Selective coideals on \((FIN_k^{[\infty]}, \leq)\)

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Abstract

A notion of selective coideal on \((FIN_k^{[\infty]}, \leq)\) (see definitions below) is given. The natural versions of the local Ramsey property and the abstract Baire property relative to this context are proven to be equivalent, and it is also shown that the family of subsets of \(FIN_k^{[\infty]}\) having the local Ramsey property relative to a selective coideal on \((FIN_k^{[\infty]}, \leq)\) is closed under the Souslin operation. Finally, it is proven that such selective coideals satisfy a sort of canonical partition property, in the sense of Taylor [16].

1 Introduction

Let \(\mathbb{N}\) be the set of nonnegative integers. For a given \(A \subseteq \mathbb{N}\), let \(A^{[\infty]} = \{X \subseteq A : |X| = \infty\}\). Consider the sets of the form:
\[ [a, A] = \{ B \in \mathbb{N}^\infty : a \sqsubseteq B \subseteq a \cup A \} \]

where \( a \) is a finite subset of \( \mathbb{N} \), \( A \in \mathbb{N}^\infty \) and \( a \sqsubseteq B \) means that \( a \) is an initial segment of \( B \). The relativized version of the completely Ramsey property (see [3]) for subsets of \( \mathbb{N}^\infty \), known as **local Ramsey property**, is described as follows:

For a family \( \mathcal{H} \subseteq \mathbb{N}^\infty \), a set \( \mathcal{X} \subseteq \mathbb{N}^\infty \) is said to be **\( \mathcal{H} \)-Ramsey** if for every \([a, A]\) with \( A \in \mathcal{H} \) there exits \( B \in \mathcal{H} \cap [a, A] \) such that \([a, B] \subseteq \mathcal{X}\) or \([a, B] \cap \mathcal{X} = \emptyset\). \( \mathcal{X} \) is said to be **\( \mathcal{H} \)-Ramsey null** if for every \([a, A]\) with \( A \in \mathcal{H} \) there exits \( B \in \mathcal{H} \cap [a, A] \) such that \([a, B] \cap \mathcal{X} = \emptyset\).

In [7], Mathias introduces the **happy families** (or selective coideals) of subsets of \( \mathbb{N} \) and study the local Ramsey property relative to such families. He proved that the analytic subsets of \( \mathbb{N}^\infty \) are **\( U \)-Ramsey** when \( U \) is a Ramsey ultrafilter and generalized this result for arbitrary happy families. In this work, a notion of selective coideal on \( (FIN_k^\infty, \leq) \) is given in order to obtain results which are analog to those of Mathias, in this context. The structure of this work is as follows: in section 2 we present the definition of \( FIN_k, FIN_k^\infty \) and related notions, give some notation and state some useful known results. In section 3 our notion of selective coideal on \( (FIN_k^\infty, \leq) \) is given. The corresponding **local Ramsey property** is also introduced in this context and it is proven to be equivalent to the abstract Baire property, when relativized to a selective coideal on \( (FIN_k^\infty, \leq) \). In section 4 we prove that, relative to a selective coideal on \( (FIN_k^\infty, \leq) \), the family of locally Ramsey subsets of \( FIN_k^\infty \) is closed under the Souslin operation, showing in this way that this family includes the analytic subsets of \( FIN_k^\infty \). A particular case of our results yields a selective version of Milliken’s theorem (see [11]). Finally, we use these facts to show that every selective coideal on \( (FIN_k^\infty, \leq) \) satisfies a sort of **canonical partition property**, in the sense of Taylor [16].

## 2 Preliminaries

Fix an integer \( k \geq 1 \). Given \( p: \mathbb{N} \to \{0, 1, \ldots, k\} \), denote \( supp(p) = \{n : p(n) \neq 0\} \) and \( rang(p) \) the image set of \( p \). Consider the set

\[ FIN_k = \{ p: \mathbb{N} \to \{0, 1, \ldots, k\} : |supp(p)| < \infty \text{ and } k \in rang(p) \} \]
we say that \( X = (x_n)_{n \in \mathcal{I}} \subseteq FIN_k \), with \( \mathcal{I} \in \mathcal{P}(\mathbb{N}) \) is a basic block sequence if

\[
   n < m \Rightarrow \max(\text{supp}(x_n)) < \min(\text{supp}(x_m))
\]

The length of \( X \), denoted by \(|X|\), is the cardinality of \( \mathcal{I} \). For infinite basic block sequences (i.e., basic block sequences of infinite length) we assume that \( \mathcal{I} = \mathbb{N} \). Define \( T : FIN_k \to FIN_{k-1} \) by

\[
   T(p)(n) = \max\{p(n) - 1, 0\}
\]

For \( j \in \mathbb{N} \), \( T^{(j)} \) is the \( j \)-th iteration of \( T \), i.e., \( T^{(0)}(p) = p \) and \( T^{(j+1)}(p) = T(T^{(j)}(p)) \). Given a basic block sequence \( A = (a_n)_{n \in \mathcal{I}} \) we define \([A] \subseteq FIN_k\) as the set which elements are of the form

\[
   T^{(j_0)}(a_{n_0}) + T^{(j_1)}(a_{n_1}) + \cdots + T^{(j_r)}(a_{n_r})
\]

with \( n_0 < n_1 < \cdots < n_r \in \mathcal{I} \), \( j_0 < j_1 < \cdots < j_r \in \{0, 1, \ldots, k\} \), and \( j_i = 0 \) for some \( i \in \{0, 1, \ldots, r\} \). Denote by \( FIN_k^{[\infty]} \) (resp. \( FIN_k^{<\infty}] \), the set of infinite (resp. finite) basic block sequences. Also, denote by \( FIN_k^{[n]} \) the set of finite basic block sequences of length \( n \). For \( A, B \in FIN_k^{[\infty]} \), define

\[
   A \leq B \iff A \subseteq [B]
\]

If \( A = (a_1, a_2, \ldots) \in FIN_k^{[\infty]} \), for every integer \( n \geq 1 \), denote

\[
   A \upharpoonright n := (a_1, a_2, \ldots, a_n) \in FIN_k^{[n]}
\]

and \( A \upharpoonright 0 := \emptyset \). We say that \( a \in FIN_k^{<\infty} \) is compatible with \( A \) (or \( A \) is compatible with \( a \)) if there exists \( B \leq A \) such that \( a = B \upharpoonright n \) for some \( n \). In this case we say that \( a \) is an initial segment of \( B \) and write \( a \subseteq B \). Denote by \([A]^{<\infty} \) (resp. \([A]^{[n]} \)) the set of those members of \( FIN_k^{<\infty} \) (resp. \( FIN_k^{[n]} \)) which are compatibles with \( A \). The following is a well known result:

**Theorem 1** (Gowers \[5\]). Given an integer \( n > 0 \) and

\[
   f : FIN_k \to \{0, 1, \ldots, r - 1\}
\]

there exists \( A \in FIN_k^{[\infty]} \) such that \( f \) is constant on \([A]^{[\infty]}\).
For $k = 1$, theorem 1 reduces to Hindman’s theorem [6].

Fix $A \in FIN_k^{[\infty]}$. For $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_m) \in A^{[\infty]}$, write $a <_A b$ to mean $\max(\text{supp}(a)) < \min(\text{supp}(b))$. Notice that if $a <_A b$ then we can build the “concatenation” $c = a \lhd b = (a_1, \ldots, a_n, b_1, \ldots, b_m) \in FIN_k^{[\infty]}$.

For $B \leq A$ and $a \in A^{[\infty]}$, define

$$[B]^{[\infty]} / a = \{ b \in [B]^{[\infty]} : a <_A b \}$$

$$B / a = \{ b \in B : a <_A b \}$$

and the “Ellentuck type” neighborhood

$$[a, B] := \{ C \in FIN_k^{[\infty]} : a \sqsubseteq C \text{ and } C / a \subseteq [B] \}$$

Notice that if $a \in [B]^{[\infty]}$ then

$$[a, B] = \{ C \in FIN_k^{[\infty]} : a \sqsubseteq C \text{ and } C \leq B \}$$

Also, let

$$[a, B]^{[n]} := \bigcup \{ [C]^{[n]} : C \in [a, B] \}.$$

and

$$[a, B]^{[\infty]} = \bigcup_n [a, B]^{[n]}.$$

### 3 Selectivity

The following definition is inspired on the known notion of coideal. The same name will be used:

**Definition 1.** We say that $\mathcal{H} \subseteq FIN_k^{[\infty]}$ is a **coideal** on $(FIN_k^{[\infty]}, \leq)$ if it satisfies the following:

1. If $A \leq B$ and $A \in \mathcal{H}$ then $B \in \mathcal{H}$.

2. Given $A \in \mathcal{H}$ and $O \subseteq [A]$, there exists $B \in \mathcal{H}$, with $B \leq A$, such that and $[B] \subseteq O$ or $[B] \subseteq O^c$.  


Notation. \( \mathcal{H}|A := \{B \in \mathcal{H} : B \leq A\} \).

Remark. Notice that part 2 of the previous definition implies that every coideal satisfies a local version of Gower’s theorem. Also, we are only interested in coideals satisfying the following:

\[
\forall A \in \mathcal{H} \ \forall a \in [A]^<\infty (B \in \mathcal{H}|A \Rightarrow B/a \in \mathcal{H})
\]

Intuitively, this means that \( \mathcal{H} \) is closed under finite changes.

For the next two definitions, fix a coideal \( \mathcal{H} \subseteq \text{FIN}_k^{[\infty]} \).

Definition 2. \( \mathcal{X} \subseteq \text{FIN}_k^{[\infty]} \) is \( \mathcal{H} \)-Ramsey if given \( A \in \mathcal{H} \) and \( a \in \text{FIN}_k^{[<\infty]} \) there exists \( B \in [a, A] \cap \mathcal{H} \) such that \( [a, B] \subseteq \mathcal{X} \) or \( [a, B] \subseteq \mathcal{X}^c \). If given \( A \in \mathcal{H} \) and \( a \in \text{FIN}_k^{[<\infty]} \) there exists \( B \in [a, A] \cap \mathcal{H} \) such that \( [a, B] \subseteq \mathcal{X}^c \), we say that \( \mathcal{X} \) is \( \mathcal{H} \)-Ramsey null.

Definition 3. \( \mathcal{X} \subseteq \text{FIN}_k^{[\infty]} \) is \( \mathcal{H} \)-Baire if given \( A \in \mathcal{H} \) and \( a \in \text{FIN}_k^{[<\infty]} \) there exists \( [b, B] \subseteq [a, A] \) with \( B \in \mathcal{H} \), such that \( [b, B] \subseteq \mathcal{X} \) or \( [b, B] \subseteq \mathcal{X}^c \). If given \( A \in \mathcal{H} \) and \( a \in \text{FIN}_k^{[<\infty]} \) there exists \( [b, B] \subseteq [a, A] \) with \( B \in \mathcal{H} \), such that \( [b, B] \subseteq \mathcal{X}^c \), we say that \( \mathcal{X} \) is \( \mathcal{H} \)-meager.

It is clear that if \( \mathcal{X} \) is \( \mathcal{H} \)-Ramsey then \( \mathcal{X} \) is \( \mathcal{H} \)-Baire.

We say that a sequence \( (A_n)_{n \geq 0} \subseteq \text{FIN}_k^{[\infty]} \) is decreasing if \( A_{n+1} \leq A_n \) for every \( n \). We say that \( B \in \text{FIN}_k^{[\infty]} \) is a diagonalization of the sequence \( (A_n)_{n \geq 0} \) if for every \( b \in B \), if \( n = \max(\text{supp}(b)) \) then \( B/b \subseteq [A_n] \). Notice that for such \( B \) we have \( [b, B] \subseteq [b, A_n] \) for every \( b \in [B]^{[<\infty]} \) with \( n = \max(\text{supp}(b)) > 0 \). Also, notice that every decreasing sequence has a diagonalization.

Definition 4. A coideal \( \mathcal{H} \subseteq \text{FIN}_k^{[\infty]} \) is selective if for every decreasing sequence \( (A_n)_{n \geq 0} \subseteq \mathcal{H} \) there exists \( B \in \mathcal{H} \) such that \( B \) is a diagonalization of the sequence \( (A_n)_{n \geq 0} \).

The main result of this work (theorem 3 below) consists of using our notion of selectivity to obtain a characterization of the \( \mathcal{H} \)-Ramsey property in terms of the \( \mathcal{H} \)-Baire property, in order to translate some important results concerning the locally Ramsey subsets of \( \mathbb{N}^{[\infty]} \) to the context of \( \text{FIN}_k^{[\infty]} \). For instance,
we will show that analytic subsets of $FIN^{[\infty]}_k$ (with the product topology, regarding $FIN_k$ as discrete space) are $\mathcal{H}$–Ramsey, whenever $\mathcal{H}$ is selective.

Proposition 1 below is a version for $FIN^{[\infty]}_k$ of a result due to Mathias ([7]) concerning selective coideals on $\mathbb{N}$. It provides some interesting examples of selective coideals on $(FIN^{[\infty]}_k, \subseteq)$. Our proof of it is similar to Todorcevic’s proof of Mathias’ result (see [17]).

**Definition 5.** A, $B \in FIN^{[\infty]}_k$ are **almost disjoint** if $|[A] \cap [B]| < \infty$. Also, $(B_j)_{j \leq m} \subseteq FIN^{[\infty]}_k$ **almost covers** $A$ if $A \subseteq \bigcup_{j \leq m} [B_j]$.

**Proposition 1.** Let $\mathcal{A}$ be an infinite collection of almost disjoint members of $FIN^{[\infty]}_k$. Define $\mathcal{H}$ as the family of those members of $FIN^{[\infty]}_k$ which cannot be almost covered by any finite subset of $\mathcal{A}$. Then $\mathcal{H}$ is a selective coideal.

**Proof.** It is easy to prove that $\mathcal{H}$ is a coideal. To see that it is selective, consider a decreasing sequence $(A_n)_{n \geq 0} \subseteq \mathcal{H}$. We split the discussion into two cases.

**Case 1:** There exists an infinite sequence $(C_m)_m$ of distinct members of $\mathcal{A}$ such that $|[C_m] \cap [A_n]| = \infty$, for every $m$ and $n$. Build $B \in FIN^{[\infty]}_k$ as follows: $b_0 = A_0 \upharpoonright 1$. Let $n_0 = max(supp(b_0))$ and choose $b_1 \in [A_{n_0}] \cap [C_0]/b_0$. Suppose that $b_1$ has been defined and let $n_i = max(supp(b_i))$. Consider

$$m_i = max\{ j : 2^j \text{ is a divisor of } i \}$$

and choose $b_{i+1} \in [A_{m_i}] \cap [C_{m_i}]/b_i$. Then $B = (b_1, b_2, \ldots) \in FIN^{[\infty]}_k$ is a diagonalization: since $(A_n)_{n \geq 0}$ is decreasing, if $b \in [B]^{<\infty}$ and $max(supp(b)) = n$ then $B/b \subseteq [A_n]$. Also $B \in \mathcal{H}$, since $|B \cap [C_m]| = \infty$ for infinitely many $m$’s.

**Case 2:** There is no $(C_m)_m \subseteq \mathcal{A}$ as in case 1. Choose a diagonalization $B_0$ of $(A_n)_{n \geq 0}$. If $B_0 \in \mathcal{H}$ we are done. Otherwise there is $(D_i)_{i \leq j} \subseteq \mathcal{A}$ such that $B_0 \subseteq^* \bigcup_{i \leq j} [D_i]$. Choose a diagonalization $B_1$ of $(A_n \setminus \bigcup_{i \leq j} [D_i])_{n \geq 0} \subseteq \mathcal{H}$. If $B_1 \in \mathcal{H}$ we are done. Otherwise there is $(F_i)_{i \leq j} \subseteq \mathcal{A}$ such that $B_1 \subseteq^* \bigcup_{i \leq j} [F_i]$. And so on. This process stops after finitely many of steps, otherwise we would be in case 1.

Before stating and proving our main result, we will prove theorem 2 below, which is a sort of local version of the corresponding Galvin lemma (or Nash-Williams theorem) for selective coideals on $(FIN^{[\infty]}_k, \subseteq)$. First, we need to define the following combinatorial forcing:
Fix $\mathcal{F} \subseteq \text{FIN}_k^{[<\omega]}$. We say that $B \in \mathcal{H}$ accepts $a \in \text{FIN}_k^{[<\omega]}$ if for every $B' \in [a, B]$ there exists $b \in \mathcal{F}$ such that $b \subseteq B'$. We say that $B$ rejects $a$ if no element of $[a, B] \cap \mathcal{H}$ accepts $a$; and we say that $B$ decides $a$ if $B$ either accepts or rejects $a$. This combinatorial forcing has the following features:

**Lemma 1.**
1. If $B$ accepts (rejects) $a$, then every $B' \in \mathcal{H} \upharpoonright B$ accepts (rejects) $a$.
2. Given $B \in \mathcal{H}$ and $a \in \text{FIN}_k^{[<\omega]}$, there exists $B' \in \mathcal{H} \upharpoonright B$ which decides $a$.
3. If $B$ accepts $a$ then $B$ accepts every $b \in [a, B'][[a]+1]$.
4. If $B$ rejects $a$ then there exists $B' \in [a, B] \cap \mathcal{H}$ such that $B$ does not accept any $b \in [a, B'][[a]+1]$.

**Proof.** 1–3 follow from the definitions. To see 4 let $O = \{b \in \text{FIN}_k^{[a]+1} : B \text{ accepts } b\}$

Notice that, being $\mathcal{H}$ a coideal, there exists $B' \in [a, B] \cap \mathcal{H}$ such that $[a, B'][[a]+1] \subseteq O$ or $[a, B'][[a]+1] \subseteq O^c$. Suppose that $[a, B'][[a]+1] \subseteq O$. Since

$$[a, B'] = \bigcup_{b \in [a, B'][[a]+1]} [b, B']$$

we have that $B'$ accepts $a$, which contradicts the fact that $B$ rejects $a$. Therefore, $[a, B'][[a]+1] \subseteq O^c$ and hence $B$ does not accept any $b \in [a, B'][[a]+1]$.

**Lemma 2.** Given a selective coideal $\mathcal{H}$ on $(\text{FIN}_k^{[\omega]}, \leq)$, $A \in \mathcal{H}$ and $\mathcal{F} \subseteq \text{FIN}_k^{[<\omega]}$, there exists $B \in \mathcal{H} \upharpoonright A$ which decides every $b \in [B][<\omega]$.

**Proof.** We shall build a decreasing sequence $(A_j)_{j \geq 0} \subseteq \mathcal{H} \upharpoonright A$, in the following way: by applying part 2 of lemma [1] to $A$ and $\emptyset$ we find $A_0 \in \mathcal{H} \upharpoonright A$ which decides $\emptyset$. Suppose that we have defined $A_j$. List

$$\mathcal{F}_j = \{a_0, a_1, \ldots, a_{n_j}\} = \{a \in [A][<\omega] : \max(\text{supp}(a)) = j\}$$

By applying part 2 of lemma [1] to $A_j$ and $a_0$ we find $B_1 \in \mathcal{H} \upharpoonright A_j$ which decides $a_0$. Now, apply part 2 of lemma [1] to $B_1$ and $a_1$ to obtain $B_2 \in \mathcal{H} \upharpoonright B_1$
which decides $a_1$. Following in this way we obtain $B_{n_j} \in \mathcal{H} \upharpoonright B_{n_j-1}$ which decides $a_{n_j}$. Define $A_{j+1} = B_{n_j}$. It is clear that $A_{m+1}$ decides every $b \in (\bigcup_{m \leq j+1} \mathcal{F}_m) \cup \{\emptyset\}$. Let $B \in \mathcal{H}$ be a diagonalization of $(A_j)_{j \geq 0}$. Then if $b \in [B]^{< \infty}$ and $\max(\text{supp}(b)) = n$, since $[b, B] \subseteq [b, A_n]$ and $A_n$ decides $b$, we have that $B$ decides $b$.

**Theorem 2.** Given a selective coideal $\mathcal{H} \subseteq \text{FIN}_k^{[\mathbb{N}]}$, $A \in \mathcal{H}$ and $\mathcal{F} \subseteq \text{FIN}_k^{[< \infty]}$, there exists $B \in \mathcal{H}[A]$ such that one of the following holds:

1. $[B]^{< \infty} \cap \mathcal{F} = \emptyset$, or

2. $\forall C \in [\emptyset, B]$ ($\exists a \in \mathcal{F}$) $(a \subseteq C)$.

**Proof.** Consider $B$ as in lemma 2. If $B$ accepts $\emptyset$ part 2 of the theorem holds. Assume that $B$ rejects $\emptyset$ and define $A_0 = B$. Apply part 4 of lemma 1 to $B$ and $\emptyset$ to obtain $A_1 \in \mathcal{H} \upharpoonright B$ which rejects every $b \in [A_1] \cup \{\emptyset\}$. By an argument similar to that of the proof of lemma 2 we can find $A_2 \in \mathcal{H} \upharpoonright A_1$ which rejects every $b$ in

$$\bigcup_{a \in [B]} [a, A_2]^{[2]} \cup [A_1] \cup \{\emptyset\}$$

In the same way we can obtain $A_3 \in \mathcal{H} \upharpoonright A_2$ which rejects every $b$ in

$$\bigcup_{a \in [B]} [a, A_3]^{[3]} \cup \bigcup_{a \in [B]} [a, A_2]^{[2]} \cup [A_1] \cup \{\emptyset\}$$

Following in this way we build a decreasing sequence $(A_n)_{n \geq 0} \subseteq \mathcal{H} \upharpoonright A$ such that for every $n$, $A_n$ rejects every $b$ in

$$\bigcup_{j \geq 0} \bigcup_{a \in [B]} [a, A_{j+1}]^{[j+1]} \cup [A_1] \cup \{\emptyset\}$$

Let $B' \in \mathcal{H} \upharpoonright B$ be a diagonalization of $(A_n)_{n \geq 0}$. Then $B'$ is as required. In fact, if $b \in [B']^{< \infty}$ and $\max(\text{supp}(b)) = n$ then, since $n \geq |b|$, we have

$$[b, B'] \subseteq [b, A_n] \subseteq [b, A_|b|] \subseteq \bigcup_{a \in [B]|b|^{-1}} [a, A_|b|]$$

Thus $A_{|b|}$ rejects $b$ and hence $B'$ rejects $b$, too. Therefore, no element of $[B']^{< \infty}$ is in $\mathcal{F}$. This gives us part 1 of the theorem. \qed
In a similar way we can prove the following generalization of theorem 2.

**Lemma 3.** Given a selective coideal \( \mathcal{H} \subseteq FIN_k^{[\infty]} \), \( A \in \mathcal{H} \), \( \mathcal{F} \subseteq FIN_k^{[<\infty]} \) and \( a \in FIN_k^{[<\infty]} \), there exists \( B \in [a, A] \cap \mathcal{H} \) such that one of the following holds:

1. \([a, B]^{[<\infty]} \cap \mathcal{F} = \emptyset\), or
2. \( \forall C \in [a, B] \) \( (\exists b \in \mathcal{F}) \ (b \sqsubset C) \).

Now we state the main result of this work.

**Theorem 3.** Let \( \mathcal{H} \) be a selective coideal on \((FIN_k^{[\infty]}, \leq)\).

1. \( X \subseteq FIN_k^{[\infty]} \) is \( \mathcal{H} \)-Ramsey iff \( X \) is \( \mathcal{H} \)-Baire.
2. \( X \subseteq FIN_k^{[\infty]} \) is \( \mathcal{H} \)-Ramsey null iff \( X \) is \( \mathcal{H} \)-meager.

**Proof.** Given a selective coideal \( \mathcal{H} \), let \( X \) be an \( \mathcal{H} \)-Baire subset of \( FIN_k^{[\infty]} \). Fix \([a, A]\) with \( A \in \mathcal{H} \) and define

\[
\mathcal{F}_0 = \{ b \in FIN_k^{[<\infty]} : a \sqsubseteq b \text{ and } [b, A] \subseteq X \}
\]

Consider \( B_0 \) as in lemma 3 applied to \( \mathcal{F}_0 \), \( a \) and \( B \). If part 2 of lemma 3 holds then \([a, B_0] \subseteq X \) and we are done. Otherwise define

\[
\mathcal{F}_1 = \{ b \in [B_0]^{[<\infty]} : a \sqsubseteq b \text{ and } [b, B_0] \cap X = \emptyset \}
\]

Consider \( B_1 \) as in lemma 3 applied to \( \mathcal{F}_1 \), \( a \) and \( B_0 \). If part 2 of lemma 3 holds for \( B_1 \) then \([a, B_1] \cap X = \emptyset \) and we are done. We claim that 1 from lemma 3 is not possible for \( B_1 \): assuming 1 for \( B_1 \) we obtain \( B_2 \) as in lemma 3 applied to \( \mathcal{F}_0 \cup \mathcal{F}_1 \), \( a \) and \( B_1 \). Suppose part 2 holds for \( B_2 \). If \( b \in \mathcal{F}_0 \cup \mathcal{F}_1 \) and \( b = C \upharpoonright n \) for some \( C \in [a, B_2] \) and some \( n \), then \( b \in [a, B_0]^{[<\infty]} \cap \mathcal{F}_0 \) or \( b \in [a, B_1]^{[<\infty]} \cap \mathcal{F}_1 \) which contradicts our assumptions. Therefore part 1 of lemma 3 holds for \( B_2 \), but this contradicts the fact that \( X \) is \( \mathcal{H} \)-Baire. This concludes the proof.

For \( k = 1 \), theorem 3 gives us a selective version of Milliken’s theorem [11]. Remember that \( FIN_k^{[\infty]} \) can be viewed as a metric subspace of \( FIN_k^\mathbb{N} \), the space of sequences of elements of \( FIN_k \), with the product topology regarding
as a discrete space. The basic open sets for the metric topology on $FIN_k^{[\infty]}$ are of the form:

$$[b] = \{A \in FIN_k^{[\infty]} : b \subseteq A\}.$$ 

For $b \in FIN_k^{[<\infty]}$. Then, we have the following:

**Corollary 1.** If $\mathcal{H}$ is a selective coideal on $(FIN_k^{[\infty]}, \leq)$ then every metric open subset of $FIN_k^{[\infty]}$ is $\mathcal{H}$–Ramsey.

**Proof.** Let $\mathcal{X}$ be a metric open subset of $FIN_k^{[\infty]}$ and fix a nonempty $[a, A]$. Without a loss of generality, we can assume $a = \emptyset$. Since $\mathcal{X}$ is open, there exists $\mathcal{F} \subseteq FIN_k^{[<\infty]}$ such that $\mathcal{X} = \bigcup_{b \in \mathcal{F}} [b]$. Let $B \leq A$ be as in lemma 2. If (1) holds then $[0, B] \subseteq \mathcal{X}^c$ and if (2) holds then $[0, B] \subseteq \mathcal{X}$. 

The following could be easily obtained from theorem 3 (or corollary 1). Yet, we prefer to give a proof of it by means of theorem 2 to avoid a “detour through the infinite”:

**Theorem 4.** Suppose that $\mathcal{H} \subseteq FIN_k^{[\infty]}$ is a selective coideal and fix $n \in \mathbb{N}$. Then, given a partition $f : FIN_k^{[n]} \to \{0, 1\}$ and $A \in \mathcal{H}$, there exists $B \in \mathcal{H}|A$ such that $f$ is constant on $[B]^{[n]}$.

**Proof.** Define $\mathcal{F} = f^{-1}(\{0\})$ and consider $B \in \mathcal{H}|A$ as in theorem 2. If part 1 of theorem 2 holds, then $f([B]^{[n]}) = 1$. Otherwise, $f([B]^{[n]}) = 0$.

**Definition 6.** A coideal $\mathcal{H}$ on $(FIN_k^{[\infty]}, \leq)$ is said to be **Ramsey** if for every subset $\mathcal{S}$ of $FIN_k^{[2]}$, there exists $B \in \mathcal{H}$ such that $[B]^{[2]} \subseteq \mathcal{S}$ or $[B]^{[2]} \cap \mathcal{S} = \emptyset$.

Thus, theorem 4 assures us that every selective coideal on $(FIN_k^{[\infty]}, \leq)$ is Ramsey.

## 4 The Souslin operation

Recall that given a set $X$ and a family $\mathcal{F}$ of subsets of $X$, two subsets $A, B$ of $X$ are said to be compatible with respect to $\mathcal{F}$ if there exists $C \in \mathcal{F}$ such that $C \subseteq A \cap B$. The family $\mathcal{F}$ is said to be $M$-like if for $\mathcal{G} \subseteq \mathcal{F}$ with $|\mathcal{G}| < |\mathcal{F}|$, every member of $\mathcal{F}$ which is not compatible with any member of $\mathcal{G}$ is compatible with $X \setminus \bigcup \mathcal{G}$. Also, recall that a $\sigma$-algebra $\mathcal{A}$ of subsets of
X together with a $\sigma$-ideal $A_0 \subseteq A$ is a Marczewski pair if for every $A \subseteq X$ there exists $\Phi(A) \in A$ such that $A \subseteq \Phi(A)$ and for every $B \subseteq \Phi(A) \setminus A$, $B \in A \Rightarrow B \in A_0$.

The goal of this section is to show that the family of $\mathcal{H}$–Ramsey subsets of $FIN_k^{[\infty]}$ is closed under the Souslin operation when $\mathcal{H}$ is a selective coideal. Given a family $(\mathcal{X}_a)_{a \in FIN_{k}^{[<\infty]}}$ of subsets of $FIN_k^{[\infty]}$, the result of applying the Souslin operation to this family is:

$$\bigcup_{A \in FIN_k^{[\infty]}} \bigcap_{n \in \mathbb{N}} \mathcal{X}_{A|n}$$

The following is a well known fact:

**Theorem 5** (Marczewski). *Every $\sigma$-algebra of sets which together with a $\sigma$-ideal is a Marczewski pair, is closed under the Souslin operation.*

The following proposition shows that the family $\mathcal{R}(\mathcal{H})$ of $\mathcal{H}$–Ramsey subsets of $FIN_k^{[\infty]}$ is a $\sigma$-algebra and the collection $\mathcal{R}_0(\mathcal{H})$ of $\mathcal{H}$–Ramsey null subsets of $FIN_k^{[\infty]}$ is a $\sigma$-ideal of it.

**Proposition 2.** If $\mathcal{H} \subseteq FIN_k^{[\infty]}$ is a selective coideal on $(FIN_k^{[\infty]}, \leq)$ then the families of $\mathcal{H}$–Ramsey and $\mathcal{H}$–Ramsey null subsets of $\mathcal{R}$ are closed under countable union.

*Proof.* Fix $A \in \mathcal{H}$. The proof will be made for $[\emptyset, A]$ without a loss of generality. Suppose that $(\mathcal{X}_n)_{n \geq 0}$ is a sequence of $\mathcal{H}$–Ramsey null subsets of $\mathcal{R}$. We can also assume that $\mathcal{X}_n \subseteq \mathcal{X}_{n+1}$ for all $n$ without a loss of generality. Since every $\mathcal{X}_n$ is $\mathcal{H}$–Ramsey null, by an argument similar to that of the proof of lemma 2 we can build a decreasing sequence $(B_n)_{n \geq 0} \subseteq \mathcal{H} \setminus A$ such that $[a, B_n] \cap \mathcal{X}_n = \emptyset$ for every $a \in [A]^{[<\infty]}$ with $\max(\text{supp}(a)) = n$. Let $B \in \mathcal{H}$ a diagonalization of $(B_n)_{n \geq 0}$. Then, for every $n$ we have that $[\emptyset, B] \cap \mathcal{X}_n = \emptyset$. In fact, for a fixed $n$ and $C \in [\emptyset, B]$ choose $a \sqsubseteq C$ such that $\max(\text{supp}(a)) \geq n$. If $m = \max(\text{supp}(a))$ then $[a, B_m] \cap \mathcal{X}_m = \emptyset$ and therefore $[a, B_m] \cap \mathcal{X}_n = \emptyset$. Hence $C \not\subseteq \mathcal{X}_n$; i.e., $[\emptyset, B] \cap \bigcup_n \mathcal{X}_n = \emptyset$.

Now, suppose that $(\mathcal{X}_n)_{n \geq 0}$ is a sequence of $\mathcal{H}$–Ramsey subsets of $FIN_k^{[\infty]}$ and consider $[a, A] \neq \emptyset$ with $A \in \mathcal{H}$. If there exists $B \in \mathcal{H} \setminus A$ such that $[a, B] \subseteq \mathcal{X}_n$ for some $n$, we are done. Otherwise, using an argument similar to the one above, we prove that $\bigcup \mathcal{X}_n$ is $\mathcal{H}$–Ramsey null. \qed
Given a selective coideal \( H \subseteq FIN_k^{[\infty]} \), in order to show that \((R(H), R_0(H))\) forms a Marczeswki pair it is sufficient to prove the following (see [12], [14] or [3]):

**Proposition 3.** Let \( H \) be a selective coideal on \((FIN_k^{[\infty]}, \leq)\). Assuming CH, the family

\[
Exp(H) := \{[a, A] : a \in FIN_k^{[\infty]}, A \in H\}
\]

is \( M \)-like.

**Proof.** Consider \( B \subseteq Exp(H) \) with \(|B| < |Exp(H)| = 2^{\aleph_0}\) and suppose that \([a, A]\) is not compatible with any member of \( B \), i.e. for every \( B \in B \), \( B \cap [a, A] \) does not contain any member of \( Exp(H) \). Notice that every \( X \in Exp(H) \) is \( H \)-Baire and therefore it is \( H \)-Ramsey. We claim that \([a, A]\) is compatible with \( FIN_k^{[\infty]} \setminus \bigcup B \). In fact: By proposition 2 \( \bigcup B \) is \( H \)-Ramsey. So, there exist \( B \in H \upharpoonright A \) such that:

1. \([a, B] \subseteq \bigcup B \) or
2. \([a, B] \subseteq FIN_k^{[\infty]} \setminus \bigcup B \)

1 is not possible because \([a, A]\) is not compatible with any member of \( B \). This completes the proof.

**Corollary 2.** Assuming CH, if \( H \) is a selective coideal on \((FIN_k^{[\infty]}, \leq)\) then \((R(H), R_0(H))\) forms a Marczeswki pair.

Now we use the following result due to Platek [15]:

**Theorem 6.** The use of CH can be eliminated from the proof of any statement involving only quantification over the reals and possibly some fixed set of reals as a predicate.

Since the statement

\[
\bigcup_{A \in FIN_k^{[\infty]}} \bigcap_{n \in \mathbb{N}} X_{A|n} \text{ is not } H \text{-Ramsey}
\]

is false under CH by theorem 5 and it has the required form in theorem 6, we have the following:
Corollary 3. If $H$ is a selective coideal on $(FIN^\{[\infty]\}_k, \leq)$ then $(\mathcal{R}(H), \mathcal{R}_0(H))$ forms a Marczewski pair.

□

Corollary 4. The family of $H$–Ramsey subsets of $FIN^\{[\infty]\}_k$ is closed under the Souslin operation, if $H$ is a selective coideal.

□

In the following result, “analytic” refers to the metric topology on $FIN^\{[\infty]\}_k$.

In virtue of corollaries 3 and 4 we have the following:

Corollary 5. Analytic subsets of $FIN^\{[\infty]\}_k$ are $H$–Ramsey.

□

5 Parameterized versions

In this section we give parameterized versions of theorems 2, 3 and corollary 4 (see [10]). Denote by $\mathbb{P}$, the family of perfect sets of $2^\infty$. For $x = (x_n)_n \in 2^\infty$, $x|_k = (x_0, x_1, \ldots, x_{k-1})$. For $u \in 2^{<\infty}$, let $[u] = \{x \in 2^\infty : x \subseteq u\}$ and let $|u|$ denote the length of $u$. If $Q \in \mathbb{P}$, we denote by $T_Q$ its associated perfect tree. Given $u \in 2^{<\infty}$, let $Q(u) = Q \cap [u(\bar{Q})]$, where $u(\bar{Q})$ is defined as follows: $\emptyset(\bar{Q}) = \emptyset$. If $u(\bar{Q})$ is defined, find $\sigma \in T_Q$ such that $\sigma$ is the $\subseteq$-extension of $u(\bar{Q})$ where the first ramification occurs. Set $(u^\bar{i})(\bar{Q}) = \sigma^\bar{i}$, $i = 1, 0$. Where “$^\bar{}$” is concatenation. Then, for every $n$, $Q = \bigcup\{Q(u) : u \in 2^n\}$.

Given $n \in \mathbb{N}$, we consider the following partial ordering on $\mathbb{P}$: given perfect sets $P$ and $Q$ we say that $P \subseteq_n Q$ if $P(u) \subseteq Q(u)$, for every $u \in 2^n$. If for every $u \in 2^n$ we have chosen a $P_u \subseteq Q(u)$, then $P = \bigcup_{u \in 2^n} P_u$ is perfect and we have $P(u) = P_u$ and $P \subseteq_n Q$. Property of fusion: if $Q_{n+1} \subseteq_n Q_n$, $n \in \mathbb{N}$, then the fusion $Q = \bigcap_n Q_n$ is a perfect set and $Q \subseteq_n Q_n$, for each $n$.

Recall the notions of perfectly Ramsey and perfectly Baire sets.

Definition 7. $\mathcal{X} \subseteq 2^{[\infty]} \times FIN^\{[\infty]\}_k$ is perfectly Ramsey if given $A \in FIN^\{[\infty]\}_k$, $a \in FIN^\{[<\infty]\}_k$ and $M \in \mathbb{P}$ there exist $P \in \mathbb{P}|M$ and $B \in [a, A]$ such that $P \times [a, B] \subseteq \mathcal{X}$ or $P \times [a, B] \subseteq \mathcal{X}^c$. If given $A \in FIN^\{[\infty]\}_k$, $a \in FIN^\{[<\infty]\}_k$ and $M \in \mathbb{P}$ there exist $P \in \mathbb{P}|M$ and $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}^c$, we say that $\mathcal{X}$ is perfectly Ramsey null.
Definition 8. \( \mathcal{X} \subseteq \text{FIN}^{[\infty]}_k \) is perfectly Baire if given \( A \in \text{FIN}^{[\infty]}_k, a \in \text{FIN}^{[<\infty]}_k \) and \( M \in \mathbb{P} \), there exist \( P \in \mathbb{P}|M \) and \( [b, B] \subseteq [a, A] \), such that \( P \times [b, B] \subseteq \mathcal{X} \) or \( P \times [b, B] \subseteq \mathcal{X}^c \). If given \( A \in \text{FIN}^{[\infty]}_k, a \in \text{FIN}^{[<\infty]}_k \) and \( M \in \mathbb{P} \), there exist \( P \in \mathbb{P}|M \) and \( [b, B] \subseteq [a, A] \), such that \( P \times [b, B] \subseteq \mathcal{X}^c \), we say that \( \mathcal{X} \) is perfectly meager.

Consider the following local versions of these notions: fix a coideal \( \mathcal{H} \subseteq \text{FIN}^{[\infty]}_k \).

Definition 9. \( \mathcal{X} \subseteq 2^{[\infty]} \times \text{FIN}^{[\infty]}_k \) is perfectly \( \mathcal{H} \)-Ramsey if given \( A \in \mathcal{H}, a \in \text{FIN}^{[<\infty]}_k \) and \( M \in \mathbb{P} \) there exist \( P \in \mathbb{P}|M \) and \( B \in [a, A] \cap \mathcal{H} \) such that \( P \times [b, B] \subseteq \mathcal{X} \) or \( P \times [b, B] \subseteq \mathcal{X}^c \). If given \( A \in \mathcal{H}, a \in \text{FIN}^{[<\infty]}_k \) and \( M \in \mathbb{P} \) there exist \( P \in \mathbb{P}|M \) and \( B \in [a, A] \cap \mathcal{H} \) such that \( P \times [b, B] \subseteq \mathcal{X}^c \), we say that \( \mathcal{X} \) is perfectly \( \mathcal{H} \)-Ramsey null.

Definition 10. \( \mathcal{X} \subseteq \text{FIN}^{[\infty]}_k \) is perfectly \( \mathcal{H} \)-Baire if given \( A \in \mathcal{H}, a \in \text{FIN}^{[<\infty]}_k \) and \( M \in \mathbb{P} \), there exist \( P \in \mathbb{P}|M \) and \( [b, B] \subseteq [a, A] \), with \( B \in \mathcal{H} \), such that \( P \times [b, B] \subseteq \mathcal{X} \) or \( P \times [b, B] \subseteq \mathcal{X}^c \). If given \( A \in \mathcal{H}, a \in \text{FIN}^{[<\infty]}_k \) and \( M \in \mathbb{P} \), there exist \( P \in \mathbb{P}|M \) and \( [b, B] \subseteq [a, A] \) with \( B \in \mathcal{H} \), such that \( P \times [b, B] \subseteq \mathcal{X}^c \), we say that \( \mathcal{X} \) is perfectly \( \mathcal{H} \)-meager.

To obtain a parameterized version of theorem 2, we define the following combinatorial forcing: Fix a selective coideal \( \mathcal{H} \subseteq \text{FIN}^{[\infty]}_k \) and \( \mathcal{F} \subseteq 2^{<\infty} \times \text{FIN}^{[<\infty]}_k \). Given \( Q \in \mathbb{P}, A \in \mathcal{H} \) and \( (u, a) \in 2^{<\infty} \times [A]^{<\infty} \), we say that \((Q, A)\) accepts \((u, a)\) if for every \( x \in Q(u) \) and every \( B \in [a, A] \) there exist integers \( n, m \) such that \((x|_n, [B]^m) \) \( \in \mathcal{F} \). We say that \((Q, A)\) rejects \((u, a)\) if \((M, B)\) does not accept \((u, a)\) for every \( M \in \mathbb{P}|Q \) and \( B \in \mathcal{H}[A] \). And we say that \((Q, A)\) decides \((u, a)\) if it either accepts or rejects it.

Theorem 7. Given a selective coideal \( \mathcal{H} \subseteq \text{FIN}^{[\infty]}_k \) and \( \mathcal{F} \subseteq 2^{<\infty} \times \text{FIN}^{[<\infty]}_k \), there exist \( P \in \mathbb{P} \) and \( A \in \mathcal{H} \) such that one of the following holds:

1. \( T_P \times [A]^{<\infty} \cap \mathcal{F} = \emptyset \), or
2. \( \forall x \in P \forall B \in [A]^{[\infty]} (\exists n, m \in \mathbb{N}((x|_n, B|_m) \in \mathcal{F})) \).

Proof. Proceeding as in lemma 2 we can prove that there exists a pair \((Q, B)\) which decides every \((u, a) \in T_Q \times [B]^{<\infty} \) with \(\max(\text{supp}(a)) \leq |u| \) (see lemma
2 in [8], lemma 3 in [10] and theorem 2.4 in [3]. If \((Q, B)\) accepts \((\langle \rangle, \emptyset)\) (where \(\langle \rangle\) is the empty sequence in \(2^{<\omega}\)) then part 2 of the theorem holds for \(P = Q\) and \(A = B\). If \((Q, B)\) rejects \((\langle \rangle, \emptyset)\) then we can proceed as in the proof of theorem 2 to obtain \((R, C)\) with \(R \in \mathbb{P}|Q\) and \(C \in \mathcal{H}|B\), and such that \((R, C)\) rejects every \((u, a) \in T_R \times [C][^{<\omega}]\) with \(\max(\text{supp}(a)) \leq |u|\) (see lemma 3 in [8] and lemma 4 in [10]). Hence, part 1 of the theorem holds for \(P = R\) and \(A = C\).

**Theorem 8.** For \(\mathcal{X} \subseteq 2^{\omega} \times \text{FIN}^{[\omega]}_k\) and a selective coideal \(\mathcal{H}\) on \((\text{FIN}^{[\omega]}_k, \leq)\), we have:

1. \(\mathcal{X}\) is perfectly \(\mathcal{H}\)-Ramsey iff it is perfectly \(\mathcal{H}\)-Baire.
2. \(\mathcal{X}\) is perfectly \(\mathcal{H}\)-Ramsey null iff it is perfectly \(\mathcal{H}\)-meager.

**Proof.** Let \(\mathcal{X}\) be a perfectly \(\mathcal{H}\)-Baire subset of \(2^{\omega} \times \text{FIN}^{[\omega]}_k\). Fix \(A \in \mathcal{H}, Q \in \mathbb{P}\) and define

\[
\mathcal{F}_0 = \{(u, b) \in 2^{<\omega} \times \text{FIN}^{[<\omega]}_k: Q \times [A][^{\leq\omega}] \subseteq \mathcal{X}\}
\]

Consider \(P_0\) and \(A_0\) as in theorem 7 applied to \(\mathcal{F}_0\) bellow \(A\). If part 2 of theorem 7 holds then \(P_0 \times [B_0][^{<\omega}] \subseteq \mathcal{X}\) and we are done. Otherwise define

\[
\mathcal{F}_1 = \{(u, b) \in 2^{<\omega} \times [B_0][^{<\omega}]: P_0 \times [B_0][^{\leq\omega}] \subseteq \mathcal{X}^c\}
\]

Consider \(P_1\) and \(B_1\) as in theorem 7 applied to \(\mathcal{F}_1\), bellow \(B_0\). If part 2 of theorem 7 holds, we are done. As in the proof of theorem 3, we can prove that part 1 from theorem 7 is not possible.

**Proposition 4.** The perfectly \(\mathcal{H}\)-Ramsey null subsets of \(2^{\omega} \times \text{FIN}^{[\omega]}_k\) form a \(\sigma\)-ideal.

**Proof.** Let \((\mathcal{X}_n)_n\) be a sequence of perfectly Ramsey null subsets of \(2^{\omega} \times \text{FIN}^{[\omega]}_k\) and fix \(P \times [a, A]\). We can assume \(a = \emptyset\). Also notice that the finite union of perfectly Ramsey null sets yields a perfectly Ramsey null set; so we will assume \((\forall n)\) \(\mathcal{X}_n \subseteq \mathcal{X}_{n+1}\). Proceeding as in the proof of proposition 2 we build sequences \((Q_n)_n \subseteq \mathbb{P}\) and \((B_n)_n \subseteq \mathcal{H}\) as follows: take \(Q_0 \subseteq P, B_0 \mathcal{H}|A\) such that \(Q_0 \times [0, B_0]\cap \mathcal{X}_0 = \emptyset\). Suppose \(Q_n, B_n\) have been defined such that

\[
Q_n \times [b, B_n] \cap \mathcal{X}_n = \emptyset
\]
for every $b \in [B_n]$ with $\text{max}(\text{supp}(b)) = n$. Since $\mathcal{X}_{n+1}$ is perfectly $\mathcal{H}$–Ramsey null, we can find $Q_{n+1} \subseteq Q_n$, and $B_{n+1}\mathcal{H}\mid B_n$ such that

$$Q_{n+1} \times [b, B_{n+1}] \cap \mathcal{X}_{n+1} = \emptyset$$

for every $b \in [B_{n+1}]$ with $\text{max}(\text{supp}(b)) = n + 1$. Let $Q = \bigcap_n Q_n$ and $B \in \mathcal{H}$ be a diagonalization of $(B_n)_n$. Then $Q \times [0, B] \cap \bigcup_n \mathcal{X}_n = \emptyset$: take $(x, C) \in Q \times [0, B]$ and fix arbitrary $n$. To show that $(x, C) \notin \mathcal{X}_n$ let $b \subseteq C$ such that $\text{max}(\text{supp}(b)) = m \geq n$. Then by construction $Q \times [b, B] \cap \mathcal{X}_m = \emptyset$ and hence, since $\mathcal{X}_n \subseteq \mathcal{X}_m$, we have $(x, C) \notin \mathcal{X}_n$. This completes the proof.

\[\square\]

Notice that $\mathbb{P}$ is $M$-like. Then, in virtue of proposition 3, proposition 4 and Lemma 2.7 of [14] we have the following:

**Theorem 9.** If $\mathcal{H}$ is a selective coideal on $(\text{FIN}_k^{[\infty]}, \leq)$, then the families of perfectly $\mathcal{H}$–Ramsey and perfectly $\mathcal{H}$–Ramsey null subsets of $2^{\infty} \times \text{FIN}_k^{[\infty]}$ are closed under the Souslin operation.

\[\square\]

## 6 Partition properties

In this final section, we show that the Ramsey property for coideals (see definition 6 above) characterizes the notion of “selective ultrafilter” in the context of $\text{FIN}_k^{[\infty]}$. Also, we show that selective coideals on $(\text{FIN}_k^{[\infty]}, \leq)$ satisfy a sort of canonical partition property, in the sense of Taylor [16].

### 6.1 Ramsey property

We shall say that $\mathcal{U} \subseteq \text{FIN}_k^{[\infty]}$ is an ultrafilter if it is a maximal filter on $(\text{FIN}_k^{[\infty]}, \leq)$. Then we have:

**Proposition 5.** Let $\mathcal{F}$ be an filter on $(\text{FIN}_k^{[\infty]}, \leq)$. Then $\mathcal{F}$ is maximal if and only if it satisfies: Given $A \in \mathcal{F}$ and $\mathcal{O} \subseteq [A]$, there exists $B \in \mathcal{F}$, with $B \leq A$, such that $[B] \subseteq \mathcal{O}$ or $[B] \subseteq [A] \setminus \mathcal{O}$.  

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Proof. ($\Rightarrow$) Consider $A \in \mathcal{F}$ and $O \subseteq [A]$. By Gower’s theorem (see [17]) there exists $B \leq A$ such that $[B] \subseteq O$ or $[B] \cap O = \emptyset$. Let $\mathcal{G}$ be the filter generated by $\{B\} \cup \mathcal{F}$. Since $\mathcal{F}$ is maximal, $\mathcal{G} = \mathcal{F}$, hence $B \in \mathcal{F}$.

($\Leftarrow$) Let $\mathcal{G}$ be a filter which contains $\mathcal{F}$. Assume that $\mathcal{G} \setminus \mathcal{F} \neq \emptyset$ and consider $C \in \mathcal{G} \setminus \mathcal{F}$. For a given $A \in \mathcal{F}$, define $O = [A] \cap [C]$. Let $B \leq A$ be such that $B \subseteq O$ or $B \subseteq [A] \setminus O$. If $B \subseteq O$ then $B \leq C$, which is not possible since $A \in \mathcal{F}$ and $C \in \mathcal{G} \setminus \mathcal{F}$. On the other hand, since $B \in \mathcal{G}$, there exists $D \in \mathcal{G}$ such that $D \leq R$ and $D \leq B$. Therefore $[B] \cap O \neq \emptyset$. This completes the proof.

6.2 Canonical partition property

Let us state the following result due to Taylor [16]:

Theorem 10 (Canonical partition theorem; Taylor [16]). For every function $f : FIN \rightarrow \mathbb{N}$ there exists $S \in FIN^{[\infty]}$ such that one of the following holds for all $s, t \in [S]$:

Corollary 6. Every ultrafilter on $(FIN_k^{[\infty]}, \leq)$ is a coideal.

We say that an ultrafilter $U \subseteq FIN_k^{[\infty]}$ is Ramsey if it is a Ramsey coideal. We have the following:

Corollary 7. Let $U \subseteq FIN_k^{[\infty]}$ be an ultrafilter. Then, $U$ is selective if and only if it is Ramsey.

Proof. In virtue of corollary 6 and theorem 4 we only need to prove the converse implication. So, let $U \subseteq FIN_k^{[\infty]}$ be a Ramsey ultrafilter and consider a (decreasing) sequence $(A_n)_n \subseteq U$. Define a partition of $c : FIN_k^{[2]} \rightarrow \{0, 1\}$ as follows:

$$c((a, b)) = 1 \iff b \in A_{\max(supp(a))}.$$  

Since $U$ is Ramsey there exists $B \in U$ such that $c$ is constant on $[B]^{[2]}$. Fix $a \in B$ and let $n = \max(supp(a))$. Let $C \in U$ be such that $C \leq B$ and $C \leq A_n$, and pick $b \in [C]$ with $n < \min(supp(b))$. Notice that $c((a, b)) = 1$. Hence $c$ is constantly equal to 1 on $[B]^{[2]}$, and therefore $B$ is a diagonalization of $(A_n)_n$.

6.2 Canonical partition property

Let us state the following result due to Taylor [16]:

Theorem 10 (Canonical partition theorem; Taylor [16]). For every function $f : FIN \rightarrow \mathbb{N}$ there exists $S \in FIN^{[\infty]}$ such that one of the following holds for all $s, t \in [S]$:
i) \( f(s) = f(t) \).

ii) \( f(s) = f(t) \iff \min(s) = \min(t) \).

iii) \( f(s) = f(t) \iff \max(s) = \max(t) \).

iv) \( f(s) = f(t) \iff \min(s) = \min(t) \) and \( \max(s) = \max(t) \).

v) \( f(s) = f(t) \iff s = t \).

The canonical partition theorem can be easily extended to the context of \( FIN_k \) as follows:

**Theorem 11.** For every function 
\[ f : FIN_k \to \mathbb{N} \]
there exists \( A \in FIN_k^{[\infty]} \) such that one of the following holds for all \( a, b \in [A] \):

i) \( f(a) = f(b) \).

ii) \( f(a) = f(b) \iff \min(\text{supp}(a)) = \min(\text{supp}(b)) \).

iii) \( f(a) = f(b) \iff \max(\text{supp}(a)) = \max(\text{supp}(b)) \).

iv) \( f(a) = f(b) \iff \min(\text{supp}(a)) = \min(\text{supp}(b)) \) and \( \max(\text{supp}(a)) = \max(\text{supp}(b)) \).

v) \( f(a) = f(b) \iff a = b \).

**Proof.** Given \( f : FIN_k \to \mathbb{N} \) we consider the function \( f_1 : FIN \to \mathbb{N} \) defined by \( f_1(s) = f(k\chi_s) \) where \( \chi_s \) is the characteristic function of \( s \). Then apply theorem 10 to obtain \( S = (s_1, s_2, \ldots) \in FIN^{[\infty]} \) such that one of (i)...(v) holds for all \( s, t \in [S] \). Define \( a_j := k\chi_{s_j} \) and let \( A = (a_1, a_2, \ldots) \in FIN_k^{[\infty]} \). Then \( A \) is as required.

**Definition 11.** A coideal \( \mathcal{H} \subseteq FIN_k^{[\infty]} \) is said to have the **canonical partition property** if for every \( f : FIN_k \to \mathbb{N} \) there exists \( A \in \mathcal{H} \) satisfying the conclusion of theorem 11.

We have the following:
Theorem 12. If $\mathcal{H}$ is a selective coideal on $(FIN_k^{[\infty]}, \leq)$ then it has the canonical partition property.

Proof. Let $f : FIN_k \to \mathbb{N}$ and let $\mathcal{X}$ be set of all those $A \in FIN_k^{[\infty]}$ as in theorem 11. By corollary 5, $\mathcal{X}$ is $\mathcal{H}$–Ramsey. Consider $A \in FIN_k^{[\infty]}$ such that $[\emptyset, A] \subseteq \mathcal{X}$ or $[\emptyset, A] \cap \mathcal{X} = \emptyset$. By theorem 11, $[\emptyset, A] \cap \mathcal{X} \neq \emptyset$. Hence $A \in \mathcal{H}$.

Corollary 8. Every selective ultrafilter on $(FIN_k^{[\infty]}, \leq)$ has the canonical partition property.

Final Comment. Taylor’s Canonical Partition Theorem (theorem 10 above) is strongly related to the notion of stable ordered-union ultrafilter introduced by Blass in [1]. The connection between stability in the sense of Blass and selectivity as presented in this work is currently being researched by us.

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