The Gravitational Field of Massive Non-Charged Point Source in General Relativity

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Abstract

Utilizing various gauges of the radial coordinate we give a description of static spherically symmetric space-times with point singularity at the center and vacuum outside the singularity. We show that in general relativity (GR) there exist a two-parameters family of such solutions to the Einstein equations which are physically distinguishable but only some of them describe the gravitational field of a single massive point particle with nonzero bare mass \( M_0 \). In particular, the widespread Hilbert’s form of Schwarzschild solution, which depends only on the Keplerian mass \( M < M_0 \), does not solve the Einstein equations with a massive point particle’s stress-energy tensor as a source. Novel normal coordinates for the field and a new physical class of gauges are proposed, in this way achieving a correct description of a point mass source in GR. We also introduce a gravitational mass defect of a point particle and determine the dependence of the solutions on this mass defect. The result can be described as a change of the Newton potential \( \varphi_N = -G_N M/r \) to a modified one: \( \varphi_G = -G_N M/ (r + G_N M/c^2 \ln \frac{r}{r+M_0}) \) and a corresponding modification of the four-interval. In addition we give invariant characteristics of the physically and geometrically different classes of spherically symmetric static space-times created by one point mass. These space-times are analytic manifolds with a definite singularity at the place of the matter particle.

1 Introduction

According to the remarkable sentence by Poincaré, the real problems can never be considered as solved or unsolved ones. Instead, they are always only more or less solved [1].

It is hard to believe that more then 88 years after the pioneer article by Schwarzschild (1916) [2] on the gravitational field of a point particle with gravitational mass \( M \) there still exist serious open problems, despite of the large number of papers and books on this problem [3].

The well known Schwarzschild [2] metric in Hilbert (1917) [4] gauge:

\[
d s^2 = \left(1 - \frac{2M}{\rho}\right) dt^2 + \frac{d\rho^2}{1 - \frac{2M}{\rho}} - \rho^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]
solves the vacuum Einstein equations
\[ G^\nu_\mu = 0 \]
in the spherically symmetric static case.

It owns an event horizon at
\[ \rho = \rho_G = 2M \]
and a strong hidden singularity at
\[ \rho = 0. \]

The singularity at \( \rho = 0 \) does not describe a massive point particle with proper mass \( M_0 \), because it does not solve the Einstein equations
\[ G^\nu_\mu = \kappa T^\nu_\mu \]
in presence of matter with stress-energy tensor
\[ T^\nu_\mu \sim M_0 \delta(r). \]
Here \( \delta(r) \) is the 3D Dirac function, needed to describe the mass distribution of the point particle with proper mass \( M_0 \).

According to the widespread common opinion, one is not able to use such distributions in Einstein equations (1), because these are a nonlinear differential equations [5].

In his pioneering article Schwarzschild has used a different radial variable \( r \) and a different gauge for the spherically symmetric static metric
\[ ds^2 = g_{tt}(r)^2 dt^2 + g_{rr}(r)^2 dr^2 - \rho(r)^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]
Using the gauge
\[ \det ||g_{\mu\nu}|| = 1 \]
he had fixed the three unknown functions
\[ g_{tt}(r) = 1 - \frac{2M}{\rho(r)} > 0, \quad g_{rr}(r) = -1/g_{tt}(r) < 0, \quad \rho(r) \]

obtaining
\[ \rho(r) = \sqrt[3]{r^3 + \rho_G^3}. \]

This solution has no event horizon. Its peculiar feature is that it describes a point particle with zero radius, zero volume, but nonzero area \( A_{\rho} = 4\pi\rho_G^2 \).

These strange properties of the Schwarzschild solution have been discussed by Marcel Brillouin [6] in 1923 and by Georgi Manev [7] in 1934.

The quantity \( \rho \) has a clear geometrical and physical meaning:
- It is well known that \( \rho \) defines the area
\[ A_{\rho} = 4\pi\rho^2 \]
of a centered at \( r = 0 \) sphere with "area radius" \( \rho \) and the length of a big circle on it \( l_\rho = 2\pi \rho \).

- It measures, too, the curvature of the 4D-space-time and of the 3D-space - "curvature radius". In the spherically symmetric case:

\[
^{(4)}R = ^{(4)}R(\rho), \quad \text{and} \quad ^{(3)}R = ^{(3)}R(\rho).
\]

- One can refer to \( \rho \) as an optical "luminosity variable", because the luminosity \( L \) of the distant physical objects is reciprocal to \( A_\rho \):

\[
L \sim \frac{1}{\rho^2}
\]

In contrast, the physical and geometrical meaning of the coordinate \( r \) is not defined by the spherical symmetry of the problem and is unknown \textit{a priori}. The only clear thing is that its value \( r = 0 \) corresponds to the center of the symmetry, where one must place the physical source of the gravitational field.

2 Gauge Fixing in General Relativity

General relativity is a gauge theory. The fixing of the gauge in GR is described by a proper choice of the quantities

\[
\tilde{\Gamma}_\mu = -\frac{1}{\sqrt{|g|}}g_{\mu\nu}\partial_\lambda \left( \sqrt{|g|}g^{\lambda\nu} \right)
\]

in the 4D d’Alembert operator

\[
g^{\mu\nu}\nabla_\mu \nabla_\nu = g^{\mu\nu} \left( \partial_\mu \partial_\nu - \tilde{\Gamma}_\mu \partial_\nu \right).
\]

In our problem the choice of spherical coordinates and static metric dictates the form of three of them:

\[
\tilde{\Gamma}_t = 0, \quad \tilde{\Gamma}_\theta = -\cot \theta, \quad \tilde{\Gamma}_\phi = 0,
\]

but the function \( \rho(r) \) and, equivalently, the form of the quantity

\[
\tilde{\Gamma}_r = \left( \ln \left( \frac{\sqrt{-g_{rr}}}{\sqrt{g_{tt}} \rho^2} \right) \right)'
\]

are still not fixed. Here and further on, the prime denotes differentiation with respect to the variable \( r \).

We refer to the freedom of choice of the function \( \rho(r) \) as a \textit{rho-gauge freedom} in a large sense, and to the choice of the \( \rho(r) \) function as a \textit{rho-gauge fixing}.

The strong believe in the independence of the GR results in the choice of coordinates \( x \) in the space-time \( M^{(1,3)} \{ g_{\mu\nu}(x) \} \) predisposes us to a somewhat
light-head attitude of mind towards the choice of the coordinates for a given specific problem.

Indeed, it is obvious that physical results of any theory must not depend on the choice of the variables and, in particular, these results must be invariant under changes of coordinates. This requirement is a basic principle in GR. It is fulfilled for any already fixed mathematical problem.

Nevertheless, the change of the interpretation of the variables may change the formulation of the mathematical problem and thus, the physical results, because we are using the variables according to their meaning. For example, if we are considering the luminosity variable $\rho$ as a radial variable of the problem, it seems natural to put the point source at the point $\rho = 0$. In general, we may obtain a physically different model, if we are considering another variable $r$ as a radial one. In this case we shall place the source at a different geometrical point $r = 0$, which now seems to be the natural position for the center $C$.

The relation between these two geometrical "points" and between the corresponding physical models strongly depends on the choice of the function $\rho(r)$, i.e. on the radial gauge. Thus, applying the same physical requirements in different "natural" variables, we arrive at different physical theories, because we are solving EE under different boundary conditions, coded in corresponding Dirac $\delta$-functions. One has to find a theoretical or an experimental reasons to resolve this essential ambiguity.

We shall see, that the choice of the radial coordinate in the one particle problem in GR in the above sense is essential for the description of its gravitational field and needs a careful analysis. The different coordinates in a given frame are equivalent only locally. Especially, a well known mathematical fact is that in the vicinity of a definite singular points of mathematical functions one must use a definite special type of coordinates for the adequate description of the character of singularity.

In the literature one can find a large number of useful gauges:

- Schwarzschild gauge (1916) \[2\];
- Hilbert gauge (1917) \[4\];
- Droste gauge (1917) \[9\];
- Weyl gauge (1917) \[10\];
- Einstein-Rosen gauge (1935) \[11\];
- Isotropic gauge;
- Harmonic gauge;
- Pugachev-Gun’ko-Menzel gauge (1974-76) \[13\], e.t.c.

In the last case

$$\bar{\Gamma}_r = -\frac{2}{r}$$

in a complete coherent way with the flat space-time spherical coordinates. Then

$$ds^2 = e^{2\phi_N(r)} \left( dt^2 - \frac{dr^2}{N(r)} \right) - \frac{r^2}{N(r)^2} \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),$$

where $N(r) = \frac{r}{\rho G} \left( 1 - e^{-\frac{\rho G}{r}} \right)$. 

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It is remarkable that in this solution the exact classical Newton gravitational potential
\[ \varphi_N(r) = -\frac{G_N M}{r} \]
defines an exact GR result.

The component \( g_{tt} \) has an essentially singular point at \( r = 0 \) in the complex plane.

It is curious that a representation of the space-time metric of that type was considered by Georgi Manev [8] already in 1924, making use of definite physical motivation.

One has to stress two apparent facts:
1) An event horizon \( \rho_H \) exists in the physical domain only under Hilbert choice of the function \( \rho(r) \equiv r \), not in the other gauges, discussed above.
   This demonstrates that the existence of black holes in the theory strongly depends on this choice of the rho-gauge in a large sense.
2) The choice of the function \( \rho(r) \) can change drastically the character of the singularity at the place of the point source of the metric field in GR, because, actually, this way we are changing the corresponding boundary conditions for Einstein equations.

3 Normal Coordinates for Gravitational Field of a Point Particle in General Relativity

Let us represent the metric \( ds^2 \) of the problem at hand in a specific form:
\[ ds^2 = e^{2\varphi_1} dt^2 - e^{-2\varphi_1+4\varphi_2} dr^2 - \rho^2 e^{-2\varphi_1+2\varphi_2} (d\theta^2 + \sin^2 \theta d\phi^2) \tag{2} \]
where \( \varphi_1(r) \), \( \varphi_2(r) \) and \( \bar{\varphi}(r) \) are unknown functions of the variable \( r \) and \( \rho \) is a constant – the unit for luminosity distance \( \rho = \rho e^{-\varphi_1+\bar{\varphi}_2} \).

For the restriction of the gravitational action and the mechanical action on the orbits of the group \( SO(3) \otimes Tr_t \) one obtains:
\[ A_{GR} = \frac{1}{2G_N} \int dt \int dr \left( e^{\bar{\varphi}} (-(\bar{\rho}\bar{\varphi}_1')^2 + (\bar{\rho}\bar{\varphi}_2')^2) + e^{-\bar{\varphi}} e^{2\bar{\varphi}_2} \right), \]
\[ A_{M0} = -\int dt \int dr M_0 e^{\bar{\varphi}_1} \delta(r). \tag{3} \]

Thus we see that the field variables \( \varphi_1(r) \), \( \varphi_2(r) \) and \( \bar{\varphi}(r) \) play the role of a normal fields’ coordinates for our problem.

In these variables the field equations read:
\[ \bar{\Delta}_r \varphi_1(r) = \frac{G_N M_0}{\rho^2} e^{\bar{\varphi}_1(r)-\bar{\varphi}(r)} \delta(r), \]
\[ \bar{\Delta}_r \varphi_2(r) = \frac{1}{\rho^2} e^{2(\varphi_2(r)-\bar{\varphi}(r))} \tag{4} \]
where $\Delta_r = e^{-\bar{\varphi}} \frac{d}{dr} \left(e^{\bar{\varphi}} \frac{d}{dr}\right)$ is related with the radial part of the 3D-Laplacean.

The variation of the total action with respect to the auxiliary variable $\bar{\varphi}$ gives the constraint:

$$e^{\bar{\varphi}} \left( - \left( \bar{\rho} \varphi'_1 \right)^2 + \left( \bar{\rho} \varphi'_2 \right)^2 \right) - e^{-\bar{\varphi}} e^{2 \varphi_2} \equiv 0.$$  

(5)

4 Regular Gauges and General Regular Solutions of the Problem

The advantage of the above normal fields’ coordinates is that in them the field equations (4) are linear with respect to the derivatives of the unknown functions $\varphi_{1,2}(r)$. This circumstance legitimates the correct application of the mathematical theory of distributions [14] and makes our normal coordinates a privileged field variables.

The choice of the function $\bar{\varphi}(r)$ fixes the gauge in the normal coordinates. We have to choose this function in a way that makes the equations (4) with $\delta(r)$ function meaningful. We call this class a regular gauges. The simplest one is the basic regular gauge:

$$\bar{\varphi}(r) \equiv 0 \quad (or \quad \bar{\Gamma} = 0).$$

Other regular gauges describe diffeomorphysms of the fixed by the basic regular gauge manifold $\mathcal{M}^{(3)} \{g_{mn}(r)\}$.

Under this gauge the field equations (4) acquire the simplest (linear with respect to derivatives) form:

$$\varphi''_1(r) = \frac{G_N M_0}{\bar{\rho}^2} e^{\bar{\varphi}(0)} \delta(r),$$

(6)

$$\varphi''_2(r) = \frac{1}{\bar{\rho}^2} e^{2 \varphi_2(r)}.$$  

(7)

The constraint (5) acquires the simple form, too:

$$- \left( \bar{\rho} \varphi'_1 \right)^2 + \left( \bar{\rho} \varphi'_2 \right)^2 - e^{2 \varphi_2} \equiv 0.$$  

(8)

As one sees, the basic regular gauge $\bar{\varphi}(r) \equiv 0$ has the unique property to split into three independent relations the system of field equations (4) and the constraint (5).

An unexpected feature of this two parametric variety of solutions for the gravitational field of a point particle is that each solution must be considered only in a definite finite domain $r \in \left[0, r_\infty\right)$, if we wish to have a monotonic increase of the luminosity variable in the interval $[\rho_0, \infty)$. One easily can overcome this problem using the regular radial gauge transformation

$$r \rightarrow \tilde{r} = \frac{r/\bar{r}}{r/\bar{r} + 1}$$

(9)
with a proper scale $\hat{r}$ of the new radial variable $r$ (Note that in the present article we are using the same notation $r$ for different radial variables.) We call this new regular gauge a regular physical gauge.

The above linear fractional diffeomorphism, which carries us from basic to the physical regular gauge, does not change the number and the character of the singular points of the solutions in the whole compactified complex plane of the variable $r$. The transformation (6) simply places the point $r = r_\infty$ at infinity: $r = \infty$, at the same time preserving the initial place of the origin $r = 0$. Now the new variable $r$ varies in the standard interval $r \in [0, \infty)$ and the regular solutions acquire the final form. Under proper additional conditions we obtain the solution of the problem at hand outside the source in the form, similar to that one, written at first by Georgi Manev [8]:

$$
\frac{ds^2}{e^{2\varphi_G}} = \left(dt^2 - \frac{dr^2}{N_G(r)^2}\right) - \rho(r)^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right).
$$

Here we are using a modified (Newton-like) gravitational potential:

$$
\varphi_G(r; M, M_0) := -\frac{G_N M}{r + G_N M/\ln(M_0/M)},
$$

a coefficient

$$
N_G(r) = \left(2\varphi_G\right)^{-1}\left(e^{2\varphi_G} - 1\right),
$$

and an optical luminosity variable

$$
\rho(r) = \frac{2G_N M}{1 - e^{2\varphi_G}} = \frac{r + G_N M/\ln(M_0/M)}{N_G(r)}.
$$

As a result we reach the relations:

$$
\varphi_0(0) = e^{2\varphi_1(0)} = \left(\frac{M}{M_0}\right)^2 < 1.
$$

The ratio

$$
\varrho = \frac{M}{M_0} \in (0, 1)
$$

describes the gravitational mass defect of the point particle. This is the second physical parameter in the problem at hand.

The Keplerian mass $M$ and the ratio $\varrho$ define completely the solutions.

Outside of the source (for $r > 0$) the solutions coincide with solution in Hilbert gauge and strictly respect the Birkhoff theorem. One must apply the Birkhoff theorem only in the interval $\rho \in [\rho_0, \infty)$. It is remarkable that the minimal value of the luminosity variable is

$$
\rho_0 = \frac{2G_N M}{1 - \varrho^2} \geq \rho_G.
$$

This changes the Gauss theorem and leads to important physical consequences.
5 Total energy of a point source and its gravitational field

In the problem at hand we have an extreme example of an "island universe". In it a privileged reference system and a well defined global time exist. It is well known that under these conditions the energy of the gravitational field can be defined unambiguously [3]. Moreover, we can calculate the total energy of the aggregate of a mechanical particle and its gravitational field in a canonical way. Indeed, the canonical procedure produces a total Hamilton density

\[ H_{tot} = \sum_{a=1,2,\mu=t,r} \pi^a_\mu \varphi_{a,\mu} - L_{tot} = \frac{1}{2} G_N \left( -\bar{\rho}^2 \varphi'_1{}^2 + \bar{\rho}^2 \varphi'_2{}^2 - e^{2\varphi_2} \right) + M_0 e^{\varphi_1} \delta(r). \]

For the total energy of the aggregate of the point source and its gravitational field one obtains:

\[ E_{tot} = M_0 + E_{GR} = M = \varrho M_0 < M_0. \] (13)

This result follows from the relation \( E_{tot} = \int_0^\infty H_{tot} dr \) and completely agrees with the strong equivalence principle of GR.

The energy of the very gravitational field, created by the point particle, is a negative quantity:

\[ E_{GR} = E_{tot} - E_0 = M - M_0 < 0. \]

6 Local Singularities of the Point Sources

One can write down an exact relativistic Poisson equation:

\[ \Delta_r \varphi_1(r) = 4\pi G_N M_0 \delta_g(r) = \gamma(r) \delta(r). \] (14)

This is a specific realization of a Fock's (1964) idea [3] in our normal fields' coordinates.

Under diffeomorphisms of the space \( \mathcal{M}^{(3)} \{ g_{\mu\nu}(r) \} \) the singularities of the right hand side remain unchanged.

One can distinguish the physically different solutions of Einstein equations [4] by investigation of the asymptotic of the coefficient

\[ \gamma(r) = 1 / \left( 4\pi \rho(r)^2 \sqrt{-g_{rr}(r)} \right). \]

For our regular solutions the limit \( r \to +0 \) of this coefficient is a constant:

\[ \gamma(0) = \frac{1}{4\pi \rho G} \left( \frac{1}{\rho} - \varrho \right). \] (15)
For other solutions one obtains as follows:

Schwarzschild solution: \( \gamma_S(r) \sim \frac{1}{4\pi G} \left( \frac{\rho_G}{r} \right)^{1/2} \);

Hilbert solution: \( \gamma_H(r) \sim \frac{i}{4\pi G} \left( \frac{\rho_G}{r} \right)^{5/2} \);

Droste solution: \( \gamma_D(r) \sim \frac{1}{4\pi G} \);

Weyl solution: \( \gamma_W(r) \sim \frac{16}{\pi G} \left( \frac{r}{\rho_G} \right)^4 \);

Einstein-Rosen solution: \( \gamma_{ER}(r) \sim \frac{1}{4\pi G} \left( \frac{\rho_G}{r} \right)^{1/2} \);

Isotropic (t-r) solution: \( \gamma_{ER}(r) \sim \frac{1}{4\pi G} \left( \frac{r}{\rho_G} \right)^{3/2} \);

Pugachev-Gun’ko-Menzel solution: \( \gamma_{P GU}(r) \sim \frac{1}{4\pi G} \left( \frac{\rho_G}{r} \right)^{2/3} \).

As we see:

- Most of the listed solutions are physically different.
- Only two of them: Schwarzschild and Einstein-Rosen ones, have the same singularity at the place of the point source. As a result, these solutions describe diffeomorphic spaces \( \mathcal{M}^{(3)} \{ g_{mn}(r) \} \).
- As a result of the alteration of the physical meaning of the variable \( \rho = r \) inside the sphere of radius \( \rho_H \), in Hilbert gauge the coefficient \( \gamma(r) \) tends to an imaginary infinity for \( r \to 0 \). This means that the corresponding point source of Hilbert solution is space-like, instead of time-like. Such property is not allowable for a source of any physical field. This is in a sharp contrast to the real asymptotic of all other solutions in the limit \( r \to 0 \).

7 Scalar Invariants of the Riemann Tensor

The use of scalar invariants of the Riemann tensor allows a manifestly coordinate independent description of the geometry of space-time.

For our regular solutions one is able to derive the following simple invariants:

\[
I_1 = \frac{1}{8\rho_G^2} \frac{(1 - \rho^2)^4}{\theta^2} \delta \left( \frac{r}{\rho_G} \right) = \frac{\pi (1 - \rho^2)^3}{2\rho} \delta \left( \frac{r}{\rho_G} \right) = -\frac{1}{2} R(r),
\]

\[
I_2 = \frac{\Theta(r/\rho_G)^{-1}}{8\rho_G^2 \theta^2} \left( 1 - \rho^2 e^{4r/\rho_G} \right)^4 = \frac{\rho_G^2}{8\rho_G^2} \frac{\Theta(r/\rho_G)^{-1}}{\rho(r)^4},
\]

\[
I_3 = \frac{\Theta(r/\rho_G)}{4\rho_G^2} \left( 1 - \rho^2 e^{4r/\rho_G} \right)^3 = \frac{\rho_G}{4} \frac{\Theta(r/\rho_G)}{\rho(r)^3}.
\]

As seen:

- The invariants \( I_1, \ldots, I_3 \) of the Riemann tensor are a well defined distributions.
• Three of them are independent on the real axes $r \in (-\infty, \infty)$ and this is the true number of the independent invariants in the problem of the single point source of gravity.

• On the real physical interval $r \in [0, \infty)$ one has $I_2 = 0$ and we remain only with two independent invariants.

• For $r \in (0, \infty)$ the only independent invariant is $I_3$, as is well known from the case of Hilbert solution.

• The geometry of the space-time depends essentially on both of the two parameters, $M$ and $\varrho$, which define the regular solutions of Einstein equations.

8 Concluding Remarks

As we have shown, there exist infinitely many static solutions of Einstein equations of spherical symmetry, which describe a physically and geometrically different space-times.

Using a novel normal coordinates for gravitational field of a single point particle with bare mechanical mass $M_0$ we are able to describe correctly the massive static non-charged point source of gravity in GR.

It turns out that this problem has a two-parametric family of regular solutions, parameterized by Keplerian mass $M$ and by gravitational mass defect ratio $\varrho = \frac{M}{M_0} \in (0, 1)$.

For the regular solutions the physical values of the optical luminosity variable $\rho$ are in the semi-constraint interval $\rho \in \left[\frac{\rho_G}{1 - \varrho^2}, \infty\right)$.

Outside the source, i.e. for $\rho > \frac{\rho_G}{1 - \varrho^2}$, the Birkhoff theorem is strictly respected for all regular solutions.

For the class of our regular solutions the non-physical interval of the optical luminosity distance

$$\rho \in \left[0, \frac{\rho_G}{1 - \varrho^2}\right],$$

which includes the luminosity radius $\rho_H = \rho_G$ (the event horizon), is to be considered as an optical illusion.

For the regular solution the 3D-volume of a ball centered at the source, is

$$Vol(r_b) = \frac{4}{3} \pi \rho_G^3 \frac{12 \varrho}{(1 - \varrho^2)^2} \frac{r_b}{\rho_G} + O\left(\frac{r_b}{\rho_G}\right).$$

It goes to zero linearly with respect to the radius $r_b \to 0$.

Hence, a point source of gravity can be surrounded by a sphere with an arbitrary small volume $Vol(r_b)$ in it and with an arbitrary small radius $r_b$. In contrast, when $r_b \to 0$ the area of the ball’s surface has a finite limit:

$$\frac{4 \pi \rho_G^2}{(1 - \varrho^2)^2} > 4 \pi \rho_G^2,$$

and the radius of the big circles on this surface tends to a finite number $\frac{2 \pi \rho_G}{1 - \varrho^2} > 2 \pi \rho_G$. 

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Such unusual Riemannian geometry of the point source in GR may have an interesting physical consequences. It appeared at first in this problem in the original Schwarzschild article [2] and was discussed in different aspects by Marcel Brillouin [6] and by Georgi Manev [7].

As we see, in GR we are forced to introduce an essentially new notion for massive matter points, prescribing to them quite unusual geometrical properties. The geometry of space-time around such matter points must be essentially different then the geometry around the "empty" geometrical points, or around the point with finite density of mass-energy.

Further details and developments of present article can be found in [15].

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