\section{Introduction and preliminaries}

Let $A$, $B$ be two rings (algebras). An additive map $h : A \to B$ is called an $n$-Jordan homomorphism if $h(a^n) = (h(a))^n$ for all $a \in A$. Every Jordan homomorphism is an $n$-Jordan homomorphism, for all $n \geq 2$, but the converse is false in general. In this paper we investigate the $n$-Jordan homomorphisms on Banach algebras. Some results related to continuity are given as well.

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uniform algebra on a compact metric space, then there are exactly $2^{\text{Card}(\mathbb{C})}$ complex-valued ring homomorphisms on $A$ whose kernels are nonmaximal prime ideals (see [4, Corollary 2.4]). As an example, take

$$A := \begin{bmatrix} 0 & \mathbb{R} & \mathbb{R} \\ 0 & 0 & \mathbb{R} \\ 0 & 0 & 0 & \mathbb{R} \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

then $A$ is an algebra equipped with the usual matrix-like operations. It is easy to see that

$$A^3 \neq 0 = A^4.$$ 

So any additive map from $A$ into itself is a 4-Jordan homomorphism, but its kernel does not need to be an ideal of $A$. Now let $B$ be the algebra of all $A$-valued continuous functions from $[0, 1]$ into $A$ with supremum norm. Then $B$ is an infinite-dimensional Banach algebra, and the product of any four elements of $B$ is 0. Since $B$ is infinite-dimensional, there are linear discontinuous maps which are 4-Jordan homomorphisms from $B$ into itself (see [3]). In this paper we study the continuity of linear $n$-Jordan homomorphisms on $C^*$-algebras.

2. Main result

By definition, it is obvious that $n$-ring homomorphisms are $n$-Jordan homomorphisms. Conversely, under a certain condition, $n$-Jordan homomorphisms are ring homomorphisms. For example, each Jordan homomorphism $h$ from a commutative Banach algebra $A$ into $\mathbb{C}$ is a ring homomorphism: Fix $a, b \in A$ arbitrarily. Since $h((a + b)^2) = h(a + b)^2$ a simple calculation shows that $h(ab + ba) = 2h(a)h(b)$. The commutativity of $A$ implies that $h(ab) = h(a)h(b)$ and hence $h$ is a ring homomorphism. In 1968, Zelazko [8] proved the following theorem (see also [5, Theorem 1.1]).

**Theorem 2.1.** Suppose that $A$ is a Banach algebra, which need not be commutative, and suppose that $B$ is a semisimple commutative Banach algebra. Then each Jordan homomorphism $h : A \rightarrow B$ is a ring homomorphism.

We prove the following result for 3-Jordan homomorphisms and 4-Jordan homomorphisms on commutative algebras.

**Theorem 2.2.** Let $n \in \{3, 4\}$ be fixed, $A, B$ be two commutative algebras, and let $h : A \rightarrow B$ be an $n$-Jordan homomorphism. Then $h$ is an $n$-ring homomorphism.

**Proof.** First, let $n = 3$. Recall that $h$ is additive mapping such that $h(a^3) = (h(a))^3$ for all $a \in A$. Replacement of $a$ by $x + y$ results in

$$h(x^2 y + xy^2) = h(x)^2 h(y) + h(x)h(y)^2.$$  \hfill (2.1)
Hence, for every \( x, y, z \in A \),

\[
h(xyz) = \frac{1}{2} h((x + z)^2y + (x + z)y^2 - (x^2y + xy^2 + z^2y + zy^2))
\]

\[
= \frac{1}{2} [h((x + z)^2y + (x + z)y^2] - h(x^2y + xy^2] - h(z^2y + zy^2)]
\]

\[
= \frac{1}{2} [\{ h((x + z)^2(y) + (x + z)[h(y)]^2 - h(x)^2h(y) + h(x)[h(y)]^2
\]

\[
- [h(z)^2h(y) + h(z)[h(y)]^2]
\]

\[
= h(x)h(y)h(z).
\]

This means that \( h \) is a 3-ring homomorphism. Now suppose that \( n = 4 \). Then \( h \) is additive and \( h(a^4) = (h(a))^4 \) for all \( a \in A \). Replace \( a \) by \( x + y \) in the equality above to get

\[
h(4x^3y + 6x^2y^2 + 4xy^3) = 4h(x)^3h(y) + 6h(x)^2h(y)^2 + 4h(x)h(y)^3.
\]

(2.2)

Replacing \( x \) by \( x + z \) in (2.2), we obtain

\[
h((4x^3y + 6x^2y^2 + 4xy^3) + (4x^3y + 6x^2y^2 + 4xy^3) + 12(x^2zy + xz^2y + xzy^2))
\]

\[
= (4h(x)^3h(y) + 6h(x)^2h(y)^2 + 4h(x)h(y)^3) + (4h(z)^3h(y) + 6h(z)^2h(y)^2
\]

\[
+ 4h(z)h(y)^3) + 12(h(x)^2h(z)h(y) + h(x)h(z)^2h(y)
\]

\[
+ h(x)h(z)h(y)^2).
\]

(2.3)

Combining (2.2) and (2.3) gives

\[
h(xyz)(x + y + z) = (h(x)h(y)h(z))(h(x) + h(y) + h(z)).
\]

(2.4)

Replace \( z \) by \(-x\) in (2.4) to obtain

\[
h(x^2y^2) = h(x)^2h(y)^2
\]

(2.5)

and replace \( y \) by \( y + w \) in (2.5) to get

\[
h(x^2yw) = h(x)^2h(y)h(w).
\]

(2.6)

Now replace \( x \) by \( x + t \) to obtain

\[
h(xtyw) = h(x)h(t)h(y)h(w).
\]

Hence, \( h \) is a 4-ring homomorphism. \( \Box \)

By Theorem 2.2 and [1, Theorem 3.2] we deduce the following result.

**Corollary 2.3.** Let \( h : A \to B \) be a linear involution preserving 3-Jordan homomorphism between commutative C*-algebras. Then \( h \) is norm contractive (that is, \( \|h\| \leq 1 \)).

Also, by Theorem 2.2 and [7, Theorem 2.3], we have the following corollary.
Corollary 2.4. Let $h : A \rightarrow B$ be a linear involution preserving 4-Jordan homomorphism between commutative $C^*$-algebras; then $h$ is completely positive. Thus $h$ is bounded.

Now we prove our main theorem.

Theorem 2.5. Suppose that $A$ is a Banach algebra, which need not be commutative, and suppose that $B$ is a semisimple commutative Banach algebra. Then each 3-Jordan homomorphism $h : A \rightarrow B$ is a 3-ring homomorphism.

Proof. We prove the theorem in two steps as follows.

Step I. Suppose $B = C$. We have $h(a^3) = h(a)^3$ for all $a \in A$. Replace $a$ by $x + y$ to obtain

$$h(xy y + yx^2 + y^2 x + x^2 y + xy^2 + yxy) = 3(h(x)^2 h(y) + h(x)h(y)^2)$$

(2.7)

and replace $y$ by $-y$ in (2.7) to get

$$h(-xy x - yx^2 + y^2 x - x^2 y + xy^2 + yxy) = 3(-h(x)^2 h(y) + h(x)h(y)^2).$$

(2.8)

By (2.7) and (2.8) we obtain the relation

$$h(xy y^2 + y^2 x + yxy) = 3(h(x)h(y)^2).$$

(2.9)

Replacing $y$ by $y - z$ in (2.9), we get

$$h(xy y^2 + xz^2 - 2xy z + yxy - yxz - zxy + zxz + z^2 x + y^2 x - 2yzx)$$

$$= 3(h(x)^2 h(y) + h(x)h(y)^2) - 6h(x)h(y)h(z).$$

(2.10)

By (2.9) and (2.10), we obtain

$$h(yxz + zxy + 2xyz + 2yzx) = 6h(x)h(y)h(z).$$

(2.11)

Replacing $z$ by $x$ in (2.11), we get

$$h(3y x^2 + x^2 y + 2y x x) = 6h(x)^2 h(y),$$

(2.12)

and combining (2.9) and (2.12), we obtain

$$h(xy x + 2y x^2) = 3h(x)^2 h(y).$$

(2.13)

From (2.8) and (2.13), we conclude that

$$h(yx^2 - x^2 y) = 0.$$  

(2.14)

Replacing $x$ by $x + z$ in (2.14), we get

$$h(yx^2 + yz^2 + 2yx z - x^2 y - z^2 y - 2x z y) = 0.$$
and from this equality and (2.14) it follows that
\[ h(yxz - xzy) = 0. \] (2.15)

Combining (2.11) and (2.15) gives
\[ h(yxz + 3xyz + 2yzx) = 6h(x)h(y)h(z), \] (2.16)
and then replacing \( z \) by \( x \) in (2.16) leads to
\[ h(xy + xy^2) = 2h(x)^2h(y). \] (2.17)

Finally, combining (2.13) and (2.17) to obtain
\[ h(yx^2) = h(y)h(x)^2 \] (2.18)
and then replacing \( x \) by \( x + z \) in (2.18), we conclude that
\[ h(yxz) = h(y)h(x)h(z); \]
hence, \( h \) is a 3-ring homomorphism.

**Step II.** \( B \) is arbitrary semisimple and commutative. Let \( M_B \) be the maximal ideal space of \( B \). We associate with each \( f \in M_B \) a function \( h_f : A \to \mathbb{C} \) defined by
\[ h_f(a) := f(h(a)) \]
for all \( a \in A \). It is easy to see that \( h_f \) is additive and \( h_f(a^3) = (h_f(a))^3 \) for all \( a \in A \). So step I applied to \( h_f \) implies that \( h_f \) is a 3-ring homomorphism. By the definition of \( h_f \), we obtain that
\[ f(h(abc)) = f(h(a))f(h(b))f(h(c)) = f(h(a)h(b)h(c)). \]

Hence
\[ h(abc) - h(a)h(b)h(c) \in \text{Ker}(f) \]
for all \( a, b, c \in A \) and all \( f \in M_B \). Since \( B \) is assumed to be semisimple, we get \( h(abc) = h(a)h(b)h(c) \) for all \( a, b, c \in A \). We thus conclude that \( h \) is a 3-ring homomorphism, and the proof is complete.

From now on we consider such \( n \)-Jordan homomorphisms as are linear.

**Corollary 2.6.** Suppose that \( A, B \) are \( C^* \)-algebras, where \( A \) need not be commutative, and suppose that \( B \) is semisimple and commutative. Then every involution preserving 3-Jordan homomorphism \( h : A \to B \) is norm contractive (that is, \( \|h\| \leq 1 \)).

**Proof.** It follows from Theorem 2.5 and [1, Theorem 2.1].
THEOREM 2.7. Let $h : A \rightarrow B$ be a bounded involution preserving $k$-Jordan homomorphism between $C^*$-algebras such that $h(a^*a) = h(a)^*h(a)$ for all $a \in A$. Then $h$ is norm contractive (that is, $\|h\| \leq 1$).

PROOF. From [7, Lemma 2.4],

$$\|h(a)\|^{4k+2} = \|(h(a)^*h(a))^{2k+1}\| = \|(h(a)^*h(a))^k(h(a)^*h(a))(h(a)^*h(a))^k\|$$

$$= \|[(h(a)(h(a)^*h(a))^k]h(a)(h(a)^*h(a))^k]\|$$

$$= \|h(a)(h(a)^*h(a))^k\|^2 = \|h(a)(h(a)^*h(a))^k\|^2$$

$$= \|h(a)(h(a)^*h(a))^k\|^2 \leq \|h(a)\|^2 \|h((a^*a)^k)\|^2$$

$$\leq \|h\|^2 \|a\|^2 \|h\|^2 (a^*a)^k \|^2$$

$$\leq \|h\|^4 \|a\|^{4k+2},$$

for all $a \in A$, which implies that $\|h\| \leq 1$ by taking $(4k + 2)$th roots. \qed

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