A non standard approach of spectral theory of self-adjoint operators (generalized Gelfand eigenvectors)

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Abstract
Let $u = \int_{-\infty}^{+\infty} \lambda dE_{\lambda}$ be a self-adjoint operator in a Hilbert space $H$. Our purpose is to provide a non-standard description of the spectral family $(E_{\lambda})$ and the generalized Gelfand eigenvectors.

1 The non standard framework
We consider here the nonstandard analysis framework (see [Goldblatt], for a brief overview) and apply it to set universes : Let $A$ be a set of $\in$-independent elements (i.e. a non empty set such as there is not any chain $x_0 \in x_1 \in \ldots \in x_n$ whose ends belong to $A$ ) and let us associate to $A$ its superstructure $U_A$, i.e. the union of the sequence $(A_n)$ defined by $A_0 = A$ and $A_{n+1} = A_n \cup P'(A_n)$, where $P'$ denotes the set of non empty subsets. The couple made of $U_A$ and the restriction of $\in$ to $U_A$ is a standard model of Zermelo theory with atoms and where Zorn's theorem holds.

Note :
1. Zermelo theory is ZF without the substitution axiom (see [Jech]).

2. We need to choose once for all a $\emptyset_A \in A$ that will play the role of the empty set in $U_A$; $U_A$ atoms are then the elements of $A\setminus\{\emptyset_A\}$. Such a $U_A$ universe contains all necessary sets to describe the structures (in the sense of N.Bourbaki) over elements of $U_A$.

3. We may assume that $A$ is chosen such that $C$ and $H$ are its subsets; even if this requires to replace $C$ and $H$ by isomorphic copies. $C$ and $H$ have then their binary operation graphs in $U_A$. One can associate to $U_A$ a first order language $L(U_A)$ whose relational symbols are $=$ and $\in$ and whose constant set is $U_A$. 
The non-standard method relies on the existence of a function called enlargement defined from $U_A$ to the set class as $a \mapsto ^* a$ with the following properties:

i. $^*a = a$ for each atom,

ii. if $R$ is a relation of $L(U_A)$ without any free variable and where all quantifiers are limited - i.e. in the form of $\exists x \in t$ or $\forall x \in t$ - and if $^*R$ is the relation obtained by replacing every outstanding constant $a$ by its image $^*a$, $R$ and $^*R$ relations are either both true or both false (transfer principle),

iii. if $F \in U_A$ is a filter on a set $u \in U_A$, the intersection of $^*F$ sets (which is a filter over $^*u$) is not empty.

In particular, given i and ii, $\mathbb{C}$ could be enlarged into a set $^*\mathbb{C}$ containing $\mathbb{C}$. We will call $^*\mathbb{C}$ elements hypercomplexes and define over $^*\mathbb{C}$ an addition and a multiplication operations, whose graphs are the enlargements of the addition and multiplication graphs over $\mathbb{C}$. For these operations $^*\mathbb{C}$ is an algebraically closed field. The same analysis applies to $\mathbb{R}$ and leads to the subfield $^*\mathbb{R}$ of $^*\mathbb{C}$. $^*\mathbb{R}$ is totally ordered and generates with element $i$ the field $^*\mathbb{C}$ (This allows to consider the real and imaginary parts of a hypercomplex and to introduce the absolute hypervalue of a hypercomplex).

An element $\xi \in ^* \mathbb{R}$ is infinitesimal (resp. limited), if $-1 < n\xi < 1$ for every natural number $n$ (resp. $-n < \xi < n$ for at least one natural number). A hypercomplex is infinitesimal (resp. limited) if its absolute hypervalue is infinitesimal (resp. limited). The subset $^*\mathbb{C}_b$ made of bounded elements is a valuation ring whose residual field could be identified with $\mathbb{C}$: more precisely the image of $z \in ^* \mathbb{C}_b$ is the unique complex number $^\circ z$ such that $z - ^\circ z$ is infinitesimal.

From now on, we consider a fixed universe $U$ and an enlargement $^*U$ of $U$.

2 Hyperhermitian spaces

2.1 Internal vector spaces A set is called internal if it belongs to an enlargement. A $^*\mathbb{C}$-vector space $F$ is internal if the set $F$ is internal and if the graphs of its binary operations are internal. A linear function $u : F \rightarrow G$ between internal vector spaces is internal if its graph is internal.
Theorem 1 Internal $^\ast \mathbb{C}$ vector spaces and internal linear applications make up an abelian category (denoted $\text{Ivs}$).

Let $E$ be a complex vector space of $U$ and let us assume that binary operations are in $U$. If $\text{vss}_f(E)$ is the set of finite-dimensional subspaces of $E$, its enlargement $^\ast \text{vss}(E)$ is made of $^\ast E$ internal subspaces (and is itself an internal linear space for the enlargement of binary operations of $E$).

An internal vector space $F$ is said to be hyperfinite-dimensional, if there is a complex vector space $E$ in $U$ such that $F$ is isomorphic (through an internal isomorphism) to an element of $^\ast \text{vss}(E)$.

Corollary 2 The full subcategory of $\text{Ivs}$ set made of hyperfinite-dimensional subspaces is a thick subcategory.

The importance of these subspaces lies in the following proposition :

Proposition 3 Let $H$ be a complex vector space of $U$. Since $\mathbb{C} \subset ^\ast \mathbb{C}$, $^\ast H$ is also a complex space and the function :

$$f : H \rightarrow ^\ast H$$

$$h \mapsto ^\ast h$$

is an injective linear map whose image is included in an internal hyperfinite-dimensional subspace.

2.2 Definition of hyperhermitian spaces and orthonormal hyperbases

The concept of positive-definite sesquilinear form extends easily to hypercomplex vector spaces. We call hyperhermitian space an internal hyperfinite-dimensional vector space provided with an internal positive-definite sesquilinear form. A number of properties are preserved by transfer, when passing from hermitian to hyperhermitian spaces. Let us mention in particular the orthogonal property and moreover the decomposition of a hyperhermitian space into a direct sum of an internal linear subspace and its orthogonal.

Orthonormal hyperbases play a very important role here : a subset $B$ of a hyperhermitian space $F$ is an orthonormal hyperbasis, if $B$ is an orthonormal set and $0$ is the only element of $F$ to be orthogonal to every element of $B$.

Theorem 4 Let $F$ be a hyperhermitian vector space with scalar product $\langle , \rangle$ and let $u$ be a symmetric internal linear function for $\langle , \rangle$ over $F$. There exists an orthonormal hyperbasis $B$ of $F$ made of eigenvectors of $u$. 

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Such a hyperbasis $B$ has an important property: it is hyperfinite i.e. it belongs to the enlargement of the set of finite subsets of some set in $U_A$. We can then define by enlargement the sum operation $\sum$ for a hyperfinite set of indices. This internal operation has by transfer the following characteristic properties:

1. $\sum_{b \in B} \lambda_b b = 0$
2. $\sum_{b \in B_1 \cup B_2} \lambda_b b + \sum_{b \in B_1 \cap B_2} \lambda_b b = \sum_{b \in B_1} \lambda_b b + \sum_{b \in B_2} \lambda_b b$, for $B_1$ and $B_2$ hyperfinite sets.

This leads to:

**Proposition 5** Let $B$ be an orthonormal hyperbasis of a hyperhermitian space $F$. Every $x \in F$ could be expressed as: $x = \sum_{b \in B} < b, x > b$

**Remark 6** Infinite hyperfinite sets are uncountable.

### 2.3 Hull of a hyperhermitian space

Let $F$ be a hyperhermitian space with a scalar product $<, >$. We define a hypernorm $|||$(i.e. a norm with hyperreal values) by $||x|| = \sqrt{< x, x >}$.

Let us consider now the set $F_b \subset F$ of bounded hypernorm elements and define over $F_b$ an equivalence relation $\sim$ by: $x \sim y$ iff $x - y$ has an infinitesimal norm. The operations over $F$ are preserved by passing to the quotient space and make of $^eF = F/\sim$ a complex vector space. The scalar product $<, >$ is also preserved by passing to the quotient and make of $^eF$ a prehilbertian space. A classic result (see [Luxemburg Stroyan]) shows that $^eF$ is a Hilbert space. For $x \in ^eF$, we call shadow of $x$ the class of $x$ in $^eF$ and denote it also $^o x$.

Let $B$ be an orthonormal hyperbasis of $F$. If $F$ is a non finite-dimensional space, set $B$ is uncountable, while for every bounded $x$, the subset $\{ b \in B / < b, x > \text{ is not negligible} \}$ is countable. This prevents the extension of the equality $x = \sum_{b \in B} < x, b >$ to the hull of $F$.

### 2.4 The Loeb spectral measure

Let $F$ be an infinite-dimensional hyperhermitian space and $B$ an orthonormal basis. The set $\mathcal{F}_0(B)$ of hyperfinite subsets of $B$ is a Boolean algebra where the union of a increasing sequence $(B_n)$ does not belong to the Boolean algebra unless this sequence is stationary. For every $\xi \in F_b$ we define a positive measure on $\mathcal{F}_0(B)$ by $\nu_{\xi}(C) = ^o \langle \sum_{b \in C} | < b, \xi > |^2 \rangle$. By the fundamental theorem of P. Loeb, this measure extends to the Boolean $\sigma$-algebra $\mathcal{F}(B)$ generated by $\mathcal{F}_0(B)$ and depends only on the shadow $h$ of $\xi$.

**Theorem 7** There is a unique mapping $\varpi$ from the Boolean $\sigma$-algebra $\mathcal{F}(B)$ to the set of orthogonal projectors of $^eF$ such that for any $h \in ^eF$ and any $C \in \mathcal{F}(B)$: $\nu_h(C) = < h, \varpi(C)h >$. 

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2.5 Hull of a symmetric internal linear function over a hyperhermitian space

Let \( v \) be this function. The linear mapping \( v - iI \) (where \( I \) is the identity function) is bijective and allows us to define the Cayley transform \( c_v = (v + iI)(v - iI)^{-1} \) which is a unitary function that becomes, by passing to the quotient, a unitary function \( ^e c_v \) over \( ^e F \). Let \( ^e F_{v,f} \) be the orthogonal of the eigenspace of \( ^e c_v \) associated to the eigenvalue 1.

The graph \( \Gamma_v \subset F \times F \) is an internal subset. The shadows of \( \Gamma_v \) bounded elements in \( ^e (F \times F) \) make up a closed subspace \( G_v \). Note here that \( ^e (F \times F) \) could be easily identified with \( ^e F \times ^e F \).

**Proposition 8** The intersection \( G_v \cap ^e F_{v,f} \times ^e F_{v,f} \) is the graph of a self-adjoint operator \( t_v \) in \( ^e F_{v,f} \).

Let us define an orthonormal hyperbasis \( B \) of \( F \) made of \( v \) eigenvectors. The mapping \( b \mapsto \lambda_b \) (which associates to an eigenvector \( b \) its eigenvalue) is an internal hyperreal function. The set \( B_{v,f} \) of \( v \) eigenvectors with bounded eigenvalues is a borelian set in the sense of 2.4.

**Proposition 9** The orthogonal projector over \( ^e F_{v,f} \) is the orthogonal projector associated to the borelian set \( B_{v,f} \) by the spectral measure \( \varpi \) defined in 2.4.

For any \( \lambda \in \mathbb{R} \) the set \( \Omega_{\lambda} = \{ b \in B_{v,f} \mid ^e \lambda_b \leq \lambda \} \), is a borelian set. We can thus define \( E_{\lambda} = \varpi(\Omega_{\lambda}) \) as an orthogonal projector in \( ^e F_{v,f} \).

**Proposition 10** We have \( t_v = \int_{-\infty}^{+\infty} \lambda dE_{\lambda} \), which means that \( (E_{\lambda}) \) is the spectral family of \( t_v \).

3 Renormalization of an orthonormal basis

We still consider a hyperhermitian space \( F \) and an orthonormal hyperbasis \( B \) of \( F \).

3.1 \( ^e F_{B,h} \) Spaces

For any element \( h \in ^e F \), \( h =^o x \) and \( ^e F_{B,h} \) denotes the closed subspace generated by \( \varpi(C)(h) \), where \( C \) evolves in the Borel algebra \( \mathfrak{F}(B) \).

\( \nu_h \) is the unique measure over \( \mathfrak{F}(B) \) such that

\[
\nu_h(C) = ^o \left( \sum_{b \in C} |< b, x >|^2 \right)
\]

for every hyperfinite subset \( C \).
Proposition 11  a. There is a unique isometry $J_{B,h} : {}^eF_{B,h} \to L^2(B, \nu_h)$ that links $\varpi(C)(h)$ to the characteristic function class of $C$ in $L^2(B, \nu_h)$.

b. Let $z$ be a bounded element of $F$ and $k$ its shadow in $^eF_{B,h}$. The image $k$ of $k$ is the class of the function $f_z : b \mapsto \left( \frac{<b,z>}{<b,x>} \right)$ if this ratio is bounded and 0 otherwise.

c. Let $\zeta$ be the measure $\nu_k - |k|^2 \nu_h$. $\zeta$ and $\nu_h$ are mutually singular (and $\nu_k = |k|^2 \nu_h + \zeta$ is the Lebesgue decomposition of $\nu_k$ with respect to $\nu_h$).

Remark 12  a. The quantity $\frac{<b,z>}{<b,x>}$ is in general the ratio of two infinitesimals. We cannot indeed express it through the shadows $h$ and $k$ of $x$ and $z$.

b. We will determine $J_{B,h}^{-1}$ in the next paragraph (Theorem 14).

3.2 Introduction of a second scalar product  Let us consider a second internal scalar product $[,]$ over the hyperhermitian space $F$. There exists a symmetric positive definite internal linear function $j$ such that $[x,y] = < j(x), j(y) >$. The dual scalar product $\{,\}$ is defined by $\{x,y\} = < j^{-1}(x), j^{-1}(y) >$ and we have $|<x,y>|^2 \leq [x,x]\{y,y\}$ for every $x,y$.

We now introduce the mapping:

$${}^eF_{[,]} \times {}^eF_{\{,\}} \to \mathbb{C}$$

where $\langle {}^o x_{[,]}, {}^o y_{\{,\} \rangle} = {}^o < x, y >$,

$${}^eF_{[,]}$$ is the hull of $F$ with a scalar product $[,]$,

$${}^eF_{\{,\}}$$ is the hull of $F$ with a scalar product $\{,\}$,

$${}^o x_{[,]}$$ is the shadow of $x$ in $^eF_{[,]}$,

$${}^o y_{\{,\}}$$ is the shadow of $y$ in $^eF_{\{,\}}$.

Reminder - Let $G$ be a complex vector space. A mapping $f : G \to \mathbb{C}$ is called antilinear if $f(x + y) = f(x) + f(y)$ and $f(\lambda x) = \lambda f(x)$. This property is equivalent to have $\overline{f} : x \mapsto \overline{f(x)}$ linear.

A linear function $f$ over a normed space $G$ is bounded, if the linear form $\overline{f}$ is bounded. The norm of $f$ is then defined as the norm of $\overline{f}$. The vector space of bounded antilinear functions from $G$ to $\mathbb{C}$, is called the antilinear dual of $G$ and denoted $\tilde{G}$. 

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Proposition 13 Every linear form \( f \) over \( {^e}_F \{, \} \) can be defined as \( y \mapsto \langle x_f, y \rangle \) with a unique \( x_f \in {^e}_F [,] \). The mapping \( f \mapsto x_f \) is an antilinear bijective isometry from the dual of \( {^e}_F [,] \) into \( {^e}_F [,] \).

Every antilinear form \( \phi \) over \( {^e}_F [,] \) can be defined as \( x \mapsto \langle x | \phi, y \rangle \) with a unique \( y_\phi \in {^e}_F \{, \} \). The mapping \( \phi \mapsto y_\phi \) is an antilinear bijective isometry from the dual of \( {^e}_F [,] \) into \( {^e}_F \{, \} \).

In other words, the sesquilinear form \( \langle | \rangle \) puts in duality \( {^e}_F [,] \) and \( {^e}_F \{, \} \).

In addition we can find an orthonormal hyperbasis \( \Gamma \) of \( F [,] \) which is also orthogonal for \( <, > \).

The hypersum \( \sigma = \sum_{\gamma \in \Gamma} \| \langle \gamma, \gamma \rangle \| = \sum_{\gamma \in \Gamma} \sqrt{\langle \gamma, \gamma \rangle} \) is independent of the choice of \( \Gamma \) basis.

Theorem 14 Assume \( \sigma \) bounded.

a. The identity mapping from \( F \) to \( F \) transforms bounded elements for \( [,] \) into bounded elements for \( <, > \) and bounded elements for \( <, > \) into bounded elements for \( \{, \} \). This induces, by passing to the quotient, continuous mappings : \( {^e}_F [,] \to {^e}_F \) and \( {^e}_F \to {^e}_F \{, \} \).

b. The complementary of the set \( \{ b | \frac{b}{<x,b>} \text{ is bounded for } \{, \} \} \) is of null \( \nu_h \)-measure. Let us define the following function :

\[
\phi_x : B \to {^e}_F \{, \} \quad b \mapsto \begin{cases} \circ \left( \frac{b}{<x,b>} \right) & \text{if } \frac{b}{<x,b>} \text{ is bounded} \\ 0 & \text{if not} \end{cases}
\]

c. Let \( f \in L^2(B, \nu_h) \) and \( J_{B,h} : {^e}_F_{B,h} \to L^2(B, \nu_h) \) the isometry defined in proposition \( \ref{prop:Isometry} \). The mapping :

\[
B \to \quad {^e}_F \{, \} \\
b \mapsto f(b) \phi_x(b)
\]

is weakly integrable. For every \( q \in {^e}_F_h \) we have :

\[
\langle \tilde{q}, J_{B,h}^{-1}(f) \rangle = \int_B f(b) \langle q, \phi_x(b) \rangle d\nu_h(b)
\]

where \( \tilde{q} \) is the image of \( q \) by the mapping : \( {^e}_F [,] \to {^e}_F \) defined in a.

Remark 15 a. Point c. could be also expressed as : the weak integral \( \int_B f(b) \langle q, \phi_x(b) \rangle d\nu_h(b) \) is the image of \( J_{B,h}^{-1}(f) \) by the mapping : \( {^e}_F \to {^e}_F \{, \} \) defined in point a.
b. With a possible exception for a countable set of elements \(b\), the scalar product \(<x, b>\) is infinitesimal and \(\frac{b}{<x, b>}\) is thus unbounded for \(<, >\). The introduction of the \(\{, \}\) associated norm (renormalization procedure) allows however to extract a finite value.

4 Application to spectral theory

Let \(G\) be a complex vector space with a locally convex topology, \(<, >\) a continuous scalar product over \(G\), and \(u\) a continuous linear function over \(G\), symmetric and essentially selfadjoint relative to \(<, >\). We have the following results :

a. \(< u(x), y > = < x, u(y) >\) for every \(x\) and \(y\) in \(G\),

b. If \(\hat{G}\) is the completion of \(G\) for the scalar product \(<, >\), the closure of the graph of \(u\) in \(\hat{G} \times \hat{G}\) is the graph of a selfadjoint operator \(\tilde{u}\).

Note

1. A scalar product \([,]\) over \(G\) is nuclear (relative to \(<, >\)) if there is a constant \(C\) such that \(\sum_{i=1}^{n} \|b_i\|^2 = \sum_{i=1}^{n} < b_i, b_i > \leq C\) for any finite sequence \(b_1, ..., b_n\) of vectors of \(G\) that is orthonormal for \([,]\).

2. \([,]\) is called strongly nuclear (relative to \(<, >\)) if there is an auxiliary nuclear scalar product \([,]_{aux}\) such that \([,]\) is nuclear relative to \([,]_{aux}\).

In what follows we assume a fixed scalar product \([,]\) over \(G\) continuous and strongly nuclear relative to \(<, >\). The existence of such a scalar product is guaranteed if \(G\) is a nuclear space.

We introduce the enlargement \(*G\) of \(G\) and provide it with the internal scalar product obtained by extending the scalar product \(<, >\) (with the same notation). We then extend \(u\) into an internal linear application \(*u\) over \(*G\). By proposition 3 there exists a hyperfinite-dimensional internal subspace \(F\) of \(*G\) that contains the image of \(G\) in \(*G\) by the mapping \(*\).

Let us fix such a subspace \(F\) that we assume hyperhermitian for the scalar product \(<, >\). To simplify notations, we identify \(G\) with its image in \(\overline{F}\).

The canonical injection of \(G\) into \(F\) induces an isometry from \(G\) into \(\overline{F}\); and the closure \(\overline{G}\) of \(G\) in \(\overline{F}\) is a completion of \(G\) for \(<, >\).

Let \(v\) be a mapping over \(F\) that associates to an element \(x\) the orthogonal projection of \(u(x)\) over \(F\). \(v\) is an internal linear application, symmetric for \(<, >\), that extends \(u\). We can thus apply to \(v\) propositions 8 and 9 using same notations :
Proposition 16  a. The mapping $t_v$ associated to $v$ in proposition 8 is defined everywhere in $G$; and $t_v = u(x)$.

b. Let $(E_\lambda)$ be the spectral family associated to $t_v$ in proposition 10. Every $E_\lambda$ preserves the closure $\overline{G}$ and the family $\left( E_\lambda|_G \right)$ of $E_\lambda$ restrictions to $\overline{G}$ is the spectral family of projectors of the closure $\overline{u}$ of $u$.

Remark 17 Passing from $G, u$ to $F, v$ can be seen as the nonstandard variant of the Ritz projection method.

4.1 Disintegration of the measure $\nu_h$ Let us fix an orthonomal hyper-basis $B$ of eigenvectors of $v$ and denote $\lambda_b$ the eigenvalue associated to eigenvector $b$ and $\varpi$ the Loeb spectral measure associated to $B$ in theorem 7. Let $h$ be an element of $\overline{G}$. Propositions 9 and 16 show that $\sigma_h(\lambda) = \langle h, E_\lambda h \rangle$ is equal to $\langle h, \varpi(\Omega_\lambda)(b) \rangle$ where $\Omega_\lambda$ is the set of $b$ such that $\lambda_b$ is bounded and $\circ \lambda_b \leq \lambda$.

Proposition 18 Let $q$ be a real function defined on $B$ as follows :

$$
q(b) = \begin{cases} 
\circ \lambda_b & \text{if } \lambda_b \text{ is limited} \\
0 & \text{otherwise}
\end{cases}
$$

The Stieljes measure generated by the increasing real function $\sigma_h$ is the image of measure $\nu_h$ by $q$.

Since a linear raising $j : L^\infty(\mathbb{R}, d\sigma_\lambda) \rightarrow L^\infty(\mathbb{R}, d\sigma_\lambda)$ (i.e. a linear mapping such as $\phi$ is the class of $j(\phi)$) exists, with $L^\infty(\mathbb{R}, d\sigma_\lambda)$ the space of essentially bounded numerical functions, we have the following proposition :

Proposition 19 We can associate to every hyperfinite subset $C$ of $B$ an element $\omega_C$ of the space of essentially bounded numerical functions $L^\infty(\mathbb{R}, d\sigma_\lambda)$, that meets the following conditions :

a. The function $\omega_C$ takes only positive real values

b. $\omega_{\varnothing}(\lambda) = 0$ and $\omega_{C_1 \cup C_2}(\lambda) = \omega_{C_1}(\lambda) + \omega_{C_2}(\lambda)$ if $C_1$ and $C_2$ are disjoint.

c. Every numerical function $f$ integrable with respect to $d\sigma_\lambda$ verifies: $\int_C f(q(b))d\nu_h(b) = \int_{-\infty}^{+\infty} f(\lambda)\omega_C(\lambda)d\sigma_h(\lambda)$.

Statements a. and b. show that we can extend the mapping $\tau_\lambda : C \mapsto \omega_C(\lambda)$ to a Loeb measure over the Boolean $\sigma$-algebra $\mathfrak{S}(B)$.
Proposition 20  a. \( \nu_h(C) = \int_{-\infty}^{+\infty} \tau_\lambda(C) d\sigma_h(\lambda) \) for every borelian set \( C \).

b. For \( \sigma_h \)-almost every \( \lambda \), the measure \( \tau_\lambda \) is carried by the set \\
\( q^{-1}(\lambda) = \{ b \in B \mid \lambda_b \text{ is bounded and } ^o\lambda_b = \lambda \} \)

This means that the family \( (\tau_\lambda) \) is a disintegration of the measure \( \nu_h \) with respect to its image by \( q \).

We assumed so far the existence of a scalar product \([ , ]\) that is strongly nuclear over \( G \). This scalar product extends to \( ^*G \) and induces a scalar product on \( F \) with hypercomplex values. We will keep denoting it \([ , ]\).

We can now introduce the dual scalar product \([ , ]\) (see 3.2):

**Lemma 21**  a. Let \( \Gamma \) be an orthonormal basis for \([ , ]\) that is also orthogonal for \( <, > \). The hypersum \( \sum_{\gamma \in \Gamma} \| \gamma \| \) is bounded – and we can thus apply theorem 14.

b. Let \( \phi \) be an element of \( F \) limited for the dual scalar product \([ , ]\). For every \( g \in G \) the hypercomplex \( < g, \phi > \) is limited and the function \( \phi^G : g \mapsto ^o( < g, \phi >) \) is a continuous antilinear form on \( G \).

**Consequences of lemma 21**

a. \( \eta_b = \frac{b}{\sqrt{\langle b,b \rangle}} \) is of norm 1 for the dual scalar product \([ , ]\) and defines therefore the continuous antilinear form \( \eta^G_b \) on \( G \).

b. Assume again theorem 14 notations. If \( h \) is an element of \( ^cF \) and a shadow of an element \( x \), we know that \( \nu_h \)-almost every \( \frac{b}{\langle x,b \rangle} \) is limited for \([ , ]\) and we have therefore a continuous antilinear form \( \phi^G_x : g \mapsto ^o( \frac{\langle g,b \rangle}{\langle x,b \rangle} ) \).

Consider an element \( h \) of \( G \). \( h \) is its own shadow. Since \\
\( \frac{b}{\langle h,b \rangle} = \frac{\sqrt{\langle h,b \rangle}}{\langle h,b \rangle} \cdot \frac{b}{\sqrt{\{b,b\}}} \) is limited for \([ , ]\) iff \( < h, \frac{b}{\sqrt{\{b,b\}}} > \) is not infinitesimal (i.e. if \( \eta^G_h(h) \neq 0 \)). And we have : \( \phi^G_b(b)(g) = \frac{\eta^G_b(g)}{\eta^G_b(h)} \).

**4.2 Main Theorem** Let \( G'_a \) be the antilinear dual of \( G \) (i.e. the space of continuous antilinear forms on \( G \)). \( G'_a \) is obviously a complex subspace of the space of complex functions defined on \( G \).

The function \( u \) admits an adjoint \( u^* \) which is the linear function from \( G'_a \) into itself defined by \( u^*(\phi)(g) = \phi(u(g)) \). We will call eigenfunctional of \( u \) any element of \( G'_a \) which is an eigenvector of \( u^* \). Those of \( \eta^G_b \) which
are not null are such eigenfuctionals of $u$ (for the eigenvalues $\lambda_b$).

We finally get to the last and main theorem of this work (with the same notations used so far):

**Theorem 22** Let $h$ be an element of $G$ and $B_h$ the set of $b \in B$ such that $\eta^G_h(h) \neq 0$.

a. The set $B_h$ is a borelian subset of $B$ whose complementary is of null $\sigma_h$-measure. For $b \in B_h$ the antilinear form $\eta^G_h$ is an eigenfuctional of $u$ for the eigenvalue $\lambda_b$.

b. The family $\left( \phi^G_h(b) = \frac{\eta^G_h(b)}{\eta^G_h(h)} \right), b \in B_h$, is weakly $\tau_\lambda$-integrable for $\sigma_h$-almost every $\lambda$.

c. For almost every $\lambda$ the weak integral $\omega_{h,\lambda} = \int_{B_h} \frac{1}{\eta^G_h(h)} \eta^G_h b d\tau_\lambda(b)$ is, if not null, an eigenfuctional of $u$ for the value $\lambda$.

d. Let $(E_\lambda)$ be the spectral projector family of the closure of $u$ in the completion of $G$ for $\langle \cdot,\rangle$ (see proposition 16). For any $g \in G$ we have $\langle\cdot, E_\lambda h \rangle = \int_{-\infty}^{\lambda} \omega_\mu(g) d\sigma_h(\mu)$

**Variant** – The antilinear form $g \mapsto \langle g, E_\lambda h \rangle$ associated to an element $E_\lambda h$ of the completion of $G$, is the weak integral $\int_{-\infty}^{\lambda} \omega_\mu d\sigma_h(\mu)$ where for almost every $\mu$ the antilinear form $\omega_\mu$ is, if not null, an eigenfuctional of $u$ for the eigenvalue $\mu$ of $u$.

**Remark 23** The statement c. of the preceding theorem is a nonstandard approach of the integral representation of the spectral projectors of an essentially selfadjoint operator with eigenfunctionals (see [Gelfand Chilov Tome 3 Chapter 4 n° 4.2 Theorem 1]).

The outcome of this approach is the description of these eigenfuctional $\omega_i$ with the eigenvectors of an extension $v$ of the considered operator $u$ in hyperfinite dimension. These eigenvectors do not have an immediate interpretation in the initial space $G$; but their normalised forms for the scalar product $\langle \cdot,\rangle$ define eigenfunctionals.

We aggregate through integration those eigenfunctionals which are associated to the same eigenvalue; and here again the Loeb’s technique comes to make things much easier.

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