Pin-STRUCTURES ON SURFACES
AND QUADRATIC FORMS

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Abstract. A correspondence between different Pin-type structures on a compact surface and quadratic (linear) forms on its homology is constructed. Addition of structures is defined and expressed in terms of these quadratic forms.

In this paper we try to clarify the relation between Pin-type structures on a compact surface and quadratic forms on its homology. This relation is well-known and useful in the case of Spin- and Pin\(^{-}\)-structures ([J], [KT]). It makes it much easier to understand the nature of some fundamental in low-dimensional topology objects such as \(\mathbb{Z}/2\)-Seifert form on a surface in an oriented 3-manifold and Rokhlin form on a characteristic surface in an oriented 4-manifold (see, e.g., [F], [DFM]).

Remark. A slightly more general approach of [DFM] also explains these forms in the case of a non-oriented ambient manifold, as well as the newly found Benedetti-Marin form [BM].

We will show that a similar correspondence between quadratic forms and structures can also be defined for other Pin-type structures. In this short paper we restrict ourselves to a geometrical description in the simplest case, when the structural group is a \(\mathbb{Z}/2\)-extension of the orthogonal group \(O_n\). Up to isomorphism, there are four such extensions corresponding to the four elements of \(H^2(BO_n;\mathbb{Z}/2)\), each element being the obstruction to existence of a structure in an \(O_n\)-bundle \(P \to X\). When non-empty, the set of all the structures of a given type forms an affine space over \(H^1(X;\mathbb{Z}/2)\). If the obstruction is the trivial element of \(H^2(BO_n;\mathbb{Z}/2)\), this set obviously coincides with \(H^1(X;\mathbb{Z}/2)\). The classes \(w_2\) and \(w_2 + w_1^2\) characterize Pin\(^{+}\)- and Pin\(^{-}\)-structures respectively. Finally, the structures corresponding to the remaining class \(w_1^2\) are not (to the best of our knowledge) mentioned in literature and did

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not receive any special name. We will call them $\tilde{O}_n$-structures, $\tilde{O}_n$ standing for the nontrivial semi-direct product $\mathbb{Z}/4 \rtimes SO_n$ (topologically, $\tilde{O}_n \to O_n$ is a trivial double covering).

Remark. Actually, $\tilde{O}_n$-structures do appear in literature implicitly, e.g., as framings in the complexification of a vector bundle [A], or as linear forms on a real algebraic variety [N]. Besides, they complete the descending table in [KT, Corol. 2.15]: a Pin$^-$-structure on a manifold $M$ descends to an $\tilde{O}_n$-structure on a codimension one submanifold whose normal bundle is isomorphic to the determinant of the tangent bundle of $M$.

Thus, we show that for each of these four classes there is a one-to-one correspondence between the set of structures on a compact surface and the set of specific quadratic (or, in special cases, linear) forms on the 1-homology of the surface. (This correspondence is known in the Pin$^-$-case (see, e.g., [KT]) and is obvious in the case of the trivial extension. The two others are defined in §2.) Another subject of the paper, which has never (as far as we know) been mentioned explicitly (see, though, a slightly different approach in [D]), is addition of structures. This operation, defined in §3, naturally extends the canonical affine action of $H_1(X; \mathbb{Z}/2)$ and covers addition of the characteristic classes. (By the way, this gives one more reason for considering $\tilde{O}_n$-structures: they are sums of Pin$^-$ and Pin$^+$-structures.) We give an interpretation of this operation in terms of quadratic forms.

§2. Quadratic forms

1. Pin$^-$-structures—quadratic forms $H_1(F; \mathbb{Z}/2) \to \mathbb{Z}/4$. We start with reminding the standard construction of the quadratic form $q$ corresponding to a Pin$^-$-structure on a compact surface $F$. Pick an integral class $\alpha \in H_1(F; \mathbb{Z})$ and realize it by an immersed collection of oriented circles $S \to F$. Let $n(S)$ and $i(S)$ be the numbers of components and self-intersection points of $S$ respectively. The tangent vector field to $S$ defines a Pin$_1^-$-reduction of the restriction to $S$ of the given Pin$_2^-$-bundle on $F$. Since Pin$_1^- \cong \mathbb{Z}/4$ is a discrete abelian group, one can consider the total holonomy $h(S) \in$ Pin$_1^-$ of this bundle along $S$ (which, by definition, is the sum in Pin$_1^-$ of the holonomies along the components of $S$). We let $q(\alpha) = h(S) + 2(n(S) + i(S)) \pmod{4}$. Now standard arguments apply to show that $q(\alpha)$ does not depend on $S$ and satisfies the identity $q(\alpha + \beta) = q(\alpha) + q(\beta) + 2\langle \alpha, \beta \rangle$, where $\langle \cdot, \cdot \rangle$ is the intersection form on $F$ and $2: \mathbb{Z}/2 \to \mathbb{Z}/4$ is the unique inclusion. ($q(\alpha)$ obviously does not change during a regular homotopy of $S$, and elementary transformations like Reidemeister move I and smoothing a self-intersection point can easily be controlled; see, e.g., [KT].) Since the mod 2 reduction
of \( q \) coincides with \( w_1 : H_1(F; \mathbb{Z}/2) \to \mathbb{Z}/2 \), the above formula implies, in particular, that \( q \) factors through \( \mathbb{Z}/2 \)-homology of \( F \).

2. \( \tilde{O}_2 \)-structures—linear forms \( H_1(F; \mathbb{Z}/4) \to \mathbb{Z}/4 \). Since the corresponding 1-dimensional group \( \tilde{O}_1 \) is also \( \mathbb{Z}/4 \), this case is similar to the previous one. The only difference is that one should not adjust holonomy by the numbers of components and self-intersection points, i.e., \( q(\alpha) = h(S) \). The result is a linear form \( q \) which factors through \( \mathbb{Z}/4 \)-homology, \( q : H_1(F; \mathbb{Z}/4) \to \mathbb{Z}/4 \), and whose restriction mod 2 coincides with \( w_1 : H_1(F; \mathbb{Z}/2) \to \mathbb{Z}/2 \).

3. Pin\(^+\)-structures—quadratic forms \( H_1(F; \mathbb{Z}/4) \to \mathbb{Z}/2 \). The construction goes similar to the case of Pin\(^-\)-structures. Now Pin\(^+\) \( \cong O_1 \times \mathbb{Z}/2 \), the projection of \( h(S) \) to the first factor \( O_1 \) being just the value of \( w_1 \) on \( \alpha \). In order to drop this standard component, we consider the projection \( p_2 h(S) \) to the second factor \( \mathbb{Z}/2 \); then we let \( q(\alpha) = p_2 h(S) + n(S) + i(S) \mod 2 \). This form satisfies the identity \( q(\alpha + \beta) = q(\alpha) + q(\beta) + \langle \alpha, \beta \rangle \), which, in particular, implies that it factors through \( H_1(F; \mathbb{Z}/4) \).

4. Trivial structures—linear forms \( H_1(F; \mathbb{Z}/2) \to \mathbb{Z}/2 \). This case, when structures just are cohomology classes, admits a description similar to the previous three: the total holonomy \( h(S) \) is an element of \( O_1 \times \mathbb{Z}/2 \), and we let \( q(\alpha) = p_2 h(S) \).

We can now uniformize all the four cases and consider quadratic (linear) forms \( H_1(F; \mathbb{Z}/4) \to \mathbb{Z}/4 \). (In the case of Pin\(^+\) and trivial structures \( \mathbb{Z}/2 \) is embedded in \( \mathbb{Z}/4 \) via multiplication by 2.) This gives the following result:

**Theorem A.** Given a compact surface \( F \), there is a canonical affine one-to-one correspondence between structures on \( F \) with the characteristic class \( aw_2 + bw_1^2 \) (for some fixed \( a, b \in \mathbb{Z}/2 \)) and functions \( q : H_1(F; \mathbb{Z}/4) \to \mathbb{Z}/4 \) satisfying the following conditions:

1. \( q(\alpha + \beta) = q(\alpha) + q(\beta) + 2a \langle \alpha, \beta \rangle \);
2. \( q(\alpha) = bw_1(\alpha) \mod 2 \).

**Proof.** The only thing that needs proof is the fact that the constructed map \( \{\text{structures}\} \to \{\text{forms}\} \) is one-to-one. Since both the sets are affine over \( H^1(F; \mathbb{Z}/2) \), it suffices to show that existence of forms implies existence of structures. This is obvious for Pin\(^-\)- and trivial structures, or if the surface is not closed (since structures always exist in such cases). For Pin\(^+\)- and \( \tilde{O}_2 \)-structures on closed surfaces one can easily see that desired forms exist if and only if elements of order 2 in \( H_1(F; \mathbb{Z}/4) \) annihilate \( w_1 \) (or, equivalently, have trivial self-intersection). This is the case when the surface is the connected sum of an even number of \( \mathbb{R}P^2 \)'s, i.e., exactly when \( w_2 = w_1^2 = 0 \). \( \Box \)
Corollary (classification of Pin\(^+\)-structures up to isomorphism). Two Pin\(^+\)-structures on a closed surface \(F\) are isomorphic (i.e., can be transformed into each other by a diffeomorphism of the surface) if and only if the values of the corresponding quadratic forms on the (unique) 2-torsion element of \(H_1(F;\mathbb{Z})\) coincide. In particular, two structures are isomorphic if and only if they are cobordant.

Proof. The mentioned value is the only algebraic invariant of forms (an easy exercise), and, as usual in 2-dimensional topology, one can find an automorphism of the lattice \(H_1(F;\mathbb{Z})\) which is accompanied by a diffeomorphism of \(F\). □

Remark. Note that we have to consider integral homology here, since otherwise one cannot distinguish between different 2-torsion elements, and the algebraic invariant disappears.

§3. Addition of structures

Given two \(\mathbb{Z}/2\)-extensions \(G_1 \to O_n, G_2 \to O_n\), one can define their sum \(G_1 \lor G_2\) to be the quotient \(G_1 \times_{O_n} G_2 / \text{Diag}(\mathbb{Z}/2)\), where Diag is the canonical diagonal map

\[
\text{Diag} : \mathbb{Z}/2 = \text{Ker}[G_i \to O_n] \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 = \text{Ker}[G_1 \times_{O_n} G_2 \to O_n].
\]

(In fact, this is one of the standard algebraic approaches to definition of the group structure on the set of isomorphism classes of \(\mathbb{Z}/2\)-extensions of \(O_n\), which is isomorphic to \(H^2(BO_n;\mathbb{Z}/2)\).) To apply this procedure to structures, one should fix first some representatives \(G(\omega)\) of the isomorphism classes of extensions, one for each characteristic class \(\omega \in H^2(BO_n;\mathbb{Z}/2)\), and some maps \(G(\omega_1) \lor G(\omega_2) \to G(\omega_1 + \omega_2)\). To do that uniformly in all dimensions, it suffices to just pick some isomorphisms \(\text{Pin}^-_1 = \widetilde{O}_1 = \mathbb{Z}/4, \ \text{Pin}^+_1 = O_1 \times \mathbb{Z}/2,\) and \(\mathbb{Z}/4 \lor \mathbb{Z}/4 = O_1 \times \mathbb{Z}/2\) (see [DFM]). Such maps can certainly be chosen and fixed once and forever. Then one can give the following definition:

Definition. Let \(P \to X\) be an \(O_n\)-bundle. Then, given two structures \(\Phi_1 \to P, \Phi_2 \to P\) with characteristic classes \(\omega_1, \omega_2 \in H^2(BO_n;\mathbb{Z}/2)\) respectively, we define their sum \(\Phi_1 \lor \Phi_2 \to P\) to be the \((\omega_1 + \omega_2)\)-structure associated with the fibered product \(\Phi_1 \times_P \Phi_2 \to P\) via the composed map

\[
G(\omega_1) \times_{O_n} G(\omega_2) \to G(\omega_1) \lor G(\omega_2) \xrightarrow{\cong} G(\omega_1 + \omega_2).
\]

Theorem B below is proved in [DFM].
Theorem B. $\lor$ is a group operation on the set of all structures on a given $O_n$-bundle $P \to X$, which extends the canonical affine action of $H^1(X; \mathbb{Z}/2)$ on this set (i.e., $\lor$-sum with an $(O_n \times \mathbb{Z}/2)$-structure coincides with the affine shift by the corresponding cohomology class).

Theorem C. $\lor$-sum of structures on a compact surface corresponds to the following pointwise operation of quadratic forms: $(q_1, q_2) \mapsto q_1 + q_2 + 2q_1q_2$.

Proof. Due to the uniform construction of §2 it suffices to consider structures on the normal bundle to a circle, when the statement is obvious. □

The introduced $\lor$-sum operation admits an interpretation in terms of Spin-structures. Given an $O_n$-bundle $\xi: P \to X$, let us denote by Spin$(\xi)$, Pin$^\pm(\xi)$, etc. the set of all the Spin-, Pin$^\pm$, etc. structures on $\xi$ respectively. Then, according to [KT], there are natural isomorphisms Pin$^-$$(\xi) = \operatorname{Spin}(\xi \oplus \det \xi)$ and Pin$^+(\xi) = \operatorname{Spin}(\xi \oplus 3 \det \xi)$. Similar arguments show that, besides, there are isomorphisms $\tilde{O}_n(\xi) = \operatorname{Spin}(2\xi) = \operatorname{Spin}(2 \det \xi)$. Consider the Whitney sum of the above three bundles:

$$(\xi \oplus \det \xi) \oplus (\xi \oplus 3 \det \xi) \oplus (2\xi) = 4(\xi \oplus \det \xi).$$

This bundle has a canonical Spin-structure (the quaternion Spin-structure, which is defined on $4\eta$ for any bundle $\eta$, see [DFM]). Hence, Spin-structures on any two of the three summands define a Spin-structure on the third one, and one can easily see that the obtained maps Pin$^-$$(\xi) \times \operatorname{Pin}^+(\xi) \to \tilde{O}_n(\xi)$, etc. coincide with the $\lor$-sum. This gives an alternative description of this operation in the most interesting cases which are not reduced to the affine action of $H^1(X; \mathbb{Z}/2)$.

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