Linear-time generation of specifiable combinatorial structures:
general theory and first examples

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Abstract
Various specifiable combinatorial structures, with \(d\) extensive parameters, can be exactly sampled both by the recursive method, with linear arithmetic complexity if a heavy preprocessing is performed, or by the Boltzmann method, with average complexity \(\Theta(n^{1+d/2})\).

We discuss a modified recursive method, crucially based on the asymptotic expansion of the associated saddle-point integrals, which can be adopted for a large number of such structures (e.g. partitions, permutations, lattice walks, trees, random graphs, all with a variety of prescribed statistics and/or constraints). The new algorithm requires no preprocessing, still it has linear complexity on average. In terms of bit complexity, instead of the arithmetic one, we only have extra logarithmic factors. For many families of structures, this provides, at our knowledge, the only known quasi-linear generators.

We present the general theory, and detail a specific example: the partitions of \(n\) elements into \(k\) non-empty blocks, counted by the Stirling numbers of the second kind. These objects are involved in the exact sampling of minimal automata with prescribed alphabet size and number of states, which is thus performed here with average \(\Theta(n \ln n)\) bit complexity, outbreaking all previously known \(\Theta(n^{3/2})\) algorithms.

Keywords: Random combinatorial structures, Random generation, Recursive method, Random minimal automata.
1 Introduction

This paper deals with the exact sampling of random combinatorial structures $X$, from measures on statistical ensembles with multiple size parameters, $X \in X_n$, $n = (n_1, \ldots, n_d) \in \mathbb{N}^d$. We address the case in which the structures have a combinatorial specification (see [6], sec. I.2 and references therein), i.e. are described in terms of elementary constructors (disjoint union, cartesian product, sequence, set, multiset, cycle, ...), a situation in which, under some mild further hypotheses, there exist already two general algorithmic strategies: the recursive [9, 11] and the Boltzmann methods [5, 7]. Setting $N = \sum_j n_j$ the sum of the size parameters, the recursive method has bit complexity $\Theta(N)$ or $\Theta(N \log N)$ in most cases whenever the coefficients of the generating function have explicit fast-computable formulas. However, if this is not the case, it has only a poor $\Theta(N^{d+1})$ time and space complexity. On the other side, the Boltzmann method has a time complexity $\Theta(N^{d/2+1})$ on average, quasi-linear space complexity, and a wider range of applicability.

A natural goal is to fill this gap, and provide a 'mixed' algorithm that achieves a quasi-linear space and (average-)time complexity, with no preprocessing, essentially in every context for which a Boltzmann sampling is available. Within this paper we shall require an extra property, pertinent to the recursive method, namely that we have a linear recursion at the level of the generating functions that implies an algorithmic step-by-step construction of the structure (in particular, to a certain extent, the recursion must have non-negative coefficients). This is in fact quite often the case for objects within the symbolic method framework.

For various special cases of combinatorial structures, linear or quasi-linear algorithms have been designed. We mention in particular, as prototype examples, the Remy algorithm [13], for generating random planar binary trees of a given size, and a recent extension [1] for unary-binary trees. These algorithms are elegant, and intrinsically combinatorial. The drawback is that they are rare gems, and exist only for very few specific problems. On the contrary, the strategy we present here aims to be quite general, and extend to weighted objects with a minimal amount of extra work.

Within the theory developed here, and supplied with the (easy) verification of the conditions in Section 5, one can produce quasi-linear algorithms for sampling: (1) partitions of a set, constrained to the number of blocks, and possibly the set of allowed cardinalities (that we discuss here in detail); (2) permutations, constrained to the number of cycles, and possibly the set of allowed cycle lengths; (3) walks and directed walks, constrained to their endpoints, and to other statistics, e.g., in $\mathbb{Z}^2$, the area encircled by the path (these further statistics make the problem non-trivial); (4) Various families of trees, e.g. with prescribed number of nodes for each degree. In particular, in conjunction with the results in [2, 3], our algorithm for the first example implies the quasi-linear uniform generation of random $n$-state minimal automata over a $k$-symbol alphabet, for any $k \geq 2$.

2 Two examples

Before setting up a general theory, let us illustrate with some specific examples how the ‘classical’ recursive method works, and why one should expect that our enhancement is feasible. Our first example is ‘too easy’ for us to improve on previous complexity: sampling a random directed walk on $\mathbb{N}^2$, from $(0,0)$ to $(n,m)$. There are $\binom{n+m}{n}$ such walks, satisfying the binomial relation

\[ \binom{n+m}{n} = \binom{n+m-1}{n-1} + \binom{n+m-1}{n}. \]

\[ ^1 \text{We say quasi-linear to denote the two possibilities altogether.} \]

\[ ^2 \text{Here and in the whole paper, we neglect } \ln N \text{ factors in complexity, when this is } \Theta(N^\gamma), \gamma > 1. \]

\[ ^3 \text{To some extent, one can reduce the space complexity, while degrading the time complexity, see later on.} \]
We stress the fact, important at our aims, that this relation can be rephrased into an algorithmic construction, based on a branching procedure: if one could sample uniformly from the ensembles \( \mathcal{X}_{n-1,m} \) and \( \mathcal{X}_{n,m-1} \), and could efficiently toss a biased coin with parameter \( p_{n,m} = \binom{n+m-1}{n-1} / \binom{n+m}{n} \), then one could sample uniformly from \( \mathcal{X}_{n,m} \), by first tossing the coin, then, depending from the result, appending “north” or “east” in the list of steps, and sampling uniformly from the first or the second ensemble, respectively. Let \( T_{E,N} \) the complexity needed to add one (east or north) step to our constructed object. This is thus a constant, independent of \( n \) and \( m \), and the overall average and worst-case complexities satisfy the associated linear relations

\[
\begin{align}
T_{n,m}^{\text{aver}} &= T_{n,m}^{\text{coin}} + (p_{n,m}(T_{n-1,m}^{\text{aver}} + T_{E}^{\text{step}}) + (1 - p_{n,m})(T_{n-1,m}^{\text{aver}} + T_{N}^{\text{step}})) ; \\
T_{n,m}^{\text{worst}} &= T_{n,m}^{\text{coin}} + \max \left( T_{n-1,m}^{\text{worst}} + T_{E}^{\text{step}} , T_{n,m-1}^{\text{worst}} + T_{N}^{\text{step}} \right).
\end{align}
\]

We can thus recursively push our calculation of complexity to the sole delicate point, the complexity of producing the properly-biased coin. The crucial fact that makes this problem easy is that, although the involved binomials are by themselves huge numbers (with \( \mathcal{O}(N \ln N) \) digits), the ratio \( p_{n,m} \) is just the simple rational function \( \frac{n}{n+m} \), and various performing Buffon machines \([8,10]\) can simulate this coin. So, the classical recursive method has a ‘good’ linear complexity.

The use of the Boltzmann method would go as follows. Consider random walks of length \( N = n + m \), not constrained to the final position, with i.i.d. steps going east or north with probabilities \( p \) and \( 1 - p \). These walks are trivially generated in linear time, and reach \( (n',N-n') \) with probability \( \binom{N}{n'} p^{n'} (1-p)^{N-n'} \). Thus \( n' \) is a random variable, centered around \( pN \). However, even using a biased coin at the optimal value for \( p \) (and neglecting the bit complexity of producing this biased coin), the probability that \( n' = n \) is only of the order of \( N^{-1/2} \), thus we need to perform on average \( N^{1/2} \) independent runs of the algorithm, and we have a ‘bad’ overall average complexity \( \Theta(N^{3/2}) \).

Now let us move on to an apparently similar structure: the partitions of \( n + m \) elements into \( n \) non-empty parts. These structures are counted by the Stirling numbers of the second kind, \( \{n\}^{n+m}_n \) [4] chapt. 5, and satisfy the linear recurrence relation

\[
\binom{n+m}{n} = \binom{n+m}{n-1} + n \binom{n+m-1}{n}.
\]

We stress again that this recursion has an algorithmic counterpart: the element \( n+m \) can either be a singleton (first summand), or can be inserted in one of the \( n \) previous blocks (second summand). If we had a biased coin of parameter \( p_{n,m} = \binom{n+m-1}{n-1} / \binom{n+m}{n} \), and could sample from the ensembles of size up to \( n+m-1 \), we could grow our partition by tossing our coin, and, if the second summand is selected, toss a further integer uniformly in \( \{1, \ldots, n\} \), for choosing the block receiving the new element (this is done with small complexity \( \Theta(\ln n) \)). We thus have a formula for average and worst-case complexities completely analogous to (2), and yet again the whole complexity estimate is pushed towards the determination of the complexity for the biased coin, \( T_{n,m}^{\text{coin}} \).

Now, despite the apparent similarity of the underlying recursions \([11]\) and \([3]\), in this case there is no simple formula for \( p_{n,m} \). The recursive method would have as only resort a painful preprocessing of the values \( \binom{n'+m'}{n'} \) for all \( n' \leq n, m' \leq m \), which is expensive, namely \( \Theta(N^{3} \ln N) \), in terms of both time and space complexities. One could reach \( \Theta(N \ln N) \) space complexity, by recalculating the exact Stirling tables at all rounds, in small congruence classes, and then using the chinese remainder theorem, at a price of a \( \Theta(N^{4} \ln N) \) time complexity.

On the other side, the Boltzmann method works along the same lines as for random walks, thus within linear space, and a time complexity \( \Theta(N^{3/2}) \) [2]. This can be seen, e.g., from the simple
generating function in which we do not fix the number of elements, but only the number of parts

\[ \sum_{m \geq 0} \binom{n+m}{n} x^n = \prod_{y=1}^{n} \frac{1}{1-xy}; \quad \binom{n+m}{n} = \oint_{y=1}^{n} \frac{z^{-m}}{2\pi i} \frac{1}{1-zy}. \tag{4} \]

Here we made use of the Cauchy residue theorem, and obtained a prototype example of saddle-point integral [6] ch. VIII. Note how (4) agrees with (3), as \( \oint_{y=1}^{n} \frac{z^{-m}}{2\pi i} \frac{1}{1-zy} = 0. \)

These partitions are in bijection with certain rectangular \((n+m) \times n\) tableaux [2], whose profile is described by a sequence of \(n\) independent geometric variables \(c_y\), with average \(xy\), and total sum \(\sum y c_y = m\). Tableaux with a given profile are easily uniformly sampled. The sum over \(m\) makes these variables independent, thus providing with a simple efficient sampling, at the price of having at most a \(\Theta(n)\) acceptance probability, a quantity maximised when \(x\) is the unique solution in \([0, n^{-1}]\) of \(x \frac{d}{dx} \ln \left( \prod_{y=1}^{n} \frac{1}{1-xy} \right) = m\), that for large \(N\) leads to the transcendental equation [3]

\[ \frac{m+n}{n} = \frac{-\ln(1-nx)}{nx}. \tag{5} \]

Here comes our crucial observation: the saddle-point formula (4), besides being at the heart of the Boltzmann method for this problem, can also efficiently provide good (and automatisable) estimates for our biased coins \(p_{n,m}\), the missing ingredient in the recursive algorithm. The complex-analysis justification of this claim is well known (see e.g. [6, secs. VIII.2, 3]). What is less known is that, with some extra work (still automatisable), it is possible to convert these estimates into rigorous upper and lower bounds. Better and better estimates will be more and more computationally expensive, but, for most of our coin tossings, we will not need a high precision (knowing \(d\) binary digits of \(p_{n,m}\) is enough for a fraction \(1-2^{-d}\) of the recursive steps).

In our example we have

\[ 1 - p_{n,m} = \left( \oint \frac{dx}{2\pi i x} \left( \frac{x^{-m}}{\prod_{y=1}^{n} (1-xy)} \right) \right) / \left( \oint \frac{dx}{2\pi i x} \left( \frac{x^{-m}}{\prod_{y=1}^{n} (1-xy)} \right) \right), \tag{6} \]

a quantity which is approximatively given by \(x_* n\), where \(x_*\) is the position of the saddle point [5].

There exists also an alternate saddle-point expression for Stirling numbers of the second kind. As we deal with "unlabeled sets of non-empty sets", we also have

\[ \binom{n+m}{n} = \frac{(n+m)!}{n!} \oint \frac{dz}{2\pi i z} \frac{(e^z - 1)^n}{z^{n+m}}. \tag{7} \]

The position of the saddle point satisfies

\[ \frac{m+n}{n} = \frac{z}{1-e^{-z}}. \tag{8} \]

(This is the same equation as (5), if we identify \(1-e^{-z} = nx\).

In this case the recursion (3) is a bit more hidden. We should use the fact that \( \oint \frac{dz}{2\pi i z} (f(z)) = 0\), to get an equivalence with the relation

\[ 0 = \frac{(n+m)!}{n!} \oint \frac{dz}{2\pi i dz} \frac{(e^z - 1)^n}{z^{n+m}}. \tag{9} \]

In this framework we find an expression for \(p_{n,m}\), alternate w.r.t. (6)

\[ 1 - p_{n,m} = \left( \oint \frac{dz}{2\pi i z} \frac{z n z}{n+m} \frac{(e^z - 1)^n}{z^{n+m}} \right) / \left( \oint \frac{dz}{2\pi i z} \frac{(e^z - 1)^n}{z^{n+m}} \right), \tag{10} \]

3
a quantity which is approximatively \( \frac{\zeta}{m + n} \), where \( z_\ast \) is the position of the saddle point (S).

We have thus arrived at the intuition that, through the idea of saddle-point estimates, we can improve the recursive method. In order to make this precise, we need to address three issues:

i) We need to translate equations like (10) into exact bounds, of the form \( \xi - \epsilon \leq p \leq \xi + \epsilon \), where the functions \( \xi = \xi(n, m, z_\ast) \) and \( \epsilon = \epsilon(n, m) \) are sufficiently explicit to admit fast bit-complexity evaluations at the required \( o(\epsilon) \) precision (roughly speaking, these functions can be defined in terms of special functions such as exponentials or logarithms, but not through transcendental equations).

ii) In most of the interesting cases, including (S), the expression \( z_\ast = z_\ast(n, m) \) is the solution of a transcendental equation, so we need an efficient numerical approximation method, and we must control the propagation of the error in \( \xi(n, m, z_\ast(n, m)) \).

iii) We branch “left” or “right” if the random value \( x \in [0, 1] \) is \( x < \xi - \epsilon \) or \( x > \xi + \epsilon \). We need to resolve the case \( x \in [\xi - \epsilon, \xi + \epsilon] \). This may be done through a tighter bound, that uses one more term in the Taylor or Euler-Maclaurin expansions pertinent to the saddle point analysis, or even through a standard step of the recursive method, with the exact construction of the branching probabilities, when \( 2\epsilon \) is small enough so that the associated average complexity is negligible.

In the following sections we outline a general strategy to address these issues, for large families of specifiable combinatorial structures, and describe sufficient conditions for our strategy to apply, easy to verify on any given problem. All along the paper, we illustrate this automated construction on the example of partitions discussed above.

### 3 Linear-time recursive method with oracles

In order to pursue the idea above, it is convenient to separate the study into two parts. In this section, we show that, given a hierarchy of oracles for these bounds, assumed to cost a certain complexity, the average complexity of the recursive method would be quasi-linear. Next, in Section 4 we show how these oracles are implemented, with the announced complexity, from the saddle point expressions.

We assume to have statistical ensembles of combinatorial structures \( X \), with \( d \) size parameters, \( X \in \mathcal{X}_n, n = (n_1, \ldots, n_d) \in \mathbb{N}^d \), and we want to sample from some measure, that could possibly depend on further real-positive weight parameters (that will be considered as fixed, and whose dependence is left implicit). We call \( N = \sum j n_j \).

We also assume to have some generating functions \( Z_n \) associated to these measures, for which we know saddle-point expressions. We suppose to have a recurrence relation, of the form

\[
Z_n = \sum_{j=1}^{k+1} c_j(n) Z_{n-v_j}
\]

where the \( v_j \)'s are vectors in \( \mathbb{N}^d \setminus \mathbf{0} \), and the \( c_j(n) \)'s can be computed easily. We assume that the relation above is associated to a recursive construction of the objects: one can sample from the ensemble \( \mathcal{X}_n \), by choosing \( 1 \leq j \leq k + 1 \) with probability \( \xi_j - \xi_{j-1} \), where, \( \xi_j = Z_{n-1}^{-1} \sum_{i=1}^{j} c_j(n) Z_{n-v_j} \), then sampling from \( \mathcal{X}_{n-v_j} \), and finally performing a further algorithmic step for growing the structure, of complexity \( T^{\text{step}}_n \leq P \cdot (\ln N)^p \) (as discussed above, for the examples of directed walks and partitions of \( n \) elements into \( k \) blocks we have \( p = 0 \) and \( p = 1 \) respectively). Thus, we have an ‘intrinsic complexity’ of the recursive method, \( T_{\text{intr}}(N) \leq \sum M P \cdot (\ln M)^p \leq P N (\ln N)^p \), that would be the complexity in the idealised paradigm in which the oracle and the sampling of random numbers for the branching procedure have zero cost. This complexity summand is inherent to the recursive method, and ineliminable (unless one changes completely the algorithm, and e.g. finds a
more efficient construction). As our main point here is the optimisation of the branching procedure, we will not address the issue of optimising $T_{\text{intr}}(N)$.

Let us denote $\mathcal{N}_{\text{easy}}(N) \subset \mathbb{N}^d$ the set of values $n$ for which the sampling is performed more efficiently with some different method, with complexity $T_{\text{easy}}(n)$, such that $T_{\text{easy}}(n) = o(N)$ for all $n \in \mathcal{N}_{\text{easy}}(N)$. A simple general choice is $\mathcal{N}_{\text{easy}} = \{n \mid \sum_j n_j \leq N_0\}$, where $N_0 = o(N^{1/\alpha})$, and $\alpha$ is the smallest complexity among the ordinary recursive and Boltzmann algorithms. Note that, as a result, the recursive construction always halts at sizes $\gg 1$, and our uncertainty on the thresholds $\xi$ will be $\ll 1$ at all steps. For some special problems, $\mathcal{N}_{\text{easy}}$ could be larger. For example, it could include certain extreme ranges of parameters, $n_j/N = o(1)$ for certain $j$, even for large $N$.

We assume that, for some integer $s_{\text{Max}}$, we have a hierarchy of estimates, of the form $\xi_{j,s}^\ell \leq \xi_j \leq \xi_{j,s}^u$ with $\xi_{j,s}^u - \xi_{j,s}^\ell \leq g_s N^{-s}$, for each level $s < s_{\text{Max}}$. These intervals of uncertainty may overlap, e.g. it may be that $\xi_{j,s}^u > \xi_{j+1,s}^\ell$, although we know that, by construction, $\xi_j < \xi_{j+1}$.

Our algorithm goes as follows: when at size $M$, sample $x \in [0,1]$, and evaluate $\xi_{j}^{\ell,1}$ and $\xi_{j}^{u,1}$ for all $1 \leq j \leq k + 1$. If $j$ is determined univocally, i.e. $\xi_{j-1}^{u,1} < x < \xi_{j}^{\ell,1}$ for some $j$, (this happens with probability at least $1 - k g_1/M$), we go on along the appropriate branch, as in an ordinary recursive algorithm. Otherwise, we need to consider the tighter bound at $s = 2$, and so on. The probability of having to consider a bound of level $s \geq 2$ is at most $k g_{s-1} M^{-s+1}$. We set $g_0 = 1$, in order to make this formula valid at all $s$. If not even the last bound at $s = s_{\text{Max}} - 1$ is tight enough, we perform an ordinary, exact recursive method.

Let $T_s(M)$ be an upper bound to the complexity for the evaluation of the level-$s$ bounds, and $T_{\text{exact}}(M)$ be a bound to the exact recursive method. With respect to the idealised recursive method, with zero-cost oracles, the average complexity has extra terms, of order $\sum M = \sum_{M = N_0}^N g_s M^{-s} T_{s+1}(M)$, from the use of the level-$s$ bound, and of order $\sum_{M = N_0}^N g_s^{s_{\text{Max}}} M^{-s_{\text{Max}}} T_{\text{exact}}(M)$, from the use of our ‘last resort’ exact method. Under moderate assumptions on our complexities, $T_s(M) \leq Q_{s'}(\ln M)^{g_s}$ and $T_{\text{exact}}(M) \leq M^\gamma$, where choosing $s_{\text{Max}} > \gamma + 1$, the overall cost is dominated either by the intrinsic complexity, $T_{\text{intr}}(N) = \Theta(N (\ln N) \rho)$, or by the determination of the level-1 bounds, which takes $\Theta(N (\ln N)^\eta)$. In some cases, one can ensure that the latter logarithmic pre-factor $(\ln N)^\eta$ does not exceed the intrinsic one $(\ln N)^\rho$, by producing the bits of the level-1 bounds as long as they are needed, and performing a realistic analysis at the level of bit complexity. We do not do this here and we just summarise the result of the analysis.

**Proposition 3.1.** Consider a recursive algorithm, with complexities:

1) $R$ for extracting a random bit;
2) $T_s(M) \leq Q_{s'}(\ln M)^{g_s}$ for producing $[\log_2 M^s/g_s]$ digits of the bounds $\xi_{j,s}^\ell$ and $\xi_{j,s}^u$;
3) $T_{\text{exact}}(M) \leq M^\gamma$ for performing an ordinary exact recursive step;
4) $C_d$ for querying the $d$-th digit of $\xi_{j,1}^\ell$, $\xi_{j,1}^u$, if the first $d - 1$ ones are known, with $C_d \leq H d^b e^\gamma d$.

---

4This is the case for Stirling numbers of the second kind. If the number of parts $k$ is sub-linear w.r.t. the number of elements $n$, we can try to randomly colour our elements, with labels from 1 to $k$ and thus with complexity $n \ln k$, and reject the result if any colour is not used, event of probability bounded by $k \exp(-n/k)$ and thus of order 1 if $k \ln n \ll n$. In the opposite regime, the number of parts being $n - k$, with $k$ sub-linear, we can randomly sample $k$ edges of $K_n$, use the connected components as parts of the partition, and accept the resulting configurations: (i) never, if the graph contains any loop; (ii) otherwise, with probability $p = \prod (j^{j-2}-C_j$ if we have $C_j$ tree components of size $j$. Note that the factors for $j = 1, 2$ are just 1. The probability of having any loop at all is bounded by classical results on Erdős-Rényi graphs, while the expected $C_j$ for $j \geq 3$ is of order $k^{j-1}/n^{j-2}$, thus $o(1)$ as long as $k \ll \sqrt{n}$.

5See Appendix [A] for a partial discussion.
If \( \gamma < s_{\text{Max}} - 1 \), and \( \eta < \ln 2 \), then the algorithm runs with average bit complexity bounded by

\[
T \leq T_{\text{intr}}(N) + K N + o(N); \quad K = 3kR + kHH!(1 - e^{\eta-\ln 2})^{-h}-1.
\]

If the hypothesis (4) does not hold, we still have

\[
T \leq T_{\text{intr}}(N) + (3kR + Q_1(\ln N)^{q_1})N + o(N).
\]

4 Construction of the oracles

In this section we explain how one can systematically construct a hierarchy of oracles satisfying the complexity constraints of Proposition 3.1 when the unnormalised measures \( Z_n \) are expressed through a saddle-point integral, in which the integrand has a sufficiently simple form. This is done in subsection 4.3.6 In order to do so, we need some preliminary technical results, discussed in subsections 4.1 and 4.2. A subtle issue on how to determine efficiently the position of the saddle point is discussed in subsection 4.4.

4.1 Formal solution of \( S(x(y)) = y^2 \)

While the systematic expansion in \( n^{-1} \) of saddle point integrals can be performed in several equivalent ways, and among them through the brute-force Taylor expansion of the non-quadratic part of the action, the resulting bounds are more or less performing, depending on the used construction, and some new special tricks come into play.

One of them is the solution of the equation \( S(x(y)) = y^2 \), given that \( x(y) = y + a_2y^2 + a_3y^3 + \ldots \) and \( S(x) = x^2 + b_3x^3 + b_4x^4 + \ldots \) \( (x(y) \text{ and } S(x) \text{ are formal power series}) \). There exist two versions of the problem: finding the appropriate series \( a \), given \( b \), or finding \( b \) given \( a \). We thus need to solve, for all \( k \geq 3 \), \( C_k(a, b) := [y^k]S(x(y)) = 0 \) (lower degrees are matched automatically).

The relevant observation is that \( C_k(a, b) = 2a_k + b_k + C'_k(a, b) \), where \( C'_k \) is a polynomial depending only on the indeterminates \( a_2, \ldots, a_{k-2} \) and \( b_3, \ldots, b_{k-1} \). Thus the system of equations is triangular, for both versions of the problem.

The first few terms for \( b(a) \) read

\[
\begin{align*}
  b_3 &= -2a_2; & b_4 &= 5a_2^2 - 2a_3; \\
  b_5 &= -14a_3^2 + 12a_2a_3 - 2a_4; & b_6 &= 42a_3^2 - 56a_2^2a_3 + 7a_3^2 + 14a_2a_4 - 2a_5; \\
  b_7 &= -132a_5^2 + 240a_3^2a_3 - 72a_2a_3^2 - 72a_2^2a_4 + 16a_3a_4 + 16a_2a_5 - 2a_6. 
\end{align*}
\]

The solution for \( a(b) \) is best visualised separating even and odd coefficients. The first terms are

\[
\begin{align*}
  2a_2 &= -b_3; & 2a_4 &= -2b_3^2 + 3b_3b_4 - b_5; \\
  2a_6 &= -7b_3^3 + 20b_3b_4 - 10b_3b_5^2 - 10b_3b_5 + 4b_4b_5 + 4b_3b_6 - b_7; 
\end{align*}
\]

and

\[
\begin{align*}
  2^3a_3 &= 5b_3^2 - 4b_4; & 2^7a_5 &= 231b_3^4 - 504b_3^2b_4 + 112b_4^2 + 224b_3b_5 - 64b_6; \\
  2^{11}a_7 &= 14586b_3^6 - 51480b_3^4b_4 + 41184b_3^2b_4^2 - 4224b_4^3 + 27456b_3^3b_5 - 25344b_3b_4b_5 \\
  &\quad + 2304b_5^2 - 12672b_3b_6 + 4608b_4b_6 + 4608b_3b_7 - 1024b_8. 
\end{align*}
\]

Note that, in our applications, we will only need the solution \( a(b) \) up to order \( s_{\text{Max}} \); thus, for every problem, where \( s_{\text{Max}} \) is fixed and determined by the complexity of the ordinary recursive step, this is a fixed \( O(1) \) preprocessing.

\[\text{An extension to a larger class of integrands is discussed in Appendix B}\]
4.2 Polynomial bounds to analytic functions

We introduce here a convenient notation for calculating error bounds in the complex plane, that generalizes the standard "±" notation for error propagation from elementary statistics on \( \mathbb{R} \). For \( A, B \in \mathbb{C} \), denote as customary \( f(A) \equiv \{ f(a) \}_A \), \( AB = \{ ab \}_{a \in A, b \in B} \) and \( A + B = \{ a + b \}_{a \in A, b \in B} \). For \( a \in \mathbb{C} \) and \( b \in \mathbb{R}^+ \), let \( a \mp b \) denote the disk in \( \mathbb{C} \) of center \( a \) and radius \( b \). This notation has several nice properties, such as

\[
\begin{align*}
(17) & \quad (a \pm b) + (c \pm d) = (a + c) \pm (b + d); \\
(18) & \quad c(a \pm b) = ca \pm |c|b;
\end{align*}
\]

and, when \( f(z) \) is analytic, as an analytic function on \( D \) always takes its maximum on \( \partial D \), \( f(a \mp b) \subseteq f(a) \mp b' \), with \( b' = \max_\partial |f(a + be^{\theta}) - f(a)| \).

Among the corollaries of this fact, we have for any real positive \( b \)

\[
\exp(\pm b) \subseteq 1 \mp (e^b - 1),
\]

and for real positive values \( a, b, c, d \) such that \( a > b, c > d \)

\[
\begin{align*}
(20) & \quad \frac{a \pm b}{c \pm d} \subseteq \frac{1}{e^2 - d^2}((ac + bd) \pm (ad + bc)).
\end{align*}
\]

We also have, for \( P(z) = p_1z + p_2z^2 + \ldots + p_dz^d \) a polynomial,

\[
e^{P(z)} \in e^{p_1z}e^{\pm(\sum |p_2|z^2 + \ldots + |p_d|z^d)} \subseteq e^{p_1z}\left(1 \pm 2e^{\sum |p_j|z^j - 1}\right) \quad |z| \leq \eta.
\]

We need a similar result for generic functions. Consider the function \( f(z) = f_0 + f_1z + f_2z^2 + \ldots \), analytic and with radius of convergence \( \rho \), and call \( f^{[k]}(z) = f_0 + f_1z + f_2z^2 + \ldots + f_kz^k \). For \( \eta < \rho \), we want to determine a function \( r(\eta) \) such that \( f(z) \in f^{[k]}(z) \pm r(\eta)\eta^k \). Assume that all coefficients \( f_j \) are real positive, for \( j \geq k \). Then the maximum on the disk \( D \) of radius \( \eta \) is realised for \( z = +\eta \), and we have

\[
|f(z) - f^{[k]}(z)| = \sum_{j \geq k} f_j|z|^j \leq |z|^k \sum_{j \geq k} f_j|\eta|^{j-k} = |z|^k \frac{f(\eta) - f^{[k]}(\eta)}{\eta^k};
\]

so that we can state

\[
f(z) \in f^{[k]}(z) \pm |z|^k \frac{f(\eta) - f^{[k]}(\eta)}{\eta^k}.
\]

If \( f_j \)'s are all negative, or have alternating sign, it suffices to take \( f^{[k]}(\eta) - f(\eta) \) or \( f^{[k]}(-\eta) - f(-\eta) \). If we have an explicit decomposition \( f(z) = \sum_{\sigma, \tau = \pm} f_{\sigma\tau}(z) \), where, for \( j \geq k \), \( [z_j] f_{\sigma\tau}(z) \) has sign \( \sigma \) or \( \tau \) depending if \( j \) is even or odd, we can use the previous estimates separately on the four terms, and recombine them using (17). Let us call \( \mathcal{F}^k_{\sigma, \tau} \) the convex cone of analytic functions \( f \), non-singular in \( z = 0 \), such that even/odd coefficients \( f_j \) with \( j \geq k \) have sign \( \sigma \) and \( \tau \), respectively. We say that \( \{ f_{\sigma, \tau}(x) \}_{\sigma, \tau = \pm} \) is a \( k \)-sign-decomposition of \( f(z) \), if \( f(x) = \sum_{\sigma, \tau = \pm} f_{\sigma, \tau}(x) \) and \( f_{\sigma, \tau}(x) \in \mathcal{F}^k_{\sigma, \tau} \). Of course, if \( P \) is a polynomial with real coefficients, of degree \( d \), and we have a \( k \)-sign-decomposition of \( f \), we have a straightforward \( k \)-sign-decomposition of \( f + P \) (if \( d > k \), just attribute positive and negative coefficients of \( P \) to \( f^+ + f^- \), respectively), a fact that we use later on.
Sign-decompositions may look abstract, but are in fact easily obtained in various concrete circumstances. Let us illustrate this within our case example, i.e. for the action $S(z)$ (logarithm of the integrand) in equation (7). Let us parametrise this function according to the position $\zeta$ of the saddle point. We thus have, using (8), $\frac{\hat{r}_{n+1}}{n!} = \frac{\zeta}{1-e^{-\zeta}}$ (so that $\zeta \in \mathbb{R}^+$), and, up to a rescaling,

$$
S_\zeta(x) = (1 - e^{-\zeta}) \ln(e^{\zeta x} - 1) - \zeta \ln(\zeta + x).
$$

We want to present a sign-decomposition (in the variable $x$) that holds simultaneously for all values of $\zeta$ in the range. It is not evident a priori that this is possible. But in fact the two summands of (24) are in $\mathcal{F}_{-1}^1$ and $\mathcal{F}_{1,1}$, respectively. This is obvious for the second one. For the first one, use the striking fact

$$
\ln \frac{e^x - y}{1 - y} = \frac{x}{1 - y} + y \sum_{n,k} \frac{(-1)^{n-1}}{n!(1 - y)^n} T_{n,k} x^n y^k
$$

where the coefficients $T_{n,k}$ are the Eulerian numbers (number of permutations of $n + 1$ objects with $k$ rises) [4 sect. 6.5], and in particular they are all positive integers. Our first summand is related to the expression above, identifying $y = e^{-\zeta} \in [0, 1]$, thus it is in $\mathcal{F}_{-1}^1$.

### 4.3 The hierarchy of saddle-point bounds

Suppose you want to evaluate a hierarchy of bounds to the quantity, analogous to equation (6),

$$
\xi_n = \left( \int \frac{dz}{2\pi i} A(z) \exp(nS(z)) \right) / \left( \int \frac{dz}{2\pi i} B(z) \exp(nS(z)) \right).
$$

Suppose that $A$ and $B$ are polynomials, that the dominant saddle point $z_*$ is isolated, on the positive real axis, that $S''(z_*) > 0$ (so that the steepest-descent Cauchy contour is vertical near to the saddle point), and that we have an allowed topology of contour around the origin, of finite length. Essentially all of these requirements can be relaxed, at the price of making the discussion more convoluted. In particular, the requirement of singularity on $\mathbb{R}^+$ is not really restrictive, as in fact it is essentially implied by the requirement of having a recursive description with positive coefficients. Up to a rescaling of the variables $z$ and $n$, we can set $z_* = S''(z_*)/2 = 1$.

We want to determine a finite sequence of complex numbers $a_s$, and functions $r_s(\eta) : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all $s \leq s_{\text{Max}}$, setting $x(y) = y + a_2 y^2 + \cdots + a_s y^s$,

$$
|S(1 + ix(y)) + y^2| \leq r_s(\eta)|y|^{s+1} \quad \forall |y| \leq \eta;
$$

e.i., $S(1 + ix(y)) \in -y^2 \pm r_s(\eta)|y|^{s+1}$.

From Section 4.1 we know explicitly the unique candidate series $a_i$, in terms of the first $s_{\text{Max}}$ derivatives of $S(z)$ at the saddle point. We need $r_s(\eta) < \infty$ when $\eta$ is large enough for our purposes. As we will see, since we assumed that $z_*$ is isolated, this will always be the case for $n$ large enough. However, the existence of $r_s(\eta)$ is not by itself sufficient. As the bounds are expressed in terms of this function, we need it to be computable. At the light of the results of Section 4.2 we have an automatised construction if we know an explicit sign-decomposition of $S(1 + x)$.

Let us concentrate on the numerator of (26). Fix $\eta$ such that $r_s(\eta) < \infty$, and call $x_\pm = x(\pm \eta)$.

---

\footnote{For example, the treatment is easily extended to the case of $A$ and $B$ with a sign-decomposition, each summand of the decomposition having no singularities in a neighbourhood of the saddle point, as these can be rephrased into polynomials, up to bounds of the form $\pm r(\eta)|z - z_*|^k$ for arbitrary large $k$.}
Divide the contour path $\gamma$ into the path $\gamma_{\text{gauss}}$ image of $[-\eta, \eta]$ w.r.t. $z_s + x(y)$, and a path $\gamma_{\text{rest}}$ from $x_+$ to $x_-$ encircling the origin from the left. We thus have

\[
(28) \quad \oint_\gamma \frac{dz}{2\pi i} A(z) \exp(nS(z)) = \left( \oint_{\gamma_{\text{gauss}}} + \oint_{\gamma_{\text{rest}}} \right) \frac{dz}{2\pi i} A(z) \exp(nS(z))
\]

and we will concentrate on the first summand (we easily bound the second summand at the end). Write $z = z_0 + ix(y)$. Make a change of variables from $z$ to $y$, with Jacobian $J(y) = idx(y)/dy = i(1 + 2a_2 y + \cdots + sa_n y^{n-1})$, to obtain an integral proportional to

\[
(29) \quad \int_{-\eta}^{\eta} dy J(y) A(z_0 + iy(y)) e^{n(-y^2 + rs(y)|y|^s+1)}.
\]

The quantity $J(y)A(z_0 + iy(y))$ is a certain polynomial $P(y) = p_0 + p_1 y + \cdots + p_c y^c$, where the coefficients $p_j$ depend on $z_s$ and the $a_j$’s. We can use (19) on the remainder term in the exponential. So, calling $R = (e^{nr(y)|y|^s+1} - 1)n^{-s+1}$, the integral above is inside the disk

\[
(30) \quad \sum_{j=0}^{c} p_j \int_{-\eta}^{\eta} dy \cdot (1 \pm R|y|^s+1)e^{-ny^2}.
\]

We sum and subtract the integral on the intervals $(-\infty, -\eta]$ and $[\eta, +\infty)$ (we consider the subtracted quantities together with the integral $\oint_{\gamma_{\text{rest}}}$). We are thus led to the study of integrals of the form

\[
(31) \quad I_j^-(n) = \int_{-\infty}^{\infty} dy \cdot y^j e^{-ny^2}; \quad I_j^+(n) = \int_{-\infty}^{\infty} dy \cdot |y|^j e^{-ny^2};
\]

which give the well-known formula

\[
(32) \quad I_j^+(n) = \frac{1 + e^j}{2} n^{-\frac{j+1}{2}} \Gamma\left(\frac{j + 1}{2}\right).
\]

In most cases one can easily bound the portion of the integral associated to $\gamma_{\text{rest}}$ by some quantity of the form $T(n, \eta) = T(z_0^* \exp(-n|\eta^2 - r_s(\eta)|\eta^{s+1}))$, for some finite $\tau$. This is discussed, e.g., in [6, sec. VIII.3], and, for the most frequent problems, this issue has already been solved explicitly in the literature concerning asymptotic enumeration. As the function $r_s(\eta)$ is smooth and finite near $\eta = 0$, the function $\eta^2 - r_s(\eta)\eta^{s+1}$ has a positive maximum for some $\eta > 0$, which is a locally optimal value for our bounds of these terms.

So we have

\[
(33) \quad \xi_n = \frac{A \pm (\delta A + A_{\text{rest}})}{B \pm (\delta B + B_{\text{rest}})}
\]

with

\[
(34) \quad A = \sum_{j=0}^{[c]} p_{2j} n^{-j} \frac{1}{2} \Gamma\left(\frac{j + 1}{2}\right); \quad \delta A = R \sum_{j=0}^{[c]} p_j n^{-\frac{j+s+3}{2}} \Gamma\left(\frac{j + s + 3}{2}\right);
\]

and $A_{\text{rest}} = T(n, \eta)$. We proceed similarly for $B$, and simplify the ratio using (20), to produce upper and lower bounds.

---

*A simple criterium, applicable in many cases including our case example, is given in Appendix C.*
4.4 Following the saddle point

As we have seen, we are in general able to construct analytic expressions for our bounds, that depend on the rational parameters $\alpha_j = n_j/N$ both directly and through the location of the saddle point $z_*(\alpha)$. The equation that determines $z_*$ from $\alpha$, however, is often transcendental. Thus, apparently it should be solved numerically at each round, at the appropriate precision, this being computationally expensive. We can improve on this, by exploiting two facts:

1. Even if we miss the location of the saddle point ‘by a tiny bit’, a less tight version of bounds still exist. In this case one could treat the error factor $\exp(\mathcal{N}e \times (y))$ through equation (21). If we have $\epsilon = o(N^{-1})$, essentially nothing happens at the level of the first bound, which is the dominant source of complexity, as we know from Section 3. More generally, $\epsilon = o(N^{-\gamma/1})$ suffices to have no effect on the level-$s$ bound.

2. The saddle point moves slowly. Namely, if the singularity is isolated, the proper root of the equation $S'_n(z) = 0$ has no multiplicity, and its variation is linear, $|z_*(n-v_j) - z_*(n)|/|z_*(n)| = \mathcal{O}(N^{-1})$. If we determined $z_*(n)$ up to an error $\mathcal{O}(N^{-\gamma})$, we already know $z_*(n-v_j)$ up to an error $\mathcal{O}(\max(N^{-\gamma}, N^{-1})) = \mathcal{O}(N^{-\min(\gamma,1)})$. Let us then perform just one step of Newton iteration [12, sec. 6]. The error is squared, i.e. $\mathcal{O}(N^{-2\min(\gamma,1)})$. Thus, as $2\min(\gamma,1) = \gamma$ is only solved by $\gamma = 0$ and 2, if we find the value $z_*$ for the initial step of our algorithm, at precision $\mathcal{O}(N^{-2})$, we will keep this level of precision at all times just by performing a single Newton iteration at each step. When we need to use a higher level bound (this happens on average finitely many times on the full run), it suffices to perform a few more Newton iterations to determine $z_*$ at higher precision. The average total number of required Newton iterations is thus $\mathcal{O}(N)$.

5 Summary of sufficient conditions for applying our method

We now evince, from the construction of the previous sections, a list of sufficient conditions to be verified on a given combinatorial problem, for it to be amenable to our method. These conditions are essentially analytic, certified by finite expressions, and normally easily achieved from any given explicit combinatorial specification. Once the conditions are established, the construction of the algorithm is automatized.

1) You need a recursion relation $Z_n = \sum_{j=1}^{k+1} c_j(n) Z_{n-v_j}$, with positive $c_j$’s, that translates into an algorithmic recursive construction.

2) You need to establish a bound on a single ‘ordinary’ recursive step, of the form $T_{\text{exact}}(M) \leq M^\gamma (\ln M)^\gamma$. You can then set $s_{\text{Max}}$ to the smallest integer strictly larger than $\gamma + 1$.

3) You should establish a set $\mathcal{N}_{\text{easy}}(N)$ where the generation is sublinear, and the pertinent alternate algorithm. This may consist of all $n$ such that $\sum_j n_j = o(\alpha^{1/\alpha})$, with notations as on page 5.

4) You need a saddle-point expression for the partition functions, $Z_n = \oint \frac{dz}{2\pi i} A(z) \exp(S_n(z))$. You shall determine the associated saddle-point equation, the appropriated topology of the contour, and a bound on the tail terms.

5) You must write $S_n(z)$ as a sum, where each summand depends from a unique size variable $n_j$, and produce a sign-decomposition for each of these summands.

Many examples of specifiable combinatorial structures arising in the literature (in particular, in the extensive compendium of [6]) are accessible to the criteria above, and their systematic analysis opens up a wide range of applications.
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A Some aspects of bit complexity for the branching procedure

In Section 3 we analysed the complexity of the branching part of the recursive method, under the arithmetic paradigm. This accounted for attributing a unit cost to the evaluation of functions up to precision $\Theta(N^{-a})$, and thus requiring $O(\ln N)$ bits, and, similarly, allowed to sample “random real numbers in $[0, 1]$”, and compare them to these functions, again within a unit cost.

Clearly, a more scrupulous analysis of the bit complexity is mandatory. It is nowadays a standard result that the evaluation with $d$ digits of precision of an expression involving certain classes of elementary functions requires a complexity scaling as $d^3$, where $\gamma$ is a finite exponent depending on the class of functions entering the expression (see e.g. [D.E. Knuth, The Art of Computer Programming, Addison-Wesley, 1998], in particular vol. 2, chapt. 4). We do not enter here in the details of this wide branch of Theoretical Computer Science.

Instead, the paradigm for the extraction of a “random real number” $x \in [0, 1]$ is quite easy to describe: we extract the binary digits of $x$ one at the time, as long as needed. Let $\xi \in [0, 1]$ be a threshold value, of which we can query the binary digits, one by one, with complexity $C_j$ for the $j$-th query, and let $R$ be the complexity for querying a random bit. The average complexity $T(\xi)$, to determine if $x \in [0, 1]$ is smaller or larger than $\xi$, is in fact independent of $\xi$, as the probability of halting at the $k$-th digit is always $2^{-k}$, and thus we have $T(\xi) = (R + C_1) + \frac{1}{2}(R + C_2) + \frac{1}{4}(R + C_3) + \ldots = 2R + \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} C_i$.

In our situation, the result is only slightly different. We have two thresholds, $\xi^u$ and $\xi^l$, and we need to determine if $x \leq \xi^l$, $\xi^l < x \leq \xi^u$ or $x > \xi^u$. We now know that the thresholds are dyadic numbers, with $d$ digits of precision, which differ by $2^{-d}$. In this case the complexity is not uniform anymore, and the worst case is

\begin{equation}
\xi^l = 0.1 \overline{0} \ldots 1; \quad \xi^u = 0.1 \overline{0} \ldots 0;
\end{equation}

for which we have the slightly modified expression $T(\xi) = (R + 2C_1) + (R + C_2) + \frac{1}{2}(R + C_3) + \ldots + \frac{1}{2^{d-1}} (R + C_d) \leq 3R + \sum_{i=1}^{d} 2^{-i+2} C_i$. The best case, obtained for $\xi^l = 0.0 \overline{0} \ldots 0$ and $\xi^u = 0.0 \overline{0} \ldots 1$, gives a quite similar lower bound, $T(\xi) \geq (2 - 2^{-d})R + \sum_{i=1}^{d} 2^{-i+2} C_i$.

Under the assumption, presented in the hypothesis (4) of Proposition 3.1 and coherent with our discussion on the complexity of evaluating expressions at a given precision, that $C_d \leq H d^{\alpha} e^{\eta d}$, with $\eta < \ln 2$, the sums above converge, and we obtain the bound $T(\xi) \leq 3R + H h!(1 - e^{-\eta \ln 2})^{-h-1}$ appearing in the forementioned proposition. At this aim it is useful to perform the approximation

$$\sum_{j \geq 0} j^h e^{-aj} \leq h! \sum_{j \geq 0} \binom{j+h}{h} e^{-aj} = h!(1 - e^{-a})^{-h-1}.$$

B Sums of logs and Euler–Maclaurin

Our construction of the bounds presented in Section 4 as a function of $n$ but performed once and for all in a preprocessing phase of constant complexity, assumes that we can present a sign-decomposition, and the analytic evaluation of the derivatives of the action, valid for all ranges of $\alpha = n/N$. A case in which this is possible is when the action has the form

\begin{equation}
S_n(z) = \sum_j n_j S_j(z),
\end{equation}

and for each $S_j$ a sign-decomposition is produced. As we have seen, the saddle point integral (7) is already in this form. But this does not necessarily occur in all the problems we aim to analyse. For
example, already in our alternate expression we encountered a different situation, as we have (we change variables from $z$ to $z/n$, and omit an overall constant).

$$S_{(n,m)} = -m \ln(z) - \sum_{y=1}^{n} \ln \left(1 - \frac{zy}{n}\right).$$

Apparenty, we need to construct dynamically our bounds for all the different values of $n$ we encounter, a procedure that would be too expensive. In fact this computation can be avoided, and we can convert (37) in a form analogue to (36). To see this heuristically, remark that, for large $n$,

$$n \sum_{y=1}^{n} \ln \left(1 - \frac{zy}{n}\right) \simeq n \int_{0}^{1} dy \ln (1 - zy) = n \left(-1 - \frac{(1 - z) \ln(1 - z)}{z}\right).$$

We can transform the ‘$\simeq$’ sign into a systematic hierarchy of bounds, in inverse powers of $n$, using the customary Euler-Maclaurin expansion (see e.g. [14, Chapt. 7]).

Let us write this more explicitly for the case “sum of logs” that often occurs in saddle point integrals associated to specifiable combinatorial structures. Let $f(z, x)$ a smooth function of two variables, let $f', f'', \ldots$ denote differentiation w.r.t. the second argument, and let $g(z, x) = f'(z, x)/f(z, x)$. We have

$$\sum_{y=1}^{n} \ln f(z, y/n) \in n \int_{0}^{1} dy \ln f(z, y) + \ln \sqrt{\frac{f(z, 1)}{f(z, 0)}} + \sum_{k=1}^{k_{\text{Max}}} (-1)^{k-1} \frac{B_{k}}{(2k)!} \left(g^{(2k-2)}(z, 1) - g^{(2k-2)}(z, 0)\right) + n^{-2k_{\text{Max}}+2} \frac{2\zeta(2n)}{(2\pi)^{2n}} \int_{0}^{1} dy \vert g^{(2k_{\text{Max}}-1)}(z, y)\vert;$$

(where the $B_{j}$’s are the Bernoulli numbers). If one has even a moderate control on $g^{(k)}(z, y)$ for $y \in [0, 1]$, an estimate in this form is easily integrated in the general construction of the bounds performed in Section 4.3.

\section{A criterium for tails pruning}

Here we present a simple criterium for bounding the contour integral on the open path $\gamma_{\text{rest}}$, discussed in subection 4.3. We also discuss a very elementary bound on the “tail completion”, the extra term arising from the fact that we replace the Gaussian integrals on a finite interval by the complete integral, by adding and subtracting a correction term.

Let us first consider the integral over $\gamma_{\text{rest}}$. Suppose that $\overline{S(z)} = S(\overline{z})$, and $A(z) = z^{i}$. In this case $x_{\pm}$ are complex conjugates. Suppose that, at the radius $\rho = |z_{*} + x_{\pm}|$, $\Re S(\rho \exp(i\theta))$ is monotone for $\theta \in [\arg(z_{*} + x_{\pm}), \pi]$. In such a case we have

$$\left| \int_{\gamma_{\text{rest}}} \frac{dz}{2\pi i z} A(z) \exp\left(n(S(z) - S(z_{*})\right) \leq \rho^{2} \exp\left(n \Re(S(z_{*} + x_{\pm}) - S(z_{*}))\right).$$

\footnote{Note that we already know that, in this case, we should perform the integral on a contour with $|z| < 1$, so that we have no troubles with the radius of convergence of the log.}
For the tails of the Gaussian integrals, any customary bound on the \( \text{erf} \) error function makes the game. A good compromise between simplicity and tightness is based on the following calculation, valid for all \( k \in \mathbb{N} \) and \( a \in \mathbb{R}^+ \):

\[
\int_a^\infty dx x^k e^{-\frac{x^2}{2}} = \int_0^{\infty} dx (a + x)^k e^{-\frac{a^2}{2} - ax - \frac{x^2}{2}} \leq e^{-\frac{a^2}{2}} \int_0^{\infty} dx (a + x)^k e^{-ax}
\]

\[
= e^{-\frac{a^2}{2}} \sum_{h=0}^k \binom{k}{h} a^{k-h} \cdot h! a^{-h-1} \leq a^{k-1} e^{-\frac{a^2}{2}} \sum_{h=0}^k \left( \frac{k}{a^2} \right)^h \leq a^{k-1} e^{-\frac{a^2}{2} R},
\]

where \( R \) may be chosen to be \( \max (k, (k/a)^2) \), or also, if \( k < a^2 \), extending the geometric sum to infinity, \( (1 - k/a^2)^{-1} \). For \( a \gg 1 \) and \( k = \mathcal{O}(1) \), as in our application (where \( a \sim \sqrt{N} \)), the second estimate is the more tight.