Toeplitz Determinants for a Class of Holomorphic Mappings in Higher Dimensions

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Abstract
In this paper, we establish the sharp bounds of certain Toeplitz determinants formed over the coefficients of holomorphic mappings from a class defined on the unit ball of a complex Banach space and on the unit polydisc in \( \mathbb{C}^n \). Derived bounds provide certain new results for the subclasses of normalized univalent functions and extend some known results in higher dimensions.

Keywords Quasi-convex mappings · Toeplitz determinants · Coefficient inequalities

Mathematics Subject Classification 32H02 · 30C45

1 Introduction
Let \( \mathcal{S} \) be the class of analytic univalent functions in the unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) having the form \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \) and \( \mathcal{K}(\alpha) \subset \mathcal{S} \) denote the class of convex functions of order \( \alpha \), \( 0 \leq \alpha < 1 \). A function \( g \in \mathcal{K}(\alpha) \) if and only if

\[
\text{Re} \left( 1 + \frac{zg''(z)}{g'(z)} \right) > \alpha, \quad z \in \mathbb{D}.
\]
For $\alpha = 0$, the class $K(\alpha)$ reduces to the class of convex functions $K := K(0)$. A function $g \in S$ is said to be starlike of order $\alpha$ if and only if

$$\text{Re}\left(\frac{zg'(z)}{g(z)}\right) > \alpha, \quad z \in \mathbb{U}.$$ 

The class of all starlike functions of order $\alpha$ is denoted by $S^*(\alpha)$ and let $S^* := S^*(0)$. Recently, Ali et al. [2] obtained the bounds of certain Toeplitz determinants whose entries are the Taylor series coefficients of functions in $S$ and some of its subclasses. For $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Toeplitz matrix is given by

$$T_{m,n}(g) = \begin{bmatrix} b_n & b_{n+1} & \cdots & b_{n+m-1} \\ b_{n+1} & b_n & \cdots & b_{n+m-2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n+m-1} & b_{n+m-2} & \cdots & b_n \end{bmatrix}.$$ 

In particular, the second order Toeplitz determinant is

$$\det T_{2,2}(g) = b_2^2 - b_3^2$$ 

(1)

and the third order Toeplitz determinant is given by

$$\det T_{3,1}(g) = \begin{vmatrix} 1 & b_2 & b_3 \\ b_2 & 1 & b_2 \\ b_3 & b_2 & 1 \end{vmatrix} = 2b_2^2b_3 - 2b_2^2 - b_3^2 + 1.$$ 

(2)

Toeplitz matrices and Toeplitz determinants have various applications in pure as well as in applied mathematics. They occur in a variety of fields including partial differential equations, image processing and differential geometry. For more details and applications, we refer [21].

Numerous articles have recently focused on finding sharp estimates for the Toeplitz and Hermitian–Toeplitz determinants for various classes, but in one dimensional complex plane. Ahuja et al. [1] established the sharp bounds of $|\det T_{2,2}(g)|$ and $|\det T_{3,1}(g)|$ for the class $K$ and its subclasses. The problem of estimating determinants of Hermitian Toeplitz matrices was initiated with the papers [4, 10, 12–14]. For the class $K$ and $K(\alpha)$, the following bounds are proved in [1].

**Theorem A** [1] If $f \in K$, then $|\det T_{2,2}(f)| \leq 2$. The bound is sharp.

**Theorem B** [1] If $f \in K$, then $|\det T_{3,1}(f)| \leq 4$. The bound is sharp.

**Theorem C** [1] If $f \in K(\alpha)$, then the following sharp inequality hold:

$$|\det T_{2,2}(f)| \leq \frac{2(1-\alpha)^2(2\alpha^2 - 6\alpha + 9)}{9}.$$
Theorem D [1] If \( f \in \mathcal{K}(\alpha) \) and \( \alpha \in [0, 1/2] \), then the following sharp inequality hold:

\[
|\det T_{3,1}(f)| \leq \frac{8\alpha^4 - 34\alpha^3 + 71\alpha^2 - 72\alpha + 36}{9}.
\]

In this paper, we generalize the above results in higher dimensions for a class of holomorphic mappings defined on the unit ball in complex Banach space and the unit polydisc in \( \mathbb{C}^n \). Let \( X \) be a complex Banach space with respect to norm \( \| \cdot \| \) and \( \mathbb{B} = \{ z \in X : \| z \| < 1 \} \) be the unit ball. When \( X = \mathbb{C} \), \( \mathbb{B} \) is denoted by \( \mathbb{U} \).

Let \( C^n \) denote the space of \( n \)-complex variables \( z = (z_1, z_2, \ldots, z_n)' \) and \( \mathbb{U}^n \) be the Euclidean unit ball in \( \mathbb{C}^n \). The boundary and distinguished boundary of \( \mathbb{U}^n \) are denoted by \( \partial \mathbb{U}^n \) and \( \partial_0 \mathbb{U}^n \), respectively.

Let \( L(X, Y) \) denote the set of all continuous linear operators from \( X \) into a complex Banach space \( Y \). For each \( z \in X \{ 0 \} \), let

\[
T_z = \{ l_z \in L(X, \mathbb{C}) : l_z(z) = \| z \|, \| l_z \| = 1 \}.
\]

By the Hahn-Banach theorem, this set is non-empty.

By \( \mathcal{H}(\Omega, \Omega') \), we denote the set of holomorphic mappings from a domain \( \Omega \subseteq X \) into a domain \( \Omega' \subseteq Y \) and let \( \mathcal{H}(\Omega) = \mathcal{H}(\Omega, X) \). If \( g \in \mathcal{H}(\mathbb{B}) \) and \( z \in \mathbb{B} \), then for each \( k = 1, 2, \ldots, \) there is a bounded symmetric \( k \)-linear mapping

\[
D^k g(z) : \prod_{j=1}^k X \to X,
\]

called the \( k^{th} \) order Fréchet derivative of \( g \) at \( z \) such that

\[
g(w) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k g(z)((w - z)^k)
\]

for all \( w \) in some neighborhood of \( z \).

A mapping \( g \in \mathcal{H}(\Omega) \) is said to be biholomorphic if \( g(\Omega) \) is a domain in \( X \) and the inverse \( g^{-1} \) exists and is holomorphic on \( g(\Omega) \). If the Fréchet derivative \( Dg(z) \) has a bounded inverse for each \( z \in \Omega \), then \( g \in \mathcal{H}(\Omega) \) is called locally biholomorphic mapping on \( \Omega \). Analog of the class \( S \), let \( S(\mathbb{B}) \) denote the class of biholomorphic mappings \( g \) from \( \mathbb{B} \) into \( X \), satisfying \( g(0) = 0 \) and \( Dg(0) = I \), where \( I \) represents the linear identity operator from \( X \) into \( X \). It is easily seen that \( S(\mathbb{B}) \) is not a normal family when the dimension is greater than one [3]. A mapping \( g \in S(\mathbb{B}) \) is said to be starlike if \( g(\mathbb{B}) \) is starlike with respect to the origin. Also, a mapping \( g \in S(\mathbb{B}) \) is said to be convex if \( g(\mathbb{B}) \) is convex.

On a bounded circular domain \( \Omega \subseteq \mathbb{C}^n \), the first and the \( m^{th} \) Fréchet derivative of a holomorphic mapping \( g : \Omega \to X \) are written by \( Dg(z) \) and \( D^m g(z)(a^{m-1}, \cdot) \).
respectively. The matrix representations are

\[
Dg(z) = \left( \frac{\partial g_j}{\partial z_k} \right)_{1 \leq j, k \leq n},
\]

\[
D^m g(z)(a^{m-1}, \cdot) = \left( \sum_{p_1, p_2, \ldots, p_{m-1}=1}^{n} \frac{\partial^m g_j(z)}{\partial z_k \partial z_{p_1} \ldots \partial z_{p_{m-1}}} a_{p_1} \ldots a_{p_{m-1}} \right)_{1 \leq j, k \leq n},
\]

where \( g(z) = (g_1(z), g_2(z), \ldots g_n(z))' \) and \( a = (a_1, a_2, \ldots a_n)' \in \mathbb{C}^n \).

Liu and Liu [15] defined the following class:

Definition 1 [15] Suppose \( \alpha \in [0, 1) \) and \( g : \mathbb{B} \to X \) is a normalized locally biholomorphic mapping. If

\[
\text{Re} \left\{ l_z [(Dg(z))^{-1}(D^2 g(z)(z^2) + Dg(z)(z))] \right\} \geq \alpha \|z\|, \quad l_z \in T_z, \quad z \in \mathbb{B} \setminus \{0\},
\]

then \( f \) is called a quasi convex mapping of type \( B \) and order \( \alpha \) on \( \mathbb{B} \).

If \( \mathbb{B} = \mathbb{U}^n \) and \( X = \mathbb{C}^n \), then the above condition reduces to

\[
\left| \frac{q_k(z)}{z_k} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}, \quad \forall z \in \mathbb{U}^n \setminus \{0\},
\]

where

\[
q(z) = (q_1(z), q_2(z), \ldots, q_n(z))' = (Dg(z))^{-1}(D^2 g(z)(z^2) + Dg(z)(z))
\]

is a column vector in \( \mathbb{C}^n \) and \( k \) satisfies

\[
|z_k| = \|z\| = \max_{1 \leq j \leq n} \{|z_j|\}.
\]

For \( \mathbb{B} = \mathbb{U} \) and \( X = \mathbb{C} \), the relation is equivalent to

\[
\text{Re} \left( 1 + \frac{zg''(z)}{g'(z)} \right) > \alpha, \quad z \in \mathbb{U}.
\]

Let \( \mathcal{K}_{\alpha}(\mathbb{B}) \) denote the class of quasi convex mappings of type \( B \) and order \( \alpha \).

When \( \alpha = 0 \), Definition 1 is the definition of quasi convex mapping of type \( B \) introduced by Roper and Suffridge [17].

Definition 2 Let \( \Phi : \mathbb{U} \to \mathbb{C} \) be a biholomorphic function such that \( \text{Re} \Phi(z) > 0 \), \( \Phi(0) = 1 \), \( \Phi'(0) > 0 \) and \( \Phi''(0) \in \mathbb{R} \). Let \( \mathcal{M}_\Phi \) be the class of mappings given by

\[
\mathcal{M}_\Phi = \left\{ p \in \mathcal{H}(\mathbb{B}) : p(0) = Dp(0) = I, \frac{l_z(p(z))}{\|z\|} \in \Phi(\mathbb{U}), \quad z \in \mathbb{B} \setminus \{0\}, l_z \in T_z \right\}.
\]
In case of $\mathbb{B} = \mathbb{U}^n$ and $X = \mathbb{C}^n$, we have

$$\mathcal{M}_\Phi = \left\{ p \in \mathcal{H}(\mathbb{B}) : p(0) = Dp(0) = I, \frac{p_j(z)}{z_j} \in \Phi(\mathbb{U}), z \in \mathbb{U}^n \setminus \{0\} \right\},$$

where $p(z) = (p_1(z), p_2(z), \ldots, p_n(z))^t$ is a column vector in $\mathbb{C}^n$ and $j$ satisfies $|z_j| = \|z\| = \max_{1 \leq k \leq n} \{|z_k|\}$.

In 1999, Roper and Suffridge [17] gave a sufficient condition for a normalized biholomorphic convex mapping on the Euclidean unit ball in $\mathbb{C}^n$. Later, Zhu [22] provided a brief proof of this theorem. Xu et al. [19] obtained the sharp bounds of Fekete-Szegö inequality for the class of quasi-convex mappings of type $B$ and order $\alpha$ defined on the unit ball $\mathbb{B}$ and on the unit polydisc in $\mathbb{C}^n$. Liu and Liu [16] derived the sharp estimates of all homogenous expansions for a subclass of holomorphic mappings of quasi-convex mappings of type $B$ and order $\alpha$ in higher dimensions. Contrary to the coefficient inequalities for many subclasses of $S$, only few are known for homogeneous expansions for subclasses of biholomorphic mappings in the case of several complex variables [5, 6, 8, 9, 11, 20].

In case of one complex variable, many coefficient problems are studied for the class $K$ such as Theorems A-D. A natural question arises that how to retain these results in higher dimensions. Providing an answer to this question is the aim of this study.

2 Main Results

The main results of the paper are stated and proved in this section.

**Theorem 2.1** Let $g \in \mathcal{H}(\mathbb{B}, \mathbb{C})$ with $g(0) = 1$, $g(z) \neq 0$, $z \in \mathbb{B}$ and suppose that $G(z) = zg(z)$. If $(DG(z))^{-1} \left(D^2 G(z)(z^2) + DG(z)(z)\right) \in \mathcal{M}_\Phi$ such that $\Phi$ satisfies

$$|\Phi''(0) + 2(\Phi'(0))^2| \geq 2\Phi'(0) > 0,$$

then

$$\left| \left( \frac{l_z(D^2 G(0)(z^2))}{2||z||^2} \right)^2 - \left( \frac{l_z(D^3 G(0)(z^3))}{3||z||^3} \right)^2 \right| \leq \left( \frac{\Phi'(0))^2}{4} + \frac{(\Phi'(0))^2}{36} \left( \frac{1}{2} \Phi''(0) + \Phi'(0) \right)^2 \right.$$

The bound is sharp.

**Proof** Fix $z \in X \setminus \{0\}$ and let $h : \mathbb{U} \rightarrow \mathbb{C}$ be defined by

$$h(\xi) = \begin{cases} \frac{l_z((DG(\xi z_0))^{-1}(D^2 G(\xi z_0)((\xi z_0)^2) + DG(\xi z_0)\xi z_0))}{\xi}, & \xi \neq 0, \\ 1, & \xi = 0, \end{cases}$$
where \( z_0 = \frac{\dot{z}}{\|z\|} \). Then \( h \in \mathcal{H}(U) \) and \( h(0) = \Phi(0) = 1 \). Since 
\[
(DG(z))^{-1}(D^2 G(z)(z^2) + DG(z)(z)) \in \mathcal{M}_{\Phi},
\]
therefore, we have 
\[
h(\xi) = \frac{l_{z_0}((DG(\xi z_0))^{-1}(D^2 G(\xi z_0)((\xi z_0)^2) + DG(\xi z_0)\xi z_0))}{\xi} = \frac{l_{z_0}((DG(\xi z_0))^{-1}(D^2 G(\xi z_0)((\xi z_0)^2) + DG(\xi z_0)\xi z_0))}{\|\xi z_0\|} = \frac{l_{z_0}((DG(\xi z_0))^{-1}(D^2 G(\xi z_0)((\xi z_0)^2) + DG(\xi z_0)\xi z_0))}{\|\xi z_0\|} \in \Phi(U), \quad \xi \in U.
\]
Applying a similar method used in [7, Theorem 7.1.14], we get 
\[
(DG(z))^{-1} = \frac{1}{g(z)} \left( I - \frac{zDg(z)}{g(z)} \right).
\]
A simple computation using the fact \( G(z) = zg(z) \) yields 
\[
D^2 G(z)(z^2) + DG(z)(z) = (D^2 g(z)(z^2) + 3Dg(z)(z) + g(z))z.
\]
By using (3) and (4), it follows 
\[
(DG(z))^{-1}(D^2 G(z)(z^2) + DG(z)(z)) = \frac{D^2 g(z)(z^2) + 3Dg(z)(z) + g(z)}{g(z) + Dg(z)(z)}z. \tag{5}
\]
Consequently 
\[
l_{z}((DG(z))^{-1}(D^2 G(z)(z^2) + DG(z)(z))) = \frac{D^2 g(z)(z^2) + 3Dg(z)(z) + g(z)}{g(z) + Dg(z)(z)} \|z\|. \tag{6}
\]
Using (6), we obtain 
\[
h(\xi) = \frac{l_{z_0}((DG(\xi z_0))^{-1}(D^2 G(\xi z_0)((\xi z_0)^2) + DG(\xi z_0)\xi z_0))}{\|\xi z_0\|} = \frac{D^2 g(\xi z_0)((\xi z_0)^2) + 3Dg(\xi z_0)(\xi z_0) + g(\xi z_0)}{g(\xi z_0) + Dg(\xi z_0)(\xi z_0)}.
\]
Equivalently, we can write 
\[
h(\xi)(g(\xi z_0) + Dg(\xi z_0)(\xi z_0)) = D^2 g(\xi z_0)((\xi z_0)^2) + 3Dg(\xi z_0)(\xi z_0) + g(\xi z_0).
\]
The series expansion in terms of \( \xi \) gives 
\[
\left( 1 + h'(0)\xi + \frac{h''(0)}{2}\xi^2 + \cdots \right) \left( 1 + 2Dg(0)(z_0)\xi + \frac{3Dg(0)(z_0)^2}{2}\xi^2 + \cdots \right)
\]
\[ = 1 + 4Dg(0)(z_0)z + \frac{9Dg(0)(z_0^2)}{2}z^2 + \cdots. \]

Comparison of the homogenous expansions of either sides of the above equality provide \( h'(0) = 2Dg(0)(z_0). \) That is

\[ h'(0)\|z\| = 2Dg(0)(z). \] (7)

Also, we have

\[ \frac{D^2G(0)(z^2)}{2!} = Dg(0)(z)z, \]

which gives

\[ \frac{l_z(D^2G(0)(z^2))}{2!} = Dg(0)(z]\|z\|. \]

Now, using \(|h'(0)| \leq \Phi'(0)\) with (7), we obtain

\[ \left| \frac{l_z(D^2G(0)(z^2))}{2!}\|z\|^2 \right| \leq \frac{\Phi'(0)}{2}. \] (8)

Moreover, for \( \lambda \in \mathbb{C}, \) Xu et al. [19, Theorem 3.1] proved that

\[ \left| \frac{l_z(D^3G(0)(z^3))}{3!}\|z\|^3 - \lambda \left( \frac{l_z(D^2G(0)(z^2))}{2!}\|z\|^2 \right)^2 \right| \leq \frac{|\Phi'(0)|}{6} \max \left\{ 1, \left| \frac{1}{2} \Phi''(0) + \left( 1 - \frac{3}{2}\lambda \right) \Phi'(0) \right| \right\}, \quad z \in \mathbb{B} \setminus \{0\}. \]

(9)

Since \(|\Phi''(0) + 2(\Phi'(0))^2| \geq 2\Phi'(0)\), therefore the above inequality gives

\[ \left| \frac{l_z(D^3G(0)(z^3))}{3!}\|z\|^3 \right| \leq \frac{\Phi'(0)}{6} \left( \frac{1}{2} \Phi''(0) + \Phi'(0) \right). \] (10)

Also, note that

\[ \left| \left( \frac{l_z(D^3G(0)(z^3))}{3!}\|z\|^3 \right)^2 - \left( \frac{l_z(D^2G(0)(z^2))}{2!}\|z\|^2 \right)^2 \right| \leq \left| \frac{l_z(D^3G(0)(z^3))}{3!}\|z\|^3 \right|^2 + \left| \frac{l_z(D^2G(0)(z^2))}{2!}\|z\|^2 \right|^2. \]

The required bound follows from the above inequality together with the bounds given in (8) and (10).
The result is sharp for the mapping $G$ given by

$$DG(z) = I \exp \int_0^{T_u(z)} \frac{\Phi(it) - 1}{t} dt, \quad z \in B, \quad \|u\| = 1.$$  \quad (11)

Clearly, $(DG(z))^{-1}(D^2 G(z)(z^2) + DG(z)(z)) \in \mathcal{M}_\Phi$ and

$$D^3 G(0)(z^3) = \frac{1}{6} \left( \frac{\Phi''(0)}{2} + (\Phi'(0))^2 \right) (l_u(z))^2 z$$

which immediately gives

$$\frac{l_z(D^2 G(0)(z^2))}{2!} = \frac{i \Phi'(0)}{2} l_u(z) \|z\|$$

and

$$\frac{l_z(D^3 G(0)(z^3))}{3! \|z\|^3} = -\frac{1}{6} \left( \frac{\Phi''(0)}{2} + (\Phi'(0))^2 \right) (l_u(z))^2 \|z\|^2.$$  \quad (12)

Taking $z = ru \ (0 < r < 1)$, we get

$$\frac{l_z(D^3 G(0)(z^3))}{3! \|z\|^3} = -\frac{1}{6} \left( \frac{\Phi''(0)}{2} + (\Phi'(0))^2 \right) \text{ and } \frac{l_z(D^2 G(0)(z^2))}{2! \|z\|^2} = \frac{i \Phi'(0)}{2}.$$  \quad (12)

According to the above equations, we have

$$\left( \frac{l_z(D^3 G(0)(z^3))}{3! \|z\|^3} \right)^2 - \left( \frac{l_z(D^2 G(0)(z^2))}{2! \|z\|^2} \right)^2 = \frac{(\Phi'(0))^2}{4} + \frac{1}{36} \left( \frac{\Phi''(0)}{2} + (\Phi'(0))^2 \right)^2,$$

which establishes the sharpness of the bound and completes the proof. \hfill \Box

**Theorem 2.2** Let $g \in \mathcal{H}(B, \mathbb{C})$ with $g(0) = 1$, $g(z) \neq 0$, $z \in B$ and suppose that $G(z) = zg(z)$. If $(DG(z))^{-1}(D^2 G(z)(z^2) + DG(z)(z)) \in \mathcal{M}_\Phi$ such that $\Phi$ satisfy

$$2\Phi'(0) - 2(\Phi'(0))^2 \leq \Phi''(0) \leq 4(\Phi'(0))^2 - 2\Phi'(0),$$

then

$$|2a_2^2a_3 - a_2^2 - 2a_2^2 + 1| \leq 1 + 2(\Phi'(0))^2 + \frac{(\Phi'(0))^2}{4} \left( \frac{\Phi''(0)}{2\Phi'(0)} - 3\Phi'(0) \right) \left( \frac{\Phi''(0)}{2\Phi'(0)} + \Phi'(0) \right),$$

where

$$a_3 = \frac{l_z(D^3 G(0)(z^3))}{3! \|z\|^3} \text{ and } a_2 = \frac{l_z(D^2 G(0)(z^2))}{2! \|z\|^2}.$$
The bound is sharp.

Proof Since $2\Phi'(0) \leq 4(\Phi'(0))^2 - \Phi''(0)$, inequality (9) gives

$$\left| \frac{l_z(D^3 G(0)(z^3))}{3!||z||^3} - 2 \left( \frac{l_z(D^2 G(0)(z^2))}{2!||z||^2} \right)^2 \right| \leq \frac{\Phi'(0)}{6} \left( 2\Phi'(0) - \frac{1}{2} \frac{\Phi''(0)}{\Phi'(0)} \right)$$

for $z \in \mathbb{B} \setminus \{0\}$. Also, since $\Phi$ satisfy $\Phi''(0) + 2(\Phi'(0))^2 \geq 2\Phi'(0)$, therefore by (9), we have

$$\left| \frac{l_z(D^3 G(0)(z^3))}{3!||z||^3} \right| \leq \frac{\Phi'(0)}{6} \left( \frac{1}{2} \frac{\Phi''(0)}{\Phi'(0)} + \Phi'(0) \right), \quad z \in \mathbb{B} \setminus \{0\}. \quad (14)$$

Also, we have

$$|2a_2^2a_3 - 2a_2^2 - a_3^2 + 1| \leq 1 + 2|a_2|^2 + |a_3||a_3 - 2a_2^2|.$$  

The required bound follows directly from the above inequality along with the bounds given in (8) and (14), and the bound of $|a_3 - 2a_2^2|$ given by (13).

Equality case holds for the mapping $G(z)$ defined in (11) as for this mapping, we have $a_2 = i\frac{\Phi'(0)}{2}$, $a_3 = -\frac{1}{6} \left( \frac{\Phi''(0)}{2} + (\Phi'(0))^2 \right)$ and hence

$$2a_2^2a_3 - 2a_2^2 - a_3^2 + 1 = 2(\Phi'(0))^2 + \frac{(\Phi'(0))^2}{12} \left( \frac{\Phi''(0)}{2\Phi'(0)} - 3\Phi'(0) \right) \left( \frac{\Phi''(0)}{2\Phi'(0)} + \Phi'(0) \right) + 1,$$

which establish the sharpness of the result. $\square$

Theorem 2.3 Let $g \in \mathcal{H}(\mathbb{U}^n, \mathbb{C})$ with $g(0) = 1$, $g(z) \neq 0$, $z \in \mathbb{U}^n$ and suppose that $G(z) = zg(z)$. If $(DG(z))^{-1}(D^2 G(z)(z^2) + DG(z)(z) \in \mathcal{M}_\Phi$ such that $\Phi$ satisfies

$$|\Phi''(0) + 2(\Phi'(0))^2| \geq 2\Phi'(0),$$

then

$$\left\| \frac{(D^3 G(0)(z^3))}{3!} \right\|^2 - \left\| \frac{(D^2 G(0)(z^2))}{2!} \right\|^2 \leq \frac{\Phi'(0)^2}{36} \left( \frac{\Phi''(0)}{2\Phi'(0)} + \Phi'(0) \right)^2 \frac{\|z\|^4}{4}, \quad z \in \mathbb{U}^n. \quad (15)$$

The bound is sharp.
Proof For $z \in \mathbb{U}^n \setminus \{0\}$ and $z_0 = \frac{z}{\|z\|}$, define $h_k : \mathbb{U} \to \mathbb{C}$ such that

$$h_k(\xi) = \begin{cases} \frac{\xi z_k}{p_k(\xi z_0)\|z_0\|}, & \xi \neq 0, \\ 1, & \xi = 0, \end{cases}$$

where $p(z) = (DG(z))^{-1}(D^2 G(z)(z^2) + DG(z)(z))$ and $k$ satisfies $|z_k| = \|z\| = \max_{1 \leq j \leq n} |z_j|$. By (5), we have

$$h_k(\xi) = \frac{D^2 g(\xi z_0)((\xi z_0)^2) + 3Dg(\xi z_0)(\xi z_0) + g(\xi z_0)}{g(\xi z_0) + Dg(\xi z_0)(\xi z_0)},$$

or, equivalently

$$h_k(\xi)(g(\xi z_0) + Dg(\xi z_0)(\xi z_0)) = D^2 g(\xi z_0)((\xi z_0)^2) + 3Dg(\xi z_0)(\xi z_0) + g(\xi z_0).$$

Comparison of same homogeneous expansions in the Taylor series expansions in terms of $\xi$ yield

$$h_k'(0) = 2Dg(0)(z_0).$$

Furthermore, from $G(z_0) = z_0g(z_0)$, we have

$$\frac{D^2 G_k(0)(z_0^2)}{2!} = Dg(0)(z_0)\frac{z_j}{\|z\|}.$$  

Combining (17) and (18) with the fact $|h_k'(0)| \leq \Phi'(0)$ gives

$$\left| \frac{D^2 G_k(0)(z_0^2)}{2!} \frac{\|z\|}{z_j} \right| \leq \frac{\Phi'(0)}{2}.$$  

If $z_0 \in \partial \mathbb{U}^n$, then we get

$$\left| \frac{D^2 G_k(0)(z_0^2)}{2!} \right| \leq \frac{\Phi'(0)}{2}.$$  

Since

$$\frac{D^2 G_k(0)(z_0^2)}{2!}, \ k = 1, 2, \ldots n$$

are holomorphic functions on $\overline{\mathbb{U}}^n$, therefore by the maximum modulus theorem of holomorphic functions on the unit polydisc, we have

$$\left| \frac{D^2 G_k(0)(z_0^2)}{2!} \right| \leq \frac{\Phi'(0)}{2}, \ z_0 \in \mathbb{U}^n, k = 1, 2, \ldots n.$$
That is
\[
\left| \frac{D^2 G_k(0)(z^2)}{2!} \right| \leq \frac{\Phi'(0)\|z\|^2}{2}, \quad z \in \partial \mathbb{U}^n, \ k = 1, 2, \ldots, n.
\] (19)

For \( \lambda \in \mathbb{C}, \) Xu et al. [19, Theorem 3.2] established that
\[
\left| \frac{D^3 G_k(0)(z^3)}{3!} - \frac{1}{2} \lambda D^2 G_k(0)(z, \frac{D^2 G(0)(z^2)}{2!}) \right| \leq \frac{|\Phi'(0)|\|z\|^3}{6} \max \left\{ 1, \left| \frac{1}{2} \Phi''(0) + \left(1 - \frac{3}{2}\lambda\right) \Phi'(0) \right| \right\}.
\] (20)

Since \(|\Phi''(0) + 2(\Phi'(0))^2| \geq 2\Phi'(0)\), therefore, from (20), we get
\[
\left| \frac{D^3 G_k(0)(z^3)}{3!} \right| \leq \frac{\Phi'(0)\|z\|^3}{6} \left( \frac{1}{2} \Phi''(0) + \Phi'(0) \right)
\] (21)

for \( z \in \mathbb{U}^n \) and \( k = 1, 2, \ldots, n \). Using the bounds from (19) and (21), we have
\[
\left| \left( \frac{D^3 G(0)(z^3)}{3!} \right)^2 - \left( \frac{D^2 G(0)(z^2)}{2!} \right)^2 \right| \leq \frac{(\Phi'(0))^2\|z\|^6}{36} \left( \frac{1}{2} \Phi''(0) + \Phi'(0) \right)^2 + \frac{(\Phi'(0))^2\|z\|^4}{4}, \quad z \in \mathbb{U}^n,
\]

which is the required bound.

To prove the sharpness of the bound, consider the mapping \( G \) given by
\[
DG(z) = I \exp \int_0^{z_1} \frac{\Phi(it) - 1}{t} \, dt.
\] (22)

It can be showed that \((DG(z))^{-1}(D^2 G(z)(z^2) + DG(z)(z)) \in \mathcal{M}_\Phi\) and for \( z = (r, 0, \ldots, 0)' \) in (22), the equality case holds in (15). \( \square \)

**Theorem 2.4** Let \( g \in \mathcal{H}(\mathbb{U}^n, \mathbb{C}), \) \( g(0) = 1, \) \( g(z) \neq 0, \) \( z \in \mathbb{U}^n \) and suppose that \( G(z) = zg(z). \) If \((DG(z))^{-1}(D^2 G(z)(z^2) + DG(z)(z)) \in \mathcal{M}_\Phi\) and \( \Phi \) satisfy
\[
2\Phi'(0) - 2(\Phi'(0))^2 \leq \Phi''(0) \leq 4(\Phi'(0))^2 - 2\Phi'(0),
\]
\[ \|2a_2^2a_3 - a_3^2 - 2a_2^3 + 1\| \leq 1 + \frac{(\Phi''(0))^2\|z\|^6}{36} \left( 2\Phi'(0) - \frac{1}{2} \Phi''(0) \right) \left( \frac{1}{2} \Phi'(0) + \Phi'(0) \right) \] 

where

\[ a_3 = \frac{D^3G(0)(z^3)}{3!} \quad \text{and} \quad a_2^2 = \frac{1}{2} D^2G(0) \left( z, \frac{D^2G(0)(z^2)}{2!} \right). \]

The bound is sharp.

**Proof** Since \( G(z) = zg(z) \), we have

\[ \frac{1}{2} D^2G_k(0) \left( z_0, \frac{D^2G(0)(z_0^2)}{2!} \right) \frac{z_k}{\|z\|} = \left( \frac{D^2G_k(0)(z_0^2)}{2!} \right)^2, \quad k = 1, 2, \ldots, n, \]

where \( z_0 = \frac{z_k}{\|z\|} \) and \( k \) satisfies \( |z_k| = \|z\| = \max_{1 \leq j \leq n} |z_j| \) (see [18]). If \( z_0 \in \partial \mathbb{U}^n \), then

\[ \left| \frac{1}{2} D^2G_k(0) \left( z_0, \frac{D^2G(0)(z_0^2)}{2!} \right) \right| \leq \left( \Phi'(0) \right)^2 \|z\|^4. \]

Considering the function \( h_k(\zeta) \) given in (16) and following the same methodology as in the proof of Theorem 2.3, we obtain (19), which together with the above relation yields

\[ \left| \frac{1}{2} D^2G_k(0) \left( z_0, \frac{D^2G(0)(z_0^2)}{2!} \right) \right| \leq \frac{(\Phi'(0))^2 \|z\|^4}{4}. \]

Also, since \( \Phi \) satisfy \( 2\Phi'(0) \leq 4(\Phi'(0))^2 - \Phi''(0) \), therefore from (20), we obtain

\[ \left| \frac{D^3G_k(0)(z^3)}{3!} - D^2G_k(0) \left( z, \frac{D^2G(0)(z^2)}{2!} \right) \right| \leq \frac{|\Phi'(0)| \|z\|^3}{6} \left( 2\Phi'(0) - \frac{1}{2} \Phi''(0) \right), \quad k = 1, 2, \ldots, n. \]

Thus, from (21), (23) and (24), we have

\[ \left| 1 + D^2G_k(0) \left( z, \frac{D^2G(0)(z^2)}{2!} \right) \left( \frac{D^3G_k(0)(z^3)}{3!} \right) - D^2G_k(0) \left( z, \frac{D^2G(0)(z^2)}{2!} \right) \right| \]

\[ - \left( \frac{D^3G_k(0)(z^3)}{3!} \right)^2 \leq 1 + \left| \frac{D^3G_k(0)(z^3)}{3!} - \frac{D^3G_k(0)(z^3)}{3!} \right| \leq 1 + \left| \frac{D^3G_k(0)(z^3)}{3!} \right| \]
In case of Remark 1 which is the required bound.

Let $g \in H_1$ give different subclasses of holomorphic mappings. For instance, various choices of $g$ yield the following results.

For $z \in U^n$ and $k = 1, 2, \ldots n$. Therefore

\[
\| 1 + D^2 G(0) \left( z, \frac{D^2 G(0)(z^2)}{2!} \right) \| \leq 1 + \frac{(\Phi'(0))^2 \|z\|^6}{36} \left( 2 \Phi'(0) - \frac{1}{2} \Phi''(0) \right) \left( \frac{1}{2} \Phi'(0) + \Phi'(0) \right) + \frac{(\Phi'(0))^2 \|z\|^4}{2},
\]

which is the required bound.

Sharpness of the bound can be seen from the mapping $G(z)$ defined in (22) by taking $z = (r, 0, \ldots, 0)$, which completes the proof. \hfill \Box

Note that if $g \in H(B)$ and $(Dg(z))^{-1}(D^2 g(z)(z^2) + Dg(z)(z)) \in M_\Phi$, then various choices of $\Phi$ give different subclasses of holomorphic mappings. For instance, when $\Phi(z) = (1 + (1 - 2\alpha)z)/(1 - z)$ and $\Phi(z) = (1 + z)/(1 - z)$, we easily obtain $g \in K_{\alpha}(B)$ and $g \in K(B)$, respectively. For these classes, Theorem 2.1 to Theorem 2.4 yield the following results.

**Corollary 2.1** Let $g \in H(B, C)$ and $G(z) = zg(z) \in C_{\alpha}(B)$. Then the following inequality holds:

\[
\left( \frac{l_z(D^2 G(0)(z^2))}{2! \|z\|^2} \right)^2 - \left( \frac{l_z(D^3 G(0)(z^3))}{3! \|z\|^3} \right)^2 \leq \frac{2(1 - \alpha)^2(2\alpha^2 - 6\alpha + 9)}{9}, \quad l_z \in T_z, \ z \in B \setminus \{0\}.
\]

If $B = U^n$ and $X = C^n$, then

\[
\left\{ \left( \frac{D^3 G(0)(z^3)}{3!} \right)^2 - \left( \frac{D^2 G(0)(z^2)}{2!} \right)^2 \right\} \leq (1 - \alpha)^2 \|z\|^4 + \frac{(2\alpha^2 - 5\alpha + 3)^2 \|z\|^6}{9}, \quad z \in U^n.
\]

All these bounds are sharp.

**Remark 1** In case of $n = 1$, $B = U$ and (25) reduces to the following:

\[
\left( \frac{G(3)(0)}{3!} \right)^2 - \left( \frac{G''(0)}{2!} \right)^2 \leq \frac{2(1 - \alpha)^2(2\alpha^2 - 6\alpha + 9)}{9},
\]
which is equivalent to Theorem C.

**Corollary 2.2** Let \( g \in \mathcal{H}(\mathbb{B}, \mathbb{C}) \) and \( G(z) = zg(z) \in \mathcal{C}_\alpha(\mathbb{B}) \). Then for \( \alpha \in [0, 1/2] \), the following sharp bound holds:

\[
|2a_2^2a_3 - a_2^2 - 2a_2^2 + 1| \leq \frac{8\alpha^4 - 34\alpha^3 + 71\alpha^2 - 72\alpha + 36}{9}, \quad z \in \mathbb{B} \setminus \{0\},
\]

where

\[
a_3 = \frac{l_z(D^3G(0)(z^3))}{3!||z||^3} \quad \text{and} \quad a_2 = \frac{l_z(D^2G(0)(z^2))}{2!||z||^2}.
\]

**Corollary 2.3** Let \( g \in \mathcal{H}(\mathbb{U}^n, \mathbb{C}) \) and \( G(z) = zg(z) \in \mathcal{C}_\alpha(\mathbb{U}^n) \). Then for \( \alpha \in [0, 1/2] \), the following sharp inequality holds:

\[
\|2a_2^2a_3 - a_2^2 - 2a_2^2 + 1\| \leq 1 + 2\|z\|^4(1 - \alpha)^2 + \frac{(1 - \alpha)^2(9 - 18\alpha + 8\alpha^2)||z||^6}{9},
\]

for \( z \in \mathbb{U}^n \), where

\[
a_3 = \frac{D^3G(0)(z^3)}{3!} \quad \text{and} \quad a_2 = \frac{1}{2} D^2G(0)\left(z, \frac{D^2G(0)(z^2)}{2!}\right).
\]

**Remark 2** When \( n = 1 \), Corollary 2.3 is equivalent to Theorem D.

In particular, for \( \alpha = 0 \), we obtain the following results for the class \( \mathcal{C} \) in higher dimensions.

**Corollary 2.4** Let \( g \in \mathcal{H}(\mathbb{B}, \mathbb{C}) \) and \( G(z) = zg(z) \in \mathcal{C}(\mathbb{B}) \). Then the following holds:

\[
\left| \left( \frac{l_z(D^2G(0)(z^2))}{2!||z||^2} \right)^2 - \left( \frac{l_z(D^3G(0)(z^3))}{3!||z||^3} \right)^2 \right| \leq 2, \quad l_z \in T_z, \quad z \in \mathbb{B} \setminus \{0\}.
\]

If \( \mathbb{B} = \mathbb{U}^n \) and \( X = \mathbb{C}^n \), then

\[
\left\| \left( \frac{D^3G(0)(z^3)}{3!} \right)^2 - \left( \frac{D^2G(0)(z^2)}{2!} \right)^2 \right\| \leq ||z||^4 + ||z||^6, \quad z \in \mathbb{U}^n.
\]

All these bounds are sharp.

**Remark 3** For \( n = 1 \), (27) is equivalent to Theorem A.

**Corollary 2.5** Let \( g \in \mathcal{H}(\mathbb{B}, \mathbb{C}) \) and \( G(z) = zg(z) \in \mathcal{C}(\mathbb{B}) \). Then the following sharp bound holds:

\[
|2a_2^2a_3 - a_2^2 - 2a_2^2 + 1| \leq 4, \quad z \in \mathbb{B} \setminus \{0\},
\]
where

\[ a_3 = \frac{l_z(D^3G(0)(z^3))}{3!||z||^3} \quad \text{and} \quad a_2 = \frac{l_z(D^2G(0)(z^2))}{2!||z||^2}. \]

**Corollary 2.6** Let \( g \in \mathcal{H}(\mathbb{U}^n, \mathbb{C}) \) and \( G(z) = zg(z) \in \mathcal{C}(\mathbb{U}^n) \). Then for \( \alpha \in [0, 1/2] \), the following sharp estimation holds:

\[ |2a_2a_3 - a_3^2 - 2a_2^2 + 1| \leq 1 + 2||z||^4 + ||z||^6 \]

for \( z \in \mathbb{U}^n \), where

\[ a_3 = \frac{D^3G(0)(z^3)}{3!} \quad \text{and} \quad a_2^2 = \frac{1}{2} D^2G(0) \left( z, \frac{D^2G(0)(z^2)}{2!} \right). \]

**Remark 4** when \( n = 1 \), Corollary 2.6 is equivalent to Theorem B.

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**Declarations**

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