A finite temperature investigation of dual superconductivity in the modified SO(3) lattice gauge theory

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Abstract

We study the $SO(3)$ lattice gauge theory in 3+1 dimensions with the adjoint Wilson action modified by a $\mathbb{Z}_2$ monopole suppression term and by means of the Pisa disorder operator. We find evidence for a finite temperature deconfinement transition driven by the condensation of $U(1)$ magnetic charges. A finite-size scaling test shows consistency with the critical exponents of the 3D Ising model.

Lattice $SU(N)$ pure gauge theories in the fundamental representation undergo a finite temperature deconfinement phase transition [1,2] signalled by the spontaneous breaking of the global center symmetry $\mathbb{Z}_N$ [3,4]. It is an interesting question whether this happens also for the theory in the adjoint representation $SU(N)/\mathbb{Z}_N$ [5]. In the latter case the center is trivial and naively there is no global symmetry to be broken. Moreover, although both the fundamental and the adjoint $SU(N)$ lattice theories have the same naive continuum limit,
non-perturbative investigations carried out for Wilson [6,7] as well as for Villain discretizations [8] showed a different behavior. For the adjoint cases no evidence for a finite temperature phase transition was found, whereas a bulk transition separating the strong from the weak coupling regions appeared for both kinds of discretizations.

The bulk transition was explained in terms of a condensation of lattice artifacts - $\mathbb{Z}_2$ monopoles [8]. It was argued that the finite temperature transition could be overshadowed by the bulk one and $\mathbb{Z}_2$ monopole suppression terms were thus proposed [9]. More recently the Villain mixed fundamental-adjoint $SU(2)$ model with a monopole (and vortex) suppression has been investigated [10,11,12] and first evidence for a deconfinement transition in the Ising 3D universality class, at least for strong coupling ($N_\tau = 2$), was given. Moreover, in the weak coupling region negative states of the Polyakov loop in the adjoint representation were found [10,13] and later linked to the non-trivial twist sectors of the theory [14], given that on the hypertorus $T^4$ the partition function of the $SO(3)$ theory in the Villain formulation with complete $\mathbb{Z}_2$ monopole suppression was shown to be equivalent to that of the $SU(2)$ theory in the fundamental representation when summed over all twist sectors [15,16,17,18].

In a recent paper we have reinvestigated the $SO(3)$ theory using the Wilson action and a $\mathbb{Z}_2$ monopole suppression term with a “chemical potential” $\lambda$ [19,20,21,22]. The phase diagram in the $\beta_A - \lambda$ plane was studied at zero and finite temperature monitoring the tunneling between twist sectors and its disappearance for strong enough $\mathbb{Z}_2$ monopole suppression. For $\lambda \geq .85$ we have found a strong indication for the existence of a finite temperature deconfinement transition although having restricted the simulation to a fixed
(e.g. trivial) twist sector. The proposed phase diagram is redrawn in Fig. 1. Since no proper order parameter was available a determination of critical exponents was not intended, and there was no answer given to the question about the underlying confinement mechanism.

A disorder parameter related to Abelian monopole condensation in the dual superconductivity picture of confinement [23,24,25] has been devised by the Pisa group some time ago [26,27,28,29,30]. It is the vacuum expectation value of a magnetically charged operator \( \langle \mu \rangle \) shown to be different from zero in the confined phase, thus signaling dual superconductivity, and going to zero at the deconfining phase transition. Similar parameters have been constructed more recently by Fröhlich and Marchetti [31] as well as in the framework of the lattice Schrödinger functional [32,33] leading to analogous results. The main advantage of these parameters is that they can be applied also to full QCD, where the center symmetry is explicitly broken by the fermionic degrees
of freedom and - as we shall show - to the adjoint pure gauge theory, where center symmetry becomes trivial. In this letter we will use the Pisa disorder operator in order to answer the questions raised above for the $SO(3)$ lattice gauge theory. In particular we will check the critical exponents and whether the dual superconductor scenario applies also to this case.

We will study the $SU(2)$ adjoint representation Wilson action modified by a $Z_2$ monopole suppression term

$$S = \frac{4}{3} \beta_A \sum_P \left( 1 - \frac{\text{Tr}^2 F_U P}{4} \right) + \lambda \sum_c (1 - \sigma_c), \quad (1)$$

where the product $\sigma_c = \prod_{P \in \partial c} \text{sign}(\text{Tr}_F U_P)$, taken around elementary 3-cubes $c$, defines the $Z_2$ magnetic charges. Their density can be introduced as $M = 1 - \langle \frac{1}{N_c} \sum_c \sigma_c \rangle$ normalized such that it tends to one in the strong coupling region and to zero in the weak coupling limit, $N_c$ denoting the total number of elementary 3-cubes. Although $\sigma_c$ is constructed in terms of fundamental representation plaquettes, it is a natural $SO(3)$ quantity ensuring that the action (1) is center-blind in the entire $\beta_A - \lambda$ plane. The link variables can be represented both by $SO(3)$ or $SU(2)$ matrices, exploiting the property $\text{Tr}_A = \text{Tr}^2_F - 1$ for the Wilson term or picking a random $SU(2)$ representative of the $SO(3)$ link to construct the $Z_2$ monopole contribution. A standard Metropolis algorithm has been used to update the links. In [21] we have found a strong indication in favor of finite temperature phase transitions with lines moving up with the time-like lattice extent $N_r$ and running away from the bulk transition approximately parallel to the $\lambda$ axis (see Fig. 1).

The Pisa disorder operator [26,27,28,29,30] was shown to be a reliable order parameter for $SU(2)$ and $SU(3)$ gauge theories in the fundamental repre-
sentation, with and without dynamical quarks, giving critical exponents in agreement with other order parameters. Moreover, it can give important informations about the mechanism which confines quarks into hadrons. The Pisa disorder operator is motivated by the dual superconductor scenario for the QCD vacuum [23,24,25], driven by the condensation of $U(1)$ magnetic charges. The construction of the operator in the case of the modified $SO(3)$ theory follows the same line as in the fundamental case, so we will avoid going into the details and we will refer to the original papers for further details [26,27,28,29,30].

The idea is to construct a magnetically charged operator $\mu$ which shifts the quantum field at a given time slice by a classical external field corresponding to a magnetic monopole. The $U(1)$ subgroup of the gauge group which defines the magnetic charge is selected by an Abelian projection, usually fixed by diagonalizing an operator $X$ in the adjoint representation. The disorder parameter is defined as

$$\langle \mu(t) \rangle = \frac{\int (DU) e^{-S_M(t)} f(DU)}{\int (DU) e^{-S}} ,$$

(2)

where $S_M(t)$ denotes the Wilson action with the space-time plaquettes $U_{i4}(\vec{x},t)$ at a fixed time-slice $t$ modified by an insertion of an external monopole field

$$\tilde{U}_{i4}(\vec{x}, t) = U_i(\vec{x}, t) \Phi_i(\vec{x} + \hat{i}, \vec{y}) U_4(\vec{x} + \hat{i}, t) U_i^\dagger(\vec{x}, t + 1) U_4^\dagger(\vec{x}, t)) ,$$

(3)

where $\Phi_i(\vec{x}, \vec{y}) = \Omega e^{iT_a \tilde{b}_a(\vec{x}-\hat{i}, \vec{y})} \Omega^\dagger$, with $\Omega$ the gauge transformation which diagonalizes an operator $X$ in the adjoint representation. $T_a$ denote the generators of the Cartan subalgebra and $\tilde{b}$ the discretized transverse field generated at the lattice spatial point $\vec{x}$ by a magnetic monopole sitting at $\vec{y}$. We decided to work with the completely random Abelian projection (RAP) [29,30] in which
we do not diagonalize any operator $X$: it can be thought as a kind of averaging over a continuous infinity of Abelian projections. It should be stressed that only the plaquette contribution to the action (1) is modified by the insertion of the monopole field and not the chemical potential term. From the definition of $\mu$, making use of an iterated change of variables, it can be shown that the correlation function $D(\Delta t) = \langle \bar{\mu}(\vec{y}, t + \Delta t)\mu(\vec{y}, t) \rangle$ describes the creation of a monopole at $(\vec{y}, t)$ and its propagation from $t$ to $t + \Delta t$ [26,27,28,29,30]. At large $\Delta t$, by cluster property, $D(\Delta t) \simeq A \exp(-M\Delta t) + \langle \mu \rangle^2$. A non-vanishing $\langle \mu \rangle$ indicates spontaneous breaking of the $U(1)$ magnetic symmetry and hence dual superconductivity. In the thermodynamical limit one expects $\langle \mu \rangle \neq 0$ for $T < T_c$, while $\langle \mu \rangle = 0$ for $T > T_c$ if the deconfining phase transition is associated with a transition from a dual superconductor to a trivial vacuum.

At finite temperature there is no way to put a monopole and an antimonopole at large distance along the $t$-axis as it is done at $T = 0$, since at $T \sim T_c$ the temporal extent $N_\tau a$ is comparable to the correlation length. Therefore, one computes $\langle \mu \rangle$ but with $C^*$-periodic boundary conditions in time direction imposed to the numerator in Eq. (2) in order to ensure magnetic charge conservation: $U_i(\vec{x}, N_\tau) = U_i^*(\vec{x}, 0)$, where $U_i^*$ is the complex conjugate of $U_i$.

These boundary conditions have been indicated in Eq. (2) by the index $M$ at the integration measure. They change the sign of the term proportional to $\sigma_3$ in the links, creating a dislocation with magnetic charge -1 at the boundary which annihilates the positive magnetic charge created by the operator $\mu$.

As a consequence the magnetic charge is conserved and everything is consistent. Of course, the denominator in Eq. (2) is computed using standard periodic boundary conditions. We used links in the fundamental representation, so we implemented exactly the above condition, but an analogous con-
dition holds also for link variables defined in the adjoint representation, i.e. 
\[ U_i(\vec{x}, N_\tau) = (\mathbb{I}_3 + 2T_2^2)U_i(\vec{x}, 0)(\mathbb{I}_3 + 2T_2^2); \] charge conjugation is realized in both representations through rotations by an angle \( \pi \) around the color 2-axis.

Since \( \langle \mu \rangle \) is the average of the exponential of a sum over the physical volume, it is affected by huge fluctuations which make it difficult to be measured in Monte Carlo simulations. A way out is to compute the derivative with respect to the coupling parameter \( \beta \equiv \beta_A \), which contains all the relevant information
\[
\rho = \frac{d}{d\beta} \log \langle \mu \rangle. \tag{4}
\]

It is given by the difference between the Wilson plaquette action term \( \langle \Pi \rangle \) averaged with the usual measure and the modified plaquette action term \( \langle \Pi_M \rangle \) averaged with the modified measure \( (DU)_M e^{-SM} / \int (DU)_M e^{-SM}. \) The order parameter can be reconstructed from
\[
\langle \mu \rangle = \exp \left( \int_0^\beta \rho(\beta') d\beta' \right). \tag{5}
\]

Eq. (5) tells us that in order to have \( \langle \mu \rangle \neq 0 \) in a confined phase, where the dual magnetic symmetry becomes broken, \( \rho \) should stay finite for \( \beta < \beta_c \) in the thermodynamical limit (i.e. in the limit of spatial lattice extent \( N_s \to \infty \)), while a sharp negative peak for \( \rho \) occurring at \( \beta_c \) and diverging for \( N_s \to \infty \) should signal the phase transition associated with the restoration of the dual magnetic symmetry. Above the transition a sufficient condition for \( \langle \mu \rangle \) to vanish would be to have \( \rho \to -\infty \) in the thermodynamical limit. In [26,27,28,29,30] for the \( SU(2) \) case it was argued using perturbation theory that \( \rho \) for \( \beta \to \infty \) reaches negative plateau values linearly scaling with \( N_s \). In [34] a more detailed numerical analysis has been performed for both \( SU(2) \) and

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SU(3) pure gauge theories, showing that, in the case where $\mu$ is magnetically charged, $\rho$ diverges (to negative values) linearly in the weak coupling limit, and more and more rapidly as $T \to T_c$ from above, where it diverges as $N_s^{1/\nu}$, thus proving that $\langle \mu \rangle$ is strictly zero for every temperature $T > T_c$, as follows from Eq. (5). Therefore, when simulating the theory at accessibly large $\beta$ values and lattice sizes, a good criterion for $\langle \mu \rangle$ being exactly zero above the transition temperature, is that $\rho$ keeps diverging at least linearly with $N_s$ in a wide range of $\beta$ values above the transition.

Fig. 2. Disorder operator $\rho$ computed at different values of the chemical potential $\lambda$ at finite temperature in the asymmetric volume $V = 4 \times 12^3$. 

(a) $\lambda = 0.0$  
(b) $\lambda = 0.4$  
(c) $\lambda = 0.7$  
(d) $\lambda = 0.8$
We will now present our numerical results. First of all we studied the bulk transition (compare with Fig. 1) for some \( \lambda \)-values between 0 (no monopole suppression) and 0.8 (partial monopole suppression) by varying \( \beta_A \). We do this for finite temperature \( (N_r = 4) \) to investigate the interplay between the bulk transition and the finite temperature one. Fig. 2 shows a clear dip at the location of the bulk transition, in agreement with what was found in previous works with other observables. Although the dip in \( \rho \) decreases in magnitude with increasing \( \lambda \), i.e. as we are suppressing more and more the \( \mathbb{Z}_2 \) monopoles, the lattice artifacts are still present and overshadow the finite temperature transition, assuming the latter exists. Therefore, one cannot yet see neither the scaling of the physical transition with the temperature nor the finite-size scaling of the operator with the spatial volume, although one expects that on large enough lattices the two transitions should decouple also for these values of the couplings [21]. Anyway we can conclude that a condensation of \( U(1) \) magnetic charges takes place below the bulk transition line.

Let us now turn to the more interesting case of stronger monopole suppression \( (\lambda = 1.0) \), where the decoupling of the bulk from the finite temperature transition occurs already at reasonable volumes. Again we vary the temperature through \( \beta_A \). We have kept the system fixed in the trivial twist sector so to compare with the results for the adjoint Polyakov loop and the distribution of the fundamental Polyakov loop variable in [21]. As one can see from Fig. 3 (a), at fixed \( N_r = 4 \) the parameter \( \rho \) shows a dip around \( \beta_A \simeq 1.0 \) which becomes deeper and deeper with increasing spatial volume. \( \rho \) stays finite in the low \( \beta_A \) region as the spatial volume is increased, as can be inferred from Fig. 3 (c), while the data for high \( \beta_A \) are consistent with its saturation at negative plateau values diverging more than linearly with \( N_s \), as Fig. 3 (d) shows.
Fig. 3. $\rho$ computed in the trivial twist sector at finite temperature ($N_T = 4$) for different values of the spatial volume and for chemical potential $\lambda = 1.0$ (a). Finite-size scaling analysis for $\rho$ (b). The strong and weak coupling regions are highlighted in (c) and (d) respectively.

This gives us clear evidence that also in this case, like for the lattice theory in the fundamental representation, condensation of $U(1)$ magnetic charge takes place below the deconfinement phase transition and disappears above $T_c$. The vacuum is a dual superconductor in the confined phase and becomes trivial in the deconfined one. It shows that the Pisa disorder operator is a meaningful order parameter also in a center-blind theory and for fixed twist. Moreover,
the position of the dip is well-consistent with the results found in our previous work [21]. The statistics is not sufficient in order to determine the critical

dependents independently, but as one can see from Fig. 3 (b), using the known critical exponent for the 3D Ising model, $\nu = 0.63$ and estimating the critical adjoint coupling $\beta_c^A = 0.98$ the data show a reasonable finite-size scaling behavior with the spatial volume. We have also checked the scaling with respect to the continuum limit for varying $N_r$, making some simulations by fixing $N_r = 6$ and $N_s = 16, 20, 24$. As one can see from Fig. 4 (a) the dip occurs at a larger $\beta_A$ value than for $N_r = 4$ and becomes again deeper by increasing $N_s$, in agreement with a finite temperature phase transition. In this case we estimate $\beta_c^A = 1.19$. Fig. 4 (b) shows the quality of finite-size scaling assuming the Ising model value for the critical index. Our results are in agreement with the finite-size scaling observed for the specific heat in the Villain action case for $N_r = 2$ [10,11,12].

We can conclude that our investigation of the modified lattice $SO(3)$ gauge
theory with Wilson action and $\mathbb{Z}_2$ monopole suppression using the Pisa disorder operator has confirmed our previous results [21]. Although having restricted to the fixed trivial twist case we find a clear indication for the existence of a finite temperature transition decoupled from the notorious bulk transition. The critical behavior at such a transition reasonably agrees with the critical exponents of the 3D Ising model. The nature of the Pisa disorder parameter we used tells us that the transition is related to a condensation of $U(1)$ magnetic charges for $T < T_c$ which disappears above the transition, proving that the dual superconductor scenario is a good model of confinement also for the adjoint theory. Of course, a final answer can be given only after all twist sectors will be taken into account simultaneously. This study is currently under way. Moreover, it would certainly be interesting to compute also the free energy of an (extended) center vortex in order to check how the vortex condensation mechanism works in this case.

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