Models of inflation and the spectral index of the density perturbation

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Abstract

If an adiabatic density perturbation is responsible for large scale structure and the cmb anisotropy, its spectral index \( n \) will be measured in the foreseeable future with an accuracy \( \Delta n \sim 0.01 \). This is precisely the kind of accuracy required to distinguish between many models of inflation. Most of them have an inflationary potential \( V \simeq V_0(1 \pm \mu \phi^p) \) with the constant term dominating. Except for \( 0 < p < 2 \), the prediction is \( n = 1 \pm (2/N)(p-1)/(p-2) \), where \( N \) is the number of e-folds of inflation after cosmological scales leave the horizon. It typically lies in the range \( 0.9 \lesssim n \lesssim 1.1 \). For \( p = 2 \) one has \( n = 1 \pm 2M_{Pl}^2\mu \) where \( M_{Pl} = (8\pi G)^{-1/2} \). A generic supergravity theory gives contributions of order \( \pm H^2 \) to the inflaton mass-squared \( m^2 \) whereas inflation requires \( |m^2| \ll H^2 \). Proposals for keeping \( m^2 \) small are surveyed with emphasis on their implications for \( n \). Finally, the case of a multi-component inflaton (as in ‘double’ inflation) is discussed. In all cases, the best method of calculating the spectrum of the adiabatic density perturbation starts with the assumption that, after smoothing on a super-horizon scale, the evolution of the universe along each comoving worldline will be practically the same as in an unperturbed universe with the same initial inflaton field. The possible isocurvature density perturbation is briefly discussed, with a simple derivation of the fact that the low multipoles of the cmb anisotropy are six times as big as for an adiabatic density perturbation.

I. INTRODUCTION

Inflation generates an adiabatic density perturbation, which is generally thought to be responsible for large scale structure and the cosmic microwave background (cmb) anisotropy, and it also generates gravitational waves which might give a significant contribution to the latter \[4,5\].

The spectrum of the density perturbation is conveniently specified by a quantity \( \delta_H^2 \) and the spectral index \( n \) is defined by \( \delta_H^2 \propto k^{n-1} \) where \( k \) is the comoving wavenumber.
At present there is no detectable scale dependence, and observational limits on $n$ are only mildly constraining for inflationary models, the most notable result being that ‘extended’ inflation \[4\] is ruled out, except for rather contrived versions \[5\]. But in the foreseeable future one can expect a good measurement of $n$ and it is reasonable to ask what it will tell us.

At a purely phenomenological level the answer to the question is simple and well known. The spectral index $n$ is given by \[6–8\]

$$n - 1 \simeq -3M_{Pl}^2(V'/V)^2 + 2M_{Pl}^2(V''/V)$$

(1)

where $M_{Pl} = (8\pi G)^{-1/2} = 2.4 \times 10^{18}$ GeV is the reduced Planck mass. The right hand side of this expression is a combination of the potential and its derivatives, evaluated at the epoch when cosmological scales leave the horizon. If $n$ is distinguishable from 1 the combination will be measured, otherwise it will be constrained to near zero. Moreover, the relative contribution of gravitational waves to the mean-square low multipoles of the cmb anisotropy is given by \[9–12\] $r \simeq 5M_{Pl}^2(V'/V)^2$. Their eventual detection or non-detection will determine or constrain the first term, so that measuring $n$ determines or constrains $V''/V$. Finally, with more data one might measure the effective $n$ over a range of scales and also justify the use of more accurate formulas \[13\], to obtain limited additional information on the shape of $V$ while cosmological scales are leaving the horizon \[14\].

This ‘reconstruction’ approach is interesting, but it relies to a large extent on detecting gravitational waves and will in any case always be limited by the narrow range of scales accessible to cosmology. Meanwhile, since the advent of hybrid inflation \[15,16\], inflation model-building is beginning to come back into the fold of particle physics. Like the earliest models of inflation, but unlike those proposed in the intervening years, hybrid inflation models work in the regime where all relevant fields are small on the Planck scale \[17\]. If one accepts the usual view that field theory beyond the standard model involves supergravity \[18\], this represents a crucial simplification because in the context of supergravity the potential is expected to have an infinite power series expansion in the fields. For field values much bigger than $M_{Pl}$ one has no idea what form the potential will take, expect perhaps in the direction of the moduli fields of superstring theory \[13,20\]. But in the small-field regime one is entitled to make the usual assumption, that the expansion is dominated by a very few low-order terms. As a result, inflation model-building has become a more varied, yet better controlled activity.

Given this situation it is reasonable to go beyond the purely phenomenological level, and ask how well a measurement of the spectral index will discriminate between different models of inflation. That question is the main focus of the present paper.

There are of course many other aspects of inflation model-building. The prediction Eq. (17) of the normalization of $\delta_H$ provides another constraint on the parameters of the inflationary potential, which can crucially affect the viability of a given potential. It may be regarded as fixing the magnitude of $V$ as opposed to its derivatives, a quantity whose theoretical significance is not yet clear (for some recent discussions see Refs. \[17,21,22,23\]). Another theoretical issue is the difficulty of implementing slow-roll inflation in the context of supergravity \[14,24\]. These two things will be considered briefly in the present paper, but other aspects of inflation model building, in particular reheating and ‘preheating’ (see for instance Ref. [25]) will not be discussed at all.

In most of the discussion we assume that the slowly-rolling inflaton field is essentially
unique. This is the case in most of the models so far proposed, and is necessary for the validity of the above formula for $n$. But in Section VII we discuss the case where there is a family of inflationary trajectories lying in the space of two or more fields; in other words, where there is a multi-component inflaton. The quantum fluctuation kicks the field from one trajectory to another, and if the trajectories are physically inequivalent this gives a contribution to the spectrum of the density perturbation which has to be added to the usual one, coming from the fluctuation back and forth along the same trajectory. In all cases, the best method of calculating the spectrum of the adiabatic density perturbation starts with the assumption that, after smoothing on a super-horizon scale, the evolution of the universe along each comoving worldline will be the same as in an unperturbed universe with the same initial inflaton field. We also discuss briefly the case of an isocurvature density perturbation, giving a simple derivation of the fact that the low multipoles of the CMB anisotropy are six times as big as for an adiabatic density perturbation.

We assume that the scale dependence of $\delta_H$ can be represented by a spectral index, $\delta_H^2 \propto k^{n-1}$, at least while cosmological scales are leaving the horizon. This is the case if slow-roll inflation is continuously maintained, unless there is a cancellation between the terms of Eq. (1) for $n - 1$. A feature, such as a shoulder in the spectrum, can easily be generated if slow-roll inflation is briefly interrupted at the epoch when the relevant scale is leaving the horizon [26–30,11]. However to get any kind of feature during the few $e$-folds corresponding to cosmological scales requires a delicate balance of parameters. Observation does suggest a shoulder in the spectrum of the present density perturbation, in the tens of Mpc range, though at present the evidence is also consistent with a smooth rise corresponding to $n \approx 0.8$. But the spectrum $\delta_H$ refers to the density perturbation in the early universe, and the most natural explanation for a shoulder in the spectrum of the observed density perturbation would be that it is in the transfer function relating the present density perturbation to the early one. Indeed, any departure from the pure cold dark matter hypothesis leads to a shoulder on just the required scales. The conclusion is that there is little motivation to depart from the slow-roll inflation paradigm, and its attendant scale dependence $\delta_H^2 \propto k^{n-1}$.

The discussion is confined to models of inflation leading to a spatially flat universe on observable scales. This is the generic case, though at the expense of moderate fine-tuning the universe can be open (negatively curved) if we live inside a bubble formed at the end of a preliminary era of inflation [31].

Throughout the paper, $a$ denotes the scale factor of the universe normalized to 1 at present and $H = \dot{a}/a$ is the Hubble parameter, with present value $H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}$. The corresponding Hubble distance is $cH_0^{-1} = 3000h^{-1} \text{ Mpc}$. From now on the units are $c = h = 1$. The comoving wavenumber in a Fourier series expansion is denoted by $k/a$, so that $k$ is the present wavenumber and $k^{-1}$ conveniently specifies the corresponding comoving scale. The scale is said to be outside the horizon when $aH/k$ is bigger than 1 and inside it otherwise. A given scale of interest leaves the horizon during inflation and re-enters it long afterwards during radiation or matter domination.

The plan of the paper is as follows. In Section II the eventual accuracy of the measurement of $n$ is discussed. In Section III some general issues relating to inflation model-building are discussed, and specific models are discussed in Section IV and V. In Section VI various ways of implementing slow-roll inflation in the context of supergravity are briefly discussed, focussing on their implications for the spectral index. Section VII discusses inflation with a
multi-component inflaton, and the calculation of the spectrum of the density perturbation in both this and the single-component case. A summary is provided in Section VIII.

II. HOW ACCURATELY WILL THE SPECTRAL INDEX BE DETERMINED?

Before discussing the theoretical predictions we need an idea of the eventual accuracy to expect for the determination of the spectral index of the adiabatic density perturbation. First let us define this perturbation.

A. The spectrum of the adiabatic density perturbation

The evolution of small density perturbations in the universe is governed by linear equations, with each Fourier mode evolving independently. On scales well outside the horizon the evolution is governed purely by gravity, and in each comoving region its evolution with respect to the local proper time is is the same as it would be in an unperturbed universe.

For the moment we are not interested in the evolution during inflation and in the very early universe. Rather, we focus on the relatively late era when the non-baryonic dark matter has its present form, and the radiation-dominated era immediately preceding the present matter-dominated era has begun. During this era, consider the perturbations \( \delta \rho_i \) in the energy densities of the various particle species. They are said to be adiabatic if each of the densities \( \rho_i \) is a unique function of the total density \( \rho \). The perturbations all vanish if we slice the universe into hypersurfaces of constant energy density. With any other slicing, all species of matter have a common density contrast and so do all species of radiation, with \( \delta \rho_m / \rho_m = \frac{3}{4} \delta \rho_r / \rho_r \).

A different type of density perturbation would be an isocurvature one, for which the total density perturbation vanishes but the separate ones do not. The vacuum fluctuation during inflation will generate an adiabatic perturbation, which is the subject of the present paper. At the end of Section VII we consider briefly the isocurvature perturbation that might also be produced.

The adiabatic density perturbation is determined by the total density perturbation \( \delta \rho / \rho \). This in turn is determined by a quantity \( \mathcal{R} \), which defines the curvature of hypersurface orthogonal to comoving worldlines (‘comoving hypersurfaces’).\(^1\) At least during the relatively late era that we are focussing on at the moment, \( \mathcal{R} \) is time-independent, and

\[
\frac{\delta \rho}{\rho} = \frac{4}{9} \left( \frac{k}{aH} \right)^2 \mathcal{R} \tag{2}
\]

\(^1\)For each Fourier mode, the curvature scalar \( R^{(3)} \) is given by \( \mathcal{R} = \frac{1}{4} (a/k)^2 R^{(3)} \) (it is a perturbation since the unperturbed universe is supposed to be flat). There is unfortunately no standard notation for \( \mathcal{R} \). It was first defined in Ref. 32 where it was called \( \phi_m \). It was called \( R_m \) in Ref. 33, and is a factor \( \frac{3}{2} k^{-2} \) times the quantity \( \delta K \) of Ref. 34. On the scales far outside the horizon where it is constant (the only regime where it is of interest) it coincides with the quantity \( \xi/3 \) of Ref. 35 and the quantity \( \xi \) of Ref. 27.
As always we focus on a single Fourier component, well before horizon entry.)

After horizon entry the evolution becomes more complicated, and for radiation as opposed to matter it involves the full phase-space densities. The main observational quantities are the present matter density perturbation, which we can get a handle by observing the distribution and motion of galaxies, and the cmb anisotropy. There is also some information on the matter density perturbation at earlier times.

Inflation typically predicts that $\mathcal{R}$ is a Gaussian random field, which is usually assumed and is consistent with observation. Then all stochastic properties of the perturbations are determined by the spectrum $\mathcal{P}_\mathcal{R}$ of $\mathcal{R}$. We are supposing that any scale dependence of the spectrum has the form $\mathcal{P}_\mathcal{R} \propto k^{-n}$ where $n$ is the spectral index.

Instead of $\mathcal{P}_\mathcal{R}$ one usually considers

$$\delta_H^2 \equiv (4/25)\mathcal{P}_\mathcal{R}$$  \hspace{1cm} (3)

Then, on scales $k^{-1} \gtrsim 10$ Mpc where its evolution is still linear, the spectrum of the present density perturbation $\delta \equiv \delta \rho / \rho$ is given by

$$\mathcal{P}_\delta = \delta_H^2 (k/H_0)^4 T^2(k)$$  \hspace{1cm} (4)

The transfer function $T(k)$ represents non-gravitational effects, and is equal to 1 on the large scales $k^{-1} \gg 100$ Mpc where such effects are absent. Information on $\mathcal{P}_\delta$ on scales of order 10 to 100 Mpc is provided by a variety of observations on the abundance, distribution and motion of galaxies, and if the transfer function is known one can deduce $\delta_H$.

The $l$th multipole of the cmb anisotropy probes the scale $k^{-1} \simeq 2/(H_0 l) = 6000 h^{-1} l^{-1}$ Mpc, corresponding to the distance subtended by angle $1/l$ at the surface of last scattering. Since this ‘surface’ is of order 10 Mpc thick the anisotropy will be wiped out on smaller scales, corresponding to $l \gtrsim 10^4$. It thus lies entirely within the linear regime. The mean-square multipoles seen by a randomly placed observer can be calculated in terms of $\delta_H^2 (k)$ at $k^{-1} \simeq 2/(H_0 l)$, given another transfer function encoding the effect of non-gravitational interactions. At present observations exist only for $l \lesssim 100$, corresponding to $k^{-1} \gtrsim 100$ Mpc.

The COBE measurement corresponds to multipoles $l \sim 30$, and gives an accurate determination of $\delta_H$ on the corresponding scales because the evolution is purely gravitational (dominated by the Sachs-Wolfe effect). On the scale $k \simeq 5H_0$ corresponding to the $l \sim 10$ multipoles which define the centre of the measured range one finds \[\delta_H = 1.94 \times 10^{-5}\] \hspace{1cm} (5)

with a 2σ uncertainty of 15%. This assumes that gravitational waves give a negligible contribution. Otherwise, the result is multiplied by a factor $(1 + r)^{-1/2}$ which can be calculated in a given model of inflation.

In order to arrive at an observational value for $n$ we need measurements of $\delta_H$ also on smaller scales. Here transfer functions are needed and they depend strongly on the

\[2\text{The spectrum } \mathcal{P}_f(k) \text{ of a quantity } f \text{ is the modulus-squared of its Fourier coefficient, times a factor which we choose so that } f \text{ has mean-square value } f^2 = \int_0^\infty \mathcal{P}_f dk / k.\]
cosmology. Since we are taking particle physics seriously it is reasonable to assume that the cosmological constant vanishes and (in the interests of simplicity) that the density is critical. Then the main uncertainties are the value of the Hubble constant \( H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1} \), the value of the baryon density \( \Omega_B \), and the nature of the dark matter.

The simplest assumption is pure cold dark matter. This case is viable at present \( h \simeq 0.5, \Omega_B \simeq 0.12 \) and

\[
0.7 \lesssim n \lesssim 0.9
\]

(6)

This assumption will be invalidated if a shoulder is observed in the spectrum, or if \( h \) or \( \Omega_B \) differ from the above values by more than 10% or so.

A more robust assumption is that there is an admixture of hot dark matter in the form of a single neutrino species \( \Omega_B < 0.15 \) and \( h > 0.4 \) a result at something like 95% confidence-level is \( 0.7 < n < 1.3 \)

(7)

The uncertainty in \( n \) comes mostly from the uncertainties in \( H_0, \Omega_B \) and the nature of the dark matter. The quoted lower bound comes from the cmb anisotropy in the region of the first Doppler peak and depends only on \( \Omega_B \), though other data point to a lower bound that is not much weaker. The upper bound depends strongly on \( h \) and the nature of the dark matter. With the same hypothesis about the dark matter one finds \( n < 1.04 \) if \( h > 0.5 \) and \( n < 0.9 \) if \( h > 0.6 \), a very dramatic improvement. The bounds on \( n \) under other hypotheses about the dark matter have not been explored in detail but one does not expect wildly different results since the effect of modifying pure cold dark matter is always to reduce the density perturbation on small scales.

The bounds on \( n \) become tighter if there are significant gravitational waves, but that is not the case in small-field models, nor is it the case in most other models that have been proposed \( \Omega_B \). In models where \( r \simeq 5(1 - n) \) (corresponding to an exponential potential, and approximately to a power-law potential) the lower bound becomes \( n > 0.8 \) \( \Omega_B \), which rules out extended inflation \( \Omega_B \) except for rather contrived versions \( \Omega_B \).

To get an idea of the eventual accuracy to be expected for \( n \), suppose that \( \delta_H \) has been measured at just two wavenumbers \( k \) and \( q \), with \( k > q \). If the values of \( \delta_H \) have fractional uncertainties \( \Delta_k \) and \( \Delta_q \), the uncertainty in \( n \) is

\[
\Delta n = (\Delta_k^2 + \Delta_q^2)^{1/2}/\ln(k/q)
\]

(8)

Other things being equal one will use the longest possible ‘lever arm’ \( k/q \), so the idealization of using only two wavenumbers is not too far from reality.

The longest useful lever arm at present is provided by \( k = (10h^{-1} \text{ Mpc})^{-1} \), the smallest scale that is linear now, and \( q = 5H_0 = (600h^{-1} \text{ Mpc})^{-1} \), the scale effectively probed by COBE. If each piece of data has a 10% uncertainty this gives \( \Delta n = .034 \). At present the uncertainties are \( \Delta_q = 15\% \) as quoted above, and \textit{excluding the uncertainty in the transfer function} \( \Delta_k = 30\% \) \( \Omega_B \). Thus we would already be not too far from the situation of 10% errors if only the cosmological parameters and the nature of the dark matter were understood. It seems quite reasonable to suppose that good progress in this direction will
have been made in a few years, giving an accuracy of order 10% on both pieces of data and an accuracy $\Delta n \simeq 0.04$ for the spectral index.

Looking for much higher accuracy, one cannot use data corresponding to multipoles of order 10. The reason is that the value of $\delta_H$ adduced even from perfect data is subject to cosmic variance, arising from the fact that the multipoles are measured only at our position. Using multipoles of order $l$ (over unit interval of $\ln l$) the uncertainty due to cosmic variance is of order $1/l$. To achieve something like 1% accuracy, one needs data centered on $l = 100$, corresponding to $q = (60h^{-1}\text{Mpc})^{-1}$. With 1% uncertainty also at $k = (10h^{-1}\text{Mpc})^{-1}$, one gets $\Delta n = 0.008$. We see that improving the accuracy by a factor 10 only increases the accuracy on $\Delta n$ by a factor 4, because of the necessarily shorter lever arm.

In the future one will be able to go to smaller scales, with the best determination of the spectral index coming from the cmb anisotropy with good angular resolution, but an accuracy better than $\Delta n \sim 0.01$ will be difficult to achieve if only because of the uncertainty in the cosmological parameters [42,44]. Fortunately, this accuracy turns out to be all that is needed to discriminate sharply between different models.

B. Scale-dependence of the spectral index?

If the power-law parametrization $\delta_H^2 \propto k^{n-1}$ is imperfect one can still define an effective spectral index by $n-1 \equiv d\ln \delta_H/d\ln k$ and it then makes sense to ask whether the variation of the spectral index will ever be observable.

In simple models $n-1$ is proportional to $1/N$ if it varies at all, where $N$ is the number of $e$-folds of inflation remaining when the relevant scale leaves the horizon, and we consider only that case.

Although the smallest scale on which $\delta_H(k)$ can be measured is at present about 10 Mpc, this limit will go down an order of magnitude or two in the future, making the total accessible range perhaps four decades ranging from $6000h^{-1}\text{Mpc}$ to say $0.6h^{-1}\text{Mpc}$. This would correspond to $\Delta \ln k = 4.5$ (measured from the centre of the range) and therefore to $\Delta N = 4.5$.

As we shall discuss in the next section, the number of $e$-folds after (say) the scale in the centre of this range leaves the horizon depends on the history of the universe after inflation, as well as the energy scale at which inflation ends. For a high (ordinary) inflationary energy scale $N \simeq 50$ is reasonable, but a low scale combined with an era of thermal inflation [57] will give $N \simeq 25$. One or more additional bouts of thermal inflation are not unlikely from a theoretical viewpoint, but since each one reduces $N$ by $\simeq 10$ there had better be at most one or two.

Assume first at most one bout of thermal inflation. Then the fractional changes in $n$ and $N$ are small so that

$$\frac{\Delta n}{|1-n|} = \frac{\Delta N}{N}$$

Let us suppose that the range is split in half to yield two separate determinations of $n$, corresponding to $\Delta N = 4.5$. Then one finds that

$$\frac{\Delta n}{.01} = \frac{|1-n| \cdot N}{0.11 \cdot 50}$$
From what has been said above, the scale dependence $\Delta n$ will have to be bigger than .01 if it is to be detectable in the foreseeable future. As a result its variation will be detectable only if it is quite far away from 1.

Now suppose that there are two bouts of thermal inflation so that $N$ varies from say 10 to 20 while cosmological scales leave the horizon. Then $n - 1$ will double over the observable range so that its variation might be detectable even if $|n - 1|$ is of order .01. Three bouts would of course have a dramatic effect, with four probably forbidden because cosmological scales would re-enter the horizon during thermal inflation which would lose the prediction for the density perturbation.

If the spectral index is strongly scale-dependent it is more useful to consider the spectrum itself. The case that we are considering, $n - 1 = q/N$, corresponds to $\delta_H^2 \propto N^q$. As discussed in Section VIII, the strongest scale dependence obtained from a reasonably motivated potential corresponds to $q = -4$, coming from a cubic potential. In order to produce enough early structure, $\delta_H$ should fall by no more than a factor 5 or so as one goes down through the four orders of magnitude available to observation \[14\], which with the cubic potential corresponds to requiring $N \gtrsim 11$ in the center of the range. Thus the cubic potential permits at most one additional bout of thermal inflation, at least with a low scale for ordinary inflation. Even if $N$ is not low enough to make it dramatic, the scale-dependence of $n$ in this model might still be enough to improve the viability of a pure cold dark matter model \[15\].

C. Observational constraints on very small scales

The above considerations apply only on cosmological scales. On smaller scales there are only upper limits on $\delta_H$. The most useful limit, from the viewpoint of constraining models of inflation, is the one on the smallest relevant scale which is the one leaving the horizon just before the end of inflation. It has been considered in Refs. \[16\] and \[22\], and according to the latter reference it corresponds to $\delta_H \lesssim 0.1$. In models where $n$ is constant this corresponds (using Eq. (5)) to $n - 1 < 0.33(50/N)$, which is no stronger than the constraint coming from cosmological scales. It represent a powerful additional constraint in models where $n$ increases on small scales, as discussed in an extreme case in Ref. \[22\].

III. SLOW-ROLL INFLATION

Before discussing particular models it will be useful to deal with some generalities. Practically all models are of the slow-roll variety \[4\]. Such models are the simplest, and at least while cosmological scales are leaving the horizon they are motivated by the observed fact that the spectrum has mild scale dependence, which is consistent with a power law. Until Section VII we focus on models in which the slow-rolling inflaton field $\phi$ is essentially unique.

During inflation the potential $V(\phi)$ is supposed to satisfy the flatness conditions $\epsilon \ll 1$ and $|\eta| \ll 1$, where

$$\epsilon \equiv \frac{1}{2} M_{Pl}^2 (V''/V)^2$$  \hspace{1cm} (11)
\[ \eta \equiv M_{\text{Pl}}^2 V''/V \]  

(12)

Given these conditions, the evolution

\[ \ddot{\phi} + 3H \dot{\phi} = -V' \]  

(13)

typically settles down to the slow-roll evolution

\[ 3H \dot{\phi} = -V' \]  

(14)

Slow roll and the flatness condition \( \epsilon \ll 1 \) ensure that the energy density \( \rho = V + \frac{1}{2} \dot{\phi}^2 \) is close to \( V \) and is slowly varying. As a result \( H \) is slowly varying, which implies that one can write \( a \propto e^{Ht} \) at least over a Hubble time or so.\(^3\) The flatness condition \( |\eta| \ll 1 \) then ensures that \( \dot{\phi} \) and \( \epsilon \) are slowly varying.

A crucial role is played by the number of Hubble times \( N(\phi) \) of inflation, still remaining when \( \phi \) has a given value. By definition \( dN = -H dt \), and the slow-roll condition together with the flatness condition \( \epsilon \ll 1 \) lead to

\[ \frac{dN}{d\phi} = -\frac{H}{\dot{\phi}} = \frac{V}{M_{\text{Pl}}^2 V'} \]  

(15)

This leads to

\[ N = \left| \int_{\phi_{\text{end}}}^{\phi} M_{\text{Pl}}^{-2} \frac{V}{V'} d\phi \right| \]  

(16)

The slow-roll paradigm is motivated, while cosmological scales are leaving the horizon, by the observed fact that \( \delta_H \) does not vary much on such scales. In typical models slow-roll persists until almost the end of inflation, but it can happen that it fails much earlier. The loop-correction model of Ref. \[48\] is the only one so far exhibited with this property, and as discussed there the only effect of the failure of slow roll is that one needs to replace the above expression for \( N(\phi) \) with something more accurate, when it is used to work out the value of \( \phi \) corresponding to cosmological scales.\(^4\)

A. The predictions

The quantum fluctuation of the inflaton field gives rise to an adiabatic density perturbation, whose spectrum is

\(^3\)Here and in what follows, slow variation of a function \( f \) of time means \( |d \ln f/d \ln a| \ll 1 \), and slow variation of a function \( g \) of wavenumber means that \( |d \ln g/d \ln k| \ll 1 \).

\(^4\)In principle one might be concerned that the quantum fluctuation during non-slow-roll inflation might become big and produce unwanted black holes. But in the slow-roll regime the spectrum \( \delta_H \) becomes smaller as the speed of rolling is increased (i.e., as \( \epsilon \) is increased) and it seems reasonable to suppose that this trend persists beyond the slow-roll regime.
\[ \delta_H^2(k) = \frac{1}{75\pi^2 M_{Pl}^6 V^3} \frac{V}{V'} = \frac{1}{150\pi^2 M_{Pl}^4} \epsilon \]  

(17)

In this expression, the potential and its derivative are evaluated at the epoch of horizon exit for the scale \( k \), which is defined by \( k = aH \).

This result was in essence derived at about the same time in Refs. [49–51,35]. To be precise, these authors give results which lead more or less directly to the desired one after the spectrum has been defined, though that last step is not explicitly made and except for the last work only a particular potential is discussed. (The last three works give results equivalent to the one we quote, and the first gives a result which is approximately the same.) Soon afterwards it was derived again, this time with an explicitly defined spectrum [34].

Inflation also generates gravitational waves, whose relative contribution to the mean-squared low multipoles of the cmb anisotropy is

\[ r = 5M_{Pl}^2 (V'/V)^2 \]  

(18)

The contribution of the gravitational waves was estimated roughly in Ref. [9], and in Ref. [10] a result equivalent to \( r = 6.2M_{Pl}^2 (V'/V)^2 \) was obtained under certain approximations. The more correct factor \( \simeq 5 \), for the multipoles most relevant for the determination of the COBE normalization of \( \delta_H \), was given in Refs. [11,12].

Returning to the spectrum of the density perturbation, comparison of the prediction (17) with the value deduced from the COBE observation of the cmb anisotropy gives [5], ignoring the gravitational waves

\[ M_{Pl}^{-3} V^{3/2} / V' = 5.3 \times 10^{-4} \]  

(19)

This relation provides a useful constraint on the parameters of the potential. It can be written in the equivalent form

\[ V^{1/4}/\epsilon^{1/4} = 6.7 \times 10^{16} \text{GeV} \]  

(20)

Since \( \epsilon \) is much less than 1, the inflationary energy scale \( V^{1/4} \) is at least a couple of orders of magnitude below the Planck scale [34].

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5Strictly speaking none of these five derivations is completely satisfactory. The first three make simplifying assumptions, and all except the third assume something equivalent to the constancy of \( \mathcal{R} \) without adequate proof. But as discussed in Section VII the constancy of \( \mathcal{R} \), and therefore the validity of the result, is easy to establish for a single-component inflaton. Also, none of them properly considers the effect of the inflaton field perturbation on the metric, but again this is easily justified to first order in the slow-roll approximation [3]. A little later a formalism was given that takes the metric perturbation into account [52,53], which formed the basis for the more accurate calculation of [13].

6In this context there is no point in including the effect of gravitational waves on the cmb anisotropy, since the prediction for \( \delta_H \) that is being used has an error of at least the same order. If necessary one could include the effect of the gravitational waves using the more accurate formula for \( \delta_H \) [13,14].
Our main focus is on the spectral index of the density perturbation. It may be defined by
\[ n - 1 \equiv 2 d \ln \delta_H / d \ln k. \]
Since the right hand side of Eq. (17) is evaluated when \( k = aH \), and since the rate of change of \( H \) is negligible compared with that of \( a \) we have \( d \ln k = H dt \). From the slow-roll condition \( dt = d\phi / \dot{\phi} = -(3H/V')d\phi \) and therefore
\[
\frac{d}{d \ln k} = -M_{Pl}^2 \frac{V'}{V} \frac{d}{d \phi}
\]  
This gives \([6–8]\) the formula advertised in the introduction,
\[ n - 1 = -6\epsilon + 2\eta \]  
More accurate versions of the formulas for \( \delta_H \) and \( n \) have been derived \([13]\) which provide estimates of the errors in the usual formulas. They are
\[ \frac{\Delta \delta_H}{\delta_H} = O(\epsilon, \eta) \]  
\[ \Delta n = O(\epsilon^2, \eta^2, \gamma^2) \]  
where \( \gamma \) is a third flatness parameter defined by \( \gamma^2 \equiv M_{Pl}^4 V''/V^2 \). We see that the usual formula \( n - 1 = -6\epsilon + 2\eta \) is valid (barring cancellations) if and only if there is a third flatness condition (in addition to \( \epsilon \ll 1 \) and \( |\eta| \ll 1 \)) which is
\[ \gamma^2 \ll \max\{\epsilon, \eta\} \]  
In typical models it is an automatically consequence of the other two, but it is in principle an additional requirement.
In most models, \( \epsilon \) and \( \gamma \) are both negligible compared with \( \eta \). Then \( n = 1 + 2\eta \), and
\[ \Delta n \sim |1 - n|^2 \]  
This theoretical error will be comparable with the eventual observational uncertainty (say \( \Delta n \sim .01 \)) if it turns out that \( |1 - n| \gtrsim 0.1 \), in which case the use of the more accurate formula would be needed to fully interpret the observation. However, as will become clear an accuracy \( \Delta n \) of order .01 is (in the models proposed so far) needed only if \( |1 - n| \lesssim 0.1 \). Thus the more accurate formula for the spectral index is not required for the purpose of distinguishing between models.

Differentiating the usual expression one finds \([54]\)
\[ \frac{d \ln(n - 1)}{d \ln k} = \left[ -16\epsilon\eta + 24\epsilon^2 + 2\gamma^2 \right]/(n - 1) \]  
Barring cancellations, we see that \( n - 1 \) varies little over an interval \( \Delta \ln k \sim 1 \) if and only if all three flatness conditions are satisfied. Cosmological scales correspond to only \( \Delta \ln k \sim 10 \), and in most models \( n - 1 \) varies little over this interval too. Then one can write \( \delta_H^2 \propto k^{n-1} \), the usual definition of \( n \). (A model where the two terms of \( n - 1 \) cancel, allowing it to vary significantly, is mentioned in Section V(D).)
B. A simpler formula for the spectral index

In almost all models proposed so far, $\epsilon$ is negligible so that one can write $n - 1 = 2\eta$.

Consider first the potential $V \simeq V_0(1 \pm \mu \phi^p)$, with the constant term dominating. We shall encounter potentials of this form frequently, and since the constant term dominates they give

$$\frac{\epsilon}{\eta} = \frac{p}{p-1} \frac{\mu}{2p} \phi^p \ll \frac{p}{p-1}$$

(28)

Except in the case $p \simeq 1$, we see that $\epsilon$ is indeed negligible in these models.

This result applies whether or not $\phi$ is small on the Planck scale. If it is small, one can argue that $\epsilon$ is negligible irrespectively of the form of the potential. To see this, use the estimate $\Delta \ln k \simeq 9$ for the range of cosmological scales. It corresponds to 9 $e$-folds of inflation. In slow-roll inflation the quantity $V'/V$ has negligible variation over one $e$-fold and in typical models it has only small variation over the 9 $e$-folds. Taking that to be the case, one learns from (15) that in small-field models the contribution of $(V'/V)$ to the spectral index is $3M_{Pl}^2 (V'/V)^2 \ll (1/9)^2 = .04$, which is indeed negligible.

This upper bound on $(V'/V)$ means [41] that the effect of gravitational waves on the cmb anisotropy will be too small ever to detect, because owing to cosmic variance the relative contribution $r \simeq 5M_{Pl}^2 (V'/V)^2$ has to be $\approx 0.1$ if it is to be detectable [55]. The effect is actually undetectable in most large-field models as well [11].

C. The number of $e$-folds

A model of inflation will give us an inflationary potential $V(\phi)$, and a prescription for the value $\phi_{end}$ of the field at the end of inflation. What we need to obtain the predictions is the value of $\phi$ when cosmological scales leave the horizon, which is given in terms of $N$ by Eq. (16). This quantity can in turn be determined if the history of the universe after inflation is known. Consider first the epoch when the scale $k^{-1} = H_0^{-1} = 3000h^{-1}$ Mpc leaves the horizon, which can be taken to mark the beginning of cosmological inflation. Using a subscript 1 to denote this epoch, $N_1 = \ln(a_{end}/a_1)$. Since this scale is the one entering the horizon now, $a_1 H_1 = a_0 H_0$ where the subscript 0 indicates the present epoch and

$$N_1 = \ln \left( \frac{a_{end} H_{end}}{a_0 H_0} \right) - \ln \left( \frac{H_{end}}{H_1} \right)$$

(29)

The second term will be given by the model of inflation and is usually $\lesssim 1$; for simplicity let us ignore it. The first term is known if we know the evolution of the scale factor between the

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7 This is roughly the biggest cosmologically interesting scale. The absolute limit of direct observation is $2H_0^{-1}$, the distance to the particle horizon. Since the prediction is made for a randomly placed observer in a much bigger patch, bigger scales in principle contribute to it, but sensitivity rapidly decreases outside our horizon. Only if the spectrum increases sharply on very large scales [56] might there be a significant effect.
end of cosmological inflation and the present. Assume first that the end of inflation gives way promptly to matter domination, which is followed by a radiation dominated era lasting until the present matter dominated era begins, one has

\[ N_1 = 62 - \ln(10^{16} \text{ GeV}/V_{\text{end}}^{1/4}) - \frac{1}{3} \ln(V_{\text{end}}^{1/4}/\rho_{\text{reh}}^{1/4}) \]  

\[ (\rho_{\text{reh}} \text{ is the ‘reheat’ temperature, when radiation domination begins.}) \]  

With \( V^{1/4} \sim 10^{16} \text{ GeV} \) and instant reheating this gives \( N_1 \simeq 62 \), the biggest possible value. However, \( \rho_{\text{reh}} \) should probably be no bigger than \( 10^{10} \text{ GeV} \) to avoid too many gravitinos, and using that value gives \( N_1 = 56 \), perhaps the biggest reasonable value. With \( V^{1/4} = 10^{10} \text{ GeV} \), the lowest scale usually considered, one finds \( N_1 = 47 \) with instant reheating, and \( N_1 = 39 \) if reheating is delayed to just before nucleosynthesis. If there is in addition an era of thermal inflation \([57]\) lasting about \( 10 e \)-folds this figure is reduced to \( N_1 \simeq 29 \). Subsequent eras of the thermal inflation are not particularly unlikely from a theoretical viewpoint, but since each era would subtract \( \simeq 10 \) from \( N_1 \) this would have dramatic observational consequences with more than two eras practically forbidden.

The smallest scale that will be directly probed in the foreseeable future is perhaps four orders of of magnitude lower than \( H_0^{-1} \), which corresponds to replacing \( N_1 \) by \( N_1 - 9 \). A central value, appropriate for use when calculating the spectral index, is therefore \( N \simeq N_1 - 4.5 \), and assuming at most one bout of thermal inflation it probably lies in the range

\[ 24 \lesssim N \lesssim 51 \]  

(31)

In the models that have been proposed, the predicted value of \( n - 1 \) is either independent of \( N \), or is proportional to \( 1/N \). In the latter case, taking \( N = 25 \) instead of the usual \( N = 50 \) or so doubles the predicted value of \( n - 1 \).

D. Hybrid and non-hybrid models of inflation

In non-hybrid models of inflation, the potential \( V(\phi) \) is dominated by the inflaton field \( \phi \). The potential has a minimum corresponding to the vacuum value (vev) of \( \phi \), at which \( V \) vanishes (or anyhow is much smaller than during inflation). Inflation ends when the minimum is approached, leading to a failure of one of the flatness conditions, and \( \phi \) then oscillates about its vev. In hybrid models, the potential \( V(\phi, \psi) \) is dominated by some other field \( \psi \), which is held in place by its interaction with \( \phi \). In ordinary hybrid models \( \psi \) is fixed, whereas in ‘mutated’ models it is slowly varying as it adjusts to minimize the potential at fixed \( \phi \), but either way we again end up with an effective potential \( V(\phi) \) during inflation. However, inflation now typically ends when the other field is destabilized, as \( \phi \) passes through some critical value \( \phi_c \), rather than by failure of the flatness conditions. After inflation ends both \( \phi \) and the other field oscillate about their vevs, though they might be very inhomogeneous depending on the model.

E. The power-series expansion

In the context of supergravity one expects the potential \( V \) to have an infinite power series expansion in the fields \([18]\). It vanishes at its minimum, which corresponds to the
vacuum expectation values (vevs) of the fields. Assuming for simplicity that odd terms are forbidden by some symmetry, the expansion for a single field with all others fixed at their vev will be of the form (with coefficients of either sign)

\[ V(\phi) = V_0 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4 + \lambda' M_{Pl}^{-2} \phi^6 + \cdots \]  

(32)

The minimum, at which \( V \) vanishes, corresponds to the vev of \( \phi \).

In the usual applications of particle theory, to such things as the standard model, neutrino masses, Peccei-Quinn symmetry or a GUT, one is interested only in field values \( \phi \ll M_{Pl} \). In this regime it is reasonable to expect the series to be dominated by one, two or at most three terms. To be precise, one expects the quadratic and/or the quartic terms to be important, plus at most one higher term. As a result there is good control over the form of the potential, and plenty of predictive power. It is attractive to suppose that inflation model-building lies in the same regime, and indeed one may even hope that the fields occurring in the inflation model are ones that already occur in some particle physics application.

As one considers bigger field values the justification for keeping only a few terms becomes weaker. For values of order \( M_{Pl} \) one may still argue for keeping only a few terms, giving for instance a vev of order \( M_{Pl} \). For field values one or more orders of magnitude bigger there is no theoretical argument for such an assumption. Equally there is none against it (except in the case of the moduli fields of superstring theory).

Returning to the small-field case, not all terms need be significant. In particular the \( \phi^4 \) term can be absent, or at least suppressed by factors like \((100 \text{ GeV} / M_{Pl})^2\) which make it completely negligible. If that happens, \( \phi \) is said to correspond to a flat direction in field space (see for example Ref. [57] and references cited there). As we shall note later a ‘flat’ direction in this sense is a good candidate for ordinary hybrid inflation. (Note though that this use of the word ‘flat’ is not directly related to the ‘flatness conditions’ \( \epsilon \ll 1 \) and \( |\eta| \ll 1 \) that are needed for slow-roll inflation per se.)

The above form of the expansion is appropriate if \( \phi \) is the modulus of a complex field charged under a \( U(1) \) (or higher) symmetry, the vevs of other fields charged under the symmetry being negligible. In general odd terms will appear, though the linear term is still forbidden if the origin is the fixed point of some lower symmetry, such as a subgroup \( Z_n \) of a \( U(1) \).

In non-hybrid models, where only the inflaton field is relevant, the above expansion is what we need, with \( \phi \) the inflaton field. In hybrid inflation, where the dominant contribution to the potential comes from a different field \( \psi \), we need a similar expansion in both fields simultaneously.

If two or more fields are significant we have to worry about the fact that in the context of supergravity one does not expect the fields to be canonically normalized. Rather, the kinetic term will be of the form

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8For a recent proposal for identifying the inflaton field with a flat direction of a supersymmetric extension of the standard model, see Ref. [22]; note though that the non-inflaton field is a distance of order \( M_{Pl} \) from its vacuum value in the models considered there.
\[ \mathcal{L}_{\text{kin}} = \frac{1}{2} \sum_{a,b} h_{ab} \partial_{\mu} \phi_a \partial^{\mu} \phi_b \]  

(33)

If the metric \( h_{ab} \) has vanishing curvature then one can redefine the fields to get the canonical normalization \( h_{ab} = \delta_{ab} \). In supergravity the curvature does not vanish, but it will be negligible in the small-field regime. For field values \( \gtrsim M_{\text{Pl}} \) it will become significant and in general must be taken into account. But even in a hybrid inflation model, the non-inflaton field is fixed (or nearly so) during inflation, and one can always impose canonical normalization on a single-component inflaton field even though that might not be the most natural choice in the context of supergravity. Unless otherwise stated, the inflationary potential \( V(\phi) \) is supposed to refer to a canonically normalized inflaton field in what follows.

In the above discussion we have in mind the case of Einstein gravity, corresponding to \( \mathcal{L} = -\frac{1}{2} M_{\text{Pl}}^2 R + \tilde{\mathcal{L}} \). The second term is the Lagrangian defining the particle theory, which we have in mind is some gauge theory formulated in the usual context of essentially flat spacetime. This is a reasonable assumption because the energy scale when cosmological scales leave the horizon during inflation is at least two orders of magnitude below the Planck scale. Modifications of Einstein gravity are sometimes considered though, such as replacing \( M_{\text{Pl}} \) by a field or adding an \( R^2 \) term. In these cases, and many others, one can recover Einstein gravity by redefining the fields and the spacetime metric, at the expense of making the non-gravitational part of the Lagrangian more complicated. In this way one typically arrives at a model of inflation with large fields and a non-polynomial potential, such as the one in Eq. (45). Models of this kind are in practice constructed without reference to supergravity, and ignore our lack of understanding of the large-field behaviour of the potential just as much as large-field models invoking say a power-law potential.

F. The difficulty of keeping \( \eta \) small in supergravity

In small-field models of inflation, where the theory is under control, it is reasonable to work in the context of supergravity. This is a relatively recent activity because although small-field models were the first to be proposed [24,60], they were soon abandoned in favour of models with fields first of order the Planck scale [61] and then much bigger [62]. Activity began again after hybrid inflation was proposed [15,16], with the realization [17] that the model is again of the small-field type. In Ref. [17] supersymmetric implementations of hybrid inflation were considered, in the context of both global supersymmetry and of supergravity.

According to supergravity, the potential can be written as a ‘\( D \) term’ plus an ‘\( F \) term’, and it is usually supposed that the \( D \) term vanishes during inflation. All models proposed so far are of this kind, except for hybrid inflation models with the \( D \) term dominating [24,63,64]; such models are quite special and are expected to give a distinctive \( n = 0.96 \) to \( 0.98 \) as briefly mentioned in Section V.

For models where the \( D \) term vanishes, the flatness parameter \( \eta = M_{\text{Pl}}^2 V''/V \) generically receives various contributions of order \( \pm 1 \). This crucial point was first emphasized in Ref. [17], though it is essentially a special case of the more general result, noted much earlier [65,66], that there are contributions of order \( \pm H^2 \) to the mass-squared of every scalar field. Indeed, in a small-field model the troublesome contributions to \( \eta \) may be regarded as contributions to the coefficient \( m^2 \) in the expansion (32) of the inflaton potential.
Since slow-roll inflation requires $|\eta| \ll 1$, either there is an accidental cancellation, or the form of the $F$ term is non-generic. An accident is reasonable only if $\eta$ is not too small, corresponding to $n = 1 + 2\eta$ not too close to 1.

Pending a full discussion in Section VII, the $\eta$ problem will be mentioned as appropriate when discussing the various models.

G. Before cosmological inflation

The only era of inflation that is directly relevant for observation is the one beginning when the presently observable universe leaves the horizon. This era of ‘cosmological inflation’ will undoubtedly be preceded by more inflation, but all memory of earlier inflation is lost apart from the starting values of the fields at the beginning of cosmological inflation. Nevertheless, one ought to try to understand the earlier era if only to check that the assumed starting values are not ridiculous.

A complete history of the universe will presumably start when the energy density is at the Planck scale. (Recall that $V^{1/4}$ is at least two orders of magnitude lower during cosmological inflation.) The generally accepted hypothesis is that the scalar fields at that epoch take on chaotically varying values as one moves around the universe, inflation occurring where they have suitable values $[62,1]$. It is indeed desirable to start descending from the Planck scale with an era of inflation for at least two reasons. One is to avoid the universe either collapsing or becoming practically empty, and the other is to have an event horizon so that the homogeneous patch within which we are supposed to live is not eaten up by its inhomogeneous surroundings. However, there is no reason to suppose that this initial era of inflation is of the slow-roll variety. The motivation for slow-roll comes from the observed fact that $\delta H$ is almost scale-independent, which applies only during the relatively brief era when cosmological scales are leaving the horizon. In the context of supergravity, where achieving slow-roll inflation requires rather delicate conditions, it might be quite attractive to suppose that non-slow-roll inflation takes the universe down from the Planck scale with slow-roll setting in only much later. A well known potential that can give non-slow-roll inflation is $V \propto \exp(\sqrt{2/p\phi/M_{Pl}})$, which gives $a \propto t^p$ and corresponds to non-slow-roll inflation in the regime where $p$ is bigger than 1 but not much bigger.

If slow-roll is established well before cosmological scales leave the horizon it is possible to have an era of ‘eternal inflation’ during which the motion of the inflaton field is dominated by the quantum fluctuation. For eternal inflation taking place at large field values see Ref. $[77]$. The corresponding phenomenon for inflation near a maximum was noted earlier by a number of authors. The condition for this to occur is that the predicted spectrum $\delta_H$ be formally bigger than 1 $[68]$.

Coming to the beginning of cosmological inflation, the possibilities are actually so varied that one can contemplate a wide range of initial field values. Going back in time, one might find a smooth inflationary trajectory going all the way back to an era when $V$ is at the Planck scale (or at any rate much bigger than its value during cosmological inflation). In that case the inflaton field will probably be decreasing during inflation. Another natural possibility is for the inflaton to find itself near a maximum of the potential before cosmological inflation starts. Then there may be eternal inflation followed by slow-roll inflation. If the maximum is a fixed point of the symmetries it is quite natural for the field to have been driven there.
by its interaction with other fields. Otherwise it could arrive there by accident, though this is perhaps only reasonable if the distance from the maximum to the minimum is \( \sim M_{\text{Pl}} \) (see for instance Ref. [69] for an example). In this latter case, the fact that eternal inflation occurs near the maximum may help to enhance the probability of inflation starting there.

### IV. NON-HYBRID MODELS

Non-hybrid models of inflation proposed so far fall into two broad classes. Either the inflaton field is of order \( 10 M_{\text{Pl}} \) when cosmological scales leave the horizon, moving towards the origin under a power-law potential \( V \propto \phi^p \), or it is \( \sim M_{\text{Pl}} \) and moving away from the origin under a potential \( V \approx V_0 (1 - \mu \phi^p) \). The first case is often called ‘chaotic’ inflation because it provides a way of descending from the Planck scale with chaotic initial field conditions. The second case is often called ‘new’ inflation because that was the name given to the first example of it.

#### A. Power-law and exponential potentials

The simplest potential giving inflation is

\[
V = \frac{1}{2} m^2 \phi^2 \tag{34}
\]

and obvious generalizations are \( V = \frac{1}{4} \lambda \phi^4 \), and \( V = \lambda M_{\text{Pl}}^4 \phi^p \) with \( p/2 \geq 3 \). Potentials of this form were proposed as the simplest realizations of chaotic initial conditions at the Planck scale [62].

Inflation ends at \( \phi_{\text{end}} \simeq p M_{\text{Pl}} \), after which \( \phi \) starts to oscillate about its vev \( \phi = 0 \). When cosmological scales leave the horizon \( \phi = \sqrt{2Np} M_{\text{Pl}} \). Since the inflaton field is of order 1 to \( 10 M_{\text{Pl}} \), there is no particle physics motivation for a power-law potential though equally there is no argument against it.

The model gives \( n - 1 = -(2 + p)/(2N) \), and gravitational waves are big enough to be eventually observable with \( r = 2.5p/N = 5(1 - n) - 2.5/N \). The COBE normalization corresponds to \( m = 1.8 \times 10^{13} \text{ GeV} \) for the quadratic case. For \( p = 4, 6, 8 \) it gives respectively \( \lambda = 2 \times 10^{-14} \), \( \lambda = 8 \times 10^{-17} \), \( \lambda = 6 \times 10^{-20} \) and so on. The energy scale is \( V^{1/4} \sim 10^{16} \text{ GeV} \) (corresponding to \( \epsilon \) not very small) but as we are not dealing with a broken symmetry there is no question of relating this to a GUT scale.

The same prediction is obtained for a more complicated potential, provided that it is proportional to \( \phi^p \) during cosmological inflation, and in particular \( \phi \) could have a nonzero vev \( \ll M_{\text{Pl}} \) [71, 24]. The more general case, where more than one power of \( \phi \) is important even while cosmological scales are leaving the horizon, has also been considered [72]. It can

\[ \text{\footnotesize 9} \]

The idea that inflation could start at a maximum of the potential through non-thermal effects was first promoted in Ref. [70]. If the maximum is a fixed point the field might alternatively be placed there through thermal corrections to the potential [59], but this ‘new inflation’ hypothesis is difficult to realize and has fallen out of favour now that alternatives have been recognized.
give rise to a variety of predictions \cite{3} for the scale-dependence of $\delta_H$ but of course requires a delicate balance of coefficients.

The limit of a high power is an exponential potential, of the form $V = \exp(\sqrt{2/q}\phi)$.

This gives $\epsilon = \eta/2 = 1/q$ which lead to $n - 1 = -2/q$ and $r = 10/q$. This is the case of ‘extended inflation’, where the basic interaction involves non-Einstein gravity but the exponential potential occurs after transforming to Einstein gravity \cite{3}. However, simple versions of this proposal are ruled out by observation, because the end of inflation corresponds to a first order phase transition, and in order for the bubbles not to spoil the cmb isotropy one requires $n \lesssim 0.75$ in contradiction with observation \cite{3,3}.

\section*{B. The inverted quadratic potential}

Another simple potential leading to inflation is \cite{70,19,69,20,74–80}

$$
V = V_0 - \frac{1}{2} m^2 \phi^2 + \cdots 
$$

(35)

We shall call this the ‘inverted’ quadratic potential, to distinguish it from the same potential with the plus sign which comes from the simplest version of hybrid inflation. The dots indicate the effect of higher powers, that are supposed to come in after cosmological scales leave the horizon. As long as they are negligible, $V_0$ has to dominate for inflation to occur.

This potential gives $1 - n = 2M^2_{Pl}m^2/V_0$. If $m$ and $V_0$ are regarded as free parameters, the region of parameter space permitting slow-roll inflation corresponds to $1 - n \ll 1$. Thus $n$ is indistinguishable from 1 except on the edge of parameter space. However, in the context of supergravity there are two reasons why the edge might be regarded as favoured. One is the fact that $\eta$ generically receives contributions of order 1. Since slow-roll inflation requires $|\eta| \ll 1$, either $\eta$ is somewhat reduced from its natural value by accident, or it is suppressed because the theory has a non-generic form. In the first case one does not expect $\eta$ to be tiny, and in the present context this means that $1 - n = 2\eta$ will not be tiny either.

The other reason for expecting $n$ to be significantly below 1, which is specific to this potential, has to do with the magnitude of the vev $\phi_m$. If the inverted quadratic form for the potential holds until $V_0$ ceases to dominate, one expects

$$
\phi_m \sim \frac{V_0^{1/2}}{m} = \left( \frac{2}{1 - n} \right)^{1/2} M_{Pl}
$$

(36)

(This is also an estimate of $\phi_{end}$ in that case.) To understand the potential within the context of particle theory the vev should not be more than a few times $M_{Pl}$, which requires $n$ to be well below 1.

We see that to have $n$ indistinguishable from 1, the potential should steepen drastically after cosmological scales leave the horizon. Even given that condition, one also requires that the smallness of $\eta$ is not an accident in the context of supergravity. So far no model has been proposed that satisfies both of these conditions, though a steepening of the potential with an accidental suppression of $\eta$ has recently been considered \cite{80}.

In the next section we shall also consider the inverted quadratic potential in the context of hybrid inflation, where some other field is responsible for $V_0$. In that case inflation ends
at some $\phi_{\text{end}} \ll M_{\text{Pl}}$, and one can be dealing with a version of supergravity that keeps $\eta$ tiny in which case $n$ will be indistinguishable from 1.

When cosmological scales leave the horizon $\phi \sim \phi_{\text{end}} e^{-x}$ where $x \equiv N(1-n)/2$. The COBE normalization gives

$$\frac{V_0^{1/2}}{M_{\text{Pl}}^2} = 5.3 \times 10^{-4} N^{-1} x e^{-x} \frac{\phi_{\text{end}}}{M_{\text{Pl}}}$$

(37)

For $\phi_{\text{end}} \sim M_{\text{Pl}}$ this gives $V_0^{1/4} \sim 5 \times 10^{14}$ GeV (using $n > 0.7$). But in a hybrid inflation model $\phi_{\text{end}}$ and $V_0^{1/4}$ could be much lower.

Staying with non-hybrid models, there have been several proposals for realizing the inverted quadratic potential without steepening. Since there is no steepening they will all give $n$ significantly below 1, and are not expected to occupy the regime $0.9 \lesssim n < 1$ populated by models to be discussed later. Let us list them briefly.

1. **Modular inflation**

If $\phi$ is the real or imaginary part of one of the superstring moduli, its potential is generally expected \cite{19,20} to be of the form $V = \Lambda^4 f(\phi/M_{\text{Pl}})$, with $f(x)$ and its derivatives roughly of order 1 in the regime $|x| \sim 1$. In that region one expects the flatness parameters $\eta \equiv M_{\text{Pl}}^2 V''/V$ and $\epsilon \equiv M_{\text{Pl}}^2 (V'/V)^2/2$ to be both roughly of order 1, and one might hope to find within it a region where they both happen to be significantly below 1 so that slow-roll inflation can occur. In the regime $|x| \gg 1$ one expects either a potential that rises too steeply to support inflation, or one that becomes periodic.

It is usually supposed that inflation takes place near a maximum. (As the maximum is not usually supposed to be a fixed point of a symmetry there is no reason why the interaction of the modulus should drive it there, but the existence of ‘eternal’ inflation may still be held to favour it as a starting point.) One then expects the potential to be of the inverted quadratic form, with no significant steepening after cosmological scales leave the horizon. So far, investigations using specific models \cite{19,76,81} have actually concluded that viable inflation does not occur.\cite{19}

Many complex fields other than the modulus are expected in the context of supersymmetry, and one could take $\phi$ to be the real or imaginary part of one of them. Although the potential for such fields is not understood for $\phi \gg M_{\text{Pl}}$ it is not unreasonable to suppose that there is a vev at $\phi \sim M_{\text{Pl}}$, and if so one can again expect an inverted quadratic form for the potential.

2. **‘Natural’ inflation**

The prediction for $n$ depends only on the form of the potential while cosmological scales are leaving the horizon. A potential reducing to the inverted quadratic one in this regime is

\footnote{Ref. \cite{82} claims to have been successful, but an analytic calculation of that model reported in Ref. \cite{83} finds that it is not viable.}
This potential typically arises if $\phi$ is the pseudo-Goldstone boson of a $U(1)$ symmetry broken by non-perturbative effects. Inflation using it was first discussed in Ref. [19], and then in Ref. [74] where it was called ‘natural’ inflation, and has subsequently been considered by several authors [69,75–77,47,84].

Suggestions as to the identity of $\phi$ have been that it is the imaginary part of a modulus such as the dilaton [19,76], the angular part of an ordinary complex field [74,76,69,77,47,84] or something else [75]. For any of them to be reasonable $M$ should be not too far above the Planck scale, corresponding to $\eta$ not too small and $n$ significantly below 1.

Even when $\eta$ is not tiny, one still feels more comfortable with a definite mechanism for keeping it well below 1 in the context of supergravity, and the $U(1)$ symmetry is such a mechanism provided that it is broken only by the superpotential (Section VII).

3. A loop-corrected potential

It was pointed out in Ref. [48] that a different mechanism for keeping $\eta$ small might be provided by a loop correction of the form $A\phi^2 \ln(\phi/B)$ [48]. Since the loop correction has to be large one needs to allow a renormalization group modification of it. The most reasonable model to emerge has an inverted quadratic potential without steepening, leading again to $n$ well below 1.

C. Cubic and higher potentials

If the quadratic term is heavily suppressed or absent, one will have

$$V \simeq V_0(1 - \mu \phi^p + \cdots)$$

(39)

with $p \geq 3$. For this potential one expects that the integral (14) for $N$ is dominated by the limit $\phi$ leading to (38)

$$\phi^{p-2} = [p(p-2)\mu NM_{Pl}^2]^{-1}$$

(40)

and

$$n \simeq 1 - 2 \left( \frac{p-1}{p-2} \right) \frac{1}{N}$$

(41)

It is easy to see that the integral is typically dominated by the $\phi$ limit, if higher terms in the potential (38) become significant only when $V_0$ ceases to dominate at $\phi^p \sim \mu^{-1}$. Then, in the regime where $V_0$ dominates, $\eta = [(p(p-1)M_{Pl}^2/\phi^2)\mu\phi^p$, and if this expression becomes of order 1 in that regime inflation presumably ends soon after. Otherwise inflation ends when $V_0$ ceases to dominate. At the end of inflation one therefore has $M_{Pl}^2 \mu\phi_{end}^{-2} \sim 1$ if $\phi_{end} \ll M_{Pl}$, otherwise one has $\mu\phi_{end}^p \sim 1$. (We are supposing for simplicity that $p$ is not enormous, and dropping it in these rough estimates.) The integral (14) is dominated by the limit $\phi$ provided that
This is always satisfied in the first case, and is satisfied in the second case provided that $\phi_{\text{end}} \ll \sqrt{N M_{\text{Pl}}}$ which we shall assume. If higher order terms come in more quickly than we have supposed, or if inflation ends through a hybrid inflation mechanism then $\phi_{\text{end}}$ will be smaller than these estimates, and one will have to see whether the criterion (42) is satisfied.

Continuing with the assumption that it is satisfied, the COBE normalization is $5.3 \times 10^{-4} = (p \mu M_{\text{Pl}})^{\frac{1}{p-1}} [N(p-2)]^{\frac{1}{p-1}} V_0^{\frac{1}{2}} M_{\text{Pl}}^{-2}$ (43)

In particular, for $p = 4$ the dimensionless coupling is $\lambda \equiv 4 V_0 \mu = 2.8 \times 10^{-13}(50/N)^3$.

As we noted earlier, a strong suppression of the quadratic term may well occur in a supergravity theory. An explicit example has been given recently \[45\]. In this proposal, one finds a potential

$$V = V_0(1 + \beta \phi^2 \psi - \gamma \phi^3 + \cdots)$$

where $\psi$ is another field. Then, with $\beta$ and $\gamma$ of order 1 in Planck units, and initial values $\psi \sim M_{\text{Pl}}$ and $\phi \simeq 0$ one can check that the quadratic term is driven to a negligible value before cosmological inflation begins.

This proposal gives $\phi$ a vev of order $M_{\text{Pl}}$. Some particle-physics motivation for small-field models with $p \geq 3$ is given in Refs. \[77,78\], though not in the context of supergravity.

One could contemplate models in which more than one power of $\phi$ is significant while cosmological scales leave the horizon, but this requires a delicate balance of coefficients. Models of this kind were also discussed a long time ago \[61,86\], again with a vev of order $M_{\text{Pl}}$, but their motivation was in the context of setting the initial value of $\phi$ through thermal equilibrium and has disappeared with the realization that this ‘new inflation’ mechanism is not needed. They could give a range of predictions for $n$.

D. Another potential

Potentials have been proposed that are of the form

$$V \simeq V_0(1 - e^{-q\phi/M_{\text{Pl}}})$$

with $q$ of order 1. This form is supposed to apply in the regime where $V_0$ dominates, which is $\phi \gtrsim M_{\text{Pl}}$. Inflation ends at $\phi_{\text{end}} \sim M_{\text{Pl}}$, and when cosmological scales leave the horizon one has

$$\phi = \frac{1}{q} \ln(q^2 N) M_{\text{Pl}}$$

$$n - 1 = -2 \eta = -2/N$$

21
Gravitational waves are negligible.

The most reasonable-looking derivation of a potential of this form in the context of particle theory [24] starts with a highly non-minimal kinetic term. In that case \( q \) can have different values such as 1 or \( \sqrt{2} \). Another derivation [5] modifies Einstein gravity by adding a large \( R^2 \) term to the usual \( R \) term, but with a huge coefficient, and a third [88] uses a variable Planck mass. In both cases, after transforming back to Einstein gravity one obtains the above form with \( q = \sqrt{2/3} \).

This potential is mimicked by \( V = V_0(1 - \mu \phi^{-p}) \) with \( p \rightarrow \infty \) (Table 1).

V. HYBRID INFLATION MODELS

In hybrid inflation models [15,16], the slowly rolling inflaton field \( \phi \) is not the one responsible for most of the energy density. That role is played by another field \( \psi \), which is held in place by its interaction with \( \phi \) until the latter reaches a critical value \( \phi_c \). When that happens \( \psi \) is destabilized and inflation ends. As was pointed out in Ref. [17], \( \phi_c \) is typically small in Planck units which means that hybrid inflation models are typically small-field models.

The end of inflation can correspond to either a first [15] or a second order [16] phase transition, but the second-order case (corresponding to the absence of a potential barrier) is simpler and is the only one that will be considered here. It has been considered by many authors [3,9,24,22,18,6,21,88,17,10,11,17]. (The first-order case amounts to an Einstein gravity version of extended inflation [4,5], and has been considered in Refs. [98,17]. For a non-Einstein version of second-order hybrid inflation see Ref. [19].)

A. Ordinary hybrid inflation

The potential for the original model of (second-order) hybrid inflation is [16]

\[ V = \frac{1}{4} \lambda (M^2 - \psi^2)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \lambda' \psi^2 \phi^2 \]  

\[ = V_0 - \frac{1}{2} m_\psi^2 \psi^2 + \frac{1}{4} \lambda \psi^4 + \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \lambda' \psi^2 \phi^2 \]  

The field \( \psi \) is fixed at the origin if \( \phi > \phi_c \), where

\[ \phi_c^2 = \frac{m^2}{\lambda' - \lambda M^2} \]  

In this regime slow-roll inflation can take place, with the quadratic potential

\[ V = V_0 + \frac{1}{2} m^2 \phi^2 \]  

In the context of particle theory one expects the couplings \( \lambda \) and \( \lambda' \) to be \( \lesssim 1 \). In order to have inflation at small \( \phi \) the first term must dominate after cosmological scales leave

\[ ^{11} \text{We take this to be the definition of the term 'hybrid', which is coming to be the standard usage though somewhat narrower than in Refs. [16,90] where it was first introduced.} \]
the horizon, because at the other extreme one arrives at $V \propto \phi^p$ which requires $\phi \sim 10$ to $20M_{\text{Pl}}$. To fully justify the low-order polynomial form of the potential one would also like to have $M \ll M_{\text{Pl}}$, which also turns out to be a requirement for inflation to end promptly at $\phi_c$. It is reasonable to also consider $M \sim M_{\text{Pl}}$, which has been done in Refs. [22,17].

The same inflationary potential is obtained with other forms for the last term of the potential (49), an example being [22] $\lambda M^{-2} \psi^2 \phi^4$, leading to

$$\phi_c^2 = M_{\text{Pl}} m \psi / \lambda' = M_{\text{Pl}} M \lambda^{1/2} / \lambda'$$

Other prescriptions for $\phi_c$ are provided by mutated hybrid inflation as described later.

When cosmological scales leave the horizon

$$\frac{\phi}{\phi_c} = e^{\frac{n-1}{2} N}$$

At least with the above two prescriptions for $\phi_c$ this implies that we are dealing with a model in which the inflaton field is small [17].

The quadratic inflationary potential gives [3] $n = 1 + 2M_{\text{Pl}}^2 m^2 / V_0$. From the flatness condition $\eta \ll 1$, the allowed region of parameter space corresponds to $n - 1 \ll 1$, so one expects $n$ to be indistinguishable from 1 unless there is some reason for the parameters to be near the edge of the allowed region. Two reasons might be cited for wanting to be near the edge, but neither is very compelling. One would be the fact that in $\eta \equiv M_{\text{Pl}}^2 V'' / V$ is of order 1 in a *generic* supergravity theory. If the smallness of $\eta$ is due to an accident it should not be too small. However, in contrast with the inverted quadratic potential discussed earlier, we are now dealing with a model in which the inflaton field is small. As discussed in Section VII this makes it more attractive to suppose that the smallness of $\eta$ is ensured from the start, in which case there is no reason why it should not be tiny. A more subtle consideration is the proposal of Ref. [22] that $m \sim m_\psi \sim 100$ GeV and $M \sim M_{\text{Pl}}$ are favoured values, which taken literally does indeed give $\eta \sim 1$. However the COBE normalization, to be discussed in a moment, would then require a precise choice of $\phi_c$, and in particular with either of the prescriptions one would need a precise value of $\lambda'$ which is not of order 1. Thus, while the suggested orders of magnitude may be reasonable as rough estimates, it cannot be said that the extreme edge of the allowed region is really favoured, and indeed of the six examples displayed in the Figures of Ref. [22] all except one have $n$ indistinguishable from 1.

The conclusion is that in the ordinary hybrid inflation model one expects $n$ to be indistinguishable from 1, though a value significantly above 1 is not out of the question.

The COBE normalization is

$$5.3 \times 10^{-4} = M_{\text{Pl}}^{-3} \frac{V_0^{3/2}}{m^2 \phi}$$

\[12\] Since the actual models used in Ref. [22] are formulated in the context of global supersymmetry the smallness of $\eta$ should presumably be viewed as an accident for them. But one can imagine that a different implementation of the proposal might keep $\eta$ small automatically and we have proceeded with the discussion on this assumption. If the smallness of $\eta$ is really accidental, very small values are disfavoured and one should reject the other five examples in favour of the last one. This point seems to have been overlooked in Ref. [22].
where $\phi$ is given by Eq. (53). With $\phi_c$ given by either of the above prescriptions this imposes a limit $n \lesssim 1.3$ (assuming that $V_0$ dominates the potential and that $M \lesssim M_{Pl}$).

What about higher powers of $\phi$ in the inflationary potential? They can be ignored if their contribution is insignificant when $\phi$ has the value given by (54). For a quartic term $\frac{1}{4} \lambda \phi^4$ this requires $\lambda \ll 10^{-7} (1 - n)^3$ which seems very small. However, the quartic term can be eliminated altogether, by identifying $\phi$ with one of the ‘flat’ directions of particle theory.

If a higher power $\phi^p$ does dominate we have

$$n = 1 + 2 \left( \frac{p - 1}{p - 2} \right) \frac{1}{N}$$

(55)

For $p = 4$ the COBE normalization requires a very small dimensionless coupling $\lambda \equiv 4 \mu V_0 \lesssim 10^{-12}$. This case is therefore less attractive than the $p = 2$ case. With higher powers, inflation can be interrupted and the requirement that the interruption is over before cosmological scales leave the horizon places an additional restriction on the parameter space. The case where quadratic, cubic and quartic terms may all be important has also been discussed, though the possibility of inflation being interrupted was ignored. As with the example discussed at the end of the next subsection (and the large-field model of Ref. [72]) there are enough parameters to allow a variety of possibilities for the scale dependence of $\delta_H(k)$, though a delicate balance between the parameters is required to achieve this. A linear term $p = 1$ is forbidden if the origin is a fixed point of appropriate symmetries, and even if a linear term is present the case that it dominates is somewhat artificial (see the example at the end of the next subsection). For the record, it gives a spectral index indistinguishable from 1 in a small-field model, because $V'' = 0$.

To summarize, ordinary hybrid inflation leading to a a quadratic inflationary potential $V(\phi)$ is a very attractive model. The full potential $V(\phi, \psi)$ need contain only quadratic and quartic terms, and need have no small couplings after the quartic term in $\phi$ has been eliminated by identifying $\phi$ with a ‘flat’ direction. Of course, whether the model can be successfully embedded in a more complete theory is a bigger question, which is not the subject of the present work, and is still a long way from being answered.

**B. Inverted hybrid inflation**

Instead of $\phi$ rolling towards the origin it might roll away from it. The simplest way of achieving this ‘inverted’ hybrid inflation is to have

$$V = V_0 - \frac{1}{2} m_\phi^2 \phi^2 + \frac{1}{2} m_\psi^2 \psi^2 - \frac{1}{2} \lambda \phi^2 \psi^2 + \frac{1}{4} \lambda_\phi \phi^4 + \frac{1}{4} \lambda_\psi \psi^4$$

(56)

There is supposed to be a minimum at nonzero $\phi$ and $\psi$, at which $V$ vanishes, but at fixed $\phi$ there is a minimum at $\psi = 0$ provided that

13 As was pointed out a long time ago, a potential of this kind with the fields in thermal equilibrium leads to high temperature symmetry restoration, the inverse of the usual case. The same thing is happening in our non-equilibrium situation.
\[ \phi < \phi_c = \frac{m_\psi}{\sqrt{\lambda}} \]  

(57)

In this regime one can have inflation with the inverted quadratic potential

\[ V = V_0 - \frac{1}{2} m^2 \phi^2 \]  

(58)

This is the same potential that we discussed already, with the hybrid inflation mechanism now ending inflation while the field is still small, instead of the steepening of the potential that has to be postulated in a non-hybrid model. The hybrid mechanism looks more natural, because it involves only quadratic and quartic terms, with no need for the dimensionless couplings of quartic terms to be small.

One could replace the quadratic term \(-\frac{1}{2} m^2 \phi^2\) by a higher order term, but the hybrid inflation mechanism would not then offer any simplification compared with the corresponding single-field models. If two or more terms are comparable while cosmological scales leave the horizon, there are more complicated possibilities including a significant variation of the spectral index. For example, one could have

\[ V = V_0 - \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4 \]  

(59)

This has a minimum at \(\phi_m = m/\sqrt{\lambda}\) and we suppose that \(V_0\) still dominates there. Then, if \(\phi_c \sim \phi_m\) the spectral index could flip between the values \(n = 1 \pm 2 M_{Pl}^2 m^2/V_0\) on cosmological scales. The number of \(e\)-folds taken to flip is \(\Delta N \sim M_{Pl}^2 (V/V')\phi_m \sim (\Delta n)^{-1}\) where \(\Delta n\) is the magnitude of \(1 - n\) before and after the flip. For the flip to occur with \(\Delta N < 10\) (the range of cosmological scales) requires \(\Delta n > 0.1\).

For \(\Delta n \lesssim 0.1\), the whole of cosmological inflation could take place with \(\phi\) midway between the maximum and minimum. In particular it could be near the point of inflexion so that the potential is practically linear. By choosing the same point as the origin we recover the case \(p = 1\) discussed earlier. We see that at least in this particular realization that case is either unnatural because the cosmological epoch has to occupy a special place on the trajectory, or is trivial because the entire trajectory (not just its linear part) gives \(n\) indistinguishable from 1.

C. Mutated hybrid inflation

In both ordinary and inverted hybrid inflation, the other field \(\psi\) is precisely fixed during inflation. If it varies, an effective potential \(V(\phi)\) can be generated even if the original potential contains no piece that depends only on \(\phi\). This mechanism was first proposed in Ref. [21], where it was called mutated hybrid inflation. The potential considered was

\[ V = V_0 - A\psi + B \psi^2 \phi^2 + C \psi^2 \]  

(60)

The last term serves only to give \(V\) a minimum at which it vanishes, and is negligible during inflation. All of the other terms are significant, with \(V_0\) dominating. For suitable choices of the parameters inflation takes place with \(\psi\) held at the instantaneous minimum, leading to a potential
Shortly afterwards the mechanism was rediscovered\cite{96} and called ‘smooth’ hybrid inflation emphasizing that any topological defects associated with $\psi$ will never be produced (in contrast with the case of ordinary and inverted hybrid inflation). The potential considered there was \( V = V_0 - A\psi^4 + B\psi^6 \phi^2 + C\psi^8 \), where again the last term is negligible during inflation and \( V_0 \) dominates the remainder. It leads to \( V = V_0 (1 - \mu \phi^{-2}) \).

Retaining the original name, the most general mutated hybrid inflation model with only two significant terms is\cite{83}

\[
V = V_0 - \frac{\sigma}{p} M_{Pl}^{4-p} \psi^p + \frac{\lambda}{q} M_{Pl}^{4-q-r} \psi^q \phi^r + \ldots
\]

In a suitable regime of parameter space, $\psi$ adjusts itself to minimize $V$ at fixed $\phi$, and $\psi \ll \phi$ so that the slight curvature of the inflaton trajectory does not affect the field dynamics. Then, provided that $V_0$ dominates the energy density, the effective potential during inflation is

\[
V = V_0 (1 - \mu \phi^{-2})
\]

where

\[
\mu = M_{Pl}^{4+\alpha} \left( \frac{q-p}{pq} \right) \frac{\sigma q^q \lambda^{\frac{p}{q-p}}}{V_0} > 0
\]

\[
\alpha = \frac{pr}{q-p}
\]

For $q > p$, the exponent $\alpha$ is positive as in the examples already mentioned, but for $p > q$ it is negative with $\alpha < -1$. In both cases it can be non-integral, though integer values are the most common for low choices of the integers $p$ and $q$.

The situation in the regime $-2 \lesssim \alpha < -1$ is similar to the one that we discussed already for the case $\alpha = -2$; the prediction for $n$ covers a continuous range below 1 because it depends on the parameters, but to have a small-field model the potential has to be steepened after cosmological scales leave the horizon. An example of such steepening is provided in the next subsection.

For choices of $\alpha$ outside the range $-2 \lesssim \alpha < 0$, the integral (66) is dominated by the limit $\phi$ provided that we are dealing with a small-field model, corresponding to

\[
\phi_{\text{end}} \simeq \mu^{1/\alpha} \ll M_{Pl}
\]

Then one has\cite{83}

\[
n = 1 - 2 \left( \frac{\alpha + 1}{\alpha + 2} \right) \frac{1}{N}
\]

This prediction is listed in the Table for some integer values of $\alpha$, along with the limiting cases $\alpha = \pm \infty$ and $\alpha = 0$.

Of the various possibilities regarding $\alpha$, some are preferred over others in the context of supersymmetry. One would prefer\cite{83} $q$ and $r$ to be even if $\alpha > 0$ (corresponding to $q > p$)
and $p$ to be even if $\alpha < 0$. Applying this criterion with $p = 1$ or 2 and $q$ and $r$ as low as possible leads \[83\] to the original mutated hybrid model, along with the cases $\alpha = -2$ and $\alpha = -4$ that we discussed earlier in the context of inverted hybrid and single-field models. In the original model (at least) the order of magnitude of the inflationary energy scale $V_0^{1/4}$ can be understood if supersymmetry is broken by gaugino condensation in a hidden sector.

A different example of a mutated hybrid inflation potential is given in Ref. \[47\], where $\psi$ is a pseudo-Goldstone boson with the potential \[38\]. Depending on the parameter values it might reduce in practice to a potential of the form discussed above, to one of the kind discussed in the next subsection or to something different.

D. Mutated hybrid inflation with explicit $\phi$ dependence

So far we have assumed that the original potential has no piece that depends only on $\phi$. If there is such a piece it has to be added to the inflationary potential \(63\). If it dominates while cosmological scales leave the horizon, the only effect that the $\psi$ variation has on the inflationary prediction is to determine $\phi_c$ through Eq. \(66\).

E. Hybrid inflation with a loop-corrected potential

In the context of supersymmetry one expects the correction to be of the form $A\phi^2 \ln(\phi/B)$ or $A\ln(\phi/B)$, depending on the mechanism of supersymmetry breaking during inflation. We discussed an application of the first case earlier, and now consider the second.

The prediction for $n$ with a loop correction of the form $A\ln(\phi/B)$ is the same as for a potential $V_0(1 - \mu \phi^{-\alpha})$ with $\alpha \simeq 0$, and as in the Table it is between .96 and .98.

It was first proposed \[94,95,97\] to give a slope to the flat classical potential coming out of a globally supersymmetric model that had been written down in Ref. \[17\]. The superpotential is

\[
W = \sigma(\Phi_1\Phi_2 + \Lambda^2)\Phi_3
\]

(68)

where $\sigma$ is a dimensionless coupling, $\Lambda$ is a mass scale and the $\Phi_n$ are complex fields. This gives the classical potential

\[
V = \sum_n |\frac{\partial W}{\partial \Phi_n}|^2
\]

(69)

With $\Phi_1 = \Phi_2 = 0$, $V$ has the constant value $\sigma^2 \Lambda^4$, and the loop correction gives it a small slope making $|\Phi_3|$ the inflaton of a hybrid inflation model.

As had been pointed out in the earlier reference, one expects that in this model supergravity corrections will give $\eta \equiv M_{Pl}^2|V''/V| \sim 1$, preventing inflation\[14\]. However, it has recently been noted \[83,84\] that a loop correction of the same form is likely to be responsible

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\[14\]Sufficient accidental suppression of $\eta$ is unlikely because the loop correction gives $\eta = .01$ to .02, and $W$ is not of a form that would guarantee the smallness of $\eta$ for a reasonable Kahler potential.
for the slope of a hybrid inflation model dominated by the $D$ term, of the type that was proposed at tree level in Ref. [24]. As pointed out in that reference, such a $D$ term model has the nice feature that there is no problem about keeping $\eta$ small.

VI. THE DIFFICULTY OF INFLATION MODEL-BUILDING IN SUPERGRAVITY

In this section the problem of keeping $\eta$ small in the context of supergravity is explained, and various proposed solutions to it are listed. Most of them have been encountered already.

A. The problem

The kinetic terms of the complex scalar fields $\Phi_n$ are given in terms of the Kahler potential $K$ (a real function of $\Phi_n$ and its complex conjugate $\bar{\Phi}_n$) by

$$\mathcal{L}_{\text{kin}} = (\partial_\mu \Phi_n) K_{\bar{n}m} (\partial^\mu \Phi_m)$$

Here and in the following expressions, a subscript $n$ denotes the derivative with respect to the $\Phi_n$ ($\bar{n}$ the derivative with respect to $\bar{\Phi}_n$) and a summation over repeated indices is implied. Following Ref. [24], focus on a given point on the inflationary trajectory, and choose it as the origin $\Phi_n = 0$. For the analysis leading to the usual slow-roll predictions to be valid we need the relevant fields to be canonically normalized in a small region around this point, which is ensured by choosing the fields so that

$$K_{\bar{n}m} = \delta_{nm} + O(\Phi_n^2)$$

This is analogous to using a locally inertial frame in general relativity. The ‘curvature scale’, beyond which higher order terms become significant, is expected to be of order the Planck scale $|\Phi_n| \sim M_{\text{Pl}}$.

The potential $V$ consists of a ‘$D$’ term and an ‘$F$’ term, and the problem arises when the $F$ term dominates, which is usually taken to be the case. The $F$ term involves the superpotential $W$ (a holomorphic function of the complex fields $\Phi_n$) and is

$$V = e^{K/M_{\text{Pl}}^2} \tilde{V}$$

where

$$\tilde{V} = (W_n + M_{\text{Pl}}^{-2}WK_n) K^{\bar{m}\bar{n}} (\bar{W}_m + M_{\text{Pl}}^{-2}\bar{W}K_m) - 3M_{\text{Pl}}^{-2}|W|^2$$

15Strictly speaking this choice ensures only that equations involving at most second derivatives of the fields are valid. It ensures the validity of the prediction for $\delta_H$ but not, strictly speaking, of the prediction for $n$. The correction analogous to $\gamma$ in Eqs. (24) and (27) will involve both $V'''$ and the higher order contributions to $K_{\bar{n}m}$. It could be worked out in a particular model from the general formalism of Ref. [103], specialized to a single field, and we are working on the assumption that it is negligible.
The matrix $K^{nm}$ is the inverse of $K_{nm}$.

This expression depends on $K$ and $W$ only through the combination $G \equiv K + \ln |W|^2$, so it is invariant under the transformation $K \rightarrow K - F - \bar{F}, \ W \rightarrow e^F W$ where $F$ is any holomorphic function. As a result one can choose $K$ and $W$ so that about the given point on the trajectory

$$K = \sum_n |\Phi_n|^2 + \cdots$$

The inflaton field $\phi$ can be chosen to be $2^{-1/2}$ times the real part of one of the $\Phi_n$, and one then finds

$$\eta \equiv M_{Pl}^2 V''/V = 1 + M_{Pl}^2 \tilde{V''}/\tilde{V}$$

The first term, which alone would give $\eta = 1$, comes from the factor $e^{K/M_{Pl}^2}$. To have $|\eta| \ll 1$ it must be cancelled by the second term. Examination of $\tilde{V}$ shows that there are indeed contributions of order 1, which may be positive or negative, but generically their total will not be $-1$ to high accuracy. They come from the quadratic term in the expansion (74) of $K$, and from the quartic $|\Phi|^4$ term through $K^{nm}$. Thus the minimal supergravity approximation, of keeping only the quadratic term, cannot be used in this context.

B. Solving the problem

How severe the problem is depends on the magnitude of $\eta$. If $\eta$ is not too small then its smallness could be due to accidental cancellations. Having $\eta$ not too small requires that $n - 1 = -6\epsilon + 2\eta$ be not too small (unless $3\epsilon \approx \eta$ which is not the case in models that have been proposed) so the observational bound $|n - 1| < .3$ is already beginning to make an accident look unlikely. An accidental cancellation is being assumed by proponents of modular inflation as discussed earlier.

A toy model for accidental cancellation would be the superpotential $W = A(\Phi - \Phi_0)^2$. The inflaton is supposed to be the real part of $\Phi$ and with minimal supergravity its mass-squared vanishes provided that $\Phi_0 = M_{Pl}$. Including the quartic term in $K$ the same thing will occur for some other value $\Phi_0 = B$, where $B$ depends on the coefficient of the quartic term but is still of order $M_{Pl}$. So $\eta$ will be accidentally suppressed if $|\Phi_0 - B| \ll M_{Pl}$.

Barring an accidental cancellation, the smallness of $\eta$ requires a non-generic form for the supergravity potential. With minimal supergravity, a simple choice that works is $W = V_0^{1/2} \Phi$, which leads to the global supersymmetry result, corresponding to an exactly flat potential $V = V_0$. Taking $W = V_0^{1/2} \Phi(1 - \frac{1}{2} \mu \Phi^p)$ could then give $V \approx V_0(1 - \mu \phi^p)$. Alternatively, taking $W$ of the form $W_{\Phi_0}$ could give $\eta$ hybrid inflation with $\Phi_1 = \Phi_2 = 0$, leading again to $W = V_0^{1/2} \Phi$ and an absolutely flat potential whose slope might come from a loop correction $[6,14,17]$. But the quartic contribution to $K$ is expected to spoil minimal supergravity, in which case these choices will not work.$^{16}$ In the context of nonminimal supergravity there are the following proposals, mostly involving things that have been mentioned earlier.

$^{16}$A possible intermediate strategy might be to use one of these forms to eliminate the contri-
1. The potential might be dominated by the $D$ term rather than by the $F$ term \[24\] in which case its variation will probably be dominated by a loop correction \[63,64\] of the form $A \ln(\phi/B)$, giving $n = 0.96$ to $0.98$.

2. One can impose restrictions on the form of the $F$ term \[24\], which are of a type that might emerge from superstring theory provided that it is in a perturbative regime. This scheme is sufficiently flexible that it can accomodate practically all versions of hybrid inflation \[24,21,83\], as well as the large kinetic term model of Ref. \[24\]. Thus it can accomodate more or less any measured value of $n$.

3. It has been known for a long time that supergravity models of the ‘no-scale’ type possess a ‘Heisenberg symmetry’ that can eliminate the mass-squared of order $H^2$ for generic scalar fields. The problem has been to stabilize the ‘Polonyi’ or (in the modern setting) modulus field occurring in such theories. It was known from the beginning that an ad hoc prescription can fix the modulus to give viable models of inflation, though the focus was not on small-field models (for reviews see Refs. \[86,1\] and for a different ad hoc prescription see Ref. \[102\]). Recently it has been noted \[105\] that a loop correction might provide the stabilization automatically. As yet no model of inflation based on this latter scheme has been proposed. Since the modulus will adjust itself to the current minimum of the potential during inflation \[106\] (instead of being absolutely fixed as in the early ad hoc schemes) one may be looking at something resembling a mutated hybrid inflation model.

4. The tree-level contribution to $\eta$ might be cancelled, within a limited interval of $\phi$, by a loop correction \[48\], leading to an inverted quadratic potential with $n$ considerably below 1.

5. Identifying the inflaton field with a pseudo-Goldstone boson keeps its potential absolutely flat in the limit of unbroken symmetry. Explicitly breaking the symmetry \[74,76,78,47,84\] then gives a nonzero $\eta$, which can be small if the symmetry is broken only by $W$ and not by $K$, or if most of the potential comes from another field \[84\]. This solution to the problem would probably give $n$ significantly below 1 in the former case, but indistinguishable from 1 in the latter.

6. The form of the potential $V(\phi)$ might depend on some other field, which is driven to a value corresponding to negligible $\eta$ before cosmological inflation starts. This is the proposal of Ref. \[45\] mentioned in Section IV, leading to $n = 0.84$ to $0.92$. For the proposal to work one still needs a mechanism for keeping the potential flat in the

| \[86x710]1. The potential might be dominated by the $D$ term rather than by the $F$ term \[24\] in which case its variation will probably be dominated by a loop correction \[63,64\] of the form $A \ln(\phi/B)$, giving $n = 0.96$ to $0.98$.  
2. One can impose restrictions on the form of the $F$ term \[24\], which are of a type that might emerge from superstring theory provided that it is in a perturbative regime. This scheme is sufficiently flexible that it can accomodate practically all versions of hybrid inflation \[24,21,83\], as well as the large kinetic term model of Ref. \[24\]. Thus it can accomodate more or less any measured value of $n$.  
3. It has been known for a long time that supergravity models of the ‘no-scale’ type possess a ‘Heisenberg symmetry’ that can eliminate the mass-squared of order $H^2$ for generic scalar fields. The problem has been to stabilize the ‘Polonyi’ or (in the modern setting) modulus field occurring in such theories. It was known from the beginning that an ad hoc prescription can fix the modulus to give viable models of inflation, though the focus was not on small-field models (for reviews see Refs. \[86,1\] and for a different ad hoc prescription see Ref. \[102\]). Recently it has been noted \[105\] that a loop correction might provide the stabilization automatically. As yet no model of inflation based on this latter scheme has been proposed. Since the modulus will adjust itself to the current minimum of the potential during inflation \[106\] (instead of being absolutely fixed as in the early ad hoc schemes) one may be looking at something resembling a mutated hybrid inflation model.  
4. The tree-level contribution to $\eta$ might be cancelled, within a limited interval of $\phi$, by a loop correction \[18\], leading to an inverted quadratic potential with $n$ considerably below 1.  
5. Identifying the inflaton field with a pseudo-Goldstone boson keeps its potential absolutely flat in the limit of unbroken symmetry. Explicitly breaking the symmetry \[74,76,78,47,84\] then gives a nonzero $\eta$, which can be small if the symmetry is broken only by $W$ and not by $K$, or if most of the potential comes from another field \[84\]. This solution to the problem would probably give $n$ significantly below 1 in the former case, but indistinguishable from 1 in the latter.  
6. The form of the potential $V(\phi)$ might depend on some other field, which is driven to a value corresponding to negligible $\eta$ before cosmological inflation starts. This is the proposal of Ref. \[45\] mentioned in Section IV, leading to $n = 0.84$ to $0.92$. For the proposal to work one still needs a mechanism for keeping the potential flat in the |
direction of the other field (a global $U(1)$ in the above model) so it might become the inflaton in a different regime of parameter space.

C. Inflation as a probe of supergravity

From these considerations we see that inflation is a very powerful probe of supergravity. In most models, the $F$ term dominates the potential, and $\eta$ generically receives various contributions of order 1 which must cancel. To keep it small requires either an accidental cancellation (reasonable only if $\eta$ is not too small) or non-generic forms for $K$ and $W$. In constructing suitable forms, it is not permissible to make the ‘minimal supergravity’ approximation of ignoring the higher order terms in the expansion (74) of $K$.

Having ensured the smallness of $\eta$, one is not in general left with the global supergravity result $V = \sum |W_n|^2$. It does indeed hold for solution 2 of the problem, but both $W$ and $K$ need to have very special forms.

Finally, if the $D$ term dominates there is no problem about keeping $\eta$ small, but to require this is itself a strong constraint on the model.

VII. INFLATION WITH A MULTI-COMPONENT INFATON

So far we have assumed that the slow-rolling inflaton field is essentially unique. What does ‘essentially’ mean in this context? A strictly unique inflaton trajectory would be one lying in a steep-sided valley in field space. This is not very likely in a realistic model. Rather there will be a whole family of possible inflaton trajectories, lying in the space of two or more real fields $\phi_1, \phi_2, \cdots$. In this case, it quite useful to think of the inflaton field as a multi-component object with components $\phi_a$. But even for a multi-component inflaton field it may well be the case that different choices of trajectory lead to the same universe, and if that is the case we still have an ‘essentially’ unique inflaton field.

A familiar example of the essentially unique case is if the inflaton field is the modulus of a complex field charged under a global $U(1)$ symmetry. Then the possible inflaton trajectories are the radial lines and the $U(1)$ symmetry ensures that all trajectories correspond to identical universes during inflation. Moreover, the transition to a universe composed of matter plus radiation will be the same for all of them. Thus the inflaton field is essentially unique.

More generally, the inflaton field will be essentially unique if the inflaton trajectories are practically straight in the space of canonically normalized fields, and if also the transition from inflation to a matter/radiation universe is the same for each trajectory.

The essentially non-unique case is when different possible trajectories correspond to differently evolving universes. Then trajectories near the classical one cannot be ignored, because the quantum fluctuation kicks the inflaton field onto them in a random way. From now on, ‘multi-component’ will refer to this case.

Even in the single-component case there is a quantum fluctuation, which kicks the inflaton back and forth along the classical trajectory. This causes an adiabatic density perturbation, and in general the effect of the orthogonal fluctuation onto nearby trajectories
is to cause an additional adiabatic density perturbation. (Exceptionally it might cause an isocurvature perturbation as mentioned at the end of this section.)

Multi-component inflaton models generally have just two components, and are called double inflation models because the trajectory can lie first in the direction of one field, then in the direction of the other. They were first proposed in the context of non-Einstein gravity \[107–114\]. By redefining the fields and the spacetime metric one can recover Einstein gravity, with fields that are not small on the Planck scale and in general non-canonical kinetic terms and a non-polynomial potential. Then models with canonical kinetic terms were proposed \[27–30,11,115–121\], with potentials such as \( V = \lambda_1 \phi_1^n + \lambda_2 \phi_2^q \). These potentials too inflate in the large-field regime where theory provides no guidance about the form of the potential. However there seems to be no bar to having a small-field multi-component model, and one may yet emerge in a well-motivated particle theory setting. In that case a hybrid model might emerge, though the models proposed so far are all of the non-hybrid type (ie., the multi-component inflaton is entirely responsible for the potential).

In this brief survey we have focussed on the era when cosmological scales leave the horizon. In the hybrid inflation model of Ref. \[22,47\], the ‘other’ field is responsible for the last several \( e \)-folds of inflation, so one is really dealing with a two-component inflaton (in a non-hybrid model). The scales corresponding to the last few \( e \)-folds are many orders of magnitude shorter than the cosmological scales, but it turns out that the perturbation on them is big so that black holes can be produced. This phenomenon was investigated in Refs. \[22,47\]. The second reference also investigated the possible production of topological defects, when the first field is destabilized.

A. Calculating the spectrum of the curvature perturbation

We noted at the beginning of Section II that the adiabatic density perturbation on scales well outside the horizon is specified by a quantity \( \mathcal{R} \), which defines the curvature of comoving hypersurfaces. Going back in time from the epoch of horizon entry, it soon achieves a constant value, which is maintained at least until the beginning of the radiation-dominated era preceding the present matter-dominated one, and unless otherwise stated \( \mathcal{R} \) denotes this constant value. On the assumption that it is a gaussian random field, which is generally the case if it is generated by a vacuum fluctuation of the inflaton field, its stochastic properties are completely determined by its spectrum \( P_\mathcal{R} \).

In this section we will see how to calculate the spectrum, first for a single-component inflaton and then for a multi-component one. In both cases we use an approach that has only recently been developed \[121,122\], though its starting point can already be seen in the first derivations of the spectrum \[49–51\]. This is the assumption that after smoothing on a scale well outside the horizon (Hubble distance) the evolution of the universe along each comoving worldline will be practically the same as in a Robertson-Walker universe.\footnote{‘Smoothing’ on a scale \( R \) means that one replaces (say) \( \rho(x) \) by \( \int d^3x' W(|x' - x|)\rho(x') \) with \( W(y) \approx 1 \) for \( y \lesssim R \) and \( W \approx 0 \) for \( y \gtrsim R \). A simple choice is to take \( W = 1 \) for \( y < R \) and \( W = 0 \) for \( y > R \) (top-hat smoothing).}
One always makes such an assumption when doing cosmological perturbation theory. When a quantity is split into an average plus a perturbation, the average is identified with the ‘background’ quantity that is supposed to correspond to a Robertson-Walker universe (i.e., to an absolutely homogeneous and isotropic one). And it is accepted without question that the criterion for the size of the averaging region is that it be much bigger than the horizon.

The averaging scale will of course be chosen to be a comoving one. When splitting a quantity into an average and a perturbation it is usually taken to be much bigger than the one corresponding to the whole presently observable universe. But in the early universe it makes equal sense to make it much smaller. This is indeed often done implicitly (sometimes explicitly [123]) for a single small region, namely the one around us. The crucial idea behind the present approach [121,122] is to recognize that the comparison of different regions provides a simple and powerful technique for calculating the density perturbation.

As we discuss later, it is quite different from the usual one of writing down, and then solving, a closed set of equations for the perturbations in the relevant degrees of freedom (for instance the components of the inflaton field during inflation). Roughly speaking the present approach replaces the sequence ‘perturb then solve’ by the far simpler sequence ‘solve then perturb’, though it is actually more general than the other approach. For the case of a single-component inflaton it gives a very simple, and completely general, proof of the constancy of $R$ on scales well outside the horizon. For the multi-component case it allows one to follow the evolution of $R$, knowing only the evolution of the unperturbed universe corresponding to a given value of the initial inflaton field. So far it has been applied to three multi-component models [121,114,47].

B. The case of a single-component inflaton

We begin with a derivation of the usual result for the single-component case. The assumption about the evolution along each comoving worldline is invoked only at the very end, when it is used to establish the constancy of $R$ which up till now has only been demonstrated for special cases. Otherwise the proof is the standard one [3], but it provides a useful starting point for the multi-component case.

A few Hubble times after horizon exit during inflation, when $R$ can first be regarded as a classical quantity, its spectrum can be calculated using the relation [33,3]

$$R(x) = H\Delta\tau(x)$$

where $\Delta\tau$ is the separation of the comoving hypersurface (with curvature $\mathcal{R}$) from a spatially flat one coinciding with it on average. The relation is generally true, but we apply it at an epoch a few Hubble times after horizon exit during inflation.

On a comoving hypersurface the inflaton field $\phi$ is uniform, because the momentum density $\dot{\phi}\nabla\phi$ vanishes. It follows that

$$\Delta\tau(x) = -\delta\phi(x)/\dot{\phi}$$

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**In [3] there is an incorrect minus sign on the right hand side.**
where now $\delta \phi$ is defined on the flat hypersurface. Note that the comoving hypersurfaces become singular (infinitely distorted) in the slow-roll limit $\dot{\phi} \to 0$, so that to first order in slow-roll any non-singular choice of hypersurface could actually be used to define $\delta \phi$.

The spectrum of $\delta \phi$ is calculated by assuming that well before horizon entry (when the particle concept makes sense) $\delta \phi$ is a practically free field in the vacuum state. Using the flatness and slow-roll conditions one finds, a few Hubble times after horizon exit, the famous result $P_\phi = (H/2\pi)^2$, which leads to the usual formula (17) for the spectrum.

However, this result refers to $R$ a few Hubble times after horizon exit, and we need to check that $R$ remains constant until the radiation dominated era where we need it. To calculate the rate of change of $R$ we proceed as follows [124,3].

In addition to the energy density $\rho$ and the pressure $P$, we consider a locally defined Hubble parameter $H = \frac{1}{3} D_\mu u^\mu$ where $u^\mu$ is the four-velocity of comoving worldlines and $D_\mu$ is the covariant derivative. Our $3H$ is often called $\theta$ in the literature. The universe is sliced into comoving hypersurfaces, and each quantity is split into an average (‘background’) plus a perturbation, $\rho(x,t) = \rho + \delta \rho(x,t)$ and so on. (We use the same symbol for the local and the background quantity since there is no confusion in practice.) As usual, $x$ is the Cartesian position-vector of a comoving worldline and $t$ is the time. As we are working to first order in the perturbations they ‘live’ in unperturbed space-time.

The locally defined quantities satisfy [124,34,125,3]

$$H^2 = M_{Pl}^2 \rho/3 - \frac{2 k^2}{3 a^2 R}$$  \hspace{1cm} (78)

(The equation is valid for each Fourier mode, and also for the full quantities with the identification $(k/a)^2 = -\nabla^2$. From now on that is taken for granted.) This is the Friedmann equation except that $K \equiv (2/3)k^2R$ need not be constant. The evolution along each world-line is

$$\frac{d\rho}{d\tau} = -3H(\rho + P)$$  \hspace{1cm} (79)

$$\frac{dH}{d\tau} = -H^2 - \frac{1}{2} M_{Pl}^{-2}(\rho + 3P) + \frac{1}{3} \frac{(k/a)^2 \delta P}{\rho + P}$$  \hspace{1cm} (80)

Except for the last term these are the same as in an unperturbed universe. If it vanishes $R$ is constant, but otherwise one finds

$$\dot{R} = -H \delta P/(\rho + P)$$  \hspace{1cm} (81)

In this equation we have in mind that $\rho$ and $P$ are background quantities, though as we are working to first order in the perturbations it would make no difference if they were the locally defined quantities.

The equation shows that $R$ will be constant if $\delta P$ is negligible. We now show that this is so, by first demonstrating that $\delta \rho$ is negligible, and then using the new viewpoint to see that $P$ will be a practically unique function of $\rho$ making $\delta P$ also negligible.

Extracting the perturbations from Eq. (78) gives

$$2 \frac{\delta H}{H} = \frac{\delta \rho}{\rho} - \frac{2}{3} \left( \frac{k}{aH} \right)^2 R$$  \hspace{1cm} (82)
This allows one to calculate the evolution of $\delta \rho$ from Eq. (79), but we have to remember that the proper-time separation of the hypersurfaces is position-dependent. Writing $\tau(x, t) = t + \delta \tau(x, t)$ we have \[33, 124, 126\]

$$\delta(\dot{\tau}) = -\delta P/\left(\rho + P\right) \tag{83}$$

Writing $\delta \rho/\rho \equiv (k/aH)^2 Z$ one finds \[125\]

$$\left(fZ\right)' = f(1 + w)\mathcal{R} \tag{84}$$

Here a prime denotes $d/d(\ln a)$ and $f'/f \equiv (5 + 3w)/2$ where $w \equiv P/\rho$. With $w$ and $\mathcal{R}$ constant, and dropping a decaying mode, this gives

$$Z = \frac{2 + 2w}{5 + 3w}\mathcal{R} \tag{85}$$

More generally, integrating Eq. (84) will give $|Z| \sim |\mathcal{R}|$ for any reasonable variation of $w$ and $\mathcal{R}$. Even for a bizarre variation there is no scale dependence in either $w$ (obviously) or in $\mathcal{R}$ (because Eq. (91) gives it in terms of $\delta P$, and we will see that if $\delta P$ is significant it is scale-independent). In all cases $\delta \rho/\rho$ becomes negligible on scales sufficiently far outside the horizon.

The discussion so far applies to each Fourier mode separately, on the assumption that the corresponding perturbation is small. To make the final step, of showing that $\delta P$ is also negligible, we need to consider the full quantities $\rho(x, t)$ and so on. But we still want to consider only scales that are well outside the horizon, so we suppose that all quantities are smoothed on a comoving scale somewhat shorter than the one of interest. The smoothing removes Fourier modes on scales shorter than the smoothing scale, but has practically no effect on the scale of interest.

Having done this, we invoke the assumption that the evolution of the universe along each worldline is practically the same as in an unperturbed universe. In the context of slow-roll inflation, this means that the evolution is determined by the inflaton field at the ‘initial’ epoch a few Hubble times after horizon exit. To high accuracy, $\rho$ and $P$ are well defined functions of the initial inflaton field and if it has only one component this means that they are well defined functions of each other. Therefore $\delta P$ will be very small on comoving hypersurfaces because $\delta \rho$ is. \[19\]

Finally, we note for future reference that $\delta H$ is also negligible because of Eq. (82).

C. The multi-component case

It is assumed that while cosmological scales are leaving the horizon all components of the inflaton have the slow-roll behaviour,

\[19\] If $k/a$ is the smoothing scale, the assumption that the evolution is the same as in an unperturbed universe with the same initial inflaton field has in general errors of order $(k/aH)^2$. In the single-component case, where $\delta P$ is also of this order, we cannot use the assumption to actually calculate it, but neither is it of any interest.
3Hφ_a = −V_a \tag{86}

(The subscript ,a denotes the derivative with respect to φ_a.) Differentiating this and comparing it with the exact expression \ddot{φ}_a + 3H \dot{φ}_a + V_{,a} = 0 gives consistency provided that

\begin{align}
M_{Pl}^2(V_{,a}/V)^2 &\ll 1 \tag{87} \\
M_{Pl}^2|V_{,ab}/V| &\ll 1 \tag{88}
\end{align}

(The second condition could actually be replaced by a weaker one but let us retain it for simplicity.) One expects slow-roll to hold if these flatness conditions are satisfied. Slow-roll plus the first flatness condition imply that \( H \) (and therefore \( \rho \)) is slowly varying, giving quasi-exponential inflation. The second flatness condition ensures that \( \dot{φ}_a \) is slowly varying.

It is not necessary to assume that all of the fields continue to slow-roll after cosmological scales leave the horizon. For instance, one or more of the fields might start to oscillate, while the others continue to support quasi-exponential inflation, which ends only when slow-roll fails for all of them. Alternatively, the oscillation of some field might briefly interrupt inflation, which resumes when its amplitude becomes small enough. (Of course these things might happen while cosmological scales leave the horizon too, but that case will not be considered.)

The expression (76) for \( \mathcal{R} \) still holds in the multi-component case. Also, one still has \( \Delta \tau = -\delta \phi / \dot{\phi} \) if \( \delta \phi \) denotes the component of the vector \( \delta \phi_a \) parallel to the trajectory. (The momentum density seen by an observer orthogonal to an arbitrary hypersurface is \( \dot{\phi}_a \nabla \phi_a \).) A few Hubble times after horizon exit the spectrum of every inflaton field component, in particular the parallel one, is still \( (H/2\pi)^2 \). If \( \mathcal{R} \) had no subsequent variation this would lead to the usual prediction, but we are considering the case where the variation is significant. It is given in terms of \( \delta P \) by Eq. (81), and when \( \delta P \) is significant it can be calculated from the assumption that the evolution along each worldline is the same as for an unperturbed universe with the same initial inflaton field. This will give

\[ \delta P = P_a \delta \phi_a \tag{89} \]

where \( \delta \phi_a \) is evaluated at the initial epoch and the function \( P(\phi_1, \phi_2, \cdots, t) \) represents the evolution of \( P \) in an unperturbed universe. Choosing the basis so that one of the components is the parallel one, and remembering that all components have spectrum \( (H/2\pi)^2 \), one can calculate the final spectrum of \( \mathcal{R} \). The only input is the evolution of \( P \) in the unperturbed universe corresponding to a generic initial inflaton field (close to the classical initial field).

In this discussion we started with Eq. (76) for the initial \( \mathcal{R} \), and then invoked Eq. (81) to evolve it. The equations can actually be combined to give

\[ \mathcal{R} = \delta N \tag{90} \]

where \( N = \int H d\tau \) is the number of Hubble times between the initial flat hypersurface and the final comoving one on which \( \mathcal{R} \) is evaluated. This remarkable expression was given in Ref. [107] and proved in Refs. [121,122]. The approach we are using is close to the one in the last reference.

The proof that Eqs. (76) and (81) lead to \( \mathcal{R} = \delta N \) is very simple. First combine them to give
\[ R(x, t) = H_1 \Delta \tau_1(x) - \int_{t_1}^{t} H(t) \frac{\delta P}{p + P} dt \]  

where \( t_1 \) is a few Hubble times after horizon exit. Then use Eq. (83) to give

\[ R(x, t) = H_1 \Delta \tau_1(x) + \int_{t_1}^{t} H(t) \delta \dot{\tau}(x, t) dt \]  

Next note that because \( \delta H \) is negligible this can be written

\[ R(x, t) = H_1 \Delta \tau_1(x) + \delta \int_{t_1}^{t} H(x, t) \dot{\tau}(x, t) dt \]  

Finally redefine \( \tau(x, t) \) so that it vanishes on the initial flat hypersurface, which gives the desired relation \( R = \delta \dot{N} \).

In Ref. [122] this relation is derived using an arbitrary smooth interpolation of hypersurfaces between the initial and final one, rather than by making the sudden jump to a comoving one. Then \( H \) is replaced by the corresponding quantity \( \tilde{H} \) for worldlines orthogonal to the interpolation (incidentally making \( \delta \tilde{H} \) non-negligible). One then finds \( R = \delta \tilde{N} \).

One also finds that the right hand side is independent of the choice of the interpolation, as it must be for consistency. If the interpolating hypersurfaces are chosen to be comoving except very near the initial one, \( \tilde{N} \simeq N \) which gives the desired formula \( R = \delta N \).\(^{20}\)

D. Calculating the spectrum and the spectral index

Now we derive explicit formulas for the spectrum and the spectral index, following [122]. Since the evolution of \( H \) along a comoving worldline will be the same as for a homogeneous universe with the same initial inflaton field, \( \dot{N} \) is a function only of this field and we have

\[ R = N_a \delta \phi_a \]  

(Repeated indices are summed over and the subscript \( a \) denotes differentiation with respect to \( \phi_a \).) The perturbations \( \delta \phi_a \) are Gaussian random fields generated by the vacuum fluctuation, and have a common spectrum \( (H/2\pi)^2 \). The spectrum \( \delta_H^2 \equiv (4/25)P_R \) is therefore

\[ \delta_H^2 = \frac{V}{45\pi^2 M_{\text{Pl}}^2} N_a N_a \]  

In the single-component case, \( N' = M_{\text{Pl}}^2 V/V' \) and we recover the usual expression. In the multi-component case we can always choose the basis fields so that while cosmological scales are leaving the horizon one of them points along the inflaton trajectory, and then

\(^{20}\)The last step is not spelled out in Ref. [122]. The statement that \( \tilde{N} \) is independent of the interpolation is true only on scales well outside the horizon, and its physical interpretation is unclear though it drops out very simply in the explicit calculation.
its contribution gives the standard result with the orthogonal directions giving an additional contribution. Since the spectrum of gravitational waves is independent of the number components (being equal to a numerical constant times $V$) the relative contribution $r$ of gravitational waves to the cmb is always smaller in the multi-component case.

The contribution from the orthogonal directions depends on the whole inflationary potential after the relevant scale leaves the horizon, and maybe even on the evolution of the energy density after inflation as well. This is in contrast to the contribution from the parallel direction which depends only on $V$ and $V'$ evaluated when the relevant scale leaves the horizon. The contribution from the orthogonal directions will be at most of order the one from the parallel direction provided that all $N, a$ are at most of order $M_{Pl}^{-2}V/V'$. We shall see later that this is a reasonable expectation at least if $R$ stops varying after the end of slow-roll inflation.

To calculate the spectral index we need the analogue of Eqs. (21) and (13). Using the chain rule and $dN = -H dt$ one finds

$$\frac{d}{d \ln k} = -\frac{M_{Pl}^2 V_a}{V} \frac{\partial}{\partial \phi_a}$$  (96)

$$N_a V_a = M_{Pl}^{-2}V$$  (97)

Differentiating the second expression gives

$$V_a N_{ab} + N_a V_{ab} = M_{Pl}^{-2}V_b$$  (98)

Using these results one finds

$$n - 1 = -\frac{M_{Pl}^2 V_a V_a}{V^2} - \frac{2}{M_{Pl}^2 N_a N_a} + 2 \frac{M_{Pl}^2 N_a N_b V_{ab}}{V N_d N_d}$$  (99)

Again, we recover the single field case using $N' = M_{Pl}^{-2}V/V'$. Differentiating this expression and setting $M_{Pl} = 1$ for clarity gives

$$\frac{dn}{d \ln k} = -\frac{2}{V^2} V_a V_b V_{ab} + \frac{2}{V^4} (V_a V_a)^2 + \frac{4 (V - N_a N_b V_{ab})^2}{V (N_d N_d)^2}
+ \frac{2 N_a N_b V_{ab} V_{abc}}{V N_d N_d} + \frac{4 (V_c - N_a V_{ac}) N_b V_{bc}}{V N_d N_d}$$  (100)

A correction to the formula for $n - 1$ has also been worked out [103]. Analogously with the single-component case, both this correction and the variation of $n - 1$ involve the first, second and third derivatives of $V$. Provided that the derivatives of $N$ in the orthogonal directions are not particularly big, and barring cancellations, a third flatness condition $V_{abc}V_{c}/V^2 \ll \max\{\sum_a (V_a)^2, \sum_{ab} |V_{ab}|\}$ ensures that both the correction and the variation of $n - 1$ in a Hubble time are small. (One could find a weaker condition that would do the same job.)

These formulas give the spectrum and spectral index of the density perturbation, if one knows the evolution of the homogeneous universe corresponding both to the classical inflaton trajectory and to nearby trajectories. An important difference in principle from the single-component case, is that the classical trajectory is not uniquely specified by the potential,
but rather has to be given as a separate piece of information. However, if there are only two components the classical trajectory can be determined from the COBE normalization of the spectrum, and then there is still a prediction for the spectral index.

This treatment can be generalized straightforwardly to the case of non-canonical kinetic terms of the form (33), that is expected in supergravity. However, in the small-field regime one expects the curvature associated with the ‘metric’ $h_{ab}$ to be negligible, and then one can recover the canonical normalization $h_{ab} = \delta_{ab}$ by redefining the fields.

**E. When will $R$ become constant?**

We need to evaluate $N$ up to the epoch where $R = \delta N$ has no further time dependence. When will that be?

As long as all fields are slow-rolling, $R$ is constant if and only if the inflaton trajectory is straight. If it turns through a small angle $\theta$, and the trajectories have not converged appreciably since horizon exit, the fractional change in $R$ is in fact $2\theta$. Since slow-roll requires that the change in the vector $\phi_a$ during one Hubble time is negligible, the total angle turned is $\ll N$. Hence the relative contribution of the orthogonal directions cannot be orders of magnitude bigger than the one from the parallel direction, if it is generated during slow-roll inflation. (In two dimensions the angle turned cannot exceed $2\pi$ of course, but there could be a corkscrew motion in more dimensions.) Later slow-roll may fail for one or more of the fields, with or without interrupting inflation, and things become more complicated, but in general there is no reason why $R$ should stop varying before the end of inflation.

Now let us ask what happens after the end of inflation (or to be more precise, after significant particle production has spoiled the above analysis, which may happen a little before the end). The simplest case is if the relevant trajectories have practically converged to a single trajectory $\phi_a(\tau)$, as in Ref. [47]. Then $R$ will not vary any more (even after inflation is over) as soon as the trajectory has been reached. Indeed, setting $\tau = 0$ at the end of inflation, this unique trajectory corresponds to a post-inflationary universe depending only on $\tau$. The fluctuation in the initial field values causes a fluctuation $\Delta \tau$ in the arrival time at the end of inflation, leading to a time-independent $R = \delta N = H_{\text{end}} \Delta \tau$.

What if the trajectory is not unique at the end of inflation? After the completion of the transition from inflation to a universe of radiation and matter, Eq. (81) tells us that $R$ will be constant provided that $\delta P$ and $\delta \rho$ are related in a definite way. This is the case during

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21 Thinking in two dimensions and taking the trajectory to be an arc of a circle, a displacement $\delta \phi$ towards the center decreases the length of the trajectory by an amount $\delta \phi$, to be compared with the decrease $\delta \phi$ for the same displacement along the trajectory. (The rms displacements will indeed be the same if the trajectories have not converged.) Since the speed along the new trajectory is faster in inverse proportion to the length since it is proportional to $V'$ and $V$ is fixed at the initial and final points on the trajectory. Thus the perpendicular displacement increases $N$ by $2\theta$ times the effect of a parallel displacement, for $\theta \ll 1$. 

39
matter domination ($\delta P = 0$) or radiation domination ($\delta P = \frac{1}{3} \delta \rho$). 22

Immediately following inflation there might be a quite complicated situation, with ‘pre-
heating’ 25 or else the quantum fluctuation of the ‘other’ field in hybrid models 17 con-
verting most of the inflationary potential energy into marginally relativistic particles in
much less than a Hubble time. But after at most a few Hubble times one expects to arrive
at a matter-dominated era so that $R$ is constant. Subsequent events will not cause $R$ to
vary provided that they occur at definite values of the energy density, since again $P$ will
have a definite relation with $\rho$. This is indeed the case for the usually-considered events,
such as the decay of matter into radiation and thermal phase transitions (including thermal
inflation). The conclusion is that it is reasonable to suppose that $R$ achieves a constant
value at most a few Hubble times after inflation. On the other hand one cannot exclude
the possibility that one of the orthogonal components of the inflaton provides a significant
additional degree of freedom, allowing $R$ to have additional variation before we finally arrive
at the radiation-dominated era preceding the present matter-dominated era.

This leaves the transition, lasting probably at most a few Hubble times, from inflation
to the first epoch of matter domination. It is conjecture in Ref. [103] that the variation
of $R$ during the transition may still be negligible if slow-roll holds almost until the end
of inflation. (This is in the present context of the first-order slow-roll calculation, not the
second-order one that is the main focus of Ref. [103].) The question requires detailed study
however.

F. Working out the perturbation generated by slow-roll inflation

If $R$ stops varying by the end of inflation, the final hypersurface can be located just
before the end (not necessarily at the very end because that might not correspond to a
hypersurface of constant energy density). Then, knowing the potential and the hypersurface
in field space that corresponds to the end of inflation, one can work out $N(\phi_1, \phi_2, \cdots)$ using
the equations of motion for the fields, and the expression

$$3M_{pl}^2 H = \rho = V + \frac{1}{2} \frac{d\phi_0}{d\tau} \frac{d\phi_0}{d\tau}$$

To perform such a calculation it is not necessary that all of the fields continue to slow-roll
after cosmological scales leave the horizon. In particular, the oscillation of some field might
briefly interrupt inflation, which resumes when its amplitude becomes small enough. If that
happens it may be necessary to take into account ‘preheating’ during the interruption.

In general all this is quite complicated, but there is one case that may be extremely
simple, at least in a limited regime of parameter space. This is the case

$$V = V_1(\phi_1) + V_2(\phi_2) + \cdots$$

22If the ‘matter’ consists of a nearly homogeneous oscillating scalar field one actually has $\delta P = \delta \rho$
($=\frac{1}{2} \delta (d\phi/d\tau)^2$), since on comoving hypersurfaces $\phi$ and therefore $V$ is constant. But this still
makes $\delta P$ negligible.
with each $V_a$ proportional to a power of $\phi_a$. For a single-component inflaton this gives inflation ending at $\phi_{\text{end}} \simeq M_{\text{Pl}}$, with cosmological scales leaving the horizon at $\phi \gg \phi_{\text{end}}$. If the potentials $V_a$ are identical we recover that case. If they are different, slow-roll may fail in sequence for the different components, but in some regime of parameter space the result for $N$ (at least) might be the same as if it failed simultaneously for all components. If that is the case one can derive simple formulas \[28,121\], provided that cosmological scales leave the horizon at $\phi_a \gg \phi_{\text{end}}$ for all components.

One has

$$Hdt = -M_{\text{Pl}}^{-2} \frac{V}{V'} d\phi_1 = -M_{\text{Pl}}^{-2} \sum_a \frac{V_a}{V'_a} d\phi_a$$  \hspace{1cm} (103)$$

It follows that

$$N = M_{\text{Pl}}^{-2} \sum_a \int \frac{\phi_a}{V'_a} d\phi_a$$  \hspace{1cm} (104)$$

Since each integral is dominated by the endpoint $\phi_a$, we have $N_a = M_{\text{Pl}}^{-2} V_a/V'_a$ and

$$\delta_H^2 = \frac{V}{75 \pi^2 M_{\text{Pl}}^6} \sum_a \left(\frac{V_a}{V'_a}\right)^2$$  \hspace{1cm} (105)$$

The spectral index is given by Eq. (99), which simplifies slightly because $V_{,ab} = \delta_{ab} V''_a$.

The simplest case is $V = \frac{1}{2} m_1^2 \phi_1^2 + \frac{1}{2} m_2^2 \phi_2^2$. Then $n$ is given by the following formula

$$1 - n = \frac{1}{N} \left[ \frac{(1 + r)(1 + \mu^2 r)}{(1 + \mu r)^2} + 1 \right]$$  \hspace{1cm} (106)$$

where $r = \phi_2^2/\phi_1^2$ and $\mu = m_2^2/m_1^2$. If $\mu = 1$ this reduces to the single-component formula $1 - n = 2/N$. Otherwise it can be much bigger, but note that our assumptions will be valid if at all in a restricted region of the $r$-$\mu$ plane.

G. Comparison with the usual approach

The usual approach is to use cosmological perturbation theory to find a closed system of linear equations, for perturbations in the relevant degrees of freedom. For each Fourier mode there is a set of coupled differential equations, which can be solved with enough effort. This approach has been used to establish the constancy of $R$ in special cases for a single-component inflaton \[128\], and to calculate $R$ at the end of inflation with a multi-component inflaton \[115,116,27,113,114\]. When it can be formulated, the usual approach will lead to the same result as the present one, since on scales far outside the horizon all spatial gradients in the equations will become negligible.

On the other hand it is not clear that the desired closed system of equations will always exist. It might for instance happen that the small-scale quantum fluctuation of the ‘other’ field in a hybrid model generates field gradients which play a crucial role in the transition from inflation to a matter-dominated universe \[17\]. In that case the evolution of (say) $P$ on
large scales is sensitive to the small-scale behaviour, and one will not be able to develop a closed system of equations for a given large-scale Fourier component.

A striking demonstration of the greater power of the present approach is provided by the issue of the constancy of $\mathcal{R}$ for a single-component inflaton. In the usual approach one has to make simplifying assumptions, and even then the equations are sufficiently complicated that it is possible to make mistakes and arrive at the incorrect conclusion that $\mathcal{R}$ is not constant [27]. The present approach makes redundant all proofs of the constancy of $\mathcal{R}$ based on the usual approach.

H. An isocurvature density perturbation?

Following the astrophysics usage, we classify a density perturbation as adiabatic or isocurvature with reference to its properties well before horizon entry, but during the radiation-dominated era preceding the present matter-dominated era. For an adiabatic density perturbation, the density of each particle species is a unique function of the total energy density. For an isocurvature density perturbation the total density perturbation vanishes, but those of the individual particle species do not. The most general density perturbation is the sum of an adiabatic and an isocurvature perturbation, with $\mathcal{R}$ specifying the adiabatic density perturbation only.

For an isocurvature perturbation to exist the universe has to possess more than the single degree of freedom provided by the total energy density. If the inflaton trajectory is unique, or has become so by the end of inflation, there is only the single degree of freedom corresponding to the fluctuation back and forth along the trajectory and there can be no isocurvature perturbation. Otherwise one of the orthogonal fields can provide the necessary degree of freedom. The simplest way for this to happen is for the orthogonal field to survive, and acquire a potential so that it starts to oscillate and becomes matter. The start of the oscillation will be determined by the total energy density, but its amplitude will depend on the initial field value so there will be an isocurvature perturbation in the axion density. It will be compensated, for given energy density, by the perturbations in the other species of matter and radiation which will continue to satisfy the adiabatic condition $\delta \rho_m / \rho_m = \frac{3}{4} \delta \rho_r / \rho_r$.

The classic example of this is the axion field [12,23], which is simple because the fluctuation in the direction of the axion field causes no adiabatic density perturbation, at least in the models proposed so far. The more general case, where one of the components

23In some of the theoretical literature this kind of classification is made also at earlier times, in particular during inflation, and also during the present matter-dominated era. From that viewpoint the density perturbation always starts out with an isocurvature component in the multi-component case, and it always ends up as adiabatic by the present matter-dominated era.

24If the potential of the ‘orthogonal’ field already exists during inflation the inflaton trajectory will have a tiny component in its direction, so that it is not strictly orthogonal to the inflaton trajectory. This makes no practical difference. In the axion case the potential is usually supposed to be generated by QCD effects long after inflation.
of the inflaton may cause both an adiabatic and an isocurvature perturbation has been looked at in for instance Ref. [120], though not in the context of specific particle physics. If an isocurvature perturbation in the non-baryonic dark matter density is generated during inflation, it must not conflict with observation and this imposes strong constraints on, for instance, models of the axion [123].

An isocurvature perturbation in the density of one a species of matter may be defined by the 'entropy perturbation' [33, 126, 3]

$$S = \frac{\delta \rho_m}{\rho_m} - \frac{3}{4} \frac{\delta \rho_r}{\rho_r}$$  \hspace{1cm} (107)

where $\rho_m$ is the non-baryonic dark matter density. Equivalently, $S = \delta y/y$, where $y = \rho_m/\rho_r^{3/4}$. Since we are dealing with scales far outside the horizon, $\rho_m$ and $\rho_r$ evolve as they would in an unperturbed universe which means that $y$ is constant and so is $S$. Provided that the field fluctuation is small $S$ will be proportional to it, and so will be a Gaussian random field with a nearly flat spectrum [123].

For an isocurvature perturbation, $R$ vanishes during the radiation dominated era preceding the present matter dominated era. But on the very large scales entering the horizon well after matter domination, $S$ generates a nonzero $R$ during matter domination, namely $R = \frac{1}{3}S$. A simple way of seeing this, which has not been noted before, is through the relation (81). Since $\delta \rho = 0$, one has $S = -(\rho_m^{-1} + \frac{3}{2} \rho_r^{-1}) \delta \rho_r$. Then, using $\delta P = \delta \rho_r/3$, $\rho_r/\rho_m \propto a$ and $H dt = da/a$ one finds the quoted result by integrating Eq. (81).

As discussed for instance in Ref. [8], the large-scale cmb anisotropy coming from an isocurvature perturbation is $\Delta T/T = -(\frac{1}{3} + \frac{1}{15})S$, where $S$ is evaluated on the last-scattering surface. The second term is the Sachs-Wolfe effect coming from the curvature perturbation we just calculated, and the first term is the anisotropy $\frac{1}{4} \delta \rho_r/\rho_r$ just after last scattering (on a comoving hypersurface). By contrast the anisotropy from an adiabatic perturbation comes only from the Sachs-Wolfe effect, so for a given large-scale density perturbation the isocurvature perturbation gives an anisotropy six times bigger. As a result an isocurvature perturbation cannot be the dominant contribution to the cmb, though one could contemplate a small contribution [129].

VIII. DISCUSSION AND CONCLUSION

Let us summarize. Except in the last section we have focussed on models involving a single-component inflaton field, since they are simple and give a relatively clean prediction for $n$. The models considered include non-hybrid ones where the inflaton field $\phi$ dominates the potential during inflation, and hybrid ones where this role is played by a different field $\psi$. In the latter case $\psi$ minimizes the potential during inflation, so that one still ends up with an effective potential $V(\phi)$. It is usually assumed that $V(\phi)$ (non-hybrid) or $V(\phi, \psi)$ (hybrid) consists of one or a few low-order terms in a power-series expansion, but this has motivation in the context of supergravity only if the fields are at most of order $M_{Pl}$. In the hybrid case one can still obtain a non-polynomial $V(\phi)$ during inflation since $\psi$ may be a function of $\phi$ (the case of mutated hybrid inflation). A non-polynomial potential can also emerge in a theory starting out with non-canonical kinetic terms or non-Einstein gravity, as well as from a quantum correction.
A mathematically simple potential giving inflation is the one first proposed to implement chaotic initial conditions at the Planck scale, \( V \propto \phi^p \), but with this potential cosmological scales leave the horizon when \( \phi \sim 10M_{\text{Pl}} \). It gives \( n - 1 = -(2 + p)/(2N) \). It also gives a significant gravitational wave contribution to the cmb anisotropy, being practically the only viable potential proposed so far that does. (Extended inflation, which also gives a significant contribution, is ruled out by observation except in contrived versions.)

Virtually all other models so far proposed give, during inflation, a potential effectively of the form \( V = V_0(1 \pm \mu \phi^p) \) with the constant term dominating. With this potential the fields can be \( \lesssim M_{\text{Pl}} \), and in many cases \( \ll M_{\text{Pl}} \). For the plus sign \( p \) is a positive integer, but for the minus sign it can be negative, or even non-integral. In the last case, \( p \) just below zero mimics the case of the potential \( V_0[1 - A \ln(B/\phi)] \) and \( p \to -\infty \) mimics \( V_0(1 - e^{-q\phi}) \), both of which have good particle physics motivation (respectively from a quantum correction or a theory starting out with non-canonical kinetic terms).

Except for \( 0 \lesssim p \lesssim 2 \) the prediction for \( n \) depends only on the exponent, as is shown in the Table for some integer values. The only important exceptions are the quadratic potentials \( V = V_0 \pm \frac{1}{2}m^2\phi^2 \) which give \( n = 1 \pm 2M_{\text{Pl}}^2m^2/V_0 \). The flatness conditions require \( M_{\text{Pl}}^2m^2/V_0 \) to be considerably less than 1. Unless there is a reason why the parameters should be on the edge of the allowed region, one therefore expects \( n \) to be indistinguishable from 1 in this case. In the context of supergravity there may or may not be such a reason, depending on the model.

Looking at the Table, the most striking thing is that most of the potentials make \( n \) close to 1, but not very close. In fact practically all of the listed potentials are ruled out unless \( n \) lies in one of the intervals \( .84 < n < .98 \) or \( 1.04 < n < 1.16 \). Another interesting feature is the dividing line between positive and negative values of \( p \), which occurs somewhere in the range \( n = .92 \) and \( n = .96 \).

Although any selection is as yet tentative, there is no doubt that some of these potentials are more favoured than others. Values of \( n \) significantly bigger than 1 seem unlikely; the potentials \( V = V_0(1 + \mu \phi^p) \) with \( p > 2 \) are not favoured by particle theory, whereas the quadratic potential is likely to give \( n \) indistinguishable from 1. Values \( n < .84 \) are also unlikely, unless the inverted quadratic potential emerges from one of the non-hybrid settings discussed in Section IV.

This leaves the regime \( .84 < n \leq 1.00 \), and within that are a few potentials that might be regarded as favoured theoretically. A very subjective selection corresponds to the five rows marked of the Table marked by **. The first and last cases are the quadratic and inverted quadratic potentials \( V_0 \pm \frac{1}{2}m^2\phi^2 \), thought of as being derived respectively from ordinary and inverted hybrid inflation; since there is no reason for the parameters to be on the edge of the region allowed by the flatness condition \( \eta \ll 1 \) one expects \( n \) to be indistinguishable from 1 in these models. The second case is the loop-corrected potential that might arise if the \( D \) term dominates, mimicked by the potential \( V_0(1 - \mu \phi^{-p}) \) with \( p \simeq 0 \). The third case is the potential \( V_0(1 - \mu \phi^{-2}) \) which comes out of the original mutated hybrid inflation model [21]. The fourth is the cubic potential advocated in Ref. [45].

With this last potential, for small \( N \), the scale dependence of the spectrum (not well-represented by constant \( n \)) becomes strong enough to give a useful lower bound on \( N \). It was estimated at the end of Section II as \( N \gtrsim 11 \).

To summarize the situation regarding \( n \) in these models, it is clear that a measurement
of it will give valuable discrimination between different potentials. Values of $n$ significantly above 1 are disfavoured theoretically.

We have looked briefly at inflation model-building in the demanding context of supergravity, focussing on the problem of keeping the inflaton mass small enough in the face of generic contributions of order $\pm H^2$.

We also considered models with a multi-component inflaton, and we have looked in some detail at the calculation of the spectrum in both this and the single-component case. In both cases the most powerful calculational technique $[121,122]$ starts with the observation that after smoothing the relevant quantities on scales far outside the horizon, the evolution of the universe along each comoving worldline will be the same as for an unperturbed universe with the same initial inflaton field. In the single-component case this justifies the usual assumption that $\mathcal{R}$ is constant. In the multi-component case it leads to a simple formula for $n$, whose only input is the unperturbed evolution. Finally, we looked at the case of an isocurvature perturbation, giving a simple derivation of the previously obscure fact that the low multipoles of the cmb anisotropy are six times as big as for an adiabatic density perturbation.
TABLE I. Predictions for $n$ are displayed for inflationary potentials of the form $V = V_0(1\pm \mu \phi^p)$, with the first term dominating. The sixth row represents the limiting case $p \to 0$ from below, with the minus sign. In most cases the prediction is proportional to $1/N$, where $N$ is the number of $e$-folds of inflation after cosmological scales leave the horizon. Results are given for $N = 50$ and $N = 25$, corresponds to different cosmologies after inflation. The predictions for $n$ are quoted to only two decimal places, because a better observational accuracy would be very hard to achieve. The symbols ** mark the rows corresponding to five potentials that might be favoured on the basis of theory. The first of them is expected to give $n$ indistinguishable from 1, but the last may or may not depending on how the potential is derived.

| $V(\phi)/V_0$ | $(n - 1)(N/50)$ | $N = 50$ | $N = 25$ |
|---------------|----------------|---------|---------|
| $1 + \mu \phi$ | .00            | 1.00    | 1.00    |
| $1 + \mu \phi^2$ | 1.00?          |         |         |
| $1 + \mu \phi^3$ | .08            | 1.08    | 1.16    |
| $1 + \mu \phi^4$ | .06            | 1.06    | 1.12    |
| $1 + \mu \phi^\infty$ | .04            | 1.04    | 1.08    |
| **            |                |         |         |
| $1 - \mu \phi^{-1}$ | -.02          | .98     | .96     |
| **            |                |         |         |
| $1 - \mu \phi^{-2}$ | -.03          | .97     | .94     |
| $1 - \mu \phi^{-4}$ | -.03          | .97     | .93     |
| $1 - \mu \phi^{\pm \infty}$ | -.04          | .96     | .92     |
| $1 - \mu \phi^4$ | -.06           | .94     | .88     |
| **            |                |         |         |
| $1 - \mu \phi^3$ | -.03           | .92     | .84     |
| **            |                |         |         |
| $1 - \mu \phi^2$ | 1.00??         |         |         |
| **            |                |         |         |
| $1 - \mu \phi$ | .00            | 1.00    | 1.00    |
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