On competition indices and periods of multipartite tournaments

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Abstract

In this paper, we compute competition indices and periods of multipartite tournaments. We first show that the competition period of an acyclic digraph $D$ is one and $\zeta(D) + 1$ is a sharp upper bound of the competition index of $D$ where $\zeta(D)$ is the sink elimination index of $D$. Then we prove that, especially, for an acyclic $k$-partite tournament $D$, the competition index of $D$ is $\zeta(D)$ or $\zeta(D) + 1$ for an integer $k \geq 3$. By developing useful tools to create infinitely many directed walks in a certain regular pattern from given directed walks, we show that the competition period of a multipartite tournament with sinks and directed cycles is at most three. We also prove that the competition index of a primitive digraph does not exceed its exponent.

Keywords. $m$-step competition graph, multipartite tournament, competition index, competition period, sink sequence, exponent of a primitive digraph

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1 Introduction

The underlying graph of each digraph in this paper is assumed to be simple unless otherwise mentioned.

Given a digraph $D$, the \textit{competition graph} $C(D)$ of $D$ has the same vertex set as $D$ and has an edge between vertices $u$ and $v$ if and only if there exists a common prey of $u$ and $v$ in $D$. If $(u, v)$ is an arc of a digraph $D$, then we call $v$ a prey of $u$ (in $D$) and call $u$ a predator of $v$ (in $D$). The notion of competition graph is due to Cohen \cite{8} and has arisen from ecology. Competition graphs also have applications in coding, radio transmission, and modeling of complex economic systems. (See \cite{21} and \cite{22} for a summary of these applications.) Various variants of notion of competition graphs have been introduced and
of a digraph $D$, denoted by $C^m(D)$, has the same vertex set as $D$ and has an edge between vertices $u$ and $v$ if and only if there exists an $m$-step common prey of $u$ and $v$. The notion of $m$-step competition graph is introduced by Cho et al. [7] as a generalization of competition graph. By definition, it is obvious that $C^1(D)$ is the competition graph $D$. Since its introduction, it has been extensively studied (see for example [2, 6, 12, 13, 14, 20, 24]). Cho et al. [7] showed that for any digraph $D$ and a positive integer $m$, $C^m(D) = C(D^m)$.

For the two-element Boolean algebra $\mathcal{B} = \{0, 1\}$, $\mathcal{B}_n$ denotes the set of all $n \times n$ (Boolean) matrices over $\mathcal{B}$. Under the Boolean operations, we can define matrix addition and multiplication in $\mathcal{B}_n$. A graph $G$ is called the row graph of a matrix $A \in \mathcal{B}_n$ and denoted by $R(A)$ if the rows of $A$ are the vertices of $G$, and two vertices are adjacent in $G$ if and only if their corresponding rows have a nonzero entry in the same column of $A$. This notion was studied by Greenberg et al. [11]. As noted in [11], the competition graph of a digraph $D$ is the row graph of its adjacency matrix.

Cho and Kim [5] introduced the notions of competition index and competition period of $D$ for a strongly connected digraph $D$, and Kim [14] extended these notions to a general digraph $D$. Consider the graph sequence $C^1(D), C^2(D), C^3(D), \ldots, C^m(D), \ldots$ for a digraph $D$. Note that for a digraph $D$ and its adjacency matrix $A$, the graph sequence $C^1(D), C^2(D), \ldots, C^m(D), \ldots$ is equivalent to the row graph sequence $R(A), R(A^2), \ldots, R(A^m), \ldots$. Since the cardinality of the Boolean matrix set $\mathcal{B}_n$ is equal to a finite number $2^{n^2}$, there is a smallest positive integer $q$ such that $C^{q+i}(D) = C^{q+r+i}(D)$ equivalently $R(A^{q+i}) = R(A^{q+r+i})$ for some positive integer $r$ and all nonnegative integers $i$. Such an integer $q$ is called the competition index of $D$ and is denoted by cindex($D$). For $q = \text{cindex}(D)$, there is also a smallest positive integer $p$ such that $C^{q}(D) = C^{q+p}(D)$ equivalently $R(A^q) = R(A^{q+p})$. Such an integer $p$ is called the competition period of $D$ and is denoted by cperiod($D$). Refer to [15, 16, 17] for some results of competition indices and competition periods of digraphs.

Eoh et al. [9] studied the $m$-step competition graphs of orientations of complete bipartite graphs for an integer $m \geq 2$. They introduced a notion of sink sequences of digraphs, which played a key role in the paper. Given a digraph $D$, we call a vertex of outdegree zero a sink in $D$. We define a nonnegative integer $\zeta(D)$ and sequences

$$\{W_0, W_1, \ldots, W_{\zeta(D)}\} \quad \text{and} \quad \{D_0, D_1, \ldots, D_{\zeta(D)}\}$$
of subsets of \( V(D) \) and subdigraphs of \( D \), respectively, as follows. Let \( D_0 = D \) and \( W_0 \) be the set of sinks in \( D \). If \( W_0 = V(D) \) or \( W_0 = \emptyset \), then let \( \zeta(D) = 0 \). Otherwise, let \( D_1 = D_0 - W_0 \) and let \( W_1 \) be the set of sinks in \( D_1 \). If \( W_1 = V(D_1) \) or \( W_1 = \emptyset \), then let \( \zeta(D) = 1 \). Otherwise, let \( D_2 = D_1 - W_1 \) and let \( W_2 \) be the set of sinks in \( D_2 \). If \( W_2 = V(D_2) \) or \( W_2 = \emptyset \), then let \( \zeta(D) = 2 \). We continue in this way until we obtain \( W_k = V(D_k) \) or \( W_k = \emptyset \) for some nonnegative integer \( k \). Then we let \( \zeta(D) = k \). By definition, \( 0 \leq \zeta(D) \leq |V(D)| - 1 \). We call \( \zeta(D) \) the sink elimination index of \( D \), the sequence \((W_0, W_1, \ldots, W_{\zeta(D)})\) the sink sequence of \( D \), and the sequence \((D_0, D_1, \ldots, D_{\zeta(D)})\) the digraph sequence associated with the sink sequence of \( D \).

In this paper, we study competition indices and competition periods of multipartite tournaments in terms of sink sequences of digraphs. A \( k \)-partite tournament is an orientation of a complete \( k \)-partite graph for a positive integer \( k \). In particular, if \( k \geq 2 \), then we call it a multipartite tournament.

Two vertices \( u \) and \( v \) in a digraph \( D \) are said to be strongly connected if there are directed walks from \( u \) to \( v \) and from \( v \) to \( u \). We say that a digraph \( D \) is strongly connected if each pair in \( V(D) \) is strongly connected. A digraph \( D \) is said to be primitive if \( D \) is strongly connected and the greatest common divisor of lengths of its directed cycles is equal to 1. It is known \([4]\) that if a digraph \( D \) is primitive, then there exists a positive integer \( t \) such that there is a directed walk of length exactly \( t \) from each vertex \( u \) to each vertex \( v \) (possibly \( u = v \)). The smallest integer \( t \) is called the exponent of the primitive digraph \( D \) and it is denoted by \( \exp(D) \).

In Section 2 we deal with acyclic multipartite tournaments. We show that the competition period of an acyclic digraph \( D \) is one and \( \zeta(D) + 1 \) is a sharp upper bound of the competition index of \( D \) (Theorem 2.3). Especially, it turns out that the competition index of an acyclic \( k \)-partite tournament \( D \) is \( \zeta(D) \) or \( \zeta(D) + 1 \) for an integer \( k \geq 3 \) (Theorem 2.8). In Section 3 we handle multipartite tournaments with sinks and directed cycles. We introduce types of a directed walk and types of a vertex in \( \bigcup_{i=0}^{\zeta(D)-1} W_i \) where \((W_0, \ldots, W_{\zeta(D)})\) is the sink sequence of a multipartite tournament \( D \) with sinks and directed cycles, and then show that each vertex in \( \bigcup_{i=0}^{\zeta(D)-1} W_i \) is of Type 1 or Type 2 (Theorem 3.8). We show that the existence of \((u, w)\)-directed walk of Type 1 or Type 2 of a certain length for a vertex \( u \) in \( D_{\zeta(D)} \) and \( w \in \bigcup_{i=0}^{\zeta(D)-1} W_i \) guarantees the existence of a \((u, w)\)-directed walk of length \( m \) in \( D \) for infinitely many integers \( m \) in a specific form (Lemmas 3.4-3.7). By integrating these results, we show that the competition period of a multipartite tournament with sinks and directed cycles is at most three (Theorem 3.9). In Section 4 we show that the competition index of a primitive digraph is at most its exponent (Theorem 4.2). In Section 5 we take care of it to eventually compute the competition period and competition index of a tournament with sinks (Theorem 5.4).

For a positive integer \( n \), we denote the set \( \{1, 2, \ldots, n\} \) by \([n]\) for simplicity.
2 Acyclic multipartite tournaments

In this section, we compute the competition period and competition index of an acyclic multipartite tournament.

Proposition 2.1 ([9]). For a digraph $D$, the following are equivalent.

(i) $D$ is acyclic.

(ii) $W_{\zeta(D)} = V(D_{\zeta(D)}) \neq \emptyset$.

(iii) $\bigcup_{i=0}^{\zeta(D)} W_i = V(D)$.

The following lemma is a stronger version of Proposition 2.3 given by Eoh et al. [9] in the sense that, regarding the set $\mathcal{L}$ of lengths of directed walks with an initial vertex in $W_i$, $\max \mathcal{L} \leq i$ is replaced with $\mathcal{L} = \{0, \ldots, i\}$. As a matter of fact, their proof asserted this stronger version.

Lemma 2.2 (Restatement of Proposition 2.3 in [9]). Let $D$ be a digraph with $\zeta(D) \geq 1$ and $(W_0, \ldots, W_{\zeta(D)})$ be the sink sequence of $D$. Then, for each $i = 0, \ldots, \zeta(D) - 1$ and each vertex $v$ in $W_i$, among the directed walks starting from $v$, there exist directed walks of lengths $0, \ldots, i$ and no directed walks of length greater than $i$. Furthermore, if $D$ is acyclic, then the statement is true for even $i = \zeta(D)$.

Theorem 2.3. Let $D$ be an acyclic digraph and $(W_0, \ldots, W_{\zeta(D)})$ be its sink sequence. Then, for $\zeta(D) \geq 1$, we have the following:

(i) $C^m(D)$ is an empty graph for any integer $m > \zeta(D)$;

(ii) $\text{cperiod}(D) = 1$;

(iii) if $|W_i| = 1$ for some integer $i \in \{0, \ldots, \zeta(D) - 1\}$, then $C^{\zeta(D)}(D)$ is the union of the complete graph with the vertex set $W_{\zeta(D)}$ and the empty graph with the vertex set $V(D) \setminus W_{\zeta(D)}$;

(iv) $c_{\text{index}}(D) \leq \zeta(D) + 1$ where the equality holds if $|W_{\zeta(D)}| > |W_i|$ for some integer $i \in \{0, \ldots, \zeta(D) - 1\}$.

Proof. Take an integer $m > \zeta(D)$. Since $D$ is acyclic, $\bigcup_{i=0}^{\zeta(D)} W_i = V(D)$ by Proposition 2.1 and so no vertex in $D$ has an $m$-step prey in $D$ by Lemma 2.2. Therefore $C^m(D)$ is an empty graph and so the statement (i) is true. Then, by the definition of competition period, the statement (ii) is immediately true.

To show the statement (iii), suppose that $|W_i| = 1$ for some integer $i \in \{0, \ldots, \zeta(D) - 1\}$. Let $W_i = \{w\}$. Since every vertex in $W_j$ has at least one out-neighbor in $W_{j-1}$ for each integer $1 \leq j \leq \zeta(D)$, there is a directed walk of length $\zeta(D) - i$ from $v$ to $w$ for each
vertex $v \in W_{\zeta(D)}$. Moreover, there is a directed walk of length $i$ from $w$ to a vertex $u \in W_0$. By concatenating those directed walks, we obtain a directed walk of length $\zeta(D)$ from each vertex in $W_{\zeta(D)}$ to $u$. Thus $W_{\zeta(D)}$ forms a clique in $C^{(D)}(D)$. By Proposition 2.1(iii), every vertex in $V(D) \setminus W_{\zeta(D)}$ belongs to $W_j$ for some $j \in \{0, \ldots, \zeta(D) - 1\}$. Therefore every vertex in $V(D) \setminus W_{\zeta(D)}$ is isolated in $C^{(D)}(D)$ by Lemma 2.2 Thus the statement (iii) is true.

The inequality $\text{cindex}(D) \leq \zeta(D) + 1$ immediately follows from (i). To figure out when the equality holds, suppose that $|W_{\zeta(D)}| > |W_i|$ for some integer $i \in \{0, \ldots, \zeta(D) - 1\}$. As we have observed above, there are at least $|W_{\zeta(D)}|$ directed walks starting from distinct vertices in $W_{\zeta(D)}$ to a vertex in $W_i$. Since $|W_{\zeta(D)}| > |W_i|$, there are at least two directed walks terminating at the same vertex in $W_i$ by the pigeonhole principle. Then the origins of those directed walks form a clique in $C^{(D)}(D)$. Thus $C^{(D)}(D)$ is not an empty graph. Hence, by the definition of competition index and (i), $\text{cindex}(D) = \zeta(D) + 1$.

It is easy to check that an acyclic multipartite tournament has sink elimination index $\zeta(D) \geq 1$.

**Theorem 2.4.** For an integer $k \geq 2$, let $D$ be an acyclic $k$-partite tournament with a $k$-partition $(V_1, \ldots, V_k)$. Then $(W_0, \ldots, W_{\zeta(D)})$ is the sink sequence of $D$ if and only if $(W_0, \ldots, W_{\zeta(D)})$ is a partition of $V(D)$ satisfying the following:

(i) for each $i = 0, \ldots, \zeta(D)$, $W_i$ is a subset of a partite set of $D$;

(ii) if there is an arc from a vertex in $W_j$ to a vertex in $W_i$ for some $i, j \in \{0, \ldots, \zeta(D)\}$, then $i < j$ and $W_i$ and $W_j$ are included in different partite sets.

(iii) for each $i = 0, \ldots, \zeta(D) - 1$, there is an arc from each vertex in $W_{i+1}$ to each vertex in $W_i$.

**Proof.** Suppose that $(W_0, \ldots, W_{\zeta(D)})$ is the sink sequence of $D$. Let $(D_0, \ldots, D_{\zeta(D)})$ be the digraph sequence associated with it. Suppose, to the contrary, that there are two vertices $u, v \in W_i$ for some $i \in \{0, \ldots, \zeta(D)\}$ such that $u \in V_p$ and $v \in V_q$ with $p \neq q$. Since $D$ is a $k$-partite tournament, $(u, v)$ or $(v, u)$ is an arc in $D$. Without loss of generality, we may assume that $(u, v)$ is an arc in $D$. By the definition of $D_i$, $(u, v)$ is an arc in $D_i$, which contradicts the assumption that $u \in W_i$. Thus, for each integer $0 \leq i \leq \zeta(D)$, $W_i \subseteq V_s$ for some $s \in [k]$ and so the statement (i) is true.

Suppose there is an arc from a vertex $W_j$ to a vertex $W_i$ for some $i, j \in \{0, \ldots, \zeta(D)\}$. Then, by the definition of sink sequence, $i < j$. By (i), $W_i \subseteq V_p$ and $W_j \subseteq V_q$ for some $p, q \in [k]$. Since an arc goes from $W_j$ to $W_i$, $p \neq q$ and so the statement (ii) is true.

By definition, $W_i$ and $W_{i+1}$ are not empty sets for each $i = 0, \ldots, \zeta(D) - 1$, so there exists an arc from a vertex in $W_{i+1}$ to a vertex in $W_i$. Therefore by (ii), $W_i$ and $W_{i+1}$ are included in different partite sets. Since $D$ is a multipartite tournament, there is an arc from each vertex in $W_{i+1}$ to each vertex in $W_i$. Hence (iii) is true and so the ‘only if’ part is valid.
Conversely, consider a partition \((W_0, \ldots, W_{\zeta(D)})\) of \(V(D)\) satisfying (i), (ii), and (iii). By (ii), every vertex in \(W_0\) is a sink of \(D\). Suppose that there is a sink \(v\) of \(D\) in \(V(D) \setminus W_0\). Then, since \(\bigcup_{j=0}^{\zeta(D)} W_j = V(D), v \in W_i\) for some integer \(i \in \{\zeta(D)\}\). By (iii), \(W_i\) and \(W_{i-1}\) are included in distinct partite sets. Since \(D\) is a multipartite tournament and (ii) is true, there is an arc from \(v\) to a vertex in \(W_{i-1}\), which contradicts the choice of \(v\). Therefore \(W_0\) is the set of sinks in \(D\). By applying a similar argument to \(W_1\) of the subdigraph \(D_1\) induced by \(V(D) \setminus W_0\), we may show that \(W_1\) is the set of sinks of \(D_1\). Inductively, we may show that \(W_i\) is the set of sinks of the subdigraph of \(D\) induced by \(V(D) \setminus \bigcup_{j=0}^{i-1} W_j\) for \(i = 2, \ldots, \zeta(D)\). Since \((W_0, \ldots, W_{\zeta(D)})\) is a partition of \(V(D)\), we may conclude that \((W_0, \ldots, W_{\zeta(D)})\) is the sink sequence of \(D\) and so the ‘if’ part is true.

\[\square\]

**Corollary 2.5.** For an integer \(k \geq 2\), let \(D\) be an acyclic \(k\)-partite tournament with a \(k\)-partition \((V_1, \ldots, V_k)\). If \((W_0, \ldots, W_{\zeta(D)})\) is the sink sequence of \(D\), then \(\zeta(D) \geq k - 1\) where the equality holds if and only if \(W_i\) is a partite set of \(D\) for each \(i = 0, \ldots, \zeta(D)\);

**Proof.** Since \(D\) is acyclic, \(\bigcup_{i=0}^{\zeta(D)} W_i = V(D)\) by Proposition 2.1. By Theorem 2.4(i), \(W_i\) is a subset of a partite set of \(D\) for each integer \(0 \leq i \leq \zeta(D)\). If \(\zeta(D) < k - 1\), then there exists a partite set not containing \(W_i\) for any \(i \in \{0, \ldots, \zeta(D)\}\), which is impossible. Thus \(\zeta(D) \geq k - 1\). It is obvious that \(\zeta(D) = k - 1\) if and only if \((W_0, \ldots, W_{\zeta(D)}) = \{V_1, \ldots, V_k\}\).

If \(D\) is an acyclic \(k\)-partite tournament for an integer \(k \geq 3\), then \(\zeta(D) \geq 2\) by Corollary 2.5 and we have the following result.

**Corollary 2.6.** Let \(D\) be an acyclic \(k\)-partite tournament for an integer \(k \geq 3\) and let \((W_0, \ldots, W_{\zeta(D)})\) be the sink sequence of \(D\). Then there is an integer \(i \in \{0, \ldots, \zeta(D) - 2\}\) such that \(W_i\) and \(W_{i+2}\) are included in different partite sets.

**Proof.** Let \((V_1, \ldots, V_k)\) be a \(k\)-partition of \(D\). By Theorem 2.4(iii), \(W_0\) and \(W_1\) are included in different partite sets. Without loss of generality, we may assume that \(W_0 \subseteq V_1\) and \(W_1 \subseteq V_2\). Suppose, to the contrary, that \(W_i\) and \(W_{i+2}\) are included in the same partite set for each integer \(0 \leq i \leq \zeta(D) - 2\). Then

\[\bigcup_{0 \leq i \leq \zeta(D)/2} W_{2i} \subseteq V_1 \quad \text{and} \quad \bigcup_{0 \leq i \leq \zeta(D) - 1/2} W_{2i+1} \subseteq V_2.\]

Therefore \(\bigcup_{i=0}^{\zeta(D)} W_i \subseteq V_1 \cup V_2\). Since \(k \geq 3\), \(V_3 \neq \emptyset\). Thus \(\bigcup_{i=0}^{\zeta(D)} W_i \subseteq V_1 \cup V_2 \cup V_3 \subseteq V(D)\), which contradicts Proposition 2.1(iii).

\[\square\]

**Lemma 2.7.** Let \(D\) be an acyclic \(k\)-partite tournament for an integer \(k \geq 3\) and let \((W_0, \ldots, W_{\zeta(D)})\) be the sink sequence of \(D\). For each integer \(1 \leq s \leq \zeta(D)\), if there exist integers \(p, q \in \{0, \ldots, s\}\) with \(p > q\) such that \(W_p\) and \(W_q\) are included in different partite sets, then there exists a directed path of length \(s - p + q + 1\) from each vertex in \(W_s\) to each vertex in \(W_0\).
Proof. Fix $s \in \{1, \ldots, \zeta(D)\}$. Suppose that there exist integers $p, q \in \{0, \ldots, s\}$ with $p > q$ such that $W_p$ and $W_q$ are included in different partite sets. Take vertices $u \in W_s$ and $x \in W_0$. Now we take two vertices $v \in W_p$ and $w \in W_q$ so that $u = v$ if $p = s$ and $w = x$ if $q = 0$. Since $D$ is a multipartite tournament, every vertex in $W_{i-1}$ is an out-neighbor of each vertex in $W_i$ for each integer $1 \leq i \leq \zeta(D)$. Therefore, there exist a $(u, v)$-directed path $P$ of length $s - p$ and a $(w, x)$-directed path $Q$ of length $q$. Since $W_p$ and $W_q$ are included in different partite sets, $v$ and $w$ are linked by an arc. Then, by Theorem 2.4(ii), $(v, w)$ is an arc of $D$. Thus $P \rightarrow Q$ is a $(u, x)$-directed path of length $(s - p) + 1 + q$. Since $u$ and $x$ were arbitrarily chosen, the statement is true.

We recall that if $D$ is an acyclic $k$-partite tournament for an integer $k \geq 3$, then $\zeta(D) \geq 2$.

Theorem 2.8. For an integer $k \geq 3$, let $D$ be an acyclic $k$-partite tournament with the sink sequence $(W_0, \ldots, W_{\zeta(D)})$. Then the following are true:

(i) $C^{(D)}(D)$ is the union of the complete graph with the vertex set $W_{\zeta(D)}$ and the empty graph with the vertex set $V(D) \setminus W_{\zeta(D)}$;

(ii) $C^{(D)}_1(D)$ is the union of the complete graph with the vertex set $W_{\zeta(D)} \cup W_{\zeta(D)-1}$ and the empty graph with the vertex set $\bigcup_{i=0}^{\zeta(D)-2} W_i$;

(iii) if $|W_{\zeta(D)}| \geq 2$, then $\text{cin}(D) = \zeta(D) + 1$, otherwise $\text{cin}(D) = \zeta(D)$.

Proof. Take a vertex $z \in W_0$. By Theorem 2.4(iii), $W_0$ and $W_1$ are included in different partite sets. Then, since $0 < 1 \leq \zeta(D) - 1$, by Lemma 2.4 there exist

(a) a directed path of length $\zeta(D)$ from any vertex in $W_{\zeta(D)}$ to $z$ and

(b) a directed path of length $\zeta(D) - 1$ from any vertex in $W_{\zeta(D)-1}$ to $z$.

By (a), $W_{\zeta(D)}$ forms a clique in $C^{(D)}(D)$. By Lemma 2.4 every vertex in $\bigcup_{i=0}^{\zeta(D)-1} W_i$ is isolated in $C^{(D)}(D)$. By Proposition 2.3 $\bigcup_{i=0}^{\zeta(D)-1} W_i = V(D) \setminus W_{\zeta(D)}$ and so the statement (i) is true.

Since $D$ is a $k$-partite tournament with $k \geq 3$, there is an integer $i \in \{0, \ldots, \zeta(D) - 2\}$ such that $W_i$ and $W_{i+2}$ are included in different partite sets by Corollary 2.6. Thus, by Lemma 2.4 there is a directed path of length $\zeta(D) - 1$ from each vertex in $W_{\zeta(D)}$ to $z$. Hence, by (b), $z$ is a $(\zeta(D) - 1)$-step common prey of each vertex in $W_{\zeta(D)} \cup W_{\zeta(D)-1}$ and so $W_{\zeta(D)} \cup W_{\zeta(D)-1}$ forms a clique in $C^{(D)}_1(D)$. By Lemma 2.4 every vertex in $\bigcup_{i=0}^{\zeta(D)-2} W_i$ is isolated in $C^{(D)}_1(D)$ and so, by Proposition 2.3 the statement (ii) is true.

Suppose $|W_{\zeta(D)}| \geq 2$. Then, by the statement (i), $C^{(D)}_1(D)$ is not empty. Since $C^m(D)$ is empty for each integer $m > \zeta(D)$ by Theorem 2.3(i), $\text{cin}(D)$ is $\zeta(D) + 1$. Now suppose $|W_{\zeta(D)}| \leq 1$. Then, since $D$ is acyclic, $|W_{\zeta(D)}| = 1$ and so, by the statements (i) and (ii), $C^{(D)}_1(D)$ is empty and $C^{(D)}_1(D)$ is not empty, respectively. Since $C^m(D)$ is empty for each integer $m > \zeta(D)$ by Theorem 2.3(i), $\text{cin}(D)$ is $\zeta(D)$. Therefore the statement (iii) is true.

\[ \square \]
3 Multipartite tournaments with sinks and directed cycles

If a digraph $D$ is acyclic, then $C^m(D)$ is empty for any integer $m > \zeta(D)$ and so the competition period of $D$ is 1 (Theorem 2.3). In addition, Cho and Kim [5] showed that a digraph without sinks has competition period 1. In this vein, this section studies the competition period of a multipartite tournament having a sink and a directed cycle. By the way, Eoh et al. [9] showed that if $C^M(D)$ is an empty graph for a digraph $D$ and a positive integer $M$, then so is $C^m(D)$ for any positive integer $m \geq M$. Therefore, if $C^M(D)$ is an empty graph for a digraph $D$ and a positive integer $M$, then the competition period of $D$ is 1. As a matter of fact, in the case of a multipartite tournament $D$, having a sink and a directed cycle guarantees that $C^m(D)$ is not empty for every positive integer $m$ by the following proposition.

**Proposition 3.1.** If a multipartite tournament $D$ has a sink and a directed cycle, then $C^m(D)$ is not empty for every positive integer $m$.

**Proof.** Suppose that a $k$-partite tournament $D$ for an integer $k \geq 2$ has a sink $x$ and a directed cycle $C := v_0v_1 \cdots v_{l-1}v_0$ for some integer $l \geq 3$. Let $(V_1, \ldots, V_k)$ be a $k$-partition of $D$. Without loss of generality, we may assume that $x \in V_1$. Since $V_1$ is a partite set of $D$, there exist at least two vertices of $V(C) \setminus V_1$. Let $v_i$ and $v_j$ be vertices of $V(C) \setminus V_1$ for some two distinct integers $i, j \in \{0, 1, \ldots, l-1\}$. Since $D$ is a $k$-partite tournament and $x$ is a sink, $(v_i, x)$ and $(v_j, x)$ are arcs of $D$. Then $v_i$ and $v_j$ are adjacent in $C(D)$. Since $v_i$ and $v_j$ are on $C$, $v_{i-m+1}$ and $v_{j-m+1}$ are adjacent in $C^m(D)$ for every positive integer $m$ (all the subscripts are reduced to modulo $l$). Hence $C^m(D)$ is not empty for every positive integer $m$. ~

In the following, we shall show that if a multipartite tournament $D$ has a sink and a directed cycle, then the competition period of $D$ is at most three, which is our main result. To do so, we need several theorems and lemmas.

**Theorem 3.2 ([10]).** Let $D$ be a $k$-partite tournament with $k \geq 3$. Then $D$ contains a directed cycle of length 3 if and only if there exists a directed cycle in $D$ which contains vertices from at least three partite sets.

Given a digraph $D$, we call a directed cycle of length at least 4 in $D$, a directed hole if it is an induced subdigraph of $D$.

We note that a directed hole of length 4 in a multipartite tournament $D$ is of the form $v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_0$ such that $\{v_0, v_2\} \subseteq X$ and $\{v_1, v_3\} \subseteq Y$ for some distinct partite sets $X$ and $Y$ of $D$.

Given a multipartite tournament $D$, if a directed walk contains vertices which induce a directed cycle of length 3 in $D$ (resp. a directed hole of length 4 in $D$), then we say that it is of Type 1 (resp. Type 2).
Lemma 3.3. Let $D$ be a multipartite tournament of order $n \geq 3$. Then any directed walk of length at least $n$ in $D$ is of Type 1 or Type 2.

Proof. Let $Q$ be a directed walk of length at least $n$ in $D$. Since the length of $Q$ is greater than or equal to the number of vertices of $D$, $Q$ contains a directed cycle $C := u_0 \to u_1 \to u_2 \to \cdots \to u_{l-1} \to u_0$ for an integer $l \geq 3$.

Case 1. There are at least 3 vertices on $C$ which belong to distinct partite sets in $D$. We may apply Theorem 3.2 to the multipartite tournament induced by $V(C)$ to conclude that there is a directed cycle of length 3 all of whose vertices are on $C$. Then it is easy to check that $Q$ is of Type 1.

Case 2. There exist two distinct partite sets $X$ and $Y$ in $D$ such that $V(C) \subseteq X \cup Y$. Then $l$ is even, so $l \geq 4$. Without loss of generality, we may assume that a vertex on $C$ with an even index belongs to $X$ and a vertex on $C$ with an odd index belongs to $Y$. Since $D$ is a $k$-partite tournament, there is an arc between $u_i$ and $u_{l-i-1}$ for each integer $0 \leq i \leq l/2-1$. Since $(u_{l-i}, u_0) \in A(D)$ and $C$ is a directed cycle, $(u_i, u_{l-i-1})$ is an arc in $D$ for some $i \in \{1, \ldots, l/2-1\}$. We may regard $i$ as the smallest index among $1, \ldots, l/2-1$ such that $(u_i, u_{l-i-1})$ is an arc in $D$. Then $(u_{l-i}, u_{l-i-1}) \in A(D)$ and $C' := u_{l-i} \to u_i \to u_{l-i-1} \to u_{l-i} \to u_{l-1}$ is a directed hole of length 4 in $D$. Thus $Q$ is of Type 2. \hfill $\square$

We note that a multipartite tournament has a sink if and only if the sink elimination index is greater than or equal to one. In the following five results prior to our main result, we examine lengths of directed walks from a vertex in $D$ to a vertex in $C$ with the sink elimination index $\zeta$.

Lemma 3.4. Let $D$ be a multipartite tournament. Suppose that there exists a $(u, v)$-directed walk of Type 1 of length $\ell$ for some vertices $u$ and $v$. Then there is a $(u, v)$-directed walk of length $\ell + 3m$ for each nonnegative integer $m$.

Proof. Let $Z$ be a $(u, v)$-directed walk of length $\ell$ that contains three vertices which induce a directed cycle $C$ of length 3 in $D$. Let $x$ be a vertex on $C$, $Z_1$ be a $(u, x)$-section of $Z$, and $Z_2$ be the $(x, v)$-section of $Z$ obtained by cutting $Z_1$ away from $Z$. We may assume that the sequence representing $C$ starts at $x$. Then, for a nonnegative integer $m$, we may create a $(u, v)$-directed walk of length $\ell + 3m$ in such a way that we traverse $Z_1$, $C$ as many as $m$ times, and then $Z_2$. \hfill $\square$

Lemma 3.5. Let $D$ be a multipartite tournament with the sink elimination index $\zeta(D) \geq 1$ and the sink sequence $(W_0, \ldots, W_{\zeta(D)})$. For a vertex $u$, suppose that there exists a $(u, w)$-directed walk $Z$ of Type 2 of length $\ell$ for some vertex $w$ such that (i) $w$ belongs to $\bigcup_{i=0}^{\zeta(D)-1} W_i$ or (ii) there exists a directed hole of length 4 such that its four vertices are on $Z$ and $w$ does not belong to any of two partite sets which the four vertices belong to. Then there exists a positive integer $N$ such that for each integer $m \geq N$, there is a $(u, w)$-directed walk of length $\ell + 2m$. 

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Proof. Since $Z$ is of Type 2, there are four vertices on $Z$ which induce a directed hole $H := v_0 \to v_1 \to v_2 \to v_3 \to v_0$ of length 4 in $D$. Then there exist two distinct partite sets $X$ and $Y$ of $D$ such that $\{v_0, v_2\} \subseteq X$ and $\{v_1, v_3\} \subseteq Y$. If the case (ii) of the lemma statement happens, we may assume that $H$ is the hole mentioned in the case. Let $Z_1$ be a $(u, v_0)$-section of $Z$, and $Z_2$ be the $(v_0, w)$-section of $Z$ obtained by cutting $Z_1$ away from $Z$. We denote the lengths of $Z_1$ and $Z_2$ by $\ell_1$ and $\ell_2$, respectively.

Case 1. There exists a vertex $z$ on $Z_2$ which does not belong to $X \cup Y$. Let $Z_3$ be a $(v_0, z)$-section of $Z_2$, and $Z_4$ be the $(z, w)$-section of $Z_2$ obtained by cutting $Z_3$ away from $Z_2$. We denote the lengths of $Z_3$ and $Z_4$ by $\ell_3$ and $\ell_4$, respectively. We consider the two subcases: $(v_i, z) \in A(D)$ for each $i = 0, 1, 2, 3$; there is an arc $(z, v)$ for some vertex $v$ on $H$. Suppose $(v_i, z) \in A(D)$ for each $i = 0, 1, 2, 3$ and fix a positive integer $\alpha$. Then, traverse the directed walk $Z_1$, go around $H$ as many times as desired, depart it at an appropriate vertex on $H$ to reach $z$ by an arc, and traverse $Z_4$. This creates a $(u, w)$-directed walk of length $\ell_1 + \alpha + \ell_4$.

Now suppose that there is an arc $(z, v)$ for some vertex $v$ on $H$. Let $D^*$ be the subdiagram of $D$ induced by $V(Z_3) \cup V(H)$. Then $D^*$ contains vertices from at least three partite sets. Since $D^*$ is a multipartite tournament, $D^*$ contains a directed cycle $C$ of length 3 by Theorem 5.2. Moreover, the directed walk $Z_3$, the arc $(z, v)$, and $(v, v_0)$-section of $H$ form a closed directed walk $Q$ in $D^*$. Then $V(Q) \cup V(H) = V(D^*)$ and $V(Q) \cap V(H) \neq \emptyset$, so $D^*$ is strongly connected. Since $D^*$ contains the directed cycles $C$ of length 3 and $H$ of length 4, we may conclude that $D^*$ is primitive. Thus, for any $\beta \geq \exp(D^*)$, there is a $(v_0, z)$-directed walk of length $\beta$ and so there is a $(u, w)$-directed walk of length $\ell_1 + \beta + \ell_4$.

Since $\ell_1 + \ell_4 \leq \ell$ and $\alpha$ and $\beta$ were arbitrarily chosen among the positive integers bounded below, we have shown in both subcases that there exists a sufficiently large $N$ such that there is a $(u, w)$-directed walk of length $\ell + 2m$ for each integer $m \geq N$.

Case 2. Every vertex on $Z_2$ belongs to $X \cup Y$. Then the case (i) of the lemma statement happens, that is, $w \in \bigcup_{i=0}^{\ell(D)-1} W_i$. Let $j = 0$ if $w \in Y$ and $j = 1$ if $w \in X$. Then $2m + j + 1$ and $\ell_2$ have the same parity. For, $\ell_2$ is odd if $w \in Y$ and $\ell_2$ is even if $w \in X$ since $v_0 \in X$ and $Z_2$ is a $(v_0, w)$-directed walk whose vertices belong to $X \cup Y$. Furthermore, $w$ and any of $v_j, v_{j+2}$ belong to distinct partite sets. Since $w \in \bigcup_{i=0}^{\ell(D)-1} W_i$ and $\{v_0, v_1, v_2, v_3\} \subseteq V(D_{\ell(D)})$, there are arcs $(v_j, w)$ and $(v_{j+2}, w)$ in $D$. Now let

$$\Theta_i = Z_1 \to H_i \to H' \to w$$

where $(v_0, v_j)$-section of $H$ and $H_i$ means the directed walk obtained by going around $H$ $i$ times, for a nonnegative integer $i$. Then $\Theta_i$ is a $(u, w)$-directed walk in $D$. For a nonnegative integer $i$, we denote by $\Lambda_i$ the directed walk obtained from $\Theta_i$ by replacing the arc $(v_j, w)$ with the directed path $v_j \to v_{j+1} \to v_{j+2} \to w$. Then the lengths of $\Theta_i$ and $\Lambda_i$ are $\ell_1 + 4i + j + 1$ and $\ell_1 + 4i + j + 3$, respectively, for each nonnegative integer $i$. Accordingly, we have shown that there is a $(u, w)$-directed walk of length $\ell_1 + 2m + j + 1$ for each nonnegative integer $m$. Since $2m + j + 1$ and $\ell_2$ have the same parity, we have
actually shown that there exists \((u, w)\)-directed walk of length \(\ell + 2m\) for each nonnegative integer \(m\).

Let \(p_1, \ldots, p_t\) be positive integers with \(\gcd(p_1, \ldots, p_t) = 1\). The Frobenius number of \(p_1, \ldots, p_t\) is the largest integer \(b\) for which the Frobenius equation

\[
p_1x_1 + \cdots + p_tx_t = b
\]

has no nonnegative integer solution \((x_1, \ldots, x_t)\). The number \(b\) is denoted by \(F(p_1, \ldots, p_t)\).

**Lemma 3.6.** Let \(D\) be a multipartite tournament with the sink elimination index \(\zeta(D) \geq 1\) and the sink sequence \((W_0, \ldots, W_{\zeta(D)})\). For a vertex \(u \in V(D_{\zeta(D)})\), suppose that there exists a \((u, w)\)-directed walk of length \(\ell\) for some vertex \(w \in \bigcup_{i=0}^{\zeta(D)-1} W_i\) such that its sequence contains both a directed cycle of length 3 and a directed hole of length 4. Then there exists a positive integer \(N\) such that for each integer \(m \geq N\), there is a \((u, w)\)-directed walk of length \(\ell + m\).

**Proof.** Let \(Z\) be a \((u, w)\)-directed walk of length \(\ell\) whose sequence contains a directed cycle \(C\) of length 3 and a directed hole \(H\) of length 4. If (i) the sequence of \(C\) appears before that of \(H\) on \(Z\), then there exists a vertex on \(H\) such that, on \(Z\), all the vertices on \(C\) appear before it and all the vertices on \(H\) appear after it. We denote such a vertex by \(x\). Suppose that (ii) the sequence of \(H\) appears before that of \(C\) on \(Z\). Since the vertices on \(H\) belong to two distinct partite sets and the vertices on \(C\) belong to three distinct partite sets, there exists a vertex \(y\) on \(C\) which does not belong to any of two partite sets which contain the vertices on \(H\).

Now we denote by \(v\) the vertex \(x\) if (i) happens, and the vertex \(y\) if (ii) occurs. Let \(Z_1\) be a \((u, v)\)-section of \(Z\), and \(Z_2\) be the \((v, w)\)-section of \(Z\) obtained by cutting \(Z_1\) away from \(Z\). We denote the lengths of \(Z_1\) and \(Z_2\) by \(\ell_1\) and \(\ell_2\), respectively. Fix a nonnegative integer \(t\). Since \(F(2, 3) = 1\), there exist nonnegative integers \(m_1, m_2\) such that \(t = 3m_1 + 2m_2 - 2\).

Consider the case (i). Then \(v = x\). Since \(Z_1\) is a \((u, v)\)-directed walk of Type 1, by Lemma 3.4 there exists a \((u, v)\)-directed walk \(Q_1\) of length \(\ell_1 + 3m_1\). Since \(Z_2\) is a \((v, w)\)-directed walk of Type 2, by Lemma 3.5 there exists a positive integer \(N'\) such that there is a \((v, w)\)-directed walk \(Q_2\) of length \(\ell_2 + 2(m_2 + N')\). Then \(Q_1 \rightarrow Q_2\) is a \((u, w)\)-directed walk of length

\[
(\ell_1 + 3m_1) + (\ell_2 + 2(m_2 + N')) = \ell + 2N' + 2 + (3m_1 + 2m_2 - 2) = \ell + 2N' + 2 + t.
\]

Consider the case (ii). Then \(v = y\). By Lemma 3.5 there exists a positive integer \(N''\) such that there is a \((u, v)\)-directed walk \(Q'_1\) of length \(\ell_1 + 2(m_2 + N'')\). By applying Lemma 3.4 to the \((v, w)\)-directed walk \(C \rightarrow Z_2\), there exists a \((v, w)\)-directed walk \(Q'_2\) of length \(3 + \ell_2 + 3m_1\). Then \(Q'_1 \rightarrow Q'_2\) is a \((u, w)\)-directed walk of length

\[
(\ell_1 + 2(m_2 + N'')) + (3 + \ell_2 + 3m_1) = \ell + 2N'' + 5 + (3m_1 + 2m_2 - 2) = \ell + 2N'' + 5 + t.
\]
Let
\[
N = \begin{cases} 
2N' + 2, & \text{if } v = x; \\
2N'' + 5, & \text{if } v = y.
\end{cases}
\]

Then, since \( t \) was chosen as an arbitrary nonnegative integer, we have shown that, for each integer \( m \geq N \), there is a \((u, w)\)-directed walk of length \( \ell + m \). \( \square \)

**Lemma 3.7.** Let \( D \) be a multipartite tournament with a directed cycle, the sink elimination index \( \zeta(D) \geq 1 \), and the sink sequence \((W_0, \ldots, W_{\zeta(D)})\). For a vertex \( u \in V(D_{\zeta(D)}) \), suppose that there exist a \((u, w_1)\)-directed walk \( Q_1 \) of Type 1 and a \((u, w_2)\)-directed walk \( Q_2 \) of Type 2 for some vertices \( w_1, w_2 \in \bigcup_{i=0}^{\zeta(D)-1} W_i \). Then there exists a positive integer \( N \) such that there is a \((u, w)\)-directed walk of length \( \ell \) for each vertex \( w \in \bigcup_{i=0}^{\zeta(D)-1} W_i \) and any integer \( \ell \geq N \).

**Proof.** Take \( w \in \bigcup_{i=0}^{\zeta(D)-1} W_i \). Let \( C \) be a directed cycle of length 3 in \( D \) induced by three vertices in \( Q_1 \) and \( H \) be a directed hole of length 4 in \( D \) induced by four vertices in \( Q_2 \). Then there are three distinct partite sets of \( D \) to which the vertices on \( C \) belong. Since the vertices on \( H \) belong to two distinct partite sets, there must be an arc linking a vertex \( x \) on \( C \) and a vertex \( y \) on \( H \). Moreover, there exist an arc from a vertex \( y' \) on \( H \) to \( w \) and an arc from a vertex \( x' \) on \( C \) to \( w \) since \( w \in \bigcup_{i=0}^{\zeta(D)-1} W_i \) and the vertices on directed cycles belong to \( V(D_{\zeta(D)}) \).

If \((x, y) \in A(D)\) (resp. \((y, x) \in A(D)\)), then a \((u, x)\)-section of \( Q_1 \), \( C \) starting at \( x \), the arc \((x, y)\), \( H \) starting at \( y \), the \((y, y')\)-section of \( H \), and the arc \((y', w)\) (resp. a \((u, y)\)-section of \( Q_2 \), \( H \) starting at \( y \), the arc \((y, x)\), \( C \) starting at \( x \), the \((x, x')\)-section of \( C \), and the arc \((x', w)\)) form a \((u, w)\)-directed walk \( Z_w \) whose sequence contains both a directed cycle of length 3 and a directed hole of length 4. Then there exists a positive integer \( N_w \) such that, for each integer \( m \geq N_w \), there is a \((u, w)\)-directed walk of length \( \ell_w + m \) by Lemma 3.6 where \( \ell_w \) is the length of \( Z_w \). If we denote by \( N \) the maximum of such integers \( \ell_w + N_w \), there exists a \((u, w)\)-directed walk of length \( \ell \) for each vertex \( w \in \bigcup_{i=0}^{\zeta(D)-1} W_i \) and each integer \( \ell \geq N \). \( \square \)

**Theorem 3.8.** Let \( D \) be a multipartite tournament with a directed cycle, the sink elimination index \( \zeta(D) \geq 1 \), and the sink sequence \((W_0, \ldots, W_{\zeta(D)})\). Then, for each vertex \( w \in \bigcup_{i=0}^{\zeta(D)-1} W_i \), one of the following properties is true:

(i) for every vertex \( u \) in \( D_{\zeta(D)} \), there is a \((u, w)\)-directed walk of Type 1;

(ii) for every vertex \( u \) in \( D_{\zeta(D)} \), there is a \((u, w)\)-directed walk of Type 2.

**Proof.** Suppose that \( D \) has \( n \) vertices and fix vertex \( w \in \bigcup_{i=0}^{\zeta(D)-1} W_i \). Since \( D \) has a directed cycle, \( n \geq 3 \). We fix \( u \in V(D_{\zeta(D)}) \). Since \( D_{\zeta(D)} \) has no sinks, there is a directed walk \( Q_u \) in \( D_{\zeta(D)} \) of length at least \( n \) starting at \( u \). Since \( D_{\zeta(D)} \) is a subdigraph of \( D \), \( Q_u \) is a directed walk in \( D \). By Lemma 3.3, \( Q_u \) is of Type 1 or Type 2. Then \( Q_u \) contains
vertices which induce a directed cycle \( C_u \) in \( D \) which has a length 3 or is a directed hole of length 4. Since the vertices on \( C_u \) belong to at least two distinct partite sets, there must be a vertex \( y_u \) on \( C_u \) which belongs to a partite set distinct from the one to which \( w \) belongs. Since \( D \) is a multipartite tournament, there must be an arc linking \( y_u \) and \( w \). Yet, \( w \in \bigcup_{i=0}^{\zeta(D)-1} W_i \) implies \((y_u, w) \in A(D)\). Then the \((u, y_u)\)-section of \( Q_u, C_u \) starting at \( y_u \), and the arc \((y_u, w)\) form a \((u, w)\)-directed walk of Type 1 or Type 2 in \( D \). Thus we have shown that

\[
(*) \text{ for every vertex } u \text{ in } D_{\zeta(D)}, \text{ there is a } (u, w)\text{-directed walk of Type 1 or Type 2.}
\]

For every vertex \( u \) in \( D_{\zeta(D)} \), if there exist a \((u, w)\)-directed walk of Type 1 and a \((u, w)\)-directed walk of Type 2, then the theorem statement is immediately true.

Now suppose that there exists a vertex \( v \) in \( D_{\zeta(D)} \) such that either there is no \((v, w)\)-directed walk of Type 1 or there is no \((v, w)\)-directed walk of Type 2. Then, by Lemma 3.3 every \((v, w)\)-directed walk of length at least \( n \) is of Type 2 or every \((v, w)\)-directed walk of length at least \( n \) is of Type 1. We assume the latter. Then, by \((*)\), there exists a \((v, w)\)-directed walk \( Z_v \) of Type 1. Then \( Z_v \) contains vertices which induce a directed cycle \( C \) of length 3 in \( D \). In the following, we will show that for every vertex \( u \) in \( D_{\zeta(D)} \), there is a \((u, w)\)-directed walk of Type 1. Take a vertex \( u \) in \( D_{\zeta(D)} \). By \((*)\), there is a \((u, w)\)-directed walk of Type 1 or Type 2. If there is a \((u, w)\)-directed walk of Type 1, then there is nothing to prove. Now suppose that there exists a \((u, w)\)-directed walk \( Z_u \) of Type 2. Then \( Z_u \) contains vertices which induce a directed hole \( H \) of length 4 in \( D \) and each vertex on \( H \) belongs to one of two distinct partite sets of \( D \). Since the vertices on \( C \) belong to three distinct partite sets, there must be an arc between a vertex \( x \) on \( C \) and a vertex \( y \) on \( H \). Yet, since no directed walk of Type 2 starting at \( v \) exists, \((y, x)\) is an arc in \( D \). Now, a \((u, y)\)-section of \( Z_u \), the arc \((y, x)\), \( C \) starting at \( x \), an \((x, w)\)-section of \( Z_v \) form a \((u, w)\)-directed walk of Type 1. Therefore we have shown that \((i)\) is true. By applying a similar argument, one can show that \((ii)\) is true in the former case.

Let \( D \) be a multipartite tournament with a directed cycle, the sink elimination index \( \zeta(D) \geq 1 \), and the sink sequence \((W_0, \ldots, W_{\zeta(D)})\). Then we say that a vertex in \( \bigcup_{i=0}^{\zeta(D)-1} W_i \) is of Type 1 (resp. Type 2) if \((i)\) (resp. \((ii)\)) of the above theorem is true. By the same theorem, each vertex in \( \bigcup_{i=0}^{\zeta(D)-1} W_i \) is of Type 1 or Type 2.

**Theorem 3.9.** If a \( k \)-partite tournament \( D \) has a sink and a directed cycle for an integer \( k \geq 2 \), then the competition period of \( D \) is at most three. Especially, if \( k = 2 \), then the competition period of \( D \) is at most two.

**Proof.** Let \( D \) be a \( k \)-partite tournament with a sink and a directed cycle for an integer \( k \geq 2 \). Since \( D \) has a sink and a directed cycle, \( \zeta(D) \geq 1 \). Let \((W_0, \ldots, W_{\zeta(D)})\) be the sink sequence of \( D \) and \( U_{\zeta(D)} = \bigcup_{i=0}^{\zeta(D)-1} W_i \). Since \( D \) has a directed cycle, \( W_{\zeta(D)} = \emptyset \) by Proposition 2.1. By Proposition 3.4, \( C^m(D) \) is not empty for every positive integer \( m \). If \( u \in U_{\zeta(D)} \), then \( u \) has no \( m \)-step prey for any integer \( m \geq \zeta(D) \) by Lemma 2.2. Therefore
every vertex in $U_{\zeta(D)}$ is isolated in $C^m(D)$ for any integer $m \geq \zeta(D)$. Thus it is sufficient to consider the vertices in $V(D_{\zeta(D)})$ when determining the competition period of $D$.

Suppose that there exist vertices $w_1$ and $w_2$ in $U_{\zeta(D)}$ of Type 1 and Type 2, respectively. Fix $w \in U_{\zeta(D)}$. Then, by Lemma 3.4, for each vertex $u$ in $D_{\zeta(D)}$, there exists a $(u, w)$-directed walk of length $\ell$ for each integer $\ell \geq N_u$ for some positive integer $N_u$. If we denote by $N$ the maximum of such integers $N_u$, then there exists a $(u, w)$-directed walk of length $\ell$ for each vertex $u$ in $D_{\zeta(D)}$ and each integer $\ell \geq N$. Thus, for each integer $m \geq N$, $w$ is an $m$-step prey of each vertex $u$ in $D_{\zeta(D)}$ and so $V(D_{\zeta(D)})$ forms a clique in $C^m(D)$ and $D$ has the competition period one.

Now it remains to consider the case that every vertex in $U_{\zeta(D)}$ is only of Type 1 or every vertex in $U_{\zeta(D)}$ is only of Type 2. We first consider the case in which every vertex in $U_{\zeta(D)}$ is only of Type 1. Suppose that there are two vertices $u_1$ and $u_2$ in $D_{\zeta(D)}$ which are adjacent in infinitely many step competition graphs of $D$. Then there exist a $(u_1, w)$-directed walk $Z_1$ and a $(u_2, w)$-directed walk $Z_2$ of the same length $\ell(u_1, u_2) \geq |V(D)|$ for a vertex $w$ in $D$. If $w \in V(D_{\zeta(D)})$, then $u_1$ and $u_2$ are adjacent in $C^m(D)$ for each integer $m \geq \ell(u_1, u_2)$ since $D_{\zeta(D)}$ has no sink. Now suppose that $w \in U_{\zeta(D)}$. Since $\ell(u_1, u_2) \geq |V(D)|$, each of $Z_1$ and $Z_2$ is of Type 1 or Type 2 by Lemma 3.3. By the case assumption, there exist a $(u_1, w)$-directed walk $Z_3$ and a $(u_2, w)$-directed walk $Z_4$ of Type 1. Then, by considering the following three cases:

(a) both of $Z_1$ and $Z_2$ are of Type 1;

(b) both of $Z_1$ and $Z_2$ are of Type 2;

(c) $Z_1$ and $Z_2$ are of different types

and by applying Lemmas 3.4 and 3.7 to $Z_1, Z_2, Z_3, Z_4$ whichever suitable, we may deduce one of the following:

(i) for some positive integer $L(u_1, u_2)$ and any integer $m \geq L(u_1, u_2)$, $u_1$ and $u_2$ have an $(\ell(u_1, u_2) + 3m)$-step common prey;

(ii) $u_1$ and $u_2$ have an $m$-step common prey for any integer $m \geq N(u_1, u_2)$ for some positive integer $N(u_1, u_2)$.

Suppose (i) happens and $u_1$ and $u_2$ have an $(\ell(u_1, u_2) + 3m^* + i)$-prey for some integer $m^* \geq L(u_1, u_2)$ and some $i$ in $\{1, 2\}$. Then, by repeating the above argument for $\ell(u_1, u_2) + 3m^* + i$ instead of $\ell(u_1, u_2)$, we may guarantee the existence of a positive integer $L(u_1, u_2)$ such that $u_1$ and $u_2$ have an $(\ell(u_1, u_2) + 3m^* + i + 3m)$-step common prey for any integer $m \geq L(u_1, u_2)$. Now $u_1$ and $u_2$ have an $(\ell(u_1, u_2) + 3m)$-step common prey and an $(\ell(u_1, u_2) + 3m + i)$-step common prey for any integer $m \geq L(u_1, u_2)$. Even if $u_1$ and $u_2$ have an $(\ell(u_1, u_2) + 3m^* + j)$-step common prey for some $m^* \geq L(u_1, u_2)$ and some $j \in \{1, 2\} \setminus i$, we may apply the same argument to find a positive integer $L''(u_1, u_2)$ such that $u_1$ and $u_2$ have an $(\ell(u_1, u_2) + 3m)$-step common prey,
an \((\ell(u_1, u_2) + 3m + i)\)-step common prey, and an \((\ell(u_1, u_2) + 3m + j)\)-step common prey for any integer \(m \geq L''(u_1, u_2)\).

We let \(M(u_1, u_2)\) stands for one of \(L(u_1, u_2), L'(u_1, u_2), L''(u_1, u_2), N(u_1, u_2)\), whichever appropriate. Let \(L\) be the maximum of \(M(u_1, u_2)\) over the pairs \(\{u_1, u_2\}\) in \(D_\zeta(D)\) which are adjacent in infinitely many step competition graphs of \(D\) (\(L\) exists since there are at most \(\binom{|V(D_\zeta(D))|}{2}\) pairs to consider). Then it is easy to check that \(C^{L+i}(D) = C^{L+3+i}(D)\) for each nonnegative integer \(i\), so the competition period of \(D\) is 1 or 3. Using Lemma 3.5 one can show that the competition period of \(D\) is at most two by a similar argument if every vertex in \(U_\zeta(D)\) is of Type 2, from which the ‘especially’ part follows.

4 Strongly connected multipartite tournaments

In this section, we study competition indices of strongly connected multipartite tournaments. Cho and Kim \[5\] showed that a digraph without sinks has competition period 1. In this section, we show that the competition index of a primitive digraph is at most its exponent.

**Proposition 4.1.** Let \(D\) be a digraph without sinks. If two vertices are adjacent in \(C^M(D)\) for a positive integer \(M\), then they are also adjacent in \(C^m(D)\) for any positive integer \(m \geq M\).

**Proof.** Let \(x\) and \(y\) are adjacent in \(C^M(D)\). Then \(x\) and \(y\) have an \(M\)-step common prey \(z\) in \(D\). Since \(D\) has no sinks, \(z\) has an out-neighbor \(w\) in \(D\). Then \(w\) is an \((M+1)\)-step common prey of \(x\) and \(y\). Hence \(x\) and \(y\) are adjacent in \(C^{(M+1)}(D)\). We may repeat this argument to show that \(x\) and \(y\) are adjacent in \(C^{(M+2)}(D)\). In this way, we may show that \(x\) and \(y\) are adjacent in \(C^m(D)\) for any positive integer \(m \geq M\). \(\square\)

**Theorem 4.2.** Let \(D\) be a primitive digraph. Then

(i) \(C^m(D)\) is a complete graph for each integer \(m \geq \exp(D)\);

(ii) \(\text{cindex}(D) \leq \exp(D)\);

(iii) \(\text{cperiod}(D) = 1\).

**Proof.** Since we assumed that the underlying graph of each digraph \(D\) dealt in this paper is simple, \(D\) contains neither a loop nor a directed cycle of length 2. Then, by the hypothesis that \(D\) is primitive, \(|V(D)| \geq 4\). Now take two distinct vertices \(u\) and \(v\) in \(D\) and let \(t = \exp(D)\). Then there exist a \((u, w)\)-directed walk and a \((v, w)\)-directed walk of length \(t\) for any vertex \(w\) in \(D\). Therefore \(u\) and \(v\) are adjacent in \(C^t(D)\). Since \(u\) and \(v\) are arbitrarily chosen, \(C^t(D)\) is a complete graph. Since \(D\) is primitive, \(D\) does not contain a sink and so, by Proposition 4.1 the statement (i) is true. By the definitions of cindex and cperiod, the statements (ii) and (iii) are true. \(\square\)
Let $D$ be a strongly connected $k$-partite tournament of order $n$ such that the length of a longest directed cycle is $k$ for an integer $4 \leq k \leq n$ and take two vertices $u$ and $v$ in $D$. Bondy \[23\] showed that for an integer $k \geq 3$, a strongly connected $k$-partite tournament contains a directed cycle of length $m$ for each integer $3 \leq m \leq k$. Therefore $D$ is primitive. On the other hand, Volkmann \[23\] showed that for an integer $k \geq 3$, every vertex of any strongly connected $k$-partite tournament with a longest cycle of length $k$ belongs to a directed cycle of length $m$ for each integer $3 \leq m \leq k$. Thus there is a directed cycle $C_m$ of length $m$ which contains the vertex $u$ for each integer $3 \leq m \leq k$. Now, since $D$ is strongly connected, there is a $(u, v)$-directed walk $W_1$ of length $l \leq n - 1$ in $D$. Since $l \leq n - 1$, $F(3, 4, \ldots, k) + n - l > F(3, 4, \ldots, k)$. Therefore, by concatenating the directed cycles, we obtain a closed directed walk $W_2$ of length $F(3, 4, \ldots, k) + n - l$. Now $W_2 \rightarrow W_1$ is a $(u, v)$-directed walk of length $F(3, 4, \ldots, k) + n$. Since $u$ and $v$ were arbitrarily chosen, $\exp(D) \leq F(3, 4, \ldots, k) + n$. It is known that $F(3, 4) = 5$ and $F(3, 4, \ldots, k) = 2$ for any integer $k \geq 5$. Thus

$$\exp(D) \leq \begin{cases} 5 + n, & \text{if } k = 4; \\ 2 + n, & \text{otherwise}. \end{cases}$$

Now, by Theorem 4.2(ii), we have the following proposition.

**Proposition 4.3.** Let $D$ be a strongly connected $k$-partite tournament of order $n$ such that the length of a longest directed cycle is $k$ for an integer $4 \leq k \leq n$. Then

$$\text{cindex}(D) \leq \begin{cases} 5 + n, & \text{if } k = 4; \\ 2 + n, & \text{otherwise}. \end{cases}$$

### 5 Tournaments

A tournament with $n$ vertices, denoted by $T_n$, is a digraph resulting from orienting the edges of a complete graph $K_n$. The outdegree of a vertex $v_i$ in $T_n$ is called the score of $v_i$, denoted by $s_i$. If the vertices of an $n$-tournament $T_n$ are labeled $v_1, v_2, \ldots, v_n$ so that $0 \leq s_1 \leq s_2 \leq \cdots \leq s_n$, then the sequence $(s_1, s_2, \ldots, s_n)$ is called the **score sequence** of $T_n$.

Let $D$ be a $k$-partite tournament with $n$ vertices. Then clearly $k$ is a positive integer with $k \leq n$. One can easily see that $k = n$ if and only if $D$ is a tournament.

**Theorem 5.1.** \[24\] Let $D$ be a tournament with $n$ vertices and $(s_1, s_2, \ldots, s_n)$ be its score sequence. Then $C^m(D)$ is as follows:

\[
C^2(D) = \begin{cases} K_{n-2} \cup I_2, & \text{if } s_1 = 0, s_2 = 1; \\ K_{n-1} \cup I_1, & \text{if } s_1 = 0, s_2 \geq 2; \\ K_n \text{ or } K_n - P_2 \text{ or } K_n - P_3, & \text{if } s_1 = 1, s_2 \geq 2; \\ K_n - P_3 \text{ or } K_n - P_4, & \text{if } s_1 = s_2 = 1, s_3 \geq 2; \\ K_n, & \text{if } s_1 \geq 2; \end{cases}
\]
(ii) \( C^3(D) = \begin{cases} K_n, & \text{if either } s_1 = 1, s_2 \geq 2 \text{ or } s_1 \geq 2; \\ K_n - P_2, & \text{if } s_1 = s_2 = 1, s_3 \geq 2; \\ K_n - C_3, & \text{if } s_1 = s_2 = s_3 = 1; \end{cases} \)

(iii) for \( m \geq 4 \), \( C^m(D) = \begin{cases} K_n, & \text{if } s_1 = 1, s_2 \geq 2 \text{ or } s_1 = s_2 = 1, s_3 \geq 2 \text{ or } s_1 \geq 2; \\ K_n - C_3, & \text{if } s_1 = s_2 = s_3 = 1. \end{cases} \)

Theorem 5.1 left out the characterization of the \( m \)-step competition graph of a tournament with sinks for \( m \geq 3 \). In this section, we take care of it to eventually compute the competition period and competition index of a tournament with sinks.

**Proposition 5.2.** There is at most one sink in any tournament.

*Proof.* Suppose to the contrary that there is a tournament \( D \) with at least two sinks. Let \( x \) and \( y \) be sinks of \( D \). Since \( D \) is a tournament, one of \((x, y)\) or \((y, x)\) must be in \( A(D) \), which is a contradiction. Hence there is at most one sink in any tournament. \( \square \)

**Corollary 5.3.** Let \( n \) be an integer with \( n \geq 3 \). Then a tournament of order \( n \) is acyclic if and only if its sink elimination index is \( n - 1 \).

**Theorem 5.4.** Let \( D \) be a tournament of order \( n \geq 2 \) with a sink and let \( \zeta(D) \) be the sink elimination index of \( D \). Then, for a positive integer \( m \), the following are true:

(i) \( 1 \leq \zeta(D) \leq n - 1 \) with \( \zeta(D) \neq n - 2 \). Moreover, for each integer \( i \) satisfying \( 1 \leq i \leq n - 1 \) and \( i \neq n - 2 \), there exists a tournament with the sink elimination index \( i \);

(ii) if \( 1 \leq m < \zeta(D) \), then \( C^m(D) \) is the union of the complete graph with vertex set \( V(D_{m-1}) \setminus W_{m-1} \) and the empty graph with the vertex set \( \bigcup_{i=0}^{m-1} W_i \);

(iii) if \( m \geq \zeta(D) \), then \( C^m(D) \) is the union of the complete graph with vertex set \( V(D_{\zeta(D)}) \) and the empty graph with the vertex set \( \bigcup_{i=0}^{\zeta(D)-1} W_i \);

(iv) \( \text{cperiod}(D) = 1 \);

(v) \( \text{cindex}(D) = \zeta(D) \).

*Proof.* Let \( (W_0, \ldots, W_{\zeta(D)}) \) be the sink sequence of \( D \). By the hypothesis that \( D \) has a sink, \( W_0 \neq \emptyset \). Then \( |W_0| = 1 \) by Proposition 5.2. Since \( n \geq 2 \), \( W_0 \neq V(D) \). Thus \( \zeta(D) \neq 0 \) and so \( 1 \leq \zeta(D) \leq n - 1 \). Now \( |W_i| = 1 \) for each integer \( 0 \leq i \leq \zeta(D) - 1 \) by Proposition 5.2.

Suppose \( \zeta(D) = n - 2 \). Then \( D_{\zeta(D)} \) is a tournament of order 2, which has a sink and a non-sink, and we reach a contradiction. Therefore \( \zeta(D) \neq n - 2 \).
To show the ‘moreover’ part of the statement (i), fix $i$ such that $1 \leq i \leq n - 1$ and $i \neq n - 2$. We consider a tournament $D'$ defined by $V(D') = \{v_1, v_2, \ldots, v_n\}$ and $A(D') = \{(v_i, v_k) \mid 1 \leq k < l \leq n\} \cup \{(v_{i+1}, v_n)\}$. It is easy to check that $W_{j-1} = \{v_j\}$ for each $j = 1, \ldots, i$ and so $\zeta(D') \geq i$. On the other hand, there exists a directed cycle $v_n \rightarrow v_{n-1} \rightarrow v_{n-2} \cdots \rightarrow v_{i+1} \rightarrow v_n$ in $D'$ and so we may conclude that $\zeta(D') = i$. Thus the statement (i) is true.

To show the statements (ii) and (iii), we denote the vertices of $D$ as $v_1, v_2, \ldots, v_n$ so that $W_i = \{v_{i+j}\}$ for each integer $0 \leq i \leq \zeta(D) - 1$. Then, for each integer $1 \leq i \leq \zeta(D)$, the length of a longest directed walk with an initial vertex $v_i$ is $i - 1$ by Lemma 2.2. Take a positive integer $m$. Suppose $1 \leq m < \zeta(D)$. Then, for every vertex $v$ in $V(D_m) \setminus W_{m-1}$, there is an arc from $v$ to $v_m$. Concatenating this arc with the directed walk of length $m - 1$ from $v_m$ to $v_1$, every vertex in $V(D_m) \setminus W_{m-1}$ has $v_1$ as an $m$-step prey. Since $V(D_m) \setminus W_{m-1} = V(D) \setminus \bigcup_{i=0}^{m-1} W_i$, $V(D_m) \setminus W_{m-1}$ forms a clique of size $|V(D) \setminus \bigcup_{i=0}^{m-1} W_i| = n - m$ in $C^m(D)$. Since no vertex in $\bigcup_{i=0}^{m-1} W_i$ has an $m$-step prey in $D$ by Lemma 2.2, the vertices in $\bigcup_{i=0}^{m-1} W_i$ are isolated in $C^m(D)$. Thus the statement (ii) is true.

Suppose $m \geq \zeta(D)$. By (i), $1 \leq \zeta(D) \leq n - 1$ and $\zeta(D) \neq n - 2$. If $\zeta(D) = n - 1$, then there is at most one vertex which has an $m$-step prey in $D$ and so $C^m(D) = I_n = K_1 \cup I_{n-1}$. Suppose $1 \leq \zeta(D) \leq n - 3$. Then $D_{\zeta(D)}$ is a tournament of order at least 3 and so $W_{\zeta(D)} \neq V(D_{\zeta(D)})$ by Proposition 5.2. By the definition of $\zeta(D)$, $W_{\zeta(D)} = \emptyset$, that is, $D_{\zeta(D)}$ is a tournament without sinks. Thus, for each vertex $v \in D_{\zeta(D)}$, there exists a directed walk $X_v$ in $D_{\zeta(D)}$ of length $k$ from $v$ to a vertex in $D_{\zeta(D)}$ where $k = m - \zeta(D)$. Since $D_{\zeta(D)}$ is a subdigraph of $D$, $X_v$ is a directed walk of length $k$ in $D$. By the definition of sink sequence, there is a directed walk of length $\zeta(D)$ from each vertex in $D_{\zeta(D)}$ to $v_1$ in $D$. By concatenating those two directed walks, we obtain a $(v, v_1)$-directed walk of length $m$ for each vertex $v \in D_{\zeta(D)}$ in $D$. Hence the vertices in $D_{\zeta(D)}$ forms a clique of size $n - \zeta(D)$ in $C^m(D)$. Since no vertex in $\bigcup_{i=0}^{\zeta(D)-1} W_i$ has an $m$-step prey in $D$ by Lemma 2.2, the vertices in $\bigcup_{i=0}^{\zeta(D)-1} W_i$ are isolated in $C^m(D)$. Therefore the statement (iii) is valid. Thus the statements (iv) and (v) immediately follow from (ii) and (iii).

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