EIGENVALUE ESTIMATE AND COMPACTNESS FOR CLOSED $f$-MINIMAL SURFACES

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Abstract. Let $\Omega$ be a bounded domain with convex boundary in a complete noncompact Riemannian manifold with Bakry-Émery Ricci curvature bounded below by a positive constant. We prove a lower bound of the first eigenvalue of the weighted Laplacian for closed embedded $f$-minimal hypersurfaces contained in $\Omega$. Using this estimate, we prove a compactness theorem for the space of closed embedded $f$-minimal surfaces with the uniform upper bounds of genus and diameter in a complete 3-manifold with Bakry-Émery Ricci curvature bounded below by a positive constant and admitting an exhaustion by bounded domains with convex boundary.

1. Introduction

A hypersurface $\Sigma$ immersed in a Riemannian manifold $(M, g)$ is said to be $f$-minimal if its mean curvature $H$ satisfies that, for any $p \in \Sigma$,

$$H = \langle \nabla f, \nu \rangle,$$

where $\nu$ is the unit normal at $p \in \Sigma$, $f$ is a smooth function defined on $M$, and $\nabla f$ denotes the gradient of $f$ on $M$. When $f$ is a constant function, an $f$-minimal hypersurface is just a minimal hypersurface. One nontrivial class of $f$-minimal hypersurfaces is self-shrinker for mean curvature flow in the Euclidean space $(\mathbb{R}^{n+1}, g_{can})$. Recall that a self-shrinker is a hypersurface immersed in $(\mathbb{R}^{n+1}, g_{can})$ satisfying that

$$H = \frac{1}{2} \langle x, \nu \rangle,$$

where $x$ is the position vector in $\mathbb{R}^{n+1}$. Hence a self-shrinker is an $f$-minimal hypersurface $\Sigma$ with $f = \frac{|x|^2}{4}$ (see, for instance [10] and the references therein the work about self-shrinkers).

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In the study of $f$-minimal hypersurfaces, it is convenient to consider the ambient space as a smooth metric measure space $(M, \overline{g}, e^{-f}d\mu)$, where $d\mu$ is the volume form of $\overline{g}$. For $(M, \overline{g}, e^{-f}d\mu)$, an important and natural tensor is the Bakry-Émery Ricci curvature $\overline{\text{Ric}}_f := \overline{\text{Ric}} + \nabla^2 f$. There are many interesting examples of smooth metric measure spaces $(M, \overline{g}, e^{-f}d\mu)$ with $\overline{\text{Ric}}_f \geq k$, where constant $k$ is positive. A non-trivial class of examples is shrinking gradient Ricci soliton. It is known that after a normalization, a shrinking gradient Ricci soliton $(M, \overline{g}, f)$ satisfies the equation $\overline{\text{Ric}} + \nabla^2 f = \frac{1}{2} \overline{g}$ or equivalently $\overline{\text{Ric}}_f = \frac{1}{2}$. We refer to [2], a survey of this topic where some compact and noncompact examples are explained. Even though the asymptotic growth of the potential function $f$ of a noncompact shrinking gradient Ricci soliton is close to that of Gaussian shrinking soliton [3], both geometry and topology can be quite different. We may consider $f$-minimal hypersurfaces in a shrinking gradient Ricci soliton. For instance, a self-shrinker in $\mathbb{R}^{n+1}$ can be viewed as an $f$-minimal hypersurface in a Gaussian shrinking soliton $(\mathbb{R}^{n+1}, g_{\text{can}}, \frac{1}{4}(\cdot)^2)$.

There are other examples of $f$-minimal hypersurfaces. Let $M$ be the hyperbolic space $\mathbb{H}^{n+1}(-1)$. Let $r$ denote the distance function from a fixed point $p \in M$ and $f(x) = nar^2$, where $a > 0$ is a constant. Then $\overline{\text{Ric}}_f \geq n(2a - 1)$ and the geodesic sphere of radius $r$ centered at $p$ in $\mathbb{H}^{n+1}(-1)$ is an $f$-minimal hypersurface if it satisfies $2ar = \coth r$.

An $f$-minimal hypersurface $\Sigma$ has two aspects to view. One is that $\Sigma$ is $f$-minimal if and only if $\Sigma$ is a critical point of the weighted volume functional $e^{-f}d\sigma$, where $d\sigma$ is the volume element of $\Sigma$. Another one is that $\Sigma$ is $f$-minimal if and only if $\Sigma$ is minimal in the new conformal metric $\tilde{g} = e^{-2f/n} \overline{g}$ (see Section 2). $f$-minimal hypersurfaces have been studied before, even more general stationary hypersurfaces for parametric elliptic functionals, see for instance the work of White [21] and Colding-Minicozzi [12].

In this paper, we will first estimate the lower bound of the first eigenvalue of the weighted Laplacian $\Delta_f = \Delta - \langle \nabla f, \nabla \cdot \rangle$ for closed (i.e. compact and without boundary) embedded $f$-minimal hypersurfaces in a complete metric measure space $(M, \overline{g}, e^{-f}d\mu)$. Subsequently using the eigenvalue estimate,
we study the compactness for the space of closed embedded $f$-minimal surfaces in a complete noncompact 3-manifold. To explain our result, we recall some backgrounds and related results.

In 1983, Choi-Wang [8] estimated the lower bound for the first eigenvalue of closed minimal hypersurfaces in a complete Riemannian manifold with Ricci curvature bounded below by a positive constant and proved the following

**Theorem 1.** [8] If $M$ is a simply connected complete Riemannian manifold with Ricci curvature bounded below by a constant $k > 0$ and $\Sigma$ is a closed embedded minimal hypersurface, then the first eigenvalue of the Laplacian $\Delta$ on $\Sigma$ is at least $\frac{k}{2}$.

Later, using a covering argument, Choi-Schoen [7] proved that the assumption that $M$ is simply connected is not needed. Recently, Du-Ma [15] extended Theorem 1 to the first eigenvalue of the weighted Laplacian $\Delta_f$ on a closed embedded $f$-minimal hypersurface in a simply connected compact manifold with positive Bakry-Émery Ricci curvature $\text{Ric}_f$. Very recently, Li-Wei [14] also used the covering argument to delete the assumption that the ambient space is simply connected in the result of Du-Ma.

Observe that a complete manifold with Ricci curvature bounded below by a positive constant must be compact. But this conclusion is not true for complete manifolds with Bakry-Émery Ricci curvature $\text{Ric}_f$ bounded below by a positive constant. One example is Gaussian shrinking soliton $(\mathbb{R}^{n+1}, g_{\text{can}}, e^{-\frac{|x|^2}{4}}d\mu)$ with $\text{Ric}_f = \frac{1}{2}$. Hence the theorems of Du-Ma and Li-Wei cannot be applied to self-shrinkers.

For self-shrinkers, Ding-Xin [13] recently obtained a lower bound of the first eigenvalue $\lambda_1(\mathcal{L})$ of the weighted Laplacian $\mathcal{L} = \Delta - \frac{1}{2}\langle x, \nabla \cdot \rangle$ (i.e. $\Delta_f$) on a closed $n$-dimensional embedded self-shrinker in the Euclidean space $\mathbb{R}^{n+1}$, that is, $\lambda_1(\mathcal{L}) \geq \frac{1}{4}$.

We will discuss the lower bound of the first eigenvalue of $\Delta_f$ of a closed embedded $f$-minimal hypersurface in the case that the ambient space is complete noncompact. Precisely, we prove the following
Theorem 2. Let \((M, g, e^{-f}d\mu)\) be a complete noncompact smooth metric measure space with Bakry-Émery Ricci curvature \(\overline{\text{Ric}}_f \geq k\), where \(k\) is a positive constant. Let \(\Sigma\) be a closed embedded \(f\)-minimal hypersurface in \(M\). If there is a bounded domain \(D\) in \(M\) with convex boundary \(\partial D\) so that \(\Sigma\) is contained in \(D\), then the first eigenvalue \(\lambda_1(\Delta_f)\) of the weighted Laplacian \(\Delta_f\) on \(\Sigma\) satisfies

\[
\lambda_1(\Delta_f) \geq \frac{k}{2}.
\]

The boundary \(\partial D\) is called convex if, for any \(p \in \partial D\), the second fundamental form \(A\) of \(\partial D\) at \(p\) is nonnegative with respect to outer unit normal of \(\partial D\).

A closed self-shrinker \(\Sigma^n\) in \(\mathbb{R}^{n+1}\) satisfies the assumption of Theorem 2 since there always exists a ball \(D\) containing \(\Sigma\). Therefore Theorem 2 implies the result of Ding-Xin for self-shrinkers mentioned before. Besides, we give a different and hence alternative proof of their result.

Remark 1. If \(M\) is a Cartan-Hadamard manifold, all geodesic balls are convex. If \(M\) is a complete noncompact Riemannian manifold with nonnegative sectional curvature, the work of Cheeger-Gromoll [4] asserts that \(M\) admits an exhaustion by convex domains.

In [8], Choi-Wang used the lower bound estimate of the first eigenvalue in Theorem 1 to obtain an upper bound of area of a simply connected closed embedded minimal surface \(\Sigma\) in a 3-manifold, depending on the genus \(g\) of \(\Sigma\) and the positive lower bound \(k\) of Ricci curvature of \(M\). Further the lower bound of the first eigenvalue and the upper bound of area were used by Choi-Schoen [7] to prove a smooth compactness theorem for the space of closed embedded minimal surfaces of genus \(g\) in a closed 3-manifold \(M^3\) with positive Ricci curvature. Very recently, Li-Wei [14] proved a compactness theorem for closed embedded \(f\)-minimal surfaces in a compact 3-manifold with Bakry-Émery Ricci curvature \(\overline{\text{Ric}}_f \geq k\), where constant \(k > 0\).

Recently Ding-Xin [13] applied the lower bound estimate of the first eigenvalue of the weighted Laplacian on a self-shrinker to prove a compactness theorem for closed self-shrinkers with uniform bounds of genus and diameter.
As it was mentioned before, a self-shrinker in $\mathbb{R}^3$ is an $f$-minimal surface in a complete noncompact $\mathbb{R}^3$ with $\overline{\text{Ric}}_f \geq \frac{1}{2}$. Motivated by this example, we consider the compactness for $f$-minimal surfaces in a complete noncompact manifold. We prove that

**Theorem 3.** Let $(M^3, \overline{g}, e^{-f}d\mu)$ be a complete noncompact smooth metric measure space with $\overline{\text{Ric}}_f \geq k$, where $k$ is a positive constant. Assume that $M$ admits an exhaustion by bounded domains with convex boundary. Then the space, denoted by $S_{D,g}$, of closed embedded $f$-minimal surface in $M$ with genus at most $g$ and diameter at most $D$ is compact in the $C^m$ topology, for any $m \geq 2$.

Namely, any sequence in $S_{D,g}$ has a subsequence that converges in the $C^m$ topology on compact subsets to a surface in $S_{D,g}$, for any $m \geq 2$.

Theorem 3 implies especially the compactness theorem of Ding-Xin for self-shrinkers. We also prove the following compactness theorem, which implies Theorem 3.

**Theorem 4.** Let $(M^3, \overline{g}, e^{-f}d\mu)$ be a complete noncompact smooth metric measure space with $\overline{\text{Ric}}_f \geq k$, where $k$ is a positive constant. Given a bounded domain $\Omega$, let $S$ be the space of closed embedded $f$-minimal surface in $M$ with genus at most $g$ and contained in the closure $\overline{\Omega}$. If there is a bounded domain $U$ with convex boundary so that $\overline{\Omega} \subset U$, then $S$ is compact in the $C^m$ topology, for any $m \geq 2$.

Namely, any sequence in $S$ has a subsequence that converges in the $C^m$ topology on compact subsets to a surface in $S$, for any $m \geq 2$.

If $M$ admits an exhaustion by bounded domains with convex boundary, such $U$ as in Theorem 4 always exists. Also, the assumption that $f$-minimal surfaces are contained in the closure of a bounded domain $\Omega$ in Theorem 4 is equivalent to that there is a uniform upper bound of extrinsic diameter of $f$-minimal surfaces (see Remark 4 in Section 6).
It is worth of mentioning that for self-shrinkers in $\mathbb{R}^3$, Colding-Minicozzi [11] proved a smooth compactness theorem for complete embedded self-shrinkers with the uniform upper bound of genus and the uniform scale-invariant area growth. In [5], we generalized their result to the complete embedded $f$-minimal surfaces in a complete noncompact smooth metric measure space with $\text{Ric}_f \geq k$, where constant $k > 0$.

From Theorems 3 and 4 we immediately have the following uniform curvature estimates respectively.

**Corollary 1.** Let $(M^3, g, e^{-f}d\mu)$ be a complete smooth metric measure space with $\text{Ric}_f \geq k$, where $k$ is a positive constant. Then for any integer $g$ and a positive constant $D$, there exists a constant $C$ depending only on $M$, $g$ and $D$ such that if $\Sigma$ is a closed embedded $f$-minimal surface of genus $g$ and diameter at most $D$ in $M$, the norm $|A|$ of the second fundamental form of $\Sigma$ satisfies

$$\max_{x \in \Sigma} |A| \leq C.$$

**Corollary 2.** Let $(M^3, g, e^{-f}d\mu)$ be a complete noncompact smooth metric measure space with $\text{Ric}_f \geq k$, where $k$ is a positive constant. Let $\Omega$ be a bounded domain whose closure is contained in a bounded domain $U$ with convex boundary. Then for any integer $g$, there exists a constant $C$ depending only on $U$, $g$ such that if $\Sigma$ is a closed embedded $f$-minimal surface of genus $g$ contained in $\overline{\Omega}$, the norm $|A|$ of the second fundamental form of $\Sigma$ satisfies

$$\max_{x \in \Sigma} |A| \leq C.$$

On the other hand, observe that a similar argument of Theorem 2 also works for the case that the ambient space is a compact manifold with convex boundary. Hence we have the following estimate:

**Theorem 5.** Let $(M, g)$ be a simply connected compact manifold with convex boundary $\partial M$, and $f$ a smooth function on $M$. Assume that $\text{Ric}_f \geq k$, where $k$ is a positive constant. If $\Sigma$ is a closed $f$-minimal hypersurface embedded in $M$ and does not intersect the boundary $\partial M$, then the first eigenvalue of
the weighted Laplacian on $\Sigma$ satisfies

$$\lambda_1(\Delta f) \geq \frac{k}{2}. \quad (2)$$

A special case of Theorem 5 is that

**Corollary 3.** Let $(M, \bar{g})$ be a simply connected compact manifold with convex boundary $\partial M$. Assume that $M$ has the Ricci curvature $\text{Ric} \geq k$, where $k$ is a positive constant. If $\Sigma$ is a closed embedded minimal hypersurface in $M$ and does not intersect the boundary $\partial M$, then the first eigenvalue of the Laplacian on $\Sigma$ satisfies

$$\lambda_1(\Delta) \geq \frac{k}{2}.$$ 

Corollary 3 can be viewed as an extension of Theorem 1 by Choi-Wang if we consider the empty boundary as a convex one.

The rest of this paper is organized as follows: In Section 2 some definitions and notation are given; In Section 3 we give some facts which will be used later; In Section 4 we prove Theorems 2 and 5; In Section 6 we prove Theorems 3 and 4; In Appendix we give the proof of the known Reilly formula for the weighted metric measure space, for the sake of completeness of proof.

## 2. Definitions and notation

In general, a smooth metric measure space, denoted by $(N, g, e^{-w}d\text{vol})$, is a Riemannian manifold $(N, g)$ together with a weighted volume form $e^{-w}d\text{vol}$ on $N$, where $w$ is a smooth function on $N$ and $d\text{vol}$ the volume element induced by the Riemannian metric $g$. The associated weighted Laplacian $\Delta_w$ is defined by

$$\Delta_w u := \Delta u - (\nabla w, \nabla u)$$

where $\Delta$ and $\nabla$ are the Laplacian and gradient on $(N, g)$ respectively.

The second order operator $\Delta_w$ is a self-adjoint operator on the space of square integrable functions on $N$ with respect to the measure $e^{-w}d\text{vol}$. For a closed manifold $N$, the first eigenvalue of $\Delta_w$, denoted by $\lambda_1(\Delta_w)$, is the lowest nonzero real number $\lambda_1$ satisfying

$$\Delta_w u = -\lambda_1 u, \quad \text{on } N.$$
It is well known that the definition of $\lambda_1(\Delta_w)$ is equivalent to that

$$
\lambda_1(\Delta_w) = \inf_{f_N \cdot w e^{-w}d\sigma = 0, u \not\equiv 0} \frac{\int_N |\nabla u|^2 e^{-w}d\sigma}{\int_N u^2 e^{-w}d\sigma}.
$$

The $\infty$-Bakry-Émery Ricci curvature tensor $\mathrm{Ric}_w$ (for simplicity, Bakry-Émery Ricci curvature) on $(N, g, e^{-w}d\text{vol})$ is defined by

$$
\mathrm{Ric}_w := \mathrm{Ric} + \nabla^2 w
$$

where $\mathrm{Ric}$ denotes the Ricci curvature of $(N, g)$ and $\nabla^2 w$ is the Hessian of $w$ on $N$. If $w$ is constant, $\Delta_w$ and $\mathrm{Ric}_w$ are the Laplacian $\Delta$ and Ricci curvature $\mathrm{Ric}$ on $N$ respectively.

Now, let $(M^{n+1}, \overline{g})$ be an $(n + 1)$-dimensional Riemannian manifold. Assume that $f$ is a smooth function on $M$ so that $(M^{n+1}, \overline{g}, e^{-f}d\mu)$ is a smooth metric measure space, where $d\mu$ is the volume element induced by $\overline{g}$.

Let $i : \Sigma^n \to M^{n+1}$ be an $n$-dimensional smooth immersion. Then $i : (\Sigma^n; i^* \overline{g}) \to (M^{n+1}, \overline{g})$ is an isometric immersion with the induced metric $i^* \overline{g}$. For simplicity, we still denote $i^* \overline{g}$ by $\overline{g}$ whenever there is no confusion. Let $d\sigma$ denote the volume element of $(\Sigma, \overline{g})$. Then the function $f$ induces a weighted measure $e^{-f}d\sigma$ on $\Sigma$. Thus we have an induced smooth metric measure space $(\Sigma^n, \overline{g}, e^{-f}d\sigma)$.

In this paper, unless otherwise specified, we denote by a bar all quantities on $(M, \overline{g})$, for instance by $\overline{\nabla}$ and $\overline{\nabla}^2$ the Levi-Civita connection and the Ricci curvature tensor of $(M, \overline{g})$ respectively. Also we denote for example by $\nabla$, $\Delta$ and $\Delta_f$, the Levi-Civita connection, the Ricci curvature tensor, the Laplacian, and the weighted Laplacian on $(\Sigma, \overline{g})$ respectively.

Let $p \in \Sigma$ and $\nu$ a unit normal at $p$. The second fundamental form $A$, the mean curvature $H$, and the mean curvature vector $\overline{H}$ of hypersurface $(\Sigma, \overline{g})$ are defined respectively by:

$$
A : T_p \Sigma \to T_p \Sigma, \quad A(X) := \overline{\nabla}_X \nu, \quad X \in T_p \Sigma;
$$

$$
H := \text{tr}A = -\sum_{i=1}^n \langle \overline{\nabla}_{e_i} e_i, \nu \rangle,
$$

$$
\overline{H} := -H \nu.
$$
Define the weighted mean curvature vector $\mathbf{H}_f$ and the weighted mean curvature $H_f$ of $(\Sigma, \bar{g})$ by

$$\mathbf{H}_f := \mathbf{H} - (\nabla f)^\perp,$$

$$H_f = -H_f \nu,$$

where $\perp$ denotes the projection to the normal bundle of $\Sigma$. It holds that

$$H_f = H - \langle \nabla f, \nu \rangle.$$

**Definition 1.** A hypersurface $\Sigma$ immersed in $(M^{n+1}, \bar{g}, e^{-f}d\mu)$ with the induced metric $\bar{g}$ is called an $f$-minimal hypersurface if its weighted mean curvature $H_f$ vanishes identically, or equivalently if it satisfies

$$H = \langle \nabla f, \nu \rangle.$$

**Definition 2.** The weighted volume of $(\Sigma, \bar{g})$ is defined by

$$V_f(\Sigma) := \int_\Sigma e^{-f}d\sigma.$$

It is well known that $\Sigma$ is $f$-minimal if and only if $\Sigma$ is a critical point of the weighted volume functional. Namely, it holds that

**Proposition 1.** If $T$ is a compactly supported normal variational field on $\Sigma$ (i.e. $T = T^\perp$), then the first variation formula of the weighted volume of $(\Sigma, \bar{g})$ is given by

$$\frac{d}{dt}V_f(\Sigma_t)\bigg|_{t=0} = -\int_\Sigma \langle T, \mathbf{H}_f \rangle \bar{g} e^{-f}d\sigma.$$

On the other hand, an $f$-minimal submanifold can be viewed as a minimal submanifold under a conformal metric. Precisely, define the new metric $\tilde{g} = e^{-\frac{f}{2}}\bar{g}$ on $M$, which is conformal to $\bar{g}$. Then the immersion $i : \Sigma \to M$ induces a metric $i^*\tilde{g}$ on $\Sigma$ from $(M, \tilde{g})$. In the following, $i^*\tilde{g}$ is still denoted by $\tilde{g}$ for simplicity of notation. The volume of $(\Sigma, \tilde{g})$ is

$$\tilde{V}(\Sigma) := \int_\Sigma d\tilde{\sigma} = \int_\Sigma e^{-f}d\sigma = V_f(\Sigma).$$

Hence Proposition 1 and (3) imply that

$$\int_\Sigma \langle T, \tilde{\mathbf{H}} \rangle \tilde{g} d\tilde{\sigma} = \int_\Sigma \langle T, \mathbf{H}_f \rangle \bar{g} e^{-f}d\sigma,$$

where $d\tilde{\sigma} = e^{-f}d\sigma$ and $\tilde{\mathbf{H}}$ denote the volume element and the mean curvature vector of $\Sigma$ with respect to the conformal metric $\tilde{g}$ respectively.
implies that $\bar{H} = e^{2f}H_f$. Therefore $(\Sigma, \bar{g})$ is $f$-minimal in $(M, \bar{g})$ if and only if $(\Sigma, \bar{g})$ is minimal in $(M, \bar{g})$.

In this paper, for closed hypersurfaces, we choose $\nu$ to be the outer unit normal.

3. SOME FACTS ON THE WEIGHTED LAPLACIAN AND $f$-MINIMAL HYPERSURFACES

In this section we give some known results which will be used later in this paper. Recall that Reilly [19] proved an integral version of the Bochner formula for compact domains of a Riemannian manifold, which is called Reilly formula. In [15], Du-Ma obtained a Reilly formula for metric measure spaces, which is the following proposition. We include its proof in Appendix for the sake of completeness.

**Proposition 2.** [15] Let $\Omega$ be a compact Riemannian manifold with boundary $\partial \Omega$ and $(\Omega, \bar{g}, e^{-f}d\mu)$ be a smooth metric measure space. Then we have

$$
\int_\Omega (\Delta_f u)^2 e^{-f} = \int_\Omega |\nabla u|^2 e^{-f} + \int_\Omega \overline{\text{Ric}}_f(\nabla u, \nabla u)e^{-f} + 2\int_{\partial \Omega} u_\nu \Delta_f(u)e^{-f}
+ \int_{\partial \Omega} A(\nabla u, \nabla u)e^{-f} + \int_{\partial \Omega} u_\nu^2 H_f e^{-f},
$$

where $\nu$ is the outward pointing unit normal to $\partial \Omega$ and $A$ is the second fundamental form of $\partial \Omega$ with respect to the normal $\nu$, the quantities with bar denote the ones on $(\Omega, \bar{g})$ (for instance, $\overline{\text{Ric}}_f$ denotes the Bakry-\'Emery Ricci curvature on $(\Omega, \bar{g})$), and $\Delta_f$ and $H_f$ denote the weighted Laplacian on $\partial \Omega$ and the weighted mean curvature of $\partial \Omega$ respectively.

A Riemannian manifold with Bakry-\'Emery Ricci curvature bounded below by a positive constant has some properties similar to a Riemannian manifold with Ricci curvature bounded below by a positive constant. We refer to the work of Wei-Wylie [20], and Munteanu-Wang [17, 18] and the references therein.

We will use the following proposition by Morgan [16] (see also [20] Corollary 5.1).
Proposition 3. If a complete smooth metric measure space \((N, g, e^{-\omega}d\text{vol})\) has \(\text{Ric}_w \geq k\), where \(k\) is a positive constant, then \(N\) has finite weighted volume and finite fundamental group.

For \(f\)-minimal hypersurfaces, the following intersection theorem holds, which was proved by Wei-Wylie ([20] Theorem 7.4).

Proposition 4. [20] Any two closed \(f\)-minimal hypersurfaces in a smooth metric measure space \((M, \overline{g}, e^{-f}d\mu)\) with \(\text{Ric}_f > 0\) must intersect. Thus a closed \(f\)-minimal hypersurface in \(M\) must be connected.

The first and third authors [6] of the present paper proved that the finite weighted volume of a self-shrinker immersed in \(\mathbb{R}^m\) implies it is properly immersed. In [5], we generalizes this result to \(f\)-minimal submanifolds.

Proposition 5. [5] Let \(\Sigma^n\) be an \(n\)-dimensional complete \(f\)-minimal submanifold immersed in an \(m\)-dimensional Riemannian manifold \(M^m\), \(n < m\). If \(\Sigma\) has finite weighted volume, then \(\Sigma\) is properly immersed in \(M\).

Remark 2. We studied \(f\)-minimal submanifolds in [5]. An \(f\)-minimal hypersurface is an \(f\)-minimal submanifold with codimension 1.

4. Lower Bound for \(\lambda_1(\Delta_f)\)

In this section, we apply Reilly formula for metric measure space to prove Theorems [2] and [5].

Proof of Theorem [2]. Since \(\overline{\text{Ric}}_f \geq k\), where constant \(k > 0\), Proposition [3] implies that \(M\) has finite fundamental group. We first assume that \(M\) is simply connected. Since \(\Sigma\) is connected (Proposition [4] and embedded in \(M\), \(\Sigma\) is orientable and divides \(M\) into two components (see its proof in [7]). Thus \(\Sigma\) divides \(D\) into bounded two components \(\Omega_1\) and \(\Omega_2\). That is \(D \setminus \Sigma = \Omega_1 \cup \Omega_2\) with \(\partial \Omega_1 = \Sigma\) and \(\partial \Omega_2 = \partial D \cup \Sigma\).

For simplicity, we denote by \(\lambda_1\) the first eigenvalue \(\lambda_1(\Delta_f)\) of the weighted Laplacian \(\Delta_f\) on \(\Sigma\). Let \(h\) be a corresponding eigenfunction so that on \(\Sigma\),

\[
\Delta_f h + \lambda_1 h = 0 \quad \text{with} \quad \int_{\Sigma} h^2 e^{-f} = 1.
\]
Consider the solution of the Dirichlet problem on $\Omega_1$ so that
\begin{align}
\begin{cases}
\Delta f u = 0 & \text{in } \Omega_1 \\
u = h & \text{on } \partial \Omega_1 = \Sigma.
\end{cases}
\end{align}
(9)
Substituting $\Omega_1$ for $\Omega$ and putting the solution $u$ of (9) in Proposition 2. Then the assumption of Ric implies that
\[0 \geq k \int_{\Omega_1} |\nabla u|^2 e^{-f} - 2\lambda_1 \int_{\Sigma} u \nu e^{-f} + \int_{\Sigma} A(\nabla h, \nabla h) e^{-f},\]
where $\nu$ is the outer unit normal of $\Sigma$ with respect to $\Omega_1$. By the Stokes' theorem and (9),
\[\int_{\Sigma} u \nu e^{-f} = \int_{\Omega_1} (|\nabla u|^2 + u \Delta f u) e^{-f} = \int_{\Omega_1} |\nabla u|^2 e^{-f}.\]
Thus
\[0 \geq (k - 2\lambda_1) \int_{\Omega_1} |\nabla u|^2 e^{-f} + \int_{\Sigma} A(\nabla h, \nabla h) e^{-f}.\]
If $\int_{\Sigma} A(\nabla h, \nabla h) e^{-f} \geq 0$, by $u \not\equiv C$, we have
\[\lambda_1 \geq \frac{k}{2}.
\]
If $\int_{\Sigma} A(\nabla h, \nabla h) e^{-f} < 0$, we consider the compact domain $\Omega_2$ with the boundary $\partial \Omega_2 = \Sigma \cup \partial D$. Let $u$ be the solution of the mixed problem
\begin{align}
\begin{cases}
\Delta f u = 0 & \text{in } \Omega_2 \\
u = h & \text{on } \Sigma \\
u \bar{\nu} = 0 & \text{on } \partial D,
\end{cases}
\end{align}
(10)
where $\bar{\nu}$ denotes the outer unit normal of $\partial D$ with respect to $\Omega_2$.
Substituting $\Omega_2$ for $\Omega$ and putting the solution $u$ of (10) in Proposition 2 we have
\[0 \geq \int_{\Omega_2} |\nabla u|^2 e^{-f} + k \int_{\Omega_2} |\nabla u|^2 e^{-f} - 2\lambda_1 \int_{\Sigma} h \nu \bar{\nu} e^{-f}
+ \int_{\Sigma} \tilde{A}(\nabla h, \nabla h) e^{-f} + \int_{\partial D} \tilde{A}(\nabla u, \nabla u) e^{-f},\]
where $\tilde{\nu}$ denotes the outer unit normal of $\Sigma$ with respect to $\Omega_2$, and $\tilde{A}$ denotes the second fundamental form of $\Sigma$ with respect to normal $\tilde{\nu}$.

On the other hand, the Stokes' theorem and (10) imply
\[\int_{\Omega_2} |\nabla u|^2 e^{-f} = \int_{\partial \Omega_2} u \nu \bar{\nu} e^{-f} = \int_{\Sigma} h \nu \bar{\nu} e^{-f}.\]
Thus, we have
\[0 \geq (k - 2\lambda_1) \int_{\Omega_2} |\nabla u|^2 e^{-f} + \int_{\Sigma} \tilde{A}(\nabla h, \nabla h) e^{-f} + \int_{\partial D} \tilde{A}(\nabla u, \nabla u) e^{-f}.\]
(11)
Since $\partial D$ is assumed convex, the last term on the right side of (11) is nonnegative. Observe that the orientations of $\Sigma$ are opposite for $\Omega_1$ and $\Omega_2$. Namely, $\tilde{\nu} = -\nu$. Then $\tilde{A}(\nabla u, \nabla u) = -A(\nabla u, \nabla u)$ on $\Sigma$. This implies that the second term on the right side of (11) is nonnegative. Thus

$$0 \geq (k - 2 \lambda_1) \int_{\Omega_2} |\nabla u|^2 e^{-f}.$$

Since $u$ is not constant function, we conclude that $k - 2 \lambda_1 \leq 0$. Again, we have

$$\lambda_1 \geq \frac{k}{2}.$$

Therefore we obtain that $\lambda_1(\Delta_f) \geq \frac{k}{2}$ if $M$ is simply connected.

Second, if $M$ is not simply connected, we consider the universal covering $\hat{M}$, which is a finite $|\pi_1|$-fold covering. $\hat{M}$ is simply connected and the covering map $\pi : \hat{M} \to M$ is a locally isometry.

Take $\hat{f} = f \circ \pi$. Obviously $\hat{M}$ has $\hat{Ric}_\hat{f} \geq k$, and the lift $\hat{\Sigma}$ of $\Sigma$ is also $\hat{f}$-minimal, embedded and closed. By Proposition 3, $\hat{\Sigma}$ must be connected. Since $\hat{M}$ is simply connected, the closed embedded connected $\hat{\Sigma}$ must be orientable and thus divides $\hat{M}$ into two components. Moreover the connectedness of $\hat{\Sigma}$ implies that the lift $\hat{D}$ of $D$ is also connected. Also $\partial \hat{D} = \hat{\partial D}$ is smooth and convex. Hence the assertion obtained for the simply connected ambient space can be applied here. Thus the first eigenvalue of the weighted Laplacian $\hat{\Delta}_f$ on $\hat{\Sigma}$ satisfies $\lambda_1(\hat{\Delta}_f) \geq \frac{k}{2}$.

Observing the lift of the first eigenfunction of $\Sigma$ is also an eigenfunction of $\hat{M}$, we have

$$\lambda_1(\Delta_f) \geq \lambda_1(\hat{\Delta}_f) \geq \frac{k}{2}.$$  

\[\square\]

Remark 3. In Theorem 2 the boundary $\partial D$ is not necessarily smooth. $\partial D$ can be assumed to be $C^1$, which is sufficient to the existence of the solution of the mixed problem (10).

Theorem 5 holds by the same argument as that of Theorem 2.
5. Upper bound on Area and total curvature of \( f \)-minimal surfaces

In this section, we study surfaces in a 3-manifold. First we estimate the corresponding upper bounds on the area and weighted area of an embedded closed \( f \)-minimal surface by applying the first eigenvalue estimate in Section 4. Next we discuss the upper bound on the total curvature. We begin with a result of Yang and Yau [22]:

**Proposition 6.** [22] Let \( \Sigma^2 \) be a closed orientable Riemannian surface with genus \( g \). Then the first eigenvalue \( \lambda_1(\Delta) \) of the Laplacian \( \Delta \) on \( \Sigma \) satisfies that

\[
\lambda_1(\Delta) \text{Area}(\Sigma) \leq 8\pi(1 + g).
\]

Using Theorem 2 and Proposition 6 we obtain the following area estimates for closed embedded \( f \)-minimal surfaces if the ambient space is simply connected.

**Proposition 7.** Let \((M^3, \bar{g}, e^{-f}d\mu)\) be a simply connected complete smooth metric measure space with \( \text{Ric}_f \geq k \), where \( k \) is a positive constant. Let \( \Sigma^2 \subset M \) be a closed embedded \( f \)-minimal surface with genus \( g \). If \( \Sigma \) is contained in a bounded domain \( D \) with convex boundary \( \partial D \), then its area and weighted area satisfy the following inequalities respectively.

\[
\text{Area}(\Sigma) \leq \frac{16\pi(1 + g)}{k} e^{\text{osc}_\Sigma f},
\]

\[
\text{Area}_f(\Sigma) \leq \frac{16\pi(1 + g)}{k} e^{-\inf_{\Sigma} f},
\]

where \( \text{osc}_\Sigma f = \sup_{\Sigma} f - \inf_{\Sigma} f \).

**Proof.** Consider the conformal metric \( \tilde{g} = e^{-f}\bar{g} \) on \( M \). Let \( \lambda_1(\tilde{\Delta}) \) be the first eigenvalue of the Laplacian \( \tilde{\Delta} \) on \((\Sigma, \tilde{g})\), which satisfies

\[
\lambda_1(\tilde{\Delta}) = \inf_{\int_{\Sigma} u d\tilde{\sigma} = 0, u \neq 0} \frac{\int_{\Sigma} |\tilde{\nabla} u|_{\tilde{g}}^2 d\tilde{\sigma}}{\int_{\Sigma} u^2 d\tilde{\sigma}},
\]

where \( \tilde{\Delta}, \tilde{\nabla} \) and \( d\tilde{\sigma} \) are the Laplacian, gradient and area element of \( \Sigma \) with respect to the metric \( \tilde{g} \) respectively.
On the other hand, the first eigenvalue of the weighted Laplacian \( \lambda_1(\Delta_f) \) on \((\Sigma, \tilde{g})\) satisfies
\[
\lambda_1(\Delta_f) = \inf_{\int_\Sigma ue^{-f} d\sigma = 0, u \neq 0} \frac{\int_\Sigma |\nabla u|^2 e^{-f} d\sigma}{\int_\Sigma u^2 e^{-f} d\sigma}.
\]
Since \( \tilde{\nabla} u = e^f \nabla u \), \( d\tilde{\sigma} = e^{-f} d\sigma \) and \( \tilde{g} = e^{-f} g \),
\[
\lambda_1(\tilde{\Delta}) = \inf_{\int_\Sigma ue^{-f} d\sigma = 0, u \neq 0} \frac{\int_\Sigma |\nabla u|^2 e^{-f} d\sigma}{\int_\Sigma u^2 e^{-f} d\sigma} \geq \inf_{\int_\Sigma ue^{-f} d\sigma = 0, u \neq 0} \frac{\int_\Sigma |\nabla u|^2 e^{-f + \inf_\Sigma f} d\sigma}{\int_\Sigma u^2 e^{-f} d\sigma} \geq e^{\inf_\Sigma f} \lambda_1(\Delta_f).
\]
Combining Theorem 2 with Proposition 6 gives the following estimate
\[
(14) \quad \text{Area}(\Sigma, \tilde{g}) \leq \frac{16\pi (1 + g)}{k} e^{-\inf_\Sigma f}.
\]
Since \( \text{Area}_f(\Sigma) = \int_\Sigma e^{-f} d\sigma = \text{Area}(\Sigma, \tilde{g}) \),
\[
\text{Area}_f(\Sigma) \leq \frac{16\pi (1 + g)}{k} e^{-\inf_\Sigma f},
\]
which is \(13\). Thus
\[
\text{Area}(\Sigma) \leq \frac{16\pi (1 + g)}{k} e^{\sup_\Sigma f - \inf_\Sigma (f)} = \frac{16\pi (1 + g)}{k} e^{\text{osc}_\Sigma (f)}.
\]
That is, \(12\) holds.

Now, suppose that \( M \) is not simply connected. We use a covering argument as in [7].

**Proposition 8.** Let \((M^3, \tilde{g}, e^{-f} d\mu)\) be a complete smooth metric measure space with \( \text{Ric} \tilde{g} \geq k > 0 \), where \( k \) is a positive constant. Let \( \Sigma^2 \) be a closed embedded \( f \)-minimal surface. If \( \Sigma \) is contained in a bounded domain \( D \) of \( M \) with convex boundary \( \partial D \), then
\[
(15) \quad \text{Area}_f(\Sigma) \leq \frac{16\pi}{k} \left( \frac{2}{|\pi|_1} - \frac{1}{2} \chi(\Sigma) \right) e^{-\inf_\Sigma f}.
\]
\begin{equation}
\text{Area}(\Sigma) \leq \frac{16\pi}{k} \left( \frac{2}{|\pi_1|} - \frac{1}{2} \chi(\Sigma) \right) e^{\text{osc}_{\Sigma} f},
\end{equation}

where $|\pi_1|$ is the order of the first fundamental group of $M$, and $\chi(\Sigma)$ is the Euler characteristic of $\Sigma$.

**Proof.** Let $\hat{M}$ be the universal covering manifold of $M$. By Proposition 3, the covering is a finite $|\pi_1|$-fold covering. Let $\hat{\Sigma}$ be the lifting of $\Sigma$. In the proof of Theorem 2, we have shown that $\hat{\Sigma}$ is orientable and satisfies the assumption of Theorem 2. Hence Theorem 2 implies that the first eigenvalue of the weighted Laplacian of $\hat{\Sigma}$ satisfies

$$\lambda_1(\hat{\Delta}_{\hat{f}}) \geq \frac{k}{2},$$

where $\hat{f}$ is the lift of $f$. By Proposition 7, we conclude that

$$\text{Area}(\hat{\Sigma}) \leq \frac{16\pi}{k} \left( 2 - \frac{1}{2} \chi(\hat{\Sigma}) \right) e^{\text{osc}_{\hat{\Sigma}}(\hat{f})},$$

and

$$\text{Area}_{\hat{f}}(\hat{\Sigma}) = \int_{\hat{\Sigma}} e^{-\hat{f}} d\sigma \leq \frac{16\pi}{k} \left( 2 - \frac{1}{2} \chi(\hat{\Sigma}) \right) e^{-\inf_{\hat{\Sigma}}(\hat{f})}.$$

Thus (15) and (16) follow from the facts that $\chi(\hat{\Sigma}) = |\pi_1| \cdot \chi(\Sigma)$, $\text{Area}(\hat{\Sigma}) = |\pi_1| \cdot \text{Area}(\Sigma)$, $\text{Area}_{\hat{f}}(\hat{\Sigma}) = |\pi_1| \cdot \text{Area}_{\hat{f}}(\Sigma)$, $\inf_{\hat{\Sigma}}(\hat{f}) = \inf_{\Sigma}(f)$ and $\text{osc}_{\hat{\Sigma}}(\hat{f}) = \text{osc}_{\Sigma}(f)$.

\[\square\]

In the following, we will give the upper bound for the total curvature of $f$-minimal surfaces. Here the term the total curvature of $\Sigma$ means $\int_{\Sigma} |A|^2 d\sigma$ not $\int_{\Sigma} Kd\sigma$.

**Proposition 9.** If $(M^3, g, e^{-f}d\mu)$ is a smooth metric measure space with $\text{Ric}_{\hat{f}} \geq k$, where $k$ is a positive constant. Let $\Sigma^2 \subset M$ be a closed embedded $f$-minimal surface with genus $g$. If $\Sigma$ is contained in a bounded domain $D$ of $M$ with convex boundary $\partial D$, then $\Sigma$ satisfies

\begin{equation}
\int_{\Sigma} |A|^2 d\sigma \leq C,
\end{equation}

where $A$ is the second fundamental form of $(\Sigma, g)$ and $C$ is a constant depending on the genus $g$ of $\Sigma$, the order $|\pi_1|$ of the first fundamental group of $M$, the maximum $\sup_{\Sigma} K$ of the sectional curvature of $M$ on $\Sigma$, the lower bound $k$ of the Bakry-Émery Ricci curvature of $M$, the oscillation $\text{osc}_{\Sigma}(f)$ and the maximum $\sup_{\Sigma} |\nabla f|$ on $\Sigma$. 
Proof. By the Gauss equation and Gauss-Bonnet formula,

\[ \int_{\Sigma} |A|^2 d\sigma = \int_{\Sigma} H^2 - 2 \int_{\Sigma} (K - \overline{K}) = \int_{\Sigma} \langle \nabla f, n \rangle^2 - 4\pi \chi(\Sigma) + 2 \int_{\Sigma} \overline{K} \leq (\sup_{\Sigma} |\nabla f|)^2 \text{Area}(\Sigma) + 8\pi (g - 1) + 2(\sup_{\Sigma} \overline{K}) \text{Area}(\Sigma). \]

Substituting (16) for Area(\Sigma), we have the conclusion of theorem. \( \square \)

To prove the compactness theorem in Section 6, we need the following total curvature estimate for \((\Sigma, \tilde{g})\), which is a minimal surface in \((M, \tilde{g})\).

**Proposition 10.** If \((M^3, \overline{g}, e^{-f} d\mu)\) is a smooth metric measure space with \(\text{Ric}_f \geq k\), where \(k\) is a positive constant. Let \(\Sigma^2 \subset M\) be a closed embedded \(f\)-minimal surface with genus \(g\). If \(\Sigma\) is contained in a bounded domain \(D\) of \(M\) with convex boundary \(\partial \Omega\), then \(\Sigma\) satisfies that

\[ \int_{\Sigma} |\tilde{A}|^2 d\tilde{\sigma} \leq C, \]

where \(\tilde{A}\) is the second fundamental form of \((\Sigma, \tilde{g})\) with respect to the conformal metric \(\tilde{g} = e^{-f} \overline{g}\) of \(M\) and \(C\) is a constant depending on the genus \(g\) of \(\Sigma\), the order \(|\pi_1|\) of the first fundamental group of \(M\), the maximum \(\sup_{\Sigma} \tilde{K}\) of the sectional curvature of \((M, \tilde{g})\) on \(\Sigma\), the lower bound \(k\) of the Bakry-Émery Ricci curvature of \(M\), the oscillation \(\text{osc}_{\Sigma}(f)\) on \(\Sigma\).

Proof. By the Gauss equation and Gauss-Bonnet formula, we have

\[ \int_{\Sigma} |\tilde{A}|^2 d\tilde{\sigma} = \int_{\Sigma} \overline{H}^2 - 2 \int_{\Sigma} (\tilde{K} - \overline{K}) d\tilde{\sigma} = -4\pi \chi(\Sigma) + 2 \int_{\Sigma} \tilde{K} d\tilde{\sigma} \leq 8\pi (g - 1) + 2(\sup_{\Sigma} \tilde{K}) \text{Area}_{\Sigma}(\Sigma). \]

In the above, we used \(\tilde{H} = e^f H = 0\) and \(\text{Area}_{\Sigma}(\Sigma, \tilde{g}) = \text{Area}_{\Sigma}(\Sigma)\). From (15) in Proposition 8, Inequality (18) holds. \( \square \)
6. Compactness of compact f-minimal surfaces

We will prove some compactness theorems for closed embedded f-minimal surfaces in 3-manifolds. We have two ways to prove Theorem 4.

The first proof roughly follows the one in [11] (cf [7]) with some modifications. The modifications can be made because we have the assumptions that f-minimal surfaces are contained in the closure of a bounded domain Ω of M and Ω is contained in a bounded domain U with convex boundary. The second proof will need a compactness theorem of complete embedded f-minimal surfaces that was proved by us in [5].

We prefer to give two proofs here since the first one is independent of the compactness theorem of complete embedded f-minimal surfaces. But the compactness theorem of complete embedded f-minimal surfaces needs a theorem about non-existence of $L_f$-stable minimal surfaces ([5] Theorem 3).

First proof.

We first prove a singular compactness theorem, which is a variation of Choi-Schoen’s [7] singular compactness theorem (cf Proposition 7.14 in [11], Anderson [1] and White [21]). Namely,

**Proposition 11.** Let $(M^3, g)$ be a 3-manifold. Assume that Ω is bounded domain in M. Let $Σ_i$ be a sequence of closed embedded minimal surfaces contained in $\overline{Ω}$, with genus $g$, and satisfying

\[(19) \quad \text{Area}(Σ_i) \leq C_1\]

and

\[(20) \quad \int_{Σ_i} |A_{Σ_i}|^2 \leq C_2,\]

Then there exists a finite set of points $S \subset \overline{Ω}$ and a subsequence, still denoted by $Σ_i$, that converges uniformly in the $C^m$ topology ( $m \geq 2$ ) on compact subsets of $M \setminus S$ to a complete minimal surface $Σ \subset \overline{Ω}$ (possibly with multiplicity).

The subsequence also converges to $Σ$ in extrinsic Hausdorff distance. $Σ$ is smooth, embedded in M, has genus at most $g$ and satisfies (19) and (20).
Proof. We may use the same argument as that of Proposition 7.14 in [11]. Moreover, \( \Sigma_i \subset \overline{\Omega} \) implies that the singular set \( S \subset \overline{\Omega} \) and the smooth surface \( \Sigma \subset \overline{\Omega} \). Here we omit the details of proof.

\[ \square \]

We can apply Proposition [11] to the \( f \)-minimal surfaces which are minimal in the conformal metric.

**Lemma 1.** Let \( (M^3, \overline{g}, e^{-f} d\mu) \) be a smooth metric measure space. Assume that \( \Omega \) is a bounded domain in \( M \). Let \( \Sigma_i \subset \overline{\Omega} \) be a sequence of closed embedded \( f \)-minimal surfaces of genus \( g \). Suppose that \( \tilde{g} = e^{-f} \overline{g} \) on \( M \) and \( (\Sigma_i, \tilde{g}) \) satisfy that

\[
\text{Area}(\Sigma_i, \tilde{g}) = \text{Area}_f(\Sigma_i) \leq C_1
\]

and

\[
\int_{\Sigma_i} |\tilde{A}_{\Sigma_i}|^2 d\tilde{\sigma} \leq C_2,
\]

where \( \tilde{A}_{\Sigma_i} \) and \( d\tilde{\sigma} \) denote the second fundamental form and the volume element of \( (\Sigma_i, \tilde{g}) \) respectively. Then there exists a finite set of points \( S \subset \overline{\Omega} \) and a subsequence, still denoted by \( \Sigma_i \), that converges uniformly in the \( C^m \) topology \( (m \geq 2) \) on compact subsets of \( M \setminus S \) to a complete \( f \)-minimal surface \( \Sigma \subset \overline{\Omega} \) (possibly with multiplicity).

The subsequence also converges to \( \Sigma \) in extrinsic Hausdorff distance. \( \Sigma \) is smooth, embedded in \( M \), has genus at most \( g \), and satisfies \((21)\) and \((22)\).

**Proof.** Since an \( f \)-minimal surface in the original metric \( \overline{g} \) is equivalent to it is minimal in the conformal metric \( \tilde{g} \), we can apply Proposition [11] to get the conclusion of the lemma.

\[ \square \]

**Proof of Theorem 4.** We first consider that \( M \) is simply connected. Since \( \Sigma_i \subset \overline{\Omega} \subset U \). By Proposition [7] and Proposition [10]

\[
\text{Area}(\Sigma_i, \tilde{g}) = \text{Area}_f(\Sigma_i) \leq C_1
\]

and

\[
\int_{\Sigma_i} |\tilde{A}_{\Sigma_i}|^2 d\tilde{\sigma} \leq C_2,
\]
where \( C_1 \) and \( C_2 \) depend on \( g, \sup_{\Omega_j} f, \sup_{\tilde{\Omega}_j} \tilde{K} \) and \( k \).

By Lemma \([1]\) there exists a finite set of points \( S \subset \tilde{\Omega} \) and a subsequence \( \Sigma_{i'} \) that converges uniformly in the \( C^m \) topology (any \( m \geq 2 \)) on compact subsets of \( M \setminus S \) to a complete \( f \)-minimal surface \( \Sigma \subset \overline{\Omega} \) without boundary (possibly with multiplicity). \( \Sigma \) is smooth, embedded in \( M \) and has genus at most \( g \). Equivalently, with respect to the conformal metric \( \tilde{g} \), a subsequence \( \Sigma_{i'} \) of minimal surfaces converges uniformly in the \( C^m \) topology on compact subsets of \( M \setminus S \) to a complete minimal surface \( \Sigma \subset \overline{\Omega} \).

Since complete embedded \( \Sigma \subset \overline{\Omega} \) satisfies (21), it must be properly embedded (Proposition \([5]\)), thus closed, and orientable.

We need to prove that the convergence is smooth across the points \( S \).

By Allard’s regularity theorem, it suffices to prove that the convergence has multiplicity one. If the multiplicity is not one, by a proof similar to that of Choi-Schoen \([7]\) (also cf \([11]\) P. 249), we can show that there is an \( i \) big enough and a \( \Sigma_i \) in the convergent subsequence, so that the first eigenvalue of the Laplacian \( \tilde{\Delta}_{\Sigma_i} \) on \( \Sigma_i \) with the conformal metric \( \tilde{g} \) satisfies
\[
\lambda_1(\tilde{\Delta}_{\Sigma_i}) < \frac{k}{2} e^{\inf f}. \]

We have
\[
\lambda_1(\tilde{\Delta}_{\Sigma_i}) = \inf \left\{ \frac{\int_{\Sigma_i} |\nabla \varphi|^2 d\tilde{\sigma}}{\int_{\Sigma_i} \varphi^2 d\tilde{\sigma}}, \int_{\Sigma_i} \varphi d\tilde{\sigma} = 0 \right\} 
= \inf \left\{ \frac{\int_{\Sigma_i} |\nabla \varphi|^2 d\sigma}{\int_{\Sigma_i} \varphi^2 e^{-f} d\sigma}, \int_{\Sigma_i} \varphi e^{-f} d\sigma = 0 \right\} 
\geq \lambda_1(\Delta_{\Sigma_i}) e^{\inf f}.
\]

By Theorem \([2]\) \( \Sigma_i \subset \overline{\Omega} \subset U \) implies \( \lambda_1(\Delta_{\Sigma_i}) \geq \frac{k}{2} \). Thus we have a contradiction.

When \( M \) is not simply connected, we use a covering argument. The assumption of \( \text{Ric}_f \geq k \), where constant \( k > 0 \), implies that \( M \) has finite fundamental group \( \pi_1 \) (Proposition \([8]\) \( \text{Proposition} \) \([5]\) \( \text{Proposition} \) \([10]\)). We consider the finite-fold universal covering \( \tilde{M} \). By the proof of Theorem \([2]\) we know that the corresponding lifts of \( \Sigma_i, \overline{\Omega} \) and \( U \) satisfy that \( \tilde{\Sigma}_i \subset \tilde{\Omega} \subset \tilde{U} \). Then Propositions \([8]\) and \([10]\) give the uniform bounds of area and total curvature in the conformal metric \( \tilde{g} \) on \( \tilde{M} \). By the assertion on the simply connected ambient manifold before,
we have the smooth convergence of a subsequence of $\hat{\Sigma}_i$. This implies the smooth convergence of a subsequence of $\Sigma_i$.

\[ \square \]

**Second Proof.** In [5], we proved the following

**Theorem 6.** [5] Let $(M^3, \overline{g}, e^{-f}d\mu)$ be a complete smooth metric measure space and $\text{Ric}_f \geq k$, where $k$ is a positive constant. Given an integer $g \geq 0$ and a constant $V > 0$, the space $S_{g,V}$ of smooth complete embedded $f$-minimal surfaces $\Sigma \subset M$ with

- genus at most $g$,
- $\partial \Sigma = \emptyset$,
- $\int_{\Sigma} e^{-f}d\sigma \leq V$

is compact in the $C^m$ topology, for any $m \geq 2$. Namely, any sequence of $S_{g,V}$ has a subsequence that converges in the $C^m$ topology on compact subsets to a surface in $S_{D,g}$, for any $m \geq 2$.

**Proof of Theorem 4.** Since a surface in $S$ is contained in $\overline{\Omega} \subset U$, by Proposition 8 we have the uniform bound $V$ of the weighted volume of closed embedded $f$-minimal surfaces in $S$. Hence Theorem 6 can be applied. Moreover $\Sigma_i \subset \overline{\Omega}$ implies that the smooth limit surface $\Sigma \subset \overline{\Omega}$. Otherwise, since the subsequence $\{\Sigma_i\}$ converges uniformly in the $C^m$ topology ($m \geq 2$) on any compact subset of $M$ to $\Sigma$, there is a surface $\Sigma_i$ (with index $i$ big enough) in the subsequence would not satisfy $\Sigma_i \subset \overline{\Omega}$.

By Proposition 5, $\Sigma$ must be properly embedded. Thus $\Sigma$ must be closed.

\[ \square \]

Using Theorem 4, we may prove Theorem 3 in Introduction.

**Lemma 2.** Let $(M^3, \overline{g}, e^{-f}d\mu)$ be a complete noncompact smooth metric measure space with $\text{Ric}_f \geq k > 0$. If $\Sigma$ is any closed $f$-minimal surface in $M$ with genus at most $g$ and diameter at most $D$, then $\Sigma \subset B_r(p)$ for some $r > 0$ (indepedent of $\Sigma$), where $B_r(p)$ is a ball in $M$ with radius $r$ centered at $p \in M$. 
Proof. Fix a closed $f$-minimal surface $\Sigma_0$. Obviously, $\Sigma_0 \subset B_{r_0}(p)$ for some $r_0 > 0$. Proposition 4 says that $\Sigma$ and $\Sigma_0$ must intersect. Then for $x \in \Sigma$, $d(p, x) \leq d(p, x_0) + d(x_0, x) \leq r_0 + D, x_0 \in \Sigma_0$.

Taking $r = r_0 + D$, we have $\Sigma \subset B_{r_0 + D}$.

Remark 4. In Lemma 2 and hence in Theorem 3, $D$ is a bound of intrinsic diameter of closed $f$-minimal surfaces or a bound of extrinsic diameter of closed $f$-minimal surfaces. Also by Proposition 4, the assumption that $f$-minimal surfaces are contained in the closure of a bounded domain $\Omega$ in Theorem 5 is equivalent to that the uniform upper bound of the extrinsic diameter of $f$-minimal surfaces.

Proof of Theorem 3. By Lemma 2, we may apply Theorem 4 to the space $S_{D,g}$. Next, the closed embedded limit $\Sigma$ must have diameter at most $D$. Otherwise, since the subsequence $\{\Sigma_i\}$ converges uniformly in the $C^m$ topology ($m \geq 2$) on any compact subset of $M$ to $\Sigma$, there is a surface $\Sigma_i$ (with the index $i$ big enough) in the subsequence would have diameter greater than $D$. So $\Sigma$ must in $S_{D,g}$.

7. Appendix

In this section, we include the proof Proposition 2 in Section 3.

Proof of Proposition 2. Recall the Bochner formula

$$\frac{1}{2} \Delta_f |\nabla u|^2 - (\nabla u, \nabla (\Delta_f u)) = |\nabla^2 u|^2 + \text{Ric}_f(\nabla u, \nabla u).$$

Integrating this equation on $\Omega$ with respect to weighted measure $e^{-f}d\mu$, we obtain

$$\int_{\Omega} \left( \frac{1}{2} \Delta_f |\nabla u|^2 - (\nabla u, \nabla (\Delta_f u)) \right) e^{-f} = \int_{\Omega} |\nabla^2 u|^2 e^{-f} + \int_{\Omega} \text{Ric}_f(\nabla u, \nabla u) e^{-f}.$$

On the other hand, by the divergence formula, it holds that

$$\frac{1}{2} \Delta_f |\nabla u|^2 - (\nabla u, \nabla (\Delta_f u)) = \frac{1}{2} \text{div}(e^{-f}|\nabla u|^2) e^{-f} - \text{div}(e^{-f} \Delta_f(u) \nabla u) e^{-f} + (\Delta_f u)^2.$$
Integrating and applying the Stokes’ theorem, we have

\[
\int_{\Omega} \left( \frac{1}{2} \Delta f |\nabla u|^2 - \langle \nabla u, \nabla (\Delta f u) \rangle \right) e^{-f} = \int_{\partial \Omega} \frac{1}{2} |\nabla u|^2_{\nu} - (\Delta f u) u_{\nu} e^{-f} + \int_{\Omega} (\Delta f)^2 u e^{-f}.
\]

Then

\[
\frac{1}{2} |\nabla u|^2_{\nu} - (\Delta f u) u_{\nu} = \langle \nabla u, \nabla (\Delta f u) \rangle e^{-f}
\]

By substituting (24) into (24), we obtain

\[
\int_{\Omega} \left( \frac{1}{2} \Delta f |\nabla u|^2 - \langle \nabla u, \nabla (\Delta f u) \rangle \right) e^{-f}
\]

\[
= - \int_{\partial \Omega} (\Delta f u) u_{\nu} e^{-f} - \int_{\partial \Omega} H f u_{\nu}^2 e^{-f}
\]

\[
+ \int_{\partial \Omega} \left( \langle \nabla u, \nabla u_{\nu} \rangle - A(\nabla u, \nabla u) \right) e^{-f} + \int_{\Omega} (\Delta f)^2 u e^{-f}
\]

\[
= -2 \int_{\partial \Omega} (\Delta f u) u_{\nu} e^{-f} - \int_{\partial \Omega} H f u_{\nu}^2 e^{-f}
\]

\[
- \int_{\partial \Omega} A(\nabla u, \nabla u) e^{-f} + \int_{\Omega} (\Delta f)^2 u e^{-f}.
\]

Consequently

\[
\int_{\Omega} (\Delta f)^2 u e^{-f} = \int_{\Omega} |\nabla u|^2 e^{-f} + \int_{\Omega} \text{Ric}(\nabla u, \nabla u) e^{-f} + 2 \int_{\partial \Omega} u_{\nu} \Delta f (u) e^{-f}
\]

\[
+ \int_{\partial \Omega} A(\nabla u, \nabla u) e^{-f} + \int_{\partial \Omega} u_{\nu}^2 H f e^{-f}.
\]

References

[1] M Anderson, Curvature estimates and compactness theorems for minimal surfaces in 3-manifolds, Ann. Sci. École Norm. Sup. IV (1985), no. 18, 89–105.
[2] Huai-Dong Cao, *Recent progress on Ricci solitons*, Recent advances in geometric analysis, Adv. Lect. Math. (ALM), vol. 11, Int. Press, Somerville, MA, 2010, pp. 1–38. MR2648937 (2011d:53061)

[3] Huai-Dong Cao and Detang Zhou, *On complete gradient shrinking Ricci solitons*, J. Differential Geom. **85** (2010), no. 2, 175–185. MR2732975 (2011k:53040)

[4] Jeff Cheeger and Detlef Gromoll, *On the Structure of Complete Manifolds of Nonnegative Curvature*, Annals of Mathematics **96** (1972), 413–443, DOI 10.2307/1970819.MR 0309010

[5] Xu Cheng, Tito Mejia, and Detang Zhou, *Stability and compactness for complete f-minimal surfaces*, arXiv:1210.8076 [math.DG] (2012).

[6] Xu Cheng and Detang Zhou, *Volume estimate about self-shrinkers*, Proc. Amer. Math. Soc., DOI 10.1090/S0002-9939-2011-11922-7 , (to appear in print).

[7] Hyeong In Choi and Richard Schoen, *The space of minimal embeddings of a surface into a three-dimensional manifold of positive Ricci curvature*, Invent. Math. **81** (1985), no. 3, 387–394, DOI 10.1007/BF01388577. MR807063 (87a:58040)

[8] Hyeong In Choi and Ai Nung Wang, *A first eigenvalue estimate for minimal hypersurfaces*, J. Differential Geom. **18** (1983), no. 3, 559–562. MR723817 (85d:53028)

[9] Tobias H. Colding and William P. Minicozzi II, *Smooth Compactness of self-shrinkers*, Comment. Math. Helv. **87** (2012), 463–475, DOI 10.4171/CMH/260.

[10] Guofang Wei and Will Wylie, *Comparison geometry for the Bakry-Emery Ricci tensor*, J. Differential Geom. **83** (2009), no. 2, 377–405. MR2577473 (2011a:53064)

[11] B. White, *Curvature estimates and compactness theorems in 3-manifolds for surfaces that are stationary for parametric elliptic functionals*, Invent. Math. **88** (1987), no. 2, 243–256. DOI 10.1007/BF01388908. MR880951 (88g:53037)

[12] Paul C. Yang and Shing Tung Yau, *Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **7** (1980), no. 1, 55–63. MR577325 (81m:58084)
