalized $T$-invariant measure on $\Omega$ (unique ergodicity). Of course, in this case $\omega(n)$ has to be replaced by $f(T^n \omega)$ in (1), (4) and (5), where $f : \Omega \to \mathbb{R}$ is a continuous function.

Having studied ($\ast$) of the introduction in the above theorem, we will now state our result on ($\ast\ast$).

**Theorem 2.** If $(\Omega, T)$ satisfy (PW), the function $M^E$ is uniform for each $E \in \mathbb{R}$.

**Remark 3.**

(a) Uniform existence of the Lyapunov exponent is rather unusual. This is, of course, clear from Theorem 1. Alternatively, it is not hard to see directly that it already fails for discrete almost periodic operators. Namely, it is known that the Almost-Mathieu-Operator with coupling bigger than 2 has uniform positive Lyapunov exponent (cf. [23]). By a deterministic version of the theorem of Oseledec (cf. Theorem 8.1 of [35] for example), this would force pure point spectrum for all these operators, if the Lyapunov exponent existed uniformly on the spectrum. However, there are examples of such Almost-Mathieu Operators without point spectrum [3, 8].

(b) Uniform existence of the Lyapunov exponent has been shown for primitive substitutions by Hof in [23] based on earlier work of Geerse/Hof [24]. For certain Sturmian subshifts it has been shown in [17]. For arbitrary linearly repetitive systems it has been investigated in the authors thesis (cf. [15, 17, 30] as well). The above theorem contains all these results.

(c) The theorem is a rather direct consequence of the subadditivetheorem of [37].

The two theorems allow one to infer some interesting conclusions.

As validity of (ii) is known for arbitrary Sturmian dynamical subshifts [3, 11] (cf. [11] as well), we have the following corollary of Theorem 1.

**Corollary 2.1.** Let $(\Omega(\alpha), T)$ be a Sturmian dynamical system with rotation number $\alpha$. Then $M^E$ is uniform for every $E \in \mathbb{R}$.

**Remark 4.** So far uniformity of $M^E$ for Sturmian systems could only be established for rotation numbers with bounded continued fraction expansion [17]. This set of rotation numbers has measure zero. Thus, the corollary considerably extends one of the two main results of [17]. Moreover, the corollary is remarkable as it is known that a general type of uniform ergodic theorem (or equivalently (PW) [37]) actually fails as soon as the continued fraction expansion of $\alpha$ is unbounded [37, 38].

Combining Theorem 1 and Theorem 3, we find the following corollary.

**Corollary 2.2.** Let $(\Omega, T)$ satisfy (PW). Then $\Sigma = \{ E \in \mathbb{R} : \gamma(E) = 0 \}$. If $(\Omega, T)$ is furthermore aperiodic, then $\Sigma$ is a Cantor set of Lebesgue measure zero.

**Remark 5.** For aperiodic $(\Omega, T)$ satisfying (PW), this gives an alternative proof of (S).
As discussed above primitive substitutions satisfy (PW) (and even (LR)). As validity of \((Z)\) for primitive Substitutions has been a particular focus of earlier investigations, we explicitly state the following consequence of the foregoing corollary.

**Corollary 2.3.** Let \((\Omega, T)\) be aperiodic and associated to a primitive substitution, then \(\Sigma\) is a Cantor set of Lebesgue measure zero.

**Remark 6.** As discussed in the introduction, zero measure spectrum for substitutions has been considered by several authors [5, 7, 13, 42]. The most general result has been the result of Bovier/Ghez [7]. They require existence of a square as well as a trace map satisfying a semifiniteness condition [7]. This is not satisfied by every substitution. In particular, they could not treat the Rudin-Shapiro substitution (cf. Section 5 for further discussion).

### 3. Key results

In this section, we present results of Furman [22] and of the author [37] and adopt them to our setting. In the sequel \((\Omega, T)\) will always be a subshift over a finite alphabet as discussed in the introduction. However, let us emphasize that the results and proofs given below (with the only exception of Lemma 3.5) are valid for arbitrary dynamical systems over metric compact spaces (cf. Remark 2 (d)).

We start with some simple facts concerning uniquely ergodic systems. Define for a continuous \(b : \Omega \to \mathbb{R}\) and \(n \in \mathbb{Z}\) the averaged function \(A_n(b) : \Omega \to \mathbb{R}\) by

\[
A_n(b)(\omega) = \begin{cases} 
 n^{-1} \sum_{k=0}^{n-1} b(T^k \omega) & : \ n > 0 \\
 0 & : \ n = 0 \\
 |n|^{-1} \sum_{k=1}^{|n|} b(T^{-k} \omega) & : \ n < 0 
\end{cases}
\]

(7)

The following proposition is well known see e.g. [43].

**Proposition 3.1.** Let \((\Omega, T)\) be uniquely ergodic with invariant probability measure \(\mu\). Let \(b\) be a continuous function on \(\Omega\). Then the averaged functions \(A_n(b)\) converge uniformly towards the constant function with value \(\mu(b)\) for \(|n|\) tending to infinity.

The following result by A. Furman is crucial to our approach. It describes the structure of uniform functions with positive average. It can be seen as a continuous version of an Oseledec type theorem.

**Lemma 3.2.** Let \((\Omega, T)\) be strictly ergodic with invariant probability measure \(\mu\). Let \(B : \Omega \to \text{SL}(2, \mathbb{R})\) be uniform with \(\Lambda(B) > 0\). Then, there exists a continuous function \(G : \Omega \to \text{GL}(2, \mathbb{R})\) and a continuous function \(b : \Omega \to \mathbb{R}\) with \(\mu(b) = \Lambda(B)\) such that for every \(n \in \mathbb{Z}\) the following holds:

\[
B(n, \omega) = G(T^n \omega)^{-1} \begin{pmatrix} \exp(-nA_n(b)(\omega)) & 0 \\ 0 & \exp(nA_n(b)(\omega)) \end{pmatrix} G(\omega).
\]

**Proof.** It suffices to show the result for \(n = 1\). The other cases then follow by multiplication and inversion. Now, the result is essentially given by Theorem 4 in [22]. While this is quite clear, it is not completely explicit from the actual statement of the theorem. Thus, for the convenience of the reader and as we will use a similar reasoning later on, we include a discussion.

Theorem 4 of [22] states that uniformity of \(B\) implies that (in the notation of [22]) either \(\Lambda(B) = 0\) or \(B\) is continuously diagonalizable. As we have \(\Lambda(B) > 0\), we
infer that $B$ is continuously diagonalizable. This means that there exist continuous functions $C : \Omega \to GL(2, \mathbb{R})$ and $a, d : \Omega \to \mathbb{R}$ with
\begin{equation}
B(1, \omega) = C(T\omega)^{-1} \begin{pmatrix}
\exp(a(\omega)) & 0 \\
0 & \exp(d(\omega))
\end{pmatrix} C(\omega).
\end{equation}
Setting, $h(\omega) \equiv \frac{1}{2} \log |\det C(\omega)|$ and $G(\omega) \equiv |\det C(\omega)|^{-\frac{1}{2}} C(\omega)$, we can rewrite $B(1, \omega)$ as
\begin{equation}
G(T\omega)^{-1} \begin{pmatrix}
\exp(a(\omega) - h(T\omega) + h(\omega)) & 0 \\
0 & \exp(d(\omega) - h(T\omega) + h(\omega))
\end{pmatrix} G(\omega).
\end{equation}
Using $1 = |\det G| = |\det B|$, we find
\begin{equation}
a(\omega) - h(T\omega) + h(\omega) = -(d(\omega) - h(T\omega) + h(\omega))
\end{equation}
Thus, defining $b : \Omega \to \mathbb{R}$ by $b(\omega) \equiv d(\omega) - h(T^{-1}\omega) + h(\omega)$, we arrive at the desired equation
\begin{equation}
B(1, \omega) = G(T\omega)^{-1} \begin{pmatrix}
\exp(-b(\omega)) & 0 \\
0 & \exp(b(\omega))
\end{pmatrix} G(\omega).
\end{equation}
It remains to show the statement about $\mu(b)$. By compactness of $\Omega$ and continuity of $G$ we have $\Lambda(B) = \Lambda(G(T\cdot)BG^{-1})$. Using this, we infer from (8) and the previous Proposition
\begin{equation}
\Lambda(B) = \Lambda\left( \begin{pmatrix}
\exp(-b(\omega)) & 0 \\
0 & \exp(b(\omega))
\end{pmatrix} \right) = |\mu(b)|.
\end{equation}
If $\mu(b) = |\mu(b)|$ the proof is finished. If $\mu(b) = -|\mu(b)|$, the claim follows after a suitable conjugation.

\textbf{Lemma 3.3.} Let $(\Omega, T)$ be strictly ergodic. Let $A : \Omega \to SL(2, \mathbb{R})$ be uniform. Let $(A_n)$ be a sequence of continuous $SL(2, \mathbb{R})$-valued functions converging to $A$ in the sense that $d(A_n, A) \equiv \sup_{\omega \in \Omega} \{\|A_n(\omega) - A(\omega)\|\} \to 0$, $n \to \infty$. Then, $\Lambda(A_n) \to \Lambda(A)$, $n \to \infty$.

\textbf{Proof.} Again this is essentially a result of [22]. More precisely, Theorem 5 of [22] shows that $\Lambda(A_n)$ converges to $\Lambda(A)$ whenever the following holds: $A$ is a uniform $GL(2, \mathbb{R})$-valued function and $d(A_n, A) \to 0$ and $d(A_n^{-1}, A^{-1}) \to 0$, $n \to \infty$. Now, for functions $A_n, A$ with values in $SL(2, \mathbb{R})$, it is easy to see that $d(A_n^{-1}, A^{-1}) \to 0$, $n \to \infty$ if $d(A_n, A) \to 0, n \to \infty$. The proof of the lemma is finished.

\textbf{Lemma 3.4.} Let $(\Omega, T)$ be strictly ergodic. Let $A : \Omega \to GL(2, \mathbb{R})$ be continuous. Then, the inequality $\limsup_{n \to \infty} n^{-1} \log \|A(n, \omega)\| \leq \Lambda(A)$ holds uniformly on $\Omega$.

\textbf{Proof.} This follows from Theorem 1 of [22].

Finally, we need the following lemma providing a large supply of uniform functions if $(\Omega, T)$ satisfies (PW).

\textbf{Lemma 3.5.} Let $(\Omega, T)$ satisfy (PW). Let $F : W \to \mathbb{R}$ satisfy $F(xy) \leq F(x) + F(y)$ (i.e. $F$ is subadditive). Then, the limit $\lim_{|x| \to \infty} \frac{F(x)}{|x|}$ exists.

\textbf{Proof.} This is just one half of Theorem 2 of [22].
4. PROOFS OF THE MAIN RESULTS

In this section, we use the results of the foregoing section to prove the theorems stated in Section 3. Again let us point out that the lemmas given below and their proofs are valid for arbitrary topological dynamical systems over metric compact spaces (cf. Remark 3 (d)).

We start with some lemmas needed for the proof of Theorem 3. We will denote the Euclidean norm of an element $v \in \mathbb{R}^2$ by $\|v\|$ i.e. $\|v\|^2 = v_1^2 + v_2^2$ for $v$ with components $v_1$ and $v_2$.

**Lemma 4.1.** Let $(\Omega, T)$ be strictly ergodic. If $M^E$ is uniform for every $E \in \mathbb{R}$ then $\Sigma = \{E \in \mathbb{R} : \gamma(E) = 0\}$ and the Lyapunov exponent $\gamma : \mathbb{R} \rightarrow [0, \infty)$ is continuous.

**Proof.** We start by showing continuity of the Lyapunov exponent. Consider a sequence $(E_n)$ in $\mathbb{R}$ converging to $E \in \mathbb{R}$. As the function $M^E$ is uniform by assumption, by Lemma 3.3, it suffices to show that $d(M^{E_n}, M^E) \rightarrow 0, n \rightarrow \infty$.

This is clear from the definition of $M^E$ in (4).

Set $\Gamma \equiv \{E \in \mathbb{R} : \gamma(E) = 0\}$. The inclusion $\Gamma \subset \Sigma$ follows from general principles (cf. e.g. [10]). Thus, it suffices to show the opposite inclusion $\Sigma \subset \Gamma$. By (3), it suffices to show $\sigma(H_\omega) \subset \Gamma$ for a fixed $\omega \in \Omega$.

Assume the contrary. Then there exists spectrum of $H_\omega$ in the complement $\Gamma^c \equiv \mathbb{R} \setminus \Gamma$ in $\mathbb{R}$. As $\gamma$ is continuous, the set $\Gamma^c$ is open. Thus, spectrum of $H_\omega$ can only exist in $\Gamma^c$, if spectral measures of $H_\omega$ give actually weight to $\Gamma^c$.

By standard results on generalized eigenfunction expansion [8], there exists then an $E \in \Gamma^c$ admitting a polynomially bounded solution $u \neq 0$ of (3). This gives a contradiction in the following way:

By (3) and Lemma 3.2 applied to $M^E$ (use $E \in \Gamma^c$ to obtain $\Lambda(M^E) \equiv \gamma(E) > 0$), there exist continuous functions $G : \Omega \rightarrow GL(2, \mathbb{R})$ and $b : \Omega \rightarrow \mathbb{R}$ with $\mu(b) = \Lambda(M^E) = \gamma(E) > 0$ with

\[
\begin{pmatrix}
  u(n+1) \\
  u(n)
\end{pmatrix} = G(T^n \omega)^{-1} \begin{pmatrix}
  \exp(-nA_n(b)(\omega)) & 0 \\
  0 & \exp(nA_n(b)(\omega))
\end{pmatrix} G(\omega) \begin{pmatrix}
  u(1) \\
  u(0)
\end{pmatrix}
\]

for every $n \in \mathbb{Z}$. As $G : \Omega \rightarrow GL(2, \mathbb{R})$ is continuous on the compact space $\Omega$, there exist constants $\rho_1, \rho_2, 0 < \rho_1, \rho_2 < \infty$, with

\[
\|G(\omega)\| \leq \rho_1, \quad \|G(\omega)^{-1}v\| \geq \rho_2\|v\|
\]

for every $\omega \in \Omega$ and every $v \in \mathbb{R}^2$. Moreover, as $b$ is continuous and $(\Omega,T)$ is strictly ergodic, we infer from Proposition 3.1

\[
\lim_{|n| \rightarrow \infty} A_n(b)(\omega) = \mu(b) = \Lambda(M^E) = \gamma(E) > 0.
\]

Set $\begin{pmatrix}
  x \\
  y
\end{pmatrix} \equiv G(\omega) \begin{pmatrix}
  u(1) \\
  u(0)
\end{pmatrix}$. As $u$ is polynomially bounded, we infer $y \equiv 0$ from (10), (11) and (12) by considering the right half-axis i.e. large values of $n$. Similarly, considering the left half-axis i.e. small values of $n$, we infer $x \equiv 0$. As $G(\omega)$ is invertible this means $u(0) = u(1) = 0$ and this gives the contradiction $u \equiv 0$. \hfill \Box
Lemma 4.2. If \((\Omega, T)\) is strictly ergodic, \(M^E\) is uniform for each \(E \in \mathbb{R}\) with \(\gamma(E) = 0\).

Proof. By \(\det M^E(\omega) = 1\), we have \(1 \leq \|M^E(n, \omega)\|\) and therefore \(0 \leq \liminf_{n \to \infty} n^{-1} \log \|M^E(n, \omega)\| \leq \limsup_{n \to \infty} n^{-1} \log \|M^E(n, \omega)\|.\) Now, the statement follows from Lemma 3.3. \(\square\)

Moreover, we have the following lemma. The lemma is certainly well known. However, we could not find a proof in the literature. Thus, for the convenience of the reader, we include a proof.

Lemma 4.3. If \((\Omega, T)\) is strictly ergodic, \(M^E\) is uniform with \(\gamma(E) > 0\) for each \(E \in \mathbb{R} \setminus \Sigma\).

Proof. Let \(E \in \mathbb{R} \setminus \Sigma\) be given. The proof will be split in four steps. Recall that \(\Sigma\) is the spectrum of \(H_\omega\) for every \(\omega \in \Omega\) by (2) and thus \(E\) belongs to the resolvent of \(H_\omega\) for all \(\omega \in \Omega\).

Step 1. For every \(\omega \in \Omega\), there exist unique (up to a sign) normalized \(U(\omega), V(\omega) \in \mathbb{R}^2\) such that \(\|M^E(n, \omega)U(\omega)\|\) is exponentially decaying for \(n \to \infty\) and \(\|M^E(n, \omega)V(\omega)\|\) is exponentially decaying for \(n \to -\infty\). The vectors \(U(\omega), V(\omega)\) are linearly independent. For fixed \(\omega \in \Omega\) they can be chosen to be continuous in a neighborhood of \(\omega\).

Step 2. Define the matrix \(C(\omega)\) by \(C(\omega) \equiv (U(\omega), V(\omega))\). Then \(C(\omega)\) is invertible and there exist functions \(a, b : \Omega \to \mathbb{R} \setminus \{0\}\) such that
\[
(13) \quad C(T\omega)^{-1}M^E(\omega)C(\omega) = \begin{pmatrix} a(\omega) & 0 \\ 0 & b(\omega) \end{pmatrix}.
\]

Step 3. The functions \(|a|, |b|, ||C||, ||C^{-1}|| : \Omega \to \mathbb{R}\) are continuous.

Step 4. \(M^E\) is uniform with \(\gamma(E) > 0\).

Ad Step 1. This is standard up to the continuity statement. Here is a sketch of the construction: Fix \(\omega \in \Omega\). Set \(u_0(n) \equiv (H_\omega - E)^{-1}\delta_0(n)\) and \(u_{-1}(n) \equiv (H_\omega - E)^{-1}\delta_{-1}(n)\), where \(\delta_k, k \in \mathbb{Z}\), is the sequence in \(l^2(\mathbb{Z})\) vanishing everywhere except in \(k\), where it takes the value 1. By Combes/Thomas arguments, see e.g. (10), the vectors \((u_0(0), u_0(1))^t\) and \((u_{-1}(0), u_{-1}(1))^t\) give rise to solutions of (12) which decay exponentially for \(n \to \infty\). Here, \((x, y)^t\) denotes the transpose of \((x, y)\). It is easy to see that not both of these solutions can vanish identically. Thus, after normalizing, we find a vector \(U(\omega)\) with the desired properties. By continuity of \(\omega \mapsto (H_\omega - E)^{-1}\delta_k, k \in \mathbb{Z}\), we can choose \(U(\omega)\) continuous in a neighborhood of a fixed \(\omega\). The construction for \(V(\omega)\) is similar focusing on behaviour of solutions to the left i.e. for \(n \to -\infty\).

Uniqueness follows by standard arguments from constancy of the Wronskian. Linear independence is clear as \(E\) is not an eigenvalue of \(H_\omega\).

Ad Step 2. The matrix \(C\) is invertible by linear independence of \(U\) and \(V\). The uniqueness statements of Step 1, show that there exist functions \(a, b : \Omega \to \mathbb{R}\) with \(M^E(\omega)U(\omega) = a(\omega)U(T\omega)\) and \(M^E(\omega)V(\omega) = b(\omega)V(T\omega)\). This easily yields (13). As the left hand side of this equation is invertible, the right hand side is invertible as well. This shows that \(a\) and \(b\) do not vanish anywhere.

Ad Step 3. Direct calculations show that the functions in question do not change if \(U(\omega)\) or \(V(\omega)\) or both are replaced by \(-U(\omega)\) resp. \(-V(\omega)\). By Step 1, such a
replacement can be used to provide a version of $V$ and $U$ continuous around an arbitrary $\omega \in \Omega$. This gives the desired continuity.

Ad Step 4. As $\|C\|$ and $\|C^{-1}\|$ are continuous by Step 3 and $\Omega$ is compact, there exist constants $\kappa_1, \kappa_2$ with $0 < \kappa_1, \kappa_2 < \infty$ with $\kappa_1 \leq \|C(\omega)\|, \|C^{-1}(T\omega)\| \leq \kappa_2$ for every $\omega \in \Omega$. Thus, uniformity of $M^E$ will follow from uniformity of $\omega \mapsto C^{-1}(T\omega)M^E(\omega)C(\omega)$, which in turn will follow by Step 2 from uniformity of

$$\omega \mapsto D(\omega) \equiv \begin{pmatrix} |a(\omega)| & 0 \\ 0 & |b(\omega)| \end{pmatrix}.$$ 

As $|a|$ and $|b|$ are continuous by Step 3 and do not vanish by Step 2, the functions $\ln |a|, \ln |b| : \Omega \to \mathbb{R}$ are continuous. The desired uniformity of $D$ follows now by Proposition 3.1 (see proof of Lemma 3.2 for a similar reasoning). Positivity of $\gamma(E)$ is immediate from Step 1. \hfill \Box

A simple but crucial step in the proof of Theorem 2 is to relate the transfer matrices to subadditive functions. This will allow us to use Lemma 3.5 to show that the uniformity assumption of Lemma 3.2 and Lemma 3.3 holds for subshifts satisfying (PW). We proceed as follows. Let $(\Omega, T)$ be a strictly ergodic subshift and let $E \in \mathbb{R}$ be given. To the matrix valued function $M^E$ we associate the function $F^E : \mathcal{W} \to \mathbb{R}$ by setting

$$F^E(x) \equiv \log \|M^E(|x|, \omega)\|,$$

where $\omega \in \Omega$ is arbitrary with $\omega(1)\ldots\omega(|x|) = x$. It is not hard to see that this is well defined. Moreover, by submultiplicativity of the norm $\|\cdot\|$, we infer that $F^E$ satisfies $F^E(xy) \leq F^E(x) + F^E(y)$.

Proposition 4.4. $M^E$ is uniform if and only if the limit $\lim_{|x| \to \infty} \frac{F^E(x)}{|x|}$ exists.

Proof. This is straightforward. \hfill \Box

Now, we can prove the results stated in Section 2.

Proof of Theorem 2. The implication (i)$\Rightarrow$(ii) is an immediate consequence of Lemma 4.1. This lemma also shows continuity of the Lyapunov exponent. The implication (ii)$\Rightarrow$(i) follows from Lemma 4.2 and Lemma 4.3.

It remains to show the statement on aperiodic subshifts. As $\Sigma$ is closed and has no discrete points by general principles on random operators, the Cantor property will follow if $\Sigma$ has measure zero. But this follows from the first statement of the theorem, as the set $\{E \in \mathbb{R} : \gamma(E) = 0\}$ has measure zero by the results of Kotani theory discussed in the introduction. \hfill \Box

Proof of Theorem 3. This is immediate from Lemma 3.5 and Proposition 4.4. \hfill \Box

5. Further discussion

In this section we will present some discussion and comments on the results proven in the previous sections.

As shown in the introduction and the proof of Theorem 1, the problem (Z) can essentially be reduced to establishing the inclusion $\Sigma \subset \{E \in \mathbb{R} : \gamma(E) = 0\}$. This has been investigated for various models by various authors \cite{5, 6, 7, 13, 19, 42}. All these proofs rely on the same tool viz trace maps. Trace maps are very powerful
as they capture the underlying hierarchical structures. Besides being applicable in the investigation of \((\mathcal{Z})\), trace maps are extremely useful because

- trace map bounds are an important tool to prove absence of eigenvalues.

As already mentioned, most of the cited literature actually studies both \(\mathcal{A}\) and \(\mathcal{Z}\). In fact, \(\mathcal{Z}\) can even be shown to follow from a strong version of \(\mathcal{A}\) \cite{13} (cf. \cite{13} as well). While this makes the trace map approach to \(\mathcal{Z}\) very attractive, it has two drawbacks:

- The analysis of the actual trace maps may be quite hard or even impossible.
- The trace map formalism only applies to substitution-like subshifts.

Thus, trace map methods can not be expected to establish zero-measure spectrum in a generality comparable to the validity of the underlying Kotani result.

Let us now compare this with the method presented above. Essentially, our method has a complementary profile: It does not seem to give information concerning absence of eigenvalues. But on the other hand it only requires a weak ergodic type condition.

This condition is in particular met by subshifts satisfying (PW). This class of subshifts is rather large. It contains all linearly repetitive subshifts ("perfectly ordered quasicrystals" in the sense of \cite{14}) and a fortiori all primitive substitutions. In particular, it gives information on the Rudin-Shapiro substitution which so far had been unattainable. Moreover, quite likely, the condition (PW) will be satisfied for certain circle maps, where \(\mathcal{Z}\) could not be proven by other means (except of course Sturmian ones s. below).

Thus, (PW) provides a natural framework for validity of \(\mathcal{Z}\) providing a large class of examples.

All the same, it seems worthwhile pointing out that (PW) does not contain the class of Sturmian systems whose rotation number has unbounded continued fraction expansion. This is in fact the only class known to satisfy \(\mathcal{Z}\) (and much more \cite{13,12,10,16,28,29,41}) not covered by (PW). For this class, one can use the implication \((ii) \Rightarrow (i)\) of Theorem \cite{1} to conclude uniform existence of the Lyapunov exponent as done in Corollary \cite{2.1}. Still it seems desirable to give a direct proof of uniform existence of the Lyapunov exponent for these systems.

Finally, let us briefly discuss the case of operators associated to arbitrary strictly ergodic dynamical systems (cf. Remark \cite{2} (dl)). As discussed above, Theorem \cite{1} remains valid in this case. But it might be of rather limited use. Thus, the following result (which is essentially a corollary of the proof of Theorem \cite{1}) might be a more appropriate formulation in this case.

**Theorem 3.** Let \((\Omega, T)\) be an arbitrary strictly ergodic dynamical system. Then, \(\mathbb{R} \setminus \Sigma\) is open and \(\mathcal{M}^E\) is uniform with \(\gamma(E) > 0\) for every \(E \in \mathbb{R} \setminus \Sigma\). Conversely, if \(I \subset \mathbb{R}\) is open such that \(\mathcal{M}^E\) is uniform with \(\gamma(E) > 0\) for every \(E \in I\), then \(I\) is contained in \(\mathbb{R} \setminus \Sigma\).

**Proof.** Openness of \(\mathbb{R} \setminus \Sigma\) is clear. Now, the first statement follows from Lemma \cite{43}. The second statement follows from the proof of Lemma \cite{4.1} after replacing \(\Gamma^c\) by \(I\). \(\square\)

The theorem characterizes the complement of the spectrum in \(\mathbb{R}\) as the largest open set in \(\mathbb{R}\) on which uniformity and positivity of the Lyapunov exponent hold.
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