The adiabatic theorem [1–3] is a fundamental ingredient in a number of applications in quantum mechanics. Under adiabatic dynamics, a quantum system evolves obeying a sufficiently slowly-varying Hamiltonian, which prevents changes in the populations of the energy eigenlevels. In particular, if the system is prepared in an eigenstate $|E_n(0)\rangle$ of the Hamiltonian $H(t)$ at a time $t = 0$, it will evolve to the corresponding instantaneous eigenstate $|E_n(t)\rangle$ at later times. The concept of adiabaticity plays a relevant role in a vast array of fields, such as energy-level crossings in molecules [4, 5], quantum field theory [6], geometric phases [7, 8], quantum computation [9–13], quantum thermodynamics [14, 15], among others. However, despite such a wide range of applications, both sufficiency and necessity of quantitative conditions for the adiabatic behavior have been challenged [16]. In particular, inconsistencies in the application of the adiabatic theorem may appear for oscillating Hamiltonians as a consequence of resonant transitions between their energy levels [17, 18]. Such inconsistencies have led to a reevaluation of the adiabatic theorem, yielding many new proposals of adiabatic conditions (ACs) in more general settings (see, e.g., Refs. [19–22]).

The first experimental investigation on the comparison among these proposals has been considered by Du et al. [23], where the authors considered a single nuclear spin-1/2 particle in a rotating magnetic field manipulated by nuclear magnetic resonance (NMR) techniques. It is then shown the violation of both the sufficiency and necessity of the traditional AC, with partial success via some other generalized ACs. It is remarkable such possible violations are already apparent for a single quantum bit (qubit) system. More specifically, the ACs analyzed in Ref. [23] can be cast in the form of adiabatic co-efficients $C_n(t)$ which, for a qubit, are given by

$$C_1 = \max_{t \in [0,\tau]} \left| \frac{\langle E_0(t) | H(t) | E_1(t) \rangle}{|E_0(t) - E_1(t)|^2} \right|^2,$$

$$C_2 = \max_{t \in [0,\tau]} \left| \frac{d}{dt} \left( \frac{\langle E_0(t) | H(t) | E_1(t) \rangle}{|E_0(t) - E_1(t)|^2} \right) \right|,$$

$$C_3 = \max_{t \in [0,\tau]} \left| \frac{|E_1(t) - E_0(t) - \Delta(t)|}{|dE_0(t)|} \right|,$$

$$C_4 = \max_{t \in [0,\tau]} \left\{ \frac{\dot{\tau}^2 \|H(t)\|}{\|E_0(t) - E_1(t)\|^3} \left( \frac{\dot{\tau}^2 \|H(t)\|}{\|E_0(t) - E_1(t)\|^3} \right) \right\},$$

where $|E_n(t)\rangle$ are eigenstates of $H(t)$ with energies $E_n(t)$, $\tau$ is the total evolution time, $dE_0(t) = (E_0(t)/\partial t)(E_0(t)]$ (1b), $\Delta(t) = i\gamma_1(t) - i\gamma_0(t) + \frac{\dot{\gamma}_1(t)}{\gamma_1(t)}$, and $\dot{E}_0(t) = \langle E_0(t) | E_0(t) \rangle$, with the dot symbol denoting time derivative and $\| \cdot \|$ denoting the usual operator norm. The adiabaticity coefficient $C_1$ is the well-known standard (traditional) adiabatic condition [3, 18, 24], while the conditions $C_2$, $C_3$ and $C_4$ as shown above were derived by Tong et al. [20], Wu et al. [21, 22] and Ambainis-Regev [19], respectively. In general, the adiabatic behavior in a quantum system is achieved when $C_n \ll 1$. In Ref. [23], it is shown that the condition $C_1$ is neither sufficient nor necessary for guaranteeing the adiabatic behavior, while the conditions $C_2$, $C_3$ and $C_4$ seem to successfully indicate the resonant phenomena observed in their specific experiment (even though their validity is debatable for more general quantum systems).

Here, instead of looking for new proposals of ACs, we adopt a different strategy to analyze the conditions in Eq. (1). More specifically, we consider the dynamics of the system in
The system is initialized in the state $|0\rangle$ with optical pumping, so that the adiabatic dynamics is achieved if the system evolves as $|\psi_{ad}(t)\rangle = |E_1(t)\rangle$, up to a global phase. It is possible to show that the Hamiltonian presents a near-to-resonance situation when we set $|\omega - \omega_0| \ll |\omega_1|$. Thus, to study the adiabaticity validity conditions in our Hamiltonian in Eq. (2) we need to compute the coefficients in Eq. (1) for different values of the $\omega$. In our experiment, we set the detuning $\omega_0 = 2\pi \times 1.0$ MHz, the coupling strength $\omega_{rt} = 2\pi \times 20.0$ KHz, and $\omega = a \times \omega_0$ ($a = 10.0, 1.0173, 1.0, 0.9827$ and $0.1$, respectively).

In Fig. 2a, we experimentally compute the fidelity of obtaining the system in the $|\psi_{ad}(t)\rangle$, where we use the fidelity as $F(t) = |\text{Tr}[\rho(t)\rho_{ad}(t)]|$, where $\rho(t)$ is solution of the Eq. (3) and $\rho_{ad}(t) = |E_1(t)\rangle \langle E_1(t)|$. We show the experimental results for three different situations, where we have $\omega \gg \omega_0, \omega \ll \omega_0$ and $\omega \approx \omega_0$. When we have $|\omega - \omega_0| \gg |\omega_1|$, the condition should provide us $C_n \ll 1$. However, looking at Fig. 2b we can see that such result is not obtained in case $\omega \gg \omega_0$.

Therefore, all the ACs provided by Eq. (1) are not necessary, once we have adiabaticity even in case where the ACs are not obeyed. On the other hand, in the near-to-resonance situation we no have adiabaticity (once the fidelity is much smaller than 1), in contrast with Fig. 2b, where we get $C_n \approx 10^{-2}$. Therefore, the ACs are not sufficient for studying the adiabatic behavior of our system. In conclusion, rather differently from the system considered in Ref. [23], all the ACs provided by Eq. (1) are not applicable to the dynamics governed by the Hamiltonian in Eq. (2). These results imply that a direct application of the ACs yields neither sufficient nor necessary.

At this point, providing a new condition for adiabaticity could be a natural path to follow. Nevertheless, in order to investigate the applicability of ACs, we will implement, similarly as in classical mechanics, a transformation to a non-inertial frame in Schrödinger equation. By considering frame representation in quantum mechanics, Eq. (3) can be taken as Schrödinger equation in an inertial frame [27]. To introduce a non-inertial frame, we can perform a rotation using the unitary time-dependent operator $\hat{O}(t) = e^{i\omega t\hat{S}}$. In this frame, the dynamics is given by

$$\rho(t) = (-i/\hbar)[\hat{H}(t), \rho(t)] ,$$

where $\hat{H}(t) = O(t)\hat{H}(t)\hat{O}(t) + i\hbar \dot{O}(t)\hat{O}(t)$ and $\rho_0(t) = O(t)\rho(t)\hat{O}(t)$. The contribution $i\hbar \dot{O}(t)\hat{O}(t)$ in $H(t)$ can be interpreted as a “fictitious potential” [27]. This procedure is a common strategy, e.g., in nuclear magnetic resonance, where we use the non-inertial frame to describe the system dynamics [28, 29]. By computing the non-inertial Hamiltonian $\hat{H}(t)$ we find $H_0(t) = (\omega_0 - \omega)\hat{S}_z/2 + \sin(\omega t) \tan \theta \hat{S}_x \hat{S}_y$, with $\hat{S}_z(t) = \omega_0 \cos(\omega t)\hat{S}_x - \sin(\omega t)\hat{S}_y/2$ and $\hat{S}_x = \sigma_x, \hat{S}_y = \sigma_y, \hat{S}_z = \sigma_z$. Now, if we compute the conditions $C_n$ considering the set of eigenstates and energies of the new Hamiltonian $H(t)$ we obtain the curves shown in Fig. 2c. Thus, considering the results in Figs. 2a and 2c, it is possible to conclude that the coefficients $C_n$ computed in the non-inertial frame allow us to successfully
describe the adiabaticity of the inertial frame. This is in contrast with previous results, which indicated that the ACs may be problematic as we consider oscillating or rotating fields in resonant conditions [17, 18]. In particular, notice that even the traditional AC, when analyzed in this non-inertial frame, becomes sufficient and necessary for the adiabatic behavior of the single-qubit oscillating Hamiltonian in Eq. (2).

Validation mechanism for ACs and frame-dependent adiabaticity – We now establish a general validation mechanism for ACs connecting inertial and non-inertial frames, with special focus on cases under resonant conditions. This approach is applicable beyond the single-qubit system previously considered, holding for more general multi-particle quantum systems. Let us consider a Hamiltonian $H(t)$ in an inertial reference frame and its non-inertial counterpart $H_{O}(t)$, where the change of reference frame is provided by a generic time-dependent unitary operator $O(t)$. The Hamiltonians $H(t)$ and $H_{O}(t)$ obey eigenvalue equations given by $H(t)|E_{n}(t)⟩ = E_{n}(t)|E_{n}(t)⟩$ and $H_{O}(t)|E_{n}^{O}(t)⟩ = E_{n}^{O}(t)|E_{n}^{O}(t)⟩$, with $[H(t), H(t')] ≠ 0$ and $[H_{O}(t), H_{O}(t')] ≠ 0$, in general. The adiabatic dynamics in the inertial frame, which is governed by $H(t)$, can be defined through its corresponding evolution operator $U(t, t_0) = \sum_{n} e^{i \theta_{n}(t) / \hbar}[E_{n}(t)|E_{n}(t_0)⟩⟨E_{n}(t)|]$, where $\theta_{n}(t) = -E_{n}(t)/\hbar + i[E_{n}(t)](dt/dt)E_{n}(t)$) is the adiabatic phase, which composed by its dynamic and geometric contributions, respectively. Then, we can connect the adiabatic evolution in the inertial and non-inertial frames through the theorem below.

**Theorem 1** Consider a Hamiltonian $H(t)$ and its non-inertial counterpart $H_{O}(t) = O(t)[H(t)O(t) + i\hbar\dot{O}(t)O(t)]$, with $O(t)$ an arbitrary unitary transformation. The eigenstates of $H(t)$ and $H_{O}(t)$ are denoted by $|E_{n}(t)⟩$ and $|E_{n}^{O}(t)⟩$, respectively. Then, if a quantum system $S$ is prepared at time $t = t_0$ in a particular eigenstate $|E_{k}(t_0)⟩$ of $H(t_0)$, then the adiabatic evolution of $S$ in the inertial frame, governed by $H(t)$, is associated with the adiabatic evolution of $S$ in the non-inertial frame, governed by $H_{O}(t)$, if and only if

$$
|\langle E_{m}^{O}(t)|O(t)|E_{n}(t)⟩| = |\langle E_{m}^{O}(t_0)|O(t_0)|E_{n}(t_0)⟩| \quad \forall t, m ,
$$

where $t \in [t_0, \tau]$, with $\tau$ denoting the total time of evolution. Conversely, if the adiabatic dynamics in the non-inertial frame starts in $|E_{m}^{O}(t_0)⟩$, then the dynamics in the inertial frame is also adiabatic if and only if Eq. (5) is satisfied.

The proof is provided in the Supplementary Material. Notice that Theorem 1 establishes that, if Eq. (5) is satisfied, then a non-adiabatic behavior in the non-inertial frame ensures a non-adiabatic behavior in the original frame and vice-versa, provided that the evolution starts in a single eigenstate of the initial Hamiltonian. Then, we can apply this result to general resonant Hamiltonians. A typical scenario exhibiting resonance phenomena appears when a physical system is coupled to both a static high intensity field $\vec{B}_{0}$ and a time-dependent transverse field $\vec{B}_{T}(t)$, where $||\vec{B}_{T}(t)|| < ||\vec{B}_{0}||$. Here, we will consider that the transverse field $\vec{B}_{T}(t)$ is associated to a single rotating or oscillating field with frequency $\omega$. In this context, we can write a general multi-qubit Hamiltonian as

$$
H(\omega, t) = \hbar \omega_{0}H_{0} + \hbar \omega_{T}H_{T}(\omega, t) ,
$$

where the contributions $\hbar \omega_{0}H_{0}$ and $\hbar \omega_{T}H_{T}(\omega, t)$ depend on the fields $\vec{B}_{0}$ and $\vec{B}_{T}(t)$, respectively. Since $\vec{B}_{0} \perp \vec{B}_{T}(t)$, we observe that $[H_{T}(\omega, t), H_{0}] ≠ 0$. In the case $||\vec{B}_{T}(t)|| ≪ ||\vec{B}_{0}||$, the eigenstates $|E_{n}(t)⟩$ of the Hamiltonian $H(\omega, t)$ can be written as $|E_{n}(t)⟩ \approx |E_{n}^{0}⟩$, where $|E_{n}^{0}⟩$ is a stationary eigenstate of the Hamiltonian $\hbar \omega_{0}H_{0}$.

If we have a far-from-resonance situation, we can approximate the dynamics obtained from $H(\omega, t)$ as that one driven by $\hbar \omega_{0}H_{0}$. However, in a near-to-resonance field configuration the most convenient way to study the system dynamics is by adopting a change of reference frame. A general approach to frame change is obtained by the choice $O(\omega, t) = e^{i\omega_{T}t\hat{H}_{T}}$. Then, from Eq. (6), we can show that

$$
H_{O}(\omega, t) = h(\omega_{0} - \omega)H_{0} + \hbar \omega_{T}H_{O,T}(\omega, t) .
$$
where $H_{O,T}(\omega, t) = O(t)H_T(t)O^\dagger(t)$. It is worth mentioning that $[H_{O,T}(\omega, t), H_0] \neq 0$, once $[H_T(\omega, t), H_0] \neq 0$. In addition, since $H_{O,T}(\omega, t)$ is constrained to $H_T(t)$ through a unitary transformation, $\|H_T(\omega, t)\| = \|H_{O,T}(\omega, t)\|$. Therefore, due to the quantity $\omega \equiv \omega_1$ in the first term of $H_{O}(\omega, t)$, the contribution of $H_T(\omega, t)$ cannot be ignored in this new frame.

As shown in the Supplementary Material, by considering the generic Hamiltonian in Eq. (6), we obtain that Eq. (5) in Theorem 1 is automatically satisfied if the quantum system is in a far-from resonance configuration $|\omega - \omega_0| \gg |\alpha\tau|$, so that the adiabatic dynamics in the inertial frame can be always predicted from the adiabaticity analysis in the non-inertial frame. For this reason, the curves in Fig. 2a, yielding $C_n \ll 1$ for $|\omega| \gg |\omega_0|$ and $|\omega| \ll |\omega_0|$. On the other hand, at resonance (or near-to-resonance) configuration $|\omega - \omega_0| \ll |\alpha\tau|$, Eq. (5) in Theorem 1 reduces to the rather simple condition $[E_x(0)E_y(0)] = [E_x^{(0)}(0)E_y^{(0)}]$. Hence, provided a generic Hamiltonian given by Eq. (6) at resonance (or near-to-resonance) situation, if the corresponding Hamiltonian in the non-inertial frame has time-dependent eigenstates obeying $[E_x^m(0)|E_x^m|] = \text{constant}$, $\forall t, m$, for a particular initial state $|E_x^m(0)|$, then a non-adiabatic evolution in the non-inertial frame implies non-adiabatic evolution in the inertial frame. This is exactly the case for the Hamiltonian in Eq. (2), with the violation of adiabaticity at resonance illustrated in Fig. 2c for all the ACs considered.

Revisiting the problem of the spin-1/2 particle in a rotating magnetic field – We now apply our general treatment to the NMR Hamiltonian discussed by Du et al. [23]. The dynamics describes a single spin-1/2 particle coupled to a static field $\vec{B}_0 = B_0\hat{z}$ and a transverse radio-frequency field $\vec{B}_\text{rf}(t) = B_\text{rf}[\cos(\omega_0)t + \sin(\omega_0)t]$, with Hamiltonian given by

$$H_{\text{nnr}}(t) = \frac{\omega_0}{2}\sigma_z + \frac{\omega_\text{rf}}{2}\left[\cos(\omega_0)t\sigma_x + \sin(\omega_0)t\sigma_z\right],$$

where $|\omega_0| \gg |\omega_0|$. The system is prepared in an eigenstate of $\sigma_z$, and the frequencies are chosen such that the standard AC is satisfied [23]. In this scenario, the violations and agreements about ACs for this system have widely been discussed in literature [30–33]. Here we analyze this Hamiltonian from a different point of view. By writing the system dynamics in the non-inertial frame through $O(t) = e^{i\frac{\omega}{2}t\sigma_z}$, we obtain $H_{\text{nnr}}^\text{nnr} = (\omega_0 - \omega)\sigma_z/2 + (\omega_\text{rf}/2)\sigma_x$. Since this Hamiltonian is time-independent, the dynamics under $H_{\text{nnr}}^\text{nnr}$ is trivially adiabatic, with all ACs in Eq. (1) satisfied. Therefore, the is no direct visualization of the resonant point. However, Theorem 1 cannot be directly applied here near to resonance because the initial state in this case is not an individual eigenstate of $H_{\text{nnr}}^\text{nnr}$, since $H_{\text{nnr}}^\text{nnr}$ is approximately proportional to $\sigma_x$. We can circumvent this problem by taking advantage of the time-independence of $H_{\text{nnr}}^\text{nnr}$. More specifically, we start from the evolution operator $U_{\text{nnr}}(t, t_0) = e^{-iH_{\text{nnr}}(t-t_0)}$ in the non-inertial frame and investigate under which conditions we may obtain an adiabatic dynamics in the inertial frame. This can be suitably addressed by Theorem 2 below.

**Theorem 2** Consider a Hamiltonian $H(t)$ and its non-inertial counterpart $H_0 = O(t)H(t)O^\dagger(t) + i\hbar O(t)O^\dagger(t)$, with $O(t)$ an arbitrary unitary transformation and $H_0$ a constant Hamiltonian. The eigenstates of $H(t)$ and $H_0$ are denoted by $|E_x(t)|$ and $|E_m^\text{nnr}|$, respectively. Then, if a quantum system $S$ is prepared at time $t = t_0$ in a particular eigenstate $|E_x(t_0)|$ of $H(t)$, then the adiabatic evolution of $S$ in the inertial frame, governed by $H(t)$, occurs if and only if

$$|\langle E_x(t)\langle U(t,t_0)|E_x(t_0)\rangle| \neq 0,$$

where $t \in [t_0, \tau]$, with $\tau$ denoting the total time of evolution, and $U_{\text{nnr}}(t, t_0) = O(t)e^{-iH_0|t-t_0|}/O(t_0)$. The proof is provided in the Supplementary Material. The experimental results in Ref. [23] can be validated by Theorem 2, since the Hamiltonian in Eq. (8) satisfies Eq. (9) in a far-from resonance situation and violates it at resonance. In fact, the initial state $|\psi(0)\rangle$ can be approximated written as $|\psi(0)\rangle = |E_x(0)\rangle \approx |n\rangle$ with $|\sigma^x_n\rangle = (-1)^{n+1}|n\rangle$. Thus, Eq. (9) provides the condition $|\langle k|e^{-iH_{\text{nnr}}|t\rangle}| = |\langle k|n\rangle| \neq 0$. For $k$ and $\forall t \in [0, \tau]$. In a far-from-resonance situation, we have $H_{\text{nnr}}^\text{nnr} \approx \omega_0/2\sigma_z$, and we conclude that $|\langle k|e^{-iH_{\text{nnr}}|t\rangle}| \neq 0$. This shows that the dynamics in the inertial frame is (approximately) adiabatic far from resonance. On the other hand, near to resonance, we get $H_{\text{nnr}}^\text{nnr} \approx \omega_0/2\sigma_z$, where we can immediately conclude that $|\langle k|e^{-iH_{r}|t\rangle}| \neq 0$ for any $t \in [0, \tau]$.

**Conclusions** – We have introduced a framework to validate ACs in generic discrete multi-particle Hamiltonians, which is rather convenient to analyze quantum systems at resonance. This is based on the analysis of ACs in a suitably designed non-inertial reference frame. In particular, we have both theoretically and experimentally shown that several relevant ACs [provided by Eq. (1)], which include the traditional AC, are sufficient and necessary to describe the adiabatic behavior of a qubit in an oscillating field given by Eq. (2). In this case, sufficiency and necessity are fundamentally obtained through the non-inertial frame map, with all the conditions failing to point out the adiabatic behavior in the original reference frame. The experimental realization has been performed through a single trapped Ytterbium ion, with excellent agreement with the theoretical results. More generally, the validation of ACs has been expanded to arbitrary Hamiltonians through Theorems 1 and 2, with detailed conditions provided for a large class of Hamiltonians in the form of Eq. (6). Therefore, instead of looking for new approaches for defining ACs, we have introduced a mechanism based on “fictitious potentials” (associated with non-inertial frames) to reveal a correct indication of ACs, both at resonance and off-resonant situations. In addition, as a further example, we discuss how the validation mechanism through non-inertial frames can be useful to describe the results presented in Ref. [23], where the adiabatic dynamics of a single spin-1/2 in NMR had been previously investigated. More general settings, such as decoherence effects, are left for future research.

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Proof of Theorem 1

Let us consider two Hamiltonians, an inertial frame Hamiltonian $H(t)$ and its non-inertial counterpart $H_0(t)$, which are related by a time-dependent unitary $O(t)$. The dynamics associated with Hamiltonians $H(t)$ and $H_0(t)$ are given by

$$\dot{\rho}(t) = \frac{1}{\hbar} [H(t), \rho(t)], \quad (S1)$$

$$\dot{\rho}_0(t) = \frac{1}{\hbar} [H_0(t), \rho_0(t)], \quad (S2)$$

where $H_0(t) = O(t)H(t)O^\dagger(t) + i\hbar \dot{O}(t)O^\dagger(t)$ and $\rho_0(t) = O(t)\rho(t)O^\dagger(t)$. Then, the connection between the evolved states $|\psi(t)\rangle$ and $|\psi_0(t)\rangle$ in inertial and non-inertial frames, respectively, is given by $|\psi_0(t)\rangle = O(t)|\psi(t)\rangle$, $\forall t \in [t_0, \tau]$. By considering the initial state in inertial frame given by a single eigenstate of $H(t)$, namely $\omega \equiv |\phi(t_0)\rangle = |E_k(t_0)\rangle$, the adiabatic dynamics in this frame is written as

$$|\psi(t)\rangle = e^{i \int_{t_0}^{t} \theta(t') dt'} |E_k(t)\rangle, \quad (S3)$$

where $\theta(t) = -E_k(t)/\hbar + i(E_k(t))(d/dt)|E_k(t)\rangle$ is the adiabatic phase collected in this frame. The dynamics will be adiabatic if and only if

$$|E_m(t)|\langle \psi(t)| = |E_m(t_0)|\langle \psi_0(t_0)|, \quad \forall m, \forall t \in [t_0, \tau]. \quad (S4)$$

Therefore, we can write

$$|E_m^O(t)|\langle \psi_0(t)| = |E_m^O(t_0)|\langle \psi_0(t_0)|, \quad (S5)$$

Thus, Eq. (S5) establishes a necessary and sufficient condition to obtain an adiabatic evolution in the non-inertial frame, assuming an adiabatic evolution in the original frame. To conclude our proof, let us consider the converse case, where the system starts in a eigenstate of $|E_m^O(t_0)\rangle$ in non-inertial frame. If the dynamics is adiabatic we write

$$|\psi_O(t)\rangle = e^{i \omega_{\tau} (t_{\tau})} |E_m^O(t)\rangle, \quad (S6)$$

where $\omega_{\tau}(t)$ is the adiabatic phase collected in this frame. This dynamics will be adiabatic in the inertial frame if and only if

$$|E_m(t)|\langle \psi(t)| = |E_m(t_0)|\langle \psi(t_0)|, \quad \forall m, \forall t \in [t_0, \tau]. \quad (S7)$$

Therefore, by using the same procedure as before, we get the condition

$$|E_k^O(t)|\langle \psi(t)| = |E_k^O(t_0)|\langle \psi_0(t_0)|, \quad (S8)$$

which is equivalent to Eq. (S5). This ends the proof of Theorem 1.

Application of Theorem 1 to the Hamiltonian in Eq. (6)

Let us consider a generic system under action of a single time-dependent oscillating/rotating field with characteristic frequency $\omega$, whose Hamiltonian reads

$$H(\omega, t) = \hbar \omega H_0 + \hbar \omega T H_T(\omega, t), \quad (S9)$$

where we consider the transverse term $\omega T H_T(\omega, t)$ as a perturbation, so that $|\omega T H_0| \gg |\omega T H_T(\omega, t)|, \forall t \in [0, \tau]$. In this case, the eigenstates $|E_n(t)\rangle$ and energies $E_n(t)$ of $H(\omega, t)$ can be obtained as perturbation of eigenstates $|E_n^0\rangle$ and energies $E_n^0$ of $\hbar \omega H_0$ as (up to a normalization coefficient)

$$|E_n(t)\rangle = |E_n^0\rangle + O(|\hbar \omega T H_T(\omega, t)|), \quad (S10)$$

$$E_n(t) = E_n^0 + O(|\hbar \omega T H_T(\omega, t)|). \quad (S11)$$

On the other hand, in the non-inertial frame, we have $H_O(t) = O(t)H(t)O^\dagger(t) + i\hbar \dot{O}(t)O^\dagger(t)$, which yields

$$H_O(\omega, t) = \hbar (\omega_0 - \omega) H_0 + \hbar \omega T H_{O,T}(\omega, t). \quad (S12)$$
where $H_{O,T}(\omega, t) = O(t)H_T(\omega, t)O^\dagger(t)$. Now, we separately consider two specific cases:

- **Far-from resonance situation** $|\omega_0 - \omega| \gg |\omega_T|$: In this case, the term $\hbar \omega_T H_{O,T}(\omega, t)$ in Eq. (S12) works as a perturbation. Therefore the set of eigenvectors of $H_0(\omega, t)$ reads

$$|E_n^0(t)\rangle = |E_n^0\rangle + \mathcal{O}(\hbar \omega_T H_T(\omega, t)),$$  \hspace{1cm} (S13)

where we have used that the energy gaps $E_n^0 - E_0^0$ of the Hamiltonian $\hbar (\omega - \omega) H_0$ are identical to energy gaps $E_n^0 - E_0^0$ of $\hbar \omega_T H_0$ and $\|\hbar \omega_T H_{O,T}(\omega, t)\| = \|\hbar \omega_T H_T(\omega, t)\|$. Thus, from Eqs. (S10) and (S13) we conclude, for any eigenstate $|E_k(t)\rangle$,

$$\langle E_m^0(t)|O(t)|E_k(t)\rangle \approx e^{i\frac{\delta m}{\hbar}\sqrt{\tau}}\delta_{mk},$$ \hspace{1cm} (S14)

so that we get $|\langle E_m^0(t)|O(t)|E_k(t)\rangle| = \text{constant}$. \forall m, \forall t \in [t_0, \tau].

- **Resonance situation** $|\omega_0 - \omega| \ll |\omega_T|$: Now, we have a more subtle situation. Firstly, we can use Eqs. (S10) and (S11) to write

$$O(t)|E_n(t)\rangle = e^{i\frac{\delta_n}{\hbar}\sqrt{\tau}}|E_n^0\rangle + \mathcal{O}(\hbar \omega_T H_T(\omega, t)),$$ \hspace{1cm} (S15)

$$\int_{t_0}^{t} \delta_n(\xi) d\xi = -\frac{E_n^0}{\hbar}(t - t_0) + \mathcal{O}(\hbar \omega_T H_T(\omega, t)),$$ \hspace{1cm} (S16)

so that

$$\langle E_m^0(t)|O(t)|E_k(t)\rangle \approx e^{i\frac{\delta_m}{\hbar}\sqrt{\tau}}\langle E_m^0(t)|E_k^0\rangle.$$ \hspace{1cm} (S17)

Now, it is possible to see that if $|\langle E_m^0(t)|E_k^0\rangle| = |\langle E_m^0(t_0)|E_k^0\rangle|$, \forall $t \in [t_0, \tau]$, then we obtain $|\langle E_m^0(t)|O(t)|E_k(t)\rangle| = \text{constant}$. Therefore the set of eigenvectors of $H_0(\omega, t)$ reads

$$|E_n^0(t)\rangle = |E_n^0\rangle + \mathcal{O}(\hbar \omega_T H_T(\omega, t)),$$ \hspace{1cm} (S13)

Moreover, assuming adiabatic dynamics in the inertial frame, we get

$$|\langle E_k(t)|\psi(t)\rangle| = |\langle E_k(t_0)|\psi(t_0)\rangle|.$$ \hspace{1cm} (S19)

By using the relationship between inertial and non-inertial frames as $|\psi(t)\rangle = O(t)|\psi(t)\rangle$, we can write

$$|\langle E_k(t)|O(t)|\psi(t)\rangle| = |\langle E_k(t_0)|\psi(t_0)\rangle|.$$ \hspace{1cm} (S20)

Thus, by inserting the initial state $|\psi(t_0)\rangle = |E_n(t_0)\rangle$ in Eq. (S21), we get

$$|\langle E_k(t)|O(t)|e^{-i\hbar \omega_T(t-t_0)}O(t_0)|E_n(t_0)\rangle| = |\langle E_k(t)|E_n(t_0)\rangle|.$$ \hspace{1cm} (S22)

This concludes the proof of Theorem 2.