MAX-STABLE PROCESSES AND THE FUNCTIONAL $D$-NORM REVISITED

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Abstract. Aulbach et al. (2013) introduced a max-domain of attraction approach for extreme value theory in $C[0,1]$ based on functional distribution functions, which is more general than the approach based on weak convergence in de Haan and Lin (2001). We characterize this new approach by decomposing a process into its univariate margins and its copula process. In particular, those processes with a polynomial rate of convergence towards a max-stable process are considered. Furthermore we investigate the concept of differentiability in distribution of a max-stable processes.

1. Introduction

A stochastic process $\xi = (\xi_t)_{t \in [0,1]}$ on the interval $[0,1]$, whose sample paths belong to the space $C[0,1]$ of continuous functions on $[0,1]$, is called max-stable (MSP), if there are norming functions $a_n, b_n \in C[0,1]$, $a_n > 0$, such that the distribution of the process $\max_{1 \leq i \leq n} (\xi_i - b_n)/a_n$ coincides with that of $\xi$ for each $n \in \mathbb{N}$. By $\xi^{(1)}, \xi^{(2)}, \ldots$ we denote independent copies of $\xi$.

Aulbach et al. (2013) established a characterization of the distribution of an MSP via a norm on $E[0,1]$, the space of bounded functions on $[0,1]$ which have finitely many discontinuities. This norm is called $D$-norm and it is defined by means of a so-called generator process.

An MSP $\eta = (\eta_t)_{t \in [0,1]} \in C[0,1]$ with standard negative exponential margins $P(\eta_t \leq x) = \exp(x)$, $x \leq 0$, will be called a standard max-stable process (SMSP). As the distribution of a stochastic process on $[0,1]$ is determined by its finite dimensional marginal distributions, a process $\eta \in C[0,1]$ with identical marginal distribution function (df) $F(x) = \exp(x)$, $x \leq 0$, is standard max-stable if and only

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If
\[ P\left( \eta \leq \frac{f}{n} \right)^n = P(\eta \leq f), \quad f \in E^{-}[0, 1], \quad n \in \mathbb{N}, \]
where \( E^{-}[0, 1] \) denotes the subset of those functions in \( E[0, 1] \), which attain only non positive values. Note that it would be sufficient in equation (1) to consider \( f \in C^{-}[0, 1] \) of non positive continuous functions. The extension to \( E^{-}[0, 1] \), however, provides the inclusion of the finite dimensional marginal distributions (fidis) of \( \eta \), as
\[ P(\eta_i \leq x_i, \ 1 \leq i \leq d) = P(\eta \leq f), \]
where \( 0 \leq t_1 < \cdots < t_d \leq 1 \) and \( f \in E^{-}[0, 1] \) is given by \( f(t_i) = x_i < 0 \) for \( i \in \{1, \ldots, d\} \) and \( f(t) = 0 \) for \( t \in [0, 1] \setminus \{t_1, \ldots, t_d\} \).

From Aulbach et al. (2013) we know that (1) is equivalent with the condition that there is some norm \( \|\cdot\|_D \) on \( E[0, 1] \), called \( D \)-norm, satisfying
\[ P(\eta \leq f) = \exp(-\|f\|_D), \quad f \in E^{-}[0, 1]. \]
Precisely, there exists a stochastic process \( Z = (Z_t)_{t \in [0, 1]} \in C[0, 1] \) with
\[ 0 \leq Z_t \leq m, \quad E(Z_t) = 1, \quad t \in [0, 1], \]
for some number \( m \geq 1 \), such that
\[ \|f\|_D = E\left( \sup_{t \in [0, 1]} (|f(t)|Z_t) \right), \quad f \in E[0, 1]. \]
The condition \( Z_t \leq m, \ t \in [0, 1], \) can be weakened to \( E\left( \sup_{t \in [0, 1]} Z_t \right) < \infty \). Observe that property [2] corresponds to the spectral representation of a max-stable process given in Resnick and Roy (1991) and de Haan (1984) since \( P(-\eta^{-1} \leq y) = \exp(-y^{-1}) \) for \( y > 0 \) and \( t \in [0, 1] \).

Based on this characterization, Aulbach et al. (2013) introduced a functional domain of attraction approach for stochastic processes in terms of convergence of their distribution functions, which is more general than the one based on weak convergence as investigated in de Haan and Lin (2001). In Section 2 of the present paper we will carry de Haan and Lin’s (2001) characterization of max-domain of attraction for stochastic processes in \( C[0, 1] \) in terms of weak convergence over to our domain of attraction approach based on convergence of df.

Buishand et al. (2008) suggested the definition of generalized Pareto processes (GPP), which extends the multivariate approach to function spaces. This particular approach was investigated and settled in Ferreira and de Haan (2012), Aulbach et al. (2013) and Dombry and Ribatet (2013). In Section 3 we will introduce certain \( \delta \)-neighborhoods of GPP, which can be characterized by their rate of convergence.
towards a max-stable process. This is in complete accordance with the multivariate case.

Finally, we establish the concept of differentiability in distribution of an SMSP in Section 4. To this end, we investigate some properties of SMSP such as the partial derivatives of a $D$-norm, the distribution of the increments of an SMSP and the conditional distribution of an SMSP given one point being observed.

To improve the readability of this paper we use bold face such as $\xi$, $X$ for stochastic processes and default font $f$, $a_n$ etc. for non stochastic functions. Operations on functions such as $\xi < f$ or $(\xi - b_n)/a_n$ are meant pointwise. The usual abbreviations iid, a.s. and rv for the terms independent and identically distributed, almost surely and random variable, respectively, are used.

2. A Characterization of Max-Domain of Attraction

In the multivariate framework, it is well-known that a rv $(X_1, \ldots, X_d)$ is in the domain of attraction of a multivariate max-stable distribution if and only if its copula has this property and the distribution of $X_i$ is in the univariate domain of attraction of a max-stable distribution for each $i = 1, \ldots, n$. We refer to Galambos (1978), Deheuvels (1978, 1984) and Aulbach et al. (2012) for details.

De Haan and Lin (2001) extended this result to stochastic processes, where domain of attraction is now meant in the sense of weak convergence; condition (3), see below, is part of their characterization. We will carry de Haan and Lin’s (2001) result over to our domain of attraction approach based on convergence of dfs of stochastic processes.

Let $X = (X_t)_{t \in [0, 1]} \in C[0, 1]$ be a stochastic process with continuous marginal df $F_t(x) = P(X_t \leq x)$, $x \in \mathbb{R}$, $t \in [0, 1]$, and let $\xi = (\xi_t)_{t \in [0, 1]} \in C[0, 1]$ be an MSP with marginal df $G_t$, $t \in [0, 1]$. Suppose that there exist norming functions $a_n, b_n \in C[0, 1]$, $a_n > 0$, $n \in \mathbb{N}$, such that

$$
\sup_{t \in [0, 1]} |n\{F_t(a_n(t)f(t) + b_n(t)) - 1\} - \log \{G_t(f(t))\}| \to_{n \to \infty} 0
$$

for each $f \in E[0, 1]$ with $\inf_{t \in [0, 1]} G_t(f(t)) > 0$. This is essentially condition (3.11) in de Haan and Lin (2001). Using Taylor expansion $\log(1 + \varepsilon) = \varepsilon + O(\varepsilon^2)$ as $\varepsilon \to 0$, condition (3) in particular implies weak convergence of the univariate margins

$$
F_t(a_n(t)x + b_n(t))^n \to_{n \to \infty} G_t(x), \quad x \in \mathbb{R}, t \in [0, 1].
$$

Put $U := (U_t)_{t \in [0, 1]} := (F_t(X_t))_{t \in [0, 1]}$, which is the copula process corresponding to $X$. Let $U^{(1)}, U^{(2)}, \ldots$ be independent copies of $U$, and let $X^{(1)}, X^{(2)}, \ldots$ be independent copies of $X$. The following theorem is the main result of this section.
Theorem 2.1. We have under condition 3

\[ P \left( \max_{1 \leq i \leq n} \frac{X^{(i)} - b_n}{a_n} \leq f \right) \rightarrow_{n \to \infty} P(\xi \leq f), \quad f \in E[0, 1], \]

if and only if

\[ P \left( n \max_{1 \leq i \leq n} \left( \frac{U^{(i)} - 1}{\max t, x, 1} \right) \leq g \right) \rightarrow_{n \to \infty} P(\eta \leq g), \quad g \in E^{-}[0, 1], \]

where for the implication \[ \Rightarrow \] we set \( \eta_t := \log(G_t(\xi_t)), \quad t \in [0, 1], \) and for the reverse conclusion \( \xi_t := G_t^{-1}(\exp(\eta_t)), \quad t \in [0, 1]. \) In both cases the processes \( \xi := (\xi_t)_{t \in [0, 1]} \) and \( \eta := (\eta_t)_{t \in [0, 1]} \in C[0, 1] \) are max-stable, \( \eta \) being an SMSP.

By Lemma 1 in Aulbach et al. (2013) or the elementary arguments as in the proof of Theorem 9.4.1 in de Haan and Ferreira (2006), one obtains that \( P(G_t(\xi_t) = 0 \text{ for some } t \in [0, 1]) = 0, \) i.e., the processes \( \eta \) and \( \xi \) are well defined.

Proof. As \( X \) has continuous sample paths, we have continuity of the function \( [0, 1] \ni t \mapsto G_t(x) \) for each \( x \in \mathbb{R} \) and, thus, continuity of the function \( [0, 1] \times \mathbb{R} \ni (t, x) \rightarrow G_t(x) \) as well as its monotonicity in \( x \) for a fixed \( t. \)

We first establish the implication \[ \Rightarrow \]. Choose \( g \in E^{-}[0, 1] \) with \( \sup_{t \in [0, 1]} g(t) < 0 \) and put \( f(t) := G_t^{-1}(\exp(g(t))). \) Then \( f \in E[0, 1] \) and we obtain from assumption \[ 3 \]

\[ P \left( \max_{1 \leq i \leq n} X^{(i)} \leq a_n f + b_n \right) \rightarrow_{n \to \infty} P(\xi \leq f) = P(\eta \leq g) = \exp(-\|g\|_D), \]

where \( \|\cdot\|_D \) is the \( D \)-norm corresponding to the SMSP \( \eta. \)

We have, on the other hand, by condition \[ 3 \]

\[ P \left( \max_{1 \leq i \leq n} X^{(i)} \leq a_n f + b_n \right) \]

\[ = P \left( n \max_{1 \leq i \leq n} \left( U^{(i)} - 1 \right) \leq n(F_t(a_n(t)f(t) + b_n(t)) - 1), \quad t \in [0, 1] \right) \]

\[ = P \left( n \max_{1 \leq i \leq n} \left( U^{(i)} - 1 \right) \leq g(t) + r_n(t), \quad t \in [0, 1] \right), \]

where \( r_n(t) = o(1) \) as \( n \to \infty, \) uniformly for \( t \in [0, 1]. \) We claim that

\[ P \left( n \max_{1 \leq i \leq n} \left( U^{(i)} - 1 \right) \leq g \right) \rightarrow_{n \to \infty} P(\eta \leq g). \]

Replace \( g \) by \( g + \varepsilon \) and \( g - \varepsilon \) for \( \varepsilon > 0 \) small enough such that \( g + \varepsilon < 0, \) and put

\[ f_\varepsilon(t) := G_t^{-1}(\exp(g(t) + \varepsilon)), \quad f_{-\varepsilon}(t) := G_t^{-1}(\exp(g(t) - \varepsilon)), \quad t \in [0, 1]. \]
Then $f_\varepsilon, f_{-\varepsilon} \in E[0,1]$, and we obtain from condition 3 and equation 6 for $n \geq n_0$

$$P\left( n \max_{1 \leq i \leq n} \left( U^{(i)}_t - 1 \right) \leq n(F_t(a_n(t)f_\varepsilon(t) + b_n(t)) - 1), t \in [0,1] \right)$$

$$\geq P\left( n \max_{1 \leq i \leq n} \left( U^{(i)} - 1 \right) \leq g \right)$$

$$\geq P\left( n \max_{1 \leq i \leq n} \left( U^{(i)}_t - 1 \right) \leq n(F_t(a_n(t)f_{-\varepsilon}(t) + b_n(t)) - 1), t \in [0,1] \right),$$

where the upper bound converges to $\exp(-\|g + \varepsilon\|_D)$ and the lower bound to $\exp(-\|g - \varepsilon\|_D)$. As both converge to $\exp(-\|g\|_D)$ as $\varepsilon \to 0$, we have established 7.

Next we claim that 7 is true for each $g \in E^-[0,1]$, i.e., we drop the assumption $\sup_{t \in [0,1]} g(t) < 0$. We prove this by a contradiction. Suppose first that there exist $s$ such that

$$\lim \inf_{n \to \infty} P\left( n \max_{1 \leq i \leq n} \left( U^{(i)} - 1 \right) \leq g \right) \leq \exp(-\|g\|_D) - \delta$$

for some $\delta > 0$. From 7 we deduce that for each $\varepsilon > 0$

$$\lim_{n \to \infty} P\left( n \max_{1 \leq i \leq n} \left( U^{(i)} - 1 \right) \leq g - \varepsilon \right) = \exp(-\|g - \varepsilon\|_D)$$

and, thus,

$$\exp(-\|g\|_D) - \delta \geq \lim \inf_{n \to \infty} P\left( n \max_{1 \leq i \leq n} \left( U^{(i)} - 1 \right) \leq g \right) \geq \lim \inf_{n \to \infty} P\left( n \max_{1 \leq i \leq n} \left( U^{(i)} - 1 \right) \leq g - \varepsilon \right) = \exp(-\|g - \varepsilon\|_D).$$

As $\varepsilon > 0$ was arbitrary, we have reached a contradiction and, thus, we have established that

$$\lim \inf_{n \to \infty} P\left( n \max_{1 \leq i \leq n} \left( U^{(i)} - 1 \right) \leq g \right) \geq \exp(-\|g\|_D), \quad g \in E^-[0,1].$$

Suppose next that there exists $g \in E^-[0,1]$ such that

$$\lim \sup_{n \to \infty} P\left( n \max_{1 \leq i \leq n} \left( U^{(i)} - 1 \right) \leq g \right) \geq \exp(-\|g\|_D) + \delta$$

for some $\delta > 0$. We have by 7 for $\varepsilon > 0$

$$\lim_{n \to \infty} P\left( n \max_{1 \leq i \leq n} \left( U^{(i)} - 1 \right) \leq -\varepsilon \right) = \exp(-\varepsilon\|1\|_D) \to_{\varepsilon \to 0} 1,$$
and, thus,

\[
\exp(-\|g\|_D) + \delta 
\leq \limsup_{n \to \infty} P\left(n \max_{1 \leq i \leq n} \left(U^{(i)} - 1\right) \leq g\right)
\]

\[
\leq \limsup_{n \to \infty} \left(P\left(n \max_{1 \leq i \leq n} \left(U^{(i)} - 1\right) \leq g, n \max_{1 \leq i \leq n} \left(U^{(i)} - 1\right) \leq -\varepsilon\right)
+ P\left(n \max_{1 \leq i \leq n} \left(U^{(i)} - 1\right) \leq -\varepsilon\right)\right)
\]

\[
= \exp(-\|(\min(g(t), -\varepsilon)\)_{t \in [0,1]}\|_D) + 1 - \exp(-\|1\|_D)
\]

by (7). As the first term in the final line above converges to \(\exp(-\|g\|_D)\) as \(\varepsilon \downarrow 0\) and the second one to zero, we have established another contradiction and, thus,

\[
\limsup_{n \to \infty} P\left(n \max_{1 \leq i \leq n} \left(U^{(i)} - 1\right) \leq g\right) \leq \exp(-\|g\|_D), \quad g \in E^-[0,1].
\]

This proves equation (7) for arbitrary \(g \in E^-[0,1]\) and completes the proof of the conclusion (4) \(\Rightarrow\) (5).

Next we establish the implication (5) \(\Rightarrow\) (4). Choose \(f \in E[0,1]\) with \(\inf_{t \in [0,1]} G_t(f(t)) > 0\) and put \(g(t) := \log(G_t(f(t)))\), \(t \in [0,1]\). From the assumption (5) we obtain

\[
P\left(n \max_{1 \leq i \leq n} \left(U^{(i)} - 1\right) \leq g\right) \Rightarrow_{n \to \infty} P(\eta \leq f) = P(\xi \leq f) = \exp(-\|g\|_D).
\]

On the other hand, we have by condition (5)

\[
P\left(n \max_{1 \leq i \leq n} \left(U^{(i)} - 1\right) \leq g\right)
= P\left(n \max_{1 \leq i \leq n} \left(U^{(i)} - 1\right) \leq n(F_t(a_n(t)f(t) + b_n(t)) - 1) + r_n(t), t \in [0,1]\right),
\]

where \(r_n(t) = o(1)\) as \(n \to \infty\), uniformly for \(t \in [0,1]\). We claim that

\[
P\left(n \max_{1 \leq i \leq n} \left(U^{(i)} - 1\right) \leq n(F_t(a_n(t)f(t) + b_n(t)) - 1), t \in [0,1]\right)
\to_{n \to \infty} P(\eta \leq g)
\]

\[
= \exp(-\|g\|_D).
\]

Replace \(g\) by \(\min(g+\varepsilon,0)\) and \(g - \varepsilon\), where \(\varepsilon > 0\) is arbitrary. Then we obtain from (5) and condition (3) for \(n \geq n_0 = n_0(\varepsilon)\)

\[
P(n \max_{1 \leq i \leq n} \left(U^{(i)} - 1\right) \leq \min(g+\varepsilon,0))
= P(n \max_{1 \leq i \leq n} \left(U^{(i)} - 1\right) \leq g + \varepsilon)
\]
\[ P \left( \max_{1 \leq i \leq n} (U^{(i)} - 1) \leq n (F_t(a_n(t) f(t) + b_n(t)) - 1), \; t \in [0, 1] \right) \]
\[ \geq P \left( \max_{1 \leq i \leq n} (U^{(i)} - 1) \leq g - \varepsilon \right). \]

Since
\[ P \left( \max_{1 \leq i \leq n} (U^{(i)} - 1) \leq \min(g + \varepsilon, 0) \right) \to_{n \to \infty} \exp(-\|\min(g + \varepsilon, 0)\|_D) \]
as well as
\[ P \left( \max_{1 \leq i \leq n} (U^{(i)} - 1) \leq g - \varepsilon \right) \to_{n \to \infty} \exp(-\|g - \varepsilon\|_D) \]
and \( \varepsilon > 0 \) was arbitrary, \( \square \) follows.

From the equality
\[ P \left( \max_{1 \leq i \leq n} (U^{(i)} - 1) \leq n (F_t(a_n(t) f(t) + b_n(t)) - 1), \; t \in [0, 1] \right) \]
\[ = P \left( \max_{1 \leq i \leq n} X^{(i)} \leq a_n f + b_n \right) \]
and from \( \square \) we obtain
\[ \lim_{n \to \infty} P \left( \max_{1 \leq i \leq n} X^{(i)} \leq a_n f + b_n \right) = P(\xi \leq f) \]
for each \( f \in E[0, 1] \) with \( \inf_{t \in [0, 1]} G_t(f(t)) > 0 \). If \( \inf_{t \in [0, 1]} G_t(f(t)) = 0 \), then, for \( \varepsilon > 0 \), there exists \( t_0 \in [0, 1] \) such that \( G_{t_0}(f(t_0)) \leq \varepsilon \). We, thus, have \( P(\xi \leq f) \leq P(\xi_{t_0} \leq f(t_0)) = G_{t_0}(f(t_0)) \leq \varepsilon \) and, by condition \( \Box \),
\[ P \left( \max_{1 \leq i \leq n} X^{(i)} \leq a_n f + b_n \right) \leq P \left( \max_{1 \leq i \leq n} X_{t_0}^{(i)} \leq a_n(t_0) f(t_0) + b_n(t_0) \right) \]
\[ \to_{n \to \infty} G_{t_0}(f(t_0)) \leq \varepsilon. \]

As \( \varepsilon > 0 \) was arbitrary, we have established
\[ \lim_{n \to \infty} P \left( \max_{1 \leq i \leq n} X^{(i)} \leq a_n f + b_n \right) = 0 = P(\xi \leq f) \]
in that case, where \( \inf_{t \in [0, 1]} G_t(f(t)) = 0 \) and, thus, \( \square \) for each \( f \in E[0, 1] \). This completes the proof of Theorem 2.1.

\[ \square \]

Corollary 2.2. Let \( X = (X_t)_{t \in [0, 1]} \) be a stochastic process with identical continuous marginal \( df(x) = P(X_t \leq x), x \in \mathbb{R}, t \in [0, 1] \), and let \( \xi = (\xi_t)_{t \in [0, 1]} \in C[0, 1] \) be an MSP with identical marginal \( df \) of \( G \). Denote by \( U = (U_t)_{t \in [0, 1]} := (F(X_t))_{t \in [0, 1]} \) the copula process pertaining to \( X \). Then we have \( X \in D(\xi) \) if and only if \( U \in D(\eta) \) and \( F \in D(G) \).

For a characterization of the condition \( U \in D(\eta) \) in terms of certain neighborhoods of generalized Pareto processes see Proposition 3.3 below.
Proof of Corollary 2.2. The assumption $F \in D(G)$ yields $\sup_{x \in \mathbb{R}} |F^n(a_n x + b_n) - G(x)| \to_{n \to \infty} 0$ for some sequence of norming constants $a_n > 0$, $b_n \in \mathbb{R}$, $n \in \mathbb{N}$. Taking logarithms and using Taylor expansion of $\log(1 + x)$ for $x \in [x_0, x_1]$ with $0 < x_0 \leq x_1$ implies

$$\sup_{x \in [x_0, x_1]} |n(F(a_n x + b_n) - 1) - \log(G(x))| \to_{n \to \infty} 0$$

and, thus, condition (b) is satisfied. Corollary 2.2 is now an immediate consequence of Theorem 2.1 together with the fact that the assumption $X \in D(\xi)$ implies in particular that $F \in D(G)$.

We conclude this section with a short remark. Choose $f \in E[0,1]$ which is not the constant function zero. If $\eta \in C^-[0,1]$ is an SMSP, then the univariate rv

$$\eta_f := \sup_{t \in [0,1]} \frac{\eta}{|f(t)|}$$

is by equation (2) negative exponential distributed with parameter $\|f\|_D$. In particular we obtain that $P\left(\sup_{t \in [0,1]} \eta < 0\right) = 1$.

3. $\delta$-Neighborhood of a Generalized Pareto Process

First we recall some facts from Aulbach et al. (2013). A univariate generalized Pareto distribution (GPD) $W$ is simply given by $W(x) = 1 + \log(G(x))$, $G(x) \geq 1/e$, where $G$ is a univariate extreme value distribution (EVD). It was established by Pickands (1975) and Balkema and de Haan (1974) that, roughly, the maximum of $n$ iid univariate observations, linearly standardized, converges in distribution to an EVD as $n$ increases if, and only if, the exceedances above an increasing threshold follow a generalized Pareto distribution (GPD). The multivariate analogon is due to Rootzén and Tajvidi (2006). It was observed by Buishand et al. (2008) that a $d$-dimensional GPD $W$ with ultimately standard Pareto margins can be represented in its upper tail as $W(x) = P(U^{-1} Z \leq x)$, $x_0 \leq x \leq 0 \in \mathbb{R}^d$, where the rv $U$ is uniformly on $(0,1)$ distributed and independent of the rv $Z = (Z_1, \ldots, Z_d)$ with $0 \leq Z_i \leq m$ for some $m \geq 1$ and $E(Z_i) = 1$, $1 \leq i \leq d$. The following definition extends this approach to function spaces.

**Definition 3.1.** Let $U$ be a rv which is uniformly distributed on $[0,1]$ and independent of the generator process $Z \in C[0,1]$ that is characterized by the two properties

$$0 \leq Z_t \leq m \quad \text{and} \quad E(Z_t) = 1, \quad t \in [0,1],$$

for some constant $m \geq 1$. Then the stochastic process

$$Y := \frac{1}{U} Z \in C^+[0,1].$$
is a generalized Pareto process (GPP) (Buishand et al., 2008; Ferreira and de Haan, 2012; Dombry and Ribatet, 2013).

The univariate margins $Y_t$ of $Y$ have ultimately standard Pareto tails:

$$P(Y_t \leq x) = P\left(\frac{1}{x} Z_t \leq U\right)$$

$$= \int_0^m P\left(\frac{1}{x} z \leq U\right) (P \ast Z_t)(dz)$$

$$= 1 - \frac{1}{x} \int_0^m z (P \ast Z_t)(dz)$$

$$= 1 - \frac{1}{x} E(Z_t)$$

$$= 1 - \frac{1}{x}, \quad x \geq m, \ 0 \leq t \leq 1.$$  

Put $V := \max(-1/Y, M)$, where $M < 0$ is an arbitrary constant, which ensures that $V > -\infty$. Then, by Fubini’s Theorem,

$$P(V \leq f) = P\left(\sup_{t \in [0,1]} (|f(t)|Z_t) \leq U\right)$$

$$= 1 - \int_0^1 P\left(\sup_{t \in [0,1]} (|f(t)|Z_t) > u\right) du$$

$$= 1 - E\left(\sup_{t \in [0,1]} (|f(t)|Z_t)\right)$$

$$= 1 - \|f\|_D$$

for all $f \in E^{-}[0,1]$ with $\|f\|_\infty \leq \min(1/m, |M|)$, i.e., $V$ has the property that its functional df is in its upper tail equal to

$$W(f) := P(V \leq f)$$

$$= 1 - \|f\|_D$$

$$= 1 + \log(\exp(-\|f\|_D))$$

$$= 1 + \log(G(f)), \quad f \in E^{-}[0,1], \ \|f\|_\infty \leq \min(1/m, |M|),$$

where $G(f) = P(\eta \leq f)$ is the functional df of the MSP $\eta$ with $D$-norm $\|\cdot\|_D$ and generator $Z$. This representation of the upper tail of a GPP in terms of $1 + \log(G)$ is in complete accordance with the uni- and multivariate case (see, for example, Falk et al., 2011, Chapter 5). We write $W = 1 + \log(G)$ in short notation and call $V$ a GPP as well.

**Remark 3.2.** Due to representation (10), the GPD process $V$ is obviously in the functional domain of attraction of an SMSP $\eta$ with $D$-norm $\|\cdot\|_D$ and generator $Z$:
Take \( a_n = 1/n \) and \( b_n = 0 \). We have for \( f \in E^[-0,1] \) and large enough \( n \in \mathbb{N} \):

\[
P \left( V \leq \frac{1}{n} f \right)^n = \left( 1 - \frac{1}{n} \| f \|_D \right)^n \to_{n \to \infty} \exp( -\| f \|_D ) = P( \eta \leq f ).
\]

The following result is a functional version of the well-known fact that the spectral df of a GPD random vector is equal to a uniform df in a neighborhood of zero.

**Lemma 3.3.** We have for \( f \in E^[-0,1] \) with \( \| f \|_\infty \leq m \) and some \( s_0 < 0 \)

\[
W_f(s) := P(V \leq s | f |) = 1 + s \| f \|_D, \quad s \leq s_0 \leq 0.
\]

Let \( U \) be a copula process. The next result is established in Aulbach et al. (2013).

**Proposition 3.4.** The property \( U \in D(\eta) \) is equivalent with the condition

\[
\lim_{s \to 0} \frac{1 - H_f(s)}{1 - W_f(s)} = 1, \quad f \in E^[-0,1],
\]

i.e., the spectral df \( H_f(s) = P(U - 1 \leq s | f |) \), \( s \leq 0 \), of \( U - 1 \) is tail equivalent with that of the GPD \( W_f = 1 + \log( G_f ) \), \( G(\cdot) = \exp( -\| \cdot \|_D ) \).

This characterization of the domain of attraction of an SMSP in terms of a certain GPP suggests to focus on the following standard case. Recall that Section 2 justified to consider SMSPs in place of general MSPs.

**Definition 3.5.** A stochastic process \( V \in C^[-0,1] \) is a **standard generalized Pareto process** (SGPP), if there exists a \( D \)-norm \( \| \cdot \|_D \) on \( E^[0,1] \) and some \( c_0 > 0 \) such that

\[
P(V \leq f) = 1 - \| f \|_D
\]

for all \( f \in E^[-0,1] \) with \( \| f \|_\infty \leq c_0 \).

The same arguments as at the end of Section 2 suggest to consider the rv

\[
V_f := \sup_{t \in [0,1]} \frac{V_t}{|f(t)|}
\]

if we want to test whether a given process \( V \in C^[-0,1] \) actually is an SGPP. Again \( f \in E^[-0,1] \) is not the constant function zero. In the case that \( V \) is an SGPP we obtain

\[
P(V_f > x) = \| f \|_D |x|, \quad -1 \leq x \leq 0,
\]

if \( \| f \|_\infty \leq c_0 \), i.e., \( V_f \) follows in its upper tail, precisely on \((-1,0)\), a uniform distribution.

Using the spectral decomposition of a stochastic process in \( C^[-0,1] \), we can easily extend the definition of a spectral \( \delta \)-neighborhood of a multivariate GPD as in Falk
et al. (2011, Section 5.5) to the spectral $\delta$-neighborhood of an SGPP. Denote by $E_{1}^{-}[0,1] = \{ f \in E^{-}[0,1] : \|f\|_{\infty} = 1 \}$ the unit sphere in $E^{-}[0,1]$.

**Definition 3.6.** We say that a stochastic process $Y \in C^{-}[0,1]$ belongs to the spectral $\delta$-neighborhood of the SGPP $V$ for some $\delta \in (0, 1]$, if we have uniformly for $f \in E_{1}^{-}[0,1]$ the expansion

$$1 - P(Y \leq cf) = (1 - P(V \leq cf))(1 + O(c^{\delta}))$$

$$= c\|f\|_{D}(1 + O(c^{\delta}))$$

as $c \downarrow 0$.

An SMSP is, for example, in the spectral $\delta$-neighborhood of the corresponding GPP with $\delta = 1$.

The following result extends Theorem 5.5.5 in Falk et al. (2011) on the rate of convergence of multivariate extremes. It shows that $\delta$-neighborhoods collect, roughly, all processes which have a polynomial rate of convergence towards an SMSP.

**Proposition 3.7.** Let $Y$ be a stochastic process in $C^{-}[0,1]$, $V$ an SGPP with $D$-norm $\|\cdot\|_{D}$ and $\eta$ a corresponding SMSP.

(i) Suppose that $Y$ is in the spectral $\delta$-neighborhood of $V$ for some $\delta \in (0, 1]$. Then we have

$$\sup_{f \in E_{1}^{-}[0,1]} \left| P\left( Y \leq \frac{f}{n} \right)^{n} - P(\eta \leq f) \right| = O(n^{-\delta}).$$

(ii) Suppose that $H_{f}(c) = P(Y \leq cf)$ is differentiable with respect to $c$ in a left neighborhood of 0 for any $f \in E_{1}^{-}[0,1]$, i.e., $h_{f}(c) \equiv (\partial/\partial c)H_{f}(c)$ exists for $c \in (-\varepsilon, 0)$ and any $f \in E_{1}^{-}[0,1]$. Suppose, moreover, that $H_{f}$ satisfies the von Mises condition

$$\frac{-ch_{f}(c)}{1 - H_{f}(c)} =: 1 + r_{f}(c) \rightarrow_{c\downarrow 0} 1,$$

$f \in E_{1}^{-}[0,1]$, with remainder term $r_{f}$ satisfying

$$\sup_{f \in E_{1}^{-}[0,1]} \left| \int_{c}^{0} \frac{r_{f}(t)}{t} \, dt \right| \rightarrow_{c\downarrow 0} 0.$$

If

$$\sup_{f \in E^{-}[0,1]} \left| P\left( Y \leq \frac{f}{n} \right)^{n} - P(\eta \leq f) \right| = O(n^{-\delta})$$

for some $\delta \in (0, 1]$, then $Y$ is in the spectral $\delta$-neighborhood of the GPP $V$. 
Proof. Note that
\[
\sup_{f \in E^{[0,1]}} \left| P\left( Y \leq \frac{f}{n} \right)^n - P(\eta \leq f) \right|
= \sup_{f \in E^{[0,1]}} \left| P\left( Y \leq \frac{\|f\|_\infty}{n} \frac{f}{\|f\|_\infty} \right)^n - P(\eta \leq \|f\|_\infty \frac{f}{\|f\|_\infty}) \right|
= \sup_{c<0} \sup_{f \in E^{[0,1]}} \left| P\left( Y \leq \frac{c}{n} |f| \right)^n - P(\eta \leq c |f|) \right|
= \sup_{f \in E^{[0,1]}} \sup_{c<0} \left| P\left( Y \leq \frac{c}{n} |f| \right)^n - P(\eta \leq c |f|) \right|
\]
The assertion now follows by repeating the arguments in the proof of Theorem 1.1 in [Falk and Reiss (2002)], where the bivariate case has been established. □

4. DISTRIBUTIONAL DIFFERENTIABILITY OF AN SMSP

In this section, our aim is to establish the concept of distributional differentiability of an SMSP \( \eta = (\eta_t)_{t \in [0,1]} \), that is, the convergence in distribution of the difference quotient \( (\eta_{t+h} - \eta_t)/h \) to some rv on the real line for \( h \to 0 \). To this end, we need to determine the distribution of the increments \( \eta_s - \eta_t, \ s \neq t \). This can be achieved by the calculation of the conditional distribution of \( \eta_t \), given \( \{\eta_{t_0} = x\} \) for some \( t_0 \in [0,1] \) and \( x < 0 \), which is part of an interesting and challenging open problem by itself: Suppose the distribution of the SMSP \( (\eta_t)_{t \in [0,1]} \) is known, and the process has already been observed at some points \( \{\eta_{t_k} = x_k, \ k = 1, \ldots, n\} \), how can one determine the conditional distribution of \( (\eta_t)_{t \in [0,1]} \) given these observations?

As an auxiliary result, we first compute the partial derivatives of a functional \( D \)-norm \( \|\cdot\|_D \). For this purpose, we need the following definition. Let \( \mathcal{X} \) be a normed function space, and \( J : \mathcal{X} \to \mathbb{R} \) a functional. The first variation (or the Gâteaux differential) of \( J \) at \( u \in \mathcal{X} \) in the direction \( v \in \mathcal{X} \) is defined as
\[
\nabla J(u)(v) := \lim_{\varepsilon \to 0} \frac{J(u + \varepsilon v) - J(u)}{\varepsilon} = \frac{d}{d \varepsilon} J(u + \varepsilon v) \bigg|_{\varepsilon = 0}.
\]
Moreover, the right-hand (left-hand) first variation of \( J \) at \( u \) in the direction \( v \) is defined as
\[
\nabla^+ J(u)(v) := \lim_{\varepsilon \downarrow 0} \frac{J(u + \varepsilon v) - J(u)}{\varepsilon} \quad \text{and} \quad \nabla^- J(u)(v) := \lim_{\varepsilon \downarrow 0} \frac{J(u) - J(u - \varepsilon v)}{\varepsilon}.
\]
Considering a \( D \)-norm \( \|\cdot\|_D \) a functional on the space \( E[0,1] \), we can calculate the first variation of \( \|\cdot\|_D \). The choice of the space \( E[0,1] \) allows us the incorporation of the fidis and therefore yields the partial derivatives of a multivariate \( D \)-norm. This finite-dimensional version of the following result has already been observed by
Therefore we obtain by the dominated convergence theorem of the one point set
\[ \{ t \} \]
with
\[ \| \cdot \| : = \| \cdot \|_{\mathbb{R}^d} \]
Denote further by \( \text{sgn} \) the sign function, i.e. \( \text{sgn}(x) = 1 \) for \( x > 0 \) and \( \text{sgn}(x) = -1 \) for \( x < 0 \).

**Lemma 4.1.** Let \( \| \cdot \|_D \) be a D-norm on the function space \( E[0, 1] \) with generator \( Z = (Z_t)_{t \in [0, 1]} \in C[0, 1] \). Let \( t_0 \in [0, 1] \) and \( 1_{\{t_0\}} \in E[0, 1] \) be the indicator function of the one point set \( \{t_0\} \). Then for every \( f \in E[0, 1] \) with \( f(t_0) \neq 0 \)
\[
\nabla^+ \| f \|_D (1_{\{t_0\}}) = \lim_{\varepsilon \downarrow 0} \frac{\| f + \varepsilon 1_{\{t_0\}} \|_D - \| f \|_D}{\varepsilon}
= \text{sgn}(f(t_0)) E \left( \{ \sup_{t \neq t_0} |f(t)| Z_t \leq |f(t_0)| Z_{t_0} \} | Z_{t_0} \right)
= \text{sgn}(f(t_0)) E \left( \{ \sup_{t \neq t_0} |f(t)| Z_t < |f(t_0)| Z_{t_0} \} | Z_{t_0} \right).
\]
and
\[
\nabla^- \| f \|_D (1_{\{t_0\}}) = \lim_{\varepsilon \downarrow 0} \frac{\| f \|_D - \| f - \varepsilon 1_{\{t_0\}} \|_D}{\varepsilon}
= \text{sgn}(f(t_0)) E \left( \{ \sup_{t \neq t_0} |f(t)| Z_t \leq |f(t_0)| Z_{t_0} \} | Z_{t_0} \right).
\]
In particular, the left-side first variation of \( \| f \|_D \) in the direction \( 1_{\{t_0\}} \) always equals zero if \( f \) is continuous in \( t_0 \).

**Proof.** Let \( f \in E[0, 1] \) with \( f(t_0) > 0 \). We have
\[
\| f \|_D = E \left( \sup_{t \in [0, 1]} |f(t)| Z_t \right).
\]
First we calculate the right-hand first variation of \( \| \cdot \|_D \). For \( \varepsilon > 0 \) there exists a disjoint decomposition of the underlying probability space \( (\Omega, \mathcal{A}, P) \) via
\[
\Omega = A_1 + A_2^\varepsilon + A_3^\varepsilon
\]
with
\[
A_1 := \left\{ \sup_{t \neq t_0} |f(t)| Z_t \leq f(t_0) Z_{t_0} = \sup_{t \in [0, 1]} |f(t)| Z_t \right\},
\]
\[
A_2^\varepsilon := \left\{ f(t_0) Z_{t_0} < \sup_{t \neq t_0} |f(t)| Z_t = \sup_{t \in [0, 1]} |f(t)| Z_t \leq (f(t_0) + \varepsilon) Z_{t_0} \right\} \downarrow_{\varepsilon \downarrow 0} \emptyset,
\]
\[
A_3^\varepsilon := \left\{ (f(t_0) + \varepsilon) Z_{t_0} < \sup_{t \neq t_0} |f(t)| Z_t = \sup_{t \in [0, 1]} |f(t)| Z_t \right\}.
\]

Therefore we obtain by the dominated convergence theorem
\[
\nabla^+ \| f \|_D (1_{\{t_0\}}) = \lim_{\varepsilon \downarrow 0} \frac{\| f + \varepsilon 1_{\{t_0\}} \|_D - \| f \|_D}{\varepsilon}
\]
Hence we obtain again by the dominated convergence theorem
\[
\| \cdot \| = \lim_{\varepsilon \downarrow 0} E \left( \frac{1}{\varepsilon} \left( \max_{t \neq t_0} \| f(t) \|_{Z_t, (f(t_0) + \varepsilon)Z_{t_0}} - \sup_{t \in [0,1]} \| f(t) \|_{Z_t} \right) \right)
\]
\[
= \lim_{\varepsilon \downarrow 0} E \left( \frac{1}{\varepsilon} \left( (f(t_0) + \varepsilon)Z_{t_0} - f(t_0)Z_{t_0} \right) \cdot 1_{A_1} \right)
\]
\[
+ \lim_{\varepsilon \downarrow 0} E \left( \frac{1}{\varepsilon} \left( (f(t_0) + \varepsilon)Z_{t_0} - \sup_{t \neq t_0} \| f(t) \|_{Z_t} \right) \cdot 1_{A_2} \right)
\]
\[
+ \lim_{\varepsilon \downarrow 0} E \left( \frac{1}{\varepsilon} \left( \sup_{t \neq t_0} \| f(t) \|_{Z_t} - \sup_{t \neq t_0} \| f(t) \|_{Z_t} \right) \cdot 1_{A_3} \right)
\]
\[= E(Z_{t_0} \cdot 1_{A_1}) \]

since the second summand after the second to last equality vanishes as
\[
\frac{1}{\varepsilon} \left( (f(t_0) + \varepsilon)Z_{t_0} - \sup_{t \neq t_0} \| f(t) \|_{Z_t} \right) \cdot 1_{A_2} < \frac{1}{\varepsilon} \left( (f(t_0) + \varepsilon)Z_{t_0} - f(t_0)Z_{t_0} \right) \cdot 1_{A_2} \to_{\varepsilon \downarrow 0} Z_{t_0} \cdot 1_{\emptyset} = 0.
\]

Note that
\[
\sup_{t \neq t_0} \| f(t) \|_{Z_t} \leq f(t_0)Z_{t_0} \iff \sup_{t \in [0,1]} \| f(t) \|_{Z_t} = f(t_0)Z_{t_0}.
\]

In order to calculate the left-side first variation of \( \| \cdot \|_{D} \), we find for \( \varepsilon > 0 \) a disjoint decomposition of \( (\Omega, \mathcal{A}, P) \) via
\[
\Omega = B_1^\varepsilon + B_2^\varepsilon + B_3
\]

with
\[
B_1^\varepsilon := \left\{ \sup_{t \neq t_0} \| f(t) \|_{Z_t} \leq (f(t_0) - \varepsilon)Z_{t_0} \right\} \uparrow_{\varepsilon \downarrow 0} \sup_{t \neq t_0} \| f(t) \|_{Z_t} < (f(t_0) - \varepsilon)Z_{t_0} \right\} =: B_1,
\]
\[
B_2^\varepsilon := \left\{ (f(t_0) - \varepsilon)Z_{t_0} < \sup_{t \neq t_0} \| f(t) \|_{Z_t} \leq (f(t_0) - \varepsilon)Z_{t_0} \right\} \downarrow_{\varepsilon \downarrow 0} \emptyset
\]
\[
B_3 := \left\{ f(t_0)Z_{t_0} < \sup_{t \neq t_0} \| f(t) \|_{Z_t} \right\}.
\]

Hence we obtain again by the dominated convergence theorem
\[
\nabla^ {-} \| f \|_{D} (1_{(t_0)}) = \lim_{\varepsilon \downarrow 0} \frac{\| f \|_{D} - \| f - \varepsilon 1_{(t_0)} \|_{D}}{\varepsilon}
\]
\[
= \lim_{\varepsilon \downarrow 0} E \left( \frac{1}{\varepsilon} \left( \sup_{t \in [0,1]} \| f(t) \|_{Z_t} - \max \left( \sup_{t \neq t_0} \| f(t) \|_{Z_t}, (f(t_0) - \varepsilon)Z_{t_0} \right) \right) \right)
\]
\[
= \lim_{\varepsilon \downarrow 0} E \left( \frac{1}{\varepsilon} \left( f(t_0)Z_{t_0} - (f(t_0) - \varepsilon)Z_{t_0} \right) \cdot 1_{B_1^\varepsilon} \right)
\]
by basic rules of conditional distributions for almost all \( x < t \).

Now define the function \( g \).
The random variable \( f(t) \) argument as above. The case \( f(t_0) < 0 \) works analogously.

The finite-dimensional version of the following Lemma is part of Proposition 5 in Dombry and Éyi-Minko (2012).

**Lemma 4.2.** Let \( \eta = (\eta_t)_{t \in [0,1]} \) be an SMSP with D-norm \( \| \cdot \|_D \) generated by \( Z = (Z_t)_{t \in [0,1]} \). Choose an arbitrary \( t_0 \in [0,1] \). Then for every \( f \in \mathcal{E}^-_{t} [0,1] \) with \( f(t_0) = 0 \) and almost all \( x < 0 \)

\[
P(\eta \leq f | \eta_{t_0} = x) = \exp \left( - (x + \| f + x \cdot 1_{\{t_0\}} \|_D) \right) \cdot E \left( 1_{\{\sup_{t \in [0,1]} |f(t)| Z_t \leq x | Z_{t_0} \}} Z_t \right).
\]

**Proof.** The random variable \( \eta_{t_0} \) has Lebesgue-density \( e^y, y \leq 0 \). Therefore, we have by basic rules of conditional distributions for almost all \( x < 0 \)

\[
P(\eta \leq f | \eta_{t_0} = x) = \lim_{\varepsilon \downarrow 0} \frac{e^{-1} P(\eta \leq f, \eta_{t_0} \in (x, x + \varepsilon])}{e^{-1} P(\eta_{t_0} \in (x, x + \varepsilon])}
= \exp(-x) \lim_{\varepsilon \downarrow 0} \frac{P(\eta \leq f, \eta_{t_0} \leq x + \varepsilon) - P(\eta \leq f, \eta_{t_0} \leq x)}{\varepsilon}.
\]

Now define the function \( g \in \mathcal{E}^-_{t} [0,1] \) by \( g(t) = f(t), t \neq t_0 \), and \( g(t_0) = x \). Then we have by Lemma 4.1.

\[
P(\eta \leq f | \eta_{t_0} = x) = \exp(-x) \lim_{\varepsilon \downarrow 0} \frac{\exp(-\| g + \varepsilon 1_{\{t_0\}} \|_D) - \exp(-\| g \|_D)}{\varepsilon}
= - \exp(-x) \exp(-\| g \|_D) \cdot \nabla^+ \| g \|_D (1_{\{t_0\}})
= \exp(-x + \| f + x 1_{\{t_0\}} \|_D) \cdot E \left( 1_{\{\sup_{t \in [0,1]} |f(t)| Z_t \leq x | Z_{t_0} \}} \right).
\]

The preceding result can be used for the derivation of the distribution of the increments of an SMSP, which is the content of the next lemma.
Lemma 4.3. Consider an SMSP $\eta = (\eta_t)_{t \in [0,1]}$ with generator process $Z = (Z_t)_{t \in [0,1]}$ and choose arbitrary $s, t \in [0,1]$, $s \neq t$. Denote by $\|\cdot\|_D$ the $D$-norm pertaining to $(\eta_s, \eta_t)$. Then for every $x \in \mathbb{R}$

$$P(\eta_s - \eta_t \leq x) = \begin{cases} \int_{-\infty}^{0} \exp(-\|x + y, y\|_D) \cdot E(1_{\{y Z_t \leq (x+y)Z_s\}}) \, dy, & x < 0 \\ \int_{-\infty}^{\infty} \exp(-\|x + y, y\|_D) \cdot E(1_{\{y Z_t \leq (x+y)Z_s\}}) \, dy + 1 - \exp(-x), & x \geq 0. \end{cases}$$

Proof. We have by basic rules of conditional distributions for every $x \in \mathbb{R}$

$$P(\eta_s \leq x + \eta_t) = \begin{cases} \int_{-\infty}^{0} P(\eta_s \leq x + y|\eta_t = y) \exp(y) \, dy, & x < 0 \\ \int_{-\infty}^{0} P(\eta_s \leq x + y|\eta_t = y) \exp(y) \, dy + \int_{-\infty}^{0} \exp(y) \, dy, & x \geq 0. \end{cases}$$

On the other hand, we obtain from Lemma 4.2 for almost all $y < 0$ and every $x \in \mathbb{R}$ with $x + y \leq 0$

$$P(\eta_s \leq x + y|\eta_t = y) = \exp(-(y + \|x + y, y\|_D)) \cdot E(1_{\{\max(|x+y|Z_s, |y|Z_t) = |y|Z_t\}} \cdot Z_t)$$

$$= \exp(-(y + \|x + y, y\|_D)) \cdot E(1_{\{x+y \geq yZ_t\}} \cdot Z_t),$$

which completes the proof.

The preceding lemma allows us to introduce the following differentiability concept. We call a stochastic process $(X_t)_{t \in [0,1]}$ differentiable in distribution in $t_0 \in [0,1]$, if the difference quotient $(X_{t_0+h} - X_{t_0})/h$ converges in distribution to some rv on the real line for $h \to 0$.

**Proposition 4.4** (Differentiability in Distribution of SMSP). Let $\eta = (\eta_t)_{t \in [0,1]}$ be an SMSP with generator process $Z = (Z_t)_{t \in [0,1]} \in C[0,1]$. Suppose that for some $t_0 \in [0,1]$

$$Z_{t_0+h} - Z_{t_0} \to_{h \to 0} \xi_{t_0} \quad a.s.$$  \hspace{1cm} (12)

Then we have for $x \in \mathbb{R}$

$$P\left(\frac{\eta_{t_0+h} - \eta_{t_0}}{h} \leq x\right) \to_{h \to 0} H(x) := \int_{-\infty}^{0} \exp(y) E\left(1_{\{\xi_{t_0} \leq -\frac{y}{h}Z_{t_0}\}} \cdot Z_{t_0}\right) \, dy.$$

Condition (12) means that $(\partial/\partial t)Z_t$ exists for $t = t_0$ a.s. and, therefore, $\xi_{t_0} = Z_{t_0}'$. 


Proof of Proposition 4.4. We have for $x \in \mathbb{R}$ and $h > 0$ by Lemma 4.3

\[
P(\eta_{t_0+h} - \eta_{t_0} \leq hx) = \int_{-\infty}^{-h|x|} \exp(-\| (hx + y, y) \|_{D(h)}) \cdot E\left( \mathbf{1}_{\{ y Z_{t_0} \leq (hx+y)Z_{t_0+h} \}} Z_{t_0} \right) dy + o(1)
\]

as $h \downarrow 0$, where $\| \cdot \|_{D(h)}$ is the $D_h$-norm generated by $(Z_{t_0+h}, Z_{t_0})$. Now we obtain for almost all $y < -h|x|

\[
E\left( \mathbf{1}_{\{ y Z_{t_0} \leq (hx+y)Z_{t_0+h} \}} Z_{t_0} \right) = E\left( \mathbf{1}_{\{ y \xi_{t_0} \leq -\frac{y}{h} Z_{t_0} \}} Z_{t_0} \right)
\]

by condition (12) which implies the assertion if $h \downarrow 0$. On the other hand, we have for $x \in \mathbb{R}$ and $h < 0$ by Lemma 4.3, condition (12), and the fact that $E(Z_{t_0}) = 1

\[
P(\eta_{t_0+h} - \eta_{t_0} \geq hx) = 1 - P(\eta_{t_0+h} - \eta_{t_0} < hx)
\]

\[
= 1 - \int_{-\infty}^{h|x|} \exp(-\| (hx + y, y) \|_{D(h)}) \cdot E\left( \mathbf{1}_{\{ y Z_{t_0} \leq (hx+y)Z_{t_0+h} \}} Z_{t_0} \right) dy + o(1)
\]

\[
\rightarrow_{h \uparrow 0} 1 - \int_{-\infty}^{0} \exp(y) E\left( \mathbf{1}_{\{ \xi_{t_0} \geq -\frac{y}{h} Z_{t_0} \}} Z_{t_0} \right) dy
\]

\[
= 1 - \int_{-\infty}^{0} \exp(y) dy + \int_{-\infty}^{0} \exp(y) E\left( \mathbf{1}_{\{ \xi_{t_0} \leq -\frac{y}{h} Z_{t_0} \}} Z_{t_0} \right) dy
\]

\[
= \int_{-\infty}^{0} \exp(y) E\left( \mathbf{1}_{\{ \xi_{t_0} \leq -\frac{y}{h} Z_{t_0} \}} Z_{t_0} \right) dy.
\]

□

Proposition 4.4 does not imply differentiability of the path of $\eta$ at $t_0$. But if $\eta$ is differentiable at $t_0$, then $H$ is the df of the derivative $(\partial/\partial t)\eta_t$ of $\eta$ at $t = t_0$. We, therefore, denote by $\eta'_{t_0}$ a rv which follows the df $H$.

Suppose that $(\partial/\partial t)Z_t$ exists for $t = t_0$ a.s. Then

\[
F_{t_0}(x) := E\left( \mathbf{1}_{\{ Z'_{t_0} \leq xZ_{t_0} \}} Z_{t_0} \right), \quad x \in \mathbb{R},
\]

defines a common df on $\mathbb{R}$. Denote by $\zeta_{t_0}$ a rv, which follows this df and which is independent of $\eta_{t_0}$. Then we obtain the equation

\[
H(x) = P(-\eta_{t_0} \zeta_{t_0} \leq x), \quad x \in \mathbb{R},
\]
i.e., we have
\[ \eta'_t = -\eta_0 \zeta_t. \]

The pathwise derivative of \( \eta \) at \( t_0 \), if it exists, coincides, therefore, in distribution with \( -\eta_0 \zeta_{t_0} \).

**Lemma 4.5.** Suppose that \( E(Z'_{t_0}) \) exists. Then the mean value of \( F_{t_0} \) exists as well and coincides with \( E(Z'_{t_0}) \).

**Proof.** The expectation of an arbitrary rv \( \xi \) exists iff
\[
\int_0^\infty P(\xi > x) \, dx + \int_{-\infty}^0 P(\xi < x) \, dx < \infty,
\]
and in this case
\[
E(\xi) = \int_0^\infty P(\xi > x) \, dx - \int_{-\infty}^0 P(\xi < x) \, dx.
\]

As a consequence we obtain from Fubini’s theorem
\[
\int x F_{t_0}(dx)
= \int_0^\infty 1 - E\left(\mathbf{1}_{\{Z'_{t_0} \leq xZ_{t_0}\}} Z_{t_0}\right) dx - \int_{-\infty}^0 E\left(\mathbf{1}_{\{Z'_{t_0} \geq xZ_{t_0}\}} Z_{t_0}\right) dx
= \int_0^\infty E\left(\mathbf{1}_{\{Z'_{t_0} > xZ_{t_0}\}} Z_{t_0}\right) dx - \int_{-\infty}^0 E\left(\mathbf{1}_{\{Z'_{t_0} \leq xZ_{t_0}\}} Z_{t_0}\right) dx
= \int z \int_0^\infty \mathbf{1}_{\{z' > xz\}} dx \left( P \ast (Z_{t_0}, Z'_{t_0}) \right)(d(z, z'))
\]
\[
\quad - \int z \int_0^\infty \mathbf{1}_{\{z' \leq xz\}} dx \left( P \ast (Z_{t_0}, Z'_{t_0}) \right)(d(z, z'))
= \int z \max\left(\frac{z'}{z}, 0\right) \left( P \ast (Z_{t_0}, Z'_{t_0}) \right)(d(z, z'))
\quad + \int z \min\left(\frac{z'}{z}, 0\right) \left( P \ast (Z_{t_0}, Z'_{t_0}) \right)(d(z, z'))
= E(Z'_{t_0}).
\]

As a consequence we obtain in particular
\[
E(\eta'_{t_0}) = -E(\eta_0 \zeta_{t_0}) = -E(\eta_0) E(\zeta_{t_0}) = E(Z'_{t_0}).
\]

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