THE COSMOLOGICAL TIME FUNCTIONS AND LIGHTLIKE RAYS

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Abstract. It is proved that all discontinuity points of a finite cosmological time function, \( \tau \), are on past lightlike rays. As a result, it is proved that if \((M, g)\) is a chronological space-time without past lightlike rays then there is a representation of \( g \) such that its cosmological time function is regular. In addition, by reducing conditions of regularity sufficient conditions for causal simplicity and causal pseudoconvexity of space-time is given. It is also proved that the second condition of regularity can be reduced to satisfies only on inextendible past-directed causal rays if \((M, g)\) be a space-time, conformal with an open subspace of Minkowski space-time or \( \tau \) be continuous.

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1. Introduction

The concept of cosmological time function was defined in [1]. Time functions are important in study of the global causal theory of space-time. They can be defined arbitrary and may have little physical significance. But the cosmological time function is defined canonically and consequently, study of it gives us important information about space-time. Let us recall its definition.

Let \((M, g)\) be a space-time and \( d : M \times M \to [0, \infty] \) be the Lorentzian distance function. The cosmological time function \( \tau : M \to [0, \infty] \) is defined by:

\[
\tau(q) := \sup_{p \leq q} d(p, q),
\]

where \( p \leq q \) means that \( q \in J^+(p) \).

A time function on the space-time \((M, g)\) in the usual sense is a real valued continuous function which is strictly increasing on causal curves. The existence of such a function on \( M \) requires causal stability (no closed causal curve in any Lorentzian metric sufficiently near the space-time metric exists). Although \( \tau \) is not always well behaved, if it is regular then it is a Cauchy time function [1].
Definition 1. \[1\] The cosmological time function $\tau$ of $(M, g)$ is regular if and only if:

- $\tau(q) < \infty$, for all $q \in M$,
- $\tau \to 0$ along every past inextendible causal curve.

The first condition is an assertion that for each point $q$ of the space-time any particle that passes through $q$ has been in existence for a finite time (the space-time has an initial singularity in the strong sense). The second condition asserts that every particle came into existence at the initial singularity.

It is proved \[1\] that if the cosmological time function of $(M, g)$ is regular then:

- it is an almost everywhere differentiable time function;
- $(M, g)$ is globally hyperbolic;
- For each $q \in M$ there is a future-directed timelike ray $\gamma_q : (0, \tau(q)] \to M$ that realizes the distance from the initial singularity to $q$, that is $\gamma_q$ is future-directed timelike unit geodesic which is maximal on each segment, such that:

$$\gamma_q(\tau(q)) = q, \quad \tau(\gamma_q(t)) = t, \quad \text{for } t \in (0, \tau(q)]. \quad (1)$$

We recall that since the Lorentzian distance is not conformally invariant, $\tau$ is not too and even if $(M, g)$ is globally hyperbolic, $\tau$ is not necessarily regular. But it is proved in \[4\] that if $(M, g)$ is globally hyperbolic then there is a smooth real function $\Omega > 0$ such that $\tau$ is regular, on $(M, \Omega g)$. This gives a characterization for global hyperbolicity.

It is also proved \[14\] that the regular $\tau$ is differentiable at $q$ iff there exists only one timelike ray of the form (1).

In the whole of this paper we suppose that $\tau$ is finite. We reduce the second condition of regularity and investigate its effects on the cosmological time function and causal properties of space-time.

In the second section, we recall some definitions and theorems which are used widely in this paper. In the third section of this paper it is proved that a point of discontinuity appears on a past lightlike ray. Hence $\tau$ is a time function in space-times with no past lightlike rays. Using this time function it can be shown by a simple proof that non-totally vicious space-times without lightlike rays are globally hyperbolic. A different proof was given in \[10\].

In addition, it can be asked that is it possible to reduce the second condition in the following way: $\tau \to 0$, along every past inextendible causal geodesic.

The following example shows that this is not the case in general.

Example 1. \[1\] Let $M := \{(x, y, t) \in S^1 \times \mathbb{R} \times \mathbb{R} : t > -1\}$ with the metric
\[ g := dy^2 + e^{2y}(dx dt + (|t|^{2\alpha} + (e^{y^2} - 1))dx^2). \]

Although \( \tau \) is going to zero along past inextendible causal geodesics it does not go to zero along causal curves.

In section 4, we investigate some conditions on \((M, g)\) and \(\tau\) which imply that \(\tau\) is regular in this situation. In addition, Using the cosmological time function sufficient conditions for causal simplicity and causal pseudoconvexity of space-time is given.

2. Preliminaries

The standard notations from Lorentzian geometry are used in this paper. The reader is referred to [2] [11]. We denote with \((M, g)\) a \(C^\infty\) space-time (a connected, Hausdorff, time oriented Lorentzian manifold) of dimension \(n \geq 2\) and signature \((-+,+,...,+\)). If \(p, q \in M\), then \(q \in I^+(p)\) (resp. \(q \in J^+(p)\)) means that there is a future directed timelike (resp. causal) curve from \(p\) to \(q\). \(I^+(p)\) is called the chronological future and \(J^+(p)\) the causal future of \(p\). Likewise \(I^-(p)\) and \(J^-(p)\) are defined and are called chronological and causal past of \(p\). \((M, g)\) is causal (resp. chronological) if there is no closed causal (resp. timelike) curve in it. A causal space-time is globally hyperbolic if \(J^+(p) \cap J^-(q)\) be compact, for every \(p, q \in M\) and is causally simple if \(J^\pm(p)\) be closed, for every \(p \in M\). In addition, a space-time is strongly causal if every point of it has arbitrary small causally convex neighbourhoods. The interested reader is referred to [8] for more details.

If \(q \in J^+(p)\) the Lorentzian distance \(d(p, q)\) is the supremum of the length of all causal curves from \(p\) to \(q\) and if \(q \notin J^+(p)\) then \(d(p, q) = 0\). If \((M, g)\) is strongly causal then there is a conformal class of \(g\) such that \(d\) and \(\tau\) are bounded.

**Lemma 1.** [6] Let \(h\) be an auxiliary complete Riemannian metric on \(M\) and let \(\rho\) be the associated distance. Let \(q \in M\) and \(B_n(q) = \{r : \rho(q, r) < n\}\) be the open ball of radius \(n\) centered at \(q\). If \((M, g)\) is strongly causal, then there is a smooth function \(\Omega > 0\), such that \(\text{diam}(M, \Omega g) = \sup\{d(p, q), p, q \in M\}\) is finite and for every \(\epsilon > 0\) there is \(n \in N\) such that if \(\gamma : I \rightarrow M\) is any \(C^1\) causal curve,

\[
\int_{I \cap \gamma^{-1}(M - B_n(q))} \sqrt{-g(\gamma', \gamma')} dt < \epsilon
\]

that is, its many connected pieces contained in the open set \(M - \overline{B_n(q)}\) have a total Lorentzian length less than \(\epsilon\).
The limit curve theorem will be used several times. The reader is referred to [9] for a strong formulation. All the curves are parametrized by $h$-arc length, where $h$ is a complete Riemannian metric. A past (future) ray in $(M, g)$ is a maximal past (future) inextendible causal geodesic $\gamma : [0, \infty) \to M$.

A sequence of causal curves $\gamma_n : [a_n, b_n] \to M$ is called limit maximizing if:

$$L(\gamma_n) \geq d(\gamma_n(a_n), \gamma_n(b_n)) - \epsilon_n,$$

where $\epsilon_n \to 0$.

The following lemma is used in section 3 and 4.

**Lemma 2.** [5] Let $z_n$ be a sequence in $M$ with $z_n \to z$. Let $z_n \in I^+(p_n)$ with finite $d(z_n, p_n)$. Let $\gamma_n : [0, a_n] \to M$ be a limit maximizing sequence of causal curves with $\gamma_n(0) = z_n$ and $\gamma_n(a_n) = p_n$. Let $\tilde{\gamma}_n : [0, \infty) \to M$ be any future extension of $\gamma_n$. Suppose either:

- $p_n \to \infty$, i.e. no subsequent is convergent,
- $d(z_n, p_n) \to \infty$.

Then any limit curve $\gamma : [0, \infty) \to M$ of the sequence $\tilde{\gamma}_n$ is a causal ray starting at $z$.

3. **Continuity of cosmological time function and lightlike rays**

The cosmological time function $\tau$ is not a time function in general, but if $\tau < \infty$ it has the following property:

$$q \in J^+(p) \Rightarrow \tau(p) + d(p, q) \leq \tau(q).$$  \hfill (2)

This implies that $\tau$ is isotone.

**Definition 2.** A function $t : M \to R$ which satisfies $q \in J^+(p) \Rightarrow t(p) \leq t(q)$ is said to be isotone.

It is proved in [8] that isotones are almost everywhere continuous and differentiable.

**Theorem 1.** Every isotone function $f : M \to R$ on $(M, g)$ is almost everywhere continuous and almost everywhere differentiable. Moreover, it is differentiable at $p \in M$ iff it is Gâteaux-differentiable at $p$. Finally, if $x : I \to M$ is a timelike curve, the isotone function $f$ is upper/lower semi-continuous at $x_0 = x(t_0)$ iff $f \circ x$ has the same property at $t_0$.

Hence the cosmological time function is almost everywhere continuous. The question is where a point of discontinuity appears.
Theorem 2. Let \((M, g)\) be a space-time with finite cosmological time function \(\tau\). If \(\tau\) be discontinuous at \(q \in M\) then it lies on a past lightlike ray.

Proof. Since \(p \mapsto d(p, q)\), for every \(q \in M\), is lower semi-continuous, \(\tau\) is lower semi-continuous too. Hence it is not upper semi-continuous at \(q\). Consequently, there is a sequence \(\{q_n\}\) and \(\epsilon > 0\) that \(q_n \to q\) and \(\tau(q_n) \geq \tau(q) + \epsilon\). Suppose that \(\{p_n\}\) be a sequence in \(M\) such that \(d(p_n, q_n) \geq \tau(q_n) - 1/n\). (2) implies that \(\tau(p_n) + d(p_n, q_n) \leq \tau(q_n)\). Since \(\tau\) is finite we have
\[
d(p_n, q_n) \leq \tau(q_n) - \tau(p_n)\]
and consequently:
\[
\tau(q_n) - 1/n \leq d(p_n, q_n) \leq \tau(q_n) - \tau(p_n).
\]
This implies that \(\tau(p_n) \to 0\). Let \(\{\gamma_n\}\) be a limit maximizing sequence such that for each \(n\), \(\gamma_n : [0, a_n] \to M\), is a past-directed timelike curve, \(\gamma_n(0) = q_n\), \(\gamma_n(a_n) = p_n\), i.e. \(L(\gamma_n) > d(p_n, q_n) - \epsilon_n\) where \(\epsilon_n \to 0\).

Let \(\hat{\gamma}_n : [0, \infty) \to M\) be any past inextendible extension of \(\gamma_n\) and \(\gamma : [0, a] \to M\) be the limit curve of \(\hat{\gamma}_n\) with \(\gamma(0) = q\). \(\{p_n\}\) diverges to infinity (if \(p_n \to p\) then by lower semi continuity of \(\tau\), \(\tau(p) \leq \liminf(\tau(p_n))\)). This implies that \(\tau(p) = 0\), which is a contradiction). Lemma 2 implies that \(\gamma\) is a causal ray. If \(\gamma\) is not a null ray, it is timelike. As all the curves are parametrized by arc-length, \(\limsup(l(\gamma_n|_{[0, b]})) \leq l(\gamma|_{[0, b]})\). There are \(\delta, b > 0\) such that \(l(\gamma|_{[0, b]}) + \delta \leq \epsilon/2\).

Consequently,
\[
l(\gamma_n|_{[b, a_n]}) = l(\gamma_n) - l(\gamma_n|_{[0, b]}) \geq \tau(q) + \epsilon/2 - 1/n - \epsilon_n.
\]
Hence for sufficiently large \(n\), \(l(\gamma_n|_{[b, a_n]}) > \tau(q)\). Since \(\gamma\) is timelike, \(\gamma_n(b) \in I^-(q)\), for sufficiently large \(n\). Hence \(\tau(q) \geq l(\gamma_n|_{[b, a_n]}) > \tau(q)\) which is a contradiction. \(\square\)

As an application of Theorem 2 the following two theorems can be proved.

Theorem 3. Let \((M, g)\) be a space-time without past lightlike rays and \(\tau < \infty\) then for each \(q\) there is a future-directed unit speed timelike ray \(\gamma_q : (0, \tau(q)] \to M\) which is maximal in each segment, such that:

\[
\gamma_q(\tau(q)) = q, \quad \tau(\gamma_q(t)) = t, \quad t \in (0, \tau(q)].
\]

Proof. Let \(q \in M\) and \(\{p_n\}\) be a sequence such that \(d(p_n, q) \geq \tau(q) - 1/n\). \(p_n\) diverges to infinity as it is proved in the previous theorem. Let \(\gamma_n : [0, a_n] \to M\), \(\gamma_n(0) = q\), \(\gamma_n(a_n) = p_n\) be a limit maximizing sequence of curves:
\[
\epsilon_n = d(\gamma_n(a_n), \gamma_n(0) - l(\gamma_n|_{[0, a_n]})), \epsilon_n \to 0.
\]
and \(\hat{\gamma}_n\) be a past inextendible extension of \(\gamma_n\), for \(n \in \mathbb{N}\). By using of Lemma 2 and assumption \(\hat{\gamma}_n\) converges to a timelike ray \(\gamma : [0, \infty) \to M\). It suffices to prove that,
\[d(\gamma(b), q) = \tau(q) - \tau(\gamma(b)), \text{ for every } b \in [0, \infty).\]
\[l(\gamma_n([0, b])) = l(\gamma_n) - l(\gamma_n[b, a_n]) \geq \tau(q) - (1/n) - \epsilon_n - \tau(\gamma_n(b)).\]
Since \(\tau\) is continuous we have,
\[d(\gamma(b), q) \geq l(\gamma_n([0, b])) \geq \limsup(l(\gamma_n([0, b]))) \geq \tau(q) - \limsup(\tau(\gamma_n(b))) = \tau(q) - \tau(\gamma(b)),\]
and the proof is complete. \(\square\)

The proof of the following theorem is similar to what is given for regular cosmological time functions in [1].

**Theorem 4.** Let \((M, g)\) be a space-time without past lightlike rays. If \(\tau < \infty\) then it is a time function.

**Proof.** If \(q \in I^+(p)\) then by using of (2) it is clear that \(\tau(p) < \tau(q)\). Assume that \(q \in J^+(p) - I^+(p)\) then there is a lightlike geodesic from \(p\) to \(q\). Let \(\gamma_n\) be the timelike ray to \(p\) guaranteed by Theorem 3 and \(x \in \gamma_p\). By cutting the corner argument near \(p\) we have:
\[d(x, q) > d(x, p) + d(p, q).\]
Consequently, \(\tau(q) - \tau(p) \geq d(x, q) > d(x, p) = \tau(p) - \tau(x) > 0\) and the proof is complete. \(\square\)

The following theorem was proved in [10].

**Theorem 5.** Non-totally vicious space-time \((M, g)\) with no lightlike rays is a globally hyperbolic space-time.

It is proved in [10] that a chronological space-time without lightlike lines is stably causal and consequently has a time function. Then using this time function the above theorem is proved. In this paper, by using of Theorem 4 and the following lemmas it can be shown that there is a representation of \((M, g)\) such that the cosmological time function is regular.

**Lemma 3.** Let \((M, g)\) be a chronological space-time without past (or future) lightlike rays then \((M, g)\) is strongly causal.

**Proof.** Suppose by contradiction that \((M, g)\) is not strongly causal. Then there is \(p \in M\) and a sequence of arbitrary small relatively compact neighbourhoods \(U_n, n \in \mathbb{N}\), of \(p\) which are not causally convex, i.e., for every \(U_n\) there exist \(p_n, q_n \in U_n\) and a causal curve \(\gamma_n\) from \(p_n\) to \(q_n\) which are not contained in \(U_n\), \(p_n \to p\), \(q_n \to q\) and \(p = q\). The second part of the limit curve theorem, Theorem 3.1 [9], implies that one of the following cases occur:
1) \(\gamma_n\)s are contained in a compact set. Since \(M\) is chronological, the limit curve \(\gamma_q\) is a closed maximal lightlike curve. The curve \(\gamma\) making infinite rounds around \(\gamma_q\) is a past lightlike ray which
is a contradiction.

2) $\gamma_n$s are not contained in a compact set. Then $\gamma_p \circ \gamma_q = \gamma$ is the limit curve of $\gamma_n$s. Since $\gamma$ is not a lightlike line, the chronology violating set is non-empty. This is a contradiction since $\tau < \infty$. □

Lemma 2 and Lemma 3 imply the following theorem.

**Theorem 6.** Let $(M, g)$ be a chronological space-time without past lightlike rays then there is a positive real function $\Omega$, that the cosmological time function of $(M, \Omega g)$ is regular.

**Proof.** $(M, g)$ is strongly causal by using of Lemma 3. Let $q_0 \in M$ and $\Omega > 0$ be given as in Lemma 1. It is clear that $\tau$ is finite. Suppose by contradiction that there is a past inextendible causal curve $\eta : [0, \infty) \to M$, $\eta(0) = p$ such that $\tau \to a > 0$, along $\eta$. Choose $n$ such that the length of any causal curve out of $B_n(q_0)$ is less than $\epsilon < a$ and $p \in B_n(q_0)$.

Since $M$ is strongly causal it is non-total imprisoning and consequently there is $t_0 \in \mathbb{R}$ such that $\eta(t) \in M - B_n(q_0)$, for $t > t_0$.

In addition suppose that $\{t_n, t_n \to \infty\}$, be a sequence of real numbers and $p_n = \eta(t_n)$. Let $\gamma_{p_n}$ be the maximal timelike ray guaranteed by Theorem 3. Since $l(\gamma_{p_n}) = \tau(p_n) > \epsilon$, $\gamma_{p_n} \cap B_n(q_0) \neq \emptyset$.

Again by using of non-total imprisoning condition, $\gamma_{p_n}$ escape $B_n(q_0)$ in a point $q_n = \gamma_{p_n}(s_n)$.

Since $B_n(q_0)$ is compact $q_n \to q$. let $\gamma'_n$ be a reparametrization of $\gamma_{p_n}|[0, s_n]$ with $h$ arc length in such a way that $\gamma'_n(t) = \gamma_{p_n}(s_n - t)$. $\gamma'_n \to \gamma$, $\gamma(0) = q$.

Since $\gamma_{p_n}$ is maximal, for every $n$, Lemma 2 implies that $\gamma$ is a timelike ray.

Indeed, we have:

\[ \tau(\gamma(b)) - \tau(\gamma(0)) = \limsup_{n} (\tau(\gamma'_n(b)) - \tau(\gamma'_n(0))) = \limsup_{n} (l(\gamma'_n||[0,b])) - l(\gamma||[0,b])) = d(\gamma(0), \gamma(b)) \leq \liminf_{n} (d(\gamma'_n(0), \gamma'_n(0)) = \tau(\gamma(b)) - \tau(\gamma(0)), \text{Since $\tau$ is continuous. In addition, $\tau$ is a time function and consequently $\tau(\gamma(b)) - \tau(\gamma(0)) > 0$.}

Let $\eta_n = \gamma'_n \circ \eta$. $\gamma$ and $\eta$ are the limit curves of $\eta_n$ and non of them are lightlike ray. The limit curve theorem implies that $\eta \subset I^+(q)$. This means that $M$ has a TIF, which is a contradiction since $M$ has no past lightlike ray and the boundary of a TIF is generated by past lightlike rays. □

A non- totally vicious space-time without lightlike rays is chronologival [2] and Theorem 5 is easily given by using of Theorem 6.

4. **The cosmological time function and causal rays**

In this section, we reduce the second condition of regularity and investigate the causality properties of space-times which satisfy:

- $\tau$ is finite;
- $\tau \to 0$ on past inextendible causal geodesics (or on past null rays).
Example 1 shows that in this case \( \tau \) is not necessarily regular. Indeed, it can be checked that it is not globally hyperbolic. Causal pseudoconvexity and causal simplicity are weaker conditions than global hyperbolicity. We will prove that in a reflective space-time the above conditions imply causal pseudoconvexity (or causal simplicity). The space-time \((M, g)\) is called causally pseudoconvex if for each compact set \( K \) there exists a compact set \( K' \) such that each geodesic with both end points in \( K \) has its image in \( K' \).

**Definition 3.** \[12\] Assume \( p_n \to p \) and \( q_n \to q \) for distinct points \( p \) and \( q \) in space-time \( M \). We say that space-time \( M \) has causal limit geodesic segment property (LGS), if each pair \( p_n \) and \( q_n \) can be joined by a geodesic segment, then there is a limit geodesic segment from \( p \) to \( q \).

We can define causal, null or maximally null (LGS) property by restricting the condition of the above definition to causal, null or maximally null geodesics, respectively. Theorem 7, Theorem 8 and Lemma 4 are used to prove the main results of this section. The following theorem gives a characterization for pseudoconvexity.

**Theorem 7.** \[12\] Let \((M, g)\) be a strongly causal space-time. Then it is (null or maximal null) causal pseudoconvex if and only if it has (null or maximally null) causal LGS property.

**Lemma 4.** If the cosmological time function \((M, g)\) satisfies the following conditions:

- \( \tau < \infty \);
- \( \tau \to 0 \), along past lightlike rays,

then \((M, g)\) is non-totally imprisoning.

**Proof.** Since \( \tau < \infty \), \((M, g)\) is chronological. Suppose by contradiction that there is a past inextendible causal curve which is imprisoned in a compact set \( K \). Hence there is a lightlike line which is imprisoned in \( K \) \[7\]. Since \( K \) is compact there are \( p_i \in M \), \( i = 1, ..., n \), that \( K \subset \cup I^+(p_i) \) and consequently \( \tau(x) \geq \min(\tau(p_i)) \), \( i = 1, ..., n \), for all \( x \in K \). But it is a contradiction to assumption. \( \square \)

**Theorem 8.** \[11\] Let \( S \subset M \), and set \( B = \partial I^+(S) \). Then if \( x \in B \), there exists a null geodesic \( \eta \subset B \) with future endpoint \( x \) and which is either past-endless or has a past endpoint on \( S \).

**Theorem 9.** \[13\] Assume that \((M, g)\) be a causal space-time, but not causally simple. Then there are (1) \( p, q \in M \) such that \( p \) has a future inextendible maximal null geodesic ray in \( I^-(p) \) or (2) \( q \) has a past inextendible maximal null geodesic ray in \( \partial I^+(q) \). Conversely, assume \((M, g)\) has a \( p \) (resp. \( q \)) such that \( \partial I^-(p) \) has a future null geodesic ray (resp. \( \partial I^+(p) \) has a past inextendible maximal null geodesic ray), then \((M, g)\) is not causally simple.
Remark. We recall that \((M, g)\) is reflecting iff \(p \in \overline{I^{-}(q)} \iff q \in \overline{I^{+}(p)}\), for \(p, q \in M\).

**Theorem 10.** Let \((M, g)\) be a reflecting space-time such that its cosmological time function \(\tau\) has the following properties:

- \(\tau\) is finite;
- \(\tau \to 0\), along every past lightlike ray \(\gamma\);

then \((M, g)\) is causally simple.

**Proof.** Lemma 4 implies that \((M, g)\) is causal. Suppose by contradiction that \((M, g)\) is not causally simple. Hence there are \(p, q \in M\) such that \(q \in J^{+}(p) - J^{+}(p)\) and (1) or (2) in Theorem 9 occurs:

In the first case there is a past lightlike ray \(\gamma\) which is contained \(q\) and lies in \(\partial I^{+}(p)\). Since \((M, g)\) is reflecting, there is a sequence \(\{p_n\}, p_n \to p\), such that \(p_n \in I^{-}(q)\). Lower semi continuity of cosmological time function implies that \(\tau(p) \leq \liminf \tau(p_n) \leq \tau(q)\). \(q\) can be choosen arbitrary on the causal ray \(\gamma\) and consequently \(\tau(\gamma) \geq \tau(p)\) which is a contradiction, since \(\tau \to 0\) along it.

Again in the second case reflectivity of \((M, g)\) implies that \(q\) lies in \(\partial I^{+}(p)\), hence by Theorem 8 there is a past lightlike ray \(\alpha\) with end point \(q\) in \(\partial I^{+}(p)\). Indeed, it can not be a null geodesic from \(p\) to \(q\) since \(q \notin J^{+}(p)\). The proof in this case is similar to case (1). \(\square\)

**Theorem 11.** Let \((M, g)\) be a reflective space-time such that:

- \(\tau\) is finite,
- \(\tau \to 0\) along past inextendible causal geodesics,

then \((M, g)\) is causally pseudoconvex.

**Proof.** Theorem 10 implies that \((M, g)\) is causally simple. Suppose by contradiction that \((M, g)\) is not causally pseudoconvex then Theorem 7 implies that there are sequences \(\{p_n\}, \{q_n\}\) and geodesics \(\gamma_n\) from \(p_n\) to \(q_n\) such that \(p_n \to p\), \(q_n \to q\) but there limit curve \(\gamma\) with \(\gamma(0) = q\) does not contain \(p\). Hence limit curve theorem implies that \(\gamma(t) \in J^{+}(p)\) and consequently \(\tau(p) \leq \tau(\gamma(t))\) which is a contradiction, since \(\gamma\) is an inextendible past geodesic. \(\square\)

**Theorem 12.** Let \((M, g)\) be a space-time which satisfies the following properties:

- \(\tau < \infty\);
- \(\tau \to 0\), along past inextendible causal rays;
- \(\tau\) is continuous;

then \(\tau\) is regular.
Proof. Suppose by contradiction that \( \tau \not\to 0 \) along a past inextendible causal curve \( \gamma : [0, \infty) \to M \). Let \( p_n = \gamma(t_n) \), where \( t_n \to \infty \) and \( \gamma_n : [0, a_n] \to M \), \( \gamma_n(0) = p \) and \( \gamma_n(a_n) = p_n \) be a limit maximizing sequence. Since by Lemma 4 \((M, g)\) is non-total imprisoning, \( \{p_n\} \) escapes to infinity. Consequently, the limit curve of \( \gamma_n \), \( \eta : [0, \infty) \to M \), \( \eta(0) = p \) is a causal ray. \( \gamma_n(t) \in \overline{I^+(p_m)} \), for \( m \geq n \). Hence \( \eta \subset \overline{I^+(\gamma)} \). Since \( \tau > a \) on \( I^+(\gamma) \) and it is continuous, \( \tau(\eta(t)) \geq a \), for every \( t \), which is a contradiction. \( \Box \)

Theorem 13. Let \((M, g)\) be a space-time of dimension \( n + 1 \), conformally embedded as an open subset in Minkowski space-time \((E^{n,1}, h)\), and its cosmological time function satisfies the following properties:

- \( \tau < \infty \);
- \( \tau \to 0 \), along past causal rays;

then \( \tau \) is regular.

Proof. Suppose that \( \tau \to a \), \( a \neq 0 \), along a past inextendible causal curve \( \gamma : [0, \infty) \to M \). Let \( p_n \), \( \gamma_n \) and \( \eta \) be as in the pervious theorem. It is trivial that \( \eta \subset \overline{I^+(\gamma)} \). \( \partial(I^+(\gamma)) \) is an achronal boundary. \( \tau > a \) on \( I^+(\gamma) \), but \( \tau \to 0 \) on \( \eta \). Hence \( \eta \) has to escape \( I^+(\gamma) \) in a point \( x = \eta(s) \) and \( \eta|[s, \infty) \subset \partial I^+(\gamma) \). This implies that \( \eta \) is not a timelike ray. In addition, if \( \eta'(0) = w \) then \( < w, \alpha' >_h = 0 \), for any lightlike \( \alpha \) in \( \partial I^+(\gamma) \), since \( M \) is an open subset of Minkowski space-time. Hence \( < w, \eta' >_h = 0 \), which is a contradiction. \( \Box \)

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