Low thrust spacecraft transfers optimization method with the stepwise control structure in the Earth-Moon system in terms of the L1-L2 transfer

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Abstract. The paper outlines the method for determination of the locally optimal stepwise control structure in the problem of the low thrust spacecraft transfer optimization in the Earth-Moon system, including the L1-L2 transfer. The total flight time as an optimization criterion is considered. The optimal control programs were obtained by using the Pontryagin’s maximum principle. As a result of optimization, optimal control programs, corresponding trajectories, and minimal total flight times were determined.

1. Introduction
Nowadays the spacefaring nations are developing the missions to achieve the libration points of the Earth-Moon system, especially L1 and L2. The optimal interplanetary trajectories and the trajectories of flights to the Moon pass near libration point L1 of the Earth-Moon system, as shown in works [1-2]. Moreover, the usage of the libration points will help to decrease the fuel expenses for orbit maintaining, to start from the Earth at any moment without choosing the date of start, to monitor the solar wind and to avoid the radiation from the Earth. One of the main problems of such missions is to determine the optimal control structure of the spacecraft transfers.

2. Mathematical Model of the Mission
Let us formulate the general statement of the optimization problem. The following parameters are considered:

- $x(t) = (r(t), V(t), m_f(t), r_E(t), r_M(t), r_S(t))^T \in X$ is a system state vector corresponding to boundary conditions, defined by the purpose of the transfer and possible restrictions, where $X$ is a set of admissible state area;
- $u(t) = (\delta(t), e(t))^T \in U$ is a vector of control functions, where $U$ is a set of admissible control parameters;
- $p = (a_0, j_{sp})^T \in P$ is a vector of optimized design parameters. It is limited by a set of admissible area of design parameters $P$.

Here $t$ is the current time, $r(t)$ is a radius vector of the spacecraft (SC), $V(t)$ is a vector of the SC velocity, $m_f(t)$ is an expended fuel mass, $r_E(t), r_M(t), r_S(t)$ are the radius-vectors of the Earth, the Moon and the Sun, $\delta(t)$ is a function of thrust switching, $e(t)$ is a thrusting direction unit vector, $a_0$ is a nominal acceleration of the SC in the initial orbit, $j_{sp}$ is a specific impulse of the propulsion system.
The boundary conditions of the flight are shown in Table 1.

**Table 1. Boundary conditions of the flights in the Earth-Moon system**

| Finishing time | Radius-vector | Velocity vector | Radius-vector of the Earth | Radius-vector of the Moon | Radius-vector of the Sun | Set of admissible state area |
|----------------|---------------|-----------------|--------------------------|--------------------------|--------------------------|-----------------------------|
| L1 (t₁)        | r₁(t₁)       | V₁(t₁)         | rₑ(t₁)                   | rₘ(t₁)                   | rₛ(t₁)                   | Xₐ₁                        |
| L2 (t₂)        | r₂(t₂)       | V₂(t₂)         | rₑ(t₂)                   | rₘ(t₂)                   | rₛ(t₂)                   | Xₐ₂                        |

Optimizing these space transfers with low thrust we need to determine vectors \( u_{\text{opt}}(t) \) and \( p_{\text{opt}} \) (vectors of optimal control functions and optimal design parameters correspondingly) that provide the minimum duration of flight \( T \) to perform the mission purposes according to Table 1.

\[
T = \min_{u \in U, p \in P} T | m = \text{unfixed}, x \in X
\]  

(1)

The transfers are considered in terms of the barycentric combined inertial (BCI) frame with corresponding motion equations [3]. The following assumptions are made: the eccentricity of the Moon and the Earth orbits around barycenter is neglected; the eccentricity of the gravitational fields of the Earth, the Moon and the Sun are neglected, the SC sometimes moves in the Earth and the Moon shadow.

### 3. Control laws

In this work we use Fedorenko successful linearization method [1] that accepts the limitation on the composed functions that have Frechet derivatives. The method is based on bringing the variation optimal control problem to the iteration problem of linear programming. The functional for optimization was chosen as a sum of the fuel used during the transfer and the components accounting for the conditions of the spacecraft final orbit insertion:

\[
I = m(t_i^{(n)}) + (r_i^{(n)}(t_i^{(n)}) - R_k)^2 + (\varphi_i^{(n)}(t_i^{(n)}) - \omega_m t_i^{(n)})^2 + v_i^{(n)}(t_i^{(n)}))^2 + (\varphi_i^{(n)}(t_i^{(n)}) - \varphi_i)^2 \rightarrow \min
\]  

(2)

where \( R_k \) is a radius of the final orbit, \( \omega_m \) – Moon rotation rate around the barycenter of the system, \( \varphi_i \) – angular rotation rate on the final orbit, \( n \) – the number of the final segment of the transfer.

To solve the assigned problem we should find the following variables:

\[
\frac{\partial I}{\partial t} \frac{\partial I}{\partial I} \frac{\partial I}{\partial I} \frac{\partial I}{\partial I} \frac{\partial I}{\partial I} \frac{\partial I}{\partial I} 
\]  

(3)

The exact solution of the problem was established with the use of the Pontryagin maximum method and the numerical integration. The analysis of that solution shows, that the transfer trajectory has three general segments of work (Figure 1). Each of the general steps of the transfer control program was divided into several segments of work to provide a better accuracy.
The thrust is directed at the angle of \( \lambda \) to the radius-vector of the SC. Thus, \( u \) is a piecewise continuous function which is determined by the following parameters: \( \lambda, \partial T_i, \partial T_e \) (each \( \lambda, \partial T_i, \partial T_e \) is relevant to the corresponding segment of the trajectory).

So, according to notations of the Fedorenko method we have:

\[
\begin{align*}
    u^{i_{1}} &= \{ \lambda_i \}, \quad p^{i_{1}} = \{ T_i \}, \quad q = \{ a_i, c_o \},
\end{align*}
\]

where \( a_i \) – spacecraft acceleration, \( c_o \) – exhaust velocity. As one can see from (2) the functional does not include the integral component, it consists only of the terminal component and depends on the following state vector:

\[
\begin{align*}
    x^{i_{1}}(t^{i_{1}}) &= \{ r, \varphi, v_r, v_{\varphi}, m \} = \{ x_1, x_2, x_3, x_4, x_5 \}^T.
\end{align*}
\]

After obtaining of motion equations [3] with the boundary conditions (Table 1) integrated we get the state vector components relevant to the final transfer segment determined.

Let us derive the Hamiltonian for costate vectors \( \psi, \psi_r, \psi_{\varphi}, \psi_w \):}

\[
H = \frac{dr}{dt} \psi_r + \frac{d\varphi}{dr} \psi_{\varphi} + \frac{dv_r}{dt} \psi_{w_r} + \frac{dv_{\varphi}}{dt} \psi_w + \frac{dm}{dt} \psi_w.
\]

Then we introduce the right parts of the motion equations into (6):

\[
H = v_r \psi_r + \frac{v_r}{r} \psi_{w_r} + \left( \frac{v^2}{r^2} - \frac{a_i}{1-m} \delta \cos \lambda \right) \psi_{\varphi} + \left( -\frac{v}{r} + \frac{a_i}{1-m} \delta \sin \lambda \right) \psi_{w_r} + \beta \psi_w.
\]

Here we can estimate the costate functions derivatives by deriving the Hamiltonian with respect to the corresponding state coordinates:

\[
\begin{align*}
    \frac{d\psi_r}{dt} &= \frac{\partial H}{\partial r}, \quad \frac{d\psi_{\varphi}}{dt} = \frac{\partial H}{\partial \varphi}, \quad \frac{d\psi_{w_r}}{dt} = \frac{\partial H}{\partial v_r}, \quad \frac{d\psi_w}{dt} = \frac{\partial H}{\partial \psi_w}, \quad \frac{dm}{dt} = \frac{\partial H}{\partial m}.
\end{align*}
\]
Scalar function $\mu$ determines the finishing time of each segment. In this case each of five $\mu$ functions is a null vector, because there are no discontinuous jumps of the state coordinates on the transfer segments boundaries.

To determine the needed derivatives we need to find the following variables:

$$a^{i(s)} = -\frac{1}{\mu^{i(s)} f^{i(s)} + \mu^{i(s)}} = -1$$

$$d = a^{i(s)} (F_i + F_i f^{i(s)} \mu^i) = -1 \left( \beta + 2 (r^{i(s)} (t_0^{i(s)}) - r_z) \right) v + 2 \left( \phi^{i(s)} (t_0^{i(s)}) - \omega_m t_0^{i(s)} \right) v + 2 \left( v^{i(s)} (t_0^{i(s)}) - v_{e,z} \right) v$$

Then the right parts of motion equations for the state coordinates should be found:

$$f_i(T_i) = v_i(T_i)$$
$$f_s(T_i) = v_s(T_i)$$
$$f_{v_i}(T_i) = \left( \frac{v_i^2}{r} + \frac{a_s}{1-m} - \delta \sin \lambda \right) T_i$$
$$f_{v_s}(T_i) = \left( -\frac{v_s v_i}{r} + \frac{a_s}{1-m} - \delta \sin \lambda \right) T_i$$
$$f_f(T_i) = \beta$$

Now the costate functions values in the final points could be found:

$$\psi^{i(s)} = F_i + d \cdot \mu^i$$

$$\left( \begin{array}{c}
\psi_s \\
\psi_s \\
\psi_{v_i} \\
\psi_{v_s} \\
\psi_v
\end{array} \right) = \left( \begin{array}{c}
2 (r^{i(s)} (t_0^{i(s)}) - r_z) \\
2 (\phi^{i(s)} (t_0^{i(s)}) - \omega_m t_0^{i(s)}) \\
2 v^{i(s)} (t_0^{i(s)}) \\
2 v^{i(s)} (t_0^{i(s)}) - v_{e,z} \\
1
\end{array} \right)$$

Let us find $\frac{\partial J}{\partial \lambda_i}$.

Firstly, let us solve $\omega(s)$ (analytical derivatives of the functional with respect to $u^{i(s)}$):

$$\omega^i(s) = \psi^i s f^i(s), \quad i = 1, ..., K$$

where costate functions $\psi_i$ are the solution of the equations:

$$\frac{d \psi_i}{dt} = -(f^i)^j \psi_j, \quad i = 1, ..., K$$

So, $\omega(s)$ would be as follows:

$$\omega = -\psi_{v_i} \frac{a_s}{1-m} \sin \lambda_i + \psi_{v_s} \frac{a_v}{1-m} \cos \lambda_i$$

$$\omega_{\lambda_i} = \int_0^1 \omega(s) ds$$

Now let us find $\frac{\partial J}{\partial q} = \left( \frac{\partial J}{\partial a_0}; \frac{\partial J}{\partial c_0} \right)$.
\[
\frac{\partial f}{\partial q} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \cos \lambda_i \delta & 0 \\
0 & \frac{1-m}{\delta} & 0 & 0 \\
0 & 0 & -\frac{1-m}{\delta} & 0 \\
0 & c_0 & -\frac{a_0}{c_0} & c_0 \\
0 & \frac{\partial f}{\partial a_0} & \frac{\partial f}{\partial c_0}
\end{pmatrix}.
\]

(18)

\[
\frac{\partial J}{\partial q} = D = \sum_{i=1}^{n} \left( (\psi^{(i)})^T \cdot \frac{\partial f^{(i)}}{\partial q} \right) ds + \frac{\partial F}{\partial q} + d \frac{\partial \mu^{(i)}}{\partial q};
\]

(19)

\[
\frac{\partial J}{\partial q} = \int_{0}^{q} (\psi_{\lambda} \cdot \frac{\cos \lambda \cdot \delta}{1-m} + \psi_{\mu} \cdot \frac{\sin \lambda \cdot \delta}{1-m} + \psi_{c} \cdot \frac{\delta}{c_0}) ds;
\]

(20)

\[
\frac{\partial J}{\partial a_0} = \int_{0}^{q} (\psi_{\lambda} \cdot \frac{\cos \lambda \cdot \delta}{1-m} + \psi_{\mu} \cdot \frac{\sin \lambda \cdot \delta}{1-m} + \psi_{c} \cdot \frac{\delta}{c_0}) ds;
\]

(21)

\[
\frac{\partial J}{\partial c_0} = \int_{0}^{q} (-\psi_{\lambda} \cdot \frac{a_0 \cdot \delta}{c_0}) ds.
\]

(22)

Now let us find \( \frac{\partial J}{\partial T} = \left( \frac{\partial J}{\partial T_0} \frac{\partial J}{\partial T_i} \right) \cdot \left( \frac{\partial T_0}{\partial T} \frac{\partial T_i}{\partial T} \right) \).

As all the initial boundary conditions are constants, we have:

\[
\Pi^{(i)}(t_0^{(i)}) = 0,
\]

(23)

\[
\Pi^{(i)}(t_0^{(i)}) = (\psi^{(i)}(t_0^{(i)}))^T \cdot \frac{\partial f^{(i)}}{\partial p^{(i)}}(s) \cdot ds = (\psi^{(i)}(t_0^{(i)}))^T \cdot [f^{(i)}(t_0^{(i)}) - f^{(i)}(t_0^{(i)})];
\]

(24)

Thus, we have found the analytical expressions for all the functional derivatives (3), therefore, the problem is solved.

4. Results of the optimization

The example of the optimization process is shown in Figure 2. The red points show the terminal position of the spacecraft, the grey points show the terminal position of the Moon and the black line corresponds to the L2 orbit. As one can see from Figure 2, the optimization process makes the spacecraft reach the L2 libration point.
The optimal control program for the L1-L2 transfer and the corresponding trajectory have been obtained. In Figure 3 the red steps correspond to the active segments of the trajectory, and the green lines correspond to the passive segments, where the engine is turned off. In Figure 3 we can see, that the first passive segment of the trajectory is fully degenerated.

5. Conclusion

As we can see from the paper the usage of the Fedorenko method to estimate the derivatives and the gradient method to optimize the control laws allows determining the optimal control programs and the corresponding trajectories for the space missions by the example of the L1-L2 mission in the Earth-Moon system. The obtained results are in good agreement with the results obtained by the usage of Pontryagin's maximum principle in the three-body task framework [4]. The accuracy of the
calculations was lower than 4%. The applied methods demonstrate their effectiveness for the complex optimization of the SC transfers. The findings may be used to calculate the required design-ballistic parameters of the future space missions.

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