Hadron Properties in a Chiral Quark-Sigma Model

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Abstract

Within a chiral quark sigma model in which quarks interact via the exchange of $\sigma$- and $\vec{\pi}$-mesons, hadron properties are investigated. This model of the nucleon and delta is based on the idea that strong QCD forces on very short distances (a small length scales 0.2-1 fm) result in hidden chiral $SU(2) \times SU(2)$ symmetry and that there is a separation of roles between these forces which are responsible for binding quarks in hadrons and the forces which produce absolute confinement. We have solved the field equations in the mean field approximation for the hedgehog baryon state with different sets of model parameters. A new parameterization which well describe the nucleon properties has been introduced and compared with experimental data.

1 INTRODUCTION

The fundamental constituents of hadrons are quarks. Interaction of these quarks is described by quantum chromodynamics (QCD) in terms of the exchange of gluons. The QCD theory is a non-abelian gauge theory and has the properties of chiral symmetry, asymptotic freedom, where the effective coupling constant can be shown to tend to zero at short distances and confinement, where at long distances the coupling constant in QCD grows. QCD Lagrangian in $u$, $d$ and to some extent $s$ sector is invariant under chiral symmetry if we neglect the current masses ($m_u \approx 2$MeV, $m_d \approx 5$MeV, $m_s \approx 150$MeV) of these quarks in comparison to the QCD scale parameter $\Lambda (\approx 250$ MeV). It is becoming clear now that chiral symmetry of QCD Lagrangian and its spontaneous breaking \cite{1} play a very important role in determining the structure of low mass hadrons which consist of $u$, $d$ and $s$ quarks, and instantaneously play a crucial role in hadron correlators in mediating the spontaneous chiral symmetry breaking \cite{2, 3}. Physical confinement of quarks seems to play a lesser role. The spontaneous breaking of the chiral symmetry is signaled by the non-vanishing values in physical vacuum of the quark and gluon condensates \cite{4-6} and we describe the solution of the mean-field equations for the so-called hedgehog \cite{7} baryon state. Contrary to the claims made by other authors \cite{7, 8} we wish to stress that the hedgehog is not a mythical creature. This property has been interpreted as an intrinsic wave function containing a mixture. The mean-field equations are a straightforward extension of that outlined by Goldflam and Wilets \cite{9} and Birse and Banerjee \cite{10}. Description of the model is given Sec.2. The results and discussion are given in Sec.3.
2 CHIRAL QUARK-SIGMA MODEL

We describe the interactions of quarks with \( \sigma \) and \( \vec{\pi} \) - mesons by Brise and Banerjee [10]. The Lagrangian density is,

\[
L (r) = i \overline{\Psi} \partial_\mu \gamma^\mu \Psi + \frac{1}{2} (\partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi}) + g \overline{\Psi} (\sigma + i\gamma_5 \vec{\tau} \cdot \vec{\pi}) \Psi - U (\sigma, \vec{\pi}) ,
\]

with

\[
U (\sigma, \vec{\pi}) = \frac{\lambda^2}{4} (\sigma^2 + \vec{\pi}^2 - \nu^2)^2 + m_\pi^2 f_\pi \sigma ,
\]

is the meson-meson interaction potential where the \( \Psi \), \( \sigma \) and \( \vec{\pi} \) are the quark, (scalar, isoscalar) sigma and (pseudoscalar, isovector) pion fields, respectively. In the semiclassical or mean-field approximation the meson fields are treated as time-independent classical fields. This means that we are replacing power and products of the meson fields by corresponding powers and products of their expectation values. The meson-meson interactions in equ. (2) lead to hidden chiral \( SU(2) \times SU(2) \) symmetry with \( \sigma (r) \) taking on a vacuum expectation value

\[
\langle \sigma \rangle = - f_\pi ,
\]

where \( f_\pi = 93 \text{ MeV} \) is the pion decay constant. The final term in equ. (2) is included to break the chiral symmetry. It leads to partial conservation of axial-vector isospin current (PCAC). The parameters \( \lambda^2, \nu^2 \) can be expressed in terms of \( f_\pi \), the masses of mesons as,

\[
\lambda^2 = \frac{m_\sigma^2 - m_\pi^2}{2 f_\pi^2} ,
\]

\[
\nu^2 = f_\pi^2 - \frac{m_\pi^2}{\lambda^2} .
\]

Now we expand the extremum, with the shifted field defined as

\[
\sigma = \sigma' - f_\pi ,
\]

substituting by equation(6) into equation (1) we get :

\[
L (r) = i \overline{\Psi} \partial_\mu \gamma^\mu \Psi + \frac{1}{2} (\partial_\mu \sigma' \partial^\mu \sigma' + \partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi}) - g \overline{\Psi} f_\pi \Psi + g \overline{\Psi} \sigma' \Psi + ig \overline{\Psi} \gamma_5 \vec{\tau} \cdot \vec{\pi} \Psi
\]

\[
- U (\sigma', \vec{\pi}) ,
\]

with

\[
U (\sigma', \vec{\pi}) = \frac{\lambda^2}{4} \left( (\sigma' - f_\pi)^2 + \vec{\pi}^2 - \nu^2 \right)^2 + m_\pi^2 f_\pi (\sigma' - f_\pi) ,
\]

the time-independent fields \( \sigma' (r) \)and \( \vec{\pi} (r) \) are to satisfy the Euler-Lagragian equations, and the quark wave function satisfies the Dirac eigenvalue equation. Substituting by equation (7) in Euler-Lagragian equation we get:

\[
\Box \sigma' = g \overline{\Psi} - \lambda^2 \left( (\sigma' - f)^2 + \vec{\pi}^2 - \nu^2 \right) (\sigma' - f) - m_\pi^2 f_\pi ,
\]

\[
\Box \vec{\pi} = ig \overline{\Psi} \gamma_5 \vec{\tau} \vec{\pi} - \lambda^2 \left( (\sigma' - f)^2 + \vec{\pi}^2 - \nu^2 \right) \vec{\pi} .
\]
where $\vec{\tau}$ refers to Pauli isospin-matrices, $\gamma_5 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$. 

By using Hedgehog Ansatz [10] where

$$\vec{\pi}(r) = \hat{r} \pi(r).$$

(11)

The chiral Dirac equation for the quarks are

$$\frac{du}{dr} = -p(r) u + (E + m_q - S(r)) w,$$

(12)

where $S(r) = g \langle \sigma' \rangle$, $P(r) = \langle \vec{\pi} \cdot \hat{r} \rangle$, $E$ are the scalar potential, the pseudoscalar potential and the eigenvalue of the quarks spinor $\Psi$

$$\frac{dw}{dr} = -(E - m_q + S(r)) u - \left( \frac{2}{r} - p(r) \right) w,$$

(13)

The set of equations( 9,10,12,13) is solved following the method used by Goldflam and Wilets [9] and Birse and Banarje [10] for the Soliton Bag model [13]. Including the color degree of freedom, one has $g \overline{\Psi} \Psi \rightarrow N_c g \overline{\Psi} \Psi$ where $N_c = 3$ colors, $g$ is coupling constant. Thus

$$\Psi(r) = \frac{1}{\sqrt{4\pi}} \left[ \begin{array}{c} u(r) \\ iw(r) \end{array} \right] \quad \text{and} \quad \overline{\Psi}(r) = \frac{1}{\sqrt{4\pi}} \left[ \begin{array}{c} u(r) \\ iw(r) \end{array} \right]$$

(14)

then

$$\rho_s = N_c g \overline{\Psi} \Psi = \frac{3g}{4\pi} (u^2 - w^2),$$

(15)

$$\rho_p = i N_c g \overline{\Psi} \gamma_5 \vec{\tau} \Psi = \frac{3g}{4\pi} g(-2uw),$$

(16)

$$\rho_v = \frac{3g}{4\pi} (u^2 + w^2),$$

(17)

where $\rho_s$, $\rho_p$ and $\rho_v$ are the sigma density, pion density, and vector density respectively. These equations are subject to the boundary conditions that asymptotically the fields approach their vacuum values,

$$\sigma(r) \sim -f_\pi \text{ MeV} \quad \pi(r) \sim 0 \text{ at } r \rightarrow \infty.$$ 

(18)

Finally, we have solved the equations (12, 13) using fourth order Rung-Kutta. Due to the implicit nonlinearity of our equations (9, 10) it is necessary to iterate the solution until self-consistency is achieved. To start this iteration process we use the chiral circle form for the meson fields:

$$S(r) = m_q (1 - \cos \theta)$$

(19)

$$P(r) = -m_q \sin \theta$$

(20)

where

$$\theta = \pi \tanh r$$

(21)
3 Numerical Results and Discussion

The set of equations (9-13) are solved numerically by iteration as [10] for different values of the sigma mass, quarks mass, quark-meson coupling constant and pion decay coupling constant. The dependence of the nucleon properties on the sigma and quark masses and pion decay coupling constant are listed in tables(1), (2), (3) and (4). As seen from table(2) increasing the sigma mass the quark eigenvalue decreases, while the quark and pion kinetic and interaction energies increase. The experimental value of the pion decay coupling constant is of the order 93 MeV. Decreasing this value decreases more the quark eigenvalue and slowly decrease the quark, sigma and pion kinetic energies, while the quark-sigma and pion interaction energies increase, as seen from table(1). The constituent quark mass has a wider range $\sim$ 313-470 MeV, depending on the model parameters. In fact, there no experiment value of this parameter since the quark can not be isolated. Birse and Banerjee [10] suggested a value 500 MeV which is very large. Decreasing this value has the effect to increase the quark eigenvalue, while strongly decreases the quark, sigma and pion kinetic energies, as seen from table(3). In addition, the quark-sigma and pion interaction energies decrease. Recently, topological and fractal models have been introduced to calculate the constituent quark mass [14-20]. A value 336 MeV has been introduced by these models [21]. However, using this value in the present sigma model did not lead to any convergence in the iteration procedure for solving the field equations. We thus suggest a value of the quark-meson coupling constant 92.3 MeV, which lead to a value 461.5 MeV for the constituent quark mass($M_q = g f_\pi$) [22]. This value is very close to that derived from NJL model which is of the order 463 MeV [22]. The sigma mass is taken to be 900 MeV which is about twice the value of the quark mass, as derived from NJL model. The pion mass is taken from experiment which is of the order 139.6 MeV. This set of masses and coupling constant predicted the correct nucleon mass (after subtracting the CM correction), where we get a value 939 MeV, as seen from table (4) this table shows also that the experimental magnetic moments of the proton and neutron are well reproduced. The Goldberger-Trieman pion-nucleon coupling constant which is of the order 1.4 is also very well reproduced.

Table(1) Details of energy calculations and magnetic moments for $m_q = 500$ MeV, $m_\sigma = 1200$ MeV $m_\pi = 139.6$ MeV at different value of $f_\pi$. All quantities in MeV.

| $f_\pi$ | 93   | 91.77 | 91  | 90  | 89  | 88  | 87  | [10] |
|--------|------|-------|-----|-----|-----|-----|-----|------|
| Quark eigenvalue | 31.995 | 24.11 | 19.20 | 12.88 | 6.59 | 3.45 | -5.83 | 30.5 |
| Quark kinetic energy | 1225.011 | 1220.99 | 1218 | 1214.71 | 1210 | 1206 | 1202 | 1219 |
| Sigma kinetic energy | 359.66 | 357.67 | 356.31 | 354.4 | 352 | 350 | 347 | 358 |
| Pion kinetic energy | 562.92 | 560.98 | 559.61 | 557.6 | 555 | 553 | 550 | 565 |
| Sigma interaction energy | -183 | -193 | -198 | -205 | -212 | -219 | -226 | -184 |
| pion interaction energy | -945 | -955 | -961 | -970 | -978 | -989 | -994 | -943 |
| Meson interaction energy | 100.5361 | 100.725 | 100.899 | 101.175 | 101.5 | 101.9 | 102.3 | 101 |
| Hedgehog mass baryon | 1119.12 | 1091 | 1075.45 | 1051.86 | 1029 | 1006 | 983 | 1116 |
| Total moment of proton | 2.87 | 2.90 | 2.92 | 2.951 | 2.79 | 3.02 | (2.79)Exp. |
| Total moment of neutron | -2.29 | -2.32 | -2.33 | -2.355 | -2.37 | -2.39 | -2.41 | (-1.91)Exp. |
| Total $g_A (0)$ | 1.527 | 1.553 | 1.56 | 1.589 | 1.85 | 1.854 | 1.84 | (1.25)Exp. |
| Total $g_{\pi NN \frac{m_\pi}{2M_B}}$ | 1.8606 | 1.859 | 1.857 | 1.856 | 1.61 | 1.63 | 1.65 | (1.00)Exp. |
Table(2) Details of energy calculations and magnetic moments for $m_q = 500\text{MeV}, m_\sigma = 139.6\text{MeV}$ at different values of $m_\sigma$ parameter. All quantities in MeV.

| $m_\sigma$ | 1200 | 1100 | 1000 | 900  | 800  | 500  | 10  |
|------------|------|------|------|------|------|------|-----|
| Quark eigenvalue | 31.995 | 33.26 | 34.96 | 37.26 | 40.411 | 58.09 | 30.5 |
| Quark kinetic energy | 1225.01 | 1221.97 | 1217.36 | 1210.54 | 1200.72 | 1137.96 | 1219 |
| Sigma kinetic energy | 359.66 | 360.33 | 360.65 | 360.48 | 359.55 | 348.51 | 358 |
| Pion kinetic energy | 562.92 | 553.7 | 542.61 | 528.86 | 511 | 428.23 | 565 |
| Sigma interaction energy | -183 | -182 | -180 | -178 | -174 | -148 | -184 |
| Pion interaction energy | -945 | -939 | -931 | -920 | -905 | -815 | -943 |
| Meson interaction energy | 100.536 | 102.04 | 103.81 | 105.91 | 108.35 | 117 | 101 |
| Hedgehog mass baryon | 119.12 | 1115.95 | 1111.99 | 1107.07 | 1100.83 | 831.80 | 1116 |
| Total moment of proton | 2.876 | 2.875 | 2.874 | 2.872 | 2.871 | 2.87 | (2.79) Exp. |
| Total moment of neutron | -2.29 | -2.289 | -2.282 | -2.273 | -2.264 | -2.22 | (-1.91) Exp. |
| Total $g_A (0)$ | 1.8606 | 1.85 | 1.84 | 1.83 | 1.82 | 1.78 | (1.25) Exp. |
| Total $g_{\pi NN} \frac{m_\pi}{2M_B}$ | 1.5276 | 1.5278 | 1.5279 | 1.528 | 1.5289 | 1.53 | (1.00) Exp. |

Table(3) Details of energy calculations and magnetic moments for $m_\sigma = 1200\text{MeV}, m_\pi = 139.6\text{MeV}$ at different values of $m_q$. All quantities MeV.

| $m_q$ | 500  | 490  | 480  | 470  | 460  | 450  | 10  |
|-------|------|------|------|------|------|------|-----|
| Quark eigenvalue | 31.995 | 44.77 | 57.58 | 70.47 | 83.46 | 96.58 | 30.5 |
| Quark kinetic energy | 1225.01 | 1209.51 | 1193.33 | 1176.42 | 1158.6 | 1139.71 | 1219 |
| Sigma kinetic energy | 359.66 | 353.8 | 347.53 | 340.58 | 332.97 | 289.74 | 358 |
| Pion kinetic energy | 562.92 | 553.86 | 547.3 | 540.58 | 532.97 | 511 | 565 |
| Sigma interaction energy | -183 | -164 | -144 | -123.35 | -101 | -77 | -184 |
| Pion interaction energy | -945 | -910 | -876 | -841.66 | -806 | -772 | -943 |
| Meson interaction energy | 100.536 | 100.057 | 99.6822 | 99.411 | 99.244 | 99.20 | 101 |
| Hedgehog mass baryon | 119.12 | 1141.74 | 1163.67 | 1184.91 | 1205.36 | 1224 | 1116 |
| Total moment of proton | 2.876 | 2.867 | 2.85 | 2.83 | 2.824 | 2.80 | (2.79) Exp. |
| Total moment of neutron | -2.29 | -2.289 | -2.282 | -2.273 | -2.264 | -2.22 | (-1.91) Exp. |
| Total $g_A (0)$ | 1.8606 | 1.85 | 1.84 | 1.83 | 1.82 | 1.78 | (1.25) Exp. |
| Total $g_{\pi NN} \frac{m_\pi}{2M_B}$ | 1.5276 | 1.5278 | 1.5279 | 1.528 | 1.5289 | 1.53 | (1.00) Exp. |

Table(4) Details of energy calculations and magnetic moments for $m_q = 461\text{MeV}, m_\pi = 139.6\text{MeV}, m_\sigma = 900\text{MeV}, f_\pi = 92.3\text{MeV}$. All quantities MeV.

| Quark kinetic energy | 1134.58 |
| Sigma kinetic energy | 331.9 |
| Pion kinetic energy | 477.09 |
| Sigma interaction energy | -98 |
| Pion interaction energy | -778 |
| Total without C. M. correction | 939.3 |
| Meson interaction energy | 106.62 |
| Hedgehog mass baryon | 1172 |
| Total moment of proton | 2.82 |
| Total moment of neutron | -2.22 |
| Total $g_A (0)$ | 1.822 |
| Total $g_{\pi NN} \frac{m_\pi}{2M_B}$ | 1.484 |
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