MODIFIED SCATTERING FOR A DISPERSION-MANAGED
NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. We prove sharp $L^\infty$ decay and modified scattering for a one-dimensional dispersion-managed cubic nonlinear Schrödinger equation with small initial data chosen from a weighted Sobolev space. Specifically, we work with an averaged version of the dispersion-managed NLS in the strong dispersion management regime. The proof adapts techniques from [9, 13], which established small-data modified scattering for the standard 1d cubic NLS.

1. Introduction

We study the long-time behavior of small solutions for a ‘dispersion-managed’ nonlinear Schrödinger equation (NLS) in one space dimension. Such models arise as the envelope equation for electromagnetic wave propagation in fiber-optics communication systems in which the dispersion is varied periodically along the optical fiber (see e.g. [1, 4, 14]). This may be modeled via an equation of the form

$$i\partial_t w + d(t)\partial_{xx} w = c|w|^2 w$$

(1.1)

for some 1-periodic function $d(t)$ and some $c \in \mathbb{R} \setminus \{0\}$. One may then decompose

$$d(t) = d_{av} + d_0(t),$$

with $d_0$ 1-periodic and mean zero, and $d_{av}$ giving the average dispersion.

In the present paper, we will not work directly with the model (1.1), but rather with an ‘averaged’ version in the so-called strong dispersion management regime (as introduced in [7]). In particular, we consider the equation

$$i\partial_t u + d_{av}\partial_{xx} u = c \int_0^1 e^{-iD(\tau)\Delta} \{ |e^{iD(\tau)\Delta} u|^2 e^{iD(\tau)\Delta} u \} \, d\tau,$$

(1.2)

where

$$D(\tau) := \int_0^\tau d_0(\sigma) \, d\sigma$$

(see e.g. [4] Section 1.2) for a derivation of this model). Here we work with the specific choice

$$d(t) = \begin{cases} d_+, & 0 \leq t < t_+, \\ -d_-, & t_+ < t \leq 1 \end{cases}$$

(1.3)

with $d_+$ and $t_+$ chosen so that $d_{av} \neq 0$. As a matter of fact, because we consider only small initial conditions, the sign of $d_{av}$ and $c$ play no essential role, and so we assume without loss of generality that $d_{av} = c = 1$.

Dispersion-managed nonlinear Schrödinger equations have been the subject of a great deal of recent research, due largely to their connection with applications in fiber-optics based communications. This includes both numerical investigations and rigorous mathematical studies (see e.g. [2, 4, 6–8, 10, 11, 17, 18]). Much of the work to
date has centered on questions of well-posedness and the existence and properties of soliton solutions. In this work, we will show that for small initial data chosen from a weighted Sobolev space, the corresponding solutions to (1.2) are global in time and decay as \( |t| \to \infty \). Moreover, such solutions exhibit modified scattering, that is, asymptotically linear behavior up to a logarithmic phase correction (as in the case of the standard 1d cubic NLS). In particular, our result demonstrates the absence of small coherent structures for (1.2).

Our main result is the following.

**Theorem 1.** Let \( u_0 \in H^{1,1} \) satisfy \( \|u_0\|_{H^{1,1}} = \varepsilon > 0 \). If \( \varepsilon \) is sufficiently small, then there exists a unique solution \( u \in C_{t}H^{1,1}_{\mathbb{R}}([0, \infty) \times \mathbb{R}) \) to (1.2) with \( u|_{t=0} = u_0 \). Furthermore, the solution obeys

\[
\|u(t)\|_{L^{\infty}} \lesssim \varepsilon (1 + |t|)^{-\frac{1}{2}}
\]  

for all \( t \geq 0 \), and there exists \( W \in L^{\infty}(\mathbb{R}) \) such that

\[
u(t, x) = (2it)^{-1/2}e^{ix^2/4t}\left[\exp\left\{-\frac{1}{2}|W(x)|^2 \log t\right\}W(x)\right] + O(t^{-\frac{1}{2}-\frac{3}{48}})
\]

in \( L^{\infty} \) as \( t \to \infty \).

Theorem 1 fits in the general context of modified scattering for long-range non-linear Schrödinger equations. In particular, many previous works have considered the standard 1d cubic NLS

\[
i\partial_t u + \partial_{xx} u = \pm|u|^2 u,
\]

for which one also obtains sharp \( L^{\infty} \) and modified scattering for small data in \( H^{1,1} \). In fact, in the defocusing case, one can capitalize on the completely integrable structure of (1.5) to obtain this result without any size restriction on the initial data [5]. Several different approaches have been utilized in order to establish small-data modified scattering for (1.5) (see e.g. [9, 12, 13, 15], as well as [16] for a review). Essentially, each of these approaches are based off of a bootstrap argument involving some ‘dispersive’ type norm and some ‘energy’ type norm, using an ODE argument to obtain estimates for the dispersive part and a chain-rule type estimate and Grönwall to control the energy part. We follow the same general strategy, adapting techniques particularly from [13] and [9]. In particular, we use the Fourier representation and a ‘space-time non-resonance’ type approach as in [13] to control the dispersive norm, while the energy-type estimate relies on the chain-rule type identity satisfied by the Galilean operator \( \mathcal{J}(t) = x + 2it\partial_x \). The estimate for the dispersive norm follows largely as in [13], while the estimate of energy norm requires some modifications. In particular, while the energy estimate for the standard NLS relies directly on the \( L^{\infty} \)-decay for solutions, we must instead rely on the factorization of the free propagator (see (2.1)) in order to exhibit suitable decay in the nonlinearity. For the details, see [9,13] and [9,13] below.

The problem for (1.2) (as opposed to the original model (1.1)) is simplified by the fact that the underlying linear part of the equation is still given by the standard Schrödinger equation. In the setting of (1.1), the situation is more complicated due to the time-dependence in the linear part of the equation, which affects the underlying linear dispersion (see e.g. [2]). We plan to study the model (1.1) in a future work.

The rest of the paper is organized as follows: In Section 2 we collect some notation, a few useful identities and inequalities, and discuss the \( H^1 \) and \( H^{1,1} \)
well-posedness for (1.2). In Section 3, we prove the main result, Theorem 1. In particular, in Section 3.1, we establish global existence and $L^\infty$ decay, while in Section 3.2, we establish the long-time asymptotic behavior of solutions.

2. Preliminaries

In this section, we introduce some notation that will be used throughout the rest of the paper. First, we write $A \lesssim B$ or $B \gtrsim A$ to denote the inequality $A \leq CB$ for some $C > 0$. If $A \lesssim B$ and $B \gtrsim A$, we write $A \sim B$. We also utilize the standard ‘big oh’ notation $O$. Finally, we use the Japanese bracket notation $\langle x \rangle := (1+|x|^2)^{1/2}$.

Our notation for the Fourier transform is $\mathcal{F}[f](\xi) = \hat{f}(\xi) := (2\pi)^{-1/2} \int_{\mathbb{R}} e^{ix\xi} f(x) \, dx$

We define the spaces $H^1$ and $H^{1,1}$ via the norms $\|u\|_{H^1} = \|\langle \partial_x \rangle u\|_{L^2}$ and $\|u\|_{H^{1,1}} = \|u\|_{H^1} + \|xu\|_{L^2}$, where $\langle \partial_x \rangle = \mathcal{F}^{-1}(\xi) \mathcal{F}$.

Solutions to the linear Schrödinger equation with $u|_{t=0} = u_0$ are given by $u(t,x) = e^{it\Delta} u_0(x)$, where $\mathcal{F}^{-1} e^{-i\xi^2} \mathcal{F}$.

The free propagator $e^{it\Delta}$ admits the integral kernel $e^{it\Delta}(x,y) := (4\pi it)^{-\frac{1}{2}} e^{i(x-y)^2 / 4t}$

which implies the factorization identity $e^{it\Delta} = \mathcal{M}(t) \mathcal{D}(t) \mathcal{F} \mathcal{M}(t)$, \hspace{2cm} (2.1)

where $[\mathcal{M}(t)f](x) = e^{ix^2 / 4t} f(x)$ and $[\mathcal{D}(t)f](x) = (2it)^{-\frac{1}{2}} f(x/t)$.

We will make use of the Galilean operator $J(t) = x + 2it\partial_x$. On the one hand, one can directly compute and show that $J(t) = \mathcal{M}(t) [2it\partial_x] \mathcal{M}(-t)$.

On the other hand, an ODE argument leads to the identity $J(t) = e^{it\Delta} xe^{-it\Delta}$. \hspace{2cm} (2.2)

We will make use of the following chain rule identity for $J(t)$, which follows from a direct computation: for any $p > 0$,

$J(t) |z|^p z = \frac{p+2}{2} |z|^{p+2} [J(t)z] - \frac{p}{2} |z|^{p-2} z \overline{[J(t)z]}$. \hspace{2cm} (2.3)

We will also utilize the following elementary estimate several times below: for $0 \leq \alpha \leq 1$,

$|e^{ix} - 1| \lesssim |x|^\alpha$. \hspace{2cm} (2.4)
2.1. Well-posedness. In this section we discuss the $H^1$ and $H^{1,1}$ well-posedness for (1.2) (see e.g. [24] for other well-posedness results for dispersion-managed NLS). As much of what follows is standard, we focus only on the main points and the new estimates needed to treat the specific model (1.2); we refer the reader to [3] for a general introduction to well-posedness for nonlinear Schrödinger equations.

We construct solutions to (1.2) as solutions to the following Duhamel formula:

$$u(t) = e^{i\Delta}u_0 - i \int_1^t \int_0^1 e^{i(t-s)\Delta} \left\{ e^{iD(\tau)\Delta} \left\{ u(s)\right\} d\tau \right\} ds.$$  

We apply the standard contraction mapping argument, with the key nonlinear estimate given as follows: By the chain rule, the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, and the fact that $e^{i\Delta}$ is unitary on $H^1$, we have

$$\| \int_0^1 \langle \partial_x \rangle \left\{ e^{-iD(\tau)\Delta} \left\{ u(s)\right\} d\tau \right\} ds \|_{L^1_t L^2_x([0,T] \times \mathbb{R})} \lesssim \sup_{\tau \in [0,1]} T \| \langle \partial_x \rangle \left\{ e^{-iD(\tau)\Delta} \left\{ u(s)\right\} d\tau \right\} ds \|_{L^\infty_t L^2_x} \lesssim \sup_{\tau \in [0,1]} T \| \langle \partial_x \rangle u \|_{L^\infty_t L^2_x} \lesssim \| \langle \partial_x \rangle u \|_{L^3_t L^2_x}^3.$$  

This allows us to establish local existence for $u_0 \in H^1$ for times $T \sim \| u_0 \|_{H^1}^{-2}$. Similarly, by utilizing the chain rule for $J(t)$ (see (2.3)), we may establish local existence for $u_0 \in H^{1,1}$, again with $T \sim \| u_0 \|_{H^{1,1}}^{-2}$, albeit only with the crude bound

$$\| xu(t) \|_{L^2_x} \lesssim \| J(t)u(t) \|_{L^2_x} + t \| \partial_x u(t) \|_{L^2_x} \lesssim (1 + t) \| u_0 \|_{H^{1,1}}.$$  

This $H^1$-subcritical well-posedness also includes the usual blowup alternative, that is, either $u$ is forward-global or there exists $T_* < \infty$ such that

$$\lim_{t \to T_*} \| u(t) \|_{H^1} = \infty.$$  

We will eventually obtain explicit bounds on the growth of the $H^1$-norm of solutions, which in particular imply that the solutions may be extended to be forward-global in time.

3. Proof of Main Result

We let $u$ be a solution to (1.2) with $\| u_0 \|_{H^{1,1}} = \varepsilon$. By the local theory and Sobolev embedding, we may assume that

$$\| u(t) \|_{H^{1,1}} \leq 2\varepsilon.$$  

for $t \in [0,1]$, and that the solution exists on some maximal interval $(0, T_*)$ with $T_* > 1$.

3.1. Global existence and decay. The first part of the proof of Theorem 1 is based on a bootstrap argument for times $t \geq 1$ involving the following ‘dispersive’ and ‘energy’ norms:

$$\| u(t) \|_{X_D} := \| \hat{f}(t) \|_{L^\infty_x}, \quad \text{where} \quad f(t) = e^{-i\Delta}u(t),$$

$$\| u(t) \|_{X_E} := \| \partial_x u(t) \|_{L^2} + \| J(t)u(t) \|_{L^2}.$$  

Note that by (2.2), we may also write

$$\| J(t)u(t) \|_{L^2} = \| xf(t) \|_{L^2}.$$  

We then define
\[ ||u(t)||_X = \sup_{s \in [1, T]} \{ ||u(s)||_{X_D} + ||u(s)||_{X_E} \}. \]

Control over these norms will imply the desired $L^\infty$ decay.

**Lemma 1.** For any $t \geq 1$,
\[ ||u(t)||_{L^\infty} \lesssim t^{-\frac{1}{2}} \{ ||u(t)||_{X_D} + ||u(t)||_{X_E} \}. \]

**Proof.** We write $f(t) = e^{-it\Delta}u(t)$ as above. By (2.1), Hausdorff–Young, and Cauchy–Schwarz, we have
\begin{align*}
||u(t)||_{L^\infty} &= ||M(t)D(t)F(M(t)f(t))||_{L^\infty} \\
&\lesssim t^{-\frac{1}{2}} \{ ||\hat{f}(t)||_{L^\infty} + ||F[M(t) - 1]f(t)||_{L^\infty} \} \\
&\lesssim t^{-\frac{1}{2}} \{ ||u(t)||_{X_D} + t^{-\frac{1}{2}} ||\xi||_\infty ||f(t)||_{L^2} \} \\
&\lesssim t^{-\frac{1}{2}} \{ ||u(t)||_{X_D} + ||u(t)||_{X_E} \}.
\end{align*}

□

The first part of Theorem 1 will follow from the following bootstrap estimate.

**Proposition 2.** Let $u : [1, T] \times \mathbb{R} \to \mathbb{C}$ be a solution to (1.2) satisfying (3.1). Then there exists $C > 0$ (independent of $T$) so that
\[ ||u(t)||_X \leq 8\varepsilon + C||u(t)||^3_X \]
for all $t \in [1, T]$.

We split this proposition into two lemmas. We begin by estimating the dispersive norm.

**Lemma 2** (Dispersive bound). For any $t \geq 1$,
\[ ||u(t)||_{X_D} \leq 2\varepsilon + C \int_1^t s^{-1-\frac{1}{2}} ||u(s)||_X^3 ds. \]

**Proof of Lemma 2** We begin with the Duhamel formula for the profile $f(t) = e^{-it\Delta}u(t)$:
\[ f(t) = f(1) - i \int_1^t \int_0^1 e^{-is\Delta}e^{iD(\tau)\Delta}F(e^{iD(\tau)\Delta}u(s)) d\tau ds, \]
where $F(z) = |z|^2 z$. Taking the Fourier transform yields the following:
\[ \hat{f}(t) = \hat{f}(1) - i (2\pi)^{-1} \int_1^t \int_0^1 \int \int \left[ e^{i(s+D(\tau))(\xi^2 - (\xi - \eta)^2 + (\eta - \sigma)^2 - \sigma^2)} \right. \\
\left. \times \hat{f}(s, \xi - \eta)\hat{f}(s, \eta - \sigma)\hat{f}(s, \sigma) \right] d\eta d\sigma d\tau ds \]
Changing variables via $\xi - \sigma \mapsto \sigma$, we find that
\[ \hat{f}(t) = \hat{f}(1) - i (2\pi)^{-1} \int_1^t \int_0^1 \int \int e^{2i(s+D(\tau))\eta\sigma}G(s, \xi, \eta, \sigma) d\sigma d\eta d\tau ds, \]
where
\[ G(s, \xi, \eta, \sigma) := \hat{f}(\xi - \eta)\hat{f}(\eta - \xi + \sigma)\hat{f}(\xi - \sigma). \]
By Plancherel, we may write
\[ \hat{f}(t) = \hat{f}(1) - i(2\pi)^{-1} \int_1^t \int_0^1 \int_0^1 \mathcal{F}_{\eta,\sigma} \left[ e^{2i(s+D(\tau))\eta\sigma} \right] \mathcal{F}_{\eta,\sigma}^{-1} [G(s, \xi, \eta, \sigma)] \, d\sigma \, d\sigma \, ds \] (3.7)

Noting the identity
\[ \mathcal{F}_{\eta,\sigma} \left[ e^{2i(s+D(\tau))\eta\sigma} \right] = \frac{1}{2(s+D(\tau))} e^{\frac{-i\eta\sigma}{2(s+D(\tau))}} \] (3.8)

and the fact that
\[ G(s, \xi, 0, 0) = |\hat{f}(s, \xi)|^2 \hat{f}(s, \xi), \]
we may therefore write
\[ \hat{f}(t, \xi) = \hat{f}(1, \xi) - i(2\pi)^{-1} \int_1^t \int_0^1 \frac{1}{2(s+D(\tau))} |\hat{f}(s, \xi)|^2 \hat{f}(s, \xi) \, d\tau \, ds \]
\[ + \int_1^t \int_0^1 \frac{1}{2(s+D(\tau))} \left[ \int \left[ e^{\frac{-i\eta\sigma}{2(s+D(\tau))}} - 1 \right] \mathcal{F}_{\eta,\sigma}^{-1} [G] \, d\sigma \, d\eta \right] \, d\tau \, ds. \]

In particular, this implies
\[ i\partial_t \hat{f}(t, \xi) = \int_0^1 \frac{1}{2(t+D(\tau))} |\hat{f}(t, \xi)|^2 \hat{f}(t, \xi) \, d\tau + \int_0^1 \frac{1}{2(t+D(\tau))} \mathcal{R}(t, \tau, \xi) \, d\tau, \] (3.9)

where
\[ \mathcal{R}(t, \tau, \xi) = (2\pi)^{-1} \int \left[ e^{\frac{-i\eta\sigma}{2(t+D(\tau))}} - 1 \right] \mathcal{F}_{\eta,\sigma}^{-1} [G] \, d\sigma \, d\eta. \]

We now employ an integrating factor to remove the first term on the right-hand side of (3.9). With
\[ \Theta(t) = \int_1^t \int_0^1 \frac{1}{2(s+D(\tau))} |\hat{f}(s, \xi)|^2 \, d\tau \, ds \] (3.10)
and \( g = e^{i\Theta(t)} \hat{f} \), we obtain
\[ i\partial_t g = e^{i\Theta(t)} \int_0^1 \frac{1}{2(t+D(\tau))} \mathcal{R}(t, \tau, \xi) \, d\tau. \]

We want estimate this quantity in \( L^\infty_\xi \). Using the definition of \( \mathcal{R} \) from above, Cauchy–Schwarz and the bound (2.4), we find that
\[ |\partial_t g| \lesssim \int_0^1 \int \int \|t+D(\tau)|^{-1-\frac{1}{2}} |\eta| \| |\sigma| \| \mathcal{F}_{\eta,\sigma}^{-1} [G](s, \xi, \eta, \sigma) \| \, d\sigma \, d\eta \, d\tau \] (3.11)

To proceed, we now need to invert the Fourier transform appearing in (3.11). Writing
\[ \mathcal{F}_{\eta,\sigma}^{-1} [G] = \mathcal{F}_{\sigma}^{-1} \left[ \mathcal{F}_{\eta}^{-1} \left[ \hat{f}(\xi - \eta) \hat{f}(\eta - \xi + \sigma) \right] \hat{f}(\xi - \sigma) \right], \]

a direct computation leads to the estimate
\[ |\mathcal{F}_{\eta,\sigma}^{-1} [G](\eta, \sigma)| \lesssim \int |f(z - \eta)| |f(z)| |f(z - \sigma)| \, dz. \] (3.12)
Thus, by the triangle inequality and Cauchy–Schwarz, and the fact that $D(\tau) \geq 0$,

$$|s.11| \lesssim \int_0^1 \int \int \int \left| \tau + D(\tau) \right|^{-\frac{1}{2}} \left| \left| z - \eta \right| + \left| z \right| \right| \left| \left| z - \sigma \right| + \left| z \right| \right| \times \left| f(z - \eta) \right| \left| f(z) \right| \, dz \, d\tau$$

$$\lesssim t^{-1/2} \left\| f \right\|_{L^1} \left| \left( x \right) \right|^{1/2} \left\| f \right\|_{L^2},$$

$$\lesssim t^{-1/2} \left\| \left\langle x \right\rangle f \right\|_{L^2} \lesssim t^{-1/2} \left\| u(t) \right\|_{X^1}.$$

The desired estimate then follows from the fundamental theorem of calculus and the triangle inequality:

$$g(t) = |f(t)| = |f(1)| + \int_1^t |\partial_s g(s)| \, ds \implies \left\| u(t) \right\|_{X^D} \leq 2\varepsilon + C \int_1^t s^{-1/2} \left\| u(s) \right\|_{X^1}^3 \, ds.$$

We next estimate the energy norm.

**Lemma 3** (Energy bound). For $u$ as above, we have the following estimate:

$$\left\| u(t) \right\|_{X^E} \leq 2t^{-1/2} \varepsilon + Ct^{-1/2} \int_1^t s^{-1/2} \left\| u(s) \right\|_{X^1}^3 \, ds. \quad (3.13)$$

*Proof of Lemma 3* The starting point is the Duhamel formula

$$u(t) = e^{i(t-1)D} u(1) - i \int_1^t \int_0^1 e^{i(t-s)D} e^{-iD(\tau)} \Delta F(e^{iD(\tau)} \Delta u(s)) \, d\tau \, ds$$

with $F(z) = |z|^2 z$.

We first estimate the weighted component of the $X^E$-norm. Using \textcolor{red}{\textbf{[29]}}, we deduce

$$J(t)u(t) = e^{itD} x f(1)$$

$$- i \int_1^t \int_0^1 e^{i(t-s)D} e^{-iD(\tau)} \Delta \left[ J(s + D(\tau)) F(e^{iD(\tau)} \Delta u(s)) \right] \, d\tau \, ds.$$

Thus, using \textcolor{red}{\textbf{[28]}}, \textcolor{red}{\textbf{[31]}}, \textcolor{red}{\textbf{[29]}}, Sobolev embedding, and the unitarity of $e^{itD}$, we obtain

$$\left\| J(t)u(t) \right\|_{L^2} \leq 2\varepsilon + C \int_1^t \int_0^1 \left\| e^{iD(\tau)} \Delta u(s) \right\|_{L^2} \left\| J(s + D(\tau)) e^{iD(\tau)} \Delta u(s) \right\|_{L^2} \, d\tau \, ds$$

$$\leq 2\varepsilon + C \int_1^t \int_0^1 \left\| e^{iD(\tau)} \Delta u(s) \right\|_{L^2}^2 \left\| e^{iD(\tau)} \Delta J(s) u(s) \right\|_{L^2} \, d\tau \, ds$$

$$\leq 2\varepsilon + C \int_1^t s^{1/2} \left\| \left( \int_0^1 \left| e^{i(s+D(\tau))} f(s) \right|_{L^\infty}^2 \, d\tau \right) \right\| \left\| u(s) \right\|_{X_E}.$$

\textsuperscript{1}The fact that $D(\tau) \geq 0$ is a convenient consequence of our particular choice of parameters above. To treat situations in which $D(\tau)$ may take negative values, one only needs to observe that $\sup_{s \in [0,1]} \left| D(\tau) \right| \leq T_0$ for some $T_0 > 0$ and then begin the bootstrap estimate at times $t \geq 2T_0$, for example.
To proceed, we use the factorization identity (2.1) and estimate as we did in the proof of Lemma 1. This yields
\[
\|e^{i(s+D(\tau))\Delta}f(s)\|_{L^\infty_x}
\lesssim |s+D(\tau)|^{-\frac{1}{2}} \left\{ \|\hat{f}\|_{L^\infty_x} + \|\mathcal{F}[M(s+D(\tau))-1]f\|_{L^1_x} \right\}
\lesssim |s+D(\tau)|^{-\frac{1}{2}} \left\{ \|\hat{f}\|_{L^\infty_x} + \|M(s+D(\tau))-1\|_{\mathcal{L}_1} \right\}
\lesssim |s+D(\tau)|^{-\frac{1}{2}} \left\{ \|\hat{f}\|_{L^\infty_x} + |s+D(\tau)|^{-\frac{1}{2}} \|x\| \|\hat{f}\|_{L^1_x} \right\}
\lesssim |s+D(\tau)|^{-\frac{1}{2}} \|\hat{f}\|_{L^\infty_x} + |s+D(\tau)|^{-\frac{1}{2}} \|x\| \|\hat{f}\|_{L^2_x}
\lesssim \left\{ |s+D(\tau)|^{-\frac{1}{2}} + |s+D(\tau)|^{-\frac{1}{2}} \right\} \|u(s)\|_{X} \lesssim |s|^{-\frac{1}{2}} \|u(s)\|_{X},
\] where we have again used the fact that \(D(\tau) \geq 0\) (cf. the footnote on page 7). Inserting this into (3.14), we obtain the desired estimate for the weighted component of the \(X_E\)-norm.

For the \(H^1\) component of the \(X_E\)-norm, we estimate in much the same way, using the chain rule directly in place of (2.3). \(\square\)

With Lemma 2 and Lemma 3 in place, we readily obtain the estimate appearing in Proposition 2. Using a standard continuity argument, the well-posedness theory discussed in Section 2.1 and Lemma 1, we deduce the first part of Theorem 1. In particular, we have the following:

**Corollary 3.** For \(\|u_0\|_{H^{1.1}} = \varepsilon\) sufficiently small, there exists a unique, forward-global solution \(u \in C_t H^{1.1}_x(0,\infty) \times \mathbb{R}\) to (1.2) with \(u|_{t=0} = u_0\) obeying
\[
\|u(t)\|_{X} \lesssim \varepsilon \quad \text{for all} \quad t \geq 1.
\]
In particular,
\[
\|u(t)\|_{L^\infty} \lesssim \varepsilon (1 + |t|)^{-\frac{1}{2}} \quad \text{for all} \quad t \geq 0.
\]

### 3.2. Asymptotic behavior.

We now turn to the second part of Theorem 1 namely, the asymptotic behavior of small solutions to (1.2).

**Proof of 1.3.** To begin, we return to the setting of the proof of Lemma 2, this time with the bounds provided by Corollary 3 in hand. In particular, recalling
\[
g(t) = e^{i\Theta(t)} \hat{f}(t), \quad \text{with} \quad \Theta(t) = \int_{1}^{t} \int_{0}^{1} \frac{1}{2(s+D(\tau))} |\hat{f}(s,\xi)|^2 \, d\tau \, ds,
\]
the estimate of (3.11) now yields the bound
\[
\|\partial_t g\|_{L^\infty_x} \lesssim t^{-1-\frac{1}{2\varepsilon}} \varepsilon^{3}.
\]
It follows that
\[
\|g(t) - W_0\|_{L^\infty} \lesssim \varepsilon^3 t^{-\frac{1}{2\varepsilon}}
\]
for some \(W_0 \in L^\infty\). In particular, we have \(|\hat{f}(t)| \to |W_0|\) in \(L^\infty\), with the same rate of convergence.

We next observe that
\[
\left| \int_{0}^{1} \frac{1}{2(s+D(\tau))} \, d\tau - \frac{1}{2s} \right| \lesssim s^{-2}
\]
for all \(s \geq 1\). Using this, we deduce that
\[
\Theta(t) = \frac{1}{2} |W_0|^2 \log t + \Phi(t),
\]
where $\Phi(t)$ converges to a real-valued limit $\Phi_\infty$ in $L^\infty$ (with a rate of $t^{-\frac{1}{10}}$). Thus, setting $W = e^{-i\Phi_\infty} W_0$, we may obtain

$$\hat{f}(t) = e^{-\frac{i}{2} W_0^2 \log t} e^{-i\Phi_\infty} W_0 + O(t^{-\frac{1}{20}}) = e^{-\frac{i}{2} |W|^2 \log t} W + O(t^{-\frac{1}{20}})$$

(3.16)
in $L^\infty$ as $t \to \infty$. Finally, using (2.1) and estimating as we did for Lemma 1, we obtain

$$u(t) = e^{it\Delta} \hat{f}(t) = M(t)D(t)\hat{f}(t) + O(t^{-\frac{1}{20}}).$$

Inserting the asymptotic behavior for $\hat{f}$ obtained in (3.16), we obtain the desired asymptotic behavior for $u(t)$. □

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