Stokes-type Integral Equalities for Scalarly Essentially Integrable Locally Convex Vector Valued Forms which are Functions of an Unbounded Spectral Operator

Benedetto Silvestri
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Abstract. In this work we establish a Stokes-type integral equality for scalarly essentially integrable forms on an orientable smooth manifold with values in the locally convex linear space $⟨B(G), σ(B(G), N)⟩$, where $G$ is a complex Banach space and $N$ is a suitable linear subspace of the norm dual of $B(G)$. This result widely extends the Newton-Leibnitz-type equality stated in one of our previous articles. To obtain our equality we generalize the main result of that article, and employ the Stokes theorem for smooth locally convex vector valued forms established in a prodromic paper. Two facts are remarkable. Firstly these forms need not be smooth nor even continuously differentiable. Secondly these forms need not be smooth nor even continuously differentiable.

Introduction 1. In this work we establish in Thm. 13 a Stokes-type integral equality for scalarly essentially integrable $(B(G), σ(B(G), N))$-valued forms on an orientable smooth manifold, where $G$ is a complex Banach space. This result widely extends the Newton-Leibnitz-type equality established in [3] Cor. 2.33. To obtain the equality we employ the Extension Thm. 4 a generalization of [3] Thm. 2.25 along with the Stokes theorem for smooth locally convex vector valued forms [4] Thm. 2.54. Two facts are remarkable. Firstly these forms are functions of a possibly unbounded scalar type spectral operator in $G$. Secondly these forms need not be smooth nor even continuously differentiable.

Notation 2. In the present work we employ the notation of [3] and these of [4], with the following two remarks. First what in [3] is called “Radon measure” and meant measure in the sense of Bourbaki [1] Ch. III, §1, no.3, Def. 2, here accordingly will be called simply “measure”. Second if $Z$ is a $K$-locally convex vector space with $K ∈ \{R, C\}$, then we let $Z' = L(Z, K)$ denote the topological dual of $Z$.

If $G$ is a $C$-Banach space, then let $CIO(G)$ denote the set of closed operators in $G$. If $X$ is a locally compact space and $µ$ is a measure on $X$, then a map $f : X → C$ is scalarly essentially $µ$-integrable or simply essentially $µ$-integrable iff $R ∘ i^C(f) ∲ f$ and $I ∘ i^C(f) ∲ f$ are essentially $µ$-integrable, where $R, I ∈ L(C_R, R)$ are the real and imaginary part respectively.

We recall from [3] pg. 39-40 that if $(Z, τ)$ is a Hausdorff locally convex space over $K ∈ \{R, C\}$, then by definition $f : X → (Z, τ)$ is scalarly essentially $(µ, Z)$-integrable, or $f : X → Z$ is scalarly essentially $(µ, Z)$-integrable with respect to the topology $τ$, iff $ψ ∋ f$ is essentially $µ$-integrable for every $ψ ∈ (Z, τ)'$ and the weak integral of $f$ belongs to $Z$, namely there exists a necessarily unique element $s ∈ Z$ such that $ψ(s) = \int (ψ ∋ f) dµ$ for every $ψ ∈ (Z, τ)'$. In such a case we shall define $\int f dµ := s$.

Let $N ∈ Z^*_∗$, define $P^{[N]} : R^N → R^{N-1}$, $x ↦ x ↣ [1, N - 1] ∩ Z$ if $N > 1$; $x ↦ 0$ if $N = 1$. Let $M$ be a nonzero dimensional manifold with boundary and let $(U, φ)$ be a boundary chart of
M, define $\phi^M := (\mathcal{D}[\dim M] \circ \mathcal{R}^M) \circ \phi \circ \mathcal{U}_{\mathcal{L}(\partial M)}$, where $f_x = f \upharpoonright \text{Range}(f)$ for any map $f$. Let $\mathcal{U}$ be a collection of charts of $M$, and let $\mathcal{U}_{\theta}$ be the subcollection of those elements in $\mathcal{U}$ that are boundary charts, define $\mathcal{U}^\theta := \{(U \cap \partial M, \phi^M) \mid (U, \phi) \in \mathcal{U}_{\theta}\}$. If $\mathcal{U}$ is an atlas of $M$, then $\mathcal{U}^\theta$ is an atlas of $\partial M$, moreover if $M$ is oriented and $\mathcal{U}$ is oriented, then $\mathcal{U}^\theta$ is oriented and $(U \cap \partial M, \phi^M)$ is $\gamma$-oriented iff $(U, \phi) \in \mathcal{U}$ is $\gamma$-oriented, with $\gamma \in \{1, -1\}$.

We fix the following data. A $C$-Banach space $G$; a possibly unbounded scalar type spectral operator $R$ in $G$, let $\sigma(R)$ be its spectrum and let $E$ be its resolution of identity; an $E$–appropriate set $N$ [3, Def.2.11]; a scalar type spectral operator $T \in B(G)$ and let $\sigma(T)$ denote its spectrum; locally compact spaces $X, Y$ and measures $\mu$ and $\nu$ on $X$ and $Y$ respectively; a finite dimensional smooth manifold $M$, with or without boundary, such that $N := \dim M \neq 0$.

**Theorem 3.** Let $\{\sigma_n\}_{n \in \mathbb{N}}$ be an $E$–sequence, let the maps $X \ni x \mapsto f_x \in \text{Bor}(\sigma(R))$ and $Y \ni y \mapsto u_y \in \text{Bor}(\sigma(R))$ be such that $f_x \in L^\infty(\sigma(R))$, $\mu - \text{l.a.e.}(X)$ and $u_y \in L^\infty(\sigma(R))$, $\nu - \text{l.a.e.}(Y)$. Let $X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G), N) \rangle$ and $Y \ni y \mapsto u_y(R) \in \langle B(G), \sigma(B(G), N) \rangle$ be scalarly essentially $(\mu, B(G))$–integrable and $(\nu, B(G))$–integrable respectively, while let $g, h \in \text{Bor}(\sigma(R))$. If for all $n \in \mathbb{N}$,

\[ g(R) \int f_x(R) d \mu(x) \subseteq h(R) \int u_y(R) d \nu(y), \]

\[ g(R) \int f_x(R) d \mu(x) \uparrow \Theta = h(R) \int u_y(R) d \nu(y) \uparrow \Theta. \]

In (1) the weak-integrals are with respect to the measures $\mu$ and $\nu$ and with respect to the $\sigma(B(G, \sigma_n), N_{\sigma_n})$-topology, while in (2)

\[ \Theta \doteq \text{Dom}(g(R) \int f_x(R) d \mu(x)) \cap \text{Dom}(h(R) \int u_y(R) d \nu(y)), \]

and the weak-integrals are with respect to the measures $\mu$ and $\nu$ and with respect to the $\sigma(B(G), N)$-topology.

**Proof.** (1) is meaningful by [3, Thm. 2.22]. By [3, (1.18)], for all $z \in \Theta$

\[ g(R) \int f_x(R) d \mu(x) z = \lim_{n \in \mathbb{N}} E(\sigma_n) g(R) \int f_x(R) d \mu(x) z \]

by [2] Thm. 18.2.11(g)] and [3] (2.25)]

\[ = \lim_{n \in \mathbb{N}} g(R) \int f_x(R) d \mu(x) E(\sigma_n) z \]

by [3] (2.31)] and [3] Lemma 1.7] applied to $g(R)$

\[ = \lim_{n \in \mathbb{N}} g(R) \upharpoonright G_{\sigma_n} \int f_x(R) \upharpoonright G_{\sigma_n} d \mu(x) E(\sigma_n) z \]
by hypothesis (1)
\[
\lim_{n \in \mathbb{N}} h(R_{\sigma_n} \uparrow G_{\sigma_n}) \int u_y(R_{\sigma_n} \uparrow G_{\sigma_n}) \, d\nu(y) E(\sigma_n) z
\]
by what above proven and by replacing \( g \) with \( h, f \) with \( u \) and \( \mu \) with \( \nu \)

(3)
\[
h(R) \int u_y(R) \, d\nu(y) z.
\]

\[
\square
\]

**Theorem 4 (\( \sigma(B(G), N) \)–Extension Theorem).** Let \( X \ni x \mapsto f_x \in \text{Bor}(\sigma(R)) \) be such that \( \tilde{f_x} \in L^\infty E(\sigma(R)), \mu - \text{l.a.e.}(X) \) and \( X \ni x \mapsto f_x(R) \in \langle B(G), \sigma(B(G), N) \rangle \) be scalarly essentially \((\mu, B(G))\)-integrable. Moreover let \( Y \ni y \mapsto u_y \in \text{Bor}(\sigma(R)) \) be such that \( \tilde{u_y} \in L^\infty E(\sigma(R)), \nu - \text{l.a.e.}(Y) \) and \( Y \ni y \mapsto u_y(R) \in \langle B(G), \sigma(B(G), N) \rangle \) be scalarly essentially \((\nu, B(G))\)-integrable. Finally let \( g, h \in \text{Bor}(\sigma(R)) \) and assume that \( \exists \]

(4)
\[
h(R) \int u_y(R) \, d\nu(y) \in B(G).
\]

If \( \{\sigma_n\}_{n \in \mathbb{N}} \) is an \( E \)-sequence and for all \( n \in \mathbb{N} \)

(5)
\[
g(R_{\sigma_n} \uparrow G_{\sigma_n}) \int f_x(R_{\sigma_n} \uparrow G_{\sigma_n}) \, d\mu(x) \subseteq h(R_{\sigma_n} \uparrow G_{\sigma_n}) \int u_y(R_{\sigma_n} \uparrow G_{\sigma_n}) \, d\nu(y),
\]
then

(6)
\[
g(R) \int f_x(R) \, d\mu(x) = h(R) \int u_y(R) \, d\nu(y).
\]

In (5) the weak-integral are with respect to the measures \( \mu \) and \( \nu \) and with respect to the \( \sigma(B(G_{\sigma_n}), N_{\sigma_n}) \)-topology, while in (6) the weak-integral is with respect to the measures \( \mu \) and \( \nu \) and with respect to the \( \sigma(B(G), N) \)-topology.

Notice that \( g(R) \) and \( h(R) \) are possibly **unbounded** operators in \( G \).

Proof. (4) and (2) imply

(7)
\[
g(R) \int f_x(R) \, d\mu(x) \subseteq h(R) \int u_y(R) \, d\nu(y).
\]

Let us set

(8)
\[
(\forall n \in \mathbb{N})(\delta_n := \frac{1}{|g|([0, n]))}.
\]

We claim that

(9)
\[
\begin{cases}
\bigcup_{n \in \mathbb{N}} \delta_n = \sigma(R) \\
n \geq m \Rightarrow \delta_n \supseteq \delta_m \\
(\forall n \in \mathbb{N}) (g(\delta_n) \text{ is bounded}).
\end{cases}
\]

\[\text{For instance but not necessarily when } \tilde{h} \in L^\infty E(\sigma(R)) \text{ since in such a case Thm. 18.2.11 implies } h(R) \in B(G)\]
Since $|g| \in \text{Bor}(\sigma(R))$ we have $\delta_n \in \mathcal{B}(\mathbb{C})$ for all $n \in \mathbb{N}$, so $\{\delta_n\}_{n \in \mathbb{N}}$ is an $E$–sequence, hence by [3] (1.18)

$$\lim_{n \to \infty} E(\delta_n) = 1;$$

with respect to the strong operator topology on $B(G)$. Indeed the first equality follows by

$$\bigcup_{n \in \mathbb{N}} \delta_n = \bigcup_{n \in \mathbb{N}} |g|([0, n]) = |g|\left(\bigcup_{n \in \mathbb{N}} [0, n]\right) = |g|(\mathbb{R}^+) = \text{Dom}(g) \supseteq \sigma(R),$$

the second by the fact that $|g|$ preserves the inclusion, the third by the inclusion $|g|(\delta_n) \subseteq [0, n]$. Hence our claim. By the third statement of (9), $\delta_n \in \mathcal{B}(\mathbb{C})$ and [3] Lemma 1.16 we obtain

$$\text{(11)} \quad (\forall n \in \mathbb{N})(E(\delta_n)G \subseteq \text{Dom}(g(R))).$$

By [3] (2.25) and (11) for all $n \in \mathbb{N}$

$$\int f_\lambda(R) \, d\mu(x) E(\delta_n)G \subseteq E(\delta_n)G \subseteq \text{Dom}(g(R)).$$

Therefore

$$(\forall n \in \mathbb{N})(\forall v \in G) \left(E(\delta_n)v \in \text{Dom}\left(g(R) \int f_\lambda(R) \, d\mu(x)\right)\right).$$

Hence by (10)

$$\text{(12)} \quad D \supseteq \text{Dom}\left(g(R) \int f_\lambda(R) \, d\mu(x)\right) \text{ is dense in } G.$$

Now $\int f_\lambda(R) \, d\mu(x) \in B(G)$ and $g(R)$ is closed by [2] Thm. 18.2.11, so by [3] Lemma 1.15 we find that

$$\text{(13)} \quad g(R) \int f_\lambda(R) \, d\mu(x) \text{ is closed}.$$ 

Next (4) and (7) imply

$$\text{(14)} \quad g(R) \int f_\lambda(R) \, d\mu(x) \in B(D, G).$$

Now (13), (14) and [3] Lemma 1.16 imply that $D$ is closed in $G$, therefore by (12)

$$D = G;$$

therefore the statement follows by (7).

### Definition 5

Let $V$ be an open neighbourhood of $\sigma(R)$, $l \in \mathbb{R}_+ \cup \{+\infty\}$ such that $]-l, l[ \subseteq V$, and $F : V \to \mathbb{C}$ be analytic. Moreover let $W$ be a set and $g : W \to \mathbb{R}$ such that $g(W) \subseteq ]-l, l[$. Let $F_t : V \ni \lambda \mapsto F(t\lambda) \in \mathbb{C}$ with $t \in ]-l, l[$, then define the following operator valued map originating by the Borel functional calculus of the operator $R$

$$F_{t,\lambda}^R : W \ni x \mapsto F_{g(x)}(R) \in \text{ClO}(G).$$
Corollary 6. Let $V$ be an open neighbourhood of $\sigma(T)$, $l \in \mathbb{R}_+ \cup \{+\infty\}$ such that $]-l,l[ \subseteq V$, and $F : V \to \mathbb{C}$ be analytic. Moreover let $n, p \in \mathbb{Z}^+$, $W$ be an open set of $\mathbb{R}^n$, and $g \in \mathcal{C}^p(W, \mathbb{R})$ such that $g(W) \subseteq ]-l,l[$. Thus $\zeta^{T}_{F,g} \in \mathcal{C}^p(W, B(G))$, and for every $i \in [1, n] \cap \mathbb{Z}$ we have

$$\frac{\partial \zeta^{T}_{F,g}}{\partial e_i} = \frac{\partial g}{\partial e_i} \cdot T\zeta^{T}_{F,g}.$$  

Proof. $]-l,l[ \ni t \mapsto F_t(T) \in B(G)$ is smooth since [3, Thm. 1.21], therefore the first sentence of the statement follows since composition of $\mathcal{C}^p$-maps is a $\mathcal{C}^p$-map, while the equality follows by the Chain Rule and by [3, Thm. 1.21].

Definition 7. Let $k \in \mathbb{Z}_+$, $\omega \in \text{Alt}^k(M)$ and $(U, \phi : U \to W)$ be a chart of $M$. Define

$$\begin{cases}
\omega^\phi : M(k, N, <) \to \mathcal{A}(W); \\
I \mapsto (\tau^M_{U})^*(\omega)(\partial^\phi_{I_1}, \ldots, \partial^\phi_{I_k}) \circ \phi^{-1}.
\end{cases}$$

Moreover let $V$ be an open neighbourhood of $\sigma(R)$ such that $R \cdot V \subseteq V$, $F : V \to \mathbb{C}$ be analytic and let $\delta \in \mathcal{B}(\mathbb{C})$ be such that

$$\text{Range}(\zeta^{R_\delta \mid G_0}_{F,\omega^\phi}) \subseteq B(G_0);$$

$$\zeta^{R_\delta \mid G_0}_{F,\omega^\phi} \in \mathcal{L}^1(\langle W, B(G_0), \sigma(B(G_0), N_0) \rangle, \lambda).$$

Define

$$f^\delta_{\omega^\phi} : M(k, N, <) \ni I \mapsto f^\delta_{\omega^\phi} := \zeta^{R_\delta \mid G_0}_{F,\omega^\phi} \circ \phi,$$

and then define $[\omega, \phi, \delta, F] \in \text{Alt}^k(U, M; \langle B(G_0), \sigma(B(G_0), N_0) \rangle, \lambda)$ such that

$$[\omega, \phi, \delta, F] := \sum_{I \in M(k, N, <)} f^\delta_{\omega^\phi} \wedge k \wedge x^\phi_{I_{s-1}}.$$  

Remark 8. Let $k \in \mathbb{Z}_+$, $\omega \in \text{Alt}^k(M)$ and $(U, \phi : U \to W)$ be a chart of $M$. Let $\sigma \in \mathcal{B}(\mathbb{C})$ be bounded, thus $R_\sigma \mid G_0 \in B(G_0)$ since [3, Lemma 1.7]. Hence $\zeta^{R_\sigma \mid G_0}_{F,\omega^\phi} \in \mathcal{A}_c(W, \langle B(G_0), \| \cdot \| \rangle)$ by Cor. 6, so $f^\delta_{\omega^\phi} \in \mathcal{A}_c(U, \langle B(G_0), \| \cdot \| \rangle)$ and then $[\omega, \phi, \sigma, F]$ is smooth w.r.t. the norm topology, namely $[\omega, \phi, \sigma, F] \in \text{Alt}^k(U, M; \langle B(G_0), \| \cdot \| \rangle)$. Finally as a result $\zeta^{R_\delta \mid G_0}_{F,\omega^\phi}$ is norm continuous and compactly supported, therefore $\zeta^{R_\delta \mid G_0}_{F,\omega^\phi}$ is Lebesgue integrable w.r.t. the norm topology and its integral belongs to $B(G_0)$.

Remark 9. Let $\delta \in \mathcal{B}(\mathbb{C})$. The norm topology on $B(G_0)$ is stronger than the topology $\sigma(B(G_0), N_0)$ since this last is the weakest topology on $B(G_0)$ among those for which $N_0$ is a set of continuous functionals, and since $N_0 \subseteq B(G_0)'$. Thus we can

\[\text{for instance when } \delta \text{ is bounded see Rmk.} 8\]
and shall identify $A(U, \langle B(G_0), \| \cdot \| \rangle)$ as a $A(U)$-submodule of $A(U, \langle B(G_0), \sigma(B(G_0), N_0) \rangle)$ and $\text{Alt}^k(U, M; \langle B(G_0), \| \cdot \| \rangle)$ as a $A(U)$-submodule of $\text{Alt}^k(U, M; \langle B(G_0), \sigma(B(G_0), N_0) \rangle)$.

**Remark 10.** Let $\delta \in \mathcal{B}(\mathbb{C})$, then any map defined on $X$ and with values in $B(G_0)$ that is scalarly essentially $\mu$-integrable w.r.t. the norm topology it is also scalarly essentially $\mu$-integrable w.r.t. the $\sigma(B(G_0), N_0)$-topology since $N_0 \subseteq B(G_0)$.

**Definition 11.** Let $k \in \mathbb{Z}_+$, $\omega \in \text{Alt}^k(M)$ and $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in D}$ be an atlas of $M$. Let $V$ be an open neighbourhood of $\sigma(R)$ such that $R \cdot V \subseteq V$, $F : V \to \mathbb{C}$ be analytic and $\delta \in \mathcal{B}(\mathbb{C})$ be such that $(15)$ holds for $\phi = \phi_\alpha$ and for every $\alpha \in D$. Define $[\omega, \delta, F] \in \text{Alt}^k(M; \langle B(G_0), \sigma(B(G_0), N_0) \rangle)$, $\lambda$ such that for all $\alpha \in D$

$$(i_{U_\alpha}^M)^*([\omega, \delta, F]) = [\omega, \phi_\alpha, \delta, F].$$

**Definition 12.** Let $k \in \mathbb{Z}_+^*$, $\omega \in \text{Alt}^{k-1}(M)$ and $i \in [1, k] \cap \mathbb{Z}$. Define $d_i(\omega) \in A(M)$ and $n_i(\omega) \in \text{Alt}^{k}(M)$ such that for any given atlas $\mathcal{U}$ of $M$ we have for every $(U, \phi) \in \mathcal{U}$

$$(i_{U}^M)^*(d_i(\omega)) := \phi_i^a [(i_{U}^M)^*(\omega)(\partial^\phi_1, \ldots, \partial^\phi_{i-1}, \partial^\phi_{i+1}, \ldots, \partial^\phi_k)],$$

$$(i_{U}^M)^*(n_i(\omega)) := (i_{U}^M)^*(\omega)(\partial^\phi_1, \ldots, \partial^\phi_{i-1}, \partial^\phi_{i+1}, \ldots, \partial^\phi_k) \wedge \frac{d}{dx^s},$$

where $\widehat{z}$ stands for $z$ missing.

The above two definitions are well-set since the usual gluing lemma for smooth forms, since the extension of the gluing lemma via charts at scalarly essentially integrable locally convex vector valued maps [4, Rmk.1.2], and since the extension of the gluing lemma via charts at smooth locally convex vector valued maps [4, Notation], where the compatibility in both the definitions is ensured by the following simple fact

$$(i_{U_{a,\beta}}^M)^*(\omega)(\partial^\phi_{a,\beta}) \circ (i_{U_{a,\beta}}^M)^* = (i_{U_{a,\beta}}^M)^*(\omega)(\partial^\phi_{a,\beta}, \ldots, \partial^\phi_{b,\beta}),$$

where $U_{a,\beta} = U_a \cap U_\beta$ and $\phi_{a,\beta} = (\phi_a \circ i_{U_{a,\beta}}^M)_\beta$.

**Theorem 13 (Stokes equality for $\sigma(B(G), N)$-integrable forms functions of an unbounded operator).** Let $M$ be oriented with boundary and $\omega \in \text{Alt}_{\text{loc}}^{N-1}(M)$. Let $V$ be an open neighbourhood of $\sigma(R)$ such that $R \cdot V \subseteq V$ and $F : V \to \mathbb{C}$ be analytic. Assume that there exists a finite family $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in D}$ of oriented charts of $M$ such that $\{U_\alpha\}_{\alpha \in D}$ is a covering of $\text{supp}(\omega)$ and

1. $\tilde{F}_t \in \Omega_{E}^\infty(\sigma(R))$ for every $t \in \mathbb{R}$, and for all $\alpha \in D$ such that $\phi_\alpha$ is a boundary chart, the map

$$\zeta_R^E \phi_\alpha : \phi_\alpha(U_\alpha \cap \partial M) \to \langle B(G), \sigma(B(G), N) \rangle,$$

is scalarly essentially $(\lambda_{\phi_\alpha(U_\alpha \cap \partial M)}, B(G))$-integrable,
(2) $(\tilde{\omega}^R)_{t} ∈ \Omega^\infty_{E}(\sigma(\mathbb{R}))$ for every $t ∈ \mathbb{R}$, and for all $α ∈ D$ and $i ∈ [1, N] \cap \mathbb{Z}$, the map

$$\tilde{c}^R_{\frac{dF}{dt}} φ_i^α : ω_t(U_a) → ⟨B(\mathcal{G}) , σ(B(\mathcal{G}), N)⟩,$$

is scalarly essentially $(φ_α(U_a), B(\mathcal{G}))-integrable$;

where

$$ω_i^α := (i^M_{U_a})^*(ω(φ_i^α, \ldots , φ_i^α, \ldots , φ_i^α) ∧ φ_α^{-1}) .$$

Thus

$$R \int \sum_{i=1}^{N} (-1)^{i-1} b_i(ω) \cdot [n_i(ω), σ(R), \frac{dF}{dt}] = \int (t^M_{\partial M})^*(\{ω, σ(R), F\});$$

where the integrals belong to $B(\mathcal{G})$ and are with respect to the $σ(B(\mathcal{G}), N)$ topology and in case $\partial M = ∅$, then the integral in the right-hand side has to be understood equal to 0.

Remark 14. Let $U = \{(U_α, φ_α)\}_{α ∈ D}$ and $U^\partial$ be as in Notation. Thus $U^\partial$ is a family of oriented charts of $\partial M$ such that $\{Q_α\}_{α ∈ D}$, with $Q_α = U_α ∩ \partial M$ for every $α ∈ D$, is a collection of open sets of $\partial M$ and a covering of $\text{supp}(ω) ∩ \partial M$ compact set of $\partial M$. Next set $D^+ = D ∪ \{↑\}$, $U_+ = \bigcup_{α} \text{Supp}(ω)$, $Q_+ = \bigcup_{α} \text{Supp}(ω) ∩ \partial M$ and let $\{(ψ_α)\}_{α ∈ D^+}$ be a smooth partition of unity of $\partial M$ subordinate to $\{(U_α)\}_{α ∈ D}$ and $\{(k_α)\}_{α ∈ D^+}$ be a smooth partition of unity of $\partial M$ subordinate to $\{Q_α\}_{α ∈ D}$. Thus since (31) and (30) applied to $δ = σ(R)$ the statement of Thm. [13] reads as follows

$$R \int \sum_{α ∈ D} γ_{φ_α}(i^M_{U_α} ∩ \partial M) \cdot \sum_{i=1}^{N} (-1)^{i-1} \frac{∂α_i^φ}{∂e_i} \zeta^R_{\frac{dF}{dt}} φ_i^α dλ_{φ_α(U_α)} =$$

$$\int \sum_{α ∈ D} γ_{φ_α}(i^M_{U_α} ∩ \partial M) \cdot (φ_i^α)^{-1})^*(k_α) \left(\zeta^R_{\frac{dF}{dt}} φ_i^α \circ i^M_{φ_α(U_α ∩ \partial M)} \circ (i^{R-1}_{\partial M} \circ i^{R-1}_{φ_α(U_α ∩ \partial M)})^h \right) dλ_{φ_α(U_α ∩ \partial M)} ;$$

where $i^{R-1}_{\partial M} : ℝ^{N-1} → ℝ^N$ is such that if $N > 1$, then $Pr^{N}_{k} \circ i^{R-1}_{\partial M} = Pr^{R-1}_{k}$ if $k ∈ [1, N-1] \cap \mathbb{Z}$, and $Pr^{N}_{0} \circ i^{R-1}_{\partial M} = 0_{ℝ^{N-1}}$ the constant map on $ℝ^{N-1}$ equal to 0; while $i^{R}_{0} : ℝ → 0$. Notice that $(i^{R}_{\partial M} ∩ i^{R}_{φ_α(U_α ∩ \partial M)})^h$ is a diffeomorphism of $φ_α(U_α ∩ \partial M)$ onto $φ_α(U_α ∩ \partial M)$ thus the right-hand side of the above equality is well-set since hypothesis [1] and the theorem of change of variable in multiple integrals.

Remark 15. The strategy employed to obtain Thm. [13] is as follows: Given an $E$-sequence of bounded sets $\{(σ_α)_{m ∈ \mathbb{N}}$ we apply for every $n ∈ \mathbb{N}$ the Stokes Thm. for locally convex vector-valued forms [4, Thm. 2.54] to the $⟨B(\mathcal{G}_α), σ(B(\mathcal{G}_α), N_α)⟩$-valued form $[ω, σ_α, F]$ which is smooth as a result of Rmk. [8]. Then develop the terms of these equalities by employing the families of oriented charts $U$ and $U^\partial$, and the families of smooth maps $\{(ψ_α)\}_{α ∈ D}$ and $\{(k_α)\}_{α ∈ D}$. Finally we apply the Extension Thm. [4] to the sequence of the resulting equalities.
Remark 16. Thm. [13] establishes a Stokes-type equality for \( \langle B(G), \sigma(B(G), N) \rangle \)-valued integrable forms: (1) that arise from the Borelian functional calculus of the possibly unbounded operator \( R \); (2) that might be not smooth nor even continuously differentiable. To this regard we notice that the rigidity of analytic functions prevents any reasonable attempt to use the strong operator derivability on \( \text{Dom}(R) \) in [3 Thm. 1.23(2)] in order to prove regularity of these forms.

Proof of Thm. [13] We maintain the data and notation introduced in Rmk. [14] in addition we let \((U, \phi)\) be an oriented chart of \( M \) and \( h \in \mathcal{A}(M) \) and \( k \in \mathcal{A}(\partial M) \) be such that

\[
\begin{cases}
\text{supp}(h) \subseteq U; \\
\text{supp}(k) \subseteq U \cap \partial M.
\end{cases}
\]

Let \( \{\sigma_n\}_{n \in \mathbb{N}} \) be an \( E \)-sequence of bounded sets and \( n \in \mathbb{N} \), let \( \delta \in \{\sigma_n, \sigma(R)\} \), let \( R_\delta \) denote \( R_\delta \upharpoonright G_\delta \) and let \( \psi \in \mathcal{N} \). By Rmk. [8] and Rmk. [9] we have that \([\omega, \sigma_n, F] \in \text{Alt}^{N-1}(M, \langle B(G_{\sigma_n}), \sigma(B(G_{\sigma_n})), N_{\sigma_n} \rangle) \) so by [4 Thm. 2.42, (16)], since the unique element of a smooth partition of the unity subordinated to the open covering \( \{U\} \) of \( U \) equals 1 when evaluated on \( U \), and finally by [4 Prp. 1.45], we have

\[
\int \psi_x(hd[\omega, \sigma_n, F]) = \int h(\psi_x[\omega, \sigma_n, F]) = \gamma_\phi \int (i_{U_1}^M \circ \phi^{-1})^* (h)(i_{U_1}^M \circ \phi^{-1})^* (d\psi_x[\omega, \sigma_n, F]).
\]

Next

\[
(i_{U_1}^M \circ \phi^{-1})^* d\psi_x[\omega, \sigma_n, F] = (\phi^{-1})^* (i_{U_1}^M)^* d\psi_x[\omega, \sigma_n, F]
\]

\[
= \psi_x d(\phi^{-1})^* (i_{U_1}^M)^* [\omega, \sigma_n, F]
\]

\[
= \psi_x d(\phi^{-1})^* [\omega, \phi, \sigma_n, F];
\]

where the second equality follows by [4 Thm. 2.42], the third one by Def. [11] applied to any atlas containing \((U, \phi)\). Now by definition

\[
[\omega, \phi, \delta, F] = \sum_{i=1}^{N} \left( \zeta_{R^m, \omega_i}^\phi \circ \phi \right) \otimes (dx_1^\phi \wedge \ldots \wedge dx_i^\phi \wedge \ldots dx_N^\phi);
\]

thus

\[
d(\phi^{-1})^* ([\omega, \phi, \sigma_n, F]) = \sum_{i=1}^{N} (-1)^{i-1} \frac{\partial \zeta_{R^m, \omega_i}^\phi}{\partial e_i} \otimes (dx_1^{\text{Id}_{\phi(\Omega)}} \wedge \ldots dx_i^{\text{Id}_{\phi(\Omega)}} \wedge \ldots dx_N^{\text{Id}_{\phi(\Omega)}})
\]

\[
= \sum_{i=1}^{N} (-1)^{i-1} R^m_i \frac{\partial \omega_i^\phi}{\partial e_i} \frac{\partial \omega_i^\phi}{\partial \omega_i^\phi} \otimes (dx_1^{\text{Id}_{\phi(\Omega)}} \wedge \ldots dx_i^{\text{Id}_{\phi(\Omega)}} \wedge \ldots dx_N^{\text{Id}_{\phi(\Omega)}});
\]
where the second equality follows since Cor. 6. Next $R^{\sigma_n}$ is norm continuous, thus by the end of Rmk. 8 we have that

$$
(21) \quad \int (i_U^M \circ \phi^{-1})^*(h) R^{\sigma_n} \frac{\partial \omega_i}{\partial e_i} \int_{\mathbb{R}^n} \frac{R^{\sigma_n}}{\partial e_i} \omega_i \cdot d \lambda_{\phi(U)} = R^{\sigma_n} \int (i_U^M \circ \phi^{-1})^*(h) \frac{\partial \omega_i}{\partial e_i} \int_{\mathbb{R}^n} \frac{R^{\sigma_n}}{\partial e_i} \omega_i \cdot d \lambda_{\phi(U)} \in B(G_{\sigma_n});
$$

the integrals being w.r.t. the norm topology on $B(G_{\sigma_n})$, then also w.r.t. the $\langle B(G_{\sigma_n}), \sigma(B(G_{\sigma_n}), N_{\tau_n}) \rangle$ topology since Rmk. 10. Now (17), (18), (20) and (21) yield

$$
\int h d[\omega, \sigma_n, F] = R^{\sigma_n} \gamma_{\phi} \int (i_U^M \circ \phi^{-1})^*(h) \sum_{i=1}^{N} (-1)^{i-1} \frac{\partial \omega_i}{\partial e_i} \int_{\mathbb{R}^n} \frac{R^{\sigma_n}}{\partial e_i} \omega_i \cdot d \lambda_{\phi(U)}
$$

(22)

integrals w.r.t. the $\langle B(G_{\sigma_n}), \sigma(B(G_{\sigma_n}), N_{\tau_n}) \rangle$ topology, where the second equality follows by the next equality obtained by direct calculation

$$
(23) \quad \int h \sum_{i=1}^{N} (-1)^{i-1} \frac{\partial \omega_i}{\partial e_i} \cdot [\nu_i(\omega), \sigma_n, \frac{dF}{d\lambda}] = \gamma_{\phi} \int (i_U^M \circ \phi^{-1})^*(h) \sum_{i=1}^{N} (-1)^{i-1} \frac{\partial \omega_i}{\partial e_i} \int_{\mathbb{R}^n} \frac{R^{\sigma_n}}{\partial e_i} \omega_i \cdot d \lambda_{\phi(U)}.
$$

Now by (22) applied to $(U, \phi) = (U_\alpha, \phi_\alpha)$ and $h = \psi_\alpha$ for every $\alpha \in D$ and since [4 Cor.2.53] we obtain

$$
\int d[\omega, \sigma_n, F] = R^{\sigma_n} \int \sum_{i=1}^{N} (-1)^{i-1} \frac{\partial \omega_i}{\partial e_i} \cdot [\nu_i(\omega), \sigma_n, \frac{dF}{d\lambda}]
$$

(24)

$$
R^{\sigma_n} \int \sum_{\alpha \in D} \gamma_{\phi_\alpha} (i_U^M \circ \phi_{\alpha}^{-1})^*(\psi_\alpha) \sum_{i=1}^{N} (-1)^{i-1} \frac{\partial \omega_i}{\partial e_i} \int_{\mathbb{R}^n} \frac{R^{\sigma_n}}{\partial e_i} \omega_i \cdot d \lambda_{\phi_\alpha(U_\alpha)};
$$

where all the three integrals are w.r.t. the $\langle B(G_{\sigma_n}), \sigma(B(G_{\sigma_n}), N_{\tau_n}) \rangle$ topology. Now if $\partial M = \emptyset$ the statement follows by the above equality, [4 Thm. 2.54] and by our Extension Thm. 4. Thus in what follows assume in addition that $\partial M \neq \emptyset$ and that $(U, \phi)$ is a boundary chart, therefore $(U, \phi_{\partial M})$ is a chart of $\partial M$ such that $\gamma_{\phi M} = \gamma_{\phi}$. Next since the unique element of a smooth partition of the unity subordinated to the open covering $\{U \cap \partial M\}$, w.r.t. the topological space $\partial M$, of $U \cap \partial M$ equals 1 when evaluated on $U \cap \partial M$, we have
by [4] Thm. 2.42, (16), [4] Thm. 1.45 and \( \gamma_{\phi M} = \gamma_\phi \)

(25)

\[
\int \psi_x (k(t^M_{\partial M})^x[\omega, \delta, F]) = \int k\psi_x (t^M_{\partial M})^x[\omega, \delta, F]
\]

\[
= \gamma_\phi \int \left( (t^M_{\partial M} \circ (\phi^{\partial M})^{-1})^x \right) \left( t^M_{\partial M} \circ (\phi^{\partial M})^{-1} \right)^x \psi_x (t^M_{\partial M})^x[\omega, \delta, F]
\]

\[
= \gamma_\phi \int \left( (t^M_{\partial M} \circ (\phi^{\partial M})^{-1})^x \right) \psi_x ((\phi^{\partial M})^{-1})^x (t^M_{\partial M})^x[\omega, \delta, F]
\]

\[
= \gamma_\phi \int \left( (t^M_{\partial M} \circ (\phi^{\partial M})^{-1})^x \right) \psi_x ((\phi^{\partial M})^{-1})^x (t^M_{\partial M})^x[\omega, \delta, F]
\]

\[
= \gamma_\phi \int \left( (t^M_{\partial M} \circ (\phi^{\partial M})^{-1})^x \right) \psi_x ((\phi^{\partial M})^{-1})^x (t^M_{\partial M})^x[\omega, \delta, F].
\]

Next by (19) and since \((I^M_{U \cap \partial M})^x \psi^x_N = 0\), we obtain

\[
(I^M_{U \cap \partial M} \circ (\phi^{\partial M})^{-1})^x (t^M_{U \cap \partial M})^x[\omega, \delta, F] = (I^M_{U \cap \partial M})^x[\omega, \phi, \delta, F]
\]

\[
= \left( C_{F, \delta, \phi} \circ \phi \circ I^M_{U \cap \partial M} \right) \times \bigwedge_{s=1}^{N-1} (I^M_{U \cap \partial M})^x (dx_s^\phi)
\]

\[
= \left( C_{F, \delta, \phi} \circ \phi \circ I^M_{U \cap \partial M} \right) \times \bigwedge_{s=1}^{N-1} dx_s^{\phi M}
\]

and by letting \( Z := \phi^{\partial M}(U \cap \partial M) \)

(26)

\[
((\phi^{\partial M})^{-1})^x (t^M_{U \cap \partial M})^x[\omega, \delta, F] = \left( C_{F, \delta, \phi} \circ \phi \circ I^M_{U \cap \partial M} \circ (\phi^{\partial M})^{-1} \right) \times \bigwedge_{s=1}^{N-1} dx_s^{\widehat{\phi M}}.
\]

Next

\[
i_{R^{N+1}}(Z) = \phi(U \cap \partial M) \subseteq \partial H^N;
\]

by definition of boundary chart of \( M \). Define \( P_{[N]} := i_{R^{N+1}} \circ P_{[N]} \), thus by letting \( \text{Pr}(R^N) \) be the set of projectors of \( R^N \), we have

(27)

\[
\begin{cases}
P_{[N]} \in \text{Pr}(R^N), \\
\partial H^N = P_{[N]}(R^N);
\end{cases}
\]

moreover by definition of \( \phi^{\partial M} \) we have

(28)

\[
(\forall x \in Z) \left( P_{[N]} \left( (\phi \circ I^M_{U \cap \partial M} \circ (\phi^{\partial M})^{-1})(x) \right) = i_{R^{N-1}}(x) \right).
\]

Now \((\phi \circ I^M_{U \cap \partial M} \circ (\phi^{\partial M})^{-1})(x) \in \partial H^N\) since \( \phi \) is a boundary chart of \( M \), therefore by (28) and (27) we obtain

\[
\phi \circ I^M_{U \cap \partial M} \circ (\phi^{\partial M})^{-1} = i_{\phi(U \cap \partial M)} \circ (i_{\phi Z})_{R^{N-1}}.
\]
Therefore by (26)
\[ ((\phi^M)^{-1})^\times (t_{U \cup \partial M})^\times (t_{U})^\times [\omega, \delta, F] = \left( \zeta_{F,\omega}^{R} \circ i_{\phi(U \cup \partial M)}^{\phi(U)} \circ (i_{\delta}^{R_{N-1}} \circ i_{Z}^{R_{N-1}})_{2} \right) \otimes \bigwedge_{s=1}^{N-1} dx_{s}^{1}; \]

hence by (25) we obtain
\[ (29) \int k(t_{\partial M})^\times [\omega, \delta, F] = \gamma_{\phi} \int ((\partial M)^{-1})^\times (t_{\partial M})^\times \left( \zeta_{F,\omega}^{R} \circ i_{\phi(U \cup \partial M)}^{\phi(U)} \circ (i_{\delta}^{R_{N-1}} \circ i_{Z}^{R_{N-1}})_{2} \right) d\lambda_{Z}; \]

where the integrals are w.r.t. the \( (B(G_{0}), \sigma(B(G_{0}), N_{0})) \) topology. Now by (29) applied to \((U, \phi) = (U_{\alpha}, \phi_{\alpha})\) and \(k = k_{\alpha}\) for every \( \alpha \in D \) and since [4] Cor. 2.53] we obtain by letting \( Z_{\alpha} := (\phi_{\alpha}^{\partial M})(U_{\alpha} \cap \partial M) \)
\[ (30) \int (t_{\partial M})^\times [\omega, \delta, F] = \sum_{\alpha \in D} \gamma_{\phi_{\alpha}}(t_{\partial M}^{\partial M}(U_{\alpha} \cap \partial M) \circ (\phi_{\alpha}^{-1})^{\partial M}) \left( \zeta_{F,\omega}^{R} \circ i_{\phi(U_{\alpha} \cap \partial M)}^{\phi(U_{\alpha})} \circ (i_{\delta}^{R_{N-1}} \circ i_{Z}^{R_{N-1}})_{2} \right) d\lambda_{Z_{\alpha}}. \]

Next by (23) applied to \((U, \phi) = (U_{\alpha}, \phi_{\alpha})\) and \(h = \psi_{\alpha}\) for every \( \alpha \in D \) and since [4] Cor. 2.53] we obtain
\[ (31) \int (\sum_{i=1}^{N} (-1)^{i-1} \partial_{t_{\alpha}}(\omega) \cdot [\eta_{i}(\omega), \delta, D] = \sum_{\alpha \in D} \gamma_{\phi_{\alpha}}(t_{\partial M}^{\partial M}(U_{\alpha} \cap \partial M) \circ (\phi_{\alpha}^{-1})^{\partial M}) \left( \zeta_{F,\omega}^{R} \circ i_{\phi(U_{\alpha} \cap \partial M)}^{\phi(U_{\alpha})} \circ (i_{\delta}^{R_{N-1}} \circ i_{Z}^{R_{N-1}})_{2} \right) d\lambda_{Z_{\alpha}}. \]

Now since [4] Thm. 2.54] applied to the form \([\omega, \sigma, F]\), since (24) and since (30) applied to \( \delta = \sigma_{\alpha} \) we obtain
\[ R^{\sigma_{\alpha}} \sum_{\alpha \in D} \gamma_{\phi_{\alpha}}(t_{\partial M}^{\partial M}(U_{\alpha} \cap \partial M) \circ (\phi_{\alpha}^{-1})^{\partial M}) \left( \zeta_{F,\omega}^{R} \circ i_{\phi(U_{\alpha} \cap \partial M)}^{\phi(U_{\alpha})} \circ (i_{\delta}^{R_{N-1}} \circ i_{Z}^{R_{N-1}})_{2} \right) d\lambda_{Z_{\alpha}}. \]

Now we can employ our Extension Thm. [4] to the above sequence of equality to obtain
\[ R \int (\sum_{\alpha \in D} \gamma_{\phi_{\alpha}}(t_{\partial M}^{\partial M}(U_{\alpha} \cap \partial M) \circ (\phi_{\alpha}^{-1})^{\partial M}) \left( \zeta_{F,\omega}^{R} \circ i_{\phi(U_{\alpha} \cap \partial M)}^{\phi(U_{\alpha})} \circ (i_{\delta}^{R_{N-1}} \circ i_{Z}^{R_{N-1}})_{2} \right) d\lambda_{Z_{\alpha}}; \]

and the statement follows since (31) and (30) applied to \( \delta = \sigma(R) \). 

**Corollary 17.** Let \( M \) be oriented with boundary and \( \omega \in \text{Alt}^{N-1}(M) \). Let \( V \) be an open neighbourhood of \( \sigma(R) \) such that \( R \cdot V \subseteq V \) and \( F : V \rightarrow \mathbb{C} \) be analytic. Assume that there exists a finite collection \( U = \{ (U_{\alpha}, \phi_{\alpha}) \}_{\alpha \in D} \) of oriented charts of \( M \) such that \( \{ U_{\alpha} \}_{\alpha \in D} \) is a covering of the support of \( \omega \) and
(1) \( \bar{F}_t \in \Omega_\infty^E(\sigma(R)) \) for every \( t \in \mathbb{R} \), and for all \( \alpha \in D \) such that \( \phi_\alpha \) is a boundary chart, the map
\[
\psi \circ \zeta^R_{E,u_\alpha} \circ i_{\phi_\alpha(U_\alpha)}^{\phi_\alpha(U_\alpha)} \quad \text{is } \lambda_{\phi_\alpha(U_\alpha \cap \partial M)}\text{-measurable for every } \psi \in \mathbb{N}; \text{ and}
\]
\[
\int \| \cdot \|_\infty \circ \bar{F}_{\omega_\alpha} \circ i_{\phi_\alpha(U_\alpha)}^{\phi_\alpha(U_\alpha)} d\lambda_{\phi_\alpha(U_\alpha \cap \partial M)} < \infty;
\]

(2) \( (\bar{dF}/d\lambda)_t \in \Omega_\infty^E(\sigma(R)) \) for every \( t \in \mathbb{R} \), and for all \( \alpha \in D \) and \( i \in [1, N] \cap \mathbb{Z} \), the map
\[
\psi \circ \zeta^R_{\omega_i} \quad \text{is } \lambda_{\phi_\alpha(U_\alpha)}\text{-measurable for every } \psi \in \mathbb{N}; \text{ and}
\]
\[
\int \| \cdot \|_\infty \circ (\bar{dF}/d\lambda)_{\omega_i} d\lambda_{\phi_\alpha(U_\alpha)} < \infty.
\]

Thus the statement of Thm. [13] holds true. Moreover if in addition \( \mathbb{N} \) is an \( E \)-appropriate set with the isometric duality property and \( C \doteq \sup_{\alpha \in B(\mathbb{C})} \| E(\alpha) \|_E \), then we obtain the following estimates
\[
\left\| \int \zeta^R_{E,u_\alpha} \circ i_{\phi_\alpha(U_\alpha)}^{\phi_\alpha(U_\alpha)} d\lambda_{\phi_\alpha(U_\alpha \cap \partial M)} \right\| \leq 4C \int \| \cdot \|_\infty \circ \bar{F}_{\omega_\alpha} \circ i_{\phi_\alpha(U_\alpha)}^{\phi_\alpha(U_\alpha)} d\lambda_{\phi_\alpha(U_\alpha \cap \partial M)};
\]
and
\[
\left\| \int \zeta^R_{\omega_i} \circ i_{\phi_\alpha(U_\alpha)}^{\phi_\alpha(U_\alpha)} d\lambda_{\phi_\alpha(U_\alpha)} \right\| \leq 4C \int \| \cdot \|_\infty \circ (\bar{dF}/d\lambda)_{\omega_i} d\lambda_{\phi_\alpha(U_\alpha)}.
\]

**Remark 18.** By letting \( \text{Bor}(\mathbb{C}) \) be the set of complex valued Borelian maps on \( \mathbb{C} \), we have
\[
\left\{ \bar{F}_{\omega_\alpha} : \phi_\alpha(U_\alpha \cap \partial M) \rightarrow \text{Bor}(\mathbb{C}) \right\}
\]
and
\[
\left\{ (\bar{dF}/d\lambda)_{\omega_i} : \phi_\alpha(U_\alpha) \rightarrow \text{Bor}(\mathbb{C}) \right\}
\]
where we recall that \( L_t : \lambda \mapsto L(t, \lambda) \) for any \( L \in \text{Bor}(\mathbb{C}) \) and any \( t \in \mathbb{R} \). Therefore the upper integrals in hypotheses (1) and (2) are well-set.

**Proof.** By [3] (1.42), hypotheses, and [2] Thm. 18.2.11(c) we obtain that
\[
\| \cdot \| \circ \zeta^R_{E,u_\alpha} \circ i_{\phi_\alpha(U_\alpha)}^{\phi_\alpha(U_\alpha)} \in \tilde{\mathcal{F}}_1(\phi_\alpha(U_\alpha \cap \partial M), \lambda_{\phi_\alpha(U_\alpha \cap \partial M)}),
\]
and
\[ \| \cdot \| \circ \zeta^R_{\hat{\theta}^R_{\phi^2}} \in \mathfrak{F}_1(\phi_{\alpha}(U_{\alpha}), \lambda_{\phi_{\alpha}(U_{\alpha})}). \]

Then the hypotheses of Thm. [13] are satisfied by [3, footnote 1 pg. 39] and by [3, Thm. 2.2] and the first part of the statement follows. The estimates in the statement follow by the estimate in [3, Thm. 2.2] and by [2, Thm. 18.2.11(c)].

**Corollary 19.** The statement of Cor. [17] holds if $G$ is a complex Hilbert space and $N$ is replaced by $N_{pd}(G)$.

**Proof.** By the end of [3, Rmk. 2.12] and by Cor. [17].

**Corollary 20.** The statement of Cor. [17] holds if $G$ is reflexive and $N$ is replaced by $N_{st}(G)$.

**Proof.** By employing [3, Cor. 2.6] instead of [3, Thm. 2.2] the proof runs exactly as the one in Cor. [17].
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