Signatures of modified dispersion relation of graviton in the cosmic microwave background

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Abstract. The dispersion relation of graviton is a fundamental issue for fundamental physics about gravity. In this paper we investigate how the modified dispersion relation of graviton affects the cosmic microwave background (CMB) power spectra, in particular the B-mode polarization. Our results will be useful to test the dispersion relation of graviton at the energy scale around $10^{-29}$ eV.

Keywords: CMBR theory, gravitational waves and CMBR polarization, modified gravity

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1 Introduction

General Relativity is widely accepted as a fundamental theory to govern the dynamics of space-time itself. It governs the behaviors of black holes, the generation and propagation of gravitational waves, the formation of all structures and so on in the Universe. However, the rapid developments of observations, in particular the evidence for the dark energy and dark matter, inspire some alternative theories to general relativity at some scales. Various attempts to modify gravity have been present in literatures. For example, see [1–4] for some recent reviews on modified gravity theories. Testing modified gravity theories in solar system, binary pulsar systems and cosmological scales has become one of the core tasks. In this paper we focus on the effects of modified gravity on the cosmic scales.

The cosmic microwave background (CMB) is a probe to physics in the early universe and the cosmological evolution that followed. Thomson scattering in the presence of primordial fluctuations affects not only the temperature but also its polarization. The CMB polarization which can be decomposed into E-mode and B-mode can be affected by both scalar and tensor perturbations. In particular, the B-mode component mainly comes from the tensor perturbation on very large scales and encodes the information of the gravitational waves [5–9]. High multipoles of CMB B-modes (e.g. $\ell \gtrsim 100$) are significantly affected by CMB lensing. The detection of a B-mode signal would provide valuable information and would help us to test modified gravity theories [10, 11].

In modified gravity theory, the evolution equation of gravitational waves is presented in a nonstandard way [12–20]. In general, the evolution of the gravitational wave amplitude can be changed by three ways: (1) the damping rate of gravitational waves; (2) the dispersion relation; (3) an additional source term on the perfect fluid. Any modification of the tensor wave equation can potentially lead to observable effects on the CMB on both temperature and polarization spectrum. In this paper, we explore the imprint of the modified dispersion relation of graviton in the CMB angular power spectra. Phenomenologically we suppose that the dispersion relation of graviton is parametrized by

$$E^2 = p^2 + \sum_n \lambda_n p^n M^{n-2},$$

where $p$ is the physical momentum of graviton and $M$ is an energy scale. For $n = 0$, it implies that the graviton has a mass $\sqrt{\lambda_0} M$. Since the CMB itself reflects the physics at very large scale, the terms with $n > 0$ which encode the modification at UV scales should not significantly affect the CMB power spectra, and thus we focus on the cases with $n < 0$ which may
imply the non-locality of gravity. Note that \( n = 0 \) has been investigated in \cite{12}. Here we suppose that the evolution of the background cosmology is still described by the usual Friedmann equation and explore how the evolution of gravitational waves corresponding to the above dispersion relation affect the CMB power spectra. For \( n < 0 \), the modes with small momenta are significantly changed by the modified dispersion relation and acquire time-dependent oscillations which enhance the B-mode spectrum at large scales (or low multipoles).

This paper is organized as follows. In section 2, we provide the detail to illustrate the effects of the modified dispersion relation of graviton on CMB power spectra by semi-analytical and full numerical methods by modifying the publicly available codes CAMB \cite{21, 22}. A short summary and discussion will be given in section 3.

2 The tensor contribution to the cosmic microwave background spectra

In the synchronous gauge, the metric with perturbations is given by

\[
\frac{ds^2}{a^2(\tau)} = -d\tau^2 + (\delta_{ij} + h_{ij})dx^idx^j, \tag{2.1}
\]

where the components \( g_{00} \) and \( g_{0i} \) are unperturbed. The temperature and polarization perturbations generated by gravitational waves satisfy the following Boltzmann equations, \cite{23–28},

\[
\dot{\Delta}_T^{(T)} + ik\mu \Delta_T^{(T)} = -\dot{h} - \dot{\kappa}[\Delta_T^{(T)} - \Psi], \tag{2.2}
\]

\[
\dot{\Delta}_P^{(T)} + ik\mu \Delta_P^{(T)} = -\dot{\kappa}[\Delta_P^{(T)} + \Psi], \tag{2.3}
\]

\[
\Psi = \left[ \frac{1}{10} \Delta_T^{(T)} + \frac{1}{7} \Delta_T^{(P)} + \frac{3}{70} \Delta_T^{(4)} - \frac{3}{5} \Delta_P^{(T)} + \frac{6}{7} \Delta_P^{(2)} - \frac{3}{70} \Delta_P^{(4)} \right], \tag{2.4}
\]

where the dots denote the derivative with respect to the conformal time \( \tau \equiv \int dt/a(t) \), \( \mu = \hat{n} \cdot \hat{k} \) is the angle between the photon direction and wave vector, and the superscript \( (T) \) denotes the contributions from the tensor perturbations. Here \( \dot{\kappa} = an_e x_e \sigma_T \) is the differential optical depth for Thomson scattering, where \( a(\tau) \) is the expansion factor normalized to unity today, \( n_e \) is the electron density, \( x_e \) is the ionization fraction and \( \sigma_T \) is the Thomson cross section. Temperature anisotropies have additional source in the metric perturbation \( h \). The only external source is that of the tensor metric perturbation which evolves according to the Einstein equations.

These Boltzmann equations can be integrated along the line of sight to give, \cite{25},

\[
\Delta_{(T,P)}^{(T)} = \int_0^{\tau_0} d\tau S_{T,P}(k,\tau) \chi_k^{(T)}(\tau_0 - \tau), \tag{2.5}
\]

\[
S_T^{(T)}(k,\tau) = -\dot{h} e^{-\kappa} + g\Psi, \tag{2.6}
\]

\[
S_P^{(T)}(k,\tau) = -g\Psi, \tag{2.7}
\]

where \( g(\tau) = \dot{\kappa} \exp(-\kappa) \) is the visibility function, \( \chi_k^{(T)}(\tau) = \sqrt{\frac{(\ell+2)!}{2(\ell-2)!}} \frac{\mu(k\tau)}{(k\tau)^2} \) is the spherical Bessel function. The anisotropy spectrum or the polarization spectrum is then obtained by integrating over the initial power spectrum of the metric perturbation

\[
C_{(T,P)}^{(T)} = \langle 4\pi \rangle^2 \int k^2 dk P_h(k) \left| \Delta_{(T,P)}^{(T)}(k,\tau=\tau_0) \right|^2. \tag{2.8}
\]
The expressions for the E and B power spectrum are given by
\[ C_{E\ell}^{(T)} = (4\pi)^2 \int k^2 dk P_h(k) \left[ \int_0^{\tau_0} d\tau P_{SP}^{(T)} \left( -j_\ell(x) + j_\ell''(x) + \frac{2j_\ell(x)}{x^2} + \frac{4j_\ell'(x)}{x} \right) \right]^2, \quad (2.9) \]
\[ C_{B\ell}^{(T)} = (4\pi)^2 \int k^2 dk P_h(k) \left[ \int_0^{\tau_0} d\tau P_{SP}^{(T)} \left( 2j_\ell'(x) + \frac{4j_\ell(x)}{x} \right) \right]^2, \quad (2.10) \]
where \( x = k(\tau_0 - \tau) \). The two-point correlations of the temperature anisotropies and polarization patterns at different points in the sky are presented by these equations. In the limit of vanishing momentum, only \( \tilde{\Delta}_{T,0}^{(T)} \) and \( \tilde{\Delta}_{P,0}^{(T)} \) are non-zero and the solutions of Boltzmann equations are given by, \[29\],
\[ \tilde{\Delta}_{T,0}^{(T)}(\tau_0) = \int_0^{\tau_0} d\tau \left( -\frac{6}{7}e^{-\kappa(\tau)} - \frac{1}{7}e^{-\frac{3}{10}\kappa(\tau)} \right) \hat{h}, \quad (2.11) \]
\[ \tilde{\Delta}_{P,0}^{(T)}(\tau_0) = \int_0^{\tau_0} d\tau \left( -\frac{1}{7}e^{-\kappa(\tau)} + \frac{1}{7}e^{-\frac{3}{10}\kappa(\tau)} \right) \hat{h}, \quad (2.12) \]
which clearly show that the transfer functions \( \tilde{\Delta}_{T,0}^{(T)} \) and \( \tilde{\Delta}_{P,0}^{(T)} \) are closely related to the time evolution of tensor perturbations.

We mainly focus on the effects of evolution of gravitational waves with modified dispersion relation on the CMB. Since the typical energy scale relevant to CMB is the Hubble scale \( H_\tau \approx 2.3 \times 10^4 H_0 \approx 3.3 \times 10^{-29} \text{eV} \) at recombination, for convenience, the energy scale \( M \) in eq. (1.1) can be taken as \( M = H_\tau \). Here, for example, we only pick up one term in the second part of eq. (1.1), and then the equation of motion for the gravitational wave becomes
\[ \ddot{h}_k + 2\frac{\dot{a}}{a}\dot{h}_k + \left( k^2 + \lambda_\alpha \frac{k^\alpha}{H_\tau^{2-\alpha}} a^{2-\alpha} \right) h_k = 0, \quad (2.13) \]
where contributions on the right hand side due to the anisotropic stress generated by neutrinos and photons are ignored, and \( k \) is the comoving mode. For \( \alpha = 0 \), it nothing but the case for a massive graviton with mass \( m_g = \sqrt{\lambda_0} H_\tau \). Similarly, we introduce a parameter
\[ m_k^2 \equiv \lambda_\alpha \left( \frac{k}{k_\tau} \right)^\alpha H_\tau^2 \quad (2.14) \]
which is the physical effective mass of graviton with comoving mode \( k \) at recombination, where \( k_\tau \equiv a_\tau H_\tau \) is the mode which re-entered horizon at recombination and \( a_\tau \) is the scale factor. Since CMB measures the structures at cosmic scales and then is a powerful tool to probe gravity in the infrared (IR) limit, we focus on the case of \( \alpha \leq 0 \) in this paper. The short-wavelength modes satisfying
\[ m_k \ll \frac{k}{a_\tau}, \quad (2.15) \]
or equivalently
\[ k \gg k_0 \equiv \lambda_0^{\frac{1}{2-\alpha}} k_\tau, \quad (2.16) \]
are not affected by modified term in the dispersion relation, no matter \( m_k \) is below the Hubble rate \( H_\tau \) or not. In terms of the multipole number, the transition to the massless regime corresponds to
\[ \ell_0 \equiv k_0(\tau_0 - \tau_\ell) \approx \frac{k_\tau}{H_0} \int_0^{\tau_\ell} \frac{dz}{E(z)} \approx 66.2 \times \frac{H_\tau}{H_0}^{\frac{1}{2-\alpha}}, \quad (2.17) \]
where \( E(z) \equiv H(z)/H_0 = \sqrt{\Omega_\Lambda + \Omega_m(1+z)^3 + \Omega_r(1+z)^4} \). Therefore the CMB power spectra are expected to be the same as those in the massless case for \( \ell \gg \ell_0 \). If \( \lambda_\alpha < 1 \), there is a parameter space for

\[
\frac{k}{a_r} < m_k < H_r,
\]

or equivalently

\[
\lambda^{-\frac{1}{\alpha}}_\alpha k < k < \lambda^{\frac{1}{1-\alpha}}_\alpha k_r.
\]

Even though the second term in the bracket of eq. (2.13) becomes dominant, these modes do not oscillate because of the Hubble friction term. Finally, the modes start to oscillate before the recombination if their wavelengths are long enough, namely

\[
m_k > \frac{k}{a_r} \quad \text{and} \quad m_k > H_r,
\]

which implies

\[
k < \min \left( \lambda^{-\frac{1}{\alpha}}_\alpha, \lambda^{\frac{1}{1-\alpha}}_\alpha \right) k_r.
\]

For \( \alpha = 0 \), this condition becomes \( k < \lambda_0^{1/2} k_r \) and \( \lambda_0 > 1 \).

At the matter dominated stage, \( a(t) = a_r (t/t_r)^{2/3} \), and for the long wavelength modes satisfying \( k \ll \min \left( \lambda^{-\frac{1}{\alpha}}_\alpha, \lambda^{\frac{1}{1-\alpha}}_\alpha \right) k_r \), eq. (2.13) can be simplified to be

\[
h''_k + \frac{2}{\bar{x}} h'_k + (m_k t_r)^2 \bar{x}^{-2\alpha/3} h_k = 0,
\]

where the prime denotes the derivative with respect to \( x \equiv t/t_r \). Note \( m_k t_r = \frac{2 m_k}{3 H_r} = \frac{2}{3} \lambda_\alpha^{1/2} \left( \frac{k}{k_r} \right)^{\alpha/2} \) because of \( H_r = 2/(3 t_r) \). Here we assume \( \lambda_\alpha > 0 \) and the case with \( \lambda_\alpha < 0 \) is briefly discussed in the appendix. A. Eq. (2.22) can be analytically solved and the solution is given by

\[
\frac{h(m_k, x)}{h_0} = 2^\beta \Gamma (1 + \beta) \bar{x}^{-\beta} J_\beta(\bar{x}),
\]

and then

\[
\frac{h'(m_k, x)}{h_0} = -2^\beta \Gamma (1 + \beta) m_k t_r \bar{x}^{\frac{1}{2\alpha}} \bar{x}^{-\beta} J_{1+\beta}(\bar{x}),
\]

where

\[
\beta = \frac{3}{6 - 2\alpha},
\]

\[
\bar{x} = 2 \beta m_k t_r \bar{x}^{\frac{1}{2\alpha}}.
\]

and \( J_\beta(\bar{x}) \) is the Bessel function of the first kind. For \( \alpha = 0 \), \( h(m_k, x)/h_0 = \sin(m_g t)/(m_g t) \) which is that same as that in [12]. The behaviors of \( h(m_k, x)/h_0 \) and \( h'(m_k, x)/h_0 \) for \( \alpha = -1 \) and \( \alpha = 0 \) are illustrated in figure 1.
In the vanishing momentum limit, the non-zero solutions of Boltzmann equations are given in eqs. (2.11) and (2.12) which indicates that the oscillatory solution of gravitational wave in eq. (2.23) will enhance CMB angular power spectra at large scales. Let us consider a mode with \( k \ll k_r \), e.g. \( k = 0.035k_r \) which roughly corresponds to the CMB quadrupole. Integrating over the conformal time in eq. (2.12), the function of \( \left| \tilde{\Delta}_{P,0}(\tau_0) \right|^2 \) is illustrated in figure 2 for \( \alpha = -1 \). The oscillation signature comes from the gravitational waves and the envelope gradually decreases with \( \lambda_{-1} \) owning to the damping term in the wave equation. Roughly speaking, \( \left| \tilde{\Delta}_{P,0}(\tau_0) \right|^2 \) monotonously increases from \( \lambda_{-1} = 0 \) to 1, and then drops from 1 to 3. Because of

\[
C^{(T)}_{BB,2} \propto \left| \tilde{\Delta}_{P,0}(\tau_0) \right|^2,
\]

the oscillatory behavior in figure 2 is expected to yield oscillatory \( C^{(T)}_{BB,2} \) for different values of \( \lambda_{-1} \). These signatures match the numerical results for BB tensor angular power spectrum at \( \ell = 2 \) quite well in figure 3. In addition, for \( 0 < \lambda_{-1} \lesssim 3 \), \( C^{(T)}_{BB,\ell} \) is oscillatory for \( \ell \lesssim 5 \), but
keeps increasing for $5 \lesssim \ell \lesssim 100$. See the upper plot in figure 3. For $3 \lesssim \lambda_{-1} \lesssim 9$, $C_{BB,\ell = 2}^{(T)}$ is also oscillatory, but keeps decreasing for $3 \lesssim \ell \lesssim 8$ and increasing for $8 \lesssim \ell \lesssim 100$. See the bottom plot in figure 3.

In order to obtain the numerical results for the CMB angular power spectra for the modified dispersion relation of graviton, we modify CAMB [21] by taking into account the evolution equation for the tensor perturbation in eq. (2.13), where we add two parameters $\alpha$ and $\lambda_\alpha$. By varying these two parameters, we can explore how the modified dispersion relation of graviton affects the CMB power spectra. Keeping $\lambda_\alpha = 10^{-6}$, we plot TT, TE, EE and BB angular power spectra contributed from the tensor perturbations for $\alpha = -1$ and $\alpha = 0$ in figure 4. We find that the CMB polarization power spectra are more sensitive to the modification of the dispersion relation of graviton. Keeping $m_k$ fixed, the oscillation of gravitational wave for $\alpha = -1$ is slightly stronger than $\alpha = 0$ in figure 1, and thus all of the quadrupoles of TT, TE, EE and BB power spectra for $\alpha = -1$ is higher than those for $\alpha = 0$.

3 Summary and discussion

In this paper we explore how the modified dispersion relation of graviton affects the shape of CMB angular power spectra. Due to the oscillations of the amplitude of gravitational waves, the modification of dispersion relation enhances the quadrupoles of CMB power spectra, and may lead to different oscillatory behaviors for different multipoles with $\ell \lesssim 100$.

Once the CMB B-modes contributed by primordial gravitational waves are detected in the future, we can test the dispersion relation of graviton at the energy scales around $10^{-29}$ eV. In particular, the low CMB multipoles ($\ell \lesssim 10$) are very sensitive to the modification of dispersion relation of graviton, and it implies that the low CMB multipoles are very useful to test the dispersion relation of graviton.
\[ l(l+1)C_T^{BB,\ell}/2\pi \left[ \mu K^2 \right] \]

Figure 3. The plots of \( C_T^{BB,\ell} \) for different values of \( \lambda_\alpha = -1 \). The black dotted curve corresponds to the massless case.

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Figure 4. The plots of TT, TE, EE and BB angular power spectra contributed from tensor perturbations. Here we take \( r = 0.1 \) and compare the results for \( \alpha = 0 \) (blue dashed curves) and \( \alpha = -1 \) (red solid curves) with \( \lambda_\alpha = 10^{-6} \). The black dotted curves correspond to \( \lambda_\alpha = 0 \).

A The case with \( \lambda_\alpha < 0 \)

For \( \lambda_\alpha < 0 \), eq. (2.22) becomes

\[
h''_k + \frac{2}{x} h'_k - \left( m_k t_r \right)^2 x^{-2\alpha/3} h_k = 0, \tag{A.1}
\]

where \( m_k t_r = \frac{2}{3} |\lambda_\alpha|^{1/2} \left( \frac{k}{k_r} \right)^{\alpha/2} \). Since the last term on the left hand side of the above equation is negative, we expect that the solution becomes unstable and grows up even faster than exponential growth due to the factor \( x^{-2\alpha/3} \) for \( \alpha < 0 \). For \( x \gg 1 \), the solution of the above equation is approximately given by

\[
h_k \sim x^{-(1-\alpha/6)} \exp \left[ \frac{m_k t_r}{\gamma} x^{\gamma} \right], \tag{A.2}
\]

where \( \gamma = 1 - \alpha/3 \). This solution indicates that the modes for \( \lambda_\alpha < 0 \) are unstable and should be un-physical.

References

[1] T. Clifton, P.G. Ferreira, A. Padilla and C. Skordis, Modified gravity and cosmology, *Phys. Rept.* **513** (2012) 1 [arXiv:1106.2476] [insPIRE].

- 8 -
[2] S. Nojiri and S.D. Odintsov, *Unified cosmic history in modified gravity: from F(R) theory to Lorentz non-invariant models*, Phys. Rept. 505 (2011) 59 [arXiv:1011.0544] [nSPIRE].

[3] S. Capozziello and M. De Laurentis, *Extended theories of gravity*, Phys. Rept. 509 (2011) 167 [arXiv:1108.6266] [nSPIRE].

[4] S. Nojiri, S.D. Odintsov and V.K. Oikonomou, *Modified gravity theories on a nutshell: inflation, bounce and late-time evolution*, Phys. Rept. 692 (2017) 1 [arXiv:1705.11098] [nSPIRE].

[5] M. Kamionkowski, A. Kosowsky and A. Stebbins, *A probe of primordial gravity waves and vorticity*, Phys. Rev. Lett. 78 (1997) 2058 [astro-ph/9609132] [nSPIRE].

[6] P. Cabella and M. Kamionkowski, *Theory of cosmic microwave background polarization*, astro-ph/0403392 [nSPIRE].

[7] A. Kosowsky, *Introduction to microwave background polarization*, New Astron. Rev. 43 (1999) 157 [astro-ph/9904102] [nSPIRE].

[8] M. Kamionkowski and E.D. Kovetz, *The quest for B-modes from inflationary gravitational waves*, Ann. Rev. Astron. Astrophys. 54 (2016) 227 [arXiv:1510.06042] [nSPIRE].

[9] S. Chandrasekhar, *Radiation transfer*, Dover, U.S.A., (1960).

[10] BICEP2 collaboration, P.A.R. Ade et al., *Detection of B-mode polarization at degree angular scales by BICEP2*, Phys. Rev. Lett. 112 (2014) 241101 [arXiv:1403.3985] [nSPIRE].

[11] POLARBEAR collaboration, P.A.R. Ade et al., *A measurement of the cosmic microwave background B-mode polarization power spectrum at sub-degree scales with POLARBEAR*, Astrophys. J. 794 (2014) 171 [arXiv:1403.2369] [nSPIRE].

[12] S. Dubovsky, R. Flauger, A. Starobinsky and I. Tkachev, *Signatures of a graviton mass in the cosmic microwave background*, Phys. Rev. D 81 (2010) 023523 [arXiv:0907.1658] [nSPIRE].

[13] M. Fasiello and R.H. Ribeiro, *Mild bounds on bigravity from primordial gravitational waves*, JCAP 07 (2015) 027 [arXiv:1505.00404] [nSPIRE].

[14] W. Lin and M. Ishak, *Testing gravity theories using tensor perturbations*, Phys. Rev. D 94 (2016) 123011 [arXiv:1605.03504] [nSPIRE].

[15] P. Brax, S. Cespedes and A.-C. Davis, *Signatures of graviton masses on the CMB*, arXiv:1710.09818 [nSPIRE].

[16] L. Amendola, G. Ballesteros and V. Pettorino, *Effects of modified gravity on B-mode polarization*, Phys. Rev. D 90 (2014) 043009 [arXiv:1405.7004] [nSPIRE].

[17] L. Xu, *Gravitational waves: a test for modified gravity*, Phys. Rev. D 91 (2015) 103520 [arXiv:1410.6977] [nSPIRE].

[18] V. Pettorino and L. Amendola, *Friction in gravitational waves: a test for early-time modified gravity*, Phys. Lett. B 742 (2015) 353 [arXiv:1408.2224] [nSPIRE].

[19] M. Raveri, C. Baccigalupi, A. Silvestri and S.-Y. Zhou, *Measuring the speed of cosmological gravitational waves*, Phys. Rev. D 91 (2015) 061501 [arXiv:1405.7974] [nSPIRE].

[20] L. Boubekeur, E. Giusarma, O. Mena and H. Ramírez, *Current status of modified gravity*, Phys. Rev. D 90 (2014) 103512 [arXiv:1407.6837] [nSPIRE].

[21] A. Højjati, L. Pogosian and G.-B. Zhao, *Testing gravity with CAMB and CosmoMC*, JCAP 08 (2011) 005 [arXiv:1106.4543] [nSPIRE].

[22] J.N. Dossett, M. Ishak and J. Moldenhauer, *Testing general relativity at cosmological scales: implementation and parameter correlations*, Phys. Rev. D 84 (2011) 123001 [arXiv:1109.4583] [nSPIRE].
[23] C.-P. Ma and E. Bertschinger, *Cosmological perturbation theory in the synchronous and conformal Newtonian gauges*, *Astrophys. J.* **455** (1995) 7 [astro-ph/9506072] [inspire]

[24] M. Zaldarriaga and U. Seljak, *An all sky analysis of polarization in the microwave background*, *Phys. Rev. D* **55** (1997) 1830 [astro-ph/9609170] [inspire]

[25] U. Seljak and M. Zaldarriaga, *A line of sight integration approach to cosmic microwave background anisotropies*, *Astrophys. J.* **469** (1996) 437 [astro-ph/9603033] [inspire]

[26] M. Zaldarriaga and D.D. Harari, *Analytic approach to the polarization of the cosmic microwave background in flat and open universes*, *Phys. Rev. D* **52** (1995) 3276 [astro-ph/9504085] [inspire]

[27] S. Weinberg, *Cosmology*, Oxford University Press, New York U.S.A., (2008) [inspire]

[28] S. Dodelson, *Modern cosmology*, Academic Press, San Diego U.S.A., (2003) [inspire]

[29] M.M. Basko and A.G. Polnarev, *Polarization and anisotropy of the relict radiation in an anisotropic universe*, *Mon. Not. Roy. Astron. Soc.* **191** (1980) 207.